

# Weak Dependence of Functional INGARCH Processes

Jürgen Franke

University of Kaiserslautern, Department of Mathematics.

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Corresponding Author: Jürgen Franke, University of Kaiserslautern, Department of Mathematics, Erwin-Schroedinger-Str., 67663 Kaiserslautern, Germany. E-mail: franke@mathematik.uni-kl.de

## Abstract

We introduce a class of models for time series of counts which include INGARCH-type models as well as log linear models for conditionally Poisson distributed data. For those processes, we formulate simple conditions for stationarity and weak dependence with a geometric rate. The coupling argument used in the proof serves as a role model for a similar treatment of integer-valued time series models based on other types of thinning operations.

**Keywords:** count data, integer-valued time series, integer GARCH, Poisson regression, weak dependence, Bernstein inequality, Rosenthal inequality

## 1 Introduction

We study a general class of models for time series of counts which is motivated by the so-called integer-valued GARCH or INGARCH model. For an integer-valued time series  $\{Y_t, t \in \mathbb{Z}\}$  with values in  $\mathbb{N}_0 = \{0, 1, 2, \dots\}$ , let  $\mathcal{F}_t$  denote the  $\sigma$ -algebra generated by  $Y_s, s \leq t$ . Then, the time series is called an integer-valued GARCH process of order (1,1) if the conditional distribution of  $Y_t$  given  $\mathcal{F}_{t-1}$  is of the form

$$\mathcal{L}(Y_t | \mathcal{F}_{t-1}) = \text{Poisson}(\lambda_t), \quad \lambda_t = d + a\lambda_{t-1} + bY_{t-1}, \quad (1)$$

for some parameters  $d > 0, a, b \geq 0$ . This model has been considered by (Rydborg and Shephard, 2000) and (Streett, 2000), and a more detailed analysis has been given by (Ferland et al, 2006), (Weiß, 2009) and (Fokianos et al, 2009). The latter have also extended the INGARCH model in an additive nonlinear manner by considering

$$\mathcal{L}(Y_t|\mathcal{F}_{t-1}) = \text{Poisson}(\lambda_t), \quad \lambda_t = f(\lambda_{t-1}) + b(Y_{t-1}) \quad (2)$$

for some positive functions  $f, b$  which are known up to finitely many parameters. Related models have been considered by (Davis et al, 2003, 2005) and (Zhu et al, 2010). Mark that the names of those models are motivated by analogy to the popular GARCH model for time series as  $\lambda_t$  is the conditional variance of  $Y_t$  given  $\mathcal{F}_{t-1}$ .

As usual for time series, standard asymptotics for, e.g., parameter estimates depends on weak dependence properties of such processes. However, classical weak dependence in the sense of mixing or geometric ergodicity is hard to show for processes of INGARCH type, compare e.g. (Fokianos et al, 2009), if at all possible. An alternative would be the rather new weak dependence concept of (Doukhan and Louhichi, 1999) for which now major tools for deriving asymptotic results are available, compare, e.g., (Dedecker and Doukhan, 2003), (Doukhan and Neumann, 2007) or (Kallabis and Neumann, 2007). A recommended survey of weak dependence and its consequences is given by (Dedecker et al, 2007). The application of the concept in a nonparametric framework is described in Nze et al. (Nze et al, 2002).

Weak dependence has been shown and applied for a bilinear integer-valued process by (Doukhan et al, 2006), however using the special structure of that model. In this note, we use an intuitive coupling argument to prove weak dependence for a general class of INGARCH models which includes (1) and (2) as well as log linear models for time series of counts, compare (Fokianos and Kedem, 2004) and the discussion in the introduction of (Fokianos et al, 2009). It turns out that, for integer-valued time series, weak dependence with geometrically decreasing coefficients can be shown in a much easier way than geometric ergodicity. To illustrate the use of that concept we formulate a Bernstein inequality and a Rosenthal inequality for general INGARCH models which follow immediately from general results of (Doukhan and Neumann, 2007).

