Modification of Simpson moduli spaces of 1-dimensional sheaves by vector bundles, an experimental example

Oleksandr Iena

Vom Fachbereich Mathematik der Technischen Universität Kaiserslautern
zur Verleihung des akademischen Grades Doktor der Naturwissenschaften
(Doctor rerum naturalium, Dr. rer. nat.)
genehmigte Dissertation

1. Gutachter: Prof. Dr. Günther Trautmann
2. Gutachter: Prof. Dr. Alexander Tikhomirov

Vollzug der Promotion 07.05.2009

D 386

Oleksandr Iena. *"Modification of Simpson moduli spaces of* 1*-dimensional sheaves by vector bundles, an experimental example".*

Abstract. This thesis deals with the following question. Given a moduli space of coherent sheaves on a projective variety with a fixed Hilbert polynomial, to find a natural construction that replaces the subvariety of the sheaves that are not locally free on their support (we call such sheaves singular) by some variety consisting of sheaves that are locally free on their support. We consider this problem on the example of the coherent sheaves on $\mathbb{P}_2$ with Hilbert polynomial $3m+1$.

Given a singular coherent sheaf $\mathcal{F}$ with singular curve C as its support we replace $\mathcal{F}$ by locally free sheaves $\mathcal{E}$ supported on a reducible curve $C_0 \cup C_1$, where C_0 is a partial normalization of C and C_1 is an extra curve bearing the degree of $\mathcal{E}$. These bundles resemble the bundles considered by Nagaraj and Seshadri (cf. [19], [20], [26]). Many properties of the singular $3m+1$ sheaves are inherited by the new sheaves we introduce in this thesis (we call them R-bundles). We consider R-bundles as natural replacements of the singular sheaves.

R-bundles refine the information about $3m+1$ sheaves on $\mathbb{P}_2$. Namely, for every isomorphism class of singular $3m+1$ sheaves there are $\mathbb{P}_1$ many equivalence classes of R-bundles.

There is a variety $\widetilde{M}$ of dimension 10 that may be considered as the space of all the isomorphism classes of the non-singular $3m+1$ sheaves on $\mathbb{P}_2$ together with all the equivalence classes of all R-bundles. This variety is obtained by blowing up the moduli space of $3m+1$ sheaves on $\mathbb{P}_2$ along the subvariety of singular sheaves.

We modify the definition of a $3m+1$ family and obtain a notion of a new family over an arbitrary variety S. In particular $3m+1$ families of the non-singular sheaves on $\mathbb{P}_2$ are families in this sense. New families over one point are either non-singular $3m+1$ sheaves or R-bundles. For every variety S we introduce an equivalence relation on the set of all new families over S. The notion of equivalence for families over one point coincides with isomorphism for non-singular $3m+1$ sheaves and with equivalence for R-bundles.

We obtain a moduli functor $\widetilde{\mathcal{M}} : (\mathrm{Sch}) \to (\mathrm{Sets})$ that assigns to every variety S the set of the equivalence classes of the new families over S. There is a natural transformation of functors $\widetilde{\mathcal{M}} \to \mathcal{M}$ that establishes a relation between $\widetilde{\mathcal{M}}$ and the moduli functor $\mathcal{M}$ of the $3m+1$ moduli problem on $\mathbb{P}_2$. There is also a natural transformation $\widetilde{\mathcal{M}} \to \mathrm{Hom}(\_\ , \widetilde{M})$, inducing a bijection $\widetilde{\mathcal{M}}(\mathrm{pt}) \cong \widetilde{M}$, which means that $\widetilde{M}$ is a coarse moduli space of the moduli problem $\widetilde{\mathcal{M}}$.

Oleksandr Iena. *"Modifizierung von Simpson-Modulräumen* 1*-dimensionaler Garben durch Vektorbündel, ein experimentelles Beispiel".*

Zusammenfassung. In dieser Dissertation wird die folgende Frage erörtert. Gegeben sei ein Modulraum von kohärenten Garben auf einer projektiven Varietät mit festem Hilbertpolynom, zu finden ist eine natürliche Konstruktion, die die Untervarietät der Garben, die nicht lokal frei auf ihrem Träger sind (solche Garben nennen wir singulär), durch eine andere, aus lokal freien Garben bestehende Varietät ersetzt. Wir betrachten diese Frage am Beispiel der kohärenten Garben auf $\mathbb{P}_2$ mit Hilbertpolynom $3m+1$.

Sei $\mathcal{F}$ eine singuläre kohärente Garbe mit singulärer Kurve C als Träger. Wir ersetzen $\mathcal{F}$ durch 1-dimensionale lokal freie Garben $\mathcal{E}$, deren Träger eine reduzible Kurve $C_0 \cup C_1$ ist, so dass C_0 eine partielle Normalisierung von C ist und C_1 eine zusätzliche, den Grad von $\mathcal{E}$ tragende Kurve ist. Diese Vektorbündel ähneln den von Nagaraj und Seshadri betrachteten Vektorbündeln (siehe [19], [20], [26]). Die in dieser Dissertation eingeführten neuen Garben (wir nennen sie R-Bündel) behalten viele Eigenschaften der singulären $3m+1$ Garben. Wir betrachten R-Bündel als einen natürlichen Ersatz für die singulären Garben.

R-Bündel präzisieren die Informationen über $3m+1$ Garben auf $\mathbb{P}_2$. Es gibt nämlich $\mathbb{P}_1$ viele verschiedene Äquivalenzklassen für jede Isomorphieklasse von singulären $3m+1$ Garben.

Es gibt eine Varietät $\widetilde{M}$ der Dimension 10, die als Raum aller Isomorphieklassen der nicht singulären Garben und aller Äquivalenzklassen von R-Bündeln betrachtet werden kann. Diese Varietät entsteht durch die Aufblasung des Modulraums von $3m+1$ Garben auf $\mathbb{P}_2$ entlang der Untervarietät der singulären Garben.

Wir modifizieren die Definition einer $3m+1$ Familie und bekommen für jede Varietät S einen neuen Begriff einer Familie über S. $3m+1$ Familien der nicht singulären Garben auf $\mathbb{P}_2$ sind Familien dieser Art. Neue Familien über einem Punkt sind entweder nicht singuläre $3m+1$ Garben oder R-Bündel. Für jede Varietät S wird auf der Menge aller R-Bündel über S eine Äquivalenzrelation eingeführt. Der Äquvalenzbegriff für die Familien über einem Punkt stimmt mit dem Isomorphiebegriff für nicht singuläre $3m+1$ Garben und mit dem Äquivalenzbegriff für R-Bündel überein.

Wir konstruieren einen Modulfunktor $\widetilde{\mathcal{M}} : (\mathrm{Sch}) \to (\mathrm{Sets})$, der jeder Varietät S die Menge der Äquivalenzklassen von R-Bündeln über S zuordnet. Eine Beziehung zwischen $\widetilde{\mathcal{M}}$ und dem Modulfunktor $\mathcal{M}$ des $3m+1$ Modulproblems auf $\mathbb{P}_2$ wird durch eine natürliche Transformation der Funktoren $\widetilde{\mathcal{M}} \to \mathcal{M}$ festgelegt. Es gibt auch eine natürliche Transformation $\widetilde{\mathcal{M}} \to \mathrm{Hom}(\_\,, \widetilde{M})$, die eine Bijektion $\widetilde{\mathcal{M}}(\mathrm{pt}) \cong \widetilde{M}$ induziert, was $\widetilde{M}$ zu einem groben Modulraum des Modulproblems $\widetilde{\mathcal{M}}$ macht.

Acknowledgements

I thank my adviser Prof. Dr. Günther Trautmann for providing me with the theme of this dissertation. The results of this thesis would not have been possible without his support, advices, and our fruitful discussions. I wish to thank him for being my teacher.

Contents

Basic notations

$\mathrm{Bl}_Y X$	blow up of X at $Y \subseteq X$
$\det A$, $\lvert A\rvert$	determinant of A
A^{T}	transpose of A
ev	evaluation morphism
$\mathrm{GL}_m(\Bbbk)$	group of invertible $m \times m$ matrices over $\Bbbk$
$J(a_1, \ldots, a_m)$	Jacobian matrix of polynomials $a_1, \ldots, a_m$
$\Bbbk$	base field
$\mathrm{Mat}_{m\times n}(\Bbbk)$	set of all $m \times n$ matrices over $\Bbbk$
$N_{Y/X}$	normal bundle to a subvariety Y in X
N_x, $N_x Y$	normal space to a subvariety Y in X at a point $x \in Y$
$\mathcal{O}_X$	structure sheaf of X
$\mathrm{Pic}\, X$	Picard group of X
$\mathbb{P}_n$	projective space of dimension n
$\mathbb{P}V$	projective space associated to a vector space V
$\mathbb{P}\mathcal{E}$	projective bundle associated to a vector bundle $\mathcal{E}$ (classical notation)
$\mathbf{P}\mathcal{E}$	projective bundle associated to a quasi-coherent sheaf $\mathcal{E}$ (Grothendieck's notation)
(Sets)	category of sets
(Sch)	category of separated schemes of finite type over $\Bbbk$
$\mathrm{Sing}\, \mathcal{F}$	set (closed subvariety) of $x \in X$ where $\mathcal{F}_x$ is not a locally free $\mathcal{O}_{X,x}$ module
$\mathrm{Span}(W)$	subspace generated by a subset W of a vector space V
$\mathrm{Supp}\, \mathcal{F}$	support of a sheaf $\mathcal{F}$
$T_x X$	tangent space at $x \in X$ in X
$\mathrm{T}\, X$	tangent bundle of X
$\rightarrowtail$, $\hookrightarrow$	injective morphisms
$\twoheadrightarrow$	surjective morphisms

Introduction

Important conventions

In this dissertation $\Bbbk$ is an algebraically closed field of characteristic zero.

We work in the category of separated schemes of finite type over $\Bbbk$. We denote this category by (Sch). The objects of this category are referred to as schemes or varieties interchangeably. We consider only closed points of them. Note that we do not restrict ourselves to reduced or irreducible varieties. All the schemes in (Sch) are automatically noetherian.

Dealing with homomorphism between vector bundles and identifying them with matrices we consider the matrices acting on elements from the right, i. e, the composition $X \xrightarrow{A} Y \xrightarrow{B} Z$ is given by the matrix $A \cdot B$.

Some historical remarks and general references

Classification is one of the important problems mathematics deals with. It is often useful to have a geometrical structure on the space of objects to be classified. This way one comes to the notion of a moduli space. It was Riemann who already studied moduli of curves.

The study of moduli spaces of sheaves on curves began in Atiyah's paper [2]. Narasimhan, Seshadri, Ramanan, and many others studied the moduli of sheaves on Riemann surfaces (cf. [22], [26], and [21]).

Takemoto in [28] and [29], Gieseker in [7], and Maruyama in [15], [16], and [17] started the study of moduli spaces of semi-stable sheaves on higher-dimensional varieties. Their constructions were improved by Simpson.

Simpson showed in [27] that for an arbitrary smooth projective variety $\mathfrak{X}$ and for an arbitrary numerical polynomial $P \in \mathbb{Q}[m]$ there is a coarse moduli space $M_P(\mathfrak{X})$ of semi-stable sheaves on $\mathfrak{X}$ with Hilbert polynomial P. The result of Simpson is an existence result. It is not much known about the structure of $M_P(\mathfrak{X})$ for concrete $\mathfrak{X}$ and P.

For $\mathfrak{X} = \mathbb{P}_2$ and for linear polynomial P the spaces $M_{am+b}(\mathbb{P}_2)$ were studied in [14]. Moduli spaces $M_{3m+1}(\mathbb{P}_3)$ and $M_{3m+1}(\mathbb{P}_2)$ have been described in [5] and [4].

The modern formulation of a moduli problem in terms of moduli functors is due to Grothendieck (cf. [10]). Dealing with moduli spaces requires techniques from the geometric invariant theory. The main reference on this subject is [18]. Newstead's book [23] may also be useful. A nice overview of the theory of moduli spaces on surfaces is presented by Huybrechts and Lehn in [13].

Initially posed problem

In general $M_P(\mathfrak{X})$ contains isomorphism or S-equivalence classes of sheaves that are not locally free on their support. Since locally free sheaves are more convenient to work with, it seems natural to ask for a construction which gives a natural possibility to replace sheaves that are not locally free on their support by sheaves that are locally free on their support. This dissertation aims to present such a construction for the moduli space $M_{3m+1}(\mathbb{P}_2)$. Initially the following questions have been posed to the author.

- To find sheaves that are locally free on their one-dimensional support (new sheaves) and could be considered as natural replacements of singular $3m+1$ sheaves on $\mathbb{P}_2$.
- To describe the isomorphism classes of the new sheaves and to find if possible a parameter space for them.
- To replace the notion of a Simpson $3m+1$ family on $\mathbb{P}_2$ by a new one, where the singular $3m+1$ sheaves on $\mathbb{P}_2$ are replaced by new sheaves, i. e., to define the corresponding moduli problem (the corresponding functor).
- To study the relations between the new moduli problem and the Simpson moduli problem for $3m+1$ sheaves on $\mathbb{P}_2$.
- To investigate the question concerning the existence of a moduli space (fine or coarse) for the new moduli problem.
- To see how the Simpson moduli space $M = M_{3m+1}(\mathbb{P}_2)$ is related to this question.

Structure of the dissertation

In **Chapter 1** we construct R-bundles and discuss their properties. In Section 1.1 we make an overview of some results from [5] and prove some important statements about $3m+1$ sheaves on $\mathbb{P}_2$. In Section 1.2 we consider a construction of R-bundles. New objects related to the construction of the R-bundles are studied in Section 1.3.

In **Chapter 2** we describe R-bundles up to isomorphisms (Section 2.1). We introduce also a notion of equivalence for R-bundles and give the description of the equivalence classes in Section 2.2. Section 2.3 is intended to illustrate the results of this chapter and to develop some intuition in dealing with R-bundles.

In **Chapter 3** families over arbitrary varieties are defined. In Section 3.1 we consider spaces $\widetilde{X}$ and $\widetilde{M}$ that parameterize R-bundles and study their properties. Further we construct a family over $\widetilde{X}$ in Section 3.2. Section 3.3 contains a definition of a family over an arbitrary S. Some properties of such families are studied.

Appendix A contains some general results that are used in this dissertation. Section A.1 presents some statements about flatness and base change, in Section A.2 we collect some facts about blow ups. Section A.3 deals with conics in $\mathbb{P}_2$ and their relation to $2m+2$ sheaves on $\mathbb{P}_2$. In Section A.4 we discuss some questions concerning gluing locally free sheaves.

Overview of results

Recalling some results from [5]. We consider semi-stable sheaves on $\mathbb{P}_2$ with Hilbert polynomial $3m+1$ and call them simply $3m+1$ sheaves. A family of $3m+1$ sheaves over S (or simply a $3m+1$ family over S) is by definition a flat sheaf $\mathcal{F}$ on $S \times \mathbb{P}_2$ such that for every $s \in S$ the restriction $\mathcal{F}_s$ of $\mathcal{F}$ to the fibre $\{s\} \times \mathbb{P}_2 \cong \mathbb{P}_2$ is a $3m+1$ sheaf on $\mathbb{P}_2$. It is known that all $3m+1$ sheaves are stable and there exists a fine moduli space $M = M_{3m+1}(\mathbb{P}_2)$

of $3m+1$ sheaves. The latter means that M represents the functor $\mathcal{M}$ of this moduli problem. One sees that $3m+1$ sheaves on $\mathbb{P}_2$ occur exactly as the non-split extensions

$$0 \to \mathcal{O}_C \to \mathcal{F} \to \Bbbk_p \to 0,$$

where $C = \operatorname{Supp} \mathcal{F}$ is the cubic curve in $\mathbb{P}_2$ supporting $\mathcal{F}$ and p is a point on C. The moduli space M is isomorphic to the universal cubic curve

$$\{(\langle f \rangle, \langle x \rangle) \in \mathbb{P}_9 \times \mathbb{P}_2 \mid f(x) = 0\},$$

where $\mathbb{P}_9$ is identified with the space of cubic curves in $\mathbb{P}_2$.

The $3m+1$ sheaves on $\mathbb{P}_2$ are exactly the sheaves given by resolutions

$$0 \to 2\mathcal{O}_{\mathbb{P}_2}(-2) \xrightarrow{A} \mathcal{O}_{\mathbb{P}_2}(-1) \oplus \mathcal{O}_{\mathbb{P}_2} \to \mathcal{F} \to 0,$$

where $A = \begin{pmatrix} z_1 & q_1 \\ z_2 & q_2 \end{pmatrix}$ with linear independent linear forms $z_1, z_2 \in \Gamma(\mathbb{P}_2, \mathcal{O}_{\mathbb{P}_2}(1))$ and non-zero determinant. The space of all such matrices is a parameter space of M and is denoted by X. X is isomorphic to an open subset in $\Bbbk^{18}$ and is acted on by the group $G = \mathrm{GL}_2(\Bbbk) \times H$, where H is the group of 2×2 matrices

$$\begin{pmatrix} \lambda & z \\ 0 & \mu \end{pmatrix}, \quad \lambda, \mu \in \Bbbk, \quad \lambda\mu \neq 0, \quad z \in \Gamma(\mathbb{P}_2, \mathcal{O}_{\mathbb{P}_2}(1)).$$

The action is defined by the rule $(g, h) \cdot A = gAh^{-1}$. M is a geometric quotient of X by G, the quotient morphism is

$$A = \begin{pmatrix} z_1 & q_1 \\ z_2 & q_2 \end{pmatrix} \mapsto \langle \det A \rangle \times \langle z_1 \wedge z_2 \rangle.$$

Singular objects. A $3m+1$ sheaf is called singular if it is not locally free on its support. A point $\langle f \rangle \times p \in M$ represents an isomorphism class of a singular sheaf if and only if p is a singular point of the curve $\{f = 0\} \subseteq \mathbb{P}_2$. The subvariety of all singular sheaves in M is denoted by M_8. The corresponding subvariety in X is denoted by X_8. Both M_8 and X_8 are smooth of codimension 2 in M and X respectively. Moreover X_8 is a global complete intersection in X (cf. Lemma 1.7).

New objects. We replace every singular sheaf $\mathcal{F}$ supported on a cubic curve C by invertible sheaves on curves of the type $C_0 \cup C_1$, where C_0 is a partial normalization of C. We call the new objects ***R*-bundles**. They occur as flat limits of non-singular $3m+1$ sheaves.

Considering invertible sheaves supported on C_0 is not enough since they will not be flat limits of $3m+1$ sheaves. So the curve C_1 is important as it guarantees flatness of families of sheaves that have as their fibres either R-bundles or non-singular $3m+1$ sheaves.

Since $3m+1$ sheaves come together with an embedding of the supporting curve into $\mathbb{P}_2$, our construction of R-bundles comes together with an embedding of the curve $C_0 \cup C_1$ into a reducible surface $D_0 \cup D_1$ containing two irreducible components D_0 and D_1. This surface is a flat limit of $\mathbb{P}_2$.

Construction of new objects. We consider the singular $3m+1$ sheaves as one-dimensional flat limits of non-singular $3m+1$ sheaves and describe a construction that substitutes the singular sheaves by sheaves that are locally free on their support and that are also flat limits of non-singular $3m+1$ sheaves.

Let us outline some details. For a matrix $A \in X_8$ representing a singular $3m+1$ sheaf on $\mathbb{P}_2$ and for $B \in T_A X$ we consider an open set $U \subseteq \Bbbk^1$ containing 0 such that $A + tB \in X$ for all t from U. We obtain the one parameter family of $3m+1$ sheaves given by the resolution

$$0 \to 2\mathcal{O}_{U\times\mathbb{P}_2}(-2H) \xrightarrow{A+tB} \mathcal{O}_{U\times\mathbb{P}_2}(-H) \oplus \mathcal{O}_{U\times\mathbb{P}_2} \to \mathcal{E} \to 0,$$

where the divisor H is the pull back of a line $h \subseteq \mathbb{P}_2$.

Let $Z \xrightarrow{\sigma} U \times \mathbb{P}_2$ be the blow up of $U \times \mathbb{P}_2$ at the point $0 \times p$, where $p \in \mathbb{P}_2$ is the point where the $3m+1$ sheaf $\mathcal{E}_0 = \mathcal{E}|_{\{0\}\times\mathbb{P}_2}$ given by the matrix A is not free on its support. Let D_1 be the exceptional divisor of σ. We obtain that the composed morphism $Z \xrightarrow{\sigma} U \times \mathbb{P}_2 \xrightarrow{p_1} U$ is flat.

The fibres Z_t of $Z \to U$ are isomorphic to $\mathbb{P}_2$ for $t \neq 0$, and $Z_0 \cong \hat{\mathbb{P}}_2 := D_0 \cup D_1$, where $D_0 = \widetilde{\mathbb{P}}_2$ is the blowing up of the projective plane $\mathbb{P}_2$ at the point p, and $D_1 = \mathbb{P}_2$ is attached to $\widetilde{\mathbb{P}}_2$ along the exceptional divisor L of the blowing up $\widetilde{\mathbb{P}}_2 \to \mathbb{P}_2$ (cf. Definition 1.11). The fibres Z_t can be considered as closed subvarieties in $\mathbb{P}_2 \times \mathbb{P}_2$.

On $Z_0 = \hat{\mathbb{P}}_2$ we define a divisor H as the pull back of a line $h \subseteq \mathbb{P}_2$ from the first $\mathbb{P}_2$ and F is defined as the pull back of a line $f \subseteq \mathbb{P}_2$ in the second $\mathbb{P}_2$, i. e., $\mathcal{O}_{Z_0}(H) = \mathcal{O}_{\mathbb{P}_2\times\mathbb{P}_2}(1,0)|_{Z_0}$ and $\mathcal{O}_{Z_0}(F) = \mathcal{O}_{\mathbb{P}_2\times\mathbb{P}_2}(0,1)|_{Z_0}$. The Picard group of Z_0 is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$, as free generators one can take the sheaves $\mathcal{O}_{Z_0}(H)$ and $\mathcal{O}_{Z_0}(F)$.

The sheaf $\sigma^*\mathcal{E}$ has torsion along D_1 which is a subsheaf $2\mathcal{O}_{D_1}(-1)$. We obtain a commutative diagram

$$\begin{tikzcd}
 & & & 0 \arrow[d] & \\
 & 0 \arrow[d] & & 2\mathcal{O}_{D_1}(-1) \arrow[d] & \\
0 \arrow[r] & 2\mathcal{O}_Z(-2H) \arrow[r, "\sigma^*(A+tB)"] \arrow[d] & \mathcal{O}_Z(-H) \oplus \mathcal{O}_Z \arrow[r] \arrow[d, equal] & \sigma^*\mathcal{E} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & 2\mathcal{O}_Z(-2H + D_1) \arrow[r] \arrow[d] & \mathcal{O}_Z(-H) \oplus \mathcal{O}_Z \arrow[r] & \widetilde{\mathcal{E}} \arrow[r] \arrow[d] & 0 \\
 & 2\mathcal{O}_{D_1}(-1) \arrow[d] & & 0 & \\
 & 0 & & &
\end{tikzcd}$$

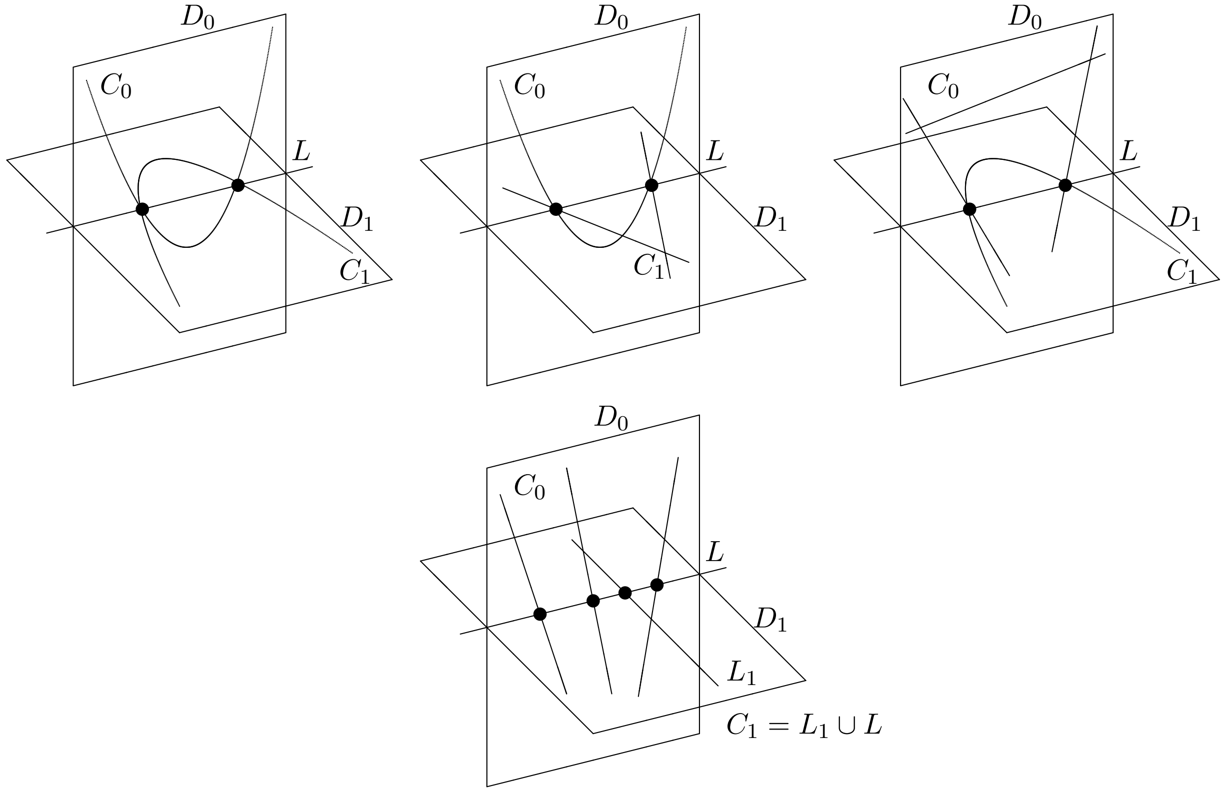
and a sheaf $\widetilde{\mathcal{E}}$ on Z.

Proposition 1.24. *$\widetilde{\mathcal{E}}$ is locally free on its support if and only if B is a normal vector to X_8 at A, i. e., if and only if $B \in T_A X \setminus T_A X_8$.*

We call the one parameter families $\widetilde{\mathcal{E}}$ that are locally free on their support new one parameter families. New one parameter families $\widetilde{\mathcal{E}}$ are flat over U. We call the fibres $\widetilde{\mathcal{E}}_0$ of new one parameter families $\widetilde{\mathcal{E}}$ over $t = 0$ R-**bundles** on $\hat{\mathbb{P}}_2$.

Properties of R-bundles on $\hat{\mathbb{P}}_2$. R-bundles are supported on reducible curves of the type $C_0 \cup C_1$, where C_0 is a curve in D_0, a partial normalization of the curve $C' = \operatorname{Supp}\mathcal{E}_0$, and C_1 is a conic in $D_1 = \mathbb{P}_2$.

The following pictures give an illustration how the curves C_0 and C_1 look like.



- It turns out that in all cases the restriction of $\widetilde{\mathcal{E}}_0$ to C_0 is isomorphic to the structure sheaf $\mathcal{O}_{C_0}$ of the curve C_0.
- The restriction to C_1 is a $2m+2$ semi-stable sheaf on $D_1 = \mathbb{P}_2$ which is locally free on C_1 and has degree 1.
- R-bundles are non-trivial extensions

$$0 \to \mathcal{O}_C \to \widetilde{\mathcal{E}}_0 \to \Bbbk_q \to 0, \tag{$*$}$$

  where $C = C_0 \cup C_1$ and q is a point of C_1.
- For an R-bundle $\mathcal{E}$ in generic situation (when the line L is not contained in C) there is a gluing exact sequence

$$0 \to \mathcal{E} \to \mathcal{E}_{C_0} \oplus \mathcal{E}_{C_1} \to \mathcal{E}_{C_0 \cap C_1} \to 0.$$

- One may consider $Z_0 = \hat{\mathbb{P}}_2$ as a closed subvariety in $\mathbb{P}_2 \times \mathbb{P}_2$ (cf. Definition 1.11). The Hilbert polynomials of an R-bundle $\mathcal{E}$ and its restrictions to the curves C_0 and C_1 with respect to the invertible sheaf $\mathcal{L} = \mathcal{O}_{\mathbb{P}_2 \times \mathbb{P}_2}(1,1)$ are

$$\chi(\mathcal{E} \otimes \mathcal{L}^m) = 6m+1, \quad \chi(\mathcal{E}_{C_0} \otimes \mathcal{L}^m) = 4m+1, \quad \chi(\mathcal{E}_{C_1} \otimes \mathcal{L}^m) = 2m+2.$$

We find some characteristic properties of R-bundles.

- We show that R-bundles are exactly those sheaves on Z_0 given by a resolution (cf. Proposition 1.37 and Proposition 1.57)

$$0 \to 2\mathcal{O}_{Z_0}(-H-F) \xrightarrow{\Phi} \mathcal{O}_{Z_0}(-H) \oplus \mathcal{O}_{Z_0} \to \widetilde{\mathcal{E}}_0 \to 0, \tag{$**$}$$

  with $\Phi = \begin{pmatrix} l_1 & \widetilde{q}_1 \\ l_2 & \widetilde{q}_2 \end{pmatrix}$ such that $\det(\Phi|_{D_0}) \neq 0$, $(\Phi|_{D_1})(q) \neq 0$ for all $q \in D_1$, and the linear forms l_1 and l_2 are linear independent and their common zero point in D_1 does not belong to L.

- The restrictions of resolutions (∗∗) to the components D_0 and D_1 of Z_0 are resolutions of Beilinson type (cf. Remarks 1.55 and 1.56). Moreover the morphisms of R-bundles are in one-to-one correspondence with the morphisms of the corresponding resolutions (see Proposition 1.40).

- Propositions 1.48 and 1.60 say that R-bundles are exactly the bundles on C that occur as non-trivial extensions (∗), where $q \in D_1 \setminus L$ and C is a curve in Z_0 given by a resolution

$$0 \to \mathcal{O}_{Z_0}(-2F - H) \to \mathcal{O}_{Z_0} \to \mathcal{O}_C \to 0.$$

- Let $\sigma_0 : Z_0 \to \mathbb{P}_2$ be a contraction of D_1 to a point $p \in \mathbb{P}_2$. Propositions 1.49 and 1.61 state that a sheaf $\mathcal{E}$ on Z_0 is an R-bundle if and only if $\mathcal{E}$ is locally free on its support, the Hilbert polynomial of its restriction $\mathcal{E}|_{D_0}$ to D_0 is $4m + 1$, and there is an exact sequence

$$0 \to 2\mathcal{O}_{D_1}(-L) \to \sigma_0^*\mathcal{F} \to \mathcal{E} \to 0$$

  for some $3m + 1$ sheaf $\mathcal{F}$ that is not locally free at the point p. In fact, as we already noticed, the restriction of an R-bundle to D_0 is always isomorphic to the structure sheaf of a curve C_0 given by a resolution

$$0 \to \mathcal{O}_{D_0}(-2F - H) \to \mathcal{O}_{D_0} \to \mathcal{O}_{C_0} \to 0.$$

Isomorphism classes of R-bundles. If we fix a curve $C_0 \cup C_1$ then the isomorphism classes of R-bundles supported on this curve are in one-to-one correspondence with an open subset of C_1 (cf. Corollary 2.11).

There are non-isomorphic R-bundles $\mathcal{E}$ and $\mathcal{E}'$ with isomorphic restrictions $\mathcal{E}_{C_1}$ and $\mathcal{E}'_{C_1}$.

There is a parameter space $\mathbb{X}$ of all isomorphism classes of R-bundles on Z_0 (see Definition 2.1). $\mathbb{X}$ is an open subvariety of $\mathbb{k}^{18}$. There is a natural action of the group G on $\mathbb{X}$.

Proposition 2.2. *The orbits of G in $\mathbb{X}$ are in one-to-one correspondence with the isomorphism classes of R-bundles on $\hat{\mathbb{P}}_2$.*

Corollary 2.9 says that there is an orbit space $\mathbb{Y}''$ of the action $G \times \mathbb{X} \to \mathbb{X}$. So $\mathbb{Y}''$ is the variety of all isomorphism classes of R-bundles on Z_0. The variety $\mathbb{Y}''$ is a quasi projective variety, it may be realized as an open subset of a hypersurface in $\mathbb{P}_9 \times \mathbb{P}_2$, in particular the dimension of $\mathbb{Y}''$ is 10.

Equivalence classes of R-bundles. To be able to consider R-bundles and the non-singular $3m + 1$ sheaves simultaneously it is necessary to introduce an equivalence relation on the set of R-bundles. For a point $A \in X_8$ we introduce the following equivalence relation on the set of R-bundles constructed at $A \in X_8$ (cf. Definition 2.12). Two R-bundles $\mathcal{E}_1$ and $\mathcal{E}_2$ on $\hat{\mathbb{P}}_2$ constructed at the same point $A \in X_8$ are called equivalent if there exists an automorphism $\widetilde{\phi}$ of Z_0 that acts identically on $D_0 = \widetilde{\mathbb{P}}_2$ and such that $\widetilde{\phi}^*(\mathcal{E}_1) \cong \mathcal{E}_2$. Our notion of equivalence corresponds to the notion of equivalence given in Definition 4.1, (ii) from [26].

Theorem 2.19. *There is a one-to-one correspondence between the equivalence classes of R-bundles constructed at $A \in X_8$ and points of $\mathbb{P}N_A$.*

For a generic $A \in X_8$ (when the corresponding singular $3m + 1$ sheaf is defined by an ordinary double point singularity on a cubic curve in $\mathbb{P}_2$) there are only two equivalence classes with a singular conic C_1 as a support in D_1. Degenerations of A with double-point singularity give us only one equivalence class with the curve C_1 being singular. If the singularity is a triple point, all the equivalence classes have singular curve C_1. In this case one could identify the line $L = D_0 \cap D_1$ with the set of all equivalence classes of R-bundles constructed at A.

If we fix a curve $C_0 \cup C_1$, then in a generic situation for every equivalence class of R-bundles supported on $C_0 \cup C_1$ there are two isomorphism classes of R-bundles on $C_0 \cup C_1$. There are curves for which we have a one-to-one correspondence between the isomorphism classes and the equivalence classes. There are also situations when there is a one-dimensional variety of isomorphism classes of R-bundles corresponding to a given equivalence class of R-bundles on $C_0 \cup C_1$. See also Section 2.3 for concrete examples.

Parameter spaces for the new objects. Since X_8 and M_8 are smooth subvarieties of codimension 2 in X and M respectively, we may consider the blow up $\widetilde{M} = \mathrm{Bl}_{M_8} M$ as the space whose points are all the isomorphism classes of non-singular $3m+1$ sheaves on $\mathbb{P}_2$ and also all the equivalence classes of R-bundles. Note that $\widetilde{M}$ is obtained from M by replacing each point from M_8 by $\mathbb{P}_1$. Analogously $\widetilde{X} = \mathrm{Bl}_{X_8} X$ may be seen as a variety parameterizing the above objects. We give concrete descriptions of $\widetilde{X}$ and $\widetilde{M}$ as subvarieties in $\mathbb{P}_1$-bundles over X and M respectively.

It turns out that the action of the group G on X can be uniquely lifted along the blow up $\widetilde{X} \to X$ to an action on $\widetilde{X}$ (cf. Lemma 3.3). The restriction of this action to the exceptional divisor $E_X \cong \mathbb{P}N_{X_8/X}$ of $\widetilde{X} \to X$ is a natural action of G on $\mathbb{P}N_{X_8/X}$ (cf. Lemma 3.5). The quotient morphism $X \xrightarrow{\nu} M$ lifts uniquely to a morphism $\widetilde{X} \xrightarrow{\widetilde{\nu}} \widetilde{M}$ and we obtain the commutative diagram

$$\begin{tikzcd} \widetilde{X} \arrow[r, "\widetilde{\nu}"] \arrow[d] & \widetilde{M} \arrow[d] \\ X \arrow[r, "\nu"] & M. \end{tikzcd}$$

Proposition 3.9. *$\widetilde{X}$ is a principal vector bundle over $\widetilde{M}$ with fibre $\mathbb{P}G$. In particular $\widetilde{\nu} : \widetilde{X} \to \widetilde{M}$ is a quotient of the action of G on $\widetilde{X}$.*

"Universal" family over $\widetilde{X}$. We construct a flat morphism $Y \to \widetilde{X}$ and a sheaf $\widetilde{\mathcal{U}}$ on Y locally free on its support and flat over $\widetilde{X}$ (cf. Propositions 3.18 and 3.20) such that the fibres of $\widetilde{\mathcal{U}}$ are either non-singular $3m+1$ sheaves on $\mathbb{P}_2$ or R-bundles on $\hat{\mathbb{P}}_2$. Moreover, the isomorphism (for $3m+1$ sheaves) or equivalence (for R-bundles) class of $\widetilde{\mathcal{U}}_x$ corresponds to the point $x \in \widetilde{X}$.

So every R-bundle on $\hat{\mathbb{P}}_2$ up to equivalence may be realized as a fibre of $\widetilde{\mathcal{U}}$. Therefore, one may consider the sheaf $\widetilde{U}$ as a "universal" family of R-bundles together with the non-singular $3m+1$ sheaves.

General families, functor $\widetilde{\mathcal{M}}$. In Definition 3.21 we define a family over an arbitrary S. In particular $3m+1$ families of the non-singular sheaves on $\mathbb{P}_2$ are families in the sense of Definition 3.21. For every $S \in \mathrm{Ob}\,(\mathrm{Sch})$ we introduce an equivalence relation on the set of all families over S. For families over one point this relation coincides either with the isomorphism for non-singular $3m+1$ sheaves or with the equivalence for R-bundles.

For a morphism $f : T \to S$ and for a family over S we define a family over T. We obtain this way the map from the set of all families over S to the set of all families over T. This map is compatible with the equivalence relations and therefore we obtain a functor $\widetilde{\mathcal{M}} : (\mathrm{Sch}) \to (\mathrm{Sets})$ that assigns to every $S \in \mathrm{Ob}\,(\mathrm{Sch})$ the set of the equivalence classes of the families over S.

There is a natural transformation $\widetilde{\mathcal{M}} \to \mathcal{M}$, where $\mathcal{M}$ denotes the functor of the $3m+1$ moduli problem on $\mathbb{P}_2$.

Proposition 3.40. *There is a natural transformation of functors*

$$\widetilde{\mathcal{M}} \to \mathrm{Hom}(\_\,, \widetilde{M})$$

and the commutative diagram

$$\begin{tikzcd} \widetilde{\mathcal{M}} \arrow[r] \arrow[d, "\mu"'] & \mathrm{Hom}(\_\,, \widetilde{M}) \arrow[d] \\ \mathcal{M} \arrow[r, "\cong"] & \mathrm{Hom}(\_\,, M). \end{tikzcd}$$

Main difficulties

Gluing sheaves. The variety $\hat{\mathbb{P}}_2$ is reducible, it consists of two components D_0 and D_1. To define a sheaf on $\hat{\mathbb{P}}_2$ one needs to describe its restrictions to the components D_0 and D_1 and the "gluing" data that describe how the sheaves on D_0 and D_1 are glued together. There are non-isomorphic sheaves on $\hat{\mathbb{P}}_2$ with isomorphic restrictions to D_0 and D_1. Therefore, one needs some gluing statements to describe sheaves on $\hat{\mathbb{P}}_2$.

- A naive description of gluing would be to describe a sheaf $\mathcal{F}$ on $\hat{\mathbb{P}}_2$ using the exact sequence
$$0 \to \mathcal{F} \to \mathcal{F}_{D_0} \oplus \mathcal{F}_{D_1} \to \mathcal{F}_L \to 0,$$
where $\mathcal{F}_{D_0}$, $\mathcal{F}_{D_1}$, and $\mathcal{F}_L$ denote the restrictions of $\mathcal{F}$ to D_0, D_1, and $L = D_0 \cap D_1$ respectively. But such a sequence exists in general only for vector bundles on $\hat{\mathbb{P}}_2$. We describe in particular the Picard group of $\hat{\mathbb{P}}_2$ by gluing invertible sheaves on D_0 and D_1 (cf. page 22). The gluing sequence is one of the main tools for the calculations of the cohomology groups of sheaves on $\hat{\mathbb{P}}_2$ (cf. page 25).

- R-bundles are obtained by gluing together a structure sheaf of a curve C_0 in D_0 and of a semi-stable $2m+2$ sheaf on $D_1 = \mathbb{P}_2$ that is locally free on its support.

  A naive description of gluing for an R-bundle $\mathcal{E}$ works only if the support of $\mathcal{E}$ does not contain the line $L = D_0 \cap D_1$ (cf. Remark 1.59). We obtain in this case the exact sequence
$$0 \to \mathcal{E} \to \mathcal{E}_{C_0} \oplus \mathcal{E}_{C_1} \to \mathcal{E}_{C_0 \cap C_1} \to 0.$$

  If the support $\operatorname{Supp} \mathcal{E}$ of an R-bundle $\mathcal{E}$ contains the line L, it is not only reducible but also non-reduced, there is a "double" structure on the line L (see page 47). It seems difficult do describe the gluing data in this case. Therefore, we describe R-bundles by means of locally free resolutions of $\hat{\mathbb{P}}_2$.

Locally free resolutions. For coherent sheaves on $\hat{\mathbb{P}}_2$ there are no standard resolutions of Beilinson type. Nevertheless the properties of locally free resolutions of R-bundles of the type ($**$) are similar to the properties of Beilinson resolutions.

Namely the homomorphisms between the sheaves given by resolutions of this type are in one-to-one correspondence with the morphisms of the corresponding resolutions and the restrictions of ($**$) to D_0 and D_1 are Beilinson resolutions. One can consider ($**$) as a gluing of its restrictions to D_0 and D_1 (cf. Proposition 1.58).

Computing cohomology. Computing cohomology of invertible sheaves on $\hat{\mathbb{P}}_2$ we use the gluing exact sequence and reduce the question to computing cohomology of D_0 and D_1. To compute cohomology of D_0 we consider D_0 as a hypersurface in $\mathbb{P}_2 \times \mathbb{P}_1$. For $\mathcal{O}_{D_0}(aH + bF)$ there is a resolution (cf. (1.24))

$$0 \to \mathcal{O}_{\mathbb{P}_2\times\mathbb{P}_1}(a-1, b-1) \to \mathcal{O}_{\mathbb{P}_2\times\mathbb{P}_1}(a, b) \to \mathcal{O}_{D_0}(aH + bF) \to 0.$$

Hence the problem reduces to computing cohomology of $\mathbb{P}_2 \times \mathbb{P}_1$, which is done by using the Künneth formula (cf. (1.25)) from [25].

Equivalence of R-bundles. One should define an equivalence relation on the set of R-bundles to be able to consider them along with the non-singular $3m+1$ sheaves. There are non-isomorphic R-bundles which are equivalent. In a generic case there are two isomorphism classes for a given equivalence class of R-bundles on a fixed curve $C_0 \cup C_1$.

Definition of a family. Both R-bundles and the non-singular $3m+1$ sheaves are coherent sheaves on $\mathbb{P}_2 \times \mathbb{P}_2$ with Hilbert polynomial $6m+1$ with respect to the invertible sheaf $\mathcal{O}_{\mathbb{P}_2\times\mathbb{P}_2}(1,1)$. The family of sheaves $\widetilde{\mathcal{U}}$ over $\widetilde{X}$ we construct in Section 3.2 is defined as a sheaf on Y, where Y is a non-trivial $\mathbb{P}_2$-bundle over $\widetilde{X} \times \mathbb{P}_2$. We consider $\widetilde{\mathcal{U}}$ as a "universal family" for a moduli problem we want to define.

So despite the fact that each fibre $\widetilde{\mathcal{U}}_x$, $x \in \widetilde{X}$, of $\widetilde{\mathcal{U}}$ may be considered as a sheaf on $\mathbb{P}_2 \times \mathbb{P}_2$ with Hilbert polynomial $6m+1$ there is no way to consider all fibres of $\widetilde{\mathcal{U}}$ in the same ambient space as it has been done for Simpson moduli problems. We achieve however that locally over the base the general families defined in Section 3.3 may be considered as families of sheaves in $\mathbb{P}_2 \times \mathbb{P}_2$ with Hilbert polynomial $6m+1$ (cf. Proposition 3.35).

Chapter 1

Construction of R-bundles

Summary

In this chapter we construct R-bundles and discuss their properties.

In Section 1.1 we make an overview of some results from [5] and prove some important statements about $3m+1$ sheaves on $\mathbb{P}_2$, i. e., (semi-)stable sheaves on $\mathbb{P}_2$ with Hilbert polynomial $3m+1$. We consider the moduli space $M = M_{3m+1}(\mathbb{P}_2)$ of $3m+1$ sheaves on $\mathbb{P}_2$. It is known (cf. [5]) that M is isomorphic to the universal cubic curve.

There is a parameter space X such that M is a geometrical quotient of X. X is isomorphic to an open subset in $\mathbb{k}^{18}$. Elements of X are the matrices A defining the resolutions

$$0 \to 2\mathcal{O}_{\mathbb{P}_2}(-2) \xrightarrow{A} \mathcal{O}_{\mathbb{P}_2}(-1) \oplus \mathcal{O}_{\mathbb{P}_2} \to \mathcal{F} \to 0.$$

A $3m+1$ sheaf is called singular if it is not locally free on its support. We describe the closed subvariety M_8 in M of all singular $3m+1$ sheaves on $\mathbb{P}_2$ and the corresponding subvariety X_8 in X that parameterizes the singular sheaves. The subvarieties M_8 and X_8 are smooth of codimension 2 in M and X respectively. We give a description of the subbundle T_{X_8} in T_X.

In Section 1.2 we propose a construction of R-bundles. We consider the singular $3m+1$ sheaves as one-dimensional limits of non-singular $3m+1$ sheaves and describe a construction that substitutes the singular sheaves by sheaves locally free on their support that may be also considered as limits of non-singular $3m+1$ sheaves.

Namely, for $A \in X_8$ and $B \in T_A X$ we consider an open set $U \subseteq \mathbb{k}^1$ containing 0 such that $A + tB \in X$ for all t from U. We obtain the one parameter family of $3m+1$ sheaves given by the resolution

$$0 \to 2\mathcal{O}_{U\times\mathbb{P}_2}(-2H) \xrightarrow{A+tB} \mathcal{O}_{U\times\mathbb{P}_2}(-H) \oplus \mathcal{O}_{U\times\mathbb{P}_2} \to \mathcal{E} \to 0.$$

Let $Z \xrightarrow{\sigma} U \times \mathbb{P}_2$ be the blow up of $U \times \mathbb{P}_2$ at the point $0 \times p$, where $p \in \mathbb{P}_2$ is the point where the $3m+1$ sheaf $\mathcal{E}_0 = \mathcal{E}|_{\{0\}\times\mathbb{P}_2}$ given by the matrix A is not free on its support. Let D_1 be the

exceptional divisor of σ. We obtain then a commutative diagram

$$
\begin{tikzcd}
 & & & 0 \arrow[d] & \\
 & 0 \arrow[d] & & 2\mathcal{O}_{D_1}(-1) \arrow[d] & \\
0 \arrow[r] & 2\mathcal{O}_Z(-2H) \arrow[r, "\sigma^*(A+tB)"] \arrow[d] & \mathcal{O}_Z(-H)\oplus\mathcal{O}_Z \arrow[r] \arrow[d, equal] & \sigma^*\mathcal{E} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & 2\mathcal{O}_Z(-2H+D_1) \arrow[r] \arrow[d] & \mathcal{O}_Z(-H)\oplus\mathcal{O}_Z \arrow[r] & \widetilde{\mathcal{E}} \arrow[r] \arrow[d] & 0 \\
 & 2\mathcal{O}_{D_1}(-1) \arrow[d] & & 0 & \\
 & 0 & & &
\end{tikzcd}
$$

and a sheaf $\widetilde{\mathcal{E}}$. It turns out that $\widetilde{\mathcal{E}}$ is flat over U. Its fibre $\widetilde{\mathcal{E}}_0$ over 0 is a sheaf on $Z_0 = \hat{\mathbb{P}}_2 \cong \widetilde{\mathbb{P}}_2 \cup \mathbb{P}_2$, where $\widetilde{\mathbb{P}}_2$ is the blow up of $\mathbb{P}_2$ at a point and $\widetilde{\mathbb{P}}_2$ is glued together with $\mathbb{P}_2$ along the exceptional line of $\widetilde{\mathbb{P}}_2$ (cf. Definition 1.11). The sheaves $\widetilde{\mathcal{E}}_0$ that are locally free on their support are considered to be replacements of the singular $3m+1$ sheaves. The sheaf $\widetilde{\mathcal{E}}_0$ is locally free on its support if and only if B is a normal direction to X_8.

New objects related to the construction of R-bundles are studied in Section 1.3. In particular we describe the Picard group of $\hat{\mathbb{P}}_2$, we compute for some invertible sheaves on $\hat{\mathbb{P}}_2$ their cohomology groups and the direct images with respect to the canonical projection $\hat{\mathbb{P}}_2 \to \mathbb{P}_2$. We use those calculations to obtain some properties of R-bundles.

We find some characteristic properties of R-bundles, i. e., properties that may be used to define them. We see that R-bundles are exactly those sheaves on Z_0 given by a resolution (cf. Proposition 1.37 and Proposition 1.57)

$$0 \to 2\mathcal{O}_{Z_0}(-H-F) \xrightarrow{\Phi} \mathcal{O}_{Z_0}(-H) \oplus \mathcal{O}_{Z_0} \to \widetilde{\mathcal{E}}_0 \to 0,$$

with $\Phi = \left(\begin{smallmatrix} l_1 & \widetilde{q}_1 \\ l_2 & \widetilde{q}_2 \end{smallmatrix}\right)$ such that $\det(\Phi|_{D_0}) \neq 0$, $(\Phi|_{D_1})(q) \neq 0$ for all $q \in D_1$, and the linear forms l_1 and l_2 are linear independent and their common zero point in D_1 does not belong to L.

Propositions 1.48 and Proposition 1.60 say that R-bundles are exactly the non-trivial extensions of $\Bbbk_q$, $q \in D_1 \setminus L$, by $\mathcal{O}_C$, where C is a curve in Z_0 given by a resolution

$$0 \to \mathcal{O}_{Z_0}(-2F-H) \to \mathcal{O}_{Z_0} \to \mathcal{O}_C \to 0.$$

We see that the behavior of R-bundles is quite similar to the behavior of $3m+1$ sheaves, which is exactly what one could expect.

1.1 $3m+1$ sheaves, overview

In this section we will give an overview about semi-stable sheaves on $\mathbb{P}_2$ with Hilbert polynomial $3m+1$. We will call such sheaves $3m+1$ sheaves. We will briefly repeat some results from [5] and also prove some useful lemmata. One could find some useful details in [4].

1.1.1 Review of [5].

Let us recall that for an arbitrary smooth projective variety $\mathfrak{X}$ and for an arbitrary numerical polynomial $P \in \mathbb{Q}[m]$ there is a coarse moduli space $M_P(\mathfrak{X})$ of semi-stable sheaves on $\mathfrak{X}$ with Hilbert polynomial P. This has been proven by Simpson in [27].

Moduli space $M_{3m+1}(\mathbb{P}_2)$.

In the case of $\mathfrak{X} = \mathbb{P}_2$ and $P = 3m + 1$ the space $M = M_{3m+1}(\mathbb{P}_2)$ was described in [5] and has been mentioned in [14]. In particular all $3m+1$ sheaves are stable and M is a fine moduli space, i. e., M represents the functor $\mathcal{M}$ of this moduli problem. This holds since the coefficients 3 and 1 are coprime. One can show that $3m+1$ sheaves on $\mathbb{P}_2$ are exactly the non-split extensions

$$0 \to \mathcal{O}_C \to \mathcal{F} \to \Bbbk_p \to 0, \tag{1.1}$$

where $C = \operatorname{Supp} \mathcal{F}$ is the cubic curve in $\mathbb{P}_2$ supporting $\mathcal{F}$ and p is a point on C. The moduli space M is isomorphic to the universal cubic curve

$$\{(\langle f \rangle, \langle x \rangle) \in \mathbb{P}_9 \times \mathbb{P}_2 \mid f(x) = 0\},$$

where $\mathbb{P}_9 = \{\langle C_{00}, C_{10}, \dots, C_{03} \rangle\}$ is identified with the space of cubic curves in $\mathbb{P}_2$ by

$$\langle C_{00}, C_{10}, \dots, C_{03} \rangle \leftrightarrow f,$$

where

$$\begin{aligned} f =& C_{00}x_0^3 + C_{10}x_0^2x_1 + C_{01}x_0^2x_2 + C_{20}x_0x_1^2 + \\ & C_{11}x_0x_1x_2 + C_{02}x_0x_2^2 + C_{30}x_1^3 + C_{21}x_1^2x_2 + C_{12}x_1x_2^2 + C_{03}x_2^3. \end{aligned} \tag{1.2}$$

Let us recall that a family of $3m + 1$ sheaves over S (or simply a $3m + 1$ family over S) is by definition a flat sheaf $\mathcal{F}$ on $S \times \mathbb{P}_2$ such that for every $s \in S$ the restriction $\mathcal{F}_s$ of $\mathcal{E}$ to the fibre $\{s\} \times \mathbb{P}_2 \cong \mathbb{P}_2$ is a $3m + 1$ sheaf on $\mathbb{P}_2$.

Parameter space X.

Let us consider the set of the matrices $\begin{pmatrix} z_1 & q_1 \\ z_2 & q_2 \end{pmatrix}$, where

$$z_1, z_2 \in \Gamma(\mathbb{P}_2, \mathcal{O}_{\mathbb{P}_2}(1)), \quad q_1, q_2 \in \Gamma(\mathbb{P}_2, \mathcal{O}_{\mathbb{P}_2}(2)).$$

Let us fix some coordinates $\langle x_0, x_1, x_2 \rangle$ in $\mathbb{P}_2$. We can now identify z_1 and z_2 with some linear forms

$$z_1 = a_0x_0 + a_1x_1 + a_2x_2, \quad z_2 = b_0x_0 + b_1x_1 + b_2x_2, \quad a_i, b_i \in \Bbbk,$$

and q_1, q_2 may be identified with quadratic forms

$$q_1 = A_{00}x_0^2 + A_{01}x_0x_1 + \dots + A_{22}x_2^2, \quad q_2 = B_{00}x_0^2 + B_{01}x_0x_1 + \dots + B_{22}x_2^2, \quad A_{ij}, B_{ij} \in \Bbbk.$$

Thus one can identify the set of all matrices $\begin{pmatrix} z_1 & q_1 \\ z_2 & q_2 \end{pmatrix}$ with the affine variety $\Bbbk^{18}$.

Let us consider

$$X = \left\{ \begin{pmatrix} z_1 & q_1 \\ z_2 & q_2 \end{pmatrix} \mid z_1 \wedge z_2 \neq 0, z_1q_2 - z_2q_1 \neq 0 \right\},$$

i. e., the set of matrices $\begin{pmatrix} z_1 & q_1 \\ z_2 & q_2 \end{pmatrix}$ with linear independent forms z_1 and z_2 and non-zero determinant. Then X is an open subset in $\Bbbk^{18}$ (in the variety of all matrices), hence a quasi-affine variety.

In [5] it was shown that $3m+1$ sheaves $\mathcal{F}$ on $\mathbb{P}_2$ are exactly those possessing a resolution

$$0 \to 2\mathcal{O}_{\mathbb{P}_2}(-2) \xrightarrow{A} \mathcal{O}_{\mathbb{P}_2}(-1) \oplus \mathcal{O}_{\mathbb{P}_2} \to \mathcal{F} \to 0 \tag{1.3}$$

with the matrix A from X.

Quotient $X \to M$.

Let H be the group of 2×2 matrices

$$\begin{pmatrix} \lambda & z \\ 0 & \mu \end{pmatrix}, \quad \lambda, \mu \in \Bbbk, \quad \lambda\mu \neq 0, \quad z \in \Gamma(\mathbb{P}_2, \mathcal{O}_{\mathbb{P}_2}(1)).$$

Then H is an algebraic group, as algebraic variety it is isomorphic to $\Bbbk^* \times \Bbbk^* \times \Bbbk^3$. Consider then the algebraic group

$$G = \mathrm{GL}_2(\Bbbk) \times H. \tag{1.4}$$

Then G acts on X from the left by the rule

$$(g, h) \cdot A = gAh^{-1}.$$

This action corresponds to isomorphisms of exact sequences given by $g \in \mathrm{GL}_2(\Bbbk)$, $h \in H$:

$$\begin{tikzcd} 0 \arrow[r] & 2\mathcal{O}_{\mathbb{P}_2}(-2) \arrow[r, "A"] & \mathcal{O}_{\mathbb{P}_2}(-1) \oplus \mathcal{O}_{\mathbb{P}_2} \arrow[r] & \mathcal{E} \arrow[r] & 0 \\ 0 \arrow[r] & 2\mathcal{O}_{\mathbb{P}_2}(-2) \arrow[r] \arrow[u, "g"] & \mathcal{O}_{\mathbb{P}_2}(-1) \oplus \mathcal{O}_{\mathbb{P}_2} \arrow[r] \arrow[u, "h"] & \mathcal{F} \arrow[r] \arrow[u, "\cong"] & 0. \end{tikzcd} \tag{1.5}$$

Since $\mathrm{Hom}(\mathcal{O}_{\mathbb{P}_2}(-1) \oplus \mathcal{O}_{\mathbb{P}_2}, 2\mathcal{O}_{\mathbb{P}_2}(-2)) = \mathrm{Ext}^1(\mathcal{O}_{\mathbb{P}_2}(-1) \oplus \mathcal{O}_{\mathbb{P}_2}, 2\mathcal{O}_{\mathbb{P}_2}(-2)) = 0$, every morphism of $3m+1$ sheaves lifts uniquely to a morphism of resolutions of the type (1.3). In particular the isomorphisms of $3m+1$ sheaves lift to the morphisms of the type (1.5). This implies that the orbits of the action of G on X are in one-to-one correspondence with the isomorphism classes of $3m+1$ sheaves, i. e., with the points of $M = M_{3m+1}(\mathbb{P}_2)$. Moreover, it has been shown in [5] that M is a geometric quotient of X by G. For an arbitrary point $A \in X$ its stabilizer under the action of G is the subgroup

$$St = \left\{ \left(\begin{smallmatrix} \lambda & 0 \\ 0 & \lambda \end{smallmatrix}\right) \times \left(\begin{smallmatrix} \lambda & 0 \\ 0 & \lambda \end{smallmatrix}\right) \mid \lambda \in \Bbbk^* \right\}.$$

Hence the action of $\mathbb{P}G := G/St$ on X is free and one can show that X is a principal bundle over M with fibre $\mathbb{P}G$.

As we already mentioned that M is isomorphic to the universal cubic curve, the quotient morphism $X \xrightarrow{\nu} M$, which sends A to the isomorphism class $[\mathcal{F}_A]$ of the corresponding sheaf $\mathcal{F}_A$ (for a matrix $A \in X$ we denote by $\mathcal{F}_A$ the sheaf defined by A as in the resolution (1.3)) is just the morphism given by

$$A = \begin{pmatrix} z_1 & q_1 \\ z_2 & q_2 \end{pmatrix} \mapsto \langle \det A \rangle \times \langle z_1 \wedge z_2 \rangle. \tag{1.6}$$

Note that the determinant of A is a cubic curve and the point $z_1 \wedge z_2$ is a point on this curve. We identify here $\langle z_1 \wedge z_2 \rangle$ with the point p where both z_1 and z_2 vanish.

Lemma 1.1. *Let $z_1 = a_0x_0 + a_1x_1 + a_2x_2$ and $z_2 = b_0x_0 + b_1x_1 + b_2x_2$, $a_i, b_j \in \Bbbk$, be two linear independent linear forms. Then $\langle z_1 \wedge z_2 \rangle = \langle d_0, d_1, d_2 \rangle$, where d_i are the minors of the matrix $\begin{pmatrix} a_0 & a_1 & a_2 \\ b_0 & b_1 & b_2 \end{pmatrix}$, i. e.,*

$$d_0 = \det \begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix}, \quad d_1 = -\det \begin{pmatrix} a_0 & a_2 \\ b_0 & b_2 \end{pmatrix}, \quad d_2 = \det \begin{pmatrix} a_0 & a_1 \\ b_0 & b_1 \end{pmatrix}. \tag{1.7}$$

Proof. The matrices $\left(\begin{smallmatrix} a_0 & a_1 & a_2 \\ a_0 & a_1 & a_2 \\ b_0 & b_1 & b_2 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} b_0 & b_1 & b_2 \\ a_0 & a_1 & a_2 \\ b_0 & b_1 & b_2 \end{smallmatrix}\right)$ are clearly degenerate. Therefore, their determinants are zero. Using the expansion formula for determinant along the first row one obtains $a_0d_0 + a_1d_1 + a_2d_2 = 0$ and $b_0d_0 + b_1d_1 + b_2d_2 = 0$, i. e., $z_1(d_0, d_1, d_2) = z_2(d_0, d_1, d_2) = 0$. So, $p = \langle d_0, d_1, d_2 \rangle$ is the point where both z_1 and z_2 vanish. $\square$

1.1.2 Singular sheaves.

Definition-characterization of singular sheaves.

Lemma 1.2. *Let $A = \begin{pmatrix} z_1 & q_1 \\ z_2 & q_2 \end{pmatrix}$ be a matrix from X. Let p be the point where both linear forms z_1 and z_2 vanish. Then the following statements are equivalent.*

1) The sheaf $\mathcal{F}$ defined by the matrix A as in (1.3) *is locally free on its support;*

2) at least one of the quadratic forms q_1 and q_2 does not vanish at p;

3) p is a nonsingular point of the curve $C = \{\det A = 0\}$.

Proof. Since $\mathcal{F}$ is supported on the curve C given by the determinant of the matrix A, one concludes that $\mathcal{F}_x$, $x \in C$, is not free if and only the rank of the matrix A at the point x is zero, i. e., if and only if all the entries of the matrix A vanish at x. The only point where this could hold true is $x = p$. Therefore, we conclude that $\mathcal{F}$ is not locally free if and only if $q_1(p) = q_2(p) = 0$. This we proved the equivalence of 1) and 2).

Let $f = \det A$, then $f = z_1q_2 - z_2q_1$ and one sees that p is a singular point of C if and only if

$$\frac{\partial f}{\partial z_1}(p) = \frac{\partial f}{\partial z_2}(p) = 0.$$

Since

$$\frac{\partial f}{\partial z_1}(p) = \left(q_2 + z_1\frac{\partial q_1}{\partial z_1} - z_2\frac{\partial q_1}{\partial z_1} \right)(p) = q_2(p)$$

and

$$\frac{\partial f}{\partial z_2}(p) = \left(z_1\frac{\partial q_2}{\partial z_2} - q_1 - z_2\frac{\partial q_1}{\partial z_2} \right)(p) = -q_1(p),$$

we obtain the equivalence of 2) and 3). This proves the lemma. $\square$

Definition 1.3. *Following [5] we call the sheaves that are not locally free on their support **singular sheaves**. Sheaves that are not singular are called then **non-singular**.*

From Lemma 1.2, using the description of the quotient map (1.6), one obtains the following corollary.

Corollary 1.4. *A point $\langle f \rangle \times p \in M$ represents an isomorphism class of a singular sheaf if and only if p is a singular point of the curve $\{f = 0\} \subseteq \mathbb{P}_2$.*

As we already noticed, M is given in $\mathbb{P}_9 \times \mathbb{P}_2$ by the equation

$$
\begin{aligned}
f(x_0, x_1, x_2) = & C_{00}x_0^3 + C_{10}x_0^2x_1 + C_{01}x_0^2x_2 + C_{20}x_0x_1^2 + \\
& C_{11}x_0x_1x_2 + C_{02}x_0x_2^2 + C_{30}x_1^3 + C_{21}x_1^2x_2 + C_{12}x_1x_2^2 + C_{03}x_2^3 = 0,
\end{aligned}
$$

where $\mathbb{P}_9 = \{\langle C_{00}, C_{10}, \ldots, C_{03}\rangle\}$ and $\mathbb{P}_2 = \{\langle x_0, x_1, x_2\rangle\}$.

Space M_8 and its defining equations.

Let M_8 be the subset of M that consists of the isomorphism classes of singular sheaves. By Corollary 1.4 $(\langle f\rangle, \langle x\rangle)$ belongs to M_8 if and only if $\langle x\rangle$ is a singular point of $\{f = 0\}$. The latter holds if and only if the partial derivatives of f vanish at $\langle x\rangle$. Since

$$
\frac{\partial f}{\partial x_0} = 3C_{00}x_0^2 + 2C_{10}x_0x_1 + 2C_{01}x_0x_2 + C_{20}x_1^2 + C_{11}x_1x_2 + C_{02}x_2^2,
$$

$$
\frac{\partial f}{\partial x_1} = C_{10}x_0^2 + 2C_{20}x_0x_1 + C_{11}x_0x_2 + 3C_{30}x_1^2 + 2C_{21}x_1x_2 + C_{12}x_2^2,
$$

$$
\frac{\partial f}{\partial x_2} = C_{01}x_0^2 + C_{11}x_0x_1 + 2C_{02}x_0x_2 + C_{21}x_1^2 + 2C_{12}x_1x_2 + 3C_{03}x_2^2,
$$

we obtain that M_8 is given by the equations e_0, e_1, and e_2, where

$$
\begin{aligned}
e_1 &= 3C_{00}x_0^2 + 2C_{10}x_0x_1 + 2C_{01}x_0x_2 + C_{20}x_1^2 + C_{11}x_1x_2 + C_{02}x_2^2, \\
e_1 &= C_{10}x_0^2 + 2C_{20}x_0x_1 + C_{11}x_0x_2 + 3C_{30}x_1^2 + 2C_{21}x_1x_2 + C_{12}x_2^2, \\
e_2 &= C_{01}x_0^2 + C_{11}x_0x_1 + 2C_{02}x_0x_2 + C_{21}x_1^2 + 2C_{12}x_1x_2 + 3C_{03}x_2^2.
\end{aligned}
\tag{1.8}
$$

Since

$$
x_0\frac{\partial f}{\partial x_0} + x_1\frac{\partial d}{\partial x_1} + x_2\frac{\partial f}{\partial x_2} = 3f,
$$

we conclude that M_8 is given in M locally by two equations. Namely in $M(x_0) := M \cap \{x_0 \neq 0\}$ the equations of M_8 are e_1 and e_2. In $M(x_1) := M \cap \{x_1 \neq 0\}$ the equations are e_0 and e_2 and in $M(x_2) := M \cap \{x_2 \neq 0\}$ they are e_0 and e_1.

Lemma 1.5. *M_8 is smooth of codimension 2 in M, i. e. the dimension of M_8 is 8. In particular M_8 is a locally complete intersection.*

Proof. The part of the jacobian matrix of e_0, e_1, e_2 with respect to the variables C_{00}, …, C_{03} is

$$
\begin{pmatrix}
3x_0^2 & 2x_0x_1 & 2x_0x_2 & x_1^2 & x_1x_2 & x_2^2 & 0 & 0 & 0 & 0 \\
0 & x_2^2 & 0 & 2x_0x_1 & x_0x_2 & 0 & 2x_1^2 & 2x_1x_2 & x_2^2 & 0 \\
0 & 0 & x_0^2 & 0 & x_0x_1 & 2x_0x_2 & 0 & x_1^2 & 2x_1x_2 & 3x_2^2
\end{pmatrix}.
$$

One sees that its rank is always 3. Therefore, M_8 is smooth of codimension 3 in $\mathbb{P}_9 \times \mathbb{P}_2$. The codimension of M_8 in M is then 2. This proves the lemma. $\square$

Space X_8.

Let X_8 be the subset of all the matrices in X defining singular sheaves.

Consider a matrix $A = \begin{pmatrix} z_1 & q_1 \\ z_2 & q_2 \end{pmatrix}$ from X. Let

$$\begin{aligned} z_1 =& a_0x_0 + a_1x_1 + a_2x_2, \\ z_2 =& b_0x_0 + b_1x_1 + b_2x_2, \\ q_1 =& A_{00}x_0^2 + A_{01}x_0x_1 + \dots + A_{22}x_2^2, \\ q_2 =& B_{00}x_0^2 + B_{01}x_0x_1 + \dots + B_{22}x_2^2, \quad a_i, b_i, A_{ij}, B_{ij} \in \Bbbk. \end{aligned}$$

Then $p = \langle d_0, d_1, d_2 \rangle$ is the point where both z_1 and z_2 vanish (we use the notations from (1.7)). By Lemma 1.2, A lies in X_8 if and only if $q_1(p) = q_2(p) = 0$. Therefore, we obtain equations defining X_8 in X:

$$\begin{aligned} f_3 = q_2(p) = A_{00}d_0^2 + A_{01}d_0d_1 + \dots + A_{22}d_2^2, \\ f_4 = q_2(p) = B_{00}d_0^2 + B_{01}d_0d_1 + \dots + B_{22}d_2^2, \end{aligned} \tag{1.9}$$

where a_i, b_i, A_{ij}, and B_{ij} are considered as variables. Hence

$$X_8 = X \cap V(f_3, f_4), \tag{1.10}$$

where $V(f_3, f_4) \subseteq \Bbbk^{18}$ is the affine subvariety given by the polynomials f_3 and f_4.

Let us see how this equations are connected with those of M_8. We have the geometrical quotient of X by the group G

$$X \xrightarrow{\nu} M, \quad A = \begin{pmatrix} z_1 & q_1 \\ z_2 & q_2 \end{pmatrix} \mapsto (\langle \det A \rangle, \langle z_1 \wedge z_2 \rangle).$$

It is clear that X_8 is the preimage of M_8 under ν. This means that the liftings of the equations of M_8 are equations of X_8. So let us calculate the liftings $(e_i \circ \nu)(A)$ of the equations e_0, e_1, and e_2 to X.

Lemma 1.6. *$(e_i \circ \nu)(A) = -b_if_3 + a_if_4, \quad i = 0, 1, 2.$*

Proof. From

$$\begin{aligned} \det \begin{pmatrix} z_1 & q_1 \\ z_2 & q_2 \end{pmatrix} = & (a_0x_0 + a_1x_1 + a_2x_2) \cdot \\ & \quad (A_{00}x_0^2 + A_{01}x_0x_1 + A_{02}x_0x_2 + A_{11}x_1^2 + A_{12}x_1x_2 + A_{22}x_2^2) - \\ & (b_0x_0 + b_1x_1 + b_2x_2) \cdot \\ & \quad (B_{00}x_0^2 + B_{01}x_0x_1 + B_{02}x_0x_2 + B_{11}x_1^2 + B_{12}x_1x_2 + B_{22}x_2^2) \end{aligned}$$

one obtains (cf. (1.2)),

$$\begin{aligned} C_{00} =& a_0B_{00} - b_0A_{00}, \\ C_{10} =& a_0B_{01} + a_1B_{00} - b_0A_{01} - b_1A_{00}, \\ C_{01} =& a_0B_{02} + a_2B_{00} - b_0A_{02} - b_2A_{00}, \\ C_{20} =& a_0B_{11} + a_1B_{01} - b_0A_{11} - b_1A_{01}, \\ C_{11} =& a_0B_{12} + a_1B_{02} + a_2B_{01} - b_0A_{12} - b_1A_{02} - b_2A_{01}, \\ C_{02} =& a_0B_{22} + a_2B_{02} - b_0A_{22} - b_2A_{02}, \\ C_{30} =& a_1B_{11} - b_1A_{11}, \\ C_{21} =& a_1B_{12} + a_2B_{11} - b_1A_{12} - b_2A_{11}, \\ C_{12} =& a_1B_{22} + a_2B_{12} - b_1A_{22} - b_2A_{12}, \\ C_{03} =& a_2B_{22} - b_2A_{22}. \end{aligned}$$

The lifting of e_0 is

$$
\begin{aligned}
(e_0\circ\nu)(A) =& 3(a_0B_{00}-b_0A_{00})d_0^2+\\
&2(a_0B_{01}+a_1B_{00}-b_0A_{01}-b_1A_{00})d_0d_1+\\
&2(a_0B_{02}+a_2B_{00}-b_0A_{02}-b_2A_{00})d_0d_2+\\
&(a_0B_{11}+a_1B_{01}-b_0A_{11}-b_1A_{01})d_1^2+\\
&(a_0B_{12}+a_1B_{02}+a_2B_{01}-b_0A_{12}-b_1A_{02}-b_2A_{01})d_1d_2+\\
&(a_0B_{22}+a_2B_{02}-b_0A_{22}-b_2A_{02})d_2^2=\\
&A_{00}(-3b_0d_0^2-2b_1d_0d_1-2b_2d_0d_2)+\\
&A_{01}(-2b_0d_0d_1-b_1d_1^2-b_2d_1d_2)+\\
&A_{02}(-2b_0d_0d_2-b_1d_1d_2-b_2d_2^2)+\\
&A_{11}(-b_0d_1^2)+A_{12}(-b_0d_1d_2)+A_{22}(-b_0d_2^2)+\\
&B_{00}(3a_0d_0^2+2a_1d_0d_1+2a_2d_0d_2)+\\
&B_{01}(2a_0d_0d_1+a_1d_1^2+a_2d_1d_2)+\\
&B_{02}(2a_0d_0d_2+a_1d_1d_2+a_2d_2^2)+\\
&B_{11}(a_0d_1^2)+B_{12}(a_0d_1d_2)+B_{22}(a_0d_2^2)=\\
&-b_0(A_{00}d_0^2+A_{01}d_0d_1+A_{02}d_0d_2+A_{11}d_1^2+A_{12}d_1d_2+A_{22}d_2^2)+\\
&a_0(B_{00}d_0^2+B_{01}d_0d_1+B_{02}d_0d_2+B_{11}d_1^2+B_{12}d_1d_2+B_{22}d_2^2)=\\
&-b_0q_1(d_0,d_1,d_2)+a_0q_2(d_0,d_1,d_2)=-b_0f_3+a_0f_4.
\end{aligned}
$$

We used here that

$$a_0d_0+a_1d_1+a_2d_2=0 \quad\text{and}\quad b_0d_0+b_1d_1+b_2d_2=0.$$

Analogously one obtains $(e_1\circ\nu)(A)=-b_1f_3+a_1f_4$ and $(e_2\circ\nu)(A)=-b_2f_3+a_2f_4$. □

Since in X locally at least one of d_0, d_1, d_2 is a unit, we conclude that the zero set of the liftings of e_0, e_1, e_2 coincides with the zero set of f_3 and f_4.

Lemma 1.7. *0) X_8 is an algebraic subvariety of codimension 2 in X given by the equations $f_3=f_4=0$;*
1) X_8 is smooth;
2) X_8 is a global complete intersection in X.

Proof. 0) Follows from (1.10).
1)Let $J(f_3,f_4)$ be the Jacobian matrix of f_3 and f_4. We can consider this 2×18 matrix as a block matrix

$$J(f_3,f_4)=\begin{pmatrix} J_{a_i} & J_{b_i} & J_{A_{ij}} & J_{B_{ij}}\end{pmatrix},$$

where

$$J_{a_i}=J_{(a_0,a_1,a_2)}(f_3,f_4)$$

is the jacobian matrix of f_3 and f_4 with respect to the variables (a_i), $i=0,1,2$;

$$J_{b_i}=J_{(b_0,b_1,b_2)}(f_3,f_4)$$

is the jacobian matrix of f_3 and f_4 with respect to the variables (b_i), $i=0,1,2$;

$$J_{A_{ij}}=J_{(A_{00},\dots,A_{22})}(f_3,f_4)$$

is the jacobian matrix of f_3 and f_4 with respect to the variables (A_{ij}), $i, j \in \{0, 1, 2\}$, and

$$J_{B_{ij}} = J_{(B_{00},\dots,B_{22})}(f_3, f_4)$$

is the jacobian matrix of f_3 and f_4 with respect to the variables (B_{ij}), $i, j \in \{0, 1, 2\}$.

Then

$$\begin{pmatrix} J_{A_{ij}} & J_{B_{ij}} \end{pmatrix} = \begin{pmatrix} d_0^2 & d_0d_1 & \dots & d_2^2 & 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & d_0^2 & d_0d_1 & \dots & d_2^2 \end{pmatrix}.$$

Since A is an element of $X_8 \subseteq X$, the forms z_1 and z_2 ale linear independent, i. e., at least one of the minors d_0, d_1, d_2 is not zero. Therefore, one concludes, that the matrix $J(f_3, f_4)$ has full rank, i. e., X_8 is a smooth subvariety of X.

2) X_8 is a locally complete intersection in X as a smooth subvariety of a smooth variety (see for example [12], II, Example 8.22.1). Since there are two global equation (1.9) of X_8, we conclude that X_8 is a global complete intersection. □

Let us calculate the tangent equations of X_8 in X at a point $A \in X_8$,

$$A = \begin{pmatrix} a_0x_0 + a_1x_1 + a_2x_2 & A_{00}x_0^2 + A_{01}x_0x_1 + \dots + A_{22}x_2^2 \\ b_0x_0 + b_1x_1 + b_2x_2 & B_{00}x_0^2 + B_{01}x_0x_1 + \dots + B_{22}x_2^2 \end{pmatrix}.$$

Then the Jacobian matrix $J(f_3, f_4)$ of f_3 and f_4 is

$$\begin{pmatrix} \left(\frac{\partial f_3}{\partial a_k}\right)_{k=0,1,2} & \left(\frac{\partial f_3}{\partial b_k}\right)_{k=0,1,2} & \left(\frac{\partial f_3}{\partial A_{ij}}\right)_{ij} & \left(\frac{\partial f_3}{\partial B_{ij}}\right)_{ij} \\ \\ \left(\frac{\partial f_4}{\partial a_k}\right)_{k=0,1,2} & \left(\frac{\partial f_4}{\partial b_k}\right)_{k=0,1,2} & \left(\frac{\partial f_4}{\partial A_{ij}}\right)_{ij} & \left(\frac{\partial f_4}{\partial B_{ij}}\right)_{ij} \end{pmatrix}.$$

Since $\frac{\partial f_3}{\partial B_{ij}} = 0$ and $\frac{\partial f_4}{\partial A_{ij}} = 0$, we get

$$J(f_3, f_4) = \begin{pmatrix} \left(\frac{\partial f_3}{\partial a_k}\right)_{k=0,1,2} & \left(\frac{\partial f_3}{\partial b_k}\right)_{k=0,1,2} & \left(\frac{\partial f_3}{\partial A_{ij}}\right)_{ij} & 0 \\ \\ \left(\frac{\partial f_4}{\partial a_k}\right)_{k=0,1,2} & \left(\frac{\partial f_4}{\partial b_k}\right)_{k=0,1,2} & 0 & \left(\frac{\partial f_4}{\partial B_{ij}}\right)_{ij} \end{pmatrix}.$$

One clearly has $\frac{\partial f_3}{\partial A_{ij}} = d_id_j$ and $\frac{\partial f_4}{\partial B_{ij}} = d_id_j$. We have also

$$\frac{\partial f_3}{\partial a_k} = \sum_{ij} A_{ij} \left(\frac{\partial d_i}{\partial a_k} d_j + \frac{\partial d_j}{\partial a_k} d_i \right), \quad \frac{\partial f_3}{\partial b_k} = \sum_{ij} A_{ij} \left(\frac{\partial d_i}{\partial b_k} d_j + \frac{\partial d_j}{\partial b_k} d_i \right),$$

$$\frac{\partial f_4}{\partial a_k} = \sum_{ij} B_{ij} \left(\frac{\partial d_i}{\partial a_k} d_j + \frac{\partial d_j}{\partial a_k} d_i \right), \quad \frac{\partial f_4}{\partial b_k} = \sum_{ij} B_{ij} \left(\frac{\partial d_i}{\partial b_k} d_j + \frac{\partial d_j}{\partial b_k} d_i \right).$$

Let us denote

$$\alpha_{ij,k} := \left(\frac{\partial d_i}{\partial a_k} d_j + \frac{\partial d_j}{\partial a_k} d_i \right) \quad \text{and} \quad \beta_{ij,k} := \left(\frac{\partial d_i}{\partial b_k} d_j + \frac{\partial d_j}{\partial b_k} d_i \right).$$

In this notations

$$\frac{\partial f_3}{\partial a_k} = \sum_{ij} A_{ij}\alpha_{ij,k}, \quad \frac{\partial f_3}{\partial b_k} = \sum_{ij} A_{ij}\beta_{ij,k}, \quad \frac{\partial f_4}{\partial a_k} = \sum_{ij} B_{ij}\alpha_{ij,k}, \quad \frac{\partial f_4}{\partial b_k} = \sum_{ij} B_{ij}\beta_{ij,k}.$$

Thus
$$B = \begin{pmatrix} \xi_0 x_0 + \xi_1 x_1 + \xi_2 x_2 & \xi_{00} x_0^2 + \cdots + \xi_{22} x_2^2 \\ \eta_0 x_0 + \eta_1 x_1 + \eta_2 x_2 & \eta_{00} x_0^2 + \cdots + \eta_{22} x_2^2 \end{pmatrix}$$
is a tangent vector if and only if
$$\sum_k \left(\sum_{ij} A_{ij} \alpha_{ij,k} \right) \xi_k + \sum_k \left(\sum_{ij} A_{ij} \beta_{ij,k} \right) \eta_k + \sum_{ij} d_i d_j \xi_{ij} = 0$$
and
$$\sum_k \left(\sum_{ij} B_{ij} \alpha_{ij,k} \right) \xi_k + \sum_k \left(\sum_{ij} B_{ij} \beta_{ij,k} \right) \eta_k + \sum_{ij} d_i d_j \eta_{ij} = 0.$$
Hence the equations
$$\begin{gathered} T_1(A) := \sum_k \left(\sum_{ij} A_{ij} \alpha_{ij,k} \right) \xi_k + \sum_k \left(\sum_{ij} A_{ij} \beta_{ij,k} \right) \eta_k + \sum_{ij} d_i d_j \xi_{ij} \\ \text{and} \\ T_2(A) := \sum_k \left(\sum_{ij} B_{ij} \alpha_{ij,k} \right) \xi_k + \sum_k \left(\sum_{ij} B_{ij} \beta_{ij,k} \right) \eta_k + \sum_{ij} d_i d_j \eta_{ij} \end{gathered} \tag{1.11}$$
are tangent equations at point A.

Let us calculate $\alpha_{ij,k}$ and $\beta_{ij,k}$. One easily calculates
$$\left(\frac{\partial d_i}{\partial a_k} \right)_{ik} = \begin{pmatrix} 0 & b_2 & -b_1 \\ -b_2 & 0 & b_0 \\ b_1 & -b_0 & 0 \end{pmatrix}, \quad \left(\frac{\partial d_i}{\partial b_k} \right)_{ik} = \begin{pmatrix} 0 & -a_2 & a_1 \\ a_2 & 0 & -a_0 \\ -a_1 & a_0 & 0 \end{pmatrix}.$$
Let
$$(s_{ik}) = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}$$
and let us use the following notations. For i and j from $\{0,1,2\}$ put
$$a_{\bar{ij}} := \begin{cases} 0, & i = j \\ a_{\nu(i,j)}, & i \neq j \end{cases},$$
where $\nu(i,j)$ is the only element in the set $\{0,1,2\} \setminus \{i,j\}$ for $i \neq j$. Analogously let us define
$$b_{\bar{ij}} := \begin{cases} 0, & i = j \\ b_{\nu(i,j)}, & i \neq j \end{cases}.$$
Then
$$\frac{\partial d_i}{\partial a_k} = s_{ik} b_{\bar{ik}}, \quad \frac{\partial d_i}{\partial b_k} = s_{ki} a_{\bar{ik}},$$
and one finally obtains
$$\alpha_{ij,k} = s_{ik} b_{\bar{ik}} d_j + s_{jk} b_{\bar{jk}} d_i \quad \text{and} \quad \beta_{ij,k} = s_{ki} a_{\bar{ik}} d_j + s_{kj} a_{\bar{jk}} d_i. \tag{1.12}$$

Example 1.8. *Let*

$$A = \begin{pmatrix} x_1 & q_1 \\ x_2 & q_2 \end{pmatrix}, \quad q_1 = A_{01}x_0x_1 + \cdots + A_{22}x_2^2, \quad q_2 = B_{01}x_0x_1 + \cdots + B_{22}x_2^2.$$

Then $p = \langle 1, 0, 0\rangle$ and we claim then the Jacobian matrix $J(f_3, f_4)$ at point A is

$$\begin{pmatrix} -A_{01} & 0 & 0 & -A_{02} & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -B_{01} & 0 & 0 & -B_{02} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

i. e., tangent equations at A for

$$B = \begin{pmatrix} \xi_0x_0 + \xi_1x_1 + \xi_2x_2 & \xi_{00}x_0^2 + \ldots \xi_{22}x_2^2 \\ \eta_0x_0 + \eta_1x_1 + \eta_2x_2 & \eta_{00}x_0^2 + \ldots \eta_{22}x_2^2 \end{pmatrix},$$

are

$$\begin{cases} \xi_{00} = A_{01}\xi_0 + A_{02}\eta_0 \\ \eta_{00} = B_{01}\xi_0 + B_{02}\eta_0. \end{cases} \tag{1.13}$$

Proof. We have $f_3 = q_2(p) = A_{00}d_0^2 + A_{01}d_0d_1 + \cdots + A_{22}d_2^2$. Note that $d_0(A) = 1$ and $d_1(A) = d_2(A) = 0$. Since $d_0 = a_1b_2 - a_2b_1$, we have $d_0(A) = 1$ and

$$\begin{aligned} &\frac{\partial d_0}{\partial a_0}(A) = 0, && \frac{\partial d_0}{\partial a_1}(A) = b_2(A) = 1, && \frac{\partial d_0}{\partial a_2}(A) = -b_1(A) = 0, \\ &\frac{\partial d_0}{\partial b_0}(A) = 0, && \frac{\partial d_0}{\partial b_1}(A) = -a_2(A) = 0, && \frac{\partial d_0}{\partial b_2}(A) = a_1(A) = 1. \end{aligned}$$

From $d_1 = -(a_0b_2 - a_2b_0)$, we have $d_1(A) = 0$ and

$$\begin{aligned} &\frac{\partial d_1}{\partial a_0}(A) = -b_2(A) = -1, && \frac{\partial d_1}{\partial a_1}(A) = 0, && \frac{\partial d_1}{\partial a_2}(A) = b_0(A) = 0, \\ &\frac{\partial d_1}{\partial b_0}(A) = a_2(A) = 0, && \frac{\partial d_1}{\partial b_1}(A) = 0, && \frac{\partial d_1}{\partial b_2}(A) = -a_0(A) = 0. \end{aligned}$$

From $d_2 = a_0b_1 - a_1b_0$, we have $d_2(A) = 0$ and

$$\begin{aligned} &\frac{\partial d_2}{\partial a_0}(A) = b_1(A) = 0, && \frac{\partial d_2}{\partial a_1}(A) = -b_0(A) = 0, && \frac{\partial d_2}{\partial a_2}(A) = 0, \\ &\frac{\partial d_2}{\partial b_0}(A) = -a_1(A) = -1, && \frac{\partial d_2}{\partial b_1}(A) = a_0(A) = 0, && \frac{\partial d_2}{\partial b_2}(A) = 0. \end{aligned}$$

Straightforward calculations lead to $\frac{\partial f_3}{\partial a_0} = -A_{01}$, $\frac{\partial f_3}{\partial b_0} = -A_{02}$, $\frac{\partial f_3}{\partial A_{00}} = 1$ and all the other derivatives are zero.

