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On the Large Time Behavior of Diffusions

Results Between Analysis and Probability

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Preface

This thesis contains results which have been established during my Phd-studies under supervision of Prof. Dr. Heinrich von Weizsäcker. I was deeply impressed by his broad interest and mathematical knowledge. This work would definitely not have been possible without the stimulating research environment provided by the research group led by Prof. Dr. Heinrich von Weizsäcker.

This thesis does not contain all results which have been obtained during the last years. The interested reader will find further results in the domain of intersection of probability and analysis in [1] and [58]. Included are mainly those results, which really reflect the influence of the 'Kaiserslautern school of probability'. The investigation of processes conditioned on unlikely events has been a recurring theme in Kaiserslautern. In this thesis we aim to continue this tradition.

Several outcomes concerning quasistationary distributions for one-dimensional diffusions result from very stimulating discussions with Dr. David Steinsaltz during a pleasent stay at Oxford University (Worcester college). Furthermore, some inspiring results of his collaboration with Prof. Dr. Steve Evans initiated my present joint investigation with Robert Grummt of an extremely interesting super-process, which has been constructed by Fleischmann and Mueller.

At this point I would like to take the opportunity to thank Prof. Dr. Carl Mueller for his joint work with Klaus Fleischmann and for serving as external referee for this work. Moreover it is a great pleasure to thank Prof. Dr. Fritz Gesztesy for patiently answering several questions concerning his joint work with Dr. Maxim Zinchenko. Moreover I am particularly grateful to him for having read and discussed the analytic part of Chapter 3.

It is planned to submit several results of this work for publication in part jointly with Dr. David Steinsaltz, Leif Döring and Robert Grummt, respectively. The main ideas and results presented in this work are due to myself.

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Publications

• *On the strong uniqueness of some finite dimensional Dirichlet operators.* Infin. Dimens. Anal. Quantum Probab. Relat. Top. **11** (2008), no. 2, 279–293.

• (with A. Bassi and D. Dürr) *On the tong time behavior of free stochastic Schrödinger Evolutions*, Rev. Math. Phys. **22** (2010), 55–89

Chapter 1

Introduction

The study of quasistationary distributions is a long standing problem in several areas of probability theory and a complete understanding of the structure of quasistationary distributions seems to be available only in rather special situations such as Markov chains on finite sets or more general processes with compact state space. For a regularly updated extensive bibliography with about 380 entries concerning the topic of quasistationary distributions we refer to [77]. In this work we consider one-dimensional diffusions on the half-line and study the problem of convergence to quasistationarity. The starting point of our investigation is the recent contribution [84] of S. Evans and D. Steinsaltz. It might be rather surprising that despite the key contributions [25], [65], [66], [84] and [20] the structure of quasistationary distribution of one-dimensional killed diffusions has not been completely clarified. Even worse, since the proof of the main result of [25] has a serious gap[1] even for the case of one-dimensional diffusions with trivial internal killing the general picture is still rather incomplete. Similar problems for other classes of stochastic processes have also been investigated quite frequently (see e.g. the important contributions [41] and [56] of H. Kesten and his co-authors and the work [89] of E. van Doorn, where classes of Markov-chains on the integers and in particular birth and death chains are considered) but often a complete understanding is still missing.
The first work concerning the Yaglom limit for special one-dimensional diffusions including non-trivial internal killing seems to be [54]. Some results for one-dimensional diffusions with a compact state space have been established by N. Sidorova in chapter 4 of her phd-thesis under supervision of Prof. H. von Weizsäcker (see also [80]). The case of general one-dimensional diffusions with a non-compact state space seems to be considered for the first time in [84]. Steinsaltz and Evans establish in [84] several results concerning the quasistationary convergence of one-dimensional diffusions killed at the boundary and in the interior of the state space. In particular they prove an interesting dichotomy. Under quite general conditions a one-dimensional diffusion conditioned on long survival either runs off to infinity or converges to a quasistationary distribution given by the lowest eigenfunction

[1] J. San Martín communicated to the author several new ideas, which might after providing some further arguments finally lead to a rigorous new proof of the results in [25]. Moreover we should stress that the ideas developed in [25] played an important role in further developments.

of the generator. In this thesis we complete some of the results of Steinsaltz and Evans by giving conditions which allow to decide whether convergence to quasistationarity or escape to infinity occurs. Furthermore, we will be able to remove some unnecessary conditions posed in [84]. Unfortunately there are still several natural questions, which we leave open. These are collected at the end of each chapter.

The dichotomy of Evans and Steinsaltz is derived mainly by purely probabilistic arguments. In contrast to this we include several basic facts from the analytic theory of Sturm-Liouville operators. Although it is well-known since the pioneering work of Mandl [63] that the convergence to quasistationary distributions for one-dimensional diffusions is intimately connected to the bottom of the spectrum of the diffusion generator, only elementary rudiments and in many proceeding works on the problem nothing more than definitions and very basic results of the rich spectral theory have been used. The inclusion of some analytic methods allows to provide a more transparent picture. The use of these techniques even seems to be necessary since in contrast to the case of trivial internal killing methods based on the scale function are no longer available. But we have to stress that we strongly rely on Theorem 3.3 of [84] which seems to be difficult to prove by purely analytic methods. Thus the study of quasistationary distributions represents an area, where probabilistic and analytic methods cross-fertilize each other.

Let us formulate the following theorem, which follows from our general analysis presented in chapter 1, which might be considered as the main part of this work.

Theorem 1.0.1. *Let $(X_t)_{t\geq 0}$ be a diffusion process in $(0,\infty)$ corresponding to the generator $L^\kappa = -\frac{1}{2}\frac{d^2}{dx^2} + b(x)\frac{d}{dx} + \kappa(x)$ with Dirichlet boundary conditions at 0, where $b \in L^1_{loc}([0,\infty))$ and $0 \leq \kappa \in C([0,\infty))$. Moreover, assume that $\int_0^\infty e^{\int_0^x 2b(s)\,ds}\,dx = \infty$. Then the following assertions hold*

1. *Assume that ∞ is a natural boundary point and in case of $\kappa \neq 0$ suppose that $\lim_{x\to\infty}\kappa(x) \neq \lambda_0^\kappa$, where λ_0^κ denotes the bottom of the spectrum of the Sturm-Liouville operator L^κ. Then X_t converges to quasistationarity if and only if the lowest eigenfunction $\varphi(\lambda_0^\kappa,\cdot)$ is integrable with respect to the measure ρ.*

2. *If ∞ is an entrance boundary point, then X_t converges to quasistationarity if and only if the lowest eigenfunction $\varphi(\lambda_0^\kappa,\cdot)$ is integrable with respect to ρ.*

This theorem answers a question which was posed to me in a private communication with David Steinsaltz. The assertion of the above theorem can be interpreted as a 'quasistationary analogue' of the classical recurrence/transience dichotomy. In the classical case a process with a transition function $p(t,\cdot,\cdot)$ is transient iff the integral $\int_0^\infty p(t,x,y)\,dt$ is finite, and recurrent else. In the quasistationary setting, we have convergence to quasistationarity iff $\int_0^\infty \varphi(\lambda_0^\kappa,y)\,\rho(dy)$ is finite and escape to infinity, else. The analogy with the recurrence/transience dichotomy does not perfectly fit and we refer the reader to [66], where different ideas and connections to ergodic properties of h-transformed processes are presented.

We actually prove more than Theorem 1.0.1, e.g. we show that $\lambda_0^\kappa < \lim_{x\to\infty}\kappa(x)$ always implies convergence to quasistationarity. In order to prove our results we borrow some methods and ideas from mathematical physics, especially the theory of Schrödinger operators. Extensions and adaptions of these ideas in combination with methods already available in the literature on quasistationary distribution of one-dimensional diffusions play an important role in this work.

In the recent paper [20] the authors consider diffusions on the half-line including those, whose coefficients are strongly singular, in particular they are not integrable near 0. In this work the authors are interested in establishing results concerning existence and uniqueness of quasistationary distributions. Under the assumption that $\kappa \equiv 0$ and that a quasistationary distribution exists the uniqueness question is completely solved in [20], but the existence question is only answered under a whole bunch of conditions, which in particular ensure the discreteness of the spectrum of the diffusion operator. This condition simplifies the existence problem considerably: the up to multiplicative constants uniquely determined ground state gives rise to a quasistationary distribution as soon as it is integrable. Thus the existence problem is reduced to the question of the integrability of the ground state. If the bottom of the spectrum is not a L^2-eigenvalue the existence question becomes even more involved. Thus it might be surprising that a very elementary inequality turns out to play an important role. This inequality is a direct consequence of the fundamental theorem of calculus and will be used many times in this work.

In chapter 3 we consider similar to [20] generators of diffusions, whose coefficients are non-regular at 0 without assuming the discreteness of the spectrum. We are able to prove the existence of quasistationary distributions also in cases, where the spectrum is not necessarily discrete. This will done via a mixture of analytic and probabilistic techniques similar to the regular case. But the spectral theoretic considerations require more advanced tools.

Let us summarize the progress which is achieved in this work. In the regular case we are able to complete recent results of D. Steinsaltz and S. Evans in a non-trivial way. Even in the case $\kappa \equiv 0$ complete proofs of our main results are still missing in the literature. In the non-regular case the results of this work extend several of the recent results of [20]. Indeed for a large class of diffusions we will be able to show the existence of quasistationary distribution under the assumption of the strict positivity of the spectrum of the generator associated to the diffusion. Moreover our arguments are in large parts also applicable to higher dimensional problems. We hope that we are able to contribute to the higher dimensional case in near future. For first steps towards a better understanding of quasistationary distributions in higher dimensions with non-compact state space we refer to the very recent preprint [21].

In the third chapter of this work we consider a problem, which might seem quite unrelated to the proceeding results at least at the first glance. A second look at this problem already indicates that tools and ideas very similar to the ones used in the previous chapters might lead to new and in part rather precise results. We consider a rather exotic superprocess, which was constructed quite recently by Klaus Fleischmann and Carl Mueller in [42]. This process, which is called the super Brownian motion with a single point source, is strongly related to a certain family of selfadjoint extensions of the restriction of the Laplacian to smooth functions whose compact support does not contain zero. In order to get a better understanding of this process we derive results concerning the large time behavior of the expected mass. Moreover a form of the strong law of large numbers for this process is established under the assumption that a certain formula for the variance holds true. Thus it only remains to derive this formula for the second moment. Our interest in this project comes from two sources. First the process is interesting in its own since until now the probabilistic meaning of the process is quite unclear. In his review of [43] Peter Mörters calls this process 'probabilistically somewhat mysterious'. We hope that our study of the large time behavior will be a first step towards a better probabilistic understanding of the Super-Brownian motion with a single point source. This process might also be useful in order to extend results concerning probabilistic approaches to the scattering length. In [87] Michael Taylor investigates several analytic questions concerning Schrödinger operators with positive potentials. In particular he extends the path integral representation for the scattering length, which originally goes back to Kac and Luttinger (see [55]), to a large class of positive potentials. In this work M. Taylor also poses the question whether similar results are possible for potentials with a non-trivial negative part. Since negative potentials generate mass one should look for probabilistic representations of the scattering length via branching or super-processes. Since the scattering length can be easily calculated for the selfadjoint extensions of the Laplacian, we hope to be able to give a path integral representation for the scattering length in this special case and to use this as a guideline for further extensions. On the other hand part of the interest in the study of large time behavior of superprocesses stems from the work [86], where Steinsaltz and Evans use some of their results on quasistationary distribution in order to investigate the large time behavior of a superprocess modeling damage segregation in cell branching. Their analysis is restricted to a one dimensional spatial state-space. In order to design a more realistic biological model the inclusion of a higher dimensional spatial state-space becomes necessary. Our investigation of the large time behavior of the two dimensional Super-Brownian motion with a single point source can be seen as a first small step toward this direction.

The last chapter is rather unrelated to the previous ones. It is included only for reasons of completeness in order to describe a topic which has also been part of my research during the last years. Here we establish essential selfadjointness of a singular magnetic Schrödinger operator on arbitrary complete Riemannian manifolds without boundary. This result is new and extends a previous result of M. Shubin. Moreover, it turned out that the same method can also be used in order to establish a new result concerning strong uniqueness of a class of finite dimensional Dirichlet operators (see [58]).

Let us end this introduction with a few words concerning the prerequisites, which we believe to be necessary in order to understand this work. Since this work is partly analytic and partly probabilistic we assume that the reader has basic knowledge in both subjects. The analytic background which we use without specific references can be found in [95], for the probabilistic background we refer to [39] and [45]. A certain familiarity with the most basic parts of the theory of Sturm-Liouville operators (as presented e.g. in [53], [19] and [96]) will certainly be helpful. For other results used in this work we will usually give precise references.

1.1 Demographic Motivation

In this section we sketch the original motivation for the work [84] of Steinsaltz and Evans. We borrow several arguments from [83], where a much more complete discussion is presented. From the early days of demography it was known that there is an exponential increase in mortality through a wide range of the lifespan. As mentioned in [83] this observation dates back at least to the 1820s and is usually attributed to B. Gompertz. Much more recently demographers found out that the exponential rate of increase in mortality rates tends to slow down at extreme old ages. Let us be more precise and give some definitions used in demography. We mainly follow Appendix A in [88]. For a given group of individuals we denote by l_x the number of individuals who survive to reach the age x. Among these there are individuals who survive for 12 months. The number of these survivors is l_{x+1}. The mortality rate μ_x (also force of mortality or hazard rate) is on a purely formal level defined as

$$\mu_x = -\frac{1}{l_x}\frac{dl_x}{dx} = -\frac{d}{dx}\log(l_x). \tag{1.1.1}$$

Integration of this equation gives

$$\frac{l_{x+1}}{l_x} = e^{-\int_x^{x+1}\mu_z\,dz}. \tag{1.1.2}$$

Gombertz's law of mortality states that the force of mortality increases with age exponentially, i.e.

$$\mu_x = a\,e^{bx}.$$

For a large part of the age range this seems to be a good approximation, but the following figure 1.1 indicates that at extreme old ages this is no longer the case. Figure 1.1 is copied

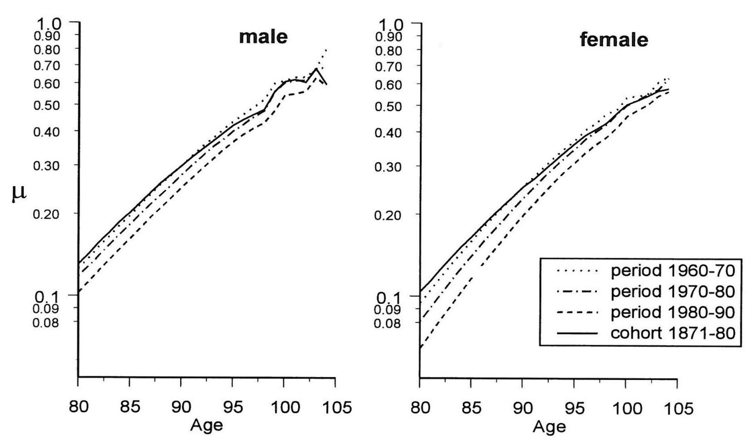

Figure 1.1: The force of mortality μ by age from [88], page 82

from page 82 in [88] and displays the force of mortality μ by age. In order to give a clean picture at the extreme ages above 95 data of Japan and 13 Western European countries are pooled. It is visible that the exponential increase of the death rates slows down at extreme ages. Thus for very high ages the usual Gombertz law is not valid anymore.

We should also point out again that the slow down of mortality rates at extreme ages is a rather universal phenomenon. It is also observed in large cohorts of drosophila or med-flies. The graphic 1.1 as well as the following graphic should only serve as an illustration of the demographic phenomenon described above. For details concerning the data set and other statistical details we therefore refer to the original sources.

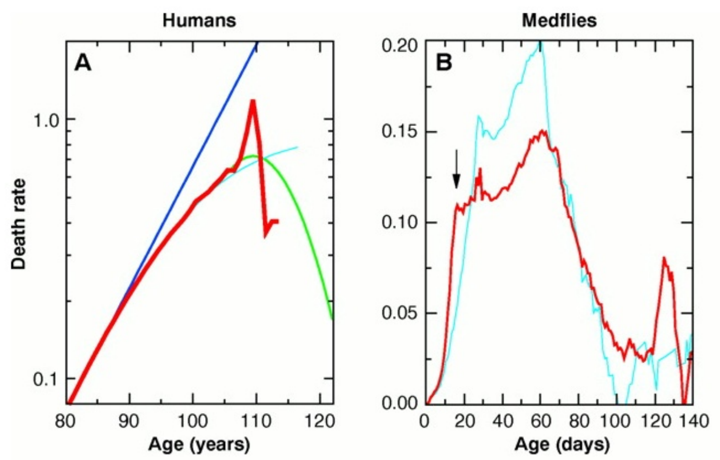

Figure 1.2: In the left picture again the death rates for human females are displayed. Moreover it includes the exponential curve that best fits the data at age 80, the logistic cure that best fits the entire data set. Moreover a curve. the logarithm of which is quadratic. is fit to the data at ages 105 and higher. The right diagram displays death rates for a cohort of 1,203,646 medflies taken from from [90]. The dark curve is for female flies and the bright curve for males.

There exist at least two different common types of explanations for this phenomenon. One type of explanation is usually referred to as temporal heterogeneity. The advocates of this type of explanation argue that the slow down of the exponential increase of the mortality rate is caused by a slow down of the aging process itself at extremely large ages. On the other side there is a type of explanation, which is referred to as population heterogeneity. Population heterogeneity refers to differences in the initial population, which cause the occurrence of mortality plateaus. Thus according to this type of explanation an individuum which reaches a large age has just been intrinsically healthier.
Quite recently J. Weitz and H. Fraser presented in [98] a completely different explanation for this phenomenon. They describe a rather simple toy model, which uses a Markov process in order to model the aging process. Such models are often referred to as Markov mortality models. Weitz and Fraser model 'vitality' X_t at age t as a one dimensional

Brownian motion with constant drift b towards 0. The time of death is the first hitting time of 0. In this rather specific model the time of death has a distribution which has a closed analytic form. Let T_0 be the time of death then the probability of survival until age t with initial vitality $y \in (0,\infty)$ is

$$\mathbb{P}_y\big(T_0 > t\big) = \int_t^\infty \frac{y}{\sqrt{2\pi r^3}}\, e^{-\frac{(y-br)^2}{2r}}\, dr.$$

A rather straightforward computation shows that in the model of Weitz and Fraser

$$\lim_{t\to\infty} \frac{\mathbb{P}_y(T_0 > t+1)}{\mathbb{P}_y(T_0 > t)} = \lim_{t\to\infty} \mathbb{P}_y(T_0 > t+1 \mid T_0 > t) = e^{-\frac{b^2}{2}}.$$

Asymptotically the model of Weitz and Fraser therefore predicts constant hazard rates. The important point in the work of Weitz and Fraser consists in giving an example where mortality plateaus arise ex nihilo. What Weitz and Fraser did not observe is the fact that mortality plateaus are a quite common feature of a large class of Markov mortality models. Steinsaltz and Evans showed that mortality plateaus are a very general phenomenon, which should be expected in many Markov models and is related to convergence to quasistationary distributions. In the Weitz-Fraser model this means that conditioned on survival the vitality converges in the large age limit to a non-trivial distribution, i.e. the distribution $\mathbb{P}_y(X_t \in \cdot \mid T_0 > t)$ has a non-trivial limit as $t \to \infty$. Therefore the demographic phenomenon of mortality plateaus leads to the interesting mathematical question of convergence to quasistationarity of one-dimensional diffusions on the half-line. And the mathematical analysis indicates that in quite arbitrary Markov mortality models the occurrence of mortality plateaus is unavoidable. Let us finally mention that in the review article [26] written by the biologist J.R. Carey the work of Steinsaltz and Evans is referred to as one of the most innovative works in biodemographic modeling. Thus the problem considered in this work seems to be not entirely of purely mathematical interest.

Chapter 2

Quasistationary Distributions in the Regular Case

In this chapter we aim to complete previous results of David Steinsaltz and Steve Evans in several directions. In order to do this we have to explore some analytic consequences of several of their assumptions. These consequences will already indicate how we have to proceed. In the next section we summarize several known facts and apply them in section 2.2 to our situation. The probabilistic conclusion is finally drawn in section 2.3, where the main results are presented. In the final section 2.4 a short summery of our strategy is given and some open problems are formulated and discussed. During this chapter standard notation will be used. For a Radon measure μ on $(0,\infty)$ $L^p((0,\infty),\mu)$ denotes as usual the space of all (equivalence classes of) functions f which satisfy $\int_0^\infty |f(y)|^p\,\mu(dy) < \infty$. If μ is the Lebesgue measure we also denote this space just by $L^p((0,\infty))$.

2.1 Assumptions, Definitions and Previous Results

In this section we describe our setting and basic assumptions. Let us start with the necessary analytic terminology. In general a Sturm-Liouville operator is any formal differential operator of the form $\tau = \tau_{p,q,V} = -\frac{1}{2p}\frac{d}{dx}q\frac{d}{dx} + V$, where $p, q : (a,b) \to (0,\infty)$ and $V : (a,b) \to \mathbb{R}$ are sufficiently well behaved functions. In this work we consider only operators where $p = q = \rho$, $V = \kappa \geq 0$ and $a = 0$, $b = \infty$. Moreover we always assume in this chapter that $\rho(x) = e^{-2\int_0^x b(s)\,ds}$ for some continuous $b \in L^1_{loc}([0,\infty))$ and $0 \leq \kappa \in C([0,\infty)$. These conditions are not at all necessary but the inclusion of even bigger classes of diffusion generators seems to be only of purely academic interest. Concerning the assumptions on b we could replace the condition $b \in L^1_{loc}([0,\infty))$ by the condition that for some $c \in (0,\infty)$ $\int_0^1 e^{-\int_c^x 2b(s)\,ds}\,dx < \infty$ and $\int_0^1 e^{\int_c^x 2b(s)\,ds}\,dx < \infty$, which exactly means that the boundary point 0 is regular in the sense of Feller and also in the sense of Weyl (see Definition 2.1.1 and Definition 2.1.2). The formal differential operator $L^\kappa = -\frac{1}{2\rho}\frac{d}{dx}\rho\frac{d}{dx} + \kappa$ gives rise to a closable densely defined quadratic form $\tilde{q}^\kappa$ in $L^2((0,\infty),\rho(y)\,dy)$ by

$$C_c^\infty((0,\infty)) \ni \varphi \mapsto \tilde{q}^\kappa(\varphi) = \frac{1}{2}\int_0^\infty |\varphi'(y)|^2 \rho(y)\,dy + \int_0^\infty \kappa(y)|\varphi(y)|^2\,\rho(y)\,dy.$$

The closure of this quadratic form will be denoted by q^κ. In the sequel ρ will also denote the measure $\rho(y)\,dy$. To the quadratic form q^κ there corresponds a uniquely defined positive selfadjoint operator L^κ. This selfadjoint extension of the Sturm-Liouville differential expression is the so called Friedrichs extension. It is easy to see that the action of the operator L^κ is given by

$$L^\kappa\varphi(x) = -\frac{1}{2}\varphi''(x) + b(x)\varphi'(x) + \kappa(x)\varphi(x).$$

The bottom of the spectrum of L^κ will be denoted by λ_0^κ. $\sigma(L^\kappa)$ will denote the spectrum of the selfadjoint operator L^κ. The corresponding objects with $\kappa \equiv 0$ are often denoted by q, L and λ_0 instead of q^0, L^0 and λ_0^0, respectively. Since L^κ and L are selfadjoint operators the spectral theorem implies the existence of spectral resolutions $(E_\lambda^\kappa)_{\lambda\in[\lambda_0^\kappa,\infty)}$ and $(E_\lambda)_{\lambda\in[\lambda_0^\kappa,\infty)}$, respectively. For the basic facts concerning spectral theory of selfadjoint operators the reader should consult [95]. Furthermore, the spectral theorem for selfadjoint operators allows to consider functions $f(L^\kappa)$ of the operator. As explained in section 8.2 of [95] for a given Borel-measurable function $f : \mathbb{R} \to \mathbb{R}$ the operator $f(L^\kappa)$ is defined via

$$\begin{aligned} \mathcal{D}(f(L^\kappa)) &= \left\{ u \in L^2((0,\infty),\rho) \mid \int_{\sigma(L^\kappa)} |f(\lambda)|^2\, d\|E_\lambda^\kappa u\|^2_{L^2(\rho)} < \infty \right\} \\ f(L^\kappa)u &= \int_{\sigma(L^\kappa)} f(\lambda)\, dE_\lambda^\kappa u. \end{aligned} \tag{2.1.1}$$

Observe that for a Borel-measurable function $f : [0,\infty) \to \mathbb{R}$ and $\alpha \geq 0$ we have $Ran(f(L^\kappa)) \subset \mathcal{D}((L^\kappa)^\alpha)$ if $[0,\infty) \ni \lambda \mapsto |\lambda^\alpha f(\lambda)|$ is bounded. This implies in particular that the range of e^{-tL^κ} is contained in the domain of all powers of L^κ. Moreover, the spectral theorem allows to clarify further the connection between the quadratic form q^κ and the associated non-negative operator L^κ. Let $\sqrt{L^\kappa}$ denote the unique non-negative square root of L^κ which is defined via the spectral theorem. Then we have $\mathcal{D}(q^\kappa) = \mathcal{D}(\sqrt{L^\kappa})$ and for every $f \in \mathcal{D}(L^\kappa)$ we have

$$q^\kappa(f,g) = \big(\sqrt{L^\kappa}f, \sqrt{L^\kappa}g\big)_{L^2((0,\infty),\rho)}. \tag{2.1.2}$$

Using the 'elliptic' Harnack inequality it is not difficult (see [63] and Lemma 2.2 in [84]) to see that

$$\begin{aligned} \lambda_0^\kappa = \max\{\lambda \in \mathbb{R}\ | &\text{there is a positive solution of } (L^\kappa - \lambda)u = 0 \\ &\text{with } u(0) = 0,\ u'(0) = 1\}. \end{aligned} \tag{2.1.3}$$

Equation (2.1.3) already indicates that for $0 \leq \lambda \leq \lambda_0^\kappa$ solutions of $(L^\kappa - \lambda)u = 0$ might have a probabilistic significance.
In the sequel we usually denote by $\varphi(\lambda,\cdot)$ the solution of the ordinary differential equation

$$(L^\kappa - \lambda)\varphi(\lambda,\cdot) = 0, \;\; \varphi(\lambda,0) = 0,\ \varphi'(\lambda,0) = 1. \tag{2.1.4}$$

It might be important to note that solutions in (2.1.3) and (2.1.4) are solutions in the sense of the theory of ordinary differential equations. Despite the fact that we write L^κ in (2.1.3) and (2.1.4) we do not suppose in (2.1.3) and (2.1.4) that u and $\varphi(\lambda,\cdot)$ belongs to $L^2((0,\infty),\rho)$. In the sequel we usually try to make clear whether we consider solutions in the sense of ordinary differential equation or solutions in the sense of Hilbert space theory. Thus we do not require that the solution also belongs to the Hilbert space $L^2((0,\infty),\rho)$ and thus is an eigenfunction in the sense of spectral theory. Usually we denote solutions of (2.1.4) which are also eigenfunctions in the sense of spectral theory by u_λ. In this case we always assume u_λ to be normalized, i.e. $\|u_\lambda\|_{L^2((0,\infty),\rho)} = 1$.
At one point we will refer to the operator L_N which is associated to the closure of the quadratic form

$$C_c^\infty([0,\infty)) \ni \varphi \mapsto \tilde{q}^N(\varphi) = \frac{1}{2}\int_0^\infty |\varphi'(y)|^2 \rho(y)\,dy,$$

i.e. L_N is the selfadjoint realization of the differential expression $-\frac{1}{2\rho}\frac{d}{dx}\big(\rho\frac{d}{dx}\big)$ in $L^2((0,\infty),\rho)$, which has Neumann-boundary conditions at 0. The quadratic form q is a Dirichlet form and the canonically associated Markov-process is a solution for the martingale problem associated to the operator L with pure killing at 0. This means that there exists a family of measures $(\mathbb{P}_x)_{x\in(0,\infty)}$ on the space $C([0,\infty),\mathbb{R})$ of real valued continuous functions on $[0,\infty)$ such that for every $f \in L^2((0,\infty),\rho)$ and every $x \in (0,\infty)$ (due to the Feller property)

$$(e^{-tL}f)(x) = \mathbb{E}_x[f(X_t), T_0 > t],$$

where $(X_t)_t$ is the canonical process on $C([0,\infty),\mathbb{R})$ and $T_0 = \inf\{t > 0 \mid X_t = 0\}$. One dimensional diffusions have the great advantage that several important probabilities can be calculated quite explicitly (see chapter 5 in [75]). We just recall that $\mathbb{P}_x\big(T_0 < \infty\big) = 1$ for $x > 0$, if and only if for some $c > 0$ $\int_c^\infty e^{\int_0^t 2b(s)\,ds}\,dt = \infty$. If $\int_c^\infty e^{\int_0^t 2b(s)\,ds}\,dt < \infty$ then

$$\mathbb{P}_x\big(T_0 < \infty\big) = \frac{\int_x^\infty e^{\int_0^t 2b(s)\,ds}\,dt}{\int_0^\infty e^{\int_0^t 2b(s)\,ds}\,dt}.$$

Several other qualitative properties can be deduced in a similar way. In the next remark we summarize another important fact from the theory of one-dimensional diffusions.

Remark 2.1.1. *The diffusion corresponding to L is recurrent (see [75] chapter 5) if and only if for $x_0 \in (0,\infty)$*

$$\int_0^{x_0} \exp\Big(\int_{x_0}^x 2b(s)\,ds\Big)\,dx = \infty \quad \textit{and} \quad \int_{x_0}^\infty \exp\Big(\int_{x_0}^\infty 2b(s)\,ds\Big)\,dx = \infty.$$

In analytic terms recurrence means that the associated generator is critical (see [46] and [75]). Recall that L^κ is called critical iff there exists a unique (up to constant multiples) positive solution ψ of $L^\kappa\psi = 0$. Otherwise L^κ is called subcritical. The following fact from the criticality theory will be used later in Lemma 2.2.2: If 0 is an eigenvalue in the L^2-sense, then the generator is necessarily critical. This follows e.g. from Theorem 3.15 *of [46] or Theorem 4.2 in [67]. The fact that the technical assumptions of Theorem 4.2 in [67] are satisfied follows from the proof of Lemma 2.2.2 (1). In order to apply Theorem 4.2 in [67] observe assuming further that the bottom of the spectrum of L^κ is an eigenvalue in the L^2-sense. Then for every negative $0 \neq W \in L^\infty((0,\infty))$ with compact support in $(0,\infty)$ we get*

$$\inf_{\substack{\phi \in C_c^\infty((0,\infty)) \\ \|\varphi\|_{L^2((0,\infty),\rho)}=1}} \left(q^\kappa(\varphi) + \int_0^\infty W(x)|\varphi(x)|^2 \,\rho(dx) \right) \leq \int_0^\infty W(x)|u_{\lambda_0^\kappa}(x)|^2 \,\rho(dx) < 0.$$

Also the semigroup e^{-tL^κ} has a probabilistic representation: We consider the product space

$$C([0,\infty)) \times [0,\infty) = \{(\omega,\xi) \in C([0,\infty)) \times [0,\infty)\}$$

endowed with the natural product σ-field. Let $(\tilde{\mathbb{P}}_x)_{x\in(0,\infty)}$ denote the family of measures which is induced by the Dirichlet form q^0. For $x \in (0,\infty)$ we define the measures

$$\tilde{\mathbb{P}}_x \otimes e^{-\xi}\, d\xi.$$

and the stopping time

$$T_\kappa(\omega,\xi) = \inf\left\{ s \geq 0 \mid \int_0^s \kappa(\omega_s)\, ds \geq \xi \right\}.$$

If we set

$$\tau_\partial = \min\bigl(T_0, T_\kappa\bigr)$$

then

$$(e^{-tL^\kappa} f)(x) = \tilde{\mathbb{E}}_x\left[f(X_t), \tau_\partial > t\right] = \mathbb{E}_x\left[e^{-\int_0^t \kappa(X_s)\,ds} f(X_t), T_0 > t\right] \tag{2.1.5}$$

Since it will be clear from the context, which probability measure is meant we omit the tilde. It is rather straightforward to justify the Feynman-Kac representation (2.1.5) using standard methods which are explained in great detail in the first three chapters of [29].

Let us recall the usual Feller classification (see e.g. chapter 3 in [10]) for diffusion generators $-\frac{1}{2}\frac{d^2}{dx^2} + b(x)\frac{d}{dx}$ in an open interval $(0,r)$.

Definition 2.1.1. *Let $c \in (0,r)$ be given and set $\rho(t) = e^{-\int_c^t 2b(s)\,ds}$. The point r is called **accessible**, if $\int_c^r \rho(y)^{-1} \int_c^y \rho(z)\,dz\,dy < \infty$, and otherwise **inaccessible**. If r is an accessible boundary point, then it is called **regular** iff $\int_c^r \int_c^y \rho(t)^{-1}\,dt\rho(y)\,dy < \infty$. If r is accessible and $\int_c^r \int_c^y \rho(t)^{-1}\,dt\rho(y)\,dy = \infty$, then r is called **exit boundary**. If r is inaccessible then it is called **entrance boundary**, iff $\int_c^r \int_c^y \rho(t)^{-1}\,dt\rho(y)\,dy < \infty$. If r is inaccessible and $\int_c^r \int_c^y \rho(t)^{-1}\,dt\rho(y)\,dy = \infty$, then r is called **natural**. Of course the same classification holds for 0.*

In this chapter we will always assume that the boundary point infinity is inaccessible. As mentioned in [84] the case of two accessible boundary points will offer no additional novelty.

The names of the boundaries can be justified by the behavior of the associated diffusion process, e.g. if the boundary point r is accessible, then $\mathbb{P}_x(T_r < \infty) > 0$ for every $r \geq x > 0$. On the other hand if $r = \infty$ is natural then for any $y \in (0, \infty)$

$$\lim_{x \to \infty} \mathbb{P}_x\big(T_y > s\big) = 0 \text{ for any } s > 0.$$

It is easy to check that the boundary point r is regular if and only if $\int_c^r \rho(t)\,dt < \infty$ and $\int_c^r \rho(t)^{-1}\,dt < \infty$. If r is an entrance boundary, then $\int_c^r \rho(t)\,dt < \infty$ but $\int_c^r \rho(t)^{-1}\,dt = \pm\infty$. If r is exit then $\int_c^r \rho(t)^{-1}\,dt$ is finite but $\int_c^r \rho(t)\,dt$ is not finite.

A boundary point is thus regular in the sense of Feller if and only if it is regular in the sense of Weyl. For convenience of the reader let us recall the relevant definition of the well-known Weyl theory of selfadjoint extensions of singular Sturm-Liouville operators $L^\kappa = -\frac{1}{2\rho}\frac{d}{dx}(\rho\frac{d}{dx}) + \kappa$ in $(0, r)$ adapted to our special situation.

Definition 2.1.2. *Let $z \in \mathbb{C}$. We say that boundary r is of **limit point type**, if there exists $c \in (0, r)$ and a solution f of $(L^\kappa - z)f = 0$ such that $\int_c^r |f(r)|^2 \rho(y)\,dy = \infty$. If there exists $c \in (0, \infty)$, such that for every solution of the equation $(L^\kappa - z)f = 0$ the integral $\int_c^r |f(y)|^2 \rho(y)\,dy$ is finite, then we say that r is of **limit-circle type**. An analogous notation applies to the boundary point 0.*

A fundamental result in the theory of Sturm-Liouville operators is the so called Weyl-alternative, which states that exactly one of the above situations holds and that the limit-point/limit-circle classification is independent of $z \in \mathbb{C}$ (see [53]). Moreover if we are in the limit point case at r then for every $z \in \mathbb{C} \setminus \mathbb{R}$ there exists exactly one solution of the equation $(L^\kappa - z)f = 0$ which satisfies $\int_c^r |f(s)|^2 \rho(dy) < \infty$. Roughly limit-circle case at a boundary point r means that we have to specify boundary conditions at r in order to get a selfadjoint realization, whereas in the limit-point case at r no boundary conditions at r are necessary.

As we mentioned in the introduction we are mainly interested in the asymptotic behavior of the diffusion process $(X_t)_{t \geq 0}$ corresponding to the operator L^κ on the half line conditioned on survival. Therefore let us summarize the relevant definitions. Observe that our formulation slightly differs from the one in [84] as we use the measure ρ as a reference measure instead of the Lebesgue measure.

Definition 2.1.3. *We say that X_t **converges from the initial distribution** ν **to the quasistationary distribution** φ **on compacta** if for any positive z, and any Borel $A \subset [0, z]$*

$$\lim_{t \to \infty} \mathbb{P}_\nu\big(X_t \in A \mid X_t \leq z\big) = \frac{\int_A \varphi(y)\,\rho(dy)}{\int_0^z \varphi(y)\,\rho(dy)};$$

X_t ***converges from the initial distribution*** ν ***to the quasistationary distribution*** φ *if for any Borel subset* $A \subset [0,\infty)$

$$\lim_{t\to\infty} \mathbb{P}_\nu\big(X_t \in A \mid \tau_\partial > t\big) = \frac{\int_A \varphi(y)\,\rho(dy)}{\int_0^\infty \varphi(y)\,\rho(dy)}.$$

Finally we say that X_t ***escapes from the initial distribution*** ν ***to infinity*** *if*

$$\lim_{t\to\infty} \mathbb{P}_\nu\big(X_t \le z \mid \tau_\partial > t\big) = 0.$$

Remark 2.1.2. *In the literature there is no commonly accepted definition of quasistationary distributions. The probability measure* $\frac{\varphi(y)\,\rho(dy)}{\int_0^\infty \varphi(y)\,\rho(dy)}$ *in Definition 2.1.3 is sometimes also called a quasi-limiting distribution. A quasistationary distribution* $\tilde{\nu}$ *is often defined as a probability measure* $\tilde{\nu}$ *supported in* $(0,\infty)$ *satisfying*

$$\mathbb{P}_{\tilde{\nu}}\big(X_t \in A \mid \tau_\partial > t\big) = \tilde{\nu}(A), \ \forall \text{ Borel sets } A \subset (0,\infty),\, t > 0.$$

Quasi-limiting distributions are also called Yaglom limits. It is not difficult to see that quasi-limiting distributions are in particular quasistationary distributions. Most of the time we use both terms interchangeable since we are mainly interested in the quasi-limiting distribution.

Let us now describe one of the main results of Steinsaltz and Evans (Theorem 3.3 in [84]). Observe that we have due to equation (2.1.3) that $\varphi(\lambda_0^\kappa,\cdot)$ is positive.

Theorem 2.1.1 (Theorem 3.3 in [84])**.** *Let* $(X_t)_{t\ge 0}$ *denote the diffusion which is associated to the Dirichlet form* q^κ *and let* L^κ *denote the operator associated to* q^κ*. Moreover let* $\lambda_0^\kappa = \inf\sigma(L^\kappa)$ *and the* $\varphi(\lambda_0^\kappa,\cdot)$ *solve the ordinary differential equation* $L^\kappa\varphi(\lambda_0^\kappa,\cdot) = \lambda_0^\kappa\varphi(\lambda_0^\kappa,\cdot)$ *with* $\varphi(\lambda_0^\kappa,0) = 0$ *and* $\varphi'(\lambda_0^\kappa,0) = 1$*. Assume that* ∞ *is a natural boundary point and that we are in the limit-point case at* ∞*. Suppose that either*

$$\liminf_{x\to\infty} \kappa(x) > \lambda_0^\kappa \quad \text{or} \quad \limsup_{x\to\infty} \kappa(x) < \lambda_0^\kappa.$$

Then either X_t *converges to the quasistationary distribution* $\varphi(\lambda_0^\kappa,\cdot)$*, or* X_t *escapes to infinity. In the case* $\liminf \kappa(x) > \lambda_0^\kappa$*,* X_t *converges to the quasistationary distribution* $\varphi(\lambda_0^\kappa,\cdot)$ *if and only if* $\int_0^\infty \varphi(\lambda_0^\kappa,y)\,\rho(dy)$ *is finite.*

We would like to stress that the assertion of Theorem 2.1.1 is not at all trivial. A priori it is neither clear that the conditional distribution converges nor that the mass can not split with part of the mass remaining on $(0,\infty)$ and part of the mass escaping to infinity. Missing are obviously simple general conditions, which allow to decide which case actually occurs. Steinsaltz and Evans are able to specify one case, in which X_t converges to quasistationarity. They prove that in the case $\lambda_0^\kappa < K =: \lim_{t\to\infty}\kappa(t)$ one has convergence to quasistationarity if the condition

$$\text{(GB')} \quad \exists \tilde{b},\tilde{\kappa} \ge 0\, \forall y \text{ large enough } : |b(y)| \le \tilde{b}y \text{ and } \kappa(y) \le \tilde{\kappa}y$$

or the related condition

$$\text{(GB")} \quad \exists \bar{b}_1, \bar{b}_2, \bar{\kappa}, \beta \geq 0 \, \forall y \text{ large enough } : \bar{b}_1 y^\beta \geq b(y) \geq -\bar{b}_1 y, \, b'(y) \geq -\bar{b}_2 y^2 \quad \text{and } \kappa(y) \leq \tilde{\kappa} y$$

are satisfied (Notice, that our sign convention differs from that of [84]). We would like to point out that though these conditions are satisfied in many applications they are from a theoretical point of view somewhat unnatural, e.g. there seems to be no natural reason, why an upper bound on the killing rate as in (GB') should be necessary. On the contrary if one strengthens the killing rate κ then from a heuristic point of view it should be easier to prove convergence to quasistationarity.

Remark 2.1.3. *We make use of Theorem 2.1.1 only in the case* $\lambda_0^\kappa > \lim_{x\to\infty} \kappa(x)$ *and* $\rho((0,\infty)) < \infty$. *In the other cases we use different techniques. In the next section we show that* ∞ *is always in the limit-point case. As emphasized in [84] in this case the heuristic behind Theorem 2.1.1 is quite clear, but the translation of this idea into formal mathematics is not trivial.*

2.2 Analytic Results

In this section we present several results, which are mainly derived via analytic techniques. We use standard methods from the theory of Sturm-Liouville operators but the inclusion of these methods in the study of quasistationary distributions seems to be new. The results are most probably not surprising for or even known to experts in the Sturm-Liouville theory but since our results will mainly be of interest for the probability community we present rather complete proofs. We start by establishing a connection between the Feller classification and the Weyl classification of boundary points. This has already been investigated in [99] for the case $\kappa = 0$, but in this work the author introduces the notion of weak entrance boundary and shows that one is in the limit-circle case if the boundary point is of weak entrance type. In particular we show that there are no weak entrance boundaries at ∞ by proving that ∞ is of limit-point case if and only if it is not a regular boundary point. The proof we give is rather well-known in the Schrödinger case (see [13] and [79] for similar ideas in a much more general context and section 13.4 in [96]). The required regularity of the coefficients of the Sturm-Liouville expression is far from being optimal and is assumed only for convenience.

Lemma 2.2.1. *Let the Sturm-Liouville expression* $\tau f(x) = \frac{1}{r(x)}\big(r(x)f'(x)\big)' + q(x)f(x)$ *be given. Assume that* r *is strictly positive and locally Lipschitz in* $(0,\infty)$ *and* $q \in L^2_{loc}([0,\infty))$ *such that* $q(x) \geq -C|x|^2 + D$ *for some constants* $C, D \geq 0$ *Then we are in the limit point case at* ∞.

Proof. Obviously we can assume that $D = 0$. As usual in the theory of Sturm-Liouville operators we define the maximal operator T and the minimal operator $\tilde{T}$ associated to the differential expression τ via

$$\mathcal{D}(T) := \big\{ f \in L^2((0,\infty), r(x)dx) \mid f,\, rf' \text{ absolutely continuous in } (0,\infty), \\ \tau f \in L^2((0,\infty), r(x)dx) \big\}$$

$$Tf := \tau f \ \text{ for } f \in \mathcal{D}(T)$$

and

$$\mathcal{D}(\tilde{T}) := \big\{ f \in \mathcal{D}(T) \mid \text{f has compact support in } (0,\infty) \big\}, \quad \tilde{T} f := Tf \ \text{ for } f \in \mathcal{D}(\tilde{T}),$$

respectively. Let T_γ be the restriction of the maximal operator T to the domain

$$\mathcal{D} = \big\{ f \in \mathcal{D}(T) \mid f(0)\cos(\gamma) - rf'(0)\sin(\gamma) = 0 \big\},$$

where $\gamma \in [0,\pi)$. The reader may even set $\gamma = 0$ in the sequel. Then $(T_\gamma, \mathcal{D})$ is a 1-dimensional restriction of the maximal operator. We will prove that T_γ defines a symmetric operator, i.e. $(f, T_\gamma f)_{L^2(\rho)} \in \mathbb{R}$ for every $f \in \mathcal{D}$. This proves the assertion of the Lemma, since if we are in the limit-circle case at ∞ then the deficiency indices of the minimal operator $\tilde{T}$ would be $(2,2)$ and the operator T_γ would be a 3-dimensional extension of $\tilde{T}$. This is a contradiction to the symmetry of T_γ since every maximal symmetric extension of $\tilde{T}$ is a 2-dimensional extension. Let $\varphi \in C_c^\infty(\mathbb{R})$ such that $0 \le \varphi \le 1$ and

$$\varphi(x) = \begin{cases} 1 & \text{if } |x| \le 1 \\ 0 & \text{if } |x| \ge 2. \end{cases}$$

Further we set $\varphi_k(x) = \varphi(\frac{x}{k})$ $(k \in \mathbb{N})$. Then we get

$$\begin{aligned} \int_0^\infty \varphi_k(x)^2 \overline{f(x)} T_\gamma f(x) r(x)\, dx &= \int_0^\infty \varphi_k(x)^2 \overline{f(x)} \big((rf'(x))' + r(x)q(x) \big)\, dx \\ &= \int_0^\infty \varphi_k(x)^2 \big(|f'(x)|^2 + q(x)|f(x)|^2 \big) r(x)\, dx \\ &+ 2 \int_0^\infty \varphi_k(x)\varphi_k'(x) \overline{f(x)} f'(x) r(x)\, dx. \end{aligned}$$

This gives for $f \in \mathcal{D}(T_\gamma)$ and $k \in \mathbb{N}$

$$\begin{aligned} (f, T_\gamma f)_{L^2(0,b,r)} &= \lim_{k\to\infty} \int_0^\infty \varphi_k(x)^2 \overline{f(x)} T_\gamma f(x)\, r(x) dx \\ &= \lim_{r\to\infty} \Big\{ \int_0^\infty \varphi_k^2(x) \big(|f'(x)|^2 + q|f|^2 \big)\, r(x) dx \\ &+ 2 \int_0^\infty \varphi_r(x)\varphi_r'(x) \overline{f(x)} f'(x) r(x)\, dx \Big\}. \end{aligned} \tag{2.2.1}$$

The first term on the right hand side of equation (2.2.1) is real and therefore is suffices to prove that the second term converges to 0 as $k \to \infty$. For the second term on the right hand side of equation (2.2.1) we have for some constant $M > 0$

$$\left|\int_0^\infty \varphi_k(x)\varphi_k'(x)\overline{f(x)}f'(x)\,r(x)dx\right| \leq \left(\int_0^\infty \varphi_k(x)^2|f'(x)|^2 r(x)\,dx \int_0^\infty |\varphi_k'(x)|^2|f(x)|^2\,r(x)dx\right)^{\frac{1}{2}}$$
$$\leq Mk^{-1}\left(\int_0^\infty \varphi_k(x)^2|f'(x)|^2\,r(x)dx \int_k^{2k} |f(x)|^2 r(x)\,dx\right)^{\frac{1}{2}}. \tag{2.2.2}$$

For the first integral on the right hand side of the inequality (2.2.2) we get with the help of (2.2.1)

$$\begin{aligned}\int_0^\infty \varphi_k(x)^2|f'(x)|^2 r(x)\,dx &= \int_0^\infty \varphi_k^2(x)\overline{f(x)}(T_\gamma f(x))\,r(x)dx - \int_0^\infty \varphi_k^2(x)q(x)|f(x)|^2\,r(x)dx\\ &- 2\int_0^\infty \varphi_k\varphi_k'(x)\overline{f(x)}f(x)'\,r(x)dx\\ &\leq \int_0^\infty \varphi_k(x)|f(x)||T_\gamma f(x)|\,r(x)dx + C\int_0^\infty \varphi_k^2(x)x^2|f(x)|^2\,r(x)dx\\ &+ \frac{1}{2}\int_0^\infty \varphi_k(x)^2|f'(x)|^2\,r(x)dx + 4M^2k^{-2}\|f\|^2_{L^2(r(x)dx)}\\ &\leq \|f\|_{L^2(r(x)dx)}\|T_\gamma f\|_{L^2(r(x)dx)} + C(2k)^2\|f\|_{L^2(r(x)dx)}\\ &+ \frac{1}{2}\int_0^\infty \varphi_k(x)^2|f'(x)|^2\,r(x)dx + 4M^2k^{-2}\|f\|_{L^2(r(x)dx)}.\end{aligned}$$

This gives

$$\begin{aligned}\int_0^\infty \varphi_k(x)^2|f'(x)|^2\,r(x)dx \leq 2\|f\|_{L^2(r(x)dx)}\|T_\gamma f\|_{L^2(r(x)dx)} + 2C(2k)^2\|f\|^2_{L^2(r(x)dx)}\\ + 8M^2k^{-2}\|f\|^2_{L^2(r(x)dx)}\end{aligned}$$

and therefore for large k

$$\int_0^\infty \varphi_k(x)^2|f'(x)|^2\,dx \leq C_1 + C_2k^2 \leq C_3k^2.$$

Using (2.2.2) this implies that as $k \to \infty$

$$\left|\int_0^\infty \varphi_k(x)\varphi_k'(x)\overline{f(x)}f'(x)\,r(x)dx\right| \leq Mk^{-1}\left(C_3k^2\int_k^{2k} |f(x)|^2\,r(x)dx\right)^{\frac{1}{2}} \to 0.$$

This finishes the proof of the assertion. □

The above result in particular shows that the assumption (LP) (= limit point case at infinity) in [84] can always be dropped and we are allowed to use the results established by Steinsaltz and Evans. Steinsaltz and Evans proved that (LP) holds true if $\kappa \equiv 0$ or if the assumption

$$\text{(LP')}\ \liminf_{z\to\infty} z^{-2}\big(b(z)^2 - b'(z) + 2\kappa(z)\big) > -\infty$$

is satisfied. The occurrence of (LP') is explained by the fact that Steinsaltz and Evans do not use the space $L^2((0,\infty),\rho)$ from the very beginning but instead work with $L^2((0,\infty),dx)$. Let us recall another fundamental result from the theory of Sturm-Liouville operators with one regular boundary point. A proof of it can be found in [47], [19] or any other textbook on the theory of Sturm-Liouville operators.

Theorem 2.2.1. *Let $\tau = -\frac{1}{2\rho}\frac{d}{dx}\big(\rho\frac{d}{dx}\big)+\kappa$ be a Sturm-Liouville expression which is regular at 0 and in the limit point case at infinity and let H be the selfadjont realization of τ in $L^2((0,\infty),\rho(x)dx)$ with Dirichlet boundary conditions at 0, where $\rho(x) = e^{-\int_0^x 2b(s)\,ds}$. Let $\varphi(z,\cdot)$ be the unique solution of the ordinary differential equation $\tau\varphi(z,\cdot) = z\varphi(z,\cdot)$ satisfying $\varphi(z,0) = 0$ and $\varphi'(z,0) = 1$. Then there exists a measure σ such that supp$(\sigma) = \sigma(H)$ an such that*

$$U : L^2((0,\infty),\rho) \to L^2(\sigma(H),\sigma),\ h \mapsto \hat{h}(\cdot) = \int_0^\infty h(x)\varphi(\cdot,x)\,\rho(x)dx$$

uniquely extends to a unitary mapping with the property that for every $F \in C(\mathbb{R})$

$$U\,F(H)U^{-1} = M_F$$

in $L^2(\mathbb{R},\mu)$, where M_F denotes the maximal operator acting by multiplication with F. Moreover the spectrum of H is simple and $\sigma(F(H)) = \text{ess ran}_\sigma(F)$.

The spectrum of a selfadjoint operator can be classified at least into two parts. Recall that the essential spectrum $\sigma_{ess}(A)$ of a selfadjoint operator A consists of all limit points of the spectrum $\sigma(A)$ and all eigenvalues of A of infinite multiplicity. The discrete part $\sigma_d(A)$ of the spectrum $\sigma(A)$ of A consists of all isolated eigenvalues of finite multiplicity. In the Sturm-Liouville case every eigenvalue has finite multiplicity and therefore the essential spectrum of Sturm-Liouville operators consists of the limit points of the spectrum. It is well-known that the essential part of the spectrum of selfadjoint operators is invariant with respect to relatively compact perturbations. It might be useful to recall the definition of relative compactness. Let $T : X \to X$ be an operator acting in a Banach space X. Then an operator $V : X \to X$ is called relatively compact with respect to T, if $\mathcal{D}(T) \subset \mathcal{D}(V)$ and if for some $z \in \mathbb{C}\setminus\sigma(T)$ the operator $V(T-z)^{-1}$ is compact. We refer to section 9.2 of [95] for further details, which will be used in the next Lemma, where several consequences of the condition $\lambda_0^\kappa \neq \lim_{t\to\infty}\kappa(t)$ are investigated.

Lemma 2.2.2. *Let the Sturm-Liouville expression $\tau f = -\frac{1}{2\rho}(\rho f')'$ in $(0,\infty)$ be given and let L be the selfadjoint realization of τ in the Hilbert space $L^2((0,\infty),\rho(x)\,dx)$ satisfying*

Dirichlet boundary conditions at 0. *For some continuous and bounded function* $0 \leq \kappa \in C([0,\infty))$ *let* L^κ *denote the selfadjoint realization of the Sturm-Liouville expression* $\tau + \kappa$ *satisfying Dirichlet boundary conditions at* 0. *As usual set* $\lambda_0 = \inf(\sigma(L))$ *and* $\lambda_0^\kappa = \inf(\sigma(L^\kappa))$. *Then the following assertions are true:*

(1) *If* $\lim_{t\to\infty} \kappa(t) = 0$, *then* $\sigma_{ess}(L) = \sigma_{ess}(L^\kappa)$

(2) $\lambda_0 > 0$ *and* $\int_0^\infty \rho(t)^{-1}\, dt = \infty$ *imply* $\rho(\mathbb{R}_+) < \infty$.

(3) $\lambda_0 > 0$ *and* $\rho([0,\infty)) = \infty$ *imply* $\lim_{r\to\infty} \frac{1}{r} \log \rho([0,r)) > 0$.

