Hypersurface singularities in positive characteristic

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Introduction

The study of algebroid singularities lies on the cross-roads of many different areas of mathematics. Initially, during the nineteenth and early twentieth century, algebraic geometers worked on plane curve singularities. Since the late 1960s, new methods in singularity theory have been rapidly developed. One of the fundamental results is the fibration theorem of Milnor [Mil68]. It deals with hypersurface singularities related to functions of several complex variables. This book has been extremely influential and since then the development of the theory over the field $\mathbb{C}$ of complex numbers is ongoing. Besides, the interaction between the different methods makes the study of hypersurface singularities particularly fruitful.
Nevertheless, it was soon observed that these methods cannot be carried offhand in the case of positive characteristic. For example, purely topological reasoning cannot be used here since fields of positive characteristic have only the trivial valuation.
Moreover, unlike the complex case, a systematic development of a general theory of hypersurface singularities in the context of algebraically closed fields of arbitrary characteristic is scarce in the literature. To the knowledge of the author, the first extensive development on curve singularities in positive characteristic has been worked out in [Cam80]. It is well-known that the Puiseux theorem does not hold in finite characteristic. In his book Campillo used an algebraic reasoning, defining a curve singularity as a local ring $\mathcal{O}$ of Krull dimension 1. Moreover, he considered the completion $\hat{\mathcal{O}}$ in the $\mathfrak{m}$-adic sense, and showed the existence of a parametrization. Furthermore, he established that the Hamburger-Noether expansion is the most effective replacement for the Puiseux theorem. Furthermore, he introduces the equivalence relation of equisingularity in finite characteristic (cf. also [CGL07]).
A further central topic in singularity theory is the classification of hypersurface singularities. In the early 1970's Arnold introduced the notion of modality and developed the classification over $\mathbb{C}$ with respect to right equivalence [Arn72]. First singularities of modality 0 are then classified. These are mostly known as simple or ADE-singularites. Also Arnold and especially Brieskorn [Bri71] established the coincidence of this classification with that of simple Lie Groups. In subsequent papers Arnold classified singularities of modality 1 [Arn73] and 2 [Arn75]. In [AGV85], the reader is refered to a complete list of normal forms of simple, unimodular and bimodular singularities. Types of singularities of modality 3 have been discussed by Wall in [Wal99b]. In [Sch90], unimodular plane curve singularities are classified for contact equivalence. In positive characteristic, a complete list of simple singularities for contact equivalence ($\overset{c}{\sim}$) is presented in [GrK90]. The consideration of $\overset{c}{\sim}$ for the classification in finite

characteristic was motivated by the intention to keep some analogy with the results established in characteristic zero. To illustrate this, let us consider the following example given in [GrK90]. If $char(K) = 5$, then E_8 is not simple for right equivalence but it is simple for contact equivalence.
In [Hol98] and [Bou02] follow the classifications of T-singularities and W-singularities from Schappert's list, in arbitrary characteristic for the first class, and in $char(K) \neq 2$ for the last one.

The goal of this dissertation is to give a systematic treatment of hypersurface singularities in arbitrary characteristic which provides the necessary tools, theoretically and computationally, for the purpose of classification.

Throughout this work, K denotes an algebraically closed field of arbitrary characteristic. We consider the ring $K[[\boldsymbol{x}]] := K[[x_1, \dots, x_n]]$ of formal power series.
Following Campillo in [Cam80], we define a hypersurface singularity as a local K-algebra of the form $R_f := K[[\boldsymbol{x}]]/\langle f \rangle$ where $f \in \mathfrak{m} \subset K[[\boldsymbol{x}]]$ and $\mathfrak{m}$ is the maximal ideal of $K[[\boldsymbol{x}]]$. We should mention that in characteristic zero, isolated hypersurface singularities are mostly known as those having finite Milnor number. This definition has to be modified in arbitrary characteristic since the Milnor number is not an invariant for contact equivalence in positive characteristic.
Hence, in arbitrary characteristic, we define isolated hypersurface singularities R_f as those for which $\tau(f) < \infty$ holds, where τ denotes the Tjurina number.

Our approach to deal with the subject of our work relies mainly on the methods developed among others in [Arn74], [AGV85], [GLS06], [GrK90], [Kou73] and [Wal99a] for the study of invariants of hypersurface singularities and computation of normal forms over $\mathbb{C}$. We shall discuss thoroughly how these results have to be modified in the context of positive characteristic with the concern to keep some analogy with the characteric zero case. Also, we shall widely use the notations elaborated in [Wal99a].

Analogous to the notion of semiquasihomogeneity (SQH) considered by Arnold in his important paper [Arn74], we consider finite set of weights $\boldsymbol{W} \subset \mathbb{Z}^n_{>0}$ and their related valuations $v_{\boldsymbol{W}}$ and we formalize Arnold's discussion by introducing the notion of semipiecewise-homogeneity. More precisely, we say that $f \in K[[\boldsymbol{x}]]$ is semipiecewise-homogeneous and we write (SPH) if $f = f_{\boldsymbol{P}} + f_1$ where $f_{\boldsymbol{P}}$ is piecewise-homogeneous (PH) with respect to $\boldsymbol{W}$, $\tau(f_{\boldsymbol{P}}) < \infty$ and $v_{\boldsymbol{W}}(f_1) > v_{\boldsymbol{W}}(f_{\boldsymbol{P}})$.
In the particular case where $f_{\boldsymbol{P}} = f_\Gamma$ is the truncation of f with respect to its Newton polytope Γ, Kouchnirenko in [Kou73] looked for conditions which would imply the finiteness of $\mu(f)$. He introduced an important geometrical feature of the Newton polytope which he called the Newton number $\mu_N(f)$ and established that $\mu(f) \geq \mu_N(f)$. Furthermore, if a certain condition of non-degeneracy holds, then $\mu(f)$ is finite. His main results in positive characteristic though, are shown only for the cases where the Newton polytope meets all coordinate subspaces. One of the central results in this case is that $\mu(f) = \mu_N(f)$ provided that a condition of non-degeneracy holds.
In his paper [Wal99a], Wall did slightly modify the notion of Newton polytope allowing all its facets to be extended to meet all coordinate subspaces. He introduced the

notion of *strict non-degeneracy* with respect to the so-called C*-polytopes* and he called this $NPND^*$. This condition of non-degeneracy turns out to be an appropriate one. Indeed, on the one hand, Wall showed that any semiquasihomogeneous hypersurface singularity is strictly non-degenerate with respect to some C-polytope. On the other hand, he asserted that this condition implies the finiteness of the Milnor number.
Following Wall's proof over $\mathbb{C}$ fairly closely, we show that the condition $NPND^*$ does also make sense in finite characteristic. Indeed, also in this case, if f is $NPND^*$ with respect to some polytope $\boldsymbol{P}$, then $\mu(f) < \infty$ which yields $\tau(f) < \infty$ and therefore R_f is isolated. Moreover, we establish the following result dealing with (SQH), where $f \in \mathfrak{m}$ is called (SQH), if $f = f_\Delta + f_1$ where f_Δ is quasihomogeneous, $\tau(f_\Delta) < \infty$ and the weighted order of f_1 is strictly bigger than that of f_Δ.

Proposition 2.3.23. Let $f \in \mathfrak{m}^3 \subset K[[\boldsymbol{x}]]$ be (SQH) with principal part f_Δ having weighted degree $d \in \mathbb{Z}_{>0}$. Then, the following are equivalent

1. f is $NPND^*$ with respect to some C-polytope $\boldsymbol{P}$ of $\mathbb{R}^n_{\geq 0}$,
2. $\mu(f_\Delta)$ is finite,
3. $char(K)$ does not divide d.

Furthermore, we show in this case that $\mu(f) = \mu(f_\Delta)$ (cf. Proposition 2.1.41).

Also, over $\mathbb{C}$, it is well-known that for reduced elements $f \in K[[x, y]]$, the invariants $\mu(f)$, the delta invariant $\delta(f)$ of f and the number of irreducible factors $r(f)$ of f are closely related. More precisely

$$\mu(f) = 2\delta(f) - r(f) + 1. \tag{1}$$

In positive characteristic though, it turns out that (1) is false.
Nevertheless, using the results established in [BeP00], [Kou73] and [Wal99a], we show that (1) holds whenever f is non-degenerate with respect to some C-polytope $\boldsymbol{P}$.

In characteristic zero yet, it is widely accepted that (1) holds in the same way as over $\mathbb{C}$. However, we are not aware of any proof of it in the literature.
Using the Lefschetz principle, we give a proof of this claim in characteristic zero (cf. Proposition 5.3.2). Also, we transfer the following known results about the invariants μ and τ over $\mathbb{C}$ to algebraically closed fields of characteristic zero.
Let K be an algebraically closed field such that $char(K) = 0$ and let $f \in K[[\boldsymbol{x}]]$. Then, we have

- $\mu(f) < \infty$, if and only if, $\tau(f) < \infty$.
- Arnold's statement on (SQH) hypersurface singularities:
  If f is (SQH) with principal part f_Δ, then $\mu(f) = \mu(f_\Delta)$.
- The Milnor number is an invariant for contact equivalence:
  For $g \in K[[\boldsymbol{x}]]$, if $f \overset{c}{\sim} g$, then $\mu(f) = \mu(g)$.

A further fundamental invariant for hypersurface singularities is the determinacy.
In this thesis, we give an extensive development of determinacy for both right and contact equivalences.
It is established over $\mathbb{C}$ that each isolated hypersurface singularity is finitely determined (that is, it has a polynomial normal form) and the converse does also hold.
In this work, we show the same claim in arbitrary characteristic (cf. Corollary 3.1.22). Nevertheless, we should notice that the bounds of determinacy which are established over $\mathbb{C}$ have to be modified in the context of positive characteristic. In [GrK90], it is stated that each $f \in \mathfrak{m}^2$ having finite Tjurina number $\tau(f)$ is $2\tau(f)$-contact determined. In this thesis though, we establish the following finite determinacy theorem in arbitrary characteristic.

Theorem 3.1.15. Let $f \in \mathfrak{m}^2 \subset K[[x_1, \ldots, x_n]]$ such that $n \geq 2$.

1. f is **right** $(2k - ord(f) + 2)$-determined if
$$\mathfrak{m}^k \subset j(f).$$
2. f is **contact** $(2k - ord(f) + 2)$-determined if
$$\mathfrak{m}^k \subset tj(f).$$

where $j(f) = \langle f_{x_1}, \ldots, f_{x_n} \rangle$ is the Jacobian ideal of f and $tj(f) = \langle f \rangle + j(f)$ is the Tjurina ideal of f.

To deal with normal forms in arbitrary characteristic, we follow the methods developed over $\mathbb{C}$ by Arnold and discussed by Wall in his paper [Wal99a]. Nevertheless, we should notice that the restrictions imposed by Arnold in terms of condition (A) and by Wall in terms of condition $NPND^*$ do not apply to all the cases related to the classification in finite characteristic.
In this thesis, we formalize this development by elaborating new objects and imposing new conditions which are weaker than (A) and $NPND^*$ but yet provide a more general setting for the theory in arbitrary characteristic. Also we should notice that the results about normal forms which are established in this work yield very often an improvement of the so far introduced bounds in finite characteristic.
Our approach is the following:
Considering a finite set of weights $\boldsymbol{W} \subset \mathbb{Z}^n_{>0}$, this gives rise to a filtration of ideals $(F_d)_{d \in \mathbb{Z}_{\geq 0}}$ of $K[[\boldsymbol{x}]]$ where $F_d := \{g \in K[[\boldsymbol{x}]] : v_{\boldsymbol{W}}(g) \geq d\}$.
In addition, we associate to each local K-algebra $K[[\boldsymbol{x}]]/I$, where $I \subset K[[\boldsymbol{x}]]$ is an ideal, a graded K-algebra

$$gr_{\boldsymbol{W}}(K[[\boldsymbol{x}]]/I) := \bigoplus\nolimits_{d \geq 0} F_{\geq d}/(F_{>d} + (F_{\geq d} \cap I)).$$

Besides, we observe that, if I is a zero-dimensional ideal, then $gr_{\boldsymbol{W}}(K[[\boldsymbol{x}]]/I)$ surjects onto $K[[\boldsymbol{x}]]/I$ as K-vector spaces and also $dim_K(gr_{\boldsymbol{W}}(K[[\boldsymbol{x}]]/I))$ is finite.
For our subsequent discussion, we reformulate Arnold's condition (A) as follows:
Let $f \in K[\boldsymbol{x}]$ be (PH) of type $(\boldsymbol{W}\,;\, d)$. We say that f is (A) with respect to $\boldsymbol{W}$ if for any non zero $g \in j(f)$ there exists a derivation ξ such that

(A1) $v_{\boldsymbol{W}}(g) = v_{\boldsymbol{W}}(\xi) + v_{\boldsymbol{W}}(f)$ and

(A2) $v_{\boldsymbol{W}}(g - \xi f) > v_{\boldsymbol{W}}(g)$.

In other words, we say that f is (A) with respect to $\boldsymbol{W}$ if any non zero $g \in j(f)$ satisfies conditions $(A1)$ and $(A2)$ with respect to f and $\boldsymbol{W}$.

The key idea of Arnold for the computation of normal forms is to consider for each d, arising in the filtration $(F_d)_d$, all monomials $M \in K[\boldsymbol{x}]$ such that $v_{\boldsymbol{W}}(M) = d$ and which are independent modulo terms in $F_{\geq d}$ satisfying $(A1)$ and $(A2)$ with respect to f and $\boldsymbol{W}$.

In arbitrary characteristic, we elaborate in analogy to condition (A) a new condition, which we call (AC):
Let $f \in K[\boldsymbol{x}]$ be (PH) of type $(\boldsymbol{W}\,;\,d)$. We say that f is (AC) with respect to $\boldsymbol{W}$ if for any non zero $g \in tj(f)$ there exist a formal power series $b_0 \in K[[\boldsymbol{x}]]$ and a derivation $\xi \in Der_K(K[[\boldsymbol{x}]])$ such that

(AC1) $v_{\boldsymbol{W}}(g) = min\big\{v_{\boldsymbol{W}}(b_0) + v_{\boldsymbol{W}}(f)\,;\, v_{\boldsymbol{W}}(\xi) + v_{\boldsymbol{W}}(f)\big\}$ and

(AC2) $v_{\boldsymbol{W}}(g - b_0 f - \xi f) > v_{\boldsymbol{W}}(g)$.

Hence, f is (AC) with respect to $\boldsymbol{W}$ if any non zero $g \in tj(f)$ satisfies conditions $(AC1)$ and $(AC2)$ with respect to f and $\boldsymbol{W}$.

We should mention that (A) is related to right equivalence while (AC) is related to contact equivalence (therefore the letter C in (AC)).

We should notice that each quasihomogeneous polynomial with respect to a weight $\boldsymbol{w} \in \mathbb{Z}^n_{>0}$ is both (A) and (AC) with respect to $\{\boldsymbol{w}\}$.

On the other hand, we formalize Arnold's key idea as follows:
For a positive integer d, we consider the following ideals in $K[[\boldsymbol{x}]]$

$$j^A_{\boldsymbol{W}}(f,d) := \big\langle g \in j(f) :\ v_{\boldsymbol{W}}(g) = d \ \text{ and } \ g \ \text{ is } \ (A1) \ \text{ with respect to } \ f \text{ and } \boldsymbol{W} \big\rangle,$$

$$tj^{AC}_{\boldsymbol{W}}(f,d) := \big\langle g \in tj(f) :\ v_{\boldsymbol{W}}(g) = d \ \text{ and } \ g \ \text{ is } \ (AC1) \ \text{ with respect to } \ f \text{ and } \boldsymbol{W} \big\rangle,$$

and the graded K-algebras

$$gr^A_{\boldsymbol{W}}(M_f) := \bigoplus_{d \geq 0} F_{\geq d}/(j^A_{\boldsymbol{W}}(f,d) + F_{>d})$$

and

$$gr^{AC}_{\boldsymbol{W}}(T_f) := \bigoplus_{d \geq 0} F_{\geq d}/(tj^{AC}_{\boldsymbol{W}}(f,d) + F_{>d}).$$

We should also mention that $gr^{A}_{\boldsymbol{W}}(M_f)$ and $gr^{AC}_{\boldsymbol{W}}(T_f)$ respectively may have infinite dimension as K-vector spaces even though $\mu(f) < \infty$ and $\tau(f) < \infty$ respectively.

Moreover (A) and (AC) are charcterized via these new objects as follows:

Proposition 3.2.9. Let $f \in K[\boldsymbol{x}]$ be (PH) of type $(\boldsymbol{W}\,;\,d)$.

1. If $\mu(f) < \infty$, then f is (A) with respect to $\boldsymbol{W}$, if and only if, $gr^{A}_{\boldsymbol{W}}(M_f) \cong M_f$ as K-vector spaces, i.e $dim_K(gr^{A}_{\boldsymbol{W}}(M_f)) = \mu(f)$
2. If $\tau(f) < \infty$, then f is (AC) with respect to $\boldsymbol{W}$, if and only if, $gr^{AC}_{\boldsymbol{W}}(T_f) \cong T_f$ as K-vector spaces, i.e $dim_K(gr^{AC}_{\boldsymbol{W}}(T_f)) = \tau(f)$.

In [Wal99a], Wall observed that the condition (A) imposed by Arnold for the computation of normal forms is on the one hand restrictive since it does not apply to all cases and on the other hand not necessary for the proof of the main results.
Based on these observations, we elaborate new conditions, which we call (AA) and (AAC). Let $f \in K[\boldsymbol{x}]$ be (PH) of type $(\boldsymbol{W}\,;\,d)$.

1. We say that f is **almost** (A) and we write f is (AA) with respect to $\boldsymbol{W}$, if $dim_K(gr^{A}_{\boldsymbol{W}}(M_f)) < \infty$.
2. We say that f is **almost** (AC) and we write f is (AAC) with respect to $\boldsymbol{W}$, if $dim_K(gr^{AC}_{\boldsymbol{W}}(T_f)) < \infty$.

Furthermore, we call a K-basis of $gr^{A}_{\boldsymbol{W}}(M_f)$ (resp. $gr^{AC}_{\boldsymbol{W}}(T_f)$) consisting of monomials a **regular basis** of M_f (resp. T_f).

It is clear that (AA) and (AAC) are weaker than (A) and (AC), respectively. Also, it turns out that both of these new conditions enclose Wall's condition $NPND^*$.

With these tools at our disposal, we get the following results about normal forms:

Theorem 3.3.2. Let $f \in \mathfrak{m} \subset K[[\boldsymbol{x}]]$ be such that $\tau(f)$ is finite and let $\boldsymbol{W} \subset \mathbb{Z}^n_{>0}$ be a finite set of weights corresponding to the Newton polytope Γ of f.
Further, let $\{e_{\boldsymbol{\alpha}} :\ \boldsymbol{\alpha} \in \Lambda\}$ be a K-basis of $gr^{AC}_{\boldsymbol{W}}(T_{f_\Gamma})$ consisting of monomials. Then,

$$f \overset{c}{\sim} f_\Gamma \,+\, \sum_{\boldsymbol{\alpha}\in\Lambda^*} c_{\boldsymbol{\alpha}} e_{\boldsymbol{\alpha}},$$

where

$$\Lambda^* \text{ is a finite subset of } \Big\{\boldsymbol{\alpha} \in \Lambda :\ v_{\boldsymbol{W}}(e_{\boldsymbol{\alpha}}) > v_{\boldsymbol{W}}(f)\Big\}$$

and the coefficients $c_{\boldsymbol{\alpha}} \in K$ are suitable.

Clearly, for all $\boldsymbol{\alpha} \in \Lambda^*$, the monomials $e_{\boldsymbol{\alpha}}$ have total degrees which are smaller than the degree of contact determinacy of f.

In Theorem 3.3.4, we give a similar statement for right equivalence.

If (AAC) holds, then we get the following result on normal forms in arbitrary characteristic.

Theorem 3.3.6. Let $f \in \mathfrak{m} \subset K[[\boldsymbol{x}]]$ be (SPH) with respect to a C-polytope $\boldsymbol{P}$ and let $\boldsymbol{W} \subset \mathbb{Z}^n_{>0}$ be a finite set of weights corresponding to $\boldsymbol{P}$. If $f_{\boldsymbol{P}}$ is (AAC) with respect to $\boldsymbol{P}$ and $\left\{e_{\boldsymbol{\alpha}} : \boldsymbol{\alpha} \in \Lambda\right\}$ is a regular basis of $T_{f_{\boldsymbol{P}}}$, then f is **finitely contact determined** and

$$f \overset{c}{\sim} f_{\boldsymbol{P}} + \sum_{\boldsymbol{\alpha} \in \Lambda^*} c_{\boldsymbol{\alpha}} e_{\boldsymbol{\alpha}},$$

where

$$\Lambda^* \subset \left\{\boldsymbol{\alpha} \in \Lambda : v_{\boldsymbol{W}}(e_{\boldsymbol{\alpha}}) \geq v_{\boldsymbol{W}}(f - f_{\boldsymbol{P}})\right\}$$

and the coefficients $c_{\boldsymbol{\alpha}} \in K$ are suitable.

Theorem 3.3.14 establishes the same for right equivalence whenever (AA) holds.

Altogether, this yields interesting results on bounds of determinacy in arbitrary characteristic.

Theorem 3.3.18. Let $f \in \mathfrak{m} \subset K[[\boldsymbol{x}]]$ be (SPH) such that $f_{\boldsymbol{P}}$ is (AAC) with respect to $\boldsymbol{P}$. Further, let $\boldsymbol{W} \subset \mathbb{Z}^n_{>0}$ be a finite set of weights corresponding to $\boldsymbol{P}$ and let $\{e_{\boldsymbol{\alpha}} : \boldsymbol{\alpha} \in \Lambda\}$ be a K-basis of $gr^{AC}_{\boldsymbol{W}}(T_{f_{\boldsymbol{P}}})$ consisting of monomials.
Then f is k-contact determined if $\mathfrak{m}^{k+1} \subset F_{>D}$ where

$$D := max\Big\{v_{\boldsymbol{W}}(f_{\boldsymbol{P}}) \,,\ max\left\{v_{\boldsymbol{W}}(e_{\boldsymbol{\alpha}}) : \boldsymbol{\alpha} \in \Lambda\right\}\Big\}.$$

In the particular case where (AC) holds we have

Corollary 3.3.21 Let $f \in \mathfrak{m} \subset K[[\boldsymbol{x}]]$ be (SPH) such that $f_{\boldsymbol{P}}$ is (AC) with respect to $\boldsymbol{P}$. Further let $\boldsymbol{W} \subset \mathbb{Z}^n_{>0}$ be a finite set of weights corresponding to $\boldsymbol{P}$ and let $d = v_{\boldsymbol{W}}(f)$. If D and k are positive integers such that $\mathfrak{m}^{k+1} \subset F_{\geq D} \subset tj(f_{\boldsymbol{P}}) \cap F_{>d}$, then f is k-contact determined.

Similar statements for right equivalence are given in Theorem 3.3.20 and Corollary 3.3.25. In the last part of Chapter 3, we shall give examples for application of these results.

In the final part of this work, we discuss the so far presented results from the computational viewpoint. In chapter 4, we shall present algorithms which we implementes in the computer algebra system SINGULAR. We use this to obtain explicit regular bases and normal forms for right and for contact equivalence. There are two key observations for our algorithms. Given a finite set of weights $\boldsymbol{W} \subset \mathbb{Z}^n_{>0}$, we notice:

1. the related valuation $v_{\boldsymbol{W}}$ to $\boldsymbol{W}$ does not give rise to an admissible degree ordering in the sense of standard bases. For this reason, the key idea for our computa-

tions, is to perform separate calculations for the different weights of $\boldsymbol{W}$ and then fit them together.

2. If $I \subset K[[\boldsymbol{x}]]$ is an ideal, then we establish in Proposition 2.1.50 that

$$gr_{\boldsymbol{W}}(K[[\boldsymbol{x}]]/I) \cong \bigoplus_{d\geq 0} K[\boldsymbol{x}]_d/In_{\boldsymbol{W}}(I)_d,$$

where $In_{\boldsymbol{W}}(I)$ is the initial ideal of I with respect to $\boldsymbol{W}$.
This observation is of essential use for our algorithms when computing regular bases.

Organization of the material

In Chapter 1, we introduce the background on isolated hypersurface singularities, equivalence relations and invariants needed for our work.
Chapter 2 is the first main part of this dissertation. In this chapter, we formalize the notions of semipiecewise-homogeneous hypersurface singularities and piecewise-homogeneous grading and present related results which are needed for the subsequent chapters. In the second part, we discuss thoroughly non-degeneracy in arbitrary characteristic.
Chapter 3 is devoted to determinacy and normal forms of isolated hypersurface singularities. In the first part, we give finite determinacy theorems in arbitrary characteristic with respect to right and to contact equivalence. Furthermore, we show that "isolated" and finite determinacy properties are equivalent. In the second part, we formalize Arnold's key ideas in [Arn74] for the computation of normal forms and define the conditions (AA) and (AAC). We thoroughly discuss these conditions as well as some related results for cases occuring in Schappert's list of normal forms. The last part of Chapter 3 is devoted to the study of normal forms in the general setting of isolated hypersurface singularities imposing neither condition (A) nor condition $NPND^*$. Finally, we discuss the cases where (AA) and (AAC) hold and present the related results on normal forms and bounds of determinacy in this case.
In Chapter 4, we present algorithms which we implement in SINGULAR for the purpose of explicit computation of regular bases and normal forms.
In Chapter 5, we transfer some classical results on invariants over $\mathbb{C}$ to algebraically closed fields of characteristic zero known as Lefschetz principle.
For the convenience of the reader, we review in appendix A some results from field theory which are needed in Chapter 5 and finally in appendix B, we present our SINGULAR library $gradalg.lib$ where the algorithms presented in Chapter 4 are implemented.

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Chapter 1

Preliminaries

The first chapter is an overview of the main objects of interest in this dissertation.
After some notation is fixed, we define isolated hypersurface singularities in the general context of arbitrary characteristic and discuss related results.
Afterwards, we overview briefly right and contact equivalence and then we deal with the mostly relevant invariants for this work.
The last part is devoted to algebroid plane curve singularities.
We introduce the parametrization equivalence and show how this is closely related to contact equivalence.

1.1 Notations

Throughout this whole thesis we shall use the following conventions and notations.
We deal with fields K of arbitrary characteristic $p \geq 0$ and we assume in general, unless otherwise stated, the fields to be algebraically closed.

We denote by $\mathbb{Z}_{>0}$ the set of strictly positive integers, that is $\mathbb{N} \setminus \{0\}$.

For $n \in \mathbb{Z}_{>0}$, we denote by $\mathbb{R}^n_{\geq 0}$ (resp. $\mathbb{R}^n_{>0}$), the positive (resp. the strictly positive) orthant.
On the other hand, if $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n)$, $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_n) \in \mathbb{R}^n$, then we write

$$\langle \boldsymbol{\alpha}\, ,\, \boldsymbol{\beta} \rangle = \sum_{i=1}^{n} \alpha_i \beta_i$$

for the scalar product of $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$.

For a subset $I \subset \{1, \ldots, n\}$, we write

$$\mathbb{R}^I = \{\boldsymbol{x} \in \mathbb{R}^n \,:\, x_i = 0 \;\text{ if }\; i \notin I\},$$

and similarly for $\mathbb{R}^{*I}$, $\mathbb{C}^I$ and $\mathbb{C}^{*I}$.

We write $K[\boldsymbol{x}] := K[x_1, \ldots, x_n]$ for the ring of polynomials over K, having n variables, and we denote by $Mon(K[\boldsymbol{x}])$ its semigroup of monomials.
Also, for $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n{}_{\geq 0}$,we denote the monomial $x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ by $\boldsymbol{x}^{\boldsymbol{\alpha}}$. Moreover the positive integer $|\boldsymbol{\alpha}| = \alpha_1 + \ldots + \alpha_n$ is called the total degree of $\boldsymbol{x}^{\boldsymbol{\alpha}}$ and is denoted by $deg(\boldsymbol{x}^{\boldsymbol{\alpha}})$.

We write $K[[\boldsymbol{x}]] := K[[x_1, \ldots, x_n]]$ for the local ring of formal power series over K, having n variables and we denote by $\mathfrak{m}$ its maximal ideal.

Let $f = \sum_{\boldsymbol{\alpha} \in \mathbb{Z}^n_{\geq 0}} a_{\boldsymbol{\alpha}} \boldsymbol{x}^{\boldsymbol{\alpha}} \in K[[\boldsymbol{x}]]$ be a formal power seies.
The *support* of f is the set

$$supp(f) := \{\boldsymbol{\alpha} : \quad a_{\boldsymbol{\alpha}} \neq 0\}.$$

Furthermore, the order of f is

$$ord(f) := inf\{|\boldsymbol{\alpha}| : \quad \boldsymbol{\alpha} \in supp(f)\}.$$

For $i = 1, \ldots, n$, we write $f_{x_i} := \frac{\partial f}{\partial x_i}$.

We denote by $Der_K(K[[\boldsymbol{x}]])$ the space of K-derivations on $K[[\boldsymbol{x}]]$. Furthermore, we observe that $Der_K(K[[\boldsymbol{x}]])$ is isomorphic to the $K[[\boldsymbol{x}]]$-module $\sum_{i=1}^{n} K[[\boldsymbol{x}]]\partial_{x_i}$.

Let $\xi = \sum_{i=1}^{n} g_i \partial_{x_i} \in Der_K(K[[\boldsymbol{x}]])$, where $g_i \in K[[\boldsymbol{x}]]$, for $i = 1, \ldots, n$, and let $f \in K[[\boldsymbol{x}]]$. Then we write

$$\xi f := \sum_{i=1}^{n} g_i f_{x_i}.$$

We denote by $K[[\boldsymbol{x}]]^*$ for the group of units of $K[[\boldsymbol{x}]]$ (i.e formal power series with non zero constant terms) and $Aut(K[[\boldsymbol{x}]])$ denotes the group of automorphisms defined on $K[[\boldsymbol{x}]]$.

If $\mathcal{A} \subset \mathbb{R}^n$, then $Conv(\mathcal{A})$ denotes the convex hull of $\mathcal{A}$.

If E is an arbitrary finite set, then we denote by $\sharp(E)$ the number of elements of E, which is also the cardinality of E.

1.2 Hypersurface Singularities

1.2.1 Preliminary Concepts

Following Campillo in [Cam80], we shall give an algebraic definition of singularities.

Definition 1.2.1. *An (***algebroid***)* **singularity** *is a local K-algebra R which is isomorphic to $K[[\boldsymbol{x}]]/I$, where I is a proper ideal of $K[[\boldsymbol{x}]]$.*
*If $I = \langle f \rangle$, with $f \in \mathfrak{m} \smallsetminus \{0\}$ is a formal power series, then R is called an (***algebroid***)* **hypersurface singularity** .

Let $I \subset \mathfrak{m}$ be a proper ideal in $K[[\boldsymbol{x}]]$ and let $R = K[[\boldsymbol{x}]]/I$. If we consider the affine scheme $Spec(R)$, we see that it has only one closed point ξ which corresponds to the unique maximal ideal $\bar{\mathfrak{m}}$ in the local algebra R. Investigating the local properties of the closed point ξ in the affine scheme is the same as studying the localisation $R_{\bar{\mathfrak{m}}}$ which is just isomorphic to R.

Definition 1.2.2. *Let $f \in \langle x\, ,\, y \rangle$ be a non-zero element of $K[[x, y]]$. Then the hypersurface singularity $R_f = K[[x, y]]/\langle f \rangle$ is called plane curve singularity.*

Definition 1.2.3. *Let $f \in \mathfrak{m} \smallsetminus \{0\}$ be a formal power series.*

1. *The ideal*
$$j(f) := \left\langle f_{x_1}, \ldots, f_{x_n} \right\rangle \subset K[[\boldsymbol{x}]]$$
is called the **Jacobian ideal***, or the* **Milnor ideal** *of f, and*
$$tj(f) := \langle f \rangle + j(f) \subset K[[\boldsymbol{x}]]$$
is called the **Tjurina ideal** *of f.*

2. *The K-algebras*
$$M_f := K[[\boldsymbol{x}]]/j(f), \qquad T_f := K[[\boldsymbol{x}]]/tj(f)$$
are called the **Milnor** *and* **Tjurina algebra** *of f, respectively.*

3. *The numbers*
$$\mu(f) := dim_K(M_f), \qquad \tau(f) = dim_K(T_f)$$
are called the **Milnor** *and* **Tjurina numbers** *of f, respectively .*

The Milnor and the Tjurina algebras and, in particular, their dimension play an important role in the sudy of isolated hypersurface singularities as we shall see later in this chapter.

Remark 1.2.4. *Let $f \in \mathfrak{m} \subset K[[\boldsymbol{x}]]$ be a non-zero element. It is straightforward from Definition 1.2.3 that, if $\mu(f)$ is finite, then $\tau(f)$ is also finite too.*
If $K = \mathbb{C}$, it is well-known that $\mu(f) < \infty \Leftrightarrow \tau(f) < \infty$ (cf. [GLS06, lemma 2.3]). Also, this claim is widely accepted in characteristic zero and we shall give a proof of it

in Proposition 5.2.1 of the last chapter of this work. In finite characteristic though, the latter claim does not hold in general as the following example shows.
Let $char(K) = 5$, and let $f = x^5 + y^4 \in K[[x, y]]$ be an equation of type W_{12}. Using the computer algebra system SINGULAR, *we obtain $\tau(f) = 15$ while $\mu(f) = \infty$.*

In the following, we briefly review the notions of right and contact equivalence.

Definition 1.2.5. *Let $f, g \in \mathfrak{m} \subset K[[\boldsymbol{x}]]$.*

1. *f is called* **right equivalent** *to g, $f \overset{r}{\sim} g$, if there exists an automorphism φ of $K[[\boldsymbol{x}]]$ such that $g = \varphi(f)$.*
2. *f is called* **contact equivalent** *to g, $f \overset{c}{\sim} g$, if there exists an automorphism φ of $K[[\boldsymbol{x}]]$ and a unit $u \in K[[\boldsymbol{x}]]^*$ such that $g = u \cdot \varphi(f)$.*

It is straightforward from the above definition that the right and the contact equivalence are equivalence relations on the set of formal power series.

Remark 1.2.6. *It is clear, that $f \overset{r}{\sim} g$ implies $f \overset{c}{\sim} g$. However, it is well-known, that the converse does not hold even though the characteristic is zero.*

In the subsequent parts of this work, we should very often make use of the following lemma.

Lemma 1.2.7. *Let $f, g \in \mathfrak{m} \subset K[[\boldsymbol{x}]]$. Furthermore, let $\varphi \in Aut(K[[\boldsymbol{x}]])$ be an automorphism of $K[[\boldsymbol{x}]]$ and let $u \in (K[[\boldsymbol{x}]])^*$ be a unit. Then*

1. *$j(\varphi(f)) = \varphi(j(f))$.*
2. *$\langle uf \rangle + j(uf) = \langle f \rangle + j(f)$, or shortly $tj(uf) = tj(f)$*
3. *$f \overset{r}{\sim} g$ implies that $M_f \cong M_g$ and $T_f \cong T_g$ as K-algebras.*
   In particular, $\mu(f) = \mu(g)$ and $\tau(f) = \tau(g)$.
4. *$f \overset{c}{\sim} g$ implies that $T_f \cong T_g$ and hence $\tau(f) = \tau(g)$.*

Proof. 1. If $h_i = \varphi(x_i)$, $i = 1, \ldots, n$, then we can write for all i

$$h_i = \Big(\sum_{j=1}^{n} (h_{i,x_j} \mod \mathfrak{m}) x_j\Big) + g_i \quad \text{where} \quad g_i \in \mathfrak{m}^2.$$

Hence for every $l = 1, \ldots, n$, $h_{i,x_l} = (h_{i,x_l} \mod \mathfrak{m}) + g_{i,x_l}$.
On the other hand, we have

$$((\varphi(f))_{x_1}, \ldots, (\varphi(f))_{x_n}) = (\varphi(f_{x_1}), \ldots, \varphi(f_{x_n})) \cdot J(\varphi), \quad \text{where}$$

$$J(\varphi) := \begin{pmatrix} h_{1,x_1} & \ldots & h_{1,x_n} \\ \vdots & & \vdots \\ h_{n,x_1} & \ldots & h_{n,x_n} \end{pmatrix} \in K[[\boldsymbol{x}]]^{n \times n}.$$

It follows then from the above equation that $j(\varphi(f)) \subset \varphi(j(f))$.
Besides, we have that for a matrix $A = (a_{i,j})_{1\leq i,j\leq n} \in K[[\boldsymbol{x}]]^{n\times n}$ the following holds: A is invertible in $K[[\boldsymbol{x}]]^{n\times n}$, if and only if, the matrix $(a_{i,j} \mod \mathfrak{m})_{i,j}$ is invertible in $K^{n\times n}$. As φ is an automorphism of $K[[\boldsymbol{x}]]$, it follows that the jacobian matrix of φ, which is $((h_{i,x_j} \mod \mathfrak{m}))_{i,j}$, is invertible in $K^{n\times n}$ and consequently $J(\varphi)$ is invertible in $K[[\boldsymbol{x}]]^{n\times n}$.
Therefore $\varphi(j(f)) \subset j(\varphi(f))$.
Hence $j(\varphi(f)) = \varphi(j(f))$ and

$$\langle \varphi(f)\rangle + j(\varphi(f)) = \langle \varphi(f)\rangle + \varphi(j(f)) = \varphi(\langle f\rangle + j(f)).$$

2. By the product rule we have $\langle uf\rangle + j(uf) = \langle f\rangle + j(f)$.

3. and 4. follow immediately from 1. and 2.

□

Remark 1.2.8. *Given $f \in K[[\boldsymbol{x}]]$, $\varphi \in Aut(K[[\boldsymbol{x}]])$ and $u \in (K[[\boldsymbol{x}]])^*$, it follows clearly from the first two assertions of Lemma 1.2.7 that*

$$tj(u\varphi(f)) = \varphi(tj(f)).$$

1.2.2 Isolated Hypersurface Singularities

In the sequel, we deal with the "isolated" property in arbitrary characteristic.

Definition 1.2.9. *Let $f \in \mathfrak{m}$ and let $R_f = K[[\boldsymbol{x}]]/\langle f\rangle$.*

1. *0 is called an* **isolated singularity of f**, *if there exists a $k > 0$ such that*

$$\mathfrak{m}^k \subset j(f).$$

2. *R_f is called an* **isolated hypersurface singularity**, *if there exists a $k > 0$ such that*

$$\mathfrak{m}^k \subset tj(f).$$

Lemma 1.2.10. *Let $f \in \mathfrak{m}$ and let $R_f = K[[\boldsymbol{x}]]/\langle f\rangle$. Then, 0 is an isolated singularity of f (resp. R_f is an isolated hypersurface singularity) if and only if $\mu(f) < \infty$ (resp. $\tau(f) < \infty$).*

Proof. The proof is straightforward from Definition 1.2.9. □

Proposition 1.2.11. *Let $f \in \mathfrak{m} \setminus \{0\} \subset K[[x_1, \dots, x_n]]$ and let $R_f = K[[\boldsymbol{x}]]/\langle f\rangle$.*

1. *If R_f is an isolated hypersurface singularity singularity, then R_f is reduced.*

2. *If $n = 2$, then R_f is an isolated singularity, if and only if, R_f is reduced.*

Remark 1.2.12. *We should mention, that the claim of Proposition 1.2.11 does also hold if we generally admit K to be a* **perfect** *field (cf. the proof of Lemma 1.2.13). If K is not perfect though, then it is not true in general that a reduced plane curve singularity is isolated. For example, let $K = \mathbb{F}_2(t)$ be the field of rational functions over $\mathbb{F}_2$. K is not perfect and $f = x^2 + ty^2 \in \langle x\,,\,y\rangle \subset K[[x,y]]$ is reduced but $\tau(f)$ is infinite.*

The proof of Proposition 1.2.11 uses the subsequent two lemmas.

Lemma 1.2.13. *Let K be a perfect field and let $f \in \mathfrak{m} \subset K[[x_1,\ldots,x_n]]$.*

1. *If $char(K) = p > 0$, then*

$$\begin{aligned} j(f) \subset \langle f\rangle \Leftrightarrow & \text{ there exists a unit } \ u \in K[[\boldsymbol{x}]]^* \\ & \text{ such that } \ uf \in K[[{x_1}^p,\ldots,{x_n}^p]]. \end{aligned}$$

2. *If $char(K) = 0$, then $j(f) \subset \langle f\rangle \Leftrightarrow f = 0$.*

Proof. Let $f \in \mathfrak{m} \subset K[[\boldsymbol{x}]]$. We write $K[[\boldsymbol{x}']] := K[[x_1,\ldots,x_{n-1}]]$.
We show in the following that we can, witout loss of generality, assume that f is a Weierstrass polynomial. Indeed, the Weierstrass preparation theorem asserts the existence of $\varphi \in Aut(K[[\boldsymbol{x}]])$ and $u \in K[[\boldsymbol{x}]]^*$ and $a_1,\ldots,a_b \in K[[\boldsymbol{x}^{'}]]$ for some integer $b \geq 0$ such that

$$f = u\cdot\varphi({x_n}^b + a_1{x_n}^{b-1} + \ldots + a_b). \tag{1.1}$$

If $g = {x_n}^b + a_1{x_n}^{b-1} + \ldots + a_b$, we claim that

- $j(g) \subset \langle g\rangle \Leftrightarrow j(f) \subset \langle f\rangle$ and
- if $char(K) = p > 0$, then the following are equivalent
  (1) $u_1.g \in K[[{x_1}^p,\ldots,{x_n}^p]]$ for some unit $u_1 \in K[[\boldsymbol{x}]]^*$.
  (2) $u_2.f \in K[[{x_1}^p,\ldots,{x_n}^p]]$ for some unit $u_2 \in K[[\boldsymbol{x}]]^*$.

Indeed, since φ is an automorphism, Lemma 1.2.7 yields

$$\begin{aligned} j(g) \subset \langle g\rangle \quad &\Leftrightarrow \quad \langle g\rangle + j(g) = \langle g\rangle \\ &\Leftrightarrow \quad \varphi(\langle g\rangle + j(g)) = \varphi(\langle g\rangle) \\ &\Leftrightarrow \quad \langle\varphi(g)\rangle + j(\varphi(g)) = \langle\varphi(g)\rangle \\ &\Leftrightarrow \quad \langle u\cdot\varphi(g)\rangle + j(u\cdot\varphi(g)) = \langle u\cdot\varphi(g)\rangle \\ &\Leftrightarrow \quad \langle f\rangle + j(f) = \langle f\rangle \\ &\Leftrightarrow \quad j(f) \subset \langle f\rangle \end{aligned}$$

Let $u_1 \in K[[\boldsymbol{x}]]^*$ be such that $u_1 \cdot g \in K[[{x_1}^p,\ldots,{x_n}^p]]$. Hence, since K is perfect (i.e $K^p = K$), there exists $h \in K[[\boldsymbol{x}]]$ such that $u_1 \cdot g = h^p$. Thus, $u^{-1}\cdot\varphi(u_1)\cdot f \in K[[{x_1}^p,\ldots,{x_n}^p]]$ follows by (1.1). Moreover $u^{-1}\cdot\varphi(u_1)$ is obviously a unit. Conversely, if there exists a unit u_2 such that $u_2 \cdot f \in K[[{x_1}^p,\ldots,{x_n}^p]]$, then we show in the same way that $u_1 \cdot g \in K[[{x_1}^p,\ldots,{x_n}^p]]$.

Therefore, we can assume without loss of generality that $f = g$.
Now, we suppose that $j(f) \subset \langle f\rangle$. Then, there exist $g_1, \ldots, g_n \in K[[\boldsymbol{x}]]$ such that

$$a_{1,x_i}{x_n}^{b-1} + \ldots + a_{b,x_i} = f_{x_i} = g_i f, \quad i = 1, \ldots, n-1.$$

and

$$b{x_n}^{b-1} + \ldots + a_{b-1,x_n} = f_{x_n} = g_n f.$$

Hence, considering for $i = 1, \ldots, n$ the x_n-degrees, shows $deg_{x_n}(f_{x_i}) \geq b$ on the one hand and $deg_{x_n}(f_{x_i}) \leq b-1$ on the other hand. Thus, $f_{x_i} = 0$ clearly follows and therefore $a_{j,x_i} = 0$ for all $i = 1, \ldots, n-1$ and $j = 1, \ldots, b$. If $char(K) = p > 0$, this yields $a_j \in K[[{x_1}^p, \ldots, x_{n-1}^p]]$ for all j and so $f \in K[[{x_1}^p, \ldots, x_{n-1}^p]][x_n]$. As also $f_{x_n} = 0$, we obtain $f \in K[[{x_1}^p, \ldots, x_{n-1}^p]][{x_n}^p] \subset K[[{x_1}^p, \ldots, x_n^p]]$.
In characteristic 0 however, $f_{x_i} = 0$ for all i implies that $f = 0$ since $f \in \mathfrak{m}$.
Conversely, if $f \in K[[{x_1}^p, \ldots, x_n^p]]$ and $char(K) = p$, then it follows clearly, that $f_{x_i} = 0$ for all $i = 1, \ldots, n$, and hence the inclusion $j(f) \subset \langle f\rangle$ obviously follows.
Now let $char(K) = 0$, and f=0. It is then trivial that $j(f) \subset \langle f\rangle$. □

Lemma 1.2.14. *Let $f \in \mathfrak{m} \subset K[[x_1, \ldots, x_n]]$ be such that $f \neq 0$ and f is reduced. Furthermore, let R be its associated hypersurface singularity. Then*

$$dim(T_f) < dim(R).$$

Proof. We have in general $dim(T_f) \leq dim(R)$.
We take first f irreducible and suppose that $dim(T_f) = dim(R)$. But this is equivalent to $j(f) \subset \langle f\rangle$ which is a contradiction to the assumptions f is reduced and $f \neq 0$, (cf. Lemma 1.2.13). Therefore $dim(T_f) < dim(R)$.
Now let $f = u \cdot f_1 \cdots f_r$ be the decomposition of f into irreducible components, where $u \in K[[\boldsymbol{x}]]^*$ and $f_i \in K[[\boldsymbol{x}]]$ is irreducible for all $i = 1, \ldots n,$. As f is reduced, then it follows that $\langle f_i\rangle \neq \langle f_j\rangle$ for all $i \neq j$. Moreover we have

$$Spec(T_f) = \bigcup_i Spec(T_{f_i}) \cup \bigcup_{i<j} (Spec(R_i) \cap (Spec(R_j))$$

where for all i, R_i is the associated hypersurface singularity to f_i.
As for all i, f_i is irreducible, it follows then from the above that

$$dim(Spec(T_{f_i})) < dim(Spec(R_i)) \leq dim(Spec(R)).$$

On the other hand, we have for all $i \neq j$,

$$dim(Spec(R_i) \cap Spec(R_j)) < dim(Spec(R_i)),$$

i.e $dim(K[[\boldsymbol{x}]]/\langle f_i, f_j\rangle) < dim(R_i)$, for otherwise that means $\langle f_i, f_j\rangle \subset \langle f_i\rangle$, which implies that f_i divides f_j. But this is a contradiction to $gcd(f_i, f_j) = 1$ since both of them are irreducible and $f_i \neq f_j$.
Therefore $dim(Spec(R_i)) \cap Spec(R_j)) < dim(Spec(R_i)) \leq dim(Spec(R))$.
Hence $dim(T_f) < dim(R)$. □

With these tools at our disposal, we give in the following the proof of Proposition 1.2.11.

Proof. 1. Suppose that f is not reduced, which means that we can write $f = g^r h$ for some g and $h \in K[[\boldsymbol{x}]]$ and some integer $r \geq 2$.
Therefore g divides f_{x_i} for all $1 \leq i \leq n$. Hence $\langle f \rangle + j(f) \subset \langle g \rangle$. Then, it follows that $dim_K(T_f) \geq dim_K(K[[\boldsymbol{x}]]/\langle g \rangle)$.
As g is a nonzero element of the integral domain $K[[\boldsymbol{x}]]$, it is therefore a nonzerodivisor in $K[[\boldsymbol{x}]]$. Thus, by the Krull's principal ideal theorem, we have $dim_K(K[[\boldsymbol{x}]]/\langle g \rangle) = n - 1 \geq 1$, which leads to $dim_K(T_f) \geq 1$ and consequently $\tau(f) = \infty$, which means that R is not an isolated singularity.

2. Here, it is enough to show that a reduced plane curve singularity is isolated. Suppose f is reduced, then it follows by lemma 1.1.14 that $dim(T_f) < dim(R) = 1$. Therefore $dim(T_f) = 0$ and hence $\tau(f) < \infty$. Thus R is isolated.

□

In the following, we reformulate the well-known **curve selection lemma** in arbitrary characteristic.

Lemma 1.2.15. (The curve selection lemma)
Let K be an algebraically closed field of arbitrary characteristic. Further, let I be a proper ideal of $K[[\boldsymbol{x}]]$ and let $R = K[[\boldsymbol{x}]]/I$ be the corresponding algebroid singularity. If $dim(R) \geq 1$, then there exists a reduced **curve** *singularity $R^{'}$ such that*

$$R \twoheadrightarrow R^{'}.$$

Furthermore, there exists a K-algebra homomorphism

$$\begin{aligned} \psi : \quad K[[\boldsymbol{x}]] &\rightarrow K[[t]] \\ x_i &\mapsto x_i(t) \end{aligned}$$

such that $I \subset Ker(\phi)$.

Proof. Let $I \subset K[[\boldsymbol{x}]]$ be a proper ideal and let $R = K[[\boldsymbol{x}]]/I$ such that $dim(R) \geq 1$. As $I \subset \sqrt{I}$ and hence $R \twoheadrightarrow K[[\boldsymbol{x}]]/\sqrt{I}$, we can assume without loss of generality that the algebroid singularity R is reduced.
Let $\mathfrak{p} \supset I$ be a minimal prime ideal belonging to I and let $d := dim(R) = dim(K[[\boldsymbol{x}]]/\mathfrak{p})$. Further, let $f \in K[[\boldsymbol{x}]]$ and $f \notin \mathfrak{p}$. Then, it is clear that f is a non zerodivisor in $K[[\boldsymbol{x}]]/\mathfrak{p}$ and it follows by the Krull's principal ideal theorem that

$$dim(K[[\boldsymbol{x}]]/\mathfrak{p} + \langle f \rangle) = d - 1.$$

On the other hand, we have

$$R \twoheadrightarrow K[[\boldsymbol{x}]]/\mathfrak{p} \twoheadrightarrow K[[\boldsymbol{x}]]/\mathfrak{p} + \langle f \rangle.$$

This shows that R surjects onto a ring where the dimension drops by 1. In this way, we can show after finitely many steps that R surjects onto a ring $R^{'}$ of dimension 1. Moreover, we have

$$R \twoheadrightarrow R^{'} \twoheadrightarrow R^{'}_{red}.$$

Now, if we consider the normalization of an arbitrary irreducible component of the curve $R^{'}_{red}$, we get clearly a non zero K-algebra homomorphism

$$\begin{aligned} \psi: \quad K[[\boldsymbol{x}]] &\to K[[t]] \\ x_i &\mapsto x_i(t) \end{aligned}$$

with $I \subset Ker(\phi)$.

□

Invariants of hypersurface singularities

Let $f \in \mathfrak{m}$ be non-zero in $K[[\boldsymbol{x}]]$. Considering an equivalence relation $\mathcal{E}$ on $K[[\boldsymbol{x}]]$, we call (numerical) **invariant** of f with respect to $\mathcal{E}$, a number which depends only on the orbit of f with respect to $\mathcal{E}$. Moreover, observing that for non-zero $f, g \in \mathfrak{m}$, we have

$$K[[\boldsymbol{x}]]/\langle f\rangle \cong K[[\boldsymbol{x}]]/\langle g\rangle \Longleftrightarrow f \overset{c}{\sim} g, \tag{1.2}$$

we define as follows the invariants of hypersurface singularities in arbitrary characteristic.

Definition 1.2.16. *Let $f \in \mathfrak{m} \subset K[[\boldsymbol{x}]]$ be such that $f \neq 0$ and let $R_f = K[[\boldsymbol{x}]]/\langle f\rangle$. We call invariant of the hypersurface singularity R_f any numerical invariant of f with respect to contact equivalence.*

It is straightforward from (1.2) that Definition 1.2.16 makes sense.

In this subsection, we present briefly some invariants which are relevant for our development.

Let $f \in \mathfrak{m} \subset K[[\boldsymbol{x}]]$ be such that $f \neq 0$ and let $R_f = K[[\boldsymbol{x}]]/\langle f\rangle$. It is straightforward from Lemma 1.2.7 that $\tau(f)$ is an invariant of f.
Nevertheless, $\mu(f)$ is not an invariant in finite characteristic as the following example due to [GrK90] shows.

Example 1.2.17. *Let $char(K) = p > 0$ and let $f = x^p + y^{p+1} \in K[[x,y]]$. Then, $\mu(f) = \infty$ while $\mu((1+x)\cdot f)$ is finite. Hence, $\mu(f)$ is not an invariant of the plane curve singularity R_f.*

If $K = \mathbb{C}$ though, it is established that, if f, $g \in \mathbb{C}[[\boldsymbol{x}]]$ and $f \overset{c}{\sim} g$, then $\mu(f) = \mu(g)$. In the last chapter of this work, we shall present a proof of this claim over algebraically closed fields of characteristic zero (cf. Proposition 5.3.1).

Further invariants of isolated hypersurface singularities of the form R_f, where $f \in \mathfrak{m} \setminus \{0\}$ are given by

$$\begin{aligned} m(f) &:= max\left\{k \in \mathbb{Z}:\ f \in \mathfrak{m}^k\right\}, \\ \delta(f) &:= dim_K(\bar{R}_f/R_f), \text{ where } \bar{R}_f \text{ is the normalization of } R_f. \end{aligned}$$

$m(f)$ is mostly known as the multiplicity of f and $\delta(f)$ as the delta invariant of f. Besides, it is well-known, that the number of irreducible factors of f is an invariant. This is is usually denoted by $r(f)$ and also called the number of branches of the hypersurface singularity R_f.

Remark 1.2.18. *For plane curve singularities over $\mathbb{C}$, it is established that*

$$\mu(f) = 2\delta(f) - r(f) + 1.$$

In Chapter 5 of the present work, we shall show the same claim in the more general context of characteristic zero (cf. Proposition 5.3.2). In finite characteristic though, this claim is false. Indeed, Let $char(K) = p > 0$ and let $f = (1+x)\cdot(x^p + y^{p+1})$ as in Example 1.2.17. It is not difficult to see that $\mu(f) = p^2$ and $\delta(f) = \frac{p(p-1)}{2}$. Clealy f is irreducible. Hence $2\delta(f) - r(f) + 1 = p(p-1) \neq \mu(f)$.
However, we shall show in the last section of Chapter 2, that under a certain condition of non-degeneracy the above formula for the Milnor number does also hold in finite characteristic.
In arbitrary characteristic yet, we have more generally for reduced non-zero $f \in K[[x,y]]$ that

$$\mu(f) \geq 2\delta(f) - r(f) + 1$$

(cf. [Del73] and [MHW01]).

In Chapter 3 of the present work, we shall discuss thoroughly a further invariant which is the degree of contact determinacy.

1.3 $\mathcal{P}$-Action on Plane Curve Singularties

In this section, we introduce a further fundamental equivalence relation, the parametrization equivalence, which is of big use in the classification of unimodal plane curve singularities that are defined via their parametrizations. (cf. [Hol98] and [Bou02]).

Remark 1.3.1. *Let K be an algebraically closed field of charactristic zero. Furthermore, let $f \in \langle x,y\rangle \subset K[[x,y]]$ be such that $f \neq 0$ and f is reduced. It is established in [Cam80], that the plane curve singularity $R = K[[x,y]]/\langle f\rangle$ has a parametrization.*

1. *If f is irreducible, then a parametrization of R is given by a map*

$$\begin{aligned} \psi:\ K[[x,y]] &\longrightarrow K[[t]] \\ x &\mapsto x(t) \\ y &\mapsto y(t) \end{aligned}$$

such that $Ker(\psi) = \langle f\rangle$, and the induced map

$$R \hookrightarrow K[[t]]$$

is the normalization map.
More precisely, ψ satisfies the following factorization property:

If $\psi' : K[[x,y]] \longrightarrow K[[t]]$ is another parametrization of R, then ψ' factors in a unique way through ψ, that is there exists an isomorphism $\phi : K[[t]] \longrightarrow K[[t]]$ making the following diagram commutative:

$$\begin{tikzcd} K[[x,y]] \arrow[r, "\psi'"] \arrow[dr, "\psi"'] & K[[t]] \\ & K[[t]] \arrow[u, "\phi"'] \end{tikzcd}$$

2. *If f decomposes into several branches, then a parametrization of R is given by a set of parametrizations of the branches.*
   More precisely, if $f = f_1 \dots f_s$ is the decomposition of f into irreducible factors, then $\bar{R} \cong \bigoplus_{i=1}^{s} K[[t_i]]$ is the normalization of R and a parametrization ψ of R can be represented as a matrix of the form:

$$\psi = \begin{pmatrix} x(t_1) & \Big| & y(t_1) \\ \vdots & & \vdots \\ x(t_s) & & y(t_s) \end{pmatrix}$$

   where for $i = 1, \dots, s$, $(\ x(t_i) \,|\, y(t_i)\ )$ represents a parametrization of the ith branch.

Let R be a reduced plane curve singularity and let $\bar{R} = \bigoplus_{i=1}^{s} K[[t_i]]$ be the normalization of R.
Considering $\bar{R}$ as a K-algebra, let $\boldsymbol{\phi} \in Aut_K(\bar{R})$, then we can write $\boldsymbol{\phi} = (\phi_1, \dots, \phi_s)$, where

$$(\phi_1, \dots, \phi_s) \in Aut_K(K[[t_1]]) \times \dots \times Aut_K(K[[t_s]]).$$

More precisely, for all $1 \leq i \leq s$, we have

$$\begin{aligned} \phi_i : \quad & K[[t_i]] \longrightarrow K[[t_i]] \\ & t_i \mapsto \sum_{j=1}^{\infty} \phi_{ij} t_i^j \end{aligned}$$

where for all $j \geq 1$, $\phi_{ij} \in K$ and $\phi_{i1} \neq 0$.

Definition 1.3.2. *Let R be a reduced plane curve singularity and let $\bar{R} = \bigoplus_{i=1}^{s} K[[t_i]]$ be the normalization of the local ring R.*

1. **Reparametrization of the branches:**
   Let $\psi = \begin{pmatrix} x(t_1) & \Big| & y(t_1) \\ \vdots & & \vdots \\ x(t_s) & & y(t_s) \end{pmatrix} \in \bar{R}^2$, where for $i = 1, \dots, s$,

$$(x(t_i) \mid y(t_i)) = \left(\sum_{j=1}^{\infty} a_{ij} t_i^j \mid \sum_{j=1}^{\infty} b_{ij} t_i^j \right).$$

Let $\phi = (\phi_1, \ldots, \phi_s) \in Aut_K \bar{R}$*, such that for* $i = 1, \ldots, s$*,*

$$\begin{aligned} \phi_i : K[[t_i]] &\longrightarrow K[[t_i]] \\ t_i &\mapsto \sum_{j=1}^{\infty} \phi_{ij} t_i^j \end{aligned}$$

with $\phi_{i1} \neq 0$*. Then,*

$$(\phi_i(x(t_i)) \mid \phi_i(y(t_i))) = \left(\sum_{j=1}^{\infty} a_{ij}(\phi_i(t_i))^j \mid \sum_{j=1}^{\infty} b_{ij}(\phi_i(t_i))^j \right)$$

is called a **reparametrization of the** i**th branch of** ψ *, and the element*

$$\phi \circ \psi = \begin{pmatrix} \phi_1(x(t_1)) & \Big| & \phi_1(y(t_1)) \\ \vdots & & \vdots \\ \phi_s(x(t_s)) & \Big| & \phi_s(y(t_s)) \end{pmatrix} \in \bar{R}^2$$

is called a **reparametrization** *of* ψ*.*

2. **Coordinate change:**
Let $\psi = \begin{pmatrix} x(t_1) & \Big| & y(t_1) \\ \vdots & & \vdots \\ x(t_s) & \Big| & y(t_s) \end{pmatrix} = (x(t) \mid y(t)) \in \bar{R}^2$*, and* $\Phi \in Aut_K(K[[x, y]])$*.*
We can write

$$\begin{aligned} \Phi : \quad K[[x,y]] &\longrightarrow K[[x,y]] \\ x &\mapsto Ax + By + o(2) \\ y &\mapsto Cx + Dy + o(2) \end{aligned}$$

such that, det $\begin{pmatrix} A & C \\ B & D \end{pmatrix} \neq 0.$

We define

$$\psi \circ \Phi := \quad (Ax(t) + By(t) + o(2) \mid Cx(t) + Dy(t) + o(2)).$$

3. *Let* ψ *and* $\psi' \in \bar{R}^2$*. Then,* ψ *is said to be* **parametrization equivalent** *to* ψ'*,* $\psi \overset{\mathrm{p}}{\sim} \psi'$*, if there exists a reparametrization* ϕ*, and a coordinate change* Φ *as above, such that the following diagram commutes:*

$$\begin{tikzcd} K[[x,y]] \arrow[r, "\psi"] \arrow[d, "\Phi"'] & \bar{R} \arrow[d, "\phi"] \\ K[[x,y]] \arrow[r, "\psi'"'] & \bar{R} \end{tikzcd}$$

4. *Let $\mathcal{P} := Aut_K(\bar{R}) \times Aut_K(K[[x,y]])$, endowed with the multiplication:*
   $(\phi', \Phi')(\phi, \Phi) = (\phi' \circ \phi, \Phi \circ \Phi')$.
   $\mathcal{P}$ is called the **parametrization group** .

A group action of $\mathcal{P}$ on $\bar{R}^2$ is defined as follows:

$$\begin{gathered} \mathcal{P} \times \bar{R}^2 \longrightarrow \bar{R}^2 \\ ((\phi, \Phi), \psi) \mapsto \phi \circ \psi \circ \Phi \end{gathered}$$

Note that, if ψ and ψ' are given elements in $\bar{R}^2$, then

$$\psi \underset{p}{\sim} \psi' \Longleftrightarrow \psi' \in \mathcal{P}\psi,$$

where $\mathcal{P}\psi$ denotes the orbit of ψ under the above group action.

