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Optimal Investment in the Face of Adversity

Taxes, Crashes, and Illiquidity

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Risk comes from not knowing what you're doing.

Warren Buffett

Preface

This thesis summarizes most of my recent research in the mathematical theory of optimal investment. The theme of this work is the extension of the standard framework to include specific real-world features, and the evaluation of their significance. My interest focuses in particular on portfolio choice in the presence of taxes and under the threat of catastrophic events or illiquidity.

The core of this work represents a composition of 4 articles, which are grouped into 3 chapters on

- portfolio choice with *capital gains taxes*,
- worst-case optimal investment for *crash scenarios*, and
- asset allocation under the threat of *illiquidity*.

The structure and style of the thesis reflect its origins: Each of the 4 main parts consists of a revised and partly extended version of a corresponding original research article that has been submitted or accepted for publication in a scientific journal. The 4 parts are independent of each other, and each features an abstract as well as its own numbering and a separate list of references. In addition, an introductory chapter gives a more detailed account of the background and contents of the thesis.

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Contents

Introduction

This thesis deals with 3 important aspects of optimal investment in real-world financial markets: taxes, crashes, and illiquidity.
The purpose of this introductory section is to explain, in general terms, the necessary background and to outline the contributions of this thesis.

What is Optimal Investment? From a naïve perspective, one might be mislead to believe that optimal investment is concerned with finding undervalued securities, i.e. securities whose 'fundamental value' exceeds their price. These assets would then be bought and held until the market recognizes their 'true' value, when their price would rise and the security would be sold with a profit. This sounds like a recipe for making money, and, hence, there must be something wrong with it. Indeed, there are good reasons to believe in the efficiency of financial markets, which essentially means that prices reflect all available information. Therefore, every asset is worth exactly its price: No asset is under-, and none is overvalued. Once this is accepted, the focus shifts from the underlying of a specific security to the statistics of its price. As soon as a mathematical model for the latter is available, the investor's optimal portfolio decision can be determined as the solution of a purely formal optimization problem.
Investment decisions thus become amenable to mathematical analysis. In addition to exact quantifiable results, this formal approach often leads to crucial qualitative insights. Hence, from a mathematical modeling perspective, the challenge is to set up a suitable framework that captures some (if not all) facets of the optimal investment problem and to develop methods to solve it. When the mathematical analysis is successfully completed, it remains to interpret the formal results and reconcile them with economic intuition.
To summarize, the mathematical theory of optimal investment generally deals with the following questions:

- What are appropriate mathematical models for the dynamics of security prices and for investors' preferences?
- Given a mathematical model for the investment problem, what is the optimal portfolio allocation?

History of the Portfolio Problem. It was Nobel laureate[1] Harry Markowitz who in his pathbreaking work [Markowitz 1952] first bridged the gap between qualitative economic and quantitative mathematical theory. Markowitz' fundamental qualitative insight was that it is not a security's own risk that is important to an investor, but rather the contribution it makes to the risk of her (the investor's) portfolio.
While Markowitz assumed mean-variance preferences, it soon became apparent that a more realistic and sensible approach would have to be based on the notion of expected utility. John von Neumann and Oskar Morgenstern had

[1] The *Sveriges Riksbank Prize in Economic Sciences in Memory of Alfred Nobel* is briefly referred to as the *Nobel Prize in Economics* here.

introduced this concept in [von Neumann, Morgenstern 1944] to model risk attitudes of rational agents, and had justified it by a mathematical derivation from simple axiomatic foundations. It was also clear that Markowitz' one-period analysis asked for a generalization to decision problems with multiple periods. In this formulation, Paul A. Samuelson, himself a Nobel Prize laureate, determined optimal portfolio strategies in [Samuelson 1969].
On a different tack, Robert C. Merton left the discrete-time framework altogether and investigated the optimal investment problem in a continuous-time setting with expected utility preferences.[2] In his seminal contributions [Merton 1969] and [Merton 1971], he applied techniques from stochastic dynamic programming to solve the optimal investment problem when stock prices follow the dynamics of [Black, Scholes 1973]. Although Merton's mathematical modeling differed substantially from that of Markowitz, his results confirmed Markowitz' qualitative insights. In the years to follow, Merton's formulation turned out to be rich enough to become the paradigm of a new discipline: the mathematical theory of optimal investment.

Aspects of Real-World Investment. Ever since the publication of Merton's influential papers, mathematicians have tried to extend Merton's framework in order to capture interesting effects that Merton neglected. Indeed, despite its popularity, it is clear that Merton's original formulation fails to include several important aspects of trading in real-world financial markets. These include, for instance, taxation, transactions costs, market crashes, incomplete information, illiquid markets, trading restrictions, investors with market impact, and stock price dynamics beyond the Black-Scholes-Merton specification.
A large part of the existing literature focuses on generalized price dynamics, while the remainder of Merton's original modeling is essentially maintained. As this may considerably complicate the formal analysis, it became necessary to develop new methods beyond the dynamic programming technique, the duality approach of [Cox, Huang 1989] being the most prominent example. On the very abstract end, this strand of literature culminated in the work of Walter Schachermayer and Dmitry Kramkov, who investigated the Merton problem for general semimartingale dynamics in [Kramkov, Schachermayer 1999]. There is also a well-established approach to modeling transactions costs based on techniques from the theory of stochastic impulse control, while models with incomplete information are generally treated with the help of stochastic filtering theory. However, there do not seem to exist standard unified frameworks to address each of the issues mentioned above.
In this thesis, we will be concerned with the mathematical modeling of capital gains taxes, financial market crashes, and illiquidity.

Investors' Preferences. Following Merton, the overwhelming part of the literature assumes that investors' risk attitudes can be modeled by the theory of von

[2] Together with Myron Scholes, Merton was awarded the Nobel Prize in Economics for his contributions to the theory of rational option pricing.

Neumann and Morgenstern. However, as pointed out by Amos Tversky and Daniel Kahneman, see e.g. [Tversky, Kahneman 1974], expected utility may not be adequate as a descriptive theory of the behavior towards risk of real-world agents. Important extensions of the expected utility framework include recursive utility, see e.g. [Kreps, Porteus 1989], and worst-case preferences, see [Gilboa, Schmeidler 1989]. While the concept of recursive utility is concerned with the temporal dimension of dynamic choice problems under uncertainty, the criterion introduced in [Gilboa, Schmeidler 1989] can be understood as a combination of expected utility and worst-case attitudes: The agent aims to maximize expected utility in the most adverse scenario.
In the second part of this thesis, we investigate an alternative specification of investors' preferences under the threat of a major catastrophic event that causes asset prices to drop sharply: Following [Korn, Wilmott 2002], we assume that the investor takes on a worst-case attitude towards the occurrence of the crash. In the spirit of Kahneman and Tversky, this emphasis on the worst-case scenario reflects a large aversion to sudden losses.

Mathematical Methods. In the course of studying taxes, crashes, and illiquidity, we get to apply a variety of mathematical methods in this thesis.
Optimal investment in the illiquid market model that we propose can be analyzed with a variant of Merton's classical stochastic control technique. In the traditional style of dynamic programming, the validity of this approach is established by a corresponding verification theorem.
The portfolio problem with taxation can be rephrased in such a way that an appropriate generalization of the duality method becomes applicable. We develop this extension and use it to solve the investment problem.
After outlining alternative methods to treat the optimal investment problem for worst-case crash scenarios, we introduce a novel martingale approach to the worst-case portfolio problem. The martingale approach is based on a change-of-measure technique and is not directly related to dynamic programming or duality methods.

Economic Insights. Let us briefly outline some qualitative conclusions that can be derived from our analysis.
Firstly, we show that an investor that is liable to deferred capital gains taxes acts as though she possessed a specific derivative contract written on her own terminal wealth. To avoid misunderstandings, we wish to emphasize that this derivative is *not* the investor's tax liability.
Secondly we demonstrate that, under the threat of a crash, it is optimal to reallocate wealth from risky to riskless securities towards the end of the time horizon. This rationalizes a characteristic trait of real-world investment behavior. Moreover, the classical Merton strategy appears as a limit for large time horizons.
Finally, we show that illiquidity, i.e. the inability to trade in certain critical periods, can have a severe impact on the investor's welfare and portfolio

choice. As both the threat of a crash and illiquidity reduce the investor's optimal holdings in risky securities, these approaches may contribute to a further understanding of the equity premium puzzle of [Mehra, Prescott 1985] in a general equilibrium framework.

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Optimal Investment
with Deferred Capital Gains Taxes

Background

In this chapter, we analyze the optimal portfolio problem with capital gains taxes. We assume that taxes are deferred until the end of the investment horizon. The problem is solved with the help of a modification of the classical martingale method.

This chapter is essentially a reprint of [Seifried 2009a], which has been accepted for publication in *Mathematical Methods of Operations Research*.

References

[Seifried 2009a] Seifried, F.T.: *Optimal Investment with Deferred Capital Gains Taxes: A Simple Martingale Method Approach*, to appear in Mathematical Methods of Operations Research.

Optimal Investment with Deferred Capital Gains Taxes
A Simple Martingale Method Approach

Abstract We solve the optimal portfolio problem of an investor in a complete market who is liable to deferred taxes due on capital gains, irrespective of their origin. In a Brownian framework we explicitly determine optimal strategies. Our analysis is based on a modification of the standard martingale method applied to the after-tax utility function, which exhibits a kink at the level of initial wealth, and Clark's formula. Numerical results show that the Merton strategy is close to optimal under taxation.

Keywords optimal investment · capital gains taxes · deferred taxes · martingale method · Clark's formula

Mathematics Subject Classification (2000) 91B28

1 Introduction and Overview

Typically tax legislation treats dividends, capital gains from stocks, and interest gains differently, and taxation of capital gains from stocks is sophisticated. This makes finding optimal portfolio decisions under taxation a mathematically challenging problem. However, a novel legislation in Germany introduces a so-called compensation tax ('Abgeltungssteuer') of 25% due on capital gains irrespective of their origin.

Motivated by this scheme of taxation, we study the corresponding optimal investment problem with taxes due on the gains in total wealth at the end of the time horizon. This is a stylization in that for private investors taxes on gains are typically levied immediately when they are realized. Our model applies directly, however, to *tax-deferred pension accounts*, which are designed to make investment into stocks and mutual funds attractive for retirement savings of private investors. Thus taxes are due exclusively on total capital gains as accrued until the date of retirement, i.e. the liquidation of the portfolio,[1] and

[1] Early withdrawals for consumption may be taxed, an issue which we also address.

there are no transactions costs for portfolio reallocations. These investment products are readily available both in the U.S. and in Germany.

We propose a simple model for the taxation scheme described above: Since we assume taxes to be due on total capital gains only, we can absorb them into the utility function, which then exhibits a characteristic kink at the level of initial wealth. The basic structure of the mathematical problem thus being preserved, we can resort to the martingale method of portfolio optimization. We establish suitable extensions of well-known results and use them to solve the optimal portfolio problem in a complete financial market for a general semimartingale setting. In a Merton model, we can determine the optimal portfolio strategy with the help of Clark's formula and obtain completely explicit solutions for CRRA and CARA investors involving a particular financial derivative on optimal terminal wealth. Our numerical results show that while taxation reduces optimal expected utility, the impact of taxes on asset allocation is negligible: The classical Merton strategy yields a nearly optimal performance.

The main contributions of this paper are the novel *idea* of absorbing taxes into utility, the complete solution to the investment problem, and the representation of optimal strategies. Moreover the martingale method for kinked utility functions in a semimartingale framework is only addressed in few papers such as [Bouchard, Touzi, Zeghal 2004] and [Westray, Zheng 2009], but with a different focus. In contrast to the literature on optimal investment with taxes, our static tax model does not involve timing decisions, tax bases, wash sales, etc., compare [Constantinides 1983], [Constantinides 1984], [Cadenillas, Pliska 1999], [Dammon, Spatt, Zhang 2001] and many others, or questions of asset location as discussed in [Dammon, Spatt, Zhang 2004] or [Huang 2008]. Another strand of literature is based on dynamic programming, see e.g. [Jouini, Koehl, Touzi 1999], [Tahar, Soner, Touzi 2007] or [DeMiguel, Uppal 2005], and also emphasizes intertemporal aspects of taxation. Hence our approach is hardly comparable with the existing literature on portfolio optimization with taxes; we view its simplicity as one of its benefits. Finally, our model separates long-term *strategic* effects from the *tactical* short-term impact of taxation on portfolio choice: By disregarding the latter, we are in a position to study the exclusive influence of the former.

This paper is organized as follows: In Section 2, we formulate the portfolio problem with capital gains taxes. Section 3 treats abstract concave optimality results, which are applied to identify optimal terminal wealth and optimal consumption in Section 4. In Section 5, we compute the corresponding portfolio strategies, Section 6 contains numerical results, and Section 7 concludes.

2 The Portfolio Problem

Financial Market. We assume as given a financial market model on a probability space $(\Omega, \mathfrak{F}, \mathbb{P})$ equipped with a filtration $\{\mathfrak{F}_t\}_{t\in[0,T]}$ satisfying the usual conditions of completeness and right-continuity, $\mathfrak{F}_0$ being $\mathbb{P}$-trivial. The financial market consists of a locally riskless bond $P^0 = \{P^0_t\}$, a strictly positive process of finite variation with $P^0_0 = 1$, and d risky assets $P^i = \{P^i_t\}$, $i = 1, \dots, d$, which we suppose to be semimartingales.
To exclude arbitrage, we further assume there exists an equivalent martingale measure $\mathbb{Q}$, so the discounted price process $\frac{P^i}{P^0}$ is a $\mathbb{Q}$-local martingale for every $i = 1, \dots, d$. We let $\zeta = \{\zeta_t\}$ be a càdlàg modification of the Radon-Nikodým density martingale $\{\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}|_{\mathfrak{F}_t}\}$, and denote by $Z = \{Z_t\}$,

$$Z_t \triangleq \frac{\zeta_t}{P^0_t}\,,\ t \in [0, T],$$

the corresponding state-price deflator.
The crucial hypothesis we make is that of **completeness**: We assume that for any positive $\mathfrak{F}_T$-measurable random variable X with $x \triangleq \mathbb{E}_{\mathbb{Q}}[\frac{X}{P^0_T}] = \mathbb{E}[Z_T X] < \infty$ there exists $\varphi \in \mathcal{A}(x)$ such that

$$X = x + \textstyle\int_0^T \langle \varphi_t, \mathrm{d}P_t\rangle \text{ a.s.},$$

where $P = (P^0, P^1, \dots, P^d)'$ is viewed as an $\mathbb{R}^{d+1}$-valued process, $\mathcal{A}(x)$ denotes the class of admissible trading strategies available for initial wealth $x \in (0, \infty)$, and $\langle\,\cdot\,,\,\cdot\,\rangle$ is the Euclidean scalar product. Intuitively, φ^i_t is the amount invested into risky asset i at time t. We need and shall not specify $\mathcal{A}(x)$ in more detail, but it is clear that each $\varphi \in \mathcal{A}(x)$ must be a (multi-dimensional) semimartingale integrand for P, and to preclude doubling strategies we impose the no-bankruptcy requirement

$$X^\varphi_t \triangleq x + \textstyle\int_0^t \langle \varphi_s, \mathrm{d}P_s\rangle \geq 0 \text{ a.s. for all } t \in [0, T],\ \text{ for any } \varphi \in \mathcal{A}(x).$$

Utility and Taxes. Let $u : (0, \infty) \to \mathbb{R}$ be the investor's utility function, which we assume to be increasing, strictly concave, and of class C^2. Suppose the investor seeks to maximize expected utility from terminal wealth[2]

$$\mathbb{E}[u(\hat{X}^\varphi_T)],$$

where $\hat{X}^\varphi_T$ denotes the after-tax wealth corresponding to the untaxed wealth X^φ_T that is achieved by the portfolio strategy $\varphi \in \mathcal{A}(x_0)$. We set $u(0) \triangleq \lim_{x\downarrow 0} u(x) \in [-\infty, \infty)$, $u'(0) \triangleq \lim_{x\downarrow 0} u'(x) \in (0, \infty]$ and assume that marginal utility vanishes at infinity, $u'(\infty) \triangleq \lim_{x\uparrow\infty} u'(x) = 0$.
Now we introduce deferred capital gains taxes of the type explained in Section 1 and let $x_0 \in (0, \infty)$ denote the investor's initial wealth and $k \in [0, 1)$ her

[2] The investment problem with consumption can be analyzed similarly and will be considered below.

personal tax rate. If $x \in (0, \infty)$ is the investor's untaxed wealth at $t = T$, then since taxes are due on capital gains, irrespective of their origin, the after-tax wealth $\hat{x}$ is given by

$$\hat{x} = x \text{ if } x \leq x_0 \text{ and } \hat{x} = x_0 + q(x - x_0) \text{ if } x > x_0,$$

where $q \triangleq 1 - k \in (0, 1]$. The investor obtains utility $u(\hat{x})$ from *taxed* wealth $\hat{x}$; thus the effective utility $\hat{u}(x)$ obtained from *untaxed* terminal wealth x is computed via $\hat{u} : (0, \infty) \to \mathbb{R}$,

$$\hat{u}(x) \triangleq u(x) \text{ if } x \leq x_0 \text{ and } \hat{u}(x) \triangleq u(x_0 + q(x - x_0)) \text{ otherwise,} \tag{1}$$

see Figure 1. As above we put $\hat{u}(0) \triangleq \lim_{x \downarrow 0} \hat{u}(x)$.

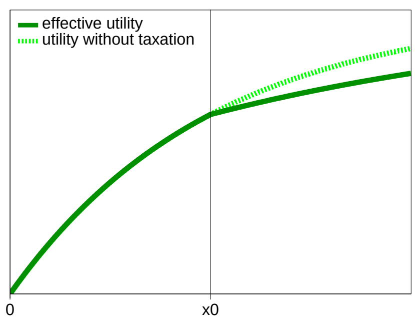

Fig. 1 Effective utility function $\hat{u}$.

Then we may formulate the **optimal portfolio problem** with deferred capital gains taxes as the problem to

$$\text{maximize } \mathbb{E}[\hat{u}(X_T^\varphi)] \text{ over } \varphi \in \mathcal{A}(x_0), \tag{P}$$

where $X_T^\varphi = x_0 + \int_0^T \langle \varphi_t, \mathrm{d}P_t \rangle$ is the terminal wealth corresponding to $\varphi \in \mathcal{A}(x_0)$. This is a standard portfolio problem, except for the fact that the utility function $\hat{u}$ fails to be continuously differentiable: It exhibits *first-order risk aversion* at x_0 in the sense of [Segal, Spivak 1990].

Note that the tax code can be interpreted to say that the investor is short k call options with strike x_0 on her own terminal wealth. This is non-standard because the underlying of the derivative is affected by the investor's portfolio choice. [Carpenter 2000] investigates the formally similar problem of optimal investment with incentive fees, where the investor's (i.e., the manager's) terminal wealth is

$$\hat{X}_T^\varphi = \alpha(X_T^\varphi - B_T)^+ + K$$

for some $\alpha \in (0, 1)$, a benchmark $B = \{B_t\}$ and a *constant* base salary $K > 0$.

3 Optimization by Concave Duality

In order to solve problem (P), we adapt the classical martingale method. Following the lines of [Pliska 1986], [Karatzas, Lehoczky, Shreve 1987] and [Cox, Huang 1991], we first establish an abstract concave optimization result for the associated static problem.

Young's Inequality. Since the marginal utility function $u' :\ (0,\infty) \to (0, u'(0))$ is strictly decreasing and bijective with $\frac{\mathrm{d}\hat{u}}{\mathrm{d}x}(x) = u'(x)$ for $x < x_0$ and $\frac{\mathrm{d}\hat{u}}{\mathrm{d}x}(x) = qu'(x_0+q(x-x_0))$ for $x > x_0$, it is clear that $\frac{\mathrm{d}\hat{u}}{\mathrm{d}x}$ maps $(0,\infty)$ onto $(0, qu'(x_0)) \cup (u'(x_0), u'(0))$ in a one-to-one fashion. We define $\hat{\iota} :\ (0,\infty) \to [0,\infty)$ to be the uniquely determined continuous function that is positive and decreasing and coincides with the inverse of $\frac{\mathrm{d}\hat{u}}{\mathrm{d}x}$ on $(0, qu'(x_0)) \cup (u'(x_0), u'(0))$, see Figure 2.

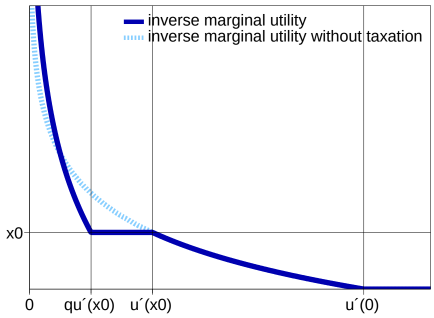

Fig. 2 Inverse marginal utility $\hat{\iota}$.

To be precise, $\hat{\iota}$ is given by

$$\hat{\iota}(\lambda) = x_0 + \tfrac{1}{q}[\iota(\tfrac{1}{q}\lambda) - x_0],\ \ \lambda \in (0, qu'(x_0)),$$
$$\hat{\iota}(\lambda) = x_0,\ \ \lambda \in [qu'(x_0), u'(x_0)], \quad \hat{\iota}(\lambda) = \iota(\lambda),\ \ \lambda \in (u'(x_0), \infty), \tag{2}$$

where $\iota \triangleq (u')^{-1}$ denotes the inverse marginal utility of u and by convention $\iota(\lambda) = 0$ for $\lambda \in [u'(0), \infty)$. Thus $\hat{\iota}$ is of class C^1, except possibly at $\lambda = qu'(x_0), u'(x_0), u'(0)$. The Fenchel-Legendre transform $\hat{u}^*$ of $\hat{u}$ is then given by[3]

$$\hat{u}^* :\ (0,\infty) \to \mathbb{R}, \quad \hat{u}^*(\lambda) \triangleq \lambda\hat{\iota}(\lambda) - \hat{u}(\hat{\iota}(\lambda)).$$

We fix $\lambda \in (0,\infty)$ and consider the function $g :\ [0,\infty) \to \mathbb{R} \cup \{\infty\}$, $g(x) \triangleq \lambda x - \hat{u}(x)$. Clearly g is continuous and of class C^1 on $(0,\infty)$ except at x_0, with $g' = \lambda - \frac{\mathrm{d}\hat{u}}{\mathrm{d}x}$. Hence we have $\lim_{x\downarrow 0} g'(x) = \lambda - u'(0)$ and $\lim_{x\uparrow\infty} g'(x) = \lambda > 0$, so g attains its minimum. Elementary arguments now show that

$$\operatorname*{arg\,min}_{x\in[0,\infty)} g(x) = \hat{\iota}(\lambda) \text{ if } \lambda \notin [qu'(x_0), u'(x_0)], \quad \operatorname*{arg\,min}_{x\in[0,\infty)} g(x) = x_0 = \hat{\iota}(\lambda) \text{ else},$$

[3] Note that $\hat{u}(0) = u(0) > -\infty$ if $u'(0) < \infty$, i.e. if $\hat{\iota}$ takes on the value 0.

and we conclude that $\hat{u}^*$ admits the Fenchel-Legendre representation

$$\hat{u}^*(\lambda) = \inf_{x\in[0,\infty)} \{\lambda x - \hat{u}(x)\} \text{ for all } \lambda \in (0,\infty), \tag{3}$$

where the infimum is uniquely attained. In particular

$$\lambda\hat{\imath}(\lambda) - \hat{u}(\hat{\imath}(\lambda)) = \hat{u}^*(\lambda) \le \lambda x - \hat{u}(x) \text{ for all } x \in [0,\infty),$$

so we obtain the crucial

Lemma 1 (Young's Inequality) *We have*

$$\hat{u}(x) \le \hat{u}(\hat{\imath}(\lambda)) + \lambda[x - \hat{\imath}(\lambda)] \textit{ for every } x \in [0,\infty) \textit{ and any } \lambda \in (0,\infty),$$

with equality if and only if $x = \hat{\imath}(\lambda)$.

As Figure 3 shows, taxes manifest in a linear section of the dual of the effective utility function.

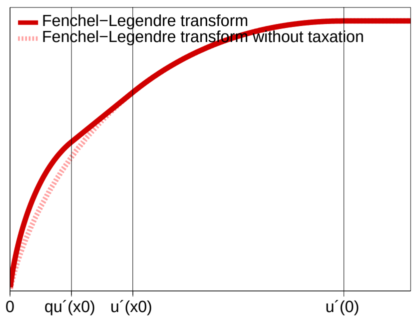

Fig. 3 Fenchel-Legendre transform $\hat{u}^*$.

Optimization by Concave Duality. We let $Z \in \mathrm{L}^1(\mathbb{P})$ be a given strictly positive $\mathfrak{F}_T$-measurable random variable and put

$$\mathcal{X}(x) \triangleq \left\{X \in \mathrm{L}^0(\mathbb{P},\mathfrak{F}_T) \,:\, X \text{ positive and } \mathbb{E}[\hat{u}(X)^-] < \infty,\ \mathbb{E}[ZX] \le x\right\}.$$

We can now extend the classical method of optimization by concavity and state

Theorem 1 (Optimization by Concave Duality) *Suppose* $x_0 \in (0,\infty)$ *and* $\gamma_0 \in (0,\infty)$ *are such that* $\mathbb{E}[Z\hat{\imath}(\gamma_0 Z)] = x_0$. *Then* $X^\star \triangleq \hat{\imath}(\gamma_0 Z) \in \mathcal{X}(x_0)$ *and*

$$\mathbb{E}[\hat{u}(X^\star)] = \sup_{X\in\mathcal{X}(x_0)} \mathbb{E}[\hat{u}(X)].$$

In fact, $X^\star$ *is the a.s. unique maximizer in the above maximization problem.*

Proof With the help of Lemma 1, the claim follows along well-known lines of reasoning: Clearly $X^\star = \hat{\imath}(\gamma_0 Z)$ is a positive $\mathfrak{F}_T$-measurable random variable, in fact even strictly positive if $u(0) = -\infty$; the budget constraint $\mathbb{E}[ZX^\star] \leq x_0$ is satisfied (with equality) by assumption; and Young's inequality yields

$$\hat{u}(\hat{\imath}(\gamma_0 Z)) \geq \hat{u}(1) + \gamma_0 Z[\hat{\imath}(\gamma_0 Z) - 1], \text{ so } \hat{u}(X^\star)^- \leq |\hat{u}(1)| + \gamma_0 Z,$$

so $X^\star \in \mathcal{X}(x_0)$. Now let $X \in \mathcal{X}(x_0)$ be arbitrary. Again using Young's inequality, we have

$$\hat{u}(X^\star) = \hat{u}(\hat{\imath}(\gamma_0 Z)) \geq \hat{u}(X) + \gamma_0 Z[\hat{\imath}(\gamma_0 Z) - X] = \hat{u}(X) + \gamma_0 Z[X^\star - X],$$

whence upon taking expectations and using the fact that $\mathbb{E}[ZX^\star] = x_0$, we obtain

$$\mathbb{E}[\hat{u}(X^\star)] = \mathbb{E}[\hat{u}(X)] + \gamma_0 \left(x_0 - \mathbb{E}[ZX]\right) \geq \mathbb{E}[\hat{u}(X)].$$

Uniqueness follows from the second part of Young's inequality. □

Since $\hat{\imath}$ is continuous, the proof of the standard criterion for the existence of Lagrange multipliers remains valid, and we have

Proposition 1 (Existence of Lagrange Multipliers) *If x_0 and γ_0 are as in the preceding theorem, then for any $x_1 \in (0, x_0)$ there does exist some $\gamma_1 \in (\gamma_0, \infty)$ such that $\mathbb{E}[Z\hat{\imath}(\gamma_1 Z)] = x_1$.*

Remark 1 (Uniqueness of Lagrange Multipliers) *As $X^\star = \hat{\imath}(\gamma_0 Z)$ is a.s. unique and $\hat{\imath}$ is strictly decreasing off $[qu'(x_0), u'(x_0)]$, for typical specifications of the state-price deflator the Lagrange multiplier γ_0 is uniquely determined by x_0.*

4 Solution of the Portfolio Problem

Solution of the Portfolio Problem. Using the results of Section 3, we are now in a position to solve the optimal investment problem (P).

Theorem 2 (Solution of the Portfolio Problem) *Suppose that $x_0 \leq \mathbb{E}[Z_T \hat{\imath}(\gamma Z_T)] < \infty$ for some $\gamma \in (0, \infty)$. Then there exists $\gamma_0 \in (0, \infty)$ with $\mathbb{E}[Z_T \hat{\imath}(\gamma_0 Z_T)] = x_0$, and the a.s. uniquely determined optimal terminal wealth in problem (P) is given by*

$$X^\star \triangleq \hat{\imath}(\gamma_0 Z_T),$$

where $\hat{\imath}$ is given by equation (2).

Proof If in Theorem 1 we choose $Z \triangleq Z_T$, then by our completeness assumption $\mathbb{E}[ZX] \leq x_0$ if and only if

$$X = x + \int_0^T \langle \varphi_t, \mathrm{d}P_t \rangle \text{ for some } x \leq x_0 \text{ and } \varphi \in \mathcal{A}(x_0),$$

i.e. $\mathcal{X}(x_0)$ is the set of all terminal payoffs attainable with initial wealth x_0. Since the condition $\mathbb{E}[\hat{u}(X)^-] < \infty$ is obviously necessary for optimality in (P), it follows that

$$\sup_{\varphi \in \mathcal{A}(x_0)} \mathbb{E}\left[\hat{u}(X_T^\varphi)\right] = \sup_{X \in \mathcal{X}(x_0)} \mathbb{E}\left[\hat{u}(X)\right].$$

Now Proposition 1 ensures the existence of γ_0, and the concave optimization result of Theorem 1 applies to yield the assertion. □

Remark 2 *In fact our approach applies, as does the standard martingale method, to state-dependent utility functions satisfying appropriate integrability conditions. More interestingly, an inspection of the above arguments reveals that the taxation threshold, i.e. the wealth level above which taxes apply, need not coincide with the initial capital, and may in fact be an arbitrary $\mathfrak{F}_T$-measurable random variable.*

The structure of the solution in Theorem 1 is similar to the optimal terminal wealth arising in problems with risk constraints, or more precisely limited expected loss, as investigated in [Basak, Shapiro 2001]. However, the two problems are fundamentally different: In optimal investment with limited expected losses, the location of the flat section of the inverse marginal utility function is determined *endogenously*, whereas in optimal investment with capital gains taxes it is the *exogenously* given interval $[qu'(x_0), u'(x_0)]$. Moreover an inspection of the corresponding inverse marginal utilities on $(0, qu'(x_0))$ reveals that the two problems cannot be transformed into one another.

Portfolio Problem with Consumption. Suppose that the investor can additionally consume at any rate she desires, so she chooses a consumption process $c = \{c_t\}$, which we assume to be positive and predictable, and an admissible portfolio strategy $\varphi \in \mathcal{A}(x_0)$ such that

$$X_t^{c,\varphi} \triangleq x_0 + \int_0^t \langle \varphi_s, \mathrm{d}P_s \rangle - \int_0^t c_s \mathrm{d}s \geq 0 \text{ a.s. for all } t \in [0, T],$$

in which case we write $(c, \varphi) \in \mathcal{A}_c(x_0)$. We let u and u_t denote the investor's utility function for terminal wealth and time-t consumption, $t \in [0, T]$, respectively; we impose the same conditions on u and u_t as in Section 2, and use analogous notation.[4] She aims to maximize expected utility from consumption and terminal wealth, i.e. the quantity

$$\mathbb{E}\left[\int_0^T u_t(\hat{c}_t)\mathrm{d}t + u(\hat{X}_T^{c,\varphi})\right],$$

where $\hat{c} = \{\hat{c}_t\}$ and $\hat{X}_T^{c,\varphi}$ denote, respectively, the after-tax consumption stream and the after-tax terminal wealth associated to $(c, \varphi) \in \mathcal{A}_c(x_0)$.
We assume as before that taxes are due on capital gains at $t = T$, and that furthermore consumption rates above an exogenously given threshold level

[4] Of course we also require $[0, T] \times (0, \infty) \to \mathbb{R}$, $(t, x) \mapsto u_t(x)$ to be Borel measurable.

$\bar{c} = \{\bar{c}_t\}$, a predictable positive process, are subject to taxation: Thus early withdrawals below a personal tax allowance are free of charge, whereas excess consumption is taxed. Furthermore, we allow for a dynamic tax rate $k = \{k_t\}$, which is modeled as a predictable process taking values in $[0, 1)$.
It develops as in Section 2 that the investor's effective utility from an untaxed consumption stream $c = \{c_t\}$ and untaxed terminal wealth $X_T^{c,\varphi}$ is

$$\int_0^T \hat{u}_t(c_t)\mathrm{d}t + \hat{u}(X_T^{c,\varphi}),$$

where $\hat{u}$ is given by (1) as before and $\hat{u}_t : (0, \infty) \to \mathbb{R}$,

$$\hat{u}_t(c) \triangleq u_t(c) \text{ if } c \leq \bar{c}_t, \ \hat{u}_t(c) \triangleq u_t(\bar{c}_t + q_t(c - \bar{c}_t)) \text{ otherwise},$$

with $q = \{q_t\}$ defined as $q_t \triangleq 1 - k_t$, $t \in [0, T]$. Therefore the **optimal consumption-portfolio problem** is to

$$\text{maximize } \mathbb{E}\left[\int_0^T \hat{u}_t(c_t)\mathrm{d}t + \hat{u}(X_T^{c,\varphi})\right] \text{ over } (c, \varphi) \in \mathcal{A}_c(x_0). \tag{P$_c$}$$

Using arguments analogous to those presented above, we may then establish

Theorem 3 (Solution of the Consumption-Portfolio Problem) *If for some $\gamma \in (0, \infty)$ we have $x_0 \leq \mathbb{E}[\int_0^T Z_t \hat{\imath}_t(\gamma Z_t)\mathrm{d}t + Z_T \hat{\imath}(\gamma Z_T)] < \infty$, then the optimal terminal wealth and consumption rate in problem (P$_c$) are given by*

$$X_T^\star \triangleq \hat{\imath}(\gamma_0 Z_T) \text{ and } c_t^\star \triangleq \hat{\imath}_t(\gamma_0 Z_t) \text{ for } t \in [0, T],$$

where for each $t \in [0, T]$ the (possibly random) function $\hat{\imath}_t$ is given by

$$\hat{\imath}_t(\lambda) = \bar{c}_t + \tfrac{1}{q_t}[\iota_t(\tfrac{1}{q_t}\lambda) - \bar{c}_t], \ \lambda \in (0, q_t u_t'(\bar{c}_t)),$$
$$\hat{\imath}_t(\lambda) = \bar{c}_t, \ \lambda \in [q_t u_t'(\bar{c}_t), u_t'(\bar{c}_t)], \quad \hat{\imath}_t(\lambda) = \iota_t(\lambda), \ \lambda \in (u'(\bar{c}_t), \infty),$$

and $\gamma_0 \in (0, \infty)$ is such that $\mathbb{E}[\int_0^T Z_t \hat{\imath}_t(\gamma_0 Z_t)\mathrm{d}t + Z_T \hat{\imath}(\gamma_0 Z_T)] = x_0$.

Proposition 1 and Remark 1 on existence and uniqueness of Lagrange multipliers above apply *mutatis mutandis.*

5 Optimal Portfolio Strategy

In this section, we determine the portfolio strategy that gives rise to the optimal terminal wealth in Theorem 2 in the classical setting of [Merton 1969], [Merton 1971]. We consider a d-dimensional Black-Scholes market modeled on canonical path space, i.e. we suppose that $\Omega = \mathrm{C}([0, T], \mathbb{R}^d)$, that $\mathbb{P}$ is Wiener measure, and that $\{\mathfrak{F}_t\}$ is the natural (completed) filtration of the coordinate process $W = \{W_t\}$. Moreover we let

$$\mathrm{d}P_t^0 = P_t^0 r \mathrm{d}t, \quad \mathrm{d}P_t = \mathrm{diag}(P_t)(r\underline{1} + \eta)\mathrm{d}t + \mathrm{diag}(P_t)\sigma \mathrm{d}W_t,$$

where $P = (P^1, \ldots, P^d)$ is regarded as an $\mathbb{R}^d$-valued process, $r \in \mathbb{R}$, $\eta \in \mathbb{R}^d$, and $\sigma \in \mathbb{R}^{d \times d}$ is such that $\sigma\sigma^{\mathrm{t}}$ is strictly positive definite.[5]

[5] σ^{t} denotes the transpose of σ, and $\sigma^{-\mathrm{t}} \triangleq (\sigma^{\mathrm{t}})^{-1}$.

Wealth Equation. In the above framework the state-price density is given by

$$Z_t = \exp\{-(r + \tfrac{1}{2}|\theta|^2)t - \langle \theta, W_t \rangle\},\ t \in [0,T] \text{ a.s.},$$

where $\theta \triangleq \sigma^{-1}\eta$ denotes the market price of risk. If the investor employs the admissible portfolio strategy $\varphi \in \mathcal{A}(x_0)$, then her wealth satisfies

$$\mathrm{d}X_t^\varphi = rX_t^\varphi \mathrm{d}t + \langle \varphi_t, \eta \mathrm{d}t + \sigma \mathrm{d}W_t \rangle,\ X_0^\varphi = x_0;$$

Itô's formula shows that this is equivalent to

$$Z_t X_t^\varphi = x_0 + \int_0^t Z_s \langle \sigma^{\mathrm{t}} \varphi_s - \theta X_s^\varphi, \mathrm{d}W_s \rangle \text{ for } t \in [0,T] \text{ a.s.} \tag{4}$$

Here we take $\mathcal{A}(x)$ to be the class of progressive processes with the property that X^φ is well-defined by the preceding stochastic differential equation with initial condition $X_0^\varphi = x$ and $X_t^\varphi \geq 0$ for all $t \in [0,T]$ a.s. It is then well-known, see Proposition 1.6.2 in [Karatzas, Shreve 1998] or Theorem 48 in [Korn, Korn 2001], that the assumptions stated at the beginning of Section 2 are fulfilled.

Lemma 2 (Representation Formula) *Let $X^\star$ be a positive $\mathfrak{F}_T$-measurable random variable with $\mathbb{E}[Z_T X^\star] < \infty$, and suppose $\psi = \{\psi_t\}$ is a progressive process in $\mathbb{R}^d$ with*

$$\mathbb{E}[Z_T X^\star | \mathfrak{F}_t] = \mathbb{E}[Z_T X^\star] + \int_0^t \langle \psi_s, \mathrm{d}W_s \rangle \textit{ a.s. for each } t \in [0,T]. \tag{5}$$

Let $\{X_t^\star\}$ be the continuous process determined via $Z_t X_t^\star = \mathbb{E}[Z_T X^\star | \mathfrak{F}_t]$ for $t \in [0,T]$ a.s. Then the $\mathbb{R}^d$-valued progressive process $\varphi^\star = \{\varphi_t^\star\}$ given by

$$\varphi_t^\star \triangleq \sigma^{-\mathrm{t}} \left(\frac{\psi_t}{Z_t} + \theta X_t^\star \right) \textit{ for } t \in [0,T] \tag{6}$$

is an admissible portfolio strategy, and $X_t^{\varphi^\star} = X_t^\star$ for all $t \in [0,T]$ a.s.

If $X^\star$ is the optimal terminal wealth of Theorem 2, then the process $\{X_t^\star\}$ describes the investor's wealth if she acts according to the optimal strategy.

Proof Substituting formula (6) into (4) and using (5) for $t = T$ immediately yields the result; see also Theorem 3.7.6 in [Karatzas, Shreve 1998]. □

Optimal Strategy. Lemma 2 reduces the problem of finding the optimal portfolio strategy $\varphi^\star$ to that of finding a suitable integrand for the martingale representation in equation (5), and thus makes the problem amenable to Clark's formula [Clark 1970]. We need the following straightforward extension of Theorem E.2 in [Karatzas, Shreve 1998] to a multi-dimensional framework.