(4) *If* $\lambda_0^\kappa < \lim_{t\to\infty} \kappa(t)$ *then* λ_0^κ *is a simple isolated eigenvalue with a unique positive eigenfunction.*

(5) *If* $\lambda_0^\kappa > \lim_{t\to\infty} \kappa(t)$ *then* $\lambda_0 > 0$

Proof. *Assertion (1)* can be derived from the fact that the essential spectra of two selfadjoint operators T_1 and T_2 coincide, if the for some $z \in \mathbb{C} \setminus (\sigma(T_1) \cup \sigma(T_2))$ the difference

$$(T_1 - z)^{-1} - (T_2 - z)^{-1}$$

is a compact operator. Set $\kappa_n(t) = \mathbf{1}_{[0,n]}(t)\kappa(t)$. The resolvent equation gives for $z \in \mathbb{C} \setminus \mathbb{R}$

$$(L^{\kappa_n} - z)^{-1} - (L - z)^{-1} = (L^{\kappa_n} - z)^{-1}(L - L^{\kappa_n})(L - z)^{-1} = -(L^{\kappa_n} - z)^{-1}\kappa_n(L - z)^{-1}.$$

Observe now that the operator $\kappa_n(L - z)^{-1}$ is compact, i.e. the operator acting by multiplication with κ_n is relatively compact with respect to the operator L. This can be seen by considering the explicit form of the resolvent (see chapter 3.3 in [53], similar results can be found in [24]). We have

$$[\kappa_n(L - z)^{-1}]g(x) = \kappa_n(x)\frac{1}{W(v,u)}\left(v(x)\int_0^x u(y)g(y)\rho(y)dy + u(x)\int_x^\infty v(y)g(y)\rho(y)dy\right),$$

where the Wronskian $W(f,g)$ of two locally absolutely continuous functions f and g is defined by $W(f,g)(x) = f(x)\rho g'(x) - \rho f'(x)g(x)$ and u,v are linearly independent solutions of $(\tau - z)w = 0$ satisfying $u(0) = 0$,$u'(0) = 1$ and $\int_1^\infty |v(y)|^2\, \rho(dy) < \infty$. As is well known the Wronskian $W(v,u)$ is actually independent of $x \in (0,\infty)$ and is not zero due to the linear independence of v and u. Observe that we are using the fact that we are in the limit point case at infinity. Thus $\kappa_n(L - z)^{-1}$ is an integral operator in $L^2(\rho)$ with kernel $k(\cdot,\cdot)$ given by

$$k(x,y) = \begin{cases} \frac{\kappa_n(x)}{W(v,u)} v(x)u(y) & \text{if } y \leq x, \\ \frac{\kappa_n(x)}{W(v,u)} v(y)u(x) & \text{if } y \geq x, \end{cases}$$

which is Hilbert-Schmidt and therefore in particular compact. In order to prove the Hilbert-Schmidt property observe that due to the mentioned properties of u and v

$$\int_0^\infty \int_0^\infty |k(x,y)|^2\, \rho(dy)\, \rho(dx) = \frac{1}{W(v,u)^2} \int_0^n \left(\int_0^x |u(y)|^2\, \rho(dy) |\kappa_n(x)v(x)|^2 + \int_x^\infty |v(y)|^2\, \rho(dy) |\kappa_n(x)u(x)|^2 \right) \rho(dx) < \infty.$$

It remains to show that L^{κ_n} converges in the norm-resolvent sense to L^κ since in this case

$$(L-z)^{-1} - (L^{\kappa_n} - z)^{-1} \to (L-z)^{-1} - (L^\kappa - z)^{-1}$$

with respect to the operator norm. But this follows from the resolvent equation

$$(L^{\kappa_n} - z)^{-1} - (L^\kappa - z)^{-1} = (L^{\kappa_n} - z)^{-1}(\kappa - \kappa_n)(L-z)^{-1}$$

since

$$\begin{aligned}\left\|(L^{\kappa_n} - z)^{-1} - (L^\kappa - z)^{-1}\right\| &\leq \|(L^{\kappa_n} - z)^{-1}\|\|\kappa - \kappa_n\|_\infty\|(L^\kappa - z)^{-1}\| \\ &\leq \frac{1}{(\Im z)^2}\|\kappa - \kappa_n\|_\infty \to 0\end{aligned}$$

as $n \to \infty$. In the second inequality we used the fact that the operator norm of an operator, which acts by multiplication with a function f, coincides with the supremum norm of the function f.

Assertion (2) is contained in [65] and also follows from Theorem 1 of the recent work [76] (In [76] somewhat stronger conditions on the drift are imposed, but an inspection of the proof shows that these are not necessary).

Assertion (3) follows e.g. from the work of Notarantonio [69]. His result implies that the bottom of the essential spectrum of the operator L_N with Neumann boundary conditions at 0 is bounded above by $\limsup_{r\to\infty} \frac{1}{r} \log \rho((0,r))$. This is 0 if the volume growth is subexponential. In order to prove assertion (3) it is therefore enough to show that $\lambda_0 > 0$ implies the strict positivity of the bottom of the spectrum of L_N. Since the difference $(L_N + 1)^{-1} - (L+1)^{-1}$ is obviously compact, the bottom of the essential spectrum of L_N coincides with the bottom of the essential spectrum of L. Thus if the bottom of the spectrum of L is strictly positive, then also the bottom of the essential spectrum of L_N is strictly positive. If $\lambda_0^N := \inf \sigma(L_N) = 0$ then $\lambda_0^N = 0$ is necessarily an isolated eigenvalue of the operators L_N. Let us assume that $\lambda_0^N = 0$. The unique (up to positive multiples) non-trivial and non-negative eigenfunction $v^N \in L^2((0,\infty), \rho(x)dx)$ associated to $\lambda_0^N = 0$ therefore solves the boundary value problem

$$L_N v^N = \lambda_0^N v^N = 0,\ v^N(0) > 0 \text{ and } \rho\frac{dv^N}{dx}(0) = 0.$$

Since this ordinary differential equation has a unique solution and since for some constant $c > 0$ the constant function $c\mathbf{1}$ is also a solution of this equation, we conclude that $v_N = c\mathbf{1}$. Since due to the assumption $\rho((0,\infty)) = \infty$ we finally arrive at the contradiction $v_N \notin L^2((0,\infty), \rho(dx))$ and therefore $\lambda_0^N > 0$.

Assertion (4): If $\lambda_0^\kappa < \lim_{t\to\infty} \kappa(t) = K$ then an application of the result in (1) shows that $L^\kappa = L + K + (\kappa - K)$ has the same essential spectrum as $L + K$. Since L is a positive operator the bottom of the essential spectrum of $L + K$ has to be bigger than or equal to K. The assumption $\lambda_0^\kappa < K$ therefore implies

$$\lambda_0^\kappa < K \leq \lambda_{ess}(L+K) = \lambda_{ess}(L^\kappa),$$

which implies the assertion of the theorem.
Assertion (5): Again the application of (1) implies that $L^\kappa = L + K + (\kappa - K)$ and $L + K$ have the same essential spectrum. In particular we conclude that $\inf \sigma_{ess}(L) + K = \inf \sigma_{ess}(L + K) \geq \lambda_0^\kappa$ and therefore $\inf \sigma_{ess}(L) \geq \lambda_0^\kappa - K > 0$. If $0 \leq \lambda_0 < \inf \sigma_{ess}(L)$ then λ_0 is an isolated eigenvalue. Because of the transience of the diffusion corresponding to the generator L (see Remark 2.1.1) and criticality theory (see Remark 2.1.1 and [46], [75]) 0 can not be an isolated eigenvalue. Thus $\lambda_0 > 0$. □

In some sense it is Lemma 2.2.2, which allows us to go beyond the assertions of Theorem 2.1.1, since it allows us to separate the influence of the drift from the effect of the killing term. Moreover it clearly shows, why the case $\lambda_0^\kappa < K$ will turn out to be easier than the case $\lambda_0^\kappa > K$.

Remark 2.2.1. *In Lemma 2.2.2 we usually worked with the assumption that the limit* $\lim_{t\to\infty} \kappa(t)$ *exists. Since Theorem 2.1.1 does not assume such a condition it is natural to ask, which assertions of Lemma 2.2.2 really rely on the existence of the limit* $\lim_{t\to\infty} \kappa(t)$*. In assertion (4) of Lemma 2.2.2 we assumed that* $\lim_{t\to\infty} \kappa(t) > \lambda_0^\kappa$*. Let us now assume only that* $\liminf_{t\to\infty} \kappa(t) > \lambda_0^\kappa$*. By the so called decomposition principle (as in the proof of Theorem 14.11 in [51]) it is not difficult to see that the essential spectrum of* L^κ *and the essential spectrum of the operator* L_a^κ *(*$a > 0$*) do not differ, where the operator* L_a^κ *is the selfadjoint extension of* τ^κ *in* $L^2((a,\infty),\rho)$ *satisfying Dirichlet boundary conditions at a. Therefore we have*

$$\inf \sigma_{ess}(L^\kappa) \geq \lim_{a\to\infty} \inf \sigma(L_a^\kappa).$$

If $a_0 > 0$ *and* $\varepsilon > 0$ *are such that* $\inf_{t\geq a_0} \kappa(t) > \lambda_0^\kappa + \varepsilon$ *we thus conclude that*

$$\begin{aligned}\inf \sigma_{ess}(L^\kappa) &\geq \lim_{a\to\infty} \inf \sigma(L_a^\kappa) \\ &\geq \inf_{\substack{\varphi\in C_c^\infty(a_0,\infty) \\ \|\varphi\|_{L^2((a_0,\infty),\rho)}=1}} \left(\frac{1}{2}\int_{a_0}^\infty |\varphi'(x)|^2 \,\rho(dx) + \int_{a_0}^\infty \kappa(x)|\varphi(x)|^2 \rho(dx)\right) \\ &\geq \lambda_0^\kappa + \varepsilon.\end{aligned}$$

Summarizing we have shown that already the condition $\liminf_{t\to\infty} \kappa(t) > \lambda_0^\kappa$ *implies the bottom of the spectrum of* L^κ *is an isolated eigenvalue. A similar argument applies also to assertion (5). We preferred to present Lemma 2.2.2 in the given form, since in this case the necessary arguments are very straightforward.*

Remark 2.2.2. *The assertion (2) of the above lemma can be made much more precise. Assuming that absorption is certain Ross Pinsky proves in [76] that*

$$\frac{1}{8A(b)} \leq \lambda_0 \leq \frac{1}{2A(b)}, \tag{2.2.3}$$

where

$$A(b) = \sup_{x>0} \rho([x,\infty)) \int_0^x \rho(t)^{-1}\, dt.$$

Related analytic inequalities, which are usually referred to as weighted Hardy inequalities, can be found in [68]. Indeed the results of [68] can be used in order to deduce Pinsky's bounds in a rather straightforward way.

Remark 2.2.3. *The fact that the bottom of the spectrum is an isolated eigenvalue is also of practical interest, because in this case the associated eigenfunction can be approximated in a quite accurate way by the ground states of regular Sturm-Liouville operators on bounded intervals (see the recent survey [97]). Such a result is reproved in the recent preprint [91] in order to provide an approximation of the minimal quasistationary distribution of a diffusion generator with discrete spectrum via a Fleming-Viot type interacting particle system.*

Remark 2.2.4. *Assume that λ_0^κ is an eigenvalue with associated eigenfunction $u_{\lambda_0^\kappa} \in L^2((0,\infty),\rho)$, which by general theory is strictly positive and simple (this is a direct consequence of standard theorems of Perron-Frobenius type, see e.g. Satz 17.6 in [96]). Then*

$$\lim_{t\to\infty} e^{\lambda_0^\kappa t} p^\kappa(t,x,y) = c\, u_{\lambda_0^\kappa}(x) u_{\lambda_0^\kappa}(y),$$

where c is a normalizing constant, which is 1 if $\|u_{\lambda_0^\kappa}\|_{L^2((0,\infty),\rho)} = 1$. This was proved in [82] for the transition function of Brownian motion on Riemannian manifolds but the proof carries over without essential changes to our case. The interested reader will find the necessary arguments later in the proof of Lemma 4.3.1.

We will also make use of the following result which is a special case of Theorem 3.1 in [84].

Lemma 2.2.3 (Theorem 3.1 in [84])**.** *Let $0 \le f \in L^2((0,\infty),\rho)$ with compact support $supp(f) \subset [0,\infty)$ be given and let ν_f denote the measure $f(x)\rho(dx)$. Let L^κ be as in Lemma 2.2.2 and let $p^\kappa(t,\cdot,\cdot)$ denote the integral of e^{-tL^κ}. Then for arbitrary measurable bounded sets $A, B \subset (0,\infty)$*

$$\lim_{t\to\infty} \frac{\int_0^\infty f(x) \int_B p^\kappa(t,x,y)\,\rho(dy)\,\rho(dx)}{\int_0^\infty f(x) \int_A p^\kappa(t,x,y)\,\rho(dy)\,\rho(dx)} = \frac{\int_B \varphi(\lambda_0^\kappa, y)\,\rho(dy)}{\int_A \varphi(\lambda_0^\kappa, y)\,\rho(dy)},$$

i.e. X_t converges from the initial distributions $\frac{\nu_f}{\int_0^\infty f(s)\,\rho(ds)}$ on compacta to the quasistationary distribution $\varphi(\lambda_0^\kappa,\cdot)$.

The above lemma can be proved rather directly using the spectral representation for Sturm-Liouville operators. The reader will see the necessary arguments later in this work in the proof of Theorem 2.2.3. Our first goal consists in extending this result to the case of general compactly supported initial distributions ν. Before we prove such an extension we will deduce some consequences of Lemma 2.2.3. This will lead to Theorem 2.2.2, which is often referred to as a strong ratio limit theorem. Before we begin with the proof of the strong ratio limit theorem we explain another analytic fact which has no direct relation to spectral theory but which will turn out to be very useful.

Remark 2.2.5. *We will make good use of the local parabolic Harnack inequality which quite general holds for second order parabolic differential equations. The local parabolic Harnack inequality ([62]) states that for fixed* $x_0 \in (0,\infty), t \in (0,\infty)$ *and* $R > 0$ *there is a constant* C *such that for every weak solution* u *of* $(\partial_t - L^\kappa)u = 0$ *which is non-negative in* $Q((x_0,t_0),4R) \subset (0,\infty)\times(0,\infty)$

$$\sup_{\Theta((x_0,t_0),R/2)} u \le C \inf_{Q((x_0,t_0),R)} u,$$

where

$$Q((x_0,t_0),R) = \{X \in \mathbb{R}^2 \mid \max\big(|x-x_0|, \sqrt{|t-t_0|}\big) < R, t < t_0\}$$

and $\Theta((x_0,t_0),R/2) = Q((x_0,t_0-R^2),R)$. *As in Theorem 10 of [28] this inequality can be applied to the transition kernel* $p^\kappa(t,x,y)$ *in order to prove that for every compact* $K \subset (0,\infty)$ *and* $T > 0$ *there is a constant* $c = c(K,T) > 0$ *such that for* $t \ge T$, $x_1,x_2,x_3,x_4 \in K$

$$c^{-1}p^\kappa(t,x_1,x_2) \le p^\kappa(t,x_3,x_4) \le c p^\kappa(t,x_1,x_2).$$

Moreover the local parabolic Harnack inequality shows that there exists a locally bounded function $F : (0,\infty) \to (0,\infty)$ *such that for every* $t \ge 1$, $y \in (0,\infty)$ *and* $|z-x| < \frac{1}{2} \wedge \frac{|x|}{4}$

$$p^\kappa(t,x,y) \le F(x)p^\kappa(t+1,z,y)$$

Some very important ingredients for the following proofs go back to [5], where it played an important role in a different context. Since we do not assume criticality of the operator and since no Hölder-continuity of the coefficients is required we can not directly copy the proof of Theorem 2.2 in [5].

Lemma 2.2.4. *For any* $x_0 \in (0,\infty)$ *the family of functions*

$$\left\{[0,\infty)\times\mathbb{R}_+\times\mathbb{R}_+ \ni (t,x,y) \mapsto \frac{p^\kappa(t+s,x,y)}{p^\kappa(s,x_0,x_0)} \mid s \ge 1\right\}$$

is relatively compact in the space $C((0,\infty)^2,\mathbb{R})$ *of real-valued continuous functions on* $(0,\infty)^2$.

Proof. Let $(s_n)_{n\in\mathbb{N}}$ be a sequence with $1 \le s_n \to \infty$ and set for $t \in [0,\infty)$, $x,y \in (0,\infty)$

$$r_n(t,x,y) = \frac{p^\kappa(s_n+t,x,y)}{p^\kappa(s_n,a,a,)},$$

where $a \in (0,\infty)$ is fixed. The functions $(t,x,y) \mapsto r_n(t,x,y)$ $(n \in \mathbb{N})$ are solutions to the parabolic equation

$$(2\partial_t + L^\kappa_x + L^\kappa_y)r_n(t,x,y) = 0,$$

where the operator L^κ_x and L^κ_y act as L^κ on x- and y-variable, respectively. By the local parabolic Harnack inequality (compare Remark 2.2.5) we conclude that for each compact set $K \subset (0,\infty)$ there exists a constant C_K such that for all $n \in \mathbb{N}$, $t \ge 0$ and $x,y,a \in K$

$$p^\kappa(s_n+t,x,y) \le C_K p^\kappa(s_n+t,a,a)$$

By general spectral theory it is proved by Davies in Lemma 4.3.1 of his work [28] that $r \mapsto p^\kappa(r, x_0, x_0)$ is non-increasing. Therefore we conclude that for $t \geq 0$ and $x, y \in K$

$$\frac{p^\kappa(s_n + t, x, y)}{p^\kappa(s_n, a, a)} \leq C_K.$$

Theorem 6.28 in [62] shows that the set $\{r_n \mid n \in \mathbb{N}\}$ is locally uniformly equicontinuous. Therefore by the theorem of Arzela-Ascoli there exists a subsequence $(r_{n_k})_{k\in\mathbb{N}}$ which converges locally uniformly. □

In the following proof we compare the quotient of the heat kernel at different time and different spatial points. In the probability literature one usually refers to such results as strong ratio limit theorems. These are known for several types of stochastic processes. Results for certain branching processes can be found in the book [6] of Athreya and Ney. A proof of the strong ratio property for certain Markov chains on the integers was given in the work [57] of Kesten.

Theorem 2.2.2. *Let $p^\kappa(t, \cdot, \cdot)$ denote the integral kernel of the selfadjoint operator e^{-tL^κ} then for any $x_0 \in (0, \infty)$*

$$\lim_{s\to\infty} \frac{p(t+s, x, y)}{p(s, a, a)} = e^{-\lambda_0 t} \frac{\varphi(\lambda_0^\kappa, x)\varphi(\lambda_0^\kappa, y)}{\varphi(\lambda_0^\kappa, a)\varphi(\lambda_0^\kappa, a)}$$

Proof. For every sequence $(s_n)_{n\in\mathbb{N}} \subset (0, \infty)$ converging to infinity we know by Lemma 2.2.4 that for some subsequence $(s_{n_k})_k$ of (s_n) there exists a function ψ such that

$$\frac{p^\kappa(s_{n_k} + t, x, y)}{p^\kappa(s_{n_k}, a, a)} \to \psi(t, x, y),$$

where the convergence is locally uniform in $[0, \infty) \times (0, \infty)^2$. Since by Lemma 7.7 in [84] (see also the proof of Theorem 25 in [28]) for every $f \in L^2((0, \infty), \rho)$ with compact support

$$\lim_{t\to\infty} \frac{\langle e^{-(t+s)L^\kappa} f, f\rangle_{L^2(\rho)}}{\langle e^{-sL^\kappa} f, f\rangle_{L^2(\rho)}} = e^{-\lambda_0^\kappa t}$$

one easily concludes that

$$\psi(t, x, y) = e^{-\lambda_0^\kappa(t)} \psi(0, x, y).$$

Lemma 2.2.3 shows that for every $f, g, h \in C_c^\infty(0, \infty)$

$$\begin{aligned}
\frac{\int_0^\infty g(y)\varphi(\lambda_0^\kappa, y)\,\rho(dy)}{\int_0^\infty h(y)\varphi(\lambda_0^\kappa, y)\,\rho(dy)} &= \lim_{k\to\infty} \frac{\int_0^\infty f(x)\int_0^\infty g(y)p^\kappa(s_{n_k}, x, y)\,\rho(dy)\,\rho(dx)}{\int_0^\infty f(x)\int_0^\infty h(y)p^\kappa(s_{n_k}, x, y)\,\rho(dy)\,\rho(dx)} \\
&= \lim_{k\to\infty} \frac{\int_0^\infty f(x)\int_0^\infty g(y)\frac{p^\kappa(s_{n_k},x,y)}{p^\kappa(s_{n_k},x_0,x_0})\,\rho(dy)\,\rho(dx)}{\int_0^\infty f(x)\int_0^\infty h(y)\frac{p^\kappa(s_{n_k},x,y)}{p^\kappa(s_{n_k},x_0,x_0)}\,\rho(dy)\,\rho(dx)} \\
&= \frac{\int_0^\infty f(x)\int_0^\infty g(y)\psi(0, x, y)\,\rho(dy)}{\int_0^\infty f(x)\int_0^\infty h(y)\psi(0, x, y)\,\rho(dy)}
\end{aligned}$$

This implies that for $x \in (0,\infty)$, $g,h \in C_c^\infty((0,\infty)$

$$\frac{\int_0^\infty g(y)\varphi(\lambda_0^\kappa,y)\,\rho(dy)}{\int_0^\infty h(y)\varphi(\lambda_0^\kappa,y)\,\rho(dy)}\int_0^\infty h(y)\psi(0,x,y)\,\rho(dy) = \int_0^\infty g(y)\psi(0,x,y)\,\rho(dy)$$

and furthermore for every $h \in C_c^\infty(0,\infty)$

$$\psi(0,x,y) = \varphi(\lambda_0^\kappa,y)\frac{\int_0^\infty h(y)\psi(0,x,y)\,\rho(dy)}{\int_0^\infty h(y)\varphi(\lambda_0^\kappa,y)\,\rho(dy)}$$

Due to the symmetry of $\psi(0,\cdot,\cdot)$ we conclude that for some constant $c \geq 0$

$$\psi(0,x,y) = c\,\varphi(\lambda_0^\kappa,x)\varphi(\lambda_0^\kappa,y).$$

Because of $\psi(0,a,a) = 1$ we arrive at $c^{-1} = \varphi(\lambda_0^\kappa,a)\varphi(\lambda_0^\kappa,a)$. Since this is true for every subsequence the assertion of the theorem is proved. □

Corollary 2.2.1. *Let $A \subset (0,\infty)$ and $B \subset (0,\infty)$ be precompact sets and let $s \geq 0$ be given. Then*

$$\lim_{t\to\infty}\frac{\mathbb{P}_\nu(X_{t+s}\in A)}{\mathbb{P}_\nu(X_t\in B)} = e^{-\lambda_0^\kappa s}\frac{\int_A \varphi(\lambda_0^\kappa,x)\,\rho(dx)}{\int_B \varphi(\lambda_0^\kappa,y)\,\rho(dy)}$$

for every initial distribution ν, which is compactly supported in $(0,\infty)$

Proof. We have seen in Theorem 2.2.2 that for fixed $x \in (0,\infty)$ we have

$$\lim_{t\to\infty}\frac{p(t+s,x,y)}{p(t,a,a)} = e^{-\lambda_0^\kappa s}\frac{\varphi(\lambda_0^\kappa,x)\varphi(\lambda_0^\kappa,y)}{\varphi(\lambda_0^\kappa,a)\varphi(\lambda_0^\kappa,a)},$$

where the convergence is locally uniform on $(0,\infty)\times(0,\infty)$. For every Borel measure ν with compact support in $[0,\infty)$ we therefore get

$$\begin{aligned}
\mathbb{P}_\nu\big(X_{t+s}\in A \mid X_t\in B\big) &= \frac{\mathbb{P}_\nu\big(X_{t+s}\in A\big)}{\mathbb{P}_\nu\big(X_t\in B\big)} = \frac{\int_0^\infty \nu(dx)\int_0^\infty \rho(dy)p(t+s,x,y)\mathbf{1}_A(y)}{\int_0^\infty \nu(dx)\int_0^\infty \rho(dy)p^\kappa(t,x,y)\mathbf{1}_B(y)} \\
&= \frac{\int_0^\infty \nu(dx)\int_0^\infty \rho(dy)p^\kappa(t+s,x,y)\mathbf{1}_A(y)}{p^\kappa(t,a,a)} \\
&\cdot\left(\frac{\int_0^\infty \nu(dx)\int_0^\infty \rho(dy)p^\kappa(t,x,y)\mathbf{1}_B(y)}{p^\kappa(t,a,a)}\right)^{-1} \\
&= \int_0^\infty \nu(dx)\int_0^\infty \rho(dy)\frac{p^\kappa(t+s,x,y)}{p^\kappa(t,a,a)}\mathbf{1}_A(y) \\
&\cdot\left(\int_0^\infty \nu(dx)\int_0^\infty \rho(dy)\frac{p^\kappa(t,x,y)}{p^\kappa(t,x,a)}\mathbf{1}_B(y)\right)^{-1} \\
&\to e^{-\lambda_0^\kappa s}\frac{\int_A \varphi(\lambda_0^\kappa,y)\,\rho(dy)}{\int_B \varphi(\lambda_0^\kappa,y)\,\rho(dy)}
\end{aligned}$$

□

Remark 2.2.6. *Theorem 2.2.2 has an intimate relation to the parabolic Martin boundary theory. In terms of this theory it says that every sequence* $(s_n, x) \subset (0,\infty)\times(0,\infty)$ *with* $\lim_{n\to\infty} s_n = -\infty$ *converges in the parabolic Martin topology to the parabolic Martin boundary point corresponding to the minimal parabolic function* $h_{\lambda_0^\kappa}(t,x) = e^{\lambda_0^\kappa t}\varphi(\lambda_0^\kappa, x)$. *The parabolic Martin theory probably allows to conclude that the parabolic function* $h_{\lambda_0^\kappa}$ *is even invariant, since it corresponds to a point in the parabolic Martin boundary, whose time coordinate is* ∞, *and therefore the* $h_{\lambda_0^\kappa}$*-transformed process should be conservative. It might be worth to look at the relation to the parabolic Martin boundary in more detail.*

The next result is an extension of Theorem 3.1 of [84]. Steinsaltz and Evans had to pose an additional condition on the initial distribution ν and they stated the general case as an open problem. Their most general condition reads

(ID') If X_0 has distribution ν, then $\exists s \geq 0$ for which the distribution of X_s has a density $f \in L^2((0,\infty),\rho)$, with $\liminf_{\lambda\downarrow\lambda_0^\kappa} Uf(\lambda) > -\infty$.

Obviously it is not at all easy to check, whether an initial distribution ν with compact support satisfies the condition (ID'). Using some results from spectral theory and several ideas of Steinsaltz and Evans we are able to allow arbitrary compactly supported initial distributions. We give a rather detailed proof of this theorem and only sketch the proof of the analogous result for the case of an exit boundary at 0 in the next section, since it is virtually the same. Observe however that for our main results concerning the convergence to quasistationary distributions Theorem 2.2.3 is not needed at least in this chapter. One main ingredient in the proof is the insight that the 'high-energy' spectrum can be cut out without changing anything. For a Radon measure ν on $(0,\infty)$ and a Borel measurable function $f:(0,\infty)\mapsto\mathbb{C}$ we use the notation $\langle \nu, f\rangle := \int_0^\infty f(s)\,\nu(ds)$.

Theorem 2.2.3. *Let* ν *be an initial distribution which is compactly supported in* $(0,\infty)$. *Let* $(X_t)_{t\geq 0}$ *be the diffusion corresponding to the Dirichlet form* q^κ. *Then* X_t *converges from the initial distribution* ν *on compacta to the quasistationary distribution* $\varphi(\lambda_0^\kappa,\cdot)$.

Proof. For reasons of convenience let us denote during this proof the spectral resolution of L^κ by $(E_\lambda)_\lambda$ instead of $(E_\lambda^\kappa)_\lambda$. Let us first prove that for every $\lambda_1 \in (\lambda_0^\kappa,\infty)$

$$\lim_{t\to\infty} \frac{\langle \nu, E([\lambda_0^\kappa,\lambda_1])e^{-tL^\kappa}\mathbf{1}_A\rangle}{\langle \nu, E([\lambda_0^\kappa,\lambda_1])e^{-tL^\kappa}\mathbf{1}_{[0,z]}\rangle} = \frac{\int_A \varphi(\lambda_0^\kappa,y)\,\rho(dy)}{\int_0^z \varphi(\lambda_0^\kappa,y)\,\rho(dy)}. \tag{2.2.4}$$

Observe that the operator $E([\lambda_0^\kappa,\lambda_1))e^{-tL^\kappa}$ has an continuous integral kernel $(t,x,y)\mapsto h^{\lambda_1}(t,x,y)$ with respect to the measure $\rho(dx)$, where

$$h^{\lambda_1}(t,x,y) = \int_{[\lambda_0^\kappa,\lambda_1]} e^{-t\lambda}\varphi(\lambda,x)\varphi(\lambda,y)\,\sigma(d\lambda). \tag{2.2.5}$$

This implies that for every pre-compact subset $K\subset[0,\infty)$ the function $E([\lambda_0^\kappa,\lambda_1])e^{-tL^\kappa}\mathbf{1}_K$ is continuous and therefore $\langle\nu, E([\lambda_0^\kappa,\lambda_1])e^{-tL^\kappa}\mathbf{1}_K\rangle = \int_{\mathbb{R}} E([\lambda_0^\kappa,\lambda_1])e^{-tL^\kappa}\mathbf{1}_K(x)\,\nu(dx)$ is welldefined. The formula (2.2.5) follows directly from the eigenfunction expansion theorem of Weyl as for compactly supported function $f, g \in L^\infty((0,\infty),\rho)$

$$(g, E([\lambda_0^\kappa,\lambda_1])e^{-tL^\kappa} f)_{L^2((0,\infty),\rho)} = \int_{[\lambda_0^\kappa,\infty)} \mathbf{1}_{[\lambda_0^\kappa,\lambda_1]} e^{\lambda t}\hat{f}(\lambda)\hat{g}(\lambda)\,\sigma(d\lambda)$$
$$= \int_0^\infty f(x)\int_0^\infty g(y)\int_{[\lambda_0^\kappa,\lambda_1]} e^{\lambda t}\varphi(\lambda,x)\varphi(\lambda,y)\,\sigma(d\lambda)\,\rho(dy)\rho(dx),$$

where we used Fubini's theorem and the notation

$$\hat{f}(\lambda) = \int_0^\infty \varphi(\lambda,x)f(x)\,\rho(dx) \text{ and } \hat{g}(\lambda) = \int_0^\infty \varphi(\lambda,x)g(x)\,\rho(dx).$$

Observe that the use of Fubini's theorem in (2.2) is easily justified by the fact that the bounded functions f, g have compact support and that $(\lambda, x) \mapsto \varphi(\lambda, x)$ is continuous. Using (2.2.5) we have that for every Borel set $A \subset [0, z]$

$$\langle \nu, E([\lambda_0^\kappa,\lambda_1])e^{-tL^\kappa}\mathbf{1}_A\rangle = \int_{[\lambda_0^\kappa,\lambda_1]} e^{-t\lambda}\int \varphi(\lambda,x)\,\nu(dx)\int_A \varphi(\lambda,y)\,\rho(dy)\,\sigma(d\lambda) \tag{2.2.6}$$

The assertion (2.2.4) follows from the facts that for continuous functions $g, h : \mathbb{R} \to \mathbb{R}$

$$\frac{\int_{[\lambda_0^\kappa,\lambda_1]} e^{-t\lambda}g(\lambda)\,\sigma(d\lambda)}{\int_{[\lambda_0^\kappa,\lambda_1]} e^{-t\lambda}h(\lambda)\,\sigma(d\lambda)} = \frac{\int_{[\lambda_0^\kappa,\lambda_1]} e^{-t\lambda}g(\lambda)\,\sigma(d\lambda)}{\int_{[\lambda_0^\kappa,\lambda_1]} e^{-t\lambda}\,\sigma(d\lambda)} \cdot \left(\frac{\int_{[\lambda_0^\kappa,\lambda_1]} e^{-t\lambda}h(\lambda)\,\sigma(d\lambda)}{\int_{[\lambda_0^\kappa,\lambda_1]} e^{-t\lambda}\,\sigma(d\lambda)}\right)^{-1} \tag{2.2.7}$$

and that

$$\lim_{t\to\infty}\left|\frac{\int_{[\lambda_0^\kappa,\lambda_1]} e^{-t\lambda}g(\lambda)\,\sigma(d\lambda)}{\int_{[\lambda_0^\kappa,\lambda_1]} e^{-t\lambda}\,\sigma(d\lambda)} - g(\lambda_0^\kappa)\right| = 0. \tag{2.2.8}$$

Equation (2.2.7) is obvious. In order to establish equation (2.2.8) we first show that for $0 \le f \in L^1([\lambda_0^\kappa,\infty), d\sigma) \cap C([\lambda_0^\kappa,\infty))$

$$\limsup_{t\to\infty}\frac{\int_{[\lambda_0^\kappa,\lambda_1]} e^{\lambda t}f(\lambda)\,\sigma(d\lambda)}{\int_{[\lambda_0^\kappa,\lambda_2]} e^{\lambda t}\,\sigma(d\lambda)} \text{ and } \liminf_{t\to\infty}\frac{\int_{[\lambda_0^\kappa,\lambda_1]} e^{\lambda t}f(\lambda)\,\sigma(d\lambda)}{\int_{[\lambda_0^\kappa,\lambda_2]} e^{\lambda t}\,\sigma(d\lambda)} \tag{2.2.9}$$

are independent of $\lambda_1, \lambda_2 \in (\lambda_0^\kappa, \infty)$. As in the proof of Theorem 3.1 of [84] (page 1299-1300) this directly follows from the estimate

$$\left|\frac{\int_{[\lambda_0^\kappa,\lambda_1]} e^{-t\lambda}f(\lambda)\,\sigma(d\lambda)}{\int_{[\lambda_0^\kappa,\lambda_2]} e^{-t\lambda}\,\sigma(d\lambda)} - \frac{\int_{[\lambda_0^\kappa,\tilde{\lambda}_1]} e^{-t\lambda}f(\lambda)\,\sigma(d\lambda)}{\int_{[\lambda_0^\kappa,\lambda_2]} e^{-t\lambda}\,\sigma(d\lambda)}\right| \le e^{(\lambda_0^\kappa-\lambda_1\wedge\tilde{\lambda}_1\wedge\lambda_2)t}\frac{\int_{[\lambda_0^\kappa,\lambda_1\vee\tilde{\lambda}_1\vee\lambda_2]} f(\lambda)\sigma(d\lambda)}{\int_{[\lambda_0^\kappa,\lambda_2]}\sigma(d\lambda)},$$

where $\lambda_1, \tilde{\lambda}_1, \lambda_2 > \lambda_0^\kappa$. Because of

$$\begin{aligned}
\limsup_{t\to\infty}&\left|\frac{\int_{[\lambda_0^\kappa,\lambda_1]} e^{-t\lambda} g(\lambda)\,\sigma(d\lambda)}{\int_{[\lambda_0^\kappa,\lambda_1]} e^{-t\lambda}\,\sigma(d\lambda)} - g(\lambda_0^\kappa)\right| \\
&= \limsup_{t\to\infty}\left|\frac{\int_{[\lambda_0^\kappa,\lambda_1]} e^{-t\lambda} g(\lambda)\,\sigma(d\lambda)}{\int_{[\lambda_0^\kappa,\lambda_1]} e^{-t\lambda}\,\sigma(d\lambda)} - \frac{\int_{[\lambda_0^\kappa,\lambda_1]} e^{-t\lambda} g(\lambda_0^\kappa)\,\sigma(d\lambda)}{\int_{[\lambda_0^\kappa,\lambda_1]} e^{-t\lambda}\,\sigma(d\lambda)}\right| \\
&= \limsup_{t\to\infty}\frac{\int_{[\lambda_0^\kappa,\lambda_1]} e^{-t\lambda}|g(\lambda)-g(\lambda_0^\kappa)|\,\sigma(d\lambda)}{\int_{[\lambda_0^\kappa,\lambda_1]} e^{-t\lambda}\,\sigma(d\lambda)} \\
&\le \sup_{[\lambda_0^\kappa,\lambda_1]} |g(\lambda)-g(\lambda_0^\kappa)|
\end{aligned}$$

(2.2.9) together with the continuity of g implies (2.2.8). The assertion of the Theorem follows from (2.2.4) once it is shown that

$$\lim_{t\to\infty}\frac{\mathbb{P}_\nu(X_t\in A)}{\mathbb{P}_\nu(X_t\le z)} = \lim_{t\to\infty}\frac{\langle \nu, E([\lambda_0^\kappa,\lambda_1])e^{-tL^\kappa}\mathbf{1}_A\rangle}{\langle \nu, E([\lambda_0^\kappa,\lambda_1])e^{-tL^\kappa}\mathbf{1}_{[0,z]}\rangle} \tag{2.2.10}$$

Here observe that

$$\begin{aligned}
\frac{\langle \nu, e^{-tL^\kappa}\mathbf{1}_A\rangle}{\langle \nu, E([0,\lambda_1])e^{-tL^\kappa}\mathbf{1}_A\rangle} &= \frac{\langle \nu, E([0,\lambda_1])e^{-tL^\kappa}\mathbf{1}_A\rangle + \langle \nu, E((\lambda_1,\infty))e^{-tL^\kappa}\mathbf{1}_A\rangle}{\langle \nu, E([0,\lambda_1])e^{-tL^\kappa}\mathbf{1}_A\rangle} \\
&= 1 + \frac{\langle \nu, E((\lambda_1,\infty))e^{-tL^\kappa}\mathbf{1}_A\rangle}{\langle \nu, E([0,\lambda_1])e^{-tL^\kappa}\mathbf{1}_A\rangle}.
\end{aligned} \tag{2.2.11}$$

Again we should point out that the function $E((\lambda_1,\infty))e^{-tL^\kappa}\mathbf{1}_A$ is continuous, since $e^{-tL^\kappa}\mathbf{1}_A$ and $E([0,\lambda_1])e^{-tL^\kappa}\mathbf{1}_A$ are continuous. Thus $\langle \nu, E((\lambda_1,\infty))e^{-tL^\kappa}\mathbf{1}_A\rangle$ is welldefined. Observe that there is a constant $C_\nu > 0$ such that

$$\begin{aligned}
\sup_{x\in\operatorname{supp}(\nu)}\left|E((\lambda_1,\infty))e^{-tL^\kappa}\mathbf{1}_A(x)\right| &= \sup_{x\in\operatorname{supp}(\nu)}\left|e^{-tL^\kappa}E((\lambda_1,\infty))\mathbf{1}_A(x)\right| \\
&\le C_\nu\left(\int_0^\infty \left|\sqrt{L^\kappa}e^{-tL^\kappa}E((\lambda_1,\infty))\mathbf{1}_A(x)\right|^2\rho(dx)\right)^{\frac{1}{2}} \\
&= C_\nu\left(\int_{(\lambda_1,\infty)} e^{-2t\lambda}\lambda\, d\|E_\lambda\mathbf{1}_A\|^2_{L^2((0,\infty),\rho)}\right)^{\frac{1}{2}}.
\end{aligned} \tag{2.2.12}$$

Here we used the inequality which has been mentioned in the introduction: for every $f\in C^1([0,\infty)$ vanishing at 0 and $x\in(0,a)$ we have

$$\begin{aligned}
|f(x)| &\le \int_0^x |f'(x)|\sqrt{\rho(t)}^{-1}\sqrt{\rho(t)}\,dt \\
&\le \left(2\int_0^a \rho(t)^{-1}\,dt\right)^{\frac{1}{2}}\left(\frac{1}{2}\int_0^\infty |f'(x)|^2\rho(dt) + \int_0^\infty \kappa(x)|f(x)|^2\rho(t)\,dt\right)^{\frac{1}{2}}.
\end{aligned}$$

Equation (2.2.12) shows that for every $\varepsilon > 0$

$$-\lim_{t\to\infty}\frac{1}{t}\log\left|\langle\nu, E((\lambda_1,\infty))e^{-tL^\kappa}\mathbf{1}_A\rangle\right| \geq \lambda_1 - \varepsilon. \tag{2.2.13}$$

As $\varphi(\lambda_0^\kappa, x) > 0$ for every $x \in (0,\infty)$ there is by continuity $\lambda_1 > \lambda_0^\kappa$ such that for every $\lambda \in [\lambda_0^\kappa, \lambda_1]$

$$\int_0^\infty \varphi(\lambda,x)\nu(dx),\ \int_A \varphi(\lambda,x)\rho(dx) > 0.$$

Then it is easy to see that

$$-\lim_{t\to\infty}\frac{1}{t}\log\int_{[\lambda_0^\kappa,\lambda_1]} e^{-\lambda t}\int_0^\infty \varphi(\lambda,x)\nu(dx)\int_A \varphi(\lambda,y)\rho(dx)\sigma(d\lambda) \leq \lambda_0^\kappa. \tag{2.2.14}$$

Equation (2.2.11), (2.2.13) and (2.2.14) clearly show

$$\lim_{t\to\infty}\frac{\langle\nu, e^{-tL^\kappa}\mathbf{1}_A\rangle}{\langle\nu, E([0,\lambda_1])e^{-tL^\kappa}\mathbf{1}_A\rangle} = 1$$

and therefore (2.2.10). □

As mentioned above with exception of the recent work [20] all authors consider the case, where 0 is at least regular and ∞ natural. We will show that in some sense the situation is much more transparent, if ∞ is an entrance boundary. This is due to the fact that the spectrum of the operator L^κ is purely discrete. This important fact has been overseen by previous authors (see e.g. section 3 in [20]) working on quasistationary distributions for one-dimensional diffusions, though the proof is very simple. In the course of the proof we use some ideas, which are rather standard in the literature concerning spectral theory of differential operators. We point out that this result also appears with a different proof in the preprint [78].

Theorem 2.2.4. *Assume that* 0 *is regular and that* ∞ *is an entrance boundary point and let* $0 \leq \kappa \in C([0,\infty))$ *be an arbitrary non-negative continuous real-valued function. As usual let* L^κ *denote the selfadjoint extension of the Sturm-Liouville expression* $\tau_\kappa = -\frac{1}{2\rho}\frac{d}{dx}\big(\rho\frac{d}{dx}\big) + \kappa$ *with Dirichlet boundary conditions at* 0. *Then the spectrum of* L^κ *is discrete.*

Proof. Assume that 0 is a regular boundary point. Observe that the spectrum of L^κ is discrete if the canonical imbedding

$$\iota_\kappa : \mathcal{D}(q^\kappa) \to L^2((0,\infty),\rho)$$

is compact. This follows from the fact that the spectrum of L^κ is discrete iff the resolvent $(L^\kappa+1)^{-1}$ is a compact operator and the fact that $Ran((L^\kappa+1)^{-1}) \subset \mathcal{D}(q^\kappa)$ such that $(L^\kappa+1)^{-1}$ can be factorized as $\iota_\kappa \circ (L^\kappa+1)^{-1}$. It is enough to prove that the spectrum of L is discrete since we have $\iota_\kappa = \iota_0 \circ j$, where $j : \mathcal{D}(q^\kappa) \to \mathcal{D}(q)$, $f \mapsto f$ is continuous. Therefore we may assume that $\kappa = 0$. In order to prove the discreteness of the spectrum of L we will use oscillation theory. We show that for every $\lambda \geq 0$ every solution f of $(L-\lambda)f = 0$ has only finitely many zeroes and then apply a well-known result of Hartmann[1] (see e.g. Satz

[1] We thank Prof. Dr. Hubert Kalf for directing us to the result of P. Hartmann

1.1 in [94]). Let f be a solution of the eigenvalue equation

$$-\frac{1}{2}f''(x)+b(x)f'(x)=\lambda f(x),\ x\in(0,\infty),\ \lambda>0$$

and assume that f has infinitely many zeros in $(1,\infty)$, which are denoted by $(z_k)_{k\in\mathbb{N}}$. Between two successive zeros the function has a local extremum. Let $x_1>1$ be a minimum between two successive zeros, then denote by $\tilde{x}_1>x_1$ the first local maximum following x_1. Iterating this procedure we arrive at two sequence $(x_k)_{k\in\mathbb{N}}$ and $(\tilde{x}_k)_{k\in\mathbb{N}}$. Since f solves the equation $\tau u=\lambda u$ one easily sees that local maxima are positive and local minima are negative. Observe now that a direct calculation gives for $x>1$

$$\begin{aligned} f(x) &= f(1)+f'(x)\rho(x)\int_1^x \rho(s)^{-1}\,ds+\int_1^x \rho(s)^{-1}\,ds\int_s^x 2\lambda f(t)\rho(t)\,dt \\ &= f(1)+f'(x)\rho(x)\int_1^x \rho(s)^{-1}\,ds+\int_1^x dt\, f(t)\rho(t)\int_1^t \rho(s)^{-1}\,ds. \end{aligned} \tag{2.2.15}$$

The fastest way to prove equation (2.2.15) consists in the calculation of

$$\int_1^x \rho(s)^{-1}\,ds\int_s^x 2\lambda f(t)\rho(t)\,dx=\int_1^x \rho(s)^{-1}\,ds\int_s^x \frac{-1}{\rho(t)}\big(\rho(t)f'(t)\big)'\,\rho(t)dt$$

using the fundamental theorem of calculus. Equation (2.2.15) gives

$$\begin{aligned} 0<f(\tilde{x}_k)-f(x_k) &= 2\lambda\int_{x_k}^{\tilde{x}_k} dt\,\rho(t)f(t)\int_1^t \rho(s)^{-1}\,ds \\ &\le 2\lambda\big(f(\tilde{x}_k)-f(x_k)\big)\int_{x_k}^{\tilde{x}_k} dt\,\rho(t)f(t)\int_1^t \rho(s)^{-1}\,ds \end{aligned}$$

and therefore

$$\frac{1}{2\lambda}\le\int_{x_k}^{\tilde{x}_k} dt\,\rho(t)\int_1^t \rho(s)^{-1}\,ds.$$

This is a contradiction to our assumption that infinity is an entrance boundary. □

Remark 2.2.7. *The above result concerning the spectrum can certainly also be deduced from general necessary and sufficient conditions for the discreteness of the spectrum of Sturm-Liouville operators obtained in [27]. Since the application of the main result in [27] seems to be not immediate, we decided to present the above simple proof. Theorem 1 in [76] also does not apply directly.*

2.3 Convergence to Quasistationarity

In this section we consider the problem of the existence of the Yaglom limit. More precisely we ask for conditions, which ensure that X_t converges to the quasistationary distribution $\varphi(\lambda_0^\kappa,\cdot)$. Recall that we always assume that 0 is regular.

2.3.1 0 Regular and ∞ Natural

In this section we assume that 0 is regular (as mentioned in section 2.1 we even assume for simplicity that $b \in L^1_{loc}([0,\infty))$ and if not otherwise stated that infinity is natural in the sense of Feller. We are now ready to draw our probabilistic conclusions from the analytic results of the last section.

The Case $\lim_{x\to\infty}\kappa(x) > \lambda_0^\kappa$

In this section we consider the case where the asymptotic killing rate $K = \lim_{x\to\infty}\kappa(x)$ is strictly bigger than λ_0^κ. Theorem 2.1.1 shows that one has convergence to the quasistationary distribution if and only if the lowest eigenfunction is integrable. We give a new proof of this assertion and moreover prove that the lowest eigenfunction is actually always integrable. Therefore $K > \lambda_0^\kappa$ always implies convergence to the quasistationary distribution. In contrast to Steinsaltz and Evans we do not have to assume that ∞ is a natural boundary. The important new ingredient is the fact that in this case the bottom of the spectrum is an isolated eigenvalue This already implies the square-integrability and the λ_0^κ-invariance of the corresponding eigenfunction.

Theorem 2.3.1. *Consider the Sub-Markov semigroup e^{-tL^κ} and denote by $(X_t)_{t\geq 0}$ the process which is associated to the Dirichlet form q^κ. Assume that $\lim_{x\to\infty}\kappa(x) > \lambda_0^\kappa$. Then we have*

$$\lim_{t\to\infty} e^{\lambda_0^\kappa t}\mathbb{P}_x(\tau_\partial > t) = u_{\lambda_0^\kappa}(x)\int_0^\infty u_{\lambda_0^\kappa}(y)\,d\rho(y),$$

where $u_{\lambda_0^\kappa} \in L^2((0,\infty),\rho)$ denotes the up to positive multiples uniquely determined eigenfunction associated to the eigenvalue λ_0^κ. Furthermore, X_t converges to the quasistationary distribution $u_{\lambda_0^\kappa}$.

Proof. As we have already seen λ_0^κ is an isolated eigenvalue and therefore the eigenfunction $u_{\lambda_0^\kappa}$ is square integrable and satisfies

$$e^{-tL^\kappa}u_{\lambda_0^\kappa} = e^{-t\lambda_0^\kappa}u_{\lambda_0^\kappa}.$$

Notice that we really use the fact that λ_0^κ is an discrete eigenvalue in the L^2-sense in this step. In general at lest without any bounds on the coefficients it is not at all clear that $e^{-tL^\kappa}\varphi(\lambda,\cdot) = e^{-t\lambda_0^\kappa}\varphi(\lambda,\cdot)$.
Assume that $u_{\lambda_0^\kappa} \in L^1((0,\infty),\rho)$. By the local parabolic Harnack inequality (see Remark 2.2.5) there exists a locally bounded function $F : (0,\infty) \mapsto (0,\infty)$ such that for $t \geq 1$, $y, x \in (0,\infty)$ and every $\tilde{x}$ with $|x-\tilde{x}| < \frac{1}{2}\wedge\frac{x}{4} = r(x)$

$$p^\kappa(t,x,y) \leq F(x)p^\kappa(t+1,\tilde{x},y).$$

Denoting by $B(x, r(x))$ the ball with center x and radius $r(x)$ this gives

$$\begin{aligned} p^\kappa(t,x,y) &\leq \frac{\int_{B(x,r(x))} p^\kappa(t,x,y) u_{\lambda_0^\kappa}(\tilde{x})\, d\rho(\tilde{x})}{\int_{B(x,r(x))} u_{\lambda_0^\kappa}(\tilde{x})\, d\rho(\tilde{x})} \leq C(x) \frac{\int_{B(x,r(x))} p^\kappa(t,x,y) u_{\lambda_0^\kappa}(\tilde{x})\, d\rho(\tilde{x})}{\int_{B(x,r(x))} u_{\lambda_0^\kappa}(\tilde{x})\, d\rho(\tilde{x})} \\ &\leq C(x) \frac{\int_{B(x,r(x))} p^\kappa(t,\tilde{x},y) u_{\lambda_0^\kappa}(\tilde{x})\, d\rho(\tilde{x})}{\int_{B(x,r(x))} u_{\lambda_0^\kappa}(\tilde{x})\, d\rho(\tilde{x})} \leq C(x) \frac{e^{-t\lambda_0^\kappa} u_{\lambda_0^\kappa}(y)}{\int_{B(x,r(x))} u_{\lambda_0^\kappa}(\tilde{x})\, d\rho(\tilde{x})} \end{aligned}$$

The dominated convergence theorem together with (see Remark 2.2.4)

$$\lim_{t \nearrow \infty} e^{\lambda_0^\kappa t} p^\kappa(t,x,y) = u_{\lambda_0^\kappa}(x) u_{\lambda_0^\kappa}(y)$$

implies that

$$\lim_{t\to\infty} e^{\lambda_0^\kappa t} \mathbb{P}_x\big(\tau_\partial > t\big) = \lim_{t\to\infty} \int_0^\infty e^{\lambda_0^\kappa t} p^\kappa(t,x,y)\, d\rho(y) = u_{\lambda_0^\kappa}(x) \int_0^\infty u_{\lambda_0^\kappa}(y)\, d\rho(y). \tag{2.3.1}$$

In order to show the convergence to the quasistationary distribution $u_{\lambda_0^\kappa}$ let a measurable subset $U \subset (0, \infty)$ be given. Equation (2.3.1) together with the dominated convergence theorem give

$$\begin{aligned} \lim_{t\to\infty} e^{\lambda_0^\kappa t} \mathbb{P}_x\big(X_t \in U, \tau_\partial > t\big) &= \lim_{t\to\infty} e^{\lambda_0^\kappa t} \int_0^\infty \mathbf{1}_U(y) p^\kappa(t,x,y)\, d\rho(y) \\ &= u_{\lambda_0^\kappa}(x) \int_U u_{\lambda_0^\kappa}(y)\, d\rho(y). \end{aligned} \tag{2.3.2}$$

Using equation (2.3.2) we arrive at

$$\begin{aligned} \lim_{t\to\infty} \mathbb{P}_x\big(X_t \in U \mid \tau_\partial > t\big) &= \lim_{t\to\infty} \frac{\mathbb{P}_x\big(X_t \in U, \tau_\partial > t\big)}{\mathbb{P}_x\big(\tau_\partial > t\big)} = \frac{\lim_{t\to\infty} e^{\lambda_0^\kappa t} \mathbb{P}_x\big(X_t \in U, \tau_\partial > t\big)}{\lim_{t\to\infty} e^{\lambda_0^\kappa t} \mathbb{P}_x\big(\tau_\kappa > t\big)} \\ &= \frac{\int_U u_{\lambda_0^\kappa}(y)\, d\rho(y)}{\int_0^\infty u_{\lambda_0^\kappa}(y)\, d\rho(y)}. \end{aligned}$$

It remains to prove the integrability of $u_{\lambda_0^\kappa}$. This will be done in the following Lemma. □

Remark 2.3.1. *A closer look at the above proof shows that under the assumption that the bottom of the spectrum of L^κ is an eigenvalue with associated eigenfunction $u_{\lambda_0^\kappa}$ then X_t converges to the quasistationary distribution $\varphi(\lambda_0^\kappa, \cdot)$ if and only if $\int_0^\infty u_{\lambda_0^\kappa}(y)\, \rho(dy) < \infty$. Let us stress that this equivalence is already contained in Theorem 3.3 of [84] but our proof is new.*

In the following Lemma we will use a variation of a very nice argument, which we learned from the book [19] of Carmona and Lacroix, where a similar strategy was used in order to derive properties of eigenfunctions of Schrödinger operators. That similar ideas are also applicable in our situation is not obvious for at least two reasons. First in contrast to the situation in [19] we have a domain with boundary and second we do not know a priori that eigenfunctions are bounded. We will show, how the one-dimensional character of our problem helps us to overcome these problems. In the course of the proof we use a well-known result form the potential theory of Markov-processes. Namely we use that for a right continuous Markov process $(\Omega, \mathcal{F}, (\mathbb{Q}_x)_{x\in E}, (X_t)_{t\geq 0})$ with state space $(E, \mathcal{E})$ with life time ζ, which satisfies certain conditions (which hold in our case) the following fomrula holds (see Proposition D.15 in [29]): For every Borel set $B \subset E$ and $\lambda > 0$ there exists a measure μ_B^λ, called the λ-equilibrium measure, such that

$$\int g^\lambda(x,y)\, \mu_B^\lambda(dy) = \mathbb{E}_{\mathbb{Q}_x}\big[e^{-\lambda T_B} - e^{-\lambda\zeta}, T_B < \infty\big], \tag{2.3.3}$$

where $T_B = \inf\{t > 0 \mid X_t \in B\}$ denotes the first hitting time of B and $g^\lambda(x,y)$ the λ-potential. The measure μ_B^λ is a Radon measure and (for further details see appendix D in [29] and the monograph [11]) its support is always contained in the closure of B. In our case we consider a one-dimensional diffusion with life time T_0 and $B = \{a\}$. Then for $x > a$ by the strong Markov property we arrive at

$$\begin{aligned} \mathbb{E}_x\big[e^{-\lambda T_a} - e^{-\lambda T_0}; T_a < \infty\big] &= \mathbb{E}_x\big[e^{-\lambda T_a} - e^{-\lambda (T_a + T_0\circ\theta_{T_a})}; T_a < \infty\big] \\ &= \mathbb{E}_x\big[e^{-\lambda T_a}(1 - e^{-\lambda T_0\circ\theta_{T_a}}; T_a < \infty\big] \\ &= \mathbb{E}_x\big[\mathbb{E}_x\big[e^{-\lambda T_a}(1 - e^{-\lambda T_0\circ\theta_{T_a}}); T_a < \infty \mid \mathcal{F}_{T_a}\big]\big] \\ &= \mathbb{E}_x\big[e^{-\lambda T_a}\mathbf{1}_{\{T_a<\infty\}}\mathbb{E}_x\big[(1 - e^{-\lambda T_0\circ\theta_{T_B}} \mid \mathcal{F}_{T_a}\big]\big] \\ &= \mathbb{E}_x\big[e^{-\lambda T_a}\mathbf{1}_{\{T_a<\infty\}}\mathbb{E}_{X_{T_a}}\big[1 - e^{-\lambda T_0}\big]\big] \\ &= \mathbb{E}_a\big[1 - e^{-\lambda T_0}\big]\mathbb{E}_x\big[e^{-\lambda T_a}\mathbf{1}_{\{T_a<\infty\}}\big] \\ &=: B(a)\mathbb{E}_x\big[e^{-\lambda T_a}\mathbf{1}_{\{T_a<\infty\}}\big]. \end{aligned} \tag{2.3.4}$$

These potential theretic facts will be used in the proof of the next Lemma. During the proof we use methods, which might also be applicable to higher dimensional problems. Therefore we do not use specific one-dimensional features at several points in the proof.

