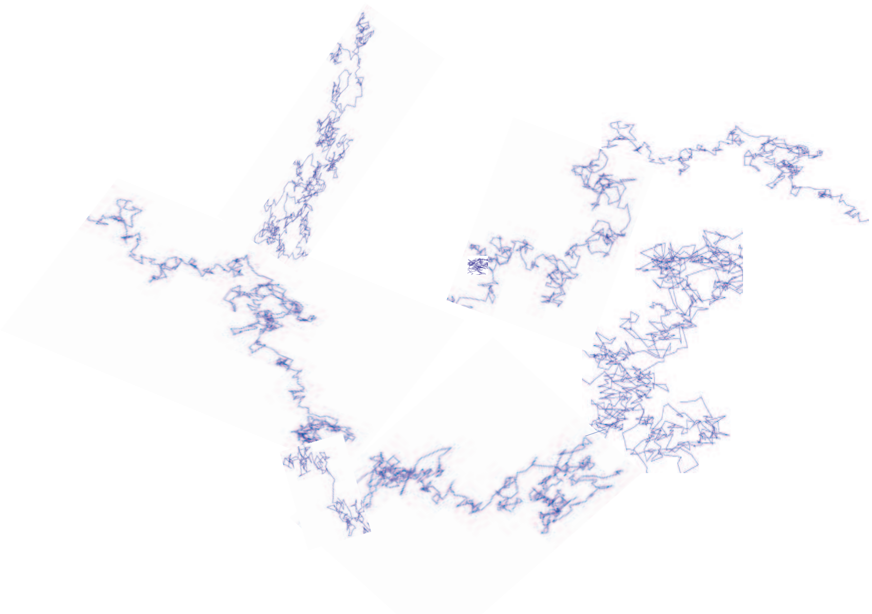
Stochastic Impulse Control
and
Asset Allocation with Liquidity Breakdowns

Peter M. Diesinger

January 20, 2009



Supervised by

Prof. Dr. Ralf Korn

Datum der Disputation: 14. Mai 2009

Vom Fachbereich Mathematik der Technischen Universität Kaiserslautern
zur Verleihung des akademischen Grades Doktor der Naturwissenschaften
(Doctor rerum naturalium, Dr. rer. nat.) genehmigte Dissertation.

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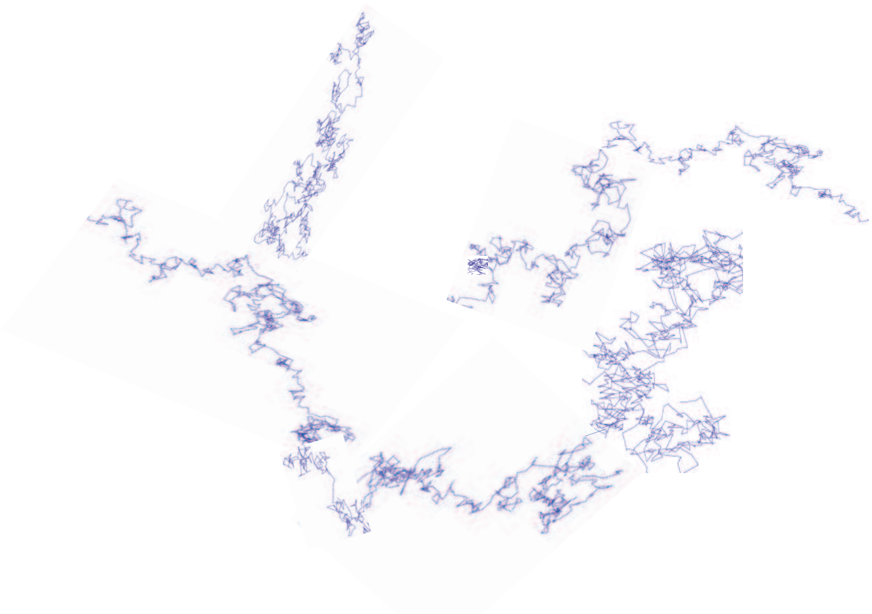

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My thanks to Ralf Korn for his excellent mentoring over the past few years. He always found the time to help me with his good advice and constructive criticism. Many thanks to Holger Kraft for the good collaboration. The second chapter of this thesis is based on an idea of his; parts of this chapter will be published in Finance and Stochastics and are the result of the cooperation with Holger Kraft and Frank Seifried. I thank Frank Seifried for many valuable discussions. Frank is always a good ally to tackle all kinds of mathematical problems. Thanks to the other colleagues in the Financial Mathematics and Stochastic Control group for the pleasant working atmosphere, in particular to Stefanie Müller and Jörn Sass. I would like to thank Nicole Bäuerle for refereeing this thesis. My thanks to my friends and family for their constant support. Special, heartfelt thanks to my parents Aloys and Margrit Diesinger.

Introduction

Continuous stochastic control theory has found many applications in optimal investment. However, it lacks some reality, as it is based on the assumption that interventions are costless, which yields optimal strategies where the controller has to intervene at every time instant. This thesis consists of the examination of two types of more realistic control methods with possible applications.

In the first chapter, we study the stochastic impulse control of a diffusion process. We suppose that the controller minimizes expected discounted costs accumulating as running and controlling cost, respectively. Each control action causes costs which are bounded from below by some positive constant. This makes a continuous control impossible as it would lead to an immediate ruin of the controller. In comparison to continuous control, apart from the pioneering work by Bensoussan and Lions [3] and [2], Menaldi [19], Richard [25], and Harrison, Selke and A. Taylor [11], there is only very few literature on this problem. The objective of the first part of Chapter 1 is to give a rigorous development of the relevant theory, where our guideline is to establish verification and convergence results under minimal assumptions, without focusing on the existence of solutions to the corresponding (quasi-)variational inequalities. If the impulse control problem can be characterized or approximated by (quasi-)variational inequalities, it remains to solve these equations. For many problems, such as applications in portfolio selection for stock markets, an impulse control approach is the appropriate model. However, it is very difficult to obtain explicit analytic solutions. Papers dealing with applications of stochastic impulse control to financial market models include Buckley and Korn [5], Eastham and Hastings [8], Jeanblanc-Picqué [14], Korn [16], and Morton and Pliska [23]. In Section 1.2, we solve the stochastic impulse control problem for a one-dimensional diffusion process with constant coefficients and convex running costs. Further, in Section 1.3, we solve a particular multi-dimensional example, where the uncontrolled process is given by an at least two-dimensional Brownian motion and the cost functions are rotationally symmetric. By symmetry, this problem can be reduced to a one-dimensional problem. In the last section of the first chapter, we suggest a new impulse control problem, where the controller is in addition allowed to invest his initial capital into a market consisting of a money market account and a risky asset. Trading in this market involves transaction costs. The costs which arise upon controlling the diffusion process and upon trading in this market have to be paid out of the controller's bond holdings. The aim of the controller is to minimize the running costs, caused by the abstract diffusion process, without getting ruined. This combines the general theory of stochastic impulse control with the particular case of optimal investment in a market with transaction costs. The linkage arises by the restriction of the set of admissible strategies.

As opposed to papers dealing with the extension of the standard market model by including transaction costs, there is another strand of literature extending the standard market model by taking liquidity constraints into account. For instance, Longstaff [20] considers the portfolio problem of an investor who can only implement portfolio strategies with finite variation. Schwartz and Tebaldi [27] assume that an investor cannot trade a risky asset at all, i.e. the trading interruption is permanent. Rogers [26] analyzes the portfolio decision of an investor who is constrained to change his strategy at discrete points in time only, although trading takes place continuously. Kahl, Liu and Longstaff [15], and Longstaff [21] consider an investment problem where the advent of a trading interruption is known. These papers are related to the second main aspect of this thesis presented in the second chapter. There, we suggest a new model for illiquidity. This chapter is based on a paper which is joint work with Holger Kraft and Frank Seifried [7]. We analyze the portfolio decision of an investor trading in a market where the economy switches randomly between two possible states, a normal state where trading takes place continuously, and an illiquidity state where trading is not allowed at all. We allow for jumps in the market prices at the beginning and at the end of a trading interruption. Section 2.1 provides an explicit representation of the investor's portfolio dynamics in the illiquidity state in an abstract market consisting of two assets. In Section 2.2 we specify this market model and assume that the investor maximizes expected utility from terminal wealth. We establish convergence results, if the maximal number of liquidity breakdowns goes to infinity. In the Markovian framework of Section 2.3, we provide the corresponding Hamilton-Jacobi-Bellman equations and prove a verification result. We apply these results to study the portfolio problem for a logarithmic investor and an investor with a power utility function, respectively. Further, we extend this model to an economy with three regimes. For instance, the third state could model an additional financial crisis where trading is still possible, but the excess return is lower and the volatility is higher than in the normal state.

Contents

1 Stochastic impulse control

In stochastic control one considers a diffusion which depending on its state causes costs, the so-called running costs. Whenever the process is controlled additional costs occur. The aim of the controller is to find a control strategy that minimizes the expected discounted costs accumulating as running and controlling cost, respectively. In impulse control one assumes that each control action causes costs which are bounded from below by some *positive* constant, i.e. one embeds an additional fixed cost component such as transaction costs. This fixed component makes a continuous control impossible as it would lead to an immediate ruin of the controller. Consequently, the controller does not only have to choose the control actions, but also some non accumulating intervention times, which justifies the terminology *impulse* control.

In this chapter, we study the stochastic impulse control problem with an infinite time horizon. The uncontrolled process is given by a time-homogeneous diffusion process. Section 1.1 provides the corresponding theory. In particular, Theorem 1.25 allows us to approximate the impulse control problem by impulse control problems where only finitely many interventions are possible. Our guideline is to establish verification and convergence results under minimal assumptions, without focusing on the existence of solutions to the corresponding (quasi-)variational inequalities. If the impulse control problem can be characterized or approximated by (quasi-)variational inequalities, it remains to solve these equations. For instance this is done in Sections 1.2 and 1.3. In Section 1.2 we suppose that the uncontrolled process is given by a one-dimensional diffusion process with constant coefficients, and that the running costs are convex. We solve the corresponding impulse control problem by applying the results of the previous section, and present some examples illustrating the shape of the value functions and the corresponding optimal strategies. In Section 1.3 we consider a particular multi-dimensional example. We assume that the uncontrolled process is given by an at least two-dimensional Brownian motion and that the cost functions are rotationally symmetric. Thus we may reduce this problem to a one-dimensional setting. Again, applying the results of Section 1.1 we solve the corresponding impulse control problem. We illustrate the dependance of the value functions and their optimal strategies on fixed costs, proportional costs and different running cost functions, respectively. In the last section of this chapter, we suggest an impulse control problem, where the controller is in addition allowed to invest his initial capital into a market consisting of a money market account and a risky asset. Trading in this market also involves transaction costs. The costs which arise upon controlling the abstract diffusion process and upon trading in this market have to be paid out of the controller's bond holdings. His aim is to minimize the running costs, caused by the abstract diffusion process, without getting ruined. In particular, this time, his optimal strategy will depend on his current wealth.

1.1 Optimal stochastic impulse control

Let $\mathcal{W}^m$ be the Wiener space $C^0(\mathbb{R}_0^+, \mathbb{R}^m) = \{f : \mathbb{R}_0^+ \to \mathbb{R}^m,\ f \text{ continuous}\}$ topologized by uniform convergence on compact intervals and endowed with the corresponding Borel σ-field. We will denote the coordinate mappings by $\pi_t : C^0(\mathbb{R}_0^+, \mathbb{R}^m) \to \mathbb{R}^m$, $\varphi \mapsto \pi_t(\varphi) = \varphi(t)$, $t \geq 0$. The underlying complete probability space will be denoted by $(\Omega, \mathcal{F}, P)$. Let $(B_t)_{t\geq 0}$ be a m-dimensional standard Brownian motion with state space $\mathcal{W}^m$. The completed natural filtration of the Brownian motion B will be denoted by $(\mathcal{F}_t)_{t\geq 0}$. We assume that $0 \in \mathbb{N}$, and $|.|$ denotes the Euclidean norm.

Definition 1.1. *An **impulse control strategy** $S = (\tau_n, \delta_n)_{n\in\mathbb{N}}$ is a sequence such that for all $n \in \mathbb{N}$*

- ◇ $0 = \tau_0 \leq \tau_1 \leq \cdots \leq \tau_n \leq \cdots$ *almost surely*
- ◇ $\tau_n : \Omega \to [0, \infty]$ *is a stopping time w.r.t $(\mathcal{F}_t)_{t\geq 0}$* *"intervention times"*
- ◇ $\delta_n : \Omega \to \mathbb{R}^d$ *is measurable w.r.t. the σ-algebra of τ_n-past $\mathcal{F}_{\tau_n}$* *"control actions"*

*An impulse control strategy $(\tau_n, \delta_n)_{n\in\mathbb{N}}$ will be called **admissible** if we have*

- ◇ $\lim\limits_{n\to\infty} \tau_n = \infty$ *almost surely*

The set of all admissible impulse control strategies will be denoted by $\mathcal{A}$.

We interpret τ_n as the n-th time at which a controller enforces a jump in the state of the system, with δ_n being the size of the jump enforced. As mentioned above, the admissibility condition takes care that the intervention times do not accumulate and will allow us to introduce fixed costs.

Remark 1.2. *Let $\tau_0, \ldots, \tau_n : \Omega \to [0, \infty]$ be $(\mathcal{F}_t)_{t\geq 0}$-stopping times such that $0 = \tau_0 \leq \tau_1 \leq \cdots \leq \tau_n$ almost surely and let $\delta_k : \Omega \to \mathbb{R}^d$ be $\mathcal{F}_{\tau_k}$-measurable for all $0 \leq k \leq n$. Then we will identify the vector*

$$((\tau_0, \delta_0), \ldots, (\tau_n, \delta_n))$$

with an admissible impulse control strategy by setting $\tau_l \equiv \infty$ and $\delta_l \equiv 0$ for all $l > n$.

Let us now state the definition of the controlled process corresponding to an impulse control strategy.

Definition 1.3. *Let $S = (\tau_n, \delta_n)_{n\in\mathbb{N}}$ be an impulse control strategy, let $x \in \mathbb{R}^d$ and let*

$$b : \mathbb{R}^d \to \mathbb{R}^d \qquad\qquad \sigma : \mathbb{R}^d \to \mathbb{R}^{d\times m}$$

*be Borel measurable and locally bounded. A stochastic process $X = (X_t)_{t\geq 0}$ adapted to the filtration $(\mathcal{F}_t)_{t\geq 0}$ is called a **controlled diffusion process** corresponding to the strategy S if it solves the following stochastic integral equation*

$$X_t = x + \int_0^t b(X_{s-})ds + \int_0^t \sigma(X_{s-})dB_s + \sum_{j=1}^{\infty} 1_{\{\tau_j \leq t\}} \delta_j$$

with infinitesimal drift b and infinitesimal covariance $a = \sigma\sigma^t$. Writing out the coordinates we have

$$X_t^{(i)} = x^{(i)} + \int_0^t b_i(X_{s-})ds + \sum_{j=1}^{m} \int_0^t \sigma_{i,j}(X_{s-})dB_s^{(j)} + \sum_{j=1}^{\infty} 1_{\{\tau_j \le t\}} \delta_j^{(i)}, \quad 1 \le i \le d$$

with $X = (X^{(1)}, \dots, X^{(d)})^t$, $x = (x^{(1)}, \dots, x^{(d)})^t$, $b = (b_1, \dots, b_d)^t$, $\sigma = (\sigma_{i,j})_{1\le i\le d,\, 1\le j\le m}$, $B = (B^{(1)}, \dots, B^{(m)})^t$ and $\delta = (\delta^{(1)}, \dots, \delta^{(d)})^t$. We will also use the notation

$$dX_t = b(X_{t-})dt + \sigma(X_{t-})dB_t + \sum_{j=1}^{\infty} 1_{\{\tau_j = t\}} \delta_j, \quad X_{0-} = x$$

and refer to x as the ***starting point*** *of the process X.*

When there is a risk of ambiguity, we will write X^S rather then X for a controlled diffusion process corresponding to a strategy S. As we will only consider diffusion processes, for sake of brevity we will prefer the notion *controlled process* to *controlled diffusion process.* Note that a controlled process is in general *not* Markovian, since the control may depend on the past. However as we shall see in the following theorem, X is strong Markovian on each stochastic interval given by two successive intervention times.

Theorem 1.4. *Let $S = (\tau_n, \delta_n)_{n\in\mathbb{N}}$ be an admissible impulse control strategy and suppose that b and σ are Lipschitz. Then there exists a unique càdlàg, $(\mathcal{F}_t)_{t\ge 0}$-adapted strong solution of the stochastic integral equation*

$$X_t = x + \int_0^t b(X_{s-})ds + \int_0^t \sigma(X_{s-})dB_s + \sum_{j=1}^{\infty} 1_{\{\tau_j \le t\}} \delta_j.$$

Further, the controlled process X is a strong $(\mathcal{F}_t)_{t\ge 0}$-Markov process on each stochastic interval $[\tau_k, \tau_{k+1})$, i.e. for each $k \in \mathbb{N}$ we have

$$E(\psi|\mathcal{F}_\tau) = E(\psi|X_\tau)$$

for every finite $(\mathcal{F}_t)_{t\ge 0}$-stopping time τ with $\tau_k \le \tau < \tau_{k+1}$ and every bounded $\sigma(X_{(\tau+\cdot)\wedge\tau_{k+1}})$-measurable functional ψ.

In Theorem 1.25 we will give a sharper version of the strong Markov property stated above and therefore omit a proof of Theorem 1.4 here. Regarding existence of a solution we state the following remark.

Remark 1.5. *For the construction of a controlled process X corresponding to an impulse control strategy $S = (\tau_n, \delta_n)_{n\in\mathbb{N}}$, we first solve the following set of equations with random* *initial conditions*

$$X_t^{0,(i)} = x^{(i)} + \int_0^t b_i(X_s^0)ds + \sum_{j=1}^m \int_0^t \sigma_{i,j}(X_s^0)dB_s^{(j)}, \quad t \geq 0$$

$$X_t^{k,(i)} = X_{\tau_k}^{k-1,(i)} + \delta_k^{(i)} + \int_{\tau_k}^t b_i(X_s^k)ds + \sum_{j=1}^m \int_{\tau_k}^t \sigma_{i,j}(X_s^k)dB_s^{(j)}, \quad t \geq \tau_k$$

where $1 \leq i \leq d$ *and* $k \geq 1$. *Then, for* $n \in \mathbb{N}$ *we set*

$$X_t = X_t^n, \quad \forall\, t \in [\tau_n, \tau_{n+1}).$$

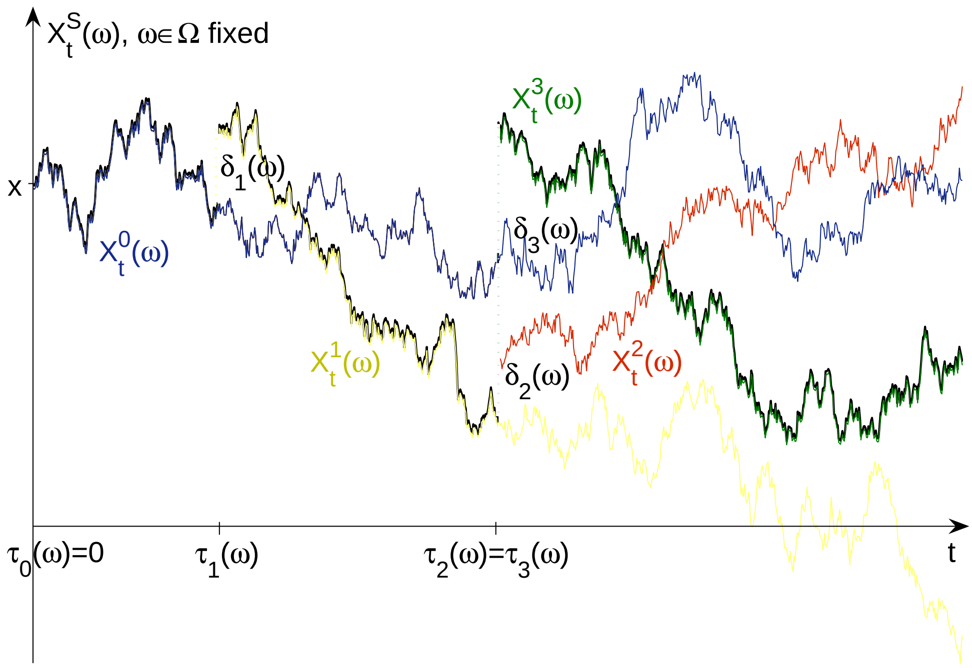

Figure 1: Construction of the process X.

From now on we always assume that b and σ are Lipschitz.
If $S = (\tau_n, \delta_n)_{n\in\mathbb{N}}$ is admissible then at most finitely many impulses may occur within each finite time interval. Thus $(\sum_{j=1}^{\infty} 1_{\{\tau_j \leq t\}}\delta_j)_{t\geq 0}$ is locally of finite variation and X is a càdlàg semimartingale with $X_{\tau_n} = X_{\tau_n -} + \sum_{j=1}^{\infty} 1_{\{\tau_j = \tau_n\}}\delta_j$ and

$$X_t^c = x + \int_0^t b(X_{s-})ds + \int_0^t \sigma(X_{s-})dB_s + \sum_{j=1}^{\infty} 1_{\{\tau_j = 0\}}\delta_j.$$

Note also, that in this setting, X being a càdlàg semimartingale is equivalent to the jump process ΔX defined by $\Delta X_t = X_t - X_{t-}$ being locally of finite variation.

Remark 1.6. *Since $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ is the filtered probability space of a BM B, where $(\mathcal{F}_t)_{t\geq 0}$ is the completed natural filtration of B, each càdlàg semimartingale X on this space verifies*

$$\sum_{0<s\leq t} |\Delta X_s| < \infty \quad \textit{almost surely, for all } t > 0.$$

Indeed, if $X = M + A$ is a decomposition of the càdlàg semimartingale X with M a local martingale and A locally of finite variation, then M is continuous. Thus, the jump processes ΔX and ΔA are equal and therefore

$$\sum_{0<s\leq t} |\Delta X_s| = \sum_{0<s\leq t} |\Delta A_s| < \infty \quad \textit{almost surely, for all } t > 0,$$

since A is locally of finite variation.

In the following, $(P_x)_{x\in\mathbb{R}^d}$ denotes a family of probability measures such that under P_x the process X starts in x. Furthermore, let E_x be the expectation operator associated with P_x. The action of the controller consists of the choice of the parameters τ_n and δ_n which causes costs. His control problem includes intervention cost C and running cost f, where we assume that the former consist of a fixed positive component K and a variable component c.

Definition 1.7. *Let $K > 0$ and $c : \mathbb{R}^d \to \mathbb{R}_0^+$ such that*

- ◇ *c is continuous and $c(0) = 0$*
- ◇ *$c(y) \to \infty$ as $|y| \to \infty$*
- ◇ *c is subadditive: $c(y_1 + y_2) \leq c(y_1) + c(y_2) \quad \forall\, y_1, y_2 \in \mathbb{R}^d$*

*Then $C : \mathbb{R}^d \to [K, \infty)$, $C(y) = K + c(y)$ is called **controlling cost** with fixed costs K and variable cost c.*

The impulse control problem consists of minimizing the expected discounted cost over the set of admissible impulse control strategies

$$v(x) = \inf_{S=(\tau_n,\delta_n)_{n\in\mathbb{N}}\in\mathcal{A}} E_x\Big(\int_0^\infty e^{-\alpha s} f(X_s^S)ds + \sum_{n=1}^\infty 1_{\{\tau_n<\infty\}} e^{-\alpha\tau_n} C(\delta_n)\Big) \tag{1}$$

where $f : \mathbb{R}^d \to \mathbb{R}_0^+$ is Borel measurable and $\alpha > 0$. With the interpretation of f as the **running cost** and α as a **discount factor**, the function v is called the **value function** of

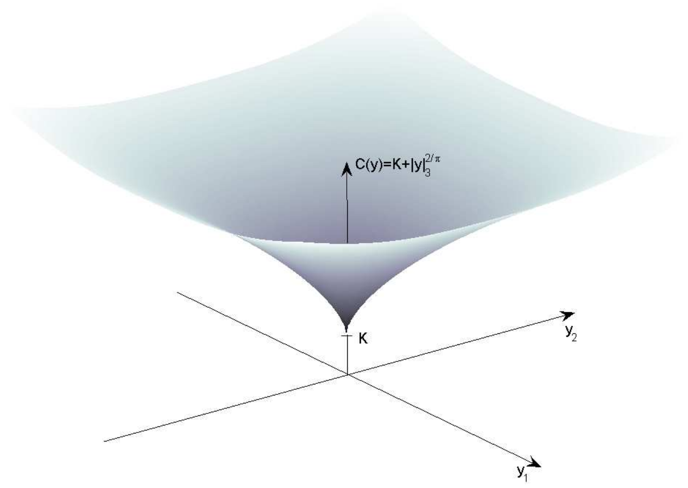

Figure 2: Example for a controlling cost function.

our impulse control problem. An admissible strategy S such that the infimum is attained will be called **optimal**. Further, for $S = (\tau_n, \delta_n)_{n\in\mathbb{N}} \in \mathcal{A}$ let us write

$$J^S(x) = E_x(\int_0^\infty e^{-\alpha s} f(X_s^S)ds + \sum_{n=1}^\infty 1_{\{\tau_n<\infty\}} e^{-\alpha\tau_n} C(\delta_n)).$$

In words, the subadditivity of the variable cost c means that it is not more expensive to control by $x + y$ at once, rather than first to control by x and afterwards to control by y. This assumption copes with the degression of the indirect costs and takes care that the intervention times of an optimal impulse control strategy, respectively those of the qvi-control, are strictly increasing (see the following remark as well as Lemma 1.13 (ii) below), which in turn allows us to avoid some "counting" in the proof of Theorem 1.14. However, this assumption is by no means necessary and for a proof of Theorem 1.14 which does not use the subadditivity of the cost function c compare to the proof of Theorem 1.34.

Remark 1.8. *Due to the subadditivity of c, a controlling cost $C = K + c$ is strictly subadditive and therefore, an optimal impulse control strategy $S = (\tau_n, \delta_n)_{n\in\mathbb{N}}$ has to satisfy $\tau_n < \tau_{n+1}$ whenever $\tau_n < \infty$, for all $n \geq 1$.*

We might consider $-E_x(\int_0^\infty e^{-\alpha s} f(X_s^S)ds + \sum_{n=1}^\infty 1_{\{\tau_n<\infty\}} e^{-\alpha\tau_n} C(\delta_n))$ as the profit which can be realized by starting in x and applying the control S. Then

$$-v(x) = \sup_{S=(\tau_n,\delta_n)_{n\in\mathbb{N}}\in\mathcal{A}} -E_x(\int_0^\infty e^{-\alpha s} f(X_s^S)ds + \sum_{n=1}^\infty 1_{\{\tau_n<\infty\}} e^{-\alpha\tau_n} C(\delta_n))$$

corresponds to the profit from the disposition of the state x. Thus, according to the law of supply and demand, the revenue corresponding to having an additional unit of each of the components, $-\partial_x v$, represents the market price when the quantity available is x.

In order to construct an optimal strategy it will be interesting to consider

$$\inf_{y\in\mathbb{R}^d}(v(x+y)+C(y)), \quad x\in\mathbb{R}^d$$

which represents the value of the strategy that consists of doing the best immediate action when starting in x and behaving optimally afterwards. More generally let us define a minimum operator M as follows

Definition 1.9. *Let C be a controlling cost and let $u:\mathbb{R}^d\to\mathbb{R}$ be bounded from below. Then we set*

$$Mu(x)=\inf_{y\in\mathbb{R}^d}(u(x+y)+C(y)), \quad x\in\mathbb{R}^d.$$

By our assumption on u and the properties of C, the minimum operator M is well defined. The following lemma summarizes some important properties of M. It is an extension of Baccarin and Sanfelici [1, Theorem 2].

Lemma 1.10 (Properties of the minimum operator)**.** *Assume that $C=K+c$ is a controlling cost and let $u:\mathbb{R}^d\to\mathbb{R}$ be bounded from below. Then M satisfies the following properties:*

(i) $M0=K$, Mu is bounded from below and for $z:\mathbb{R}^d\to\mathbb{R}$ with $u\le z$ we have $Mu\le Mz$.

(ii) If u is continuous then there exists a Borel measurable function $\varphi_u:\mathbb{R}^d\to\mathbb{R}^d$ such that

$$Mu(x)=u(x+\varphi_u(x))+C(\varphi_u(x)) \quad \forall\, x\in\mathbb{R}^d.$$

(iii) If u is continuous then Mu is continuous.

(iv) Let $u_n:\mathbb{R}^d\to\mathbb{R}$ continuous for all $n\in\mathbb{N}$ with $u_0\le u_1\le\cdots\le u_n\le\cdots\le u$, let u_0 be bounded from below and let u be continuous.

- *Let $x\in\mathbb{R}^d$ and set $r=|x|+\sup_{n\in\mathbb{N}}|\varphi_{u_n}(x)|$. If $\sup_{|z|\le r}|u_n(z)-u(z)|\to 0$, as $n\to\infty$, then $Mu_n(x)\to Mu(x)$, as $n\to\infty$.*
- *On each compact subset of $\mathbb{R}^d$, $(Mu_n)_{n\in\mathbb{N}}$ converges pointwise if and only if $(Mu_n)_{n\in\mathbb{N}}$ converges uniformly.*

In particular, if $\sup_{|x|\le r}|u_n(x)-u(x)|\to 0$, as $n\to\infty$ for all $r>0$, then $\sup_{|x|\le r}|Mu_n(x)-Mu(x)|\to 0$, as $n\to\infty$ for all $r>0$.

Proof. (i) By definition of M, $u\le z$ implies $Mu\le Mz$ and we have $M0=C(0)=K+c(0)=K$ since $c\ge 0$ and $c(0)=0$. Moreover, Mu is bounded from below since u is bounded from below and $C\ge K$.

(ii) Let $g_u : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$, $(x,y) \mapsto g_u(x,y) = u(x+y) + C(y)$ and let $\mu_u : \mathbb{R}^d \to \mathcal{P}(\mathbb{R}^d)$ be a set-valued function defined by

$$x \mapsto \mu_u(x) = \{v \in \mathbb{R}^d : g_u(x,v) = \inf_{y \in \mathbb{R}^d} g_u(x,y)\}.$$

Note that g_u is continuous since u and c are continuous and that $\lim_{|y|\to\infty} g_u(x,y) = \infty$ for all $x \in \mathbb{R}^d$, since u is bounded from below and $\lim_{|y|\to\infty} c(y) = \infty$. Thus, for all $x \in \mathbb{R}^d$ there exists a compact set $K_u(x)$ such that $\mu_u(x) \subset K_u(x)$ and $g_u(x,\cdot)$ achieves its infimum.
Let us observe that μ_u is upper semicontinuous. Let $x \in \mathbb{R}^d$, let $(x_n)_{n\in\mathbb{N}}$ be a sequence such that $\lim_{n\to\infty} x_n = x$, let $v_n \in \mu_u(x_n)$ each $n \in \mathbb{N}$ and suppose that $\lim_{n\to\infty} v_n = v$ for some v. Note that by the above we have $v \in \mathbb{R}^d$. Assume that $v \notin \mu_u(x)$. Then there exists some $\tilde{v} \in \mathbb{R}^d$ and some $\varepsilon > 0$ such that

$$g_u(x,\tilde{v}) = g_u(x,v) - \varepsilon.$$

Thus, by continuity of g_u there exists some $n_1 \in \mathbb{N}$ such that

$$g_u(x_n,\tilde{v}) < g_u(x,v) - \frac{\varepsilon}{2}, \quad \text{for all } n \geq n_1.$$

On the other hand, the continuity of g_u implies the existence of some $n_2 \in \mathbb{N}$ such that

$$g_u(x_n,v_n) > g_u(x,v) - \frac{\varepsilon}{2}, \quad \text{for all } n \geq n_2.$$

Hence, for $N = n_1 \vee n_2$ we have

$$g_u(x_N,v_N) > g_u(x,v) - \frac{\varepsilon}{2} > g_u(x_N,\tilde{v}),$$

which is a contradiction to v_N being an element of $\mu_u(x_N)$. This implies that $v \in \mu_u(x)$ and we have shown that μ_u is upper semicontinuous.
Thus, by Hildenbrand [13, Part I Sec. D Lemma 1] there exists a Borel measurable mapping $\varphi_u : \mathbb{R}^d \to \mathbb{R}^d$ such that

$$\varphi_u(x) \in \mu_u(x), \quad x \in \mathbb{R}^d.$$

Let us present an alternative illustrative proof for the existence of a Borel measurable function φ_u such that the above holds true in dimension one. Let $d = 1$ and note that for each $x \in \mathbb{R}$, $\mu_u(x)$ is compact in $\mathbb{R}$, since $\mu_u(x)$ is a closed subset of $K_u(x)$. Set $\varphi_u : \mathbb{R} \to \mathbb{R}$, $x \mapsto \varphi_u(x) = \inf \mu_u(x)$. By the upper semicontinuity of μ_u we will now establish that φ_u is lower semicontinuous. Assume that φ_u is not lower semicontinuous. Then there exist $x \in \mathbb{R}$, $\varepsilon > 0$ and a sequence $(x_n)_{n\in\mathbb{N}}$ such that $|x_n - x| < \frac{1}{n}$ but

$$\varphi_u(x_n) \leq \varphi_u(x) - \varepsilon$$

for all $n \in \mathbb{N}$. For each $n \in \mathbb{N}$, we have $\varphi_u(x_n) \in \mu_u(x_n)$. Thus, since μ_u is upper semicontinuous, for v given by

$$\liminf_{n\to\infty} \varphi_u(x_n) = v$$

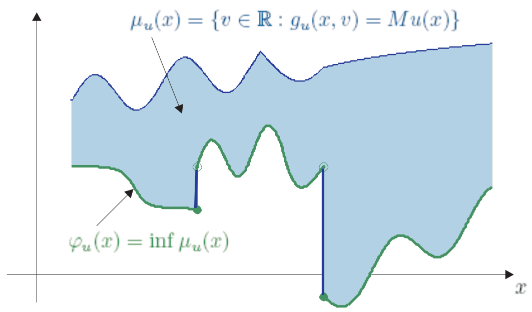

Figure 3: Definition of φ_u in dimension one.

we have $v \in \mu_u(x)$ and therefore

$$\varphi_u(x) \leq v.$$

But on the other hand, since $\varphi_u(x_n) \leq \varphi_u(x) - \varepsilon$ for all $n \in \mathbb{N}$, we have

$$v \leq \varphi_u(x) - \varepsilon < \varphi_u(x).$$

Hence the function φ_u is lower semicontinuous. Thus, φ_u is a Borel measurable function such that

$$Mu(x) = u(x + \varphi_u(x)) + C(\varphi_u(x))$$

for all $x \in \mathbb{R}^d$.

(iii) Let $x \in \mathbb{R}^d$ and $(x_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}^d$ such that $x_n \to x$ as $n \to \infty$. For $y \in \mathbb{R}^d$ we have

$$Mu(x_n) \leq u(x_n + y) + C(y)$$

and from the continuity of u it follows that

$$\begin{aligned}\limsup_{n\to\infty} Mu(x_n) &\leq \limsup_{n\to\infty} u(x_n + y) + C(y)\\ &= \lim_{n\to\infty} u(x_n + y) + C(y)\\ &= u(x + y) + C(y).\end{aligned}$$

Since $y \in \mathbb{R}^d$ is arbitrary, we obtain

$$\limsup_{n\to\infty} Mu(x_n) \leq Mu(x).$$

From (ii) it holds that

$$Mu(x_n) = u(x_n + \varphi_u(x_n)) + C(\varphi_u(x_n)).$$

By continuity of u, for $r > 0$ there exists some $N(r)$ such that

$$Mu(x_n) \leq u(x_n) + C(0) \leq u(x) + C(0) + r \quad \forall\, n \geq N(r),$$

which yields that $(Mu(x_n))_{n\in\mathbb{N}}$ is bounded. Thus $(\varphi_u(x_n))_{n\in\mathbb{N}}$ is bounded, since otherwise there would be a subsequence $(n_k)_{k\in\mathbb{N}}$ in $\mathbb{N}$ such that $|\varphi_u(x_{n_k})| \to \infty$ as $k \to \infty$ which would imply $Mu(x_{n_k}) \to \infty$ as $k \to \infty$ since $c(\varphi_u(x_{n_k})) \to \infty$ when $k \to \infty$ and since u is bounded from below.

Let $(n_k)_{k\in\mathbb{N}}$ be a subsequence in $\mathbb{N}$ such that $(Mu(x_{n_k}))_{k\in\mathbb{N}}$ converges. Since $(\varphi_u(x_{n_k}))_{k\in\mathbb{N}}$ is bounded, by Bolzano-Weierstrass there exists a subsequence $(n_{k_l})_{l\in\mathbb{N}}$ of $(n_k)_{k\in\mathbb{N}}$ and some $y \in \mathbb{R}^d$ such that $\varphi_u(x_{n_{k_l}}) \to y$ as $l \to \infty$. Thus, by our assumption on $(n_k)_{k\in\mathbb{N}}$ and the continuity of u and c we get

$$\begin{aligned}\lim_{k\to\infty} Mu(x_{n_k}) &= \lim_{l\to\infty} Mu(x_{n_{k_l}}) \\ &= \lim_{l\to\infty} u(x_{n_{k_l}} + \varphi_u(x_{n_{k_l}})) + C(\varphi_u(x_{n_{k_l}})) \\ &= u(x+y) + C(y) \\ &\geq Mu(x).\end{aligned}$$

Hence for all subsequences $(n_k)_{k\in\mathbb{N}}$ such that $(Mu(x_{n_k}))_{k\in\mathbb{N}}$ converges we have $\lim\limits_{k\to\infty} Mu(x_{n_k}) \geq Mu(x)$, yielding that

$$\liminf_{n\to\infty} Mu(x_n) \geq Mu(x).$$

(iv) Let $x \in \mathbb{R}^d$. Since $u_n(x) \leq u(x)$, by (i) we get that $(Mu_n(x))_{n\in\mathbb{N}}$ is bounded. Thus $(\varphi_{u_n}(x))_{n\in\mathbb{N}}$ is bounded as u_0 is bounded from below and $c(y) \to \infty$ as $|y| \to \infty$. This implies that

$$\sup_{n\in\mathbb{N}} |\varphi_{u_n}(x)| < \infty.$$

By using (i) and setting $r = |x| + \sup\limits_{n\in\mathbb{N}} |\varphi_{u_n}(x)|$ we obtain

$$\begin{aligned}Mu(x) \geq Mu_n(x) &= u_n(x + \varphi_{u_n}(x)) + C(\varphi_{u_n}(x)) \\ &\geq u(x + \varphi_{u_n}(x)) + C(\varphi_{u_n}(x)) - \sup_{|z|\leq r} |u_n(z) - u(z)| \\ &\geq Mu(x) - \sup_{|z|\leq r} |u_n(z) - u(z)|\end{aligned}$$

and therefore, since by assumption $\sup\limits_{|z|\leq r} |u_n(z) - u(z)| \to 0$ as $n \to \infty$, we have

$$\lim_{n\to\infty} Mu_n(x) = Mu(x).$$

Finally, let $A \subset \mathbb{R}^d$ be compact and assume that $Mu_n \to Mu$ pointwise on A as $n \to \infty$. Note that by (iii), $Mu_n - Mu$ is continuous for all $n \in \mathbb{N}$. Moreover, by (i) we have $Mu - Mu_n \geq 0$ and therefore, by Dini's theorem we obtain

$$\lim_{n\to\infty} \sup_{x\in A} |Mu(x) - Mu_n(x)| = 0.$$

The last assertion is an immediate consequence of these results. □

When for all $x \in \mathbb{R}^d$, $\mu_u(x)$ consists of a single element, the definition of upper semicontinuity is equivalent to the definition of continuity for a function. Thus, in this case, if u is continuous then $\varphi_u = \mu_u$ is continuous.

Let us now give a heuristic derivation of the so called quasi-variational inequalities for problem (1) in order to get an intuitive understanding of these relations. They are the analogue of the Hamilton-Jacobi-Bellman equations in instantaneous stochastic control. Let C be a controlling cost, f a running cost and α a discount factor. Let v be the corresponding value function and note that

$$Mv(x) = \inf_{y \in \mathbb{R}^d} (v(x+y) + C(y)), \quad x \in \mathbb{R}^d$$

is well defined since $v \geq 0$ and $C \geq K$. As mentioned above, $Mv(x)$ represents the value of the strategy that consists of doing the best immediate action when starting in x and behaving optimally afterwards. An immediate action does not need to be optimal which yields that

$$v \leq Mv,$$

where equality holds in case that an immediate action is indeed optimal. Let $x \in \mathbb{R}^d$, assume that there exists an optimal impulse control strategy $S = (\tau_n, \delta_n)_{n \in \mathbb{N}}$, let X be the associated controlled process and assume that the following variant of the Bellman principle holds

$$v(x) = E_x\Big(\int_0^t e^{-\alpha s} f(X_s) ds + \sum_{i=1}^{\infty} 1_{\{\tau_i \leq t\}} e^{-\alpha \tau_i} C(\delta_i) + e^{-\alpha t} v(X_t)\Big),$$

for all $t > 0$. If an immediate impulse is not given, the system is left to evolve freely in some small interval of length $\delta > 0$, i.e. $\tau_1 > \delta$. Suppose that v is sufficiently smooth to apply Itô's formula to obtain that

$$\begin{aligned} v(x) &= E_x\Big(\int_0^\delta e^{-\alpha s} f(X_s) ds + e^{-\alpha \delta} v(X_\delta)\Big) \\ &= E_x\Bigg(v(x) + \int_0^\delta e^{-\alpha s}(f(X_s) - \alpha v(X_s)) ds + \sum_{i=1}^{d} \sum_{j=1}^{m} \int_0^\delta e^{-\alpha s} \partial_{x_i} v(X_s) \sigma_{i,j}(X_s) dB_s^{(j)} \\ &\qquad + \sum_{i=1}^{d} \int_0^\delta e^{-\alpha s} \left\{ \partial_{x_i} v(X_s) b_i(X_s) + \frac{1}{2} \sum_{j=1}^{d} \partial_{x_i, x_j} v(X_s) a_{i,j}(X_s) \right\} ds \Bigg). \end{aligned}$$

Suppose that $E_x \int_0^\delta e^{-\alpha s} \partial_x v(X_s) \sigma(X_s) dB_s = 0$ and subtract $v(x)$ on both sides of the equation above to get

$$0 = E_x\Bigg(\int_0^\delta e^{-\alpha s}(f(X_s) - \alpha v(X_s))ds$$

$$+ \sum_{i=1}^d \int_0^\delta e^{-\alpha s} \left\{ \partial_{x_i} v(X_s) b_i(X_s) + \frac{1}{2} \sum_{j=1}^d \partial_{x_i,x_j} v(X_s) a_{i,j}(X_s) \right\} ds \Bigg).$$

Now, dividing by δ, letting $\delta \downarrow 0$, interchanging the order of taking this limit and taking the expectation and applying the mean value theorem, formally leads to

$$0 = Lv(x) + f(x),$$

where L is defined by

$$Lv(x) = -\alpha v(x) + \sum_{i=1}^d \partial_{x_i} v(x) b_i(x) + \frac{1}{2} \sum_{i,j=1}^d \partial_{x_i,x_j} v(x) a_{i,j}(x). \tag{2}$$

By applying a variant of Itô's formula for càdlàg semimartingales, we can also derive the equality above without assuming that $\tau_1 > \delta$ but under the weaker assumption that $\tau_1 > 0$ (replace T by δ in the proof of Theorem 1.14). However, if an immediate impulse is optimal, then for $S = (\tau_n, \delta_n)_{n \in \mathbb{N}} \in \mathcal{A}$ such that $\tau_1 > \delta$ we have

$$v(x) < E_x(\int_0^\delta e^{-\alpha s} f(X_s^S) ds + e^{-\alpha \delta} v(X_\delta^S))$$

and therefore

$$Lv(x) + f(x) > 0.$$

Furthermore, note that we either exercise an immediate impulse or we leave the system to evolve freely, which implies that

$$(v(x) - Mv(x))(Lv(x) + f(x)) = 0.$$

For convenience let us make the agreement that from now on, whenever we write C, f, α, M or L we implicitly assume that these are controlling cost, running cost, a discount factor, the minimum operator from Definition 1.9, the operator defined by (2), respectively. Starting directly to exploit this convention we give the following definition.

Definition 1.11. *Let $v^* : \mathbb{R}^d \to \mathbb{R}_0^+$. The following three relations are called the* ***quasi-variational inequalities*** *(abbreviated* ***qvi****) for the impulse control problem* (1)

- $\diamond$ $Lv^* + f \geq 0$

◇ $v^* \leq Mv^*$

◇ $(v^* - Mv^*)(Lv^* + f) = 0$

A function $v^* : \mathbb{R}^d \to \mathbb{R}_0^+$ *which satisfies these quasi-variational inequalities is said to be a* ***solution of the qvi*** *for problem* (1).

Instead of specifying the smoothness assumption on v^* directly in Definition 1.11, we will preferably mention it explicitly whenever dealing with solutions of the above qvi. The advantage being that we do not have to repeat this definition when considering different types of differentiability. As a minimal assumption on v^*, we will always assume that it is continuous.

The dependence of the right hand side of the inequality $v^* \leq Mv^*$ upon the solution v^* justifies the terminology *quasi*-variational inequality. Note that the assumption $v^* \geq 0$ in the definition above implies that Mv^* is welldefined and corresponds to the property $v \geq 0$ of the value function v. Recall that in this setting both, the running cost f and the controlling cost C are nonnegative. Let us define a special impulse control strategy, constructed with the help of a solution of the qvi.

Definition 1.12. *Let* v^* *be a solution of the quasi-variational inequalities, let* $(\tau_0, \delta_0) \equiv (0, 0)$ *and for* $n \geq 1$ *set*

$$\begin{aligned} S_{n-1} &= ((\tau_0, \delta_0), \ldots, (\tau_{n-1}, \delta_{n-1})) \\ \tau_n &= \inf\{t \geq \tau_{n-1} : v^*(X_t^{S_{n-1}}) = Mv^*(X_t^{S_{n-1}})\} \\ \delta_n &= \begin{cases} \varphi_{v^*}(X_{\tau_n}^{S_{n-1}}) & \text{if } \tau_n < \infty \\ 0 & \text{if } \tau_n = \infty. \end{cases} \end{aligned}$$

We then call $S = (\tau_n, \delta_n)_{n\in\mathbb{N}}$ *a* ***qvi-control***.

According to our interpretation of the minimum operator M, for $v = v^*$ this definition merely states that when following a qvi-control we intervene whenever our process hits a state where it is indeed optimal to intervene and we then control by a best possible jump size. Thus, we are locally behaving in an optimal way and therefore, given some growth assumptions on v^*, we will expect a qvi-control to be (globally) optimal. Even more, it will turn out that given the existence of a smooth solution v^* of the qvi, we have $v = v^*$.

Let D denote the set of all states, where we intervene when following a qvi-strategy, i.e. set

$$D = \{x \in \mathbb{R}^d : v^*(x) = Mv^*(x)\}.$$

Note that this **intervention region** D is a closed subset of $\mathbb{R}^d$, since by the continuity of v^* and Mv^*, its complement D^c is open in $\mathbb{R}^d$.

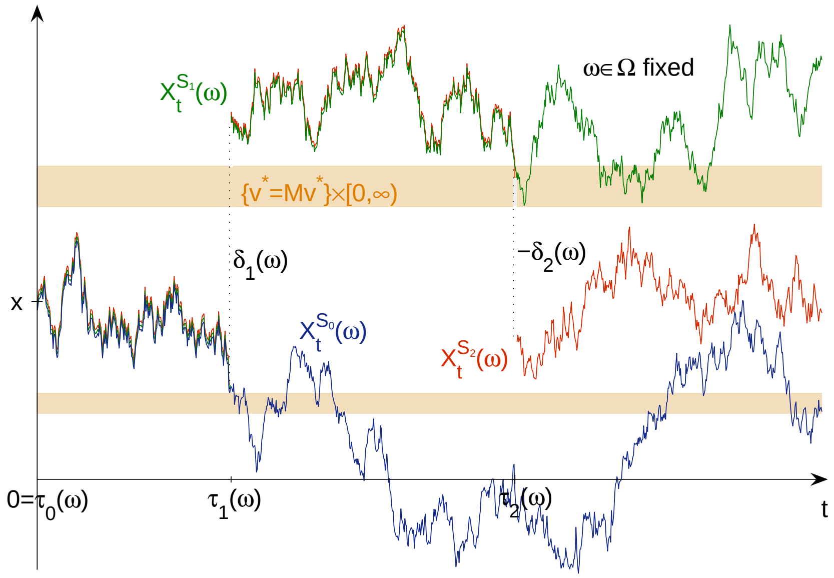

Figure 4: Example for a qvi-control.

The following lemma summarizes some basic properties of qvi-controls, which are immediate consequences of Lemma 1.10. Note that for the proof of this lemma we only need that v^* is a continuous function which is bounded from below.

Lemma 1.13. *Let v^* be a solution of the quasi-variational inequalities and let $S = (\tau_n, \delta_n)_{n\in\mathbb{N}}$ be the corresponding qvi-control. Then*

(i) S is an impulse control strategy.

(ii) For $n \geq 1$ we have $\tau_n < \tau_{n+1}$ whenever $\tau_n < \infty$.

Proof. (i) Let $n \geq 1$. By the continuity of $X^{S_{n-1}}$ on $[\tau_{n-1}, \infty)$, the continuity of v^* and Lemma 1.10 we get that $v^*(X^{S_{n-1}})$ and $Mv^*(X^{S_{n-1}})$ are continuous on $[\tau_{n-1}, \infty)$, which implies that τ_n is a $(\mathcal{F}_t)_{t\geq 0}$ stopping time. Furthermore, Lemma 1.10 implies the existence of $\mathcal{F}_{\tau_n}$-measurable δ_n.

(ii) This is a consequence of Lemma 1.10 as well as the subadditivity of c. Let $n \geq 1$ and suppose that $\tau_n < \infty$. By the definition of S_n, δ_n and by Lemma 1.10, we have

$$\begin{aligned} v^*(X_{\tau_n}^{S_n}) + C(\delta_n) &= v^*(X_{\tau_n}^{S_{n-1}} + \delta_n) + C(\delta_n) \\ &= v^*(X_{\tau_n}^{S_{n-1}} + \varphi_{v^*}(X_{\tau_n}^{S_{n-1}})) + C(\varphi_{v^*}(X_{\tau_n}^{S_{n-1}})) \\ &= Mv^*(X_{\tau_n}^{S_{n-1}}). \end{aligned}$$

Thus, for each $y \in \mathbb{R}^d$ we have

$$\begin{aligned} v^*(X_{\tau_n}^{S_n}) &= Mv^*(X_{\tau_n}^{S_{n-1}}) - C(\delta_n) \\ &\leq v^*(X_{\tau_n}^{S_{n-1}} + \delta_n + y) + C(\delta_n + y) - C(\delta_n) \\ &\leq v^*(X_{\tau_n}^{S_n} + y) + C(y) - K. \end{aligned}$$

Taking the infimum over $y \in \mathbb{R}^d$ yields

$$v^*(X_{\tau_n}^{S_n}) \leq Mv^*(X_{\tau_n}^{S_n}) - K < Mv^*(X_{\tau_n}^{S_n})$$

and the assertion follows by the definition of τ_{n+1}.

□

The following theorem is a justification for considering qvi and qvi-controls. It states that *given* the assumptions (3) and (4) as well as the existence of a smooth solution of the qvi, the value function is characterized by this solution. Moreover, in this setting it verifies that the qvi-control is optimal, in particular admissible, for our underlying impulse control problem.

Theorem 1.14 (Verification theorem). *Assume that there exists a solution $v^* \in C^2$ of the quasi-variational inequalities for the impulse control problem (1) such that*

$$E_x\Big(\int_0^T |e^{-\alpha s}\partial_x v^*(X_s^S)\sigma(X_s^S)|^2 ds\Big) < \infty \quad \forall\, T > 0 \tag{3}$$

$$\liminf_{T\to\infty} E_x(e^{-\alpha T} v^*(X_T^S)) = 0 \tag{4}$$

for all $S \in \mathcal{A}$. Then, the qvi-control to v^* is an optimal impulse control strategy and $v(x) = v^*(x)$.

Assumption (3) yields that the stochastic integral $(\int_0^t e^{-\alpha s}\partial_x v^*(X_{s-}^S)\sigma(X_{s-}^S)dB_s)_{t\geq 0}$ is a continuous martingale. Intuitively, the transversality condition (4) rules out those impulse control strategies which involve accumulating debt. More precisely the remaining costs at time T have to grow slower than α such that their current value is pushed to zero.
We will prove this theorem in three steps. At first we show that $v^*(x) \leq v(x)$. Then, under the assumption that the qvi-control to v^* is admissible, we derive that it is optimal and $v^*(x) = v(x)$. Finally we prove that the qvi-control is indeed admissible. In step one it is advantageous to apply a suitable version of Itô's formula (see P. Protter [24, Sec. II.7.]) directly to the controlled processes X^S instead of first applying Itô's formula repeatedly to the restrictions $X_{|(\tau_k,\tau_{k+1})}$ for k smaller than some natural number n, adding the jumps and afterwards taking the limit in n.