Theorem 4 (Clark's Formula) *Let F be an $\mathfrak{F}_T$-measurable random variable on $(\Omega, \mathfrak{F}, \mathbb{P})$ such that $\mathbb{E}[F^2] < \infty$, and suppose that*[6]

[6] Recall that in the present 'canonical' setting, W is the identity mapping on $\Omega = \mathrm{C}([0,T], \mathbb{R}^d)$; we write $\|\omega\| = \sup_{t \in [0,T]} |\omega_t|$ for each $\omega \in \Omega$. Condition (a) differs slightly from the formulation in [Karatzas, Shreve 1998], but the proof goes through literally.

(a) there exist a function $g: [0,1] \to [0,\infty)$ *with* $\limsup_{\varepsilon\downarrow 0} \frac{g(\varepsilon)}{\varepsilon} < \infty$ *and an* $\mathfrak{F}_T$*-measurable random variable* h *with* $\mathbb{E}[h^2] < \infty$ *such that*

$$|F(\omega+\phi) - F(\omega)| \le h(\omega) g(\|\phi\|) \text{ for all } \omega, \phi \in \Omega \text{ with } \|\phi\| \le 1,$$

(b) there is an $\mathfrak{F}_T$*-measurable mapping*[7] $\nabla F(\,\cdot\,,\,\cdot\,): \Omega \to \mathcal{M}([0,T])^d$, $\omega \mapsto \nabla F(\omega,\,\cdot\,)$ *such that for* $\mathbb{P}$*-a.e.* $\omega \in \Omega$ *we have*

$$\tfrac{\mathrm{d}}{\mathrm{d}\varepsilon}\Big|_{\varepsilon=0} F(\omega+\varepsilon\phi) = \textstyle\int_0^T \langle \phi_t, \nabla F(\omega, \mathrm{d}t)\rangle \text{ whenever } \phi \in \mathrm{C}^1([0,T],\mathbb{R}^d).$$

Then the Leb$\otimes\mathbb{P}$*-a.e. unique progressive process* $\psi = \{\psi_t\}$ *with* $\int_0^T \psi_t \mathrm{d}W_t = F$ *a.s. satisfies*

$$\psi_t = \mathbb{E}\left[\nabla F(\,\cdot\,,(t,T])\,\middle|\,\mathfrak{F}_t\right] \text{ a.s. for Leb-a.e. } t \in [0,T]. \tag{7}$$

In the following let us say that a function $h: (0,\infty) \to \mathbb{R}$ is *polynomially bounded at* 0 *and* ∞ if there exist $c, \kappa \in (0,\infty)$ such that

$$|h(x)| \le c\left(x + \tfrac{1}{x}\right)^\kappa \text{ for all } x \in (0,\infty).$$

Then combining Clark's formula with Lemma 2 leads to

Theorem 5 (Optimal Strategy) *In the situation of Theorem 2 and under this section's standing assumptions, suppose that* ι *and* $\frac{\mathrm{d}\iota}{\mathrm{d}\lambda}$ *are polynomially bounded at* 0 *and* ∞*. Then the optimal portfolio strategy* $\varphi^\star = \{\varphi^\star_t\}$ *in problem (P) is given by*

$$\varphi^\star_t = -\sigma^{-\mathrm{t}}\theta \mathrm{e}^{-r(T-t)}\mathbb{E}_{\mathbb{Q}}\left[\gamma_0 Z_T \tfrac{\mathrm{d}\hat\iota}{\mathrm{d}\lambda}(\gamma_0 Z_T)\,\middle|\,\mathfrak{F}_t\right] \text{ a.s., Leb-a.e. } t \in [0,T]. \tag{8}$$

Proof We apply Clark's formula to $F = Z_T X^\star = Z_T \hat\iota(\gamma_0 Z_T)$; we have $\mathbb{E}[F^2] < \infty$ by polynomial boundedness of ι and the fact that $\mathbb{E}[(Z_T + \frac{1}{Z_T})^\kappa] < \infty$ for any $\kappa \in (0,\infty)$. For arbitrary $\omega, \phi \in \Omega$ with $\|\phi\| \le 1$, we note that

$$F(\omega+\phi) - F(\omega) = Z_T(\omega)\left[\mathrm{e}^{-\langle\theta,\phi_T\rangle}\hat\iota(\gamma_0 Z_T(\omega)\mathrm{e}^{-\langle\theta,\phi_T\rangle}) - \hat\iota(\gamma_0 Z_T(\omega))\right]$$

and $\frac{\mathrm{d}}{\mathrm{d}p}\mathrm{e}^{-\langle\theta,p\rangle}\hat\iota(z\mathrm{e}^{-\langle\theta,p\rangle}) = -\theta\mathrm{e}^{-\langle\theta,p\rangle}[\hat\iota(z\mathrm{e}^{-\langle\theta,p\rangle}) + z\mathrm{e}^{-\langle\theta,p\rangle}\frac{\mathrm{d}\hat\iota}{\mathrm{d}\lambda}(z\mathrm{e}^{-\langle\theta,p\rangle})]$. Thus condition (a) of Theorem 4 also follows from the mean value theorem, the assumptions on ι and $\frac{\mathrm{d}\iota}{\mathrm{d}\lambda}$, and the fact that $0 < \mathrm{e}^{-|\theta|} \le \mathrm{e}^{-\langle\theta,\phi_T\rangle} < \mathrm{e}^{|\theta|}$. Finally $\nabla F(\,\cdot\,,\,\cdot\,)$ of (b) is found to equal[8]

$$\nabla F(\omega,\,\cdot\,) = -\theta\left[\hat\iota(\gamma_0 Z_T(\omega)) + \gamma_0 Z_T(\omega)\tfrac{\mathrm{d}\hat\iota}{\mathrm{d}\lambda}(\gamma_0 Z_T(\omega))\right]\delta_T$$

for any $\omega \in \Omega$ such that $\gamma_0 Z_T(\omega) \notin \{qu'(x_0), u'(x_0), u'(0)\}$, i.e. for $\mathbb{P}$-a.e. $\omega \in \Omega$. Hence using the martingale property of $\{Z_t X^\star_t\}$ we obtain

$$\psi_t = -\theta Z_t X^\star_t - \theta\mathbb{E}\left[Z_T\gamma_0 Z_T\tfrac{\mathrm{d}\hat\iota}{\mathrm{d}\lambda}(\gamma_0 Z_T)\,\middle|\,\mathfrak{F}_t\right] \text{ a.s., Leb-a.e. } t \in [0,T].$$

Now Lemma 2 and Bayes' formula yield the desired representation of $\varphi^\star$. □

[7] $\mathcal{M}([0,T])$ is the linear space of finite Borel measures on $[0,T]$, endowed with the topology of weak convergence and the associated Borel σ-field.

[8] $\delta_T \in \mathcal{M}([0,T])$ denotes the Dirac point measure at T.

In Proposition 5.2 of [Lakner, Nygren 2006] a similar result is established under different technical conditions. For the specific utility functions considered below, the optimal strategy can alternatively be computed explicitly with Markov and martingale techniques; for more details, we refer the reader to [Gabih, Grecksch, Richter, Wunderlich 2006] and the references therein.

CRRA Utility. We say that the investor's utility function u is CRRA, or exhibits constant relative risk aversion ρ, if

$$u(x) = \tfrac{1}{1-\rho} x^{1-\rho}, \ x \in (0,\infty), \ \text{with } \rho > 0,$$

where it is understood that $u(x) = \log(x)$, $x \in (0,\infty)$, if $\rho = 1$.

Lemma 3 *If u is a* CRRA *utility function with relative risk aversion ρ, then*

$$\lambda \tfrac{\mathrm{d}\hat{\iota}}{\mathrm{d}\lambda}(\lambda) = -\tfrac{1}{\rho}\left\{ \hat{\iota}(\lambda) + \tfrac{k}{1-k} x_0 1_{\{\hat{\iota}(\lambda) > x_0\}} - x_0 1_{\{\hat{\iota}(\lambda) = x_0\}} \right\}$$

for $\lambda \in (0,\infty)$, $\lambda \neq qu'(x_0), u'(x_0)$.

Proof The inverse marginal utility ι of u being given by $\iota(\lambda) = \lambda^{-\frac{1}{\rho}}$, it follows that ι satisfies $\lambda \frac{\mathrm{d}\iota}{\mathrm{d}\lambda}(\lambda) = -\frac{1}{\rho}\iota(\lambda)$, $\lambda \in (0,\infty)$. An explicit calculation via equation (2) yields

$$\lambda \tfrac{\mathrm{d}\hat{\iota}}{\mathrm{d}\lambda}(\lambda) = -\tfrac{1}{\rho}\left\{ \hat{\iota}(\lambda) + \tfrac{k}{1-k} x_0 1_{\{\lambda \in (0, qu'(x_0))\}} - x_0 1_{\{\lambda \in (qu'(x_0), u'(x_0))\}} \right\}$$

for $\lambda \neq qu'(x_0), u'(x_0)$. Since $\lambda < qu'(x_0)$ if and only if $\hat{\iota}(\lambda) > x_0$ and $qu'(x_0) < \lambda < u'(x_0)$ if and only if $\hat{\iota}(\lambda) = x_0$, the claim follows. □

For brevity we cast a name for the financial derivative implicit in the above formula.

Definition 1 *If $X_T^\star$ denotes the optimal terminal wealth in the portfolio problem (P), then the European derivative on $X_T^\star$ with payoff function $g: [0,\infty) \to \mathbb{R}$ given by*

$$g(x) \triangleq 0 \ \textit{if } x < x_0, \ g(x) \triangleq -1 \ \textit{if } x = x_0, \ g(x) \triangleq \tfrac{k}{1-k} \ \textit{if } x > x_0,$$

*is referred to as the **tax derivative**.*[9]

In combination with Theorem 5, we are now able to explicitly determine the CRRA investor's optimal portfolio strategy.

Corollary 1 (Optimal Portfolio for CRRA Investors) *Under the conditions of Theorem 2, suppose that u is a* CRRA *utility function with relative risk aversion ρ. Then the optimal portfolio strategy in the investment problem (P) is given by*

$$\varphi_t^\star = \pi^M \left(X_t^\star + x_0 C_t\right) \ \textit{for } t \in [0,T], \tag{9}$$

where $\pi^M \triangleq \frac{1}{\rho}(\sigma\sigma^{\mathrm{t}})^{-1}\eta$ denotes the vector of Merton proportions and C_t is the fair time-t price of the corresponding tax derivative.

[9] Note that in the presence of taxes we have $X_T^\star = x_0$ with strictly positive probability.

To interpret (9) note that if $X_t^\star \ll x_0$, then the tax derivative is deep out of the money and the investor behaves as if there were no taxes. If $X_t^\star \gg x_0$, i.e. the tax derivative is deep in the money, terminal wealth will very likely be subject to taxation, and additional funds are invested to compensate for tax liabilities. Put differently, since for $x > x_0$ the investor's utility is approximately $u(qx)$, which has constant relative risk aversion ρ, she behaves like a Merton investor who will obtain a tax refund $C_T \approx \frac{k}{1-k}$ on each of the first x_0 units of her terminal wealth.
The tax derivative also represents the effect of first-order risk aversion induced by the kink in the utility function, which makes the investor extraordinarily reluctant to taking risk at wealth level x_0: For example, if $t \approx T$ and $X_t^\star \approx x_0$, then $C_t \approx -1$ and the investor refrains completely from investing into risky assets.

Note that *a priori*, i.e. before the static optimization problem has been resolved, the option price C_t is unknown. If it were given exogenously, the investor would face an optimal investment problem with a lower bound on terminal wealth and would simply set aside a part of her wealth to hedge the option; the optimal strategy for this (sub-)problem would be given by (9), see e.g. [Korn 2005]. In this sense, the investor behaves as though she possessed a derivative written on her own terminal wealth. We wish to stress, however, that this derivative does *not* coincide with the call option discussed in Section 2.

CARA Utility. Next assume that the investor has constant absolute risk aversion, i.e.

$$u(x) = -\mathrm{e}^{-\varrho x},\ x \in (0,\infty),\ \text{for some } \varrho > 0.$$

Lemma 4 *If u is a* CARA *utility function with absolute risk aversion ϱ, then*

$$\lambda \tfrac{\mathrm{d}\hat\iota}{\mathrm{d}\lambda}(\lambda) = -\tfrac{1}{\varrho}\left\{1_{\{\hat\iota(\lambda)>0\}} + \tfrac{k}{1-k}1_{\{\hat\iota(\lambda)>x_0\}} - 1_{\{\hat\iota(\lambda)=x_0\}}\right\}$$

for $\lambda \in (0,\infty)$, $\lambda \neq qu'(x_0), u'(x_0), u'(0)$.

Proof Using the fact that the inverse marginal utility ι of u satisfies $\iota(\lambda) = \frac{1}{\varrho}(\log\frac{\lambda}{\varrho})^-$ for $\lambda \in (0,\infty)$, we see similarly as in Lemma 3 that

$$\lambda \tfrac{\mathrm{d}\hat\iota}{\mathrm{d}\lambda}(\lambda) = -\tfrac{1}{\varrho}\left\{\tfrac{1}{q}1_{\{\lambda\in(0,qu'(x_0))\}} + 1_{\{\lambda\in(u'(x_0),u'(0))\}}\right\}$$

for $\lambda \neq qu'(x_0), u'(x_0), u'(0)$, whence the asserted identity follows. $\square$

Let us set $\varphi^M \triangleq \frac{1}{\varrho}(\sigma\sigma^{\mathrm{t}})^{-1}\eta$. In the untaxed benchmark case $k = 0$, the optimal strategy is given by

$$\bar\varphi_t^\star = \varphi^M \bar C_t^0 \text{ a.s. with } \bar C_t^0 \triangleq \mathrm{e}^{-r(T-t)}\mathbb{Q}(\bar X_T^\star > 0|\mathfrak{F}_t) \text{ for } t \in [0,T], \tag{10}$$

i.e. $\bar{C}_t^0$ is the fair time-t price of a cash-or-nothing call on optimal terminal wealth $\bar{X}_T^\star$ with strike 0. This differs from the well-known strategy of constantly investing φ^M into risky assets because of our positivity constraint on admissible wealth processes, as explained in Section 3 of [Cox, Huang 1989]. In the presence of capital gains taxes, we have

Corollary 2 (Optimal Portfolio for CARA Investors) *In the situation of Theorem 2, if u is a CARA utility function with absolute risk aversion ϱ, the optimal strategy in problem (P) is given by*

$$\varphi_t^\star = \varphi^M \left(C_t^0 + C_t\right) \quad \text{for } t \in [0,T], \tag{11}$$

where $C_t^0 \triangleq \mathrm{e}^{-r(T-t)}\mathbb{Q}(X_T^\star > 0|\mathfrak{F}_t)$ is the fair time-t price of a European cash-or-nothing call on optimal terminal wealth with strike 0, see (10), and C_t is the fair time-t price of the associated tax derivative.

Observe that for $X_t^\star \ll x_0$ the investor applies the classical strategy $\varphi_t^\star \approx \varphi^M C_t^0$, whereas for $X_t^\star \gg x_0$ we have $C_t \approx \frac{k}{1-k} C_t^0$, so $\varphi_t^\star \approx \frac{1}{(1-k)\varrho}(\sigma\sigma^\mathfrak{t})^{-1}\eta C_t^0$. This makes sense: For $x > x_0$ we have $u(x) = u(x_0 + q(x - x_0)) = -c\mathrm{e}^{-(1-k)\varrho x}$, i.e. the investor's utility function has constant absolute risk aversion $(1-k)\varrho$.

6 Numerical Analysis

To illustrate our results, we consider a Brownian framework as described in Section 5 and focus on CRRA utility.[10] It is clear by scaling that it suffices to consider the case $x_0 = 1$.

Tax Derivative. Valuation of the tax derivative discussed in the previous section yields the following Black-Scholes type formula.

Proposition 2 (Price of the Tax Derivative) *The fair time-t price of the tax derivative is given by*

$$C_t = \mathrm{e}^{-r(T-t)} \left\{ \tfrac{1}{1-k}\Phi\left(d_t + \tfrac{\log(1-k)}{|\theta|\sqrt{T-t}}\right) - \Phi(d_t) \right\} \quad \text{for } t \in [0,T],$$

where Φ denotes the cumulative distribution function of the standard normal distribution and $d_t \triangleq \frac{\log(\frac{u'(x_0)}{\gamma_0 Z_t}) + (r - \frac{1}{2}|\theta|^2)(T-t)}{|\theta|\sqrt{T-t}}$, $t \in [0,T]$, is a function of the state-price deflator.

Figures 4 and 5 depict the fair price of the tax derivative for different times to maturity and different levels of the tax rate as a function of the state-price deflator, reflecting the interpretations of optimal strategies given in the previous section. To get more pronounced effects, Figure 4 assumes a tax rate of $k = 50\%$.

[10] For our numerical results, unless stated otherwise we use a single-asset Black-Scholes model with parameter values $T = 5$, $r = 0.05$, $\eta = 0.07$, $\sigma = 0.2$, $\rho = 3$, $k = 25\%$.

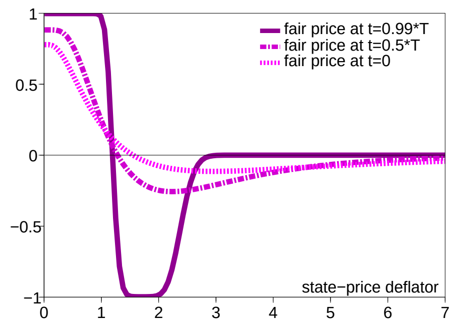

Fig. 4 Fair price C_t of the tax derivative as a function of the state-price deflator Z_t.

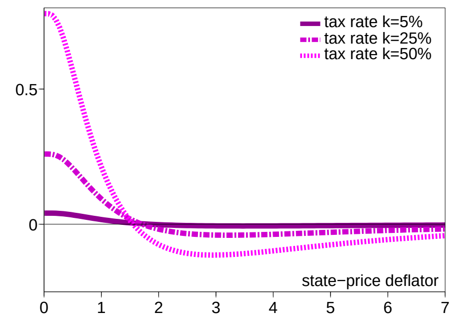

Fig. 5 Fair price C_t of the tax derivative as a function of the state-price deflator Z_t.

Wealth Effect. To assess the loss induced by taxes, we define the **tax burden** to be the fraction of initial wealth an investor would be willing to give up in order to be exempt from taxation. In formulas, the tax burden is $\frac{x_0-x_1}{x_0}$, where x_1 is the solution to

$$v(x_1) = \sup_{\varphi \in \mathcal{A}(x_0)} \mathbb{E}[\hat{u}(X_T^\varphi)]$$

and $v: \ (0,\infty) \to \mathbb{R}$, $v(x) \triangleq \sup_{\varphi \in \mathcal{A}(x)} \mathbb{E}[u(X_T^\varphi)]$, is the value function of the untaxed portfolio problem, which is known explicitly.

For a CRRA investor with relative risk aversion ρ, the tax burden is plotted as a function of the tax rate in Figure 6; Figure 7 depicts the dependence of the tax burden on the investor's risk aversion. It is seen that for realistic tax rates the tax burden is very well approximated as a linear function of the tax rate. The slope of this function, which depends on the investor's risk aversion, can be read off Figure 7. Not surprisingly, *ceteris paribus* the tax burden is higher for less risk-averse investors. Finally, the dependence of the tax burden on the investment horizon is depicted in Figure 8. Since for a large time horizon the investor is very likely to end up with profits, the tax burden tends to the tax

rate.
To summarize, the loss induced by taxation is considerable: The *wealth effect* of taxation cannot be neglected.

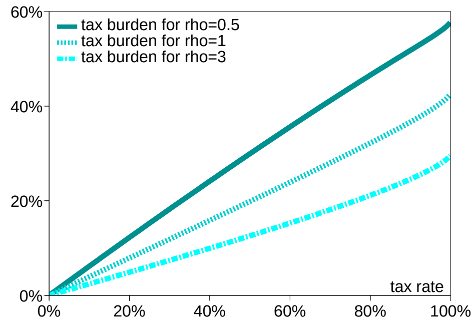

Fig. 6 Tax burden as a function of the tax rate k.

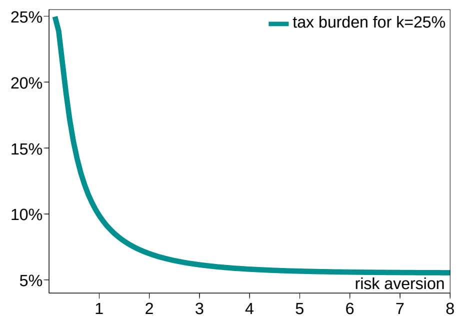

Fig. 7 Tax burden as a function of relative risk aversion ρ.

Strategy Effect. In order to study the impact of taxation on the investor's portfolio choice, we compute the expected utility obtained by applying the classical Merton strategy in the *taxed* portfolio problem (P) and determine the amount x_1 of initial wealth required to achieve the same level of utility by means of the *optimal* strategy under taxation, i.e.

$$\mathbb{E}[\hat{u}(X_T^M)] = \sup_{\varphi \in \mathcal{A}(x_1)} \mathbb{E}[\hat{u}(X_T^\varphi)],$$

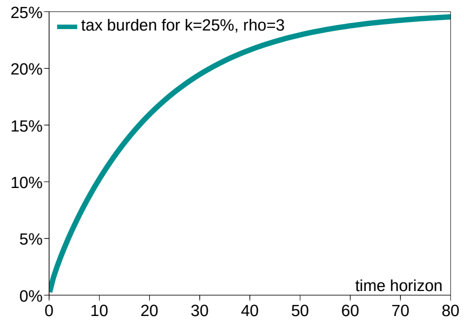

Fig. 8 Tax burden as a function of the time horizon T.

where X_T^M denotes the terminal wealth obtained with the Merton strategy. Then the **ignorance cost** $\frac{x_0 - x_1}{x_0}$ quantifies the loss encountered by ignoring tax liability in investment decisions.

For CRRA investors with relative risk aversion $\rho = 0.5, 1, 3$, the ignorance cost is depicted in Figure 9. Given realistic tax rates it is well below 0.5%; interestingly, for $\rho > 1$ the relative performance of the Merton strategy is worst for an (unreasonably) high tax rate *below* 100%.
It follows that the *strategy effect* of taxation is negligible; note that this is not obvious *a priori*: One cannot make the above comparison unless one has solved the optimal portfolio problem with taxes. Thus we arrive at the reassuring conclusion that *the Merton strategy is almost optimal also under deferred taxation*.

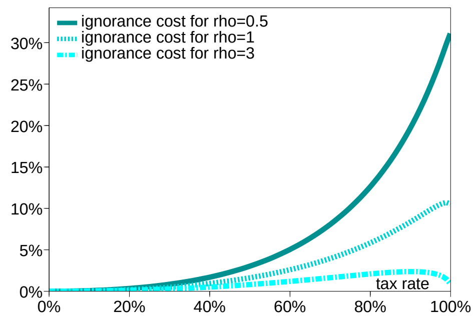

Fig. 9 Ignorance cost as a function of the tax rate k.

7 Conclusion

We have suggested a novel martingale approach to optimal investment with deferred capital gains taxes, and provided a complete solution to the portfolio problem. In a Merton model, optimal trading strategies have been determined; for CRRA and CARA investors we have obtained explicit formulae involving a derivative on optimal terminal wealth. Our numerical analysis demonstrates that, while the wealth effect of deferred capital gains taxation is considerable, the classical Merton strategy performs virtually optimally in the presence of taxes. Hence the *strategic* effect of tax liability on asset allocation is negligible, and only short-term *tactical* considerations will influence portfolio choice significantly.

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Portfolio Optimization under the Threat of a Crash

Background

This chapter is concerned with optimal asset allocation under the threat of a financial market crash. The investor takes a worst-case attitude towards the crash, so her investment objective is to be best off in the most adverse crash scenario.

The first part of the present chapter is largely based on the article [Korn, Seifried 2009], which will appear in the *Radon Series on Computational and Applied Mathematics*, and provides a general survey of the worst-case approach to optimal investment. In the second part, which is an extended version of [Seifried 2009b], we present in detail the novel martingale approach to optimal portfolio choice for worst-case crash scenarios.

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A Worst-Case Approach to Continuous-Time Portfolio Optimization

Abstract We survey the main ideas, results and methods behind the worst-case approach to portfolio optimization in continuous time. This will cover the indifference approach, the HJB-system approach and the very recent martingale approach. We illustrate the difference to conventional portfolio optimization with explicitly solved examples.

Keywords optimal portfolios · worst-case approach · utility indifference · HJB-equation

Mathematics Subject Classification (2000) 93E20

1 Introduction

Stock price models that abandon the continuity of sample paths to include the possibility of asset price jumps have (re-)gained an enormous interest in recent years with the introduction of Lévy processes to financial mathematics (see the monograph [Cont, Tankov 2004] and their impressive list of references). Their main motivation is the inability of the standard geometric Brownian motion based models to explain large stock price moves, which are often observed at the markets. In particular, sudden price falls of the whole market, so-called crashes, are not incorporated into the standard continuous-path framework.

While many of those recently introduced Lévy process models exhibit a very good fit to observed market prices, they have the drawback that their analytical handling is not easy. Even more seriously, estimating the necessary input parameters from market data is not at all trivial, sometimes not even very stable from a statistical point of view. Motivated by this and also by the desire to be able to model market crashes, [Hua, Wilmott 1997] introduced their so-called *crash model*. Its distinctive feature is that stock prices are assumed to follow geometric Brownian motions in *normal times*; at a *crash time* they

suddenly fall by an unknown factor, which they assume to be bounded by an explicitly known constant. Besides the height of the crash, the time and the number of crashes up to a given time horizon are also unknown, but not explicitly modeled in a stochastic way. [Hua, Wilmott 1997] obtain so-called *worst-case option prices* by figuring out the crash scenario that generates the worst case with respect to the option price.
In the context of portfolio optimization, looking at the worst case is also an interesting alternative to the focus on expected utility or on the mean-variance criterion. Of course, such a consideration of the worst case needs two essential components: an exact definition of the worst case and a concept how this worst case enters the portfolio decisions. Examples for worst-case approaches that appeared in the continuous-time portfolio optimization literature are, among others, [Talay, Zheng 2002], [Riedel 2009], [Schied 2005] and [Hernández-Hernández, Schied 2006]. These approaches typically focus on the parameters of asset prices, i.e. on the market coefficients. The worst case is then modeled as the parameter setting leading to an optimal portfolio process with the lowest utility. In [Talay, Zheng 2002] the market is explicitly regarded as an opponent to the investor that chooses the market coefficients. However, the price processes still remain diffusion processes. [Schied 2005] formalizes the idea by considering a whole set of probability measures that are candidates to govern the evolution of stock prices. In light of this setting, he determines the portfolio strategy that yields the highest lower bound for the expected utility from terminal wealth over all those possible probability measures.

In contrast to the approaches mentioned above, [Korn, Wilmott 2002] have taken up the [Hua, Wilmott 1997] framework. They focus on the uncertainty of the number, time and height of possible market crashes. By an indifference argument, they show how to derive a characterization of the worst-case optimal portfolio process. This approach is extended to a more general market setting by [Korn, Menkens 2005] and to problems including insurance risk processes in [Korn 2005]. [Korn, Steffensen 2007] relate the indifference approach of [Korn, Wilmott 2002] to a system of inequalities that they call the HJB-system and thereby obtain optimality of the worst-case portfolio process in a wider class of strategies. [Seifried 2009b] provides a novel martingale approach to worst-case optimal investment; see also the next chapter of this thesis.

In this survey paper, we will focus on the worst-case approach in the sense of [Korn, Wilmott 2002]. The indifference approach will be considered in Section 2, the HJB-systems approach will be the subject of Section 3, while the martingale approach will be presented in Section 4. Section 5 concludes.

2 The Indifference Approach to Worst-Case Portfolio Optimization in the Log-Utility Case

2.1 Motivation and Model

In this section, we consider the simplest case of the continuous-time worst-case market model introduced by [Hua, Wilmott 1997] and taken up by [Korn, Wilmott 2002]. We look at a market consisting of a riskless bond and one risky security whose price dynamics are given by

$$\mathrm{d}P_0(t) = P_0(t)r\mathrm{d}t,\ P_0(0) = 1 \tag{1}$$

$$\mathrm{d}P_1(t) = P_1(t)\left[b\mathrm{d}t + \sigma \mathrm{d}W(t)\right],\ P_1(0) = p_1 \tag{2}$$

with constant market coefficients $b > r$ and $\sigma \neq 0$ in *normal times*. At a so-called **crash time** τ, which is modeled as a stopping time, the stock price can suddenly fall by a relative amount k with $0 \leq k \leq k^* < 1$. Here, k^* is assumed to be the biggest possible crash height. Thus in a crash scenario (τ, k) we shall have

$$P_1(\tau) = (1-k)P_1(\tau-).$$

In this section, we restrict ourselves to the case that at most one such crash can happen before the investment horizon T.

We assume the crash to be unknown a priori, but observable, so an investor will specify her actions completely by a pre-crash portfolio strategy $\underline{\pi}$ and a post-crash portfolio strategy $\overline{\pi}$, both of which we assume to be progressive processes. For ease of exposition, we assume throughout that pre-crash strategies are *bounded* and *continuous*. As an abbreviation we introduce $\pi \triangleq (\underline{\pi}, \overline{\pi})$. Then for a possible crash scenario (τ, k) the dynamics of the investor's wealth process $X = X^\pi = \{X^\pi(t) : t \in [0,T]\}$ are governed by the stochastic differential equation

$$\begin{aligned}
\frac{\mathrm{d}X^\pi(t)}{X^\pi(t)} &= (r + \underline{\pi}(t)(b-r))\mathrm{d}t + \underline{\pi}(t)\sigma \mathrm{d}W(t) \text{ on } [\![0,\tau)\!),\ X^\pi(0) = x \\
X^\pi(\tau) &= (1 - \underline{\pi}(\tau)k)X^\pi(\tau-) \\
\frac{\mathrm{d}X^\pi(t)}{X^\pi(t)} &= (r + \overline{\pi}(t)(b-r))\mathrm{d}t + \overline{\pi}(t)\sigma \mathrm{d}W(t) \text{ on } (\!(\tau, T]\!]
\end{aligned}$$

where $x > 0$ denotes the initial wealth. Thus, in accordance with the intended interpretation, the pre-crash strategy $\underline{\pi}$ is valid up to and including the crash time, whereas $\overline{\pi}$ is only applied starting immediately afterwards. All portfolio strategies π that guarantee a corresponding non-negative wealth process starting from an initial wealth of $x > 0$ form the class $\mathcal{A}(x)$ of admissible strategies with initial wealth x. If we consider only the time interval $[t,T]$, we use the obviously modified notation $\mathcal{A}(t,x)$ for the class of admissible strategies starting at time t with wealth $x > 0$.

Before we state the worst-case portfolio problem, we define $\tilde{X}^\pi = \{\tilde{X}^\pi(t) : t \in [0,T]\}$ as the wealth process in the standard crash-free market model given by equations (1), (2) that corresponds to the portfolio process π.

Definition 1 (Worst-Case Portfolio Problem) *Let U be an $\mathbb{R}$-valued strictly concave, increasing and differentiable function. U will be called a utility function.*
1. *The problem*

$$\sup_{\pi\in\mathcal{A}(x)} \inf_{0\leq\tau\leq T,\ 0\leq k\leq k^*} \mathbb{E}\left[U(X^\pi(T))\right] \tag{P}$$

with final wealth $X^\pi(T)$ in the case of a crash of size k at time τ given by

$$X^\pi(T) = (1-\underline{\pi}(\tau)k)\,\tilde{X}^\pi(T)$$

*is called the **worst-case portfolio problem** with **value function***

$$\nu^1(t,x) \triangleq \sup_{\pi\in\mathcal{A}(t,x)} \inf_{t\leq\tau\leq T,\ 0\leq k\leq k^*} \mathbb{E}\left[U\left(X^\pi(T)\right)\right].$$

2. *We denote by $\nu^0(t,x)$ the value function of the optimization problem in the standard (crash-free) Black-Scholes setting; it is given by*

$$\nu^0(t,x) = \sup_{\pi\in\mathcal{A}(t,x)} \mathbb{E}[U(\tilde{X}^\pi(T))].$$

To allow for explicit computations, we consider the special case of the logarithmic utility function

$$U(x) = \ln(x),\ x>0$$

in this section. We then have the following representation of the value function in the Black-Scholes setting (see e.g. [Korn 1997]):

$$\nu^0(t,x) = \ln(x) + r(T-t) + \tfrac{1}{2}(\tfrac{b-r}{\sigma})^2(T-t) \tag{3}$$

as well as the corresponding optimal portfolio process

$$\pi^* = \tfrac{b-r}{\sigma^2}\,. \tag{4}$$

We motivate the basic ideas of our worst-case concept by looking at two extreme strategies. Note first that the (worst-case) optimal post-crash strategy is π^*. This is simply due to the fact that this is the optimal portfolio process in the then relevant market. If we also chose the portfolio process π^* before the crash (provided that it satisfies $\pi^* < \frac{1}{k^*}$), the worst case would be a crash of maximal height k^* (recall that due to the assumption $b>r$ the log-optimal portfolio process is positive!). One can easily verify that the exact time of this crash would have no impact on the resulting final expected utility. It can therefore be obtained from the worst crash happening immediately and equals

$$\nu^0\left(t,(1-\pi^*k^*)x\right) = \ln(x) + r(T-t) + \tfrac{1}{2}(\tfrac{b-r}{\sigma})^2(T-t) + \ln(1-\pi^*k^*). \tag{5}$$

If, instead, we consider a very prudent investor that chooses $\underline{\pi}(t)=0$ before the crash, the worst case for her is the no-crash scenario. To see this, note that a crash would not harm the investor; however, she could never switch to the strategy π^* after the crash (such a switch would result in a higher expected terminal utility!). Hence, she can never benefit from the knowledge that no further crash can happen. Her corresponding final utility would simply be

$$\mathbb{E}[\ln(x\mathrm{e}^{r(T-t)})] = \ln(x) + r(T-t). \tag{6}$$

Comparing the representations (5) and (6) one can draw the following conclusions:

- It depends on the investment time left $T-t$ which of the two extreme strategies yields a higher worst-case bound.
- While the first strategy takes too much risk (especially when the remaining investment time is small), the second one is too risk averse (especially when the remaining investment time is big). An optimal strategy should in a way balance this out.
- A portfolio process that consists of two constant parts $\underline{\pi}$ and $\overline{\pi}$ cannot be optimal with respect to the worst-case criterion.

2.2 Indifference Strategies: Characterization and Optimality

We take up the conclusions from the end of the preceding section and look for a portfolio process that attains a balance between good performance of the wealth process when no crash happens and a (just) acceptable loss in the crash scenario. For this we try to find a pre-crash portfolio process making us indifferent between the two scenarios:

- The worst crash happens immediately.
- No crash occurs at all.

Such a portfolio process $\hat{\pi} = (\underline{\hat{\pi}}, \pi^*)$ has to satisfy the following identity between the expected utilities corresponding to the two different scenarios:

$$\nu^0\left(t, (1-\underline{\hat{\pi}}(t)k^*)x\right) = \mathbb{E}^{t,x}[\ln(\tilde{X}^{\hat{\pi}}(T))].$$

Applying Itô's formula to the right-hand side of this equality and using the explicit form of $\nu^0(t,x)$ on the left-hand side results in

$$\begin{aligned} &\ln(x) + r(T-t) + \tfrac{1}{2}(\tfrac{b-r}{\sigma})^2(T-t) + \ln\left(1-\underline{\hat{\pi}}(t)k^*\right) \\ &= \ln(x) + r(T-t) + \mathbb{E}\left[\int_t^T \left\{\underline{\hat{\pi}}(s)(b-r) - \tfrac{1}{2}\underline{\hat{\pi}}(s)^2\sigma^2\right\} \mathrm{d}s\right] \\ &\quad + \mathbb{E}\left[\int_t^T \underline{\hat{\pi}}(s)\sigma \mathrm{d}W(s)\right]. \end{aligned} \tag{7}$$

If we assume existence of a deterministic indifference portfolio process $\hat{\pi}$, the stochastic integral has mean zero and the expectation in front of the ds-integral can be dropped. Eliminating identical terms on both sides of equation (7) yields

$$\tfrac{1}{2}(\tfrac{b-r}{\sigma})^2(T-t) + \ln\left(1-\underline{\hat{\pi}}(t)k^*\right) = \int_t^T \left\{\underline{\hat{\pi}}(s)(b-r) - \tfrac{1}{2}\underline{\hat{\pi}}(s)^2\sigma^2\right\} \mathrm{d}s.$$

Assuming that $\hat{\underline{\pi}}$ is differentiable, differentiation of this identity with respect to t leads to the ordinary differential equation

$$\hat{\underline{\pi}}'(t) = -\tfrac{\sigma^2}{2k^*}\left[1 - \hat{\underline{\pi}}(t)k^*\right]\left[\hat{\underline{\pi}}(t) - \pi^*\right]^2 \tag{8}$$

while the obvious final condition

$$\hat{\underline{\pi}}(T) = 0 \tag{9}$$

follows directly from (7). It is now straightforward to verify that there is a unique solution to equations (8) and (9). Even more, one can directly prove that the strategy determined by (8) and (9) solves the worst-case problem. The following result is taken from [Korn, Wilmott 2002], but we will give a somewhat shorter proof.