Similarly one shows that $\frac{\partial f_4}{\partial a_0} = -B_{01}$, $\frac{\partial f_4}{\partial b_0} = -B_{02}$, $\frac{\partial f_4}{\partial B_{00}} = 1$ and all the others derivatives are zero. This proves the required statement $\square$

1.2 R-bundles

We propose here a construction of sheaves which we will use as replacements of the singular $3m+1$ sheaves on $\mathbb{P}_2$.

1.2.1 Construction of new one parameter families.

Singular $3m+1$ sheaves as one-dimensional limits of non-singular sheaves.

Consider some matrix $A = \begin{pmatrix} z_1 & q_1 \\ z_2 & q_2 \end{pmatrix}$ from X_8. Let p denote the point where both z_1 and z_2 vanish. Take a matrix $B \in \mathbb{k}^{18}$, $B = \begin{pmatrix} w_1 & p_1 \\ w_2 & p_2 \end{pmatrix}$, and consider the morphism

$$l_B : \mathbb{k} \to \mathbb{k}^{18}, \quad t \mapsto A + tB.$$

Let $U = l_B^{-1}(X)$. Since l_B is a morphism, U is an open set in $\mathbb{k}$. In this way we obtain the morphism

$$l_B|_U : U \to X. \tag{1.14}$$

By abuse of notation we will also denote it by l_B.

So, we obtain a (one parameter) family $3m+1$ sheaves over $U \subseteq \mathbb{k}$, i. e., the sheaf $\mathcal{E}$ on $U \times \mathbb{P}_2$, such that for every t from U the restriction $\mathcal{E}_t$ of $\mathcal{E}$ to the fibre $\{t\} \times \mathbb{P}_2$ is given by the matrix $A_t := A + tB \in X$. In other words the sheaf $\mathcal{E}$ is given by the resolution

$$0 \to 2\mathcal{O}_{U\times\mathbb{P}_2}(-2H) \xrightarrow{A_t} \mathcal{O}_{U\times\mathbb{P}_2}(-H) \oplus \mathcal{O}_{U\times\mathbb{P}_2} \to \mathcal{E} \to 0. \tag{1.15}$$

Here H is the pull back of a line $h \subseteq \mathbb{P}_2$. We choose h such that the point p does not lie on h.

Remark 1.9. *Note that by shrinking U we can also assume that $\mathcal{E}_t$ is locally free on its support for all $t \in U$, $t \neq 0$. We can interpret now the singular fibre $\mathcal{E}_0$ as a limit for $t \to 0$ of nonsingular sheaves $\mathcal{E}_t$.*

Blow up of $U \times \mathbb{P}_2$.

Consider the point $(0, p) \in U \times \mathbb{P}_2$. Let

$$\sigma : \widetilde{U \times \mathbb{P}_2} \to U \times \mathbb{P}_2$$

be the blowing up of $U \times \mathbb{P}_2$ at $(0, p)$. Let us denote $Z := \widetilde{U \times \mathbb{P}_2}$. Let D_1 denote the exceptional divisor of σ.

One can describe Z explicitly in coordinates as a subvariety of $U \times \mathbb{P}_2 \times \mathbb{P}_2$. Let us fix some complementary to z_1 and z_2 linear form z_0, i. e, a linear form z_0 such that the forms z_0, z_1, z_2 constitute a basis of the space of linear forms on $\mathbb{P}_2$. Then we can consider Z as a subvariety in

$$U \times \mathbb{P}_2 \times \mathbb{P}_2 = \{(t, \langle x_0, x_1, x_2\rangle, \langle u_0, u_1, u_2\rangle)\},$$

given by the 2×2 minors of the matrix $\begin{pmatrix} tz_0 & z_1 & z_2 \\ u_0 & u_1 & u_2 \end{pmatrix}$, i. e., by the equations

$$\begin{cases} tz_0u_1 = u_0z_1, \\ tz_0u_2 = u_0z_2, \\ z_1u_2 = z_2u_1. \end{cases} \tag{1.16}$$

Let us consider the map

$$Z \xrightarrow{\sigma} U \times \mathbb{P}_2 \xrightarrow{p_1} U$$

and its fibres Z_t over $t \in U$. Since the restriction of σ to $Z \setminus Z_0$ is an isomorphism, we conclude that $Z_t \cong \mathbb{P}_2$ for $t \neq 0$.

Taking $t = 0$ in (1.16)), we obtain that $Z_0 \subseteq \{0\} \times \mathbb{P}_2 \times \mathbb{P}_2$ is given by the equations

$$\begin{cases} u_0 z_1 = 0 \\ u_0 z_2 = 0 \\ z_1 u_2 - z_2 u_1 = 0. \end{cases}$$

Thus

$$Z_0 = \{(0, \langle x_0, x_1, x_2\rangle, \langle 0, u_1, u_2\rangle) |\ z_1 u_2 - z_2 u_1 = 0\} \cup \{u_0 \neq 0, t = z_1 = z_2 = 0\}.$$

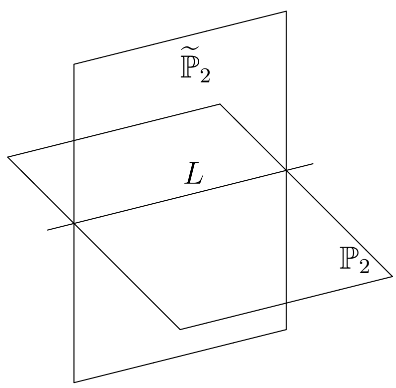
One easily sees that the first set is the proper transform of $\{0\} \times \mathbb{P}_2$ under σ, it is isomorphic to the blow up $\widetilde{\mathbb{P}}_2$ of $\mathbb{P}_2$ at the point p (recall that p is given by $z_1 = z_2 = 0$). We will also denote it by D_0.

The second set is the projective plane (exceptional divisor of $\sigma : Z \to U \times \mathbb{P}_2$)

$$D_1 \cong \mathbb{P}_2 = \{(0, p, \langle u_0, u_1, u_2\rangle)\}$$

without the line given by $u_0 = 0$. Therefore, Z_0 is isomorphic to the blowing up of $\mathbb{P}_2$ at the point p with the projective plane $\mathbb{P}_2$ attached along the exceptional divisor of this blowing up: $Z_0 = \widetilde{\mathbb{P}}_2 \cup D_1$ and $\widetilde{\mathbb{P}}_2 \cap D_1$ is the line $L := \{t = u_0 = z_1 = z_2\} = \{(0, p, \langle 0, u_1, u_2\rangle)\}$. We proved the following

Lemma 1.10. *The fibres Z_t of $Z \xrightarrow{\sigma} U \times \mathbb{P}_2 \xrightarrow{p_1} U$ are all of dimension 2. Moreover Z_t is isomorphic to $\mathbb{P}_2$ for $t \neq 0$, and $Z_0 \cong \widetilde{\mathbb{P}}_2 \cup \mathbb{P}_2$, where $\widetilde{\mathbb{P}}_2$ is the blowing up of the projective plane $\mathbb{P}_2$ at point p, and $\mathbb{P}_2$ is attached to $\widetilde{\mathbb{P}}_2$ along the exceptional divisor of the blowing up $\widetilde{\mathbb{P}}_2 \to \mathbb{P}_2$.*



Space Z_0.

Definition 1.11. *We will denote by $\hat{\mathbb{P}}_2$ the space Z_0 we described above, i. e.,*

$$\hat{\mathbb{P}}_2 = \widetilde{\mathbb{P}}_2 \cup \mathbb{P}_2 \subseteq \mathbb{P}_2 \times \mathbb{P}_2$$

given by the equations $u_0 z_1 = u_0 z_2 = z_1 u_2 - z_2 u_1 = 0$, where $\langle u_0, u_1, u_2\rangle$ and $\langle x_0, x_1, x_2\rangle$ are points in the first and the second $\mathbb{P}_2$ from the product $\mathbb{P}_2 \times \mathbb{P}_2$ respectively. We will also denote by D_0 the component $\widetilde{\mathbb{P}}_2$ and by D_1 the component $\mathbb{P}_2$ of $\hat{\mathbb{P}}_2$.

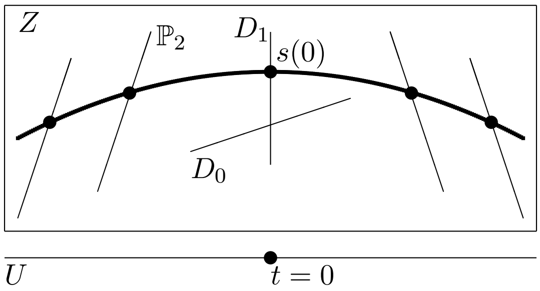
Proposition 1.12. *The morphism* $Z \xrightarrow{\sigma} U \times \mathbb{P}_2 \xrightarrow{p_1} U$ *is flat.*

Proof. Since both Z and U are regular, $\dim Z = 3$, $\dim U = 1$, and $\dim Z_t = 2 = \dim Z - \dim U$ for all $t \in U$, the required statement follows from Theorem A.2. □

Remark 1.13. *Note that there is a natural section of the projection* $U \times \mathbb{P}_2 \xrightarrow{p_1} U$*. It is given by*

$$U \ni t \mapsto (z_1 + tw_1) \wedge (z_1 + tw_1) \in \mathbb{P}_2.$$

This lifts uniquely to a section $s : U \to Z$ *of* $Z \xrightarrow{\sigma} U \times \mathbb{P}_2 \xrightarrow{p_1} U$*. In particular we obtain a point* $s(0) \in D_1 \setminus L$*.*



New one parameter families, main construction.

Applying σ^* to the sequence (1.15) we obtain the sequence

$$0 \to 2\mathcal{O}_Z(-2H) \xrightarrow{\sigma^*(A_t)} \mathcal{O}_Z(-H) \oplus \mathcal{O}_Z \to \sigma^*(\mathcal{E}) \to 0,$$

which remains exact because the sheaf $O_Z(-2H)$ is locally free and, therefore, has no torsion (morphism $\sigma^*(A_t)$ is injective outside of D_1, therefore its kernel may leave only on D_1).

Note that the support of $\sigma^*(\mathcal{E})$ on the exceptional divisor $D = D_1$ is the whole plane D_1. We are going now to modify this sheaf in order to obtain a sheaf with one-dimensional support.

Note that there is a canonical section $s \in \Gamma(Z, \mathcal{O}_Z(D))$, which gives us the exact sequence

$$0 \to \mathcal{O}_Z(-D) \xrightarrow{s} \mathcal{O}_Z \to \mathcal{O}_D \to 0.$$

Tensoring with $\mathcal{O}(D)$, one gets the exact sequence

$$0 \to \mathcal{O}_Z \xrightarrow{s} \mathcal{O}_Z(D) \to \mathcal{O}_D \otimes \mathcal{O}_Z(D) \to 0.$$

Tensoring this once more with $2\mathcal{O}_Z(-2H)$, we obtain the injective map

$$0 \to 2\mathcal{O}_Z(-2H) \xrightarrow{\left(\begin{smallmatrix} s & 0 \\ 0 & s \end{smallmatrix}\right)} 2\mathcal{O}_Z(-2H + D).$$

Lemma 1.14. $\sigma^*(A_t)$ *factorizes uniquely through* s*, i. e., there exists*

$$2\mathcal{O}_Z(-2H + D) \xrightarrow{\widetilde{A}_t} \mathcal{O}_Z(-H) \oplus \mathcal{O}_Z$$

such that the diagram

$$\begin{tikzcd}
2\mathcal{O}_Z(-2H) \arrow[r, "\sigma^*(A_t)"] \arrow[d, "\left(\begin{smallmatrix} s & 0 \\ 0 & s \end{smallmatrix}\right)"'] & \mathcal{O}_Z(-H) \oplus \mathcal{O}_Z \\
2\mathcal{O}_Z(-2H + D) \arrow[ur, "\widetilde{A}_t"] &
\end{tikzcd}$$

commutes.

Proof. Follows from Lemma 1.15. □

Lemma 1.15. *Let $\mathfrak{X}$ be a variety and let $D \subseteq \mathfrak{X}$ be a divisor given by the sequence*

$$0 \to \mathcal{O}_{\mathfrak{X}}(-D) \xrightarrow{s} \mathcal{O}_{\mathfrak{X}} \to \mathcal{O}_D \to 0.$$

Let $g : \mathcal{O}_{\mathfrak{X}} \to \mathcal{L}$ be a section of a (line) bundle $\mathcal{L}$ on $\mathfrak{X}$. Suppose that g vanishes on D. Then g factors uniquely through s, in other words there exists a unique $\widetilde{g} \in \Gamma(\mathfrak{X}, \mathcal{L} \otimes \mathcal{O}_{\mathfrak{X}}(-D))$ such that the diagram

$$\begin{tikzcd}
\mathcal{O}_{\mathfrak{X}} \arrow[r, "g"] \arrow[d, "s"'] & \mathcal{L} \\
\mathcal{O}_{\mathfrak{X}}(D) \arrow[ur, "\widetilde{g}"'] &
\end{tikzcd}$$

commutes.

Proof. The statement follows from the diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{O}_{\mathfrak{X}}(-D) \arrow[r, "s"] \arrow[d, "g"'] & \mathcal{O}_{\mathfrak{X}} \arrow[r] \arrow[d, "g"'] \arrow[dl, dotted, "\exists!\widetilde{g}"'] \arrow[dr, dotted, "0"] & \mathcal{O}_D \arrow[r] \arrow[d, "g|_D=0"] & 0 \\
0 \arrow[r] & \mathcal{O}_{\mathfrak{X}}(-D) \otimes \mathcal{L} \arrow[r, "s"] & \mathcal{O}_{\mathfrak{X}} \otimes \mathcal{L} \arrow[r] & \mathcal{O}_D \otimes \mathcal{L} \arrow[r] & 0
\end{tikzcd}$$

by using the universal property of kernel. □

Remark 1.16. *Note that $\widetilde{A}_t$ is injective since $2\mathcal{O}_Z(-2H + D)$ is torsion free and since $\left(\begin{smallmatrix} s & 0 \\ 0 & s \end{smallmatrix}\right)$ is an isomorphism outside of the exceptional divisor D.*

Lemma 1.17. *There is the following commutative diagram with exact rows and columns:*

$$\begin{tikzcd}
 & & & 0 \arrow[d] & \\
 & 0 \arrow[d] & & 2\mathcal{O}_D(-1) \arrow[d] & \\
0 \arrow[r] & 2\mathcal{O}_Z(-2H) \arrow[r, "\sigma^*(A_t)"] \arrow[d, "\left(\begin{smallmatrix} s & 0 \\ 0 & s \end{smallmatrix}\right)"] & \mathcal{O}_Z(-H) \oplus \mathcal{O}_Z \arrow[r] \arrow[d, equal] & \sigma^*\mathcal{E} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & 2\mathcal{O}_Z(-2H + D) \arrow[r, "\widetilde{A}_t"] \arrow[d] & \mathcal{O}_Z(-H) \oplus \mathcal{O}_Z \arrow[r] & \widetilde{\mathcal{E}} \arrow[r] \arrow[d] & 0. \\
 & 2\mathcal{O}_D(-1) \arrow[d] & & 0 & \\
 & 0 & & &
\end{tikzcd}$$

Proof. By Lemma 1.15, using the snake lemma one obtains the following commutative diagram with exact rows and columns

$$
\begin{tikzcd}
 & & & 0 \arrow[d] & \\
 & 0 \arrow[d] & & \mathcal{C} \arrow[d] & \\
0 \arrow[r] & 2\mathcal{O}_Z(-2H) \arrow[r, "\sigma^*(A_t)"] \arrow[d, "\left(\begin{smallmatrix} s & 0 \\ 0 & s \end{smallmatrix}\right)"] & \mathcal{O}_Z(-H) \oplus \mathcal{O}_Z \arrow[r] \arrow[d, equal] & \sigma^*\mathcal{E} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & 2\mathcal{O}_Z(-2H+D) \arrow[r, "\widetilde{A}_t"] \arrow[d] & \mathcal{O}_Z(-H) \oplus \mathcal{O}_Z \arrow[r] & \widetilde{\mathcal{E}} \arrow[r] \arrow[d] & 0, \\
 & \mathcal{C} \arrow[d] & & 0 & \\
 & 0 & & &
\end{tikzcd}
$$

where $\mathcal{C} = \mathcal{O}_D \otimes \mathcal{O}_Z(-2H+D)$. Since by Lemma A.3 $\mathcal{O}_D \otimes \mathcal{O}_Z(D) = \mathcal{O}_D(-1)$, we obtain that $\mathcal{C} = 2\mathcal{O}_D \otimes \mathcal{O}_Z(D) \otimes \mathcal{O}_Z(-2H) = 2\mathcal{O}_D(-1) \otimes \mathcal{O}_Z(-2H)$. Since the line $h \subseteq \mathbb{P}_2$ does not contain the point p, we conclude that H and D do not intersect. Therefore, $\mathcal{C} = 2\mathcal{O}_D(-1) \otimes \mathcal{O}_Z(-2H) = 2\mathcal{O}_D(-1)$. This proves the lemma. $\square$

Proposition 1.18. $\sigma_*(\widetilde{\mathcal{E}}) \cong \mathcal{E}$.

Proof. By Lemma A.4 $R^p\sigma_*(\mathcal{C}) = R^p\sigma_*(\mathcal{O}_D(-1)) = 0$, $p \geqslant 0$. Therefore, after applying σ_* to the exact sequence

$$0 \to \mathcal{C} \to \sigma^*\mathcal{E} \to \widetilde{\mathcal{E}} \to 0,$$

we obtain $\sigma_*(\sigma^*\mathcal{E}) \cong \sigma_*(\widetilde{\mathcal{E}})$. By Lemma A.8 $\sigma_*(\sigma^*\mathcal{E}) \cong \mathcal{E}$, which proves the lemma. $\square$

Remark 1.19. *In fact we have even more. Applying σ_* to the diagram defining $\widetilde{\mathcal{E}}$ we obtain*

$$
\begin{tikzcd}
0 \arrow[r] & 2\mathcal{O}_{U\times\mathbb{P}_2}(-2H) \arrow[r, "\sigma_*(\sigma^*(A_t))"] \arrow[d, "\sigma_*\left(\begin{smallmatrix} s & 0 \\ 0 & s \end{smallmatrix}\right)", "\cong"'] & \mathcal{O}_{U\times\mathbb{P}_2}(-H) \oplus \mathcal{O}_{U\times\mathbb{P}_2} \arrow[r] \arrow[d, equal] & \sigma_*(\sigma^*\mathcal{E}) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & 2\mathcal{O}_{U\times\mathbb{P}_2}(-2H) \arrow[r, "\sigma_*(\widetilde{A}_t)"] & \mathcal{O}_{U\times\mathbb{P}_2}(-H) \oplus \mathcal{O}_{U\times\mathbb{P}_2} \arrow[r] & \sigma_*(\widetilde{\mathcal{E}}) \arrow[r] & 0,
\end{tikzcd}
$$

where the above row is isomorphic by Lemma A.8 to the resolution defining $\mathcal{E}$:

$$0 \to 2\mathcal{O}_{U\times\mathbb{P}_2}(-2H) \xrightarrow{A_t} \mathcal{O}_{U\times\mathbb{P}_2}(-H) \oplus \mathcal{O}_{U\times\mathbb{P}_2} \to \mathcal{E} \to 0.$$

We can assume without loss of generality that $z_1 = x_1$, $z_2 = x_2$ (make if necessary the change of coordinates in $\mathbb{P}_2$). Take as a complementary form $z_0 = x_0$. In this case $p = \langle 1, 0, 0 \rangle$. The matrix A is then

$$\begin{pmatrix} x_1 & q_1 \\ x_2 & q_2 \end{pmatrix}, \quad q_1 = A_{01}x_0x_1 + \cdots + A_{22}x_2^2, \quad q_2 = B_{01}x_0x_1 + \cdots + B_{22}x_2^2. \tag{1.17}$$

Note, that since both q_1 and q_2 vanish at $p = \langle 1, 0, 0 \rangle$ there are no monomials x_0^2 in the expressions of q_1 and q_2. Let

$$B = \begin{pmatrix} w_1 & p_1 \\ w_2 & p_2 \end{pmatrix},$$

and let

$$\begin{aligned} w_1 &= \xi_0 x_0 + \xi_1 x_1 + \xi_2 x_2, \\ w_1 &= \eta_0 x_0 + \eta_1 x_1 + \eta_2 x_2, \\ p_1 &= \xi_{00} x_0^2 + \cdots + \xi_{22} x_2^2, \\ p_2 &= \eta_{00} x_0^2 + \cdots + \eta_{22} x_2^2. \end{aligned}$$

Lemma 1.20. *$\mathcal{O}_Z(D) \cong \mathcal{O}_Z(H - F) \cong \mathcal{O}_Z(H) \otimes \mathcal{O}_Z(-F)$, where $\mathcal{O}_Z(F) \cong \pi^*\mathcal{O}_{\mathbb{P}_2}(1)$ and $\pi : Z \to \mathbb{P}_2$ is the canonical projection onto the second $\mathbb{P}_2$ (recall that $Z \subseteq U \times \mathbb{P}_2 \times \mathbb{P}_2$).*

Proof. Note that $\mathcal{O}_Z(H)$ is given by the cocycle $h_{ij} = \frac{x_j}{x_i}$ on the open covering $Z = \bigcup U_i$, $U_i = Z(x_i) = \{x_i \neq 0\}$. The sheaf $\mathcal{O}_Z(F)$ is given by the cocycle $f_{kl} = \frac{u_l}{u_k}$ on the covering $Z = \bigcup V_k$, $V_k = Z(u_k) = \{u_k \neq 0\}$. Then $\mathcal{O}_Z(-F)$ is given by the cocycle $f_{kl}^{-1} = \frac{u_k}{u_l}$ and therefore $\mathcal{O}_Z(H) \otimes \mathcal{O}_Z(-F)$ is given by the cocycle $g_{ik,jl} = \frac{x_j}{x_i} \cdot \frac{u_k}{u_l}$ on the covering $Z = \bigcup W_{ik}$, $W_{ik} = Z(x_i, u_k) = \{x_i \neq 0, u_k \neq 0\}$.

Note that the local defining functions of the divisor D are

$$\gamma_{ik} = \begin{cases} \frac{tx_0}{x_i} & \text{on } Z(x_i, u_0), \\ \frac{x_1}{x_i} & \text{on } Z(x_i, u_1), \\ \frac{x_2}{x_i} & \text{on } Z(x_i, u_2). \end{cases}$$

Thus the cocycle of $\mathcal{O}_Z(D)$ is $\frac{\gamma_{ik}}{\gamma_{jl}}$. Using the defining equations of the blow up Z one easily calculates $\frac{\gamma_{ik}}{\gamma_{jl}} = \frac{x_j}{x_i} \cdot \frac{u_k}{u_l}$. This coincides with the cocycle $g_{ik,jl}$ of $\mathcal{O}_Z(H) \otimes \mathcal{O}_Z(-F)$. This proves the required statement. □

Remark 1.21. *In this lemma we proved the equivalence of divisors $D \sim H - F$.*

As a consequence of Lemma 1.20 after the substitution of D by $H - F$ in the diagram from Lemma 1.17 we obtain the diagram with exact rows and columns:

$$\begin{tikzcd} & & & 0 \arrow[d] & \\ & 0 \arrow[d] & & 2\mathcal{O}_D(-1) \arrow[d] & \\ 0 \arrow[r] & 2\mathcal{O}_Z(-2H) \arrow[r, "\sigma^*(A_t)"] \arrow[d, "\left(\begin{smallmatrix} s & 0 \\ 0 & s \end{smallmatrix}\right)"] & \mathcal{O}_Z(-H) \oplus \mathcal{O}_Z \arrow[r] \arrow[d, equal] & \sigma^*\mathcal{E} \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & 2\mathcal{O}_Z(-H-F) \arrow[r, "\widetilde{A}_t"] \arrow[d] & \mathcal{O}_Z(-H) \oplus \mathcal{O}_Z \arrow[r] & \widetilde{\mathcal{E}} \arrow[r] \arrow[d] & 0. \\ & 2\mathcal{O}_D(-1) \arrow[d] & & 0 & \\ & 0 & & & \end{tikzcd} \tag{1.18}$$

Lemma 1.22. *x_1 factorizes as $x_1 = u_1 \cdot s$, analogously $x_2 = u_2 \cdot s$, $tx_0 = u_0 \cdot s$.*

Proof. We know already that for each of the sections of $\mathcal{O}_Z(H)$ above there is a factorization

$$\begin{tikzcd} \mathcal{O}_Z(-H) \arrow[r] \arrow[d, "s"] & \mathcal{O}_Z \, . \\ \mathcal{O}_Z(-H+D) \arrow[ur, dotted, "\exists!"'] & \end{tikzcd}$$

In Lemma 1.20 we have just shown that $\mathcal{O}_Z(-H+D) \cong \mathcal{O}_Z(-F)$. We shall show that the section $\mathcal{O}_Z(-F) \to \mathcal{O}_Z$ in the commutative diagram

$$\begin{tikzcd} \mathcal{O}_Z(-H) \arrow[r] \arrow[d, "s"] & \mathcal{O}_Z \\ \mathcal{O}_Z(-F) \arrow[ur] & \end{tikzcd}$$

equals u_1, u_2, u_0 respectively. Note that factorizing through s means dividing locally by the local defining equation γ_{ik} of D. On $Z(x_i, u_k)$ the section x_1 is given by the function $\frac{x_1}{x_i}$. The defining function of D in this chart is $\frac{x_k}{x_i}$ if $k \neq 0$ and $\frac{tx_0}{x_i}$ if $k = 0$. Thus

$$\frac{x_1}{x_i}\Big/\gamma_{ik} = \begin{cases} \frac{x_1}{x_i}\Big/\frac{x_k}{x_i} = \frac{u_1}{u_k} & \text{if } k \neq 0, \\ \frac{x_1}{x_i}\Big/\frac{tx_0}{x_i} = \frac{u_1}{u_0} & \text{if } k = 0, \end{cases}$$

i. e., $\frac{x_1}{x_i}\big/\gamma_{ik} = \frac{u_1}{u_k}$. This is a description of the section u_1 of $\mathcal{O}_Z(F)$. We proved that $x_1 = u_1 \cdot s$. Analogously one proves the statement for x_2 and tx_0. □

As a corollary we obtain the following

Lemma 1.23. *Let A be as in (1.17) and let $B = \begin{pmatrix} w_1 & p_1 \\ w_2 & p_2 \end{pmatrix}$ with*

$$\begin{aligned} w_1 &= \xi_0 x_0 + \xi_1 x_1 + \xi_2 x_2, \\ w_1 &= \eta_0 x_0 + \eta_1 x_1 + \eta_2 x_2, \\ p_1 &= \xi_{00} x_0^2 + \cdots + \xi_{22} x_2^2, \\ p_2 &= \eta_{00} x_0^2 + \cdots + \eta_{22} x_2^2. \end{aligned}$$

Then $\widetilde{A}_t$ equals

$$\begin{pmatrix} u_1 & u_1(A_{01}x_0 + A_{11}x_1 + A_{12}x_2) + u_2(A_{02}x_0 + A_{22}x_2) \\ u_2 & u_1(B_{01}x_0 + B_{11}x_1 + B_{12}x_2) + u_2(B_{02}x_0 + B_{22}x_2) \end{pmatrix} +$$
$$\begin{pmatrix} \xi_0 u_0 + t\xi_1 u_1 + t\xi_2 u_2 & \xi_{00} x_0 u_0 + t\xi_{01} u_1 x_0 + \cdots + t\xi_{22} u_2 x_2 \\ \eta_0 u_0 + t\eta_1 u_1 + t\eta_2 u_2 & \eta_{00} x_0 u_0 + t\eta_{01} u_1 x_0 + \cdots + t\eta_{22} u_2 x_2 \end{pmatrix}$$

and can be treated as a morphism $2\mathcal{O}_Z(-H-F) \to \mathcal{O}_Z(-H) \oplus \mathcal{O}_Z$. We have thus the exact sequence

$$0 \to 2\mathcal{O}_Z(-H-F) \xrightarrow{\widetilde{A}_t} \mathcal{O}_Z(-H) \oplus \mathcal{O}_Z \to \widetilde{\mathcal{E}} \to 0. \tag{1.19}$$

1.2.2 First properties of new one parameter families.

Conditions for local freeness on support.

We constructed a sheaf $\widetilde{\mathcal{E}}$ on Z. Let us consider the map $Z \xrightarrow{\sigma} U \times \mathbb{P}_2 \xrightarrow{p_1} U$ and the restrictions $\widetilde{\mathcal{E}}_t$ of the sheaf $\widetilde{\mathcal{E}}$ to the fibres Z_t over $t \in U$. Since the restriction of σ to $Z \setminus Z_0$ is an isomorphism, we conclude that $Z_t \cong \mathbb{P}_2$ and $\widetilde{\mathcal{E}}_t \cong \mathcal{E}_t$ for $t \neq 0$. So outside of Z_0 the sheaf $\widetilde{\mathcal{E}}$ is basically the same as $\mathcal{E}$. In particular $\widetilde{\mathcal{E}}_t$ are locally free on their support for $t \neq 0$. Thus one could consider the fibre $\widetilde{\mathcal{E}}_0$ as a limit of nonsingular sheaves $\mathcal{E}_t$ for $t \to 0$. This way we have so to say replaced the singular sheaf $\mathcal{E}_0$ by the sheaf $\widetilde{\mathcal{E}}_0$ on the reducible variety Z_0.

The sheaf $\widetilde{\mathcal{E}}$ is not locally free (on its support) at some point if and only if the matrix $\widetilde{A}_t$ vanishes at this point. Since A_t vanishes only at point $(0,p)$, and since the preimage of $(0,p)$ is the exceptional divisor $D = D_1$, we conclude that $\widetilde{A}_t$ may only vanish at points lying in D_1.

Suppose that the matrix $\widetilde{A}_t$ vanishes at some point from D_1. Since from Lemma 1.23 we have

$$(\widetilde{A}_t)_{t=x_1=x_2=0} = \begin{pmatrix} u_1 + \xi_0 u_0 & A_{01}x_0u_1 + A_{02}x_0u_2 + \xi_{00}x_0u_0 \\ u_2 + \eta_0 u_0 & B_{01}x_0u_1 + B_{02}x_0u_2 + \eta_{00}x_0u_0 \end{pmatrix}, \tag{1.20}$$

vanishing of this matrix is equivalent to

$$\begin{cases} \xi_{00} = A_{01}\xi_0 + A_{02}\eta_0, \\ \eta_{00} = B_{01}\xi_0 + B_{02}\eta_0. \end{cases}$$

But these equations are by Example 1.8 exactly the tangent equations at A. We obtained

Proposition 1.24. *$\widetilde{\mathcal{E}}$ is locally free on its support if and only if B is a normal vector to X_8 at A, i. e., if and only if $B \in T_AX \setminus T_AX_8$.*

So using the normal directions B to X_8 at A we obtain new one parameter families $\widetilde{\mathcal{E}}$ that are isomorphic to the initial one parameter families $\mathcal{E}$ for $t \neq 0$ such that the new limit value "$\widetilde{\mathcal{E}}_0 = \lim\limits_{t\to 0} \widetilde{\mathcal{E}}_t$" is a sheaf that is locally free on its support .

Let us call the one parameter families $\widetilde{\mathcal{E}}$ that are locally free on their support **new families** or families of **new type**.

From now on we consider only the sheaves that are locally free on their support, i. e., those obtained by the help of normal directions.

Flatness.

Lemma 1.25. *Let $\widetilde{\mathcal{E}}$ be a new one parameter family.*

1) $\mathcal{T}or_1^{\mathcal{O}_Z}(\widetilde{\mathcal{E}}, \mathcal{O}_{D_1}) = 0$ and the restriction of $\widetilde{\mathcal{E}}$ to D_1 is given by the matrix

$$\widetilde{A}_{D_1} = (\widetilde{A}_t)_{t=x_1=x_2=0} = \begin{pmatrix} u_1 + \xi_0 u_0 & A_{01}u_1 + A_{02}u_2 + \xi_{00}u_0 \\ u_2 + \eta_0 u_0 & B_{01}u_1 + B_{02}u_2 + \eta_{00}u_0 \end{pmatrix},$$

i. e., the restriction of $\widetilde{\mathcal{E}}$ to D_1 is given by the resolution

$$0 \to 2\mathcal{O}_{D_1}(-1) \xrightarrow{\widetilde{A}_{D_1}} 2\mathcal{O}_{D_1} \to \widetilde{\mathcal{E}}|_{D_1} \to 0,$$

in particular $\widetilde{\mathcal{E}}|_{D_1}$ is a $2m+2$ sheaf on $D_1 = \mathbb{P}_2$.

2) $\mathcal{T}or_1^{\mathcal{O}_Z}(\widetilde{\mathcal{E}}, \mathcal{O}_{D_0}) = 0$ and the restriction of $\widetilde{\mathcal{E}}$ to $\widetilde{\mathbb{P}}_2$ is given by the matrix

$$\widetilde{A} = (\widetilde{A}_t)_{t=u_0=0} = \begin{pmatrix} u_1 & u_1(A_{01}x_0 + A_{11}x_1 + A_{12}x_2) + u_2(A_{02}x_0 + A_{22}x_2) \\ u_2 & u_1(B_{01}x_0 + B_{11}x_1 + B_{12}x_2) + u_2(B_{02}x_0 + B_{22}x_2) \end{pmatrix},$$

i. e., the restriction of $\widetilde{E}$ to $\widetilde{\mathbb{P}}_2$ is given by the resolution

$$0 \to 2\mathcal{O}_{\widetilde{\mathbb{P}}_2}(-H-F) \xrightarrow{\widetilde{A}} \mathcal{O}_{\widetilde{\mathbb{P}}_2}(-H) \oplus \mathcal{O}_{\widetilde{\mathbb{P}}_2} \to \widetilde{\mathcal{E}}|_{\widetilde{\mathbb{P}}_2} \to 0,$$

where $\mathcal{O}_{\widetilde{\mathbb{P}}_2}(F) \cong \pi^*\mathcal{O}_{\mathbb{P}_1}(1)$ and $\pi : \widetilde{\mathbb{P}}_2 \to \mathbb{P}_1$ is the canonical projection.

Proof. 1) Let us restrict the resolution (1.19) to D_1. We obtain this way the exact sequence

$$0 \to \mathscr{T}or_1^{\mathcal{O}_Z}(\widetilde{\mathcal{E}}, \mathcal{O}_{D_1}) \to 2\mathcal{O}_{D_1}(-L) \xrightarrow{\widetilde{A}_t|_{D_1}} 2\mathcal{O}_{D_1} \to \widetilde{\mathcal{E}}_{D_1} \to 0.$$

We used here that $\mathscr{T}or_1^{\mathcal{O}_Z}(\mathcal{O}_Z(-H) \oplus \mathcal{O}_Z, \mathcal{O}_{D_1}) = 0$ because $\mathcal{O}_Z(-H) \oplus \mathcal{O}_Z$ is a locally free sheaf. By Lemma 1.23 we have

$$\widetilde{A}_{D_1} := \widetilde{A}_t|_{D_1} = \begin{pmatrix} u_1 + \xi_0 u_0 & A_{01}u_1 + A_{02}u_2 + \xi_{00}u_0 \\ u_2 + \eta_0 u_0 & B_{01}u_1 + B_{02}u_2 + \eta_{00}u_0 \end{pmatrix}.$$

This morphism is injective if and only if $\det \widetilde{A}_{D_1} \neq 0$. One can write

$$\det \widetilde{A}_{D_1} = (B_{01}u_1 + B_{02}u_2 + \eta_{00}u_0, -(A_{01}u_1 + A_{02}u_2 + \xi_{00}u_0)) \cdot \begin{pmatrix} u_1 + \xi_0 u_0 \\ u_2 + \eta_0 u_0 \end{pmatrix}.$$

Suppose $\det \widetilde{A}_{D_1} = 0$. Then, since $u_1 + \xi_0 u_0$ and $u_2 + \eta_0 u_0$ are linear independent, one concludes that

$$\begin{pmatrix} A_{01}u_1 + A_{02}u_2 + \xi_{00}u_0 \\ B_{01}u_1 + B_{02}u_2 + \eta_{00}u_0 \end{pmatrix} = \lambda \begin{pmatrix} u_1 + \xi_0 u_0 \\ u_2 + \eta_0 u_0 \end{pmatrix}, \quad \lambda \in \Bbbk^*.$$

This holds if and only if $A_{02} = B_{01} = 0$, $x_{00} = A_{01}\xi_0$, and $\eta_{00} = B_{02}\eta_0$. Therefore, B is a tangent direction, which is a contradiction. This proves that the determinant of $\widetilde{A}_{D_1}$ is non-zero and hence $\widetilde{A}_{D_1}$ is injective. Therefore, $\mathscr{T}or_1^{\mathcal{O}_Z}(\widetilde{\mathcal{E}}, \mathcal{O}_{D_1}) = 0$. This proves the first part of the lemma.

2) Let us restrict now the sequence (1.19) to $D_0 = \widetilde{\mathbb{P}}_2$. We get the exact sequence

$$0 \to \mathscr{T}or_1^{\mathcal{O}_Z}(\widetilde{\mathcal{E}}, \mathcal{O}_{\widetilde{\mathbb{P}}_2}) \to 2\mathcal{O}_{\widetilde{\mathbb{P}}_2}(-H-F) \xrightarrow{\widetilde{A}_t|_{\widetilde{\mathbb{P}}_2}} \mathcal{O}_{\widetilde{\mathbb{P}}_2}(-H) \oplus \mathcal{O}_{\widetilde{\mathbb{P}}_2} \to \widetilde{\mathcal{E}}_{\widetilde{\mathbb{P}}_2} \to 0.$$

By Lemma 1.23 we have

$$\widetilde{A} := \widetilde{A}_t|_{\widetilde{\mathbb{P}}_2} = \begin{pmatrix} u_1 & u_1(A_{01}x_0 + A_{11}x_1 + A_{12}x_2) + u_2(A_{02}x_0 + A_{22}x_2) \\ u_2 & u_1(B_{01}x_0 + B_{11}x_1 + B_{12}x_2) + u_2(B_{02}x_0 + B_{22}x_2) \end{pmatrix}.$$

Let us prove the injectivity of $\widetilde{A}$. For an arbitrary point x from $\widetilde{\mathbb{P}}_2$, let us consider the restriction $\widetilde{A}_x$ of $\widetilde{A}$ to the stalk at x:

$$2\mathcal{O}_{\widetilde{\mathbb{P}}_2,x} \xrightarrow{\widetilde{A}_x} 2\mathcal{O}_{\widetilde{\mathbb{P}}_2,x}.$$

First of all one sees that for $x \in \widetilde{\mathbb{P}}_2 \setminus L$ the map $\widetilde{A}_x$ is the same as $A_{\sigma(x)}$, $\sigma(x) \in \mathbb{P}_2$. We just use here that σ is an isomorphism on $\widetilde{\mathbb{P}}_2 \setminus L$. Therefore, we conclude that $\widetilde{A}$ is injective outside of L, thus its kernel may only be supported on L. This is impossible since $2\mathcal{O}_{\widetilde{\mathbb{P}}_2}(-H-F)$ is a locally free sheaf and hence has no torsion. This proves the second part of the lemma. □

Proposition 1.26. *The sheaf $\widetilde{\mathcal{E}}$ is flat over U.*

Proof. Since $\widetilde{\mathcal{E}}|_{U\setminus\{0\}}$ is a $3m+1$ family over $U \setminus \{0\}$, we conclude that $\widetilde{\mathcal{E}}$ is flat over each point $t \in U$, $t \neq 0$. It remains to prove the flatness for $t = 0$. The sheaf $\widetilde{\mathcal{E}}$ is flat over $t = 0$ if and only if the restriction of the resolution

$$0 \to 2\mathcal{O}_Z(-H-F) \xrightarrow{\widetilde{A}_t} \mathcal{O}_Z(-H) \oplus \mathcal{O}_Z \to \widetilde{\mathcal{E}} \to 0$$

to Z_0 remains exact, i. e., if the morphism

$$2\mathcal{O}_Z(-H-F)|_{Z_0} \xrightarrow{\widetilde{A}_t|_{Z_0}} (\mathcal{O}_Z(-H) \oplus \mathcal{O}_Z)|_{Z_0}$$

is injective. But this is true because the restrictions $2\mathcal{O}_Z(-H-F)|_{D_1} \xrightarrow{\widetilde{A}_t|_{D_1}} (\mathcal{O}_Z(-H)\oplus\mathcal{O}_Z)|_{D_1}$ and $2\mathcal{O}_Z(-H-F)|_{\widetilde{\mathbb{P}}_2} \xrightarrow{\widetilde{A}_t|_{\widetilde{\mathbb{P}}_2}} (\mathcal{O}_Z(-H)\oplus\mathcal{O}_Z)|_{\widetilde{\mathbb{P}}_2}$ are injections by Lemma 1.25. Therefore, the kernel of $\widetilde{A}_t|_{Z_0}$ can only be supported on L. But $2\mathcal{O}_Z(-H-F)|_{Z_0}$ is a locally free sheaf, which has no torsion. Therefore, $\widetilde{A}_t|_{Z_0}$ is injective. This proves the required statement. $\square$

1.3 New objects on $\hat{\mathbb{P}}_2$

We study here the new objects on $Z_0 = \hat{\mathbb{P}}_2$. The main goal of this section is to give different characterizations of R-bundles on $\hat{\mathbb{P}}_2$.

1.3.1 Invertible sheaves on Z_0 and their cohomology.

The Picard group of Z_0.

Recall that $\hat{\mathbb{P}}_2$ may be considered as a closed subvariety in $\mathbb{P}_2 \times \mathbb{P}_2$ (cf. Definition 1.11). We define a divisor H as the pull back of a line $h \subseteq \mathbb{P}_2$ from the first $\mathbb{P}_2$ and F is defined as the pull back of a line $f \subseteq \mathbb{P}_2$ in the second $\mathbb{P}_2$. In other words we have $\mathcal{O}_{Z_0}(H) = \mathcal{O}_{\mathbb{P}_2\times\mathbb{P}_2}(1,0)|_{Z_0}$ and $\mathcal{O}_{Z_0}(F) = \mathcal{O}_{\mathbb{P}_2\times\mathbb{P}_2}(0,1)|_{Z_0}$.

Recall that Z_0 consists of the two components D_0 and D_1. We will denote by H_i and F_i the restrictions to D_i, $i = 0, 1$, of the divisors H and F respectively. Since it does not cause any misunderstandings, we will often write just H and F for the restrictions H_i and F_i.

The intersection $L = D_0 \cap D_1$ is isomorphic to $\mathbb{P}_1$ and is a divisor both in D_0 and D_1. Of course, one has $L \sim F_1$ as divisors in D_1. It holds also $F_0 + L \sim H_0$ as divisors in D_0, equivalently $L \sim H_0 - F_0$. Note also that $H_1 \sim 0$.

Note that by Lemma A.16 and Lemma A.17 a locally free sheaf on Z_0 is uniquely defined by its restrictions to the components $D_0 = \widetilde{\mathbb{P}}_2$ and $D_1 = \mathbb{P}_2$.

The Picard group of $D_1 \cong \mathbb{P}_2$ is isomorphic to $\mathbb{Z}$ and the isomorphism is given by

$$\mathbb{Z} \to \mathrm{Pic}(D_1), \quad b \mapsto [\mathcal{O}_{D_1}(bL)].$$

The Picard group of D_0 is $\mathbb{Z} \oplus \mathrm{Pic}(\mathbb{P}_2) \cong \mathbb{Z} \oplus \mathbb{Z}$ (cf. [12], V, Proposition 3.2), the isomorphism is given by the map

$$\mathbb{Z} \oplus \mathbb{Z} \to \mathrm{Pic}(D_0), \quad (a, b) \mapsto [\mathcal{O}_{D_0}(aH + bL)].$$

Let $\mathcal{O}_{D_0}(aH + b_0L)$ be an invertible sheaf on D_0 and let $\mathcal{O}_{D_1}(b_1L)$ an invertible sheaf on D_1. Their restrictions to L are $\mathcal{O}_L(-b_0)$ and $\mathcal{O}_L(b_1)$ respectively, so they define an invertible sheaf on L if and only if $b_0 = -b_1 = b$. The gluing of these two sheaves is isomorphic to $\mathcal{O}_{Z_0}((a-b)H + bF)$ because

$$\mathcal{O}_{Z_0}((a-b)H + bF)|_{D_0} \cong \mathcal{O}_{D_0}((a-b)H_0 + bF_0) \cong \mathcal{O}_{D_0}(aH_0 + b(F_0 - H_0)) \cong \mathcal{O}_{D_0}(aH - bL)$$

and

$$\mathcal{O}_{Z_0}((a-b)H + bF)|_{D_1} \cong \mathcal{O}_{D_1}(bF_1) \cong \mathcal{O}_{D_1}(bL).$$

We proved the following lemma.

Lemma 1.27. *The Picard group of Z_0 is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$. The isomorphism is given by*

$$\mathbb{Z} \oplus \mathbb{Z} \to \mathrm{Pic}(Z_0), \quad (a, b) \mapsto [\mathcal{O}_{Z_0}(aH + bF)].$$

There is also the following isomorphism of $\mathrm{Pic}(D_0)$ *and* $\mathrm{Pic}(Z_0)$*:*

$$\mathrm{Pic}(D_0) \to \mathrm{Pic}(Z_0), \quad [\mathcal{O}_{D_0}(aH + bL)] = [\mathcal{O}_{D_0}((a+b)H - bF_0)] \mapsto [\mathcal{O}_{Z_0}((a+b)H - bF)].$$

Euler characteristic of line bundles on $\widetilde{\mathbb{P}}_2$.

For a divisor δ on $\widetilde{\mathbb{P}}_2$ there is the Riemann-Roch formula (cf. [12], V, Theorem 1.6)

$$\chi\mathcal{O}_{\widetilde{\mathbb{P}}_2}(\delta) = \frac{1}{2}\delta^2 - \frac{1}{2}\delta K_{\widetilde{\mathbb{P}}_2} + \chi\mathcal{O}_{\widetilde{\mathbb{P}}_2},$$

where $K_{\widetilde{\mathbb{P}}_2}$ is a canonical divisor on $\widetilde{\mathbb{P}}_2$. We know that $K_{\widetilde{\mathbb{P}}_2} = -3H + L$ (cf. [12], V, Proposition 3.3) and that $\chi\mathcal{O}_{\widetilde{\mathbb{P}}_2} = \chi\mathcal{O}_{\mathbb{P}_2} = 1$ (cf. [12], V, Corollary 3.5). Thus one obtains

$$\chi\mathcal{O}_{\widetilde{\mathbb{P}}_2}(\delta) = \frac{1}{2}\delta^2 + \frac{1}{2}\delta(3H - L) + 1.$$

For $\delta = \alpha H - \beta L$, using $H^2 = 1$, $L^2 = -1$, and $H \cdot L = 0$, we get

$$\begin{aligned}\chi\mathcal{O}_{\widetilde{\mathbb{P}}_2}(\alpha H - \beta L) =& \frac{1}{2}(\alpha H - \beta L)^2 + \frac{1}{2}(\alpha H - \beta L)(3H - L) + 1 = \\ & \frac{1}{2}(\alpha^2 H^2 + \beta^2 L^2) + \frac{1}{2}(3\alpha H^2 + \beta L^2) + 1 = \\ & \frac{1}{2}(\alpha^2 - \beta^2) + \frac{1}{2}(3\alpha - \beta) + 1 = \frac{1}{2}\alpha^2 + \frac{3}{2}\alpha - \frac{1}{2}\beta^2 - \frac{1}{2}\beta + 1.\end{aligned}$$

Thus

$$\begin{aligned}\chi(\mathcal{O}_{\widetilde{\mathbb{P}}_2}(aH + bF_0)\otimes\mathcal{O}_{\widetilde{\mathbb{P}}_2}(mH + mF_0)) = \chi\mathcal{O}_{\widetilde{\mathbb{P}}_2}((a + b + 2m)H - (b + m)L) = \\ \frac{1}{2}(a + b + 2m)^2 + \frac{3}{2}(a + b + 2m) - \frac{1}{2}(b + m)^2 - \frac{1}{2}(b + m) + 1 = \\ \frac{3}{2}m^2 + [2(a + b) + 3 - b - \frac{1}{2}]m + \frac{1}{2}(a + b)^2 + \frac{3}{2}(a + b) - \frac{1}{2}b^2 - \frac{1}{2}b + 1 = \\ \frac{3}{2}m^2 + \left[2a + b + \frac{5}{2}\right] m + \frac{1}{2}(a + b)^2 + \frac{3}{2}(a + b) - \frac{1}{2}b^2 - \frac{1}{2}b + 1.\end{aligned}$$

We proved the following Lemma.

Lemma 1.28. *The Hilbert polynomial of the invertible sheaf $\mathcal{O}_{\widetilde{\mathbb{P}}_2}(a, b) = \mathcal{O}_{\widetilde{\mathbb{P}}_2}(aH + bF)$ on $\widetilde{\mathbb{P}}_2$ with respect to the sheaf $\mathcal{O}_{\widetilde{\mathbb{P}}_2}(1, 1) = \mathcal{O}_{\widetilde{\mathbb{P}}_2}(H + F)$ equals*

$$\frac{3}{2}m^2 + \left[2a + b + \frac{5}{2}\right] m + \frac{1}{2}(a + b)^2 + \frac{3}{2}(a + b) - \frac{1}{2}b^2 - \frac{1}{2}b + 1. \tag{1.21}$$

In particular we obtain the following Hilbert polynomials on $\widetilde{\mathbb{P}}_2$ with respect to the sheaf $\mathcal{O}_{\widetilde{\mathbb{P}}_2}(1, 1) = \mathcal{O}_{\widetilde{\mathbb{P}}_2}(H + F)$.

$$\begin{array}{llllll}(a, b) = (-2, 0) & \Longrightarrow & \frac{3}{2}m^2 - \frac{3}{2}m, & (a, b) = (0, 0) & \Longrightarrow & \frac{3}{2}m^2 + \frac{5}{2}m + 1, \\ (a, b) = (-1, -1) & \Longrightarrow & \frac{3}{2}m^2 - \frac{1}{2}m, & (a, b) = (0, 1) & \Longrightarrow & \frac{3}{2}m^2 + \frac{7}{2}m + 2, \\ (a, b) = (-1, 0) & \Longrightarrow & \frac{3}{2}m^2 + \frac{1}{2}m, & (a, b) = (1, 0) & \Longrightarrow & \frac{3}{2}m^2 + \frac{9}{2}m + 3, \\ (a, b) = (0, -1) & \Longrightarrow & \frac{3}{2}m^2 + \frac{3}{2}m, & (a, b) = (1, -1) & \Longrightarrow & \frac{3}{2}m^2 + \frac{7}{2}m + 1.\end{array} \tag{1.22}$$

Lemma 1.29. *Let $\mathcal{F}$ be a $3m+1$ sheaf on $\mathbb{P}_2$ such that $\mathcal{F}$ is not locally free at point p. Let $\sigma_0 : \widetilde{\mathbb{P}}_2 \to \mathbb{P}_2$ be the blowing up of p. Let $\mathcal{L} = \mathcal{O}_{\widetilde{\mathbb{P}}_2}(H + F_0) \cong \mathcal{O}_{\widetilde{\mathbb{P}}_2}(2H - L)$. Then $\sigma_0^*\mathcal{F}$ has the Hilbert polynomial $6m+1$ with respect to $\mathcal{L}$.*

Proof. Since there is a resolution

$$0 \to 2\mathcal{O}_{\widetilde{\mathbb{P}}_2}(-2H) \to \mathcal{O}_{\widetilde{\mathbb{P}}_2}(-H) \oplus \mathcal{O}_{\widetilde{\mathbb{P}}_2} \to \sigma_0^*\mathcal{F} \to 0,$$

the Hilbert polynomial of $\sigma_0^*\mathcal{F}$ is

$$\left(\frac{3}{2}m^2 + \frac{1}{2}m\right) + \left(\frac{3}{2}m^2 + \frac{5}{2}m + 1\right) - 2\left(\frac{3}{2}m^2 - \frac{3}{2}m\right) = 6m + 1.$$

This proves the lemma. $\square$

Euler characteristic of line bundles on Z_0.

Lemma 1.30. $\chi\mathcal{O}_{Z_0}(aH + bF) = \frac{1}{2}(a+b)^2 + \frac{3}{2}(a+b) + 1.$

Proof. For a line bundle $\mathcal{O}_{Z_0}(aH + bF)$ by Lemma A.16 there is the gluing sequence

$$0 \to \mathcal{O}_{Z_0}(aH + bF) \to \mathcal{O}_{\widetilde{\mathbb{P}}_2}((a+b)H - bL) \oplus \mathcal{O}_{D_1}(b) \to \mathcal{O}_L(b) \to 0.$$

We are using here that F is equivalent to $H - L$ on $\widetilde{\mathbb{P}}_2$.

By formula (1.22) we obtain

$$\chi\mathcal{O}_{\widetilde{\mathbb{P}}_2}((a+b)H - bL) = \frac{1}{2}(a+b)^2 + \frac{3}{2}(a+b) - \frac{1}{2}b^2 - \frac{1}{2}b + 1$$

Since

$$\chi\mathcal{O}_{D_1}(b) = \binom{b+2}{2} = \frac{1}{2}(b+2)(b+1) = \frac{1}{2}b^2 + \frac{3}{2}b + 1$$

and

$$\chi\mathcal{O}_L(b) = b + 1,$$

we calculate

$$\begin{aligned}\chi\mathcal{O}_{Z_0}(aH + bF) =& \chi\mathcal{O}_{\widetilde{\mathbb{P}}_2}((a+b)H - bL) + \chi\mathcal{O}_{D_1}(b) - \chi\mathcal{O}_L(b) = \\ & \frac{1}{2}(a+b)^2 + \frac{3}{2}(a+b) - \frac{1}{2}b^2 - \frac{1}{2}b + 1 + \frac{1}{2}b^2 + \frac{3}{2}b + 1 - (b+1) = \\ & \frac{1}{2}(a+b)^2 + \frac{3}{2}(a+b) + 1.\end{aligned}$$

This completes the proof. $\square$

Lemma 1.31. *The Hilbert polynomial of $\mathcal{O}_{Z_0}(aH + bF)$ with respect to the invertible sheaf $\mathcal{L} = \mathcal{O}_{Z_0}(H + F)$ equals*

$$2m^2 + [2(a+b) + 3] \cdot m + \frac{1}{2}(a+b)^2 + \frac{3}{2}(a+b) + 1. \tag{1.23}$$

Note that the result depends only on the sum $a+b$.

Proof. Using Lemma 1.30 we obtain that

$$\begin{aligned}\chi(\mathcal{O}_{Z_0}(aH + bF) \otimes \mathcal{L}^{\otimes m}) =& \chi\mathcal{O}_{Z_0}((a+m)H + (b+m)F) = \\ & \frac{1}{2}(a+b+2m)^2 + \frac{3}{2}(a+b+2m) + 1 = \\ & 2m^2 + [2(a+b) + 3] \cdot m + \frac{1}{2}(a+b)^2 + \frac{3}{2}(a+b) + 1.\end{aligned}$$

This proves the required statement. $\square$

Some cohomology groups of sheaves on Z_0.

We collect here calculations of cohomology groups for some locally free sheaves on Z_0 and D_0.

Table of results. We collect some results about cohomology groups in the following table.

Proposition 1.32.

	h^0	h^1	h^2
$\mathcal{O}_{Z_0}(-2F)$	0	0	0
$\mathcal{O}_{Z_0}(-H)$	0	0	0
$\mathcal{O}_{Z_0}(-F)$	0	0	0
$\mathcal{O}_{Z_0}$	1	0	0
$\mathcal{O}_{Z_0}(H-F)$	1	0	0
$\mathcal{O}_{Z_0}(-H+F)$	1	0	0
$\mathcal{O}_{Z_0}(H)$	3	0	0
$\mathcal{O}_{Z_0}(F)$	3	0	0
$\mathcal{O}_{Z_0}(-H-F)$	0	0	0
$\mathcal{O}_{Z_0}(H+F)$	6	0	0
$\mathcal{O}_{Z_0}(H+2F)$	10	0	0

The computations below constitute the proof of Proposition 1.32.

Some key tools for computing cohomologies on Z_0. Let us collect here some short exact sequences. We will make use of the corresponding long exact cohomology sequences.

Recall that for a locally free sheaf $\mathcal{G}$ on Z_0 by Lemma A.16 there is the gluing exact sequence

$$0 \to \mathcal{G} \to \mathcal{G}|_{D_0} \oplus \mathcal{G}|_{D_1} \to \mathcal{G}|_L \to 0.$$

Note also that $\mathcal{O}_{D_0}$ has the locally free resolution

$$0 \to \mathcal{O}_{\mathbb{P}_2\times\mathbb{P}_1}(-1,-1) \xrightarrow{x_1u_2 - x_2u_1} \mathcal{O}_{\mathbb{P}_2\times\mathbb{P}_1} \to \mathcal{O}_{D_0} \to 0.$$

For arbitrary $a, b \in \mathbb{Z}$ this gives the resolutions

$$0 \to \mathcal{O}_{\mathbb{P}_2\times\mathbb{P}_1}(a-1, b-1) \to \mathcal{O}_{\mathbb{P}_2\times\mathbb{P}_1}(a,b) \to \mathcal{O}_{D_0}(aH+bF) \to 0. \tag{1.24}$$

Note also that the restriction homomorphism

$$H^0(D_1, \mathcal{O}_{D_1}(nL)) \to H^0(L, \mathcal{O}_L(n))$$

is always surjective.

To compute the cohomology groups of $\mathcal{O}_{\mathbb{P}_2\times\mathbb{P}_1}(\mu,\nu)$ we will use the **Künneth** formula from [25]:

$$H^q(\mathbb{P}_n \times \mathbb{P}_m, \mathcal{O}_{\mathbb{P}_n\times\mathbb{P}_m}(\mu,\nu)) \cong \bigoplus_{i+j=q} H^i(\mathbb{P}_n, \mathcal{O}_{\mathbb{P}_n}(\mu)) \otimes H^j(\mathbb{P}_m, \mathcal{O}_{\mathbb{P}_m}(\nu)). \tag{1.25}$$

Sheaf $\mathcal{O}_{Z_0}$. Consider the gluing exact sequence

$$0 \to \mathcal{O}_{Z_0} \to \mathcal{O}_{D_0} \oplus \mathcal{O}_{D_1} \to \mathcal{O}_L \to 0.$$

This gives the long exact cohomology sequence

$$\begin{aligned} 0 \to H^0(\mathcal{O}_{Z_0}) \to H^0(\mathcal{O}_{D_0}) \oplus H^0(\mathcal{O}_{D_1}) \to H^0(\mathcal{O}_L) \xrightarrow{0} \\ \to H^1(\mathcal{O}_{Z_0}) \to H^1(\mathcal{O}_{D_0}) \oplus H^1(\mathcal{O}_{D_1}) \to H^1(\mathcal{O}_L) \to \\ \to H^2(\mathcal{O}_{Z_0}) \to H^2(\mathcal{O}_{D_0}) \oplus H^2(\mathcal{O}_{D_1}) \to H^2(\mathcal{O}_L) \to 0. \end{aligned}$$

Note that the map $H^0(\mathcal{O}_L) \to H^1(\mathcal{O}_{Z_0})$ is zero because the restriction homomorphism

$$H^0(D_1, \mathcal{O}_{D_1}) \to H^0(L, \mathcal{O}_L)$$

is an isomorphism, it holds $H^0(D_1, \mathcal{O}_{D_1}) \cong H^0(L, \mathcal{O}_L) \cong \Bbbk$. Using that $H^i(\mathcal{O}_{D_0}) = H^i(\mathcal{O}_L) = 0$ for $i = 1, 2$, we conclude that $H^i(\mathcal{O}_{Z_0}) \cong H^i(\mathcal{O}_{D_0})$, $i = 0, 1, 2$.

Consider the locally free resolution of $\mathcal{O}_{D_0}$ on $\mathbb{P}_2 \times \mathbb{P}_1$

$$0 \to \mathcal{O}_{\mathbb{P}_2\times\mathbb{P}_1}(-1,-1) \xrightarrow{x_1u_2 - x_2u_1} \mathcal{O}_{\mathbb{P}_2\times\mathbb{P}_1} \to \mathcal{O}_{D_0} \to 0$$

and the corresponding long exact cohomology sequence

$$\begin{aligned} 0 \to H^0(\mathcal{O}_{\mathbb{P}_2\times\mathbb{P}_1}(-1,-1)) \to H^0(\mathcal{O}_{\mathbb{P}_2\times\mathbb{P}_1}) \to H^0(\mathcal{O}_{D_0}) \to \\ \to H^1(\mathcal{O}_{\mathbb{P}_2\times\mathbb{P}_1}(-1,-1)) \to H^1(\mathcal{O}_{\mathbb{P}_2\times\mathbb{P}_1}) \to H^1(\mathcal{O}_{D_0}) \to \\ \to H^2(\mathcal{O}_{\mathbb{P}_2\times\mathbb{P}_1}(-1,-1)) \to H^2(\mathcal{O}_{\mathbb{P}_2\times\mathbb{P}_1}) \to H^2(\mathcal{O}_{D_0}) \to \\ \to H^3(\mathcal{O}_{\mathbb{P}_2\times\mathbb{P}_1}(-1,-1)) \to H^3(\mathcal{O}_{\mathbb{P}_2\times\mathbb{P}_1}) \to 0. \end{aligned}$$

By Künneth formula $H^i(\mathcal{O}_{\mathbb{P}_2\times\mathbb{P}_1}(-1,-1)) = 0$, for all i, $H^i(\mathcal{O}_{\mathbb{P}_2\times\mathbb{P}_1}) = 0$ for $i \neq 0$, and $H^0(\mathcal{O}_{\mathbb{P}_2\times\mathbb{P}_1}) \cong \Bbbk$. Therefore, $H^1(\mathcal{O}_{D_0}) = H^2(\mathcal{O}_{D_0}) = 0, \quad H^0(\mathcal{O}_{D_0}) \cong \Bbbk$. This proves

$$H^1(\mathcal{O}_{Z_0}) = H^2(\mathcal{O}_{Z_0}) = 0, \quad H^0(\mathcal{O}_{Z_0}) \cong \Bbbk.$$

Sheaf $\mathcal{O}_{Z_0}(H - F)$. Consider the gluing exact sequence

$$0 \to \mathcal{O}_{Z_0}(H-F) \to \mathcal{O}_{D_0}(H-F) \oplus \mathcal{O}_{D_1}(-L) \to \mathcal{O}_L(-1) \to 0.$$

Since all the cohomology groups of $\mathcal{O}_{D_1}(-L)$ and $\mathcal{O}_L(-1)$ are zero, using the long exact cohomology sequence we conclude of the above short exact sequence we conclude that $\mathcal{O}_{Z_0}(H-F) \cong \mathcal{O}_{D_0}(H-F)$ for all i. So it remains to compute the cohomology groups of $\mathcal{O}_{D_0}(H-F)$.

Consider the locally free resolution

$$0 \to \mathcal{O}_{\mathbb{P}_2\times\mathbb{P}_1}(0,-2) \xrightarrow{x_1u_2 - x_2u_1} \mathcal{O}_{\mathbb{P}_2\times\mathbb{P}_1}(1,-1) \to \mathcal{O}_{D_0}(H-F) \to 0$$

and the corresponding long exact cohomology sequence

$$\begin{aligned} 0 \to H^0(\mathcal{O}_{\mathbb{P}_2\times\mathbb{P}_1}(0,-2)) \to H^0(\mathcal{O}_{\mathbb{P}_2\times\mathbb{P}_1}(1,-1)) \to H^0(\mathcal{O}_{D_0}(H-F)) \to \\ \to H^1(\mathcal{O}_{\mathbb{P}_2\times\mathbb{P}_1}(0,-2)) \to H^1(\mathcal{O}_{\mathbb{P}_2\times\mathbb{P}_1}(1,-1)) \to H^1(\mathcal{O}_{D_0}(H-F)) \to \\ \to H^2(\mathcal{O}_{\mathbb{P}_2\times\mathbb{P}_1}(0,-2)) \to H^2(\mathcal{O}_{\mathbb{P}_2\times\mathbb{P}_1}(1,-1)) \to H^2(\mathcal{O}_{D_0}(H-F)) \to \\ \to H^3(\mathcal{O}_{\mathbb{P}_2\times\mathbb{P}_1}(0,-2)) \to H^3(\mathcal{O}_{\mathbb{P}_2\times\mathbb{P}_1}(1,-1)) \to 0. \end{aligned}$$

By Künneth formula $H^i(\mathcal{O}_{\mathbb{P}_2\times\mathbb{P}_1}(-1,-1)) = 0$, for all i, $H^i(\mathcal{O}_{\mathbb{P}_2\times\mathbb{P}_1}(0,-2)) = 0$ for $i \neq 1$, and $H^1(\mathcal{O}_{\mathbb{P}_2\times\mathbb{P}_1}(0,-2)) \cong \Bbbk$. Therefore,

$$H^1(\mathcal{O}_{D_0}(H-F)) = H^2(\mathcal{O}_{D_0}(H-F)) = 0, \quad H^0(\mathcal{O}_{D_0}(H-F)) \cong \Bbbk.$$

Sheaf $\mathcal{O}_{Z_0}(-H+F)$. Consider the gluing exact sequence

$$0 \to \mathcal{O}_{Z_0}(-H+F) \to \mathcal{O}_{D_0}(-H+F) \oplus \mathcal{O}_{D_1}(L) \to \mathcal{O}_L(1) \to 0.$$

This gives the long exact cohomology sequence

$$\begin{aligned} 0 &\to H^0(\mathcal{O}_{Z_0}(-H+F)) \to H^0(\mathcal{O}_{D_0}(-H+F)) \oplus H^0(\mathcal{O}_{D_1}(L)) \to H^0(\mathcal{O}_L(1)) \xrightarrow{0} \\ &\to H^1(\mathcal{O}_{Z_0}(-H+F)) \to H^1(\mathcal{O}_{D_0}(-H+F)) \oplus H^1(\mathcal{O}_{D_1}(L)) \to H^1(\mathcal{O}_L(1)) \to \\ &\to H^2(\mathcal{O}_{Z_0}(-H+F)) \to H^2(\mathcal{O}_{D_0}(-H+F)) \oplus H^2(\mathcal{O}_{D_1}(L)) \to H^2(\mathcal{O}_L(1)) \to 0. \end{aligned}$$

Since $H^i(\mathcal{O}_L(1)) = H^i(\mathcal{O}_{D_1}(L)) = 0$ for $i \neq 0$, we obtain the isomorphisms

$$H^i(\mathcal{O}_{Z_0}(-H-F)) \cong H^i(\mathcal{O}_{D_0}(-H-F)), \quad i \neq 0.$$

From the resolution

$$0 \to \mathcal{O}_{\mathbb{P}_2\times\mathbb{P}_1}(-2,0) \to \mathcal{O}_{\mathbb{P}_2\times\mathbb{P}_1}(-1,1) \to \mathcal{O}_{D_0}(-H+F) \to 0$$

we obtain the long exact cohomology sequence

$$\begin{aligned} 0 &\to H^0(\mathcal{O}_{\mathbb{P}_2\times\mathbb{P}_1}(-2,0)) \to H^0(\mathcal{O}_{\mathbb{P}_2\times\mathbb{P}_1}(-1,1)) \to H^0(\mathcal{O}_{D_0}(-H+F)) \to \\ &\to H^1(\mathcal{O}_{\mathbb{P}_2\times\mathbb{P}_1}(-2,0)) \to H^1(\mathcal{O}_{\mathbb{P}_2\times\mathbb{P}_1}(-1,1)) \to H^1(\mathcal{O}_{D_0}(-H+F)) \to \\ &\to H^2(\mathcal{O}_{\mathbb{P}_2\times\mathbb{P}_1}(-2,0)) \to H^2(\mathcal{O}_{\mathbb{P}_2\times\mathbb{P}_1}(-1,1)) \to H^2(\mathcal{O}_{D_0}(-H+F)) \to \\ &\to H^3(\mathcal{O}_{\mathbb{P}_2\times\mathbb{P}_1}(-2,0)) \to H^3(\mathcal{O}_{\mathbb{P}_2\times\mathbb{P}_1}(-1,1)) \to 0. \end{aligned}$$

By Künneth formula we obtain that all the cohomologies of $\mathcal{O}_{\mathbb{P}_2\times\mathbb{P}_1}(-2,0)$ and $\mathcal{O}_{\mathbb{P}_2\times\mathbb{P}_1}(-1,1)$ are zero. Therefore, $H^i(\mathcal{O}_{D_0}(-H+F)) = 0$ for all i.

We obtain now

$$H^i(\mathcal{O}_{Z_0}(-H-F)) \cong H^i(\mathcal{O}_{D_0}(-H-F)) = 0, \quad i \neq 0.$$

From the exact sequence

$$0 \to H^0(\mathcal{O}_{Z_0}(-H+F)) \to H^0(\mathcal{O}_{D_1}(L)) \to H^0(\mathcal{O}_L(1)) \to 0$$

using $H^0(\mathcal{O}_{D_1}(L)) \cong \Bbbk^3$ and $H^0(\mathcal{O}_L(1)) \cong \Bbbk^2$ we conclude that $H^0(\mathcal{O}_{Z_0}(-H+F)) \cong \Bbbk$.

We obtained

$$H^1(\mathcal{O}_{D_0}(-H+F)) = H^2(\mathcal{O}_{D_0}(-H+F)) = 0, \quad H^0(\mathcal{O}_{D_0}(-H+F)) \cong \Bbbk.$$

Sheaf $\mathcal{O}_{Z_0}(-H)$. Consider the gluing exact sequence

$$0 \to \mathcal{O}_{Z_0}(-H) \to \mathcal{O}_{D_0}(-H) \oplus \mathcal{O}_{D_1} \to \mathcal{O}_L \to 0.$$

This gives the long exact cohomology sequence

$$\begin{aligned} 0 &\to H^0(\mathcal{O}_{Z_0}(-H)) \to H^0(\mathcal{O}_{D_0}(-H)) \oplus H^0(\mathcal{O}_{D_1}) \to H^0(\mathcal{O}_L) \xrightarrow{0} \\ &\to H^1(\mathcal{O}_{Z_0}(-H)) \to H^1(\mathcal{O}_{D_0}(-H)) \oplus H^1(\mathcal{O}_{D_1}) \to H^1(\mathcal{O}_L) \to \\ &\to H^2(\mathcal{O}_{Z_0}(-H)) \to H^2(\mathcal{O}_{D_0}(-H)) \oplus H^2(\mathcal{O}_{D_1}) \to H^2(\mathcal{O}_L) \to 0. \end{aligned}$$

We obtain $H^i(\mathcal{O}_{Z_0}(-H)) \cong H^i(\mathcal{O}_{D_0}(-H))$, $i = 0,1,2$. Using the resolution

$$0 \to \mathcal{O}_{\mathbb{P}_2\times\mathbb{P}_1}(-2,-1) \to \mathcal{O}_{\mathbb{P}_2\times\mathbb{P}_1}(-1,0) \to \mathcal{O}_{D_0}(-H) \to 0$$

we obtain the long exact cohomology sequence

$$\begin{aligned}
0 \to H^0(\mathcal{O}_{\mathbb{P}_2\times\mathbb{P}_1}(-2,-1)) \to H^0(\mathcal{O}_{\mathbb{P}_2\times\mathbb{P}_1}(-1,0)) \to H^0(\mathcal{O}_{D_0}(-H)) \to \\
\to H^1(\mathcal{O}_{\mathbb{P}_2\times\mathbb{P}_1}(-2,-1)) \to H^1(\mathcal{O}_{\mathbb{P}_2\times\mathbb{P}_1}(-1,0)) \to H^1(\mathcal{O}_{D_0}(-H)) \to \\
\to H^2(\mathcal{O}_{\mathbb{P}_2\times\mathbb{P}_1}(-2,-1)) \to H^2(\mathcal{O}_{\mathbb{P}_2\times\mathbb{P}_1}(-1,0)) \to H^2(\mathcal{O}_{D_0}(-H)) \\
\to H^3(\mathcal{O}_{\mathbb{P}_2\times\mathbb{P}_1}(-2,-1)) \to H^3(\mathcal{O}_{\mathbb{P}_2\times\mathbb{P}_1}(-1,0)) \to 0.
\end{aligned}$$

By Künneth formula we conclude that $H^i(\mathcal{O}_{\mathbb{P}_2\times\mathbb{P}_1}(-2,-1)) = H^j(\mathcal{O}_{\mathbb{P}_2\times\mathbb{P}_1})(-1,0) = 0$ for all i and j. Therefore, $H^i(\mathcal{O}_{D_0}(-H)) = 0$ for all i and hence

$$H^i(\mathcal{O}_{D_0}(-H)) = 0 \quad \text{for all} \quad i.$$

Sheaf $\mathcal{O}_{Z_0}(-F)$. Consider the gluing exact sequence

$$0 \to \mathcal{O}_{Z_0}(-F) \to \mathcal{O}_{D_0}(-F) \oplus \mathcal{O}_{D_1}(-L) \to \mathcal{O}_L(-1) \to 0.$$

This gives the long exact cohomology sequence

$$\begin{aligned}
0 \to H^0(\mathcal{O}_{Z_0}(-F)) \to H^0(\mathcal{O}_{D_0}(-F)) \oplus H^0(\mathcal{O}_{D_1}(-L)) \to H^0(\mathcal{O}_L(-1)) \to \\
\to H^1(\mathcal{O}_{Z_0}(-F)) \to H^1(\mathcal{O}_{D_0}(-F)) \oplus H^1(\mathcal{O}_{D_1}(-L)) \to H^1(\mathcal{O}_L(-1)) \to \\
\to H^2(\mathcal{O}_{Z_0}(-F)) \to H^2(\mathcal{O}_{D_0}(-F)) \oplus H^2(\mathcal{O}_{D_1}(-L)) \to H^2(\mathcal{O}_L(-1)) \to 0.
\end{aligned}$$

Since all cohomology groups of the sheaves $\mathcal{O}_{D_1}(-L)$ and $\mathcal{O}_L(-1)$ are zero, one obtains the isomorphisms $H^i(\mathcal{O}_{Z_0}(-F)) \cong H^i(\mathcal{O}_{D_0}(-F))$ for all i.

Using the resolution

$$0 \to \mathcal{O}_{\mathbb{P}_2\times\mathbb{P}_1}(-1,-1) \to \mathcal{O}_{\mathbb{P}_2\times\mathbb{P}_1}(0,-1) \to \mathcal{O}_{D_0}(-F) \to 0$$

we obtain the long exact cohomology sequence

$$\begin{aligned}
0 \to H^0(\mathcal{O}_{\mathbb{P}_2\times\mathbb{P}_1}(-1,-2)) \to H^0(\mathcal{O}_{\mathbb{P}_2\times\mathbb{P}_1}(0,-1)) \to H^0(\mathcal{O}_{D_0}(-F)) \to \\
\to H^1(\mathcal{O}_{\mathbb{P}_2\times\mathbb{P}_1}(-1,-2)) \to H^1(\mathcal{O}_{\mathbb{P}_2\times\mathbb{P}_1}(0,-1)) \to H^1(\mathcal{O}_{D_0}(-F)) \to \\
\to H^2(\mathcal{O}_{\mathbb{P}_2\times\mathbb{P}_1}(-1,-2)) \to H^2(\mathcal{O}_{\mathbb{P}_2\times\mathbb{P}_1}(0,-1)) \to H^2(\mathcal{O}_{D_0}(-F)) \to \\
\to H^3(\mathcal{O}_{\mathbb{P}_2\times\mathbb{P}_1}(-1,-2)) \to H^3(\mathcal{O}_{\mathbb{P}_2\times\mathbb{P}_1}(0,-1)) \to 0.
\end{aligned}$$

All cohomology groups of the sheaves $\mathcal{O}_{\mathbb{P}_2\times\mathbb{P}_1}(-1,-2)$ and $\mathcal{O}_{\mathbb{P}_2\times\mathbb{P}_1}(0,-1)$ vanish by Künneth formula, thus the sheaf $\mathcal{O}_{D_0}(-F)$ has zero cohomology groups. We obtain finally

$$H^i(\mathcal{O}_{D_0}(-F)) = 0 \quad \text{for all} \quad i.$$

Sheaf $\mathcal{O}_{Z_0}(H)$. Consider the gluing exact sequence

$$0 \to \mathcal{O}_{Z_0}(H) \to \mathcal{O}_{D_0}(H) \oplus \mathcal{O}_{D_1} \to \mathcal{O}_L \to 0.$$

This gives the long exact cohomology sequence

$$\begin{aligned}
0 \to H^0(\mathcal{O}_{Z_0}(H)) \to H^0(\mathcal{O}_{D_0}(H)) \oplus H^0(\mathcal{O}_{D_1}) \to H^0(\mathcal{O}_L) \xrightarrow{0} \\
\to H^1(\mathcal{O}_{Z_0}(H)) \to H^1(\mathcal{O}_{D_0}(H)) \oplus H^1(\mathcal{O}_{D_1}) \to H^1(\mathcal{O}_L) \to \\
\to H^2(\mathcal{O}_{Z_0}(H)) \to H^2(\mathcal{O}_{D_0}(H)) \oplus H^2(\mathcal{O}_{D_1}) \to H^2(\mathcal{O}_L) \to 0.
\end{aligned}$$

As in the case of $\mathcal{O}_{Z_0}$ we obtain the isomorphisms $H^i(\mathcal{O}_{Z_0}(H)) \cong H^i(\mathcal{O}_{D_0}(H))$, $i = 0, 1, 2$.

From the resolution

$$0 \to \mathcal{O}_{\mathbb{P}_2\times\mathbb{P}_1}(0,-1) \to \mathcal{O}_{\mathbb{P}_2\times\mathbb{P}_1}(1,0) \to \mathcal{O}_{D_0}(H) \to 0$$

we obtain the long exact cohomology sequence

$$\begin{aligned} 0 &\to H^0(\mathcal{O}_{\mathbb{P}_2\times\mathbb{P}_1}(0,-1)) \to H^0(\mathcal{O}_{\mathbb{P}_2\times\mathbb{P}_1}(1,0)) \to H^0(\mathcal{O}_{D_0}(H)) \to \\ &\to H^1(\mathcal{O}_{\mathbb{P}_2\times\mathbb{P}_1}(0,-1)) \to H^1(\mathcal{O}_{\mathbb{P}_2\times\mathbb{P}_1}(1,0)) \to H^1(\mathcal{O}_{D_0}(H)) \to \\ &\to H^2(\mathcal{O}_{\mathbb{P}_2\times\mathbb{P}_1}(0,-1)) \to H^2(\mathcal{O}_{\mathbb{P}_2\times\mathbb{P}_1}(1,0)) \to H^2(\mathcal{O}_{D_0}(H)) \to \\ &\to H^3(\mathcal{O}_{\mathbb{P}_2\times\mathbb{P}_1}(0,-1)) \to H^3(\mathcal{O}_{\mathbb{P}_2\times\mathbb{P}_1}(1,0)) \to 0. \end{aligned}$$

All the cohomologies of $\mathcal{O}_{\mathbb{P}_2\times\mathbb{P}_1}(0,-1)$ are zero by Künneth formula. Therefore $H^i(\mathcal{O}_{D_0}(H)) \cong H^i(\mathcal{O}_{\mathbb{P}_2\times\mathbb{P}_1}(1,0))$ for all i. Again using Künneth formula we conclude that $H^i(\mathcal{O}_{\mathbb{P}_2\times\mathbb{P}_1}(1,0)) = 0$ for $i \neq 0$ and $H^0(\mathcal{O}_{\mathbb{P}_2\times\mathbb{P}_1}(1,0)) = H^0(\mathcal{O}_{\mathbb{P}_2}(1)) \otimes H^0(\mathcal{O}_{\mathbb{P}_1}) \cong \Gamma(\mathbb{P}_2, \mathcal{O}_{\mathbb{P}_2}(1)) \cong \mathbb{k}^3$. We obtain finally

$$H^0(\mathcal{O}_{Z_0}(H)) \cong \mathbb{k}^3, \quad H^i(\mathcal{O}_{Z_0}(H)) = 0, \ i \neq 0.$$

Sheaf $\mathcal{O}_{Z_0}(F)$. Consider the gluing exact sequence

$$0 \to \mathcal{O}_{Z_0}(F) \to \mathcal{O}_{D_0}(F) \oplus \mathcal{O}_{D_1}(L) \to \mathcal{O}_L(1) \to 0.$$

This gives the long exact cohomology sequence

$$\begin{aligned} 0 &\to H^0(\mathcal{O}_{Z_0}(F)) \to H^0(\mathcal{O}_{D_0}(F)) \oplus H^0(\mathcal{O}_{D_1}(L)) \to H^0(\mathcal{O}_L(1)) \xrightarrow{0} \\ &\to H^1(\mathcal{O}_{Z_0}(F)) \to H^1(\mathcal{O}_{D_0}(F)) \oplus H^1(\mathcal{O}_{D_1}(L)) \to H^1(\mathcal{O}_L(1)) \to \\ &\to H^2(\mathcal{O}_{Z_0}(F)) \to H^2(\mathcal{O}_{D_0}(F)) \oplus H^2(\mathcal{O}_{D_1}(L)) \to H^2(\mathcal{O}_L(1)) \to 0. \end{aligned}$$

Note that the map $H^0(\mathcal{O}_{D_1}(L)) \to H^0(\mathcal{O}_L(1))$ is surjective, therefore the homomorphism $H^0(\mathcal{O}_L(1)) \to H^1(\mathcal{O}_{Z_0}(F))$ is zero. We have $H^1(\mathcal{O}_{D_1}(L)) = H^2(\mathcal{O}_{D_1}(L)) = 0$ and $H^1(\mathcal{O}_L(1)) = H^2(\mathcal{O}_L(1)) = 0$. Therefore, $H^i(\mathcal{O}_{Z_0}(F)) \cong H^i(\mathcal{O}_{D_0}(F))$, $i = 1, 2$, and we have the exact sequence

$$0 \to H^0(\mathcal{O}_{Z_0}(F)) \to H^0(\mathcal{O}_{D_0}(F)) \oplus H^0(\mathcal{O}_{D_1}(L)) \to H^0(\mathcal{O}_L(1)) \to 0. \tag{1.26}$$

From the resolution

$$0 \to \mathcal{O}_{\mathbb{P}_2\times\mathbb{P}_1}(-1,0) \to \mathcal{O}_{\mathbb{P}_2\times\mathbb{P}_1}(0,1) \to \mathcal{O}_{D_0}(F) \to 0$$

we obtain the long exact cohomology sequence

$$\begin{aligned} 0 &\to H^0(\mathcal{O}_{\mathbb{P}_2\times\mathbb{P}_1}(-1,0)) \to H^0(\mathcal{O}_{\mathbb{P}_2\times\mathbb{P}_1}(0,1)) \to H^0(\mathcal{O}_{D_0}(F)) \to \\ &\to H^1(\mathcal{O}_{\mathbb{P}_2\times\mathbb{P}_1}(-1,0)) \to H^1(\mathcal{O}_{\mathbb{P}_2\times\mathbb{P}_1}(0,1)) \to H^1(\mathcal{O}_{D_0}(F)) \to \\ &\to H^2(\mathcal{O}_{\mathbb{P}_2\times\mathbb{P}_1}(-1,0)) \to H^2(\mathcal{O}_{\mathbb{P}_2\times\mathbb{P}_1}(0,1)) \to H^2(\mathcal{O}_{D_0}(F)) \to \\ &\to H^3(\mathcal{O}_{\mathbb{P}_2\times\mathbb{P}_1}(-1,0)) \to H^3(\mathcal{O}_{\mathbb{P}_2\times\mathbb{P}_1}(0,1)) \to 0. \end{aligned}$$

The cohomology groups of $\mathcal{O}_{\mathbb{P}_2\times\mathbb{P}_1}(-1,0)$ are all zero, hence for all i we obtain the isomorphisms $H^i(\mathcal{O}_{D_0}(F)) \cong H^i(\mathcal{O}_{\mathbb{P}_2\times\mathbb{P}_1}(0,1))$. Since $H^i(\mathcal{O}_{\mathbb{P}_2\times\mathbb{P}_1}(0,1)) = 0$, $i \neq 0$, and $H^0(\mathcal{O}_{\mathbb{P}_2\times\mathbb{P}_1}(0,1)) \cong \mathbb{k}^2$, we conclude that $H^i(\mathcal{O}_{D_0}(F)) = 0$ for $i \neq 0$ and $H^0(\mathcal{O}_{D_0}(F)) \cong \mathbb{k}^2$. Therefore, one gets $H^i(\mathcal{O}_{Z_0}(F)) = 0$ for $i \neq 0$ and from the exact sequence (1.26), using $H^0(\mathcal{O}_{D_1}(L)) \cong \mathbb{k}^3$ and $H^0(\mathcal{O}_L(1)) \cong \mathbb{k}^2$, we conclude that $H^i(\mathcal{O}_{Z_0}(F))$ has dimension 3. We obtained

$$H^0(\mathcal{O}_{Z_0}(F)) \cong \mathbb{k}^3, \quad H^i(\mathcal{O}_{Z_0}(F)) = 0, \ i \neq 0.$$

Sheaf $\mathcal{O}_{Z_0}(-H-F)$. Consider the gluing exact sequence

$$0 \to \mathcal{O}_{Z_0}(-H-F) \to \mathcal{O}_{D_0}(-H-F) \oplus \mathcal{O}_{D_1}(-L) \to \mathcal{O}_L(-1) \to 0.$$

This gives the long exact cohomology sequence

$$\begin{aligned} 0 &\to H^0(\mathcal{O}_{Z_0}(-H-F)) \to H^0(\mathcal{O}_{D_0}(-H-F)) \oplus H^0(\mathcal{O}_{D_1}(-L)) \to H^0(\mathcal{O}_L(-1)) \xrightarrow{0} \\ &\to H^1(\mathcal{O}_{Z_0}(-H-F)) \to H^1(\mathcal{O}_{D_0}(-H-F)) \oplus H^1(\mathcal{O}_{D_1}(-L)) \to H^1(\mathcal{O}_L(-1)) \to \\ &\to H^2(\mathcal{O}_{Z_0}(-H-F)) \to H^2(\mathcal{O}_{D_0}(-H-F)) \oplus H^2(\mathcal{O}_{D_1}(-L)) \to H^2(\mathcal{O}_L(-1)) \to 0. \end{aligned}$$

Since $H^i(\mathcal{O}_L(-1)) = H^i(\mathcal{O}_{D_1}(-L)) = 0$ for all i, we obtain for all i the isomorphisms

$$H^i(\mathcal{O}_{Z_0}(-H-F)) \cong H^i(\mathcal{O}_{D_0}(-H-F)).$$

From the resolution

$$0 \to \mathcal{O}_{\mathbb{P}_2\times\mathbb{P}_1}(-2,-2) \to \mathcal{O}_{\mathbb{P}_2\times\mathbb{P}_1}(-1,-1) \to \mathcal{O}_{D_0}(-H-F) \to 0$$

we obtain the long exact cohomology sequence

$$\begin{aligned} 0 &\to H^0(\mathcal{O}_{\mathbb{P}_2\times\mathbb{P}_1}(-2,-2)) \to H^0(\mathcal{O}_{\mathbb{P}_2\times\mathbb{P}_1}(-1,-1)) \to H^0(\mathcal{O}_{D_0}(-H-F)) \to \\ &\to H^1(\mathcal{O}_{\mathbb{P}_2\times\mathbb{P}_1}(-2,-2)) \to H^1(\mathcal{O}_{\mathbb{P}_2\times\mathbb{P}_1}(-1,-1)) \to H^1(\mathcal{O}_{D_0}(-H-F)) \to \\ &\to H^2(\mathcal{O}_{\mathbb{P}_2\times\mathbb{P}_1}(-2,-2)) \to H^2(\mathcal{O}_{\mathbb{P}_2\times\mathbb{P}_1}(-1,-1)) \to H^2(\mathcal{O}_{D_0}(-H-F)) \to \\ &\to H^3(\mathcal{O}_{\mathbb{P}_2\times\mathbb{P}_1}(-2,-2)) \to H^3(\mathcal{O}_{\mathbb{P}_2\times\mathbb{P}_1}(-1,-1)) \to 0. \end{aligned}$$

By Künneth formula we compute $H^i(\mathcal{O}_{\mathbb{P}_2\times\mathbb{P}_1}(-2,-2)) = H^i(\mathcal{O}_{\mathbb{P}_2\times\mathbb{P}_1}(-1,-1)) = 0$ for all i. Therefore, $H^i(\mathcal{O}_{D_0}(-H-F)) = 0$ for all i.