Lemma 2.3.1. *Assume that $\lambda_0^\kappa < K := \lim_{t\to\infty}\kappa(t)$. Then the square integrable non-negative eigenfunction $u_{\lambda_0^\kappa}$ associated to the isolated eigenvalue λ_0^κ is integrable with respect to the measure ρ.*

Proof. We have already seen that $\lambda_0^\kappa < K$ implies that λ_0^κ is an isolated eigenvalue and therefore the non-negative function $u_{\lambda_0^\kappa}$ belongs to $L^2((0,\infty),\rho)$. Thus we have for every $x \in [0,\infty)$

$$e^{-\lambda_0^\kappa t}u_{\lambda_0}(x) = \mathbb{E}_x\left[e^{-\int_0^t \kappa(X_s)\,ds}u_{\lambda_0^\kappa}(X_s), T_0 > t\right]. \tag{2.3.5}$$

For $t \geq 0$ set

$$M_t = e^{-\int_0^t(\kappa-\lambda_0^\kappa)(X_s)\,ds}u_{\lambda_0^\kappa}(X_t)\mathbf{1}_{\{T_0>t\}}.$$

Then $(M_t)_{t\geq 0}$ is a positive martingale, since for $s<t$

$$\begin{aligned}\mathbb{E}_x\left[e^{-\int_0^t(\kappa-\lambda_0^\kappa)(X_r)\,dr}u_{\lambda_0^\kappa}(X_t)\mathbf{1}_{\{T_0>t\}}\mid\mathcal{F}_s\right] &= e^{-\int_0^s(\kappa-\lambda_0^\kappa)(X_r)\,dr}\mathbf{1}_{\{T_0>s\}}\\ &\cdot\mathbb{E}_x\left[e^{-\int_0^{t-s}(\kappa-\lambda_0^\kappa)(X_r)\,dr}u_{\lambda_0^\kappa}(X_{t-s})\circ\theta_s\mathbf{1}_{\{T_0>t-s\}}\circ\theta_s\mid\mathcal{F}_s\right]\\ &= e^{-\int_0^s(\kappa-\lambda_0^\kappa)(X_s)\,ds}\mathbf{1}_{\{T_0>s\}}\\ &\cdot\mathbb{E}_{X_s}\left[e^{-\int_0^{t-s}(\kappa-\lambda_0^\kappa)(X_r)\,dr}u_{\lambda_0^\kappa}(X_{t-s})\mathbf{1}_{\{T_0>t-s\}}\right]\\ &= e^{-\int_0^s(\kappa-\lambda_0^\kappa)(X_r)\,dr}u_{\lambda_0^\kappa}(X_s)\mathbf{1}_{\{T_0>s\}},\end{aligned}$$

where we used equation (2.3.5) in the last step. By the assumption $\lambda_0^\kappa<K$ there exist positive real numbers a and ε such that $\kappa(x)-\lambda_0^\kappa>\varepsilon$ for every $x\in[a,\infty)$. Let T_a be the first hitting time of the set $\{a\}$. By the optional sampling theorem we get for every $T>0$ and $x>a$

$$\begin{aligned}u_{\lambda_0^\kappa}(x) &= \mathbb{E}_x\left[e^{-\int_0^{T_a\wedge T}(\kappa-\lambda_0^\kappa)(X_s)\,ds}u_{\lambda_0^\kappa}(X_{T_a\wedge T})\mathbf{1}_{\{T_0>T_a\wedge T\}}\right]\\ &= \mathbb{E}_x\left[e^{-\int_0^{T}(\kappa-\lambda_0^\kappa)(X_s)\,ds}u_{\lambda_0^\kappa}(X_T)\mathbf{1}_{\{T_0>T\}}\mathbf{1}_{\{T_a>T\}}\right]\\ &+ \mathbb{E}_x\left[e^{-\int_0^{T_a}(\kappa-\lambda_0^\kappa)(X_s)\,ds}u_{\lambda_0^\kappa}(X_{T_a})\mathbf{1}_{\{T_0>T_a\}}\mathbf{1}_{\{T_a\leq T\}}\right]\\ &= \mathbb{E}_x\left[e^{-\int_0^{T}(\kappa-\lambda_0^\kappa)(X_s)\,ds}u_{\lambda_0^\kappa}(X_T)\mathbf{1}_{\{T_a>T\}}\right]\\ &+ \mathbb{E}_x\left[e^{-\int_0^{T_a}(\kappa-\lambda_0^\kappa)(X_s)\,ds}u_{\lambda_0^\kappa}(X_{T_a})\mathbf{1}_{\{T_a\leq T\}}\right]\end{aligned} \tag{2.3.6}$$

In contrast to [19] we do not know that $u_{\lambda_0^\kappa}$ is bounded (in general this will indeed not be true), since in our situation the semigroup e^{-tL^κ} is not necessarily ultracontractive. Thus we can not directly refer to the optional stopping theorem. Instead of this we will use the fact that our problem is one-dimensional, again. We prove that for each fixed $x>a$

$$\lim_{T\to\infty}\mathbb{E}_x\left[e^{-\int_0^{T}(\kappa-\lambda_0^\kappa)(X_s)\,ds}u_{\lambda_0^\kappa}(X_T)\mathbf{1}_{\{T_a>T\}}\right]=0. \tag{2.3.7}$$

First observe that by the choice of a we obtain

$$\mathbb{E}_x\left[e^{-\int_0^{T}(\kappa-\lambda_0^\kappa)(X_s)\,ds}u_{\lambda_0^\kappa}(X_T)\mathbf{1}_{\{T_a>T\}}\right]\leq e^{-\varepsilon T}\mathbb{E}_x\left[u_{\lambda_0^\kappa}(X_T)\mathbf{1}_{\{T_a>T\}}\right].$$

Thus it is enough to show, that

$$\mathbb{E}_x\left[u_{\lambda_0^\kappa}(X_T)\mathbf{1}_{\{T_a>T\}}\right]$$

remains bounded as $T \to \infty$. Denote by L_a the positive selfadjoint operator which is associated to the closure of the quadratic form

$$C_c^\infty((a,\infty)) \ni \varphi \mapsto \frac{1}{2}\int_a^\infty |\varphi'(t)|^2\,\rho(dt).$$

Then we get

$$(e^{-tL_a}u_{\lambda_0^\kappa})(x) = \mathbb{E}_x\big[u_{\lambda_0^\kappa}(X_t)\mathbf{1}_{\{T_a>t\}}\big]. \tag{2.3.8}$$

In order to convert the L^2-bounds into pointwise estimates we again use the following simple estimate

$$\begin{aligned}|f(x)| &= \left|\int_0^x f'(t)\,dt\right| \le \left|\int_0^x f'(t)\sqrt{\rho(t)}^{-1}\sqrt{\rho(t)}\,dt\right| \\ &\le \left(\int_0^x \rho(t)^{-1}\,dt\right)^{\frac12}\left(\int_0^\infty |f'(t)|^2\rho(t)\,dt\right)^{\frac12} \\ &= C(x)\left(\frac{1}{2}\int_0^\infty |f'(t)|^2\rho(t)\,dt\right)^{\frac12}.\end{aligned}$$

By equation (2.1.2) this simple inequality allows us to conclude that $a \le x \le M$ and $T > t_0$ for some $t_0 > 0$

$$\begin{aligned}\mathbb{E}_x\big[u_{\lambda_0^\kappa}(X_T), T_a > T\big] &= \big|\big(e^{-TL_a}u_{\lambda_0^\kappa})\big)(x)\big| \\ &\le C(M)\left(\int_a^\infty \big|\frac{d}{dx}(e^{-TL_a}u_{\lambda_0^\kappa})(x)\big|^2\rho(x)\,dx\right)^{\frac12} \\ &= C(M)\|\sqrt{L_a}e^{-TL_a}u_{\lambda_0^\kappa}\|_{L^2((a,\infty),\rho)} \\ &\le C(M)\sup_{t\in[t_0,\infty)}\|\sqrt{L_a}e^{-tL_a}u_{\lambda_0^\kappa}\|_{L^2((a,\infty),\rho)} < \infty\end{aligned} \tag{2.3.9}$$

Here we used the fact that $u_{\lambda_0^\kappa}$ is square integrable in order to be able to use the Hilbert space spectral theory. Using (2.3.9) we arrive at

$$\lim_{T\to\infty}\mathbb{E}_x\left[e^{-\int_0^T(\kappa-\lambda_0^\kappa)(X_s)\,ds}u_{\lambda_0^\kappa}(X_T)\mathbf{1}_{\{T_a>T\}}\right] = 0. \tag{2.3.10}$$

(2.3.6) and (2.3.10) imply that

$$\begin{aligned}0 \le u_{\lambda_0}(x) &= \mathbb{E}_x\left[e^{\int_0^{T_a}(\kappa-\lambda_0^\kappa)(X_s)\,ds}u_{\lambda_0^\kappa}(X_{T_a})\right] \\ &\le C_a\mathbb{E}_x\big[e^{-\varepsilon T_a}; T_a < \infty\big].\end{aligned} \tag{2.3.11}$$

In order to show the integrability of the right-hand side we apply the basic facts (2.3.3) and (2.3.4) from potential theory. This gives that for some compactly supported measure μ_a^ε

$$\mathbb{E}_x\big[e^{-\varepsilon T_a}; T_a < \infty\big] = B(a)^{-1}\int g^\varepsilon(x,y)\mu_a^\varepsilon(dy).$$

The ε-potential g^ε is defined by

$$g^\varepsilon(x,y) = \int_0^\infty e^{-\varepsilon t} p(t,x,y)\,dt,$$

where $p(t,x,y) = p^0(t,x,y)$ denotes the integral kernel of the operator e^{-tL^0}. Observe that due to the selfadjointness of the operator e^{-tL} the integral kernel $p(t,x,y)$ of e^{-tL} is necessarily symmetric with respect to the measure ρ. This shows that

$$\begin{aligned}\int_0^\infty \int_0^\infty e^{-\varepsilon t} p(t,x,y)\,dt\,\rho(dx) &= \int_0^\infty \int_0^\infty e^{-\varepsilon t} p(t,x,y)\,\rho(dx)\,dt \\ &= \int_0^\infty e^{-\varepsilon t} \mathbb{P}_y\big(T_0 > t\big)\,dt \le \int_0^\infty e^{-\varepsilon t}\,dt < \infty\end{aligned}$$

Since the measure μ_a^ε is compactly supported the function

$$x \mapsto \int g^\varepsilon(x,y)\mu_a^\varepsilon(dy)$$

thus belongs to $L^1(\mathbb{R}_+,\rho)$. This finally finishes the proof. □

Remark 2.3.2. *The above result displays a general principle, which seems to be well-known to analysts and mathematical physicists. The decay of eigenfunctions associated to isolated eigenvalues is dictated by the decay of the Green's function at least in regions where the potential κ is negligible .*

We have observed in Remark 2.2.1 that assertion (4) of Lemma 2.2.2 remains valid if one replaces the condition $\lim_{t\to\infty}\kappa(t) > \lambda_0^\kappa$ by the condition $\liminf_{t\to\infty}\kappa(t) > \lambda_0^\kappa$. Thus as in the proof of Theorem 2.3.1 one shows that X_t converges to quasistationarity if and only if the eigenfunction $u_{\lambda_0^\kappa} \in L^2((0,\infty),\rho)$ corresponding to the eigenvalue λ_0^κ belongs to $L^1((0,\infty),\rho)$.
In the proof of Lemma 2.3.1, where the integrability of $u_{\lambda_0^\kappa}$ was shown, the condition $\lim_{t\to\infty}\kappa(t) > \lambda_0^\kappa$ entered only in order to find a real number $a \in (0,\infty)$ such that $\kappa(t) - \lambda_0^\kappa > \varepsilon$ for every $t \in [a,\infty)$ and some $\varepsilon > 0$. Thus we arrive at the following result.

Corollary 2.3.1. *Consider the Sub-Markov semigroup e^{-tL^κ} and let $(X_t)_{t\ge 0}$ denote the process associated to the Dirichlet form q^κ. Moreover, assume that* $\liminf_{x\to\infty}\kappa(x) > \lambda_0^\kappa$. *Then we have*

$$\lim_{t\to\infty} e^{\lambda_0^\kappa t}\mathbb{P}_x(\tau_\partial > t) = u_{\lambda_0^\kappa}(x)\int_0^\infty u_{\lambda_0^\kappa}(y)\,d\rho(y),$$

where $u_{\lambda_0^\kappa} \in L^2((0,\infty),\rho)$ denotes the up to positive multiples uniquely determined eigenfunction associated to the eigenvalue λ_0^κ. Moreover, X_t converges to the quasistationary distribution $u_{\lambda_0^\kappa}$.

Remark 2.3.3. *Observe that in the situation $\lambda_0^\kappa < K$ it is essentially the killing rate that is responsible for the existence of the Yaglom limit. The process conditioned on survival will in general be located with high probability in regions where the killing rate κ is small. It will turn out that in the situation $\lambda_0^\kappa > K$ it is the drift which decides whether convergence to quasistationarity or escape to infinity occur. Thus the condition $\lambda_0^\kappa \neq K$ allows to separate the effects of the drift and the killing.*

The Case $\lim_{x\to\infty} \kappa(x) < \lambda_0^\kappa$: Part I

In this section we consider the case $\lim_{x\to\infty} \kappa(x) < \lambda_0^\kappa$ and $\int_0^\infty \rho(t)^{-1}\,dt = \infty$. In the case $\lim_{x\to\infty} \kappa(x) < \lambda_0^\kappa$ the situation is more subtle than in the case $\lim_{x\to\infty} \kappa(x) > \lambda_0^\kappa$. This is due to the fact that in contrast to the case $\lim_{x\to\infty} \kappa(x) > \lambda_0^\kappa$ one usually has

$$\lim_{t\to\infty} e^{\lambda_0^\kappa t}\,\mathbb{P}_x(X_t \in A, \tau_\partial > t) = 0 \tag{2.3.12}$$

for every bounded Borel set $A \subset [0,\infty)$. This can be seen for a Brownian motion with constant drift by a direct computation. Equation (2.3.12) remains true for every diffusion, if the bottom of the spectrum of the diffusion generator is not an eigenvalue in the L^2-sense. Therefore we cannot expect a longtime behavior which is precisely exponential. The key to a complete understanding of this situation is to distinguish two cases. We show that the additional assumption $\int_0^\infty \rho(t)^{-1}\,dt = \infty$ implies actually convergence to quasistationarity. In particular the lowest eigenfunction $\varphi(\lambda_0^\kappa, \cdot)$ is integrable.
As in [84] we set

$$F_t(\nu, \cdot) = \mathbb{P}_\nu\big(X_t \in \cdot \mid \tau_\partial > t\big)$$

and

$$a_t(\nu, r) = \mathbb{P}_\nu\big(\tau_\partial > t + r \mid \tau_\partial > t\big) = \int F_t(\nu, dy)\mathbb{P}_y\big(\tau_\partial > r\big).$$

If the process X_t started from the compactly supported initial distribution ν escapes to infinity, then for any sequence $(t_n)_{n\in\mathbb{N}}$ converging to infinity the measures $F_{t_n}(\nu, dy)$ converge weakly to point measure δ_∞. If the process X_t started from ν converges to the quasistationary disdistribution φ then then the limit of $F_{t_n}(\nu, dy)$ is concentrated on $\mathbb{R}_+$ and has the density $\frac{\varphi(\lambda_0^\kappa,\cdot)}{\int_0^\infty \varphi(\lambda_0^\kappa,y)\,\rho(dy)}$ with respect to the measure ρ. The next Lemma is a just a combination of Lemma 5.3 and Theorem 3.3 in [84] together with Theorem 2.2.3.

Lemma 2.3.2. *Assume that ∞ is a natural boundary point and suppose that $\lambda_0^\kappa \neq \lim_{x\to\infty} \kappa(x)$. Then the limit $a(\nu, r) = \lim_{t\to\infty} a_t(\nu, r)$ exists and is equal to*

$$a(\nu, r) = F(\nu, \mathbb{R}_+) \int \varphi(\lambda_0^\kappa, y)\mathbb{P}_y\big(\tau_\partial > r\big)\,\rho(dy) + (1 - F(\nu, \mathbb{R}_+)e^{-Kr}. \tag{2.3.13}$$

Either $F(\nu, \mathbb{R}_+) = 0$ for every compactly supported initial distribution ν or $F(\nu, \mathbb{R}_+) = 1$ for every such ν.

Proof. Let ν be a compactly supported initial distribution. Let $(t_n)_n \subset (0,\infty)$ be a sequence converging to infinity. On the compactification $[0,\infty]$ of $(0,\infty)$ the sequence of measures $F_{t_n}(\nu, dy)$ has a limit point. By Theorem 2.1.1 this limit point is either a measure on $(0,\infty)$ which has the density $\frac{\varphi(\lambda_0^\kappa,\cdot)}{\int_0^\infty \varphi(\lambda_0^\kappa,y)\,\rho(dy)}$ with respect to the measure ρ or is the point mass at ∞. Theorem 2.1.1 shows that there is only one limit point and that the limit point is independent of the sequence (t_n) and the initial distribution ν. Thus $F_t(\nu, dy)$ converges weakly. If ∞ is natural than one has

$$\lim_{y\to\infty} \mathbb{P}_y\big(\tau_\partial > t\big) = e^{-Kt},$$

where $K = \lim_{t\to\infty} \kappa(t)$. This shows that

$$\lim_{t\to\infty} \int F_t(\nu, dy)\mathbb{P}_y\big(\tau_\partial > r\big) = F(\nu, \mathbb{R}_+) \int \varphi(\lambda_0^\kappa, y)\mathbb{P}_y\big(\tau_\partial > r\big)\,\rho(dy) + (1 - F(\nu, \mathbb{R}_+)e^{-Kr}$$

□

In order to decide, whether X_t converges to the quasistationary distribution, we investigate the asymptotic behavior $r \mapsto \mathbb{P}_\nu\big(\tau_\partial > r\big)$ for $r \to \infty$.

Lemma 2.3.3. *Assume that the bottom of the spectrum λ_0^κ is strictly positive and that $\rho(dy)$ is a finite measure. Then for any compactly supported initial distribution ν we have*

$$-\lim_{t\to\infty} \frac{1}{t} \log \mathbb{P}_\nu\big(\tau_\partial > t\big) = \lambda_0^\kappa$$

Proof. Due to the assumption on ρ we conclude that $\mathbf{1}$ belongs to $L^2((0,\infty),\rho)$. This allows us to use tools from the Hilbert-space theory. First as in the proof of Lemma 2.3.1 that for all $f \in \{g \in L^2(\mathbb{R}_+,\rho) \mid g' \in L^2(\mathbb{R}_+,\rho)\}$

$$|f(x)| \le \sqrt{\int_0^x \rho(s)^{-1}\,ds}\left(\int_0^\infty |f'(t)|^2 \rho(t)\,dt\right)^{\frac{1}{2}}.$$

Since $\mathbf{1} \in L^2(\mathbb{R}_+,\rho)$ we have $e^{-tL^\kappa}\mathbf{1} \in \mathcal{D}(q[\kappa]) \subset \{f \in L^2(\mathbb{R}_+,\rho) \mid f' \in L^2(\mathbb{R}_+,\rho)\}$ and therefore for $f(x) = \mathbb{P}_x\big(\tau_\partial > t\big) = (e^{-tL^\kappa}\mathbf{1})(x)$

$$\begin{aligned}
\mathbb{P}_x\big(\tau_\partial > t\big) &\le \sqrt{2\int_0^x \rho(s)^{-1}\,ds}\left(\frac{1}{2}\int_0^\infty |f'(t)|^2\,\rho(dt)\right)^{\frac{1}{2}} \\
&\le \sqrt{2\int_0^x \rho(s)^{-1}\,ds}\left(\frac{1}{2}\int_0^\infty |f'(t)|^2\,\rho(dt) + \int_0^\infty \kappa(y)|f(y)|^2\,\rho(dy)\right)^{\frac{1}{2}} \\
&\le \sqrt{2\int_0^x \rho(s)^{-1}\,ds}\|\sqrt{L^\kappa}e^{-tL^\kappa}\mathbf{1}\|_{L^2(\rho)} \\
&= \sqrt{2\int_0^x \rho(s)^{-1}\,ds}\left(\int_{[\lambda_0^\kappa,\infty)} \lambda e^{-2\lambda t}\,d\|E_\lambda^\kappa \mathbf{1}\|^2_{L^2(\rho)}\right)^{\frac{1}{2}}
\end{aligned} \tag{2.3.14}$$

This shows that with $A(\nu) = \sqrt{2\int_0^{\mathrm{supp}\nu} \rho(s)^{-1}\,ds}$

$$\begin{aligned}
\limsup_{t\to\infty} \frac{1}{t}\log \mathbb{P}_\nu\big(\tau_\partial > t\big) &\le \limsup_{t\to\infty} \frac{1}{t}\log\left(A(\nu)\left(\int_{[\lambda_0^\kappa,\infty)} \lambda e^{-2\lambda t}\,d\|E_\lambda^\kappa\mathbf{1}\|^2_{L^2(\rho)}\right)^{\frac{1}{2}}\right) \\
&\le \limsup_{t\to\infty} \frac{1}{t}\log A(\nu) + \frac{1}{2}\limsup_{t\to\infty}\frac{1}{t}\log \int_{[\lambda_0^\kappa,\infty)} \lambda e^{-2\lambda t}\,d\|E_\lambda^\kappa\mathbf{1}\|^2_{L^2(\rho)} \\
&\le -\lambda_0
\end{aligned} \tag{2.3.15}$$

Now we need a matching lower bound but we want to stress that for our main application of the assertion of this lemma the upper bound already suffices. By the local parabolic Harnack inequality (see Remark 2.2.5) there exists a locally bounded function $F:(0,\infty)\to(0,\infty)$ such that for every z with $|z-x|<\frac{1}{2}\wedge\frac{x}{4}$

$$p^\kappa(t,x,y)\le F(x)p^\kappa(t+1,z,y).$$

For given $z\in(0,\infty)$ and every $x\in(0,\infty)$ with $|x-z|<\frac{1}{2}\wedge\frac{z}{5}$ we therefore have

$$p^\kappa(t,x,y)\le\Big(\sup_{\{x\in(0,\infty)||x-z|<\frac{1}{2}\wedge\frac{z}{5}\}}F(x)\Big)p^\kappa(t+1,z,y). \tag{2.3.16}$$

Thus for the open ball $B:=B(z,\frac{1}{2}\wedge\frac{z}{5})\subset(0,\infty)$ around z with radius $\frac{1}{2}\wedge\frac{z}{5}$ and some constant C with the help of (2.3.16) we arrive at

$$\begin{aligned}\int_{[\lambda_0^\kappa,\infty)}e^{-\lambda t}\,d(E_\lambda^\kappa\mathbf{1}_B,\mathbf{1}_B)_{L^2(\rho)}&=\int_B\int_B p^\kappa(t,x,y)\,\rho(dy)\,\rho(dz)\\&\le\int_B\int p^\kappa(t,x,y)\,\rho(dy)\,\rho(dz)\\&\le C\int p^\kappa(t+1,z,y)\,\rho(dy)\\&=C\,\mathbb{P}_z\big(\tau_\partial>t+1\big).\end{aligned} \tag{2.3.17}$$

If we can show that the infimum of the support of the finite measure $d\mu=d\|E_\lambda^\kappa\mathbf{1}_B\|^2_{L^2(\rho)}$ equals λ_0^κ then we are done since then (2.3.17) implies that the exponential rate of $\mathbb{P}_z\big(\tau_\partial>t\big)$ is bigger or equal to λ_0^κ and another application of the local parabolic Harnack inequality shows that the exponential rate of decay of $\mathbb{P}_z\big(\tau_\partial>t\big)$ is locally uniform in z. Therefore assume that $\lambda_0^\kappa<\mu_0=\inf\operatorname{supp}(\mu)$. Then we necessarily have $\|E^\kappa(I)\mathbf{1}_B\|^2_{L^2(\rho)}=0$ for every interval $I\subset[\lambda_0^\kappa,\mu_0)$. Because of

$$\begin{aligned}\int_{[\mu_0,\infty)}e^{-\lambda t}\,d(E_\lambda^\kappa\mathbf{1}_B,\mathbf{1}_B)_{L^2((0,\infty),\rho)}&=\int_{[\lambda_0^\kappa,\infty)}e^{-\lambda t}\,d(E_\lambda^\kappa\mathbf{1}_B,\mathbf{1}_B)_{L^2((0,\infty),\rho)}\\&=\int_B\int_B p^\kappa(t,x,y)\,\rho(dy)\,\rho(dz)\\&\ge\int_{\tilde B}\int_{\tilde B}p^\kappa(t,x,y)\,\rho(dy)\,\rho(dz)\\&=\int_{[\lambda_0^\kappa,\infty)}e^{-\lambda t}\,d(E_\lambda^\kappa\mathbf{1}_{\tilde B},\mathbf{1}_{\tilde B})_{L^2((0,\infty),\rho)}\end{aligned}$$

for every Borel measurable $\tilde B\subset B$ we conclude $\inf\operatorname{supp}\big(d\|E_\lambda\mathbf{1}_{\tilde B}\|^2_{L^2((0,\infty),\rho)}\big)\ge\mu_0$. For every $\tilde B\subset B$ and every interval $I\subset[\lambda_0^\kappa,\mu_0)$ we thus have

$$E^\kappa(I)\mathbf{1}_{\tilde B}\equiv 0.$$

This implies that for every Borel subset $\tilde{B} \subset B$ and $\lambda \in (0, \infty)$

$$0 = (U\, E^{\kappa}(I)\mathbf{1}_{\tilde{B}})(\lambda) = \mathbf{1}_I(\lambda)(U\mathbf{1}_{\tilde{B}})(\lambda) = \mathbf{1}_I(\lambda) \int_0^\infty \varphi(\lambda, y)\mathbf{1}_{\tilde{B}}(y)\rho(dy)$$

This implies that for every $\lambda \in I$ and $y \in B$ $\varphi(\lambda, y) = 0$. Obviously this is impossible since the function $\varphi(\lambda, \cdot)$ solves the initial value problem $L^{\kappa}\varphi(\lambda, \cdot) = \lambda\varphi(\lambda, \cdot)$ with $\varphi(\lambda, 0) = 0$ and $\varphi'(\lambda, 0) = 1$ in the sense of the theory of ordinary differential equations and therefore can not be the trivial solution. □

Lemma 2.3.4. *Assume that $K := \lim_{x\to\infty} \kappa(x) \neq \lambda_0^{\kappa}$ and let ν be a compactly supported initial distributions, then there exists $\eta_\nu \in \mathbb{R}$ such that*

$$a(\nu, r) = e^{-\eta_\nu r} \tag{2.3.18}$$

If X_t escapes to infinity then $\eta_\nu = K$.

Proof. By Lemma 2.3.2 for every sequence $(t_n)_n$ converging to ∞ the limit

$$\begin{aligned} a(\nu, r) &= \lim_{n\to\infty} \frac{\mathbb{P}_\nu(\tau_\partial > t_n + r)}{\mathbb{P}_\nu(\tau_\partial > t_n)} \\ &= \lim_{n\to\infty} \mathbb{P}_\nu(\tau_\partial > t_n + r \mid \tau_\partial > t_n) \\ &= F(\nu, \mathbb{R}_+) \int_0^\infty \varphi(\lambda_0^\kappa, y)\mathbb{P}_y\big(\tau_\partial > r\big)\, \rho(dy) + (1 - F(\nu, \mathbb{R}_+)e^{-Kr} \end{aligned}$$

exists and defines a continuous function. Moreover we have

$$\begin{aligned} a(\nu, r+s) &= \lim_{n\to\infty} \frac{\mathbb{P}_\nu(\tau_\partial > t_n + r + s)}{\mathbb{P}(\tau_\partial > t_n)} \\ &= \lim_{n\to\infty} \frac{\mathbb{P}_\nu(\tau_\partial > t_n + r + s)}{\mathbb{P}_\nu(\tau_\partial > t_n + s)} \frac{\mathbb{P}_\nu(\tau_\partial > t_n + s)}{\mathbb{P}_\nu(\tau_\partial > t_n)} \\ &= \lim_{n\to\infty} \frac{\mathbb{P}_\nu(\tau_\partial > t_n + r + s)}{\mathbb{P}_\nu(\tau_\partial > t_n + s)} \lim_{n\to\infty} \frac{\mathbb{P}_\nu(\tau_\partial > t_n + s)}{\mathbb{P}_\nu(\tau_\partial > t_n)} \\ &= a(\nu, r)a(\nu, s). \end{aligned}$$

This gives for some $\eta_\nu \in \mathbb{R}$

$$a(\nu, r) = e^{-\eta_\nu r}.$$

The last assertion follows from (2.3.13). □

The following definition is obviously motivated by the demographic phenomenon, which is explained in the introduction and in part stimulated the results of [84] and this work.

Definition 2.3.1. *The quantity η_ν in (2.3.18) will be called the* ***asymptotic mortality rate*** *for the process $(X_t)_{t\geq 0}$ started from the initial distribution ν.*

Theorem 2.3.2. *Let infinity be a natural boundary point for the diffusion $(X_t)_t$, which is associated to the Dirichlet form q^κ. Assume that $K := \lim_{x\to\infty} \kappa(x) < \lambda_0^\kappa$ and $\Lambda(\infty) = \int_0^\infty \rho(t)^{-1}\,dt = \infty$. Then X_t started from an arbitrary compactly supported initial distribution ν converges to the quasistationary distribution $\varphi(\lambda_0^\kappa, \cdot)$. Moreover the asymptotic mortality rate η_ν is independent of ν and equals λ_0^κ.*

Proof. If X_t escapes to infinity then we know from Lemma 2.3.4 that

$$a(\nu, r) = \lim_{t\to\infty} \frac{\mathbb{P}_\nu(\tau_\partial > t + r)}{\mathbb{P}_\nu(\tau_\partial > t)} = e^{-K\,r}.$$

Since by assumption $\lambda_0^\kappa > K$ we conclude using Lemma 2.2.2 (5) the strict positivity of λ_0. The strict positivity of λ_0 together with the assumption $\int_0^\infty \rho(t)^{-1}\,dt = \infty$ allows to apply Lemma 2.2.2 (2) in order to conclude that the symmetrizing measure ρ is finite. Therefore Lemma 2.3.3 shows that for every compactly supported measure ν

$$-\lim_{t\to\infty} \frac{1}{t} \log \mathbb{P}_\nu\big(\tau_\partial > t\big) = \lambda_0^\kappa.$$

In the case of escape to infinity equation (2.3.18) implies

$$-\lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}_\nu(\tau_\partial > n) = \eta_\nu = K := \lim_{x\to\infty} \kappa(x) \neq \lambda_0^\kappa.$$

Therefore the assumption $F(\nu, \mathbb{R}_+) = 0$ can not be true and thus by Theorem 2.1.1 we conclude $F(\nu, \mathbb{R}_+) = 1$ and $F(\nu, \infty) = 0$. Thus X_t converges from every compactly supported initial distribution ν to the quasistationary distribution $\varphi(\lambda_0^\kappa, \cdot)$. □

The above theorem has the following Corollary, which in a slightly more restrictive form already appears in the work [25] of Collet, Martínez and San Martín. But since the proof given in [25] has a serious gap it seems to be worth to point out that the assertion is in fact correct.

Corollary 2.3.2. *Let $\kappa \equiv 0$ and let $(X_t)_{t\geq 0}$ be the diffusion which is associated to the Dirichlet form q. Furthermore assume that ∞ is a natural boundary point for $(X_t)_{t\geq 0}$. If $\lambda_0 > 0$ and $\int_0^\infty \rho(t)^{-1}\,dt = \infty$ then X_t converges from every compactly supported initial distribution ν to the quasistationary distribution $\varphi(\lambda_0, \cdot)$. If $\lambda_0 = 0$ and $\int_0^\infty \rho(t)^{-1}\,dt = \infty$ then X_t started from ν escapes to infinity.*

Proof. The first part of the assertion follows directly from Theorem 2.3.2. In order to prove the second assertion we use the fact that the assumption concerning the boundary point infinity implies $\int_0^\infty \varphi(\lambda_0, x)\,\rho(dy) = \int_0^\infty \varphi(0, y)\,\rho(dy) = \int_0^\infty \int_0^x \rho^{-1}(t)\,dt\,\rho(dx) = \infty$. □

Remark 2.3.4. *The heuristics behind Theorem 2.3.2 are quite clear. If $\lambda_0^\kappa > K = \lim_{t\to\infty} \kappa(t)$ and if $\int_0^\infty \rho(t)^{-1} = \infty$, then already the drift prevents escape to infinity and therefore we get convergence to quasistationarity.*

The Case $\lim_{t\to\infty}\kappa(t) < \lambda_0^\kappa$: Part II

In this section we consider the case where $\lim_{t\to\infty}\kappa(t) > \lambda_0^\kappa$ and $\int_0^\infty \rho(t)^{-1}\,dt < \infty$. Intuition for this case can be gained from the analytic result of Lemma 2.2.2 (3). The assertion $\limsup_{r\to\infty}\frac{1}{r}\log\rho((0,r)) > 0$ implies a strong drift towards infinity. Thus one should expect that the process, conditioned on survival, follows the tendency of the original process and escapes to infinity. In the proof of this assertion we will play the same game as above. We use information from the L^2-theory and turn it into pointwise results by an elementary inequality.

Theorem 2.3.3. *Assume that ∞ is a natural boundary point and that $\int_0^\infty e^{\int_0^t 2b(s)\,ds}\,dt < \infty$. Let $(X_t)_{t\geq 0}$ be the diffusion corresponding to the Dirichlet form q^κ and assume that $\lambda_0^\kappa > \lim_{x\to\infty}\kappa(x)$. Then X_t escapes from every initial distribution ν, which is compactly supported in $(0,\infty)$, to infinity. The rate of escape is exponential.*

Proof. Let $z \in (0,\infty)$ be given. We will show that

$$\mathbb{P}_x\big(X_t \leq z \mid \tau_\partial > t\big) \to 0$$

as $t\to\infty$. Observe that the condition $\int_0^\infty \rho(t)^{-1}\,dt < \infty$ implies that for each $a\in(0,\infty)$ and each $x\in(a,\infty)$ the probability that the diffusion corresponding to L starting from x does not hit a is positive. For $\varepsilon > 0$. We can choose $a = a_\varepsilon \in (0,\infty)$ such that $\kappa(t)\in(K-\varepsilon, K+\varepsilon)$ for every $t\in[a,\infty)$. Then we have for every $x\in(2a,\infty)$

$$\begin{aligned}\mathbb{P}_x\big(\tau_\partial > t\big) &= \mathbb{E}_x\left[e^{-\int_0^t \kappa(X_s)\,ds},\, T_0 > t\right] \geq \mathbb{E}_x\left[e^{-\int_0^t \kappa(X_s)\,ds},\, T_a > t\right]\\ &\geq e^{-(K+\varepsilon)t}\mathbb{P}_x\big(T_a > t\big) \geq e^{-(K+\varepsilon)t}\mathbb{P}_x\big(T_a = \infty\big)\end{aligned} \tag{2.3.19}$$

Therefore the large time asymptotic of $\mathbb{P}_x(\tau_\partial > t)$ in t is slower than $e^{-(K+\varepsilon)t}$ at least if $x > 2a$. Another application of the local parabolic Harnack principle shows that the inequality $\mathbb{P}_z(\tau_\partial > t) \geq C_z e^{-(K+\varepsilon)t}$ implies that for some constant $C^1 > 0$ and $C^2 > 0$ for every x with $|z-x| < \frac{1}{2}\wedge\frac{|z|}{5}$

$$C^2\,e^{-t(K+\varepsilon)} \leq C^1\,\mathbb{P}_z\big(\tau_\partial > t\big) \leq \mathbb{P}_x\big(\tau_\partial > t+1\big).$$

Thus we have for every $x_0 > 0$

$$-\lim_{t\to\infty}\sup_{x\geq x_0}\mathbb{P}_x\big(\tau_\partial > t\big) < (K+\varepsilon).$$

Furthermore, we have

$$\begin{aligned}\mathbb{P}_x\big(X_t \leq z \mid \tau_\partial > t\big) &= \frac{\mathbb{P}_x\big(X_t\leq z,\,\tau_\partial > t\big)}{\mathbb{P}_x\big(\tau_\partial > t\big)}\\ &\leq \frac{\mathbb{P}_x\big(X_t\leq z,\,\tau_\partial > t\big)}{e^{-(K+\varepsilon)t}\mathbb{P}_x(T_a=\infty)}\\ &= \frac{e^{(K+\varepsilon)t}\mathbb{P}_x\big(X_t\leq z,\,\tau_\partial > t\big)}{\mathbb{P}_x\big(T_a=\infty\big)}.\end{aligned} \tag{2.3.20}$$

Thus it remains to show that

$$\mathbb{P}_x\big(X_t \le z,\, \tau_\partial > t\big)$$

vanishes faster than $e^{-(K+\varepsilon)t}$ for some $\varepsilon > 0$. Once more we use the inequality (valid for $f \in \mathcal{D}(q[\kappa])$)

$$\begin{aligned} |f(x)|^2 \le \left(\int_0^x |f'(t)|\,dt\right)^2 &= \left(\int_0^x |f'(t)|\sqrt{\rho(t)}\sqrt{\rho(t)^{-1}}\,dt\right)^2 \\ &\le 2\int_0^\infty \rho(t)^{-1}\,dt \left(\frac{1}{2}\int_0^\infty |f'(t)|^2\,\rho(dt) + \int_0^\infty \kappa(t)|f(t)|^2\,\rho(dt)\right) \\ &= 2\int_0^\infty \rho(t)^{-1}\,dt\,\|\sqrt{L^\kappa} f\|^2_{L^2(\rho)} \end{aligned}$$

Hence we get

$$\begin{aligned} \mathbb{P}_x\big(X_t \le z,\, \tau_\partial > t\big) &= (e^{-tL^\kappa}\mathbf{1}_{[0,z]})(x) \\ &\le \left(2\int_0^\infty \rho(t)^{-1}\,dt\right)^{\frac{1}{2}} \|\sqrt{L^\kappa} e^{-tL^\kappa}\mathbf{1}_{[0,z]}\|_{L^2(\rho)} = \\ &= \left(2\int_0^\infty \rho(t)^{-1}\,dt\right)^{\frac{1}{2}} \left(\int_{[\lambda_0^\kappa,\infty)} \lambda e^{-2t\lambda}\, d\|E_\lambda^\kappa \mathbf{1}_{[0,z]}\|^2_{L^2(\rho)}\right)^{\frac{1}{2}} \end{aligned} \tag{2.3.21}$$

The inequality (2.3.21) implies that for every $l < \lambda_0^\kappa$

$$\lim_{t\to\infty} e^{lt} \sup_{x\in(0,\infty)} \mathbb{P}_x\big(X_t \le z,\, \tau_\partial > t\big) = 0.$$

□

Remark 2.3.5. *Again we want to stress that in the proof of Theorem 2.3.3 we did not use the existence of the limit* $\lim_{t\to\infty}\kappa(t) =: K$*. What we did use is the existence of a real number* $a \in (0,\infty)$ *such that for some* $\varepsilon > 0$ *and every* $x \in [0,\infty)$ *we have* $\kappa(x) < \lambda_0^\kappa - \varepsilon$*. Therefore it is possible to replace the assumption* $\lim_{t\to\infty}\kappa(t) < \lambda_0^\kappa$ *by the weaker*

$$\limsup_{t\to\infty}\kappa(t) < \lambda_0^\kappa.$$

The sufficiency of such a condition is strongly suggested by Theorem 2.1.1, again.

The above proof shows more than stated in the theorem. It shows that the condition $lim_{x\to\infty}\kappa(x) < \lambda_0^\kappa$ and $\int_0^\infty \rho(r)^{-1}\,dt < \infty$ imply that X_t escapes to infinity exponentially fast with rate $\lambda_0^\kappa - K$. In order to compare this with previous results we formulate the following Corollary.

Corollary 2.3.3. *Assume that* $\kappa \equiv 0$*,* $\int_0^\infty \rho(t)^{-1}\,dt < \infty$ *and that* $\lambda_0 > 0$*. Then for every* $x_0, z \in (0,\infty)$

$$\lim_{t\to\infty} e^{\lambda_0 t} \sup_{x\in[x_0,\infty)} \mathbb{P}_x\big(X_t \le z \mid T_0 > t\big) < \infty.$$

If $\kappa \equiv 0$, $\int_0^\infty \rho(t)^{-1}\,dt < \infty$ and $\lambda_0 = 0$ then

$$\lim_{t\to\infty} \sup_{x\in[x_0,\infty)} \mathbb{P}_x\big(X_t \le z \mid T_0 > t\big) = 0.$$

Proof. This follows from (2.3.21) as for $x \in [x_0,\infty)$

$$\begin{aligned}\mathbb{P}_x\big(X_t \le z \mid T_0 > t\big) &= \frac{\mathbb{P}_x(X_t \le z; T_0 > t)}{\mathbb{P}_x(T_0 > t)} \le \frac{\sup_{x\in[x_0,\infty)} \mathbb{P}_x(X_t \le z; T_0 > t)}{\mathbb{P}_{x_0}(T_0 = \infty)} \\ &\le \mathbb{P}_{x_0}(T_0 = \infty)^{-1} \Big(\int_0^\infty \rho(t)^{-1}\,dt\Big)^{\frac{1}{2}} \Big(\int_{[\lambda_0,\infty)} \lambda e^{-2t\lambda}\, d\|E_\lambda \mathbf{1}_{[0,z]}\|^2_{L^2(\rho)}\Big)^{\frac{1}{2}}\end{aligned}$$

and

$$\lim_{t\to\infty} e^{\lambda_0 t} \Big(\int_{[\lambda_0,\infty)} \lambda e^{-2t\lambda}\, d\|E_\lambda \mathbf{1}_{[0,z]}\|^2_{L^2(\rho)}\Big)^{\frac{1}{2}} < \infty.$$

Assume now that $\lambda_0 = 0$. Using Remark 2.1.1 conclude that 0 is not an eigenvalue in the L^2-sense and therefore the for every $z \in (0,\infty)$ the measure $d\|E_\lambda \mathbf{1}_{[0,z]}\|^2_{L^2(\rho)}$ puts no mass on $\lambda_0 = 0$. This implies that

$$\lim_{t\to\infty} \Big(\int_{[0,\infty)} \lambda e^{-2t\lambda}\, d\|E_\lambda \mathbf{1}_{[0,z]}\|^2_{L^2(\rho)}\Big)^{\frac{1}{2}} \le \|E(\{0\})\mathbf{1}_{[0,z]}\|^2_{L^2(\rho)} = 0.$$

The one-dimensional inequality of Sobolev-type finishes the proof of this assertion. □

Remark 2.3.6. *The above theorem shows that in the situation of the theorem the L^2-eigenvalue λ_0^κ gives the exponential convergence rate at which X_t escapes to infinity. In the situation of Corollary 2.3.3 a similar but slightly weaker result has been obtained by Martínez and San-Martín (see Theorem 4 in [65]) by completely different methods. Our proof seems to be simpler and more transparent.*

As another consequence of Theorem 17 we note the following Corollary concerning the non-existence of general quasistationary distributions. Until now we did not distinguish between the Yaglom limit or a quasi-limiting distributions and the general notion of quasistationary distributions. As already mentioned in Remark 2.1.2 a quasi-limiting distribution $\tilde\nu$, which is in our case a probability measure on $(0,\infty)$ is always quasistationary in the sense that for every Borel set $A \subset (0,\infty)$

$$\mathbb{P}_{\tilde\nu}\big(X_t \in A \mid \tau_\partial > t\big) = \tilde\nu(A).$$

Observe that for every such quasistationary distribution $\tilde\nu$ there is a $\lambda \in \mathbb{R}$ such that $\mathbb{P}_\nu(\tau_\partial > t) = e^{-\lambda t}$.

Corollary 2.3.4. *Let infinity be a natural boundary and assume that $\lambda_0^\kappa > K$ and $\int_0^\infty \rho(t)^{-1}\,dt < \infty$. Then there is no quasistationary distribution $\tilde\nu$ satisfying $\sup_{x\le c} \int p(t,x,y)\tilde\nu(dy) < \infty$ for every $c > 0$.*

Proof. Assume that $\tilde{\nu}$ is a general quasistationary distribution. It is easy to see, that the measure $\tilde{\nu}$ has to be absolutely continuous with respect to ρ with a positive locally bounded density $r : [0,\infty) \to (0,\infty)$. In order to see this observe first that due to the quasistationarity of $\tilde{\nu}$ we have for some $\lambda \in \mathbb{R}$ and every measurable $A \subset (0,\infty)$ and $t > 0$

$$\begin{aligned} e^{-\lambda t}\tilde{\nu}(A) = \mathbb{P}_{\tilde{\nu}}\big(X_t \in A\big) &= \int_0^\infty \tilde{\nu}(dx) \int_A p(t,x,y)\,\rho(dy) \\ &= \int_A \int_0^\infty \rho(dy) \int_0^\infty p(t,x,y)\,\tilde{\nu}(dx). \end{aligned}$$

This implies that $e^{\lambda t}\int_0^\infty p(t,x,y)\,\tilde{\nu}(dx)$ is independent of $t > 0$ and that $\tilde{\nu}$ is absolutely continuous with density $r(y) := e^{\lambda t}\int_0^\infty p(t,x,y)\,\tilde{\nu}(dx)$ with respect to $\rho(dy)$. In order to prove the desired properties of r fix $y_0 \in (0,\infty)$ and observe that by the local parabolic Harnack inequality again we have for some $F(y)$ and all $|y - y_0| < \frac{1}{2} \wedge \frac{|x|}{4}$ and $x \in (0,\infty)$

$$\int_0^\infty p(t,x,y_0)\,\nu(dx) \le F(y_0) \int_0^\infty p(t+1,x,y)\,\nu(dx).$$

Thus if $r(y_0)$ is strictly positive then also is $r(y)$ for all $|y - y_0| < \frac{1}{2} \wedge \frac{|x|}{4}$. For $c > 1$ set $\nu_1^c(dx) = \mathbf{1}_{[0,c]}(x)r(x)\rho(dx)$ and $\nu_2^c(dx) = \mathbf{1}_{(c,\infty)}r(x)\rho(dx)$. Since

$$\mathbb{P}_{\mathbf{1}_{[1/2,1]}d\nu}(\tau_\partial > t) \le \mathbb{P}_{\nu_1^c}(\tau_\partial > t) \le \mathbb{P}_\nu(\tau_\partial > t) = e^{-\lambda t} \quad\text{and}\quad -\lim_{t\to\infty}\frac{1}{t}\log \mathbb{P}_{\mathbf{1}_{[1/2,1]}d\nu}(\tau_\partial > t) = K$$

we conclude that $\lambda \le K < \lambda_0^\kappa$. Let ε be smaller than $(\lambda_0^\kappa - K)/2$ and choose c such that $\sup_{t\ge c/2}|K - \kappa(t)| < \varepsilon$. Then we have for every $a, b \in (0,\infty)$

$$\begin{aligned} \nu((a,b]) &= \mathbb{P}_{\tilde{\nu}}\big(X_t \in (a,b] \mid \tau_\partial > t\big) \\ &= e^{\lambda t}\mathbb{P}_{\nu_1^c}\big(X_t \in (a,b]\big) + \mathbb{P}_{\nu_2^c}\big(X_t \in (a,b] \mid \tau_\partial > t\big)\frac{\mathbb{P}_{\nu_2^c}\big(\tau_\partial > t\big)}{\mathbb{P}_\nu\big(\tau_\partial > t\big)} \\ &\le e^{\lambda t}\mathbb{P}_{\nu_1^c}\big(X_t \in (a,b]\big) + \mathbb{P}_{\nu_2^c}\big(X_t \in (a,b] \mid \tau_\partial > t\big) \to 0 \end{aligned}$$

as $t \to \infty$. Here we used

$$\mathbb{P}_{\nu_1^c}(X_t \in (a,b]) = \big(\mathbf{1}_{[0,c]}r, e^{-tL^\kappa}\mathbf{1}_{(a,b]}\big)_{L^2((0,\infty),\rho)} \le e^{-\lambda_0^\kappa t}\|\mathbf{1}_{[0,c]}r\|_{L^2((0,\infty),\rho)}\,\|\mathbf{1}_{(a,b]}\|_{L^2((0,\infty),\rho)}$$

and that by the proof of Theorem 2.3.3 for some constant C

$$\mathbb{P}_{\nu_2^c}\big(X_t \in (a,b] \mid \tau_\partial > t\big) \le \frac{e^{(K+\varepsilon)t}\sup_{x\ge c}\mathbb{P}_x(X_t \in (a,b])}{\mathbb{P}_c(T_a = \infty)} \le C\,e^{-t(\lambda_0^\kappa - K - \varepsilon)}.$$

This proves the assertion □

It seems very reasonable to conjecture that the additional assumption

$$\sup_{x\le c}\int p(t,x,y)\tilde{\nu}(dy) < \infty \text{ for every } c > 0$$

automatically holds true. Since we are mainly interested in the quasilimiting distributions we did not try to remove this additional assertion.

Remark 2.3.7. *Let us consider a diffusion with generator given by* $-\frac{1}{2}\frac{d^2}{dx^2} + b(x)\frac{d}{dx}$. *If* $\mathbb{P}_x(T_0 = \infty) > 0$, *then there are good reasons to consider the long-time asymptotics of*

$$\mathbb{P}_x(X_t \in \cdot \mid T_0 \in (t, \infty)).$$

Thus one includes the condition that absorption happens in finite time. Conditions of this kind can often be found in the analogous problems in the theory of branching processes. In our setting this problem can be reduced to the above by an h-transform. The function $h(x) = \mathbb{P}_x(T_0 < \infty)$ *is harmonic and by general theory (see e.g. [75], chapter 4, sections 3 and 10) the process* (X_t) *conditioned to hit* 0 *is corresponds to the generator* L^h *whose action is given by*

$$(\frac{1}{h}L(h\,f))(x) = -\frac{1}{2}\frac{d^2 f}{dx^2}(x) + \left(b(x) - \frac{h'(x)}{h(x)}\right)\frac{df}{dx}(x)$$

The process associated to the operator L^h *can again be defined by Dirichlet form techniques and the associated measures on the path space is denoted with* $\tilde{\mathbb{P}}_x$. *As explained above we have*

$$\mathbb{P}_x(\cdot \mid T_0 < \infty) = \tilde{\mathbb{P}}_x(\cdot).$$

The operator L^h *can be realized as a selfadjoint operator in the Hilbert space* $L^2((0,\infty), h(x)\rho(dx))$. *Since the spectrum is invariant under h-transforms, we conclude that the positivity of the bottom of the spectrum of* L *implies the positivity of the spectrum of* L^h. *Since absorption is certain with respect to the measure* $\tilde{\mathbb{P}}_x$ *we can apply our previous results in order to conclude that for every Borel set* $A \subset (0, \infty)$

$$\lim_{t\to\infty} \mathbb{P}_x(X_t \in A\cdot \mid T_0 \in (t,\infty)) = \frac{\int_0^\infty \varphi^h(\lambda_0, x)h(X)\rho(dx)}{\int_0^\infty \varphi^h(\lambda_0, x)h(X)\rho(dx)},$$

where $\varphi^h(\lambda_0, x)$ *is the unique solution of* $(L^h - \lambda_0)u = 0$, *which satisfies* $\varphi^h(\lambda_0, 0) = 0$ *and* $(\varphi^h)'(\lambda_0, 0) = 1$.

Remark 2.3.8. *It seems to be a rather general principle that there are three possibilities. The first possibility is the non-existence of quasistationary distributions. If there exists a quasistationary distribution then it is either unique or there is a whole continuum of quasistationary distributions parameterized by an real interval. This is at least true for birth and death processes on the non-negative integers*

2.3.2 The Case of an Entrance Boundary at Infinity

Observe that $\int_0^\infty e^{\int_0^t 2b(s)\,ds}\,dt = \infty$ if ∞ is an entrance boundary. This follows e.g. from the fact that in this situation $\int_0^\infty e^{-\int_0^\infty 2b(s)\,ds}\,dt < \infty$ and therefore

$$x = \int_0^x e^{\int_0^t b(s)\,ds} e^{-\int_0^t b(s)\,ds}\,dt \le \left(\int_0^x e^{\int_0^t 2b(s)\,ds}\,dt\right)^{\frac{1}{2}} \left(\int_0^x e^{-\int_0^t 2b(s)\,ds}\,dt\right)^{\frac{1}{2}}$$

This clearly implies $\int_0^\infty e^{\int_0^t 2b(s)\,ds}\,dt = \infty$. Thus we are not in the case, where absorption of the unkilled process is certain and the reasoning used in the proof of Theorem 2.3.3 does not apply. In deed in the next theorem we prove that we always have convergence to quasistationarity if infinity is an entrance boundary.