Theorem 1 (Worst-Case Optimal Portfolio for Logarithmic Utility) *The portfolio process $\hat{\pi} = (\hat{\underline{\pi}}, \pi^*)$ determined by* (8)*,* (9) *and* (4) *solves the worst-case investment problem* (P) *with logarithmic utility.*

Proof Let $\hat{\underline{\pi}}$ be the unique pre-crash portfolio process determined by (8), (9).
STEP 1. We first show that the worst-case scenario for $\hat{\pi}$ is attained by a jump of maximum size k^* at any time $t \in [0, T]$. This obviously is the case if the corresponding expectation function

$$\hat{\nu}\left(t, X^{\hat{\pi}}(t)\right) = \nu^0\left(t, (1 - \hat{\underline{\pi}}(t)k^*)X^{\hat{\pi}}(t)\right)$$

is a martingale. However, by the explicit form of $\nu^0(t, x)$ given in equation (3) and the fact that $\hat{\underline{\pi}}$ satisfies (8), (9), we obtain

$$\begin{aligned}
&\nu^0\left(t, (1 - \hat{\underline{\pi}}(t)k^*)X^{\hat{\pi}}(t)\right) = \ln\left(x\left(1 - \hat{\underline{\pi}}(0)k\right)\right) + \left(r + \tfrac{1}{2}(\tfrac{b-r}{\sigma})^2\right)T \\
&\quad + \int_0^t \frac{1}{1 - \hat{\underline{\pi}}(s)k^*}\left[\hat{\underline{\pi}}(s)(b-r) - \tfrac{1}{2}\sigma^2\left(\hat{\underline{\pi}}(s)^2 + (\pi^*)^2\right)\right]\mathrm{d}s + \int_0^t \sigma\hat{\underline{\pi}}(s)\mathrm{d}W(s) \\
&= \ln\left(x\left(1 - \hat{\underline{\pi}}(0)k\right)\right) + \left(r + \tfrac{1}{2}(\tfrac{b-r}{\sigma})^2\right)T + \int_0^t \sigma\hat{\underline{\pi}}(s)\mathrm{d}W(s).
\end{aligned}$$

As the integrand of the stochastic integral is deterministic and bounded, the martingale property is established.
STEP 2. Let now $\pi = (\underline{\pi}, \pi^*)$ be an admissible portfolio process with a better worst-case performance than $\hat{\pi}$; without loss of generality suppose that the portfolio process π^* is used in the Black-Scholes setting after the crash. Due to continuity it must be constant in $t = 0$. Thus, to obtain a higher worst-case bound than $\hat{\pi}$, it must satisfy

$$\underline{\pi}(0) < \hat{\underline{\pi}}(0).$$

Further, as we have

$$\begin{aligned}
&\mathbb{E}[\ln(\tilde{X}^{\pi}(T))] = \ln(x) + \left(r + \tfrac{1}{2}\left(\tfrac{b-r}{\sigma}\right)^2\right)T + \int_0^T \mathbb{E}\left[\underline{\pi}(s)(b-r) - \tfrac{1}{2}\sigma^2\underline{\pi}(s)^2\right]\mathrm{d}s \\
&\le \ln(x) + \left(r + \tfrac{1}{2}\left(\tfrac{b-r}{\sigma}\right)^2\right)T + \int_0^T \left\{\mathbb{E}\left[\underline{\pi}(s)\right](b-r) - \tfrac{1}{2}\sigma^2\left(\mathbb{E}\left[\underline{\pi}(s)\right]\right)^2\right\}\mathrm{d}s \quad (10)
\end{aligned}$$

by Jensen's inequality, due to continuity of $\underline{\hat{\pi}}$ there has to be a smallest deterministic time $\bar{t} \in [0, T]$ with

$$\mathbb{E}\left[\underline{\pi}(\bar{t})\right] \geq \mathbb{E}\left[\underline{\hat{\pi}}(\bar{t})\right] = \underline{\hat{\pi}}(\bar{t})$$

if in the no-crash scenario the portfolio process π delivers a higher worst-case bound than $\hat{\pi}$. Note that due to the indifference construction $\hat{\pi}$ attains its worst-case bound also in the no-crash-scenario.
We now look at the worst-crash scenario at time $\bar{t}$. In this situation we obtain

$$\begin{aligned}
&\mathbb{E}[\ln\left(X^{\pi}(T)\right)] = \mathbb{E}\left[\ln\left(X^{\pi}(\bar{t})\right) + \textstyle\int_{\bar{t}}^{T}\left\{r + \pi^*(b-r) + \frac{1}{2}\sigma^2(\pi^*)^2\right\}\mathrm{d}s\right] \\
&= \mathbb{E}[\ln(\tilde{X}^{\pi}(\bar{t}))] + \mathbb{E}[\ln\left(1 - \underline{\pi}(\bar{t})k^*\right)] + \textstyle\int_{\bar{t}}^{T}\left\{r + \pi^*(b-r) + \frac{1}{2}\sigma^2(\pi^*)^2\right\}\mathrm{d}s \\
&\leq \mathbb{E}[\ln(\tilde{X}^{\underline{\pi}}(\bar{t}))] + \ln\left(1 - \mathbb{E}\left[\underline{\pi}(\bar{t})\right]k^*\right) + \textstyle\int_{\bar{t}}^{T}\left\{r + \pi^*(b-r) + \frac{1}{2}\sigma^2(\pi^*)^2\right\}\mathrm{d}s \\
&\leq \mathbb{E}[\ln(\tilde{X}^{\underline{\hat{\pi}}}(\bar{t}))] + \ln\left(1 - \left(\underline{\hat{\pi}}(\bar{t})\right)k^*\right) + \textstyle\int_{\bar{t}}^{T}\left\{r + \pi^*(b-r) + \frac{1}{2}\sigma^2(\pi^*)^2\right\}\mathrm{d}s \\
&= \mathbb{E}[\ln\left(X^{\hat{\pi}}(T)\right)].
\end{aligned}$$

Note that in the first inequality, we have used Jensen's inequality. The second inequality is a consequence of (10), the fact that for $\hat{\pi}$ (10) is satisfied with equality, and of course the defining property of $\bar{t}$. Hence, we arrive at a contradiction to the assumption that π attains a higher worst-case bound than $\hat{\pi}$. □

Remark 1 (Analysis of the Worst-Case Optimal Portfolio Process) *From the explicit form of the differential equation* (8) *and* (9) *for the worst-case optimal pre-crash strategy $\hat{\pi}$, we can see that*

$$0 \leq \underline{\hat{\pi}}(t) \leq \min\left\{\pi^*, \tfrac{1}{k^*}\right\} \triangleq \pi_{\min} \textit{ for } t \in [0, T].$$

More precisely, under the change of variable $t \mapsto T - t$ the differential equation (8), (9) *takes the form*

$$h'(t) = \tfrac{\sigma^2}{2k^*}\left[1 - h(t)k^*\right]\left[h(t) - \pi^*\right]^2, \; h(0) = 0$$

with $\underline{\hat{\pi}}(t) = h(T-t)$. It is then clear that starting in 0*, in particular below $\pi_{\min}$, h cannot cross either* 0*, π^* or $\frac{1}{k^*}$. Therefore, even in the case $\pi^* > \frac{1}{k^*}$, the worst-case optimal portfolio process avoids a negative wealth at any time.*

As constant portfolio processes often play a very prominent role in portfolio optimization, one might ask for the best constant portfolio process under the worst-case setting. As it is clear that the best constant portfolio process *after* the crash is $\overline{\pi}(t) = \pi^*$, we refer to a constant (worst-case) portfolio process as a pair of the form

$$\pi(t) = (\underline{\pi}(t), \overline{\pi}(t)) = (c, \pi^*) \text{ for all } t \in [0, T]$$

with c a constant. As shown in [Korn, Wilmott 2002], the optimal constant c depends on the time horizon T. We therefore introduce the optimal constant

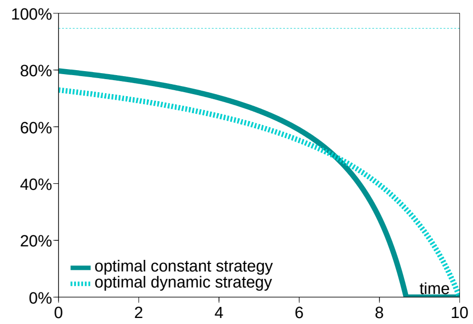

Fig. 1 Worst-case optimal strategy (dotted) and worst-case constant best strategy as a function of the time horizon (straight).

portfolio process as a function of time $c(t)$ (where the time variable actually denotes the time horizon) as

$$c(t) = \left(\frac{1}{2}\left(\frac{b-r}{\sigma^2} + \frac{1}{k^*}\right) - \sqrt{\frac{1}{4}\left(\frac{b-r}{\sigma^2} - \frac{1}{k^*}\right)^2 + \frac{1}{\sigma^2 t}} \right)^+ .$$

Obviously, this constant $c(t)$ converges towards $\pi_{\min}$ as $t \to \infty$. Note in particular that

$$c(t) = 0 \text{ if and only if } b - r \leq \frac{k^*}{t} ,$$

i.e. if the "crash height per time unit" exceeds the excess return of the stock.

Example 1 *To demonstrate the performance of the worst-case strategy together with the worst-case optimal constant portfolio process, we look at an example where we have chosen the following data:*

$$r = 5\%,\ b = 20\%,\ \sigma = 40\%,\ k^* = 20\%,\ T = 10.$$

As long as no crash has happened, the worst-case optimal portfolio process $\hat{\pi}$ is given by the dotted curved line which shows $\underline{\hat{\pi}}$. After the jump the investor has to switch to the line parallel to the x-axis with $\overline{\pi} = \pi^* = 0.9375$. For reasons of comparison, the dark line shows the optimal constant portfolio $c(T-t)$ which would be chosen if the portfolio problem started at time t. One clearly sees that the constant portfolio function c differs from the worst-case optimal portfolio $\underline{\hat{\pi}}$. It is below the worst-case optimal portfolio close to the investment horizon, and above it if the investment horizon is far away.

2.3 Indifference Strategies: Generalizations

The central result of the previous section can be generalized in various ways by simply using indifference arguments. Here we list some of them.

A Finite Number of Possible Crashes. In [Korn, Menkens 2005] we allow for more than one crash until the time horizon T. In such a situation of at most n crashes, a portfolio process is specified by an $(n+1)$-vector $\pi = (\pi_0, \ldots, \pi_n)$ where π_j is the portfolio process that will be used by the investor if still at most j crashes can occur. Then an optimal portfolio process $\hat{\pi}$ exists and is given as the solution of the following sequence of ordinary differential equations for $j = 1, \ldots, n$:

$$\begin{aligned} &\hat{\pi}_0(t) = \tfrac{b-r}{\sigma^2} \\ &\hat{\pi}_j{}'(t) = -\tfrac{\sigma^2}{2k^*}\left[1 - \hat{\pi}_j(t)k^*\right]\left[\hat{\pi}_j(t) - \hat{\pi}_{j-1}(t)\right]^2, \ \hat{\pi}_j(T) = 0. \end{aligned}$$

Note that each such differential equation has a unique solution that satisfies

$$0 \leq \hat{\pi}_j(t) \leq \min\left\{\hat{\pi}_{j-1}(t), \tfrac{1}{k^*}\right\} \text{ for } t \in [0,T], \ j = 1, \ldots, n.$$

Indeed, the arguments used to ensure the corresponding properties in the one-crash setting are valid here, too.

More General Utility Functions. If instead of the logarithmic utility function we choose a general utility function U, then [Korn, Menkens 2005] and [Korn 2005] contain verification results that are only valid under very restrictive assumptions. These assumptions are hard to verify, and they are by far not necessary conditions. However, by restricting to deterministic strategies it can be shown that similar differential equations as (8), (9) characterize the worst-case optimal *deterministic* portfolio process $\hat{\pi}$. In the case of the negative exponential utility function

$$U(x) = 1 - \mathrm{e}^{-\lambda x}, \ x \in \mathbb{R}, \text{ for some } \lambda > 0$$

there is even a completely explicit result. By assuming $r = 0$, $b > r$ and allowing for a possibly negative wealth it is shown in [Korn 2005] that we have:

Theorem 2 (Worst-Case Optimal Portfolio for Exponential Utility) *The optimal deterministic amount of money $A = \{A(t) : t \in [0,T]\}$ invested in the stock before the crash is given by*

$$A(t) = A^* - \frac{2}{\lambda\sigma^2}\frac{\lambda k^*}{T - t + 2k^*/b} \quad \textit{for } t \in [0,T]$$

while after the crash it is optimal to hold the constant amount of money

$$A^* = \frac{b}{\lambda\sigma^2}$$

in the stock.

Changing Market Conditions. As we have modeled the impact of a crash so far, its only consequence is a drop of the stock price. However, in real-world financial markets, the occurrence of a crash might have a more persistent effect. In [Korn 2005] and [Korn, Menkens 2005] this is modeled by a change of the market coefficients after the crash. In such a situation, one can still insist on being indifferent between the worst possible crash and the no-crash scenario. Such a point of view is taken in [Menkens 2006]. However, under certain relations between the market situations before and after the crash, it is shown in [Korn 2005] that one has to sacrifice indifference to obtain worst-case optimality.
In addition to the model considered so far, we assume that in the crash scenario (τ, k) *after* the crash the price dynamics are given by

$$\begin{aligned}
&\mathrm{d}P_0^1(t) = P_0^1(t) r_1 \mathrm{d}t,\ P_0^1(\tau) = P_0(\tau) \\
&\mathrm{d}P_1^1(t) = P_1^1(t) \left[b_1 \mathrm{d}t + \sigma_1 \mathrm{d}W(t) \right],\ P_1^1(\tau) = (1-k) P_1(\tau)
\end{aligned}$$

with constant market coefficients r_1, b_1, and $\sigma_1 \neq 0$. To illustrate the possible new effect, we look again at the situation of Theorem 2, i.e. at the negative exponential utility case with $r = r_1 = 0$ and the notation

$$A^* = \tfrac{b}{\lambda \sigma^2}\,, \quad A_1^* = \tfrac{b_1}{\lambda \sigma_1^2}\,.$$

Note that after a crash we are in the new market. Thus, if we compare the crash-free scenario with a crash scenario we always have to use the value function in the crash-free scenario of the new market. Further, if it is more attractive to invest in the stock in the new market than in the original market, the possible loss caused by a crash might be overcompensated by the better market conditions in the new market. It can therefore be optimal *not* to insist on indifference. This is the content of the following theorem from [Korn 2005]:

Theorem 3 *Under the assumptions of Theorem 2, $r = r_1 = 0$ and with the post-crash stock price dynamics given by*

$$\mathrm{d}P_1^1(t) = P_1^1(t) \left[b_1 \mathrm{d}t + \sigma_1 \mathrm{d}W(t) \right]$$

we have the following assertions:

a) For $A_1^* \leq A^*$ the results of Theorem 2 remain valid if we replace A^* by A_1^*.

b) For $A_1^* > A^*$ the optimal deterministic amount of money invested in the stock before the crash is given by

$$A(t) = \min \left\{ A^*, A_1^* - \frac{2k^*}{\lambda \sigma_1^2 (T-t) + 2k^*/A_1^*} \right\} \text{ for } t \in [0,T]. \tag{11}$$

The optimal amount of money invested in the stock after a crash equals A_1^*.

As part b) of the theorem shows, it can thus be better to invest optimally in the market before the crash than to insist on indifference. Following the (deterministic) indifference strategy before the crash would lead to a loss in terms of expected utility compared to A^* if no crash occurs. In the crash scenario, if there is still much time to the investment horizon T, equation (11) shows that the strategy A is below the indifference strategy and would thus also lead to a smaller loss.

3 HJB-Systems for Worst-Case Portfolio Optimization

The classical method to solve continuous-time portfolio problems is to apply the basic tool of continuous-time stochastic control theory, the Hamilton-Jacobi-Bellman equation (for short: HJB-equation). This approach has been introduced by Merton (see e.g. [Merton 1969], [Merton 1971]). Since then numerous papers have been written on this subject (see e.g. the monograph by [Korn 1997]).
The purpose of this section is to outline the approach of [Korn, Steffensen 2007] who derive a system of inequalities that can be regarded as an analog to the HJB-equation in the worst-case setting. The main achievement of the introduction of this **HJB-inequality system** is that one can prove that the optimal deterministic strategies derived in [Korn 2005] and [Korn, Menkens 2005] are indeed optimal among all admissible portfolio processes.

The conceptually new aspect of [Korn, Steffensen 2007] is the interpretation of the worst-case setting as a game between the market and the investor. While the market is "allowed" to choose a crash sequence, the investor chooses the portfolio process. The stock price dynamics are modeled by

$$\mathrm{d}P_1(t) = P_1(t-)\left[b\mathrm{d}t + \sigma\mathrm{d}W(t) - k^*\mathrm{d}N(t)\right], \; P_1(0) = p_1.$$

Here, $N = \{N(t) : t \in [0,T]\}$ is a process that counts the number of jumps such that

$$N(t) = \#\left\{0 < s \le t : P_1(s) \ne P_1(s-)\right\} \text{ for } t \in [0,T]$$

and k^* is the (maximal) crash height. For simplicity, we always assume that a crash of maximum size happens (for more on this, see [Korn, Steffensen 2007]). While in the indifference approach we simply ignored the modeling of jumps, we now assume that the market chooses a jump strategy N with a maximum number of jumps n and $N(t) - N(t-) \in \{0,1\}$. This strategy can also be characterized as a sequence of jump times $(\tau_1, \ldots, \tau_n)$. We denote by $\mathcal{B}(n)$ the class of crash scenarios with at most n jumps.
As before, we assume the portfolio process π to be adapted (now with respect to the filtration generated by the stock price and the counting process N, which models the investor's ability to know how many crashes can still occur!), and we suppose that portfolio processes take values in a subset A of $\mathbf{R}$. Further, we use the notation $\pi = (\pi_0, \ldots, \pi_n)$ where $\pi_j(t)$ denotes the part of the portfolio

process that the investor chooses if still at most j crashes can occur. To apply standard arguments from stochastic control theory and to avoid a negative wealth due to a crash, we also assume

$$\mathbb{E}\left[\int_0^T |\pi_j(s)|^m \mathrm{d}s\right] < \infty \text{ for } m = 1, 2, \dots \text{ and } \pi_j(t)k^* < -1,\ j = 1, \dots, n.$$

Then for a given "control" (π, N) the wealth process follows the dynamics

$$\begin{aligned} X^{(\pi,N)}(0) &= x, \\ \mathrm{d}X^{(\pi,N)}(t) &= X^{(\pi,N)}(t)\left[(r + \pi_j(t)(b - r))\,\mathrm{d}t + \sigma \mathrm{d}W(t)\right] \text{ on } (\!(\tau_{j-1}, \tau_j]\!] \\ X^{(\pi,N)}(\tau_j) &= (1 - \pi_j(\tau_j)k^*)\, X^{(\pi,N)}(\tau_j-),\ j = 1, \dots, n. \end{aligned}$$

We assume that the investor chooses a portfolio process to maximize worst-case expected utility of terminal wealth in the sense of the optimization problem

$$\sup_{\pi \in \mathcal{A}(x)} \inf_{N \in \mathcal{B}(n)} \mathbb{E}[U(X^{(\pi,N)}(T))].$$

For $\nu \in \mathrm{C}^{1,2}$ we define the differential operator $\mathcal{L}^\pi \nu$ by

$$\mathcal{L}^\pi \nu(t,x) \triangleq \nu_t(t,x) + \nu_x(t,x)(r + \pi(b - r))x + \tfrac{1}{2}\nu_{xx}(t,x)\pi^2\sigma^2 x^2$$

and for $n \in \mathbf{N}$ we define the **value function** $V^n(t,x)$ by

$$V^n(t,x) \triangleq \sup_{\pi \in \mathcal{A}(t,x)} \inf_{N \in \mathcal{B}(t,n)} \mathbb{E}^{t,x,n}[U(X^{(\pi,N)}(T))].$$

Here as above $\mathcal{A}(t,x)$ and $\mathcal{B}(t,n)$ denote, respectively, admissible strategies and possible crash sequences on $[t,T]$, given that the investor's wealth is x and n crashes are possible. With this notation we can now formulate the main result of this section.

Theorem 4 (Verification Theorem) *The worst-case optimization problem can be solved via the following recursive system of HJB-equations.*
STEP 0. *Assume that $\nu^0(t,x)$ is a polynomially bounded classical solution of*

$$0 = \sup_{\pi \in A}\left[\mathcal{L}^\pi \nu^0(t,x)\right],\ \nu^0(T,x) = U(x)$$

and that

$$p(t,x) \triangleq \arg\sup_{\pi \in A}\left[\mathcal{L}^\pi \nu^0(t,x)\right]$$

is an admissible control function. Then we have

$$V^0(t,x) = \nu^0(t,x)$$

and the optimal control function with no crash remaining exists and is given by

$$\pi_0^*(t) = p(t, X^{(\pi^*,N)}(t)).$$

Step n. *For* $n \in \mathbf{N}$ *and every function* $\nu^n \in \mathrm{C}^{1,2}$*, define* $A_n'(t,x)$ *and* $A_n''(t,x)$ *by*

$$\begin{aligned} A_n'(t,x) &\triangleq \left\{\pi \in A : \pi < \tfrac{1}{k^*},\ 0 \le \mathcal{L}^\pi \nu^n(t,x)\right\} \\ A_n''(t,x) &\triangleq \left\{\pi \in A : \pi < \tfrac{1}{k^*},\ 0 \le \nu^{n-1}\left(t,(1-\pi k^*)x\right) - \nu^n(t,x)\right\}. \end{aligned}$$

Assume that there exists a polynomially bounded $\mathrm{C}^{1,2}$*-solution of*

$$\begin{aligned} 0 &\le \sup_{\pi \in A_n''(t,x)} \left[\mathcal{L}^\pi \nu^n(t,x)\right] \\ 0 &\le \sup_{\pi \in A_n'(t,x)} \left[\nu^{n-1}\left(t,(1-\pi k^*)x\right) - \nu^n(t,x)\right] \\ 0 &= \sup_{\pi \in A_n''(t,x)} \left[\mathcal{L}^\pi \nu^n(t,x)\right] \sup_{\pi \in A_n'(t,x)} \left[\nu^{n-1}\left(t,(1-\pi k^*)x\right) - \nu^n(t,x)\right] \\ \nu^n(T,x) &= U(x) \end{aligned}$$

and that

$$p^n(t,x) \triangleq \operatorname*{arg\,sup}_{\pi \in A_n''(t,x)} \left[\mathcal{L}^\pi \nu^n(t,x)\right]$$

is an admissible control function. Then

$$V^n(t,x) = \nu^n(t,x)$$

and the optimal control function with n *crashes remaining exists and is given by*

$$\pi_n^*(t) = p^n(t, X^{(\pi^*,N)}(t)).$$

Moreover, with n *crashes remaining it is optimal for the market to intervene at the first time* t *when* $\pi_n^*(t)$ *is in* $A_n''(t, X^{(\pi^*,N)}(t))$*.*

Remark 2 (Form of the HJB-System) *The form of the HJB-system characterizing the value functions* $\nu^n(t,x)$ *needs explanation, as it differs in certain aspects from the HJB-equation or HJB-inequalities of related problems. For this, note first that if we looked at the portfolio problem where the jump process is a Poisson process with constant intensity* λ *and jump size* k^**, then the corresponding HJB-equation would read*

$$\begin{aligned} 0 &= \sup_{\pi \in A} \tilde{\mathcal{L}}^\pi \nu^0(t,x) \\ &= \sup_{\pi \in A} \left[\nu_t^0 + \tfrac{1}{2}\sigma^2\pi^2x^2\nu_{xx}^0 + \left(r + \pi(b-r)\right)x\nu_x^0 \right. \\ &\qquad \left. + \lambda\left(\nu^0\left(t,(1-\pi k^*)x\right) - \nu^0(t,x)\right)\right] \\ &= \sup_{\pi \in A} \left[\mathcal{L}^\pi \nu^0(t,x) + \lambda\left(\nu^0\left(t,(1-\pi k^*)x\right) - \nu^0(t,x)\right)\right]. \end{aligned} \tag{12}$$

As for a utility function of class C^2 *and* $b > r$ *the optimal portfolio process ought to be non-negative, we would expect from* (12) *that*

$$0 \le \sup_{\pi \in A} \left[\mathcal{L}^\pi \nu^0(t,x)\right] \tag{13}$$

which also motivates this requirement for $\mathcal{L}^\pi \nu^n$ in the verification theorem. This inequality also characterizes the set $A'_n(t,x)$. It further suggests that the investor should only search among those π that satisfy this inequality when she considers the optimal performance (with respect to the no-crash scenario). On the other hand, she should not give the market a chance to hit her more by a crash than necessary. Therefore, she ought to restrict π to those strategies that satisfy

$$\nu^n(t,x) \le \nu^{n-1}\left(t,(1-\pi k^*)x\right) \tag{14}$$

which is the requirement that characterizes the set $A''_n(t,x)$. The assumption that both inequalities (14) *and* (13) *are strict would intuitively contradict the idea of ν^n being a value function, as it would not be in line with the form of the HJB-equation* (12). *This motivates the presence of the complementarity condition that (at least) one of the two inequalities always has to be satisfied with equality.*

Finally, let us add some remarks on how to solve the HJB-inequality system. As the HJB-approach is a verification technique, one needs to start with a reasonable idea of the structure of the solution. This is in particular true for the sets $A'_n(t,x)$ and $A''_n(t,x)$, which are defined in terms of the solution to the HJB-inequality system. [Korn, Steffensen 2007] provide a heuristic argument to motivate considering the set

$$\mathcal{N}^n \triangleq \Big\{(t,x) : V_t^n(t,x) = -V_x^n(t,x)(r+\pi(b-r))x - \tfrac{1}{2}V_{xx}^n(t,x)\pi^2\sigma^2x^2, \\ \text{where } \pi = -\frac{V_x^n(t,x)}{V_{xx}^n(t,x)x}\frac{b-r}{\sigma^2}\Big\}$$

and its complement, which is characterized by the requirement that π satisfies $V^n(t,x) = V^{n-1}(t,(1-\pi k^*)x)$. Intuitively, $\mathcal{N}^n$ is the region where Merton behavior is optimal, while outside $\mathcal{N}^n$ indifference is optimal. If, as in [Korn, Steffensen 2007], pre- and post-crash coefficients coincide, then $\mathcal{N}^n$ is typically empty for $n \ge 1$, and the HJB-system can be solved with a separation ansatz. By contrast, if neither $\mathcal{N}$ nor its complement are empty, the corresponding boundary is in general difficult to determine.

4 A Martingale Approach to Worst-Case Portfolio Optimization

In contrast to the dynamic programming approach, the martingale approach to the worst-case portfolio problem is based on martingale optimality arguments; it also models the market as an opponent to the investor. In the following we briefly outline its main components: the Change-of-Measure Device, the Indifference-Optimality Principle, and the notion of an Indifference Frontier.

4.1 The Change-of-Measure Device

We consider the worst-case portfolio problem (P) and assume that

$$U(x) = \tfrac{1}{\gamma}x^{\gamma},\ x > 0,\ \text{with } \gamma < 1,\ \gamma \neq 0.$$

Moreover, suppose that if a crash occurs, it has maximum size k^*. We let Θ denote the class of $[0,T] \cup \{\infty\}$-valued stopping times and interpret the event $\{\tau = \infty\}$ as there being no crash at all. We recall that admissible strategies are assumed to be bounded and continuous before the crash. Then we may equivalently reformulate the worst-case portfolio problem (P) as the problem to optimally choose a *pre-crash* strategy so as to obtain

$$\sup_{\pi \in \mathcal{A}(x)} \inf_{\tau \in \Theta} \mathbb{E}\left[\nu^0\left(\tau, (1 - \pi(\tau)k^*)X^{\pi}(\tau)\right)\right] \tag{$\mathrm{P_{pre}}$}$$

where as above ν^0 denotes the value function of the post-crash optimization problem, which is known explicitly:

$$\nu^0(t,x) = \tfrac{1}{\gamma}x^{\gamma} \exp\left\{[\gamma r + \tfrac{1}{2}(\tfrac{b-r}{\sigma})^2 \tfrac{\gamma}{1-\gamma}](T-t)\right\}. \tag{15}$$

This is intuitively completely obvious because no further crash can occur, and can be shown formally with the following trick:

Theorem 5 (Change-of-Measure Device) *Consider the classical optimal portfolio problem with random initial time τ and time-τ initial wealth ξ*

$$\sup_{\pi \in \mathcal{A}(\tau,\xi)} \mathbb{E}^{\tau,\xi}[U(\tilde{X}^{\pi}(T))] \tag{$\mathrm{P_{post}}$}$$

where τ is a stopping time and $\mathcal{A}(\tau,\xi)$ denotes the corresponding class of admissible strategies on $[\tau, T]$. Then for any $\pi \in \mathcal{A}(\tau,\xi)$ we can write

$$U(\tilde{X}^{\pi}(T)) = U(\xi) \exp\left\{\gamma \textstyle\int_{\tau}^{T} \Phi(\pi(s))\mathrm{d}s\right\} M_{\pi}(T) \tag{16}$$

with a martingale $M_{\pi} = \{M_{\pi}(t) : t \in [0,T]\}$ satisfying $M_{\pi}(\tau) = 1$ and

$$\Phi(y) \triangleq r + (b-r)y - \tfrac{1}{2}(1-\gamma)\sigma^2 y^2.$$

Thus the optimal solution to problem $(\mathrm{P_{post}})$ *is given by* $\pi^* = \frac{b-r}{(1-\gamma)\sigma^2}$.

Proof The first part is a consequence of Itô's formula and Novikov's condition, making use of the boundedness assumption on π. To establish the second note that clearly π^* maximizes Φ. Hence, if $\pi \in \mathcal{A}(\tau,\xi)$ is an arbitrary strategy, we have from (16) and the martingale property of M_{π}

$$\begin{aligned}
\mathbb{E}^{\tau,\xi}[U(\tilde{X}^{\pi}(T))] &= \mathbb{E}^{\tau,\xi}\left[U(\xi)\exp\left\{\gamma \textstyle\int_{\tau}^{T} \Phi(\pi(s))\mathrm{d}s\right\} M_{\pi}(T)\right] \\
&\leq \mathbb{E}^{\tau,\xi}\left[U(\xi)\exp\left\{\gamma \textstyle\int_{\tau}^{T} \Phi(\pi^*)\mathrm{d}s\right\} M_{\pi}(T)\right] \\
&= \mathbb{E}^{\tau,\xi}\left[U(\xi)\exp\left\{\gamma \textstyle\int_{\tau}^{T} \Phi(\pi^*)\mathrm{d}s\right\} M_{\pi^*}(T)\right] \\
&= \mathbb{E}^{\tau,\xi}[U(\tilde{X}^{\pi^*}(T))]
\end{aligned}$$

so π^* is optimal. □

The Change-of-Measure Device allows to transform the stochastic optimization problem to a *pathwise* maximization, quite similar to the log-case. Note that changing market coefficients are subsumed by the above framework, and that Theorem 5 also adapts immediately to situations with deterministic trading constraints.

4.2 Abstract Indifference Strategies

The form of ($\mathrm{P}_{\mathrm{pre}}$) suggests a reformulation of the worst-case portfolio problem as a zero-sum stochastic game; this is the motivation for the martingale approach. Let us consider an abstract controller-and-stopper game played between two players A (the controller) and B (the stopper). Player A controls a stochastic process

$$W = W^{\lambda} = \{W^{\lambda}(t) : t \in [0,T]\}$$

by choosing λ from a given class of admissible controls Λ, and player B decides on the duration of the game by choosing a stopping time $\tau \in \Theta$. The controller and stopper aim to maximize or minimize, respectively, the expectation

$$\mathbb{E}[W^{\lambda}(\tau)].$$

Assuming that player A has to choose her strategy first, she faces the problem to obtain

$$\sup_{\lambda \in \Lambda} \inf_{\tau \in \Theta} \mathbb{E}[W^{\lambda}(\tau)]. \qquad (\mathrm{P}_{\mathrm{abstract}})$$

Now if player A can choose her strategy $\hat{\lambda} \in \Lambda$ in such a way that $W^{\hat{\lambda}}$ is a martingale, then player B's actions become irrelevant to her because by optional stopping

$$\mathbb{E}[W^{\hat{\lambda}}(\sigma)] = \mathbb{E}[W^{\hat{\lambda}}(\tau)] \text{ for all stopping times } \sigma, \tau.$$

Thus it makes sense to call such a strategy $\hat{\lambda}$ an **indifference strategy**. The crucial benefit of indifference strategies is formulated in

Proposition 1 (Indifference-Optimality Principle) *If $\hat{\lambda}$ is an indifference strategy, and for all $\lambda \in \Lambda$ there exists a single $\tau \in \Theta$ such that $\mathbb{E}[W^{\hat{\lambda}}(\tau)] \geq \mathbb{E}[W^{\lambda}(\tau)]$, then $\hat{\lambda}$ is optimal for player A in* ($\mathrm{P}_{\mathrm{abstract}}$).

4.3 Optimality and the Indifference Frontier

In the framework of the previous section, observe that if we call player A the investor and player B the market, then setting $\Lambda \triangleq \mathcal{A}(x)$ and

$$W^{\pi}(t) \triangleq \nu^0(t, (1-\pi(t)k^*)X^{\pi}(t)) \text{ for } t \in [0,T] \text{ and } W^{\pi}(\infty) \triangleq \nu^0(T, X^{\pi}(T))$$

we obtain the worst-case portfolio problem ($\mathrm{P}_{\mathrm{pre}}$). Note also that the seemingly obvious terminal condition (9) is in fact a consequence of the martingale property of $W^{\hat{\pi}}$ between T and ∞. To construct an indifference strategy $\hat{\pi}$, one goes through the same calculation as in the first part of the proof of Theorem 1 to obtain the ordinary differential equation

$$\hat{\pi}'(t) = -\tfrac{\sigma^2}{2k^*}(1-\gamma)[1-\hat{\pi}(t)k^*]\left[\hat{\pi}(t)-\pi^*\right]^2, \ \hat{\pi}(T)=0 \tag{17}$$

for $\hat{\pi}$, making use of the explicit form (15) of ν^0. Here and in the following, we assume for simplicity that market coefficients do not change after a crash; in particular one sees as in Remark 1 that $0 \le \hat{\pi}(t) \le \min\{\pi^*, \frac{1}{k^*}\}$ for all $t \in [0,T]$.

Lemma 1 (Indifference Frontier) *Let $\pi \in \mathcal{A}(x)$ be an admissible strategy, let $\hat{\pi}$ be determined by equation* (17)*, set $\sigma \triangleq \inf\{t : \pi(t) > \hat{\pi}(t)\}$ and define*

$$\tilde{\pi}(t) \triangleq \pi(t) \ \textit{if}\ t < \sigma \ \textit{and}\ \tilde{\pi}(t) \triangleq \hat{\pi}(t) \ \textit{if}\ t \ge \sigma.$$

Then $\tilde{\pi} \in \mathcal{A}(x)$ and the worst-case bound attained by $\tilde{\pi}$ is at least as big as that achieved by π.

Proof Let τ be an arbitrary stopping time. By continuity we have $\tilde{\pi}(t) = \hat{\pi}(t)$ if $0 \le t \le \sigma$, and since $\hat{\pi}$ is an indifference strategy the process $W^{\tilde{\pi}}$ is a martingale on $[\sigma, T] \cup \{\infty\}$. Thus we obtain

$$\mathbb{E}[W^{\tilde{\pi}}(\tau)] = \mathbb{E}[W^{\tilde{\pi}}(\tau \wedge \sigma)] = \mathbb{E}[W^{\pi}(\tau \wedge \sigma)] \ge \inf_{\tau' \in \Theta} \mathbb{E}[W^{\pi}(\tau')].$$

Since τ is arbitrary, the conclusion follows. $\square$

Remark 3 *Lemma 1 implies that it suffices to search for optimal strategies which are dominated by the indifference strategy. Hence $\hat{\pi}$ represents a frontier which rules out too optimistic investment, i.e. a too great exposure to the risk of a crash.*

Now it is not hard to see that the strategy $\hat{\pi}$ is worst-case optimal. Indeed, by the Change-of-Measure Device (and the fact that Φ is a quadratic function) the indifference strategy yields an optimal performance for the no-crash scenario in the class of all strategies that remain below the Indifference Frontier. Hence, optimality follows from the Indifference-Optimality Principle.

Theorem 6 (Solution of the Worst-Case Portfolio Problem) *The optimal strategy in the pre-crash market for the worst-case portfolio problem* (P) *is given by the indifference strategy $\hat{\pi}$ determined from* (17)*. After the crash, the Merton strategy $\pi^* = \frac{b-r}{(1-\gamma)\sigma^2}$ is optimal.*

The indifference strategy has been verified to be optimal in [Korn, Steffensen 2007] by means of the dynamic programming methods presented in the previous section. The martingale approach provides a simpler and more direct way to analyze the problem, as it focuses directly on the crucial notion of indifference.

4.4 Extensions

The approach outlined above applies to more general settings than that considered here. For instance we can consider general Lévy-driven asset price models, we can remove the continuity assumption imposed on admissible trading strategies, we can allow for changing market coefficients, and we can consider multiple crashes. Although this complicates the formal analysis, the concepts developed above remain valid and provide the key to solve the worst-case optimal portfolio problem. For a detailed exposition of the martingale approach to worst-case portfolio problems in a more general framework, we refer to [Seifried 2009b] and the next chapter of this thesis.

5 Conclusion and Further Aspects

The worst-case approach to continuous-time portfolio optimization represents, on the one hand, a generalization of the classical Merton setting, and on the other hand, an alternative to technically involved frameworks such as the Lévy process setting. Its main strength lies in the fact that for standard utility functions we can derive fully explicit optimal portfolio strategies. Their specific form is appealing, in particular the reduction of risky investments when the time horizon gets near while there is still crash risk. Of course, the strategies depend heavily on the assumed upper bound k^* for the jump height and on the maximum number of jumps n.
We believe that there is a lot of potential in the worst-case approach from both the scientific and the application-oriented perspective.

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Optimal Investment for Worst-Case Crash Scenarios

A Martingale Approach

Abstract We investigate the optimal portfolio problem under the threat of a financial market crash in a multi-dimensional jump-diffusion framework. We set up a non-probabilistic crash model and consider an investor that seeks to maximize CRRA utility in the worst possible crash scenario. We recast the problem as a stochastic differential game; with the help of the fundamental notion of indifference strategies, we completely solve the portfolio problem using martingale arguments.

Keywords optimal investment · worst-case scenario · market crash · indifference strategy · controller-vs-stopper game

Mathematics Subject Classification (2000) 91B28 · 91A15 · 93E20

1 Introduction and Overview

Financial Market Crashes and Knightian Uncertainty. One of the inherent hazards of investing in financial markets is the risk of a disastrous event, i.e. a sudden and sharp decrease in asset prices, possibly affecting future investment opportunities. In this paper, we study the optimal investment problem in a financial market that, in addition to ordinary jumps in asset prices, is threatened by the possibility of such a catastrophic decline. We argue that a market crash represents an uncertainty which is fundamentally different from the risk of stock price movements. In order to clarify this, we follow F. Knight and distinguish between two notions of 'risk'. In his seminal work [Knight 1921], he provides the following definition:

> *Uncertainty* must be taken in a sense radically distinct from the familiar notion of *Risk*, from which it has never been properly separated. The term "risk," as loosely used in everyday speech and in economic discussion, really covers two things which [...] are categorically different.

> [...] The essential fact is that "risk" means in some cases a quantity susceptible of measurement, while at other times it is something distinctly not of this character [...]. It will appear that a measurable uncertainty, or "risk" proper [...], is so far different from an unmeasurable one that it is not in effect an uncertainty at all.

Knight explicitly acknowledges the fact that both risk and uncertainty may be present in a given situation.[1] In particular, while financial market crashes or economic crises are largely unique events, there is ample time series data on 'regular' fluctuation of asset prices. Hence, the latter can be regarded as subject to risk rather than uncertainty. In this paper, we will therefore take the point of view that from the perspective of a "small investor", market crashes are subject to (Knightian, 'true') uncertainty, while ordinary price fluctuations are a matter of (quantifiable, stochastic, probabilistic) risk.
The (non-stochastic) uncertainty inherent in financial market crashes may arise from several sources: A large economic or political crisis can cause asset prices to drop unexpectedly and sharply; natural disasters may also lead to a market collapse. Financial assets issued by parties with limited liability may partly default due to unknown exogenous reasons. Irrational exuberance can prompt a typical crash scenario: An investor facing a bubble market does not have access to appropriate statistical data, and although she may not be willing to discard the investment opportunity a priori, she is well advised to be very cautious about a possible burst of the bubble. Historical examples of such crash scenarios abound, including the terrorist attacks of 9/11 and the recent Credit Meltdown. It is therefore clear that financial market crashes are of crucial importance for asset allocation, in particular to long-term investors.
Finally, we wish to point out that the worst-case approach to optimal investment rationalizes a characteristic trait of real-world investor behavior: We show that under the threat of a crash, the optimal portfolio strategy reallocates wealth from risky to riskless assets towards the end of the time horizon. The Merton strategy appears as the limiting solution for large time horizons, provided its crash exposure is below 100%. Moreover, the worst-case approach can contribute to the understanding of the equity premium puzzle.

Related Literature. The worst-case approach to market crashes was introduced by [Hua, Wilmott 1997] in the context of option pricing for a discrete-time framework. [Korn, Wilmott 2002] study the associated optimal investment problem, and their analysis is extended in [Korn, Menkens 2005] and [Korn, Steffensen 2007]. We refer to [Korn, Seifried 2009] and the previous chapter of this thesis for an overview.
An alternative worst-case approach to portfolio optimization assumes that market coefficients or the probability measure governing asset prices are subject to uncertainty. Thus [Talay, Zheng 2002] explicitly model the market as an opponent to the investor that adversely chooses drift and volatility parameters, and the strand of literature originating with [Schied 2005] studies

[1] Moreover, uncertainty can turn into risk if sufficient data have accumulated.

optimal investment under ambiguity for robust preference specifications; we refer to [Schied 2008] and the references therein. [Riedel 2009] investigates the optimal stopping problem in such multiple-prior models.
Finally, [Liu, Longstaff, Pan 2003] and [Das, Uppal 2004] use a probabilistic model of systemic event risk to study crash-related jumps in prices and volatilities and international portfolio choice, respectively. We refer to the monograph [Cont, Tankov 2004] for a comprehensive account of stochastic modeling for asset prices with jumps. [Barro 2006] investigates the macroeconomic impact of disastrous events on asset pricing in a general equilibrium framework and shows that the inclusion of major market crashes can explain many asset pricing puzzles.