Proof. Let $S = (\tau_n, \delta_n)_{n\in\mathbb{N}} \in \mathcal{A}$ be an admissible impulse control strategy such that $\tau_n < \tau_{n+1}$ whenever τ_n is finite and $n \geq 1$ (see Remark 1.8). By Itô's formula applied to the $C^{1,2}$ function $(t,x) \mapsto e^{-\alpha t} v^*(x)$ and the càdlàg semimartingale X^S, for $T > 0$ we obtain

$$\begin{aligned}
&e^{-\alpha T} v^*(X_T^S) - v^*(X_0^S) - \sum_{0<s\leq T} e^{-\alpha s}(v^*(X_s^S) - v^*(X_{s-}^S)) \\
&\quad = \int_0^T -\alpha e^{-\alpha s} v^*(X_{s-}^S) ds + \sum_{k=1}^d \sum_{l=1}^m \int_0^T e^{-\alpha s} \partial_{x_k} v^*(X_{s-}^S) \sigma_{k,l}(X_{s-}^S) dB_s^{(l)} \\
&\qquad + \sum_{k=1}^d \int_0^T e^{-\alpha s} \{\partial_{x_k} v^*(X_{s-}^S) b_k(X_{s-}^S) + \frac{1}{2} \sum_{l=1}^d \partial_{x_k,x_l} v^*(X_{s-}^S) a_{k,l}(X_{s-}^S)\} ds.
\end{aligned}$$

Hence, by the definition of L, we have

$$\begin{aligned}
&e^{-\alpha T} v^*(X_T^S) - v^*(X_0^S) - \sum_{0<s\leq T} e^{-\alpha s}(v^*(X_s^S) - v^*(X_{s-}^S)) \\
&\qquad = \int_0^T e^{-\alpha s} \partial_x v^*(X_{s-}^S) \sigma(X_{s-}^S) dB_s + \int_0^T e^{-\alpha s} L v^*(X_{s-}^S) ds.
\end{aligned} \tag{5}$$

Note that since $\tau_n < \tau_{n+1}$ for $\tau_n < \infty$ and $n \geq 1$, the sum on the left hand side of equation (5) is given by

$$1_{\{0<\tau_1\leq T\}} e^{-\alpha \tau_1}(v^*(X_{\tau_1}^S) - v^*(X_{\tau_1-}^S)) + \sum_{i=2}^\infty 1_{\{\tau_i \leq T\}} e^{-\alpha \tau_i}(v^*(X_{\tau_i}^S) - v^*(X_{\tau_i-}^S)).$$

Therefore as

$$-v^*(X_0^S) + 1_{\{0=\tau_1\}}(v^*(X_0^S) - v^*(X_{0-}^S)) = -v^*(X_{0-}^S),$$

by equation (5) we have

$$\begin{aligned}
&e^{-\alpha T} v^*(X_T^S) - v^*(X_{0-}^S) - \sum_{i=1}^\infty 1_{\{\tau_i \leq T\}} e^{-\alpha \tau_i}(v^*(X_{\tau_i}^S) - v^*(X_{\tau_i-}^S)) \\
&\qquad = \int_0^T e^{-\alpha s} \partial_x v^*(X_{s-}^S) \sigma(X_{s-}^S) dB_s + \int_0^T e^{-\alpha s} L v^*(X_{s-}^S) ds.
\end{aligned}$$

Since v^* is a solution of the qvi it holds that

$$-Lv^* \leq f$$

as well as

$$v^*(X_{\tau_i-}^S) \leq M v^*(X_{\tau_i-}^S) \leq v^*(X_{\tau_i-}^S + \delta_i) + C(\delta_i) = v^*(X_{\tau_i}^S) + C(\delta_i),$$

whenever $\tau_i < \infty$ and $i \geq 1$, where we again use that the stopping times are strictly increasing. Consequently we have

$$v^*(X^S_{0-}) - e^{-\alpha T} v^*(X^S_T) \leq \int_0^T e^{-\alpha s} f(X^S_s) ds + \sum_{i=1}^{\infty} 1_{\{\tau_i \leq T\}} e^{-\alpha \tau_i} C(\delta_i) - \int_0^T e^{-\alpha s} \partial_x v^*(X^S_{s-}) \sigma(X^S_{s-}) dB_s. \tag{6}$$

Now, by assumption (3), taking expectations in (6) yields

$$v^*(x) - E_x(e^{-\alpha T} v^*(X^S_T)) \leq E_x(\int_0^T e^{-\alpha s} f(X^S_s) ds + \sum_{i=1}^{\infty} 1_{\{\tau_i \leq T\}} e^{-\alpha \tau_i} C(\delta_i)).$$

Finally, by monotone convergence and assumption (4) we have

$$v^*(x) \leq E_x(\int_0^{\infty} e^{-\alpha s} f(X^S_s) ds + \sum_{i=1}^{\infty} 1_{\{\tau_i < \infty\}} e^{-\alpha \tau_i} C(\delta_i)).$$

Thus we have

$$v^*(x) \leq v(x).$$

Moreover, if the qvi-control S to v^* is admissible, then by Lemma 1.13 and since v^* is a solution of the qvi equality holds in (6), which yields that $v^*(x) = v(x)$ and S is optimal. It remains to show that the qvi-control $S = (\tau_n, \delta_n)_{n \in \mathbb{N}}$ to v^* is admissible. Let $n \geq 1$ and define an admissible impulse control strategy by

$$S_n = ((\tau_0, \delta_0), \ldots, (\tau_n, \delta_n)).$$

Let $T > 0$ and set $\hat{\tau}_n = \tau_n \wedge T$. An application of Itô's formula to the càdlàg semimartingale X^{S_n} yields

$$e^{-\alpha \hat{\tau}_n} v^*(X^{S_n}_{\hat{\tau}_n}) - v^*(X^{S_n}_0) - \sum_{0 < s \leq \hat{\tau}_n} e^{-\alpha s}(v^*(X^{S_n}_s) - v^*(X^{S_n}_{s-})) = \int_0^{\hat{\tau}_n} e^{-\alpha s} \partial_x v^*(X^{S_n}_{s-}) \sigma(X^{S_n}_{s-}) dB_s + \int_0^{\hat{\tau}_n} e^{-\alpha s} L v^*(X^{S_n}_{s-}) ds.$$

By Lemma 1.13, for $i \geq 1$ we have $\tau_i < \tau_{i+1}$ whenever $\tau_i < \infty$ and therefore

$$\sum_{0 < s \leq \hat{\tau}_n} e^{-\alpha s}(v^*(X^{S_n}_s) - v^*(X^{S_n}_{s-})) = 1_{\{0 < \tau_1 \leq T\}} e^{-\alpha \tau_1}(v^*(X^{S_n}_{\tau_1}) - v^*(X^{S_n}_{\tau_1 -})) + \sum_{i=2}^{n} 1_{\{\tau_i \leq T\}} e^{-\alpha \tau_i}(v^*(X^{S_n}_{\tau_i}) - v^*(X^{S_n}_{\tau_i -})).$$

Thus we have

$$
\begin{aligned}
e^{-\alpha\hat{\tau}_n}v^*(X^{S_n}_{\hat{\tau}_n}) - v^*(X^{S_n}_{0-}) - \sum_{i=1}^{n} 1_{\{\tau_i\le T\}}e^{-\alpha\tau_i}(v^*(X^{S_n}_{\tau_i}) - v^*(X^{S_n}_{\tau_i-}))& \\
= \int\limits_0^{\hat{\tau}_n} e^{-\alpha s}\partial_x v^*(X^{S_n}_{s-})\sigma(X^{S_n}_{s-})dB_s + \int\limits_0^{\hat{\tau}_n} e^{-\alpha s}Lv^*(X^{S_n}_{s-})ds.&
\end{aligned}
$$

By the definition of S_n we have

$$
v^*(X^{S_n}_{\tau_i-}) = v^*(X^{S_n}_{\tau_i}) + C(\delta_i)
$$

whenever $\tau_i < \infty$ and $1 \le i \le n$. Thus, since v^* is a solution of the quasi-variational inequalities and $\hat{\tau}_n \le \tau_n$ we get

$$
\begin{aligned}
v^*(X^{S_n}_{0-}) - e^{-\alpha\hat{\tau}_n}v^*(X^{S_n}_{\hat{\tau}_n}) = \int\limits_0^{\hat{\tau}_n} e^{-\alpha s}f(X^{S_n}_s)ds + \sum_{i=1}^{n} 1_{\{\tau_i\le T\}}e^{-\alpha\tau_i}C(\delta_i)& \\
- \int\limits_0^{\hat{\tau}_n} e^{-\alpha s}\partial_x v^*(X^{S_n}_{s-})\sigma(X^{S_n}_{s-})dB_s.&
\end{aligned}
\tag{7}
$$

By assumption (3), the stopped process $(\int\limits_0^{t\wedge T} e^{-\alpha s}\partial_x v^*(X^{S_n}_{s-})\sigma(X^{S_n}_{s-})dB_s)_{t\ge 0}$ is uniformly integrable. Thus taking expectations in (7) yields

$$
v^*(x) = E_x(e^{-\alpha\hat{\tau}_n}v^*(X^{S_n}_{\hat{\tau}_n}) + \int\limits_0^{\hat{\tau}_n} e^{-\alpha s}f(X^{S_n}_s)ds + \sum_{i=1}^{n} 1_{\{\tau_i\le T\}}e^{-\alpha\tau_i}C(\delta_i)).
$$

Now since v^*, f and C are nonnegative, by monotone convergence and the finiteness of $v^*(x)$ we obtain

$$
E_x(\sum_{i=1}^{\infty} 1_{\{\tau_i<\infty\}}e^{-\alpha\tau_i}C(\delta_i)) < \infty
$$

and therefore

$$
K\sum_{i=1}^{\infty} 1_{\{\tau_i<\infty\}}e^{-\alpha\tau_i} \le \sum_{i=1}^{\infty} 1_{\{\tau_i<\infty\}}e^{-\alpha\tau_i}C(\delta_i) < \infty \quad \text{a.s.}
$$

Hence the qvi-control is admissible. □

As mentioned above, we may circumvent the assumption that the function c is subadditive. However, the price to pay is the loss of the strict monotonicity of the intervention times of an optimal respectively qvi-control. The proof above can of course be generalized to this situation by merely thinning out the sequence of stopping times to make them strictly increasing, again. This will be exhibited in detail in the proof of Theorem 1.34, where we do *not* have strictly increasing optimal respectively qvi intervention times even though we are still tacitly assuming that c is subadditive.

In particular, under the assumptions of Theorem 1.14, we have proved a variant of the Bellman principle, which we used in the heuristic derivation of the qvi. More generally, we have

Corollary 1.15 (Bellman principle)**.** *Assume that the value function $v \in C^2$ is a solution of the quasi-variational inequalities. Let the qvi-control $S = (\tau_n, \delta_n)_{n\in\mathbb{N}}$ to v be admissible and assume that*

$$E_x(\int_0^t |e^{-\alpha s}\partial_x v(X_s)\sigma(X_s)|^2 ds) < \infty,$$

where X denotes the corresponding controlled process and $t > 0$. Then

$$v(x) = E_x(\int_0^t e^{-\alpha s} f(X_s)ds + \sum_{i=1}^{\infty} 1_{\{\tau_i \le t\}} e^{-\alpha\tau_i} C(\delta_i) + e^{-\alpha t} v(X_t)).$$

Proof. This is an immediate consequence of the proof of Theorem 1.14, since equality holds in (6), where $v(x)$ is equal to $v^*(x)$. □

Summarizing the assumptions of Theorem 1.14 that were used to establish the admissibility of the qvi-control, we state

Corollary 1.16. *Let $v^* \in C^2$ be a solution of the quasi-variational inequalities and let $S = (\tau_n, \delta_n)_{n\in\mathbb{N}}$ be the corresponding qvi-control. For each $n \in \mathbb{N}$ set $S_n = ((\tau_0, \delta_0), \ldots, (\tau_n, \delta_n))$ and suppose that*

$$E_x(\int_0^\infty |e^{-\alpha s}\partial_x v^*(X_s^{S_n})\sigma(X_s^{S_n})|^2) ds < \infty.$$

Then S is admissible.

Proof. Again, this follows from the proof of Theorem 1.14. □

Remark 1.17 (Generalized Itô formula and verification theorem)**.** *Let Z be a continuous semimartingale and let $h : \mathbb{R} \to \mathbb{R}$ be a function whose derivative is absolutely continuous. Let $h_1(x) = h(0) + xh'(0) + \int_0^x \int_0^y (h''(z))^+ dzdy$ and let $h_2(x) = \int_0^x \int_0^y (h''(z))^- dzdy$. These functions are convex and we have $h = h_1 - h_2$. By the Meyer-Tanaka formula we obtain the Itô formula*

$$h(Z_t) - h(Z_0) = \int_0^t h'(Z_s)dZ_s + \frac{1}{2}\int_0^t h''(Z_s)d\langle Z\rangle_s.$$

Thus for $d = 1$ we may weaken the assumption $v^* \in C^2$ of Theorem 1.14 by only requiring that the first derivative of v^* is absolutely continuous. In particular, for $d = 1$ we may replace the assumption $v^* \in C^2$ by $v^* \in C^1$ and v^* twice continuously differentiable up to a finite number of points.

For latter reference, we state a simple generalization of Itô's formula for a d-dimensional continuous semimartingale Z and a function $x \mapsto h(|x|)$, where h is a function such that its first derivative is absolutely continuous. As we concatenate h with $x \mapsto |x|$, the process becomes one-dimensional and we may apply the above remark. The following proposition makes this precise.

Proposition 1.18. *Let $Z = (Z^{(1)}, \dots, Z^{(d)})^t$ be a continuous semimartingale such that 0 is nonattainable, i.e. $P(Z_t = 0$ for some $t \geq 0) = 0$. Let $h : \mathbb{R}_0^+ \to \mathbb{R}$ be a function whose derivative is absolutely continuous and suppose that $g : \mathbb{R}^d \to \mathbb{R}$ is given by $g(x) = h(|x|)$. Then we have*

$$g(Z_t) - g(Z_0) = \sum_{i=1}^{d} \int_0^t \partial_{x_i} g(Z_s) dZ_s^{(i)} + \frac{1}{2} \sum_{1 \leq i,j \leq d} \int_0^t \partial_{x_i,x_j} g(Z_s) d\langle Z^{(i)}, Z^{(j)} \rangle_s.$$

Proof. Since 0 is nonattainable, an application of Itô's formula for $x \mapsto |x|$ and the continuous semimartingale Z yields.

$$|Z_t| - |Z_0| = \sum_{i=1}^{d} \int_0^t \partial_{x_i} |Z_s| dZ_s^{(i)} + \frac{1}{2} \sum_{1 \leq i,j \leq d} \int_0^t \partial_{x_i,x_j} |Z_s| d\langle Z^{(i)}, Z^{(j)} \rangle_s.$$

By Remark 1.17 we may apply Itô's formula for h and the continuous semimartingale $|Z|$

$$\begin{aligned}
h(|Z_t|) - h(|Z_0|) &= \int_0^t h'(|Z_s|) d|Z_s| + \frac{1}{2} \int_0^t h''(|Z_s|) d\langle |Z| \rangle_s \\
&= \sum_{i=1}^{d} \int_0^t h'(|Z_s|) \partial_{x_i} |Z_s| dZ_s^{(i)} \\
&\quad + \frac{1}{2} \sum_{1 \leq i,j \leq d} \int_0^t h'(|Z_s|) \partial_{x_i,x_j} |Z_s| + h''(|Z_s|) \partial_{x_i} |Z_s| \partial_{x_j} |Z_s| d\langle Z^{(i)}, Z^{(j)} \rangle_s \\
&= \sum_{i=1}^{d} \int_0^t \partial_{x_i} g(Z_s) dZ_s^{(i)} + \frac{1}{2} \sum_{1 \leq i,j \leq d} \int_0^t \partial_{x_i,x_j} g(Z_s) d\langle Z^{(i)}, Z^{(j)} \rangle_s.
\end{aligned}$$

□

The following lemma states a sufficient condition on σ such that 0 is nonattainable for the uncontrolled process X^{S_0}. Note that b and σ are Lipschitz.

Lemma 1.19. *Suppose that rank$(a(0)) \geq 2$. Then, by Friedman [9, Theorem 4.1] the uncontrolled process satisfies*

$$P_x(X_t^{S_0} = 0 \text{ for some } t > 0) = 0 \quad \text{for any } x \neq 0.$$

By Proposition 1.18 and Lemma 1.19 we obtain the following generalization of the verification theorem for radial symmetric functions.

Remark 1.20 (Generalized verification theorem)**.** *Let $h : \mathbb{R}_0^+ \to \mathbb{R}$ be a function whose derivative is absolutely continuous and suppose that the solution of the quasi-variational inequalities is given by $v^*(x) = h(|x|)$. If rank$(a(0)) \geq 2$ then the assertion of Theorem (1.14) holds true. In particular, the radial symmetric function v^* does not need to be twice differentiable on a finite number of balls centered at the origin.*

Next, we will consider two impulse control problems, where we are only allowed to intervene for at most n times. In both cases, the corresponding value functions can be obtained by iteratively solving variational inequalities. Under some natural restrictions we will establish convergence of the associated sequences of value functions for the problems with at most n interventions to the value function v of our original impulse control problem.

Let $n \in \mathbb{N}$ and consider the impulse control problem (1), with the restriction that this time we are only allowed to intervene for at most n times:

$$v_n(x) = \inf_{S=(\tau_k,\delta_k)_{0\leq k\leq n}\in\mathcal{A}} E_x\Big(\int_0^\infty e^{-\alpha s} f(X_s^S)ds + \sum_{k=1}^n 1_{\{\tau_k<\infty\}} e^{-\alpha\tau_k} C(\delta_k)\Big). \tag{8}$$

The function v_n is called the **value function** for the impulse control problem above. A strategy $S_n = (\tau_k^{(n)}, \delta_k^{(n)})_{0\leq k\leq n} \in \mathcal{A}$ such that the infimum is attained will be called **optimal**.

Remark 1.21. *Due to the strict subadditivity of a controlling cost C, an optimal impulse control strategy $S_n = (\tau_k^{(n)}, \delta_k^{(n)})_{0\leq k\leq n}$ has to satisfy $\tau_k^{(n)} < \tau_{k+1}^{(n)}$ whenever $\tau_k^{(n)} < \infty$, for all $k \geq 1$.*

As before, let us give a brief motivation for the following definition of the variational inequalities for the impulse control problem (8). For $n \geq 1$ let v_n be the value function when we are allowed to intervene for at most n times. Note that Mv_{n-1} represents the value of the strategy that consists of doing the best immediate action when we are allowed to intervene for at most n times and behaving optimally afterwards when at most $n-1$ interventions are possible. Thus, in general we have

$$v_n \leq Mv_{n-1}$$

where equality holds in case that an immediate intervention is optimal. Now let $n \geq 0$, let $x \in \mathbb{R}^d$, assume that there exists an optimal strategy $S_n = (\tau_k^{(n)}, \delta_k^{(n)})_{0\leq k\leq n}$ and let

X be the correspondingly controlled process. If the first optimal intervention time $\tau_1^{(n)}$ is bounded from below by some positive δ, we suppose that

$$v_n(x) = E_x(\int_0^\delta e^{-\alpha s} f(X_s)ds + e^{-\alpha\delta} v_n(X_\delta))$$

and Itô's formula yields that

$$Lv_n(x) + f(x) = 0.$$

If an immediate impulse is optimal, then for $S = (\tau_k, \delta_k)_{0\le k\le n} \in \mathcal{A}$ such that $\tau_1 > \delta$ we expect that

$$v_n(x) < E_x(\int_0^\delta e^{-\alpha s} f(X_s^S)ds + e^{-\alpha\delta} v_n(X_\delta^S))$$

and therefore

$$Lv_n(x) + f(x) > 0.$$

Finally note that v_0 is the cost of the nonintervention strategy $S_0 = (0, 0)$ and therefore, by the above, we have

$$Lv_0 + f \equiv 0.$$

Definition 1.22. *Let $1 \le n \in \mathbb{N}$ and let $v_n^* : \mathbb{R}^d \to \mathbb{R}_0^+$. We may say that v_n^* solves the* ***variational inequalities*** *(abbreviated* $\boldsymbol{vi_n}$*) corresponding to the impulse control problem (8), if we have $v_n^* \le v_{n-1}^*$ and*

- ◇ $Lv_n^* + f \ge 0$
- ◇ $v_n^* \le Mv_{n-1}^*$
- ◇ $(v_n^* - Mv_{n-1}^*)(Lv_n^* + f) = 0,$

where v_{n-1}^* solves vi_{n-1} and $v_0^* : \mathbb{R}^d \to \mathbb{R}_0^+$ solves $Lv_0^* + f = 0$ (for short: v_0^* solves $\boldsymbol{vi_0}$*).*

Observe that the right hand side of the inequality $v_n^* \le Mv_{n-1}^*$ is explicitly known at step n of the iteration and therefore we only have to deal with variational inequalities. The assumptions $v_n^* \ge 0$ and $v_n^* \le v_{n-1}^*$ in the definition above correspond to the properties $v_n \ge 0$ and $v_n \le v_{n-1}$ of the value function v_n, where the former implies that Mv_n^* is welldefined. The following definition gives rise to an optimal impulse control strategy for problem (8), constructed with the help of solutions of the vi_k.

Definition 1.23. *Let $n \in \mathbb{N}$ and set $(\tau_0^{(n)}, \delta_0^{(n)}) \equiv (0, 0)$. Let v_k^* be a solution of the variational inequalities vi_k, for each $0 \le k \le n$. Then, for $1 \le k \le n$ set*

$$S_{k-1}^{(n)} = ((\tau_0^{(n)}, \delta_0^{(n)}), \ldots, (\tau_{k-1}^{(n)}, \delta_{k-1}^{(n)}))$$
$$\tau_k^{(n)} = \inf\{t \ge \tau_{k-1}^{(n)} : v_{n-(k-1)}^*(X_t^{S_{k-1}^{(n)}}) = Mv_{n-k}^*(X_t^{S_{k-1}^{(n)}})\}$$
$$\delta_k^{(n)} = \begin{cases} \varphi_{v_{n-k}^*}(X_{\tau_k^{(n)}}^{S_{k-1}^{(n)}}) & \text{if } \tau_k^{(n)} < \infty \\ 0 & \text{if } \tau_k^{(n)} = \infty. \end{cases}$$

We then call $S_n = (\tau_k^{(n)}, \delta_k^{(n)})_{0\le k\le n}$ *a* $\boldsymbol{vi_n}$***-control***.

For $l \ge 1$ let D_l denote the set of all points, where it is optimal to intervene for the first time when we are allowed to intervene for at most l times, i.e. set

$$D_l = \{x \in \mathbb{R}^d : v_l^*(x) = Mv_{l-1}^*(x)\}.$$

These **intervention regions** D_l are closed subsets of $\mathbb{R}^d$, since by the continuity of v_l^* and Mv_{l-1}^*, their complements D_l^c are open in $\mathbb{R}^d$. By Lemma 1.10 and the monotonicity of

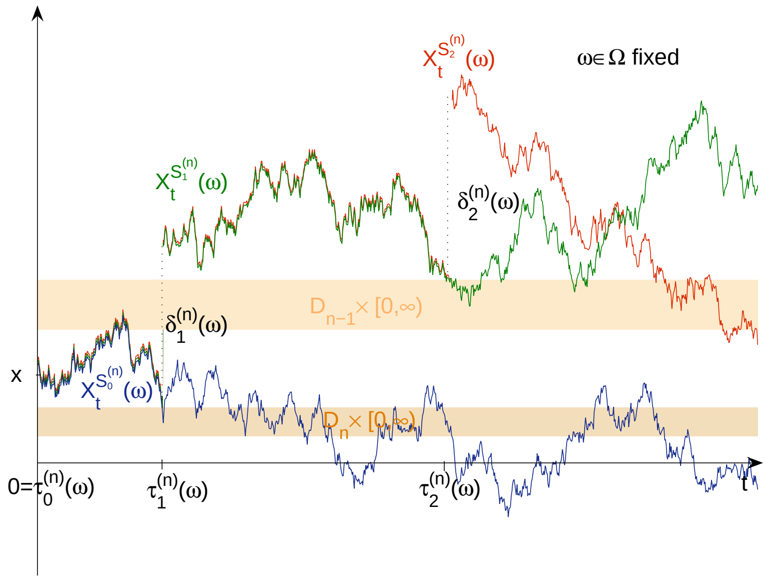

Figure 5: Example for a vi_n-strategy.

the sequence $(v_l^*)_{l\in\mathbb{N}}$ we get

Lemma 1.24. *Let* $n \in \mathbb{N}$ *and for each* $0 \le k \le n$ *let* v_k^* *be a solution of the variational inequalities* vi_k*. Let* $S_n = (\tau_k^{(n)}, \delta_k^{(n)})_{0\le k\le n}$ *be the associated* vi_n*-control, then*

(i) S_n *is an impulse control strategy.*

(ii) For $1 \le k \le n-1$ *we have* $\tau_k^{(n)} < \tau_{k+1}^{(n)}$ *whenever* $\tau_k^{(n)} < \infty$.

Proof. (i) As before, this is an immediate consequence of Lemma 1.10.

(ii) Let $1 \le k \le n-1$ and suppose that $\tau_k^{(n)} < \infty$. By the definition of the vi_n-control S_n, $\delta_k^{(n)}$ and by Lemma 1.10, we have

$$\begin{aligned} v_{n-k}^*(X_{\tau_k^{(n)}}^{S_k^{(n)}}) + C(\delta_k^{(n)}) &= v_{n-k}^*(X_{\tau_k^{(n)}}^{S_{k-1}^{(n)}} + \delta_k^{(n)}) + C(\delta_k^{(n)}) \\ &= v_{n-k}^*(X_{\tau_k^{(n)}}^{S_{k-1}^{(n)}} + \varphi_{v_{n-k}^*}(X_{\tau_k^{(n)}}^{S_{k-1}^{(n)}})) + C(\varphi_{v_{n-k}^*}(X_{\tau_k^{(n)}}^{S_{k-1}^{(n)}})) \\ &= Mv_{n-k}^*(X_{\tau_k^{(n)}}^{S_{k-1}^{(n)}}). \end{aligned}$$

Thus, for each $y \in \mathbb{R}^d$ we have

$$\begin{aligned} v^*_{n-k}(X^{S^{(n)}_k}_{\tau^{(n)}_k}) &= Mv^*_{n-k}(X^{S^{(n)}_{k-1}}_{\tau^{(n)}_k}) - C(\delta^{(n)}_k) \\ &\le v^*_{n-k}(X^{S^{(n)}_{k-1}}_{\tau^{(n)}_k} + \delta^{(n)}_k + y) + C(\delta^{(n)}_k + y) - C(\delta^{(n)}_k) \\ &\le v^*_{n-k}(X^{S^{(n)}_k}_{\tau^{(n)}_k} + y) + C(y) - K \end{aligned}$$

where the last inequality is due to the subadditivity of c. Now, taking the infimum over $y \in \mathbb{R}^d$ and using the monotonicity of the sequence $(v^*_l)_l$, we obtain

$$\begin{aligned} v^*_{n-k}(X^{S^{(n)}_k}_{\tau^{(n)}_k}) &\le Mv^*_{n-k}(X^{S^{(n)}_k}_{\tau^{(n)}_k}) - K \\ &\le Mv^*_{n-(k+1)}(X^{S^{(n)}_k}_{\tau^{(n)}_k}) - K. \end{aligned}$$

Thus, by the definition of $\tau^{(n)}_{k+1}$ we have

$$\tau^{(n)}_k < \tau^{(n)}_{k+1}.$$

$\square$

Before verifying the optimality of the vi-control, we will strengthen the strong Markov property stated in Theorem 1.4. The following theorem states that after the last intervention time we may not only disregard the information on the past life of a finite controlled process except for its present state, but also are allowed to shift the process back to starting time zero. Recall that we are always assuming that b and σ are Lipschitz.

Theorem 1.25 (Strong Markov property involving time shift)**.** *Let $x \in \mathbb{R}^d$, let $S_n = (\tau_k, \delta_k)_{0\le k\le n}$ be a finite impulse control strategy and let X^{S_n} denote the corresponding controlled process with starting point x. Let $h : \mathcal{W}^d \to \mathbb{R}$ be bounded and measurable. Then we have*

$$E(h(X^{S_n}_{\tau_n+\cdot})|\mathcal{F}_{\tau_n}) = E(h(X^{S_0,y}))|_{y=X^{S_n}_{\tau_n}} \quad a.s.\ on\ \{\tau_n < \infty\}$$

where $X^{S_0,y}$ denotes the uncontrolled process X^{S_0} with $X^{S_0}_0 = y$ and we set $X^{S_n}_\infty \equiv 0$.

Note that since $\{\tau_n < \infty\} \in \mathcal{F}_{\tau_n}$ the assertion is equivalent to

$$E(h(X^{S_n}_{\tau_n+\cdot})1_{\{\tau_n<\infty\}}|\mathcal{F}_{\tau_n}) = E(h(X^{S_0,y}))|_{y=X^{S_n}_{\tau_n}} 1_{\{\tau_n<\infty\}}$$

and consequently does not depend on our convention for $X^{S_n}_\infty$.

Proof. [1] Let W be the canonical m-dimensional Brownian motion on the Wiener space $\mathcal{W}^m$, i.e. let

$$W_t = \pi_t : (\mathcal{W}^m, (\mathcal{G}_t)_{t\ge0}, \nu) \to \mathbb{R}^m, \quad t \ge 0$$

[1] Thanks to Frank Seifried for pointing out the relevance of this result and for the main idea of its proof.

where ν is the Wiener measure on $\mathcal{W}^m$ and $(\mathcal{G}_t)_{t\geq 0}$ denotes the standard extension of the natural filtration $(\sigma(W_s, s \leq t))_{t\geq 0}$. By Hackenbroch-Thalmaier [10, Satz 6.25] there exists a measurable function

$$\phi : \mathbb{R}^d \times \mathcal{W}^m \to \mathcal{W}^d$$

which is continuous in $z \in \mathbb{R}^d$ and solves

$$\phi(z,\cdot)(t) = z + \int_0^t b(\phi(z,\cdot)(s))ds + \int_0^t \sigma(\phi(z,\cdot)(s))dW_s.$$

For $t \geq 0$ let $\tilde{\mathcal{F}}_t = \mathcal{F}_{\tau_n+t}$ and note that $(\tilde{\mathcal{F}}_t)_{t\geq 0}$ is again a standard filtration. Set $\tilde{B}_t = B_{\tau_n+t} - B_{\tau_n}$ with the convention $B_\infty \equiv 0$. The time shifted process $X^{S_n}_{\tau_n+\cdot}$ is the $(\tilde{\mathcal{F}}_t)_{t\geq 0}$-adapted solution of

$$X^{S_n}_{\tau_n+t} = X^{S_n}_{\tau_n} + \int_0^t b(X^{S_n}_{\tau_n+s})ds + \int_0^t \sigma(X^{S_n}_{\tau_n+s})d\tilde{B}_s$$

on $\{\tau_n < \infty\}$. Therefore, by a result of Blagoveščensky-Freidlin (see Hackenbroch-Thalmaier [10, Satz 6.26], we have

$$X^{S_n}_{\tau_n+\cdot}1_{\{\tau_n<\infty\}} = \phi(X^{S_n}_{\tau_n}, \tilde{B})1_{\{\tau_n<\infty\}}.$$

Thus, since $\{\tau_n < \infty\} \in \mathcal{F}_{\tau_n}$ we find

$$\begin{aligned}
E(h(X^{S_n}_{\tau_n+\cdot})|\mathcal{F}_{\tau_n})1_{\{\tau_n<\infty\}} &= E(h(\phi(X^{S_n}_{\tau_n}, \tilde{B}))|\mathcal{F}_{\tau_n})1_{\{\tau_n<\infty\}} \\
&= E(h(\phi(y,\tilde{B})))|_{y=X^{S_n}_{\tau_n}}1_{\{\tau_n<\infty\}} \\
&= \int_{\mathcal{W}^m} h(\phi(y,\omega))\nu(d\omega)|_{y=X^{S_n}_{\tau_n}}1_{\{\tau_n<\infty\}} \\
&= E(h(\phi(y,B)))|_{y=X^{S_n}_{\tau_n}}1_{\{\tau_n<\infty\}}
\end{aligned}$$

where the second equality follows by a monotone class argument as $X^{S_n}_{\tau_n}$ is $\mathcal{F}_{\tau_n}$-measurable and since $\tilde{B}$ is independent of $\mathcal{F}_{\tau_n}$. Again, by Blagoveščensky-Freidlin we have

$$\phi(y,B) = X^{S_0,y}$$

and therefore, we end up with

$$E(h(X^{S_n}_{\tau_n+\cdot})|\mathcal{F}_{\tau_n}) = E(h(X^{S_0,y}))|_{y=X^{S_n}_{\tau_n}}$$

almost surely on $\{\tau_n < \infty\}$. □

Clearly, by monotone convergence the above assertion holds also true for nonnegative measurable functions h.

Corollary 1.26. *Under the assumptions of the theorem above we have*

$$E(\int_{\tau_n}^{\infty} e^{-\alpha s} f(X_s^{S_n})ds) = E(1_{\{\tau_n<\infty\}} e^{-\alpha\tau_n} v_0(X_{\tau_n}^{S_n})).$$

Proof. Since τ_n is $\mathcal{F}_{\tau_n}$-measurable, an application of Theorem 1.25 yields

$$\begin{aligned} E(\int_{\tau_n}^{\infty} e^{-\alpha s} f(X_s^{S_n})ds) &= E(1_{\{\tau_n<\infty\}} e^{-\alpha\tau_n} E(\int_0^{\infty} e^{-\alpha s} f(X_{\tau_n+s}^{S_n})ds|\mathcal{F}_{\tau_n})) \\ &= E(1_{\{\tau_n<\infty\}} e^{-\alpha\tau_n} E(\int_0^{\infty} e^{-\alpha s} f(X_s^{S_0,y})ds)|_{y=X_{\tau_n}^{S_n}}) \\ &= E(1_{\{\tau_n<\infty\}} e^{-\alpha\tau_n} v_0(X_{\tau_n}^{S_n})). \end{aligned}$$

□

Let us now verify the optimality of the vi-control.

Theorem 1.27 (Verification theorem)**.** *Let $n \in \mathbb{N}$ and assume that for $0 \le k \le n$ there exist solutions $v_k^* \in C^2$ of the variational inequalities vi_k for the impulse control problem (8) such that*

$$E_x(\int_0^T |e^{-\alpha s}\partial_x v_k^*(X_s^S)\sigma(X_s^S)|^2 ds) < \infty \quad \forall\, T > 0 \tag{9}$$

$$\liminf_{T\to\infty} E_x(e^{-\alpha T} v_0^*(X_T^S)) = 0 \tag{10}$$

for all $S = (\tau_k, \delta_k)_{0\le k\le n} \in \mathcal{A}$. Then, the vi_n-control to v_n^* is optimal and $v_n(x) = v_n^*(x)$.

We will first consider the case where $n = 0$ and show that $v_0^*(x) = v_0(x)$. Then we will prove that for $n \ge 1$ we have $v_n^*(x) \le v_n(x)$. Finally we derive that the vi_n-control is optimal and $v_n^*(x) = v_n(x)$.

Proof. Let S_0 be the nonintervention strategy and apply Itô's formula to the $C^{1,2}$ function $(t, x) \mapsto e^{-\alpha t} v_0^*(x)$ and the continuous semimartingale X^{S_0}. For $T > 0$ we obtain

$$e^{-\alpha T} v_0^*(X_T^{S_0}) - v_0^*(X_0^{S_0}) = \int_0^T e^{-\alpha s}\partial_x v_0^*(X_s^{S_0})\sigma(X_s^{S_0})dB_s + \int_0^T e^{-\alpha s} L v_0^*(X_s^{S_0})ds.$$

Thus, since v_0^* solves vi_0 we get

$$e^{-\alpha T} v_0^*(X_T^{S_0}) - v_0^*(X_0^{S_0}) = \int_0^T e^{-\alpha s}\partial_x v_0^*(X_s^{S_0})\sigma(X_s^{S_0})dB_s - \int_0^T e^{-\alpha s} f(X_s^{S_0})ds$$

and by assumption (9), taking expectations yields

$$v_0^*(x) - E_x(e^{-\alpha T} v_0^*(X_T^{S_0})) = E_x(\int_0^T e^{-\alpha s} f(X_s^{S_0}) ds).$$

Finally, by monotone convergence and assumption (10) we have

$$v_0^*(x) = E_x(\int_0^\infty e^{-\alpha s} f(X_s^{S_0}) ds) = v_0(x).$$

Now, let $n \geq 1$ and let $S = (\tau_k, \delta_k)_{0 \leq k \leq n} \in \mathcal{A}$ such that $\tau_k < \tau_{k+1}$ whenever τ_k is finite and $k \geq 1$. Let $T > 0$ and for $0 \leq k \leq n$ set $\hat{\tau}_k = \tau_k \wedge T$. For $0 \leq k \leq n-1$, an application of Itô's formula to the $C^{1,2}$ function $(t, x) \mapsto e^{-\alpha t} v_{n-k}^*(x)$ and the càdlàg semimartingale X^S yields

$$\begin{aligned} e^{-\alpha \hat{\tau}_{k+1}} v_{n-k}^*(X_{\hat{\tau}_{k+1}}^S) - e^{-\alpha \hat{\tau}_k} v_{n-k}^*(X_{\hat{\tau}_k}^S) - \sum_{\hat{\tau}_k < s \leq \hat{\tau}_{k+1}} e^{-\alpha s} (v_{n-k}^*(X_s^S) - v_{n-k}^*(X_{s-}^S)) \\ = \int_{\hat{\tau}_k}^{\hat{\tau}_{k+1}} e^{-\alpha s} \partial_x v_{n-k}^*(X_{s-}^S) \sigma(X_{s-}^S) dB_s + \int_{\hat{\tau}_k}^{\hat{\tau}_{k+1}} e^{-\alpha s} L v_{n-k}^*(X_{s-}^S) ds. \end{aligned}$$

Thus, since

$$-v_n^*(X_0^S) + 1_{\{0=\tau_1\}}(v_n^*(X_0^S) - v_n^*(X_{0-}^S)) = -v_n^*(X_{0-}^S)$$

for $k = 0$ we have

$$\begin{aligned} e^{-\alpha \hat{\tau}_1} v_n^*(X_{\hat{\tau}_1}^S) - v_n^*(X_{0-}^S) - 1_{\{\tau_1 \leq T\}} e^{-\alpha \tau_1} (v_n^*(X_{\tau_1}^S) - v_n^*(X_{\tau_1 -}^S)) \\ = \int_0^{\hat{\tau}_1} e^{-\alpha s} \partial_x v_n^*(X_{s-}^S) \sigma(X_{s-}^S) dB_s + \int_0^{\hat{\tau}_1} e^{-\alpha s} L v_n^*(X_{s-}^S) ds \end{aligned}$$

whereas for $k \geq 1$ we obtain

$$\begin{aligned} e^{-\alpha \hat{\tau}_{k+1}} v_{n-k}^*(X_{\hat{\tau}_{k+1}}^S) - e^{-\alpha \hat{\tau}_k} v_{n-k}^*(X_{\hat{\tau}_k}^S) - 1_{\{\tau_{k+1} \leq T\}} e^{-\alpha \tau_{k+1}} (v_{n-k}^*(X_{\tau_{k+1}}^S) - v_{n-k}^*(X_{\tau_{k+1}-}^S)) \\ = \int_{\hat{\tau}_k}^{\hat{\tau}_{k+1}} e^{-\alpha s} \partial_x v_{n-k}^*(X_{s-}^S) \sigma(X_{s-}^S) dB_s + \int_{\hat{\tau}_k}^{\hat{\tau}_{k+1}} e^{-\alpha s} L v_{n-k}^*(X_{s-}^S) ds. \end{aligned}$$

Since v_{n-k}^* is a solution of the vi_{n-k} it holds that

$$-L v_{n-k}^* \leq f$$

as well as

$$v_{n-k}^*(X_{\tau_{k+1}-}^S) \leq Mv_{n-(k+1)}^*(X_{\tau_{k+1}-}^S) \leq v_{n-(k+1)}^*(X_{\tau_{k+1}-}^S + \delta_{k+1}) + C(\delta_{k+1})$$
$$= v_{n-(k+1)}^*(X_{\tau_{k+1}}^S) + C(\delta_{k+1}),$$

whenever $\tau_{k+1} < \infty$. Thus, for $k = 0$ we have

$$v_n^*(X_{0-}^S) - e^{-\alpha\hat{\tau}_1} v_n^*(X_{\hat{\tau}_1}^S) \leq \int_0^{\hat{\tau}_1} e^{-\alpha s} f(X_s^S) ds - \int_0^{\hat{\tau}_1} e^{-\alpha s} \partial_x v_n^*(X_{s-}^S) \sigma(X_{s-}^S) dB_s$$
$$+ 1_{\{\tau_1 \leq T\}} e^{-\alpha\tau_1} (v_{n-1}^*(X_{\tau_1}^S) - v_n^*(X_{\tau_1}^S) + C(\delta_1))$$

and for $k \geq 1$ we get

$$e^{-\alpha\hat{\tau}_k} v_{n-k}^*(X_{\hat{\tau}_k}^S) - e^{-\alpha\hat{\tau}_{k+1}} v_{n-k}^*(X_{\hat{\tau}_{k+1}}^S)$$
$$\leq \int_{\hat{\tau}_k}^{\hat{\tau}_{k+1}} e^{-\alpha s} f(X_s^S) ds - \int_{\hat{\tau}_k}^{\hat{\tau}_{k+1}} e^{-\alpha s} \partial_x v_{n-k}^*(X_{s-}^S) \sigma(X_{s-}^S) dB_s$$
$$+ 1_{\{\tau_{k+1} \leq T\}} e^{-\alpha\tau_{k+1}} (v_{n-(k+1)}^*(X_{\tau_{k+1}}^S) - v_{n-k}^*(X_{\tau_{k+1}}^S) + C(\delta_{k+1})).$$

Let us rewrite the left hand sides of the inequalities above as

$$v_n^*(X_{0-}^S) - e^{-\alpha\hat{\tau}_1} v_n^*(X_{\hat{\tau}_1}^S) = v_n^*(X_{0-}^S) - 1_{\{\tau_1 \leq T\}} e^{-\alpha\tau_1} v_n^*(X_{\tau_1}^S) - 1_{\{\tau_1 > T\}} e^{-\alpha T} v_n^*(X_T^S)$$

respectively

$$e^{-\alpha\hat{\tau}_k} v_{n-k}^*(X_{\hat{\tau}_k}^S) - e^{-\alpha\hat{\tau}_{k+1}} v_{n-k}^*(X_{\hat{\tau}_{k+1}}^S)$$
$$= 1_{\{\tau_k \leq T\}} e^{-\alpha\tau_k} v_{n-k}^*(X_{\tau_k}^S) - 1_{\{\tau_{k+1} \leq T\}} e^{-\alpha\tau_{k+1}} v_{n-k}^*(X_{\tau_{k+1}}^S)$$
$$- 1_{\{\tau_k \leq T < \tau_{k+1}\}} e^{-\alpha T} v_{n-k}^*(X_T^S).$$

Now, summing up over $0 \leq k \leq n-1$, by assumption (9) and optional stopping we have

$$v_n^*(x) - E_x(1_{\{\tau_n \leq T\}} e^{-\alpha\tau_n} v_0^*(X_{\tau_n}^S) - \sum_{k=1}^{n} 1_{\{\tau_{k-1} \leq T < \tau_k\}} e^{-\alpha T} v_{n-(k-1)}^*(X_T^S)) \tag{11}$$
$$\leq E_x(\int_0^{\hat{\tau}_n} e^{-\alpha s} f(X_s^S) ds + \sum_{k=1}^{n} 1_{\{\tau_k \leq T\}} e^{-\alpha\tau_k} C(\delta_k)).$$

Note that for $1 \leq k \leq n$ by assumption (10) and the monotonicity of $(v_l^*)_l$ we have

$$\liminf_{T\to\infty} E_x(1_{\{\tau_{k-1} \leq T < \tau_k\}} e^{-\alpha T} v_{n-(k-1)}^*(X_T^S)) = 0$$

hence, by monotone convergence we establish

$$\begin{aligned}v_n^*(x) &- E_x(1_{\{\tau_n<\infty\}}e^{-\alpha\tau_n}v_0^*(X^S_{\tau_n}))\\ &\le E_x(\int_0^{\tau_n} e^{-\alpha s}f(X_s^S)ds + \sum_{k=1}^n 1_{\{\tau_k<\infty\}}e^{-\alpha\tau_k}C(\delta_k))\end{aligned}$$

and the strong Markov property (see Corollary 1.26 and recall that we have already shown that $v_0^* = v_0$) implies that

$$v_n^*(x) \le E_x(\int_0^{\infty} e^{-\alpha s}f(X_s^S)ds + \sum_{k=1}^n 1_{\{\tau_k<\infty\}}e^{-\alpha\tau_k}C(\delta_k)).$$

Thus taking the infimum over all $S = (\tau_k, \delta_k)_{0\le k\le n} \in \mathcal{A}$ we have

$$v_n^*(x) \le v_n(x).$$

Finally consider the vi_n-control S_n (recall Lemma 1.24). In this case, equality holds in (11) since v_{n-k}^* solves the vi_{n-k} for $0 \le k \le n-1$ and therefore $v_n^*(x) = v_n(x)$ and S_n is optimal. □

As before, notice that we may circumvent the subadditivity assumption on c which results in the loss of the strict monotonicity of the optimal intervention times and the vi_n stopping times, respectively, by modifying the proof above according to the one of Theorem 1.34.

In particular, we have proved the following variant of the Bellman principle.

Corollary 1.28 (Bellman principle)**.** *Let $n \in \mathbb{N}$ and assume that for $0 \le k \le n$ the value functions v_k are C^2 solutions of the variational inequalities vi_k. Let $S_n = (\tau_k^{(n)}, \delta_k^{(n)})_{0\le k\le n}$ be the vi_n-control and assume that*

$$E_x(\int_0^t |e^{-\alpha s}\partial_x v_k(X_s)\sigma(X_s)|^2 ds) < \infty$$

each $0 \le k \le n$, where X denotes the corresponding controlled process and $t > 0$. Then

$$\begin{aligned}v_n(x) = E_x(&\int_0^{\tau_n^{(n)}\wedge t} e^{-\alpha s}f(X_s)ds + \sum_{k=1}^n 1_{\{\tau_k^{(n)}\le t\}}e^{-\alpha\tau_k^{(n)}}C(\delta_k^{(n)})\\ &+ \sum_{k=0}^n 1_{\{\tau_k^{(n)}\le t<\tau_{k+1}^{(n)}\}}e^{-\alpha(\tau_n^{(n)}\wedge t)}v_{n-k}(X_{\tau_n^{(n)}\wedge t})).\end{aligned}$$

Proof. This follows from the proof of Theorem 1.27, since under the assumptions above equality holds in (11). □

Next, we provide two convergence results which show that the impulse control problem with possibly infinitely many interventions can be approximated by impulse control problems where only finitely many impulses are allowed.

Lemma 1.29 (Convergence from above). *If the nonintervention cost v_0 is bounded, then for each $1 \leq n \in \mathbb{N}$ we have*

$$||v_n - v||_{\sup} \leq \frac{||v_0||^2_{\sup}}{nK}.$$

Proof. Let $x \in \mathbb{R}^d$ and let $1 \leq n \in \mathbb{N}$. Let $S = (\tau_k, \delta_k)_{k \in \mathbb{N}} \in \mathcal{A}$ such that $J^S(x) \leq v_0(x)$ and set $S_n = (\tau_k, \delta_k)_{0 \leq k \leq n}$. By Corollary 1.26 we have

$$\begin{aligned} J^{S_n}(x) - J^S(x) &\leq E_x\Big(\int_{\tau_n}^{\infty} e^{-\alpha s} f(X_s^{S_n}) ds\Big) \\ &= E_x(1_{\{\tau_n < \infty\}} e^{-\alpha \tau_n} v_0(X_{\tau_n}^{S_n})) \\ &\leq ||v_0||_{\sup} E_x(1_{\{\tau_n < \infty\}} e^{-\alpha \tau_n}). \end{aligned}$$

Note that since $J^S(x) \leq v_0(x)$ we have

$$\begin{aligned} nK E_x(1_{\{\tau_n < \infty\}} e^{-\alpha \tau_n}) &\leq E_x\Big(\sum_{k=1}^{n} 1_{\{\tau_k < \infty\}} e^{-\alpha \tau_k} C(\delta_k)\Big) \\ &\leq J^S(x) \leq v_0(x) \leq ||v_0||_{\sup}. \end{aligned}$$

Thus we have

$$J^{S_n}(x) - J^S(x) \leq \frac{||v_0||^2_{\sup}}{nK}$$

and hence

$$v_n(x) - J^S(x) \leq v_n(x) - J^{S_n}(x) + \frac{||v_0||^2_{\sup}}{nK} \leq \frac{||v_0||^2_{\sup}}{nK}.$$

Taking the infimum over all admissible strategies yields

$$0 \leq v_n(x) - v(x) \leq \frac{||v_0||^2_{\sup}}{nK}.$$

□

The above lemma is also a hint to the fact that the cost reduction by using control actions decreases with growing fixed control costs. Even more, this decrease is inversely proportional to the fixed cost component. Given the existence of an optimal strategy, we may rewrite the previous lemma as follows.

Lemma 1.30 (Convergence from above)**.** *Let κ be a compact subset of $\mathbb{R}^d$ and suppose that for each $x \in \kappa$ there exists an optimal strategy $S = (\tau_k, \delta_k)_{k\in\mathbb{N}}$ corresponding to problem (1) such that $(X^S_{\tau_k})_{k\geq 1} \subseteq \kappa$. Then we have*

$$||v_n - v||_{\sup(\kappa)} \leq \frac{||v_0||^2_{\sup(\kappa)}}{nK}, \quad \forall\, 1 \leq n \in \mathbb{N}.$$

Proof. Let $x \in \kappa$ and let $S = (\tau_k, \delta_k)_{k\in\mathbb{N}}$ be an admissible impulse control strategy such that $v(x) = J^S(x)$ and $(X^S_{\tau_k})_{k\geq 1} \subseteq \kappa$. Let $1 \leq n \in \mathbb{N}$ and set $S_n = (\tau_k, \delta_k)_{0\leq k\leq n}$. As before, by Corollary 1.26 we have

$$J^{S_n}(x) - v(x) \leq ||v_0||_{\sup(\kappa)} E_x(1_{\{\tau_n<\infty\}} e^{-\alpha\tau_n}).$$

Thus, since

$$nKE_x(1_{\{\tau_n<\infty\}} e^{-\alpha\tau_n}) \leq ||v_0||_{\sup(\kappa)},$$

we have

$$J^{S_n}(x) - v(x) \leq \frac{||v_0||^2_{\sup(\kappa)}}{nK}$$

and therefore

$$0 \leq v_n(x) - v(x) \leq v_n(x) - J^{S_n}(x) + \frac{||v_0||^2_{\sup(\kappa)}}{nK} \leq \frac{||v_0||^2_{\sup(\kappa)}}{nK}.$$

□

Under the combined assumptions of Theorem 1.27 and Lemma 1.29 or Lemma 1.30 we are now able to approximate the value function v, by solutions of variational inequalities. More importantly, for each $\varepsilon > 0$ we may choose n sufficiently large, such that the vi_n-strategy S_n becomes ε-optimal for the impulse control problem (1) where infinitely many interventions are allowed, i.e

$$||J^{S_n} - v||_{\sup} < \varepsilon.$$

Let us now consider the impulse control problem (8) where we are only charging running costs until the last intervention has been made i.e. for $n \in \mathbb{N}$ set

$$\hat{v}_n(x) = \inf_{S=(\tau_k,\delta_k)_{0\leq k\leq n}\in\mathcal{A}} E_x\left(\int_0^{\tau_n} e^{-\alpha s} f(X^S_s)ds + \sum_{k=1}^{n} 1_{\{\tau_k<\infty\}} e^{-\alpha\tau_k} C(\delta_k)\right). \tag{12}$$

As before, we call $\hat{v}_n$ the **value function** for the impulse control problem above. An impulse control strategy with at most n interventions such that the infimum is attained will be called **optimal** for problem (12). Further, for $S = (\tau_n, \delta_n)_{0\leq k\leq n} \in \mathcal{A}$ let us write

$$\hat{J}^S(x) = E_x\left(\int_0^{\tau_n} e^{-\alpha s} f(X^S_s)ds + \sum_{k=1}^{n} 1_{\{\tau_k<\infty\}} e^{-\alpha\tau_k} C(\delta_k)\right).$$

This time, the stopping times of an optimal strategy do *not* have to be strictly increasing. Indeed, it might be optimal to intervene immediately for n times, pay the fixed costs nK and back out. More generally whenever two optimal intervention times before the last intervention time τ_n coincide, then from this time onwards all intervention times are equal to τ_n and the corresponding control actions are equal to zero. In other words whenever we intervene repeatedly at the same time, we are intending to quit the game by sufficiently often paying the fixed costs K.