Theorem 2.3.4. *Assume that* 0 *is regular and that* ∞ *is an entrance boundary point. Let* $0 \le \kappa \in C([0,\infty))$ *be given and let* $(X_t)_{t\ge 0}$ *denote the prosses associated to the Dirichlet form* q^κ. *Then the bottom of the spectrum is an isolated eigenvalue with associated non-negative eigenfunction* $u_{\lambda_0^\kappa}$ *and* X_t *converges to the quasistationary distribution* $u_{\lambda_0^\kappa}$ *from every compactly supported initial distribution* ν.

Proof. The first assertion follows from Theorem 2.2.4. Remark 2.3.1 directly implies that X_t converges to the quasistationary distribution $u_{\lambda_0^\kappa}$ from every compactly supported initial distribution if and only if $\int_0^\infty u_{\lambda_0^\kappa}(y)\,\rho(dy)$. Since we are assuming that 0 is regular and infinity is an entrance boundary we have

$$\int_0^\infty \rho(x)\,dx < \infty$$

and therefore

$$\int_0^\infty u_{\lambda_0^\kappa}(y)\rho(dy) \le \left(\int_0^\infty \rho(x)\,dx\right)^{\frac{1}{2}} \left(\int_0^\infty |u_{\lambda_0^\kappa}(y)|^2\rho(dy)\right)^{\frac{1}{2}}.$$

Since by Theorem 2.2.4 $u_{\lambda_0^\kappa}$ is an L^2-eigenfunction this shows that $u_{\lambda_0^\kappa}$ is integrable. □

Remark 2.3.9. *The uniqueness of quasistationary distributions (assuming their existence) was successfully addressed in the recent paper [20] at least under the assumption* $\kappa \equiv 0$. *In [20] the authors show that in this case the uniqueness of quasistationary distributions is equivalent to the assertion that for any* $a > 0$ *there exists* $y_a > 0$ *such that* $\sup_{x>y_a} \mathbb{E}_x\left[e^{aT_{y_a}}\right] < \infty$, *where* T_{y_a} *denotes the first hitting time of* y_a. *Thus uniqueness of quasistationary distributions is equivalent to the fact that 'time of implosion from infinity into the interior' has arbitrary large exponential moments. It is also proved in [20] that this is equivalent to infinity being an entrance boundary. If* $\kappa \equiv 0$ *and if absorption is certain our results show that the existence of a quasistationary distribution is equivalent to the existence of exponential moments of the first hitting time of* 0. *Both results together explain the macrostructure of the set of quasistationary distributions.*

2.3.3 Concluding Remarks and Open Problems

It is useful to summarize the basic strategy which has been applied in the previous chapter in order to study the large time behavior of one dimensional diffusions conditioned on extended survival. Our main strategy can be divided into three steps. In the first step one establishes a local version of the desired limit theorems. Via ideas of Steinsaltz and Evans this local result allows one to prove a dichotomy as a first step. This dichotomy asserts that the conditioned law has a limit and the limit is either the point mass at infinity or a measure $\frac{\varphi(\lambda_0^\kappa, y)\rho(dy)}{\int_0^\infty \varphi(\lambda_0^\kappa, y)\rho(dy)}$ on $(0, \infty)$. In the next step one investigates the tail behavior of the distribution of the killing time in order to conclude which case occurs. This strategy is quite robust and might even apply to certain multidimensional problems.

We end this chapter with a discussion of natural questions, which are left open. Assume that 0 regular, ∞ natural and $\lambda_0^\kappa = \lim_{x\to\infty} \kappa(x)$. We conjecture that still X_t **either** converges to quasistationarity **or** escapes to ∞. A rather simple case, where this is true is the Brownian motion with killing $\kappa \geq 0$. It is shown in [84] that $\varphi(\lambda_0^\kappa, \cdot)$ is λ_0^κ invariant, i.e. for $L^\kappa = -\frac{1}{2}\frac{d^2}{dx^2} + \kappa$

$$e^{-tL^\kappa}\varphi(\lambda_0^\kappa, \cdot) = e^{-\lambda_0^\kappa t}\varphi(\lambda_0^\kappa, \cdot).$$

Assume that $\varphi(\lambda_0^\kappa, \cdot) \in L^1((0, \infty), dx)$, then due to the ultracontractivity of e^{-tL^κ} one easily concludes $\varphi(\lambda_0^\kappa, \cdot) \in L^2((0, \infty), dx)$, which means that λ_0^κ belongs to the point spectrum of L^κ and therefore X_t converges to the quasistationary distribution $\varphi(\lambda_0^\kappa, \cdot)$. In general we do not know how to prove the required dichotomy. Even if one is able to prove that the process either escapes to infinity or converges to a quasistationary distribution, it seem to be difficult to decide, which case actually occurs. If $\lambda_0^\kappa \neq \lim_{x\to\infty}$ we used the fact the decay of $\mathbb{P}_x(\tau_\partial > t)$ on an exponential scale is λ_0^κ, if X_t converges to quasistationary and strictly smaller than λ_0^κ if X_t escapes to infinity. This will no longer be true if $\lambda_0^\kappa = \lim_{x\to\infty} \kappa(x)$. Since we now that in many situation a whole set of quasistationary distributions exists it would be clearly desirable to study the role of quasistationary distributions whose domain of attraction does not contain the set of compactly supported initial distributions. We will come to this problem again in the next chapter.

Chapter 3

Quasistationary Distributions : the Non-regular Case

In this chapter we consider diffusions on the half-line, for which the boundary point 0 is an exit boundary. Since we just have seen how one can handle non-trivial killing rates κ in the interior and since this does not change the general picture we assume for simplicity that $\kappa \equiv 0$. The spectral analysis of Chapter 2 which was necessary in order to separate the effects of the drift and the killing is more ore less the same in the case of an exit boundary at 0. As mentioned in the introduction the only results which are available in the literature are contained in the recent work [20][1] of Cattiaux, Collet et al.. We are aiming in the extension of some of their results. Our strategy is similar to the one used in the regular case, but in contrast to the regular case a more detailed investigation of the Weyl spectral representation is necessary. This will be done in the first section. Our main strategy follows the one used in the regular case but the technical problems become more difficult. In comparison to [20] we are able to remove several technical assumptions. The most important point is the fact that in our results the diffusion generator has not necessarily a purely discrete spectrum. We have already seen that existence of continuous spectral types in general complicates the problem of existence of quasistationary distributions. The spectral representation theorem of Weyl was the key in the proof of the convergence to quasistationary distributions on compacta.
Since 0 is no longer a regular boundary point the situation becomes even more subtle now. Since this time we are in the limit point case at both boundary points we generally have to expect that the spectrum does not have multiplicity 1 and therefore a spectral *matrix* is involved in the spectral representation. Therefore it is not clear whether and how the convergence to quasistationary distributions on compacta can be proved. We avoid this additional complication by showing that for a large class of diffusions the spectrum will still have multiplicity one. Her we heavily rely on ideas of Fritz Gesztesy and Maxim Zinchenko developed in [47] in the case of Schrödinger operators. At this point we emphasize that the

[1] We want to stress that we greatly benefited from ideas developed in [20]. From [20] we learned e.g. the importance of the Harnack inequality in the context of quasistationary distributions

investigation of diffusions which have a singular behavior at 0 is interesting for applications to biology. The reader is referred to [20] for a detailed explanation of the role of these diffusions in certain models of population dynamics. We just mention that these diffusions arise as scaling limits of birth-death processes.

3.1 One-dimensional Diffusions on the half-line

In this section we describe the basic setting of this work and recall very shortly some of the basic facts from the probabilistic and analytic theory of one dimensional Sturm-Liouville operators, though they have already been collected in greater detail in chapter 1. In contrast to chapter 1 we generally work with diffusion which are killed only at 0, i.e. we assume that $\kappa \equiv 0$. This is mainly for reasons of convenience, since we have already seen in chapter 1 how one can include non-trivial internal killing. Let $b : (0,\infty) \to \mathbb{R}$ be a continuous function. We consider the quadratic form q, which is defined as the closure of the form

$$C_c^\infty(\mathbb{R}) \ni \varphi \mapsto \frac{1}{2}\int_0^\infty |\varphi'(x)|^2 \, \rho(dx),$$

where $\rho(dx) = \rho(x)\,dx = e^{-\int_1^x 2b(s)\,ds}\,dx$. It is easy to see that q is a Dirichlet form, to which there is an associated strong Markov process $(X_t)_{T_0 > t \geq 0}$, which has continuous paths. The generator of this diffusion is given by the unique selfadjoint extension of associated to the form q in $L^2((0,\infty),\rho)$. This selfadjoint realization of the formal differential expression

$$\tau = -\frac{1}{2}\frac{d^2}{dx^2} + b(x)\frac{d}{dx}$$

will be denoted by L. We define e^{-tL} via the spectral calculus, i.e.

$$e^{-tL}g = \int_{\sigma(L)} e^{-t\lambda}\, E_L(d\lambda)g, \text{ for } g \in L^2((0,\infty),\rho),$$

where $(E_L(\lambda))_{\lambda\in\mathbb{R}}$ denotes the spectral resolution of L. By Stone's formula the spectral resolution of L can be calculated from the resolvent $(L-z)^{-1}$ $(z \in \mathbb{C}\setminus\sigma(L))$ via (compare Satz 8.11 in [95] and Lemma 2.5 in [47])

$$\begin{aligned}(f,&E_L((\lambda_1,\lambda_2])g)_{L^2((0,\infty),\rho)} \\ &= \lim_{\delta\to 0+}\lim_{\varepsilon\to 0+}\int_{\lambda_1+\delta}^{\lambda_2+\delta} d\lambda\left(f, \left[(L-(\lambda+i\varepsilon))^{-1} - (L-(\lambda-i\varepsilon))^{-1}\right]g\right)_{L^2((0,\infty,\rho)}.\end{aligned} \tag{3.1.1}$$

Let λ_0 denote the bottom of the spectrum of L, i.e. $\lambda_0 = \inf\sigma(L)$. From the general theory of symmetric Markov semigroups we know that e^{-tL} generates a consistent family of strongly continuous semigroups on $L^p((0,\infty),\rho)$ $(1 \leq p < \infty$ and by duality also give rise to a family of operators on $L^\infty((0,\infty),\rho)$ (see [45]). Moreover there exists a continuous function $(0,\infty)^3 \ni (t,x,y) \mapsto p(t,x,y) \in (0,\infty)$, which is symmetric in x and y, such that for all $f \in L^p((0,\infty),\rho)$

$$e^{-tL} f(x) = \int_0^\infty p(t,x,y) f(y) \rho(dy).$$

The general theory of Dirichlet forms (see e.g. [45]) implies that there is a ρ-symmetric Hunt process $(C([0,\infty),(0,\infty)), (X_t)_{t\geq 0}(\mathcal{F}_t)_{t\geq 0}, (\mathbb{P}_x)_{x\in(0,\infty)})$ such that for every $x \in (0,\infty)$ and every $f \in C_c^\infty((0,\infty))$ the process

$$\left(f(X_t) - f(X_0) - \int_0^t Lf(X_s)\, ds \right)_{t\geq 0}$$

is a martingale up to the explosion time with quadratic variation $\int_0^t |f'(X_s)|^2\, ds$. Moreover it is easy to see that for all $f \in C_c^\infty((0,\infty))$

$$e^{-tL} f(x) = \mathbb{E}_x[f(X_t), T_\infty \wedge T_0 > t],$$

where $T_a = \inf\{t > 0 \mid X_t = a\}$. In the sequel we will assume that ∞ is inaccessible, in which case $\mathbb{P}_x(T_\infty = \infty) = 1$. Under the above assumptions there exists one and only one symmetric Markov semigroup corresponding to L and in this sense the process is unique. The reader, who is not familiar with the theory of Dirichlet forms should assume that b is locally Lipschitz and should think of the process (X_t) as solution of the corresponding stochastic differential equation killed when hitting 0. We consider the case, where the boundary point 0 is in the Feller classification an exit boundary point and infinity is inaccessible. This means, that

$$\int_1^0 \left(\int_1^t \rho(s)\, ds \right) \rho(t)^{-1}\, dt < \infty \quad \text{and} \quad \int_1^0 \left(\int_1^t \rho(s)^{-1}\, ds \right) \rho(t)\, dt = \infty \tag{3.1.2}$$

and that

$$\int_1^\infty \left(\int_1^t \rho(s)\, ds \right) \rho(t)^{-1}\, dt = \infty. \tag{3.1.3}$$

The condition (3.1.2) implies that $\int_0^1 \rho(t)\, dt = \infty$ and $\int_0^1 \rho(t)^{-1}\, dt < \infty$. Recall that an inaccessible boundary point is called natural if

$$\int_1^\infty \left(\int_1^t \rho(s)^{-1}\, ds \right) \rho(t)\, dt = \infty$$

and entrance if the integral is finite. As seen in Theorem 2.2.1 (see also [99]) the differential expression τ is in the limit point case at 0 and at infinity if 0 is exit and ∞ is inaccessible, i.e. the restriction of L to $C_c^\infty((0,\infty))$ is even essentially selfadjoint and hence has a unique selfadjoint extension.

3.2 Spectral decomposition of L

We are interested in extending the approach used in [63], [84] and chapter 1 to the case, where 0 is allowed to be an exit boundary. One first uses spectral theoretic tools in order to prove a local result and then establishes the global result similar to the regular case. In the case of a regular boundary point at 0 it is known that the spectrum is of multiplicity one and therefore the Weyl-Titchmarsh spectral representation has a particular simple scalar form. If 0 is an exit boundary infinity is inaccessible then in the Weyl terminology we are in the limit point case at both endpoints and therefore a priori the spectrum might also have multiplicity two. This means that one has to consider 2×2-matrix valued measures instead of scalar measures. For more information concerning the classical spectral theory of Sturm-Liouville operators the reader is referred to [19]. In this section we establish, that for $\varepsilon > 0$ at least the operator $E_L((0, \varepsilon))L$ has a scalar spectral representation, which is more than enough for the solution of our problem. We will strongly rely on some ideas which are due to Gesztesy and Zinchenko in the context of Schrödinger operators with strongly singular potentials. New is the offered connection to the Feller classification. Since Gesztesy and Zinchenko presented their results for Schrödinger operators only and since we mainly have probabilists as readers in mind we include complete proofs of the necessary analytic machinery in a form which is adapted to our situation. In the final part of this section we show that for a large class of diffusions the results of Gesztesy and Zinchenko directly apply. We need some definitions.

Definition 3.2.1. *A solution $\tilde{\varphi}$ of*

$$(\tau\psi)(z, x) = z\psi(z, x) \tag{3.2.1}$$

*is called **analytic Weyl-Titchmarsh solution in** B_R, if $\tilde{\varphi}$ satisfies the following conditions for every $z \in B_R$:*

i) for every fixed $z \in B_R$ $\tilde{\varphi}(z, \cdot)$ is a non-trivial solution and for every fixed $x \in (0, \infty)$ the function $\tilde{\varphi}(z, x)$ is analytic in $z \in B_R$

ii) $\tilde{\varphi}(z, x) \in \mathbb{R}$ for every $(z, x) \in (-R, R) \times (0, \infty)$

iii) $\tilde{\varphi}(z, x)$ satisfies the following L^2-condition near 0

$$\int_0^b |\tilde{\varphi}(z, x)|^2 \rho(dx) < \infty$$

for every $b \in (0, \infty), z \in B_R$

For $z \in \mathbb{C}$ and $x_0 \in (0, \infty)$ we denote by $\varphi(z, \cdot, x_0)$ and $\theta(z, \cdot, x_0)$ the solutions of

$$(\tau\psi)(z, x) = z\psi(z, x)$$

satisfying

$$\varphi(z, x_0, x_0) = \rho(x_0)\theta'(z, x_0, x_0) = 0,\ \rho(x_0)\varphi'(z, x_0, x_0) = \theta(z, x_0, x_0) = 1.$$

For every fixed $x \in (0,\infty)$ the solutions $\varphi(z,x,x_0)$ and $\theta(z,x,x_0)$ are analytic with respect to $z \in \mathbb{C}$. Moreover we have

$$W(\theta(z,\cdot,x_0),\varphi(z,\cdot,x_0))(x) = 1,$$

where $W(f,g)(x) = f(x)\rho(x)g'(x) - \rho(x)f'(x)g(x)$ denotes the Wronskian. Later we will use the fact that for any fixed $x \in (0,\infty)$ the function $\tilde{\varphi}(z,x)$ is analytic in $z \in B_R$. For this observe that

$$\tilde{\varphi}(z,x) = \bar{\varphi}(z,x) := \tilde{\varphi}(z,x_0)\rho(x_0)\varphi(z,x,x_0) + \tilde{\varphi}(z,x_0)\theta(z,x,x_0),$$

as both sides are solutions of $\tau u = zu$ satisfying $\tilde{\varphi}(z,x_0) = \bar{\varphi}(z,x_0)$ and $\rho(x_0)\tilde{\varphi}(z,x_0) = \rho(x_0)\bar{\varphi}(z,x_0)$. Differentiating both sides shows the required analyticity of $\tilde{\varphi}(z,x)$ in $z \in B_R$. Moreover we introduce the Weyl-Titchmarsh solutions $\psi_\pm(z,\cdot,x_0)$ $(x_0 \in (0,\infty), z \in \mathbb{C}\setminus\mathbb{R})$ of the equation (3.2.1). Since we are in the limit-point case at 0 and at ∞ the Weyl-Titchmarsh solutions are up to constant multiples characterized by

$$\psi_-(z,\cdot,x_0) \in L^2((0,x_0),\rho),\ \psi_+(z,\cdot,x_0) \in L^2((x_0,\infty),\rho),\ z \in \mathbb{C}\setminus\mathbb{R}.$$

We normalize $\psi_\pm(z,\cdot,x_0)$ by requiring

$$\psi_\pm(z,x_0,x_0) = 1.$$

This gives

$$\psi_\pm(z,x,x_0) = \theta(z,x,x_0) + m_\pm(z,x_0)\varphi(z,x,x_0),\ x,x_0 \in (0,\infty), z \in \mathbb{C}\setminus\mathbb{R},$$

where $m_\pm(z,x_0)$ is given by

$$m_\pm(z,x) = \frac{\rho(x)\psi'_\pm(z,x,x_0)}{\psi_\pm(z,x,x_0)}\ (x,x_0 \in (0,\infty), z \in \mathbb{C}\setminus\mathbb{R}).$$

It is known that $m_\pm(z,x)$ are Herglotz- and anti-Herglotz-functions, respectively (details about Herglotz function can be found in our standard reference [95] and in the appendix of [47]).

Lemma 3.2.1. *Assume that there exists an analytic Weyl-Titchmarsh solution $\tilde{\varphi}(z,x)$ in B_R. Then there exists a solution $\tilde{\theta}(z,x)$ of (3.2.1), which for every $x \in (0,\infty)$ is analytic in $z \in B_R$, real-valued for $z \in \mathbb{R}$, such that for every $z \in B_R$*

$$W(\tilde{\theta}(z,\cdot),\tilde{\varphi}(z,\cdot)) \equiv 1.$$

Proof. We basically follow the proof of Lemma 3.3 in [47]. Fix $x_0 \in (0,\infty)$ and consider

$$\tilde{\theta}(z,x) = \frac{\rho(x_0)\tilde{\varphi}'(z,x_0)}{(\tilde{\varphi}(z,x_0))^2 + (\rho(x_0)\tilde{\varphi}'(z,x_0))^2}\theta(z,x,x_0) - \frac{\tilde{\varphi}(z,x_0)}{(\tilde{\varphi}(z,x_0))^2 + (\rho(x_0)\tilde{\varphi}'(z,x_0))^2}\varphi(z,x,x_0).$$

Since the solution $\tilde{\varphi}(z,\cdot)$ is non-trivial for every $z \in B_R$ we have $(\tilde{\varphi}(z,x_0), \rho(x_0)\tilde{\varphi}'(z,x_0)) \neq (0,0)$ for every every $z \in B_R$. Thus we conclude that $\tilde{\theta}(z,\cdot)$ is well defined for $z \in B_R$ and due to the properties of $\tilde{\varphi}(z,x_0)$, $\theta(z,x,x_0)$ and $\varphi(z,x,x_0)$ analytic in $z \in B_R$ for every fixed $x \in (0,\infty)$. Moreover we have

$$\begin{aligned} W(\tilde{\theta}(z,\cdot),\tilde{\varphi}(z,\cdot))(x) &= W(\tilde{\theta}(z,\cdot),\tilde{\varphi}(z,\cdot))(x_0) \\ &= \tilde{\theta}(z,x_0)\rho(x_0)\tilde{\varphi}'(z,x_0) - \rho(x_0)\tilde{\theta}'(z,x_0)\tilde{\varphi}(z,x_0) \\ &= \frac{(\rho\tilde{\varphi}'(z,x_0))^2}{(\tilde{\varphi}(z,x_0))^2 + (\rho(x_0)\tilde{\varphi}'(z,x_0))^2} + \frac{(\tilde{\varphi}(z,x_0))^2}{(\tilde{\varphi}(z,x_0))^2 + (\rho(x_0)\tilde{\varphi}'(z,x_0))^2} = 1 \end{aligned}$$

□

Introduce $\tilde{m}_+(z)$ in such a way that

$$\tilde{\psi}_+(z,x) = \tilde{\theta}(z,x) + \tilde{m}(z)\tilde{\varphi}(z,x), x \in (0,\infty)$$

satisfies for $z \in B_R \setminus \mathbb{R}$

$$\tilde{\psi}_+(z,\cdot) \in L^2((a,\infty),\rho), a \in (0,\infty),$$

since the differential expression is in the limit point case at infinity the solution $\tilde{\psi}_+(z,\cdot)$ is proportional to $\psi_+(z,\cdot,x_0)$. This gives

$$m_+(z,x) = \frac{\rho\tilde{\theta}'(z,x) + \tilde{m}_+(z)\rho\tilde{\varphi}'(z,x)}{\tilde{\theta}(z,x) + \tilde{m}(z)\tilde{\varphi}(z,x)}.$$

Thus by a simple calculation we get

$$\begin{aligned} \tilde{m}_+(z) &= \frac{m_+(z,x)\theta(z,x) - \rho\,\tilde{\theta}'(z,x)}{\rho\,\tilde{\varphi}(z,x) - m_+(z,x)\tilde{\varphi}(z,x)} = \frac{W(\tilde{\theta}(z,\cdot),\psi_+(z,\cdot,x_0))}{W(\psi_+(z,\cdot,x_0),\tilde{\varphi}(z,\cdot))} \\ &= \frac{\tilde{\theta}(z,x)}{\tilde{\varphi}(z,x)}\frac{m_+(z,x)}{m_-(z,x) - m_+(z,x)} - \frac{\rho\,\tilde{\theta}'(z,x)}{\tilde{\varphi}(z,x)}\frac{1}{m_-(z,x) - m_+(z,x)} \end{aligned} \tag{3.2.2}$$

Since m_- and m_+ are Herglotz- and Anti-Herglotz functions respectively and $\tilde{\varphi}(x,z)$, $\tilde{\theta}(z,x)$ are analytic and since $\tilde{\varphi}(z,x) \neq 0$ we see that $\tilde{m}_+(z)$ is analytic in $z \in B_R$. Note that $\tilde{\varphi}(z,x)$ is not 0, since if there would be an x_0 with $\tilde{\varphi}(z,x_0) = 0$, then the function $\tilde{\varphi}(z,\cdot)$ would be an eigenfunction to a non-real eigenvalue of the selfadjoint realization of τ in $L^2((0,x_0),\rho)$ with Dirichlet boundary condition at x_0. A direct computation shows that the Green's function $G(z,x,y)$ ($z \in B_R \setminus \sigma(L)$, $x,y \in (0,\infty)$) is given by

$$G(z,x,y) = \begin{cases} \tilde{\varphi}(z,x)\tilde{\psi}_+(z,y) & \text{if } 0 < x \leq y, \\ \tilde{\varphi}(z,y)\tilde{\psi}_+(z,x) & \text{if } 0 < y \leq x \end{cases}. \tag{3.2.3}$$

This means that for every $x \in (0,\infty)$ and $f \in L^2((0,\infty),\rho)$

$$((L-z)^{-1}f)(x) = \int_0^\infty G(z,x,y)f(y)\rho(dy). \tag{3.2.4}$$

The following Lemma corresponds to Lemma 3.4 in [47]

Lemma 3.2.2. *Assume that there exists an analytic Weyl-Titchmarsh solution in B_R. Then the function $\tilde{m}_+$ satisfies the following conditions.*

i) $\tilde{m}_+(z) = \overline{\tilde{m}_+(\overline{z})}$ *for* $z \in B_R$

ii $\varepsilon|\tilde{m}_+(\lambda + i\varepsilon)| \leq C(\lambda_1, \lambda_2, \varepsilon_0)$ *for* $\lambda \in [\lambda_1, \lambda_2], 0 < \varepsilon \leq \varepsilon_0$

iii) $\varepsilon|\Re(\tilde{m}_+(\lambda + i\varepsilon)| =_{\varepsilon \downarrow 0} o(1)$ *for* $\lambda \in [\lambda_1, \lambda_2], 0 < \varepsilon \leq \varepsilon_0$

iv) $\lim_{\varepsilon \downarrow 0}(i\varepsilon)\tilde{m}_+(\lambda + i\varepsilon) = \lim_{\varepsilon \downarrow 0} \varepsilon \Im \tilde{m}_+(\lambda + i\varepsilon)$ *exists for all* $\lambda \in (-R, R)$ *and is non-negative.*

v) $\tilde{m}_+(\lambda + i0) = \lim_{\varepsilon \downarrow 0} \tilde{m}_+(\lambda + i\varepsilon)$ *exists for a.e.* $\lambda \in [\lambda_1, \lambda_2]$ *and* $\Im(\tilde{m}_+(\lambda + i0)) \geq 0$ *for a.e.* $\lambda \in [\lambda_1, \lambda_2]$.

Moreover there exists a measure σ such that

$$\int_{(\lambda_1, \lambda_2]} d\sigma(\lambda) = \sigma((\lambda_1, \lambda_2]) = \lim_{\delta \downarrow 0} \lim_{\varepsilon \downarrow 0} \frac{1}{\pi} \int_{\lambda_1 + \delta}^{\lambda_2 + \delta} d\lambda \Im(\tilde{m}_+(\lambda + i\varepsilon)) \tag{3.2.5}$$

Proof. Since for $(\lambda, x) \in (-R, R) \times (0, \infty)$ the numbers $\tilde{\varphi}(\lambda, x)$ and $\tilde{\theta}(\lambda, x)$ are real, we get for $(z, x) \in B_R \times (0, \infty)$

$$\tilde{\varphi}(z, x) = \overline{\tilde{\varphi}(\overline{z}, x)},\ \tilde{\theta}(z, x) = \overline{\tilde{\theta}(\overline{z}, x)}.$$

Let $c, d \in (0, \infty)$ with $c < d$ then

$$\begin{aligned} \int_{\sigma(L)} \frac{d\|E_L(\lambda)\chi_{[c,d]}\|^2_{L^2((0,\infty),\rho)}}{\lambda - z} &= (\chi_{[c,d]}, (L - z)^{-1}\chi_{[c,d]})_{L^2((0,\infty),\rho)} \\ &= \int_c^d \rho(dx) \int_c^x \rho(dy)\, \tilde{\theta}(z, x)\tilde{\varphi}(z, y) \\ &+ \int_c^d \rho(dx) \int_x^d \rho(dy)\, \tilde{\varphi}(z, x)\tilde{\theta}(z, y) \\ &+ \tilde{m}_+(z)\Big[\int_c^d \rho(dx)\tilde{\varphi}(z, x)\Big]^2 \end{aligned} \tag{3.2.6}$$

Observe that we can find c, d in such a way that

$$\int_c^d \tilde{\varphi}(z, x)\, \rho(dx) \neq 0$$

for all z in sufficiently small complex open neighborhood of $[\lambda_1, \lambda_2]$. Equation (3.2.6) already implies i). Now remark that the function H given by

$$H : \mathbb{C} \setminus \sigma(L) \ni z \mapsto \int_{\sigma(L)} \frac{d\|E_L(\lambda)\chi_{[c,d]}\|^2_{L^2((0,\infty),\rho)}}{\lambda - z}$$

is a Herglotz function which together with the basic properties of $\tilde{\theta}(z,x)$ and $\tilde{\varphi}(z,x)$ directly implies ii)-v). Due to the Herglotz property of H we have

$$\lim_{\delta\downarrow 0}\lim_{\varepsilon\downarrow 0}\frac{1}{\pi}\int_{\lambda_1+\delta}^{\lambda_2+\delta} dx \Im \int_{\sigma(L)} \frac{d\|E_L(\lambda)\chi_{[c,d]}\|^2_{L^2((0,\infty),\rho)}}{\lambda-(x+i\varepsilon)} = \|E_L((\lambda_1,\lambda_2])\chi_{[c,d]}\|^2_{L^2((0,\infty),\rho)}.$$

Then (3.2.6) applied to $\lambda+i\varepsilon$ with $\lambda\in(\lambda_1,\lambda_2)$ and $\varepsilon>0$ small enough implies that σ, defined in (3.2.5) satisfies

$$\begin{aligned}\sigma((\lambda_1,\lambda_2]) &= \lim_{\delta\downarrow 0}\lim_{\varepsilon\downarrow 0}\frac{1}{\pi}\int_{\lambda_1+\delta}^{\lambda_2+\delta} d\lambda \Im(\tilde{m}_+(\lambda+i\varepsilon))\\ &= \lim_{\delta\downarrow 0}\lim_{\varepsilon\downarrow 0}\frac{1}{\pi}\int_{\lambda_1+\delta}^{\lambda_2+\delta} d\lambda \Im\Big\{\int_{\sigma(L)} \frac{d\|E_L(\tilde{\lambda})\chi_{[c_0,d_0]}\|^2_{L^2((0,\infty)\rho)}}{\tilde{\lambda}-\lambda-i\varepsilon}\\ &\cdot\Big[\Big(\int_{c_0}^{d_0}\rho(dx)\tilde{\varphi}(\lambda,x)\Big)^2 + 2i\varepsilon\Big(\int_{c_0}^{d_0}\rho(dx)(d/dz)\tilde{\varphi}(z,x)|_{z=\lambda}\Big)+O(\varepsilon^2)\Big]^{-1} + O(\varepsilon)\Big\}\\ &= \int_{(\lambda_1,\lambda_2]} d\|E_L(\lambda)\chi_{[c_0,d_0]}\|^2_{L^2((0,\infty),\rho)}\Big[\int_{c_0}^{d_0}\rho(dx)\tilde{\varphi}(\lambda,x)\Big]^{-2}.\end{aligned}$$

and thus really defines a measure on $\mathbb{R}$. □

The following theorem connects the measure σ with the spectral measure

Theorem 3.2.1. *Assume that there exists an analytic Weyl-Titchmarsh solution in B_R. Let $f,g\in C_c^\infty(0,\infty)$ and $F\in C(\mathbb{R})$ be given. Moreover let $\lambda_1,\lambda_2\in(-R,R)$ with $\lambda_1<\lambda_2$. Then we have*

$$(f,F(L)E_L((\lambda_1,\lambda_2])g)_{L^2((0,\infty),\rho)} = (\hat{f}, M_{F\chi_{(\lambda_1,\lambda_2]}}\hat{g})_{L^2(\mathbb{R},\sigma)}, \tag{3.2.7}$$

where $M_{F\chi_{(\lambda_1,\lambda_2]}}$ denotes the bounded operator, which acts by multiplication with $F\chi_{(\lambda_1,\lambda_2]}$.

Proof. As in [47] we use Stone's formula in order to connect σ with the spectral resolution E_L of L. Stone's formula implies that

$$\begin{aligned}(f,F(L)E_L((\lambda_1,\lambda_2])g)_{L^2((0,\infty),\rho)} &= \lim_{\delta\downarrow 0}\lim_{\varepsilon\downarrow 0}\int_{\lambda_1+\delta}^{\lambda_2+\delta}\frac{d\lambda}{2\pi i}F(\lambda)\Big[(f,(L-(\lambda+i\varepsilon))^{-1}g)_{L^2((0,\infty),\rho)}\\ &\qquad - (f,(L-(\lambda-i\varepsilon))^{-1}g)_{L^2((0,\infty),\rho)}\Big].\end{aligned}$$

Using the formula (3.2.3) for the kernel of the resolvent we get

$$
\begin{aligned}
&(f,F(L)E_L((\lambda_1,\lambda_2])g)_{L^2((0,\infty),\rho)} \\
&\quad = \lim_{\delta\downarrow 0}\lim_{\varepsilon\downarrow 0}\frac{1}{2\pi i}\int_{\lambda_1+\delta}^{\lambda_2+\delta} d\lambda F(\lambda)\Big\{\int_0^\infty \rho(dx)\Big[\overline{f(x)}\tilde\psi_+(\lambda+i\varepsilon,x)\int_0^x \tilde\varphi(\lambda+i\varepsilon,y)g(y)\rho(dy) \\
&\quad + \overline{f(x)}\tilde\varphi(\lambda+i\varepsilon,x)\int_x^\infty \rho(dy)\tilde\psi_+(\lambda+i\varepsilon,y)g(y)\Big] \\
&\quad - \Big[\overline{f(x)}\tilde\psi_+(\lambda-i\varepsilon,x)\int_0^x \rho(dy)\tilde\varphi(\lambda-i\varepsilon,y)g(y) \\
&\quad + \overline{f(x)}\tilde\varphi(\lambda-i\varepsilon,x)\int_x^\infty \rho(dy)\tilde\psi_+(\lambda-i\varepsilon,y)g(y)\Big]\Big\}.
\end{aligned}
$$

Since all integrals are on bounded sets and since the integrands are continuous we are allowed to interchange integrations to get

$$
\begin{aligned}
&(f,F(L)E_L((\lambda_1,\lambda_2])g)_{L^2((0,\infty),\rho)} \\
&\quad = \int_0^\infty \rho(dx)\,\overline{f(x)}\Big\{\int_0^x \rho(dy)\,g(y)\lim_{\delta\downarrow 0}\lim_{\varepsilon\downarrow 0}\frac{1}{2\pi i}\int_{\lambda_1+\delta}^{\lambda_2+\delta} d\lambda\,F(\lambda)\big[\tilde\psi_+(\lambda+i\varepsilon,x)\tilde\varphi(\lambda+i\varepsilon,y) \\
&\quad - \tilde\psi_+(\lambda-i\varepsilon,x)\tilde\varphi(\lambda-i\varepsilon,y)\big] \\
&\quad + \int_x^\infty \rho(dy)\,g(y)\lim_{\delta\downarrow 0}\lim_{\varepsilon\downarrow 0}\frac{1}{2\pi i}\int_{\lambda_1+\delta}^{\lambda_2+\delta} d\lambda\,F(\lambda)\big[\tilde\varphi(\lambda+i\varepsilon,x)\tilde\psi_+(\lambda+i\varepsilon,y) \\
&\quad - \tilde\varphi(\lambda-i\varepsilon,x)\tilde\psi_+(\lambda-i\varepsilon,y)\big]\Big\}
\end{aligned}
\tag{3.2.8}
$$

Due to the regularity properties of $\tilde\varphi$ and $\tilde\theta$ we have

$$
\begin{aligned}
\tilde\theta(\lambda\pm i\varepsilon) &=_{\varepsilon\downarrow 0} \tilde\theta(\lambda,x)\pm i\varepsilon\frac{d}{dz}\tilde\theta(z,x)|_{z=\lambda}+O(\varepsilon^2) \\
\tilde\varphi(\lambda\pm i\varepsilon) &=_{\varepsilon\downarrow 0} \tilde\varphi(\lambda,x)\pm i\varepsilon\frac{d}{dz}\tilde\varphi(z,x)|_{z=\lambda}+O(\varepsilon^2),
\end{aligned}
\tag{3.2.9}
$$

where $O(\varepsilon)$ is locally uniform in (λ, x). Therefore we get by (3.2.9) and by ii) and iii) of Lemma 3.2.2

$$
\begin{aligned}
\tilde\varphi(\lambda+i\varepsilon,x)\tilde\psi_+(\lambda+i\varepsilon,y)-\tilde\varphi(\lambda-i\varepsilon)\tilde\psi_+(\lambda-i\varepsilon,y) &= \tilde\varphi(\lambda,x)[\tilde\theta(\lambda,y)+\tilde m_+(\lambda-i\varepsilon)\tilde\varphi(\lambda,y)] \\
&\quad + \tilde\varphi(\lambda,x)[\tilde\theta(\lambda,y)+\tilde m_+(\lambda-i\varepsilon)\tilde\varphi(\lambda,y)] \\
&\quad + o(1),
\end{aligned}
\tag{3.2.10}
$$

where $o(1)$ is locally uniform. Equation (3.2.10) together with (3.2.8) gives

$$(f,F(L)E_L((\lambda_1,\lambda_2])g)_{L^2((0,\infty),\rho)} = \\ = \int_0^\infty \rho(dx)\,\overline{f(x)} \int_0^\infty \rho(dy)\,g(y) \lim_{\delta\downarrow 0}\lim_{\varepsilon\downarrow 0}\frac{1}{\pi}\int_{\lambda_1+\delta}^{\lambda_+\delta} d\lambda\,F(\lambda)\tilde{\varphi}(\lambda,x)\tilde{\varphi}(\lambda,y)\Im(\tilde{m}_+(\lambda+i\varepsilon)) \tag{3.2.11}$$

By definition of the measure σ we have for every $h\in C(\mathbb{R})$

$$\int_{(\lambda_1,\lambda_2]} h(\lambda)\,\sigma(d\lambda) = \lim_{\delta\downarrow 0}\lim_{\varepsilon\downarrow 0}\frac{1}{\pi}\int_{\lambda_1+\delta}^{\lambda_2+\delta} d\lambda\,\Im(\tilde{m}_+(\lambda+i\varepsilon))h(\lambda)$$

and therefore we arrive at

$$(f,F(L)E_L((\lambda_1,\lambda_2])g)_{L^2((0,\infty),\rho)} \\ = \int_0^\infty \rho(dx)\overline{f(x)}\int_0^\infty \rho(dy)g(y)\int_{(\lambda_1,\lambda_2]}\sigma(d\lambda)F(\lambda)\tilde{\varphi}(\lambda,x)\tilde{\varphi}(\lambda,y).$$

□

In the next Theorem we apply our results to diffusion generators, for which the boundary point 0 is an exit boundary.

Theorem 3.2.2. *Assume that 0 is an exit boundary. Then for every $R>0$ there exists an analytic Weyl-Titchmarsh solution $\tilde{\varphi}^R(z,x)$ for L in B_R. Therefore there is a measure σ_R on $\mathbb{R}$, such that for every $F\in C(\mathbb{R})$, every $f,g\in C_c^\infty((0,\infty))$ and every $\lambda_1,\lambda_2\in(-R,R)$ with $\lambda_1<\lambda_2$*

$$(f,F(L)E_L((\lambda_1,\lambda_2])g)_{L^2((0,\infty),\rho)} = (\hat{f},M_F M_{\chi_{(\lambda_1,\lambda_2]}}\hat{g})_{L^2(\mathbb{R},\sigma_R)},$$

where for $h\in C_c^\infty((0,\infty))$

$$\hat{h}(\lambda) = \int_0^\infty \tilde{\varphi}^R(\lambda,x)g(x)\,\rho(dx)$$

Proof. We have to show that there exists an analytic Weyl-Titchmarsh function $\tilde{\varphi}(z,\cdot)$ $(z\in B_R)$. For $z\in B_R$ we construct a solution solution $u^R(z,x)$ of the equation $(L-z)u=0$ in a certain neighborhood U of 0, which for every fixed $x\in U$ is analytic in $z\in B_R$ and which for every fixed $z\in B_R$ satisfies $\lim_{x\to 0}u_R(z,x)=1$. Since the ordinary differential equation $(L-z)u=0$ is linear each $u_R(z,\cdot)$ $(z\in B_R)$ gives rise to a solution of $(L-z)u=0$ not only in U but also in $(0,\infty)$. Thus it is enough to construct $u_R(z,\cdot)$ only in U. This will be done by the usual iteration procedure. For $1>\varepsilon>0$ we choose $\delta=\delta_R>0$ such that for every $z\in B_R$ and $x\in(0,\delta)$

$$\int_0^x \rho(r)^{-1}\,dr\int_r^\delta \rho(s)\,ds<\varepsilon.$$

Set $u_0^R(z,x) = 1$ and recursively $u_{n+1}^R(z,x) = 1 + z\int_0^x \rho(r)^{-1}\,dr \int_r^\delta u_n^R(z,s)\rho(s)\,ds$. By induction one easily shows that the sequence $(u_n^R(z,x))_{n\in\mathbb{N}}$ converges uniformly in $x \in [0,\delta_R]$ and $z \in B_R$. This implies that for every $x \in [0,\delta_R]$ the limit $u^R(z,x) = \lim_{n\to\infty} u_n^R(z,x)$ is continuous and for every fixed x is analytic in $z \in B_R$. Moreover the limit satisfies

$$u_R(z,x) = 1 + 2z\int_0^x \rho(r)^{-1}\,dr \int_r^\delta u_n^R(z,s)\rho(s)\,ds$$

and solves for every $z \in B_R$ the equation $(L-z)u = 0$ in $(0,\delta_R)$ with $\lim_{x\to 0} u^R(z,x) = 1$ $(z \in B_R)$. Moreover $u^R(\lambda,x) \in \mathbb{R}$ for $\lambda \in \mathbb{R}\cap B_R$. Now choose $\tilde{\delta}_R > 0$ such that $u^R(z,x) \neq 0$ for every $(z,x) \in B_R \times (0,\tilde{\delta}_R)$ and set for every $x \in (0,\tilde{\delta}_R)$

$$\tilde{\varphi}^R(z,x) = u^R(z,x)\int_0^x u^R(z,y)^{-2}\rho(y)^{-1}\,dy.$$

Observe that $\tilde{\varphi}^R(\lambda,x) \in \mathbb{R}$ for real λ and that $\lim_{x\to 0}\tilde{\varphi}^R(z,x) = 0$. Furthermore $\tilde{\varphi}^R(z,x)$ is analytic in $z \in B_R$ since for any simple closed path γ in B_R due to the analyticity of $u^R(z,x)$ in $z \in B_R$

$$\int_\gamma \left(\int_0^x u^R(z,s)^{-2}\rho(s)^{-1}\,ds\right)dz = \int_0^x \left(\int_\gamma u^R(z,s)^{-2}\,dz\right)\rho(s)^{-1}\,ds = 0$$

Morera's theorem then gives the analyticity of $\tilde{\varphi}^R(z,x)$ in $z \in B_R$. The two solutions $u^R(z,\cdot)$ and $\tilde{\varphi}^R(z,\cdot)$ are linear independent and the solution $\tilde{\varphi}(z,\cdot)$ is moreover integrable since

$$\left|\int_0^x \tilde{\varphi}^R(z,t)\rho(t)\,dt\right| \le \frac{\max_{y\in[0,x]}|u^R(z,y)|}{[\min_{y\in[0,x]}|u^R(z,y)|]^2}\int_0^x \rho(t)\int_0^t \rho(s)^{-1}\,ds < \infty$$

by the definition of an exit boundary. It remains to observe that $\tilde{\varphi}^R(z,\cdot) \in L^\infty((0,1),\rho)$. Since bounded and integrable functions are square integrable we get for every $z \in B_R$

$$\int_0^a |\tilde{\varphi}^R(z,x)|^2\,\rho(dx) < \infty$$

for every $a \in (0,\infty)$. The remaining assertion follows directly from Theorem 3.2.1. □

Remark 3.2.1. *The assertion of Theorem 3.2.2 is important for our problem since it provides the function $\tilde{\varphi}^R(\lambda_0^\kappa(,cdot)$ as a natural candidate for the quasistationary distribution of the diffusion associated to the diffusion operator L. Without an assertion similar to Theorem 3.2.2 such a natural choice is only possible in the case, where the bottom of the spectrum is an eigenvalue in the L^2-sense.*

The proof of the above theorem has the following Corollary

Corollary 3.2.1. *Assume that 0 is an exit boundary. Then for every $\lambda \in \mathbb{C}$ the ordinary differential equation $(\tau-\lambda)u = 0$ has a fundamental system $(u_\lambda^1, u_\lambda^2)$ of solutions with*

$$\lim_{x\to 0} u_\lambda^1(x) = 0 \ \ \textit{and} \ \ \lim_{x\to 0} u_\lambda^2(x) = 1.$$

Moreover one has $u_\lambda^1 \in L^1((0,\infty),\rho)\cap L^2((0,\infty),\rho)$.

Proposition 3.2.1. *Let $\tau = -\frac{1}{2}\frac{d^2}{dx^2} + b(x)\frac{d}{dx}$ be a Sturm-Liouville expression, which is in the limit-point case at infinity. Moreover assume that 0 is and exit boundary and let L denote the uniquely determined selfadjoint extension of τ in $L^2((0,\infty),\rho)$. If $\lambda_0 = \inf \sigma(L)$ and $R > 2\lambda_0$ then the function $\tilde{\varphi}^R(\lambda_0,\cdot)$ does not change sign in $(0,\infty)$.*

Proof. Assume that $\lambda_0 = \inf\sigma(L)$ is an eigenvalue in the sense of spectral theory. Then since e^{-tL} is positivity improving we conclude that u_{λ_0} can be chosen to be strictly positive. Thus assume that λ_0 is the bottom of the essential spectrum and suppose that $\tilde{\varphi}^R(\lambda_0,\cdot)$ has a zero in $(0,\infty)$, say $\tilde{\varphi}^R(\lambda_0,x_0) = 0$. Let us consider the selfadjoint realization L_{x_0} of τ in $L^2((x_0,\infty),\rho)$, which satisfies Dirichlet boundary conditions at x_0. Then $\lambda_0 = \inf\sigma(L_{x_0})$ and $\tilde{\varphi}^R(\lambda_0,\cdot)$ solves the boundary value problem $(\tau-\lambda_0)\tilde{\varphi}^R(\lambda_0,\cdot) = 0$ in (x_0,∞) with $\varphi(\lambda_0,x_0) = 0$ and $\rho\tilde{\varphi}^R(\lambda_0,x_0) \neq 0$. It follows from Theorem 3.2.4 (see also [78]) later in this work that the spectrum of L_x is discrete and therefore λ_0 is an eigenvalue. Thus $\tilde{\varphi}^R(\lambda_0,\cdot)$ has no zero in $(0,x_0)$. Using Lemma 2.2 of [84] we conclude that $\tilde{\varphi}(\lambda_0,\cdot)$ does not change sign in (x_0,∞). Therefore we may assume that in $(0,x_0)$ $\tilde{\varphi}(\lambda_0,\cdot)$ is positive and in (x_0,∞) is negative. Choose $0 \le f \in C_c^\infty((0,x_0))$ and $g \in C_c^\infty((x_0,\infty))$. Then we have $\lim_{R>\lambda\downarrow\lambda_0}\int\tilde{\varphi}^R(\lambda,x)f(x)\,\rho(dx)\int\tilde{\varphi}^R(\lambda,x)g(x)\,\rho(dx) < 0$. Choose $\varepsilon > 0$ and $\lambda_1 < R$ such that for every $\lambda \in [\lambda_0,\lambda_1]$ we have

$$\int\tilde{\varphi}^R(\lambda,x)f(x)\,\rho(dx)\int\tilde{\varphi}^R(\lambda,x)g(x)\,\rho(dx) < -\varepsilon < 0.$$

Then we get the contradiction

$$\begin{aligned}0 \le (g,e^{-tL}f)_{L^2((0,\infty),\rho)} &= \int_{[\lambda_0,\lambda_1)} e^{-t\lambda}\hat{f}(\lambda)\hat{g}(\lambda)\,\sigma(d\lambda)\\ &\quad + (E((\lambda_1,\infty)g, e^{-tL}E((1\lambda_0,\infty)f)_{L^2((0,\infty),\rho)} < 0.\end{aligned}$$

for sufficiently large $t > 0$ since

$$|(E((\lambda_1,\infty)g, e^{-tL}E((\lambda_1,\infty)f)_{L^2((0,\infty),\rho)}| \le e^{-t\lambda_1}\|f\|_{L^2((0,\infty),\rho)}\|g\|_{L^2(((0,\infty),\rho)}.$$

□

In the sequel we comment on the direct applicability of the results of Gesztesy and Zinchenko. Let for some $b \in C^((0,\infty)$ ρ be given by $\rho(x) = e^{-2\int_1^x b(s)\,ds} = e^{-B(x)}$ and let $\tau f = -\frac{1}{2\rho}(\rho f')'$ be a Sturm-Liouville expression, for which 0 is an exit boundary and infinity is inaccessible. Then we know already that the differential expression τ is in the limit point case at 0 and infinity. Thus τ has a unique selfadjoint extension L. Let the unitary mapping $U : L^2((0,\infty)) \to L^2((0,\infty),\rho)$ be given by $Uf(x) = e^{B(x)/2}f(x)$. The unitary U transforms the operator L into the operator $\tilde{L}$, which acts on $f \in C_c^\infty((0,\infty))$ by

$$\tilde{L}f = -\frac{1}{2}f'' + \frac{1}{2}\big(b^2 - b'\big)f. =: \frac{1}{2}\big(-f'' + Vf\big). \tag{3.2.12}$$

Thus we end up we a differential expression of Schrödinger type, for which Gesztesy and Zinchenko established their results. As already remarked above we are interested in drifts

b, which are singular at 0 with a singularity, which is of Bessel type. Thus we consider drifts b of the form

$$b(x) = \frac{a}{x} + c(x), \tag{3.2.13}$$

where $x \in \mathbb{R}$ and $c \in C^1((0,\infty)) \cap L^2_{loc}([0,\infty))$ with $c' \in L^1_{loc}([0,\infty, x\,dx)$. For such a drift b our potential V in (3.2.12) has the form

$$V(x) = \frac{a^2}{x^2} + \frac{a}{x^2} + c(x)\frac{2a}{x} + c(x)^2 - c'(x) = \frac{(a+1/2)^2 - 1/4}{x^2} + W(x). \tag{3.2.14}$$

Using the fact that $\tilde{L} \restriction C_c^\infty((0,\infty))$ is essentially selfadjoint, Example 3.10 in [47], Theorem 3.5 in [47] and Theorem 3.6 in [47] we arrive at the analogue of the Weyl spectral representation for the regular case (see Theorem 2.2.1)

Theorem 3.2.3. *Let V be given as in 3.2.14 and assume that $a \geq \frac{1}{2}$. Let $f, g \in C_c^\infty((0,\infty))$, $F \in C(\mathbb{R})$ and $\lambda_1, \lambda_2 \in \mathbb{R}$, $\lambda_1 < \lambda_2$. Then there exists a continuous function $\mathbb{R} \times (0,\infty) \ni (\lambda, x) \mapsto \tilde{\psi}(\lambda, x)$ satisfying $\tilde{L}\tilde{\psi}(\lambda, x) = \tilde{\psi}(\lambda, x)$ in the sense of the theory of ordinary differential equations and a measure σ such that*

$$\big(f, F(\tilde{L})E_{\tilde{L}}((\lambda_1,\lambda_2])g\big)_{L^2((0,\infty),dx)} = \big(\hat{f}, M_F M_{\mathbf{1}_{(\lambda_1,\lambda_2]}}\hat{g}\big)_{L^2(\mathbb{R},\sigma)},$$

where for $h \in C_c^\infty((0,\infty))$

$$\tilde{U}h(\lambda) = \hat{h}(\lambda) = \int_0^\infty dx\, \tilde{\psi}(\lambda, x)h(x)$$

and M_G denotes again the maximally defined operator of multiplication by the $d\sigma$-measurable function G in the Hilbert space $L^2(\mathbb{R},\sigma)$. The operator $\tilde{U}$ extends canonically to a unitary mapping such that

$$\tilde{U}F(\tilde{L})\tilde{U}^{-1} = M_F.$$

in $L^2(\mathbb{R},\sigma)$. Moreover the spectrum is simple and

$$\sigma(F(\tilde{L})) = ess.ran_\sigma(F) \text{ and } \sigma(\tilde{L}) = supp(\sigma).$$

Using our unitary operator U we can now transform the spectral represention of $\tilde{L}$ obtained in Theorem 3.2.3 back in order to obtain a spectral representation of L. Since L is unitarily equivalent to $\tilde{L}$ we conclude first that the spectrum of the operator L is simple. Setting $Y(z,x) = W(x) - 2z$ we conclude by formula (3.106) in [47] that for every compact $K \subset \mathbb{C} \times [0,\infty)$ and $(z,x) \in K$

$$\big|\tilde{\psi}(z,x)\big| \leq x^{1/2+\gamma} \exp\left(\frac{1}{\gamma}\int_0^x y|Y(z,y)|\,dy\right), \tag{3.2.15}$$

where $\gamma = (a+1/2)^2$. This estimate implies that $|\tilde{\psi}(z,\cdot)| \in L^2((0,a))$ for every $a \in (0,\infty)$. An application of Theorem 3.2.3 implies that for $f, g \in C_c^\infty((0,\infty))$, $F \in C(\mathbb{R})$, $\lambda_1, \lambda_2 \in \mathbb{R}$,

$\lambda_1 < \lambda_2$

$$\begin{aligned}\big(f, F(L)E_L((\lambda_1,\lambda_2])g\big)_{L^2((0,\infty),\rho)} &= \big(U^{-1}f, U^{-1}F(L)E_L((\lambda_1,\lambda_2])UU^{-1}g\big)_{L^2((0,\infty))} \\ &= \big(U^{-1}f, F(\tilde{L})E_{\tilde{L}}((\lambda_1,\lambda_2])U^{-1}g\big)_{L^2((0,\infty))} \\ &= \big(\overline{U^{-1}f}, M_F M_{\mathbf{1}_{(\lambda_1,\lambda_2]}}\overline{U^{-1}g}\big)_{L^2(\mathbb{R},\sigma)},\end{aligned} \tag{3.2.16}$$

where[2]

$$\overline{U^{-1}f}(\lambda) := \int_0^\infty dx\, \tilde{\psi}(\lambda,x)e^{-B(x)/2}f(x) = \int_0^\infty \rho(dx)\, \tilde{\psi}(\lambda,x)e^{B(x)/2}f(x). \tag{3.2.17}$$

Notice that the function $\xi(\lambda,\cdot) := \tilde{\psi}(\lambda,\cdot)e^{B(\cdot)/2}$ solves the ordinary differential equation $L(\tilde{\psi}(\lambda,x)e^{B(x)/2}) = \lambda\tilde{\psi}(\lambda,x)e^{B(x)/2}$ and that (3.2.15) gives bounds on the generalized eigenfunctions $\xi(\lambda,\cdot)$ in a neighborhood of 0. Thus we arrive at an expansion into generalized eigenfunction as in the regular case with the advantage that there is no restriction of the type $|\lambda_1|, |\lambda_2| < R$ but with the drawback that we have to assume that (3.2.13) hold. We hope that in a future work Theorem 3.2.2 will allow to remove hypotheses concerning the drift further, since this result only uses assumptions which are formulated in terms of the Feller classification.