The approach illustrated in this note with INGARCH-type models serves as a role model for proving weak dependence for integer-valued time series models based on other thinning operations, compare (Weiß, 2008) for a review of the latter. In particular for binomial thinning, which leads to the class of general integer-valued autoregressions (INAR), (Triebisch, 2008) has proven weak dependence and discussed the asymptotic behaviour of nonparametric sieve

estimates.

## 2 Functional INGARCH Models and Weak Dependence

In this section, we introduce a general class of INGARCH models of arbitrary orders  $p, q \geq 0$  which we call *functional* INGARCH(p,q) or FINGARCH(p,q) models as it is related to the parametric INGARCH class in a similar manner as the functional autoregressive models of (Chen and Tsay, 1993) to linear autoregressions. This class includes (1) and (2) as well as the log linear Poisson autoregressions of (Fokianos and Tjøstheim, 2009) as special cases where the latter corresponds to the choice  $g(z, y) = e^d z^a (1 + y)^b$  below.

To specify the generation of the Poisson variables appearing in the formulation of those models, we follow (Fokianos et al, 2009) and start from a sequence of independent Poisson processes  $\{N_t(\cdot), t \in \mathbb{Z}\}$  of unit intensity. Then, a FINGARCH(p,q) process  $Y_t, t \in \mathbb{Z}$ , satisfies the recursion

$$Y_t = N_t(\lambda_t), \quad \lambda_t = g(\lambda_{t-1}, \dots, \lambda_{t-p}, Y_{t-1}, \dots, Y_{t-q}), \quad (3)$$

where  $g : (0, \infty)^p \times \mathbb{N}_0^q \rightarrow (0, \infty)$  is some measurable function. To simplify notation, we set

$$\lambda_{t-1}^{(p)} = (\lambda_{t-1}, \dots, \lambda_{t-p}), \quad Y_{t-1}^{(q)} = (Y_{t-1}, \dots, Y_{t-q}) \quad \text{such that } \lambda_t = g(\lambda_{t-1}^{(p)}, Y_{t-1}^{(q)}).$$

We assume in the following that  $g$  is Lipschitz in each argument with Lipschitz constants summing up to a constant less than 1, i.e. for  $z, \tilde{z} \in (0, \infty)^p, y, \tilde{y} \in \mathbb{N}_0^q$  we have

$$|g(z, y) - g(\tilde{z}, \tilde{y})| \leq \sum_{i=1}^p a_i |z_i - \tilde{z}_i| + \sum_{i=1}^q b_i |y_i - \tilde{y}_i| \quad (4)$$

with  $a_1 + \dots + a_p + b_1 + \dots + b_q = L < 1$ . Mark that for the special cases (1) and (2), (4) reduces to the conditions for geometric ergodicity given by (Fokianos et al, 2009).

First, we remark that, under condition (4), there exist strictly stationary FINGARCH processes. We postpone the proof to the appendix.

**Theorem 1** *If  $g(z, y)$  satisfies the Lipschitz condition (4) then there exists a strictly stationary FINGARCH(p,q) process  $\{Y_t\}_{t \in \mathbb{Z}}$  satisfying (3) and having a finite mean  $E Y_t < \infty$ .*

There are several variants of the weak dependence concept. Here, we consider the notion of  $\theta$ -weak dependence for univariate time series in the formulation of (Doukhan and Neumann, 2007). Let  $\{Z_t\}$  be a stationary time series with values in  $\mathbb{R}$ . For an arbitrary function  $h : \mathbb{R}^u \rightarrow \mathbb{R}$ , let

$$\text{Lip } h = \sup \left\{ \frac{|h(x_1, \dots, x_u) - h(y_1, \dots, y_u)|}{|x_1 - y_1| + \dots + |x_u - y_u|} : (x_1, \dots, x_u) \neq (y_1, \dots, y_u) \right\}.$$

$\Lambda$  denotes the set of functions  $h : \mathbb{R}^u \rightarrow \mathbb{R}$ , such that  $\text{Lip } h < \infty$  and  $\Lambda^{(1)} = \{h \in \Lambda : \|h\|_\infty \leq 1\}$ .

