EQUIVARIANT CHARACTER BIJECTIONS IN GROUPS OF LIE TYPE

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Introduction

A classical conjecture in the representation theory of finite groups, the McKay conjecture, states that for any finite group G and for any prime p, the number $|\mathrm{Irr}_{p'}(G)|$ of complex irreducible characters of p'–degree coincides with the number $|\mathrm{Irr}_{p'}(N_G(P))|$ of complex irreducible characters of p'–degree of the normalizer $N_G(P)$ of any Sylow p–subgroup P of G.

Recently a reduction theorem for the McKay conjecture was proved by Isaacs, Malle and Navarro [IMN07, Theorem B]: The general McKay conjecture is true if a certain stronger version holds for all finite simple groups and their covering groups. A simple group admitting the stronger conditions of [IMN07, Section 10] is called "good". This result has sparked a lot of research in an attempt to prove that all finite simple groups are good. Malle has considered the sporadic groups, alternating groups and exceptional covering groups of Lie type in [Mal08a]. For the finite simple groups of Lie type and p not the defining characteristic there are partial results by Malle and Späth [Mal07, Mal08b, Spä09, Spä10a, Spä10b], in particular they show that the exceptional groups of Lie type are good for those primes.

In this work the goal is to make progress towards proving that the finite simple groups of Lie type are good for their defining characteristic. It was shown in [Bru09a] that most groups of Lie type that coincide with their universal cover are good for their defining characteristic, mainly by counting characters. This covers many of the exceptional groups of Lie type. In the last months further results by Brunat [Bru09b, Bru10] and Brunat and Himstedt [BH09] have appeared on the arXiv, extending the earlier results and showing among other things that those groups of Lie type for which the center of the universal cover is of order two or three, are good for their defining characteristic.

A major ingredient necessary to show that a simple group G is good is the construction of an automorphism–equivariant bijection for the universal covering group of G that respects central characters. We generalize a construction of Alexandre Turull for $\mathrm{SL}_n(q)$ [Tur08, Section 4] to all types of finite reductive groups, making use of the so called Steinberg map. This yields more structural bijections than those obtained by Brunat and we obtain new results for groups with more complicated covering and automorphism groups (e.g. types A_n and D_n), as well as recovering many of the results of Brunat by far simpler methods.

We don't consider Suzuki and Ree groups and related exceptional automorphisms of the untwisted groups $\mathsf{B}_2(2^f)$, $\mathsf{F}_4(2^f)$ and $\mathsf{G}_2(3^f)$, as well as certain very small cases given in Table 13.2. With the exception of $\mathsf{D}_n(2)$ and $^2\mathsf{D}_n(2)$ the groups that we don't consider have already been treated with other methods elsewhere, see [Mal08a], [Bru09a] and [Cab08]. However, it is possible to generalize our method to the Suzuki and Ree groups and the exceptional automorphisms, see Example 9.5. Combining the earlier work on these special cases with our results, we obtain:

Theorem 1. *Let G be a finite simple group of Lie–type[1], $\mathbf{G}_{\mathrm{sc}}$ a simply connected algebraic group defined over an algebraically closed field of characteristic p and F a Frobenius map of $\mathbf{G}_{\mathrm{sc}}$, such that $\mathbf{G}_{\mathrm{sc}}^F$ is the universal central extension of G. Let $\mathbf{U}^F$ be a Sylow p–subgroup of $\mathbf{G}_{\mathrm{sc}}^F$, $\mathbf{B}_{\mathrm{sc}}^F = N_{\mathbf{G}_{\mathrm{sc}}^F}(\mathbf{U}^F)$ and $Z := \mathbf{Z}(\mathbf{G}_{\mathrm{sc}}^F)$. Then there exists a bijection*
$$f : \mathrm{Irr}_{p'}(\mathbf{B}_{\mathrm{sc}}^F) \to \mathrm{Irr}_{p'}(\mathbf{G}_{\mathrm{sc}}^F),$$
with the following properties for all $\eta \in \mathrm{Irr}_{p'}(\mathbf{B}_{\mathrm{sc}}^F)$:

- *Let η_Z denote the unique character of Z below η, then $\eta_Z = f(\eta)_Z$.*
- *For any diagonal automorphism d stabilizing $\mathbf{B}_{\mathrm{sc}}^F$ we have $f(\eta^d) = f(\eta)^d$.*
- *Let N_η denote the orbit of η under the action of the group of diagonal automorphisms, then $f(N_\eta^\sigma) = f(N_\eta)^\sigma$ for an arbitrary automorphism σ of $\mathbf{G}_{\mathrm{sc}}^F$ stabilizing $\mathbf{B}_{\mathrm{sc}}^F$.*

As a special case we recover the results of [Bru09a] and [Bru09b]: When G coincides with $\mathbf{G}_{\mathrm{sc}}^F$, i.e., if $\mathbf{Z}(\mathbf{G}_{\mathrm{sc}}^F)$ is trivial, the sets N_η contain exactly one character and the theorem then states that $\mathbf{G}_{\mathrm{sc}}^F$ must be "good" in most cases (see the proof of Theorem 5 and Remark 2 in [Bru09a]). In [Bru09b] the numbers $|\mathrm{Irr}_{p'}(\mathbf{G}_{\mathrm{sc}}^F)|$ are computed explicitly for most types, we obtain them in Theorem 11.8. The identity dubbed "relative McKay" of [Bru09b, Theorem A] given there only for trivial or prime order of $H^1(F, \mathbf{Z}(\mathbf{G}_{\mathrm{sc}}))$, namely

$$|\mathrm{Irr}_{p'}(\mathbf{G}_{\mathrm{sc}}^F \,\big|\, \nu)| = |\mathrm{Irr}_{p'}(\mathbf{B}_{\mathrm{sc}}^F \,\big|\, \nu)| \text{ for } \nu \in \mathrm{Irr}(\mathbf{Z}(\mathbf{G}_{\mathrm{sc}}^F))$$

is an immediate consequence of Theorem 1. Suppose $|\mathbf{Z}(\mathbf{G}_{\mathrm{sc}}^F)| = 2, 3$ and that any field or graph automorphism σ fixes at least one character of every σ–invariant set $f(N_\eta)$ (which is shown in this work only on the side of $\mathbf{B}_{\mathrm{sc}}^F$, see Proposition 11.13), then we can recover many of the results of [BH09] and [Bru10].

To show that all the finite simple groups of Lie type are good in their defining characteristic much work remains to be done. In Section 16 we provide an interesting example in the group $\mathrm{SL}_9(\mathbb{F}_{7^3})$ concerning extension properties of characters and pose a conjecture on the action of $\mathrm{Aut}(\mathbf{G}_{\mathrm{sc}}^F)$ on the sets N_η.

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[1]except $\mathsf{D}_n(2)$ or $^2\mathsf{D}_n(2)$

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Summary of contents

In Section 1 we introduce the simply connected simple algebraic groups $\mathbf{G}_{\mathrm{sc}}$ using the Steinberg presentation. Instead of directly considering the groups $\mathbf{G}_{\mathrm{sc}}$, we will construct "universal" groups $\mathbf{G}_{\mathrm{u}}$ that contain $\mathbf{G}_{\mathrm{sc}}$ as their derived subgroup in Section 2. Using Frobenius maps in Section 3 we obtain the finite groups $\mathbf{G}_{\mathrm{sc}}^F$ and $\mathbf{G}_{\mathrm{u}}^F$, whose automorphism groups we describe in Section 4. We conclude the stating of well known results by recalling the most important results of Clifford theory in Section 5. In the last two sections of the preparatory Chapter I we go on to explore properties of the universal groups in Section 6 and define the dual universal group $\mathbf{G}_{\mathrm{u}}^*$ in Section 7.

In Chapter II we develop a parametrization for the set $\mathrm{Irr}_{p'}(N_{\mathbf{G}^F}(\mathbf{U}^F))$ where $\mathbf{G} = \mathbf{G}_{\mathrm{u}}$ or $\mathbf{G} = \mathbf{G}_{\mathrm{sc}}$. For a standard Frobenius map F_q of $\mathbf{G}_{\mathrm{u}}$ we construct labels for the elements of $\mathrm{Irr}_{p'}(N_{\mathbf{G}_{\mathrm{u}}^{F_q}}(\mathbf{U}^{F_q}))$ in Section 8, see Theorem 8.6. We investigate the action of automorphisms on the labels in Section 9 and give complete results in Proposition 9.4. Next we generalize the construction of the labels to the twisted groups in Section 10, see Theorem 10.8. We use those results for the universal groups to describe $\mathrm{Irr}_{p'}(N_{\mathbf{G}_{\mathrm{sc}}^F}(\mathbf{U}^F))$ in Section 11 relative to the labels constructed earlier, see Proposition 11.3 and Proposition 11.4. Furthermore we obtain complete results on the action of automorphisms on $\mathrm{Irr}_{p'}(N_{\mathbf{G}_{\mathrm{sc}}^F}(\mathbf{U}^F))$, see Proposition 11.13. As an application we infer the cardinalities $|\mathrm{Irr}_{p'}(N_{\mathbf{G}_{\mathrm{sc}}^F}(\mathbf{U}^F))|$ (and thus $|\mathrm{Irr}_{p'}(\mathbf{G}_{\mathrm{sc}}^F)|$ by Theorem 1) by combinatorial methods from the labels (Theorem 11.8). The underlying characters of $\mathbf{Z}(\mathbf{G}_{\mathrm{sc}}^F)$ are determined by Proposition 12.1 in Section 12.

The final Chapter III is concerned with the parametrization of $\mathrm{Irr}_{p'}(\mathbf{G}^F)$ and tying this together with the results of Chapter II. We recall a few facts from Deligne–Lusztig theory in Section 13 to parametrize $\mathrm{Irr}_{p'}(\mathbf{G}_{\mathrm{u}}^F)$ by the semisimple conjugacy classes of a dual group. The action of automorphisms of the dual group on the semisimple classes is known, and compatible with the action of the corresponding automorphisms on $\mathrm{Irr}_{p'}(\mathbf{G}_{\mathrm{u}}^F)$. However the author is not aware of any general results on the action of non–diagonal automorphisms on the p'–characters contained in a single Lusztig–series $\mathcal{E}(\mathbf{G}_{\mathrm{sc}}^F, [\bar{s}])$, which would be required to extend the bijection f from Theorem 1 to be $\mathrm{Aut}(\mathbf{G}_{\mathrm{sc}}^F)$–equivariant if $\mathbf{Z}(\mathbf{G}_{\mathrm{sc}}^F)$ is non–trivial. To make use of the results of Deligne–Lusztig theory we require a suitable parametrization for the semisimple conjugacy classes of the dual groups. It turns out that the central tool here is the parametrization of the semisimple classes of $\mathbf{G}_{\mathrm{sc}}$ given by the Steinberg map. This is explained in Section 14. Putting everything together in Section 15, we obtain a natural $\mathrm{Aut}(\mathbf{G}_{\mathrm{u}}^F)$–equivariant bijection of $\mathrm{Irr}_{p'}(\mathbf{B}_{\mathrm{u}}^F)$ with $\mathrm{Irr}_{p'}(\mathbf{G}_{\mathrm{u}}^F)$ in Theorem 15.1. This bijection is compatible with the multiplication by linear characters (Theorem 15.3), a property we use to construct the bijection f from Theorem 1 in Theorem 15.4. In Section 16 we briefly discuss remaining open problems.

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I. Fundamentals and preparation

Firstly we describe the simple algebraic group $\mathbf{G}_{\mathrm{sc}}$ of simply connected type realized by the Steinberg presentation in Section 1. Using $\mathbf{G}_{\mathrm{sc}}$ we construct the *universal group* $\mathbf{G}_{\mathrm{u}}$, an algebraic group with connected center, containing $\mathbf{G}_{\mathrm{sc}}$ as the derived subgroup in Section 2. Then we introduce the related finite groups $\mathbf{G}_{\mathrm{sc}}^F$ and $\mathbf{G}_{\mathrm{u}}^F$ in Section 3, whose automorphism groups we describe in Section 4. We introduce the concept of duality of algebraic groups and investigate the universal groups in this regard in Section 7 using an explicit description of the dual fundamental weights of $\mathbf{G}_{\mathrm{u}}$ from Section 6. Furthermore we recall facts from Clifford theory in Section 5 that shall be needed later on.

1. Simply connected algebraic groups

The statements of this section about the Steinberg presentation can be found in [Ste68] and [Car72], those about algebraic groups in [Hum75].

Notation 1.1. We shall keep the following fixed throughout the work. Let p be a fixed prime number, $\mathbf{k}$ an algebraic closure of $\mathbb{F}_p$, Φ an irreducible root system, $\Delta = \{\alpha_1, \ldots, \alpha_n\}$ a basis of Φ and $\Phi^+ := \Phi \cap \{\sum_{i=1}^n a_i \alpha_i \mid a_i \in \mathbb{N}_0\}$ the set of positive roots. The associated Dynkin diagrams, Cartan matrices and our chosen numbering of the simple roots Δ can be found in Appendix 17.

We define $\mathbf{G}_{\mathrm{sc}} = \mathbf{G}_{\mathrm{sc}}(\Phi, \mathbf{k})$ as the abstract group generated by symbols $x_\alpha(t)$, where $\alpha \in \Phi$, $t \in \mathbf{k}$, subjected to the Steinberg relations:

$$
\begin{aligned}
x_\alpha(t_1)x_\alpha(t_2) &= x_\alpha(t_1 + t_2), \\
h_\alpha(t_1)h_\alpha(t_2) &= h_\alpha(t_1 t_2) && \text{for } t_1 t_2 \neq 0, \\
[x_\alpha(t_1), x_\beta(t_2)] &= \prod_{i,j>0} x_{i\alpha+j\beta}\left(c_{ij\alpha\beta}(t_2)^i t_1^j\right) && \text{for } \alpha \neq \pm\beta,
\end{aligned}
$$

where $t_1, t_2 \in \mathbf{k}$, $\alpha, \beta \in \Phi$,

$$
h_\alpha(t) := n_\alpha(t)n_\alpha(-1), \; n_\alpha(t) := x_\alpha(t)x_{-\alpha}(-t^{-1})x_\alpha(t) \text{ for } t \neq 0
$$

and the $c_{ij\alpha\beta} \in \{\pm 1, \pm 2, \pm 3\}$ are as defined in [Ste68, Lemma 15]. Steinberg has shown [Ste68, Theorem 8] that this gives a presentation for a simple algebraic group over $\mathbf{k}$ of simply connected type with root system Φ.

The group $\mathbf{T}_{\mathrm{sc}} := \{h_\alpha(t) \mid \alpha \in \Phi^+, \ t \in \mathbf{k}^\times\}$ is a direct product of the groups $h_{\alpha_i}(\mathbf{k}^\times)$ for $i = 1, \ldots, n$ [Ste68, Lemma 28], hence it is a maximal torus of $\mathbf{G}_{\mathrm{sc}}$. To $\mathbf{T}_{\mathrm{sc}}$ belong the $\mathbb{Z}$–lattices of algebraic homomorphisms $X(\mathbf{T}_{\mathrm{sc}}) := \mathrm{Hom}(\mathbf{T}_{\mathrm{sc}}, \mathbf{k}^\times)$ and $Y(\mathbf{T}_{\mathrm{sc}}) := \mathrm{Hom}(\mathbf{k}^\times, \mathbf{T}_{\mathrm{sc}})$ of rank n. Composition of maps from $X(\mathbf{T}_{\mathrm{sc}})$ and $Y(\mathbf{T}_{\mathrm{sc}})$ yields algebraic group homomorphisms $\mathbf{k}^\times \to \mathbf{k}^\times$ and those are of the form $v \mapsto v^k$ for some $k \in \mathbb{Z}$. For $\omega \in X(\mathbf{T}_{\mathrm{sc}})$ and $\beta \in Y(\mathbf{T}_{\mathrm{sc}})$ with $(\omega \circ \beta)(v) = v^k$ for all $v \in \mathbf{k}^\times$ we set $\langle \omega, \beta \rangle := k$. This is a perfect pairing between $X(\mathbf{T}_{\mathrm{sc}})$ and $Y(\mathbf{T}_{\mathrm{sc}})$.

The root system Φ can be recovered in $X(\mathbf{T}_{\mathrm{sc}})$. We define the *root subgroups*

$$\mathbf{U}_\alpha := \{x_\alpha(u) \mid u \in \mathbf{k}\}$$

each of which is isomorphic to the algebraic group $(\mathbf{k}, +)$ [Car72, 5.1]. The conjugation action of $\mathbf{T}_{\mathrm{sc}}$ normalizes $\mathbf{U}_\alpha$ by the proof of [Car72, 12.1.1] and thus yields an automorphism of $(\mathbf{k}, +)$. Those are of the form $u \mapsto vu$ for some $v \in \mathbf{k}^\times$. Hence we can relate to every root subgroup $\mathbf{U}_\alpha$ a homomorphism $\alpha' : \mathbf{T}_{\mathrm{sc}} \to \mathbf{k}^\times$ such that $x_\alpha(u)^t = x_\alpha(\alpha'(t)u)$ for all $t \in \mathbf{T}_{\mathrm{sc}}$ and $u \in \mathbf{k}$. By the the proof of [Car72, 12.1.1] we have $\langle \alpha_i', h_{\alpha_j} \rangle = C_{ij}$, where C is the Cartan matrix of Φ. We can construct an isomorphism of root systems, mapping α to α' and the $h_\alpha \in Y(\mathbf{T}_{\mathrm{sc}})$ can be considered to be the roots of a root system dual to Φ [Car85, 1.8]. We shall identify α' with α from now on.

Since $\mathbf{T}_{\mathrm{sc}}$ is the direct product $\prod_{i=1}^n h_{\alpha_i}(\mathbf{k}^\times)$, a basis of the the $\mathbb{Z}$–lattice $Y(\mathbf{T}_{\mathrm{sc}})$ is given by the $\alpha_i^\vee := h_{\alpha_i}$. In $X(\mathbf{T}_{\mathrm{sc}})$ we obtain a basis by considering the *fundamental weights* $\omega_i \in X(\mathbf{T}_{\mathrm{sc}})$, defined by $\omega_i(t) := v_i$ for $t = \prod_{j=1}^n h_{\alpha_j}(v_j) \in \mathbf{T}_{\mathrm{sc}}$, hence the fundamental weights are a dual basis of the $\alpha_i^\vee$ with respect to the perfect pairing, i.e., $\langle \omega_i, \alpha_j^\vee \rangle = \delta_{ij}$. We can obtain the ω_i as $\mathbb{Q}$–linear combinations of the simple roots, with coefficients given by the rows of the inverse Cartan matrix:

$$\langle \sum_{k=1}^n a_{ik}\alpha_k, \alpha_j^\vee \rangle = \delta_{ij} \text{ for all } i, j \in \{1, \ldots, n\} \Longleftrightarrow (a_{ij})_{i,j} \, C = E_n.$$

We would like to have a basis $\omega_i^\vee$ of $Y(\mathbf{T}_{\mathrm{sc}})$ with $\langle \alpha_i, \omega_j^\vee \rangle = \delta_{ij}$, but usually $Y(\mathbf{T}_{\mathrm{sc}})$ does not contain such a basis. We shall construct one for the universal groups in Section 6.

The $\mathbb{Z}$–span $\mathbb{Z}\Phi$ of $\Phi \subseteq X(\mathbf{T}_{\mathrm{sc}})$ is called the *root lattice* and $\Lambda := X(\mathbf{T}_{\mathrm{sc}})/\mathbb{Z}\Phi$ is the *fundamental group* of $\mathbf{G}_{\mathrm{sc}}$. Since $X(\mathbf{T}_{\mathrm{sc}})$ and $\mathbb{Z}\Phi$ both have rank n the fundamental group is a finite abelian group. Let l denote the exponent of Λ, i.e., the least common multiple of all orders of elements in Λ. Straight letters such as l, r or k will always

denote some fixed constant, that may only depend on the type and rank of the considered root system.

Let $\mathbf{U} := \prod_{\alpha \in \Phi^+} \mathbf{U}_\alpha$. The set $\mathbf{U}$ is a *maximal connected unipotent* subgroup and $\mathbf{B}_{\mathrm{sc}} := \mathbf{T}_{\mathrm{sc}} \mathbf{U} = N_{\mathbf{G}_{\mathrm{sc}}}(\mathbf{U})$ a *Borel* subgroup of $\mathbf{G}_{\mathrm{sc}}$, see [Hum75, Corollary 23.1 D and Theorem 30.4].

2. Universal algebraic group

The center Z of $\mathbf{G}_{\mathrm{sc}}$ is a finite subgroup of $\mathbf{T}_{\mathrm{sc}}$ given by $Z = \cap_{i=1}^{n} \ker \alpha_i$ [DM91, 0.35]. Let r_p be the minimal number of generators required to generate Z, where p is the characteristic of $\mathbf{k}$. By convention $\mathsf{r}_p = 0$ if Z is trivial. Let r be the maximal r_p occuring for any characteristic p and fixed root system Φ. For an injective group homomorphism $\rho : Z \to \mathcal{Z}$ where $\mathcal{Z} := (\mathbf{k}^\times)^{\mathsf{r}}$ is a torus of rank r we define a *universal group* of type Φ:

$$\mathbf{G}_{\mathrm{u}} := (\mathbf{G}_{\mathrm{sc}} \times \mathcal{Z}) \, / \, \left\{ \left(z, \rho(z)^{-1}\right) \mid z \in Z \right\}$$

This depends on the choice of ρ, we shall proof the existence of and fix specific ρ at the end of Section 6. Since $\mathbf{G}_{\mathrm{sc}}$ and $\mathcal{Z}$ are connected algebraic groups with finite intersection this construction yields a connected algebraic group.

Usually we have $\mathsf{r} = 1$. When Φ is a root system of type D_n and n is even then $\mathsf{r} = 2$. We shall denote this case by $\mathsf{D}_{\mathrm{even}}$. The case $\mathsf{r} = 0$ occurs only in a few exceptional root systems, of course $\mathbf{G}_{\mathrm{u}} \cong \mathbf{G}_{\mathrm{sc}}$ in this case.

Example 2.1. If l is a p–power, we have $\mathsf{r}_p = 0$ since $|Z| = \mathsf{l}_{p'}^{\mathsf{r}}$ [DM91, 13.14] and Z contains only elements of p'–order. Thus $\mathbf{G}_{\mathrm{u}} = \mathbf{G}_{\mathrm{sc}} \times \mathcal{Z}$ in this case. The universal algebraic group is isomorphic to $\mathrm{GL}_{n+1}(\mathbf{k})$ in type A_n and to $\mathrm{CSp}_{2n}(\mathbf{k})$ in type C_n: Identify $\mathbf{G}_{\mathrm{sc}}$ with $\mathrm{SL}_{n+1}(\mathbf{k})$, resp. $\mathrm{Sp}_{2n}(\mathbf{k})$ as in [Car85, Section 1.11], then identify $\mathcal{Z}$ with the one–dimensional torus of scalar matrices of $\mathrm{GL}_{n+1}(\mathbf{k})$, resp. $\mathrm{GL}_{2n}(\mathbf{k})$. Take ρ as the natural identification and the assertion follows.

The universal group $\mathbf{G}_{\mathrm{u}}$ contains $\mathbf{G}_{\mathrm{sc}}$ via $g \mapsto \overline{(g, 1)}$ and $\mathcal{Z}$ via $z \mapsto \overline{(1, z)}$ where $\overline{\cdot} : \mathbf{G}_{\mathrm{sc}} \times \mathcal{Z} \longrightarrow \mathbf{G}_{\mathrm{u}}$ denotes the canonical epimorphism. For convenience we identify elements of $\mathbf{G}_{\mathrm{sc}}$ and $\mathcal{Z}$ with their images in $\mathbf{G}_{\mathrm{u}} = \mathbf{G}_{\mathrm{sc}}\mathcal{Z}$. We shall write $sz = \overline{(s, z)}$ for $s \in \mathbf{G}_{\mathrm{sc}}$ and $z \in \mathcal{Z}$. We denote the r canonical homomorphisms from $\mathbf{k}^\times$ to $\mathcal{Z}$ by $\mathbf{z}_i$ for $i \in \{1, \ldots, \mathsf{r}\}$, when $\mathsf{r} = 1$ we omit the index. We also write $s\mu$ for $s\,\mathbf{z}(\mu)$ or $s(\mu_1, \mu_2)$ for $s\,\mathbf{z}_1(\mu_1)\,\mathbf{z}_2(\mu_2)$ and $\mu, \mu_i \in \mathbf{k}^\times$.