In the rest of this chapter we only consider diffusions whose drift b has the form (3.2.13), since later we even have to restrict our class of diffusion further by adding another condition. Thus we can use Theorem 3.2.2 as well as Theorem 3.2.3. Finally let us end this section with a result concerning the case, where 0 is an exit boundary and ∞ is an entrance boundary. In Theorem 2.2.4 we included a simple proof of the fact that the spectrum of L is discrete, if 0 is regular and ∞ is entrance. After having worked out the proof of this theorem and the next theorem we found out that in the preprint [78] I. Shigekawa establishes a similar result using quite different techniques.

Theorem 3.2.4. *Assume that* 0 *is an exit boundary and* ∞ *is entrance. Then the spectrum of* L *is purely discrete and the eigenfunctions belong to* $L^1((0,\infty),\rho)$.

Proof. Since we already know from Theorem 2.2.4 that for every $a \in (0,\infty)$ the spectrum of L_a is discrete, where L_a denotes the operator associated to the closure of the quadratic form q_a

$$C_c^\infty((0,\infty)) \ni f \mapsto q_a(f) = \frac{1}{2}\int_a^\infty |f(t)|^2\rho(dt),$$

it is enough to prove that the spectrum of the operator L^a is discrete, where the operator L^a is associated to closure of the quadratic form

$$C_c^\infty((0,\infty)) \ni f \mapsto q^a(f) = \int_0^a |f(t)|^2\rho(dt).$$

[2]In contrast to the following chapters $\overline{z}$ does not denote complex conjugation of a complex number z

Without loss of generality assume that $1 \in (0,a)$. The associated diffusion has a regular boundary at a and an exit boundary at 0. In order to simplify the notation we prove the result for the converse situation, i.e. we assume that 0 is regular and a is exit. Again, the main tool is the result of Hartmann (see e.g. Satz 1.1 in [94]), which says that the essential spectrum is empty if for every $\lambda \in \mathbb{R}$ every solution of the equation $(\tau - \lambda)u = 0$ has only finitely many zeros in $(0,a)$. It is enough to prove this for $\lambda > 0$ since the assertion for $\lambda \leq 0$ then follows from Korollary 14.20 b) in [94]. Thus assume that $\lambda > 0$ and that v is a non-trivial solution of the equation $(\tau - \lambda)u = 0$ and assume that v has infinitely many zeros in $(0,a)$. Observe that no x_0 with $v(x_0) = 0$ is a local extremum since v is assumed to be a nontrivial solution. Between two successive zeros there is necessarily an extremum. As $\lambda > 0$ and v satisfies $\tau v = \lambda v$ local maxima of v are necessarily positive and local minima negative. Thus we can choose a sequence $(x_n)_n$ converging to a and $(\tilde{x}_n)_{n\in\mathbb{N}}$ such that x_n is a maximum with $v(x_n) > 0$ between two successive zeros and a sequence $(\tilde{x}_n)_{n\in\mathbb{N}}$ such that $\tilde{x}_n$ is the first zero of v which is bigger than x_n and that v is non-increasing in $(x_n, \tilde{x}_n)$. This gives

$$0 < v(x_n) = \int_{x_n}^{\tilde{x}_n} \rho(s)^{-1}\, ds \int_{x_n}^{s} \rho(t) 2\lambda v(t)\, dt \leq 2\lambda v(x_n) \int_{x_n}^{\tilde{x}_n} \rho(s)^{-1}\, ds \int_{x_n}^{s} \rho(s)\, dt$$

and therefore

$$\frac{1}{2\lambda} \leq \int_{x_n}^{\tilde{x}_n} \rho(s)^{-1}\, ds \int_{x_n}^{s} \rho(t)\, dt. \tag{3.2.18}$$

Since a is assumed to be an exit boundary we have $\int_1^a \rho(s)^{-1} \int_1^s \rho(t)\, dt < \infty$, (3.2.18) can not be true. Thus every solution of the eigenvalue equation $(\tau - \lambda)u = 0$ has only a finite number of zeros in $(0,a)$. Summarizing we have shown that the spectrum of L is purely discrete.

It remains to prove the integrability of the eigenfunction u_{λ_0} corresponding to the lowest eigenvalue λ_0 of L. First observe that $\lim_{x\to 0} u_{\lambda_0}(x) = 0$. This follows from the fact that by definition of L there exists a sequence $(\varphi_n)_{n\in\mathbb{N}} \subset C_c^\infty((0,\infty))$ which converges to u_{λ_0} with respect to the norm

$$\mathcal{D}(q) \ni f \mapsto \left(\|f\|^2_{L^2((0,\infty),\rho)} + \int_0^\infty |f'(x)|^2\, \rho(dx) \right)^{\frac{1}{2}}.$$

The elementary inequality

$$\sup_{x\in([0,a])} |\varphi(x)| \leq C_a \left(\int_0^\infty |\varphi'(x)|^2\, \rho(dx) \right)^{\frac{1}{2}}$$

implies that for $a \in (0,\infty)$ $\|\varphi_n - \varphi_m\|_{C([0,a))} \to 0$ as $n, m \to \infty$ and therefore $\lim_{x\to 0} u_{\lambda_0}(x) = 0$. We have seen in the proof of Theorem 3.2.2, that there exists a solution v_{λ_0} of the equation $(L - \lambda_0)u = 0$ with the property that $\lim_{x\to 0} v_{\lambda_0}(x) = 1$ and a solution $\tilde{v}_{\lambda_0}$ with $\lim_{x\to 0} \tilde{v}_{\lambda_0}(x) = 0$ and $\tilde{v}_{\lambda_0} \in L^1((0,a),\rho)$ $(a \in (0,\infty))$ such that $(v_{\lambda_0}, \tilde{v}_{\lambda_0})$ forms a basis of the space of solutions of $(L-\lambda_0)u=0$. Due to the basis property of $(v_{\lambda_0},\tilde{v}_{\lambda_0})$ we have $u_{\lambda_0}=av_{\lambda_0}+b\tilde{v}_{\lambda_0}$ for some scalars a,b. But because of

$$0=\lim_{x\to 0}u_{\lambda_0}(x)=\lim_{x\to 0}\big(av_{\lambda_0}(x)+b\tilde{v}_{\lambda_0}(x)\big)=a$$

we get $u_{\lambda_0}=b\tilde{v}_{\lambda_0}$. Thus we conclude that $u_{\lambda_0}\in L^1((0,a),\rho)$. Since ∞ is entrance we moreover have $\int_1^\infty \rho(dy)<\infty$ and therefore

$$u_{\lambda_0}\in L^1((1,\infty),\rho)\subset L^2((1,\infty),\rho).$$

Here we used that u_{λ_0} is an L^2-eigenvalue. All in all we have shown that $u_{\lambda_0}\in L^1((0,\infty),\rho)$. □

In the following Corollary we formulate a direct consequence of the above proof.

Corollary 3.2.2. *Assume that* 0 *is an exit boundary and that the boundary point infinity is inaccessible. Let L be the unique selfadjoint realization of the Sturm-Liouville expression* τ *defined by* $\tau f=-\frac{1}{2\rho}\frac{d}{dx}\big(\rho\frac{df}{dx}\big)$. *For any* $f\in\mathcal{D}(\sqrt{L})$ *we have*

$$\lim_{x\downarrow 0}|f(x)|=0.$$

Observe that this Corollary allows us to use methods as in the regular case. If $f\in\mathcal{D}(\sqrt{L})$ then we have for every $x>\varepsilon>0$

$$\begin{aligned}|f(x)|&=\lim_{\varepsilon\to 0}\big|f(x)-f(\varepsilon)\big|\le\lim_{\varepsilon\to 0}\int_\varepsilon^x|f'(s)|\,ds\\&\le\lim_{\varepsilon\to 0}\left[\left(2\int_\varepsilon^x\rho(t)^{-1}\,dt\right)^{\frac12}\left(\frac12\int_\varepsilon^x|f'(s)|^2\,\rho(ds)\right)^{\frac12}\right]\\&=\left(2\int_0^x\rho(t)^{-1}\,dt\right)^{\frac12}\left(\frac12\int_0^x|f'(s)|^2\,\rho(ds)\right)^{\frac12},\end{aligned}$$

where we used the fact that $\int_0^x\rho(t)^{-1}\,dt<\infty$ since 0 is exit.

Remark 3.2.2. *In [20] the authors proved that the spectrum is discrete under the assumptions* $b\in C^1((0,\infty))$ *and*

$$C=-\inf_{y\in(0,\infty)}(b(y)^2-b'(y))<\infty\;\;\textit{and}\;\;\lim_{y\to\infty}(b(y)^2-b'(y))=\infty.$$

applying some facts from the theory of Schrödinger operators to the Schrödinger operator $-\frac12\frac{d^2}{dx^2}+\frac{b^2-b'}{2}$, *which is unitarily equivalent to L. Similarly we also used ideas from the theory of Schrödinger operators, but translated them in a suitable form. In order to get a complete characterization of the existence of quasistationary distributions our result is more useful.*

3.3 Existence of Quasistationary Distributions

As in the regular case we are interested in the existence of quasistationary distributions. As explained we will assume that 0 is an exit boundary and thus have to work with diffusions with two singular boundary points. As in chapter 1 we prove existence of quasistationary distributions by proving the existence of the Yaglom limit. At several points the technical problems become much more involved and we are not able to solve the problem in complete generality. Still our results are new and extend the only existing work [20] on this problem in some respect and together with the uniqueness result in [20] a better understanding of the problem is obtained.

We assume for the rest of this chapter that the condition

(H) $\exists z \in (0,\infty)\,\forall t>0 : e^{-tL}\mathbf{1}_{[0,z]} \in L^2((0,\infty),\rho)$
and $\lim_{\varepsilon\to 0} e^{-tL}\mathbf{1}_{[\varepsilon,z]} = e^{-tL}\mathbf{1}_{[0,z]}$ in $L^2((0,\infty),\rho)$

is satisfied.

Assuming certain absorption at 0 condition (H) will allow to establish the existence of quasistationary distributions under the assumption of strict positivity of the bottom of the spectrum. In the following Proposition we will find a class of diffusions which satisfies (H). We make use of the same methods as in Proposition 2.2 of [20].

Proposition 3.3.1. *Assume that $b \in C^1(0,\infty)$ and that for some $c>0$ $\inf_{s>0}(|b(s)|^2 - b'(s)) > c$. If*

$$\int_0^1 s\sqrt{\rho(s)}\,ds < \infty$$

then

$$e^{-tL}\mathbf{1}_{[0,\varepsilon]} \in L^2((0,\infty),\rho).$$

Moreover for any sequence $(a_n)_{n\in\mathbb{N}} \subset (0,,\varepsilon)$ with $lim_{n\to\infty} a_n = 0$ we have

$$\lim_{n\to\infty} e^{-tL}\mathbf{1}_{[a_n,\varepsilon]} = e^{-tL}\mathbf{1}_{[0,\varepsilon]}$$

in $L^2((0,\infty),\rho)$.

Proof. First observe that it is enough to prove $e^{-tL}\mathbf{1}_{[0,\varepsilon]} \in L^2((0,\infty),\rho)$ for $1>\varepsilon>0$. By the Girsanov theorem (see also proposition 2.2 in [20]) we have

$$\begin{aligned} e^{-tL}\mathbf{1}_{[0,\varepsilon]}(x) &= \mathbb{E}_x^{BM}\left[\mathbf{1}_{[0,\varepsilon]}(X_t)e^{\frac{1}{2}Q(x)-\frac{1}{2}Q(X_t)-\frac{1}{2}\int_0^t(b^2-b')(X_s)\,ds}, t<T_0\right] \\ &\le e^{-\frac{c}{2}t}e^{\frac{1}{2}Q(x)}\int_0^\varepsilon p^D(t,x,y)e^{-\frac{1}{2}Q(y)}\,dy \end{aligned} \tag{3.3.1}$$

where $\mathbb{E}^{BM}$ denotes the expectation with respect to the Brownian motion, $Q(x) = 2\int_1^x b(s)\,ds$ and $p^D(t,x,y)$ denotes the heat kernel for the Laplacian on $(0,\infty)$ with Dirichlet boundary conditions at 0, i.e.

$$p^D(t,x,y) = \frac{1}{\sqrt{2\pi t}}\left(e^{\frac{(x-y)^2}{2t}} - e^{\frac{(x+y)^2}{2t}}\right) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}} e^{-\frac{y^2}{2t}} = \frac{\sqrt{2}}{\sqrt{\pi t}} e^{-\frac{x^2}{2t}} e^{-\frac{y^2}{2t}} \sinh\left(\frac{xy}{t}\right)$$

Using $\sinh\left(\frac{xy}{t}\right) \le y \sinh\left(\frac{x}{t}\right) \le \frac{y}{2} e^{\frac{x}{t}}$ (for $y \in (0,1)$ and $x \in (0,\infty)$) we therefore get by 3.3.1

$$e^{-tL}\mathbf{1}_{[0,\varepsilon]}(x) \le e^{-\frac{c}{2}t} e^{\frac{1}{2}Q(x)} \frac{\sqrt{2}}{\sqrt{\pi t}} e^{-\frac{x^2}{2t}} e^{\frac{x}{t}} \int_0^\varepsilon y e^{-\frac{1}{2}Q(y)}\,dy \tag{3.3.2}$$

The estimate 3.3.2 implies $e^{-tL}\mathbf{1}_{[0,\varepsilon]} \in L^2((0,\infty),\rho)$. □

As explained in Remark 4.6 in [20] Proposition 3.3.1 is applicable to drifts of the form $b(s) = \frac{a}{x} + g(x)$, where $g \in C^1([0,\infty))$ and $0 \le a < 2$. In order to get a more complete picture it is desirable to include regular perturbations of arbitrary Bessel processes, for which 0 is an exit boundary. A step in this direction is the following proposition. The general idea used in the proof is similar to the idea used in Proposition 3.3.1

Proposition 3.3.2. *Assume that $\nu \ge 1$ that $b(s) = -\frac{-2\nu+1}{2s} + c(s)$ where $c \in C^1([0,\infty))$ is such that $\inf_{x>0}\left[\frac{1}{2}(c^2(x) - c'(x)) - \frac{(-2\nu+1)c(x)}{x}\right] > -\infty$. Moreover assume that $\mathbb{P}_x(T_0 < \infty) = 1$. Then*

$$e^{-tL}\mathbf{1}_{[0,z]} \in L^2((0,\infty),\rho)$$

for every $z \in (0,\infty)$. Moreover we have

$$\lim_{\varepsilon\to 0} e^{-tL}\mathbf{1}_{[\varepsilon,z]} = e^{-tL}\mathbf{1}_{[0,z]}$$

in $L^2((0,\infty),\rho)$.

Proof. Obviously we can without loss of generality assume $0 < z < 1$, since we already know that for every $z' \in [1,\infty)$

$$e^{-tL}\mathbf{1}_{[z,z']} \in L^2((0,\infty),\rho).$$

Denote by I_ν the modified Bessel function of the first kind, i.e. (see [12] page 638)

$$I_\nu(x) = \sum_{k=0}^{\infty} \frac{(x/2)^{\nu+2k}}{k!\Gamma(\nu+k+1)},$$

where Γ denotes the Γ-function. Let R^ν denote the Bessel process, i.e. the minimal diffusion associated to the generator $-\frac{1}{2}\frac{d^2}{dx^2} - \frac{2\nu+1}{2x}\frac{d}{dx}$. For $\nu \ge 1$ the process R^ν has the transition function (see [12] page 73 formula 44)

$$p^\nu(t,x,y) = \frac{1}{2t}(xy)^{-\nu} e^{-\frac{x^2+y^2}{2t}} I_\nu\left(\frac{xy}{t}\right)$$

with respect to the measure $m^\nu(dx) = 2x^{2\nu+1}dx$. Let $\mathbb{P}^{\pm\nu}$ denote the measure associated to $R^{\pm\nu}$. From the h-transform property (compare [12] page 75 formula 43) we get that the transition function $p^{-\nu}(t,x,y)$ for the process $R^{-\nu}$ with respect to the measure $m^\nu(dx) = 2x^{2\nu+1}dx$ is given by

$$\begin{aligned} p^{-\nu}(t,x,y) &= \frac{1}{h(x)} p^\nu(t,x,y)h(y) = x^{2\nu}\frac{1}{2t}(xy)^{-\nu}e^{-\frac{x^2+y^2}{2t}} I_\nu\left(\frac{xy}{t}\right)y^{-2\nu} \\ &= \frac{1}{2t}\, x^\nu e^{-\frac{x^2+y^2}{2t}} I_\nu\left(\frac{xy}{t}\right)y^{-3\nu}, \end{aligned} \tag{3.3.3}$$

where $h(x) = \frac{1}{2\nu}x^{-2\nu}$. Set $C(x) = \int_1^x c(s)\,ds$. Let $\tilde\rho$ denote the symmetrizing measure of the Sturm-Liouville expression $\tilde\tau_B = -\frac{1}{2}\frac{d^2}{dx^2} - \frac{-2\nu+1}{2s}\frac{d}{dx}$. For $f \in L^2((0,\infty),\tilde\rho)$ define the operators $\tilde P_t$ by $\tilde P_t f(x) = e^{-C(x)/2}(P_t(e^{C/2}f))(x)$. Since $\int_0^\infty |fe^{-C/2}(x)|^2\,\tilde\rho(dx) = \int_0^\infty |f(x)|^2\,\rho(dx) < \infty$ the expression $P_t(e^{C/2}f)$ is well-defined and we have $\tilde P_t f \in L^2((0,\infty),\tilde\rho)$. We see by a simple calculation that the generator $\tilde L$ of $\tilde P_t$ acts on smooth functions with compact support via

$$\tilde L f(x) = \tau_B f(x) + \left[\frac{1}{2}(c^2(x) - c'(x)) - \frac{(-2\nu+1)c(x)}{x}\right]f(x).$$

Since $\tilde L \restriction C_c^\infty((0,\infty)$ is essentially selfadjoint if this is true for the operator L we conclude that it is enough to check the action of $\tilde L$ on $C_c^\infty((0,\infty)$. Applying the Feynman-Kac formula and the assumptions of the proposition and (3.3.3) we get for some constants K_t, $\tilde K_t$

$$\begin{aligned} F_t(x) := e^{-tL}\mathbf{1}_{[0,\varepsilon]}(x) &= e^{C(x)/2}\tilde P_t(e^{-C(x)/2}\mathbf{1}_{[0,\varepsilon]})(x) = e^{C(x)/2}e^{-t\tilde L}(e^{-C(x)/2}\mathbf{1}_{[0,\varepsilon]})(x) \\ &= e^{C(x)/2}\mathbb{E}_x^{-\nu}\left[e^{-\int_0^t[\frac{1}{2}(c^2(X_s)-c'(X_s))-\frac{(-2\nu+1)c(X_s)}{X_s}]\,ds}e^{-C(x)/2}\mathbf{1}_{[0,\varepsilon]}(X_t), T_0 > t\right] \\ &\le e^{C(x)/2}K_t\mathbb{E}_x^{-\nu}\left[\mathbf{1}_{[0,\varepsilon]}(X_t), T_0 > t\right] \\ &= e^{C(x)/2}K_t\int_0^\varepsilon p^{-\nu}(t,x,y)(y)y^{2\nu+1}\,dy \\ &\le e^{C(x)/2}K_t\int_0^\varepsilon \frac{1}{2t}\,x^\nu e^{-\frac{x^2+y^2}{2t}}I_\nu\left(\frac{xy}{t}\right)y^{-3\nu}y^{2\nu+1}\,dy \\ &\le e^{\frac{C(x)}{2}}\tilde K_t\frac{x^\nu}{2t}e^{-\frac{x^2}{2t}}\int_0^\varepsilon e^{-\frac{y^2}{2t}}I_\nu\left(\frac{xy}{t}\right)y^{-3\nu}y^{2\nu+1}\,dy. \end{aligned}$$

Using

$$I_\nu\left(\frac{xy}{t}\right) = \sum_{k=0}^\infty \frac{(xy/2t)^{\nu+2k}}{k!\Gamma(\nu+k+1)} \le y^\nu\sum_{k=0}^\infty\frac{(x/2t)^{\nu+2k}}{k!\Gamma(\nu+k+1)} = y^\nu I_\nu(x/t)$$

we get for some constant $K_t^1 > 0$

$$F_t(x) \le K_t^1 e^{\frac{1}{2}C(x)}\frac{x^\nu}{2t}e^{-\frac{x^2}{2t}}I_\nu(x/t). \tag{3.3.4}$$

Near 0 the density $\rho(x) = e^{-2\int_1^t \frac{-(-2\nu+1)}{2s} + c(s)\, ds}$ behaves as $e^{-C(x)}x^{-2\nu+1}$ and near ∞ the density ρ behaves as $e^{-C(x)}x^{-2\nu+1}$. Using the asymptotics (see page 638 in [12])

$$I_\nu(z) \sim \frac{1}{\Gamma(\nu+1)}\left(\frac{z}{2}\right)^\nu \text{ for small } z \text{ and } I_\nu(z) \sim \frac{e^z}{\sqrt{2\pi z}} \text{ for large } z$$

we obtain

$$\int_0^\infty |F_t(x)|^2 \,\rho(dx) \le \int_0^\infty |K_t^1 e^{\frac{1}{2}C(x)} \frac{x^\nu}{2t} e^{-\frac{x^2}{2t}} I_\nu(x/t)|^2 \rho(dx) < \infty.$$

□

Remark 3.3.1. *Despite our permanent effort we have not been able to give necessary and sufficient conditions, which ensure* $e^{-tL}\mathbf{1}_{[0,z]} \in L^2((0,\infty),\rho)$. *Such a result would clarify the status of our hypotheses. In the above results we have seen, that this holds in many interesting cases, in particular it is always satisfied under the assumptions of [20]. In a subsequent joint work with Leif Döring we try to extend the above proposition further by making use of Khas'minskii's Lemma. It is an interesting task to search for more general and if possible even optimal assumptions which imply the validity of (H).*

The following result is a local version of the Yaglom limit and is the first important step in the analysis of the full problem. Recall that we always assume the validity of (H) and notice that in the right side of the assertion does not depend on R. This is so since for different R, R' the solutions $\tilde{\varphi}^R(\lambda_0^\kappa,\cdot)$ and $\tilde{\varphi}^{R'}(\lambda_0,\cdot)$ are linearly dependent by Corollary 3.2.1 and therefore the quotient in the assertion does not change, if one replaces in the assertion $\tilde{\varphi}^R(\lambda_0^\kappa,\cdot)$ by $\tilde{\varphi}^{R'}(\lambda_0,\cdot)$. Moreover as observed in (3.2.16) and (3.2.17) $\tilde{\varphi}^R(\lambda_0^\kappa,\cdot)$ can replaced by $\tilde{\psi}(\lambda_0,\cdot)e^{B(\cdot)/2}$.

Theorem 3.3.1. *Assume that* 0 *is an exit boundary and* $A \subset B$ *are compact subsets of* $[0,\infty)$. *Let* ν *be an initial distribution, which is compactly supported in* $(0,\infty)$.

$$\lim_{t\to\infty} \mathbb{P}_\nu\big(X_t \in A \mid X_t \in B\big) = \frac{\int_A \tilde{\varphi}^R(\lambda_0,x)\rho(dx)}{\int_B \tilde{\varphi}^R(\lambda_0,x)\rho(dx)}.$$

Proof. Let us fix $R > 10|\lambda_0|$ and denote the for $\lambda \in \mathbb{R}$ with $|\lambda| < R$ $\tilde{\varphi}^R(\lambda,\cdot)$ just by $\tilde{\varphi}(\lambda,\cdot)$. The proof relies on the above spectral theoretic results. First observe that it is easy to see that for $z > 0$ and a subsets $A, B \subset\subset (0,z]$ with $A \subset B$ we have

$$\lim_{t\to\infty} \mathbb{P}_\nu\big(X_t \in A \mid X_t \in B) = \frac{\int_A \tilde{\varphi}(\lambda_0,z)\,\rho(dz)}{\int_B \tilde{\varphi}(\lambda_0,z)\,\rho(dz)}.$$

This can be done exactly as in the proof of the corresponding result for a diffusion with a regular boundary. An additional problem occurs if $A \subset [0,z]$ due to the fact that $\mathbf{1}_A$ does not necessarily belong to $L^2((0,\infty),\rho)$. Here condition (H) will be used. Let for a subset $A \subset [0,z]$ and $\varepsilon > 0$ A_ε denote the set $\{x \in A \mid |x| > \varepsilon\}$. Then by condition (H) we have for every $t > 0$ $\lim_{\varepsilon\to 0} e^{-tL}\mathbf{1}_{A_\varepsilon} = e^{-tL}\mathbf{1}_A$ in $L^2((0,\infty),\rho)$. This allows to decompose $e^{-tL}\mathbf{1}_A$ as

$$\mathbb{P}_x(X_t \in A) = e^{-tL}\mathbf{1}_A(x) = E([\lambda_0,\lambda_1])e^{-tL}\mathbf{1}_A(x) + E((\lambda_1,\infty))e^{-tL}\mathbf{1}_A(x)$$

for almost every $x \in (0,\infty)$ and due to continuity for every $x \in (0,\infty)$. Observe moreover that for $\lambda_1 \in (0,\infty)$ the continuous integral kernel $h^{\lambda_1}(t,x,y)$ of the operator $E((0,\lambda_1))e^{-tL}$ is given by

$$h^{\lambda_1}(t,x,y) = \int_{[\lambda_0,\lambda_1]} e^{-t\lambda}\tilde{\varphi}(\lambda,x)\tilde{\varphi}(\lambda,y)\,\sigma(d\lambda)$$

If ν is an initial distribution with compact support in $(0,\infty)$, then due to continuity of the functions $e^{-tL}\mathbf{1}_A$, $E([\lambda_0,\lambda_1])e^{-tL}\mathbf{1}_A$ and $E((\lambda_1,\infty))e^{-tL}\mathbf{1}_A$ the expressions

$$\langle e^{-tL}\mathbf{1}_A,\nu\rangle,\ \langle E([\lambda_0,\lambda_1])e^{-tL}\mathbf{1}_A,\nu\rangle \text{ and } \langle E((\lambda_1,\infty))e^{-tL}\mathbf{1}_A,\nu\rangle$$

are welldefined and we get

$$\langle e^{-tL}\mathbf{1}_A,\nu\rangle = \langle E([\lambda_0,\lambda_1])e^{-tL}\mathbf{1}_A,\nu\rangle + \langle E((\lambda_1,\infty))e^{-tL}\mathbf{1}_A,\nu\rangle.$$

and using Fubini's theorem we arrive at

$$\begin{aligned} E([\lambda_0,\lambda_1])e^{-tL}\mathbf{1}_A(x) &= \int_0^\infty h^{\lambda_1}(t,x,y)\mathbf{1}_A\,\rho(dy) \\ &= \int_{[\lambda_0,\lambda_1]} e^{-\lambda t}\int_0^\infty \tilde{\varphi}(\lambda,x)\,\nu(dx)\int_A \tilde{\varphi}(\lambda,y)\,\rho(dy)\sigma(d\lambda) \end{aligned} \tag{3.3.5}$$

The use of Fubini's theorem may be easily justified by using properties of the generalized eigenfunctions $\tilde{\varphi}(\lambda,x)$. Now observe that

$$\lim_{t\to\infty} \frac{\langle E([\lambda_0,\lambda_1])e^{-tL}\mathbf{1}_A,\nu\rangle}{\langle E([\lambda_0,\lambda_1])e^{-tL}\mathbf{1}_{[0,z]},\nu\rangle} = \frac{\int_A \tilde{\varphi}(\lambda_0,x)\,\rho(dx)}{\int_0^z \tilde{\varphi}(\lambda_0,x)\,\rho(dx)} \tag{3.3.6}$$

Using the integrability properties of $\varphi(z,\cdot)$ established in Corollary 3.2.1 and the proof of Theorem 3.2.2 this follows exactly as in the regular case (see Theorem 2.2.3 chapter 2). Thus it remains to show that

$$\lim_{t\to\infty} \mathbb{P}_\nu(X_t \in A \mid X_t \le z) = \lim_{t\to\infty} \frac{\langle E([\lambda_0,\lambda_1])e^{-tL}\mathbf{1}_A,\nu\rangle}{\langle E([\lambda_0,\lambda_1])e^{-tL}\mathbf{1}_{[0,z]},\nu\rangle} \tag{3.3.7}$$

This will be proved using ideas, which are similar to the case of a regular boundary at 0. In order to do so observe that

$$\begin{aligned} \frac{\mathbb{P}_\nu(X_t \in A)}{\langle E([\lambda_0,\lambda_1])e^{-tL}\mathbf{1}_A,\nu\rangle} &= \frac{\langle E([\lambda_0,\lambda_1])e^{-tL}\mathbf{1}_A,\nu\rangle + \langle E((\lambda_1,\infty))e^{-tL}\mathbf{1}_A,\nu\rangle}{\langle E([\lambda_0,\lambda_1])e^{-tL}\mathbf{1}_A,\nu\rangle} \\ &= 1 + \frac{\langle E((\lambda_1,\infty))e^{-tL}\mathbf{1}_A,\nu\rangle}{\langle E([\lambda_0,\lambda_1])e^{-tL}\mathbf{1}_A,\nu\rangle}. \end{aligned}$$

Using $e^{-tL}\mathbf{1}_A \in L^2((0,\infty),\rho)$ and the elementary inequality

$$|g'(x)| \le \left(2\int_0^x \rho(x)^{-1}\,dx\right)^{\frac{1}{2}} \left(\frac{1}{2}\int_0^\infty |g'(x)|^2\rho(dx)\right)^{\frac{1}{2}}$$

valid for $g \in \mathcal{D}(\sqrt{L})$ gives

$$\begin{aligned}\langle E((\lambda_1,\infty))e^{-tL}\mathbf{1}_A,\nu\rangle &\le C_\nu\left(\frac{1}{2}\int_0^\infty |g_t'(x)|^2\rho(dx)\right)^{\frac{1}{2}} \\ &= \left(\int_{[\lambda_1,\infty)} e^{-2(t-\varepsilon)\lambda}\,\|E(d\lambda)e^{-\varepsilon L}\mathbf{1}_A\|^2_{L^2((0,\infty),\rho)}\right)^{\frac{1}{2}},\end{aligned}$$

where $g_t = E((\lambda_1,\infty))e^{-tL}\mathbf{1}_A$ and $\varepsilon > 0$ is small enough. Therefore $t \mapsto \langle E((\lambda_1,\infty))e^{-tL}\mathbf{1}_A,\nu\rangle$ decays exponentially with an exponential rate which is strictly bigger than the exponential rate of decay of $\langle E([\lambda_0,\lambda_1])e^{-tL}\mathbf{1}_A,\nu\rangle$ □

Remark 3.3.2. *Let $\tau f = -\frac{1}{2\rho}(\rho f')'$ be a Sturm-Liouville operator for which is 0 is the exit point case and infinity is inaccessible and let L be the unique selfadjoint realization of τ. Exactly as in the regular case it is possible to deduce the strong ratio limit property if the transition function $p(t,\cdot,\cdot)$ corresponding to e^{-tL}, i.e. for $s > 0$, $x_0 \in (0,\infty)$*

$$\lim_{t\to\infty} \frac{p(t+s,x,y)}{p(t,x_0,x_0)} = e^{-\lambda_0 s}\frac{\tilde\varphi(\lambda_0,x)\tilde\varphi(\lambda_0,y)}{\tilde\varphi(\lambda_0,x_0)\tilde\varphi(\lambda_0,x_0)}, \tag{3.3.8}$$

where is the non-trivial function $\tilde\varphi(\lambda_0,\cdot)$ is an element of the one-dimensional space of solutions of $\tau\tilde\varphi(z,\cdot) = z\tilde\varphi(t,\cdot)$ satisfying $\lim_{x\downarrow 0}\tilde\varphi(z,x) = 0$. Observe that the hypothesis (H) is not necessary in order to deduce the strong ratio limit property.

Having established Mandl's local version of the convergence to quasistationarity we are now able to prove with the methods of Steinsaltz and Evans the dichotomy In the following we use a beautiful argument, which under the assumption, that 0 is regular, is due to Steinsaltz and Evans. Due to Theorem 3.3.1 their reasoning works without substantial changes. In order to make this work self contained we present a full proof.

Lemma 3.3.1. *Let 0 be an exit boundary and let ∞ be natural. Suppose the initial distribution ν is compactly supported and $\lambda_0 > 0$. Then either*

- *X_t converges to the quasistationary distribution $\tilde\varphi(\lambda_0,\cdot)$; or*
- *X_t escapes to infinity.*

Proof. Let

$$f(z,t) := \frac{\mathbb{P}_\nu(X_t > z)}{\mathbb{P}_\nu(X_t \le z)},$$

then

$$\begin{aligned}
\lim_{t\to\infty} f(z,t) = \infty, \ \ \forall z \geq 0 &\iff X_t \text{ escapes to infinity,} \\
\lim_{z\to\infty}\lim_{t\to\infty} f(z,t) = 0 &\iff X_t \text{ converges to the qsd } \varphi(\lambda_0,\cdot).
\end{aligned}$$

Hence, to prove the dichotomy it has to be ensured that no other limits of f than zero and infinity are possible. First note that for fixed $0 \leq a \leq z$

$$\begin{aligned}
\mathbb{P}_\nu(X_{n+1} > z) &= \mathbb{P}_\nu(X_{n+1} \geq a) - \mathbb{P}_\nu(a \leq X_{n+1} \leq z) \\
&\geq \mathbb{P}_\nu(X_{n+1} \geq a | X_n > z)\mathbb{P}_\nu(X_n > z) - \mathbb{P}_\nu(a \leq X_{n+1} \leq z) \\
&\geq \mathbb{P}_z(\forall t \in [0,1] : X_t \geq a)\mathbb{P}_\nu(X_n > z) - \mathbb{P}_\nu(a \leq X_{n+1} \leq z),
\end{aligned}$$

where we used the strong Markov property. Due to naturality of ∞, $\mathbb{P}_z(\forall t \in [0,1] : X_t \geq a)$ converges to 1 as z tends to infinity. Further, using convergence to the quasistationary distribution $\varphi(\lambda_0,\cdot)$ on compacta we get

$$\frac{\mathbb{P}_\nu(a \leq X_{n+1} \leq z)}{\mathbb{P}_\nu(X_{n+1} \leq z)} \overset{n\to\infty}{\to} \frac{\int_a^z \tilde{\varphi}(\lambda_0,x)\,\rho(dx)}{\int_0^z \tilde{\varphi}(\lambda_0,x)\,\rho(dx)} \leq \frac{\int_a^\infty \tilde{\varphi}(\lambda_0,x)\,dx}{\int_0^z \varphi(\lambda_0,x)\,\rho(dx)}.$$

For each $\epsilon > 0$ we can find n', z_0, a such that $\mathbb{P}_z(\forall t \in [0,1] : X_t \geq a) > 1-\epsilon$ and $\frac{\mathbb{P}_\nu(a\leq X_{n+1}\leq z)}{\mathbb{P}_\nu(X_{n+1}\leq z)} < \epsilon$ for $n \geq n', z \geq z_0$. Further, since $\lambda_0 > 0$, Lemma ... can be used to find some n'' such that $\frac{\mathbb{P}_\nu(X_n\leq z)}{\mathbb{P}_\nu(X_{n+1}\leq z)} \geq q > 1/(1-\epsilon)$ for $n \geq n''$ and ϵ small enough. Altogether, for a, n, z large enough

$$\begin{aligned}
f(z,n+1) &= \frac{\mathbb{P}_\nu(X_{n+1} > z)}{\mathbb{P}_\nu(X_{n+1} \leq z)} \\
&\geq \frac{\mathbb{P}_z(\forall t \in [0,1] : X_t \geq a)\mathbb{P}_\nu(X_n > z)}{\mathbb{P}_\nu(X_{n+1} \leq z)} - \frac{\mathbb{P}_\nu(a \leq X_{n+1} \leq z)}{\mathbb{P}_\nu(X_{n+1} \leq z)} \\
&\geq \mathbb{P}_z(\forall t \in [0,1] : X_t \geq a)\frac{\mathbb{P}_\nu(X_n > z)}{\mathbb{P}_\nu(X_n \leq z)}\frac{\mathbb{P}_\nu(X_n \leq z)}{\mathbb{P}_\nu(X_{n+1} \leq z)} - \frac{\mathbb{P}_\nu(a \leq X_{n+1} \leq z)}{\mathbb{P}_\nu(X_{n+1} \leq z)} \\
&\geq q(1-\epsilon)f(z,n) - \epsilon.
\end{aligned}$$

Since ϵ is arbitrary small taking limits on both sides yields

- $\limsup_{n\to\infty} f(n,z) = \infty, \ \ \forall z \geq 0$; or
- $\lim_{z\to\infty}\limsup_{n\to\infty} f(n,z) = 0$.

We still have to extend the above to real times $t \geq 0$. First note that

$$\begin{aligned}
&\lim_{z\to\infty}\limsup_{t\to\infty} \frac{\mathbb{P}_\nu(X_t > z)}{\mathbb{P}_\nu(X_t \leq z)} \\
=\ &\lim_{z\to\infty}\limsup_{t\to\infty} \frac{\mathbb{P}_\nu(X_t > z)}{\mathbb{P}_\nu(X_n \leq z)}\frac{\mathbb{P}_\nu(X_n \leq z)}{\mathbb{P}_\nu(X_t \leq z)} \\
\leq\ &\lim_{z\to\infty}\limsup_{n\to\infty} \frac{\mathbb{P}_\nu(\exists t \in [n,n+1) : X_t > z)}{\mathbb{P}_\nu(X_n \leq z)} \sup_{t\in[n,n+1)} \frac{\mathbb{P}_\nu(X_n \leq z)}{\mathbb{P}_\nu(X_t \leq z)}.
\end{aligned}$$

Further,

$$\begin{aligned}
&\mathbb{P}_\nu(\exists t \in [n, n+1) : X_t > z) \\
= \;& \mathbb{P}_\nu(\exists t \in [n, n+1) : X_t > z, X_n \geq z) + \mathbb{P}_\nu(\exists t \in [n, n+1) : X_t > z, X_n < z) \\
\leq \;& \mathbb{P}_\nu(X_n \geq z) + \mathbb{P}_\nu(\exists t \in [n, n+1) : X_t > z | X_n < z)\mathbb{P}(X_n \leq z) \\
\leq \;& \mathbb{P}_\nu(X_n \geq z) + \mathbb{P}_\nu(\exists t \in [n, n+1) : X_t > z | X_n < z)
\end{aligned}$$

The second summand tends to zero due to convergence to the quasistationary distribution $\varphi(\lambda_0, \cdot)$ on compacta. Indeed,

$$\mathbb{P}_\nu(\exists t \in [n, n+1) : X_t > z | X_n < z) \overset{n\to\infty}{\to} \frac{\int_0^z \mathbb{P}_x(\exists t \in [0,1) : X_t > z)\tilde{\varphi}(\lambda_0, x)\, \rho(dx)}{\int_0^\infty \tilde{\varphi}(\lambda_0, x)\, \rho(dx)}.$$

Without loss of generality we may assume $\int_0^\infty \tilde{\varphi}(\lambda_0, x)\, \rho(dx) < \infty$ since otherwise X_t escapes to infinity. Hence, due to dominated convergence the limit in z can be taken in the inside which tends to zero since ∞ is inaccessible. In total from the above follows $\lim_{z\to\infty} \lim_{t\to\infty} \frac{\mathbb{P}_\nu(X_t > z)}{\mathbb{P}_\nu(X_t \leq z)} = 0$. Finally, the second case is similar. For any $a > z$

$$\begin{aligned}
&\liminf_{t\to\infty} f(z,t) \\
\geq \;& \liminf_{n\to\infty} \frac{\mathbb{P}_\nu(X_n > a)) - \mathbb{P}_\nu(a < X_n, \exists t \in [n, n+1) : X_t \leq z)}{\mathbb{P}_\nu(X_n \leq a) + \mathbb{P}_\nu(a < X_n, \exists t \in [n, n+1) : X_t \leq z)} \\
\geq \;& \liminf_{n\to\infty} \frac{f(a,n)(1 - \mathbb{P}_\nu(\exists t \in [n, n+1) : X_t \leq z | a < X_n)}{1 + f(a,n)\mathbb{P}_\nu(\exists t \in [n, n+1) : X_t \leq z | a < X_n)} \\
= \;& \liminf_{n\to\infty} \mathbb{P}_\nu(\exists t \in [n, n+1) : X_t \leq z | a < X_n)^{-1} - 1,
\end{aligned}$$

where the last equality is true since $\lim_{n\to\infty} f(a,n) = \infty$. Hence, the right-hand side diverges as a tends to infinity since ∞ is natural. Finally, we proved that for all $z \geq 0$, $\lim_{t\to\infty} f(z,t) = \infty$. □

Lemma 3.3.1 gives us an important dichotomy. Either we have escape to infinity or convergence to quasistationarity. Let us first comment on the case where absorption at 0 is not certain.

Theorem 3.3.2. *Assume that* 0 *is an exit boundary and that* $\int_1^\infty \rho(t)^{-1}\, dt < \infty$. *Then we have for every* $a \in (0, \infty)$ *and every compact subset* $K \subset [0, \infty)$

$$\limsup_{t\to\infty} e^{\lambda_0 t} \sup_{x\in[a,\infty)} \mathbb{P}_x\big(X_t \in K \mid T_0 > t\big) < \infty$$

Proof. As we did quite often we use the elementary fact that for fixed $a \in (0, \infty)$ and for every $f \in \mathcal{D}(q)$

$$\begin{aligned}
\sup_{c\in(0,a)} |f(x)| &\leq \left(2\int_0^a \rho(t)^{-1}\, dt\right)^{\frac{1}{2}} \left(\frac{1}{2}\int_0^\infty |f(t)|^2 \rho(dt)\right)^{\frac{1}{2}} \\
&\leq \left(2\int_0^\infty \rho(t)^{-1}\, dt\right)^{\frac{1}{2}} \left(\frac{1}{2}\int_0^\infty |f(t)|^2 \rho(dt)\right)^{\frac{1}{2}},
\end{aligned}$$

where we use our assumption and that 0 is assumed to be an exit boundary. By the spectral calculus we conclude that $e^{-tL}g \in \mathcal{D}(q)$ for every $g \in L^2((0,\infty),\rho)$ and therefore for $t > \varepsilon > 0$, $f := e^{-\varepsilon L}\mathbf{1}_K \in L^2((0,\infty),\rho)$ (by hypothesis (H)) and $C = \int_0^\infty \rho(t)^{-1}\,dt < \infty$

$$\begin{aligned}\sup_{x\in(0,\infty)} \mathbb{P}_x\big(X_t \in K\big) &= \sup_{x\in(0,\infty)} |(e^{-tL}\mathbf{1}_K)'(x)| \le C\Big(\int_0^\infty |(e^{-tL}\mathbf{1}_K)'(x)|^2\rho(dx)\Big)^{\frac12}\\ &= C\Big(\int_0^\infty |(e^{-(t-\varepsilon)L}e^{-\varepsilon L}\mathbf{1}_K)'(x)|^2\rho(dx)\Big)^{\frac12} \qquad (3.3.9)\\ &= C\Big(\int_{[\lambda_0,\infty)} \lambda e^{-2(t-\varepsilon)\lambda}\,d\|E_\lambda f\|^2_{L^2((0,\infty),\rho)}\Big)^{\frac12}.\end{aligned}$$

This inequality therefore gives us

$$\limsup_{t\to\infty} e^{\lambda_0 t}\mathbb{P}_x\big(X_t \in K\big) \le C\limsup_{t\to\infty}\Big(\int_{[\lambda_0,\infty)} \lambda e^{2\lambda_0 t}e^{-2(t-\varepsilon)\lambda}\,d\|E_\lambda f\|^2_{L^2((0,\infty),\rho)}\Big)^{\frac12}.$$

The right hand side is finite and is even equal to 0 if λ_0 does not belong to the point spectrum of L and in particular in the case $\lambda_0 = 0$. Therefore we get for any $\varepsilon > 0$

$$\limsup_{t\to\infty} e^{\lambda_0 t}\sup_{x\in(\varepsilon,\infty)} \mathbb{P}_x\big(X_t \in K \mid T_0 > t\big) < \infty.$$

□

In the sequel we consider only diffusions which are absorbed in 0 with probability one. We are interested in finding sufficient conditions ensuring the existence of quasistationary distributions. In the following Theorem we show in particular that the behavior of $\mathbb{P}_\nu(X_t \in \cdot \mid T_0 > t)$ does not depend on the initial distribution ν. Moreover we prove that convergence to quasistationarity is implied by the strict positivity of the generator of the diffusion.

Theorem 3.3.3. *Assume that* 0 *is an exit boundary and infinity is natural and that* $\mathbb{P}_x(T_0 < \infty) = 1$. *If* $\lambda_0 > 0$ *then* X_t *converges to the quasistationary distribution* $\tilde\varphi(\lambda_0,\cdot)$.

Proof. By Lemma 3.3.1 we know that X_t converges either to the quasistationary distribution or escapes to infinity. This means that the family of measures

$$F_t(\nu,\cdot) = \mathbb{P}_\nu(X_t \in \cdot \mid T_0 > t)$$

converges weakly either to the measure $\tilde\varphi(\lambda_0,\cdot)\,d\rho$ or to δ_∞. In the second case we have for every distribution ν which is compactly supported in $(0,\infty)$

$$\begin{aligned}\lim_{s\to\infty} \mathbb{P}_\nu\big(T_0 > t+s \mid T_0 > s\big) &= \lim_{s\to\infty}\int_0^\infty \mathbb{P}_y(T_0 > t)\mathbb{P}_\nu(X_s \in dy \mid T_0 > s)\\ &= \lim_{y\to\infty} \mathbb{P}_y\big(T_0 > t\big) = 1,\end{aligned}$$

where we used the assumption that ∞ is a natural boundary point. If X_t converges to the quasistationary distribution, then we have for some $\alpha > 0$

$$\lim_{t\to\infty} \frac{\mathbb{P}_x\big(T_0 > t+s\big)}{\mathbb{P}_x\big(T_0 > t\big)} = e^{-\alpha s}$$

and therefore

$$-\lim_{t\to\infty} \frac{1}{t} \log \mathbb{P}_x\big(T_0 > t\big) = \alpha.$$

Thus in order to prove the assertion of the theorem it is enough to show that

$$-\lim_{t\to\infty} \frac{1}{t} \log \mathbb{P}_\nu(T_0 > t) > 0.$$

As in the regular case it remains to investigate the exponential asymptotic behavior of $\mathbb{P}_\nu(T_0 > t)$ but in contrast to the regular case the function $\mathbf{1}$ is not in $L^2((0,\infty),\rho)$ and therefore we have to use a slightly different route. We will apply a well-known martingale technique. The bottom of the spectrum of the operator $L - \lambda_0/2$ is $\lambda_0/2$. Therefore the operator $L - \lambda_0/2$ is subcritical or equivalently the Greens function

$$g(x,y) = \int_0^\infty e^{s\lambda_0/2} p(s,x,y)\, ds$$

exists. Thus by Proposition 5.1.3 in [75] there exist two linearly independent positive solutions u_1, u_2 of the equation $(L - \lambda_0/2)u = 0$. Thus by Corollary 3.2.1 there exists a positive solution $\tilde{u}$ of $(L-\lambda_0/2)u = 0$ with $\lim_{x\to 0+} u(x) = 1$. Consider the stochastic process $(Y_t)_{t\geq 0} = (e^{\lambda_0/2(t\wedge T_0)} u(X_{t\wedge T_0}))_{t\geq 0}$. Then by the Itô formula $(Y_{t\wedge T_M})_{t>0}$ is a martingale with respect to $\mathbb{P}_x$. Therefore we conclude that for $x \in (0,M)$

$$\mathbb{E}_x\big[u(X_{t\wedge T_0 \wedge T_M}) e^{\lambda_0/2(t\wedge T_0)}\big] = u(x).$$

Sending M to ∞ and then t to ∞ we conclude by double application of Fatou's Lemma

$$\mathbb{E}_x\big[e^{\lambda_0/2 T_0}\big] \leq u(x). \tag{3.3.10}$$

Observe that we used the assumption $\mathbb{P}_x(T_0 < \infty) = 1$ in this step in order to conclude that $\lim_{t\to\infty} t \wedge T_0 = T_0$ $\mathbb{P}_x$-almost surely. The integrability of $e^{\frac{\lambda_0}{2} T_0}$ implies the desired

$$\forall x \in (0,\infty) : -\lim_{t\to\infty} \frac{1}{t} \log \mathbb{P}_x(T_0 > t) \geq \frac{\lambda_0}{2} > 0,$$

as the exponential Markov inequality in combination with (3.3.10) gives

$$\lim_{t\to\infty} \frac{1}{t} \log \mathbb{P}_x\big(T_0 > t\big) \leq \lim_{t\to\infty} \frac{1}{t} \log\big[e^{-\frac{\lambda_0 t}{2}} \mathbb{E}_x\big[e^{\frac{\lambda_0 T_0}{2}}\big] \leq -\frac{\lambda_0}{2}.$$

Another approach to the exponential decay of the tails of T_0 is to use assumption (H) and similar arguments as in the regular case. The ideas used above do not make use of

hypothesis (H).
Now we can apply the parabolic Harnack principle in order to conclude that for every compact $K \subset (0,\infty)$
$$-\lim_{t\to\infty}\frac{1}{t}\log\sup_{x\in K}\mathbb{P}_x(T_0>t)>\frac{\lambda_0}{2},$$
which finishes the proof. □

In our final theorem we consider the case, where infinity is an entrance boundary. In the proof we make again use of the following fact: Assume that λ_0 is an eigenvalue in the spectral theoretic sense then
$$\lim_{t\to\infty}e^{\lambda_0 t}p(t,x,y)=u_{\lambda_0}(x)u_{\lambda_0}(y),$$
where u_{λ_0} denotes the unique nonnegative normalized eigenfunction associated to λ_0 and the convergence is locally uniform in $(0,\infty)$.

Theorem 3.3.4. *Assume that* 0 *is an exit boundary and that* ∞ *is entrance. Then there is only one quasistationary distribution and* X_t *converges from every initial distribution* ν*, which is compactly supported in* $(0,\infty)$ *to the quasistationary distribution* $v_{\lambda_0}(x)\rho(dx)$*, where* v_{λ_0} *is the unique (up to multiples) positive* L^2*-eigenfunction to the eigenvalue* λ_0*.*

Proof. The proof[3] of the next theorem does not differ from the corresponding result in the regular case. We have seen in Theorem (3.2.4) that the spectrum of L is purely discrete and the lowest eigenfunction is integrable with respect to ρ. Moreover by [82] (see also Remark 2.2.4)
$$\lim_{t\to\infty}e^{\lambda_0 t}p(t,x,y)=u_{\lambda_0}(x)u_{\lambda_0}(y),\tag{3.3.11}$$
where u_{λ_0} is the unique positive eigenfunction, normalized to $\|u_{\lambda_0}\|_{L^2((0,\infty),\rho)}=1$, corresponding to the lowest eigenvalue λ_0. Assume that the initial distribution ν is compactly supported in $(0,\infty)$. Using again the parabolic Harnack principle we get for some locally bounded function $\Theta:(0,\infty\to(0,\infty)$, every $z\in(0,\infty)$ with $|z-x|<\delta(x)=\frac{1}{2}\wedge\frac{x}{4}$ and every $y\in(0,\infty)$
$$p(t,x,y)\le\Theta(x)e^{\lambda_0 t}p(t+1,z,y)$$
and therefore
$$\begin{aligned}p(t,x,y)&=\Theta(x)\frac{\int_{|z-x|<\delta(x)}p(t,z,y)u_{\lambda_0}(z)\,\rho(dz)}{\int_{|z-x|<\delta(x)}u_{\lambda_0}(z)\,\rho(dz)}\\&=\frac{\Theta(x)}{\int_{|z-x|<\delta(x)}u_{\lambda_0}(z)\rho(dz)}e^{-\lambda_0(t+1)}u_{\lambda_0}(y).\end{aligned}\tag{3.3.12}$$
Therefore dominated convergence directly implies
$$\lim_{t\to\infty}\mathbb{P}_\nu\big(X_t\in A\mid T_0>t\big)=\lim_{t\to\infty}\frac{e^{\lambda_0 t}\int_0^\infty\nu(dx)\int_A\rho(dy)\,p(t,x,y)}{e^{\lambda_0 t}\int_0^\infty\nu(dx)\int_0^\infty\rho(dy)\,p(t,x,y)}=\frac{\int_A u_{\lambda_0}(x)\rho(dx)}{\int_0^\infty u_{\lambda_0}(x)\,\rho(dx)}.$$
The remaining assertions are already contained in chapter 7 of [20]. □

[3] A similar strategy is already used in [20] in the case of a purely discrete spectrum

The following Corollary gives a short summary of our results. The reader should have in mind that we still work under the assumption (H).

Corollary 3.3.1. *Assume that* 0 *is an exit boundary point and that infinity is inaccessible.*

a) If infinity is an entrance boundary point then we have $\eta = \lambda_0 > 0$ *and* X_t *converges to the quasistationary distribution* $u_{\lambda_0}(x)$*, where* u_{λ_0} *is the unique (up to constant multiples) positive* L^2*-eigenfunction corresponding ton the bottom of the spectrum* λ_0*. In this case there is exactly one quasistationary distribution.*

b) If infinity is natural and absorption is certain and $\lambda_0 > 0$ *then* $\eta = \lambda_0$ *and* X_t *converges to the quasistationary distribution* $\tilde{\varphi}(\lambda_0, \cdot)$*, where* $\tilde{\varphi}(\lambda_0, \cdot)$ *is the unique (up to positive multiples) positive solution of* $(\tau - \lambda_0)u = 0$ *with* $\lim_{x\to 0} \tilde{\varphi}(\lambda_0, x) = 0$

b) If infinity is natural and if $\mathbb{P}_x(T_0 < \infty) \neq 1$ *then* X_t *escapes to infinity exponentially fast, where the exponential rate is given by* λ_0*.*

3.4 Concluding Remarks and Open Problems

In this chapter we established a characterization for the existence of quasistationary distributions for class of one-dimensional diffusions on the halfline, for which 0 is accessible and ∞ is inaccessible. We have not been able to work without further conditions, though we have been able to extend results of the recent paper [20] significantly. Our assumptions allowed us to give sufficient conditions for the existence of quasistationary distributions for diffusions, whose drift behaves like the drift of a Bessel diffusion at 0 and satisfies a mild condition at infinity. Under these conditions let us assume that absorption is certain. Then we know by the above results that there is a unique quasistationary distribution if and only if the bottom of the spectrum is strictly positive and infinity is an entrance boundary. Thus if the bottom of the spectrum is strictly positive and if infinity is natural then there are quasistationary distribution, which are not associated to the bottom of the spectrum. One quasistationary distribution μ_0 is given by the Yaglom limit. Let μ be another quasistationary distribution. Then μ is absolutely continuous with respect to ρ with a continuous non-negative density f and there is $\alpha > 0$ such that $\mathbb{P}_\mu(T_0 > t) = e^{-\alpha t}$. In the case of birth and death processes on $\mathbb{N}_0$ it is known that $0 \leq \alpha \leq \lambda_0$. The quasistationary distribution given by the Yaglom limit μ_0 is thus minimal under the quasistationary distributions in the sense that that the expected absorption $\mathbb{E}_{\mu_0}[T_0]$ is minimal. Moreover we see that quasistationary distributions μ different from μ_0 somehow correspond to positive values below λ_0. This is rather interesting, since these values have no meaning in the L^2-theory of the diffusion operator and until now little seems to be known about these quasistationary distributions in the case of diffusions. As we have shown above the minimal quasistationary distribution occurs naturally as the Yaglom limit

$$\lim_{t\to\infty} \mathbb{P}_\nu(X_t \in \cdot \mid T_0 > t), \tag{3.4.1}$$

where ν is compactly supported in $(0, \infty)$. This means that the compactly supported initial distributions belong to the domain of attraction of the minimal quasistationary distributions. Moreover it is known that every initial distribution belongs o the domain of attraction of the minimal quasistationary distributions, if there is a unique quasistationary distribution, i.e. if infinity is an entrance boundary. It is thus natural to expect that other quasistationary distributions might occur as limits (3.4.1), if one starts with initial distributions whose tails decay not too rapidly. A further analysis of the whole set of quasistationary distribution constitutes an interesting project for further research.