**Definition:** The time series  $\{Z_t\}_{t \in \mathbb{Z}}$  is called  $\theta$ -weak dependent if there exists a sequence  $\theta = (\theta_r)_{r \in \mathbb{N}}$  decreasing to zero at infinity such that, for any  $g_1 : \mathbb{R}^u \rightarrow \mathbb{R}$ ,  $g_2 : \mathbb{R}^v \rightarrow \mathbb{R}$ ,  $u, v \in \mathbb{N}$ , satisfying  $g_1, g_2 \in \Lambda^{(1)}$ , and for any  $u$ -tuple  $(s_1, \dots, s_u)$  and any  $v$ -tuple  $(t_1, \dots, t_v)$  of integers with  $s_1 \leq \dots \leq s_u < s_u + r \leq t_1 \leq \dots \leq t_v$ , the following inequality is fulfilled:

$$|\text{Cov}(g_1(Z_{s_1}, \dots, Z_{s_u}), g_2(Z_{t_1}, \dots, Z_{t_v}))| \leq v \text{Lip } g_2 \theta_r. \quad (5)$$

Functions of weak dependent time series are usually also weak dependent. For later reference, we state the following precise lemma based on remark 7 of (Doukhan and Neumann, 2007).

**Lemma 2** *Let  $\{X_t\}_{t \in \mathbb{Z}}$  be a stationary real-valued  $\theta$ -weak dependent process with coefficients  $\theta_t^X$ , satisfying  $EX_t^2 < \infty$ . Let  $F : \mathbb{R}^d \rightarrow \mathbb{R}$  be a Lipschitz continuous function such that  $F(0) = 0$ . Then,*

$$Y_t = F(X_t, X_{t-1}, \dots, X_{t-d+1})$$

*is  $\theta$ -weak dependent with coefficients satisfying  $\theta_t^Y = O(\theta_{t-d+1}^X)$ .*

**Proof:** From the definition above it is immediate that  $Z_t = (X_t, \dots, X_{t-d+1})$  is weak dependent with coefficients  $\theta_t^Z = \theta_{t-d+1}^X$ . Now,  $Y_t = F(Z_t)$ ,  $E\|Z_t\|^2 < \infty$ , such that we can apply Proposition 2.2 of (Dedecker et al, 2007) for the special case  $p = 2, a = 1$  to get the desired result. ■

To communicate the main idea avoiding a somewhat cumbersome notation, we first consider the case  $p = q = 1$ .

**Proposition 3** *Let  $\{Y_t\}_{t \in \mathbb{Z}}$  be a stationary FINGARCH(1,1) process satisfying (3) for  $p = q = 1$ . If  $g(z, y)$  satisfies the Lipschitz condition*

$$|g(z, y) - g(\tilde{z}, \tilde{y})| \leq a|z - \tilde{z}| + b|y - \tilde{y}|, \quad z \in (0, \infty), y \in \mathbb{N}_0 \quad (6)$$

*with  $L = a + b < 1$ , then,  $\{Y_t\}_{t \in \mathbb{Z}}$  is  $\theta$ -weak dependent with geometrically decreasing coefficients*

$$\theta_t \leq cL^t \quad \text{for some } c > 0.$$

**Proof:** For the proof, we need the concept of  $\tau$ -weak dependence. By Proposition 2.3. of (Dedecker et al, 2007), the  $\theta$ -weak dependence coefficients of a time series  $\{Z_t\}$  can be equivalently defined as

$$\theta_r = \max_{v \geq 1} \frac{1}{v} \sup_{s < s+r \leq t_1 < \dots < t_v} \theta(\mathcal{M}_s, Z_{t_1}, \dots, Z_{t_v}), \quad r \geq 1,$$

where  $\mathcal{M}_s$  is the  $\sigma$ -algebra generated by  $Z_k, k \leq s$ , and

$$\theta(\mathcal{M}_s, Z_{t_1}, \dots, Z_{t_v}) = \sup_{g \in \Lambda^{(1)}} \mathbb{E} \left| \mathbb{E}\{g(Z_{t_1}, \dots, Z_{t_v}) \mid \mathcal{M}_s\} - \mathbb{E}g(Z_{t_1}, \dots, Z_{t_v}) \right|$$