By definition $\mathcal{Z} \subseteq \mathbf{Z}(\mathbf{G}_{\mathrm{u}})$ and $\mathcal{Z} \cap \mathbf{G}_{\mathrm{sc}} = \mathbf{Z}(\mathbf{G}_{\mathrm{sc}})$, thus the universal group has connected center $\mathbf{Z}(\mathbf{G}_{\mathrm{u}}) = \mathcal{Z}$. A maximal torus in $\mathbf{G}_{\mathrm{u}}$ is given by $\mathbf{T}_{\mathrm{u}} := \mathbf{T}_{\mathrm{sc}}\mathcal{Z}$ (not a direct product!). We set $\mathbf{B}_{\mathrm{u}} := \mathbf{T}_{\mathrm{u}}\mathbf{U} = N_{\mathbf{G}_{\mathrm{u}}}(\mathbf{U})$.

Example 2.2. Consider type D_n and p odd. By Proposition 6.3 we obtain

$$\mathbf{Z}(\mathbf{G}_{\mathrm{sc}}) = \left\{ 1, \; h_{\alpha_{n-1}}(-1)h_{\alpha_n}(-1), a, b \right\}$$

where

$$a = \begin{cases} h_{\alpha_{n-1}}(-1) \prod_{i=1}^{n-2} h_{\alpha_i}((-1)^i) & n \text{ even}, \\ h_{\alpha_{n-1}}(-\mathtt{i})h_{\alpha_n}(\mathtt{i}) \prod_{i=1}^{n-2} h_{\alpha_i}((-1)^i) & n \text{ odd}, \end{cases}$$

$$b = \begin{cases} h_{\alpha_n}(-1) \prod_{i=1}^{n-2} h_{\alpha_i}((-1)^i) & n \text{ even,} \\ h_{\alpha_{n-1}}(\mathtt{i}) h_{\alpha_n}(-\mathtt{i}) \prod_{i=1}^{n-2} h_{\alpha_i}((-1)^i) & n \text{ odd,} \end{cases}$$

for a primitive 4–th root of unity $\mathtt{i} \in \mathbf{k}^{\times}$.

If n is odd we define $\rho(a) = \mathtt{i}$, $\rho(b) = -\mathtt{i}$. If n is even we set $\rho(a) = (-1, 1)$, $\rho(b) = (1, -1)$ and $\rho(ab) = (-1, -1)$. Recall our convention to omit the map(s) $\mathbf{z}$, then

$$\mathbf{Z}(\mathbf{G}_{\mathrm{sc}}) = \begin{cases} \{1, -1, \mathtt{i}, -\mathtt{i}\} & n \text{ odd or} \\ \{(1, 1), (-1, 1), (1, -1), (-1, -1)\} & n \text{ even.} \end{cases}$$

3. Finite groups

Let $q := p^k$ for some $k \in \mathbb{N}$. We define the *standard Frobenius map* $F_q : \mathbf{G}_{\mathrm{sc}} \to \mathbf{G}_{\mathrm{sc}}$ to be the map induced by $x_\alpha(u) \mapsto x_\alpha(u^q)$ for all $\alpha \in \Phi$, this defines a homomorphism of algebraic groups [Ste68, Chapter 10]. We can extend F_q to $\mathbf{G}_{\mathrm{sc}} \times \mathcal{Z}$ by $z \mapsto z^q$ on $\mathcal{Z}$. Since F_q acts on $Z(\mathbf{G}_{\mathrm{sc}})$ by $z \mapsto z^q$ as well, this also defines a standard Frobenius map F_q on the factor group $\mathbf{G}_{\mathrm{u}}$. The set of fix points of F_q, denoted by $\mathbf{G}_{\mathrm{sc}}^{F_q}$ resp. $\mathbf{G}_{\mathrm{u}}^{F_q}$ is a finite group [Car85, 1.17].

By "twisting" the standard Frobenius map, we can obtain further finite groups. For this construction we assume that all the simple roots of Δ are of the same length, which means we won't consider the Suzuki and Ree groups, those are treated in [IMN07] and [Bru09a]. To the root system Φ belongs a Dynkin diagram. A symmetry τ of that diagram can be be used to realize an automorphism Γ of $\mathbf{G}_{\mathrm{sc}}$, a *graph automorphism* [Ste68, Theorem 29]. By the same Theorem we can choose Γ such that $\Gamma(x_{\alpha_i}(u)) = x_{\tau(\alpha_i)}(u)$, i.e., it acts on the root subgroups $\mathbf{U}_{\alpha_i}$ belonging to simple roots in the same way τ does on the nodes of the corresponding Dynkin diagram. Whenever we speak of a graph automorphism of $\mathbf{G}_{\mathrm{sc}}$ we shall mean one with that property.

Example 3.1. In type A_n, $n \geq 2$, the only graph automorphism is induced by $\tau : \alpha_i \mapsto \alpha_{n-i+1}$. In type D_n we have one graph automorphism induced by $\alpha_{n-1} \mapsto \alpha_n$ for all $n \geq 3$ and another one by $\alpha_1 \mapsto \alpha_4 \mapsto \alpha_3 \mapsto \alpha_1$ if $n = 4$.

Let Γ be a graph automorphism of $\mathbf{G}_{\mathrm{sc}}$ with associated symmetry of the Dynkin diagram τ. For $\alpha \in \Delta$ and $u \in \mathbf{k}^\times$ we have $\Gamma(h_\alpha(u)) = h_{\tau(\alpha)}(u)$ by the definitions of Γ and h_α, so Γ stabilizes $\mathbf{T}_{\mathrm{sc}}$. Thus we can define an action of Γ on $X(\mathbf{T}_{\mathrm{sc}})$ and $Y(\mathbf{T}_{\mathrm{sc}})$ by $\Gamma(\chi) := \chi \circ \Gamma$ and $\Gamma(\gamma) := \Gamma \circ \gamma$ for $\chi \in X(\mathbf{T}_{\mathrm{sc}})$ and $\gamma \in Y(\mathbf{T}_{\mathrm{sc}})$, of course $\langle \Gamma(\chi), \gamma \rangle = \langle \chi, \Gamma(\gamma) \rangle$.

In Section 1 we identified the root system Φ with a subset of $X(\mathbf{T}_{\mathrm{sc}})$. Now we have an action of Γ on $\Delta \subseteq \Phi \subseteq X(\mathbf{T}_{\mathrm{sc}})$ defined above and one of τ on Δ. For $t \in \mathbf{T}_{\mathrm{sc}}$ and $\alpha \in \Delta$ we have

$$\Gamma^{-1}\left(x_\alpha(u)\right)^t = \Gamma^{-1}\left(\Gamma(t)x_\alpha(u)\Gamma(t)^{-1}\right)$$

so $\Gamma(t)$ acts on $\mathbf{U}_\alpha$ by conjugation as does t on $\Gamma^{-1}(\mathbf{U}_\alpha) = \mathbf{U}_{\tau^{-1}(\alpha)}$, thus Γ acts on $\Delta \subseteq X(\mathbf{T}_{\mathrm{sc}})$ as the inverse of τ. When τ acts on the root system by $\tau(\alpha_i) = \alpha_j$ we shall also write $\tau(i) = j$, $\Gamma(j) = i$.

It is always possible to find an extension of a graph automorphisms of $\mathbf{G}_{\mathrm{sc}}$ to $\mathbf{G}_{\mathrm{u}}$, see Table 6.9. For example in type A_n such an extension is given by $z \mapsto z^{-1}$ on $\mathcal{Z}$ and the corresponding Frobenius map would be $z \mapsto z^{-q}$ on $\mathcal{Z}$. We call such an extension a graph automorphism of $\mathbf{G}_{\mathrm{u}}$ and fix specific extensions in Table 6.9.

Definition 3.2. *A Frobenius map of* $\mathbf{G} = \mathbf{G}_{\mathrm{sc}}$ *or* $\mathbf{G} = \mathbf{G}_{\mathrm{u}}$ *is a homomorphism* $F : \mathbf{G} \to \mathbf{G}$ *of algebraic groups, such that* $F = F_q \circ \Gamma$, *where* F_q *is a standard Frobenius map und* Γ *is a graph automorphism of* $\mathbf{G}$.

The groups $\mathbf{G}^F$ where F involves a non–trivial graph automorphism are called *twisted groups of Lie type*.

Proposition 3.3. *A universal group* $\mathbf{G}_{\mathrm{u}}$ *in the sense of the previous section, is a connected algebraic group with connected center,* $[\mathbf{G}_{\mathrm{u}}, \mathbf{G}_{\mathrm{u}}] = \mathbf{G}_{\mathrm{sc}}$ *and for every Frobenius morphism F of* $\mathbf{G}_{\mathrm{sc}}$ *there exists one of* $\mathbf{G}_{\mathrm{u}}$ *that restricts to F.*

Proof. The universal group $\mathbf{G}_{\mathrm{u}} = \mathbf{G}_{\mathrm{sc}}\mathcal{Z}$ is algebraic since $\mathbf{G}_{\mathrm{sc}}$ and $\mathcal{Z}$ are algebraic groups with finite intersection Z. A normal subgroup of finite index of $\mathbf{G}_{\mathrm{u}}$ would yield one of $\mathbf{G}_{\mathrm{sc}}$ or $\mathcal{Z}$ by intersecting, so $\mathbf{G}_{\mathrm{u}}$ must be connected. Since $\mathbf{Z}(\mathbf{G}_{\mathrm{sc}})$ can be considered a subset of $\mathcal{Z}$ by construction, we have $\mathbf{Z}(\mathbf{G}_{\mathrm{u}}) = \mathcal{Z}$ which is a connected torus of rank r. Furthermore we have $[\mathbf{G}_{\mathrm{sc}}, \mathbf{G}_{\mathrm{sc}}] = \mathbf{G}_{\mathrm{sc}}$ by [DM91, 0.37] and since $\mathcal{Z}$ is central $[\mathbf{G}_{\mathrm{sc}}\mathcal{Z}, \mathbf{G}_{\mathrm{sc}}\mathcal{Z}] \subseteq \mathbf{G}_{\mathrm{sc}}$, so $[\mathbf{G}_{\mathrm{u}}, \mathbf{G}_{\mathrm{u}}] = \mathbf{G}_{\mathrm{sc}}$. The extension property is discussed above. $\qquad\square$

Proposition 3.4. *A Sylow p–subgroup of* $\mathbf{G}_{\mathrm{sc}}^F$ *and* $\mathbf{G}_{\mathrm{u}}^F$ *is given by* $\mathbf{U}^F$.

Proof. From the Steinberg relations it follows that $\mathbf{U}^F$ is a p–group. It is a Sylow p–subgroup of $\mathbf{G}_{\mathrm{sc}}^F$ and $\mathbf{G}_{\mathrm{u}}^F$ by [Car72, Theorem 9.4.10] in the untwisted case and by [Car72, Lemma 14.1.2 and Proposition 14.1.3] in the twisted case. $\qquad\square$

4. Automorphisms

Let $F = F_q \circ \Gamma$ or $F = F_q$ be a Frobenius map and $q = p^k$. The automorphism group of the finite group $\mathbf{G}_{\mathrm{sc}}^F$ is generated by the following automorphisms [Ste68, Theorem 10.30]:

- The diagonal automorphisms: those are given by conjugation with elements of $\mathbf{G}_{\mathrm{u}}^F$.

- The field automorphisms: those are induced by the Frobenius maps F_{p^i} with $i \mid k$. They are generated by F_p, and $(F_p)^k = F_q$.

- The graph automorphisms. Those only occur if the algebraic group $\mathbf{G}_{\mathrm{sc}}$ admits such an automorphism. If a non–trivial graph automorphism was used in the definition of F then this coincides with a field automorphism since then $F_q(g) = \Gamma^{-1}(g)$ for all $g \in \mathbf{G}_{\mathrm{sc}}^F$.

- In types B_2 and F_4 with $p = 2$ and G_2 with $p = 3$ there exist exceptional automorphisms of the finite group $\mathbf{G}_{\mathrm{sc}}^F$. Those arise from the symmetry of the Coxeter diagram of those types [Car72, 12.3 and 12.4]. We won't consider them, since these groups and automorphisms are already treated in [Bru09a]. Our methods should apply to them as well though, see Example 9.5.

We only consider diagonal automorphisms that stabilize $\mathbf{B}_{\mathrm{sc}}^F$, i.e., those that are obtained by conjugating with elements of $\mathbf{T}_{\mathrm{u}}^F$, since those already contain representatives of all classes of diagonal automorphisms modulo conjugation in $\mathbf{G}_{\mathrm{sc}}^F$ (see [Ste68] before Theorem 30).

For $\sigma \in \langle F_q, \Gamma \rangle$ we can also define actions on the lattices $X(\mathbf{T}_{\mathrm{sc}})$ and $Y(\mathbf{T}_{\mathrm{sc}})$ by

$$\sigma(\beta) := \sigma \circ \beta \text{ for } \beta \in Y(\mathbf{T}_{\mathrm{sc}}) \text{ and } \sigma(\omega) := \omega \circ \sigma \text{ for } \omega \in X(\mathbf{T}_{\mathrm{sc}}).$$

We can act from the left since $\langle \Gamma, \sigma \rangle$ is an abelian group. Remember the fundamental weights $\omega_i \in X(\mathbf{T}_{\mathrm{sc}})$ with $\langle \omega_i, \alpha_j^\vee \rangle = \delta_{ij}$.

Lemma 4.1. *We have*
$$\Gamma(\omega_i) = \omega_{\Gamma(i)}$$
for a graph automorphism Γ and a fundamental weight $\omega_i \in X(\mathbf{T}_{\mathrm{sc}})$, $i = 1, \ldots, n$.

Proof. Recall that $\Gamma(i) = j$ if and only if $\Gamma(\alpha_i) = \alpha_j$ and $\tau(j) = i$, where τ is the symmetrie of Δ we used in defining Γ. Since $\Gamma(x_{\alpha_i}(u)) = x_{\tau(\alpha_i)}(u)$, we also have $\Gamma(h_{\alpha_i}(t)) = h_{\tau(\alpha_i)}(t)$ by the definition of $h_{\alpha_i} = \alpha_i^\vee$. Consider

$$\langle \Gamma(\omega_i), \alpha_j^\vee \rangle = \langle \omega_i, \Gamma(\alpha_j^\vee) \rangle = \langle \omega_i, \alpha_{\tau(j)}^\vee \rangle = 1 \Longleftrightarrow j = \tau^{-1}(i),$$

thus $\Gamma(\omega_i) = \omega_{\tau^{-1}(i)} = \omega_{\Gamma(i)}$. $\qquad\square$

There are extensions of all those automorphism to $\mathrm{Aut}(\mathbf{G}_\mathrm{u}^F)$: We extended field and graph automorphisms in the previous section and the diagonal automorphisms are restrictions of inner automorphisms of $\mathbf{G}_\mathrm{u}$. For the graph automorphisms the choice of the extension can be ambiguous as the following example demonstrates.

Example 4.2. Let $\mathbf{G}_\mathrm{u}$ be the universal group of type D_4 and Γ the graph automorphism of order 3 of $\mathbf{G}_{\mathrm{sc}}$ that maps $x_{\alpha_1}(u)$ to $x_{\alpha_3}(u)$. Then

$$\mathbf{G}_\mathrm{u} \to \mathbf{G}_\mathrm{u}, \quad s(z_1, z_2) \mapsto \Gamma(s)(z_2^{-1}, z_2^{-1} z_1) \text{ and } s(z_1, z_2) \mapsto \Gamma(s)(z_2, z_2^{-1} z_1^{-1})$$

for $s \in \mathbf{G}_{\mathrm{sc}}$ and $z_i \in \mathbf{k}^\times$ both define extensions of Γ to $\mathbf{G}_\mathrm{u}$ and together with extensions of Γ^{-1} this yields four distinct outer automorphisms of $\mathbf{G}_\mathrm{u}$.

5. Clifford theory

In this section we state well known results of Clifford theory and describe our conventions for discussing the involved objects. Clifford theory describes the characters of a finite group G relative to those of a normal subgroup H (and vice versa). We say a character $\chi \in \mathrm{Irr}(G)$ lies above $\eta \in \mathrm{Irr}(H)$ if η is a constituent of the restriction $\chi|_H$ of χ to H. We also say η is a character below or under χ. The set of all irreducible characters of G above some fixed $\eta \in \mathrm{Irr}(H)$ is denoted by $\mathrm{Irr}(G \,|\, \eta)$.

Since H is a normal subgroup there is an action of G on H and $\mathrm{Irr}(H)$ by conjugation. Conjugation in G fixes $\chi \in \mathrm{Irr}(G)$ and so it must permute the constituents of the restriction of χ to H. In fact the set of all irreducible characters of H below χ is given by $\{\eta^g \,|\, g \in G\}$, [Isa94, Theorem 6.2]. The fix–point group of the action of G on that set is called the *inertia group* $I_G(\eta) := \{g \in G \,|\, \eta^g = \eta\}$.

Notation 5.1. In the special situation that $G = \mathbf{G}_\mathrm{u}^F$ (or $G = \mathbf{B}_\mathrm{u}^F$), $H = \mathbf{G}_\mathrm{sc}^F$ (resp. $H = \mathbf{B}_\mathrm{sc}^F$), $\eta \in \mathrm{Irr}(H)$ and $\chi \in \mathrm{Irr}(G \,|\, \eta)$, we usually denote the set $\mathrm{Irr}(G \,|\, \eta)$ by $M_\chi = M_\eta$ and the set $\{\eta^g \,|\, g \in G\}$ by $N_\eta = N_\chi$.

Theorem 5.2. *For fixed $\eta \in \mathrm{Irr}(H)$ the sets $\mathrm{Irr}(I_G(\eta) \,|\, \eta)$ and $\mathrm{Irr}(G \,|\, \eta)$ are in bijection. A bijection is given by induction of characters. If $\psi^G = \chi$ for $\psi \in \mathrm{Irr}(I_G(\eta) \,|\, \eta)$ and $\chi \in \mathrm{Irr}(G \,|\, \eta)$, then ψ is the unique irreducible character of $I_G(\eta)$ below χ that lies above η.*

Proof. [Isa94, Theorem 6.11]. □

If there is a character ψ of the inertia group that restricts to η, we say η *extends to* $I_G(\eta)$ and ψ is *an extension* of η.

Theorem 5.3. *If $\psi \in \mathrm{Irr}(I_G(\eta))$ is an extension of $\eta \in \mathrm{Irr}(H)$ then*

$$\mathrm{Irr}(I_G(\eta) \,|\, \eta) = \{\beta\psi \,|\, \beta \in \mathrm{Irr}(I_G(\eta)/H)\}\,.$$

Proof. [Isa94, Corollary 6.17]. □

The situation is particularly favourable if G/H is cyclic. In this case it is also easy to describe $\mathrm{Irr}(I_G(\eta)/H)$.

Theorem 5.4. *If G/H is cyclic, all irreducible characters of H extend to their respective inertia groups.*

Proof. [Isa94, Corollary 11.22]. □

6. Dual fundamental weights

In this section we fix the map $\rho : Z \to \mathcal{Z}$ used in the construction of the universal group in Section 2 and define the *dual fundamental weights* $\omega_j^\vee \in Y(\mathbf{T}_\mathrm{u})$ with $\langle \alpha_i, \omega_j^\vee \rangle = \delta_{ij}$.

Notation 6.1. For a natural number $n \in \mathbb{N}$ we denote by n_p the p–part of n, i.e., the largest p–power that divides n. By $n_{p'}$ we denote the p–prime part of n, i.e., $n_{p'} := n/n_p$. For an arbitrary fixed element $\mu \in \mathbf{k}^\times$ let $\nu \in \mathbf{k}$ denote an l–th root of μ of order $\mathsf{l}|\langle\mu\rangle|$, where l is the the exponent of $X(\mathbf{T}_\mathrm{sc})/\mathbb{Z}\Phi$ as in Section 1. The number of l–th roots of unity in $\mathbf{k}$ is $\mathsf{l}_{p'}$. If μ generates $\mathbb{F}_q^\times$ let ζ be the primitive $\mathsf{l}_{p'}$–th root of unity such that $\nu^q = \zeta\nu$.

Let $(a_{ij})_{ij} = C^{-1} \in \mathbb{Q}^{n \times n}$ be the inverse Cartan matrix of Φ. For every $j \in \{1, \dots, n\}$ we define

$$\beta_j : \mathbf{k}^\times \to \mathbf{T}_\mathrm{sc}, \ \beta_j(\mu) := \prod_{k=1}^n h_{\alpha_k}(\mu^{\mathsf{l}a_{kj}}) \in \mathbf{T}_\mathrm{sc}.$$

This is well defined: l is the exponent of $\Lambda = X(\mathbf{T}_\mathrm{sc})/\mathbb{Z}\Phi$, thus $(\mathsf{l}\omega_i)\mathbb{Z}\Phi \subseteq \mathbb{Z}\Phi$, i.e. $\mathsf{l}a_{ij} \in \mathbb{Z}$ since $\Delta = \{\alpha_1, \dots, \alpha_n\}$ is a basis of $\mathbb{Z}\Phi$.

Lemma 6.2. *The action of $\beta_j(\mu)$ on the root subgroup $\mathbf{U}_{\alpha_i}$ is trivial if $i \neq j$, otherwise*

$$x_{\alpha_i}(u)^{\beta_i(\mu)} = x_{\alpha_i}(\mu^\mathsf{l} u).$$

Proof. From Section 1 we know how $\mathbf{T}_\mathrm{sc}$ acts on the U_α:

$$h_{\alpha_j}(t)x_{\alpha_i}(u)h_{\alpha_j}(t)^{-1} = x_{\alpha_i}(t^{C_{ij}}u),$$

where (C_{ij}) is the Cartan matrix. Acting with $\beta_j(v) = \prod_{k=1}^n h_{\alpha_k}(v^{\mathsf{l}a_{kj}})$ on $x_{\alpha_i}(u)$ yields

$$x_{\alpha_i}\left(v^{\mathsf{l}\sum_{k=1}^n C_{ik}a_{kj}}u\right) = x_{\alpha_k}(v^{\mathsf{l}\delta_{ij}}u),$$

since $(a_{kj})_k$ is the j–th column of the inverse Cartan matrix. $\square$

Of course $\beta_j \in \mathrm{Hom}(\mathbf{k}^\times, \mathbf{T}_\mathrm{sc}) = Y(\mathbf{T}_\mathrm{sc})$ and by the above lemma we have

$$\langle \alpha_i, \beta_j \rangle = \delta_{ij}\mathsf{l}$$

Here $\langle \cdot, \cdot \rangle : X(\mathbf{T}_\mathrm{sc}) \times Y(\mathbf{T}_\mathrm{sc}) \to \mathbb{Z}$ is the perfect pairing between $X(\mathbf{T}_\mathrm{sc})$ and $Y(\mathbf{T}_\mathrm{sc})$ (see Section 1).

Proposition 6.3. *Let ζ be as in Notation 6.1. The $\beta_j(\zeta)$ generate $Z := \mathbf{Z}(\mathbf{G}_\mathrm{sc})$.*

Proof. Given any $z \in Z$ there exist $v_j \in \mathbf{k}^\times$ with $z = \prod h_{\alpha_j}(v_j)$ since $Z \leq \mathbf{T}_{\mathrm{sc}}$. The torus $\mathbf{T}_{\mathrm{sc}}$ is a direct product of images of the h_{α_j}, and since $|Z| = l_{p'}$ [DM91, Lemma 13.14] the v_j must all be $l_{p'}$–th roots of unity. We write $z = \prod h_{\alpha_j}(\zeta^{b_j})$. Therefore $z = \beta(\zeta)$ for $\beta = \sum b_j \alpha_j^\vee \in Y(\mathbf{T}_{\mathrm{sc}})$. For $z \in Z$ we must have $\alpha_i(z) = 1$ for all i, so $(\alpha_i \circ \beta)(\zeta) = 1$. Then we have $\langle \alpha_i, \beta \rangle \equiv 0 \bmod l_{p'}$, it follows $\sum b_j \langle \alpha_i, \alpha_j^\vee \rangle \equiv 0 \bmod l_{p'}$, thus $Cb \equiv 0 \bmod l_{p'}$.