Chapter 4

Large time behavior of the two-dimensional Super-Brownian motion with a single point source

In this chapter we study the multi-dimensional super-Brownian motion with a single point source as introduced by Fleischmann and Mueller in [42]. More precisely we mainly investigate the large time behavior of the two-dimensional super-Brownian motion with a single point source and finite variance. This was mentioned as an open problem by Fleischmann, Mueller and Vogt in [43]. Heuristically, the class of processes that is studied in this chapter are super-Brownian motions with additional birth at the origin. Formally these processes are defined via the log-Laplace equation

$$\mathbb{E}_\mu\Big[\exp\big(-\langle X_t,\varphi\rangle\big)\Big] = \exp\Big(-\int_{\mathbb{R}^n} v(t,x)\,\mu(dx)\Big),$$

where v is the unique positive solution of the non-linear equation

$$\partial_t v(t,x) = -\Delta_\alpha v(t,x) - v(t,x)^{1+\beta},\ v(0,\cdot) = \varphi(\cdot),$$

where $\alpha \in \mathbb{R}$, $\beta \in (0,1]$ $(d=2)$, $\beta \in (0,1)$ $(d=3)$ and φ belongs to a suitable class of testfunctions. Here the operator $-\Delta_\alpha$ $(\alpha \in \mathbb{R})$ is not a fractional power of the Laplacian, but is an element of the 1-parameter family of selfadjoint extensions of $-\Delta \restriction C_c^\infty(\mathbb{R}^d \setminus \{0\})$ with $d = 2,3$. Heuristically, the operators $-\Delta_\alpha$ represent the Laplacian perturbed by a certain δ_0-potential and are sometimes called Hamiltonians with point interaction in the analytic literature. The superprocess associated to $-\Delta_\alpha$ is rather interesting, since a first thought could indicate that in dimensions greater than 1 such a process does not exist just because in higher dimensions single points are polar. This might be the reason, why these operators have not been considered from a probabilistic point of view (apart from [42]) despite of the enormous analytic literature about this subject. The situation is different in one dimension, where a rather good understanding of the super-Brownian motion with a single point source has been obtained. We only mention that, after the derivation of the large time behavior of the expected total mass in [34] by Engländer and Fleischmann, Engländer and Turaev proved in [36] a (weak) law of large numbers for this process. As in one dimension single points are non-polar for the Brownian motion a probabilistic approach via a Feynman-Kac formula is rather straightforward. In higher dimensions the semigroups corresponding to the selfadjoint extensions $-\Delta_\alpha$ ($\alpha \in \mathbb{R}$) are positivity preserving, but they are not transition functions of a Markov process with state space $\mathbb{R}^d$ ($d = 2, 3$). Using purely analytic techniques Fleischmann and Mueller constructed in a quite non-trivial way a superprocess associated to $-\Delta_\alpha$ and thus in principle opened the way for a probabilistic investigation of these operators. Unfortunately the probabilistic meaning and almost all qualitative aspects concerning the process seem to be unclear at the moment. In [43] Fleischmann, Mueller and Vogt started to study the large scale behavior of this process in three dimensions. Their results concerning the large time scaling of the expectation are still incomplete. Moreover, the scaling behavior in the two dimensional case is mentioned as an open problem and it seems to be a common belief that in two dimensions the derivation of the large time behavior of the expectation is even more difficult (see e.g. [32] p. 500). In this work we complement the results of [43] concerning the scaling behavior of the expectation in the three dimensional case. Furthermore, we prove that the process in two dimensions exhibits a behavior, which is quite different from the three dimensional case. More precisely we prove that in two dimensions the large time behavior of the expected mass is always precisely exponential, whereas in the three dimensional case the exact scaling of the expectation depends on the parameter α. If $\alpha < 0$, the expectation has precise exponential long time asymptotics. If $\alpha > 0$, the long time behavior of $\mathbb{E}_\mu[\langle X_t, \varphi\rangle]$ ($\varphi \in C_c(\mathbb{R}^3)$) is comparable to the one of the usual super Brownian motion, i.e. it is given by $t^{-3/2}$. If $\alpha = 0$, we find that $\mathbb{E}_\mu[\langle X_t, \varphi\rangle]$ behaves like $t^{-1/2}$ as $t \to \infty$.

On the one hand we hope that our results will help to reveal the probabilistic mechanism underlying the process and to understand the scope of probabilistic representations of rather exotic operators such as the above mentioned Hamiltonians with point interaction. Further research in this direction will be necessary and our results are only a small step. On the other hand we want to demonstrate at a concrete example that our main tool, which is a Fourier type analysis, has a broader scope than usually thought in the probabilistic literature.

This chapter is organized in the following way. In section 2 we recall and provide some analytic facts concerning the operator $-\Delta_\alpha$. In particular we describe the spectrum and show how the operator can be diagonalized. In section 3 we recall the precise definition of the Super-Brownian motion with a single point source. The main results of this work can be found in section 4, where we give the precise long time behavior of the expectation and also establish the law of large numbers in the case of dimension two *under the assumption that a formula for the second moment, known to hold true for more regular superprocesses, is also valid in our case.* More details concerning the heuristics for the law of large numbers can be found in [37] and [44]. Our main result is Theorem 4.3.3 where we prove that there is a random variable N_α depending only on α and the initial measure μ such that

$$\lim_{t\to\infty} \frac{\langle f, X_t\rangle}{\mathbb{E}_\mu\big[\langle f, X_t\rangle\big]} = \frac{N_\alpha}{\int_{\mathbb{R}^2} \psi_\alpha(x)\,\mu(dx)},$$

where $f \in C_c(\mathbb{R}^2 \setminus \{0\})$, ψ_α is a certain explicitly known function and the limit denotes almost sure convergence.
We stress that in many previous results only convergence in probability or even weaker kinds of convergence have been established (see [36], [37] and [32]). Almost sure convergence results are given in [93], [44] and [35] for certain branching diffusions, in [92] for super-Brownian motion and [22] for a quite large class of superprocesses. Assuming that a certain formula for the second moment holds also true in our case we prove the strong law of large numbers. As it is quite common to most of the above mentioned results concerning the large time behavior of super-processes our method is a combination of analytic facts with some probabilistic (mainly martingale) arguments. An exception is the work [35] where Engländer, Harris and Kyprianou use a more probabilistic approach which is still not extended to superprocesses. Therefore this technique which is often called spine technique is not applicable to our setting.

4.1 Analytic Results

In this section we present some known analytic results concerning the formal operator $-\frac{1}{2}\Delta - \delta_0^\alpha$. These operators are often referred to as Hamiltonians with point interaction in the mathematical physics literature. We present only the most basic facts. For additional details we refer to the comprehensive work [4]. First, we have to clarify how we rigorously define the formal operator $-\Delta - \delta_0^\alpha$. This will be done via the theory of selfadjoint extensions of symmetric operators. Though this topic is well-known in the analytic community we present some basic results for the convenience of the reader. In order to use all spectral theoretic results it will sometimes be necessary to use the complex Hilbert space $L^2(\mathbb{R}^d) = L^2(\mathbb{R}^d, \mathbb{C})$. The operators $-\Delta - \delta_0^\alpha$ are rigorously defined as a selfadjoint extension $-\Delta_\alpha$ of the symmetric operator $-\Delta \restriction C_c^\infty(\mathbb{R}^d \setminus \{0\})$. Since in dimensions $d \geq 4$ the operator $-\Delta \restriction C_c^\infty(\mathbb{R}^d \setminus \{0\})$ is essentially selfadjoint, point interactions do not exist. If the dimension is strictly smaller than four, this is no longer true and there are selfadjoint extensions which differ from the Friedrichs extension. These extensions will play an important role in this work. The operator $-\Delta_\alpha$ should not be confused with fractal powers of the ordinary Laplacian, which occur as generators of certain symmetric Levý processes. Since each operator $-\Delta_\alpha$ is selfadjoint we can associate to $-\Delta_\alpha$ a projection valued spectral measure $E^\alpha(d\lambda)$, where the subscript α is sometimes omitted. At several points we will make use of the Sobolev embedding theorem (see chapter 4 in [17]). As in [4] we denote by $H^2(U) = H^{2,2}(U)$ the standard Sobolev space consisting of all $L^2(U)$ functions whose weak derivatives up to second order also belong to $L^2(U)$. One version of the Sobolev embedding theorem states that for $d = 2, 3$ and for every open set $U \subset \mathbb{R}^d$

with a smooth boundary there exists a constant $C > 0$ such that for every $f \in H^{2,2}(U)$

$$\sup_{x\in U} |f(x)| \le C \, \|f\|_{H^{2,2}(U)} = C\bigg(\|f\|_{L^2(U)} + \|\nabla f\|_{L^2(U)} + \sum_{i,j=1}^{d} \|\partial_i \partial_j f\|_{L^2(U)} \bigg).$$

In case of $U = \mathbb{R}^d$ the above inequality implies that form some $\tilde{C} > 0$ and every $f \in H^{2,2}(U)$

$$\sup_{x\in U} |f(x)| \le \tilde{C}\big(\|f\|_{L^2(U)} + \|\Delta f\|_{L^2(U)}\big).$$

The last inequality can be read as the fact that for every $\lambda \in \mathbb{C} \setminus [0,\infty)$ the operator $(-\Delta - \lambda)^{-1}$ maps $L^2(\mathbb{R}^d)$ continuously to the space $C_b(\mathbb{R}^d)$ of all bounded continuous functions on $\mathbb{R}^d$.

4.1.1 Point Interactions in Three Dimensions

We start with an explanation of the 3-dimensional problem. Consider the operator $\tilde{H} = -\Delta \restriction C_c^\infty(\mathbb{R}^3 \setminus \{0\})$. $\tilde{H}$ is a densely defined, symmetric and non-negative operator in the Hilbert space $L^2(\mathbb{R}^3)$, which has self-adjoint extensions given by the Krein-theory. Since the deficiency indices are $(1,1)$ there exists a one-parameter, parameterized by $\alpha \in (-\infty,\infty]$, family of selfadjoint extension. The selfadjoint extension corresponding to $\alpha = \infty$ is the Friedrichs extension. The domain $\mathcal{D}(-\Delta_\alpha)$ $(\alpha \in \mathbb{R})$ consists of all elements ψ of the form

$$\psi(x) = \varphi_k(x) + (\alpha - ik/4\pi)^{-1}\varphi_k(0)G_k(x) \;\; x \in \mathbb{R}^d \setminus \{0\}, \tag{4.1.1}$$

where φ_k belongs to the Sobolev space $H^{2,2}(\mathbb{R}^3)$ and $G_k(x) = (4\pi|x|)^{-1}e^{ik|x|}$ $(\Im k > 0)$ denotes the free Greens function. Observe that, due to the Sobolev-embedding, φ_k is continuous and therefore $\varphi_k(0)$ is well-defined. The decomposition in (4.1.1) is unique and with such a ψ one has

$$(-\Delta_\alpha - k^2)\psi = (-\Delta - k^2)\varphi_k. \tag{4.1.2}$$

The spectral analysis of the operator $H^\alpha := -\Delta_\alpha$ in $\mathbb{R}^3$ is rather straightforward (see Theorem 1.1.4. in [4]). Its essential spectrum coincides with the absolutely continuous spectrum and is given by $[0,\infty)$. The point spectrum $\sigma_p(H^\alpha)$ is empty if $\alpha > 0$ and $\sigma_p(H^\alpha) = \{-8\pi^2\alpha^2\}$ if $\alpha < 0$. If $\alpha = 0$, the operator H^α exhibits a resonance at zero. If $\alpha < 0$, the unique (up to constant multiples) eigenfunction associated to the discrete eigenvalue $\lambda_\alpha = -8\pi^2\alpha^2$ is given by

$$\psi_\alpha(x) = \frac{1}{4\pi|x|} e^{4\pi\alpha|x|}.$$

Thus the eigenfunction corresponding to the bottom of the spectrum belongs to $L^2(\mathbb{R}^2)$ and is exponentially decreasing at infinity and therefore it is in particular integrable. The strongly continuous semigroup $(e^{-tH^\alpha})_{t>0}$ in $L^2(\mathbb{R}^3)$ consists of integral operators e^{-tH^α} $(t > 0)$, whose integral kernels $p^\alpha(t,x,y)$ are given by (see formula array (3.4) in [1])

$$p^\alpha(t,x,y) = p(t,x,y) + \frac{2t}{|x||y|}\frac{1}{(4\pi t)^{\frac{3}{2}}} e^{-\frac{(|x|+|y|)^2}{4t}} - \frac{8\pi\alpha t}{|x||y|}\int_0^\infty du\, \frac{e^{-4\pi\alpha u}}{(4\pi t)^{\frac{3}{2}}} e^{-\frac{(u+|x|+|y|)^2}{4t}}. \tag{4.1.3}$$

This explicit expression for the heat kernel has been derived in [1]. There is another fact concerning the operator which will be used extensively. It can be diagonalized in a way which is quite similar to the diagonalization of the Laplacian via Fourier transform. Set

$$\varphi_\alpha(k,x) = e^{ik\cdot x} + \frac{1}{4\pi\alpha - i|k|}\,\frac{e^{i|k||x|}}{|x|} \tag{4.1.4}$$

and let

$$\mathcal{F}_\alpha : \mathcal{H}_{ac} \to \mathcal{H}_{ac},\ \mathcal{F}_\alpha f(k) = \int_{\mathbb{R}^3} \varphi_\alpha(k,x) f(x)\,\frac{dx}{(2\pi)^{\frac{3}{2}}}, \tag{4.1.5}$$

where $\mathcal{H}_{ac} = E([0,\infty))L^2(\mathbb{R}^3)$ denotes the absolutely continuous subspace of $L^2(\mathbb{R}^3)$ (see e.g. equation (3) in [72] and section I.1.4 in [4]). It is known that $\mathcal{F}_\alpha$ defines unitary operators, which diagonalize the absolutely continuous part of the operator $-\Delta_\alpha$ in the sense that $\mathcal{F}_\alpha(-\Delta_\alpha)\mathcal{F}_\alpha^{-1}$ acts as multiplication with $|\cdot|^2$. We stress that $\mathcal{F}_\alpha$ acts as an ordinary Lebesgue integral only for functions, which are not too singular and decay at infinity. For a general L^2-function $f \in \mathcal{H}_{ac}$, $\mathcal{F}f$ is given as the L^2-limit of the sequence $\left(\int_{|x|<n} f(x)\varphi(k,x)\,dx\right)_{n\in\mathbb{N}}$. Notice that we often omit the subscript α in the quantities introduced above. This should not cause any confusion since in our results $\alpha \in \mathbb{R}$ is usually fixed.
The fact that in the case $\alpha = 0$ the generalized eigenfunction $\varphi_\alpha(\cdot,x)$ $(x \in \mathbb{R}^3 \setminus \{0\})$ has a 'pole of first order' will be responsible for the fact that the large time behavior of P_t^0 differs from the large time behavior of P_t^β $(\beta > 0)$. In the spectral theoretic literature one says that 0 is a resonance if $\alpha = 0$.

4.1.2 Point Interactions in Two Dimensions

Let us finally describe the situation in two dimensions. We stress that our parameterization of the family of selfadjoint extensions is the same as in [1] and thus differs from the one used in [4] (see the footnote on page 225 of [1]). Our α and their parameter, which will be denoted by $\tilde{\alpha}$, are related by $\alpha = 4\pi\tilde{\alpha} - 2\Psi(1) - 2\ln 2$, where Ψ denotes the Digamma-function. The non-negative operator $-\Delta \restriction C_c^\infty(\mathbb{R}^2 \setminus \{0\})$ has deficiency indices $(1,1)$. Thus there is a one-parameter family of selfadjoint extension $(-\Delta_\alpha)_{\alpha\in\mathbb{R}}$ of the symmetric operator $-\Delta \restriction C_c^\infty(\mathbb{R}^2 \setminus \{0\})$. The construction of all selfadjoint extensions is analogous to the three-dimensional case. Essentially one has to replace the free three-dimensional Greens function by the free two-dimensional Greens function, i.e. as formulated in [4] as Theorem 5.3 the domain $\mathcal{D}(-\Delta_\alpha)$ of the selfadjoint operator $-\Delta_\alpha$ is given by

$$\begin{aligned}\mathcal{D}(-\Delta_\alpha) = \big\{\psi \mid \psi = \phi_k + 2\pi[2\pi\tilde{\alpha} - \Psi(1) + \ln(k/2i)]^{-1}\phi_k(0)G_k,\\ \text{where } \phi_k \in \mathcal{D}(-\Delta), G_k(x) = (i/4)H_0^{(1)}(k|x|)\big\}\end{aligned} \tag{4.1.6}$$

and

$$(-\Delta_\alpha - k^2)^{-1} = (-\Delta - k^2)^{-1} + 2\pi\big[2\pi\tilde{\alpha} - \Psi(1) - \ln(k/2i)\big]^{-1}(\overline{G_k(\cdot)},\,\cdot\,)_{L^2(\mathbb{R}^2)}G_k(\cdot) \tag{4.1.7}$$

The spectrum of the operator $\sigma(-\Delta_\alpha)$ consists of an absolutely continuous part $\sigma_{ac}(-\Delta_\alpha) = [0,\infty)$ and the point spectrum $\sigma_p(-\Delta_\alpha) = \{-e^{-\alpha}\} = \{\lambda_\alpha\}$ (recall that our parameterization of the selfadjoint extensions differs from [4]). As remarked on page 100 in [4] this bound state is determined by the pole structure of (4.1.7). $(-\Delta_\alpha - k^2)^{-1}(x,y)$ $(x \neq y)$ has a meromorphic continuation to the entire logarithmic Riemann surface such that in the cut plane $\{k \in \mathbb{C} \setminus \{0\} \mid -\pi < \arg k < \pi\}$ $-\Delta_\alpha$ there is only the pole at λ_α. The multiplicity of the eigenvalue is one and the unique positive unnormalized eigenfunction $\tilde{\psi}_\alpha$ is given .

$$\tilde{\psi}_\alpha(x) = G_{ie^{-\alpha/2}}(x),$$

where $G_\lambda(x-y)$ denotes the integral kernel of $(-\lambda^2 - \Delta)^{-1}$ $(\Im\lambda > 0)$. Observe that the lowest eigenfunction $\psi_\alpha(x)$ inherits the properties of the two-dimensional Greens function. It decays exponentially as $x \to \infty$ and has a logarithmic singularity at 0. Set $\psi_\alpha := \|\tilde{\psi}_\alpha\|^{-1}_{L^2(\mathbb{R}^2)}\tilde{\psi}_\alpha$. Thus in contrast to the three-dimensional case the point spectrum is nonempty for every $\alpha \in \mathbb{R}$ and there is never a resonance. As $-\Delta_\alpha$ is selfadjoint we may consider the operators $e^{-t(-\Delta_\alpha)} = P_t^\alpha$, again. These are integral operators and the kernel of P_t^α is given by (see equation (3.12) and equation (3.15) in [1])

$$\begin{aligned}
p^\alpha(t,x,y) &:= \frac{1}{4\pi t} e^{-\frac{(x-y)^2}{4t}} \\
&+ \frac{\sqrt{4\pi t}}{\sqrt{|x||y|}} \frac{1}{4\pi t} e^{-\frac{(|x|+|y|)^2}{4t}} \int_0^\infty du\, \frac{t^u e^{-\alpha u}}{\Gamma(u)} \int_0^\infty dr\, \frac{r^{u-1} e^{-r(|x|+|y|)^2/4t}}{(r+1)^{u+\frac{1}{2}}} \tilde{K}_0\Big(\frac{|x||y|}{2t}(r+1)\Big) \\
&= \frac{1}{4\pi t} e^{-\frac{(x-y)^2}{4t}} \\
&+ \frac{1}{2\pi} \int_0^\infty t^{u-1} \frac{e^{-\alpha u}}{\Gamma(u)} \int_1^\infty (z-1)^{u-1} z^{-u} e^{-z\frac{|x|^2+|y|^2}{4t}} K_0\Big(\frac{|x||y|}{2t} z\Big)\, dz\, du \\
&=: p(t,x,y) + \tilde{p}^\alpha(t,x,y)
\end{aligned} \tag{4.1.8}$$

Here Γ denotes the Gamma function and $\tilde{K}_0$ is defined by

$$\tilde{K}_0(z) = e^z (2z/\pi)^{\frac{1}{2}} K_0(z),$$

where $K_0 \geq 0$ is the Macdonald function (modified Bessel function of the third kind) of order zero. Observe that in Lemma 2.6 of [42] it is shown that for every $T > 0$ there exists a constant $C = C(\alpha, T)$ such that for $t \in (0,T]$ and $x,y \in \mathbb{R}^2 \setminus \{0\}$

$$p^\alpha(t,x,y) \leq p(t,x,y) + C\, t^{-\frac{1}{2}} \frac{1}{\sqrt{|x||y|}} e^{-\frac{|x|^2}{4t}} e^{-\frac{|y|^2}{4t}}. \tag{4.1.9}$$

Furthermore we use expansions in generalized eigenfunctions again. In two dimensions the generalized eigenfunctions or scattering wave functions $\varphi_\alpha(k,x)$ (see formula (2.36) in [3]) read

$$\varphi_\alpha(k,x) = e^{ik\cdot x} + \frac{i\pi}{2}\big(2\pi\tilde{\alpha} - \Psi(1) + \ln(|k|/2i)\big)^{-1} H_0^{(1)}(|k||x|), \tag{4.1.10}$$

where $H_0^{(1)}$ denotes the Hankel function of first kind and of order zero. Their asymptotic behavior is

$$H_0^{(1)}(s) \approx i\frac{2}{\pi}\ln(s) \text{ as } s \to 0 \text{ and } H_0^{(1)}(s) \approx \sqrt{\frac{2}{\pi s}}e^{-i(\pi+\frac{3\pi}{4})}e^{is} \text{ as } s \to \infty. \tag{4.1.11}$$

Observe that $A(\alpha,k) := \frac{i\pi}{2}\big(2\pi\tilde{\alpha} - \Psi(1) + \ln(|k|/2i)\big)^{-1}$ is bounded as a function of k. Using the generalized eigenfunctions we arrive, as in the three-dimensional case, at a generalized Fourier transform $\mathcal{F}_\alpha$, which diagonalizes the absolutely continuous part of the operator $-\Delta_\alpha$. In the sequel we will often omit the subscript α.

Remark 4.1.1. *In the previous chapters we have already seen that expansions in generalized eigenfunctions are an extremely useful tool. In one dimension this is much better known due to the Weyl theory of Sturm-Liouville operators part of which is formulated in Theorem 2.2.1 above. But we want to stress that in higher dimensions there exist several useful results concerning expansions in generalized eigenfunctions as well and the existence of eigenfunction expansions is not restricted to the class of exactly solvable problems such as Hamiltonians with point interactions. These analytic results are useful in probabilistic problems, too. For example the results of [56] can be used in order to recover some of the interesting results of Collet et Al. (see [26]) concerning the asymptotic behavior of a Brownian motion on exterior domains.*

In order to clarify whether the local behavior, i.e. the mass in bounded regions, and the global behavior, i.e. the mass in unbounded regions have the same large time asymptotics it is important to consider the large time behavior of P_t^α on $L^\infty(\mathbb{R}^2)$. Roughly the large time behavior of the mass in bounded regions is given by the L^2-behavior of P_t^α. The large time behavior of the mass in unbounded regions is more closely connected to the L^∞-behavior of P_t^α. The investigation of the large time behavior of $\sup_{x\in\mathbb{R}^2}(P_t^\alpha\mathbf{1})(x)$ is the content of Lemma 2.1 in [9]. Observe first that the authors of [9] seem to have missed that the parameters in [4] and [1] differ. This becomes clear e.g. in formula (17) of [9]. Moreover the assertion of the following lemma is *not correct*. We explain the hidden mistake made by Blanchard and Ben Amor in Remark (4.1.2). In a first approach to several of our results we used the results stated in the work of Blanchard and Ben-Amor and only recently found out that several of these results are not correct.

Lemma 4.1.1 (Lemma 2.1 in [9])**.** *In two dimensions the integral kernels $p^\alpha(t,x,y)$ induce bounded positivity preserving operators P_t^α on $L^\infty(\mathbb{R}^2)$, whose norm is given by*

$$\|P_t^\alpha\|_{\infty,\infty} \leq 1 + \nu(te^{-\alpha})$$

where the ν-function is defined (see [38] p. 219) by

$$\nu(t) = \int_0^\infty \frac{t^r}{\Gamma(r+1)}\,dr$$

Remark 4.1.2. *In this remark we comment on Lemma 4.1.1. The mistake Blanchard and Ben Amor did is not easy to detect for two reasons. The first reason is the following: On might expect that exponential asymptotic behavior of $\|P_t^\alpha\|_{\infty,\infty}$ is similar to the one of the semigroup in L^2. And indeed this expectancy is actually met by the asymptotic behavior of the ν-function. Indeed in [38] (p. 210) one finds Ramanujan's formula*

$$\nu(x) = e^x - \int_0^\infty \frac{e^{-xt}}{t\big[\pi^2 + (\log t)^2\big]}\,dt, \tag{4.1.12}$$

which directly implies that

$$\lim_{t\to\infty} \frac{1}{t}\log \nu(t) = 1.$$

Assuming that Lemma 4.1.1 is correct the exponential growth rate $\lim_{t\to\infty}\frac{1}{t}\log\|P_t^\alpha\|_{\infty,\infty}$ of the norms $\|P_t^\alpha\|_{\infty,\infty}$ is given by

$$\begin{aligned} -\lambda_\alpha(\infty) :&= \lim_{t\to\infty}\frac{1}{t}\log\|P_t^\alpha\|_{\infty,\infty} = \lim_{t\to\infty}\frac{1}{t}\log \sup_{x\in\mathbb{R}^2\setminus\{0\}} P_t^\alpha\mathbf{1}(x) = \\ &= \lim_{t\to\infty}\frac{1}{t}\log\nu(te^{-\alpha}) = e^{-\alpha} = -\lambda_\alpha. \end{aligned}$$

Thus the assertion of Lemma 4.1.1 beautiful fits in the general picture and it is tempting to believe the truth of this propostion.
The second reason, why the mistake in Lemma 4.1.1 is difficult to detect, is the fact that the calculation in [9] is indeed correct. What is not correct is their declaration of the Whittaker function $W_{-\frac{1}{2},0}$. The authors use the formula $W_{-\frac{1}{2},0}(z) = \sqrt{z}e^{z/2}$ for which they refer to [64] but in this reference a different formula for $W_{-\frac{1}{2},0}(z)$ is given.
Let us finally point out that also Lemma 2.2 in [9] contains a serious mistake. In this Lemma Ben Amor and Blanchard claimed to have shown, that the operators P_t^α ($t > 0$) are ultracontractive in two dimensions. If $P_t^\alpha = e^{-t(-\Delta_\alpha)}$ is ultracontractive then all eigenfunctions have to be bounded, but the eigenfunction ψ_α of $-\Delta_\alpha$ is unbounded. Thus several basic properties such as the p-independence ($p \in (1,\infty)$) of the generator of the two-dimensional semigroup $(P_t^\alpha)_{t\geq 0}$ considered in $L^p(\mathbb{R}^2)$ are still open.

Remark 4.1.3. *The situation in three dimensions the situation is the following. A direct calculation shows that the operator P_t^α does not map bounded functions to bounded functions. Furthermore, it is shown in [2] that in three dimensions the operators $-\Delta_\alpha$ are generators of strongly continuous semigroups in $L^p(\mathbb{R}^3)$ iff $p \in (3/2, 3)$. In two dimensions $e^{-t\Delta_\alpha}$ is strongly continuous in $L^p(\mathbb{R}^2)$ for every $p \in (1,\infty)$. The integral kernels $p^\alpha(t,\cdot,\cdot)$ probably do not induce bounded operators on $L^\infty(\mathbb{R}^2\setminus\{0\})$. However using the bound in (4.1.9) a direct calculation shows that for every $x \in \mathbb{R}^2\setminus\{0\}$, $t > 0$, $\alpha\in\mathbb{R}$ and $f \in L^\infty(\mathbb{R})$ the expression*

$$P_t^\alpha f(x) = \int_{\mathbb{R}^d\setminus\{0\}} p^\alpha(t,x,y)f(y)\,dy$$

is well-defined and even continuous in $x \in \mathbb{R}^2\setminus\{0\}$.

In the next proposition we state some basic mapping properties of the generalized Fourier transform.

Proposition 4.1.1. *Let $d = 2$ and assume that the element $g \in \mathcal{H}_{ac}$ belongs to $\bigcap_{n\in\mathbb{N}_0}\big(\mathcal{D}((-\Delta_\alpha)^n)\cap L^1(\mathbb{R}^2)\big)$, where $\mathcal{D}((-\Delta_\alpha)^n)$ denotes the domain of the n-th power of the operator $-\Delta_\alpha$. Then*

$$|\hat{g}(k)| \le c_n\, |k|^{-2n} \ \text{ for } |k| \text{ large enough }.$$

Moreover, if additionally $g \in L^p(\mathbb{R}^2)$ for every $p \in (1,\infty)$ then

$$\lim_{k\to 0} |k|^\gamma \hat{g}(k) = 0$$

for every $\gamma > 0$.

Proof. The general principle is the same as in the case of the classical Fourier transform. The Fourier transform maps smooth functions to decaying ones. In our case smoothness means roughly that g belongs to the domain of high powers of $-\Delta_\alpha$. First observe that for $k \in \mathbb{R}^2 \setminus \{0\}$ and $n \in \mathbb{N}_0$ the assumption $-\Delta_\alpha^n g \in L^2(\mathbb{R}^d)$ implies that

$$\int_{|x|\le 1} |\varphi(k,x)(-\Delta_\alpha g(x))|\, dx \le \left(\int_{|x|\le 1} |\varphi(k,x)|^2\, dx\right)^{\frac12} \left(\int_{\mathbb{R}^2} |(-\Delta_\alpha g(x))|^2\, dx\right)^{\frac12} < \infty.$$

Moreover observe that for some constants $C_1, C_2, C_3 > 0$

$$\int_{|x|\le 1} |\varphi(k,x)|^2\, dx \le C_1 + C_2|k|^{-2} \int_{|x|\le |k|} |H_0^{(1)}(|y|)|\, dy \le C_3,$$

where we made use of the explicit form of the generalized eigenfunctions, again. This gives

$$\sup_{|k|>1} \int_{|x|\le 1} |\varphi(k,x)(-\Delta_\alpha g(x))|\, dx < \infty.$$

Since we also assume that $-\Delta_\alpha g \in L^1(\mathbb{R}^2)$ we conclude using (4.1.10) that

$$\sup_{|k|>1} \int_{|x|\ge 1} |\varphi(k,x)(-\Delta_\alpha)^n g(x)|\, dx < \infty$$

and finally that

$$\sup_{|k|>1} \int_{\mathbb{R}^2} |\varphi(k,x)(-\Delta_\alpha)^n g(x)|\, dx < \infty. \tag{4.1.13}$$

In particular for every fixed $k \in \mathbb{R}^2 \setminus \{0\}$ the function $\varphi(k,\cdot)(-\Delta_\alpha)^n g(\cdot)$ is absolutely integrable. Since $\mathcal{F}$ diagonalizes the operator $-\Delta_\alpha$ we have

$$|k|^{2n} \int_{\mathbb{R}^2} \varphi(k,x) g(x)\, dx = \int_{\mathbb{R}^2} \varphi(k,x)(-\Delta_\alpha)^n g(x)\, dx.$$

Together with (4.1.13) we arrive at ($|k| > 1$)

$$\int_{\mathbb{R}^2} \varphi(k,x) g(x)\, dx \le |k|^{-2n} \sup_{|k|>1} \int_{\mathbb{R}^2} |\varphi(k,x)(-\Delta_\alpha)^n g(x)|\, dx$$

The last assertion is a direct consequence of the dominated convergence theorem and the special form of the generalized eigenfunctions given in (4.1.10). □

Usually we apply Proposition 4.1.1 to functions of the following rather special type. If $f \in C_c^\infty(\mathbb{R}^2 \setminus \{0\})$ then by the very definition of the operator $-\Delta_\alpha$ we have $f \in \bigcap_{n\in\mathbb{N}_0} \mathcal{D}((-\Delta_\alpha)^n)$ and since ψ_α is an eigenfunction we also have $\psi_\alpha \in \bigcap_{n\in\mathbb{N}_0} \mathcal{D}((-\Delta_\alpha)^n)$. Because of

$$E([0,\infty))f = f - \big(\psi_\alpha, f\big)_{L^2(\mathbb{R}^2)} \psi_\alpha \in L^1(\mathbb{R}^2)$$

we therefore see that also $E([0,\infty))f \in \bigcap_{n\in\mathbb{N}_0} \mathcal{D}((-\Delta_\alpha)^n)$. Due to the decay properties of ψ_α we also get $E([0,\infty))f \in L^1(\mathbb{R}^2)$ and using the fact that $-\Delta_\alpha E([0,\infty))f = E([0,\infty))(-\Delta_\alpha f) = E([0,\infty))(-\Delta f)$. Since $-\Delta f$ has the same basic properties as f we can iterate this and finally conclude that $E([0,\infty))f$ satisfies the conditions of Proposition 4.1.1.

The assertion of the following lemma seems to be folklore but we have not been able to find a suitable reference for it. Therefore and since most of our readers will have a probabilistic background we decided to present an abstract proof though a verification of the assertion might also be possible by a direct (but probably tedious) calculation. But the abstract argument has the obvious advantage of being applicable to situations, where bounds for the heat kernel and the generalized eigenfunctions are available.

Lemma 4.1.2. *Let $d = 2, 3$ and $\alpha \in \mathbb{R}$ fixed. For every $k, x \in \mathbb{R}^d \setminus \{0\}$ we have*

$$P_t^\alpha \varphi(k,\cdot)(x) = e^{-t|k|^2} \varphi(k,x).$$

Proof. For $t > 0$ the expression $P_t^\alpha f(x)$ are well-defined for $x \in \mathbb{R}^d \setminus \{0\}$ and $f \in L^2(\mathbb{R}^d) + L^\infty(\mathbb{R}^d)$ due to Remark 4.1.3. Moreover it is easy to see from the properties of the heat kernel, the bounds in Lemma 2.6 [42] (see (4.1.9) for the two-dimensional bound) that

$$(k,x) \mapsto (P_t^\alpha \varphi(k,\cdot))(x)$$

is continuous in $\mathbb{R}^d \setminus \{0\} \times \mathbb{R}^d \setminus \{0\}$. Let $\psi \in C_c^\infty(\mathbb{R}^2 \setminus \{0\})$ be given. Decompose ψ as $\psi = (\psi, \psi_\alpha)\psi_\alpha + E([0,\infty))\psi = \psi_1 + \psi_2$. Assume we have shown that $(\psi_\alpha, \varphi(k,\cdot))_{L^2(\mathbb{R}^d)} = 0$ then the following arguments imply the assertion of the theorem. Due to our assumption on ψ the expression $(\psi, P_t^\alpha \varphi(k,\cdot))_{L^2(\mathbb{R}^2\setminus\{0\})}$ is well-defined and due to the symmetry of the integral kernel of P_t^α we have

$$\begin{aligned}
\big(\psi, P_t^\alpha \varphi(k,\cdot)\big)_{L^2(\mathbb{R}^d)} &= \big(P_t^\alpha \psi, \varphi(k,\cdot)\big)_{L^2(\mathbb{R}^d)} = \big(P_t^\alpha \psi_1, \varphi(k,\cdot)\big)_{L^2(\mathbb{R}^d)} + \big(P_t^\alpha \psi_2, \varphi(k,\cdot)\big)_{L^2(\mathbb{R}^d)} \\
&= \big(P_t^\alpha \psi_2, \varphi(k,\cdot)\big)_{L^2(\mathbb{R}^d)} = \int_{\mathbb{R}^2} P_t^\alpha \psi_2(x) \varphi(k,x)\, dx \\
&= e^{-t|k|^2} (\psi_2, \varphi(k,\cdot))_{L^2(\mathbb{R}^d)} = e^{-t|k|^2} (\psi, \varphi(k,\cdot))_{L^2(\mathbb{R}^d)},
\end{aligned}$$

where the penultimate equality holds for Lebesgue almost every $k \in \mathbb{R}^d$. Since this is true for every $\psi \in C_c^\infty(\mathbb{R}^d \setminus \{0\})$, we first conclude that the assertion of the Lemma holds for Lebesgue almost every $(k,x) \in \mathbb{R}^d \times \mathbb{R}^d$ and due to the continuity for every $(k,x) \in (\mathbb{R}^d \times \mathbb{R}^d) \setminus \{(0,0)\}$. Thus it remains to prove that $(\psi_\alpha, \varphi(k,\cdot))_{L^2(\mathbb{R}^d \setminus \{0\})} = 0$ for every $k \in \mathbb{R}^d \setminus \{0\}$. Heuristically this is obvious since $\varphi(k,\cdot)$ is an 'eigenfunction' to the 'eigenvalue' $|k|^2 \neq \lambda_\alpha$. Since $\varphi(k,\cdot)$ does not belong to $L^2(\mathbb{R}^d)$ the required 'orthogonality' does not follow directly from the spectral theorem. In order to prove this 'orthogonality' first observe that due to the decay properties of the ground state ψ_α the expression

$$(\psi_\alpha, \varphi(k,\cdot))_{L^2(\mathbb{R}^d)} := \int_{\mathbb{R}^d} \psi_\alpha(x)\varphi(k,x)\,dx$$

is welldefined for every $k \in \mathbb{R}^d \setminus \{0\}$. Fix $k_0 \in \mathbb{R}^d \setminus \{0\}$ and consider the balls $B_n := B_{\frac{1}{n}}(k_0)$ with center k_0 and radius $\frac{1}{n}$ for $n \in \mathbb{N}$ such that $\frac{1}{n} \in (0, |k_0|/2)$. Denote by $|B_n|$ the Lebesgue measure of the set B_n. Notice that for $g \in \mathcal{H}_{ac}$ the $L^2(\mathbb{R}^d)$-function

$$x \mapsto \int_{B_n} dk\,\overline{\varphi(k,x)}\hat{g}(k)$$

again belongs to the absolutely continuous spectral subspace and therefore is orthogonal to the ground state ψ_α. For every $g \in \mathcal{H}_{ac}$ such that $\hat{g} \in C(\mathbb{R}^d \setminus \{0\})$ we thus have

$$0 = \frac{1}{|B_n|}\int_{\mathbb{R}^d} dx\,\psi_\alpha(x)\int_{B_n} dk\,\overline{\varphi(k,x)}\hat{g}(k) \to \int_{\mathbb{R}^d} \psi_\alpha(x)\overline{\varphi(k_0,x)}\,dx \cdot \hat{g}(k_0),$$

where we used that $\psi_\alpha(\cdot)\sup_{k \in B_{|k_0|/2}(k_0)} \varphi(k,\cdot) \in L^1(\mathbb{R}^d)$,

$$\lim_{n\to\infty} \frac{1}{|B_n|}\int_{B_n} dk\,\overline{\varphi(k,x)}\hat{g}(k) = \overline{\varphi(k_0,x)}\hat{g}(k_0) \tag{4.1.14}$$

for every $x \in \mathbb{R}^2 \setminus \{0\}$ and dominated convergence. Equation (4.1.14) follows from Since there exists an $g \in \mathcal{H}_{ac}$ such that $\hat{g} \in C(\mathbb{R}^d \setminus \{0\})$ and $\hat{g}(k_0) \neq 0$ we arrive at

$$\int_{\mathbb{R}^d} \psi_\alpha(x)\varphi(k_0,x)\,dx = 0.$$

□

Remark 4.1.4. *Let $F : \mathbb{R}^3 \to [0,\infty]$ given by $F(x) = |x|^{-1}$ and let $(B_t)_{t\geq 0}$ denote a standard three-dimensional Brownian motion. It is a well known textbook example that the process $(F(B_t))_{t\geq 0} = (\langle F, \delta_{B_t}\rangle)_{t\geq 0}$ is a local martingale but not a martingale. This is obviously connected to the fact, that F is an invariant density for the semigroups P_t^0, which can be shown by an explicit calculation. Since we will not use this fact, we omit the proof.*

4.2 Super-Brownian Motion with a Single Point Source

The construction of super-Brownian motions with a single point source was carried out in dimension 2 and 3 in the work [42] of Fleischmann and Mueller. As we already mentioned in the introduction it is a priori not at all clear, that the selfadjoint extensions $-\Delta_\alpha$ correspond in any way to a well defined Markov process. Define

$$\Phi^\rho = \left\{ \varphi : \mathbb{R}^d \backslash \{0\} \to \mathbb{R} \mid 0 \le \varphi \le C_\varphi \, |\cdot|^{-(d-1)/2} \text{ for some } C_\varphi > 0 \text{ and } \int_{\mathbb{R}^d} dx \, |\varphi(x)|^\rho < \infty \right\}$$

and let $\mathcal{M}(\mathbb{R}^d \setminus \{0\})$ denote the set of all Radon measures in $\mathbb{R}^d \setminus \{0\}$ such that $\langle \mu, \varphi \rangle < \infty$ for all continuous $\varphi \in \Phi^\rho$ with $0 \le \varphi \le C \, |\cdot|^{-(d-1)/2}$. Fleischmann and Mueller constructed a measure-valued process $(X_t)_{t \ge 0}$ associated to the operator $-\Delta_\alpha$ in the following sense.

Theorem 4.2.1 (Theorem 4.4 in [42])**.** *If $d = 2$, let $0 < \beta \le 1$, and if $d = 3$ let $0 < \beta < 1$ and assume furthermore that*

$$\frac{1}{1 - \beta(d-1)/(d+1)} < \rho < \frac{d+1}{d-1}.$$

Then for each $\alpha \in \mathbb{R}$, there is a (unique in law) non-degenerate $\mathcal{M}(\mathbb{R}^d \setminus \{0\})$-valued time-homogeneous Markov process $X = (X, \mathbb{P}_\mu, \mu \in \mathcal{M}(\mathbb{R}^d \setminus \{0\})$ with log-Laplace transition functional

$$-\log \mathbb{E}\big(e^{-\langle \varphi, X_t \rangle}\big) = \langle v(t, \cdot), X_0 \rangle, \ t > 0,$$

where v solves

$$v(t,x) = P_t^\alpha \varphi(x) - \int_0^t ds \, P_{t-s}^\alpha (v^{1+\beta}(s, \cdot))(x)$$

and $\varphi \in \Phi^\rho$.

A characterization of the process in terms of a martingale problem seems to be still missing in the literature. Observe that in contrast to $d = 2$ the case $\beta = 1$ is excluded in three dimensions. This seems to be a technical artefact of the proof in [42], which occurs due to the additional singularity at zero. Due to this singularity the solution of the log-Laplace equation was constructed in a highly non-trivial way via Picard iteration in weighted L^p-spaces. For $d = 2$ and $\beta = 1$ recall the important formula

$$\mathbb{E}\big[\langle \varphi, X_t \rangle\big] = \langle P_t^\alpha \varphi, X_0 \rangle. \tag{4.2.1}$$

Moreover we *assume* the validity of the following formula

$$\mathbb{E}\big[\langle \varphi, X_t \rangle \langle \tilde{\varphi}, X_t \rangle\big] = \langle P_t^\alpha \varphi, X_0 \rangle \langle P_t^\alpha \tilde{\varphi}, X_0 \rangle + \left\langle \int_0^t P_s^\alpha \big(P_{t-s}^\alpha \varphi \, P_{t-s}^\alpha \tilde{\varphi} \big) \, ds, X_0 \right\rangle. \tag{4.2.2}$$

We hope to be able to derive equation (4.2.2) from the log-Laplace equation in a subsequent work. In this thesis we show that (4.2.2) implies some kind of strong law of large numbers.

Note again that in two dimensions we always take $\beta = 1$. Equation (4.2.1) remains also true if $d = 3$ and $\beta \in (0,1)$. In the sequel we will derive some additional basic properties of this superprocess. After having established existence and uniqueness of a stochastic process one is usually interested in its large time behavior. First steps towards a better understanding of the large scale behavior have been carried out in [43]. Assuming that $d = 3$ and that for some sequence $(\lambda_k)_{k\in\mathbb{N}}$ the limit $\alpha^* := \lim_{k\to\infty} k^{\frac{1}{2}}\lambda_k\alpha \in [-\infty, \infty]$ exists the authors of [43] proved in Theorem 3 that for a large class of starting measures μ

$$\lim_{k\to\infty} k^{-\frac{1}{2}}\mathbb{E}_\mu\left[X_{kt}^{\lambda_k\alpha}(k^{\frac{1}{2}}dy)\right] = \langle\mu, \theta_t^{\alpha^*}(\cdot, y)\rangle dy, \tag{4.2.3}$$

where $(X_t^\beta)_{t\geq 0}$ denotes the superprocess from Theorem 4.2.1 corresponding to the parameter β and

$$\theta_t^\alpha(x,y) = \begin{cases} \frac{2t}{|x||y|}\frac{e^{-\frac{|y|^2}{4t}}}{(4\pi t)^{\frac{3}{2}}} - \frac{8\pi\alpha}{|x||y|}\int_0^\infty du\,\frac{e^{-4\pi\alpha u}}{(4\pi t)^{\frac{3}{2}}}e^{-\frac{(u+|y|)^2}{4t}} & \text{if } \alpha \in \mathbb{R} \\ 0 & \text{if } \alpha = +\infty \\ +\infty & \text{if } \alpha = -\infty. \end{cases}$$

Their techniques show in particular that for $\alpha = 0$ and a large class of initial distributions μ

$$\lim_{k\to\infty} k^{-\frac{1}{2}}\mathbb{E}_\mu\left[\langle X_{kt}, \mathbf{1}\rangle\right] = 2t\langle\mu, |\cdot|^{-1}\rangle \int_{\mathbb{R}^3} \frac{e^{\frac{|y|^2}{4t}}}{|y|(4\pi t)^{\frac{3}{2}}}\,dy. \tag{4.2.4}$$

The results of [43] are derived using the explicit scaling behavior of the three-dimensional heat kernel $p^\alpha(t,x,y)$ (see Lemma 1 in [43]). Since the two dimensional heat kernel is analytically more subtle this technique does not apply directly. Thus we are forced to use different techniques in order to understand the large time behavior of the two dimensional superprocess. Observe that in the results of [43] concerning the three-dimensional superprocess together with time and space also the parameter α varies. We believe that it is also interesting to derive results on the large time behavior of the expected mass in bounded regions for fixed values of α. In three dimensions it will turn out that the long time behavior of the expected mass in bounded regions differ. In particular we show that for $\alpha = 0$ the limit on the left hand side of (4.2.4) will be 0 if the constant function $\mathbf{1}$ is replaced by $f \in C_c^\infty(\mathbb{R}^2 \setminus \{0\})$.

4.3 Law of Large Numbers

Several authors obtained versions of the law of large numbers for superprocesses. The first such law of large numbers for branching diffusions seems to be the pioneering work [93] of Watanabe (see also the recent work [92], where Watanabe's ideas are presented in detail for super-Brownian motion). More recently Engländer and coauthors established weak laws of large number for classes of superprocesses (see [36], [37] and [33]). Their methods can not be used in our case, since they strongly rely on analytic and probabilistic results which are not known to hold for the superprocess with a single point source. Quite recently Chen et Al. proved in [22] the strong law of large numbers for a large class of superprocesses, where the underlying spatial motion is a symmetric Hunt process satisfying certain assumptions. The results of [37], [32] and [22] are applicable to different situations and it is not clear at the moment, how to establish results which contain all previous ones. Our setting differs from the above mentioned works again and none of the existing results is applicable in our case. Therefore we use a different route but we want to point out that we greatly benefit from all of the above mentioned works. Our approach to the law of large numbers is motivated by a similar result of Steinsaltz and Evans in [86] concerning superprocesses whose spatial motion is a one-dimensional diffusion on the half-line. In order to establish results concerning the large time behavior of the super-diffusion Steinsaltz and Evans make use of their results concerning quasistationary distributions of one-dimensional diffusions on the half-line, where expansions in generalized eigenfunctions play a central role. Later we found out that Watanabe already used Fourier-analytic methods in the branching process context. In this work we will demonstrate that such a Fourier-analytic method is not restricted to the case of super-Brownian motions. In our setting the technical details are more involved than in the case of a classical Super-Brownian motion and some care is needed in order to handle several singularities at zero. Let us finally stress that the case of a one-dimensional super-Brownian motion with a single point source can be treated in a very similar way and is technically more elementary since all appearing objects are smooth.

4.3.1 Scaling of the Expectation

The analytic situation already indicates that the large time behavior of the two-dimensional superprocess should be similar to the large-time behavior of the three-dimensional superprocess with $\alpha < 0$. In both cases the point spectrum is not empty and one should expect precise exponential decay. We start with a simple Lemma, which is well known for symmetric differential operators having regular lower order terms (see e.g. [82]). Since our operator has a highly singular potential we provide the proof for convenience of the reader. Let us remark that it might be possible (though probably quite tedious) to derive the following assertion by a direct calculation.

Lemma 4.3.1. *Let $p^\alpha(t,x,y)$ denote the kernel of the semigroup P_t^α. Then for every $x, y \in \mathbb{R}^d \setminus \{0\}$*

$$\lim_{t\to\infty} e^{\lambda_\alpha t} p^\alpha(t,x,y) = \psi_\alpha(x)\psi_\alpha(y) \tag{4.3.1}$$

in two dimensions for every $\alpha \in \mathbb{R}$ and in three dimensions for every $\alpha < 0$.

Proof. Proving the assertion in three dimensions is very similar to proving it in two dimensions. Therefore we restrict ourselves to the latter case. The strategy is to derive first an L^2-version of the assertion and then use a Sobolev estimate to convert this into the pointwise result. Using the spectral theorem and the nature of the spectrum of $-\Delta_\alpha$ it is easy to see that for every $f \in L^2(\mathbb{R}^2)$

$$\begin{aligned}\lim_{t\to\infty} e^{\lambda_\alpha t} P_t^\alpha f &= \lim_{t\to\infty} e^{\lambda_\alpha t} \int_{\sigma(-\Delta_\alpha)} e^{-t\lambda}\, dE_\lambda f \\ &= \lim_{t\to\infty} e^{\lambda_\alpha t} \Big(e^{-t\lambda_\alpha} E(\{\lambda_\alpha\}) f + \int_{[0,\infty]} e^{-\lambda t}\, dE_\lambda f \Big) = E(\{\lambda_\alpha\}) f\end{aligned} \tag{4.3.2}$$

in the L^2-norm. Due to the simplicity of the eigenvalue λ_α we have

$$E(\{\lambda_\alpha\}) f = (\psi_\alpha, f)_{L^2(\mathbb{R}^2)} \psi_\alpha.$$

This already proves that for every $f \in L^2(\mathbb{R}^2)$

$$L^2 - \lim_{t\to\infty} e^{-\lambda_\alpha t} \int_{\mathbb{R}^2} p^\alpha(t, \cdot, y) f(y)\, dy = \psi_\alpha(\cdot)(\psi_\alpha, f)_{L^2(\mathbb{R}^2)}.$$

In order to use the Sobolev inequality we also need

$$\lim_{t\to\infty} \big(\|e^{\lambda_\alpha t} P_t^\alpha f - E(\{\lambda_\alpha\}) f\|_{L^2(\mathbb{R}^2)} + \|(-\Delta_\alpha)^k (e^{\lambda_\alpha t} P_t^\alpha f - E(\{\lambda_\alpha\}) f)\|_{L^2(\mathbb{R}^2)} \big) = 0,$$

which can be seen to hold true for all $f \in \mathcal{D}((-\Delta_\alpha)^k)$ and $k \in \mathbb{N}$ using equation (4.3.2) and the fact that P_t^α, $E(\{\lambda_\alpha\})$ and $-\Delta_\alpha$ commute. To convince ourselves that these results apply to the heat kernel of $-\Delta_\alpha$ as well we notice that for $x, y \in \mathbb{R}^d \setminus \{0\}$ $p^\alpha(t, \cdot, y), p^\alpha(t, x, \cdot) \in L^2(\mathbb{R}^2)$ and

$$p^\alpha(t+s, x, y) = \int_{\mathbb{R}^2} p^\alpha(t, x, z) p^\alpha(s, z, y)\, dz.$$

The first assertion can be easily proved by an abstract principle using spectral theory and elliptic regularity but also follows from the bound derived in Lemma 2.6 of [42] the second assertion is just the semigroup property. If we define $p_t^{\alpha,y}(x) = p^\alpha(t+1, x, y)$ for all $y \in \mathbb{R}^2 \setminus \{0\}$ and $t > 0$, the latter fact implies on the one hand $P_t^\alpha p_0^{\alpha,y} = p_t^{\alpha,y}$. And on the other hand it implies that $p_t^{\alpha,y} \in \mathcal{D}((-\Delta_\alpha)^k)$ for every $k \in \mathbb{N}$ since by the spectral calculus $\mathrm{Ran}(P_t^\alpha) \subset \mathcal{D}\big((-\Delta_\alpha)^k\big)$. Hence

$$\lim_{t\to\infty} \|(-\Delta_\alpha)^k (e^{\lambda_\alpha t} p_t^{\alpha,y} - E(\{\lambda_\alpha\}) p_0^{\alpha,y})\|_{L^2(\mathbb{R}^2)} = 0$$

for all $k \in \mathbb{N}_0$. Now we are ready to derive the pointwise assertion. In order to do so, recall from section 4.1 that every element ψ in the domain of $-\Delta_\alpha$ can be written as

$$\psi(x) = \phi_k(x) + 2\pi[2\pi\tilde{\alpha} - \Psi(1) + \log(k/2i)]^{-1} \phi_k(0) G_k(x),$$

where $k \in \mathbb{C}$ with $\Im k > 0$ and $\phi_k \in H^2(\mathbb{R}^2)$. From this section we also know that $(-\Delta_\alpha - k^2)\psi(x) = (-\Delta - k^2)\phi_k(x)$ holds true for every $x \in \mathbb{R}^2 \setminus \{0\}$. Therefore the

Sobolev-embedding theorem implies that for every $x_0 \in \mathbb{R}^2 \setminus \{0\}$ and $r < |x_0|/2$ there is a constant $C_{r,x_0} > 0$ such that

$$\sup_{x \in B_r(x_0)} |\psi(x)| \le C_{r,x_0}\big(\|(-\Delta_\alpha)\psi\|_{L^2(\mathbb{R}^2)} + \|\psi\|_{L^2(\mathbb{R}^2)}\big)$$

for all $\psi \in \mathcal{D}(-\Delta_\alpha)$. Since this implies that for every compact set $K \subset \mathbb{R}^2 \setminus \{0\}$ and some constant $C_K > 0$

$$\begin{aligned}\sup_{x\in K}\big|(e^{\lambda_\alpha t}p_t^{\alpha,y} - E(\{\lambda_\alpha\})p_0^{\alpha,y})(x)\big| &= \sup_{x\in K}\big|(e^{\lambda_\alpha t}p_t^{\alpha,y}(x) - e^{-\lambda_\alpha}\psi_\alpha(y)\psi_\alpha(x)\big| \\ &\le C_K\big(\|(-\Delta_\alpha)(e^{\lambda_\alpha t}p_t^{\alpha,y} - E(\{\lambda_\alpha\})p_0^{\alpha,y})\|_{L^2(\mathbb{R}^2)} \\ &\quad + \|e^{\lambda_\alpha t}p_t^{\alpha,y} - E(\{\lambda_\alpha\})p_0^{\alpha,y}\|_{L^2(\mathbb{R}^2)}\big) \to 0\end{aligned}$$

as $t \to \infty$, we conclude

$$\lim_{t\to\infty} e^{\lambda_\alpha t}p^\alpha(t,x,y) = \psi_\alpha(x)\psi_\alpha(y)$$

for every $y \in \mathbb{R}^2$ locally uniformly in $x \in \mathbb{R}^2$. Due to the symmetry of $p^\alpha(t,\cdot,\cdot)$ this holds also locally uniformly in (x, y). □

Remark 4.3.1. *An alternative route to the assertion of Lemma 4.3.1 is via an eigenvalue expansion of the heat kernel. It is rather easy to see, that the operator* P_t^α *acts on* $E((-\infty,0))L^2(\mathbb{R}^2) \oplus E([0,\infty))L^2(\mathbb{R}^2)$ *as the direct sum of the operator with integral kernel* $e^{-\lambda_\alpha t}\psi_\alpha(x)\psi_\alpha(y)$ *and an operator with integral kernel* $p_+^\alpha(t,x,y)$*, where for fixed* $x, y, \in \mathbb{R}^2 \setminus \{0\}$

$$p_+^\alpha(t,x,y) = \int_{\mathbb{R}^2} e^{-t|k|^2}\overline{\varphi(k,x)}\varphi(k,y)\,dk.$$

Using this formula one easily derives Lemma 4.3.1 using the fact that $p_+^\alpha(t,x,y)$ *decays locally uniformly in* $x, y \in \mathbb{R}^2 \setminus \{0\}$ *as* $t \to \infty$*.*

In the proof of the following Corollary we use an argument, which is often used in the theory of quasistationary distributions (see the previous chapters and e.g. Lemma 5.3 in [20]).