Definition 1.3.3. *Let $k \in \mathbb{Z}_{>0}$, and $\bar{R}^2 = \bigoplus_{i=1}^{s} K[[t_i]]$.*

1. *We define $\bar{R}^2_k := (\bar{R}/\langle(t_1, \ldots, t_s)\rangle^{k+1})^2$.*
2. *Let $\psi = \left(\begin{array}{c|c} x(t_1) & y(t_1) \\ \vdots & \vdots \\ x(t_s) & y(t_s) \end{array}\right) \in \bar{R}^2$.*
   We define $\psi_k = \left(\begin{array}{c|c} j^k x(t_1) & j^k y(t_1) \\ \vdots & \vdots \\ j^k x(t_s) & j^k y(t_s) \end{array}\right)$
   where for $i = 1, \ldots, s$,
   $$j^k x(t_i) \equiv x(t_i) \quad mod\, \langle t_i \rangle^{k+1}, \quad \textit{and}$$
   $$j^k y(t_i) \equiv y(t_i) \quad mod\, \langle t_i \rangle^{k+1}.$$
3. *Let $\phi = (\phi_1, \ldots, \phi_s) \in Aut_K(\bar{R})$.*
   We define $\phi_k = (\phi_{1,k}, \ldots, \phi_{s,k})$,
   where for i= $1, \ldots, s$
   $$\phi_{i,k} \equiv \phi_i \quad mod\, \langle t_i \rangle^{k+1}.$$
   (Recall that $Aut_K(K[[t_i]]) \cong \langle t_i \rangle$.)
4. *We define $\mathcal{P}_k := \{(\phi_k, \Phi_k) : \quad (\phi, \Phi) \in \mathcal{P}\}$.*

Hence, we get a group action induced by the action of the group $\mathcal{P}$ on $\bar{R}^2$, as follows:

$$\begin{gathered} \mathcal{P}_k \times \bar{R}^2_k \longrightarrow \bar{R}^2_k \\ ((\phi_k, \Phi_k), \psi_k) \mapsto (\phi \circ \psi \circ \Phi)_k. \end{gathered}$$

Definition 1.3.4. *Let $f, g \in K[[x, y]]$ be two plane curve singularities, having the same number of branches s.*
One says that f is **parametrization equivalent** *to g, $f \underset{p}{\sim} g$, if there exist a parametrization ψ of f, and a parametrization ψ' of g, such that $\psi \underset{p}{\sim} \psi'$.*

Lemma 1.3.5. *(***Lifting lemma** *)*
Let ϕ be a morphism of K-algebras,

$$\phi : K[[x_1, \ldots, x_n]]/I \longrightarrow K[[y_1, \ldots, y_m]]/J,$$

where I and J are ideals of $K[[x_1, \ldots, x_n]]$ and of $K[[y_1, \ldots, y_m]]$ respectively.
If $n = m$ and ϕ is an isomorphism, then there exists a lifting

$$\tilde{\phi} : K[[x_1, \ldots, x_n]] \longrightarrow K[[y_1, \ldots, y_m]]$$

of ϕ which is an isomorphism.
If $n \geq m$, and ϕ is surjective, then there exists a lifting $\tilde{\phi}$ of ϕ which is surjective too. (See [GLS06]).

Lemma 1.3.6. *Let $f, g \in K[[x, y]]$ be two given plane curve singularities. Then*

$$f \overset{c}{\sim} g \Longleftrightarrow f \overset{p}{\sim} g.$$

Proof. First, suppose that $f \overset{p}{\sim} g$. Then, f and g have the same number of branches s, therefore f and g have the same normalization ring $\bar{R} = \bigoplus_{i=1}^{s} K[[t_i]]$.
Moreover, there exist parametrizations ψ, and ψ' of R_f and R_g respectively, such that $\psi \overset{p}{\sim} \psi'$, which implies the existence of $\phi \in Aut_K(\bar{R})$ and
$\Phi \in Aut_K(K[[x, y]])$, such that the following diagram commutes:

$$\begin{tikzcd} K[[x,y]] \arrow[r, "\psi"] \arrow[d, "\Phi"'] & \bar{R} \arrow[d, "\phi"] \\ K[[x,y]] \arrow[r, "\psi'"'] & \bar{R} \end{tikzcd}$$

Besides, as $Ker(\psi) = \langle f \rangle$, we have $\phi \circ \psi(f) = 0$.
On the other hand, $\phi \circ \psi = \psi' \circ \Phi$, hence $\psi' \circ \Phi(f) = 0$ which implies that $\Phi(f) \in ker(\psi') = \langle g \rangle$. Thus, $\langle \Phi(f) \rangle \subset \langle g \rangle$.
Similarly, we show that $\Phi^{-1}(g) \in \langle f \rangle$, that is $\langle g \rangle \subset \langle \Phi(f) \rangle$.
Hence, $\langle \Phi(f) \rangle = \langle g \rangle$, and then $f \overset{c}{\sim} g$.
Conversely, suppose that $f \overset{c}{\sim} g$, then in particular, there exists an isomorphism of K-algebras $\Phi : K[[x, y]]/\langle f \rangle \longrightarrow K[[x, y]]/\langle g \rangle$. Hence, the local rings related to the singularities f and g respectively have isomorphic normalization rings.
Furthermore, by the lifting lemma, there exists an isomorphism $\tilde{\Phi} \in Aut_K(K[[x, y]])$

such that the following diagram commutes:

$$\begin{tikzcd} K[[x,y]] \arrow[r, "s_1"] \arrow[d, "\tilde{\Phi}"'] & K[[x,y]]/\langle f\rangle \arrow[d, "\Phi"] \\ K[[x,y]] \arrow[r, "s_2"'] & K[[x,y]]/\langle g\rangle \end{tikzcd}$$

It can easily be checked that the following diagram also commutes:

$$\begin{tikzcd} K[[x,y]] \arrow[r, "s_1"] \arrow[d, "\tilde{\Phi}"'] & K[[x,y]]/\langle f\rangle \arrow[r, hook, "n_1"] \arrow[d, "\Phi"] & \bar{R} \arrow[d, "\phi"] \\ K[[x,y]] \arrow[r, "s_2"'] & K[[x,y]]/\langle g\rangle \arrow[r, hook, "n_2"'] & \bar{R} \end{tikzcd}$$

Moreover, by definition of a parametrization of a plane curve singularity; we have
$\psi := n_1 \circ s_1$ is a parametrization of R_f, and
$\psi' := n_2 \circ s_2$ is a parametrization of R_g.
Hence, the last commutative diagram is equivalent to the fact that $f \overset{\mathrm{p}}{\sim} g$.

$\square$

Chapter 2

C-Polytopes and Non-Degeneracy

The first examples arising in the classification of hypersurface singularities belong to the *semiquasihomogeneous* class whose elements are represented by equations of the form $f = f_\Delta + h \in K[[\boldsymbol{x}]]$ where Δ is an $(n-1)$-dimensional face of the *Newton polytope* of f, f_Δ is a *quasihomogeneous* polynomial having finite Tjurina number and all elements of $supp(h)$ lie strictly above Δ.
For the purpose of computation of normal forms and motivated by the classification, the investigation of this particular class giving rise to *quasihomogeneous* filtrations of $K[[\boldsymbol{x}]]$ plays the central role in the important paper [Arn74] of Arnold. However, it was already noticed in that paper that it is often useful to consider *piecewise-homogeneous* filtrations in which the role of Δ is played by the Newton polytope and where a finite set of weights has to be considered.
The theory of these was rapidly developed, culminating in the work [Kou76]. Kouchnirenko considered an arbitrary subset $\mathcal{M} \in \mathbb{N}^n$, looked for conditions for the existence of $f \in K[[\boldsymbol{x}]]$ such that $supp(f) \subset \mathcal{M}$ and $\mu(f) < \infty$ and found out the minimal value of the Milnor number in case that such an f exists. His answer was given in terms of certain geometrical features of the Newton polytope which is related to the set $\mathcal{M}$. He introduced the notion of *non-degeneracy* in arbitrary characteristic. His main results in positive characteristic though, are proved only for the cases where the Newton polytope meets all coordinate subspaces. Of course, these cases do not include all semiquasihomogeneous ones.
In his paper [Wal99a] about Newton polytopes and non-degeneracy over $\mathbb{C}$, Wall did slightly modify the notion of Newton polytope allowing its facets to be extended out to meet all coordinate subspaces. So he introduced the notion of *strict non-degeneracy* with respect to the so called *C-polytopes*. This condition of non-degeneracy turns out to be an appropriate one. Indeed, on one hand, Wall showed that any semiquasihomogeneous hypersurface singularity is strictly non-degenerate with respect to some C-polytope. On the other hand, he asserted that this condition implies the finiteness of the Milnor number. The results that we present in the last section of the present chapter

shows that strict non-degeneracy does also make sense in positive characteristic.

Our attempt in this chapter is to give an explicit development in arbitrary characteristic of these notions which are crucial for the subsequent chapters. Throughout this chapter, we shall use widely the notation elaborated by Wall in [Wal99a]. The chapter is organized as follows. Section 2.1 is devoted to the study of objects and notions which are closely related to C-polytopes. In part 2.1.1, we briefly review the definitions of Newton polytopes, C-polytopes and Newton number. Moreover, we explain the one to one correspondence between finite sets of weights and C-polytopes.
Although the notion of semipiecewise-homogeneity is merely a generalization of that one of semiquasihomogeneity, it was not explicitly defined in the literature. It deserves a closer look for it provides a more systematical and efficient development of the theory. In part 2.1.2, we start by defining the piecewise-homogeneous order of a formal power series. Afterwards semipiecewise-homogeneous hypersurface singularities are introduced. These can be represented by equations of the form $f = f_{\boldsymbol{P}} + h \in K[[\boldsymbol{x}]]$, where $f_{\boldsymbol{P}}$ is a piecewise-homogeneous polynomial with respect to a C-polytope $\boldsymbol{P}$, $\tau(f_{\boldsymbol{P}})$ is finite and any element of $supp(h)$ lies strictly above $\boldsymbol{P}$. We notice that the condition $\tau(f_{\boldsymbol{P}}) < \infty$ is to the case of arbitrary characteristic as the condition $\mu(f_{\boldsymbol{P}})$ to the case of characteristic zero. In both cases the principal part of the (hypersurface) singularity is isolated. The semiquasihomogeneous case over $\mathbb{C}$ is thoroughly discussed in the literature. It is well-known, amongst others, that a semiquasihomogeneous hypersurface singularity is isolated, besides it has the same Milnor number as its principal part. In positive characteristic, we show that this result remains true, if and only if, the characteristic does not divide the weighted degree of the principal part $f_{\triangle}$.
In subsection 2.1.3, we describe how finite sets of weights in $\mathbb{Z}^n_{>0}$ give rise to a piecewise-homogeneous grading of algebroid singularities $K[[\boldsymbol{x}]]/I$, where I is a proper ideal of $K[[\boldsymbol{x}]]$. Afterwards, we study their associated graded K-algebras and show that these are finite dimensional K-vector spaces in the case of zero dimensional ideals.
Although we do not make essential use of toric varieties in this dissertation, for completeness and to supplement the picture of C-polytopes, we present in part 2.1.4 the relation between these two notions.
In section 2.2, we deal with piecewise-homogeneous orders on the set $Der_K(K[[\boldsymbol{x}]])$ of derivations and their properties. This notion was introduced by Arnold in [Arn74] over $\mathbb{C}$ as a tool for the computation of normal forms and hence for the purpose of classification. We discuss this in detail in the more general setting of arbitrary characteristic.
The last section 2.3 is devoted to the notion of non-degeneracy. We start by recalling the definitions and the main results which are related to the non-degeneracy elaborated by Kouchnirenko in [Kou76]. Afterwards, we present in arbitrary characteristic Wall's notion of strict non-degeneracy over $\mathbb{C}$. Based on the observations of Wall, we compare the two notions. Also, we notice that the main result proved by Wall over $\mathbb{C}$ in [Wal99a] remains true in positive characteristic, namely that strict non-degeneracy implies finite Milnor number and finally we show that any semiquasihomogeneous hypersurface singularity for which the characteristic does not divide the weighted degree of its associated principal part is strictly non-degenerate.

Throughout the present chapter K denotes an algebraically closed field of arbitrary chracteristic. Further, for $f \in K[[\boldsymbol{x}]]$, we denote by $R_f := K[[\boldsymbol{x}]]/\langle f \rangle$ the associated hypersurface singularity to f.

2.1 C-Polytopes and Piecewise-Homogeneous Graded Algebroid Singularities

2.1.1 C-Polytopes and Newton Polytopes

We recall the definitions of a C-polytope, Newton diagram and Newton polytope of a formal power series and introduce some notations.

Definition 2.1.1. *A C-***polytope** *is a polytope $\boldsymbol{P} \subset \mathbb{R}^n_{\geq 0}$ such that*

1. *each ray through the origin in $\mathbb{R}^n_{\geq 0}$ meets $\boldsymbol{P}$ in just one point, and*
2. *the region in $\mathbb{R}^n_{\geq 0}$ lying above $\boldsymbol{P}$ (i.e not containing 0) is convex.*

Remark 2.1.2. *We would like to observe that a C-polytope divides the positive orthant in 2 connected components where actually the one not containing zero is even convex.*

Notation 2.1.3. *Let $\boldsymbol{P}$ be a C-polytope. For each face δ of $\boldsymbol{P}$, we set*

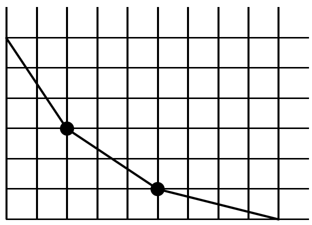
$$I_\delta^{'} := \{i,\ \ 1 \leq i \leq n : x_i = 0 \ \ \textit{on} \ \ \delta\}$$

and I_δ denotes the complement of $I_\delta^{'}$ in $\{1, \ldots, n\}$.

Definition 2.1.4. *Let $\boldsymbol{P} \subset \mathbb{R}^n_{\geq 0}$ be a C-polytope*

1. *A top-dimensional (i.e $(n-1)$-dimensional) face Δ of $\boldsymbol{P}$ is called a* **facet***.*
2. *We call a face δ of $\boldsymbol{P}$* **inner face** *if it lies in no proper coordinate subspace, that is if no coordinate x_i vanishes identically on δ (i.e $I_\delta^{'} = \emptyset$).*

Example 2.1.5. *In the following figure, the union of the thick lines represent a C-polytope in the plane.*



C-polytope

This C-polytope hat 3 facets which are the three line segments that compose it and 2 further inner faces which are the 2 vertices of the C-polytope not lying on the coordinate axes.

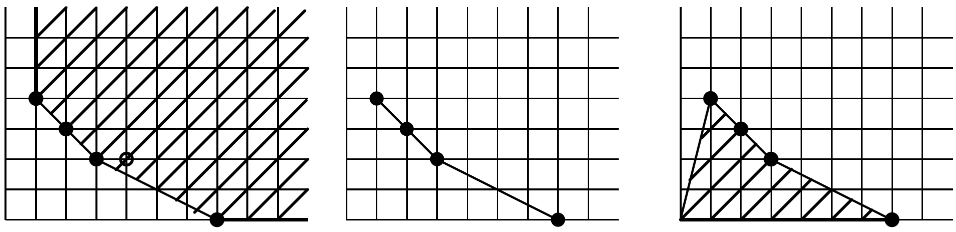
Definition 2.1.6. 1. *Let $f \in K[[\boldsymbol{x}]]$. Then, we call the set*

$$\Gamma_+(f) := conv(supp(f) + \mathbb{R}^n_{\geq 0})$$

the **Newton diagram** *of f and the boundary of $\Gamma_+(f)$ is called the* **Newton boundary** *of f.*
The union of the compact faces of the boundary of $\Gamma_+(f)$ is called the **Newton polytope** $\Gamma(f)$ *of f.*
Further, we denote the cone joining the origin and the Newton polytope $\Gamma(f)$ by $\Gamma_-(f)$.

2. *A formal power series $f \in K[[\boldsymbol{x}]]$ is called* **convenient** (CO) *if its Newton polytope $\Gamma(f)$ meets all the coordinate subspaces, i.e none of the elements x_i, $i = 1, \dots n$, divides f.*

Example 2.1.7. *Let $f = x(y^4 + xy^3 + x^2y^2 - x^3y^2 + x^6)$.*



$\Gamma_+(f)$ $\Gamma(f)$ $\Gamma_-(f)$

In particular, f is not convenient and the Newton polytope $\Gamma(f)$ has two facets, with slopes -1 and $-\frac{1}{2}$.

Remark 2.1.8. *We observe that the Newton polytope $\Gamma(f)$ of a convenient power series $f \in K[[\boldsymbol{x}]]$ is a C-$polytope$. Moreover, it is easy to notice that $\Gamma(f)$ is not an invariant of the orbit of f under the $\mathcal{R}$-action or the $\mathcal{K}$-action.*

In the sequel, we shall often use the following notation.

Notation 2.1.9. *Let $f = \sum_{\boldsymbol{\alpha} \in \mathbb{Z}^n_{\geq 0}} a_{\boldsymbol{\alpha}} \boldsymbol{x}^{\boldsymbol{\alpha}} \in K[[\boldsymbol{x}]]$.*
For a non empty subset $A \subset \mathbb{R}^n_{\geq 0}$, we write $f_A = \sum_{\boldsymbol{\alpha} \in A \cap \mathbb{Z}^n_{\geq 0}} a_{\boldsymbol{\alpha}} \boldsymbol{x}^{\boldsymbol{\alpha}}$ and we set $f_\emptyset = 0$.

Definition 2.1.10. *Let $\boldsymbol{P}$ be a compact polytope in $\mathbb{R}^n_{\geq 0}$ and let $f \in K[[x_1, \dots, x_n]]$.*

1. *We define the* **Newton number** *of $\boldsymbol{P}$ as*

$$V_N(\boldsymbol{P}) := n!\, V_n(\boldsymbol{P}) + \sum_{r=1}^{n-1} (-1)^r (n-r)!\, V_{n-r}(\boldsymbol{P}) + (-1)^n,$$

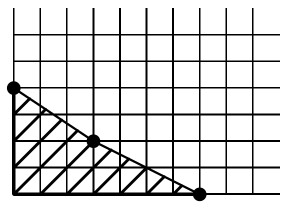
where $V_n(\boldsymbol{P})$ is the euclidian n-dimensional volume of the polytope $\boldsymbol{P}$ and, for $q = 1, \dots, n-1$, $V_q(\boldsymbol{P})$ denotes the sum of the euclidian q-dimensional volumes of the intersection of $\boldsymbol{P}$ with the coordinate subspaces of dimension q.

2. *We define the Newton number $\mu_N(f)$ of f as follows:*

   (a) *If f is (CO), we have*
   $$\mu_N(f) = V_N(\Gamma_-(f)),$$

   (b) *otherwise, we set*
   $$f_m := f + \sum_{i=1}^{n} {x_i}^m$$
   and we take
   $$\mu_N(f) = \sup_{m\in\mathbb{N}} \mu_N(f_m) \in \mathbb{N} \cup \{\infty\}.$$

Example 2.1.11. *We consider $f = y^4 + x^3y^2 + x^7 \in K[[x,y]]$. Clearly f is (CO). For $q \in \{1,2\}$, we denote for short $V_q := V_q(\Gamma_-(f))$. Then,*
$$\mu_N(f) = 2V_2 - V_1 + 1,$$
with $V_2 = \frac{3\cdot 2}{2} + 3\cdot 2 + \frac{4\cdot 2}{2} = 13$ and $V_1 = 4 + 7 = 11$.
Hence, $\mu_N(f) = 16$.



$\Gamma_-(f)$

In particular, we notice in this example that $\mu(f) = 16 = \mu_N(f)$. Indeed, in his paper [Kou76], Kouchnirenko shows that in arbitrary characteristic, the Milnor number and the Newton number coincide for any non-degenerate hypersurface singularity. This notion of non-degeneracy shall be defined in Section 2.3 of the present chapter.

Remark 2.1.12. 1. *For $f \in K[[\boldsymbol{x}]]$, Kouchnirenko shows in [Kou76, Theorem I] that the Milnor number and the Newton number satisfy in general the following relation*
$$\mu(f) \geq \mu_N(f).$$

2. *It is clear from Definition 2.1.10 that the Newton number of a convenient power series is finite. For the non-convenient ones, we have the following result given by Kouchnirenko in his paper [Kou76].*
   Let $f \in K[[x_1,\dots,x_n]]$ be non-convenient and let $q \in \mathbb{Z}_{>0}$ with $1 \leq q < n$, such that
   $$\mathbb{R}^{\{i\}} \cap supp(f) = \emptyset \text{ for } i \in \{1,\dots,q\}, \text{ and}$$

$$\mathbb{R}^{\{i\}} \cap supp(f) \neq \emptyset \ \textit{ for } \ i \in \{q+1, \ldots, n\}.$$

Further, as $\Gamma(f)$ is compact in $\mathbb{R}^n_{>0}$, we set $d(f) = \underset{\boldsymbol{r}\in\Gamma(f)}{max} |\boldsymbol{r}|$.

If $dim(\Gamma(f)) = n-1$, then

$$\mu_N(f) = \infty \Longleftrightarrow \mu_N\left(f + \sum_{i=1}^{q} {x_i}^{d(f)^n+1}\right) > (d(f)-1)^n.$$

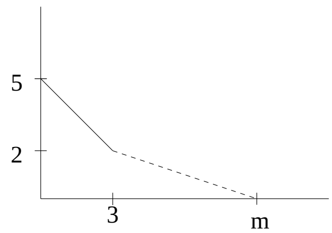
Example 2.1.13. *Let $f = y^5 + x^3y^2 \in K[[x,y]]$. We notice that f is not (CO). Moreover, for $m, m' \in \mathbb{Z}_{>0}$ such that $m > m'$, we have $\mu_N(f_m) > \mu_N(f_{m'})$. Hence, we can write*

$$\mu_N(f) = \sup_{m\in\mathbb{N}} \mu_N(f_m) = \sup_{m\geq 9} \mu_N(f_m).$$

Thus for $m \geq 9$, we get

$$\begin{array}{rcl} \mu(f) & \geq & \mu_N(f_m) \\ & = & 2\left(\frac{5\cdot 3}{2} + \frac{m\cdot 2}{2}\right) - 5 - m + 1 \\ & = & 11 + m. \end{array}$$

This shows that $\mu_N(f) = \infty$.



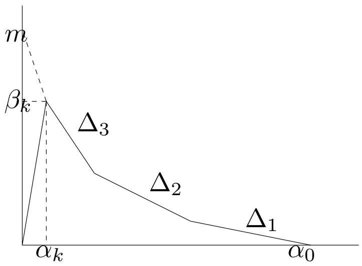
Lemma 2.1.14. *Let $f \in \mathfrak{m}^2 \subset K[[x,y]]$ be reduced. Then*

$$\mu_N(f) = V_N(\Gamma_-(f)).$$

Proof. We denote the facets of $\Gamma(f)$ by $\Delta_i, 1 \leq i \leq k$, listed in order with decreasing slopes. We denote the lattice points at the ends of Δ_i by $(\alpha_{i-1}, \beta_{i-1})$ and (α_i, β_i), $i = 1, \ldots, k$, so that $\alpha_0 > \alpha_1 > \cdots > \alpha_k$ while $(\beta_i)_i$ is increasing. If f is (CO), then the claim follows clearly from Definition 2.1.10, otherwise x divides f or y divides f. Without loss of generality, we can assume that x divides f.
Furthermore, we notice that the function

$$\begin{array}{ccc} \mathbb{N} & \longrightarrow & \mathbb{Z}_{>0} \\ m & \mapsto & \mu_N(f_m) \end{array}$$

is increasing.
If $\Gamma(f)$ intersects the x–axis, that is $\alpha_0 \neq 0$ and $\beta_0 = 0$, then, as f is reduced, we have necessarily that $\alpha_k = 1$.



Hence, we can write

$$f_\Gamma = c_0 x^{\alpha_0} + c_1 x^{\alpha_1} y^{\beta_1} + \cdots + c_k x y^{\beta_k},$$

where the coefficients $c_i \in K$, $0 \leq i \leq k$. Thus, for $m \in \mathbb{N}$, we have

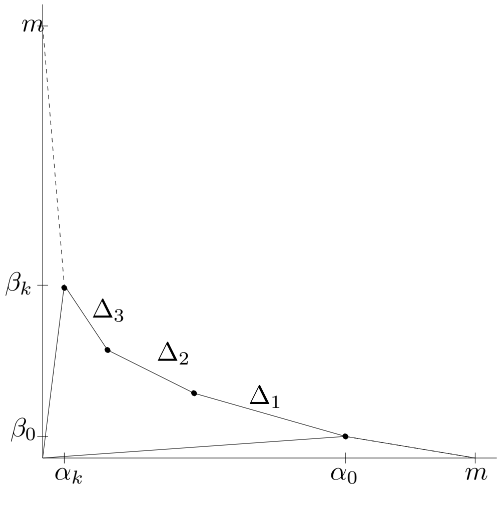
$$f_m = f + y^m$$

and, since f_m is (CO), we have by definition $\mu_N(f_m) = V_{\mathbb{N}}(\Gamma_-(f_m))$.
For m large enough, we get clearly

$$\begin{aligned} \mu_N(f_m) &= V_N(\Gamma_-(f_m)) \\ &= V_N(\Gamma_-(f)) + 2\left(\tfrac{m}{2}\right) - m \\ &= V_N(\Gamma_-(f)). \end{aligned}$$

Finally, if $\Gamma(f)$ does not intersect the x–axis, then we have $\beta_0 = 1$ and $\alpha_k = 1$, since f is reduced. Hence, we have

$$f_\Gamma = c_0 x^{\alpha_0} y + c_1 x^{\alpha_1} y^{\beta_1} + \cdots + c_k x y^{\beta_k}\ .$$



Thus $f_m = f + x^m + y^m$, $m \in \mathbb{N}$.

For m large enough, we get

$$\begin{aligned} \mu_N(f_m) &= V_N(\Gamma_-(f_m)) \\ &= V_N(\Gamma_-(f)) + 2\left(\frac{m}{2}+\frac{m}{2}\right) - (m+m) \\ &= V_N(\Gamma_-(f)). \end{aligned}$$

This proves the claim. $\square$

In the following, we would like to describe the correspondence between C-polytopes of $\mathbb{R}^n_{>0}$ and finite sets of weights.

Let $\boldsymbol{W} = \left\{\boldsymbol{w} = (w_1, \dots, w_n) \in \mathbb{R}^n_{>0}\right\}$ be a **finite** set in $\mathbb{R}^n_{>0}$. Then, $\boldsymbol{W}$ gives rise to a finite set

$$\mathcal{L}_{\boldsymbol{W}} := \{\lambda_{\boldsymbol{w}} = \langle \boldsymbol{w}, . \rangle : \mathbb{R}^n \to \mathbb{R} \mid \boldsymbol{w} \in \boldsymbol{W}\},$$

of linear functions given by

$$\lambda_{\boldsymbol{w}}(\boldsymbol{\alpha}) := \langle \boldsymbol{w}, \boldsymbol{\alpha} \rangle := \sum_{i=1}^{n} w_i \alpha_i,$$

with $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_n) \in \mathbb{R}^n$ and $\boldsymbol{w} = (w_1, \dots, w_n)$.

Notation 2.1.15. *If $\boldsymbol{W}$ is a finite set of weights and $\mathcal{L}_{\boldsymbol{W}}$ is its associated set of linear functions, then we define the function $\lambda_{\boldsymbol{W}} : \mathbb{R}^n \longrightarrow \mathbb{R}$ by*

$$\lambda_{\boldsymbol{W}}(\boldsymbol{\alpha}) := \min_{\boldsymbol{w} \in \boldsymbol{W}} \{\lambda_{\boldsymbol{w}}(\boldsymbol{\alpha})\}.$$

Definition 2.1.16.

1. *A finite set $\boldsymbol{W} \subset \mathbb{R}^n_{>0}$ is called a finite set of* **weights**.
2. *A non-empty finite set of weights $\boldsymbol{W}$ is called* **irredundant** *if for any proper subset $\boldsymbol{W}' \subset \boldsymbol{W}$, we have $\lambda_{\boldsymbol{W}} < \lambda_{\boldsymbol{W}'}$.*

Throughout the whole chapter, we consider only irredundant finite sets $\boldsymbol{W}$ of weights. On the other hand, we would like to mention that the weights we shall consider in practice lie in $\mathbb{Q}^n_{>0}$.

Remark 2.1.17. *We should notice that there is a one to one correspondence between C-polytopes and irredundant finite sets of weights. This can be described as follows:*

1. *Let $\boldsymbol{W}$ be an irredundant finite set of weights.*
   Then, $\boldsymbol{W}$ defines a C-polytope $\boldsymbol{P}_{\boldsymbol{W}} := \{\boldsymbol{\alpha} \in \mathbb{R}^n_{\geq 0} : \lambda_{\boldsymbol{W}}(\boldsymbol{\alpha}) = 1\}$.
   We can write
   $$\boldsymbol{P}_{\boldsymbol{W}} = \bigcup_{\boldsymbol{w} \in \boldsymbol{W}} \Delta_{\boldsymbol{w}},$$
   where $\Delta_{\boldsymbol{w}} := \{\boldsymbol{\alpha} \in \mathbb{R}^n_{\geq 0} : \lambda_{\boldsymbol{W}}(\boldsymbol{\alpha}) = \lambda_{\boldsymbol{w}}(\boldsymbol{\alpha}) = 1\}$. These are the facets of the polytope $\boldsymbol{P}_{\boldsymbol{W}}$. Indeed, since $\boldsymbol{W}$ is irredundant, each facet is non-empty and $(n-1)$-dimensional.

2. *Conversely, a $C\text{-}polytope$ $\boldsymbol{P}$ gives naturally rise to a finite set $\boldsymbol{W_P} \subset \mathbb{R}^n_{>0}$ of weights. Indeed, if we consider the collection of the facets Δ of $\boldsymbol{P}$, then we can associate to it the following finite set of linear functions*

$$\mathcal{L}_{\boldsymbol{P}} = \{\lambda_{\scriptscriptstyle\Delta} : \lambda_{\scriptscriptstyle\Delta}(\boldsymbol{\alpha}) = 1, \text{ for all } \boldsymbol{\alpha} \in \Delta, \ \Delta \text{ facet of } \boldsymbol{P}\}\,.$$

In this way, we obtain a finite set of weights

$$\boldsymbol{W_P} = \left\{w_\Delta \in \mathbb{R}^n_{>0} : \ \lambda_\Delta = \langle w_\Delta\,,\,.\rangle,\ \Delta \text{ facet of } \boldsymbol{P}\right\},$$

for which we have clearly that $\boldsymbol{P_{W_P}} = \boldsymbol{P}$.

2.1.2 Semipiecewise-Homogeneous Hypersurface Singularities

Definition 2.1.18. *Let $\boldsymbol{W} \subset \mathbb{Q}^n_{>0}$ be a finite set of weights.*

1. *Let $\boldsymbol{\alpha} \in \mathbb{Z}^n_{\geq 0}$. We call the positive rational number*

$$\boldsymbol{W}\text{-}deg(\boldsymbol{x}^{\boldsymbol{\alpha}}) := \lambda_{\boldsymbol{W}}(\boldsymbol{\alpha})$$

the **piecewise-homogeneous degree** *or the* $\boldsymbol{W}$**-degree** *of the monomial $\boldsymbol{x}^{\boldsymbol{\alpha}}$.*

2. *Let $f \in K[[\boldsymbol{x}]]$. Then,*

$$\boldsymbol{W}\text{-}ord(f) := v_{\boldsymbol{W}}(f) := min\{\lambda_{\boldsymbol{W}}(\alpha) : \ \alpha \in supp(f)\}$$

is called the **piecewise-homogeneous order** *or the $\boldsymbol{W}$-order of f.*
We set $v_{\boldsymbol{W}}(0) = \infty$.

Notation 2.1.19. *If $\boldsymbol{W}$ contains only one weight $\boldsymbol{w} \in \mathbb{R}^n_{>0}$, then we denote $v_{\boldsymbol{W}} := v_{\boldsymbol{w}}$.*

Remark 2.1.20. *Let $\boldsymbol{W} \subset \mathbb{Q}^n_{>0}$ be a finite set of weights.*

1. *We have clearly by Definition 2.1.18 that*

$$v_{\boldsymbol{W}}(f) = \underset{\boldsymbol{w}\in\boldsymbol{W}}{min}\,\{v_{\boldsymbol{w}}(f)\}\,.$$

2. *For $i = 1, \ldots, n$, if we set*

$$\epsilon_i = \ (\delta_{i,k})_{1\leq k\leq n} = \ (0 \ \ \ldots \ \ 0, \ \ \overset{i}{1}, \ \ 0 \ \ \ldots \ \ 0)$$

where $\delta_{i,k}$ is the Kronecker symbol, then we have

$$\lambda_{\boldsymbol{W}}(\epsilon_i) = \underset{\boldsymbol{w}\in\boldsymbol{W}}{min}\,\{\lambda_{\boldsymbol{w}}(\epsilon_i)\} = \underset{\boldsymbol{w}\in\boldsymbol{W}}{min}\,\{w_i\}\,.$$

Remark 2.1.21. *Let $f, g \in K[[\boldsymbol{x}]]$. It follows clearly from Definition 2.1.18 that*

1. $v_{\boldsymbol{W}}(f+g) \geq min\{v_{\boldsymbol{W}}(f)\,,\, v_{\boldsymbol{W}}(g)\}$.

2. *$v_{\boldsymbol{W}}(fg) \geq v_{\boldsymbol{W}}(f) + v_{\boldsymbol{W}}(g)$ and the equality does always hold if $\sharp(\boldsymbol{W}) = 1$.*

If $\sharp(\boldsymbol{W}) \geq 2$ however, then it is of interest to notice that the equality holds if and only if the $\boldsymbol{W}$-order can be reduced to a $\boldsymbol{w}$-order for some weight $\boldsymbol{w} \in \boldsymbol{W}$. This is precisely the statement of the next lemma.

Lemma 2.1.22. *Let $f, g \in K[[\boldsymbol{x}]]$ and let $\boldsymbol{W} \subset \mathbb{Q}^n_{>0}$ be a finite set of weights. Then, $v_{\boldsymbol{W}}(fg) = v_{\boldsymbol{W}}(f) + v_{\boldsymbol{W}}(g)$, if and only if, for some $\boldsymbol{w} \in \boldsymbol{W}$ we have $v_{\boldsymbol{W}}(f) = v_{\boldsymbol{w}}(f)$ and $v_{\boldsymbol{W}}(g) = v_{\boldsymbol{w}}(g)$.*

Proof. Let $\boldsymbol{w}$, $\boldsymbol{w'}$ and $\boldsymbol{w}''$ be weights in $\boldsymbol{W}$ so that

$$v_{\boldsymbol{W}}(fg) = v_{\boldsymbol{w}}(fg), \quad v_{\boldsymbol{W}}(f) = v_{\boldsymbol{w'}}(f) \text{ and } v_{\boldsymbol{W}}(g) = v_{\boldsymbol{w}''}(g).$$

Hence, we have clearly that

$$\begin{aligned} v_{\boldsymbol{W}}(fg) = v_{\boldsymbol{W}}(f) + v_{\boldsymbol{W}}(g) \quad &\Longleftrightarrow \quad v_{\boldsymbol{w}}(fg) = v_{\boldsymbol{w'}}(f) + v_{\boldsymbol{w}''}(g) \\ &\Longleftrightarrow \quad v_{\boldsymbol{w}}(f) + v_{\boldsymbol{w}}(g) = v_{\boldsymbol{w'}}(f) + v_{\boldsymbol{w}''}(g) \\ &\Longleftrightarrow \quad v_{\boldsymbol{w}}(f) - v_{\boldsymbol{w'}}(f) = v_{\boldsymbol{w}''}(g) - v_{\boldsymbol{w}}(g). \end{aligned}$$

As $0 \leq v_{\boldsymbol{w}}(f) - v_{\boldsymbol{w'}}(f) = v_{\boldsymbol{w}''}(g) - v_{\boldsymbol{w}}(g) \leq 0$, then it follows clearly that $v_{\boldsymbol{w'}}(f) = v_{\boldsymbol{w}}(f)$ and $v_{\boldsymbol{w}''}(g) = v_{\boldsymbol{w}}(g)$. This shows the lemma. □

Definition 2.1.23. *Let $\boldsymbol{W} \subset \mathbb{Q}^n_{>0}$ be a finite set of weights.*

1. *A polynomial $f = \sum_{\boldsymbol{\alpha} \in \mathbb{Z}^n_{\geq 0}} a_{\boldsymbol{\alpha}} \boldsymbol{x}^{\boldsymbol{\alpha}} \in K[\boldsymbol{x}]$ is called* **piecewise-homogeneous** *or (PH) of type $(\boldsymbol{W}\,;\, d)$ if*

$$\boldsymbol{W}\text{-}deg(\boldsymbol{x}^{\boldsymbol{\alpha}}) = d, \quad \textit{for all } \ \boldsymbol{\alpha} \in supp(f).$$

   d is called the **piecewise degree** *or the $\boldsymbol{W}$-degree of f.*

2. *If the set $\boldsymbol{W}$ has only one weight, then we call a piecewise-homogeneous polynomial f of type $(\boldsymbol{W}\,;\, d)$* **quasihomogeneous** *or (QH).*

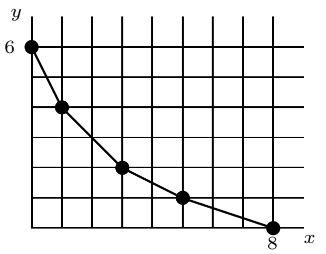
Remark 2.1.24. 1. *It is clear that any (QH) polynomial is (PH).*

2. *Obviously a quasihomogeneous (QH) polynomial of type $(\boldsymbol{w}\,;\, d)$, where $\boldsymbol{w} = (w_1, \ldots, w_n) \in \mathbb{Q}^n_{>0}$ is also (QH) of type $(\bar{\boldsymbol{w}}\,,\, 1)$ for $\bar{\boldsymbol{w}} = (\frac{w_1}{d}, \ldots, \frac{w_n}{d})$.*

Example 2.1.25. *The polynomial $f = xy^4 + x^3y^2 + x^5y + y^6 + x^8 \in K[x, y]$ is (PH) of type $(\boldsymbol{W}\,;\, 1)$ where*

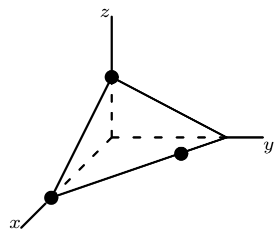
$$\boldsymbol{W} = \left\{ \boldsymbol{w_1} = \left(\frac{1}{8}, \frac{3}{8}\right) \,;\, \boldsymbol{w_2} = \left(\frac{1}{7}, \frac{2}{7}\right) \,;\, \boldsymbol{w_3} = \left(\frac{1}{5}, \frac{1}{5}\right) \,;\, \boldsymbol{w_4} = \left(\frac{1}{3}, \frac{1}{6}\right) \right\}.$$

The polynomial $f = x^3 + xy^3 + z^2 \in K[x, y, z]$ is (QH) of type $(\boldsymbol{w}\,;\, 1)$ where $\boldsymbol{w} = (\frac{1}{3}, \frac{2}{9}, \frac{1}{2})$.



$f = xy^4 + x^3y^2 + x^5y + y^6 + x^8$

$\mathbf{\Gamma}(f) = \boldsymbol{P}_{\boldsymbol{W}}$



$f = x^3 + xy^3 + z^2$

$\mathbf{\Gamma}(f) \neq \boldsymbol{P}_{\boldsymbol{W}}$

Remark 2.1.26. *Let f be a (PH) polynomial. We observe that in general, there exist infinitely many C-polytopes $\boldsymbol{P}$ of $\mathbb{R}^n_{\geq 0}$ such that the polynomial $f_{\boldsymbol{P}}$ is equal to f as the next Example 2.1.27 shows.*

Example 2.1.27. *Let $f = x^{r_0} + x^{r_1}y^{s_1} + y^{s_2} \in K[[x,y]]$ such that $r_0 > r_1$ and $s_2 > s_1$ and $r_0s_1 + s_2r_1 < r_0s_2$. Obviously f is a convenient (PH) polynomial of type $(\boldsymbol{W}\,;\,1)$ where $\boldsymbol{W}$ is the set of two weights arising from the two facets of the Newton polygon $\mathbf{\Gamma}$ of f (cf. Remark 2.1.17).*
Further, we denote by σ_1 the facet of $\mathbf{\Gamma}$ meeting the x-axis and by σ_2 the other facet. Moreover, we denote $\tilde{\sigma}_2$ the extension of σ_2 to the x-axis and we consider the set of points

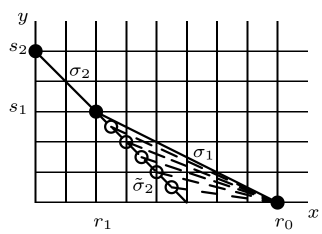
$$\mathcal{C} := \{M = (r,s) \in \mathbb{R}^2_{\geq 0} \,:\, (r,s) \in \tilde{\sigma}_2 \;\text{ and }\; 0 < s \leq s_1\}.$$

Obviously the set $\mathcal{C}$ is infinite.
On the other hand, for $M = (r,s) \in \mathcal{C}$, we consider respectively the edge $\sigma_{1,M}$ with end points $(r_0, 0)$, $(r,s)$and the edge $\sigma_{2,M}$ with end points (r,s), $(0, s_2)$.
Now, let $\boldsymbol{P}_M = \sigma_{1,M} \cup \sigma_{2,M}$. It is clear that

1. *$\boldsymbol{P}_M$ is a C-polytope of $\mathbb{R}^2_{\geq 0}$.*

2. *No point of $supp(f)$ lies below $\boldsymbol{P}_M$.*

3. *f is (PH) of type $(\boldsymbol{W}_{\boldsymbol{P}_M}\,;\,1)$ where $\boldsymbol{W}_{\boldsymbol{P}_M}$ is the set of two weights arising from $\boldsymbol{P}_M$.*



Definition 2.1.28. *A hypersurface singularity R is called* **piecewise-homogeneous** *or (PH) if there exists a piecewise-homogeneous polynomial $f \in K[\boldsymbol{x}]$ such that $R \cong R_f$.*

In the following, we deal with some examples of (PH) hypersurface singularities.

Example 2.1.29. *Let $g = y^4 + xy^4 + x^3y^2 + x^4y^2 + x^7 + x^8$. The associated hypersurface singularity R_g is (PH). Indeed, let $f = y^4 + x^3y^2 + x^7$. We have f is (PH) of type $(\boldsymbol{W}\,;\,1)$, where $\boldsymbol{W} = \left\{\boldsymbol{w_1} = (\frac{1}{7}, \frac{2}{7}), \boldsymbol{w_2} = (\frac{1}{6}, \frac{1}{4})\right\}$. On the other hand, we have $g = (1+x)f$.*
Hence, $g \overset{c}{\sim} f$ and therefore $R_g \cong R_f$ obviously follows.

Example 2.1.30. *Let $f = y^4 + x^2y^3 + x^3y^2 + x^7$ and let $R_f = K[[\boldsymbol{x}]]/\langle f\rangle$ be the hypersurface singularity associated to f.*

- *If $char(K) \neq 3$, then we claim that R_f is (PH).*

Indeed, we can show later in Example 3.3.9) that
$f \overset{c}{\sim} y^4 + x^3y^2 + x^7$. On the other hand, the latter polynomial is (PH) as Example 2.1.29 shows. Therefore R_f is (PH) by Definition 2.1.28.

- *We show however, that R_f is not (PH) whenever $char(K) = 3$.*

Indeed, we show in the following that there is no (PH) polynomial which is contact equivalent to f.
Let $u \in K[[\boldsymbol{x}]]^*$ be a unit in $K[[\boldsymbol{x}]]$ and let $\varphi \in Aut(K[[\boldsymbol{x}]])$ be an automorphism of $K[[\boldsymbol{x}]]$. Then, we can write

$$u = e + h \;\; \textit{and} \;\; \varphi: \;\; x \mapsto ax + by + h_1, \;\; y \mapsto cx + dy + h_2,$$

where

1. *$e \in K \setminus \{0\}$,*
2. *$h \in \mathfrak{m}$,*
3. *a, b, c and $d \in K$ such that $ad - bc \neq 0$ and*
4. *h_1 and h_2 are in $\mathfrak{m}^2$.*

On the other hand, we can show using SINGULAR, *that any monomial having a $\boldsymbol{W}$-degree strictly bigger than $13/12$ lies in the ideal $tj(f)$. Thus, in particular the ideal $\mathfrak{m}^8$ is a subset of I. Hence, according to Corollary $3.4.?$ of the next chapter, for any $g \in K[[\boldsymbol{x}]]$ such that $f - g \in \mathfrak{m}^8$, we have $f \overset{c}{\sim} g$.*
Then, we can write

$$u\varphi(f) \overset{c}{\sim} (c^4x^4 + c^3dx^3y + d^3cxy^3 + d^4y^4) + c^2a^3x^5 + d^2b^3y^5 + \tilde{h},$$

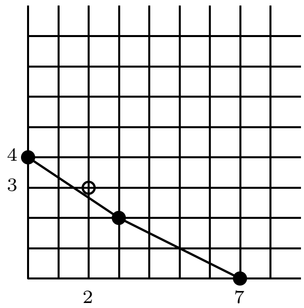
where $\tilde{h}$ is in $\mathfrak{m}^5 \subset K[[\boldsymbol{x}]]$.

- *Suppose that $c \neq 0$ and $d \neq 0$. Then, $u\varphi(f) = g + g_1$, where g is a homogeneous polynomial of degree 4 and g_1 is a nonzero polynomial in $\mathfrak{m}^5 \setminus \{0\}$.*
  We see clearly that in this case the polynomial, $g + g_1$ is not (PH).

- *Now suppose that $c = 0$. Then, it follows from the condition $ad - bc \neq 0$, that $a \neq 0$ and $d \neq 0$. Furthermore, we have that*
$$u\varphi(f) = d^4y^4 + a^3d^2x^3y^2 + a^7x^7 + g_1, \text{ such that}$$
$g_1 = a^2d^3x^2y^3 - abd^3xy^4 + (d^2b^3 + b^2d^3)y^5 - ad^3xy^3h_1 - bd^3y^4h_1 + a^6bx^6y + b^6axy^6 + b^7y^7 + d^3y^3h_1^2 + dh_2^3y$.
The polynomial $g := d^4y^4 + a^3d^2x^3y^2 + a^7x^7$ is (PH) of degree 1 (c.f Example 2.1.29) and the polynomial g_1 is not zero as it has the nonzero term $a^2d^3x^2y^3$. Besides, the piecewise degree of g_1 is strictly greater than 1 and therefore the polynomial $g + g_1$ is not (PH).

- *The case $d = 0$ is analogous to the case $c = 0$.*

So, the claim clearly follows.



Newton polygone of
$f = y^4 + x^3y^2 + x^7$

In the following remark, we would like to formulate in arbitrary characteristic some known facts about quasihomogeneous hypersurface singularities.

Remark 2.1.31. *1. Let $char(K)$ be arbitrary and let $f \in K[\boldsymbol{x}]$ be (QH) of type $(\boldsymbol{w}\,;\, d)$ where $\boldsymbol{w} = (w_1, \ldots, w_n) \in \mathbb{R}^n_{>0}$. Then f satisfies obviously the* **Euler relation**
$$df = \sum_{i=1}^{n} w_i x_i f_{x_i}, \text{ in } K[\boldsymbol{x}],$$
and the relation
$$f(t^{w_1}x_1, \ldots, t^{w_n}x_n) = t^d f(x_1, \ldots, x_n) \text{ in } K[\boldsymbol{x}, t].$$

2. *It is easy to notice that, if $char(K)$ does not divide the degree d of quasi-homogeneity, then it follows from the Euler relation that $f \in j(f)$ and thus $\tau(f) = \mu(f)$.*
On the other hand, it has been established in a theorem of K. Saito [Sai71] that for $f \in \mathbb{C}\{\boldsymbol{x}\}$ having finite Milnor number, the converse does also hold. More precisely, let $f \in \mathbb{C}\{\boldsymbol{x}\}$ be such that 0 is an isolated singularity of f. Then
$$R_f \text{ is } (QH) \Longleftrightarrow \mu(f) = \tau(f).$$

Using the particular relations fulfilled by a (QH) polynomial $f \in K[\boldsymbol{x}]$ quoted in the first part of Remark 2.1.31, we present in the sequel some important properties of (QH) hypersurface singularities.

Lemma 2.1.32. *Let $f \in K[\boldsymbol{x}]$ be (QH) of type $(\boldsymbol{w}\,;\, d)$ and let $g \in K[[\boldsymbol{x}]]$ be arbitrary. If $char(K)$ does not divide d, then*

$$f \overset{c}{\sim} g \Longleftrightarrow f \overset{r}{\sim} g.$$

Proof. The proof repeats the same arguments used in [GLS06, 2.13], replacing the field of complex numbers $\mathbb{C}$ by an algebraically closed field such that $char(K) \nmid d$. ☐

Lemma 2.1.33. *Let $\boldsymbol{w} = (w_1, \ldots, w_n) \in \mathbb{Z}^n_{>0}$ such that $gcd(w_1, \ldots, w_n) = 1$ and let $d \in \mathbb{Z}_{>0}$. Further, let $f \in \mathfrak{m}^3 \subset K[[\boldsymbol{x}]]$ be (QH) of type $(\boldsymbol{w}\,,\, d)$ such that $\tau(f)$ is finite. Then the following are equivalent*

(1) $char(k) \nmid d$

(2) $\mu(f)$ *is finite.*

Proof. If $char(K)$ does not divide d, then it follows clearly from the Euler's relation that $f \in j(f)$ (cf. Remark 2.1.31) and hence $\mu(f) = \tau(f) < \infty$.
To show the implication $(2) \Rightarrow (1)$, we suppose that $char(K)$ divides d. Hence the Euler relation reads

$$w_1 x_1 f_{x_1} + \ldots + w_n x_n f_{x_n} = 0.$$

As $gcd(w_1, \ldots, w_n) = 1$, we can suppose for example that $char(K) \nmid w_n$. Thus, we can write

$$x_n f_{x_n} = -\frac{w_1}{w_n} f_{x_1} - \ldots - \frac{w_{n-1}}{w_n} f_{x_{n-1}}.$$

On the other hand it is easy to see that x_n is not zero in $K[[\boldsymbol{x}]]/\langle f_{x_1}, \ldots, f_{x_n}\rangle$. Indeed, otherwise we would have $x_n \in \langle f_{x_1}, \ldots, f_{x_n}\rangle \subset \mathfrak{m}^2$ which is impossible.
Hence f_{x_n} is a zero divisor in $K[[\boldsymbol{x}]]/\langle f_{x_1}, \ldots, f_{x_n}\rangle$. Therefore the sequence $f_{x_1}, \ldots, f_{x_n}$ is not regular in the Cohen-Macaulay ring $K[[\boldsymbol{x}]]$. Then it follows that $dim(M_f) \geq 1$, where M_f is the Milnor algebra associated to f (cf. [GLS06, B.8.3]). But this contradicts $\mu(f) < \infty$. Hence $char(K) \nmid d$. ☐

Remark 2.1.34. *Let f be a (QH) of type $(\boldsymbol{w}\,; d)$ where $\boldsymbol{w} \in \mathbb{Z}^n_{>0}$ and $d \in \mathbb{Z}_{>0}$. We should notice that in arbitrary characteristic, the partial derivations of f are either 0 or non-zero (QH) polynomials. More precisely, for $i = 1, \ldots, n$, we have*

$$f_{x_i} = 0 \text{ or } f_{x_i} \text{ is } (QH) \text{ of type } (\boldsymbol{w}\,;\, d - w_i).$$

So we get the following lemma:

Lemma 2.1.35. *Let $f \in \mathfrak{m}$ be a (QH) polynomial. If $\mu(f)$ is finite, then*

$$\{\boldsymbol{r} \in K^n :\ f_{x_1}(\boldsymbol{r}) = \ldots = f_{x_n}(\boldsymbol{r}) = 0\} = \{0\}.$$

Proof. Let $\boldsymbol{w} = (w_1, .., w_n) \in \mathbb{Z}^n_{>0}$ be the weight associated to f and let

$$N = \{\boldsymbol{r} \in K^n : \ f_{x_1}(\boldsymbol{r}) = \ldots = f_{x_n}(\boldsymbol{r}) = 0\}$$

We suppose that there exists $\boldsymbol{r} = (r_1, \ldots, r_n) \in N$ such that $r \neq 0$. Then for any $t \in K$, we have clearly $(t^{w_1} r_1, \ldots, t^{w_n} r_n) \in N$.
Further let I be the ideal associated to the set of points

$$\{(t^{w_1} r_1, \ldots, t^{w_n} r_n)) : \ t \in K\}\,.$$

Obviously $I \supset j(f)$ and $dim(K[[\boldsymbol{x}]])/I = 1$, this yields $dim(M_f) \geq 1$ which contradicts $\mu(f) < \infty$. □

Remark 2.1.36. *Let $\boldsymbol{P} \subset \mathbb{R}^n_{\geq 0}$ be a C-polytope. Let $f = \sum_{\boldsymbol{\alpha}} a_{\boldsymbol{\alpha}} x^{\boldsymbol{\alpha}} \in K[[\boldsymbol{x}]]$ be such that the truncation $f_{\boldsymbol{P}} = \sum_{\boldsymbol{\alpha} \in \boldsymbol{P}} a_{\boldsymbol{\alpha}} x^{\boldsymbol{\alpha}} \neq 0$. Then, it follows clearly that*

$$f_{\boldsymbol{P}} \ \text{ is a } \ (PH) \ \text{ polynomial of type } \ (\boldsymbol{W}\,;\,1),$$

where $\boldsymbol{W}$ is the finite set of weights associated to $\boldsymbol{P}$ (cf. Remark 2.1.17). If we have further that no point of $supp(f)$ lies below $\boldsymbol{P}$, then we can write

$$f = f_{\boldsymbol{P}} + f_1, \ \text{with } v_{\boldsymbol{W}}(f_1) > 1.$$

Furthermore, $f_{\boldsymbol{P}}$ is called the **principal part** *of f.*

Definition 2.1.37. *Let $f \in \mathfrak{m} \subset K[[\boldsymbol{x}]]$.*

1. *We call f* **semipiecewise-homogeneous** *or (SPH) if there exists a C-polytope $\boldsymbol{P}$ in $\mathbb{R}^n_{\geq 0}$ such that no point of $supp(f)$ lies below $\boldsymbol{P}$ and the (PH) polynomial $f_{\boldsymbol{P}}$ has a finite Tjurina number.*

2. *A formal power series $f \in K[[\boldsymbol{x}]]$ is called* **semiquasihomogeneous** *(SQH) if there is a face Δ of $\boldsymbol{\Gamma}(f)$ of dimension $n-1$ such that the (QH) truncation f_Δ has finite Tjurina number. This face Δ is then uniquely determined and f_Δ is called the principal part of f.*

3. *A hypersurface singularity is called* **semipiecewise-homogeneous** *(resp.* **semiquasihomogeneous***) or (SPH) (resp. (SQH)) if there exists a (SPH) (resp. (SQH)) power series f such that $R \cong R_f$.*

Obviously, any (SQH) hypersurface singularity is (SPH) too. One has only to consider the extension of the face Δ to the coordinate hypersurfaces to get a C-polytope in $\mathbb{R}^n_{\geq 0}$.

Remark 2.1.38. *Let $f \in K[[\boldsymbol{x}]]$ be (SQH) and let f_Δ be the principal part of f. Considering the weight vector associated to the facet Δ, we observe easily that f_Δ is a (QH)-polynomial of type $(\boldsymbol{w}\,;\,d)$ where $d \in \mathbb{Z}_{>0}$. Moreover, we can write*

$$f = f_\Delta + g, \ \ \text{with } \ \tau(f_\Delta) < \infty \ \text{ and } \ v_{\boldsymbol{w}}(g) > d.$$

In chapter 3 of the present dissertation, Corollary 3.3.13 establishes the following.

Lemma 2.1.39. *Any semiquasihomogeneous hypersurface singularity is isolated.*

Proof. cf. Corollory 3.3.13. ☐

The investigation of (SQH) singularities over the field of complex numbers $\mathbb{C}$ plays a central role in the important paper [Arn74] of Arnold where he shows for example that a (SQH) hypersurface singularity has the same Milnor number as its associated principal part. Of course, we would like to investigate in how far this remains true in arbitrary characteristic.
The following example however shows that this property does not hold in general when $char(K) > 0$.

Example 2.1.40. *Let $char(K) = 7$ and let $f = x^7 + x^6y + y^4 \in K[[x, y]]$. Further, let Δ be the line segment with end points $(7, 0)$ and $(0, 4)$. It is clear that f is (SQH) of principal part $f_\Delta = x^7 + y^4$ (note that f_Δ is reduced and hence $\tau(f_\Delta) < \infty$). On the other hand $\mu(f_\Delta)$ is infinite while $\mu(f) = 21 < \infty$.*

We notice that in this example, $char(K) = 7$ divides the weighted degree of f_Δ which is 28.

Proposition 2.1.41. *Let K be an algebraically closed field of arbitrary charactersic and let $\boldsymbol{w} = (w_1, \dots, w_n) \in \mathbb{Z}^n_{>0}$. Further, let $f \in K[[\boldsymbol{x}]]$ be (SQH) with principal part f_Δ of type $(\boldsymbol{w}\,;\, d)$, $d \in \mathbb{Z}_{>0}$. If $\mu(f_\Delta)$ is finite, then $\mu(f) = \mu(f_\Delta)$.*

Proof. Throughout the whole proof, we use the following notation:
$K[[\boldsymbol{y}, t]] := K[[y_1, \dots, y_n, t]]$ and $K[[\boldsymbol{x}, t]] := K[[x_1, \dots, x_n, t]]$.
We can write $f = f_\Delta + g$, where $g \in K[[\boldsymbol{x}]]$ and $v_{\boldsymbol{w}}(g) > d$.
Further, we assume $\mu(f_\Delta) < \infty$ and we set

$$\hat{f}(\boldsymbol{x}, t) := \frac{f(t^{w_1}x_1, \dots, t^{w_n}x_n)}{t^d} \in K[[\boldsymbol{x}, t]].$$

Thus $\hat{f}(\boldsymbol{x}, t) = f_\Delta(\boldsymbol{x}) + t^m g(\boldsymbol{x}, t)$, $m \geq 1$. Hence, we can write

$$\hat{f}_{x_i}(\boldsymbol{x}, t) = f_{\Delta, x_i}(\boldsymbol{x}) + t^m g_{x_i}(\boldsymbol{x}, t) \in K[[\boldsymbol{x}, t]]. \tag{2.1}$$

We consider the following K-algebra homomorphism

$$\begin{array}{rccc} \Phi: & K[[\boldsymbol{y}, t]] & \longrightarrow & K[[\boldsymbol{x}, t]] \\ & y_i & \mapsto & \hat{f}_{x_i}(\boldsymbol{x}, t) \\ & t & \mapsto & t \end{array}$$

Clearly, it follows from (2.1) that

$$dim_K(K[[\boldsymbol{x}, t]]/\langle \hat{f}_{x_1}(\boldsymbol{x}, t), \dots, \hat{f}_{x_n}(\boldsymbol{x}, t)\,,\, t\rangle) = dim_K(K[[\boldsymbol{x}]]/j(f_\Delta)) = \mu(f_\Delta). \tag{2.2}$$

Thus, as $\mu(f_\Delta)$ is finite, we see that the morphism Φ is quasifinite and even finite (cf. [GLS06, 1.13]).
Moreover, it follows from (2.2) that $dim(K[[\boldsymbol{x},t]]/\langle \hat{f}_{x_1}(\boldsymbol{x},t),\ldots,\hat{f}_{x_n}(\boldsymbol{x},t)\,,\,t\rangle)$ is zero. Then we can write obviously

$$dim(K[[\boldsymbol{x},t]]) = dim(K[[\boldsymbol{y},t]]) + dim(K[[\boldsymbol{x},t]]/\langle \hat{f}_{x_1}(\boldsymbol{x},t),\ldots,\hat{f}_{x_n}(\boldsymbol{x},t)\,,\,t\rangle.$$

Besides, for $K[[\boldsymbol{x},t]]$ is Cohen-Macaulay, it follows from [Eis96, 18.16] that Φ is flat. Altogether, and using the Nakayama lemma, we obtain that $K[[\boldsymbol{x},t]]$ is a free $K[[\boldsymbol{y},t]]$-module of rank $\mu(f_\Delta)$. Hence

$$K[[\boldsymbol{x},t]]\otimes_{K[[y,t]]}K[[\boldsymbol{y},t]]/\langle y\rangle = K[[\boldsymbol{x},t]]/\langle \hat{f}_{x_1}(\boldsymbol{x},t),\ldots,\hat{f}_{x_n}(\boldsymbol{x},t)\rangle$$

is a free $K[[t]]$-module of rank $\mu(f_\Delta)$. Over the field of fractions $K((t))$, we consider the morphism

$$\begin{array}{rcll} \varphi:\ K((t))[[\boldsymbol{x}]] & \longrightarrow & K((t))[[\boldsymbol{x}]] & \\ x_i & \mapsto & t^{w_i}x_i, & \qquad i=1,\ldots,n \\ t & \mapsto & t. & \end{array}$$

It is straightforward that φ is an isomorphism of local algebras and in $K((t))[[\boldsymbol{x}]]$, we have

$$\hat{f}(\boldsymbol{x},t) = \frac{1}{t^d}\varphi(f(\boldsymbol{x})).$$

Writing K' for $K((t))$, we have cleary

$$\langle \hat{f}(\boldsymbol{x},t)\rangle_{K'[[\boldsymbol{x}]]} = \langle \varphi(f(\boldsymbol{x}))\rangle_{K'[[\boldsymbol{x}]]}.$$

Since φ is an isomorphism, we have by Lemma 1.2.7

$$j(\hat{f})K'[[\boldsymbol{x}]] = j(\varphi(f))K'[[\boldsymbol{x}]] = \varphi(j(f))K'[[\boldsymbol{x}]].$$

Due to the above, we get

$$\begin{array}{rcl} K[[\boldsymbol{x},t]]/\langle \hat{f}_{x_1}(\boldsymbol{x},t),\ldots,\hat{f}_{x_n}(\boldsymbol{x},t)\rangle\otimes_{K[[t]]}K((t)) & \cong & K'[[\boldsymbol{x}]]/\varphi(j(f))K'[[\boldsymbol{x}]] \\ & \cong & K'[[\boldsymbol{x}]]/j(f)K'[[\boldsymbol{x}]] \end{array}$$

is a K'-vector space of finite dimension $\mu(f_\Delta)$.
Finally, it follows by Theorem 5.1.7 that

$$dim_K(K[[\boldsymbol{x}]]/j(f)) = dim_{K'}(K'[[\boldsymbol{x}]]/j(f)K'[[\boldsymbol{x}]]) = \mu(f_\Delta).$$

□

If $char(K)=0$, we shall give an alternative proof of Proposition 2.1.41 in Chapter 5 using Lefschetz principle (cf. Proposition 5.2.2).