Lemma 1.31 (Backing out). *Let $n \in \mathbb{N}$ be fixed and assume that there exists an optimal strategy $S = (\tau_j, \delta_j)_{0 \le j \le n}$ for problem (12). Then for almost all $\omega \in \Omega$ there exists some $0 \le \iota \le n$ such that $\tau_0 < \cdots < \tau_\iota = \cdots = \tau_n$.*

Proof. Let $n \in \mathbb{N}$, assume that $S = (\tau_j, \delta_j)_{0 \le j \le n}$ is optimal for (12) and set

$$\kappa = \inf\{j \in \mathbb{N} : \tau_j = \tau_{j+1} < \tau_n\}.$$

We have to show that $P(N) = 0$, where $N = \{\kappa < \infty\}$. Let us define a new strategy S_1 by setting

$$S_1 = ((\tau_0, \delta_0), \ldots, (\tau_{\kappa-1}, \delta_{\kappa-1}), (\tau_\kappa, \delta_\kappa + \delta_{\kappa+1}), (\tau_{\kappa+1}, 0), (\tau_{\kappa+2}, \delta_{\kappa+2}), \ldots, (\tau_n, \delta_n))$$

on N and $S_1 = S$ on N^c. In words, if $\kappa < \infty$ we modify S such that at time τ_κ we control by $\delta_\kappa + \delta_{\kappa+1}$ and then by 0 instead of first controlling by δ_κ and then by $\delta_{\kappa+1}$. Note that S_1 is an impulse control strategy. Due to the subadditivity of the function c we have

$$C(\delta_\kappa + \delta_{\kappa+1}) + K \le C(\delta_\kappa) + C(\delta_{\kappa+1})$$

and therefore

$$\hat{J}^{S_1}(x) \le \hat{J}^S(x) = v(x).$$

Let us now modify S_1 on $\{\kappa < \infty\}$ by shifting the jump of size zero to the last intervention time τ_n, i.e set

$$S_2 = ((\tau_0, \delta_0), \ldots, (\tau_{\kappa-1}, \delta_{\kappa-1}), (\tau_\kappa, \delta_\kappa + \delta_{\kappa+1}), (\tau_{\kappa+2}, \delta_{\kappa+2}), \ldots, (\tau_n, \delta_n), (\tau_n, 0))$$

on N and set $S_2 = S_1 = S$ on N^c. Again, note that S_2 is an impulse control strategy. Suppose that $P(N) > 0$. Then due to the discounting with $\alpha > 0$ we have

$$\hat{J}^{S_2}(x) < \hat{J}^{S_1}(x)$$

which is a contradiction to S being optimal. Thus we have $P(N) = 0$. $\square$

Note that the sequence $(\hat{v}_n(x))_{n \in \mathbb{N}}$ is monotonously increasing. Furthermore, we have $\hat{v}_n(x) \le v(x)$ and as already mentioned $\hat{v}_n \le nK$, each $n \in \mathbb{N}$.

For $n \ge 1$, the variational inequalities for problem (12) (and their derivation) are the same as in case of problem (8). However, here we do not start the iteration with the costs of the nonintervention strategy but with $\hat{v}_0 \equiv 0$.

Definition 1.32. *Let $1 \leq n \in \mathbb{N}$. A function $\hat{v}_n^* : \mathbb{R}^d \to \mathbb{R}_0^+$ is called a solution of the* ***variational inequalities*** *(for short: $\hat{\boldsymbol{vi}}_{\boldsymbol{n}}$) for the impulse control problem (12), if we have $\hat{v}_n^* \geq \hat{v}_{n-1}^*$ and*

- $L\hat{v}_n^* + f \geq 0$
- $\hat{v}_n^* \leq M\hat{v}_{n-1}^*$
- $(\hat{v}_n^* - M\hat{v}_{n-1}^*)(L\hat{v}_n^* + f) = 0$

where $\hat{v}_{n-1}^*$ is a solution of $\hat{vi}_{n-1}$ and $\hat{v}_0^* \equiv 0$ (for short: $\hat{v}_0^*$ solves $\hat{\boldsymbol{vi}}_{\boldsymbol{0}}$).

Again, the following definition gives rise to an optimal impulse control strategy.

Definition 1.33. *Let $n \in \mathbb{N}$, let $(\tau_0^{(n)}, \delta_0^{(n)}) \equiv (0,0)$ and for $0 \leq k \leq n$ let $\hat{v}_k^*$ be a solution of the variational inequalities $\hat{vi}_k$. For $1 \leq k \leq n$ set*

$$
\begin{aligned}
S_{k-1}^{(n)} &= ((\tau_0^{(n)}, \delta_0^{(n)}), \ldots, (\tau_{k-1}^{(n)}, \delta_{k-1}^{(n)})) \\
\tau_k^{(n)} &= \inf\{t \geq \tau_{k-1}^{(n)} : v_{n-(k-1)}^*(X_t^{S_{k-1}^{(n)}}) = Mv_{n-k}^*(X_t^{S_{k-1}^{(n)}})\} \\
\delta_k^{(n)} &= \begin{cases} \varphi_{v_{n-k}^*}(X_{\tau_k^{(n)}}^{S_{k-1}^{(n)}}) & \text{if } \tau_k^{(n)} < \infty \\ 0 & \text{if } \tau_k^{(n)} = \infty. \end{cases}
\end{aligned}
$$

We then call $S_n = (\tau_k^{(n)}, \delta_k^{(n)})_{0 \leq k \leq n}$ a ***$\hat{vi}_n$-control****.*

Having settled the definition of $\hat{vi}$-controls let us now verify their optimality.

Theorem 1.34 (Verification theorem)**.** *Let $n \in \mathbb{N}$ and assume that for $1 \leq k \leq n$ there exist solutions $\hat{v}_k^* \in C^2$ of the variational inequalities $\hat{vi}_k$ for the impulse control problem (12) such that*

$$
E_x\left(\int_0^T |e^{-\alpha s}\partial_x \hat{v}_k^*(X_s^S)\sigma(X_s^S)|^2 ds\right) < \infty \quad \forall\, T > 0 \tag{13}
$$

$$
\liminf_{T\to\infty} E_x(e^{-\alpha T}\hat{v}_n^*(X_T^S)) = 0 \tag{14}
$$

for all $S = (\tau_k, \delta_k)_{0 \leq k \leq n} \in \mathcal{A}$. Then, the $\hat{vi}_n$-control to $\hat{v}_n^*$ is optimal and $\hat{v}_n(x) = \hat{v}_n^*(x)$.

As opposed to the proof of Theorem 1.27, here we do not need the strong Markov property, since as there are no running costs beyond the last intervention time, we do not have to deal with the tails of our controlled processes. For $n = 0$ the assertion now follows immediately from the definition of $\hat{v}_0^*$. Though having established the lemma above, in the proof of this theorem for the case where $n \geq 1$ we will not use the subadditivity of c. With regard to this the following proof will be more general, yet also more technical, than the ones of Theorem 1.27 and Theorem 1.14 respectively.

Proof. For $n = 0$ there is nothing to prove. Let $n \geq 1$ and let $S = (\tau_k, \delta_k)_{0 \leq k \leq n} \in \mathcal{A}$. Set $m_0 = 0$, $m_1 = 1$ and for $i \geq 2$ define $m_i : \Omega \to \mathbb{N}$ by

$$m_i = \inf\{j \in \mathbb{N} : \tau_j > \tau_{m_{i-1}}\} \wedge n + 1.$$

Note that m_i is $\mathcal{F}_{\tau_{m_{i-1}}}$-measurable and τ_{m_i} is a $(\mathcal{F}_t)_{t\geq 0}$-stopping time. Furthermore, define a $\mathcal{F}$-measurable function $l : \Omega \to \mathbb{N}$ by

$$l = \inf\{k \in \mathbb{N} : \tau_{m_k} = \tau_n\}.$$

Finally, for $T > 0$ and $k \in \mathbb{N}$ set

$$\hat{\tau}_{m_k} = \tau_{m_k} \wedge T.$$

Let $0 \leq k \leq l - 1$ and note that by definition of m we have $0 = \tau_{m_0} \leq \tau_{m_1} < \cdots < \tau_{m_k} < \tau_{m_{k+1}}$ whenever τ_{m_k} is finite. The remaining part of the proof is analogous to the one of

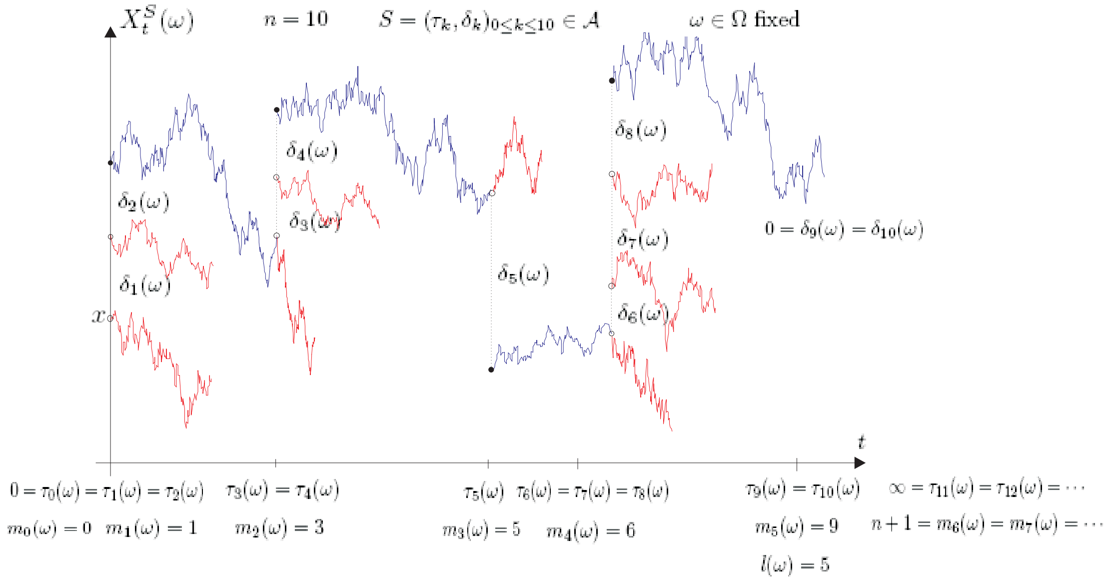

Figure 6: Some notation.

Theorem 1.27. Applying Itô's formula to the càdlàg semimartingale X^S yields

$$\begin{aligned}
& e^{-\alpha \hat{\tau}_{m_{k+1}}} \hat{v}^*_{n-(m_{k+1}-1)}(X^S_{\hat{\tau}_{m_{k+1}}}) - e^{-\alpha \hat{\tau}_{m_k}} \hat{v}^*_{n-(m_{k+1}-1)}(X^S_{\hat{\tau}_{m_k}}) \\
& - \sum_{\hat{\tau}_{m_k} < s \leq \hat{\tau}_{m_{k+1}}} e^{-\alpha s} \big(\hat{v}^*_{n-(m_{k+1}-1)}(X^S_s) - \hat{v}^*_{n-(m_{k+1}-1)}(X^S_{s-})\big) \\
= & \int_{\hat{\tau}_{m_k}}^{\hat{\tau}_{m_{k+1}}} e^{-\alpha s} \partial_x \hat{v}^*_{n-(m_{k+1}-1)}(X^S_{s-}) \sigma(X^S_{s-}) dB_s + \int_{\hat{\tau}_{m_k}}^{\hat{\tau}_{m_{k+1}}} e^{-\alpha s} L \hat{v}^*_{n-(m_{k+1}-1)}(X^S_{s-}) ds.
\end{aligned}$$

Let us rewrite the equation above by replacing the sum of the jumps of $\hat{v}^*_{n-(m_{k+1}-1)}(X^S)$. Taking into account that τ_1 might be equal to zero, for $k=0$ this yields

$$\begin{aligned} &e^{-\alpha\hat{\tau}_1}\hat{v}^*_n(X^S_{\hat{\tau}_1}) - \hat{v}^*_n(X^S_{0-}) - 1_{\{\tau_1\le T\}}e^{-\alpha\tau_1}(\hat{v}^*_n(X^S_{\tau_1}) - \hat{v}^*_n(X^S_{\tau_1-}))\\ &\qquad\qquad = \int\limits_0^{\hat{\tau}_1} e^{-\alpha s}\partial_x\hat{v}^*_n(X^S_{s-})\sigma(X^S_{s-})dB_s + \int\limits_0^{\hat{\tau}_1} e^{-\alpha s}L\hat{v}^*_n(X^S_{s-})ds. \end{aligned}$$

whereas for $k\ge 1$ we have

$$\begin{aligned} &e^{-\alpha\hat{\tau}_{m_{k+1}}}\hat{v}^*_{n-(m_{k+1}-1)}(X^S_{\hat{\tau}_{m_{k+1}}}) - e^{-\alpha\hat{\tau}_{m_k}}\hat{v}^*_{n-(m_{k+1}-1)}(X^S_{\hat{\tau}_{m_k}})\\ &-1_{\{\tau_{m_{k+1}}\le T\}}e^{-\alpha\tau_{m_{k+1}}}(\hat{v}^*_{n-(m_{k+1}-1)}(X^S_{\tau_{m_{k+1}}}) - \hat{v}^*_{n-(m_{k+1}-1)}(X^S_{\tau_{m_{k+1}}-}))\\ &= \int\limits_{\hat{\tau}_{m_k}}^{\hat{\tau}_{m_{k+1}}} e^{-\alpha s}\partial_x\hat{v}^*_{n-(m_{k+1}-1)}(X^S_{s-})\sigma(X^S_{s-})dB_s + \int\limits_{\hat{\tau}_{m_k}}^{\hat{\tau}_{m_{k+1}}} e^{-\alpha s}L\hat{v}^*_{n-(m_{k+1}-1)}(X^S_{s-})ds. \end{aligned}$$

Since $\hat{v}^*_j$ is a solution of the $\hat{vi}_j$ for each $1\le j\le n$ we have

$$-L\hat{v}^*_{n-(m_{k+1}-1)} \le f$$

as well as

$$\begin{aligned} \hat{v}^*_{n-(m_{k+1}-1)}(X^S_{\tau_{m_{k+1}}-}) &\le \hat{v}^*_{n-(m_{k+2}-1)}(X^S_{\tau_{m_{k+1}}-} + \delta_{m_{k+1}} + \ldots + \delta_{m_{k+2}-1})\\ &\quad + C(\delta_{m_{k+1}}) + \ldots + C(\delta_{m_{k+2}-1})\\ &= \hat{v}^*_{n-(m_{k+2}-1)}(X^S_{\tau_{m_{k+1}}}) + C(\delta_{m_{k+1}}) + \ldots + C(\delta_{m_{k+2}-1}) \end{aligned}$$

whenever $\tau_{m_{k+1}}$ is finite. Accordingly for $k=0$ we have

$$\begin{aligned} &\hat{v}^*_n(X^S_{0-}) - e^{-\alpha\hat{\tau}_1}\hat{v}^*_n(X^S_{\hat{\tau}_1})\\ &\qquad \le \int\limits_0^{\hat{\tau}_1} e^{-\alpha s}f(X^S_s)ds - \int\limits_0^{\hat{\tau}_1} e^{-\alpha s}\partial_x\hat{v}^*_n(X^S_{s-})\sigma(X^S_{s-})dB_s\\ &\qquad\quad + 1_{\{\tau_1\le T\}}e^{-\alpha\tau_1}(\hat{v}^*_{n-(m_2-1)}(X^S_{\tau_1}) - \hat{v}^*_n(X^S_{\tau_1}) + C(\delta_1) + \ldots + C(\delta_{m_2-1})) \end{aligned}$$

and for $k\ge 1$ obtain

$$\begin{aligned} &e^{-\alpha\hat{\tau}_{m_k}}\hat{v}^*_{n-(m_{k+1}-1)}(X^S_{\hat{\tau}_{m_k}}) - e^{-\alpha\hat{\tau}_{m_{k+1}}}\hat{v}^*_{n-(m_{k+1}-1)}(X^S_{\hat{\tau}_{m_{k+1}}})\\ &\qquad \le \int\limits_{\hat{\tau}_{m_k}}^{\hat{\tau}_{m_{k+1}}} e^{-\alpha s}f(X^S_s)ds - \int\limits_{\hat{\tau}_{m_k}}^{\hat{\tau}_{m_{k+1}}} e^{-\alpha s}\partial_x\hat{v}^*_{n-(m_{k+1}-1)}(X^S_{s-})\sigma(X^S_{s-})dB_s\\ &\qquad\quad + 1_{\{\tau_{m_{k+1}}\le T\}}e^{-\alpha\tau_{m_{k+1}}}(\hat{v}^*_{n-(m_{k+2}-1)}(X^S_{\tau_{m_{k+1}}}) - \hat{v}^*_{n-(m_{k+1}-1)}(X^S_{\tau_{m_{k+1}}})\\ &\qquad\qquad\qquad + C(\delta_{m_{k+1}}) + \ldots + C(\delta_{m_{k+2}-1})). \end{aligned}$$

A distinction of the cases where τ_k is less or greater than T, respectively, on the left hand sides of the inequalities above yields

$$\begin{aligned}
&\hat{v}_n^*(X_{0-}^S) - 1_{\{\tau_1 \le T\}} e^{-\alpha\tau_1}\hat{v}_n^*(X_{\tau_1}^S) - 1_{\{\tau_1 > T\}} e^{-\alpha T}\hat{v}_n^*(X_T^S) \\
&\qquad \le \int_0^{\hat{\tau}_1} e^{-\alpha s} f(X_s^S) ds - \int_0^{\hat{\tau}_1} e^{-\alpha s}\partial_x \hat{v}_n^*(X_{s-}^S)\sigma(X_{s-}^S) dB_s \\
&\qquad\quad + 1_{\{\tau_1 \le T\}} e^{-\alpha\tau_1}(\hat{v}_{n-(m_2-1)}^*(X_{\tau_1}^S) - \hat{v}_n^*(X_{\tau_1}^S) + C(\delta_1) + \ldots + C(\delta_{m_2-1}))
\end{aligned}$$

and in case where $k \ge 1$ we have

$$\begin{aligned}
&1_{\{\tau_{m_k} \le T\}} e^{-\alpha\tau_{m_k}} \hat{v}_{n-(m_{k+1}-1)}^*(X_{\tau_{m_k}}^S) - 1_{\{\tau_{m_{k+1}} \le T\}} e^{-\alpha\tau_{m_{k+1}}} \hat{v}_{n-(m_{k+1}-1)}^*(X_{\tau_{m_{k+1}}}^S) \\
&\qquad\qquad - 1_{\{\tau_{m_k} \le T < \tau_{m_{k+1}}\}} e^{-\alpha T} \hat{v}_{n-(m_{k+1}-1)}^*(X_T^S) \\
&\le \int_{\hat{\tau}_{m_k}}^{\hat{\tau}_{m_{k+1}}} e^{-\alpha s} f(X_s^S) ds - \int_{\hat{\tau}_{m_k}}^{\hat{\tau}_{m_{k+1}}} e^{-\alpha s}\partial_x \hat{v}_{n-(m_{k+1}-1)}^*(X_{s-}^S)\sigma(X_{s-}^S) dB_s \\
&\quad + 1_{\{\tau_{m_{k+1}} \le T\}} e^{-\alpha\tau_{m_{k+1}}}(\hat{v}_{n-(m_{k+2}-1)}^*(X_{\tau_{m_{k+1}}}^S) - \hat{v}_{n-(m_{k+1}-1)}^*(X_{\tau_{m_{k+1}}}^S) \\
&\qquad\qquad\qquad\qquad + C(\delta_{m_{k+1}}) + \ldots + C(\delta_{m_{k+2}-1})).
\end{aligned}$$

Adding up these inequalities over $0 \le k \le l-1$ yields

$$\begin{aligned}
&\hat{v}_n^*(X_{0-}^S) - \sum_{k=0}^{l-1} 1_{\{\tau_{m_k} \le T < \tau_{m_{k+1}}\}} e^{-\alpha T} \hat{v}_{n-(m_{k+1}-1)}^*(X_T^S) \\
&\qquad \le \int_0^{\hat{\tau}_n} e^{-\alpha s} f(X_s^S) ds - \sum_{k=0}^{l-1} \int_{\hat{\tau}_{m_k}}^{\hat{\tau}_{m_{k+1}}} e^{-\alpha s}\partial_x \hat{v}_{n-(m_{k+1}-1)}^*(X_{s-}^S)\sigma(X_{s-}^S) dB_s \\
&\qquad\quad + \sum_{k=0}^{l-1} 1_{\{\tau_{m_{k+1}} \le T\}} e^{-\alpha\tau_{m_{k+1}}}(C(\delta_{m_{k+1}}) + \ldots + C(\delta_{m_{k+2}-1}))
\end{aligned}$$

equivalently we have

$$\begin{aligned}
&\hat{v}_n^*(X_{0-}^S) - \sum_{k=0}^{n-1} 1_{\{\tau_k \le T < \tau_{k+1}\}} e^{-\alpha T} \hat{v}_{n-k}^*(X_T^S) \\
&\le \int_0^{\hat{\tau}_n} e^{-\alpha s} f(X_s^S) ds - \sum_{k=0}^{n-1} \int_{\hat{\tau}_k}^{\hat{\tau}_{k+1}} e^{-\alpha s}\partial_x \hat{v}_{n-k}^*(X_{s-}^S)\sigma(X_{s-}^S) dB_s + \sum_{k=1}^{n} 1_{\{\tau_k \le T\}} e^{-\alpha\tau_k} C(\delta_k).
\end{aligned}$$

Next, by assumption (13) and optional stopping we have

$$\hat{v}_n^*(x) - E_x(\sum_{k=0}^{n-1} 1_{\{\tau_k \le T < \tau_{k+1}\}} e^{-\alpha T} \hat{v}_{n-k}^*(X_T^S)) \tag{15}$$

$$\le E_x(\int_0^{\hat{\tau}_n} e^{-\alpha s} f(X_s^S) ds + \sum_{k=1}^{n} 1_{\{\tau_k \le T\}} e^{-\alpha \tau_k} C(\delta_k)).$$

By assumption (14) and the monotonicity of $(\hat{v}_j^*)_j$ we have

$$\liminf_{T\to\infty} E_x(1_{\{\tau_k \le T < \tau_{k+1}\}} e^{-\alpha T} \hat{v}_{n-k}^*(X_T^S)) \le \liminf_{T\to\infty} E_x(e^{-\alpha T} \hat{v}_n^*(X_T^S)) = 0$$

each $0 \le k \le n-1$. Thus, by monotone convergence we obtain

$$\hat{v}_n^*(x) \le E_x(\int_0^{\tau_n} e^{-\alpha s} f(X_s^S) ds + \sum_{k=1}^{n} 1_{\{\tau_k < \infty\}} e^{-\alpha \tau_k} C(\delta_k)).$$

Hence we have

$$\hat{v}_n^*(x) \le \hat{v}_n(x).$$

Eventually consider the $\hat{vi}_n$-strategy $S_n = (\tau_k^{(n)}, \delta_k^{(n)})_{0 \le k \le n}$. Then equality holds in (15), thus we have

$$\hat{v}_n^*(x) = E_x(\int_0^{\tau_n^{(n)}} e^{-\alpha s} f(X_s^{S_n}) ds + \sum_{k=1}^{n} 1_{\{\tau_k^{(n)} < \infty\}} e^{-\alpha \tau_k^{(n)}} C(\delta_k^{(n)})) = \hat{v}_n(x)$$

and therefore S_n is optimal. □

As an immediate consequence of the proof of the theorem above we get the following variant of the Bellman principle.

Corollary 1.35 (Bellman principle)**.** *Let $n \in \mathbb{N}$ and assume that the value functions $\hat{v}_k$ are C^2 solutions of the variational inequalities $\hat{vi}_k$ for all $1 \le k \le n$. Let $S_n = (\tau_k^{(n)}, \delta_k^{(n)})_{0 \le k \le n}$ be the $\hat{vi}_n$-control to $\hat{v}_n$. Fix $t > 0$ and assume that*

$$E_x(\int_0^t |e^{-\alpha s} \partial_x \hat{v}_k(X_s) \sigma(X_s)|^2 ds) < \infty$$

each $1 \le k \le n$, where X denotes the corresponding controlled process. Then we have

$$\hat{v}_n(x) = E_x(\int_0^{\tau_n^{(n)} \wedge t} e^{-\alpha s} f(X_s) ds + \sum_{k=1}^{n} 1_{\{\tau_k^{(n)} \le t\}} e^{-\alpha \tau_k^{(n)}} C(\delta_k^{(n)})$$

$$+ \sum_{k=0}^{n-1} 1_{\{\tau_k^{(n)} \le t < \tau_{k+1}^{(n)}\}} e^{-\alpha t} \hat{v}_{n-k}(X_t)).$$

Proof. Under the assumptions above equality holds in (15). □

Regarding convergence of $(\hat{v}_n)_{n\in\mathbb{N}}$ towards v, we state the following lemma.

Lemma 1.36 (Convergence from below)**.** *Let $1 \leq n \in \mathbb{N}$ and suppose that the nonintervention cost v_0 is bounded. Then we have*

$$||v - \hat{v}_n||_{\sup} \leq \frac{||v_0||^2_{\sup}}{nK}.$$

Proof. Let $\varepsilon > 0$, let $x \in \mathbb{R}^d$ and let $S_n = (\tau_k, \delta_k)_{0\leq k\leq n} \in \mathcal{A}$ such that $\hat{J}^{S_n}(x) - \hat{v}_n(x) \leq \varepsilon$. Then, by Corollary 1.26 we have

$$\begin{aligned} 0 &\leq v(x) - \hat{v}_n(x) \leq v(x) - \hat{J}^{S_n}(x) + \varepsilon \\ &\leq J^{S_n}(x) - \hat{J}^{S_n}(x) + \varepsilon = E_x(\int_{\tau_n}^{\infty} e^{-\alpha s} f(X_s^{S_n})ds) + \varepsilon \\ &= E_x(1_{\{\tau_n<\infty\}}e^{-\alpha\tau_n}v_0(X_{\tau_n}^{S_n})) + \varepsilon. \end{aligned}$$

Thus, since

$$\begin{aligned} nKE_x(1_{\{\tau_n<\infty\}}e^{-\alpha\tau_n}) &\leq E_x(\sum_{k=1}^{n} 1_{\{\tau_k<\infty\}}e^{-\alpha\tau_k}C(\delta_k)) \\ &\leq \hat{J}^{S_n}(x) \leq \hat{v}_n(x) + \varepsilon \leq v_0(x) + \varepsilon \end{aligned}$$

we obtain

$$0 \leq v(x) - \hat{v}_n(x) \leq \frac{||v_0||^2_{\sup}}{nK} + (\frac{||v_0||_{\sup}}{nK} + 1)\varepsilon.$$

Letting ε tend to zero, the desired conclusion follows. □

As a consequence of this lemma, given the assumption of Theorem 1.34 and the boundedness of the nonintervention cost v_0, we have

$$||v - \hat{J}^{S_n}||_{\sup} \leq \frac{||v_0||^2_{\sup}}{nK}$$

where S_n denotes the $\hat{vi}_n$-control. Thus, as before, we are able to *construct* ε-optimal impulse control strategies. Further, as an immediate consequence of Lemma 1.29 and Lemma 1.36 we get

Corollary 1.37. *Let $1 \leq n \in \mathbb{N}$ and assume that v_0 is bounded. Then we have*

$$||v_n - \hat{v}_n||_{\sup} \leq 2\frac{||v_0||^2_{\sup}}{nK}.$$

Note that in general, without any explicit assumption on v_0, since $\hat{v}_n \leq v \leq v_n$, we have the following remark.

Remark 1.38. *Let $x \in \mathbb{R}^d$ and let $\varepsilon > 0$. If there exists some $n \in \mathbb{N}$ such that $v_n(x) - \hat{v}_n(x) < \varepsilon$ then $v_n(x) - v(x) < \varepsilon$ and $v(x) - \hat{v}_n(x) < \varepsilon$. Thus, given the assumptions of the verification theorems for v_n and $\hat{v}_n$, and if n is such that $v_n^*(x) - \hat{v}_n^*(x) < \varepsilon$, then the vi_n-control and the $\widehat{vi}_n$-control are both ε-optimal for the impulse control problem (1).*

Given the existence of optimal strategies, we may reformulate the previous lemma in the following way.

Lemma 1.39 (Convergence from below)**.** *Let $1 \leq n \in \mathbb{N}$ and let κ be a compact subset of $\mathbb{R}^d$. Suppose that for each $x \in \kappa$ there exists an optimal strategy $S_n = (\tau_k, \delta_k)_{0 \leq k \leq n}$ corresponding to problem (12) such that $X_{\tau_n}^{S_n} \in \kappa$. Then we have*

$$||v - \hat{v}_n||_{\sup(\kappa)} \leq \frac{||v_0||_{\sup(\kappa)}^2}{nK}.$$

Proof. Let $1 \leq n \in \mathbb{N}$ and let $x \in \kappa$. By assumption, there exists an optimal strategy $S_n = (\tau_k, \delta_k)_{0 \leq k \leq n}$ for (12) such that $X_{\tau_n}^{S_n} \in \kappa$. By Corollary 1.26 we have

$$\begin{aligned} 0 \leq v(x) - \hat{v}_n(x) &= v(x) - \hat{J}^{S_n}(x) \leq J^{S_n}(x) - \hat{J}^{S_n}(x) \\ &= E_x\Big(\int_{\tau_n}^{\infty} e^{-\alpha s} f(X_s^{S_n}) ds\Big) = E_x(1_{\{\tau_n < \infty\}} e^{-\alpha \tau_n} v_0(X_{\tau_n}^{S_n})). \end{aligned}$$

Thus, since

$$nK E_x(1_{\{\tau_n < \infty\}} e^{-\alpha \tau_n}) \leq E_x\Big(\sum_{k=1}^{n} 1_{\{\tau_k < \infty\}} e^{-\alpha \tau_k} C(\delta_k)\Big) \leq \hat{J}^{S_n}(x) = \hat{v}_n(x) \leq v_0(x)$$

we obtain

$$0 \leq v(x) - \hat{v}_n(x) \leq \frac{||v_0||_{\sup(\kappa)}^2}{nK}.$$

$\square$

1.2 The one-dimensional stochastic impulse control problem

In this section, we apply the convergence result, Lemma 1.30, in order to derive the solution to the impulse control problem (1) for convex running costs and a one-dimensional diffusion process with constant coefficients. That is, we assume that $m = d = 1$ and the infinitesimal drift b and the infinitesimal variance σ are constant with $\sigma \neq 0$. The running costs $f : \mathbb{R} \to \mathbb{R}_0^+$ are supposed to be convex and such that condition (10) is satisfied. Further, we assume that the controlling costs consist of fixed and proportional costs $C : \mathbb{R} \to [K, \infty)$, $C(y) = K + k|y|$, where $k \in \mathbb{R}_0^+$. Then, the value functions corresponding to (1) with the restriction that at most $n \in \mathbb{N}$ interventions are allowed are given by

$$v_n(x) = \inf_{S=(\tau_k,\delta_k)_{0\le k\le n}\in\mathcal{A}} E_x\Big(\int_0^\infty e^{-\alpha s} f(X_s^S)ds + \sum_{k=1}^n 1_{\{\tau_k<\infty\}} e^{-\alpha\tau_k}(K + k|\delta_k|)\Big), \tag{16}$$

where

$$X_t^S = x + bt + \sigma B_t + \sum_{k=1}^n 1_{\{\tau_k \le t\}}\delta_k.$$

The following verification result states a sufficient condition for optimality. The intervention regions will be denoted by $D_n = (\underline{c}_n, \bar{c}_n)^c$. For simplicity of notation we will set $\underline{c}_0 = -\infty$ and $\bar{c}_0 = \infty$ with the convention that $(-\infty, \underline{c}_0] = [\bar{c}_0, \infty) = \emptyset$. Further, in the following, $w_n''(\bar{c}_n)$ will denote a left-sided derivative and $w_n''(\underline{c}_n)$ will denote a right-sided derivative, respectively.

Lemma 1.40. *Let $1 \le N \in \mathbb{N}$ and let $w_0 = v_0$. Suppose that there exists a sequence of functions $(w_n)_{1\le n\le N}$, $w_n : \mathbb{R} \to \mathbb{R}_0^+$ with $w_n \le w_{n-1}$ and two pairs of sequences $(\underline{c}_n, \bar{c}_n)_{1\le n\le N}$ and $(\underline{b}_n, \bar{b}_n)_{1\le n\le N+1}$ with*

$$\bar{b}_{N+1} \le \bar{c}_N, \quad \underline{b}_{N+1} \ge \underline{c}_N, \quad \underline{c}_{n-1} \le \underline{c}_n \le \underline{b}_n \le \bar{b}_n \le \bar{c}_n \le \bar{c}_{n-1} \quad \forall\, 1 \le n \le N,$$

such that for all $1 \le n \le N$ we have

(i) $\quad w_n \in C^2([\underline{c}_n, \bar{c}_n])$ *and* $Lw_n(x) + f(x) = 0 \quad \forall\, x \in [\underline{c}_n, \bar{c}_n]$

(ii)
$$\begin{aligned} w_n(x) &= w_n(\bar{c}_n) + k(x - \bar{c}_n) \quad \forall\, x \ge \bar{c}_n \\ w_n(x) &= w_n(\underline{c}_n) + k(\underline{c}_n - x) \quad \forall\, x \le \underline{c}_n \end{aligned}$$

(iii)
$$\begin{aligned} w_n(\bar{c}_n) &= w_{n-1}(\bar{b}_n) + K + k(\bar{c}_n - \bar{b}_n) \\ w_n(\underline{c}_n) &= w_{n-1}(\underline{b}_n) + K + k(\underline{b}_n - \underline{c}_n) \end{aligned}$$

(iv)
$$\begin{aligned} w_n'(x) = k &\iff x \in \{\bar{b}_{n+1}\} \cup [\bar{c}_n, \infty) \quad \text{and} \quad w_0'(x) = k \iff x = \bar{b}_1 \\ w_n'(x) = -k &\iff x \in \{\underline{b}_{n+1}\} \cup (-\infty, \underline{c}_n] \quad \text{and} \quad w_0'(x) = -k \iff x = \underline{b}_1 \end{aligned}$$

(v) $\qquad w''_{n-1}(x) > 0 \quad \forall\, x \in [\underline{b}_n, \bar{b}_n] \quad \textit{and} \quad w''_N(x) > 0 \quad \forall\, x \in \{\underline{b}_{N+1}, \bar{b}_{N+1}\}$

(vi)
$$\begin{aligned} -\alpha k x + f(x) &\geq -\alpha k \bar{c}_n + f(\bar{c}_n) \quad \forall\, x \geq \bar{c}_n \\ \alpha k x + f(x) &\geq \alpha k \underline{c}_n + f(\underline{c}_n) \quad \forall\, x \leq \underline{c}_n \end{aligned}$$

(vii)
$$\begin{aligned} w_n(x) &< w_{n-1}(\bar{b}_n) + K + k(x - \bar{b}_n) \quad \forall\, x \in [\bar{b}_n, \bar{b}_{n+1}), \quad \textit{if} \quad \bar{b}_n < \bar{b}_{n+1} \\ w_n(x) &< w_{n-1}(\underline{b}_n) + K + k(\underline{b}_n - x) \quad \forall\, x \in (\underline{b}_{n+1}, \underline{b}_n], \quad \textit{if} \quad \underline{b}_{n+1} < \underline{b}_n. \end{aligned}$$

Then, w_n is in $C^1 \cap C^2(\mathbb{R} \setminus \{\bar{c}_n,\ \underline{c}_n\})$ and it coincides with the value function v_n, that is

$$v_n(x) = w_n(x),$$

for each $x \in \mathbb{R}$ and $1 \leq n \leq N$. Further, the optimal strategy, when at most n interventions are allowed, is to shift X to $\bar{b}_n$ if it is greater or equal than $\bar{c}_n$ and to shift X to $\underline{b}_n$ if it is less or equal than $\underline{c}_n$, respectively. As long as X stays in $(\underline{c}_n, \bar{c}_n)$ it is optimal to do nothing.

Proof. Let $1 \leq n \leq N$. By the assumptions (i), (ii) and since $w'_n(\bar{c}_n) = k$ as well as $w'_n(\underline{c}_n) = -k$, we have $w_n \in C^1 \cap C^2(\mathbb{R} \setminus \{\bar{c}_n,\ \underline{c}_n\})$. If w_n solves the variational inequalities corresponding to (16), then by Remark 1.17 and Theorem 1.27 it coincides with the value function v_n and the optimal strategy is given by the corresponding vi_n-control.

First we show that $Lw_n + f \geq 0$. By (i) we only have to show this on $(\underline{c}_n, \bar{c}_n)^c$. Let $x \geq \bar{c}_n$.

$$\begin{aligned} Lw_n(x) + f(x) &\overset{(ii)}{=} -\alpha(w_n(\bar{c}_n) + k(x - \bar{c}_n)) + bk + f(x) \\ &\overset{(i)}{=} -\alpha k(x - \bar{c}_n) + f(x) - f(\bar{c}_n) - \frac{1}{2}\sigma^2 w''_n(\bar{c}_n) + b(k - w'_n(\bar{c}_n)) \\ &\overset{(iv)}{=} -\alpha k(x - \bar{c}_n) + f(x) - f(\bar{c}_n) - \frac{1}{2}\sigma^2 w''_n(\bar{c}_n). \end{aligned}$$

By the assumption (iv) and (v) we have $w''_n(\bar{c}_n) \leq 0$ and therefore, by (vi) we get

$$Lw_n(x) + f(x) \geq 0.$$

Now, let $x \leq \underline{c}_n$.

$$\begin{aligned} Lw_n(x) + f(x) &\overset{(ii)}{=} -\alpha(w_n(\underline{c}_n) + k(\underline{c}_n - x)) - bk + f(x) \\ &\overset{(i)}{=} -\alpha k(\underline{c}_n - x) + f(x) - f(\underline{c}_n) - \frac{1}{2}\sigma^2 w''_n(\underline{c}_n) - b(k + w'_n(\underline{c}_n)) \\ &\overset{(iv)}{=} -\alpha k(\underline{c}_n - x) + f(x) - f(\underline{c}_n) - \frac{1}{2}\sigma^2 w''_n(\underline{c}_n). \end{aligned}$$

As before, by the assumptions (iv) and (v) we have $w''_n(\underline{c}_n) \leq 0$. Thus, by (vi) we obtain

$$Lw_n(x) + f(x) \geq 0.$$

Next, we show that $w_n \leq Mw_{n-1}$ within five steps.
1. Let $x \geq \bar{c}_n$. We have

$$w_n(x) \overset{(ii)}{=} w_n(\bar{c}_n) + k(x - \bar{c}_n) \overset{(iii)}{=} w_{n-1}(\bar{b}_n) + K + k(x - \bar{b}_n).$$

By assumptions (iv) and (v), w_{n-1} is monotonously increasing on $[\bar{b}_n, \infty)$. Further, w_{n-1} is affine linear on $[\bar{c}_{n-1}, \infty)$ (recall the convention that $\bar{c}_0 = \infty$) and therefore, we have

$$Mw_{n-1}(x) = \inf_{y \leq 0 \wedge \bar{c}_{n-1} - x} (w_{n-1}(x + y) + K - ky).$$

Since $\bar{b}_n \leq x \wedge \bar{c}_{n-1}$, $w'_{n-1}(\bar{b}_n) = k$ and $w''_{n-1}(\bar{b}_n) > 0$, a local minimum of $y \mapsto w_{n-1}(x + y) + K - ky$ on $(-\infty, 0 \wedge \bar{c}_{n-1} - x]$ is attained at $y = \bar{b}_n - x$. By assumption (iv), (v) and (ii), it is the global minimum, i.e we have

$$Mw_{n-1}(x) = w_{n-1}(\bar{b}_n) + K + k(x - \bar{b}_n).$$

Thus, for all $x \geq \bar{c}_n$ we have

$$w_n(x) = Mw_{n-1}(x).$$

2. Let $x \leq \underline{c}_n$. Similar to 1. we have

$$w_n(x) \overset{(ii)}{=} w_n(\underline{c}_n) + k(\underline{c}_n - x) \overset{(iii)}{=} w_{n-1}(\underline{b}_n) + K + k(\underline{b}_n - x).$$

Since w_{n-1} is monotonously decreasing on $(-\infty, \underline{b}_n]$ and affine linear on $(-\infty, \underline{c}_{n-1}]$ (with the convention that $\underline{c}_0 = -\infty$), we may write

$$Mw_{n-1}(x) = \inf_{y \geq 0 \vee \underline{c}_{n-1} - x} (w_{n-1}(x + y) + K + ky).$$

Since $\underline{b}_n \geq x \vee \underline{c}_{n-1}$, $w'_{n-1}(\underline{b}_n) = -k$ and $w''_{n-1}(\underline{b}_n) > 0$, a local minimum of $y \mapsto w_{n-1}(x + y) + K + ky$ on $[0 \vee \underline{c}_{n-1} - x, \infty)$ is attained at $y = \underline{b}_n - x$. By assumption (iv), (v) and (ii), it is the global minimum, i.e we have

$$Mw_{n-1}(x) = w_{n-1}(\underline{b}_n) + K + k(\underline{b}_n - x).$$

Thus, for all $x \leq \underline{c}_n$ we have

$$w_n(x) = Mw_{n-1}(x).$$

3. Let $\bar{b}_n \leq x < \bar{c}_n$. As in case 1. we have

$$Mw_{n-1}(x) = \inf_{y \leq 0} (w_{n-1}(x + y) + K - ky) = w_{n-1}(\bar{b}_n) + K + k(x - \bar{b}_n).$$

Further, by (iv) and (v) we have $w'_n(z) > k$ for all $z \in (\bar{b}_{n+1}, \bar{c}_n)$ and therefore, since $w_n(\bar{c}_n) = w_{n-1}(\bar{b}_n) + K + k(\bar{c}_n - \bar{b}_n)$ we have

$$w_n(z) < w_{n-1}(\bar{b}_n) + K + k(z - \bar{b}_n) \quad \forall z \in [\bar{b}_{n+1}, \bar{c}_n).$$

Thus, if $\bar{b}_n \geq \bar{b}_{n+1}$, then $x \in [\bar{b}_{n+1}, \bar{c}_n)$ and we have $w_n(x) < Mw_{n-1}(x)$. If $\bar{b}_n < \bar{b}_{n+1}$, then by assumption (vii) we have

$$w_n(z) < w_{n-1}(\bar{b}_n) + K + k(z - \bar{b}_n) \quad \forall z \in [\bar{b}_n, \bar{c}_n)$$

which implies that $w_n(x) < Mw_{n-1}(x)$. Thus, for all $\bar{b}_n \leq x < \bar{c}_n$ we obtain

$$w_n(x) < Mw_{n-1}(x).$$

4. Let $\underline{c}_n < x \leq \underline{b}_n$. This case is similar to 3.

$$Mw_{n-1}(x) = \inf_{y \geq 0}(w_{n-1}(x+y) + K + ky) = w_{n-1}(\underline{b}_n) + K + k(\underline{b}_n - x).$$

Since $w_n'(z) < -k$ for all $z \in (\underline{c}_n, \underline{b}_{n+1})$ and $w_n(\underline{c}_n) = w_{n-1}(\underline{b}_n) + K + k(\underline{b}_n - \underline{c}_n)$ we have

$$w_n(z) < w_{n-1}(\underline{b}_n) + K + k(\underline{b}_n - z) \quad \forall z \in (\underline{c}_n, \underline{b}_{n+1}].$$

Thus, if $\underline{b}_{n+1} \geq \underline{b}_n$ we get $w_n(x) < Mw_{n-1}(x)$. If $\underline{b}_{n+1} < \underline{b}_n$, then by assumption (vii) we have

$$w_n(z) < w_{n-1}(\underline{b}_n) + K + k(\underline{b}_n - z) \quad \forall z \in (\underline{c}_n, \underline{b}_n].$$

and consequently $w_n(x) < Mw_{n-1}(x)$. Thus, for all $\underline{c}_n < x \leq \underline{b}_n$ we obtain

$$w_n(x) < Mw_{n-1}(x).$$

5. Let $\underline{b}_n < x < \bar{b}_n$. By assumption (v), w_{n-1} is strictly convex on $[\underline{b}_n, \bar{b}_n]$. Further, we have $w_{n-1}'(\underline{b}_n) = -k$ and $w_{n-1}'(\bar{b}_n) = k$. Thus, there exists a unique minimizer of w_{n-1} on $[\underline{b}_n, \bar{b}_n]$, which we denote by x_0. By definition of x_0, we have

$$Mw_{n-1}(x) = \inf_{y \leq 0}(w_{n-1}(x+y) + K - ky), \quad \text{if} \quad x \geq x_0$$

$$Mw_{n-1}(x) = \inf_{y \geq 0}(w_{n-1}(x+y) + K + ky), \quad \text{if} \quad x \leq x_0.$$

Due to assumption (iv), these infima are attained at $y = 0$, i.e. $Mw_{n-1}(x) = w_{n-1}(x) + K$. Thus, since $w_n \leq w_{n-1}$, for all $\underline{b}_n < x < \bar{b}_n$ we obtain

$$w_n(x) < Mw_{n-1}(x).$$

Finally, $(w_n - Mw_{n-1})(Lw_n + f) = 0$ follows immediately from 1. and assumption (i). $\square$

Note that if there exists a solution to $(i) - (vii)$ then it is unique, since it coincides with the value function. The following remark states a sufficient condition for (vi) and an explicit representation of v_0 in case of quadratic running costs.

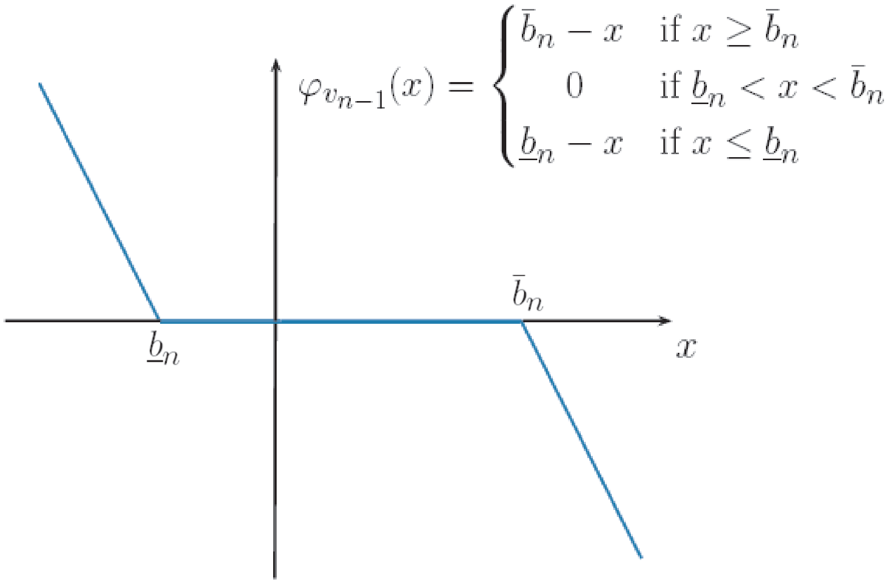

Figure 7: Argmin $\varphi_{v_{n-1}}$ of Mv_{n-1}.

Remark 1.41. *In case of quadratic running costs, a sufficient condition for* (vi) *is given by*

$$\underline{c}_N \leq -\frac{\alpha k}{2} \quad \text{and} \quad \bar{c}_N \geq \frac{\alpha k}{2}.$$

Further, in this case, the nonintervention costs are given by

$$v_0(x) = E(\int_0^\infty e^{-\alpha s} f(x + bs + \sigma B_s) ds) = \frac{1}{\alpha}\left(x^2 + \frac{1}{\alpha}\left(2xb + \sigma^2 + \frac{2b^2}{\alpha}\right)\right).$$

In view of Lemma 1.40 and Lemma 1.30, it remains to find the solution to the conditions $(i) - (vii)$. We will construct the solution to the conditions (i), (iii) and (iv) on the intervals $[\underline{c}_n, \bar{c}_n]$. The extension to $\mathbb{R}$ is then given by (ii). Finally, we will have to verify the remaining assumptions $(v) - (vii)$.

The following proposition states the solution to the differential equation in condition (i).

Proposition 1.42. *The general solution to the linear second order ordinary differential equation*

$$u''(x) + \frac{2b}{\sigma^2} u'(x) - \frac{2\alpha}{\sigma^2} u(x) = -\frac{2}{\sigma^2} f(x)$$

is given by

$$u(x) = \mu e^{\lambda_1 x} + \nu e^{\lambda_2 x} + \frac{2}{\sigma^2(\lambda_2 - \lambda_1)} \int_x^c (e^{\lambda_2 (x-t)} - e^{\lambda_1 (x-t)}) f(t) dt$$

where μ, ν, $c \in \mathbb{R}$ *and*

$$\lambda_{1,2} = -\frac{b}{\sigma^2} \pm \frac{1}{\sigma}\sqrt{\frac{b^2}{\sigma^2} + 2\alpha}.$$

Proof. The solution to the homogeneous equation

$$u''(x) + \frac{2b}{\sigma^2}u'(x) - \frac{2\alpha}{\sigma^2}u(x) = 0$$

is given by

$$u_h(x) = \mu e^{\lambda_1 x} + \nu e^{\lambda_2 x},$$

where μ, $\nu \in \mathbb{R}$ and $\lambda_{1,2}$ as in the assertion. By means of the method of variation of constants, a particular solution to the inhomogeneous differential equation is given by

$$u_p(x) = \frac{2}{\sigma^2}\int_x^c \frac{f(t)e^{\lambda_1 t}}{W(t)}dt\ e^{\lambda_2 x} - \frac{2}{\sigma^2}\int_x^c \frac{f(t)e^{\lambda_2 t}}{W(t)}dt\ e^{\lambda_1 x},$$

where $c \in \mathbb{R}$ and the Wronskian determinant W is given by $W(t) = (\lambda_2 - \lambda_1)e^{(\lambda_1+\lambda_2)t}$. Note that since $\alpha > 0$ we have $\lambda_1 > 0$, $\lambda_2 < 0$ and therefore $W < 0$. Thus, the general solution to the inhomogeneous differential equation is given by

$$u(x) = u_h(x) + u_p(x).$$

□

Let $1 \leq n \leq N \in \mathbb{N}$. By the proposition above, the general solution to (i) is given by

$$w_n(x) = \mu_n e^{\lambda_1 x} + \nu_n e^{\lambda_2 x} + \frac{2}{\sigma^2(\lambda_2 - \lambda_1)}\int_x^{\bar{c}_n}(e^{\lambda_2(x-t)} - e^{\lambda_1(x-t)})f(t)dt, \tag{17}$$

where μ_n, $\nu_n \in \mathbb{R}$ and $\lambda_{1,2}$ as in the assertion of the proposition above. Note that w_0 is given by the nonintervention costs

$$w_0(x) = E(\int_0^\infty e^{-\alpha s}f(x + bs + \sigma B_s)ds).$$

Next, given w_{n-1}, we will determine μ_n, ν_n, $\underline{c}_n$, $\bar{c}_n$, $\underline{b}_n$ and $\bar{b}_n$ such that the conditions (iii) and (iv) are satisfied. We will tacitly assume that there exist solutions to the equations (18) and (21) – (22). Later on, we will provide some examples where these equations are solved numerically.

First, we choose $\underline{b}_n$ and $\bar{b}_n$ such that we have

$$w'_{n-1}(\underline{b}_n) = -k \quad \text{and} \quad w'_{n-1}(\bar{b}_n) = k. \tag{18}$$

Next, we determine μ_n and ν_n as functions of $\bar{c}_n$. The equation (17) together with the condition

$$w_n(\bar{c}_n) = w_{n-1}(\bar{b}_n) + K + k(\bar{c}_n - \bar{b}_n)$$

implies that

$$\mu_n e^{\lambda_1 \bar{c}_n} + \nu_n e^{\lambda_2 \bar{c}_n} = w_{n-1}(\bar{b}_n) + K + k(\bar{c}_n - \bar{b}_n).$$

Further, the condition $w'_n(\bar{c}_n) = k$ implies that

$$\mu_n = \frac{k - \lambda_2 \nu_n e^{\lambda_2 \bar{c}_n}}{\lambda_1 e^{\lambda_1 \bar{c}_n}}.$$

The previous two equations yield ν_n as a function of $\bar{c}_n$

$$\nu_n = \frac{\lambda_1}{\lambda_1 - \lambda_2} \left[w_{n-1}(\bar{b}_n) + K + k(\bar{c}_n - \bar{b}_n) - \frac{k}{\lambda_1} \right] e^{-\lambda_2 \bar{c}_n} \tag{19}$$

and accordingly, we also obtain μ_n in dependance of $\bar{c}_n$

$$\mu_n = \left[\frac{k}{\lambda_1} - \frac{\lambda_2}{\lambda_1 - \lambda_2} (w_{n-1}(\bar{b}_n) + K + k(\bar{c}_n - \bar{b}_n) - \frac{k}{\lambda_1}) \right] e^{-\lambda_1 \bar{c}_n}. \tag{20}$$

Next, we will determine $\underline{c}_n$ and $\bar{c}_n$. The condition $w'_n(\underline{c}_n) = -k$ together with (17) implies

$$\lambda_1 \mu_n e^{\lambda_1 \underline{c}_n} + \lambda_2 \nu_n e^{\lambda_2 \underline{c}_n} + \frac{2}{\sigma^2(\lambda_2 - \lambda_1)} \int_{\underline{c}_n}^{\bar{c}_n} (\lambda_2 e^{\lambda_2(\underline{c}_n - t)} - \lambda_1 e^{\lambda_1(\underline{c}_n - t)}) f(t) dt = -k. \tag{21}$$

Further, by (17) and

$$w_n(\underline{c}_n) = w_{n-1}(\underline{b}_n) + K + k(\underline{b}_n - \underline{c}_n)$$

we have

$$\mu_n e^{\lambda_1 \underline{c}_n} + \nu_n e^{\lambda_2 \underline{c}_n} + \frac{2}{\sigma^2(\lambda_2 - \lambda_1)} \int_{\underline{c}_n}^{\bar{c}_n} (e^{\lambda_2(\underline{c}_n - t)} - e^{\lambda_1(\underline{c}_n - t)}) f(t) dt = w_{n-1}(\underline{b}_n) + K + k(\underline{b}_n - \underline{c}_n). \tag{22}$$

Using (19) and (20) in the equations (21) and (22), we obtain two equations for $\underline{c}_n$ and $\bar{c}_n$. Finally, having determined $\underline{c}_n$ and $\bar{c}_n$, we get μ_n and ν_n by (19) and (20). The function w_n is then given by (17).