Outline. This paper extends the existing literature on worst-case optimal investment for crash scenarios to multi-asset frameworks, discontinuous price dynamics, and arbitrary crash heights.[2] Moreover, it presents a systematic alternative approach based on martingale optimality arguments rather than dynamic programming. Our analysis is based on three key concepts: The Change-of-Measure Device, the Indifference-Optimality Principle, and the Indifference Frontier.
This paper is organized as follows: In Section 2, we set up the mathematical framework and formulate the optimal investment problem for worst-case crash scenarios. Section 3 discusses the Change-of-Measure Device. Section 4 introduces indifference strategies, both in abstract settings and for crash scenarios. In particular, the Indifference Frontier is shown to provide a natural barrier for the exposure to the threat of a crash. The investment problem is then solved completely in Section 5, and Section 6 provides illustrations of our results. Finally, Section 7 contains extensions of our basic model, and Section 8 concludes.

2 Portfolio Optimization in Worst-Case Scenarios

In this section, we construct a mathematical model of a financial market that is threatened by a major catastrophic event.

Mathematical Framework. We fix a time horizon $T > 0$ and write $\mathfrak{T} \triangleq [0, T]$. All random quantities to be considered in the sequel are defined on the probability space $(\Omega, \mathfrak{F}, \mathbb{P})$, which is endowed with a filtration $\{\mathfrak{F}_t\}$ satisfying the usual conditions of right-continuity and completeness. Stochastic processes have time horizon $\mathfrak{T}$, unless otherwise stated; for technical reasons we let $\bar{\mathfrak{T}} \triangleq \mathfrak{T} \cup \{\infty\}$ and extend $\{\mathfrak{F}_t\}$ by setting $\mathfrak{F}_\infty \triangleq \mathfrak{F}_T$. By default, a stochastic process $\{X_t\}$ on $\mathfrak{T}$ is extended to $\bar{\mathfrak{T}}$ by letting $X_\infty \triangleq X_T$. We use càdlàg versions of semimartingales and omit a.s. qualifiers throughout.

[2] In particular, it resolves the open problems posed in [Korn, Steffensen 2007].

Crash Scenario. The financial market we consider consists of $n+1$ assets: The first asset is a money market account, and the remaining n assets are risky and vulnerable to market crashes. Following [Korn, Wilmott 2002], we propose a non-probabilistic model of crashes: A financial market **crash scenario** is defined as a pair (τ, ℓ), where the $\bar{\mathfrak{T}}$-valued $\{\mathfrak{F}_t\}$-stopping time τ describes the time when the crash occurs, and the $\mathfrak{F}_\tau$-measurable $[0, \ell^\infty]$-valued[3] random variable ℓ is the vector of relative crash sizes of the n risky assets. Here, the event $\{\tau = \infty\}$ is interpreted as there being no crash at all, and the event $\{\ell = 0\}$ corresponds to a credible all-clear announcement; the vector $\ell^\infty \in [0,1)^n$ of maximum crash heights is known a priori. We denote by Θ the class of all such crash scenarios (τ, ℓ).

Asset Price Dynamics. We specify two models for the temporal evolution of asset prices, one valid before and the other after a possible crash time τ. Before the crash, the locally riskless **money market account** $B = \{B_t\}$ satisfies

$$\mathrm{d}B_t = r_t B_t \mathrm{d}t$$

for some deterministic interest rate $r = \{r_t\}$, and the n-dimensional vector of **risky assets** $P = \{P_t\}$ is driven by a Brownian motion and a compound Poisson process. Formally, P satisfies

$$\mathrm{d}P_t = \mathrm{diag}(P_{t-}) \left[(r_t \underline{1} + \eta_t)\mathrm{d}t + \sigma_t.\mathrm{d}W_t - \textstyle\int_{\mathbb{R}^n} \ell \nu(\mathrm{d}t, \mathrm{d}\ell) \right],$$

where $W = \{W_t\}$ is a standard $\mathbb{R}^m$-dimensional Wiener process; ν is a Poisson random measure with finite intensity ϑ supported in $[0, \ell^{\max}]$ for some $\ell^{\max} \in [0,1)^n$ with[4]

$$\ell^{\max,i} < \ell^{\infty,i} \text{ for } i = 1, \ldots, n;$$

the deterministic functions $\eta = \{\eta_t\}$ and $\sigma = \{\sigma_t\}$ are continuous and take values in $\mathbb{R}^n$ and $\mathbb{R}^{n\times m}$, respectively; and $\sigma_t \sigma_t^\dagger$ is positive definite for $t \in \mathfrak{T}$. We denote by $\tilde{\nu}$ the compensated random measure associated to ν. In the regime after the crash an analogous model is valid, whose quantities we denote by $\bar{B}$, $\bar{P}$, $\bar{r}$, $\bar{W}$, etc.; as a general rule, we denote by $\bar{\aleph}$ the post-crash quantity corresponding to $\aleph$.

Portfolio Strategies. The investor is endowed with an initial capital $x > 0$; she can observe a possible market crash and react accordingly. Thus she chooses a **portfolio strategy** $\pi = \{\pi_t\}_{t\in\mathfrak{T}}$ to be applied before the crash, and a family $\bar{\pi} = \{\bar{\pi}^{\pi,\tau,\ell}\}_{\pi\in\Pi,\,(\tau,\ell)\in\Theta}$ of contingent portfolio strategies $\bar{\pi}^{\pi,\tau,\ell} = \{\bar{\pi}_t^{\pi,\tau,\ell}\}_{t\in[\tau,T]}$ to be applied afterwards if the realized crash scenario is (τ, ℓ) and $\pi \in \Pi$ was used before the crash. Although this seems to give the investor access to too much information on the crash,[5] we shall see that she does not

[3] For $a, b \in \mathbb{R}^n$, we set $[a, b] \triangleq [a^1, b^1] \times \cdots \times [a^n, b^n] \subseteq \mathbb{R}^n$.

[4] Thus the crash may be more grave than a 'regular' price jump for each asset.

[5] She cannot only observe the crash itself, i.e. $\tau(\omega_0)$ and $\ell(\omega_0)$ for the 'true' $\omega_0 \in \Omega$, but also learns when and how it would have occurred under different circumstances, i.e. $\tau(\omega)$ and $\ell(\omega)$ for *every* $\omega \in \Omega$.

take it into account. Technically we require both π and $\bar{\pi}^{\pi,\tau,\ell}$ for $\pi \in \Pi$, $(\tau,\ell) \in \Theta$ to be $\{\mathfrak{F}_t\}$-predictable processes with values in $\mathbb{R}^{1\times n}$.
We interpret π_t and $\bar{\pi}_t^{\pi,\tau,\ell}$ as the vector of fractions of wealth held in risky assets at time t, given the crash (τ,ℓ) has not occurred as yet or that it has occurred, respectively. Since we suppose that the regime change is associated with price jumps in risky assets of relative size ℓ, it follows that the investor's wealth $X = \{X_t\}$ satisfies

$$\mathrm{d}X_t = (1-\pi_t.\underline{1})X_{t-}\frac{\mathrm{d}B_t}{B_t} + \pi_t X_{t-}.\frac{\mathrm{d}P_t}{P_{t-}} \text{ on } [\![0,\tau)\!),\ X_0 = x,$$

$$\mathrm{d}X_t = (1-\bar{\pi}_t^{\pi,\tau,\ell}.\underline{1})X_{t-}\frac{\mathrm{d}\bar{B}_t}{\bar{B}_t} + \bar{\pi}_t^{\pi,\tau,\ell} X_{t-}.\frac{\mathrm{d}\bar{P}_t}{\bar{P}_{t-}} \text{ on } [\![\tau,T]\!],\ X_\tau = (1-\pi_\tau.\ell)X_{\tau-}.$$

Observe that the portfolio π is valid during the crash, and the strategy $\bar{\pi}$ is only applied afterwards. The relative wealth loss equals $\pi_\tau.\ell$, and hence bankruptcy can occur if $\pi_t.\ell^\infty \geq 1$ for some $t \in \mathfrak{T}$. Moreover we suppose that shorting of risky assets is prohibited.
Thus we say that a pre-crash strategy π is **admissible** if it takes values in

$$\Sigma^\infty \triangleq \{\pi \in [0,\infty)^n \ :\ \pi.\ell^\infty < 1\},$$

and we denote by Π the set of admissible pre-crash strategies. We consider admissible any post-crash strategy $\{\bar{\pi}^{\pi,\tau,\ell}\}_{\pi\in\Pi,\,(\tau,\ell)\in\Theta}$ with values in

$$\bar{\Sigma}^{\max} \triangleq \{\bar{\pi} \in [0,\infty)^n \ :\ \bar{\pi}.\bar{\ell}^{\max} < 1\}$$

such that for each $\pi \in \Pi$ and $(\tau,\ell) \in \Theta$

$$\mathbb{E}[\sup\nolimits_{t\in[\![\tau,T]\!]} u(X_t^{\pi,\bar{\pi},\tau,\ell})^-] < \infty, \tag{1}$$

and we write $\bar{\Pi}$ for the collection of admissible contingent portfolio strategies in the regime after the crash. Here, u is the investor's utility function and $X^{\pi,\bar{\pi},\tau,\ell}$ is her wealth process, which are specified below.

Remark 1 *If we set*

$$\Sigma^{\max} \triangleq \{\pi \in [0,\infty)^n \ :\ \pi.\ell^{\max} < 1\},$$

then it is clear that $\Sigma^\infty \subseteq \Sigma^{\max}$. In fact, as $\ell^{\max,i} < \ell^{\infty,i}$ for $i = 1,\ldots,n$, there exists some $\delta > 0$ such that $\pi.\ell^{\max} \leq 1-\delta$ for all $\pi \in \Sigma^\infty$, i.e. $\pi.\ell^{\max}$ is bounded away from 1 for $\pi \in \Sigma^\infty$. In particular, the analog of (1) in the pre-crash market is trivially satisfied.

We write $X^{\pi,\bar{\pi},\tau,\ell}$ for the **wealth process** associated to admissible portfolio strategies $\pi \in \Pi$ and $\bar{\pi} \in \bar{\Pi}$ and a crash scenario (τ,ℓ), i.e. the uniquely determined solution $X = \{X_t\}$ to

$$\mathrm{d}X_t = X_{t-}\left[(r_t + \pi_t.\eta_t)\mathrm{d}t + \pi_t.\sigma_t.\mathrm{d}W_t - \textstyle\int_{\mathbb{R}^n}\pi_t.\ell\nu(\mathrm{d}t,\mathrm{d}\ell)\right] \text{ on } [\![0,\tau)\!),\ X_0 = x$$

$$\begin{aligned}\mathrm{d}X_t &= X_{t-}\left[(\bar{r}_t + \bar{\pi}_t^{\pi,\tau,\ell}.\bar{\eta}_t)\mathrm{d}t + \bar{\pi}_t^{\pi,\tau,\ell}.\bar{\sigma}_t.\mathrm{d}\bar{W}_t - \textstyle\int_{\mathbb{R}^n}\bar{\pi}_t^{\pi,\tau,\ell}.\ell\bar{\nu}(\mathrm{d}t,\mathrm{d}\ell)\right] \text{ on } [\![\tau,T]\!],\\ X_\tau &= (1-\pi_\tau.\ell)X_{\tau-}.\end{aligned}$$

Worst-Case Optimization Problem. We assume that the investor's attitude towards risk is captured by a CRRA utility function $u : (0,\infty) \to \mathbb{R}$ with relative risk aversion $1-\rho$, i.e.

$$u(x) = \tfrac{1}{\rho}x^{\rho},\ x \in (0,\infty),\ \text{for some}^{6}\ \rho < 1,\ \rho \neq 0.$$

By contrast, she takes a worst-case point of view towards the threat of a crash. Thus the investor faces the **worst-case optimal investment problem** to obtain

$$\sup_{\pi\in\Pi,\,\bar\pi\in\bar\Pi}\ \inf_{(\tau,\ell)\in\Theta}\ \mathbb{E}[u(X_T^{\pi,\bar\pi,\tau,\ell})]. \tag{P}$$

With $\pi^M = \{\pi_t^M\}$ denoting the classical optimal strategy for the pre-crash market in the absence of crash risk, we assume throughout that

$$\pi_t^M.\ell^\infty > 0 \text{ for } t \in \mathcal{T}. \tag{H$_{\text{pre}}$}$$

This means that the crash is disadvantageous to investors that decide to ignore it, and therefore a priori the crash is perceived as a threat.

Interpretation and Applications. Problem (P) reflects an extraordinarily cautious attitude to the uncertainty concerning the crash. In particular, the investor is not able or willing to assign numerical probabilities to the disastrous event. As explained in Section 1, there are various situations when such an extreme attitude is justified: economic, political, and natural crises, bubble markets, defaults of securities with fractional recovery, and many others. More generally, focusing on the worst-case scenario is a reasonable, albeit conservative, approach to major disasters for which reliable statistical models are not available.

3 The Change-of-Measure Device

The Change-of-Measure Device is established in this section. This directly leads to the identification of the optimal post-crash strategy. Moreover, the Change-of-Measure Device is crucial for the indifference and optimality arguments in Sections 4 and 5.

Merton's Problem with Random Initial Time. We investigate the Merton problem, see [Merton 1969] and [Merton 1971], with random initial time τ, initial wealth X_τ and terminal time T. Note that this is an optimal investment problem with random duration. However it is the *initial* rather than the terminal time which is random, which renders it fairly simple.
Formally, in the setting of Section 2, we consider the situation when $\pi \in \Pi$ and $(\tau,\ell) \in \Theta$ are fixed and only $\bar\pi \in \bar\Pi$ is at the investor's discretion. Since the only relevant member of the family $\bar\pi = \{\bar\pi^{\pi',\tau',\ell'}\}_{\pi'\in\Pi,\,(\tau',\ell')\in\Theta}$ is $\bar\pi^{\pi,\tau,\ell}$, we slightly abuse notation and identify $\bar\pi$ and $\bar\pi^{\pi,\tau,\ell}$ in what follows. Moreover

for any $t \in \mathfrak{T}$ let the closed set $K_t \subseteq \bar{\Sigma}^{\max}$ describe a trading constraint and put

$$\bar{\Pi}_0 \triangleq \{\bar{\pi} \in \bar{\Pi} \, : \, \bar{\pi}_t \in K_t \text{ for all } t \in [\tau, T]\}.$$

Then the investor tries to solve the **post-crash problem**

$$\sup_{\bar{\pi} \in \bar{\Pi}_0} \mathbb{E}[u(X_T^{\pi,\bar{\pi},\tau,\ell})], \tag{P$_{\text{post}}$}$$

where the wealth process $X^{\pi,\bar{\pi},\tau,\ell} = X$ satisfies

$$\begin{aligned} \mathrm{d}X_t &= X_{t-}\left[(\bar{r}_t + \bar{\pi}_t.\bar{\eta}_t)\mathrm{d}t + \bar{\pi}_t.\bar{\sigma}_t.\mathrm{d}\bar{W}_t - \textstyle\int_{\mathbb{R}^n} \bar{\pi}_t.\ell\bar{\nu}(\mathrm{d}t, \mathrm{d}\ell)\right], \\ X_\tau &= (1 - \pi_\tau.\ell)X_{\tau-}; \end{aligned}$$

for brevity we write $X^{\bar{\pi}} \triangleq X^{\pi,\bar{\pi},\tau,\ell}$.

Lemma 1 (Moments of Stochastic Exponentials) *If $F = \{F_t\}$, $G = \{G_t\}$, and $H = \{H_t(\,\cdot\,)\}$ are bounded predictable processes*[7] *with values in $\mathbb{R}$, $\mathbb{R}^{1\times m}$ and $(-1, \infty)$, respectively, we have*

$$\mathbb{E}\left[\sup_{t\in\mathfrak{T}} \mathcal{E}_t\left(\textstyle\int_0^{\cdot} F_s \mathrm{d}s + \int_0^{\cdot} G_s.\mathrm{d}W_s + \int_{(0,\,\cdot\,]\times\mathbb{R}^n} H_s(\ell)\nu(\mathrm{d}s, \mathrm{d}\ell)\right)\right] < \infty.$$

In particular, the exponential process $\mathcal{E}(\int_0^{\cdot} G_s.\mathrm{d}W_s + \int_{(0,\,\cdot\,]\times\mathbb{R}^n} H_s(\ell)\tilde{\nu}(\mathrm{d}s, \mathrm{d}\ell))$ is a martingale.

Proof Writing $Z \triangleq \mathcal{E}(\int_0^{\cdot} F_s \mathrm{d}s + \int_0^{\cdot} G_s.\mathrm{d}W_s + \int_{(0,\,\cdot\,]\times\mathbb{R}^n} H_s(\ell)\nu(\mathrm{d}s, \mathrm{d}\ell))$ we have

$$Z_t = \exp\left\{\textstyle\int_0^t F_s \mathrm{d}s\right\} \mathcal{E}_t\left(\textstyle\int_0^{\cdot} G_s.\mathrm{d}W_s\right) \prod_{s\in(0,t],\ \Delta Y_s \neq 0} (1 + H_s(\Delta Y_s)) \text{ for } t \in \mathfrak{T},$$

where $Y = \{Y_t\}$ is given by $Y_t \triangleq \int_{(0,t]\times\mathbb{R}^n} \ell\nu(\mathrm{d}s, \mathrm{d}\ell)$, $t \in \mathfrak{T}$, and $\mathcal{E}(\int_0^{\cdot} G_s.\mathrm{d}W_s)$ is an L^2-bounded martingale by Novikov's condition. Then the counting process $N = \{N_t\}$ associated to Y, $N_t \triangleq \sharp\{s \in (0, t] \, : \, \Delta Y_s \neq 0\}$ for $t \in \mathfrak{T}$, is a Poisson process with intensity $\vartheta(\mathbb{R}^n)$, and thus Cauchy's and Doob's inequality yield

$$\begin{aligned} \mathbb{E}\left[\sup_{t\in\mathfrak{T}} Z_t\right] &\leq \mathrm{e}^{\kappa T}\mathbb{E}\left[\sup_{t\in\mathfrak{T}} \mathcal{E}_t\left(\textstyle\int_0^{\cdot} G_s.\mathrm{d}W_s\right)^2\right]^{\frac{1}{2}} \mathbb{E}\left[\sup_{t\in\mathfrak{T}} (1 + \kappa)^{2N_t}\right]^{\frac{1}{2}} \\ &\leq 2\mathrm{e}^{\kappa T}\mathbb{E}\left[\mathcal{E}_T\left(\textstyle\int_0^{\cdot} G_s.\mathrm{d}W_s\right)^2\right]^{\frac{1}{2}} \mathbb{E}\left[(1 + \kappa)^{2N_T}\right]^{\frac{1}{2}}. \end{aligned}$$

Here, $\kappa > 0$ is such that $|F_t|, |H_t(\ell)| \leq \kappa$ for $t \in \mathfrak{T}$, $\ell \in \mathbb{R}^n$, and $\mathbb{E}[(1+\kappa)^{2N_T}] = \mathrm{e}^{\kappa(\kappa+2)T\vartheta(\mathbb{R})} < \infty$. This yields the desired result. $\square$

[7] H is predictable in the sense that the mapping $\mathfrak{T} \times \Omega \times \mathbb{R}^n$, $(t, \omega, \ell) \mapsto H_t(\ell)(\omega)$ is $\mathfrak{P} \otimes \mathfrak{B}(\mathbb{R}^n)$-measurable, where $\mathfrak{P}$ is the predictable σ-field on $\mathfrak{T} \times \Omega$; see [Jacod, Shiryayev 1987].

Change-of-Measure Device. With the help of Lemma 1 it is possible to decompose terminal utility into a nearly deterministic and a martingale part.

Theorem 1 (Change-of-Measure Device) *In the setting of the post-crash problem (P_{post}), for any admissible post-crash strategy $\bar{\pi} \in \bar{\Pi}_0$ we have*

$$u(X_T^{\bar{\pi}}) = u(X_\tau^{\bar{\pi}}) \exp\left\{\rho \int_\tau^T \bar{\Phi}_t(\bar{\pi}_t) \mathrm{d}t\right\} M_T(\bar{\pi}), \tag{2}$$

where $X_\tau^{\bar{\pi}} = (1 - \pi_\tau.\ell) X_\tau^{\pi}$ is $\mathfrak{F}_\tau$-measurable and not influenced by the choice of $\bar{\pi}$; the mapping $\bar{\Phi} : \mathfrak{T} \times \bar{\Sigma}^{\max} \to \mathbb{R}$ is given by

$$\bar{\Phi}_t(\bar{\pi}) \triangleq \bar{r}_t + \bar{\pi}.\bar{\eta}_t - \tfrac{1}{2}(1-\rho)\bar{\pi}.\bar{\sigma}_t.\bar{\sigma}_t^{\mathfrak{t}}.\bar{\pi}^{\mathfrak{t}} + \tfrac{1}{\rho}\int_{\mathbb{R}^n} \left[(1-\bar{\pi}.\ell)^\rho - 1\right] \bar{\vartheta}(\mathrm{d}\ell); \tag{3}$$

and $M(\bar{\pi}) = \{M_t(\bar{\pi})\}$ is a local $\{\mathfrak{F}_t\}$-martingale with $M_\tau(\bar{\pi}) = 1$. Moreover, $M(\bar{\pi})$ is an $\{\mathfrak{F}_t\}$-martingale if $\bar{\pi}.\bar{\ell}^{\max}$ is bounded away from 1.

Proof Using Itô's formula and the fact that $xu'(x) = \rho u(x)$, $x^2 u''(x) = -\rho(1-\rho)u(x)$ for $x \in (0,\infty)$, the dynamics of $\{u(X_t^{\bar{\pi}})\}$ on $[\![\tau, T]\!]$ evaluate to

$$\begin{aligned} \mathrm{d}u(X_t^{\bar{\pi}}) &= u(X_{t-}^{\bar{\pi}}) \left\{\rho(\bar{r}_t + \bar{\pi}_t.\bar{\eta}_t)\mathrm{d}t + \rho\bar{\pi}_t.\bar{\sigma}_t.\mathrm{d}\bar{W}_t - \tfrac{1}{2}\rho(1-\rho)\bar{\pi}_t.\bar{\sigma}_t.\bar{\sigma}_t^{\mathfrak{t}}.\bar{\pi}_t^{\mathfrak{t}}\mathrm{d}t\right\} \\ &\quad + u(X_{t-}^{\bar{\pi}}) \int_{\mathbb{R}^n} \left[(1-\bar{\pi}_t.\ell)^\rho - 1\right] \bar{\nu}(\mathrm{d}t, \mathrm{d}\ell) \\ &= u(X_{t-}^{\bar{\pi}}) \left\{\rho\bar{\Phi}_t(\bar{\pi}_t)\mathrm{d}t + \rho\bar{\pi}_t.\bar{\sigma}_t.\mathrm{d}\bar{W}_t + \int_{\mathbb{R}^n} \left[(1-\bar{\pi}_t.\ell)^\rho - 1\right] \tilde{\bar{\nu}}(\mathrm{d}t, \mathrm{d}\ell)\right\}. \end{aligned}$$

Thus we obtain equation (2) with

$$M(\bar{\pi}) \triangleq \mathcal{E}\left(\rho \int_\tau^{\cdot} \bar{\pi}_t.\bar{\sigma}_t.\mathrm{d}\bar{W}_t + \int_{(\tau,\,\cdot\,]\times\mathbb{R}^n} \left[(1-\bar{\pi}_t.\ell)^\rho - 1\right] \tilde{\bar{\nu}}(\mathrm{d}t, \mathrm{d}\ell)\right).$$

Lemma 1 implies that $M(\bar{\pi})$ is an $\{\mathfrak{F}_t\}$-martingale if $\bar{\pi}.\bar{\ell}^{\max}$ is bounded away from 1, and the proof is complete. □

With the help of Theorem 1 the Merton problem ($\mathrm{P}_{\mathrm{post}}$) is now easily solved. We construct $\bar{\pi}^\circ = \{\bar{\pi}_t^\circ\}$ in such a way that

$$\bar{\pi}_t^\circ = \underset{\bar{\pi} \in K_t}{\arg\max}\, \bar{\Phi}_t(\bar{\pi}) \text{ for any } t \in \mathfrak{T}. \tag{4}$$

A standard selection argument shows that $\bar{\pi}^\circ$ is a continuous function provided the mapping $t \mapsto K_t$ is continuous;[8] in particular, $\bar{\pi}^\circ.\bar{\ell}^{\max}$ is uniformly bounded away from 1. The key observation at this point is that the maximizer $\bar{\pi}^\circ$ is deterministic, hence so is $\{\bar{\Phi}_t(\bar{\pi}_t^\circ)\}$. Therefore, the Change-of-Measure

[8] See, for instance, Theorem 3.6 in [Stokey, Lucas, Prescott 1989]. A set-valued mapping is continuous if it is both upper and lower semi-continuous. Note that the function $\bar{\Phi}_t(\,\cdot\,)$ is strictly concave, so the maximizer is uniquely determined.

Device shows that for an arbitrary admissible strategy $\bar{\pi} \in \bar{\Pi}$, with $\{\tau_n\}$ a localizing sequence of stopping times that reduce the local martingale $M(\bar{\pi})$,

$$\begin{aligned}
\mathbb{E}[u(X_T^{\bar{\pi}})] &\le \liminf_{n\to\infty} \mathbb{E}[u(X_{\tau_n}^{\bar{\pi}})] \\
&= \liminf_{n\to\infty} \mathbb{E}\left[u(X_\tau^{\bar{\pi}}) \exp\left\{\rho \textstyle\int_\tau^{\tau_n} \bar{\Phi}_t(\bar{\pi}_t)\mathrm{d}t\right\} M_{\tau_n}(\bar{\pi})\right] \\
&\le \liminf_{n\to\infty} \mathbb{E}\left[u(X_\tau^{\bar{\pi}}) \exp\left\{\rho \textstyle\int_\tau^{T} \bar{\Phi}_t(\bar{\pi}_t^\circ)\mathrm{d}t\right\} M_{\tau_n}(\bar{\pi})\right] \\
&= \mathbb{E}\left[u(X_\tau^{\bar{\pi}}) \exp\left\{\rho \textstyle\int_\tau^{T} \bar{\Phi}_t(\bar{\pi}_t^\circ)\mathrm{d}t\right\} M_{\tau}(\bar{\pi})\right] \\
&= \mathbb{E}\left[u(X_\tau^{\bar{\pi}^\circ}) \exp\left\{\rho \textstyle\int_\tau^{T} \bar{\Phi}_t(\bar{\pi}_t^\circ)\mathrm{d}t\right\} M_{\tau}(\bar{\pi}^\circ)\right] \\
&= \mathbb{E}\left[u(X_\tau^{\bar{\pi}^\circ}) \exp\left\{\rho \textstyle\int_\tau^{T} \bar{\Phi}_t(\bar{\pi}_t^\circ)\mathrm{d}t\right\} M_{T}(\bar{\pi}^\circ)\right] = \mathbb{E}[u(X_T^{\bar{\pi}^\circ})].
\end{aligned}$$

Here, we make use of Fatou's lemma and dominated convergence with (1) in the first inequality; the martingale properties of $\{M_{t\wedge\tau_n}(\bar{\pi})\}$ and $M(\bar{\pi}^\circ)$; the fact that $M_\tau(\bar{\pi}) = M_\tau(\bar{\pi}^\circ) = 1$; and $\mathfrak{F}_\tau$-measurability of the random variables $X_\tau^{\bar{\pi}} = X_\tau^{\bar{\pi}^\circ}$ and $\int_\tau^T \bar{\Phi}_t(\bar{\pi}_t^\circ)\mathrm{d}t$. Hence, we have established

Corollary 1 (Solution of the Merton Problem with Random Initial Time) *The optimal strategy in problem (P_{post}) is given by $\bar{\pi}^\circ = \{\bar{\pi}_t^\circ\}$ as defined in equation (4), i.e.*

$$\mathbb{E}[u(X_T^{\pi,\bar{\pi}^\circ,\tau,\ell})] = \sup_{\bar{\pi}\in\bar{\Pi}_0} \mathbb{E}[u(X_T^{\pi,\bar{\pi},\tau,\ell})].$$

In particular, the Change-of-Measure Device can be applied to directly solve the classical Merton problem with CRRA preferences. In the present context, it follows that the Merton strategy in the pre-crash market without the threat of a crash, as used in assumption ($\mathrm{H}_{\mathrm{pre}}$), is given by

$$\pi_t^M = \arg\max_{\pi\in\Sigma^{\max}} \Phi_t(\pi) \text{ for } t \in \mathfrak{T},$$

where $\Phi : \mathfrak{T} \times \Sigma^{\max} \to \mathbb{R}$ is defined in analogy to (3).

Remark 2 *We refer to [Karatzas, Shreve 1998] or [Korn, Korn 2001] for standard approaches to the Merton problem. To clarify the relationship to the above, note that from the point of view of* dynamic programming*, the Change-of-Measure Device builds on the separable form $v(t,x) = g(t)u(x)$ of the value function.*[9] *From the perspective of the* duality method *of [Cox, Huang 1989], [Kallsen 2000], and many others, the Change-of-Measure Device exploits the fact that the state-price density is of the form $xu'(x) = \rho u(x)$ for a* CRRA *utility function u. The direct change-of-measure approach above is also applied in portfolio optimization with risk-sensitive criteria or incomplete information, see [Nagai, Runggaldier 2006] and the references therein.*

[9] Note that this can also be seen a priori from an elementary scaling argument.

Reformulation of the Worst-Case Portfolio Problem. It follows from Theorem 1 that given any strategy $\pi \in \Pi$, the optimal strategy to be applied after an arbitrary crash scenario $(\tau, \ell) \in \Theta$ is given by $\bar{\pi}^\star = \bar{\pi}^M$, i.e.

$$\bar{\pi}_t^\star = \bar{\pi}_t^M = \arg\max_{\bar{\pi} \in \bar{\Sigma}^{\max}} \bar{\Phi}_t(\bar{\pi}).$$

In particular, the optimal post-crash strategy does actually not depend on (τ, ℓ), and hence as above we simply write $\bar{\pi}^\star \in \bar{\Pi}$. Letting $\bar{v}$ denote the value function of the corresponding Merton problem ($\mathrm{P}_{\mathrm{post}}$), we may now equivalently recast the worst-case portfolio problem (P) as the **pre-crash worst-case problem**

$$\sup_{\pi \in \Pi} \inf_{\tau \in \Theta} \mathbb{E}\left[\bar{v}(\tau, (1 - \pi_\tau . \ell^\infty) X_\tau^\pi)\right]. \tag{$\mathrm{P}_{\mathrm{pre}}$}$$

Here, we use monotonicity of $\bar{v}$ to conclude that $\ell = \ell^\infty$ is the worst-case crash height, and write $\tau \in \Theta$ instead of $(\tau, \ell^\infty) \in \Theta$ for brevity. With the help of equation (2) the value function $\bar{v}$ can be expressed explicitly as

$$\begin{aligned} \bar{v}(t, x) &= \bar{g}(t) u(x) \text{ for } t \in \mathcal{T},\ x \in (0, \infty), \\ &\text{where } \bar{g}(t) = \exp\left\{\rho \int_t^T \bar{\Psi}_s \mathrm{d}s\right\} \text{ for } t \in \mathcal{T}. \end{aligned} \tag{5}$$

Here and in the following, as in [Korn, Menkens 2005]

$$\Psi_t \triangleq \max_{\pi \in \Sigma^{\max}} \Phi_t(\pi) = \Phi_t(\pi_t^M) \text{ and } \bar{\Psi}_t \triangleq \max_{\bar{\pi} \in \bar{\Sigma}^{\max}} \bar{\Phi}_t(\bar{\pi}) = \bar{\Phi}_t(\bar{\pi}_t^\star),\ t \in \mathcal{T},$$

are the utility growth potentials of the markets before and after the crash.

4 Indifference Strategies

In this section, we reformulate the worst-case investment problem as a stochastic game and introduce the fundamental notion of indifference strategies.

Abstract Indifference Strategies. Consider an abstract zero-sum stochastic differential game between two agents A and B with the following protocol: Agent A controls a stochastic process $W = W^\lambda = \{W_t^\lambda\}_{t \in \bar{\mathcal{T}}}$ by choosing λ from a class of admissible controls Λ, whereas Agent B decides on the duration of the game by choosing a stopping time $\tau \in \Theta$; Agent A has to choose her strategy first, and her choice is reported to Agent B, who then makes his choice. Agents A and B aim to maximize or minimize, respectively, the expected value $\mathbb{E}[W_\tau^\lambda]$. Thus Agent A faces the problem to obtain

$$\sup_{\lambda \in \Lambda} \inf_{\tau \in \Theta} \mathbb{E}[W_\tau^\lambda]. \tag{$\mathrm{P}_{\mathrm{abstract}}$}$$

Problem ($\mathrm{P}_{\mathrm{abstract}}$) is known as the **controller-vs-stopper game**. This has been studied in abstract settings by [Karatzas, Sudderth 2001] for linear diffusions, and by [Karatzas, Zamfirescu 2008] for problems when the control affects only the drift. For the worst-case portfolio problem, we now provide a direct approach based on the concept of indifference.

Remark 3 *Observe that if we call Agent A the 'investor' and Agent B the 'market', put $\Lambda \triangleq \Pi$ and set*

$$W_t^\pi \triangleq \bar{v}(t, (1-\pi_t.\ell^\infty)X_t^\pi) \text{ for } t \in \mathfrak{T},\ W_\infty^\pi \triangleq \bar{v}(T, X_T^\pi),$$

then we recover the pre-crash worst-case problem (P_{pre}).

If Agent A can choose her strategy $\hat{\lambda} \in \Lambda$ in such a way as to render the process $W^{\hat{\lambda}}$ an $\{\mathfrak{F}_t\}$-martingale, then her opponent's actions become irrelevant to her; indeed by optional stopping

$$\mathbb{E}[W_\sigma^{\hat{\lambda}}] = \mathbb{E}[W_\tau^{\hat{\lambda}}] \text{ for any } \sigma, \tau \in \Theta.$$

Therefore we refer to such a strategy $\hat{\lambda}$ as an **indifference strategy**. Note that since $\tau \in \Theta$ can attain the value ∞, Agent B has the option of not intervening at all.
The crucial property of indifference strategies is formalized in the following simple result.

Proposition 1 (Indifference-Optimality Principle) *If $\hat{\lambda}$ is an indifference strategy and for all $\lambda \in \Lambda$ we have*

$$\mathbb{E}[W^{\hat{\lambda}}(\tau_0)] \geq \mathbb{E}[W^\lambda(\tau_0)] \text{ for at least one } \tau_0 \in \Theta,$$

then $\hat{\lambda}$ is optimal for Agent A in problem ($P_{abstract}$).

Proof Given an arbitrary strategy $\lambda \in \Lambda$, we have

$$\inf_{\tau \in \Theta} \mathbb{E}[W_\tau^\lambda] \leq \mathbb{E}[W_{\tau_0}^\lambda] \leq \mathbb{E}[W_{\tau_0}^{\hat{\lambda}}] = \inf_{\tau \in \Theta} \mathbb{E}[W_\tau^{\hat{\lambda}}]$$

where $\tau_0 \in \Theta$ is as above and the last identity makes use of the fact that $\hat{\lambda}$ is an indifference strategy. □

Indifference at ∞. As a consequence of the martingale property between T and ∞ and the fact that both $W_T^{\hat{\lambda}}$ and $W_\infty^{\hat{\lambda}}$ are $\mathfrak{F}_T$-measurable, any indifference strategy $\hat{\lambda}$ satisfies

$$W_T^{\hat{\lambda}} = W_\infty^{\hat{\lambda}}.$$

In particular for problem ($\mathrm{P_{pre}}$) it follows that any indifference strategy $\hat{\pi} \in \Pi$ must satisfy the boundary condition

$$\hat{\pi}_T.\ell^\infty = 0.$$

Conversely this condition trivially implies the martingale property of $W^{\hat{\pi}}$ on $\{T, \infty\}$.

Indifference Strategies for Crash Scenarios. We now return to the setting of the worst-case portfolio problem. Writing $\bar{v}(t,x) = \bar{g}(t)u(x)$ as in (5), we deduce that

$$\frac{\dot{\bar{g}}(t)}{\bar{g}(t)} = -\rho\bar{\Psi}_t = -\rho\bar{\Phi}_t(\bar{\pi}^\star_t) \text{ for } t \in \mathfrak{T}.$$

Now assume $\pi \in \Pi$ is such that $\beta = \{\beta_t\}$, $\beta_t \triangleq \pi_t.\ell^\infty$ for $t \in \mathfrak{T}$, has paths of class C^1. Then $\{\bar{v}(t,(1-\pi_t.\ell^\infty)X^\pi_t)\}$ is a semimartingale, so Itô's formula is applicable and yields

$$\begin{aligned}
&\mathrm{d}\bar{v}(t,(1-\pi_t.\ell^\infty)X^\pi_t) \\
&\quad= \bar{v}(t,(1-\pi_t.\ell^\infty)X^\pi_{t-})\Big\{\frac{\dot{\bar{g}}(t)}{\bar{g}(t)}\mathrm{d}t - \rho\frac{\dot{\beta}_t}{1-\beta_t}\mathrm{d}t + \rho(r_t+\pi_t.\eta_t)\mathrm{d}t \\
&\qquad+\rho\pi_t.\sigma_t.\mathrm{d}W_t - \tfrac{1}{2}\rho(1-\rho)\pi_t.\sigma_t.\sigma^{\mathrm{t}}_t.\pi^{\mathrm{t}}_t\mathrm{d}t + \textstyle\int^n_{\mathbb{R}}\left[(1-\pi_t.\ell)^\rho - 1\right]\nu(\mathrm{d}t,\mathrm{d}\ell)\Big\} \\
&\quad= \bar{v}(t,(1-\pi_t.\ell^\infty)X^\pi_{t-})\Big\{-\rho\bar{\Phi}_t(\bar{\pi}^\star_t) - \rho\frac{\dot{\beta}_t}{1-\beta_t} + \rho\Phi_t(\pi_t)\Big\}\mathrm{d}t \\
&\qquad+\bar{v}(t,(1-\pi_t.\ell^\infty)X^\pi_{t-})\left\{\rho\pi_t.\sigma_t.\mathrm{d}W_t + \textstyle\int_{\mathbb{R}^n}\left[(1-\pi_t.\ell)^\rho - 1\right]\tilde{\nu}(\mathrm{d}t,\mathrm{d}\ell)\right\}.
\end{aligned}$$

In addition, suppose $\beta = \{\beta_t\}$ satisfies, path by path, the ordinary differential equation

$$\dot{\beta}_t = (1-\beta_t)\left\{\Phi_t(\pi_t) - \bar{\Phi}_t(\bar{\pi}^\star_t)\right\} \text{ on } \mathfrak{T},\ \beta_T = 0. \tag{6}$$

Then by Remark 1 and Lemma 1 the local martingale $\mathcal{E}(\rho\int_0^\cdot \pi_t.\sigma_t.\mathrm{d}W_t + \int_{(0,\,\cdot\,]\times\mathbb{R}^n}[(1-\pi_t.\ell)^\rho - 1]\tilde{\nu}(\mathrm{d}t,\mathrm{d}\ell))$ is in fact an $\{\mathfrak{F}_t\}$-martingale on $\mathfrak{T}$, and in combination with the boundary condition we infer that $\{\bar{v}(t,(1-\pi_t.\ell^\infty)X^\pi_t)\}$ is an $\{\mathfrak{F}_t\}$-martingale on $\bar{\mathfrak{T}}$, i.e. π is an indifference strategy.