2.1.3 Piecewise-Homogeneous Grading of Algebroid Singularities

In the first part of the present section, we shall show how C-polytopes (or equivalently finite sets of weights) give rise to particular filtrations of algebroid singularities. These are the so called *piecewise* filtrations and generalize the well-known quasihomogeneous filtrations which are induced by only one weight. After that, we shall deal with the main properties of the associated graded K-algebras.

Lemma 2.1.42. *Let $\boldsymbol{W} \subset \mathbb{Z}^n_{>0}$ be a finite set of weights and let $d \in \mathbb{N}$. Then the sets*

$$F_{\geq d} := \{g \in K[[\boldsymbol{x}]] : \;\; v_{\boldsymbol{W}}(g) \geq d\},$$

and

$$F_{>d} := \{g \in K[[\boldsymbol{x}]] : \;\; v_{\boldsymbol{W}}(g) > d\}.$$

are ideals of $K[[\boldsymbol{x}]]$. Moreover, we have

1. *$F_{\geq 0} = K[[\boldsymbol{x}]]$ and $F_{>0} = \mathfrak{m}$,*
2. *$F_{>d} \subset F_{\geq d}$ and $F_{\geq d'} \subset F_{\geq d}$ for any $d^{'} \geq d$ and*
3. *for any $d^{'} \in \mathbb{N}$, we have $F_{\geq d}F_{\geq d'} \subset F_{\geq d+d'}$.*

Proof. The proof is obvious. ☐

Remark 2.1.43. 1. *For any $d \in \mathbb{N}$, it is easy to see that the K-linear spaces $K[[\boldsymbol{x}]]/F_{\geq d}$ and $K[[\boldsymbol{x}]]/F_{>d}$ are finite dimensional.*

2. *We observe clearly from Lemma 2.1.42 that the ideals $F_{\geq d}$, $d \in \mathbb{N}$, give rise to a decreasing filtration*

$$F_{\geq 0} \supset F_{\geq 1} \supset \ldots \supset F_{\geq d} \supset \ldots$$

of $K[[\boldsymbol{x}]]$. On the other hand, if I is a proper ideal of $K[[\boldsymbol{x}]]$, then we see clearly that

$$(F_{\geq 0} + I)/I \supset (F_{\geq 1} + I)/I \supset \ldots \supset (F_{\geq d} + I)/I \supset \ldots$$

is the induced quotient filtration on the algebroid singularity $K[[\boldsymbol{x}]]/I$.

Definition 2.1.44. *Let $\boldsymbol{W} \subset \mathbb{Q}^n_{>0}$ be a finite set of weights and let $I \subset K[[\boldsymbol{x}]]$ be a proper ideal. We call the decreasing filtration*

$$(F_{\geq 0} + I)/I \supset (F_{\geq 1} + I)/I \supset \ldots \supset (F_{\geq d} + I)/I \supset \ldots$$

where for $d \in \mathbb{N}$, $F_{\geq d} := \{g \in K[[\boldsymbol{x}]] : \;\; v_{\boldsymbol{W}}(g) \geq d\}$ the **piecewise-homogeneous filtration** *or* **$\boldsymbol{W}$-filtration** *of the algebroid singularity $K[[\boldsymbol{x}]]/I$.*
Further, the ideals $(F_{\geq d} + I)/I$, $d \in \mathbb{N}$, are called the $\boldsymbol{W}$-ideals of $K[[\boldsymbol{x}]]/I$.

In the following, we shall study the associated grading to a given piecewise filtration of an algebroid singularity. Next, we consider the associated graded K-algebra $gr_{\boldsymbol{W}}(K[[\boldsymbol{x}]]/I)$, namely

$$gr_{\boldsymbol{W}}(K[[\boldsymbol{x}]]/I) := \bigoplus_{d\in\mathbb{N}} gr_{\boldsymbol{W}\,,\,d}(K[[\boldsymbol{x}]]/I),$$

where

$$gr_{\boldsymbol{W}\,,\,d}(K[[\boldsymbol{x}]]/I) := F_{\geq d}/(F_{>d} + (F_{\geq d} \cap I)).$$

Remark 2.1.45. *We observe that the monomials of the K-algebra $gr_{\boldsymbol{W}}(K[[\boldsymbol{x}]])$ are of the form*

$$\delta_{\boldsymbol{\alpha}} = \boldsymbol{x}^{\boldsymbol{\alpha}} + F_{\geq \lambda_{\boldsymbol{W}}(\boldsymbol{\alpha})} \in F_{\geq \lambda_{\boldsymbol{W}}(\boldsymbol{\alpha})}/F_{>\lambda_{\boldsymbol{W}}(\boldsymbol{\alpha})},$$

that is $\delta_{\boldsymbol{\alpha}}$ is the residue class of the monomial $\boldsymbol{x}^{\boldsymbol{\alpha}}$ of $K[[\boldsymbol{x}]]$ modulo the ideal $F_{>\lambda_{\boldsymbol{W}}(\boldsymbol{\alpha})}$. Considering Remark 2.1.21 and Lemma 2.1.22, the multiplication on $gr_{\boldsymbol{W}}(K[[\boldsymbol{x}]]$ is defined as follows:

$$\delta_{\boldsymbol{\alpha_1}} \cdot \delta_{\boldsymbol{\alpha_2}} := \begin{cases} \delta_{\boldsymbol{\alpha_1}+\boldsymbol{\alpha_2}} & , \ \textit{if for some } \boldsymbol{w} \in \boldsymbol{W},\ \lambda_{\boldsymbol{W}}(\boldsymbol{\alpha_1}) = \lambda_{\boldsymbol{w}}(\boldsymbol{\alpha_1}) \\ & \quad \textit{and } \lambda_{\boldsymbol{W}}(\boldsymbol{\alpha_2}) = \lambda_{\boldsymbol{w}}(\boldsymbol{\alpha_2}), \\ 0 & , \ \textit{otherwise.} \end{cases}$$

Following [GrP02, 5.5.10], we define the initial ideal of I associated to $\boldsymbol{W}$.

Definition 2.1.46. *Let $\boldsymbol{W}$ be a finite set of weights.*

1. *For $f = \sum_{\boldsymbol{\alpha}} a_{\boldsymbol{\alpha}}\boldsymbol{x}^{\boldsymbol{\alpha}} \in K[[\boldsymbol{x}]]$ such that $d := v_{\boldsymbol{W}}(f)$, we call*

$$In_{\boldsymbol{W}}(f) := \sum_{\boldsymbol{W}-deg(\boldsymbol{x}^{\boldsymbol{\alpha}})=d} a_{\boldsymbol{\alpha}}\boldsymbol{x}^{\boldsymbol{\alpha}}$$

   the **initial form** *of f with respect to $\boldsymbol{W}$.*

2. *Let $I \subset K[[\boldsymbol{x}]]$ be an ideal. The ideal*

$$In_{\boldsymbol{W}}(I) := \langle In_{\boldsymbol{W}}(f) :\ f \in I \setminus \{0\}\rangle \subset K[\boldsymbol{x}]$$

   is called the **initial ideal** *of I with respect to $\boldsymbol{W}$.*

Notation 2.1.47. *If $\boldsymbol{W}$ contains only one weight, then for $f \in K[[\boldsymbol{x}]]$, we denote $In_{\boldsymbol{W}}(f) := In(f)$.*

Remark 2.1.48.

1. *$In_{\boldsymbol{W}}(f)$ is a (PH) polynomial of type $(\boldsymbol{W}\,,\, v_{\boldsymbol{W}}(f))$.*
2. *It is of interest to note that in general $In_{\boldsymbol{W}}(fg) \neq In_{\boldsymbol{W}}(f)In_{\boldsymbol{W}}(g)$ while the equality holds when the set $\boldsymbol{W}$ contains only one weight.*

Example 2.1.49. *Let $\boldsymbol{W} = \{(1,2),(3,1)\}$. We consider $f = x^5 + xy^2 + y^5$ and $g = x^7 + y^7$. It is clear that f is (PH) of type $(\boldsymbol{W}\,;5)$ and g is is (PH) of type $(\boldsymbol{W}\,;7)$. Moreover $f = In_{\boldsymbol{W}}(f)$ and $g = In_{\boldsymbol{W}}(g)$. But $In_{\boldsymbol{W}}(fg) = x^{12} + x^8y^2 + xy^9 + y^{12} \neq In_{\boldsymbol{W}}(f)In_{\boldsymbol{W}}(g)$.*

Proposition 2.1.50. *Let $I \subset K[[\boldsymbol{x}]]$ be an ideal and let $\boldsymbol{W}$ be a finite set of weights in $\mathbb{Z}^n_{>0}$. Then*

$$gr_{\boldsymbol{W}}(K[[\boldsymbol{x}]]/I) \cong \bigoplus_{d\geq 0} K[\boldsymbol{x}]_d/In_{\boldsymbol{W}}(I)_d$$

as K-vector spaces.

Proof. Let $d \in \mathbb{Z}_{>0}$, we define the following two K-vector spces:
$K[\boldsymbol{x}]_d := \langle \boldsymbol{x}^{\boldsymbol{\alpha}} :\ v_{\boldsymbol{W}}(\boldsymbol{x}^{\boldsymbol{\alpha}}) = d\rangle_K$ and $In_{\boldsymbol{W}}(I)_d := In_{\boldsymbol{W}}(I) \cap K[\boldsymbol{x}]_d$.
Cleary, we have $K[\boldsymbol{x}]_d \cong F_{\geq d}/F_{>d}$.
Hence, we can consider the canonical K-linear surjection

$$\begin{array}{rccl} \varphi_d :& K[\boldsymbol{x}]_d & \twoheadrightarrow & F_{\geq d}/(F_{>d} + (F_{\geq d} \cap I)) \\ & f & \mapsto & \bar{f}. \end{array}$$

In the following, we show that $In_{\boldsymbol{W}}(I)_d = Ker(\varphi_d)$. First, let $f \in I$ be such that $v_{\boldsymbol{W}}(f) = d$. Thus, we can write $f = In_{\boldsymbol{W}}(f) + g$, with $g \in F_{>d}$.
Hence, $f - g \in F_{>d} + (F_{\geq d} \cap I)$ and therefore $\varphi_d(In_{\boldsymbol{W}}(f)) = 0$. This yields $In_{\boldsymbol{W}}(I)_d \subset Ker(\varphi_d)$. On the other hand, let $f \in K[\boldsymbol{x}]_d$ be such that $\varphi_d(f) = 0$. Then $f \in F_{>d} + (F_{\geq d} \cap I)$, that is there exist $g \in F_{>d}$ and $h \in F_{\geq d} \cap I$ with $f = g + h$. But $f \in K[\boldsymbol{x}]_d$ and $g \in F_{>d}$ yield to $f = In_{\boldsymbol{W}}(h) \in In_{\boldsymbol{W}}(I)_d$ and thus the inclusion $In_{\boldsymbol{W}}(I)_d \supset Ker(\varphi_d)$ follows.
So φ_d is an isomorphism of K-vector spaces. Hence, we have

$$K[\boldsymbol{x}]_d/In_{\boldsymbol{W}}(I)_d \cong F_{\geq d}/(F_{>d} + (F_{\geq d} \cap I))$$

and the K-vector space isomorphism

$$\bigoplus_{d\geq 0} K[\boldsymbol{x}]_d/In_{\boldsymbol{W}}(I)_d \cong gr_{\boldsymbol{w}}(K[[\boldsymbol{x}]]/I)$$

clearly follows.

□

Using the computer algebra system SINGULAR, the computation of the initial ideal is almost immediate if we deal with only one weight as the following lemma shows.

Lemma 2.1.51. *Let $I \subset \mathfrak{m}$ be an ideal, and let let $\boldsymbol{w} \subset \mathbb{Z}^n_{>0}$. Further, let $\{f_1, \ldots, f_s\}$ be a standard basis of I with respect to a local weighted ordering associated to $\boldsymbol{w}$. Then*

$$In(I) = \langle In(f_1), \ldots, In(f_s)\rangle.$$

Proof. The proof that we shall give is similar to the one given in [GrP02, 5.5.11].
Let $f \in I$. For $\{f_1, \ldots, f_s\}$ is a standard basis of I with respect to a $\boldsymbol{w}$- local weighted ordering, there exist a unit $u \in K[[\boldsymbol{x}]]^*$ and $g_1, \ldots, g_s \in K[[\boldsymbol{x}]]$ such that

$$uf = \sum_{i=1}^{s} g_i f_i \quad \text{and} \quad v_{\boldsymbol{w}}(In(uf)) \leq v_{\boldsymbol{w}}(In(g_i f_i)),$$

for all i. Now, let

$$N := \{1 \leq i \leq s : \ v_{\boldsymbol{w}}(In(uf)) = v_{\boldsymbol{w}}(In(g_i f_i))\}\,.$$

Finally Remark 2.1.21 yields

$$In(f) = \sum_{i \in N} In(g_i) In(f_i).$$

□

Remark 2.1.52. *In general, Lemma 2.1.51 fails when the finite set of weights $\boldsymbol{W}$ contains more than one element, for it is not possible to construct a monomial ordering which is compatible with the piecewise ordering $v_{\boldsymbol{W}}$. Indeed, let for example $\boldsymbol{W} = \{(1,2)\,,(3,1)\}$. Although $v_{\boldsymbol{W}}(xy^2) = 5 > v_{\boldsymbol{W}}(x^4) = 4$, we have $v_{\boldsymbol{W}}(y^2 \cdot xy^2) = 7 < v_{\boldsymbol{W}}(y^2 \cdot x^4) = 8$.*

In the last part of this subsection, we shall investigate (piecewise-homogeneous) graded algebroid singularities associated to zero dimensional ideals of $K[[\boldsymbol{x}]]$

Proposition 2.1.53. *Let $\boldsymbol{W} \subset \mathbb{Z}^n_{>0}$ be a finite set of weights and let $I \subset K[[\boldsymbol{x}]]$ be a proper ideal of $K[[\boldsymbol{x}]]$. If $dim_K(K[[\boldsymbol{x}]]/I)$ is finite, then $dim_K(gr_{\boldsymbol{W}}(K[[\boldsymbol{x}]]/I))$ is also finite.*

Proof. $K[[\boldsymbol{x}]]/I$ is a finite dimensional vector space means that the Krull dimension of the K-algebra $K[[\boldsymbol{x}]]/I$ is zero. Hence, there exists $k \in \mathbb{Z}_{>0}$ such that $\mathfrak{m}^k \subset I$ and thus there is a d_0 such that $F_{\geq d} \subset F_{\geq d_0} \subset \mathfrak{m}^k \subset I$ for any $d \geq d_0$. But then

$$(I \cap F_{\geq d}) + F_{>d} = F_{\geq d}$$

for $d \geq d_0$, and hence

$$gr_{\boldsymbol{W}}(K[[\boldsymbol{x}]]/I) = \bigoplus_{d=0}^{d_0} F_{\geq d}/(F_{>d} + (F_{\geq d} \cap I)).$$

It thus suffices to see that each $F_{\geq d}/(F_{>d} + (F_{\geq d} \cap I))$ has finite dimension. However, there is an integer m such that $\mathfrak{m}^m \subset F_{>d} \subset (F_{\geq d} \cap I) + F_{>d}$, so that the dimension is bounded by $dim_K(F_{\geq d}/\mathfrak{m}^m)$ which is clearly finite. □

Corollary 2.1.54. *Let $\boldsymbol{W} \subset \mathbb{Z}^n_{>0}$ be a finite set of weights and let $I \subset K[[\boldsymbol{x}]]$ be a proper ideal of $K[[\boldsymbol{x}]]$. If $dim_K(K[[\boldsymbol{x}]]/I)$ is finite, then there exists an epimorphism of K-vector spaces*

$$gr_{\boldsymbol{W}}(K[[\boldsymbol{x}]]/I) \twoheadrightarrow K[[\boldsymbol{x}]]/I.$$

Proof. By Proposition 2.1.53, we know that $dim_K(gr_{\boldsymbol{W}}(K[[\boldsymbol{x}]]/I))$ is finite. Hence, we can write the graded K-algebra as a finite sum

$$\bigoplus_{d=0}^{m} F_{\geq d}/(F_{>d} + (F_{\geq d} \cap I)).$$

For $d \in \mathbb{Z}_{>0}$, it is clear that the monomials of $\boldsymbol{W}$-degree precisely d generate the K-space $F_{\geq d}/F_{>d}$. Thus, their images in $F_{\geq d}/(F_{>d}+F_{\geq d}\cap I)$ span this linear space. Hence, a set of monomials $\{e_{\boldsymbol{\alpha}} : \boldsymbol{\alpha} \in \Lambda\}$ is a basis of $gr_{\boldsymbol{W}}(K[[\boldsymbol{x}]]/I)$ if for each value d of $v_{\boldsymbol{W}}$ lying between 0 and m, those $e_{\boldsymbol{\alpha}}$ of $\boldsymbol{W}$-degree precisely d are independant modulo the ideal $F_{>d} + F_{\geq d} \cap I$.
Let $\mathcal{B} = \{e_{\boldsymbol{\alpha}} : \boldsymbol{\alpha} \in \Lambda\}$ be a basis of $gr_{\boldsymbol{W}}(K[[\boldsymbol{x}]]/I)$ consisting of monomials.
We claim that the set $\{e_{\boldsymbol{\alpha}} \text{ mod}(I) : e_{\boldsymbol{\alpha}} \in \mathcal{B}\}$ span the linear space $K[[\boldsymbol{x}]]/I$.

Indeed, let $g \in K[[\boldsymbol{x}]]$ such that $v_{\boldsymbol{W}}(g) = d$. We write $g = g_d + g_{>d}$ where g_d is (PH) of $\boldsymbol{W}$-degree equal to d and $g_{>d} \in F_{>d}$.
Let $\{e_{\boldsymbol{\alpha}} : \boldsymbol{\alpha} \in \Lambda_d\}$ be the subset of monomials of $\mathcal{B}$ of $\boldsymbol{W}$-degree precisely d. Then, we can write

$$g_d = \sum_{\boldsymbol{\alpha} \in \Lambda_d} c_{\boldsymbol{\alpha}} e_{\boldsymbol{\alpha}} + h + h_1,$$

where the coefficients $c_{\boldsymbol{\alpha}}$ are in K, $h \in F_{\geq d} \cap I$ and $h_1 \in F_{>d}$.
Therefore, it follows clearly that

$$g\ mod(I)\ = \sum_{\boldsymbol{\alpha} \in \Lambda_d} c_{\boldsymbol{\alpha}}(e_{\boldsymbol{\alpha}}\ mod(I)) + h_1\ mod(I).$$

If we denote $d_1 = v_{\boldsymbol{W}}(h_1)$, then we have clearly $d_1 > d$ and using the same considerations as for g_d leads to

$$g\ mod(I)\ = \sum_{\boldsymbol{\alpha} \in \Lambda_d \cup \Lambda_{d_1}} c_{\boldsymbol{\alpha}}(e_{\boldsymbol{\alpha}}\ mod(I)) + h_2\ mod(I),$$

where $\{e_{\boldsymbol{\alpha}} : \boldsymbol{\alpha} \in \Lambda_{d_1}\}$ is the subset of monomials of $\mathcal{B}$ with $\boldsymbol{W}$-degree precisely d_1 and $h_2 \in F_{>d_1}$. As the ideal I is zero dimensional, we see clearly that the claim follows after finitely many iterations. ☐

Remark 2.1.55. *Let $\boldsymbol{W} \subset \mathbb{Z}^n_{>0}$ be a finite set of weights and let $f \in \mathfrak{m} \subset K[[\boldsymbol{x}]]$. We consider $M_f := K[[\boldsymbol{x}]]/j(f)$ and $T_f := K[[\boldsymbol{x}]]/I$ the Milnor algebra and the Tjurina algebra of f respectively. Hence, if $\mu(f) < \infty$ (resp. $\tau(f) < \infty$), then it follows by Proposition 2.1.53*

$$\mu(f) \leq dim_K(gr_{\boldsymbol{W}}(M_f)) < \infty,$$

$$(resp.\ \ \tau(f) \leq dim_K(gr_{\boldsymbol{W}}(T_f)) < \infty).$$

2.1.4 Toric Varieties and C-polytopes

In the last part of the present section, we shall discuss how we associate to any C-polytope a toric variety.
Let K be an algebraically closed field of positive characteristic.
We write K^* for the set $K \setminus \{0\}$.
We denote the ring of Laurent polynomials $K[x_1, {x_1}^{-1}, x_2, {x_2}^{-1}, \ldots, x_n, {x_n}^{-1}]$ by $K[\boldsymbol{x}, \boldsymbol{x}^{-1}]$.
Furthermore, we consider the algebraic torus

$$(\mathbb{K}^*)^n := Spec(K[\boldsymbol{x}, \boldsymbol{x}^{-1}]).$$

Definition 2.1.56. *A toric variety is an irreducible algebraic variety X over K equipped with an action of an algebraic torus $(\mathbb{K}^*)^n$ having an open dense orbit.*

For the sequel, let $\boldsymbol{P} \subset \mathbb{R}^n_{>0}$ be a C-polytope and let

$$\boldsymbol{W} := \boldsymbol{W_P} = \{w_\Delta : \Delta \text{ facet of } \boldsymbol{P}\}$$

be the finite set of weights associated to $\boldsymbol{P}$ (cf. Remark 2.1.17). Furthermore, let $\mathcal{L}_{\boldsymbol{P}} = \{\lambda_\Delta : \Delta \text{ facet of } \boldsymbol{P}\}$ be the set of linear functions associated to $\boldsymbol{P}$.
Following Wall in [Wal99a], we shall use the following notation:

Notation 2.1.57. *For any face Δ of $\boldsymbol{P}$,*

1. *we write $P[\Delta]$ for the cone over Δ (with base 0),*
2. *we denote*
$$R_\Delta := \{f \in K[\boldsymbol{x}] : supp(f) \subset P[\Delta]\}$$
for the ring spanned by the monomials which correspond to the lattice points of $P[\Delta]$, and finally
3. *we write M_Δ for the semigroup $Mon(R_\Delta)$ of monomials in R_Δ*

Remark 2.1.58. *Let $\boldsymbol{\alpha}$ be a lattice point in $\mathbb{Z}^n_{\geq 0}$. Then, it is easy to notice that*

$$\boldsymbol{x}^{\boldsymbol{\alpha}} \in M_\Delta \Longleftrightarrow v_{\boldsymbol{W}}(\boldsymbol{x}^{\boldsymbol{\alpha}}) = v_\Delta(\boldsymbol{x}^{\boldsymbol{\alpha}}) \Longleftrightarrow \lambda_{\boldsymbol{W}}(\boldsymbol{\alpha}) = \lambda_\Delta(\boldsymbol{\alpha}).$$

We summarize the main properties of the ring R_Δ in the following proposition due to Kouchnirenko.

Proposition 2.1.59. *Let $\boldsymbol{P}$ be a C-polytope in $\mathbb{R}^n_{>0}$ and let Δ be any face of $\boldsymbol{P}$. Then,*

1. *R_Δ is a graded Cohen-Macaulay ring.*
2. *Any inclusion $\delta \subset \Delta$ of faces of $\boldsymbol{P}$ induces an epimorphism*
$$\pi_{\Delta,\delta} : \quad R_\Delta \twoheadrightarrow R_\delta.$$

Proof. See [Kou76]. $\square$

Remark 2.1.60. *Let $\boldsymbol{P}$ be a C-polytope in $\mathbb{R}^n_{>0}$ and let $\boldsymbol{W} := \boldsymbol{W}_{\boldsymbol{P}}$ be the finite set of weights associated to $\boldsymbol{P}$. Further, let Δ be a facet of $\boldsymbol{P}$.*

1. *If we denote by R the graded K-algebra $gr_{\boldsymbol{W}}(K[[\boldsymbol{x}]])$, then it is clear that we can identify the ring R_Δ with a subring of R. Moreover, the grading of R_Δ is induced by the one of R. More precisely, it is induced by the linear function $\lambda_\Delta \in \mathcal{L}_{\boldsymbol{P}}$. Hence, for any inclusion $\delta \subset \Delta$, we see easily that the grading on R_δ is induced by the restriction of λ_Δ on the cone $P[\delta]$.*

2. *Let*
$$J_\Delta = \bigoplus\nolimits_{\boldsymbol{\alpha} \notin P[\Delta]} K \cdot \delta_{\boldsymbol{\alpha}} \subset R,$$
where $\delta_{\boldsymbol{\alpha}} = \boldsymbol{x}^{\boldsymbol{\alpha}} + F_{>\lambda_{\boldsymbol{W}}(\boldsymbol{\alpha})}$ is the residue class of the monomial $\boldsymbol{x}^{\boldsymbol{\alpha}}$ modulo the ideal $F_{>\lambda_{\boldsymbol{W}}(\boldsymbol{\alpha})}$. It is easy to see that J_Δ is an ideal of R. Furthermore, we have obviously that
$$R_\Delta \cong R/J_\Delta.$$
Hence, we have clearly an epimorphism of K-algebras
$$\pi_\Delta : \quad R \longrightarrow R_\Delta.$$
On the other hand, R is clearly isomorphic to the polynomial ring $K[\boldsymbol{x}]$ for it is generated by monomials. Then, it follows that R_Δ is a finitely generated K-algebra.

3. *For any inclusion $\delta \subset \Delta$ of faces and with the notations of Proposition 2.1.59, we have*
$$\pi_\delta = \pi_{\Delta,\delta} \circ \pi_\Delta.$$

Before giving the main proposition of this subsection, let us fix some notations following Wall in [Wal99a].

Notation 2.1.61. *Let $\boldsymbol{P}$ be a C-polytope and let Δ be any face of $\boldsymbol{P}$, then we write*
$$T_\Delta := Spec(R_\Delta)$$
for the affine spectrum associated to R_Δ.

Proposition 2.1.62. *Let $\boldsymbol{P}$ be a C-polytope in $\mathbb{R}^n_{\geq 0}$ and let Δ be a face of $\boldsymbol{P}$. Then, the affine spectrum $T_\Delta := Spec(R_\Delta)$ of R_Δ is a toric variety. Furthermore, $(\mathbb{K}^*)^n$ acts on T_Δ with one orbit corresponding to each face of Δ.*

Proof. For the proof, we use the analogy that exists with the well-known case where ($K = \mathbb{C}$) and we quote for example [GKZ94] and [Wal99a].
Let $\boldsymbol{P}$ be a C-polytope in $\mathbb{R}^n_{>0}$ and let Δ be a face of $\boldsymbol{P}$. Further, let $\boldsymbol{W} := \boldsymbol{W}_{\boldsymbol{P}}$ be the finite set of weights associated to $\boldsymbol{P}$.
Clearly, we can consider R_Δ as a subring of $K[x_1, {x_1}^{-1}, \ldots, x_n, {x_n}^{-1}]$. Then, it follows that the image of the associated map $(\mathbb{K}^*)^n \longrightarrow T_\Delta$ is dense in T_Δ.
On the other hand, each point ξ of T_Δ corresponds to a ring homomorphism
$$\phi_\xi : \quad R_\Delta \longrightarrow K.$$

Hence, we have clearly the following action of $(\mathbb{K}^*)^n$ on T_Δ.

$$\begin{array}{rccl} \chi_\Delta: & (\mathbb{K}^*)^n \times T_\Delta & \longrightarrow & T_\Delta \\ & (\boldsymbol{\mu} \ , \ \xi) & \mapsto & \chi_\Delta(\boldsymbol{\mu}, \xi) := \boldsymbol{\mu} \cdot \xi, \end{array}$$

where for $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_n)$, the point $\boldsymbol{\mu} \cdot \xi$ corresponds to the ring homomorphism $\boldsymbol{\mu} \cdot \phi_\xi : \ R_\Delta \longrightarrow K$ defined by

$$(\boldsymbol{\mu} \cdot \phi_\xi)(f(x_1, \ldots, x_n)) = f(\mu_1 x_1, \ldots, \mu_n x_n), \quad f \in R_\Delta.$$

On the other hand, we have by Proposition 2.1.59 that any inclusion $\delta \subset \Delta$ of faces induces an epimorphism $R_\Delta \twoheadrightarrow R_\delta$ and therefore an inclusion $T_\delta \hookrightarrow T_\Delta$. Thus the subset of T_Δ given by

$$U_\Delta := T_\Delta \setminus \bigcup \{T_\delta : \quad \delta \ \text{ is a proper face of } \ \Delta\}$$

is open in T_Δ.
Besides, U_Δ can be characterized as follows:
$\xi \in U_\Delta$, if and only if, ϕ_ξ defines a homomorphism from the semigroup of monomials M_Δ to K^*.
For a proof of this intermediate result we refer to [Wal99a] since the arguments used there are independent of the characteristic.
Furthermore, if $\xi \in U_\Delta$, then it turns out that the corresponding homomorphism ϕ_ξ is induced by evaluating on a point $\boldsymbol{r}_\xi \in (\mathbb{K}^*)^n$. Indeed, ϕ_ξ can be extended (non-uniquely) to a homomorphism $\tilde{\phi}_\xi : \mathbb{Z}^n \to K^*$. Moreover let $\boldsymbol{r}_\xi$ be the point of $(\mathbb{K}^*)^n$ with corrdinates

$$r_{\xi,i} := \tilde{\phi}(\epsilon_i),$$

where for $i = 1, \ldots, n$,

$$\begin{array}{rccccccc} & & & & i & & & \\ \epsilon_i = & (0 & \ldots & 0, & 1, & 0 & \ldots & 0). \end{array}$$

Thus, we see easily that for any monomial $m = \boldsymbol{x}^{\boldsymbol{\alpha}} \in M_\Delta$, we have

$$\phi_\xi(\boldsymbol{x}^{\boldsymbol{\alpha}}) = \boldsymbol{r}^{\boldsymbol{\alpha}}.$$

Then, it follows that there is a surjective homomorphism

$$(\mathbb{K}^*)^n \twoheadrightarrow U_\Delta,$$

and therefore U_Δ is dense in T_Δ.
Finally, we shall show that U_Δ corresponds to one orbit of the action χ_Δ.
Let $\xi \in U_\Delta$, we denote by O_ξ the orbit of the point ξ under the action χ_Δ, that is

$$O_\xi := \{\boldsymbol{\mu} \cdot \phi_\xi : \quad \boldsymbol{\mu} \in (\mathbb{K}^*)^n\}.$$

Further, let $\boldsymbol{r}_\xi$ be the a point of $(\mathbb{K}^*)^n$ corresponding to the homomorphism ϕ_ξ. Hence, by definition of the action χ_Δ and the characterization of ϕ_ξ via $\boldsymbol{r}_\xi$, we have for any monomial $\boldsymbol{x}^{\boldsymbol{\alpha}} \in M_\Delta$

$$\begin{array}{rcl} \boldsymbol{\mu} \cdot \phi_\xi(\boldsymbol{x}^{\boldsymbol{\alpha}}) & = & \phi_\xi(\boldsymbol{\mu}^{\boldsymbol{\alpha}} \boldsymbol{x}^{\boldsymbol{\alpha}}) \\ & = & \boldsymbol{\mu}^{\boldsymbol{\alpha}} \boldsymbol{r}_\xi{}^{\boldsymbol{\alpha}} \\ & = & (\boldsymbol{\mu} \boldsymbol{r}_\xi)^{\boldsymbol{\alpha}}. \end{array}$$

Hence, the homomorphism $\boldsymbol{\mu} \cdot \phi_\xi$ takes its values in K^* and therefore it corresponds to a point of U_Δ. This shows the inclusion

$$O_\xi \subset U_\Delta.$$

Conversely, let ξ' be an arbitrary point of U_Δ. Then, the corresponding homomorphism $\phi_{\xi'}$ is induced by evaluating on a point $\boldsymbol{r}_{\xi'} \in (\mathbb{K}^*)^n$. Further, let $\boldsymbol{\mu}_{\xi,\xi'} := \boldsymbol{r}_{\xi'} \boldsymbol{r}_\xi{}^{-1} \in (\mathbb{K}^*)^n$ and let $\boldsymbol{x}^{\boldsymbol{\alpha}} \in M_\Delta$ be a monomial. Then, we have

$$\begin{array}{rcl} \phi_{\xi'}(\boldsymbol{x}^{\boldsymbol{\alpha}}) & = & \boldsymbol{r}_{\xi'}{}^{\boldsymbol{\alpha}} \\ & = & \boldsymbol{\mu}_{\xi,\xi'} \boldsymbol{r}_\xi{}^{\boldsymbol{\alpha}} \\ & = & \boldsymbol{\mu}_{\xi,\xi'} \cdot \phi_\xi(\boldsymbol{x}^{\boldsymbol{\alpha}}). \end{array}$$

Hence, $\xi' \in O_\xi$ and therefore $U_\Delta \subset O_\xi$.
Thus, $U_\Delta = O_\xi$ and similarly each face of Δ corresponds to one orbit of the action χ_Δ.

□

Corollary 2.1.63. *Let $\boldsymbol{P}$ be a C-polytope in $\mathbb{R}^n_{>0}$ such that the associated set of weights $\boldsymbol{W}_{\boldsymbol{P}}$ is a subset of $\mathbb{Q}^n_{>0}$ and let Δ be a face of $\boldsymbol{P}$. Then, the projective spectrum $Proj(R_\Delta)$ of the graded ring R_Δ is a toric variety.*

Proof. Let $\boldsymbol{P}$ be a C-polytope in $\mathbb{R}^n_{>0}$ and let Δ be a face of $\boldsymbol{P}$. Further, let λ_Δ be the linear function of $\mathcal{L}_{\boldsymbol{P}}$ associated to Δ with $\lambda_\Delta = \langle w_\Delta \,, \cdot \rangle$ and $w_\Delta = (w_1, \ldots, w_n) \in \mathbb{Q}^n_{>0}$. Then, it is a well-known fact that $Proj(R_\Delta)$ can be considered as the quotient by K^* (with action induced by λ_Δ) of the toric variety T_Δ. See [Wal99a].
More precisely, let $N \in \mathbb{Z}_{>0}$ be such that $Nw_\Delta \in \mathbb{Z}^n_{>0}$. On the other hand, we recall that in the proof of Theorem 2.1.62, we associate to any point $\xi \in T_\Delta$ a morphism of rings

$$\phi_\xi : \quad R_\Delta \longrightarrow K$$

and we define an action χ_Δ of $(\mathbb{K}^*)^n$ on T_Δ. Then, we get an action of $\mathbb{K}^*$ on T_Δ as follows:

$$\begin{array}{ccc} \mathbb{K}^* \times T_\Delta & \longrightarrow & T_\Delta \\ (t \;\;,\;\; \xi) & \mapsto & \chi_\Delta((t^{Nw_1}, \ldots, t^{Nw_n}) \,, \phi_\xi). \end{array}$$

Moreover, we have

$$Proj(R_\Delta) \cong T_\Delta / \mathbb{K}^*.$$

Thus, the claim follows. □

2.2 C-Polytopes and derivations

In the following, we define the filtred order of a K-derivation with respect to a finite set $\boldsymbol{W}$ of weights.
For this purpose, we associate to any derivation of the form $\boldsymbol{x}^{\boldsymbol{\alpha}}\partial_{x_i}$ the n-tuple $\boldsymbol{\alpha} - \epsilon_i$ of $\mathbb{Z}^n_{\geq 0}$ obtained from $\boldsymbol{\alpha}$ by decreasing the coordinate α_i by 1.

Definition 2.2.1. *1. Let $\xi = g\partial_{x_i}$ where $i \in \{1, \ldots, n\}$ and $g \in K[[\boldsymbol{x}]]$.*

$$\boldsymbol{W}\text{-}ord(\xi) := v_{\boldsymbol{W}}(\xi) := min\{\lambda_{\boldsymbol{W}}(\boldsymbol{\alpha} - \epsilon_i) : \quad \alpha \in supp(g)\}$$

is called the piecewise-homogeneous order or the $\boldsymbol{W}$-order of ξ.

2. More generally, we define the $\boldsymbol{W}$-order of a derivation $\xi = \sum_{i=1}^n g_i\partial_{x_i}$ as follows

$$\boldsymbol{W}\text{-}ord(\xi) := v_{\boldsymbol{W}}(\xi) := min\{v_{\boldsymbol{W}}(g_i\partial_{x_i}) : \quad 1 \leq i \leq n\}.$$

We set $\boldsymbol{W}$-ord$(0) := \infty$.

Remark 2.2.2. *Let $\xi = \sum_{i=1}^n g_i\partial_{x_i} \in Der_K(K[[\boldsymbol{x}]])$ and let $\boldsymbol{W}$ be a finite set of weights. Then, for $i = 1, \ldots, n$, we have*

$$v_{\boldsymbol{W}}(g_i) \geq v_{\boldsymbol{W}}(\xi) + \lambda_{\boldsymbol{W}}(\epsilon_i).$$

Indeed, for i, $1 \leq i \leq n$, let $\boldsymbol{\alpha} \in supp(g_i)$. As the functions $\lambda_{\boldsymbol{w}}$, $\boldsymbol{w} \in \boldsymbol{W}$, are linear, then we can write

$$\lambda_{\boldsymbol{w}}(\boldsymbol{\alpha}) = \lambda_{\boldsymbol{w}}(\boldsymbol{\alpha} - \epsilon_i) + \lambda_{\boldsymbol{w}}(\epsilon_i).$$

Hence by Definition of $\lambda_{\boldsymbol{W}}$, we have

$$\lambda_{\boldsymbol{w}}(\boldsymbol{\alpha}) \geq \lambda_{\boldsymbol{W}}(\boldsymbol{\alpha} - \epsilon_i) + \lambda_{\boldsymbol{W}}(\epsilon_i) \text{ and so}$$

$$\lambda_{\boldsymbol{W}}(\boldsymbol{\alpha}) \geq \lambda_{\boldsymbol{W}}(\boldsymbol{\alpha} - \epsilon_i) + \lambda_{\boldsymbol{W}}(\epsilon_i).$$

Thus, $v_{\boldsymbol{W}}(g_i) \geq v_{\boldsymbol{W}}(\xi) + v_{\boldsymbol{W}}(\epsilon_i)$ follows obviously from Definitions 2.1.18 and 2.2.1.

Lemma 2.2.3. *Let $f \in K[[\boldsymbol{x}]]$ and $\xi \in Der_K(K[[\boldsymbol{x}]])$. Further, let $\boldsymbol{W}$ be a finite set of weights. Then,*

$$v_{\boldsymbol{W}}(\xi f) \geq v_{\boldsymbol{W}}(\xi) + v_{\boldsymbol{W}}(f).$$

Before starting the proof, we would like to mention that also in characteristic zero, the equality does not necessarily hold as the following example shows.

Example 2.2.4. *Let $char(K) = 0$ and let $\boldsymbol{W} = \{(\frac{1}{7}, \frac{2}{7}), (\frac{1}{6}, \frac{1}{4})\}$.*
Further,let $\xi = x\partial_y \in Der_K(K[[x, y]])$ and let $f = y$.
Obviously, $\xi f = x$ and hence $v_{\boldsymbol{W}}(\xi f) = v_{\boldsymbol{W}}(x) = \frac{1}{7}$.
Clearly, we have $v_{\boldsymbol{W}}(f) = v_{\boldsymbol{W}}(y) = \frac{1}{4}$. On the other hand, we associate ξ to the point $(1, -1)$ and thus we have $v_{\boldsymbol{W}}(\xi) = -\frac{1}{7}$. Hence,

$$v_{\boldsymbol{W}}(\xi f) = \frac{1}{7} > \frac{3}{28} = v_{\boldsymbol{W}}(\xi) + v_{\boldsymbol{W}}(f).$$

Now, we give the proof of Lemma 2.2.3.

Proof. Let $f \in K[[\boldsymbol{x}]]$ and let $\xi \in Der_K(K[[\boldsymbol{x}]])$.
First, we suppose that ξ is of the form $\xi = g\partial_{x_i}$, where $i \in \{1, \ldots, n\}$ and $g \in K[[\boldsymbol{x}]]$. Hence, $\xi f = gf_{x_i}$ and we observe that if $f_{x_i} = 0$, then the lemma is trivial as we have $v_{\boldsymbol{W}}(0) = \infty$.
We assume $f_{x_i} \neq 0$, then we have

$$supp(\xi f) \subseteq \{\beta + \alpha - \epsilon_i : \quad \beta \in supp(g) \text{ and } \alpha \in supp(f)\}.$$

Hence, by Definition 2.1.18, we have

$$\begin{aligned} v_{\boldsymbol{W}}(\xi f) &= v_{\boldsymbol{W}}(gf_{x_i}) \\ &\geq min\,\{\lambda_{\boldsymbol{W}}(\beta + \alpha - \epsilon_i) : \beta \in supp(g) \text{ and } \alpha \in supp(f)\} \end{aligned}$$

On the other hand, we have for all $\boldsymbol{w} \in \boldsymbol{W}$

$$\begin{aligned} \lambda_j(\beta + \alpha - \epsilon_i) &= \lambda_j(\beta - \epsilon_i) + \lambda_j(\alpha) \\ &\geq \lambda_{\boldsymbol{W}}(\beta - \epsilon_i) + \lambda_{\boldsymbol{W}}(\alpha). \end{aligned}$$

Thus,

$$\lambda_{\boldsymbol{W}}(\beta + \alpha - \epsilon_i) \geq \lambda_{\boldsymbol{W}}(\beta - \epsilon_i) + \lambda_{\boldsymbol{W}}(\alpha).$$

This leads to

$$\begin{aligned} v_{\boldsymbol{W}}(\xi f) &\geq min\,\{\lambda_{\boldsymbol{W}}(\beta - \epsilon_i) + \lambda_{\boldsymbol{W}}(\alpha) : \quad \beta \in supp(g) \text{ and } \alpha \in supp(f)\} \\ &= v_{\boldsymbol{W}}(\xi) + v_{\boldsymbol{W}}(f). \end{aligned}$$

Now, suppose ξ is of the form $\xi = \sum_{i=1}^{n} g_i\partial_{x_i}$.
It follows by Definitions 2.1.18 and 2.2.1, that

1. $v_{\boldsymbol{W}}(\xi) = min\{v_{\boldsymbol{W}}(g_i\partial_{x_i}) : i = 1, \ldots, n\}$.
2. $v_{\boldsymbol{W}}(\xi f) = min\{v_{\boldsymbol{W}}(g_i f_{x_i}) : \quad i = 1, \ldots n\}$.

Moreover, it follows from the first part of our proof that for any i, $1 \leq i \leq n$, we have

$$\begin{aligned} v_{\boldsymbol{W}}(g_i f_{x_i}) &\geq v_{\boldsymbol{W}}(g_i\partial_{x_i}) + v_{\boldsymbol{W}}(f) \\ &\geq v_{\boldsymbol{W}}(\xi) + v_{\boldsymbol{W}}(f). \end{aligned}$$

Hence, $v_{\boldsymbol{W}}(\xi f) \geq v_{\boldsymbol{W}}(\xi) + v_{\boldsymbol{W}}(f)$ follows clearly. ☐

Following Arnold [Arn74 6.6], we give in the final part of the first section a technical lemma which we need later for the proof of our central theorem about normal forms. For this purpose, let again $\boldsymbol{W}$ be a finite set of weights.

Lemma 2.2.5. *Let $f \in \mathfrak{m}^2$ be a formal power series and let $\varphi \in Aut(K[[\boldsymbol{x}]])$ be an automorphism of the form $\varphi : \quad x_i \mapsto x_i + g_i, \quad i = 1, \ldots n$, such that*

$$v_{\boldsymbol{W}}(g_i) > v_{\boldsymbol{W}}(x_i),$$

for all $i = 1, \ldots, n$. *Further, let* $\boldsymbol{W}$ *be a finite set of weights. Then,*

$$\varphi(f) = f + \xi f + R,$$

where $\xi = \sum_{i=1}^{n} g_i \partial_{x_i} \in Der_K(K[[\boldsymbol{x}]])$ *and* $R \in K[[\boldsymbol{x}]]$ *satisfies*

$$v_{\boldsymbol{W}}(R) > v_{\boldsymbol{W}}(\xi) + v_{\boldsymbol{W}}(f).$$

Proof. We consider a finite set of weights $\boldsymbol{W}$. Further, let $f \in \mathfrak{m}^2$ and let $\varphi \in Aut(K[[\boldsymbol{x}]])$ be defined by
$\varphi: \quad x_i \mapsto x_i + g_i, \quad i = 1, \ldots n$, such that $v_{\boldsymbol{W}}(g_i) > v_{\boldsymbol{W}}(x_i)$ for all $i = 1, \ldots, n$.
We can assume by linearity of φ and the linearity of the action of a derivation on the set of power series that f is a monomial in $K[[\boldsymbol{x}]]$ and we write $f = x^{\boldsymbol{\alpha}}$, where $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n_{\geq 0}$. Thus, we have

$$\varphi(f) = (x_1 + g_1)^{\alpha_1} \ldots (x_n + g_n)^{\alpha_n}.$$

By developing the right hand side of the equation, we obtain

$$\begin{aligned}
\varphi(f) &= x^{\boldsymbol{\alpha}} + \sum_{k=1}^{n} \alpha_k g_k {x_1}^{\alpha_1} \ldots {x_k}^{\alpha_k - 1} \ldots {x_n}^{\alpha_n} + \\
&\quad \sum_{|\boldsymbol{m}|=2}^{|\boldsymbol{\alpha}|} \binom{\alpha_1}{m_1} \ldots \binom{\alpha_n}{m_n} {g_1}^{m_1} \ldots {g_n}^{m_n} x_1^{\alpha_1 - m_1} \ldots x_n^{\alpha_n - m_n} \\
&= f + \xi f + R,
\end{aligned}$$

where $\xi = \sum_{i=1}^{n} g_i \partial_{x_i} \in Der_K(K[[\boldsymbol{x}]])$ and
$R = \sum_{|\boldsymbol{m}|=2}^{|\boldsymbol{\alpha}|} \binom{\alpha_1}{m_1} \ldots \binom{\alpha_n}{m_n} {g_1}^{m_1} \ldots {g_n}^{m_n} x_1^{\alpha_1 - m_1} \ldots x_n^{\alpha_n - m_n}$.
If $R = 0$, then the claim of Lemma 2.2.5 follows obviously.
If $R \neq 0$, we denote for any $m \in \mathbb{Z}^n_{\geq 0}$ such that $2 \leq\mid \boldsymbol{m} \mid\leq\mid \boldsymbol{\alpha} \mid$,

$$R_{\boldsymbol{m}} := \binom{\alpha_1}{m_1} \ldots \binom{\alpha_n}{m_n} {g_1}^{m_1} \ldots {g_n}^{m_n} x_1^{\alpha_1 - m_1} \ldots x_n^{\alpha_n - m_n}.$$

Moreover, for $h \in K[[\boldsymbol{x}]]$ and for any $\boldsymbol{w} \in \boldsymbol{W}$, we write $v_{\boldsymbol{w}}(h)$ for the weighted order of h with respect to the weight $\boldsymbol{w}$. We have clearly from Definition 2.1.18

$$v_{\boldsymbol{W}}(h) = min\{v_{\boldsymbol{w}}(h) : \quad \boldsymbol{w} \in \boldsymbol{W}\}.$$

On the other hand, using Remark 2.2.2, we get for any $\boldsymbol{w} \in \boldsymbol{W}$

$$\begin{aligned}
v_{\boldsymbol{w}}(R_{\boldsymbol{m}}) &\geq \left(\sum_{i=1}^{n} m_i v_{\boldsymbol{w}}(g_i)) + \lambda_{\boldsymbol{w}}(\boldsymbol{\alpha} - \boldsymbol{m} \right) \\
&\geq \left(\sum_{i=1}^{n} m_i (v_{\boldsymbol{w}}(\xi) + \lambda_{\boldsymbol{w}}(\epsilon_i))) + \lambda_{\boldsymbol{w}}(\boldsymbol{\alpha} - \boldsymbol{m} \right) \\
&= \mid \boldsymbol{m} \mid v_{\boldsymbol{w}}(\xi) + \sum_{i=1}^{n} m_i \lambda_{\boldsymbol{w}}(\epsilon_i) + \lambda_{\boldsymbol{w}}(\boldsymbol{\alpha} - \boldsymbol{m}).
\end{aligned}$$

Clearly, $\lambda_{\boldsymbol{w}}(\boldsymbol{\alpha}-\boldsymbol{m}) = \sum_{i=1}^{n}(\alpha_i - m_i)\lambda_{\boldsymbol{w}}(\epsilon_i)$ follows by linearity of the function $\lambda_{\boldsymbol{w}}$. Therefore, we have

$$\begin{aligned} v_{\boldsymbol{w}}(R_{\boldsymbol{m}}) &\geq \mid \boldsymbol{m} \mid v_{\boldsymbol{w}}(\xi) + \sum_{i=1}^{n} \alpha_i \lambda_{\boldsymbol{w}}(\epsilon_i) \\ &= \mid \boldsymbol{m} \mid v_{\boldsymbol{w}}(\xi) + \lambda_{\boldsymbol{w}}(\boldsymbol{\alpha}) \end{aligned}$$

Then, it follows that

$$v_{\boldsymbol{W}}(R_{\boldsymbol{m}}) \geq \mid \boldsymbol{m} \mid v_{\boldsymbol{W}}(\xi) + v_{\boldsymbol{W}}(f).$$

As $\mid \boldsymbol{m} \mid \geq 2$, we obtain then

$$v_{\boldsymbol{W}}(R_{\boldsymbol{m}}) > v_{\boldsymbol{W}}(\xi) + v_{\boldsymbol{W}}(f).$$

On the other hand, by Remark 2.1.21 we have that,

$$v_{\boldsymbol{W}}(R) \geq min\{v_{\boldsymbol{W}}(R_{\boldsymbol{m}}) : \ 2 \leq \mid \boldsymbol{m} \mid \leq \mid \boldsymbol{\alpha} \mid\}.$$

Hence, the claim $v_{\boldsymbol{W}}(R) > v_{\boldsymbol{W}}(\xi) + v_{\boldsymbol{W}}(f)$ clearly follows. □

2.3 Non-Degenerate Hypersurface Singularities

In this section, we give the definitions of non-degeneracy. These are essentially standard, and were established amongst others in [Arn74] and [Kou76].
We recall that K is an algebraically closed field of arbitrary characteristic.
First, we shall fix some necassary notations for the sequel.

For a subset $I \subset \{1, \ldots, n\}$, we recall that

$$\mathbb{R}^I = \{\boldsymbol{x} \in \mathbb{R}^n : \, x_i = 0 \ \text{ if } \ i \notin I\} = \bigcap_{i \notin I} \{x_i = 0\} \, .$$

Hence, we have obviously:

- $\mathbb{R}^{\emptyset} = \{0\}$ and
- $\mathbb{R}^{\{1,\ldots,n\}} = \mathbb{R}^n$.

Similarly, we define

$$K^I = \{\boldsymbol{r} \in K^n : \, r_i = 0 \ \text{ if } \ i \notin I\} \, .$$

Notation 2.3.1. *If $f \in K[[\boldsymbol{x}]]$ and δ is a face of a C-polytope, then we write f_{δ, x_i} $(1 \leq i \leq n)$ for the partial derivations of the power series f_δ.*

In the following, we shall generalize Wall's definition of non-degeneracy for arbitrary characteristic. But before going into the details, we would like to notice that in [Kou76] Kouchnirenko defines non-degeneracy only with respect to *Newton polytopes* of *convenient* hypersurface singularities, while in [Wal99a], non-degeneracy is defined in the more general setting of arbitrary C-polytopes.

Definition 2.3.2. *Let $f \in K[[\boldsymbol{x}]]$ and let $\boldsymbol{P}$ be a C-polytope such that $supp(f)$ has no point below $\boldsymbol{P}$. Furthermore, let δ be any face of $\boldsymbol{P}$.*

1. *We say that f is* **non-degenerate** *or f satisfies $(ND1)$ with respect to δ if*

$$\{\boldsymbol{r} \in \mathbb{K}^n : \ f_{\delta,x_1}(\boldsymbol{r}) = \ldots = f_{\delta,x_n}(\boldsymbol{r}) = 0\} \subset \bigcup_{1 \le i \le n} \{x_i = 0\} \, .$$

   That is, there is no common zero of the f_{δ,x_i}, $1 \le i \le n$, in the open torus K^{*n}.

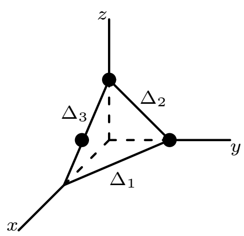
2. *f is called $(ND1)$ with respect to $\boldsymbol{P}$ if f is $(ND1)$ with respect to each face of $\boldsymbol{P}$.*

3. *If f is (CO) and moreover the condition $(ND1)$ holds for each face of the Newton polytope $\Gamma(f)$, then we say that f is $NPND$ (non-degenerate with respect to the Newton polytope).*

4. *A hypersurface singularity R is called $(ND1)$ with respect to δ (resp. $NPND$) if there exists $f \in K[[\boldsymbol{x}]]$ such that f is $(ND1)$ with respect to δ (resp. $NPND$) and moreover $R_f \cong R$.*

Remark 2.3.3. *Let $f \in K[[\boldsymbol{x}]]$ and let $\boldsymbol{P}$ be a C-polytope such that $supp(f)$ has no point below $\boldsymbol{P}$. It is of interest to notice that from Definition 2.3.2, we have clearly that f is $(ND1)$ with respect to a face δ of $\boldsymbol{P}$ (resp. with respect to $\boldsymbol{P}$), if and only if, $f_{\boldsymbol{P}}$ is $(ND1)$ with respect to δ (resp. with respect to $\boldsymbol{P}$).*

Example 2.3.4. *Let $char(K) = 0$ and we consider the A_1-singularity given by the equation $f = y^2 + xz + z^2 \in K[[x, y, z]]$. Further, we consider the C-polytope $\boldsymbol{P}$ in $\mathbb{R}^3_{\ge 0}$ which is the triangle with the vertices of coordinates $(2, 0, 0)$, $(0, 2, 0)$ and $(0, 0, 2)$.*



We observe that all points of $supp(f)$ lie on $\boldsymbol{P}$. On the other hand, $\boldsymbol{P}$ has the following inner faces:

- *The facet of the triangle wich is the whole C-polytope $\boldsymbol{P}$.*
- *The line segment $\Delta_1 = [(2, 0, 0), (0, 2, 0)]$.*
- *The line segment $\Delta_2 = [(0, 2, 0), (0, 0, 2)]$.*
- *The line segment $\Delta_3 = [(2, 0, 0), (0, 0, 2)]$.*

We show in the following, that A_1 satisfies $(ND1)$ at any inner face of $\boldsymbol{P}$. Indeed, we have

- $f_{\boldsymbol{P}} = y^2 + xz + z^2$, $f_{\boldsymbol{P},x} = z$, $f_{\boldsymbol{P},y} = 2y$ *and* $f_{\boldsymbol{P},z} = x + 2z$. *It follows clearly that* $(0,0,0)$ *is the unique common zero of the functions* $f_{\boldsymbol{P},x}$, $f_{\boldsymbol{P},y}$ *and* $f_{\boldsymbol{P},z}$.
- $f_{\Delta_1} = y^2$ *and hence, we see that any common zero of the functions* $f_{\Delta_1,x}$, $f_{\Delta_1,y}$ *and* $f_{\Delta_1,z}$ *lies on the coordinate hyperplane* $\{y = 0\}$.
- $f_{\Delta_2} = y^2 + z^2$. *Thus, we have* $f_{\Delta_2,x} = 0$, $f_{\Delta_2,y} = 2y$ *and* $f_{\Delta_2,z} = 2z$. *Then, it follows that any common zero of these functions lies on the intersection of the coordinate hyperplanes* $\{y = 0\}$ *and* $\{z = 0\}$.
- $f_{\Delta_3} = z^2 + xz$ *and as the above, we show that any singular point of this truncation lies on* $\{x = 0\} \cap \{z = 0\}$.

Then, it follows from Definition 2.3.2 that $(ND1)$ holds at each inner face of $\boldsymbol{P}$. However, if we consider the 0-dimensional face $\delta = \{(2,0,0)\}$ of $\boldsymbol{P}$ belonging to the x-axis, we have $f_\delta = 0$ and thus the property $(ND1)$ fails at this face.
Nevertheless, if we consider the triangle with vertices $(1,0,1), (0,2,0)$ and $(0,0,2)$, we see easily that it represents the Newton polytope $\Gamma(f)$ and moreover $(ND1)$ holds at any face of this compact polytope (which is not a C-polytope). Nevertheless, f is not $NPND$ as f is not (CO).

Lemma 2.3.5. *Let $f \in \mathfrak{m}^3 \subset K[[\boldsymbol{x}]]$ be (SQH) with principal part f_Δ having weighted degree d. If $char(K)$ does not devide d, then the hypersurface singularity $R_f = K[[\boldsymbol{x}]]/\langle f \rangle$ is $(ND1)$ with respect to Δ.*

Proof. By definition of semiquasihomogeneity, we have $\tau(f_\Delta) < \infty$. Moreover, as $char(K) \nmid d$, it follows by Lemma 2.1.33 that $\mu(f_\Delta) < \infty$. Hence, by Lemma 2.1.35

$$\{\boldsymbol{r} \in K^n : \ f_{\Delta,x_1}(\boldsymbol{r}) = \ldots = f_{\Delta,x_n} = 0\} = \{0\}\,.$$

Therefore, f is $(ND1)$ with respect to Δ and so R_f is also. □

Remark 2.3.6. *Let $f \in \mathfrak{m}^3 \subset K[\boldsymbol{x}]$ be (QH) of weighted degree d and moreover (CO). If $char(K)$ does not divide d, attention should be drawn to the following:*
The above Lemma 2.3.5 asserts only that f is $(ND1)$ with respect to the $(n-1)$-dimensional face of $\Gamma(f)$. To claim that f is $NPND$, we should show that f is $(ND1)$ with respect to each face δ of $\Gamma(f)$. The following example shows, that this is not necessarily the case.

Example 2.3.7. *Let $char(K) = 0$ and let $q \in \mathbb{Z}_{>0}$ be such that $q \geq 2$. Furthermore, let $g = (x+y)^q + x^{q-1}z + z^q \in K[x,y,z]$. Clearly g is homogeneous of degree q and $\tau(g)$ is finite. Moreover, let $\boldsymbol{P} \in \mathbb{R}^3_{\geq 0}$ be the convex hull of the points $\{(q,0,0),(0,q,0),(0,0,q)\}$. It is easy to see that $\boldsymbol{P} = \Gamma(g)$. Let Δ be the face of $\boldsymbol{P}$ which is the line segment of the (x,y)-hyperplane having the end points of coordinates $(q,0,0)$ and $(0,q,0)$. We consider the truncation $g_\Delta = (x+y)^q$. We have $g_{\Delta,x} = g_{\Delta,y} = q(x+y)^{q-1}$ and $g_{\Delta,z} = 0$. Thus*

$$\{\boldsymbol{r} \in K^n : \ g_{\Delta,x}(\boldsymbol{r}) = g_{\Delta,y}(\boldsymbol{r}) = g_{\Delta,z}(\boldsymbol{r}) = 0\} \subset \{x + y = 0\}\,.$$

So for example the point $(1,-1,1)$ *is a common zero of the partial derivations of* g_Δ *and therefore, g does not satisfy condition* $(ND1)$ *at the face* Δ*. We observe however that* g *satisfies* $(ND1)$ *at the unique facet of* $\Gamma(g)$*.*

Remark 2.3.8. *We would like to mention that the property* $(ND1)$ *is in general preserved neither under* $\mathcal{R}$*-actions nor under* $\mathcal{K}$*-actions. Indeed, let* $g = (x+y)^2 + xz + z2 \in K[[x,y,z]]$*. Let* Δ *be the line segment with the end points* $(2,0,0)$ *and* $(0,2,0)$*. In Example 2.3.7 we showed that* g *is not* $(ND1)$ *with respect to* Δ*. On the other hand, if we consider the following* K*-automorphism on* $K[[x,y,z]]$

$$\phi:\ x \mapsto x,\ y \mapsto x+y,\ z \mapsto z,$$

then we see easily that $g = \phi(f)$*, where* $f = y^2 + xz + z^2 \in K[[x,y,z]]$*. In Example 2.3.4 though, we showed that* f *is* $(ND1)$ *with respect to* Δ*.*

In his paper [Kou76] about Newton polytopes and Milnor numbers, Kouchnirenko establishes the following important property resulting from non-degeneracy.

Proposition 2.3.9. *Let* K *be an algebraically closed field of arbitrary characteristic and let* $f \in K[[\boldsymbol{x}]]$*. If* f *is* $NPND$*, then* f *has a finite Milnor number. Moreover,* $\mu(f) = \mu_N(f)$*.*

Proof. See [Kou76]. □

Remark 2.3.10. *Example 2.3.7 shows that the converse of Proposition 2.3.9 is not true in general. Indeed, for* $q \geq 2$*, the homogeneous polynomial* $g = (x+y)^q + x^{q-1}z + z^q \in K[[x,y,z]]$ *has finite Milnor number but is not* $NPND$ *since it is not* $(ND1)$ *with respect to a face of* $\Gamma(g)$*.*

In characteristic zero however, Kouchnirenko shows that the statement of Proposition 2.3.9 does also hold for non-degenerate elements wich are not necessary convenient (CO).

Proposition 2.3.11. *Let* K *be an algebraically closed field of characteristic zero and let* $f \in K[[\boldsymbol{x}]]$*. If* f *satisfies* $(ND1)$ *at each face of the Newton polytope* $\Gamma(f)$*, then* $\mu(f)$ *is finite and* $\mu(f) = \mu_N(f)$*.*

Proof. See [Kou76]. □

In his paper [Wall99a] on Newton polytopes and non-degeneracy, Wall manages to establish on the field $\mathbb{C}$ a condition of non-degeneracy which includes the case of all semiquasihomogeneous hypersurface singularities and where the principal results proved in [Kou76] still hold. Wall calls this condition **strict non-degeneracy**. In the following, we formulater Wall's definition in arbitrary characteristic.