As a particular example, we consider a situation where the running costs are quadratic, i.e. $f(x) = x^2$. Further, we assume that the infinitesimal drift and the infinitesimal variance of the uncontrolled process are given by $b = \sigma = 1$ and that $K = 2$, $k = 0.5$ and $\alpha = 0.3$. It is sufficient to choose $N = 10$ as for greater values of n, the optimal strategies and the value functions are virtually identical. First, we determine a sequence of functions $(w_n)_{0\leq n\leq 10}$ and two pairs of sequences $(\underline{c}_n, \bar{c}_n)_{1\leq n\leq 10}$ and $(\underline{b}_n, \bar{b}_n)_{1\leq n\leq 11}$ as described above. Then, for each $1 \leq n \leq 10$, we extend the function w_n to the whole real line, according to (ii). Thus, the conditions $(i) - (iii)$ as well as the "$\Leftarrow$" implication in condition (iv) are satisfied. Condition (vi) can easily be verified by applying Remark 1.41. Figure 9 depicts the first derivative of the function w_n for $1 \leq n \leq 9$, as well as the condition $w_n \leq Mw_{n-1}$ for $1 \leq n \leq 10$. It indicates that the remaining assumptions of Lemma 1.40 hold true. Hence the function w_n coincides with the value function v_n for each $0 \leq n \leq 10$, which in turn approximate the value function v of problem (1) uniformly on compact sets. The value functions as well as the corresponding optimal strategies are depicted in Figure 8. Note that the sequence $(\underline{b}_n)_n$ is increasing, which is due to the drift $b = 1$. It can be seen that the distances $\bar{c}_n - \bar{b}_n$ and $\underline{b}_n - \underline{c}_n$ become smaller as n gets greater. This is due to the fact that if the controller is allowed to intervene more often, then his interventions will become smaller in order to pay less proportional costs. As prescribed in the assumptions of Lemma 1.40 the sequence $(\bar{c}_n)_n$ is decreasing and the sequence $(\underline{c}_n)_n$ is increasing, respectively. This reflects that a controller who is allowed to intervene more often, will intervene earlier in order to pay less running costs. Further, Figure 8 suggests that the convergence of the value functions and the optimal strategies is very fast. Indeed, as mentioned above, it is sufficient to choose $N = 10$.

Figure 10 depicts the value functions and the optimal strategies for $f(x) = x^4$ and $b = 0$. The remaining parameters are chosen as in the previous example. Since the running costs are symmetric and the infinitesimal drift b vanishes, the value functions v_n are also symmetric and we have $\underline{c}_n = -\bar{c}_n$ as well as $\underline{b}_n = -\bar{b}_n$.

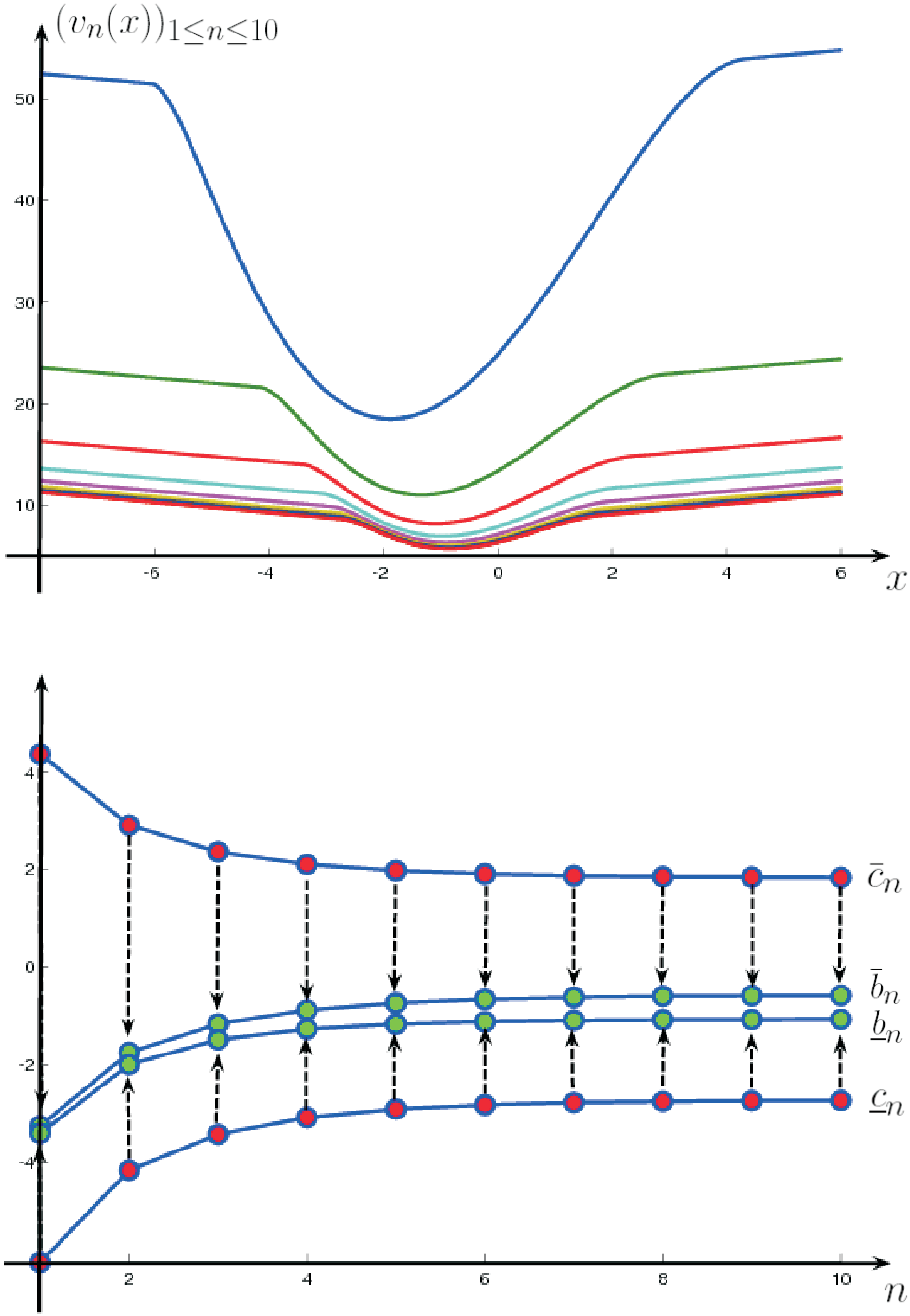

Figure 8: Value functions v_n and the corresponding optimal strategies.

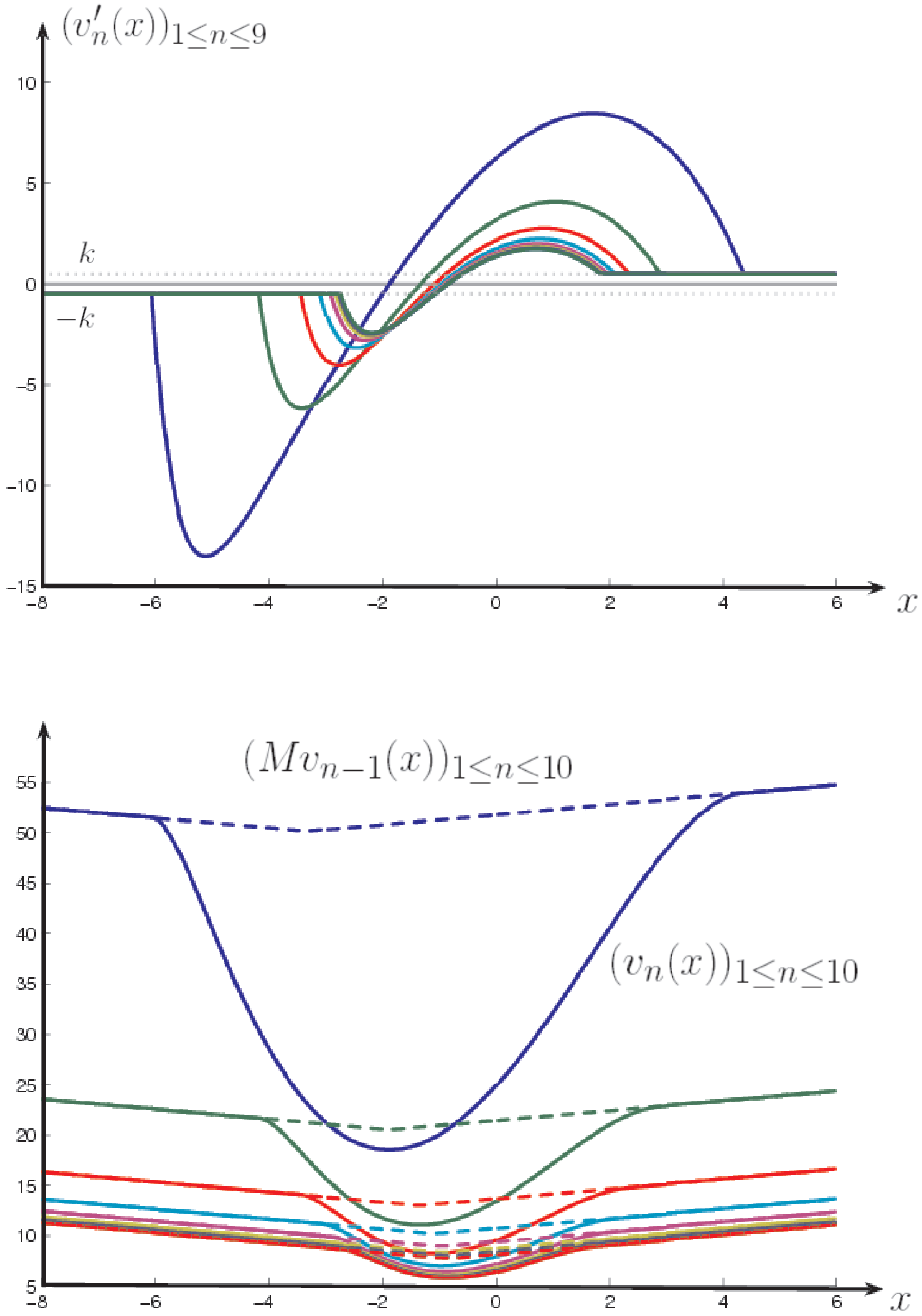

Figure 9: First derivatives of the value functions v_n and the conditions $v_n \le Mv_{n-1}$.

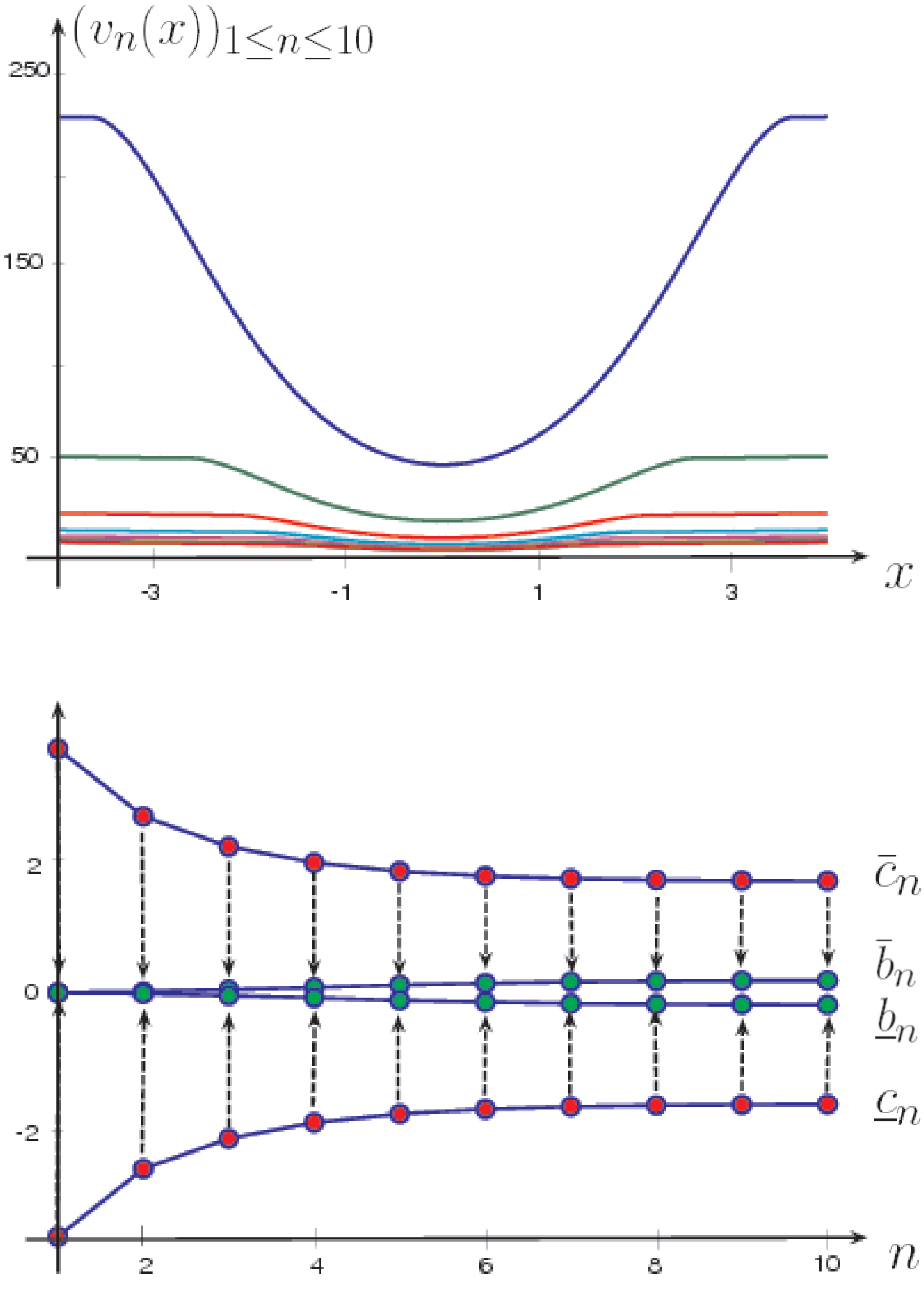

Figure 10: Value functions v_n and the corresponding optimal strategies.

Next, we consider a situation as in the first example, however, this time we assume that there are no proportional costs. That is we set $k = 0$, $K = 2$, $b = \sigma = 1$, $f(x) = x^2$, $\alpha = 0.3$ and $N = 10$. Figure 11 depicts the value functions v_n as well as the value functions $\hat{v}_n$ corresponding to problem (12). The latter can be determined similar to the value functions v_n. Figure 12 depicts the optimal strategies. On top we have the optimal strategies corresponding to the value functions v_n, whereas the optimal strategies corresponding to $\hat{v}_n$ are illustrated below. As there are no proportional costs, in both cases we have $\underline{b}_n = \bar{b}_n$ except for the optimal control corresponding to the value function $\hat{v}_1$. In this case we have $\bar{c}_1 = \bar{b}_1$ and $\underline{c}_1 = \underline{b}_1$ as upon intervention, the controller backs out of the game and thus chooses the cheapest control action, which consists in shifting the underlying process by 0. Further, the values of $\bar{c}_n$ and $\bar{b}_n$ corresponding to v_n are smaller than those corresponding to the first example where $k = 0.5$. This is because a controller who, in addition, has to pay proportional costs upon each intervention, has to pay strictly more for each non trivial intervention. Thus he intervenes later and also by a smaller amount, than a controller who only has to pay fixed costs.

Figure 13 depicts the optimal strategies for $b = 0$. The remaining parameters are chosen as in the previous example. Here, except for $\hat{v}_1$, the optimal intervention is to shift the paths to the origin, which is the minimum of the running cost function.

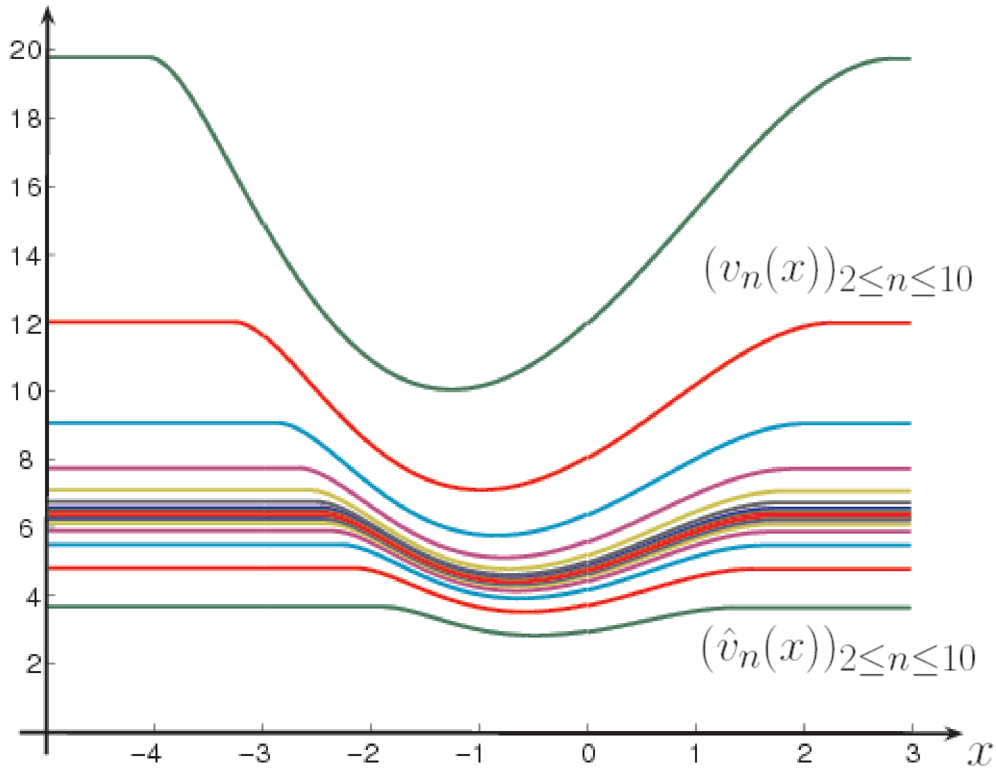

Figure 11: Value functions v_n and $\hat{v}_n$.

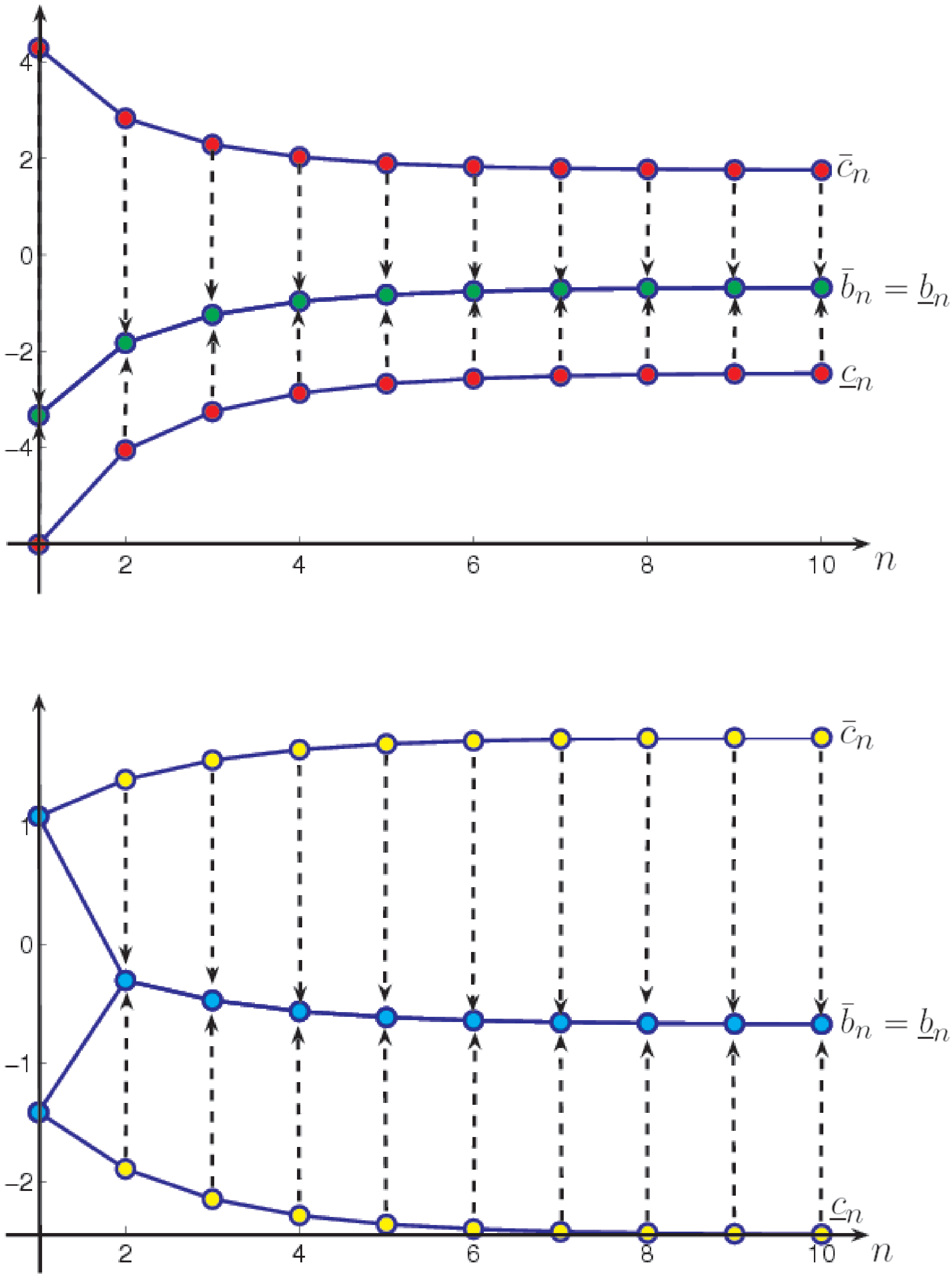

Figure 12: Optimal strategies.

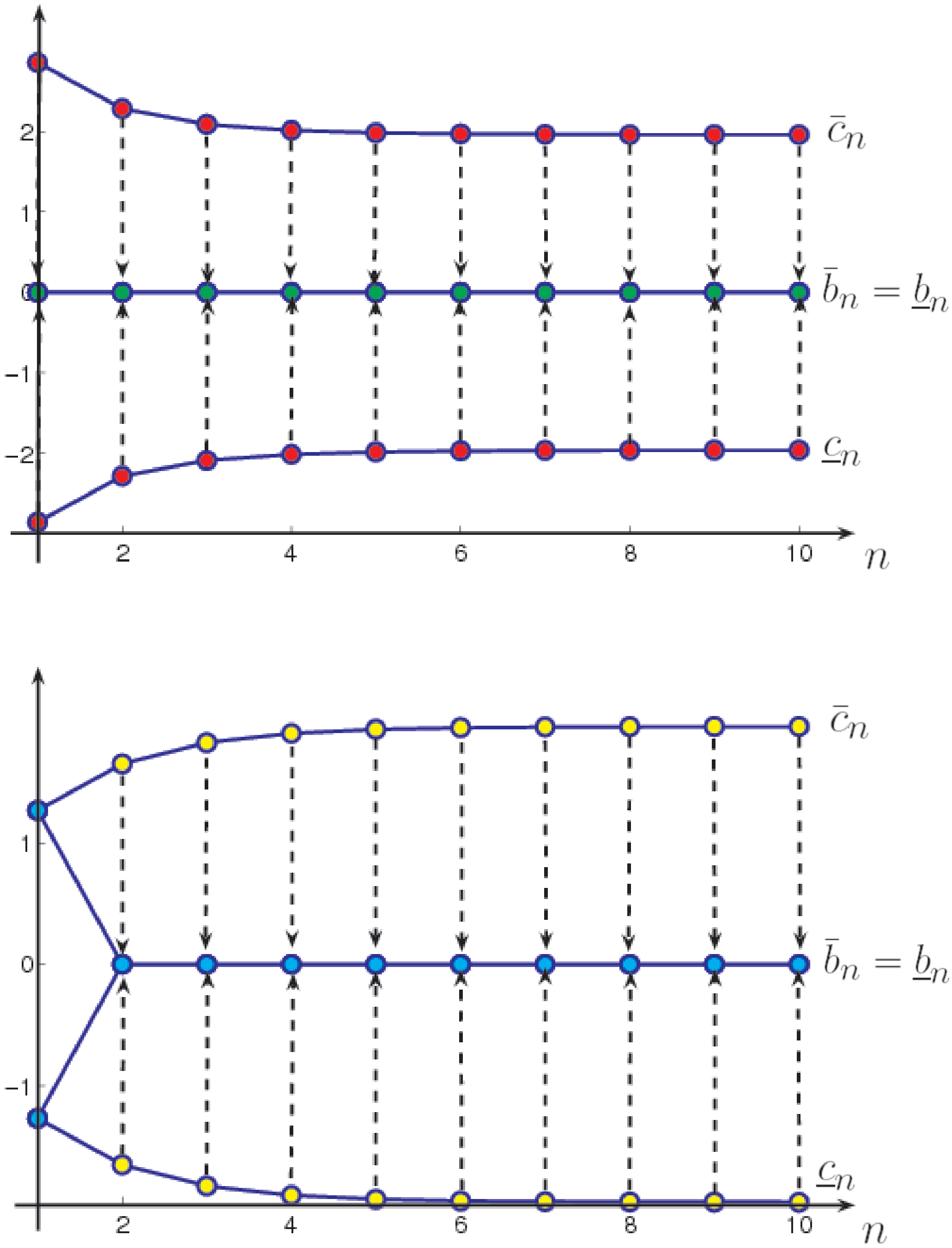

Figure 13: Optimal strategies.

1.3 A multi-dimensional example

In this section, we study a particular multi-dimensional example and apply the impulse control techniques introduced in Section 1.1. We assume that the uncontrolled process is a d-dimensional Brownian motion B with $d \geq 2$, i.e. $m = d \geq 2$, the infinitesimal drift b vanishes, and the infinitesimal variance σ is given by the identity matrix E^d. Further, we assume that the controlling costs consist of fixed and proportional costs $C : \mathbb{R}^d \to [K, \infty)$, $C(y) = K + k|y|$, where $k \in \mathbb{R}_0^+$. In addition, we assume that the running costs $f : \mathbb{R}^d \to \mathbb{R}_0^+$ are given by $f(y) = g(|y|)$, where $g : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is a continuous nondecreasing function. As before, $|.|$ denotes the Euclidean norm. The value function v is given by

$$v(x) = \inf_{S=(\tau_j,\delta_j)_{j\in\mathbb{N}}\in\mathcal{A}} E_x\Big(\int_0^\infty e^{-\alpha s} g(|X_s^S|)ds + \sum_{j=1}^\infty 1_{\{\tau_j<\infty\}} e^{-\alpha\tau_j}(K + k|\delta_j|)\Big), \tag{23}$$

where

$$X_t^S = x + B_t + \sum_{j=1}^\infty 1_{\{\tau_j \leq t\}}\delta_j.$$

Since the running costs and the controlling costs only depend on the Euclidean norm of the controlled process and since the uncontrolled process is a Brownian motion, we conjecture that v is rotationally symmetric. An application of Itô's formula to the d-dimensional Bessel process $|x + B|$ yields

$$|x + B_t| = |x| + \sum_{i=1}^d \int_0^t \frac{x^{(i)} + B_s^{(i)}}{|x + B_s|} dB_s^{(i)} + \int_0^t \frac{d-1}{2|x + B_s|} ds,$$

where the first term on the right-hand side is a one-dimensional Brownian motion W by Lévy's theorem. We introduce the following one-dimensional impulse control problem.

$$\nu(r) = \inf_{S=(\tau_j,\delta_j)_{j\in\mathbb{N}}\in\mathcal{A}} E_r\Big(\int_0^\infty e^{-\alpha s} g(\chi_s^S)ds + \sum_{j=1}^\infty 1_{\{\tau_j<\infty\}} e^{-\alpha\tau_j}(K + k|\delta_j|)\Big), \tag{24}$$

where

$$\chi_t^S = r + W_t + \int_0^t \frac{d-1}{2\chi_s^S} ds + \sum_{j=1}^\infty 1_{\{\tau_j \leq t\}}\delta_j$$

and δ_j is such that $\chi_{\tau_j}^S \geq 0$ for each $j \in \mathbb{N}$. Let $\varpi : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ be a continuous function. The infinitesimal generator and the minimum operator corresponding to this one-dimensional problem are given by

$$\Lambda\varpi(r) = -\alpha\varpi(r) + \frac{1}{2}\left(\frac{d-1}{r}\varpi'(r) + \varpi''(r)\right)$$
$$\mu\varpi(r) = \inf_{l\geq -r}(\varpi(r+l) + K + k|l|),$$

where the derivatives have to exist in some reasonable sense.

Proposition 1.43. *Let ϖ be a solution to the quasi-variational inequalities corresponding to (24) given above and set*

$$w : \mathbb{R}^d \to \mathbb{R}_0^+, \quad w(x) = \varpi(|x|).$$

Then w solves the quasi-variational inequalities corresponding to the original problem (23) on $\mathbb{R}^d \setminus \{0\}$.

We show that $Lw(x) = \Lambda\varpi(|x|)$ as well as $Mw(x) = \mu\varpi(|x|)$, for each $x \in \mathbb{R}^d \setminus \{0\}$.

Proof. Let ϖ be a solution to the quasi-variational inequalities corresponding to (24). For $x \in \mathbb{R}^d \setminus \{0\}$ and $1 \leq i \leq d$ we have

$$\partial_{x_i,x_i} w(x) = \varpi''(|x|)\frac{x_i^2}{|x|^2} + \varpi'(|x|)\left(\frac{1}{|x|} - \frac{x_i^2}{|x|^3}\right)$$

and therefore

$$\begin{aligned} Lw(x) &= -\alpha w(x) + \frac{1}{2}\sum_{i=1}^d \partial_{x_i,x_i} w(x) \\ &= -\alpha\varpi(|x|) + \frac{1}{2}\left(\frac{d-1}{|x|}\varpi'(|x|) + \varpi''(|x|)\right) = \Lambda\varpi(|x|), \end{aligned}$$

which implies that

$$Lw(x) + f(x) = \Lambda\varpi(|x|) + g(|x|) \geq 0.$$

Further, we have

$$\begin{aligned} w(x) = \varpi(|x|) \leq \mu\varpi(|x|) &= \inf_{l \geq -|x|}(\varpi(|x| + l) + K + k|l|) \\ &= \inf_{y \in \mathbb{R}^d}(\varpi(|x + y|) + K + k|y|) = \inf_{y \in \mathbb{R}^d}(w(x + y) + K + k|y|) \\ &= Mw(x). \end{aligned}$$

By the above

$$(w(x) - Mw(x))(Lw(x) + f(x)) = (\varpi(|x|) - \mu\varpi(|x|))(\Lambda\varpi(|x|) + g(|x|)) = 0.$$

and the assertion follows. □

As a sufficient condition for optimality, we give the following verification result.

Lemma 1.44. *If there exists a nondecreasing function $\varpi : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ and a pair (b_0, r_0) with $0 \leq b_0 < r_0$ such that we have*

(i) $\varpi \in C^2((0, r_0])$ *and* $\Lambda\varpi(r) + g(r) = 0$ *for all* $r \in (0, r_0]$

(ii) $\varpi(r) = \varpi(r_0) + k(r - r_0)$ *for all* $r \geq r_0$

(iii) $\varpi(r_0) = \varpi(b_0) + K + k(r_0 - b_0)$

(iv) $\varpi'(r) = k \iff r \in \{b_0\} \cup [r_0, \infty)$ *and* $\varpi'(0) = 0$

(v) $\varpi''(b_0) > 0$

(vi) $-\alpha kr + \frac{(d-1)k}{2r} + g(r) \geq -\alpha kr_0 + \frac{(d-1)k}{2r_0} + g(r_0)$ *for all* $r \geq r_0$

then ϖ *is in* $C^1 \cap C^2(\mathbb{R}^+ \setminus \{r_0\})$ *and the value functions (23) and (24) are given by*

$$v(x) = \nu(|x|) = \varpi(|x|),$$

for each $x \in \mathbb{R}^d \setminus \{0\}$*. Further, the optimal strategy for problem (24) is to shift* χ *to* b_0 *if it is greater or equal than* r_0*, and to do nothing as long as* χ *is less than* r_0*. The optimal strategy for (23) is to do nothing a long as* $|X| < r_0$*. If* $|X| = r_0$*, one has to shift* X *towards* $\frac{b_0}{r_0}X$*, which is the nearest point (w.r.t. the Euclidean norm) of the circle centered at zero with radius* b_0*.*

Proof. By the assumptions (i), (ii) and since $\varpi'(r_0) = k$ we have $\varpi \in C^1 \cap C^2(\mathbb{R}^+ \setminus \{r_0\})$. If ϖ solves the quasi-variational inequalities corresponding to (24), then by Remark 1.17 it coincides with the value function ν and the qvi-control is optimal. Further, by Proposition 1.43 the function $\varpi(|\cdot|)$ solves the quasi-variational inequalities for problem (23) and therefore, since $d \geq 2$ by Remark 1.20 it coincides with the value function v. Again, the optimal strategy is given by the corresponding qvi-control. Thus, we only have to show that ϖ is a solution to the quasi-variational inequalities corresponding to (24).

First, we show that $\Lambda\varpi + g \geq 0$. By construction of ϖ we only have to show this for $r \geq r_0$. Therefore, let $r \geq r_0$.

$$\begin{aligned}
\Lambda\varpi(r) + g(r) &\overset{(ii)}{=} -\alpha(\varpi(r_0) + k(r - r_0)) + \frac{(d-1)k}{2r} + g(r) \\
&\overset{(i)}{=} -\alpha k(r - r_0) + \frac{(d-1)k}{2r} + g(r) - \frac{1}{2}\left(\frac{d-1}{r_0}\varpi'(r_0) + \varpi''(r_0)\right) - g(r_0) \\
&\overset{(iv)}{=} -\alpha k(r - r_0) + \frac{(d-1)k}{2}\left(\frac{1}{r} - \frac{1}{r_0}\right) + g(r) - g(r_0) - \frac{1}{2}\varpi''(r_0).
\end{aligned}$$

By the assumptions (iv) and (v) we have $\varpi''(r_0) \leq 0$ and thus, by (vi) we have

$$\Lambda\varpi(r) + g(r) \geq 0.$$

Next, we show that $\varpi \leq M\varpi$ in three steps. Note that since ϖ is nondecreasing, we have

$$\mu\varpi(r) = \inf_{l \in [0,r]}(\varpi(r - l) + K + kl).$$

1. Let $r \geq r_0$. We have

$$\varpi(r) \overset{(ii)}{=} \varpi(r_0) + k(r - r_0) \overset{(iii)}{=} \varpi(b_0) + K + k(r - b_0).$$

Due to the affine linear form of ϖ on $[r_0, \infty)$, we have

$$\mu\varpi(r) = \inf_{l \in [r-r_0, r]} (\varpi(r-l) + K + kl).$$

Since $b_0 \in [0, r_0]$, $\varpi'(b_0) = k$ and $\varpi''(b_0) > 0$, a local minimum of $l \mapsto \varpi(r-l) + K + kl$ on $[r - r_0, r]$ is attained at $l = r - b_0$. By assumption (iv) and (v), it is the global minimum, i.e we have

$$\mu\varpi(r) = \varpi(b_0) + K + k(r - b_0).$$

Thus, for all $r \geq r_0$ we have

$$\varpi(r) = \mu\varpi(r).$$

2. Let $b_0 \leq r < r_0$. As in the previous case, by (iv) and (v) we have

$$\mu\varpi(r) = \varpi(b_0) + K + k(r - b_0).$$

Further, by (iv) and (v) we have $\varpi'(l) > k$ for all $l \in (r, r_0)$ and therefore, since $\varpi(r_0) = \varpi(b_0) + K + k(r_0 - b_0)$ it follows that

$$\varpi(r) < \varpi(b_0) + K + k(r - b_0).$$

3. Let $0 \leq r < b_0$. By assumption (iv), the minimum of the function $l \mapsto \varpi(r-l) + K + kl$ on $[0, r]$ is attained at the boundary. By (iv) and (v) we have $\varpi'(l) < k$ for all $l \in [0, r]$. Thus we have $\varpi(r) + K < \varpi(0) + K + kr$ and therefore

$$\mu\varpi(r) = \varpi(r) + K > \varpi(r).$$

Finally, $(\varpi - \mu\varpi)(L\varpi + g) = 0$ on $\mathbb{R}^+$ follows immediately from 1. and (i). $\square$

The following remark states a sufficient condition for (vi) in case of quadratic running costs.

Remark 1.45. *If $k = 0$ then (vi) is satisfied since g is nondecreasing. For general k and $g(r) = r^2$ we have*

$$-\alpha kr + \frac{(d-1)k}{2r} + g(r) = -\alpha kr_0 + \frac{(d-1)k}{2r_0} + g(r_0) \iff r \in \{r_0, r_1, r_2\}$$

where

$$r_{1,2} = \frac{1}{2}(\alpha k - r_0) \pm \frac{1}{4r_0}\sqrt{8k(d-1)r_0 + (2\alpha kr_0 - 2r_0^2)^2}.$$

Further, we have $\lim_{r\to\infty} -\alpha kr + \frac{(d-1)k}{2r} + r^2 = \infty$ and therefore, a sufficient condition for (vi) is given by $r_0 \geq r_1 \vee r_2 = r_1$.

Regarding existence of a function ϖ satisfying (i), (iii) and (iv), we state the following lemma (e.g. see H. Heuser [12, Satz 21.3]).

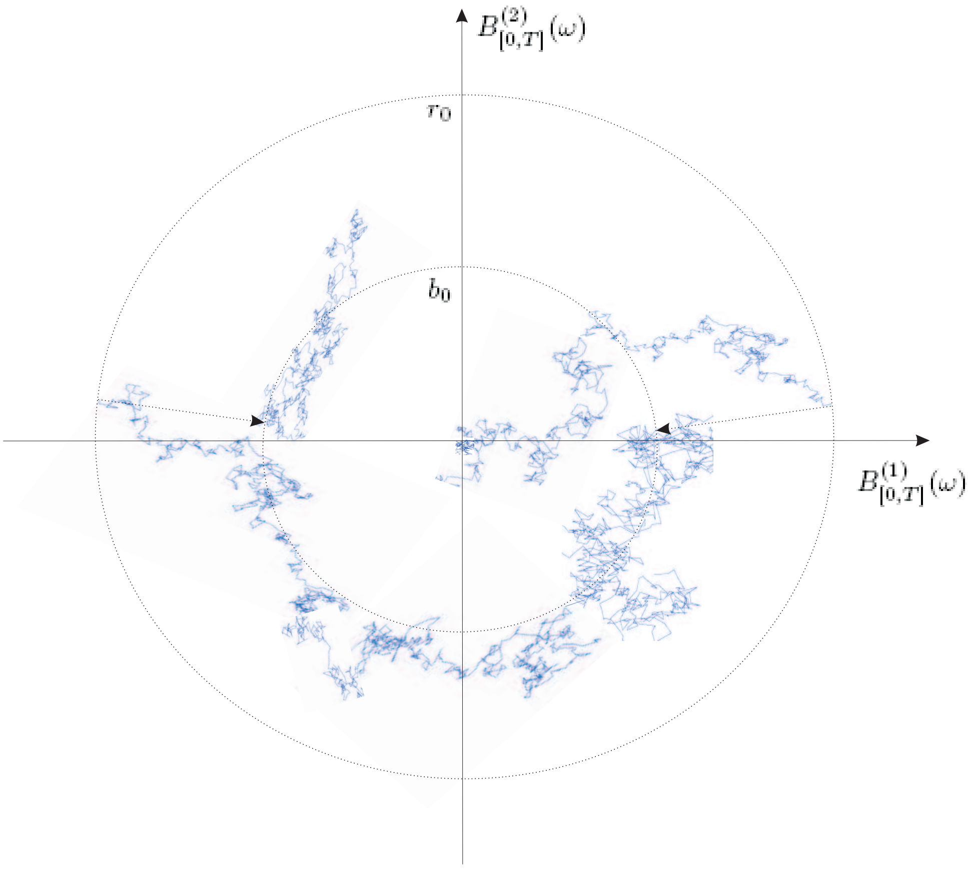

Figure 14: Optimal strategy for v with $d = 2$ and $g(r) = r^2$.

Lemma 1.46. *Let r_0, b_0 and ϖ_{b_0} in $\mathbb{R}_0^+$. There exists a unique solution $\varpi : (0, r_0] \to \mathbb{R}$ of the constraint ordinary differential equation*

$$\Lambda\varpi + g = 0, \quad \varpi(r_0) = \varpi_{b_0} + K + k(r_0 - b_0), \quad \varpi'(r_0) = k.$$

It is given by

$$\varpi(r) = \varpi_{b_0} + K + k(r - b_0) + \int_r^{r_0} (s - r)V(s)ds,$$

where $V \in C((0, r_0])$ is the solution of the Volterra integral equation

$$V(r) + \int_r^{r_0} \kappa(r, s)V(s)ds = h(r),$$

with $h(r) = -2g(r) - \frac{(d-1)k}{r} + 2\alpha(\varpi_{b_0} + K + k(r - b_0))$ and $\kappa(r, s) = -\frac{d-1}{r} - 2\alpha(s - r)$.

An explicit representation of V is given by

$$V(r) = h(r) - \int_r^{r_0} \sum_{k=1}^{\infty} \kappa_k(r,s)h(s)ds,$$

where $\kappa_1 = \kappa$ and $\kappa_k(r,s) = -\int_r^s \kappa_1(r,u)\kappa_{k-1}(u,s)du$ for $k \geq 2$.

Now, for a numerical analysis of the value function, the remaining task is to construct the solution ϖ as in the lemma above and to choose ϖ_{b_0}, r_0 and b_0 such that $\varpi'(0) = 0$, $\varpi'(b_0) = k$ and $\varpi(b_0) = \varpi_{b_0}$, i.e we have to solve three fixed point problems. Afterwards, we expand ϖ such that $\varpi(r) = \varpi(r_0) + k(r - r_0)$ for all $r \geq r_0$. Finally, we have to check the conditions (v), (vi) and with regards to (iv), we must check whether $\varpi'(s) \neq k$ for all $s \in (0, r_0) \setminus \{b_0\}$. If ϖ is also nondecreasing, then by Lemma 1.44 it coincides with the value function ν and the optimal strategy is characterized by (b_0, r_0).

In order to illustrate the form and the slope of the value function ν, we plot them in Figure 15 for the choice of $d = 2$, $g(r) = r^2$, $\alpha = 0.3$, $K = 2$ and $k = 0.2$. The dependance of r_0 and b_0 on proportional and fixed costs is illustrated in Figure 16. If the proportional or fixed costs increase, then the optimal intervention radius r_0 becomes greater. Further, a rise in the proportional costs also leads to a greater optimal target radius b_0, since the optimal intervention $r_0 - b_0$ becomes smaller and r_0 becomes greater. On the other hand, a rise in the fixed costs yields a smaller value for b_0, since it becomes better to intervene less often, although one has to pay some additional proportional costs. For $k = 0$, the optimal strategy is to shift χ to the point with smallest possible running costs and therefore $b_0 = 0$.

Figure 17 depicts the value function ν in dependance of proportional and fixed costs, respectively. The optimal strategies are illustrated by vertical lines, corresponding to r_0 and b_0. Clearly, if the controlling costs increase, then also the corresponding value functions increase.

Eventually, Figure 18 depicts the value function ν and its first derivative ν' for several running cost functions g. Note that the optimal intervention radius r_0 corresponding to the running cost function $g(r) = e^r$ is between those corresponding to the running cost functions $g(r) = r^2$ and $g(r) = r^4$. This is reasonable, since on this interval we have $r^2 \leq e^r \leq r^4$. Here, the optimal target radii b_0 are almost the same, since the proportional costs are very small in comparison to the running costs.

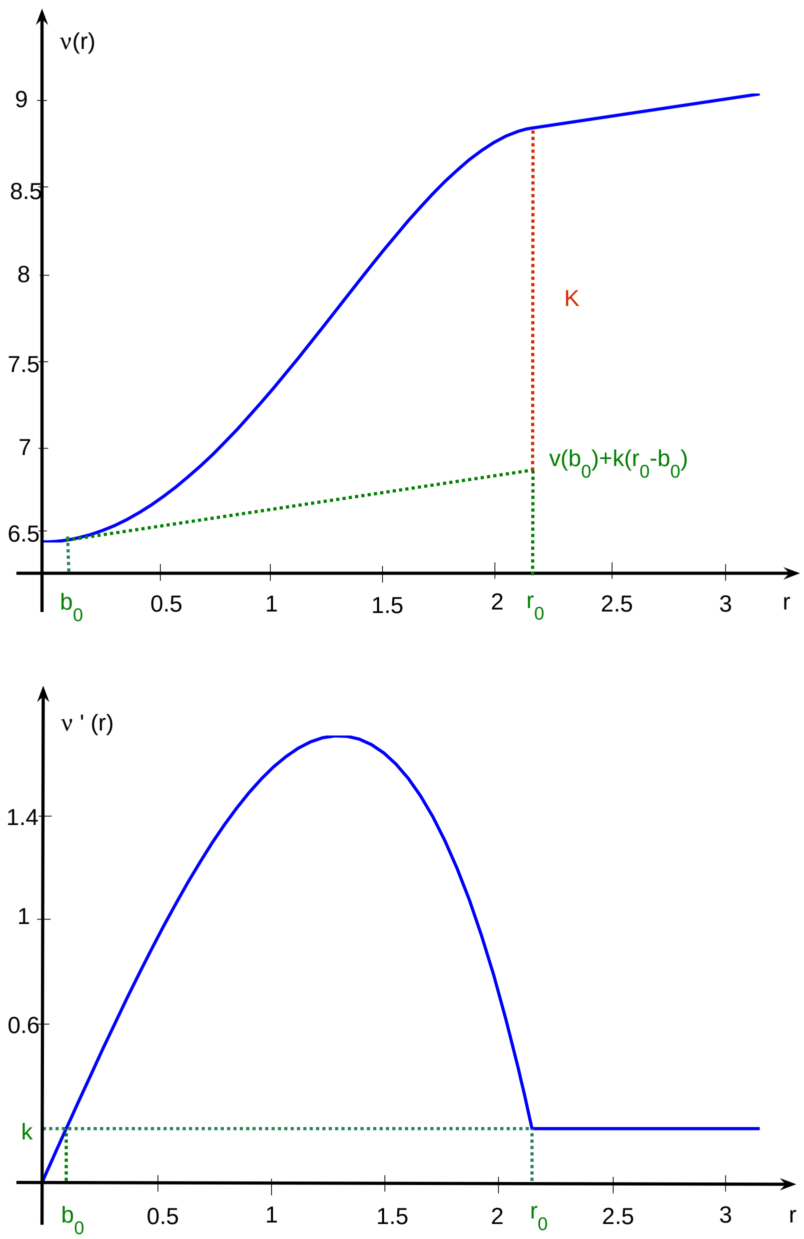

Figure 15: Value function ν.

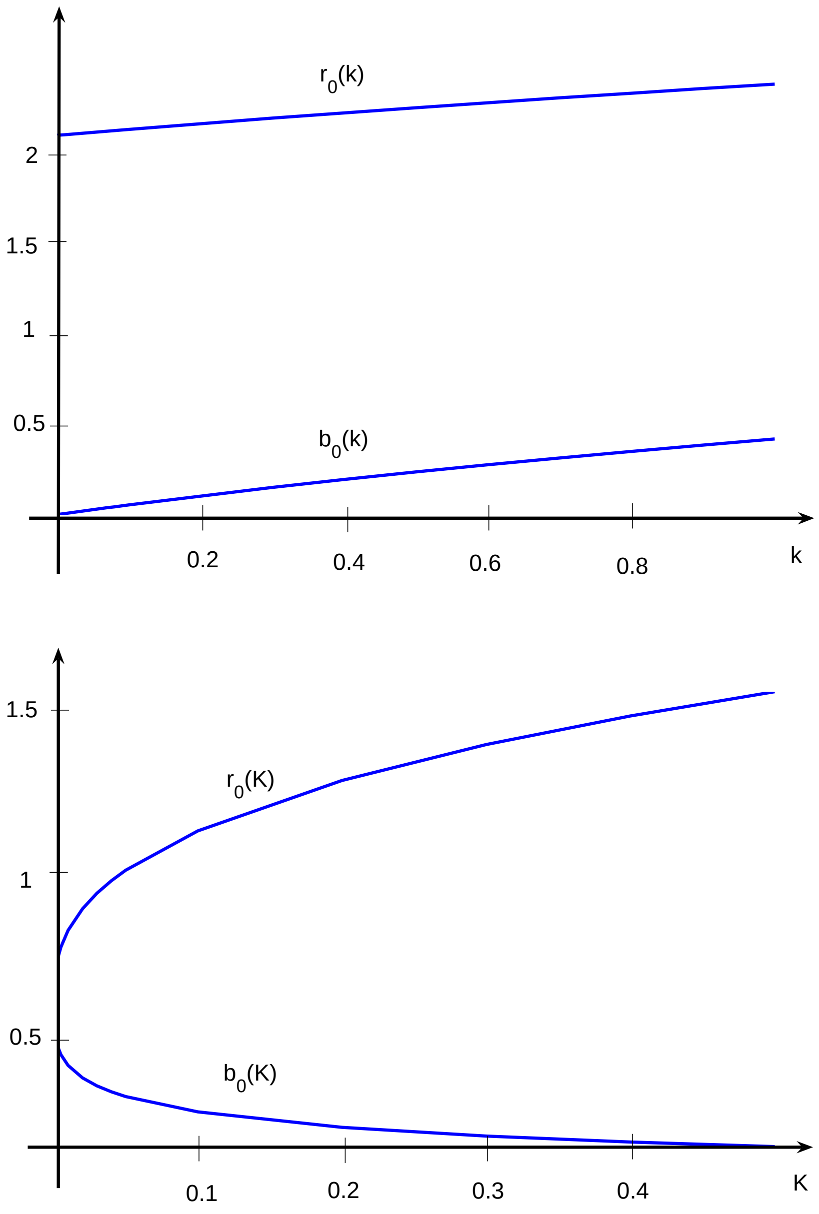

Figure 16: Optimal strategy in dependance of proportional and fixed costs.