Sheaf $\mathcal{O}_{Z_0}(-2F)$. Consider the gluing exact sequence

$$0 \to \mathcal{O}_{Z_0}(-2F) \to \mathcal{O}_{D_0}(-2F) \oplus \mathcal{O}_{D_1}(-2L) \to \mathcal{O}_L(-2) \to 0.$$

This gives the long exact cohomology sequence

$$\begin{aligned} 0 &\to H^0(\mathcal{O}_{Z_0}(-2F)) \to H^0(\mathcal{O}_{D_0}(-2F)) \oplus H^0(\mathcal{O}_{D_1}(-2L)) \to H^0(\mathcal{O}_L(-2)) \to \\ &\to H^1(\mathcal{O}_{Z_0}(-2F)) \to H^1(\mathcal{O}_{D_0}(-2F)) \oplus H^1(\mathcal{O}_{D_1}(-2L)) \to H^1(\mathcal{O}_L(-2)) \to \\ &\to H^2(\mathcal{O}_{Z_0}(-2F)) \to H^2(\mathcal{O}_{D_0}(-2F)) \oplus H^2(\mathcal{O}_{D_1}(-2L)) \to H^2(\mathcal{O}_L(-2)) \to 0. \end{aligned}$$

All cohomology groups of $\mathcal{O}_{D_1}(-2L)$ are zero, we have also that $H^0(\mathcal{O}_L(-2)) = H^2(\mathcal{O}_L(-2)) = 0$ and $H^1(\mathcal{O}_L(-2)) \cong \Bbbk$. Therefore, $H^0(\mathcal{O}_{Z_0}(-2F)) \cong H^0(\mathcal{O}_{D_0}(-2F))$ and we obtain the exact sequence

$$0 \to H^1(\mathcal{O}_{Z_0}(-2F)) \to H^1(\mathcal{O}_{D_0}(-2F)) \to \Bbbk \to H^2(\mathcal{O}_{Z_0}(-2F)) \to H^2(\mathcal{O}_{D_0}(-2F)) \to 0$$

From the resolution

$$0 \to \mathcal{O}_{\mathbb{P}_2\times\mathbb{P}_1}(-1,-3) \to \mathcal{O}_{\mathbb{P}_2\times\mathbb{P}_1}(0,-2) \to \mathcal{O}_{D_0}(-2F) \to 0$$

we obtain the long exact cohomology sequence

$$\begin{aligned} 0 &\to H^0(\mathcal{O}_{\mathbb{P}_2\times\mathbb{P}_1}(-1,-3)) \to H^0(\mathcal{O}_{\mathbb{P}_2\times\mathbb{P}_1}(0,-2)) \to H^0(\mathcal{O}_{D_0}(-2F)) \to \\ &\to H^1(\mathcal{O}_{\mathbb{P}_2\times\mathbb{P}_1}(-1,-3)) \to H^1(\mathcal{O}_{\mathbb{P}_2\times\mathbb{P}_1}(0,-2)) \to H^1(\mathcal{O}_{D_0}(-2F)) \to \\ &\to H^2(\mathcal{O}_{\mathbb{P}_2\times\mathbb{P}_1}(-1,-3)) \to H^2(\mathcal{O}_{\mathbb{P}_2\times\mathbb{P}_1}(0,-2)) \to H^2(\mathcal{O}_{D_0}(-2F)) \to \\ &\to H^3(\mathcal{O}_{\mathbb{P}_2\times\mathbb{P}_1}(-1,-3)) \to H^3(\mathcal{O}_{\mathbb{P}_2\times\mathbb{P}_1}(0,-2)) \to 0. \end{aligned}$$

By Künneth formula $H^i(\mathcal{O}_{\mathbb{P}_2\times\mathbb{P}_1}(-1,-3)) = 0$ for all i. We have also $H^i(\mathcal{O}_{\mathbb{P}_2\times\mathbb{P}_1}(0,-2))$ for $i \neq 1$, and $H^1(\mathcal{O}_{\mathbb{P}_2\times\mathbb{P}_1}(0,-2)) \cong \Bbbk$. Thus $H^0(\mathcal{O}_{D_0}(-2F)) = H^2(\mathcal{O}_{D_0}(-2F)) = 0$ and $H^1(\mathcal{O}_{D_0}(-2F)) \cong \Bbbk$. We get this way $H^0(\mathcal{O}_{Z_0}(-2F)) = 0$ and the exact sequence

$$0 \to H^1(\mathcal{O}_{Z_0}(-2F)) \to \Bbbk \to \Bbbk \to H^2(\mathcal{O}_{Z_0}(-2F)) \to 0,$$

where the morphism $\Bbbk \to \Bbbk$ is the morphism $H^1(\mathcal{O}_{D_0}(-2F)) \to H^1(\mathcal{O}_L(-2))$ induced by the restriction map $\mathcal{O}_{D_0}(-2F) \to \mathcal{O}_L(-2)$.

Claim. *The restriction homomorphism*

$$\Bbbk \cong H^1(\mathcal{O}_{D_0}(-2F)) \to H^1(\mathcal{O}_L(-2)) \cong \Bbbk$$

is an isomorphism.

Proof. Consider the exact sequence

$$0 \to \mathcal{O}_{D_0}(-H-3F) \xrightarrow{(u_2\ -u_1)} 2\mathcal{O}_{D_0}(-H-2F) \xrightarrow{\binom{x_1}{x_2}} \mathcal{O}_{D_0}(-2F) \to \mathcal{O}_L(-2) \to 0$$

and its splitting in the two short exact sequences

$$0 \to \mathcal{O}_{D_0}(-H-3F) \to 2\mathcal{O}_{D_0}(-H-2F) \to \mathcal{A} \to 0$$

and

$$0 \to \mathcal{A} \to \mathcal{O}_{D_0}(-2F) \to \mathcal{O}_L(-2) \to 0.$$

First of all note that all the cohomologies of $\mathcal{O}_{D_0}(-H-aF)$ are zero. This follows by Künneth formula from the long exact cohomology sequence that corresponds to the resolution

$$0 \to \mathcal{O}_{\mathbb{P}_2\times\mathbb{P}_1}(-2,-a-1) \to \mathcal{O}_{\mathbb{P}_2\times\mathbb{P}_1}(-1,-a) \to \mathcal{O}_{D_0}(-H-aF) \to 0.$$

Then from the long exact cohomology sequence

$$H^i(2\mathcal{O}_{D_0}(-H-2F)) \to H^i(\mathcal{A}) \to H^{i+1}(\mathcal{O}_{D_0}(-H-3F))$$

one concludes that $H^i(\mathcal{A}) = 0$ for all i. From the long exact cohomology sequence

$$H^1(\mathcal{A}) \to H^1(\mathcal{O}_{D_0}(-2F)) \to H^1(\mathcal{O}_L(-2)) \to H^2(\mathcal{A}).$$

we obtain that the homomorphism $H^1(\mathcal{O}_{D_0}(-2F)) \to H^1(\mathcal{O}_L(-2))$ is an isomorphism. $\square$

From this claim and from the exact sequence

$$0 \to H^1(\mathcal{O}_{Z_0}(-2F)) \to H^1(\mathcal{O}_{D_0}(-2F)) \to H^1(\mathcal{O}_L(-2)) \to H^2(\mathcal{O}_{Z_0}(-2F)) \to 0$$

we conclude that $H^1(\mathcal{O}_{Z_0}(-2F)) = 0$, $i = 1, 2$. We obtained that

$$H^i(\mathcal{O}_{Z_0}(-2F)) = 0, \quad \text{for all } i.$$

Sheaf $\mathcal{O}_{Z_0}(H+2F)$. Consider the gluing exact sequence

$$0 \to \mathcal{O}_{Z_0}(H+2F) \to \mathcal{O}_{D_0}(H+2F) \oplus \mathcal{O}_{D_1}(2L) \to \mathcal{O}_L(2) \to 0.$$

This gives the long exact cohomology sequence

$$\begin{aligned} 0 &\to H^0(\mathcal{O}_{Z_0}(H+2F)) \to H^0(\mathcal{O}_{D_0}(H+2F)) \oplus H^0(\mathcal{O}_{D_1}(2L)) \to H^0(\mathcal{O}_L(2)) \xrightarrow{0} \\ &\to H^1(\mathcal{O}_{Z_0}(H+2F)) \to H^1(\mathcal{O}_{D_0}(H+2F)) \oplus H^1(\mathcal{O}_{D_1}(2L)) \to H^1(\mathcal{O}_L(2)) \to \\ &\to H^2(\mathcal{O}_{Z_0}(H+2F)) \to H^2(\mathcal{O}_{D_0}(H+2F)) \oplus H^2(\mathcal{O}_{D_1}(2L)) \to H^2(\mathcal{O}_L(2)) \to 0. \end{aligned}$$

Note that the restriction homomorphism $H^0(\mathcal{O}_{D_1}(2L)) \to H^0(\mathcal{O}_L(2))$ is surjective. Since $H^i(\mathcal{O}_{D_1}(2L)) = H^i(\mathcal{O}_L(2)) = 0$ for $i \neq 0$ we conclude that $H^i(\mathcal{O}_{Z_0}(H+2F)) \cong H^i(\mathcal{O}_{D_0}(H+2F))$ for $i \neq 0$.

From the resolution

$$0 \to \mathcal{O}_{\mathbb{P}_2\times\mathbb{P}_1}(0,1) \to \mathcal{O}_{\mathbb{P}_2\times\mathbb{P}_1}(1,2) \to \mathcal{O}_{D_0}(H+2F) \to 0$$

we obtain the long exact cohomology sequence

$$\begin{aligned} 0 &\to H^0(\mathcal{O}_{\mathbb{P}_2\times\mathbb{P}_1}(0,1)) \to H^0(\mathcal{O}_{\mathbb{P}_2\times\mathbb{P}_1}(1,2)) \to H^0(\mathcal{O}_{D_0}(H+2F)) \to \\ &\to H^1(\mathcal{O}_{\mathbb{P}_2\times\mathbb{P}_1}(0,1)) \to H^1(\mathcal{O}_{\mathbb{P}_2\times\mathbb{P}_1}(1,2)) \to H^1(\mathcal{O}_{D_0}(H+2F)) \to \\ &\to H^2(\mathcal{O}_{\mathbb{P}_2\times\mathbb{P}_1}(0,1)) \to H^2(\mathcal{O}_{\mathbb{P}_2\times\mathbb{P}_1}(1,2)) \to H^2(\mathcal{O}_{D_0}(H+2F)) \to \\ &\to H^3(\mathcal{O}_{\mathbb{P}_2\times\mathbb{P}_1}(0,1)) \to H^3(\mathcal{O}_{\mathbb{P}_2\times\mathbb{P}_1}(1,2)) \to 0. \end{aligned}$$

By Künneth formula one concludes that $H^i(\mathcal{O}_{\mathbb{P}_2\times\mathbb{P}_1}(0,1)) = 0$ for $i \neq 0$ and that

$$H^0(\mathcal{O}_{\mathbb{P}_2\times\mathbb{P}_1}(0,1)) \cong H^0(\mathcal{O}_{\mathbb{P}_2}) \otimes H^0(\mathcal{O}_{\mathbb{P}_1}(1)) \cong \mathbb{k}^2.$$

Again by Künneth formula we obtain $H^i(\mathcal{O}_{\mathbb{P}_2\times\mathbb{P}_1}(1,2)) = 0$ for $i \neq 0$ and

$$H^0(\mathcal{O}_{\mathbb{P}_2\times\mathbb{P}_1}(1,2)) \cong H^0(\mathcal{O}_{\mathbb{P}_2}(1)) \otimes H^0(\mathcal{O}_{\mathbb{P}_1}(2)) \cong \mathbb{k}^3 \otimes \mathbb{k}^3 \cong \mathbb{k}^9.$$

We conclude this way that $H^1(\mathcal{O}_{D_0}(H+2F)) = H^2(\mathcal{O}_{D_0}(H+2F)) = 0$ and that $H^0(\mathcal{O}_{D_0}(H+2F)) \cong \mathbb{k}^7$.

Since $\dim H^0(\mathcal{O}_{D_1}(2L)) = 6$ and $\dim H^0(\mathcal{O}_L(2)) = 3$, then using the exact sequence

$$0 \to H^0(\mathcal{O}_{Z_0}(H+2F)) \to H^0(\mathcal{O}_{D_0}(H+2F)) \oplus H^0(\mathcal{O}_{D_1}(2L)) \to H^0(\mathcal{O}_L(2)) \to 0$$

we conclude that $H^0(\mathcal{O}_{Z_0}(H+2F)) \cong \mathbb{k}^{10}$. We proved that

$$H^0(\mathcal{O}_{Z_0}(H+2F)) \cong \mathbb{k}^{10}, \quad H^i(\mathcal{O}_{Z_0}(H+2F)) = 0, \quad i \neq 0.$$

Sheaf $\mathcal{O}_{Z_0}(H+F)$. Consider the gluing exact sequence

$$0 \to \mathcal{O}_{Z_0}(H+F) \to \mathcal{O}_{D_0}(H+F) \oplus \mathcal{O}_{D_1}(L) \to \mathcal{O}_L(1) \to 0.$$

This gives the long exact cohomology sequence

$$\begin{aligned} 0 &\to H^0(\mathcal{O}_{Z_0}(H+F)) \to H^0(\mathcal{O}_{D_0}(H+F)) \oplus H^0(\mathcal{O}_{D_1}(L)) \to H^0(\mathcal{O}_L(1)) \xrightarrow{0} \\ &\to H^1(\mathcal{O}_{Z_0}(H+F)) \to H^1(\mathcal{O}_{D_0}(H+F)) \oplus H^1(\mathcal{O}_{D_1}(L)) \to H^1(\mathcal{O}_L(1)) \to \\ &\to H^2(\mathcal{O}_{Z_0}(H+F)) \to H^2(\mathcal{O}_{D_0}(H+F)) \oplus H^2(\mathcal{O}_{D_1}(L)) \to H^2(\mathcal{O}_L(1)) \to 0 \end{aligned}$$

and from $H^i(\mathcal{O}_{D_1}(L)) = H^i(\mathcal{O}_L(1)) = 0$, $i \neq 0$, we obtain the isomorphisms $H^i(\mathcal{O}_{Z_0}(H+F)) \cong H^i(\mathcal{O}_{D_0}(H+F))$, $i \neq 0$, and the exact sequence

$$0 \to H^0(\mathcal{O}_{Z_0}(H+F)) \to H^0(\mathcal{O}_{D_0}(H+F)) \oplus H^0(\mathcal{O}_{D_1}(L)) \to H^0(\mathcal{O}_L(1)) \to 0$$

From the resolution

$$0 \to \mathcal{O}_{\mathbb{P}_2 \times \mathbb{P}_1} \to \mathcal{O}_{\mathbb{P}_2 \times \mathbb{P}_1}(1,1) \to \mathcal{O}_{D_0}(H+F) \to 0$$

we obtain the long exact cohomology sequence

$$\begin{aligned} 0 &\to H^0(\mathcal{O}_{\mathbb{P}_2\times\mathbb{P}_1}) \to H^0(\mathcal{O}_{\mathbb{P}_2\times\mathbb{P}_1}(1,1)) \to H^0(\mathcal{O}_{D_0}(H+F)) \to \\ &\to H^1(\mathcal{O}_{\mathbb{P}_2\times\mathbb{P}_1}) \to H^1(\mathcal{O}_{\mathbb{P}_2\times\mathbb{P}_1}(1,1)) \to H^1(\mathcal{O}_{D_0}(H+F)) \to \\ &\to H^2(\mathcal{O}_{\mathbb{P}_2\times\mathbb{P}_1}) \to H^2(\mathcal{O}_{\mathbb{P}_2\times\mathbb{P}_1}(1,1)) \to H^2(\mathcal{O}_{D_0}(H+F)) \to \\ &\to H^3(\mathcal{O}_{\mathbb{P}_2\times\mathbb{P}_1}) \to H^3(\mathcal{O}_{\mathbb{P}_2\times\mathbb{P}_1}(1,1)) \to 0. \end{aligned}$$

Using Künneth formula we compute $H^0(\mathcal{O}_{\mathbb{P}_2\times\mathbb{P}_1}(1,1)) \cong \mathbb{k}^6$, $H^0(\mathcal{O}_{\mathbb{P}_2\times\mathbb{P}_1}) \cong \mathbb{k}$ and all the other cohomology groups of the sheaves $\mathcal{O}_{\mathbb{P}_2\times\mathbb{P}_1}(1,1)$ and $\mathcal{O}_{\mathbb{P}_2\times\mathbb{P}_1}$ are zero. Therefore, $H^i(\mathcal{O}_{D_0}(H+F)) = 0$, $i \neq 0$, and $H^0(\mathcal{O}_{D_0}(H+F)) \cong \mathbb{k}^5$. Since $\dim H^0(\mathcal{O}_{D_1}(L)) = 3$ and $\dim H^0(\mathcal{O}_L(1)) = 2$, we conclude that $H^0(\mathcal{O}_{Z_0}(H+F)) \cong \mathbb{k}^6$.

We proved that

$$H^0(\mathcal{O}_{Z_0}(H+F)) \cong \mathbb{k}^6, \quad H^i(\mathcal{O}_{Z_0}(H+F)) = 0, \quad i \neq 0.$$

Some direct images.

Let us calculate some direct images with respect to the contraction

$$\sigma_0 : Z_0 \to \mathbb{P}_2.$$

We will consider $\mathcal{O}_{Z_0}$ as a sheaf on $\mathbb{P}_2 \times \mathbb{P}_2$ given by the ideal sheaf $\mathcal{I}_{Z_0}$:

$$0 \to \mathcal{I}_{Z_0} \to \mathcal{O}_{\mathbb{P}_2\times\mathbb{P}_2} \to \mathcal{O}_{Z_0} \to 0. \tag{1.27}$$

Lemma 1.33. *The ideal sheaf $\mathcal{I}_{Z_0}$ of Z_0 is given by the resolution*

$$0 \to \mathcal{O}_{\mathbb{P}_2\times\mathbb{P}_2}(-2,-1) \oplus \mathcal{O}_{\mathbb{P}_2\times\mathbb{P}_2}(-1,-2) \xrightarrow{\left(\begin{smallmatrix} u_0 & -u_2 & u_1 \\ 0 & -x_2 & x_1 \end{smallmatrix}\right)} 3\mathcal{O}_{\mathbb{P}_2\times\mathbb{P}_2}(-1,-1) \xrightarrow{\left(\begin{smallmatrix} x_1u_2 - x_2u_1 \\ u_0x_1 \\ u_0x_2 \end{smallmatrix}\right)} \mathcal{I}_{Z_0} \to 0.$$

Proof. It is clear that this sequence is a complex. Suppose that

$$(a,b,c)\left(\begin{smallmatrix} x_1u_2 - x_2u_1 \\ u_0x_1 \\ u_0x_2 \end{smallmatrix}\right) = a(x_1u_2 - x_2u_1) + bu_0x_1 + cu_0x_2 = 0.$$

Then $a = \xi u_0$ and $\xi(x_1u_2 - x_2u_1) + bx_1 + cx_2 = x_1(\xi u_2 + b) + x_2(c - \xi u_1) = 0$ and therefore $c - \xi u_1 = \eta x_1$ and $(\xi u_2 + b) = -\eta x_2$. We obtain that

$$c = \xi u_1 + \eta x_1, \quad b = -\xi u_2 - \eta x_2,$$

thus

$$\begin{pmatrix} a & b & c \end{pmatrix} = \begin{pmatrix} \xi & \eta \end{pmatrix} \begin{pmatrix} u_0 & -u_2 & u_1 \\ 0 & -x_2 & x_1 \end{pmatrix}.$$

This proves that the sequence is exact in the middle term. One sees also that $\left(\begin{smallmatrix} u_0 & -u_2 & u_1 \\ 0 & -x_2 & x_1 \end{smallmatrix}\right)$ is injective. $\square$

Consider the diagram

$$\begin{tikzcd} Z_0 \arrow[r, "\iota"] \arrow[dr, "\sigma_0"'] & \mathbb{P}_2 \times \mathbb{P}_2 \arrow[d, "p"] \\ & \mathbb{P}_2 \end{tikzcd},$$

where ι is the inclusion of Z_0 in $\mathbb{P}_2 \times \mathbb{P}_2$ and $p = p_1$ is the projection on the first factor. By abuse of notation we identify the sheaves on Z_0 with their images under ι_*, i. e., we consider them as sheaves on $\mathbb{P}_2 \times \mathbb{P}_2$. Then for every $\mathcal{O}_{Z_0}$ module $\mathcal{F}$ we have $R^k\sigma_{0*}\mathcal{F} \cong R^k p_*\mathcal{F}$ for all k.

Lemma 1.34. $R^k p_*\mathcal{O}_{\mathbb{P}_2\times\mathbb{P}_2}(m,n) \cong \mathcal{O}_{\mathbb{P}_2}(m) \otimes H^k(\mathbb{P}_2, \mathcal{O}_{\mathbb{P}_2}(n))$.

Proof. Recall (cf. [12], III, Proposition 8.1) that $R^k p_*\mathcal{O}_{\mathbb{P}_2\times\mathbb{P}_2}(m,n)$ is the sheaf associated to the presheaf

$$U \mapsto H^k(U \times \mathbb{P}_2, \mathcal{O}_{\mathbb{P}_2\times\mathbb{P}_2}(m,n)|_{U\times\mathbb{P}_2}).$$

For an affine open set $U \subseteq \mathbb{P}_2$ using the Künneth formula we obtain

$$\begin{aligned} H^k(U \times \mathbb{P}_2, \mathcal{O}_{\mathbb{P}_2\times\mathbb{P}_2}(m,n)|_{U\times\mathbb{P}_2}) \cong H^0(U, \mathcal{O}_{\mathbb{P}_2}(m)) \otimes H^k(\mathbb{P}_2, \mathcal{O}_{\mathbb{P}_2}(n)) = \\ = \mathcal{O}_{\mathbb{P}_2}(m)(U) \otimes H^k(\mathbb{P}_2, \mathcal{O}_{\mathbb{P}_2}(n)). \end{aligned}$$

Therefore, we conclude that $R^k p_*\mathcal{O}_{\mathbb{P}_2\times\mathbb{P}_2}(m,n) \cong \mathcal{O}_{\mathbb{P}_2}(m) \otimes H^k(\mathbb{P}_2, \mathcal{O}_{\mathbb{P}_2}(n))$. This proves the required statement. □

Sheaf $\mathcal{O}_{Z_0}$. From the resolution of $\mathcal{I}_{Z_0}$ we obtain the long exact sequence

$$\begin{aligned} 0 &\to p_*\mathcal{O}_{\mathbb{P}_2\times\mathbb{P}_2}(-2,-1) \oplus p_*\mathcal{O}_{\mathbb{P}_2\times\mathbb{P}_2}(-1,-2) \to 3p_*\mathcal{O}_{\mathbb{P}_2\times\mathbb{P}_2}(-1,-1) \to p_*\mathcal{I}_{Z_0} \to \\ &\to R^1p_*\mathcal{O}_{\mathbb{P}_2\times\mathbb{P}_2}(-2,-1) \oplus R^1p_*\mathcal{O}_{\mathbb{P}_2\times\mathbb{P}_2}(-1,-2) \to 3R^1p_*\mathcal{O}_{\mathbb{P}_2\times\mathbb{P}_2}(-1,-1) \to R^1p_*\mathcal{I}_{Z_0} \to \\ &\to R^2p_*\mathcal{O}_{\mathbb{P}_2\times\mathbb{P}_2}(-2,-1) \oplus R^2p_*\mathcal{O}_{\mathbb{P}_2\times\mathbb{P}_2}(-1,-2) \to 3R^2p_*\mathcal{O}_{\mathbb{P}_2\times\mathbb{P}_2}(-1,-1) \to R^2p_*\mathcal{I}_{Z_0} \to \dots \end{aligned}$$

Using Lemma 1.34 we conclude that $R^k p_*\mathcal{I}_{Z_0} = 0$ for all $k \geqslant 0$.

From the sequence (1.27) one obtains the long exact sequence

$$\begin{aligned} 0 &\to p_*\mathcal{I}_{Z_0} \to p_*\mathcal{O}_{\mathbb{P}_2\times\mathbb{P}_2} \to p_*\mathcal{O}_{Z_0} \to \\ &\to R^1p_*\mathcal{I}_{Z_0} \to R^1p_*\mathcal{O}_{\mathbb{P}_2\times\mathbb{P}_2} \to R^1p_*\mathcal{O}_{Z_0} \to \\ &\to R^2p_*\mathcal{I}_{Z_0} \to R^2p_*\mathcal{O}_{\mathbb{P}_2\times\mathbb{P}_2} \to R^2p_*\mathcal{O}_{Z_0} \to \dots \end{aligned}$$

Since $R^k p_*\mathcal{I}_{Z_0} = 0$ for all k, we conclude

$$R^k\sigma_{0*}\mathcal{O}_{Z_0} \cong R^k p_*\mathcal{O}_{\mathbb{P}_2\times\mathbb{P}_2} \cong \begin{cases} \mathcal{O}_{\mathbb{P}_2} & \text{if } k = 0, \\ 0 & \text{if } k \neq 0\ . \end{cases}$$

Sheaf $\mathcal{O}_{Z_0}(-H)$. Tensoring the sequence (1.27) with $\mathcal{O}_{\mathbb{P}_2\times\mathbb{P}_2}(-1,0)$ we obtain the exact sequence

$$0 \to \mathcal{I}_{Z_0}(-1,0) \to \mathcal{O}_{\mathbb{P}_2\times\mathbb{P}_2}(-1,0) \to \mathcal{O}_{Z_0}(-H) \to 0.$$

The sequence

$$0 \to \mathcal{O}_{\mathbb{P}_2\times\mathbb{P}_2}(-3,-1) \oplus \mathcal{O}_{\mathbb{P}_2\times\mathbb{P}_2}(-2,-2) \to 3\mathcal{O}_{\mathbb{P}_2\times\mathbb{P}_2}(-2,-1) \to \mathcal{I}_{Z_0}(-1,0) \to 0.$$

is also exact. Hence, using Lemma 1.34, from the long exact sequence

$$\begin{aligned} 0 &\to p_*(\mathcal{O}_{\mathbb{P}_2\times\mathbb{P}_2}(-3,-1) \oplus \mathcal{O}_{\mathbb{P}_2\times\mathbb{P}_2}(-2,-2)) \to 3p_*\mathcal{O}_{\mathbb{P}_2\times\mathbb{P}_2}(-2,-1) \to p_*\mathcal{I}_{Z_0}(-1,0) \to \\ &\to R^1p_*(\mathcal{O}_{\mathbb{P}_2\times\mathbb{P}_2}(-3,-1) \oplus \mathcal{O}_{\mathbb{P}_2\times\mathbb{P}_2}(-2,-2)) \to 3R^1p_*\mathcal{O}_{\mathbb{P}_2\times\mathbb{P}_2}(-2,-1) \to R^1p_*\mathcal{I}_{Z_0}(-1,0) \to \\ &\to R^2p_*(\mathcal{O}_{\mathbb{P}_2\times\mathbb{P}_2}(-3,-1) \oplus \mathcal{O}_{\mathbb{P}_2\times\mathbb{P}_2}(-2,-2)) \to 3R^2p_*\mathcal{O}_{\mathbb{P}_2\times\mathbb{P}_2}(-2,-1) \to R^2p_*\mathcal{I}_{Z_0}(-1,0) \to \dots \end{aligned}$$

we conclude that $R^k p_* \mathcal{I}_{Z_0}(-1,0) = 0$ for all $k \geqslant 0$. Therefore, using the long exact sequence

$$\begin{aligned} 0 \to p_* \mathcal{I}_{Z_0}(-1,0) \to p_* \mathcal{O}_{\mathbb{P}_2\times\mathbb{P}_2}(-1,0) \to p_* \mathcal{O}_{Z_0}(-H) \to \\ \to R^1 p_* \mathcal{I}_{Z_0}(-1,0) \to R^1 p_* \mathcal{O}_{\mathbb{P}_2\times\mathbb{P}_2}(-1,0) \to R^1 p_* \mathcal{O}_{Z_0}(-H) \to \\ \to R^2 p_* \mathcal{I}_{Z_0}(-1,0) \to R^2 p_* \mathcal{O}_{\mathbb{P}_2\times\mathbb{P}_2}(-1,0) \to R^2 p_* \mathcal{O}_{Z_0}(-H) \to \dots \end{aligned}$$

we obtain

$$R^k \sigma_{0*} \mathcal{O}_{Z_0}(-H) \cong R^k p_* \mathcal{O}_{\mathbb{P}_2\times\mathbb{P}_2}(-1,0) \cong \begin{cases} \mathcal{O}_{\mathbb{P}_2}(-1) & \text{if } k = 0, \\ 0 & \text{if } k \neq 0\ . \end{cases}$$

Sheaf $\mathcal{O}_{Z_0}(-H-F)$. Tensoring the sequence (1.27) with $\mathcal{O}_{\mathbb{P}_2\times\mathbb{P}_2}(-1,-1)$ we obtain the exact sequence

$$0 \to \mathcal{I}_{Z_0}(-1,-1) \to \mathcal{O}_{\mathbb{P}_2\times\mathbb{P}_2}(-1,-1) \to \mathcal{O}_{Z_0}(-H-F) \to 0.$$

The sequence

$$0 \to \mathcal{O}_{\mathbb{P}_2\times\mathbb{P}_2}(-3,-2) \oplus \mathcal{O}_{\mathbb{P}_2\times\mathbb{P}_2}(-2,-3) \to 3\mathcal{O}_{\mathbb{P}_2\times\mathbb{P}_2}(-2,-2) \to \mathcal{I}_{Z_0}(-1,-1) \to 0.$$

is also exact. Hence, using Lemma 1.34, from the long exact sequence

$$\begin{aligned} 0 \to p_* \mathcal{O}_{\mathbb{P}_2\times\mathbb{P}_2}(-3,-2) \oplus p_* \mathcal{O}_{\mathbb{P}_2\times\mathbb{P}_2}(-2,-3) \to 3p_* \mathcal{O}_{\mathbb{P}_2\times\mathbb{P}_2}(-2,-2) \to p_* \mathcal{I}_{Z_0}(-1,-1) \to \\ \to R^1 p_* \mathcal{O}_{\mathbb{P}_2\times\mathbb{P}_2}(-3,-2) \oplus R^1 p_* \mathcal{O}_{\mathbb{P}_2\times\mathbb{P}_2}(-2,-3) \to 3R^1 p_* \mathcal{O}_{\mathbb{P}_2\times\mathbb{P}_2}(-2,-2) \to R^1 p_* \mathcal{I}_{Z_0}(-1,-1) \to \\ \to R^2 p_* \mathcal{O}_{\mathbb{P}_2\times\mathbb{P}_2}(-3,-2) \oplus R^2 p_* \mathcal{O}_{\mathbb{P}_2\times\mathbb{P}_2}(-2,-3) \to 3R^2 p_* \mathcal{O}_{\mathbb{P}_2\times\mathbb{P}_2}(-2,-2) \to R^2 p_* \mathcal{I}_{Z_0}(-1,-1) \to \dots \end{aligned}$$

we conclude $R^k p_* \mathcal{I}_{Z_0}(-1,-1) = 0$ for $k \neq 1$ and $R^1 p_* \mathcal{I}_{Z_0}(-1,-1) \cong \mathcal{O}_{\mathbb{P}_2}(-2)$. Lemma 1.34 implies also $R^k p_* \mathcal{O}_{\mathbb{P}_2\times\mathbb{P}_2}(-1,-1) = 0$ for all k. Hence from the exact sequence

$$\begin{aligned} 0 \to p_* \mathcal{I}_{Z_0}(-1,-1) \to p_* \mathcal{O}_{\mathbb{P}_2\times\mathbb{P}_2}(-1,-1) \to p_* \mathcal{O}_{Z_0}(-H-F) \to \\ \to R^1 p_* \mathcal{I}_{Z_0}(-1,-1) \to R^1 p_* \mathcal{O}_{\mathbb{P}_2\times\mathbb{P}_2}(-1,-1) \to R^1 p_* \mathcal{O}_{Z_0}(-H-F) \to \\ \to R^2 p_* \mathcal{I}_{Z_0}(-1,-1) \to R^2 p_* \mathcal{O}_{\mathbb{P}_2\times\mathbb{P}_2}(-1,-1) \to R^2 p_* \mathcal{O}_{Z_0}(-H-F) \to \dots \end{aligned}$$

we conclude that

$$R^k p_* \mathcal{O}_{Z_0}(-H-F) \cong R^{k+1} p_* \mathcal{I}_{Z_0}(-1,-1), \quad k \geqslant 0.$$

Thus

$$R^k p_* \mathcal{O}_{Z_0}(-H-F) = \begin{cases} \mathcal{O}_{\mathbb{P}_2}(-2) & \text{if } k = 0, \\ 0 & \text{if } k \neq 0\ . \end{cases}$$

We proved the following lemma.

Lemma 1.35.

$$\begin{aligned} & R^0 \sigma_{0*} \mathcal{O}_{Z_0} \cong \mathcal{O}_{\mathbb{P}_2}, \quad R^0 \sigma_{0*} \mathcal{O}_{Z_0}(-H) \cong \mathcal{O}_{\mathbb{P}_2}(-1), \quad R^0 \sigma_{0*} \mathcal{O}_{Z_0}(-H-F) \cong \mathcal{O}_{\mathbb{P}_2}(-2), \\ & R^k \sigma_{0*} \mathcal{O}_{Z_0} = R^k \sigma_{0*} \mathcal{O}_{Z_0}(-H) = R^k \sigma_{0*} \mathcal{O}_{Z_0}(-H-F) = 0, \quad k > 0. \end{aligned}$$

1.3.2 R-bundles on $\hat{\mathbb{P}}_2$, their properties.

Definition 1.36. *Let us call the fibres $\widetilde{\mathcal{E}}_0$ of new one parameter families $\widetilde{\mathcal{E}}$ over $t = 0$ R-**bundles** or R-**sheaves** on Z_0. R stays here for "replacement". One could call them simply **new sheaves on** Z_0.*

As it has been shown above that R-bundles are exactly those constructed by the help of normal to X_8 directions.

We are going now to describe different exact sequences with R-bundles.

Locally free resolutions of R-bundles on Z_0.

Let σ_0 denote the restriction of σ to Z_0, i. e., σ_0 is a map $Z_0 \to \mathbb{P}_2$. Let us recall the meaning of H and F. H is given by a line in $D_0 = \widetilde{\mathbb{P}}_2$ that does not meet L. The divisor F is given, say, by equation $u_1 = 0$ and has as components the line $F_1 := \{u_1 = 0\}$ in $D_1 \cong \mathbb{P}_2$ and the line $F_0 := \{u_1 = 0\}$ in $D_0 = \widetilde{\mathbb{P}}_2$.

Since $\widetilde{\mathcal{E}}$ is flat over U, restricting (1.19) to Z_0 we obtain the exact sequence

$$0 \to 2\mathcal{O}_Z(-H-F)|_{Z_0} \xrightarrow{\widetilde{A}_t|_{Z_0}} \mathcal{O}_Z(-H)|_{Z_0} \oplus \mathcal{O}_Z|_{Z_0} \to \widetilde{\mathcal{E}}_0 \to 0.$$

As $\mathcal{O}_Z|_{Z_0} \cong \mathcal{O}_{Z_0}$, $\mathcal{O}_Z(-H)|_{Z_0} \cong \mathcal{O}_{Z_0}(-H)$, $\mathcal{O}_Z(-H-F)|_{Z_0} \cong \mathcal{O}_{Z_0}(-H-F)$, we obtain the exact sequence

$$0 \to 2\mathcal{O}_{Z_0}(-H-F) \xrightarrow{\widetilde{A}_t|_{Z_0}} \mathcal{O}_{Z_0}(-H) \oplus \mathcal{O}_{Z_0} \to \widetilde{\mathcal{E}}_0 \to 0, \tag{1.28}$$

where

$$\widetilde{A}_t|_{Z_0} = \begin{pmatrix} u_1 + \xi_0 u_0 & u_1(A_{01}x_0 + A_{11}x_1 + A_{12}x_2) + u_2(A_{02}x_0 + A_{22}x_2) + \xi_{00}x_0u_0 \\ u_2 + \eta_0 u_0 & u_1(B_{01}x_0 + B_{11}x_1 + B_{12}x_2) + u_2(B_{02}x_0 + B_{22}x_2) + \eta_{00}x_0u_0 \end{pmatrix}$$

and we interpret the entries of this matrix as sections of the corresponding locally free sheaves on Z_0. Let us collect our observations in the following proposition.

Proposition 1.37. *Every R-bundle on Z_0 has a resolution*

$$0 \to 2\mathcal{O}_{Z_0}(-H-F) \xrightarrow{\begin{pmatrix} l_1 & \widetilde{q}_1 \\ l_2 & \widetilde{q}_2 \end{pmatrix}} \mathcal{O}_{Z_0}(-H) \oplus \mathcal{O}_{Z_0} \to \widetilde{\mathcal{E}}_0 \to 0, \tag{1.29}$$

where $l_1, l_2 \in \Gamma(Z_0, \mathcal{O}_{Z_0}(F)) \cong \Gamma(D_1, \mathcal{O}_{D_1}(F)) \cong \mathbb{k}^3$, and $\widetilde{q}_1, \widetilde{q}_2 \in \Gamma(Z_0, \mathcal{O}_{Z_0}(H+F)) \cong \mathbb{k}^6$.

Moreover, the matrix $\Phi = \begin{pmatrix} l_1 & \widetilde{q}_1 \\ l_2 & \widetilde{q}_2 \end{pmatrix}$ has the following properties:

- *l_1 and l_2 are linear independent and their common zero point $l_1 \wedge l_2$ in $D_1 \cong \mathbb{P}_2$ does not belong to L;*
- $\det(\Phi|_{D_0}) \neq 0$*;*
- $(\Phi|_{D_1})(q) \neq 0$ *for all $q \in D_1$, in particular* $\det(\Phi|_{D_1}) \neq 0$.

Remark 1.38. *Note that the point $l_1 \wedge l_2 \in D_1$ is exactly the point $s(0)$ from Remark 1.13.*

Proposition 1.39. *Let $\mathcal{E}$ be a sheaf on Z_0 given by a resolution*

$$0 \to 2\mathcal{O}_{Z_0}(-H-F) \xrightarrow{\Phi} \mathcal{O}_{Z_0}(-H) \oplus \mathcal{O}_{Z_0} \to \mathcal{E} \to 0.$$

Then applying σ_{0*} to this sequence gives the exact sequence

$$0 \to 2\mathcal{O}_{\mathbb{P}_2}(-2) \xrightarrow{\sigma_{0*}\Phi} \mathcal{O}_{\mathbb{P}_2}(-1) \oplus \mathcal{O}_{\mathbb{P}_2} \to \sigma_{0*}\mathcal{E} \to 0.$$

In particular push forwards of R-bundles on Z_0 are $3m+1$ sheaves on $\mathbb{P}_2$.

Proof. Follows from Lemma 1.35. □

R-bundles on Z_0 have the following important property, which makes resolutions (1.28) similar to the Beilinson resolutions of coherent sheaves on $\mathbb{P}_n$ (cf. [3]).

Proposition 1.40. *Every morphism between sheaves possessing resolutions of the type* (1.28) *can be uniquely lifted to a morphism of the corresponding resolutions. In particular this holds for R-bundles.*

Proof. A lifting exists if and only if

$$\mathrm{Ext}^1(\mathcal{O}_{Z_0}(-H) \oplus \mathcal{O}_{Z_0}, 2\mathcal{O}_{Z_0}(-H-F)) = 0.$$

There is a unique lifting if and only if

$$\mathrm{Ext}^1(\mathcal{O}_{Z_0}(-H) \oplus \mathcal{O}_{Z_0}, 2\mathcal{O}_{Z_0}(-H-F)) = \mathrm{Hom}(\mathcal{O}_{Z_0}(-H) \oplus \mathcal{O}_{Z_0}, 2\mathcal{O}_{Z_0}(-H-F)) = 0.$$

Since

$$\begin{aligned}\mathrm{Ext}^1(\mathcal{O}_{Z_0}(-H) \oplus \mathcal{O}_{Z_0}, 2\mathcal{O}_{Z_0}(-H-F)) = \\ = 2\,\mathrm{Ext}^1(\mathcal{O}_{Z_0}(-H), \mathcal{O}_{Z_0}(-H-F)) \oplus 2\,\mathrm{Ext}^1(\mathcal{O}_{Z_0}, \mathcal{O}_{Z_0}(-H-F)),\end{aligned}$$

it is enough to show that $\mathrm{Ext}^1(\mathcal{O}_{Z_0}(-H), \mathcal{O}_{Z_0}(-H-F)) = \mathrm{Ext}^1(\mathcal{O}_{Z_0}, \mathcal{O}_{Z_0}(-H-F)) = 0$. Since the sheaf $\mathcal{O}_{Z_0}(-H)$ is locally free, one obtains (cf. [12], III, Proposition 6.7 and Proposition 6.3, (c))

$$\mathrm{Ext}^i(\mathcal{O}_{Z_0}(-H), \mathcal{O}_{Z_0}(-H-F)) = H^i(\mathcal{O}_{Z_0}(H) \otimes \mathcal{O}_{Z_0}(-H-F)) = H^i(\mathcal{O}_{Z_0}(-F))$$

and

$$\mathrm{Ext}^i(\mathcal{O}_{Z_0}, \mathcal{O}_{Z_0}(-H-F)) = H^i(\mathcal{O}_{Z_0}(-H-F)).$$

By Proposition 1.32 all cohomology groups of $\mathcal{O}_{Z_0}(-F)$ and $\mathcal{O}_{Z_0}(-H-F)$ are zero. One concludes that

$$\mathrm{Ext}^1(\mathcal{O}_{Z_0}(-H) \oplus \mathcal{O}_{Z_0}, 2\mathcal{O}_{Z_0}(-H-F)) = \mathrm{Hom}(\mathcal{O}_{Z_0}(-H) \oplus \mathcal{O}_{Z_0}, 2\mathcal{O}_{Z_0}(-H-F)) = 0.$$

This completes the proof. □

Remark 1.41. *1) Since* $\Gamma(Z_0, \mathcal{O}_{Z_0}) = \Bbbk$, *we conclude that the set of endomorphisms of the sheaf* $2\mathcal{O}_{Z_0}(-H-F)$ *is just the set* $\mathrm{Mat}_{2\times 2}(\Bbbk)$ *of* 2×2 *matrices over* $\Bbbk$.

2) The set of endomorphism of $\mathcal{O}_{Z_0}(-H) \oplus \mathcal{O}_{Z_0}$ *can be identified with the set of the matrices* $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, *where* $a \in \mathrm{Hom}(\mathcal{O}_{Z_0}(-H), \mathcal{O}_{Z_0}(-H)) \cong \Bbbk$, $b \in \mathrm{Hom}(\mathcal{O}_{Z_0}(-H), \mathcal{O}_{Z_0}) \cong \Gamma(Z_0, \mathcal{O}_{Z_0}(H))$, $c \in \mathrm{Hom}(\mathcal{O}_{Z_0}, \mathcal{O}_{Z_0}(-H)) \cong \Gamma(Z_0, \mathcal{O}_{Z_0}(-H))$, $d \in \mathrm{Hom}(\mathcal{O}_{Z_0}, \mathcal{O}_{Z_0}) \cong \Bbbk$. *We have also an isomorphism* $\Gamma(Z_0, \mathcal{O}_{Z_0}(H)) \cong \Gamma(\mathbb{P}_2, \mathcal{O}_{\mathbb{P}_2}(1))$ *and* $\Gamma(Z_0, \mathcal{O}_{Z_0}(H)) \cong \Gamma(\mathbb{P}_2, \mathcal{O}_{\mathbb{P}_2}(-1)) = 0$. *That is why we can interpret the endomorphisms of* $\mathcal{O}_{Z_0}(-H) \oplus \mathcal{O}_{Z_0}$ *as matrices* $\begin{pmatrix} \lambda & z \\ 0 & \mu \end{pmatrix}$, *where* $\lambda, \mu \in \Bbbk$ *and* z *is a linear form over* $\mathbb{P}_2$.

Remark 1.42. *Note that from the uniqueness of the lifting it follows that isomorphisms between R-bundles lift to automorphisms of* $\mathcal{O}_{Z_0}(-H) \oplus \mathcal{O}_{Z_0}$ *and thus the induced endomorphisms of* $2\mathcal{O}_{Z_0}(-H-F)$ *are also automorphisms in this case. In other words, isomorphisms of R-bundles give rise to invertible matrices.*

Lemma 1.43. *1) Let* $\mathcal{E}$ *be an R-bundle on* Z_0. *Then* $\mathscr{T}\!or_1^{\mathcal{O}_{Z_0}}(\mathcal{E}, \mathcal{O}_{D_1}) = 0$.

2) Let $\mathcal{E}$ *be an R-bundle on* Z_0. *Then* $\mathscr{T}\!or_1^{\mathcal{O}_{Z_0}}(\mathcal{E}, \mathcal{O}_{\widetilde{\mathbb{P}}_2}) = 0$.

Proof. 1) The proof is absolutely the same as that of Lemma 1.25, 1). We restrict the resolution of the type (1.28) to D_1 and obtain the exact sequence

$$0 \to \mathscr{T}or_1^{\mathcal{O}_{Z_0}}(\mathcal{E}, \mathcal{O}_{D_1}) \to 2\mathcal{O}_{D_1}(-L) \xrightarrow{\widetilde{A}_{D_1}} 2\mathcal{O}_{D_1} \to \mathcal{E}_{D_1} \to 0,$$

where

$$\widetilde{A}_{D_1} = \begin{pmatrix} u_1 + \xi_0 u_0 & A_{01}u_1 + A_{02}u_2 + \xi_{00}u_0 \\ u_2 + \eta_0 u_0 & B_{01}u_1 + B_{02}u_2 + \eta_{00}u_0 \end{pmatrix}.$$

This matrix is injective because it has been obtained by the help of a normal direction.

2) The proof is absolutely the same as that of Lemma 1.25, 2). We consider the resolution of the type (1.28). Restricting this resolution to $\widetilde{\mathbb{P}}_2$ we obtain the exact sequence

$$0 \to \mathscr{T}or_1^{\mathcal{O}_{Z_0}}(\mathcal{E}, \mathcal{O}_{\widetilde{\mathbb{P}}_2}) \to 2\mathcal{O}_{\widetilde{\mathbb{P}}_2}(-H-F) \xrightarrow{\widetilde{A}} \mathcal{O}_{\widetilde{\mathbb{P}}_2}(-H) \oplus \mathcal{O}_{\widetilde{\mathbb{P}}_2} \to \mathcal{E}_{\widetilde{\mathbb{P}}_2} \to 0,$$

where

$$\widetilde{A} = \begin{pmatrix} u_1 & u_1(A_{01}x_0 + A_{11}x_1 + A_{12}x_2) + u_2(A_{02}x_0 + A_{22}x_2) \\ u_2 & u_1(B_{01}x_0 + B_{11}x_1 + B_{12}x_2) + u_2(B_{02}x_0 + B_{22}x_2) \end{pmatrix}.$$

Let us prove the injectivity of $\widetilde{A}$. For an arbitrary point x from $\widetilde{\mathbb{P}}_2$, let us consider the restriction $\widetilde{A}_x$ of $\widetilde{A}$ to the stalk at x:

$$2\mathcal{O}_{\widetilde{\mathbb{P}}_2,x} \xrightarrow{\widetilde{A}_x} 2\mathcal{O}_{\widetilde{\mathbb{P}}_2,x}.$$

First of all one sees that for $x \in \widetilde{\mathbb{P}}_2 \setminus L$ the map $\widetilde{A}_x$ is the same as $A_{\sigma(x)}$, $\sigma(x) \in \mathbb{P}_2$. We just use here that σ is an isomorphism on $\widetilde{\mathbb{P}}_2 \setminus L$. Therefore, we conclude that $\widetilde{A}$ is injective outside of L, thus its kernel may only be supported on L. This is impossible since $2\mathcal{O}_{\widetilde{\mathbb{P}}_2}(-H-F)$ is a locally free sheaf and hence has no torsion. This proves the second part of the lemma. □

Hilbert polynomials of R-bundles on $\hat{\mathbb{P}}_2$, comparison with $3m+1$ sheaves on $\mathbb{P}_2$.

Let us compute Hilbert polynomials of R-bundles with respect to the sheaf $\mathcal{L} := \mathcal{O}_{Z_0}(1,1) = \mathcal{O}_{Z_0}(H+F)$.

Lemma 1.44. *Let $\mathcal{E}$ be an R-bundle on $Z_0 = \hat{\mathbb{P}}_2$, then its Hilbert polynomial with respect to $\mathcal{L}$ equals* $6m+1$.

Proof. Note that $\mathcal{E}$ have a resolution of the type

$$0 \to 2\mathcal{O}_{Z_0}(-H-F) \to \mathcal{O}_{Z_0}(-H) \oplus \mathcal{O}_{Z_0} \to \mathcal{E} \to 0.$$

Therefore, to compute the Hilbert polynomial of $\mathcal{E}$ it is enough to compute the Hilbert polynomials of the sheaves $\mathcal{O}_{Z_0}(-H-F)$, $\mathcal{O}_{Z_0}(-H)$ and $\mathcal{O}_{Z_0}$.

In Lemma 1.31 we computed the Hilbert polynomials of the sheaves $\mathcal{O}_{Z_0}(a,b)$. In particular for $a+b=0$ we obtain the Hilbert polynomial of $\mathcal{O}_{Z_0}$:

$$2m^2 + 3m + 1.$$

For $a+b=-1$ we obtain the Hilbert polynomial of $\mathcal{O}_{Z_0}(-H)$ and $\mathcal{O}_{Z_0}(-F)$:

$$2m^2 + m.$$

For $a+b=-2$ we obtain the Hilbert polynomial of $\mathcal{O}_{Z_0}(-2H)$, $\mathcal{O}_{Z_0}(-2F)$ and of $\mathcal{O}_{Z_0}(-H-F)$:

$$2m^2 - m.$$

We compute now

$$\begin{aligned}\chi(\mathcal{E}\otimes\mathcal{L}^{\otimes m}) &= \chi(\mathcal{O}_{Z_0}(-H)\otimes\mathcal{L}^{\otimes m}) + \chi(\mathcal{O}_{Z_0}\otimes\mathcal{L}^{\otimes m}) - \chi(\mathcal{O}_{Z_0}(-H-F)\mathcal{L}^{\otimes m})\\ &= (2m^2+m) + (2m^2+3m+1) - 2(2m^2-m)\\ &= 6m+1.\end{aligned}$$

This completes the proof. □

Note that our fibres $\mathbb{P}_2$ and Z_0 lie in the product $\mathbb{P}_2\times\mathbb{P}_2$. We have just calculated the Hilbert polynomial of R-bundles on Z_0 with respect to the sheaf $\mathcal{O}_{Z_0}(H+F)\cong\mathcal{O}_{\mathbb{P}_2\times\mathbb{P}_2}(1,1)|_{Z_0}$. To be able to compare the Hilbert polynomials of "new" and "old" sheaves one must calculate the Hilbert polynomials of "old" sheaves with respect to the sheaf $\mathcal{O}_{\mathbb{P}_2\times\mathbb{P}_2}(1,1)|_{\mathbb{P}_2}$.

Lemma 1.45. $\mathcal{O}_{\mathbb{P}_2\times\mathbb{P}_2}(1,1)|_{\mathbb{P}_2}\cong\mathcal{O}_{\mathbb{P}_2}(2)$.

Proof. First of all note that the fibres $\mathbb{P}_2$ are embedded into $\mathbb{P}_2\times\mathbb{P}_2$ by the maps

$$\mathbb{P}_2\xrightarrow{j_t}\mathbb{P}_2\times\mathbb{P}_2,\quad \langle x_0,x_1,x_2\rangle\to(\langle x_0,x_1,x_2\rangle,\langle tx_0,x_1,x_2\rangle)$$

for $t\neq 0$. Let π_1 and π_2 be the projections $\mathbb{P}_2\times\mathbb{P}_2\to\mathbb{P}_2$. Then

$$\mathcal{O}_{\mathbb{P}_2\times\mathbb{P}_2}(1,1)|_{\mathbb{P}_2}\cong j_t^*(\pi_1^*\mathcal{O}_{\mathbb{P}_2}(1)\otimes\pi_2^*\mathcal{O}_{\mathbb{P}_2}(1))\cong j_t^*\pi_1^*\mathcal{O}_{\mathbb{P}_2}(1)\otimes j_t^*\pi_2^*\mathcal{O}_{\mathbb{P}_2}(1)\cong$$
$$(\pi_1 j_t)^*\mathcal{O}_{\mathbb{P}_2}(1)\otimes(\pi_2 j_t)^*\mathcal{O}_{\mathbb{P}_2}(1).$$

But the morphisms $\pi_1 j_t=\mathrm{id}_{\mathbb{P}_2}$ and $\pi_2 j_t=(\langle x_0,x_1,x_2\rangle\mapsto\langle tx_0,x_1,x_2\rangle)$ are automorphisms of $\mathbb{P}_2$. Therefore, $(\pi_1 j_t)^*\mathcal{O}_{\mathbb{P}_2}(1)\cong(\pi_2 j_t)^*\mathcal{O}_{\mathbb{P}_2}(1)\cong\mathcal{O}_{\mathbb{P}_2}(1)$ and we obtain

$$\mathcal{O}_{\mathbb{P}_2\times\mathbb{P}_2}(1,1)|_{\mathbb{P}_2}\cong\mathcal{O}_{\mathbb{P}_2}(1)\otimes\mathcal{O}_{\mathbb{P}_2}(1)\cong\mathcal{O}_{\mathbb{P}_2}(2).$$

This proves the statement of the lemma. □

For every $3m+1$ sheaf $\mathcal{F}$ on $\mathbb{P}_2$ its Hilbert polynomial with respect to $\mathcal{O}_{\mathbb{P}_2\times\mathbb{P}_2}(1,1)|_{\mathbb{P}_2}$ is

$$\chi(\mathcal{F}\otimes(\mathcal{O}_{\mathbb{P}_2}(2))^{\otimes m})=\chi(\mathcal{F}\otimes\mathcal{O}_{P_2}(2m))=3(2m)+1=6m+1.$$

We see that "new" and "old" sheaves have the same Hilbert polynomial $6m+1$.

Remark 1.46. *Note that if we consider $\mathbb{P}_2$ embedded as above into the product $\mathbb{P}_2\times\mathbb{P}_2$, then there are two different twisting sheaves $\mathcal{O}_{\mathbb{P}_2}(1,0):=\mathcal{O}_{\mathbb{P}_2\times\mathbb{P}_2}(1,0)|_{\mathbb{P}_2}$ and $\mathcal{O}_{\mathbb{P}_2}(0,1):=\mathcal{O}_{\mathbb{P}_2\times\mathbb{P}_2}(0,1)|_{\mathbb{P}_2}$ that are both isomorphic to $\mathcal{O}_{\mathbb{P}_2}(1)$.*

If we consider $Z_0=\hat{\mathbb{P}}_2$ embedded into $\mathbb{P}_2\times\mathbb{P}_2$, then $\mathcal{O}_{Z_0}(H)=\mathcal{O}_{Z_0}(1,0)=\mathcal{O}_{\mathbb{P}_2\times\mathbb{P}_2}(1,0)|_{Z_0}$ and $\mathcal{O}_{Z_0}(F)=\mathcal{O}_{Z_0}(0,1)=\mathcal{O}_{\mathbb{P}_2\times\mathbb{P}_2}(0,1)|_{Z_0}$

R-bundles as extensions.

First of all let us prove the following lemma.

Lemma 1.47. *1) Let $l_1,l_2\in\Gamma(Z_0,\mathcal{O}_{Z_0}(F))$ be two linear independent forms. Then the homomorphism*

$$\mathcal{O}_{Z_0}\xrightarrow{(l_1\ l_2)}2\mathcal{O}_{Z_0}(F)$$

is injective.

2) Let q be the common zero point of l_1 and l_2 in D_1. If q lies outside of L, then there is the exact sequence

$$0\to\mathcal{O}_{Z_0}(-2F)\xrightarrow{(l_2\ -l_1)}2\mathcal{O}_{Z_0}(-F)\xrightarrow{\binom{l_1}{l_2}}\mathcal{O}_{Z_0}\to\Bbbk_q\to 0.$$

Proof. 1) For $l \in \Gamma(Z_0, \mathcal{O}_{Z_0}(F))$, the kernel of $\mathcal{O}_{Z_0} \xrightarrow{l} \mathcal{O}_{Z_0}(F)$ may only be supported on the zero set of l. For all l different from αu_0, $\alpha \in \Bbbk$, this zero set is a subscheme of dimension 1 in Z_0. But there is no 1-dimensional torsion of $\mathcal{O}_{Z_0}$. Therefore, l is injective for all $l \neq \alpha u_0$.

Suppose that the kernel of (l_1, l_2) is different from zero. Then from the considerations above it follows that both l_1 and l_2 are multiples of u_0, hence they are linear dependent, which is a contradiction.

2) The sequence

$$2\mathcal{O}_{Z_0}(F) \xrightarrow{\binom{l_1}{l_2}} \mathcal{O}_{Z_0} \to \Bbbk_q \to 0$$

is exact as the pull back of the exact sequence

$$2\mathcal{O}_{D_1}(F) \xrightarrow{\binom{l_1}{l_2}} \mathcal{O}_{D_0} \to \Bbbk_q \to 0$$

on $D_1 \cong \mathbb{P}_2$. Let $\mathcal{K}$ be the kernel of $2\mathcal{O}_{Z_0}(F) \xrightarrow{\binom{l_1}{l_2}} \mathcal{O}_{Z_0}$. By part 1) of this lemma and by the universal property of kernel we conclude that $\mathcal{O}_{Z_0}(-2F)$ is a submodule of $\mathcal{K}$ and there is the commutative diagram with injective arrows

$$\begin{tikzcd} \mathcal{K} \arrow[r, tail] & 2\mathcal{O}_{Z_0}(F)\ . \\ \mathcal{O}_{Z_0}(-2F) \arrow[u, tail, "i"] \arrow[ur, tail, "(l_2\ -l_1)"'] & \end{tikzcd}$$

From the exact sequence

$$0 \to \mathcal{K} \to 2\mathcal{O}_{Z_0}(F) \xrightarrow{\binom{l_1}{l_2}} \mathcal{O}_{Z_0} \to \Bbbk_q \to 0$$

using Lemma 1.31 we conclude that the Hilbert polynomial of $\mathcal{K}$ is

$$2(2m^2 + m) - (2m^2 + 3m + 1) + 1 = 2m^2 - m,$$

which coincides with the Hilbert polynomial of $\mathcal{O}_{Z_0}(-2F)$. Therefore, the inclusion

$$\mathcal{O}_{Z_0}(-2F) \xrightarrow{i} \mathcal{K}$$

is an isomorphism. This proves the second statement of the lemma. $\square$

Proposition 1.48. *Let $\widetilde{\mathcal{E}}_0$ be an R-bundle, and let $\begin{pmatrix} l_1 & \widetilde{q}_1 \\ l_2 & \widetilde{q}_2 \end{pmatrix}$ be as in (1.29). Let C be the support of $\widetilde{\mathcal{E}}_0$, i. e., the curve given by the equation $l_1\widetilde{q}_2 - l_2\widetilde{q}_1 = 0$. Let $q = \langle l_1 \wedge l_2 \rangle \in D_1 \setminus L$ be the point where l_1 and l_2 vanish (cf. Proposition 1.37), then there is a non-trivial extension*

$$0 \to \mathcal{O}_C \to \widetilde{\mathcal{E}}_0 \to \Bbbk_q \to 0.$$

Proof. By Lemma 1.47 there is the exact sequence

$$0 \to \mathcal{O}_{Z_0}(-2F - H) \xrightarrow{(l_2\ -l_1)} 2\mathcal{O}_{Z_0}(-F - H) \xrightarrow{\binom{l_1}{l_2}} \mathcal{O}_{Z_0}(-H) \to \Bbbk_q \to 0.$$

Let us split this exact sequence into two short exact sequences

$$0 \to \mathcal{A} \to \mathcal{O}_{Z_0}(-H) \to \Bbbk_q \to 0$$

and

$$0 \to \mathcal{O}_{Z_0}(-2F-H) \xrightarrow{\left(\,l_2\ -l_1\,\right)} 2\mathcal{O}_{Z_0}(F-H) \to \mathcal{A} \to 0.$$

We obtain then the commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & \mathcal{O}_{Z_0}(-2F-H) \arrow[r, "-(l_1\widetilde{q}_2-l_2\widetilde{q}_1)"] \arrow[d, "\left(\,l_2\ -l_1\,\right)"'] & \mathcal{O}_{Z_0} \arrow[r] \arrow[d, "\left(\,0\ 1\,\right)"] & \mathcal{O}_C \arrow[r] & 0 \\
0 \arrow[r] & 2\mathcal{O}_{Z_0}(-H-F) \arrow[r, "\begin{pmatrix} l_1 & \widetilde{q}_1 \\ l_2 & \widetilde{q}_2 \end{pmatrix}"] \arrow[d] \arrow[dr, "\binom{l_1}{l_2}"'] & \mathcal{O}_{Z_0}(-H)\oplus\mathcal{O}_{Z_0} \arrow[r] \arrow[d, "\binom{1}{0}"] & \widetilde{\mathcal{E}}_0 \arrow[r] & 0 \\
0 \arrow[r] & \mathcal{A} \arrow[r] \arrow[d] & \mathcal{O}_{Z_0}(-H) \arrow[r] \arrow[d] & \Bbbk_q \arrow[r] & 0 \\
 & 0 & 0 & &
\end{tikzcd}$$

with exact rows and columns. By snake lemma this induces the exact sequence

$$0 \to \mathcal{O}_C \to \widetilde{\mathcal{E}}_0 \to \Bbbk_q \to 0$$

that makes the above diagram a 9-diagram. We proved that $\widetilde{\mathcal{E}}$ is an extension of $\Bbbk_q$ by $\mathcal{O}_C$. If this extension is trivial, then $\widetilde{\mathcal{E}} \cong \mathcal{O}_C \oplus \Bbbk_q$ and hence $\sigma_{0*}\Bbbk_q \cong \Bbbk_p$ is a direct summand of the $3m+1$ sheaf $\sigma_{0*}\widetilde{\mathcal{E}}$ on $\mathbb{P}_2$ (cf. Proposition 1.39). This contradicts the stability of $3m+1$ sheaves on $\mathbb{P}_2$. Therefore, $\widetilde{\mathcal{E}}$ is a non-trivial extension of $\Bbbk_q$ by $\mathcal{O}_C$. We proved the required statement. $\square$

Factor of a pull back of a $3m+1$ sheaf.

Restricting (1.18) to Z_0 and using the flatness of $\widetilde{\mathcal{E}}$ over U one obtains the commutative diagram with exact rows and columns

$$\begin{tikzcd}
 & & & 0 \arrow[d] & \\
 & & & 2\mathcal{O}_{D_1}(-L) \arrow[d] & \\
 & 2\mathcal{O}_{Z_0}(-2H) \arrow[r, "\sigma_0^*A"] \arrow[d, "\left(\begin{smallmatrix} s & 0 \\ 0 & s \end{smallmatrix}\right)"] & \mathcal{O}_{Z_0}(-H)\oplus\mathcal{O}_{Z_0} \arrow[r] \arrow[d, equal] & \sigma_0^*\mathcal{E} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & 2\mathcal{O}_{Z_0}(-H-F) \arrow[r, "\widetilde{A}_t|_{Z_0}"] \arrow[d] & \mathcal{O}_{Z_0}(-H)\oplus\mathcal{O}_{Z_0} \arrow[r] & \widetilde{\mathcal{E}}_0 \arrow[r] \arrow[d] & 0. \\
 & 2\mathcal{O}_{D_1}(-L) \arrow[d] & & 0 & \\
 & 0 & & &
\end{tikzcd} \tag{1.30}$$

In particular there is an exact sequence

$$0 \to 2\mathcal{O}_{D_1}(-L) \to \sigma_0^*\mathcal{E} \to \widetilde{\mathcal{E}}_0 \to 0.$$

Therefore, we obtain the following property of R-bundles on $\hat{\mathbb{P}}_2$.

Proposition 1.49. *For every R-bundle $\mathcal{E}$ on $\hat{\mathbb{P}}_2$ there exists a $3m+1$ sheaf $\mathcal{F}$ on $\mathbb{P}_2$ singular at a point $p \in \mathbb{P}_2$ and the exact sequence*

$$0 \to 2\mathcal{O}_{D_1}(-L) \to \sigma_0^*\mathcal{F} \to \mathcal{E} \to 0,$$

where $\sigma_0 : \hat{\mathbb{P}}_2 \to \mathbb{P}_2$ is the contraction of D_1 to the point p.

Let us return back to the diagram (1.30).

Lemma 1.50. *The kernel of σ_0^*A is isomorphic to $2\mathcal{O}_{D_1}(-L)$ and there is the following locally free resolution of $\sigma_0^*\mathcal{E}$:*

$$2\mathcal{O}_{Z_0}(-2H-F) \xrightarrow{\left(\begin{smallmatrix} u_0 & 0 \\ 0 & u_0 \end{smallmatrix}\right)} 2\mathcal{O}_{Z_0}(-2H) \xrightarrow{\sigma_0^*A} \mathcal{O}_{Z_0}(-H) \oplus \mathcal{O}_{Z_0} \to \sigma_0^*\mathcal{E} \to 0. \tag{1.31}$$

Proof. As $u_0x_1 = u_0x_2 = 0$ at Z_0, it is clear that the image of

$$2\mathcal{O}_{Z_0}(-2H-F) \xrightarrow{\left(\begin{smallmatrix} u_0 & 0 \\ 0 & u_0 \end{smallmatrix}\right)} 2\mathcal{O}_{Z_0}(-2H)$$

lies in the kernel of σ_0^*A. Let us show that the kernel consists of multiples of u_0.

As A is an injective morphism on $\mathbb{P}_2$, we conclude that the kernel of σ_0^*A can be supported only on D_1. Let us consider the open sets

$$W_{0i} = \{x_0 \neq 0, u_i \neq 0\} \tag{1.32}$$

in $\mathbb{k} \times \mathbb{P}_2 \times \mathbb{P}_2$. Then D_1 is covered by this sets. The morphism σ_0^*A looks as

$$\begin{pmatrix} \frac{x_1}{x_0} & \frac{x_1}{x_0}(\dots) + \frac{x_2}{x_0}(\dots) \\ \frac{x_2}{x_0} & \frac{x_1}{x_0}(\dots) + \frac{x_2}{x_0}(\dots) \end{pmatrix}$$

We will calculate the kernel in each of these sets. Every W_{0i} can be identified with $\mathbb{k} \times \mathbb{k}^2 \times \mathbb{k}^2 \cong \mathbb{k}^5$. Then $Z \cap W_{0i}$ can be identified with $\mathbb{k}^3$. As Z is given in by the minors of the matrix

$$\begin{pmatrix} tx_0 & x_1 & x_2 \\ u_0 & u_1 & u_2 \end{pmatrix},$$

the local coordinates for $Z \cap W_{00}$ are t, $\frac{u_1}{u_0}$, and $\frac{u_2}{u_0}$. We have in this case

$$\frac{x_1}{x_0} = t\frac{u_1}{u_0}, \quad \frac{x_2}{x_0} = t\frac{u_2}{u_0}. \tag{1.33}$$

The equation for Z_0 in this chart is $t = 0$.

The tuple (a, b) belongs to the kernel of σ_0^*A if and only if $(a, b)\sigma_0^*A$ is a multiple of t but this is always the case because in this case σ_0^*A is a multiple of t by (1.33). Thus (a, b) is generated by the identity matrix.

The local coordinates for $Z \cap W_{01}$ are $\frac{x_1}{x_0}$, $\frac{u_0}{u_1}$, and $\frac{u_2}{u_1}$. We have in this case

$$t = \frac{x_1}{x_0}\frac{u_0}{u_1}, \quad \frac{x_2}{x_0} = \frac{x_1}{x_0}\frac{u_2}{u_1}. \tag{1.34}$$

The equation for Z_0 in this chart is $\frac{x_1}{x_0}\frac{u_0}{u_1} = 0$.

The tuple (a, b) belongs to the kernel of σ_0^*A if and only if $(a, b)\sigma_0^*A$ is a multiple of $\frac{x_1}{x_0}\frac{u_0}{u_1}$. By (1.34) the matrix σ_0^*A is a multiple of $\frac{x_1}{x_0}$. Therefore, $(a, b) = (\frac{u_0}{u_1}a', \frac{u_0}{u_1}b')$, i. e., (a, b) is generated by the matrix $\begin{pmatrix} \frac{u_0}{u_1} & 0 \\ 0 & \frac{u_0}{u_1} \end{pmatrix}$.

The local coordinates for $Z \cap W_{02}$ are $\frac{x_2}{x_0}$, $\frac{u_0}{u_2}$, and $\frac{u_1}{u_2}$. We have in this case

$$t = \frac{x_2}{x_0}\frac{u_0}{u_2}, \quad \frac{x_1}{x_0} = \frac{x_2}{x_0}\frac{u_1}{u_2}. \tag{1.35}$$

The equation for Z_0 in this chart is $\frac{x_2}{x_0}\frac{u_0}{u_2} = 0$.

The tuple (a, b) belongs to the kernel of σ_0^*A if and only if $(a, b)\sigma_0^*A$ is a multiple of $\frac{x_2}{x_0}\frac{u_0}{u_2}$. By (1.35) the matrix σ_0^*A is a multiple of $\frac{x_2}{x_0}$. Therefore, $(a, b) = (\frac{u_0}{u_2}a', \frac{u_0}{u_1}b')$, i. e., (a, b) is generated by the matrix $\begin{pmatrix} \frac{u_0}{u_2} & 0 \\ 0 & \frac{u_0}{u_2} \end{pmatrix}$.

We have shown that every element from the kernel of σ_0^*A is generated by

$$2\mathcal{O}_{Z_0}(-2H - F) \xrightarrow{\left(\begin{smallmatrix} u_0 & 0 \\ 0 & u_0 \end{smallmatrix}\right)} 2\mathcal{O}_{Z_0}(-2H).$$

Since $\left(\begin{smallmatrix} u_0 & 0 \\ 0 & u_0 \end{smallmatrix}\right)$ is annihilated by the equations of D_1, there is a factorization

$$\begin{tikzcd}
2\mathcal{O}_{Z_0}(-2H - F) \arrow[rr, "\left(\begin{smallmatrix} u_0 & 0 \\ 0 & u_0 \end{smallmatrix}\right)"] \arrow[dr] & & 2\mathcal{O}_{Z_0}(-2H). \\
& 2\mathcal{O}_{D_1}(-L) \arrow[ur, "\phi"'] &
\end{tikzcd}$$

Moreover the restriction of ϕ to D_1 is equal to

$$2\mathcal{O}_{D_1}(-L) \xrightarrow{\left(\begin{smallmatrix} u_0 & 0 \\ 0 & u_0 \end{smallmatrix}\right)} 2\mathcal{O}_{D_1},$$

thus is injective. This implies that ϕ is injective outside of L. But $2\mathcal{O}_{D_1}(-L)$ has no torsion as a locally free sheaf on D_1. Therefore, ϕ is injective. By abuse of notation we will write $\left(\begin{smallmatrix} u_0 & 0 \\ 0 & u_0 \end{smallmatrix}\right)$ for ϕ. $\square$

Proposition 1.51. *Every R-bundle $\widetilde{\mathcal{E}}_0$ possesses the commutative diagram*

$$\begin{tikzcd}
 & & 0 \arrow[d] & & 0 \arrow[d] & \\
 & & 2\mathcal{O}_{D_1}(-L) \arrow[d, "\left(\begin{smallmatrix} u_0 & 0 \\ 0 & u_0 \end{smallmatrix}\right)"] & & 2\mathcal{O}_{D_1}(-L) \arrow[d] & \\
0 \arrow[r] & 2\mathcal{O}_{D_1}(-L) \arrow[r, "\left(\begin{smallmatrix} u_0 & 0 \\ 0 & u_0 \end{smallmatrix}\right)"'] \arrow[ur, equal] & 2\mathcal{O}_{Z_0}(-2H) \arrow[r, "\sigma_0^*A"] \arrow[d, "\left(\begin{smallmatrix} s & 0 \\ 0 & s \end{smallmatrix}\right)"] & \mathcal{O}_{Z_0}(-H) \oplus \mathcal{O}_{Z_0} \arrow[r] \arrow[d, equal] & \sigma_0^*\mathcal{E} \arrow[r] \arrow[d] & 0 \\
 & 0 \arrow[r] & 2\mathcal{O}_{Z_0}(-H - F) \arrow[r, "\widetilde{A}_t|_{Z_0}"] \arrow[d] & \mathcal{O}_{Z_0}(-H) \oplus \mathcal{O}_{Z_0} \arrow[r] & \widetilde{\mathcal{E}}_0 \arrow[r] \arrow[d] & 0 \\
 & & 2\mathcal{O}_{D_1}(-L) \arrow[d] & & 0 & \\
 & & 0 & & &
\end{tikzcd} \tag{1.36}$$

with exact rows and columns.

Proof. Follows from diagram (1.30) by applying Lemma 1.50. $\square$

Corollary 1.52. *We obtain also a locally free resolution of $\mathcal{O}_{D_1}(-L)$*

$$\mathcal{O}_{Z_0}(-2H - F) \xrightarrow{u_0} \mathcal{O}_{Z_0}(-2H) \xrightarrow{s} \mathcal{O}_{Z_0}(-H - F) \to \mathcal{O}_{D_1}(-L) \to 0. \tag{1.37}$$

There is also the following exact sequence we will use later:

$$0 \to \mathcal{O}_{D_1}(-L) \xrightarrow{u_0} \mathcal{O}_{Z_0}(-2H) \xrightarrow{s} \mathcal{O}_{Z_0}(-H - F) \to \mathcal{O}_{D_1}(-L) \to 0. \tag{1.38}$$

Restrictions of R-bundles to D_0 and D_1.

From Lemma 1.25, 1) we already know that the restriction $\mathcal{E}|_{D_1}$ of an R-bundle $\mathcal{E}$ on Z_0 to D_1 is a $2m+2$ sheaf on $\mathbb{P}_2$. In particular we obtain (cf. section A.3) that the support C_1 of the sheaf $\mathcal{E}|_{D_1}$ is a conic in $\mathbb{P}_2$.

Let us describe the restrictions of R-bundles to D_0.

Lemma 1.53. *1) For a fixed matrix $A \in X_8$, the restriction to $\widetilde{\mathbb{P}}_2$ of each R-bundle $\mathcal{E}$ constructed at the point A is isomorphic to $\mathcal{O}_{C_0}$, where the curve $C_0 = \{\det \widetilde{A} = 0\} \subseteq \widetilde{\mathbb{P}}_2$ is the support of $\mathcal{E}_{\widetilde{\mathbb{P}}_2}$.*

2) This isomorphism is unique up to multiplication by a non-zero constant.

Proof. 1) We will show that there is a surjective morphism $\mathcal{O}_{\widetilde{\mathbb{P}}_2} \to \mathcal{E}_{\widetilde{\mathbb{P}}_2}$ with kernel isomorphic to the ideal sheaf of C_0. Therefore, $\mathcal{E}_{\widetilde{\mathbb{P}}_2} \cong \mathcal{O}_{C_0}$.

We have the resolution

$$0 \to 2\mathcal{O}_{\widetilde{\mathbb{P}}_2}(-H-F) \xrightarrow{\widetilde{A}} \mathcal{O}_{\widetilde{\mathbb{P}}_2}(-H) \oplus \mathcal{O}_{\widetilde{\mathbb{P}}_2} \xrightarrow{\beta} \mathcal{E}_{\widetilde{\mathbb{P}}_2} \to 0, \quad \widetilde{A} = \left(\begin{smallmatrix} u_1 & Q_1 \\ u_2 & Q_2 \end{smallmatrix}\right).$$

The morphism

$$2\mathcal{O}_{\widetilde{\mathbb{P}}_2}(-H-F) \xrightarrow{\left(\begin{smallmatrix} u_1 \\ u_2 \end{smallmatrix}\right)} \mathcal{O}_{\widetilde{\mathbb{P}}_2}(-H)$$

is obtained by the pulling back from $\mathbb{P}_1$ to $\widetilde{\mathbb{P}}_2$ of the exact Euler sequence

$$2\mathcal{O}_{\mathbb{P}_1}(-1) \xrightarrow{\left(\begin{smallmatrix} u_1 \\ u_2 \end{smallmatrix}\right)} \mathcal{O}_{\mathbb{P}_1} \to 0$$

and by tensoring it with $\mathcal{O}_{\widetilde{\mathbb{P}}_2}(-H)$. Therefore, $2\mathcal{O}_{\widetilde{\mathbb{P}}_2}(-H-F) \xrightarrow{\left(\begin{smallmatrix} u_1 \\ u_2 \end{smallmatrix}\right)} \mathcal{O}_{\widetilde{\mathbb{P}}_2}(-H)$ is a surjection and the canonical map

$$\alpha = (\mathcal{O}_{\widetilde{\mathbb{P}}_2} \hookrightarrow \mathcal{O}_{\widetilde{\mathbb{P}}_2}(-H) \oplus \mathcal{O}_{\widetilde{\mathbb{P}}_2} \xrightarrow{\beta} \widetilde{\mathcal{E}}|_{\widetilde{\mathbb{P}}_2})$$

is then surjective as well. We obtained a surjection $\mathcal{O}_{\widetilde{\mathbb{P}}_2} \xrightarrow{\alpha} \widetilde{\mathcal{E}}|_{\widetilde{\mathbb{P}}_2}$. Straightforward calculations show that its kernel consists of those sections which are multiples of the determinant of $\widetilde{A}$. Therefore, the kernel of α coincides with the ideal sheaf of C_0. Let us see this on stalks. For a point $x \in \widetilde{\mathbb{P}}_2$ consider $f \in \mathcal{O}_{\widetilde{\mathbb{P}}_2,x}$ such that $\alpha(f) = 0$. This means $\beta(0,f) = 0$ and therefore from the exactness of the sequence above it follows that $(0,f) = (\xi,\eta)\widetilde{A}_x$ for some $(\xi,\eta) \in 2\mathcal{O}_{\widetilde{\mathbb{P}}_2}(-H-F)_x$. Let

$$\widetilde{A}_x = \begin{pmatrix} l_1 & w_1 \\ l_2 & w_2 \end{pmatrix}, \quad l_1, l_2, w_1, w_2 \in \mathcal{O}_{\widetilde{\mathbb{P}}_2,x}.$$

Then $(0,f) = (\xi,\eta)\widetilde{A}_x = (\xi l_1 + \eta l_2, \xi w_1 + \eta w_2)$, i. e., $\xi l_1 + \eta l_2 = 0$ and $f = \xi w_1 + \eta w_2$. Note that since the sections u_1 and u_2 do not vanish simultaneously, at least one of l_1 and l_2 is invertible in $\mathcal{O}_{\widetilde{\mathbb{P}}_2,x}$. Let us assume without loss of generality that l_1 is invertible. Then $\xi = -l_1^{-1}\eta l_2$ and thus

$$f = \xi w_1 + \eta w_2 = -l_1^{-1}\eta l_2 w_1 + \eta w_2 = l_1^{-1}\eta(-l_2 w_1 + l_1 w_2) = l_1^{-1}\eta \cdot \det \widetilde{A}_x,$$

which proves the required statement.

2) Since C_0 is a compact curve, one gets $\operatorname{Hom}(\widetilde{\mathcal{E}}_0|_{\widetilde{\mathbb{P}}_2}, \widetilde{\mathcal{E}}_0|_{\widetilde{\mathbb{P}}_2}) \cong \operatorname{Hom}(\mathcal{O}_{C_0}, \mathcal{O}_{C_0}) \cong \Bbbk$, i. e., $\widetilde{\mathcal{E}}_0|_{\widetilde{\mathbb{P}}_2}$ and $\mathcal{O}_{C_0}$ are simple sheaves. This implies $\operatorname{Hom}(\widetilde{\mathcal{E}}_0|_{\widetilde{\mathbb{P}}_2}, \mathcal{O}_{C_0}) \cong \Bbbk$, which proves the second part of the lemma. $\square$

Remark 1.54. *Note that the Hilbert polynomial with respect to $\mathcal{O}_{D_0}(H+F)$ of the sheaf $\mathcal{B}$ given by the resolution*

$$0 \to 2\mathcal{O}_{\widetilde{\mathbb{P}}_2}(-H-F) \to \mathcal{O}_{\widetilde{\mathbb{P}}_2}(-H) \oplus \mathcal{O}_{\widetilde{\mathbb{P}}_2} \to \mathcal{B} \to 0,$$

is $4m+1$. In particular this holds for the sheaf $\mathcal{O}_{C_0}$ from Lemma 1.25.

Proof. Using (1.22) and the resolution of $\mathcal{B}$ we obtain that Hilbert polynomial of $\mathcal{B}$ is

$$\left(\frac{3}{2}m^2 + \frac{1}{2}m\right) + \left(\frac{3}{2}m^2 + \frac{5}{2}m + 1\right) - 2\left(\frac{3}{2}m^2 - \frac{1}{2}m\right) = 4m + 1.$$

This proves the required statement. □

Remark 1.55. *Note that the restriction of resolution* (1.29) *to the component $D_1 = \mathbb{P}_2$ is a Beilinson resolution of $\widetilde{\mathcal{E}}_0$ on $\mathbb{P}_2$.*

Remark 1.56. *One can also show that the restriction of* (1.29) *to $D_0 = \widetilde{\mathbb{P}}_2$ is the resolution of Beilinson type from [1], Theorem 8.*

Proof. Recall that $\widetilde{\mathbb{P}}_2 \cong \mathbf{P}(\mathcal{O}_{\mathbb{P}_1} \oplus \mathcal{O}_{\mathbb{P}_1}(-1))$ is a $\mathbb{P}_1$-bundle over $\mathbb{P}_1$ (cf. [12], V, Example 2.11.5). Let $\pi : \widetilde{\mathbb{P}}_2 \to \mathbb{P}_1$ be the projection.

The relative sheaf $\mathcal{O}_{\mathrm{rel}}(1)$ defined by the sheaf $\mathcal{O}_{\mathbb{P}_1} \oplus \mathcal{O}_{\mathbb{P}_1}(-1)$ is isomorphic to $\mathcal{O}_{\widetilde{\mathbb{P}}_2}(L) \cong \mathcal{O}_{\widetilde{\mathbb{P}}_2}(H-F) = \mathcal{O}_{\widetilde{\mathbb{P}}_2}(1,-1)$ (cf. [12], V, Proposition 2.8). We obtain then the sheaf

$$\mathcal{O}_{\mathrm{rel}}(1) \cong \mathcal{O}_{\widetilde{\mathbb{P}}_2}(1,-1).$$

Since $\Omega^1_{\mathbb{P}_1} \cong \mathcal{O}_{\mathbb{P}_1}(-2)$, we obtain $\Omega^1_{\mathbb{P}_1}(1) \cong \mathcal{O}_{\mathbb{P}_1}(-1)$ and

$$\pi^*(\Omega^1(1)) \cong \pi^*\mathcal{O}_{\mathbb{P}_1}(-1) = \mathcal{O}_{\widetilde{\mathbb{P}}_2}(0,-1).$$

The dual $\mathcal{Q}^*$ of the canonical quotient sheaf $\mathcal{Q}$ is defined by the exact sequence

$$0 \to \mathcal{Q}^* \to \pi^*(\mathcal{O}_{\mathbb{P}_1} \oplus \mathcal{O}_{\mathbb{P}_1}(-1)) \to \mathcal{O}_{\mathrm{rel}}(1) \to 0.$$

Since $\pi^*(\mathcal{O}_{\mathbb{P}_1} \oplus \mathcal{O}_{\mathbb{P}_1}(-1)) \cong \mathcal{O}_{\widetilde{\mathbb{P}}_2} \oplus \mathcal{O}_{\widetilde{\mathbb{P}}_2}(0,-1)$ and $\mathcal{O}_{\mathrm{rel}}(1) \cong \mathcal{O}_{\widetilde{\mathbb{P}}_2}(1,-1)$, we get

$$0 \to \mathcal{Q}^* \to \mathcal{O}_{\widetilde{\mathbb{P}}_2} \oplus \mathcal{O}_{\widetilde{\mathbb{P}}_2}(0,-1) \to \mathcal{O}_{\widetilde{\mathbb{P}}_2}(1,-1) \to 0.$$

Using (1.22) we compute the Hilbert Polynomial of $\mathcal{Q}^*$ with respect to $\mathcal{O}_{\widetilde{\mathbb{P}}_2}(1,1)$

$$\left(\frac{3}{2}m^2 + \frac{5}{2}m + 1\right) + \left(\frac{3}{2}m^2 + \frac{3}{2}m\right) - \left(\frac{3}{2}m^2 + \frac{7}{2}m + 1\right) = \frac{3}{2}m^2 + \frac{1}{2}m.$$

Since $\mathcal{Q}^*$ is an invertible sheaf, it has the form $\mathcal{O}_{\widetilde{\mathbb{P}}_2}(a,b)$ for some a and b. By (1.21) its Hilbert polynomial is

$$\frac{3}{2}m^2 + \left[2a + b + \frac{5}{2}\right]m + \frac{1}{2}(a+b)^2 + \frac{3}{2}(a+b) - \frac{1}{2}b^2 - \frac{1}{2}b + 1.$$

Therefore, comparing the coefficients we obtain

$$2a + b + \frac{5}{2} = \frac{1}{2} \quad \text{and} \quad \frac{1}{2}(a+b)^2 + \frac{3}{2}(a+b) - \frac{1}{2}b^2 - \frac{1}{2}b + 1 = 0.$$

Thus $b = -2a - 2$ and

$$\frac{1}{2}(a+2)^2 - \frac{3}{2}(a+2) - \frac{1}{2}(4a^2+8a+4) + \frac{1}{2}(2a+2) + 1 = -\frac{3}{2}a^2 - \frac{5}{2}a - 1 = 0.$$

So a must satisfy $3a^2+5a+2=0$. The only integer root of this equation is $a=-1$ (the second root is $-\frac{2}{3}$). So $a=-1$ and $b=-2a-2=0$. We obtained

$$\mathcal{Q}^* \cong \mathcal{O}_{\widetilde{\mathbb{P}}_2}(-1,0).$$

Since $\widetilde{\mathbb{P}}_2 = \mathbf{P}(\mathcal{O}_{\mathbb{P}_1} \oplus \mathcal{O}_{\mathbb{P}_1}(-1)) = \mathbb{P}(\mathcal{O}_{\mathbb{P}_1} \oplus \mathcal{O}_{\mathbb{P}_1}(1))$, we may apply now Theorem 8 from [1]. Let us formulate the statement of that theorem here for the case of $\widetilde{\mathbb{P}}_2 \xrightarrow{\pi} \mathbb{P}_1$.