Definition 2.3.12. *Let* $f \in K[[\boldsymbol{x}]]$ *and let* δ *be any face of a* C*-polytope* $\boldsymbol{P}$ *such that no point of* $supp(f)$ *lies below* $\boldsymbol{P}$*. Further, for any* $\boldsymbol{r} = (r_1,\ldots,r_n) \in K^n$*, let* $I_{\boldsymbol{r}} := \{i : r_i \neq 0\}$*.*

1. *We say that* f *is* **strictly non-degenerate** *or* f *satisfies* $(ND2)$ *at* δ *if, for any common zero* $\boldsymbol{r}$ *of the functions* f_{δ,x_i} $(1 \leq i \leq n)$*, we have* $\delta \cap \mathbb{R}^{I_{\boldsymbol{r}}} = \emptyset$*.*

2. *We say that f satisfies $NPND^*$ with respect to $\boldsymbol{P}$ if f satisfies $(ND2)$ for every inner face of $\boldsymbol{P}$.*

3. *A hypersurface singularity R is called $(ND2)$ with respect to δ (resp. $NPND^*$ with respect to $\boldsymbol{P}$) if there is $f \in K[[\boldsymbol{x}]]$ such that f satisfies $(ND2)$ at δ (resp. f satisfies $NPND^*$ with respect to $\boldsymbol{P}$) and moreover $R_f \cong R$.*

Remark 2.3.13. *1. First, we would like to mention that according to the lemma 1.1 in [Wal99a], if $(ND2)$ holds at any* **inner** *face then it also holds for any face of the C-polytope. Indeed, this lemma establishes that for any face δ of a C-polytope $\boldsymbol{P}$, there exists an inner face $\delta^{'}$ of $\boldsymbol{P}$ with $\delta^{'} \cap \mathbb{R}^{I_\delta} = \delta$ (for the notations, we refer to 2.1.3). Therefore, for any subset $I \in \{1, \ldots, n\}$, we see clearly that the condition $\delta \cap \mathbb{R}^I = \emptyset$ for any* **inner** *face of $\boldsymbol{P}$ implies the same condition for any face of $\boldsymbol{P}$.*

2. If the condition $(ND2)$ holds for an inner face δ, then we should have necessarily that $f_\delta \neq 0$. Otherwise, any $\boldsymbol{r} \in (K \setminus \{0\})^n$ is a common zero of f_{δ,x_i}, $1 \leq i \leq n$. As $\mathbb{R}^{I_{\boldsymbol{r}}} = \mathbb{R}^n$, we have therefore $\delta \cap \mathbb{R}^{I_{\boldsymbol{r}}} \neq \emptyset$ wich contradicts the condition $(ND2)$ at δ.

The following lemma helps understanding condition $(ND2)$.

Lemma 2.3.14. *Let $f \in \mathfrak{m} \subset K[[\boldsymbol{x}]]$ and let δ be a face of a C-polytope $\boldsymbol{P}$ such that no point of $supp(f)$ lies below $\boldsymbol{P}$. Furthermore, let $\boldsymbol{r} \in \mathbb{K}^n$ be a common zero of f_{δ,x_i}, $1 \leq i \leq n$. If f is $(ND2)$ with respect to δ, then f_δ vanishes identically on $K^{I_{\boldsymbol{r}}}$ and therefore f_{δ,x_i}, $1 \leq i \leq n$, do so too.*

Proof. Let $\boldsymbol{r} \in K^n$ be a common zero of f_{δ,x_i}, $1 \leq i \leq n$. Throughout this proof, we denote $I := I_{\boldsymbol{r}}$. If we write $f = \sum\limits_{\boldsymbol{\alpha} \in supp(f)} a_{\boldsymbol{\alpha}} x^{\boldsymbol{\alpha}}$, then we get $f_\delta = \sum\limits_{\boldsymbol{\alpha} \in \delta} a_{\boldsymbol{\alpha}} x^{\boldsymbol{\alpha}}$.
On the other hand, as $\delta \cap \mathbb{R}^I = \emptyset$, then it follows that for any $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n) \in \delta$ there exists $i \notin I$ such that x_i divides $x^{\boldsymbol{\alpha}}$. Indeed, since $\boldsymbol{\alpha} \in \delta$, then $\boldsymbol{\alpha} \notin \mathbb{R}^I$. Hence, there exists $i \notin I$ such that $\boldsymbol{\alpha}_i \neq 0$ and thus we get $x_i \mid x^{\boldsymbol{\alpha}}$.
Now let $\boldsymbol{s} = (s_1, \ldots, s_n) \in K^I$, that is $s_i = 0$ for all $i \notin I$. Then, for any $\boldsymbol{\alpha} \in \delta$, we have clearly $\boldsymbol{s}^{\boldsymbol{\alpha}} = 0$ and therefore $f_\delta(\boldsymbol{s}) = 0$. This means that f_δ vanishes identically on K^I and obviously all derivations of f_δ too. □

Here, we notice that K^I is a union of torus. Of course, $r \in K^I$ and Lemma 2.3.14 asserts that condition $(ND2)$ implies that $\boldsymbol{r}$ is not an isolated singularity of f (see Definition 1.2.9).

The next lemma compares condition $(ND2)$ to condition $(ND1)$.

Lemma 2.3.15. *Let $f \in K[[\boldsymbol{x}]]$ and let $\boldsymbol{P} \subset \mathbb{R}^n_{\geq 0}$ be a C-polytope. Further, let δ be any* **inner** *face of $\boldsymbol{P}$. If f satisfies $(ND2)$ at δ then f satisfies also $(ND1)$ at δ.*

Proof. Let $f \in K[[\boldsymbol{x}]]$ and let δ be an **inner** face of a C-polytope $\boldsymbol{P} \subset \mathbb{R}^n_{\geq 0}$. On the other hand, let $\boldsymbol{r} = (r_1, \ldots, r_n)$ be a common zero to the f_{δ,x_i} $1 \leq i \leq n$.

We assume $(ND2)$ holds at δ and we suppose that $r_i \neq 0$ for all $i = 1, \ldots, n$, which means f does not satisfy $(ND1)$ with respect to δ.
Hence, if we denote $I = \{i : r_i \neq 0\}$, then we have $\mathbb{R}^I = \mathbb{R}^n$ and

$$\emptyset = \delta \cap \mathbb{R}^I = \delta \cap \mathbb{R}^n = \delta$$

which is impossible. Therefore f is $(ND1)$ with respect to δ.

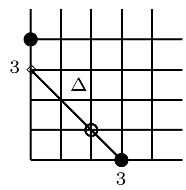
□

The converse of Lemma 2.3.15 is not true in general as the following example shows

Example 2.3.16. *In $char(K) = 0$, we consider the isolated plane curve singularity of equation $f = x^3 + x^2y + y^4$. Let Δ be the line segment joining the points $(3, 0)$ and $(0, 3)$. Obviously, Δ is a C-polytope and moreover no point of $supp(f)$ lies below $\boldsymbol{P}$. On the other hand, we have*

$$f_\Delta = x^3 + x^2y, \quad f_{\Delta,x} = 3x^2 + 2xy, \quad \textit{and} \quad f_{\Delta,y} = x^2.$$

Hence a common zero to $f_{\Delta,x}$ and $f_{\Delta,y}$ must lie on the y-axis and thus f is $(ND1)$ at Δ. However, the point $\boldsymbol{r} = (0, 1)$ is a common zero of $f_{\Delta,x}$ and $f_{\Delta,y}$, while $\Delta \cap \mathbb{R}^{I_{\boldsymbol{r}}} = \Delta \cap (\{0\} \times \mathbb{R}) = \{(0, 3)\} \neq \emptyset$. This shows that f is not $(ND2)$ at Δ.



Nevertheless, in the special case where the inner face is disjoint from the coordinate subspaces, we show that conditions $(ND2)$ and $(ND1)$ are equivalent (cf. [Wal99a]). We formulate this in the next lemma.

Lemma 2.3.17. *Let $f \in K[[\boldsymbol{x}]]$ and let $P \subset \mathbb{R}^n_{>0}$ be a C-polytope. Then, for a face δ disjoint from the coordinate subspaces, conditions $(ND1)$ and $(ND2)$ coincide.*

Proof. Let $\boldsymbol{P}$ ba a C-polytope in $\mathbb{R}^n_{>0}$ and let δ be a face of $\boldsymbol{P}$ that is disjoint from the coordinate subspaces, that is δ lies in $(\mathbb{R} \setminus 0)^n$. Hence the implication $(ND2) \Longrightarrow (ND1)$ follows from Lemma 2.3.15 as in particular δ is an inner face of $\boldsymbol{P}$.
Conversely, let f be a power series in $K[[\boldsymbol{x}]]$ and let $\boldsymbol{r}$ be a common zero of the equations f_{δ,x_i}, $1 \leq i \leq n$. We suppose that f satisfies the condition $(ND1)$ with respect to $\boldsymbol{P}$. Then, it follows that the set $I = \{i : r_i \neq 0\}$ is **strictly** contained in the set of all indices $\{1, \ldots, n\}$ and hence $\mathbb{R}^I$ is contained in the complement of $(\mathbb{R} \setminus 0)^n$. Therefore, by assumption on δ, we have that $\delta \cap \mathbb{R}^I = \emptyset$ and so the condition $(ND2)$ follows. □

Lemma 2.3.18. *Let $f \in K[[\boldsymbol{x}]]$ and let $P \subset \mathbb{R}^n_{>0}$ ba a C-polytope. Further, let $\delta = \{\boldsymbol{a} = (a_1, \ldots, a_n) \in \mathbb{Z}^n\}$ be an inner 0-dimensional face of $\boldsymbol{P}$. Then, f satisfies $(ND2)$ at δ, if and only if, $\boldsymbol{a} \in supp(f)$ and moreover $char(K)$ does not divide $gcd(a_1, \ldots, a_n)$.*

Proof. Let $\boldsymbol{a} = (a_1, \ldots, a_n) \in K^n$ and let $\delta = \{\boldsymbol{a}\}$ be an inner vertex of $\boldsymbol{P}$.
Hence, we have clearly $f_\delta = c \cdot x^{\boldsymbol{a}}$, where $c \in K$, besides $a_i \neq 0$ for all $i = 1, \ldots, n$ (see Definition 2.1.4). If f is $(ND2)$ at δ, then it follows by Remark 2.3.13 that $f_\delta \neq 0$ and hence $\boldsymbol{a} \in supp(f)$. On the other hand, if we suppose that $char(K)$ divides $gcd(a_1, \ldots, a_n)$, then all partial derivations of f_δ would be zero.
Hence, in particular, $\boldsymbol{r} = (r_1, \ldots, r_n) \in K^n$, with $r_i = 1$ for all $i = 1, \ldots, n$, is a common zero of f_{δ, x_i}, $1 \leq i \leq n$, which yields $\delta \cap \mathbb{R}^{I_{\boldsymbol{r}}} = \delta \neq \emptyset$. Thus, the contradiction to the condition $(ND2)$ at δ follows.
Conversely, if $\boldsymbol{a} \in supp(f)$ and $char(K)$ does not divide the gcd of the coordinates of $\boldsymbol{a}$, we see that $f_\delta \neq 0$ and moreover, if $\boldsymbol{r} = (r_1, \ldots, r_n)$ is a common zeroe of the partial derivations of f_δ, then $r_j = 0$ for soome $j \in \{1, \ldots, n\}$. There again, if we suppose $\delta \cap R^{I_{\boldsymbol{r}}} \neq \emptyset$, then we get $a_j = 0$ which is impossible. Therefore, f is $(ND2)$ at δ. $\square$

In [Wal99a, 1.2], Wall establishes that over the field $\mathbb{C}$, the $NPND^*$ property implies the "isolated" property. The following proposition shows that Wall's statement holds in arbitrary characteristic too.

Proposition 2.3.19. *If $f \in K[[\boldsymbol{x}]]$ satisfies $NPND^*$ for some C-polytope $\boldsymbol{P}$, then the origin is an isolated singularity of f, that is $\mu(f)$ is finite.*
Moreover $\mu(f) = V_N(\Gamma_-(f))$.

Proof. The proof that we give in the following is an adaptation to arbitrary characteristic of the one given by Wall in [Wal99a].
Let $f \in K[[\boldsymbol{x}]]$ and let $\boldsymbol{P} \subset \mathbb{R}^n_{\geq 0}$ be a C-polytope such that f is $NPND^*$ with respect to $\boldsymbol{P}$. We claim that, the set

$$\Lambda = \{i : \; f_{x_i} \neq 0\}$$

is not empty.
Otherwise, it follows that any point $\boldsymbol{r} = (r_1, \ldots, r_n) \in K^n$ with $r_i = 1$, for all $i = 1, \ldots, n$, is a common zero of the functions f_{x_i}, $1 \leq i \leq n$. Thus, for the set $I_{\boldsymbol{r}} = \{i : \; r_i \neq 0\}$, we have $\mathbb{R}^{I_{\boldsymbol{r}}} = \mathbb{R}^n$. Then, it is clear that for any inner face δ of $\boldsymbol{P}$, $\delta \cap \mathbb{R}^{I_{\boldsymbol{r}}} \neq \emptyset$ which is a contradiction to $NPND^*$ for f with respect to $\boldsymbol{P}$.
Moreover, for any inner face δ of $\boldsymbol{P}$, we notice that $f_\delta \neq 0$ (cf. Remark 2.3.13).
We suppose that 0 is not an isolated singular point of f. Hence, $dim(M_f) \geq 1$ and it follows by the **curve selection lemma** (cf. Lemma 1.2.15) that there exists a reduced irreducible curve $K[[\boldsymbol{x}]]/J$, where J is a proper ideal, such that

$$M_f \twoheadrightarrow K[[\boldsymbol{x}]]/J.$$

Let

$$I^{'} = \{i : \; x_i \in J\} \;\text{ and }\; I = \{1, \ldots, n\} \setminus I^{'}.$$

We have $I \neq \emptyset$, otherwise $I^{'} = \{1, \ldots, n\}$ which implies $\mathfrak{m} \subset J$ against the assumption $dim(K[[\boldsymbol{x}]]/J) = 1$. Thus, we have $\{0\} \subsetneq \mathbb{R}^I$.
On the other hand, as $\boldsymbol{P}$ is a C-polytope, we have necessarily that

$$\mathbb{R}^n_{>0} \supset \boldsymbol{P} \cap \mathbb{R}^I \neq \emptyset.$$

Moreover, it follows by the curve selection lemma that there exists a K-algebra homomorphism

$$\psi, \quad K[[\boldsymbol{x}]] \longrightarrow K[[t]],$$

such that $j(f) \subset Ker(\psi)$.
More precisely, for any $i \in I$, the exists $m_i \in \mathbb{Z}_{>0}$ and $b_i \in K \setminus \{0\}$, such that

$$\psi(x_i) = b_i t^{m_i} + \text{ higher terms.}$$

We consider on $\boldsymbol{P} \cap \mathbb{R}^I$ the minimum of the linear function λ defined on $\mathbb{R}^n$ by

$$\lambda(\boldsymbol{a}) = \sum_{i \in I} m_i a_i,$$

and we write

$$\nu = min_{\boldsymbol{a} \in \boldsymbol{P} \cap \mathbb{R}^I} \lambda(\boldsymbol{a}).$$

Let δ be the face of $\boldsymbol{P} \cap \mathbb{R}^I$ along which the value ν is attained.
We recall that

$$I^{'}_{\delta} = \{i : x_i = 0 \text{ on } \delta\} \text{ and } I_\delta = \{1, \ldots, n\} \setminus I^{'}_{\delta}.$$

We have

$$I^{'} = \{i : x_i = 0 \text{ on } \mathbb{R}^I\} \quad \subset \quad \{i : x_i = 0 \text{ on } \delta\} = I^{'}_{\delta}.$$

Indeed, the inclusion follows because $\delta \subset \mathbb{R}^I$. Hence, $I_\delta \subset I$.
Moreover, we know by [Wal99a, Lemma 1.1] that we can choose an **inner** face $\delta^{'}$ of $\boldsymbol{P}$ such that

$$\delta^{'} \cap \mathbb{R}^{I_\delta} = \delta. \tag{2.3}$$

We define an algebroid curve singularity R_0 by the parametrization

$$\psi_0 : \quad K[[\boldsymbol{x}]] \to K[[t]]$$

given by

$$\psi_0(x_i) = b_i t^{m_i} \text{ if } i \in I_\delta \text{ and } \psi_0(x_i) = 0 \text{ otherwise.}$$

As no point of $supp(f)$ lies below $\boldsymbol{P}$, then we have clearly that

$$\psi(f) = at^\nu + \text{ higher terms},$$

where $a \in K$. Similarly, we have for any $i \in \Lambda$,

$$\psi(f_{x_i}) = c_i t^{\nu - m_i} + \text{ higher terms}, \text{ where } c_i \in K. \tag{2.4}$$

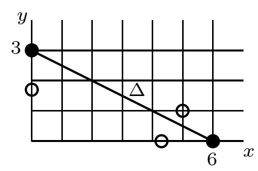
And we get
$$\psi_0(f_{\delta,x_i}) = c_i t^{\nu - m_i}. \tag{2.5}$$
On the other hand, we have $j(f) \subset Ker(\psi)$ by the curve selection lemma, so it follows in particular that in equation (2.9), $c_i = 0$, for all $i \in \Lambda$. Hence, we have obviously by equation (2.10)
$$j(f_\delta) \subset Ker(\psi_0).$$
Let $t \in K \setminus \{0\}$ and let $\boldsymbol{r} = (r_1, \dots, r_n) \in K^n$ be such that
$$r_i = b_i t^{m_i} \ \text{ if } \ i \in I_\delta \ \text{ and } \ r_i = 0 \ \text{ otherwise},$$
then we have clearly that, for $i = 1, \dots, n$, $f_{\delta,x_i}(\boldsymbol{r}) = 0$. On the other hand, we have
$$I_{\boldsymbol{r}} = \{i : r_i \neq 0\} = I_\delta.$$
Hence, it follows from (2.8) that
$$\delta' \cap \mathbb{R}^{I_{\boldsymbol{r}}} = \delta' \cap \mathbb{R}^{I_\delta} = \delta \neq \emptyset.$$
But this contradicts $(ND2)$ for δ' and thus the claim $\mu(f) < \infty$ follows.
Finally, the claim $\mu(f) = V_N(\Gamma_-(f))$ shall be shown at the end of Section 3.1 of the next Chapter 3 as a corollary of Proposition 2.3.9 and Theorem 3.1.15 on finite determinacy in arbitrary characteristic. □

Corollary 2.3.20. *Let $f \in K[[\boldsymbol{x}]]$ and let $R = R_f$ be the hypersurface singularity associated to f. If R is $NPND^*$ with respect to some polytope $\boldsymbol{P}$, then R is isolated.*

Proof. For $f \in K[[\boldsymbol{x}]]$, we know that $\tau(f) \leq \mu(f)$. Thus, Corollary 2.3.20 is a trivial consequence from Definition 2.3.12 and Proposition 2.3.19. □

Remark 2.3.21. *The converse of Proposition 2.3.19 does not hold in general as the following example shows.*

Example 2.3.22. *The converse of Proposition 2.3.19 does not hold in general as the following example shows. Let $char(K) = 2$, and let $f = x^6 + x^5 y + y^3 \in K[[x, y]]$. We have $\mu(f) = 13$, moreover we can see easily that f is (SQH) of principal part $f_\Delta = x^6 + y^3$ which is a (QH) polynomial of type $((1,2)\,;6)$. As $char(K) = 2$ divides the degree of quasihomogeneity (6), the subsequent Proposition 2.3.23 asserts that there is no C-polytope $\boldsymbol{P} \subset \mathbb{R}^2_{>0}$ with respect to which f is $NPND^*$.*



$f = x^6 + x^5 y + y^3$

Proposition 2.3.23. *Let $f \in \mathfrak{m}^3 \subset K[[\boldsymbol{x}]]$ be (SQH) with principal part f_Δ having weighted degree $d \in \mathbb{Z}_{>0}$. Then, the following are equivalent*

1. *f is $NPND^*$ with respect to some C-polytope $\boldsymbol{P}$ of $\mathbb{R}^n_{\geq 0}$,*
2. *$\mu(f_\Delta)$ is finite,*
3. *$char(K)$ does not divide d.*

Proof. The implication $(1) \Rightarrow (2)$ follows by Proposition 2.3.19 because f_Δ does also satisfy $NPND^*$ with respect to $\boldsymbol{P}$.
$(2) \Leftrightarrow (3)$ follows by Lemma 2.1.33.
It remains only to show the implication $(2) \Rightarrow (1)$. To do so, we consider the extension of the facet Δ to the coordinate hypersurfaces which we denote by $\bar{\Delta}$. It is clear that $\bar{\Delta}$ is a C-polytope in $\mathbb{R}^n_{\geq 0}$.Furthermore it has a unique inner face, which is itself, and the associated truncation $f_{\bar{\Delta}}$ is equal to f_Δ. On the other hand, we have from Lemma 2.1.35, that 0 is the unique common zero of f_{Δ,x_i}, $i = 1, \ldots, n$. Hence, the associated set $\mathbb{R}^I$ is equal to $\{0\}$. Moreover, we claim that $0 \notin \bar{\Delta}$. Indeed, if we assume the contrary, then we have necessarily that the set $\{\boldsymbol{\alpha} \in \mathbb{R}^n : \ \boldsymbol{\alpha} \in supp(f_\Delta)\} \subset \mathbb{R}^J$, where J is **strictly** contained in $\{1, \ldots, n\}$.
This means that there exists i, $1 \leq i \leq n$, such that the polynomial f_Δ does not depend on the indeterminate x_i. Therefore $f_{\Delta,x_i} = 0$ but this is a contradiction to $\mu(f_\Delta) < \infty$. Thus, we have obviously that $\bar{\Delta} \cap \mathbb{R}^I = \emptyset$ which shows that f is $NPND^*$ with respect to $\bar{\Delta}$. □

In [Wal99a], Wall deals in part with complex plane curves fulfilling $NPND^*$ with respect to a C-polytope $\boldsymbol{P}$ and considers how $\boldsymbol{P}$ compares with the Newton polytope. He comes to the conclusion that for reduced **plane** curve singularities there is always a way to make the condition $NPND^*$ satisfied. Moreover, after investigating Wall's observations, we observe easily and without any need to further proofs that his conclusions hold also in characteristic zero. We summarize this observations in the following two lemmas.

Lemma 2.3.24. *Let $char(K) = 0$ and let $f \in K[[x, y]]$ be reduced. Let $\boldsymbol{P} \subset \mathbb{R}^2_{\geq 0}$ be a C-polytope. Further, let δ be an inner vertex of $\boldsymbol{P}$ and let Δ be an inner edge of $\boldsymbol{P}$ with end points (a, b) and (c, d). Then, we have*

1. *f satisfies $(ND2)$ at δ , if and only if, δ is a vertex of $\Gamma(f)$.*
2. *If Δ is disjoint from the coordinate axes (i.e (a, b) and (c, d) are inner vertices of $\boldsymbol{P}$), then f satisfies $(ND2)$ at Δ, if and only if, Δ is an edge of $\Gamma(f)$.*
3. *If one end of Δ - say (a,b)- is an inner vertex , and the other - (c,d) - lies on the x-axis (i.e $d = 0$), then f satisfies $(ND2)$ at Δ, if and only if, Δ is an edge of $\Gamma(f)$ or there is a point $(\tilde{c}, \tilde{d})$ of $\Gamma(f)$ on Δ with $\tilde{d} = 1$ and so the line segment $[(a, b), (\tilde{c}, 1)]$ is an edge of $\Gamma(f)$.*
   Inverting the roles of c and d, we have the same statement if the point (c, d) lies on the y-axis.

4. *If each end point of Δ lies on a coordinate axis, then f satisfies $(ND2)$ at Δ, if and only if, f is semiquasihomogeneous (SQH).*

Proof. cf. [Wal99a]. □

Remark 2.3.25. *First, we observe that the first statement of Lemma 2.3.24 can be considered as a corollary of Lemma 2.3.18 in characteristic zero. Moreover, we note that Lemma 2.3.24 establishes that any (SQH) bivariate power series in characteristic zero is $NPND^*$ with respect to its Newton polytope.*

Lemma 2.3.26. *Let $char(K) = 0$ and let $f \in K[[x, y]]$ be reduced. Then there is a C-polytope $\boldsymbol{P} \subset \mathbb{R}^2_{\geq 0}$ with respect to which f satisfies $NPND^*$. Moreover if the term xy does not appear in f, then $\boldsymbol{P}$ can be uniquely determined by a minimal set of linear functions λ_j.*

Proof. cf. [Wal99a]. □

Remark 2.3.27. *Example 2.3.22 shows that the claim of Lemma 2.3.26 does not hold in arbitrary characteristic.*

In the last part of this chapter, we shall investigate in finite characteristic, how non-degeneracy affects the well-known formula

$$\mu(f) = 2\delta(f) - r(f) + 1$$

for reduced plane curve singularities over $\mathbb{C}$ and which is in general not true in finite characteristic (cf. Remark 1.2.18).
For this purpose, we should present in the following the condition of non-degeneracy introduced in [BeP00].

Definition 2.3.28. *Let $f \in \mathfrak{m}^2 \subset K[[x, y]]$ and let Γ be the Newton polytope of f. After Beelen und Pellikaan, f is non-degenerate in the weak sense (WND), if*

1. *f is (CO) and*
2. *for every line segment δ of Γ*

$$\{\boldsymbol{r} \in \mathbb{K}^n : f_\delta(\boldsymbol{r}) = f_{\delta,x}(\boldsymbol{r}) = f_{\delta,y}(\boldsymbol{r}) = 0\} \subset \{x = 0\} \cup \{y = 0\}.$$

Lemma 2.3.29. *Let $f \in \mathfrak{m}^2 \subset K[[x, y]]$ be (CO). Furthermore, let Γ be the Newton polytope of f and let δ be a line segment of Γ.*

1. *If f is $(ND1)$ with respect to δ, then f is (WND) with respect to δ too.*
2. *If f is $(ND2)$ with respect to δ, then f is (WND) with respect to δ too.*

Proof. The first assertion is straightforward from Definition 2.3.2 and 2.3.28. Hence, the second assertion follows clearly by Lemma 2.3.15. □

Let $f \in \mathfrak{m}$, we recall that $\mu_N(f)$ denotes the Milnor number of f (cf. Definition 2.1.10). The following claim is due to [BeP00]

Theorem 2.3.30. *Let $f \in \mathfrak{m}^2 \subset K[[x,y]]$ be (CO). If is non-degenearte in the weak sense, then*

$$\mu_N(f) = 2\delta(f) - r(f) + 1,$$

where $\delta(f)$ is the delta invariant of f and $r(f)$ is the number of irreducible factors of f.

Proof. cf. [Bep00, 3.11 and 3.17] □

Corollary 2.3.31. *Let $f \in \mathfrak{m}^2 \subset K[[x,y]]$ be $NPND$, then*

$$\mu(f) = 2\delta(f) - r(f) + 1.$$

Proof. The claim is straightforward from Proposition 2.3.9 and Theorem 2.3.30. □

Chapter 3

Finite Determinacy and Normal Forms

This chapter deals with the main results related to determinacy and computation of normal forms in arbitrary characteristic.
In the first part, we recall the notions of jets and finite determinacy for right and for contact equivalence. Moreover, we show that the well-known theorem about finite determinacy over $\mathbb{C}$ (cf. [GLS06, 2.23]) does also hold in characteristic zero. Afterwards, we formulate a new theorem on finite determinacy in arbitrary characteristic. Moreover, as it is the case over $\mathbb{C}$, we show that the properties "isolated" and "finitely determined" for hyersurface singularities are also equivalent in arbitrary characteristic.
For the purpose of providing a general setting to the computation of normal forms in finite characteristic, we formalize Arnold and Wall methods over $\mathbb{C}$ in the second section of the present chapter. Also, we introduce new objects and formulate the new conditions (AA) and (AAC) and show that they are weaker than those imposed by Arnold and Wall for their development of the theory.
With these preparations made, we formulate in the last section our results about normal forms and bounds of determinacy in arbitrary characteristic.

Throughout the present chapter K denotes an algebraically closed field of arbitrary chracteristic.

3.1 Finite Determinacy of Isolated Hypersurface Singularities

We review briefly the definitions of jets and finite determinacy.

Definition 3.1.1. *Let $f \in K[[\boldsymbol{x}]]$ and k be a positive integer. Then*
$f^{(k)} :=$ image of f in $K[[\boldsymbol{x}]]/\mathfrak{m}^{k+1}$ denotes the ***k*-jet** *of f and we write*
$J^{(k)} := K[[\boldsymbol{x}]]/\mathfrak{m}^{k+1}$ for the K-vector space of all k-jets.

Definition 3.1.2. *1. $f \in K[[\boldsymbol{x}]]$ is called* **contact k-determined** *(resp.* **right k-determined***) if for each $g \in K[[\boldsymbol{x}]]$ with $f^{(k)} = g^{(k)}$ we have $f \overset{c}{\sim} g$ (resp. $f \overset{r}{\sim} g$). We say then that f is determined by its k-jet up to contact (resp. right) equivalence.*

2. $f \in K[[\boldsymbol{x}]]$ is called **finitely contact determined** *(resp.***finitely right determined***) if f is contact (resp. right) k-determined for some positive integer k.*

3. The minimal such k is called **the degree of contact determinacy** *(resp.* **the degree of right determinacy***) of f.*

Proposition 3.1.3. *The degree of contact (resp. right) determinacy is an invariant of the $\mathcal{K}$-orbit (resp. $\mathcal{R}$-orbit).*

Proof. The proof is straightforward from the above Definition 3.1.2. ☐

We recall that a hypersurface singularity is a local K-algebra of the form

$$R_f = K[[\boldsymbol{x}]]/\langle f \rangle \ \text{ where } \ f \in \mathfrak{m} \subset K[[\boldsymbol{x}]].$$

In the next definition of finitely determined hypersurface singularities, the choice of the contact equivalence is motivated by the following observation:
For $f, g \in \mathfrak{m}$, we have $R_f \cong R_g$, if and only if, $f \overset{c}{\sim} g$.

Definition 3.1.4. *Let $f \in \mathfrak{m} \subset K[[\boldsymbol{x}]]$ and let $k \in \mathbb{Z}_{>0}$. The hypersurface singularity R_f is called* **finitely** **k-determined** *if f is finitely* **contact** *k-determined. The minimal such k is called the* **degree of determinacy** *of R_f.*

We would like to mention that in the above definitions, we consider the total degree on $K[[\boldsymbol{x}]]$. Now, considering a finite set of weights, we introduce in the following the notion of **piecewise finite determinacy**.

Definition 3.1.5. *Let $\boldsymbol{W} \subset \mathbb{Q}^n_{>0}$ be a finite set of weights and let $f \in K[[\boldsymbol{x}]]$. Further, let $d \in \mathbb{Q}_{\geq 0}$. Then $f^{(\boldsymbol{W},d)} :=$ image of f in $K[[\boldsymbol{x}]]/F_{>d}$ is called the $(\boldsymbol{W}, d)$-jet of f (or the* **piecewise-homogeneous** **d-jet of f with respect to $\boldsymbol{W}$***) and we write $J^{(\boldsymbol{W},d)} := K[[\boldsymbol{x}]]/F_{>d}$ for the K-vector space of all $(\boldsymbol{W}, d)$-jets.*

Definition 3.1.6. *Let $\boldsymbol{W} \subset \mathbb{Q}^n_{>0}$ be a finite set of weights.*

1. $f \in K[[\boldsymbol{x}]]$ is called **contact piecewise d-determined** *(resp.* **right piecewise d-determined***) with respect to $\boldsymbol{W}$ if for each $g \in K[[\boldsymbol{x}]]$ with $f^{(\boldsymbol{W},d)} = g^{(\boldsymbol{W},d)}$ we have $f \overset{c}{\sim} g$ (resp. $f \overset{r}{\sim} g$).*

2. $f \in K[[\boldsymbol{x}]]$ is called **finitely contact piecewise determined** *(resp.* **finitely right piecewise determined***) if f is contact (resp. right) piecewise $(\boldsymbol{W}, d)$-determined for some finite set of weights $\boldsymbol{W} \subset \mathbb{Q}^n_{>0}$ and some $d \in \mathbb{Q}_{>0}$.*

3. The minimal such d is called **the piecewise-homogeneous degree of contact determinacy** *(resp.* **the piecewise-homogeneous degree of right determinacy***) of f with respect to $\boldsymbol{W}$.*

Definition 3.1.7. *Let $f \in \mathfrak{m} \subset K[[\boldsymbol{x}]]$. Further, let $\boldsymbol{W} \subset \mathbb{Q}^n_{>0}$ be a finite set of weights and let $d \in \mathbb{Q}_{>0}$. We call the hypersurface singularity R_f* **finitely piecewise d-determined with respect to $\boldsymbol{W}$** *if f is finitely* **contact** *piecewise d-determined with respect to $\boldsymbol{W}$. The minimal such d is called the degree of piecewise determinacy of R_f with respect to $\boldsymbol{W}$.*

The following observation on zero-dimensional ideals is very useful for the sequel.

Lemma 3.1.8. *Let I be a proper ideal in $K[[\boldsymbol{x}]]$ and let k be a positive integer. Then*

$$\mathfrak{m}^k \subset I \iff \mathfrak{m}^k \subset I + \mathfrak{m}^{k+1}.$$

Proof. The implication ($\Rightarrow$) is obvious.
The converse ($\Leftarrow$) follows by applying Nakayama's lemma to the ideal $\langle \mathfrak{m}^{k+1}, I\rangle / I$ of $K[[\boldsymbol{x}]]/I$. ☐

Remark 3.1.9. *The filtred version of Lemma 3.1.8 is wrong. In other words, if we consider the filtration of $K[[\boldsymbol{x}]]$ associated to a given finite set of weights and if I is a proper ideal of $K[[\boldsymbol{x}]]$, then*

$$F_{\geq d} \subset I + F_{>d} \not\Longrightarrow F_{\geq d} \subset I.$$

For example, let $char(K) = 3$ and let $f = x^7 + x^3y^2 + y^4 \in K[[x, y]]$. We consider the ideal $tj(f)$ and the finite set of weights $\boldsymbol{W} := \{(1/7\,, 2/7)\,, (1/6\,, 1/4)\} \subset \mathbb{Q}^2_{>0}$. Using the SINGULAR *function $grideal$ from the library $gradalg.lib$, we compute*

$$F_{\geq 1} = \langle x^7,\, x^5y,\, x^2y^3,\, x^3y^2,\, y^4\rangle.$$

Again using SINGULAR*, we show that x^7, x^3y^2, $y^4 \in tj(f)$. On the other hand, we have $v_{\boldsymbol{W}}(x^2y^3) = v_{\boldsymbol{W}}(x^5y) = 13/12 > 1$, thus x^2y^3 and x^5y are in $F_{>1}$. Altogether, we see that $F_{\geq 1} \subset tj(f) + F_{>1}$. Nevertheless $F_{\geq 1} \not\subset tj(f)$ as $x^5y \notin tj(f)$.*

In (analytic) singularity theory over the field $\mathbb{C}$ of complex numbers, it is established that any isolated (analytic) hypersurface singularity is right as well as contact finite determined. This is for example the statement of Theorem 2.23 in [GLS06] where the proof uses mainly the so called *infinitesimal characterization of local triviality*. Nevertheless, we observe that the arguments used by the authors in [GLS06] for the proofs show actually that all these statements hold also over algebraically closed fields of characteristic zero. Indeed, we need only to prove the following claim about the existence and uniqueness of solutions of ordinary differential equations in characteristic zero.

Lemma 3.1.10. *Let R be a commutative ring of characteristic zero.*

1. *Let $G = (g_1, \ldots, g_n) \in R[[\boldsymbol{x}, t]]^n = R[[x_1, \ldots, x_n, t]]^n$.
   For a given $\boldsymbol{a} = (a_1, \ldots, a_n) \in \langle \boldsymbol{x}\rangle^n$, the differential equation*

$$\frac{\partial Y}{\partial t}(\boldsymbol{x}, t) = G(Y(\boldsymbol{x}, t), t) \;\; \textit{with initial condition} \;\; Y(\boldsymbol{x}, 0) = \boldsymbol{a}$$

   has a unique solution.

2. *Let $u \in R[[t]]$. For a given $a \in R$, the differential equation*

$$\frac{\partial y}{\partial t} = u \cdot y$$

has a unique solution $y \in R[[t]]$ with $y(0) = a$.

Proof. Let R be a commutative ring such that $char(R) = 0$.

1. To show the first claim, we use induction on n.
For $n = 1$, let $g \in R[[x, t]]$ and let $a \in \langle \boldsymbol{x} \rangle$. We consider the ordinary differential equation
$$\frac{\partial y}{\partial t}(x, t) = g(y(x, t), t) \tag{3.1}$$
with initial condition $y(x, 0) = a$.
We write $g = \sum_{j,k \geq 0} b_{j,k} x^j t^k$ and $y = \sum_{i \geq 0} c_i(x) t^i$. Comparing both sides of the equation (3.1), we show that this differential equation has a unique solution. Indeed, the condition $y(x, 0) = a$ yields $c_0(x) = a$. On the other hand, equation (3.1) is equivalent to
$$\begin{aligned} \sum_{i \geq 0} (i+1) c_{i+1}(x) t^i &= g(y(x, t), t) \\ &= \sum_{j,k,l \geq 0} b_{j,k} d_l(x) t^{k+l} \end{aligned}$$
where
$$d_l(x) = \sum_{i_1 + \ldots + i_j = l} c_{i_1}(x) \cdot \ldots \cdot c_{i_j}(x).$$
Hence, since $char(R) = 0$, we get for $i \geq 0$
$$c_{i+1} = \left(\frac{1}{i+1}\right) \cdot \left(\sum_{k+l=i} b_{j,k} \sum_{i_1 + \ldots + i_j = l} c_{i_1}(x) \cdot \ldots \cdot c_{i_j}(x) \right) \tag{3.2}$$
Clearly, the recursive formula (3.2) determines uniquely the coefficients of y.
Let $G = (g_1, \ldots, g_n) \in R[[\boldsymbol{x}, t]]^n = R[[x_1, \ldots, x_n, t]]^n$.
For a given $\boldsymbol{a} = (a_1, \ldots, a_n) \in \langle \boldsymbol{x} \rangle^n$, we consider the differential equation
$$\frac{\partial Y}{\partial t}(\boldsymbol{x}, t) = G(Y(\boldsymbol{x}, t), t), \quad Y(\boldsymbol{x}, 0) = \boldsymbol{a}. \tag{3.3}$$
We denote $R' := R[[x_n]]$ and $R'[[\boldsymbol{x}', t]] = R[[x_n]][[x_1, \ldots, x_{n-1}, t]]$. Writing

   - $G' = (g_1, \ldots, g_{n-1}) \in R'[[\boldsymbol{x}', t]]^{n-1}$ and
   - $\boldsymbol{a}' = (a_1, \ldots, a_{n-1}) \in \langle \boldsymbol{x}' \rangle^{n-1} \subset R'[[\boldsymbol{x}']]^{n-1}$,

the induction hypothesis yields that the differential equation

$$\frac{\partial Y'}{\partial t}(\boldsymbol{x}', t) = G'(Y'(\boldsymbol{x}', t), t), \quad Y'(\boldsymbol{x}', 0) = \boldsymbol{a}' \tag{3.4}$$

has a unique solution $Y' = (y_1, \ldots, y_{n-1}) \in R'[[\boldsymbol{x}', t]]^{n-1} = R[[\boldsymbol{x}, t]]^{n-1}$.
On the other hand, we observe that $R[[\boldsymbol{x}, t]] = R[[\boldsymbol{x}']][[x_n, t]]$. Hence, if we set $g = g_n(y_1, \ldots, y_{n-1}, x_n, t)$, we see easily that the existence and the unicity in $R[[\boldsymbol{x}']][[x_n, t]]$ of the solution of the ordinary differential equation

$$\frac{\partial \varphi}{\partial t}(x_n, t) = g(\varphi(x_n, t), t), \quad \varphi(x_n, 0) = a_n \tag{3.5}$$

follow again by the induction hypothesis. Finally, let $y_n(\boldsymbol{x}, t) = \varphi(x_n, t)$. Altogether, we get that $Y = (y_1, \ldots, y_n) \in R[[\boldsymbol{x}, t]]^n$ is the unique solution of the differential equation (3.3).

2. Let $u \in R[[t]]$. For a given $a \in R$, we see easily that the differential equation

$$\frac{\partial y}{\partial t} = u \cdot y, \quad y(0) = a \tag{3.6}$$

can be considered as an equation of the form (3.3), where $n = 0$. Indeed, it suffices to take $G(Y(t)) = u \cdot Y(t)$.

$\square$

In the following we give the fundamental theorem on infinitesimal characterization of local triviality in characteristic zero.

Theorem 3.1.11. (Infinitesimal characterization of local triviality). *Let K be a field of characteristic zero. Further, let $F \in \langle x_1, \ldots, x_n \rangle \subset K[[x_1, \ldots, x_n, t]]$ and let $b \geq 0$, $c \geq 0$ be integers.*

1. *The following are equivalent*
   (a) $\frac{\partial F}{\partial t} \in \langle x_1, \ldots, x_n \rangle^b \cdot \left\langle \frac{\partial F}{\partial x_1}, \ldots, \frac{\partial F}{\partial x_n} \right\rangle + \langle x_1, \ldots, x_n \rangle^c \cdot \langle F \rangle$.
   (b) *There exists $\phi = (\phi_1, \ldots, \phi_n) \in K[[x, t]]^n$, $u \in K[[x, t]]$ satisfying*
      i. $u(\boldsymbol{x}, 0) = 1$,
      ii. $u(\boldsymbol{x}, t) - 1 \in \langle x_1, \ldots, x_n \rangle^c \cdot K[[\boldsymbol{x}, t]]$,
      iii. $\phi_i(\boldsymbol{x}, 0) = x_i$, $i = 1, \ldots, n$,
      iv. $\phi_i(\boldsymbol{x}, t) - x_i \in \langle x_1, \ldots, x_n \rangle^b \cdot K[[\boldsymbol{x}, t]]$, $i = 1, \ldots, n$,
      v. $u(\boldsymbol{x}, t) \cdot F(\phi(\boldsymbol{x}, t), t) = F(\boldsymbol{x}, 0)$.
2. *Moreover, the condition*

$$\frac{\partial F}{\partial t} \in \langle x_1, \ldots, x_n \rangle^b \cdot \left\langle \frac{\partial F}{\partial x_1}, \ldots, \frac{\partial F}{\partial x_n} \right\rangle$$

is equivalent to $1.(b)$ *with* $u = 1$.

Proof. Using the claim of Lemma 3.1.10 which holds in characteristic zero, we notice that the arguments used in the proof of [Theorem 2.22, GLS06] of infinitesimal characterization of local triviality over $\mathbb{C}$ show in the same way the claim of Theorem 3.1.11. □

At the end of the present section, we shall give a characterization of finitely determined hypersurface singularities. For this task, the following observation is crucial.

Remark 3.1.12. *Actually, the proof given in [GLS06, Theorem 2.22] shows that the implication $(b) \Rightarrow (a)$ of Theorem 3.1.11 holds even in positive characteristic.*

The finite determinacy theorem, asserting that isolated hypersurface singularies are finitely determined, follows in characteristic zero from Theorem 3.1.11 and Lemma 3.1.8.

Theorem 3.1.13. (Finite determinacy theorem in characteristic zero).
Let $f \in \mathfrak{m} \in K[[\boldsymbol{x}]]$ and let $char(K) = 0$.

1. *f is* **right** *k-determined if*

$$\mathfrak{m}^{k+1} \subset \mathfrak{m}^2 \cdot j(f). \tag{3.7}$$

2. *f is* **contact** *k-determined if*

$$\mathfrak{m}^{k+1} \subset \mathfrak{m}^2 \cdot j(f) + \mathfrak{m} \cdot \langle f \rangle. \tag{3.8}$$

Proof. cf. [GLS06, Theorem 2.23] □

Remark 3.1.14. *Theorem 3.1.13 does not hold in finite characteristic as the following example shows: Let $char(K) = 2$ and let $f = y^2 + x^3 y$. Using* SINGULAR, *we show that $\tau(f) = 5$, hence R_f is an isolated plane curve singularity. Further, we write I for the ideal $\mathfrak{m}\langle f \rangle + \mathfrak{m}^2 j(f)$. We have $\mathfrak{m}^5 \subset I$. Nevertheless, f is not contact 4-determined as it would follow from Theorem 3.1.13. Otherwise, we would have for example $f \overset{c}{\sim} f + x^5$ but this is impossible since f has two irreducible components while $f + x^5$ has only one.*

In the sequel, we assume the field K to have an arbitrary characteristic.

Theorem 3.1.15. (Finite determinacy theorem in arbitrary characteristic).
Let $f \in \mathfrak{m}^2 \subset K[[x_1, \ldots, x_n]]$ such that $n \geq 2$.

1. *f is* **right** *$(2k - ord(f) + 2)$-determined if*

$$\mathfrak{m}^k \subset j(f). \tag{3.9}$$

2. *f is* **contact** *$(2k - ord(f) + 2)$-determined if*

$$\mathfrak{m}^k \subset tj(f). \tag{3.10}$$

Proof. We start by giving the proof of the second assertion of the theorem concerning the contact determinacy. Let $f \in K[[\boldsymbol{x}]]$ be such that $ord(f) \geq 2$ and let $k \in \mathbb{Z}_{>0}$ be such that $\mathfrak{m}^k \subset tj(f)$. We denote $s := ord(f)$ and, for all i, $1 \leq i \leq n$, $r_i := ord(f_{x_i})$.
It follows from (3.10) that $\tau(f) < \infty$. Hence the set $\{i \;:\; f_{x_i} \neq 0\}$ is not empty since $n \geq 2$. Moreover, it is clear that $r_i \geq s-1$, for all $1 \leq i \leq n$. Thus, it follows from (3.10) that $\mathfrak{m}^k \subset \mathfrak{m}^s + \mathfrak{m}^{s-1} \subset \mathfrak{m}^{s-1}$. Therefore, $k \geq s-1$ follows. Throughout this proof, we denote $N := 2k - s + 2$ and we consider $g \in K[[\boldsymbol{x}]]$ such that $g - f \in \mathfrak{m}^{N+1}$. We show in the following that $g \overset{c}{\sim} f$. For this purpose, we construct inductively sequences

- $(u_p)_{p\geq 1} \subset K[[\boldsymbol{x}]]^*$,
- $(\varphi_p)_{p\geq 1} \subset Aut(K[[\boldsymbol{x}]])$ and
- $(f_p)_{p\geq 0} \subset K[[\boldsymbol{x}]]$, such that $f_0 = f$ and for all $p \geq 1$, we have
  (a) $f_p = u_p\varphi_p(f_{p-1})$,
  (b) $f_p \overset{c}{\sim} f$ and
  (c) $g - f_p \in \mathfrak{m}^{N+p+1}$.

In the following, we describe the first step of our construction. First of all, it easy to notice that $N + 1 = 2k - s + 3 \geq k + s - 1 - s + 3 = k + 2$. On the other hand, as $\mathfrak{m}^{N+1} = \mathfrak{m}^{N+1-k}\mathfrak{m}^k$, then it follows from (3.10) that $\mathfrak{m}^{N+1} \subset \mathfrak{m}^{N+1-k}tj(f)$, and thus we can write

$$g - f = \sum_{1\leq i\leq n} b_i^{(1)} f_{x_i} \;\;+\;\; b_0^{(1)} f,$$

with $b_i^{(1)} \in \mathfrak{m}^{N+1-k}$, for all $i = 0, \ldots, n$.
Moreover, we have $N+1-k = k-(s-1)+2 \geq 2$. Therefore, if we set $u_1 := 1+b_0^{(1)}$, then we see clearly that u_1 is a unit in $K[[\boldsymbol{x}]]$. Besides,

$$\begin{array}{rccll} \varphi_1 : & K[[\boldsymbol{x}]] & \longrightarrow & K[[\boldsymbol{x}]] & \\ & x_i & \mapsto & x_i + b_i^{(1)} & \qquad\qquad \text{for } \; i = 1, \ldots, n \end{array}$$

is a K-algebra automorphism on $K[[\boldsymbol{x}]]$ and

$$\varphi_1(f) = f + \sum_{1\leq i\leq n} b_i^{(1)} f_{x_i} + h_1, \;\;\text{with}\;\; h_1 \in \mathfrak{m}^{N+2}.$$

Indeed, h_1 has the following form:

$$h_1 = \sum_{1\leq l\leq t\leq s}\;\sum_{1\leq i_1\leq\ldots\leq i_s\leq n} \alpha_{i_1,\ldots,i_s} x_{i_1}\cdot\ldots\cdot x_{i_{l-1}}\cdot b_{i_l}^{(1)}\cdot x_{i_{l+1}}\cdot\ldots\cdot x_{i_{t-1}}\cdot b_{i_t}^{(1)}\cdot x_{i_{t+1}}\cdot\ldots\cdot x_{i_s} + H_1,$$

where the coefficients $\alpha_{i_1,\ldots,i_s} \in K$ and $ord(H_1) \geq ord(h_1)$. Clearly, we have

$$\begin{array}{rcl} ord(h_1) & \geq & \underset{1\leq i_1\leq\ldots\leq i_s\leq n}{min} ord(\alpha_{i_1,\ldots,i_s} x_{i_1}\cdot\ldots\cdot b_{i_l}^{(1)}\cdot\ldots\cdot b_{i_t}^{(1)}\cdot\ldots\cdot x_{i_s}) \\ & \geq & 2(N+1-k) + (s-2) \\ & = & N+2. \end{array}$$

Now let $f_1 := u_1\varphi_1(f)$. We have

$$\begin{aligned} f_1 &= \left(1+b_{0}^{(1)}\right)\left(f+\sum_{i=1}^{n} b_{i}^{(1)} f_{x_i} + h_1\right) \\ &= g+\sum_{i=1}^{n} b_{0}^{(1)} b_{i}^{(1)} f_{x_i} \quad + \quad \left(1+b_{0}^{(1)}\right) h_1. \end{aligned}$$

On the other hand, we have for all $i = 1, \ldots, n$

$$\begin{aligned} ord(b_{0}^{(1)} b_{i}^{(1)} f_{x_i}) &\geq 2(N+1-k)+r_i \\ &\geq 2(N+1-k)+s-1 \\ &= N+2+(N-2k+s-1) \\ &= N+3. \end{aligned}$$

Hence $f_1 \overset{c}{\sim} f$ and $g - f_1 \in \mathfrak{m}^{N+2}$. Altogether yields

(a) $ord(f_1) = ord(f) = s$ and

(b) $\mathfrak{m}^k \subset tj(f_1)$ since $tj(f_1) = \varphi_1(tj(f))$ follows by Lemma 1.2.7).

In this way, we get $f_1 \in K[[\boldsymbol{x}]]$ having the same properties as f and moreover $g - f_1$ lies in a higher power of the maximal ideal $\mathfrak{m}$ as $g - f$. Proceeding recursively we construct the sequences $(u_p)_{p\geq 1}$, $(\varphi_p)_{p\geq 1}$ and $(f_p)_{p\geq 0}$ as required. Now it is clear that the sequence $(f_p)_{p\geq 0}$ converges to g in the $\mathfrak{m}$-adic topology of $K[[\boldsymbol{x}]]$ since for any positive integer $M \geq 1$, there exists by our construction an integer $M' \geq 1$ such that $g - f_p \in \mathfrak{m}^M$ for all $p \geq M'$. Hence the claim $g \overset{c}{\sim} f$ clearly follows.

Finally to show the first assertion of the theorem, we assume that $\mathfrak{m}^k \subset j(f)$. Similarly, we construct sequences

- $(\varphi_p)_{p\geq 1} \subset Aut(K[[\boldsymbol{x}]])$ and
- $(f_p)_{p\geq 0} \subset K[[\boldsymbol{x}]]$, such that $f_0 = f$ and for all $p \geq 1$, we have
  - (a) $f_p = \varphi_p(f_{p-1})$,
  - (b) $f_p \overset{r}{\sim} f$ and
  - (c) $g - f_p \in \mathfrak{m}^{N+p+1}$.

Observing again that $\mathfrak{m}^{N+1} = \mathfrak{m}^{N+1-k}\mathfrak{m}^k \subset \mathfrak{m}^{N+1-k} j(f)$, then we can write

$$g - f = \sum_{1\leq i\leq n} b_{i}^{(1)} f_{x_i}$$

with $b_{i}^{(1)} \in \mathfrak{m}^{N+1-k}$, for all $i = 1, \ldots, n$. Hence, arguing in the same way as in the first part of our proof shows that

$$\begin{aligned} \varphi_1 : \quad K[[\boldsymbol{x}]] &\longrightarrow K[[\boldsymbol{x}]] \\ x_i &\mapsto x_i + b_{i}^{(1)} \qquad \text{for } i = 1, \ldots, n \end{aligned}$$

is a K- algebra automorphism on $K[[\boldsymbol{x}]]$ and

$$\varphi_1(f) = f + \sum_{1\le i\le n} b_i^{(1)} f_{x_i} + h_1,$$

with $h_1 \in \mathfrak{m}^{N+2}$. Thus setting $f_1 := \varphi_1(f)$ shows that $g - f_1 = -h_1 \in \mathfrak{m}^{N+2}$. Proceeding recursively as in the proof of the second assertion of the theorem, we construct the sequences $(\varphi_p)_{p\ge 1}$ and $(f_p)_{p\ge 0}$ such that the latter converges to g in the $\mathfrak{m}$-adic topology. Thus $g \overset{r}{\sim} f$ clearly follows. □

Remark 3.1.16. *If $char(K) = 0$, we notice that the bound for determinacy given in Theorem 3.1.15 is in general higher than the one provided by Theorem 3.1.13. Indeed, (3.10) implies that $\mathfrak{m}^{k+2} \subset \mathfrak{m}^2 j(f) + \mathfrak{m}\langle f\rangle$. Hence, it follows by Theorem 3.1.13 that f is $k+1$-determined. Instead, Theorem 3.1.15 asserts that f is $2k-s+2$-determined and $2k - s + 2 \ge k + 1$ follows as $k \ge s - 1$.*

From the viewpoint of calculations, there is a handy way to compute the smallest bound of determinacy that one can obtain from Theorem 3.1.15. To do so, we need to compute the smallest positive integer k for which condition (3.9) or (3.10) holds. Using SINGULAR this computation can be accomplished by the function $highcorner$ of an ideal (in our case the Milnor or the Tjurina ideal) when a local degree ordering is predefined. The output is a monomial and the integer k is then the total degree of this monomial added to 1. For more details, we refer to [GrP02].

We attempt in the following to compare the bound of determinacy that we get from Theorem 3.1.15, and other well-known bounds of determinacy in positive characteristic. For this purpose, we make first the following observation.

Proposition 3.1.17. *Let $f \in \mathfrak{m}^2 \subset K[[\boldsymbol{x}]]$.*

1. *If $\mu(f) < \infty$, then $j(f) \supset \mathfrak{m}^{\mu(f)}$.*
2. *If $\tau(f) < \infty$, then $tj(f) \supset \mathfrak{m}^{\tau(f)}$.*

Proof. The assertions of Proposition 3.1.17 can both be proved in the same way. Therefore and for the reason of size, we only show the second assertion.
By assumption, T_f is a finite dimensional K-vector space of dimension $\tau(f)$. We set $\tau := \tau(f)$ and $I := tj(f)$. Furthermore, for $s \in \mathbb{Z}_{>0}$, let

$$\bar{\mathfrak{m}}^s := (\mathfrak{m}^s + I)/I,$$

be the image of $\mathfrak{m}^s$ in T_f. It is clear that for any s, $\bar{\mathfrak{m}}^s$ is a finite dimensional K-vector subspace of T_f. We claim that for all $1 \le s \le \tau$, we have $dim_K(\bar{\mathfrak{m}}^s) \le \tau - s$.
We argue by induction on s.
For $s = 1$, as $\bar{\mathfrak{m}}$ is the maximal ideal of the local K-algebra T_f, we have then $dim_K(T_f/\bar{\mathfrak{m}}) = 1$ and therefore $dim_K(\bar{\mathfrak{m}}) = \tau - 1$.
Now, let s be such that $1 \le s < \tau$ and we suppose that $dim_K(\bar{\mathfrak{m}}^s) \le \tau - s$. We have to consider the following two possibilities:

- $\bar{\mathfrak{m}}^{s+1} = \bar{\mathfrak{m}}^s$. Then, it follows by Nakayama's lemma that $\bar{\mathfrak{m}}^s = 0$ and hence $\mathfrak{m}^\tau \subset \mathfrak{m}^s \subset I$.
- $\bar{\mathfrak{m}}^{s+1}$ is a proper subspace of $\bar{\mathfrak{m}}^s$. Thus, $dim_K(\bar{\mathfrak{m}}^{s+1}) \leq dim_K(\bar{\mathfrak{m}}^s) - 1 \leq \tau - (s+1)$.

Therefore, we have $dim_K(\bar{\mathfrak{m}}^\tau) = 0$ and hence $\mathfrak{m}^\tau \subset I$. □

In [GrK90], the authors established the following bounds of determinacy in positive characteristic.

Theorem 3.1.18. *Let $char(K) \geq 0$ and let $f \in K[[\boldsymbol{x}]]$.*

1. *If $\mu(f) < \infty$, then f is right $2\mu(f)$-determined.*
2. *If $\tau(f) < \infty$, then f is contact $2\tau(f)$-determined.*

Proof. See [GrK90]. □

Remark 3.1.19. *It turns out that the bounds given in Theorem 3.1.15 are in generel better than those given by Theorem 3.1.18. Indeed, let $f \in \mathfrak{m}^2 \subset K[[\boldsymbol{x}]]$ be such that $\tau(f) < \infty$. Then, it follows from Proposition 3.1.17 that $\mathfrak{m}^\tau \subset tj(f)$. Hence, if we consider the smallest positive integer k such that $\mathfrak{m}^k \subset tj(f)$, we have clearly*

$$2\tau(f) \geq 2k \geq 2k - (ord(f) - 2).$$

Similarly, we notice that the same claim holds for the bounds of right determinacy.

Example 3.1.20. *Let $char(K) = 23$ and let $f = y^8 + x^8y^4 + x^{23} \in K[[x, y]]$. Using* SINGULAR, *we get $\tau(f) = 105$ and $\mathfrak{m}^{25} \subset tj(f)$. While Theorem 3.1.18 asserts that 210 is a bound of contact determinacy of f, we obtain from Theorem 3.1.15 that f is contact 44-determined.*

It is established over $\mathbb{C}$ that isolated hypersurface singularities are finitely determined and the converse does also hold ([GLS06, Corollary 2.39]). The last part of the present section is devoted to the study of this claim in positive characteristic. It is straightforward from Theorems 3.1.15 and 3.1.18 that in arbitrary characteristic, any isolated hypersurface singularity (resp. any $f \in K[[\boldsymbol{x}]]$ for which 0 is an isolated singularity) is finitely contact (resp. right) determined. The following proposition asserts that, conversely, the claim does also hold.

Theorem 3.1.21. *Let $f \in \mathfrak{m} \subset K[[\boldsymbol{x}]]$.*

1. *If f is right k-determined, then $\mathfrak{m} \cdot j(f) \supset \mathfrak{m}^{k+1}$.*
2. *If f is contact k-determined, then $\mathfrak{m} \cdot j(f) + \langle f \rangle \supset \mathfrak{m}^{k+1}$.*

Proof. We show only the second assertion of the theorem as the first one can be proved in the same way.
Let $f \in \mathfrak{m}$ be contact k-determined, and let $l \in \mathbb{Z}_{>0}$ be such that $l \geq k+1$.
Let $f^{(l)}$ be the l-jet of f. Furthermore, let $\mathcal{K}^{(l)}$ be the l-jet of the contact group $\mathcal{K}$.

Throughout this proof we shall write G for the algebraic group $\mathcal{K}^{(l)}$.
Considering the regular algebraic action, where by abuse of notation u, Φ and h denote k-jets and their representations at the same time,

$$\begin{array}{rccl} \psi_{(l)}: & G \times J^{(l)} & \longrightarrow & J^{(l)} \\ & ((u\,,\,\phi)\,,\,h) & \mapsto & (u\,,\,\phi)\cdot h := (u \cdot \phi(h))^{(l)} \end{array}$$

of G on the smooth variety $J^{(l)}$, Luna's slices theorem yields the existence in $J^{(l)}$ of a slice S to the orbit $Gf^{(l)}$ at the point $f^{(l)}$ under the action $\psi_{(l)}$ (cf. [Slo80, 5.1 Lemma 1]). This means by definition that

1. $f^{(l)} \in S$ and

2. the morphism

$$\begin{array}{rccl} \psi_{_S}: & G \times S & \longrightarrow & J^{(l)} \\ & ((u\,,\,\phi)\,,\,h) & \mapsto & (u\,,\,\phi)\cdot h \end{array}$$

   is smooth, which is equivalent to $\psi_{_S}$ is flat and all fibres are smooth (cf. [Har77, Theorem 10.2]).

Let $g \in \mathfrak{m}^{k+1}$, we shall show in the following that $g \in \mathfrak{m}^2 \cdot j(f) + \mathfrak{m} \cdot \langle f \rangle$. First we notice that, as f is k-determined, $f^{(l)} + tg^{(l)} \in Gf^{(l)}$ follows obviously for any $t \in K$. Hence, $L = \left\{ f^{(l)} + tg^{(l)} : t \in K \right\}$ is a line in $Gf^{(l)}$ and $\psi_{_S}^{-1}(L)$ is smooth in $G \times S$. Moreover, $\psi_{_S}^{-1}(L) \subset G \times \left\{ f^{(l)} \right\}$. Indeed, let $((u\,,\,\phi), h) \in \psi_{_S}^{-1}(L) \subset G \times S$, then $(u\,,\,\phi)\cdot h \in L \subset Gf^{(l)}$. Hence, for G is a group, we get $h \in Gf^{(l)}$. Altogether, we get $h \in Gf^{(l)} \cap S = \left\{ f^{(l)} \right\}$.
Furthermore, we have obviously $\psi_{_S}^{-1}(f^{(l)}) = G_{f^{(l)}} \times \left\{ f^{(l)} \right\}$ where $G_{f^{(l)}}$ is the stabilizer of $f^{(l)}$. On the other hand, as $\psi_{_S}^{-1}(L)$ is smooth, then we can write it as a product

$$\psi_{_S}^{-1}(L) \cong G_{f^{(l)}} \times L.$$

Thus, it follows that the morphism $\psi_{_S}^{-1}(L) \longrightarrow L$ is smooth. Moreover, we see clearly that $((1\,,\,id), f^{(l)}) \in \psi_{_S}^{-1}(L)$. Then by the curve selection lemma there exists a smooth locally closed variety T in G of dimension 1 and such that $(1\,,\,id) \in T$. Besides, the morphism $T \times \left\{ f^{(l)} \right\} \longrightarrow L$ is smooth and locally an isomorphism. Thus, for any $t \in K$, there exists locally a unique $(u_t\,,\,\phi_t) \in T$ such that $u_t \cdot \phi_t(f^{(l)}) = f^{(l)} + tg^{(l)}$. Moreover $(u_0\,,\,\phi_0) = (1\,,\,id)$ holds.
Recall that each automorphism ϕ of $K[[x]]$ is uniquely represented by a tuple $(\phi_1, \ldots, \phi_n) \in K[[x]]^n$ of power series such that

$$\phi_i(0) = 0 \quad \text{for all } i = 1, \ldots, n$$

and

$$\det \left(\frac{\partial \phi_i}{\partial x_j}(0) \right)_{i,j=1,\ldots,n} \neq 0.$$

Since the operation

$$inv:\ G \longrightarrow G : (u,\phi) \mapsto \big(\phi^{-1}(u^{-1}), \phi^{-1}\big)$$

of taking inverses is a self-inverse morphism of the algebraic group G, its restriction to T

$$T \longrightarrow G : (u_t,\phi_t) \mapsto \big(\phi_t^{-1}(u_t^{-1}), \phi_t^{-1}\big)$$

is an isomorphism from T onto its image, both of which are smooth curves in G.
In particular, parametrizing the image there are power series $u \in K[[x,t]]$ and $\phi = (\phi_1,\ldots,\phi_n) \in \mathbb{K}[[x,t]]^n$ such that

- $u(x,t) = \phi_t^{-1}\left(u_t^{-1}(x)\right)$ and
- $\phi_i(x,t) = \phi_t^{-1}(x_i)$ for $i=1,\ldots,n$.