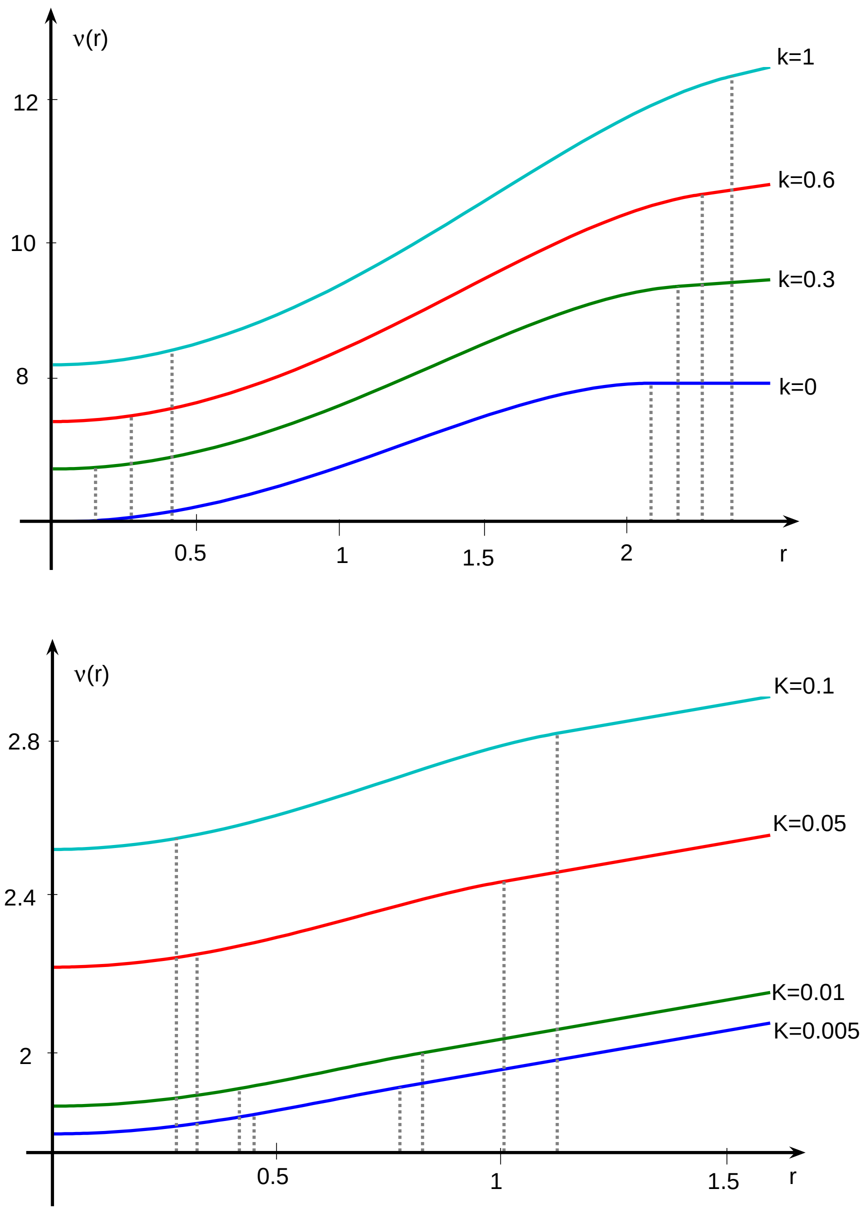

Figure 17: Value function in dependance of proportional and fixed costs.

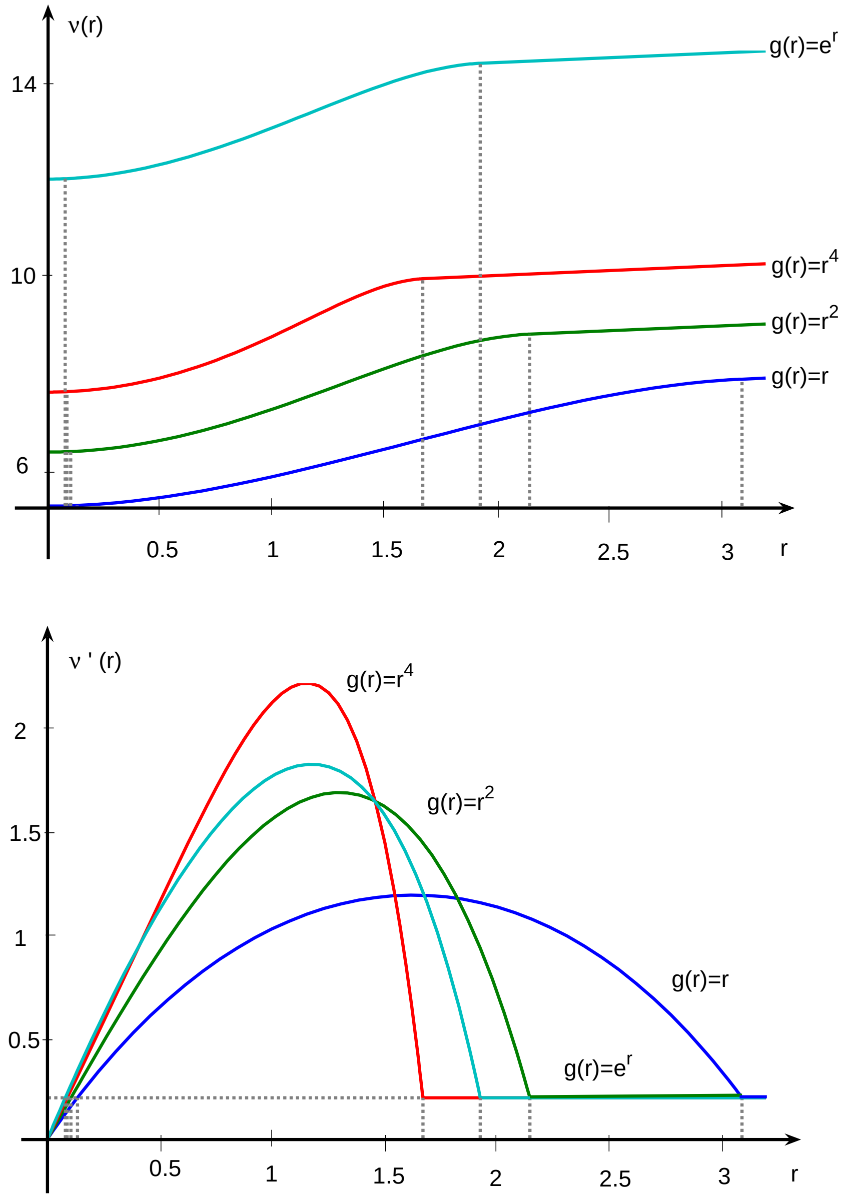

Figure 18: Value function in dependance of running costs.

1.4 Self-financing stochastic impulse control

In this section, we consider the impulse control problem (1) where the controller minimizes the expected discounted running costs of a diffusion process X. However, this time, we additionally assume that the controller is allowed to invest his initial capital into a market consisting of two assets. The first one being riskfree and paying a constant interest $r > 0$ ("bond"), the second one being risky ("stock") with drift $\beta > r$ and volatility $\xi > 0$. Controlling costs upon shifting the paths of the process X have to be paid out of the bond holdings. Further, whenever the controller changes his portfolio by selling or buying shares of the stock, he also faces costs including a positive fixed cost component. For sake of simplicity we assume that X is one-dimensional and that the fixed cost component does not depend on whether the investor controls the process X or whether he rebalances his portfolio. In order to exclude arbitrage strategies we assume that the wealth invested in the bond as well as the wealth invested in the stock remain nonnegative. We assume that B is a two-dimensional Brownian motion.

This time, the action of the controller is modeled as follows.

Definition 1.47. *An **impulse control strategy** $S = (\tau_n, \zeta_n, \delta_n)_{n\in\mathbb{N}}$ is a sequence such that for all $n \in \mathbb{N}$*

- $\diamond$ $0 = \tau_0 \leq \tau_1 \leq \cdots \leq \tau_n \leq \cdots$ *almost surely*
- $\diamond$ $\tau_n : \Omega \to [0, \infty]$ *is a stopping time w.r.t* $(\mathcal{F}_t)_{t\geq 0}$ *"intervention times"*
- $\diamond$ $\zeta_n : \Omega \to \mathbb{R}$ *is measurable w.r.t. the σ-algebra of τ_n-past $\mathcal{F}_{\tau_n}$* *"transactions"*
- $\diamond$ $\delta_n : \Omega \to \mathbb{R}$ *is measurable w.r.t. the σ-algebra of τ_n-past $\mathcal{F}_{\tau_n}$* *"control actions"*

We interpret τ_n as the n-th time at which the controller intervenes. At each intervention, he controls the process X by shifting its path by δ_n or he changes his portfolio by investing ζ_n of his money from the bond into the stock.

Remark 1.48. *Let $\tau_0, \ldots, \tau_n : \Omega \to [0, \infty]$ be $(\mathcal{F}_t)_{t\geq 0}$-stopping times such that $0 = \tau_0 \leq \tau_1 \leq \cdots \leq \tau_n$ almost surely and let ζ_k, $\delta_k : \Omega \to \mathbb{R}$ be $\mathcal{F}_{\tau_k}$-measurable for all $0 \leq k \leq n$. Then we will identify the vector*

$$((\tau_0, \zeta_0, \delta_0), \ldots, (\tau_n, \zeta_n, \delta_n))$$

with an impulse control strategy by setting $\tau_l \equiv \infty$, $\zeta_l \equiv 0$ and $\delta_l \equiv 0$ for all $l > n$ (see Definition 1.51).

Let P^0 and P^1 denote the wealth invested in the bond and in the stock, respectively. Their dynamics are given by

$$dP_t^0 = rP_t^0 dt \qquad \text{"bond holdings"}$$

$$dP_t^1 = P_t^1(\beta dt + \xi dB_t^{(1)}) \qquad \text{“stock holdings”}$$

Each intervention causes costs, consisting of a positive fixed component K as well as costs c which depend on the size of the control and the size of the stock transaction.

Definition 1.49. *Let $K > 0$ and let $c : \mathbb{R}^2 \to \mathbb{R}_0^+$ such that*

- ⋄ *c is continuous and $c(0) = 0$*
- ⋄ *$c(y) \to \infty$ as $|y| \to \infty$*
- ⋄ *c is subadditive*

*Then $C : \mathbb{R}^2 \to [K, \infty)$, $C(y) = K + c(y)$ is called **intervention cost** with fixed costs K and variable cost c.*

Definition 1.50. *Let $S = (\tau_n, \zeta_n, \delta_n)_{n\in\mathbb{N}}$ be an impulse control strategy and let*

$$b : \mathbb{R} \to \mathbb{R} \qquad\qquad \sigma : \mathbb{R} \to \mathbb{R}$$

*be Borel measurable and locally bounded. A stochastic process $(P_t^0, P_t^1, X_t)_{t\geq 0}$ adapted to the filtration $(\mathcal{F}_t)_{t\geq 0}$ is called a **controlled process** corresponding to the strategy S if it solves the following stochastic differential equations*

$$dP_t^0 = rP_{t-}^0 dt - \sum_{j=1}^{\infty} 1_{\{\tau_j = t\}}(\zeta_j + C(\zeta_j, \delta_j)), \quad P_{0-}^0 = p_0$$

$$dP_t^1 = P_{t-}^1(\beta dt + \xi dB_t^{(1)}) + \sum_{j=1}^{\infty} 1_{\{\tau_j = t\}}\zeta_j, \quad P_{0-}^1 = p_1$$

$$dX_t = b(X_{t-})dt + \sigma(X_{t-})dB_t^{(2)} + \sum_{j=1}^{\infty} 1_{\{\tau_j = t\}}\delta_j, \quad X_{0-} = x$$

*where $x \in \mathbb{R}$ and p_0, $p_1 \in \mathbb{R}_0^+$. We refer to (p_0, p_1, x) as the **starting point** of the process (P^0, P^1, X).*

Let $S = (\tau_n, \zeta_n, \delta_n)_{n\in\mathbb{N}}$ be an impulse control strategy such that $\lim_{n\to\infty} \tau_n = \infty$ almost surely. If b and σ are Lipschitz then there exists a unique càdlàg, $(\mathcal{F}_t)_{t\geq 0}$-adapted strong solution of the stochastic differential equations above. From now on we always assume that b and σ are Lipschitz. As before, when there is a risk of ambiguity, we will also write $(P^{0,S}, P^{1,S}, X^S)$ for a controlled process corresponding to a strategy S.

Definition 1.51. *Let $S = (\tau_n, \zeta_n, \delta_n)_{n\in\mathbb{N}}$ be an impulse control strategy and let (P^0, P^1, X) denote the corresponding controlled process. S will be called **admissible** if we have*

- ⋄ $\lim_{n\to\infty} \tau_n = \infty$ *almost surely*

$\diamond$ $P^0_{\tau_i} \geq 0$, $P^1_{\tau_i} \geq 0$ *for each* $i \in \mathbb{N}$.

The set of all admissible impulse control strategies will be denoted by $\mathcal{A}(p_0, p_1)$.

Let $f : \mathbb{R} \to \mathbb{R}^+_0$ be Borel measurable and let $\alpha > 0$. We interpret f as the **running cost** and α as a **discount factor**. We consider the following impulse control problem of minimizing the expected discounted running costs of X over the set of admissible impulse control strategies. The controller has to pay the primary costs, that is his expenses resulting from controlling X and buying or selling shares of the stock, immediately out of the bond. Upon minimizing the running costs of X, he has to take care that his wealth invested in the bond and in the stock remains nonnegative. However, he does not necessarily have to maximize P^0 or P^1.

$$v(p_0, p_1, x) = \inf_{S=(\tau_n,\zeta_n,\delta_n)_{n\in\mathbb{N}}\in\mathcal{A}(p_0,p_1)} E_x\Big(\int_0^\infty e^{-\alpha s} f(X^S_s)ds\Big) \tag{25}$$

The function v is called the **value function** corresponding to this impulse control problem. An admissible strategy S such that the infimum is attained will be called **optimal**. In particular, note that an optimal control for X will depend on the performance of the risky asset P^1. Given the existence of an optimal strategy S, similar as in the proof of Lemma 1.31, by the subadditivity of C, we can construct an admissible strategy $\hat{S} = (\tau_n, \zeta_n, \delta_n)_{n\in\mathbb{N}}$ such that $\tau_n < \tau_{n+1}$ whenever $\tau_n < \infty$, which yields the same running costs as S. From now on we always assume that the intervention times of an optimal strategy are strictly increasing whenever they are finite. Note that since the components of C are even *strictly* increasing, following the strategy $\hat{S}$ yields a higher wealth in the bond.

Definition 1.52. *Let C be an intervention cost function and let $u : (\mathbb{R}^+_0)^2 \times \mathbb{R} \to \mathbb{R}$ be bounded from below. Then we set*

$$Mu(p_0, p_1, x) = \inf_{(\zeta,\delta)\in\Theta(p_0,p_1)} u(p_0 - \zeta - C(\zeta, \delta), p_1 + \zeta, x + \delta),$$

where $\Theta(p_0, p_1) = \{(\zeta, \delta) \in \mathbb{R}^2 : \zeta + C(\zeta, \delta) \leq p_0, \ \zeta \geq -p_1\}$.

Note the difference between the definition of Θ and the admissibility conditions $P^0_{\tau_i} \geq 0$, $P^1_{\tau_i} \geq 0$. However, they coincide for strategies which are such that their intervention times are strictly increasing. $Mv(p_0, p_1, x)$ represents the value of the strategy that consists of doing the best immediate control of the process X or doing the best immediate stock investment, when starting in (p_0, p_1, x) and behaving optimally afterwards.

Definition 1.53. *Let $v^* : (\mathbb{R}^+_0)^2 \times \mathbb{R} \to \mathbb{R}^+_0$ be twice continuously differentiable and for $(p_0, p_1, x) \in (\mathbb{R}^+_0)^2 \times \mathbb{R}$ set*

$$\begin{aligned} Lv^*(p_0, p_1, x) = &-\alpha v^*(p_0, p_1, x) + \partial_{p_0} v^*(p_0, p_1, x) r p_0 + \partial_{p_1} v^*(p_0, p_1, x)\beta p_1 \\ &+ \frac{1}{2}\partial_{p_1,p_1} v^*(p_0, p_1, x)\xi^2 p_1^2 + \partial_x v^*(p_0, p_1, x)b(x) + \frac{1}{2}\partial_{x,x} v^*(p_0, p_1, x)\sigma^2(x). \end{aligned}$$

If G is the generator of the process $(s, P_s^0, P_s^1, X_s)_{s\geq 0}$ when there are no interventions, then we have $e^{-\alpha s}Lv^*(p_0,p_1,x) = G(e^{-\alpha \cdot}v^*)(s,p_0,p_1,x)$. The quasi-variational inequalities corresponding to (25) are the same as in Definition 1.11. Note that generalized Itô formulas involving local time may not be applied in this setting as we are dealing with a (at least) three-dimensional problem. The following definition introduces our best guess for an optimal intervention strategy.

Definition 1.54. *Let v^* be a solution to the quasi-variational inequalities, let $(\tau_0, \zeta_0, \delta_0) \equiv 0$ and for $n \geq 1$ set*

$$
\begin{aligned}
&S_{n-1} = ((\tau_0,\zeta_0,\delta_0),\ldots,(\tau_{n-1},\zeta_{n-1},\delta_{n-1}))\\
&\tau_n = \inf\{t \geq \tau_{n-1} : v^*(P_t^{0,S_{n-1}}, P_t^{1,S_{n-1}}, X_t^{S_{n-1}}) = Mv^*(P_t^{0,S_{n-1}}, P_t^{1,S_{n-1}}, X_t^{S_{n-1}})\}\\
&(\zeta_n,\delta_n) = \begin{cases} \arg Mv^*(P_{\tau_n}^{0,S_{n-1}}, P_{\tau_n}^{1,S_{n-1}}, X_{\tau_n}^{S_{n-1}}) & \text{if } \tau_n < \infty\\ \qquad (0,0) & \text{if } \tau_n = \infty. \end{cases}
\end{aligned}
$$

We then call $S = (\tau_n,\zeta_n,\delta_n)_{n\in\mathbb{N}}$ a ***qvi-control****.*

Next, we state a sufficient condition for optimality of the qvi-control.

Theorem 1.55 (Verification theorem)**.** *Assume that there exists a solution $v^* \in C^2$ to the quasi-variational inequalities for the impulse control problem (25) such that*

$$E_{p_0,p_1,x}\Big(\int_0^T (e^{-\alpha s}\partial_{p_1}v^*(P_s^{0,S}, P_s^{1,S}, X_s^S)\xi P_s^{1,S})^2 ds\Big) < \infty \quad \forall\, T > 0 \tag{26}$$

$$E_{p_0,p_1,x}\Big(\int_0^T (e^{-\alpha s}\partial_x v^*(P_s^{0,S}, P_s^{1,S}, X_s^S)\sigma(X_s^S))^2 ds\Big) < \infty \quad \forall\, T > 0 \tag{27}$$

$$\liminf_{T\to\infty} E_{p_0,p_1,x}(e^{-\alpha T}v^*(P_T^{0,S}, P_T^{1,S}, X_T^S)) = 0 \tag{28}$$

for all $S \in \mathcal{A}(p_0,p_1)$. Then we have $v^*(p_0,p_1,x) \leq v(p_0,p_1,x)$. Further, if the qvi-control to v^* is an admissible impulse control strategy then it is optimal and we have $v(p_0,p_1,x) = v^*(p_0,p_1,x)$.

Without the admissibility assumption of nonnegative wealth, the transversality condition (28) would not be satisfied.

Proof. Let $S = (\tau_n,\zeta_n,\delta_n)_{n\in\mathbb{N}} \in \mathcal{A}(p_0,p_1)$ be an admissible impulse control strategy such that we have $\tau_n < \tau_{n+1}$, whenever τ_n is finite and $n \geq 1$. Further, let (P^0,P^1,X) be the corresponding controlled process and let $T > 0$. By Itô's formula applied to the $C^{1,2}$ function $(s,p_0,p_1,x) \mapsto e^{-\alpha s}v^*(p_0,p_1,x)$ and the càdlàg semimartingale (P^0,P^1,X) we obtain

$$e^{-\alpha T}v^*(P_T^0,P_T^1,X_T)-v^*(p_0,p_1,x)-\sum_{0\leq s\leq T}e^{-\alpha s}(v^*(P_s^0,P_s^1,X_s)-v^*(P_{s-}^0,P_{s-}^1,X_{s-}))$$

$$=\int_0^T -\alpha e^{-\alpha s}v^*(P_s^0,P_s^1,X_s)ds+\int_0^T e^{-\alpha s}\partial_{p_0}v^*(P_s^0,P_s^1,X_s)rP_s^0 ds$$

$$+\int_0^T e^{-\alpha s}\partial_{p_1}v^*(P_s^0,P_s^1,X_s)\beta P_s^1 ds+\int_0^T e^{-\alpha s}\partial_{p_1}v^*(P_{s-}^0,P_{s-}^1,X_{s-})\xi P_{s-}^1 dB_s^{(1)}$$

$$+\int_0^T e^{-\alpha s}\partial_x v^*(P_s^0,P_s^1,X_s)b(X_s)ds+\int_0^T e^{-\alpha s}\partial_x v^*(P_{s-}^0,P_{s-}^1,X_{s-})\sigma(X_{s-})dB_s^{(2)}$$

$$+\frac{1}{2}\int_0^T e^{-\alpha s}\partial_{p_1,p_1}v^*(P_s^0,P_s^1,X_s)\xi^2(P_s^1)^2 ds+\frac{1}{2}\int_0^T e^{-\alpha s}\partial_{x,x}v^*(P_s^0,P_s^1,X_s)\sigma^2(X_s)ds.$$

Thus, by the definition of L, we have

$$\begin{aligned}
&v^*(p_0,p_1,x)-e^{-\alpha T}v^*(P_T^0,P_T^1,X_T)\\
&\qquad=-\int_0^T e^{-\alpha s}\partial_{p_1}v^*(P_{s-}^0,P_{s-}^1,X_{s-})\xi P_{s-}^1 dB_s^{(1)}\\
&\qquad\quad-\int_0^T e^{-\alpha s}\partial_x v^*(P_{s-}^0,P_{s-}^1,X_{s-})\sigma(X_{s-})dB_s^{(2)}\\
&\qquad\quad-\int_0^T e^{-\alpha s}Lv^*(P_s^0,P_s^1,X_s)ds\\
&\qquad\quad-\sum_{i=1}^{\infty}1_{\{\tau_i\leq T\}}e^{-\alpha\tau_i}(v^*(P_{\tau_i}^0,P_{\tau_i}^1,X_{\tau_i})-v^*(P_{\tau_i-}^0,P_{\tau_i-}^1,X_{\tau_i-})).
\end{aligned}$$

Since v^* is a solution to the qvi, it holds that

$$-Lv^*\leq f$$

as well as

$$\begin{aligned}
v^*(P_{\tau_i-}^0,P_{\tau_i-}^1,X_{\tau_i-})&\leq Mv^*(P_{\tau_i-}^0,P_{\tau_i-}^1,X_{\tau_i-})\\
&\leq v^*(P_{\tau_i-}^0-\zeta_i-C(\zeta_i,\delta_i),P_{\tau_i-}^1+\zeta_i,X_{\tau_i-}+\delta_i)\\
&=v^*(P_{\tau_i}^0,P_{\tau_i}^1,X_{\tau_i}),
\end{aligned}$$

whenever $\tau_i < \infty$ and $i \geq 1$. The second inequality is due to the fact that since the stopping times of S are strictly increasing and $S \in \mathcal{A}(p_0, p_1)$, we have $(\zeta_i, \delta_i) \in \Theta(P^0_{\tau_i -}, P^1_{\tau_i -})$. Consequently, the assumptions (26) and (27) yield

$$v^*(p_0, p_1, x) - E_{p_0, p_1, x}(e^{-\alpha T} v^*(P^0_T, P^1_T, X_T)) \leq E_x(\int_0^T e^{-\alpha s} f(X_s) ds). \tag{29}$$

Finally, by monotone convergence and assumption (28) we have

$$v^*(p_0, p_1, x) \leq E_x(\int_0^\infty e^{-\alpha s} f(X_s) ds).$$

Thus we have

$$v^*(p_0, p_1, x) \leq v(p_0, p_1, x).$$

Let $S = (\tau_n, \zeta_n, \delta_n)_{n \in \mathbb{N}}$ be the qvi-control, assume that $S \in \mathcal{A}(p_0, p_1)$ and let (P^0, P^1, X) be the correspondingly controlled process. Let $l = \inf\{k \in \mathbb{N} : \tau_k = \infty\}$. Set $m_0 = 0$, $m_1 = 1$ and for $i \geq 2$ set

$$m_i = \inf\{j \in \mathbb{N} : \tau_j > \tau_{m_{i-1}}\} \wedge l.$$

Note that $\tau_{m_0} \leq \tau_{m_1} < \cdots < \tau_{m_i}$ for all $i \in \mathbb{N}$ such that $\tau_{m_{i-1}} < \infty$. For $T > 0$ we have

$$\begin{aligned}
&v^*(p_0, p_1, x) - e^{-\alpha T} v^*(P^0_T, P^1_T, X_T) \\
&\qquad = -\int_0^T e^{-\alpha s} \partial_{p_1} v^*(P^0_{s-}, P^1_{s-}, X_{s-}) \xi P^1_{s-} dB^{(1)}_s \\
&\qquad\quad - \int_0^T e^{-\alpha s} \partial_x v^*(P^0_{s-}, P^1_{s-}, X_{s-}) \sigma(X_{s-}) dB^{(2)}_s \\
&\qquad\quad - \int_0^T e^{-\alpha s} L v^*(P^0_s, P^1_s, X_s) ds \\
&\qquad\quad - \sum_{i=1}^\infty 1_{\{\tau_{m_i} \leq T\}} e^{-\alpha \tau_{m_i}} (v^*(P^0_{\tau_{m_i}}, P^1_{\tau_{m_i}}, X_{\tau_{m_i}}) - v^*(P^0_{\tau_{m_i}-}, P^1_{\tau_{m_i}-}, X_{\tau_{m_i}-})).
\end{aligned}$$

Since v^* is a solution to the qvi and by the definition of S we get

$$\begin{aligned}
&v^*(P^0_{\tau_{m_i}-}, P^1_{\tau_{m_i}-}, X_{\tau_{m_i}-}) \\
=& v^*(P^0_{\tau_{m_i}-} - \zeta_{m_i} - C(\zeta_{m_i}, \delta_{m_i}) - \cdots - \zeta_{m_{i+1}-1} - C(\zeta_{m_{i+1}-1}, \delta_{m_{i+1}-1}), \\
&\qquad P^1_{\tau_{m_i}-} + \zeta_{m_i} + \cdots + \zeta_{m_{i+1}-1}, X_{\tau_{m_i}-} + \delta_{m_i} + \cdots + \delta_{m_{i+1}-1}) \\
=& v^*(P^0_{\tau_{m_i}}, P^1_{\tau_{m_i}}, X_{\tau_{m_i}}).
\end{aligned}$$

Thus, again since v^* solves the qvi and since S is the qvi control, we have

$$\begin{aligned} v^*(p_0, p_1, x) - e^{-\alpha T} v^*(P_T^0, P_T^1, X_T) \\ = - \int_0^T e^{-\alpha s} \partial_{p_1} v^*(P_{s-}^0, P_{s-}^1, X_{s-}) \xi P_{s-}^1 dB_s^{(1)} \\ - \int_0^T e^{-\alpha s} \partial_x v^*(P_{s-}^0, P_{s-}^1, X_{s-}) \sigma(X_{s-}) dB_s^{(2)} \\ + \int_0^T e^{-\alpha s} f(X_s) ds. \end{aligned}$$

By monotone convergence and by the assumptions (26), (27) and (28), we obtain

$$v^*(p_0, p_1, x) = E_x(\int_0^\infty e^{-\alpha s} f(X_s) ds). \tag{30}$$

Thus we have $v^*(p_0, p_1, x) = v(p_0, p_1, x)$ and the qvi-control S is optimal. $\square$

2 Asset allocation with liquidity breakdowns

In this chapter, we analyze the portfolio decision of an investor facing the threat of illiquidity, where illiquidity is understood as a state of the economy in which the investor is not able to trade at all. Some of the results of this chapter are accepted for publication in Finance and Stochastics [7].

In Section 2.1, we introduce an abstract model of a financial market consisting of two assets, and provide an explicit solution to the stochastic differential equation describing the investor's portfolio process in the illiquidity state. In Section 2.2 we specify our market model and introduce the investor's portfolio problem. We show that the value function of a model in which only finitely many liquidity breakdowns can occur converges uniformly to the value function of a model with infinitely many breakdowns if the number of possible breakdowns goes to infinity. Furthermore, we show how the optimal security demands of the model with finitely many breakdowns can be used to approximate the optimal solution of the model with infinitely many breakdowns. In the Markovian framework of Section 2.3, the Hamilton-Jacobi-Bellman equations are provided and a verification result is proved. We apply this result in Section 2.4 in order to derive the optimal investment strategy, as well as the value function of an investor with a logarithmic utility function. In particular, we show that in this case, the optimal strategy does not depend on the maximal number of illiquidity regimes. Further, we give an alternative proof of the convergence of the value functions, this time by considering the corresponding Hamilton-Jacobi-Bellman equations. Eventually we generalize our model to an economy with three regimes. For instance, the third state could model an additional financial crisis where trading is still possible, but the excess return is lower and the volatility is higher than in the normal state. In Section 2.5 we derive the optimal strategy and the value function of an investor with a power utility function. In this case, in addition to our convergence results for general utility functions, we may show that the optimal strategies converge pointwise if the maximal number of liquidity breakdowns goes to infinity. The last section illustrates our results by a numerical analysis.

2.1 Continuous-time portfolio dynamics with illiquidity

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, P)$ be a filtered probability space satisfying the usual conditions, where $T > 0$ is a finite time horizon. In the following, all random variables and stochastic processes will be defined on this stochastic basis. In this section we provide an abstract model of a financial market with two assets if liquidity breakdowns can occur. One asset is given as an $\mathbb{R}^+$-valued stochastic process M with continuous paths of locally finite variation. The second asset is given by an $\mathbb{R}^+$-valued càdlàg semimartingale which we denote by S. We think of M as the price of a bond, whereas S models the price of a risky asset. The economy is supposed to be in one of two possible states which we denote as state 0 and state 1, respectively. We assume that regime shifts from state i to state $1-i$ are possible as long as a maximal number $k_0 \in \mathbb{N} \cup \{\infty\}$ of illiquidity regimes is not exceeded. Asset prices may depend on the current state of the market.

Definition 2.1. *Let $N^{0,1}$, $N^{1,0}$ be counting processes and assume that they are non-explosive, i.e. there are finitely many jumps in $[0,T]$ a.s. The $\{0,1\}$-valued càdlàg process I given by*

$$\begin{aligned} dI &= 1_{\{I_-=0,\ K_-<k_0\}} dN^{0,1} - 1_{\{I_-=1,\ K_-<k_0\}} dN^{1,0}, \\ dK &= 1_{\{I_-=1\}} dN^{1,0} \end{aligned}$$

with $I_{t_0} = 0$ and $K_{t_0} = 0$ is called the current ***state of the market****. The process K counts the number of jumps into the liquidity state since initial time $t_0 \in [0,T]$.*

The solutions to these stochastic differential equations will be denoted by I^{t_0,k_0} and K^{t_0,k_0} and we omit the superscripts if there is no ambiguity. We interpret state 0 as the **normal** state of the market, in which trading takes place continuously, whereas state 1 represents an **illiquidity** state, in which trading is not possible. Let $\tau_{1,0}^0 = t_0$ and for $1 \le k \in \mathbb{N}$ set

$$\tau_{0,1}^k = \inf\{t \in (\tau_{1,0}^{k-1}, T] \,:\, I_t = 1\} \quad \text{and} \quad \tau_{1,0}^k = \inf\{t \in (\tau_{0,1}^k, T] \,:\, I_t = 0\}.$$

These stopping times are marking the regime shifts from one state into the other and we have

$$t_0 = \tau_{1,0}^0 < \tau_{0,1}^1 < \tau_{1,0}^1 < \tau_{0,1}^2 < \tau_{1,0}^2 < \dots$$

whenever they are finite. We may rewrite I in the following way

$$I^{t_0,k_0} = \sum_{k=1}^{k_0} 1_{[\tau_{0,1}^k, \tau_{1,0}^k)}.$$

We consider an investor who is restricted to choose self-financing strategies such that his wealth dynamics

$$dX = \varphi_- dS + (X_- - \varphi_- S_-) \frac{dM}{M_-}, \quad X_{t_0} = x_0 > 0,$$

have a unique solution X with $X_t \ge 0$ for all $t \in [t_0, T]$ a.s. The càdlàg process φ denotes the number of stocks in the investor's portfolio. It is given by

$$\varphi = \varphi^I = \{\varphi_t^{I_t}\}_{t \in [t_0,T]} \quad \text{with} \quad \varphi^1 = \sum_{k=1}^{k_0} 1_{[\tau_{0,1}^k, \tau_{1,0}^k)} \ \varphi^0_{\tau_{0,1}^k -}.$$

In the normal state of the economy, state 0, the investor can choose his portfolio strategy φ^0 according to the above restrictions. However, in the illiquidity regime, state 1, trading is not allowed and the investor is forced to hold the number of assets φ^1 which he has chosen before the liquidity breakdown. The investor's portfolio process π corresponding to φ is given by

$$\pi = \pi^I = \{\pi_t^{I_t}\}_{t\in[t_0,T]} \quad \text{with} \quad \pi^i = \frac{\varphi^i S}{X} \quad \text{for} \quad i = 0, 1.$$

Note that the process π^1 is exogenously determined by the market. We may rewrite the wealth dynamics in the following way

$$dX = X_-\left(\pi_-\frac{dS}{S_-} + (1-\pi_-)\frac{dM}{M_-}\right), \quad X_{t_0} = x_0 > 0.$$

To avoid bankruptcy shortselling is not allowed, hence the class of admissible portfolio strategies consists of all càdlàg processes π^0 which take values in $[0,1]$. As for the processes I and K, when there is a risk of ambiguity, we will write X^{π^0,t_0,x_0,k_0} instead of X. The following lemma derives a stochastic differential equation for the dynamics of π^1 and provides an explicit solution.

Lemma 2.2 (Portfolio dynamics in illiquidity). *For every $k \in \mathbb{N}$ with $1 \le k \le k_0$, the dynamics of the portfolio process π on the stochastic interval $[\tau_{0,1}^k, \tau_{1,0}^k)$ are given by*

$$d\pi = \pi_-(1-\pi_-)\left(\frac{dS}{S_-} - \frac{dM}{M_-} - \pi_-\frac{d\langle S\rangle^c}{S_-^2} - \pi_- d\sum\frac{(\frac{\Delta S}{S_-})^2}{1+\pi_-\frac{\Delta S}{S_-}}\right)$$

with

$$\pi_{\tau_{0,1}^k} = \frac{\pi_{\tau_{0,1}^k-}\left(1+\frac{\Delta S_{\tau_{0,1}^k}}{S_{\tau_{0,1}^k-}}\right)}{1+\pi_{\tau_{0,1}^k-}\frac{\Delta S_{\tau_{0,1}^k}}{S_{\tau_{0,1}^k-}}}.$$

This stochastic differential equation admits the closed-form solution $\pi = \frac{1}{1+Z}$, where Z is given by

$$dZ = Z_-\left(-\frac{dS}{S_-} + \frac{dM}{M_-} + \frac{d\langle S\rangle^c}{S_-^2} + d\sum\frac{(\frac{\Delta S}{S_-})^2}{1+\frac{\Delta S}{S_-}}\right), \quad Z_{\tau_{0,1}^k} = \frac{1}{\pi_{\tau_{0,1}^k}} - 1.$$

Proof. By the definition of π, we have $\pi_{\tau_{0,1}^k} = \frac{\varphi_{\tau_{0,1}^k-}S_{\tau_{0,1}^k}}{X_{\tau_{0,1}^k}}$ since $\varphi_{\tau_{0,1}^k} = \varphi^1_{\tau_{0,1}^k} = \varphi^0_{\tau_{0,1}^k-} = \varphi_{\tau_{0,1}^k-}$. Therefore,

$$\pi_{\tau_{0,1}^k} = \pi_{\tau_{0,1}^k-}\frac{S_{\tau_{0,1}^k}}{S_{\tau_{0,1}^k-}}\frac{X_{\tau_{0,1}^k-}}{X_{\tau_{0,1}^k}} = \pi_{\tau_{0,1}^k-}\frac{S_{\tau_{0,1}^k-}+\Delta S_{\tau_{0,1}^k}}{S_{\tau_{0,1}^k-}}\frac{X_{\tau_{0,1}^k-}}{X_{\tau_{0,1}^k-}+\Delta X_{\tau_{0,1}^k}}.$$

Due to the wealth equation we have $\Delta X = \varphi_-\Delta S = \pi_- X_-\frac{\Delta S}{S_-}$ and thus we obtain

$$\pi_{\tau_{0,1}^k} = \frac{\pi_{\tau_{0,1}^k-}\left(1+\frac{\Delta S_{\tau_{0,1}^k}}{S_{\tau_{0,1}^k-}}\right)}{1+\pi_{\tau_{0,1}^k-}\frac{\Delta S_{\tau_{0,1}^k}}{S_{\tau_{0,1}^k-}}}.$$

Upon applying Itô's formula we obtain

$$d\frac{1}{X}=\frac{1}{X_-}\left(-\pi_-\frac{dS}{S_-}-(1-\pi_-)\frac{dM}{M_-}\right)+\frac{1}{X_-}\pi_-^2\frac{d\langle S\rangle^c}{S_-^2}+d\sum\Delta\frac{1}{X}+\frac{1}{X_-^2}\Delta X$$

and therefore, since

$$\Delta\frac{1}{X}+\frac{1}{X_-^2}\Delta X=\frac{1}{X_-+\Delta X}-\frac{1}{X_-}+\frac{1}{X_-^2}\Delta X=\frac{(\frac{\Delta X}{X_-})^2}{X_-+\Delta X}=\frac{(\pi_-\frac{\Delta S}{S_-})^2}{X_-(1+\pi_-\frac{\Delta S}{S_-})}$$

we have

$$d\frac{1}{X}=\frac{1}{X_-}\left(-\pi_-\frac{dS}{S_-}-(1-\pi_-)\frac{dM}{M_-}+\pi_-^2\frac{d\langle S\rangle^c}{S_-^2}+d\sum\frac{(\pi_-\frac{\Delta S}{S_-})^2}{1+\pi_-\frac{\Delta S}{S_-}}\right).$$

By the product rule, we have $d\frac{S}{X}=S_-d\frac{1}{X}+\frac{1}{X}dS+d\langle S,\frac{1}{X}\rangle$ with

$$d\langle S,\frac{1}{X}\rangle=-\frac{1}{X_-}\pi_-\frac{d\langle S\rangle}{S_-}+\frac{1}{X_-}d\langle S,\sum\frac{(\pi_-\frac{\Delta S}{S_-})^2}{1+\pi_-\frac{\Delta S}{S_-}}\rangle=\frac{1}{X_-}\left(-\pi_-\frac{d\langle S\rangle}{S_-}+S_-d\sum\frac{\pi_-^2(\frac{\Delta S}{S_-})^3}{1+\pi_-\frac{\Delta S}{S_-}}\right)$$

and since $\frac{\pi X}{S}=\varphi^1=\varphi^0_{\tau^k_{0,1}-}$ on $[\tau^k_{0,1},\tau^k_{1,0})$ it follows that

$$\begin{aligned}
d\pi&=d\left(\frac{\pi X}{S}\frac{S}{X}\right)=\frac{\pi_-X_-}{S_-}d\frac{S}{X}=\pi_-X_-d\frac{1}{X}+\pi_-\frac{dS}{S_-}-\pi_-^2\frac{d\langle S\rangle}{S_-^2}+d\sum\frac{(\pi_-\frac{\Delta S}{S_-})^3}{1+\pi_-\frac{\Delta S}{S_-}}\\
&=-\pi_-^2\frac{dS}{S_-}+\pi_-\frac{dS}{S_-}-\pi_-(1-\pi_-)\frac{dM}{M_-}+\pi_-^3\frac{d\langle S\rangle^c}{S_-^2}-\pi_-^2\frac{d\langle S\rangle}{S_-^2}+d\sum\frac{\pi_-^3(\frac{\Delta S}{S_-})^2(1+\frac{\Delta S}{S_-})}{1+\pi_-\frac{\Delta S}{S_-}}\\
&=\pi_-(1-\pi_-)\left(\frac{dS}{S_-}-\frac{dM}{M_-}-\pi_-\frac{d\langle S\rangle^c}{S_-^2}\right)-\pi_-^2d\sum(\frac{\Delta S}{S_-})^2+d\sum\frac{\pi_-^3(\frac{\Delta S}{S_-})^2(1+\frac{\Delta S}{S_-})}{1+\pi_-\frac{\Delta S}{S_-}}\\
&=\pi_-(1-\pi_-)\left(\frac{dS}{S_-}-\frac{dM}{M_-}-\pi_-\frac{d\langle S\rangle^c}{S_-^2}-\pi_-d\sum\frac{(\frac{\Delta S}{S_-})^2}{1+\pi_-\frac{\Delta S}{S_-}}\right)
\end{aligned}$$

on the stochastic interval $[\tau^k_{0,1},\tau^k_{1,0})$, making use of the fact that

$$\begin{aligned}
-\pi_-^2\left(\frac{\Delta S}{S_-}\right)^2+\frac{\pi_-^3(\frac{\Delta S}{S_-})^2(1+\frac{\Delta S}{S_-})}{1+\pi_-\frac{\Delta S}{S_-}}&=-\pi_-^2\left(\frac{\Delta S}{S_-}\right)^2\left(1-\frac{\pi_-(1+\frac{\Delta S}{S_-})}{1+\pi_-\frac{\Delta S}{S_-}}\right)\\
&=-\pi_-^2(1-\pi_-)\frac{(\frac{\Delta S}{S_-})^2}{1+\pi_-\frac{\Delta S}{S_-}}.
\end{aligned}$$

This proves that π satisfies the stochastic differential equation stated in the assertion. Next, writing

$$dV=\frac{dS}{S_-}-\frac{dM}{M_-}-\pi_-\frac{d\langle S\rangle^c}{S_-^2}-\pi_-d\sum\frac{(\frac{\Delta S}{S_-})^2}{1+\pi_-\frac{\Delta S}{S_-}}$$

we have $d\pi = \pi_-(1-\pi_-)dV$ and $\Delta\pi = \pi_-(1-\pi_-)\Delta V$ with

$$\Delta V = \frac{\Delta S}{S_-} - \pi_- \frac{(\frac{\Delta S}{S_-})^2}{1+\pi_- \frac{\Delta S}{S_-}} = \frac{\frac{\Delta S}{S_-}}{1+\pi_- \frac{\Delta S}{S_-}}$$

as well as

$$\Delta\frac{1}{\pi} + \frac{1}{\pi_-^2}\Delta\pi = \frac{(\frac{\Delta\pi}{\pi_-})^2}{\pi_- + \Delta\pi} = \frac{(1-\pi_-)^2(\Delta V)^2}{\pi_-(1+(1-\pi_-)\Delta V)}.$$

Therefore, an application of Itô's formula to the process $Z = \frac{1}{\pi} - 1$ yields

$$\begin{aligned} dZ = d\frac{1}{\pi} &= -\frac{1}{\pi_-^2}d\pi + \frac{1}{\pi_-^3}d\langle\pi\rangle^c + d\sum \Delta\frac{1}{\pi} + \frac{1}{\pi_-^2}\Delta\pi \\ &= \frac{1-\pi_-}{\pi_-}\left(-dV + (1-\pi_-)d\langle V\rangle^c + d\sum \frac{(1-\pi_-)(\Delta V)^2}{1+(1-\pi_-)\Delta V}\right) \\ &= Z_-\Bigg\{-\frac{dS}{S_-} + \frac{dM}{M_-} + \pi_-\frac{d\langle S\rangle^c}{S_-^2} + \pi_- d\sum \frac{(\frac{\Delta S}{S_-})^2}{1+\pi_-\frac{\Delta S}{S_-}} + (1-\pi_-)d\frac{\langle S\rangle^c}{S_-^2} \\ &\qquad + d\sum \frac{(1-\pi_-)(\frac{\Delta S}{S_-})^2}{(1+\pi_-\frac{\Delta S}{S_-})^2\left(1+(1-\pi_-)\frac{\frac{\Delta S}{S_-}}{1+\pi_-\frac{\Delta S}{S_-}}\right)}\Bigg\} \end{aligned}$$

on $[\tau_{0,1}^k, \tau_{1,0}^k)$. Since

$$\begin{aligned} &\pi_- d\sum \frac{(\frac{\Delta S}{S_-})^2}{1+\pi_-\frac{\Delta S}{S_-}} + d\sum \frac{(1-\pi_-)(\frac{\Delta S}{S_-})^2}{(1+\pi_-\frac{\Delta S}{S_-})^2\left(1+(1-\pi_-)\frac{\frac{\Delta S}{S_-}}{1+\pi_-\frac{\Delta S}{S_-}}\right)} \\ &= \pi_- d\sum \frac{(\frac{\Delta S}{S_-})^2}{1+\pi_-\frac{\Delta S}{S_-}} + d\sum \frac{(1-\pi_-)(\frac{\Delta S}{S_-})^2}{(1+\pi_-\frac{\Delta S}{S_-})(1+\frac{\Delta S}{S_-})} = d\sum \frac{(\frac{\Delta S}{S_-})^2}{1+\frac{\Delta S}{S_-}}, \end{aligned}$$

it follows that

$$dZ = Z_-\left(-\frac{dS}{S_-} + \frac{dM}{M_-} + \frac{d\langle S\rangle^c}{S_-^2} + d\sum \frac{(\frac{\Delta S}{S_-})^2}{1+\frac{\Delta S}{S_-}}\right).$$

□

Note that the jumps of the associated driving processes for π and Z satisfy

$$\frac{\Delta S}{S_-} - \pi_- \frac{(\frac{\Delta S}{S_-})^2}{1+\pi_-\frac{\Delta S}{S_-}} = \frac{\frac{\Delta S}{S_-}}{1+\pi_-\frac{\Delta S}{S_-}}$$

as well as

$$-\frac{\Delta S}{S_-} + \frac{(\frac{\Delta S}{S_-})^2}{1+\frac{\Delta S}{S_-}} = -\frac{\frac{\Delta S}{S_-}}{1+\frac{\Delta S}{S_-}}.$$

Thus, we obtain the following remark.

Remark 2.3. *The lemma shows in particular that π takes values in $[0,1]$ only, since Z is a stochastic exponential and*

$$-\frac{\frac{\Delta S}{S_-}}{1+\frac{\Delta S}{S_-}} > -1,$$

since $\Delta S > -S_-$ as S is positive. Further, if $\sum\limits_{[0,t]} |\Delta S| < \infty$ a.s. for each $t \in [0,T]$, then the dynamics of π and Z simplify to

$$d\pi = \pi_-(1-\pi_-)\left(\frac{dS^c}{S_-} - \frac{dM}{M_-} - \pi_-\frac{d\langle S\rangle^c}{S_-^2} + d\sum \frac{\frac{\Delta S}{S_-}}{1+\pi_-\frac{\Delta S}{S_-}}\right)$$

$$dZ = Z_-\left(-\frac{dS^c}{S_-} + \frac{dM}{M_-} + \frac{d\langle S\rangle^c}{S_-^2} - d\sum \frac{\frac{\Delta S}{S_-}}{1+\frac{\Delta S}{S_-}}\right)$$

because $\frac{dS}{S_-} = \frac{dS^c}{S_-} + d\sum \frac{\Delta S}{S_-}$.

2.2 Portfolio problem with illiquidity and convergence

From now on, we specify our two-asset securities market in the following way. We assume that the bond dynamics are given by

$$dM = M_- r_{I_-} dt$$

for constant riskless interest rates r_0, $r_1 > 0$ and that the dynamics of the risky asset S are given by

$$dS = S_-[(r_{I_-} + \alpha_{I_-})dt + \sigma_{I_-} dB - L_{I_-} dN^{I_-} - 1_{\{K_- < k_0\}} L_{I_-,1-I_-} dN^{I_-,1-I_-}]$$

on $[t_0, T)$ with $S_T = \left(1 - 1_{\{I_T=1\}}\ell\right) S_{T-}$. Here, α_0, $\alpha_1 \in \mathbb{R}$ are excess returns, σ_0, $\sigma_1 \geq 0$ are volatilities, L_0, L_1, $L_{0,1}$, $L_{1,0}$, $\ell \in [0,1)$ are loss rates, B is a standard Brownian motion, and N^0, N^1 are Poisson processes with constant intensities λ_0, $\lambda_1 \geq 0$. The constant ℓ models liquidation costs if at the investment horizon T the economy is in the illiquidity state. The wealth dynamics can then be rewritten more explicitly, as

$$dX = X_-\left[(r_{I_-} + \pi_-\alpha_{I_-})dt + \pi_-\sigma_{I_-} dB - \pi_- L_{I_-} dN^{I_-} - 1_{\{K_- < k_0\}} \pi_- L_{I_-,1-I_-} dN^{I_-,1-I_-}\right]$$

on $[t_0, T)$ with $X_{t_0} = x_0 > 0$ and $X_T = \left(1 - 1_{\{I_T=1\}}\pi_{T-}\ell\right) X_{T-}$. Further rewriting the previous lemma, we get the following result for the investor's portfolio process in the illiquidity state.

Corollary 2.4. *For every $k \in \mathbb{N}$ with $1 \leq k \leq k_0$, the dynamics of the portfolio process π on the stochastic interval $[\tau_{0,1}^k, \tau_{1,0}^k)$ are given by*

$$d\pi = \pi_-(1-\pi_-)\left((\alpha_1 - \pi_-\sigma_1^2)dt + \sigma_1 dB - \frac{L_1}{1-\pi_- L_1} dN^1\right)$$

with

$$\pi_{\tau_{0,1}^k} = \frac{\pi_{\tau_{0,1}^k -}(1-L_{0,1})}{1-\pi_{\tau_{0,1}^k -}L_{0,1}}.$$

This stochastic differential equation has the closed-form solution $\pi = \frac{1}{1+Z}$ *where*

$$dZ = Z_-\left((\sigma_1^2-\alpha_1)dt - \sigma_1 dB + \frac{L_1}{1-L_1}dN^1\right), \quad Z_{\tau_{0,1}^k} = \frac{1}{\pi_{\tau_{0,1}^k}} - 1.$$

Proof. This follows immediately from Remark 2.3 and $\frac{\frac{\Delta S}{S_-}}{1+\pi_-\frac{\Delta S}{S_-}} = -\frac{L_1}{1-\pi_-L_1}dN^1$ as well as $-\frac{\frac{\Delta S}{S_-}}{1+\frac{\Delta S}{S_-}} = \frac{L_1}{1-L_1}dN^1$. □

Note that Z is a geometric Brownian motion if $L_1 = 0$.

For an admissible strategy π^0, the solution to the wealth equation is explicitly given by

$$X_t^{\pi^0,t_0,x_0,k_0} = x_0 \exp\Big(\int_{t_0}^{t} r_{I_-} + \pi_-\alpha_{I_-} - \frac{1}{2}\pi_-^2\sigma_{I_-}^2\,ds + \int_{t_0}^{t} \pi_-\sigma_{I_-}dB\Big)$$
$$\prod_{[t_0,t]}(1-\pi_-L_{I_-})^{\Delta N^{I_-}}(1-1_{\{K_-<k_0\}}\pi_-L_{I_-,1-I_-})^{\Delta N^{I_-,1-I_-}}$$

for all $t \in [t_0, T)$. This implies the following lemma.