Claim. *Every sheaf* $\mathcal{F}$ *on* $\widetilde{\mathbb{P}}_2 = \mathbf{P}(\mathcal{O}_{\mathbb{P}_1} \oplus \mathcal{O}_{\mathbb{P}_1}(-1)) = \mathbb{P}(\mathcal{O}_{\mathbb{P}_1} \oplus \mathcal{O}_{\mathbb{P}_1}(1))$ *is obtained as a cohomology of the complex* $C^{\bullet}_{\mathcal{F}}$, *where*

$$C^p_{\mathcal{F}} = \bigoplus_{q+h=s-p} H^s(\mathcal{F} \otimes \pi^*(\mathcal{O}_{\mathbb{P}_1}(-q)) \otimes \mathcal{O}_{\mathrm{rel}}(-h)) \otimes \pi^*(\Omega^q_{\mathbb{P}_1}(q)) \otimes \wedge^h \mathcal{Q}^*. \tag{1.39}$$

Using that $\pi^*(\Omega^q_{\mathbb{P}_1}(q))$ and $\wedge^h \mathcal{Q}^*$ are zero for q and h different from 0 or 1, using that

$$\pi^*(\Omega^1_{\mathbb{P}_1}(1)) \cong \mathcal{O}_{\widetilde{\mathbb{P}}_2}(0,-1), \quad \mathcal{O}_{\mathrm{rel}}(1) \cong \mathcal{O}_{\widetilde{\mathbb{P}}_2}(1,-1), \quad \text{and} \quad \mathcal{Q}^* \cong \mathcal{O}_{\mathbb{P}_1}(-1,0)$$

we obtain the formula

$$C^p_{\mathcal{F}} = \bigoplus_{\substack{q+h=s-p \\ q,\ h\in\{0,1\}}} H^s(\mathcal{F}(-h,h-q)) \otimes \mathcal{O}_{\widetilde{\mathbb{P}}_2}(-h,-q), \tag{1.40}$$

where $\mathcal{F}(-h,h-q) = \mathcal{F} \otimes \mathcal{O}_{\widetilde{\mathbb{P}}_2}(-h,h-q)$. We need the cohomologies of the sheaves $\mathcal{F}$, $\mathcal{F}(-1,1)$, $\mathcal{F}(0,-1)$, $\mathcal{F}(-1,0)$.

Claim. *The cohomologies of the sheaves* $\mathcal{F}$, $\mathcal{F}(-1,1)$, $\mathcal{F}(0,-1)$, $\mathcal{F}(-1,0)$ *are described in the following table.*

	h^0	h^1	h^2
$\mathcal{F}$	1	0	0
$\mathcal{F}(-1,0)$	0	2	0
$\mathcal{F}(0,-1)$	0	0	0
$\mathcal{F}(-1,1)$	0	1	0

Proof of the Claim. The restrictions of R-bundles to $D_0 = \widetilde{\mathbb{P}}_2$ are the structure sheaves of their support, i. e., every such a sheaf is given by the resolution

$$0 \to \mathcal{O}_{\widetilde{\mathbb{P}}_2}(-H-2F) \to \mathcal{O}_{\widetilde{\mathbb{P}}_2} \to \mathcal{F} \to 0.$$

Using this resolution and computing the cohomology of the invertible sheaves on $\widetilde{\mathbb{P}}_2$ by means of the exact sequences (1.24) and the corresponding long exact sequences (using Künneth formula for the product $\mathbb{P}_2 \times \mathbb{P}_1$) we obtain the required statement. $\square$

Using (1.40) we obtain finally $C^{-2}_{\mathcal{F}} = C^1_{\mathcal{F}} = C^2_{\mathcal{F}} = 0$ and

$$C^{-1}_{\mathcal{F}} = H^1(\widetilde{\mathbb{P}}_2, \mathcal{F}(-1,0)) \otimes \mathcal{O}_{\widetilde{\mathbb{P}}_2}(-1,-1),$$
$$C^0_{\mathcal{F}} = H^1(\widetilde{\mathbb{P}}_2, \mathcal{F}(-1,1)) \otimes \mathcal{O}_{\widetilde{\mathbb{P}}_2}(-1,0) \oplus H^0(\widetilde{\mathbb{P}}_2, \mathcal{F}) \otimes \mathcal{O}_{\widetilde{\mathbb{P}}_2}.$$

Therefore, the resolution of the sheaf $\mathcal{F}$ is

$$0 \to H^1(\mathcal{F}(-1,0)) \otimes \mathcal{O}_{\widetilde{\mathbb{P}}_2}(-1,-1) \to H^1(\mathcal{F}(-1,1)) \otimes \mathcal{O}_{\widetilde{\mathbb{P}}_2}(-1,0) \oplus H^0(\mathcal{F}) \otimes \mathcal{O}_{\widetilde{\mathbb{P}}_2} \to \mathcal{F} \to 0,$$

i.e.,

$$0 \to 2\mathcal{O}_{\widetilde{\mathbb{P}}_2}(-1,-1) \to \mathcal{O}_{\widetilde{\mathbb{P}}_2}(-1,0) \oplus \mathcal{O}_{\widetilde{\mathbb{P}}_2} \to \mathcal{F} \to 0$$

But the restriction of (1.29) to $D_0 = \widetilde{\mathbb{P}}_2$ is also a resolution of this type. We proved the required statement. □

Double structure.

We want to show here that an R-bundle on Z_0 is not always defined only by its restrictions to $D_0 = \widetilde{\mathbb{P}}_2$ and $D_1 = \mathbb{P}_2$. Let us consider an example.

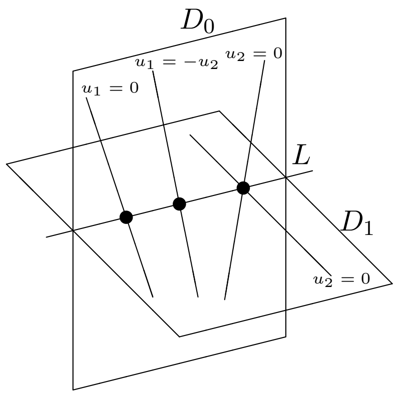
Let $A = \begin{pmatrix} x_1 & 0 \\ x_2 & x_2(x_1+x_2) \end{pmatrix}$ and let $B = \begin{pmatrix} 0 & x_0^2 \\ 0 & 0 \end{pmatrix}$. Note that B is a normal direction to X_8 at the point A in this case (cf. (1.13), Example 1.8). The matrix on Z_0 is then

$$\widetilde{A}_B = \begin{pmatrix} u_1 & u_0 x_0 \\ u_2 & u_2(x_1+x_2) \end{pmatrix}.$$

The support of the sheaf is given by the determinant of $\widetilde{A}_B$, by the equation

$$u_1 u_2 (x_1 + x_2) - u_0 u_2 x_0 = 0.$$

In this case the support of the R-bundle defined by $\widetilde{A}_B$ consists of lines and L is one of those lines.



Let us consider the situation in the chart $Z \cap W_{01} \cong \mathbb{k}^3$ (cf. (1.32)). In this case the local coordinates are

$$y_1 := \frac{x_1}{x_0}, \quad v_0 := \frac{u_0}{u_1}, \quad \text{and } v_2 := \frac{u_2}{u_1}.$$

The equation for Z_0 in this chart is $y_1 v_0 = 0$. The matrix $\widetilde{A}_B$ is then

$$\begin{pmatrix} 1 & v_0 \\ v_2 & v_2 y_1 (1+v_2) \end{pmatrix}.$$

Its determinant is $v_2 y_1 (1+v_2) - v_0 v_2$. The ideal of the support is then

$$(y_1 v_0, v_2 y_1 (1+v_2) - v_0 v_2).$$

Take some point z at L such that z is different from the intersection points with other lines of the support and let us consider the situation at the stalk at z. Then $v_2 \neq 0$ and $1+v_2 \neq 0$, i. e., are invertible. We obtain the ideal of the support at z:

$$(y_1 v_0, y_1(1+v_2) - v_0) = (y_1^2(1+v_2), y_1(1+v_2) - v_0) = (y_1^2, y_1(1+v_2) - v_0).$$

It is obviously contained in the ideal (y_1, v_0) of L but is not equal to it. This means we have a "double" structure on L, i. e, the support of the R-bundle in this case is not a reduced variety.

1.3.3 The inverse constructions.

We have already proven many properties of R-bundles. In fact some of them may be used as characteristic properties, i. e., one could use them to define R-bundles.

Locally free resolutions.

Let us show that the converse of Proposition 1.37 holds true.

Proposition 1.57. *Let* $\Phi : 2\mathcal{O}_{Z_0}(-H-F) \to \mathcal{O}_{Z_0}(-H) \oplus \mathcal{O}_{Z_0}$, $\Phi = \left(\begin{smallmatrix} l_1 & \tilde{q}_1 \\ l_2 & \tilde{q}_2 \end{smallmatrix}\right)$, *have the following properties:*

- l_1 *and* l_2 *are linear independent and their common zero point* $l_1 \wedge l_2$ *in* $D_1 \cong \mathbb{P}_2$ *does not belong to* L*;*
- $\det(\Phi|_{D_0}) \neq 0$*;*
- $(\Phi|_{D_1})(q) \neq 0$ *for all* $q \in D_1$*, in particular* $\det(\Phi|_{D_1}) \neq 0$.

Then the cokernel of Φ*, i.e., the sheaf* $\mathcal{E}$ *defined by the exact sequence*

$$0 \to 2\mathcal{O}_{Z_0}(-H-F) \xrightarrow{\Phi} \mathcal{O}_{Z_0}(-H) \oplus \mathcal{O}_{Z_0} \to \mathcal{E} \to 0$$

is an R*-bundle.*

Proof. It is enough to show that $\mathcal{E}$ is a fibre of a new one-dimensional family over $t=0$ (cf. Definition 1.36 on page 35).

First of all note that Φ never vanishes, hence $\mathcal{E}$ is locally free on its support.

Let

$$\Phi = \begin{pmatrix} \xi_0 u_0 + a_1 u_1 + a_2 u_2 & \xi_{00} x_0 u_0 + u_1(A_{01}x_0 + A_{11}x_1 + A_{12}x_2) + u_2(A_{02}x_0 + A_{22}x_2) \\ \eta_0 u_0 + b_1 u_1 + b_2 u_2 & \eta_{00} x_0 u_0 + u_1(B_{01}x_0 + B_{11}x_1 + B_{12}x_2) + u_2(B_{02}x_0 + B_{22}x_2) \end{pmatrix}.$$

Take

$$A = \begin{pmatrix} a_1 x_1 + a_2 x_2 & x_1(A_{01}x_0 + A_{11}x_1 + A_{12}x_2) + x_2(A_{02}x_0 + A_{22}x_2) \\ b_1 x_1 + b_2 x_2 & x_1(B_{01}x_0 + B_{11}x_1 + B_{12}x_2) + x_2(B_{02}x_0 + B_{22}x_2) \end{pmatrix}, \quad B = \begin{pmatrix} \xi_0 x_0 & \xi_{00} x_0^2 \\ \eta_0 x_0 & \eta_{00} x_0^2 \end{pmatrix}.$$

Since $l_1 \wedge l_2$ lies outside of L, we conclude that $a_1 b_2 - a_2 b_1 \neq 0$ and hence $a_1 x_1 + a_2 x_2$ and $b_1 x_1 + b_2 x_2$ are linear independent. Moreover, $\det A = 0$ if and only if $\det(\Phi|_{D_0}) = 0$. Therefore, we conclude that A lies in X and then in X_8.

Then Φ is obtained applying the construction at $A \in X_8$ along B. Note that B is a normal direction by Proposition 1.24. $\square$

In Proposition 1.57 we use the properties of $\Phi|_{D_0}$ and $\Phi|_{D_1}$ on D_0 and D_1 respectively. So it would be natural if R-bundles were defined just by morphisms

$$2\mathcal{O}_{D_0}(-H-F) \xrightarrow{\Phi|_{D_0}} \mathcal{O}_{D_0}(-H) \oplus \mathcal{O}_{D_0} \quad \text{and} \quad 2\mathcal{O}_{D_1}(-F) \xrightarrow{\Phi|_{D_1}} 2\mathcal{O}_{D_1}.$$

Let is formulate this idea properly.

Let

$$0 \to 2\mathcal{O}_{D_0}(-H-F) \xrightarrow{\Phi_0} \mathcal{O}_{D_0}(-H) \oplus \mathcal{O}_{D_0} \to \mathcal{E}_0 \to 0$$

and

$$0 \to 2\mathcal{O}_{D_1}(-F) \xrightarrow{\Phi_1} 2\mathcal{O}_{D_1} \to \mathcal{E}_1 \to 0$$

be two exact sequences and let Φ_0 and Φ_1 be compatible on $L = D_0 \cap D_1$. Consider the gluing sequences (cf. Lemmata A.16 and A.17)

$$0 \to 2\mathcal{O}_{Z_0}(-H-F) \to 2\mathcal{O}_{D_0}(-H-F) \oplus 2\mathcal{O}_{D_1}(-F) \to 2\mathcal{O}_L(-1) \to 0$$

and

$$0 \to \mathcal{O}_{Z_0}(-H) \oplus \mathcal{O}_{Z_0} \to (\mathcal{O}_{D_0}(-H) \oplus \mathcal{O}_{D_0}) \oplus 2\mathcal{O}_{D_1} \to 2\mathcal{O}_L \to 0.$$

Using the compatibility of Φ_0 and Φ_1, the universal property of kernel, and the snake lemma we obtain the commutative diagram with exact rows and columns

$$\begin{tikzcd}
 & & & & 0 \arrow[d] & \\
 & & 0 \arrow[d] & 0 \arrow[d] & \mathcal{C} \arrow[d] & \\
0 \arrow[rr] & & 2\mathcal{O}_{Z_0}(-H-F) \arrow[r] \arrow[d, dotted, "\exists!\ \Phi"'] & 2\mathcal{O}_{D_0}(-H-F) \oplus 2\mathcal{O}_{D_1}(-F) \arrow[r] \arrow[d, "\begin{pmatrix} \Phi_0 & 0 \\ 0 & \Phi_1 \end{pmatrix}"] & 2\mathcal{O}_L(-1) \arrow[r] \arrow[d, "\Phi_0|_L = \Phi_1|_L"] & 0 \\
0 \arrow[rr] & & \mathcal{O}_{Z_0}(-H) \oplus \mathcal{O}_{Z_0} \arrow[r] \arrow[d, dotted] & (\mathcal{O}_{D_0}(-H) \oplus \mathcal{O}_{D_0}) \oplus 2\mathcal{O}_{D_1} \arrow[r] \arrow[d] & 2\mathcal{O}_L \arrow[r] \arrow[d] & 0. \\
0 \arrow[r] & \mathcal{C} \arrow[r] & \mathcal{E} \arrow[r, dotted] \arrow[d] & \mathcal{E}_0 \oplus \mathcal{E}_1 \arrow[r, dotted, "\exists"] \arrow[d] & \mathcal{E}_{01} \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 & 0 &
\end{tikzcd}$$

So, Φ_0 and Φ_1 define an injective homomorphism $2\mathcal{O}_{Z_0}(-H-F) \xrightarrow{\Phi} \mathcal{O}_{Z_0}(-H) \oplus \mathcal{O}_{Z_0}(-H-F)$ with $\Phi|_{D_0} = \Phi_0$ and $\Phi|_{D_1} = \Phi_1$.

We obtained the following "gluing" of resolutions.

Proposition 1.58. *A resolution of the type* (1.29) *is uniquely defined by its restrictions to D_0 and D_1, which are resolutions of Beilinson type (cf. Remark 1.56 and Remark 1.55).*

Let $\Phi_{01} := \Phi_0|_L = \Phi_1|_L$. If Φ_0 and Φ_1 satisfy the conditions of Proposition 1.57, then $\Phi_{01} = \begin{pmatrix} w_1 & w_3 \\ w_2 & w_4 \end{pmatrix}$, where the linear forms $w_1, w_2 \in \Gamma(L, \mathcal{O}_L(1))$ are linear independent.

If at least one of w_3 and w_4 is different from zero, then Φ_{01} is injective and $\mathcal{C} = 0$. In this case we have the "gluing" exact sequence

$$0 \to \mathcal{E} \to \mathcal{E}_0 \oplus \mathcal{E}_1 \to \mathcal{E}_{01} \to 0$$

and the sheaf $\mathcal{E}_{01}$ is supported on the zero set of the determinant $w_1 w_4 - w_3 w_2$ of the matrix Φ_{01}, i. e., at most on two points from L.

If both w_3 and w_4 are zero, we get $\Phi_{01} = \begin{pmatrix} w_1 & 0 \\ w_2 & 0 \end{pmatrix}$ and therefore the morphism

$$\mathcal{O}_L(-2) \xrightarrow{(\, w_2 \ -w_1 \,)} 2\mathcal{O}_L(-1)$$

is the kernel of Φ_{01}. Hence $\mathcal{C} \cong \mathcal{O}_L(-2)$. In this case $\mathcal{E}_{01}$ is isomorphic to $\mathcal{O}_L$ and the support of the sheaf $\mathcal{E}$ contains the line L. In this case we have a "double structure" on L (cf. page 47).

Remark 1.59. *In particular we see that the gluing for an R-bundle exists if and only if the line L is not contained in the support of that sheaf.*

Extensions.

We are going to prove here the converse of Proposition 1.48.

Proposition 1.60. *Let $C \subseteq Z_0$ be a curve defined by the exact sequence*

$$0 \to \mathcal{O}_{Z_0}(-2F-H) \xrightarrow{f} \mathcal{O}_{Z_0} \xrightarrow{p_1} \mathcal{O}_C \to 0.$$

Let q be a point from $D_1 \setminus L$ and let $\mathcal{E}$ be an invertible sheaf on C such that there is a non-trivial extension

$$0 \to \mathcal{O}_C \xrightarrow{\alpha} \mathcal{E} \xrightarrow{\beta} \Bbbk_q \to 0.$$

Then the sheaf $\mathcal{E}$ is an R-bundle.

Proof. We are going to show that $\mathcal{E}$ has a resolution of the type (1.29). Then by Proposition 1.57 we will conclude that $\mathcal{E}$ is an R-bundle. The proof basically repeats with minor modifications the second part of the proof of Lemma 5.3 from [4].

Recall that $\Bbbk_q$ has a resolution as in Lemma 1.47. Tensoring that resolution $\mathcal{O}_{Z_0}(-H)$, we obtain the exact sequence

$$0 \to \mathcal{O}_{Z_0}(-2F-H) \xrightarrow{(\, l_2 \ -l_1 \,)} 2\mathcal{O}_{Z_0}(-F-H) \xrightarrow{\binom{l_1}{l_2}} \mathcal{O}_{Z_0}(-H) \xrightarrow{p_2} \Bbbk_q \to 0.$$

Claim. $\operatorname{Ext}^1(\mathcal{O}_{Z_0}(-H), \mathcal{O}_C) = 0$.

Proof. Since $\mathcal{O}_{Z_0}(-H)$ is a locally free sheaf, we have $\operatorname{Ext}^1(\mathcal{O}_{Z_0}(-H), \mathcal{O}_C) \cong H^1(\mathcal{O}_C \otimes \mathcal{O}_{Z_0}(H))$. Therefore, it is enough to show that $H^1(\mathcal{O}_C \otimes \mathcal{O}_{Z_0}(H)) = 0$. Tensoring the defining sequence of C with $\mathcal{O}_{Z_0}(H)$ we obtain a resolution of $\mathcal{O}_C \otimes \mathcal{O}_{Z_0}(H)$:

$$0 \to \mathcal{O}_{Z_0}(-2F) \to \mathcal{O}_{Z_0}(H) \to \mathcal{O}_C \otimes \mathcal{O}_{Z_0}(H) \to 0.$$

From the corresponding long exact cohomology sequence using that $H^2(Z_0, \mathcal{O}_{Z_0}(-2F)) = H^1(Z_0, \mathcal{O}_{Z_0}(H)) = 0$ (see Proposition 1.32) we conclude that $H^1(\mathcal{O}_C \otimes \mathcal{O}_{Z_0}(H)) = 0$. $\square$

Thus there exists a lifting $\mathcal{O}_{Z_0}(-H) \xrightarrow{b} \mathcal{E}$ of p, i. e., $b\beta = p$. We obtain thus a surjective homomorphism $\mathcal{O}_{Z_0}(-H) \oplus \mathcal{O}_{Z_0} \xrightarrow{p} \mathcal{E}$ defined by $p = \binom{b}{a}$, where $a = p_1\alpha$. There is the diagram with commutative squares and exact rows and columns

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{O}_{Z_0} \arrow[r, "(\,0\ 1\,)"] \arrow[d, "p_1"] \arrow[drr, "a"] & \mathcal{O}_{Z_0}(-H) \oplus \mathcal{O}_{Z_0} \arrow[r, "\binom{1}{0}"] \arrow[d, "p"] & \mathcal{O}_{Z_0}(-H) \arrow[r] \arrow[d, "p_2"] \arrow[dl, dotted, "\exists\, b"'] & 0 \\
0 \arrow[r] & \mathcal{O}_C \arrow[r, "\alpha"] \arrow[d] & \mathcal{E} \arrow[r, "\beta"] \arrow[d] & \Bbbk_q \arrow[r] \arrow[d] & 0. \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

Note that since α is injective the kernel of a coincides with the kernel of p_1, i.e., with the image of f. As $\left(\begin{smallmatrix} l_1 \\ l_2 \end{smallmatrix}\right) b\beta = \left(\begin{smallmatrix} l_1 \\ l_2 \end{smallmatrix}\right) p_2 = 0$, we conclude that the image of $\left(\begin{smallmatrix} l_1 \\ l_2 \end{smallmatrix}\right) b$ lies in the kernel of β, which coincides with the image of α. Since by Proposition 1.32

$$\operatorname{Ext}^1(\mathcal{O}_{Z_0}(-H-F), \mathcal{O}_{Z_0}(-2F-H)) \cong H^1(Z_0, \mathcal{O}_{Z_0}(-F)) = 0,$$

we conclude that there is a lifting $\left(\begin{smallmatrix} q_1 \\ q_2 \end{smallmatrix}\right)$ of $-\left(\begin{smallmatrix} l_1 \\ l_2 \end{smallmatrix}\right) b$, i.e., the diagram

$$\begin{tikzcd}
 & & 2\mathcal{O}_{Z_0}(-H-F) \arrow[r, "\left(\begin{smallmatrix} l_1 \\ l_2 \end{smallmatrix}\right)"] \arrow[d, dotted, "\exists\ \left(\begin{smallmatrix} q_1 \\ q_2 \end{smallmatrix}\right)"'] & \mathcal{O}_{Z_0}(-H) \arrow[d, "-b"] & \\
0 \arrow[r] & \mathcal{O}_{Z_0}(-H-2F) \arrow[r, "f"] & \mathcal{O}_{Z_0} \arrow[r, "a"] & \operatorname{Im}(a) \arrow[r] & 0
\end{tikzcd}$$

commutes. In other words this means that $q_1a + l_1b = q_2a + l_2b = 0$. We obtain a homomorphism

$$\mathcal{O}_{Z_0}(-2F-H) \oplus 2\mathcal{O}_{Z_0}(-F-H) \xrightarrow{\Psi} \mathcal{O}_{Z_0}(-H) \oplus \mathcal{O}_{Z_0}, \quad \Psi = \begin{pmatrix} 0 & f \\ l_1 & q_1 \\ l_2 & q_2 \end{pmatrix}.$$

Since it holds $\Psi \cdot p = \begin{pmatrix} 0 & f \\ l_1 & q_1 \\ l_2 & q_2 \end{pmatrix} \begin{pmatrix} b \\ a \end{pmatrix} = \begin{pmatrix} fa \\ l_1b + q_1a \\ l_2b + q_2a \end{pmatrix} = 0$, one concludes that $\operatorname{Im}\Psi$ is contained in $\ker p$. Standard diagram chasing shows that $\ker p \subseteq \operatorname{Im}\Psi$. Therefore, the sequence

$$\mathcal{O}_{Z_0}(-2F-H) \oplus 2\mathcal{O}_{Z_0}(-F-H) \xrightarrow{\Psi} \mathcal{O}_{Z_0}(-H) \oplus \mathcal{O}_{Z_0} \xrightarrow{p} \mathcal{E} \to 0$$

is exact. We obtain the diagram with commutative squares and with exact rows and columns:

$$\begin{tikzcd}
 & & & 0 \arrow[d] & \\
 & 0 \arrow[d] & & \mathcal{O}_{Z_0}(-2F-H) \arrow[d, "(\, l_2\ -l_1 \,)"] & \\
0 \arrow[r] & \mathcal{O}_{Z_0}(-2F-H) \arrow[r, "(\,1\ 0\,)"] \arrow[d, "f"] & \mathcal{O}_{Z_0}(-2F-H) \oplus 2\mathcal{O}_{Z_0}(-F-H) \arrow[r, "\left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)"] \arrow[d, "\Psi"] & 2\mathcal{O}_{Z_0}(-F-H) \arrow[r] \arrow[d, "\left(\begin{smallmatrix} l_1 \\ l_2 \end{smallmatrix}\right)"] & 0 \\
0 \arrow[r] & \mathcal{O}_{Z_0} \arrow[r, "(\,0\ 1\,)"] \arrow[d, "p_1"] \arrow[dr, "a"] & \mathcal{O}_{Z_0}(-H) \oplus \mathcal{O}_{Z_0} \arrow[r, "\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)"] \arrow[d, "p"'] & \mathcal{O}_{Z_0}(-H) \arrow[r] \arrow[d, "p_2"] \arrow[dl, dotted, "\exists\ b"'] & 0 \\
0 \arrow[r] & \mathcal{O}_C \arrow[r, "\alpha"] \arrow[d] & \mathcal{E} \arrow[r, "\beta"] \arrow[d] & \Bbbk_q \arrow[r] \arrow[d] & 0. \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

From $-(l_1q_2 - l_2q_1)a = (\, l_2\ -l_1 \,) \left(\begin{smallmatrix} q_1 \\ q_2 \end{smallmatrix}\right) a = -(\, l_2\ -l_1 \,) \left(\begin{smallmatrix} l_1 \\ l_2 \end{smallmatrix}\right) b = 0$ by the universal property of kernel it follows that there exists an endomorphism λ of $\mathcal{O}_{Z_0}(-H-2F)$ such that the diagram

$$\begin{tikzcd}
 & & 2\mathcal{O}_{Z_0}(-H-2F) \arrow[dl, dotted, "\exists\ \lambda"'] \arrow[d, "-(l_1q_2-l_2q_1)"] & & \\
0 \arrow[r] & \mathcal{O}_{Z_0}(-H-2F) \arrow[r, "f"] & \mathcal{O}_{Z_0} \arrow[r, "a"] & \operatorname{Im}(a) \arrow[r] & 0
\end{tikzcd}$$

commutes. We can treat λ as an element from $\Bbbk$ because the endomorphism group of $\mathcal{O}_{Z_0}(-H-2F)$ is isomorphic to $\Bbbk$. We obtained $\lambda f = -(l_1q_2 - l_2q_1)$.

Suppose $\lambda = 0$. Then $\left(\begin{smallmatrix} -q_2 & q_1 \end{smallmatrix}\right)\left(\begin{smallmatrix} l_1 \\ l_2 \end{smallmatrix}\right) = 0$ and by the universal property of kernel $\left(\begin{smallmatrix} -q_2 & q_1 \end{smallmatrix}\right)$ factors through the kernel of $\left(\begin{smallmatrix} l_1 \\ l_2 \end{smallmatrix}\right)$. This means that there is some $\xi \in \Gamma(Z_0, \mathcal{O}_{Z_0}(F))$ that makes the diagram

$$\begin{tikzcd} & & \mathcal{O}_{Z_0}(-H-F) \arrow[dll, dotted, "\exists\ \xi"'] \arrow[d, "\left(\begin{smallmatrix} -q_2 & q_1 \end{smallmatrix}\right)"] & & \\ 0 \arrow[r] & \mathcal{O}_{Z_0}(-F) \arrow[r, "\left(\begin{smallmatrix} l_2 & -l_1 \end{smallmatrix}\right)"] & 2\mathcal{O}_{Z_0} \arrow[r, "\left(\begin{smallmatrix} l_1 \\ l_2 \end{smallmatrix}\right)"] & \mathcal{O}_{Z_0}(F) \arrow[r] & 0 \end{tikzcd}$$

commute. Therefore, $\Psi = \left(\begin{smallmatrix} 0 & f \\ l_1 & -\xi l_1 \\ l_2 & -\xi l_2 \end{smallmatrix}\right)$ and its image coincides with the image of $\left(\begin{smallmatrix} 0 & f \\ l_1 & 0 \\ l_2 & 0 \end{smallmatrix}\right)$. This implies that $\mathcal{E}$ is a direct sum of the cokernels of $\mathcal{O}_{Z_0}(-H-2F) \xrightarrow{f} \mathcal{O}_{Z_0}$ and of $2\mathcal{O}_{Z_0}(-H-F) \xrightarrow{\left(\begin{smallmatrix} l_1 \\ l_2 \end{smallmatrix}\right)} \mathcal{O}_{Z_0}(-h)$, i.e., $\mathcal{E} \cong \mathcal{O}_C \oplus \Bbbk_q$. This is a contradiction because we assumed that the extension

$$0 \to \mathcal{O}_C \xrightarrow{\alpha} \mathcal{E} \xrightarrow{\beta} \Bbbk_q \to 0.$$

is non-trivial. This proves $\lambda \neq 0$ and therefore $f = \mu(l_1q_2 - l_2q_1)$ for $\mu = -\lambda^{-1}$. In particular we also obtain that $l_1q_2 - l_2q_1 \neq 0$ (because otherwise $f = 0$).

Consider the automorphism $\left(\begin{smallmatrix} 1 & \mu l_2 & -\mu l_1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right)$ of $\mathcal{O}_{Z_0}(-2F-H) \oplus 2\mathcal{O}_{Z_0}(-F-H)$. Then

$$\begin{pmatrix} 1 & \mu l_2 & -\mu l_1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \Psi = \begin{pmatrix} 1 & \mu l_2 & -\mu l_1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & \mu(l_1q_2 - l_2q_1) \\ l_1 & q_1 \\ l_2 & q_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ l_1 & q_1 \\ l_2 & q_2 \end{pmatrix}$$

and we finally conclude that $\mathcal{E}$ is given by the resolution

$$0 \to 2\mathcal{O}_{Z_0}(-H-F) \xrightarrow{\left(\begin{smallmatrix} l_1 & q_1 \\ l_2 & q_2 \end{smallmatrix}\right)} \mathcal{O}_{Z_0}(-H) \oplus \mathcal{O}_{Z_0} \xrightarrow{p} \mathcal{E} \to 0.$$

We used here that the morphism $\mathcal{O}_{Z_0}(-H-F) \xrightarrow{\left(\begin{smallmatrix} l_1 & q_1 \\ l_2 & q_2 \end{smallmatrix}\right)} \mathcal{O}_{Z_0}(-H) \oplus \mathcal{O}_{Z_0}$ is injective because the Hilbert polynomial of its kernel is zero. This completes the proof of Proposition 1.60. $\square$

Factor of a pull back of a $3m+1$ sheaf.

We are going to prove here the converse of Proposition 1.49.

Let $\sigma_0 : Z_0 \to \mathbb{P}_2$ be a contraction of D_1 to a point $p \in \mathbb{P}_2$, say $p = \langle 1, 0, 0 \rangle = \{x_1 = x_2 = 0\}$. Let us consider a sheaf $\mathcal{E}$ on Z_0 such that there is an exact sequence

$$0 \to 2\mathcal{O}_{D_1}(-L) \xrightarrow{\varrho} \sigma_0^*\mathcal{F} \xrightarrow{\theta} \mathcal{E} \to 0 \tag{1.41}$$

for some $3m+1$ sheaf $\mathcal{F}$ that is not locally free at the point p. Let us also assume that

- $\mathcal{E}$ is locally free on its support;
- Hilbert polynomial of $\mathcal{E}|_{D_0}$ is $4m+1$.

Using (1.21) one sees that the last condition holds in particular if $\mathcal{E}|_{D_0}$ is the structure sheaf of a curve C_0 given by a resolution

$$0 \to \mathcal{O}_{D_0}(-2F-H) \to \mathcal{O}_{D_0} \to \mathcal{O}_{C_0} \to 0.$$

We will prove the following proposition.

Proposition 1.61. *The sheaf $\mathcal{E}$ as described above is an R-bundle.*

The considerations below constitute a proof of Proposition 1.61 As $\mathcal{F}$ is a $3m+1$ sheaf, there is a resolution of $\mathcal{F}$

$$0 \to 2\mathcal{O}_{\mathbb{P}_2}(-2h) \xrightarrow{A} \mathcal{O}_{\mathbb{P}_2}(-h) \oplus \mathcal{O}_{\mathbb{P}_2} \to \mathcal{F} \to 0$$

for some matrix $A \in X_8$, $A = \begin{pmatrix} x_1 & x_1(\dots)+x_2(\dots) \\ x_2 & x_1(\dots)+x_2(\dots) \end{pmatrix}$. By pulling back this resolution we obtain the exact sequence

$$2\mathcal{O}_{Z_0}(-2H) \xrightarrow{\sigma_0^* A} \mathcal{O}_{Z_0}(-H) \oplus \mathcal{O}_{Z_0} \xrightarrow{\pi} \sigma_0^*\mathcal{F} \to 0.$$

By Lemma 1.50 the kernel of $\sigma_0^* A$ is isomorphic to $2\mathcal{O}_{D_1}(-L)$ and is given by the map

$$2\mathcal{O}_{D_1}(-L) \xrightarrow{\left(\begin{smallmatrix} u_0 & 0 \\ 0 & u_0 \end{smallmatrix}\right)} 2\mathcal{O}_{Z_0}(-2H).$$

Using the snake lemma we obtain thus the following commutative diagram with exact rows and columns

$$\begin{tikzcd}
 & & 0 \arrow[d] & & 0 \arrow[d] & \\
 & & 2\mathcal{O}_{D_1}(-L) \arrow[d, "\left(\begin{smallmatrix} u_0 & 0 \\ 0 & u_0 \end{smallmatrix}\right)"] \arrow[dl, equal] & & 2\mathcal{O}_{D_1}(-L) \arrow[d, "\varrho"] & \\
0 \arrow[r] & 2\mathcal{O}_{D_1}(-L) \arrow[r, "\left(\begin{smallmatrix} u_0 & 0 \\ 0 & u_0 \end{smallmatrix}\right)"'] & 2\mathcal{O}_{Z_0}(-2H) \arrow[r, "\sigma_0^*A"] \arrow[d] & \mathcal{O}_{Z_0}(-H)\oplus\mathcal{O}_{Z_0} \arrow[r, "\pi"] \arrow[d, equal] & \sigma_0^*\mathcal{F} \arrow[r] \arrow[d, "\theta"] & 0 \\
 & 0 \arrow[r] & \mathcal{K} \arrow[r] \arrow[d] & \mathcal{O}_{Z_0}(-H)\oplus\mathcal{O}_{Z_0} \arrow[r, "\pi'"] & \mathcal{E} \arrow[r] \arrow[d] & 0. \\
 & & 2\mathcal{O}_{D_1}(-L) \arrow[d] & & 0 & \\
 & & 0 & & &
\end{tikzcd} \tag{1.42}$$

This diagram is similar to diagram (1.36). We are going to prove that the sheaf $\mathcal{K}$ is isomorphic to $2\mathcal{O}_{Z_0}(-H-F)$ and that the sheaf $\mathcal{E}$ is an R-bundle.

Let us consider the resolution of $2\mathcal{O}_{D_1}(-L)$ (cf. Corollary 1.52)

$$0 \to 2\mathcal{O}_{D_1}(-L) \xrightarrow{\left(\begin{smallmatrix} u_0 & 0 \\ 0 & u_0 \end{smallmatrix}\right)} 2\mathcal{O}_{Z_0}(-2H) \xrightarrow{\left(\begin{smallmatrix} s & 0 \\ 0 & s \end{smallmatrix}\right)} 2\mathcal{O}_{Z_0}(-H-F) \to \mathcal{O}_{D_1}(-L) \to 0, \tag{1.43}$$

and the resolution

$$0 \to 2\mathcal{O}_{D_1}(-L) \xrightarrow{\left(\begin{smallmatrix} u_0 & 0 \\ 0 & u_0 \end{smallmatrix}\right)} 2\mathcal{O}_{Z_0}(-2H) \xrightarrow{\sigma_0^*A} \mathcal{O}_{Z_0}(-H) \oplus \mathcal{O}_{Z_0} \xrightarrow{\pi} \sigma_0^*\mathcal{F} \to 0. \tag{1.44}$$

of $\sigma_0^*\mathcal{F}$. Let us split the long exact sequence (1.44) into two short exact sequences:

$$\begin{tikzcd}
0 \arrow[r] & 2\mathcal{O}_{D_1}(-L) \arrow[r] & 2\mathcal{O}_{Z_0}(-2H) \arrow[rr, "\sigma_0^*A"] \arrow[dr, two heads] & & \mathcal{O}_{Z_0}(-H)\oplus\mathcal{O}_{Z_0} \arrow[r] & \sigma_0^*\mathcal{F} \arrow[r] & 0. \\
 & & & \mathcal{A} \arrow[ur, tail] & & &
\end{tikzcd} \tag{1.45}$$

Lemma 1.62. *1)* $(\sigma_0^*\mathcal{F})|_{D_1} \cong 2\mathcal{O}_{D_1}$*;*

2) The restriction of (1.44) *to* D_1 *remains exact. In particular* $\mathcal{E}_{D_1}$ *is given by the resolution*

$$0 \to 2\mathcal{O}_{D_1}(-L) \xrightarrow{\varrho_{D_1}} 2\mathcal{O}_{D_1} \xrightarrow{\theta_{D_1}} \mathcal{E}_{D_1} \to 0.$$

and has Hilbert polynomial $2m+2$*.*

Proof. 1) Consider the restriction of (1.44) to D_1. Then $(\mathcal{O}_{Z_0}(-H) \oplus \mathcal{O}_{Z_0})|_{D_1} \cong 2\mathcal{O}_{D_1}$ since H and D_1 do not intersect. Moreover, $(\sigma_0^*A)|_{D_1} = 0$. This implies that π_{D_1} is an isomorphism. We proved the first part.

2) Let us restrict the sequence (1.41) to D_1. Using 1) we obtain the exact sequence

$$2\mathcal{O}_{D_1}(-L) \xrightarrow{\varrho_{D_1}} 2\mathcal{O}_{D_1} \xrightarrow{\theta_{D_1}} \mathcal{E}_{D_1} \to 0.$$

Note that ϱ_{D_1} is the same as ϱ outside of L. Therefore, ϱ_{D_1} is injective outside of L. Thus the kernel of ϱ_{D_1} can be only supported on L. But $2\mathcal{O}_{D_1}(-L)$ has no torsion. That is why ϱ_{D_1} is injective. We obtained the exact sequence

$$0 \to 2\mathcal{O}_{D_1}(-L) \xrightarrow{\varrho_{D_1}} 2\mathcal{O}_{D_1} \xrightarrow{\theta_{D_1}} \mathcal{E}_{D_1} \to 0.$$

This proves the lemma. □

Lemma 1.63. *1) The restriction of* (1.44) *to* D_0 *remains exact;*

2) $\mathscr{T}or_1^{\mathcal{O}_{Z_0}}(\sigma_0^*\mathcal{F}, \mathcal{O}_{D_0}) = 0$ *and* $\mathscr{T}or_1^{\mathcal{O}_{Z_0}}(\mathcal{E}, \mathcal{O}_{D_0}) = 0$*.*

Proof. 1) Let us consider the restriction of the resolution (1.41) to $D_0 = \widetilde{\mathbb{P}}_2$. We get the exact sequence

$$2\mathcal{O}_{D_1}(-L)|_{\widetilde{\mathbb{P}}_2} \xrightarrow{\varrho_{D_0}} (\sigma_0^*\mathcal{F})|_{\widetilde{\mathbb{P}}_2} \xrightarrow{\theta_{D_0}} \mathcal{E}_{\widetilde{\mathbb{P}}_2} \to 0.$$

It holds also $2\mathcal{O}_{D_1}(-L)|_{\widetilde{\mathbb{P}}_2} \cong 2\mathcal{O}_L(-1)$, thus there is an exact sequence

$$2\mathcal{O}_L(-1) \xrightarrow{\varrho_{D_0}} (\sigma_0^*\mathcal{F})_{D_0} \xrightarrow{\theta_{D_0}} \mathcal{E}_{D_0} \to 0.$$

Since we assumed that $\mathcal{E}_{\widetilde{\mathbb{P}}_2}$ has the Hilbert polynomial $4m+1$, since the Hilbert polynomial of $(\sigma_0^*\mathcal{F})_{D_0}$ is $6m+1$ by Lemma 1.29 and the Hilbert polynomial of $2\mathcal{O}_L(-1)$ is $2m$, we conclude that the Hilbert polynomial of the kernel of ϱ_{D_0} is zero. That is why ϱ_{D_0} is injective.

2) Restricting (1.31) to $\widetilde{\mathbb{P}}_2$ we obtain the complex

$$2\mathcal{O}_{\widetilde{\mathbb{P}}_2}(-2H-F) \xrightarrow{0} 2\mathcal{O}_{\widetilde{\mathbb{P}}_2}(-2H) \xrightarrow{(\sigma_0^*A)|_{\widetilde{\mathbb{P}}_2}} \mathcal{O}_{\widetilde{\mathbb{P}}_2}(-H) \oplus \mathcal{O}_{\widetilde{\mathbb{P}}_2} \to (\sigma_0^*\mathcal{F})|_{\widetilde{\mathbb{P}}_2} \to 0.$$

Thus $\mathscr{T}or_1^{\mathcal{O}_{Z_0}}(\sigma_0^*\mathcal{F}, \mathcal{O}_{D_0})$ is just the kernel of $(\sigma_0^*A)|_{\widetilde{\mathbb{P}}_2}$. The kernel of σ_0^*A can only live on L because the morphism $\sigma_0 : Z_0 \to \mathbb{P}_2$ is an isomorphism outside of D_1. Since the sheaf $\mathcal{O}_{D_0}(-2H)$ is locally free, this means that the kernel is zero, therefore $\mathscr{T}or_1^{\mathcal{O}_{Z_0}}(\sigma_0^*\mathcal{F}, \mathcal{O}_{D_0}) = 0$.

As $\mathscr{T}or_1^{\mathcal{O}_{Z_0}}(\sigma_0^*\mathcal{F}, \mathcal{O}_{D_0}) = 0$, and since we have shown that ϱ_{D_0} is injective, there is an exact sequence

$$0 \to \mathscr{T}or_1^{\mathcal{O}_{Z_0}}(\mathcal{E}, \mathcal{O}_{\widetilde{\mathbb{P}}_2}) \xrightarrow{0} 2\mathcal{O}_L(-1) \xrightarrow{\varrho_{D_0}} (\sigma_0^*\mathcal{F})_{D_0}.$$

This proves that $\mathscr{T}or_1^{\mathcal{O}_{Z_0}}(\mathcal{E}, \mathcal{O}_{\widetilde{\mathbb{P}}_2}) = 0$. We proved the second statement of the lemma. □

We will use the following lemma.

Lemma 1.64. $\mathrm{Hom}(2\mathcal{O}_{Z_0}(-H-F),\mathcal{A}) = \mathrm{Ext}^1(2\mathcal{O}_{Z_0}(-H-F),\mathcal{A}) = 0.$

Proof. Since $2\mathcal{O}_{Z_0}(-H-F)$ is a locally free sheaf on Z_0, it holds

$$\mathrm{Hom}(2\mathcal{O}_{Z_0}(-H-F),\mathcal{A}) \cong H^0(2\mathcal{A}(H+F)), \quad \mathrm{Ext}^1(2\mathcal{O}_{Z_0}(-H-F),\mathcal{A}) \cong H^1(2\mathcal{A}(H+F)),$$

where $\mathcal{A}(H+F) := \mathcal{A}\otimes\mathcal{O}_{Z_0}(H+F)$. Let us use the left part of the sequence (1.45):

$$0 \to 2\mathcal{O}_{D_1}(-L) \to 2\mathcal{O}_{Z_0}(-2H) \to \mathcal{A} \to 0.$$

After tensoring by $\mathcal{O}_{Z_0}(H+F)$ we obtain the exact sequence

$$0 \to 2\mathcal{O}_{D_1} \to 2\mathcal{O}_{Z_0}(-H+F) \to \mathcal{A}(H+F) \to 0.$$

Therefore, there is a long exact cohomology sequence

$$\begin{aligned} 0 \to H^0(2\mathcal{O}_{D_1}) \to H^0(2\mathcal{O}_{Z_0}(-H+F)) \to H^0(\mathcal{A}(H+F)) \to \\ \to H^1(2\mathcal{O}_{D_1}) \to H^1(2\mathcal{O}_{Z_0}(-H+F)) \to H^1(\mathcal{A}(H+F)) \to H^2(2\mathcal{O}_{D_1}) \to \dots. \end{aligned}$$

Since it holds $H^1(2\mathcal{O}_{D_1}) = H^2(2\mathcal{O}_{D_1}) = 0$ and $H^0(2\mathcal{O}_{D_1}) \cong \Bbbk^2$, since by Proposition 1.32 we have $H^0(\mathcal{O}_{Z_0}(-H+F)) \cong \Bbbk$ and $H^1(\mathcal{O}_{Z_0}(-H+F)) = 0$, we conclude that $H^0(\mathcal{A}(H+F)) = H^1(\mathcal{A}(H+F)) = 0$. This proves the lemma. $\square$

Proposition 1.65. *Every morphism* $\alpha : 2\mathcal{O}_{D_1} \to \sigma_0^*\mathcal{F}$ *lifts uniquely to the morphism of the resolutions* (1.43) *and* (1.44).

Proof. Let us consider the exact sequence

$$0 \to \mathcal{A} \to \mathcal{O}_{Z_0}(-H)\oplus\mathcal{O}_{Z_0} \to \sigma_0^*\mathcal{F} \to 0.$$

Applying $\mathrm{Hom}(2\mathcal{O}_{Z_0}(-H-F), \_\_\,)$ we obtain the long exact sequence

$$\begin{aligned} 0 \to \mathrm{Hom}(2\mathcal{O}_{Z_0}(-H-F),\mathcal{A}) \to \mathrm{Hom}(2\mathcal{O}_{Z_0}(-H-F),\mathcal{O}_{Z_0}(-H)\oplus\mathcal{O}_{Z_0}) \to \\ \to \mathrm{Hom}(2\mathcal{O}_{Z_0}(-H-F),\sigma_0^*\mathcal{F}) \to \mathrm{Ext}^1(2\mathcal{O}_{Z_0}(-H-F),\mathcal{A}). \end{aligned}$$

The existence of the lifting of α to the morphism $\widetilde{A} : 2\mathcal{O}_{Z_0}(-H+F) \to \mathcal{O}_{Z_0}(-H)\oplus\mathcal{O}_{Z_0}$ follows from the first part of Lemma 1.64, i. e., from $\mathrm{Ext}^1(2\mathcal{O}_{Z_0}(-H-F),\mathcal{A}) = 0$. The uniqueness of this lifting follows from the second part of the lemma, i. e, from $\mathrm{Hom}(2\mathcal{O}_{Z_0}(-H-F),\mathcal{A}) = 0$. We obtained a commutative diagram

$$\begin{tikzcd} 2\mathcal{O}_{Z_0}(-H-F) \arrow[r] \arrow[d, dotted, "\exists!\ \widetilde{A}"'] \arrow[dr] & 2\mathcal{O}_{D_1}(-L) \arrow[d, "\alpha"] \\ \mathcal{O}_{Z_0}(-H)\oplus\mathcal{O}_{Z_0} \arrow[r] & \sigma_0^*\mathcal{F}. \end{tikzcd}$$

This induces uniquely a morphism $2\mathcal{O}_{Z_0}(-2H) \to \mathcal{A}$ such that the diagram

$$\begin{tikzcd} 2\mathcal{O}_{Z_0}(-2H) \arrow[r, "\left(\begin{smallmatrix} s & 0 \\ 0 & s \end{smallmatrix}\right)"] \arrow[d, dotted, "\exists! B'"] & 2\mathcal{O}_{Z_0}(-H-F) \arrow[r] \arrow[d, "\widetilde{A}"'] & 2\mathcal{O}_{D_1}(-L) \arrow[r] \arrow[d, "\alpha"] & 0 \\ 0 \arrow[r] & \mathcal{A} \arrow[r] & \mathcal{O}_{Z_0}(-H)\oplus\mathcal{O}_{Z_0} \arrow[r, "\pi"] & \sigma_0^*\mathcal{F} \arrow[r] & 0. \end{tikzcd}$$

Applying $\mathrm{Hom}(2\mathcal{O}_{Z_0}, \_\_\,)$ to the exact sequence

$$0 \to 2\mathcal{O}_{D_1}(-L) \to 2\mathcal{O}_{Z_0}(-2H) \to \mathcal{A} \to 0$$

we obtain the long exact sequence

$$\begin{aligned} 0 \to \mathrm{Hom}(2\mathcal{O}_{Z_0}(-2H), 2\mathcal{O}_{D_1}(-L)) \to \mathrm{Hom}(2\mathcal{O}_{Z_0}(-2H), 2\mathcal{O}_{Z_0}(-2H)) \to \\ \to \mathrm{Hom}(2\mathcal{O}_{Z_0}(-2H), \mathcal{A}) \to \mathrm{Ext}^1(2\mathcal{O}_{Z_0}(-2H), 2\mathcal{O}_{D_1}(-L)). \end{aligned}$$

Since

$$\mathrm{Hom}(2\mathcal{O}_{Z_0}(-2H), 2\mathcal{O}_{D_1}(-L)) \cong H^0(4\mathcal{O}_{D_1}(-L)) = 0$$

and

$$\mathrm{Ext}^1(2\mathcal{O}_{Z_0}(-2H), 2\mathcal{O}_{D_1}(-L)) \cong H^1(4\mathcal{O}_{D_1}(-L)) = 0,$$

we obtain the isomorphism

$$\mathrm{Hom}(2\mathcal{O}_{Z_0}(-2H), 2\mathcal{O}_{Z_0}(-2H)) \cong \mathrm{Hom}(2\mathcal{O}_{Z_0}(-2H), \mathcal{A}).$$

This means there is a unique lifting of B' to a morphism $B : 2\mathcal{O}_{Z_0}(-2H) \to 2\mathcal{O}_{Z_0}(-2H)$, i. e., there is the following commutative diagram.

$$\begin{array}{ccccccccc} 0 \to 2\mathcal{O}_{D_1}(-L) & \xrightarrow{\left(\begin{smallmatrix} u_0 & 0 \\ 0 & u_0 \end{smallmatrix}\right)} & 2\mathcal{O}_{Z_0}(-2H) & \xrightarrow{\left(\begin{smallmatrix} s & 0 \\ 0 & s \end{smallmatrix}\right)} & 2\mathcal{O}_{Z_0}(-H-F) & \longrightarrow & 2\mathcal{O}_{D_1}(-L) \to 0 \\ \downarrow B & & \downarrow \exists ! B & & \widetilde{A} \downarrow & & \downarrow \alpha \\ 0 \to 2\mathcal{O}_{D_1}(-L) & \xrightarrow{\left(\begin{smallmatrix} u_0 & 0 \\ 0 & u_0 \end{smallmatrix}\right)} & 2\mathcal{O}_{Z_0}(-2H) & \xrightarrow{\sigma_0^* A} & \mathcal{O}_{Z_0}(-H) \oplus \mathcal{O}_{Z_0} & \xrightarrow{\pi} & \sigma_0^*\mathcal{F} \longrightarrow 0. \end{array} \tag{1.46}$$

We proved the proposition. □

Let us now return to the situation given in (1.41), i. e., as α we consider the morphism ϱ. By Proposition 1.65 there is a unique lifting of ϱ to a morphism of resolutions. Let $\widetilde{A}$ and B be the lifting morphisms as in (1.46).

Proposition 1.66. *$\widetilde{A}$ is injective and the sequence*

$$0 \to 2\mathcal{O}_{Z_0}(-H-F) \xrightarrow{\widetilde{A}} \mathcal{O}_{Z_0}(-H) \oplus \mathcal{O}_{Z_0} \xrightarrow{\pi'} \mathcal{E} \to 0$$

is exact. In particular $\mathcal{K} \cong 2\mathcal{O}_{Z_0}(-H-F)$.

Proof. Let us restrict the diagram (1.46) to D_1. We obtain the commutative diagram

$$\begin{array}{ccccc} 2\mathcal{O}_{D_1} & \xrightarrow{0} & 2\mathcal{O}_{D_1}(-L) & \longrightarrow & 2\mathcal{O}_{D_1}(-L) \to 0 \\ \downarrow B & & \widetilde{A}_{D_1} \downarrow & & \downarrow \varrho_{D_1} \\ 2\mathcal{O}_{D_1} & \xrightarrow{0} & 2\mathcal{O}_{D_1} & \xrightarrow{\pi_{D_1}} & (\sigma_0^*\mathcal{F})_{D_1} \longrightarrow 0 \end{array}$$

with exact rows. This means that the horizontal arrows in the right square are isomorphisms. By Lemma 1.62 ϱ_{D_1} is injective. Therefore, $\widetilde{A}_{D_1}$ is injective as well.

Let us consider the restriction of (1.46) to D_0. This gives us the commutative diagram

$$\begin{array}{ccccc} 0 \longrightarrow 2\mathcal{O}_{D_0}(-2H) & \xrightarrow{\left(\begin{smallmatrix} s & 0 \\ 0 & s \end{smallmatrix}\right)} & 2\mathcal{O}_{D_0}(-H-F_0) & \longrightarrow & 2\mathcal{O}_L(-1) \to 0 \\ \downarrow B & & \widetilde{A}_{D_0} \downarrow & & \downarrow \varrho_{D_0} \\ 0 \longrightarrow 2\mathcal{O}_{D_0}(-2H) & \xrightarrow{\sigma_0^* A} & \mathcal{O}_{D_0}(-H) \oplus \mathcal{O}_{Z_0} & \xrightarrow{\pi} & (\sigma_0^*\mathcal{F})_{D_0} \to 0. \end{array}$$

with exact rows. The lower row is exact because $\sigma_0^* A$ is injective on $D_0 \setminus L$ and hence its kernel may only be supported on L, which is impossible as $2\mathcal{O}_{D_0}(-2H)$ is a locally free sheaf. The upper row is just the locally free resolution of $2\mathcal{O}_L$

$$0 \to 2\mathcal{O}_{D_0}(-L) \xrightarrow{\left(\begin{smallmatrix} s & 0 \\ 0 & s \end{smallmatrix}\right)} 2\mathcal{O}_{D_0} \to 2\mathcal{O}_L \to 0$$

twisted by $\mathcal{O}_{D_0}(-H - F_0)$.

If $B = 0$, then $\left(\begin{smallmatrix} s & 0 \\ 0 & s \end{smallmatrix}\right) \widetilde{A}_{D_0} = 0$. Since the section s is non-zero (in local charts) over $D_0 \setminus L$, we conclude that $\widetilde{A}_{D_0}$ is zero on $D_0 \setminus L$. Therefore, $\widetilde{A}_{D_0} = 0$ and we get $\varrho_{D_0} = 0$. But this is a contradiction, since ϱ_{D_0} is injective by Lemma 1.63.

If B has rank 1, then we can write

$$B = \begin{pmatrix} \lambda a & \lambda b \\ \mu a & \mu b \end{pmatrix}, \qquad (\lambda, \mu) \in \mathbb{k}^2, (a, b) \neq (0, 0), (\lambda, \mu) \neq (0, 0).$$

In this case the kernel of B is isomorphic to $\mathcal{O}_{Z_0}(-2H)$ and is generated by the matrix $\begin{pmatrix} \mu & -\lambda \end{pmatrix}$. We know also that $\sigma_0^* A = \begin{pmatrix} x_1 & q_1 \\ x_2 & q_2 \end{pmatrix}$. Let $\widetilde{A}_{D_0} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$. Note that by Lemma 1.15 there is a unique factorization

$$\begin{pmatrix} x_1 & q_1 \\ x_2 & q_2 \end{pmatrix} = \begin{pmatrix} s & 0 \\ 0 & s \end{pmatrix} \begin{pmatrix} u_1 & \widetilde{q}_1 \\ u_2 & \widetilde{q}_2 \end{pmatrix}.$$

From $B\sigma_0^*(A) = \left(\begin{smallmatrix} s & 0 \\ 0 & s \end{smallmatrix}\right) \widetilde{A}_{D_0}$ it follows that

$$\left(\begin{smallmatrix} s & 0 \\ 0 & s \end{smallmatrix}\right) \left(B \left(\begin{smallmatrix} u_1 & \widetilde{q}_1 \\ u_2 & \widetilde{q}_2 \end{smallmatrix}\right) - \widetilde{A} \right) = 0,$$

hence by a similar argument as above

$$\widetilde{A}_{D_0} = B \begin{pmatrix} u_1 & \widetilde{q}_1 \\ u_2 & \widetilde{q}_2 \end{pmatrix} = \begin{pmatrix} \lambda a & \lambda b \\ \mu a & \mu b \end{pmatrix} \begin{pmatrix} u_1 & \widetilde{q}_1 \\ u_2 & \widetilde{q}_2 \end{pmatrix} = \begin{pmatrix} \lambda(au_1 + bu_2) & \lambda(a\widetilde{q}_1 + b\widetilde{q}_2) \\ \mu(au_1 + bu_2) & \mu(a\widetilde{q}_1 + b\widetilde{q}_2) \end{pmatrix}.$$

Note that $(a, b) \neq (0, 0)$ and u_1 and u_2 are linear independent. Therefore, $(au_1 + bu_2) \neq 0$ and the kernel of $\widetilde{A}$ is generated by $\begin{pmatrix} \mu & -\lambda \end{pmatrix}$ and is isomorphic to $\mathcal{O}_{Z_0}(-H - F)$. Since by Lemma 1.63 ϱ_{D_0} is injective, by snake Lemma we conclude that the kernels of B and of $\widetilde{A}_{D_0}$ are isomorphic. But $\mathcal{O}_{Z_0}(-2H) \ncong \mathcal{O}_{Z_0}(-H - F)$, because $\mathcal{O}_{D_0}(L) \ncong \mathcal{O}_{Z_0}$ (we use $F \sim H - L$). The contradiction we obtained shows that B may not have rank 1.

We showed that B is an invertible matrix. This implies in particular that $\widetilde{A}_{D_0}$ is injective. Applying the 5-lemma to the diagram (1.46) we conclude that $\widetilde{A}$ is an injective morphism. By construction of $\widetilde{A}$ as a lifting of ϱ we have $\pi' \circ \widetilde{A} = 0$, hence $\widetilde{A}$ factorizes uniquely through $\mathcal{K}$, i. e., there is a commutative diagram

$$\begin{tikzcd}
 & 2\mathcal{O}_{Z_0}(-H - F) \arrow[dl, dotted, "\exists!\, \imath"'] \arrow[d, tail, "\widetilde{A}"] \\
\mathcal{K} \arrow[r, tail] & \mathcal{O}_{Z_0}(-H) \oplus \mathcal{O}_{Z_0}.
\end{tikzcd}$$

The injectivity of $\widetilde{A}$ implies the injectivity of the induced morphism $\imath$. The cokernel of $\imath$ has zero Hilbert polynomial because from (1.42) it follows that $\mathcal{K}$ has the same Hilbert polynomial as $2\mathcal{O}_{Z_0}(-H - F)$, namely $2(2m^2 - m)$ (cf. page 38). Therefore, $\imath$ is an isomorphism. This proves the required statement. $\square$

Corollary 1.67. $\mathscr{T}\!or_1^{\mathcal{O}_{Z_0}}(\mathcal{E}, \mathcal{O}_{D_1}) = 0.$

Proof. Restricting

$$0 \to 2\mathcal{O}_{Z_0}(-H-F) \xrightarrow{\widetilde{A}} \mathcal{O}_{Z_0}(-H) \oplus \mathcal{O}_{Z_0} \xrightarrow{\pi'} \mathcal{E} \to 0$$

to D_1 we obtain the exact sequence

$$0 \to \mathscr{T}\!or_1^{\mathcal{O}_{Z_0}}(\mathcal{E}, \mathcal{O}_{D_1}) \to 2\mathcal{O}_{D_1}(-L) \xrightarrow{\widetilde{A}_{D_1}} 2\mathcal{O}_{D_1} \xrightarrow{\pi'_{D_1}} \mathcal{E}_{D_1} \to 0.$$

As we noticed in the proof of Proposition 1.66, $\widetilde{A}_{D_1}$ is injective. Therefore, $\mathscr{T}\!or_1^{\mathcal{O}_{Z_0}}(\mathcal{E}, \mathcal{O}_{D_1}) = 0$. We proved the required statement. $\square$

Remark 1.68. *Note that since B in* (1.46) *is invertible, then acting on the upper row with* $\mathrm{GL}_2(\Bbbk)$ *we can always attain that B is an identity matrix.*

$$\begin{tikzcd}
0 \arrow[r] & 2\mathcal{O}_{D_1}(-L) \arrow[loop above, "\mathrm{GL}_2(\Bbbk)"] \arrow[r, "\left(\begin{smallmatrix} u_0 & 0 \\ 0 & u_0 \end{smallmatrix}\right)"] \arrow[d, "B"] & 2\mathcal{O}_{Z_0}(-2H) \arrow[loop above, "\mathrm{GL}_2(\Bbbk)"] \arrow[r, "\left(\begin{smallmatrix} s & 0 \\ 0 & s \end{smallmatrix}\right)"] \arrow[d, "B"] & 2\mathcal{O}_{Z_0}(-H-F) \arrow[loop above, "\mathrm{GL}_2(\Bbbk)"] \arrow[r] \arrow[d, "\widetilde{A}"'] & 2\mathcal{O}_{D_1}(-L) \arrow[loop above, "\mathrm{GL}_2(\Bbbk)"] \arrow[r] \arrow[d, "\alpha"] & 0 \\
0 \arrow[r] & 2\mathcal{O}_{D_1}(-L) \arrow[r, "\left(\begin{smallmatrix} u_0 & 0 \\ 0 & u_0 \end{smallmatrix}\right)"] & 2\mathcal{O}_{Z_0}(-2H) \arrow[r, "\sigma_0^*A"] & \mathcal{O}_{Z_0}(-H) \oplus \mathcal{O}_{Z_0} \arrow[r, "\pi"] & \sigma_0^*\mathcal{F} \arrow[r] & 0.
\end{tikzcd}$$

Lemma 1.69. *Let B be an identity matrix, let*

$$A = \begin{pmatrix} x_1 & x_1(A_{01}x_0 + A_{11}x_1 + A_{12}x_2) + x_2(A_{02}x_0 + A_{22}x_2) \\ x_2 & x_1(B_{01}x_0 + B_{11}x_1 + B_{12}x_2) + x_2(B_{02}x_0 + B_{22}x_2) \end{pmatrix}.$$

Then

$$\widetilde{A} = \begin{pmatrix} u_1 + \xi_0 u_0 & u_1(A_{01}x_0 + A_{11}x_1 + A_{12}x_2) + u_2(A_{02}x_0 + A_{22}x_2) + \xi_{00}x_0u_0 \\ u_2 + \eta_0 u_0 & u_1(B_{01}x_0 + B_{11}x_1 + B_{12}x_2) + u_2(B_{02}x_0 + B_{22}x_2) + \eta_{00}x_0u_0 \end{pmatrix}$$

for some ξ_0, η_0, ξ_{00} and η_{00}.

Proof. Straightforward calculations with cocycles of the corresponding locally free sheaves. $\square$

By Proposition 1.57 we conclude that $\mathcal{E}$ is an R-bundle. This completes the proof of Proposition 1.61.

Proposition 1.70. *Every morphism of R-bundles $\mathcal{E}_1 \to \mathcal{E}_2$ lifts uniquely to a morphism of resolutions of the type* (1.41).

In particular isomorphisms of R-bundles lift to isomorphisms of resolutions.

Proof. By Proposition 1.49 R-bundles have resolutions of the type (1.41). To prove the required statement it is enough to show that for every $3m+1$ sheaf $\mathcal{F}$ on $\mathbb{P}_2$ it holds

$$\mathrm{Hom}(\sigma_0^*\mathcal{F}, 2\mathcal{O}_{D_1}(-L)) = \mathrm{Ext}^1(\sigma_0^*\mathcal{F}, 2\mathcal{O}_{D_1}(-L)) = 0.$$

Let us consider again the exact sequence (1.45). From the short exact sequence

$$0 \to \mathcal{A} \to \mathcal{O}_{Z_0}(-H) \oplus \mathcal{O}_{Z_0} \to \sigma_0^*\mathcal{F} \to 0$$

we obtain the long exact sequence

$$\begin{aligned} 0 &\to \mathrm{Hom}(\sigma_0^*\mathcal{F}, 2\mathcal{O}_{D_1}(-L)) \to \mathrm{Hom}(\mathcal{O}_{Z_0}(-H) \oplus \mathcal{O}_{Z_0}, 2\mathcal{O}_{D_1}(-L)) \to \mathrm{Hom}(\mathcal{A}, 2\mathcal{O}_{D_1}(-L)) \to \\ &\to \mathrm{Ext}^1(\sigma_0^*\mathcal{F}, 2\mathcal{O}_{D_1}(-L)) \to \mathrm{Ext}^1(\mathcal{O}_{Z_0}(-H) \oplus \mathcal{O}_{Z_0}, 2\mathcal{O}_{D_1}(-L)) \to \dots \end{aligned}$$

Since the sheaf $\mathcal{O}_{Z_0}(-H) \oplus \mathcal{O}_{Z_0}$ is locally free we obtain

$$\mathrm{Hom}(\mathcal{O}_{Z_0}(-H) \oplus \mathcal{O}_{Z_0}, 2\mathcal{O}_{D_1}(-L)) \cong H^0(4\mathcal{O}_{D_1}(-L)) = 0,$$
$$\mathrm{Ext}^1(\mathcal{O}_{Z_0}(-H) \oplus \mathcal{O}_{Z_0}, 2\mathcal{O}_{D_1}(-L)) \cong H^1(4\mathcal{O}_{D_1}(-L)) = 0.$$

Therefore, $\mathrm{Hom}(\sigma_0^*\mathcal{F}, 2\mathcal{O}_{D_1}(-L)) = 0$ and $\mathrm{Ext}^1(\sigma_0^*\mathcal{F}, 2\mathcal{O}_{D_1}(-L)) \cong \mathrm{Hom}(\mathcal{A}, 2\mathcal{O}_{D_1}(-L))$.

Using the exact sequence

$$0 \to 2\mathcal{O}_{D_1}(-L) \to 2\mathcal{O}_{Z_0}(-2H) \to \mathcal{A} \to 0$$

we obtain the injective homomorphism

$$0 \to \mathrm{Hom}(\mathcal{A}, 2\mathcal{O}_{D_1}(-L)) \to \mathrm{Hom}(2\mathcal{O}_{Z_0}(-2H), 2\mathcal{O}_{D_1}(-L)).$$

From $\mathrm{Hom}(2\mathcal{O}_{Z_0}(-2H), 2\mathcal{O}_{D_1}(-L)) \cong H^0(4\mathcal{O}_{D_1}(-L)) = 0$ we get $\mathrm{Ext}^1(\sigma_0^*\mathcal{F}, 2\mathcal{O}_{D_1}(-L)) = 0$. This completes the proof. $\square$

Chapter 2

Equivalence of R-bundles

Summary

In this chapter we consider R-bundles on $\hat{\mathbb{P}}_2$ up to equivalence.

In Section 2.1 we describe the set of the isomorphism classes of R-bundles on $\hat{\mathbb{P}}_2$. We show that on the set of the classes of isomorphism of R-bundles there is a natural structure of a quasi-projective variety.

In Section 2.2 we introduce the following equivalence relation on the set of R-bundles constructed at the same point $A \in X_8$ (cf. Definition 2.12). Two R-bundles $\mathcal{E}_1$ and $\mathcal{E}_2$ be on $\hat{\mathbb{P}}_2$ constructed at the same point $A \in X_8$ are called equivalent if there exists an automorphism $\widetilde{\phi}$ of Z_0 that acts identically on $D_0 = \widetilde{\mathbb{P}}_2$ and such that $\widetilde{\phi}^*(\mathcal{E}_1) \cong \mathcal{E}_2$. Our notion of equivalence is similar to the notion of equivalence given in Definition 4.1, (ii) from [26]. We show in Theorem 2.19 that the equivalence classes are in one-to-one correspondence with the points in projective normal space $\mathbb{P}N_A = \mathbb{P}(T_AX/T_AX_8)$.

For a given $A \in X_8$ we consider also the question about the number of the equivalence classes of R-bundles constructed at A with singular curve C_1, where C_1 denotes the supporting curve in D_1. For a generic $A \in X_8$ (when the corresponding singular $3m+1$ sheaf is defined by an ordinary double point singularity on a cubic curve in $\mathbb{P}_2$) there are only two equivalence classes with a singular conic C_1 as a support in D_1. Degenerations of A with double-point singularity give us only one equivalence class with the curve C_1 being singular. If the singularity is a triple point, all the equivalence classes have singular curve C_1. In this case one could identify the line $L = D_0 \cap D_1$ with the set of all equivalence classes of R-bundles constructed at A.

In Section 2.3 we present for every type of singular $3m+1$ sheaves on $\mathbb{P}_2$ a detailed illustration to Theorem 2.19 and compare the equivalence and the isomorphism classes of R-bundles. We see that in the generic case R-bundles are line bundles on the curves of the types X_1 and X_2 considered in [26].

2.1 Classes of isomorphism of R-bundles

We will describe here the set of the isomorphism classes of R-bundles on $\hat{\mathbb{P}}_2$. We will show that this space may be identified with the set of orbits of some group action on some quasi-affine variety. We will also show that there exists an orbit space of that action.

2.1.1 Isomorphism classes as orbits of a group action.

Let us consider the set of the matrices

$$\Phi = \begin{pmatrix} l_1 & \widetilde{q}_1 \\ l_2 & \widetilde{q}_2 \end{pmatrix}, \quad l_1, l_2 \in \Gamma(Z_0, \mathcal{O}_{Z_0}(F)), \quad \widetilde{q}_1, \widetilde{q}_2 \in \Gamma(Z_0, \mathcal{O}_{Z_0}(F+H)). \tag{2.1}$$

Note that by Proposition 1.32 $\Gamma(Z_0, \mathcal{O}_{Z_0}(F)) \cong \Gamma(D_1, \mathcal{O}_{D_1}(F_0)) \cong \Gamma(\mathbb{P}_2, \mathcal{O}_{\mathbb{P}_2}(1))$ is a 3-dimensional vector space which may be identified with the space of linear forms in variables u_0, u_1, u_2 (coordinates of $D_1 = \mathbb{P}_2$).

We have also $\Gamma(Z_0, \mathcal{O}_{Z_0}(H)) \cong \Gamma(\widetilde{\mathbb{P}}_2, \mathcal{O}_{\widetilde{\mathbb{P}}_2}(H)) \cong \Gamma(\mathbb{P}_2, \mathcal{O}_{\mathbb{P}_2}(1))$. This means that the vector space $\Gamma(Z_0, \mathcal{O}_{Z_0}(H))$ may be considered as the space of linear forms in x_0, x_1, x_2.

The dimension of $\Gamma(Z_0, \mathcal{O}_{Z_0}(F+H))$ is 6, as a basis of $\Gamma(Z_0, \mathcal{O}_{Z_0}(F+H))$ one could choose the set

$$\{x_0u_0, x_0u_1, x_0u_2, x_1u_1, x_1u_2, x_2u_2\}.$$

Therefore, one can consider the space $\mathbb{A}$ of all matrices (2.1) just as the affine variety $\mathbb{k}^{18}$.

Definition 2.1. *Let $\mathbb{X}'$ be the open set in the set of all matrices* (2.1) *defined by the following conditions:*

- *the determinant $\Delta = l_1\widetilde{q}_2 - l_2\widetilde{q}_1$ and its restrictions to D_0 and D_1 are non-zero;*
- *$l_1 \wedge l_2 \neq 0$, i. e., l_1 and l_2 are linear independent;*
- *the common zero point of l_1 and l_2 in D_1 lies outside of L.*

Let $\mathbb{X}$ be the open set in $\mathbb{X}'$ given by an extra condition:

- *Φ does not vanish on D_1.*

There is a natural action of the group G (cf. (1.4)) on the set $\mathbb{A}$ of all matrices of the type (2.1) and also on $\mathbb{X}'$ and $\mathbb{X}$. G acts from the left by the rule

$$(g, h) \cdot \Phi = g\Phi h^{-1}.$$

On $\mathbb{X}'$ this action corresponds to isomorphisms of exact sequences given by $g \in \mathrm{GL}_2(\mathbb{k})$, $h \in H$:

$$\begin{tikzcd}
0 \arrow[r] & 2\mathcal{O}_{Z_0}(-H-F) \arrow[r, "\Phi"] & \mathcal{O}_{Z_0}(-H) \oplus \mathcal{O}_{Z_0} \arrow[r] & \mathcal{E} \arrow[r] & 0 \\
0 \arrow[r] & 2\mathcal{O}_{Z_0}(-H-F) \arrow[r] \arrow[u, "g"] & \mathcal{O}_{Z_0}(-H) \oplus \mathcal{O}_{Z_0} \arrow[r] \arrow[u, "h"] & \mathcal{F} \arrow[r] \arrow[u, "\cong"] & 0.
\end{tikzcd}$$

By Proposition 1.40 isomorphisms of such exact sequences are in one-to-one correspondence with the isomorphisms $\mathcal{E} \xrightarrow{\cong} \mathcal{F}$. In other words, this means that the orbits of G in $\mathbb{X}'$ are in one-to-one correspondence with the isomorphism classes of the sheaves on Z_0 given by resolutions of the type

$$0 \to 2\mathcal{O}_{Z_0}(-H-F) \xrightarrow{\Phi} \mathcal{O}_{Z_0}(-H) \oplus \mathcal{O}_{Z_0} \to \mathcal{E} \to 0, \quad \Phi \in \mathbb{X}'.$$

The sheaves given by such resolutions for $\Phi \in \mathbb{X}$ are by Propositions 1.37 and 1.57 exactly R-bundles on $Z_0 = \hat{\mathbb{P}}_2$. We proved the following proposition.

Proposition 2.2. *The orbits of G in $\mathbb{X}$ are in one-to-one correspondence with the isomorphism classes of R-bundles on $\hat{\mathbb{P}}_2$.*

2.1.2 Orbit space.

It would be nice to understand whether there is a natural geometrical structure on the space of isomorphism classes of R-bundles on $\hat{\mathbb{P}}_2$.

Lemma 2.3. *Let $\Phi \in \mathbb{X}'$, then its stabilizer is the group*

$$St = \{ \left(\begin{smallmatrix} \lambda & 0 \\ 0 & \lambda \end{smallmatrix}\right) \times \left(\begin{smallmatrix} \lambda & 0 \\ 0 & \lambda \end{smallmatrix}\right) \mid \lambda \in \mathbb{k}^* \} \cong \mathbb{k}^*.$$

Proof. Let $\Phi = \left(\begin{smallmatrix} l_1 & q_1 \\ l_2 & q_2 \end{smallmatrix}\right)$ and suppose that $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2(k)$ and $\left(\begin{smallmatrix} \lambda & z \\ 0 & \mu \end{smallmatrix}\right) \in H$ are such that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} l_1 & q_1 \\ l_2 & q_2 \end{pmatrix} = \begin{pmatrix} l_1 & q_1 \\ l_2 & q_2 \end{pmatrix} \begin{pmatrix} \lambda & z \\ 0 & \mu \end{pmatrix}.$$

Comparing the entries one obtains

$$al_1 + bl_2 = \lambda l_1, \quad cl_1 + dl_2 = \lambda l_2, \quad aq_1 + bq_2 = zl_1 + \mu q_1, \quad cq_1 + dq_2 = zl_2 + \mu q_2$$

and therefore $a = d = \lambda$, $b = c = 0$, and

$$(\lambda - \mu)q_1 = zl_1, \quad (\lambda - \mu)q_2 = zl_2.$$

If $\lambda \neq \mu$, then $\det \Phi = 0$, which is a contradiction since $\Phi \in \mathbb{X}$. Thus $\lambda = \mu$ and $zl_1 = zl_2 = 0$. We obtain $z \cdot (l_1, l_2) = 0$ and conclude then by Lemma 1.47 that $z = 0$. Finally we obtain

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \lambda & z \\ 0 & \mu \end{pmatrix} = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}.$$

This proves the statement of the lemma. □

Corollary 2.4. *The induced action of the group $\mathbb{P}G = G/St$ on $\mathbb{X}'$ is free.*

Consider the vector space $V = \Gamma(Z_0, \mathcal{O}_{Z_0}(H + 2F))$. By Proposition 1.32 its dimension is 10. Then $\mathbb{P}V \cong \mathbb{P}_9$. One can represent an element $f \in V$ as a polynomial

$$f = C_0 x_0 u_0^2 + C_1 x_0 u_0 u_1 + C_2 x_0 u_0 u_2 + C_3 x_0 u_1^2 + \\ C_4 x_0 u_1 u_2 + C_5 x_0 u_2^2 + C_6 x_1 u_1^2 + C_7 x_1 u_1 u_2 + C_8 x_1 u_2^2 + C_9 x_2 u_2^2$$

with relations $x_1 u_0 = x_2 u_0 = x_1 u_2 - x_2 u_1 = 0$.

The set V_1 of those f that vanish on

$$D_1 = \{x_1 = x_2 = 0\}$$

consists of $f = C_6 x_1 u_1^2 + C_7 x_1 u_1 u_2 + C_8 x_1 u_2^2 + C_9 x_2 u_2^2$, i. e., is a subspace of dimension 4 in V.

The set V_0 of those f that vanish on $D_0 = \{u_0 = 0\}$ consists of

$$f = C_3 x_0 u_1^2 + C_4 x_0 u_1 u_2 + C_5 x_0 u_2^2 + C_6 x_1 u_1^2 + C_7 x_1 u_1 u_2 + C_8 x_1 u_2^2 + C_9 x_2 u_2^2$$

and hence form a subspace of dimension 7 in V.

Consider the morphism

$$\mathbb{X}' \xrightarrow{\vartheta} \mathbb{P}V \times D_1, \quad \Phi = \begin{pmatrix} l_1 & \widetilde{q}_1 \\ l_2 & \widetilde{q}_2 \end{pmatrix} \mapsto \langle l_1 \widetilde{q}_2 - l_2 \widetilde{q}_1 \rangle \times \langle l_1 \wedge l_2 \rangle.$$

First of all we have the following obvious lemma.

Lemma 2.5. *ϑ is G-invariant, i. e., $\vartheta(g \cdot \Phi) = \vartheta(\Phi)$ for all $g \in G$ and $\Phi \in \mathbb{X}'$.*

Let $\mathbb{Y} \subseteq \mathbb{P}V \times D_1$ be the universal curve defined by

$$\mathbb{Y} = \{\langle f \rangle \times \langle q \rangle \mid f(q) = 0\}.$$

Let us consider the open set of $\mathbb{Y}$ given by

$$\mathbb{Y}' = \mathbb{Y} \cap (\mathbb{P}V \setminus (\mathbb{P}V_0 \cup \mathbb{P}V_1)) \times (D_1 \setminus L)).$$

In other words $\mathbb{Y}'$ consists of those $\langle f \rangle \times \langle q \rangle$ from $\mathbb{Y}$ such that q does not lie on the line L and f does not vanish identically on D_0 or on D_1.

Note that the image of ϑ is contained in $\mathbb{Y}'$ by definition of $\mathbb{X}'$.

Lemma 2.6. *There is a section $s : \mathbb{Y}' \to \mathbb{X}'$ of $\vartheta : \mathbb{X}' \to \mathbb{Y}'$, i. e., a morphisms that satisfies $\vartheta \circ s = \mathrm{id}_{\mathbb{Y}'}$. In particular this implies that $\vartheta(\mathbb{X}') = \mathbb{Y}'$.*

Proof. Let us show that $\mathbb{X}' \xrightarrow{\vartheta} \mathbb{Y}'$ is surjective. Take any point $\langle f \rangle \times \langle q \rangle$ in $\mathbb{Y}'$. Let $\langle q \rangle = \langle u_0, u_1, u_2 \rangle$ and let

$$\begin{aligned} f =& C_0x_0u_0^2 + C_1x_0u_0u_1 + C_2x_0u_0u_2 + C_3x_0u_1^2 + \\ & C_4x_0u_1u_2 + C_5x_0u_2^2 + C_6x_1u_1^2 + C_7x_1u_1u_2 + C_8x_1u_2^2 + C_9x_2u_2^2 = 0. \end{aligned}$$

Since $q \in D_1 \setminus L$, we conclude that $u_1 = \xi u_0$ and $u_2 = \eta u_0$ for some $\xi, \eta \in \Bbbk$. As $q \in D_1$, we have $x_1 = x_2 = 0$. Therefore,

$$\begin{aligned} 0 = f(q) =& C_0x_0u_0^2 + C_1\xi x_0u_0^2 + C_2\eta x_0u_0^2 + C_3\xi^2x_0u_0^2 + C_4\xi\eta x_0u_0^2 + C_5\eta^2x_0u_0^2 = \\ & u_0^2x_0(C_0 + C_1\xi + C_2\eta + C_3\xi^2 + C_4\xi\eta + C_5\eta^2), \end{aligned}$$

and we obtain $C_0 = -(C_1\xi + C_2\eta + C_3\xi^2 + C_4\xi\eta + C_5\eta^2)$. Using this and the equalities $u_0x_1 = u_0x_2 = 0$ we obtain

$$\begin{aligned} f =& -(C_1\xi + C_2\eta + C_3\xi^2 + C_4\xi\eta + C_5\eta^2)x_0u_0^2 + C_1x_0u_0u_1 + C_2x_0u_0u_2 + C_3x_0u_1^2 + \\ & C_4x_0u_1u_2 + C_5x_0u_2^2 + C_6x_1u_1^2 + C_7x_1u_1u_2 + C_8x_1u_2^2 + C_9x_2u_2^2 = \\ & C_1x_0u_0(u_1 - \xi u_0) + C_2x_0u_0(u_2 - \eta u_0) + C_3x_0(u_1^2 - \xi^2u_0^2) + C_4x_0(u_1u_2 - \xi\eta u_0^2) + \\ & C_5x_0(u_2^2 - \eta^2u_0) + C_6x_1u_1^2 + C_7x_1u_1u_2 + C_8x_1u_2^2 + C_9x_2u_2^2 = \\ & (u_1 - \xi u_0)(C_1x_0u_0 + C_3x_0(u_1 + \xi u_0)) + (u_2 - \eta u_0)(C_2x_0u_0 + C_5x_0(u_2 + \eta u_0)) + \\ & C_4x_0(u_1u_2 - \xi u_2u_0 + \xi u_2u_0 - \xi\eta u_0^2) + C_6x_1u_1^2 + C_7x_1u_1u_2 + C_8x_1u_2^2 + C_9x_2u_2^2 = \\ & (u_1 - \xi u_0)(C_1x_0u_0 + C_3x_0(u_1 + \xi u_0) + C_4x_0u_2) + \\ & (u_2 - \eta u_0)(C_2x_0u_0 + C_5x_0(u_2 + \eta u_0) + \xi C_4x_0u_0) + \\ & C_6x_1u_1(u_1 - \xi u_0) + C_7x_1u_1(u_2 - \eta u_0) + C_8x_1u_2(u_2 - \eta u_0) + C_9x_2u_2(u_2 - \eta u_0) = \\ & (u_1 - \xi u_0)(C_1x_0u_0 + C_3x_0(u_1 + \xi u_0) + C_4x_0u_2 + C_6x_1u_1) + \\ & (u_2 - \eta u_0)(C_2x_0u_0 + C_5x_0(u_2 + \eta u_0) + \xi C_4x_0u_0 + C_7x_1u_1 + C_8x_1u_2 + C_9x_2u_2). \end{aligned}$$

The latter means that f is the determinant of the matrix

$$\Phi = \begin{pmatrix} u_1 - \xi u_0 & \widetilde{q}_1 \\ u_2 - \eta u_0 & \widetilde{q}_1 \end{pmatrix},$$

where

$$\widetilde{q}_1 = -(C_2x_0u_0 + C_5x_0(u_2 + \eta u_0) + \xi C_4x_0u_0 + C_7x_1u_1 + C_8x_1u_2 + C_9x_2u_2)$$

and

$$\widetilde{q}_2 = C_1 x_0 u_0 + C_3 x_0 (u_1 + \xi u_0) + C_4 x_0 u_2 + C_6 x_1 u_1.$$

Note that $\vartheta(\Phi) = \langle f \rangle \times \langle q \rangle$ and that Φ belongs to $\mathbb{X}'$. This shows that we have constructed the morphism

$$s : \mathbb{Y}' \to \mathbb{X}', \quad \langle f \rangle \times \langle 1, \xi, \eta \rangle \mapsto \begin{pmatrix} u_1 - \xi u_0 & \widetilde{q}_1 \\ u_2 - \eta u_0 & \widetilde{q}_1 \end{pmatrix},$$

such that $\Phi \circ s = \mathrm{id}_{\mathbb{Y}'}$. This proves the required statement. $\square$

Let us consider the morphism $\vartheta : \mathbb{X}' \to \mathbb{Y}'$.

Proposition 2.7. *$(\mathbb{Y}', \vartheta)$ is an orbit space of the action $G \times \mathbb{X}' \to \mathbb{X}'$, i. e., the fibres of $\mathbb{X}' \xrightarrow{\vartheta} \mathbb{Y}'$ coincide with the orbits of this action and for every G-invariant morphism $\rho : \mathbb{X}' \to T$ there exists a unique morphism $\mathbb{Y}' \to T$ such that the diagram*

$$\begin{tikzcd} \mathbb{X}' \arrow[rr, "\vartheta"] \arrow[dr, "\rho"'] & & \mathbb{Y}' \arrow[dl, dotted, "\exists!"] \\ & T & \end{tikzcd}$$

commutes.

Proof. Let us prove that the fibres of ϑ coincide with the orbits of G. Suppose $\vartheta(\Phi) = \vartheta(\Psi)$, for some matrices

$$\Phi = \begin{pmatrix} l_1 & \widetilde{q}_1 \\ l_2 & \widetilde{q}_2 \end{pmatrix} \quad \text{and} \quad \Psi = \begin{pmatrix} l'_1 & \widetilde{q}'_1 \\ l'_2 & \widetilde{q}'_2 \end{pmatrix}.$$

We will show that Φ and Ψ lie in the same orbit.

Since $\langle l_1 \wedge l_2 \rangle = \langle l'_1 \wedge l'_2 \rangle$, we conclude that the spaces $\Bbbk l_1 + \Bbbk l_2$ and $\Bbbk l'_1 + \Bbbk l'_2$ coincide. Therefore, there exists $g \in \mathrm{GL}_2(\Bbbk)$ such that

$$\left(\begin{smallmatrix} l_1 \\ l_2 \end{smallmatrix}\right) = g \cdot \left(\begin{smallmatrix} l'_1 \\ l'_2 \end{smallmatrix}\right)$$

and we may assume that $l_i = l'_i$, $i = 1, 2$.

Multiplying one of the matrices Φ and Ψ by $\left(\begin{smallmatrix} \lambda & 0 \\ 0 & \lambda \end{smallmatrix}\right)$ for an appropriate $\lambda \in \Bbbk^*$ we may also assume that the determinants of Φ and Ψ are equal, i. e., that

$$l_1 \widetilde{q}_2 - l_2 \widetilde{q}_1 = l_1 \widetilde{q}'_2 - l_2 \widetilde{q}'_1.$$

This means that

$$(\widetilde{q}_2 - \widetilde{q}'_2, -\widetilde{q}_1 + \widetilde{q}'_1) \begin{pmatrix} l_1 \\ l_2 \end{pmatrix} = 0$$

and hence by Lemma 1.47 we conclude that

$$(\widetilde{q}_2 - \widetilde{q}'_2, -\widetilde{q}_1 + \widetilde{q}'_1) = z \cdot (l_2, -l_1),$$

for some $z \in \Gamma(Z_0, \mathcal{O}_{Z_0}(H))$. Thus $\widetilde{q}_2 = \widetilde{q}'_2 + z l_2$, $\widetilde{q}_1 = \widetilde{q}'_1 + z l_1$, and one concludes that

$$\Phi = \Psi \cdot \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}.$$

This proves that Φ and Ψ lie in the same orbit of G.