Since $u_0 = 1$ and $\phi_0 = id_{K[[x]]^n}$ we have

- $u(x,0) = \phi_0^{-1}\left(u_0^{-1}(x)\right) = \phi_0^{-1}(1) = 1$ and
- $\phi_i(x,0) = \phi_0^{-1}(x_i) = x_i$ for $i=1,\ldots,n$.

Altogether with $F^{(l)} = f^{(l)} + t\cdot g^{(l)}$ and

$$u_t \cdot \phi_t\big(f^{(l)}\big) = f^{(l)} + t\cdot g^{(l)}$$

yields

$$F(x,0) = f^{(l)} = \phi_t^{-1}(u_t^{-1})\cdot\phi_t\big(F^l\big) = u(x,t)\cdot F^{(l)}\big(\phi(x,t),t\big).$$

Applying the derivation $\frac{\partial}{\partial t}$ to both sides of the equation we get

$$\begin{aligned} 0 \quad = \quad & \frac{\partial u}{\partial t}(x,t)\cdot F^{(l)}\big(\phi(x,t),t\big) \\ & +u(x,t)\cdot\left(\sum_{i=1}^n \frac{\partial F^{(l)}}{\partial x_i}\big(\phi(x,t),t\big)\cdot\frac{\partial\phi_i}{\partial t}(x,t) + \frac{\partial F^{(l)}}{\partial t}\big(\phi(x,t),t\big)\right). \end{aligned}$$

Evaluating the right hand side for $t=0$ and applying the above relations for $u(x,0)$ and $\phi_i(x,0)$ we get

$$0 = \frac{\partial u}{\partial t}(x,0)\cdot f^{(l)} + \sum_{i=1}^n \frac{\partial f^{(l)}}{\partial x_i}\cdot\frac{\partial\phi_i}{\partial t}(x,0) + g^{(l)},$$

or equivalently

$$g^{(l)} = -\frac{\partial u}{\partial t}(x,0)\cdot f^{(l)} - \sum_{i=1}^n \frac{\partial f^{(l)}}{\partial x_i}\cdot\frac{\partial\phi_i}{\partial t}(x,0).$$

Moreover, we have

$$\frac{\partial u}{\partial t}(x,0) \in K[[\boldsymbol{x}]]$$

and

$$\frac{\partial \phi_i}{\partial t}(x,0) \in \langle x_1, \ldots, x_n \rangle$$

for $i = 1, \ldots, n$, since then it follows that

$$g \in \langle f \rangle + \mathfrak{m} \cdot j(f) + \mathfrak{m}^{l+1}$$

for any $l \geq k+1$. Hence in particular

$$g \in \langle f \rangle + \mathfrak{m} \cdot j(f) + \mathfrak{m}^{k+2}.$$

Thus the claim $\mathfrak{m}^{k+1} \subset \mathfrak{m} \cdot j(f) + \langle f \rangle$ follows by Lemma 3.1.8. □

Whith these preparation made, we give in the following a characterization of finite determinacy.

Corollary 3.1.22. *Let K be an algebraically closed field of arbitrary characteristic. Let $f \in K[[\boldsymbol{x}]]$ and let R_f be the local ring of the hypersurface singularity defined by f. Then,*

1. 0 *is an isolated singularity of f, if and only if, f is right finitely determined.*
2. R_f *is isolated, if and only if, R_f is finitely determined.*

Proof. In both assertions of Corollary 3.1.22, the *if* part is straightforward from Theorem 3.1.21 while the *only if* part follows obviously from Theorem 3.1.15. □

At the end of this first section, we give the proof of the second claim of Proposition 2.3.19.

Proof. of Proposition 2.3.19 (the sequel) Let $f \in K[[\boldsymbol{x}]]$. We suppose that f satisfies $NPND^*$ with respect to some C-polytope $\boldsymbol{P}$. We have to show that $\mu(f) = V_N(\Gamma_-(f))$. This claim was established and proved by Wall over $\mathbb{C}$ in [Wal99a, 1.5]. It turns out that his arguments show also the claim in arbitrary characteristic. Hence, to avoid repetition, we present shortly Wall's idea for the proof: As f is $NPND^*$, the first part of Proposition 2.3.19 asserts that $\mu(f)$ is finite and Corollary 3.1.22 establishes that f is right finitely determined. On the other hand, suppose that for some q, $1 \leq q \leq n$, $\Gamma_{(f)}$ intersects the x_i-axis for $q < i \leq n$ but not for $1 \leq i \leq q$. We choose $m_1, m_2, \ldots, m_q \in \mathbb{Z}_{>0}$ such that m_1 is greater than the degree of determinacy and $m_2 < \ldots < m_q$. We set $\boldsymbol{m} = (m_1, \ldots, m_q) \in \mathbb{Z}^q_{>0}$ and we write $f_{\boldsymbol{m}} = f + \sum_{1 \leq i \leq q} {x_i}^{m_i}$. Clearly $f_{\boldsymbol{m}} \overset{\mathrm{r}}{\sim} f$ and therefore $\mu(f) = \mu(f_{\boldsymbol{m}})$ follows. Wall shows that the convenient power series f_m is $NPND$. Hence, Proposition 2.3.9 yields $\mu(f_{\boldsymbol{m}}) = V_N(\Gamma_-(f_{\boldsymbol{m}}))$. On the other hand, the map $m \mapsto V_N(\Gamma_-(f_m))$ is affine in each m_i separately. Moreover, it follows from Remark 2.1.12 that for all $\boldsymbol{m}$ we have $V_N(\Gamma_-(f_{\boldsymbol{m}})) \leq \mu(f_{\boldsymbol{m}})$. Thus, for m_i large enough, $V_N(\Gamma_-(f_{\boldsymbol{m}})) \leq \mu(f)$ and so $V_N(\Gamma_-(f_{\boldsymbol{m}}))$ is constant for $m_i >> 0$. Hence $V_N(\Gamma_-(f_{\boldsymbol{m}}))$ is identically constant. Finally taking each $m_i = 0$, we get $V_N(\Gamma_-(f))$ which completes the proof. □

3.2 (AA) and (AAC)-Hypersurface Singularities

For the purpose of computation of normal forms over $\mathbb{C}$ with respect to the right equivalence, Arnold introduced in [Arn74, 9.2] a condition that he called (A). In the first part of this section, we review briefly this condition and then reformulate it to a new condition which is compatible in arbitrary characteristic with the contact equivalence. For the sequel, let $\boldsymbol{W} \subset \mathbb{Z}^n_{>0}$ be an irredundant finite set of weights and let $\boldsymbol{P}$ be its associated C-polytope (cf. Remark 2.1.17).

Definition 3.2.1. *Let $f \in K[\boldsymbol{x}]$ be (PH) of type $(\boldsymbol{W}\,;\,d)$. We say that f is (A) with respect to $\boldsymbol{W}$ or f is (A) with respect to $\boldsymbol{P}$ if for any non zero $g \in j(f)$ there exists a derivation ξ such that*

(A1) $v_{\boldsymbol{W}}(g) = v_{\boldsymbol{W}}(\xi) + v_{\boldsymbol{W}}(f)$ *and*

(A2) $v_{\boldsymbol{W}}(g - \xi f) > v_{\boldsymbol{W}}(g)$.

In other words, we say that f is (A) with respect to $\boldsymbol{W}$ (or equivalently with respect to $\boldsymbol{P}$) if any non zero $g \in j(f)$ satisfies conditions $(A1)$ and $(A2)$ with respect to f and $\boldsymbol{W}$ (or equivalently with respect to f and $\boldsymbol{P}$).

We adapt in the following the condition (A) to arbitrary characteristic and we denote it (AC) where the added letter C refers to the contact equivalence relation.

Definition 3.2.2. *Let $f \in K[\boldsymbol{x}]$ be (PH) of type $(\boldsymbol{W}\,;\,d)$. We say that f is (AC) with respect to $\boldsymbol{W}$ or f is (AC) with respect to $\boldsymbol{P}$ if for any non zero $g \in tj(f)$ there exist a formal power series $b_0 \in K[[\boldsymbol{x}]]$ and a derivation $\xi \in Der_K(K[[\boldsymbol{x}]])$ such that*

(AC1) $v_{\boldsymbol{W}}(g) = min\{v_{\boldsymbol{W}}(b_0) + v_{\boldsymbol{W}}(f)\,;\, v_{\boldsymbol{W}}(\xi) + v_{\boldsymbol{W}}(f)\}$ *and*

(AC2) $v_{\boldsymbol{W}}(g - b_0 f - \xi f) > v_{\boldsymbol{W}}(g)$.

Hence, f is (AC) with respect to $\boldsymbol{W}$ (equivalently $\boldsymbol{P}$) if any non zero $g \in tj(f)$ satisfies conditions $(AC1)$ and $(AC2)$ with respect to f and $\boldsymbol{W}$ (equivalently f and $\boldsymbol{P}$).

We use for the following lemma Notation 2.1.57.

Lemma 3.2.3. *Let $f \in K[\boldsymbol{x}]$ be (PH) of type $(\boldsymbol{W}\,;\,d)$ and let $\boldsymbol{P}$ be the C-polytope associated to $\boldsymbol{W}$. Further, let Δ be a face of $\boldsymbol{P}$ and let $\boldsymbol{x}^{\boldsymbol{\alpha}} \in R_\Delta$.*

1. *If $\boldsymbol{x}^{\boldsymbol{\alpha}} \in R_\Delta \cap j(f)$ is $(A1)$ and $(A2)$ with respect to f and $\boldsymbol{W}$, then for any $\boldsymbol{\beta} \in P[\Delta]$ the monomial $x^{\boldsymbol{\alpha}+\boldsymbol{\beta}}$ satisfies also $(A1)$ and $(A2)$ with respect to f and $\boldsymbol{W}$.*

2. *If $\boldsymbol{x}^{\boldsymbol{\alpha}} \in R_\Delta \cap tj(f)$ is $(AC1)$ and $(AC2)$ with respect to f and $\boldsymbol{W}$, then for any $\boldsymbol{\beta} \in P[\Delta]$ the monomial $x^{\boldsymbol{\alpha}+\boldsymbol{\beta}}$ satisfies also $(AC1)$ and $(AC2)$ with respect to f and $\boldsymbol{W}$.*

Proof. The claims of Lemma 3.2.3 can be both proved in the same way. Hence for the reason of size we show only the second claim.
To do so, we suppose that $x^{\boldsymbol{\alpha}} \in R_\Delta \cap tj(f)$ and besides conditions $(AC1)$ and $(AC2)$ hold with respect to f and $\boldsymbol{W}$. Moreover, for $\boldsymbol{\beta} \in P[\Delta]$, Remark 2.1.58 yields

- $v_{\boldsymbol{W}}(x^\alpha) = v_\Delta(x^\alpha)$,
- $v_{\boldsymbol{W}}(x^{\boldsymbol{\beta}}) = v_\Delta(x^{\boldsymbol{\beta}})$,
- $x^{\boldsymbol{\alpha}+\boldsymbol{\beta}} \in R_\Delta$ and therefore $v_{\boldsymbol{W}}(x^{\boldsymbol{\alpha}+\boldsymbol{\beta}}) = v_\Delta(x^{\boldsymbol{\alpha}+\boldsymbol{\beta}})$.

On the other hand there exist $\xi \in Der_K(K[[\boldsymbol{x}]])$ and b_0 and $h \in K[[\boldsymbol{x}]]$ such that $x^{\boldsymbol{\alpha}} = b_0 f + \xi f + h$ with

(AC1) $v_{\boldsymbol{W}}(x^{\boldsymbol{\alpha}}) = min\{v_{\boldsymbol{W}}(b_0) + v_{\boldsymbol{W}}(f)\,;\, v_{\boldsymbol{W}}(\xi) + v_{\boldsymbol{W}}(f)\}$ and

(AC2) $v_{\boldsymbol{W}}(h) > v_{\boldsymbol{W}}(x^{\boldsymbol{\alpha}})$.

Thus we can write

$$x^{\boldsymbol{\alpha}+\boldsymbol{\beta}} = (x^{\boldsymbol{\beta}} b_0) f + (x^{\boldsymbol{\beta}} \xi) f + x^{\boldsymbol{\beta}} h.$$

Moreover, as $x^{\boldsymbol{\alpha}}$ is $(AC2)$ with respect to f and $\boldsymbol{W}$, and using Lemma 2.1.22 we get

$$v_{\boldsymbol{W}}(x^{\boldsymbol{\beta}} h) > v_{\boldsymbol{W}}(x^{\boldsymbol{\alpha}+\boldsymbol{\beta}}) \tag{3.11}$$

Indeed

$$\begin{array}{rcl} v_{\boldsymbol{W}}(x^{\boldsymbol{\beta}} h) & \geq & v_{\boldsymbol{W}}(x^{\boldsymbol{\beta}}) + v_{\boldsymbol{W}}(h) \\ & > & v_{\boldsymbol{W}}(x^{\boldsymbol{\beta}}) + v_{\boldsymbol{W}}(x^{\boldsymbol{\alpha}}) \\ & = & v_\Delta(x^{\boldsymbol{\alpha}}) + v_\Delta(x^{\boldsymbol{\beta}}) \\ & = & v_\Delta(x^{\boldsymbol{\alpha}+\boldsymbol{\beta}}) \\ & = & v_{\boldsymbol{W}}(x^{\boldsymbol{\alpha}+\boldsymbol{\beta}}) \end{array}$$

Furthermore, since $\boldsymbol{x}^{\boldsymbol{\alpha}}$ is $(AC1)$ with respect to f and $\boldsymbol{W}$, we can suppose without loss of generality that $v_{\boldsymbol{W}}(x^{\boldsymbol{\alpha}}) = v_{\boldsymbol{W}}(b_0) + v_{\boldsymbol{W}}(f)$.
We claim that either $v_{\boldsymbol{W}}(\xi f) = v_{\boldsymbol{W}}(\xi) + v_{\boldsymbol{W}}(f)$ or $v_{\boldsymbol{W}}(b_0 f) = v_{\boldsymbol{W}}(b_0) + v_{\boldsymbol{W}}(f)$ holds. Indeed $v_{\boldsymbol{W}}(h) > v_{\boldsymbol{W}}(\boldsymbol{x}^{\boldsymbol{\alpha}})$ yields

$$\begin{array}{rcl} v_W(x^{\boldsymbol{\alpha}}) & \geq & min\{v_{\boldsymbol{W}}(b_0 f)\,;\, v_{\boldsymbol{W}}(\xi f)\} \\ & \geq & min\{v_{\boldsymbol{W}}(b_0) + v_{\boldsymbol{W}}(f)\,;\, v_{\boldsymbol{W}}(\xi) + v_{\boldsymbol{W}}(f)\} \\ & = & v_{\boldsymbol{W}}(b_0) + v_{\boldsymbol{W}}(f) = v_{\boldsymbol{W}}(x^{\boldsymbol{\alpha}}). \end{array}$$

Then, it follows that $v_{\boldsymbol{W}}(x^{\boldsymbol{\alpha}}) = min\{v_{\boldsymbol{W}}(b_0 f)\,;\, v_{\boldsymbol{W}}(\xi f)\}$.

- If $min\{v_{\boldsymbol{W}}(b_0 f)\,;\, v_{\boldsymbol{W}}(\xi f)\} = v_{\boldsymbol{W}}(b_0 f)$, then we get

$$v_{\boldsymbol{W}}(b_0) + v_{\boldsymbol{W}}(f) = v_{\boldsymbol{W}}(x^{\boldsymbol{\alpha}}) = v_{\boldsymbol{W}}(b_0 f).$$

- If $min\{v_{\boldsymbol{W}}(b_0 f)\,;\, v_{\boldsymbol{W}}(\xi f)\} = v_{\boldsymbol{W}}(\xi f)$, that is

$$\begin{aligned} v_{\boldsymbol{W}}(\xi f) &= v_{\boldsymbol{W}}(x^{\boldsymbol{\alpha}}) \\ &= v_{\boldsymbol{W}}(b_0) + v_{\boldsymbol{W}}(f) \\ &\leq v_{\boldsymbol{W}}(\xi) + v_{\boldsymbol{W}}(f) \end{aligned}$$

  Nevertheless, Lemma 2.2.3 asserts $v_{\boldsymbol{W}}(\xi) + v_{\boldsymbol{W}}(f) \leq v_{\boldsymbol{W}}(\xi f)$. Hence, we get $v_{\boldsymbol{W}}(\xi f) = v_{\boldsymbol{W}}(\xi) + v_{\boldsymbol{W}}(f)$.

And so the claim follows. Using this we show in the following that $\boldsymbol{x}^{\boldsymbol{\alpha}+\boldsymbol{\beta}}$ is $(AC1)$ with respect to f and $\boldsymbol{W}$.

(a) If $v_{\boldsymbol{W}}(b_0 f) = v_{\boldsymbol{W}}(b_0) + v_{\boldsymbol{W}}(f) = v_{\boldsymbol{W}}(x^{\boldsymbol{\alpha}})$, then it follows from Lemma 2.1.22 that $v_{\boldsymbol{W}}(f) = v_{\Delta}(f)$ and $v_{\boldsymbol{W}}(b_0) = v_{\Delta}(b_0)$. Therefore

$$\begin{aligned} v_{\boldsymbol{W}}(x^{\boldsymbol{\alpha}+\boldsymbol{\beta}}) &= v_{\boldsymbol{W}}(x^{\boldsymbol{\alpha}}) + v_{\boldsymbol{W}}(x^{\boldsymbol{\beta}}) \\ &= v_{\boldsymbol{W}}(b_0) + v_{\boldsymbol{W}}(f) + v_{\boldsymbol{W}}(x^{\boldsymbol{\beta}}) \\ &= v_{\Delta}(b_0) + v_{\Delta}(f) + v_{\Delta}(x^{\boldsymbol{\beta}}) \\ &= v_{\Delta}(x^{\boldsymbol{\beta}} \cdot b_0) + v_{\Delta}(f) \\ &= v_{\boldsymbol{W}}(x^{\boldsymbol{\beta}} \cdot b_0) + v_{\boldsymbol{W}}(f). \end{aligned}$$

(b) If $v_{\boldsymbol{W}}(\xi f) = v_{\boldsymbol{W}}(\xi) + v_{\boldsymbol{W}}(f) = v_{\boldsymbol{W}}(x^{\boldsymbol{\alpha}})$, then we show as for the above that

$$v_{\boldsymbol{W}}(x^{\boldsymbol{\alpha}+\boldsymbol{\beta}}) = v_{\boldsymbol{W}}(x^{\boldsymbol{\beta}}\xi) + v_{\boldsymbol{W}}(f).$$

Therefore $x^{\boldsymbol{\alpha}+\boldsymbol{\beta}}$ is $(AC1)$ with respect to f and $\boldsymbol{W}$. Moreover, it follows clearly from (3.11) that the condition $(AC2)$ holds which terminates the proof.

$\square$

Proposition 3.2.4. *Let $f \in K[\boldsymbol{x}]$ be (QH) of type $(\boldsymbol{w}\,;\, d)$, $\boldsymbol{w} \in \mathbb{Z}^n_{>0}$ and $d \in \mathbb{Z}_{>0}$. Then f is (A) and (AC) with respect to $\{\boldsymbol{w}\}$.*

Proof. Let $f \in K[\boldsymbol{x}]$ be a quasihomogeneous polynomial of type $(\boldsymbol{w}\,;\, d)$, $\boldsymbol{w} \in \mathbb{Z}^n_{>0}$ and $d \in \mathbb{Z}_{>0}$. We write $\boldsymbol{w} = (w_1, \ldots, w_n)$ and $I = tj(f)$.
For any $i = 1, \ldots, n$, we have clearly that

$$f_{x_i} \;\text{ is either }\; 0 \;\text{ or a }\; (QH) \;\text{ polynomial of type }\; (\boldsymbol{w}\,;\, d - w_i).$$

For $g \in I$, we show in the following the existence of power series b_0 and $g_1 \in K[[\boldsymbol{x}]]$ and a derivation ξ such that

$$g = b_0 f + \xi f + g_1$$

satisfying

- $(AC1) : v(g) := \boldsymbol{w}\text{-ord}(g) = min\{v(b_0) + v(f)\,;\, v(\xi) + v(f)\}$ and

- $(AC2) : v(g - b_0 f - \xi f) > v(g)$.

To do so, we consider the set $\mathcal{I} := \{i : 1 \leq i \leq n \text{ and } f_{x_i} \neq 0\}$.
We notice that the set $\mathcal{I}$ may be empty when $char(K) > 0$.
For the sequel, we denote the set $\mathcal{I} \cup \{0\}$ by $\mathcal{I}_0$ and f by f_{x_0}.
As $g \in I$, then we can write $g = \sum_{i \in \mathcal{I}_0} h_i f_{x_i}$, where the h_i are power series in $K[[\boldsymbol{x}]]$.
For $i \in \mathcal{I}_0$, we denote $d_i := v(h_i)$.
Hence, for $i \in \mathcal{I}_0$, we can write

$$h_i := h_i^{(1)} + h_i^{(2)},$$

where $h_i^{(1)} \in K[\boldsymbol{x}]$ is a (QH) polynomial of $\boldsymbol{w}$-degree equal to d_i and
$h_i^{(2)} \in K[[\boldsymbol{x}]]$ is such that $v(h_i^{(2)}) > d_i$. Hence

$$g = \sum_{i \in \mathcal{I}_0} h_i^{(1)} f_{x_i} + \sum_{i \in \mathcal{I}_0} h_i^{(2)} f_{x_i}.$$

We set $w_0 := 0$ and we observe that the polynomials $h_i^{(1)} f_{x_i}$, $i \in \mathcal{I}_0$, are (QH) of degree $d_i + d - w_i$. Thus we have obviously for all $i \in \mathcal{I}_0$

$$v(h_i^{(1)} f_{x_i}) = v(h_i^{(1)}) + v(f_{x_i}) = d_i + d - w_i.$$

On the other hand, we have for any $i \in \mathcal{I}_0$,

$$\begin{aligned} v(h_i^{(2)} f_{x_i}) &= v(h_i^{(2)}) + v(f_{x_i}) \\ &> d_i + d - w_i \\ &= v(h_i^{(1)} f_{x_i}). \end{aligned}$$

Then, it follows clearly that

$$\begin{aligned} v(g) &\geq min\left\{v(h_i^{(1)} f_{x_i}) : i \in \mathcal{I}_0\right\} \\ &= min\left\{d_i + d - w_i : i \in \mathcal{I}_0\right\} \end{aligned}$$

Now,let $i_0 \in \mathcal{I}_0$ such that $min\{d_i + d - w_i : i \in \mathcal{I}_0\} = d_{i_0} + d - w_{i_0}$.

- If $v(g) = d_{i_0} + d - w_{i_0}$, then the claim of the Proposition 3.2.4 follows clearly by taking $b_0 = h_0^{(1)}$, $\xi = \sum_{i \in \mathcal{I}} h_i^{(1)} \partial_{x_i}$, $g_1 = \sum_{i \in \mathcal{I}_0} h_i^{(2)} f_{x_i}$ and besides by showing that $v(\xi) = min\{d_i - w_i : i \in \mathcal{I}\}$. To do so we consider the linear function λ: $\mathbb{R}^n \to \mathbb{R}$ associated to $\boldsymbol{w}$ and defined by

$$\lambda(\boldsymbol{\alpha}) := \langle \boldsymbol{w}, \boldsymbol{\alpha} \rangle := \sum_{i=1}^{n} w_i \alpha_i,$$

with $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n$. On the other hand, Definition 2.2.1 yields

$$\begin{aligned}
v(\xi) &= min\left\{v(h_i^{(1)}\partial_{x_i}): \;\; i \in \mathcal{I}\right\} \\
&= min\Big\{min\{\lambda(\boldsymbol{\alpha} - \epsilon_i): \;\; \boldsymbol{\alpha} \in supp(h_i^{(1)})\}: \;\; i \in \mathcal{I}\Big\} \\
&= min\Big\{min\{\lambda(\boldsymbol{\alpha}): \;\; \boldsymbol{\alpha} \in supp(h_i^{(1)})\} - \lambda(\epsilon_i): \;\; i \in \mathcal{I}\Big\} \\
&= min\left\{v(h_i^{(1)}) - \lambda(\epsilon_i): \;\; i \in \mathcal{I}\right\} \\
&= min\left\{d_i - w_i: \;\; i \in \mathcal{I}\right\}.
\end{aligned}$$

- If $v(g) > d_{i_0} + d - w_{i_0}$ however, there exists a subset $\mathcal{I}_0^{(1)}$ of $\mathcal{I}_0$ containing i_0 such that

  1. $d_i + d - w_i = d_{i_0} + d - w_{i_0}$ for all $i \in \mathcal{I}_0^{(1)}$ and
  2. $\sum_{i \in \mathcal{I}_0^{(1)}} h_i^{(1)} f_{x_i} = 0$.

  Hence, we have
  $$g = \sum_{i \in \mathcal{I}_0 \setminus \mathcal{I}_0^{(1)}} h_i^{(1)} f_{x_i} + \sum_{i \in \mathcal{I}_0} h_i^{(2)} f_{x_i}.$$

  Now we have to consider two cases:

  (a) If $\mathcal{I}_0 \setminus \mathcal{I}_0^{(1)} \neq \emptyset$, then we get
  $$v(g) \geq min\left\{d_i + d - w_i : \, i \in \mathcal{I}_0 \setminus \mathcal{I}_0^{(1)}\right\}.$$
  If the equality holds, then the claim follows.
  If not, then we use the same considerations as in the above to rewrite g.

  (b) If $\mathcal{I}_0 \setminus \mathcal{I}_0^{(1)} = \emptyset$, then g has the form
  $$g = \sum_{i \in \mathcal{I}_0} h_i^{(2)} f_{x_i}.$$
  In this case we decompose the power series $h_i^{(2)}$, $i \in \mathcal{I}_0$, into their (QH) parts as we did for the power series h_i, $i \in \mathcal{I}_0$.

Thus, using again the method that we followed in the case where the equality between the weighted orders does not hold, we show that after finitely many iterations there exists a subset $\mathcal{I}_0^*$ of $\mathcal{I}_0$ such that

$$g = \sum_{i \in \mathcal{I}_0^*} b_i f_{x_i} + g_1,$$

where

1. for all $i \in \mathcal{I}_0^*$, b_i is a (QH) polynomial,

2. $v(g) = min\left\{v(b_i f_{x_i}) : i \in \mathcal{I}_0^*\right\} = min\left\{v(b_i) + v(f_{x_i}) : i \in \mathcal{I}_0^*\right\}$ and
3. $v(g_1) > v(g)$.

Hence, the claim follows by setting

$$\xi = \sum_{i \in \mathcal{I}_0^* \setminus \{0\}} b_i \partial_{x_i}.$$

Altogether, it yields f is (AC) with respect to $\{\boldsymbol{w}\}$. Finally, we should notice that the so far used arguments in the present proof show in the same way that f is (A) with respect to $\{\boldsymbol{w}\}$. Hence in order to avoid repetition, we decide to omit the proof of the last claim. □

Considering a (PH) polynomial f, we discuss in the following how condition (A) (resp. (AC)) is related to the piecewise-homogeneous grading of the K-algebras M_f (resp. T_f). For this purpose we consider:

Notation 3.2.5. *Let $f \in K[\boldsymbol{x}]$ be (PH) of type $(\boldsymbol{W}\,;\,d)$.*

1. *We write*

$$j^A_{\boldsymbol{W}}(f) := \langle g \in j(f) :\ g \ \textit{ is } \ (A1) \ \textit{ with respect to } \ f,\ \boldsymbol{W}\rangle$$

and

$$tj^{AC}_{\boldsymbol{W}}(f) := \langle g \in tj(f) :\ g \ \textit{ is } \ (AC1) \ \textit{ with respect to } \ f,\ \boldsymbol{W}\rangle\,.$$

2. *For $d \in \mathbb{N}$, we write*

$$j^A_{\boldsymbol{W}}(f,d) := \langle g \in j(f) :\ v_{\boldsymbol{W}}(g) = d \ \textit{ and } \ g \ \textit{ is } \ (A1) \ \textit{ with respect to } \ f,\ \boldsymbol{W}\rangle$$

and

$$tj^{AC}_{\boldsymbol{W}}(f,d) := \langle g \in tj(f) :\ v_{\boldsymbol{W}}(g) = d \ \textit{ and } \ g \ \textit{ is } \ (AC1) \ \textit{ with respect to } \ f,\ \boldsymbol{W}\rangle\,.$$

3. *We denote*

$$gr^A_{\boldsymbol{W}}(M_f) := \bigoplus_{d \geq 0} F_{\geq d}/(j^A_{\boldsymbol{W}}(f,d) + F_{>d}),$$

and

$$gr^{AC}_{\boldsymbol{W}}(T_f) := \bigoplus_{d \geq 0} F_{\geq d}/(tj^{AC}_{\boldsymbol{W}}(f,d) + F_{>d}).$$

Refering to Definition 2.1.44, we should mention that $gr^A_{\boldsymbol{W}}(M_f)$ (resp. $gr^{AC}_{\boldsymbol{W}}(T_f)$) is a K-algebra in the same way as $gr_{\boldsymbol{W}}(M_f)$ (resp. $gr_{\boldsymbol{W}}(T_f)$). Nevertheless, it is of interest to notice the following.

Remark 3.2.6. *If $\mu(f) < \infty$ (resp. $\tau(f) < \infty$), then Proposition 2.1.53 establishes that $gr_{\boldsymbol{W}}(M_f)$ (resp. $gr_{\boldsymbol{W}}(T_f)$) has finite dimension as K-vector space. For $gr^A_{\boldsymbol{W}}(M_f)$ (resp. $gr^{AC}_{\boldsymbol{W}}(T_f)$) yet, the dimension can be infinite as Example 3.2.16 shows.*

Furthermore, it is not difficult to establish the following relations between the so far defined K-algebras.

Lemma 3.2.7. *Let $f \in K[\boldsymbol{x}]$ be (PH) of type $(\boldsymbol{W}\,;\,d)$. Then, there exist canonical epimorphisms of K-vector spaces*

$$gr^A_{\boldsymbol{W}}(M_f) \twoheadrightarrow gr^{AC}_{\boldsymbol{W}}(T_f)\ ,\ gr^A_{\boldsymbol{W}}(M_f) \twoheadrightarrow gr_{\boldsymbol{W}}(M_f)\ ,\ gr^{AC}_{\boldsymbol{W}}(T_f) \twoheadrightarrow gr_{\boldsymbol{W}}(T_f).$$

If moreover f is (A) (resp. (AC)) with respect to $\boldsymbol{W}$, then $gr^A_{\boldsymbol{W}}(M_f) \cong gr_{\boldsymbol{W}}(M_f)$ (resp. $gr^{AC}_{\boldsymbol{W}}(T_f) \cong gr_{\boldsymbol{W}}(T_f)$) as K-vector spaces.

Proof. The claim of lemma 3.2.7 is straightforward from Definition 2.1.44 and Notation 3.2.5. This is why we choose to omit the proof. □

For the sequel, we consider a (PH) polynomial f such that $\mu(f) < \infty$ (resp. $\tau(f) < \infty$). The emphasis is put on the relations between the Milnor (resp. Tjurina) algebra of f and their associated piecewise-homogeneous gradings.

Lemma 3.2.8. *Let $f \in K[\boldsymbol{x}]$ be (PH) of type $(\boldsymbol{W}\,;\,d)$.*

1. *If $\tau(f) < \infty$, then*
$$gr^{AC}_{\boldsymbol{W}}(T_f) \twoheadrightarrow gr_{\boldsymbol{W}}(T_f) \twoheadrightarrow T_f.$$
2. *If $\mu(f) < \infty$, then*
$$gr^A_{\boldsymbol{W}}(M_f) \twoheadrightarrow gr_{\boldsymbol{W}}(M_f) \twoheadrightarrow M_f.$$

Proof. The claim is straightforward from Corollary 2.1.54 and Lemma 3.2.7. □

From the computational point of view, the following proposition is crucial for it provides a characterization of conditions (A) and (AC) by means of finite dimensional K-vector spaces.

Proposition 3.2.9. *Let $f \in K[\boldsymbol{x}]$ be (PH) of type $(\boldsymbol{W}\,;\,d)$.*

1. *If $\mu(f) < \infty$, then f is (A) with respect to $\boldsymbol{W}$, if and only if, $gr^A_{\boldsymbol{W}}(M_f) \cong M_f$ as K-vector spaces, i.e $dim_K(gr^A_{\boldsymbol{W}}(M_f)) = \mu(f)$*
2. *If $\tau(f) < \infty$, then f is (AC) with respect to $\boldsymbol{W}$, if and only if, $gr^{AC}_{\boldsymbol{W}}(T_f) \cong T_f$ as K-vector spaces, i.e $dim_K(gr^{AC}_{\boldsymbol{W}}(T_f)) = \tau(f)$.*

Proof. In the following we only show the second assertion of proposition 3.2.9 as the first one can be proved in the same way. Hence, we consider a piecewise-homogeneous polynomial $f \in K[\boldsymbol{x}]$ such that $\tau(f) < \infty$. We denote $I_f := tj(f)$ and $I^{AC}_{f,d} := tj^{AC}_{\boldsymbol{W}}(f,d)$. Moreover, let $\mathcal{B} = \{e_{\boldsymbol{\alpha}}, \boldsymbol{\alpha} \in \Lambda\}$ be a basis of the K-vector space $gr^{AC}_{\boldsymbol{W}}(T_f)$ consisting of monomials of $K[[\boldsymbol{x}]]$.

As $gr_{\boldsymbol{W}}^{AC}(T_f) \twoheadrightarrow gr_{\boldsymbol{W}}(T_f) \twoheadrightarrow T_f$ follows by Lemma 3.2.8, then $\mathcal{B}$ projects to a generating system of T_f. Namely, the set $\{e_{\boldsymbol{\alpha}} \;\; mod(I_f),\, \boldsymbol{\alpha} \in \Lambda\}$ is a generating system of T_f. First, we suppose that f satisfies the condition (AC) and we show that the system $\{e_{\boldsymbol{\alpha}}\text{mod}(I_f), \boldsymbol{\alpha} \in \Lambda\}$ is linearly independant in T_f. Indeed, considering a relation $\sum_{\boldsymbol{\alpha}\in\Lambda} c_{\boldsymbol{\alpha}} e_{\boldsymbol{\alpha}} \, mod\,(I_f) \equiv 0$, where for $\boldsymbol{\alpha} \in \Lambda$, $c_{\boldsymbol{\alpha}} \in K$, means that

$$\sum_{\boldsymbol{\alpha}\in\Lambda} c_{\boldsymbol{\alpha}} e_{\boldsymbol{\alpha}} \in I_f. \tag{3.12}$$

If the least $\boldsymbol{W}$-degree of a monomial occuring in the relation (3.12) with non zero coefficient is $d \in \mathbb{Z}_{>0}$, then we have

$$v_{\boldsymbol{W}}\left(\sum_{\boldsymbol{\alpha}\in\Lambda} c_{\boldsymbol{\alpha}} e_{\boldsymbol{\alpha}}\right) = d.$$

Hence,

$$\sum_{\boldsymbol{\alpha}\in\Lambda} c_{\boldsymbol{\alpha}} e_{\boldsymbol{\alpha}} \in F_{\geq d} \cap I_f.$$

Moreover, using Notation 3.2.5, condition $(AC2)$ yields

$$\sum_{\boldsymbol{\alpha}\in\Lambda} c_{\boldsymbol{\alpha}} e_{\boldsymbol{\alpha}} \in I_{f,d}^{AC} + F_{>d}.$$

Hence, in the K-space $F_{\geq d}/(I_{f,d}^{AC} + F_{>d})$, we have $\sum_{\boldsymbol{\alpha}\in\Lambda} c_{\boldsymbol{\alpha}} e_{\boldsymbol{\alpha}} = 0$. Thus, the set $\{e_{\boldsymbol{\alpha}} : \boldsymbol{\alpha} \in \Lambda\}$ is dependant in the K-space $F_{\geq d}/(I_{f,d}^{AC} + F_{>d})$ against the choice of the $e_{\boldsymbol{\alpha}}$ and so the claim follows.
Now, we suppose that the surjection $gr_{\boldsymbol{W}}^{AC}(T_f) \twoheadrightarrow T_f$ is an isomorphism of K-linear spaces. Hence, the set $\{e_{\boldsymbol{\alpha}} \text{ mod}(I_f) : \boldsymbol{\alpha} \in \Lambda\}$ is a basis of the linear space T_f. For the sequel we consider $g \in K[[\boldsymbol{x}]]$ such that $v_{\boldsymbol{W}}(g) = d$. Hence, we can write $g = g_d + g_{>d}$ where g_d is (PH) of type $(\boldsymbol{W}\,;\,d)$ and $v_{\boldsymbol{W}}(g_{>d}) > d$. We denote

$$\Lambda_{(d)} := \Big\{\boldsymbol{\alpha} \in \Lambda : \boldsymbol{W}\text{-deg}(e_{\boldsymbol{\alpha}}) = d\Big\} \text{ and } \Lambda_{(>d)} := \Big\{\boldsymbol{\alpha} \in \Lambda : \boldsymbol{W}\text{-deg}(e_{\boldsymbol{\alpha}}) > d\Big\}.$$

Hence, we can write

$$g\, mod\,(I_f) = \sum_{\boldsymbol{\alpha}\in\Lambda_{(d)}} c_{\boldsymbol{\alpha}} e_{\boldsymbol{\alpha}}\, mod\,(I_f) + \sum_{\boldsymbol{\alpha}\in\Lambda_{(>d)}} c_{\boldsymbol{\alpha}} e_{\boldsymbol{\alpha}}\, mod\,(I_f). \tag{3.13}$$

On the other hand, we have

$$g_d - \sum_{\boldsymbol{\alpha}\in\Lambda_{(d)}} c_{\boldsymbol{\alpha}} e_{\boldsymbol{\alpha}} \in I_{f,d}^{AC} + F_{>d}. \tag{3.14}$$

Now we assume $g \in I_f$. As $\Big\{e_{\boldsymbol{\alpha}} \text{ mod}(I_f) : \boldsymbol{\alpha} \in \Lambda\Big\}$ is a basis of T_f, then it follows in particular from (3.13) that all the coefficients $c_{\boldsymbol{\alpha}}$, $\boldsymbol{\alpha} \in \Lambda_{(d)}$, are 0. Thus, (3.14) becomes

$$g_d \in I_{f,d}^{AC} + F_{>d}.$$

Therefore, $g \in I_{f,d}^{AC} + F_{>d}$ and so the claim follows. $\square$

Although the central result of Arnold in [Arn74, 9.5] on normal forms supposes that the condition (A) holds, Wall observed in his paper [Wal99a] that this condition is not necessary. We shall reformulate Wall's discussion in the next section of the present chapter and give an explicit development about the computation of normal forms. For this purpose, it deserves to elaborate the following new conditions.

Definition 3.2.10. *Let $f \in K[\boldsymbol{x}]$ be (PH) of type $(\boldsymbol{W}\,;\,d)$. We say that f is* **almost** *(A) and we write f is (AA) (resp. f is* **almost** *(AC) and we write f is (AAC)) with respect to $\boldsymbol{W}$ if $dim_K(gr^A_{\boldsymbol{W}}(M_f) < \infty$ (resp. $dim_K(gr^{AC}_{\boldsymbol{W}}(T_f)) < \infty$. Furthermore, we call a K-basis of $gr^A_{\boldsymbol{W}}(M_f)$ (resp. $gr^{AC}_{\boldsymbol{W}}(T_f)$) consisting of monomials a* **regular basis** *of M_f (resp. T_f).*

Notation 3.2.11. *If f is (AA) (resp. (AAC)) with respect to $\boldsymbol{W}$ and if $\boldsymbol{P}$ is the C-polytope associated to $\boldsymbol{W}$, then we say also that f is (AA) (resp. (AAC)) with respect to $\boldsymbol{P}$.*

Lemma 3.2.12. *Let $f \in K[\boldsymbol{x}]$ be (PH) of type $(\boldsymbol{W}\,;\,d)$. If f is (AA) with respect to $\boldsymbol{W}$ then f is (AAC) with respect to $\boldsymbol{W}$.*

Proof. It follows from Lemma 3.2.7 that $dim_K(gr^{AC}_{\boldsymbol{W}}(T_f)) \leq dim_K(gr^A_{\boldsymbol{W}}(M_f))$ which shows obviously the claim. ☐

Remark 3.2.13. *Let $f \in K[\boldsymbol{x}]$ be (PH) of type $(\boldsymbol{W}\,;\,d)$. The following observations are straightforward from Proposition 3.2.9:*

1. *If $\mu(f) < \infty$ and f is (A) with respect to $\boldsymbol{W}$ then f is also (AA) with respect to $\boldsymbol{W}$.*
2. *If $\tau(f) < \infty$ and f is (AC) with respect to $\boldsymbol{W}$ then f is also (AAC) with respect to $\boldsymbol{W}$.*

Proposition 3.2.14. *Let $char(K) = 0$ and let $f = x^a + \lambda x^2y^2 + y^b \in K[[x, y]]$, where $\lambda \neq 0$, $a \geq 4$ and $b \geq 5$. If $\mu(f) < \infty$ (resp. $\tau(f) < \infty$), then f is (A) (resp. (AC)) with respect to $\Gamma(f)$. Furthermore, there exists a regular basis of M_f (resp. of T_f) consisting of monomials lying strictly below $\Gamma(f)$.*

Proof. Let $char(K) = 0$ and let $f = x^a + \lambda x^2y^2 + y^b \in K[[x, y]]$, where $\lambda \neq 0$, $a \geq 4$ and $b \geq 5$. Without loss of generality, we can suppose that $b \geq a$. We write $a = da'$ and $b = db'$ where $d = gcd(a, b)$. Clearly f is (PH) of type $\left(\boldsymbol{W}\,;\,\bar{d}\right)$ where $\boldsymbol{W} = \{(2b', (a-2)\cdot b')\,,\,((b-2)\cdot a', 2a')\}$ and $\bar{d} = 2da'b'$.
Arnold established that f is (A) with respect to $\Gamma(f)$ and showed the existence of a regular basis of M_f such that any monomial in it lies strictly below $\Gamma(f)$. For the proof of this claim we refer to [Arn74, 9.8 and 9.9].
For the sequel, we assume $\tau(f) < \infty$ and we show that f is (AC). We have

$$f_x = ax^{a-1} + 2\lambda xy^2\ ,\ f_y = 2\lambda x^2y + by^{b-1}.$$

So, it is not difficult to see that the set of monomials

$$\mathcal{B} = \left\{1, x, \ldots, x^{a-1}, y, xy, y^2, \ldots, y^{b-1}\right\}$$

is a K-vector space basis of T_f. Thus $\tau(f) = a + b$.
Moreover, we claim that the monomials xy^2, x^2y, x^2y^2, x^a and y^b fulfill conditions $(AC1)$ and $(AC2)$ with respect to f and $\boldsymbol{W}$. Indeed, we can write

(1) $xy^2 = \xi_1 f + h_1$ where $\xi_1 = \frac{1}{2\lambda}\partial_x$ and $h_1 = -\frac{1}{2\lambda}ax^{a-1}$.

(2) $x^2y = \xi_2 f + h_2$ where $\xi_2 = \frac{1}{2\lambda}\partial_y$ and $h_1 = -\frac{1}{2\lambda}by^{b-1}$.

(3) $x^2y^2 = \alpha_1 f + \xi_3 f$, where α_1 is a non zero constant and $\xi_3 = \alpha_2 x\partial_x + \alpha_3 y\partial_y$ with also $\alpha_2, \alpha_3 \in K \setminus \{0\}$.

(4) $x^a = \beta_1 f + \xi_4 f$, where $\xi_4 = \beta_2 x\partial_x + \beta_3 y\partial_y$ and $\beta_i \in K \setminus \{0\}$ for $1 \leq i \leq 3$.

(5) $y^b = \gamma_1 f + \xi_5 f$, where $\xi_5 = \gamma_2 x\partial_x + \gamma_3 y\partial_y$ and $\gamma_i \in K \setminus \{0\}$ for $1 \leq i \leq 3$.

On the other hand, we have

(1) $v_{\boldsymbol{W}}(xy^2) = a'(b+2)$ and $v_{\boldsymbol{W}}(\xi_1) = -(b-2)a'$. Thus, we get clearly $v_{\boldsymbol{W}}(xy^2) = v_{\boldsymbol{W}}(\xi_1) + v_{\boldsymbol{W}}(f)$. Moreover

$$v_W(h_1) = 2b'(a-1) = v_{\boldsymbol{W}}(xy^2) + (b'(a-2) - 2a') > v_{\boldsymbol{W}}(xy^2).$$

Therefore the claim follows for xy^2.

(2) In the same way x^2y satisfies $(AC1)$ and $(AC2)$ with respect to f and $\boldsymbol{W}$.

(3) Obviously $v_{\boldsymbol{W}}(\xi_3) = 0$ and $v_{\boldsymbol{W}}(x^2y^2) = v_{\boldsymbol{W}}(f)$. This implies clearly the claim for x^2y^2. Besides we see easily in the same way that x^a and y^b satisfy $(AC1)$ and $(AC2)$ with respect to f and $\boldsymbol{W}$.

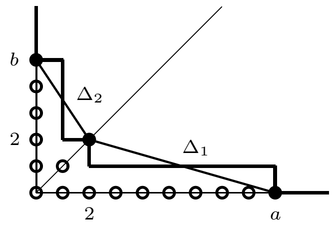
In the following, we denote by Δ_1 the line segment of $\Gamma(f)$ with end points $(a, 0)$ and $(2, 2)$ and we write Δ_2 for the line segment of with end points $(2, 2)$ and $(0, b)$. Besides, let $\delta_{1,2} = \{(2, 2)\}$. It is evident that Δ_1, Δ_2 and $\delta_{1,2}$ are faces of $\Gamma(f)$. Moreover, x^{a-1}, x^2y and x^2y^2 are in the cone $P[\Delta_1]$. Thus, as $x^n \in P[\Delta_1]$ for any $n \in \mathbb{N}$, then it follows from Lemma 3.2.3 that any monomial in the set

$$\left\{x^{2+n}y\,,\, x^{2+n}y^2\,,\, x^{a+n} :\ n \in \mathbb{N}\right\}$$

is $(AC1)$ and $(AC2)$ with respect to f and $\boldsymbol{W}$. In the same way, since xy^2, x^2y^2 and y^b in $P[\Delta_2]$, it follows from Lemma 3.2.3 that any monomial in

$$\left\{xy^{2+n}\,,\, x^2y^{2+n}\,,\, y^{b+n} :\ n \in \mathbb{N}\right\}$$

is $(AC1)$ and $(AC2)$ with respect to f and $\boldsymbol{W}$.
Altogether, this shows that $dim_K(gr^{AC}_{\boldsymbol{W}}(T_f)) < \infty$ and a **regular** basis of T_f is contained in the set $\mathcal{B}$. Therefore $dim_K(gr^{AC}_{\boldsymbol{W}}(T_f)) \leq \sharp(\mathcal{B}) = \tau(f)$. However, Lemma 3.2.8 states that $dim_K(gr^{AC}_{\boldsymbol{W}}(T_f)) \geq \tau(f)$. Hence the claim f is (AC) follows from Propsition 3.2.9. Finally, it is easy to see that all monomials in $\mathcal{B}$ lie strictly above $\Gamma(f)$. This terminates the proof.



Regular basis of
$f = x^a + \lambda x^2 y^2 + y^b$

□

In arbitrary characteristic, the following claim generalizes Proposition 3.2.14.

Proposition 3.2.15. *Let $char(K) \neq 2$ and let $f = x^a + \lambda x^2 y^2 + y^b \in K[[x, y]]$, where $\lambda \neq 0$, $a \geq 4$ and $b \geq 5$. If $\mu(f) < \infty$ (resp. $\tau(f) < \infty$), then f is (AC) with respect to $\Gamma(f)$. Furthermore, there exists a regular basis of T_f lying below $\Gamma(f)$.*

Proof. The proof repeats the so far used arguments in the one of Proposition 3.2.14. Hence for the reason of size we discuss shortly the following cases:

(i) If $char(K) \nmid a$, $char(K) \nmid b$ and $char(K) \nmid ab - 2 \cdot (a + b)$, we observe that f_x and f_y are equal to the respective partial derivatives of f in characteristic zero. Thus, the proof of Proposition 3.2.14 shows in the same way the claim of the present proposition.

(ii) If $char(K) \nmid a$, $char(K) \nmid b$, but $char(K) \mid ab - 2 \cdot (a + b)$, then we can see in this case that $x^a \notin tj(f)$ and $dim_K(gr^{AC}_{\boldsymbol{W}}(T_f)) = \tau(f) = a + b + 1$.

(iii) If $char(K) \mid a$ and $char(K) \nmid b$, then we have

$$f_x = 2\lambda x y^2\ ,\ f_y = 2\lambda x^2 y + b y^{b-1}.$$

Nevertheless, in this case also, it is not difficult to see that the monomials xy^2, x^2y, x^2y^2, x^a and y^b do fulfill conditions $(AC1)$ and $(AC2)$ with respect to f and $\boldsymbol{W}$ and the claim follows in the same way as in the above.

(iv) The claim in the case $char(K) \mid b$ and $char(K) \nmid a$ can be easily derived from the case (iii).

(v) If $char(K) \mid a$ and $char(K) \mid b$, then

$$f_x = 2\lambda x y^2\ ,\ f_y = 2\lambda x^2 y.$$

Arguing as in the proof of Proposition 3.2.14, we can show that $\tau(f) = a+b+1$ and the set

$$\mathcal{B} = \left\{1, x, \ldots, x^{a-1}, y, xy, y^2, \ldots, y^b\right\}$$

is at the same time a K-vector space basis and a regular basis of T_f. This shows the claim.

□

We should mention that the claim of Proposition 3.2.15 is in general not true when $char(K) = 2$.

Example 3.2.16. *Let* $char(K) = 2$ *and let* $f = x^5 + x^2y^2 + y^4 \in K[[x, y]]$. f *is* (PH) *of type* $(\boldsymbol{W}, d)$*, where* $\boldsymbol{W} = \{(4, 6); (5, 5)\}$ *and* $d = 20$. *Using* SINGULAR*, we can compute* $\tau(f) = 16$ *and show that* $\langle x, y\rangle^7 \subset tj(f)$. *However, we claim that* f *is not even* (AAC). *Indeed, let* $n \in \mathbb{Z}_{>0}$ *be such that* $n \geq 2$*, then obviously* $y^{4n} \subset \mathfrak{m}^7 \subset tj(f)$. *On the other hand, we show in the following that* y^{4n} *does not satisfy* $(AC2)$ *with respect to* f *and* $\boldsymbol{W}$.
We have $f_x = x^4$ *and* $f_y = 0$*, then we can write* y^{4n} *as*

$$y^{4n} = y^{4n-4}f + (xy^{4n-4})\partial_x f + x^2y^{4n-2}.$$

Besides, setting $\xi = (xy^{4n-4})\partial_x$*, we have* $v_{\boldsymbol{W}}(y^{4n}) = 20n = v_{\boldsymbol{W}}(y^{4n-4}) + v_{\boldsymbol{W}}(f)$ *and* $v_{\boldsymbol{W}}(\xi) = 20n - 20$. *Thus, clearly* $v_{\boldsymbol{W}}(y^{4n}) = v_{\boldsymbol{W}}(\xi) + v_{\boldsymbol{W}}(f)$. *Nevertheless,* $v_{\boldsymbol{W}}(y^{4n} - y^{4n-4}f - \xi f) = v_{\boldsymbol{W}}(x^2y^{4n-2}) = 20n$ *and this shows that* f *is not* $(AC2)$. *Hence the infinite set* $\{y^{4n} : n \geq 2\}$ *is contained in a* K*-basis of the vector space* $gr^{AC}_{\boldsymbol{W}}(T_f)$ *and so the claim that* f *is not* (AAC) *clearly follows.*

In [Wal99a], Wall established over $\mathbb{C}$ that if $f \in K[[\boldsymbol{x}]]$ is $NPND^*$ with respect to some C-polytope $\boldsymbol{P}$, then $f_{\boldsymbol{P}}$ is (AA) with respect to $\boldsymbol{P}$. To show this claim, Wall presented a pure algebraic proof based on the observations of Kouchnirenko in [Kou76, 4,6] and which is independant of the characteristic. Therefore the same claim does hold in arbitrary characteristic.

Proposition 3.2.17. *Let* $f \in K[[\boldsymbol{x}]]$ *such that* $char(K) \geq 0$. *If* f *satisfies* $NPND^*$ *with respect to some* C*-polytope* $\boldsymbol{P}$*, then* $f_{\boldsymbol{P}}$ *is* (AA) *and* (AAC) *with respect to* $\boldsymbol{P}$.

Proof. See [Wall99a, 2.2 and 2.3] for a proof of the claim that $f_{\boldsymbol{P}}$ is (AA) with respect to $\boldsymbol{P}$. Finally, $f_{\boldsymbol{P}}$ is (AAC) with respect to $\boldsymbol{P}$ follows from Lemma 3.2.12. □

Corollary 3.2.18. *Let* $f \in \mathfrak{m}^3$ *be* (SQH) *with principal part* f_Δ *having weighted degree* $d \in \mathbb{Z}_{>0}$. *If* $char(K)$ *does not divide* d*, then* f *is* (AA) *and* (AAC) *with respect to its Newton polytope.*

Proof. The proof is straightforward from Proposition 2.3.23 and Proposition 3.2.17. □

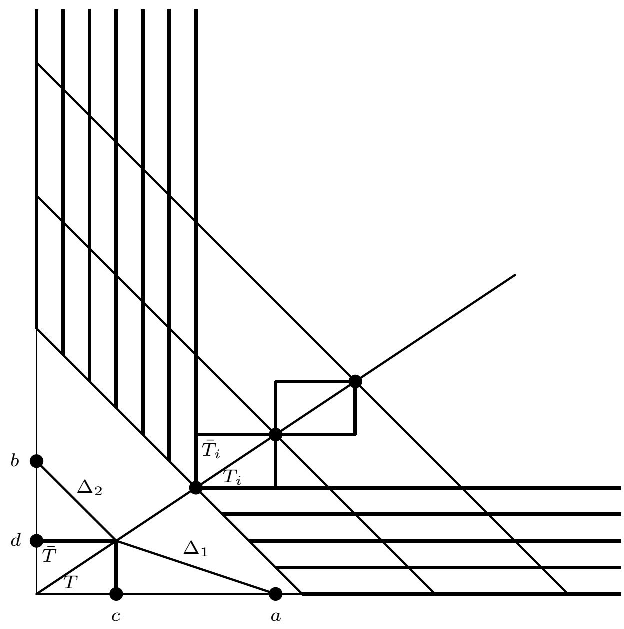
The next proposition was motivated by the following observations: In the classification of simple and unimodal plane curve singularities, the cases which mostly occur are those of elements of $K[[x, y]]$ which are (SQH) or (SPH) with respect to a 2-facet Newton polytope (see for example [AGV85], [Sch90], [GrK90], [DrG98]). Moreover, as we shall see in the next section, the computation of regular bases provides an important tool to the computation of normal forms.

Proposition 3.2.19. *Let* $f = x^a + \lambda x^c y^d + y^b \in K[[x, y]]$ *be reduced such that* $\lambda \in K \setminus \{0\}$, $a > c$, $b > d$ *and* $ad + bc < ab$. *Then* f *is* (AAC) *with respect to* $\Gamma(f)$*, if and only if, there exists* $k \in \mathbb{Z}_{>0}$ *such that any monomial of total degree* $k \cdot (c + d)$ *satisfies* $(AC1)$ *and* $(AC2)$ *with respect to* f *and* $\Gamma(f)$.

Proof. Throughout this proof, we denote $\Gamma := \Gamma(f)$, $\Delta_1 = [(a,0),(c,d)]$ and moreover $\Delta_2 = [(c,d),(0,b)]$. Clearly Δ_1 and Δ_2 are the two facets of Γ. Furthermore, let $\boldsymbol{W} := \{w_1\,;\,w_2\} \subset \mathbb{Z}^2_{>0}$ where $w_1 = bc \cdot (d\,;\,a-c)$ and $w_2 = ad \cdot (b-d\,;\,c)$. It is easy to see that f is (PH) with respect to $\boldsymbol{W}$ of degree $abcd$. Moreover we say for short that a monomial is $(AC1)$ and $(AC2)$ if it satisfies these conditions with respect to f and $\boldsymbol{W}$. Furthermore, we write $deg(M)$ for the total degree of a monomial M.
If f is (AAC) with respect to Γ, that is $dim_K(gr^{AC}_{\boldsymbol{w}}(T_f)) < \infty$, then there exists $N \in \mathbb{Z}_{>0}$ such that any monomial in $\mathfrak{m}^N$ is $(AC1)$ and $(AC2)$. We set k the smallest positive integer such that $k \cdot (c+d) \geq N$.
Conversely, we suppose that there exists $k \in \mathbb{Z}_{>0}$ such that any monomial $x^\alpha y^\beta$ with $\alpha + \beta = k \cdot (c+d)$ is $(AC1)$ and $(AC2)$. We claim that any monomial in $\mathfrak{m}^{k\cdot(c+d)}$ is also $(AC1)$ and $(AC2)$. Indeed, we consider $i \in \mathbb{N}$ and

$$B_i = \Big\{ M \in Mon(K[[\boldsymbol{x}]]) :\ (k+i)\cdot(c+d) \leq deg(M)) \leq (k+i+1)\cdot(c+d) \Big\},$$

and we show by induction that any monomial $M \in B_i$, $i \in \mathbb{N}$, is $(AC1)$ and $(AC2)$.



$f = x^a + \lambda x^c y^d + y^b$
is (AAC)

For this purpose we consider the following triangles in $\mathbb{R}^2_{\geq 0}$:

(1) T has the vertices $(0,0)$, $(c,0)$ and (c,d).

(2) $\bar{T}$ has the vertices $(0,0)$, $(d,0)$ and (c,d).

(3) T_i has the vertices $((k+i)\cdot c,(k+i)\cdot d)$, $((k+i+1)\cdot c,(k+i)\cdot d)$ and $((k+i+1)\cdot c,(k+i+1)\cdot d)$.

(4) $\bar{T}_i$ has the vertices $((k+i)\cdot c,(k+i)\cdot d)$, $((k+i)\cdot c,(k+i+1)\cdot d)$ and $((k+i+1)\cdot c,(k+i+1)\cdot d)$.

Let M be a monomial such that $deg(M)=k\cdot(c+d)$. We have by assumption that M is $(AC1)$ and $(AC2)$. Hence, it follows by Lemma 3.2.3 that

1. if $M\in P[\Delta_1]$, then for any $r\in\mathbb{N}$, x^rM is $(AC1)$ and $(AC2)$ follows,
2. if $M\in P[\Delta_2]$, then for any $r\in\mathbb{N}$, we have y^rM is $(AC1)$ and $(AC2)$.

Thus, in order to prove the claim for $i=0$ and based on these observations, it is enough to show that any monomial having its support in T_0 (resp. in $\bar{T}_0$) is $(AC1)$ and $(AC2)$. Nevertheless, such monomials can be written as the product of $x^{kc}y^{kd}$ and a monomial having its support either on $T\subset P[\Delta_1]$ or $\bar{T}\subset P[\Delta_2]$.
On the other hand $x^{kc}y^{kd}\in P[\Delta_1]\cap P[\Delta_2]$. Hence, it follows from Lemma 3.2.3 that the lattice points of T_0 and $\bar{T}_0$ correspond to monomials which are $(AC1)$ and $(AC2)$. Hence again by Lemma 3.2.3, we deduce that any monomial M, for which it holds $k\cdot(c+d)\leq\deg(M)\leq(k+1)\cdot(c+d)$, is $(AC1)$ and $(AC2)$.
The induction step $i\Rightarrow i+1$ can be proved in the same way by considering the triangles T_i, $\bar{T}_i$, T and $\bar{T}$.
Altogether, this shows that any monomial M such that $deg(M)\geq k\cdot(c+d)$ is $(AC1)$ and $(AC2)$. Consequently $gr^{AC}_{\boldsymbol{W}}(T_f)$ is finite dimensional as K-vector space and this terminates the proof. ☐

Example 3.2.20. *Let $char(K)=3$ and we consider a plane curve singularity of type $E_{3,3}$ corresponding to the equation $f=x^{12}+x^3y^2+y^3\in K[[x,y]]$. Furthermore, let $\boldsymbol{W}=\Big\{(6;27)\,,\,(8;24)\Big\}$ and let $d=72$. Clearly, f is reduced and f is (PH) of type $\{\boldsymbol{W}\,;\,d\}$. Using in* SINGULAR *the function $isAC$ from the library $gradalg.lib$ (cf. Algorithm 4.3.4 in Chapter 4), we show that any monomial of total degree 15 satisfies both of $(AC1)$ and $(AC2)$ with respect to f and $\boldsymbol{W}$. Thus Proposition 3.2.19 yields f is (AAC) with respect to $\boldsymbol{W}$. Moreover, using the function $ACgrbase$ from $gradalg.lib$ (cf. Algorithm 4.2.4 in Chapter 4) shows that*

$$\mathcal{B}=\left\{1,x,\ldots,x^{12},y,xy,x^2y,y^2,xy^2,x^2y^2,xy^3,x^2y^3,x^2y^4\right\}$$

is a K-basis of the vector space $gr^{AC}_{\boldsymbol{W}}(T_f)$. Hence $dim_K(gr^{AC}_{\boldsymbol{W}}(T_f))=22$ while $\tau(f)=21$. Therefore f is not (AC).

For $f\in K[[x,y]]$ arising in the same way as in Proposition 3.2.19, we notice that the same claim holds for the condition (AA).