Remark 4 *In the* 1*-dimensional case, equation (6) can be formulated in terms of* π *only. One thus obtains an ordinary differential equation characterizing the indifference strategy* $\pi = \hat{\pi}$*; details are presented below. However, in a multi-dimensional framework, there are in general infinitely many such indifference strategies.*

Construction of the Indifference Frontier. For the further analysis, we require two homogeneity conditions on the pre- and post-crash markets, see $(\mathrm{H}^1_{\mathrm{post}})$ and $(\mathrm{H}^2_{\mathrm{post}})$ below. We henceforth assume that

$$\{r_t - \bar{\Psi}_t\} \text{ does not change sign.} \tag{$\mathrm{H}^1_{\mathrm{post}}$}$$

Intuitively, hypothesis $(\mathrm{H}^1_{\mathrm{post}})$ states that the post-crash market is either always better or always worse than the riskless investment before the crash.
We now construct a specific indifference strategy $\hat{\pi}$ with associated crash exposure $\hat{\beta}$. For this purpose, we need to distinguish the cases

$$r_t \leq \bar{\Psi}_t \text{ for } t \in \mathfrak{T}, \tag{$\mathrm{C}_{\mathrm{bull}}$}$$

when the post-crash market is at least as good as the riskless asset before the crash; and

$$r_t \geq \bar{\Psi}_t \text{ for } t \in \mathcal{T}. \tag{$\mathrm{C_{bear}}$}$$

We will see below that in the bear market case ($\mathrm{C_{bear}}$), the worst-case investment problem can be solved directly, and indifference arguments are not required. Hence, in case ($\mathrm{C_{bear}}$) we simply set

$$\hat{\beta}_t \triangleq 0 \text{ for } t \in \mathcal{T}. \tag{$\mathrm{I_{bear}}$}$$

For bull markets ($\mathrm{C_{bull}}$), define $\psi : \mathcal{T} \times [0,1] \to \mathbb{R}$ by

$$\psi_t(\beta) \triangleq \max_{\pi \in \Sigma^{\max},\, \pi.\ell^\infty = \beta} \Phi_t(\pi),$$

note that ψ is continuous, and define $\hat{\beta} = \{\hat{\beta}_t\}$ by the ordinary differential equation

$$\dot{\hat{\beta}}_t = (1 - \hat{\beta}_t)\left\{\psi_t(\hat{\beta}_t) - \bar{\Phi}_t(\bar{\pi}^\star_t)\right\} \text{ on } \mathcal{T},\ \hat{\beta}_T = 0. \tag{$\mathrm{I_{bull}}$}$$

Since a solution to ($\mathrm{I_{bull}}$) cannot exit from the interval $[0,1]$, it follows that $\hat{\beta}$ is well-defined and $0 \leq \hat{\beta}_t \leq 1$ for $t \in \mathcal{T}$. Moreover, $\hat{\beta}$ is uniquely determined. In fact, we have $\hat{\beta}_t < 1$ for $t \in \mathcal{T}$ because of the following elementary result.

Lemma 2 *Suppose that $F : \mathcal{T} \times [0,1] \to \mathbb{R}$ is continuous with $F(t,0) \leq 0$ for $t \in \mathcal{T}$, and let $\beta = \{\beta_t\}$ satisfy*

$$\dot{\beta}_t = (1 - \beta_t)F(t, \beta_t) \text{ on } \mathcal{T},\ \beta_T = 0.$$

Then $\beta_t < 1$ for all $t \in \mathcal{T}$.

Proof Note that $0 \leq \beta_t \leq 1$ for $t \in \mathcal{T}$. We have

$$\tfrac{\mathrm{d}}{\mathrm{d}t} \log(1 - \beta_t) = -F(t, \beta_t) \text{ if } \beta_t < 1,$$

and hence $\log(1 - \beta_t) = -\int_t^T F(s, \beta_s)\mathrm{d}s$ if $\beta_s < 1$ for $s \in [t,T]$. It follows that

$$1 - \beta_t = \exp\left\{-\int_t^T F(s,\beta_s)\mathrm{d}s\right\} \geq \mathrm{e}^{-M(T-t)} \geq 2\delta \text{ if } \beta_s < 1 \text{ for } s \in [t,T],$$

where $\delta \triangleq \frac{1}{2}\mathrm{e}^{-MT}$ and $M > 0$ is such that $|F(t,\beta)| \leq M$ for all $t \in \mathcal{T}$, $\beta \in [0,1]$. Hence, $t_0 \triangleq \inf\{t \in \mathcal{T} : \beta_t \leq 1 - \delta\}$ satisfies $t_0 = 0$, for otherwise the preceding argument would imply $\beta_{t_0} \leq 1 - 2\delta$, hence $\beta_t < 1 - \delta$ for some $t < t_0$ by continuity, which contradicts the definition of t_0. □

It follows that we can construct an admissible indifference strategy $\hat{\pi} = \{\hat{\pi}_t\}$ by setting

$$\hat{\pi}_t \triangleq \operatorname*{arg\,max}_{\pi \in \Sigma^{\max},\, \pi.\ell^\infty = \hat{\beta}_t} \Phi_t(\pi) \text{ for } t \in \mathcal{T}. \tag{7}$$

In fact it is not hard to see, using the same arguments as for (4) above, that $\hat{\pi}$ is a continuous function of time.

Remark 5 *Note that due to the definition of $\psi_t(\,\cdot\,)$, $\hat\beta$ is the smallest exposure to the threat of a crash that can be attained with indifference strategies.*

In addition to hypothesis ($\mathrm{H}^1_{\mathrm{post}}$), in case ($\mathrm{C}_{\mathrm{bull}}$) we impose the condition that

$$\{\Psi_t - \bar\Psi_t\} \text{ does not change sign if } \hat\beta_t \geq \pi^M_t.\ell^\infty \text{ for some } t \in \mathcal{T}. \qquad (\mathrm{H}^2_{\mathrm{post}})$$

Since $\hat\beta_T = 0 < \pi^M_T.\ell^\infty$ and for $\hat\beta_t = \pi^M_t.\ell^\infty$ the right-hand side of ($\mathrm{I}_{\mathrm{bull}}$) evaluates to $(1-\hat\beta_t)\{\Psi_t - \bar\Psi_t\}$, we can subdivide ($\mathrm{C}_{\mathrm{bull}}$) into the cases

$$\hat\beta_t \leq \pi^M_t.\ell^\infty \text{ for } t \in \mathcal{T}, \qquad (\mathrm{C}_{\mathrm{bull}}\text{-a})$$

which is certainly true if $\Psi_t \geq \bar\Psi_t$ for all $t \in \mathcal{T}$; and

$$\hat\beta_t \geq \pi^M_t.\ell^\infty \text{ for some } t \in \mathcal{T}, \qquad (\mathrm{C}_{\mathrm{bull}}\text{-b})$$

in which case $\Psi_t \leq \bar\Psi_t$ for $t \in \mathcal{T}$ by ($\mathrm{H}^2_{\mathrm{post}}$), and hence there is some $T_0 \in \mathcal{T}$ such that $\hat\beta_t \geq \pi^M_t.\ell^\infty$ for $t \in [0, T_0]$ and $\hat\beta_t \leq \pi^M_t.\ell^\infty$ for $t \in [T_0, T]$.

Remark 6 *Hypotheses (H^1_{post}) and (H^2_{post}) are trivially satisfied for constant market coefficients. Moreover, we are never in the bear market case (C_{bear}) if the post-crash interest rate is at least as big as that before the crash. Finally, the benchmark situation when the pre- and post-crash coefficients coincide is always subsumed by case (C_{bull}-a).*

The 1*-Dimensional Case.* If there is only a single risky asset, the indifference strategy in (7) can be obtained directly as the solution to the 1-dimensional analog of (6),

$$\dot{\hat\pi}_t = \frac{1 - \hat\pi_t \ell^\infty}{\ell^\infty} \left\{ \Phi_t(\hat\pi_t) - \bar\Phi_t(\bar\pi^\star_t) \right\} \text{ on } \mathcal{T},\ \hat\pi_T = 0. \qquad (8)$$

This can be rearranged as

$$\dot{\hat\pi}_t = \frac{1 - \hat\pi_t \ell^\infty}{\ell^\infty} \left\{ [\Psi_t - \bar\Psi_t] + [\Phi_t(\hat\pi_t) - \Phi_t(\pi^M_t)] \right\} \text{ on } \mathcal{T},\ \hat\pi_T = 0.$$

When there are no jumps before the crash, the function $\Phi_t(\,\cdot\,)$ is purely quadratic and maximal at π^M_t, so that $\Phi_t(\hat\pi_t) - \Phi_t(\pi^M_t) = -\frac{1}{2}(1-\rho)\sigma^2_t(\hat\pi_t - \pi^M_t)^2$ and we have

$$\dot{\hat\pi}_t = \frac{1 - \hat\pi_t \ell^\infty}{\ell^\infty} \left\{ \Psi_t - \bar\Psi_t - \tfrac{1}{2}(1-\rho)\sigma^2_t \left(\hat\pi_t - \pi^M_t\right)^2 \right\} \text{ on } \mathcal{T},\ \hat\pi_T = 0.$$

This is the formulation of [Korn, Menkens 2005] and [Korn, Steffensen 2007].

5 Optimal Strategies for Worst-Case Scenarios

Our next goal is to determine optimal strategies for the worst-case portfolio problem (P), or equivalently problem (P_{pre}), and to establish their optimality. More precisely, we demonstrate that $\pi^\star = \{\pi_t^\star\}$,

$$\pi_t^\star \triangleq \operatorname*{arg\,max}_{\pi \in \Sigma^{\max},\, \pi.\ell^\infty \le \hat{\beta}_t} \Phi_t(\pi) \text{ for } t \in \mathcal{T} \tag{9}$$

is a worst-case optimal portfolio strategy before a possible crash.

Bear Markets. Let us first investigate the bear market case (C_{bear}), i.e. $\bar{\Psi}_t \le r_t$ for $t \in \mathcal{T}$. It is clear from (9) and (I_{bear}) that $\pi_t^\star = 0$ for $t \in \mathcal{T}$. An explicit computation shows that

$$W^{\pi^\star}_{\tau,\ell} = \exp\left\{\rho\left(\int_0^\tau r_s \mathrm{d}s + \int_\tau^T \bar{\Psi}_s \mathrm{d}s\right)\right\} u(x) \text{ for any } (\tau,\ell) \in \Theta,$$

so $(0,\ell^\infty) \in \Theta$ is a worst-case scenario for $\pi^\star$. Hence, for any $\pi \in \Pi$ the inequality $W^{\pi}_{0,\ell^\infty} = \bar{v}(0,(1-\pi_0.\ell^\infty)x) \le \bar{v}(0,x) = W^{\pi^\star}_{0,\ell^\infty}$ implies

$$\inf_{(\tau,\ell)\in\Theta} \mathbb{E}[W^{\pi}_{\tau,\ell}] \le \mathbb{E}[W^{\pi}_{0,\ell^\infty}] \le \mathbb{E}[W^{\pi^\star}_{0,\ell^\infty}] = \inf_{(\tau,\ell)\in\Theta} \mathbb{E}[W^{\pi^\star}_{\tau,\ell}],$$

so the worst-case performance of π is not better than that of $\pi^\star$. It follows that the optimal strategy to be applied before the crash in (C_{bear}) is the no-participation strategy $\pi^\star = 0$, as intuition suggests.

Indifference Frontier. In the remainder of this section, we address the substantially more interesting case (C_{bull}), when $\hat{\beta}_t \ge 0$ for $t \in \mathcal{T}$. In a first step we show that, given an arbitrary strategy $\pi \in \Pi$, it is always advantageous to switch to the indifference strategy $\hat{\pi}$ once the crash exposure $\beta_t \triangleq \pi_t.\ell^\infty$ exceeds the indifference level $\hat{\beta}_t$.
The proof of this result is surprisingly complicated,[10] so let us first explain the main idea. It is natural to consider the stopping time[11] $\varrho \triangleq \inf\{t \in \mathcal{T} : \beta_t > \hat{\beta}_t\}$, i.e. the first time the crash vulnerability exceeds the indifference level $\hat{\beta}$, and to switch from π to $\hat{\pi}$ at ϱ. However, it is possible that $\beta_\varrho < \hat{\beta}_\varrho$, so using $\hat{\pi}$ *at time* ϱ *already* results in an increase of vulnerability to the threat of a crash; on the other hand, if $\hat{\pi}$ is used only *just after* ϱ, then β may already have returned to a level below $\hat{\beta}$, and we encounter the same difficulty. Hence, we must carefully decide whether to switch at time ϱ or immediately afterwards; in addition, we must keep track of the 'regular' jumps of the price process, which may coincide with the crash scenario τ under consideration.

[10] A simple proof under restrictive assumptions in the 1-dimensional case (in particular, admissible strategies are assumed to be *continuous*) can be found in [Korn, Seifried 2009]; see also the previous chapter of this thesis.
[11] We use the standard convention $\inf \varnothing \triangleq \infty$.

Proposition 2 (Indifference Frontier) *Suppose $\pi \in \Pi$ is an arbitrary admissible strategy, let $\beta = \{\beta_t\}$ be given by $\beta_t \triangleq \pi_t.\ell^\infty$ for $t \in \mathfrak{T}$, set $\varrho \triangleq \inf\{t \in \mathfrak{T} : \beta_t > \hat{\beta}_t\}$, and define $\tilde{\pi} = \{\tilde{\pi}_t\}$ by*

$$\tilde{\pi}_t \triangleq \pi \text{ if } t < \rho \text{ or } (t = \rho,\ \beta_\rho \le \hat{\beta}_\rho) \text{ and } \tilde{\pi}_t \triangleq \hat{\pi} \text{ if } t > \rho \text{ or } (t = \rho,\ \beta_\rho > \hat{\beta}_\rho).$$

Then $\tilde{\pi} \in \Pi$ and the worst-case bound for problem (P_{pre}) attained by $\tilde{\pi}$ is at least as great as that achieved by π, i.e.

$$\inf_{(\tau,\ell^\infty)\in\Theta} \mathbb{E}[W_\tau^{\tilde{\pi}}] \ge \inf_{(\tau,\ell^\infty)\in\Theta} \mathbb{E}[W_\tau^{\pi}].$$

Note that $\tilde{\pi}$ switches to the indifference strategy $\hat{\pi}$ immediately only if $\beta_\varrho > \hat{\beta}_\varrho$.

Proof For the sake of clarity the proof is divided into 5 steps. It is helpful to distinguish the cases

a) $\beta_\varrho \ge \hat{\beta}_\varrho$, when immediate switching to $\hat{\pi}$ is advantageous and no price jump occurs;
b) $\beta_\varrho < \hat{\beta}_\varrho$, when immediately after ϱ there are points of time t with $\beta_t > \hat{\beta}_t$, which can be exhausted by stopping times.

Step 1. We note that ϱ is an $\{\mathfrak{F}_t\}$-stopping time by virtue of the début theorem, and hence $\tilde{\pi} \in \Pi$. We define an $\{\mathfrak{F}_t\}$-stopping time σ_0 by setting

$$\sigma_0 \triangleq \varrho \text{ on } \{\beta_\varrho \ge \hat{\beta}_\varrho\},\ \sigma_0 \triangleq \infty \text{ elsewhere.}$$

Then σ_0 is predictable; indeed its graph $[\![\sigma_0]\!] = [\![0, \varrho]\!] \cap \{\beta \ge \hat{\beta}\}$ is predictable. Further we use the predictable section theorem to choose, for each $n \in \mathbb{N}$, a predictable $\{\mathfrak{F}_t\}$-stopping time σ_n such that

$$\varrho < \sigma_n \le \varrho + \tfrac{1}{n},\ \beta_{\sigma_n} > \hat{\beta}_{\sigma_n} \text{ on } \{\sigma_n < \infty\},\ \sigma_n = \infty \text{ on } \{\beta_\varrho \ge \hat{\beta}_\varrho\},$$

and $\mathbb{P}(\beta_\varrho < \hat{\beta}_\varrho,\ \sigma_n = \infty) < 2^{-n}$. By the Borel-Cantelli lemma we have

$$\sigma_n \downarrow \varrho \text{ as } n \to \infty \text{ on } \{\beta_\varrho < \hat{\beta}_\varrho\}.$$

Step 2. Let $\{W_t^{\tilde{\pi}}\}$ and $\{W_t^{\pi}\}$ be defined as in Remark 3, i.e.

$$W_t^\pi = \bar{v}(t, (1 - \pi_t.\ell^\infty)X_t^\pi) = \bar{v}(t, (1 - \beta_t)X_t^\pi) \text{ for } t \in \mathfrak{T},\ W_\infty^\pi = \bar{v}(T, X_T^\pi),$$

and similarly for $\{W_t^{\tilde{\pi}}\}$. Further let $\tilde{\beta} = \{\tilde{\beta}_t\}$ be given by $\tilde{\beta}_t \triangleq \tilde{\pi}_t.\ell^\infty$, $t \in \mathfrak{T}$. Note that

$$\tilde{\beta} = \beta \text{ and } X^{\tilde{\pi}} = X^\pi \text{ on } [\![0, \varrho)\!) \text{ and on } [\![0, \varrho]\!] \cap \{\beta < \hat{\beta}\}, \tag{10}$$

and by definition and (right-)continuity of $\hat{\beta}$

$$W^{\tilde{\pi}} \text{ is a martingale on } (\!(\varrho, \infty]\!] \text{ and } W_{\varrho+}^{\tilde{\pi}} = W_\varrho^{\tilde{\pi}} \text{ on } \{\beta_\varrho \ge \hat{\beta}_\varrho\}. \tag{11}$$

In the following we fix an arbitrary crash scenario $\tau \in \Theta$, and for every $n \in \mathbb{N}$ we define the random time τ_n by

$$\tau_n \triangleq \sigma_n \text{ on } \{\beta_\varrho < \hat{\beta}_\varrho,\ \tau > \varrho\},\ \tau_n \triangleq \tau \wedge \varrho \text{ elsewhere;}$$

since $\{\beta_\varrho < \hat{\beta}_\varrho,\ \tau > \varrho\} \in \mathfrak{F}_\varrho$ and $\varrho < \sigma_n$ it is readily seen that each τ_n is an $\{\mathfrak{F}_t\}$-stopping time, hence a possible crash scenario. We shall prove that

$$\mathbb{E}[W_\tau^{\tilde{\pi}}] \geq \inf_{n \in \mathbb{N}} \mathbb{E}[W_{\tau_n}^\pi],$$

which will establish the assertion.
Step 3. We consider case a) in the third step and demonstrate that for arbitrary $n \in \mathbb{N}$

$$\mathbb{E}\left[1_{\{\beta_\varrho \geq \hat{\beta}_\varrho\}} W_\tau^{\tilde{\pi}}\right] \geq \mathbb{E}\left[1_{\{\beta_\varrho \geq \hat{\beta}_\varrho\}} W_{\tau_n}^\pi\right]. \tag{12}$$

Note that $\tau_n = \tau \wedge \varrho$ on $\{\beta_\varrho \geq \hat{\beta}_\varrho\}$. Using property (11) in the first identity and property (10) in the second, we find

$$\begin{aligned}
\mathbb{E}\left[1_{\{\beta_\varrho \geq \hat{\beta}_\varrho\}} W_\tau^{\tilde{\pi}}\right] &= \mathbb{E}\left[1_{\{\beta_\varrho \geq \hat{\beta}_\varrho\}} W_{\tau \wedge \varrho}^{\tilde{\pi}}\right] \\
&= \mathbb{E}\left[1_{\{\beta_\varrho \geq \hat{\beta}_\varrho,\, \tau < \varrho\}} W_\tau^{\tilde{\pi}}\right] + \mathbb{E}\left[1_{\{\beta_\varrho \geq \hat{\beta}_\varrho,\, \tau \geq \varrho\}} W_\varrho^{\tilde{\pi}}\right] \\
&= \mathbb{E}\left[1_{\{\beta_\varrho \geq \hat{\beta}_\varrho,\, \tau < \varrho\}} W_\tau^{\pi}\right] + \mathbb{E}\left[1_{\{\beta_\varrho \geq \hat{\beta}_\varrho,\, \tau \geq \varrho\}} W_\varrho^{\tilde{\pi}}\right] \\
&= \mathbb{E}\left[1_{\{\beta_\varrho \geq \hat{\beta}_\varrho,\, \tau < \varrho\}} W_{\tau_n}^{\pi}\right] + \mathbb{E}\left[1_{\{\beta_\varrho \geq \hat{\beta}_\varrho,\, \tau \geq \varrho\}} W_\varrho^{\tilde{\pi}}\right].
\end{aligned}$$

Recalling that σ_0 is predictable and that jump times of Poisson processes are totally inaccessible, it follows that $\Delta P_{\sigma_0} = 0$, so by virtue of (10) $X_\varrho^{\tilde{\pi}} = X_{\varrho-}^{\tilde{\pi}} = X_{\varrho-}^{\pi} = X_\varrho^\pi$ on $\{\varrho = \sigma_0\}$. As $\sigma_0 = \varrho$ on $\{\beta_\varrho \geq \hat{\beta}_\varrho\}$, we deduce that

$$W_\varrho^{\tilde{\pi}} = \bar{v}(\varrho, (1 - \tilde{\beta}_\varrho) X_\varrho^{\tilde{\pi}}) \geq \bar{v}(\varrho, (1 - \beta_\varrho) X_\varrho^\pi) = W_\varrho^\pi \text{ on } \{\beta_\varrho \geq \hat{\beta}_\varrho\}.$$

Putting things together, we obtain

$$\begin{aligned}
\mathbb{E}\left[1_{\{\beta_\varrho \geq \hat{\beta}_\varrho\}} W_\tau^{\tilde{\pi}}\right] &\geq \mathbb{E}\left[1_{\{\beta_\varrho \geq \hat{\beta}_\varrho,\, \tau < \varrho\}} W_{\tau_n}^{\pi}\right] + \mathbb{E}\left[1_{\{\beta_\varrho \geq \hat{\beta}_\varrho,\, \tau \geq \varrho\}} W_\varrho^{\pi}\right] \\
&= \mathbb{E}\left[1_{\{\beta_\varrho \geq \hat{\beta}_\varrho\}} W_{\tau_n}^{\pi}\right]
\end{aligned}$$

as asserted in (12).
Step 4. We address case b) in the fourth step and show that given any $\varepsilon > 0$ there exists some $n \in \mathbb{N}$ such that

$$\mathbb{E}\left[1_{\{\beta_\varrho < \hat{\beta}_\varrho\}} W_\tau^{\tilde{\pi}}\right] \geq \mathbb{E}\left[1_{\{\beta_\varrho < \hat{\beta}_\varrho\}} W_{\tau_n}^\pi\right] - \varepsilon. \tag{13}$$

Properties (10) and (11) successively imply that

$$\begin{aligned}\mathbb{E}\left[1_{\{\beta_\varrho<\hat\beta_\varrho\}}W^{\tilde\pi}_\tau\right] &= \mathbb{E}\left[1_{\{\beta_\varrho<\hat\beta_\varrho,\,\tau\le\varrho\}}W^{\tilde\pi}_\tau\right] + \mathbb{E}\left[1_{\{\beta_\varrho<\hat\beta_\varrho,\,\tau>\varrho\}}W^{\tilde\pi}_\tau\right]\\ &= \mathbb{E}\left[1_{\{\beta_\varrho<\hat\beta_\varrho,\,\tau\le\varrho\}}W^{\pi}_\tau\right] + \mathbb{E}\left[1_{\{\beta_\varrho<\hat\beta_\varrho,\,\tau>\varrho\}}W^{\tilde\pi}_\tau\right]\\ &= \mathbb{E}\left[1_{\{\beta_\varrho<\hat\beta_\varrho,\,\tau\le\varrho\}}W^{\pi}_\tau\right] + \mathbb{E}\left[1_{\{\beta_\varrho<\hat\beta_\varrho,\,\tau>\varrho\}}W^{\tilde\pi}_{\sigma_n}\right]\\ &= \mathbb{E}\left[1_{\{\beta_\varrho<\hat\beta_\varrho,\,\tau\le\varrho\}}W^{\pi}_{\tau_n}\right] + \mathbb{E}\left[1_{\{\beta_\varrho<\hat\beta_\varrho,\,\tau>\varrho\}}W^{\tilde\pi}_{\sigma_n}\right].\end{aligned}$$

Here, the third identity uses the martingale property in (11) together with the fact that $\sigma_n > \varrho$ on $\{\beta_\varrho < \hat\beta_\varrho\}$. We are now going to show that the second summand satisfies, for large enough $n \in \mathbb{N}$,

$$\mathbb{E}\left[1_{\{\beta_\varrho<\hat\beta_\varrho,\,\tau>\varrho\}}W^{\tilde\pi}_{\sigma_n}\right] \ge \mathbb{E}\left[1_{\{\beta_\varrho<\hat\beta_\varrho,\,\tau>\varrho\}}W^{\pi}_{\sigma_n}\right] - \varepsilon. \tag{14}$$

Recalling that $\sigma_n \downarrow \varrho$ as $n \to \infty$ on $\{\beta_\varrho < \hat\beta_\varrho\}$, note that

$$\lim_{n\to\infty} X^{\tilde\pi}_{\sigma_n} = X^{\tilde\pi}_\varrho = X^{\pi}_\varrho = \lim_{n\to\infty} X^{\pi}_{\sigma_n} \text{ on } \{\beta_\varrho<\hat\beta_\varrho\}, \tag{15}$$

where $X^{\tilde\pi}_\varrho = X^\pi_\varrho$ by virtue of (10). Then using Remark 1 we choose $\delta > 0$ such that $\hat\beta_t \le 1-\delta$ for $t \in \mathcal{T}$ and estimate

$$\begin{aligned}\mathbb{E}\left[1_{\{\beta_\varrho<\hat\beta_\varrho,\,\tau>\varrho\}}W^{\tilde\pi}_{\sigma_n}\right] &= \mathbb{E}\left[1_{\{\beta_\varrho<\hat\beta_\varrho,\,\tau>\varrho\}}\bar v\big(\sigma_n, (1-\hat\beta_{\sigma_n})X^{\tilde\pi}_{\sigma_n}\big)\right]\\ &= \mathbb{E}\left[1_{\{\beta_\varrho<\hat\beta_\varrho,\,\tau>\varrho\}}\bar v\big(\sigma_n, [(1-\hat\beta_{\sigma_n})\vee\delta]X^{\tilde\pi}_{\sigma_n}\big)\right]\\ &\ge \mathbb{E}\left[1_{\{\beta_\varrho<\hat\beta_\varrho,\,\tau>\varrho\}}\bar v\big(\sigma_n, [(1-\beta_{\sigma_n})\vee\delta]X^{\tilde\pi}_{\sigma_n}\big)\right].\end{aligned}$$

Since $\pi, \tilde\pi$ are admissible, the processes $\{u(X^\pi_t)\}$ and $\{u(X^{\tilde\pi}_t)\}$ are L^1-bounded by Lemma 1. Thus, together with equation (15) and the explicit form (5) of $\bar v$, the dominated convergence theorem applies to yield

$$\mathbb{E}\left[1_{\{\beta_\varrho<\hat\beta_\varrho,\,\tau>\varrho\}}\left|\bar v\left(\sigma_n, [(1-\beta_{\sigma_n})\vee\delta]X^{\tilde\pi}_{\sigma_n}\right) - \bar v\left(\sigma_n, [(1-\beta_{\sigma_n})\vee\delta]X^{\pi}_{\sigma_n}\right)\right|\right] \to 0$$

as $n \to \infty$. Hence, for large enough $n \in \mathbb{N}$ we have

$$\begin{aligned}&\mathbb{E}\left[1_{\{\beta_\varrho<\hat\beta_\varrho,\,\tau>\varrho\}}W^{\tilde\pi}_{\sigma_n}\right] \ge \mathbb{E}\left[1_{\{\beta_\varrho<\hat\beta_\varrho,\,\tau>\varrho\}}\bar v\left(\sigma_n, [(1-\beta_{\sigma_n})\vee\delta]X^{\pi}_{\sigma_n}\right)\right] - \varepsilon\\ &\quad\ge \mathbb{E}\left[1_{\{\beta_\varrho<\hat\beta_\varrho,\,\tau>\varrho\}}\bar v\left(\sigma_n, (1-\beta_{\sigma_n})X^{\pi}_{\sigma_n}\right)\right] - \varepsilon = \mathbb{E}\left[1_{\{\beta_\varrho<\hat\beta_\varrho,\,\tau>\varrho\}}W^{\pi}_{\sigma_n}\right] - \varepsilon,\end{aligned}$$

i.e. (14) holds. Thus, we have verified that

$$\begin{aligned}\mathbb{E}\left[1_{\{\beta_\varrho<\hat\beta_\varrho\}}W^{\tilde\pi}_\tau\right] &\ge \mathbb{E}\left[1_{\{\beta_\varrho<\hat\beta_\varrho,\,\tau\le\varrho\}}W^{\pi}_{\tau_n}\right] + \mathbb{E}\left[1_{\{\beta_\varrho<\hat\beta_\varrho,\,\tau>\varrho\}}W^{\pi}_{\sigma_n}\right] - \varepsilon\\ &= \mathbb{E}\left[1_{\{\beta_\varrho<\hat\beta_\varrho\}}W^{\pi}_{\tau_n}\right] - \varepsilon,\end{aligned}$$

provided $n \in \mathbb{N}$ is chosen sufficiently large, which establishes (13).

STEP 5. Upon combining (12) and (13) and recalling that $\varepsilon > 0$ was arbitrary, we are now in a position to conclude that

$$\mathbb{E}[W_\tau^{\hat\pi}] \geq \inf_{n\in\mathbb{N}} \mathbb{E}[W_{\tau_n}^\pi],$$

which completes the proof. $\square$

The previous result shows that it suffices to search for a worst-case optimal pre-crash strategy $\pi^\star \in \Pi$ which satisfies $\pi_t^\star.\ell^\infty \leq \hat\beta_t$ for all $t \in \mathfrak{T}$. Thus $\hat\beta$ represents a *frontier* which rules out too blue-eyed investment, i.e. a too great exposure to the threat of a financial market crash.

Bull Markets. Next we show, still assuming situation ($\mathrm{C}_{\mathrm{bull}}$), that the strategy $\pi^\star$ in (9) is optimal for the pre-crash worst-case investment problem ($\mathrm{P}_{\mathrm{pre}}$). Let us first consider the case ($\mathrm{C}_{\mathrm{bull}}$-a), when $\pi^\star = \hat\pi$. By the Indifference-Optimality Principle, it suffices to exhibit, for each admissible strategy $\pi \in \Pi$, a *single* crash scenario $\tau \in \Theta$ such that

$$\mathbb{E}[W_\tau^{\hat\pi}] \geq \mathbb{E}[W_\tau^\pi].$$

However, if we consider the no-crash scenario $(\infty, \ell^\infty) \in \Theta$, then we are in the setting of the Merton problem ($\mathrm{P}_{\mathrm{post}}$) with initial time 0 and constraints given by the Indifference Frontier. Hence, setting

$$K_t \triangleq \{\pi \in \Sigma^{\max} : \pi.\ell^\infty \leq \hat\beta_t\} \text{ for } t \in \mathfrak{T}$$

we are in a position to apply the Change-of-Measure Device, and the desired result follows immediately.
Now consider the case ($\mathrm{C}_{\mathrm{bull}}$-b) and recall that $\Psi_t - \bar\Psi_t \leq 0$, $t \in \mathfrak{T}$. Obviously $W^{\pi^\star} = \{\bar v(t, (1-\pi_t^\star.\ell^\infty)X_t^{\pi^\star})\}$ is a martingale on $[T_0, T] \cup \{\infty\}$, and applying Itô's formula exactly as in the derivation of equation (6) we obtain

$$\begin{aligned} \mathrm{d}W_t^{\pi^\star} &= \rho W_{t-}^{\pi^\star}(\Psi_t - \bar\Psi_t)\mathrm{d}t \\ &\quad + W_{t-}^{\pi^\star}\left\{\rho\pi_t^\star.\sigma_t.\mathrm{d}W_t + \textstyle\int_{\mathbb{R}^n}\left[(1-\pi_t^\star.\ell)^\rho - 1\right]\tilde\nu(\mathrm{d}t, \mathrm{d}\ell)\right\} \text{ on } [0, T_0]. \end{aligned}$$

Thus Lemma 1 implies that $W_t^{\pi^\star} = W_0^{\pi^\star}\exp\{\rho\int_0^t\{\Psi_s - \bar\Psi_s)\mathrm{d}s\}M_t$ for $t \in [0, T_0]$ a.s, where $M = \{M_t\}$ is a positive martingale. Since $\Psi_t - \bar\Psi_t \leq 0$ by assumption, $W^{\pi^\star}$ is an $\{\mathfrak{F}_t\}$-supermartingale. Put differently, the pair $(\infty, \ell^\infty) \in \Theta$ is a worst-case scenario for $\pi^\star$, and the Change-of-Measure Device applies as above to yield the result.

Remark 7 *Intuitively, case (C_{bull}-b) corresponds to a situation when the post-crash market is so attractive that in order to compensate for the gains to be expected after a crash, the investor must take a large crash risk exposure in order to be indifferent: The indifference strategy deliberately worsens great perspectives.*

Worst-Case Optimal Strategy. Combining the above results, the investment problem for worst-case crash scenarios can now be solved completely.

Theorem 2 (Solution of the Worst-Case Portfolio Problem) *Provided the homogeneity conditions (H^1_{post}) and (H^2_{post}) are satisfied, the optimal strategy in the worst-case investment problem (P) is given by*

$$\pi^\star_t \triangleq \underset{\pi \in \Sigma^{\max},\, \pi.\ell^\infty \le \hat{\beta}_t}{\arg\max} \varPhi_t(\pi) \text{ for } t \in \mathfrak{T}$$

before the crash, while after the crash it is optimal to use the Merton strategy

$$\bar{\pi}^\star_t = \underset{\bar{\pi} \in \bar{\Sigma}^{\max}}{\arg\max} \bar{\varPhi}_t(\bar{\pi}) \text{ for } t \in \mathfrak{T}.$$

Moreover, in case (C_{bull}) the associated worst-case optimal terminal utility $v:\ \mathfrak{T} \times (0,\infty) \to \mathbb{R}$ is given by

$$v(t,x) = g(t)u(x) \text{ for } t \in \mathfrak{T},\ x \in (0,\infty),$$
$$\text{where } g(t) = \exp\left\{\rho \int_t^T \varPhi_s(\pi^\star_s)\mathrm{d}s\right\} \text{ for } t \in \mathfrak{T}.$$

Proof After splitting the problem (P) into the subproblems ($\mathrm{P_{post}}$) and ($\mathrm{P_{pre}}$), the first part of the assertion follows from the results of Section 3 and the preceding considerations.
The second part is then immediate from the Change-of-Measure Device and the fact that $(\infty, \ell^\infty) \in \Theta$ is a worst-case scenario for the optimal strategy in case ($\mathrm{C_{bull}}$). □

In a 1-dimensional setting, it is easy to check that

$$\pi^\star = \pi^M \wedge \hat{\pi} \text{ in case } (\mathrm{C_{bull}}) \text{ and } \pi^\star = 0 \text{ in case } (\mathrm{C_{bear}}) \text{ before the crash,}$$

while $\bar{\pi}^\star = \bar{\pi}$ after the crash. As above, $\hat{\pi}$ denotes the indifference strategy, i.e. the solution to the differential equation (8).

Remark 8 *Theorem 2 is also valid for logarithmic preferences if we set $\rho = 0$. The above analysis applies mutatis mutandis, and in fact simplifies because the decomposition (2) becomes additive rather than multiplicative.*

Long-Time Asymptotic Behavior. Finally, we address the asymptotic behavior of the worst-case optimal pre-crash strategy $\pi^\star$ for large time horizons. For this purpose we assume that market coefficients do not depend upon time.
It clearly suffices to consider case ($\mathrm{C_{bull}}$). In case ($\mathrm{C_{bull}}$-b) it is obvious that $\pi^\star_t$ coincides with the Merton strategy for large enough $T - t$. On the other hand, in case ($\mathrm{C_{bull}}$-a) the stationary point $\tilde{\beta}$ of equation ($\mathrm{I_{bull}}$) is attained for $\psi(\tilde{\beta}) = \bar{\varPhi}(\bar{\pi}^\star)$, i.e.

$$\max_{\pi \in \Sigma^{\max},\, \pi.\ell^\infty = \tilde{\beta}} \varPhi(\pi) = \max_{\bar{\pi} \in \bar{\Sigma}^{\max}} \bar{\varPhi}(\bar{\pi}).$$

The stationary portfolio proportion $\tilde{\pi}$ is then given as the maximizer of the left-hand side in the preceding identity. In particular, we can make the following observation:

Corollary 2 (Asymptotics of Worst-Case Optimal Strategies) *Suppose the pre- and post-crash market coefficients coincide and do not depend on time. Then as the time horizon gets arbitrarily large the worst-case optimal pre-crash strategy converges to the Merton strategy* π^M *if* $\pi^M.\ell^\infty \leq 1$*, and to* $\arg\max_{\pi \in \Sigma^{\max},\, \pi.\ell^\infty = 1} \Phi(\pi)$ *if* $\pi^M.\ell^\infty > 1$.

Proof Assume that $\pi^M.\ell^\infty \leq 1$ and observe that $\tilde{\beta} = \pi^M.\ell^\infty$ and $\tilde{\pi} = \pi^M$. The stationary point $\tilde{\beta}$ is absorbing for the backward differential equation ($\mathrm{I}_{\mathrm{bull}}$) because its right-hand side is negative and bounded away from 0 as long as $\hat{\beta}$ is bounded away from $\tilde{\beta}$. Next suppose that $\pi^M.\ell^\infty > 1$ and note that the right-hand side of ($\mathrm{I}_{\mathrm{bull}}$) is negative and bounded away from 0 as long as $\hat{\beta}$ is bounded away from 1, so the stationary point $\tilde{\beta} = 1$ is absorbing. □

In particular, in a 1-dimensional setting the worst-case optimal strategy converges towards $\pi^M \wedge \frac{1}{\ell^\infty}$.

6 Discussion and Numerical Illustrations

This section provides some intuition for the worst-case optimal strategies determined in Section 5.

Discussion of Optimal Strategies. The form of worst-case optimal strategies is illustrated for a 1-dimensional framework in Figure 1 for case ($\mathrm{C}_{\mathrm{bull}}$-a) and in Figure 2 for case ($\mathrm{C}_{\mathrm{bull}}$-b), together with the corresponding Merton proportions and the asymptotic portfolio for large time horizons. Unless stated otherwise, we use the following coefficients for our numerical results:

$$\begin{aligned}
&\rho = -2,\ r = \bar{r} = 5\%,\ \eta = \bar{\eta} = 20\%,\ \sigma = \bar{\sigma} = 30\%,\\
&\vartheta = \bar{\vartheta} = 1,\ \ell = \bar{\ell} = 5\%,\ \ell^\infty = 50\%.
\end{aligned}$$

Here, by a slight abuse of notation we denote by ϑ the total intensity of jumps, and by ℓ the jump height of regular jumps; thus the intensity of ν is given by $\vartheta\delta_\ell$.

We wish to point out that the worst-case optimal strategy is in accordance with empirical observations and professional advice concerning asset allocation: Towards the end of the time horizon, the investor 'plays safe' and re-allocates her wealth from risky assets to riskless bonds in order to lock in the proceeds of trading and thus become less vulnerable to the hazard of a financial market collapse. Moreover, as holdings of risky assets are less than those corresponding to traditional preferences, in a general equilibrium model with worst-case investors the equity premium is smaller than predicted by the ordinary theory. Hence, a worst-case attitude towards catastrophic events can contribute to the understanding of the *equity premium puzzle*.

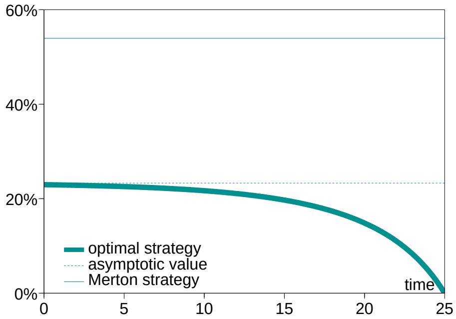

Fig. 1 Optimal strategy in case (C_{bull}-a). $\bar{r} = 7\%$, $\bar{\eta} = 15\%$, $\bar{\sigma} = 35\%$, $\bar{\vartheta} = 1.5$, $\bar{\ell} = 5\%$.