Let now $\rho : \mathbb{X}' \to T$ be a G-invariant morphism. Since the points of $\mathbb{Y}'$ are in one-to-one correspondence with the orbits of G in $\mathbb{X}'$, there exists a unique set theoretical map $\varpi : \mathbb{Y}' \to T$ such that $\varpi \circ \vartheta = \rho$. Namely, $\varpi(y) = \rho(x)$, where $x \in \mathbb{X}'$ is an arbitrary point from the preimage of $y \in \mathbb{Y}'$ under ϑ. It remains to prove that ϖ is a morphism. Using the section from Lemma 2.6 we obtain $\varpi(y) = \rho(s(y))$, i. e., $\varpi = \rho \circ s$. Therefore, ϖ is a morphism as a composition of two morphisms. This completes the proof of the proposition. $\square$

Corollary 2.8. *$\mathbb{X}' \cong \mathbb{Y}' \times \mathbb{P}G$, where $\mathbb{P}G = G/St$ is the factor group modulo the stabilizer St (cf. Lemma 2.3).*

Proof. Let s be the section from Lemma 2.6. By Corollary 2.4 we have a free action of $\mathbb{P}G$ on $\mathbb{X}'$. By Proposition 2.7 its orbit space is $\mathbb{Y}'$. Therefore, the morphism

$$\mathbb{Y}' \times \mathbb{P}G \to \mathbb{X}', \quad (y, g) \mapsto g \cdot s(y)$$

is a bijection. By Zariski main theorem (cf. [8], 6.1.14 and [9], 4.4.3) one concludes that it is an isomorphism (note that $\mathbb{Y}'$ and $\mathbb{X}'$ are smooth). □

Corollary 2.9. *$\mathbb{Y}'' = \vartheta(\mathbb{X})$ is an open subvariety of $\mathbb{Y}'$. $(\mathbb{Y}'', \vartheta|_{\mathbb{X}})$ is an orbit space of the action $G \times \mathbb{X} \to \mathbb{X}$ and $\mathbb{X} \cong \mathbb{Y}'' \times \mathbb{P}G$.*

Proof. Since $\mathbb{X} \subseteq \mathbb{X}'$ is invariant under the action of G, Proposition 2.7 implies that $(\mathbb{Y}'', \vartheta|_{\mathbb{X}})$ is an orbit space of the action of G on $\mathbb{X}$. From Corollary 2.8 we obtain $\mathbb{X} \cong \mathbb{Y}'' \times \mathbb{P}G$. Since $\mathbb{X}$ is an open subvariety in $\mathbb{X}'$ and $\mathbb{Y}'' = \vartheta(\mathbb{X}) = s^{-1}(\mathbb{X})$, one concludes that $\mathbb{Y}''$ is an open subset in $\mathbb{Y}'$. □

Corollary 2.10. *Let $\mathcal{E}'$ and $\mathcal{E}''$ be two R-bundles on $\hat{\mathbb{P}}_2$. Let $C' = \operatorname{Supp} \mathcal{E}'$ and $C'' = \operatorname{Supp} \mathcal{E}''$. By Proposition 1.48 we obtain two non-trivial extensions*

$$0 \to \mathcal{O}_{C'} \to \mathcal{E}' \to \mathbb{k}_{q'} \to 0,$$

and

$$0 \to \mathcal{O}_{C''} \to \mathcal{E}'' \to \mathbb{k}_{q''} \to 0,$$

where q' and q'' are some points on $C' \setminus D_0$ and $C'' \setminus D_0$ respectively.

Proposition 2.7 says that $\mathcal{E}'$ and $\mathcal{E}''$ are isomorphic if and only if $C' = C''$ and $q' = q''$.

Corollary 2.11. *Let us fix a curve $C \subseteq Z_0$ given by a resolution*

$$0 \to \mathcal{O}_{Z_0}(-2F - H) \to \mathcal{O}_{Z_0} \to \mathcal{O}_C \to 0.$$

Then the isomorphism classes of R-bundles supported on C are in one-to-one correspondence with the points of some open subset of $C \setminus D_0$.

2.2 A description of equivalence classes of R-bundles

We introduce here an equivalence relation on the set of R-bundles constructed at point $A \in X_8$ and describe the equivalence classes.

Definition 2.12. *Let $\mathcal{E}_1$ and $\mathcal{E}_2$ be two sheaves of the type $\widetilde{\mathcal{E}}_0$, i. e., sheaves constructed at the same point $A \in X_8$. We call them equivalent if there exists an automorphism $\widetilde{\phi}$ of Z_0 that acts identically on $\widetilde{\mathbb{P}}_2$ and such that $\widetilde{\phi}^*(\mathcal{E}_1) \cong \mathcal{E}_2$.*

Remark 2.13. *1) The relation "to be equivalent" defined in Definition 2.12 is in fact an equivalence relation on the set of R-bundles constructed at a fixed point $A \in X_8$.*

2) Definition 2.12 is similar to Definition 4.1, (ii) of equivalence for vector bundles on X_k from [26].

2.2.1 Group action on D_1.

Let us consider $D_1 = \mathbb{P}_2$ and the line $L = \{u_0 = 0\}$.

Lemma 2.14. *Automorphism of $D_1 = \mathbb{P}_2$ acting identically on the line $L = \{u_0 = 0\}$ are exactly those of the form*

$$\mathbb{P}_2 \ni \langle u_0, u_1, u_2 \rangle \mapsto \langle (u_0, u_1, u_2) \left(\begin{smallmatrix} \alpha & \beta & \gamma \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right) \rangle \in \mathbb{P}_2$$

Proof. Note that all the automorphisms of $\mathbb{P}_2$ are linear, i. e., of the form

$$\langle u_0, u_1, u_2 \rangle \mapsto \langle (u_0, u_1, u_2) \left(\begin{smallmatrix} a_{00} & a_{01} & a_{02} \\ a_{10} & a_{11} & a_{12} \\ a_{20} & a_{21} & a_{22} \end{smallmatrix}\right) \rangle$$

for some invertible matrix $\left(\begin{smallmatrix} a_{00} & a_{01} & a_{02} \\ a_{10} & a_{11} & a_{12} \\ a_{20} & a_{21} & a_{22} \end{smallmatrix}\right)$. Then the points $\langle 0, u_1, u_2 \rangle$ from the line L are mapped to $\langle a_{10}u_1 + a_{20}u_2, a_{11}u_1 + a_{21}u_2, a_{12}u_1 + a_{22}u_2 \rangle$. To obtain an automorphism acting identically on L it should necessarily hold $a_{10} = a_{20} = a_{21} = a_{12} = 0$ and $a_{11} = a_{22}$. As the matrix is defined up to multiplication by a non-zero scalar, one may take $a_{11} = a_{22} = 1$. This proves the lemma since the automorphisms given by the matrices of the type $\left(\begin{smallmatrix} \alpha & \beta & \gamma \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right)$ act identically on L. $\square$

Remark 2.15. *Note that there is a natural action of the group of the matrices $\left(\begin{smallmatrix} \alpha & \beta & \gamma \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right)$ on the set of all conics in D_1.*

Let C_1 be a conic on D_1 that is not a double line. Then there are five possible situations:

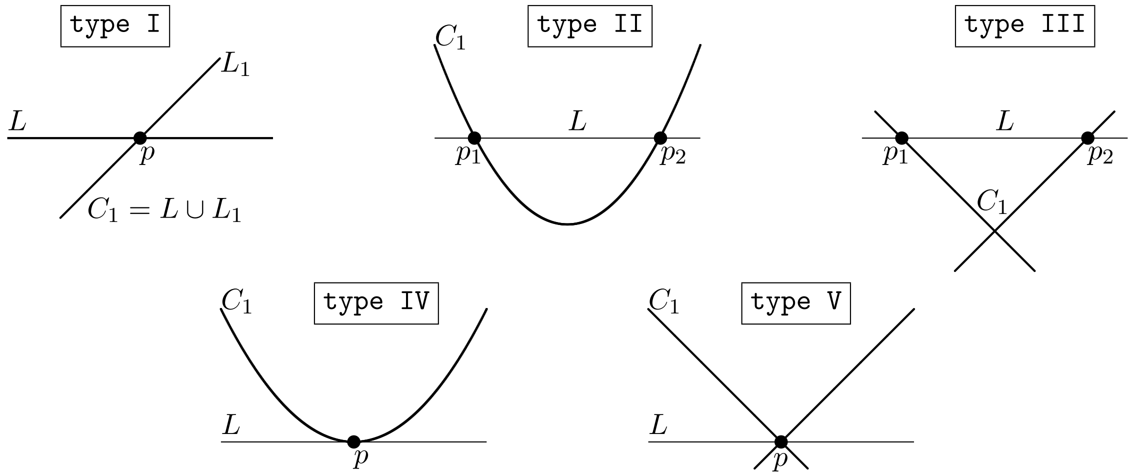
I) C_1 consist of two components one of which is the line L;

II) C_1 is smooth and $C_1 \cap L$ consists of two points;

III) C_1 is singular, i. e., has two components, and $C_1 \cap L$ consists of two points;

IV) C_1 is smooth and $C_1 \cap L$ consists of a single point, i. e., L is a tangent line to C_1;

V) C_1 is singular and $C_1 \cap L$ consists of a single point.



Proposition 2.16. *Each of the types of conics above is invariant under the action of the group of the invertible matrices* $\left(\begin{smallmatrix} \alpha & \beta & \gamma \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right)$.

Conics of the type I) lie in the same orbit of the action if and only if the intersection points of their components coincide.

Let C_1 and C_2 be two conics of the same type but not those of the type I). Then C_1 and C_2 lie in the same orbit if and only if their intersection sets with L coincide, i. e., if $C_1 \cap L = C_2 \cap L$.

Proof. Straightforward. □

2.2.2 Main result.

One can assume without loss of generality, that the coefficients A_{01}, A_{11}, and A_{12} in q_1 are zero. Indeed, applying the (affine) automorphism of X

$$X \ni \begin{pmatrix} z_1 & q_1 \\ z_2 & q_2 \end{pmatrix} \mapsto \begin{pmatrix} z_1 & q_1 \\ z_2 & q_2 \end{pmatrix} \begin{pmatrix} 1 & -(A_{01}x_0 + A_{11}x_1 + A_{12}x_2) \\ 0 & 1 \end{pmatrix},$$

which sends $A = \left(\begin{smallmatrix} x_1 & q_1 \\ x_2 & q_2 \end{smallmatrix}\right)$ to

$$\begin{pmatrix} x_1 & x_2(A_{02}x_0 + A_{22}x_2) \\ x_2 & q_2 - x_2(A_{01}x_0 + A_{11}x_1 + A_{12}x_2) \end{pmatrix},$$

one can always make A_{01}, A_{11} and A_{12} zero.

Proposition 2.17. *Let $\mathcal{E}_1$ and $\mathcal{E}_2$ be two equivalent sheaves constructed at the point A, $A_{01} = A_{11} = A_{12} = 0$ using directions*

$$B_1 = \begin{pmatrix} \xi_0 x_0 + \xi_1 x_1 + \xi_2 x_2 & \xi_{00}x_0^2 + \dots \xi_{22}x_2^2 \\ \eta_0 x_0 + \eta_1 x_1 + \eta_2 x_2 & \eta_{00}x_0^2 + \dots \eta_{22}x_2^2 \end{pmatrix}$$

and

$$B_2 = \begin{pmatrix} \mu_0 x_0 + \mu_1 x_1 + \mu_2 x_2 & \mu_{00}x_0^2 + \dots \mu_{22}x_2^2 \\ \nu_0 x_0 + \nu_1 x_1 + \nu_2 x_2 & \nu_{00}x_0^2 + \dots \nu_{22}x_2^2 \end{pmatrix}$$

respectively. We claim that B_1 and B_2 represent the same point in $\mathbb{P}N_A$, where $N_A = N_A(X_8) = T_A(X)/T_A(X_8)$ is the normal space at point A to X_8.

Proof. By Lemma 1.25 the restrictions of the sheaves $\mathcal{E}_1$ and $\mathcal{E}_2$ to D_1 are given by the matrices

$$A_1 = \begin{pmatrix} u_1 + \xi_0 u_0 & A_{02}u_2 + \xi_{00}u_0 \\ u_2 + \eta_0 u_0 & B_{01}u_1 + B_{02}u_2 + \eta_{00}u_0 \end{pmatrix},$$

and

$$A_2 = \begin{pmatrix} u_1 + \mu_0 u_0 & A_{02}u_2 + \mu_{00}u_0 \\ u_2 + \nu_0 u_0 & B_{01}u_1 + B_{02}u_2 + \nu_{00}u_0 \end{pmatrix}$$

respectively, i. e., we have the exact sequences

$$0 \to 2\mathcal{O}_{D_1}(-1) \xrightarrow{A_1} 2\mathcal{O}_{D_1} \to \mathcal{E}_1|_{D_1} \to 0$$

and

$$0 \to 2\mathcal{O}_{D_1}(-1) \xrightarrow{A_2} 2\mathcal{O}_{D_1} \to \mathcal{E}_2|_{D_1} \to 0.$$

The sheaf $\mathcal{E}_1|_{D_1}$ is supported on the curve given by the determinant

$$\begin{aligned} f_1 := \det(A_1) = & B_{01}u_1^2 - A_{02}u_2^2 + B_{02}u_1u_2 + (\xi_0\eta_{00} - \eta_0\xi_{00})u_0^2 + \\ & (\xi_0 B_{01} + \eta_{00})u_0u_1 + (\xi_0 B_{02} - \xi_{00} - \eta_0 A_{02})u_0u_2. \end{aligned}$$

The support of the sheaf $\mathcal{E}_2|_{D_1}$ is given by

$$\begin{aligned} f_2 := \det(A_2) = & B_{01}u_1^2 - A_{02}u_2^2 + B_{02}u_1u_2 + (\mu_0\nu_{00} - \nu_0\mu_{00})u_0^2 + \\ & (\mu_0 B_{01} + \nu_{00})u_0u_1 + (\mu_0 B_{02} - \mu_{00} - \nu_0 A_{02})u_0u_2. \end{aligned}$$

Let $\widetilde{\phi} : Z_0 \to Z_0$ be an isomorphism that is identical on $\widetilde{\mathbb{P}}_2$. Let $\mathcal{E}_2 \xrightarrow{\xi} \widetilde{\phi}^*(\mathcal{E}_1)$ be the isomorphism between $\mathcal{E}_2$ and $\widetilde{\phi}^*(\mathcal{E}_1)$. By Proposition 1.40 ξ can be uniquely lifted to the morphism of resolutions

$$\begin{tikzcd} 0 \arrow[r] & 2\mathcal{O}_{Z_0}(-H-F) \arrow[r, "\widetilde{A}_2"] \arrow[d, "\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)"] & \mathcal{O}_{Z_0}(-H) \oplus \mathcal{O}_{Z_0} \arrow[r] \arrow[d, "\left(\begin{smallmatrix} \bar a & \bar b \\ 0 & \bar d \end{smallmatrix}\right)"] & \mathcal{E}_2 \arrow[r] \arrow[d, "\xi"] & 0 \\ 0 \arrow[r] & 2\mathcal{O}_{Z_0}(-H-F) \arrow[r, "\widetilde{\phi}^*(\widetilde{A}_1)"] & \mathcal{O}_{Z_0}(-H) \oplus \mathcal{O}_{Z_0} \arrow[r] & \widetilde{\phi}^*(\mathcal{E}_1) \arrow[r] & 0, \end{tikzcd} \tag{2.2}$$

where $\bar b = \bar b_0 x_0 + \bar b_1 x_1 + \bar b_2 x_2$. Note that by Remark 1.42 both matrices $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} \bar a & \bar b \\ 0 & \bar d \end{smallmatrix}\right)$ are invertible.

Let $\phi := \widetilde{\phi}|_{\mathbb{P}_2} : \mathbb{P}_2 \to \mathbb{P}_2$. Since ϕ acts identically on L, it is given by

$$\langle u_0, u_1, u_2 \rangle \mapsto \langle (u_0, u_1, u_2) \left(\begin{smallmatrix} \alpha & \beta & \gamma \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right) \rangle.$$

We have also $\phi^*(\mathcal{E}_1|_{D_1}) \cong \mathcal{E}_2|_{D_1}$. Then, since isomorphic sheaves have the same support, it holds $\lambda f_2 = f_1 \circ \phi(u_0, u_1, u_2)$, $\lambda \in \mathbb{k}^*$. We have

$$\begin{aligned} & f_1 \circ \phi(u_0,u_1,u_2) = f_1(\alpha u_0, \beta u_0 + u_1, \gamma u_0 + u_2) = \\ & B_{01}(\beta u_0 + u_1)^2 - A_{02}(\gamma u_0 + u_2)^2 + B_{02}(\beta u_0 + u_1)(\gamma u_0 + u_2) + \\ & (\xi_0\eta_{00} - \eta_0\xi_{00})\alpha^2 u_0^2 + \\ & (\xi_0 B_{01} + \eta_{00})\alpha u_0(\beta u_0 + u_1) + (\xi_0 B_{02} - \xi_{00} - \eta_0 A_{02})\alpha u_0(\gamma u_0 + u_2) = \\ & B_{01}u_1^2 - A_{02}u_2^2 + B_{02}u_1u_2 + \\ & [B_{01}\beta^2 - A_{02}\gamma^2 + B_{02}\beta\gamma + (\xi_0\eta_{00} - \eta_0\xi_{00})\alpha^2 + \\ & (\xi_0 B_{01} + \eta_{00})\alpha\beta + (\xi_0 B_{02} - \xi_{00} - \eta_0 A_{02})\alpha\gamma]u_0^2 + \\ & (2B_{01}\beta + B_{02}\gamma + (\xi_0 B_{01} + \eta_{00})\alpha)u_0u_1 + \\ & (-2A_{02}\gamma + B_{02}\beta + (\xi_0 B_{02} - \xi_{00} - \eta_0 A_{02})\alpha)u_0u_2. \end{aligned}$$

Comparing the coefficients we obtain the following equations

$$\begin{cases} \lambda B_{01} = B_{01} \\ \lambda A_{02} = A_{02} \\ \lambda B_{02} = B_{02} \\ \lambda(\mu_0\nu_{00} - \nu_0\mu_{00}) = B_{01}\beta^2 - A_{02}\gamma^2 + B_{02}\beta\gamma + (\xi_0\eta_{00} - \eta_0\xi_{00})\alpha^2 + \\ \qquad\qquad + (\xi_0 B_{01} + \eta_{00})\alpha\beta + (\xi_0 B_{02} - \xi_{00} - \eta_0 A_{02})\alpha\gamma \\ \lambda(\mu_0 B_{01} + \nu_{00}) = 2B_{01}\beta + B_{02}\gamma + (\xi_0 B_{01} + \eta_{00})\alpha \\ \lambda(\mu_0 B_{02} - \mu_{00} - \nu_0 A_{02}) = -2A_{02}\gamma + B_{02}\beta + (\xi_0 B_{02} - \xi_{00} - \eta_0 A_{02})\alpha. \end{cases} \tag{2.3}$$

Consider the case $B_{01} = A_{02} = B_{02} = 0$. Then the above system of equations is equivalent to the system

$$\begin{cases} \lambda(\mu_0\nu_{00} - \nu_0\mu_{00}) = (\xi_0\eta_{00} - \eta_0\xi_{00})\alpha^2 + (\eta_{00})\alpha\beta + (-\xi_{00})\alpha\gamma \\ \lambda(\nu_{00}) = (\eta_{00})\alpha \\ \lambda(-\mu_{00}) = (-\xi_{00})\alpha. \end{cases}$$

In particular it follows that $\lambda\mu_{00} - \alpha\xi_{00} = 0$ and $\lambda\nu_{00} - \alpha\eta_{00} = 0$.

Since the tangent equations in the case $B_{01} = A_{02} = B_{02} = 0$ are just $\xi_{00} = \eta_{00} = 0$ (see Example 1.8, (1.13)), we conclude that $\lambda B_2 - \alpha B_1$ is a tangent vector to X_8 at A. Therefore, B_1 and B_2 represent the same vector in $\mathbb{P}N_A$.

We can assume now that at least one of the coefficients B_{01}, A_{02}, and B_{02} is not zero. Then in the system (2.3) $\lambda = 1$, and we can rewrite it in the form

$$\begin{cases} \mu_0\nu_{00} - \nu_0\mu_{00} = B_{01}\beta^2 - A_{02}\gamma^2 + B_{02}\beta\gamma + (\xi_0\eta_{00} - \eta_0\xi_{00})\alpha^2 + \\ \qquad\qquad + (\xi_0 B_{01} + \eta_{00})\alpha\beta + (\xi_0 B_{02} - \xi_{00} - \eta_0 A_{02})\alpha\gamma \\ 2B_{01}\beta + B_{02}\gamma = (\mu_0 B_{01} + \nu_{00}) - (\xi_0 B_{01} + \eta_{00})\alpha \\ B_{02}\beta - 2A_{02}\gamma = (\mu_0 B_{02} - \mu_{00} - \nu_0 A_{02}) - (\xi_0 B_{02} - \xi_{00} - \eta_0 A_{02})\alpha. \end{cases}$$

From the resolution

$$0 \to 2\mathcal{O}_{D_1}(-1) \xrightarrow{A_1} 2\mathcal{O}_{D_1} \to \mathcal{E}_1|_{D_1} \to 0$$

applying ϕ^* we obtain the exact sequence

$$0 \to 2\mathcal{O}_{D_1}(-1) \xrightarrow{\phi^*(A_1)} 2\mathcal{O}_{D_1} \to \phi^*(\mathcal{E}_1)|_{D_1} \to 0.$$

Restricting (2.2) to D_1 we obtain the following commutative diagram with exact rows

$$\begin{tikzcd} 0 \arrow[r] & 2\mathcal{O}_{D_1}(-1) \arrow[r, "A_2"] \arrow[d, "\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)"'] & 2\mathcal{O}_{D_1} \arrow[r] \arrow[d, "\left(\begin{smallmatrix} \bar a & \bar b_0 \\ 0 & \bar d \end{smallmatrix}\right)"] & \mathcal{E}_2|_{D_1} \arrow[r] \arrow[d, "\xi|_{D_1}"] & 0 \\ 0 \arrow[r] & 2\mathcal{O}_{D_1}(-1) \arrow[r, "\phi^*(A_1)"] & 2\mathcal{O}_{D_1} \arrow[r] & \phi^*(\mathcal{E}_1|_{D_1}) \arrow[r] & 0. \end{tikzcd} \tag{2.4}$$

Restricting this once more to L we will get the commutative square

$$\begin{tikzcd} 2\mathcal{O}_L(-1) \arrow[r, "\left(\begin{smallmatrix} u_1 & * \\ u_2 & * \end{smallmatrix}\right)"] \arrow[d, "\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)"'] & 2\mathcal{O}_L \arrow[d, "\left(\begin{smallmatrix} \bar a & \bar b_0 \\ 0 & \bar d \end{smallmatrix}\right)"] \\ 2\mathcal{O}_L(-1) \arrow[r, "\left(\begin{smallmatrix} u_1 & * \\ u_2 & * \end{smallmatrix}\right)"] & 2\mathcal{O}_L. \end{tikzcd}$$

Therefore, comparing the entries 1.1 and 2.1 we obtain

$$a = \bar{a} = d \neq 0, \quad b = c = 0.$$

Since

$$A_1 = \begin{pmatrix} u_1 + \xi_0 u_0 & A_{02}u_2 + \xi_{00}u_0 \\ u_2 + \eta_0 u_0 & B_{01}u_1 + B_{02}u_2 + \eta_{00}u_0 \end{pmatrix},$$

we obtain that

$$\phi^* A_1 = \begin{pmatrix} (u_1 + \beta u_0) + \xi_0 \alpha u_0 & A_{02}(u_2 + \gamma u_0) + \xi_{00}\alpha u_0 \\ (u_2 + \gamma u_0) + \eta_0 \alpha u_0 & B_{01}(u_1 + \beta u_0) + B_{02}(u_2 + \gamma u_0) + \eta_{00}\alpha u_0 \end{pmatrix} =$$
$$\begin{pmatrix} u_1 + (\beta + \xi_0\alpha)u_0 & A_{02}u_2 + (A_{02}\gamma + \xi_{00}\alpha)u_0 \\ u_2 + (\gamma + \eta_0\alpha)u_0 & B_{01}u_1 + B_{02}u_2 + (B_{01}\beta + B_{02}\gamma + \eta_{00}\alpha)u_0 \end{pmatrix}.$$

As

$$A_2 = \begin{pmatrix} u_1 + \mu_0 u_0 & A_{02}u_2 + \mu_{00}u_0 \\ u_2 + \nu_0 u_0 & B_{01}u_1 + B_{02}u_2 + \nu_{00}u_0 \end{pmatrix},$$

from the condition

$$\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \phi^*(A_1) = A_2 \begin{pmatrix} a & \bar{b}_0 \\ 0 & \bar{d} \end{pmatrix}$$

comparing the entries on the places 1.1 and 2.1 we get

$$\mu_0 = \beta + \xi_0\alpha \quad \text{and} \quad \nu_0 = \gamma + \eta_0\alpha.$$

Comparing the entries on the place 1.2 one obtains

$$a(A_{02}u_2 + (A_{02}\gamma + \xi_{00}\alpha)u_0) = \bar{b}_0(u_1 + \mu_0 u_0) + \bar{d}(A_{02}u_2 + \mu_{00}u_0)$$

and therefore

$$\bar{b}_0 = 0, \quad aA_{01} = \bar{d}A_{02}, \quad a(A_{02}\gamma + \xi_{00}\alpha) = \bar{d}\mu_{00}.$$

Comparing the entries on the place 2.2 and using $\bar{b}_0 = 0$ one obtains

$$a(B_{01}u_1 + B_{02}u_2 + (B_{01}\beta + B_{02}\gamma + \eta_{00}\alpha)u_0) = \bar{d}(B_{01}u_1 + B_{02}u_2 + \nu_{00}u_0)$$

and therefore

$$aB_{01} = \bar{d}B_{01}, \quad aB_{02} = \bar{d}B_{02}, \quad a(B_{01}\beta + B_{02}\gamma + \eta_{00}\alpha) = \bar{d}\nu_{00}.$$

Recall that we are considering now the case when at least one of the coefficients A_{02}, B_{01}, and B_{02} is different from zero. Therefore, we conclude that $a = \bar{d}$ and thus

$$\mu_{00} = A_{02}\gamma + \xi_{00}\alpha \quad \text{and} \quad \nu_{00} = B_{01}\beta + B_{02}\gamma + \eta_{00}\alpha.$$

We proved that

$$\begin{aligned} \mu_0 &= \beta + \xi_0\alpha \\ \nu_0 &= \gamma + \eta_0\alpha \\ \mu_{00} &= A_{02}\gamma + \xi_{00}\alpha \\ \nu_{00} &= B_{01}\beta + B_{02}\gamma + \eta_{00}\alpha \end{aligned}$$

Therefore,

$$\beta = \mu_0 - \xi_0\alpha, \quad \gamma = \nu_0 - \eta_0\alpha,$$
$$\mu_{00} - \xi_{00}\alpha = A_{02}\gamma = A_{02}(\nu_0 - \eta_0\alpha),$$
$$\nu_{00} - \eta_{00}\alpha = B_{01}\beta + B_{02}\gamma = B_{01}(\mu_0 - \xi_0\alpha) + B_{02}(\nu_0 - \eta_0\alpha).$$

The last two equalities mean that $B_2 - \alpha B_1 \in T_A(X_8)$, so B_1 and B_2 represent the same element in $\mathbb{P}N_A$. This proves Proposition 2.17. $\square$

Proposition 2.18. *Let B_1 and B_2 be two equivalent normal directions at the point $A \in X_8$, $A_{01} = A_{11} = 0$, i. e, B_1 and B_2 represent the same point in $\mathbb{P}N_A$. Then the sheaves $\mathcal{E}_1$ and $\mathcal{E}_2$ on Z_0 constructed along B_1 and B_2 respectively are equivalent.*

Proof. Let

$$B_1 = \begin{pmatrix} \xi_0 x_0 + \xi_1 x_1 + \xi_2 x_2 & \xi_{00} x_0^2 + \cdots + \xi_{22} x_2^2 \\ \eta_0 x_0 + \eta_1 x_1 + \eta_2 x_2 & \eta_{00} x_0^2 + \cdots + \eta_{22} x_2^2 \end{pmatrix}$$

and

$$B_2 = \begin{pmatrix} \mu_0 x_0 + \mu_1 x_1 + \mu_2 x_2 & \mu_{00} x_0^2 + \cdots + \mu_{22} x_2^2 \\ \nu_0 x_0 + \nu_1 x_1 + \nu_2 x_2 & \nu_{00} x_0^2 + \cdots + \nu_{22} x_2^2 \end{pmatrix}.$$

Since B_1 and B_2 define the same point in $\mathbb{P}N_A$ and since the tangent equations at A are

$$\begin{cases} \xi_{00} = A_{02}\eta_0 \\ \eta_{00} = B_{01}\xi_0 + B_{02}\eta_0, \end{cases}$$

it follows that

$$\begin{aligned} \mu_{00} - \xi_{00}\alpha &= A_{02}(\nu_0 - \eta_0\alpha) \\ \nu_{00} - \eta_{00}\alpha &= B_{01}(\mu_0 - \xi_0\alpha) + B_{02}(\nu_0 - \eta_0\alpha) \end{aligned}$$

for some $\alpha \in \mathbb{k}^*$.

Take

$$\beta = \mu_0 - \xi_0\alpha, \quad \gamma = \nu_0 - \eta_0\alpha,$$

and let

$$\phi = \left(\begin{smallmatrix} \alpha & \beta & \gamma \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right) : \mathbb{P}_2 \to \mathbb{P}_2, \quad \langle u_0, u_1, u_2 \rangle \mapsto \langle (u_0, u_1, u_2) \left(\begin{smallmatrix} \alpha & \beta & \gamma \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right) \rangle.$$

Let

$$\widetilde{A}_{B_1} = \begin{pmatrix} u_1 + \xi_0 u_0 & u_1 A_{12} x_2 + u_2(A_{02}x_0 + A_{22}x_2) + \xi_{00} x_0 u_0 \\ u_2 + \eta_0 u_0 & u_1(B_{01}x_0 + B_{11}x_1 + B_{12}x_2) + u_2(B_{02}x_0 + B_{22}x_2) + \eta_{00} x_0 u_0 \end{pmatrix},$$

and

$$\widetilde{A}_{B_2} = \begin{pmatrix} u_1 + \mu_0 u_0 & u_1 A_{12} x_2 + u_2(A_{02}x_0 + A_{22}x_2) + \mu_{00} x_0 u_0 \\ u_2 + \nu_0 u_0 & u_1(B_{01}x_0 + B_{11}x_1 + B_{12}x_2) + u_2(B_{02}x_0 + B_{22}x_2) + \nu_{00} x_0 u_0 \end{pmatrix}$$

be the matrices defining the sheaves $\mathcal{E}_1$ and $\mathcal{E}_2$. Consider $\widetilde{\phi} : Z_0 \to Z_0$ such that $\widetilde{\phi}|_{D_1} = \phi$ and $\widetilde{\phi}|_{D_0} = \mathrm{id}_{D_0}$. Then $\widetilde{\phi}^*(\widetilde{A}_{B_1})$ equals

$$\begin{pmatrix} (u_1+\beta u_0)+\xi_0\alpha u_0 & (u_1+\beta u_0)A_{12}x_2+(u_2+\gamma u_0)(A_{02}x_0+A_{22}x_2)+\xi_{00}\alpha x_0 u_0 \\ (u_2+\gamma u_0)+\eta_0\alpha u_0 & (u_1+\beta u_0)(B_{01}x_0+B_{11}x_1+B_{12}x_2)+(u_2+\gamma u_0)(B_{02}x_0+B_{22}x_2)+\eta_{00}\alpha x_0 u_0 \end{pmatrix} =$$

$$= \begin{pmatrix} u_1+(\beta+\xi_0\alpha)u_0 & u_1A_{12}x_2+u_2(A_{02}x_0+A_{22}x_2)+(\gamma A_{02}+\xi_{00}\alpha)x_0u_0 \\ u_2+(\gamma+\eta_0\alpha)u_0 & u_1(B_{01}x_0+B_{11}x_1+B_{12}x_2)+u_2(B_{02}x_0+B_{22}x_2)+(\beta B_{01}+\gamma B_{02}+\eta_{00}\alpha)x_0u_0 \end{pmatrix}.$$

Since we took

$$\beta = \mu_0 - \xi_0\alpha, \quad \gamma = \nu_0 - \eta_0\alpha,$$

we can rewrite the tangent equations as

$$\mu_{00} - \xi_{00}\alpha = A_{02}\gamma, \quad \nu_{00} - \eta_{00}\alpha = B_{01}\beta + B_{02}\gamma.$$

It follows that $\widetilde{\phi}^*(\widetilde{A}_{B_1}) = \widetilde{A}_{B_2}$. Therefore, there is an isomorphism $\widetilde{\phi}^*(\mathcal{E}_1) \cong \mathcal{E}_2$, which means that the sheaves $\mathcal{E}_1$ and $\mathcal{E}_2$ are equivalent. □

Theorem 2.19. *There is a one-to-one correspondence between the equivalence classes of R-bundles constructed at $A \in X_8$ and points of $\mathbb{P}N_A$.*

Proof. Follows from Proposition 2.17 and Proposition 2.18. □

2.2.3 R-bundles with singular curve C_1.

We have shown that the points of $\mathbb{P}N_A(X_8) \cong \mathbb{P}_1$ parameterize the equivalence classes of R-bundles at a point $A \in X_8$. For a fixed A we are going to investigate how many equivalence classes of R-bundles at A with singular curve C_1 there are. Recall that for an R-bundle $\mathcal{E}$ constructed at A we denote the support of the restriction $\mathcal{E}|_{D_1}$ by C_1.

Criterion for C_1 to be singular.

Proposition 2.20. *Let $A \in X_8$ be as in* (1.17) *and let $\mathcal{E}$ be an R-bundle constructed at A by the help of a normal direction $B \in T_AX \setminus T_AX_8$.*

Then the support of $\mathcal{E}|_{D_1}$ is a non-smooth conic (two lines) if and only if the direction B satisfies the equation

$$B_{01}T_1^2 - A_{02}T_2^2 + (B_{02} - A_{01})T_1T_2 = 0, \tag{2.5}$$

where $T_1 = \xi_{00} - A_{01}\xi_0 - A_{02}\eta_0$ and $T_2 = \eta_{00} - B_{01}\xi_0 - B_{02}\eta_0$.

Proof. From Lemma 1.25 we have that the support of $\widetilde{\mathcal{E}}|_{D_1}$ is given by the equation

$$\begin{gathered}\det\begin{pmatrix} u_1 + \xi_0 u_0 & A_{01}u_1 + A_{02}u_2 + \xi_{00}u_0 \\ u_2 + \eta_0 u_0 & B_{01}u_1 + B_{02}u_2 + \eta_{00}u_0 \end{pmatrix} = B_{01}u_1^2 - A_{02}u_2^2 + (B_{02} - A_{01})u_1u_2 + \\ (\xi_0\eta_{00} - \eta_0\xi_{00})u_0^2 + (\xi_0 B_{01} + \eta_{00} - \eta_0 A_{01})u_0u_1 + (\xi_0 B_{02} - \xi_{00} - \eta_0 A_{02})u_0u_2 = 0.\end{gathered} \tag{2.6}$$

The symmetric matrix corresponding to this quadratic form is

$$Q = \begin{pmatrix} \xi_0\eta_{00} - \eta_0\xi_{00} & \frac{1}{2}(\xi_0 B_{01} + \eta_{00} - \eta_0 A_{01}) & \frac{1}{2}(\xi_0 B_{02} - \xi_{00} - \eta_0 A_{02}) \\ \frac{1}{2}(\xi_0 B_{01} + \eta_{00} - \eta_0 A_{01}) & B_{01} & \frac{1}{2}(B_{02} - A_{01}) \\ \frac{1}{2}(\xi_0 B_{02} - \xi_{00} - \eta_0 A_{02}) & \frac{1}{2}(B_{02} - A_{01}) & -A_{02} \end{pmatrix}.$$

By Lemma A.9 we know that the conic (2.6) consists of two lines (or one doubled line) if and only if the determinant of the above matrix is zero. Straightforward calculations show that

$$-4\det Q = B_{01}T_1^2 - A_{02}T_2^2 + (B_{02} - A_{01})T_1T_2,$$

where $T_1 = \xi_{00} - A_{01}\xi_0 - A_{02}\eta_0$ and $T_2 = \eta_{00} - B_{01}\xi_0 - B_{02}\eta_0$ are tangent equations to X_8 at the point A (cf. (1.13)). This proves the required statement. $\square$

For fixed coefficients A_{01}, A_{02}, B_{01}, and B_{02} we can decompose (2.5) into linear factors:

$$B_{01}T_1^2 - A_{02}T_2^2 + (B_{02} - A_{01})T_1T_2 = (\alpha_1 T_1 + \beta_1 T_2)(\alpha_2 T_1 + \beta_2 T_2). \tag{2.7}$$

Note, that the intersection of the support of an R-bundle with the line L is given by the determinant of the matrix

$$\begin{pmatrix} u_1 & A_{01}u_1 + A_{02}u_2 \\ u_2 & B_{01}u_1 + B_{02}u_2 \end{pmatrix},$$

i. e., by the polynomial $B_{01}u_1^2 - A_{02}u_2^2 + (B_{02} - A_{01})u_1u_2$. There are three possible situations:

- $C_1 \cap L = L$;
- $C_1 \cap L$ consists of two points;
- $C_1 \cap L$ consists of a single point.

Let us investigate all these situations. Let C be the support of the singular $3m + 1$ sheaf given by A, i. e., the cubic curve in $\mathbb{P}_2$ given by the determinant of the matrix A. Let $p \in C$ be the point where the corresponding $3m + 1$ sheaf is not free on its support, i. e., the singular point of C of A given by the linear forms of A (cf. (1.6) and Lemma 1.2).

The case when $A_{02} = 0$, $B_{01} = 0$, and $B_{02} - A_{01} = 0$.

This is the case when $C_1 \cap L = L$, i. e., when the whole line L lies in the support of an R-bundle. One sees also that this is the case when $p \in C$ is a triple singular point of C. In other words p is an intersection of tree lines.

For all normal directions the curve C_1 consists in this case of two different lines one of which is L. So there are no R-bundles with smooth curve C_1 in this case.

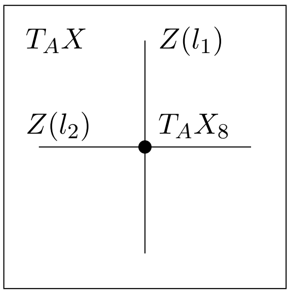
We are going now to consider the cases when at least one of the coefficients A_{02}, B_{01}, and $B_{02} - A_{01}$ is different from zero.

The case of non-zero discriminant.

Let us consider the case when the discriminant $\Delta = (B_{02} - A_{01})^2 + 4A_{01}B_{01}$ is non-zero. This means that the intersection of the support of an R-bundle with L consists of two points given by

$$B_{01}u_1^2 - A_{02}u_2^2 + (B_{02} - A_{01})u_1u_2 = 0.$$

Then the linear forms $l_1 = \alpha_1 T_1 + \beta_1 T_2$ and $l_2 = \alpha_2 T_1 + \beta_2 T_2$ in variables T_1 and T_2 from (2.7) are linear independent and we can consider them as tangent equations to X_8 at A. Then $T_A X_8 = Z(l_1) \cap Z(l_2)$. One can represent a zero set of (2.5) as a union $Z(l_1) \cup Z(l_2)$.



Let B and B' be two non-tangent to X_8 directions at the point A. Then they define the same point in $\mathbb{P}N_A(X_8)$ if and only if there exists a non-zero constant λ such that $B - \lambda B' \in T_A(X_8)$. The last condition is equivalent to the vanishing of both l_1 and l_2 on the matrix $B - \lambda B'$.

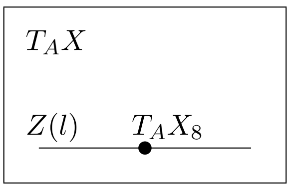
Assume that both B and B' belong to $Z(l_1)$. As we assumed them to be non-tangent, we have $l_2(B) \neq 0$ and $l_2(B') \neq 0$. Take $\lambda \neq 0$ such that $l_2(B) - \lambda l_2(B') = 0$, i. e., $\lambda = l_2(B)/l_2(B')$. Then $l_1(B - \lambda B') = 0$ and $l_2(B - \lambda B') = 0$, i. e., B and B' define the same point in $\mathbb{P}N_A(X_8)$. Analogously one shows that all the directions from $Z(l_2)$ define the same point in $\mathbb{P}N_A(X_8)$.

Let $B \in Z(l_1)$ and $B' \in Z(l_2)$ be two non-tangent to X_8 directions at the point A. Then they define the same point in $\mathbb{P}N_A(X_8)$ if and only if both l_1 and l_2 vanish on the matrix $B - \lambda B'$ for some non-zero λ. Since $B \in Z(l_1)$, we obtain $l_1(B - \lambda B') = -\lambda l_1(B')$. Since $B' \in Z(l_2)$, we get $l_2(B - \lambda B') = l_2(B)$. Finally we conclude that $B \in Z(l_1)$ and $B' \in Z(l_2)$ define the same point in $\mathbb{P}N_A(X_8)$ if and only if $l_1(B') = l_2(B) = 0$. But this is only possible for tangent directions B and B'. We obtained a contradiction and showed this way that all the normal directions to X_8 at the point A that give singular conics in $D_1 = \mathbb{P}_2$ give only two points in $\mathbb{P}N_A(X_8)$, one point corresponds to the component $Z(l_1)$ and the another one corresponds to $Z(l_2)$.

The case we have just considered is exactly the case when the point $p \in C$ is an ordinary double point singularity of C.

The case of zero discriminant.

In this case $B_{01}T_1^2 - A_{02}T_2^2 + (B_{02} - A_{01})T_1T_2 = l^2$ for a non-trivial linear form $l = \alpha T_1 + \beta T_2$.



This means that the curve C_1 is singular if and only if $l = 0$. The intersection of the support of an R-bundle with L consists of a single point given by $\alpha u_1 + \beta u_2 = 0$.

As at least one of α and β is different from zero, we can choose some T_{i_0} among T_1 and T_2 such that T_{i_0} and l are linear independent and we may consider them as tangent equations to X_8 at A. Take two normal directions $B_1, B_2 \in Z(l)$. Then $T_{i_0}(B_1) \neq 0$ and $T_{i_0}(B_2) \neq 0$. Therefore, for $\lambda = \frac{T_{i_0}(B_1)}{T_{i_0}(B_2)}$ we have that

$$l(B_1 - \lambda B_2) = 0 \quad \text{and} \quad T_{i_0}(B_1 - \lambda B_2) = 0,$$

which means that $B_1 - \lambda B_2$ is a tangent direction and thus B_1 and B_2 are equivalent normal directions.

We have shown that there is only one point in $\mathbb{P}N_A(X_8)$ with singular curve C_1.

The case of zero discriminant correspond to the case of a degenerated double point singularity $p \in C$.

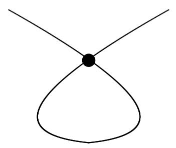
2.3 Examples

Using concrete examples we give here an illustration to Theorem 2.19.

For every singularity type of a cubic curve in $\mathbb{P}_2$ we fix a matrix $A \in X_8$ and consider the set of R-bundles constructed at A. Note that the intersection set of C_1 with L is completely described by the matrix A.

For a fixed A we describe the set of the equivalence classes of R-bundles with the curve C_1 of fixed type (see page 66). If the type of C_1 is different from I), by Proposition 2.16 it is enough to fix only one such curve (see also Definition 2.12).

2.3.1 Generic case.



$x_1^2 x_0 - x_2(x_2 + x_0) = 0$

Let us fix the matrix $A = \begin{pmatrix} x_1 & x_2(x_0+x_2) \\ x_2 & x_1x_0 \end{pmatrix}$. Let C be a curve in $\mathbb{P}_2$ given by the determinant of this matrix. This curve has an ordinary double point singularity. Then for all directions B the curve C_0 is given by the determinant of the matrix

$$\begin{pmatrix} u_1 & u_2(x_0+x_2) \\ u_2 & u_1x_0 \end{pmatrix}.$$

The intersection of C_0 with the line L is given by the equation $u_1^2 - u_2^2 = 0$ and consists of two points. Tangent equations at the point A are $\xi_{00} = \eta_0$ and $\eta_{00} = \xi_0$.

For a direction

$$B = \begin{pmatrix} \xi_0x_0 + \xi_1x_1 + \xi_2x_2 & \xi_{00}x_0^2 + \cdots + \xi_{22}x_2^2 \\ \eta_0x_0 + \eta_1x_1 + \eta_2x_2 & \eta_{00}x_0^2 + \cdots + \eta_{22}x_2^2 \end{pmatrix}$$

the restriction to D_1 of the corresponding sheaf is given by the matrix

$$\begin{pmatrix} u_1 + \xi_0u_0 & u_2 + \xi_{00}u_0 \\ u_2 + \eta_0u_0 & u_1 + \eta_{00}u_0 \end{pmatrix}.$$

The support of that sheaf on D_1 is then the curve given by the determinant of this matrix:

$$C_1 = \{u_1^2 - u_2^2 + (\xi_0\eta_{00} - \eta_0\xi_{00})u_0^2 + (\xi_0 + \eta_{00})u_0u_1 - (\eta_0 + \xi_{00})u_0u_2\}.$$

Remark 2.21 (comparison with [26])**.** *One sees that C_0 is a normalization of C (C_0 is a proper transform of C).*

If the conic C_1 is smooth, then it is isomorphic to $\mathbb{P}_1$ and thus the support of an R-bundle in this situation is a curve of type X_1 (see [26], pp. 212–213).

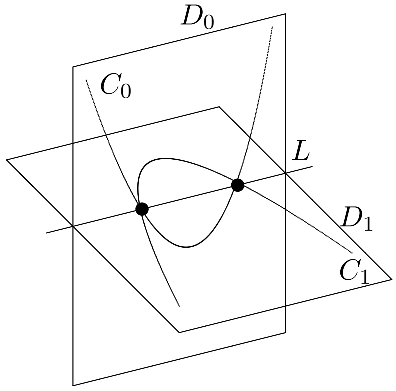
If C_1 is singular, then it is just a union of two lines and thus the whole support $C_0 \cup C_1$ is a curve of type X_2.

We conclude that R-bundles in the generic case are line bundles on the curves X_1 or X_2.

Smooth curve C_1.

We are going to describe the equivalence classes of those R-bundles constructed at the point A that have smooth curve C_1. By Proposition 2.16 it is enough to fix one such curve because the intersection of C_1 with L is completely described by the matrix A.

Let us fix some smooth conic section on $\mathbb{P}_2$ that intersects with L in the points $\{u_1^2 - u_2^2 = 0\}$, say $u_1^2 - u_2^2 - u_0u_2 = 0$, and let us see which directions B give us this conic section.



From the equations above we obtain

$$\begin{cases} \xi_0\eta_{00} - \eta_0\xi_{00} = 0, \\ \xi_0 + \eta_{00} = 0, \\ \eta_0 + \xi_{00} = 1, \end{cases}$$

thus $\xi_{00} = 1 - \eta_0$, $\eta_{00} = -\xi_0$, and $-\xi_0^2 - (1-\eta_0)\eta_0 = 0$. So we have two parameters ξ_0 and η_0 subject the relation

$$\xi_0^2 + (1-\eta_0)\eta_0 = 0.$$

We obtain that $C_1 = \{u_1^2 - u_2^2 - u_0u_1 = 0\}$ if and only if the matrix B is of the form

$$B = \begin{pmatrix} \xi x_0 + \xi_1 x_1 + \xi_2 x_2 & (1-\eta)x_0^2 + \cdots + \xi_{22}x_2^2 \\ \eta x_0 + \eta_1 x_1 + \eta_2 x_2 & -\xi x_0^2 + \cdots + \eta_{22}x_2^2 \end{pmatrix},$$

where the parameters ξ and η satisfy the equation

$$\xi^2 - \eta^2 + \eta = 0.$$

One can rewrite B as

$$B = \begin{pmatrix} 0 & x_0^2 \\ 0 & 0 \end{pmatrix} + \xi \begin{pmatrix} x_0 & 0 \\ 0 & -x_0^2 \end{pmatrix} + \eta \begin{pmatrix} 0 & -x_0^2 \\ x_0 & 0 \end{pmatrix} + \begin{pmatrix} \xi_1 x_1 + \xi_2 x_2 & \xi_{01}x_0x_1 + \cdots + \xi_{22}x_2^2 \\ \eta_1 x_1 + \eta_2 x_2 & \eta_{01}x_0x_1 + \cdots + \eta_{22}x_2^2 \end{pmatrix}.$$

This means that the directions which give us the curve $\{u_1^2 - u_2^2 - u_0u_2 = 0\}$ is a hypersurface in some 16-dimensional affine subspace of T_AX that is complementary to T_AX_8.

For a direction B with the curve $u_1^2 - u_2^2 - u_0u_2 = 0$ the matrix on Z_0 is

$$\widetilde{A}_B = \begin{pmatrix} u_1 + \xi u_0 & u_2(x_0 + x_2) + (1-\eta)u_0x_0 \\ u_2 + \eta u_0 & u_1x_0 - \xi u_0x_0 \end{pmatrix}.$$

In particular this means that for fixed parameters ξ and η we always obtain the same matrix and thus the same sheaf.

Suppose that two matrices

$$\widetilde{A}_B = \begin{pmatrix} u_1 + \xi u_0 & u_2(x_0 + x_2) + (1-\eta)u_0x_0 \\ u_2 + \eta u_0 & u_1x_0 - \xi u_0x_0 \end{pmatrix}$$

and

$$\widetilde{A}_{B'} = \begin{pmatrix} u_1 + \xi' u_0 & u_2(x_0 + x_2) + (1-\eta')u_0x_0 \\ u_2 + \eta' u_0 & u_1x_0 - \xi' u_0x_0 \end{pmatrix}$$

define isomorphic sheaves. Then by Proposition 1.40 and Remark 1.42 there is a commutative diagram

$$\begin{tikzcd} \cdot \arrow[r, "\widetilde{A}_B"] \arrow[d, "\begin{pmatrix} a & b \\ c & d \end{pmatrix}"'] & \cdot \arrow[d, "\begin{pmatrix} \lambda & z \\ 0 & \mu \end{pmatrix}"] \\ \cdot \arrow[r, "\widetilde{A}_{B'}"'] & \cdot \end{tikzcd}$$

with invertible vertical arrows. Therefore, $\widetilde{A}_B \begin{pmatrix} \lambda & z \\ 0 & \mu \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \widetilde{A}_{B'}$. Comparing the entries on the place 1.1 we obtain

$$\lambda(u_1 + \xi u_0) = a(u_1 + \xi' u_0) + b(u_2 + \eta' u_0).$$

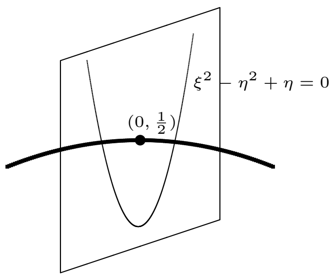
Thus $b = 0$, $a = \lambda \neq 0$, $\xi = \xi'$.

Comparing the entries on the place 2.1 we obtain

$$\lambda(u_2 + \eta u_0) = c(u_1 + \xi' u_0) + d(u_2 + \eta' u_0).$$

Thus $c = 0$, $\lambda = d = a$, $\eta = \eta'$.

We obtained that two matrices B and B' define isomorphic sheaves if and only if $\xi = \xi'$ and $\eta = \eta'$. That is why we can interpret the curve $\xi^2 - \eta^2 + \eta = 0$ in the normal plane $\Bbbk \cdot \begin{pmatrix} x_0 & 0 \\ 0 & -x_0^2 \end{pmatrix} + \Bbbk \cdot \begin{pmatrix} 0 & -x_0^2 \\ x_0 & 0 \end{pmatrix} \cong \Bbbk^2$ as the set of all isomorphism classes of the sheaves supported on $C_0 \cup C_1$, $C_1 = \{u_1^2 - u_2^2 - u_0 u_2 = 0\}$, that we obtain at the point $A = \begin{pmatrix} x_1 & x_2(x_0 + x_2) \\ x_2 & x_1 x_0 \end{pmatrix}$.



The group of invertible matrices $\left(\begin{smallmatrix} \alpha & \beta & \gamma \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right)$ acts on the set of conics in $\mathbb{P}_2$. Let us calculate the stabilizer of the conic $C_1 = u_1^2 - u_2^2 - u_0 u_2 = 0$. An element $\left(\begin{smallmatrix} \alpha & \beta & \gamma \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right)$ sends C_1 into the curve given by the polynomial

$$(\beta u_0 + u_1)^2 - (\gamma u_0 + u_2)^2 - \alpha u_0(\gamma u_0 + u_2) = u_1^2 - u_2^2 + (\beta^2 - \gamma^2 - \alpha\gamma) u_0^2 + 2\beta u_0 u_1 - (2\gamma + \alpha) u_0 u_2.$$

To obtain the same curve C_1 the equalities $\beta^2 - \gamma^2 - \alpha\gamma = 0$, $2\beta = 0$, and $2\gamma + \alpha = 1$ should hold true (we compare the coefficients). Thus either $\gamma = 0$ and $\alpha = 1$, i. e., $\left(\begin{smallmatrix} \alpha & \beta & \gamma \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right) = \mathrm{id}$ or $\gamma \neq 0$, $\alpha = -\gamma$, $\gamma = 1$, i. e., $\left(\begin{smallmatrix} \alpha & \beta & \gamma \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} -1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right)$.

Suppose that two matrices

$$\widetilde{A}_B = \begin{pmatrix} u_1 + \xi u_0 & u_2(x_0 + x_2) + (1 - \eta) u_0 x_0 \\ u_2 + \eta u_0 & u_1 x_0 - \xi u_0 x_0 \end{pmatrix}$$

and

$$\widetilde{A}_{B'} = \begin{pmatrix} u_1 + \xi' u_0 & u_2(x_0 + x_2) + (1 - \eta') u_0 x_0 \\ u_2 + \eta' u_0 & u_1 x_0 - \xi' u_0 x_0 \end{pmatrix}$$

define equivalent sheaves. Then either $\xi = \xi'$ and $\eta = \eta'$, i. e., the sheaves are isomorphic, or they are not isomorphic but there is an isomorphism of the sheaves defined by $\phi^* \widetilde{A}_B$ and $\widetilde{A}_{B'}$, where $\phi = \left(\begin{smallmatrix} -1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right)$. As

$$\phi^* \widetilde{A}_B = \begin{pmatrix} u_1 - \xi u_0 & u_2(x_0 + x_2) + \eta u_0 x_0 \\ u_2 + (1-\eta) u_0 & u_1 x_0 + \xi u_0 x_0 \end{pmatrix}$$

by the considerations above this means $\xi' = -\xi$ and $\eta' = 1 - \eta$. So for a fixed equivalence class there are exactly two points on the curve $\xi^2 - \eta^2 + \eta = 0$ defining that class. We claim that for a point (ξ, η) from the curve above the second point defining the same equivalence class is just the point on the curve defining the same normal direction. Indeed, let us fix a point (ξ, η) on the curve $\xi^2 - \eta^2 + \eta = 0$, and let us see what points on this curve define the same normal direction. Let (ξ', η') be such a point. Then

$$B = \begin{pmatrix} \xi x_0 + \dots & (1-\eta) x_0^2 + \dots \\ \eta x_0 + \dots & -\xi x_0^2 + \dots + \end{pmatrix},$$

$$B' = \begin{pmatrix} \xi' x_0 + \dots & (1-\eta') x_0^2 + \dots \\ \eta' x_0 + \dots & -\xi' x_0^2 + \dots + \end{pmatrix},$$

and $B - \alpha B' \in T_A X$ for some $\alpha \in \Bbbk^*$. As

$$B - \alpha B' = \begin{pmatrix} (\xi - \alpha\xi') x_0 + \dots & (1 - \eta - \alpha(1-\eta')) x_0^2 + \dots \\ (\eta - \alpha\eta') x_0 + \dots & (-\xi - (-\alpha\xi')) x_0^2 + \dots + \end{pmatrix},$$

using tangent equations at A we get $\xi - \alpha\xi' = 0$ and $2(\eta - \alpha\eta') = 1 - \alpha$. Thus $\xi = \alpha\xi'$ and $\eta = \frac{1-\alpha}{2} + \alpha\eta'$. Using $\xi^2 - \eta^2 + \eta = \xi'^2 - \eta'^2 + \eta' = 0$ we obtain

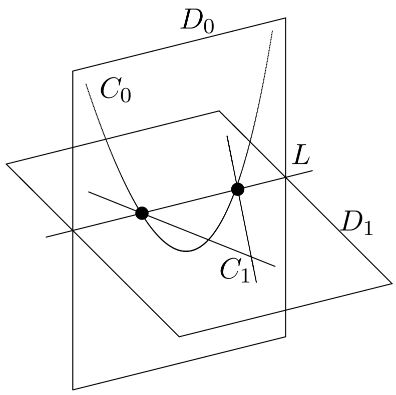
$$\begin{aligned} 0 =& \alpha^2 \xi'^2 - \left(\frac{1-\alpha}{2} + \alpha\eta'\right)^2 + \frac{1-\alpha}{2} + \alpha\eta' = \\ & \alpha^2\xi'^2 - \alpha^2\eta'^2 + \alpha^2\eta' + \frac{1-\alpha}{2} - \left(\frac{1-\alpha}{2}\right)^2 = \frac{1-\alpha}{2} - \left(\frac{1-\alpha}{2}\right)^2. \end{aligned}$$

Therefore, either $\alpha = 1$ or $\alpha = -1$. This means that either $(\xi, \eta) = (\xi', \eta')$ or $(\xi, \eta) = (-\xi', 1 - \eta')$.

Singular curve C_1.

Let us fix some singular (reducible) curve through the points $u_1 = u_2$ and $u_1 = -u_2$ on the line $L = \{u_0 = 0\}$, say

$$u_1^2 + u_0^2 + 2u_0u_1 - u_2^2 = (u_1 - u_2 + u_0)(u_1 + u_2 + u_0) = 0.$$



To obtain this curve the following equations should hold true:

$$\begin{cases} \xi_0\eta_{00} - \eta_0\xi_{00} = 1, \\ \xi_0 + \eta_{00} = 2, \\ \eta_0 + \xi_{00} = 0. \end{cases}$$

So $\eta_{00} = 2 - \xi_0$, $\xi_{00} = -\eta_0$, and $\xi_0(2 - \xi_0) + \eta_0^2 = 1$. Therefore,

$$B = \begin{pmatrix} \xi x_0 + \xi_1 x_1 + \xi_2 x_2 & -\eta x_0^2 + \cdots + \xi_{22}x_2^2 \\ \eta x_0 + \eta_1 x_1 + \eta_2 x_2 & (2-\xi)x_0^2 + \cdots + \eta_{22}x_2^2 \end{pmatrix},$$

where the parameters ξ and η satisfy the equation $\xi(2-\xi)+\eta^2-1 = (\eta-\xi+1)(\eta+\xi-1) = 0$, is the general form of the normal directions that give us the curve C_1. Of course we need $\eta \neq 0$, because otherwise, B is a tangent direction.

One can rewrite

$$B = \begin{pmatrix} 0 & 0 \\ 0 & 2x_0^2 \end{pmatrix} + \xi \begin{pmatrix} x_0 & 0 \\ 0 & -x_0^2 \end{pmatrix} + \eta \begin{pmatrix} 0 & -x_0^2 \\ x_0 & 0 \end{pmatrix} + \begin{pmatrix} \xi_1 x_1 + \xi_2 x_2 & \xi_{01}x_0x_1 + \cdots + \xi_{22}x_2^2 \\ \eta_1 x_1 + \eta_2 x_2 & \eta_{01}x_0x_1 + \cdots + \eta_{22}x_2^2 \end{pmatrix}.$$

This means that the set of directions which give us the curve $\{u_1^2 - u_2^2 + 2u_0u_2 + u_0^2 = 0\}$ is a hypersurface (two hyperplanes) in some 16-dimensional affine normal subspace of T_AX.

For a direction B with the curve $u_1^2 - u_2^2 + 2u_0u_2 + u_0^2 = 0$ the matrix on Z_0 is

$$\widetilde{A}_B = \begin{pmatrix} u_1 + \xi u_0 & u_2(x_0 + x_2) - \eta u_0 x_0 \\ u_2 + \eta u_0 & u_1 x_0 + (2-\xi)u_0 x_0 \end{pmatrix}.$$

In particular this means that for fixed parameters ξ and η we always obtain the same matrix and thus the same sheaf.

Suppose that two matrices

$$\widetilde{A}_B = \begin{pmatrix} u_1 + \xi u_0 & u_2(x_0 + x_2) - \eta u_0 x_0 \\ u_2 + \eta u_0 & u_1 x_0 + (2-\xi)u_0 x_0 \end{pmatrix}$$

and

$$\widetilde{A}_{B'} = \begin{pmatrix} u_1 + \xi' u_0 & u_2(x_0 + x_2) - \eta' u_0 x_0 \\ u_2 + \eta' u_0 & u_1 x_0 + (2-\xi')u_0 x_0 \end{pmatrix}$$

define isomorphic sheaves. Then by Proposition 1.40 and Remark 1.42 there is a commutative diagram

$$\begin{tikzcd} \cdot \arrow[r, "\widetilde{A}_B"] \arrow[d, "\begin{pmatrix} a & b \\ c & d \end{pmatrix}"'] & \cdot \arrow[d, "\begin{pmatrix} \lambda & z \\ 0 & \mu \end{pmatrix}"] \\ \cdot \arrow[r, "\widetilde{A}_{B'}"'] & \cdot \end{tikzcd}$$

with invertible vertical arrows. Therefore, $\widetilde{A}_B \begin{pmatrix} \lambda & z \\ 0 & \mu \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \widetilde{A}_{B'}$. Comparing the entries on the place 1.1 we obtain

$$\lambda(u_1 + \xi u_0) = a(u_1 + \xi' u_0) + b(u_2 + \eta' u_0).$$

Thus $b = 0$, $a = \lambda \neq 0$, $\xi = \xi'$.

Comparing the entries on the place 2.1 we obtain

$$\lambda(u_2 + \eta u_0) = c(u_1 + \xi' u_0) + d(u_2 + \eta' u_0).$$

Thus $c = 0$, $\lambda = d = a$, $\eta = \eta'$. We obtained again that two matrices B and B' define isomorphic sheaves if and only if $\xi = \xi'$ and $\eta = \eta'$.

Let us calculate the stabilizer of the conic $C_1 = \{u_1^2 - u_2^2 + u_0^2 + 2u_0u_1 = 0\}$. An element $\left(\begin{smallmatrix} \alpha & \beta & \gamma \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right)$ sends C_1 into the curve given by the polynomial

$$\begin{gathered}(\beta u_0 + u_1)^2 - (\gamma u_0 + u_2)^2 + \alpha^2 u_0 + 2\alpha u_0(\beta u_0 + u_1) = \\ u_1^2 - u_2^2 + (\beta^2 - \gamma^2 + \alpha^2 + 2\alpha\beta)u_0^2 + (2\beta + 2\alpha)u_0u_1 - 2\gamma u_0 u_2.\end{gathered}$$

To obtain the same curve, we have the equations $-2\gamma = 0$, $2\beta + 2\alpha = 2$, $\beta^2 - \gamma^2 + \alpha^2 + 2\alpha\beta = 1$. Therefore, $\beta = 1 - \alpha$ and $\gamma = 0$. So the stabilizer of C_1 is the group

$$\left\{ \left(\begin{smallmatrix} \alpha & 1-\alpha & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right) \mid \alpha \in \Bbbk^* \right\}$$

Suppose that two matrices

$$\widetilde{A}_B = \begin{pmatrix} u_1 + \xi u_0 & u_2(x_0 + x_2) - \eta u_0 x_0 \\ u_2 + \eta u_0 & u_1 x_0 + (2 - \xi) u_0 x_0 \end{pmatrix}$$

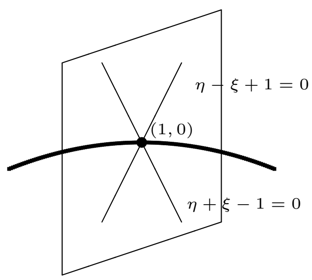
and

$$\widetilde{A}_{B'} = \begin{pmatrix} u_1 + \xi' u_0 & u_2(x_0 + x_2) - \eta' u_0 x_0 \\ u_2 + \eta' u_0 & u_1 x_0 + (2 - \xi') u_0 x_0 \end{pmatrix}$$

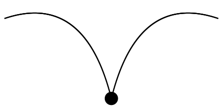
define equivalent sheaves. Then either $\xi = \xi'$ and $\eta = \eta'$, i. e., the sheaves are isomorphic, or they are not isomorphic but there is an isomorphism of the sheaves defined by $\phi^* \widetilde{A}_B$ and $\widetilde{A}_{B'}$, where $\phi = \left(\begin{smallmatrix} \alpha & 1-\alpha & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right)$. As

$$\phi^* \widetilde{A}_B = \begin{pmatrix} u_1 + (\alpha\xi + 1 - \alpha) u_0 & u_2(x_0 + x_2) - \alpha\eta u_0 x_0 \\ u_2 + \alpha\eta u_0 & u_1 x_0 + \alpha(2 - \xi) u_0 x_0 + (1 - \alpha) u_0 x_0 \end{pmatrix},$$

by the considerations above $\alpha\xi + 1 - \alpha = \xi'$ and $\alpha\eta = \eta'$. One easily sees that this holds if and only if the points (ξ, η) and (ξ', η') lies on the same line through the point $(1, 0)$. Thus we showed that the points the curve $(\eta - \xi + 1)(\eta + \xi - 1) = 0$ define the same equivalence class of R-bundles if and only if they belong to the same component of this curve. But to lie in the same components (hyperplanes) is the same as to define the same normal direction.



2.3.2 The case of a cuspidal curve.



Let us fix the matrix $A = \begin{pmatrix} x_1 & x_2^2 \\ x_2 & x_1x_0 \end{pmatrix}$. Then for all directions B the curve C_0 is given by the determinant of $\begin{pmatrix} u_1 & u_2x_2 \\ u_2 & u_1x_0 \end{pmatrix}$. The intersection of C_0 with the line L is given by the equation $u_1^2 = 0$ and consists of a single point. Tangent equations at the point A are $\xi_{00} = 0$ and $\eta_{00} = \xi_0$.

For a direction

$$B = \begin{pmatrix} \xi_0x_0 + \xi_1x_1 + \xi_2x_2 & \xi_{00}x_0^2 + \cdots + \xi_{22}x_2^2 \\ \eta_0x_0 + \eta_1x_1 + \eta_2x_2 & \eta_{00}x_0^2 + \cdots + \eta_{22}x_2^2 \end{pmatrix}$$

the restriction of the corresponding sheaf to D_1 is given by the matrix

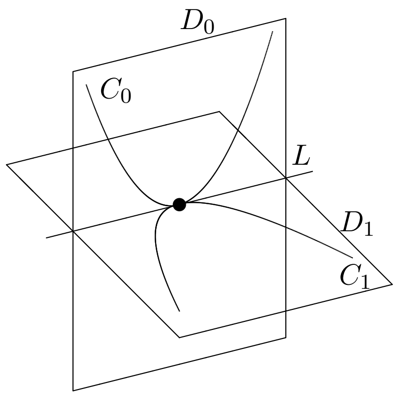
$$\begin{pmatrix} u_1 + \xi_0u_0 & \xi_{00}u_0 \\ u_2 + \eta_0u_0 & u_1 + \eta_{00}u_0 \end{pmatrix}.$$

The support of that sheaf on D_1 is then the curve given by the determinant of this matrix:

$$C_1 = \{u_1^2 + (\xi_0\eta_{00} - \eta_0\xi_{00})u_0^2 + (\xi_0 + \eta_{00})u_0u_1 - \xi_{00}u_0u_2 = 0\}.$$

Smooth curve C_1.

Let us fix some smooth conic section on $\mathbb{P}_2$ which intersects with L at the points $\{u_1 = 0\}$, say $u_1^2 - u_0u_2 = 0$, and let us see which directions B give us this conic section.



We obtain then the equations

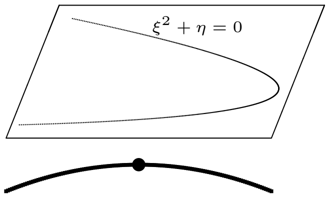
$$\begin{cases} \xi_0\eta_{00} - \eta_0\xi_{00} = 0, \\ \xi_0 + \eta_{00} = 0, \\ -\xi_{00} = -1. \end{cases}$$

Thus $\xi_{00} = 1$, $\eta_{00} = -\xi_0$, and $-\xi_0^2 - \eta_0 = 0$.

We obtained that $C_1 = \{u_1^2 - u_0u_2 = 0\}$ if and only if the matrix B is of the form

$$B = \begin{pmatrix} \xi x_0 + \xi_1x_1 + \xi_2x_2 & x_0^2 + \cdots + \xi_{22}x_2^2 \\ \eta x_0 + \eta_1x_1 + \eta_2x_2 & -\xi x_0^2 + \cdots + \eta_{22}x_2^2 \end{pmatrix},$$

where the parameters ξ and η satisfy the equation $\xi^2 + \eta = 0$.



For a direction B with the curve $u_1^2 - u_0u_2 = 0$ the matrix on Z_0 is

$$\widetilde{A}_B = \begin{pmatrix} u_1 + \xi u_0 & u_2x_2 + u_0x_0 \\ u_2 + \eta u_0 & u_1x_0 - \xi u_0x_0 \end{pmatrix}.$$

Absolutely analogously as above in the generic case one sees that the points from the curve $\xi^2 + \eta = 0$ define different isomorphism classes of sheaves.

Since an element $\left(\begin{smallmatrix} \alpha & \beta & \gamma \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right)$ sends C_1 into the curve given by the polynomial

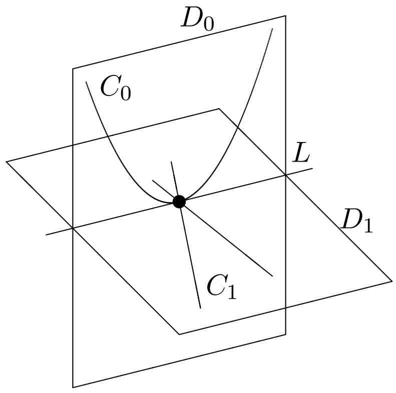
$$(\beta u_0 + u_1)^2 - \alpha u_0(\gamma u_0 + u_2) = u_1^2 + 2\beta u_0u_1 + (\beta^2 - \alpha\gamma)u_0^2 - \alpha u_0u_2,$$

to obtain the same curve, we have the equations $2\beta = 0$, $\beta^2 - \alpha\gamma = 0$, and $-\alpha = -1$. Therefore, the stabilizer of C_1 is trivial, i.e., consists only of the identity matrix. This implies that the curve $\xi^2 + \eta = 0$ in $\Bbbk^2$ parameterizes the equivalence classes of sheaves.

One easily sees that different points on the curve $\xi^2 + \eta = 0$ correspond to different normal directions.

Singular curve C_1.

We fix here a singular curve C_1 through the point $u_1 = 0$ at the line L, say $C_1 = \{u_1^2 - u_0^2 = (u_1 - u_0)(u_1 + u_0) = 0\}$.



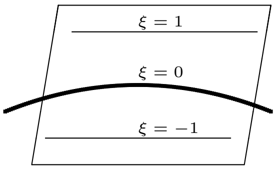
This is possible if and only if

$$\begin{cases} \xi_0\eta_{00} - \eta_0\xi_{00} = -1, \\ \xi_0 + \eta_{00} = 0, \\ -\xi_{00} = 0. \end{cases}$$

Therefore,

$$B = \begin{pmatrix} \xi x_0 + \xi_1x_1 + \xi_2x_2 & \xi_{01}x_0x_1 + \cdots + \xi_{22}x_2^2 \\ \eta x_0 + \eta_1x_1 + \eta_2x_2 & -\xi x_0^2 + \cdots + \eta_{22}x_2^2 \end{pmatrix},$$

where $\xi^2 = 1$, is the general form of normal direction defining a sheaf with the curve $C_1 = \{u_1^2 - u_0^2 = 0\}$.



For a direction B with the curve $u_1^2 - u_0^2 = 0$ the matrix on Z_0 is

$$\widetilde{A}_B = \begin{pmatrix} u_1 + \xi u_0 & u_2 x_2 \\ u_2 + \eta u_0 & u_1 x_0 - \xi u_0 x_0 \end{pmatrix}.$$

As above, we see that different points with $\xi^2 = 1$ define different isomorphism classes of sheaves. Let us have a look at the equivalence classes.

First of all we again need to compute the stabilizer group of the curve $u_1^2 - u_0^2 = 0$. Since the matrix $\left(\begin{smallmatrix} \alpha & \beta & \gamma \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right)$ sends the given curve into the curve

$$(\beta u_0 + u_1)^2 - \alpha^2 u_0^2 = u_1^2 + (\beta^2 - \alpha^2) u_0^2 + 2\beta u_0 u_1 = 0,$$

to get the same curve we get the equations $\beta = 0$ and $\alpha^2 = 1$. Thus the stabilizer group is

$$\left\{ \left(\begin{smallmatrix} 1 & 0 & \gamma \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right) \mid \gamma \in \Bbbk \right\} \cup \left\{ \left(\begin{smallmatrix} -1 & 0 & \gamma \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right) \mid \gamma \in \Bbbk \right\}.$$

Two matrices

$$\widetilde{A}_B = \begin{pmatrix} u_1 + \xi u_0 & u_2 x_2 \\ u_2 + \eta u_0 & u_1 x_0 - \xi u_0 x_0 \end{pmatrix}$$

and

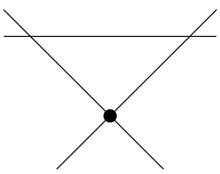
$$\widetilde{A}_{B'} = \begin{pmatrix} u_1 + \xi' u_0 & u_2 x_2 \\ u_2 + \eta' u_0 & u_1 x_0 - \xi' u_0 x_0 \end{pmatrix}$$

define equivalent sheaves if and only if there exists $\phi = \left(\begin{smallmatrix} \alpha & 0 & \gamma \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right)$, $\alpha^2 = 1$, such that the sheaves given by $\widetilde{A}_{B'}$ and $\phi^*(\widetilde{A}_B)$ are isomorphic. Since

$$\phi^*(\widetilde{A}_B) = \begin{pmatrix} u_1 + \xi \alpha u_0 & (\gamma u_0 + u_2) x_2 \\ \gamma u_0 + u_2 + \eta \alpha u_0 & u_1 x_0 - \xi \alpha u_0 x_0 \end{pmatrix} = \begin{pmatrix} u_1 + \xi \alpha u_0 & u_2 x_2 \\ u_2 + (\eta \alpha + \gamma) u_0 & u_1 x_0 - \xi \alpha u_0 x_0 \end{pmatrix},$$

the latter means $\xi' = \alpha \xi$ and $\eta' = \eta \alpha + \gamma$. But for each (ξ, η) and (ξ', η') with $\xi^2 = \xi'^2 = 1$ there exists $\alpha = \xi'/\xi$ and $\gamma = \eta' - \alpha \eta$ for which the equations hold. Therefore, all the sheaves with the curve $C_1 = \{u_1^2 - u_0^2 = 0\}$ are equivalent. One also sees that the points with $\xi^2 = 1$ define the same normal direction.

2.3.3 Three lines with simple intersections.



$x_0 x_1 x_2 = 0$

We start here from the matrix $A = \begin{pmatrix} x_1 & 0 \\ x_2 & x_2 x_0 \end{pmatrix}$. The tangent equations are in this case $\xi_{00} = 0$ and $\eta_{00} = \eta_0$. For a general direction

$$B = \begin{pmatrix} \xi_0 x_0 + \xi_1 x_1 + \xi_2 x_2 & \xi_{00} x_0^2 + \cdots + \xi_{22} x_2^2 \\ \eta_0 x_0 + \eta_1 x_1 + \eta_2 x_2 & \eta_{00} x_0^2 + \cdots + \eta_{22} x_2^2 \end{pmatrix}$$

we obtain a sheaf on Z_0 given by the matrix

$$\begin{pmatrix} u_1 + \xi_0 u_0 & \xi_{00} u_0 x_0 \\ u_2 + \eta_0 u_0 & u_2 u_0 + \eta_{00} u_0 x_0 \end{pmatrix}.$$

The restriction to $\widetilde{\mathbb{P}}_2$ is given by $\begin{pmatrix} u_1 & 0 \\ u_2 & u_2x_0 \end{pmatrix}$ and is supported on the curve $u_1u_2x_0 = 0$.

The restriction to D_1 is given by the matrix

$$\begin{pmatrix} u_1 + \xi_0 u_0 & \xi_{00}u_0 \\ u_2 + \eta_0 u_0 & u_2 + \eta_{00}u_0 \end{pmatrix}.$$

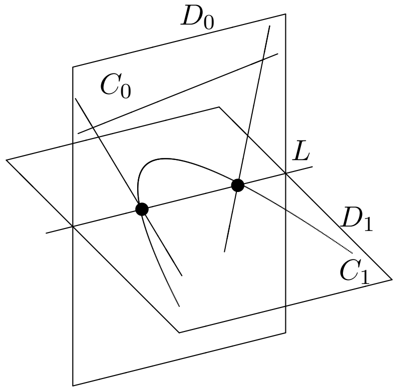
Its support on D_1 is the curve C_1 given by the polynomial

$$u_1u_2 + (\xi_0\eta_{00} - \eta_0\xi_{00})u_0^2 + \eta_{00}u_0u_1 + (\xi_0 - \xi_{00})u_0u_2.$$

The intersection of the support of the sheaf with the line L consists of two points $u_1 = 0$ and $u_2 = 0$.

Smooth curve C_1.

Let us fix some smooth curve C_1, say $C_1 = \{u_1u_2 + u_0^2 = 0\}$.



We obtain this curve if and only if

$$\begin{cases} \xi_0\eta_{00} - \eta_0\xi_{00} = 1, \\ \xi_0 - \xi_{00} = 0, \\ \eta_{00} = 0, \end{cases}$$

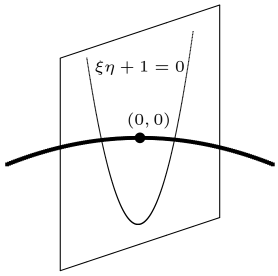
or, equivalently, if and only if

$$B = \begin{pmatrix} \xi x_0 + \xi_1 x_1 + \xi_2 x_2 & \xi x_0^2 + \cdots + \xi_{22}x_2^2 \\ \eta x_0 + \eta_1 x_1 + \eta_2 x_2 & \eta_{01}x_0x_1 + \cdots + \eta_{22}x_2^2 \end{pmatrix}$$

with $\xi\eta = -1$. For a direction B with the curve $u_1u_2 + u_0^2 = 0$ the matrix on Z_0 is

$$\widetilde{A}_B = \begin{pmatrix} u_1 + \xi u_0 & \xi u_0 x_0 \\ u_2 + \eta u_0 & u_2 x_0 \end{pmatrix}.$$

Thus we obtain a "curve" $\xi\eta = -1$ in $\Bbbk^2$ of R-bundles in this case.



Note, that this curve is isomorphic to $\mathbb{k}^*$.

As above one easily sees that different points of this curve give us non-isomorphic sheaves.

The matrix $\left(\begin{smallmatrix}\alpha & \beta & \gamma\\ 0 & 1 & 0\\ 0 & 0 & 1\end{smallmatrix}\right)$ sends the curve C_1 into the curve

$$(\beta u_0 + u_1)(\gamma u_0 + u_2) + \alpha^2 u_0^2 = u_1 u_2 + (\beta\gamma + \alpha^2)u_0^2 + \beta u_0 u_2 + \gamma u_0 u_1 = 0.$$

Thus the stabilizer of C_1 consists of two matrices $\left(\begin{smallmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix}-1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{smallmatrix}\right)$. Therefore, two sheaves given by the matrices

$$\widetilde{A}_B = \begin{pmatrix} u_1 + \xi u_0 & \xi u_0 x_0 \\ u_2 + \eta u_0 & u_2 x_0 \end{pmatrix}$$

and

$$\widetilde{A}_{B'} = \begin{pmatrix} u_1 + \xi' u_0 & \xi' u_0 x_0 \\ u_2 + \eta' u_0 & u_2 x_0 \end{pmatrix}$$

define equivalent sheaves if and only if they are either equal or if $\widetilde{A}_{B'} = \phi^*(\widetilde{A}_B)$ for $\phi = \left(\begin{smallmatrix}-1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{smallmatrix}\right)$. Since

$$\phi^*(\widetilde{A}_B) = \begin{pmatrix} u_1 - \xi u_0 & -\xi u_0 x_0 \\ u_2 - \eta u_0 & u_2 x_0 \end{pmatrix},$$

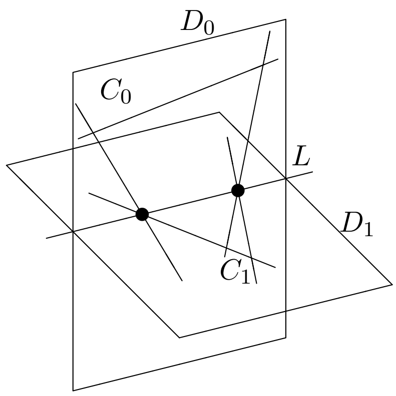
the latter means $(\xi', \eta') = (-\xi, -\eta)$. But this is the same as to say that the points (ξ, η) and (ξ', η') define the same normal direction. Indeed this is the case if and only if $B' - \alpha B \in T_A X_8$ for some non-zero scalar α. As

$$B' - \alpha B = \begin{pmatrix} (\xi' - \alpha\xi)x_0 + \dots & (\xi' - \alpha\xi)x_0^2 + \dots \\ (\eta' - \alpha\eta)x_0 + \dots & 0 \cdot x_0^2 + \dots \end{pmatrix}$$

and since the tangent equations are $\xi_{00} = 0$ and $\eta_{00} = \eta_0$, we obtain $\xi' = \alpha\xi$ and $\eta' = \alpha\eta$. Using $\xi\eta = \xi'\eta' = -1$, we get $\alpha^2 = 1$, so $(\xi', \eta') = \pm(\xi, \eta)$ is the condition for two points to define the same normal direction.

Singular curve C_1.

Here we fix a singular curve C_1, say $C_1 = \{u_1 u_2 + u_0 u_1 = u_1(u_2 + u_0) = 0\}$.



This is possible for

$$\begin{cases} \xi_0 \eta_{00} - \eta_0 \xi_{00} = 0, \\ \xi_0 - \xi_{00} = 0, \\ \eta_{00} = 1, \end{cases}$$

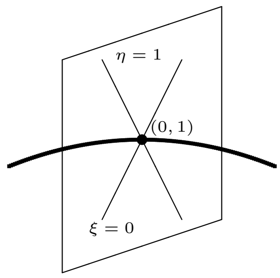
Thus

$$B = \begin{pmatrix} \xi x_0 + \xi_1 x_1 + \xi_2 x_2 & \xi x_0^2 + \cdots + \xi_{22} x_2^2 \\ \eta x_0 + \eta_1 x_1 + \eta_2 x_2 & x_0^2 + \cdots + \eta_{22} x_2^2 \end{pmatrix}$$

with $\xi - \xi\eta = 0$ and

$$\widetilde{A}_B = \begin{pmatrix} u_1 + \xi u_0 & \xi u_0 x_0 \\ u_2 + \eta u_0 & u_2 x_0 + u_0 x_0 \end{pmatrix}.$$

So, we get the "curve" $\xi(1-\eta) = 0$ (without the point $(0,1)$, which defines in this case a tangent direction) of R-bundles with the curve $C_1 = u_1(u_2 + u_0)$. All this sheaves are non-isomorphic.



One can calculate that the stabilizer of C_1 in this case consists of all the matrices of the form

$$\left(\begin{smallmatrix} \alpha & 0 & 1-\alpha \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right), \quad \alpha \in \Bbbk^*.$$

For $\phi = \left(\begin{smallmatrix} \alpha & 0 & 1-\alpha \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right)$ and

$$\widetilde{A}_B = \begin{pmatrix} u_1 + \xi u_0 & \xi u_0 x_0 \\ u_2 + \eta u_0 & u_2 x_0 + u_0 x_0 \end{pmatrix}$$

we have

$$\phi^*(\widetilde{A}_B) = \begin{pmatrix} u_1 + \xi\alpha u_0 & \xi\alpha u_0 x_0 \\ u_2 + (1-\alpha+\eta\alpha) u_0 & u_2 x_0 + (1-\alpha) u_0 x_0 + \alpha u_0 x_0 \end{pmatrix}.$$

Therefore, two sheaves given by

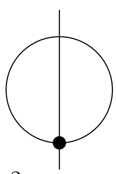
$$\widetilde{A}_B = \begin{pmatrix} u_1 + \xi u_0 & \xi u_0 x_0 \\ u_2 + \eta u_0 & u_2 x_0 + u_0 x_0 \end{pmatrix}$$

and by

$$\widetilde{A}_{B'} = \begin{pmatrix} u_1 + \xi' u_0 & \xi' u_0 x_0 \\ u_2 + \eta' u_0 & u_2 x_0 + u_0 x_0 \end{pmatrix}$$

are equivalent if and only if $\xi' = \alpha\xi$ and $\eta' = 1 - \alpha + \eta\alpha$ for some non-zero scalar α. Rewriting this equations as $\xi' = \alpha\xi$ and $1 - \eta' = \alpha(1-\eta)$ we see that the sheaves above are equivalent if and only if the points (ξ, η) and (ξ', η') lie in the same component (line) of the curve $\xi(1-\eta) = 0$. As in the examples above this corresponds to equal normal directions: each component of the curve $\xi(1-\eta) = 0$ gives us a normal direction.

2.3.4 Transversal intersection of a line with a smooth conic.



$x_1(x_1^2 - x_0 x_2) = 0$

We fix for this case the matrix $A = \begin{pmatrix} x_1 & x_0 x_1 \\ x_2 & x_1^2 \end{pmatrix}$. Tangent equations are $\xi_{00} = \xi_0$ and $\eta_{00} = 0$. For a general direction B the matrix on Z_0 is

$$\widetilde{A}_B = \begin{pmatrix} u_1 + \xi_0 u_0 & u_1 x_0 + \xi_{00} u_0 x_0 \\ u_2 + \eta_0 u_0 & u_1 x_1 + \eta_{00} u_0 x_0 \end{pmatrix}.$$

Restricting to D_1 we obtain

$$\begin{pmatrix} u_1+\xi_0 u_0 & u_1+\xi_{00}u_0 \\ u_2+\eta_0 u_0 & \eta_{00}u_0 \end{pmatrix}$$

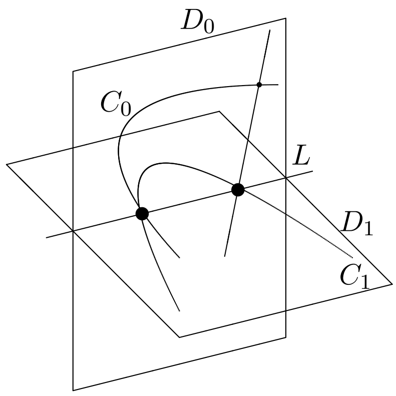
and the curve C_1 is given by the determinant of this matrix

$$(\eta_{00}-\eta_0)u_0u_1+(\xi_0\eta_{00}-\eta_0\xi_{00})u_0^2-\xi_{00}u_0u_2-u_1u_2.$$

The intersection of the support of the sheaf with the line L consists of two points $u_1=0$ and $u_2=0$.