Analogously, we define

$$\tau_r = \max_{v \geq 1} \frac{1}{v} \sup_{s < s+r \leq t_1 < \dots < t_v} \tau(\mathcal{M}_s, Z_{t_1}, \dots, Z_{t_v}), \quad r \geq 1,$$

with

$$\tau(\mathcal{M}_s, Z_{t_1}, \dots, Z_{t_v}) = \mathbb{E} \left| \sup_{g \in \Lambda^{(1)}} \left\{ \mathbb{E}\{g(Z_{t_1}, \dots, Z_{t_v}) \mid \mathcal{M}_s\} - \mathbb{E}g(Z_{t_1}, \dots, Z_{t_v}) \right\} \right|.$$

We immediately have  $\theta(\mathcal{M}_s, Z_{t_1}, \dots, Z_{t_v}) \leq \tau(\mathcal{M}_s, Z_{t_1}, \dots, Z_{t_v})$ , compare (2.2.13) of (Dedecker et al, 2007), and therefore  $\theta_r \leq \tau_r, r \geq 1$ .

By Lemma 5.2 or more generally Lemma 5.3 of (Dedecker et al, 2007), we get  $\tau$ -weak dependence and, hence, by the remarks above,  $\theta$ -weak dependence with an exponential rate if we can show that, for arbitrary initial values  $\tilde{\lambda}_0, \tilde{Y}_0$  chosen independently of  $\lambda_0, Y_0$  and having a finite first absolute moment, there exists another realization  $\tilde{Y}_t$  of the FINGARCH(1,1)-process such that for some constants  $c > 0, 0 < \rho < 1$

$$\mathbb{E}|Y_t - \tilde{Y}_t| \leq c\rho^t. \quad (7)$$

A simple coupling argument shows that kind of asymptotic closeness of  $Y_t, \tilde{Y}_t$ , i.e., vanishing influence of the initial values. Given the original process  $\{Y_t\}$ , satisfying (3) with  $p = q = 1$ , as well as  $\tilde{\lambda}_0, \tilde{Y}_0$ , we construct  $\tilde{Y}_t, t \geq 1$ , such that

$$Y_t = N_t(\lambda_t), \quad \lambda_t = g(\lambda_{t-1}, Y_{t-1}), \quad \tilde{Y}_t = N_t(\tilde{\lambda}_t), \quad \tilde{\lambda}_t = g(\tilde{\lambda}_{t-1}, \tilde{Y}_{t-1}),$$

i.e. we use the same family of independent standard Poisson processes for both time series. Let  $\mathcal{F}_t^*$  denote the  $\sigma$ -algebra generated by  $\lambda_0, Y_0, Y_1, \dots, Y_t$  and  $\tilde{\lambda}_0, \tilde{Y}_0, \tilde{Y}_1, \dots, \tilde{Y}_t$ . Then, for  $t \geq 1$ ,

$$\mathbb{E}|Y_t - \tilde{Y}_t| = \mathbb{E} \mathbb{E}\{|Y_t - \tilde{Y}_t| \mid \mathcal{F}_{t-1}^*\} = \mathbb{E} \mathbb{E}\{|N_t(\lambda_t) - N_t(\tilde{\lambda}_t)| \mid \mathcal{F}_{t-1}^*\} = \mathbb{E}|\lambda_t - \tilde{\lambda}_t|$$

as, given  $\mathcal{F}_{t-1}^*$  and therefore  $\lambda_t, \tilde{\lambda}_t, N_t(\lambda_t) - N_t(\tilde{\lambda}_t)$  is *Poisson*( $\lambda_t - \tilde{\lambda}_t$ ) for  $\lambda_t > \tilde{\lambda}_t$  and *Poisson*( $\tilde{\lambda}_t - \lambda_t$ ) else. Therefore, we have for all  $t > 1$ ,