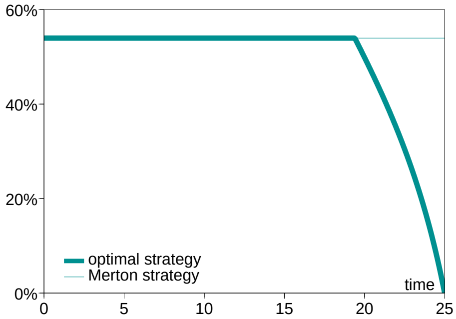

Fig. 2 Optimal strategy in case (C_{bull}-b). $\bar{r} = 7\%$, $\bar{\eta} = 20\%$, $\bar{\sigma} = 25\%$, $\bar{\vartheta} = 0.8$, $\bar{\ell} = 5\%$.

Sensitivity of Optimal Strategies. We next address the sensitivity of worst-case optimal strategies on risk aversion, crash height, and jump risk. Figure 3 depicts worst-case optimal strategies for different levels of risk aversion, together with the associated limiting values. Figure 4 illustrates the dependence of optimal strategies on the worst-case crash height.[12] It is interesting to note that the investor is not put off completely from risky investment even if there is a threat of total loss; at the beginning of a sufficiently long time horizon, she will in fact behave like a classical Merton investor.

Finally, Figure 5 shows worst-case optimal and Merton strategies for different distributions between diffusive and jump risk. The parameters are chosen in such a way that the expectation and the variance of the infinitesimal returns coincide, while the contribution of the Wiener and Poisson sources of risk changes. We see that, as in the classical Merton problem, the investor prefers diffusive risk over jump risk.

[12] See the first paragraph of Section 7 for the correct interpretation of $\ell^\infty = 100\%$.

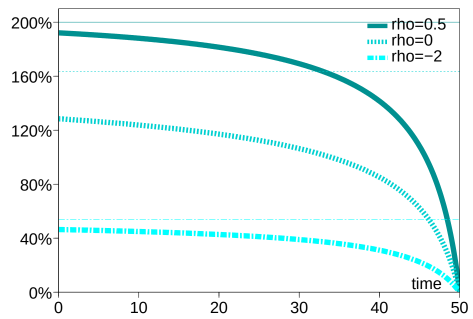

Fig. 3 Optimal strategy for different levels of risk aversion.

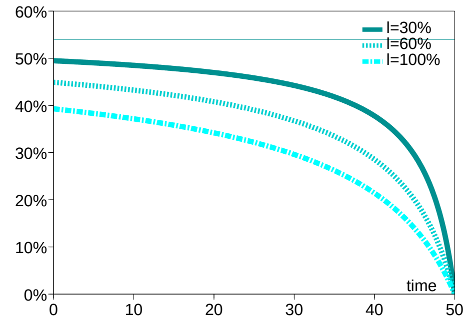

Fig. 4 Optimal strategy for different crash heights.

Worst-Case Optimal Constant Strategy. As investment strategies with constant proportions often play a prominent role in portfolio optimization, it is instructive to compute the optimal *constant* pre-crash portfolio process. Given that market coefficients are constant and do not change after a possible crash, it is not hard to see that the worst-case scenario for an arbitrary constant pre-crash strategy is given by $(T, \ell^\infty) \in \Theta$. By the Change-of-Measure Device, the worst-case optimal constant pre-crash portfolio $\pi^\star_{\text{const}}$ is therefore given as the maximizer in $\max_{\pi \in \Sigma^\infty} u(x)(1 - \pi.\ell^\infty)^\rho \exp\{\rho T \Phi(\pi)\}$. Thus, we have

$$\pi^\star_{\text{const}} = \operatorname*{arg\,max}_{\pi \in \Sigma^\infty} \log(1 - \pi.\ell^\infty) + T\Phi(\pi)$$

and due to strict concavity $\pi^\star_{\text{const}}$ is uniquely determined by the corresponding first-order condition provided that $\pi^M.\ell^\infty < 1$. In this case, it is also clear that $\pi^\star_{\text{const}}$ converges to π^M as the time horizon gets arbitrarily large. In a 1-dimensional framework without jumps, the optimal constant strategy can be computed explicitly as

$$\pi^\star_{\text{const}} = \left(\tfrac{1}{2}(\pi^M + \tfrac{1}{\ell^\infty}) - \sqrt{\tfrac{1}{4}(\pi^M - \tfrac{1}{\ell^\infty})^2 + \tfrac{1}{(1-\rho)\sigma^2 T}} \right)^+ .$$

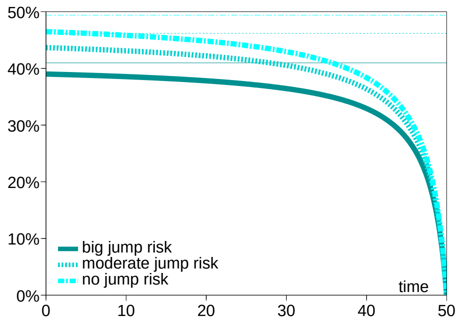

Fig. 5 Optimal strategy for different levels of jump risk. $\eta = \bar{\eta} = 100\%$, $\sigma = \bar{\sigma} = 30\%$, $\vartheta = \bar{\vartheta} = 2$, $\ell = \bar{\ell} = 30\%$ (big jump risk); $\eta = \bar{\eta} = 69\%$, $\sigma = \bar{\sigma} = 45\%$, $\vartheta = \bar{\vartheta} = 1.2$, $\ell = \bar{\ell} = 24\%$ (moderate jump risk); $\eta = \bar{\eta} = 40\%$, $\sigma = \bar{\sigma} = 52\%$, $\vartheta = \bar{\vartheta} = 0$, $\ell = \bar{\ell} = 0\%$ (no jump risk).

Figures 6 and 7 depict the worst-case optimal constant strategy $\pi^\star_{\text{const}}$ together with its dynamically optimal counterpart $\pi^\star$ and the corresponding asymptotic value.
Note that on a large time horizon, $\pi^\star_{\text{const}}$ is above $\pi^\star$, while it is smaller than $\pi^\star_0$ for short time horizons.

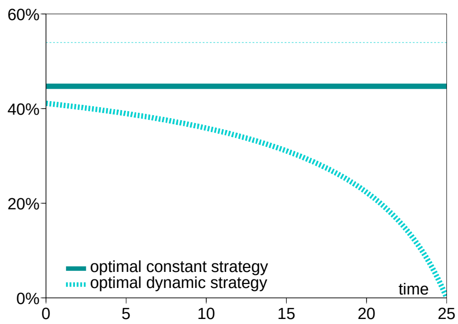

Fig. 6 Worst-case optimal constant strategy and optimal dynamic strategy.

Effective Wealth Loss. To assess the wealth loss incurred by using the Merton strategy under the threat of a crash, we determine the fraction of initial wealth a Merton investor would be willing to give up in order to be able to apply the worst-case optimal strategy. Thus we compute the worst-case expected utility attained by the Merton strategy with initial wealth x_0, and then we compute the level x_1 of initial wealth required to achieve the same worst-case utility with the optimal strategy. The **effective wealth loss** is then given by $\frac{x_0 - x_1}{x_0}$.

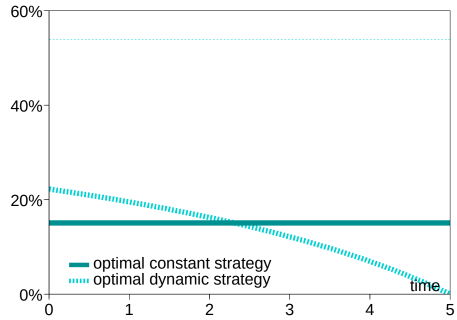

Fig. 7 Worst-case optimal constant strategy and optimal dynamic strategy.

We consider the case of constant and identical pre- and post-crash market coefficients, and recall from the previous paragraph that the worst-case scenario for any constant strategy is given by $(T, \ell^\infty) \in \Theta$. It is not hard to see that $(0, \ell^\infty) \in \Theta$ is also a worst-case scenario for the Merton strategy π^M, and due to indifference $(0, \ell^\infty)$ is a worst-case scenario for the optimal strategy $\hat{\pi} = \pi^\star$ as well. Thus the effective wealth loss is obtained from the requirement that $(1 - \pi^M.\ell^\infty)x_0 = (1 - \hat{\pi}_0.\ell^\infty)x_1$ as

$$\frac{x_0 - x_1}{x_0} = \frac{(\pi^M - \hat{\pi}_0).\ell^\infty}{1 - \hat{\pi}_0.\ell^\infty}$$

provided that $\pi^M.\ell^\infty < 1$. Figure 8 illustrates the effective wealth loss for different levels of risk aversion. Note that for $\rho = 0.5$ the Merton strategy leads to bankruptcy in the worst-case scenario. The wealth loss is huge for shorter time horizons and decreases as the time horizon gets large. However, for a time horizon $T = 50$ ($T = 25$) and $\rho = -2$, the effective wealth loss still amounts to a significant 8.1% (4.9%) of the investor's initial wealth.

Substitution and Crowding-Out Effects in Multi-Asset Markets. Finally, we illustrate the implications of the worst-case approach for asset allocation in multi-asset markets. The following figures are generated with the coefficients

$$\begin{aligned}
&\rho = -2,\ r = 5\%,\ \bar{r} = 7\%,\ \eta = (20\%, 25\%),\ \bar{\eta} = (14\%, 15\%),\\
&\sigma_1 = 30\%,\ \sigma_2 = 35\%,\ \varrho = 20\%,\ \bar{\sigma}_1 = \bar{\sigma}_2 = 20\%,\ \bar{\varrho} = 50\%,\\
&\vartheta = 1,\ \bar{\vartheta} = 1.5,\ \ell = \bar{\ell} = (5\%, 5\%),
\end{aligned}$$

using obvious vector notation and letting ϱ denote the correlation between the returns of risky assets. As the worst-case crash height varies, we can observe some interesting phenomena.

Figure 9 represents the benchmark case. Interestingly, the investor reduces her holdings of the first risky asset to 0 before the end of the time horizon. As shown in Figure 10, this **crowding-out** effect can become so strong that she

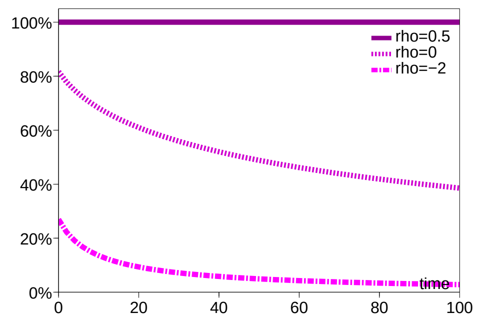

Fig. 8 Worst-case effective wealth loss incurred by the Merton strategy.

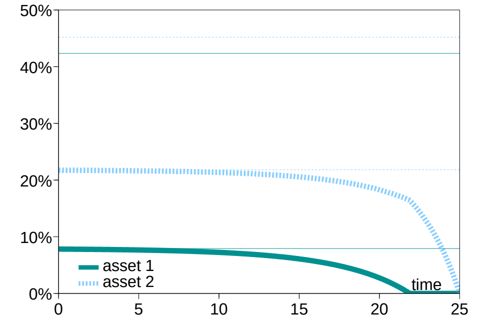

Fig. 9 Worst-case optimal portfolio proportions for $\ell^\infty = (50\%, 50\%)$.

does not invest into this asset *at all*. Note that the worst-case crash height in Figure 10 is smaller than that in Figure 9. On the other hand, Figure 11 shows that the risky assets exchange roles as the crash height of asset 2 increases.

7 Extensions

This section discusses some possible generalizations of our basic model.

Arbitrary Crash Size. If we allow for an arbitrarily large relative crash size $\ell \in [0, 1)$ for some or all risky assets, the optimality result of Theorem 2 remains valid if we formally set the corresponding coordinates of ℓ^∞ equal to 1. In fact, the worst-case portfolio problem can be reduced to the pre-crash problem ($\mathrm{P}_{\mathrm{pre}}$) as before, the only difference being that the infimum over jump heights is not necessarily attained. The remainder of the argument does not make use of the assumption $\ell^\infty \in [0, 1)^n$.

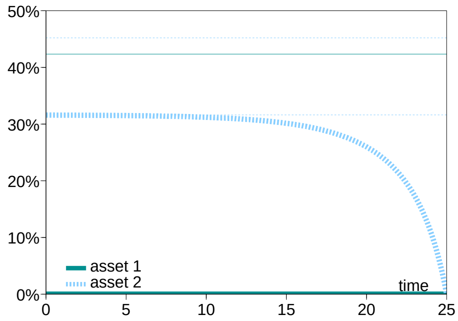

Fig. 10 Worst-case optimal portfolio proportions for $\ell^\infty = (50\%, 30\%)$.

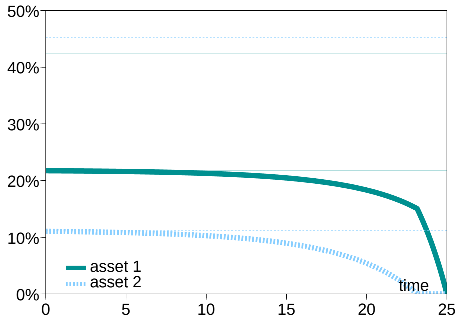

Fig. 11 Worst-case optimal portfolio proportions for $\ell^\infty = (40\%, 70\%)$.

Unknown Post-Crash Market Coefficients. The worst-case model can be extended to situations of uncertainty about post-crash market coefficients. Suppose that, in addition to the crash scenario $(\tau, \ell) \in \Theta$, the investor is concerned about a parameter $\kappa \in K$ which is used to index post-crash market coefficients. If she also takes a Knightian attitude towards the latter source of uncertainty, her optimal strategy is the same as in Theorem 2, where in conditions ($\mathrm{C}_{\mathrm{bull}}$) and ($\mathrm{C}_{\mathrm{bear}}$) each post-crash quantity $\bar{\aleph}$ is replaced by $\inf_{\kappa \in K} \bar{\aleph}^{(\kappa)}$, and where the Indifference Frontier $\hat{\beta}$ is now given by

$$\dot{\hat{\beta}}_t = (1 - \hat{\beta}_t) \left\{ \psi_t(\hat{\beta}_t) - \inf_{\kappa \in K} \bar{\Phi}_t^{(\kappa)}(\bar{\pi}_t^{(\kappa),\star}) \right\} \text{ on } \mathfrak{T}, \ \hat{\beta}_T = 0.$$

After the crash, it is optimal to use the relevant Merton strategy $\pi^{(\kappa),\star} \triangleq \pi^{(\kappa),M}$.

Shorting. If it is possible for the investor to short risky assets, the results obtained above remain valid mutatis mutandis. More precisely, we can drop nonnegativity requirements on π throughout, but continue to assume that admissible portfolio strategies are bounded. In particular, $\psi_t(0) = \max_{\pi.\ell^\infty=0} \Phi_t(\pi)$ and we need to distinguish

$$\max_{\pi.\ell^\infty=0} \Phi_t(\pi) \leq \bar{\Psi}_t \text{ for } t \in \mathfrak{T} \ (\mathrm{C}_{\mathrm{bull}}^{\mathrm{short}}) \ \text{ and } \ \max_{\pi.\ell^\infty=0} \Phi_t(\pi) \geq \bar{\Psi}_t \text{ for } t \in \mathfrak{T} \ (\mathrm{C}_{\mathrm{bear}}^{\mathrm{short}}).$$

In economic terms, with shorting a wealth loss due to a crash can be hedged with strategies different from the no-participation strategy. In case ($\mathrm{C}_{\mathrm{bear}}^{\mathrm{short}}$), the best crash hedging strategy $\pi^\star = \{\pi_t^\star\}$, $\pi_t^\star \triangleq \arg\max_{\pi.\ell^\infty=0} \Phi_t(\pi)$ for $t \in \mathfrak{T}$, is seen to be optimal, its worst-case scenario being a crash of size 0 at time 0.[13] In case ($\mathrm{C}_{\mathrm{bull}}^{\mathrm{short}}$), we can argue as before to verify that $\pi^\star = \{\pi_t^\star\}$ as defined in (9) is worst-case optimal.

Multiple Crashes. Indifference arguments can also be used to treat the worst-case investment problem when more than one crash is possible. Thus assume a financial market with a single risky asset, suppose that there can be at most $N \in \mathbb{N}$ (non-simultaneous) crashes $(\tau^{(k)}, \ell^{(k)})$, $k = 1, \ldots, N$, and denote by $\aleph^{(k)}$ the quantity $\aleph$ in the market between the k^{th} and the $(k+1)^{\text{th}}$ crash. For simplicity suppose that

$$r_t^{(k-1)} \leq \Psi_t^{(k)} \text{ for } t \in \mathfrak{T} \text{ and } k = 1, \ldots, N.$$

Then after the $(N-1)^{\text{th}}$ crash the investor faces the worst-case portfolio problem (P) with random initial time $\tau^{(N-1)}$, pre-crash market $N-1$ and post-crash market N. A modification of the arguments presented above shows that the optimal strategy is given by $\pi^{(N-1),\star} \triangleq \pi^{(N-1),M} \wedge \hat{\pi}^{(N-1)}$, where

$$\dot{\hat{\pi}}_t^{(N-1)} = \frac{1 - \hat{\pi}_t^{(N-1)} \ell^{(N),\infty}}{\ell^{(N),\infty}} \left\{ \Phi_t^{(N-1)}(\hat{\pi}_t^{(N-1)}) - \Phi_t^{(N)}(\pi_t^{(N),\star}) \right\}, \ \hat{\pi}_T^{(N-1)} = 0$$

and $\pi^{(N),\star} \triangleq \pi^{(N),M}$. In particular, it follows that the worst-case optimal utility is given by $v^{N-1}(\tau^{(N-1)}, X^\pi_{\tau^{(N-1)}})$, where $v^{N-1}(t,x) = g^{(N-1)}(t)u(x)$ with

$$g^{(N-1)}(t) = \exp\left\{ \rho \textstyle\int_t^T \Phi_s^{(N-1)}(\pi_s^{(N-1),\star}) \mathrm{d}s \right\}, \ t \in \mathfrak{T}$$

as in Theorem 2. Thus, the worst-case problem with N possible crashes is reduced to one with at most $N-1$ crashes. Arguing recursively, we find that with

$$\dot{\hat{\pi}}_t^{(k-1)} = \frac{1 - \hat{\pi}_t^{(k-1)} \ell^{(k),\infty}}{\ell^{(k),\infty}} \left\{ \Phi_t^{(k-1)}(\hat{\pi}_t^{(k-1)}) - \Phi_t^{(k)}(\pi_t^{(k),\star}) \right\}, \ \hat{\pi}_T^{(k-1)} = 0$$

for $k = 1, \ldots, N$ it is optimal to use the portfolio strategy $\pi^{(k),\star} \triangleq \pi^{(k),M} \wedge \hat{\pi}^{(k)}$ in the period between the k^{th} and the $(k+1)^{\text{th}}$ market crash. This is illustrated in Figure 12.

[13] Thus, it attains the same worst-case bound as the no-participation strategy, which is therefore also worst-case optimal. However, $\pi^\star$ is always at least as good as the no-participation strategy, and sometimes even better.

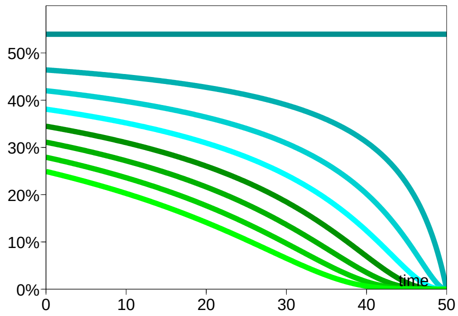

Fig. 12 Optimal strategy with multiple crashes, $k = 0, 1, \ldots, 7$ crashes remaining (top to bottom).

Individual Crashes. Consider a financial market with n risky assets, each of which is exposed to the threat of an individual crash; in a crash scenario, any one (but not more) of the risky assets may decline in value by a percentage amount at most $\ell^{\infty,i} < 1$, $i = 1, \ldots, N$, and there can be at most one crash. Then there is a unique indifference strategy $\hat{\pi}$ which is determined from the system of ordinary differential equations

$$\dot{\hat{\pi}}^i_t = \frac{1 - \hat{\pi}^i_t \ell^{\infty,i}}{\ell^{\infty,i}} \left\{ \Phi_t(\hat{\pi}_t) - \bar{\Phi}_t(\bar{\pi}^\star_t) \right\}, \; \hat{\pi}^i_T = 0 \text{ for } i = 1, \ldots, n,$$

and the optimal pre-crash strategy $\pi^\star = \{\pi^\star_t\}$ is given by

$$\pi^\star_t \triangleq \operatorname*{arg\,max}_{\pi \in [0, \hat{\pi}_t]} \Phi_t(\pi) \text{ for } t \in \mathfrak{T}.$$

Market Recovery. It is possible to incorporate more general asset price dynamics for the post-crash market. For instance, suppose that the crisis caused by a crash has only a finite duration, and the market returns to normality after an exponential waiting time. This can be subsumed by the Markov regime-switching framework of [Bäuerle, Rieder 2004] by taking one state for the stressed market and one absorbing state for the recovered market with the same coefficients as before the crash. Using the Change-of-Measure Device, it follows that the optimal strategy after recovery is the appropriate Merton strategy; similarly, one shows that the optimal strategy between the crash and recovery is given by the Merton strategy for the stressed market. In particular, the worst-case portfolio problem can be reduced to a pre-crash problem of the form ($\mathrm{P_{pre}}$),

$$\sup_{\pi \in \Pi} \inf_{\tau \in \Theta} \mathbb{E}\left[\bar{v}(\tau, (1 - \pi_\tau . \ell^\infty) X^\pi_\tau) \right],$$

where $\bar{v}(t, x) = \bar{g}(t) u(x)$, $t \in \mathfrak{T}$, $x \in (0, \infty)$, and $\bar{g}$ is the solution to

$$\dot{\bar{g}}(t) = -\rho \bar{\Psi}_t - q[g(t) - \bar{g}(t)] \text{ on } \mathfrak{T}, \; \bar{g}(T) = 1.$$

Here $\frac{1}{q}$ is the expected duration of the crisis and $g(t) = \exp\{\rho \int_t^T \Psi_s \mathrm{d}s\}$, $t \in \mathcal{T}$, compare [Bäuerle, Rieder 2004]. Under the natural assumption that $r_t \leq \bar{\Psi}_t \leq \Psi_t$, $t \in \mathcal{T}$, the Indifference Frontier is given by

$$\dot{\hat{\beta}}_t = (1 - \hat{\beta}_t) \left\{ \psi_t(\hat{\beta}_t) - \bar{\Phi}_t(\bar{\pi}_t^\star) - \tfrac{q}{\rho} \left[\frac{g(t)}{\bar{g}(t)} - 1 \right] \right\} \text{ on } \mathcal{T},\ \hat{\beta}_T = 0$$

and satisfies $0 \leq \hat{\beta}_t < 1$, $t \in \mathcal{T}$. Then the arguments given above for case (C_{bull}-a) apply to verify that the corresponding indifference strategy $\hat{\pi} = \{\hat{\pi}_t\}$ defined by (7) is worst-case optimal.

Bear Markets. If hypothesis (H_{pre}) is violated, shorting is possible, and it is assumed that a crash, if it occurs, has a certain *minimum size*, there can be more than one indifference strategy even in the 1-dimensional case. We refer the interested reader to [Menkens 2006] for a detailed investigation of such bearish pre-crash markets in a single-asset framework without jumps.

Further Generalizations. Further extensions are possible. For instance, it is possible to allow for upward jumps in the asset price dynamics, and deterministic trading constraints in the pre- and post-crash market are easily incorporated. It should also be possible to adapt our methodology to Lévy processes with infinite activity. Of course, the extensions discussed above can be combined to construct more sophisticated models for crash scenarios.

8 Conclusion

Conclusion. In this paper, we have solved the optimal investment problem for worst-case crash scenarios in a multi-asset jump-diffusion market using a novel martingale approach. The fundamental concept of indifference is highlighted, and its relation to optimality is clarified by the Indifference-Optimality Principle. The Change-of-Measure Device is combined with the concept of Indifference Frontiers to establish and interpret our optimality result.
The worst-case approach to portfolio choice under the threat of a crash leads to intuitive solutions for optimal strategies. In particular, it provides a natural rationale for real-world investor behavior: The optimal asset allocation shifts from risky to riskless as the end of the time horizon approaches, while the classical Merton strategy appears as a limit for large time horizons.

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Asset Allocation and Liquidity Breakdowns

Background

In this chapter, we investigate optimal portfolio decisions in the presence of illiquidity risk. Illiquidity is understood as a period in which it is impossible to trade on financial markets. We use dynamic programming techniques in combination with abstract convergence results to solve the corresponding optimal investment problem.

This chapter is based on [Diesinger, Kraft, Seifried 2009], which will appear in *Finance and Stochastics*. The article [Diesinger, Kraft, Seifried 2009] originates from the earlier version [Diesinger, Kraft 2007] by Holger Kraft and Peter Diesinger, where only logarithmic preferences are considered. In [Diesinger, Kraft, Seifried 2009], we establish the crucial convergence results and provide a formal proof of the Verification Theorem, with the help of which general CRRA preferences can be analyzed. The presentation here focuses on my contributions and limits itself to brief summaries elsewhere, referring to the original article where appropriate.

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[Diesinger, Kraft 2007] DIESINGER, P., KRAFT, H.: *Asset Allocation and Liquidity Breakdowns: What if your broker does not answer the phone?* preprint, available at `http://ssrn.com/abstract=994507`.

[Diesinger, Kraft, Seifried 2009] DIESINGER, P., KRAFT, H., SEIFRIED, F.T.: *Asset Allocation and Liquidity Breakdowns: What if your broker does not answer the phone?* to appear in Finance and Stochastics.

Asset Allocation and Liquidity Breakdowns

What if your broker does not answer the phone?

Abstract This paper analyzes the portfolio decision of an investor facing the threat of illiquidity. We explicitly solve the portfolio problem for a logarithmic investor. For general utility functions, an explicit solution does not seem to be available. However, under a mild growth condition we show that the value function of a model with only finitely many liquidity breakdowns converges locally uniformly to the value function of a model with infinitely many breakdowns as the number of possible breakdowns tends to infinity. Furthermore, we show how optimal security demands can be used to approximate the optimal solution in the model with infinitely many breakdowns. The results are illustrated for an investor with a power utility function.

Keywords illiquidity · portfolio decision · efficiency loss · rare disasters

Mathematics Subject Classification (2000) 91B28 · 93E20

1 Introduction

Over decades the assumption that investors can trade continuously has been central to the theory of modern finance.[1] In the history of trading at stock exchanges, there are, however, numerous examples of periods when liquidity dried out or trading was (virtually) impossible for a number of reasons, including political turmoil, war, and hyperinflation. For instance, after World War II, the Tokyo Stock Exchange closed from August 1945 until May 1949 and reopened with a loss of 95% compared to the pre-war stock prices. The stock exchanges of European countries that had been invaded by Germany

[1] See, e.g., [Merton 1969] and [Merton 1971].

were closed down for several months. The same is true of the German stock exchanges, which were closed for at least six months. Even the Swiss Stock Exchange closed from 10 May 1940 until 8 June 1940 and reopened with a loss of more than 20%. Recently, the New York Stock Exchange closed for four days after the terrorist attacks of 9/11 and reopened on 17 September with a record trade volume of 2.37 billion shares; the US stock market lost almost 10% of its value. This example shows that trading breaks can induce strong wishes to rebalance portfolios and may be accompanied by sharp price drops. The goal of this paper is to analyze the portfolio decision of an investor facing the threat of trading interruptions. The following table summarizes some historical examples.[2]

Exchange	Trading Break	Comment
London	07/1914–01/1915	WW I
New York	08/1914–11/1914	WW I
Zurich	05/1940–06/1940	WW II (Mobilization)
Frankfurt	04/1945–09/1945	Aftermath of WW II
Tokyo	08/1945–05/1949	Aftermath of WW II
New York	09/11/2001–09/14/2001	Terrorist Attack

There are several related papers that address the modeling of liquidity effects. One strand of literature weakens the assumption that investors are price takers and focuses on so-called large traders who cannot trade without affecting market prices; see, e.g., [Bank, Baum 2004] and [Çetin, Jarrow, Protter 2004]. A second strand considers models with transaction costs, in which it is not optimal to trade continuously. Papers in this area include [Davis, Norman 1990] and [Korn 1998], among others. [Longstaff 2001] looks at the portfolio problem when only strategies of finite variation can be implemented, which is interpreted as a liquidity constraint. [Schwartz, Tebaldi 2006] assume that a risky asset cannot be traded at all, i.e. the trading interruption is permanent. [Rogers 2001] analyzes portfolio decisions when strategies can only be updated at discrete points in time, although trading takes place continuously. [Kahl, Liu, Longstaff 2003] and [Longstaff 2005] are closely related to this paper. [Kahl, Liu, Longstaff 2003] consider an investor that cannot trade a risky asset for a given period of time and [Longstaff 2005] analyzes the implications for equilibrium asset prices. Finally, this paper is also related to the asset pricing literature dealing with the equity premium puzzle. As [Rietz 1988] and [Barro 2006] point out, the puzzle can be (partly) resolved if the potential for rare economic disasters is taken into account.

This paper is structured as follows. Section 2 describes the mathematical framework and analyzes the dynamics of the investor's portfolio process in periods of illiquidity. Section 3 states the portfolio problem and establishes a convergence result for value functions. In Section 4, we introduce the HJB equations and prove a verification theorem. Section 5 derives an explicit solution for logarithmic investors with infinitely many liquidity breakdowns. In

[2] See, e.g., [Jorion, Goetzmann 1999] or [Siegel 2002].

Section 6, we discuss portfolio problems where only finitely many liquidity breakdowns are possible and the investor has logarithmic or power utility. Furthermore, we analyze the relations of both problems to the situations with infinitely many periods of illiquidity. Section 7 illustrates our theoretical results by a numerical analysis and Section 8 concludes.

2 Continuous-Time Portfolio Dynamics with Illiquidity

In this section, we provide a model of a two-asset securities market in which liquidity breakdowns can occur. It is assumed that all random variables and stochastic processes are defined on a stochastic basis $(\Omega, \mathfrak{F}, \{\mathfrak{F}(t)\}_{t\in[0,T]}, \mathbb{P})$ satisfying the usual conditions where $T > 0$ is a finite time horizon. One asset is a locally risk-free money market account M, which is modeled as a continuous finite variation process taking values in $(0,\infty)$, and the second security is a risky asset with price S, which is modeled as a $(0,\infty)$-valued semimartingale.

We suppose that the economy is in one of two possible liquidity regimes which we refer to as state 0 and state 1. We interpret state 0 as the normal state of the market in which trading takes place continuously and state 1 as an illiquidity state in which trading is not possible and asset prices can have different dynamics than in state 0. We assume that regime shifts from state i to state $1-i$ are triggered by a counting process $N_{i,1-i}$ as long as a given maximal number $k_0 \in \mathbb{N} \cup \{\infty\}$ of illiquidity periods is not exceeded. The initial state is 0, and we assume that $N_{i,1-i}$ is non-explosive in that it has finitely many jumps in $[0,T]$ a.s. The current **state of the market** is then described by the $\{0,1\}$-valued càdlàg process I given by

$$\begin{aligned} \mathrm{d}I &= 1_{\{I_-=0,\ K_-<k_0\}}\mathrm{d}N_{0,1} - 1_{\{I_-=1,\ K_-<k_0\}}\mathrm{d}N_{1,0}, \\ \mathrm{d}K &= 1_{\{I_-=1\}}\mathrm{d}N_{1,0} \end{aligned}$$

with $I(t_0) = 0$ and $K(t_0) = 0$. The process K counts the number of jumps into the liquidity state since initial time $t_0 \in [0,T]$. The solutions of the above stochastic differential equations are denoted by I^{t_0,k_0} and K^{t_0,k_0} and we omit the superscripts if there is no ambiguity. Besides, we set $\tau_{1,0}^0 \triangleq t_0$, $\tau_{0,1}^k \triangleq \inf\{t \in (\tau_{1,0}^{k-1}, T] : I(t) = 1\}$ and $\tau_{1,0}^k \triangleq \inf\{t \in (\tau_{0,1}^k, T] : I(t) = 0\}$ for $k \in \mathbb{N}$. Thus, $t_0 = \tau_{1,0}^0 \le \tau_{0,1}^1 \le \tau_{1,0}^1 \le \tau_{0,1}^2 \le \tau_{1,0}^2 \le \dots$ are stopping times marking the regime shifts from one state into the other and we have

$$I^{t_0,k_0} = \sum_{k=1}^{k_0} 1_{[\![\tau_{0,1}^k, \tau_{1,0}^k[\![}.$$

The investor is restricted to choose self-financing portfolio strategies such that her wealth dynamics

$$\mathrm{d}X = \varphi_-\mathrm{d}S + (X_- - \varphi_- S_-)\,\frac{\mathrm{d}M}{M_-}\,,\ X(t_0) = x > 0,$$

have a unique solution X with $X(t) \geq 0$ for all $t \in [t_0, T]$ a.s. The càdlàg process φ denotes the number of stocks in the investor's portfolio, so

$$\varphi = \varphi_I = \{\varphi_{I(t)}(t)\}_{t\in[t_0,T]} \text{ with } \varphi_1 = \sum_{k=1}^{k_0} 1_{[\![\tau_{0,1}^k, \tau_{1,0}^k[\![} \, \varphi_0(\tau_{0,1}^k-).$$

In the liquidity state, the investor can choose her portfolio strategy φ_0 according to the above restrictions. However, when the market is illiquid, then the investor is forced to hold the portfolio that she has chosen before the liquidity breakdown. This strategy is modeled by the process φ_1. Alternatively, one can describe the investor's strategies by the wealth proportion invested in the risky asset. Therefore, we also introduce the portfolio processes π, π_0, and π_1 corresponding to φ, φ_0, and φ_1 via

$$\pi = \pi_I = \{\pi_{I(t)}(t)\}_{t\in[t_0,T]} \text{ with } \pi_i \triangleq \frac{\varphi_i S}{X} \text{ for } i = 0, 1.$$

The dynamics of π_1 are exogenously determined by the market. The wealth dynamics can then be rewritten as

$$\mathrm{d}X = X_- \left[\pi_- \frac{\mathrm{d}S}{S_-} + (1-\pi_-)\frac{\mathrm{d}M}{M_-}\right], \; X(t_0) = x_0. \tag{1}$$

To avoid bankruptcy, short-selling is not allowed. Therefore, the class of **admissible portfolio strategies** consists of all càdlàg processes π_0 that take values in $[0, 1]$. As for the processes I and K, we denote by X^{π_0,t_0,x_0,k_0} the wealth process starting at time $t_0 \in [0, T]$ with initial value $x_0 \in (0, \infty)$.

The following lemma derives a stochastic differential equation for the dynamics of the investor's portfolio process in the illiquidity state and provides an explicit solution.