Smooth curve C_1.

Let us fix $C_1=\{u_0^2-u_1u_2=0\}$,



then

$$\begin{cases} \xi_0\eta_{00}-\eta_0\xi_{00}=1, \\ \eta_{00}-\eta_0=0, \\ -\xi_{00}=0, \end{cases}$$

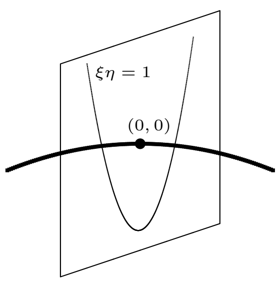
and

$$B=\begin{pmatrix} \xi x_0+\xi_1x_1+\xi_2x_2 & \xi_{01}x_0x_1+\cdots+\xi_{22}x_2^2 \\ \eta x_0+\eta_1x_1+\eta_2x_2 & \eta x_0^2+\cdots+\eta_{22}x_2^2 \end{pmatrix}$$

with $\xi\eta=1$. Thus

$$\widetilde{A}_B=\begin{pmatrix} u_1+\xi u_0 & u_1x_0 \\ u_2+\eta u_0 & u_1x_1+\eta u_0x_0 \end{pmatrix}$$

and we obtain in this case a “curve” $\xi\eta=1$ of R-bundles non-isomorphic to each other.



This curve is isomorphic to $\Bbbk^*$.

One easily sees that the stabilizer of the curve C_1 consists of two matrices $\left(\begin{smallmatrix}1&0&0\\0&1&0\\0&0&1\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix}-1&0&0\\0&1&0\\0&0&1\end{smallmatrix}\right)$. Therefore, two sheaves given by the matrices

$$\widetilde{A}_B = \begin{pmatrix} u_1 + \xi u_0 & u_1 x_0 \\ u_2 + \eta u_0 & u_1 x_1 + \eta u_0 x_0 \end{pmatrix}$$

and

$$\widetilde{A}_{B'} = \begin{pmatrix} u_1 + \xi' u_0 & u_1 x_0 \\ u_2 + \eta' u_0 & u_1 x_1 + \eta' u_0 x_0 \end{pmatrix}$$

define equivalent sheaves if and only if the matrices are either equal or if $\widetilde{A}_{B'} = \phi^*(\widetilde{A}_B)$ for $\phi = \left(\begin{smallmatrix}-1&0&0\\0&1&0\\0&0&1\end{smallmatrix}\right)$. Since

$$\phi^*(\widetilde{A}_B) = \begin{pmatrix} u_1 - \xi u_0 & u_1 x_0 \\ u_2 - \eta u_0 & u_1 x_1 - \eta u_0 x_0 \end{pmatrix},$$

the latter means $(\xi', \eta') = (-\xi, -\eta)$.

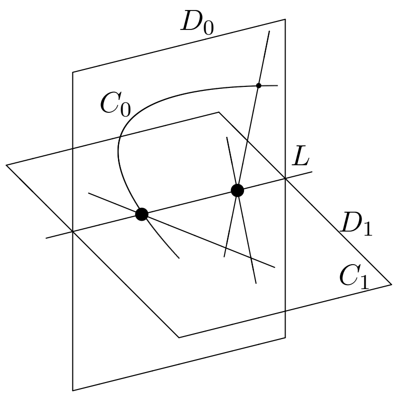
But this is the same as to say that the points (ξ, η) and (ξ', η') define the same normal direction. Indeed this is the case if and only if $B' - \alpha B \in T_A X_8$ for some non-zero scalar α. As

$$B' - \alpha B = \begin{pmatrix} (\xi' - \alpha\xi)x_0 + \dots & 0 \cdot x_0^2 + \dots \\ (\eta' - \alpha\eta)x_0 + \dots & (\eta' - \alpha\eta)x_0 + \dots \end{pmatrix}$$

and since the tangent equations are $\xi_{00} = \xi_0$ and $\eta_{00} = 0$, we obtain $\xi' = \alpha\xi$ and $\eta' = \alpha\eta$. Using $\xi\eta = \xi'\eta' = 1$, we get $\alpha^2 = 1$, so $(\xi', \eta') = \pm(\xi, \eta)$ is the condition for two points on the curve $\xi\eta = 1$ to define the same normal direction.

Singular curve C_1.

Let us fix $C_1 = \{u_1(u_0 - u_2) = 0\}$.



Then

$$\begin{cases} \xi_0 \eta_{00} - \eta_0 \xi_{00} = 0, \\ \eta_{00} - \eta_0 = 1, \\ -\xi_{00} = 0, \end{cases}$$

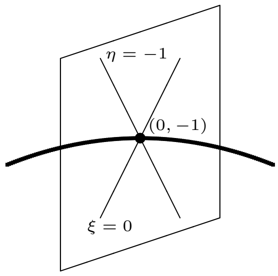
and

$$B = \begin{pmatrix} \xi x_0 + \xi_1 x_1 + \xi_2 x_2 & \xi_{01} x_0 x_1 + \dots + \xi_{22} x_2^2 \\ \eta x_0 + \eta_1 x_1 + \eta_2 x_2 & (1+\eta) x_0^2 + \dots + \eta_{22} x_2^2 \end{pmatrix}$$

with $\xi(1+\eta) = 0$. Thus

$$\widetilde{A}_B = \begin{pmatrix} u_1 + \xi u_0 & u_1 x_0 \\ u_2 + \eta u_0 & u_1 x_1 + (1+\eta) u_0 x_0 \end{pmatrix}$$

and the curve $\xi(1+\eta)$ without the point $(0,-1)$ parameterizes R-bundles with the fixed curve C_1.



The stabilizer of C_1 in this case is the group

$$\left\{ \left(\begin{smallmatrix} \alpha & 0 & \alpha - 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{smallmatrix} \right) \mid \alpha \in \Bbbk^* \right\}.$$

For the matrix

$$\widetilde{A}_B = \begin{pmatrix} u_1 + \xi u_0 & u_1 x_0 \\ u_2 + \eta u_0 & u_1 x_1 + (1+\eta) u_0 x_0 \end{pmatrix}$$

and for $\phi = \left(\begin{smallmatrix} \alpha & 0 & \alpha - 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{smallmatrix} \right)$ we have

$$\phi^*(\widetilde{A}_B) = \begin{pmatrix} u_1 + \xi \alpha u_0 & u_1 x_0 \\ u_2 + (\alpha - 1 + \eta\alpha) u_0 & u_1 x_1 + (1+\eta)\alpha u_0 x_0 \end{pmatrix}.$$

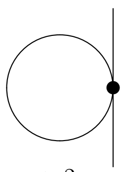
Therefore, the sheaves defined by $\widetilde{A}_B$ and by

$$\widetilde{A}_{B'} = \begin{pmatrix} u_1 + \xi' u_0 & u_1 x_0 \\ u_2 + \eta' u_0 & u_1 x_1 + (1+\eta') u_0 x_0 \end{pmatrix}$$

are equivalent if and only if $\xi' = \alpha\xi$ and $1 + \eta' = \alpha(1+\eta)$, $\alpha \in \Bbbk^*$. This means (ξ, η) and $(\xi'.\eta')$ should lie in the same component of the curve $\xi(1+\eta) = 0$.

As in the examples above this corresponds to equal normal directions: each component of the curve $\xi(1+\eta) = 0$ gives us a normal direction.

2.3.5 Tangent intersection of a line with a smooth conic.



$x_2(x_1^2 - x_0 x_2) = 0$

For this case let us take the matrix $A = \begin{pmatrix} x_1 & x_0 x_2 \\ x_2 & x_1 x_2 \end{pmatrix}$. Tangent equations at this point are $\xi_{00} = \eta_0$ and $\eta_{00} = 0$. For a general direction B the matrix on Z_0 is

$$\widetilde{A}_B = \begin{pmatrix} u_1 + \xi_0 u_0 & u_2 x_0 + \xi_{00} u_0 x_0 \\ u_2 + \eta_0 u_0 & u_1 x_2 + \eta_{00} u_0 x_0 \end{pmatrix}.$$

Restricting to D_1 we obtain

$$\begin{pmatrix} u_1 + \xi_0 u_0 & u_2 + \xi_{00} u_0 \\ u_2 + \eta_0 u_0 & \eta_{00} u_0 \end{pmatrix},$$

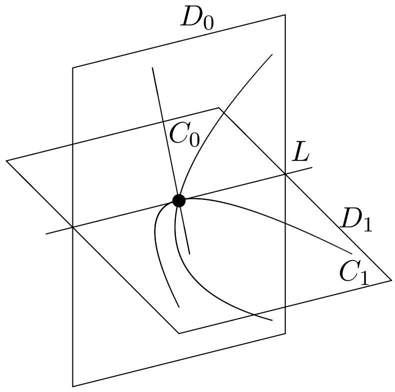
and thus the curve C_1 is in this case given by the polynomial

$$(\xi_0 \eta_{00} - \eta_0 \xi_{00}) u_0^2 - u_2^2 + \eta_{00} u_0 u_1 - (\xi_{00} + \eta_0) u_0 u_2.$$

The intersection of this curve with the line L consists of a single point $u_2 = 0$.

Smooth curve C_1.

Let us fix $C_1 = \{u_0u_1 - u_2^2 = 0\}$.



Then

$$\begin{cases} \xi_0\eta_{00} - \eta_0\xi_{00} = 0, \\ \eta_{00} = 1, \\ \xi_{00} + \eta_0 = 0, \end{cases}$$

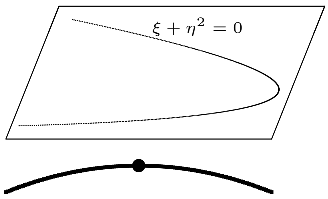
and

$$B = \begin{pmatrix} \xi x_0 + \xi_1 x_1 + \xi_2 x_2 & -\eta x_0^2 + \cdots + \xi_{22} x_2^2 \\ \eta x_0 + \eta_1 x_1 + \eta_2 x_2 & x_0^2 + \cdots + \eta_{22} x_2^2 \end{pmatrix}$$

with $\xi + \eta^2 = 0$. Therefore,

$$\widetilde{A}_B = \begin{pmatrix} u_1 + \xi u_0 & u_2 x_0 - \eta u_0 x_0 \\ u_2 + \eta u_0 & u_1 x_2 + u_0 x_0 \end{pmatrix}.$$

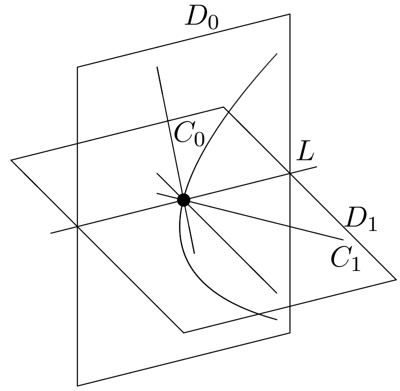
We obtain a "curve" $\xi + \eta^2 = 0$ of non-isomorphic R-bundles in this case. This curve is isomorphic to $\Bbbk$.



The stabilizer of C_1 is a trivial group in this case. Thus the points of the curve $\xi + \eta^2 = 0$ correspond to equivalence classes of R-bundles with the curve $C_1 = \{u_0u_1 - u_2^2 = 0\}$ constructed at the point $A = \begin{pmatrix} x_1 & x_0x_2 \\ x_2 & x_1x_2 \end{pmatrix}$.

Singular curve C_1.

We fix here the curve $C_1 = \{u_0^2 - u_2^2 = (u_0 - u_2)(u_0 + u_2) = 0\}$.



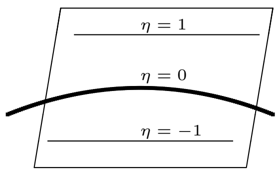
Then

$$\begin{cases} \xi_0\eta_{00} - \eta_0\xi_{00} = 1, \\ \eta_{00} = 0, \\ \xi_{00} + \eta_0 = 0, \end{cases}$$

and

$$B = \begin{pmatrix} \xi x_0 + \xi_1 x_1 + \xi_2 x_2 & -\eta x_0^2 + \cdots + \xi_{22} x_2^2 \\ \eta x_0 + \eta_1 x_1 + \eta_2 x_2 & \xi_{01} x_0 x_1 + \cdots + \eta_{22} x_2^2 \end{pmatrix}$$

with $\eta^2 = 1$.



In this case

$$\widetilde{A}_B = \begin{pmatrix} u_1 + \xi u_0 & u_2 x_0 - \eta u_0 x_0 \\ u_2 + \eta u_0 & u_1 x_2 \end{pmatrix}.$$

The matrix $\left(\begin{smallmatrix} \alpha & \beta & \gamma \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right)$ sends the curve C_1 into the curve given by the polynomial

$$\alpha^2 u_0^2 - (\gamma u_0 + u_2)^2 = (\alpha^2 - \gamma^2) u_0^2 - u_2^2 - 2\gamma u_0 u_2.$$

Therefore, the stabilizer of the curve C_1 is the group $\left\{ \left(\begin{smallmatrix} \alpha & \beta & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right) \mid \alpha = \pm 1, \beta \in \Bbbk \right\}$. Since for $\phi = \left(\begin{smallmatrix} \alpha & \beta & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right)$, $\alpha^2 = 1$, and for $\widetilde{A}_B = \begin{pmatrix} u_1 + \xi u_0 & u_2 x_0 - \eta u_0 x_0 \\ u_2 + \eta u_0 & u_1 x_2 \end{pmatrix}$ we have

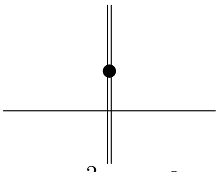
$$\phi^*(\widetilde{A}_B) = \begin{pmatrix} u_1 + (\beta + \alpha\xi) u_0 & u_2 x_0 - \eta\alpha u_0 x_0 \\ u_2 + \eta\alpha u_0 & u_1 x_2 \end{pmatrix},$$

the sheaves given by $\widetilde{A}_B$ and by

$$\widetilde{A}_{B'} = \begin{pmatrix} u_1 + \xi' u_0 & u_2 x_0 - \eta' u_0 x_0 \\ u_2 + \eta' u_0 & u_1 x_2 \end{pmatrix}$$

are equivalent if and only if there exist $\alpha = \pm 1$ and $\beta \in \Bbbk$ such that $\eta' = \alpha\eta$ and $\xi' = \alpha\xi + \beta$. But this is always possible, just take $\alpha = \eta'/\eta$ and $\beta = \xi' - \alpha\xi$.

2.3.6 Point on a double line.



$x_1^2x_0 = 0$

We consider the matrix $A = \begin{pmatrix} x_1 & 0 \\ x_2 & x_0x_1 \end{pmatrix}$. Tangent equations are $\xi_{00} = 0$ and $\xi_0 = \eta_{00}$. For a general direction B the matrix on Z_0 is

$$\widetilde{A}_B = \begin{pmatrix} u_1 + \xi_0 u_0 & \xi_{00}u_0x_0 \\ u_2 + \eta_0 u_0 & u_1x_0 + \eta_{00}u_0x_0 \end{pmatrix}.$$

Restricting to D_1 we obtain

$$\begin{pmatrix} u_1 + \xi_0 u_0 & \xi_{00}u_0 \\ u_2 + \eta_0 u_0 & u_1 + \eta_{00}u_0 \end{pmatrix},$$

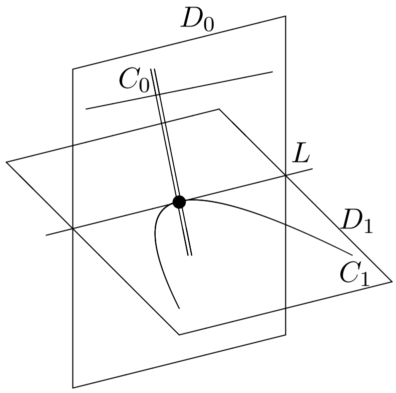
and thus the curve C_1 is in this case given by the polynomial

$$(\xi_0\eta_{00} - \eta_0\xi_{00})u_0^2 + u_1^2 + (\xi_0 + \eta_{00})u_0u_1 - \xi_{00}u_0u_2.$$

The intersection of this curve with the line L consists of a single point $u_1 = 0$.

Smooth curve C_1.

We fix here $C_1 = \{u_1^2 + u_0u_2 = 0\}$.



Then

$$\begin{cases} \xi_0\eta_{00} - \eta_0\xi_{00} = 0, \\ \xi_0 + \eta_{00} = 0, \\ -\xi_{00} = 1, \end{cases}$$

and

$$B = \begin{pmatrix} \xi x_0 + \xi_1x_1 + \xi_2x_2 & -x_0^2 + \cdots + \xi_{22}x_2^2 \\ \eta x_0 + \eta_1x_1 + \eta_2x_2 & -\xi x_0^2 + \cdots + \eta_{22}x_2^2 \end{pmatrix}$$

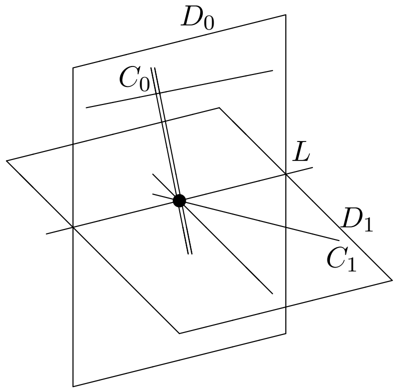
with $\eta - \xi^2 = 0$. Therefore,

$$\widetilde{A}_B = \begin{pmatrix} u_1 + \xi u_0 & -u_0x_0 \\ u_2 + \eta u_0 & u_1x_0 - \xi u_0x_0 \end{pmatrix}.$$

We have obtained a "curve" $\eta - \xi^2 = 0$ of R-bundles non-isomorphic to each other. This curve is isomorphic to $\Bbbk$. The stabilizer of C_1 in this case is trivial. Therefore, the points on the curve $\eta - \xi^2 = 0$ correspond to equivalence classes of sheaves.

Singular curve C_1.

Let us fix the curve $C_1 = \{u_1^2 - u_0^2\}$.



Then

$$\begin{cases} \xi_0\eta_{00} - \eta_0\xi_{00} = -1, \\ \xi_0 + \eta_{00} = 0, \\ -\xi_{00} = 0, \end{cases}$$

and

$$B = \begin{pmatrix} \xi x_0 + \xi_1 x_1 + \xi_2 x_2 & \xi_{01}x_0x_1 + \cdots + \xi_{22}x_2^2 \\ \eta x_0 + \eta_1 x_1 + \eta_2 x_2 & -\xi x_0^2 + \cdots + \eta_{22}x_2^2 \end{pmatrix}$$

with $\xi^2 = 1$. We have also

$$\widetilde{A}_B = \begin{pmatrix} u_1 + \xi u_0 & 0 \\ u_2 + \eta u_0 & u_1x_0 - \xi u_0 x_0 \end{pmatrix}.$$

The matrix $\left(\begin{smallmatrix} \alpha & \beta & \gamma \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right)$ sends the curve C_1 into the curve given by the polynomial

$$(\beta u_0 + u_1)^2 - \alpha^2 u_0^2 = (\beta^2 - \alpha^2)u_0^2 + u_1^2 + 2\beta u_0 u_1.$$

Therefore, the stabilizer of the curve C_1 is the group $\left\{\left(\begin{smallmatrix} \alpha & 0 & \gamma \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right) \mid \alpha = \pm 1, \gamma \in \Bbbk\right\}$. Since for $\phi = \left(\begin{smallmatrix} \alpha & 0 & \gamma \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right)$, $\alpha^2 = 1$, and for $\widetilde{A}_B = \begin{pmatrix} u_1 + \xi u_0 & 0 \\ u_2 + \eta u_0 & u_1x_0 - \xi u_0 x_0 \end{pmatrix}$ we have

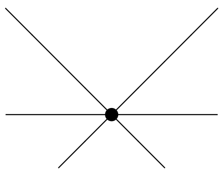
$$\phi^*(\widetilde{A}_B) = \begin{pmatrix} u_1 + \alpha\xi u_0 & 0 \\ u_2 + (\gamma + \eta\alpha)u_0 & u_1x_0 - \alpha\xi u_0 x_0 \end{pmatrix},$$

the sheaves given by $\widetilde{A}_B$ and by

$$\widetilde{A}_{B'} = \begin{pmatrix} u_1 + \xi' u_0 & 0 \\ u_2 + \eta' u_0 & u_1x_0 - \xi' u_0 x_0 \end{pmatrix}$$

are equivalent if and only if there exist $\alpha = \pm 1$ and $\gamma \in \Bbbk$ such that $\xi' = \alpha\xi$ and $\eta' = \alpha\eta + \gamma$. But this is always possible, just take $\alpha = \xi'/\xi$ and $\gamma = \eta' - \alpha\eta$. We showed that there is only one equivalence class of sheaves for a singular curve C_1 in the case $A = \begin{pmatrix} x_1 & 0 \\ x_2 & x_0x_1 \end{pmatrix}$.

2.3.7 Three lines through one point.



$x_1x_2(x_1+x_2)=0$

We start here from the matrix $A=\begin{pmatrix} x_1 & 0 \\ x_2 & x_2(x_1+x_2)\end{pmatrix}$. Tangent equations at A are $\xi_{00}=\eta_{00}=0$. For a general direction B the matrix on Z_0 is

$$\widetilde{A}_B=\begin{pmatrix} u_1+\xi_0u_0 & \xi_{00}u_0x_0 \\ u_2+\eta_0u_0 & u_2(x_1+x_2)+\eta_{00}u_0x_0\end{pmatrix}.$$

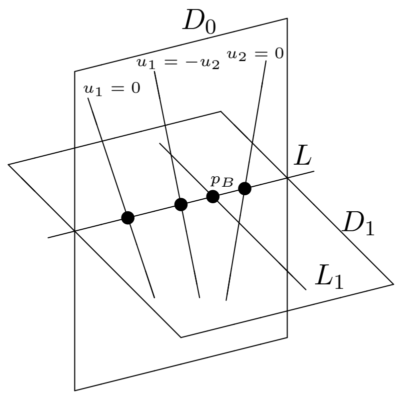
Restricting to D_1 we obtain

$$\begin{pmatrix} u_1+\xi_0u_0 & \xi_{00}u_0 \\ u_2+\eta_0u_0 & \eta_{00}u_0\end{pmatrix},$$

and thus the curve C_1 is in this case given by the polynomial

$$(\xi_0\eta_{00}-\eta_0\xi_{00})u_0^2+\eta_{00}u_0u_1-\xi_{00}u_0u_2=u_0((\xi_0\eta_{00}-\eta_0\xi_{00})u_0+\eta_{00}u_1-\xi_{00}u_2).$$

Note, that the line L is a component of this curve. Since at least one of the coefficients ξ_{00} and η_{00} is different from zero (we consider normal directions), the second component of C_1 is a line though the point $p_B:=\langle\xi_{00},\eta_{00}\rangle$ at L.



Let us denote this line L_1. It is clear that two directions B and B' with different intersection points p_B and $p_{B'}$ define non-equivalent sheaves because all the allowed automorphism of Z_0 are identities on $D_0=\widetilde{\mathbb{P}}_2$. That is why there exists at least $\mathbb{P}_1$ many equivalence classes of R-bundles in this case: each point at $L\cong\mathbb{P}_1$ defines at least one equivalence class. Let us fix such a point $\langle a,b\rangle$. This means that we fix up to multiplication by a non-zero constant the coefficients ξ_{00} and η_{00}. Let us fix a line through this point, say $L_1=\{cu_0+bu_1-au_2=0\}$. Then B defines a sheaf with this line if and only if

$$B=\begin{pmatrix} \xi x_0+\xi_1x_1+\xi_2x_2 & \alpha a x_0^2+\cdots+\xi_{22}x_2^2 \\ \eta x_0+\eta_1x_1+\eta_2x_2 & \alpha b x_0^2+\cdots+\eta_{22}x_2^2\end{pmatrix}$$

with $b\xi-a\eta=c$ for some $\alpha\in\Bbbk^*$, equivalently

$$B=\begin{pmatrix} \xi x_0+\xi_1x_1+\xi_2x_2 & \xi_{00}x_0^2+\cdots+\xi_{22}x_2^2 \\ \eta x_0+\eta_1x_1+\eta_2x_2 & \eta_{00}x_0^2+\cdots+\eta_{22}x_2^2\end{pmatrix}$$

with $b\xi-a\eta=c$ and $\langle\xi_{00},\eta_{00}\rangle=\langle a,b\rangle$. In this case

$$\widetilde{A}_B=\begin{pmatrix} u_1+\xi u_0 & \xi_{00}u_0x_0 \\ u_2+\eta u_0 & u_2(x_1+x_2)+\eta_{00}u_0x_0\end{pmatrix}.$$

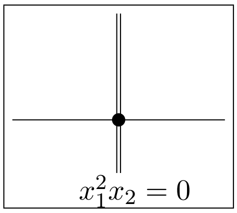
We claim that two arbitrary directions B and B' with $p_B = p_{B'}$ define equivalent sheaves. Indeed, since for $\phi = \left(\begin{smallmatrix}\alpha & \beta & \gamma\\ 0 & 1 & 0\\ 0 & 0 & 1\end{smallmatrix}\right)$ we have

$$\phi^*(\widetilde{A}_B) = \begin{pmatrix} u_1 + (\beta + \alpha\xi)u_0 & \alpha\xi_{00}u_0x_0 \\ u_2 + (\gamma + \alpha\eta)u_0 & u_2(x_1 + x_2) + \alpha\eta_{00}u_0x_0 \end{pmatrix}.$$

Since $\langle \xi'_{00}, \eta'_{00}\rangle = \langle \xi_{00}, \eta_{00}\rangle$, we define α from the equality $(\xi'_{00}, \eta'_{00}) = \alpha \cdot (\xi_{00}, \eta_{00})$ and put $\beta = \xi' - \alpha\xi$ and $\gamma = \eta' - \alpha\eta$. Then $\phi^*(\widetilde{A}_B) = \widetilde{A}_{B'}$, i.e., the sheaves defined by B and B' are equivalent.

We obtained that the line L parameterizes the equivalence classes of R-bundles constructed at the point $A = \begin{pmatrix} x_1 & 0 \\ x_2 & x_2(x_1 + x_2)\end{pmatrix}$.

2.3.8 Intersection of a line with a double line.



We consider the matrix $A = \begin{pmatrix} x_1 & 0 \\ x_2 & x_2^2\end{pmatrix}$. Tangent equations are here $\xi_{00} = \eta_{00} = 0$. For a general direction

$$B = \begin{pmatrix} \xi_0x_0 + \xi_1x_1 + \xi_2x_2 & \xi_{00}x_0^2 + \cdots + \xi_{22}x_2^2 \\ \eta_0x_0 + \eta_1x_1 + \eta_2x_2 & \eta_{00}x_0^2 + \cdots + \eta_{22}x_2^2 \end{pmatrix}$$

we obtain a sheaf on Z_0 given by the matrix

$$\begin{pmatrix} u_1 + \xi_0u_0 & \xi_{00}u_0x_0 \\ u_2 + \eta_0u_0 & u_2x_2 + \eta_{00}u_0x_0 \end{pmatrix}.$$

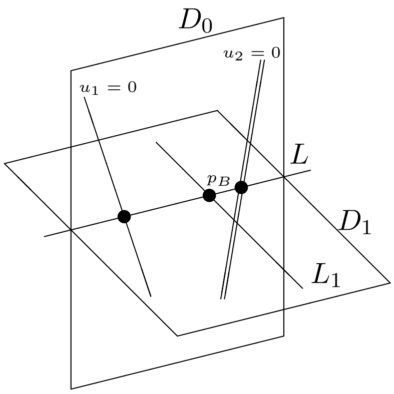
The restriction to D_1 is given by the matrix

$$\begin{pmatrix} u_1 + \xi_0u_0 & \xi_{00}u_0 \\ u_2 + \eta_0u_0 & \eta_{00}u_0 \end{pmatrix}.$$

Its support on D_1 is the curve C_1 given by the polynomial

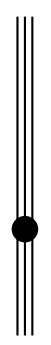
$$(\xi_0\eta_{00} - \eta_0\xi_{00})u_0^2 + \eta_{00}u_0u_1 - \xi_{00}u_0u_2 = u_0((\xi_0\eta_{00} - \eta_0\xi_{00})u_0 + \eta_{00}u_1 - \xi_{00}u_2).$$

The L is a component of this curve. The second component is the line L_1 as in the previous example.



As in the previous example we obtain that the points on the line L are in one-to-one correspondence with the equivalence classes of R-bundles in this case.

2.3.9 A line with multiplicity 3.



$x_1^3 = 0$

We consider the matrix $A = \begin{pmatrix} x_1 & 0 \\ x_2 & x_1^2 \end{pmatrix}$. Tangent equations are here $\xi_{00} = \eta_{00} = 0$. For a general direction

$$B = \begin{pmatrix} \xi_0 x_0 + \xi_1 x_1 + \xi_2 x_2 & \xi_{00} x_0^2 + \cdots + \xi_{22} x_2^2 \\ \eta_0 x_0 + \eta_1 x_1 + \eta_2 x_2 & \eta_{00} x_0^2 + \cdots + \eta_{22} x_2^2 \end{pmatrix}$$

we obtain a sheaf on Z_0 given by the matrix

$$\begin{pmatrix} u_1 + \xi_0 u_0 & \xi_{00} u_0 x_0 \\ u_2 + \eta_0 u_0 & u_1 x_1 + \eta_{00} u_0 x_0 \end{pmatrix}.$$

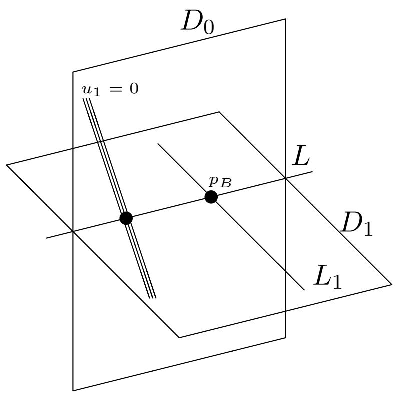
The restriction to D_1 is given by the matrix

$$\begin{pmatrix} u_1 + \xi_0 u_0 & \xi_{00} u_0 \\ u_2 + \eta_0 u_0 & \eta_{00} u_0 \end{pmatrix}.$$

Its support on D_1 is the curve C_1 given by the polynomial

$$(\xi_0\eta_{00} - \eta_0\xi_{00})u_0^2 + \eta_{00}u_0u_1 - \xi_{00}u_0u_2 = u_0((\xi_0\eta_{00} - \eta_0\xi_{00})u_0 + \eta_{00}u_1 - \xi_{00}u_2).$$

The L is a component of this curve. The second component is the line L_1 as in the previous examples.



As in the last two examples one sees that the line L is the line parameterizing the equivalence classes of R-bundles in this case.

Chapter 3

Families

Summary

We describe the blow ups $\widetilde{M} = \mathrm{Bl}_{M_8} M$ and $\widetilde{X} = \mathrm{Bl}_{X_8} X$. There is a unique lifting of the action of the group G (cf. (1.4)) on X to the action on $\widetilde{X}$. It turns out that $\widetilde{M}$ is a quotient of $\widetilde{X}$ by the group G.

We construct a morphism $Y \to \widetilde{X}$ and a sheaf $\widetilde{\mathcal{U}}$ on Y such that $\widetilde{\mathcal{U}}$ is flat over $\widetilde{X}$ and the fibres of $\widetilde{\mathcal{U}}$ are either non-singular $3m+1$ sheaves or R-bundles on $\hat{\mathbb{P}}_2$. Among the fibres of $\widetilde{\mathcal{U}}$ there are all the equivalence classes of R-bundles on $\hat{\mathbb{P}}_2$ and all isomorphism classes of non-singular $3m+1$ sheaves on $\mathbb{P}_2$.

In Definition 3.21 we define a family over an arbitrary variety S. In particular $3m+1$ families of the non-singular sheaves on $\mathbb{P}_2$ are families in the sense of Definition 3.21. For an arbitrary variety S we introduce an equivalence relation on the set of all families over S.

For a morphism $f : T \to S$ and for a family over S we define a family over T. We obtain this way the map from the set of all families over S to the set of all families over T. This map is compatible with the equivalence relations and therefore we obtain a functor $\widetilde{\mathcal{M}} : (\mathrm{Sch}) \to (\mathrm{Sets})$ that assigns to every $S \in \mathrm{Ob}\,(\mathrm{Sch})$ the set of the equivalence classes of the families over S.

There is a natural transformation $\widetilde{\mathcal{M}} \to \mathcal{M}$, where $\mathcal{M}$ denotes the functor of the $3m+1$ moduli problem on $\mathbb{P}_2$. We obtain also a natural transformation $\widetilde{\mathcal{M}} \to \mathrm{Hom}(\_\,, \widetilde{M})$ a bijection $\widetilde{\mathcal{M}}(\mathrm{pt}) \cong \mathrm{Hom}(\mathrm{pt}, \widetilde{M}) \cong \widetilde{M}$. and the commutative square

$$
\begin{tikzcd}
\widetilde{\mathcal{M}} \arrow[r] \arrow[d] & \mathrm{Hom}(\_\,, \widetilde{M}) \arrow[d] \\
\mathcal{M} \arrow[r, "\cong"] & \mathrm{Hom}(\_\,, M).
\end{tikzcd}
$$

3.1 Spaces $\mathrm{Bl}_{X_8} X$ and $\mathrm{Bl}_{M_8} M$

We investigate here the varieties $\widetilde{X} = \mathrm{Bl}_{X_8} X$ and $\widetilde{M} = \mathrm{Bl}_{M_8} M$ and their relation to each other. Since X_8 and M_8 are smooth subvarieties of codimension 2 in X and M respectively, using Theorem 2.19 we may consider the blow up $\widetilde{M}$ as the space whose points are all the isomorphism classes of non-singular $3m+1$ sheaves on $\mathbb{P}_2$ and also all the equivalence classes of R-bundles. Analogously $\widetilde{X}$ may be seen as a variety parameterizing the above objects.

3.1.1 Space $\mathrm{Bl}_{M_8} M$.

Let us consider the blow up $\widetilde{M} = \mathrm{Bl}_{M_8} M$ of M_8 in M. Recall that by Lemma 1.5 M_8 is smooth of codimension 2 in M. The exceptional divisor E_M of the blow up $\widetilde{M} \to M$ is isomorphic to the projective normal bundle $\mathbb{P}N_{M_8/M} = \mathbf{P}\mathscr{C}_{M_8/M}$.

Recall that M_8 is given in $\mathbb{P}_9 \times \mathbb{P}_2$ by equations (1.8). Since M_8 is given locally by two equations (cf. page 6) in M, we can describe $\widetilde{M}$ locally over $M(x_0) = M \cap \{x_0 \neq 0\}$ by the equation $e_1 s_2 - e_2 s_1$ in $M(x_0) \times \mathbb{P}_1 = \{(\langle f\rangle, \langle x\rangle) \times \langle s_1, s_2\rangle\}$,

$$\widetilde{M(x_0)} = \{(\langle f\rangle, \langle x\rangle) \times \langle s_1, s_2\rangle \in M(x_1) \times \mathbb{P}_1 \mid e_1 s_2 - e_2 s_1 = 0\}.$$

Analogously

$$\widetilde{M(x_1)} = \{(\langle f\rangle, \langle x\rangle) \times \langle s_1, s_2\rangle \in M(x_1) \times \mathbb{P}_1 \mid e_0 s_2 - e_2 s_1 = 0\}$$

and

$$\widetilde{M(x_2)} = \{(\langle f\rangle, \langle x\rangle) \times \langle s_1, s_2\rangle \in M(x_1) \times \mathbb{P}_1 \mid e_0 s_2 - e_1 s_1 = 0\}.$$

The gluing of $\widetilde{M(x_i)}$ to $\widetilde{M}$ are described in the following diagram.

$$\begin{tikzcd}
 & \widetilde{M(x_0)} \arrow[ddl, bend right, "{\begin{pmatrix} -\frac{x_1}{x_0} & 0 \\ -\frac{x_2}{x_0} & 1 \end{pmatrix}}"'] \arrow[d] \arrow[ddr, bend left, "{\begin{pmatrix} -\frac{x_1}{x_0} & 1 \\ -\frac{x_2}{x_0} & 0 \end{pmatrix}}"] & \\
 & M & \\
\widetilde{M(x_1)} \arrow[ur] \arrow[rr, bend right, "{\begin{pmatrix} 1 & -\frac{x_0}{x_1} \\ 0 & -\frac{x_2}{x_1} \end{pmatrix}}"'] & & \widetilde{M(x_2)} \arrow[ul]
\end{tikzcd}$$

The latter means that the gluing of $\widetilde{M(x_0)}$ and $\widetilde{M(x_1)}$ over $M(x_0 x_1)$ is given by the map

$$(\langle f\rangle, \langle x\rangle) \times \langle s_1, s_2\rangle \mapsto (\langle f\rangle, \langle x\rangle) \times \langle (s_1, s_2) \cdot \begin{pmatrix} -\frac{x_1}{x_0} & 0 \\ -\frac{x_2}{x_0} & 1 \end{pmatrix} \rangle,$$

the gluing of $\widetilde{M(x_0)}$ and $\widetilde{M(x_2)}$ over $M(x_0 x_2)$ is given by

$$(\langle f\rangle, \langle x\rangle) \times \langle s_1, s_2\rangle \mapsto (\langle f\rangle, \langle x\rangle) \times \langle (s_1, s_2) \cdot \begin{pmatrix} -\frac{x_1}{x_0} & 1 \\ -\frac{x_2}{x_0} & 0 \end{pmatrix} \rangle,$$

and the gluing of $\widetilde{M(x_1)}$ and $\widetilde{M(x_2)}$ over $M(x_1 x_2)$ is given by

$$(\langle f\rangle, \langle x\rangle) \times \langle s_1, s_2\rangle \mapsto (\langle f\rangle, \langle x\rangle) \times \langle (s_1, s_2) \cdot \begin{pmatrix} 1 & -\frac{x_0}{x_1} \\ 0 & -\frac{x_2}{x_1} \end{pmatrix} \rangle.$$

Lemma 3.1. *Let $\mathbb{P}_2 = U_0 \cup U_1 \cup U_2$ be the standard covering of $\mathbb{P}_2$, i. e., $U_i = \{x_i \neq 0\}$. Let $\mathrm{T}^* \mathbb{P}_2$ be the cotangent bundle of $\mathbb{P}_2$. Then it is given by the following cocycle g_{ij} with respect to the covering $\{U_i\}$:*

$$g_{10} = \begin{pmatrix} -\frac{x_1}{x_0} & 0 \\ -\frac{x_2}{x_0} & 1 \end{pmatrix}, \quad g_{21} = \begin{pmatrix} 1 & -\frac{x_0}{x_1} \\ 0 & -\frac{x_2}{x_1} \end{pmatrix}, \quad g_{20} = \begin{pmatrix} -\frac{x_1}{x_0} & 1 \\ -\frac{x_2}{x_0} & 0 \end{pmatrix}.$$

Proof. Let us consider the exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}_2} \xrightarrow{(\, x_0 \; x_1 \; x_2 \,)} 3\mathcal{O}_{\mathbb{P}_2}(1) \to \mathscr{T}_{\mathbb{P}_2} \to 0.$$

Then locally on U_0, U_1, and U_1 this sequence is

$$0 \to \mathcal{O}_{U_0} \xrightarrow{\begin{pmatrix}1 & \frac{x_1}{x_0} & \frac{x_2}{x_0}\end{pmatrix}} 3\mathcal{O}_{U_0} \xrightarrow{\begin{pmatrix}-\frac{x_1}{x_0} & -\frac{x_2}{x_0}\\ 1 & 0\\ 0 & 1\end{pmatrix}} 2\mathcal{O}_{U_0} \to 0,$$

$$0 \to \mathcal{O}_{U_1} \xrightarrow{\begin{pmatrix}\frac{x_0}{x_1} & 1 & \frac{x_2}{x_1}\end{pmatrix}} 3\mathcal{O}_{U_1} \xrightarrow{\begin{pmatrix}1 & 0\\ -\frac{x_0}{x_1} & -\frac{x_2}{x_1}\\ 0 & 1\end{pmatrix}} 2\mathcal{O}_{U_1} \to 0,$$

and

$$0 \to \mathcal{O}_{U_2} \xrightarrow{\begin{pmatrix}\frac{x_0}{x_2} & \frac{x_1}{x_2} & 1\end{pmatrix}} 3\mathcal{O}_{U_2} \xrightarrow{\begin{pmatrix}1 & 0\\ 0 & 1\\ -\frac{x_0}{x_2} & -\frac{x_1}{x_2}\end{pmatrix}} 2\mathcal{O}_{U_2} \to 0.$$

One calculates a cocycle g'_{ij} of $\mathscr{T}_X$ from the commutative diagrams

$$\begin{tikzcd} \arrow[r, "\begin{pmatrix}-\frac{x_1}{x_0} & -\frac{x_2}{x_0}\\ 1 & 0\\ 0 & 1\end{pmatrix}"] \arrow[d, equal] & \arrow[d, "g'_{10}"] \\ \arrow[r, "\begin{pmatrix}1 & 0\\ -\frac{x_0}{x_1} & -\frac{x_2}{x_1}\\ 0 & 1\end{pmatrix}"'] & \end{tikzcd}, \quad \begin{tikzcd} \arrow[r, "\begin{pmatrix}1 & 0\\ -\frac{x_0}{x_1} & -\frac{x_2}{x_1}\\ 0 & 1\end{pmatrix}"] \arrow[d, equal] & \arrow[d, "g'_{21}"] \\ \arrow[r, "\begin{pmatrix}1 & 0\\ 0 & 1\\ -\frac{x_0}{x_2} & -\frac{x_1}{x_2}\end{pmatrix}"'] & \end{tikzcd}, \quad \begin{tikzcd} \arrow[r, "\begin{pmatrix}-\frac{x_1}{x_0} & -\frac{x_2}{x_0}\\ 1 & 0\\ 0 & 1\end{pmatrix}"] \arrow[d, equal] & \arrow[d, "g'_{20}"] \\ \arrow[r, "\begin{pmatrix}1 & 0\\ 0 & 1\\ -\frac{x_0}{x_2} & -\frac{x_1}{x_2}\end{pmatrix}"'] & \end{tikzcd}.$$

respectively. We obtain then

$$g'_{10} = \begin{pmatrix}-\frac{x_0}{x_1} & -\frac{x_2}{x_1}\\ 0 & 1\end{pmatrix}, \quad g'_{21} = \begin{pmatrix}1 & 0\\ -\frac{x_0}{x_2} & -\frac{x_1}{x_2}\end{pmatrix}, \quad g'_{20} = \begin{pmatrix}0 & 1\\ -\frac{x_0}{x_2} & -\frac{x_1}{x_2}\end{pmatrix}.$$

To obtain a cocycle of $\mathrm{T}^*\mathbb{P}_2$ one needs to invert and to transpose the cocycle of $\mathrm{T}\,\mathbb{P}_2$. So $g_{ij} = ({g'_{ij}}^{\mathrm{T}})^{-1}$ and we obtain

$$g_{10} = \begin{pmatrix}-\frac{x_0}{x_1} & 0\\ -\frac{x_2}{x_1} & 1\end{pmatrix}^{-1} = (-\frac{x_0}{x_1})^{-1}\begin{pmatrix}1 & 0\\ \frac{x_2}{x_1} & -\frac{x_0}{x_1}\end{pmatrix} = \begin{pmatrix}-\frac{x_1}{x_0} & 0\\ -\frac{x_2}{x_0} & 1\end{pmatrix},$$

$$g_{21} = \begin{pmatrix}1 & -\frac{x_0}{x_2}\\ 0 & -\frac{x_1}{x_2}\end{pmatrix}^{-1} = (-\frac{x_1}{x_2})^{-1}\begin{pmatrix}-\frac{x_1}{x_2} & \frac{x_0}{x_2}\\ 0 & 1\end{pmatrix} = \begin{pmatrix}1 & -\frac{x_0}{x_1}\\ 0 & -\frac{x_2}{x_1}\end{pmatrix},$$

and

$$g_{20} = \begin{pmatrix}0 & -\frac{x_0}{x_2}\\ 1 & -\frac{x_1}{x_2}\end{pmatrix}^{-1} = (\frac{x_0}{x_2})^{-1}\begin{pmatrix}-\frac{x_1}{x_2} & \frac{x_0}{x_2}\\ -1 & 0\end{pmatrix} = \begin{pmatrix}-\frac{x_1}{x_0} & 1\\ -\frac{x_2}{x_0} & 0\end{pmatrix}.$$

This proves the lemma. $\square$

Thus the gluing functions of $\widetilde{M}$ are just the functions of the cocycle of $\mathrm{T}^*\,\mathbb{P}_2$.

Let us consider the projective $\mathbb{P}_1$-bundle $\mathbb{P}_9 \times \mathbb{P}(\mathrm{T}^*\,\mathbb{P}_2)$ over $\mathbb{P}_9 \times \mathbb{P}_2$. As M is a subvariety in $\mathbb{P}_9 \times \mathbb{P}_2$, we can restrict the bundle $\mathbb{P}_9 \times \mathbb{P}(\mathrm{T}^*\,\mathbb{P}_2)$ to M. Let us denote

$$P = (\mathbb{P}_9 \times \mathbb{P}(\mathrm{T}^*\,\mathbb{P}_2))|_M.$$

Then the gluing data we described above give us a closed embedding of $\widetilde{M}$ into P. There is the following commutative diagram:

$$\begin{tikzcd}
\widetilde{M} \arrow[r, tail] \arrow[dr, two heads] & P \arrow[r, tail] \arrow[d, two heads] & \mathbb{P}_9 \times \mathbb{P}(\mathrm{T}^*\,\mathbb{P}_2) \arrow[d, two heads] \\
& M \arrow[r, tail] & \mathbb{P}_9 \times \mathbb{P}_2.
\end{tikzcd}$$

In particular the exceptional divisor E_M of the blow up $\widetilde{M} \to M$ is isomorphic to the restriction $P|_{M_8}$ of P to M_8.

3.1.2 Space $\mathrm{Bl}_{X_8} X$.

Let $\widetilde{X} \xrightarrow{\alpha} X$ be the blowing up of X along X_8. Recall that X_8 is defined by two equations f_3 and f_4 (see (1.9)). Then $\widetilde{X}$ is a closed subvariety in $X \times \mathbb{P}_1$ given by the equation $t_3 f_4 - t_4 f_3 = 0$, i. e.,

$$\widetilde{X} = \{A \times \langle t_3, t_4 \rangle \in X \times \mathbb{P}_1 \mid t_3 f_4 - t_4 f_3 = 0\}. \tag{3.1}$$

Let E_X be the exceptional divisor of $\widetilde{X} \xrightarrow{\alpha} X$. Then E_X may be identified with $X_8 \times \mathbb{P}_1$.

$$\begin{tikzcd}
E_X = X_8 \times \mathbb{P}_1 \arrow[r, tail] \arrow[d] & \widetilde{X} \arrow[r, tail] \arrow[d] & X \times \mathbb{P}_1 \arrow[dl] \\
X_8 \arrow[r, tail] & X &
\end{tikzcd}$$

The restriction of α to $\widetilde{X} \setminus E_X$ gives us an isomorphism $\widetilde{X} \setminus E_X \to X \setminus X_8$.

Since X_8 is a complete intersection (see Lemma 1.7), E_X is isomorphic to the projective normal bundle $\mathbb{P}N_{X_8/X} = \mathbf{P}\mathscr{C}_{X_8/X}$ of X_8 in X. So for a point $A \in X_8$ the fibre $\alpha^{-1}(A)$ over A is isomorphic to the fibre of $\mathbb{P}N_{X_8/X}$ over A, i. e., to $\mathbb{P}N_A \cong \mathbb{P}_1$. By Theorem 2.19 we may interpret $\mathbb{P}N_A$ as a set of equivalence classes of R-bundles constructed A. So we can interpret E_X as a space that parameterizes the equivalence classes of R-bundles. Hence $\widetilde{X}$ parameterizes the classes of isomorphism of $3m+1$ sheaves on $\mathbb{P}_2$ that are locally free on their support and also the equivalence classes of R-bundles.

3.1.3 Group action on $\widetilde{X}$.

Consider the group $G = \mathrm{GL}_2(\Bbbk) \times H$, where H is the group of 2×2 matrices

$$\begin{pmatrix} \lambda & z \\ 0 & \mu \end{pmatrix}, \quad \lambda, \mu \in \Bbbk, \lambda\mu \neq 0, \quad z \in \Gamma(\mathbb{P}_2, \mathcal{O}_{\mathbb{P}_2}(1)).$$

Recall that G acts on X from the left by the rule

$$(g, h) \cdot A = gAh^{-1}.$$

As we know the orbits of this action are in one-to-one correspondence with the points of $M = M_{3m+1}(\mathbb{P}_2)$.

Note that X_8 is invariant under the action of G. Therefore, since the blowing up $\alpha : \widetilde{X} \to X$ is an isomorphism over $X \setminus X_8$ we obtain an action of G on $\widetilde{X} \setminus E$, $E = \alpha^{-1}(X_8)$. Let us describe this action explicitly.

Lemma 3.2. *The action of G on $\widetilde{X} \setminus E$ is given by the rule*

$$(g,h)\cdot(A,\langle t_3,t_4\rangle) = (gAh^{-1}, \langle (t_3,t_4)g^{\mathrm{T}}\rangle).$$

Proof. Since $\widetilde{X} \subseteq X \times \mathbb{P}_1$ is given by the equation $t_3f_4(A) - t_4f_3(A) = 0$, i. e.,

$$\widetilde{X} = \{(A,\langle t_3,t_4\rangle) \in X \times \mathbb{P}_1 \mid t_3f_4(A) - t_4f_3(A) = 0\},$$

since for a matrix A that does not lie in X_8 either $f_3(A) \neq 0$ or $f_4(A) \neq 0$, we obtain that the map $\alpha' : X \setminus X_8 \to \widetilde{X} \setminus E$ defined by

$$X \setminus X_8 \ni A \mapsto (A, \langle f_3(A), f_4(A)\rangle) \in \widetilde{X} \setminus E$$

is the inverse map to $\alpha|_{\widetilde{X}\setminus E} : \widetilde{X} \setminus E \to X \setminus X_8$. Therefore, an element $(g,h) \in G$ acts on a point $(A,\langle t_3,t_4\rangle) \in \widetilde{X} \setminus E$ by the rule

$$(A,\langle t_3,t_4\rangle) \mapsto A \mapsto (g,h)\cdot A = gAh^{-1} \mapsto \alpha'(gAh^{-1}) = (gAh^{-1}, \langle f_3(gAh^{-1}), f_4(gAh^{-1})\rangle).$$

Let us show that $\langle f_3(gAh^{-1}), f_4(gAh^{-1})\rangle = \langle (f_3(A), f_4(A))g^T\rangle$.

Let $A = \begin{pmatrix} z_1 & q_1 \\ z_2 & q_2 \end{pmatrix}$, $z_1 = a_0x_0 + a_1x_1 + a_2x_2$, $z_2 = b_0x_0 + b_1x_1 + b_2x_2$. Let d_0, d_1 and d_2 be as in (1.7). First of all note that

$$\langle f_3(Ah), f_4(Ah)\rangle = \langle f_3(A), f_4(A)\rangle$$

for all matrices $h \in H$. Clearly, let $h = \left(\begin{smallmatrix} \lambda & z \\ 0 & \mu \end{smallmatrix}\right)$. Then $Ah = \left(\begin{smallmatrix} \lambda z_1 & \mu q_1 + z_1 z \\ \lambda z_2 & \mu q_2 + z_2 z \end{smallmatrix}\right)$ and

$$\begin{aligned}\langle f_3(Ah), f_4(Ah)\rangle =& \langle (\mu q_1 + z_1 z)(\lambda^2 d_0, \lambda^2 d_1, \lambda^2 d_2), (\mu q_2 + z_2 z)(\lambda^2 d_0, \lambda^2 d_1, \lambda^2 d_2)\rangle = \\ & \langle \mu q_1(\lambda^2 d_0, \lambda^2 d_1, \lambda^2 d_2), \mu q_2(\lambda^2 d_0, \lambda^2 d_1, \lambda^2 d_2)\rangle = \\ & \langle q_1(d_0,d_1,d_2), q_2(d_0,d_1,d_2)\rangle = \langle f_3(A), f_4(A)\rangle.\end{aligned}$$

It remains to show that $\langle f_3(gA), f_4(gA)\rangle = \langle (f_3(A), f_4(A))g^T\rangle$. Since each matrix g can be decomposed in a product of elementary matrices corresponding to multiplication of a row by a scalar and to adding the multiple of a row to another row, it is enough to show this for elementary matrices. For $g = \left(\begin{smallmatrix} \lambda & 0 \\ 0 & \mu \end{smallmatrix}\right)$ we have $gA = \left(\begin{smallmatrix} \lambda z_1 & \lambda q_1 \\ \mu z_2 & \mu q_2 \end{smallmatrix}\right)$, thus

$$\begin{aligned}\langle f_3(gA), f_4(gA)\rangle = \langle \lambda q_1(\lambda\mu d_0, \lambda\mu d_1, \lambda\mu d_2), \mu q_1(\lambda\mu d_0, \lambda\mu d_1, \lambda\mu d_2)\rangle = \\ \langle \lambda q_1(d_0,d_1,d_2), \mu q_2(d_0,d_1,d_2)\rangle = \langle (f_3(A), f_4(A))\left(\begin{smallmatrix} \lambda & 0 \\ 0 & \mu \end{smallmatrix}\right)\rangle = \langle (f_3(A), f_4(A))g^{\mathrm{T}}\rangle.\end{aligned}$$

For $g = \left(\begin{smallmatrix} 1 & \mu \\ 0 & 1 \end{smallmatrix}\right)$ we have $gA = \begin{pmatrix} z_1 + \mu z_2 & q_1 + \mu q_2 \\ z_2 & q_2 \end{pmatrix}$ and

$$\begin{aligned}\langle f_3(gA), f_4(gA)\rangle =& \langle q_1(d_0,d_1,d_2) + \mu q_2(d_0,d_1,d_2), q_2(d_0,d_1,d_2)\rangle = \\ & \langle (q_1(d_0,d_1,d_2), q_2(d_0,d_1,d_2))\left(\begin{smallmatrix} 1 & 0 \\ \mu & 1 \end{smallmatrix}\right)\rangle = \langle (f_3(A), f_4(A))g^{\mathrm{T}}\rangle.\end{aligned}$$

Analogously one obtains $\langle f_3(gA), f_4(gA)\rangle = \langle (f_3(A), f_4(A))g^{\mathrm{T}}\rangle$ for $g = \left(\begin{smallmatrix} 1 & 0 \\ \mu & 1 \end{smallmatrix}\right)$.

We have finally showed

$$\langle f_3(gAh^{-1}), f_4(gAh^{-1})\rangle = \langle (f_3(A), f_4(A))g^T\rangle.$$

Since $\langle t_3,t_4\rangle = \langle f_3(A), f_4(A)\rangle$, we obtain the required statement. $\square$

Lemma 3.3. *The action of the group G on X can be uniquely lifted along α to the action on $\widetilde{X}$. In other words there exists the following commutative diagram*

$$\begin{tikzcd} G \times \widetilde{X} \arrow[r] \arrow[d, "\mathrm{id} \times \alpha"'] & \widetilde{X} \arrow[d, "\alpha"] \\ G \times X \arrow[r] & X. \end{tikzcd}$$

An element $(g,h) \in G$ acts by the rule

$$(g,h) \cdot (A, \langle t_3, t_4 \rangle) = (gAh^{-1}, \langle (t_3,t_4)g^{\mathrm{T}} \rangle).$$

Proof. For a given element $(g,h) \in G$, the automorphism of $\widetilde{X} \setminus E$

$$(A, \langle t_3, t_4 \rangle) \mapsto (gAh^{-1}, \langle (t_3,t_4)g^{\mathrm{T}} \rangle)$$

from Lemma 3.2 can be uniquely extended to an isomorphism of $\widetilde{X}$, which is defined by the same formula. $\square$

Remark 3.4. *Note that for an arbitrary point $(A, \langle t_3, t_4 \rangle) \in \widetilde{X}$ its stabilizer is as in the case of the action on X the subgroup*

$$St = \left\{ \left(\begin{smallmatrix} \lambda & 0 \\ 0 & \lambda \end{smallmatrix} \right) \times \left(\begin{smallmatrix} \lambda & 0 \\ 0 & \lambda \end{smallmatrix} \right) \mid \lambda \in \Bbbk^* \right\}.$$

Therefore, we can consider the corresponding free action of the group $\mathbb{P}G = G/St$ on $\widetilde{X}$.

We shall explain the meaning of the group action we have just described above. As the points $(A, \langle t_3, t_4 \rangle)$ in $\widetilde{X}$, $A \in X_8$, are in one-to-one correspondence with the equivalence classes of R-bundles constructed at A, the action of the group G on the exceptional divisor E_X may be interpreted as an action on the equivalence classes of R-bundles. We already noticed that E_X is isomorphic to $\mathbb{P}N_{X_8/X}$.

Note that the tangent bundle of X is trivial, i. e., $TX \cong X \times \Bbbk^{18}$. This holds because X is an open subset of $\Bbbk^{18}$ and because there are only trivial vector bundles on $\Bbbk^n$. The action of G on X induces the action of G on TX given by

$$(g,h) \cdot (A,B) = (gAh^{-1}, gBh^{-1}).$$

Since X_8 is invariant under the action of G, there is also an action of G on TX_8, which is just the restriction of the action of G on TX. This way one obtains the induced linear action of G on $N_{X_8/X}$:

$$G \times N_{X_8/X} \to N_{X_8/X}, \quad (g,h) \times (A,[B]) \mapsto (gAh^{-1}, [gBh^{-1}]).$$

Note that the notation $(A,[B])$ makes sense because $N_{X_8/X}$ is trivial (there are global tangent equations (1.11) of X_8). Then

$$(A,[B]) = A \times (T_1(A)(B), T_1(A)(B)),$$

where $T_1(A)$ and $T_2(A)$ are tangent equations at A (cf. (1.11) on page 10). Since the action of G is linear on $N_{X_8/X}$, we obtain also the action

$$G \times \mathbb{P}N_{X_8/X} \to \mathbb{P}N_{X_8/X}, \quad (g,h) \times (A, \langle [B] \rangle) \mapsto (gAh^{-1}, \langle [gBh^{-1}] \rangle).$$

Lemma 3.5. *Let $A \in X_8$ and $B \in T_A X \setminus T_A X_8$, then*

$$(gAh^{-1}, \langle [gBh^{-1}] \rangle) = (gAh^{-1}, \langle [B] g^{\mathrm{T}} \rangle).$$

In other words

$$\langle T_1(gAh^{-1})(gBh^{-1}), T_1(gAh^{-1})(gBh^{-1}) \rangle = \langle (T_1(A)(B), T_1(A)(B)) \cdot g^{\mathrm{T}} \rangle.$$

Proof. Let

$$B = \begin{pmatrix} \xi_0 x_0 + \xi_1 x_1 + \xi_2 x_2 & \xi_{00} x_0^2 + \cdots + \xi_{22} x_2^2 \\ \eta_0 x_0 + \eta_1 x_1 + \eta_2 x_2 & \eta_{00} x_0^2 + \cdots + \eta_{22} x_2^2 \end{pmatrix}$$

and let

$$A = \begin{pmatrix} a_0 x_0 + a_1 x_1 + a_2 x_2 & A_{00} x_0^2 + A_{01} x_0 x_1 + \cdots + A_{22} x_2^2 \\ b_0 x_0 + b_1 x_1 + b_2 x_2 & B_{00} x_0^2 + B_{01} x_0 x_1 + \cdots + B_{22} x_2^2 \end{pmatrix}$$

be a point from X_8.

First of all let us recall (cf. (1.11)) that

$$T_1(A)(B) = \sum_k \left(\sum_{ij} A_{ij} \alpha_{ij,k} \right) \xi_k + \sum_k \left(\sum_{ij} A_{ij} \beta_{ij,k} \right) \eta_k + \sum_{ij} d_i d_j \xi_{ij}$$

and

$$T_2(A)(B) = \sum_k \left(\sum_{ij} B_{ij} \alpha_{ij,k} \right) \xi_k + \sum_k \left(\sum_{ij} B_{ij} \beta_{ij,k} \right) \eta_k + \sum_{ij} d_i d_j \eta_{ij},$$

where $\alpha_{ij,k} = s_{ik} b_{\bar{i}\bar{k}} d_j + s_{jk} b_{\bar{j}\bar{k}} d_i$ and $\beta_{ij,k} = s_{ki} a_{\bar{i}\bar{k}} d_j + s_{kj} a_{\bar{j}\bar{k}} d_i$ (see (1.12)).

Let us show that

$$\langle T_1(gA)(gB), T_1(gA)(gB) \rangle = \langle (T_1(A)(B), T_1(A)(B)) \cdot g^{\mathrm{T}} \rangle.$$

Let $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and let

$$gA = \begin{pmatrix} a'_0 x_0 + a'_1 x_1 + a'_2 x_2 & A'_{00} x_0^2 + A'_{01} x_0 x_1 + \cdots + A'_{22} x_2^2 \\ b'_0 x_0 + b'_1 x_1 + b'_2 x_2 & B'_{00} x_0^2 + B'_{01} x_0 x_1 + \cdots + B'_{22} x_2^2 \end{pmatrix},$$

$$gB = \begin{pmatrix} \xi'_0 x_0 + \xi'_1 x_1 + \xi'_2 x_2 & \xi'_{00} x_0^2 + \cdots + \xi'_{22} x_2^2 \\ \eta'_0 x_0 + \eta'_1 x_1 + \eta'_2 x_2 & \eta'_{00} x_0^2 + \cdots + \eta'_{22} x_2^2 \end{pmatrix}.$$

Then

$$\begin{pmatrix} A'_{ij} \\ B'_{ij} \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} A_{ij} \\ B_{ij} \end{pmatrix}, \quad \begin{pmatrix} \xi'_{ij} \\ \eta'_{ij} \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \xi_{ij} \\ \eta_{ij} \end{pmatrix},$$

$$\begin{pmatrix} a'_k \\ b'_k \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} a_k \\ b_k \end{pmatrix}, \quad \begin{pmatrix} \xi'_k \\ \eta'_k \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \xi_k \\ \eta_k \end{pmatrix},$$

$$d'_i = \Delta d_i,$$

where $\Delta = ad - bc$ is the determinant of g, and also

$$\alpha'_{ij,k} = s_{ik} b'_{\bar{i}\bar{k}} d'_j + s_{jk} b'_{\bar{j}\bar{k}} d'_i = s_{ik}(c a_{\bar{i}\bar{k}} + d b_{\bar{i}\bar{k}}) \Delta d_j + s_{jk}(c a_{\bar{j}\bar{k}} + d b_{\bar{j}\bar{k}}) \Delta d_i = \Delta(-c \beta_{ij,k} + d \alpha_{ij,k})$$

and

$$\beta'_{ij,k} = s_{ki} a'_{\bar{i}\bar{k}} d'_j + s_{kj} a'_{\bar{j}\bar{k}} d'_i = s_{ki}(a a_{\bar{i}\bar{k}} + b b_{\bar{i}\bar{k}}) \Delta d_j + s_{kj}(a a_{\bar{j}\bar{k}} + b b_{\bar{j}\bar{k}}) \Delta d_i = \Delta(a \beta_{ij,k} - b \alpha_{ij,k}).$$

Then

$$\begin{aligned}
T_1(gA)(gB) = &\sum_k \left(\sum_{ij} A'_{ij}\alpha'_{ij,k} \right) \xi'_k + \sum_k \left(\sum_{ij} A'_{ij}\beta'_{ij,k} \right) \eta'_k + \sum_{ij} d'_i d'_j \xi'_{ij} = \\
&\sum_k \left(\sum_{ij} (aA_{ij} + bB_{ij})\Delta(-c\beta_{ij,k} + d\alpha_{ij,k}) \right) (a\xi_k + b\eta_k) + \\
&\sum_k \left(\sum_{ij} (aA_{ij} + bB_{ij})\Delta(a\beta_{ij,k} - b\alpha_{ij,k}) \right) (c\xi_k + d\eta_k) + \\
&\sum_{ij} \Delta^2 d_i d_j (a\xi_{ij} + b\eta_{ij}) = \\
&\sum_k \left(\sum_{ij} (aA_{ij} + bB_{ij})\Delta^2 \alpha_{ij,k} \right) \xi_k + \\
&\sum_k \left(\sum_{ij} (aA_{ij} + bB_{ij})\Delta^2 \beta_{ij,k} \right) \eta_k + \\
&\sum_{ij} \Delta^2 d_i d_j (a\xi_{ij} + b\eta_{ij}) = \Delta^2(a \cdot T_1(A)(B) + b \cdot T_2(A)(B)).
\end{aligned}$$

Analogously one calculates

$$T_2(gA)(gB) = \Delta^2(c \cdot T_1(A)(B) + c \cdot T_2(A)(B)).$$

Therefore,

$$\langle T_1(gA)(gB), T_2(gA)(gB) \rangle = \langle (T_1(A)(B), T_2(A)(B)) \cdot \begin{pmatrix} a & c \\ b & d \end{pmatrix} \rangle.$$

It remains to show that

$$\langle T_1(Ah)(Bh), T_2(Ah)(Bh) \rangle = \langle T_1(A)(B), T_2(A)(B) \rangle.$$

Let $h = \begin{pmatrix} \lambda & z \\ 0 & \mu \end{pmatrix}$, $z = c_0x_0 + c_1x_1 + c_2x_2$, let again

$$Ah = \begin{pmatrix} a'_0x_0 + a'_1x_1 + a'_2x_2 & A'_{00}x_0^2 + A'_{01}x_0x_1 + \cdots + A'_{22}x_2^2 \\ b'_0x_0 + b'_1x_1 + b'_2x_2 & B'_{00}x_0^2 + B'_{01}x_0x_1 + \cdots + B'_{22}x_2^2 \end{pmatrix},$$

$$Bh = \begin{pmatrix} \xi'_0x_0 + \xi'_1x_1 + \xi'_2x_2 & \xi'_{00}x_0^2 + \cdots + \xi'_{22}x_2^2 \\ \eta'_0x_0 + \eta'_1x_1 + \eta'_2x_2 & \eta'_{00}x_0^2 + \cdots + \eta'_{22}x_2^2 \end{pmatrix}.$$

Then we obtain the equalities

$$a'_k = \lambda a_k, \quad b'_k = \lambda b_k, \quad \xi'_k = \lambda\xi_k, \quad \eta'_k = \lambda\eta_k, \quad d'_i = \lambda^2 d_i, \quad \alpha'_{ij,k} = \lambda^3\alpha_{ij,k}, \quad \beta'_{ij,k} = \lambda^3\beta_{ij,k},$$

and

$$A'_{ij} = \mu A_{ij} + \frac{a_ic_j + a_jc_i}{1 + \delta_{ij}}, \quad B'_{ij} = \mu B_{ij} + \frac{b_ic_j + b_jc_i}{1 + \delta_{ij}},$$

$$\xi'_{ij} = \mu\xi_{ij} + \frac{\xi_ic_j + \xi_jc_i}{1 + \delta_{ij}}, \quad \eta'_{ij} = \mu\eta_{ij} + \frac{\eta_ic_j + \eta_jc_i}{1 + \delta_{ij}},$$

where

$$\delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$$

is the Kronecker symbol. Therefore, we compute

$$\begin{aligned} T_1(Ah)(Bh) = & \sum_k \left(\sum_{ij} A'_{ij}\alpha'_{ij,k} \right) \xi'_k + \sum_k \left(\sum_{ij} A'_{ij}\beta'_{ij,k} \right) \eta'_k + \sum_{ij} d'_i d'_j \xi'_{ij} = \\ & \sum_k \left(\sum_{ij} (\mu A_{ij} + \frac{a_i c_j + a_j c_i}{1+\delta_{ij}}) \lambda^3 \alpha_{ij,k} \right) \lambda \xi_k + \\ & \sum_k \left(\sum_{ij} (\mu A_{ij} + \frac{a_i c_j + a_j c_i}{1+\delta_{ij}}) \lambda^3 \beta_{ij,k} \right) \lambda \eta_k + \\ & \sum_{ij} \lambda^4 d_i d_j (\mu \xi_{ij} + \frac{\xi_i c_j + \xi_j c_i}{1+\delta_{ij}}) = \end{aligned}$$

$$\begin{aligned} =&\mu\lambda^4 \left(\sum_k \left(\sum_{ij} A_{ij}\alpha_{ij,k} \right) \xi_k + \sum_k \left(\sum_{ij} A_{ij}\beta_{ij,k} \right) \eta_k + \sum_{ij} d_i d_j \xi_{ij} \right) + \\ & \lambda^4 \sum_k \sum_{ij} \frac{a_i c_j + a_j c_i}{1+\delta_{ij}} \alpha_{ij,k}\xi_k + \\ & \lambda^4 \sum_k \sum_{ij} \frac{a_i c_j + a_j c_i}{1+\delta_{ij}} \beta_{ij,k}\eta_k + \\ & \lambda^4 \sum_{ij} d_i d_j \frac{\xi_i c_j + \xi_j c_i}{1+\delta_{ij}}. \end{aligned}$$

Claim.

$$\begin{aligned} \sum_{ij} d_i d_j \frac{\xi_i c_j + \xi_j c_i}{1+\delta_{ij}} &= \left(\sum_k d_k c_k \right) \cdot \left(\sum_k d_k \xi_k \right), \\ \sum_{ij} d_i d_j \frac{\eta_i c_j + \eta_j c_i}{1+\delta_{ij}} &= \left(\sum_k d_k c_k \right) \cdot \left(\sum_k d_k \eta_k \right), \\ \sum_{ij} \frac{a_i c_j + a_j c_i}{1+\delta_{ij}} \beta_{ij,k} &= \sum_{ij} \frac{b_i c_j + b_j c_i}{1+\delta_{ij}} \alpha_{ij,k} = 0, \\ \frac{a_i c_j + a_j c_i}{1+\delta_{ij}} \alpha_{ij,k} &= \frac{b_i c_j + b_j c_i}{1+\delta_{ij}} \beta_{ij,k} = -d_k \sum_\mu d_\mu c_\mu. \end{aligned}$$

Proof. Straightforward calculations. ☐

From this claim it follows

$$\lambda^4 \sum_k \sum_{ij} \frac{a_i c_j + a_j c_i}{1+\delta_{ij}} \alpha_{ij,k}\xi_k + \lambda^4 \sum_k \sum_{ij} \frac{a_i c_j + a_j c_i}{1+\delta_{ij}} \beta_{ij,k}\eta_k + \lambda^4 \sum_{ij} d_i d_j \frac{\xi_i c_j + \xi_j c_i}{1+\delta_{ij}} = 0$$

and we obtain $T_1(Ah)(Bh) = \mu\lambda^4 T_1(A)(B)$. Analogously using the claim above one calculates $T_2(Ah)(Bh) = \mu\lambda^4 T_2(A)(B)$. This implies finally

$$\langle T_1(Ah)(Bh), T_2(Ah)(Bh) \rangle = \langle T_1(A)(B), T_2(A)(B) \rangle.$$

This completes the proof of Lemma 3.5. ☐

Lemma 3.5 says that the natural action of G on $\mathbb{P}N_{X_8/X}$ coincides with the action on $E_X = \mathbb{P}N_{X_8/X}$ described in Lemma 3.3.

3.1.4 Quotient $\widetilde{X} \to \widetilde{M}$.

Note that since $\nu^{-1}(M_8) = X_8$, we obtain a unique lifting $\widetilde{\nu}$ of ν, i. e., the commutative diagram

$$\begin{tikzcd} \widetilde{X} \arrow[r, "\exists!\ \widetilde{\nu}"] \arrow[d] & \widetilde{M} \arrow[d] \\ X \arrow[r, "\nu"] & M. \end{tikzcd}$$

Lemma 3.6. *The morphism $\nu : X \to M$ lifts to a morphism $\widetilde{\nu} : \widetilde{X} \to \widetilde{M}$ that is locally given by*

$$\widetilde{X}(d_0) \to \widetilde{M(x_0)}, \quad A \times \langle t_3, t_4\rangle \mapsto (\langle \det A\rangle, \langle p(A)\rangle) \times \langle (t_3, t_4) \begin{pmatrix} -b_1 & -b_2 \\ a_1 & a_2 \end{pmatrix}\rangle,$$

$$\widetilde{X}(d_1) \to \widetilde{M(x_1)}, \quad A \times \langle t_3, t_4\rangle \mapsto (\langle \det A\rangle, \langle p(A)\rangle) \times \langle (t_3, t_4) \begin{pmatrix} -b_0 & -b_2 \\ a_0 & a_2 \end{pmatrix}\rangle,$$

$$\widetilde{X}(d_2) \to \widetilde{M(x_2)}, \quad A \times \langle t_3, t_4\rangle \mapsto (\langle \det A\rangle, \langle p(A)\rangle) \times \langle (t_3, t_4) \begin{pmatrix} -b_0 & -b_1 \\ a_0 & a_1 \end{pmatrix}\rangle,$$

where $p(A) = z_1 \wedge z_2 = \langle d_0, d_1, d_2\rangle$.

Proof. Let E_X be the exceptional divisor of the blow up $\widetilde{X} \to X$ and let E_M be the exceptional divisor of the blow up $\widetilde{M} \to M$. Let us consider the map $X \setminus X_8 \xrightarrow{\nu} M \setminus M_8$. Since $X \setminus X_8$ is isomorphic to $\widetilde{X} \setminus E_X$ and since $M \setminus M_8$ is isomorphic to $\widetilde{M} \setminus E_M$, we obtain the map $\widetilde{X} \setminus E_X \to \widetilde{M} \setminus E_M$ given by the formulas above. But the same formulas define a morphism $\widetilde{X} \to \widetilde{M}$. $\square$

Lemma 3.7. *Let G be the group from 3.1.3 acting on $\widetilde{X}$. Then $\widetilde{\nu} : \widetilde{X} \to \widetilde{M}$ is G invariant and the set of the orbits coincides with the set of the fibres $\widetilde{\nu}^{-1}(\xi)$, $\xi \in \widetilde{M}$.*

Proof. First of all let us show that the points from the same orbit are mapped to the same point. Consider $A \times \langle t_3, t_4\rangle$ and $gAh \times \langle (t_3, t_4)g^{\mathrm{T}}\rangle$, where

$$A = \begin{pmatrix} a_0x_0 + a_1x_1 + a_2x_2 & q_1 \\ b_0x_0 + b_1x_1 + b_2x_2 & q_2 \end{pmatrix}, \quad g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, h = \begin{pmatrix} \lambda & z \\ 0 & \mu \end{pmatrix}.$$

Assume $d_0 \neq 0$ (other two cases are absolutely analogous). Then the linear forms of gAh are

$$\lambda(\alpha a_0 + \beta b_0)x_0 + \lambda(\alpha a_1 + \beta b_1)x_1 + \lambda(\alpha a_2 + \beta b_2)x_2,$$

and

$$\lambda(\gamma a_0 + \delta b_0)x_0 + \lambda(\gamma a_1 + \delta b_1)x_1 + \lambda(\gamma a_2 + \delta b_2)x_2.$$

The image of $A \times \langle t_3, t_4\rangle$ under $\widetilde{\nu}$ is

$$(\langle \det A\rangle, \langle p(A)\rangle) \times \langle (t_3, t_4) \begin{pmatrix} -b_1 & -b_2 \\ a_1 & a_2 \end{pmatrix}\rangle.$$

The image of $gAh \times \langle (t_3,t_4)g^{\mathrm{T}}\rangle$ is

$$(\langle \det gAh\rangle, \langle p(gAh)\rangle) \times \langle (t_3,t_4)g^{\mathrm{T}} \begin{pmatrix} -\lambda(\gamma a_1+\delta b_1) & -\lambda(\gamma a_2+\delta b_2) \\ \lambda(\alpha a_0+\beta b_0) & \lambda(\alpha a_2+\beta b_2)\end{pmatrix}\rangle$$

We know that $(\langle \det A\rangle, \langle p(A)\rangle) = (\langle \det gAh\rangle, \langle p(gAh)\rangle)$. Since

$$g^{\mathrm{T}} \begin{pmatrix} -\lambda(\gamma a_1+\delta b_1) & -\lambda(\gamma a_2+\delta b_2) \\ \lambda(\alpha a_1+\beta b_1) & \lambda(\alpha a_2+\beta b_2)\end{pmatrix} = \begin{pmatrix}\alpha & \gamma \\ \beta & \delta\end{pmatrix}\begin{pmatrix}\lambda & 0 \\ 0 & \lambda\end{pmatrix}\begin{pmatrix} -\gamma a_1-\delta b_1 & -\gamma a_2-\delta b_2 \\ \alpha a_1+\beta b_1 & \alpha a_2+\beta b_2\end{pmatrix} =$$
$$\begin{pmatrix}\alpha & \gamma \\ \beta & \delta\end{pmatrix}\begin{pmatrix}\lambda & 0 \\ 0 & \lambda\end{pmatrix}\begin{pmatrix}\delta & -\gamma \\ -\beta & \alpha\end{pmatrix}\begin{pmatrix}-b_1 & -b_2 \\ a_1 & a_2\end{pmatrix} = \lambda \det g \begin{pmatrix}-b_1 & -b_2 \\ a_1 & a_2\end{pmatrix},$$

we obtain

$$\langle (t_3,t_4)\begin{pmatrix}-b_1 & -b_2 \\ a_1 & a_2\end{pmatrix}\rangle = \langle (t_3,t_4)g^{\mathrm{T}} \begin{pmatrix} -\lambda(\gamma a_1+\delta b_1) & -\lambda(\gamma a_2+\delta b_2) \\ \lambda(\alpha a_0+\beta b_0) & \lambda(\alpha a_2+\beta b_2)\end{pmatrix}\rangle.$$

Now let us assume that $A \times \langle t_3,t_4\rangle$ and $A' \times \langle t'_3,t'_4\rangle$ are mapped to the same point. Since $X \to M$ is a geometrical quotient, we obtain immediately that $A' = gAh$ for some g and h.

We assume again $A \in X(d_0)$. Let again

$$A = \begin{pmatrix} a_0x_0+a_1x_1+a_2x_2 & q_1 \\ b_0x_0+b_1x_1+b_2x_2 & q_2\end{pmatrix}, \quad g = \begin{pmatrix}\alpha & \beta \\ \gamma & \delta\end{pmatrix}, h = \begin{pmatrix}\lambda & z \\ 0 & \mu\end{pmatrix}.$$

Then the equality of the images $A \times \langle t_3,t_4\rangle$ and $A' \times \langle t'_3,t'_4\rangle$ under $\widetilde{\nu}$ means

$$\langle (t_3,t_4)\begin{pmatrix}-b_1 & -b_2 \\ a_1 & a_2\end{pmatrix}\rangle = \langle (t'_3,t'_4)\begin{pmatrix}\lambda & 0 \\ 0 & \lambda\end{pmatrix}\begin{pmatrix}\delta & -\gamma \\ -\beta & \alpha\end{pmatrix}\begin{pmatrix}-b_1 & -b_2 \\ a_1 & a_2\end{pmatrix}\rangle.$$

This implies

$$\langle t'_3,t'_4\rangle = \langle (t_3,t_4)g^{\mathrm{T}}\rangle.$$

Therefore, $A \times \langle t_3,t_4\rangle$ and $A' \times \langle t'_3,t'_4\rangle$ lie in the same orbit of G. $\square$

Lemma 3.8. *Let $A \times \langle t_3,t_4\rangle$ be a point in $\widetilde{X}$, let $\widetilde{\xi} = \widetilde{\nu}(A \times \langle t_3,t_4\rangle)$ be its image under $\widetilde{\nu}$.*

Then there exists an open neighbourhood U of $\widetilde{\xi}$ and a morphism $s_U : U \to \widetilde{X}$ such that $\widetilde{\nu} \circ s_U = i_U$ and $s_U(\widetilde{\xi}) = A \times \langle t_3,t_4\rangle$, where $i_U : U \to \widetilde{X}$ is the inclusion map of U in $\widetilde{X}$. In other words there is a local section of $\widetilde{\nu}$ through every point in $\widetilde{X}$.

Proof. First of all note that since $X \xrightarrow{\nu} M$ is a geometrical quotient there is a local section of ν through every point of X. This means there exists an open neighbourhood $V \subseteq M$ of $\xi := \nu(A)$ and a morphism $\phi_V : V \to X$ such that $\nu \circ \phi_V = i_V$ and $\phi_V(\xi) = A$, where i_V is the inclusion map of V in M. Let $\widetilde{V} = \mathrm{Bl}_{V\cap M_8}(V)$ be the blowing up of $M_8 \cap V$ in V. We may consider $\widetilde{V}$ as an open subvariety in $\widetilde{M}$ using the cartesian diagram

$$\begin{tikzcd} \widetilde{V} \arrow[r, tail, "i_{\widetilde{V}}"] \arrow[d] & \widetilde{M} \arrow[d] \\ V \arrow[r, tail, "i_V"] & M. \end{tikzcd}$$

As $\phi_V^{-1}(X_8) = M_8 \cap V$, by universal property of blowing up there is a unique lifting $\widetilde{\phi}_V$ of $\phi_V : V \to X$ that makes the diagram

$$\begin{tikzcd} \widetilde{V} \arrow[r, dotted, "\exists!\ \widetilde{\phi}_V"] \arrow[d] & \widetilde{X} \arrow[r, "\widetilde{\nu}"] \arrow[d] & \widetilde{M} \arrow[d] \\ V \arrow[r, "\phi_V"] & X \arrow[r, "\nu"] & M \end{tikzcd}$$

commutative. As $\nu \circ \phi_V = i_V$ by universal property of blow up we conclude that $\widetilde{\nu} \circ \widetilde{\phi}_V = i_{\widetilde{V}}$. Clearly $\widetilde{V}$ is an open neighbourhood of $\widetilde{\xi}$. Since it holds

$$\widetilde{\nu}(\phi_{\widetilde{V}}(\widetilde{\xi})) = \widetilde{\xi} = \widetilde{\nu}(A \times \langle t_3, t_4 \rangle),$$

by Lemma 3.7 we conclude that $A \times \langle t_3, t_4 \rangle$ and $\phi_{\widetilde{V}}(\widetilde{\xi})$ lie in the same orbit of G. As both $A \times \langle t_3, t_4 \rangle$ and $\phi_{\widetilde{V}}(\widetilde{\xi})$ lie over $A \in X$, we conclude that they are equal. Hence $\widetilde{\phi}_V$ is a local section of $\widetilde{\nu}$ through the point $A \times \langle t_3, t_4 \rangle$. This proves the statement of the lemma. □

Recall that a principal bundle with the (algebraic)group $\mathfrak{G}$ is a fibre bundle $P \xrightarrow{p} B$ with the fibre $\mathfrak{G}$ such that the transition functions are given by the right action of $\mathfrak{G}$ on itself. Let us explain this. There exists an open covering $\{U_i\}$ of B and the isomorphisms $\phi_i : p^{-1}(U_i) \to U_i \times \mathfrak{G}$ such that for all $U_{ij} = U_i \cap U_j$ the transition function $\phi_i \circ \phi_j^{-1} : U_{ij} \times G \to U_{ij} \times G$ is given by the rule $(b, g) \mapsto (b, g \cdot g_{ij}(b))$ for some morphism $g_{ij} : U_{ij} \to \mathfrak{G}$. It is known that a principle bundle P may be always realized as a left free action of the group $\mathfrak{G}$ on P.

Proposition 3.9. *$\widetilde{X}$ is a principal vector bundle over $\widetilde{M}$ with fibre $\mathbb{P}G$. In particular $\widetilde{\nu} : \widetilde{X} \to \widetilde{M}$ is a geometrical quotient.*

Proof. Let ξ_0 be an arbitrary point in $\widetilde{M}$. Then by Lemma 3.8 there exists an open neighbourhood U of $\widetilde{X}$ and a local section $s_U : U \to \widetilde{X}$ of $\widetilde{\nu}$. Then the morphism

$$U \times \mathbb{P}G \to \widetilde{\nu}^{-1}(U), \quad (\xi, g) \mapsto g(s_U(\xi))$$

is a bijection. By Zariski main theorem (cf. [8], 6.1.14 and [9], 4.4.3) it is also an isomorphism. Thus $\widetilde{X}$ is a principal vector bundle over $\widetilde{M}$ with fibre $\mathbb{P}G$. □

3.2 Construction of a family over $\widetilde{X}$

We are going to construct here a variety Y over $\widetilde{X}$ and a sheaf $\widetilde{\mathcal{U}}$ on Y such that the fibres of $\widetilde{\mathcal{U}}$ over $s \in \widetilde{X}$ are either non-singular $3m+1$ sheaves on $\mathbb{P}_2$ or R-bundles on $\hat{\mathbb{P}}_2$.

3.2.1 Space over $\widetilde{X}$.

Let $l_B : U \to X$ be an embedding of an open set of $\Bbbk$ in X along a normal direction $B \in \Bbbk^{18}$ such that $0 \in U$ is the only point in U with the image in X_8 (cf. page 12).

Claim. *l_B uniquely factorizes through $\widetilde{X} \xrightarrow{\alpha} X$, i. e., there exists the commutative diagram*

$$\begin{tikzcd} & \widetilde{X} \arrow[d, "\alpha"] \\ U \arrow[ur, dotted, "\exists!\, \widetilde{l}_B"] \arrow[r, "l_B"] & X \end{tikzcd}$$

Proof. Follows from the universal property of blow-ups. □

Let us consider the commutative diagram

$$\begin{tikzcd} U \times \mathbb{P}_2 \arrow[r, "\widetilde{l}_B \times \mathrm{id}"] \arrow[d] & \widetilde{X} \times \mathbb{P}_2 \arrow[r, "\alpha \times \mathrm{id}"] \arrow[d] & X \times \mathbb{P}_2 \arrow[d] \\ U \arrow[r, "\widetilde{l}_B"] & \widetilde{X} \arrow[r, "\alpha"] & X. \end{tikzcd}$$

On $X \times \mathbb{P}_2$ we have the universal $3m+1$ sheaf $\mathcal{U}$ (cf. [5], 6.1). By pulling back we obtain the family $\bar{\mathcal{U}} := (\alpha \times \mathrm{id})^* \mathcal{U}$ of $3m+1$ sheaves over $\widetilde{X}$. On $U \times \mathbb{P}_2$ we obtain a sheaf $\mathcal{E} = (l_B \times \mathrm{id})^* \bar{\mathcal{U}}$ of the type (1.15). Let $S_8 = \mathrm{Sing}\,\mathcal{U}$ be the closed subvariety of $X \times \mathbb{P}_2$ where $\mathcal{U}$ is not locally free, i. e.,

$$S_8 = \{z_1 = z_2 = f_3 = f_4 = 0\}.$$

Lemma 3.10. *S_8 is isomorphic to X_8. In particular S_8 is smooth.*

Proof. The restriction of $X \times \mathbb{P}_2 \xrightarrow{pr_1} X$ to S_8 gives us a morphism $S_8 \xrightarrow{pr_1} X_8$. It is enough to construct the inverse morphism. Consider the morphism $X_8 \to S_8$ given by the rule $A \mapsto (A, p(A))$, where $p(A) = \langle d_0(A), d_1(A), d_2(A) \rangle$, i. e., the point defined by the linear forms of A (cf. (1.7)). This morphism is obviously the inverse to the $S_8 \xrightarrow{pr_1} X_8$. □

We obtain that $\widetilde{S}_8 := (\alpha \times \mathrm{id})^{-1}(S_8)$ is the set of points in $\widetilde{X} \times \mathbb{P}_2$ where the sheaf $\bar{\mathcal{U}}$ is not locally free.