A generating system of the kernel of C mod $l_{p'}$ is given by the columns of $(l_{p'} C^{-1})$ which are well defined since lC^{-1} is an integer matrix and l_p is invertible modulo $l_{p'}$. Thus $(b_j)_j = l_{p'} \sum_{k=1}^n d_k (a_{jk})_j \bmod l_{p'}$ for some $d_k \in \mathbb{Z}$, but then $z = \beta(\zeta) = \sum_{k=1}^n (d_k / l_p \bmod l_{p'}) \beta_k(\zeta)$. $\qquad\square$

Notation 6.4. By $\mathbf{T}_{\mathrm{sc}}^*$ we denote the torus of the simply connected group of dual type, i.e., the group obtained by the same relations as $\mathbf{G}_{\mathrm{sc}}$ but replacing the root system Φ by a dual one Φ^*, for example the one generated by $\{\alpha_1^\vee, \dots, \alpha_n^\vee\} \subseteq Y(\mathbf{T}_{\mathrm{sc}}) \otimes_\mathbb{Z} \mathbb{R}$ considered as an abstract root system. By ω_i^*, β_i^* and Z^* we denote the objects corresponding to ω_i, β_i and Z in that group.

Note that $\mathbf{T}_{\mathrm{sc}}^*$ is not contained in the *dual group of* $\mathbf{G}_{\mathrm{sc}}$ but in the dual group $\mathbf{G}_{\mathrm{u}}^*$ of the universal group $\mathbf{G}_{\mathrm{u}}$. This will be explained in more detail in Section 7.

Lemma 6.5. *We have $\omega_i \circ \beta_j = \omega_j^* \circ \beta_i^*$.*

Proof. By the definition of the β_i we have $\omega_i(\beta_j(v)) = v^{la_{ij}}$, where the a_{ij} are the entries of the inverse Cartan matrix. Now the Cartan matrix of the dual root system Φ^* is by definition just the transpose of the Cartan matrix of Φ. Since the inverse of the transposed Cartan matrix is just the transposed inverse Cartan matrix, we have $\omega_i^*(\beta_j^*(v)) = v^{la_{ji}}$. $\qquad\square$

Using this we can define a natural bijection δ between $\mathrm{Hom}(Z^*, \mathbf{k}^\times)$ and Z. The ω_i^* are a basis of $X(\mathbf{T}_{\mathrm{sc}}^*)$ so their restrictions to Z^* are a generating system for $\mathrm{Hom}(Z^*, \mathbf{k}^\times)$. We set

$$\delta(\omega_i^*) := \prod_j h_{\alpha_j}(\omega_i^*(\beta_j^*(\zeta))).$$

The map δ is injective since $\mathbf{T}_{\mathrm{sc}}$ is a direct product of the images of the h_{α_j} and the $\beta_j^*(\zeta)$ generate Z^* by Proposition 6.3. Using Lemma 6.5 and the fact that $\prod_{j=1}^n h_{\alpha_j} \circ \omega_j$ is the identiy map on $\mathbf{T}_{\mathrm{sc}}$ (remember that $\mathbf{T}_{\mathrm{sc}}$ is a direct product of the images of the h_{α_i} and $\langle \omega_i, h_{\alpha_j} \rangle = \delta_{ij}$) we obtain

$$\delta(\omega_i^*) = \prod h_{\alpha_j}(\omega_i^*(\beta_j^*(\zeta))) = \prod h_{\alpha_j}(\omega_j(\beta_i(\zeta))) = \beta_i(\zeta) \in Z.$$

Thus δ is surjective since the $\beta_i(\zeta)$ generate Z, therefore δ is a natural group isomorphism of $X(Z^*)$ with Z. Summing up the above:

Proposition 6.6. *There is a natural isomorphism* $\delta : \mathrm{Hom}(Z^*, \mathbf{k}^\times) \to Z$ *which maps* $\omega_i^*|_{Z^*}$ *to* $\beta_i(\zeta)$.

Notation 6.7. For some *index set* $\mathcal{I}$ we call $e = [e^{(1)}, \ldots, e^{(r)}] \in \mathcal{I}^r$ a *multi index* of length r and use the following conventions. If $\mathcal{X} = \{x_i \mid i \in \mathcal{I}\}$ is some set with elements indexed by $\mathcal{I}$, then x_e denotes $[x_{e^{(1)}}, \ldots, x_{e^{(r)}}] \in \mathcal{X}^r$. We also apply this convention recursively: if f is defined on $\mathcal{X}$ then $f(x_e) = [f(x_{e^{(1)}}), \ldots, f(x_{e^{(r)}})]$.

If $\mathcal{Y}$ is a set for which y^i makes sense for $i \in \mathcal{I}$ then y^e denotes $[y^{e^{(1)}}, \ldots, y^{e^{(r)}}]$. We allow for $r = 0$, i.e., empty multi indices $e = []$ if $\mathcal{X}$ or $\{y^i \mid i \in \mathcal{I}, \ y \in \mathcal{Y}\}$ contain some kind of "trivial" element which is then denoted by $x_{[]}$, resp. $y^{[]}$.

If $y = [y_1, \ldots, y_r] \in \mathcal{Y}^r$ then y^e denotes $[y_1^{e^{(1)}}, \ldots, y_r^{e^{(r)}}]$. When discussing such a y we mean the components if no other interpretation is possible, for example, if $y = [4, 9] \in \mathbb{Z}^2$ then we say "y is a square" or "$y^{(3,2)} - 13$ is greater then 50" but we shall also say "the $\mathbb{Z}$–ideal generated by y is $\mathbb{Z}$".

Note that when $r = 1$ this always reduces to the usual meaning and that is the one we shall be most concerned with. In this work multi indices will always have $r = \mathsf{r}$ components, where r is the rank of the central torus $\mathcal{Z}$ of the universal group, as defined in Section 2. As such $\mathsf{r} = 0$ occurs only for a few exceptional groups and $\mathsf{r} \geq 2$ will only occur in the single case $\mathsf{D}_{\mathrm{even}}$.

We now use the isomorphism δ from Proposition 6.6 to fix a specific $\rho : Z \to \mathcal{Z}$ from the definition of the universal group in Section 2.

Proposition 6.8. *There exists a multi index* $\mathsf{k} = [\mathsf{k}^{(j)}] \in \{1, \ldots, n\}^{\mathsf{r}}$ *such that* Z^* *is a direct product of the groups* $\langle \beta_{\mathsf{k}^{(j)}}^*(\zeta) \rangle$, *where* r *is as defined in Section 2.*

Proof. In all cases except type A_n this is clear from Proposition 6.3 since $|Z^*|$ is a prime power [Car85, Section 1.11]. For type A_n observe $\mathrm{ord}(\beta_1(\zeta)) = \mathsf{I}_{p'} = |Z^*|$. $\qquad\square$

To be able to do concrete computations and fix appropriate extensions of graph automorphisms to $\mathbf{G}_{\mathrm{u}}$, we shall now choose k for all types. For a list of the inverse Cartan matrices see the Appendix 17.

Table 6.9: Choice of k and extensions of graph automorphisms

Type	l	k	$\Gamma(k)$	Extension of Γ to $\mathcal{Z}$
A_n	$n+1$	n	1	$z \mapsto z^{-1}$
B_n	2	n	–	
C_n	2	1	–	
D_{2m+1}	4	n	$n-1$	$z \mapsto z^{-1}$
D_4	2	$[3,4]$	$[4,1]$	$(z_1, z_2) \mapsto (z_2^{-1}, z_2^{-1}z_1)$
D_{2m}	2	$[n, n-1]$	$[n-1, n]$	$(z_1, z_2) \mapsto (z_2, z_1)$
E_6	3	1	6	$z \mapsto z^{-1}$
E_7	2	2	–	
$\mathsf{E}_8, \mathsf{F}_4, \mathsf{G}_2$	1	$[\,]$	–	

A word of warning: $\beta_k^*(\zeta)$ generates the center of the simply connected group of the **dual type**. Of course this only makes a difference for types B_n and C_n, but here k differs depending on which type we decide to look at.

If Z^* is trivial for all p, $\mathsf{r} = 0$, k is the empty multi index and we have $\mathcal{Z} = 1$. Of course only three cases can occur: $\mathsf{k} = [\,]$ (trivial center, $\mathbf{G}_u = \mathbf{G}_{sc}$), $\mathsf{k} = [\mathsf{k}^{(1)}]$ (common case) and $\mathsf{k} = [\mathsf{k}^{(1)}, \mathsf{k}^{(2)}]$ (case D_{even}). We can now fix an embedding $\rho : \mathbf{Z}(\mathbf{G}_{sc}) \to \mathcal{Z}$. For all $i \in \{1, \dots, n\}$ we set

$$\rho(\beta_i(\zeta)) := \left(\delta^{-1}(\beta_i(\zeta))(\beta_k^*(\zeta))\right)^{-1} = \omega_i^*(\beta_k^*(\zeta))^{-1}.$$

If $\mathsf{k} \in \mathbb{Z}^2$ this means $\rho(\beta_i(\zeta)) = (\omega_i^*(\beta_{\mathsf{k}^{(1)}}^*(\zeta)), \omega_i^*(\beta_{\mathsf{k}^{(2)}}^*))$ by the multi index convention 6.7 (and $\zeta \in \{1, -1\}$). This yields an injective embedding of $\mathbf{Z}(\mathbf{G}_{sc})$ into $\mathcal{Z}$ since it is the composition of the natural isomorphism δ^{-1} from Proposition 6.6 with evaluation at generators of Z^*.

Definition 6.10. *We define the dual fundamental weights* $\omega_i^\vee : \mathbf{k}^\times \to \mathbf{T}_u$ *by*

$$\omega_i^\vee(\mu) := \beta_i(\nu)\nu^{\langle \omega_k, \beta_i \rangle},$$

where ν is any l–th root of μ.

To see that this is well defined, observe that with the fixed identification ρ, Lemma 6.5 and our convention to omit the map(s) $\mathbf{z}$ we have

$$\beta_i(\zeta) = \zeta^{-\langle \omega_k, \beta_i \rangle} = \prod_{l=1}^{\mathsf{r}} \mathbf{z}_l(\zeta^{-\langle \omega_{\mathsf{k}^{(l)}}, \beta_i \rangle}).$$

Had we chosen another l–th root $\zeta\nu$ of μ the value of the dual fundamental weights would remain unchanged:

$$\beta_i(\zeta\nu)(\zeta\nu)^{\langle \omega_k, \beta_i \rangle} = \beta_i(\nu)\zeta^{\langle \omega_k, \beta_i \rangle - \langle \omega_k, \beta_i \rangle}\nu^{\langle \omega_k, \beta_i \rangle} = \beta_i(\nu)\nu^{\langle \omega_k, \beta_i \rangle}.$$

Proposition 6.11. *We have*

$$x_{\alpha_i}(u)^{\omega_i^\vee(\mu)} = x_{\alpha_i}(\mu^{\delta_{ij}} u)$$

for all $u \in \mathbf{k}$, $\mu \in \mathbf{k}^\times$ and $i, j \in \{1, \ldots, n\}$. In particular if we identify the simple root $\alpha_i \in X(\mathbf{T}_{\mathrm{sc}})$ with its unique extension to $X(\mathbf{T}_{\mathrm{u}})$ that acts trivial on $\mathcal{Z}$ we have

$$\langle \alpha_i, \omega_j^\vee \rangle = \delta_{ij}.$$

Proof. We defined $\omega_i^\vee(\mu)$ as $\beta_i(\nu)$ times some central element, so this is just Lemma 6.2. $\square$

7. Dual universal group

In the last section we fixed $\rho : Z \to \mathcal{Z}$ for $\mathbf{G}_{\mathrm{u}}$. We shall now fix a slightly different $\rho^* : Z^* \to \mathcal{Z}^*$ for $\mathbf{G}_{\mathrm{u}}^*$, where $\mathbf{G}_{\mathrm{u}}^*$ is the universal group obtained by the same construction as $\mathbf{G}_{\mathrm{u}}$ in Section 2 for a root system of dual type, see Notation 6.4. We then show that $\mathbf{G}_{\mathrm{u}}$ and $\mathbf{G}_{\mathrm{u}}^*$ are in fact dual groups.

In Table 6.9 we fixed $\mathsf{k} \in \mathbb{Z}^{\mathsf{r}}$ such that $\beta_{\mathsf{k}}^*(\zeta)$ generates Z^* and we defined ρ by

$$\rho(\beta_i(\zeta)) := \omega_{\mathsf{k}}(\beta_i(\zeta))^{-1}.$$

For $\mathbf{G}_{\mathrm{u}}^*$ we define

$$\rho^*(\beta_{\mathsf{k}}^*(\zeta)) := \zeta$$

unless in case $\mathsf{D}_{\mathrm{even}}$, here we define

$$\rho^*(\beta_{\mathsf{k}^{(1)}}^*(\zeta)) := (\zeta, 1) \text{ and } \rho^*(\beta_{\mathsf{k}^{(2)}}^*(\zeta)) := (1, \zeta).$$

Since $\beta_{\mathsf{k}}^*(\zeta)$ generates Z^* and $|Z^*| = l_{p'}$ (resp. $|Z^*| = l_{p'}^2$ in case $\mathsf{D}_{\mathrm{even}}$) this defines a monomorphism $\rho^* : Z^* \to \mathcal{Z}^*$.

To any connected reductive algebraic group $\mathbf{G}$ with a maximal torus $\mathbf{T}$ we can associate a *root datum* $(X(\mathbf{T}), \Phi, Y(\mathbf{T}), \Phi^\vee)$, where $\Phi \subseteq X(\mathbf{T}) := \mathrm{Hom}(\mathbf{T}, \mathbf{k}^\times)$ is the set of roots of $\mathbf{G}$ with respect to $\mathbf{T}$ and $\Phi^\vee \subseteq Y(\mathbf{T}) := \mathrm{Hom}(\mathbf{k}^\times, \mathbf{T})$ the set of coroots. The structure of $\mathbf{G}$ is determined up to isomorphism by the root datum, see [Hum75, 32.1].

Definition 7.1. *Let $\mathbf{G}$ and $\mathbf{G}^*$ be connected reductive algebraic groups with maximal tori $\mathbf{T}$ and $\mathbf{T}^*$. We say $\mathbf{G}$ is in duality with $\mathbf{G}^*$ if there exists a isomorphism $\delta : Y(\mathbf{T}) \to X(\mathbf{T}^*)$ that maps $\Phi^\vee$ onto Φ^*. If δ is compatible with the action of Frobenius endomorphisms F and F^* of $\mathbf{G}$ and $\mathbf{G}^*$ respectively, then we say $\mathbf{G}^F$ is in duality with $\mathbf{G}^{*F^*}$.*

We call δ from Definition 7.1 *a duality isomorphism*. We shall now describe $Y(\mathbf{T}_{\mathrm{u}})$ and $X(\mathbf{T}_{\mathrm{u}}^*)$ and show that a duality isomorphism exists.

Definition 7.2. *Any $g \in \mathbf{G}_{\mathrm{u}}$ can be written (non–uniquely) as $g = sz$ with $s \in \mathbf{G}_{\mathrm{sc}}$ and $z \in \mathcal{Z}$. We define a determinant map*

$$\det : \mathbf{G}_{\mathrm{u}} \to \mathbf{k}^\times, \ g \mapsto \det(sz) := z^{\mathsf{l}},$$

unless in case $\mathsf{D}_{\mathrm{even}}$. Here define two distinct determinant maps by

$$\det\nolimits^{(1)}(s(z_1, z_2)) := z_1^2 \text{ and } \det\nolimits^{(2)}(s(z_1, z_2)) := z_2^2.$$

The map det is well defined, since $l_{p'}$ is the exponent of $Z(\mathbf{G}_{\mathrm{sc}}) = \mathbf{G}_{\mathrm{sc}} \cap \mathcal{Z}$. As usual $\det^*$ denotes the corresponding map in the group $\mathbf{G}_{\mathrm{u}}^*$. Note that in the case of type A_n where $\mathbf{G}_{\mathrm{u}}$ is isomorphic to $\mathrm{GL}_{n+1}(\mathbf{k})$ the just defined determinant is in fact the usual determinant map. The restriction of det to $\mathbf{T}_{\mathrm{u}}$ is a prominent member of $X(\mathbf{T}_{\mathrm{u}})$: it generates the kernel of the restriction map $X(\mathbf{T}_{\mathrm{u}}) \to X(\mathbf{T}_{\mathrm{sc}})$.

Definition 7.3. *We define fundamental weights of $\mathbf{G}_{\mathrm{u}}^*$ by*

$$\hat{\omega}_i^* : \mathbf{T}_{\mathrm{u}}^* \to \mathbf{k}^\times, \ sz \mapsto \hat{\omega}_i^*(sz) := \omega_i^*(s)z^{\langle \omega_i^*, \beta_\mathsf{k}^* \rangle}$$

and $\hat{\omega}_i^*(s(z_1, z_2)) := \omega_i^*(s)z_1^{\langle \omega_i^*, \beta_{\mathsf{k}(1)}^* \rangle} z_2^{\langle \omega_i^*, \beta_{\mathsf{k}(2)}^* \rangle}$ *in case* $\mathsf{D}_{\mathrm{even}}$.

The decomposition of $t = sz$ is determined up to central elements of Z, by the choice of ρ^* we have

$$sz = (s\beta_\mathsf{k}^*(\zeta))(\zeta^{-1}z) \mapsto \omega_i^*(s\beta_\mathsf{k}^*(\zeta))\zeta^{-\langle \omega_i^*, \beta_\mathsf{k}^* \rangle} z^{\langle \omega_i^*, \beta_\mathsf{k}^* \rangle}$$

and $\omega_i^*(\beta_\mathsf{k}^*(\zeta)) = \zeta^{\langle \omega_i^*, \beta_\mathsf{k}^* \rangle}$, thus $\hat{\omega}_i^* : \mathbf{T}_{\mathrm{u}}^* \to \mathbf{k}^\times$ is well defined.

Lemma 7.4. *For $i = 1, \dots, n$ we have*

$$\langle \det, \omega_i^\vee \rangle = \langle \hat{\omega}_i^*, \mathbf{z}^* \rangle.$$

For the case $\mathsf{D}_{\mathrm{even}}$ consider the above equality to be a pair of equations over the two components of $\det$ and $\mathbf{z}^*$.

Proof. By the definitions of det and $\omega_j^\vee$ we have

$$\left(\det \circ \omega_j^\vee\right)(\mu) = \det(\mathbf{z}(\omega_\mathsf{k}(\beta_i(\nu)))) = \omega_\mathsf{k}(\beta_i(\nu^\mathsf{l})) = \omega_\mathsf{k}(\beta_i(\mu))$$

for all $\mu \in \mathbf{k}^\times$, i.e.

$$\langle \det, \omega_i^\vee \rangle = \langle \omega_\mathsf{k}, \beta_i \rangle$$

and by the definition of the $\hat{\omega}_i^*$

$$\langle \hat{\omega}_i^*, \mathbf{z}^* \rangle = \langle \omega_i^*, \beta_\mathsf{k}^* \rangle.$$

Thus the assertion follows by Lemma 6.5. $\square$

Proposition 7.5. *a) A $\mathbb{Z}$–basis of $Y(\mathbf{T}_{\mathrm{u}})$ is given by the canonical embedding(s) $\mathbf{z} : \mathbf{k}^\times \to \mathcal{Z}$ from Section 2 and the $\omega_i^\vee : \mathbf{k}^\times \to \mathbf{T}_{\mathrm{u}}$ for $i = 1, \dots, n$ defined in Definition 6.10.*

b) A $\mathbb{Z}$–basis of $X(\mathbf{T}_{\mathrm{u}}^*)$ is given by the map(s) $\det^* : \mathbf{T}_{\mathrm{u}}^* \to \mathbf{k}^\times$ and the $\hat{\omega}_i^* : \mathbf{T}_{\mathrm{u}}^* \to \mathbf{k}^\times$ for $i = 1, \dots, n$ from Definition 7.2 and Definition 7.3.

c) *The groups* $\mathbf{G}_\mathrm{u}$ *and* $\mathbf{G}_\mathrm{u}^*$ *are dual in the sense of Definition 7.1 and a duality isomorphism* $\delta : Y(\mathbf{T}_\mathrm{u}) \to X(\mathbf{T}_\mathrm{u}^*)$ *is given by* $\omega_i^\vee \mapsto \hat{\omega}_i^*$ *for* $i = 1, \ldots, n$ *and* $\mathbf{z} \mapsto \det$.

Proof. The root system $\hat{\Phi}$ of $\mathbf{G}_\mathrm{u}$ is determined by the action of $\mathbf{T}_\mathrm{u} = \mathbf{T}_\mathrm{sc}\mathcal{Z}$ on the minimal unipotent subgroups of $\mathbf{G}_\mathrm{u}$. Since $\mathcal{Z}$ is central we obtain a canonical bijection $\hat{\alpha} \mapsto \alpha$ of $\hat{\Phi}$ with the root system Φ in $X(\mathbf{T}_\mathrm{sc})$, where $\hat{\alpha}(tz) = \alpha(t)$. We shall identify α with $\hat{\alpha}$. Of course $\mathcal{Z} = \cap_{i=1}^n \ker \alpha_i$. For a fixed $y \in Y(\mathbf{T}_\mathrm{u})$ we define

$$z := y - \sum_{i=1}^n \langle \alpha_i, y \rangle \omega_i^\vee. \tag{1}$$

By Proposition 6.11 we have $\langle \alpha_i, \omega_j^\vee \rangle = \delta_{ij}$, thus $\alpha_i \circ z = 0$ for $i = 1, \ldots, n$. Hence $z \in Y(\mathcal{Z})$. A basis of $Y(\mathcal{Z})$ is given by $\mathbf{z}$. Therefore we can express y as a $\mathbb{Z}$–linear combination of the proposed basis. This proves a).

For fixed $x \in X(\mathbf{T}_\mathrm{u}^*)$ define

$$z^* := x - \sum_{j=1}^n \langle x, \alpha_j^{*\vee} \rangle \hat{\omega}_i^*.$$

We have $\langle \hat{\omega}_i^*, \alpha_j^{*\vee} \rangle = \delta_{ij}$ which is clear from the definition of the $\hat{\omega}_i^*$:

$$\hat{\omega}_i^*(\alpha_j^{*\vee}(\mu)) = \omega_i^*(\alpha_j^{*\vee}(\mu)) = \mu^{\delta_{ij}}.$$

Thus $\langle z^*, \alpha_j^{*\vee} \rangle = 0$ for $j = 1, \ldots, n$ which means that z^* is in the kernel of the restriction map $X(\mathbf{T}_\mathrm{u}) \to X(\mathbf{T}_\mathrm{sc})$. Since that kernel is generated by $\det^*$ we have b).

It is clear by a) and b) that δ is an isomorphism. To see that δ is a duality isomorphism we need to check that $\delta(\Phi^\vee) = \Phi^*$. By construction

$$\langle \alpha_i, \alpha_j^\vee \rangle = C_{ij} = \langle \alpha_j^*, \alpha_i^{*\vee} \rangle,$$

so

$$\alpha_j^\vee = \sum_{i=1}^n C_{ij} \omega_i^\vee + c_0 \, \mathbf{z}$$

and

$$\alpha_j^* = \sum_{i=1}^n C_{ij} \hat{\omega}_i^* + c_0' \det{}^*$$

for some $c_0, c_0' \in \mathbb{Z}$. Applying $\langle \det, \cdot \rangle$ and $\langle \cdot, \mathbf{z}^* \rangle$ to these equations and using Lemma 7.4 we obtain $c_0 = c_0'$. Therefore $\delta(\alpha_j^\vee) = \alpha_j^*$ and $\delta(\Phi^\vee) = \Phi^*$. $\qquad\square$

A root system of dual type is always the root system itself, except when it is of type B_n or C_n. So if we are not in these types $\mathbf{G}_{\mathrm{sc}} = [\mathbf{G}_{\mathrm{u}}, \mathbf{G}_{\mathrm{u}}] = [\mathbf{G}_{\mathrm{u}}^*, \mathbf{G}_{\mathrm{u}}^*]$ and the groups $\mathbf{G}_{\mathrm{u}}$ and $\mathbf{G}_{\mathrm{u}}^*$ are isomorphic but not "equal" since we have chosen different embeddings ρ and ρ^*. Next we consider the case of the finite groups $\mathbf{G}_{\mathrm{u}}^F$. Note that the types B_n and C_n afford no graph automorphism, so whenever a graph automorphism is involved in F we can assume $\mathbf{G}_{\mathrm{u}} \cong \mathbf{G}_{\mathrm{u}}^*$.