Lemma 1 (Portfolio Dynamics in Illiquidity) *For every $k \in \mathbb{N}$ with $k \leq k_0$, the dynamics of the portfolio process π on the stochastic interval $[\![\tau_{0,1}^k, \tau_{1,0}^k[\![$ are given by*

$$\mathrm{d}\pi = \pi_-(1-\pi_-)\left(\frac{\mathrm{d}S}{S_-} - \frac{\mathrm{d}M}{M_-} - \pi_-\frac{\mathrm{d}\langle S\rangle^{\mathfrak{c}}}{S_-^2} - \pi_- \mathrm{d}\sum \frac{(\frac{\Delta S}{S_-})^2}{1+\pi_-\frac{\Delta S}{S_-}}\right)$$

with

$$\pi(\tau_{0,1}^k) = \frac{\pi(\tau_{0,1}^k-)[1+\frac{\Delta S(\tau_{0,1}^k)}{S(\tau_{0,1}^k-)}]}{1+\pi(\tau_{0,1}^k-)\frac{\Delta S(\tau_{0,1}^k)}{S(\tau_{0,1}^k-)}}.$$

This stochastic differential equation admits the closed-form solution $\pi = \frac{1}{1+Z}$, where Z is the stochastic exponential process given by

$$\mathrm{d}Z = Z_-\left(-\frac{\mathrm{d}S}{S_-} + \frac{\mathrm{d}M}{M_-} + \frac{\mathrm{d}\langle S\rangle^{\mathfrak{c}}}{S_-^2} + \mathrm{d}\sum\frac{(\frac{\Delta S}{S_-})^2}{1+\frac{\Delta S}{S_-}}\right), \; Z(\tau_{0,1}^k) = \frac{1}{\pi(\tau_{0,1}^k)} - 1.$$

Proof Since $\pi(\tau_{0,1}^k) = \frac{\varphi(\tau_{0,1}^k-)S(\tau_{0,1}^k)}{X(\tau_{0,1}^k)}$ by definition of π, we have

$$\pi(\tau_{0,1}^k) = \pi(\tau_{0,1}^k-)\frac{S(\tau_{0,1}^k)}{S(\tau_{0,1}^k-)}\frac{X(\tau_{0,1}^k-)}{X(\tau_{0,1}^k)} = \pi_{\tau_{0,1}^k-}\frac{1+\frac{\Delta S(\tau_{0,1}^k)}{S(\tau_{0,1}^k-)}}{1+\frac{\Delta X(\tau_{0,1}^k)}{X(\tau_{0,1}^k-)}}.$$

Due to the wealth equation (1) we have $\frac{\Delta X}{X_-} = \pi_-\frac{\Delta S}{S_-}$, and the initial condition follows. Using (1) and Itô's formula we compute

$$\mathrm{d}\frac{1}{X} = \frac{1}{X_-}\left(-\pi_-\frac{\mathrm{d}S}{S_-} - (1-\pi_-)\frac{\mathrm{d}M}{M_-}\right) + \frac{1}{X_-}\pi_-^2\frac{\mathrm{d}\langle S\rangle^{\mathfrak{c}}}{S_-^2} + \mathrm{d}\sum \Delta\frac{1}{X} + \frac{1}{X_-^2}\Delta X$$

and therefore, since $\Delta\frac{1}{X} + \frac{1}{X_-^2}\Delta X = \frac{(\frac{\Delta X}{X_-})^2}{X_-+\Delta X} = \frac{(\pi_-\frac{\Delta S}{S_-})^2}{X_-(1+\pi_-\frac{\Delta S}{S_-})}$, we have

$$\mathrm{d}\frac{1}{X} = \frac{1}{X_-}\left(-\pi_-\frac{\mathrm{d}S}{S_-} - (1-\pi_-)\frac{\mathrm{d}M}{M_-} + \pi_-^2\frac{\mathrm{d}\langle S\rangle^{\mathfrak{c}}}{S_-^2} + \mathrm{d}\sum\frac{(\pi_-\frac{\Delta S}{S_-})^2}{1+\pi_-\frac{\Delta S}{S_-}}\right).$$

By the product rule, we have $\mathrm{d}\frac{S}{X} = S_-\mathrm{d}\frac{1}{X} + \frac{1}{X}\mathrm{d}S + \mathrm{d}\langle S, \frac{1}{X}\rangle$ with

$$\mathrm{d}\langle S, \tfrac{1}{X}\rangle = \frac{1}{X_-}\left(-\pi_-\frac{\mathrm{d}\langle S\rangle}{S_-} + S_-\mathrm{d}\sum\frac{\pi_-^2(\frac{\Delta S}{S_-})^3}{1+\pi_-\frac{\Delta S}{S_-}}\right).$$

Since $\frac{\pi X}{S} = \varphi^1 = \varphi^0_{\tau_{0,1}^k-}$ on $[\![\tau_{0,1}^k, \tau_{1,0}^k[\![$ it follows that

$$\begin{aligned}
\mathrm{d}\pi = \frac{\pi_-X_-}{S_-}\mathrm{d}\frac{S}{X} &= -\pi_-^2\frac{\mathrm{d}S}{S_-} - \pi_-(1-\pi_-)\frac{\mathrm{d}M}{M_-} + \pi_-^3\frac{\mathrm{d}\langle S\rangle^{\mathfrak{c}}}{S_-^2} + \mathrm{d}\sum\frac{\pi_-^3(\frac{\Delta S}{S_-})^2}{1+\pi_-\frac{\Delta S}{S_-}} \\
&\quad +\pi_-\frac{\mathrm{d}S}{S_-} - \pi_-^2\frac{\mathrm{d}\langle S\rangle}{S_-^2} + \mathrm{d}\sum\frac{\pi_-^3(\frac{\Delta S}{S_-})^3}{1+\pi_-\frac{\Delta S}{S_-}} \\
&= \pi_-(1-\pi_-)\left(\frac{\mathrm{d}S}{S_-} - \frac{\mathrm{d}M}{M_-} - \pi_-\frac{\mathrm{d}\langle S\rangle^{\mathfrak{c}}}{S_-^2}\right) - \pi_-^2\mathrm{d}\sum(\frac{\Delta S}{S_-})^2 \\
&\quad +\mathrm{d}\sum\frac{\pi_-^3(\frac{\Delta S}{S_-})^2(1+\frac{\Delta S}{S_-})}{1+\pi_-\frac{\Delta S}{S_-}} \\
&= \pi_-(1-\pi_-)\left(\frac{\mathrm{d}S}{S_-} - \frac{\mathrm{d}M}{M_-} - \pi_-\frac{\mathrm{d}\langle S\rangle^{\mathfrak{c}}}{S_-^2} - \pi_-\mathrm{d}\sum\frac{(\frac{\Delta S}{S_-})^2}{1+\pi_-\frac{\Delta S}{S_-}}\right)
\end{aligned}$$

on the stochastic interval $[\![\tau_{0,1}^k, \tau_{1,0}^k[\![$, making use of the fact that $-\pi_-^2(\frac{\Delta S}{S_-})^2 + \frac{\pi_-^3(\frac{\Delta S}{S_-})^2(1+\frac{\Delta S}{S_-})}{1+\pi_-\frac{\Delta S}{S_-}} = -\pi_-^2(\frac{\Delta S}{S_-})^2(1-\frac{\pi_-(1+\frac{\Delta S}{S_-})}{1+\pi_-\frac{\Delta S}{S_-}}) = -\pi_-^2(1-\pi_-)\frac{(\frac{\Delta S}{S_-})^2}{1+\pi_-\frac{\Delta S}{S_-}}$. This

proves that π satisfies the stochastic differential equation stated in the assertion.
Next, setting

$$\mathrm{d}V \triangleq \frac{\mathrm{d}S}{S_-} - \frac{\mathrm{d}M}{M_-} - \pi_- \frac{d\langle S\rangle^{\mathfrak{c}}}{S_-^2} - \pi_- \mathrm{d}\sum \frac{(\frac{\Delta S}{S_-})^2}{1+\pi_- \frac{\Delta S}{S_-}}$$

we have $\mathrm{d}\pi = \pi_-(1-\pi_-)\mathrm{d}V$ and $\Delta\pi = \pi_-(1-\pi_-)\Delta V$ with

$$\Delta V = \frac{\Delta S}{S_-} - \pi_- \frac{(\frac{\Delta S}{S_-})^2}{1+\pi_- \frac{\Delta S}{S_-}} = \frac{\frac{\Delta S}{S_-}}{1+\pi_- \frac{\Delta S}{S_-}}$$

as well as $\Delta \frac{1}{\pi} + \frac{1}{\pi_-^2}\Delta\pi = \frac{(\frac{\Delta\pi}{\pi_-})^2}{\pi_- + \Delta\pi} = \frac{(1-\pi_-)^2(\Delta V)^2}{\pi_-(1+(1-\pi_-)\Delta V)}$. Therefore, an application of Itô's formula to the process $Z = \frac{1}{\pi} - 1$ yields

$$\begin{aligned}
\mathrm{d}Z &= -\frac{1}{\pi_-^2}\mathrm{d}\pi + \frac{1}{\pi_-^3}\mathrm{d}\langle\pi\rangle^{\mathfrak{c}} + \mathrm{d}\sum \Delta\frac{1}{\pi} + \frac{1}{\pi_-^2}\Delta\pi \\
&= \frac{1-\pi_-}{\pi_-}\left(-\mathrm{d}V + (1-\pi_-)\mathrm{d}\langle V\rangle^{\mathfrak{c}} + \mathrm{d}\sum \frac{(1-\pi_-)(\Delta V)^2}{1+(1-\pi_-)\Delta V}\right) \\
&= Z_-\Bigg(-\frac{\mathrm{d}S}{S_-} + \frac{\mathrm{d}M}{M_-} + \pi_- \frac{\mathrm{d}\langle S\rangle^{\mathfrak{c}}}{S_-^2} + \pi_- \mathrm{d}\sum \frac{(\frac{\Delta S}{S_-})^2}{1+\pi_- \frac{\Delta S}{S_-}} + (1-\pi_-)\mathrm{d}\frac{\langle S\rangle^{\mathfrak{c}}}{S_-^2} \\
&\qquad + \mathrm{d}\sum \frac{(1-\pi_-)(\frac{\Delta S}{S_-})^2}{(1+\pi_- \frac{\Delta S}{S_-})^2[1+(1-\pi_-)\frac{\frac{\Delta S}{S_-}}{1+\pi_- \frac{\Delta S}{S_-}}]}\Bigg)
\end{aligned}$$

on $[\![\tau_{0,1}^k, \tau_{1,0}^k[\![$. Since

$$\begin{aligned}
&\pi_- \mathrm{d}\sum \frac{(\frac{\Delta S}{S_-})^2}{1+\pi_- \frac{\Delta S}{S_-}} + \mathrm{d}\sum \frac{(1-\pi_-)(\frac{\Delta S}{S_-})^2}{(1+\pi_- \frac{\Delta S}{S_-})^2[1+(1-\pi_-)\frac{\frac{\Delta S}{S_-}}{1+\pi_- \frac{\Delta S}{S_-}}]} \\
&= \pi_- \mathrm{d}\sum \frac{(\frac{\Delta S}{S_-})^2}{1+\pi_- \frac{\Delta S}{S_-}} + \mathrm{d}\sum \frac{(1-\pi_-)(\frac{\Delta S}{S_-})^2}{(1+\pi_- \frac{\Delta S}{S_-})(1+\frac{\Delta S}{S_-})} = \mathrm{d}\sum \frac{(\frac{\Delta S}{S_-})^2}{1+\frac{\Delta S}{S_-}},
\end{aligned}$$

it follows that

$$\mathrm{d}Z = Z_-\left(-\frac{\mathrm{d}S}{S_-} + \frac{\mathrm{d}M}{M_-} + \frac{\mathrm{d}\langle S\rangle^{\mathfrak{c}}}{S_-^2} + \mathrm{d}\sum \frac{(\frac{\Delta S}{S_-})^2}{1+\frac{\Delta S}{S_-}}\right).$$

This is the desired representation of π. $\square$

Remark 1 *The previous lemma shows in particular that π takes values in $[0,1]$ only since Z is a stochastic exponential and $\frac{\Delta Z}{Z_-} = -\frac{\frac{\Delta S}{S_-}}{1+\frac{\Delta S}{S_-}} > -1$. Further, if $\sum_{[0,T]} |\Delta S| < \infty$ a.s., the dynamics of π and Z simplify to*

$$\mathrm{d}\pi = \pi_-(1-\pi_-)\left(\frac{\mathrm{d}S^{\mathfrak{c}}}{S_-} - \frac{\mathrm{d}M}{M_-} - \pi_-\frac{\mathrm{d}\langle S\rangle^{\mathfrak{c}}}{S_-^2} + \mathrm{d}\sum \frac{\frac{\Delta S}{S_-}}{1+\pi_-\frac{\Delta S}{S_-}}\right)$$

$$\mathrm{d}Z = Z_-\left(-\frac{\mathrm{d}S^{\mathfrak{c}}}{S_-} + \frac{\mathrm{d}M}{M_-} + \frac{\mathrm{d}\langle S\rangle^{\mathfrak{c}}}{S_-^2} - \mathrm{d}\sum \frac{\frac{\Delta S}{S_-}}{1+\frac{\Delta S}{S_-}}\right).$$

In the remainder of this paper, we will use the following concrete specification of the **asset price dynamics**. The bond is assumed to satisfy

$$\mathrm{d}M = M_- r_{I_-}\mathrm{d}t$$

for constant riskless interest rates $r_0, r_1 > 0$, and the risky asset is assumed to satisfy

$$\mathrm{d}S = S_-\left[(r_{I_-} + \alpha_{I_-})\mathrm{d}t + \sigma_{I_-}\mathrm{d}W - L_{I_-}\mathrm{d}N_{I_-} - 1_{\{K_- < k_0\}} L_{I_-,1-I_-}\mathrm{d}N_{I_-,1-I_-}\right]$$

on $[\![t_0, T[\![$ and $S(T) = (1 - 1_{\{I(T)=1\}}\ell)S_-(T)$. Here, $\alpha_0, \alpha_1 \in \mathbb{R}$ are excess returns, $\sigma_0, \sigma_1 \geq 0$ are volatilities, $L_0, L_1, L_{0,1}, L_{1,0}, \ell \in [0,1)$ are loss rates, W is a standard Brownian motion, and N_0, N_1 are Poisson processes with constant intensities $\lambda_0, \lambda_1 \geq 0$. The constant ℓ models liquidation costs if at the investment horizon T the economy is in the illiquidity state. In this model, the wealth dynamics (1) are given more explicitly as

$$\begin{aligned}\mathrm{d}X = X_-\big[&(r_{I_-} + \pi_-\alpha_{I_-})\mathrm{d}t + \pi_-\sigma_{I_-}\mathrm{d}W - \pi_- L_{I_-}\mathrm{d}N_{I_-} \\ &-1_{\{K_- < k_0\}}\pi_- L_{I_-,1-I_-}\mathrm{d}N_{I_-,1-I_-}\big]\end{aligned}$$

on $[\![t_0, T[\![$, $X(t_0) = x_0$, and $X(T) = (1 - 1_{\{I(T)=1\}}\pi_-(T)\ell)X_-(T)$. In particular, Remark 1 yields

Corollary 1 *For every $k \in \mathbb{N}$ with $1 \leq k \leq k_0$, the dynamics of the portfolio process π on the stochastic interval $[\![\tau_{0,1}^k, \tau_{1,0}^k[\![$ are given by*

$$\mathrm{d}\pi = \pi_-(1-\pi_-)\left[(\alpha_1 - \pi_-\sigma_1^2)\mathrm{d}t + \sigma_1\mathrm{d}W - \tfrac{L_1}{1-\pi_- L_1}\mathrm{d}N_1\right]$$

with $\pi(\tau_{0,1}^k) = \frac{\pi(\tau_{0,1}^k-)(1-L_{0,1})}{1-\pi(\tau_{0,1}^k-)L_{0,1}}$. This stochastic differential equation has the closed-form solution $\pi = \frac{1}{1+Z}$ where

$$\mathrm{d}Z = Z_-\left[(\sigma_1^2 - \alpha_1)\mathrm{d}t - \sigma_1\mathrm{d}W + \tfrac{L_1}{1-L_1}\mathrm{d}N_1\right],\ Z(\tau_{0,1}^k) = \tfrac{1}{\pi(\tau_{0,1}^k)} - 1.$$

Note that Z is a geometric Brownian motion if $L_1 = 0$.

3 Portfolio Problem with Illiquidity and Convergence

We now study the portfolio for an investor that trades in the market described in the previous section. It is assumed that the investor maximizes expected utility from terminal wealth with respect to a concave non-decreasing utility function $U: (0,\infty) \to \mathbb{R}$, and thus her **portfolio problem** is to

$$\text{maximize } \mathbb{E}[U(X^{\pi_0,0,x_0,k_0}(T))] \text{ over admissible strategies } \pi_0, \tag{P}$$

where $x_0 \in (0,\infty)$ denotes her initial wealth.

Firstly we remark that, for an admissible strategy π_0, the solution to the wealth equation is explicitly given by

$$X^{\pi_0,t_0,x_0,k_0}(t) = x_0 \exp\left\{\int_{t_0}^t r_{I_-} + \pi_-\alpha_{I_-} - \tfrac{1}{2}\pi_-^2\sigma_{I_-}^2 \,\mathrm{d}s + \int_{t_0}^t \pi_-\sigma_{I_-}\,\mathrm{d}W\right\}$$
$$\prod_{[t_0,t]} \left(1-\pi_-L_{I_-}\right)^{\Delta N_{I_-}} \left(1-1_{\{K_-<k_0\}}\pi_-L_{I_-,1-I_-}\right)^{\Delta N_{I_-,1-I_-}}$$

for all $t \in [t_0,T)$ and arbitrary $t_0 \in [0,T]$, $x_0 \in (0,\infty)$, $k_0 \in \mathbb{N}\cup\{\infty\}$. This implies the following[3]

Lemma 2 (Moments of the Wealth Process) *If $\mathbb{E}[\beta^{N_{i,1-i}(T)}] < \infty$ for all $\beta \in (0,\infty)$ and $i = 0,1$, then for any $\kappa > 0$ there exists $C_\kappa \in (0,\infty)$ such that for all $t_0 \in [0,T]$ and $x_0 \in (0,\infty)$*

$$\sup_{\pi_0,\, k_0\in\mathbb{N}\cup\{\infty\}} \mathbb{E}\left[\sup_{t\in[t_0,T]} \left(1 + X^{\pi_0,t_0,x_0,k_0}(t) + \frac{1}{X^{\pi_0,t_0,x_0,k_0}(t)}\right)^\kappa\right]$$
$$\le C_\kappa\left(1 + x_0 + \tfrac{1}{x_0}\right)^\kappa.$$

Proof For $\kappa \in \mathbb{R}$, we set

$$M_\kappa \triangleq \sup_{\pi_0,\, k_0\in\mathbb{N}\cup\{\infty\}} \mathbb{E}\bigg[\sup_{t\in[t_0,T]} \exp\Big\{\kappa\int_{t_0}^t r_{I_-^{t_0,k_0}} + \pi_-\alpha_{I_-^{t_0,k_0}} - \tfrac{1}{2}\pi_-^2\sigma^2_{I_-^{t_0,k_0}}\,\mathrm{d}s$$
$$+\kappa\int_{t_0}^t \pi_-\sigma_{I_-^{t_0,k_0}}\,\mathrm{d}W\Big\}\bigg].$$

If $\kappa > 0$, $t_0 \in [0,T]$, $x_0 \in (0,\infty)$, $k_0 \in \mathbb{N}\cup\{\infty\}$, and π_0 is an admissible strategy, then the above explicit solution yields $\mathbb{E}[\sup_{t\in[t_0,T]} X^{\pi_0,t_0,x_0,k_0}(t)^\kappa] \le x_0^\kappa M_\kappa$. By Cauchy's inequality, we then obtain

$$\mathbb{E}\left[\sup_{t\in[t_0,T]} X^{\pi_0,t_0,x_0,k_0}(t)^{-\kappa}\right]$$
$$\le x_0^{-\kappa}(1-\ell)^{-\kappa} M_{-2\kappa}^{\frac{1}{2}}\mathbb{E}\bigg[\sup_{t\in[t_0,T]}\prod_{[t_0,t]}\left(1-L_{I_-^{t_0,k_0}}\right)^{-2\kappa\Delta N_{I_-^{t_0,k_0}}}$$
$$\left(1-L_{I_-^{t_0,k_0},1-I_-^{t_0,k_0}}\right)^{-2\kappa\Delta N_{I_-^{t_0,k_0},1-I_-^{t_0,k_0}}}\bigg]^{\frac{1}{2}},$$

[3] See also Lemma 1 in [Seifried 2009b].

where

$$\sup_{t\in[t_0,T]}\prod_{[t_0,t]}(1-L_{I_-^{t_0,k_0}})^{-2\kappa\Delta N_{I_-^{t_0,k_0}}}(1-L_{I_-^{t_0,k_0},1-I_-^{t_0,k_0}})^{-2\kappa\Delta N_{I_-^{t_0,k_0},1-I_-^{t_0,k_0}}}$$
$$\leq (1-L_0)^{-2\kappa N_0(T)}(1-L_1)^{-2\kappa N_1(T)}(1-L_{0,1})^{-2\kappa N_{0,1}(T)}(1-L_{1,0})^{-2\kappa N_{1,0}(T)};$$

the quantity on the right is integrable due to the assumption on $N_{i,1-i}$. The desired conclusion will thus follow from the fact that $M_\kappa < \infty$ for all $\kappa \in \mathbb{R}$. To show this, note that

$$\begin{aligned} M_\kappa = &\sup_{\pi_0,\,k_0\in\mathbb{N}\cup\{\infty\}} \mathbb{E}\Big[\sup_{t\in[t_0,T]} \exp\Big\{\kappa\textstyle\int_{t_0}^t r_{I_-^{t_0,k_0}} + \pi_-\alpha_{I_-^{t_0,k_0}} - \frac{1}{2}\pi_-^2\sigma^2_{I_-^{t_0,k_0}} \\ &\quad + \tfrac{1}{2}\kappa\pi_-^2\sigma^2_{I_-^{t_0,k_0}}\mathrm{d}s + \kappa\textstyle\int_{t_0}^t \pi_-\sigma_{I_-^{t_0,k_0}}\mathrm{d}W - \frac{1}{2}\int_{t_0}^t\kappa^2\pi_-^2\sigma^2_{I_-^{t_0,k_0}}\mathrm{d}s\Big\}\Big] \\ \leq{}& \mathrm{e}^{\rho_\infty T}\sup_{\pi_0,\,k_0\in\mathbb{N}\cup\{\infty\}} \mathbb{E}\Big[\sup_{t\in[t_0,T]}\exp\Big\{\textstyle\int_{t_0}^t\kappa\pi_-\sigma_{I_-^{t_0,k_0}}\mathrm{d}W - \frac{1}{2}\int_{t_0}^t\kappa^2\pi_-^2\sigma^2_{I_-^{t_0,k_0}}\mathrm{d}s\Big\}\Big], \end{aligned}$$

since the process $\kappa|r_{I_-^{t_0,k_0}} + \pi_-\alpha_{I_-^{t_0,k_0}} - \frac{1}{2}\pi_-^2\sigma^2_{I_-^{t_0,k_0}} + \frac{1}{2}\kappa\pi_-^2\sigma^2_{I_-^{t_0,k_0}}|$ is bounded by a constant $\rho_\infty \in (0,\infty)$ that is independent of π_0, t_0, and k_0. Recall that $\pi_{I_-^{t_0,k_0}}$ is $[0,1]$-valued by Remark 1. Next, let π_0 be an arbitrary admissible strategy, and let $t_0 \in [0,T]$, $k_0 \in \mathbb{N}\cup\{\infty\}$. Writing $\varrho \triangleq \kappa\pi_-\sigma_{I_-^{t_0,k_0}}$, it follows that ϱ is bounded by $\varrho_\infty \in (0,\infty)$, a constant independent of π_0, t_0, and k_0. Therefore, by the Novikov condition, the exponential $\exp\{\int_{t_0}^{\cdot}\varrho\mathrm{d}W - \frac{1}{2}\int_{t_0}^{\cdot}\varrho^2\mathrm{d}s\}$ is a martingale and consequently

$$\begin{aligned} &\mathbb{E}\left[\sup_{t\in[t_0,T]}\exp\left\{\textstyle\int_{t_0}^t\varrho\mathrm{d}W - \frac{1}{2}\int_{t_0}^t\varrho^2\mathrm{d}s\right\}\right] \\ &\leq \mathbb{E}\left[\sup_{t\in[t_0,T]}\exp\left\{\textstyle\int_{t_0}^t\varrho\mathrm{d}W - \frac{1}{2}\int_{t_0}^t\varrho^2\mathrm{d}s\right\}^2\right]^{\frac{1}{2}} \leq 2\mathbb{E}\left[\exp\left\{2\textstyle\int_{t_0}^T\varrho\mathrm{d}W - \int_{t_0}^T\varrho^2\mathrm{d}s\right\}\right]^{\frac{1}{2}} \\ &\leq 2\mathbb{E}\left[\exp\left\{\textstyle\int_{t_0}^T 2\varrho\mathrm{d}W - \frac{1}{2}\int_{t_0}^T(2\varrho)^2\mathrm{d}s\right\}\exp\left\{\int_{t_0}^T\varrho^2\mathrm{d}s\right\}\right]^{\frac{1}{2}} \leq 2\mathrm{e}^{\frac{\varrho_\infty^2}{2}T} < \infty \end{aligned}$$

by Doob's L^2-inequality. This yields the desired result. □

Returning to the investor's portfolio problem, we define the value function $V : [0,T]\times(0,\infty)\times(\mathbb{N}\cup\{\infty\}) \to \mathbb{R}$ corresponding to (P) by

$$V(t_0,x_0,k_0) \triangleq \sup_{\pi_0}\mathbb{E}\left[U\left(X^{\pi_0,t_0,x_0,k_0}(T)\right)\right].$$

By considering the strategy $\pi_0 = 0$, i.e. a pure bond investment, and applying Lemma 2 together with Jensen's inequality, we obtain the lower and upper bounds

$$U(x_0) \leq V(t_0,x_0,k_0) \leq U(C_1(1+x_0+\tfrac{1}{x_0})).$$

In particular, the value function is finite. We can then establish the following convergence result.

Theorem 1 (Convergence of Value Functions) *Let* $\mathbb{E}[\beta^{N_{i,1-i}(T)}] < \infty$ *for all* $\beta \in (0,\infty)$ *and* $i = 0, 1$*, and suppose that the investor's utility function* U *is polynomially bounded at* 0*, i.e. that there exist* $\kappa > 0$*,* $\rho > 0$ *and* $\delta > 0$ *such that*

$$|U(x)| \leq \rho \left(1 + \tfrac{1}{x}\right)^{\kappa} \text{ for all } x \in (0,\delta).$$

Then the value function of the investor's portfolio problem satisfies

$$\lim_{k_0 \to \infty} \sup_{t_0 \in [0,T],\, x_0 \in C} |V(t_0, x_0, k_0) - V(t_0, x_0, \infty)| = 0$$

for any compact subset C *of* $(0,\infty)$.

Proof For any utility function U, we have the concavity estimate $U(x) \leq \theta(x-1)$ for some $\theta \in \mathbb{R}$, i.e. $\theta = U'(1)$ if U is differentiable, so by the assumption on U

$$|U(x)| \leq \varrho \left(1 + x + \tfrac{1}{x}\right)^{\kappa} \text{ for all } x \in (0,\infty),$$

for suitably chosen $\kappa > 1$ and $\varrho > 0$. Thus, due to Lemma 2, compactness of C and the assumption on $N_{i,1-i}$, the family

$$\left\{U\left(X^{\pi_0,t_0,x_0,k_0}(T)\right)\right\}_{\pi_0,\, t_0 \in [0,T],\, x_0 \in C,\, k_0 \in \mathbb{N} \cup \{\infty\}} \text{ is uniformly integrable.}$$

Moreover, it is clear that

$$\sup_{\pi_0,\, t_0 \in [0,T],\, x_0 \in C} |U(X^{\pi_0,t_0,x_0,\infty}(T)) - U(X^{\pi_0,t_0,x_0,k_0}(T))| \to 0$$

in probability as $k_0 \to \infty$ since

$$\begin{aligned}
&\mathbb{P}\left(X^{\pi_0,t_0,x_0,\infty}(T) \neq X^{\pi_0,t_0,x_0,k_0}(T) \text{ for some } \pi_0,\ t_0 \in [0,T],\ x_0 \in C\right) \\
&\quad \leq \mathbb{P}\left(K^{t_0,x_0,k_0}(T) = k_0 \text{ for some } t_0 \in [0,T],\ x_0 \in C\right) \\
&\quad \leq \mathbb{P}(N_{1,0}(T) \geq k_0) \to 0 \text{ as } k_0 \to \infty.
\end{aligned}$$

To prove convergence, fix some $\varepsilon > 0$ and choose $\hat{t}_0 \in [0,T]$, $\hat{x}_0 \in C$ such that

$$\sup_{t_0 \in [0,T],\, x_0 \in C} |V(t_0, x_0, k_0) - V(t_0, x_0, \infty)| \leq \left|V(\hat{t}_0, \hat{x}_0, k_0) - V(\hat{t}_0, \hat{x}_0, \infty)\right| + \tfrac{\varepsilon}{2}.$$

For the moment, assume that $V(\hat{t}_0, \hat{x}_0, k_0) - V(\hat{t}_0, \hat{x}_0, \infty) \geq 0$. Then let $\hat{\pi}_0$ be an admissible strategy such that

$$V(\hat{t}_0, \hat{x}_0, k_0) - \mathbb{E}[U(X^{\hat{\pi}_0,\hat{t}_0,\hat{x}_0,k_0}(T))] \leq \tfrac{\varepsilon}{2}.$$

Thus we have

$$\begin{aligned}
&\sup_{t_0 \in [0,T],\, x_0 \in C} |V(t_0, x_0, k_0) - V(t_0, x_0, \infty)| \leq V(\hat{t}_0, \hat{x}_0, k_0) - V(\hat{t}_0, \hat{x}_0, \infty) + \tfrac{\varepsilon}{2} \\
&\quad \leq \mathbb{E}[U(X^{\hat{\pi}_0,\hat{t}_0,\hat{x}_0,k_0}(T))] - V(\hat{t}_0, \hat{x}_0, \infty) + \varepsilon \\
&\quad \leq \mathbb{E}[U(X^{\hat{\pi}_0,\hat{t}_0,\hat{x}_0,k_0}(T))] - \mathbb{E}[U(X^{\hat{\pi}_0,\hat{t}_0,\hat{x}_0,\infty}(T))] + \varepsilon \\
&\quad \leq \sup_{\pi_0,\, t_0 \in [0,T],\, x_0 \in C} \mathbb{E}\left[|U(X^{\pi_0,t_0,x_0,k_0}(T)) - U(X^{\pi_0,t_0,x_0,\infty}(T))|\right] + \varepsilon.
\end{aligned}$$

Applying an analogous argument in the case when $V(\hat{t}_0, \hat{x}_0, k_0) - V(\hat{t}_0, \hat{x}_0, \infty) \leq 0$, we see that the latter inequality continues to hold. Since $\varepsilon > 0$ is arbitrary, we obtain

$$\begin{aligned} &\sup_{t_0 \in [0,T],\, x_0 \in C} |V(t_0, x_0, k_0) - V(t_0, x_0, \infty)| \\ \leq &\sup_{\pi_0,\, t_0 \in [0,T],\, x_0 \in C} \mathbb{E}\left[|U(X^{\pi_0, t_0, x_0, k_0}(T)) - U(X^{\pi_0, t_0, x_0, \infty}(T))| \right], \end{aligned}$$

so that

$$\sup_{t_0 \in [0,T],\, x_0 \in C} |V(t_0, x_0, k_0) - V(t_0, x_0, \infty)| \to 0 \text{ as } k_0 \to \infty$$

by the observations made at the beginning of the proof. □

The previous result shows that the portfolio problem (P) with possibly infinitely many liquidity breakdowns can be suitably approximated by an investment problem with finitely many jumps. Moreover, due to the uniformity of convergence, the optimal strategies of problems with sufficiently many breakdowns perform arbitrarily well in the case with infinitely many breakdowns. The following corollary makes this precise.

Corollary 2 (Approximatively Optimal Strategies) *Let the assumptions of Theorem 1 be satisfied. Then for fixed $\varepsilon > 0$ and for any $t_0 \in [0, T]$ and $x_0 \in (0, \infty)$ there exists a $\hat{k}_0 \in \mathbb{N}$ such that for any admissible $\frac{\varepsilon}{3}$-optimal strategy $\hat{\pi}_0$ for $V(t_0, x_0, \hat{k}_0)$ we have*

$$\left| \mathbb{E}[U(X^{\hat{\pi}_0, t_0, x_0, \infty}(T))] - V(t_0, x_0, \infty) \right| \leq \varepsilon. \tag{2}$$

Besides, if the investor's utility function U is of the form $U(x) = \frac{1}{\gamma}x^\gamma$, then it follows that the initial wealth x_k required to achieve the given indirect utility $V(t_0, x_0, \infty)$ in the model with finitely many jumps satisfies

$$x_k = \left(\frac{V(t_0, x_0, \infty)}{V(t_0, 1, k)} \right)^{\frac{1}{\gamma}} \to x_0 \text{ as } k \to \infty. \tag{3}$$

Proof Given some $\varepsilon > 0$, for any $t_0 \in [0, T]$ and $x_0 \in (0, \infty)$ we can choose $\hat{k}_0 \in \mathbb{N}$ such that

$$\begin{aligned} &\sup_{\pi_0} |\mathbb{E}[U(X^{\pi_0, t_0, x_0, \hat{k}_0}(T))] - \mathbb{E}[U(X^{\pi_0, t_0, x_0, \infty}(T))]| \leq \tfrac{\varepsilon}{3}, \\ &|V(t_0, x_0, \hat{k}_0) - V(t_0, x_0, \infty)| \leq \tfrac{\varepsilon}{3}. \end{aligned}$$

Thus, whenever $\hat{\pi}_0$ is an admissible strategy with

$$|\mathbb{E}[U(X^{\hat{\pi}_0, t_0, x_0, \hat{k}_0}(T))] - V(t_0, x_0, \hat{k}_0)| \leq \tfrac{\varepsilon}{3},$$

we have (2). If $U(x) = \frac{1}{\gamma}x^\gamma$, then the scaling relation

$$\begin{aligned} V(t_0, x_0, k_0) &= \sup_{\pi_0} \mathbb{E}\left[U(X^{\pi_0, t_0, x_0, k_0}(T)) \right] = x_0^\gamma \sup_{\pi_0} \mathbb{E}\left[U(X^{\pi_0, t_0, 1, k_0}(T)) \right] \\ &= x_0^\gamma V(t_0, 1, k_0) \end{aligned}$$

implies (3). □

4 HJB Equations and Verification Theorem

We now investigate the optimal portfolio problem (P) applying dynamic programming techniques. To obtain Markovian dynamics, we henceforth assume that the regime shift process $N_{i,1-i}$ is a Poisson process with intensity $\lambda_{i,1-i} \geq 0$ for $i = 0, 1$ so that the integrability condition of Lemma 2 and Theorem 1 is satisfied. Consider the optimal investment problem presented in the previous sections, and suppose that there are at most $k_0 \in \mathbb{N} \cup \{\infty\}$ liquidity breakdowns. Then a collection

$$\{J^{0,k_0}, J^{1,k_0}, J^{0,k_0-1}, J^{1,k_0-1}, \ldots, J^{0,1}, J^{1,1}, J^{0,0}\},$$

where $J^{0,k}$ is a $\mathrm{C}^{1,2}$-function on $[0,T] \times (0,\infty)$ and $J^{1,k}$ is a $\mathrm{C}^{1,2,2}$-function on $[0,T] \times (0,\infty) \times [0,1]$, is said to be a solution to the **HJB equations** of the portfolio problem if the following partial differential equations are satisfied.

$$\begin{aligned}
0 = \sup_{\pi\in[0,1]} \Big\{ & J_t^{0,0}(t,x) + x(r_0 + \alpha_0\pi)J_x^{0,0}(t,x) + \tfrac{1}{2}x^2\pi^2\sigma_0^2 J_{x,x}^{0,0}(t,x) \\
& + \lambda_0 \left[J^{0,0}(t, x(1-\pi L_0)) - J^{0,0}(t,x)\right] \Big\} \\
0 = \sup_{\pi\in[0,1]} \Big\{ & J_t^{0,k}(t,x) + x(r_0 + \alpha_0\pi)J_x^{0,k}(t,x) + \tfrac{1}{2}x^2\pi^2\sigma_0^2 J_{x,x}^{0,k}(t,x) \\
& + \lambda_0 \left[J^{0,k}(t, x(1-\pi L_0)) - J^{0,k}(t,x)\right] \\
& + \lambda_{0,1} \left[J^{1,k}\left(t, x(1-\pi L_{0,1}), \tfrac{\pi(1-L_{0,1})}{1-\pi L_{0,1}}\right) - J^{0,k}(t,x)\right] \Big\} \\
0 = {} & J_t^{1,k}(t,x,\pi) + x(r_1 + \alpha_1\pi)J_x^{1,k}(t,x,\pi) + \tfrac{1}{2}x^2\pi^2\sigma_1^2 J_{x,x}^{1,k}(t,x,\pi) \\
& + x\pi^2(1-\pi)\sigma_1^2 J_{x,\pi}^{1,k}(t,x,\pi) + \pi(1-\pi)(\alpha_1 - \sigma_1^2\pi)J_\pi^{1,k}(t,x,\pi) \\
& + \tfrac{1}{2}\pi^2(1-\pi)^2\sigma_1^2 J_{\pi,\pi}^{1,k}(t,x,\pi) \\
& + \lambda_1 \left[J^{1,k}\left(t, x(1-\pi L_1), \tfrac{\pi(1-L_1)}{1-\pi L_1}\right) - J^{1,k}(t,x,\pi)\right] \\
& + \lambda_{1,0} \left[J^{0,k-1}(t, x(1-\pi L_{1,0})) - J^{1,k}(t,x,\pi)\right]
\end{aligned}$$

with boundary conditions $J^{0,k}(T,x) = U(x)$, $J^{1,k}(T,x,\pi) = U(x(1-\ell\pi))$ for all $x \in (0,\infty)$ and $\pi \in [0,1]$. If $k_0 = \infty$, then a solution to the HJB equations simply consists of a pair $\{J^{0,\infty}, J^{1,\infty}\}$, and the system above reduces to a pair of equations with $J^{0,\infty-1} = J^{0,\infty}$, etc. Note that this system can be solved iteratively if $k_0 < \infty$, whereas it does not decouple when $k_0 = \infty$. Given a solution $\{J^{0,k_0}, J^{1,k_0}, J^{0,k_0-1}, J^{1,k_0-1}, \ldots, J^{0,1}, J^{1,1}, J^{0,0}\}$ of the HJB equations, to simplify notation, we set $H^{1,k}(t,x,\pi) \triangleq 0$ and

$$\begin{aligned}
H^{0,0}(t,x,\pi) \triangleq {} & J_t^{0,0}(t,x) + x(r_0 + \alpha_0\pi)J_x^{0,0}(t,x) + \tfrac{1}{2}x^2\pi^2\sigma_0^2 J_{x,x}^{0,0}(t,x) \\
& + \lambda_0 \left[J^{0,0}(t, x(1-\pi L_0)) - J^{0,0}(t,x)\right]
\end{aligned}$$

$$
\begin{aligned}
H^{0,k}(t,x,\pi) \triangleq\; & J_t^{0,k}(t,x) + x(r_0+\alpha_0\pi)J_x^{0,k}(t,x) + \tfrac{1}{2}x^2\pi^2\sigma_0^2 J_{x,x}^{0,k}(t,x) \\
& +\lambda_0\left[J^{0,k}(t,x(1-\pi L_0)) - J^{0,k}(t,x)\right] \\
& +\lambda_{0,1}\left[J^{1,k}\left(t,x(1-\pi L_{0,1}),\tfrac{\pi(1-L_{0,1})}{1-\pi L_{0,1}}\right) - J^{0,k}(t,x)\right]
\end{aligned}
$$

for $t\in[0,T]$, $x\in(0,\infty)$, and $\pi\in[0,1]$.

We now show that $J^{0,k}$ corresponds to the value function of the optimal investment problem with k regime shifts outstanding.

Theorem 2 (Verification Theorem) *Let* $\{J^{0,k_0}, J^{1,k_0}, J^{0,k_0-1}, J^{1,k_0-1},\ldots, J^{0,1}, J^{1,1}, J^{0,0}\}$ *be a solution of the HJB equations associated to the optimal investment problem (P) with at most* $k_0\in\mathbb{N}\cup\{\infty\}$ *periods of illiquidity, and assume moreover that for each* $i=0,1$ *and* $k=1,\ldots,k_0$ *the functions* $J^{i,k}$, $J_x^{i,k}$, $J_\pi^{i,k}$ *and* $J^{0,0}$, $J_x^{0,0}$, $J_\pi^{0,0}$ *are polynomially bounded at* 0 *and* ∞ *uniformly with respect to* $t\in[0,T]$ *and* $\pi\in[0,1]$. *Then*

$$
V(t_0,x_0,k_0) \le J^{0,k_0}(t_0,x_0) \text{ for all } t_0\in[0,T] \text{ and } x_0\in(0,\infty).
$$

Moreover, if there are continuous functions $\psi_k:\ [0,T]\times(0,\infty)\to[0,1]$ *with*

$$
\psi_k(t,x)\in\operatorname*{arg\,max}_{\pi\in[0,1]} H^{0,k}(t,x,\pi) \text{ for each } k=0,\ldots,k_0,
$$

then it follows that

$$
V(t_0,x_0,k_0) = J^{0,k_0}(t_0,x_0) \text{ for all } t_0\in[0,T],\ x_0\in(0,\infty),
$$

and the optimally controlled process X^* *and the optimal strategy* π_0^* *satisfy* $\pi_0^* = \psi_{k_0-K_-}(\cdot,X^*)$.