Lemma 3.11. *$\widetilde{S}_8$ is isomorphic to the exceptional divisor $\widetilde{X}_8 = \alpha^{-1}(X_8)$ of the blow-up $\widetilde{X} \xrightarrow{\alpha} X$, in particular $\widetilde{S}_8$ is smooth.*

Proof. There is the morphism $\widetilde{S}_8 \xrightarrow{pr_1} \widetilde{X}_8$ (restriction of the projection $\widetilde{X} \times \mathbb{P}_2 \to \widetilde{X}$). The inverse morphism is given by

$$\widetilde{X}_8 \to \widetilde{S}_8, \quad (A, \langle t_3, t_4 \rangle) \mapsto (A, \langle t_3, t_4 \rangle) \times p(A).$$

This proves the lemma. □

Note that the preimage of $\widetilde{S}_8$ in $U \times \mathbb{P}_2$ is just the point $(0, p) \in U \times \mathbb{P}_2$, where $p := \{z_1(A) = z_2(A) = 0\}$.

Let

$$\tau : \widetilde{\widetilde{X} \times \mathbb{P}_2} \to \widetilde{X} \times \mathbb{P}_2$$

be the blowing up of $\widetilde{X} \times \mathbb{P}_2$ along $\widetilde{S}_8$. Then we get the cartesian diagram

$$\begin{tikzcd}
\widetilde{U \times \mathbb{P}_2} \arrow[r] \arrow[d, "\sigma"] & \widetilde{\widetilde{X} \times \mathbb{P}_2} \arrow[d, "\tau"] \\
U \times \mathbb{P}_2 \arrow[r, "\widetilde{l}_B \times \mathrm{id}"] & \widetilde{X} \times \mathbb{P}_2,
\end{tikzcd}$$

where σ is the blow-up of $U \times \mathbb{P}_2$ at the point $(0, p)$ and the horizontal arrows are embeddings. Let us denote $Y := \widetilde{\widetilde{X} \times \mathbb{P}_2}$. Let $\widetilde{D}_1$ denote the exceptional divisor of $\tau : Y \to \widetilde{X}$. Let $\widetilde{D}_0$ be the proper transform of $\widetilde{X}_8 \times \mathbb{P}_2$. It is isomorphic to the blow up of $\widetilde{X}_8 \times \mathbb{P}_2$ along $\widetilde{S}_8$.

As in Chapter 1 let $Z = \widetilde{U \times \mathbb{P}_2}$, let D_1 be the exceptional divisor of $\sigma : Z \to U \times \mathbb{P}_2$, and let D_0 be the proper transform of $\{0\} \times \mathbb{P}_2$.

Remark 3.12. *Note that $\widetilde{D}_0 \cap Z = D_0$ and $\widetilde{D}_1 \cap Z = D_1$.*

Let us describe the space Y. First of all note that there is the covering

$$\widetilde{X} \times \mathbb{P}_2 = \bigcup_{\substack{i=3,4 \\ k=0,1,2}} \widetilde{X}(t_i) \times \mathbb{P}_2(x_k),$$

where $\widetilde{X}(t_i) = \widetilde{X} \cap (X \times \mathbb{P}_1(t_i))$. Then the blow up of $\widetilde{X}(t_i) \times \mathbb{P}_2(x_k)$ is a subvariety in $\widetilde{X}(t_i) \times \mathbb{P}_2(x_k) \times \mathbb{P}_2$ given by the minors of the matrix

$$\begin{pmatrix} u_0 & u_1 & u_2 \\ f_k & \frac{z_1}{x_k} & \frac{z_2}{x_k} \end{pmatrix}.$$

The gluing maps of $\mathrm{Bl}_{\widetilde{S}_8}(\widetilde{X}(t_j) \times \mathbb{P}_2(x_l))$ and $\mathrm{Bl}_{\widetilde{S}_8}(\widetilde{X}(t_i) \times \mathbb{P}_2(x_k))$ are

$$\begin{aligned}
\mathrm{Bl}_{\widetilde{S}_8}(\widetilde{X}(t_j) \times \mathbb{P}_2(x_l)) &\xrightarrow{g_{ik,jl}} \mathrm{Bl}_{\widetilde{S}_8}(\widetilde{X}(t_i) \times \mathbb{P}_2(x_k)), \\
(\widetilde{A}, \langle x \rangle) \times \langle u_0, u_1, u_2 \rangle &\mapsto (\widetilde{A}, \langle x \rangle) \times \langle \frac{t_k}{t_l} \cdot u_0, \frac{x_j}{x_i} \cdot u_1, \frac{x_j}{x_i} \cdot u_2 \rangle.
\end{aligned}$$

Thus Y is embedded in the $\mathbb{P}_2$-bundle over $\widetilde{X} \times \mathbb{P}_2$ given by the cocycle

$$g_{ik,jl} = \begin{pmatrix} \frac{t_k}{t_l} & 0 & 0 \\ 0 & \frac{x_j}{x_i} & 0 \\ 0 & 0 & \frac{x_j}{x_i} \end{pmatrix}.$$

This is a cocycle of the vector bundle

$$\mathcal{O}_{\widetilde{X} \times \mathbb{P}_2}(0, -1) \oplus 2\mathcal{O}_{\widetilde{X} \times \mathbb{P}_2}(1, 0)$$

and also

$$\mathbb{P}(\mathcal{O}_{\widetilde{X} \times \mathbb{P}_2}(0, -1) \oplus 2\mathcal{O}_{\widetilde{X} \times \mathbb{P}_2}(1, 0)),$$

where

$$\mathcal{O}_{\widetilde{X}\times\mathbb{P}_2}(0,-1)=\mathcal{O}_{\widetilde{X}}(-1)\boxtimes\mathcal{O}_{\mathbb{P}_2}\quad\text{and}\quad\mathcal{O}_{\widetilde{X}\times\mathbb{P}_2}(1,0)=\mathcal{O}_{\widetilde{X}}\boxtimes\mathcal{O}_{\mathbb{P}_2}(1).$$

Let

$$P_Y:=\mathbb{P}(\mathcal{O}_{\widetilde{X}\times\mathbb{P}_2}(0,-1)\oplus 2\mathcal{O}_{\widetilde{X}\times\mathbb{P}_2}(1,0)),$$

then there is the commutative diagram

$$\begin{tikzcd}
Y \arrow[r, tail] \arrow[dr, two heads] & P_Y \arrow[r, tail] \arrow[d, two heads] & X\times\mathbb{P}(\mathcal{O}_{\mathbb{P}_1\times\mathbb{P}_2}(0,-1)\oplus 2\mathcal{O}_{\mathbb{P}_1\times\mathbb{P}_2}(1,0)) \arrow[d, two heads] \\
& \widetilde{X}\times\mathbb{P}_2 \arrow[r, tail] & X\times\mathbb{P}_1\times\mathbb{P}_2.
\end{tikzcd}$$

We proved the following proposition.

Proposition 3.13. *Y is a closed subvariety in a $\mathbb{P}_2$-bundle P_Y over $\widetilde{X}\times\mathbb{P}_2$. In particular the exceptional divisor D of the blow up τ is isomorphic to the restriction of P_Y to $\widetilde{S}_8$.*

There is also another possibility to describe the embedding $Y\subseteq P_Y$.

Since $\widetilde{S}_8$ is a locally complete intersection, we have $Y=\mathbf{P}(\mathcal{I}_{\widetilde{S}_8})$, where $\mathcal{I}_{\widetilde{S}_8}$ is the ideal sheaf of $\widetilde{S}_8$.

First of all note that the ideal sheaf $\mathcal{I}_{S_8}$ of $S_8\subseteq X\times\mathbb{P}_2$ is given by the surjective homomorphism

$$2\mathcal{O}_{X\times\mathbb{P}_2}(-1)\oplus 2\mathcal{O}_{X\times\mathbb{P}_2}\xrightarrow{\begin{pmatrix} z_1\\ z_2\\ f_3\\ f_4\end{pmatrix}}\mathcal{I}_{S_8}\to 0.$$

Lifting this to $\widetilde{X}\times\mathbb{P}_2$ yields the surjection

$$2\mathcal{O}_{\widetilde{X}\times\mathbb{P}_2}(-1)\oplus 2\mathcal{O}_{\widetilde{X}\times\mathbb{P}_2}\xrightarrow{\begin{pmatrix} z_1\\ z_2\\ f_3\\ f_4\end{pmatrix}}\mathcal{I}_{\widetilde{S}_8}\to 0,$$

where $\mathcal{I}_{\widetilde{S}_8}$ is the ideal sheaf of $\widetilde{S}_8\subseteq\widetilde{X}\times\mathbb{P}_2$.

Recall that $\widetilde{X}$ is a subvariety in $X\times\mathbb{P}_1$ given by the equation $t_3f_4-t_4f_3=0$ (cf. 3.1.2). Lifting Euler exact sequence

$$0\to\mathcal{O}_{\mathbb{P}_1}(-1)\xrightarrow{(t_4\ -t_3)}2\mathcal{O}_{\mathbb{P}_1}\xrightarrow{\begin{pmatrix} t_3\\ t_4\end{pmatrix}}\mathcal{O}_{\mathbb{P}_1}(1)\to 0$$

from $\mathbb{P}_1$ to $X\times\mathbb{P}_1$ and then to $\widetilde{X}$ and to $\widetilde{X}\times\mathbb{P}_2$ one obtains the exact sequence

$$\mathcal{O}_{\widetilde{X}\times\mathbb{P}_2}(0,-1)\xrightarrow{(t_4\ -t_3)}2\mathcal{O}_{\widetilde{X}\times\mathbb{P}_2}\xrightarrow{\begin{pmatrix} t_3\\ t_4\end{pmatrix}}\mathcal{O}_{\widetilde{X}\times\mathbb{P}_2}(0,1)\to 0,$$

where we use the notation $\mathcal{O}_{\widetilde{X}\times\mathbb{P}_2}(a,b):=\mathcal{O}_{\widetilde{X}}(b)\boxtimes\mathcal{O}_{\mathbb{P}_2}(a)$. As $t_3f_4-t_4f_3=0$ on $\widetilde{X}$, one obtains that the composition $\mathcal{O}_{\widetilde{X}\times\mathbb{P}_2}(0,-1)\xrightarrow{(t_4\ -t_3)}2\mathcal{O}_{\widetilde{X}\times\mathbb{P}_2}\xrightarrow{\begin{pmatrix} f_3\\ f_4\end{pmatrix}}\mathcal{I}_{\widetilde{S}_8}$ is zero. Therefore, by universal property of cokernel we obtain there is a unique factorization through $\mathcal{O}_{\widetilde{X}\times\mathbb{P}_2}(0,1)$:

$$\begin{tikzcd}
\mathcal{O}_{\widetilde{X}\times\mathbb{P}_2}(0,-1) \arrow[r, "(t_4\ -t_3)"] \arrow[dr, "0"'] & 2\mathcal{O}_{\widetilde{X}\times\mathbb{P}_2} \arrow[r, two heads, "\begin{pmatrix} t_3\\ t_4\end{pmatrix}"] \arrow[d, "\begin{pmatrix} f_3\\ f_4\end{pmatrix}"] & \mathcal{O}_{\widetilde{X}\times\mathbb{P}_2}(0,1) \arrow[r] \arrow[dl, dotted, "\exists!"] & 0. \\
& \mathcal{I}_{\widetilde{S}_8} & &
\end{tikzcd}$$

From this commutative diagram we obtain the commutative diagram

$$\begin{tikzcd}[column sep=small]
2\mathcal{O}_{\widetilde{X}\times\mathbb{P}_2}(-1,0)\oplus\mathcal{O}_{\widetilde{X}\times\mathbb{P}_2}(0,-1) \arrow[r] \arrow[dr, "\left(\begin{smallmatrix} z_1\\ z_2\\ 0\end{smallmatrix}\right)"'] & 2\mathcal{O}_{\widetilde{X}\times\mathbb{P}_2}(-1,0)\oplus 2\mathcal{O}_{\widetilde{X}\times\mathbb{P}_2} \arrow[r, two heads] \arrow[d, two heads, "\left(\begin{smallmatrix} z_1\\ z_2\\ f_3\\ f_4\end{smallmatrix}\right)"] & 2\mathcal{O}_{\widetilde{X}\times\mathbb{P}_2}(-1,0)\oplus\mathcal{O}_{\widetilde{X}\times\mathbb{P}_2}(0,1). \arrow[dl, two heads] \\
& \mathcal{I}_{\widetilde{S}_8} &
\end{tikzcd}$$

Applying the functor $\mathbf{P}$ to the right commutative triangle we obtain the commutative diagram of closed inclusions

$$\begin{tikzcd}
\mathbf{P}(2\mathcal{O}_{\widetilde{X}\times\mathbb{P}_2}(-1,0)\oplus 2\mathcal{O}_{\widetilde{X}\times\mathbb{P}_2}) & & \mathbf{P}(2\mathcal{O}_{\widetilde{X}\times\mathbb{P}_2}(-1,0)\oplus\mathcal{O}_{\widetilde{X}\times\mathbb{P}_2}(0,1)). \arrow[ll] \\
& Y=\mathbf{P}(\mathcal{I}_{\widetilde{S}_8}) \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

Note that $\mathbf{P}(2\mathcal{O}_{\widetilde{X}\times\mathbb{P}_2}(-1,0)\oplus\mathcal{O}_{\widetilde{X}\times\mathbb{P}_2}(0,1))=\mathbb{P}(2\mathcal{O}_{\widetilde{X}\times\mathbb{P}_2}(1,0)\oplus\mathcal{O}_{\widetilde{X}\times\mathbb{P}_2}(0,-1))=P_Y$.

Lemma 3.14. *1) The fibres Y_x, $x\in\widetilde{X}$, of the morphism*

$$Y\xrightarrow{\tau}\widetilde{X}\times\mathbb{P}_2\xrightarrow{p_1}\widetilde{X}$$

are isomorphic to $\mathbb{P}_2$ if x does not lie in $\widetilde{X}_8$.

2) If x belongs to $\widetilde{X}_8$, then Y_x is isomorphic to $\hat{\mathbb{P}}_2$ (see Definition 1.11).

Proof. Note that the morphisms we consider are over X, i. e., we have the commutative diagram

$$\begin{tikzcd}
Y \arrow[r, "\tau"] \arrow[dr] & \widetilde{X}\times\mathbb{P}_2 \arrow[r, "p_1"] \arrow[d] & \widetilde{X} \arrow[dl, "\alpha"] \\
& X &
\end{tikzcd}$$

The first part of the lemma holds true because τ is an isomorphism over $X\setminus X_8$.

Let now $x\in\widetilde{X}_8$. For an open set $U\subseteq\Bbbk$, $0\in U$, there exists an embedding $\widetilde{l}_B:U\to\widetilde{X}$ transversal to $\widetilde{X}_8$ such that $\widetilde{l}_B(0)=x$. The required statement follows from the commutative (cartesian) diagram

$$\begin{tikzcd}
Z \arrow[r, "L_B"] \arrow[d, "\sigma"] & Y \arrow[d, "\tau"] \\
U\times\mathbb{P}_2 \arrow[r, "\widetilde{l}_B\times\mathrm{id}"] \arrow[d, "p_1"] & \widetilde{X}\times\mathbb{P}_2 \arrow[d, "p_1"] \\
U \arrow[r, "\widetilde{l}_B"] & \widetilde{X}
\end{tikzcd} \tag{3.2}$$

and from Lemma 1.10 because in this case $Y_x\cong Z_0\cong\hat{\mathbb{P}}_2$. □

Proposition 3.15. *The morphism*

$$Y\xrightarrow{\tau}\widetilde{X}\times\mathbb{P}_2\xrightarrow{p_1}\widetilde{X}$$

is flat.

Proof. By Lemma 3.14, $\dim(Y_x)=2$ for all $x\in\widetilde{X}$. As $\dim Y=20$ and $\dim\widetilde{X}=18$, we obtain $\dim Y_x=2=\dim Y-\dim\widetilde{X}$. Since both Y and $\widetilde{X}$ are regular, by Theorem A.2 we conclude that $p_1\circ\tau$ is flat. □

3.2.2 Construction of a sheaf.

The universal sheaf $\mathcal{U}$ on $X \times \mathbb{P}_2$ is given by the resolution

$$0 \to 2\mathcal{O}_{X\times\mathbb{P}_2}(-2H) \xrightarrow{\Phi} \mathcal{O}_{X\times\mathbb{P}_2}(-H) \oplus \mathcal{O}_{X\times\mathbb{P}_2} \to \mathcal{U} \to 0.$$

Pulling back this sequence, we obtain

$$0 \to 2\mathcal{O}_{\widetilde{X}\times\mathbb{P}_2}(-2H) \xrightarrow{\bar{\Phi}} \mathcal{O}_{\widetilde{X}\times\mathbb{P}_2}(-H) \oplus \mathcal{O}_{\widetilde{X}\times\mathbb{P}_2} \to \bar{\mathcal{U}} \to 0.$$

After applying τ^* one gets the sequence

$$0 \to 2\mathcal{O}_Y(-2H) \xrightarrow{\tau^*(\bar{\Phi})} \mathcal{O}_Y(-H) \oplus \mathcal{O}_Y \to \tau^*(\bar{\mathcal{U}}) \to 0,$$

which remains exact because the sheaf $O_Y(-2H)$ is locally free and, therefore, has no torsion.

Let D be the exceptional divisor of the blowing up τ. Let s be the canonical section of $\mathcal{O}_Y(D)$ from the sequence

$$0 \to \mathcal{O}_Y(-D) \xrightarrow{s} \mathcal{O}_Y \to \mathcal{O}_D \to 0.$$

Then by Lemma 1.15 $\tau^*(\bar{\Phi})$ factorizes through s and we obtain the following commutative diagram.

$$\begin{tikzcd}
 & & & & 0 \arrow[d] & \\
 & 0 \arrow[d] & & & \mathcal{C} \arrow[d] & \\
0 \arrow[r] & 2\mathcal{O}_Y(-2H) \arrow[r, "\tau^*(\bar{\Phi})"] \arrow[d, "\begin{pmatrix} s & 0 \\ 0 & s \end{pmatrix}"] & \mathcal{O}_Y(-H) \oplus \mathcal{O}_Z \arrow[r] \arrow[d, equal] & & \tau^*\bar{\mathcal{U}} \arrow[r] \arrow[d] & 0, \\
0 \arrow[r] & 2\mathcal{O}_Y(-2H+D) \arrow[r, "\widetilde{\Phi}"] \arrow[d] & \mathcal{O}_Y(-H) \oplus \mathcal{O}_Z \arrow[rr] & & \widetilde{\mathcal{U}} \arrow[r] \arrow[d] & 0 \\
 & \mathcal{C} \arrow[d] & & & 0 & \\
 & 0 & & & &
\end{tikzcd}$$

$\mathcal{C} = \mathcal{O}_D \otimes \mathcal{O}_Y(-2H+D) \cong \mathcal{O}_D \otimes \mathcal{O}_Y(D) \otimes \mathcal{O}_Y(-2H) \cong \mathcal{O}_D(-1) \otimes \mathcal{O}_Y(-2H)$.

Remark 3.16. *Note that the restriction of $\widetilde{\mathcal{U}}$ to Z (via L_B as in diagram* (3.2)*) is isomorphic to the* 1*-parameter new family $\widetilde{\mathcal{E}}$ constructed at $\widetilde{l}_B(0)$ along B (see 1.2.1).*

In particular among the fibres of $\widetilde{\mathcal{U}}$ we obtain all the equivalence classes of R-bundles on $\hat{\mathbb{P}}_2$.

Proposition 3.17. $\tau_*(\widetilde{\mathcal{U}}) \cong \bar{\mathcal{U}}$.

Proof. By projection formula and by Lemma A.4 we have

$$R^p\tau_*(\mathcal{C}) \cong R^p\tau_*(\mathcal{O}_D(-1) \otimes \mathcal{O}_Y(-2H)) \cong R^p\tau_*(\mathcal{O}_D(-1)) \otimes \mathcal{O}_{\widetilde{X}\times\mathbb{P}_2}(-2H) = 0$$

for all $p \geqslant 0$. Therefore, after applying τ_* to the exact sequence

$$0 \to \mathcal{C} \to \tau^*\bar{\mathcal{U}} \to \widetilde{\mathcal{U}} \to 0,$$

we obtain $\tau_*(\tau^*\bar{\mathcal{U}}) \cong \tau_*(\widetilde{\mathcal{U}})$. Since by Lemma A.8 $\tau_*(\tau^*\bar{\mathcal{U}}) \cong \bar{\mathcal{U}}$, one obtains $\tau_*(\widetilde{\mathcal{U}}) \cong \bar{\mathcal{U}}$. $\square$

Proposition 3.18. *The sheaf $\widetilde{\mathcal{U}}$ is flat over $\widetilde{X}$.*

Proof. Since the sheaf $\widetilde{\mathcal{U}}$ is just a $3m+1$ family over $\widetilde{X} \setminus \alpha^{-1}(X_8)$, it is enough to prove the flatness over the points from $\alpha^{-1}(X_8)$. For each such a point x consider diagram (3.2).

Consider the sequence

$$0 \to 2\mathcal{O}_{\widetilde{X}\times\mathbb{P}_2}(-2H) \xrightarrow{\bar{\Phi}} \mathcal{O}_{\widetilde{X}\times\mathbb{P}_2}(-H) \oplus \mathcal{O}_{\widetilde{X}\times\mathbb{P}_2} \to \bar{\mathcal{U}} \to 0.$$

After applying τ^* one gets the following sequence on Y:

$$0 \to \tau^*(2\mathcal{O}_{\widetilde{X}\times\mathbb{P}_2}(-2H)) \xrightarrow{\tau^*\bar{\Phi}} \tau^*(\mathcal{O}_{\widetilde{X}\times\mathbb{P}_2}(-H) \oplus \mathcal{O}_{\widetilde{X}\times\mathbb{P}_2}) \to \tau^*\bar{\mathcal{U}} \to 0.$$

The sheaf $\widetilde{\mathcal{U}}$ is by definition a cokernel of the factorization $\widetilde{\Phi}$ as in the diagram

$$\begin{tikzcd} \tau^*(2\mathcal{O}_{\widetilde{X}\times\mathbb{P}_2}(-2H)) \arrow[r, "\tau^*\bar{\Phi}"] \arrow[d, "s"] & \tau^*(\mathcal{O}_{\widetilde{X}\times\mathbb{P}_2}(-H) \oplus \mathcal{O}_{\widetilde{X}\times\mathbb{P}_2}) \arrow[r] & \tau^*\bar{\mathcal{U}} \arrow[r] & 0 . \\ \tau^*(2\mathcal{O}_{\widetilde{X}\times\mathbb{P}_2}(-2H)) \otimes \mathcal{O}_Y(D) \arrow[ur, dotted, "\exists!\ \widetilde{\Phi}"'] & & & \end{tikzcd}$$

Therefore, the restriction $L_B^*(\widetilde{\mathcal{U}})$ of $\widetilde{\mathcal{U}}$ to Z is a cokernel of $L_B^*(\widetilde{\Phi})$. Applying L_B^* to the diagram above gives us the diagram

$$\begin{tikzcd} L_B^*\tau^*(2\mathcal{O}_{\widetilde{X}\times\mathbb{P}_2}(-2H)) \arrow[r, "L_B^*\tau^*\bar{\Phi}"] \arrow[d, "L_B^* s"] & L_B^*\tau^*(\mathcal{O}_{\widetilde{X}\times\mathbb{P}_2}(-H) \oplus \mathcal{O}_{\widetilde{X}\times\mathbb{P}_2}) \arrow[r] & L_B^*\tau^*\bar{\mathcal{U}} \arrow[r] & 0 . \\ L_B^*(\tau^*(2\mathcal{O}_{\widetilde{X}\times\mathbb{P}_2}(-2H)) \otimes \mathcal{O}_Y(D)) \arrow[ur, "L_B^*\widetilde{\Phi}"'] & & & \end{tikzcd}$$

Note that pulling back to Z the canonical section s of the exceptional divisor D of the blow up τ gives the canonical section s_Z of the exceptional divisor $D_Z = D \cap Z$ of σ. Therefore, applying the isomorphism $L_B^*\tau^* \cong \sigma^*(\widetilde{l}_B \times \mathrm{id})^*$ to the previous diagram we will obtain the diagram defining $\widetilde{\mathcal{E}}$ (note that $\mathcal{E} = (l_B \times \mathrm{id})^*\bar{\mathcal{U}}$):

$$\begin{tikzcd} \sigma^*(2\mathcal{O}_{U\times\mathbb{P}_2}(-2H)) \arrow[r] \arrow[d, "s_Z"] & \sigma^*(\mathcal{O}_{U\times\mathbb{P}_2}(-H) \oplus \mathcal{O}_{U\times\mathbb{P}_2}) \arrow[r] & \sigma^*\mathcal{E} \arrow[r] & 0 . \\ \sigma^*(2\mathcal{O}_{U\times\mathbb{P}_2}(-2H)) \otimes \mathcal{O}_Z(D_Z)) \arrow[ur] & & & \end{tikzcd}$$

This means that the restriction of the resolution of $\widetilde{\mathcal{U}}$

$$0 \to 2\mathcal{O}_Y(-2H+D) \xrightarrow{\widetilde{\Phi}} \mathcal{O}_Y(-H) \oplus \mathcal{O}_Y \to \widetilde{\mathcal{U}} \to 0$$

to the fibre Y_x is isomorphic to the restriction of the resolution of $\widetilde{\mathcal{E}}$

$$0 \to 2\mathcal{O}_Z(-2H+D) \xrightarrow{\widetilde{A}_t} \mathcal{O}_Z(-H) \oplus \mathcal{O}_Z \to \widetilde{\mathcal{E}} \to 0$$

to the fibre Z_0. Thus we conclude that $\widetilde{\Phi}|_{Y_x}$ is injective (because $\widetilde{A}_t|_{Z_0}$ is) and therefore $\mathscr{T}or_1^{\mathcal{O}_Y}(\widetilde{\mathcal{U}}, O_{Y_x}) = 0$, i. e., $\widetilde{\mathcal{U}}$ is flat over $\widetilde{X}$. $\square$

Remark 3.19. *In the proof of Proposition 3.18 we have shown in particular that the fibres of the sheaf* $\widetilde{\mathcal{U}}$ *are either non-singular* $3m+1$ *sheaves or* R*-bundles on* $\hat{\mathbb{P}}_2$.

Proposition 3.20. *The sheaf* $\widetilde{\mathcal{U}}$ *is locally free on its support.*

Proof. Since by Proposition 3.18 the sheaf $\widetilde{\mathcal{U}}$ is flat over $\widetilde{X}$, by Lemma 2.1.7 from [13] it is enough to show that the fibres of $\widetilde{\mathcal{U}}$ over the points from $\widetilde{X}$ are locally free on their support. But the fibres are either non-singular $3m+1$ sheaves on $\mathbb{P}_2$ or R-bundles. This completes the proof. $\square$

We obtained a family over $\widetilde{X}$ which has as its fibres all the non-singular $3m+1$ sheaves (up to isomorphisms) and all R-bundles on $\hat{\mathbb{P}}_2$ (up to equivalence).

3.3 New families over an arbitrary S

In this section we present the construction of a family over $\widetilde{X}$. This construction allows us to construct all R-bundles on $\hat{\mathbb{P}}_2$ simultaneously.

3.3.1 Construction of a functor.

Definition.

Definition 3.21. *A (new) family over S consists of the following data.*

- *A flat morphism $\pi : Z \to S$ such that for a point $s \in S$ the fibre $Z_s := \pi^{-1}(s)$ is either isomorphic to $\mathbb{P}_2$ or to $\hat{\mathbb{P}}_2$ (see Definition 1.11).*
- *A morphism (contraction) $Z \xrightarrow{\sigma} S \times \mathbb{P}_2$ over S such that over each point $s \in S$ the restriction $Z_s \xrightarrow{\sigma_s} \mathbb{P}_2$ is either an isomorphism $\mathbb{P}_2 \to \mathbb{P}_2$ or the contraction of $\mathbb{P}_2$ in $\hat{\mathbb{P}}_2 = \widetilde{\mathbb{P}}_2 \cup \mathbb{P}_2$ to some point in $\mathbb{P}_2$.*
- *An invertible sheaf $\mathcal{L}''$ on Z with the following properties.*
  - *The restrictions ${\mathcal{L}''}_s$ to the fibres Z_s are isomorphic to $\mathcal{O}_{Z_s}(0,1)$ (cf. Remark 1.46);*
  - *The sheaf $\mathcal{L} = \mathcal{L}' \otimes \mathcal{L}''$, where $\mathcal{L}' = \sigma^*(\mathcal{O}_S \boxtimes \mathcal{O}_{\mathbb{P}_2}(1))$, is a very ample invertible sheaf on Z relative to S (we use here Definition 4.4.2 from [8] of a relative very ample sheaf).*
- *A coherent sheaf $\mathcal{E}$ on Z that is flat over S and such that the restrictions $\mathcal{E}_s$ of $\mathcal{E}$ to the fibres Z_s are either non-singular $3m+1$ sheaves if $Z_s \cong \mathbb{P}_2$ (cf. Definition 1.3) or R-bundles on $\hat{\mathbb{P}}_2$ if $Z_s \cong \hat{\mathbb{P}}_2$.*
- *A $3m+1$ family $\mathcal{F}$ on $S \times \mathbb{P}_2$ and a surjective morphism $\sigma^*\mathcal{F} \xrightarrow{\tau} \mathcal{E}$ such that for $s \in S$ the kernel of $(\sigma^*\mathcal{F})_s \xrightarrow{\tau_s} \mathcal{E}_s$ is zero in the case $Z_s \cong \mathbb{P}_2$ and is isomorphic to $2\mathcal{O}_{D_1}(-L)$ if $Z_s \cong \hat{\mathbb{P}}_2$.*
- *Let $\operatorname{Sing} \mathcal{F}$ be the closed subvariety in $S \times \mathbb{P}_2$ where $\mathcal{F}$ is not free. We require that the restriction of σ to $\sigma^{-1}(S \times \mathbb{P}_2 \setminus \operatorname{Sing} \mathcal{F})$*

  $$\sigma^{-1}(S \times \mathbb{P}_2 \setminus \operatorname{Sing} \mathcal{F}) \xrightarrow{\sigma} S \times \mathbb{P}_2 \setminus \operatorname{Sing} \mathcal{F}$$

  is an isomorphism of open subvarieties.

We denote $\mathcal{O}_Z(1,0) := \mathcal{L}'$, $\mathcal{O}_Z(0,1) := \mathcal{L}''$, and analogously $\mathcal{O}_Z(a,b) := \mathcal{L}'^{\otimes a} \otimes \mathcal{L}''^{\otimes b}$.

Example 3.22. *1) R-bundles described in 1.3.2 are families over one point.*

2) The one parameter families constructed in 1.2.1 are families over open sets in $\Bbbk$.

3) The family over $\widetilde{X}$ that was constructed in 3.2 is a family over $\widetilde{X}$ in the sense of Definition 3.21.

Proof. Since the statements 1) and 2) are trivial, it remains to prove 3).

First of all we need the a sheaf $\mathcal{L}''$ on Y such that $\mathcal{L}''_s \cong \mathcal{O}_{Y_s}(0,1)$ for every $s \in \widetilde{X}$. Consider the exceptional divisor $\widetilde{D}_1$ of the blow up $Y \to \widetilde{X} \times \mathbb{P}_2$. Then by Remark 3.12 and by Lemma 1.20 one concludes $\mathcal{O}_Y(\widetilde{D}_1)|_{Y_s} \cong \mathcal{O}_{Y_s}(1,-1)$. Putting $\mathcal{L}'' := \mathcal{O}_Y(-\widetilde{D}_1) \otimes \mathcal{L}'$ we obtain $\mathcal{L}''_s \cong \mathcal{O}_{Y_s}(0,1)$.

It remains to prove that $\mathcal{L} = \mathcal{L}' \otimes \mathcal{L}''$ is a very ample sheaf relative to $\widetilde{X}$. Note that the sheaf $\mathcal{L}_s$ is isomorphic to $\mathcal{O}_{Y_s}(1,1)$ for every $s \in \widetilde{X}$. Therefore, one concludes that $\mathcal{L}_s$ is very ample relative to $\{s\}$ as a restriction of the sheaf $\mathcal{O}_{\mathbb{P}_2 \times \mathbb{P}_2}(1,1)$ to $Y_s \subseteq \mathbb{P}_2 \times \mathbb{P}_2$. Let π be the projection $Y \to \widetilde{X}$. Then we have a closed embedding $Y_s \to \mathbf{P}(\pi_* \mathcal{L}_s)$.

Consider the canonical morphism $\pi^* \pi_* \mathcal{L} \to \mathcal{L}$. By Proposition 1.32 we obtain

$$H^1(Z_s, \mathcal{L}_s) \cong H^1(Z_s, \mathcal{O}_{Z_s}(1,1)_s) = 0 \quad \text{and} \quad H^0(Z_s, \mathcal{L}_s) \cong H^0(Z_s, \mathcal{O}_{Z_s}(1,1)_s) \cong \Bbbk^6.$$

Therefore, by the base change theorem we conclude that the canonical map $\varphi^0(s) : \pi_* \mathcal{L}(s) \to H^0(Z_s, \mathcal{L}_s)$ is an isomorphism for every $s \in S$. Since $\mathcal{L}_s \cong \mathcal{O}_{Z_s}(1,1)$ is generated by its global sections, we conclude that the evaluation map $\mathcal{O}_{Y_s} \otimes_\Bbbk H^0(Y_s, \mathcal{L}_s) \xrightarrow{\text{ev}} \mathcal{L}_s$ is surjective for every $s \in S$.

Note that $(\pi^* \pi_* \mathcal{L})_s \cong \mathcal{O}_{Y_s} \otimes_\Bbbk (\pi_* \mathcal{L})(s)$. Under this identification the restriction of the canonical homomorphism $\pi^* \pi_* \mathcal{L} \to \mathcal{L}$ to the fibre Y_s coincides with the composition

$$\mathcal{O}_{Y_s} \otimes_\Bbbk (\pi_* \mathcal{L})(s) \xrightarrow{\text{id} \otimes \varphi^0(s)} \mathcal{O}_{Y_s} \otimes_\Bbbk H^0(Y_s, \mathcal{L}_s) \xrightarrow{\text{ev}} \mathcal{L}_s.$$

Since $\varphi^0(s)$ is a n isomorphism and since ev is surjective, we conclude that $(\pi^* \pi_* \mathcal{L})_s \to \mathcal{L}_s$ is surjective for all $s \in \widetilde{X}$ and hence $\pi^* \pi_* \mathcal{L} \to \mathcal{L}$ is surjective.

Therefore, the surjection $\pi^* \pi_* \mathcal{L} \to \mathcal{L}$ induces a morphism $Y \to \mathbf{P}(\pi_* \mathcal{L})$. We will show that this morphism is a closed embedding. Since $\mathbf{P}(\pi_* \mathcal{L}_s) \cong \mathbf{P}(\pi_* \mathcal{L})_s$ and since $Y_s \to \mathbf{P}(\pi_* \mathcal{L}_s)$ is injective, we conclude that $Y \to \mathbf{P}(\pi_* \mathcal{L})$ is injective as well. Consider the commutative diagram

$$\begin{tikzcd} Y \arrow[rr] \arrow[dr, "\pi"'] & & \mathbf{P}(\pi_* \mathcal{L}). \arrow[dl] \\ & \widetilde{X} & \end{tikzcd}$$

Note that π is a proper morphism. Note also that $\mathbf{P}(\pi_* \mathcal{L}) \to \widetilde{X}$ is separated as a projective morphism. Then by Corollary 4.8 e), II from [12] we conclude that the morphism $Y \to \mathbf{P}(\pi_* \mathcal{L})$ is proper. In particular it is closed. As we already proved that this morphism is injective, this implies that $Y \to \mathbf{P}(\pi_* \mathcal{L})$ is a closed embedding, i. e., $\mathcal{L}$ is a very ample sheaf relative to $\widetilde{X}$. $\square$

Remark 3.23. *Let $\mathcal{F}$ be a non-singular $3m+1$ family over S. Put*

$$Z = S \times \mathbb{P}_2, \quad \pi = p_1 : S \times \mathbb{P}_2 \to S, \quad \sigma = \text{id}_{S \times \mathbb{P}_2}, \quad \mathcal{L}'' = \mathcal{O}_S \boxtimes \mathcal{O}_{\mathbb{P}_2}(1), \quad \mathcal{E} = \mathcal{F}$$

and let τ be the canonical isomorphism $\epsilon_{S \times \mathbb{P}_2}(\mathcal{F}) : \text{id}^*_{S \times \mathbb{P}_2} \mathcal{F} \xrightarrow{\cong} \mathcal{F}$. Then $Z = S \times \mathbb{P}_2$ is embedded into

$$\mathbf{P}(3\mathcal{O}_{S \times \mathbb{P}_2}) \cong (S \times \mathbb{P}_2) \times \mathbb{P}_2$$

by the diagonal map $(s, \langle x \rangle) \mapsto (s, \langle x \rangle) \times \langle x \rangle$. We obtained a family from Definition 3.21. We constructed a map that sends a non-singular $3m+1$ family over S to a family from Definition 3.21.

We have now a correspondence

$$S \mapsto \text{set of the families over } S.$$

We would like to have a functor. We need "pull-backs".

Pull-backs of the families.

For an arbitrary family over S and for an arbitrary morphism $T \to S$ we will construct a family over T.

Let $(\pi : Z \to S, \sigma, \mathcal{L}'', \mathcal{E}, \mathcal{F}, \tau)$ be a family over S and let $T \xrightarrow{f} S$ be a morphism. Let

$$\begin{tikzcd} Z_T \arrow[r, "F"] \arrow[d, "\pi_T"'] & Z \arrow[d, "\pi"] \\ T \arrow[r, "f"] & S \end{tikzcd}$$

be the pull-back diagram. Note that Z_T is flat over T, the sheaf $F^*\mathcal{E}$ is flat over T as well (cf. [12], III, Proposition 9.2, (b)). Since

$$\begin{tikzcd} T \times \mathbb{P}_2 \arrow[r, "f \times \mathrm{id}"] \arrow[d, "p_1"] & S \times \mathbb{P}_2 \arrow[d, "p_1"] \\ T \arrow[r, "f"] & S \end{tikzcd}$$

is also a pull-back diagram, by universal property of pull-back there exists a unique arrow $\sigma_T : Z_T \to T \times \mathbb{P}_2$ that makes the following diagram commutative.

$$\begin{tikzcd} & T \times \mathbb{P}_2 \arrow[rr, "f \times \mathrm{id}"] \arrow[dl, "p_1"] & & S \times \mathbb{P}_2 \arrow[dl, "p_1"] \\ Z_T \arrow[ur, dotted, "\exists!\ \sigma_T"] \arrow[rr, "F"] \arrow[d, "\pi_T"'] & & Z \arrow[ur, "\sigma"] \arrow[d, "\pi"] & \\ T \arrow[rr, "f"] & & S & \end{tikzcd}$$

Note that over each point of T, the morphism σ_T is either an isomorphism $\mathbb{P}_2 \cong \mathbb{P}_2$ or a contraction $\hat{\mathbb{P}}_2 \to \mathbb{P}_2$ as described in Definition 3.21.

Applying F^* to the morphism $\sigma^*\mathcal{F} \xrightarrow{\tau} \mathcal{E}$ we obtain a surjective morphism $F^*\sigma^*\mathcal{F} \xrightarrow{F^*\tau} F^*\mathcal{E}$ Since the diagram

$$\begin{tikzcd} Z_T \arrow[r, "F"] \arrow[d, "\sigma_T"'] & Z \arrow[d, "\sigma"] \\ T \times \mathbb{P}_2 \arrow[r, "f \times \mathrm{id}"] & S \times \mathbb{P}_2 \end{tikzcd}$$

is commutative we obtain that $F^*\sigma^*\mathcal{F} \cong \sigma_T^*(f \times \mathrm{id})^*\mathcal{F}$ and in this way we get a surjective morphism $\sigma_T^*(f \times \mathrm{id})^*\mathcal{F} \to F^*\mathcal{E}$. By abuse of notation we will call this morphism $f^*\tau$. Note that $(f \times \mathrm{id})^*\mathcal{F}$ is a $3m+1$ family as a pull-back of a $3m+1$ family.

Consider the invertible sheaf $F^*\mathcal{L}''$. Note that

$$\sigma_T^*(\mathcal{O}_T \boxtimes \mathcal{O}_{\mathbb{P}_2}) \cong F^*\sigma^*(\mathcal{O}_S \boxtimes \mathcal{O}_{\mathbb{P}_2})).$$

Therefore, the invertible sheaf

$$\sigma_T^*(\mathcal{O}_T \boxtimes \mathcal{O}_{\mathbb{P}_2}) \otimes F^*\mathcal{L}'' \cong F^*\sigma^*(\mathcal{O}_S \boxtimes \mathcal{O}_{\mathbb{P}_2})) \otimes F^*\mathcal{L}'' \cong F^*(\sigma^*(\mathcal{O}_S \boxtimes \mathcal{O}_{\mathbb{P}_2})) \otimes \mathcal{L}'')$$

is very ample relative to T as a pull back of the sheaf $\sigma^*(\mathcal{O}_S \boxtimes \mathcal{O}_{\mathbb{P}_2})) \otimes \mathcal{L}''$, which is very ample relative to S by Definition 3.21. So we obtain a family

$$f^*(\pi, \sigma, \mathcal{L}'', \mathcal{E}, \mathcal{F}, \tau) := (\pi_T, \sigma_T, F^*\mathcal{L}'', F^*\mathcal{E}, (f \times \mathrm{id})^*\mathcal{F}, f^*\tau) \tag{3.3}$$

over T. We will call this family over T the pull-back of the family $(\pi, \sigma, \mathcal{L}'', \mathcal{E}, \mathcal{F}, \tau)$ along f.

Equivalence of families.

Definition 3.24. *Let $(\pi_1, \sigma_1, \mathcal{L}''_1, \mathcal{E}_1, \mathcal{F}_1, \tau_1)$ and $(\pi_2, \sigma_2, \mathcal{L}''_2, \mathcal{E}_2, \mathcal{F}_2, \tau_2)$ be two families over S, i. e., $\sigma_1 : Z_1 \to S \times \mathbb{P}_2$ and $\sigma_2 : Z_2 \to S \times \mathbb{P}_2$. Then they are called **equivalent** if the following statements hold true.*

- *There exists an isomorphism $\xi : Z_1 \to Z_2$ such that $\sigma_1 = \sigma_2 \circ \xi$ and such that $\xi^*\mathcal{L}''_2 \cong \mathcal{L}''_1$.*

$$\begin{tikzcd} & & S \times \mathbb{P}_2 \arrow[dl, bend left, "p_1"] \\ Z_1 \arrow[r, "\xi"'] \arrow[urr, bend left, "\sigma_1"] \arrow[d] & Z_2 \arrow[ur, "\sigma_2"] \arrow[dl] & \\ S & & \end{tikzcd}$$

- *There are isomorphisms $e : \mathcal{E}_1 \to \xi^*\mathcal{E}_2$ and $\phi : \mathcal{F}_1 \to \mathcal{F}_2$ such that the diagram*

$$\begin{tikzcd} \sigma_1^*\mathcal{F}_1 \arrow[dr, "\tau_1"'] & \xi^*\sigma_2^*\mathcal{F}_1 \arrow[l, "\alpha_{\xi,\sigma_2}(\mathcal{F}_1)"', "\cong"] \arrow[r, "\xi^*\sigma_2^*(\phi)"] & \xi^*\sigma_2^*\mathcal{F}_2 \arrow[dl, "\xi^*\tau_2"] \\ & \mathcal{E}_1 \arrow[r, "e"] & \xi^*\mathcal{E}_2 \end{tikzcd} \tag{3.4}$$

  is commutative.

Here $\alpha_{\xi,\sigma_2} : \xi^*\sigma_2^* \to \sigma_2^*$ is the canonical isomorphism of functors.

Remark 3.25. *For the canonical isomorphisms of functors we use here the notations from [30] (see Definition 3.10, page 47).*

Lemma 3.26. *The relation from Definition 3.24 is an equivalence relation.*

Proof. 1) Reflexivity. Consider an arbitrary family $(\pi : Z \to S, \sigma, \mathcal{L}'', \mathcal{E}, \mathcal{F}, \tau)$ over S. Put $\xi = \mathrm{id}_Z$, then $\xi^*\mathcal{L}'' \cong \mathcal{L}''$. Let $e : \mathcal{E} \to \mathrm{id}_Z^* \mathcal{E}$ be the inverse of $\epsilon_Z(\mathcal{E}) : \mathrm{id}_Z^* \mathcal{E} \to \mathcal{E}$, where $\epsilon_Z : \mathrm{id}_Z \to \mathrm{id}_{\mathscr{C}oh(Z)}$ is the canonical isomorphism of functors $\mathscr{C}oh(Z) \to \mathscr{C}oh(Z)$, put also $\phi = \mathrm{id}_{\mathcal{F}}$.

Then the commutativity of

$$\begin{tikzcd} \sigma^*\mathcal{F} \arrow[dr, "\tau_1"'] & \mathrm{id}_Z^*\,\sigma^*\mathcal{F} \arrow[l, "\alpha_{\mathrm{id}_Z,\sigma}(\mathcal{F})"', "\cong"] \arrow[r, equal, "\mathrm{id}_{\mathrm{id}_Z^*\sigma^*\mathcal{F}}"] & \mathrm{id}_Z^*\,\sigma^*\mathcal{F} \arrow[dl, "\mathrm{id}_Z^*\,\tau_2"] \\ & \mathcal{E} \arrow[r, "e"] & \mathrm{id}_Z^*\,\mathcal{E} \end{tikzcd}$$

is the same as the commutativity of

$$\begin{tikzcd} \sigma^*\mathcal{F} \arrow[d, "\tau"'] & \mathrm{id}_Z^*\,\sigma^*\mathcal{F} \arrow[l, "\epsilon_Z(\sigma^*\mathcal{F})"'] \arrow[d, "\mathrm{id}_Z^*\,\tau"] \\ \mathcal{E} & \mathrm{id}_Z^*\,\mathcal{E} \arrow[l, "\epsilon_Z(\mathcal{E})"'] \end{tikzcd}$$

because $\alpha_{\mathrm{id}_Z,\sigma}(\mathcal{F}) = \epsilon_Z(\sigma^*\mathcal{F})$. But the commutativity of the last diagram follows because ϵ_Z is a natural transformation of functors. We proved the reflexivity axiom.

2) Symmetry. Suppose that $(\pi_1 : Z_1 \to S, \sigma_1, \mathcal{L}_1'', \mathcal{E}_1, \mathcal{F}_1, \tau_1)$ is “equivalent” to $(\pi_2 : Z_2 \to S, \sigma_2, \mathcal{L}_2'', \mathcal{E}_2, \mathcal{F}_2, \tau_2)$. The latter means that there exist ξ, ϕ and e as in Definition 3.21 and the commutative diagram

$$\begin{tikzcd}
\sigma_1^*\mathcal{F}_1 \arrow[dr, "\tau_1"'] & & \xi^*\sigma_2^*\mathcal{F}_1 \arrow[ll, "\alpha_{\xi,\sigma_2}(\mathcal{F}_1)"', "\cong"] \arrow[rr, "\xi^*\sigma_2^*(\phi)"] & & \xi^*\sigma_2^*\mathcal{F}_2. \arrow[dl, "\xi^*\tau_2"] \\
& \mathcal{E}_1 \arrow[rr, "e"] & & \xi^*\mathcal{E}_2 &
\end{tikzcd}$$

Denote $\xi' = \xi^{-1}$ and apply ξ'^* to this diagram. We obtain then the commutative diagram

$$\begin{tikzcd}
\xi'^*\sigma_1^*\mathcal{F}_1 \arrow[dr, "\xi'^*\tau_1"'] & & \xi'^*\xi^*\sigma_2^*\mathcal{F}_1 \arrow[ll, "\xi'^*\alpha_{\xi,\sigma_2}(\mathcal{F}_1)"', "\cong"] \arrow[rr, "\xi'^*\xi^*\sigma_2^*(\phi)"] & & \xi'^*\xi^*\sigma_2^*\mathcal{F}_2. \arrow[dl, "\xi'^*\xi^*\tau_2"] \\
& \xi'^*\mathcal{E}_1 \arrow[rr, "\xi'^*e"] & & \xi'^*\xi^*\mathcal{E}_2 &
\end{tikzcd}$$

One can extend this diagram to the commutative diagram

$$\begin{tikzcd}
& \xi'^*\sigma_1^*\mathcal{F}_1 \arrow[r, "\xi'^*\sigma_1^*(\phi)"] \arrow[dl, equal] & \xi'^*\sigma_1^*\mathcal{F}_2 \arrow[r, "\alpha_{\xi',\sigma_1}(\mathcal{F}_2)"] & \sigma_2^*\mathcal{F}_2 \arrow[dr, equal] & \\
\xi'^*\sigma_1^*\mathcal{F}_1 \arrow[dr, "\xi'^*\tau_1"'] & \xi'^*\xi^*\sigma_2^*\mathcal{F}_1 \arrow[l, "\xi'^*\alpha_{\xi,\sigma_2}(\mathcal{F}_1)"', "\cong"] \arrow[u, "\xi'^*\alpha_{\xi,\sigma_2}(\mathcal{F}_1)"] \arrow[r, "\xi'^*\xi^*\sigma_2^*(\phi)"] & \xi'^*\xi^*\sigma_2^*\mathcal{F}_2 \arrow[u, "\xi'^*\alpha_{\xi,\sigma_2}(\mathcal{F}_2)"] \arrow[r, "\alpha_{\xi',\xi}(\sigma_2^*\mathcal{F}_2)"] \arrow[d, "\xi'^*\xi^*\tau_2"] & \mathrm{id}_{Z_2}^*\,\sigma_2^*\mathcal{F}_2 \arrow[u, "\alpha_{\mathrm{id}_{Z_2},\sigma_2}"] \arrow[r, "\epsilon_{Z_2}(\sigma_2^*\mathcal{F}_2)"] \arrow[d, "\mathrm{id}_{Z_2}^*\,\tau_2"] & \sigma_2^*\mathcal{F}_2 \arrow[d, "\tau_2"] \\
& \xi'^*\mathcal{E}_1 \arrow[r, "\xi'^*e"] & \xi'^*\xi^*\mathcal{E}_2 \arrow[r, "\alpha_{\xi',\xi}(\mathcal{E}_2)"] & \mathrm{id}_{Z_2}^*\,\mathcal{E}_2 \arrow[r, "\epsilon_{Z_2}(\mathcal{E}_2)"] & \mathcal{E}_2.
\end{tikzcd}$$

For $\phi' = \phi^{-1}$ and for $e' = (\epsilon_{Z_2}(\mathcal{E}_2) \circ \alpha_{\xi',\xi}(\mathcal{E}_2) \circ \xi'^*(e))^{-1}$ we obtain finally the commutative diagram

$$\begin{tikzcd}
\sigma_2^*\mathcal{F}_2 \arrow[dr, "\tau_2"'] & & \xi'^*\sigma_1^*\mathcal{F}_2 \arrow[ll, "\alpha_{\xi',\sigma_2}(\mathcal{F}_1)"', "\cong"] \arrow[rr, "\xi'^*\sigma_1^*(\phi')"] & & \xi'^*\sigma_1^*\mathcal{F}_1. \arrow[dl, "\xi'^*\tau_1"] \\
& \mathcal{E}_2 \arrow[rr, "e'"] & & \xi'^*\mathcal{E}_1 &
\end{tikzcd}$$

Since we also have $\xi'^*(\mathcal{L}_1'') \cong \xi'^*\xi^*(\mathcal{L}_2'') \cong \mathcal{L}_2''$, this means that the family

$$(\pi_2 : Z_2 \to S, \sigma_2, \mathcal{L}_2'', \mathcal{E}_2, \mathcal{F}_2, \tau_2)$$

is “equivalent” to the family $(\pi_1 : Z_1 \to S, \sigma_1, \mathcal{L}_1'', \mathcal{E}_1, \mathcal{F}_1, \tau_1)$. This proves the symmetry of the relation.

3) Transitivity. Let $(\pi_1 : Z_1 \to S, \mathcal{L}_1'', \sigma_1, \mathcal{E}_1, \mathcal{F}_1, \tau_1) \sim (\pi_2 : Z_2 \to S, \sigma_2, \mathcal{L}_2'', \mathcal{E}_2, \mathcal{F}_2, \tau_2)$ and $(\pi_2 : Z_2 \to S, \sigma_2, \mathcal{L}_2'', \mathcal{E}_2, \mathcal{F}_2, \tau_2) \sim (\pi_3 : Z_3 \to S, \sigma_3, \mathcal{L}_3'', \mathcal{E}_3, \mathcal{F}_3, \tau_3)$. Let ξ, e, ϕ be the data of the first equivalence, and let ξ', e', ϕ' be the data of the second equivalence. We have the commutative diagrams

$$\begin{tikzcd}
\sigma_1^*\mathcal{F}_1 \arrow[dr, "\tau_1"'] & & \xi^*\sigma_2^*\mathcal{F}_1 \arrow[ll, "\alpha_{\xi,\sigma_2}(\mathcal{F}_1)"', "\cong"] \arrow[rr, "\xi^*\sigma_2^*(\phi)"] & & \xi^*\sigma_2^*\mathcal{F}_2, \arrow[dl, "\xi^*\tau_2"] \\
& \mathcal{E}_1 \arrow[rr, "e"] & & \xi^*\mathcal{E}_2 &
\end{tikzcd}
\qquad
\begin{tikzcd}
\sigma_2^*\mathcal{F}_2 \arrow[dr, "\tau_2"'] & & \xi'^*\sigma_3^*\mathcal{F}_2 \arrow[ll, "\alpha_{\xi',\sigma_3}(\mathcal{F}_2)"', "\cong"] \arrow[rr, "\xi'^*\sigma_3^*(\phi')"] & & \xi'^*\sigma_3^*\mathcal{F}_3. \arrow[dl, "\xi'^*\tau_3"] \\
& \mathcal{E}_2 \arrow[rr, "e'"] & & \xi'^*\mathcal{E}_3 &
\end{tikzcd}$$

Applying ξ^* to the second diagram, attaching the obtained commutative diagram to the first diagram and extending the resulting diagram we obtain the commutative diagram

$$\begin{tikzcd}
 & & & (\xi'\xi)^*\sigma_3^*(\mathcal{F}_1) \arrow[lll, bend right, "\alpha_{\xi'\xi,\sigma_3}(\mathcal{F}_1)"'] \arrow[rr, bend left, "(\xi'\xi)^*\sigma_3^*(\phi'\phi)"] & & \\
\sigma_1^*\mathcal{F}_1 \arrow[dddd, "\tau_1"'] & & & \xi^*\xi'^*\sigma_3^*\mathcal{F}_1 \arrow[u, "(\xi'\xi)^*\sigma_3^*(\mathcal{F}_1)"] \arrow[dll, bend right, "\xi^*\alpha_{\xi',\sigma_3}(\mathcal{F}_1)"'] \arrow[dd, "\xi^*\xi'^*\sigma_3^*(\phi)"] \arrow[ddr, bend left, "\xi^*\xi'^*\sigma_3^*(\phi'\phi)"] & & (\xi'\xi)^*\sigma_3^*\mathcal{F}_3. \arrow[ddd, "(\xi'\xi)^*\tau_3"] \\
 & \xi^*\sigma_2^*\mathcal{F}_1 \arrow[ul, "\alpha_{\xi,\sigma_2}(\mathcal{F}_1)"] \arrow[dr, "\xi^*\sigma_2^*(\phi)"'] & & & & \\
 & & \xi^*\sigma_2^*\mathcal{F}_2 \arrow[d, "\xi^*\tau_2"] & \xi^*\xi'^*\sigma_3^*\mathcal{F}_2 \arrow[l, "\xi^*\alpha_{\xi',\sigma_3}(\mathcal{F}_2)"'] \arrow[r, "\xi^*\xi'^*\sigma_3^*(\phi')"] & \xi^*\xi'^*\sigma_3^*\mathcal{F}_3 \arrow[uur, "\alpha_{\xi,\xi'}(\sigma_3^*\mathcal{F}_3)"] \arrow[dl, "\xi^*\xi'^*\tau_3"'] & \\
\mathcal{E}_1 \arrow[rr, "e"] & & \xi^*\mathcal{E}_2 \arrow[r, "\xi^*e'"] & \xi^*\xi'^*\mathcal{E}_3 \arrow[rr, "\alpha_{\xi,\xi'}(\mathcal{E}_3)"] & & (\xi'\xi)^*\mathcal{E}_3
\end{tikzcd}$$

For $\xi'' = \xi'\xi$, $\psi'' = \psi'\psi$ and $e'' = \alpha_{\xi,\xi'}(\mathcal{E}_3) \circ \xi^* e' \circ e$ we obtain the commutative diagram

$$\begin{tikzcd}
\sigma_1^*\mathcal{F}_1 \arrow[dr, two heads, "\tau_1"] & \xi''^*\sigma_3^*(\mathcal{F}_1) \arrow[l, "\alpha_{\xi'',\sigma_3}(\mathcal{F}_1)"'] \arrow[r, "(\xi'')^*\sigma_3^*(\phi'')"] & \xi''^*\sigma_3^*\mathcal{F}_3. \arrow[d, two heads, "\xi''^*\tau_3"] \\
 & \mathcal{E}_1 \arrow[r, "e''"] & \xi''^*\mathcal{E}_3
\end{tikzcd}$$

Since $\mathcal{L}_1'' \cong \xi^*\mathcal{L}_2'' \cong \xi^*\xi'^*\mathcal{L}_3'' \cong \xi''^*\mathcal{L}_3''$, we conclude $(\pi_1 : Z_1 \to S, \sigma_1, \mathcal{L}_1'', \mathcal{E}_1, \mathcal{F}_1, \tau_1) \sim (\pi_3 : Z_3 \to S, \sigma_3, \mathcal{L}_3'', \mathcal{E}_3, \mathcal{F}_3, \tau_3)$. This proves the transitivity. In 1), 2), and 3) we proved that the relation from Definition 3.24 is in fact an equivalence relation. $\square$

Lemma 3.27. *1) Consider the map described in Remark 3.23. Then isomorphic* $3m+1$ *families are mapped to equivalent families.*

2) This gives us a map from the set of all classes of isomorphism of non-singular $3m+1$ *families over* S *to the set of the equivalence classes of new families over* S*.*

3) This map is injective. Therefore, the set of all classes of isomorphism of non-singular $3m+1$ *families can be considered as a subset in the set of the equivalence classes of families described in Definition 3.21.*

Proof. 1) Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be two isomorphic non-singular $3m+1$ families over S. And let $\phi : \mathcal{F}_1 \to \mathcal{F}_2$ be the corresponding isomorphism. Define e by the condition that the diagram

$$\begin{tikzcd}
\mathrm{id}_{S\times\mathbb{P}_2}^*\mathcal{F}_1 \arrow[dr, two heads, "\epsilon_{S\times\mathbb{P}_2}(\mathcal{F}_1)"'] & \mathrm{id}_{S\times\mathbb{P}_2}^*\,\mathrm{id}_{S\times\mathbb{P}_2}^*\mathcal{F}_1 \arrow[l, "\alpha_{\mathrm{id}_{S\times\mathbb{P}_2},\mathrm{id}_{S\times\mathbb{P}_2}}(\mathcal{F}_1)"'] \arrow[r, "\mathrm{id}_{S\times\mathbb{P}_2}^*\,\mathrm{id}_{S\times\mathbb{P}_2}^*(\phi)"] & \mathrm{id}_{S\times\mathbb{P}_2}^*\,\mathrm{id}_{S\times\mathbb{P}_2}^*\mathcal{F}_2 \arrow[d, two heads, "\mathrm{id}_{S\times\mathbb{P}_2}^*\,\epsilon_{S\times\mathbb{P}_2}(\mathcal{F}_2)"] \\
 & \mathcal{F}_1 \arrow[r, "e"] & \mathrm{id}_{S\times\mathbb{P}_2}^*\mathcal{F}_2
\end{tikzcd}$$

commutes (note that all the arrows in this diagram are isomorphisms). Then $\xi = \mathrm{id}_{S\times\mathbb{P}_2}$, e, and ϕ are data that describe an equivalence of the families that correspond to the sheaves $\mathcal{F}_1$ and $\mathcal{F}_2$.

2) Follows from 1).

3) If two families are equivalent, then in particular their $3m+1$ families are isomorphic by Definition 3.24. This implies the required injectivity. $\square$

Remark 3.28. *Note that for families over one point the notion of equivalence from Definition 3.24 coincides with the equivalence of R-bundles on* Z_0 *from Definition 2.12.*

Let us denote the set of the equivalence classes of new families over S by $\widetilde{\mathcal{M}}(S)$. For a family $(\pi, \sigma, \mathcal{L}'', \mathcal{E}, \mathcal{F}, \tau)$ we denote its equivalence class by $[(\pi, \sigma, \mathcal{L}'', \mathcal{E}, \mathcal{F}, \tau)]$.

Lemma 3.29. *Let $(\pi_1, \sigma_1, \mathcal{L}_1'', \mathcal{E}_1, \mathcal{F}_1, \tau_1)$ and $(\pi_2, \sigma_2, \mathcal{L}_2'', \mathcal{E}_2, \mathcal{F}_2, \tau_2)$ be two equivalent families over S and let $T \xrightarrow{f} S$ be a morphism. Then the pull backs of $(\pi_1, \sigma_1, \mathcal{L}_1'', \mathcal{E}_1, \mathcal{F}_1, \tau_1)$ and $(\pi_2, \sigma_2, \mathcal{L}_2'', \mathcal{E}_2, \mathcal{F}_2, \tau_2)$ along f are equivalent families over T.*

Proof. Let $\xi : Z_1 \to Z_2$, $\phi : \mathcal{F}_1 \to \mathcal{F}_2$, and $e : \mathcal{E}_1 \to \xi^*\mathcal{E}_2$ be as in Definition 3.24. Consider the commutative pull back diagram

$$\begin{tikzcd}
Z_{2T} \arrow[rrr, "F_2"] \arrow[drr, "\sigma_{2T}"] \arrow[dddr, bend right, "\pi_{2T}"'] & & & Z_2 \arrow[drr, "\sigma_2"] \arrow[dddr, bend right, "\pi_2"'] & & \\
& & T \times \mathbb{P}_2 \arrow[rrrr, "f \times \mathrm{id}_{\mathbb{P}_2}"] \arrow[ddl, bend left, "p_1"] & & & & S \times \mathbb{P}_2. \arrow[ddll, bend left, "p_1"] \\
& Z_{1T} \arrow[uul, "\xi_T"] \arrow[ur, "\sigma_{1T}"] \arrow[rrrr, "F_1"] \arrow[d, "\pi_{1T}"] & & & & Z_1 \arrow[uull, "\xi"] \arrow[ur, "\sigma_1"] \arrow[d, "\pi_1"] & \\
& T \arrow[rrrr, "f"] & & & & S &
\end{tikzcd}$$

Applying F_1^* to the diagram (3.4) and using the properties of the canonical isomorphisms we obtain the commutative diagram

$$\begin{tikzcd}
\sigma_{1T}(f \times \mathrm{id})^*\mathcal{F}_1 \arrow[dd, two heads, "f^*\tau_1"'] \arrow[drr] & & & \xi_T^*\sigma_{2T}^*(f \times \mathrm{id})^*\mathcal{F}_1 \arrow[lll, "\alpha_{\xi_T,\sigma_{2T}}((f\times\mathrm{id}_{\mathbb{P}_2})^*\mathcal{F}_1)"'] \arrow[rr, "\xi_T^*\sigma_{2T}^*(f\times\mathrm{id}_{\mathbb{P}_2})^*(\phi)"] \arrow[d] & & \xi_T^*\sigma_{2T}^*(f \times \mathrm{id})^*\mathcal{F}_2 \arrow[d] \arrow[dddd, bend left, two heads, "\xi_T^*f^*\tau_2"] \\
& F_1^*\sigma_1^*\mathcal{F}_1 \arrow[r] \arrow[dl, two heads, "F_1^*\tau_1"] & (\sigma_1F_1)^*\mathcal{F}_1 & \xi_T^*(\sigma_2F_2)^*\mathcal{F}_1 \arrow[l] \arrow[rr] & & \xi_T^*(\sigma_2F_2)^*\mathcal{F}_2 \\
F_1^*\mathcal{E}_1 \arrow[dr, "F_1^*e"] & F_1^*\xi^*\sigma_2^*\mathcal{F}_1 \arrow[u, "F_1^*\alpha_{\xi,\sigma_2}(\mathcal{F}_2)"] \arrow[rr] \arrow[dr, "F_1^*\xi^*\sigma_2^*\phi"] & & (\xi F_1)^*\sigma_2^*\mathcal{F}_1 \arrow[ul] \arrow[d] & \xi_T^*F_2^*\sigma_2^*\mathcal{F}_1 \arrow[l] \arrow[ul] \arrow[dr] & \\
& F_1^*\xi^*\mathcal{E}_2 \arrow[drr] & F_1^*\xi^*\sigma_2^*\mathcal{F}_2 \arrow[l, "F_1^*\xi^*\tau_2"'] \arrow[r] & (\xi F_1)^*\sigma_2^*\mathcal{F}_2 \arrow[d] & & \xi_T^*F_2^*\sigma_2^*\mathcal{F}_2 \arrow[ll] \arrow[uu] \arrow[d] \\
& & & (\xi F_1)^*\mathcal{E}_2 & & \xi_T^*F_2^*\mathcal{E}_2 \arrow[ll]
\end{tikzcd}$$

and finally the commutative diagram

$$\begin{tikzcd}
\sigma_{1T}^*(f \times \mathrm{id}_{\mathbb{P}_2})^*\mathcal{F}_1 \arrow[dr, two heads, "f^*\tau_1"'] & & \xi_T^*\sigma_{2T}^*(f \times \mathrm{id}_{\mathbb{P}_2})^*\mathcal{F}_1 \arrow[ll, "\alpha_{\xi_T,\sigma_{2T}}((f\times\mathrm{id}_{\mathbb{P}_2})^*\mathcal{F}_1)"'] \arrow[rr, "\xi_T^*\sigma_{2T}^*(f\times\mathrm{id}_{\mathbb{P}_2})^*(\phi)"] & & \xi_T^*\sigma_{2T}^*(f \times \mathrm{id}_{\mathbb{P}_2})^*\mathcal{F}_2. \arrow[dl, two heads, "\xi_T^*f^*\tau_2"] \\
& F_1^*\mathcal{E}_1 \arrow[rr] & & \xi_T^*F_2^*\mathcal{E}_2 &
\end{tikzcd}$$

Since

$$\xi_T^*F_2^*\mathcal{L}_2'' \cong F_1^*\xi^*\mathcal{L}_2'' \cong F_1^*\mathcal{L}_1'',$$

we conclude that the pull backs of $(\pi_1, \sigma_1, \mathcal{L}_1'', \mathcal{E}_1, \mathcal{F}_1, \tau_1)$ and $(\pi_2, \sigma_2, \mathcal{L}_2'', \mathcal{E}_2, \mathcal{F}_2, \tau_2)$ along f are equivalent families over T. $\square$

For an arbitrary morphism $f : T \to S$ Lemma 3.29 gives us a well-defined map

$$f^* : \widetilde{\mathcal{M}}(S) \to \widetilde{\mathcal{M}}(T), \quad [(\pi, \sigma, \mathcal{L}'', \mathcal{E}, \mathcal{F}, \tau)] \mapsto [f^*(\pi, \sigma, \mathcal{L}'', \mathcal{E}, \mathcal{F}, \tau)].$$

Functorial properties, construction of the functor.

Lemma 3.30. *Let* $(\pi : Z \to S, \sigma, \mathcal{L}'', \mathcal{E}, \mathcal{F}, \tau)$ *be a family over* S *and let* $T \xrightarrow{f} S$ *and* $U \xrightarrow{g} T$ *be two morphisms. Then*

1) the families $\mathrm{id}_S^*(\pi, \sigma, \mathcal{L}'', \mathcal{E}, \mathcal{F}, \tau)$ *and* $(\pi, \sigma, \mathcal{L}'', \mathcal{E}, \mathcal{F}, \tau)$ *are equivalent;*

2) the families $(fg)^*(\pi, \sigma, \mathcal{L}'', \mathcal{E}, \mathcal{F}, \tau)$ *and* $g^* f^*(\pi, \sigma, \mathcal{L}'', \mathcal{E}, \mathcal{F}, \tau)$ *are equivalent.*

Proof. 1) We have the commutative diagram

$$\begin{tikzcd}
\sigma^* \mathrm{id}_{S\times\mathbb{P}_2}^* \mathcal{F} \arrow[d, "\alpha_{\sigma,\mathrm{id}_{S\times\mathbb{P}_2}}(\mathcal{F})"] & \mathrm{id}_Z^* \sigma^* \mathrm{id}_{S\times\mathbb{P}_2}^* \mathcal{F}. \arrow[l, "\alpha_{\mathrm{id}_Z,\sigma}(\mathrm{id}_{S\times\mathbb{P}_2}^* \mathcal{F})"'] \arrow[d, "\mathrm{id}_Z^* \alpha_{\sigma,\mathrm{id}_{S\times\mathbb{P}_2}}(\mathcal{F}) = \mathrm{id}_Z^* \sigma^* \epsilon_{S\times\mathbb{P}_2}(\mathcal{F})"] \\
\sigma^* \mathcal{F} & \mathrm{id}_Z^* \sigma^* \mathcal{F} \arrow[l, "\alpha_{\mathrm{id}_Z,\sigma}(\mathcal{F})"]
\end{tikzcd}$$

Since $\mathrm{id}_S^* \tau$ is by construction the composition

$$\sigma^* \mathrm{id}_{S\times\mathbb{P}_2}^* \mathcal{F} \xrightarrow{\alpha_{\sigma,\mathrm{id}_{S\times\mathbb{P}_2}} \mathcal{F}} \sigma^* \mathcal{F} \xrightarrow{\alpha^{-1}_{\mathrm{id}_Z,\sigma}(\mathcal{F})} \mathrm{id}_Z^* \sigma^* \mathcal{F} \xrightarrow{\mathrm{id}_Z^* \tau} \mathrm{id}_Z^* \mathcal{E},$$

we obtain the commutative diagram

$$\begin{tikzcd}
\sigma^* \mathrm{id}_{S\times\mathbb{P}_2}^* \mathcal{F} \arrow[dr, two heads, "\mathrm{id}_S^* \tau"'] & & \mathrm{id}_Z^* \sigma^* \mathrm{id}_{S\times\mathbb{P}_2}^* \mathcal{F} \arrow[ll, "\alpha_{\mathrm{id}_Z,\sigma}(\mathrm{id}_{S\times\mathbb{P}_2}^* \mathcal{F})"'] \arrow[rr, "\mathrm{id}_Z^* \sigma^* \epsilon_{S\times\mathbb{P}_2}(\mathcal{F})"] & & \mathrm{id}_Z^* \sigma^* \mathcal{F}. \arrow[dl, two heads, "\mathrm{id}_Z^* \tau"] \\
& \mathrm{id}_Z^* \mathcal{E} \arrow[rr, equal] & & \mathrm{id}_Z^* \mathcal{E} &
\end{tikzcd}$$

Therefore, $\phi = \epsilon_{S\times\mathbb{P}_2}(\mathcal{F}) : \mathrm{id}_{S\times\mathbb{P}_2}^* \mathcal{F} \to \mathcal{F}$, $\xi = \mathrm{id}_Z$, $e = \mathrm{id}_{\mathrm{id}_Z^* \mathcal{E}}$ are the data that describe the equivalence of $\mathrm{id}_S^*(\pi, \sigma, \mathcal{L}'', \mathcal{E}, \mathcal{F}, \tau)$ and $(\pi, \sigma, \mathcal{L}'', \mathcal{E}, \mathcal{F}, \tau)$ (note also that $\xi^* \mathcal{L}'' = \mathrm{id}_Z^* \mathcal{L}''$).

2) Follows from the commutative diagrams

$$\begin{tikzcd}
& U \times \mathbb{P}_2 \arrow[rr, "g\times\mathrm{id}"] \arrow[ddl] & & T \times \mathbb{P}_2 \arrow[rr, "f\times\mathrm{id}"] \arrow[ddl] & & S \times \mathbb{P}_2 \arrow[ddl] \\
Z_U \arrow[ur, "\sigma_U"] \arrow[rr, "G"] \arrow[d, "\pi_U"'] & & Z_T \arrow[ur, "\sigma_T"] \arrow[rr, "F"] \arrow[d, "\pi_T"'] & & Z \arrow[ur, "\sigma"] \arrow[d, "\pi"'] & \\
U \arrow[rr, "g"] & & T \arrow[rr, "f"] & & S &
\end{tikzcd}$$

and

$$
\begin{tikzcd}
\sigma_U^*(g\times \mathrm{id})^*(f\times \mathrm{id})^*\mathcal{F} \arrow[d] \arrow[ddddd, ->>, bend right=40, "g^*(f^*\tau)"'] & \mathrm{id}^*_{Z_U}\,\sigma_U^*(g\times \mathrm{id})^*(f\times \mathrm{id})^*\mathcal{F} \arrow[l] \arrow[r] \arrow[d] & \mathrm{id}^*_{Z_U}\,\sigma_U^*(fg\times \mathrm{id})^*\mathcal{F} \arrow[dd] \arrow[ddddd, ->>, bend left=40] \\
(\sigma_T G)^*(f\times \mathrm{id})^*\mathcal{F} & \mathrm{id}^*_{Z_U}(\sigma_T G)^*(f\times \mathrm{id})^*\mathcal{F} \arrow[l] \arrow[dr] & \\
G^*\sigma_T^*(f\times \mathrm{id})^*\mathcal{F} \arrow[u] \arrow[d] \arrow[ddd, ->>, bend right=30, "G^*(f^*\tau)"'] & \mathrm{id}^*_{Z_U}\,G^*\sigma_T^*(f\times \mathrm{id})^*\mathcal{F} \arrow[l] \arrow[u] \arrow[d] & \mathrm{id}^*_{Z_U}((fg\times \mathrm{id})\sigma_U)^*\mathcal{F} \arrow[d, equal, "\mathrm{id}^*_{Z_U}(fg)^*\tau"] \\
G^*(\sigma F)^*\mathcal{F} & \mathrm{id}^*_{Z_U}\,G^*(\sigma F)^*\mathcal{F} \arrow[l] \arrow[r] & \mathrm{id}^*_{Z_U}(\sigma FG)^*\mathcal{F} \\
G^*F^*\sigma^*\mathcal{F} \arrow[u] \arrow[d, ->>, "G^*F^*\tau"] & \mathrm{id}^*_{Z_U}\,G^*F^*\sigma^*\mathcal{F} \arrow[l, "\epsilon_{Z_U}(G^*F^*\sigma^*\mathcal{F})"'] \arrow[r, "\mathrm{id}^*_{Z_U}\,\alpha_{G,F}(\sigma^*\mathcal{F})"] \arrow[u] \arrow[d, ->>, "\mathrm{id}^*_{Z_U}\,G^*F^*\tau"] & \mathrm{id}^*_{Z_U}(FG)^*\sigma^*\mathcal{F} \arrow[u] \arrow[d, ->>, "\mathrm{id}^*_{Z_U}(FG)^*\tau"'] \\
G^*F^*\mathcal{E} & \mathrm{id}^*_{Z_U}\,G^*F^*\mathcal{E} \arrow[l, "\epsilon_{Z_U}(G^*F^*\mathcal{E})"'] \arrow[r, "\mathrm{id}^*_{Z_U}\,\alpha_{G,F}(\mathcal{E})"'] & \mathrm{id}^*_{Z_U}(FG)^*\mathcal{E}.
\end{tikzcd}
$$

Of course we have $G^*F^*\mathcal{L}'' \cong \mathrm{id}^*_{Z_U}(FG)^*\mathcal{L}''$, the equivalence data are

$$
\begin{aligned}
\xi =& \mathrm{id}_{Z_U},\\
\phi =& \alpha_{g\times \mathrm{id}, f\times \mathrm{id}}(\mathcal{F}) : (g\times \mathrm{id})^*(f\times \mathrm{id})^*\mathcal{F} \to (fg\times \mathrm{id})^*\mathcal{F},\\
e =& \mathrm{id}^*_{Z_U}\,\alpha_{G,F}(\mathcal{E}) \circ (\epsilon_{Z_U}(G^*F^*\mathcal{E}))^{-1} : G^*F^*\mathcal{E} \to \mathrm{id}^*_{Z_U}(FG)^*\mathcal{E}.
\end{aligned}
$$

We have proven the lemma. □

Lemma 3.30 together with Lemma 3.29 say that $\widetilde{\mathcal{M}}$ is a functor (Sch) $\to$ (Sets). We use here the notations (Sets) and (Sch) for the category of sets and for the category of separated schemes of finite type over $\Bbbk$ respectively.

Relation between $\widetilde{\mathcal{M}}$ and $\mathcal{M}$.

Recall that by $\mathcal{M}$ we denote the functor of the $3m+1$ Simpson moduli problem on $\mathbb{P}_2$.

Proposition 3.31. *There is a natural transformation of functors $\widetilde{\mathcal{M}} \xrightarrow{\mu} \mathcal{M}$ given by the rule*

$$
\begin{gathered}
\widetilde{\mathcal{M}}(S) \ni [(\pi,\sigma,\mathcal{L}'',\mathcal{E},\mathcal{F},\tau)] \mapsto [\mathcal{F}] \in \mathcal{M}(S), \quad S \in (\mathrm{Sch})\\
\mathrm{Hom}_{(\mathrm{Sets})}(\widetilde{\mathcal{M}}(S),\widetilde{\mathcal{M}}(T)) \ni f^* \mapsto f^* \in \mathrm{Hom}_{(\mathrm{Sets})}(\mathcal{M}(S),\mathcal{M}(T)), \quad (T \xrightarrow{f} S) \in (\mathrm{Sch})
\end{gathered}
$$

Proof. Let $f: T \to S$ be an arbitrary morphism. By Definition (3.3) the $3m+1$ sheaf of the pull back family $f^*(\pi,\sigma,\mathcal{L}'',\mathcal{E},\mathcal{F},\tau)$ is the sheaf $(f\times \mathrm{id}_{\mathbb{P}_2})^*\mathcal{F}$. Therefore, the diagram

$$
\begin{tikzcd}
\widetilde{\mathcal{M}}(S) \arrow[r, "f^*"] \arrow[d, "\mu(S)"] & \widetilde{\mathcal{M}}(T) \arrow[d, "\mu(T)"] \\
\mathcal{M}(S) \arrow[r, "f^*"] & \mathcal{M}(T)
\end{tikzcd}
\qquad
\begin{tikzcd}
{[(\pi,\sigma,\mathcal{L}'',\mathcal{E},\mathcal{F},\tau)]} \arrow[r, mapsto, "f^*"] \arrow[d, mapsto, "\mu(S)"] & {[f^*(\pi,\sigma,\mathcal{L}'',\mathcal{E},\mathcal{F},\tau)]} \arrow[d, mapsto, "\mu(T)"] \\
{[\mathcal{F}]} \arrow[r, mapsto, "f^*"] & {[(f\times \mathrm{id}_{\mathbb{P}_2})^*\mathcal{F}]}
\end{tikzcd}
$$

commutes. This proves the required statement. □

3.3.2 Properties of new families.

Space Z as a subvariety in a $\mathbb{P}_2 \times \mathbb{P}_2$-bundle over S.

Let us consider a family $(\pi : Z \to S, \sigma, \mathcal{L}'', \mathcal{E}, \mathcal{F}, \tau)$ over S. Consider the invertible sheaf

$$\mathcal{L}' = \sigma^*(\mathcal{O}_S \boxtimes \mathcal{O}_{\mathbb{P}_2}(1)) \cong \sigma^*(p_1^*\mathcal{O}_S \otimes p_2^*\mathcal{O}_{\mathbb{P}_2}(1)) \cong \sigma^* p_2^* \mathcal{O}_{\mathbb{P}_2}(1)$$

on Z. Let us calculate its direct image $\mathcal{G}_1 = \pi_*\mathcal{L}'$.

Lemma 3.32. *$\pi_*\mathcal{L}' \cong 3\mathcal{O}_S$ and the canonical morphism $\pi^*\pi_*\mathcal{L}' \to \mathcal{L}'$ is surjective.*

Proof. Since $H^1(\mathbb{P}_2, \mathcal{O}_{\mathbb{P}_2}(1)) = 0$, using Proposition 1.32 and the definition of $\mathcal{L}'$ from Definition 3.21) we conclude that

$$H^1(Z_s, {\mathcal{L}'}_s) \cong H^1(Z_s, \mathcal{O}_{Z_s}(1,0)_s) = 0.$$

Therefore, by the base change theorem the canonical map $\varphi^0(s) : \pi_*\mathcal{L}'(s) \to H^0(Z_s, {\mathcal{L}'}_s)$ is an isomorphism for every $s \in S$. Since by Proposition 1.32 $H^0(Z_s, {\mathcal{L}'}_s) \cong \mathbb{k}^3$, applying the base change theorem using the surjectivity of $\varphi^{-1}(s)$ (by definition) we obtain that $\mathcal{G}_1 = \pi_*\mathcal{L}''$ is a locally free sheaf of rank 3 on S.

Note also that

$$\pi_*\mathcal{L}' \cong \pi_*\sigma^* p_2^* \mathcal{O}_{\mathbb{P}_2}(1) \cong p_{1*}p_2^*\mathcal{O}_{\mathbb{P}_2}(1).$$

Therefore, for an open set U in S we have using the Künneth formula

$$p_{1*}p_2^*\mathcal{O}_{\mathbb{P}_2}(1)(U) = p_2^*\mathcal{O}_{\mathbb{P}_2}(1)(U \times \mathbb{P}_2) \cong \mathcal{O}_S(U) \otimes \Gamma(\mathbb{P}_2, \mathcal{O}_{\mathbb{P}_2}(1)).$$

Hence $p_{1*}p_2^*\mathcal{O}_{\mathbb{P}_2}(1) \cong \mathcal{O}_S \otimes \Gamma(\mathbb{P}_2, \mathcal{O}_{\mathbb{P}_2}(1)) \cong 3\mathcal{O}_S$.

Note that $(\pi^*\pi_*\mathcal{L}')_s \cong \mathcal{O}_{Z_s} \otimes_{\mathbb{k}} (\pi_*\mathcal{L}')(s)$, where

$$(\pi_*\mathcal{L}')(s) = \pi_*\mathcal{L}'/\mathfrak{m}_{S,s} \cdot \pi_*\mathcal{L}'.$$

Under this identification the restriction of the canonical homomorphism $\pi^*\pi_*\mathcal{L}' \to \mathcal{L}'$ to the fibre Z_s coincides with the composition

$$\mathcal{O}_{Z_s} \otimes_{\mathbb{k}} (\pi_*\mathcal{L}')(s) \xrightarrow{\mathrm{id} \otimes \varphi^0(s)} \mathcal{O}_{Z_s} \otimes_{\mathbb{k}} H^0(Z_s, \mathcal{L}'_s) \xrightarrow{\mathrm{ev}} \mathcal{L}'_s,$$

where ev is the evaluation morphism and $\varphi^0(s)$ is the homomorphism from the base change theorem. We have just shown that $\varphi^0(s)$ is an isomorphism. Since $\mathcal{L}'_s \cong \mathcal{O}_{Z_s}(1,0)$ and since $\mathcal{O}_{\mathbb{P}_2}(1)$ is generated by the global sections, we conclude also that the sheaf $\mathcal{L}'_s$ is generated by its global sections as well, hence ev is surjective and we conclude that $(\pi^*\pi_*\mathcal{L}')_s \to (\mathcal{L}')_s$ is surjective for every $s \in S$. Therefore, $\pi^*\pi_*\mathcal{L}' \to \mathcal{L}'$ is surjective. This proves the statement of the lemma. $\square$

Consider now the sheaf $\mathcal{L}''$ and its direct image $\pi_*\mathcal{L}''$.

Lemma 3.33. *The sheaf $\mathcal{G}_2 = \pi_*\mathcal{L}''$ is a locally free sheaf of rank 3 on S and the canonical morphism $\pi^*\mathcal{G}_2 = \pi^*\pi_*\mathcal{L}'' \to \mathcal{L}''$ is surjective.*

Proof. Since $H^1(\mathbb{P}_2, \mathcal{O}_{\mathbb{P}_2}(1)) = 0$, using Proposition 1.32 and the properties of $\mathcal{L}''$ from Definition 3.21 we conclude that

$$H^1(Z_s, {\mathcal{L}''}_s) \cong H^1(Z_s, \mathcal{O}_{Z_s}(0,1)_s) = 0.$$

Therefore, by the base change theorem the canonical map $\varphi^0(s) : \pi_*\mathcal{L}''(s) \to H^0(Z_s, \mathcal{L}''{}_s)$ is an isomorphism for every $s \in S$. Since by Proposition 1.32 $H^0(Z_s, \mathcal{L}''{}_s) \cong \mathbb{k}^3$, applying the base change theorem using the surjectivity of $\varphi^{-1}(s)$ (by definition) we obtain that $\mathcal{G}_2 = \pi_*\mathcal{L}''$ is a locally free sheaf of rank 3 on S.

Using the same arguments as in the proof of Lemma 3.32 we conclude that $\mathcal{L}''_s$ is generated by its global sections as the pull back from $\mathbb{P}_2$ of the globally generated sheaf $\mathcal{O}_{\mathbb{P}_2}(1)$. Therefore, $(\pi^*\pi_*\mathcal{L}'')_s \to \mathcal{L}''_s$ is surjective for every $s \in S$ and one finally obtains that the canonical homomorphism $\pi^*\mathcal{G}_2 = \pi^*\pi_*\mathcal{L}'' \to \mathcal{L}''$ is surjective. This completes the proof. □

Lemma 3.34. *For $\mathcal{L} = \mathcal{L}' \otimes \mathcal{L}''$ the sheaf $\mathcal{G} = \pi_*\mathcal{L}$ is a locally free sheaf of rank 6 on S. The canonical homomorphism $\pi^*\mathcal{G} = \pi^*\pi_*\mathcal{L} \to \mathcal{L}$ is surjective.*

Proof. As above we get

$$H^1(Z_s, \mathcal{L}_s) \cong H^1(Z_s, \mathcal{O}_{Z_s}(1,1)_s) = 0 \quad \text{and} \quad H^0(Z_s, \mathcal{L}_s) \cong H^0(Z_s, \mathcal{O}_{Z_s}(1,1)_s) \cong \mathbb{k}^6.$$

Applying the base change theorem we obtain that $\mathcal{G}$ is a locally free sheaf of rank 6.

Since $\mathcal{L}$ is very ample relative to S we conclude that $\pi^*\pi_*\mathcal{L} \to \mathcal{L}$ is surjective (see also [8], 4.4 and 3.4.7). This completes the proof. □

We have already shown that the canonical homomorphisms

$$\pi^*(3\mathcal{O}_S) \cong \pi^*\pi_*\mathcal{L}' \twoheadrightarrow \mathcal{L}', \quad \pi^*\mathcal{G}_2 = \pi^*\pi_*\mathcal{L}'' \twoheadrightarrow \mathcal{L}'', \quad \pi^*\mathcal{G} = \pi^*\pi_*\mathcal{L} \twoheadrightarrow \mathcal{L}$$

are surjective. They correspond to some morphisms

$$Z \to \mathbf{P}(3\mathcal{O}_S) \cong S \times \mathbb{P}_2, \quad Z \to \mathbf{P}(\mathcal{G}_2), \quad Z \to \mathbf{P}(\mathcal{G})$$

over S. The first map coincides with σ. Since we assumed $\mathcal{L}$ relative very ample, the last morphism is a closed embedding.