Proposition 7.6. *For a Frobenius map F of $\mathbf{G}_{\mathrm{u}}$ there exists a Frobenius map F^* of $\mathbf{G}_{\mathrm{u}}^*$ such that $\mathbf{G}_{\mathrm{u}}^F$ and $\mathbf{G}_{\mathrm{u}}^{*F^*}$ are in duality by the duality isomorphism δ defined in Proposition 7.5.*

Proof. We need to construct F^* such that $\delta(F(\chi)) = F^*(\delta(\chi))$ for all $\chi \in Y(\mathbf{T}_{\mathrm{u}})$. If F is a standard Frobenius map F_q we can choose $F^* = F_q^*$ to be the standard Frobenius of $\mathbf{G}_{\mathrm{u}}^*$ for the same q. If F involves a non–trivial graph automorphism, i.e., $F = F_q \circ \Gamma$, then $\mathbf{G}_{\mathrm{sc}}$ is contained in $\mathbf{G}_{\mathrm{u}}$ and $\mathbf{G}_{\mathrm{u}}^*$. If we can find a graph automorphism Γ^* of $\mathbf{G}_{\mathrm{u}}^*$ such that

$$\delta(\Gamma(\chi)) = \Gamma^*(\delta(\chi)) \text{ for all } \chi \in Y(\mathbf{T}_{\mathrm{u}}), \tag{2}$$

then $F^* := F_q^* \circ \Gamma^*$ has the desired property. We simply define Γ^* on $X(\mathbf{T}_{\mathrm{u}}^*)$ by the equality (2), it then extends to an automorphism of $\mathbf{G}_{\mathrm{u}}^*$ by the first theorem of [Hum75, 32.1]. $\square$

Usually there is a canonical way to extend a graph automorphism of $\mathbf{G}_{\mathrm{sc}}$ to $\mathbf{G}_{\mathrm{u}}$, the only case which needs more attention is $\mathsf{D}_{\mathrm{even}}$. The graph automorphism of order two must act by permutation of the two components of the central tori of $\mathcal{Z}$ and $\mathcal{Z}^*$. However the graph automorphism of order 3 in type D_4 has multiple extensions from $\mathbf{G}_{\mathrm{sc}}$ to $\mathbf{G}_{\mathrm{u}}$ as seen in Example 4.2. We call Γ^* from the proof of Proposition 7.6 the *dual graph automorphism*. If Γ is non–trivial we have $\alpha_i^* = \alpha_i$ and thus $\delta(\alpha_i^\vee) = \alpha_i$. Since

$$\alpha_{\Gamma^{-1}(i)} = \delta(\alpha_{\Gamma^{-1}(i)}^\vee) = \delta(\Gamma(\alpha_i^\vee)) = \Gamma^*(\delta(\alpha_i^\vee)) = \alpha_{\Gamma^*(i)}$$

we see that that the restriction of Γ^* to $\mathbf{G}_{\mathrm{sc}}$ is Γ^{-1}.

The duality isomorphism induces a natural isomorphism of $\mathbf{T}_{\mathrm{u}}^{*F^*}$ with $\mathrm{Irr}(\mathbf{T}_{\mathrm{u}}^F)$ which we will now describe explicitly.

Definition 7.7. *Let w be the smallest positive integer such that F^w is a standard Frobenius (i.e. the order of the graph automorphism involved in F) and μ a $(q^w - 1)$–th root of unity in $\mathbf{k}^\times$. We define*

$$N_{F^w/F} : \mathbf{T}_{\mathrm{u}}^{F^w} \to \mathbf{T}_{\mathrm{u}}^F, \quad t \mapsto tF(t)F^2(t)\dots F^{w-1}(t)$$

For our duality isomorphism $\delta : X(\mathbf{T}_{\mathrm{u}}) \to Y(\mathbf{T}_{\mathrm{u}}^*)$ there exists a unique "dual" duality isomorphism $\delta^\vee : Y(\mathbf{T}_{\mathrm{u}}) \to X(\mathbf{T}_{\mathrm{u}}^*)$, see [Car85, Proposition 4.3.1] with similar properties as those of δ stated in Definition 7.1. We have [DM91, Proposition 13.7]:

Proposition 7.8. *There is a canonical isomorphism of $\mathrm{Irr}(\mathbf{T}_{\mathrm{u}}^F)$ with $\mathbf{T}_{\mathrm{u}}^{*F^*}$ which is given by*

$$d : \mathrm{Irr}(\mathbf{T}_{\mathrm{u}}^F) \to \mathbf{T}_{\mathrm{u}}^{*F^*}, \quad \theta \mapsto N_{F^{*m}/F^*}\left(\delta^\vee(\hat{\theta})(\mu)\right),$$

where we identify θ with an element of $X(\mathbf{T}_{\mathrm{u}}^F)$ and let $\hat{\theta}$ be any extension of $\theta \in X(\mathbf{T}_{\mathrm{u}}^F)$ to $X(\mathbf{T}_{\mathrm{u}})$.

II. The p'–characters of the normalizer

In this chapter we investigate the p'–characters of the normalizer of the fixed Sylow p–subgroup $\mathbf{U}^F$ of $\mathbf{G}_{\mathrm{sc}}^F$ and $\mathbf{G}_{\mathrm{u}}^F$. We proceed in several steps: First we consider the normalizer $\mathbf{B}_{\mathrm{u}}^F$ in the universal group where F is a standard Frobenius map in Section 8. Here we can construct labels parametrizing the characters of degree prime to p (Theorem 8.6). We then determine the action of automorphisms on our parametrization in Section 9 (Proposition 9.4). Next we generalize our construction to the twisted cases in Section 10 (Theorem 10.8). Finally we investigate the relationship between the labels of $\mathrm{Irr}_{p'}(\mathbf{B}_{\mathrm{u}}^F)$ and the p'–characters of the normalizer $\mathbf{B}_{\mathrm{sc}}^F$ in the simply connected group using Clifford theory in Section 11. Section 11 provides the two main results needed later on the action of linear characters (Proposition 11.3 and Proposition 11.4) and then goes on to further explore properties of $\mathrm{Irr}_{p'}(\mathbf{B}_{\mathrm{sc}}^F)$ for which we do not have direct proofs of equivalent statements in $\mathrm{Irr}_{p'}(\mathbf{G}_{\mathrm{sc}}^F)$ (see Theorem 11.8 and Proposition 11.13).

8. Parametrizing $\mathrm{Irr}_{p'}(\mathbf{B}_{\mathrm{u}}^{F_q})$

As explained in Chapter I the Borel subgroup $\mathbf{B}_{\mathrm{u}}^F$ is the normalizer of $\mathbf{U}^F$ in $\mathbf{G}_{\mathrm{u}}^F$ and $\mathbf{U}^F$ is a Sylow p–subgroup of $\mathbf{G}_{\mathrm{u}}^F$ and $\mathbf{B}_{\mathrm{u}}^F$. We want to parametrize the characters of $\mathbf{B}_{\mathrm{u}}^F$ of degree prime to p. Those must lie above linear characters of $\mathbf{U}^F$, since all other characters of $\mathbf{U}^F$ have degrees divisible by p. Using these characters and Clifford theory we obtain labels for $\mathrm{Irr}_{p'}(\mathbf{B}_{\mathrm{u}}^F)$. In this section we shall only be concerned with the case where $F = F_q$, i.e., when F is a standard Frobenius map, we will generalize this construction in Section 10.

First we are looking for a set of elements $t_i \in \mathbf{T}_{\mathrm{u}}^{F_q}$ that generate $\mathbf{T}_{\mathrm{u}}^{F_q}$ and act in "diagonal fashion" on the root subgroups U_{α_j}. For $i = 1, \dots, n$ we define $t_i := \omega_i^\vee(\mu)$, where $\langle \mu \rangle = \mathbb{F}_q^\times$. By the definition of F_q, the $\omega_i^\vee$, the β_i and $\rho : Z \to \mathcal{Z}$ we have

$$F_q(t_i) = \beta_i(\nu^q)\omega_{\mathsf{k}}(\beta_i(\nu^q)) = t_i\beta_i(\zeta)\omega_{\mathsf{k}}(\beta_i(\zeta)) = t_i.$$

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Proposition 8.1. *Assume $F = F_q$. Together with $t_0 := 1_{\mathbf{G}_{sc}}$ ($r = 0$), $t_0 := \mu \in \mathbf{T}_u^F$ ($r = 1$) or $t_0 := [t_{0^{(1)}}, t_{0^{(2)}}] := [(1, \mu), (\mu, 1)]$ ($r = 2$) the t_i for $i = 1, \ldots, n$ generate $\mathbf{T}_u^F$. We have*
$$x_{\alpha_j}(u)^{t_i} = x_{\alpha_j}(\mu^{\delta_{ij}} u) \text{ for } i, j \in \{1, \ldots, n\}.$$
The t_i all have order $q - 1$ (unless $t_0 = 1$, of course).

Proof. The statement about the action of the t_i is Proposition 6.11. The formula shows that the $(q - 1)^n$ products $\prod_{i=1}^n t_i^{f_i}$, $0 \leq f_i \leq q - 1$, act differently on the $x_{\alpha_j}(u)$, so they are all distinct. None of them are central except the trivial element. Multiplying with the $(q - 1)$ (resp. $(q - 1)^2$) central elements in $\langle t_0 \rangle$ we obtain $(q - 1)^{n+1}$ (resp. $(q - 1)^{n+2}$) elements of $\mathbf{T}_u^F$. Now $t_i = F_q(t_i) = t_i^q$ so $t_i^{q-1} = 1$. $\square$

Assumption 8.2. *For all that follows we require*

$$[\mathbf{U}^F, \mathbf{U}^F] = \left(\prod_\beta \mathbf{U}_\beta \right)^F$$

where β runs over $\Phi^+ \setminus \Delta$.

This is shown to be true using the commutator formula from Section 1, except when $\mathbf{G}_u^F$ is one of the groups in Table 13.2, see [How73, Lemma 7]. We shall only consider groups for which Assumption 8.2 holds. Using Assumption 8.2 and the commutator formula from Section 1 we see that $\mathbf{U}^F / [\mathbf{U}^F, \mathbf{U}^F] \cong \oplus_{i=1}^n \mathbf{U}_{\alpha_i}^F$, since $F = F_q$ stabilizes all the root subgroups. A character of $\mathbf{U}^F$ is linear if and only if it factors through
$$\tau := (\tau_1, \ldots, \tau_n) : \mathbf{U}^F / [\mathbf{U}^F, \mathbf{U}^F] \to (\mathbb{F}_q, +)^n,$$
where
$$\tau_i : \mathbf{U}^F / [\mathbf{U}^F, \mathbf{U}^F] \to \mathbb{F}_q, \ \tau_i \left(\prod_{j=1}^n x_{\alpha_j}(u_j) \right) := u_i.$$

Lemma 8.3. *For any non–trivial (complex) character ϕ of $(\mathbb{F}_q, +)$, the whole character group of $(\mathbb{F}_q, +)$ is given by the characters $u \mapsto \phi(tu)$ for $t \in \mathbb{F}_q^\times$.*

Proof. Suppose $\phi(au) = \phi(bu)$ for some $a, b \in \mathbb{F}_q^\times$ and all $u \in \mathbb{F}_q$. Multiplying with elements of $\mathbb{F}_q^\times$ is an automorphism of $(\mathbb{F}_q, +)$ therefore $\phi(ab^{-1}u) = \phi(u)$ for all $u \in \mathbb{F}_q$. It follows $ab^{-1}u - u = (ab^{-1} - 1)u \in \ker \phi$ for all $u \in \mathbb{F}_q$. So either $a = b$ or we obtain a contradiction to the fact that ϕ is non–trivial. So the $(q - 1)$ non–trivial characters of $(\mathbb{F}_q, +)$ are in fact given by $u \mapsto \phi(tu)$. $\square$

Now fix any such non–trivial character ϕ of $(\mathbb{F}_q, +)$. We shall later refine that choice of ϕ when we consider the action of field automorphisms, but for now any shall suffice.

Set
$$\phi_i := \phi \circ \tau_i$$
and for any $S \subseteq \{1, \ldots, n\}$
$$\phi_S := \prod_{i \in S} \phi_i.$$

Proposition 8.4. *The ϕ_S, where S ranges over all subsets of $\{1, \ldots, n\}$, form a complete set of representatives for the orbits of the action of $\mathbf{B}_{\mathrm{u}}^F$ on the linear characters of $\mathbf{U}^F$. (The empty product corresponds to the trivial character).*

Proof. The action of $\mathbf{B}_{\mathrm{u}}^F$ on $\mathbf{U}^F / [\mathbf{U}^F, \mathbf{U}^F]$ is given by the action of $\mathbf{T}_{\mathrm{u}}^F \cong \mathbf{B}_{\mathrm{u}}^F / \mathbf{U}^F$ on $\mathbf{U}^F / [\mathbf{U}^F, \mathbf{U}^F]$. The t_i from Proposition 8.1 are a system of generators for $\mathbf{T}_{\mathrm{u}}^F$. By Proposition 8.1 and Lemma 8.3 we know that t_i acts transitively on the characters with kernel $\ker \tau_i$ and trivial on those with kernel $\ker \tau_j$ for $i \neq j$. Every character with kernel $[\mathbf{U}^F, \mathbf{U}^F]$ is uniquely a product of characters with kernel $\ker \tau_i$ for $i = 1, \ldots, n$. $\square$

By the construction it is clear that the inertia group of ϕ_S in $\mathbf{B}_{\mathrm{u}}^F$ is given by
$$I_S := I_{\mathbf{B}_{\mathrm{u}}^F}(\phi_S) = \langle t_i \mid i \in S^c \rangle \mathbf{U}^F$$
where
$$S^c := \{0, 1, \ldots, n\} \setminus S.$$

Lemma 8.5. *The characters of I_S above ϕ_S are linear, in particular, every character ϕ_S extends to I_S.*

Proof. The characters of I_S above ϕ_S must lie above the trivial character of $\ker(\phi_S)$. But $I_S / \ker(\phi_S)$ is abelian: the t_i involved in I_S act non–trivial only on the kernel of ϕ_S. $\square$

Now any character $\chi \in \mathrm{Irr}_{p'}(\mathbf{B}_{\mathrm{u}}^F)$ lies above a linear character of $\mathbf{U}^F$ (since $\mathbf{U}^F$ is a p–group), so above some ϕ_S by Proposition 8.4 and by Clifford theory there must be a unique character ψ of I_S above ϕ_S that induces to χ. We shall use this to associate labels to the characters $\mathrm{Irr}_{p'}(\mathbf{B}_{\mathrm{u}}^F)$. Define a map $\widetilde{f} : \mathrm{Irr}_{p'}(\mathbf{B}_{\mathrm{u}}^F) \to \mathbb{C}^n$ by

$$\widetilde{f}(\chi)_i := \begin{cases} \psi(t_i) & \text{if } i \in S^c, \\ 0 & \text{if } i \in S. \end{cases}$$

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Now all the t_i are $(q-1)$–th roots of unity (by Proposition 8.1) and so must be their images. We once and for all fix an embedding of $\mathbb{F}_q^\times$ into the complex numbers (e.g. $\mu \mapsto e^{\frac{2\pi i}{q-1}}$) and define $f : \mathrm{Irr}_{p'}(\mathbf{B}_u^F) \to \mathbb{F}_q^n$ by the composition of this identification and $\widetilde{f}$.

Theorem 8.6. *The map f is a bijection of $\mathrm{Irr}_{p'}(\mathbf{B}_u^F)$ with the set of labels $\mathbb{F}_q^\times \times (\mathbb{F}_q)^n$ (or $(\mathbb{F}_q^\times)^2 \times (\mathbb{F}_q)^n$ in case D_{even}, or $\{1\} \times (\mathbb{F}_q)^n$ if $\mathsf{r} = 1$).*

Proof. The map f is clearly injective. We have $f(\chi)_0 \neq 0$ for all $\chi \in \mathrm{Irr}_{p'}(\mathbf{B}_u^F)$ since $t_0 \in \mathbf{Z}(\mathbf{B}_u)$ is always contained in the inertia group I_S. For a given label $(a_0, \dots, a_n)$, we first determine the set $S = \{i \mid a_i = 0\}$. The a_i with $i \in S^c$ then determine a linear character ψ of I_S above ϕ_S which we can induce to $\mathbf{B}_u^F$, where it must be irreducible by Clifford theory and of degree prime to p by construction. $\qquad \square$

9. Automorphisms and labels

We described the automorphisms of $\mathbf{G}_{\mathrm{sc}}^{F}$ and their extensions to $\mathbf{G}_{\mathrm{u}}^{F}$ in Section 4 and Section 7. In this section we describe the action of automorphisms on $\mathrm{Irr}_{p'}(\mathbf{B}_{\mathrm{u}}^{F})$ by computing their action on the labels constructed in the previous section. The diagonal automorphisms of $\mathbf{G}_{\mathrm{sc}}^{F}$ are inner automorphisms of $\mathbf{G}_{\mathrm{u}}^{F}$ so those act trivial on $\mathrm{Irr}_{p'}(\mathbf{B}_{\mathrm{u}}^{F})$ and the labels. This leaves us with field and graph automorphisms. We still assume $F = F_{q}$, i.e., the untwisted case.

Proposition 9.1. *Let Γ be a graph automorphism of $\mathbf{G}_{\mathrm{u}}$ and Γ^{*} the dual automorphism of $\mathbf{G}_{\mathrm{u}}^{*}$ (see Section 7). The automorphisms act on $X(\mathbf{T}_{\mathrm{u}})$ and $Y(\mathbf{T}_{\mathrm{u}})$ (resp. $X(\mathbf{T}_{\mathrm{u}}^{*})$ and $Y(\mathbf{T}_{\mathrm{u}}^{*})$) as described in Section 4. We have*

$$\Gamma^{*}(\hat{\omega}_{i}^{*}) = \hat{\omega}_{\Gamma^{-1}(i)}^{*} + \mathsf{d}_{i} \det\nolimits^{*}$$

and

$$\Gamma(\omega_{i}^{\vee}) = \omega_{\Gamma^{-1}(i)}^{\vee} + \mathsf{d}_{i} \, \mathbf{z}$$

for some $\mathsf{d}_{i} \in \mathbb{Z}^{\mathsf{r}}$. In case $\mathsf{D}_{\mathrm{even}}$ take the sum over the two components of $\mathsf{d}_{i} \det^{*}$ resp. $\mathsf{d}_{i} \, \mathbf{z}$.

Proof. Recall from Section 7 that if there exists a non–trivial graph automorphism then $\mathbf{G}_{\mathrm{u}} \cong \mathbf{G}_{\mathrm{u}}^{*}$, both contain $\mathbf{G}_{\mathrm{sc}}$, $\Gamma^{*} = \Gamma^{-1}$ on $\{1, \ldots, n\}$ and the restriction of $\hat{\omega}_{i}^{*}$ to $\mathbf{T}_{\mathrm{sc}}$ is ω_{i}. We have $\Gamma^{*}(\omega_{i}) = \omega_{\Gamma^{*}(i)} = \omega_{\Gamma^{-1}(i)}$ by Lemma 4.1. Since the kernel of the restriction map from $X(\mathbf{T}_{\mathrm{u}}^{*})$ to $X(\mathbf{T}_{\mathrm{sc}}^{*})$ is generated by $\det^{*}$ and stabilized by Γ, the first equation follows. The second is obtained by applying the duality isomorphism δ from Proposition 7.5. $\square$

By Proposition 9.1 and the definition of the t_{i} in Section 8 we have:

Corollary 9.2.

$$\Gamma(t_{i}) = \Gamma(\omega_{i}^{\vee}(\mu)) = \omega_{\Gamma^{-1}(i)}^{\vee}(\mu) \, \mathbf{z}(\mu)^{\mathsf{d}_{i}} = t_{\Gamma^{-1}(i)} t_{0}^{\mathsf{d}_{i}}.$$

Example 9.3. In type A_{n} the action of the graph automorphism on the t_{i} from Section 8 is given by

$$\Gamma(t_{0}) = t_{0}^{-1} \text{ and}$$
$$\Gamma(t_{i}) = t_{0}^{-1} t_{n+1-i} \text{ for } i = 1, \ldots, n.$$

In type D_{n} with $n = 2m + 1$ it is given by

$$\Gamma(t_{0}) = t_{0}^{-1},$$
$$\Gamma(t_{i}) = t_{0}^{-1} t_{i} \text{ for } i = 1, \ldots, n - 2 \text{ and}$$

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$$\Gamma(t_n) = t_{n-1}.$$

In type D_n with $n = 2m$ it is given by

$$\Gamma(t_{0^{(1)}}) = t_{0^{(2)}},$$
$$\Gamma(t_i) = t_i \text{ for } i = 1, \ldots, n-2 \text{ and}$$
$$\Gamma(t_n) = t_{n-1}.$$

Proposition 9.4. *Let $(a_0, \ldots, a_n)$ be the label of $\chi \in \mathrm{Irr}_{p'}(\mathbf{B}_{\mathrm{u}}^F)$ and $(a'_0, \ldots, a'_n)$ the label of χ^σ where σ is an automorphism of $\mathbf{G}_{\mathrm{u}}^F$ that stabilizes $\mathbf{B}_{\mathrm{u}}^F$ and χ^σ is defined by $\chi^\sigma(g) := \chi(\sigma(g))$.*

- *If $\sigma = F_p$ is a field automorphism, we have*

$$(a'_0, \ldots, a'_n) = (a_0^p, a_1^p, \ldots, a_n^p).$$

- *If $\sigma = \Gamma$ is a graph automorphism, we have*

$$(a'_0, \ldots, a'_n) = (\Gamma_{\mathcal{Z}}(a_0), a_0^{d_1} a_{\Gamma^{-1}(1)}, \ldots, a_0^{d_n} a_{\Gamma^{-1}(n)}),$$

  with the d_i from Proposition 9.1 and $\Gamma_{\mathcal{Z}}$ denoting the action of Γ on $\left(\mathbb{F}_q^\times\right)^r$ induced by $\mathbf{z}^{-1}$ as described explicitly in Table 6.9.

Proof. Recall the definitions of ϕ_S and ϕ_i from Section 8. Here $S := \{i \mid a_i = 0\}$, χ lies above ϕ_S and all ϕ_S^g have inertia group $I_S = \{t_i \mid i \in S^c\}$ in $\mathbf{B}_{\mathrm{u}}^F$. There is a unique linear character ψ of I_S lying above ϕ_S that induces to χ. The values of ψ on the t_i with $i \in S^c$ correspond to the $a_i \neq 0$ and $a_j = 0$ for all $j \in S$.

Since the automorphism σ stabilizes $\mathbf{U}^F$ it also acts on the characters of $\mathbf{U}^F$. For the field automorphism we have $\sigma(x_\alpha(u)) = x_\alpha(u^p)$, so the kernel of ϕ_i^σ is equal to $\ker(\phi_i)$. Therefore ϕ_i^σ must be conjugate to ϕ_i in $\mathbf{B}_{\mathrm{u}}^F$. Then χ^σ lies above ϕ_S as well. The unique character of I_S above ϕ_S that induces to χ^σ must be ψ^σ. By the construction of the a_i and using $\sigma(t_i) = t_i^p$ we obtain the stated action of σ on the label.

The action of Γ on the set $\{\phi_i\}$ is given by $\phi_i \mapsto \phi_{\Gamma(i)}$, therefore $\phi_S^\Gamma = \phi_{\Gamma(S)}$. Then χ^Γ lies above $\phi_{\Gamma(S)}$ and the unique character of $I_{\Gamma(S)}$ that induces to χ^Γ is ψ^Γ, which is indeed a character of $I_{\Gamma(S)}$: For $i \in \Gamma(S)$ we have

$$\psi^\Gamma(t_i) = \psi(t_{\Gamma^{-1}(i)} t_0^{d_i})$$

by Corollary 9.2 and $t_{\Gamma^{-1}(i)} \in I_S$. Therefore $a'_i = a_{\Gamma^{-1}(i)} a_0^{d_i}$. The statement about a'_0 is immediate from the definition of $t_0 = \mathbf{z}(\mu)$. The zero positions of the label $(a'_0, \ldots, a'_n)$ must be $\Gamma(S)$. $\qquad\square$

Since the exceptional automorphisms of $B_2(2^f)$, $F_4(2^f)$ and $G_2(3^f)$ are treated elsewhere, we don't consider them here. However our method generalizes to these as well, without proof:

Example 9.5. The exceptional automorphism γ in type B_2 with $F = F_{2^f}$ acts on the labels of characters of $\mathbf{B}_\mathrm{u}^F$ as well. Since $F = F_{2^f}$ the center of $\mathbf{G}_\mathrm{sc}^F$ is trivial and we may consider $\mathbf{G}_\mathrm{sc}^F$ as the universal group (so we don't have to define an action of γ on $\mathcal{Z}$). The action of γ and γ^{-1} on the labels is given by

$$(1, a_1, a_2) \mapsto (1, a_2, a_1^2)$$

and

$$(1, a_1, a_2) \mapsto (1, \sqrt{a_2}, a_1),$$

which is well defined, since the entries of the labels are elements of $\mathbb{F}_{2^f}$. Note $\gamma^2 = F_2$.