Proposition 3.2.21. *Let $f=x^a+\lambda x^cy^d+y^b\in K[[x,y]]$ be reduced, such that $\lambda\in K\setminus\{0\}$, $a>c$, $b>d$ and $ad+bc<ab$. Then f is (AA) with respect to $\Gamma(f)$, if and only if, there exists $k\in\mathbb{Z}_{>0}$ such that any monomial of total degree $k\cdot(c+d)$ satisfies $(A1)$ and $(A2)$ with respect to f and $\Gamma(f)$.*

Proof. In the same way as the proof of Proposition 3.2.19, this proof is also based on Lemma 3.2.3. We need only to replace conditions (AC), $(AC1)$ and $(AC2)$ by conditions (A), $(A1)$ and $(A2)$ respectively. □

Example 3.2.22. *Let $char(K) = 5$. We consider the plane curve singularity of type $W_{1,1}$ of equation $f = x^7 + x^3y^2 + y^4 \in K[[x,y]]$. It is easy to notice that f is (PH) of type $\{\boldsymbol{W}\,;\, d\}$ where $\boldsymbol{W} = \Big\{(12;24)\,,\,(14;21)\Big\}$ and $d = 84$. Using* SINGULAR, *we show that any monomial of total degree 10 is $(A1)$ and $(A2)$ with respect to f and $\boldsymbol{W}$. Thus f is (AA) with respect to $\boldsymbol{W}$ follows by Proposition 3.2.21. Moreover, $dim_K(gr^A_{\boldsymbol{W}}(M_f)) = 16$ and $\mu(f) = 16$. Hence, Proposition 3.2.9 yields f is (A) with respect to $\boldsymbol{W}$.*

3.3 Normal Forms of Isolated Hypersurface Singularities

Throughout this section K denotes an algebraically closed field of arbitrary characteristic.

Using the notions elaborated so far, we reformulate briefly the main statement given by Arnold in [Arn74] on the computation of normal forms over the field $\mathbb{C}$.

Theorem 3.3.1. *Let $f \in \mathfrak{m} \subset \mathbb{C}[[\boldsymbol{x}]]$ be such that $\mu(f)$ is finite and let $\boldsymbol{W} \subset \mathbb{Z}^n_{>0}$ be a finite set of weights corresponding to the Newton polytope Γ of f. Furthermore, let $\{e_{\boldsymbol{\alpha}} : \boldsymbol{\alpha} \in \Lambda\}$ be a K-basis of M_{f_Γ} consisting of monomials.*
If the principal part f_Γ of f satisfies condition (A), then

$$f \overset{\mathrm{r}}{\sim} f_\Gamma + \sum_{\boldsymbol{\alpha} \in \Lambda^*} c_{\boldsymbol{\alpha}} e_{\boldsymbol{\alpha}},$$

where

$$\Lambda^* \subset \Big\{\boldsymbol{\alpha} \in \Lambda : v_{\boldsymbol{W}}(e_{\boldsymbol{\alpha}}) > v_{\boldsymbol{W}}(f)\Big\}$$

and the coefficients $c_{\boldsymbol{\alpha}} \in \mathbb{C}$ are suitable.

Proof. cf. [Arn74, 9.5]. □

Nevertheless, as it was already observed by Wall in [Wal99a], the additional condition (A) in Theorem 3.3.1 is not necessary for the proof and can be omitted as we shall see in the next result. Indeed, Arnold's theorem can be reformulated as follows for the computation of normal forms with respect to the contact equivalence:

Theorem 3.3.2. *Let $f \in \mathfrak{m} \subset K[[\boldsymbol{x}]]$ be such that $\tau(f)$ is finite and let $\boldsymbol{W} \subset \mathbb{Z}^n_{>0}$ be a finite set of weights corresponding to the Newton polytope Γ of f.*
Further, let $\{e_{\boldsymbol{\alpha}} : \boldsymbol{\alpha} \in \Lambda\}$ be a K-basis of $gr^{AC}_{\boldsymbol{W}}(T_{f_\Gamma})$ consisting of monomials. Then,

$$f \overset{\mathrm{c}}{\sim} f_\Gamma + \sum_{\boldsymbol{\alpha} \in \Lambda^*} c_{\boldsymbol{\alpha}} e_{\boldsymbol{\alpha}},$$

where

$$\Lambda^* \text{ is a finite subset of} \left\{\boldsymbol{\alpha} \in \Lambda : \, v_{\boldsymbol{W}}(e_{\boldsymbol{\alpha}}) > v_{\boldsymbol{W}}(f)\right\}$$

and the coefficients $c_{\boldsymbol{\alpha}} \in K$ are suitable.

Proof. Let $f \in \mathfrak{m}$ be such that $\tau(f)$ is finite. We denote the Newton polytope Γ by $\boldsymbol{P}$ and we write $\boldsymbol{W}$ for a finite set of weights which is associated to $\boldsymbol{P}$ in $\mathbb{Z}^n_{>0}$. Clearly $f_{\boldsymbol{P}}$ is a (PH) polynomial with respect to $\boldsymbol{W}$. Let $d := v_{\boldsymbol{W}}(f)$. Then, we can write

$$f = f_{\boldsymbol{P}} + f_1, \ \text{ with } \ v_{\boldsymbol{W}}(f_1) > d.$$

Let $\left\{e_{\boldsymbol{\alpha}} \, : \, \boldsymbol{\alpha} \in \Lambda\right\}$ be a K-basis of $gr^{AC}_{\boldsymbol{W}}(T_{f_{\boldsymbol{P}}})$ consisting of monomials and let $\Lambda^{'} := \left\{\boldsymbol{\alpha} \in \Lambda : \, v_{\boldsymbol{W}}(e_{\boldsymbol{\alpha}}) > d\right\}$.
For the proof of Theorem 3.3.2, we construct inductively a sequence of power series $(g_q)_{q \in \mathbb{Z}_{\geq 0}}$ such that

- $g_0 = f$,
- $g_q \overset{c}{\sim} f$ for all q and
- the sequence $(g_q)_q$ converges in the $\mathfrak{m}$-adic topology to an element of the form $f_{\boldsymbol{P}} + \sum_{\boldsymbol{\alpha} \in \Lambda^*} c_{\boldsymbol{\alpha}} e_{\boldsymbol{\alpha}}$ where the latter sum has finitely many terms.

We describe in the following the first step of our construction. We have $f = f_{\boldsymbol{P}} + f_1$, where $d_1 := v_{\boldsymbol{W}}(f_1) > d$. Moreover, we can write

$$f_1 = f_1^{(d_1)} + f_1^{(>d_1)},$$

where

- $f_1^{(d_1)}$ is a (PH) polynomial of type $(\boldsymbol{W}\,;\, d_1)$ and
- $v_{\boldsymbol{W}}(f_1^{(>d_1)}) > d_1$.

For the sequel we denote $tj^{AC}_{\boldsymbol{W}}(f_{\boldsymbol{P}}, d_1)$ by $I^{AC}_{d_1}$ and we consider

$$\Lambda'_1 = \{\boldsymbol{\alpha} \in \Lambda^{'} : \, v_{\boldsymbol{W}}(e_{\boldsymbol{\alpha}}) = d_1\}.$$

Of course $\Lambda^{'}_1$ can be empty in the case where all monomials of piecewise-homogeneous degree d_1 satisfy $(AC1)$ and $(AC2)$ with respect to $f_{\boldsymbol{P}}$ and $\boldsymbol{W}$. If not, then $\{e_{\boldsymbol{\alpha}} : \, \boldsymbol{\alpha} \in \Lambda^{'}_{(1)}\}$ is a basis of the K-vector space $F_{\geq d_1}/(F_{>d_1} + (F_{\geq d_1} \cap I^{AC}_{d_1}))$. So we can write

$$f_1^{(d_1)} = \sum_{\boldsymbol{\alpha} \in \Lambda'_1} c_{\boldsymbol{\alpha}} e_{\boldsymbol{\alpha}} + b_0^{(1)} f_{\boldsymbol{P}} + \xi_1 f_{\boldsymbol{P}} + h_1,$$

where

- $c_{\boldsymbol{\alpha}} \in K$ for all $\boldsymbol{\alpha} \in \Lambda'_1$.

- $b_{_0}^{(1)} \in K[[\boldsymbol{x}]]$ and $\xi_1 = \sum_{i=1}^{n} b_{_i}^{(1)} \partial_{x_i} \in Der_K(K[[\boldsymbol{x}]])$ satisfy $d_1 = min\{v_{\boldsymbol{W}}(b_{_0}^{(1)}) + d\,,\, v_{\boldsymbol{W}}(\xi_1) + d\}$.
- Finally $h_1 \in K[[\boldsymbol{x}]]$ is such that $v_{\boldsymbol{W}}(h_1) > d_1$.

Moreover, for $v_{\boldsymbol{W}}(b_{_0}^{(1)}) \geq d_1 - d > 0$, we get

$$b_{_0}^{(1)} \in \mathfrak{m}, \tag{3.15}$$

On the other hand, for all $1 \leq i \leq n$, Remark 2.2.2 yields

$$v_{\boldsymbol{W}}(b_{_i}^{(1)}) \geq v_{\boldsymbol{W}}(\xi_1) + \lambda_{\boldsymbol{W}}(\epsilon_i),$$

where

$$\epsilon_i = \begin{array}{ccccccccc} & & & & i & & & \\ (0 & \ldots & 0, & 1, & 0 & \ldots & 0). \end{array}$$

As $v_{\boldsymbol{W}}(\xi_1) \geq d_1 - d > 0$, then it follows for all $i = 1, \ldots, n$, that

$$v_{\boldsymbol{W}}(b_{_i}^{(1)}) > \lambda_{\boldsymbol{W}}(\epsilon_i). \tag{3.16}$$

Furthermore, we claim that the K-algebra morphism φ_1 defined by

$$\begin{array}{rccc} \varphi_1 : & K[[x_1, \ldots, x_n]] & \longrightarrow & K[[x_1, \ldots, x_n]] \\ & x_i & \mapsto & x_i - b_{_i}^{(1)} \end{array}$$

is a K-automorphism on $K[[\boldsymbol{x}]]$.
To show the claim, we can suppose without loss of generality after permutation of the indeterminates $x_1, \ldots, x_n$ that

$$\lambda_{\boldsymbol{W}}(\epsilon_1) \geq \lambda_{\boldsymbol{W}}(\epsilon_2) \geq \ldots \geq \lambda_{\boldsymbol{W}}(\epsilon_n).$$

Using this together with the relation (3.16) shows that $b_{_1}^{(1)} \in \mathfrak{m}^2$. Furthermore, for all $i = 2, \ldots, n$, we get

$$b_{_i}^{(1)} \text{ mod } (\mathfrak{m}^2) = \sum_{l=1}^{i-1} a_{i,l} x_l,$$

where the coefficients $a_{i,l} \in K$. Hence, we can write the Jacobian matrix $J(\varphi_1)$ as follows

$$\begin{pmatrix} 1 & -a_{2,1} & -a_{3,1} & \ldots & -a_{n,1} \\ 0 & 1 & -a_{3,2} & \ldots & -a_{n,2} \\ \vdots & 0 & 1 & \ldots & \vdots \\ \vdots & 0 & \ddots & \ddots & \vdots \\ 0 & \ldots & \ldots & 0 & 1 \end{pmatrix}$$

Obviously, we have $det(J(\varphi_1)) = 1$. Therefore $\varphi_1 \in Aut(K[[\boldsymbol{x}]])$. Moreover,

$$\begin{aligned} \varphi_1(f) &= \varphi_1(f_{\boldsymbol{P}} + f_1) \\ &= \varphi_1(f_{\boldsymbol{P}}) + \varphi_1(f_1^{(d_1)}) + \varphi_1(f_1^{(>d_1)}) \\ &= \varphi_1(f_{\boldsymbol{P}}) + \varphi_1(f_1^{(d_1)}) + R_1 \end{aligned}$$

with $R_1 = \varphi_1(f_1^{(>d_1)}) \in F_{>d_1}$ for $f_1^{(>d_1)} \in F_{>d_1}$ and $\varphi_1 \in Aut(K[[\boldsymbol{x}]])$.
By Lemma 2.2.5, we can write

$$\begin{aligned} \varphi_1(f) &= f_{\boldsymbol{P}} - \xi_1 f_{\boldsymbol{P}} + f_1^{(d_1)} - \xi_1 f_1^{(d_1)} + R_1 + R_1^{'} \\ &= (1 + b_0^{(1)}) f_{\boldsymbol{P}} + \sum_{\boldsymbol{\alpha} \in {\Lambda'}_{(1)}} c_{\boldsymbol{\alpha}} e_{\boldsymbol{\alpha}} + (h_1 - \xi_1 f_1^{(d_1)} + R_1^{'}) \end{aligned}$$

where $v_{\boldsymbol{W}}(R_1^{'}) > min\{v_{\boldsymbol{W}}(\xi_1) + v_{\boldsymbol{W}}(f_{\boldsymbol{P}}),\, v_{\boldsymbol{W}}(\xi_1) + v_{\boldsymbol{W}}(f_1^{(d_1)})\} \geq d_1$.
Again by Remark 2.1.19, we have

$$\begin{aligned} v_{\boldsymbol{W}}(\xi_1 f_1^{(d_1)}) &\geq v_{\boldsymbol{W}}(\xi_1) + v_{\boldsymbol{W}}(f_1^{(d_1)}) \\ &\geq (d_1 - d) + d_1 \\ &> d_1. \end{aligned}$$

Hence, we can write

$$\varphi_1(f) = (1 + b_0^{(1)}) f_{\boldsymbol{P}} + \sum_{\boldsymbol{\alpha} \in {\Lambda'}_{(1)}} c_{\boldsymbol{\alpha}} e_{\boldsymbol{\alpha}} + R_1^{''}, \text{ with } v_{\boldsymbol{W}}(R_1^{''}) > d_1. \quad (3.17)$$

Besides, it follows by (3.15) that the power series $(1 + b_0^{(1)})$ is a unit in $K[[\boldsymbol{x}]]$. Thus, multiplying both left and right hand side of the equation (3.17) by $(1 + b_0^{(1)})^{-1}$ leads to the equation

$$(1 + b_0^{(1)})^{-1} \varphi_1(f) = f_{\boldsymbol{P}} + \sum_{\boldsymbol{\alpha} \in {\Lambda'}_{(1)}} c_{\boldsymbol{\alpha}} e_{\boldsymbol{\alpha}} + f_2, \text{ and } v_{\boldsymbol{W}}(f_2) > d_1.$$

We set $g_1 = (1 + b_0^{(1)})^{-1} \varphi_1(f)$. Obviously, we have $g_1 \overset{c}{\sim} f$ and

$$g_1 = f_{\boldsymbol{P}} + \sum_{\boldsymbol{\alpha} \in {\Lambda'}_{(1)}} c_{\boldsymbol{\alpha}} e_{\boldsymbol{\alpha}} + f_2, \text{ with } v_{\boldsymbol{W}}(f_2) > d_1 > d. \quad (3.18)$$

Note that if ${\Lambda'}_1 = \emptyset$, then the equation (3.18) changes to

$$g_1 = f_{\boldsymbol{P}} + f_2, \text{ with } v_{\boldsymbol{W}}(f_2) > d_1 > d. \quad (3.19)$$

Proceeding recursively, we construct the sequence $\{(g_q)\}_q$. On the other hand, as $\tau(f)$ is finite, then it follows by Theorem 3.1.15 that f is finitely contact determined.

Hence, there exists a finite subset Λ^* of $\Lambda^{'}$ such that the sequence $(g_q)_{q\geq 0}$ converges to $f_{\boldsymbol{P}} + \sum_{\boldsymbol{\alpha}\in\Lambda^*} c_{\boldsymbol{\alpha}} e_{\boldsymbol{\alpha}}$ in the $\mathfrak{m}$-adic topology. Thus, the claim

$$f \overset{\mathrm{c}}{\sim} f_{\boldsymbol{P}} + \sum_{\boldsymbol{\alpha}\in\Lambda^*} c_{\boldsymbol{\alpha}} e_{\boldsymbol{\alpha}}$$

clearly follows.

□

We recall that, if $M \in Mon(K[[\boldsymbol{x}]])$ is a monomial in $K[[\boldsymbol{x}]]$, then $deg(M)$ denotes the total degree of M. If we devote a closer look to the proof of Theorem 3.3.2, then we can easily see that it actually shows the following claim.

Theorem 3.3.3. *Let $f \in \mathfrak{m} \subset K[[\boldsymbol{x}]]$ and let $k \in \mathbb{Z}_{>0}$ be such that $\mathfrak{m}^k \subset tj(f)$. Further, let $\boldsymbol{W} \subset \mathbb{Z}^n_{>0}$ be a finite set of weights corresponding to the Newton polytope Γ of f and let let $\{e_{\boldsymbol{\alpha}} : \boldsymbol{\alpha} \in \Lambda\}$ be a K-basis of $gr^{AC}_{\boldsymbol{W}}(T_{f_\Gamma})$ consisting of monomials. Then,*

$$f \overset{\mathrm{c}}{\sim} f_\Gamma + \sum_{e_{\boldsymbol{\alpha}}\in\mathcal{E}(f)} c_{\boldsymbol{\alpha}} e_{\boldsymbol{\alpha}},$$

where

$$\mathcal{E}(f) \subset \Big\{M \in Mon(K[[\boldsymbol{x}]]) : deg(M) \leq 2k - ord(f) + 2 \,,\ v_{\boldsymbol{W}}(M) \geq v_{\boldsymbol{W}}(f - f_\Gamma)\Big\}$$

and the coefficients $c_{\boldsymbol{\alpha}} \in K$ are suitable.

Proof. cf. proof of Theorem 3.3.2. □

Obviously, the set $\mathcal{E}(f)$ which is defined in Theorem 3.3.3 is finite. Moreover, we can in the same way reformulate Arnold's theorem in arbitrary characteristic for right equivalence.

Theorem 3.3.4. *Let $f \in \mathfrak{m} \subset K[[\boldsymbol{x}]]$ and let $k \in \mathbb{Z}_{>0}$ be such that $\mathfrak{m}^k \subset j(f)$. Further, let $\boldsymbol{W} \subset \mathbb{Z}^n_{>0}$ be a finite set of weights corresponding to the Newton polytope Γ of f and let let $\{e_{\boldsymbol{\alpha}} : \boldsymbol{\alpha} \in \Lambda\}$ be a K-basis of $gr^{A}_{\boldsymbol{W}}(M_{f_\Gamma})$ consisting of monomials. Then,*

$$f \overset{\mathrm{r}}{\sim} f_\Gamma + \sum_{e_{\boldsymbol{\alpha}}\in\mathcal{E}(f)} c_{\boldsymbol{\alpha}} e_{\boldsymbol{\alpha}},$$

where

$$\mathcal{E}(f) \subset \Big\{M \in Mon(K[[\boldsymbol{x}]]) : deg(M) \leq 2k - ord(f) + 2 \,,\ v_{\boldsymbol{W}}(M) \geq v_{\boldsymbol{W}}(f - f_\Gamma)\Big\}$$

and the coefficients $c_{\boldsymbol{\alpha}} \in K$ are suitable.

Proof. The arguments used in the proof of Theorem 3.3.2 show in the same way the claim of Theorem 3.3.4. Thus, we decide here for the reason of size to omit the proof.

□

Example 3.3.5. *Let $char(K) = 2$ and let $f = x^2y^2 + y^4 + x^5 + x^5y^7 + x^3y^9 + x^9y^4 + x^{15} \in K[[x,y]]$. Using* SINGULAR, *we get $\tau(f) = 16$ and $\mathfrak{m}^7 \subset tj(f)$. On the other hand, it is easy to see that the set of weights $\boldsymbol{W} = \{(4,6)\,;\,(5,5)\}$ corresponds to the Newton polytope Γ of f and moreover $f_\Gamma = x^2y^2 + y^4 + x^5$. In Example 3.2.16, we have shown that the (PH) polynomial f_Γ does not satisfy (A) with respect to Γ since it is not even (AA) with respect to Γ. Hence, $gr^{AC}_{\boldsymbol{W}}(T_{f_\Gamma})$ has an infinite dimension as a K-vector space. Moreover, let*

$$\mathcal{E}(f) = \big\{M \in Mon(K[[\boldsymbol{x}]]) :\ deg(M) \leq 12 \textit{ and } v_{\boldsymbol{W}}(M) \geq 60\big\}.$$

Using the function $ACgrbase$ from the library $gradalg.lib$ in SINGULAR, *we obtain the set $\big\{xy^{11}\,,\, y^{12}\big\}$ as intersection of the set $\mathcal{E}(f)$ and a K-basis of $gr^{AC}_{\boldsymbol{W}}(T_{f_\Gamma})$ consisting of monomials. Thus, the claim*

$$f \overset{c}{\sim} x^2y^2 + y^4 + x^5 \ +\ c_1xy^{11} \ +\ c_2y^{12} \textit{ for some } c_1, c_2 \in K$$

follows clearly by Theorem 3.3.3.

In the last part of the present chapter we shall investigate the effect of the conditions (AA) and (AAC) on the computations of normal forms and bounds of determinacy. As it should be expected, it turns out that these conditions are more suited for computations.

Before going into the details, we recall that an element $f \in K[[\boldsymbol{x}]]$ is called semi--piecewise-homogeneous, if there exists a C-polytope $\boldsymbol{P}$ in $\mathbb{R}^n_{\geq 0}$ such that no point of $supp(f)$ lies below $\boldsymbol{P}$ and moreover the piecewise-homogeneous polynomial $f_{\boldsymbol{P}}$ has a finite Tjurina number (cf. Definition 2.1.37).

Theorem 3.3.6. *Let $f \in \mathfrak{m} \subset K[[\boldsymbol{x}]]$ be (SPH) with respect to a C-polytope $\boldsymbol{P}$ and let $\boldsymbol{W} \subset \mathbb{Z}^n_{>0}$ be a finite set of weights corresponding to $\boldsymbol{P}$. If $f_{\boldsymbol{P}}$ is (AAC) with respect to $\boldsymbol{P}$ and $\big\{e_{\boldsymbol{\alpha}} :\ \boldsymbol{\alpha} \in \Lambda\big\}$ is a regular basis of $T_{f_{\boldsymbol{P}}}$, then f is* **finitely contact determined** *and*

$$f \overset{c}{\sim} f_{\boldsymbol{P}} + \sum_{\boldsymbol{\alpha} \in \Lambda^*} c_{\boldsymbol{\alpha}} e_{\boldsymbol{\alpha}},$$

where

$$\Lambda^* \subset \big\{\boldsymbol{\alpha} \in \Lambda :\ v_{\boldsymbol{W}}(e_{\boldsymbol{\alpha}}) \geq v_{\boldsymbol{W}}(f - f_{\boldsymbol{P}})\big\}$$

and the coefficients $c_{\boldsymbol{\alpha}} \in K$ are suitable.

Remark 3.3.7. *We should observe that the set of indices Λ^* in Theorem 3.3.6 can be empty. Indeed, if we suppose for example that all points corresponditiong to $supp(e_{\boldsymbol{\alpha}})$, $\boldsymbol{\alpha} \in \Lambda$, lie below $\boldsymbol{P}$, then it is obvious that $\Lambda^* = \emptyset$. In this case we have $f \overset{c}{\sim} f_{\boldsymbol{P}}$.*

We give in the following a proof of Theorem 3.3.6.

Proof. Let $f \in \mathfrak{m} \subset K[[\boldsymbol{x}]]$ be (SPH) with respect to a C-polytope $\boldsymbol{P}$. Definition 2.1.37 states that the principal part $f_{\boldsymbol{P}}$ has a finite Tjurina number. Moreover, the assumption $f_{\boldsymbol{P}}$ is (AAC) with respect to $\boldsymbol{P}$ means by definition that the K-algebra $gr^{AC}_{\boldsymbol{W}}(T_{f_{\boldsymbol{P}}})$ has a finite dimension as a K-vector space. On the other hand, let $\Lambda' =$

$\Big\{\boldsymbol{\alpha} \in \Lambda : \, v_{\boldsymbol{W}}(e_{\boldsymbol{\alpha}}) \geq v_{\boldsymbol{W}}(f - f_{\boldsymbol{P}})\Big\}$. Hence, following in the same way the arguments used in the constructive proof of Theorem 3.3.2, we show the existence of a sequence $\{g_q\}_q$ such that

(1) $g_q \overset{c}{\sim} f$ for all $q \geq 0$ and

(2) for all $N \in \mathbb{Z}_{>0}$, there exists q_N and $c_{\boldsymbol{\alpha}} \in K$ such that $g_q - f_{\boldsymbol{P}} - \sum_{\boldsymbol{\alpha} \in \Lambda'} c_{\boldsymbol{\alpha}} e_{\boldsymbol{\alpha}} \in \mathfrak{m}^{q_N}$, for all $q \geq N$.

Therefore the sequence $\{g_q\}_q$ converges to $f_{\boldsymbol{P}} + \sum_{\boldsymbol{\alpha} \in \Lambda^*} c_{\boldsymbol{\alpha}} e_{\boldsymbol{\alpha}}$ in the $\mathfrak{m}$-adic topology of $K[[\boldsymbol{x}]]$. Hence, for $\Lambda^* = \Big\{\boldsymbol{\alpha} \in \Lambda^* : \, c_{\boldsymbol{\alpha}} \neq 0\Big\}$ the claim

$$f \overset{c}{\sim} f_{\boldsymbol{P}} + \sum_{\boldsymbol{\alpha} \in \Lambda^*} c_{\boldsymbol{\alpha}} e_{\boldsymbol{\alpha}}, \tag{3.20}$$

clearly follows. We still have to show that f is finitely contact determined.
Let $d = v_{\boldsymbol{W}}(f) = v_{\boldsymbol{W}}(f_{\boldsymbol{P}})$ and if $\Lambda^* \neq \emptyset$, let $d' = max\,\{v_{\boldsymbol{W}}(e_{\boldsymbol{\alpha}}) : \, \boldsymbol{\alpha} \in \Lambda^*\}$. Moreover, if $\Lambda^* \neq \emptyset$, we set $D = max\{d\,,\,d'\}$, otherwise we take $D = d$. Furthermore, let k be a positive integer such that $\mathfrak{m}^{k+1} \subset F_{>D}$ and let $h \in \mathfrak{m}^{k+1}$. Considering $g = f + h$, we see clearly that g is (SPH) with respect to $\boldsymbol{P}$ and $f_{\boldsymbol{P}}$ is its principle part. Besides, as $v_{\boldsymbol{W}}(h) > D \geq v_{\boldsymbol{W}}(e_{\boldsymbol{\alpha}})$, for all $\boldsymbol{\alpha} \in \Lambda^*$, it follows that the decomposition of the piecewise-homogeneous parts of h in the K-basis $\{e_{\boldsymbol{\alpha}} : \boldsymbol{\alpha} \in \Lambda\}$ does not change the coefficients $c_{\boldsymbol{\alpha}}$ in the relation (3.20). Thus using the same arguments as so far, we show that

$$g \overset{c}{\sim} f_{\boldsymbol{P}} + \sum_{\boldsymbol{\alpha} \in \Lambda^*} c_{\boldsymbol{\alpha}} e_{\boldsymbol{\alpha}}.$$

That is $f + h \overset{c}{\sim} f$. Hence f is k-determined and this terminates the proof. ☐

Example 3.3.8. *Let $char(K) = 3$. We recall that any plane curve singularity of type $E_{3,3}$ can be associated to a (SPH) element of $K[[x,y]]$ having the principal part $f_0 = x^{12} + x^3y^2 + y^3$. Obviously, f_0 is (PH) of type $\{\boldsymbol{W}\,;\,d\}$ where $\boldsymbol{W} = \Big\{(6;27)\,,\,(8;24)\Big\}$ and $d = 72$. Moreover, we have shown in Example 3.2.20 that f_0 is (AAC) with respect to $\boldsymbol{W}$. Hence, it follows from Theorem 3.3.6 that any $E_{3,3}$-plane curve singularity is finitely contact determined. On the other hand, using in* SINGULAR *the function $ACgrbase$ from the library $gradalg.lib$, we obtain all monomials in a K-vector space basis of $gr^{AC}_{\boldsymbol{W}}(T_f)$ having $\boldsymbol{W}$-degree bigger than 72. These are xy^3, x^2y^3 and x^2y^4. Hence, Theorem 3.3.6 asserts that any equation $f \in K[[\boldsymbol{x}]]$ corresponding to a plane curve singularity of type $E_{3,3}$ has the following normal form*

$$f \overset{c}{\sim} f_0 + c_1xy^3 + c_2x^2y^3 + c_3x^2y^4,$$

for some c_1, c_2, $c_3 \in K$.

Example 3.3.9. *We consider in the following a plane curve singularity of type $W_{1,1}$ corresponding to an equation $g \in K[[x,y]]$ such that g is (SPH) of principal part*

$f = x^7 + x^3y^2 + y^4 \in K[[x, y]]$. Let $\boldsymbol{W} = \{w_1 = (12, 24), w_2 = (14, 21)\} \subset \mathbb{Z}^2_{>0}$. Clearly f is (PH) of type $(\boldsymbol{W}\,; d)$ where $d = 84$. Hence, we can write $g = f + g_1$ with $v_{\boldsymbol{W}}(g_1) > 84$. In the following, we shall compute a normal form of g in arbitrary characteristic.

(1) If $char(K) \neq 2$ and $char(K) \neq 3$ and $char(K) \neq 7$, then it is not difficult to see that f is $NPND^*$. Hence f is (AAC) with respect to $\boldsymbol{W}$ follows by Proposition 3.2.17 and therefore there exists a finite K-basis $\mathcal{B}$ of $gr^{AC}_{\boldsymbol{W}}(T_f)$ consisting of monomials. Moreover, we notice that in this case f_x and f_y have respectively the same support as when $char(K) = 0$. Thus we can assume without loss of generality that $char(K) = 0$. On the other hand, it is easy to see that all the lattice points on the Newton polytope of f correspond to monomials which satisfy $(AC1)$ and $(AC2)$ with respect to f and $\boldsymbol{W}$. Moreover, Lemma 3.2.3 asserts that any monomial M for which $v_{\boldsymbol{W}}(M) > 84$ holds, is $(AC1)$ and $(AC2)$. Therefore no element of the basis $\mathcal{B}$ have a W-order bigger than 84. Then it follows by Theorem 3.3.6 that $g \overset{c}{\sim} f$.

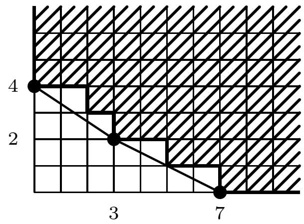
(2) If $char(K) = 7$, then we can easily show, that in this case the same claims as those of $char(K) = 0$ do also hold, especially we have $g \overset{c}{\sim} f$. So for the reason of size we decide not to go into the details.

(3) If $char(K) = 3$, then we can show in the same way as in Example 3.2.16 that f is not (AAC). Thus $dim_K(gr^{AC}_{\boldsymbol{W}}(T_f))$ is infinite. On the other hand, using SINGULAR *we get $\mathfrak{m}^7 \subset tj(f)$. Hence Theorem 3.1.15 yields f is 12-determined. On the other hand, the function $ACgrbase$ of the library $gradalg.lib$ provides all monomials in a K-basis of $gr^{AC}_{\boldsymbol{W}}(T_f)$ having total degree smaller than 12 and W-degree bigger than 84. These monomials are xy^4, x^2y^3, x^2y^4 and x^2y^5. Then, it follows by Theorem 3.3.6 that*

$$g \overset{c}{\sim} f + c_1xy^4 + c_2x^2y^3 + c_3x^2y^4 + c_4x^2y^5,$$

for some $c_i \in K$, $1 \leq i \leq 4$. Nevertheless, if we consider the parametrization equivalence $\overset{p}{\sim}$ which is equivalent to $\overset{c}{\sim}$ (cf. Definition 1.3.4 and Lemma 1.3.6), it is established in [Bou02], that in characteristic 3, $g \overset{c}{\sim} f + ax^2y^3$ for some $a \in K$

(4) If $char(K) = 2$, then we have as in the latter case that f is not (AA). Again, using SINGULAR, *we show that $\mathfrak{m}^{10} \subset tj(f)$ and moreover the monomials x^i, $8 \leq i \leq x^{18}$ and x^jy, $6 \leq j \leq 17$ and xy^4 are those monomials of a K-basis of $gr^{AC}_{\boldsymbol{W}}(T_f)$ having total degree smaller or equal 18 and $\boldsymbol{W}$-degree bigger than 84. Using the same arguments as in the above we get $g \overset{c}{\sim} f + ax^6y$ for some $a \in K \setminus \{0\}$.*



$W_{1,1}$ *in* $char(K) = 0$

The following corollary shows that (AAC) is an appropriate condition.

Corollary 3.3.10. *Let $f \in \mathfrak{m} \subset K[[\boldsymbol{x}]]$ be (SPH) with respect to a C-polytope $\boldsymbol{P}$. If the principal part $f_{\boldsymbol{P}}$ of f satisfies the condition (AAC) with respect to $\boldsymbol{P}$, then f has a finite Tjurina number. That is the hypersurface singularity R_f is isolated.*

Proof. The proof is straightforward from Theorem 3.3.6 and Corollary 3.1.22. □

Remark 3.3.11. *(1) Corollary 3.3.10 states that a (SPH) power series $f \in K[[\boldsymbol{x}]]$ having a principal part $f_{\boldsymbol{P}}$ which satisfies (AAC) with respect to the corresponding C-polytope $\boldsymbol{P}$ has a finite Tjurina number. In other words, the finiteness of $\tau(f_{\boldsymbol{P}})$ implies under the condition (AAC) the finiteness of $\tau(f)$. Nevertheless, it should be noticed, that in this case $\tau(f) \leq \tau(f_{\boldsymbol{P}})$.*

(2) In general, as the following example shows, it is not true that (SPH) elements of $K[[\boldsymbol{x}]]$ have finite Tjurina number.

Example 3.3.12. *Let $char(K) = 2$ and let $f = x^2y^2 + y^4 + x^5 + x^3y^2 \in K[[x, y]]$. Clearly f is (SPH) and $f_0 = x^2y^2 + y^4 + x^5$ is its principal part. We have $\tau(f_0) = 16$ while $\tau(f)$ is infinite. Furthermore we should notice that f_0 is not (AAC) with respect to its Newton polytope (cf. Example 3.2.16).*

In the particular case of (SQH) elements though, we obtain the following interesting result.

Corollary 3.3.13. *Let $f \in \mathfrak{m} \subset K[[\boldsymbol{x}]]$ be (SQH). Then f has a finite Tjurina number, that is the hypersurface singularity R_f is isolated.*

Proof. Let $f \in \mathfrak{m} \subset K[[\boldsymbol{x}]]$ be (SQH) with principal part f_Δ. By Definition 2.1.37, we have that f_Δ is (QH) and $\tau(f_\Delta) < \infty$. Moreover, it is established in Proposition 3.2.4 that f_Δ is (AC) with respect to its Newton polytope. Thus the claim clearly follows by Corollary 3.3.10 □

We should mention the analogy with Proposition 2.1.41 which deals with the Milnor number. Nevertheless, attention should be drawn to the fact that while $\mu(f) = \mu(f_\Delta)$, the equality does not in general hold for the Tjurina numbers $\tau(f)$ and $\tau(f_\Delta)$.

Going back to the general case of (SPH) hypersurface singularities, we formulate in the following a result on normal forms in relation with the condition (AA). In the same way as for (AAC), the following theorem shows that (AA) is an appropriate condition since it implies the finiteness of the Milnor number.

Theorem 3.3.14. *Let $f \in \mathfrak{m} \subset K[[\boldsymbol{x}]]$ be (SPH) with respect to a C-polytope $\boldsymbol{P}$ and let $\boldsymbol{W} \subset \mathbb{Z}^n_{>0}$ be a finite set of weights corresponding to $\boldsymbol{P}$. If $\mu(f_{\boldsymbol{P}}) < \infty$ and $f_{\boldsymbol{P}}$ is (AA) with respect to $\boldsymbol{P}$ and moreover $\{e_{\boldsymbol{\alpha}} : \boldsymbol{\alpha} \in \Lambda\}$ is a regular basis of $M_{f_{\boldsymbol{P}}}$, then f is* **finitely right determined** *and*

$$f \overset{\mathrm{r}}{\sim} f_{\boldsymbol{P}} + \sum_{\boldsymbol{\alpha} \in \Lambda^*} c_{\boldsymbol{\alpha}} e_{\boldsymbol{\alpha}},$$

where

$$\Lambda^* \subset \{\boldsymbol{\alpha} \in \Lambda : v_{\boldsymbol{W}}(e_{\boldsymbol{\alpha}}) \geq v_{\boldsymbol{W}}(f - f_{\boldsymbol{P}})\}$$

and the coefficients $c_{\boldsymbol{\alpha}} \in K$ are suitable.

Proof. we decide to omit the proof since it is an exact repetition of the arguments of the proof of Theorem 3.3.6. □

Remark 3.3.15. *Let $f \in \mathfrak{m} \subset K[[\boldsymbol{x}]]$ be (SPH) with respect to a C-polytope $\boldsymbol{P}$. If the principal part $f_{\boldsymbol{P}}$ has a finite Milnor number and is (A) or $f_{\boldsymbol{P}}$ is (AC)) with respect to its Newton polytope, then using Proposition 3.2.9, we can replace in Theorem 3.3.14 (resp. Theorem 3.3.6) a K-basis of the vector space $gr^A_{\boldsymbol{W}}(M_{f_{\boldsymbol{P}}})$ (resp. $gr^{AC}_{\boldsymbol{W}}(T_{f_{\boldsymbol{P}}})$) by a K-basis of the vector space $M_{f_{\boldsymbol{P}}}$ (resp. $T_{f_{\boldsymbol{P}}}$).*

Example 3.3.16. *Let $char(K) \neq 2$ and let $f \in K[[x, y]]$ be an equation corresponding to a plane curve singularity of type $T_{p,q}$, that is f is (SPH) of principal part $f_0 = x^p + \lambda x^2 y^2 + y^q$, where $\lambda \neq 0$ and $\frac{1}{p} + \frac{1}{q} < \frac{1}{2}$. Then, Proposition 3.2.15 and Theorem 3.3.6 yield $f \overset{\mathrm{c}}{\sim} f_0$.*

Example 3.3.17. *Let $char(K) = 2$ and let $f \in K[[x, y, z]]$ be associated to a hypersurface singularity of type Q_{10}, that is f is (SQH) of principal part $f_0 = x^2z+y^3+z^4$. Clearly, f_0 is (QH) of type $(\boldsymbol{W} = \{(9\,,8\,,6)\}\,;\,24)$.*
Using SINGULAR, *we show that $\tau(f_0) = 16$ and we get the following basis $\mathcal{B}$ of the K-vector space T_{f_0} consisting of monomials:*
$\mathcal{B} = \{1, x, y, xy, z, xz, yz, xyz, z^2, xz^2, yz^2, xyz^2, z^3, xz^3, yz^3, xyz^3\}$.
On the other hand, we see clearly that the four monomials xyz^2, xz^3, yz^3 and xyz^3 have $\boldsymbol{W}$-degree bigger than 24. Therefore in characteristic 2, we have

$$f \quad \overset{\mathrm{c}}{\sim} \quad x^2z + y^3 + z^4 + c_1xyz^2 + c_2xz^3 + c_3yz^3 + c_4xyz^3,$$

for some c_1, c_2, c_3 and $c_4 \in K$.
On the other hand, we observe that $v_{\boldsymbol{W}}(xyz^3) = 35$ is the biggest $\boldsymbol{W}$-degree of the monomials in $\mathcal{B}$. Furthermore, the function $degHC$ from the library $gradalg.lib$ delivers the smallest positive integer k such that $\mathfrak{m}^{k+1} \subset F_{>35}$. It turns out that $k = 5$ and therefore it follows from Theorem 3.3.6 that f and hence f_0 are 5-determined. Moreover, it is of interest to observe that this bound of determinacy is more suited for the effective computations as the one obtained by Theorem 3.1.15. Indeed, as $\mathfrak{m}^6 \subset tj(f_0)$, Theorem 3.1.15 asserts that f_0 is $12 - 3 + 2 = 11$-determined.

Based on these observations and arguing as in the proof of Theorem 3.3.6, we attempt in the last part of the present chapter to give explicit bounds of determinacy in the case of (SPH) hypersurface singularities.

For the sequel, we consider $f \in \mathfrak{m} \subset K[[\boldsymbol{x}]]$ which is (SPH) with respect to a C-polytope $\boldsymbol{P}$.

Theorem 3.3.18. *Let $f \in \mathfrak{m} \subset K[[\boldsymbol{x}]]$ be (SPH) such that $f_{\boldsymbol{P}}$ is (AAC) with respect to $\boldsymbol{P}$. Further, let $\boldsymbol{W} \subset \mathbb{Z}^n_{>0}$ be a finite set of weights corresponding to $\boldsymbol{P}$ and let $\{e_{\boldsymbol{\alpha}} : \boldsymbol{\alpha} \in \Lambda\}$ be a K-basis of $gr^{AC}_{\boldsymbol{W}}(T_{f_{\boldsymbol{P}}})$ consisting of monomials.*
Then f is k-contact determined if $\mathfrak{m}^{k+1} \subset F_{>D}$ where

$$D := max\Big\{v_{\boldsymbol{W}}(f_{\boldsymbol{P}})\ ,\ max\,\{v_{\boldsymbol{W}}(e_{\boldsymbol{\alpha}}) :\ \boldsymbol{\alpha} \in \Lambda\}\Big\}.$$

Proof. To avoid repetition, we simply refer to the last part of the proof of Theorem 3.3.6. ☐

Example 3.3.19. *Let $char(K) = 23$ and let $f = x^{23} + x^8y^4 + y^8 \in K[[x, y]]$. Clearly f is (PH) of type $\{\boldsymbol{W}\,;\,d\}$ where $\boldsymbol{W} = \Big\{(16\,;\,60)\,,\,(23\,;\,46)\Big\}$ and $d = 368$. On the other hand, $\tau(f) = 105$. Moreover, it is not difficult to see that f is $NPND^*$. Therefore, by Proposition 3.2.17, f is (AAC) with respect to its Newton polytope. Hence $dim_K(gr^{AC}_{\boldsymbol{W}}(T_f))$ is finite.*
Furthermore, using the function $ACgrbase$ from the library $gradalg.lib$, we get $dim_K(gr^{AC}_{\boldsymbol{W}}(T_f)) = 123$ and a K-basis $\mathcal{B} = \{e_{\boldsymbol{\alpha}} : \boldsymbol{\alpha} \in \Lambda\}$ of the vector space $gr^{AC}_{\boldsymbol{W}}(T_f)$ consisting of monomials such that

$$max\Big\{v_{\boldsymbol{W}}(e_{\boldsymbol{\alpha}}) :\ e_{\boldsymbol{\alpha}} \in \mathcal{B} \cap F_{>368}\Big\} = 598.$$

Moreover with the function $degHC$ from $gradalg.lib$, we get $\mathfrak{m}^{38} \subset F_{>598}$. Hence, by Theorem 3.3.18, f is 37-contact determined. Finally, refering to Example 3.1.20, it is of interest to notice that this bound of determinacy is much smaller than the one obtained by Theorem 3.1.18 (cf. also [GrK90]) and Theorem 3.1.15 respectively.

In the same way as in Theorem 3.3.18, we establish the following for right-determinacy.

Theorem 3.3.20. *Let $f \in \mathfrak{m} \subset K[[\boldsymbol{x}]]$ be (SPH) such that $\mu(f_{\boldsymbol{P}})$ is finite and $f_{\boldsymbol{P}}$ is (AA) with respect to $\boldsymbol{P}$. Further let $\boldsymbol{W} \subset \mathbb{Z}^n_{>0}$ be a finite set of weights corresponding to $\boldsymbol{P}$ and let $\{e_{\boldsymbol{\alpha}} : \boldsymbol{\alpha} \in \Lambda\}$ be a K-basis of $gr^{A}_{\boldsymbol{W}}(M_{f_{\boldsymbol{P}}})$ consisting of monomials.*
Then f is k-right determined if $\mathfrak{m}^{k+1} \subset F_{>D}$ where

$$D := max\Big\{v_{\boldsymbol{W}}(f_{\boldsymbol{P}})\ ,\ max\,\{v_{\boldsymbol{W}}(e_{\boldsymbol{\alpha}}) :\ \boldsymbol{\alpha} \in \Lambda\}\Big\}.$$

Proof. cf. the proof of Theorem 3.3.6. ☐

In the particular case where condition (AC) (resp. (A)) holds, we can reformulate Theorem 3.3.18 (resp. Theorem 3.3.20) as follows.

Corollary 3.3.21. *Let $f \in \mathfrak{m} \subset K[[\boldsymbol{x}]]$ be (SPH) such that $f_{\boldsymbol{P}}$ is (AC) with respect to $\boldsymbol{P}$. Further let $\boldsymbol{W} \subset \mathbb{Z}^n_{>0}$ be a finite set of weights corresponding to $\boldsymbol{P}$ and let $d = v_{\boldsymbol{W}}(f)$. If D and k are positive integers such that $\mathfrak{m}^{k+1} \subset F_{\geq D} \subset tj(f_{\boldsymbol{P}}) \cap F_{>d}$, then f is k-contact determined.*

Proof. Let $f \in \mathfrak{m} \subset K[[\boldsymbol{x}]]$ be (SPH) such that $f_{\boldsymbol{P}}$ is (AC) with respect to $\boldsymbol{P}$. Hence $dim_K(gr^{AC}_{\boldsymbol{W}}(T_{f_{\boldsymbol{P}}}) < \infty$. Further let $\boldsymbol{W} \subset \mathbb{Z}^n_{>0}$ be a finite set of weights corresponding to $\boldsymbol{P}$ and let $\mathcal{B} := \{e_{\boldsymbol{\alpha}} : \boldsymbol{\alpha} \in \Lambda\}$ be a K-basis of $gr^{A}_{\boldsymbol{W}}(T_{f_{\boldsymbol{P}}})$ consisting of monomials. On the other hand, setting $d := v_{\boldsymbol{W}}(f) = v_{\boldsymbol{W}}(f_{\boldsymbol{P}})$, we consider D and $k \in \mathbb{Z}_{>0}$ such that

$$F_{\geq D} \subset tj(f_{\boldsymbol{P}}) \cap F_{>d} \tag{3.21}$$

and

$$\mathfrak{m}^{k+1} \subset F_{\geq D} \tag{3.22}$$

First of all, we notice that if $N = max\,\{v_{\boldsymbol{W}}(e_{\boldsymbol{\alpha}}) : \boldsymbol{\alpha} \in \Lambda\}$, then $N < D$. Otherwise, let $\boldsymbol{\alpha} \in \Lambda$ be such that $v_{\boldsymbol{W}}(e_{\boldsymbol{\alpha}}) = N$, then $e_{\boldsymbol{\alpha}} \in F_{\geq N} \subset F_{\geq D} \subset tj(f)$. For $f_{\boldsymbol{P}}$ is (AC) with respect to $\boldsymbol{W}$, then $e_{\boldsymbol{\alpha}}$ would satisfy $(AC1)$ and $(AC2)$ with respect to $\boldsymbol{W}$ and therefore $e_{\boldsymbol{\alpha}} = 0$ in $gr^{AC}_{\boldsymbol{W}}(T_{f_{\boldsymbol{P}}})$ against the choice of $\mathcal{B}$. Hence $N < D$. Besides $d < D$ follows clearly from the relation (3.21). Hence, if we write

$$D' := max\Big\{v_{\boldsymbol{W}}(f_{\boldsymbol{P}})\ ,\ max\,\{v_{\boldsymbol{W}}(e_{\boldsymbol{\alpha}}) : \boldsymbol{\alpha} \in \Lambda\}\Big\},$$

then $D > D'$ clearly follows.
Altogether with the relation (3.22) shows that $\mathfrak{m}^{k+1} \subset F_{>D'}$. Hence, f is k-contact determined by Theorem 3.3.18. ☐

Example 3.3.22. *We consider a hypersurface singularity of type E_7 corresponding to an equation $f \in K[[x,y,z]]$. That is, f is (SQH) of principal part $f_0 = x^3 + xy^3 + z^2$. Clearly f_0 is (QH) of type $\big\{\boldsymbol{w} = (6\,,\,4\,,\,9)\,;\,18\big\}$. Hence, by Proposition 3.2.4, f_0 is (AC) with respect to its Newton polytope. We show in the following that the degree of contact determinacy of E_7 is 4 when $char(K) \neq 2$ and 5 when $char(K) = 2$.*
Observing that 2 and 3 divide the weighted degree 18 of f_0, we consider the following cases:

(1) *Case $char(K) \neq 2$ and $char(K) \neq 3$. Using* SINGULAR, *we get $\tau(f_0) = 7$ and moreover the set $\mathcal{B} = \Big\{1, x, y, xy, y^2, y^3, y^4\Big\}$ is a K-basis of T_{f_0} (cf. Remark 3.3.15). We notice that the weighted degree of any monomial in $\mathcal{B}$ is smaller than 18. Moreover, it is easy to see that $F_{\geq 19} \subset tj(f_0) \cap F_{>18}$. On the other hand, using the function $degHC$ from the library $gradalg.lib$, we get $\mathfrak{m}^5 \subset F_{\geq 19}$. Altogether, this yields by Corollary 3.3.21 that f is 4-determined. Clearly f is not 3-determined, since for example we would have $f \overset{c}{\sim} f - xy^3$ but the latter has an infinite Tjurina number. Thus 4 is the degree of determinacy of f.*

(2) *Case $char(K) = 3$. Using in the same way* SINGULAR, *we show that $F_{\geq 21} \subset tj(f_0) \cap F_{>18}$ and moreover $\mathfrak{m}^6 \subset F_{\geq 21}$.*
Thus, Corollary 3.3.21 asserts that f is 5-determined. Nevertheless, we have $\mathcal{B} = \Big\{1, x, x^2, y, xy, x^2y, y^2, xy^2, x^2y^2\Big\}$ is a K-basis of T_{f_0}. Moreover x^2y^2 is the only monomial in $\mathcal{B}$ having weighted degree bigger than 18 and $v_{\boldsymbol{w}}(x^2y^2) = 20$. Hence, by Theorem 3.3.6, we have

$$f \overset{c}{\sim} f_0 + cx^2y^2\ , c \in K.$$

On the other hand, considering the set of monomials of total degree 5*, it is easy to see that the weighted degree of all of them but* y^5 *is bigger than* 20*. Hence, if* $M \in Mon(K[[x,y,z]])$ *such that* $deg(M) = 5$ *and* $M \neq y^5$*, then Theorem 3.3.6 asserts that for any* $b \in K$*,* $f_0 + cx^2y^2 + bM \overset{c}{\sim} f_0 + cx^2y^2 \overset{c}{\sim} f$*. Moreover, we can write* $y^5 = \xi f_0$ *where* $\xi = y^2\partial_x$*. Cleary* $v_{\boldsymbol{w}}(y^5) = v_{\boldsymbol{w}}(\xi) + v_{\boldsymbol{w}}(f_0)$ *and therefore* $y^5 = 0$ *in* $gr^{AC}_{\boldsymbol{W}}(T_{f_0})$*. Thus for any* $a \in K$*,* $f + ay^5 \overset{c}{\sim} f$*. Altogether yields* 4 *is the degree of determinacy of* f*.*

(3) Case $char(K) = 2$*. Using* SINGULAR*, we show that* $\tau(f_0) = 14$ *and moreover* $\mathfrak{m}^7 \subset F_{\geq 26} \subset tj(f_0) \cap F_{>18}$*. Hence,* f *is* 6*-determined follows by Corollary 3.3.21. Moreover, the set*

$$\mathcal{B} := \left\{1, x, y, xy, y^2, y^3, y^4, z, xz, yz, xyz, y^2z, y^3z, y^4z\right\}$$

is a K*-basis of* T_{f_0}*. Clearly,* $\mathcal{B} \cap F_{>18} = \left\{y^3z, y^4z\right\}$ *and* $25 = v_{\boldsymbol{W}}(y^4z) > v_{\boldsymbol{W}}(y^3z) = 21$*. On the other hand all monomials* M *of total degree* 6 *but* y^6 *have weighted degree bigger than* 25*. Thus, by Theorem 3.3.6,* $f + M \overset{c}{\sim} f$*. Moreover,* $y^6 = \xi f_0$ *where* $\xi = y^3\partial_x$*. Since* $v_{\boldsymbol{w}}(y^6) = v_{\boldsymbol{w}}(\xi) + v_{\boldsymbol{w}}(f_0)$*,* $y^6 = 0$ *in* $gr^{AC}_{\boldsymbol{W}}(T_{f_0})$ *and Theorem 3.3.6 yields* $f + ay^6 \overset{c}{\sim} f$ *for all* $a \in K$*. Altogether, we obtain* f *is* 5*-determined. However, since* $\tau(f_0 + y^4z) = 12 \neq \tau(f_0)$*, then* f *is not* 4*-determined. This shows that the* 5 *is the degree of contact determinacy of* f*.*

Finally, it is of interest to notice that if $char(K) \neq 2$ *and* $char(K) \neq 3$*, then Lemma 2.1.32 yields for all* $g \in K[[\boldsymbol{x}]]$*,* $f \overset{c}{\sim} g$*, if and only if* $f \overset{r}{\sim} g$*. Thus due to the above, we see that in this case also, the degree of right determinacy of any hypersurface singularity of type* E_7 *is* 4*. If* $char(K) = 2$ *or* $char(K) = 3$ *though, Lemma 2.1.33 asserts that* $\mu(f_0)$ *is infinite.*

In some cases we can even give explicitly the degree of determinacy as the following result shows. First we recall that $f \in K[[\boldsymbol{x}]]$ is called convenient or (CO), if its Newton polytope meets all coordinate subspaces (cf. Definition 2.1.6).

Corollary 3.3.23. *Let* $f \in \mathfrak{m} \subset K[[\boldsymbol{x}]]$ *be* (SPH) *such that* $f_{\boldsymbol{P}}$ *satisfies* (AC) *with respect to* $\boldsymbol{P}$*. Further, let* $\boldsymbol{W} \subset \mathbb{Z}^n_{>0}$ *be a finite set of weights corresponding to* $\boldsymbol{P}$ *and let* $d := v_{\boldsymbol{W}}(f_{\boldsymbol{P}})$*. If* $f_{\boldsymbol{P}}$ *is* (CO) *and* $F_{>d} \subset tj(f_{\boldsymbol{P}})$*, then* f *is* $deg(f_{\boldsymbol{P}})$*-contact determined where* $deg(f_{\boldsymbol{P}})$ *is the total degree of* $f_{\boldsymbol{P}}$*. If moreover for any proper subset* Λ *of* $supp(f_{\boldsymbol{P}})$*, the truncation* $(f_{\boldsymbol{P}})_\Lambda$ *has an infinite Tjurina number, then* $deg(f_{\boldsymbol{P}})$ *is the* **degree** *of contact determinacy of* f*.*

Proof. Let $f \in \mathfrak{m} \subset K[[\boldsymbol{x}]]$ be (SPH) such that $f_{\boldsymbol{P}}$ satisfies (AC) with respect to $\boldsymbol{P}$. Moreover let $\boldsymbol{W} = \{\boldsymbol{w}_j : j \in J\} \subset \mathbb{Z}^n_{>0}$ be a finite set of weights corresponding to $\boldsymbol{P}$ and let $d := v_{\boldsymbol{W}}(f_{\boldsymbol{P}})$. We write $N = deg(f_{\boldsymbol{P}})$ for the total degree of $f_{\boldsymbol{P}}$.
We suppose in the following that $f_{\boldsymbol{P}}$ is (CO) and $F_{>d} \subset tj(f_{\boldsymbol{P}})$ and we claim that $\mathfrak{m}^{N+1} \subset F_{>d} \subset tj(f) \cap F_{>d}$. Nevertheless, the inclusion $F_{>d} \subset tj(f) \cap F_{>d}$ follows clearly for $F_{>d} \subset tj(f_{\boldsymbol{P}})$. On the other hand, let $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n_{\geq 0}$ such

that $\boldsymbol{x}^{\boldsymbol{\alpha}} \in \mathfrak{m}^{N+1}$. We show in the following that $v_{\boldsymbol{W}}(\boldsymbol{x}^{\boldsymbol{\alpha}}) > d$. For this purpose, we consider the n-tuples

$$\epsilon_i = \begin{pmatrix} 0 & \ldots & 0, & \overset{i}{1}, & 0 & \ldots & 0 \end{pmatrix},$$

where $i = 1, \ldots, n$. As $f_{\boldsymbol{P}}$ is (CO), then there exists $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_n) \in \mathbb{Z}^n_{>0}$ such that $\beta_i \cdot \epsilon_i \in supp(f_{\boldsymbol{P}})$ for all $1 \leq i \leq n$. Without loss of generality, we can suppose that $\beta_1 = deg(f_{\boldsymbol{P}}) = N$. On the other hand, as $f_{\boldsymbol{P}}$ is (PH) of $\boldsymbol{W}$-degree d, we can write for $i = 2, \ldots, n$,

$$\beta_1 \cdot \left(\underset{j \in J}{min}\{w^{(j)}_{1}\} \right) = \beta_i \cdot \left(\underset{j \in J}{min}\{w^{(j)}_{i}\} \right) = d.$$

For all i, since $\beta_1 \geq \beta_i$, it follows that

$$\underset{j \in J}{min}\{w^{(j)}_{1}\} \leq \underset{j \in J}{min}\{w^{(j)}_{i}\}.$$

Altogether, this yields for $j \in J$

$$\begin{aligned} \sum_{i=1}^{n} w^{(j)}_{i} \alpha_i &\geq \sum_{i=1}^{n} \alpha_i \cdot \left(\underset{j \in J}{min}\{w^{(j)}_{i}\} \right) \\ &\geq \left(\underset{j \in J}{min}\{w^{(j)}_{1}\} \right) \cdot \left(\sum_{i=1}^{n} \alpha_i \right) \\ &\geq (N+1) \left(\min_{j \in J}\{w^{(j)}_{1}\} \right) \\ &= (\beta_1 + 1) \left(\underset{j \in J}{min}\{w^{(j)}_{1}\} \right) \\ &= d + \underset{j \in J}{min}\{w^{(j)}_{1}\} \\ &> d. \end{aligned}$$

Thus the claim follows. On the other hand, Corollary 3.3.21 yields f is N-contact determined.
For the last part of the proof, we suppose that for any subset Λ of $supp(f_{\boldsymbol{P}})$, we have $\tau((f_{\boldsymbol{P}})_\Lambda) = \infty$, then it is easy to see that f cannot be $(N-1)$-determined. Otherwise $f \overset{c}{\sim} f_{\boldsymbol{P}} - x_{1}^{\beta_1}$ would follow which is impossible since $\tau(f) < \infty$ by Corollary 3.3.10 while $\tau(f_{\boldsymbol{P}} - x_{1}^{\beta_1}) = \infty$ by assumption. Hence $N = deg(f_{\boldsymbol{P}})$ is the degree of determinacy of f. $\square$

Example 3.3.24. *Let $char(K) \neq 2$ and let $f \in K[[x, y]]$, as in Example 3.3.16 be associated to a $T_{p,q}$-plane curve singularity. Further, let $f_0 = x^p + \lambda x^2 y^2 + x^q$ be the principal part of f_0, where $\lambda \neq 0$ and $\frac{1}{p} + \frac{1}{q} < \frac{1}{2}$. Clearly f_0 is (CO). On the other hand, writing d for the piecewise-homogeneous degree of f, Proposition 3.2.15 asserts that f_0 is (AC) with respect to its Newton polytope and moreover $F_{>d} \subset tj(f)$. Hence, by Corollary 3.3.23, we obtain that the positive integer $deg(f_0) = \max\{p\,,\, q\}$*

is a bound of contact determinacy of f. *Without loss of generality, we can assume* $p = deg(f_0)$. *We observe that actually* p *is the degree of contact determinacy of* f. *Indeed, if we suppose* f *is* $(p-1)$*-determined, then* $f \overset{c}{\sim} f_0 - x^p$ *would follow. Nevertheless, since* $f_0 - x^p$ *is not reduced, this yields* $\tau(f_0 - x^p) = \infty$ *whence a contradiction.*

Corollary 3.3.25. *Let* $f \in \mathfrak{m} \subset K[[\boldsymbol{x}]]$ *be* (SPH) *such that* $\mu(f_{\boldsymbol{P}})$ *is finite and moreover* $f_{\boldsymbol{P}}$ *is* (A) *with respect to* $\boldsymbol{P}$. *Further let* $\boldsymbol{W} \subset \mathbb{Z}^n_{>0}$ *be a finite set of weights corresponding to* $\boldsymbol{P}$ *and let* $d = v_{\boldsymbol{W}}(f)$. *If* D *and* k *are positive integers such that* $\mathfrak{m}^{k+1} \subset F_{\geq D} \subset j(f_{\boldsymbol{P}}) \cap F_{>d}$, *then* f *is* k*-right determined.*

Proof. The arguments used in the proof of Corollary 3.3.21 show in the same way the claim. Hence we decide to omit the proof here for the reason of size. ☐

In the particular case of (CO) elements, we get for right degree of determinacy a similar result as in Corollary 3.3.23.

Corollary 3.3.26. *Let* $f \in \mathfrak{m} \subset K[[\boldsymbol{x}]]$ *be* (SPH) *such that* $\mu(f_{\boldsymbol{P}})$ *is finite and moreover* $f_{\boldsymbol{P}}$ *is* (A) *with respect to* $\boldsymbol{P}$. *Further let* $d = v_{\boldsymbol{W}}(f)$. *If* $f_{\boldsymbol{P}}$ *is* (CO) *and* $F_{>d} \subset j(f_{\boldsymbol{P}})$, *then then* f *is* $deg(f_{\boldsymbol{P}})$*-contact determined. If moreover for any proper subset* Λ *of* $supp(f_{\boldsymbol{P}})$, *the truncation* $(f_{\boldsymbol{P}})_\Lambda$ *has an infinite Milnor number, then* $deg(f_{\boldsymbol{P}})$ *is the* **degree** *of right determinacy of* f.

Proof. cf. proof of Corollary 3.3.23. ☐

Chapter 4

Implementation in SINGULAR

In the present chapter we discuss the methods used in the so far Chapters 2 and 3 from the computational point of view. For this purpose, we shall present algorithms which we implement in the computer algebra system SINGULAR under the library $gradalg.lib$ (cf. Appendix B).
First, we show how to compute the ideals of a filtration $\left(F_l\right)_{l\in\mathbb{Z}_{\geq 0}}$ of $K[[x_1,\ldots,x_n]]$ which is related to a finite set of weights $\boldsymbol{W}\subset\mathbb{Z}^n_{>0}$.
Afterwards, we present algorithms for the computation of regular bases up to a given degree.