Corollary 4.3.1. *Let* $d = 2, 3$ *and let* $g \in L^\infty(\mathbb{R}^d)$ *be given. If* $d = 3$*, we assume additionally that* $\alpha < 0$*. Then*

$$\lim_{t\to\infty} e^{\lambda_\alpha t}e^{-t(-\Delta_\alpha)}g(x) = \psi_\alpha(x)\int_{\mathbb{R}^d}\psi_\alpha(y)g(y)\,dy$$

locally uniformly in $x \in \mathbb{R}^d \setminus \{0\}$ *and in particular*

$$\lim_{t\to\infty} e^{\lambda_\alpha t}\int_{\mathbb{R}^2}\mu(dx)\int_{\mathbb{R}^2}p^\alpha(t,x,y)\,dy = \int_{\mathbb{R}^2}\psi_\alpha(x)\mu(dx).$$

Proof. The proof of the Corollary for $d = 3$ is completely analogous to the proof for $d = 2$, therefore it will be given only for the latter case. Due to the fact that

$$\mathbb{E}_\mu\big[\langle \mathbf{1}, X_t\rangle\big] = \int_{\mathbb{R}^2} \mu(dx) P_t^\alpha \mathbf{1}(x),$$

the second assertion is a direct consequence of the first one. Thus it remains to prove the first assertion. From Corollary 2.5 of [42] it is known that the kernels $p^\alpha(t,x,y)$ satisfy the free heat equation $\partial_t p^\alpha(t,x,y) = \Delta_x p^\alpha(t,x,y)$ in $\mathbb{R}^2 \setminus \{0\}$. Therefore, we are allowed to use the parabolic Harnack inequality in order to conclude that for a locally bounded function $\theta : \mathbb{R}^2 \setminus \{0\} \to (0,\infty)$

$$p^\alpha(t,x,y) \le \theta(x)\, p^\alpha(t,z,y).$$

This gives

$$\begin{aligned} p^\alpha(t,x,y) &= \frac{\int_{B_\varepsilon(x)} p^\alpha(t,x,y)\psi_\alpha(z)\,dz}{\int_{B_\varepsilon(x)} \psi_\alpha(z)\,dz} \le \frac{\theta(x)\int_{B_\varepsilon(x)} p^\alpha(t,z,y)\psi_\alpha(z)\,dz}{\int_{B_\varepsilon(x)} \psi_\alpha(z)\,dz} \\ &\le \theta(x)\frac{e^{-\lambda_\alpha t}\psi_\alpha(y)}{\int_{B_\varepsilon(x)} \psi_\alpha(z)\,dz}. \end{aligned} \tag{4.3.3}$$

Using (4.3.3), dominated convergence and Lemma 4.3.1 we arrive at

$$\lim_{t\to\infty} e^{\lambda_\alpha t} \int_{\mathbb{R}^2} p^\alpha(t,x,y) g(y)\,dy = \psi_\alpha(x) \int_{\mathbb{R}^2} \psi_\alpha(y) g(y)\,dy.$$

□

Corollary 4.3.1 in particular shows that in two dimensions the exponential asymptotic large time behavior of the expected mass in bounded regions equals the expected total mass for every $\alpha \in \mathbb{R}$. This is no longer true in three dimensions. The fact that we have precise exponential large time behavior is analytically expressed by the fact that the bottom of the spectrum is a discrete eigenvalue if $d = 2$ or if $d = 3$ and $\alpha < 0$. In the case $d = 3$ and $\alpha \ge 0$ we cannot expect such a behavior because of $\sigma(-\Delta_\alpha) = \sigma_{ac}(-\Delta_\alpha)$, i.e. the spectrum is purely absolutely continuous. In order to investigate the expected mass in bounded regions in three dimonsion we prove the following simple Lemma.

Lemma 4.3.2. *For $d = 3$ and $\psi \in C_c(\mathbb{R}^3 \setminus \{0\})$ the generalized Fourier transform of ψ is continuous everywhere if $\alpha \neq 0$ and continuous everywhere except at $k = 0$ if $\alpha = 0$. Specifically,*

$$\mathcal{F}\psi(k) = \begin{cases} \frac{F_0(k)}{|k|} & \text{if } \alpha = 0 \\ F_\alpha(k) & \text{if } \alpha \neq 0, \end{cases}$$

where $F_0 : \mathbb{R}^3 \to \mathbb{C}$ and $F_\alpha : \mathbb{R}^3 \to \mathbb{C}$ are continuous and $F_0(0) = \frac{i}{(2\pi)^{\frac{3}{2}}} \int_{\mathbb{R}^3} \frac{\psi(x)}{|x|}\,dx$.

Proof. Due to (4.1.4) and (4.1.5), the generalized Fourier transform is given by

$$(2\pi)^{\frac{3}{2}}\mathcal{F}\psi(k) = \int \psi(x)\varphi(k,x)\,dx = \underbrace{\int \psi(x)e^{ik\cdot x}\,dx}_{I_1(k)} + \frac{1}{4\pi\alpha - i|k|}\underbrace{\int \psi(x)\frac{e^{i|k||x|}}{|x|}\,dx}_{I_2(k)}. \quad (4.3.4)$$

Since by assumption $\psi \in C_c(\mathbb{R}^3 \setminus \{0\})$ the theorem of dominated convergence justifies that I_2 as well as I_1 are continuous everywhere. For $\alpha = 0$ this implies that

$$(2\pi)^{\frac{3}{2}}\mathcal{F}\psi(k) = |k|^{-1}(|k|I_1(k) + iI_2(k)) =: |k|^{-1}\tilde{f}(k),$$

where $\tilde{f}$ is continuous everywhere. And for $\alpha \neq 0$ this implies that $\mathcal{F}\psi(k)$ is continuous everywhere. □

Now we are prepared for the case $\alpha \geq 0$. It will turn out that for $\alpha = 0$ the expectation neither scales like in the $\alpha > 0$ case nor like in the $\alpha < 0$ case. The mathematical reason being a significant change of behavior of the generalized eigenfunctions in the variable k due to the existence of a resonance at zero. The time evolution of some solutions to the Schrödinger equation would show spreading that is delayed compared to the $\alpha > 0$ case due to this changed behavior. The probabilistic counterpart of this effect is proved in the following Theorem, which shows that for $\alpha = 0$ the expected mass in bounded regions (which refers to the expression $\mathbb{E}_\mu[\langle\varphi, X_r\rangle]$ for $\varphi \in C_c^\infty(\mathbb{R}^3 \setminus \{0\})$)decays slower than in the case $\alpha > 0$. In analogy to the Schrödinger case one could therefore call the modified scaling of the expectation a resonance effect. The formulation of this result and also it proof will probably remind the reader of what we have done in the chapters on quasistationary distributions.

Theorem 4.3.1. *Let $d = 3$, $\psi \in C_c(\mathbb{R}^3 \setminus \{0\})$ and let μ be a measure with compact support in $\mathbb{R}^3 \setminus \{0\}$ then expectation $\mathbb{E}_\mu[\langle X_t, \psi\rangle]$ shows the following large time behavior*

$$\lim_{t\to\infty} t^{1/2}\mathbb{E}_\mu\big[\langle X_t, \psi\rangle\big] = \frac{\pi^{3/2}}{2}\int_{\mathbb{R}^3}\frac{\psi(y)}{|y|}\,dy \int_{\mathbb{R}^3}|x|^{-1}\,\mu(dx) \qquad \text{if } \alpha = 0$$

and

$$\lim_{t\to\infty} t^{3/2}\mathbb{E}_\mu\big[\langle X_t, \psi\rangle\big] = \frac{1}{(4\pi)^{\frac{3}{2}}}\int_{\mathbb{R}^3}\varphi(0,x)\psi(x)\,dx \int_{\mathbb{R}^3}\varphi(0,x)\,\mu(dx) \qquad \text{if } \alpha > 0.$$

Proof. The crucial step is to rewrite the expectation value such that it admits to be calculated analytically. This can be done by using the formula for the heat kernel given in Remark 4.3.1 for the case that there is no eigenvalue, that is

$$p^\alpha(t,x,y) = \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3} e^{-t|k|^2}\overline{\varphi(k,x)}\varphi(k,y)\,dk.$$

Hence we find that

$$(2\pi)^{\frac{3}{2}}\mathbb{E}_\mu\big[\langle X_t,\psi\rangle\big] = \langle X_0, P_t^\alpha \psi\rangle = \int\int \hat{\psi}(k)\overline{\varphi(k,x)}e^{-|k|^2t}\,dk\,\mu(dx).$$

Using this, the explicit form of the generalized eigenfunctions φ and Lemma 4.3.2 we can calculate how the expectation value scales in the large time limit. In the notation of the previous lemma we have for $\alpha = 0$,

$$\begin{aligned}(2\pi)^{\frac{3}{2}}P_t^\alpha\psi(x) &= \int \frac{F_0(k)}{|k|}\overline{\left(e^{ik\cdot x} + \frac{i}{|k||x|}e^{i|k||x|}\right)}e^{-|k|^2t}\,dk \\ &= \int_0^{2\pi}\int_{-1}^{1}\int_0^\infty F_0(k)e^{-i|k||x|\cos\theta}|k|e^{-|k|^2t}\,dk\,d(\cos\theta)\,d\phi \\ &\quad - \frac{i}{|x|}\int_0^{2\pi}\int_0^\pi\int_0^\infty F_0(k)e^{-i|k||x|}e^{-|k|^2t}\sin\theta\,dk\,d\theta\,d\phi,\end{aligned}$$

where we have used Lemma 4.3.2 and equation (4.1.4) in the first equality and spherical coordinates in the second. Due to the factor $e^{-|k|^2t}$, the integral is dominated by those $k \in \mathbb{R}^3$ which have modulus close to zero if $t \gg 1$. Therefore we can approximate the integral by replacing $F_0(k)$ by $F_0(0)$ for all k, which will be exact in the limit of t tending to ∞. Thus we are left with

$$\begin{aligned}(2\pi)^{\frac{3}{2}}P_t^\alpha\psi(x) &\approx 2\pi F_0(0)\int_0^\infty |k|e^{-|k|^2t}\left(\int_{-1}^1 e^{-i|k||x|\cos\theta}\,d(\cos\theta)\right)dk \\ &\quad - 4\pi F_0(0)\frac{i}{|x|}\int_0^\infty e^{-|k|^2t}e^{-i|k||x|}\,dk \\ &\approx \pi F_0(0)\left(2\int_0^\infty e^{-|k|^2t}\frac{-1}{i|x|}(e^{-i|k||x|} - e^{i|k||x|})\,dk + 4\frac{i}{|x|}\int_0^\infty e^{-|k|^2t}e^{-i|k||x|}\,dk\right) \\ &\approx \frac{-i\pi}{|x|}F_0(0)\int_0^\infty 2(e^{i|k||x|} + e^{-i|k||x|})e^{-|k|^2t}\,dk \\ &\approx \frac{-i\pi^{3/2}}{2|x|}F_0(0)t^{-1/2}e^{-\frac{|x|^2}{4t}}.\end{aligned}$$

Using the particular form of the generalized Fourier transform of ψ given in (4.3.4), this implies

$$\lim_{t\to\infty} t^{1/2}\mathbb{E}_\mu\big[\langle X_t,\psi\rangle\big] = \frac{\pi^{3/2}}{2}\int_{\mathbb{R}^3}\frac{\psi(x)}{|x|}\,dx\int |x|^{-1}\,\mu(dx).$$

In the case of $\alpha > 0$, the explicit form of the generalized eigenfunctions gives

$$(2\pi)^{\frac{3}{2}}P_t^\alpha\psi(x) = \int \hat{\psi}(k)e^{ik\cdot x}e^{-|k|^2t}\,dk + \frac{1}{|x|}\int \hat{\psi}(k)\frac{1}{4\pi\alpha - i|k|}e^{i|k||x|}e^{-|k|^2t}\,dk.$$

Since $\hat{\psi}(k)$ is continuous everywhere for $\alpha \neq 0$ we can approximate the integral for the same reason as above by

$$\begin{aligned}(2\pi)^{\frac{3}{2}} P_t^\alpha \psi(x) &\approx \hat{\psi}(0) \int e^{-|k|^2 t} e^{-ik\cdot x}\, dk + \frac{1}{|x|}\frac{\hat{\psi}(0)}{4\pi\alpha} \int e^{-|k|^2 t} e^{-i|k||x|}\, dk \\ &\approx \hat{\psi}(0) \left(\prod_{i=1}^{3} \int_{-\infty}^{\infty} e^{-k^2 t} e^{-ikx_i}\, dk + \frac{1}{\alpha|x|} \int_0^\infty |k|^2 e^{-|k|^2 t} e^{-i|k||x|}\, dk \right).\end{aligned}$$

Now, the Fourier transform of a Gaussian functions is well known. And in order to calculate the second integral, note that for $t \gg 1$ the integrand is strongly peaked around $k = \frac{1}{\sqrt{t}}$, which is the maximum of $|k|^2 e^{-|k|^2 t}$. Therefore,

$$\begin{aligned}(2\pi)^{\frac{3}{2}} P_t^\alpha \psi(x) &\approx \hat{\psi}(0) \left(\pi^{3/2} t^{-3/2} e^{-\frac{|x|^2}{4t}} + \frac{1}{\alpha|x|} e^{-i|k|\frac{|x|}{\sqrt{t}}} \int_0^\infty |k|^2 e^{-|k|^2 t}\, dk \right) \\ &\approx \hat{\psi}(0) \left(\pi^{3/2} e^{-\frac{|x|^2}{4t}} + \frac{e^{-i|k|\frac{|x|}{\sqrt{t}}}}{\alpha|x|} \frac{\sqrt{\pi}}{4} \right) t^{-3/2},\end{aligned}$$

which implies that

$$\lim_{t\to\infty} t^{3/2} \mathbb{E}_\mu \big[\langle X_t, \psi \rangle \big] = 2^{-\frac{3}{2}} \int \hat{\psi}(0) \left(1 + \frac{1}{4\pi\alpha|x|} \right) \mu(dx).$$

□

In the following remark we summarizearize our results and give some heuristic explanations. Since the rigorous probabilistic meaning of the process is not yet clear, the following heuristic explanations have a preliminary character.

Remark 4.3.2. *The difference between the long time behavior of* $\mathbb{E}_\mu\big[\langle \varphi, X_t\rangle\big]$ *(*$\varphi \in C_c(\mathbb{R}^3)$*) established in Theorem 4.3.1 and the result of [43] concerning the long time behavior of* $\mathbb{E}_\mu\big[\langle \mathbf{1}, X_t\rangle\big]$ *(see (4.2.4)) seems to be due to the transience of the three-dimensional Brownian motion. Because of the transient behavior one should expect that the mass of the three-dimensional superprocess* X_t *has some tendency to escape to infinity as* $t \to \infty$*. If the point source does not produce enough new mass at* 0*, i.e. if* $\alpha \geq 0$*, then the portion of mass which moves out to infinity is bigger than the portion of newly created mass at* 0*. Therefore the behavior of the expected total mass differs from the behavior of expected mass in bounded regions. In the case* $\alpha = 0$ *the point source seems to produce almost enough mass at* 0 *in order to compensate the transience of the underlying spatial motion. If the production of new mass at* 0 *is high enough, i.e. if* $\alpha < 0$ *then effect of the transience of the underlying spatial motion is negligible and the large time behavior of the expected mass in bounded and that in unbounded regions coincides. Due to recurrence the picture is different in two dimensions, i.e.* $\mathbb{E}_\mu\big[\langle \varphi, X_t\rangle\big]$ $(\varphi \in C_c(\mathbb{R}^3))$ *and* $\mathbb{E}_\mu\big[\langle \mathbf{1}, X_t\rangle\big]$ *exhibit the same behavior.*

4.3.2 Pathwise Large Time Behavior

We are now prepared to present the weak law of large numbers for the two-dimensional super-Brownian motion with single point source, i.e. we show that for $f \in C_c(\mathbb{R}^2 \setminus \{0\})$ the random variable $e^{\lambda_\alpha t}\langle f, X_t\rangle$ converges in $L^2(\mathbb{P}_\mu)$ and therefore also in probability. This result will later be used in the derivation of the strong law of large numbers. Recall that we only consider the two-dimensional case with $\beta = 1$ in this section. Let us point out that due to the spectral gap, which exists in the two dimensional version of our problem, one might expect to be able to use arguments similar to the recent work [22] of Chen et Al, in order to prove the law of large numbers. That is the case in so far, that we also decompose the function f into an eigenfunction part and the orthogonal part. But as already mentioned above, the semigroup $e^{-t(-\Delta_\alpha)}$ is not ultracontractive in dimensions greater than one. So the arguments of Chen et Al. are not directly applicable, which is why we take a different route and use expansions in generalized eigenfunctions.

Theorem 4.3.2. *Let $d = 2$ and $\alpha \in \mathbb{R}$ be given, let $(X_t)_{t\geq 0}$ denote the super-Brownian motion with a single point source associated to $-\Delta_\alpha$ and assume that (4.2.2) holds true. Then for every measure $\mu \in \mathcal{M}(\mathbb{R}^2 \setminus \{0\})$, having compact support in $\mathbb{R}^2 \setminus \{0\}$ there exists a non-negative, non-degenerate random variable N_α such that $\mathbb{E}_\mu[N_\alpha] = \int_{\mathbb{R}^2} \psi_\alpha(x)\,\mu(dx)$ and such that for $f \in C_c^\infty(\mathbb{R}^2 \setminus \{0\})$*

$$\lim_{t\to\infty} e^{\lambda_\alpha t}\langle X_t, f\rangle = N_\alpha \cdot \int_{\mathbb{R}^2} f(x)\psi_\alpha(x)\,dx$$

as well as

$$\lim_{t\to\infty} \frac{\langle X_t, f\rangle}{\mathbb{E}_\mu\big[\langle X_t, f\rangle\big]} = \frac{N_\alpha}{\int_{\mathbb{R}^2} \psi_\alpha(x)\,\mu(dx)}$$

in $L^2(\mathbb{P}_\mu)$. The random variable N_α is defined as

$$N_\alpha := \lim_{t\to\infty} e^{\lambda_\alpha t}\langle \psi_\alpha, X_t\rangle$$

where the limit holds almost surely and in $L^2(\mathbb{P}_\mu)$.

Remark 4.3.3. *Our results and in particular the law of large numbers allows to uncover some probabilistic features of the process. First, we conclude that also on a pathwise level the mass in every bounded region grows precisely exponential in time, where the exponential rate coincides with the bottom of the L^2-spectrum. Second, one sees by inspecting the explicit form of the eigenfunction ψ_α that most of the mass will be concentrated in a neighborhood of zero. This agrees with the heuristic interpretation of 0 as a single point source.*

Proof. Let f in $C_c^\infty(\mathbb{R}^2 \setminus \{0\})$ be given. Recall from section 4.1 that the spectrum of $-\Delta_\alpha$ is of the form $\sigma(-\Delta_\alpha) = \{\lambda_\alpha\} \cup [0,\infty)$ and that $[0,\infty)$ belongs to the absolutely continuous part of the spectrum. The eigenvalue λ_α is simple with a (modulo multiples) unique positive eigenfunction ψ_α. As an element of $L^2(\mathbb{R}^2)$ the function f can therefore be decomposed as $f = (f, \psi_\alpha)\psi_\alpha + g$, where $g \in L^2(\mathbb{R}^2)$ belongs to the absolutely continuous subspace $E([0,\infty))L^2(\mathbb{R}^2)$ of $L^2((\mathbb{R}^2)$. Thus

$$e^{\lambda_\alpha t}\langle f, X_t\rangle = e^{\lambda_\alpha t}(f, \psi_\alpha)\langle \psi_\alpha, X_t\rangle + e^{\lambda_\alpha t}\langle g, X_t\rangle.$$

Observe first that the process $(M_t^\alpha)_{t\geq 0} = (e^{\lambda_\alpha t}\langle \psi_\alpha, X_t\rangle)_{t\geq 0}$ is a non-negative martingale as according to the Markov property for $\mathcal{F}_s = \sigma(X_r; r \leq s)$ and $t > s$

$$\begin{aligned}\mathbb{E}_\mu\big[M_t^\alpha \mid \mathcal{F}_s\big] &= e^{\lambda_\alpha t}\mathbb{E}_{X_s}\big[\langle \psi_\alpha, X_{t-s}\rangle\big] = e^{\lambda_\alpha t}\langle e^{-(t-s)\Delta_\alpha}\psi_\alpha, X_s\rangle \\ &= e^{\lambda_\alpha s}\langle \psi_\alpha, X_s\rangle = M_s^\alpha.\end{aligned} \tag{4.3.5}$$

Moreover, we have by formula (4.2.2) again

$$\begin{aligned}\mathbb{E}_\mu\Big[\big(e^{\lambda_\alpha t}\langle X_t, \psi_\alpha\rangle\big)^2\Big] &= e^{2\lambda_\alpha t}\langle P_t^\alpha \psi_\alpha, X_0\rangle^2 + e^{2\lambda_\alpha t}\Big\langle \int_0^t P_s^\alpha\big((P_{t-s}^\alpha\psi_\alpha)^2\big)\,ds, X_0\Big\rangle \\ &= \langle \psi_\alpha, X_0\rangle^2 + e^{2\lambda_\alpha t}\Big\langle \int_0^t e^{-2(t-s)\lambda_\alpha} P_s^\alpha(\psi_\alpha^2)\,ds, X_0\Big\rangle \\ &= \langle \psi_\alpha, X_0\rangle^2 + \Big\langle \int_0^t e^{2\lambda_\alpha s} P_s^\alpha(\psi_\alpha^2)\,ds, X_0\Big\rangle \\ &\leq \langle \psi_\alpha, X_0\rangle^2 + \Big\langle \int_0^\infty e^{2\lambda_\alpha s} P_s^\alpha(\psi_\alpha^2)\,ds, X_0\Big\rangle.\end{aligned} \tag{4.3.6}$$

We will show that $\sup_{t\geq 0}\mathbb{E}\big[(M_t^\alpha)^2\big] = \sup_{t\geq 0}\mathbb{E}\big[(e^{\lambda_\alpha t}\langle X_t, \psi_\alpha\rangle)^2\big] < \infty$. In order to do so, observe that $\psi_\alpha \in L^4(\mathbb{R}^2)$ since the divergence at 0 is only logarithmic and that $\int_0^\infty e^{2\lambda_\alpha s}P_s^\alpha\,ds = (-\Delta_\alpha - 2\lambda_\alpha)^{-1}$. These facts together with (4.1.6) and the Sobolev embedding give for every compact set $K \subset \mathbb{R}^2 \setminus \{0\}$

$$\sup_{x\in K}\bigg(\int_0^\infty e^{2\lambda_\alpha s}P_s^\alpha\,ds\,\psi_\alpha^2\bigg)(x) = \sup_{x\in K}(-\Delta_\alpha - 2\lambda_\alpha)^{-1}\psi_\alpha^2(x) \leq C_K < \infty. \tag{4.3.7}$$

Observe that we used the fact that the resolvent is the Laplace transform of the semigroup. Equation (4.3.5) together with the inequalities (4.3.6) and (4.3.7) imply that $(M_t^\alpha)_{t\geq 0}$ is an L^2-bounded martingale. By standard martingale convergence results there exists a non-negative random variable N_α such that

$$\lim_{t\to\infty} M_t^\alpha = N_\alpha \;\; \mathbb{P}_\mu\text{-a.s. and in } L^2(\mathbb{P}_\mu). \tag{4.3.8}$$

Moreover, the random variable N_α is non-degenerate as due to the strict positivity of ψ_α

$$\mathbb{E}_\mu\big[N_\alpha\big] = \lim_{t\to\infty} e^{\lambda_\alpha t}\mathbb{E}_\mu\big[\langle \psi_\alpha, X_t\rangle\big] = \int_{\mathbb{R}^2}\psi_\alpha(x)\,\mu(dx) > 0$$

for every non-trivial measure μ with compact support in $\mathbb{R}^2 \setminus \{0\}$. In order to get the assertion of the theorem it remains to show that

$$\lim_{t\to\infty} e^{\lambda_\alpha t}\langle g, X_t\rangle = 0$$

in $L^2(\mathbb{P}_\mu)$. Heuristically, this is quite obvious since g belongs to the subspace $E([0,\infty))L^2(\mathbb{R}^2)$ and therefore $e^{t\Delta_\alpha}g$ converges to zero at least in $L^2(\mathbb{R}^2)$. Since g belongs to the absolutely continuous spectral subspace, we can expand g in generalized eigenfunctions as

$$g(x) = \mathcal{F}^{-1}\hat{g}(x) = \int_{\mathbb{R}^2} \overline{\varphi(k,x)}\hat{g}(k)\,\frac{dk}{2\pi}.$$

Therefore we get

$$\langle g, X_t\rangle = \int_{\mathbb{R}^2} \langle\overline{\varphi(k,\cdot)}, X_t\rangle \hat{g}(k)\,\frac{dk}{2\pi}.$$

The use of Fubini's theorem in this step is easily justified, since Lemma 2.6 from [42] and Proposition 4.1.1 imply

$$\mathbb{E}_\mu\left[\int_{\mathbb{R}^2} \langle|\varphi(k,\cdot)|, X_t\rangle|\hat{g}(k)|\,dk\right] = \int_{\mathbb{R}^2}\langle P_t^\alpha|\varphi(k,\cdot)|, X_0\rangle|\hat{g}(k)|\,dk < \infty. \tag{4.3.9}$$

In order to prove (4.3.9) observe that due to the asymptotic bahavior of the Hankel functions (4.1.11) the absolute value $|\varphi(k,x)|$ of the generalized eigenfunction $\varphi(k,x)$ can be dominated by a function $\tilde{\varphi}(k,x) := \frac{C_1}{\sqrt{|k||x|}} + C_2$ where $C_1, C_1 > 0$ are appropriate constants. Therefore we get with Lemma 2.6 from [42] (see (4.1.9)) for another constant C_3

$$\begin{aligned}\big(P_t^\alpha|\varphi(k,\cdot)|\big)(x) \le{}& \frac{C_1}{\sqrt{|k|}}\int_{\mathbb{R}^2} p(t,x,y)|y|^{-1/2}\,dy + C_2 \\ &+ C_3\Big(\frac{C_1}{\sqrt{|k|}}\frac{1}{\sqrt{t|x|}}e^{-|x|^2/4t}\int_{\mathbb{R}^2}\frac{1}{|y|}e^{-|y|^2/4t}\,dy \\ &+ \frac{C_2}{\sqrt{t|x|}}e^{-|x|^2/4t}\int_{\mathbb{R}^2}\frac{1}{\sqrt{|y|}}e^{-|y|^2/4t}\,dy\Big).\end{aligned} \tag{4.3.10}$$

The inequality (4.3.10) and Proposition 4.1.1 show the validity of (4.3.9). Thus by equation (4.3.9) we have $\int_{\mathbb{R}^2}\langle|\varphi(k,\cdot)|, X_t\rangle\hat{g}(k)\,dk < \infty$ almost surely and Fubini's theorem applies. Let us consider the second moment of the process $(\langle\varphi(k,\cdot), X_t\rangle)_{t\ge0}$, that is

$$\begin{aligned}\mathbb{E}_\mu\big[|\langle\varphi(k,\cdot), X_t\rangle|^2\big] &= |\langle P_t^\alpha\varphi(k,\cdot), X_0\rangle|^2 + \int_0^t \big\langle P_s^\alpha\big[|P_{t-s}^\alpha\varphi(k,\cdot)|^2\big], X_0\big\rangle\,ds \\ &= e^{-2t|k|^2}|\langle\varphi(k,\cdot), X_0\rangle|^2 + \int_0^t e^{-2(t-s)|k|^2}\big\langle P_s^\alpha\big[|\varphi(k,\cdot)|^2\big], X_0\big\rangle\,ds \\ &= e^{-2t|k|^2}I_1(k) + I_2(k,t).\end{aligned}$$

Jensen's inequality and Fubini's theorem give

$$\begin{aligned}(2\pi)^2\mathbb{E}_\mu\big[\big|e^{\lambda_\alpha t}\langle g, X_t\rangle\big|^2\big] &= e^{2\lambda_\alpha t}\,\mathbb{E}_\mu\left[\left|\int_{\mathbb{R}^2}\hat{g}(k)\langle\overline{\varphi(k,\cdot)}, X_t\rangle\,dk\right|^2\right]\\ &\le C_g e^{2\lambda_\alpha t}\,\mathbb{E}_\mu\left[\int_{\mathbb{R}^2}|\hat{g}(k)||\langle\varphi(k,\cdot), X_t\rangle|^2\,dk\right]\\ &= C_g e^{2\lambda_\alpha t}\int_{\mathbb{R}^2}|\hat{g}(k)|\mathbb{E}_\mu\big[|\langle\varphi(k,\cdot), X_t\rangle|^2\big]\,dk \qquad (4.3.11)\\ &= C_g e^{2\lambda_\alpha t}\int_{\mathbb{R}^2}|\hat{g}(k)|e^{-2t|k|^2}I_1(k)\,dk\\ &+ C_g e^{2\lambda_\alpha t}\int_{\mathbb{R}^2}|\hat{g}(k)|I_2(k,t)\,dk.\end{aligned}$$

where by Proposition 4.1.1 $C_g = \int_{\mathbb{R}^2}|\hat{g}(k)|\,dk < \infty$. The first term on the right of (4.3.11) converges obviously to zero. and due to Proposition 4.1.1 and the form of the generalized eigenfunctions we even get

$$\lim_{t\to\infty} e^{2\lambda_\alpha t}\int_{\mathbb{R}^2}|\hat{g}(k)|e^{-2t|k|^2}I_1(k)\,dk = 0.$$

Concerning the second term observe that

$$\begin{aligned}e^{2\lambda_\alpha t}\int_{\mathbb{R}^2}|\hat{g}(k)|I_2(k,t)\,dk &= e^{2\lambda_\alpha t}\int_{\mathbb{R}^2}|\hat{g}(k)|\int_0^t e^{-2(t-s)|k|^2}\big\langle P_s^\alpha\big[|\varphi(k,\cdot)|^2\big], X_0\big\rangle\,ds\,dk\\ &\le e^{(\lambda_\alpha+\varepsilon)t}\int_{\mathbb{R}^2}|\hat{g}(k)|\left\langle\int_0^t e^{(\lambda_\alpha-\varepsilon)s}P_s^\alpha\big[|\varphi(k,\cdot)|^2\big]\,ds, X_0\right\rangle dk\end{aligned}$$

In order to handle that term, divide the function $|\varphi(k,\cdot)|^2$ in two parts, by letting $|\varphi(k,\cdot)|^2 = h_1(k,\cdot) + h_2(k,\cdot)$. Here we have defined $h_1(k,\cdot) = |\varphi(k,\cdot)|^2\mathbf{1}_{B_{1/|k|}(0)}(\cdot)$ and $h_2(k,\cdot) = |\varphi(k,\cdot)|^2(1-\mathbf{1}_{B_{1/|k|}(0)})(\cdot)$. Then $\sup_{x,k\in\mathbb{R}^2\setminus\{0\}}|h_2(k,x)| < C < \infty$ and we get for $x \in \operatorname{supp}(X_0)$ and $k \in \mathbb{R}^2\setminus\{0\}$

$$\left(\int_0^t e^{(\lambda_\alpha-\varepsilon)s}P_s^\alpha h_2(k,\cdot)\,ds\right)(x) \le C\int_0^\infty e^{(\lambda_\alpha-\varepsilon)s}\sup_{x\in\operatorname{supp}(\mu)}P_s^\alpha\mathbf{1}(x)\,ds < \infty \qquad (4.3.12)$$

by Corollary 4.3.1. So it remains to consider

$$\int_0^t e^{(\lambda_\alpha-\varepsilon)s}P_s^\alpha h_1(k,\cdot)\,ds\,(x).$$

Observe that, due to the particular form of the generalized eigenfunctions, we have

$$\begin{aligned}h_1(k,\cdot) = |\varphi(k,\cdot)|^2\mathbf{1}_{B_{1/|k|}(0)}(\cdot) &= |e^{ikx} + A(\alpha,k)H_0^{(1)}(|k||x|)|^2\mathbf{1}_{B_{1/|k|}(0)}(x)\\ &\le 2 + 2|A(\alpha,k)|^2|H_0^{(1)}(|k||x|)|^2\mathbf{1}_{B_{1/|k|}(0)}(x) = 2 + \tilde{h}_1(k,\cdot),\end{aligned}$$

where $A(\alpha,k)=\frac{i\pi}{2}(2\pi\tilde{\alpha}-\Psi(1)+\ln(|k|/2i))^{-1}$. By the argument applied above, it therefore remains to consider $\tilde{h}_1(k,\cdot)$. Although $\tilde{h}_1(k,\cdot)$ is not bounded, it belongs to $L^2(\mathbb{R}^2)$. And this together with formula (4.1.7), which expresses the $-\Delta_\alpha$ resolvent in terms of the $-\Delta$ resolvent, implies that for $x\in K=\operatorname{supp}(\mu)\subset\mathbb{R}^2\setminus\{0\}$

$$\begin{aligned}\int_0^t e^{(\lambda_\alpha-\varepsilon)s}P_s^\alpha\tilde{h}_1(k,\cdot)\,ds\,(x)&\le\int_0^\infty e^{(\lambda_\alpha-\varepsilon)s}P_s^\alpha\tilde{h}_1(k,\cdot)\,ds\,(x)\\&=\big((-\Delta+(\lambda_\alpha-\varepsilon))^{-1}\tilde{h}_1(k,\cdot)\big)(x)\\&+2\pi[2\pi\tilde{\alpha}-\Psi(1)+\ln(l/2i)]^{-1}\big(\overline{G_l},\tilde{h}_1(k,\cdot)\big)_{L^2(\mathbb{R}^2)}G_l(x)\\&\le C_K\|\tilde{h}_1(k,\cdot)\|_{L^2(\mathbb{R}^2)}\\&+2\pi[2\pi\tilde{\alpha}-\Psi(1)+\ln(l/2i)]^{-1}\big(\overline{G_l},\tilde{h}_1(k,\cdot)\big)_{L^2(\mathbb{R}^2)}G_l(x)\\&=C_K\|\tilde{h}_1(k,\cdot)\|_{L^2(\mathbb{R}^2)}\\&+\frac{4\pi}{\alpha+\ln(e^{-\alpha}+\varepsilon)}\left(\frac{i}{4}H_0^{(1)}(l|x|),\tilde{h}_1(k,x)\right)_{L^2(\mathbb{R}^2)}\frac{i}{4}H_0^{(1)}(l|x|),\end{aligned}\tag{4.3.13}$$

where the representation of the resolvent as Laplace transform of the semigroup and the Sobolev embedding was used as explained in the very beginning of section 2 and where $\tilde{\alpha}$ was expressed in terms of α with the help of $l=i\sqrt{\varepsilon-\lambda_\alpha}$. Since the first term of the right hand side can be estimated by

$$C_K\|\tilde{h}_1(k,\cdot)\|_{L^2(\mathbb{R}^2)}=C_K\left(\int_{B_{|k|^{-1}}(0)}|\tilde{h}_1(k,\cdot)|^2\,dx\right)^{\frac{1}{2}}=\frac{\tilde{C}_K}{|k|}\left(\int_{B_1(0)}|H_0^{(1)}(x)|^4\,dx\right)^{\frac{1}{2}}\tag{4.3.14}$$

and since the second term has at most a logarithmic singularity in k due to $\tilde{h}_1$, we can conclude with the help of Proposition 2 that

$$\int_{\mathbb{R}^2}|\hat{g}(k)|\left\langle\int_0^\infty e^{(\lambda_\alpha-\varepsilon)s}P_s^\alpha\big[h_1(k,\cdot)\big]\,ds,X_0\right\rangle dk<\infty.$$

This finishes the proof that $e^{\lambda_\alpha t}\langle g,X_t\rangle$ converges to zero as t tends to ∞, thereby implying the assertion of the Theorem. Notice that the L^2-convergence of $e^{\lambda_\alpha t}\langle g,X_t\rangle$ is even exponentially fast with an exponential rate, which is at least $\lambda_\alpha+\varepsilon$ for every $\varepsilon>0$. □

Remark 4.3.4. *As an immediate consequence of the proof of Theorem 4.3.2 and the Borel-Cantelli Lemma, we get the following assertion: Let μ be an initial measure with compact support in $\mathbb{R}^2\setminus\{0\}$. Let $(t_n)_{n\in\mathbb{N}}$ be a sequence, such that for some $\varepsilon>0$ $\sum_{n=1}^\infty e^{t_n(\lambda_\alpha+\varepsilon)}<\infty$. Then for every $f\in C_c^\infty(\mathbb{R}^2\setminus\{0\})$ there exists a set Ω_f of full $\mathbb{P}_\mu$-measure such that for every $\omega\in\Omega_f$*

$$\lim_{n\to\infty}e^{\lambda_\alpha t_n}\langle f,X_{t_n}(\omega)\rangle=N_\alpha(\omega)\int_{\mathbb{R}^2}f(x)\psi_\alpha(x)\,dx.$$

Remark 4.3.4 makes it rather plausible that even the strong law of large numbers holds. Indeed Chen et Al. found an interesting highly non-trivial method, which allows to get rid of the assumptions on the sequence $(t_n)_n$ in Remark 4.3.4 for a large class of superprocesses. In their approach Perkin's time-dependent Itô formula (see [73]) for superprocesses plays an important role. Since to our best knowledge such an Itô formula is not known to hold for our process we proceed differently. The proof of the strong law will be given in the remaining part of this work. Choose a subset $\mathcal{D} = \{d_n \mid n \in \mathbb{N}\} \subset C_c^\infty(\mathbb{R}^2 \setminus \{0\})$ such that for every compact set $K \subset E$ there is $n_0 \in \mathbb{N}$ with $0 \leq d_{n_0} \leq 1$ and $d_{n_0} \upharpoonright K = 1$. Let $\{U_k \mid k \in \mathbb{N}\}$ be a countable basis of open sets in $\mathbb{R}^2 \setminus \{0\}$, which is closed under finite intersections. Assume that for every U_k there exists sequence $(\tilde{d}_l)_{l\in\mathbb{N}} \subset \mathcal{D}$ such that for $l \in \mathbb{N}$ $\tilde{d}_l < \tilde{d}_{l+1}$ and $\tilde{d}_l \nearrow \mathbf{1}_{U_k}$ as $l \nearrow \infty$. Then

$$\rho(\cdot,\cdot) := \sum_{n=1}^{\infty} 2^{-n} \min\big(1, \big|\langle d_n, \cdot\rangle - \langle d_n, \cdot\rangle\big|\big)$$

defines a metric for the vague topology (see the proof of Satz 31.5 and in particular equation (31.4) in [8]). Thus a sequence of measures $(\mu_k)_{k\in\mathbb{N}}$ on $\mathbb{R}^2 \setminus \{0\}$ converges vaguely to the measure $\tilde{\mu}$ iff for every $n \in \mathbb{N}$ $\lim_{k\to\infty}\langle d_n, \mu_k\rangle = \langle d_n, \tilde{\mu}\rangle$. It is well known (see Satz 31.2 in [8]) that a set $\mathcal{H}$ of measures is vaguely relatively compact iff for every $n \in \mathbb{N}$ $\sup_{\mu\in\mathcal{H}}\langle d_n, \mu\rangle < \infty$. For this and further facts concerning the vague topology we refer to section 31 of [8]. In the following we restrict ourselves to rational times. Of course the set of rationals can be replaced by any other countable dense subset of $[0, \infty)$. Such a restriction is necessary in our setting since we do not know, whether the process has a 'good' modification, which is e.g. right continuous with left limits. Though it is desirable from a aesthetical point of view to remove such a restriction we believe that our result is still interesting and clarifies the large time behavior to some extend.

Assume that we show that there exists a set Ω_1 of full $\mathbb{P}_\mu$-measure such that for $\omega \in \Omega_1$ and for every $n \in \mathbb{N}$

$$\sup_{\mathbb{Q}\ni t\geq 0} e^{\lambda_\alpha t}\langle d_n, X_t(\omega)\rangle < \infty. \tag{4.3.15}$$

Then the set $\{e^{\lambda_\alpha t}X_t(\omega) \mid \mathbb{Q} \ni t \geq 0\}$ is precompact in the vague topology and therefore has limit points. If we can show that there exists only one limit point $\mu(\omega)$, then $e^{\lambda_\alpha t}X_t(\omega)$ converges to $\mu(\omega)$ in the vague topology. By Remark 4.3.4 there exists a set of full measure Ω_2 such that for every $\omega \in \Omega_2$, $n \in \mathbb{N}$ and every sequence $(t_k)_{k\in\mathbb{N}}$ with $\sum_{k=1}^\infty e^{t_k(\lambda_\alpha+\varepsilon)} < \infty$ for some $\varepsilon > 0$

$$\lim_{k\to\infty} e^{\lambda_\alpha t_k}\langle d_n, X_{t_k}(\omega)\rangle = N_\alpha(\omega)\int_{\mathbb{R}^2} d_n(x)\psi_\alpha(x)\,dx$$

Using our assumptions concerning $\mathcal{D}$ this shows that for $\omega \in \Omega_0 = \Omega_1 \cap \Omega_2$ the only limit point $\mu(\omega)$ of $\{e^{\lambda_\alpha t}X_t(\omega) \mid t \geq 0\}$ is the measure given by $\mu(\omega)(f) = N_\alpha(\omega)\int_{\mathbb{R}^2} d_n(x)\psi_\alpha(x)\,dx$. Thus it remains to prove (4.3.15). This will be done in the following Lemma 4.3.4. In the proof of this Lemma we use the following simple Lemma

Lemma 4.3.3 (Lemma 3.4 in [93])**.** *If Y is a non-negative random variable such that $\mathbb{P}(Y > y) \leq My^{-2}$, then for every $\eta > 0$*

$$\mathbb{E}\big[Y\big] \leq \eta + M\,\eta^{-1}.$$

The proof of this Lemma consits only in writing the expectation as the integral $\int_0^\infty \mathbb{P}(Y > y)\,dy = \int_0^\eta \mathbb{P}(Y > y)\,dy + \int_\eta^\infty \mathbb{P}(Y > y)\,dy$. Now we prove and state the Lemma which establishes the validity of (4.3.15).

Lemma 4.3.4. *Assume that* (4.2.2) *holds true and let μ be a measure with compact support in $\mathbb{R}^2 \setminus \{0\}$. Then there exists a set of full $\mathbb{P}_\mu$-measure Ω_1 such that for every $n \in \mathbb{N}$ and $\omega \in \Omega_1$*

$$\sup_{\mathbb{Q}\ni t\geq 0} e^{\lambda_\alpha t}\langle d_n, X_t(\omega)\rangle < \infty$$

holds.

Proof. We show that there exists a set Ω_1 of full measure and a sequence $(g_n)_{n\in\mathbb{N}}$ of non-negative functions such that for every $n \in \mathbb{N}$ and $\omega \in \Omega_1$, $d_n \leq g_n$ and $\sup_{\mathbb{Q}\ni t\geq 0} e^{\lambda_\alpha t}\langle g_n, X_t(\omega)\rangle < \infty$. Similar to Lemma 3.1 in [93] we choose a non-negative function $h_n \in C_c^\infty(\mathbb{R}^2 \setminus \{0\})$ such that for some $\varepsilon > 0$ one has $\varepsilon + d_n \leq h_n$ in some neighborhood of the support of d_n. Then we have $P_t^\alpha h_n \geq 0$ and $P_t^\alpha h_n \to h_n$ uniformly on the support of h_n as $t \to 0$. This uniform convergence holds true as for $x \in \operatorname{supp}(h_n)$ there exists by Lemma 2.6 in [42] (see the inequality (4.1.9)) a constant C such that for $t \in (0,1)$

$$\begin{aligned}\big|P_t^\alpha h_n(x) - h_n(x)\big| &\leq \big|P_t^\alpha h_n(x) - e^{t\Delta}h_n(x)\big| + \big|e^{t\Delta}h_n(x) - h_n(x)\big| \\ &= P_t^\alpha h_n(x) - e^{t\Delta}h_n(x) + \big|e^{t\Delta}h_n(x) - h_n(x)\big| \\ &\leq Ct^{-1}\frac{1}{\sqrt{|x|}}e^{-|x|^2/4t}\int_{\mathbb{R}^2} h_n(y)\sqrt{|y|}^{-1}e^{-|y|^2/4t}\,dy + \big|e^{t\Delta}h_n(x) - h_n(x)\big| \\ &\to 0\end{aligned}$$

uniformly on compact subsets of $\mathbb{R}^2\setminus\{0\}$ as $t \to 0$. This finally shows, that $g_n := P_r^\alpha h_n \geq d_n$ for r sufficiently small.
Consider now $e^{\lambda_\alpha t}\langle g_n, X_t\rangle$ and the decomposition $g_n = (g_n, \psi_\alpha)_{L^2}\psi_\alpha + G_n$. We already know that $e^{\lambda_\alpha t}\langle \psi_\alpha, X_t\rangle$ converges almost surely and in $L^2(\mathbb{P}_\mu)$ to the random variable N_α. Thus it remains to show that there exists a set Ω^n of full $\mathbb{P}_\mu$-measure such that for every $\omega \in \Omega^n$ $\sup_{t\geq 0} e^{\lambda_\alpha t}\langle G_n, X_t(\omega)\rangle < \infty$. First observe that by Lemma 4.1.2 $(e^{t|k|^2}\langle \varphi(k,\cdot), X_t\rangle)_{t\geq 0}$ defines a martingale and that Doob's inequality applied the martingale $(e^{t|k|^2}\langle \varphi(k,\cdot), X_t\rangle)_{t\geq 0}$

gives for $|k| \neq 0$ and a constant $\mathbb{Q} \ni c > 0$ which has to be determined later

$$\begin{aligned}
&\mathbb{P}_\mu\big(\sup_{t\in[cn,c(n+1)]\cap\mathbb{Q}} |e^{t\lambda_\alpha}\langle\varphi(k,\cdot),X_t\rangle| > \varepsilon\big) \\
&\qquad = \mathbb{P}_\mu\big(\sup_{t\in[cn,c(n+1)]\cap\mathbb{Q}} |e^{t\lambda_\alpha - t|k|^2} e^{t|k|^2}\langle\varphi(k,\cdot),X_t\rangle| > \varepsilon\big) \\
&\qquad \le \frac{e^{2cn\lambda_\alpha - 2cn|k|^2}}{\varepsilon^2}\mathbb{E}_\mu\big[|e^{c(n+1)|k|^2}\langle\varphi(k,\cdot),X_{c(n+1)}\rangle|^2\big] \\
&\qquad = \frac{e^{2cn\lambda_\alpha - 2cn|k|^2}}{\varepsilon^2}\Bigg(|\langle\varphi(k,\cdot),\mu\rangle|^2 \\
&\qquad + e^{2c(n+1)|k|^2}\Big\langle\int_0^{c(n+1)} e^{-2[c(n+1)-s)|k|^2} P_s^\alpha|\varphi(k,\cdot)|^2\,ds,\mu\Big\rangle\Bigg) \\
&\qquad = \frac{e^{2cn\lambda_\alpha - 2cn|k|^2}}{\varepsilon^2}|\langle\varphi(k,\cdot),\mu\rangle|^2 \\
&\qquad + \frac{e^{2cn\lambda_\alpha + 2c|k|^2}}{\varepsilon^2}\Big\langle\int_0^{c(n+1)} e^{-2[c(n+1)-s)|k|^2} P_s^\alpha|\varphi(k,\cdot)|^2\,ds,\mu\Big\rangle \\
&\qquad \le \frac{e^{2cn\lambda_\alpha - 2cn|k|^2}}{\varepsilon^2}|\langle\varphi(k,\cdot),\mu\rangle|^2 \\
&\qquad + e^{cn(\lambda_\alpha+\varepsilon)+2c|k|^2} e^{-c(\lambda_\alpha-\varepsilon)} e^{-c(n+1)(\lambda_\alpha-\varepsilon)} \\
&\qquad \cdot\Big\langle\int_0^{c(n+1)} e^{-2[c(n+1)-s)|k|^2} P_s^\alpha|\varphi(k,\cdot)|^2\,ds,\mu\Big\rangle \\
&\qquad \le \frac{e^{2cn\lambda_\alpha - 2cn|k|^2}}{\varepsilon^2}|\langle\varphi(k,\cdot),\mu\rangle|^2 \\
&\qquad + \frac{e^{cn(\lambda_\alpha+\varepsilon)+2c|k|^2}}{\varepsilon^2} e^{-c(\lambda_\alpha-\varepsilon)}\cdot\Big\langle\int_0^{c(n+1)} e^{s(\lambda_\alpha-\varepsilon)} P_s^\alpha|\varphi(k,\cdot)|^2\,ds,\mu\Big\rangle.
\end{aligned} \tag{4.3.16}$$

In the proof of Theorem 4.3.2 via (4.3.12), (4.3.13) and (4.3.14) we have seen that for some constant C (which depends on the support of μ)

$$\Big\langle\int_0^{c(n+1)} e^{cs(\lambda-\varepsilon)} P_s^\alpha|\varphi(k,\cdot)|^2\,ds,\mu\Big\rangle \le \Big\langle\int_0^{\infty} e^{cs(\lambda-\varepsilon)} P_s^\alpha|\varphi(k,\cdot)|^2\,ds,\mu\Big\rangle \le C\big(1+\frac{1}{|k|}\big).$$

Observe moreover that the expression $|\langle\varphi(k,\cdot),\mu\rangle|^2$ can also be estimated by $K(1+|k|^{-1})$ for some constant K since the singularity of the Hankel function at 0 is logarithmic and $\sup_{x\in\text{supp}(\mu),|k|>1}\varphi(k,x) < \infty$. Therefore (4.3.16) shows that there is a constant $\tilde{C}$ such that

$$\mathbb{P}_\mu\big(\sup_{t\in[cn,c(n+1)]\cap\mathbb{Q}} |e^{t\lambda_\alpha}\langle\varphi(k,\cdot),X_t\rangle| > \varepsilon\big) \le \Big(\tilde{C}\big(1+\frac{1}{|k|}\big)\Big)\frac{e^{cn(\lambda_\alpha+\varepsilon)}}{\varepsilon^2} e^{2c|k|^2}. \tag{4.3.17}$$

Inequality (4.3.17) implies by Lemma 4.3.3 (see Lemma 3.4 in [93]) with $\eta = e^{cn(\lambda_\alpha+\varepsilon)/2}$

$$\mathbb{E}_\mu\Big[\sup_{t\in[cn,c(n+1)]\cap\mathbb{Q}} e^{\lambda_\alpha t}|\langle\varphi(k,\cdot),X_t\rangle|\Big] \le \Big(1+\tilde{C}\big(1+\frac{1}{|k|}\big)\Big)e^{\frac{cn(\lambda_\alpha+\varepsilon)}{2}} e^{2c|k|^2}.$$

Choose the constant $c > 0$ such that $2c < r$, then using $0 \le t \in \mathbb{Q}$

$$(2\pi)\langle G_l, X_t\rangle = \langle E([0,\infty))P_r h_l, X_t\rangle = \langle P_r H_l, X_t\rangle = \int_{\mathbb{R}^2} e^{-r|k|^2}\hat{H}_l(k)\langle\varphi(k,\cdot), X_t\rangle\, dk,$$

with $H_l := E([0,\infty))h_l$ we get

$$\begin{aligned}
2\pi\mathbb{E}_\mu\Big[\sup_{t\in[cn,c(n+1)]\cap\mathbb{Q}} e^{\lambda_\alpha t}|\langle G_l, X_t\rangle|\Big] &= \mathbb{E}_\mu\left[\sup_{t\in[cn,c(n+1)]\cap\mathbb{Q}} e^{\lambda_\alpha t}\left|\int_{\mathbb{R}^2} e^{-r|k|^2}\hat{H}_l(k)\langle\varphi(k,\cdot), X_t\rangle\, dk\right|\right] \\
&\le \mathbb{E}_\mu\left[\int_{\mathbb{R}^2} e^{-r|k|^2}|\hat{H}_l(k)| \sup_{t\in[cn,c(n+1)]\cap\mathbb{Q}} e^{\lambda_\alpha t}|\langle\varphi(k,\cdot), X_t\rangle|\, dk\right] \\
&= \int_{\mathbb{R}^2} e^{-r|k|^2}|\hat{H}_l(k)|\mathbb{E}_\mu\left[\sup_{t\in[cn,c(n+1)]\cap\mathbb{Q}} e^{\lambda_\alpha t}|\langle\varphi(k,\cdot), X_t\rangle|\right] dk \\
&\le \int_{\mathbb{R}^2}\left(1+\tilde{C}\big(1+\frac{1}{|k|}\big)\right)e^{-r|k|^2}|\hat{H}_l(k)|e^{2c|k|^2}\, dk \cdot e^{\frac{cn(\lambda_\alpha+\varepsilon)}{2}} \\
&= \int_{\mathbb{R}^2}\left(1+\tilde{C}\big(1+\frac{1}{|k|}\big)\right)e^{-(r-2c)|k|^2}|\hat{H}_l(k)|\, dk \cdot e^{\frac{cn(\lambda_\alpha+\varepsilon)}{2}}.
\end{aligned} \tag{4.3.18}$$

Hence by (4.3.18) and Proposition 2

$$\mathbb{E}_\mu\left[\sum_{n\in\mathbb{N}_0}\sup_{t\in[cn,c(n+1)]\cap\mathbb{Q}} e^{\lambda_\alpha t}|\langle G_l, X_t\rangle|\right] < \infty. \tag{4.3.19}$$

The inequality (4.3.19) immediately implies that there exists a set Ω^l of full measure such that for every $\omega \in \Omega^l$

$$\lim_{\mathbb{Q}\ni t\to\infty} e^{\lambda_\alpha t}\langle G_l, X_t(\omega)\rangle = 0$$

and therefore $\sup_{\mathbb{Q}\ni t\ge 0} e^{\lambda_\alpha t}\langle d_l, X_t(\omega)\rangle < \infty$. Setting $\Omega_1 := \bigcap_{n\in\mathbb{N}}\Omega^n$ finishes the proof of the assertion. □

By the discussion above we finally proved the strong law of large numbers for the two-dimensional super-Brownian motion with a single point source.