10. Parametrizing $\mathrm{Irr}_{p'}(\mathbf{B}_{\mathrm{u}}^F)$

We now consider the case when $F = \Gamma \circ F_q$ and Γ is a non-trivial graph–automorphism. Let w be the order of Γ. We have possible orders $w = 2$ (types A_n, D_n, E_6) and $w = 3$ (type D_4). The graph automorphism commutes with F_q, therefore $F^w = (F_q)^w = F_{q^w}$ and $\mathbf{G}_{\mathrm{u}}^F \le \mathbf{G}_{\mathrm{u}}^{F_{q^w}}$. This enables us to use the previous results for the untwisted case. We assume the results and notation from Section 8 for F_{q^w}, so μ is a fixed generator of $\mathbb{F}_{q^w}^\times$.

The action of Γ divides Δ into orbits. Let $\cup_{i=1}^m A_i = \{1, \ldots, n\}$ be the corresponding partition of the index set. For a Γ–orbit A, we set $\mathbf{U}_A = \langle \mathbf{U}_{\alpha_i} \mid i \in A \rangle$. The $\mathbf{U}_A$ take the role of the $\mathbf{U}_{\alpha_i}$ in the untwisted case. By the commutator relations of Section 1

$$\Gamma\left(\prod_{i=1}^n x_{\alpha_i}(u_{\alpha_i})\right) = \prod_{i=1}^n x_{\alpha_i}(u_{\Gamma^{-1}(\alpha_i)})\Theta$$

for some $\Theta \in [\mathbf{U}, \mathbf{U}]$. Therefore (using Assumption 8.2 and thus excluding the groups from Table 13.2)

$$\mathbf{U}^F / \left[\mathbf{U}^F, \mathbf{U}^F\right] \cong \oplus_{i=1}^m U_{A_i},$$

where $U_A := \mathbf{U}_A^F / \left[\mathbf{U}^F, \mathbf{U}^F\right]$.

Lemma 10.1. *Each of the U_A is isomorphic to $(\mathbb{F}_{q^{|A|}}, +)$. For every orbit A_i fix $\mathsf{a}_i \in A_i$, then such an isomorphism is given by*

$$\prod_{j=0}^{|A|-1} x_{\Gamma^{-j}(\alpha_{\mathsf{a}_i})}(u_j) \mapsto u_0.$$

Proof. Since the elements of U_A are F–invariant we have $u_0^q = u_1, \ldots, u_{|A|-2}^q = u_{|A|-1}$, $u_{|A|-1}^q = u_0$. Thus $u_0^{q^{|A|}} = u_0$ and the other u_i are just conjugates of u_0. $\qquad\square$

We set
$$t_i := N_{F^w/F}(\omega_{\mathsf{a}_i}^\vee(\mu)) \quad \text{for } i = 1, \ldots, m,$$

where $\omega_i^\vee \in X(\mathbf{T}_{\mathrm{u}})$ is defined as in Section 6 and $N_{F^w/F} : \mathbf{T}_{\mathrm{u}}^{F^w} \to \mathbf{T}_{\mathrm{u}}^F$ is defined by $N_{F^w/F}(t) := tF(t) \ldots F^{w-1}(t)$ as in Section 7. Note that this coincides with the definition of the t_i in the untwisted case if Γ is trivial.

Proposition 10.2. *For fixed $i \in \{1, \ldots, m\}$ the action of t_i on U_{A_j} with $j \ne i$ is trivial and the action on U_{A_i} is given by multiplication with $\mu^{\frac{q^w-1}{q^{|A_i|}-1}} \in \{\mu, \mu^{q+1}, \mu^{q^2+q+1}\}$ (a generator of $\mathbb{F}_{q^{|A_i|}}^\times$). Together with $t_0 := N_{F^w/F}(\mathbf{z}(\mu))$ the t_i for $i = 1, \ldots, m$ generate $\mathbf{T}_{\mathrm{u}}^F$ (in case $\mathsf{D}_{\mathrm{even}}$ we set $t_0 := N_{F^w/F}(\mathbf{z}_1(\mu))$).*

Proof. We have $\Gamma(\omega_{i}^{\vee}) = \omega_{\Gamma^{-1}(\mathsf{a}_{i})}^{\vee} + \mathsf{d}_{i}\,\mathbf{z}$ by Proposition 9.1. Thus the statement about the trivial action on $U_{A_{j}}$ with $j \neq i$ is obvious. When determining the action on $U_{A_{i}}$ we can ignore the $\mathsf{d}_{i}\,\mathbf{z}$ part since it is central. Thus t_{i} acts on $U_{A_{i}}$ in the same way as $\prod_{r=0}^{w-1} \omega_{\Gamma^{-r}(\mathsf{a}_{i})}^{\vee}(\mu^{q^{r}})$ does. We consider the isomorphism of Lemma 10.1 to describe the action. All $\omega_{\Gamma^{-r}(\mathsf{a}_{i})}^{\vee}(\mu^{q^{r}})$ act trivially on $\mathbf{U}_{\alpha_{\mathsf{a}_{i}}}$ except for those with $\Gamma^{-r}(\mathsf{a}_{i}) = \mathsf{a}_{i}$. We save the reader and ourselves the general formula and point out that for all types and orbits either $\Gamma(\mathsf{a}_{i}) = \mathsf{a}_{i}$ or $\Gamma^{r}(\mathsf{a}_{i}) \neq \mathsf{a}_{i}$ for $r = 1, \dots, w-1$. The statement about the generation of $\mathbf{T}_{\mathrm{u}}^{F}$ follows as in the untwisted case in Proposition 8.1 by counting the number of distinct actions of products of the t_{i} and finally multiplying by the number of central elements, all generated by t_{0}. $\qquad\square$

The t_{i} act as desired, but they have small defects compared to those in the untwisted case: their orders are not necessarily all the same and the first power acting trivial on all U_{A} need not be the trivial element, it can in fact be central. Using the fact that t_{0} generates $\mathcal{Z}^{F}$, we obtain relations between the t_{i} and t_{0} which shall carry over to our parametrization.

Remark 10.3. *Since* $t_{i}^{q^{|A_{i}|}-1}$ *acts trivial on all the* U_{A} *by its definition, it must be a central element of* $\mathbf{B}_{\mathrm{u}}^{F}$. *Then there exist* $\mathsf{r}_{i} \in \mathbb{Z}$ *such that* $t_{0}^{\mathsf{r}_{i}} = t_{i}^{q^{|A_{i}|}-1}$.

We illustrate our notation in the following individual examples:

Example 10.4. Type A_{n}. Here $\Gamma(\alpha_{i}) = \alpha_{n+1-i}$, $w = 2$, $A_{i} = \{\alpha_{i}, \alpha_{n+1-i}\}$ for $i \in \{1, \dots, \lceil n/2 \rceil\}$. We have $F(z) = z^{-q}$ for $z \in \mathcal{Z}$ and $t_{0} = \mu^{1-q}$. Let $n = 2m$ or $n = 2m+1$, we have $|A_{i}| = 2$ and $t_{i}^{q-1} = 1$ for $i \in \{1, \dots, m\}$. If $n = 2m+1$ then $|A_{m+1}| = 1$ and $t_{m+1}^{q-1} = t_{0}^{-1}$.

Example 10.5. Type D_{n} with Γ of order $w = 2$. Here $\Gamma(\alpha_{n-1}) = \alpha_{n}$, $A_{i} = \{\alpha_{i}\}$ for $i \in \{1, \dots, n-2\}$ and $A_{n-1} = \{\alpha_{n-1}, \alpha_{n}\}$. If $n = 2m-1$, then $F(z) = z^{-q}$ for $z \in \mathcal{Z}$ and $t_{0} = \mu^{1-q}$, $t_{n-1}^{q^{2}-1} = 1$, $t_{i}^{q-1} = t_{0}^{-1}$ for $i \in \{1, \dots, n-2\}$. If $n = 2m$, then $F((z_{1}, z_{2})) = (z_{2}^{q}, z_{1}^{q})$ for $(z_{1}, z_{2}) \in \mathcal{Z}$ and $t_{0} = (\mu, \mu^{q})$, $t_{n-1}^{q^{2}-1} = 1$, $t_{i}^{q-1} = 1$ for $i \in \{1, \dots, n-2\}$.

In type D_{4} with Γ of order $w = 3$, we have $\mathbf{G}_{\mathrm{u}}^{F} \cong \mathbf{G}_{\mathrm{sc}}^{F} \times \mathcal{Z}^{F}$, since F acts non–trivial on all non–trivial elements of $\mathbf{Z}(\mathbf{G}_{\mathrm{sc}}^{F})$ (see Example 2.2). For this particular case, it suffices to consider $\mathbf{G}_{\mathrm{sc}}$ as the "universal" group $\mathbf{G}_{\mathrm{u}}$ (so our notion of the universal group now depends on the graph–automorphism involved in the Frobenius map for D_{4}). This saves us a lot of unnecessary technicalities later on.

Definition 10.6. *When considering type* $^{3}\mathsf{D}_{4}$ *(that is* D_{4} *and* $F = F_{q} \circ \Gamma$ *where* Γ *is of order* 3*) we define the* t_{i} *differently than before. Using notation and definitions from Section 6, we set*

$$t_{0} := 1, \ t_{1} := N_{F^{3}/F}(\beta_{1}(\nu)) = \beta_{1}(\nu)\beta_{3}(\nu)^{q}\beta_{4}(\nu)^{q^{2}}, \ t_{2} := N_{F^{3}/F}(\beta_{2}(\nu)) = \beta_{2}(\nu)^{1+q+q^{2}}.$$

II. The p'–characters of the normalizer

This is well defined and all assertions of Proposition 10.2 remain true, since we changed the t_i only by a central element and $\mathbf{G}_{\mathrm{sc}}^F \cap \mathcal{Z}^F = 1$.

Example 10.7. Type D_4 with Γ of order $w = 3$.
$\Gamma(\alpha_1) = \alpha_3$, $\Gamma(\alpha_3) = \alpha_4$, $A_1 = \{\alpha_1, \alpha_3, \alpha_4\}$, $A_2 = \{\alpha_2\}$. We have $t_1^{q^3-1} = 1$ and $t_2^{q-1} = 1$.

Theorem 10.8. *There exists a map f from $\mathrm{Irr}_{p'}(\mathbf{B}_{\mathrm{u}}^F)$ to $\mathbb{F}_{q^w}^\times \times (\mathbb{F}_{q^w})^m$ that defines a bijection between $\mathrm{Irr}_{p'}(\mathbf{B}_{\mathrm{u}}^F)$ and all the tuples $(a_0, \ldots, a_m)$ that fullfill the relations $a_i^{q^{|A_i|}-1} = a_0^{\mathsf{r}_i}$ for $i = 1, \ldots, m$ and $\mathsf{r}_i \in \mathbb{Z}$ as defined in Remark 10.3. We call those tuples the labels of $\mathrm{Irr}_{p'}(\mathbf{B}_{\mathrm{u}}^F)$.*

Proof. We can construct f completely analogously as in Section 8. We replace the maps τ_i and ϕ_i by maps τ_{A_i} and ϕ_{A_i} and S by an analogously defined subset of $\{1, \ldots, m\}$. Everything holds as in the untwisted case, except for the statement about the orders of the t_i. But since ψ is a linear character of I_S the relations on the t_i carry over to the character values of ψ on the t_i and thus the labels. $\qquad\square$

It is clear that this is a complete generalization of the results of Section 8: If we simply consider a trivial graph automorphism and $A_i = \{i\}$, then the labels from Theorem 10.8 and Theorem 8.6 coincide.

11. Parametrizing $\mathrm{Irr}_{p'}(\mathbf{B}_{\mathrm{sc}}^F)$

We recall Notation 5.1. We have $\mathbf{B}_{\mathrm{sc}}^F \lhd \mathbf{B}_{\mathrm{u}}^F$ and $p \nmid |\mathbf{B}_{\mathrm{u}}^F : \mathbf{B}_{\mathrm{sc}}^F|$ so by Clifford theory (Section 5) there is a partition $\mathrm{Irr}_{p'}(\mathbf{B}_{\mathrm{u}}^F) = \cup_\chi M_\chi$ in one–to–one correspondence with a partition $\mathrm{Irr}_{p'}(\mathbf{B}_{\mathrm{sc}}^F) = \cup_\eta N_\eta$ such that

$$M_\chi := \mathrm{Irr}(\mathbf{B}_{\mathrm{u}}^F \mid \eta) \subseteq \mathrm{Irr}_{p'}(\mathbf{B}_{\mathrm{u}}^F)$$

and

$$N_\eta := \left\{ \eta^g \mid g \in \mathbf{B}_{\mathrm{u}}^F \right\} \subseteq \mathrm{Irr}_{p'}(\mathbf{B}_{\mathrm{sc}}^F),$$

for suitable $\eta \in \mathrm{Irr}(\mathbf{B}_{\mathrm{sc}}^F)$ and $\chi \in \mathrm{Irr}(\mathbf{B}_{\mathrm{u}}^F \mid \eta)$.

In this section we have two main goals: Firstly to expose the relationship between the labels of characters in M_χ and secondly to describe the related sets N_η and the action of automorphisms on those. As an application we compute the cardinalities $|\mathrm{Irr}_{p'}(\mathbf{B}_{\mathrm{sc}}^F)|$ (Theorem 11.8).

Notation 11.1. For $\eta \in \mathrm{Irr}(\mathbf{B}_{\mathrm{sc}}^F)$ and $\chi \in \mathrm{Irr}(\mathbf{B}_{\mathrm{u}}^F \mid \eta)$ from corresponding sets we shall also write $M_\chi = M_\eta$ and $N_\eta = N_\chi$. For fixed $\chi \in \mathrm{Irr}_{p'}(\mathbf{B}_{\mathrm{u}}^F)$ recall the definitions of S, ϕ_S and $I_S = I_{\mathbf{B}_{\mathrm{u}}^F}(\phi_S)$ from Section 8 and Section 10. The set $\{\phi_S^g \mid g \in \mathbf{B}_{\mathrm{u}}^F\}$ is the disjoint union of the orbits $[\phi_S^g]_{\mathbf{B}_{\mathrm{sc}}^F} := \{\phi_S^{gs} \mid s \in \mathbf{B}_{\mathrm{sc}}^F\}$ of the action of $\mathbf{B}_{\mathrm{sc}}^F$ on that set. We denote the set of all those orbits by

$$\mathcal{U}_S := \left\{ [\phi_S^g]_{\mathbf{B}_{\mathrm{sc}}^F} \mid g \in \mathbf{B}_{\mathrm{u}}^F \right\}.$$

Proposition 11.2. *For $\chi \in \mathrm{Irr}_{p'}(\mathbf{B}_{\mathrm{u}}^F)$ above ϕ_S there exists a unique $\eta \in N_\chi$ that lies above ϕ_S and we have*
$$I_{\mathbf{B}_{\mathrm{u}}^F}(\eta) = I_S \mathbf{B}_{\mathrm{sc}}^F.$$

Proof. Recall $\psi \in \mathrm{Irr}(I_S \mid \phi_S)$ the linear character from the construction of the labels, which by Clifford theory is the unique constituent of the restriction of χ to I_S above ϕ_S. Consider the restriction $\psi_{\mathrm{sc}} := \psi|_{I_{\mathrm{sc}}}$ of ψ to $I_{\mathrm{sc}} := I_S \cap \mathbf{B}_{\mathrm{sc}}^F$. The character ψ_{sc} must be the unique character of I_{sc} which is below χ and above ϕ_S, suppose ψ_{sc}' is another one. Then ψ_{sc}' must be the restriction of some ψ^g for fixed $g \in \mathbf{B}_{\mathrm{u}}^F$ since all characters of I_S below χ are given by $\{\psi^x \mid x \in \mathbf{B}_{\mathrm{u}}^F\}$, those are all linear characters by Lemma 8.5, thus ψ_{sc}' is an extension of ϕ_S. Therefore ψ^g is an extension of ϕ_S as well. But then $\psi^g = \psi$ by the uniqueness of ψ. Now we consider Clifford theory from $\mathbf{U}^F$ to $\mathbf{B}_{\mathrm{sc}}^F$. The inertia group of ϕ_S in $\mathbf{B}_{\mathrm{sc}}^F$ must be $I_{\mathrm{sc}} = I_{\mathbf{B}_{\mathrm{u}}^F}(\phi_S) \cap \mathbf{B}_{\mathrm{sc}}^F$ and there is a bijection given by induction between $\mathrm{Irr}(I_{\mathrm{sc}} \mid \phi_S)$ and $\mathrm{Irr}(\mathbf{B}_{\mathrm{sc}}^F \mid \phi_S)$. Restriction of χ to $\mathbf{B}_{\mathrm{sc}}^F$ must have a constituent $\eta \in \mathrm{Irr}(\mathbf{B}_{\mathrm{sc}}^F \mid \phi_S)$ that has ψ_{sc} as a constituent in its restriction to I_{sc}, since ψ_{sc} is an constituent of the restriction of χ to I_{sc}. We have $\psi_{\mathrm{sc}}^{\mathbf{B}_{\mathrm{sc}}^F} = \eta$. Now suppose there exists a second character η' of $\mathbf{B}_{\mathrm{sc}}^F$ which is below χ

and above ϕ_S. Then there exists $\psi'_{\mathrm{sc}} \in \mathrm{Irr}(I_{\mathrm{sc}} \,|\, \phi_S)$ with $\psi'^{\mathbf{B}_{\mathrm{sc}}^F}_{\mathrm{sc}} = \eta'$ and ψ'_{sc} is below χ and above ϕ_S as well. Thus $\psi'_{\mathrm{sc}} = \psi_{\mathrm{sc}}$ and $\eta = \eta'$ by the uniqueness of ψ_{sc}.

For the equality $I_{\mathbf{B}_{\mathrm{u}}^F}(\eta) = I_S \mathbf{B}_{\mathrm{sc}}^F$ assume $g \in I_{\mathbf{B}_{\mathrm{u}}^F}(\eta)$. Then $\eta^g = \eta$ and thus $\phi_S^g \in [\phi_S]_{\mathbf{B}_{\mathrm{sc}}^F}$, so $\phi_S^g = \phi_S^s$ for some $s \in \mathbf{B}_{\mathrm{sc}}^F$. Now $gs^{-1} \in I_S$, so $g \in I_S \mathbf{B}_{\mathrm{sc}}^F$. Conversely assume $g = ts$ with $t \in I_S$ and $s \in \mathbf{B}_{\mathrm{sc}}^F$. Then $\phi_S^t = \phi_S$ and $\phi_S^{ts} \in [\phi_S]_{\mathbf{B}_{\mathrm{sc}}^F}$. But η is the unique character of N_χ above $[\phi_S]_{\mathbf{B}_{\mathrm{sc}}^F}$, thus $ts \in I_{\mathbf{B}_{\mathrm{u}}^F}(\eta)$. $\qquad\square$

The following diagram illustrates the relationships between the various characters and groups for $g \in \mathbf{B}_{\mathrm{u}}^F$ and $s \in \mathbf{B}_{\mathrm{sc}}^F$:
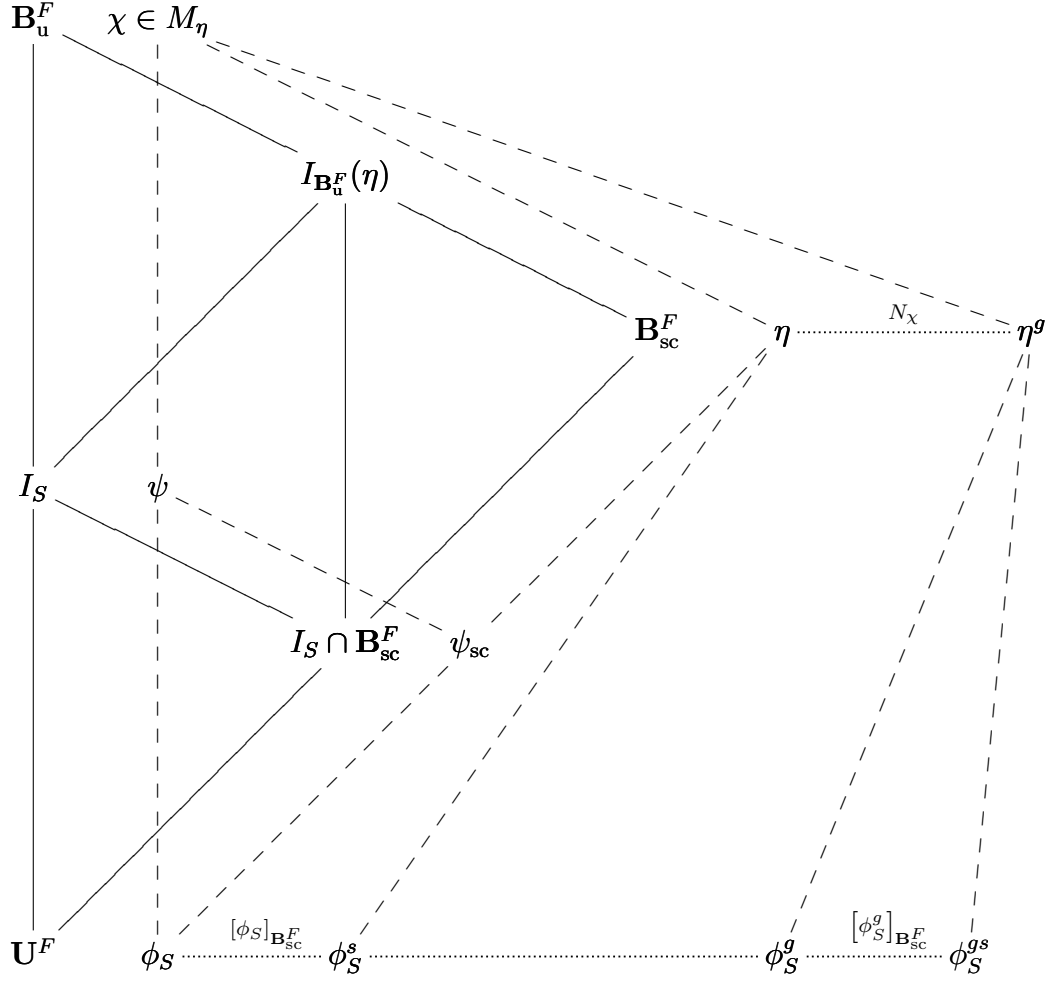
Here ψ is the unique character of I_S above ϕ_S that induces to χ as defined in Section 8 and ψ_{sc} is the restriction of ψ as in the proof of Proposition 11.2 above. The values of ψ on t_i for $i \in S^c$ define the values of the i–th position in the label of χ, whereas S is the set of zero positions in the label.

Proposition 11.3. *For* $\chi \in \mathrm{Irr}_{p'}(\mathbf{G}_{\mathrm{u}}^F)$ *we have*

$$M_\chi = \left\{ \chi\hat{\lambda} \;\middle|\; \hat{\lambda} \in \mathrm{Irr}(\mathbf{B}_{\mathrm{u}}^F/\mathbf{B}_{\mathrm{sc}}^F) \right\}.$$

Let $\hat{\lambda} \in \mathrm{Irr}(\mathbf{B}_{\mathrm{u}}^F \mid 1_{\mathbf{G}_{\mathrm{sc}}^F}) = \mathrm{Irr}(\mathbf{B}_{\mathrm{u}}^F/\mathbf{B}_{\mathrm{sc}}^F)$ *be a linear character of* $\mathbf{B}_{\mathrm{u}}^F$. *Then the label of* $\chi\hat{\lambda}$ *is just the label of* χ *multiplied componentwise by the label of* $\hat{\lambda}$.