Throughout this chapter, $\boldsymbol{W}$ denotes a finite set of weights contained in $\mathbb{Z}^n_{>0}$.

4.1 Basic Tools.

In this section, we shall present the basic algorithms which are used for the implementation of the main procedures of the library $gradalg.lib$ in SINGULAR.

Let $\boldsymbol{W}\subset\mathbb{Z}^n_{>0}$, the first algorithm computes the piecewise-homogeneous order $v_{\boldsymbol{W}}(f)$ of a polynomial $f\in K[\boldsymbol{x}]$ (cf. Definition 2.1.18). We recall that if the set $\boldsymbol{W}$ contains only one weight $\boldsymbol{w}$, then considering in SINGULAR a local weighted degree ordering with respect to $\boldsymbol{w}$, the function ord computes $v_{\boldsymbol{w}}(f)$. For details, we refer to [GrP02] and [GPS06].

Algorithm 4.1.1. (PIECEWISE-HOMOGENEOUS ORDER OF A POLYNOMIAL)

Input: A polynomial $f\in K[\boldsymbol{x}]$ and a finite set of weights $\boldsymbol{W}\subset\mathbb{Z}^n_{>0}$.
Output: $v_{\boldsymbol{W}}(f)$.
  procedure $grord(f,\boldsymbol{W})$
    choose $\boldsymbol{w}\in\boldsymbol{W}$
    $\boldsymbol{W}=\boldsymbol{W}\setminus\{\boldsymbol{w}\}$
    $N=v_{\boldsymbol{w}}(f)$
    $tmpord=0$

for all $\boldsymbol{w} \in \boldsymbol{W}$ **do**
$tmpord = v_{\boldsymbol{w}}(f)$
if $tmpord < N$ **then**
$N = tmpord$
end if
end for
return N
end procedure

Proof. Since $\boldsymbol{W}$ is a finite set, the algorithm terminates. Correctness follows obviously from Definition 2.1.18. □

Algorithm 4.1.2. (PIECEWISE-HOMOGENEOUS ORDER OF A MONOMIAL DERIVATION)

Input: A monomial $M \in K[x_1, \ldots, x_n]$, a finite set of weights $\boldsymbol{W} = \{\boldsymbol{w}_1, \ldots, \boldsymbol{w}_s\}$ and a positive integer i such that $1 \leq i \leq n$.
Output: $v_{\boldsymbol{W}}(M\partial_{x_i})$.
procedure $Dergrord(M, \boldsymbol{W}, i)$
choose $\boldsymbol{w} \in \boldsymbol{W}$
$\boldsymbol{W} = \boldsymbol{W} \setminus \{\boldsymbol{w}\}$
$N = v_{\boldsymbol{w}}(M) - \boldsymbol{w}[i]$
$tmpord = 0$
for all $\boldsymbol{w} \in \boldsymbol{W}$ **do**
$tmpord = v_{\boldsymbol{w}}(M) - \boldsymbol{w}[i]$
if $tmpord < N$ **then**
$N = tmpord$
end if
end for
return N
end procedure

Proof. Correctness of the algorithm follows from Definition 2.2.1 and termination follows since the set $\boldsymbol{W}$ is finite. □

The next algorithm computes the initial form of a polynomial with respect to a finite set of weights $\boldsymbol{W}$ (cf. Definition 2.1.46). Moreover, we recall that in SINGULAR, the functions $leadcoef$ and $leadmonom(f)$ compute respectively the leading coefficient and the leading monomial of a polynomial with respect to a given monomial ordering (cf. [GrP02] and [GPS06]).

Algorithm 4.1.3. (PIECEWISE-HOMOGENEOUS INITIAL FORM OF A POLYNOMIAL)

Input: A polynomial $f \in K[\boldsymbol{x}]$ and a finite set $\boldsymbol{W} \subset \mathbb{Z}^n_{>0}$.
Output: $In_{\boldsymbol{W}}(f)$.
procedure $grlead(f, \boldsymbol{W})$
$M = 0$
$tmplead = 0$
$N = v_{\boldsymbol{W}}(f)$

while $f \neq 0$ **do**
$M = leadcoef(f) * leadmonom(f)$
$f = f - M$
if $v_{\boldsymbol{W}}(M) = N$ **then**
$tmplead = tmplead + M$
end if
end while
return $tmplead$
end procedure

Proof. To see termination, note that f has finitely many monomials. Correctness follows clearly from Definition 2.1.46. □

For the next algorithm, we refer to Definition 3.1.5.

Algorithm 4.1.4. (PIECEWISE-HOMOGENEOUS JET OF A POLYNOMIAL)

Input: A polynomial $f \in K[\boldsymbol{x}]$, a finite set $\boldsymbol{W} \subset \mathbb{Z}^n_{>0}$ and a positive integer d.
Output: $f^{(\boldsymbol{W},d)}$.
procedure $pwjet(f, \boldsymbol{W}, d)$
if $v_{\boldsymbol{W}}(In_{\boldsymbol{W}}(f)) > d$ **then**
return 0
end if
$g = 0$
$tmpjet = 0$
while $f \neq 0$ **do**
$g = In_{\boldsymbol{W}}(f)$
$f = f - g$
if $v_{\boldsymbol{W}}(g) \leq d$ **then**
$tmpjet = tmpjet + g$
end if
end while
return $tmpjet$
end procedure

Proof. The termination follows as f has finitely many monomials.
Denoting $f = \sum_{\boldsymbol{\alpha}} a_{\boldsymbol{\alpha}} x^{\boldsymbol{\alpha}}$ and $\Lambda^* = \left\{\boldsymbol{\alpha} \in supp(f) : v_{\boldsymbol{W}}(\boldsymbol{x}^{\boldsymbol{\alpha}}) \leq d\right\}$, the correctness follows from

$$f^{(\boldsymbol{W},d)} = \sum_{\boldsymbol{\alpha} \in \Lambda^*} a_{\boldsymbol{\alpha}} x^{\boldsymbol{\alpha}}.$$

□

Remark 4.1.5. *For $f \in K[\boldsymbol{x}]$, we recall that $deg(f)$ denotes the total degree of the polynomial f. On the other hand, considering a monomial ordering $>$ and a zero-dimensional ideal $I \subset K[\boldsymbol{x}]_>$, we explain in the following the use of some* SINGULAR *functions which are relevant for the sequel.*

- *$std(I)$ computes a standard basis of the ideal I with respect to $>$.*
- *If the generators of I are a standard basis , then*
  - *$highcorner(I)$ returns the smallest monomial not contained in I.*
  - *$kbase(I)$ computes a K-vector space basis (consisting of monic monomials) of $K[\boldsymbol{x}]_>/I$.*
  - *$reduce(f, I)$ return 0, if and only if, $f \in I$.*

Algorithm 4.1.6. *($kspan(I)$)*

Input: A local degree ordering $>$ and a zero-dimensional ideal $I \subset K[\boldsymbol{x}]_>$.
Output: The maximal set (consisting of monic monomials) which generate the quotient ring $K[\boldsymbol{x}]_>/I$ as K-vector space.

procedure $kspan(I)$
  $J = std(I)$
  k is $deg(highcorner(I)) + 1$
  $\mathfrak{m}^k = std(\mathfrak{m}^k)$
  $B = kbase(\mathfrak{m}^k)$
  $tmp = 0$
  for all monomials $M \in B$ **do**
    if $reduce(M, J) \neq 0$ **then**
      $tmp = tmp, \{M\}$
    end if
  end for
  return tmp
end procedure

Proof. Let $\boldsymbol{x}^{\boldsymbol{\alpha}} = highcorner(I)$ and let $M \in Mon(K[\boldsymbol{x}])$ such that $deg(M) > deg(\boldsymbol{x}^{\boldsymbol{\alpha}})$. Since $>$ is a local degree ordering, then $M < \boldsymbol{x}^{\boldsymbol{\alpha}}$. Hence $M \in I$ follows by definition of the highcorner of I (cf. [GrP02, 1.7.11]). Setting $k = deg(\boldsymbol{x}^{\boldsymbol{\alpha}}) + 1$ yields $\langle \boldsymbol{x} \rangle^k \subset I$. Furthermore the set

$$B := kbase(\langle \boldsymbol{x} \rangle^k) = \left\{M \in Mon(K[\boldsymbol{x}]) : \, deg(M) < k\right\}$$

is a K-vector space basis of $K[\boldsymbol{x}]_>/\langle \boldsymbol{x} \rangle^k$. Obviously the set B is finite which shows the finiteness of the algorithm. Moreover, if we consider the following epimorphism of K-vector spaces

$$\begin{array}{rcl} K[\boldsymbol{x}]_>/\langle \boldsymbol{x} \rangle^k = \bigoplus_{M \in B} K \cdot M & \longrightarrow & K[\boldsymbol{x}]_>/I \\ M & \mapsto & M \, mod(I) \end{array}$$

then it is not difficult to see that the set $\left\{M \, mod(I) : \; M \notin I\right\}$ is a maximal generating system of the vector space $K[\boldsymbol{x}]_>/I$. Therefore the correctness of the algorithm follows. $\square$

Algorithm 4.1.7. *(*PIECEWISE-HOMOGENEOUS ORDER OF A POWER OF $\langle x_1, \ldots, x_n\rangle$)

Input: A local degree ordering $>$, a positive integer N and a finite set $\boldsymbol{W} \subset \mathbb{Z}^n_{>0}$.
Output: The biggest positive integer d such that $F_{\geq d} \supset \langle x_1, \ldots, x_n\rangle^N$.

procedure $degHCS((N, \boldsymbol{W}))$
  $J = std(\mathfrak{m}^N)$
  choose $M \in J$
  $D = v_{\boldsymbol{W}}(M)$
  $J = J \setminus \{M\}$
  $Dtmp = 0$
  for all monomials $M \in J$ **do**
    $Dtmp = v_{\boldsymbol{W}}(M)$
    if $Dtmp < D$ **then**
      $D = Dtmp$
    end if
  end for
  return D
end procedure

Proof. The finiteness follows since $K[\boldsymbol{x}]_>$ is Noetherian. To show the correctness, note that $J = \big\{M \in Mon(K[\boldsymbol{x}]) : deg(M) = N\big\}$ is a standard basis of $\langle \boldsymbol{x}\rangle^N$. Moreover, if $f \in \langle \boldsymbol{x}\rangle^N$, then there exists $M \in J$ such that M divides f. Hence, we can wite $f = M \cdot g$, where $g \in K[\boldsymbol{x}]_>$. On the other hand, Remark 2.1.21 yields

$$v_{\boldsymbol{W}}(f) \geq v_{\boldsymbol{W}}(g) + v_{\boldsymbol{W}}(M) \geq v_{\boldsymbol{W}}(M) \geq min\big\{M : M \in J\big\}.$$

This shows the correctness. □

Algorithm 4.1.8. *(*TOTAL DEGREE OF $highcorner(F_{\geq N})$)

Input: A local degree ordering $>$, a positive integer N and a finite set $\boldsymbol{W} \subset \mathbb{Z}^n_{>0}$.
Output: The total degree of $highcorner(F_{\geq N})$.

procedure $degHC((N, \boldsymbol{W}))$
  $k = 1$
  $D = min\{l : \mathfrak{m} \subset F_{\geq l}\}$
  while $D < N$ **do**
    for all $k \geq 2$ **do**
      $D = min\{l : \mathfrak{m}^k \subset F_{\geq l}\}$
    end for
  end while
  return $k - 1$
end procedure

Proof. It follows clearly by the definition of the ideal $F_{\geq N}$ that the K-vector space $K[\boldsymbol{x}]_>/F_{\geq N}$ has finite dimension and this shows the finiteness of the algorithm. For the correctness, we notice that

$$\langle \boldsymbol{x}\rangle^k \subset F_{\geq N} \Longleftrightarrow N = min\{l : \mathfrak{m}^k \subset F_{\geq l}\},$$

and this latter condition is checked inductively starting by $k = 1$. Thus, the first k, for which this condition is fulfilled, is obviously the smallest k where $\langle \boldsymbol{x} \rangle^k \subset F_{\geq N}$. Moreover, arguing in the same way as in the proof of Algorithm 4.1.6, we see that $k-1$ is the total degree of the highcorner of a standard basis of $F_{\geq N}$. □

Algorithm 4.1.9. *(**W**-IDEAL)*

Input: A local degree ordering $>$, a positive integer N and a finite set $\boldsymbol{W}$ of weights.
Output: A standard basis of the $\boldsymbol{W}$-ideal $F_{\geq N}$.

```
procedure grideal((N, W))
    k = degHC(N, W)
    I = std(⟨x⟩^{k+1})
    J = kbase(I)
    tmp = I
    for all M ∈ J do
        if v_W(M) ≥ N then
            tmp = tmp + ⟨M⟩
        end if
    end for
    return std(tmp)
end procedure
```

Proof. Let $k = degHC(N, \boldsymbol{W})$, then Algorithm 4.1.8 yields $\langle \boldsymbol{x} \rangle^{k+1} \subset F_{\geq N}$. Furthermore, it is easy to see that the set $J = \{M \in Mon(K[\boldsymbol{x}]) : deg(M) \leq k\}$ is a representative of a K-vector space basis of $K[\boldsymbol{x}]_> / \langle \boldsymbol{x} \rangle^{k+1}$. To see the correctness of the algorithm, we write

$$S_1 = \{M \in J : v_{\boldsymbol{W}}(M) \geq N\}$$

and

$$S_2 = \{\boldsymbol{x}^{\boldsymbol{\alpha}} \in Mon(K[\boldsymbol{x}]) : |\boldsymbol{\alpha}| = k+1 \text{ and } \boldsymbol{x}^{\boldsymbol{\alpha}} \text{ has no divisor in } S_1\}.$$

We show in the following that $S = S_1 \cup S_2$ is a standard basis of $F_{\geq N}$. Clearly, $S \subset F_{\geq N} \cap Mon(K[\boldsymbol{x}])$. On the other hand, let $g \in F_{\geq N}$ and let $LM(g)$ be the leading monomial of g with respect to $>$. If $LM(g) \notin \mathfrak{m}^{k+1}$, that is $deg(LM(g)) \leq k$, then $LM(g) \in S_1$ follows since $v_{\boldsymbol{W}}(LM(g)) \geq v_{\boldsymbol{W}}(g) \geq N$. If we suppose that $LM(g) \in \mathfrak{m}^{k+1}$, then there exists obviously a monomial $\boldsymbol{x}^{\boldsymbol{\alpha}}$ such that $|\boldsymbol{\alpha}| = k$ and $\boldsymbol{x}^{\boldsymbol{\alpha}} \mid LM(g)$. Moreover if there exists $M \in S_1$ such that $M \mid \boldsymbol{x}^{\boldsymbol{\alpha}}$, then M divides also $LM(g)$. If not, then $\boldsymbol{x}^{\boldsymbol{\alpha}} \in S_2$ by construction. Altogether yields that there exists a monomial in S such that $M \mid LM(g)$ and this shows the claim (cf. [GrP02, 1.6.1]). The termination follows obviously, since $F_{\geq N}$ is zero-dimensional and moreover the set J is finite. □

Let N be a positive integer. With the $\boldsymbol{W}$-ideal $F_{\geq N}$ at our disposal, we compute in the following a K-basis of the vector space $K[\boldsymbol{x}]_N := \langle \boldsymbol{x}^{\boldsymbol{\alpha}} : v_{\boldsymbol{W}}(\boldsymbol{x}^{\boldsymbol{\alpha}}) = N \rangle_K$.

Algorithm 4.1.10. $(grlist)$

Input: A local degree ordering $>$, a positive integer N and a finite set $\boldsymbol{W}$ of weights.
Output: A K-vector space basis of $K[\boldsymbol{x}]_N$ (consisting of monomials).

procedure $grlist((N, \boldsymbol{W}))$
    $I = grideal(N, \boldsymbol{W})$
    $tmp = 0$
    for all monomials $M \in I$ **do**
        if $v_{\boldsymbol{W}}(M)$=N **then**
            $tmp = tmp \cup \{M\}$
        end if
    end for
    return tmp
end procedure

Proof. The termination is straightforward, for $K[\boldsymbol{x}]_>$ is Noetherian. On the other hand, we notice that the set $\mathcal{B} = \left\{\boldsymbol{x}^{\boldsymbol{\alpha}} : v_{\boldsymbol{W}}(\boldsymbol{x}^{\boldsymbol{\alpha}}) = N\right\}$ represents a K-basis of the vector space $K[\boldsymbol{x}]_N$. Moreover, by Algorithm 4.1.9, the ideal $I = grideal(N, \boldsymbol{W})$ is a standard basis of $F_{\geq N}$ consisting of monomials. Let $\boldsymbol{x}^{\boldsymbol{\alpha}} \in \mathcal{B}$, then there exists a monomial $M \in I$ and $g \in K[\boldsymbol{x}]_>$ such that $\boldsymbol{x}^{\boldsymbol{\alpha}} = M \cdot g$. As $v_{\boldsymbol{W}}(M) \leq v_{\boldsymbol{W}}(M) + v_{\boldsymbol{W}}(g) \leq v_{\boldsymbol{W}}(\boldsymbol{x}^{\boldsymbol{\alpha}}) = N \leq v_{\boldsymbol{W}}(M)$, it follows that $v_{\boldsymbol{W}}(g) = 0$ and hence $g = 1$ for M and $\boldsymbol{x}^{\boldsymbol{\alpha}}$ are monic. Therefore

$$\mathcal{B} = \{M \in I : v_{\boldsymbol{W}}(M) = N\}.$$

This shows the correctness of the algorithm. $\square$

4.2 K-bases of $gr^A_{\boldsymbol{W}}(M_f)$ and $gr^{AC}_{\boldsymbol{W}}(T_f)$

Let $\boldsymbol{W}$ be a finite set of weights in $\mathbb{Z}^n_{>0}$ and let $f \in K[\boldsymbol{x}]$ be (PH) of type $(\boldsymbol{W}\,;\, d)$. We have shown in Section 3.3 of this thesis that the computation of a normal form with respect to $\overset{\mathrm{r}}{\sim}$ (resp.$\overset{\mathrm{c}}{\sim}$) for (SPH) hypersurface singularities having f as principal part is closely related to the computation of a K-vector space basis of the graded K-algebras $gr^A_{\boldsymbol{W}}(M_f)$ (resp. $gr^{AC}_{\boldsymbol{W}}(T_f)$). Nevertheless, in Remark 3.2.6, we have observed that the dimension of these as K-vector spaces is in general infinite.
In the present section, we give algorithms to compute the elements of a monomial K-basis of $gr^A_{\boldsymbol{W}}(M_f)$ (resp. $gr^{AC}_{\boldsymbol{W}}(T_f)$) up to a given degree.

For $N \in \mathbb{Z}_{\geq 0}$, we recall the ideal

$$j^A_{\boldsymbol{W}}(f,d) := \langle g \in j(f) : \ v_{\boldsymbol{W}}(g) = d \ \text{ and } \ g \ \text{ is } \ (A1) \ \text{ with respect to } \ f, \ \text{and } \boldsymbol{W} \rangle$$

and its initial ideal $In_{\boldsymbol{W}}(j^A_{\boldsymbol{W}}(f,d))$ with respect to $\boldsymbol{W}$ (cf Definition 2.1.46).
The following algorithm computes a generating sytem of the K-vector space

$$In_{\boldsymbol{W}}(j^A_{\boldsymbol{W}}(f,d))_N := In_{\boldsymbol{W}}(j^A_{\boldsymbol{W}}(f,d)) \cap K[\boldsymbol{x}]_N.$$

The idea of the algorithm is to perform separate calculations for the different weights $\boldsymbol{w} \in \boldsymbol{W}$ and then fit them together.

Algorithm 4.2.1. $(Aideal)$

Input: A local degree ordering $>$, a (PH) polynomial $f \in K[x_1, \dots, x_n]$ of type $(\boldsymbol{W}\,;\, d)$ and N a positive integer.
Output: A set of K-generators of $In_{\boldsymbol{W}}(j^A_{\boldsymbol{W}}(f))_N$ consisting of (PH) polynomials of type $(\boldsymbol{W}\,;\, N)$.

procedure $Aideal(f, \boldsymbol{W}, N)$
  $I = j(f)$
  $D = N - d$
  $J = grlist(D, \boldsymbol{W})$
  $L = 0$
  for all $i \in \{1, \dots, n\}$ **do**
    $L = L\,,\, \langle x_i f_{x_i} \rangle \cdot J$
  end for
  $tmp = I, L$
  for all $\boldsymbol{w} \in \boldsymbol{W}$ **do**
    $tmp_{\boldsymbol{w}} = 0$
    for all $i \in \{1, \dots, n\}$ **do**
      $Q_{\boldsymbol{w},i} = grlist(D + \boldsymbol{w}[i], \boldsymbol{W})$
      $tmp_{\boldsymbol{w},i} = 0$
      for all $M \in Q_{\boldsymbol{w},i}$ **do**
        if $v_{\boldsymbol{W}}(M\partial_{x_i}) = D$ **then**
          $tmp_{\boldsymbol{w},i} = tmp_{\boldsymbol{w},i}\,,\, M$
        end if
      end for
      $P_{\boldsymbol{w},i} = \langle f_{x_i} \rangle \cdot tmp_{\boldsymbol{w},i}$
      $tmp_{\boldsymbol{w}} = tmp_{\boldsymbol{w}}, P_{\boldsymbol{w},i}$
    end for
    $tmp = tmp\,,\, tmp_{\boldsymbol{w}}$
  end for
  $spantmp = 0$
  for all $g \in tmp$ **do**
    if $v_{\boldsymbol{W}}(g) = N$ **then**
      $spantmp = spantmp, In_{\boldsymbol{W}}(g)$
    end if
  end for
  return $spantmp$
end procedure

Proof. Let $f \in K[\boldsymbol{x}]$ be (PH) of type $(\boldsymbol{W}\,,\, d)$ and let N be a positive integer. First, we show the correctness of $Aideal$. For this purpose, we consider the following finite dimensional K-vector spaces:

$$V_1 := \bigoplus_{i=1}^{n} \langle \boldsymbol{x}^{\boldsymbol{\alpha}} x_i \partial_{x_i} : v_{\boldsymbol{W}}(\boldsymbol{x}^{\boldsymbol{\alpha}}) = N - d \rangle_K,$$

for $\boldsymbol{w} \in \boldsymbol{W}$

$$V_2 := \bigoplus_{i=1}^{n} K[\boldsymbol{x}]_{N-d+w_i},$$

$$V_{2,\boldsymbol{W}} := \bigoplus_{\boldsymbol{w} \in \boldsymbol{W}} V_{2,\boldsymbol{w}},$$

$$V_2 := \bigoplus_{i=1}^{n} \langle \boldsymbol{x}^{\boldsymbol{\alpha}} \partial_{x_i} : \boldsymbol{x}^{\boldsymbol{\alpha}} \in V_{2,\boldsymbol{W}} \text{ and } v_{\boldsymbol{W}}(\boldsymbol{x}^{\boldsymbol{\alpha}} \partial_{x_i}) = N - d \rangle_K$$

and

$$V := \bigoplus_{i=1}^{n} \langle \boldsymbol{x}^{\boldsymbol{\alpha}} \partial_{x_i} : v_{\boldsymbol{W}}(\boldsymbol{x}^{\boldsymbol{\alpha}} \partial_{x_i}) = N - d \rangle_K.$$

We claim that $V = V_1 + V_2$. Indeed, $V_1 \subset V$ follows clearly from Definition 2.2.1 and V_2 is obviously contained in V. Hence, $V_1 + V_2 \subset V$. Conversely, let $\boldsymbol{x}^{\boldsymbol{\alpha}} \partial_{x_i} \in V$ where $\boldsymbol{\alpha} \in (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n_{\geq 0}$.

- If $\alpha_i \neq 0$, then we can write $\boldsymbol{x}^{\boldsymbol{\alpha}} \partial_{x_i} = \boldsymbol{x}^{\boldsymbol{r}} x_i \partial_{x_i}$ with $\boldsymbol{r} = \boldsymbol{\alpha} - \epsilon_i$. Again using Definition 2.2.1 shows that $v_{\boldsymbol{W}}(\boldsymbol{x}^{\boldsymbol{r}}) = \boldsymbol{x}^{\boldsymbol{r}} x_i \partial_{x_i} = N - d$ and so $\boldsymbol{x}^{\boldsymbol{r}} x_i \partial_{x_i} \in V_1$ follows.

- If $\alpha_i = 0$ and writing for $\boldsymbol{w} \in \boldsymbol{W}$, $\boldsymbol{w} = (w_1, \ldots, w_n)$, then Definition 2.2.1 yields
$$v_{\boldsymbol{W}}(\boldsymbol{x}^{\boldsymbol{\alpha}} \partial_{x_i}) = min \left\{ v_{\boldsymbol{w}}(x^{\boldsymbol{\alpha}}) - w_i : \boldsymbol{w} \in \boldsymbol{W} \right\}.$$
  Let $w \in \boldsymbol{W}$ such that $v_{\boldsymbol{W}}(\boldsymbol{x}^{\boldsymbol{\alpha}} \partial_{x_i}) = v_{\boldsymbol{w}}(\boldsymbol{x}^{\boldsymbol{\alpha}} \partial_{x_i}) = v_{\boldsymbol{w}}(x^{\boldsymbol{\alpha}}) - w_i$. Hence, we get $v_{\boldsymbol{w}}(x^{\boldsymbol{\alpha}}) = N - d + w_i$ which implies $v_{\boldsymbol{W}}(x^{\boldsymbol{\alpha}}) \leq N - d + w_i$. On the other hand, Remark 2.2.2 yields $v_{\boldsymbol{W}}(x^{\boldsymbol{\alpha}}) \geq v_{\boldsymbol{W}}(\boldsymbol{x}^{\boldsymbol{\alpha}} \partial_{x_i}) + w_i = N - d + w_i$. Altogether, this shows that $v_{\boldsymbol{W}}(x^{\boldsymbol{\alpha}}) = N - d + w_i$. Hence $x^{\boldsymbol{\alpha}} \in V_{2,\boldsymbol{w}}$ and so $\boldsymbol{x}^{\boldsymbol{\alpha}} \partial_{x_i} \in V_2$ clearly follows.

For the sequel, we denote

$$V(f) := \sum_{i=1}^{n} \langle \boldsymbol{x}^{\boldsymbol{\alpha}} f_{x_i} : \boldsymbol{x}^{\boldsymbol{\alpha}} \partial_{x_i} \in V \rangle_{K[\boldsymbol{x}]_>}.$$

Let $\boldsymbol{x}^{\boldsymbol{\alpha}} f_{x_i} \in V(f)$, then it follows by Lemma 2.2.3 that $v_{\boldsymbol{W}}(\boldsymbol{x}^{\boldsymbol{\alpha}} f_{x_i}) \geq v_{\boldsymbol{W}}(\boldsymbol{x}^{\boldsymbol{\alpha}} \partial_{x_i}) + v_{\boldsymbol{W}}(f) = N$. We claim that

$$In_{\boldsymbol{W}}(j^A_{\boldsymbol{W}}(f, N))_N = In_{\boldsymbol{W}}(V(f)) \cap K[\boldsymbol{x}]_N.$$

Indeed, let $g \in In_{\boldsymbol{W}}(V(f)) \cap K[\boldsymbol{x}]_N$. Without loss of generality we can assume $g = In_{\boldsymbol{W}}(\boldsymbol{x}^{\boldsymbol{\alpha}} f_{x_i})$ where $\boldsymbol{x}^{\boldsymbol{\alpha}} f_{x_i} \in V(f)$ and moreover g is (PH) of type $(\boldsymbol{W}\,;\,N)$. Hence

$$\begin{aligned} v_{\boldsymbol{W}}(\boldsymbol{x}^{\boldsymbol{\alpha}} \partial_{x_i} f) &= v_{\boldsymbol{W}}(\boldsymbol{x}^{\boldsymbol{\alpha}} f_{x_i}) \\ &= v_{\boldsymbol{W}}(g) \\ &= N \\ &= (N-d)+d \\ &= v_{\boldsymbol{W}}(\boldsymbol{x}^{\boldsymbol{\alpha}} \partial_{x_i}) + v_{\boldsymbol{W}}(f) \end{aligned}$$

and the latter equality holds since $\boldsymbol{x}^{\boldsymbol{\alpha}} \partial_{x_i} \in V$. Thus, $g \in In_{\boldsymbol{W}}(j^{A}_{\boldsymbol{W}}(f,N))_N$. Convesely, let $g \in In_{\boldsymbol{W}}(j^{A}_{\boldsymbol{W}}(f,N))_N$. Here again, we can assume that $g = In_{\boldsymbol{W}}(\xi f + h)$ where $v_{\boldsymbol{W}}(g) = v_{\boldsymbol{W}}(\xi f) = v_{\boldsymbol{W}}(\xi) + v_{\boldsymbol{W}}(f) = N$ and $v_{\boldsymbol{W}}(h) > N$. Hence, we have $g = In_{\boldsymbol{W}}(\xi f)$. Moreover, as f is (PH), and writing the decomposition of ξ into its (PH) parts, we get $g = In_{\boldsymbol{W}}(In_{\boldsymbol{W}}(\xi) f)$ where $In_{\boldsymbol{W}}(\xi)$ is a (PH) derivation such that $v_{\boldsymbol{W}}(\xi) = N - d$. This yields $(\xi) \in V$ and therefore $g \in In_{\boldsymbol{W}}(V(f))_N = In_{\boldsymbol{W}}(V(f)) \cap K[\boldsymbol{x}]_N$. Altogether, this shows the correctness of the algorithm. Finally the termination is obvious since we are computing with a finite set of weights $\boldsymbol{W}$ and finitely dimensional K-vector spaces. □

Desposing of $Aideal$, we give in the following an algorithm to compute a generating system of the K-vector space $In_{\boldsymbol{W}}(tj^{AC}_{\boldsymbol{W}}(f,N))_N$ where

$$tj^{AC}_{\boldsymbol{W}}(f,d) := \langle g \in tj(f) :\ v_{\boldsymbol{W}}(g) = d \ \text{ and } \ g \ \text{ is } \ (AC1) \ \text{ with respect to } \ f \text{ and } \boldsymbol{W} \rangle\,.$$

Algorithm 4.2.2. *($ACideal$)*

Input: A local degree ordering $>$, a (PH) polynomial $f \in K[\boldsymbol{x}]$ of type $(\boldsymbol{W}\,;\,d)$ and N a positive integer.

Output: A set of K-generators of $In_{\boldsymbol{W}}(tj^{AC}_{\boldsymbol{W}}(f))_N$ consisting of (PH) polynomials of type $(\boldsymbol{W}\,;\,N)$.

procedure $ACideal(f, \boldsymbol{W}, N)$
  $I = grlist(N-d, \boldsymbol{W})$
  $J = I \cdot \langle f \rangle$
  $tmp = 0$
  for all $M \in J$ **do**
    $tmp = tmp, In_{\boldsymbol{W}}(M)$
  end for
  $spantmp = tmp, Aideal(f, \boldsymbol{W}, N)$
  return $spantmp$
end procedure

Proof. The termination follows for the same reasons as for Algorithm 4.2.1. To see correctness, we consider the ideal $J\big\langle b \cdot f \,:\, b \in K[\boldsymbol{x}]_{N-d} \big\rangle_{K[\boldsymbol{x}]_>}$ and we claim that

$$In_{\boldsymbol{W}}(tj^{AC}_{\boldsymbol{W}}(f))_N = In_{\boldsymbol{W}}(J)_N \,+\, In_{\boldsymbol{W}}(j^{A}_{\boldsymbol{W}}(f))_N.$$

First, we should observe that, as f is (PH) of type $(\boldsymbol{W}\,;\,d)$, then we have $v_{\boldsymbol{W}}(f) = v_{\boldsymbol{w}}(f) = d$ for all $\boldsymbol{w} \in \boldsymbol{W}$.
Thus, Lemma 2.1.22 asserts that $v_{\boldsymbol{W}}(b \cdot f) = v_{\boldsymbol{W}}(b) + v_{\boldsymbol{W}}(f)$ for all $b \in K[\boldsymbol{x}]_>$.
Hence the inclusion $In_{\boldsymbol{W}}(J)_N + In_{\boldsymbol{W}}(j^A_{\boldsymbol{w}}(f))_N \subset In_{\boldsymbol{W}}(tj^{AC}_{\boldsymbol{w}}(f))_N$ follows clearly.
Conversely, let $g \in In_{\boldsymbol{W}}(tj^{AC}_{\boldsymbol{w}}(f))_N$. Then in the same way as in the proof of Algorithm 4.2.1, we can write $g = In_{\boldsymbol{W}}(In_{\boldsymbol{W}}(b)f) + In_{\boldsymbol{W}}(In_{\boldsymbol{W}}(\xi)f)$ with $b \in K[\boldsymbol{x}]_>$ and $\xi \in Der_K(K[\boldsymbol{x}]_>)$ such that

$$v_{\boldsymbol{W}}(g) = min\left\{v_{\boldsymbol{W}}(In_{\boldsymbol{W}}(b_0)) + v_{\boldsymbol{W}}(f)\,;\, v_{\boldsymbol{W}}(In_{\boldsymbol{W}}(\xi)) + v_{\boldsymbol{W}}(f)\right\}.$$

This shows clearly the claim. □

Arguing similarly as in the proof of Proposition 2.1.50, we get

$$K[\boldsymbol{x}]_N / In_{\boldsymbol{W}}(j^A_{\boldsymbol{w}}(f))_N \cong F_{\geq N}/(j^A_{\boldsymbol{w}}(f,N) + F_{>N})$$

and

$$K[\boldsymbol{x}]_N / In_{\boldsymbol{W}}(tj^{AC}_{\boldsymbol{w}}(f))_N \cong F_{\geq N}/(tj^{AC}_{\boldsymbol{w}}(f,N) + F_{>N}).$$

Considering these finitely dimensional vector spaces, the following two algorithms are devoted for the computation of K-bases consisting of monomials.

Algorithm 4.2.3. *($Akbase$)*

Input: A local degree ordering $>$, a (PH) polynomial $f \in K[\boldsymbol{x}]$ of type $(\boldsymbol{W}\,;\,d)$ and N a positive integer.
Output: A basis of the K-vector space $F_{\geq N}/(j^A_{\boldsymbol{w}}(f,N) + F_{>N})$ consisting of monomials.

procedure $Akbase(f, \boldsymbol{W}, N)$
  $I = std(Aideal(f, \boldsymbol{W}, N))$
  $J = grlist(N, L)$
  $B = 0$
  for all monomials $M \in J$ **do**
    if $reduce(M, I) = M$ **then**
      $B = B\,,\,M$
    end if
  end for
  return B
end procedure

Proof. Termination of the algorithm is most easily seen since we have finitely dimensional vector spaces. For correctness, we denote the ideal $In_{\boldsymbol{W}}(j^A_{\boldsymbol{w}}(f))_N$ by I_A and we consider the epimorphism of K-vector spaces

$$K[\boldsymbol{x}]_N \twoheadrightarrow K[\boldsymbol{x}]_N / I_A.$$

Moreover $J = grlist(N, \boldsymbol{W}) = \left\{\boldsymbol{x}^{\boldsymbol{\alpha}} : \boldsymbol{\alpha} \in \Lambda\right\}$ is a K-basis of $K[\boldsymbol{x}]_N$ consisting of monomials. Hence, the set $B = \left\{\boldsymbol{x}^{\boldsymbol{\alpha}}\, mod(I_A) : \boldsymbol{x}^{\boldsymbol{\alpha}} \in J, \text{and } \boldsymbol{x}^{\boldsymbol{\alpha}} \notin I_A\right\}$ generates the vector space $K[\boldsymbol{x}]_N / I_A$. We denote Λ^* the set of indices of the elements of B and we claim that B is linearly independant in $K[\boldsymbol{x}]_N/I_A$. Indeed, we consider a zero linear combination of the elements in B in $K[\boldsymbol{x}]_N/I_A$, that is a relation $\sum_{\boldsymbol{\alpha}\in\Lambda^*} c_{\boldsymbol{\alpha}}\boldsymbol{x}^{\boldsymbol{\alpha}} = 0$. Then, there exists $g \in I_A$ such that $\sum_{\boldsymbol{\alpha}\in\Lambda^*} c_{\boldsymbol{\alpha}}\boldsymbol{x}^{\boldsymbol{\alpha}} = g$ in $K[\boldsymbol{x}]_N$. Therefore, we can write $g = \sum_{\boldsymbol{\alpha}\notin\Lambda^*} b_{\alpha}x^{\boldsymbol{\alpha}}$. Thus $c_{\boldsymbol{\alpha}} = 0$ for all $\boldsymbol{\alpha} \in \Lambda^*$ clearly follows since J is a K-basis of $K[\boldsymbol{x}]_N$. This implies that B is a K-basis of the vector space $K[\boldsymbol{x}]_N/I_A$ and therefore shows the correctness of the algorithm. □

The following algorithm computes in the same way a K-basis of the vector space $F_{\geq N}/(tj_{\boldsymbol{W}}^{AC}(f,N) + F_{>N})$.

Algorithm 4.2.4. *($ACkbase$)*

Input: A local degree ordering $>$, a (PH) polynomial $f \in K[\boldsymbol{x}]$ of type $(\boldsymbol{W}\,;\,d)$ and N a positive integer.

Output: A basis of the K-vector space $F_{\geq N}/(tj_{\boldsymbol{W}}^{AC}(f,N) + F_{>N})$ consisting of monomials.

procedure $ACkbase(f, \boldsymbol{W}, N)$
 $I = std(ACideal(f, \boldsymbol{W}, N))$
 $J = grlist(N, L)$
 $B = 0$
 for all monomials $M \in J$ **do**
  if $reduce(M, I) = M$ **then**
   $B = B\,,\,M$
  end if
 end for
 return B
end procedure

Proof. For the proof, we need only to replace the vector space $In_{\boldsymbol{W}}(j_{\boldsymbol{W}}^{A}(f))_N$ by $In_{\boldsymbol{W}}(tj_{\boldsymbol{W}}^{AC}(f))_N$ in the proof of $Akbase$ and follow the same arguments. Hence, to avoid repetition, we decide to omit the proof of Algorithm 4.2.4 □

With these tools at our disposal, we can easily compute the subset of a K-basis of $gr_{\boldsymbol{W}}^{A}(M_f)$ and $gr_{\boldsymbol{W}}^{AC}(T_f)$ respectively, consisting of monomials up to a given degree.

Algorithm 4.2.5. *($Agrbase$)*

Input: A local degree ordering $>$, a (PH) polynomial $f \in K[\boldsymbol{x}]$ of type $(\boldsymbol{W}\,;\,d)$ and N a positive integer.

Output: The elements in a monomial basis of the K-vector space $gr_{\boldsymbol{W}}^{A}(M_f)$ having a total degree smaller or equal to N.

procedure $Agrbase(f, \boldsymbol{W}, N)$
 $I = kbase(std(\langle\boldsymbol{x}\rangle^{N+1}))$
 $B = 0$

for all $M \in I$ **do**
  if $reduce(M, std(Aideal(f, \boldsymbol{W}, v_{\boldsymbol{W}}(M)))) = M$ **then**
    $B = B\ ,\ M$
  end if
end for
return B
end procedure

Proof. The proof is straightforward from the one of $Akbase$. □

Algorithm 4.2.6. *($ACgrbase$)*

Input: A local degree ordering $>$, a (PH) polynomial $f \in K[\boldsymbol{x}]$ of type $(\boldsymbol{W}\ ;\ d)$ and N a positive integer.
Output: The elements in a monomial basis of the K-vector space $gr^{AC}_{\boldsymbol{W}}(T_f)$ having a total degree smaller or equal to N.
procedure $ACgrbase$($f, \boldsymbol{W}, N$)
  $I = kbase(std(\langle \boldsymbol{x} \rangle^{N+1}))$
  $B = 0$
  for all $M \in I$ **do**
    if $reduce(M, std(ACideal(f, \boldsymbol{W}, v_{\boldsymbol{W}}(M)))) = M$ **then**
      $B = B\ ,\ M$
    end if
  end for
  return B
end procedure

Proof. The proof is straightforward from the one of $ACkbase$. □

4.3 Checking Conditions (AA) and (AAC)

The algorithms which we shall present in this last section are motivated by the characterizations established in Proposition 3.2.21 and 3.2.19 of conditions (AA) and (AAC) respectively.

We consider a piecewise-homogeneous polynomial $f \in K[\boldsymbol{x}]$ of type $(\boldsymbol{W}\ ;\ d)$ and a positive integer N.

Algorithm 4.3.1. *($Aspan$)*

Input: A local degree ordering $>$, a (PH) polynomial $f \in K[\boldsymbol{x}]$ of type $(\boldsymbol{W}\ ;\ d)$ and N a positive integer.
Output: The K-generators of $In_{\boldsymbol{W}}(j^A_{\boldsymbol{W}}(f))_N$ which belong to $j(f)$.
procedure $Aspan$($f, \boldsymbol{W}, N$)
  $I = Aideal(f, \boldsymbol{W}, N)$
  $J = std(j(f))$
  $G = 0$

for all $g \in I$ **do**
if $reduce(g, J) = 0$ **then**
$G = G\ ,\ g$
end if
end for
return G
end procedure

Proof. Termination and correctness are straightforward from the above. □

Algorithm 4.3.2. *(*$ACspan$*)*

Input: A local degree ordering $>$, a (PH) polynomial $f \in K[\boldsymbol{x}]$ of type $(\boldsymbol{W}\ ;\ d)$ and N a positive integer.
Output: The K-generators of $In_{\boldsymbol{W}}(tj^{AC}_{\boldsymbol{W}}(f))_N$ which belong to $tj(f)$.
procedure $ACspan$**(**$f, \boldsymbol{W}, N$**)**
$I = ACideal(f, \boldsymbol{W}, N)$
$J = std(tj(f))$
$G = 0$
for all $g \in I$ **do**
if $reduce(g, J) = 0$ **then**
$G = G\ ,\ g$
end if
end for
return G
end procedure

Proof. Termination and correctness are straightforward from the above. □

With these tools at our disposal, we present in the following two algorithms which check the conditions (A) and (AC) respectively for all monomials having a given total degree.

Algorithm 4.3.3. *(*isA*)*

Input: A local degree ordering $>$, a (PH) polynomial $f \in K[\boldsymbol{x}]$ of type $(\boldsymbol{W}\ ;\ d)$ and k a positive integer.
Output: 1 if all the monomials having total degree k satisfy (A) with respect to f and $\boldsymbol{W}$, 0 otherwise.
procedure isA**(**$f, \boldsymbol{W}, k$**)**
$I = std(\langle \boldsymbol{x} \rangle^k)$
for all $M \in I$ **do**
if $reduce(M, std(Aspan(f, \boldsymbol{W}, v_{\boldsymbol{W}}(M)))) \neq 0$ **then**
return 0
end if
end for
return 1
end procedure

Proof. Note that in the computer algebra system SINGULAR, a standard basis of the ideal $\langle \boldsymbol{x} \rangle^k$ consists of all monomials $M \in Mon(K[\boldsymbol{x}])$ having total degree $deg(M) = k$.
Termination follows obviously since condition (A) is checked for only finitely many elements. For Correctness, we notice that for $M \in Mon(K[\boldsymbol{x}])$, we have

M is $(A1)$ and $(A2)$ with respect to f and $\boldsymbol{W}$ if and only if $M \in Aspan(f, \boldsymbol{W}, v_{\boldsymbol{W}}(m))$.

Indeed, this claim is straightforward from Definition 3.2.1 and the Algorithms 4.2.1 and 4.3.1. □

Algorithm 4.3.4. *($isAC$)*

Input: A local degree ordering $>$, a (PH) polynomial $f \in K[\boldsymbol{x}]$ of type $(\boldsymbol{W}\,;\,d)$ and k a positive integer.
Output: 1 if all the monomials having total degree k satisfy (AC) with respect to f and $\boldsymbol{W}$, 0 otherwise.

procedure $isAC(f, \boldsymbol{W}, k)$
  $I = std(\langle \boldsymbol{x} \rangle^k)$
  for all $M \in I$ **do**
    if $reduce(M, std(ACspan(f, \boldsymbol{W}, v_{\boldsymbol{W}}(M)))) \neq 0$ **then**
      return 0
    end if
  end for
  return 1
end procedure

Proof. We decide to omit the proof since it is similar to the one of Algorithm. 4.3.3 □

Chapter 5

Some Applications of the Lefschetz Principle

In the standard references about hypersurface singularities, the main results are formulated and proved essentially over the field $\mathbb{C}$ of complex numbers. In this chapter we shall transfer theorems known for $\mathbb{C}$ to arbitrary fields of characteristic zero known as *Lefschetz principle*. In this way, we shall give explicit proofs to widely accepted claims about hypersurface singularities in characteristic zero.
The first part of the present chapter deals with the tools needed for the proofs of the main results. First we show that subfields generated by countable sets in characteristic zero are isomorphic to subfields of $\mathbb{C}$. After that, we formulate explicitly the Lefschetz principle for the study cases occuring in this chapter. Afterwards, we consider field extensions $K' \subset K$ and investigate the interrelation between algebroid singularities over K' and those over K obtained by extension of scalars.
In the second part, we deal with isolated hypersurface singularities in characteristic zero. We shall show that in this case, the finiteness of the Tjurina number is equivalent to the finiteness of the Milnor number. Furthermore, we show that the Milnor number of a semiquasihomogeneous singularity is equal to the Milnor number of its principal part. Finally, we consider well-known properties of the Milnor number which are in general not true in positive characteristic and show them in the context of characteristic zero.

5.1 Preliminaries

5.1.1 Subfields Generated by Countable Sets in Characteristic Zero

In this subsction, we consider an algebraically closed field K of characteristic zero and we shall present in the next theorem an interesting property of subfields of K which are generated by a countable number of elements. This property turns out to be a useful tool to transfer theorems known for $\mathbb{C}$ to arbitrary fields of characteristic 0.

Theorem 5.1.1. *Let K be a field of characteristic 0, and let A be a countable subset of K. Then $\mathbb{Q}(A)$ is $\mathbb{Q}$-isomorphic to a subfield of $\mathbb{C}$.*

Proof. We give here a constructive proof.
Writing $\mathbb{Q}(A) = \cup\{\mathbb{Q}(\alpha_1, \ldots, \alpha_n) : \alpha_1, \ldots, \alpha_n \in A\}$ where the union is over all finite subsets of A, we show in the following, that the subfield $\mathbb{Q}(A)$ of K is $\mathbb{Q}$-isomorphic to a subfield $\mathbb{Q}(\tilde{B})$ of $\mathbb{C}$ where $\tilde{B}$ is a countable subset $\{\tilde{\xi}_i\}_i$ of $\mathbb{C}$. We shall proceed in several steps.
First, we construct a countable subset $B := \{\xi_n : n \geq 1\}$ as follows:

$$\begin{cases} \xi_1 \in \mathbb{C}/\mathbb{Q} & \text{transcendental} \\ \xi_n \in \mathbb{Q}(\xi_1, \ldots, \xi_{n-1})/\mathbb{Q} & \text{transcendental} \end{cases}$$

Let ξ_1 be an arbitrary transcendental element of $\mathbb{C}/\mathbb{Q}$,
and for $n \geq 2$, let ξ_n be a transcendent element of $\mathbb{Q}(\xi_1, \ldots, \xi_{n-1})/\mathbb{Q}$.
This construction is possible since there is an infinite and uncountable transcendence basis for $\mathbb{C}/\mathbb{Q}$. Thus, for every $n \geq 2$, the existence of ξ_n is assured, for otherwise the extension $\mathbb{C}/\mathbb{Q}(\xi_1, \ldots, \xi_{n-1})$ would be algebraic and $\mathbb{Q}(\xi_1, \ldots, \xi_{n-1})$ would contain a transcendence basis S of the extension $\mathbb{C}/\mathbb{Q}$, which would mean by definition that the extension $\mathbb{Q}(\xi_1, \ldots, \xi_{n-1})/\mathbb{Q}(S)$ is algebraic and therefore $\mathbb{Q}(S)$ and $\mathbb{Q}(\xi_1, \ldots, \xi_{n-1})$ would have the same cardinality. But this is of course false, for $\mathbb{Q}(\xi_1, \ldots, \xi_{n-1})$ is countable while $\mathbb{Q}(S)$ is uncountable. Moreover, considering the above construction of the subset B, we notice that for all i, ξ_i is a transcendental element of $\mathbb{C}/\mathbb{Q}$.
Now, let $n, k \geq 1$ and $\{\alpha_{i_k}, \ldots, \alpha_{i_{(k+n-1)}}\}$ be an arbitrary finite subset of A with n elements.
We show by induction on n, that the subfield $\mathbb{Q}(\alpha_{i_k}, \ldots, \alpha_{i_{(k+n-1)}})$ of $\mathbb{Q}(A)$ is $\mathbb{Q}$-isomorphic to a subfield of $\mathbb{C}$.
For $n = 1$, consider the field $\mathbb{Q}(\alpha_{i_k})$.

- If α_{i_k} is transcendental over $\mathbb{Q}$, then we have

  $$\mathbb{Q}(\alpha_{i_k}) \cong \mathbb{Q}(x) \cong \mathbb{Q}(\xi_{i_k})$$

  In this case we take $\tilde{\xi}_{i_k} := \xi_{i_k}$.

- If α_{i_k} is algebraic over $\mathbb{Q}$ and $P_k := min(\mathbb{Q}, \alpha_{i_k})$, then it follows that

  $$\mathbb{Q}(\alpha_{i_k}) \cong \mathbb{Q}[x]/\langle P_k(x)\rangle \cong \mathbb{Q}[\xi_{i_k}]/\langle P_k(\xi_{i_k})\rangle$$

  It is clear that, the field $\mathbb{Q}[\xi_{i_k}]/\langle P_k(\xi_{i_k})\rangle$ is an algebraic extension of $\mathbb{Q}$ and is finitely generated by a zero of P_k in $\mathbb{C}$ which we denote by $\tilde{\xi}_{i_k}$.

Let $\tilde{K}^{(1)} := \mathbb{Q}(\tilde{\xi}_{i_k})$. In both cases $\mathbb{Q}(\alpha_{i_k})$ is $\mathbb{Q}$-isomorphic to the subfield $\tilde{K}^{(1)}$ of $\mathbb{C}$. Furthermore, we can show that, for every $l \geq 1$, we have $\xi_{i_{k+l}}$ is transcendental over $\tilde{K}^{(1)}$. Instead, we consider again both of the above cases:
In the first case where $\tilde{\xi}_{i_k} := \xi_{i_k}$, the claim follows by the construction of the subset B. In the second case, we consider the $\mathbb{Q}$-surjection

$$s : \mathbb{Q}[\xi_{i_k}] \longrightarrow \mathbb{Q}[\xi_{i_k}]/\langle P_k(\xi_{i_k})\rangle = \mathbb{Q}(\tilde{\xi}_{i_k})$$

Suppose that there exists an $l \geq 1$ such that $\xi_{i_{k+l}}$ is algebraic over $\mathbb{Q}(\tilde{\xi}_{i_k})$. Then there exists a polynomial $\tilde{P}(x) \in \mathbb{Q}(\tilde{\xi}_{i_k})[x]$ such that $\tilde{P}(\xi_{i_{k+l}}) = 0$. Writing

$$\tilde{P}(x) = \sum_{1 \leq t \leq d} a_t x^t = \sum_{1 \leq t \leq d} s(b_t) x^t,$$

where for all t, $a_t \in \mathbb{Q}(\tilde{\xi}_{i_k})$ and $b_t \in \mathbb{Q}[\xi_{i_k}]$, it turns out that the relation

$$0 = \tilde{P}(\xi_{i_{k+l}}) = \sum_{1 \leq t \leq d} s(b_t)(\xi_{i_{k+l}})^t$$

is an algebraic dependence relation between ξ_{i_k} and $\xi_{i_{k+l}}$. But this is impossible by construction of the subset B.

Let now $n \geq 2$ and suppose that $\mathbb{Q}(\alpha_{i_k}, \ldots, \alpha_{i_{(k+n-2)}})$ is $\mathbb{Q}$-isomorphic to a subfield $\tilde{K}^{(n-1)} = \mathbb{Q}(\tilde{\xi}_{i_k}, \ldots, \tilde{\xi}_{i_{(k+n-2)}})$ of $\mathbb{C}$ and for all $l \geq 1$, we have $\xi_{i_{(k+l+n-2)}}$ is transcendental over $\tilde{K}^{(n-1)}$.
Let $L_{n-1} := \mathbb{Q}(\alpha_{i_k}, \ldots, \alpha_{i_{(k+n-2)}})$ and $L_n := \mathbb{Q}(\alpha_{i_k}, \ldots, \alpha_{i_{(k+n-1)}})$.
We have $\mathbb{Q} \subset L_{n-1} \subset L_n \subset \mathbb{Q}(A)$ and $L_n = L_{n-1}(\alpha_{i_{(k+n-1)}})$.
As in the above, we consider two cases:

- If $\alpha_{i_{(k+n-1)}}$ is transcendental over L_{n-1}, then we have

  $$L_n \cong L_{n-1}(x) \cong \tilde{K}^{(n-1)}(\xi_{i_{(k+n-1)}})$$

  In this case we take $\tilde{\xi}_{i_{(k+n-1)}} := \xi_{i_{(k+n-1)}}$.

- If $\alpha_{i_{(k+n-1)}}$ is algebraic over L_{n-1} and $P_{_{(k+n-1)}} := min(L_{n-1}, \alpha_{i_{(k+n-1)}})$, then it follows that

  $$L_n \cong L_{n-1}[x]/\langle P_{_{(k+n-1)}}(x)\rangle \cong \tilde{K}^{(n-1)}[\xi_{i_{(k+n-1)}}]/\langle P_{_{(k+n-1)}}(\xi_{i_{(k+n-1)}})\rangle$$

  It is clear that, the field $\tilde{K}^{(n-1)}[\xi_{i_{(k+n-1)}}]/\langle P_{_{(k+n-1)}}(\xi_{i_{(k+n-1)}})\rangle$ is an algebraic extension of $\tilde{K}^{(n-1)}$ and is finitely generated by a zero of $P_{_{(k+n-1)}}$ in $\mathbb{C}$ which we denote by $\tilde{\xi}_{i_{(k+n-1)}}$.

Let $\tilde{K}^{(n)} := \tilde{K}^{(n-1)}(\xi_{i_{(k+n-1)}})$.
Altogether yields $\mathbb{Q}(\alpha_{i_k}, \ldots, \alpha_{i_{(k+n-1)}})$ is $\mathbb{Q}$-isomorphic to the subfield $\tilde{K}^{(n)}$ of $\mathbb{C}$. Moreover, proceeding as for the induction step $n = 1$, we have for all $l \geq 1$, $\xi_{i_{(k+l+n-1)}}$ is transcendental over $\tilde{K}^{(n)}$.
With these preparations made, we define the following $\mathbb{Q}$-morphism of fields

$$\begin{aligned} \phi: \quad \mathbb{Q}(A) &\longrightarrow \mathbb{C} \\ \alpha_i &\mapsto \tilde{\xi}_i \\ r &\mapsto r \qquad \text{for all } \ r \in \mathbb{Q} \end{aligned}$$

Considering the countable set $\tilde{B} := \{\tilde{\xi}_i\}$, it follows clearly from the above, that ϕ is a $\mathbb{Q}$-isomorphism from $\mathbb{Q}(A)$ onto the subfield $\mathbb{Q}(\tilde{B})$ of $\mathbb{C}$. Hence, the claim follows. $\square$

The Lefschetz principle
As we have already mentioned in the introduction of the present chapter, we shall bring answers in the general setting of algebraically closed fields of characteristic zero to problems on hypersurface singularities which are already solved in the case of the field $\mathbb{C}$.
The method we shall use for transfering results on hypersurface singularities over $\mathbb{C}$ to arbitrary algebraically closed fields of characteristic zero is called the *Lefschetz principle* and can be formulated as follows:
Let (P) be a problem over an algebraically closed field K of characteristic 0 such that (P) can be formulated over a subfield $K^{'}$ of K which is isomorphic to a subfield of $\mathbb{C}$. If moreover, the answer (A) of the problem (P) is in $K^{'}$ and the problem (P) is solvable over $\mathbb{C}$, then (P) is also solvable over K and has the same answer as in $\mathbb{C}$.

Schematically, the Lefschetz principle can be described as follows:

$$\underbrace{\left(\left((P) \text{ over } K^{'}\right) \wedge \left(K^{'} \cong K^{*} \subset \mathbb{C}\right)\right) \bigwedge \left((A) \text{ in } K^{'}\right) \bigwedge \left((P) \text{ solvable over } \mathbb{C}\right)}_{\Downarrow \atop \left((P) \text{ solvable over } K\right)}$$

In practice, for the problems dealt with in this chapter, the subfield $K^{'}$ of K is of the form $\mathbb{Q}(S)$ where S is a countable subset of K. On the other hand, Theorem 5.1.1 establishes that $\mathbb{Q}(S)$ is isomorphic to a subfield of $\mathbb{C}$.
Concretely, the method we shall follow to generalize known results over $\mathbb{C}$ to arbitrary algebraically closed fields of characteristic O consists of three principal steps. We consider an algebraically closed field K of characteristic zero and a problem over K which is solvable over $\mathbb{C}$.

- First, we show that the given problem over K can be formulated on a subfield $\mathbb{Q}(S)$ of K where S is a countable subset of K.
- Then, we show that the answer of the problem is in $\mathbb{Q}(S)$.
- Finally, we show that the problem which is initially solvable over $\mathbb{C}$ is also solvable over K and has the same answer as in $\mathbb{C}$.

Throughout this chapter we shall often use the following notation:

Notation 5.1.2. *Let* $f = \sum_{\boldsymbol{\alpha}} a_{\boldsymbol{\alpha}} x^{\boldsymbol{\alpha}} \in K[[\boldsymbol{x}]]$. *We write*

$$Coeff(f) := \{a_{\boldsymbol{\alpha}} : \boldsymbol{\alpha} \in supp(f)\}.$$

5.1.2 Extension of Scalars

Let $K' \subset K$ be a field extension. We consider the polynomial rings

$$K'[\boldsymbol{x}] := K'[x_1, \ldots, x_n] \text{ and } K[\boldsymbol{x}] := K[x_1, \ldots, x_n]$$

in finitely many variables. It is established that the latter ring can be obtained from the first one by extension of scalars. This is precisely formulated in the next lemma.

Theorem 5.1.3. *Let $K' \subset K$ be a field extension and let $I \subset K'[\boldsymbol{x}]$ be a proper ideal. Then, there is a canonical isomorphism*

$$(K'[\boldsymbol{x}]/I) \otimes_{K'} K \xrightarrow{\sim} K[\boldsymbol{x}]/IK[\boldsymbol{x}].$$

Proof. See [Bos00, 7.2, Satz 10] □

As the elements of polynomial algebras as well as those of tensor product algebras can be represented as finite sums, we should notice that the isomorphisms in Theorem 5.1.3 are canonical. On the other hand, replacing polynomial rings by power series rings make these finitess arguments, as we may expect, no more available. Hence, in order to generalize the statement of Theorem 5.1.3 to power series rings, we should consider the **completed** tensor product.

Definition 5.1.4. *Let K be a field and let $(A, \mathfrak{m})$ and $(B, \mathfrak{n})$ be noetherian local K-algebras. We call the K-algebra*

$$A \widehat{\otimes}_K B := \varprojlim_{(p,q)} (A/\mathfrak{m}^p \otimes_K B/\mathfrak{n}^q)$$

the completed tensor product of A and B over K.

Remark 5.1.5. *1. It is shown in [Ser00] that the completed tensor product $A \widehat{\otimes}_K B$ is naturally isomorphic as K-algebra to the completion of $A \otimes_K B$ for the $(\mathfrak{m} \otimes_K B + A \otimes_K \mathfrak{n})$-adic topology.*

2. Let $K' \subset K$ be a field extension and $I \subset K'[[\boldsymbol{x}]]$ be a proper ideal. If we consider the K'-algebra $(K'[[\boldsymbol{x}]]/I) \otimes_{K'} K$, we see easily that it has also a K-algebra structure given by

$$\lambda \cdot (g\, mod(I)) \otimes \beta = (g\, mod(I)) \otimes \lambda\beta,$$

for $g \in K'[[\boldsymbol{x}]]$, λ and β in K. Besides, this K-algebra is noetherian and local with the maximal ideal $\mathfrak{m} := (\langle \boldsymbol{x} \rangle / I) \otimes_{K'} K$.
If moreover we denote by A the K-algebra $(K'[[\boldsymbol{x}]]/I) \otimes_{K'} K$, then it follows by the first part of this remark that

$$(K'[[\boldsymbol{x}]]/I) \widehat{\otimes}_{K'} K \cong \hat{A},$$

where $\hat{A}$ is the completion of A in the $\mathfrak{m}$-adic topology.
Furthermore, we have

$$\hat{A} := \left\{ (a_1, a_2, \ldots) \in \prod_{i=1}^{\infty} A/\mathfrak{m}^i : \; a_i \equiv a_j \, mod\, \mathfrak{m}^i \;\; if \;\; j > i \right\}$$

and $\hat{A}$ has a natural ring structure, given by component wise addition and multiplication. On the other hand, we have manifestly

$$A/\mathfrak{m}^i \cong (K'[[\boldsymbol{x}]]/\langle \boldsymbol{x} \rangle^i + I) \otimes_{K'} K.$$

Theorem 5.1.6. *Let $K' \subset K$ be a field extension and let $I \subset K'[[\boldsymbol{x}]]$ be a proper ideal. Then, we have an isomorphism of K-algebras*

$$(K'[[\boldsymbol{x}]]/I)\widehat{\otimes}_{K'}K \xrightarrow{\cong} K[[\boldsymbol{x}]]/IK[[\boldsymbol{x}]].$$

Proof. Let $K' \subset K$ be a field extension and let $I \subset K'[[\boldsymbol{x}]]$ be a proper ideal. $IK[[\boldsymbol{x}]]$ denotes the ideal generated by I in $K[[\boldsymbol{x}]]$. Furthermore, let f be a formal power series and let k be a positive integer. Throughout this proof, we shall write $j_k(f)$ for the k-jet of f. Furthermore, for $k \geq 1$, we consider the following ideal of $K'[[\boldsymbol{x}]]$:

$$J_k := \left\langle j_{k-1}(f)\,,\, \boldsymbol{x}^{\boldsymbol{\alpha}} \,:\, f \in I \text{ and } \mid \boldsymbol{\alpha} \mid \geq k \right\rangle_{K'[[\boldsymbol{x}]]}$$

and we claim that $K'[[\boldsymbol{x}]]/(I + \langle \boldsymbol{x}\rangle^k) \cong K'[\boldsymbol{x}]/J_k$ as K-algebras. Indeed, let

$$\begin{array}{rcl} \phi :\, K'[[\boldsymbol{x}]] & \longrightarrow & K'[[\boldsymbol{x}]]/J_k \\ g & \mapsto & j_{k-1}(g)\, mod(J_k) \end{array}$$

Clearly, ϕ is a surjective homomorphism of local K-algebras. On the other hand, let $g \in Ker(\phi)$. Hence there exist $f_1, \ldots, f_s \in I$ and $g_1, \ldots, g_s \in K'[[\boldsymbol{x}]]$ such that $j_{k-1}(f) - \sum_{j=1}^{s} g_j \cdot j_{k-1}(f_j) \in \langle \boldsymbol{x}\rangle^k \cap K'[[\boldsymbol{x}]]$. Thus, $j_{k-1}(f - \sum_{j=1}^{s} g_j \cdot f_j) = 0$ and therefore $f \in I + \langle \boldsymbol{x}\rangle^k$. Conversely, it is straightforward that $I + \langle \boldsymbol{x}\rangle^k \subset Ker(\phi)$. Thus, the claim follows. Afterwards, we show that

$$J_k K[[\boldsymbol{x}]] := \left\langle j_{k-1}(f)\,,\, \boldsymbol{x}^{\boldsymbol{\alpha}} \,:\, f \in IK[[\boldsymbol{x}]] \text{ and } \mid \boldsymbol{\alpha} \mid \geq k \right\rangle_{K[[\boldsymbol{x}]]}.$$

We denote the ideal on the right hand side by J_k^K and we notice that the inclusion $J_k K[[\boldsymbol{x}]] \subset J_k^K$ is trivial. Conversely, let $f = \sum_{i=1}^{s} f_i g_i \in IK[[\boldsymbol{x}]]$ with $f_i \in I$ and $g_i \in K[[\boldsymbol{x}]]$, $i = 1, \ldots, s$.