Theorem 4.3.3. *Let $d = 2$ and $\alpha \in \mathbb{R}$ and assume that (4.2.2) holds true. Then the super-Brownian motion with single point source started from an initial measure μ with compact support in $\mathbb{R}^2 \setminus \{0\}$ satisfies the strong law of large numbers, i.e. there exists a set Ω_1 of full measure such that for every $\omega \in \Omega_1$*

$$\lim_{\mathbb{Q}\ni t\to\infty}\frac{X_t(\omega)}{\mathbb{E}_\mu\big[\langle 1, X_t\rangle\big]} = \frac{N_\alpha(\omega)}{\int\psi_\alpha(x)\,\mu(dx)}\,\frac{\langle\psi_\alpha,\cdot\rangle}{\int\psi_\alpha(x)\,dx}$$

with respect to the vague topology on $\mathcal{M}(\mathbb{R}^2\setminus\{0\})$.

4.4 Concluding Remarks and Open Problems

We already mentioned an open problem connected with the Super-Brownian motion with a single point source. In three dimension the possible range of β is restricted to $\beta \in (0,1)$ (see also Remark 2 in [21]). This seems to be a purely technical problem. Still it would be interesting to extend the construction to the case $\beta = 1$ since in this case our methods probably allow to study the large time behavior of the three-dimensional Super-Brownian motion with a single point source. Already our investigation of the large time behavior of the expected mass in bounded regions indicates that there is a interesting change in the behavior at $\alpha = 0$. It would be an very interesting problem to investigate the dependence on α of the pathwise behavior of the three-dimensional process. Several other basic properties of the considered Super-process seem also to be unknown at the moment. These range from regularity properties of the pathes to properties of the support such as the compact support property.

In the introduction we also referred to possible extensions of path integral formulas for the scattering length. Extensions of the results of Taylor to Schrödinger operators with potentials having non-trivial negative part are desirable.

There are several other open problems connected with this superprocess. A very interesting problem consists in the construction of a branching particle system which approximates the above super-Brownian motion with a single point source. This would shed light on the probabilistic meaning of the process. We should also point out, that some rather interesting ideas without proofs concerning the probabilistic meaning of the operator $-\Delta_\alpha$ can also be found in [31]. Though Eberle's explanations sound very reasonable it is not completely clear, whether they are completely correct. For example in equation (4.4) in [31] seems to say that the kernels $p^\alpha(t,\cdot,\cdot)$ induce a semigroup on the bounded functions on $\mathbb{R}^d$, which is not the case. A rigorous elaboration of Eberle's ideas is deferred to a subsequent project.

A related problem consists in the investigation of approximations of the Super-Brownian motion with a single point source by Super-diffusions with regular coefficients. It is known that the operator $-\Delta_\alpha$ can be approximated in strong way by sequence of Schrödinger operators $-\Delta + V_n$, where the potentials V_n are bounded functions. Indeed on can choose bounded potentials V_n such that for every $t \geq 0$ $e^{-t(-\Delta+V_n)} \to e^{-t(-\Delta_\alpha)}$ as $n \to \infty$ with respect to the operator norm. It would be interesting to know, whether also the laws of associated Super-diffusions converge in some sense. One should observe, that such results will probably require a very careful investigation, since the convergence of $e^{-t(-\Delta+V_n)}$ to $e^{-t(-\Delta_\alpha)}$ depends in a subtle way on the form of the potentials V_n.

Let us finally mention that the results of Ikebe [52] and subsequent generalizations of his work can probably be used in order to derive strong laws of large numbers for quite general superprocesses including cases, where different methods fail to apply. With the exception of the recent papers [33] and [92] most authors focused on the case, where the bottom of the spectrum is an L^2-eigenvalue. As in our investigation of the convergence to quasistationarity such a spectral theoretic assumption simplifies the problem considerably. In [33] J. Engländer proves without such an assumption a weak law of large numbers for a certain class of superdiffusions. The inclusion of general results concerning expansions in eigenfunction will most probably allow to include cases, where the bottom of the spectrum belongs to the continuous spectrum, and may even allow to prove strong laws of large numbers.

Chapter 5

Essential Selfadjointness of Singular Magnetic Schrödinger Operators on Riemannian Manifolds

In this part of the thesis we present a result, which is completely unrelated to the main topic of the work. We just included it, in order to give a flavour of the type of problems the author of this thesis was also working on during the last years. For another topic (which is actually closer to the main theme of this thesis) we refer to [7], where in a joint work with A. Bassi and D. Dürr the asymptotic large time behavior of a stochastic Schrödinger equation has been investigated.

The analysis of Schrödinger operators has been in the focus of many mathematical physicists for almost a century. In the last decade there has been a lot of interest in properties of Schrödinger operators on Riemannian manifolds (see e.g. [13], [16], [61], [30] and [70]). In these works the authors extend several results which are known in the euclidean case to larger classes of manifolds. One of the most fundamental questions concerning Schrödinger operators is the question of essential selfadjointness. In the euclidean case very general and in some respect optimal results ensuring essential selfadjointness of magnetic Schrödinger operators have been derived in the work [60] of Leinfelder and Simader. In contrast to the euclidean case existing results concerning essential selfadjointness are still incomplete. In particular the analogue of the result of Leinfelder and Simader is still missing. The most general result seems to go back to M. Shubin (see [79]), where he proves essential self-adjointness of semi-bounded magnetic Schrödinger operators under the assumption that the magnetic potential is continuously differentiable. This type of result has later been generalized in [13] to operators of Schrödinger type acting in sections of vector bundles. In this work we prove an extension of the famous result of Leinfelder and Simader on the essential selfadjointess of magnetic Schrödinger operators in $\mathbb{R}^n$. In the euclidean case the result of Leinfelder and Simader asserts that essential selfadjointness holds e.g. as long as the potential satisfies a certain lower bound and the coefficients satisfy certain minimal local regularity requirements much weaker than continuous differentiability. Due to the local character of the regularity requirements on the coefficients it is tempting to believe that the result carries over without essential changes to general complete Riemannian manifolds. Thus in particular the differentiability of the vector potential, as supposed by Shubin in [79], should not be necessary. A careful look at the original work of Leinfelder and Simader demonstrates that a sequence of cut-off functions with uniformly bounded derivatives of first and second order plays an important role in their approach. For arbitrary complete Riemannian manifolds it seems to be unknown whether such a sequence of cut-off functions exists. This technical problem might explain why the analogue of the result of Leinfelder and Simader for Riemannian manifolds has not been established, yet. In order to avoid this problem we combine ideas of Leinfelder and Simader with an old idea of Chernoff. This will allow us to give a rather transparent proof of a general result of Povzner-Wienholtz-Simader-type under more or less minimal local requirements. Our result demonstrates in contrast to the main result of [79] that essential selfadjointness on manifolds holds under conditions which parallels those of the euclidean case. Moreover, our proof is even more elementary than Shubin's since we can avoid the use a non-trivial result due to Karcher concerning the existence of smooth cut-off functions with uniformly bounded gradients. Given the work of Leinfelder and Simader our result is not at all surprising, it is much more surprising that only weaker results can be found in the literature. Let us finally remark that Shubin's result was used in [16] in order to derive continuity properties of functions of the magnetic Schrödinger operator on manifolds of bounded geometry. Our result can now be used in order to improve this result. But we should point out that the well-known probabilistic approach via the Feynman-Kac-Itô formula was used by the present author in [59] in order to derive quite sharp results which extend the results of the papers [14] and [15] from the euclidean setting to the case of manifolds with lower bounded Ricci curvature. In deed once the necessary results concerning essential selfadjointness are established, such an extension to more general manifolds is rather straightforward.

5.1 The Main Result

Let (M, g) be a complete Riemannian manifold of dimension n. The metric g induces in a canonical way a measure μ which in local coordinates is given by $d\mu(x) = \sqrt{g}dx_1 \dots dx_n$. Denote by $TM = (T_xM)_{x\in M}$ and $T^*M = (T_x^*M)_{x\in M}$ the tangent and cotangent bundle, respectively. We will work with complex valued sections of TM and T^*M and therefore have to complexify the inner product spaces T_xM and T_x^*M in the usual way. The scalar product in T_xM and T_x^*M will be denoted by $\langle \cdot, \cdot \rangle$ with induced norm $|\cdot|$. The Riemannian measure μ induces the Lebesgue spaces $L^p(M)$, $L^p(TM)$ and $L^p(T^*M)$ consisting of p-th power integrable functions, vectorfields and 1-forms, respectively. By grad and d we will denote the Riemannian gradient and the exterior derivative, respeectively. We will also use the weak adjoint d^* of the exterior derivative d. Moreover, we have to introduce the Sobolev space $H_0^1(\Lambda)$, where $\Lambda \subset M$ is an arbitrary open subset. As usual $H_0^1(\Lambda)$ is defined as the closure of $C_c^\infty(\Lambda)$ with respect to the norm $\|\cdot\|_{L^2(\Lambda)} + \|\text{grad}\cdot\|_{L^2(\Lambda)}$. Every Riemannian manifold admits a unique torsion free connection, called the Levi-Civita connection. The

Levi-Civita connection is denoted by ∇. Moreover we need the Ricci curvature, which we denote by Ric. Recall that Ric is by definition a smooth section in $T^*(M) \otimes T^*(M)$ such that for every $x \in M$ the bilinear form Ric_x on the vector space T_xM is symmetric.
We consider the class $\mathcal{M}(M)$ of admissible magnetic potentials given by

$$\mathcal{M}(M) = \{A \in L^4_{loc}(T^*M) \mid d^*A \in L^2_{loc}(M)\}.$$

Note that in this chapter $\mathcal{M}$ is used differently to chapter 3. Assume that for $A \in \mathcal{M}(M)$ and $V \in L^2_{loc}(M)$ the quadratic form

$$\begin{aligned} q[A,V] \upharpoonright C_c^\infty(M) : C_c^\infty(M) \times C_c^\infty(M) &\to \mathbb{C} \\ (\varphi, \psi) &\mapsto \int_M \langle d\varphi - iA\varphi, d\psi - iA\psi \rangle \, d\mu + \int_M V\varphi\overline{\psi} \, d\mu \end{aligned}$$

is semibounded. Then it is closable since it is the form of a semibounded symmetric operator $\tilde{H}(A,V)$, which on $C_c^\infty(M)$ is defined by

$$\tilde{H}(A,V)\psi = -\Delta\psi - 2i\langle A, d\psi\rangle + (id^*A + |A|^2 + V)\psi.$$

The closure of the form $q[A,V] \upharpoonright C_c^\infty(M)$ will be denoted by $q[A,V]$ and the uniquely associated selfadjoint operator by $H(A,V)$. The operator $H(A,V)$ is given by

$$\begin{aligned} \mathcal{D}(H(A,V)) &= \left\{u \in \mathcal{D}(q[A,V]) \mid \exists v \in L^2(M) \forall w \in \mathcal{D}(q[A,V]) : q[A,V](u,w) = (v,w)_{L^2(M)}\right\} \\ &= \left\{u \in \mathcal{D}(q[A,V]) \mid \exists v \in L^2(M) \forall w \in C_c^\infty(M) : q[A,V](u,w) = (v,w)_{L^2(M)}\right\} \end{aligned}$$

$$H(A,V)u = v.$$

Thus $H(A,V)$ is the Friedrichs extension of $\tilde{H}(A,V)$ and $H(A,V) \upharpoonright C_c^\infty(M) = \tilde{H}(A,V)$.
We will also need magnetic Schrödinger operators, which are defined on open subsets of M and satisfy Dirichlet boundary conditions. Let $\Lambda \subset M$ be an open subset and assume that the quadratic form $q_\Lambda[A,V]$

$$\begin{aligned} q_\Lambda[A,V] : C_c^\infty(\Lambda) \times C_c^\infty(\Lambda) &\to \mathbb{C} \\ (\varphi, \psi) &\mapsto \int_M \langle d\varphi - iA\varphi, d\psi - iA\psi \rangle \, d\mu + \int_M V\varphi\overline{\psi} \, d\mu \end{aligned}$$

is closable with a closure which is denoted by $q_\Lambda[A,V]$ then the uniquely associated selfadjoint operator $H_\Lambda(A,V)$ is called the magnetic Schrödinger operator in Λ satisfying Dirichlet conditions. If $\Lambda = B(o,r)$ with $o \in M$ fixed we also write $H_r(A,V)$ instead of $H_{B(o,r)}(A,V)$.
Recall that the Kato class $\mathcal{K}(M)$ (compare section E.3 and Definition 2.1 in [29]) on M consists of all function $q : M \to \mathbb{R}$ such that

$$\lim_{t\to 0} \sup_{x\in M} \int_0^t \int_M p(s,x,y)|q(y)| \, d\mu(y) \, ds = 0, \tag{5.1.1}$$

where $p(t,x,y)$ denotes the heat kernel associated to the Laplacian on the Riemannina manifold M. We define the local Kato-class $\mathcal{K}_{loc}(M)$ to consist of all functions $q : M \to \mathbb{R}$ such that for all compact $K \subset M$ $\mathbf{1}_K q \in \mathcal{K}(M)$. As is shown on page 57 in [29] Kato-class potentials are infinitesimally $-\Delta$-form bounded. This allows us to define several quadratic forms using the KLMN-theorem. Moreover Kato-class potentials preserve several important mapping properties of the free heat semigroup.

Theorem 5.1.1. *Let (M,g) be an arbitrary complete Riemannian manifold and let $A \in \mathcal{M}(M)$ and $V \in L^2_{loc}(M)$ be given. Assume that $V_- \in \mathcal{K}_{loc}(M)$ and that for some $C \in \mathbb{R}$*

$$\forall \varphi \in C_c^\infty(M) : \int_M |(d-iA)\varphi|^2 \, d\mu + \int_M V\, |\varphi|^2 \, d\mu \geq C \|\varphi\|^2_{L^2(M)}.$$

Then $H(A,V) \restriction C_c^\infty(M)$ is essentially selfadjoint.

Theorem 5.1.1 might be called a result of Povzner-Wienholtz-Simader type, since it asserts that a semibounded Schrödinger operator is already essentially selfadjoint. The above result is in some respect optimal. Observe e.g. that the formal differential operator $H(A,V)$ maps every smooth function with compact support to an element of $L^2(M,\mu)$ if and only if $|A|^2 \in L^2_{loc}(M,\mu)$, $d^*A \in L^2_{loc}(M,\mu)$ and $V \in L^2_{loc}(M,\mu)$. Thus the local requirements on A and V_+ are minimal.
We want to stress that the same type of condition occurs in a quite different situation. Consider the diffusion opoerator $D := -\Delta + \operatorname{grad} \ln F \cdot \nabla =: -\Delta + b \cdot \nabla$ in the case $M = \mathbb{R}^n$. Under some weak conditions on the non-negative function F this operator D is symmetric in $L^2(\mathbb{R}^n, F(x)dx)$. In [31] it is shown that the condition $|b|^2 \in L^2_{loc}(\mathbb{R}^n, F(x)dx)$ is necessary in order to have essential selfadjointess of $D \restriction C_c^\infty(\mathbb{R}^n)$. In [58] we proved that under some additional assumptions on F this is also sufficient. This indicates that the L^4_{loc}-condition in Theorem 5.1.1 does not only occur because of the multiplicative term $|A|^2$ but more importantly because of the first order term $\langle A, d\varphi\rangle$. This will also become clear during the proof of Theorem 5.1.1.

5.2 Proof of Theorem 5.1.1

We will need the fact that a magnetic potential $A \in \mathcal{M}(M)$ can be suitably approximated by smooth compactly supported 1-forms. For an open subset $\Lambda \subset M$ and two smooth compactly supported 1-forms α and β define

$$l_\Lambda(\alpha,\beta) = \left(\int_\Lambda |\alpha-\beta|^4 \, d\mu\right)^{\frac{1}{4}} + \left(\int_\Lambda |d^*\alpha - d^*\beta|^2 \, d\mu\right)^{\frac{1}{2}}.$$

Let $(\Lambda_n)_{n\in\mathbb{N}}$ be an exhaustion of M by bounded open subsets with smooth boundary then we set

$$l(\alpha,\beta) = \sum_{n=1}^\infty \frac{1}{2^n} \wedge l_{\Lambda_n}(\alpha,\beta).$$

It is easy to see that l defines a metric on the space $\Gamma_{comp}(T^*M)$ of smooth compactly supported 1-forms. Let $\tilde{\mathcal{M}}(M)$ be the completion of $\Gamma_{comp}(T^*M)$ with respect to the metric l.

Lemma 5.2.1. $\mathcal{M}(M) = \tilde{\mathcal{M}}(M)$

Proof. Because of the local character of the convergence this is a rather direct consequence of the Friedrichs mollification. □

The following Lemma is a geometric version of the well known Gagliardo-Nirenberg inequality

$$\forall \varphi \in C_c^\infty(\mathbb{R}^n) : \|\operatorname{grad}\varphi\|^2_{L^4(\mathbb{R}^n)} \le C_n \|\varphi\|_\infty \|\Delta\varphi\|_{L^2(\mathbb{R}^n)} \tag{5.2.1}$$

used in [60] as a fundamental tool. A closer look at the proof given in [60] shows that Leinfelder and Simader first establish the inequality

$$\forall \varphi \in C_c^\infty(\mathbb{R}^n) : \|\operatorname{grad}\varphi\|^2_{L^4(\mathbb{R}^n)} \le C_n \|\varphi\|_\infty \left(\sum_{i,j=1}^n \int_{\mathbb{R}^n} |\partial_{ij}\varphi|^2 \, dx \right)^{\frac{1}{2}} \tag{5.2.2}$$

which involves the Hessian of φ and then use the elementary equation

$$\sum_{i,j=1}^n \|\partial_{ij}\varphi\|^2_{L^2(\mathbb{R}^n)} = \|\Delta\varphi\|^2_{L^2(\mathbb{R}^n)}.$$

in order to derive the inequality (5.2.1). Our proof of the analogue of equation (5.2.2) is a coordinate free version of the one given in [60]. In the proof we use the Einstein summation convention and sum over repeated indices.

Lemma 5.2.2. *Let $\nabla d\psi$ denote the Hessian of a smooth function ψ. For all bounded $\varphi \in \mathcal{D}(-\Delta)$ with compact support one has*

1. $\|\mathit{grad}\,\varphi\|_4^2 \le C\|\varphi\|_\infty \|\nabla d\varphi\|_2$
2. $\|\mathit{grad}\,\varphi\|_4^2 \le C_\varphi \|\varphi\|_\infty (\|\Delta\varphi\|_2 + \|\varphi\|_2)$,

where C depends only on the dimension of M and C_φ depends only on the lower bound of the Ricci curvature on the support of φ.

Proof. It is enough to prove the assertions for real valued functions. For $\varphi \in C_c^\infty(M;\mathbb{R})$ we have

$$\begin{aligned}
\int_M |\operatorname{grad}\varphi|^4 \, d\mu &= \int_M \langle \operatorname{grad}\varphi, \operatorname{grad}\varphi\rangle \langle \operatorname{grad}\varphi, \operatorname{grad}\varphi\rangle \, d\mu \\
&= \int_M \langle \operatorname{grad}\varphi, \langle \operatorname{grad}\varphi, \operatorname{grad}\varphi\rangle \operatorname{grad}\varphi\rangle \, d\mu \\
&= -\int_M \langle \operatorname{grad}\varphi, \operatorname{grad}\varphi\rangle \langle \operatorname{div}\operatorname{grad}\varphi\rangle \varphi \, d\mu \\
&\quad - \int_M \langle \operatorname{grad} |\operatorname{grad}\varphi|^2, \operatorname{grad}\varphi\rangle \varphi \, d\mu \\
&= -\int_M |\operatorname{grad}\varphi|^2 (\Delta\varphi)\varphi \, d\mu - \int_M \langle \operatorname{grad} |\operatorname{grad}\varphi|^2, \operatorname{grad}\varphi\rangle \varphi \, d\mu.
\end{aligned}$$

Using the definition of the Hessian $\nabla d\varphi$ (see Definition 7.106 [50]) we have

$$\partial_l\langle \operatorname{grad}\varphi, \operatorname{grad}\varphi\rangle = 2\langle \nabla_{\partial_l}\operatorname{grad}\varphi, \operatorname{grad}\varphi\rangle = 2(\nabla d\varphi)(\partial_l, \operatorname{grad}\varphi)$$

and therefore we conclude that

$$\begin{aligned}\left|\int_M \langle \operatorname{grad}|\operatorname{grad}\varphi|^2, \operatorname{grad}\varphi\rangle\varphi\, d\mu\right| &= 2\left|\int_M \langle g^{kl}\partial_l|\operatorname{grad}\varphi|^2\partial_k, \operatorname{grad}\varphi\rangle\varphi\, d\mu\right| \\ &\leq 2\|\varphi\|_\infty \int_M |\nabla d\varphi||\operatorname{grad}\varphi|^2\, d\mu \\ &\leq 2\|\varphi\|_\infty\left(\int_M |\nabla d\varphi|^2\, d\mu\right)^{\frac{1}{2}}\left(\int_M |\operatorname{grad}\varphi|^4\, d\mu\right)^{\frac{1}{2}}.\end{aligned}$$

Hence we see that

$$\int_M |\operatorname{grad}\varphi|^4\, d\mu \leq (\sqrt{n}+2)\|\varphi\|_\infty\|\operatorname{grad}\varphi\|_4^2\|\nabla d\varphi\|_2.$$

By the Bochner-Weitzenböck formula (see e.g. formula (1.1) in [85]) we have for smooth functions with compact support $f \in C_c^\infty(M)$

$$\frac{1}{2}\Delta|\operatorname{grad} f|^2 - \langle \operatorname{grad}\Delta f, \operatorname{grad} f\rangle = |\nabla df|^2 + \operatorname{Ric}(\operatorname{grad} f, \operatorname{grad} f).$$

Using

$$\int_M \Delta|\operatorname{grad} f|^2\, d\mu = 0$$

we get after integrating by parts

$$\int_M |\nabla df|^2\, d\mu = \int_M |\Delta f|^2\, d\mu - \int_M \operatorname{Ric}(\operatorname{grad} f, \operatorname{grad} f)\, d\mu.$$

The desired result is now proved for smooth functions $\varphi \in C_c^\infty(M)$. The extension to $\varphi \in H^2(M) \cap L^\infty_{comp}(M)$ is straightforward. □

Remark 5.2.1. *Observe that without lower bound on the Ricci curvature one cannot estimate the L^4-norm of the gradiant of smooth test function u by the L^2-norms of the Laplacian of u and the function u itself. This is another difference to the euclidean case* (5.2.1)*, which is another point, which makes the general case more complicated.*

The following theorem constitutes the core of the hyperbolic approach. The finite speed of propagation property of solutions of many wave equation can effectively used in a localization of the problem. This fact was noticed by Chernoff in [26]. The next Theorem is in some sense an abstract version of his argument and was already applied in this form e.g. in [58].

Theorem 5.2.1. *Let $(N, \mathfrak{M}, \nu)$be a σ-finite measure space on a locally compact space N and let $(\tilde{S}, \mathcal{D}(\tilde{S}))$ be a symmetric densely defined operator in $L^2(N,\nu)$ which is semibounded from below. Let S denote the Friedrichs extension of $\tilde{S}$. Assume that the set of functions in $\mathcal{D}(\tilde{S})$ with compact support is dense in $L^2(N,\nu)$ and that for every $v \in \mathcal{D}(S)$ with compact support the solution $u(t) = \cos(t\sqrt{S})v$ to the abstract wave equation*

$$\frac{d^2u}{dt^2}(t) = -Su(t),\ u(0) = v \tag{5.2.3}$$

has compact support, then the restriction of S to the class of functions in $\mathcal{D}(S)$ with compact support is essentially selfadjoint.

Let us briefly comment on the proof of Theorem 5.2.1. Without loss of generality let us assume that $S \geq I$ in the form sense. It is clear that for fixed $T > 0$ the set $U_T = \{\cos(t\sqrt{S})f \mid f \in \mathcal{D}(S)_{comp},\ 0 < t \leq T\} \subset \mathcal{D}(S)$ is dense in $L^2(N,\nu)$. Let us denote by $\hat{S}$ the restriction of S to U_T. $(\hat{S}, U_T)$ is a densely defined symmetric operator in $L^2(N,\nu)$. Let $\psi \in \mathcal{D}(\hat{S}^*)$ with $\hat{S}^*\psi = 0$ be given. For any $u(t) = \cos(t\sqrt{S})f \in U_T$ the function $g(t) = (u(t),\psi)_{L^2(N,\nu)} = (\cos(t\sqrt{S})f,\psi)_{L^2(N,\nu)}$ is twice differentiable with $g(0) = (f,\psi)_{L^2(N,\nu)}$, $\frac{dg}{dt}(0) = 0$ and

$$\frac{d^2g}{dt^2} = -(S\cos(t\sqrt{S})f,\psi)_{L^2(N,\nu)} = -(\cos(t\sqrt{S})f,\hat{S}^*\psi)_{L^2(N,\nu)} = 0.$$

Thus we get for some $a, b \in \mathbb{R}$ $g(t) = a \cdot t + b$. Since $a = \frac{dg}{dt}(0) = 0$ and $b = g(0) = (f,\psi)_{L^2(N,\nu)}$ this shows that $(\cos(t\sqrt{S})f,\psi)_{L^2(N,\nu)} = (f,\psi)_{L^2(N,\nu)}$ for every $0 \leq t < T$ and $f \in \mathcal{D}(S)$ with compact support. Since these functions are dense in $L^2(N,\nu)$ we get $\cos(t\sqrt{S})\psi = \psi$ for every $0 \leq t < T$. Observe that by the spectral theorem

$$\|\psi\|^2_{L^2(N,\nu)} = \int_{[1,\infty)} |\cos(t\sqrt{\lambda})|^2\, d\|E_\lambda\psi\|^2_{L^2(N,\nu)},$$

where $(E_\lambda)_\lambda$ denotes the spectral resolution of S. If ψ is not the zero element, then the non-negative measure $\|E(\cdot)\psi\|^2_{L^2(N,\nu)}$ is non-trivial with total mass $\|\psi\|^2_{L^2(N,\nu)}$ and must therefore be concentrated on the set $\{\frac{k^2\pi^2}{t^2} \mid k \in \mathbb{N}\}$ which is not independent of $t > 1$. Thus $\psi \equiv 0$. The assertion of Theorem 2 follows from the basic criterion for essential selfadjointness. This criterion asserts (see chapter 10.1 in [95]) that a symmetric operator A in a Hilbert space $\mathcal{H}$ is essentially selfadjoint, if the deficiency indices $\gamma_+(A) := \dim\mathrm{Ker}(A^* - i)$ and $\gamma_-(A) := \dim\mathrm{Ker}(A^* + i)$ satisfy $(\gamma_+(A), \gamma_-(A)) = (0,0)$. Recall further that $\beta(A,z) := \dim\mathrm{Ran}(A - z) = \dim\mathrm{Ker}(A^* - \overline{z})$ is locally constant in z on the domain of regularity $\Gamma(A) \subset \mathbb{C}$ of A, where

$$\Gamma(A) := \Big\{z \in \mathbb{C} \mid \exists k(z) > 0 \forall x \in \mathcal{D}(A) : \|(A - z)x\| \geq k(z)\|x\|\Big\}.$$

Since $S \geq I$ by assumption we conclude that $0 \in \Gamma(S)$ and therefore $0 = \dim\mathrm{Ker}(S^*) = \gamma_+(S) = \gamma_-(S)$.
The next Lemma is known (see [26] for related results). We give a complete proof for convenience of the reader.

Lemma 5.2.3. *Let $A \in \Gamma(T^*M)$ a bounded smooth 1-form and let $V \in C^\infty(M)$ be a bounded smooth potential. Let $H(A,V)$ denote the Friedrichs extension of $\tilde{H}(A,V)$. Then for all open sets Ω_i $(i = 1,2)$ and $u_i \in L^2(M,\mu)$ with supp$(u_i) \subset \Omega_i$ $(i = 1,2)$ we have*

$$(\cos(t\sqrt{H(A,V)})u_1, u_2)_{L^2(M,\mu)} = 0$$

whenever $0 < t <$ *dist*$(\Omega_1, \Omega_2) = R$

Proof. We will use the standard approach via energy estimates. First remark that every solution to the hyperbolic problem (2.2) with the above assumptions on A and V is in fact smooth (see [40]). Let $u \in C^\infty(M)$ be such a solution. For fixed $r > 0$ and $o \in M$ consider the local energy

$$e(t) = \int_{B_{r-t}(o)} |u(t,y)|^2 + |u_t(t,y)|^2 + |(d - iA)u(t,y)|^2 \, d\mu(y).$$

Then we calculate the derivative of e

$$\begin{aligned}
\frac{de}{dt}(t) &= \int_{B_{r-t}(o)} \frac{d}{dt}(|u(t,y)|^2 + |u_s(t,y)|^2 + |(d - iA)u(t,y)|^2) \, d\mu(y) \\
&- \int_{\partial B_{r-t}(o)} |u(t,y)|^2 + |u_t(t,y)|^2 + |(d - iA)u(t,y)|^2 \, d\mathfrak{H}^{n-1}(y) \\
&= 2\int_{B_{r-t}(o)} \Re(u\overline{u_t}) + \Re(\overline{u_t}u_{tt}) - \Re\langle(d - iA)u, \overline{(d - iA)u_t}\rangle \, d\mu \\
&- \int_{\partial B_{r-t}(o)} |u(t,y)|^2 + |u_t(t,y)|^2 + |(d - iA)u(t,y)|^2 \, d\mathfrak{H}^{\mathfrak{n}-1}(y) \\
&= 2\int_{B_{r-t}(o)} \Re(u\overline{u_t}) + \Re(\overline{u_t}H(A,V)u) - \Re\langle(d - iA)u, \overline{(d - iA)u_t}\rangle \, d\mu \\
&- \int_{\partial B_{r-t}(o)} |u(t,y)|^2 + |u_t(t,y)|^2 + |(d - iA)u(t,y)|^2 \, d\mathfrak{H}^{n-1}(y) \\
&= 2\Re\int_{B_{r-t}(o)} u\overline{u_t} + \overline{u_t}H(A,V)u - \langle(d - iA)u, \overline{(d - iA)u_t}\rangle \, d\mu \\
&- \int_{\partial B_{r-t}(o)} |u(t,y)|^2 + |u_t(t,y)|^2 + |(d - iA)u(t,y)|^2 \, d\mathfrak{H}^{n-1}(y),
\end{aligned}$$

where $\mathfrak{H}^{n-1}$ denotes the $(n-1)$-dimensional Hausdorff measure. Integration by parts gives

$$\begin{aligned}\frac{de}{dt}(t) &= 2\Re\Big[\int_{B_{r-t}(o)} u\overline{u_t} + \overline{u_t}Vu\,d\mu + \int_{\partial B_{r-t}(o)} \langle(\mathrm{grad} - iA^\sharp)u, \nu\rangle_{TM} u_t\, d\mathfrak{H}^{n-1}(y)\Big]\\ &\quad - \int_{\partial B_{r-t}(o)} |u(t,y)|^2 + |u_t(t,y)|^2 + |(d-iA)u(t,y)|^2\, d\mathfrak{H}^{n-1}(y)\\ &\le 2\int_{B_{r-t}(o)} |u\overline{u_t}| + |\overline{u_t}Vu|\,d\mu + 2\int_{\partial B_{r-t}(o)} |\langle(\mathrm{grad} - iA^\sharp)u, \nu\rangle_{TM} u_t|\, d\mathfrak{H}^{n-1}(y)\\ &\quad - \int_{\partial B_{r-t}(o)} |u(t,y)|^2 + |u_t(t,y)|^2 + |(d-iA)u(t,y)|^2\, d\mathfrak{H}^{n-1}(y)\\ &\le 2\int_{B_{r-t}(o)} |u\overline{u_t}| + |\overline{u_t}Vu|\,d\mu + \int_{\partial B_{r-t}(o)} |(d-iA)u(t,y)|^2 + |u_t(t,y)|^2\, d\mathfrak{H}^{n-1}(y)\\ &\quad - \int_{\partial B_{r-t}(o)} |u(t,y)|^2 + |u_t(t,y)|^2 + |(d-iA)u(t,y)|^2\, d\mathfrak{H}^{n-1}(y)\\ &\le 2\int_{B_{r-t}(o)} |uu_t| + |u_tVu|\,d\mu,\end{aligned}$$

where ν denotes the outward normal vectorfield. In the second inequality we used the Cauchy-Schwarz inequality together with the elementary inequality $2ab \le a^2 + b^2$. Since V is assumed to be bounded we get for some $c > 0$

$$\begin{aligned}\frac{de}{dt}(t) &\le c\int_{B_{r-t}(o)} |u|^2 + |u_t|^2\,d\mu\\ &\le c\int_{B_{r-t}(o)} |u|^2 + |u_t|^2 + |(d-iA)u|^2\,d\mu \le ce(t),\end{aligned}$$

which by Gronwall's Lemma implies $e \equiv 0$. Without loss of generality we assume u_2 to be smooth. If $y \in M$ is an arbitrary point in support of u_1 then we have $u_2(x) = 0$ for every $x \in M$ with $|x - y| < R$ and therefore $\cos(t\sqrt{H(A,V)})u_2(x) = 0$ for $|x-y| \le R - t$. This proves the assertion. □

Now let for some constant $c \in \mathbb{R}$ the potentials $V_n, V \in L^2_{loc}(M)$ satisfy $V_n, V > c$. Exactly as in Lemma 5 of [60] one shows that $A_n \to A$ in $L^2_{loc}(T^*M)$ and $V_n \to V$ in $L^2_{loc}(M)$ imply that the operators $H(A_n, V_n)$ converge to $H(A,V)$ in the strong resolvent sense. An application of Theorem 3 together with the just mentioned approximation result allows to conclude that for every $A \in L^2_{loc}(T^*M)$ and every $W \in L^2_{loc}(M)$ which is semibounded from below

$$(\cos(t\sqrt{H(A,W)})u, v)_{L^2(M)} = 0 \tag{5.2.4}$$

if $u, v \in L^2(M,\mu)$ with $t < \mathrm{dist}(\mathrm{supp}(u), \mathrm{supp}(v))$. In the next Lemma we remove the restriction that the potential is semibounded from below. In the next Lemma we will see that not semiboundedness of the potential but semiboundedness of the operator matters.

Lemma 5.2.4. *Let $H(A,V)$ be as in Theorem 1. For every $u \in L^2(M)$ with compact support in $B_r(o)$ we have*

$$\forall v \in \{w \in L^2(M) \mid w \restriction B_{r+t}(o) = 0 \, a.e.\} : (\cos(t\sqrt{H(A,V)})u, v)_{L^2(M)} = 0$$

Proof. Consider the sequence $(q_n, \mathcal{D}(q_n))$ of closed non-negative quadratic forms given by $q_n = q[A, V_+ - V_- \wedge n]$ with $\mathcal{D}(q_n) = \mathcal{D}(q[A, V_+ - V_- \wedge n]) = \mathcal{D}(q[A, V_+])$. By monotone form convergence one concludes that the sequence of operators $H(A, V_n) = H(A, V_+ - V_- \wedge n)$ converges to the operator H which is associated to the regular part q^r of the quadratic form q (see [81]) where q is given by

$$q(\varphi) = \inf_n q[A, V_+ - V_- \wedge n](\varphi), \; \mathcal{D}(q) = \mathcal{D}(q[A, V_+]).$$

Recall that the regular part q^r of a quadratic form q is by definition the largest closable quadratic form which is smaller than q. Thus it is enough to show that in our case q is actually closable and that the closure of q coincides with the closure of $q \restriction C_c^\infty(M)$. For every $u \in \mathcal{D}(q[A, V_+])$ there is a sequence $(\varphi_n)_{n\in\mathbb{N}} \subset C_c^\infty(M)$ such that $\varphi_n \to \infty$ as $n \to \infty$ with respect to the norm $\sqrt{q[A, V_+](\cdot) + \|\cdot\|^2_{L^2(M)}}$. By assumption we have

$$\forall n \in \mathbb{N} : \int_M |(d - iA)\varphi_n|^2 \, d\mu + \int_M V_+ |\varphi_n|^2 \, d\mu - \int_M V_- |\varphi_n|^2 \, d\mu \geq 0.$$

Thus

$$\forall n \in \mathbb{N} : \int_M |(d - iA)\varphi_n|^2 \, d\mu + \int_M V_+ |\varphi_n|^2 \, d\mu \geq \int_M V_- |\varphi_n|^2 \, d\mu$$

and therefore

$$\limsup_{n\in\mathbb{N}} \int_M V_- |\varphi_n|^2 \, d\mu < \infty.$$

This implies that $q[A, V](\varphi_n - u) + \|\varphi_n - u\|^2_{L^2(M)} \to 0$ as $n \to \infty$. The form domain $\mathcal{D}(q)$ is therefore contained in $\overline{C_c^\infty(M)}^{q[A,V]}$. Thus q is closable and its closure $\bar{q}$ coincides with the closure of $(q[A, V], C_c^\infty(M))$. Hence we get $H(A, V_n) \to H(A, V)$ in the strong resolvent sense and we conclude that (2.3) holds with $H(A, W)$ replaced by $H(A, V)$. □

This gives us directly the following assertion

Corollary 5.2.1. *Let A and V as in Theorem 1 be given, then the subspace $\mathcal{D}_{comp}$ consisting of all compactly supported functions in the domain $\mathcal{D}(H(A,V))$ of $H(A,V)$ forms an operator core for $H(A,V)$*

Remark 5.2.2. *The above approach to the localization of the problem is also applicable to operators of Schrödinger type acting on more general vector bundles. We will apply this approach to the setting considered in [13] in a subsequent project.*

For the proof of Theorem 1 it is essential to show that bounded compactly supported functions form an operator core, This will be achieved in the Lemma 3 below. For $r > 0$ let $H_r(A,V)$ denote the operator $H(A,V)$ in the ball $B_r(o)$ with Dirichlet boundary condition, i.e. $H_r(A,V)$ is generated by the closure of $q_r[A,V]$ of the quadratic form $q[A,V] \upharpoonright C_c^\infty(B_r(o))$. The domain $\mathcal{D}(q_r[A,V])$ of the form $q_r[A,V]$ is of course contained in $\mathcal{D}(q[A,V])$. Observe that for $u \in \mathcal{D}(H(A,V))$ with support contained in $B_r(o)$ we also have $u \in \mathcal{D}(H_r(A,V))$ and $H_r(A,V)u = H(A,V)u$. On the other hand if $u \in \mathcal{D}(H_r(A,V))$ with $\mathrm{supp}(u) \subset B_r(o)$ we also have $u \in \mathcal{D}(H(A,V))$ with $H(A,V)u = H_r(A,V)u$. We further set for some fixed $T > 0$

$$\mathcal{C} = \{\varphi e^{-tH_{R+3}(A,V)}u \mid 0 < t < T,\, u \in \mathcal{D}(H(A,V)) \,\text{with}\, \mathrm{supp}(u) \subset B_R \,\text{for some}\, R > 0,\, \varphi \in C_c^\infty(B_{R+2}(o)),\, 0 \le \varphi \le 1,\, \varphi \upharpoonright B_R = 1\}$$

Lemma 5.2.5. *Let $A \in L^2_{loc}(T^*M)$ be a locally square integrable 1-form and let $V \in L^2_{loc}(M)$ be a potential such that the negative part belongs to the local Kato class $\mathcal{K}_{loc}(M)$. Then the set $\mathcal{C} \subset \mathcal{D}(H(A,V)) \cap L^\infty_{comp}(M)$ forms an operator core for $H(A,V)$*

Proof. We already know that the class of functions in $\mathcal{D}(H(A,V))$ with compact support form an operator core. Thus it is enough to prove that every $u \in \mathcal{D}(H(A,V))$ having compact support in the ball $B_R(o)$ of radius R with center o can be approximated by bounded compactly supported functions with respect to the graph norm. Let $\varphi \in C_c^\infty(M)$ be a smooth function with $0 \le \varphi \le 1$, $\varphi(x) = 1$ for $d(x,o) < R+1$ and $\varphi(x) = 0$ for $d(x,o) > R+2$. By the diamagnetic inequality we have for every $f \in L^2(B_{R+3})$ the domination property $|e^{-tH_{R+3}(A,V)}f| \le e^{-tH_{R+3}(0,V)}|f|$. By the Feynman-Kac Formula and the condition $V \in \mathcal{K}_{loc}(M)$ one concludes that the semigroup $e^{-tH_{R+3}(0,V)}$ consists of integral operators with integral kernels $p(t,x,y)$ satisfying $\sup_{x,y\in B_{R+3}(o)} p(t,x,y) \le c(t) < \infty$. In order to see this just recall that for $f \in L^2(M,\mu)$

$$e^{-tH_{R+3}(0,V)}f(x) = \mathbb{E}_x\left[e^{-\int_0^t V(X_s)\,ds} f(X_t), t < \tau_{B_{R+3}(o)}\right],$$

where the expectation is with respect to Brownian motion on M killed by exiting $B_{R+3}(o)$. Now we can use the assumption $V_- \in \mathcal{K}_{loc}(M)$ and the results in [29] (in particular Theorem 2.21 and Corollary 3.6 in [29]) in order to conclude that $e^{-tH_{R+3}(0,V)}$ has a bounded integral kernel given by

$$p(t,x,y) = \mathbb{E}_{0,x}^{t,y}\left[e^{-\int_0^t V(X_s)\,ds}, t < \tau_{B_{R+3}(o)}\right],$$

where $\mathbb{E}_{0,x}^{t,y}$ denotes expectation with respect to the unnormalized Brownian bridge measure and $\tau_{B_{R+3}(o)} = \inf\{t > 0 \mid d(o,X_t) \ge R+3\}$ the first exit time from the ball $B_{R+3}(o)$. This property give exactly as in the euclidean case $e^{-tH_{R+3}(0,V)}u \in L^\infty_{loc}(M)$. For $\psi_t = \varphi e^{-tH_{R+3}(A,V)}u = \varphi u_t \in L^\infty(M)$ we get

$$L^2 - \lim_{t\to 0} \psi_t = \varphi u = u$$

and

$$\begin{aligned}
&\Big(\int_M |H(A,V)(\psi_t - u)|^2\,d\mu\Big)^{\frac{1}{2}} = \Big(\int_M |H(A,V)(\psi_t - \varphi u)|^2\,d\mu\Big)^{\frac{1}{2}}\\
&= \Big(\int_M |\varphi H_{R+3}(A,V)(u_t-u) - 2i\langle d_A(u_t-u), d\varphi\rangle - (u_t-u)\Delta\varphi|^2\,d\mu\Big)^{\frac{1}{2}}\\
&\le \|\varphi H_{R+3}(A,V)(u_t-u)\|_{L^2(B_{R+3}(o))} + 2\Big(\int_{B_{R+3}(o)} |\langle d_A(u_t-u), d\varphi\rangle|^2\,d\mu\Big)^{\frac{1}{2}}\\
&+ \|(u_t-u)\Delta\varphi\|_{L^2(B_{R+3}(o))}\\
&\le \|H_{R+3}(A,V)(u_t-u)\|_{L^2(B_{R+3}(o))} + c\|(d-iA)(u_t-u)\|_{L^2(B_{R+3}(o))}\\
&+ \|(u_t-u)\Delta\varphi\|_{L^2(B_{R+3}(o))}
\end{aligned}$$

By the spectral theorem we conclude

$$\lim_{t\to 0}\Big(\int_M |H(A,V)(\psi_t - u)|^2\,d\mu\Big)^{\frac{1}{2}} = 0$$

This finishes the proof. □

The Gagliardo-Nirenberg inequality allows us to deduce some regularity properties of functions f belonging to $\mathcal{C}$. First observe that exactly as in Lemma 8 of [60] one deduces

Lemma 5.2.6. *Let $\Omega \subset M$ be a bounded open set and let $c > 0$. There exists a constant $d > 0$ such that for all $u \in H^2(M) \cap L^\infty(M)$ with $supp(u) \subset\subset \Omega$ and all vector potentials $A \in \mathcal{M}(M)$ satisfying the conditions of Theorem 5.1.1 and $\|d^*A\|_{L^2(\Omega)\mu)} + \||A|^2\|_{L^2(\Omega),\mu} \le c$ the inequality*

$$\|\Delta u\|_{L^2(M)} \le \|H(A,V)u\|_{L^2(M)} + d\|u\|_{L^\infty}$$

holds true.

This gives

Lemma 5.2.7. *Let $A \in \mathcal{M}(M)$ and $V \in L^2_{loc}(M)$ with $V_- \in \mathcal{K}_{loc}(M)$ be as in Theorem 5.1.1. Then we have*

$$\mathcal{C} \subset H^2(M) \cap L^\infty_{comp}(M).$$

In particular gradf $\in L^4(M)$ for every $f \in \mathcal{C}$

Proof. The proof is a modification of Lemma 9 in [60]. Assume that supp(f) is contained in the ball $B(o,R)$. In the proof of the proceeding Lemma we have seen that $\psi_t = \varphi\, e^{-tH_{R+3}(A,V)} f = \varphi f_t$ converges with respect to the graph norm to f, where $\varphi \in C_c^\infty(B(o,R+2))$ with $\varphi \restriction B(o,R) = 1$. Let $(A_n)_{n\in\mathbb{N}}$ be a sequence of smooth 1-forms such that

$$\int_{B(0,R+3)} |A_n - A|^4\,d\mu + \int_{B(o,R+3)} |d^*A_n - d^*A|^2\,d\mu \to 0$$

as $n \to \infty$ and $e^{-tH_{B(o,R+3)}(A_n,V)} \to e^{tH_{B(o,R+3)}(A,V)}$ as $n \to \infty$ with respect to the strong operator convergence (This assertion directly follows from the Feynman-Kac-Itô representation of the semigroup $e^{tH_{B(o,R+3)}(A,V)}$). Set $u_n = e^{-tH_{B(o,R+3)}(A_n,V_n)}u$. The assumption $V_- \in \mathcal{K}_{loc}(M)$ implies that there exists a $C > 0$ such that for all $u \in C_c^\infty(B(o, R+3))$ and every $n \in \mathbb{N}$

$$\begin{aligned} \|(d - iA_n)w\|^2_{L^2(B(o,R+3)} + \|w\|^2_{L^2(B(o,R+3)} \le C\big(&\|(d - iA_n)w\|^2_{L^2(B(o,R+3)} \\ &+ \|w\|^2_{L^2(B(o,R+3)} - \|\sqrt{V_-}w\|^2_{L^2(B(o,R+3)}\big). \end{aligned} \tag{5.2.5}$$

In order to prove equation (5.2.5) just recall that the potential $\chi_{B(o,R+3)}V_- \in \mathcal{K}(M)$ is $-\Delta$-form bounded with infinitesimal bound 0, i.e. for every $\varepsilon > 0$ there is $d(\varepsilon) > 0$ such that for all $\varphi \in H_0^2(B(o, R+3))$

$$\|\sqrt{V_-}\varphi\|_{L^2(B(o,R+3))} \le \varepsilon\|\operatorname{grad}\varphi\|_{L^2(B(o,R+3))} + d(\varepsilon)\|\varphi\|_{L^2(B(o,R+3)}.$$

By the quadratic form version of the diamagnetic inequality (see chapter 2.3 of [71]) this gives for every $n \in \mathbb{N}$ and every $\varphi \in H_0^1(B(o, R+3))$

$$\|\sqrt{V_-}\varphi\|_{L^2(B(o,R+3))} \le \varepsilon\|(d - iA_n)\varphi\|_{L^2(B(o,R+3))} + d(\varepsilon)\|\varphi\|_{L^2(B(o,R+3)}.$$

This implies that equation (5.2.5) holds. In particular, there is a constant $\tilde{C}$ such that for all $n \in \mathbb{N}$

$$\begin{aligned} \|(d - iA_n)u_n\|^2_{L^2(B(o,R+3)} + \|u_n\|^2_{L^2(B(o,R+3)} \le \tilde{C}\Bigg(&\|(d - iA_n)u_n\|^2_{L^2(B(o,R+3)} + \|u_n\|^2_{L^2(B(o,R+3)} \\ &+ \int_{B(o,R+3)} V|u_n|^2\, d\mu\Bigg). \end{aligned}$$

This gives

$$\begin{aligned} \|(d - iA_n)u_n\|^2_{L^2(B(o,R+3)} + \|u_n\|^2_{L^2(B(o,R+3)} &\le \tilde{C}\big(H_{B(o,R+3)}(A_n, V)u_n, u_n\big)_{L^2(B(o,R+3)} \\ &\le \tilde{C}\|H_{B(o,R+3)}(A_n, V)u_n\|_{L^2(B(o,R+3)}\|u_n\|_{L^2(B(o,R+3)}, \end{aligned}$$

which implies

$$\|u_n\|_{L^2(B(o,R+3)} \le C\|H_{R+3}(A_n, V)u_n\|_{L^2(B(o,R+3)}$$

and

$$\|(d - iA_n)u_n\|_{L^2(B(o,R+3)} \le C\|H_{R+3}(A_n, V)\|_{L^2(B(o,R+3)}.$$

Now set $v_n = \varphi u_n$ with φ as in the definition of the set $\mathcal{C}$. Then $v_n \in \mathcal{D}(H_{R+3}(A, V)) \cap L^\infty$ and therefore by the product rule

$$H_{R+3}(A_n, V)v_n = \varphi H_{R+3}(A_n, V)u_n - 2\langle du_n - iA_nu_n, d\varphi\rangle - u_n\Delta\varphi.$$

General results concerning elliptic regularity (see [40] or [48]) imply $v_n \in H^2(M)$ and by Lemma 5.2.6 we then conclude that for some constants $c, c_1, c_2, c_3, a, b > 0$

$$\begin{aligned}\|\Delta v_n\|_{L^2(B(o,R+3))} &\le 2\|H_{R+3}(A_n, V)v_n\|\|_{L^2(B(o,R+3))} + c\|v_n\|_{L^\infty(B(o,R+3))}\\ &\le c_1\|H_{R+3}(A_n, V)u_n\|_{L^2(B(o,R+3))} + c_2\|(d - iA_n)u_n\|_{Lr(B(o,R+3))}\\ &+ c_3\|u_n\|_{L^\infty(B(o,R+3))}\\ &\le 2a + b\|e^{-tH_{R+3}(0,V)}|u|\|_{L^\infty(B(o,R+3))}\end{aligned}$$

where the diamagnetic inequality $|e^{-tH_{R+3}(A_n,V)}u| \le e^{-tH_{R+3}(0,V)}|u|$ was used in the last step. Observe that as used already above because of the assumption $V_- \in \mathcal{K}_{loc}(M)$ the operator $e^{-tH_{R+3}(0,V)}$ is ultracontractive. Therefore we have $e^{-tH_{R+3}(0,V)}u \in L^\infty(B(o, R+3))$. The unit ball in the Hilbert space $L^2(B(o, R+3))$ is weakly compact and therefore there exists a weakly convergence subsequence of $(\Delta v_n)_{n\in\mathbb{N}}$. Since $v_n \to \varphi u$ in $L^2(B(o, R+3))$ we conclude that $\varphi u \in H^2(M)$. □

Now we can easily complete the proof of Theorem 5.1.1

Proof. We have already shown that the set $\mathcal{D}(H(A, V)) \cap L^\infty_{comp}(M)$ build an operator core for $H(A, V)$. Thus it remains to show that every function $f \in \mathcal{D}(H(A, V)) \cap L^\infty_{comp}(M)$ can be approximated by smooth functions $\varphi_n \in C_c^\infty(M)$ with respect to the operator norm. By Lemma 5.2.6 we conclude that $\Delta f \in L^2(M)$ and consequently by Lemma 5.2.2 $\operatorname{grad} f \in L^4(TM)$. Let $(f_n)_{n\in\mathbb{N}}$ be the sequence obtained from f by Friedrichs mollification. Since the support of f is a compact subset of M the Friedrichs mollifiers can be constructed in the standard way in local coordinates. Then we have $f_n \in C_c^\infty(U)$ for any $n \in \mathbb{N}$ and some bounded open set $U \subset M$, $f_n \to f$ in $L^\infty(M)$ and

$$\lim_{n\to\infty} \|\operatorname{grad}(f - f_n)\|_{L^4(TM)} \text{ und } \lim_{n\to\infty} \|\Delta(f_n - f)\|_{L^2(M)} = 0.$$

Therefore we get

$$\begin{aligned}\|H(A, V)f - H(A, V)f_n\|_{L^2(M)} &\le \|\Delta(f - f_n)\|_{L^2(M)} + 2\|\langle A, (f - f_n)\rangle\|_{L^2(M)}\\ &+ \|(id^*A + |A|^2 + V)(f - f_n)\|_{L^2(M)}\\ &\le \|\Delta(f - f_n)\|_{L^2(M)} + 2\||A|\|_{L^4(U)}\|\operatorname{grad}(f - f_n)\|_{L^4(TU)}\\ &+ \|(id^*A + |A|^2 + V)\|_{L^2(U)}\|f - f_n\|_{L^\infty} \to 0\end{aligned}$$

as $n \to \infty$. □

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