Proof. Unless in type $\mathsf{D}_{\mathrm{even}}$ the factor group $\mathbf{B}_{\mathrm{u}}^F/\mathbf{B}_{\mathrm{sc}}^F$ is cyclic and the assertion about M_χ then follows by Clifford theory (Theorem 5.4+Theorem 5.3+Theorem 5.2). In case $\mathsf{D}_{\mathrm{even}}$ we need to prove that all $\eta \in \mathrm{Irr}_{p'}(\mathbf{B}_{\mathrm{sc}}^F)$ extend to their respective inertia groups, in which case the assertion follows again by Theorem 5.3 and Theorem 5.2. For $\eta \in N_\chi$ from Proposition 11.2 we have

$$|\mathbf{B}_{\mathrm{u}}^F : I_S| = \chi(1) = e|\mathbf{B}_{\mathrm{u}}^F : I_{\mathbf{B}_{\mathrm{u}}^F}(\eta)|\eta(1) = e|\mathbf{B}_{\mathrm{u}}^F : I_{\mathbf{B}_{\mathrm{u}}^F}(\eta)||\mathbf{B}_{\mathrm{sc}}^F : \mathbf{B}_{\mathrm{sc}}^F \cap I_S|$$

for some $e \in \mathbb{N}$ since ϕ_S extends to I_S and $I_S \cap \mathbf{B}_{\mathrm{sc}}^F$ by Lemma 8.5. By Proposition 11.2 we have $I_{\mathbf{B}_{\mathrm{u}}^F}(\eta) = I_S\mathbf{B}_{\mathrm{sc}}^F$ and by a homomorphism theorem

$$|I_S\mathbf{B}_{\mathrm{sc}}^F/\mathbf{B}_{\mathrm{sc}}^F| = |I_S/I_S \cap \mathbf{B}_{\mathrm{sc}}^F|,$$

thus $|\mathbf{B}_{\mathrm{u}}^F : I_S| = |\mathbf{B}_{\mathrm{u}}^F : I_S\mathbf{B}_{\mathrm{sc}}^F||\mathbf{B}_{\mathrm{sc}}^F : I_S \cap \mathbf{B}_{\mathrm{sc}}^F|$ (see the above diagram). Therefore $e = 1$ and η extends to $I_S\mathbf{B}_{\mathrm{sc}}^F$.

The character $\hat{\lambda}$ lies above the trivial character $\phi_\emptyset$ of $\mathbf{U}^F$. Therefore its inertia group in $\mathbf{B}_{\mathrm{u}}^F$ is $\mathbf{B}_{\mathrm{u}}^F$ itself and the label is defined by the values of $\hat{\lambda}$ on all t_i, $i = 1, \ldots, m$ (and m is the number of distinct orbits of the graph automorphism involved in F on $\{1, \ldots, n\}$ as in the Section 10).

Both χ and $\hat{\lambda}\chi$ must lie above the the same ϕ_S of $\mathbf{U}^F$ since $\mathbf{U}^F \subseteq \mathbf{B}_{\mathrm{sc}}^F = \ker \hat{\lambda}$. If ψ is the unique character of I_S above ϕ_S that induces to χ, then $\psi\left(\hat{\lambda}|_{I_S}\right)$ must be the one that induces to $\chi\hat{\lambda}$. For $i \in S^c$ that is the assertion of the proposition. But the other entries of χ's and $\hat{\lambda}\chi$'s label are zero by definition. $\square$

The label of a linear character $\hat{\lambda}$ of $\mathbf{B}_{\mathrm{u}}^F$ with $\mathbf{B}_{\mathrm{sc}}^F \subseteq \ker \hat{\lambda}$ has a very specific form. Observe that the *determinant map* defined in Definition 7.2 composed with the embedding $\mathbf{z} : \mathbf{k}^\times \to \mathcal{Z}$ provides an isomorphism of $\mathbf{B}_{\mathrm{u}}/\mathbf{B}_{\mathrm{sc}}$ with $\mathcal{Z}$ (in case $\mathsf{D}_{\mathrm{even}}$ consider det $: s(z_1, z_2) \mapsto (\det^{(1)}(z_1, z_2), \det^{(2)}(z_1, z_2)))$. Recall our convention to omit $\mathbf{z}$ (and $\mathbf{z}^{-1}$)! The restriction of det to $\mathbf{B}_{\mathrm{u}}^F$ provides an isomorphism of $\mathbf{B}_{\mathrm{u}}^F/\mathbf{B}_{\mathrm{sc}}^F \cong (\mathbf{B}_{\mathrm{u}}/\mathbf{B}_{\mathrm{sc}})^F \cong \mathcal{Z}^F$ by [DM91, Corollary 3.13] since $\mathbf{B}_{\mathrm{sc}}$ is connected (Borel subgroups are the maximal connected solvable subgroups [Hum75, 21.3]). Note that by the same argument $\mathbf{G}_{\mathrm{u}}^F/\mathbf{G}_{\mathrm{sc}}^F \cong \mathcal{Z}^F \cong \mathbf{B}_{\mathrm{u}}^F/\mathbf{B}_{\mathrm{sc}}^F$.

II. The p′–characters of the normalizer

We can generate $\mathbf{B}_{\mathrm{u}}^F/\mathbf{B}_{\mathrm{sc}}^F$ by $\det^{-1}(t_0)$ since t_0 generates $\mathcal{Z}^F$. When in case $\mathsf{D}_{\mathrm{even}}$ untwisted $\mathbf{B}_{\mathrm{u}}^F/\mathbf{B}_{\mathrm{sc}}^F$ is generated by $[\det^{-1}(t_{0^{(1)}}), \det^{-1}(t_{0^{(2)}})]$ in conformance with the multi index convention 6.7, otherwise $\mathcal{Z}^F$ is cyclic. So $\hat{\lambda}$ is already determined by $\lambda := \hat{\lambda}(\det^{-1}(t_0)) \in (\mathbb{F}_q^{\times})^{\mathsf{r}}$ with our usual identification of the complex $(q-1)$–th roots of unity with $\mathbb{F}_q^{\times}$. To determine the label of $\hat{\lambda}$ we evaluate $\hat{\lambda}$ on the t_i for $i = 1, \ldots, m$ (by the construction of the labels) and then express those values in terms of λ. To do so we consider the relations between the $\det(t_i) \in \mathcal{Z}^F$. A generator of $\mathcal{Z}^F$ is given by t_0, so we can write

$$\det(t_i) = \prod t_0^{\mathsf{e}_i},$$

where $\mathsf{e}_i \in \mathbb{Z}^{\mathsf{r}}$ are multi indices as explained in our multi index convention 6.7 to cover the case $\mathsf{D}_{\mathrm{even}}$ and the product is over the multi index components of $t_0^{\mathsf{e}_i}$. We have

$$\hat{\lambda}(t_i) = \hat{\lambda}(\det^{-1}(\det(t_i))) = \hat{\lambda}(\det^{-1}(\prod t_0^{\mathsf{e}_i})) = \prod \hat{\lambda}(\det^{-1}(t_0))^{\mathsf{e}_i} = \prod \lambda^{\mathsf{e}_i}.$$

When not in untwisted type $\mathsf{D}_{\mathrm{even}}$ simply omit all the products to obtain the label of $\hat{\lambda}$:

$$(\lambda^{\mathsf{l}}, \lambda^{\mathsf{e}_1}, \ldots, \lambda^{\mathsf{e}_m}).$$

Otherwise with $a := \hat{\lambda}(\det^{-1}(t_{0^{(1)}}))$ and $b := \hat{\lambda}(\det^{-1}(t_{0^{(2)}}))$ the label is

$$(a^2, b^2, a^{\mathsf{e}_1^{(1)}} b^{\mathsf{e}_1^{(2)}}, \ldots, a^{\mathsf{e}_n^{(1)}} b^{\mathsf{e}_n^{(2)}}).$$

Proposition 11.4. *We have* $\mathsf{e}_i^{(j)} = \langle \omega_{\mathsf{k}(j)}, \beta_{\mathsf{a}_i} \rangle$ *for* $i \in \{1, \ldots, m\}$, $j \in \{1, \ldots, \mathsf{r}\}$ *and* $\mathsf{e}_0 = \mathsf{l}$, *resp.* $\mathsf{e}_{0^{(1)}} = [2, 0]$ *and* $\mathsf{e}_{0^{(2)}} = [0, 2]$ *in case* $\mathsf{D}_{\mathrm{even}}$. *Here the* a_i *are the fixed representatives of the orbits of the graph automorphism involved in* F *on* $\{1, \ldots, n\}$ *as in Section 10, we fixed* k *in Table 6.9 and* l *is the exponent of* $X(\mathbf{T}_{\mathrm{sc}})/\mathbb{Z}\Phi$, *as usual.*

Proof. By the definitions of the $t_i = N_{F^w/F}(\omega_i^{\vee}(\mu)) = N_{F^w/F}(\beta_i(\nu)\nu^{\langle \omega_{\mathsf{k}}, \beta_i \rangle})$ (see Definition 6.10 and Proposition 10.2) and $\det(sz) = z^{\mathsf{l}}$ (Definition 7.2):

$$\begin{aligned}
\det(t_i) &= \det\left(N_{F^w/F}\left(\omega_{\mathsf{a}_i}^{\vee}(\mu)\right)\right) \\
&= N_{F^w/F}\left(\nu^{\langle \omega_{\mathsf{k}}, \beta_{\mathsf{a}_i} \rangle}\right)^{\mathsf{l}} \\
&= N_{F^w/F}(\mu)^{\langle \omega_{\mathsf{k}}, \beta_{\mathsf{a}_i} \rangle} \\
&= t_0^{\langle \omega_{\mathsf{k}}, \beta_{\mathsf{a}_i} \rangle}.
\end{aligned}$$

$\square$

Example 11.5. In case $\mathsf{D}_{\mathrm{even}}$ with a and b as above, the label of $\hat{\lambda}$ is

$$(a^2, b^2, ab, (ab)^2, \ldots, (ab)^{n-2}, b^{\frac{n}{2}} a^{\frac{n-2}{2}}, b^{\frac{n-2}{2}} a^{\frac{n}{2}}).$$

This completes our description of the M_χ. For the characters of the N_χ we cannot give such a nice description; we can only describe them relative to the sets M_χ. However it is possible to determine the number of characters in N_χ, the corresponding central character of $\mathbf{Z}(\mathbf{G}_{\mathrm{sc}}^F)$ and the stabilizers in the group of automorphisms of $\mathbf{B}_{\mathrm{sc}}^F$.

Proposition 11.6. *Let* $\eta \in \mathrm{Irr}_{p'}(\mathbf{B}_{\mathrm{sc}}^F)$. *The set* N_η *contains* $|\mathcal{Z}^F|/|M_\eta|$ *distinct characters. By Proposition 11.3 this number can be read off the label* $(a_0, \ldots, a_n)$ *of some* $\chi \in M_\eta$. *In fact it depends only on the zero positions in the label, that is, the set* S *defined in Section 8 for* χ.

Proof. By Clifford theory

$$|N_\eta| = |\mathbf{B}_{\mathrm{u}}^F : I_{\mathbf{B}_{\mathrm{u}}^F}(\eta)| = |\mathbf{B}_{\mathrm{u}}^F/\mathbf{B}_{\mathrm{sc}}^F \ : \ I_{\mathbf{B}_{\mathrm{u}}^F}(\eta)/\mathbf{B}_{\mathrm{sc}}^F| = |\mathcal{Z}^F|/|M_\eta|,$$

where $|I_{\mathbf{B}_{\mathrm{u}}^F}(\eta)/\mathbf{B}_{\mathrm{sc}}^F| = |M_\eta|$ follows from Theorem 5.3 and Theorem 5.2 since we showed in the proof of Proposition 11.3 that η extends to $I_{\mathbf{B}_{\mathrm{u}}^F}(\eta)$. By Proposition 11.2 we have $I_{\mathbf{B}_{\mathrm{u}}^F}(\eta) = I_S \mathbf{B}_{\mathrm{sc}}^F$, which only depends on S. $\qquad\square$

We can read off the e_i of the k–th row(s) of the inverse Cartan matrices by Proposition 11.4 (see Table 6.9 for the k's and Appendix 17 for the inverse Cartan matrices), thus we can explicitly state the action on the labels and compute $|\mathrm{Irr}_{p'}(\mathbf{B}_{\mathrm{sc}}^F)|$ using Proposition 11.6.

Example 11.7. We consider type C_n and $q \equiv 1 \bmod 2$. Let $(a_0, \ldots, a_n)$ be the label of some $\chi \in \mathrm{Irr}_{p'}(\mathbf{B}_{\mathrm{u}}^F)$. The labels of linear characters of $\mathbf{B}_{\mathrm{u}}^F$ are of the form $(\lambda^2, \lambda^2, \ldots, \lambda^2, \lambda)$ for $\lambda \in \mathbb{F}_q^\times$ by Proposition 11.4. Then $|M_\chi| = q - 1$ and $|N_\chi| = 1$ if and only if $a_n \neq 0$. There are $q^{n-1}(q-1)(q-1)$ such labels, partitionened into $q^{n-1}(q-1)$ distinct sets M_χ, each lying above a unique character of $\mathbf{B}_{\mathrm{sc}}^F$.

If $a_n = 0$ then $|M_\chi| = \frac{q-1}{2}$ and $|N_\chi| = 2$. There are $q^{n-1}(q-1)$ such labels, partitioned into $2q^{n-1}$ distinct sets M_χ, each lying above two distinct characters of $\mathbf{B}_{\mathrm{sc}}^F$.

Thus we obtain a total of $q^n + q^{n-1}$ distinct sets M_χ and $|\mathrm{Irr}_{p'}(\mathbf{B}_{\mathrm{sc}}^F)| = q^n + 3q^{n-1}$.

Theorem 11.8. *The cardinality* $|\mathrm{Irr}_{p'}(\mathbf{B}_{\mathrm{sc}}^F)|$ *for various types is given in Table 11.11.*

Proof. We define a map f on the set of labels by multiplying a position $i \geq 1$ in the label $A := (a_0, \ldots, a_m)$ by some power of $a_0 \neq 0$, say $f((a_0, \ldots, a_m)) := (a_0, \ldots, a_0^k a_i, \ldots a_m)$ for some $k \in \mathbb{N}$. Then f is injective since $a_0 \neq 0$ for all labels by definition, and an inverse of f if given by $(a_0', \ldots, a_m') \mapsto (a_0', \ldots, a_i'/a_0'^k, \ldots, a_m')$. For the label L of some linear character an easy computation shows $f(AL) = f(A)f(L)$ where the product of labels is componentwise, which corresponds to the multiplication of the actual characters by Proposition 11.3. So counting the orbits of the group

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of linear characters on the set of labels and computing their lengths is equivalent to doing so in the image of f. By applying several suitable maps of this form we can bring the labels of linear characters into a "reduced" form where all the exponents $e_i = \langle \beta_{a_i}, \omega_k \rangle$ (Proposition 11.4) are reduced modulo $e_0 = 1$. The "reduced" labels of linear characters are given in Table 11.11. Using similar computations as in the above example, we enumerate the number and sizes of orbits $M_\chi \subseteq \mathrm{Irr}_{p'}(\mathbf{B}_u^F)$ and then apply Proposition 11.6. By the same argument, further transformations on labels can be applied: For example permutation of the entries and taking powers prime to $q-1$ (resp. $q+1$) of individual entries. We do this explicitly below for the harder cases. $\qquad\square$

Example 11.9. Consider type D_{2m} and $q \equiv 1 \bmod 4$. When counting orbit lengths, it is useful to define injective maps on the set of labels that make counting easier, as described in the proof of Theorem 11.8. Permuting the entries a_i with $i \geq 1$ and multiplying them by fixed powers of $a_0 \neq 0$ defines such a map. With these two operations the label of the linear character given in Example 11.5 can be brought into a form
$$(a^2, b^2, (ab), \ldots, (ab), a, b, 1, \ldots, 1).$$
We call an entry of a label an x–position if it is multiplied by x, where $x \in \{1, a, b, (ab)\}$. The number of ab– and 1–positions is equal to $m-1$. The size of M_χ is equal to

- $(q-1)^2$ if either both the a– and the b–position are not equal to zero, or one of them and an ab–position are not equal to zero. This yields
$$x := q^{n-2}(q-1)^2 + 2(q^{m-1}-1)q^{m-1}(q-1)$$
distinct M_χ.

- $(q-1)^2/4$ if all ab–, a– and b– positions are zero. There are $y := 4q^{m-1}$ such M_χ.

- $(q-1)^2/2$ otherwise. There are
$$\frac{(q-1)^2 q^n - x(q-1)^2 - y(q-1)^2/4}{(q-1)^2/2} = 2q^n - 2x - y/2 = 2q^{n-2} + 4q^m - 6q^{m-1}$$
such M_χ.

We obtain $q^n + q^{n-2} + 2q^m$ distinct M_χ and a total of $q^n + 3q^{n-2} + 6q^m + 4q^{m-1}$ characters in $\mathrm{Irr}_{p'}(\mathbf{B}_{sc}^F)$.

Example 11.10. Type A_n. Let $m := \gcd(n+1, q-1)$, then there is a bijection on the set of labels that leaves orbit lengths invariant, such that the labels of linear characters are of the form
$$(\lambda^m, \lambda^{1 \bmod m}, \lambda^{2 \bmod m}, \ldots, \lambda^{n \bmod m}).$$

The size of M_χ is $\frac{(q-1)}{d}$ where d is the smallest positive divisor of m such that all λ^k–positions with $\gcd(d, k) \neq d$ are equal to zero. Let φ be the Euler φ–function. For fixed $d < m$ there is a total of

$$P(d) := \frac{n+1}{m}\varphi(m/d)$$

λ^d–positions and

$$N(d) := \frac{n+1}{m}\sum_{l \in \mathbb{N}:d|l|m}\varphi(m/l) - 1$$

λ^k–positions such that $\gcd(d, k) = d$. Thus there is a total of

$$\sum_{d|m,\ d<m} d(q^{P(d)} - 1)q^{N(d)-P(d)} + mq^{\frac{n+1}{m}-1}$$

distinct sets M_χ and we have

$$|\mathrm{Irr}_{p'}(\mathbf{B}_{\mathrm{sc}}^F)| = \sum_{d|m,\ d<m} d^2(q^{P(d)} - 1)q^{N(d)-P(d)} + m^2 q^{\frac{n+1}{m}-1}.$$

A similar formula could be computed for the twisted case A_n.

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Table 11.11

Type	label & reduced label of $\hat{\lambda}$	$q \equiv$	distinct M_χ	$|\mathrm{Irr}_{p'}(\mathbf{B}_{\mathrm{sc}}^F)|$
A_n	$(\lambda^{(n+1)}, \lambda, \lambda^2, \ldots, \lambda^n)$	any	see Example 11.10	
$^2\mathsf{A}_n$	$(\lambda^{(n+1)}, \lambda, \lambda^2, \ldots, \lambda^{\lceil n/2\rceil})$			
B_n	$(\lambda^2, \lambda, \lambda^2, \ldots, \lambda^n)$	$1 \bmod 2$	$q^n + q^{\lfloor n/2\rfloor}$	$q^n + 3q^{\lfloor n/2\rfloor}$
	$(\lambda^2, \lambda, 1, \lambda, 1, \ldots)$	$0 \bmod 2$	q^n	q^n
C_n	$(\lambda^2, \lambda^2, \ldots, \lambda^2, \lambda)$	$1 \bmod 2$	$q^n + q^{n-1}$	$q^n + 3q^{n-1}$
	$(\lambda^2, 1, \ldots, 1, \lambda)$	$0 \bmod 2$	q^n	q^n
$\mathsf{D}_{\mathrm{odd}}$	$(\lambda^4, \lambda^2, \ldots, \lambda^{2(n-2)}, \lambda^{-1}, \lambda)$	$1 \bmod 4$	$q^n + q^{n-2} + 2q^{\left\lfloor \frac{n-2}{2}\right\rfloor}$	$q^n + 3q^{n-2} + 12q^{\left\lfloor \frac{n-2}{2}\right\rfloor}$
	$(\lambda^4, \lambda^2, 1, \ldots, \lambda^{-1}, \lambda)$	$3 \bmod 4$	$q^n + q^{n-2}$	$q^n + 3q^{n-2}$
		$0 \bmod 2$	q^n	q^n
$^2\mathsf{D}_{\mathrm{odd}}$	$(\lambda^4, \lambda^2, \ldots, \lambda^{2(n-2)}, \lambda)$	$1 \bmod 4$	$q^n + q^{n-2}$	$q^n + 3q^{n-2}$
	$(\lambda^4, \lambda^2, 1, \ldots, \lambda)$	$3 \bmod 4$	$q^n + q^{n-2} + 2q^{\left\lfloor \frac{n-2}{2}\right\rfloor}$	$q^n + 3q^{n-2} + 12q^{\left\lfloor \frac{n-2}{2}\right\rfloor}$
		$0 \bmod 2$	q^n	q^n
$\mathsf{D}_{\mathrm{even}}$	see Example 11.5	$1 \bmod 2$	$q^n + q^{n-2} + 2q^{\frac{n-2}{2}}$	$q^n + 3q^{n-2} + 6q^{\frac{n}{2}} + 4q^{\frac{n-2}{2}}$
		$0 \bmod 2$	q^n	q^n
$^2\mathsf{D}_{\mathrm{even}}$	$(\lambda^2, \lambda^2, \ldots, \lambda^{2(n-2)}, \lambda)$	$1 \bmod 2$	$q^n + q^{n-2}$	$q^n + 3q^{n-2}$
	$(\lambda^2, 1, \ldots, 1, \lambda)$	$0 \bmod 2$	q^n	q^n
$^3\mathsf{D}_4$	$(1,1,1)$		q^4	q^4
E_6	$(\lambda^3, \lambda^4, \lambda^3, \lambda^5, \lambda^6, \lambda^4, \lambda^2)$	$1 \bmod 3$	$q^6 + 2q^2$	$q^6 + 8q^2$
	$(\lambda^3, \lambda, 1, \lambda^2, 1, \lambda^2, \lambda)$	$0, 2 \bmod 3$	q^6	q^6
$^2\mathsf{E}_6$	$(\lambda^3, \lambda^4, \lambda^3, \lambda^5, \lambda^6)$	$2 \bmod 3$	$q^6 + 2q^2$	$q^6 + 8q^2$
	$(\lambda^3, \lambda, 1, \lambda^2, 1)$	$0, 1 \bmod 3$	q^6	q^6
E_7	$(\lambda^2, \lambda^4, \lambda^7, \lambda^8, \lambda^{12}, \lambda^9, \lambda^6, \lambda^3)$	$1 \bmod 2$	$q^7 + q^4$	$q^7 + 3q^4$
	$(\lambda^2, 1, \lambda, 1, 1, \lambda, 1, \lambda)$	$0 \bmod 2$	q^7	q^7
E_8	$(1, 1, 1, 1, 1, 1, 1, 1)$		q^8	q^8
F_4	$(1, 1, 1, 1, 1)$		q^4	q^4
G_2	$(1, 1, 1)$		q^2	q^2

These numbers were also computed by Olivier Brunat in [Bru09b] using different methods. Note that the λ here do not necessarily lie in the same field; they are contained in $\mathbb{F}_{q^w}$ where w is the order of the graph automorphism involved in F and have order $o(t_0) \in \{1, q-1, q+1\}$.