$$\begin{aligned} \mathbb{E}|Y_t - \tilde{Y}_t| &= \mathbb{E}|g(\lambda_{t-1}, Y_{t-1}) - g(\tilde{\lambda}_{t-1}, \tilde{Y}_{t-1})| \\ &\leq a\mathbb{E}|\lambda_{t-1} - \tilde{\lambda}_{t-1}| + b\mathbb{E}|Y_{t-1} - \tilde{Y}_{t-1}| \\ &= (a + b)\mathbb{E}|Y_{t-1} - \tilde{Y}_{t-1}| = L\mathbb{E}|Y_{t-1} - \tilde{Y}_{t-1}|. \end{aligned} \quad (8)$$

Iterating the argument, we get

$$\mathbb{E}|Y_t - \tilde{Y}_t| \leq L^{t-1}\mathbb{E}|Y_1 - \tilde{Y}_1| \leq L^{t-1} \left( a\mathbb{E}|\lambda_0 - \tilde{\lambda}_0| + b\mathbb{E}|Y_0 - \tilde{Y}_0| \right) \leq cL^t$$

for some suitably chosen constant  $c > 0$ , i.e. we have shown (7) with  $\rho = L$ . ■

**Theorem 4** *Let  $\{Y_t\}_{t \in \mathbb{Z}}$  be a stationary FINGARCH process (3). If  $g(z, y)$  satisfies the Lipschitz condition (4) then,  $\{Y_t\}_{t \in \mathbb{Z}}$  is  $\theta$ -weak dependent with geometrically decreasing coefficients*

$$\theta_t \leq c(L^{\frac{1}{\max(p,q)}})^t \quad \text{for some } c > 0.$$

**Proof:** The arguments are identical to the proof of Proposition 3 until inequality (8), which, using (4), is replaced by

$$\begin{aligned} \mathbb{E}|Y_t - \tilde{Y}_t| &= \mathbb{E}|g(\lambda_{t-1}^{(p)}, Y_{t-1}^{(q)}) - g(\tilde{\lambda}_{t-1}^{(p)}, \tilde{Y}_{t-1}^{(q)})| \\ &\leq \sum_{i=1}^p a_i \mathbb{E}|\lambda_{t-i} - \tilde{\lambda}_{t-i}| + \sum_{i=1}^q b_i \mathbb{E}|Y_{t-i} - \tilde{Y}_{t-i}| \\ &= \sum_{i=1}^p a_i \mathbb{E}|Y_{t-i} - \tilde{Y}_{t-i}| + \sum_{i=1}^q b_i \mathbb{E}|Y_{t-i} - \tilde{Y}_{t-i}| \\ &\leq L \max_{1 \leq i \leq m} \mathbb{E}|Y_{t-i} - \tilde{Y}_{t-i}| \end{aligned}$$

with  $m = \max(p, q)$ . Writing  $\mu(t) = \arg \max_{1 \leq i \leq m} \mathbb{E}|Y_{t-i} - \tilde{Y}_{t-i}|$ , we have

$$\begin{aligned} \mathbb{E}|Y_t - \tilde{Y}_t| &\leq L\mathbb{E}|Y_{t-\mu(t)} - \tilde{Y}_{t-\mu(t)}| \\ &\leq L^2 \max_{1 \leq i \leq m} \mathbb{E}|Y_{t-\mu(t)-i} - \tilde{Y}_{t-\mu(t)-i}| \\ &= L^2 \mathbb{E}|Y_{t-\mu(t)-\mu(t-\mu(t))} - \tilde{Y}_{t-\mu(t)-\mu(t-\mu(t))}| \end{aligned}$$

and so on. In each single step, the index of  $Y, \tilde{Y}$  on the right-hand side decreases by at most  $m$ , such that we have at least  $\lceil t/m \rceil$  steps in the iteration before it stops with that index becoming 0 or less. As the initial values have finite first absolute moments by assumption, we finally get, recalling  $L < 1$ , for some appropriate constant  $c > 0$

$$\mathbb{E}|Y_t - \tilde{Y}_t| \leq c(L^{1/m})^t = c\rho^t, \quad (9)$$

i.e. we have shown (7) with  $\rho = L^{1/m} < 1$ . ■

### 3 Some Inequalities for FINGARCH processes

An immediate consequence of Theorem 4, Lemma 2 and the general Bernstein inequality (Theorem 1) of (Doukhan and Neumann, 2007), is the following Bernstein inequality of bounded Lipschitz functions of a FINGARCH process.