For every $s \in S$ the canonical homomorphism

$$H^0(Z_s, \mathcal{L}'{}_s) \otimes H^0(Z_s, \mathcal{L}''{}_s) \to H^0(Z_s, \mathcal{L}_s)$$

is surjective. Therefore, one concludes that the canonical map

$$\mathcal{G}_1 \otimes \mathcal{G}_2 = \pi_*\mathcal{L}' \otimes \pi_*\mathcal{L}'' \to \pi_*\mathcal{L} = \mathcal{G}$$

is surjective as well. Therefore, the corresponding morphism

$$\mathbf{P}(\mathcal{G}) \to \mathbf{P}(\mathcal{G}_1 \otimes \mathcal{G}_2)$$

is a closed embedding and we obtain the commutative diagram

$$\begin{tikzcd}
Z \arrow[r] \arrow[d, tail] & \mathbf{P}(\mathcal{G}_1) \times_S \mathbf{P}(\mathcal{G}_2) = (S \times \mathbb{P}_2) \times_S \mathbf{P}(\mathcal{G}_2), \arrow[d, tail, "\varsigma"] \\
\mathbf{P}(\mathcal{G}) \arrow[r, tail] & \mathbf{P}(\mathcal{G}_1 \otimes \mathcal{G}_2)
\end{tikzcd}$$

where $\varsigma : \mathbf{P}(\mathcal{G}_1) \times_S \mathbf{P}(\mathcal{G}_2) \to \mathbf{P}(\mathcal{G}_1 \otimes \mathcal{G}_2)$ is the Segre embedding (cf. [8], 4.3). From the commutativity of the diagram we obtain that the morphism

$$Z \to (S \times \mathbb{P}_2) \times_S \mathbf{P}(\mathcal{G}_2) \cong \mathbb{P}_2 \times \mathbf{P}(\mathcal{G}_2)$$

is a closed embedding. We have shown that Z is a closed subvariety in some $\mathbb{P}_2 \times \mathbb{P}_2$-bundle over S (compare with Proposition 3.13).

Proposition 3.35. *Z is a closed subvariety in $\mathbb{P}_2 \times \mathbf{P}(\mathcal{G}_2)$, where $\mathcal{G}_2 = \pi_*\mathcal{L}''$ is a locally free sheaf of rank 3 on S.*

In particular this means that for every point $s \in S$ there is an open neighbourhood U of s such that Z_U is a closed subvariety in $U \times \mathbb{P}_2 \times \mathbb{P}_2$.

Local existence of resolutions.

Let $(\pi : Z \to S, \sigma, \mathcal{L}'', \mathcal{E}, \mathcal{F}, \tau)$ be a family over S. Then $\mathcal{F}$ is a $3m+1$ family over S. Let $p_1 : S \times \mathbb{P}_2 \to S$ and $p_2 : S \times \mathbb{P}_2 \to \mathbb{P}_2$ be the canonical projections. Then there exists a relative Beilinson resolution (cf. [24])

$$0 \to p_1^*\mathcal{A}_2 \otimes p_2^*\mathcal{O}_{\mathbb{P}_2}(-2) \xrightarrow{\Upsilon} (p_1^*\mathcal{A}_1 \otimes p_2^*\mathcal{O}_{\mathbb{P}_2}(-1)) \oplus (p_1^*\mathcal{A}_0 \otimes p_2^*\mathcal{O}_{\mathbb{P}_2}) \to \mathcal{F} \to 0, \tag{3.5}$$

where $\mathcal{A}_2 = R^1p_{1*}(\mathcal{F} \otimes p_2^*\Omega^2_{\mathbb{P}_2}(2))$, $\mathcal{A}_1 = R^1p_{1*}(\mathcal{F} \otimes p_2^*\Omega^1_{\mathbb{P}_2}(1))$, and $\mathcal{A}_0 = R^0p_{1*}(\mathcal{F})$ are locally free sheaves on S of rank 2, 1, and 1 respectively.

Let us apply σ^* to this resolution. Then

$$\begin{aligned}
&\sigma^*(p_1^*\mathcal{A}_0 \otimes p_2^*\mathcal{O}_{\mathbb{P}_2}) \cong \sigma^*p_1^*\mathcal{A}_0 \otimes \sigma^*p_2^*\mathcal{O}_{\mathbb{P}_2} \cong \pi^*\mathcal{A}_0 \otimes \mathcal{O}_Z,\\
&\sigma^*(p_1^*\mathcal{A}_1 \otimes p_2^*\mathcal{O}_{\mathbb{P}_2}(-1)) \cong \pi^*\mathcal{A}_1 \otimes \mathcal{O}_Z(-1,0),\\
&\sigma^*(p_1^*\mathcal{A}_2 \otimes p_2^*\mathcal{O}_{\mathbb{P}_2}(-2)) \cong \pi^*\mathcal{A}_2 \otimes \mathcal{O}_Z(-2,0).
\end{aligned}$$

We obtain the commutative diagram with exact rows and columns

$$\begin{tikzcd}
 & & 0 \arrow[d] & & 0 \arrow[d] & \\
 & & \mathcal{C}' \arrow[d] \arrow[dl, equal, bend right] & & \mathcal{C} \arrow[d] & \\
0 \arrow[r] & \mathcal{C}' \arrow[r] & \pi^*\mathcal{A}_2 \otimes \mathcal{O}_Z(-2,0) \arrow[r, "\sigma^*\Upsilon"] \arrow[d] & (\pi^*\mathcal{A}_1 \otimes \mathcal{O}_Z(-1,0)) \oplus (\pi^*\mathcal{A}_0 \otimes \mathcal{O}_Z) \arrow[r] \arrow[d, equal] & \sigma^*\mathcal{F} \arrow[r] \arrow[d, "\tau"] & 0 \\
 & 0 \arrow[r] & \mathcal{K} \arrow[r] \arrow[d] & (\pi^*\mathcal{A}_1 \otimes \mathcal{O}_Z(-1,0)) \oplus (\pi^*\mathcal{A}_0 \otimes \mathcal{O}_Z) \arrow[r] & \mathcal{E} \arrow[r] \arrow[d] & 0. \\
 & & \mathcal{C} \arrow[d] & & 0 & \\
 & & 0 & & &
\end{tikzcd}$$

Lemma 3.36. *The sheaf* $\mathcal{B} := \pi_*\mathcal{K}(1,1)$ *is a locally free sheaf of rank* 2.

Proof. Since by assumption $\mathcal{E}$ is flat over S and since the sheaf

$$(\pi^*\mathcal{A}_1 \otimes \mathcal{O}_Z(-1,0)) \oplus (\pi^*\mathcal{A}_0 \otimes \mathcal{O}_Z)$$

is flat as a locally free sheaf on Z (recall that $Z \xrightarrow{\pi} S$ is flat), we conclude that the sheaf $\mathcal{K}$ is flat over S as well.

By Proposition 1.66 we know that $\mathcal{K}_s \cong 2\mathcal{O}_{Z_s}(-1,-1)$ if $Z_s \cong \hat{\mathbb{P}}_2$. If $Z_s \cong \mathbb{P}_2$, then $\mathcal{K}_s \cong 2\mathcal{O}_{\mathbb{P}_2}(-2,0) = 2\mathcal{O}_{\mathbb{P}_2}(-1,-1)$ (cf. Remark 1.46). Thus one concludes that $\mathcal{K}_s \cong 2\mathcal{O}_{Z_s}(-1,-1)$ for all $s \in S$. The sheaf $\mathcal{K}(1,1) := \mathcal{K} \otimes \mathcal{O}_Z(1,1)$ is also flat. We have $\mathcal{K}(1,1)_s \cong 2\mathcal{O}_{Z_s}$.

One has $H^1(Z_s, \mathcal{K}(1,1)_s) \cong H^1(Z_s, 2\mathcal{O}_{Z_s}) = 0$ for every $s \in S$. Therefore, by the base change theorem the canonical homomorphism

$$\varphi^0(s) : \pi_*\mathcal{K}(1,1)(s) \to H^0(Z_s, \mathcal{K}(1,1)_s) \cong H^0(Z_s, 2\mathcal{O}_{Z_s}) \cong \mathbb{k}^2$$

is an isomorphism for every $s \in S$. Again by the base change theorem using the surjectivity of $\varphi^{-1}(s)$ (by definition) we obtain that $\pi_*\mathcal{K}(1,1)$ is a locally free sheaf of rank 2. $\square$

Proposition 3.37. $\mathcal{K} \cong \pi^*\mathcal{B} \otimes \mathcal{O}_Z(-1,-1)$, *in particular* $\mathcal{K}$ *is a locally free sheaf of rank* 2.

Proof. Consider the canonical homomorphism $\pi^*\mathcal{B} = \pi^*\pi_*\mathcal{K}(1,1) \xrightarrow{\eta} \mathcal{K}(1,1)$ and its restrictions $\pi^*\pi_*\mathcal{K}(1,1)_s \xrightarrow{\eta_s} \mathcal{K}(1,1)_s$ to the fibres Z_s, $s \in S$.

Note that $\pi^*\pi_*\mathcal{K}(1,1)_s \cong \mathcal{O}_{Z_s} \otimes_{\Bbbk} (\pi_*\mathcal{K}(1,1))(s)$, where

$$(\pi_*\mathcal{K}(1,1))(s) = \pi_*\mathcal{K}(1,1)/\mathfrak{m}_{S,s} \cdot \pi_*\mathcal{K}(1,1).$$

There is the following commutative diagram

$$\begin{tikzcd}
\mathcal{O}_{Z_s} \otimes_{\Bbbk} (\pi_*\mathcal{K}(1,1))(s) \arrow[d, "\mathrm{id} \otimes \varphi^0(s)"'] \arrow[dr, "\eta_s"] & \\
\mathcal{O}_{Z_s} \otimes_{\Bbbk} H^0(Z_s, \mathcal{K}(1,1)_s) \arrow[r, "\mathrm{ev}"] & \mathcal{K}(1,1)_s,
\end{tikzcd}$$

where ev is the evaluation morphism and $\varphi^0(s)$ is the morphism from the base change theorem. In the proof of Lemma 3.36 we have already shown that $\varphi^0(s)$ is an isomorphism for every s. Since $\mathcal{K}(1,1)_s \cong \mathcal{O}_{Z_s}$, we conclude that the map ev is an isomorphism and hence $\eta_s = \mathrm{ev} \circ (\mathrm{id} \otimes \varphi^0(s))$ is an isomorphism as well. The latter implies that the canonical homomorphism $\pi^*\mathcal{B} = \pi^*\pi_*\mathcal{K}(1,1) \to \mathcal{K}(1,1)$ is an isomorphism. This gives $\mathcal{K} \cong \pi^*\mathcal{B} \otimes \mathcal{O}_Z(-1,-1)$. Since the sheaf $\mathcal{B} = \pi_*\mathcal{K}(1,1)$ is locally free of rank 2, we obtain that $\mathcal{K}(1,1)$ is also a locally free sheaf of rank 2. This completes the proof. $\square$

We obtain finally the commutative diagram with exact rows and columns

$$\begin{tikzcd}
 & & 0 \arrow[d] & & 0 \arrow[d] & \\
 & & \mathcal{C}' \arrow[d] \arrow[dl, equal, bend right] & & \mathcal{C} \arrow[d] & \\
0 \arrow[r] & \mathcal{C}' \arrow[r] & \pi^*\mathcal{A}_2 \otimes \mathcal{O}_Z(-2,0) \arrow[r] \arrow[d, "\gamma"] & (\pi^*\mathcal{A}_1 \otimes \mathcal{O}_Z(-1,0)) \oplus (\pi^*\mathcal{A}_0 \otimes \mathcal{O}_Z) \arrow[r] \arrow[d, equal] & \sigma^*\mathcal{F} \arrow[r] \arrow[d, "\tau"] & 0 \\
 & 0 \arrow[r] & \pi^*\mathcal{B} \otimes \mathcal{O}_Z(-1,-1) \arrow[r] \arrow[d] & (\pi^*\mathcal{A}_1 \otimes \mathcal{O}_Z(-1,0)) \oplus (\pi^*\mathcal{A}_0 \otimes \mathcal{O}_Z) \arrow[r] & \mathcal{E} \arrow[r] \arrow[d] & 0. \\
 & & \mathcal{C} \arrow[d] & & 0 & \\
 & & 0 & & &
\end{tikzcd}$$

Lemma 3.38. *For every point $s \in S$ there exists an open neighbourhood U of s such that the sheaves $\mathcal{A}_2$ and $\mathcal{B}$ are both isomorphic to $2\mathcal{O}_U$. For an appropriate choice of isomorphisms the restriction of γ to $Z_U = \pi^{-1}(U)$ is then of the form*

$$2\mathcal{O}_{Z_U}(-2,0) \xrightarrow{\left(\begin{smallmatrix} e & 0 \\ 0 & e \end{smallmatrix}\right)} 2\mathcal{O}_{Z_U}(-1,-1)$$

for some $e \in \Gamma(Z, \mathcal{O}_Z(1,-1))$.

Proof. Since both sheaves $\mathcal{A}_2$ and $\mathcal{B}$ are locally free of rank 2 on S, one concludes that for every point $s \in S$ there exists an open neighbourhood U of s such that both $\mathcal{A}_2$ and $\mathcal{B}$ and are isomorphic to $2\mathcal{O}_U$ and hence their pull backs along π are free sheaves of rank 2 on Z_U. Therefore, the restriction of γ to Z_U is given by a matrix

$$2\mathcal{O}_{Z_U}(-2,0) \xrightarrow{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)} 2\mathcal{O}_{Z_U}(-1,-1), \quad a,b,c,d \in \Gamma(\mathcal{O}_{Z_U}, \mathcal{O}_{Z_U}(1,-1)).$$

It holds $\Gamma(Z_U, \mathcal{O}_{Z_U}(1,-1)) = \Gamma(U, \pi_* \mathcal{O}_{Z_U}(1,-1))$. Using Proposition 1.32, for every $s \in S$ we obtain $H^i(Z_s, \mathcal{O}_{Z_s}(1,-1)) = 0$ for $i \neq 0$ and $H^0(Z_s, \mathcal{O}_{Z_s}(1,-1)) \cong \Bbbk$. Therefore, by the base change theorem we conclude that $\pi_* \mathcal{O}_Z(1,-1)$ is a locally free sheaf of rank 1 on S. Thus making U small enough we may assume that $\pi_* \mathcal{O}_{Z_U}(1,-1)$ is isomorphic to $\mathcal{O}_U$. Therefore, we obtain the following isomorphism of $\mathcal{O}_U(U)$ modules:

$$\Gamma(Z_U, \mathcal{O}_{Z_U}(1,-1)) = \Gamma(U, \pi_* \mathcal{O}_{Z_U}(1,-1)) \cong \Gamma(U, \mathcal{O}_U).$$

Note that the canonical homomorphism

$$\mathcal{O}_U(U) \to (\pi_* \mathcal{O}_{Z_U})(U) = \mathcal{O}_{Z_U}(Z_U)$$

equips $\Gamma(Z_U, \mathcal{O}_{Z_U}(1,-1))$ with a structure of an $\mathcal{O}_U(U)$ module. As above, using the base change theorem, we conclude that the sheaf $\pi_* \mathcal{O}_Z$ is an invertible sheaf on S and shrinking U we obtain $\pi_* \mathcal{O}_{Z_U} \cong \mathcal{O}_U$. Therefore, we obtain an isomorphism of $\Bbbk$-algebras

$$\mathcal{O}_{Z_U}(Z_U) = (\pi_* \mathcal{O}_{Z_U})(U) \cong \mathcal{O}_U(U)$$

and may identify the sections from $\mathcal{O}_{Z_U}(Z_U)$ and $\mathcal{O}_U(U)$.

Let a', b', c', d' be the elements of $\Gamma(U, \mathcal{O}_U) \cong \Gamma(Z_U, \mathcal{O}_{Z_U})$ corresponding to a, b, c, d respectively. Let $e \in \Gamma(Z_U, \mathcal{O}_{Z_U}(1,-1))$ be the section that corresponds to $1 \in \Gamma(U, \mathcal{O}_U) = \mathcal{O}_U(U)$. Then

$$a = a' \cdot e, \quad b = b' \cdot e, \quad c = c' \cdot e, \quad d = d' \cdot e$$

and we obtain the commutative diagram

$$\begin{tikzcd}
2\mathcal{O}_{Z_U}(-2,0) \arrow[rr, "\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)"] \arrow[dr, "\left(\begin{smallmatrix} a' & b' \\ c' & d' \end{smallmatrix}\right)"'] & & 2\mathcal{O}_{Z_U}(-1,1). \\
& 2\mathcal{O}_{Z_U}(-2,0) \arrow[ur, "\left(\begin{smallmatrix} e & 0 \\ 0 & e \end{smallmatrix}\right)"'] &
\end{tikzcd}$$

To prove the required statement it is enough to show that the matrix $\left(\begin{smallmatrix} a' & b' \\ c' & d' \end{smallmatrix}\right)$ is invertible.

Since

$$\sigma^{-1}(S \times \mathbb{P}_2 \setminus \operatorname{Sing} \mathcal{F}) \xrightarrow{\sigma} S \times \mathbb{P}_2 \setminus \operatorname{Sing} \mathcal{F}$$

is an isomorphism, we conclude that γ is an isomorphism over $W = \sigma^{-1}(S \times \mathbb{P}_2 \setminus \operatorname{Sing} \mathcal{F})$. Then $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ is an isomorphism over the intersection of Z_U with W. Hence both $\left(\begin{smallmatrix} a' & b' \\ c' & d' \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} e & 0 \\ 0 & e \end{smallmatrix}\right)$ must be isomorphisms over this set.

Note that there exists locally a section of the morphism $W \subseteq Z \xrightarrow{\pi} S$. Therefore, the morphism

$$2\mathcal{O}_U(U) \xrightarrow{\left(\begin{smallmatrix} a' & b' \\ c' & d' \end{smallmatrix}\right)} 2\mathcal{O}_U(U)$$

is an isomorphism and the same holds for the matrix $\left(\begin{smallmatrix} a' & b' \\ c' & d' \end{smallmatrix}\right)$ considered as a morphism $2\mathcal{O}_{Z_U} \to 2\mathcal{O}_{Z_U}$. This completes the proof. $\square$

As the sheaves $\mathcal{B}$, $\mathcal{A}_0$, $\mathcal{A}_1$, and $\mathcal{A}_2$ are locally free sheaves on S of rank 2, 1, 1 and 2 respectively, the latter means that for every point $s \in S$ there exists an open neighbourhood U of s such that for $Z_U = \pi^{-1}(U)$ we have the resolution

$$0 \to 2\mathcal{O}_{U \times \mathbb{P}_2}(-2) \xrightarrow{\Psi} \mathcal{O}_{U \times \mathbb{P}_2}(-1) \oplus \mathcal{O}_{U \times \mathbb{P}_2} \to \mathcal{F}_U \to 0 \tag{3.6}$$

and the commutative diagram with exact rows and columns

$$
\begin{tikzcd}
& & 0 \arrow[d] & & 0 \arrow[d] & \\
& & \mathcal{C}' \arrow[d] \arrow[dl, equal, bend right] & & \mathcal{C} \arrow[d] & \\
0 \arrow[r] & \mathcal{C}' \arrow[r] & 2\mathcal{O}_{Z_U}(-2,0) \arrow[r, "\sigma^*\Psi"] \arrow[d, "\left(\begin{smallmatrix} e & 0 \\ 0 & e \end{smallmatrix}\right)"'] & \mathcal{O}_{Z_U}(-1,0) \oplus \mathcal{O}_{Z_U} \arrow[r] \arrow[d, equal] & (\sigma^*\mathcal{F})_U \arrow[r] \arrow[d, "\tau"] & 0 \\
& 0 \arrow[r] & 2\mathcal{O}_{Z_U}(-1,-1) \arrow[r, "\Phi"] \arrow[d] & \mathcal{O}_{Z_U}(-1,0) \oplus \mathcal{O}_{Z_U} \arrow[r] & \mathcal{E}_U \arrow[r] \arrow[d] & 0. \\
& & \mathcal{C} \arrow[d] & & 0 & \\
& & 0 & & &
\end{tikzcd}
\tag{3.7}
$$

Maps to $\widetilde{X}$ and $\widetilde{M}$.

Lemma 3.39. *Let $(\pi : Z \to S, \mathcal{L}'', \sigma, \mathcal{E}, \mathcal{F}, \tau)$ be a family over S. Then there is an open covering $S = \bigcup_i S_i$ and morphisms $\chi_i : S_i \to \widetilde{X}$ such that $\widetilde{\nu} \circ \chi_i|_{S_i \cap S_j} = \widetilde{\nu} \circ \chi_j|_{S_i \cap S_j}$ for every i and j. This defines a morphism $S \to \widetilde{M}$.*

Proof. First of all note that there is an open covering of $S = \bigcup S_i$ and morphisms $S_i \to X$ induced by the $3m+1$ family $\mathcal{F}$. All that morphisms composed with the quotient map $X \xrightarrow{\nu} M$ agree with each other on the intersections $S_i \cap S_j$ and give rise to a unique morphism $S \to M$ such that the sheaf $\mathcal{F}$ is the pull back of the universal $3m+1$ family on M with under that morphism (recall that M is a fine moduli space for the $3m+1$ moduli problem on $\mathbb{P}_2$).

Consider the diagram (3.7). From resolution (3.6) we obtain the morphism

$$a_U : U \to X, \quad s \mapsto \Psi_s \in X.$$

For every $s \in U$ consider the restriction of (3.7) to the fibre Z_s. We obtain the commutative square

$$
\begin{tikzcd}
2\mathcal{O}_{Z_s}(-2,0) \arrow[r, "(\sigma^*\Psi)_s"] \arrow[d, "\left(\begin{smallmatrix} e_s & 0 \\ 0 & e_s \end{smallmatrix}\right)"'] & \mathcal{O}_{Z_s}(-1,0) \oplus \mathcal{O}_{Z_s} \arrow[d, equal] \\
2\mathcal{O}_{Z_s}(-1,-1) \arrow[r, "\Phi_s"] & \mathcal{O}_{Z_s}(-1,0) \oplus \mathcal{O}_{Z_s}.
\end{tikzcd}
\tag{3.8}
$$

Note that $H^0(Z_s, \mathcal{O}_{Z_s}(1,-1)) \cong \mathbb{k}$. Indeed by Proposition 1.32 this holds for $Z_s \cong \hat{\mathbb{P}}_2$. If the fibre Z_s is isomorphic to $\mathbb{P}_2$, using $\mathcal{O}_{\mathbb{P}_2}(1,-1) \cong \mathcal{O}_{\mathbb{P}_2}$ (cf. Remark 1.46) we obtain $H^0(\mathbb{P}_2, \mathcal{O}_{Z_s}(1,-1)) \cong \mathbb{k}$ as well. So e_s is unique up to multiplication by a non-zero constant.

We obtain the injective morphism $\Phi_s \in \operatorname{Hom}(2\mathcal{O}_{Z_U}(-2,0), 2\mathcal{O}_{Z_U}(-1,1))$,

$$\Phi_s = \begin{pmatrix} l_1 & \widetilde{q}_1 \\ l_2 & \widetilde{q}_1 \end{pmatrix}(s) = \begin{pmatrix} l_1(s) & \widetilde{q}_1(s) \\ l_2(s) & \widetilde{q}_1(s) \end{pmatrix}, \quad l_i(s) \in \Gamma(Z_s, \mathcal{O}_{Z_s}(0,1)), \quad q_i(s) \in \Gamma(Z_s, \mathcal{O}_{Z_s}(1,1)).$$

Denote $\mathrm{pt}_s := \langle l_1(s) \wedge l_2(s) \rangle$. We obtain the morphism

$$b_U : U \to \mathbb{P}_1, \quad s \mapsto \langle \widetilde{q}_1(s)(\mathrm{pt}_s), \widetilde{q}_2(s)(\mathrm{pt}_s) \rangle.$$

Therefore, one gets the morphism

$$(a_U, b_U) : U \to X \times \mathbb{P}_1, \quad s \mapsto (a_U(s), b_U(s)).$$

Then from the commutative square (3.8) we conclude (cf. (1.9)) that

$$\widetilde{q}_1(s)(\mathrm{pt}_s) \cdot f_4(\Psi_s) - \widetilde{q}_2(s)(\mathrm{pt}_s) \cdot f_3(\Psi_s) = 0.$$

By (3.1) this means that the image of the morphism $(a_U, b_U) : U \to X \times \mathbb{P}_1$ lies in $\widetilde{X}$.

We obtained a morphism $\chi_U = (a_U, b_U) : U \to \widetilde{X}$. We can construct such a morphism in a neighbourhood of every point $s \in S$. Suppose we have two morphisms $\chi_U : U \to \widetilde{X}$ and $\chi_{U'} : U' \to \widetilde{X}$ for two open sets U and U' in S.

Consider a point $s \in U \cap U'$. We obtain two commutative squares

$$\begin{tikzcd} 2\mathcal{O}_{Z_s}(-2,0) \arrow[r, "(\sigma^*\Psi)_s"] \arrow[d, "\begin{pmatrix} e_s & 0 \\ 0 & e_s \end{pmatrix}"'] & \mathcal{O}_{Z_s}(-1,0) \oplus \mathcal{O}_{Z_s} \arrow[d, equal] \\ 2\mathcal{O}_{Z_s}(-1,-1) \arrow[r, "\Phi_s"] & \mathcal{O}_{Z_s}(-1,0) \oplus \mathcal{O}_{Z_s}, \end{tikzcd} \qquad \begin{tikzcd} 2\mathcal{O}_{Z_s}(-2,0) \arrow[r, "(\sigma^*\Psi')_s"] \arrow[d, "\begin{pmatrix} e'_s & 0 \\ 0 & e'_s \end{pmatrix}"'] & \mathcal{O}_{Z_s}(-1,0) \oplus \mathcal{O}_{Z_s} \arrow[d, equal] \\ 2\mathcal{O}_{Z_s}(-1,-1) \arrow[r, "\Phi'_s"] & \mathcal{O}_{Z_s}(-1,0) \oplus \mathcal{O}_{Z_s}. \end{tikzcd} \tag{3.9}$$

Since we have already noticed that $H^0(Z_s, \mathcal{O}_{Z_s}(1,-1)) \cong \Bbbk$, we obtain that $e_s = \lambda e'_s$ for some $\lambda \in \Bbbk^*$.

The sheaf $\mathcal{E}_s$ has the resolutions

$$0 \to 2\mathcal{O}_{Z_s}(-1,-1) \xrightarrow{\Phi_s} \mathcal{O}_{Z_s}(-1,0) \oplus \mathcal{O}_{Z_s} \to \mathcal{E}_s \to 0$$

and

$$0 \to 2\mathcal{O}_{Z_s}(-1,-1) \xrightarrow{\Phi'_s} \mathcal{O}_{Z_s}(-1,0) \oplus \mathcal{O}_{Z_s} \to \mathcal{E}_s \to 0.$$

Then (cf. Section 2.1) there is $(f, h) \in G$ such that the diagram

$$\begin{tikzcd} 2\mathcal{O}_{Z_s}(-1,1) \arrow[r, "\Phi_s"] \arrow[d, "g"'] & \mathcal{O}_{Z_s}(-1,0) \oplus \mathcal{O}_{Z_s} \arrow[d, "h"] \\ 2\mathcal{O}_{Z_s}(-1,-1) \arrow[r, "\Phi'_s"] & \mathcal{O}_{Z_s}(-1,0) \oplus \mathcal{O}_{Z_s}. \end{tikzcd}$$

commutes, i. e., $\Phi_s = g \cdot \Phi'_s \cdot h^{-1}$. In particular this implies that $b_U(s) = b_{U'}(s) \cdot g^{\mathrm{T}}$.

Using (3.9) we obtain the commutative diagram

$$\begin{tikzcd} 2\mathcal{O}_{Z_s}(-2,0) \arrow[r, "(\sigma^*\Psi)_s"] \arrow[d, "\lambda g"'] & \mathcal{O}_{Z_s}(-1,0) \oplus \mathcal{O}_{Z_s} \arrow[d, "h"] \\ 2\mathcal{O}_{Z_s}(-2,0) \arrow[r, "(\sigma^*\Psi')_s"] & \mathcal{O}_{Z_s}(-1,0) \oplus \mathcal{O}_{Z_s}. \end{tikzcd}$$

and therefore $\Psi_s = \lambda g \cdot \Psi'_s \cdot h^{-1}$ and hence $a_U(s) = \lambda g \cdot a_{U'}(s) \cdot h^{-1}$.

We obtained that $\chi_U(s) = (a_U(s), b_U(s))$ and $\chi_{U'}(s) = (a_{U'}(s), b_{U'}(s))$ lie in the same orbit of the action of G on $\widetilde{X}$. Therefore, by Lemma 3.7 $\widetilde{\nu} \circ \chi_U(s) = \widetilde{\nu}\chi_{U'}(s)$. This proves that the restrictions of $\widetilde{\nu} \circ \chi_U$ and $\widetilde{\nu} \circ \chi_{U'}$ to $U \cap U'$ coincide.

We proved that one can cover S by open sets S_i such that there are morphisms $\chi_i : S_i \to \widetilde{X}$. Moreover the morphisms $\widetilde{\nu} \circ \chi_i : S_i \to \widetilde{M}$ agree on the intersections. We obtain then a morphism $S \to \widetilde{M}$. This completes the proof. $\square$

Note that two equivalent families over S define the same morphism $S \to \widetilde{M}$. This gives us a map

$$\widetilde{\mathcal{M}}(S) \to \operatorname{Hom}(S, \widetilde{M}).$$

Consider a morphism $T \xrightarrow{f} S$ and a family $(\pi : Z \to S, \mathcal{L}'', \sigma, \mathcal{E}, \mathcal{F}, \tau)$ over S with the induced morphism $S \xrightarrow{\phi} \widetilde{M}$. From the considerations in the proof of Lemma 3.39 it follows that the morphism induced by the pull back of $(\pi : Z \to S, \mathcal{L}'', \sigma, \mathcal{E}, \mathcal{F}, \tau)$ along f coincides with the composition $\phi \circ f$. We obtained the following proposition.

Proposition 3.40. *There is a natural transformation of functors*

$$\widetilde{\mathcal{M}} \to \operatorname{Hom}(\_\,, \widetilde{M})$$

and the commutative diagram

$$\begin{tikzcd}
\widetilde{\mathcal{M}} \arrow[r] \arrow[d, "\mu"'] & \operatorname{Hom}(_\,, \widetilde{M}) \arrow[d] \\
\mathcal{M} \arrow[r] & \operatorname{Hom}(_\,, M).
\end{tikzcd}$$

Open questions

There are still questions to be answered.

- It is not clear whether the natural transformation of functors $\widetilde{\mathcal{M}} \to \mathrm{Hom}(\_\,, \widetilde{M})$ we obtained is an isomorphism (equivalence of functors), i. e., whether the space $\widetilde{M}$ is a fine moduli space of the moduli problem we defined in this thesis. This question is connected with the existence or non-existence of a descent of the sheaf $\widetilde{\mathcal{U}}$ over $\widetilde{X}$ to a sheaf over $\widetilde{M}$.

- Another question is whether $\widetilde{\mathcal{M}} \to \mathrm{Hom}(\_\,, \widetilde{M})$ is universal in the following sense:

  for every natural transformations of functors $\widetilde{\mathcal{M}} \to \mathrm{Hom}(\_\,, W)$ there exists a unique arrow $\mathrm{Hom}(\_\,, \widetilde{M}) \to \mathrm{Hom}(\_\,, W)$ (in other words a morphism $\widetilde{M} \to W$) that makes the diagram

$$\begin{tikzcd} \widetilde{\mathcal{M}} \arrow[r] \arrow[dr] & \mathrm{Hom}(\_\,, \widetilde{M}) \arrow[d, dotted, "\exists!"] \\ & \mathrm{Hom}(\_\,, W) \end{tikzcd}$$

  commute.

- It was mentioned in [4] that there is an isomorphism $M_{3m+1}(\mathbb{P}_2) \cong M_{3m+2}(\mathbb{P}_2)$. The description of the parameter space for $M_{3m+1}(\mathbb{P}_2)$ and $M_{3m+2}(\mathbb{P}_2)$ are almost the same. Therefore, the techniques presented in the dissertation may be applied almost without changes to the moduli space $M_{3m+2}(\mathbb{P}_2)$. It is however not clear whether the ideas from this thesis may be reasonably modified to work for a bigger class of Simpson's moduli spaces. We guess it may be needed a sequence of steps similar to the construction described in the dissertation to replace the singular sheaves of a moduli space $M_P(\mathbb{P}_2)$ for a general polynomial $P(m)$.

Appendix A

General statements

This chapter contains some auxiliary statements that are used in the dissertation. Some of them may be easily found in the literature and are provided with references. The statements for which the author have not found the references are provided with proofs.

A.1 Flatness and base change

The following statements are often used in the dissertation.

Theorem A.1 (Base change)**.** *Let $X \xrightarrow{f} Y$ be a projective morphism. Let $\mathcal{F}$ be a coherent sheaf on X which is flat over Y. Let $y \in Y$ be a point in Y. Then the natural map*

$$\varphi^p(y) : R^p f_*(\mathcal{F}) \otimes \Bbbk(y) \to H^p(X_y, \mathcal{F}_y)$$

is an isomorphism if and only if it is surjective. In this case it remains isomorphism in a neighbourhood of y and the following two conditions are equivalent:

1) the natural map

$$\varphi^{p-1}(y) : R^{p-1} f_*(\mathcal{F}) \otimes \Bbbk(y) \to H^{p-1}(X_y, \mathcal{F}_y)$$

is surjective;

2) $R^p f_*(\mathcal{F})$ is locally free in a neighbourhood of y.

Proof. See [12], III, Theorem 12.11. □

Theorem A.2. *Let $f : X \to Y$ be a morphism of varieties over $\Bbbk$, let Y be regular and X be Cohen-Macaulay. Assume that each fibre of X has dimension $\dim X - \dim Y$. Then f is flat.*

In particular this holds true if both X and Y are regular.

Proof. See [11], 6.1.5, or [12], III, Ex. 10.9. □

A.2 Some properties of blow ups

We collect here some facts about blow ups. The statements we refer to are provided either with references or proofs.

A.2.1 Definition and basic properties.

Let us recall the definition and some basic properties of blow ups. For details see also [6], IV, and [12], II, §7.

Let $S \subseteq Y$ be a closed embedding and let $\mathcal{I}_S = \mathcal{I}$ be the ideal sheaf of S. Then the blow up $\mathrm{Bl}_S Y$ of Y along S is by definition the scheme

$$\mathrm{Bl}_S Y := \mathbf{Proj} \bigoplus_{d \geqslant 0} \mathcal{I}^d$$

together with the canonical (structure) morphism

$$\mathbf{Proj}\ (\bigoplus_{d \geqslant 0} \mathcal{I}^d) \xrightarrow{\sigma} Y$$

and with the canonical invertible sheaf $\mathcal{O}_{\mathrm{Bl}_S Y}(1)$ on $\mathrm{Bl}_S Y$.

The subscheme $D = \sigma^{-1}(S)$ is called the **exceptional divisor** of the blow up σ. It is canonically isomorphic to the scheme

$$\mathbf{Proj} \bigoplus_{d \geqslant 0} \mathcal{I}^d / \mathcal{I}^{d+1}$$

and the embedding $D \subseteq \mathrm{Bl}_S Y$ is induced by the canonical surjective homomorphism

$$\bigoplus_{d \geqslant 0} \mathcal{I}^d \twoheadrightarrow \bigoplus_{f \geqslant 0} \mathcal{I}^d / \mathcal{I}^{d+1}.$$

D is a Cartier divisor on $\mathrm{Bl}_S Y$ with the corresponding ideal sheaf

$$\mathcal{I} \cdot \mathcal{O}_{\mathrm{Bl}_S Y} = \mathcal{O}_{\mathrm{Bl}_S Y}(-D) = \mathcal{O}_{\mathrm{Bl}_S Y}(1).$$

For every morphism $f : Z \to Y$ such that $f^{-1}(S)$ is a Cartier divisor, i. e., such that $f^{-1}\mathcal{I} \cdot \mathcal{O}_Y$ is an invertible sheaf, [1] there exists a unique morphism $g : Z \to \mathrm{Bl}_S Y$ with $f = \sigma \circ g$. This property is called the **universal property of blow up**.

There is the commutative diagram of the canonical projections

$$\begin{tikzcd}
\bigoplus_{d \geqslant 0} \mathbf{S}^d \mathcal{I} \arrow[r, two heads] \arrow[d, two heads] & \bigoplus_{d \geqslant 0} \mathbf{S}^d(\mathcal{I}/\mathcal{I}^2) \arrow[d, two heads] \\
\bigoplus_{d \geqslant 0} \mathcal{I}^d \arrow[r, two heads] & \bigoplus_{d \geqslant 0} \mathcal{I}^d/\mathcal{I}^{d+1}.
\end{tikzcd}$$

Note that if S is a **locally complete intersection**, then the vertical arrows of the above diagram are isomorphisms (cf. [6], IV, Corollary 2.4, and [12], II, Theorem 8.21A). In particular this induces the isomorphisms

$$\mathrm{Bl}_S Y = \mathbf{Proj} \bigoplus_{d \geqslant 0} \mathcal{I}^d \cong \mathbf{Proj} \bigoplus_{d \geqslant 0} \mathbf{S}^d \mathcal{I} = \mathbf{P}(\mathcal{I})$$

[1] Recall that $f^{-1}\mathcal{I} \cdot \mathcal{O}_Y = \mathcal{I} \cdot \mathcal{O}_Y$ is the image of $f^*(\mathcal{I})$ in $\mathcal{O}_Z$ under the canonical map $f^*(\mathcal{I}) \to f^*\mathcal{O}_Y = \mathcal{O}_Z$, see also [12], page 163, Caution 7.12.2.

and
$$D = \mathbf{Proj} \bigoplus_{d\geqslant 0} \mathcal{I}^d/\mathcal{I}^{d+1} \cong \mathbf{Proj} \bigoplus_{d\geqslant 0} \mathbf{S}^d(\mathcal{I}/\mathcal{I}^2) = \mathbf{P}(\mathcal{I}/\mathcal{I}^2).$$
Recall that the sheaf $\mathscr{C}_{S/Y} = \mathcal{I}/\mathcal{I}^2$ is called the **conormal sheaf** to S in Y. It is a coherent sheaf on S. If S is a locally complete intersection, then $\mathcal{I}/\mathcal{I}^2$ is a locally free sheaf on S (cf. [6], §4, Proposition 3.2).

A.2.2 Some useful statements.

Let Y be smooth and let $S \subseteq Y$ be a locally complete intersection of codimension $\operatorname{codim} S = N+1 \geqslant 2$. Let $\sigma : X \to Y$ be the blow up of the ideal $\mathcal{I}_S = \mathcal{I}$ of S. Let D be the exceptional divisor of this blow up. We obtain then a commutative diagram

$$\begin{tikzcd} D \arrow[r, hook] \arrow[d, "\sigma_D"'] & X \arrow[d, "\sigma"] \\ S \arrow[r, hook] & Y. \end{tikzcd}$$

Since S is a locally complete intersection we have that the sheaf $\mathcal{I}/\mathcal{I}^2$ is locally free, $X = \mathbf{P}(\mathcal{I})$, and $D = \mathbf{P}(\mathcal{I}/\mathcal{I}^2)$. The embedding $D \hookrightarrow X$ is induced by the projection $\mathcal{I} \to \mathcal{I}/\mathcal{I}^2$.

The ideal sheaf of D is
$$\mathcal{O}_X(-D) = \mathcal{I}\cdot\mathcal{O}_X = \mathcal{O}_X(1) = \mathcal{O}_{\mathbf{P}(\mathcal{I})}(1),$$
where $\mathcal{I}\cdot\mathcal{O}_X$ is the image of $\sigma^*(\mathcal{I})$ in $\mathcal{O}_X$ under canonical map $\sigma^*(\mathcal{I}) \to \sigma^*\mathcal{O}_Y = \mathcal{O}_X$.

Since $\mathcal{O}_D(1)$ is just a restriction of $\mathcal{O}_X(1)$ to D, i.e., $\mathcal{O}_{\mathbf{P}(\mathcal{I}/\mathcal{I}^2)}(1) = \mathcal{O}_{\mathbf{P}(\mathcal{I})}(1)|_{\mathbf{P}(\mathcal{I}/\mathcal{I}^2)}$, we conclude that
$$\mathcal{O}_D(1) = \mathcal{O}_X(1)|_D = \mathcal{O}_X(1)\otimes\mathcal{O}_D = \mathcal{O}_X(-D)\otimes\mathcal{O}_D,$$
and therefore $\mathcal{O}_X(D)\otimes\mathcal{O}_D = \mathcal{O}_D(-1)$. We proved the following simple lemma.

Lemma A.3. *$\mathcal{O}_X(D)\otimes\mathcal{O}_D = \mathcal{O}_D(-1)$.*

Lemma A.4. *1) $R^p\sigma_*(\mathcal{O}_D(-1)) = 0$ for $p \geqslant 0$.*
2) $R^p\sigma_*(\mathcal{O}_D(n)) = 0$ for $p > 0$, $n \geqslant 0$.

Proof. 1) Note that for any coherent $\mathcal{O}_D$-module $\mathcal{F}$ we have $R^p\sigma_*(\mathcal{F}) = R^p\sigma_{D*}(\mathcal{F})$. This is true because for each open set $U \subseteq Y$ we have
$$H^p(\sigma^{-1}(U), \mathcal{F}) = H^p(\sigma^{-1}(U)\cap D, \mathcal{F}) = H^p(\sigma_D^{-1}(U\cap S), \mathcal{F}),$$
i. e., the pre-sheaves defining $R^p\sigma_*(\mathcal{F})$ and $R^p\sigma_{D*}(\mathcal{F})$ are equal.

Note that $D \xrightarrow{\sigma_D} S$ is a projective bundle over S and therefore is flat. Thus $\mathcal{O}_D(-1)$ is flat over S as a locally free sheaf on D. Therefore, we can apply base change.

For each point $s \in S$ consider the fibre D_s over s.

$$\begin{tikzcd} D_s \arrow[r, hook] \arrow[d] & D \arrow[d, "\sigma_D"] \\ \{s\} \arrow[r, hook] & S. \end{tikzcd}$$

Then $D_s = \mathbb{P}_N$ (recall that $\operatorname{codim} S = N+1$) and $\mathcal{O}_D(-1)|_{D_s} = \mathcal{O}_{D_s}(-1)$. Since for all $p \geqslant 0$
$$H^p(D_s, \mathcal{O}_{D_s}(-1)) = H^p(\mathbb{P}_N, \mathcal{O}_{\mathbb{P}_N}(-1)) = 0,$$

applying Theorem A.1 we obtain $R^p\sigma_{D*}(\mathcal{F}) = 0$. As we showed that $R^p\sigma_*(\mathcal{F}) = R^p\sigma_{D*}(\mathcal{F})$, we conclude that $R^p\sigma_*(\mathcal{F}) = 0$ for $p \geqslant 0$. This proves the first part of the lemma.

2) Analogously, using that $H^p(\mathbb{P}_N, \mathcal{O}_{\mathbb{P}_N}(n)) = 0$ for $p > 0$, $n \geqslant 0$, we obtain $R^p\sigma_*(\mathcal{O}_D(n)) = 0$ for $p > 0$, $n \geqslant 0$. □

Lemma A.5. $\sigma_*(\mathcal{O}_X) \cong \sigma_*(\mathcal{O}_X(D))$.

Proof. Let us consider the exact sequence

$$0 \to \mathcal{O}_X \to \mathcal{O}_X(D) \to \mathcal{O}_X(D) \otimes \mathcal{O}_D \to 0.$$

Since by Lemma A.3 $\mathcal{O}_X(D) \otimes \mathcal{O}_D \cong \mathcal{O}_D(-1)$ we obtain the long exact sequence

$$\begin{gathered} 0 \to \sigma_*\mathcal{O}_X \to \sigma_*(\mathcal{O}_X(D)) \to \sigma_*(\mathcal{O}_D(-1)) \to R^1\sigma_*(\mathcal{O}_X) \to \dots \\ \dots \to R^p\sigma_*(\mathcal{O}_X) \to R^p\sigma_*(\mathcal{O}_X(D)) \to R^p\sigma_*(\mathcal{O}_D(-1)) \to \dots . \end{gathered}$$

From Lemma A.4 we get $R^p\sigma_*(\mathcal{O}_D(-1)) = 0$ for $p \geqslant 0$. Therefore, $\sigma_*(\mathcal{O}_X) \cong \sigma_*(\mathcal{O}_X(D))$, which proves the lemma. □

Lemma A.6. *1)* $\sigma_*(\mathcal{O}_X) \cong \mathcal{O}_Y$*;*
2) $R^p\sigma_*(\mathcal{O}_X) = 0$, $p > 0$.

Proof. 1) First of all note, that $\sigma : X \setminus D \to Y \setminus S$ is an isomorphism. Therefore, for each open set $U \subseteq Y$ we have $\mathcal{O}_Y(U \setminus S) \cong \mathcal{O}_X(\sigma^{-1}(U) \setminus D)$. Since the codimension of S in Y is $\geqslant 2$, the restriction map $\mathcal{O}_Y(U) \to \mathcal{O}_Y(U \setminus S)$ is an isomorphism.

Consider the commutative diagram

$$\begin{tikzcd} \mathcal{O}_Y(U) \arrow[r] \arrow[d, "\cong"] & \mathcal{O}_X(\sigma^{-1}(U)) = \sigma_*\mathcal{O}_X(U) \arrow[d] \\ \mathcal{O}_Y(U \setminus S) \arrow[r, "\cong"] & \mathcal{O}_X(\sigma^{-1}(U) \setminus D), \end{tikzcd}$$

where the vertical lines are the canonical restrictions and the horizontal correspond to the canonical homomorphism $\mathcal{O}_Y \to \sigma_*\mathcal{O}_X$. Since $\mathcal{O}_X$ is torsion free, the right vertical arrow is injective. It follows also from the commutativity of the diagram that it is also surjective. Therefore, the restriction map

$$\mathcal{O}_X(\sigma^{-1}(U)) \to \mathcal{O}_X(\sigma^{-1}(U) \setminus D)$$

is an isomorphism and thus we get that $\mathcal{O}_Y(U) \to \mathcal{O}_X(\sigma^{-1}(U))$ is an isomorphism, which means that $\sigma_*(\mathcal{O}_X) \cong \mathcal{O}_Y$.

2) Let $\mathcal{I}_D = \mathcal{O}_X(-D)$ be the ideal sheaf of D. Then for each $n > 0$ we have the exact sequence

$$0 \to \mathcal{I}_D^n/\mathcal{I}_D^{n+1} \to \mathcal{O}_X/\mathcal{I}_D^{n+1} \to \mathcal{O}_X/\mathcal{I}_D^n \to 0.$$

We have

$$\mathcal{I}_D/\mathcal{I}_D^2 \cong \mathcal{I}_D \otimes (\mathcal{O}_X/\mathcal{I}_D) \cong \mathcal{O}_X(-D) \otimes \mathcal{O}_D \cong \mathcal{O}_D(1)$$

and

$$\mathcal{I}_D^n/\mathcal{I}_D^{n+1} \cong (\mathcal{I}_D/\mathcal{I}_D^2)^{\otimes n} \cong \mathcal{O}_D(n).$$

Therefore, by Lemma A.4 $R^p\sigma_*(\mathcal{I}_D^n/\mathcal{I}_D^{n+1}) = R^p\sigma_*(\mathcal{O}_D(n)) = 0$, $p > 0$. From the long exact sequence

$$\dots \to R^p\sigma_*(\mathcal{I}_D^n/\mathcal{I}_D^{n+1}) \to R^p\sigma_*(\mathcal{O}_X/\mathcal{I}_D^{n+1}) \to R^p\sigma_*(\mathcal{O}_X/\mathcal{I}_D^n) \to R^{p+1}\sigma_*(\mathcal{I}_D^n/\mathcal{I}_D^{n+1}) \to \dots$$

we conclude that $R^p\sigma_*(\mathcal{O}_X/\mathcal{I}_D^{n+1}) \cong R^p\sigma_*(\mathcal{O}_X/\mathcal{I}_D^n) \cong \ldots \cong R^p\sigma_*(\mathcal{O}_X/\mathcal{I}_D) = R^p\sigma_*(\mathcal{O}_D)$, because $\mathcal{O}_X/\mathcal{I}_D = \mathcal{O}_D$. By Lemma A.4 $R^p\sigma_*(\mathcal{O}_D) = 0$, which implies that

$$R^p\sigma_*(\mathcal{O}_X/\mathcal{I}_D^n) = 0, \quad n > 0, \quad p > 0. \tag{A.1}$$

Grothendieck's comparison theorem (cf. [9], 4.1.5) states that

$$\varprojlim_n R^p\sigma_*(\mathcal{F})/\mathcal{I}_S^n R^p\sigma_*(\mathcal{F}) \cong \varprojlim_n R^p\sigma_*(\mathcal{F}/\mathcal{I}_D^n\mathcal{F})$$

for a coherent $\mathcal{O}_X$-module $\mathcal{F}$. Taking $\mathcal{F} = \mathcal{O}_X$ an using $(A.1)$ we get

$$\varprojlim_n R^p\sigma_*(\mathcal{O}_X)/\mathcal{I}_S^n R^p\sigma_*(\mathcal{O}_X) \cong \varprojlim_n R^p\sigma_*(\mathcal{O}_X/\mathcal{I}_D^n) = 0.$$

By Krull's intersection theorem this implies $R^p\sigma_*(\mathcal{O}_X) = 0$. □

Lemma A.7. *Let $\mathcal{E}$ be a locally free sheaf on Y. Then*

$$R^p\sigma_*(\sigma^*\mathcal{E}) \cong \begin{cases} \mathcal{E}, & p = 0, \\ 0, & p > 0. \end{cases}$$

Proof. Using the projection formula we obtain

$$R^p\sigma_*(\sigma^*\mathcal{E}) \cong R^p\sigma_*(\sigma^*\mathcal{E} \otimes \mathcal{O}_X) \cong \mathcal{E} \otimes R^p\sigma_*(\mathcal{O}_X).$$

By Lemma A.6 we obtain the required statement. □

Lemma A.8. *Let $\mathcal{F}$ be a coherent sheaf given by a locally free resolution*

$$0 \to \mathcal{E}_1 \to \mathcal{E}_0 \to \mathcal{F} \to 0.$$

Assume that after applying σ^* the resulting sequence

$$0 \to \sigma^*(\mathcal{E}_1) \to \sigma^*(\mathcal{E}_0) \to \sigma^*(\mathcal{F}) \to 0$$

remains exact. Then

$$R^p\sigma_*(\sigma^*\mathcal{F}) \cong \begin{cases} \mathcal{F}, & p = 0, \\ 0, & p > 0. \end{cases}$$

Proof. From the exact sequence

$$0 \to \sigma^*(\mathcal{E}_1) \to \sigma^*(\mathcal{E}_0) \to \sigma^*(\mathcal{F}) \to 0$$

we obtain the long exact sequence

$$\begin{aligned} 0 \to \sigma_*(\sigma^*(\mathcal{E}_1)) \to \sigma_*(\sigma^*(\mathcal{E}_0)) \to \sigma_*(\sigma^*(\mathcal{F})) \to \ldots \\ \ldots \to R^p\sigma_*(\sigma^*(\mathcal{E}_1)) \to R^p\sigma_*(\sigma^*(\mathcal{E}_0)) \to R^p\sigma_*(\sigma^*(\mathcal{F})) \to \ldots\ . \end{aligned}$$

Since by Lemma A.7 $R^p\sigma_*(\mathcal{E}_0) = R^p\sigma_*(\mathcal{E}_1) = 0$, $p > 0$, one immediately obtains $R^p\sigma_*(\mathcal{F}) = 0$, $p > 0$. Since we also have the commutative diagram with exact rows

$$\begin{tikzcd} 0 \arrow[r] & \sigma_*(\sigma^*(\mathcal{E}_1)) \arrow[r] & \sigma_*(\sigma^*(\mathcal{E}_0)) \arrow[r] & \sigma_*(\sigma^*(\mathcal{F})) \arrow[r] & 0 \\ 0 \arrow[r] & \mathcal{E}_1 \arrow[r] \arrow[u, "\cong"] & \mathcal{E}_0 \arrow[r] \arrow[u, "\cong"] & \mathcal{F} \arrow[r] \arrow[u] & 0, \end{tikzcd}$$

we conclude that $\sigma_*(\sigma^*(\mathcal{F})) \cong \mathcal{F}$. □

A.3 Conics in $\mathbb{P}_2$ and $2m+2$ sheaves on $\mathbb{P}_2$

We discuss here conics in $\mathbb{P}_2$ and their connection to $2m+2$ sheaves on $\mathbb{P}_2$.

Lemma A.9. *Let $C = \{f(u_0, u_1, u_2) = 0\}$ be a conic. Let Q be the corresponding symmetric matrix, i. e., $f(u_0, u_1, u_2) = (u_0, u_1, u_2)Q(u_0, u_1, u_2)^{\mathrm{T}}$. Then*
1) *C is irreducible if and only if the determinant of Q is not zero;*
2) *smooth conics are exactly those that are irreducible;*
3) *C is a union of two different lines if and only if the rank of the matrix Q equals 2;*
4) *C is a double line if and only if the rank of the matrix Q equals 1.*

Proof. 1) After an appropriate linear coordinate change we can bring the conic to the canonical form $\lambda_0 u_0^2 + \lambda_1 u_1^2 + \lambda_3 u_3^2$ that corresponds to the diagonal matrix $Q = \begin{pmatrix} \lambda_0 & 0 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_2 \end{pmatrix}$. One easily sees that $\lambda_0 u_0^2 + \lambda_1 u_1^2 + \lambda_3 u_3^2$ is irreducible if and only if all λ_i are different from zero, which is equivalent to $\det Q \neq 0$.

2) C is singular if and only if $(\frac{\partial f}{\partial u_0}, \frac{\partial f}{\partial u_1}, \frac{\partial f}{\partial u_2})$ vanishes at some point. Since $f(u_0, u_1, u_2) = (u_0, u_1, u_2)Q(u_0, u_1, u_2)^{\mathrm{T}}$, we obtain

$$\begin{pmatrix} \frac{\partial f}{\partial u_0} & \frac{\partial f}{\partial u_1} & \frac{\partial f}{\partial u_2} \end{pmatrix} = 2(u_0, u_1, u_2) \cdot Q,$$

therefore, C is singular if and only if there is a non-trivial solution of the linear equation

$$(u_0, u_1, u_2) \cdot Q = 0.$$

The latter is equivalent to $\det Q = 0$.

3) and 4) If one of λ_i equals zero, then this conic is decomposable into linear factors: $\lambda_\nu u_\nu^2 + \lambda_\mu u_\mu^2 = (\epsilon u_\nu - \varepsilon u_\mu)(\epsilon u_\nu + \varepsilon u_\mu)$, $\epsilon^2 = \lambda_\nu$, $\varepsilon^2 = -\lambda_\mu$. These factors are equal if and only if two of λ_i equal zero. But the number of the coefficients λ_i different from zero is exactly the rank of the matrix Q. This proves the statements 3) and 4). □

Let $\mathcal{E}$ be a sheaf on $\mathbb{P}_2$ with Hilbert polynomial $2m+2$ given by the resolution

$$0 \to 2\mathcal{O}_{\mathbb{P}_2}(-1) \xrightarrow{\left(\begin{smallmatrix} z_1 & z_2 \\ z_3 & z_4 \end{smallmatrix}\right)} 2\mathcal{O}_{\mathbb{P}_2} \to \mathcal{E} \to 0, \tag{A.2}$$

where z_1, z_2, z_3, z_4 are sections of $\mathcal{O}_{\mathbb{P}_2}(1)$ and the determinant $z_1 z_4 - z_2 z_3 \in \Gamma(\mathbb{P}_2, \mathcal{O}_{\mathbb{P}_2}(2))$ is non-zero. We will call such sheaves $2m+2$ sheaves. The sheaf $\mathcal{E}$ is supported on the conic

$$C = \{\det(\left(\begin{smallmatrix} z_1 & z_2 \\ z_3 & z_4 \end{smallmatrix}\right)) = z_1 z_4 - z_2 z_3 = 0\} \subseteq \mathbb{P}_2.$$

The sheaf $\mathcal{E}$ is locally free on its support if and only if the matrix $\left(\begin{smallmatrix} z_1 & z_2 \\ z_3 & z_4 \end{smallmatrix}\right)$ does not vanish, i. e., if there are no common zeros of the sections $z_1, z_2, z_3, z_4 \in \Gamma(\mathbb{P}_2, \mathcal{O}_{\mathbb{P}_2}(1))$. Equivalently, this means $\dim_{\Bbbk} \operatorname{Span}(z_1, z_2, z_3, z_4) = 3$.

Lemma A.10. $2m+2$ *sheaves on* $\mathbb{P}_2$ *with irreducible support are locally free on their support.*

Proof. If a $2m+2$ sheaf is not locally free on its support, then $\dim_{\Bbbk} \operatorname{Span}(z_1, z_2, z_3, z_4) \leqslant 2$, i. e., the determinant $z_1 z_4 - z_2 z_3$ is a homogeneous polynomial of degree 2 in at most 2 variables. Such a polynomial clearly decomposes into 2 linear factors. This proves the statement of the lemma. □

Let us clarify when a $2m+2$ sheaf with a non-smooth (reducible) support are locally free on their support.

Note that an isomorphism of $2m+2$ sheaves extends uniquely to an isomorphism of the resolutions of the type $(A.2)$. This means that isomorphism classes of $2m+2$ sheaves are in bijection with the isomorphism classes of sequences of the type $(A.2)$. But the set of the equivalence classes of these sequences is clearly the set of the orbits of the group action of $\mathrm{GL}_2(\Bbbk)\times\mathrm{GL}_2(\Bbbk)$ on the set of matrices $\{\left(\begin{smallmatrix} z_1 & z_2\\ z_3 & z_4\end{smallmatrix}\right) \mid z_1z_4 - z_2z_3 \neq 0\} \subseteq \Bbbk^{12}$ given by the rule

$$(g,h)\cdot\begin{pmatrix} z_1 & z_2\\ z_3 & z_4\end{pmatrix} = g\begin{pmatrix} z_1 & z_2\\ z_3 & z_4\end{pmatrix}h^{-1}.$$

This means that two matrices define the same isomorphism class of $2m+2$ sheaves if and only if they can be transformed into each other by Gauß algorithm applied to columns and rows.

Lemma A.11. *The determinant of the matrix $A = \left(\begin{smallmatrix} z_1 & z_2\\ z_3 & z_4\end{smallmatrix}\right)$ is decomposable into linear factors if and only if one can transform this matrix by Gauß algorithm (applied both to the rows and to the columns) to the form $\left(\begin{smallmatrix} z_1' & 0\\ z_3' & z_4'\end{smallmatrix}\right)$, $z_1', z_3', z_4' \in \Gamma(\mathbb{P}_2, \mathcal{O}_{\mathbb{P}_2}(1))$.*

Proof. If z_1 and z_2 are linear dependent, then the statement of the lemma follows immediately. So let us assume z_1 and z_2 to be linear independent. After a change of coordinates we may assume

$$A = \begin{pmatrix} x_1 & x_2\\ \alpha x_0 + \beta x_1 + \gamma x_2 & ax_0 + bx_1 + cx_2\end{pmatrix}.$$

Since there is an equivalence

$$\begin{pmatrix} x_1 & x_2\\ \alpha x_0 + \beta x_1 + \gamma x_2 & ax_0 + bx_1 + cx_2\end{pmatrix} \sim \begin{pmatrix} x_1 & x_2\\ \alpha x_0 + \gamma x_2 & ax_0 + bx_1 + (c-\beta)x_2\end{pmatrix},$$

we may assume

$$A = \begin{pmatrix} x_1 & x_2\\ \alpha x_0 + \gamma x_2 & ax_0 + bx_1 + cx_2\end{pmatrix}.$$

The determinant of this matrix is the quadratic form

$$ax_0x_1 + bx_1^2 + cx_1x_2 - \alpha x_0x_2 - \gamma x_2^2.$$

By Lemma A.9 this decomposes into linear factors if and only if the determinant of the matrix

$$\begin{pmatrix} 0 & \frac{a}{2} & -\frac{\alpha}{2}\\ \frac{a}{2} & b & \frac{c}{2}\\ -\frac{\alpha}{2} & \frac{c}{2} & -\gamma\end{pmatrix}$$

is zero. The determinant is

$$-\frac{a}{2}\cdot\begin{vmatrix} \frac{a}{2} & \frac{c}{2}\\ -\frac{\alpha}{2} & -\gamma\end{vmatrix} - \frac{\alpha}{2}\cdot\begin{vmatrix} \frac{a}{2} & b\\ -\frac{\alpha}{2} & \frac{c}{2}\end{vmatrix} = -\frac{a}{2}\left(-\frac{a}{2}\cdot\gamma + \frac{c}{2}\cdot\frac{\alpha}{2}\right) - \frac{\alpha}{2}\left(\frac{a}{2}\cdot\frac{c}{2} + b\cdot\frac{\alpha}{2}\right) =$$
$$\frac{a^2\gamma}{4} - \frac{\alpha ac}{4} - \frac{b\alpha^2}{4}.$$

So, the determinant of the matrix A has a linear factor if and only if

$$a^2\gamma - \alpha ac - b\alpha^2 = 0.$$

If $\alpha = 0$ then $a^2\gamma = 0$. So either $\gamma = 0$ or $a = 0$. If $\gamma = 0$, then

$$A = \begin{pmatrix} x_1 & x_2 \\ 0 & ax_0 + bx_1 + cx_2 \end{pmatrix} \sim \begin{pmatrix} x_2 & x_1 \\ ax_0 + bx_1 + cx_2 & 0 \end{pmatrix} \sim \begin{pmatrix} ax_0 + bx_1 + cx_2 & 0 \\ x_2 & x_1 \end{pmatrix}$$

and the statement of the lemma holds true. If $\gamma \neq 0$, then $a = 0$ and we may assume that $\gamma = 1$ then

$$A = \begin{pmatrix} x_1 & x_2 \\ x_2 & bx_1 + cx_2 \end{pmatrix}.$$

The determinant of this matrix is $bx_1^2 + cx_1x_2 - x_2^2$. Let ξ and η be such that

$$bx_1^2 + cx_1x_2 - x_2^2 = -(x_2 + \xi x_1)(x_2 + \eta x_1).$$

Then $\xi\eta = -b$ and $\xi + \eta = -c$. One gets

$$A = \begin{pmatrix} x_1 & x_2 \\ x_2 & bx_1 + cx_2 \end{pmatrix} \sim \begin{pmatrix} x_1 & x_2 + \xi x_1 \\ x_2 & bx_1 + (c+\xi)x_2 \end{pmatrix} = \begin{pmatrix} x_1 & x_2 + \xi x_1 \\ x_2 & bx_1 - \eta x_2 \end{pmatrix} \sim$$
$$\begin{pmatrix} x_1 & x_2 + \xi x_1 \\ x_2 + \eta x_1 & (b + \eta\xi)x_1 \end{pmatrix} = \begin{pmatrix} x_1 & x_2 + \xi x_1 \\ x_2 + \eta x_1 & 0 \end{pmatrix} \sim \begin{pmatrix} x_2 + \eta x_1 & 0 \\ x_1 & x_2 + \xi x_1 \end{pmatrix}.$$

It remains to consider the case $\alpha \neq 0$. If $\alpha \neq 0$, then we may assume $\alpha = 1$. We obtain this way $b = a^2\gamma - ac = a(a\gamma - c)$ and

$$A = \begin{pmatrix} x_1 & x_2 \\ x_0 + \gamma x_2 & ax_0 + a(a\gamma - c)x_1 + cx_2 \end{pmatrix} \sim$$
$$\begin{pmatrix} x_1 & x_2 - ax_1 \\ x_0 + \gamma x_2 & a(a\gamma - c)x_1 + (c - a\gamma)x_2 \end{pmatrix} = \begin{pmatrix} x_1 & x_2 - ax_1 \\ x_0 + \gamma x_2 & (c - a\gamma)(x_2 - ax_1) \end{pmatrix} \sim$$
$$\begin{pmatrix} x_1 & x_2 - ax_1 \\ x_0 + \gamma x_2 - (c - a\gamma)x_1 & 0 \end{pmatrix} \sim \begin{pmatrix} x_0 + \gamma x_2 - (c - a\gamma)x_1 & 0 \\ x_1 & x_2 - ax_1 \end{pmatrix}.$$

This completes the proof of the lemma. □

As a corollary we have the following lemma.

Lemma A.12. *A $2m+2$ sheaf supported on two lines if locally free on its support if and only if the linear forms z_1', z_3', z_4' as in the lemma above are linear independent, i. e., constitute a basis of $\Gamma(\mathbb{P}_2, \mathcal{O}_{\mathbb{P}_2}(1))$.*

Lemma A.13. *Let u_0, u_1, u_2 be a basis of $\Gamma(\mathbb{P}_2, \mathcal{O}_{\mathbb{P}_2}(1))$.*
1) Then the sheaves given by the matrices $\begin{pmatrix} u_1 & 0 \\ u_0 & u_2 \end{pmatrix}$ and $\begin{pmatrix} u_2 & 0 \\ u_0 & u_1 \end{pmatrix}$ are not isomorphic.
2) The sheaf given by the matrix $\begin{pmatrix} u_1 & 0 \\ \alpha u_0 + \beta u_1 + \gamma u_2 & u_2 \end{pmatrix}$, $\alpha \neq 0$ is isomorphic to the sheaf given by the matrix $\left(\begin{smallmatrix} u_1 & 0 \\ u_0 & u_2 \end{smallmatrix}\right)$.

Proof. 1) Suppose that the sheaves defined by the given matrices are isomorphic. This means there exist two matrices $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} \bar a & \bar b \\ \bar c & \bar d \end{smallmatrix}\right)$ from $\mathrm{GL}_2(\Bbbk)$ such that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} u_1 & 0 \\ u_0 & u_2 \end{pmatrix} = \begin{pmatrix} u_2 & 0 \\ u_0 & u_1 \end{pmatrix}\begin{pmatrix} \bar a & \bar b \\ \bar c & \bar d \end{pmatrix}.$$

After multiplying one obtains

$$\begin{pmatrix} au_1 + bu_0 & bu_2 \\ cu_1 + du_0 & du_2 \end{pmatrix} = \begin{pmatrix} \bar{a}u_2 & \bar{b}u_2 \\ \bar{a}u_0 + \bar{c}u_1 & \bar{b}u_0 + \bar{d}u_1 \end{pmatrix}.$$

Since u_0, u_1, and u_2 form a basis of linear forms on $\mathbb{P}_2$, this implies in particular $a = b = \bar{a}$, which contradicts the invertibility of $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. Therefore, the matrices $\left(\begin{smallmatrix} u_1 & 0 \\ u_0 & u_2 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} u_2 & 0 \\ u_0 & u_1 \end{smallmatrix}\right)$ define non-isomorphic sheaves.

2) Follows from the following equivalences

$$\begin{pmatrix} u_1 & 0 \\ \alpha u_0 + \beta u_1 + \gamma u_2 & u_2 \end{pmatrix} \sim \begin{pmatrix} u_1 & 0 \\ \alpha u_0 & u_2 \end{pmatrix} \sim \begin{pmatrix} u_1 & 0 \\ u_0 & \alpha^{-1} u_2 \end{pmatrix} \sim \begin{pmatrix} u_1 & 0 \\ u_0 & u_2 \end{pmatrix}.$$

□

Proposition A.14. *(1) On the union of two different lines $u_1 = 0$ and $u_2 = 0$ in $\mathbb{P}_2$ there are exactly two isomorphism classes of $2m + 2$ sheaves that are locally free on their support, this two classes are given by the matrices $\left(\begin{smallmatrix} u_1 & 0 \\ u_0 & u_2 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} u_2 & 0 \\ u_0 & u_1 \end{smallmatrix}\right)$, where u_0 is any complementary to u_1 and u_2 linear form.*

(2) All $2m + 2$ sheaves supported on a smooth conic are locally free on their support.

(3) All $2m+2$ sheaves on $\mathbb{P}_2$ that are locally free on their support are described in the statements (1) and (2) of this proposition.

Proof. Follows from the considerations above. □

A.4 Gluing of locally free sheaves

We consider here the so called gluing of locally free sheaves.

Let X be a reduced algebraic variety and let $X = X_1 \cup X_2$ be the decomposition into irreducible components. We denote $Y = X_1 \cap X_2$. Let $\mathcal{E}$ be a locally free sheaf on X. We are going to show that $\mathcal{E}$ is uniquely defined by its restrictions to X_1 and X_2, i. e., the restrictions of $\mathcal{E}$ can be "glued" together to the sheaf $\mathcal{E}$.

Let us consider the commutative diagram of closed embeddings

$$\begin{tikzcd}
 & X_2 \arrow[dr, hook, "i_2"] & \\
Y \arrow[ur, hook, "j_2"] \arrow[dr, hook, "j_1"'] & & X. \\
 & X_1 \arrow[ur, hook, "i_1"'] &
\end{tikzcd}$$

Let us consider the following commutative diagram of the restrictions

$$\begin{tikzcd}
 & i_1^*\mathcal{E} \arrow[r] \arrow[dr, "\rho_1"] & j_1^*i_1^*\mathcal{E} \arrow[d] \\
\mathcal{E} \arrow[ur, "r_1"] \arrow[rr, "r_{12}"] \arrow[dr, "r_2"'] & & (i_1j_1)^*\mathcal{E} = (i_2j_2)^*\mathcal{E}. \\
 & i_2^*\mathcal{E} \arrow[r] \arrow[ur, "\rho_1"] & j_2^*i_2^*\mathcal{E} \arrow[u]
\end{tikzcd}$$

We denote $\mathcal{E}_{X_1} := i_1^*\mathcal{E}$, $\mathcal{E}_{X_2} := i_2^*\mathcal{E}$, and $\mathcal{E}_Y := (i_1j_1)^*\mathcal{E}$. Consider the restrictions $r_1 : \mathcal{E} \to \mathcal{E}_{X_1}$ and $r_2 : \mathcal{E} \to \mathcal{E}_{X_2}$.

Lemma A.15. *The map* $\mathcal{E} \xrightarrow{(r_1,r_2)} \mathcal{E}_{X_1} \oplus \mathcal{E}_{X_2}$ *is injective.*

Proof. The map is an isomorphism on $X_1 \setminus Y$ and on $X_2 \setminus Y$. Therefore, its kernel can be only supported on Y. But the sheaf $\mathcal{E}$ is torsion free as a locally free sheaf. Hence the map is injective. □

As ρ_1 and ρ_2 are surjective, it follows that the map $\mathcal{E}_{X_1} \oplus \mathcal{E}_{X_2} \xrightarrow{\left(\begin{smallmatrix}\rho_1\\ -\rho_2\end{smallmatrix}\right)} \mathcal{E}_Y$ is surjective as well. Since $\rho_1 \circ r_1 = \rho_2 \circ \rho_2 = r_{12}$, we conclude that the image of (r_1, r_2) lies in the kernel of $\left(\begin{smallmatrix}\rho_1\\ -\rho_2\end{smallmatrix}\right)$.

Lemma A.16. *Let X be reduced, then the sequence*

$$0 \to \mathcal{E} \xrightarrow{(r_1,r_2)} \mathcal{E}_{X_1} \oplus \mathcal{E}_{X_2} \xrightarrow{\left(\begin{smallmatrix}\rho_1\\ -\rho_2\end{smallmatrix}\right)} \mathcal{E}_Y \to 0$$

is exact.

Proof. It remains to prove the exactness in the middle term, i. e., $\mathrm{Im}(r_1, r_2) = \ker\left(\begin{smallmatrix}\rho_1\\ -\rho_2\end{smallmatrix}\right)$. We can consider the question locally, i. e., in affine charts. So we may assume X, X_1 and X_2 to be affine. We may also take the open charts small enough, so we also assume $\mathcal{E}$ to be trivial, i. e., isomorphic to the direct sum of the copies of $\mathcal{O}_X$. It is also enough to prove the required statement for invertible sheaves. Let A be a coordinate ring of X, let $\mathfrak{p}_1$ and $\mathfrak{p}_2$ be the ideals in A of X_1 and X_2 respectively. Since X is reduced and since $X = X_1 \cup X_2$, the intersection $\mathfrak{p}_1 \cap \mathfrak{p}_2$ is zero. Then the map (r_1, r_2) corresponds to the injective homomorphism of rings

$$\iota : A \to A/\mathfrak{p}_1 \oplus A/\mathfrak{p}_2, \quad a \mapsto (a + \mathfrak{p}_1, a + \mathfrak{p}_2).$$

The ideal $I(Y)$ of $Y = X_1 \cap X_2$ is equal to $\mathfrak{p}_1 + \mathfrak{p}_2$. Then the map $\left(\begin{smallmatrix} \rho_1 \\ -\rho_2 \end{smallmatrix}\right)$ corresponds to the surjective ring homomorphism

$$\pi : A/\mathfrak{p}_1 \oplus A/\mathfrak{p}_2 \to A/(\mathfrak{p}_1 + \mathfrak{p}_2), \quad (a + \mathfrak{p}_1, b + \mathfrak{p}_2) \mapsto a - b + (\mathfrak{p}_1 + \mathfrak{p}_2).$$

It is clear that the image of ι lies in the kernel of π. Assume $(a+\mathfrak{p}_1, b+\mathfrak{p}_2) \mapsto a - b + (\mathfrak{p}_1 + \mathfrak{p}_2)$. Then $a - b = c_1 - c_2$, $c_1 \in \mathfrak{p}_1$, $c_2 \in \mathfrak{p}_2$, and we obtain $a - c_1 = b - c_2 =: c$. Since $c + \mathfrak{p}_1 = a + \mathfrak{p}_1$ and $c + \mathfrak{p}_2 = b + \mathfrak{p}_2$, we conclude that $\iota(c) = (a + \mathfrak{p}_1, b + \mathfrak{p}_2)$. We proved the exactness of the sequence

$$0 \to A \xrightarrow{\iota} A/\mathfrak{p}_1 \oplus A/\mathfrak{p}_2 \xrightarrow{\pi} A/(\mathfrak{p}_1 + \mathfrak{p}_2) \to 0.$$

This is equivalent to the exactness of the sequence of sheaves. We proved the lemma. □

Let $\mathcal{E}_1$ and $\mathcal{E}_2$ be locally free sheaves on X_1 and X_2 respectively. Assume that there is an isomorphism $\mathcal{E}_1|_Y \cong \mathcal{E}_2|_Y$. Let us choose some locally free on Y sheaf $\mathcal{E}_Y$ that is isomorphic to $\mathcal{E}_1|_Y \cong \mathcal{E}_2|_Y$. Let us also fix some restrictions $\rho_1 : \mathcal{E}_1 \to E_Y$ and $\rho_2 : \mathcal{E}_2 \to \mathcal{E}_Y$ and let us consider the surjective map

$$\mathcal{E}_1 \oplus \mathcal{E}_2 \xrightarrow{\left(\begin{smallmatrix} \rho_1 \\ -\rho_2 \end{smallmatrix}\right)} \mathcal{E}_Y.$$

Let us denote by $\mathcal{E}$ the kernel of $\left(\begin{smallmatrix} \rho_1 \\ -\rho_2 \end{smallmatrix}\right)$, i. e., let us consider the exact sequence

$$0 \to \mathcal{E} \xrightarrow{(\alpha_1, \alpha_2)} \mathcal{E}_1 \oplus \mathcal{E}_2 \xrightarrow{\left(\begin{smallmatrix} \rho_1 \\ -\rho_2 \end{smallmatrix}\right)} \mathcal{E}_Y \to 0. \tag{A.3}$$

Let r be the rank of the sheaf $\mathcal{E}_Y$. Then it is also the rank of the sheaves $\mathcal{E}_1$ and $\mathcal{E}_2$.

Let us consider the situation locally. Assume that $\mathcal{E}_1$ and $\mathcal{E}_2$ are locally free, assume also that X is affine, let A be the coordinate ring of X. Let $\mathfrak{p}_1$ and $\mathfrak{p}_2$ be the ideals of X_1 and X_2 respectively, i. e., $A/\mathfrak{p}_1$ is the coordinate ring of X_1 and $A/\mathfrak{p}_2$ is the coordinate ring of X_2. Then $Y = X_1 \cap X_2$ has the ideal $\mathfrak{p}_1 + \mathfrak{p}_2$ and the coordinate ring $A/(\mathfrak{p}_1 + \mathfrak{p}_2)$. The sheaf $\mathcal{E}_1$ corresponds to the module $(A/\mathfrak{p}_1)^r$ and the sheaf $\mathcal{E}_2$ corresponds to the module $(A/\mathfrak{p}_2)^r$.

Then the morphism $\left(\begin{smallmatrix} \rho_1 \\ -\rho_2 \end{smallmatrix}\right)$ corresponds to

$$(A/\mathfrak{p}_1)^r \oplus (A/\mathfrak{p}_2)^r \to (A/(\mathfrak{p}_1 + \mathfrak{p}_2))^r, \quad (\bar{a}_1, \ldots, \bar{a}_r) \oplus (\bar{b}_1, \ldots, \bar{b}_r) \mapsto (\overline{a_1 - b_1}, \ldots, \overline{a_r - b_r}).$$

One easily sees that the kernel of this homomorphism coincides with the morphism

$$(A/(\mathfrak{p}_1 \cap \mathfrak{p}_2))^r \to (A/\mathfrak{p}_1)^r \oplus (A/\mathfrak{p}_2)^r, \quad (a_1, \ldots, a_r) \mapsto (\bar{a}_1, \ldots, \bar{a}_r) \oplus (\bar{a}_1, \ldots, \bar{a}_r).$$

If $\mathfrak{p}_1 \cap \mathfrak{p}_2 = 0$, which is the case if X is reduced, then the kernel is just A^r and this means that $\mathcal{E}$ is a locally free sheaf.

Let us show that the restrictions of $\mathcal{E}$ to X_1 and to X_2 are isomorphic to $\mathcal{E}_1$ and $\mathcal{E}_2$ respectively. Let us restrict $(A.3)$ to X_1. We obtain the exact sequence

$$\mathcal{E}_{X_1} \to \mathcal{E}_1 \oplus \mathcal{E}_2|_{X_1} \xrightarrow{\left(\begin{smallmatrix} \rho_1 \\ -\rho_2|_{X_1} \end{smallmatrix}\right)} \mathcal{E}_Y \to 0.$$

morphism (α_1, α_2) is an isomorphism outside of Y, we conclude that its restriction to X_1 is an isomorphism on $X_1 \setminus Y$ as well. As $\mathcal{E}_2|_{X_1}$ is supported on Y, we conclude that $\alpha_1|_{X_1} : \mathcal{E}_{X_1} \to \mathcal{E}_1$

is an isomorphism on $X_1 \setminus Y$, hence its kernel may be only supported on Y. Since $\mathcal{E}_{X_1}$ has no torsion as a locally free sheaf, it follows that the restriction of $\alpha_1|_{X_1}$ is injective. Thus one obtains the exact sequence

$$0 \to \mathcal{E}_{X_1} \to \mathcal{E}_1 \oplus \mathcal{E}_2|_{X_1} \xrightarrow{\begin{pmatrix} \rho_1 \\ -\rho_2|_{X_1} \end{pmatrix}} \mathcal{E}_Y \to 0.$$

Note that $\rho_2|_{X_1} : \mathcal{E}_2|_{X_1} \to \mathcal{E}_Y$ is an isomorphism. This gives us a splitting of the short exact sequence above and we obtain that $\alpha_1|_{X_1} : \mathcal{E}|_{X_1} \to \mathcal{E}_1$ is an isomorphism. Analogously one shows that $\alpha_2|_{X_2} : \mathcal{E}|_{X_2} \to \mathcal{E}_2$ is an isomorphism. This proves the following lemma.

Lemma A.17 (Gluing)**.** *Let $X = X_1 \cup X_1$ be as above, let X be reduced, and let $\mathcal{E}_1$ and $\mathcal{E}_2$ be two locally free sheaves on X_1 and X_2 respectively such that their restrictions to $Y = X_1 \cap X_2$ are isomorphic. Then the exact sequence* $(A.3)$ *defines uniquely a locally free sheaf $\mathcal{E}$ on X such that the restrictions of $\mathcal{E}$ to X_i coincide with $\mathcal{E}_i$ for $i = 1, 2$.*

Index

Symbols

Bibliography

[1] Vincenzo Ancona and Giorgio Ottaviani. Canonical resolutions of sheaves on Schubert and Brieskorn varieties. In *Complex analysis (Wuppertal, 1991)*, Aspects Math., E17, pages 14–19. Vieweg, Braunschweig, 1991.

[2] M. F. Atiyah. Vector bundles over an elliptic curve. *Proc. London Math. Soc. (3)*, 7:414–452, 1957.

[3] A. A. Beĭlinson. Coherent sheaves on $\mathbf{P}^n$ and problems in linear algebra. *Funktsional. Anal. i Prilozhen.*, 12(3):68–69, 1978.

[4] Hans Georg Freiermuth. On the moduli space $M_P(\mathbb{P}_3)$ of semi-stable sheaves on $\mathbb{P}_3$ with Hilbert polynomial $P(m) = 3m + 1$. Diplomarbeit, Universutät Kaiserslautern, Germany, December 2000.

[5] Hans Georg Freiermuth and Günther Trautmann. On the moduli scheme of stable sheaves supported on cubic space curves. *Amer. J. Math.*, 126(2):363–393, 2004.

[6] William Fulton and Serge Lang. *Riemann-Roch algebra*, volume 277 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, New York, 1985.

[7] D. Gieseker. On the moduli of vector bundles on an algebraic surface. *Ann. of Math. (2)*, 106(1):45–60, 1977.

[8] A. Grothendieck. Éléments de géométrie algébrique. II. Étude globale élémentaire de quelques classes de morphismes. *Inst. Hautes Études Sci. Publ. Math.*, (8):222, 1961.

[9] A. Grothendieck. Éléments de géométrie algébrique. III. Étude cohomologique des faisceaux cohérents. I. *Inst. Hautes Études Sci. Publ. Math.*, (11):167, 1961.

[10] A. Grothendieck. Techniques de construction en géométrie analytique. I: Description axiomatique de l'espace de Teichmüller et de ses variantes; II: Généralites sur les espaces anneles et les espaces analytiques; III: Produits fibres d'espaces analytiques; IV: Formalisme général des foncteurs représentables; V: Fibres vectoriels, fibres projectifs, fibres en drapeaux. Familles d'Espaces Complexes et Fondements de la Geom. Anal., Sem. H. Cartan 13 (1960/61), No.7-8, 33 p.; No.9, 14 p.; No.10, 11 p.; No.11, 28 p.; No.12, 15 p. (1962)., 1962.

[11] A. Grothendieck. Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas. II. *Inst. Hautes Études Sci. Publ. Math.*, (24):231, 1965.

[12] Robin Hartshorne. *Algebraic geometry*. Springer-Verlag, New York, 1977. Graduate Texts in Mathematics, No. 52.

[13] Daniel Huybrechts and Manfred Lehn. *The geometry of moduli spaces of sheaves.* Aspects of Mathematics, E31. Friedr. Vieweg & Sohn, Braunschweig, 1997.

[14] J. Le Potier. Faisceaux semi-stables de dimension 1 sur le plan projectif. *Rev. Roumaine Math. Pures Appl.*, 38(7-8):635–678, 1993.

[15] Masaki Maruyama. Stable vector bundles on an algebraic surface. *Nagoya Math. J.*, 58:25–68, 1975.

[16] Masaki Maruyama. Moduli of stable sheaves. I. *J. Math. Kyoto Univ.*, 17(1):91–126, 1977.

[17] Masaki Maruyama. Moduli of stable sheaves. II. *J. Math. Kyoto Univ.*, 18(3):557–614, 1978.

[18] D. Mumford, J. Fogarty, and F. Kirwan. *Geometric invariant theory*, volume 34 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (2) [Results in Mathematics and Related Areas (2)].* Springer-Verlag, Berlin, third edition, 1994.

[19] D. S. Nagaraj and C. S. Seshadri. Degenerations of the moduli spaces of vector bundles on curves. I. *Proc. Indian Acad. Sci. Math. Sci.*, 107(2):101–137, 1997.

[20] D. S. Nagaraj and C. S. Seshadri. Degenerations of the moduli spaces of vector bundles on curves. II. Generalized Gieseker moduli spaces. *Proc. Indian Acad. Sci. Math. Sci.*, 109(2):165–201, 1999.

[21] M. S. Narasimhan and S. Ramanan. Moduli of vector bundles on a compact Riemann surface. *Ann. of Math. (2)*, 89:14–51, 1969.

[22] M. S. Narasimhan and C. S. Seshadri. Stable and unitary vector bundles on a compact Riemann surface. *Ann. of Math. (2)*, 82:540–567, 1965.

[23] P. E. Newstead. *Introduction to moduli problems and orbit spaces*, volume 51 of *Tata Institute of Fundamental Research Lectures on Mathematics and Physics.* Tata Institute of Fundamental Research, Bombay, 1978.

[24] Christian Okonek, Michael Schneider, and Heinz Spindler. *Vector bundles on complex projective spaces*, volume 3 of *Progress in Mathematics.* Birkhäuser Boston, Mass., 1980.

[25] J. H. Sampson and G. Washnitzer. A Künneth formula for coherent algebraic sheaves. *Illinois J. Math.*, 3:389–402, 1959.

[26] C. S. Seshadri. Degenerations of the moduli spaces of vector bundles on curves. In *School on Algebraic Geometry (Trieste, 1999)*, volume 1 of *ICTP Lect. Notes*, pages 205–265. Abdus Salam Int. Cent. Theoret. Phys., Trieste, 2000.

[27] Carlos T. Simpson. Moduli of representations of the fundamental group of a smooth projective variety. I. *Inst. Hautes Études Sci. Publ. Math.*, (79):47–129, 1994.

[28] Fumio Takemoto. Stable vector bundles on algebraic surfaces. *Nagoya Math. J.*, 47:29–48, 1972.

[29] Fumio Takemoto. Stable vector bundles on algebraic surfaces. II. *Nagoya Math. J.*, 52:173–195, 1973.

[30] Angelo Vistoli. Notes on grothendieck topologies, fibered categories and descent theory. `arXiv:math/0412512v4 [math.AG]`.

Wissenschaftlicher Werdegang

25. Juni 1982	Geboren in Kiew, Ukraine
Juni 1999	Volle allgemeine Mittelschulbildung (Abitur), Schulabschluss mit einer Goldmedaille an dem Kiewer naturwissenschaftlichen Lyzeum 145.
1999 – 2003	Studium der Mathematik an der Kiewer Nationalen Taras-Shewtschenko-Universität.
Juni 2003	Bachelor in Mathematik mit Prädikat an der Kiewer Nationalen Taras-Shewtschenko-Universität.
August 2003 – August 2005	Studium der Mathematik und Physik an der Technischen Universität Kaiserslautern. Studienschwerpunkt: algebraische Geometrie und Computeralgebra.
August 2005	Diplom-Mathematiker an der Technischen Universität Kaiserslautern.
September 2005 – März 2009	Doktorand, Technische Universität Kaiserslautern.

Scientific career

June 25, 1982	born in Kyiv, Ukraine
June 1999	Finished Kyiv Lyceum of Natural Sciences No.145 with a Gold Medal
September 1999 – June 2003	Student at the Faculty of Mechanics and Mathematics, National Taras Shevchenko University of Kyiv
June 2003	Bachelor degree in Mathematics with Honour at National Taras Shevchenko University of Kyiv
August 2003 – August 2005	Study of Mathematics and Physics at the Technische Universität Kaiserslautern. Specialization: algebraic geometry and computer algebra.
August 2005	Diplom-Mathematiker degree at the Technische Universität Kaiserslautern.
September 2005 – March 2009	PhD student, Technische Universität Kaiserslautern.