In addition to field and graph automorphisms, we have an action of $\mathbf{B}_{\mathrm{u}}^F$ on $\mathbf{B}_{\mathrm{sc}}^F$ by conjugation. This action corresponds to the *diagonal* automorphisms of $\mathbf{G}_{\mathrm{sc}}^F$. So, by definition, the N_η are exactly the orbits of the group of diagonal automorphism on $\mathrm{Irr}_{p'}(\mathbf{B}_{\mathrm{sc}}^F)$, which we shall describe in more detail now. For fixed $\chi \in \mathrm{Irr}_{p'}(\mathbf{B}_{\mathrm{u}}^F)$ let $\eta \in N_\chi$ be as in Proposition 11.2. Then we can parametrize N_χ by the elements of

$$\mathbf{B}_{\mathrm{u}}^F / I_{\mathbf{B}_{\mathrm{u}}^F}(\eta) = \mathbf{B}_{\mathrm{u}}^F / \langle \mathbf{B}_{\mathrm{sc}}^F, I_S \rangle \cong \mathcal{Z}^F / \langle \det(t_i) \mid i \in S^c \rangle,$$

since $I_S = I_{\mathbf{B}_{\mathrm{u}}^F}(\phi_S) = \langle t_i, \mathbf{U}^F \mid i \in S^c \rangle$ and det provides a homomorphism of $\mathbf{B}_{\mathrm{u}}^F$ onto $\mathcal{Z}^F$ with kernel $\mathbf{B}_{\mathrm{sc}}^F$, as explained earlier in this section and $I_{\mathbf{B}_{\mathrm{u}}^F}(\eta) = I_S \mathbf{B}_{\mathrm{sc}}^F$ by Proposition 11.2. We set

$$\mathcal{N}_S := \mathcal{Z}^F / \langle \det(t_i) \mid i \in S^c \rangle = \mathcal{Z}^F / \det(I_S).$$

A character η^g is then parametrized by $\det(gI_S) \in \mathcal{N}_S$. The conjugation action of some $g \in \mathbf{B}_{\mathrm{u}}^F$ on N_χ corresponds to multiplication by $\det(g)$ in $\mathcal{N}_S$. Next we investigate the other automorphisms. Suppose some automorphism γ of $\mathbf{B}_{\mathrm{sc}}^F$ stabilizes N_χ. To obtain more information on the action of γ on N_χ we take a closer look on the action of γ on the $[\phi_S^g]_{\mathbf{B}_{\mathrm{sc}}^F}$ of $\mathcal{U}_S$ (see Notation 11.1). We have a one–to–one relationship between N_χ and $\mathcal{U}_S$, which is compatible with the action of γ.

In Section 8 we arbitrarily chose some non–trivial character ϕ of $(\mathbb{F}_q, +)$ for our construction of the ϕ_S. We shall refine that choice now.

Lemma 11.12. *There are $p - 1$ non–trivial characters in $\mathrm{Irr}(\mathbb{F}_q, +)$ that are fixed by field automorphisms.*

Proof. A character of $(\mathbb{F}_q, +)$ is determined by its values on an $\mathbb{F}_p$–basis of $(\mathbb{F}_q, +) \cong (\mathbb{Z}/p\mathbb{Z})^l$. An $\mathbb{F}_p$–basis is given by powers of μ. Any character that is non–trivial on $\mu^0 = 1$ and trivial on the remaining elements of this basis has the desired property. $\square$

So without loss of generality we can assume that the ϕ_S are fixed by the field automorphism.

Proposition 11.13. *Let $\chi \in \mathrm{Irr}_{p'}(\mathbf{B}_{\mathrm{u}}^F)$ lie above ϕ_S, and γ be a field or graph automorphism stabilizing M_χ.*

1. *Then N_χ and $\mathcal{U}_S$ are γ–invariant.*

2. *The action of $\mathbf{B}_{\mathrm{u}}^F$ on N_χ is equivalent to the action of $\mathbf{B}_{\mathrm{u}}^F$ on $\mathcal{N}_S$ defined above and equivalent to the canonical action on $\mathcal{U}_S$, in particular $|\mathcal{N}_S| = |N_\chi| = |\mathcal{U}_S|$.*

3. *The character $\eta \in N_\chi$ corresponding to $\overline{1}$ in $\mathcal{N}_S$, which is the one above $[\phi_S]_{\mathbf{B}_{\mathrm{sc}}^F}$ from Proposition 11.2, is always fixed by γ. In particular the action of γ on $\eta^g \in N_\chi$, $[\phi_S^g]_{\mathbf{B}_{\mathrm{sc}}^F} \in \mathcal{U}_S$ and $\det(gI_S) \in \mathcal{N}_S$ is given by the action of γ^{-1} on the conjugating element g.*

Proof. We apply Clifford theory two times: from $\mathbf{U}^F$ to $\mathbf{B}_{\mathrm{sc}}^F$ and from $\mathbf{B}_{\mathrm{sc}}^F$ to $\mathbf{B}_{\mathrm{u}}^F$. Since χ lies above ϕ_S, the characters of N_χ must also lie above $\mathbf{B}_{\mathrm{u}}^F$–conjugates of ϕ_S. Since N_χ only contains distinct $\mathbf{B}_{\mathrm{u}}^F$–conjugates, every one of those must lie

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above a different $\mathbf{B}_{\mathrm{sc}}^F$–orbit $[\phi_S^g]_{\mathbf{B}_{\mathrm{sc}}^F} \in \mathcal{U}_S$ by Proposition 11.2. Those orbits are also parametrized by $\mathcal{N}_S$, by the discussion preceding Lemma 11.12. By definition the action of $\mathbf{B}_{\mathrm{u}}^F$ on $\mathcal{N}_S$ is equivalent to the action on N_χ via conjugation.

Now suppose γ leaves M_χ invariant. Restriction of characters and application of γ commute. Therefore γ may act on $\mathcal{U}_S$ and N_χ, but must leave them invariant as well. The fixed character exists because we have just chosen ϕ_S in such a way that it is always fixed by field automorphisms. Graph automorphisms that stabilize M_χ fix S by Proposition 9.4 and thus have to fix $\phi_S := \prod_{i \in S} \phi_i$ as well since they only permute the groups $U_{\alpha_j}/[\mathbf{U}^F, \mathbf{U}^F]$. Therefore $[\phi_S]_{\mathbf{B}_{\mathrm{sc}}^F}$ is stabilized by field and graph automorphisms, as is the unique character $\eta \in N_\chi$ above $[\phi_S]_{\mathbf{B}_{\mathrm{sc}}^F}$.

For any automorphism γ that fixes ϕ_S we have:

$$\left(\phi_S^t\right)^\gamma (u) = \phi_S(t^{-1}\gamma(u)t) = (\phi_S^\gamma)^{\gamma^{-1}(t)} (u) = \phi_S^{\gamma^{-1}(t)}(u).$$

And thus the last assertion follows. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad \square$

Remark 11.14. *To determine if an automorphism γ of $\mathbf{B}_{\mathrm{sc}}^F$ fixes $\eta \in N_\chi$ we check two conditions:*

1. *Does γ leave $M_\chi \subseteq \mathrm{Irr}_{p'}(\mathbf{B}_{\mathrm{u}}^F)$ invariant? We have given an explicit description of M_χ in this section and the action of γ on $\mathrm{Irr}_{p'}(\mathbf{B}_{\mathrm{u}}^F)$ is described by Proposition 9.4.*

2. *If so, which elements of $\mathcal{N}_S$ are fixed by γ? The answer here depends only on the type of Φ, the Frobenius map F and S, but not on the particular set M_χ. The corresponding characters of N_χ are then fixed by γ.*

Example 11.15. We consider type A_2 and $F = F_{7^2}$, that is $\mathbf{G}_{\mathrm{u}}^F = \mathrm{GL}_3(7^2)$. Let χ be the character with label $(\mu^3, 0, 0)$ where μ generates $\mathbb{F}_{7^2}^\times$. By Proposition 11.4 we have

$$M_\chi = \left\{ (\lambda^3\mu^3, 0, 0) \,\middle|\, \lambda \in \mathbb{F}_q^\times \right\} = \left\{ (\lambda^3, 0, 0) \,\middle|\, \lambda \in \mathbb{F}_q^\times \right\}.$$

Field and graph automorphisms stabilize M_χ, since by Proposition 9.4

$$\Gamma(\lambda^3, 0, 0) = (\lambda^{-3}, 0, 0) \in M_\chi$$

and

$$\sigma(\lambda^3, 0, 0) = (\lambda^{21}, 0, 0) \in M_\chi.$$

The set of zero positions is $S = \{1, 2\}$ and there are three characters in N_χ corresponding to the three elements of

$$\mathcal{N}_S = \mathbb{F}_{7^2}^\times / \langle \mu^3 = \det(t_0) \rangle = \{\bar{1}, \bar{\mu}, \bar{\mu}^2\}.$$

The graph automorphism acts as $\cdot^{-1}$ on $\mathcal{N}_S$ since $\det(t_1) = \mu$ and $\det(\Gamma(t_1)) = \det(t_2 t_0^{-1}) = \mu^{-1}$. Since $\bar{\mu}^{-1} = \bar{\mu}^{47} = \bar{\mu}^2$ we have one Γ–stable character and two

that are interchanged by Γ in N_χ. The action of the field automorphism σ is given by $\cdot^7$ on $\mathcal{N}_S$. But $\overline{\mu}^7 = \overline{\mu}\,\overline{\mu}^6 = \overline{\mu}$, so the field automorphism acts trivially on $\mathcal{N}_S$. Taking diagonal automorphisms corresponds to multiplication in $\mathcal{N}_S$: the trivial automorphism corresponding to $\overline{1}$ fixes everything, conjugation by t_1 is multiplication by $\overline{\mu}$ in $\mathcal{N}_S$ and permutes the three elements of $\mathcal{N}_S$ cyclically. The character of N_χ parametrized by $\overline{1}$ is fixed by $\langle\sigma,\Gamma\rangle$, $\overline{\mu}$ is fixed by $\langle\sigma, t_1\Gamma\rangle$ and $\overline{\mu}^2$ is fixed by $\langle\sigma, t_1^2\Gamma\rangle$.

Next consider the character χ with label $(\mu, 0, 0)$. The field automorphism σ still leaves M_χ invariant but $\mu^{-1} = \mu^{47} \notin \{\mu^{1+3k} \mid k \in \mathbb{N}\}$, so Γ does not. The zero set S is the same as before, so we do not have to repeat the computations for $\mathcal{N}_S$. The three characters of N_χ are only fixed by $\langle\sigma\rangle$.

It is clear from the above example that the situation is somewhat complicated for type A_n. For types B_n, C_n and E_7 we have $|N_\chi| \leq 2$ so in those cases a set N_χ that is stabilized by some automorphism γ of $\mathbf{G}_{\mathrm{sc}}^F$ is fixed pointwise by γ (Proposition 11.13). The complete situation for type C_n:

Proposition 11.16. *Assume type* C_n *and* $p \neq 2$.

- *There are* $(q-1)^2 q^{n-1}$ p'*–characters of* $\mathbf{B}_{\mathrm{u}}^F$ *divided into* $(q-1)q^n$ *sets* M_χ *of size* $(q-1)$, *to each of which belongs a single character of* $\mathbf{B}_{\mathrm{sc}}^F$. *A full set of labels representative for the* M_χ *is given by*

$$\{(a_0, \ldots, a_{n-1}, 1) \mid a_i \in \mathbb{F}_q\}.$$

  The field automorphism σ^i *stabilizes* M_χ *and thus the corresponding character of* $\mathbf{B}_{\mathrm{sc}}^F$ *if and only if* $a_1, \ldots, a_{n-1} \in \mathbb{F}_{p^i}$.

- *Furthermore there are* $(q-1)q^{n-1}$ *other* p'*–characters of* $\mathbf{B}_{\mathrm{u}}^F$. *Those are divided into* $2q^{n-1}$ *sets* M_χ *of size* $(q-1)/2$, *to each of which belong two characters of* $\mathbf{B}_{\mathrm{sc}}^F$. *A full set of labels representative for those* M_χ *is given by*

$$\{(1, a_1, \ldots, a_{n-1}, 0), (\mu, a_1, \ldots, a_{n-1}, 0) \mid a_i \in \mathbb{F}_q\}.$$

  The non–trivial diagonal automorphism, induced by conjugation with t_n, *permutes the two characters in each set* N_χ. *The field automorphism* σ^i *stabilizes a set* M_χ *if and only* $a_1, \ldots, a_{n-1} \in \mathbb{F}_{p^i}$. *If* M_χ *is stabilized by* σ^i *then the two characters of* N_χ *are fixed by* σ^i.

If $p = 2$ *the* $(q-1)q^n$ p'*–characters of* $\mathbf{B}_{\mathrm{u}}^F$ *are divided into* q^n *sets* M_χ *of size* $(q-1)$, *to each of which belongs a single character of* $\mathbf{B}_{\mathrm{sc}}^F$. *A full set of labels representative for the* M_χ *is given by* $(1, a_1, \ldots, a_n)$. *A set* M_χ *is stabilized by* σ^i *if and only if* $a_1, \ldots, a_{n-1} \in \mathbb{F}_{p^i}$.

12. Corresponding characters of $\mathbf{Z}(\mathbf{G}_{\mathrm{sc}}^F)$

For a given $\eta \in \mathrm{Irr}_{p'}(\mathbf{B}_{\mathrm{sc}}^F)$ we need to determine the unique character η_Z of $\mathbf{Z}(\mathbf{G}_{\mathrm{sc}}^F) \leq \mathbf{B}_{\mathrm{sc}}^F$ which lies below η. Since the characters of N_η are all conjugate in $\mathbf{B}_{\mathrm{u}}^F$, they all lie above the same η_Z and we may consider the restriction of some $\chi \in M_\eta$ to $\mathbf{Z}(\mathbf{G}_{\mathrm{sc}}^F)$ instead.

Proposition 12.1. *Let $(b_0, \ldots, b_m)$ be the label for some $\chi \in \mathrm{Irr}_{p'}(\mathbf{B}_{\mathrm{u}}^F)$. Since $Z = \mathbf{Z}(\mathbf{G}_{\mathrm{sc}}^F)$ and $\mathcal{Z}^F = \mathbf{Z}(\mathbf{G}_{\mathrm{u}}^F)$ are central, there exist unique characters $\chi_Z \in \mathrm{Irr}(Z)$ and $\chi_{\mathcal{Z}^F} \in \mathrm{Irr}(\mathcal{Z}^F)$ underlying χ. Those are uniquely determined by b_0 (the usual multi index convention 6.7 for case $\mathsf{D}_{\mathrm{even}}$ applies). With our fixed identification of $\mathbb{F}_q^\times$ with complex roots of unity:*

$$\chi_{\mathcal{Z}^F}(t_0) = b_0 \ \ and$$

$$\chi_Z(t_0) = b_0^{|\mathcal{Z}^F|/\gcd(|\mathcal{Z}^F|,\mathsf{l})}.$$

Proof. By the construction of the label χ is obtained by induction from some linear character ψ of I_S. Since $\mathbf{Z}(\mathbf{G}_{\mathrm{sc}}^F) \leq \mathcal{Z}^F \leq I_S$ we must have $\psi \mid_{\mathcal{Z}^F} = \chi_{\mathcal{Z}^F}$.

But t_0 generates $\mathcal{Z}^F$. The value of ψ on t_0 is given by b_0 of the label of χ. Therefore we only need to determine a power of t_0 that generates $\mathbf{Z}(\mathbf{G}_{\mathrm{sc}}^F)$. The character χ_Z is then determined by the same power of b_0, since ψ is linear.

The center of $\mathbf{G}_{\mathrm{sc}}^F$ is generated by $t_0^{|\mathcal{Z}^F|/\gcd(|\mathcal{Z}^F|,\mathsf{l})}$ (the multi index convention 6.7 applies for case $\mathsf{D}_{\mathrm{even}}$) and l is defined as usual as the exponent of the fundamental group. $\qquad\square$

Note that multiplication by linear characters leaves $b_0^{|\mathcal{Z}^F|/\gcd(|\mathcal{Z}^F|,\mathsf{l})}$ invariant, as it should be: $(\lambda^{\mathsf{l}})^{|\mathcal{Z}^F|/\gcd(|\mathcal{Z}^F|,\mathsf{l})} = 1$ by Proposition 11.4. The relation

$$(c_0, \ldots, c_n) \sim (b_0, \ldots, b_n) \Leftrightarrow c_0^{|\mathcal{Z}^F|/\gcd(|\mathcal{Z}^F|,\mathsf{l})} = b_0^{|\mathcal{Z}^F|/\gcd(|\mathcal{Z}^F|,\mathsf{l})}$$

partitions the set of labels into $\gcd(|\mathcal{Z}^F|, \mathsf{l})$ equivalence classes. Each contains the characters lying above a common character of $\mathbf{Z}(\mathbf{G}_{\mathrm{sc}}^F)$. In case $\mathsf{D}_{\mathrm{even}}$ there are 4 classes if $\gcd(q - 1, 2) = 2$ and one otherwise.

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We recall some facts of Deligne–Luzstig theory to describe the set of p'–characters in the universal groups $\mathbf{G}_{\mathrm{u}}^F$ in Section 13. To make use of these results we require a good parametrization of the semisimple conjugacy classes in the dual group, which we obtain by considering the Steinberg map in Section 14 (see Proposition 14.2 and Proposition 14.4). Using these results we construct an equivariant bijection between $\mathrm{Irr}_{p'}(\mathbf{B}_{\mathrm{u}}^F)$ and $\mathrm{Irr}_{p'}(\mathbf{G}_{\mathrm{u}}^F)$ (Section 15, Theorem 15.1). Finally we show that the compability of this bijection with the multiplication of linear characters (Theorem 15.3) allows us to construct a bijection of $\mathrm{Irr}_{p'}(\mathbf{B}_{\mathrm{sc}}^F)$ with $\mathrm{Irr}_{p'}(\mathbf{G}_{\mathrm{sc}}^F)$ in Theorem 15.4. In Section 16 we point out what remains to be done and provide some interesting examples.

13. Deligne–Luzstig theory

Deligne–Luzstig theory provides a powerful tool to describe the irreducible characters of the groups $\mathbf{G}_{\mathrm{u}}^F$.

Proposition 13.1. $\mathrm{Irr}(\mathbf{G}_{\mathrm{u}}^F)$ *can be partitioned into sets* $\mathcal{E}(\mathbf{G}^F, [s])$ *where* $[s]$ *runs over the F–stable, semisimple conjugacy classes of the dual group* $\mathbf{G}^*$. *If* $(\mathbf{G}_{\mathrm{u}}, F)$ *is not contained in Table 13.2, then every such set* $\mathcal{E}(\mathbf{G}^F, [s])$ *contains exactly one character of degree prime to p.*

Table 13.2

Type	Frobenius map	
D_n	$\Gamma \circ F_q$	$q = 2$ and Γ of order 2
$\mathsf{B}_n, \mathsf{C}_n, \mathsf{D}_n, \mathsf{G}_2, \mathsf{F}_4$	F_q	$q = 2$
G_2	F_q	$q = 3$

Proof. That $\mathrm{Irr}(\mathbf{G}_{\mathrm{u}}^F)$ can be partitioned in this way is well known and due to Luzstig, see [DM91, Proposition 14.41]. Since our group has connected center, the rational series and the geometric series coincide. The statement about the semisimple characters is due to the "Jordan decomposition" of characters: There is a bijection between

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a set $\mathcal{E}(\mathbf{G}^F, [s])$ and the unipotent irreducible characters of the centralizer of s in the dual group, see [DM91, Theorem 13.23]. This bijection modifies the degree only by a number prime to p, see [DM91, Remark 13.24]. We are done when all possible centralizers have exactly one unipotent character of degree prime to p, i.e., the trivial character is the only unipotent character of degree prime to p. This is true for all possible centralizers, except for those occuring in the groups listed in Table 13.2, see [Mal07, Theorem 6.8]. $\qquad\square$

From now on we shall only consider pairs $(\mathbf{G}_\mathrm{u}, F)$ that are not contained in Table 13.2. The known results on the groups of Table 13.2 are summarized in [Bru09a], most of them are shown to be "good" in [Mal08a]. We denote the unique semisimple character of $\mathbf{G}_\mathrm{u}^F$ determined by a conjugacy class $[s]$ of $\mathbf{G}_\mathrm{u}^{*F^*}$ by $\chi_{[s]}$.

Proposition 13.3. *The linear characters* $\mathrm{Irr}(\mathbf{G}_\mathrm{u} \mid 1_{\mathbf{G}_\mathrm{sc}})$ *of* $\mathbf{G}_\mathrm{u}$ *with kernel* $\mathbf{G}_\mathrm{sc}$ *are given by the set* $\left\{ \chi_{[z]} \mid z \in \mathbf{Z}(\mathbf{G}_\mathrm{u}^{*F^*}) \right\}$. *For semisimple* $s \in \mathbf{G}_\mathrm{u}^{*F^*}$ *and* $z \in \mathbf{Z}(\mathbf{G}_\mathrm{u}^{*F^*})$ *we have*

$$\chi_{[s]}\chi_{[z]} = \chi_{[sz]}.$$

Proof. [DM91, Proposition 13.30]. $\qquad\square$

Next we want to determine the underlying central characters. To do that, we need to describe the sets $\mathcal{E}(\mathbf{G}_\mathrm{u}^F, [s])$ in more detail. To every semisimple conjugacy class $[s]$ of $\mathbf{G}_\mathrm{u}^{*F^*}$ belongs a *geometric conjugacy class* of $\mathbf{G}_\mathrm{u}^F$ (see [DM91, Definition 13.2]): a class of pairs $(\mathbf{T}, \theta)$ where $\mathbf{T}$ is an F–stable torus of $\mathbf{G}_\mathrm{u}$ and $\theta \in \mathrm{Irr}(\mathbf{T}^F)$, see [DM91, Proposition 13.13]. The characters of $\mathcal{E}(\mathbf{G}_\mathrm{u}^F, [s])$ are the irreducible constituents of the Deligne–Luzstig characters $R_{\mathbf{T}^F}^{\mathbf{G}_\mathrm{u}^F}(\theta)$. The relation between s and θ is given by Proposition 7.8, i.e.

$$s = N_{F^{*m}/F^*}\left(\delta^\vee(\hat{\theta})(\mu) \right).$$

The center $\mathcal{Z}^F = \mathbf{Z}(\mathbf{G}_\mathrm{u}^F)$ is contained in every maximal torus of $\mathbf{G}_\mathrm{u}^F$, so $\mathcal{Z}^F \subseteq \mathbf{T}$.

Proposition 13.4. *With the notation as above*

$$\theta(t_0) = \det{}^*(s),$$

in case $\mathsf{D}_{\mathrm{even}}$

$$\theta(t_{0^{(i)}}) = \det{}^{*(i)}(s) \ \textit{for} \ i = 1, 2.$$

Proof. We have

$$\mu^{\langle \det^*, \delta^\vee(\hat{\theta}) \rangle} = \mu^{\langle \hat{\theta}, \mathbf{z} \rangle} = \hat{\theta}(\mathbf{z}(\mu))$$

by duality. We are done in the untwisted case since then $N_{F^{*m}/F_*} = \mathrm{id}$ and $t_0 = \mathbf{z}(\mu)$. Define $t := \delta^\vee(\hat{\theta})(\mu)$ and again by duality

$$\mathrm{det}^*(F^*(t)) = \mu^{\langle \mathrm{det}^* \circ F^*, \delta^\vee(\hat{\theta}) \rangle} = \mu^{\langle \hat{\theta}, F \circ \mathbf{z} \rangle} = \hat{\theta}(F(\mathbf{z}(\mu))),$$

and thus

$$\mathrm{det}^*(s) = \mathrm{det}^*(tF^*(t) \dots F^{*m-1}(t)) = \hat{\theta}(N_{F^m/F}(\mathbf{z}(\mu))) = \theta(t_0).$$

$\square$

Lemma 13.5. *Let $s \in \mathbf{G}_{\mathrm{u}}^*$. Then $\theta|_{\mathbf{Z}(\mathbf{G}_{\mathrm{u}}^F)}$ is the same for all pairs $(\mathbf{T}, \theta)$ in the geometric conjugacy class determined by s, and we have*

$$\chi|_{\mathbf{Z}(\mathbf{G}_{\mathrm{u}}^F)} = \chi(1) \cdot \theta|_{\mathbf{Z}(\mathbf{G}_{\mathrm{u}}^F)} \text{ for all } \chi \in \mathcal{E}(\mathbf{G}_{\mathrm{u}}^F, [s]).$$

Proof. This is [Mal07, Lemma 2.2]. $\square$

By Proposition 13.4 and Lemma 13.5, the unique underlying character of $\mathcal{Z}^F$ for $\chi_{[s]}$ must be $\theta|_{\mathcal{Z}^F}$, which is determined by $\theta(t_0) = \mathrm{det}^*(s)$.

We need information on the action of automorphisms of $\mathbf{G}_{\mathrm{u}}^F$ resp. $\mathbf{G}_{\mathrm{u}}^{*F^*}$ on the $\chi_{[s]}$, here we follow [Bru09a, Proposition 1].