**Corollary 5** (*Bernstein inequality*) *Let  $\{Y_t\}$  be a FINGARCH-process (3) satisfying the assumptions of Theorem 4. Let  $F : \mathbb{R}^d \rightarrow \mathbb{R}$  be a Lipschitz function satisfying  $F(0) = 0, |F(z)| \leq M$  for all  $z$ , and let  $X_t = F(Y_t, \dots, Y_{t-d+1})$ . Then,*

a)  $\{X_t\}$  is  $\theta$ -weak dependent with exponentially decreasing coefficients

$$\theta_t^X \leq c_X \rho^t \quad \text{with } \rho = L^{\frac{1}{\max(p,q)}} < 1 \text{ and some } c_X > 0.$$

b) With  $S_N = X_1 + \dots + X_N$ , we have

$$P(|S_N - ES_N| > t) \leq 2 \exp\left(-\frac{t^2/2}{A_N + B_N^{1/3} t^{5/3}}\right) \quad (10)$$

where  $A_N$  can be chosen as any number  $\geq \sigma_N^2 = \text{var}(S_N)$ , and

$$B_N = \frac{2M}{1-\rho} \max\left(\frac{16NMc}{A_N(1-\rho)}, 1\right).$$

**Proof:** We only have to check the conditions (1) and (2) of Theorem 1 of (Doukhan and Neumann, 2007). Condition (1) is satisfied for a  $\theta$ -weak dependent process with  $\Psi(u, v) = 2v$ ,  $K^2 = M$ , and  $\rho(r) = \theta_r/2$  from Proposition 8 and Remark 9 of (Doukhan and Neumann, 2007). Condition (2) is satisfied if  $\theta_t \leq c\rho^t$  for some  $\rho < 1$ , with  $\mu = 1$ ,  $L_1 = \frac{c}{2}L_2$ ,  $L_2 = 1/(1-\rho)$ , again by Proposition 8 of (Doukhan and Neumann, 2007). Therefore, by Theorem 4 and Lemma 2, the general Bernstein inequality of Doukhan and Neumann is applicable to FINGARCH processes. ■

The following inequality of Rosenthal type is also a direct consequence of weak dependence of FINGARCH processes and Theorem 3 of (Doukhan and Neumann, 2007). We formulate it for the time series  $Y_t$ , but it could be applied to any function of finitely many variables from the FINGARCH processes, compare Lemma 2 above, as we only need  $\theta$ -weak dependence.

**Corollary 6** (*Rosenthal inequality*) *Let  $\{Y_t\}$  be a FINGARCH-process (3) satisfying the assumptions of Theorem 4, and let  $\mu_Y = E Y_t$ ,  $S_N = Y_1 + \dots + Y_N$ ,  $\sigma_N^2 = \text{var } S_N$ , denote the mean and the cumulative sums of the time series resp.*

its variance. Moreover, let  $Z_0$  denote a standard normal variable. Assume, for some  $M \geq 1, \ell \geq 2$ , that  $E |Y_t - \mu_Y|^{\ell-2} \leq M^{\ell-2}$ . Then,

$$\left| E(S_N - ES_N)^\ell - \sigma_N^\ell EZ_0^\ell \right| \leq B_{N,\ell} \sum_{1 \leq j < \ell/2} A_{\ell,j} M^{\ell-j} N^j$$

where

$$A_{\ell,j} = \frac{1}{j!} \sum_{k_i \geq 2, i=1, \dots, j, k_1 + \dots + k_j = \ell} \frac{\ell!}{k_1! \dots k_j!},$$

$$B_{N,\ell} = 2^{\ell-1} (\ell!)^2 \max_{2 \leq k \leq \ell} \left( \sum_{s=1}^N s^{k-2} \theta_{s-1} \right)^{\ell/k},$$

with  $\theta_t$  denoting the weak dependence coefficients of  $\{Y_t\}$ .