Remark 2 *Given that* $J^{i,k}(t,x,\pi)=f^{i,k}(t,\pi)U(x)$ *or* $J^{i,k}(t,x,\pi)=f^{i,k}(t,\pi)+U(x)$, *the polynomial growth assumption is satisfied if* U *and* U' *are polynomially bounded at* 0 *and* $f^{i,k}$ *and* $f_\pi^{i,k}$ *are bounded. This is for instance the case for power or* log *utility.*

Proof (of Theorem 2) Given an admissible strategy π_0, $t_0\in[0,T]$, and $x_0\in(0,\infty)$, consider the process $J(t)\triangleq J^{I(t),k_0-K(t)}(t,X(t),\pi(t))$ for all $t\in[t_0,T]$, where the upper indices π_0,t_0,x_0,k_0 are omitted for notational convenience and, by ignoring the third coordinate, $J^{0,k}$ is interpreted as a function defined on $[0,T]\times(0,\infty)\times[0,1]$. Applying Itô's formula and using Corollary 1, we obtain

$$
\begin{aligned}
\mathrm{d}J = {} & J_t^{I_-,k_0-K_-}(\cdot,X_-,\pi_-)\mathrm{d}t + J_x^{I_-,k_0-K_-}(\cdot,X_-,\pi_-)X_-\big[(r_{I_-}+\alpha_{I_-}\pi_-)\mathrm{d}t \\
& +\sigma_{I_-}\pi_-\mathrm{d}W\big] + \tfrac{1}{2}J_{x,x}^{I_-,k_0-K_-}(\cdot,X_-,\pi_-)X_-^2\sigma_{I_-}^2\pi_-^2\mathrm{d}t \\
& +1_{\{I_-=1\}}\Big\{J_\pi^{1,k_0-K_-}(\cdot,X_-,\pi_-)\pi_-(1-\pi_-)\left[(\alpha_1-\sigma_1^2\pi_-)\mathrm{d}t+\sigma_1\mathrm{d}W\right] \\
& \quad +\tfrac{1}{2}J_{\pi,\pi}^{1,k_0-K_-}(\cdot,X_-,\pi_-)\pi_-^2(1-\pi_-)^2\sigma_1^2\mathrm{d}t \\
& \quad +J_{x,\pi}^{1,k_0-K_-}(\cdot,X_-,\pi_-)X_-\sigma_1^2\pi_-^2(1-\pi_-)\mathrm{d}t\Big\} \\
& +\left[J^{I_-,k_0-K_-}\left(\cdot,(1-\pi_-L_{I_-})X_-,\tfrac{\pi_-(1-L_{I_-})}{1-\pi_-L_{I_-}}\right)-J^{I_-,k_0-K_-}(\cdot,X_-,\pi_-)\right]\mathrm{d}N_{I_-} \\
& +1_{\{I_-=0,\,K_-<k_0\}}\Big[J^{1,k_0-K_-}\left(\cdot,(1-\pi_-L_{0,1})X_-,\tfrac{\pi_-(1-L_{0,1})}{1-\pi_-L_{0,1}}\right) \\
& \quad -J^{0,k_0-K_-}(\cdot,X_-)\Big]\mathrm{d}N_{0,1} \\
& +1_{\{I_-=1\}}\left[J^{0,k_0-K_--1}(\cdot,(1-\pi_-L_{1,0})X_-)-J^{1,k_0-K_-}(\cdot,X_-,\pi_-)\right]\mathrm{d}N_{1,0} \\
= {} & H^{I_-,k_0-K_-}(\cdot,X_-,\pi_-)\mathrm{d}t + J_x^{I_-,k_0-K_-}(\cdot,X_-,\pi_-)X_-\sigma_{I_-}\pi_-\mathrm{d}W \\
& +1_{\{I_-=1\}}J_\pi^{1,k_0-K_-}(\cdot,X_-,\pi_-)\pi_-(1-\pi_-)\sigma_1\mathrm{d}W \\
& +\left[J^{I_-,k_0-K_-}\left(\cdot,(1-\pi_-L_{I_-})X_-,\tfrac{\pi_-(1-L_{I_-})}{1-\pi_-L_{I_-}}\right)-J^{I_-,k_0-K_-}(\cdot,X_-,\pi_-)\right]\mathrm{d}\tilde{N}_{I_-} \\
& +1_{\{I_-=0,\,K_-<k_0\}}\Big[J^{1,k_0-K_-}\left(\cdot,(1-\pi_-L_{0,1})X_-,\tfrac{\pi_-(1-L_{0,1})}{1-\pi_-L_{0,1}}\right) \\
& \quad -J^{0,k_0-K_-}(\cdot,X_-)\Big]\mathrm{d}\tilde{N}_{0,1} \\
& +1_{\{I_-=1\}}\left[J^{0,k_0-K_--1}(\cdot,(1-\pi_-L_{1,0})X_-)-J^{1,k_0-K_-}(\cdot,X_-,\pi_-)\right]\mathrm{d}\tilde{N}_{1,0}
\end{aligned}
$$

on $[\![t_0,T[\![$, where $\tilde{N}_0,\tilde{N}_1,\tilde{N}_{0,1},\tilde{N}_{1,0}$ denote the compensated Poisson processes associated with $N_0,N_1,N_{0,1},N_{1,0}$. Due to the polynomial growth assumption and Lemma 2, the stochastic differentials of the local martingales in the above identity are, in fact, stochastic differentials of martingales. Therefore, by taking expectations and using the boundary conditions of the HJB equations, we arrive at

$$\mathbb{E}[U(X^{\pi_0,t_0,x_0,k_0}(T))] = J^{0,k_0}(t_0,x_0) + \mathbb{E}\left[\int_{t_0}^T H^{I,k_0-K}(t,X(t),\pi(t))\mathrm{d}t\right].$$

Since π_0, t_0, and x_0 are arbitrary, we conclude that $V(t_0,x_0,k_0)\le J^{0,k_0}(t_0,x_0)$ for all $t_0\in[0,T]$ and $x_0\in(0,\infty)$.
Now, if $\psi_k:\ [0,T]\times(0,\infty)\to[0,1]$ is a continuous function such that

$$\psi_k(t,x)\in\underset{\pi\in[0,1]}{\arg\max}\,H^{0,k}(t,x,\pi)\text{ for each }k=0,\dots,k_0,$$

then the family $\{\psi_k\}_{k=0,\dots,k_0}$ defines an optimal feedback strategy in the sense that the stochastic differential equation

$$\begin{aligned}\mathrm{d}X = X_-\big[&\big(r_{I_-} + \psi_{k_0-K_-}(\cdot, X_-)\alpha_{I_-}\big)\,\mathrm{d}t + \psi_{k_0-K_-}(\cdot, X_-)\sigma_{I_-}\mathrm{d}W \\ &-\psi_{k_0-K_-}(\cdot, X_-)L_{I_-}\mathrm{d}N_{I_-} - 1_{\{K_-<k_0\}}\psi_{k_0-K_-}(\cdot, X_-) \\ &-L_{I_-,1-I_-}\mathrm{d}N_{I_-,1-I_-}\big]\end{aligned}$$

on $[\![t_0, T[\![$, $X(t_0) = x_0$, $X(T) = (1 - 1_{\{I(T)=1\}}\psi_{k_0-K_-(T)}(T, X_-(T))\ell)X_-(T)$, admits a solution X^{ψ,t_0,x_0,k_0}, and the strategy $\pi_0^* \triangleq \psi_{k_0-K}(\cdot, X^{\psi,t_0,x_0,k_0})$ is admissible and optimal for problem (P). Of course, in this case we have $\mathbb{E}[U(X^{\pi_0^*,t_0,x_0,k_0}(T))] = V(t_0, x_0, k_0) = J^{0,k_0}(t_0, x_0)$. $\square$

5 Infinitely Many Liquidity Breakdowns and Log Utility

In this section, we solve the portfolio problem (P) with infinitely many liquidity breakdowns for $U(x) = \ln(x)$. In order to apply the Verification Theorem, we conjecture

$$\begin{aligned}J^0(t, x) &= J^{0,\infty}(t, x) = \ln(x) + f^0(t), \\ J^1(t, x, \pi) &= J^{1,\infty}(t, x, \pi) = \ln(x) + f^1(t, \pi)\end{aligned}$$

for a C^1-function f^0 on $[0, T]$ with $f^0(T) = 0$ and a $\mathrm{C}^{1,2}$-function f^1 on $[0, T] \times [0, 1]$ satisfying $f^1(T, \pi) = \ln(1 - \ell\pi)$ for all $\pi \in [0, 1]$. Furthermore, we set $H^0 \triangleq H^{0,\infty}$ and $H^1 \triangleq H^{1,\infty}$. Then the HJB equations read

$$0 = \sup_{\pi\in[0,1]}\left\{f_t^0(t) + g^0(\pi) + \lambda_{0,1}\left[f^1\left(t, \tfrac{\pi(1-L_{0,1})}{1-\pi L_{0,1}}\right) - f^0(t)\right]\right\} \tag{4}$$

$$\begin{aligned}0 = &f_t^1(t, \pi) - \lambda_{1,0}f^1(t, \pi) + \pi(1 - \pi)(\alpha_1 - \sigma_1^2\pi)f_\pi^1(t, \pi) \\ &+\tfrac{1}{2}\pi^2(1 - \pi)^2\sigma_1^2 f_{\pi,\pi}^1(t, \pi) \\ &+\lambda_1\left[f^1\left(t, \tfrac{\pi(1-L_1)}{1-\pi L_1}\right) - f^1(t, \pi)\right] + g^1(\pi) + \lambda_{1,0}f^0(t),\end{aligned} \tag{5}$$

where g^j is given by $g^j(\pi) \triangleq r_j + \alpha_j\pi - \frac{1}{2}\pi^2\sigma_j^2 + \lambda_j\ln(1 - \pi L_j) + \lambda_{j,1-j}\ln(1 - \pi L_{j,1-j})$ on $[0, 1]$, $j = 0, 1$. Equation (4) leads to the first-order condition

$$0 = \alpha_0 - \sigma_0^2\pi - \lambda_0\tfrac{L_0}{1-\pi L_0} - \lambda_{0,1}\tfrac{L_{0,1}}{1-\pi L_{0,1}} + \lambda_{0,1}f_\pi^1\left(t, \tfrac{\pi(1-L_{0,1})}{1-\pi L_{0,1}}\right)\tfrac{1-L_{0,1}}{(1-\pi L_{0,1})^2} \tag{6}$$

for the optimal stock proportion in state 0. Note that the solution of the first-order condition is a deterministic function of time (if it exists).

We now determine indirect utility in the states of illiquidity and liquidity.

Proposition 1 (Indirect Utility in Illiquidity) *For a* C^1*-function* $f^0 : [0, T] \to \mathbb{R}$*, consider the function* $f^1 : [0, T] \times [0, 1] \to \mathbb{R}$ *defined via the stochastic representation*

$$\begin{aligned}f^1(t, \pi) \triangleq &\int_t^T \left(\lambda_{1,0}f^0(s) + \mathbb{E}\left[g^1(\tilde\pi(s))\right]\right)\mathrm{e}^{-\lambda_{1,0}(s-t)}\mathrm{d}s \\ &+\mathbb{E}\left[\ln(1 - \tilde\pi(T)\ell)\right]\mathrm{e}^{-\lambda_{1,0}(T-t)},\end{aligned}$$

where $\tilde\pi$ *is given by* $\tilde\pi(s) \triangleq \frac{\pi}{\pi+(1-\pi)Z(s)}$, $s \in [t,T]$, *with* $\mathrm{d}Z = Z_-[(\sigma_1^2-\alpha_1)\mathrm{d}s - \sigma_1 \mathrm{d}W + \frac{L_1}{1-L_1}\mathrm{d}N_1]$, $Z(t)=1$. *Then* f^1 *is of class* $\mathrm{C}^{1,2}$ *on* $[0,T]\times[0,1]$, f^1 *solves the second HJB equation (5), and* f^1_π *does not depend on* f^0. *In particular, the first-order condition (6) provides an algebraic equation for the optimal stock proportion* π.

Proof The claim follows from straightforward but lengthy dominated convergence arguments, and the Feynman-Kac theorem. For details, the reader is referred to [Diesinger, Kraft, Seifried 2009] and [Diesinger 2009]. □

One central motivation for modeling the randomness of stock dynamics via Brownian motions is that continuous trading activity of market participants creates this kind of dynamics.[4] In state 1, however, trading is interrupted and thus it seems reasonable to set the diffusion term in state 1 to zero. Besides, we think of state 1 as a regime where the economy is hit by an extreme event such as a war or a political turmoil. Consequently, it may also be plausible to assume that $\alpha_1 \le 0$.

Proposition 2 (Optimal Portfolio Choice) *Assume that* $\alpha_1 \le 0$ *and* $\sigma_1 = 0$. *Then the function* f^1 *defined above is decreasing and concave. If for each* $t \in [0,T]$ *there exists a* $\pi^*(t) \in [0,1]$ *such that* $\pi^*(t)$ *is a solution to the first-order condition (6), then* $\pi^*: [0,T] \to [0,1]$ *is uniquely determined, of class* C^1, *and* $\pi^*(t) = \arg\max_{\pi\in[0,1]} H^0(t,\pi)$ *for all* $t \in [0,T]$.

Proof The first part of the claim is proved by an explicit computation of the derivatives f^1_π and $f^1_{\pi,\pi}$; the second then follows from the implicit function theorem. We refer to [Diesinger, Kraft, Seifried 2009] and [Diesinger 2009] for the details. □

Note that the requirements $\alpha_1 \le 0$ and $\sigma_1 = 0$ are not necessary for the claim in the previous proposition to hold. They however imply that f^1_π and $f^1_{\pi\pi}$ are non-positive, which is sufficient to prove the claim. The following proposition provides a representation of the value function in state 0.

Proposition 3 (Indirect Utility in Liquidity) *Suppose that there exists a continuous function* $\pi^*: [0,T] \to [0,1]$ *such that* $\pi^*(t) \in \arg\max_{\pi\in[0,1]} H^0(t,\pi)$ *for all* $t \in [0,T]$. *Consider the function* $f^0: [0,T] \to \mathbb{R}$ *given by*

$$f^0(t) \triangleq \frac{\lambda_{0,1}}{\lambda_{0,1}+\lambda_{1,0}} \mathrm{e}^{(\lambda_{0,1}+\lambda_{1,0})t} \int_t^T F(s)\mathrm{e}^{-\lambda_{0,1}s}\mathrm{d}s + \frac{\lambda_{1,0}}{\lambda_{0,1}+\lambda_{1,0}} \int_t^T F(s)\mathrm{e}^{\lambda_{1,0}s}\mathrm{d}s,$$

where $F(t) \triangleq g^0(\pi^*(t))\mathrm{e}^{-\lambda_{1,0}t} + \lambda_{0,1}\int_t^T \mathbb{E}[g^1(\tilde\pi^{t,\hat\pi_0(t)}(s))]\mathrm{e}^{-\lambda_{1,0}s}\mathrm{d}s + \lambda_{0,1}\mathbb{E}[\ln(1-\tilde\pi^{t,\hat\pi_0(t)}(T)\ell)]\mathrm{e}^{-\lambda_{1,0}T}$ *and* $\hat\pi_0(t) \triangleq \frac{(1-L_{0,1})\pi^*(t)}{1-L_{0,1}\pi^*(t)}$. *Then* f^0 *is of class* C^1 *and solves the HJB equation (4).*

[4] See, e.g., [Föllmer, Schweizer 1993] and the references therein.

Proof Substituting the representation for f^0 obtained in Proposition 1 into the HJB equation (4), we obtain an integro-differential for f^0. The latter can be reduced to a second-order linear differential equation, which is solved by variation of constants. Details can be found in [Diesinger, Kraft, Seifried 2009] and [Diesinger 2009]. □

We add that for specific parameter choices, it is possible to calculate the integrals in the above representation of f^0 explicitly. The following theorem summarizes our results in this section.

Theorem 3 (Solution of the Portfolio Problem) *Consider the portfolio problem (P) with infinitely many possible liquidity breakdowns for an investor with* $U(x) = \ln(x)$*. Suppose that there exists a continuous function* $\pi^* : [0,T] \to [0,1]$ *such that* $\pi^*(t) \in \arg\max_{\pi\in[0,1]} H^0(t,\pi)$ *for all* $t \in [0,T]$*. Then the value function is given by*

$$V(t_0, x_0, \infty) = \ln(x_0) + f^0(t_0) \text{ for } t_0 \in [0,T],\ x_0 \in (0,\infty),$$

and the optimal strategy is given by π^**.*

Proof Since $|\ln(x)| \leq \frac{1}{x}$ for $x \in (0,1)$, the assertion follows immediately from the Verification Theorem 2 and Propositions 1 and 3. □

6 Finitely Many Liquidity Breakdowns

In this section, we summarize the results for the portfolio problem (P) when only finitely many regime shifts between state 0 and state 1 are possible. We analyze the problems of investors with logarithmic and power utility functions and, for instance, provide convergence results of the optimal portfolio strategies.

6.1 Logarithmic Utility

Firstly, we assume that $U(x) = \ln(x)$. Since only finitely many breakdowns are possible, the portfolio problem can be solved recursively. Note that $J^{0,0}$ is given by

$$\begin{aligned} J^{0,0}(t,x) &= \ln(x) + f^{0,0}(t) \\ &= \ln(x) + [r_0 + \alpha_0\pi^* - \tfrac{1}{2}(\pi^*)^2\sigma_0^2 + \lambda_0 \ln(1 - \pi^* L_0)](T-t) \end{aligned}$$

where $0 = \alpha_0 - \sigma_0^2\pi^* - \lambda_0 \frac{L_0}{1-\pi^* L_0}$. As in the previous section, for $k_0 \in \mathbb{N}$, we conjecture

$$J^{0,k_0}(t,x) = \ln(x) + f^{0,k_0}(t) \text{ and } J^{1,k_0}(t,x,\pi) = \ln(x) + f^{1,k_0}(t,\pi)$$

for a C^1-function f^{0,k_0} on $[0,T]$ with $f^{0,k_0}(T)=0$ and a $C^{1,2}$-function f^{1,k_0} on $[0,T]\times[0,1]$ satisfying $f^{1,k_0}(T,\pi)=\ln(1-\ell\pi)$ for all $\pi\in[0,1]$. The HJB equations read

$$0=\sup_{\pi\in[0,1]}\left\{f_t^{0,k_0}(t)+g^0(\pi)+\lambda_{0,1}\left[f^{1,k_0}\left(t,\tfrac{\pi(1-L_{0,1})}{1-\pi L_{0,1}}\right)-f^{0,k_0}(t)\right]\right\} \quad (7)$$

$$\begin{aligned}0=&f_t^{1,k_0}(t,\pi)-\lambda_{1,0}f^{1,k_0}(t,\pi)+\pi(1-\pi)(\alpha_1-\sigma_1^2\pi)f_\pi^{1,k_0}(t,\pi)\\&+\tfrac{1}{2}\pi^2(1-\pi)^2\sigma_1^2 f_{\pi,\pi}^{1,k_0}(t,\pi)\\&+\lambda_1\left[f^{1,k_0}\left(t,\tfrac{\pi(1-L_1)}{1-\pi L_1}\right)-f^{1,k_0}(t,\pi)\right]+g^1(\pi)+\lambda_{1,0}f^{0,k_0-1}(t)\end{aligned} \quad (8)$$

with g^0 and g^1 as in the previous section. The first equation leads to the following first-order condition for the optimal stock proportion in state 0:

$$0=\alpha_0-\sigma_0^2\pi-\lambda_0\tfrac{L_0}{1-\pi L_0}-\lambda_{0,1}\tfrac{L_{0,1}}{1-\pi L_{0,1}}+\lambda_{0,1}f_\pi^{1,k_0}\left(t,\tfrac{\pi(1-L_{0,1})}{1-\pi L_{0,1}}\right)\tfrac{1-L_{0,1}}{(1-\pi L_{0,1})^2}\,. \quad (9)$$

Note that the solution of the first-order condition is a deterministic function of time given such a solution exists.

Proceeding as in Section 5, one can show

Proposition 4 (Indirect Utility in Illiquidity) *Let $k_0\in\mathbb{N}$ and let $\tilde\pi$ be as in Proposition 1. Given a C^1-function $f^{0,k_0-1}:[0,T]\to\mathbb{R}$, consider the function $f^{1,k_0}:[0,T]\times[0,1]\to\mathbb{R}$ defined via the stochastic representation*

$$\begin{aligned}f^{1,k_0}(t,\pi)\triangleq&\int_t^T\left(\lambda_{1,0}f^{0,k_0-1}(s)+\mathbb{E}[g^1(\tilde\pi(s))]\right)e^{-\lambda_{1,0}(s-t)}ds\\&+\mathbb{E}[\ln(1-\tilde\pi(T)\ell)]e^{-\lambda_{1,0}(T-t)}.\end{aligned}$$

Then f^{1,k_0} is of class $C^{1,2}$ on $[0,T]\times[0,1]$ and solves the second HJB equation (8).

In particular, we see that $f_\pi^{1,k_0}=f_\pi^1$ for all k_0, where f_π^1 is given in Proposition 1. Thus, we have

Corollary 3 (k_0-Invariance) *For any $k_0\in\mathbb{N}$, the first-order condition (9) coincides with the first-order condition (6) when infinitely many liquidity breakdowns are possible.*

Remark 3 *By Corollary 3, the convergence of optimal strategies for $k_0\to\infty$ is trivial if the investor has logarithmic utility.*

In general, a logarithmic investor makes her investment decisions myopically. If liquidity breakdowns are possible, then she adjusts her portfolio decision to take the threat of illiquidity into account. However, by the previous corollary, she remains myopic in the sense that she disregards the total number of possible breakdowns.

Proposition 5 (Indirect Utility in Liquidity) *Let $k_0 \in \mathbb{N}$ and let $\hat{\pi}_0$ be as in Proposition 3. Suppose that there exists a continuous function $\pi^* : [0,T] \to [0,1]$ such that $\pi^*(t) \in \arg\max_{\pi \in [0,1]} H^{0,k_0}(t,\pi)$ for all $t \in [0,T]$. Given a $\mathrm{C}^{1,2}$-function $f^{1,k_0} : [0,T] \times [0,1] \to \mathbb{R}$, consider the function $f^{0,k_0} : [0,T] \to \mathbb{R}$ defined via*

$$f^{0,k_0}(t) \triangleq \int_t^T (\lambda_{0,1} f^{1,k_0}(s, \hat{\pi}_0(s)) + g^0(\pi^*(s)))\, \mathrm{e}^{-\lambda_{0,1}(s-t)} \mathrm{d}s.$$

Then f^{0,k_0} is of class C^1 on $[0,T]$, and f^{0,k_0} solves the HJB equation (7).

Collecting the above results and applying the Verification Theorem 2 yields

Theorem 4 (Solution of the Portfolio Problem) *For $U(x) = \ln(x)$ we consider the portfolio problem (P) with $k_0 \in \mathbb{N}$ possible periods of illiquidity. Let $\pi_0^* \in [0,1]$ satisfy $0 = \alpha_0 - \sigma_0^2 \pi^* - \lambda_0 \frac{L_0}{1-\pi^* L_0}$, and suppose that there exists a continuous function $\pi^* : [0,T] \to [0,1]$ such that $\pi^*(t) \in \arg\max_{\pi \in [0,1]} H^{0,k}(t,\pi)$ for all $t \in [0,T]$ and $k = 1, \ldots, k_0$. Then the value function is given by*

$$V(t_0, x_0, k_0) = \ln(x_0) + f^{0,k_0}(t_0) \text{ for } t_0 \in [0,T],\ x_0 \in (0,\infty);$$

moreover it is optimal to use the strategy π^* on $\{K_- < k_0\}$ and the strategy π_0^* on $\{K_- = k_0\}$.

6.2 Power Utility

In this subsection, we consider an economy where only finitely many regime shifts between state 0 and state 1 are possible and where $U(x) = \frac{1}{\gamma} x^\gamma$ with $\gamma \neq 0$. As in the previous subsection, this problem can be solved recursively. We assume that $L_0 = L_1 = L_{0,1} = 0$ and that $\sigma_1 = 0$. Recall that $J^{0,0}$ is given by

$$J^{0,0}(t,x) = \tfrac{1}{\gamma} x^\gamma f^{0,0}(t) = \tfrac{1}{\gamma} x^\gamma \exp\left\{ \gamma (r_0 + \tfrac{1}{2} \tfrac{\alpha_0^2}{(1-\gamma)\sigma_0^2})(T-t) \right\},$$

and the optimal stock proportion is given by $\pi^* = \frac{\alpha_0}{(1-\gamma)\sigma_0^2}$. For $k_0 \in \mathbb{N}$ we conjecture

$$J^{0,k_0}(t,x) = \tfrac{1}{\gamma} x^\gamma f^{0,k_0}(t) \text{ and } J^{1,k_0}(t,x,\pi) = \tfrac{1}{\gamma} x^\gamma f^{1,k_0}(t,\pi)$$

for a C^1-function f^{0,k_0} on $[0,T]$ with $f^{0,k_0}(T) = 1$ and a $\mathrm{C}^{1,2}$-function f^{1,k_0} on $[0,T] \times [0,1]$ satisfying $f^{1,k_0}(T,\pi) = (1 - \ell\pi)^\gamma$ for all $\pi \in [0,1]$. Then the HJB equations read

$$0 = \sup_{\pi \in [0,1]} \tfrac{1}{\gamma} \left\{ f_t^{0,k_0}(t) - d^0(\pi) f^{0,k_0}(t) + \lambda_{0,1} f^{1,k_0}(t,\pi) \right\} \tag{10}$$

$$\begin{aligned} 0 = {} & f_t^{1,k_0}(t,\pi) - d^1(\pi) f^{1,k_0}(t,\pi) + \pi(1-\pi)\alpha_1 f_\pi^{1,k_0}(t,\pi) \\ & + \lambda_{1,0}(1 - \pi L_{1,0})^\gamma f^{0,k_0-1}(t), \end{aligned} \tag{11}$$

where d^0 and d^1 are given by $d^0(\pi) \triangleq \lambda_{0,1} - \gamma(r_0 + \pi\alpha_0) + \frac{1}{2}\gamma(1-\gamma)\pi^2\sigma_0^2$ and $d^1(\pi) \triangleq \lambda_{1,0} - \gamma(r_1 + \alpha_1\pi)$ on $[0,1]$. The first HJB equation leads to the first-order condition

$$0 = \gamma\alpha_0 f^{0,k_0}(t) - \gamma(1-\gamma)\pi\sigma_0^2 f^{0,k_0}(t) + \lambda_{0,1} f_\pi^{1,k_0}(t,\pi) \tag{12}$$

for the optimal stock proportion in state 0. As before, the solution to the first-order condition is a deterministic function of time given such a solution exists.

Arguing as before, we can construct the indirect utility functions recursively by a Feynman-Kac-type representation.

Proposition 6 (Indirect Utility in Illiquidity) *Let $k_0 \in \mathbb{N}$ and let $f^{0,k_0-1} : [0,T] \to \mathbb{R}$ be a given function of class C^1. Consider the function $f^{1,k_0} : [0,T]\times[0,1] \to \mathbb{R}$ defined via*

$$f^{1,k_0}(t,\pi) \triangleq \lambda_{1,0}\int_t^T \mathrm{e}^{(\gamma r_1-\lambda_{1,0})(s-t)}\left(1+\pi[\mathrm{e}^{\alpha_1(s-t)}(1-L_{1,0})-1]\right)^\gamma f^{0,k_0-1}(s)\mathrm{d}s$$
$$+\mathrm{e}^{(\gamma r_1-\lambda_{1,0})(T-t)}\left(1+\pi[\mathrm{e}^{\alpha_1(T-t)}(1-\ell)-1]\right)^\gamma.$$

Then f^{1,k_0} is of class $\mathrm{C}^{1,2}$ on $[0,T]\times[0,1]$ and solves the HJB equation (11).

Proposition 7 (Indirect Utility in Liquidity) *Let $k_0 \in \mathbb{N}$ and suppose that there exists a continuous function $\pi^*_{k_0} :\ [0,T] \to [0,1]$ such that $\pi^*_{k_0}(t) \in \arg\max_{\pi\in[0,1]} H^{0,k_0}(t,\pi)$ for all $t \in [0,T]$. Given a $\mathrm{C}^{1,2}$-function $f^{1,k_0} : [0,T]\times[0,1] \to \mathbb{R}$, consider the function $f^{0,k_0} : [0,T] \to \mathbb{R}$ defined via*

$$f^{0,k_0}(t) \triangleq \lambda_{0,1}\int_t^T \mathrm{e}^{-\int_t^v d^0(\pi^*_{k_0}(u))\mathrm{d}u} f^{1,k_0}(v,\pi^*_{k_0}(v))\mathrm{d}v + \mathrm{e}^{-\int_t^T d^0(\pi^*_{k_0}(u))\mathrm{d}u}.$$

Then f^{0,k_0} is of class C^1 on $[0,T]$ and solves the HJB equation (10).

Collecting the above results, by the Verification Theorem 2, we obtain

Theorem 5 (Solution of the Portfolio Problem) *For $U(x) = \frac{1}{\gamma}x^\gamma$, $\gamma \neq 0$, we consider the portfolio problem (P) with $k_0 \in \mathbb{N}$ possible regime shifts. Let $\pi^*_0 \triangleq \frac{\alpha_0}{(1-\gamma)\sigma_0^2}$ and assume that for each $k = 1,\dots,k_0$ there exists a continuous function $\pi^*_k :\ [0,T] \to [0,1]$ such that $\pi^*_k(t) \in \arg\max_{\pi\in[0,1]} H^{0,k}(t,\pi)$ for all $t \in [0,T]$. Then the value function of the portfolio problem is given by*

$$V(t_0,x_0,k_0) = \tfrac{1}{\gamma}x_0^\gamma f^{0,k_0}(t_0) \text{ for } t_0 \in [0,T],\ x_0 \in (0,\infty),$$

and the optimal strategy is given by $\pi^*_{k_0-K_-}$.

Finally, we derive a convergence result for the optimal strategies; in the remainder of this section, we assume that the assumptions of Theorem 5 are satisfied. In the case of infinitely many possible liquidity breakdowns, i.e. $k_0 = \infty$, there is an analog to the representation of $f^{1,\infty}$ in Proposition 6:

$$f^{1,\infty}(t,\pi) \triangleq \lambda_{1,0}\int_t^T \mathrm{e}^{(\gamma r_1-\lambda_{1,0})(s-t)}\left(1+\pi[\mathrm{e}^{\alpha_1(s-t)}(1-L_{1,0})-1]\right)^\gamma f^{0,\infty}(s)\mathrm{d}s$$
$$+\mathrm{e}^{(\gamma r_1-\lambda_{1,0})(T-t)}\left(1+\pi[\mathrm{e}^{\alpha_1(T-t)}(1-l)-1]\right)^\gamma.$$

Proposition 8 *The sequence* $\{f_\pi^{1,k}\}_{k\in\mathbb{N}}$ *converges to* $f_\pi^{1,\infty}$ *uniformly on* $[0,T]\times[0,1]$.

Proof Let $k\in\mathbb{N}$. Since we may interchange differentiation and integration in the representations of $f_\pi^{1,k}$ and $f_\pi^{1,\infty}$, we have

$$
\begin{aligned}
&\sup_{(t,\pi)\in[0,T]\times[0,1]} |f_\pi^{1,k}(t,\pi)-f_\pi^{1,\infty}(t,\pi)| \\
&= \lambda_{1,0} \sup_{(t,\pi)\in[0,T]\times[0,1]} \Big|\gamma \int_t^T \mathrm{e}^{(\gamma r_1-\lambda_{1,0})(s-t)}\Big(1+\pi[\mathrm{e}^{\alpha_1(s-t)}(1-L_{1,0})-1]\Big)^{\gamma-1} \\
&\qquad\qquad [\mathrm{e}^{\alpha_1(s-t)}(1-L_{1,0})-1](f^{0,k-1}(s)-f^{0,\infty}(s))\mathrm{d}s\Big| \\
&\le \lambda_{1,0}\, T \sup_{s\in[0,T]} |f^{0,k-1}(s)-f^{0,\infty}(s)| \sup_{\pi\in[0,1],\, s,t\in[0,T]} \Big|\gamma \mathrm{e}^{(\gamma r_1-\lambda_{1,0})(s-t)} \\
&\quad \Big(1+\pi[\mathrm{e}^{\alpha_1(s-t)}(1-L_{1,0})-1]\Big)^{\gamma-1}[\mathrm{e}^{\alpha_1(s-t)}(1-L_{1,0})-1]\Big|.
\end{aligned}
$$

Since $L_{1,0}<1$ the second supremum is finite. Thus, the assertion follows from Theorem 1. □

In the case $k_0=\infty$, the first-order condition (12) becomes

$$0=\gamma\alpha_0 f^{0,\infty}(t)-\gamma(1-\gamma)\pi\sigma_0^2 f^{0,\infty}(t)+\lambda_{0,1} f_\pi^{1,\infty}(t,\pi). \tag{13}$$

Corollary 4 (Convergence of π_k^*) *Let* $t\in[0,T]$ *and suppose that the first-order condition (12) has a solution* $\pi_k^*(t)\in[0,1]$ *for each* $k\in\mathbb{N}\cup\{\infty\}$, *where the solution for* $k=\infty$ *is unique. Then*

$$\pi_k^*(t)\to\pi_\infty^*(t) \text{ as } k\to\infty \text{ for all } t\in[0,T].$$

Proof Let $\{\pi_{k_l}^*(t)\}_{l\in\mathbb{N}}$ be a subsequence of $\{\pi_k^*(t)\}_{k\in\mathbb{N}}$. Since $\pi_k^*(t)\in[0,1]$ for each $k\in\mathbb{N}$, there exists a convergent subsequence $\{\pi_{k_{l_m}}^*(t)\}_{m\in\mathbb{N}}$. By Theorem 1, Proposition 8 and the first-order condition (12), we then obtain

$$0=\gamma\alpha_0 f^{0,\infty}(t)-\gamma(1-\gamma)\lim_{m\to\infty}\pi_{k_{l_m}}^*(t)\sigma_0^2 f^{0,\infty}(t)+\lambda_{0,1} f_\pi^{1,\infty}\Big(t,\lim_{m\to\infty}\pi_{k_{l_m}}^*(t)\Big).$$

Thus, we have shown that each subsequence of $\{\pi_k^*(t)\}_{k\in\mathbb{N}}$ has another subsequence which converges towards $\pi_\infty^*(t)$. □

Similarly as in Proposition 2, the solution of the first-order condition corresponds to the optimal strategy if, for instance, $\alpha_1\le 0$. In this case, Corollary 4 implies that the optimal strategies converge.

7 Numerical Illustrations

To illustrate the convergence of the value functions and strategies in the markets with finitely many breakdowns to the corresponding quantities in the market with infinitely many liquidity breakdowns, we consider a situation where a liquidity breakdown is rather likely. Thus we choose $\lambda_{0,1} = 0.2$, so on average a liquidity breakdown occurs every five years. Furthermore, we assume that the average duration of a liquidity breakdown is one month, i.e. $\lambda_{1,0} = 12$, and that $r_0 = r_1 = 3\%$, $\alpha_0 = 8\%$, $\alpha_1 = -3\%$, $\sigma_0 = 25\%$, and $L_{1,0} = \ell = 30\%$. The remaining parameters are assumed to be zero. This example is similar to the 9/11 case where the New York stock exchange closed for one week and reopened with a loss of 10%. To get more pronounced effects, we use higher loss rates and a longer average duration of the liquidity breakdowns. The investor is assumed to have a power utility function with $\gamma = -3$. Figure 1[5] depicts the convergence of the strategies and the (non-wealth dependent parts of) the value functions, f^{0,k_0}. As can be seen from the figure, the value functions converge extremely fast. The strategies also converge to an almost straight line that intersects the y-axis around 0.061. The upper line corresponds to the optimal strategy if at most one liquidity breakdown can occur, the second upper line to the optimal strategy if at most two breakdowns can occur, and so on. These results illustrate the theoretical results of Theorem 1 and Corollary 4. The figure also depicts the percentages Δx by which the initial capital can be reduced in order to get the same utility as in models where trading is allowed in both states. It can be seen that this percentage also converges if the number of possible breakdowns increases. This is because Δx is a function of the value functions that converge.

For a detailed numerical analysis of a model which is calibrated to data of the Tokyo Stock Exchange in the aftermath of World War II, the interested reader is referred to [Diesinger, Kraft, Seifried 2009]. As a particularly striking conclusion, one finds that a logarithmic investor with a time horizon of 30 years would be willing to give up 22.7% of her initial wealth in order to be able to trade when the market is illiquid. Thus, the threat of illiquidity can have a significant impact on asset allocation.

8 Conclusion

This paper studies the portfolio decision of an investor facing the threat of illiquidity. Illiquidity is understood as a state in which the investor is not able to trade at all. We solve the corresponding control problem explicitly, which means that we derive the solution to a system of coupled HJB equations. For investors with arbitrary utility functions, we show that a model with infinitely many liquidity breakdowns can be approximated by a model in which only finitely many breakdowns are possible. We illustrate this result for an investor

[5] This figure is taken from [Diesinger, Kraft, Seifried 2009] and [Diesinger 2009].

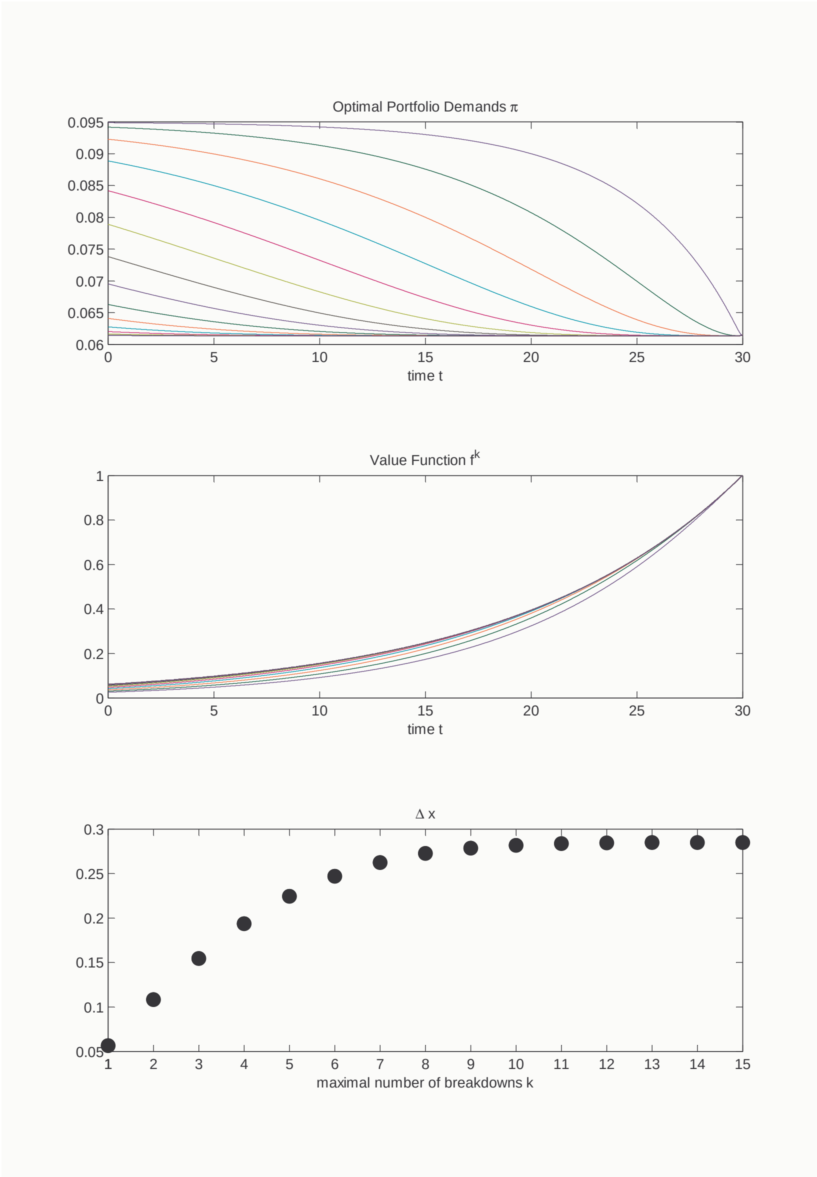

Fig. 1 Convergence of strategies, value functions, and efficiency losses as the number of breakdowns tends to infinity.

with a power utility function. Our paper also contributes to the literature dealing with the equity premium puzzle, since we introduce a model that is able to address the time dimension of an economic crisis in which trading is not possible. We remark, however, that our model is of partial equilibrium type and thus such conclusions should be viewed as suggestive rather than definitive.

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Education

03/03/1982	born in Heidelberg
08/1988–06/1992	Grundschule Laudenbach
08/1992–06/2001	Gymnasium Hemsbach
22/06/2001	Abitur, top grade with distinction
09/2001–06/2002	civil service at Deutsches Krebsforschungszentrum
11/09/2003	Vordiplom in Mathematics, top grade
04/2003–07/2004	scientific assistant at Fraunhofer Institut für Techno- und Wirtschaftsmathematik
10/2004–01/2005	visiting student at University of Oxford
29/03/2006	Diplom in Mathematics (Dipl.-Math.) at University of Kaiserslautern, specialization in probability theory, top grade with distinction
04/2006–05/2009	PhD student of Prof. Dr. Ralf Korn
02/2008–03/2008	visiting researcher at the University of Cambridge

Publications

[Diesinger, Kraft, Seifried 2009] Diesinger, P., Kraft, H., Seifried, F.T.: *Asset Allocation and Liquidity Breakdowns: What if your broker does not answer the phone?* to appear in Finance and Stochastics.

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[Seifried 2009b] Seifried, F.T.: *Optimal Investment for Worst-Case Crash Scenarios: A Martingale Approach*, preprint.

Teaching Experience: Tutorials and Seminars

summer 2006	*Mathematik für Informatiker: Statistik und Stochastik*
	(Prof. Dr. Ralf Korn)
winter 2006/2007	*Probability Theory*
	(Prof. Dr. Holger Kraft)
summer 2007	Seminar on *Alternative Models for Stock Prices*
	(Prof. Dr. Ralf Korn)
winter 2007/2008	Seminar on *Monte Carlo Methods in Finance*
	(Prof. Dr. Ralf Korn)
winter 2008/2009	*Probability Theory*
	(Prof. Dr. Heinrich von Weizsäcker)
summer 2009	*Probability Theory II*
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summer 2009	Seminar on *Financial Mathematics*
	(Prof. Dr. Jörn Saß)
summer 2009	*Statistik II für Wirtschaftswissenschaftler*
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