Lemma 2.5 (Moments of the wealth process). *If* $E(\beta^{N_T^{i,1-i}}) < \infty$ *for each* $\beta \in (0,\infty)$ *and* $i = 0, 1$, *then for any* $\kappa > 0$ *there exists* $C_\kappa \in (0,\infty)$ *such that for all* $t_0 \in [0,T]$ *and* $x_0 \in (0,\infty)$ *we have*

$$\sup_{\pi^0,\, k_0\in\mathbb{N}\cup\{\infty\}} E\left(\sup_{t\in[t_0,T]}\left(1 + X_t^{\pi^0,t_0,x_0,k_0} + \frac{1}{X_t^{\pi^0,t_0,x_0,k_0}}\right)^\kappa\right) \le C_\kappa\left(1 + x_0 + \frac{1}{x_0}\right)^\kappa.$$

Proof. For $\kappa \in \mathbb{R}$, we set

$$M_\kappa = \sup_{\pi^0,\, k_0\in\mathbb{N}\cup\{\infty\}} E\Big(\sup_{t\in[t_0,T]} \exp\{\kappa\int_{t_0}^{t} r_{I_-^{t_0,k_0}} + \pi_-\alpha_{I_-^{t_0,k_0}} - \frac{1}{2}\pi_-^2\sigma_{I_-^{t_0,k_0}}^2 ds + \kappa\int_{t_0}^{t}\pi_-\sigma_{I_-^{t_0,k_0}}dB\}\Big).$$

If $\kappa > 0$, $t_0 \in [0,T]$, $x_0 \in (0,\infty)$, $k_0 \in \mathbb{N}\cup\{\infty\}$, and π^0 is an admissible strategy, then the above explicit solution yields

$$E\big(\sup_{t\in[t_0,T]}(X_t^{\pi^0,t_0,x_0,k_0})^\kappa\big) \le x_0^\kappa M_\kappa.$$

Further, by Cauchy's inequality and since $(1-1_{\{I_T^{t_0,k_0}=1,\ t=T\}}\pi_T-\ell)^{-\kappa} \le (1-\ell)^{-\kappa}$, we obtain

$$
\begin{aligned}
&E(\sup_{t\in[t_0,T]}(X_t^{\pi^0,t_0,x_0,k_0})^{-\kappa}) \le x_0^{-\kappa}(1-\ell)^{-\kappa}M_{-2\kappa}^{\frac{1}{2}} \\
&\quad E[\sup_{t\in[t_0,T]}\prod_{[t_0,t]}(1-L_{I_-^{t_0,k_0}})^{-2\kappa\Delta N^{I_-^{t_0,k_0}}}(1-L_{I_-^{t_0,k_0},1-I_-^{t_0,k_0}})^{-2\kappa\Delta N^{I_-^{t_0,k_0},1-I_-^{t_0,k_0}}}]^{\frac{1}{2}},
\end{aligned}
$$

where

$$
\begin{aligned}
&\sup_{t\in[t_0,T]}\prod_{[t_0,t]}(1-L_{I_-^{t_0,k_0}})^{-2\kappa\Delta N^{I_-^{t_0,k_0}}}(1-L_{I_-^{t_0,k_0},1-I_-^{t_0,k_0}})^{-2\kappa\Delta N^{I_-^{t_0,k_0},1-I_-^{t_0,k_0}}} \\
&= \prod_{[t_0,T]}(1-L_{I_-^{t_0,k_0}})^{-2\kappa\Delta N^{I_-^{t_0,k_0}}}(1-L_{I_-^{t_0,k_0},1-I_-^{t_0,k_0}})^{-2\kappa\Delta N^{I_-^{t_0,k_0},1-I_-^{t_0,k_0}}} \\
&\le (1-L_0)^{-2\kappa N_T^0}(1-L_1)^{-2\kappa N_T^1}(1-L_{0,1})^{-2\kappa N_T^{0,1}}(1-L_{1,0})^{-2\kappa N_T^{1,0}};
\end{aligned}
$$

the quantity on the right is integrable due to our assumption on $N^{i,1-i}$. The desired conclusion will thus follow from the fact that $M_\kappa < \infty$ for all $\kappa \in \mathbb{R}$. To show this, note that

$$
\begin{aligned}
M_\kappa = &\sup_{\pi^0,\,k_0\in\mathbb{N}\cup\{\infty\}} E\Bigg[\sup_{t\in[t_0,T]}\exp\Bigg(\kappa\int_{t_0}^{t} r_{I_-^{t_0,k_0}} + \pi_-\alpha_{I_-^{t_0,k_0}} - \frac{1}{2}\pi_-^2\sigma_{I_-^{t_0,k_0}}^2 + \frac{1}{2}\kappa\pi_-^2\sigma_{I_-^{t_0,k_0}}^2 ds \\
&\qquad + \kappa\int_{t_0}^{t}\pi_-\sigma_{I_-^{t_0,k_0}}dB - \frac{1}{2}\kappa^2\int_{t_0}^{t}\pi_-^2\sigma_{I_-^{t_0,k_0}}^2 ds\Bigg)\Bigg] \\
\le e^{\rho_\infty T} &\sup_{\pi^0,\,k_0\in\mathbb{N}\cup\{\infty\}} E\Bigg[\sup_{t\in[t_0,T]}\exp\Bigg(\kappa\int_{t_0}^{t}\pi_-\sigma_{I_-^{t_0,k_0}}dB - \frac{1}{2}\kappa^2\int_{t_0}^{t}\pi_-^2\sigma_{I_-^{t_0,k_0}}^2 ds\Bigg)\Bigg],
\end{aligned}
$$

since the process $\kappa|r_{I_-^{t_0,k_0}} + \pi_-\alpha_{I_-^{t_0,k_0}} - \frac{1}{2}\pi_-^2\sigma_{I_-^{t_0,k_0}}^2 + \frac{1}{2}\kappa\pi_-^2\sigma_{I_-^{t_0,k_0}}^2|$ is bounded by a constant $\rho_\infty \in (0,\infty)$ that is independent of π^0, t_0, and k_0. Recall that $\pi_{I_-^{t_0,k_0}}$ is $[0,1]$-valued by Remark 2.3. Next, let π^0 be an arbitrary admissible strategy, and let $t_0 \in [0,T]$, $k_0 \in \mathbb{N}\cup\{\infty\}$. Writing $\varrho = \kappa\pi_-\sigma_{I_-^{t_0,k_0}}$, it follows that ϱ is bounded by $\varrho_\infty \in (0,\infty)$, a constant independent of π^0, t_0, and k_0. Therefore, by the Novikov condition, the exponential

$\exp(\int\limits_{t_0}^{\cdot} \varrho dB - \frac{1}{2}\int\limits_{t_0}^{\cdot} \varrho^2 ds)$ is a martingale and consequently

$$E\left[\sup_{t\in[t_0,T]} \exp\left(\int\limits_{t_0}^{t} \varrho dB - \frac{1}{2}\int\limits_{t_0}^{t} \varrho^2 ds\right)\right] \le \left(E\left[\sup_{t\in[t_0,T]} \exp\left\{\int\limits_{t_0}^{t} \varrho dB - \frac{1}{2}\int\limits_{t_0}^{t} \varrho^2 ds\right\}^2\right]\right)^{\frac{1}{2}}$$

$$\le 4\left(E\left[\exp\left\{2\int\limits_{t_0}^{T} \varrho dB - \int\limits_{t_0}^{T} \varrho^2 ds\right\}\right]\right)^{\frac{1}{2}}$$

$$\le 4\left(E\left[\exp\left\{\int\limits_{t_0}^{T} 2\varrho dB - \frac{1}{2}\int\limits_{t_0}^{T} (2\varrho)^2 ds\right\}\exp\left\{2\int\limits_{t_0}^{T} \varrho^2 ds\right\}\right]\right)^{\frac{1}{2}} \le 4e^{\varrho_\infty^2 T} < \infty$$

by Doob's L^2-inequality. This gives the desired result. □

Note that the integrability condition of the previous lemma would be satisfied, if the counting processes $N^{i,1-i}$ were Poisson processes.

We assume that our investor, trading in the market described above, maximizes expected utility from terminal wealth with respect to a concave non-decreasing utility function $U: (0,\infty) \to \mathbb{R}$. The corresponding **value function** (syn. indirect utility) is given by

$$V:\ [0,T]\times(0,\infty)\times\mathbb{N}\cup\{\infty\}\to\mathbb{R}, \quad V(t_0,x_0,k_0) = \sup_{\pi^0} E[U(X_T^{\pi^0,t_0,x_0,k_0})]. \tag{31}$$

By considering the strategy $\pi^0 = 0$, i.e. a pure bond investment, and applying the previous lemma together with Jensen's inequality, we obtain the following lower and upper bounds:

$$U(x_0) \le V(t_0,x_0,k_0) \le U\left(C_1\left(1 + x_0 + \frac{1}{x_0}\right)\right).$$

In particular, the value function is finite. The following theorem states that the value functions corresponding to problems, in which only finitely many liquidity breakdowns can occur, converge uniformly to the value function with $k_0 = \infty$, if the number of possible breakdowns goes to infinity. This is due to the fact that even if $k_0 = \infty$, almost surly there are only finitely many breakdowns before time T, since the processes $N^{i,1-i}$ which trigger the regime shifts are non-explosive.

Theorem 2.6 (Convergence of the value functions). *Suppose that $E(\beta^{N_T^{i,1-i}}) < \infty$ for all $\beta \in (0,\infty)$ and $i = 0,1$, and that the investor's utility function U is polynomially bounded at 0, i.e. that there exist $\kappa > 0$, $\rho > 0$ and $\delta > 0$ such that*

$$|U(x)| \le \rho\left(1 + \frac{1}{x}\right)^{\kappa} \quad \forall\, x \in (0,\delta).$$

Then the value function of the investor's portfolio problem satisfies

$$\lim_{k_0 \to \infty} \sup_{t_0 \in [0,T],\, x_0 \in C} |V(t_0, x_0, k_0) - V(t_0, x_0, \infty)| = 0$$

for any compact subset C *of* $(0, \infty)$.

Proof. For any utility function U, we have the concavity estimate $U(x) \leq \theta(x-1)$ for some $\theta \in \mathbb{R}$, i.e. $\theta = U'(1)$ if U is differentiable, so by our assumption on U

$$|U(x)| \leq \varrho \left(1 + x + \frac{1}{x}\right)^{\kappa} \quad \forall\, x \in (0, \infty),$$

for suitably chosen $\kappa > 1$ and $\varrho > 0$. Thus, due to Lemma 2.5, compactness of C and our assumption on $N^{i,1-i}$, the family

$$\{U(X_T^{\pi^0, t_0, x_0, k_0})\}_{\pi^0,\, t_0 \in [0,T],\, x_0 \in C,\, k_0 \in \mathbb{N} \cup \{\infty\}} \quad \text{is uniformly integrable.}$$

Moreover, it is clear that

$$\sup_{\pi^0,\, t_0 \in [0,T],\, x_0 \in C} |U(X_T^{\pi^0, t_0, x_0, \infty}) - U(X_T^{\pi^0, t_0, x_0, k_0})| \to 0 \quad \text{in probability as } k_0 \to \infty$$

since

$$\begin{aligned} &P(X_T^{\pi^0, t_0, x_0, \infty} \neq X_T^{\pi^0, t_0, x_0, k_0} \text{ for some admissible } \pi^0,\ t_0 \in [0,T],\ x_0 \in C) \\ \leq &P(K_T^{t_0, x_0, k_0} = k_0 \text{ for some } t_0 \in [0,T],\ x_0 \in C) \leq P(N_T^{1,0} \geq k_0) \to 0 \quad \text{as } k_0 \to \infty. \end{aligned}$$

To prove convergence, we fix some $\varepsilon > 0$ and choose $\hat{t}_0 \in [0,T]$, $\hat{x}_0 \in C$ such that

$$\sup_{t_0 \in [0,T],\, x_0 \in C} |V(t_0, x_0, k_0) - V(t_0, x_0, \infty)| \leq |V(\hat{t}_0, \hat{x}_0, k_0) - V(\hat{t}_0, \hat{x}_0, \infty)| + \frac{\varepsilon}{2}.$$

For the moment, assume that $V(\hat{t}_0, \hat{x}_0, k_0) - V(\hat{t}_0, \hat{x}_0, \infty) \geq 0$. Then let $\hat{\pi}^0$ be an admissible strategy such that

$$V(\hat{t}_0, \hat{x}_0, k_0) - E[U(X_T^{\hat{\pi}^0, \hat{t}_0, \hat{x}_0, k_0})] \leq \frac{\varepsilon}{2}.$$

Thus we have

$$\begin{aligned} &\sup_{t_0 \in [0,T],\, x_0 \in C} |V(t_0, x_0, k_0) - V(t_0, x_0, \infty)| \leq V(\hat{t}_0, \hat{x}_0, k_0) - V(\hat{t}_0, \hat{x}_0, \infty) + \frac{\varepsilon}{2} \\ \leq &E[U(X_T^{\hat{\pi}^0, \hat{t}_0, \hat{x}_0, k_0})] - V(\hat{t}_0, \hat{x}_0, \infty) + \varepsilon \leq E[U(X_T^{\hat{\pi}^0, \hat{t}_0, \hat{x}_0, k_0})] - E[U(X_T^{\hat{\pi}^0, \hat{t}_0, \hat{x}_0, \infty})] + \varepsilon \\ \leq &E[\sup_{\pi^0,\, t_0 \in [0,T],\, x_0 \in C} |U(X_T^{\pi^0, t_0, x_0, k_0}) - U(X_T^{\pi^0, t_0, x_0, \infty})|] + \varepsilon. \end{aligned}$$

Applying an analogous argument in the case when $V(\hat{t}_0, \hat{x}_0, k_0) - V(\hat{t}_0, \hat{x}_0, \infty) \leq 0$, we see that the latter inequality continues to hold. Since $\varepsilon > 0$ is arbitrary, we obtain

$$\begin{aligned}&\sup_{t_0 \in [0,T],\, x_0 \in C} |V(t_0, x_0, k_0) - V(t_0, x_0, \infty)| \\ \leq &E[\sup_{\pi^0,\, t_0 \in [0,T],\, x_0 \in C} |U(X_T^{\pi^0, t_0, x_0, k_0}) - U(X_T^{\pi^0, t_0, x_0, \infty})|],\end{aligned}$$

so that

$$\sup_{t_0 \in [0,T],\, x_0 \in C} |V(t_0, x_0, k_0) - V(t_0, x_0, \infty)| \to 0 \quad \text{as } k_0 \to \infty$$

by the observations made at the beginning of the proof. □

The previous result shows that the investor's portfolio problem with possibly infinitely many liquidity breakdowns can be suitably approximated by an investment problem with finitely many jumps. Moreover, due to the uniformity of convergence, the optimal strategies of problems with sufficiently many breakdowns perform arbitrarily well in the case with infinitely many breakdowns.

Corollary 2.7 (Approximately optimal strategies)**.** *Suppose that the assumptions of Theorem 2.6 are satisfied. For fixed $\varepsilon > 0$ and for any $t_0 \in [0,T]$ and $x_0 \in (0,\infty)$ there exists a $\hat{k}_0 \in \mathbb{N}$ such that for any admissible $\frac{\varepsilon}{3}$-optimal strategy $\hat{\pi}^0$ for $V(t_0, x_0, \hat{k}_0)$ we have*

$$|E[U(X_T^{\hat{\pi}^0, t_0, x_0, \infty})] - V(t_0, x_0, \infty)| \leq \varepsilon.$$

Besides, if the investor's utility function U is of the form $U(x) = \frac{1}{\gamma}x^\gamma$, then it follows that the initial wealth x_k required to achieve the given indirect utility $V(t_0, x_0, \infty)$ in the model with at most k liquidity breakdowns satisfies

$$x_k = \left(\frac{V(t_0, x_0, \infty)}{V(t_0, 1, k)}\right)^{\frac{1}{\gamma}} \to x_0 \quad \text{as } k \to \infty.$$

Proof. Given some $\varepsilon > 0$, for any $t_0 \in [0,T]$ and $x_0 \in (0,\infty)$ we can choose $\hat{k}_0 \in \mathbb{N}$ such that

$$\sup_{\pi^0} |E[U(X_T^{\pi^0, t_0, x_0, \hat{k}_0})] - E[U(X_T^{\pi^0, t_0, x_0, \infty})]| \leq \frac{\varepsilon}{3}$$

and

$$|V(t_0, x_0, \hat{k}_0) - V(t_0, x_0, \infty)| \leq \frac{\varepsilon}{3}.$$

Further, we can choose an admissible strategy $\hat{\pi}^0$ with

$$|E[U(X_T^{\hat{\pi}^0, t_0, x_0, \hat{k}_0})] - V(t_0, x_0, \hat{k}_0)| \leq \frac{\varepsilon}{3}.$$

Then it follows that

$$|E[U(X_T^{\hat{\pi}^0, t_0, x_0, \infty})] - V(t_0, x_0, \infty)| \leq \varepsilon.$$

Next, if the investor's utility function U is of the form $U(x) = \frac{1}{\gamma}x^\gamma$, then from

$$\begin{aligned} V(t_0, x_0, k_0) &= \sup_{\pi^0} E[U(X_T^{\pi^0, t_0, x_0, k_0})] = x_0^\gamma \sup_{\pi^0} E[U(X_T^{\pi^0, t_0, 1, k_0})] \\ &= x_0^\gamma V(t_0, 1, k_0) \end{aligned}$$

it follows that the initial wealth x_k such that we have $V(t_0, x_k, k) = V(t_0, x_0, \infty)$ satisfies

$$x_k = \left(\frac{V(t_0, x_0, \infty)}{V(t_0, 1, k)}\right)^{\frac{1}{\gamma}} \to x_0 \quad \text{as } k \to \infty.$$

$\square$

Remark 2.8. *We wish to stress that, without additional assumptions, the optimal strategies do not have to converge. However, for logarithmic utility and power utility convergence can be proved. Indeed, for logarithmic utility, the optimal strategies do not depend on k_0 (see Corollaries 2.19 and 2.35).*

2.3 HJB equations and verification theorem

In this section, we investigate the optimal portfolio problem (31) applying dynamic programming techniques. In order to obtain Markovian dynamics, from now on we assume that the regime shift process $N^{i,1-i}$ is a Poisson process with intensity $\lambda_{i,1-i} \geq 0$ for $i = 0, 1$. In particular, the integrability condition of Lemma 2.5 and Theorem 2.6 is thus satisfied. Let $k_0 \in \mathbb{N} \cup \{\infty\}$, then a collection

$$\{J^{0,k_0}, J^{1,k_0}, J^{0,k_0-1}, J^{1,k_0-1}, \ldots, J^{0,1}, J^{1,1}, J^{0,0}\},$$

where $J^{0,k}$ is a $C^{1,2}$-function on $[0,T] \times (0,\infty)$ and $J^{1,k}$ is a $C^{1,2,2}$-function on $[0,T] \times (0,\infty) \times [0,1]$, is said to be a solution to the Hamilton-Jacobi-Bellman equations of the portfolio problem (31) if the following partial differential equations are satisfied

$$\begin{aligned}
0 = \sup_{\pi\in[0,1]} \Big\{ & \partial_t J^{0,0}(t,x) + x(r_0+\alpha_0\pi)\partial_x J^{0,0}(t,x) + \frac{1}{2}x^2\pi^2\sigma_0^2\partial_{x,x}J^{0,0}(t,x) \\
& + \lambda_0\left[J^{0,0}(t,x(1-\pi L_0)) - J^{0,0}(t,x)\right]\Big\} \\
0 = \sup_{\pi\in[0,1]} \Big\{ & \partial_t J^{0,k}(t,x) + x(r_0+\alpha_0\pi)\partial_x J^{0,k}(t,x) + \frac{1}{2}x^2\pi^2\sigma_0^2\partial_{x,x}J^{0,k}(t,x) \\
& + \lambda_0\left[J^{0,k}(t,x(1-\pi L_0)) - J^{0,k}(t,x)\right] \\
& + \lambda_{0,1}\left[J^{1,k}\left(t,x(1-\pi L_{0,1}),\frac{\pi(1-L_{0,1})}{1-\pi L_{0,1}}\right) - J^{0,k}(t,x)\right]\Big\} \\
0 = \; & \partial_t J^{1,k}(t,x,\pi) + x(r_1+\alpha_1\pi)\partial_x J^{1,k}(t,x,\pi) + \frac{1}{2}x^2\pi^2\sigma_1^2\partial_{x,x}J^{1,k}(t,x,\pi) \\
& + x\pi^2(1-\pi)\sigma_1^2\partial_{x,\pi}J^{1,k}(t,x,\pi) + \pi(1-\pi)(\alpha_1-\sigma_1^2\pi)\partial_\pi J^{1,k}(t,x,\pi) \\
& + \frac{1}{2}\pi^2(1-\pi)^2\sigma_1^2\partial_{\pi,\pi}J^{1,k}(t,x,\pi) \\
& + \lambda_1\left[J^{1,k}\left(t,x(1-\pi L_1),\frac{\pi(1-L_1)}{1-\pi L_1}\right) - J^{1,k}(t,x,\pi)\right] \\
& + \lambda_{1,0}\left[J^{0,k-1}(t,x(1-\pi L_{1,0})) - J^{1,k}(t,x,\pi)\right]
\end{aligned}$$

subject to the boundary conditions $J^{0,k}(T,x) = U(x)$, $J^{1,k}(T,x,\pi) = U(x(1-\pi\ell))$ for all $x\in(0,\infty)$ and $\pi\in[0,1]$. If $k_0=\infty$, then a solution to the HJB equations simply consists of a pair $\{J^{0,\infty}, J^{1,\infty}\}$, and the above system reduces to a pair of equations with $J^{0,\infty-1} = J^{0,\infty}$, etc. Note that this system can be solved iteratively if $k_0<\infty$, whereas it does not decouple when $k_0=\infty$. Given a solution of the HJB equations, to simplify notation, we set

$$\begin{aligned}
H^{0,0}(t,x,\pi) &= \partial_t J^{0,0}(t,x) + x(r_0+\alpha_0\pi)\partial_x J^{0,0}(t,x) + \frac{1}{2}x^2\pi^2\sigma_0^2\partial_{x,x}J^{0,0}(t,x) \\
&\quad + \lambda_0\left[J^{0,0}(t,x(1-\pi L_0)) - J^{0,0}(t,x)\right] \\
H^{0,k}(t,x,\pi) &= \partial_t J^{0,k}(t,x) + x(r_0+\alpha_0\pi)\partial_x J^{0,k}(t,x) + \frac{1}{2}x^2\pi^2\sigma_0^2\partial_{x,x}J^{0,k}(t,x) \\
&\quad + \lambda_0\left[J^{0,k}(t,x(1-\pi L_0)) - J^{0,k}(t,x)\right] \\
&\quad + \lambda_{0,1}\left[J^{1,k}\left(t,x(1-\pi L_{0,1}),\frac{\pi(1-L_{0,1})}{1-\pi L_{0,1}}\right) - J^{0,k}(t,x)\right] \\
H^{1,k}(t,x,\pi) &= 0
\end{aligned}$$

for $t\in[0,T]$, $x\in(0,\infty)$, and $\pi\in[0,1]$. The following theorem shows that $J^{0,k}$ corresponds to the value function of the optimal investment problem with k illiquidity regimes outstanding.

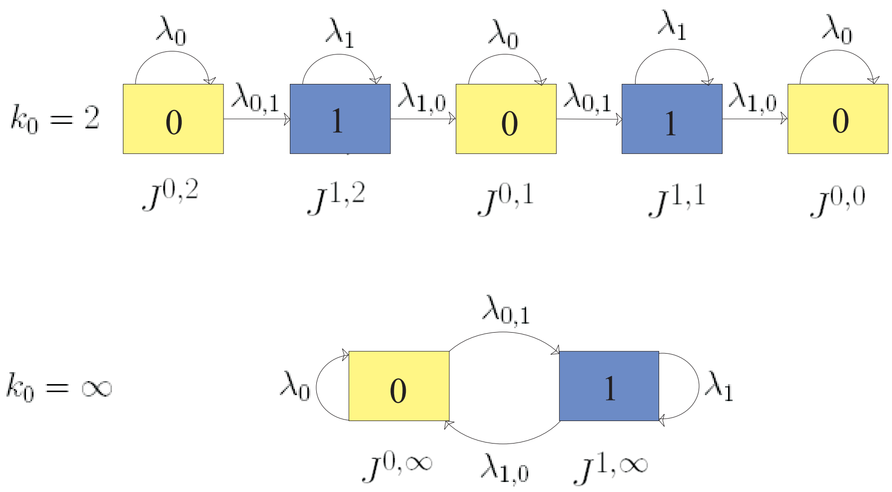

Figure 19: Economy with at most two illiquidity regimes $k_0 = 2$, and possibly infinitely many liquidity breakdowns $k_0 = \infty$, respectively.

Theorem 2.9 (Verification theorem)**.** *Let* $\{J^{0,k_0}, J^{1,k_0}, J^{0,k_0-1}, J^{1,k_0-1}, \dots, J^{0,1}, J^{1,1}, J^{0,0}\}$ *be a solution of the Hamilton-Jacobi-Bellman equations associated to the optimal investment problem (31) with at most* $k_0 \in \mathbb{N} \cup \{\infty\}$ *illiquidity regimes and assume that for each* $i \in \{0, 1\}$ *and* $k \in \{1, \dots, k_0\}$ *the functions* $J^{i,k}$, $\partial_x J^{i,k}$, $\partial_\pi J^{i,k}$ *and* $J^{0,0}$, $\partial_x J^{0,0}$, $\partial_\pi J^{0,0}$ *are polynomially bounded at* 0 *and* ∞ *uniformly with respect to* $t \in [0, T]$ *and* $\pi \in [0, 1]$*. Then*

$$V(t_0, x_0, k_0) \leq J^{0,k_0}(t_0, x_0)$$

for all $t_0 \in [0, T]$ *and* $x_0 \in (0, \infty)$*. Moreover, if there are continuous functions* $\psi_k : [0, T] \times (0, \infty) \to [0, 1]$ *such that*

$$\psi_k(t, x) \in \arg\max_{\pi \in [0,1]} H^{0,k}(t, x, \pi)$$

for each $k \in \{0, \dots, k_0\}$*, then the value function is given by*

$$V(t_0, x_0, k_0) = J^{0,k_0}(t_0, x_0)$$

for all $t_0 \in [0, T]$ *and* $x_0 \in (0, \infty)$*, and the optimally controlled process* X^* *and the optimal strategy* π^{0*} *satisfy* $\pi^{0*} = \psi_{k_0 - K_-}(\cdot, X^*)$*.*

Proof. Given an admissible strategy π^0, $t_0 \in [0, T]$, and $x_0 \in (0, \infty)$, consider the process

$$\jmath_t = J^{I_t, k_0 - K_t}(t, X_t, \pi_t)$$

for all $t \in [t_0, T]$, where the upper indices π^0, t_0, x_0, k_0 are omitted for notational convenience and, by ignoring the third coordinate, $J^{0,k}$ is interpreted as a function defined on

$[0,T]\times(0,\infty)\times[0,1]$. Applying Itô's formula and using Corollary 2.4, we obtain

$$
\begin{aligned}
d\jmath &= \partial_t J^{I_-,k_0-K_-}(.,X_-,\pi_-)dt + \partial_x J^{I_-,k_0-K_-}(.,X_-,\pi_-)X_-\left[(r_{I_-}+\alpha_{I_-}\pi_-)dt + \sigma_{I_-}\pi_- dB\right] \\
&\quad + \frac{1}{2}\partial_{x,x}J^{I_-,k_0-K_-}(.,X_-,\pi_-)X_-^2\sigma_{I_-}^2\pi_-^2 dt \\
&\quad + 1_{\{I_-=1\}}\Big\{\partial_\pi J^{1,k_0-K_-}(.,X_-,\pi_-)\pi_-(1-\pi_-)\left[(\alpha_1-\sigma_1^2\pi_-)dt+\sigma_1 dB\right] \\
&\qquad + \frac{1}{2}\partial_{\pi,\pi}J^{1,k_0-K_-}(.,X_-,\pi_-)\pi_-^2(1-\pi_-)^2\sigma_1^2 dt \\
&\qquad + \partial_{x,\pi}J^{1,k_0-K_-}(.,X_-,\pi_-)X_-\sigma_1^2\pi_-^2(1-\pi_-)dt\Big\} \\
&\quad + \left[J^{I_-,k_0-K_-}\left(.,(1-\pi_-L_{I_-})X_-,\frac{\pi_-(1-L_{I_-})}{1-\pi_-L_{I_-}}\right) - J^{I_-,k_0-K_-}(.,X_-,\pi_-)\right]dN^{I_-} \\
&\quad + 1_{\{I_-=0,\,K_-<k_0\}}\left[J^{1,k_0-K_-}\left(.,(1-\pi_-L_{0,1})X_-,\frac{\pi_-(1-L_{0,1})}{1-\pi_-L_{0,1}}\right) - J^{0,k_0-K_-}(.,X_-)\right]dN^{0,1} \\
&\quad + 1_{\{I_-=1\}}\left[J^{0,k_0-K_--1}(.,(1-\pi_-L_{1,0})X_-) - J^{1,k_0-K_-}(.,X_-,\pi_-)\right]dN^{1,0}
\end{aligned}
$$

on $[t_0,T)$. Let $\tilde{N}^0$, $\tilde{N}^1$, $\tilde{N}^{0,1}$ and $\tilde{N}^{1,0}$ denote the compensated Poisson processes associated with N^0, N^1, $N^{0,1}$ and $N^{1,0}$. Then, rewriting the previous equation, we obtain

$$
\begin{aligned}
d\jmath &= H^{I_-,k_0-K_-}(.,X_-,\pi_-)dt + \partial_x J^{I_-,k_0-K_-}(.,X_-,\pi_-)X_-\sigma_{I_-}\pi_- dB \\
&\quad + 1_{\{I_-=1\}}\partial_\pi J^{1,k_0-K_-}(.,X_-,\pi_-)\pi_-(1-\pi_-)\sigma_1 dB \\
&\quad + \left[J^{I_-,k_0-K_-}\left(.,(1-\pi_-L_{I_-})X_-,\frac{\pi_-(1-L_{I_-})}{1-\pi_-L_{I_-}}\right) - J^{I_-,k_0-K_-}(.,X_-,\pi_-)\right]d\tilde{N}^{I_-} \\
&\quad + 1_{\{I_-=0,\,K_-<k_0\}}\left[J^{1,k_0-K_-}\left(.,(1-\pi_-L_{0,1})X_-,\frac{\pi_-(1-L_{0,1})}{1-\pi_-L_{0,1}}\right) - J^{0,k_0-K_-}(.,X_-)\right]d\tilde{N}^{0,1} \\
&\quad + 1_{\{I_-=1\}}\left[J^{0,k_0-K_--1}(.,(1-\pi_-L_{1,0})X_-) - J^{1,k_0-K_-}(.,X_-,\pi_-)\right]d\tilde{N}^{1,0}
\end{aligned}
$$

on $[t_0,T)$. Due to our polynomial growth assumption and Lemma 2.5, the stochastic differentials of the local martingales in the above identity are, in fact, stochastic differentials of martingales. Therefore, by taking expectations and using the boundary conditions of the HJB equations, we arrive at

$$
E[U(X_T^{\pi^0,t_0,x_0,k_0})] = E(\jmath_{T-}) = J^{0,k_0}(t_0,x_0) + E[\int_{t_0}^{T} H^{I_{t-},k_0-K_{t-}}(t,X_{t-},\pi_{t-})dt].
$$

Since π^0, t_0, and x_0 are arbitrary, we conclude that

$$
V(t_0,x_0,k_0) \le J^{0,k_0}(t_0,x_0)
$$

for all $t_0\in[0,T]$ and $x_0\in(0,\infty)$.

Now, if $\psi_k : [0,T] \times (0,\infty) \to [0,1]$ is a continuous function such that

$$\psi_k(t,x) \in \arg\max_{\pi\in[0,1]} H^{0,k}(t,x,\pi)$$

for each $k \in \{0,\ldots,k_0\}$, then the family $\{\psi_k\}_{0\le k\le k_0}$ defines an optimal strategy in the sense that the stochastic differential equation

$$\begin{aligned} dX = X_-\big[&\big(r_{I_-} + \psi_{k_0-K_-}(\cdot,X_-)\alpha_{I_-}\big)\,dt + \psi_{k_0-K_-}(\cdot,X_-)\sigma_{I_-}dB \\ &- \psi_{k_0-K_-}(\cdot,X_-)L_{I_-}dN^{I_-} - 1_{\{K_-<k_0\}}\psi_{k_0-K_-}(\cdot,X_-) - L_{I_-,1-I_-}dN^{I_-,1-I_-}\big] \end{aligned}$$

on $[t_0,T)$ with $X_{t_0} = x_0$ and $X_T = \big(1 - 1_{\{I_T=1\}}\psi_{k_0-K_{T-}}(T,X_{T-})\ell\big)\,X_{T-}$, admits a solution X^{ψ,t_0,x_0,k_0}, and the strategy

$$\pi^{0^*} = \psi_{k_0-K_-}\left(\cdot, X^{\psi,t_0,x_0,k_0}\right)$$

is admissible and optimal for the investor's portfolio problem. Further, in this case we have

$$E[U(X_T^{\pi^{0^*},t_0,x_0,k_0})] = V(t_0,x_0,k_0) = J^{0,k_0}(t_0,x_0).$$

$\square$

Remark 2.10. *Given that $J^{i,k}(t,x,\pi) = f^{i,k}(t,\pi)U(x)$ or $J^{i,k}(t,x,\pi) = f^{i,k}(t,\pi) + U(x)$, the polynomial growth assumption is satisfied if U and U' are polynomially bounded at 0 and $f^{i,k}$ and $\partial_\pi f^{i,k}$ are bounded. This is for instance the case for power or log utility.*

2.4 Logarithmic utility

Throughout this section, we consider an investor with a logarithmic risk preference $U(x) = \ln(x)$. First, we study the portfolio problem (31) with infinitely many liquidity breakdowns, i.e. $k_0 = \infty$ and provide an explicit representation for the corresponding value function. Then, we consider the case with $k_0 \in \mathbb{N}$ and show that for a logarithmic utility function, the optimal strategy does not depend on k_0. Further, we give an alternative proof for the convergence of the value functions when the number of liquidity breakdowns goes to infinity, this time, by considering the corresponding HJB equations. Finally, we briefly state a generalization of our model with three regimes.

2.4.1 Infinitely many liquidity breakdowns

First, we consider the case with infinitely many regime shifts between state 0 and state 1. Let $k_0 = \infty$. We set $J^0 = J^{0,\infty}$ and $J^1 = J^{1,\infty}$. In order to apply the above verification theorem, we conjecture

$$J^0(t,x) = \ln(x) + f^0(t)$$

as well as

$$J^1(t,x,\pi) = \ln(x) + f^1(t,\pi)$$

for a C^1-function f^0 on $[0,T]$ with $f^0(T) = 0$ and a $\mathrm{C}^{1,2}$-function f^1 on $[0,T] \times [0,1]$ satisfying $f^1(T,\pi) = \ln(1-\pi\ell)$ for all $\pi \in [0,1]$. Furthermore, we set $H^0 = H^{0,\infty}$ and $H^1 = H^{1,\infty}$. Then the HJB equations read

$$0 = \sup_{\pi\in[0,1]} \left\{ \partial_t f^0(t) + g_0(\pi) + \lambda_{0,1} \left[f^1\left(t, \frac{\pi(1-L_{0,1})}{1-\pi L_{0,1}}\right) - f^0(t) \right] \right\} \tag{32}$$

$$\begin{aligned} 0 &= \partial_t f^1(t,\pi) - \lambda_{1,0} f^1(t,\pi) + \pi(1-\pi)(\alpha_1 - \sigma_1^2\pi)\partial_\pi f^1(t,\pi) + \frac{1}{2}\pi^2(1-\pi)^2\sigma_1^2\partial_{\pi,\pi} f^1(t,\pi) \\ &\quad + \lambda_1 \left[f^1\left(t, \frac{\pi(1-L_1)}{1-\pi L_1}\right) - f^1(t,\pi) \right] + g_1(\pi) + \lambda_{1,0} f^0(t), \end{aligned} \tag{33}$$

where g_j is given by $g_j(\pi) = r_j + \alpha_j\pi - \frac{1}{2}\pi^2\sigma_j^2 + \lambda_j \ln(1-\pi L_j) + \lambda_{j,1-j}\ln(1-\pi L_{j,1-j})$ on $[0,1]$, for $j = 0$, 1. The HJB equation (32) leads to the first-order condition

$$\begin{aligned} 0 &= \alpha_0 - \pi\sigma_0^2 - \lambda_0 \frac{L_0}{1-\pi L_0} - \lambda_{0,1}\frac{L_{0,1}}{1-\pi L_{0,1}} \\ &\quad + \lambda_{0,1}\partial_\pi f^1\left(t, \frac{\pi(1-L_{0,1})}{1-\pi L_{0,1}}\right) \frac{1-L_{0,1}}{(1-\pi L_{0,1})^2} \end{aligned} \tag{34}$$

for the optimal stock proportion in state 0. Note that if it exists, the solution of the first-order condition is a deterministic function of time.

For $t \in [0,T]$ and $\pi \in [0,1]$, let $\tilde{\pi}$ be given by $\tilde{\pi}_s = \frac{\pi}{\pi + (1-\pi)Z_s}$ for all $s \in [t,T]$, with

$$dZ = Z_- \left[(\sigma_1^2 - \alpha_1)ds - \sigma_1 dB + \frac{L_1}{1-L_1} dN^1 \right], \quad Z_t = 1.$$

We will also write $\tilde{\pi}^{t,\pi}$ for the process $\tilde{\pi}$ subject to the initial condition $\tilde{\pi}_t = \pi$. Note that Z is explicitly given by

$$Z_s = e^{(\frac{1}{2}\sigma_1^2 - \alpha_1)(s-t) - \sigma_1(B_s - B_t)} \frac{1}{(1-L_1)^{N_s^1 - N_t^1}},$$

for all $s \in [t,T]$.

Proposition 2.11 (Indirect utility in illiquidity)**.** *For a* C^1*-function* $f^0 : [0,T] \to \mathbb{R}$, *consider the function* $f^1 : [0,T] \times [0,1] \to \mathbb{R}$ *defined via the stochastic representation*

$$f^1(t,\pi) = \int_t^T (\lambda_{1,0} f^0(s) + E[g_1(\tilde{\pi}_s^{t,\pi})])e^{-\lambda_{1,0}(s-t)} ds + E[\ln(1 - \tilde{\pi}_T^{t,\pi}\ell)]e^{-\lambda_{1,0}(T-t)}.$$

(i) Then f^1 is of class $C^{1,2}$ on $[0,T]\times[0,1]$ with

$$\partial_\pi f^1(t,\pi) = \int_t^T E\left[\partial_\pi \tilde{\pi}_s^{t,\pi}\, g_1'(\tilde{\pi}_s^{t,\pi})\right] e^{-\lambda_{1,0}(s-t)} ds - E\left[\partial_\pi \tilde{\pi}_T^{t,\pi} \frac{\ell}{1-\tilde{\pi}_T^{t,\pi}\ell}\right] e^{-\lambda_{1,0}(T-t)},$$

$$\begin{aligned}\partial_{\pi,\pi} f^1(t,\pi) = & \int_t^T E\left[\partial_{\pi,\pi} \tilde{\pi}_s^{t,\pi}\, g_1'(\tilde{\pi}_s^{t,\pi}) + \left(\partial_\pi \tilde{\pi}_s^{t,\pi}\right)^2\, g_1''(\tilde{\pi}_s^{t,\pi})\right] e^{-\lambda_{1,0}(s-t)} ds \\ & - E\left[\partial_{\pi,\pi} \tilde{\pi}_T^{t,\pi} \frac{\ell}{1-\tilde{\pi}_T^{t,\pi}\ell} + \left(\partial_\pi \tilde{\pi}_T^{t,\pi}\right)^2 \frac{\ell^2}{(1-\tilde{\pi}_T^{t,\pi}\ell)^2}\right] e^{-\lambda_{1,0}(T-t)},\end{aligned}$$

where $\partial_\pi \tilde{\pi}_s^{t,\pi} = \frac{Z_s}{(\pi+(1-\pi)Z_s)^2}$ *and* $\partial_{\pi,\pi}\tilde{\pi}_s^{t,\pi} = -2\frac{Z_s(1-Z_s)}{(\pi+(1-\pi)Z_s)^3}$.

(ii) f^1 solves the HJB equation (33).

In particular, $\partial_\pi f^1$ does not depend on f^0, and thus the first-order condition (34) provides an algebraic equation for the optimal stock proportion in state 0.

Proof. (i) The explicit representation given in Remark 2.12 implies that f^1 is continuously differentiable with respect to t. Let $t\in[0,T]$, $\pi\in[0,1]$ and let $s\in[t,T]$. We have $|g_1'(\pi)| \le |\alpha_1| + \sigma_1^2 + \lambda_1 \frac{L_1}{1-L_1} + \lambda_{1,0}\frac{L_{1,0}}{1-L_{1,0}}$ and $\partial_\pi\tilde{\pi}_s^{t,\pi} \le \frac{Z_s}{(Z_s\wedge 1)^2}$ and therefore, by Remark 2.3, we obtain

$$\left|\partial_\pi \tilde{\pi}_s^{t,\pi}(\omega)\, g_1'(\tilde{\pi}_s^{t,\pi}(\omega))\right| \le \frac{Z_s(\omega)}{(Z_s(\omega)\wedge 1)^2}\left(|\alpha_1| + \sigma_1^2 + \lambda_1\frac{L_1}{1-L_1} + \lambda_{1,0}\frac{L_{1,0}}{1-L_{1,0}}\right) \tag{35}$$

for all $s\in[t,T]$ and $\omega\in\Omega$. Furthermore, we have

$$\left|\partial_\pi \tilde{\pi}_T^{t,\pi}(\omega)\frac{\ell}{1-\tilde{\pi}_T^{t,\pi}(\omega)\ell}\right| \le \frac{Z_T(\omega)}{(Z_T(\omega)\wedge 1)^2}\frac{\ell}{1-\ell} \tag{36}$$

for all $\omega\in\Omega$. Therefore, the discounted left-hand sides of (35) and (36) are uniformly bounded in π by integrable functions and we may thus interchange differentiating and integrating. This yields

$$\partial_\pi f^1(t,\pi) = \int_t^T E\left[\partial_\pi \tilde{\pi}_s^{t,\pi}\, g_1'(\tilde{\pi}_s^{t,\pi})\right] e^{-\lambda_{1,0}(s-t)} ds - E\left[\partial_\pi \tilde{\pi}_T^{t,\pi} \frac{\ell}{1-\tilde{\pi}_T^{t,\pi}\ell}\right] e^{-\lambda_{1,0}(T-t)}.$$

Again, by Remark 2.3, we have

$$\begin{aligned}\left|\partial_\pi\left[\partial_\pi \tilde{\pi}_s^{t,\pi}(\omega)\, g_1'(\tilde{\pi}_s^{t,\pi}(\omega))\right]\right| &= |\partial_{\pi,\pi}\tilde{\pi}_s^{t,\pi}(\omega) g_1'(\tilde{\pi}_s^{t,\pi}(\omega)) + (\partial_\pi\tilde{\pi}_s^{t,\pi}(\omega))^2 g_1''(\tilde{\pi}_s^{t,\pi}(\omega))| \\ &\le c_1\frac{Z_s(\omega)}{(Z_s(\omega)\wedge 1)^3}|1-Z_s(\omega)| + c_2\frac{Z_s(\omega)^2}{(Z_s(\omega)\wedge 1)^4} \qquad (37)\end{aligned}$$

for all $s \in [t,T]$ and $\omega \in \Omega$, as well as

$$\left|\partial_\pi \left[\partial_\pi \tilde{\pi}_T^{t,\pi}(\omega) \frac{\ell}{1-\tilde{\pi}_T^{t,\pi}(\omega)\ell}\right]\right| = \left|\partial_{\pi,\pi}\tilde{\pi}_T^{t,\pi}(\omega)\frac{\ell}{1-\tilde{\pi}_T^{t,\pi}(\omega)\ell} + (\partial_\pi \tilde{\pi}_T^{t,\pi}(\omega))^2 \frac{\ell^2}{(1-\tilde{\pi}_T^{t,\pi}(\omega)\ell)^2}\right|$$
$$\leq 2c_3 \frac{Z_T(\omega)}{(Z_T(\omega)\wedge 1)^3}|1-Z_T(\omega)| + c_3^2 \frac{Z_T(\omega)^2}{(Z_T(\omega)\wedge 1)^4} \tag{38}$$

for all $\omega \in \Omega$, where $c_1 = 2\left(|\alpha_1| + \sigma_1^2 + \lambda_1 \frac{L_1}{1-L_1} + \lambda_{1,0}\frac{L_{1,0}}{1-L_{1,0}}\right)$, $c_2 = \sigma_1^2 + \lambda_1 \frac{L_1^2}{(1-L_1)^2} + \lambda_{1,0}\frac{L_{1,0}^2}{(1-L_{1,0})^2}$, and $c_3 = \frac{\ell}{1-\ell}$. Thus, the discounted left-hand sides of (37) and (38) are uniformly bounded in π by integrable functions. As before, we may thus interchange differentiating and integrating. We obtain

$$\partial_{\pi,\pi} f^1(t,\pi) = \int_t^T E\left[\partial_{\pi,\pi}\tilde{\pi}_s^{t,\pi}\, g_1'(\tilde{\pi}_s^{t,\pi}) + (\partial_\pi \tilde{\pi}_s^{t,\pi})^2\, g_1''(\tilde{\pi}_s^{t,\pi})\right] e^{-\lambda_{1,0}(s-t)} ds$$
$$- E\left[\partial_{\pi,\pi}\tilde{\pi}_T^{t,\pi}\frac{\ell}{1-\tilde{\pi}_T^{t,\pi}\ell} + (\partial_\pi \tilde{\pi}_T^{t,\pi})^2 \frac{\ell^2}{(1-\tilde{\pi}_T^{t,\pi}\ell)^2}\right] e^{-\lambda_{1,0}(T-t)}.$$

(ii) The assertion follows by the Feynman-Kac formula. □

Conditioning on the number of jumps within state 1, we have the following remark.

Remark 2.12. *For f^1 defined as in the previous proposition we have*

$$f^1(t,\pi) = \sum_{n=0}^{\infty} \int_t^T e^{-\lambda_{1,0}(s-t)} p_n(t,s)[\lambda_{1,0} f^0(s) + \int_{-\infty}^{\infty} g_1(\tilde{\pi}_n(t,\pi,s,u))\psi(s-t,u)du]ds$$
$$+ e^{-\lambda_{1,0}(T-t)} \sum_{n=0}^{\infty} p_n(t,T) \int_{-\infty}^{\infty} \ln(1-\tilde{\pi}_n(t,\pi,T,u)\ell)\psi(T-t,u)du,$$

$$\partial_\pi f^1(t,\pi) = \sum_{n=0}^{\infty} \int_t^T e^{-\lambda_{1,0}(s-t)} p_n(t,s) \int_{-\infty}^{\infty} \partial_\pi \tilde{\pi}_n(t,\pi,s,u) g_1'(\tilde{\pi}_n(t,\pi,s,u))\psi(s-t,u)du\, ds$$
$$- e^{-\lambda_{1,0}(T-t)} \sum_{n=0}^{\infty} p_n(t,T) \int_{-\infty}^{\infty} \partial_\pi \tilde{\pi}_n(t,\pi,T,u) \frac{\ell}{1-\tilde{\pi}_n(t,\pi,T,u)\ell}\psi(T-t,u)du,$$

where p_n, ψ, $\tilde{\pi}_n$, $\partial_\pi \tilde{\pi}_n$ and z_n are given by

$$
\begin{aligned}
p_n(t,s) &= P(N^1_{s-t} = n) = \frac{e^{-\lambda_1(s-t)}(\lambda_1(s-t))^n}{n!}, \\
\psi(r,u) &= \frac{1}{\sqrt{2\pi r}}\, e^{-\frac{u^2}{2r}}, \\
\tilde{\pi}_n(t,\pi,s,u) &= \frac{\pi}{\pi + (1-\pi) z_n(t,s,u)}, \\
\partial_\pi \tilde{\pi}_n(t,\pi,s,u) &= \frac{z_n(t,s,u)}{(\pi + (1-\pi) z_n(t,s,u))^2}, \\
z_n(t,s,u) &= \frac{e^{(\frac{1}{2}\sigma_1^2 - \alpha_1)(s-t) - \sigma_1 u}}{(1-L_1)^n}.
\end{aligned}
$$

The volatility of the price of the risky asset can be related to the amount of trading in that asset. Thus, since trading is interrupted in state 1, it seems reasonable to set $\sigma_1 = 0$. Besides, we think of state 1 as a regime where the economy is hit by an extreme event such as a war or a political turmoil. Consequently, it may also be plausible to assume that $\alpha_1 \leq 0$. As the following proposition shows, these assumptions together with (39) are sufficient to ensure the existence of a unique smooth solution of the investor's portfolio problem (31).

Proposition 2.13 (Optimal portfolio choice)**.** *Assume that $\alpha_1 \leq 0$ and $\sigma_1 = 0$.*

(i) The function f^1 defined above is decreasing and concave, i.e. the derivatives $\partial_\pi f^1$ and $\partial_{\pi,\pi} f^1$ are non-positive.

(ii) If for each $t \in [0,T]$ there exists a $\pi^*(t) \in [0,1]$ such that $\pi^*(t)$ is a solution to the first-order condition (34), then $\pi^* : [0,T] \to [0,1]$ is uniquely determined and of class C^1. Moreover, $\pi^*(t) = \arg\max_{\pi \in [0,1]} H^0(t,\pi)$ for all $t \in [0,T]$.

(iii) A solution in (ii) exists if for all $t \in [0,T]$

$$
\begin{aligned}
\alpha_0 - \lambda_0 L_0 - \lambda_{0,1} L_{0,1} + \lambda_{0,1} \partial_\pi f^1(t,0)(1 - L_{0,1}) &\geq 0, \qquad (39) \\
\alpha_0 - \sigma_0^2 - \lambda_0 \frac{L_0}{1-L_0} - \lambda_{0,1} \frac{L_{0,1}}{1-L_{0,1}} + \lambda_{0,1} \partial_\pi f^1(t,1) \frac{1}{1-L_{0,1}} &\leq 0.
\end{aligned}
$$

Remark 2.14. *Condition (39) can be rewritten more explicitly as*

$$
\begin{aligned}
0 \leq\ & \alpha_0 - \lambda_0 L_0 - \lambda_{0,1} L_{0,1} - \lambda_{0,1}(1-L_{0,1})\ell E\left(\frac{1}{Z_T}\right) e^{-\lambda_{1,0}(T-t)} \\
& + \lambda_{0,1}(1-L_{0,1})(\alpha_1 - \lambda_1 L_1 - \lambda_{1,0} L_{1,0}) \int_t^T E\left(\frac{1}{Z_s}\right) e^{-\lambda_{1,0}(s-t)} ds, \\
0 \geq\ & \alpha_0 - \sigma_0^2 - \lambda_0 \frac{L_0}{1-L_0} - \lambda_{0,1}\frac{L_{0,1}}{1-L_{0,1}} - \lambda_{0,1}\frac{1}{1-L_{0,1}}\frac{\ell}{1-\ell} E(Z_T) e^{-\lambda_{1,0}(T-t)} \\
& + \lambda_{0,1}\frac{1}{1-L_{0,1}}\Big(\alpha_1 - \lambda_1 \frac{L_1}{1-L_1} - \lambda_{1,0}\frac{L_{1,0}}{1-L_{1,0}}\Big) \int_t^T E(Z_s) e^{-\lambda_{1,0}(s-t)} ds.
\end{aligned}
$$

Proof. (i) and (ii). Let $t \in [0,T]$, let $\pi \in [0,1]$ and let $s \in [t,T]$. We have $g_1'(\pi) = \alpha_1 - \lambda_1 \frac{L_1}{1-\pi L_1} - \lambda_{1,0}\frac{L_{1,0}}{1-\pi L_{1,0}} \leq 0$ and $\partial_\pi \tilde{\pi}_s^{t,\pi} = \frac{Z_s}{(\pi+(1-\pi)Z_s)^2} \geq 0$. Thus, by Remark 2.3 and Proposition 2.11 (i), we find $\partial_\pi f^1 \leq 0$. Since $\sigma_1 = 0$, we have

$$Z_s = \frac{e^{-\alpha_1(s-t)}}{(1-L_1)^{N_s^1 - N_t^1}} \geq 1,$$

which implies that

$$\partial_{\pi,\pi}\tilde{\pi}_s^{t,\pi} = -2\frac{Z_s(1-Z_s)}{(\pi+(1-\pi)Z_s)^3} \geq 0.$$

Furthermore,

$$g_1''(\pi) = -\lambda_1 \frac{L_1^2}{(1-\pi L_1)^2} - \lambda_{1,0}\frac{L_{1,0}^2}{(1-\pi L_{1,0})^2} \leq 0$$

and consequently

$$
\begin{aligned}
\partial_{\pi,\pi}\tilde{\pi}_s^{t,\pi}\, g_1'(\tilde{\pi}_s^{t,\pi}) + (\partial_\pi \tilde{\pi}_s^{t,\pi})^2\, g_1''(\tilde{\pi}_s^{t,\pi}) &\leq 0 \\
\partial_{\pi,\pi}\tilde{\pi}_T^{t,\pi}\frac{\ell}{1-\tilde{\pi}_T^{t,\pi}\ell} + (\partial_\pi \tilde{\pi}_T^{t,\pi})^2 \frac{\ell^2}{(1-\tilde{\pi}_T^{t,\pi}\ell)^2} &\geq 0.
\end{aligned}
$$

Thus, by Proposition 2.11 (i), we have

$$\partial_{\pi,\pi} f^1(t,\pi) \leq 0.$$

Taking the derivative with respect to π of the right hand side of the first-order condition (34), we get

$$
\begin{aligned}
0 \geq & -\sigma_0^2 - \lambda_0 \frac{L_0^2}{(1-\pi L_0)^2} - \lambda_{0,1}\frac{L_{0,1}^2}{(1-\pi L_{0,1})^2} + \lambda_{0,1}\partial_{\pi,\pi} f^1\left(t, \frac{\pi(1-L_{0,1})}{1-\pi L_{0,1}}\right)\frac{(1-L_{0,1})^2}{(1-\pi L_{0,1})^4} \\
& + 2\lambda_{0,1}\partial_\pi f^1\left(t, \frac{\pi(1-L_{0,1})}{1-\pi L_{0,1}}\right)\frac{L_{0,1}(1-L_{0,1})}{(1-\pi L_{0,1})^3}.
\end{aligned}
$$

Thus, the solution of the first-order condition (34) is unique. Furthermore, by the implicit function theorem, we conclude that for a solution π^* as detailed in (ii), the mapping π^* is continuously differentiable and maximizes the HJB equation (32).

(iii) Under our assumptions, the right hand side of the first-order condition (34) is continuous and decreasing in π. Therefore, by the intermediate value theorem, the claim follows. $\square$

Note that the requirements $\alpha_1 \leq 0$ and $\sigma_1 = 0$ are not necessary for the claim in the previous proposition to hold. They however imply that $\partial_\pi f^1$ and $\partial_{\pi,\pi} f^1$ are non-positive, which is sufficient to prove the claim. Besides, we remark that (39) is satisfied for reasonable choices of α_0. However, if α_0 is "too large" or "too small", then it can happen that this condition is not satisfied. For instance, if $\alpha_0 < 0$, i.e. an investment in stocks is strictly dominated by an investment in bonds, then the optimal number of stocks is zero. This is a corner solution and (39) excludes these kinds of degenerated cases. The following proposition provides a representation of the value function in state 0.