**Proof:** The result follows immediately from Theorem 3 of (Doukhan and Neumann, 2007), from the finiteness of  $\mu_Y$  implied by Theorem 1 and from the weak dependence established in Theorem 4. In particular, we use again  $K^2 = M$  and  $\rho(r) = \theta_r/2$  to get the specific form of the inequality in the FINGARCH case. ■

As pointed out by (Doukhan and Neumann, 2007), the last corollary can, e.g., be used to prove a central limit theorem via the method of moments. Mark that from (Ferland et al, 2006) we know in particular that all moments of the INGARCH(1,1) process (1) exist such that, here, the corollary holds for arbitrary  $\ell$ .

## 4 Appendix

To prove Theorem 1, we rely on the kind of arguments given by (Doukhan et al, 2006) in the proof of their Theorem 2.1. Throughout the appendix,  $\{N_t(\cdot)\}_{t \in \mathbb{Z}}$  is a family of independent Poisson processes with intensity 1. We define a sequence of stationary integer-valued time series  $\{Y_t(n)\}_{t \in \mathbb{Z}, n \geq 0}$ , in the following manner. We start with a sequence of i.i.d. positive random variables  $\lambda_t(0), t \in \mathbb{Z}$ , having mean 1, and we set  $Y_t(0) = N_t(\lambda_t(0)), t \in \mathbb{Z}$ . Then, using again the abbreviation

$$\lambda_{t-1}^{(p)}(n) = (\lambda_{t-1}(n), \dots, \lambda_{t-p}(n)), \quad Y_{t-1}^{(q)}(n) = (Y_{t-1}(n), \dots, Y_{t-q}(n)),$$

we set recursively

$$\lambda_t(n) = g(\lambda_{t-1}^{(p)}(n-1), Y_{t-1}^{(q)}(n-1)), \quad Y_t(n) = N_t(\lambda_t(n)), \quad n \geq 1. \quad (11)$$

To structure the proof, we first formulate some auxiliary results.



**Lemma 7** *If  $\{\lambda_t\}_{t \in \mathbb{Z}}$  is a strictly stationary time series with values in  $(0, \infty)$ , then  $\{Y_t\}_{t \in \mathbb{Z}} = \{N_t(\lambda_t)\}_{t \in \mathbb{Z}}$  is strictly stationary too.*

**Proof:** Referring to (Brockwell and Davis, 1991), p.12, it suffices to show that for any  $\ell, s$ , the random vectors  $(Y_1, \dots, Y_\ell)$  and  $(Y_{1+s}, \dots, Y_{\ell+s})$  have the same distribution. But, for any  $s$ , we have, using independence of the Poisson processes  $N_t(\cdot)$

$$\begin{aligned} P(Y_{1+s} = k_1, \dots, Y_{\ell+s} = k_\ell) &= E P\{Y_{1+s} = k_1, \dots, Y_{\ell+s} = k_\ell \mid \lambda_{1+s}, \dots, \lambda_{\ell+s}\} \\ &= E \prod_{t=1}^{\ell} P\{N_{t+s}(\lambda_{t+s}) = k_t \mid \lambda_{1+s}, \dots, \lambda_{\ell+s}\} \\ &= E \prod_{t=1}^{\ell} \left( \frac{\lambda_{t+s}^{k_t}}{k_t!} e^{-\lambda_{t+s}} \right). \end{aligned}$$

The right-hand side does not depend on  $s$  due to stationarity of  $\{\lambda_t\}_{t \in \mathbb{Z}}$ . ■

**Lemma 8** *The time series  $\{Y_t(n)\}_{t \in \mathbb{Z}}$  are strictly stationary for any  $n \geq 0$ .*

**Proof:** As in the previous proof, we have to show that the distribution of  $(Y_{1+s}(n), \dots, Y_{\ell+s}(n))$  does not depend on  $s$  which we do by induction. As  $\lambda_t(0), t \in \mathbb{Z}$ , are i.i.d. and, therefore, strictly stationary,  $\{Y_t(0)\}_{t \in \mathbb{Z}}$  is strictly stationary too by Lemma 7.