Clearly, we have $j_{k-1}(f) = \sum_{i=1}^{s} j_{k-1}(f_i) j_{k-1}(g_i)\, mod(\langle \boldsymbol{x}\rangle^k)$ which shows the claim. Altogether, this yields

$$\begin{array}{rcl} K'[[\boldsymbol{x}]]/(I + \langle \boldsymbol{x}\rangle^k)\otimes_{K'}K & \cong & K'[\boldsymbol{x}]/J_k \otimes_{K'} K \\ & \cong & K'[\boldsymbol{x}]/J_k K[[\boldsymbol{x}]] \\ & = & K'[\boldsymbol{x}]/J_k^K \\ & \cong & K[[\boldsymbol{x}]]/(IK[[\boldsymbol{x}]] + \langle \boldsymbol{x}\rangle^k). \end{array}$$

Hence, we obtain

$$\begin{array}{rcl} (K'[[\boldsymbol{x}]]/I)\widehat{\otimes}_{K'}K & = & \varprojlim_k (K'[[\boldsymbol{x}]]/(I + \langle \boldsymbol{x}\rangle^k)\otimes_{K'}K) \\ & \cong & \varprojlim_k (K[[\boldsymbol{x}]]/(IK[[\boldsymbol{x}]] + \langle \boldsymbol{x}\rangle^k)) \\ & \cong & K[[\boldsymbol{x}]]/IK[[\boldsymbol{x}]]. \end{array}$$

This terminates the proof. $\square$

Theorem 5.1.7. *Let $K' \subset K$ be a field extension and let I be a proper ideal of $K'[[\boldsymbol{x}]]$.*

1. *There is an injective K-algebra homomorphism*

$$(K'[[\boldsymbol{x}]]/I)\otimes_{K'}K \hookrightarrow K[[\boldsymbol{x}]]/IK[[\boldsymbol{x}]]. \tag{5.1}$$

2. *$dim_{K'}(K'[[\boldsymbol{x}]]/I)$ is finite, if and only if, $dim_K(K[[\boldsymbol{x}]]/IK[[\boldsymbol{x}]])$ is finite. In this case, there is an isomorphism of K-algebras*

$$(K'[[\boldsymbol{x}]]/I)\otimes_{K'}K \xrightarrow{\cong} K[[\boldsymbol{x}]]/IK[[\boldsymbol{x}]].$$

Furthermore, $dim_{K'}(K'[[\boldsymbol{x}]]/I) = dim_K(K[[\boldsymbol{x}]]/IK[[\boldsymbol{x}]])$.

Proof. Let $K' \subset K$ be a field extension and let I be a proper ideal of $K'[[\boldsymbol{x}]]$. The first assertion of Theorem 5.1.7 is an easy corollary of Theorem 5.1.6. Indeed, we have only to consider the injection of $(K'[[\boldsymbol{x}]]/I)\otimes_{K'}K$ in its completion $(K'[[\boldsymbol{x}]]/I)\widehat{\otimes}_{K'}K$ and notice that the latter K-algebra is isomorphic to $K[[\boldsymbol{x}]]/IK[[\boldsymbol{x}]]$ by Theorem 5.1.6.
In the following, we show the equivalence

$$dim_{K'}(K'[[\boldsymbol{x}]]/I) < \infty \Longleftrightarrow dim_K(K[[\boldsymbol{x}]]/IK[[\boldsymbol{x}]]) < \infty.$$

If $dim_{K'}(K'[[\boldsymbol{x}]]/I) < \infty$, then there exists a positive integer l such that

$$\langle \boldsymbol{x}\rangle^l \subset I.$$

Thus,

$$\langle \boldsymbol{x}\rangle^l \subset IK[[\boldsymbol{x}]]$$

follows clearly and therefore $dim_K(K[[\boldsymbol{x}]]/IK[[\boldsymbol{x}]]) < \infty$.
Conversely, we assume $dim_K(K[[\boldsymbol{x}]]/IK[[\boldsymbol{x}]]) < \infty$. Hence, we have by (5.1) that $dim_K((K'[[\boldsymbol{x}]]/I)\otimes_{K'}K) < \infty$. If we suppose that $dim_{K'}(K'[[\boldsymbol{x}]]/I)$ is infinite, then there exists for any positive integer d a surjection

$$K'[[\boldsymbol{x}]]/I \to K'^d \to 0.$$

Thus, it follows by the right exactness of the tensor product that

$$(K'[[\boldsymbol{x}]]/I)\otimes_{K'}K \twoheadrightarrow K^d.$$

Hence, $dim_K((K'[[\boldsymbol{x}]]/I)\otimes_{K'}K) \geq d$, for any $d \in \mathbb{Z}_{>0}$, against the finiteness of the dimension of the K-vector space $(K'[[\boldsymbol{x}]]/I)\otimes_{K'}K$.

For the sequel, we suppose that $dim_{K'}(K'[[\boldsymbol{x}]]/I)$ is finite.
Then, there exists $N \in \mathbb{Z}_{\geq 0}$ such that $\langle \boldsymbol{x}\rangle^N \subset I$. Thus we can write

$$K'[[\boldsymbol{x}]]/I + \langle \boldsymbol{x}\rangle^i = K'[[\boldsymbol{x}]]/I$$

for all $i \geq N$. Altogether, this yields

$$(K'[[\boldsymbol{x}]]/I)\widehat{\otimes}_{K'}K = (K'[[\boldsymbol{x}]]/I)\otimes_{K'}K.$$

Hence,

$$K[[\boldsymbol{x}]]/IK[[\boldsymbol{x}]] \cong (K'[[\boldsymbol{x}]]/I)\otimes_{K'}K \tag{5.2}$$

follows by Theorem 5.1.6.
It remains to show that $dim_{K'}(K'[[\boldsymbol{x}]]/I) = dim_K(K[[\boldsymbol{x}]]/IK[[\boldsymbol{x}]])$.
Let $d := dim_{K'}(K'[[\boldsymbol{x}]]/I)$ and let $d_1 := dim_K(K[[\boldsymbol{x}]]/IK[[\boldsymbol{x}]])$. It follows from the isomorphism (5.2) that $dim_K((K'[[\boldsymbol{x}]]/I)\otimes_{K'}K) = d_1$.
Moreover, tensoring by K the exact sequence of K'-vector spaces

$$0 \to K'[[\boldsymbol{x}]]/I \to {K'}^d \to 0$$

yields an isomorphism of K-vector spaces

$$(K'[[\boldsymbol{x}]]/I)\otimes_{K'}K \cong K^d,$$

which implies that $d_1 = d$ and this terminates the proof. □

5.2 Isolated Hypersurface Singularities in Characteristic Zero

5.2.1 The Milnor and the Tjurina Numbers

The following proposition gives a characterization of isolated hypersurface singularities in characteristic zero.

Proposition 5.2.1. *Let $f \in K[[x_1, \ldots, x_n]]$ where K is an algebraically closed field and $char(K) = 0$. Then $\tau(f)$ is finite, if and only if, $\mu(f)$ is finite.*

Proof. Let $f \in K[[x_1, \ldots, x_n]]$. We recall that

$$\tau(f) := dim_K(T_f) \leq \mu(f) := dim_K(M_f),$$

where T_f and M_f are the Tjurina algebra and the Milnor algebra respectively. It is clear that if $\mu(f)$ is finite, then $\tau(f)$ is also finite.
Let $A := \{Coeff(f), Coeff(f_{x_1}), \ldots, Coeff(f_{x_n}))\}$. Clearly, A has a countable number of elements.
We set $K' := \mathbb{Q}(A)$ and $I := \langle f, f_{x_1}, \ldots, f_{x_n}\rangle \subset K'[[\boldsymbol{x}]]$. It is easy to see that $IK[[\boldsymbol{x}]] = tj(f)$. Furthermore as $dim_K(K[[\boldsymbol{x}]]/tj(f)) = \tau(f)$ is finite, then it follows by Theorem 5.1.7 that $dim_{K'}(K'[[\boldsymbol{x}]]/I)$ is also finite. On the other hand, by Thorem 5.1.1, there exist a subfield $\tilde{K}$ of $\mathbb{C}$ an an isomorphism

$$\varphi : \quad K' \to \tilde{K}.$$

Clearly, φ lifts to a K-algebra isomorphism

$$\phi : \quad K'[[\boldsymbol{x}]] \to \tilde{K}[[\boldsymbol{x}]].$$

Moreover, it follows by Lemma 1.2.7 that $\phi(I) = tj(\phi(f)) \subset \tilde{K}[[\boldsymbol{x}]]$.
Hence, $K'[[\boldsymbol{x}]]/I \cong \tilde{K}[[\boldsymbol{x}]]/tj(\phi(f))$ which yields $dim_{\tilde{K}}(\tilde{K}[[\boldsymbol{x}]]/tj(\phi(f))$ is finite. Considering the field extension $\tilde{K} \subset \mathbb{C}$ and using again Theorem 5.1.7, we get

$$dim_{\mathbb{C}}\big(\mathbb{C}[[\boldsymbol{x}]]/tj(\phi(f))\mathbb{C}[[\boldsymbol{x}]]\big) < \infty.$$

But this means that the Tjurina number of $\phi(f)$ is finite over $\mathbb{C}$ and therefore the Milnor number must be also finite over $\mathbb{C}$, that is $dim_{\mathbb{C}}(\mathbb{C}[[\boldsymbol{x}]]/j(\phi(f))\mathbb{C}[[\boldsymbol{x}]])$ is finite. Hence, $dim_{\tilde{K}}(\tilde{K}[[\boldsymbol{x}]]/j(\phi(f)\tilde{K}[[\boldsymbol{x}]]) < \infty$ follows by Theorem 5.1.7 and $dim_{K'}(K'[[\boldsymbol{x}]]/j(f)K'[[\boldsymbol{x}]])$ follows as ϕ is an isomorphism.
Also, Theorem 5.1.7 implies that $dim_K(K[[\boldsymbol{x}]]/j(f))$ is finite which means that the Milnor number of f over K is finite. □

5.2.2 Semiquasihomogeneous Hypersurface Singularities in Characteristic Zero

In the following, we would like to generalize to the charecteristic zero an important property of (SQH) power series which is established over the field $\mathbb{C}$ of complex numbers. First, we should recall that, in characteristic zero, a power series $f \in K[[x_1, \ldots, x_n]]$ is called (SQH) of principal part f_Δ, if

1. f_Δ is (QH) of type $(\boldsymbol{w}\,;\, d)$ where $\boldsymbol{w}$ is a weight in $\mathbb{Q}^n_{>0}$ and $d \in \mathbb{Z}_{>0}$,
2. $f = f_\Delta + g$, where $g \in K[[\boldsymbol{x}]]$ is such that $v_{\boldsymbol{w}}(g) > d$ and finally
3. $\mu(f_\Delta)$ is finite.

Proposition 5.2.2. *Let K be an algebraically closed field of characteristic zero and let $f \in K[[\boldsymbol{x}]]$ be (SQH) with principal part f_Δ. Then $\mu(f) = \mu(f_\Delta) < \infty$.*

Proof. Let $f = f_\Delta + g$ where f_Δ is (QH) of type $(\boldsymbol{w}\,;\, d)$ and $v_{\boldsymbol{w}}(g) > d$.
Let $A := \big\{Coeff(f), Coeff(f_{x_1}) \ldots, Coeff(f_{x_n})\big\}$ and let $K' := \mathbb{Q}(A)$. Theorem 5.1.1 establishes that K' is isomorphic to a subfield $\tilde{K}$ of $\mathbb{C}$. Hence, there exists an isomorphism $\varphi :\ K' \to \tilde{K}$ which clearly lifts to an isomorphism of K-algebras

$$\begin{array}{rrcl} \phi: & K'[[x_1, \ldots, x_n]] & \to & \tilde{K}[[x_1, \ldots, x_n]] \\ & a & \mapsto & \varphi(a), \quad a \in K' \\ & x_i & \mapsto & x_i, \quad i = 1, \ldots, n \end{array}$$

It turns out that $\phi(f)$ is (SQH) in $\tilde{K}[[\boldsymbol{x}]]$ with principal part $\phi(f_\Delta)$. Indeed, write

$$f = \sum_{\langle \boldsymbol{w}\,,\,\boldsymbol{\alpha}\rangle = d} a_{\boldsymbol{\alpha}} x^{\boldsymbol{\alpha}} + \sum_{\langle \boldsymbol{w}\,,\,\boldsymbol{\alpha}\rangle > d} a_{\boldsymbol{\alpha}} x^{\boldsymbol{\alpha}} = f_\Delta + g,$$

where the coefficients $a_{\boldsymbol{\alpha}} \in K'$. Hence,

$$\phi(f) = \sum_{\langle \boldsymbol{w}\,,\,\boldsymbol{\alpha}\rangle = d} \varphi(a_{\boldsymbol{\alpha}}) x^{\boldsymbol{\alpha}} + \sum_{\langle \boldsymbol{w}\,,\,\boldsymbol{\alpha}\rangle > d} \varphi(a_{\boldsymbol{\alpha}}) x^{\boldsymbol{\alpha}} = \phi(f_\Delta) + \phi(g).$$

Clearly $\phi(f_\Delta)$ is (QH) of type $(\boldsymbol{w}\,;\,d)$. On the other hand, since $\mu(f_\Delta) < \infty$, then we have $dim_K(K[[\boldsymbol{x}]]/j(f_\Delta)K[[\boldsymbol{x}]])$ is also finite too. Thus, Theorem 5.1.7 yields $dim_{K'}(K'[[\boldsymbol{x}]]/j(f_\Delta)K'[[\boldsymbol{x}]]) < \infty$. Moreover, since ϕ is an isomorphism, we obtain that the dimension of the $\tilde{K}$-vector space $\tilde{K}[[\boldsymbol{x}]]/j(\phi(f_\Delta))\tilde{K}[[\boldsymbol{x}]]$ is finite too. Therefore, the claim $\phi(f)$ is (SQH) in $\tilde{K}[[\boldsymbol{x}]]$ follows.
As $\tilde{K}$ is a subfield of $\mathbb{C}$, then it is clear that $\phi(f)$ is (SQH) in $\mathbb{C}[[\boldsymbol{x}]]$ with principal part $\phi(f_\Delta)$. Hence,

$$\mu(\phi(f)) = \mu(f_\Delta) \;\text{ over }\; \mathbb{C} \tag{5.3}$$

Finally considering the field extensions $\tilde{K} \subset \mathbb{C}$ and $K' \subset K$, it follows by Theorem 5.1.7 and the fact that ϕ is an isomorphism that

$$\begin{aligned} dim_K(K[[\boldsymbol{x}]]/j(f)K[[\boldsymbol{x}]]) &= dim_{K'}(K'[[\boldsymbol{x}]]/j(f)K'[[\boldsymbol{x}]]) \\ &= dim_{\tilde{K}}(\tilde{K}[[\boldsymbol{x}]]/j(\phi(f))\tilde{K}[[\boldsymbol{x}]]) \\ &= dim_{\mathbb{C}}(\mathbb{C}[[\boldsymbol{x}]]/j(\phi(f))\mathbb{C}[[\boldsymbol{x}]]). \end{aligned}$$

In the same way, we have

$$\begin{aligned} dim_K(K[[\boldsymbol{x}]]/j(f_\Delta)K[[\boldsymbol{x}]]) &= dim_{K'}(K'[[\boldsymbol{x}]]/j(f_\Delta)K'[[\boldsymbol{x}]]) \\ &= dim_{\tilde{K}}(\tilde{K}[[\boldsymbol{x}]]/j(\phi(f_\Delta))\tilde{K}[[\boldsymbol{x}]]) \\ &= dim_{\mathbb{C}}(\mathbb{C}[[\boldsymbol{x}]]/j(\phi(f_\Delta))\mathbb{C}[[\boldsymbol{x}]]). \end{aligned}$$

Altogether with (5.3), this yields

$$dim_K(K[[\boldsymbol{x}]]/j(f)K[[\boldsymbol{x}]]) = dim_K(K[[\boldsymbol{x}]]/j(f_\Delta)K[[\boldsymbol{x}]]),$$

that is $\mu(f) = \mu(f_\Delta)$ over K. □

5.3 Milnor Number in Characteristic Zero

5.3.1 Milnor Number and $\mathcal{K}$-Actions

In characteristic zero, the Milnor number is an invariant under the $\mathcal{K}$-action.

Proposition 5.3.1. *Let K be an algebraically closed field of characteristic zero and let $f, g \in K[[\boldsymbol{x}]]$. Then,*

$$f \overset{c}{\sim} g \Longrightarrow \mu(f) = \mu(g).$$

Proof. Let f, $g \in K[[\boldsymbol{x}]]$ and let $u \in K[[\boldsymbol{x}]]^*$ be a unit and $\psi \in Aut(K[[\boldsymbol{x}]])$ be an automorphism, such that $g = u \cdot \psi(f)$. If $u = 1$, then we have $f \overset{r}{\sim} g$ and it is well known that $\mu(f) = \mu(g)$ holds in arbitrary characteristic. Hence, to prove the theorem, it is enough to show that for any unit u, we have $\mu(f) = \mu(u \cdot f)$.
Let $A := \left\{Coeff(u), Coeff(f), Coeff(f_{x_i})_{1\le i\le n}\right\}$. Clearly, A has a countable number of elements. Hence, it follows by Theorem 5.1.1, that the field

$K' = \mathbb{Q}(A)$ is isomorphic to a subfield $\tilde{K}$ of $\mathbb{C}$. We call φ this isomorphism and so we get an isomorphism

$$\begin{array}{rrcl} \phi: & K'[[x_1,\dots,x_n]] & \to & \tilde{K}[[x_1,\dots,x_n]] \\ & a & \mapsto & \varphi(a), \quad a \in K' \\ & x_i & \mapsto & x_i, \quad i = 1,\dots,n. \end{array}$$

Obviously, by definition of K', we have $u \cdot f \in K'[[\boldsymbol{x}]]$ and moreover, as ϕ is an isomorphism, it follows that $\phi(u)$ is a unit in $\tilde{K}[[\boldsymbol{x}]]$ and hence also a unit in $\mathbb{C}[[\boldsymbol{x}]]$. Therefore, we have

$$dim_{\mathbb{C}}(\mathbb{C}[[\boldsymbol{x}]]/j(\phi(f))\mathbb{C}[[\boldsymbol{x}]]) = dim_{\mathbb{C}}(\mathbb{C}[[\boldsymbol{x}]]/j(\phi(u)\cdot\phi(f))\mathbb{C}[[\boldsymbol{x}]]). \tag{5.4}$$

First, we suppose that over K, $\mu(f)$ is finite, that is $dim_K(K[[\boldsymbol{x}]])/j(f)K[[\boldsymbol{x}]]$ is finite. Then it follows by Theorem 5.1.7 and the fact that ϕ is an isomorphism that

$$\begin{array}{rcl} dim_K(K[[\boldsymbol{x}]]/j(f)K[[\boldsymbol{x}]]) & = & dim_{K'}(K'[[\boldsymbol{x}]]/j(f)K'[[\boldsymbol{x}]]) \\ & = & dim_{\tilde{K}}(\tilde{K}[[\boldsymbol{x}]]/j(\phi(f))\tilde{K}[[\boldsymbol{x}]]) \\ & = & dim_{\mathbb{C}}(\mathbb{C}[[\boldsymbol{x}]]/j(\phi(f))\mathbb{C}[[\boldsymbol{x}]]) \\ & < & \infty. \end{array}$$

Hence, it follows by equation (5.4) that

$$dim_{\mathbb{C}}(\mathbb{C}[[\boldsymbol{x}]]/j(\phi(f))\mathbb{C}[[\boldsymbol{x}]]) = dim_{\mathbb{C}}(\mathbb{C}[[\boldsymbol{x}]]/j(\phi(u)\cdot\phi(f))\mathbb{C}[[\boldsymbol{x}]]) < \infty.$$

Thus, using again Theorem 5.1.7 and the isomorphism ϕ, we get

$$\begin{array}{rcl} dim_{\mathbb{C}}(\mathbb{C}[[\boldsymbol{x}]]/j(\phi(u)\cdot\phi(f))\mathbb{C}[[\boldsymbol{x}]]) & = & dim_{\tilde{K}}(\tilde{K}[[\boldsymbol{x}]]/j(\phi(u)\cdot\phi(f))\tilde{K}[[\boldsymbol{x}]]) \\ & = & dim_{K'}(K'[[\boldsymbol{x}]]/j(uf)K'[[\boldsymbol{x}]]) \\ & = & dim_K(K[[\boldsymbol{x}]]/j(u\cdot f)K[[\boldsymbol{x}]]) \\ & < & \infty. \end{array}$$

Altogether, we get over K

$$\mu(f) = \mu(u\cdot f) < \infty.$$

Now, we assume $\mu(f)$ is infinite over K, i.e $dim_K(K[[\boldsymbol{x}]])/j(f)K[[\boldsymbol{x}]] = \infty$. Then, it is straightforward from the above that $\mu(u\cdot f)$ is also infinite over K, for otherwise, if $\mu(u\cdot f) < \infty$, then using the same arguments as so far, the claim $\mu(u^{-1}\cdot u\cdot f) < \infty$ would follow against the assumption. Altogether, we obtain $\mu(f) = \mu(u\cdot f)$. □

5.3.2 Equivalent Definitions in Characteristic Zero

Over $\mathbb{C}$, it is established that, if we fix the number of irreducible factors of a reduced $f \in \mathbb{C}[[x,y]]$, then the δ-invariant and the Milnor number of f determine each other. It turns out that this statement holds in the same way for algebraically closed fields of characteristic zero.

Proposition 5.3.2. *Let K be an algebraically closed field such that $char(K) = 0$ and let $f \in \mathfrak{m} \subset K[[x, y]]$ be reduced. Further, let $\delta(f)$ be the delta invariant of f and let $r(f)$ be the number of irreducible factors of f. Then*

$$\mu(f) = 2\delta(f) - r(f) + 1. \tag{5.5}$$

Proof. In the following, let K be an algebraically closed field of characteristic 0 and let $f \in \mathfrak{m} \subset K[[x, y]]$ be reduced having the following decomposition into irreducible factors

$$f = u \cdot f_1 \dots f_r, \tag{5.6}$$

where u is a unit in $K[[\boldsymbol{x}]]$ and, for $i = 1, \dots, r$, $f_i \in \mathfrak{m} \subset K[[\boldsymbol{x}]]$ is irreducible. Hence, the number $r(f)$ of irreducible factors of f is r.
Moreover, let $A = \{Coeff(u), \dots, Coeff(f_1), \dots, Coeff(f_r)\}$ and let $K' = \mathbb{Q}(A)$. By Theorem 5.1.1, K' is isomorphic to a subfield $\tilde{K}$ of $\mathbb{C}$. This field isomorphism clearly yields an isomorphism from $K'[[\boldsymbol{x}]]$ onto $\tilde{K}[[\boldsymbol{x}]]$ which we shall denote by ϕ.
First, we write r' for the number of irreducible factors of f in $K'[[\boldsymbol{x}]]$. The equality $r' = r$ is straightforward from the decomposition (5.6) of f in $K[[x, y]]$ and the definition of the subfield K' of K. Moreover, we write R' and R for the K'-algebra $K'[[\boldsymbol{x}]]/\langle f \rangle$ and for the K-algebra $K[[\boldsymbol{x}]]/\langle f \rangle$ respectively.
Besides, the normalizations of R' and R shall be denoted $\bar{R}'$ and $\bar{R}$ respectively.
We have $R' \cong \bigoplus_{i=1}^{r} K'[[t_i]]$ and $\bar{R} \cong \bigoplus_{i=1}^{r} K[[t_i]]$.
Thus, Theorem 5.1.7 yields the existence of an injective K-algebra homomorphism

$$\bar{R}' \otimes_{K'} K \hookrightarrow \bar{R}.$$

On the other hand, $\bar{R}$ is a finitely generated R-module and the K-vector space $\bar{R}/R$ has a finite dimension which is $\delta(f)$ by definition. In the same way $dim_{K'}(\bar{R}'/R')$ is finite. Hence, we can argue as in the proof of Theorem 5.1.7 to deduce the isomorphism of K-algebras

$$\bar{R}'/R' \otimes_{K'} K \cong \bar{R}/R, \tag{5.7}$$

and the equality

$$\delta(f) = dim_K(\bar{R}/R) = dim_{K'}(\bar{R}'/R'). \tag{5.8}$$

On the other hand, for ϕ is an isomorphism, it follows clearly that r is also the number of irreducible factors of $\phi(f)$ in $\tilde{K}[[x, y]]$ and moreover $\delta(f)$ is equal to the δ-invariant of $\phi(f)$ over $\tilde{K}$.
For the sequel, we write $\tilde{R}$ (resp. $\mathcal{O}$) for the associated curve singularity of $\phi(f)$ over $\tilde{K}$ (resp. over $\mathbb{C}$) and we denote by $\bar{\tilde{R}}$ and $\bar{\mathcal{O}}$ the normalizations of $\tilde{R}$ and $\mathcal{O}$ respectively.
Before going into further details, we should notice that as f is reduced over K, then it follows that the Milnor number $\mu(f) = dim_K(K[[x, y]]/j(f))$ is finite. Hence, it follows by Theorem 5.1.7 and the isomorphism ϕ that the Milnor number $\mu(\phi(f))$ of $\phi(f)$ over $\mathbb{C}$ is also finite and we have $\mu(f) = \mu(\phi(f))$. Therefore $\phi(f)$ is also reduced in $\mathbb{C}[[x, y]]$ and hence the δ-invariant δ_1 of $\phi(f)$ over $\mathbb{C}$ is finite. Thus, in the same way as for (5.7) and (5.8), we deduce that $\delta_1 = \delta(\phi(f))$. Hence it follows by the above that $\delta_1 = \delta(f)$. Moreover, if we denote by r_1 the number of irreducible branches of $\phi(f)$ over $\mathbb{C}$, we see easily that $r \leq r_1$. Hence using the formula (5.5) over $\mathbb{C}$, we can write

$$\mu(f) = 2\delta(f) - r_1 + 1.$$

On the other hand as $r \leq r_1$, we get

$$\mu(f) \leq 2\delta(f) - r + 1.$$

However, we have by Remark 1.2.18 that $\mu(f) \geq 2\delta(f) - r + 1$. Thus the equality (5.5) clearly holds. $\square$

Appendix A

Field Theory

In this appendix we review some classical results from field theory.
The fields of rational numbers and complex numbers shall be denoted by $\mathbb{Q}$ and $\mathbb{C}$ respectively.
$K[x_1, \ldots, x_n]$ denotes the polynomial ring in n variables and $K(x_1, \ldots, x_n)$ is the field of rational functions in n variables.
If $F \subset K$ are fields, then K is called a **field extension** of F. Throughout this appendix, we will refer to the pair $F \subset K$ as the field extension K/F.
If K and L are extension fields of F, then a homomorphism $\phi : K \longrightarrow L$ is an F-homomorphism if $\phi(a) = a$ for all $a \in F$. If ϕ is bijective, then it is called an F-isomorphism.

A.1 Field extensions

We recall a few general results about field extensions.

Definition A.1.1. *Let K be a field extension of F. If A is a subset of K, then the ring $F[A]$ generated by F and A is the intersection of all subrings of K that contain F and A. The field $F(A)$ generated by F and A is the intersection of all subfields of K that contain F and A. If $A = \{a_1, \ldots, a_n\}$ is finite, we will write $F[A] = F[a_1, \ldots, a_n]$ and $F(A) = F(a_1, \ldots, a_n)$. If A is finite, we call the field $F(A)$ a finitely generated extension of F.*

Proposition A.1.2. *Let K be a field extension of F and $a_1, \ldots, a_n \in K$.Then*

$$F[a_1, \ldots, a_n] = \{f(a_1, \ldots, a_n) : f \in F[x_1, \ldots, x_n]\}$$

and

$$F(a_1, \ldots, a_n) = \Big\{\frac{f(a_1, \ldots, a_n)}{g(a_1, \ldots, a_n)} : f, g \in F[x_1, \ldots, x_n], g(a_1, \ldots, a_n) \neq 0\Big\},$$

so $F(a_1, \ldots, a_n)$ is the quotient field of $F[a_1, \ldots, a_n]$.

Morandi, 1.1.9. , [Lang]. □

For arbitrary subsets A of K we can describe the field $F(A)$ in terms of finite subsets of A. This description is often convenient for turning questions about field extensions into questions about finitely generated field extensions.

Proposition A.1.3. *Let K be a field extension of F and let A be a subset of K. If $\alpha \in F(A)$, then $\alpha \in F(a_1, \ldots, a_n)$ for some $a_1, \ldots, a_n \in A$. Therefore,*

$$F(A) = \bigcup \{F(a_1, \ldots, a_n) : a_1, \ldots, a_n \in A, \}$$

where the union is over all finite subsets of A.

Morandi 1.1.10. , [Lang]. □

Definition A.1.4. (Algebraic and transcendental elements.)

Let K be a field extension of F.

1. *An element $\alpha \in K$ is* **algebraic** *over F if there is a non zero polynomial $P(x) \in F[x]$ with $P(\alpha) = 0$. If α is not algebraic over F, then α is said to be* **transcendental** *over F. If every element of K is algebraic over F, then K is said to be algebraic over F, and K/F is called an algebraic extension.*

2. *The set $\{\alpha \in K : \alpha \ \text{ is algebraic over } \ F\}$ is called* **the algebraic closure** *of F in K.*

Definition A.1.5. *If α is algebraic over a field F,* **the minimal polynomial** *of α over F is the monic polynomial $P(x)$ of least degree in $F[x]$ for which $P(\alpha) = 0$; it is denoted by $min(F, \alpha)$.*

Let us make some remarks concerning algebraic and transcendental elements.

Remark A.1.6. *Let K/F be a field extension and let $\alpha \in K$.Then, we have the following:*

1. *If α is algebraic over F, then the polynomial $min(F, \alpha)$ is irreducible over F. Furthermore,*
$$F[\alpha] = F(\alpha) \cong F[x]/\langle min(F, \alpha)\rangle$$

2. *If α is transcendental over F, then it follows that $F[\alpha] \cong F[x]$ and therefore $F(\alpha) \cong F(x)$.*

For the following remark, we refer to [Lang]

Remark A.1.7. *If F is a field which is not finite, then any algebraic extension of F has the same cardinality as F. Hence, for example, the algebraic closure $\mathbb{Q}^a$ of $\mathbb{Q}$ in $\mathbb{C}$ is countable.*

A.2 Transcendental extensions and transcendence bases

We recall some properties of field extensions that are not algebraic.

Definition A.2.1. (algebraically independent sets)
Let K/F be a field extension, and let $t_1, \ldots, t_n \in K$. the set $\{t_1, \ldots, t_n\}$ is **algebraically independent** *over F if $f(t_1, \ldots, t_n) \neq 0$ for all non zero polynomials $f \in F[x_1, \ldots, x_n]$. an arbitrary set $S \subset K$ is algebraically independent over F if any finite subset of S is algebraically independent over F. If a set is not algebraically independent over F, then it is said to be* **algebraically dependent** *over F.*

Lemma A.2.2. *Let K/F be a field extension. If $t_1, \ldots, t_n \in K$ are algebraically independent over F, then $F[t_1, \ldots, t_n]$ and $F[x_1, \ldots, x_n]$ are F-isomorphic rings, and so $F(t_1, \ldots, t_n)$ and $F(x_1, \ldots, x_n)$ are F-isomorphic fields.*

Morandi 5.19.5. □

Lemma A.2.3. *let K/F be a field extension, and let $t_1, \ldots, t_n \in K$. then the following statements are equivalent:*

1. *the set $\{t_1, \ldots, t_n\}$ is algebraically independent over F.*
2. *For each i, t_i is transcendental over $F(t_1, \ldots, t_{i-1}, t_{i+1}, \ldots, t_n)$.*
3. *For each i, t_i is transcendental over $F(t_1, \ldots, t_{i-1})$.*

Morandi 5.19.7. □

Definition A.2.4. (Transcendence basis) *If K is a field extension of F, a subset S of K is a* **transcendence basis** *for K/F if*

1. *S is algebraically independent over F, and*
2. *K is algebraic over $F(S)$.*

Note that, $\emptyset$ is a transcendence basis for K/F if and only if K/F is algebraic.

The following theorem proves the existence of a transcendence basis for any field extension.

Theorem A.2.5. *Let K/F be a field extension.*

1. *There exists a transcendence basis for K/F.*
2. *If $T \subset K$ such that $K/F(T)$ is algebraic, then T contains a transcendence basis for K/F.*
3. *If $S \subset K$ is algebraicallly independent over F, then S is contained in a transcendence basis of K/F.*
4. *If $S \subset T \subset K$ such that S is algebraically independent over F and $K/F(T)$ is algebraic, then there is a transcendence basis B for K/F with $S \subset B \subset T$.*

Morandi 5.19.14. □

It turns out that, the size of a transcendence basis for an extension K/F is unique. Indeed,

Theorem A.2.6. *Let K/F be a field extension. If S and T are transcendence bases for K/F, then $|S| = |T|$.*

Morandi 5.19.15. □

Therefore, the following definition makes sense:

Definition A.2.7. *The transcendence degree $trdeg(K/F)$ of a field extension K/F is the cardinality of any transcendence basis of K/F.*

Example A.2.8. *Consider the field extension $\mathbb{C}/\mathbb{Q}$. Since $\mathbb{Q}$ is countable and $\mathbb{C}$ is uncountable, the transcendence degree of $\mathbb{C}/\mathbb{Q}$ must be infinite; it is in fact uncountable.*

Appendix B

Singular Library ”gradalg.lib”

```
version="$Id: gradalg.lib,v 1.33 2007/05/07 $";
category="graded algebras";
info="
LIBRARY: gradalg.lib  Piecewise-Homogeneous Graded Algebras

AUTHORS: Gert-Martin Greuel   greuel@mathematik.uni-kl.de
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PROCEDURES:
grordS(f,w);      weighted order of a polynomial f
grord(f,W);       (PH)-order of f
Dergrord(f,W,n);  (PH)-order of a monomial derivation
grlead(f,W);      (PH)-initial form of f
pwjet(f,W,N);     (PH)-jet of f
kspan(I);         maximal set of generators of qring(I)
degHCS(N,W);      (PH)-order of maxideal(N)
degHC(W,N);       total deg of HC of N-th graded ideal
grideal(N,W);     standard basis of the N-th  graded ideal
grlist(N,W);      monomials with grord equal to N
AidealS(f,w,N);   (QH)-poly of grordS = N and (A)
AspanS(f,w,N);    (QH)-poly in j(f) of grordS = N and (A)
ACidealS(f,w,N);  (QH)-poly of grordS = N and (AC)
ACspanS(f,w,N);   (QH)-poly in tj(f) of grordS = N and (AC)
AkbaseS(f,w,N);   N-th space K-basis of the w-(A)-algebra
ACkbaseS(f,w,N);  N-th space K-basis of the w-(AC)-algebra
AgrbaseS(f,w,N);  K-basis up to deg <= N of the w-(A)-algebra
ACgrbaseS(f,w,N); K-basis up to deg <= N of the w-(AC)-algebra
Aideal(f,W,N);    (PH)-poly of grord = N and (A)
Aspan(f,W,N);     (PH)-poly in j(f) of grord = N and (A)
ACideal(f,W,N);   (PH)-poly of grord = N and (AC)
```

```
ACspan(f,W,N);    (PH)-poly in tj(f) of grord = N and (AC)
Akbase(f,W,N);    N-th space’s K-basis of the W-(A)-algebra
ACkbase(f,W,N);   N-th space’s K-basis of the W-(AC)-algebra
Agrbase(f,W,N);   K-basis up to deg <= N of the W-(A)-algebra
ACgrbase(f,W,N);  K-basis up to deg <= N of the W-(AC)-algebra
is_A(f,W,N);      1 if all monomials of deg = N are (A)
is_AC(f,W,N);     1 if all monomials of deg = N are (AC)
";

LIB "sing.lib";
LIB "hnoether.lib";
LIB "qhmoduli.lib";
//-----------------------------------------------------------
proc grordS(poly f, intvec w)
"USAGE:  grordS(f, w); f a polynomial, w a positive weight
RETURN:  weighted order of f with respect to w
"

  if (size(f)==1)
   return(deg(f,w));
  else

  def old = basering;
  list rl = ringlist(old);
  rl[3][1] = list("ws", w);
  def r = ring(rl);
  setring r;
  poly f = fetch(old, f);
  return(ord(f));


//-----------------------------------------------------------
proc grord(poly f, list L)
"
USAGE: grord(f, L); f a polynomial,
       L a finite list of weights
RETURN: (PH)-order of f with respect to the L
"

  int s=size(L);
  int N=grordS(f,L[1]);
  int tmpord;
  for (int i=2; i<=s; i++)

    tmpord = grordS(f,L[i]);
    if (tmpord < N)
```

```
      N = tmpord;

 return(N);

//------------------------------------------------------------
proc Dergrord(poly f, list L, int n)
"USAGE: Dergrord(f, L,n); f a monomial, L a finite list of
        weights, n a positive integer smaller than
        the number of variables of the basering
RETURN: (PH)-order of the derivation f*d_(x_n) wrt to L
"
 if (size(f) != 1)
   ERROR("the given polynomial is no monomial!");

  if (n > nvars(basering))

    "Error: last input bigger than the number of
     variables in the basering!";
     return(0);

  int s=size(L);
  int N=grordS(f,L[1])-L[1][n];
  int tmpord;
  for (int i=2; i<=s; i++)

    tmpord = grordS(f,L[i])-L[i][n];
    if (tmpord < N)
      N = tmpord;

 return(N);

//------------------------------------------------------------
proc grlead(poly f, list L)
"
USAGE: grlead(f, L); f a polynomial,
       L a finite list of weights
RETURN: (PH)-initial form of f with respect to L
"

  poly m;
  poly tmplead;
  int N=grord(f,L);
  while(f != 0)

    m=leadcoef(f)*leadmonom(f);
    f=f-m;
```

```
    if (grord(m,L)==N)
       tmplead= tmplead+m;

  return(tmplead);

//-----------------------------------------------------------
proc pwjet(poly f, list L, int N)
"
USAGE: pwjet(f, L, N); f a polynomial,
       L a finite list of weights, N a positive integer
RETURN: (PH)-jet of f
"

 if (grord(grlead(f,L),L) > N)
   return(0);
 poly m;
 poly tmpjet;
 while (f != 0)

   m=grlead(f,L);
   f=f-m;
   if (grord(m,L) <= N)
    tmpjet=tmpjet + m;
   else
    return(tmpjet);

 return(tmpjet);

//------------------------------------------------------------
proc kspan(ideal I)
"USAGE: kspan(I);   f polynomial, I ideal
ASSUME: I is a zero-dimensional ideal, the monomial ordering
        is a local degree ordering
RETURN: the maximal set consisting of monic which generate
        qring(I) as vector space
"

  if(attrib(basering,"global")==1)

    "Error: monomial ordering is not local!";
    return (0);

  ideal J=std(I);
  if (dim(J) != 0)

    "Error: ideal not zero-dimensional!";
```

```
    return (0);

  poly p = highcorner(J);
  int D = deg(p,1:nvars(basering))+1;
  ideal mD=maxideal(D);
  attrib(mD,"isSB",1);
  ideal K=kbase(mD);
  int s=ncols(K);
  ideal tmp;
  tmp[s]=0;
  for (int i=1; i <= s; i++)

    if (reduce(K[i],J,1) != 0)
     tmp[i]=K[i];

  return(simplify(tmp,2));

//-----------------------------------------------------------
proc degHCS(int N , list L)
"USAGE: degHCS(N,L); N an integer,
        L a finite list of weights
RETURN: (PH)-order of maxideal(N).
"

 ideal MN=maxideal(N);
 int s=ncols(MN);
 int D=grord(MN[1],L);
 int Dtmp;
 for (int i=2; i<=s; i++)

   Dtmp=grord(MN[i],L);
   if (Dtmp<D)  D=Dtmp;

 return(D);

//-----------------------------------------------------------
proc degHC(int N , list L)
"USAGE: degHC(N,L); N an integer,
        L a finite list of weights
ASSUME: the monomial ordering is a local degree ordering
RETURN: the total degree of the highest corner of
        the N-th graded ideal
"
 int n=1;
  int D= degHCS(1,L);
  while (D < N)
```

```
   n++;

   D=degHCS(n,L);


return(n-1);

//-----------------------------------------------------------
proc grideal(int N , list L)
"USAGE: grideal(N,L); N a positive integer,
        L a finite list of weights
ASSUME: the monomial ordering is a local degree ordering
RETURN: a standard basis of the N-th graded ideal
"

  int d=degHC(N,L);
  ideal M=maxideal(d+1);
  ideal I=kspan(M);
  int s=size(I);
  ideal tmp=M;
  for (int j=1; j<=s; j++)

   if (grord(I[j],L) >= N)
   tmp= tmp, I[j];

return(std(tmp));

//-----------------------------------------------------------
proc grlist(int N , list L)
"USAGE: grlist(N,L); N a positive integer,
        L a finite list of weights
ASSUME: the monomial ordering is a local degree ordering
RETURN: the list of minic monomials having (PH)-order = N
"
 ideal I=grideal(N,L);
  int s=ncols(I);
  ideal tmp;
  tmp[s]=0;
  poly p;
  for (int i=1; i <=s; i++)

    p=I[i];
    if(grord(p,L)==N)tmp[i]=p;

return(simplify(tmp,2));
```

```
//-----------------------------------------------------------
proc AidealS(poly f, intvec w, int N)
"USAGE: AidealS(f,w,N); f a (QH) polynomial with respect
        to w, N a positive integer
ASSUME: the monomial ordering is a local degree ordering
RETURN: A set of K-generators of (QH)-polynomials of weighted
        order N and fulfilling condition (A) for f and w
"
  ideal J=jacob(f);
   int s=size(w);
   if (s != nvars(basering))
    ERROR("weight not compatible!");
   list l=w;

   if ( f!=grlead(f,l))
    ERROR("polynomial not quasihomogeneous!");

   ideal tmp=J;
   ideal Ii,Ji;
   int i;
   for (i=1; i <= s; i++)

     Ii=grlist(N-grord(f,l)+w[i],l);
     Ji=Ii*J[i];
     tmp=tmp,Ji;


   tmp=simplify(tmp,6);
   int stmp=size(tmp);
   if (stmp != 0)

     ideal spantmp;
     spantmp[stmp]=0;
     for (i=1; i<=stmp; i++)

       if (grord(tmp[i],l)==N)
       spantmp[i]=grlead(tmp[i],l);

     return(simplify(spantmp,2));

   else
   return (ideal(0));

//-----------------------------------------------------------
proc AspanS(poly f, intvec w, int N)
```

```
"USAGE: AspanS(f,w,N); f a (QH) polynomial with respect to w,
        N a positive integer
ASSUME: the monomial ordering is a local degree ordering
RETURN: A set of K-generators of (QH)-polynomials of weighted
        order N in j(f) which are (A) with respect to f and w
"
  ideal J=jacob(f);
   ideal J1=std(J);
   int s=size(w);
   if (s != nvars(basering))
    ERROR("weight not compatible!");
   list l=w;

   if ( f!=grlead(f,l))
    ERROR("polynomial not quasihomogeneous!");

   ideal tmp=J;
   ideal Ii,Ji;
   poly fi;
   int i;
   for (i=1; i <= s; i++)

     Ii=grlist(N-grord(f,l)+w[i],l);
     Ji=Ii*J[i];
     tmp=tmp,Ji;


   tmp=simplify(tmp,6);
   int stmp=size(tmp);
   if (stmp != 0)

     ideal spantmp;
     spantmp[stmp]=0;
     for (i=1; i<=stmp; i++)

       if (grord(tmp[i],l)==N)
       fi=grlead(tmp[i],l);
         if(reduce(fi,J1)==0)
         spantmp[i]=fi;


     return(simplify(spantmp,2));

   else
   return (ideal(0));
```

```
//-----------------------------------------------------------
proc ACidealS(poly f, intvec w, int N)
"USAGE: ACidealS(f,w,N); f a (QH) polynomial with respect
        to w, N a positive integer
ASSUME: the monomial ordering is a local degree ordering
RETURN: A set of K-generators of (QH)-polynomials of
        weighted order N, and fulfilling condition (AC)
        with respect to f and w
"
  list l=w;
   ideal I=grlist(N-grord(f,l),l);
   ideal J=I*f;
   int r=size(J);
   if (r != 0)

   int i;
   ideal tmp;
   tmp[r]=0;
   for (i=1; i <= r; i++)
   tmp[i]=grlead(J[i],l);
   ideal spantmp=tmp,AidealS(f,w,N);// A_spanS checks the
                                       assumptions
   return (simplify(spantmp,6));

   else
   return(AspanS(f,w,N));

//-----------------------------------------------------------
proc ACspanS(poly f, intvec w, int N)
"USAGE: ACspanS(f,w,N); f a (QH) polynomial with respect
        to w, N a positive integer
ASSUME: the monomial ordering is a local degree ordering
RETURN: A set of K-generators of (QH)-polynomials of
        weighted order N in tj(f) which are (AC) with
        respect to f and w
"
  list l=w;
   ideal I=grlist(N-grord(f,l),l);
   ideal J=I*f;
   int r=size(J);
   if (r != 0)

   int i;
   ideal tmp;
   tmp[r]=0;
   for (i=1; i <= r; i++)
```

```
   tmp[i]=grlead(J[i],l);
   ideal spantmp=tmp,AspanS(f,w,N);// A_spanS checks the
                                     assumptions
   return (simplify(spantmp,6));

   else
   return(AspanS(f,w,N));

//-----------------------------------------------------------
proc AkbaseS(poly f, intvec w, int N)
"USAGE: AkbaseS(f,w,N); f a (QH) polynomial with respect
        to w, N a positive integer
ASSUME: the monomial ordering is a local degree ordering
RETURN: a basis (consisting of monomials) of the N-th vector
        space of the w-(A)-graded algebra of j(f)
"

 list l=w;
 ideal I=std(AidealS(f,w,N));
 ideal J=grlist(N,l);
 int s=size(J);
 if (s != 0)

  ideal L;
  L[s]=0;
  for (int i=1; i<=s; i++)

    if(reduce(J[i],I)==J[i])
    L[i]=J[i];

 return(simplify(L,2));

 else
 return(ideal(0));

//-----------------------------------------------------------
proc ACkbaseS(poly f, intvec w, int N)
"USAGE: ACkbaseS(f,w,N); f a (QH) polynomial with respect
        to w, N a positive integer
ASSUME: the monomial ordering is a local degree ordering
RETURN: a basis (consisting of monomials) of the N-th vector
        space of the w-(AC)-graded algebra of tj(f)
"

 list l=w;
 ideal I=std(ACidealS(f,w,N));
```

```
 ideal J=grlist(N,l);
 int s=size(J);
 if (s != 0)

  ideal L;
  L[s]=0;
  for (int i=1; i<=s; i++)

   if(reduce(J[i],I)==J[i])
   L[i]=J[i];

  return(simplify(L,2));

 else
 return(ideal(0));

//-----------------------------------------------------------
proc AgrbaseS(poly f, intvec w, int N)
"USAGE: AgrbaseS(f,w,N); f a (QH) polynomial with respect
        to w, N a positive integer
ASSUME: the monomial ordering is a local degree ordering
RETURN: a K-basis (consisting of monomials having total
        degree smaller or equal to N) of the w-(A)-graded
        algebra of j(f)
"

  list l=w;
  int D=invdegHCS(N+1,l);
  ideal tmp;
  for (int i=0; i <=D; i++)

     tmp=tmp,AkbaseS(f,w,i);

  return(simplify(tmp,6));

//-----------------------------------------------------------
proc ACgrbaseS(poly f, intvec w, int N)
"USAGE: ACgrbaseS(f,w,N); f a (QH) polynomial with respect
        to w, N a positive integer
ASSUME: the monomial ordering is a local degree ordering
RETURN: a K-basis (consisting of monomials having total
        degree smaller or equal to N) of the w-(AC)-graded
        algebra of tj(f)
"

  list l=w;
```

```
  int D=invdegHCS(N+1,l);
  ideal tmp;
  for (int i; i <=D; i++)

     tmp=tmp,ACkbaseS(f,w,i);

  return(simplify(tmp,6));

//-----------------------------------------------------------
proc Aideal(poly f, list L, int N)
"USAGE: Aideal(f,L,N); f a (PH) polynomial with respect
        to a finitelist L of weights, N a positive integer
ASSUME: the monomial ordering is a local degree ordering
RETURN: a set of K-generators of (PH)-polynomials having
       (PH)-order N, and fulfilling condition (A) for f and L
"
  ideal J=jacob(f);
   if ( f!=grlead(f,L))
    ERROR("polynomial not piecewise homogeneous!");
   int s=size(L);
   if ( s == 1)
    intvec w=L[1];
     return(AspanS(f,w,N));
   int D=N-grord(f,L);
   ideal Q, Ql;
   ideal m=maxideal(1);
   ideal P=grlist(D,l);
   int sm=size(m);
   for (int l=1; l<=sm; l++)

    Ql=m[l]*P*J[l];
    Q=Q,Ql;

   ideal tmp=J,Q;
   ideal tmpi,tmpij,Iij,Kij;
   int si,sij, j,k;
   for (int i=1; i <= s; i++)

     si=size(L[i]);
     if (si != nvars(basering))
      ERROR("weight not compatible!");
     tmpi=0;
     for (j=1; j <= si; j++)

       Iij=grlist(D+L[i][j],L);
       sij=size(Iij);
```

```
       if (sij != 0)
       {
         tmpij=0;tmpij[sij]=0;
         for (k=1; k <=sij; k++)
         {
            if (Dergrord(Iij[k],L,j)==D)
             tmpij[k]=Iij[k];
         }
       }
       Kij=tmpij*J[j];
       tmpi=tmpi,Kij;
     }
     tmp=tmp,tmpi;
   }
   tmp=simplify(tmp,6);
   int stmp=size(tmp);
   if (stmp != 0)
   {
     ideal spantmp;
     spantmp[stmp]=0;
     for (j=1; j<=stmp; j++)
     {
      if (grord(tmp[j],L)==N)
         spantmp[j]=grlead(tmp[j],L);
     }
     return(simplify(spantmp,2));
   }
   else
     return (ideal(0));
}
//------------------------------------------------------------
proc Aspan(poly f, list L, int N)
"USAGE: Aspan(f,L,N); f a (PH) polynomial with respect
        to a finite list L of weights, N a positive integer
ASSUME: the monomial ordering is a local degree ordering
RETURN: a set of K-generators of (PH)-polynomials having
        (PH)-order N in j(f), which are (A) with respect
        to f and L
"
{
  ideal J=jacob(f);
   ideal J1=std(J);
   if ( f!=grlead(f,L))
    ERROR("polynomial not piecewise homogeneous!");
   int s=size(L);
   if ( s == 1)
    intvec w=L[1];
```

```
  return(AspanS(f,w,N));
int D=N-grord(f,L);
ideal Q, Ql;
ideal m=maxideal(1);
ideal P=grlist(D,l);
int sm=size(m);
for (int l=1; l<=sm; l++)

 Ql=m[l]*P*J[l];
 Q=Q,Ql;

ideal tmp=J,Q;
ideal tmpi,tmpij,Iij,Kij;
poly fj;
int si,sij, j,k;
for (int i=1; i <= s; i++)

  si=size(L[i]);
  if (si != nvars(basering))
   ERROR("weight not compatible!");
  tmpi=0;
  for (j=1; j <= si; j++)

    Iij=grlist(D+L[i][j],L);
    sij=size(Iij);
    if (sij != 0)

      tmpij=0;tmpij[sij]=0;
      for (k=1; k <=sij; k++)

         if (Dergrord(Iij[k],L,j)==D)
          tmpij[k]=Iij[k];


    Kij=tmpij*J[j];
    tmpi=tmpi,Kij;

  tmp=tmp,tmpi;

tmp=simplify(tmp,6);
int stmp=size(tmp);
if (stmp != 0)

  ideal spantmp;
  spantmp[stmp]=0;
  for (j=1; j<=stmp; j++)
```

```
      if (grord(tmp[j],L)==N)
        fj=grlead(tmp[j],L);
         if(reduce(fj,J1)==0)
         spantmp[j]=fj;


     return(simplify(spantmp,2));

   else
     return (ideal(0));

//-----------------------------------------------------------
proc ACideal(poly f, list L, int N)
"USAGE: ACideal(f,L,N); f a (PH) polynomial with respect
        to a finite list L of weights, N a positive integer
ASSUME: the monomial ordering is a local degree ordering
RETURN: a set of K-generators of (PH)-polynomials having
        (PH)-order N and fulfilling (AC) with respect to
        f and L
"
  int s=size(L);
   if (s == 1)
   intvec w=L[1];
    return(ACspanS(f,w,N));
   ideal I=grlist(N-grord(f,L),L);
   ideal J=I*f;
   int j;
   int r=size(J);
   if (r !=0)

    ideal tmp;
    tmp[r]=0;
    for (j=1; j <=r; j++)
       tmp[j]=grlead(J[j],L);
    ideal spantmp=tmp,Aideal(f,L,N);  //Aideal checks
                                        the assumptions
    return (simplify(spantmp,6));

   else
   return (Aideal(f,L,N));

//-----------------------------------------------------------
proc ACspan(poly f, list L, int N)
"USAGE:  ACspan(f,L,N); f a (PH) polynomial with respect
         to a finite list L of weights, N a positive integer
```

```
ASSUME: the monomial ordering is a local degree ordering
RETURN: a set of K-generators of (PH)-polynomials having
        (PH)-order N in tj(f), which are (AC) with
        respect to f and L
"
  ideal H=f,jacob(f);
   ideal K=std(H);
   int s=size(L);
   if (s == 1)
   intvec w=L[1];
    return(ACspanS(f,w,N));
   ideal I=grlist(N-grord(f,L),L);
   ideal J=I*f;
   int i;
   int r=size(J);
   if (r !=0)

    ideal tmp;
    tmp[r]=0;
    for (i=1; i <=r; i++)
    if (reduce(grlead(J[i],L),K)==0)
       tmp[i]=grlead(J[i],L);

    ideal spantmp=tmp,Aspan(f,L,N);  //Aspan checks
                                        the assumptions
    return (simplify(spantmp,6));

   else
   return (Aspan(f,L,N));

//-----------------------------------------------------------
proc Akbase(poly f, list L, int N)
"USAGE: Akbase(f,L,N); f a (PH) polynomial with respect
        to a finite list L of weights, N a positive integer
ASSUME: the monomial ordering is a local degree ordering
RETURN: a basis (consisting of monomials) of the N-th vector
        space of the L-(A)-graded algebra of j(f)
"

 int n=size(L);
 if (n == 1)
   intvec w=L[1];
    return(AkbaseS(f,w,N));
 ideal J=grlist(N,L);
 ideal P=Aideal(f,L,N);
 ideal I=std(P);
```

```
 int s=size(J);
 if (s != 0)

   ideal Q;
   Q[s]=0;
   for (int i=1; i<=s; i++)

    if(reduce(J[i],I)==J[i])
    Q[i]=J[i];

   return(simplify(Q,2));

  else
   return (ideal(0));

//----------------------------------------------------------
proc ACkbase(poly f, list L, int N)
"USAGE: ACkbase(f,L,N); f a (PH) polynomial with respect
        to a finite list L of weights, N a positive integer
ASSUME: the monomial ordering is a local degree ordering
RETURN: a basis (consisting of monomials) of the N-th vector
        space of the L-(AC)-graded algebra of tj(f)
"

 int n=size(L);
 if (n == 1)
   intvec w=L[1];
    return(ACkbaseS(f,w,N));
 ideal J=grlist(N,L);
 ideal P=ACideal(f,L,N);
 ideal I=std(P);
 int s=size(J);
 if (s != 0)

  ideal Q;
  Q[s]=0;
  for (int i=1; i<=s; i++)

   if(reduce(J[i],I)==J[i])
   Q[i]=J[i];

  return(simplify(Q,2));

 else
 return(ideal(0));
```

```
//------------------------------------------------------------
proc Agrbase(poly f, list L, int N)
"USAGE: AgrbaseS(f,L,N); f a (PH) polynomial with respect
        to a finite list L of weights, N a positive integer
ASSUME: the monomial ordering is a local degree ordering
RETURN: a K-basis (consisting of monomials having total
        degree smaller or equal to N) of the L-(A)-graded
        algebra of j(f)
"

  ideal M=maxideal(N+1);
  attrib(M,"isSB",1);
  ideal I=kbase(M);
  int s=size(I);
  ideal tmp;
  tmp[s]=0;
  int ri;
  ideal Ji,Pi;
  for (int i=1; i <= s; i++)

   ri=grord(I[i],L);
   Ji=Aideal(f,L,ri);
   Pi=std(Ji);
   if (reduce(I[i],Pi)==I[i])
    tmp[i]=I[i];

  return(simplify(tmp,2));

//------------------------------------------------------------
proc ACgrbase(poly f, list L, int N)
"USAGE: ACgrbase(f,L,N); f a (PH) polynomial with respect
        to a finite list L of weights, N a positive integer
ASSUME: the monomial ordering is a local degree ordering
RETURN: a K-basis (consisting of monomials having total
        degree smaller or equal to N) of the L-(AC)-graded
        algebra of tj(f)
"

  ideal M=maxideal(N+1);
  attrib(M,"isSB",1);
  ideal I=kbase(M);
  int s=size(I);
  ideal tmp;
  tmp[s]=0;
  int ri;
  ideal Ji,Pi;
```

```
  for (int i=1; i <= s; i++)

   ri= grord(I[i],L);
   Ji=ACideal(f,L,ri);
   Pi=std(Ji);
   if (reduce(I[i],Pi)==I[i])
    tmp[i]=I[i];

  return(simplify(tmp,2));

//-----------------------------------------------------------
proc isA(poly f, list L, int N)
"USAGE: isA(f,L,N); f a (PH) polynomial with respect
        to a finite list L of weights, N a positive integer
ASSUME: the monomial ordering is a local degree ordering
RETURN: 1 if each monomial having total degree N is (A) with
        respect to f and L, 0 otherwise
"

 ideal M=maxideal(N);
 int s=ncols(M);
 int Ni;
 ideal Ji;
 for (int i=1; i <= s; i++)

  Ni=grord(M[i],L);
  Ji=std(A_span(f,L,Ni));
  if(reduce(M[i],Ji,1)!=0)  return(0);

 return(1);

//-----------------------------------------------------------
proc isAC(poly f, list L, int N)
"USAGE: isAC(f,L,N); f a (PH) polynomial with respect
        to a finite list L of weights, N a positive integer
ASSUME: the monomial ordering is a local degree ordering
RETURN: 1 if each monomial having total degree N is (AC) with
        respect to f and L, 0 otherwise
"

 ideal M=maxideal(N);
 int s=ncols(M);
 int Ni;
 ideal Ji,Ki;
 for (int i=1; i <= s; i++)
```

```
  Ni=grord(M[i],L);
  Ji=std(AC_span(f,L,Ni));
  Ki=reduce(M[i],Ji,1);
  if(Ki!=0)  return(0);

 return(1);

//-----------------------------------------------------------
proc limdeg_NF(poly f, intvec w1, intvec w2)
"USAGE: limdeg_NF(f,w1,w2); f a (PH) polynomial with respect
        to a set of 2 weights w1 and w2
ASSUME: the monomial ordering is a local degree ordering
RETURN: the largest power, of the monomial corresponding to
        the inner vertex of the Newton polygon of f,
        having total degree lower than the bound 2k-ord(f)+2
        of determinacy
"
 ideal I=f,jacob(f);
  ideal J=std(I);
  list l1=w1;
  list l2=w2;
  poly P=grlead(grlead(f,l1),l2);
  int k=deg(highcorner(J),1:nvars(basering))+1;
  int s=deg(lead(f),1:nvars(basering));
  int N=2*k-s+2;
  int D=deg(P,1:nvars(basering));
 return(N/D);

//-----------------------------------------------------------
proc bd_NF(poly f, intvec w1, intvec w2)
"USAGE: bd_NF(f,w1,w2); f a (PH) polynomial with respect
        to a set of 2 weights w1 and w2
ASSUME: the monomial ordering is a local degree ordering
RETURN: the bigget positive integer, lower than the bound
        2k-ord(f)+2 of determinacy, for which all monomials
        having total degree equal to this number fulfill (AC)
        with respect to f and w_1, w_2; and returns
        2k-ord(f)+2 if such integer does not exist
"

 list l1=w1;
 list l2=w2;
 list L=w1,w2;
 ideal I=f,jacob(f);
 ideal J=std(I);
 int k=deg(highcorner(J),1:nvars(basering))+1;
```

```
int s=ord(f);
int B=2*k-s+2;
int Btmp;
int d1=limdeg_NF(f,w1,w2);
poly P=grlead(grlead(f,l1),l2);
int N=deg(P,1:nvars(basering));
int i;
intvec tmp;
for (i=1; i <= d1; i++)

  if (is_AC(f,L,N*i))  return(N*i);

return(B);
```

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