Proposition 2.15 (Indirect utility in liquidity)**.** *Suppose that there exists a continuous function $\pi^*: [0,T] \to [0,1]$ such that $\pi^*(t) \in \arg\max_{\pi\in[0,1]} H^0(t,\pi)$ for all $t \in [0,T]$. Consider the function $f^0: [0,T] \to \mathbb{R}$ given by*

$$f^0(t) = \frac{\lambda_{0,1}}{\lambda_{0,1}+\lambda_{1,0}} e^{(\lambda_{0,1}+\lambda_{1,0})t} \int_t^T F(s) e^{-\lambda_{0,1}s} ds + \frac{\lambda_{1,0}}{\lambda_{0,1}+\lambda_{1,0}} \int_t^T F(s) e^{\lambda_{1,0}s} ds,$$

where

$$F(t) = g_0(\pi^*(t)) e^{-\lambda_{1,0}t} + \lambda_{0,1} \int_t^T E[g_1(\tilde{\pi}_s^{t,\hat{\pi}_0(t)})] e^{-\lambda_{1,0}s} ds + \lambda_{0,1} E[\ln(1-\tilde{\pi}_T^{t,\hat{\pi}_0(t)}\ell)] e^{-\lambda_{1,0}T}$$

and $\hat{\pi}_0(t) = \frac{\pi^*(t)(1-L_{0,1})}{1-\pi^*(t)L_{0,1}}$. Then f^0 is of class C^1 and solves the HJB equation (32).

Proof. The function F is continuous and therefore f^0 is continuously differentiable. Recall that the Hamilton-Jacob-Bellman equation (32) is given by

$$0 = \partial_t f^0(t) + g_0(\pi^*(t)) + \lambda_{0,1} f^1(t, \hat{\pi}_0(t)) - \lambda_{0,1} f^0(t).$$

For f^1 defined as in Proposition 2.11, we have

$$f^1(t,\hat{\pi}_0(t)) = \lambda_{1,0} \int_t^T f^0(s) e^{-\lambda_{1,0}(s-t)} ds + \int_t^T E[g_1(\tilde{\pi}_s^{t,\hat{\pi}_0(t)})] e^{-\lambda_{1,0}(s-t)} ds$$
$$+ E[\ln(1-\tilde{\pi}_T^{t,\hat{\pi}_0(t)}\ell)] e^{-\lambda_{1,0}(T-t)},$$

which yields the following integro-differential equation for f^0

$$0 = \partial_t f^0(t)e^{-\lambda_{1,0}t} + g_0(\pi^*(t))e^{-\lambda_{1,0}t} + \lambda_{0,1}\lambda_{1,0}\int_t^T f^0(s)e^{-\lambda_{1,0}s}ds - \lambda_{0,1}f^0(t)e^{-\lambda_{1,0}t}$$

$$+ \lambda_{0,1}\int_t^T E[g_1(\tilde{\pi}_s^{t,\hat{\pi}_0(t)})]e^{-\lambda_{1,0}s}ds \; + \lambda_{0,1}E[\ln(1-\tilde{\pi}_T^{t,\hat{\pi}_0(t)}\ell)]e^{-\lambda_{1,0}T}.$$

Substituting

$$H(t) = \int_t^T f^0(s)e^{-\lambda_{1,0}s}ds$$

into the equation above, we get

$$0 = -H''(t) - (\lambda_{1,0} - \lambda_{0,1})H'(t) + \lambda_{0,1}\lambda_{1,0}H(t) + g_0(\pi^*(t))e^{-\lambda_{1,0}t}$$

$$+ \lambda_{0,1}\int_t^T E[g_1(\tilde{\pi}_s^{t,\hat{\pi}_0(t)})]e^{-\lambda_{1,0}s}ds + \lambda_{0,1}E[\ln(1-\tilde{\pi}_T^{t,\hat{\pi}_0(t)}\ell)]e^{-\lambda_{1,0}T}.$$

Eventually, setting

$$F(t) = g_0(\pi^*(t))e^{-\lambda_{1,0}t} + \lambda_{0,1}\int_t^T E[g_1(\tilde{\pi}_s^{t,\hat{\pi}_0(t)})]e^{-\lambda_{1,0}s}ds + \lambda_{0,1}E[\ln(1-\tilde{\pi}_T^{t,\hat{\pi}_0(t)}\ell)]e^{-\lambda_{1,0}T}$$

leads to the following second-order linear inhomogeneous differential equation

$$H''(t) + (\lambda_{1,0} - \lambda_{0,1})H'(t) - \lambda_{0,1}\lambda_{1,0}H(t) = F(t) \tag{40}$$

subject to the constraints $H(T) = 0$ and $H'(T) = 0$. The characteristic equation

$$\mu^2 + (\lambda_{1,0} - \lambda_{0,1})\mu - \lambda_{0,1}\lambda_{1,0} = 0$$

has the two roots, $\mu_1 = \lambda_{0,1}$ and $\mu_2 = -\lambda_{1,0}$. Thus, the exponential conjecture yields the pair $u_1(t) = e^{\lambda_{0,1}t}$ and $u_2(t) = e^{-\lambda_{1,0}t}$ of fundamental solutions for the homogeneous differential equation. By the method of variation of constants, a particular solution of (40) is given by

$$w(t) = c_1(t)u_1(t) + c_2(t)u_2(t),$$

where c_1 and c_2 are such that

$$c_1'(t)u_1(t) + c_2'(t)u_2(t) = 0.$$

Setting w into the inhomogeneous equation (40) yields

$$\begin{aligned}F(t) &= c_1'(t)u_1'(t) + c_2'(t)u_2'(t) + c_1(t)[u_1''(t) + (\lambda_{1,0} - \lambda_{0,1})u_1'(t) - \lambda_{0,1}\lambda_{1,0}u_1(t)] \\ &\qquad + c_2(t)[u_2''(t) + (\lambda_{1,0} - \lambda_{0,1})u_2'(t) - \lambda_{0,1}\lambda_{1,0}u_2(t)] \\ &= c_1'(t)u_1'(t) + c_2'(t)u_2'(t).\end{aligned}$$

Thus, by the previous equation and the constraint for c_1 and c_2, we find

$$w(t) = u_1(t)\int_t^T \frac{F(s)u_2(s)}{W(s)}ds - u_2(t)\int_t^T \frac{F(s)u_1(s)}{W(s)}ds,$$

where the Wronskian determinant W is given by

$$W(s) = \begin{vmatrix} u_1(s) & u_2(s) \\ u_1'(s) & u_2'(s)\end{vmatrix} = -(\lambda_{0,1} + \lambda_{1,0})e^{(\lambda_{0,1}-\lambda_{1,0})s}.$$

Rewriting w we have

$$w(t) = \int_t^T \frac{F(s)}{\lambda_{0,1}+\lambda_{1,0}}e^{\lambda_{1,0}(s-t)}ds - \int_t^T \frac{F(s)}{\lambda_{0,1}+\lambda_{1,0}}e^{-\lambda_{0,1}(s-t)}ds.$$

Note that we have $w(T) = 0$ and $w'(T) = 0$. Thus, the unique solution of the constraint differential equation (40) is given by the particular solution w, i.e

$$H(t) = \int_t^T \frac{F(s)}{\lambda_{0,1}+\lambda_{1,0}}e^{\lambda_{1,0}(s-t)}ds - \int_t^T \frac{F(s)}{\lambda_{0,1}+\lambda_{1,0}}e^{-\lambda_{0,1}(s-t)}ds.$$

Differentiating H we obtain

$$\begin{aligned}H'(t) &= u_1'(t)\int_t^T \frac{F(s)u_2(s)}{W(s)}ds - u_2'(t)\int_t^T \frac{F(s)u_1(s)}{W(s)}ds \\ &= -\lambda_{0,1}\int_t^T \frac{F(s)}{\lambda_{0,1}+\lambda_{1,0}}e^{-\lambda_{0,1}(s-t)}ds - \lambda_{1,0}\int_t^T \frac{F(s)}{\lambda_{0,1}+\lambda_{1,0}}e^{\lambda_{1,0}(s-t)}ds.\end{aligned}$$

Further, by the definition of H, we have $H'(t) = -f^0(t)e^{-\lambda_{1,0}t}$ and thus f^0 is given by

$$f^0(t) = \lambda_{0,1}e^{\lambda_{1,0}t}\int_t^T \frac{F(s)}{\lambda_{0,1}+\lambda_{1,0}}e^{-\lambda_{0,1}(s-t)}ds + \lambda_{1,0}\int_t^T \frac{F(s)}{\lambda_{0,1}+\lambda_{1,0}}e^{\lambda_{1,0}s}ds.$$

□

There is a special case where the integrals in the above representation of f^0 can be calculated explicitly. Namely, if $\sigma_1 = \alpha_1 = L_1 = L_0 = L_{0,1} = 0$, the first-order condition (34) simplifies into

$$0 = \alpha_0 - \sigma_0^2\pi + \lambda_{0,1}\left[(e^{-\lambda_{1,0}(T-t)} - 1)\frac{L_{1,0}}{1-\pi L_{1,0}} - e^{-\lambda_{1,0}(T-t)}\frac{\ell}{1-\pi\ell}\right].$$

If, in addition, $\ell = L_{1,0}$, we get

$$0 = \alpha_0 - \sigma_0^2\pi - \lambda_{0,1}\frac{L_{1,0}}{1-\pi L_{1,0}}, \tag{41}$$

and the function f^0 is given by

$$\begin{aligned}
f^0(t) = {}& \frac{\lambda_{0,1}}{\lambda_{0,1}+\lambda_{1,0}} e^{(\lambda_{0,1}+\lambda_{1,0})t}\Big[\Big(g_0(\pi^*) + \frac{\lambda_{0,1}}{\lambda_{1,0}} g_1(\pi^*)\Big)\frac{1}{\lambda_{0,1}+\lambda_{1,0}}(e^{-(\lambda_{0,1}+\lambda_{1,0})t} - e^{-(\lambda_{0,1}+\lambda_{1,0})T}) \\
&+ \Big(\ln(1-\pi^* L_{1,0}) - \frac{1}{\lambda_{1,0}} g_1(\pi^*)\Big) e^{-\lambda_{1,0}T}(e^{-\lambda_{0,1}t} - e^{-\lambda_{0,1}T})\Big] \\
&+ \frac{\lambda_{1,0}}{\lambda_{0,1}+\lambda_{1,0}}\Big[\Big(\lambda_{0,1}\ln(1-\pi^* L_{1,0}) - \frac{\lambda_{0,1}}{\lambda_{1,0}} g_1(\pi^*)\Big) e^{-\lambda_{1,0}T}\frac{1}{\lambda_{1,0}}(e^{\lambda_{1,0}T} - e^{\lambda_{1,0}t}) \\
&+ \Big(g_0(\pi^*) + \frac{\lambda_{0,1}}{\lambda_{1,0}} g_1(\pi^*)\Big)(T-t)\Big],
\end{aligned}$$

where $g_0(\pi) = r_0 + \alpha_0\pi - \frac{1}{2}\pi^2\sigma_0^2$ and $g_1(\pi) = r_1 + \lambda_{1,0}\ln(1-\pi L_{1,0})$, and π^* solves the first-order condition (41). The following theorem summarizes our results in this section.

Theorem 2.16 (Solution of the portfolio problem)**.** *Suppose that there exists a continuous function $\pi^* : [0,T] \to [0,1]$ such that $\pi^*(t) \in \arg\max_{\pi\in[0,1]} H^0(t,\pi)$ for all $t \in [0,T]$. Then, for $k_0 = \infty$, the value function is given by*

$$V(t_0, x_0, \infty) = \ln(x_0) + f^0(t_0)$$

for $t_0 \in [0,T]$, $x_0 \in (0,\infty)$ and the optimal strategy is given by π^*.

Proof. Since $|\ln(x)| \le \frac{1}{x}$ for $x \in (0,1)$, the assertion follows immediately from the Verification Theorem 2.9 and Propositions 2.11 and 2.15. □

Since we wish to quantify the utility loss incurred by an investor due to the presence of illiquidity, we take a brief look at the optimal investment problem when trading is allowed in both states. In this case, a verification theorem analogous to Theorem 2.9 holds true, and the optimal portfolio strategy can be characterized by the following coupled system of Hamilton-Jacobi-Bellman equations

$$0 = \sup_{\pi_i\in[0,1]}\{\partial_t f^i(t) + g_i(\pi_i) + \lambda_{i,1-i}[f^{1-i}(t) - f^i(t)]\}, \tag{42}$$

with terminal conditions $f^i(T) = 0$, with $i \in \{0,1\}$. The associated first-order conditions then read

$$0 = \alpha_i - \pi_i \sigma_i^2 - \lambda_i \frac{L_i}{1 - \pi_i L_i} - \lambda_{i,1-i} \frac{L_{i,1-i}}{1 - \pi_i L_{i,1-i}}, \tag{43}$$

for $i \in \{0,1\}$.

Proposition 2.17 (Optimal solution without illiquidity). *Suppose that $\pi_i^* \in [0,1]$ is a solution to the first-order condition (43) for $i \in \{0,1\}$ and $f^0 : [0,T] \to \mathbb{R}$ is given by*

$$f^0(t) = \frac{\lambda_{0,1}}{\lambda_{0,1} + \lambda_{1,0}} e^{(\lambda_{1,0}+\lambda_{0,1})t} \int_t^T F(s) e^{-\lambda_{0,1} s} ds + \frac{\lambda_{1,0}}{\lambda_{0,1} + \lambda_{1,0}} \int_t^T F(s) e^{\lambda_{1,0} s} ds,$$

where

$$F(t) = g_0(\pi_0^*) e^{-\lambda_{1,0} t} + \frac{\lambda_{0,1}}{\lambda_{1,0}} g_1(\pi_1^*)(e^{-\lambda_{1,0} t} - e^{-\lambda_{1,0} T}),$$

$$g_i(\pi) = r_i + \alpha_i \pi - \frac{1}{2}\pi^2 \sigma_i^2 + \lambda_i \ln(1 - \pi L_i) + \lambda_{i,1-i} \ln(1 - \pi L_{i,1-i}).$$

Then the value function when trading is allowed in both states, is given by

$$V(t_0, x_0) = \ln(x_0) + f^0(t_0)$$

for $t_0 \in [0,T]$ and $x_0 \in (0,\infty)$.

Proof. A solution of the Hamilton-Jacobi-Bellman equation (42) for state 1 is given by

$$f^1(t) = \frac{g_1(\pi_1^*)}{\lambda_{1,0}}(1 - e^{-\lambda_{1,0}(T-t)}) + \lambda_{1,0} \int_t^T f^0(s) e^{-\lambda_{1,0}(s-t)} ds.$$

Substituting this representation of f^1 into the Hamilton-Jacobi-Bellman equation (42) for state 0, and setting

$$H(t) = \int_t^T f^0(s) e^{-\lambda_{1,0} s} ds$$

we obtain the following second-order ordinary differential equation

$$H''(t) + (\lambda_{1,0} - \lambda_{0,1}) H'(t) - \lambda_{0,1}\lambda_{1,0} H(t) = F(t),$$

with $H(T) = 0$ and $H'(T) = 0$. Note that up to the definition of F, this equation is identical to (40). Thus, as in the proof of Proposition 2.15, we have

$$f^0(t) = \frac{\lambda_{0,1}}{\lambda_{0,1} + \lambda_{1,0}} e^{(\lambda_{1,0}+\lambda_{0,1})t} \int_t^T F(s) e^{-\lambda_{0,1} s} ds + \frac{\lambda_{1,0}}{\lambda_{0,1} + \lambda_{1,0}} \int_t^T F(s) e^{\lambda_{1,0} s} ds.$$

□

2.4.2 Finitely many liquidity breakdowns

Now, we assume that $k_0 < \infty$, hence the corresponding system of HJB equations can be solved iteratively. Note that $J^{0,0}$ is given by

$$J^{0,0}(t,x) = \ln(x) + f^{0,0}(t) = \ln(x) + (r_0 + \alpha_0 \pi^* - \frac{1}{2}(\pi^*)^2\sigma_0^2 + \lambda_0 \ln(1 - \pi^* L_0))(T - t),$$

where

$$0 = \alpha_0 - \sigma_0^2 \pi^* - \lambda_0 \frac{L_0}{1 - \pi^* L_0}.$$

As in the previous section, for $1 \leq k_0 \in \mathbb{N}$, we conjecture

$$J^{0,k_0}(t,x) = \ln(x) + f^{0,k_0}(t)$$

and

$$J^{1,k_0}(t,x,\pi) = \ln(x) + f^{1,k_0}(t,\pi)$$

for a C^1-function f^{0,k_0} on $[0,T]$ with $f^{0,k_0}(T) = 0$ and a $C^{1,2}$-function f^{1,k_0} on $[0,T] \times [0,1]$ with $f^{1,k_0}(T,\pi) = \ln(1 - \pi\ell)$ for all $\pi \in [0,1]$. The corresponding Hamilton-Jacobi-Bellman equations read

$$0 = \sup_{\pi \in [0,1]} \left\{ \partial_t f^{0,k_0}(t) + g_0(\pi) + \lambda_{0,1} \left[f^{1,k_0}\left(t, \frac{\pi(1 - L_{0,1})}{1 - \pi L_{0,1}}\right) - f^{0,k_0}(t) \right] \right\} \tag{44}$$

$$\begin{aligned} 0 &= \partial_t f^{1,k_0}(t,\pi) - \lambda_{1,0} f^{1,k_0}(t,\pi) + \pi(1-\pi)(\alpha_1 - \sigma_1^2 \pi)\partial_\pi f^{1,k_0}(t,\pi) \\ &\quad + \frac{1}{2}\pi^2(1-\pi)^2\sigma_1^2 \partial_{\pi,\pi} f^{1,k_0}(t,\pi) + \lambda_1 \left[f^{1,k_0}\left(t, \frac{\pi(1-L_1)}{1-\pi L_1}\right) - f^{1,k_0}(t,\pi) \right] \\ &\quad + g_1(\pi) + \lambda_{1,0} f^{0,k_0-1}(t), \end{aligned} \tag{45}$$

with g_0 and g_1 as before. Equation (44) leads to the following first-order condition for the optimal stock proportion in state 0

$$0 = \alpha_0 - \sigma_0^2 \pi - \lambda_0 \frac{L_0}{1 - \pi L_0} - \lambda_{0,1} \frac{L_{0,1}}{1 - \pi L_{0,1}} + \lambda_{0,1} \partial_\pi f^{1,k_0}\left(t, \frac{\pi(1 - L_{0,1})}{1 - \pi L_{0,1}}\right) \frac{1 - L_{0,1}}{(1 - \pi L_{0,1})^2}.$$

Note that the solution of the first-order condition is a deterministic function of time given such a solution exists.

Proposition 2.18 (Indirect utility in illiquidity)**.** *Let $1 \leq k_0 \in \mathbb{N}$ and let $\tilde{\pi}$ as in Proposition 2.11. Given a C^1-function $f^{0,k_0-1} : [0,T] \to \mathbb{R}$, consider the function $f^{1,k_0} : [0,T] \times [0,1] \to \mathbb{R}$ defined via the stochastic representation*

$$f^{1,k_0}(t,\pi) = \int_t^T \left(\lambda_{1,0} f^{0,k_0-1}(s) + E\left[g_1(\tilde{\pi}_s^{t,\pi}) \right] \right) e^{-\lambda_{1,0}(s-t)} ds + E\left[\ln(1 - \tilde{\pi}_T^{t,\pi} \ell) \right] e^{-\lambda_{1,0}(T-t)}.$$

(i) Then f^{1,k_0} is of class $\mathrm{C}^{1,2}$ on $[0,T]\times[0,1]$ with

$$\partial_\pi f^{1,k_0}(t,\pi) = \int_t^T E\left[\partial_\pi \tilde{\pi}_s^{t,\pi}\, g_1'(\tilde{\pi}_s^{t,\pi})\right] e^{-\lambda_{1,0}(s-t)} ds - E\left[\partial_\pi \tilde{\pi}_T^{t,\pi} \frac{\ell}{1-\tilde{\pi}_T^{t,\pi}\ell}\right] e^{-\lambda_{1,0}(T-t)}.$$

(ii) f^{1,k_0} solves the HJB equation (45).

Proof. Analogously to the proof of Proposition 2.11. □

Corollary 2.19 (k_0-invariance)**.** *Let $1 \le k_0 \in \mathbb{N}$ and let f^{1,k_0} as in the previous proposition. The first-order condition for the optimal stock proportion in state 0 coincides with the first-order condition (34) when infinitely many liquidity breakdowns are possible.*

Proof. The function $\partial_\pi f^{1,k_0}$ does not depend on the maximal number of illiquidity regimes k_0, and we have $\partial_\pi f^{1,k_0} = \partial_\pi f^1$ where $\partial_\pi f^1$ is given in Proposition 2.11. □

It is well known that, in general, a logarithmic investor makes his investment decisions myopically if continuous-time trading is possible. If liquidity breakdowns are possible, then he adjusts his portfolio decision to take the threat of illiquidity into account, however, by the previous corollary, he remains myopic in the sense that he disregards the number of possible breakdowns. His optimal stock demand does not depend on the maximal number of illiquidity regimes k_0.

Proposition 2.20 (Indirect utility in liquidity)**.** *Let $1 \le k_0 \in \mathbb{N}$ and let $\hat{\pi}_0$ as in Proposition 2.15. Suppose that there exists a continuous function $\pi^* : [0,T] \to [0,1]$ such that $\pi^*(t) \in \arg\max_{\pi\in[0,1]} H^{0,k_0}(t,\pi)$ for all $t \in [0,T]$. Given a $C^{1,2}$-function $f^{1,k_0} : [0,T]\times[0,1] \to \mathbb{R}$, consider the function $f^{0,k_0} : [0,T] \to \mathbb{R}$ defined via*

$$f^{0,k_0}(t) = \int_t^T \left[\lambda_{0,1} f^{1,k_0}(s,\hat{\pi}_0(s)) + g_0(\pi^*(s))\right] e^{-\lambda_{0,1}(s-t)} ds.$$

Then f^{0,k_0} is of class C^1 on $[0,T]$, and f^{0,k_0} solves the HJB equation (44).

Proof. It is clear that f^{0,k_0} is of class C^1 on $[0,T]$, and the second claim follows by differentiating f^{0,k_0} with respect to t. □

Collecting the above results and applying the Verification Theorem 2.9 yields

Theorem 2.21 (Solution of the portfolio problem)**.** *Let $k_0 \in \mathbb{N}$. Assume that there exists some $\pi_0^* \in [0,1]$ such that $0 = \alpha_0 - \sigma_0^2\pi_0^* - \lambda_0 \frac{L_0}{1-\pi_0^* L_0}$, and suppose that there exists a continuous function $\pi^* : [0,T] \to [0,1]$ such that $\pi^*(t) \in \arg\max_{\pi\in[0,1]} H^{0,k}(t,\pi)$ for all $t \in [0,T]$ and $1 \le k \le k_0$. Then, the value function is given by*

$$V(t_0,x_0,k_0) = \ln(x_0) + f^{0,k_0}(t_0)$$

for $t_0 \in [0,T]$, $x_0 \in (0,\infty)$ and the optimal strategy is given by $\pi^* 1_{\{K_- < k_0\}} + \pi_0^* 1_{\{K_- = k_0\}}$.

2.4.3 Alternative proof of the convergence of the value functions

Next, we provide an alternative proof of the convergence of the value functions when the maximal number of illiquidity regimes goes to infinity. This time, we consider the corresponding HJB equations and apply Weissinger's refinement of the Banach fixed point theorem.

Recall Corollary 2.19 and suppose that the investor's optimal portfolio strategy is given by $\pi^* : [0,T] \to [0,1]$. Following the notation, introduced before, we define two auxiliary functions h_0 and h_1. For $t \in [0,T]$ and $s \in [t,T]$, we set

$$h_0(t) = g_0(\pi^*(t)) + \lambda_{0,1} E[\ln(1 - \tilde{\pi}_T^{t,\hat{\pi}_0(t)} \ell)] e^{-\lambda_{1,0}(T-t)}$$

as well as

$$h_1(s,t) = E[g_1(\tilde{\pi}_s^{t,\hat{\pi}_0(t)})],$$

where $\hat{\pi}_0(t) = \frac{\pi^*(t)(1-L_{0,1})}{1-\pi^*(t)L_{0,1}}$. Substituting the representation of f^{1,k_0} into the one of f^{0,k_0} yields

$$\begin{aligned} f^{0,k_0}(t) &= \int_t^T [\lambda_{0,1}\{\int_s^T (\lambda_{1,0} f^{0,k_0-1}(u) + E[g_1(\tilde{\pi}_u^{s,\hat{\pi}_0(s)})]) e^{-\lambda_{1,0}(u-s)} du \\ &\qquad + E[\ln(1 - \tilde{\pi}_T^{s,\hat{\pi}_0(s)} \ell)] e^{-\lambda_{1,0}(T-s)}\} + g_0(\pi^*(s))] e^{-\lambda_{0,1}(s-t)} ds \\ &= \int_t^T h_0(s) e^{-\lambda_{0,1}(s-t)} ds \\ &\quad + \int_t^T \int_s^T [\lambda_{0,1}\lambda_{1,0} f^{0,k_0-1}(u) + \lambda_{0,1} h_1(u,s)] e^{-\lambda_{1,0}(u-s)-\lambda_{0,1}(s-t)} du\, ds, \end{aligned}$$

for $1 \le k_0 \in \mathbb{N}$. Thus, the operator which maps f^{0,k_0-1} to f^{0,k_0} is given by

$$A[f](t) = \int_t^T h_0(s) e^{-\lambda_{0,1}(s-t)} ds + \int_t^T \int_s^T [\lambda_{0,1}\lambda_{1,0} f(u) + \lambda_{0,1} h_1(u,s)] e^{-\lambda_{1,0}(u-s)-\lambda_{0,1}(s-t)} du\, ds.$$

Lemma 2.22. *Let f and g be in $C^0([0,T],\mathbb{R})$. For each $n \in \mathbb{N}$ we have*

$$||A^n[f] - A^n[g]||_{\sup([0,T])} \le (\lambda_{0,1}\lambda_{1,0})^n \frac{T^{2n}}{n!} ||f-g||_{\sup([0,T])}.$$

Proof. By induction we show that

$$|A^n[f](t) - A^n[g](t)| \le (\lambda_{0,1}\lambda_{1,0})^n \frac{(T-t)^{2n}}{n!} ||f-g||_{\sup([0,T])},$$

for each $t \in [0,T]$ and $n \in \mathbb{N}$. The statement is trivial for $n = 0$. Let $t \in [0,T]$ and assume that the assertion holds true for some $n \in \mathbb{N}$. Then we have

$$\begin{aligned}
|A^{n+1}[f](t) - A^{n+1}[g](t)| &= |\int_t^T \int_s^T \lambda_{0,1}\lambda_{1,0}(A^n[f](u) - A^n[g](u))e^{-\lambda_{1,0}(u-s)-\lambda_{0,1}(s-t)}du\, ds| \\
&\leq (\lambda_{0,1}\lambda_{1,0})^{n+1}||f-g||_{\sup([0,T])}\int_t^T\int_s^T \frac{(T-u)^{2n}}{n!}du\, ds \\
&\leq (\lambda_{0,1}\lambda_{1,0})^{n+1}||f-g||_{\sup([0,T])}(T-t)\max_{s\in[t,T]}\int_s^T \frac{(T-u)^{2n}}{n!}du \\
&\leq (\lambda_{0,1}\lambda_{1,0})^{n+1}\frac{(T-t)^{2(n+1)}}{(n+1)!}||f-g||_{\sup([0,T])}.
\end{aligned}$$

$\square$

Next, we show that f^0 is a fixed point of A.

Lemma 2.23. *The function f^0 is a fixed point of the operator A, i.e. $A[f^0] \equiv f^0$.*

Proof. Note that we may rewrite the representation of f^0, given in Proposition 2.15, in the following way

$$\begin{aligned}
f^0(t) &= \frac{1}{\lambda_{0,1}+\lambda_{1,0}}\int_t^T [\lambda_{0,1}e^{(\lambda_{0,1}+\lambda_{1,0})t-\lambda_{0,1}s} + \lambda_{1,0}e^{\lambda_{1,0}s}]F(s)ds \\
&= \frac{1}{\lambda_{0,1}+\lambda_{1,0}}\int_t^T [\lambda_{0,1}e^{(\lambda_{0,1}+\lambda_{1,0})t-\lambda_{0,1}s} + \lambda_{1,0}e^{\lambda_{1,0}s}] \\
&\quad \{g_0(\pi^*(s))e^{-\lambda_{1,0}s} + \lambda_{0,1}\int_s^T E[g_1(\tilde{\pi}_u^{s,\hat{\pi}_0(s)})]e^{-\lambda_{1,0}u}du + \lambda_{0,1}E[\ln(1-\tilde{\pi}_T^{s,\hat{\pi}_0(s)}\ell)]e^{-\lambda_{1,0}T}\}ds \\
&= \frac{1}{\lambda_{0,1}+\lambda_{1,0}}\int_t^T h_0(s)(\lambda_{0,1}e^{-(\lambda_{0,1}+\lambda_{1,0})(s-t)} + \lambda_{1,0})ds \\
&\quad + \frac{1}{\lambda_{0,1}+\lambda_{1,0}}\int_t^T\int_s^T h_1(u,s)e^{-\lambda_{1,0}(u-s)}du\,(\lambda_{0,1}^2e^{-(\lambda_{0,1}+\lambda_{1,0})(s-t)} + \lambda_{0,1}\lambda_{1,0})ds.
\end{aligned}$$

By the definition of A, we have

$$A[f^0](t) = \int_t^T h_0(s)e^{-\lambda_{0,1}(s-t)}ds + \int_t^T \int_s^T \Big[\frac{\lambda_{0,1}\lambda_{1,0}}{\lambda_{0,1}+\lambda_{1,0}} \int_u^T h_0(v)(\lambda_{0,1}e^{-(\lambda_{0,1}+\lambda_{1,0})(v-u)} + \lambda_{1,0})dv$$

$$+ \frac{\lambda_{0,1}\lambda_{1,0}}{\lambda_{0,1}+\lambda_{1,0}} \int_u^T \int_v^T h_1(w,v)e^{-\lambda_{1,0}(w-v)}dw\,(\lambda_{0,1}^2 e^{-(\lambda_{0,1}+\lambda_{1,0})(v-u)} + \lambda_{0,1}\lambda_{1,0})dv$$

$$+ \lambda_{0,1}h_1(u,s)\Big] e^{-\lambda_{1,0}(u-s)-\lambda_{0,1}(s-t)}du\,ds.$$

Thus, setting

$$A_{h_0}[f^0](t) = \frac{\lambda_{0,1}\lambda_{1,0}}{\lambda_{0,1}+\lambda_{1,0}} \int_t^T ds \int_s^T du \int_u^T dv\, h_0(v)(\lambda_{0,1}e^{-(\lambda_{0,1}+\lambda_{1,0})(v-u)} + \lambda_{1,0})\, e^{-\lambda_{1,0}(u-s)-\lambda_{0,1}(s-t)}$$

$$+ \int_t^T h_0(s)e^{-\lambda_{0,1}(s-t)}ds$$

as well as

$$A_{h_1}[f^0](t) = \frac{\lambda_{0,1}\lambda_{1,0}}{\lambda_{0,1}+\lambda_{1,0}} \int_t^T ds \int_s^T du \int_u^T dv \int_v^T dw\, h_1(w,v)(\lambda_{0,1}^2 e^{-(\lambda_{0,1}+\lambda_{1,0})(v-u)} + \lambda_{0,1}\lambda_{1,0})$$

$$e^{-\lambda_{1,0}(w-v)-\lambda_{1,0}(u-s)-\lambda_{0,1}(s-t)} + \int_t^T \int_s^T \lambda_{0,1}h_1(u,s)e^{-\lambda_{1,0}(u-s)-\lambda_{0,1}(s-t)}du\,ds$$

yields

$$A[f^0](t) = A_{h_0}[f^0](t) + A_{h_1}[f^0](t).$$

Now, to prove $A[f^0] \equiv f^0$, we will show that

$$A_{h_0}[f^0](t) = \frac{1}{\lambda_{0,1}+\lambda_{1,0}} \int_t^T h_0(s)(\lambda_{0,1}e^{-(\lambda_{0,1}+\lambda_{1,0})(s-t)} + \lambda_{1,0})ds \tag{46}$$

and

$$A_{h_1}[f^0](t) = \frac{1}{\lambda_{0,1}+\lambda_{1,0}} \int_t^T \int_s^T h_1(u,s)e^{-\lambda_{1,0}(u-s)}du\,(\lambda_{0,1}^2 e^{-(\lambda_{0,1}+\lambda_{1,0})(s-t)} + \lambda_{0,1}\lambda_{1,0})ds. \tag{47}$$

In order to establish (46) we consider the first term in the definition of $A_{h_0}[f^0](t)$. We have

$$\int_t^T ds \int_s^T du \int_u^T dv\, h_0(v)(\lambda_{0,1}e^{-(\lambda_{0,1}+\lambda_{1,0})(v-u)} + \lambda_{1,0})\, e^{-\lambda_{1,0}(u-s)-\lambda_{0,1}(s-t)}$$

$$= \int_t^T [\int_t^v \int_t^u (\lambda_{0,1}e^{-(\lambda_{0,1}+\lambda_{1,0})(v-u)} + \lambda_{1,0})e^{-\lambda_{1,0}(u-s)-\lambda_{0,1}(s-t)} ds\, du] h_0(v)dv$$

$$= \int_t^T [\int_t^v (\lambda_{0,1}e^{-(\lambda_{0,1}+\lambda_{1,0})(v-u)} + \lambda_{1,0}) \int_t^u e^{(\lambda_{1,0}-\lambda_{0,1})s-\lambda_{1,0}u+\lambda_{0,1}t} ds\, du] h_0(v)dv$$

$$= \frac{1}{\lambda_{1,0}-\lambda_{0,1}} \int_t^T [\int_t^v (\lambda_{0,1}e^{-(\lambda_{0,1}+\lambda_{1,0})(v-u)} + \lambda_{1,0})(e^{\lambda_{0,1}(t-u)} - e^{\lambda_{1,0}(t-u)}) du] h_0(v)dv$$

$$= \frac{1}{\lambda_{1,0}-\lambda_{0,1}} \int_t^T [\int_t^v \lambda_{0,1}e^{-\lambda_{1,0}(v-u)-\lambda_{0,1}(v-t)} - \lambda_{0,1}e^{-\lambda_{0,1}(v-u)-\lambda_{1,0}(v-t)}$$

$$+ \lambda_{1,0}e^{\lambda_{0,1}(t-u)} - \lambda_{1,0}e^{\lambda_{1,0}(t-u)} du] h_0(v)dv$$

$$= \frac{1}{\lambda_{1,0}-\lambda_{0,1}} \int_t^T \left[\left(\frac{\lambda_{0,1}}{\lambda_{1,0}} - \frac{\lambda_{1,0}}{\lambda_{0,1}}\right) e^{-\lambda_{0,1}(v-t)} + \left(1 - \frac{\lambda_{0,1}}{\lambda_{1,0}}\right) e^{-(\lambda_{0,1}+\lambda_{1,0})(v-t)} + \frac{\lambda_{1,0}}{\lambda_{0,1}} - 1\right] h_0(v)dv.$$

Thus, by the definition of $A_{h_0}[f^0](t)$ we obtain

$$A_{h_0}[f^0](t) = \frac{1}{\lambda_{0,1}+\lambda_{1,0}} \int_t^T h_0(v)(\lambda_{0,1}+\lambda_{1,0})e^{-\lambda_{0,1}(v-t)} + \frac{\lambda_{0,1}\lambda_{1,0}}{\lambda_{1,0}-\lambda_{0,1}} \left[\left(\frac{\lambda_{0,1}}{\lambda_{1,0}} - \frac{\lambda_{1,0}}{\lambda_{0,1}}\right) e^{-\lambda_{0,1}(v-t)}\right.$$

$$\left. + \left(1 - \frac{\lambda_{0,1}}{\lambda_{1,0}}\right) e^{-(\lambda_{0,1}+\lambda_{1,0})(v-t)} + \frac{\lambda_{1,0}}{\lambda_{0,1}} - 1\right] h_0(v)dv$$

$$= \frac{1}{\lambda_{0,1}+\lambda_{1,0}} \int_t^T h_0(v)[(\lambda_{0,1}+\lambda_{1,0})e^{-\lambda_{0,1}(v-t)} - (\lambda_{0,1}+\lambda_{1,0})e^{-\lambda_{0,1}(v-t)}$$

$$+ \lambda_{0,1}e^{-(\lambda_{0,1}+\lambda_{1,0})(v-t)} + \lambda_{1,0}]dv$$

$$= \frac{1}{\lambda_{0,1}+\lambda_{1,0}} \int_t^T h_0(v)(\lambda_{0,1}e^{-(\lambda_{0,1}+\lambda_{1,0})(v-t)} + \lambda_{1,0})dv.$$

It remains to prove equation (47). As before, we start with rewriting the first term in the definition of $A_{h_1}[f^0](t)$.

$$\int_t^T ds \int_s^T du \int_u^T dv \int_v^T dw\, h_1(w,v) e^{-\lambda_{1,0}(w-v)} [\lambda_{0,1}^2 e^{-(\lambda_{0,1}+\lambda_{1,0})(v-u)} + \lambda_{0,1}\lambda_{1,0}] e^{-\lambda_{1,0}(u-s)-\lambda_{0,1}(s-t)}$$
$$= \int_t^T dv \int_v^T dw \int_t^v ds \int_s^v du\, h_1(w,v) e^{-\lambda_{1,0}(w-v)} [\lambda_{0,1}^2 e^{-\lambda_{0,1}(v+s-u-t)-\lambda_{1,0}(v-s)}$$
$$+ \lambda_{0,1}\lambda_{1,0} e^{-\lambda_{1,0}(u-s)-\lambda_{0,1}(s-t)}]$$
$$= \int_t^T dv \int_v^T dw\, h_1(w,v) e^{-\lambda_{1,0}(w-v)} \int_t^v ds\, [-\lambda_{0,1} e^{-\lambda_{0,1}(v-t)-\lambda_{1,0}(v-s)} + \lambda_{0,1} e^{-\lambda_{0,1}(s-t)}]$$
$$= \int_t^T dv \int_v^T dw\, h_1(w,v) e^{-\lambda_{1,0}(w-v)} \left[1 + \frac{\lambda_{0,1}}{\lambda_{1,0}} e^{-(\lambda_{0,1}+\lambda_{1,0})(v-t)} - \frac{\lambda_{0,1}+\lambda_{1,0}}{\lambda_{1,0}} e^{-\lambda_{0,1}(v-t)}\right].$$

Hence, by the definition of $A_{h_1}[f^0](t)$ we have

$$A_{h_1}[f^0](t) = \frac{1}{\lambda_{0,1}+\lambda_{1,0}} \int_t^T dv \int_v^T dw\, h_1(w,v) e^{-\lambda_{1,0}(w-v)} \Bigg[\lambda_{0,1}(\lambda_{0,1}+\lambda_{1,0}) e^{-\lambda_{0,1}(v-t)}$$
$$+ \lambda_{0,1}\lambda_{1,0}\left(1 + \frac{\lambda_{0,1}}{\lambda_{1,0}} e^{-(\lambda_{0,1}+\lambda_{1,0})(v-t)} - \frac{\lambda_{1,0}+\lambda_{0,1}}{\lambda_{1,0}} e^{-\lambda_{0,1}(v-t)}\right)\Bigg]$$
$$= \frac{1}{\lambda_{0,1}+\lambda_{1,0}} \int_t^T dv \int_v^T dw\, h_1(w,v) e^{-\lambda_{1,0}(w-v)} (\lambda_{0,1}^2 e^{-(\lambda_{0,1}+\lambda_{1,0})(v-t)} + \lambda_{0,1}\lambda_{1,0}).$$

□

Recall, the following generalization of the contraction principle.

Theorem 2.24 (Weissinger fixed point theorem)**.** *Let C be a nonempty closed subset of a Banach space $(X, ||.||)$. Suppose that $K : C \to C$ satisfies*

$$||K^n(x) - K^n(y)|| \le \theta_n ||x-y|| \quad \forall\, x, y \in C$$

with $\sum_{n=1}^{\infty} \theta_n < \infty$. Then K has a unique fixed point $\bar{x}$ such that

$$||K^j(x) - \bar{x}|| \le (\sum_{n=j}^{\infty} \theta_n) ||K(x) - x|| \quad \forall\, x \in C.$$

Thus, an application of the previous theorem yields

Theorem 2.25 (Convergence of the value functions)**.** *The sequence* $(f^{0,k_0})_{k_0\in\mathbb{N}}$ *converges uniformly towards* f^0 *and we have*

$$||f^{0,k_0} - f^0||_{\sup([0,T])} \leq \left(\sum_{n=k_0}^{\infty} (\lambda_{0,1}\lambda_{1,0})^n \frac{T^{2n}}{n!} \right) ||f^{0,1} - f^{0,0}||_{\sup([0,T])}.$$

Proof. The assertion follows from Lemma 2.22, Lemma 2.23 and Theorem 2.24, since we have $\sum\limits_{n=1}^{\infty} (\lambda_{0,1}\lambda_{1,0})^n \frac{T^{2n}}{n!} < \infty$. □

2.4.4 Generalization with three regimes

As a generalization of the model presented before, we now consider an economy with three regimes and possibly infinitely many regime shifts. We think of state 0 as the normal state of the market, state 1 corresponds to an illiquidity state in which trading is not possible at all. In addition, we introduce a third regime which we call state 2. For instance, state 2 can model an economic **crisis** where trading is possible, but the excess return is lower and the volatility is higher than in the normal state.

As before, in each state, the stock follows a jump-diffusion process where α_i denotes the excess return and σ_i denotes the volatility, $i \in \{0, 1, 2\}$. The size of a relative stock price jump within state i is denoted by L_i and its intensity is denoted by λ_i. Further, $\lambda_{i,j}$ stands for the intensity of a regime shift from state i into state $j \neq i$, and the corresponding loss rate is denoted by $L_{i,j}$. The value function is given by

$$V(t_0, x_0, \infty) = \sup_{\pi^0,\pi^2} E[\ln(X_T^{\pi^0,\pi^2,t_0,x_0,\infty})].$$

Similar to Theorem 2.9 one can show that for $i \in \{0, 2\}$, the Hamilton-Jacobi-Bellman equations corresponding to an economy with three different states are given by

$$\begin{aligned}
0 = \sup_{\pi_i\in[0,1]} \Bigg\{ & \partial_t J^i(t,x) + x(r_i + \pi_i\alpha_i)\partial_x J^i(t,x) + \frac{1}{2}x^2\pi_i^2\sigma_i^2\partial_{x,x}J^i(t,x) \\
& + \lambda_i[J^i(t, x(1-\pi_i L_i)) - J^i(t,x)] + \lambda_{i,2-i}[J^{2-i}(t, x(1-\pi_i L_{i,2-i})) - J^i(t,x)] \\
& + \lambda_{i,1} \left[J^1(t, x(1-\pi_i L_{i,1}), \frac{\pi_i(1-L_{i,1})}{1-\pi_i L_{i,1}}) - J^i(t,x) \right] \Bigg\}, \\
0 = {} & \partial_t J^1(t,x,\pi) + x(r_1 + \pi\alpha_1)\partial_x J^1(t,x,\pi) + \frac{1}{2}x^2\pi^2\sigma_1^2\partial_{x,x}J^1(t,x,\pi) \\
& + x\pi^2(1-\pi)\sigma_1^2\partial_{x,\pi}J^1(t,x,\pi) + \pi(1-\pi)(\alpha_1 - \sigma_1^2\pi)\partial_\pi J^1(t,x,\pi) \\
& + \frac{1}{2}\pi^2(1-\pi)^2\sigma_1^2\partial_{\pi,\pi}J^1(t,x,\pi) + \lambda_1 \left[J^1(t, x(1-\pi L_1), \frac{\pi(1-L_1)}{1-\pi L_1}) - J^1(t,x,\pi) \right] \\
& + \lambda_{1,0}[J^0(t, x(1-\pi L_{1,0})) - J^1(t,x,\pi)] + \lambda_{1,2}[J^2(t, x(1-\pi L_{1,2})) - J^1(t,x,\pi)],
\end{aligned}$$

subject to the boundary conditions $J^i(T,x) = \ln(x)$ and $J^1(T,x,\pi) = \ln(x(1-\pi\ell))$ for all $x \in (0,\infty)$ and $\pi \in [0,1]$, where $J^0 = J^{0,\infty}$, $J^1 = J^{1,\infty}$, $J^2 = J^{2,\infty}$ and ℓ again models liquidation costs. For $i \in \{0,2\}$, we conjecture

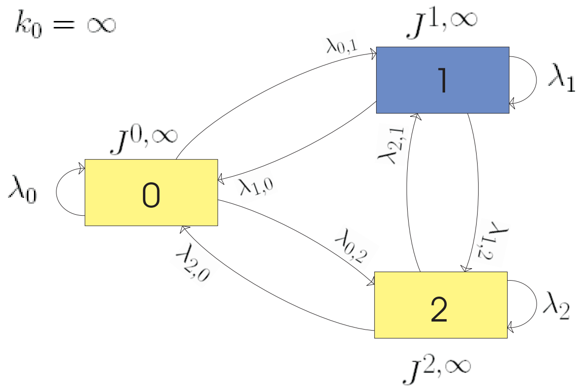

Figure 20: Economy with three regimes and possibly infinitely many liquidity breakdowns.

$$J^i(t,x) = \ln(x) + f^i(t)$$

as well as

$$J^1(t,x,\pi) = \ln(x) + f^1(t,\pi),$$

for C^1-functions f^i on $[0,T]$ with $f^i(T) = 0$, and a $C^{1,2}$-function f^1 on $[0,T] \times [0,1]$ such that $f^1(T,\pi) = \ln(1-\pi\ell)$ for all $\pi \in [0,1]$. Then the Hamilton-Jacobi-Bellman equations read

$$\begin{aligned}
0 &= \sup_{\pi_i \in [0,1]} \left\{ \partial_t f^i(t) + g_i(\pi_i) + \lambda_{i,1} \left[f^1\left(t, \frac{\pi_i(1-L_{i,1})}{1-\pi_i L_{i,1}}\right) - f^i(t) \right] + \lambda_{i,2-i}\left[f^{2-i}(t) - f^i(t)\right] \right\} \\
0 &= \partial_t f^1(t,\pi) + \pi(1-\pi)(\alpha_1 - \sigma_1^2\pi)\partial_\pi f^1(t,\pi) + \frac{1}{2}\pi^2(1-\pi)^2\sigma_1^2\partial_{\pi,\pi} f^1(t,\pi) + g_1(\pi) \\
&\quad + \lambda_1 \left[f^1\left(t, \frac{\pi(1-L_1)}{1-\pi L_1}\right) - f^1(t,\pi) \right] + \lambda_{1,0}[f^0(t) - f^1(t,\pi)] + \lambda_{1,2}[f^2(t) - f^1(t,\pi)],
\end{aligned}$$

for $i \in \{0,2\}$ and where g_j is given by

$$g_j(\pi) = r_j + \alpha_j\pi - \frac{1}{2}\pi^2\sigma_j^2 + \lambda_j \ln(1-\pi L_j) + \sum_{j \neq k \in \{0,1,2\}} \lambda_{j,k} \ln(1-\pi L_{j,k}), \quad j \in \{0,1,2\}.$$

This leads to the following first-order conditions for the investor's optimal portfolio strategies π_i^* in states $i \in \{0,2\}$

$$\begin{aligned}
0 &= \alpha_i - \sigma_i^2\pi_i - \lambda_i \frac{L_i}{1-\pi_i L_i} - \lambda_{i,1}\frac{L_{i,1}}{1-\pi_i L_{i,1}} - \lambda_{i,2-i}\frac{L_{i,2-i}}{1-\pi_i L_{i,2-i}} \\
&\quad + \lambda_{i,1}\partial_\pi f^1\left(t, \frac{\pi_i(1-L_{i,1})}{1-\pi_i L_{i,1}}\right) \frac{1-L_{i,1}}{(1-\pi_i L_{i,1})^2}.
\end{aligned}$$

As before, π_0^* and π_2^* are deterministic functions of time which only depend on $\partial_\pi f^1$.

Proposition 2.26 (Indirect utility in illiquidity)**.** *Given* C^1 *functions* $f^0, f^2 : [0,T] \to \mathbb{R}$ *consider the function* $f^1 : [0,T] \times [0,1] \to \mathbb{R}$ *defined via the stochastic representation*

$$f^1(t,\pi) = \int_t^T \left(\lambda_{1,0} f^0(s) + \lambda_{1,2} f^2(s) + E\left[g_1(\tilde{\pi}_s^{t,\pi})\right]\right) e^{-(\lambda_{1,0}+\lambda_{1,2})(s-t)} ds$$
$$+ E\left[\ln(1-\tilde{\pi}_T^{t,\pi}\ell)\right] e^{-(\lambda_{1,0}+\lambda_{1,2})(T-t)},$$

where $\tilde{\pi}$ *is given as in Proposition 2.11.*

(i) Then f^1 *is of class* $\mathrm{C}^{1,2}$ *on* $[0,T] \times [0,1]$ *with*

$$\partial_\pi f^1(t,\pi) = \int_t^T E[\partial_\pi \tilde{\pi}_s^{t,\pi}\, g_1'(\tilde{\pi}_s^{t,\pi})] e^{-(\lambda_{1,0}+\lambda_{1,2})(s-t)} ds$$
$$- E\left[\partial_\pi \tilde{\pi}_T^{t,\pi} \frac{\ell}{1-\tilde{\pi}_T^{t,\pi}\ell}\right] e^{-(\lambda_{1,0}+\lambda_{1,2})(T-t)}.$$

(ii) f^1 *solves the Hamilton-Jacobi-Bellman equation for state* 1.

In particular, $\partial_\pi f^1$ *does not depend on* f^0 *or* f^2 *and thus, as before, the first-order conditions provide algebraic equations for the optimal stock proportions.*

Proof. Analogously to the proof of Proposition 2.11. □

This time, substituting f^1 defined as in the previous proposition, into the Hamilton-Jacobi-Bellman equations for states 0 and 2 yields a linear system of two second-order differential equations. As opposed to the setting with only two different states, by reduction to first-order, we now end up with a four-dimensional system of first-order ordinary differential equations. We are able to explicitly determine the roots of the corresponding characteristic polynomial of order four and thus may derive a representation of f^0 by applying the variation of constants method and Cramer's rule. The following proposition provides such an explicit representation of the investor's indirect utility function in the normal regime, if $\lambda_{0,1} = \lambda_{2,1}$. We think of state 1 as being triggered by a catastrophic event leading to a closure of the stock exchange, whereas state 2 corresponds to an economic crisis during which the investor can still trade. Thus, the assumption $\lambda_{0,1} = \lambda_{2,1}$ means that the occurrence of a catastrophic event does not depend on whether the economy is currently in crisis or not.