If  $Y_t(n), \lambda_t(n), t \in \mathbb{Z}$ , are strictly stationary, then  $\{\lambda_t(n+1)\}_{t \in \mathbb{Z}}$  is strictly stationary too as a function of finitely observations from those processes, compare (11). The strict stationarity of  $\{Y_t(n+1)\}_{t \in \mathbb{Z}}$  follows again from Lemma 7. ■

**Lemma 9** *The distribution of  $(Y_{t+s}(n), Y_{t+s}(n-1))$  does not depend on  $s$  for all  $n \geq 1$ .*

The proof uses the same kind of arguments as in proving Lemmas 7 and 8 and is omitted.

**Proof: (Theorem 1)**

We prove convergence of the sequence  $\{Y_t(n)\}_{t \in \mathbb{Z}, n \geq 0}$ , to a strictly stationary FINGARCH process. We first remark that, by construction,  $\lambda_t(0)$  has a finite mean, and, consequently,  $EY_t(0) = E\lambda_t(0) < \infty$  too. From (4), we get immediately that

$$\begin{aligned} E\lambda_t(1) &= E g(\lambda_{t-1}^{(p)}(0), Y_{t-1}^{(q)}(0)) \\ &\leq g(1, \dots, 1) + \sum_{i=1}^p a_i E|\lambda_{t-i}(0) - 1| + \sum_{i=1}^q b_i E|Y_{t-i}(0) - 1| < \infty, \end{aligned}$$

and  $EY_t(1) = E\lambda_t(1) < \infty$ . By the same kind of argument as in deriving (9) with  $(Y_t(n), Y_t(n-1))$  replacing  $(\tilde{Y}_{t+1}, Y_{t+1})$ , and using stationarity of  $\{Y_t(m)\}_{t \in \mathbb{Z}, m \geq 0}$ , from Lemma 8, we get

$$E|Y_t(n+1) - Y_t(n)| \leq c\rho^n \quad (12)$$

with  $\rho < 1$ . We conclude that

$$P(Y_t(n+1) \neq Y_t(n)) = \sum_{i=1}^{\infty} P(|Y_t(n+1) - Y_t(n)| = i) \leq E|Y_t(n+1) - Y_t(n)| \leq c\rho^n$$

and

$$\sum_{n=1}^{\infty} P(Y_t(n+1) \neq Y_t(n)) \leq \frac{c}{1-\rho} < \infty,$$

such that by the Borel-Cantelli Lemma

$$P(Y_t(n+1) \neq Y_t(n) \text{ for infinitely many } n \geq 0) = 0.$$

Therefore, there exists a.s. an integer-valued  $Y_t$  such that  $Y_t(n) = Y_t$  for all  $n$  large enough, and, in particular,

$$Y_t(n) \rightarrow Y_t \quad \text{a.s. for } n \rightarrow \infty.$$

By the same arguments as in the proof of Theorem 2.1 of (Doukhan et al, 2006), we get that  $\{Y_t\}_{t \in \mathbb{Z}}$  is a strictly stationary time series satisfying (3).

Finally, we remark that from (12) and the triangular inequality

$$E|Y_t(n+m) - Y_t(n)| \leq c\rho^n \sum_{j=0}^{m-1} \rho^j < \frac{c\rho^n}{1-\rho} \quad \text{for all } m, n \geq 1,$$

i.e.  $Y_t(n), n \geq 1$ , is a  $L^1$ -Cauchy sequence, such that  $EY_t = \lim_{n \rightarrow \infty} EY_t(n) < \infty$ . ■

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