Proposition 2.27 (Indirect utility in normal regime)**.** *Suppose that there exist continuous functions* $\pi_i^* : [0,T] \to [0,1]$ *such that* $\pi_i^*(t)$ *maximizes the Hamilton-Jacobi-Bellman equation corresponding to state* i*, for all* $t \in [0,T]$ *and* $i \in \{0,2\}$*. Assume that* $\lambda_{0,1} = \lambda_{2,1}$

and let $\lambda_{j,\cdot} = \sum_{k\neq j} \lambda_{j,k}$ *denote the aggregate intensity of leaving state* j*. Consider the function* $f^0 : [0,T] \to \mathbb{R}$ *given by*

$$f^0(t) = e^{\lambda_{1,\cdot}t}\{c_1 \int_t^T e^{(\lambda_{0,2}+\lambda_{2,\cdot}-\lambda_{1,\cdot})(t-s)}(F_0(s) - F_2(s))ds$$
$$+ c_2 \int_t^T e^{\lambda_{2,1}(t-s)}[(\lambda_{1,0} - \lambda_{2,0})F_0(s) + (\lambda_{1,2} - \lambda_{0,2})F_2(s)]ds$$
$$+ c_3 \int_t^T e^{-\lambda_{1,\cdot}(t-s)}[(\lambda_{1,2}\lambda_{2,0} + \lambda_{1,0}\lambda_{2,\cdot})F_0(s) + (\lambda_{0,2}\lambda_{1,\cdot} + \lambda_{1,2}\lambda_{2,1})F_2(s)]ds\},$$

where c_j *for* $j \in \{1,2,3\}$ *are constants given by (51) (see below) and*

$$F_i(t) = g_i(\pi_i^*(t))e^{-\lambda_{1,\cdot}t} + \lambda_{i,1}\int_t^T E[g_1(\tilde{\pi}_s^{t,\hat{\pi}_i(t)})]e^{-\lambda_{1,\cdot}s}ds + \lambda_{i,1}E[\ln(1-\tilde{\pi}_T^{t,\hat{\pi}_i(t)}\ell)]e^{-\lambda_{1,\cdot}T}$$
$$\hat{\pi}_i(t) = \frac{\pi_i^*(t)(1-L_{i,1})}{1-\pi_i^*(t)L_{i,1}},$$

for $t \in [0,T]$ *and* $i \in \{0,2\}$*. Then* f^0 *is of class* C^1*, and* f^0 *solves the Hamilton-Jacobi-Bellman equation for state* 0*.*

Proof. Let f^1 be given as in Proposition 2.26. Then, for $i \in \{0,2\}$ we have

$$f^1(t,\hat{\pi}_i(t)) = \int_t^T \left(\lambda_{1,0}f^0(s) + \lambda_{1,2}f^2(s) + E\left[g_1(\tilde{\pi}_s^{t,\hat{\pi}_i(t)})\right]\right)e^{-\lambda_{1,\cdot}(s-t)}ds$$
$$+ E\left[\ln(1-\tilde{\pi}_T^{t,\hat{\pi}_i(t)}\ell)\right]e^{-\lambda_{1,\cdot}(T-t)}.$$

Thus, by the Hamilton-Jacobi-Bellman equations for state 0 and state 2

$$0 = \partial_t f^i(t) + g_i(\pi_i^*(t)) + \lambda_{i,1}[f^1(t,\hat{\pi}_i(t)) - f^i(t)] + \lambda_{i,2-i}[f^{2-i}(t) - f^i(t)],$$

we obtain the following integro-differential equations

$$0 = \partial_t f^i(t)e^{-\lambda_{1,\cdot}t} + \lambda_{i,1}\int_t^T (\lambda_{1,i}f^i(s) + \lambda_{1,2-i}f^{2-i}(s))e^{-\lambda_{1,\cdot}s}ds$$
$$+ \lambda_{i,2-i}f^{2-i}(t)e^{-\lambda_{1,\cdot}t} - \lambda_{i,\cdot}f^i(t)e^{-\lambda_{1,\cdot}t} + F_i(t),$$

for $i \in \{0, 2\}$. Substituting

$$H_i(t) = \int_t^T f^i(s) e^{-\lambda_{1,\cdot} s} ds,$$

where $i \in \{0, 2\}$ into these equations, we obtain a linear inhomogeneous system of two second-order constant-coefficient differential equations

$$\begin{aligned} H_0''(t) + (\lambda_{1,\cdot} - \lambda_{0,\cdot})H_0'(t) - \lambda_{0,1}\lambda_{1,0}H_0(t) - \lambda_{0,1}\lambda_{1,2}H_2(t) + \lambda_{0,2}H_2'(t) &= F_0(t) \\ H_2''(t) + (\lambda_{1,\cdot} - \lambda_{2,\cdot})H_2'(t) - \lambda_{2,1}\lambda_{1,2}H_2(t) - \lambda_{2,1}\lambda_{1,0}H_0(t) + \lambda_{2,0}H_0'(t) &= F_2(t), \end{aligned} \tag{48}$$

with terminal conditions $H_0(T) = 0$, $H_0'(T) = 0$, $H_2(T) = 0$, $H_2'(T) = 0$. By definition of H_0, the function f^0 is given by

$$f^0(t) = -H_0'(t)e^{\lambda_{1,\cdot} t}.$$

Substituting $(u_0, u_1) = (H_0, H_0')$ and $(v_0, v_1) = (H_2, H_2')$, the system (48) can be transformed into the following system of first-order ordinary differential equations

$$\begin{pmatrix} u_0' \\ u_1' \\ v_0' \\ v_1' \end{pmatrix} = A \begin{pmatrix} u_0 \\ u_1 \\ v_0 \\ v_1 \end{pmatrix} + \begin{pmatrix} 0 \\ F_0 \\ 0 \\ F_2 \end{pmatrix}, \tag{49}$$

where A is given by

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ \lambda_{0,1}\lambda_{1,0} & \lambda_{0,\cdot} - \lambda_{1,\cdot} & \lambda_{0,1}\lambda_{1,2} & -\lambda_{0,2} \\ 0 & 0 & 0 & 1 \\ \lambda_{2,1}\lambda_{1,0} & -\lambda_{2,0} & \lambda_{2,1}\lambda_{1,2} & \lambda_{2,\cdot} - \lambda_{1,\cdot} \end{pmatrix}.$$

The characteristic polynomial of the corresponding homogenous system reads

$$\chi_A(\mu) = \mu(\mu + \lambda_{1,\cdot})(\mu^2 + p\mu + q),$$

where

$$p = \lambda_{1,\cdot} - \lambda_{0,\cdot} - \lambda_{2,\cdot}$$

and

$$q = \lambda_{0,1}(\lambda_{2,\cdot} - \lambda_{1,0}) + \lambda_{2,1}(\lambda_{0,2} - \lambda_{1,2}).$$

Thus, the eigenvalues of A are explicitly given by $\mu_1 = 0$, $\mu_2 = -\lambda_{1,\cdot}$ and $\mu_{3,4} = -p/2 \pm \sqrt{(p^2/4 - q)}$. Note that in case of $\lambda_{0,1} = \lambda_{2,1}$, we obtain $\mu_3 = \lambda_{2,1}$ as well as $\mu_4 = \lambda_{0,\cdot} - \lambda_{1,\cdot} + \lambda_{2,0}$. Due to the terminal constraint $(u_0, u_1, v_0, v_1)^t(T) = 0$, the general solution to the homogeneous system vanishes. Thus, the general solution to (49) is given by a particular solution and therefore, by the variation of constants method, we have

$$(u_0, u_1, v_0, v_1)^t(t) = \sum_{j=1}^{4} w_j(t) y_j(t),$$

where $(y_j)_{1\leq j\leq 4}$ is a fundamental system of the corresponding homogenous system, i.e. $y_1,\ldots,y_4$ are linearly independent and satisfy $y_j' = Ay_j$ for each $j \in \{1,\ldots,4\}$, and $(w_j)_{1\leq j\leq 4}$ are such that

$$\sum_{j=1}^{4} w_j'(t)y_j(t) = (0, F_0, 0, F_2)^t(t).$$

By Cramer's rule we obtain

$$(u_0, u_1, v_0, v_1)^t(t) = \sum_{j=1}^{4} y_j(t) \int_t^T \frac{-D_j(s)}{\det(y_1, y_2, y_3, y_4)(s)} ds, \tag{50}$$

where D_j denotes the determinant of the matrix (y_1, y_2, y_3, y_4) where the j-th column is replaced by $(0, F_0, 0, F_2)^t$, for each $j \in \{1,\ldots,4\}$. Note that in case of four pairwise different real eigenvalues, the exponential ansatz yields that a fundamental system is given by $y_j = \nu_j e^{\mu_j \cdot}$, where ν_j denotes the eigenvector corresponding to μ_j. Eventually, rewriting (50) for $\lambda_{0,1} = \lambda_{2,1}$ we get

$$\begin{aligned}
H_0'(t) = u_1(t) = -c_1 &\int_t^T e^{(\lambda_{0,2}+\lambda_{2,\cdot}-\lambda_{1,\cdot})(t-s)}(F_0(s) - F_2(s))ds \\
&- c_2 \int_t^T e^{\lambda_{2,1}(t-s)}[(\lambda_{1,0} - \lambda_{2,0})F_0(s) + (\lambda_{1,2} - \lambda_{0,2})F_2(s)]ds \\
&- c_3 \int_t^T e^{-\lambda_{1,\cdot}(t-s)}[(\lambda_{1,2}\lambda_{2,0} + \lambda_{1,0}\lambda_{2,\cdot})F_0(s) + (\lambda_{0,2}\lambda_{1,\cdot} + \lambda_{1,2}\lambda_{2,1})F_2(s)]ds,
\end{aligned}$$

where

$$\begin{aligned}
c_1 &= \frac{(\lambda_{0,2}^2 - \lambda_{1,2}\lambda_{2,1} + \lambda_{0,2}(\lambda_{2,\cdot} - \lambda_{1,\cdot}))(\lambda_{0,2}\lambda_{2,0} - \lambda_{1,0}\lambda_{2,\cdot} + \lambda_{2,0}(\lambda_{2,\cdot} - \lambda_{1,2}))}{(\lambda_{0,2} - \lambda_{1,\cdot} + \lambda_{2,0})(\lambda_{0,2} + \lambda_{2,\cdot})(\lambda_{0,2}\lambda_{2,0} - \lambda_{1,0}\lambda_{2,\cdot} + \lambda_{2,0}(\lambda_{2,\cdot} - \lambda_{1,2}))} \qquad (51)\\
c_2 &= \frac{\lambda_{2,1}}{(\lambda_{1,\cdot} - \lambda_{0,2} - \lambda_{2,0})(\lambda_{1,\cdot} + \lambda_{2,1})} \\
c_3 &= \frac{1}{(\lambda_{1,\cdot} + \lambda_{2,1})(\lambda_{0,2} + \lambda_{2,\cdot})}.
\end{aligned}$$

□

Note that (50) also provides a representation for f^0 if $\lambda_{0,1} \neq \lambda_{2,1}$. The function f^2 possesses a similar representation as f^0.

Given the assumptions of the previous proposition, set $\lambda_{0,2} = \lambda_{1,2} = 0$. Then we have $F_0 = F$, with F given as in Proposition 2.15. Further, the constants $(c_i)_{1\leq i\leq 3}$ simplify to

$$\begin{aligned} c_1 &= 0 \\ c_2 &= \frac{\lambda_{0,1}}{(\lambda_{1,0}-\lambda_{2,0})(\lambda_{1,0}+\lambda_{0,1})} \\ c_3 &= \frac{1}{(\lambda_{1,0}+\lambda_{0,1})\lambda_{2,\cdot}} \end{aligned}$$

and therefore

$$\begin{aligned} f^0(t) &= e^{\lambda_{1,0}t}\left\{\frac{\lambda_{0,1}}{\lambda_{1,0}+\lambda_{0,1}}\int_t^T e^{\lambda_{0,1}(t-s)}F(s)ds + \frac{\lambda_{1,0}}{\lambda_{1,0}+\lambda_{0,1}}e^{-\lambda_{1,0}(t-s)}F(s)ds\right\} \\ &= \frac{\lambda_{0,1}}{\lambda_{0,1}+\lambda_{1,0}}e^{(\lambda_{0,1}+\lambda_{1,0})t}\int_t^T F(s)e^{-\lambda_{0,1}s}ds + \frac{\lambda_{1,0}}{\lambda_{0,1}+\lambda_{1,0}}\int_t^T F(s)e^{\lambda_{1,0}s}ds. \end{aligned}$$

Thus, if the third state cannot be reached, i.e. $\lambda_{0,2} = \lambda_{1,2} = 0$, then the value function coincides with the one when there are only two possible regimes, given in Proposition 2.15.

In Section 2.6 we wish to determine the initial capital which an investor who cannot trade in state 1 would be willing to give up if he were able to trade in all states. Therefore, we also derive the value function when trading is allowed in all states. A verification result analogous to Theorem 2.9 holds true and the corresponding HJB equations are given by

$$0 = \sup_{\pi_j\in[0,1]}\{\partial_t f^j(t) + g_j(\pi_j) + \sum_{j\neq k\in\{0,1,2\}}\lambda_{j,k}[f^k(t)-f^j(t)]\} \tag{52}$$

subject to the terminal constraints $f^j(T) = 0$, and where g_j is given as before, for $j \in \{0,1,2\}$. This leads to the following first-order conditions

$$0 = \alpha_j - \pi_j\sigma_j^2 - \lambda_j\frac{L_j}{1-\pi_jL_j} - \sum_{j\neq k\in\{0,1,2\}}\lambda_{j,k}\frac{L_{j,k}}{1-\pi_jL_{j,k}} \tag{53}$$

for the optimal stock proportions in states $j \in \{0,1,2\}$.

Proposition 2.28 (Optimal solution without illiquidity)**.** *Assume that* $\lambda_{0,1} = \lambda_{2,1}$ *and let* $\lambda_{j,\cdot} = \sum_{k\neq j}\lambda_{j,k}$ *for* $j \in \{0,1,2\}$*. Suppose that* $\pi_j^* \in [0,1]$ *solves the first-order condition (53)* *for $j \in \{0,1,2\}$ and let $f^0 :\ [0,T] \to \mathbb{R}$ be given by*

$$
\begin{aligned}
f^0(t) = e^{\lambda_{1,\cdot}t}\{c_1 &\int_t^T e^{(\lambda_{0,2}+\lambda_{2,\cdot}-\lambda_{1,\cdot})(t-s)}(F_0(s) - F_2(s))ds \\
&+ c_2 \int_t^T e^{\lambda_{2,1}(t-s)}[(\lambda_{1,0} - \lambda_{2,0})F_0(s) + (\lambda_{1,2} - \lambda_{0,2})F_2(s)]ds \\
&+ c_3 \int_t^T e^{-\lambda_{1,\cdot}(t-s)}[(\lambda_{1,2}\lambda_{2,0} + \lambda_{1,0}\lambda_{2,\cdot})F_0(s) + (\lambda_{0,2}\lambda_{1,\cdot} + \lambda_{1,2}\lambda_{2,1})F_2(s)]ds\},
\end{aligned}
$$

where c_j for $j \in \{1,2,3\}$ are given by (51) and

$$
F_i(t) = g_i(\pi_i^*)e^{-\lambda_{1,\cdot}t} + \frac{\lambda_{i,1}}{\lambda_{1,\cdot}} g_1(\pi_1^*)(e^{-\lambda_{1,\cdot}t} - e^{-\lambda_{1,\cdot}T}),
$$

for $t \in [0,T]$ and $i \in \{0,2\}$. Then, the value function when trading is allowed in each state, is given by

$$
V(t_0, x_0) = \ln(x_0) + f^0(t_0)
$$

for $t_0 \in [0,T]$ and $x_0 \in (0,\infty)$.

Proof. A solution to the HJB equation (52) for state 1 is given by

$$
f^1(t) = \lambda_{1,0} \int_t^T f^0(s)e^{-\lambda_{1,\cdot}(s-t)}ds + \lambda_{1,2} \int_t^T f^2(s)e^{-\lambda_{1,\cdot}(s-t)}ds + g_1(\pi_1^*)\frac{1}{\lambda_{1,\cdot}}(1 - e^{-\lambda_{1,\cdot}(T-t)}).
$$

Setting the above representation of f^1 into the HJB equations (52) for states $i \in \{0,2\}$

$$
0 = \partial_t f^i(t) + g_i(\pi_i^*) + \lambda_{i,1}[f^1(t) - f^i(t)] + \lambda_{i,2-i}[f^{2-i}(t) - f^i(t)],
$$

we obtain

$$
\begin{aligned}
0 = \partial_t f^i(t)e^{-\lambda_{1,\cdot}t} &+ \lambda_{i,1}\lambda_{1,0} \int_t^T f^0(s)e^{-\lambda_{1,\cdot}s}ds + \lambda_{i,1}\lambda_{1,2} \int_t^T f^2(s)e^{-\lambda_{1,\cdot}s}ds \\
&- \lambda_{i,1}f^i(t)e^{-\lambda_{1,\cdot}t} + \lambda_{i,2-i}f^{2-i}(t)e^{-\lambda_{1,\cdot}t} - \lambda_{i,2-i}f^i(t)e^{-\lambda_{1,\cdot}t} + F_i(t).
\end{aligned}
$$

Substituting

$$
H_i(t) = \int_t^T f^i(s)e^{-\lambda_{1,\cdot}s}ds
$$

into the previous equation, we get

$$H_i''(t) + (\lambda_{1,\cdot} - \lambda_{i,\cdot})H_i'(t) - \lambda_{i,1}\lambda_{1,i}H_i(t) - \lambda_{i,1}\lambda_{1,2-i}H_{2-i}(t) + \lambda_{i,2-i}H_{2-i}'(t) = F_i(t)$$

for $i \in \{0, 2\}$. This system of two second-order differential equations coincides with (48). Thus, as in the proof of Proposition 2.27 we obtain

$$\begin{aligned} f^0(t) = e^{\lambda_{1,\cdot}t}\{c_1 \int_t^T & e^{(\lambda_{0,2}+\lambda_{2,\cdot}-\lambda_{1,\cdot})(t-s)}(F_0(s) - F_2(s))ds \\ + c_2 \int_t^T & e^{\lambda_{2,1}(t-s)}[(\lambda_{1,0} - \lambda_{2,0})F_0(s) + (\lambda_{1,2} - \lambda_{0,2})F_2(s)]ds \\ + c_3 \int_t^T & e^{-\lambda_{1,\cdot}(t-s)}[(\lambda_{1,2}\lambda_{2,0} + \lambda_{1,0}\lambda_{2,\cdot})F_0(s) + (\lambda_{0,2}\lambda_{1,\cdot} + \lambda_{1,2}\lambda_{2,1})F_2(s)]ds\}. \end{aligned}$$

□

2.5 Power utility

In this section, we study the investor's portfolio problem (31) when only finitely many regime shifts between state 0 and state 1 are possible and where $U(x) = \frac{x^\gamma}{\gamma}$ with $\gamma \neq 0$. This problem can be solved iteratively. In addition to Theorem 2.6 and Corollary 2.7 we prove that the optimal strategies when only finitely many liquidity breakdowns can occur, converge pointwise to the optimal strategy with possibly infinitely many regime shifts. Throughout this section we assume that $L_0 = L_1 = L_{0,1} = 0$ and that $\sigma_1 = 0$. Note that $J^{0,0}$ is given by

$$J^{0,0}(t,x) = \frac{x^\gamma}{\gamma} f^{0,0}(t) = \frac{x^\gamma}{\gamma} e^{\gamma(r_0 + \frac{1}{2}\frac{\alpha_0^2}{(1-\gamma)\sigma_0^2})(T-t)}$$

and the optimal stock proportion is given by

$$\pi^* = \frac{\alpha_0}{(1-\gamma)\sigma_0^2}.$$

For $1 \leq k_0 \in \mathbb{N}$ we conjecture

$$J^{0,k_0}(t,x) = \frac{x^\gamma}{\gamma} f^{0,k_0}(t)$$

as well as

$$J^{1,k_0}(t,x,\pi) = \frac{x^\gamma}{\gamma} f^{1,k_0}(t,\pi),$$

for a C^1-function f^{0,k_0} on $[0,T]$ with $f^{0,k_0}(T)=1$ and a $\mathrm{C}^{1,2}$-function f^{1,k_0} on $[0,T]\times[0,1]$ satisfying $f^{1,k_0}(T,\pi)=(1-\pi\ell)^\gamma$ for all $\pi\in[0,1]$. The corresponding HJB equations read

$$\begin{aligned}0&=\sup_{\pi\in[0,1]}\left\{\frac{1}{\gamma}\Big(\partial_t f^{0,k_0}(t)-d_0(\pi)f^{0,k_0}(t)+\lambda_{0,1}f^{1,k_0}(t,\pi)\Big)\right\}\\0&=\partial_t f^{1,k_0}(t,\pi)-d_1(\pi)f^{1,k_0}(t,\pi)+\pi(1-\pi)\alpha_1\partial_\pi f^{1,k_0}(t,\pi)+\lambda_{1,0}(1-\pi L_{1,0})^\gamma f^{0,k_0-1}(t),\end{aligned}$$

where $d_0(\pi)=\lambda_{0,1}-\gamma(r_0+\pi\alpha_0)+\frac{1}{2}\gamma(1-\gamma)\pi^2\sigma_0^2$ and $d_1(\pi)=\lambda_{1,0}-\gamma(r_1+\alpha_1\pi)$ for $\pi\in[0,1]$. The Hamilton-Jacobi-Bellman equation for state 0 leads to the following first-order condition for the optimal stock proportion in state 0

$$0=\gamma\alpha_0 f^{0,k_0}(t)-\gamma(1-\gamma)\pi\sigma_0^2 f^{0,k_0}(t)+\lambda_{0,1}\partial_\pi f^{1,k_0}(t,\pi). \tag{54}$$

As before, the solution to the first-order condition is a deterministic function of time given such a solution exists.

Proposition 2.29 (Indirect utility in illiquidity)**.** *Let $1\le k_0\in\mathbb{N}$ and let $f^{0,k_0-1}:[0,T]\to\mathbb{R}$ be a given function which is of class C^1 on $[0,T]$. Consider the function $f^{1,k_0}:[0,T]\times[0,1]\to\mathbb{R}$ defined via*

$$\begin{aligned}f^{1,k_0}(t,\pi)=\lambda_{1,0}\int_t^T e^{(\gamma r_1-\lambda_{1,0})(s-t)}[1+\pi(e^{\alpha_1(s-t)}(1-L_{1,0})-1)]^\gamma f^{0,k_0-1}(s)ds\\+e^{(\gamma r_1-\lambda_{1,0})(T-t)}[1+\pi(e^{\alpha_1(T-t)}(1-\ell)-1)]^\gamma.\end{aligned}$$

Then f^{1,k_0} is of class $\mathrm{C}^{1,2}$ on $[0,T]\times[0,1]$ and f^{1,k_0} solves the HJB equation for state 1.

Proof. Since f^{0,k_0-1} is continuously differentiable, it follows that f^{1,k_0} is of class $C^{1,2}$. The second assertion follows by differentiation with respect to t and π. □

Proposition 2.30 (Indirect utility in liquidity)**.** *Let $1\le k_0\in\mathbb{N}$ and suppose that there exists a continuous function $\pi^*_{k_0}:\ [0,T]\to[0,1]$ such that $\pi^*_{k_0}(t)\in\arg\max_{\pi\in[0,1]}H^{0,k_0}(t,\pi)$ for all $t\in[0,T]$. Given a $C^{1,2}$-function $f^{1,k_0}:[0,T]\times[0,1]\to\mathbb{R}$, consider the function $f^{0,k_0}:[0,T]\to\mathbb{R}$ defined via*

$$f^{0,k_0}(t)=\lambda_{0,1}\int_t^T e^{-\int_t^v d_0(\pi^*_{k_0}(u))du}f^{1,k_0}(v,\pi^*_{k_0}(v))dv+e^{-\int_t^T d_0(\pi^*_{k_0}(u))du}.$$

Then f^{0,k_0} is of class C^1 on $[0,T]$, and f^{0,k_0} solves the HJB equation for state 0.

Proof. Analogously to the proof of Proposition 2.20. □

Collecting the above results, by the Verification Theorem 2.9, we obtain

Theorem 2.31 (Solution of the portfolio problem)**.** *Let $k_0 \in \mathbb{N}$ and assume that for each $k \leq k_0$ there exists a continuous function $\pi_k^* : [0,T] \to [0,1]$ such that $\pi_k^*(t) \in \arg\max_{\pi\in[0,1]} H^{0,k}(t,\pi)$ for all $t \in [0,T]$. Then the value function is given by*

$$V(t_0, x_0, k_0) = \frac{x_0^\gamma}{\gamma} f^{0,k_0}(t_0)$$

for $t_0 \in [0,T]$, $x_0 \in (0,\infty)$ and the optimal portfolio strategy is given by $\pi^*_{k_0-K_-}$.

Now, we derive a convergence result for the optimal strategies. We set $J^0 = J^{0,\infty}$, $J^1 = J^{1,\infty}$, $H^0 = H^{0,\infty}$, $H^1 = H^{1,\infty}$ and conjecture

$$J^0(t,x) = \frac{x^\gamma}{\gamma} f^0(t)$$

and

$$J^1(t,x,\pi) = \frac{x^\gamma}{\gamma} f^1(t,\pi)$$

for a C^1-function f^0 on $[0,T]$ with $f^0(T) = 1$ and a $C^{1,2}$-function f^1 on $[0,T]\times[0,1]$ such that $f^1(T,\pi) = (1-\pi\ell)^\gamma$ for all $\pi \in [0,1]$. Then, the corresponding Hamilton-Jacobi-Bellman equations read

$$0 = \sup_{\pi\in[0,1]} \left\{ \frac{1}{\gamma}\Big(\partial_t f^0(t) - d_0(\pi) f^0(t) + \lambda_{0,1} f^1(t,\pi)\Big) \right\} \tag{55}$$

$$0 = \partial_t f^1(t,\pi) - d_1(\pi) f^1(t,\pi) + \pi(1-\pi)\alpha_1 \partial_\pi f^1(t,\pi) + \lambda_{1,0}(1-\pi L_{1,0})^\gamma f^0(t), \tag{56}$$

where d_0 and d_1 are as before. Equation (55) yields the following first-order condition for the optimal stock proportion in state 0

$$0 = \gamma\alpha_0 f^0(t) - \gamma(1-\gamma)\pi\sigma_0^2 f^0(t) + \lambda_{0,1}\partial_\pi f^1(t,\pi). \tag{57}$$

Proposition 2.32 (Indirect utility in illiquidity)**.** *Let $f^0 : [0,T] \to \mathbb{R}$ be a given function which is of class C^1 on $[0,T]$. Consider the function $f^1 : [0,T]\times[0,1] \to \mathbb{R}$ defined via*

$$\begin{aligned} f^1(t,\pi) = \lambda_{1,0} \int_t^T & e^{(\gamma r_1 - \lambda_{1,0})(s-t)}[1 + \pi(e^{\alpha_1(s-t)}(1-L_{1,0}) - 1)]^\gamma f^0(s) ds \\ &+ e^{(\gamma r_1 - \lambda_{1,0})(T-t)}[1 + \pi(e^{\alpha_1(T-t)}(1-l) - 1)]^\gamma. \end{aligned}$$

Then f^1 is of class $\mathrm{C}^{1,2}$ on $[0,T]\times[0,1]$ and f^1 is a solution to the equation (56).

Proof. Analogously to the proof of Proposition 2.29. □

The following proposition states that $\partial_\pi f^{1,k_0}$ converges uniformly towards $\partial_\pi f^1$ on $[0,T] \times [0,1]$, if the maximal number of liquidity breakdowns goes to infinity.

Proposition 2.33. *Suppose that for each $k_0 \in \mathbb{N} \cup \{\infty\}$ there exists a continuous function $\pi^*_{k_0} : [0,T] \to [0,1]$ such that $\pi^*_{k_0}(t) \in \arg\max_{\pi\in[0,1]} H^{0,k_0}(t,\pi)$ for all $t \in [0,T]$. For each $k_0 \in \mathbb{N}$, let f^{1,k_0} be defined as in Proposition 2.29 and suppose that there exists a C^1-function f^0 on $[0,T]$ which satisfies (55), with f^1 as in Proposition 2.32. Then, we have*

$$\sup_{(t,\pi)\in[0,T]\times[0,1]} |\partial_\pi f^{1,k_0}(t,\pi) - \partial_\pi f^1(t,\pi)| \to 0 \quad \textit{as } k_0 \to \infty.$$

Proof. Let $1 \le k_0 \in \mathbb{N}$. Since we may interchange differentiating and integrating in the representations of $\partial_\pi f^{1,k_0}$ and $\partial_\pi f^1$, we have

$$\begin{aligned}
&\sup_{(t,\pi)\in[0,T]\times[0,1]} |\partial_\pi f^{1,k_0}(t,\pi) - \partial_\pi f^1(t,\pi)| \\
&= \lambda_{1,0} \sup_{(t,\pi)\in[0,T]\times[0,1]} \Big| \gamma \int_t^T e^{(\gamma r_1-\lambda_{1,0})(s-t)}[1+\pi(e^{\alpha_1(s-t)}(1-L_{1,0})-1)]^{\gamma-1} \\
&\qquad\qquad (e^{\alpha_1(s-t)}(1-L_{1,0})-1)(f^{0,k_0-1}(s)-f^0(s))ds\,\Big| \\
&\le \lambda_{1,0}\, T \sup_{s\in[0,T]} |f^{0,k_0-1}(s)-f^0(s)| \\
&\qquad \sup_{\pi\in[0,1],\, s,t\in[0,T]} \big| \gamma e^{(\gamma r_1-\lambda_{1,0})(s-t)}[1+\pi(e^{\alpha_1(s-t)}(1-L_{1,0})-1)]^{\gamma-1}(e^{\alpha_1(s-t)}(1-L_{1,0})-1)\,\big|.
\end{aligned}$$

The second supremum is finite, since by assumption we have $L_{1,0} < 1$. By Theorem 2.9 we have $V(t_0,x_0,\infty) = \frac{x_0^\gamma}{\gamma} f^0(t_0)$ for all $t_0 \in [0,T]$ and $x_0 \in (0,\infty)$. Thus, the assertion follows from Theorem 2.31 and Theorem 2.6. □

Remark 2.34. *Similar as in Proposition 2.13, the optimal strategies are uniquely determined by the solutions to the first-order conditions if, for instance, $\alpha_1 \le 0$.*

Thus, by comparing the first-order conditions for $k_0 \in \mathbb{N}$ and $k_0 = \infty$ we obtain pointwise convergence of the optimal strategies.

Corollary 2.35 (Convergence of the optimal strategies)**.** *Suppose that $\alpha_1 \le 0$ and that there exists a C^1-function f^0 on $[0,T]$ such that (55) holds true, where f^1 is as in Proposition 2.32. Assume that the first-order conditions (54) and (57) admit solutions $\pi^*_{k_0}(t) \in [0,1]$ for each $t \in [0,T]$ and $1 \le k_0 \in \mathbb{N} \cup \{\infty\}$. Then, for each $t \in [0,T]$ we have*

$$\pi^*_{k_0}(t) \to \pi^*_\infty(t) \quad \textit{as } k_0 \to \infty.$$

Proof. Let $t \in [0,T]$ and let $(\pi^*_{k_l}(t))_{l\ge1}$ be a subsequence of $(\pi^*_k(t))_{k\ge1}$. Since $\pi^*_k(t) \in [0,1]$ for each $k \ge 1$, there exists a subsequence $(\pi^*_{k_{l_m}}(t))_{m\ge1}$ which converges. Theorem 2.9 implies that $V(t_0,x_0,\infty) = \frac{x_0^\gamma}{\gamma} f^0(t_0)$ for all $t_0 \in [0,T]$ and $x_0 \in (0,\infty)$. Thus, by Theorem 2.31, Theorem 2.6 and Proposition 2.33 we obtain

$$\begin{aligned}
0 &= \lim_{m\to\infty} [\gamma\alpha_0 f^{0,k_{l_m}}(t) - \gamma(1-\gamma)\pi^*_{k_{l_m}}(t)\sigma_0^2 f^{0,k_{l_m}}(t) + \lambda_{0,1}\partial_\pi f^{1,k_{l_m}}(t,\pi^*_{k_{l_m}}(t))] \\
&= \gamma\alpha_0 f^0(t) - \gamma(1-\gamma)\lim_{m\to\infty}\pi^*_{k_{l_m}}(t)\sigma_0^2 f^0(t) + \lambda_{0,1}\partial_\pi f^1(t, \lim_{m\to\infty}\pi^*_{k_{l_m}}(t)).
\end{aligned}$$

The solution to the previous equation is uniquely given by $\pi^*_\infty(t)$. Thus, we have shown that each subsequence of $(\pi^*_k(t))_{k\in\mathbb{N}}$ has another subsequence which converges towards $\pi^*_\infty(t)$. $\square$

Again, since we wish to quantify the investor's loss of utility due to the presence of illiquidity, we briefly discuss the portfolio problem when trading is allowed in both states. The corresponding HJB equations are given by

$$\begin{aligned}
0 = \sup_{\pi_0\in[0,1]} &\Big\{\frac{1}{\gamma}\Big(\partial_t f^{0,k_0}(t) + \Big[\gamma(r_0+\pi_0\alpha_0) + \frac{1}{2}\gamma(\gamma-1)\pi_0^2\sigma_0^2\Big] f^{0,k_0}(t) \\
&\quad + \lambda_{0,1}[f^{1,k_0}(t) - f^{0,k_0}(t)]\Big)\Big\} \\
0 = \sup_{\pi_1\in[0,1]} &\Big\{\frac{1}{\gamma}\Big(\partial_t f^{1,k_0}(t) + \gamma(r_1+\pi_1\alpha_1) f^{1,k_0}(t) \\
&\quad + \lambda_{1,0}[(1-\pi_1 L_{1,0})^\gamma f^{0,k_0-1}(t) - f^{1,k_0}(t)]\Big)\Big\},
\end{aligned}$$

with terminal conditions $f^{i,k_0}(T) = 1$, with $i = 0, 1$.

Proposition 2.36 (Optimal solution without illiquidity)**.** *Assume that $\alpha_1 \leq 0$, let $1 \leq k_0 \in \mathbb{N}$ and for $t \in [0,T]$ let*

$$f^{0,k_0}(t) = \lambda_{0,1}\int_t^T f^{1,k_0}(s) e^{(\gamma[r_0+\frac{1}{2}\frac{\alpha_0^2}{(1-\gamma)\sigma_0^2}]-\lambda_{0,1})(s-t)} ds + e^{(\gamma[r_0+\frac{1}{2}\frac{\alpha_0^2}{(1-\gamma)\sigma_0^2}]-\lambda_{0,1})(T-t)},$$

where

$$f^{1,k_0}(t) = \lambda_{1,0}\int_t^T e^{(\gamma r_1-\lambda_{1,0})(s-t)} f^{0,k_0-1}(s) ds + e^{(\gamma r_1-\lambda_{1,0})(T-t)}.$$

Then, the value function when trading is allowed in both states is given by

$$V(t_0, x_0, k_0) = \frac{x_0^\gamma}{\gamma} f^{0,k_0}(t_0)$$

for $t_0 \in [0,T]$ and $x_0 \in (0,\infty)$.

Proof. Since we have $\sigma_1 = 0$ and $\alpha_1 \leq 0$, the optimal stock demand $\pi_1^* \in [0,1]$ for state 1 is given by $\pi_1^* = 0$. Thus, a solution to the HJB equation for state 1 is given by

$$f^{1,k_0}(t) = \lambda_{1,0}\int_t^T e^{(\gamma r_1-\lambda_{1,0})(s-t)} f^{0,k_0-1}(s) ds + e^{(\gamma r_1-\lambda_{1,0})(T-t)}.$$

The solution to the first-order condition corresponding to the HJB equation for state 0 is given by

$$\pi_0^* = \frac{\alpha_0}{(1-\gamma)\sigma_0^2}$$

and therefore,

$$f^{0,k_0}(t) = \lambda_{0,1} \int_t^T f^{1,k_0}(s) e^{(\gamma[r_0 + \frac{\alpha_0^2}{2(1-\gamma)\sigma_0^2}] - \lambda_{0,1})(s-t)} ds + e^{(\gamma[r_0 + \frac{1}{2}\frac{\alpha_0^2}{(1-\gamma)\sigma_0^2}] - \lambda_{0,1})(T-t)}$$

solves the HJB equation for state 0. □

2.6 Numerical illustrations

Firstly, we wish to illustrate the convergence of the value functions and strategies in the markets with finitely many liquidity breakdowns to the corresponding objects in the market in which infinitely many regime shifts are possible. We choose $\lambda_{0,1} = 0.2$, i.e. we consider a situation where on average a liquidity breakdown occurs every five years. Furthermore, we assume that the average duration of a liquidity breakdown is one month, i.e. $\lambda_{1,0} = 12$, and that $r_0 = r_1 = 0.03$, $\alpha_0 = 0.08$, $\alpha_1 = -0.03$, $\sigma_0 = 0.25$, and $L_{1,0} = \ell = 0.3$. The other parameters are assumed to be zero. This example is similar to the situation at 9/11, where the NYSE was closed for four days after the terrorist attacks and reopened with a loss of 10%. However, in order to get more pronounced effects, we use higher loss rates and a longer average duration of the liquidity breakdowns. The investor is assumed to have a power utility function with $\gamma = -3$. The percentage of initial wealth which an investor who cannot trade in state 1 would be willing to give up if he were able to trade in both states will be denoted by Δx. Figure 21 depicts Δx in dependance of the maximal number of liquidity breakdowns k_0. It can be seen that this percentage converges if the number of possible breakdowns increases. Figure 22 depicts the convergence of the strategies and the non-wealth dependent parts of the value functions, f^{0,k_0}. As can be seen from this figure, the value functions converge extremely fast. Since γ is negative, these functions are increasing with respect to k_0. The lowest line corresponds to $f^{0,0}$, the second lowest line to $f^{0,1}$ and so on. The portfolio strategies converge to an almost straight line that intersects the y-axis around 0.061. The upper line corresponds to the optimal strategy if at most one liquidity breakdown can occur, the second upper line to the optimal strategy if at most two breakdowns can occur, and so on. The investor's optimal stock demand for $k_0 = 0$ is given by $\frac{\alpha_0}{(1-\gamma)\sigma_0^2} = 0.32$. These figures illustrate the results of Theorem 2.6 and Corollary 2.35.

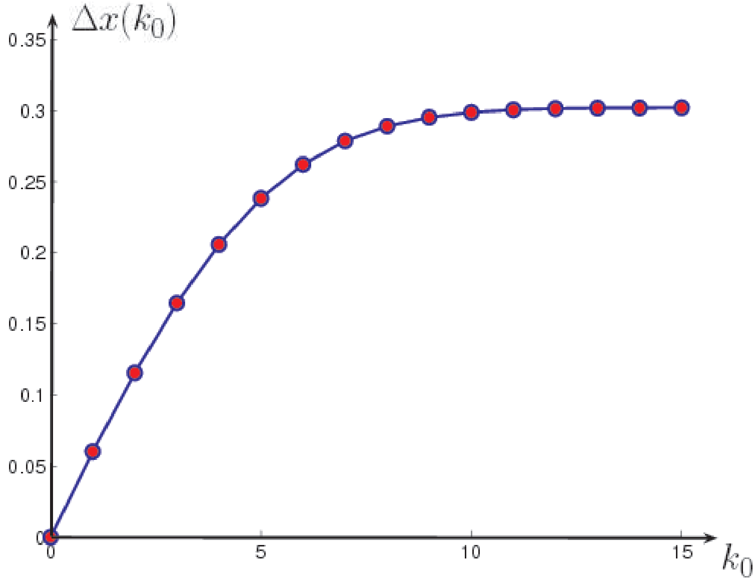

Figure 21: Percentages by which the initial capital can be reduced to get the same utility as in models where trading is allowed in both states.

$\lambda_{0,1}$	$\lambda_{1,0}$	$L_{1,0}$	ℓ	T	$\pi^*(0)(\%)$	Δx (%)
0.01	0.3	0.5	0.5	10	66.34 (16.69)	4.72 (1.20)
0.01	0.3	0.5	0.5	30	66.26 (16.66)	13.64 (3.36)
0.01	0.3	0.5	0.5	50	66.26 (16.66)	21.74 (5.99)
0.01	0.3	0.5	0	30	66.26 (16.66)	12.48 (3.30)
0.01	0.3	0.9	0.9	10	52.29 (13.66)	8.27 (2.08)
0.01	0.3	0.9	0.9	30	52.28 (13.65)	22.71 (6.11)
0.01	0.3	0.9	0.9	50	52.28 (13.65)	34.88 (9.97)
0.01	1	0.5	0.5	10	67.39 (17.00)	4.59 (1.16)
0.01	1	0.5	0.5	30	67.39 (17.00)	13.18 (3.42)
0.01	1	0.5	0.5	50	67.39 (17.00)	21.43 (5.66)
0.02	0.3	0.5	0.5	30	54.98 (13.84)	22.55 (6.20)
0.02	0.3	0.9	0.9	30	36.47 (9.44)	32.67 (9.34)

Table 1: Percentaged change of initial capital Δx and time-0 optimal portfolio demands $\pi^*(0)$. The values in brackets correspond to $U(x) = \frac{x^{-3}}{-3}$.

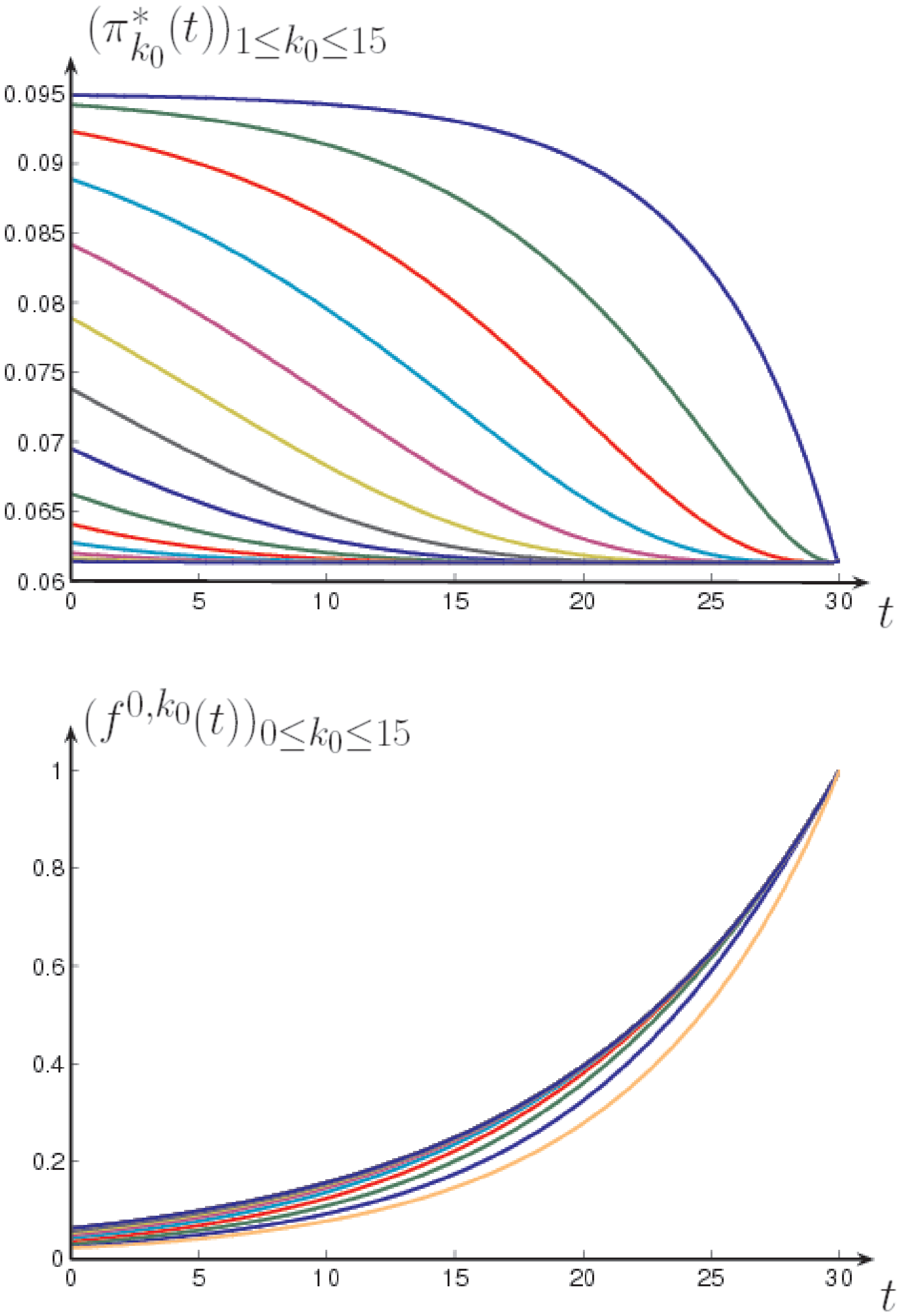

Figure 22: Optimal portfolios and non-wealth dependent part of the value functions.

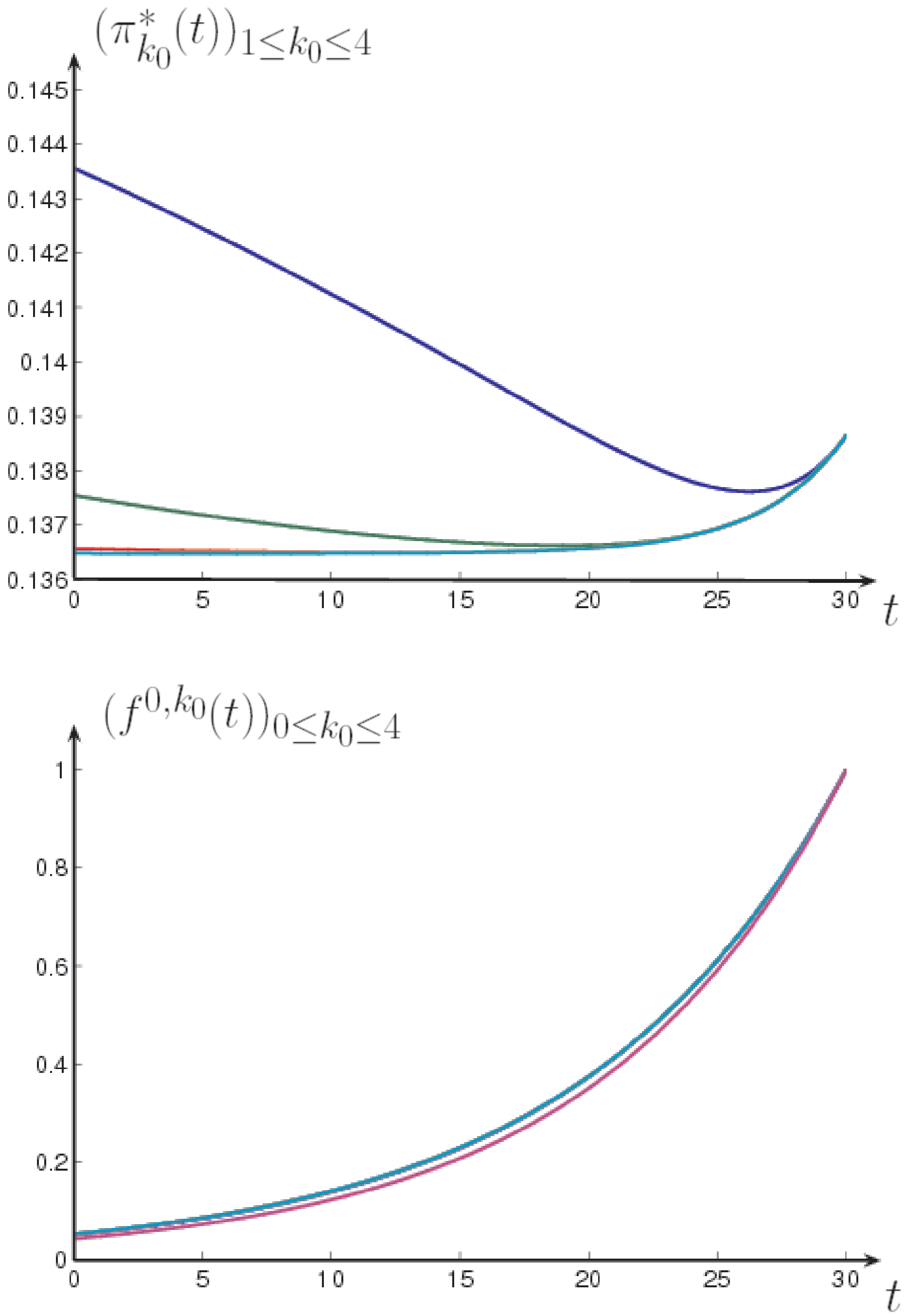

Figure 23: Optimal portfolios and non-wealth dependent part of the value functions.

Secondly, we wish to analyze an important example for a major trading break that happened in the aftermath of World War II in Japan. At that time, the Tokyo Stock Exchange was shut down for almost four years reopening with a loss of more than 90%. We calculate the percentage of initial capital Δx which a log investor and a power utility investor ($\gamma = -3$) would be willing to give up in order to be able to trade in both states. Due to our results above, we can approximate the case of $\gamma = -3$ by a model where only finitely many liquidity breakdowns are possible. Since $\lambda_{0,1}$ is small, it is sufficient to consider a model where at most four jumps into the illiquidity state can occur, i.e. $k_0 = 4$. For k_0 greater than 4, the results are virtually identical. For the log investor, we use our explicit solutions for $k_0 = \infty$. Table 1 summarizes our numerical results for different parameterizations of the model. The variable $\pi^*(0)$ denotes the time-0 optimal stock demand of an investor who is not able to trade in state 1. We assume that

$$\sigma_0 = 0.25,\ \sigma_1 = 0,\ L_0 = L_1 = L_{0,1} = 0,\ r_0 = r_1 = 0.03,\ \alpha_0 = 0.05 \text{ and } \alpha_1 = -r_1.$$

Thus, the stock dynamics are deterministic in state 1. When leaving state 1 the stock loses a fraction of its value, i.e. $L_{1,0} > 0$. Since $L_0 = L_{0,1} = 0$, the optimal stock proportion in state 0, when trading is allowed in both states, is given by $\frac{\alpha_0}{\sigma_0^2} = 80\%$ for the log investor, and $\frac{\alpha_0}{(1-\gamma)\sigma_0^2} = 20\%$ for the power utility investor, respectively. As mentioned above, since $\sigma_1 = 0$ and $\alpha_1 \leq 0$, the optimal stock demand for state 1, when trading is allowed in both states, vanishes. The parameters $\lambda_{0,1}$ and $\lambda_{1,0}$ are chosen in order to mimic situations such as in Japan after World War II. For instance, the parameterization $\lambda_{0,1} = 0.01$, $\lambda_{1,0} = 0.3$, and $L_{1,0} = 0.9$ implies that, on average, once in a century the illiquidity state is reached and, on average, this state is left after three and one-third years triggering a stock price decrease of 90%. In this particular case, a log investor with a horizon of $T = 30$ years would be willing to give up 22.71% of his initial wealth. This is due to the fact that an investor who is able to trade can avoid the loss that is triggered by a jump from state 1 to state 0. He will sell his stocks once the economy is in state 1 and thus use the money market account as a "safe harbor". If the investor cannot trade, then he will not be able to avoid this loss. For this reason, he invests considerably less of his wealth into the risky asset. Figure 23 depicts the optimal strategies and the non-wealth dependent parts of the value functions in this situation for the power utility investor. Again, the upper line corresponds to the optimal strategy if at most one liquidity breakdown can occur and the lowest line corresponds to the optimal strategy if at most $k_0 = 4$ liquidity breakdowns are possible. As for the value functions, the lowest line corresponds to $f^{0,0}$ and the upper line corresponds to $f^{0,4}$. As $\lambda_{0,1}$ is much smaller than $\lambda_{1,0}$, it is likely that at time T the economy is in state 0. Therefore, setting $\ell = 0$ has only a small impact on the percentaged change of initial capital, which can be seen in the fourth line of Table 1. However, if the loss rate $L_{1,0}$ increases from 50% to 90%, then the percentaged change of initial capital increases significantly. Increasing $\lambda_{1,0}$ to 1 results in a small change indicating that the effect of illiquidity is small if the investor does not suffer additional losses. The percentaged change of the initial capital, however, strongly depends on the intensity $\lambda_{0,1}$ modeling the probability that the exchange is closed.

Next, we reconsider the previous example of the Tokyo stock exchange, including a third regime which models an additional economic crisis, where trading is still possible. We consider an investor with a logarithmic risk preference and suppose that infinitely many regime shifts are possible. As before, we assume that

$$\sigma_0 = 0.25,\ \sigma_1 = 0,\ L_0 = L_1 = L_{0,1} = 0,\ r_0 = r_1 = 0.03,\ \alpha_0 = 0.05 \text{ and } \alpha_1 = -r_1.$$

We consider the same situation as in the fifth line of Table 1, i.e. $\lambda_{0,1} = 0.01$, $L_{1,0} = 0.9$ and $\ell = 0.9$. However, the condition $\lambda_{1,0} = 0.3$ is replaced by $\lambda_{1,0} + \lambda_{1,2} = 0.3$ such that the illiquidity state is, on average, still left after three and one-third years. Further, we set

$$\lambda_{0,2} = 0.03,\ \lambda_{2,0} = 1,\ L_{2,0} = 0, \lambda_{2,1} = \lambda_{0,1},\ L_{2,1} = 0, L_{1,2} = 0.9 \text{ and } r_2 = 0.03.$$

Note that if we had $\sigma_2 = \sigma_0$ and $\alpha_2 = \alpha_0$, then state 2 would be identical to state 0. Thus, if there were no losses except for the ones when leaving state 1, i.e. if we also had $L_2 = L_{0,2} = 0$, then the investor's indirect utility, as well as Δx, would be exactly the same as in the setting of the fifth line of Table 1 where there are only two regimes. However, here we set $\sigma_2 = 0.3$ and $\alpha_2 = 0.02$, that is we assume that the risky asset in state 2 behaves worse than in state 0. Thus, the indirect utility in this example is less than the indirect utility in the corresponding example with only two regimes. Further, since the situation in the liquidity states gets worse, the investor becomes more indifferent upon whether trading is allowed in state 1 or not, and therefore Δx becomes smaller.

$\lambda_{1,0}$	$\lambda_{1,2}$	$L_{0,2}$	λ_2	L_2	T	$\pi_0^*(0)(\%)$	$\pi_2^*(0)(\%)$	Δx (%)
0.3	0	0	0	0	30	52.28	8.30	22.23
0.3	0	0	0	0	50	52.28	8.30	34.19
0.3	0	0.2	0	0	30	44.88	8.30	18.24
0.3	0	0.2	0	0	50	44.88	8.30	28.49
0.3	0	0.2	0.5	0.1	30	44.88	0	18.17
0.3	0	0.2	0.5	0.1	50	44.88	0	28.37
0.2	0.1	0	0	0	30	52.28	8.30	22.18
0.2	0.1	0	0	0	50	52.28	8.30	34.12
0.2	0.1	0.2	0	0	30	44.88	8.30	18.21
0.2	0.1	0.2	0	0	50	44.88	8.30	28.43
0.2	0.1	0.2	0.5	0.1	30	44.88	0	18.12
0.2	0.1	0.2	0.5	0.1	50	44.88	0	28.30

Table 2: Percentaged change of initial capital Δx and time-0 optimal portfolio demands $\pi_i^*(0)$ corresponding to states $i \in \{0, 2\}$.

In the first example of Table 1, the illiquidity state cannot be followed by an economic crisis, whereas for the second example we assume that $\lambda_{1,0} = 0.2$ and $\lambda_{1,2} = 0.1$, i.e. with probability $\frac{1}{3}$ the illiquidity state is followed by state 2. In both cases, we obtain the same optimal strategy. In particular, when we also include jumps with loss rate $L_2 = 0.1$ and intensity $\lambda_2 = 0.5$ within state 2, then a pure bond investment becomes optimal as long as the economy is in the crisis state.

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