Proposition 13.6. *Let σ either be a power of the graph automorphism Γ or the standard Frobenius map F_p of $\mathbf{G}_{\mathrm{u}}$. As discussed in Section 7 we define σ^* on $\mathbf{G}_{\mathrm{u}}^*$. Then we have*

$$\sigma(\mathcal{E}(\mathbf{G}_{\mathrm{u}}^F, s)) = \mathcal{E}(\mathbf{G}_{\mathrm{u}}^F, \sigma^*(s))$$

and therefore

$$\chi_{[s]}^\sigma = \chi_{[\sigma^*(s)]}.$$

Proof. For $\sigma = F_p$ this is [Bru09a, Proposition 1], for $\sigma = \Gamma$ follow the proof given in [Bru09a] word by word and observe that all the assertions remain valid. Note that in [Bru09a] the action is from the left via $\sigma(\chi_{[s]})(g) := \chi_{[s]}(\sigma^{-1}(g))$, whereas here the action is defined by $\chi_{[s]}^\sigma(g) := \chi_{[s]}(\sigma(g))$. $\square$

14. Steinberg map

The irreducible $\mathbf{k}$–representations of $\mathbf{G}_{\mathrm{sc}}$ can be parametrized by their *heighest weight*: Given an irreducible $\mathbf{k}\mathbf{G}_{\mathrm{sc}}$ module V there exists *a maximal vector* $v^{+} \in V$ and a *dominant weight* $\lambda \in X(\mathbf{T}_{\mathrm{sc}})$ such that $\mathbf{B}_{\mathrm{sc}}.\langle v^{+}\rangle_{\mathbf{k}} \subseteq \langle v^{+}\rangle_{\mathbf{k}}$, $t.v^{+} = \lambda(t)v^{+}$ for all $t \in \mathbf{T}_{\mathrm{sc}}$ and $V = \langle \mathbf{G}_{\mathrm{sc}}v^{+}\rangle_{\mathbf{k}}$. For an introduction to this theory see [Hum75, Chapter XI] or [MT11].

Recall the fundamental weights $\omega_{i} \in X(\mathbf{T}_{\mathrm{sc}})$ from Section 1. Let π_{i} be the trace function of an irreducible representation of heighest weight ω_{i}. We call

$$\pi : \mathbf{G}_{\mathrm{sc}} \to \mathbf{k}^{n}, \ g \mapsto (\pi_{1}(g), \ldots, \pi_{n}(g))$$

the *Steinberg map*. By [Ste65, Corollary 6.6.] this map seperates the semisimple conjugacy classes of the simply connected group $\mathbf{G}_{\mathrm{sc}}$. We will use this result to obtain labels for the semisimple conjugacy classes of $\mathbf{G}_{\mathrm{u}}^{F}$ that are compatible with the labels of the characters of $\mathrm{Irr}_{p'}(\mathbf{B}_{\mathrm{u}}^{F})$.

Lemma 14.1. *Let $z \in \mathbf{Z}(\mathbf{G}_{\mathrm{sc}})$ be a central element of $\mathbf{G}_{\mathrm{sc}}$ and $s \in \mathbf{G}_{\mathrm{sc}}$ semisimple.*

a) $\pi_{i}(zs) = \omega_{i}(z)\pi_{i}(s)$

b) $\pi_{i}(\Gamma(s)) = \pi_{\Gamma(i)}(s)$

c) $\pi_{i}(F_{q}(s)) = \pi_{i}(s)^{q}$

Proof. Let $R : G \to V$ be a representation that affords π_{i} and v^{+} as above. We have $z \in \mathbf{T}_{\mathrm{sc}}$, therefore $z.v^{+} = \omega_{i}(z).v^{+}$. Let $g_{1}.v^{+}, \ldots, g_{m}.v^{+}$ be a basis of V. Since z is central $z.(g_{i}.v^{+}) = \omega_{i}(z)g_{i}.v^{+}$. Assertion a) follows.

By definition Γ stabilizes $\mathbf{B}_{\mathrm{sc}}$ so v^{+} is a maximal vector of $R \circ \Gamma$ as well. By Lemma 4.1 we have

$$\Gamma(t).v^{+} = \omega_{i}(\Gamma(t)).v^{+} = \omega_{\Gamma(i)}(t).v^{+},$$

thus $R \circ \Gamma$ is of heighest weight $\omega_{\Gamma(i)}$. This proves b).

For c) without loss of generality assume $s \in \mathbf{T}_{\mathrm{sc}}$. We can diagonalize $R(\mathbf{T}_{\mathrm{sc}})$ simultaneously and thus $\mathrm{tr}(R(F_{q}(s))) = \mathrm{tr}(R(s^{q})) = \mathrm{tr}(R(s)^{q}) = \mathrm{tr}(R(s))^{q}$. $\qquad\square$

We define

$$\hat{\pi} : \mathbf{G}_{\mathrm{u}} \to (\mathbf{k}^{\times})^{\mathrm{r}} \times \mathbf{k}^{n} \text{ by } \pi(sz) := (\det(sz), \pi_{1}(s)\hat{\omega}_{1}(z), \ldots, \pi_{n}(s)\hat{\omega}_{n}(z))$$

for $s \in \mathbf{G}_{\mathrm{sc}}$ and $z \in \mathcal{Z}$, where det is as defined in 7.2 and $\hat{\omega}_i$ is an extension of ω_i to $X(\mathbf{T}_{\mathrm{u}})$. By convention $\hat{\pi}_0 = \det$, i.e. we start indexing the components of $\hat{\pi}$ at zero. In case $\mathsf{D}_{\mathrm{even}}$ we understand this to mean $\hat{\pi}_{0^{(i)}} = \det^{(i)}$ for $i = 1, 2$.

Proposition 14.2. *The semisimple conjugacy classes of $\mathbf{G}_{\mathrm{u}}$ are seperated by $\hat{\pi}$ and the conjugacy classes mapping to $\left(\mathbb{F}_q^{\times}\right)^{\mathsf{r}} \times \mathbb{F}_q{}^n$ are precisely the $(q-1)^{\mathsf{r}} q^n$ different F_q–stable conjugacy classes of $\mathbf{G}_{\mathrm{u}}$.*

Proof. Let $\hat{\pi}(sz) = \hat{\pi}(s'z')$. Since $\det(sz) = \det(z) = \det(z')$ we have

$$z^{-1}z' \in \ker\det \cap\mathcal{Z} = \mathbf{Z}(\mathbf{G}_{\mathrm{sc}}),$$

so $s'z^{-1}z \in \mathbf{G}_{\mathrm{sc}}$. From $\pi_i(s')\omega_i(z') = \pi_i(s)\omega_i(z)$ for all $i \in \{1, \dots, n\}$ we have $\pi_i(s') = \pi_i(s)\hat{\omega}_i(z'z^{-1}) = \pi_i(sz^{-1}s')$ by Lemma 14.1. Since π seperates classes on $\mathbf{G}_{\mathrm{sc}}$ we have $[s] = [s'z^{-1}z']$ in $\mathbf{G}_{\mathrm{sc}}$ and since z is central $[sz] = [s'z']$ in $\mathbf{G}_{\mathrm{u}}$.

By Lemma 14.1 we have $\pi_i(F_q(s)) = \pi_i(s)^q$, and since the action of F_q on $\mathcal{Z}$ is given by taking q–th powers as well, we have $\hat{\pi}_i(F_q(sz)) = \hat{\pi}_i(sz)^q$. Since F_q maps semisimple classes to semisimple classes, $[sz]$ is F_q–stable if and only if $\hat{\pi}_i(sz) \in \mathbb{F}_q$ for all $i \in \{0, \dots, n\}$. $\qquad\square$

Now $\mathbf{G}_{\mathrm{u}}$ has connected center and $[\mathbf{G}_{\mathrm{u}}, \mathbf{G}_{\mathrm{u}}] = \mathbf{G}_{\mathrm{sc}}$ is simply connected, so the F_q– stable semisimple conjugacy classes are precisely the semisimple conjugacy classes of $\mathbf{G}_{\mathrm{u}}^{F_q}$ [Car85, Corollary 3.7.2 and Proposition 3.7.3]. For $sz \in \mathbf{G}_{\mathrm{u}}^{F_q}$ we call $(b_0, \dots, b_n) = \hat{\pi}(sz)$ the *label* of $[sz]$.

From now on we are considering the case of the semisimple conjugacy classes of $\mathbf{G}_{\mathrm{u}}^*$ as defined in Section 7.

Lemma 14.3. *Let Γ be a graph automorphism of $\mathbf{G}_{\mathrm{u}}$, $(b_0, \dots, b_n)$ the label of some F_q^*–stable conjugacy class $[sz]$ of $\mathbf{G}_{\mathrm{u}}^*$ and $(b_0', \dots, b_n')$ the label of $[\Gamma^*(sz)]$. Then*

- $b_0' = \Gamma_{\mathcal{Z}}(b_0)$ *where $\Gamma_{\mathcal{Z}}$ is the action of Γ on $\left(\mathbb{F}_q^{\times}\right)^{\mathsf{r}}$ induced by $\mathbf{z}^{-1}$ and*

- $b_i' = b_{\Gamma^{-1}(i)} b_0^{\mathsf{d}_i}$ *with the d_i from Proposition 9.1 for $i = 1, \dots, n$.*

Proof. By the duality of $\det^*$ and $\mathbf{z}$ (Proposition 7.5 and Proposition 7.6) we have

$$\langle \det^*, \mathbf{z}^* \rangle = \langle \det, \mathbf{z} \rangle = \mathsf{I} \text{ and } \langle \det^* \circ \Gamma^*, \mathbf{z}^* \rangle = \langle \det, \Gamma \circ \mathbf{z} \rangle.$$

Write $z = \mathbf{z}(v)$ for some $v \in (\mathbf{k}^{\times})^{\mathsf{r}}$, then $b_0 = v^{\langle \det^*, \mathbf{z}^* \rangle} = v^{\langle \det, \mathbf{z} \rangle}$ and $b_0' = v^{\langle \det^* \circ \Gamma, \mathbf{z}^* \rangle} = v^{\langle \det, \Gamma \circ \mathbf{z} \rangle} = \Gamma_{\mathcal{Z}}(b_0)$. The second statement is immediate from Lemma 14.1 and Proposition 9.1:

$$\hat{\pi}(\Gamma^*(sz))_i = \pi_i(\Gamma^*(s))\hat{\omega}_i^*(\Gamma^*(z)) = \pi_{\Gamma^*(i)}(s)\hat{\omega}_{\Gamma^*(i)}^*(z)\det^*(z)^{\mathsf{d}_i} = \hat{\pi}(sz)_{\Gamma^*(i)} b_0^{\mathsf{d}_i}.$$

Recall $\Gamma^* = \Gamma^{-1}$ on $\{1, \dots, n\}$. $\qquad\square$

III. The p'–characters of $\mathbf{G}_{\mathrm{u}}^F$ and $\mathbf{G}_{\mathrm{sc}}^F$

Next consider a Frobenius map $F = F_q \circ \Gamma$ that involves a non–trivial graph automorphism. To find the full set of semisimple classes of $\mathbf{G}_{\mathrm{u}}^*$ that are stabilized by F^* it suffices to consider the set of semisimple classes of $\mathbf{G}_{\mathrm{u}}^{*F_{q^w}^*} \supseteq \mathbf{G}_{\mathrm{u}}^{*F^*}$ that is stabilized by F^*.

We recall notation from Section 10: Let $A_1, \ldots, A_m$ be the orbits of Γ on $\{1, \ldots, n\}$ and $\mathsf{a}_i \in A_i$ the fixed representatives of each orbit.

Proposition 14.4. *Let $sz \in \mathbf{G}_{\mathrm{u}}^{*F_{q^w}^*}$ with $\hat\pi(sz) = (b_0, \ldots, b_n)$. Then $[sz]$ is an F^*–stable conjugacy class of $\mathbf{G}_{\mathrm{u}}^*$ if and only if*

- $b_0 = \Gamma_{\mathcal{Z}}(b_0)^q$ *where $\Gamma_{\mathcal{Z}}$ is the action of Γ on $\left(\mathbb{F}_q^\times\right)^{\mathsf{r}}$ induced by $\mathbf{z}^{-1}$,*

- $b_i = b_{\Gamma^{-1}(i)}^q b_0^{q\mathsf{d}_i}$ *with the $\mathsf{d}_i \in \mathbb{Z}$ from Proposition 9.1 and*

- $b_0^{\mathsf{r}_i} = b_{\mathsf{a}_i}^{q^{|A_i|}-1}$ *for $i \in \{1, \ldots, n\}$, for certain $\mathsf{r}_i \in \mathbb{Z}$.*

Proof. The first two statements follow from Lemma 14.3 and Lemma 14.1 c). Now applying the second statement $|A_i|$ times we see that a relation of the form $b_{\mathsf{a}_i}^{q^{|A_i|}} = b_{\mathsf{a}_i} b_0^{\mathsf{r}_i}$ holds for some $\mathsf{r}_i \in \mathbb{Z}$. $\qquad\square$

If some F^*–stable class $[sz]$ has $F_{q^w}^*$–label $(b_0, \ldots, b_n)$ we call $(b_0, b_{\mathsf{a}_1}, \ldots, b_{\mathsf{a}_m})$ the label of $[sz]$. If Γ is trivial both definitions coincide. In case $\mathsf{D}_{\mathrm{even}}$ and Γ non–trivial of order 2 we call $(b_{0^{(1)}}, b_1, \ldots, b_{n-1})$ the label.

Proposition 14.5. *Let $(b_0, \ldots, b_m)$ be the label of some F^*–stable conjugacy class of $\mathbf{G}_{\mathrm{u}}^*$. Then the r_i from Proposition 14.4 and the r_i from Remark 10.3 coincide. If $|A_i| = w$ then $\mathsf{r}_i = 0$, otherwise the r_i are equal to the d_i from Proposition 9.1.*

Proof. When $|A_i| = w = o(\Gamma)$ then $t_i^{q^w-1} = 1$ since $t_i = N_{F^w/F}(\omega_{\mathsf{a}_i}^\vee(\mu))$ and μ is of order $q^w - 1$. The b_i are in $\mathbb{F}_{q^w}$ so $b_i^{q^w-1} = 1$, as well. Thus we are only interested in the orbits of Γ on $\{1, \ldots, n\}$ of length smaller than w. The following cases can occur:

Type	i	$|A_i|$
A_{2m-1}	$i = m$	1
D_n	$i \leq n - 2$	1
E_6	$i = 2, 4$	1

For $w = 2$ in types A_{2m-1}, D_{2m+1} and E_6 and i from the above table we have

$$t_i = \omega_{\mathsf{a}_i}^\vee(\mu)\Gamma(\omega_{\mathsf{a}_i}^\vee(\mu))^q = \omega_{\mathsf{a}_i}^\vee(\mu)^{1+q}\mathbf{z}(\mu)^{q\mathsf{d}_i} \tag{1}$$

by Proposition 9.1. Thus

$$t_i^q = \omega_{\mathsf{a}_i}^\vee(\mu)^{q+q^2}\, \mathbf{z}(\mu)^{q^2\mathsf{d}_i} = \omega_{\mathsf{a}_i}^\vee(\mu)^{1+q}\, \mathbf{z}(\mu)^{\mathsf{d}_i} = t_i\, \mathbf{z}(\mu)^{(1-q)\mathsf{d}_i} = t_i t_0^{\mathsf{d}_i},$$

by Equation (1) and the definition of $t_0 := N_{F^w/F}(\mathbf{z}(\mu)) = \mathbf{z}(\mu)^{1-q}$. By Proposition 14.4 we have $b_i^q = b_i b_0^{-q\mathsf{d}_i}$. But $b_0^{q+1} = 1$, so $b_i^q = b_i b_0^{\mathsf{d}_i}$ in this case.

Next we consider $w = 2$ in type D_{2n}. Equation (1) still holds by the multi index convention 6.7 and $\mathsf{d}_i = [0,0]$ for $i \leq n-2$ by Example 9.3. Thus

$$t_i^q = t_i \text{ and } b_i^q = b_i.$$

When $w = 3$, $i = 2$ in type D_4, review Definition 10.6 and the comments before it. By convention $\mathbf{G}_\mathrm{u} = \mathbf{G}_\mathrm{sc}$ in this case, so $b_0 = 1$, $t_0 = 1$, $b_2^q = b_2$ and $t_2^q = t_2$. $\qquad\qquad\square$

14.1. Other parametrizations of semisimple classes

In type A_n and C_n the label of a semisimple conjugacy class could have been obtained as the coefficients of the characteristic polynomial of the class in the natural matrix representation. In those cases our label and the coefficients of the characteristic polynomial are closely related or even equal [Jan03, Part II, Chapter 2, 2.16 and 2.17]. Unfortunately no faithful matrix representation of dimension n or $2n$ of $\mathbf{G}_\mathrm{sc}$ exists for type B_n and D_n.

There is a parametrization of the semisimple conjugacy classes of $\mathrm{SO}_n^F = (\mathbf{G}_\mathrm{sc}/Z^*)^F$ by their characteristic polynomials and certain parameters ψ^+, ψ^- as described by [Wal63], but no obvious or natural relationship between our parametrization and that one is known to us. In fact one could construct a bijection of classes of $\mathbf{G}_\mathrm{u}^F$ obtained by some rather tedious counting arguments (using the tools of [CE04, Chapter 16]) with our labels, but it is not clear (to the author) how this bijection can be made into respecting central characters or automorphisms.

15. Statement of results

Let $\hat{f} : \mathrm{Irr}_{p'}(\mathbf{B}_{\mathrm{u}}^F) \to \mathrm{Irr}_{p'}(\mathbf{G}_{\mathrm{u}}^F)$ be defined as follows: First map $\psi \in \mathrm{Irr}_{p'}(\mathbf{B}_{\mathrm{u}}^F)$ to the label $(a_0, \ldots, a_m)$ obtained in sections 10 and 8. Now map to the semisimple conjugacy class $[sz]$ of $\mathbf{G}_{\mathrm{u}}^{*F^*}$ with $\hat{\pi}(sz) = (a_0, \ldots, a_m)$, which exists by Proposition 14.2 and Proposition 14.4. In the twisted case we see that the entries of the labels fulfill the same relations on both sides (Proposition 14.5). By Proposition 13.1 $[sz]$ defines a unique semisimple character $\chi \in \mathrm{Irr}_{p'}(\mathbf{G}_{\mathrm{u}}^F)$, set $\hat{f}(\psi) := \chi$.

Theorem 15.1. *The map $\hat{f}$ defined above is a bijection of $\mathrm{Irr}_{p'}(\mathbf{B}_{\mathrm{u}}^F)$ with $\mathrm{Irr}_{p'}(\mathbf{G}_{\mathrm{u}}^F)$ and we have $\hat{f}(\chi^\gamma) = \hat{f}(\chi)^\gamma$ for field and graph automorphisms γ of $\mathbf{G}_{\mathrm{u}}^F$ and $\chi \in \mathrm{Irr}_{p'}(\mathbf{B}_{\mathrm{u}}^F)$.*

Proof. The map $\hat{f}$ is a bijection as a composition of bijections as shown in all the steps used to define $\hat{f}$. The action of field automorphisms is the action of the corresponding field automorphism on the label (Proposition 9.4 and Lemma 14.1). The compability of the bijection with graph automorphisms in the untwisted case is given by Proposition 9.4 and Lemma 14.3. □

Proposition 15.2. *The map $\hat{f}$ respects central characters: We have $\mathcal{Z}^F \subseteq \mathbf{B}_{\mathrm{u}}^F$ and $\mathcal{Z}^F \subseteq \mathbf{G}_{\mathrm{u}}^F$. If $\nu \in \mathrm{Irr}(\mathcal{Z}^F)$ is the unique central character below $\chi \in \mathrm{Irr}_{p'}(\mathbf{B}_{\mathrm{u}}^F)$, then it is also the one below $\hat{f}(\chi)$.*

Proof. Proposition 12.1 and the combination of Proposition 13.4 and Lemma 13.5. □

Theorem 15.3. *The map $\hat{f}$ respects multiplication by linear characters, i.e., the restriction of $\hat{f}$ to $\mathrm{Irr}(\mathbf{B}_{\mathrm{u}}^F \,|\, 1_{\mathbf{B}_{\mathrm{sc}}^F})$ defines a bijection with $\mathrm{Irr}(\mathbf{G}_{\mathrm{u}}^F \,|\, 1_{\mathbf{G}_{\mathrm{sc}}^F})$ and $\hat{f}(\lambda\chi) = \hat{f}(\lambda)\hat{f}(\chi)$ for all $\chi \in \mathrm{Irr}_{p'}(\mathbf{B}_{\mathrm{u}}^F)$ and $\lambda \in \mathrm{Irr}(\mathbf{B}_{\mathrm{u}}^F \,|\, 1_{\mathbf{G}_{\mathrm{sc}}^F})$.*

Proof. For this proof recall once more that we usually omit the map(s) $\mathbf{z} : (\mathbf{k}^\times)^{\mathsf{r}} \to \mathbf{Z}(\mathbf{G}_{\mathrm{u}})$, that we have fixed an identification of $\mathbb{F}_q^\times$ with the $(q-1)$–th roots of unity in $\mathbb{C}$ and appeal to the multi index convention 6.7 to deal with the case $\mathsf{D}_{\mathrm{even}}$. The label of a character $\lambda \in \mathrm{Irr}_{p'}(\mathbf{B}_{\mathrm{u}}^F \,|\, 1_{\mathbf{B}_{\mathrm{sc}}^F})$ has the form

$$(\det{}^*(z), \hat{\omega}_{\mathsf{a}_1}^*(z), \ldots, \hat{\omega}_{\mathsf{a}_m}^*(z))$$

by Proposition 11.4, Lemma 6.5 and the definition of the $\hat{\omega}_i^*$ in Definition 7.3. But this is just the label of the central element $z \in \mathbf{G}_{\mathrm{sc}}^{*F^*}$. The central elements of $\mathbf{G}_{\mathrm{u}}^{*F^*}$ parametrize exactly the linear characters $\mathrm{Irr}_{p'}(\mathbf{G}_{\mathrm{u}}^F \,|\, 1_{\mathbf{G}_{\mathrm{sc}}^F})$ by Proposition 13.3. It is also clear that the label of every central element of $z \in \mathbf{Z}(\mathbf{G}_{\mathrm{u}}^{*F^*})$ defines a character of $\mathrm{Irr}(\mathbf{B}_{\mathrm{u}}^F \,|\, 1_{\mathbf{B}_{\mathrm{sc}}^F})$ with such a label by setting $\lambda(\det{}^{-1}(t_0)) := z$. The multiplication

of χ (resp. $\hat{f}(\chi)$) by λ (resp. $\hat{f}(\lambda)$) is given by componentwise multiplication of the corresponding labels on both sides (Proposition 11.3 for $\mathbf{B}_u^F$ and Proposition 13.3 and the definition of $\hat{\pi}$ for $\mathbf{G}_u^F$). $\qquad\square$

Theorem 15.4. *There exists a bijection* $f : \mathrm{Irr}_{p'}(\mathbf{B}_{\mathrm{sc}}^F) \to \mathrm{Irr}_{p'}(\mathbf{G}_{\mathrm{sc}}^F)$ *such that* f *commutes with diagonal automorphisms,* f *preserves central characters and* $f(N_\chi)^\sigma = f(N_\chi^\sigma)$ *for all* $\sigma \in \mathrm{Aut}(\mathbf{G}_{\mathrm{sc}}^F)$ *that stabilize* $\mathbf{B}_{\mathrm{sc}}^F$.

Proof. Let $G = \mathbf{G}_u^F$ (or $G = \mathbf{B}_u^F$) and $H = \mathbf{G}_{\mathrm{sc}}^F$ (resp. $H = \mathbf{B}_{\mathrm{sc}}^F$). Consider the map $d := \mathbf{z} \circ \det$ (in case $\mathsf{D}_{\mathrm{even}}$ that is $d := (\mathbf{z}_1 \circ \det^{(1)}, \mathbf{z}_2 \circ \det^{(2)})$). In both cases G/H is isomorphic to $\mathcal{Z}^F$ (see the discussion after Proposition 11.3). Fix $\chi \in \mathrm{Irr}_{p'}(G)$ and recall Notation 5.1: The set of characters of H below χ is called N_χ and the set of characters of G above any $\eta \in N_\chi$ is called M_χ.

Let $\hat{f} : \mathrm{Irr}_{p'}(\mathbf{B}_u^F) \to \mathrm{Irr}_{p'}(\mathbf{G}_u^F)$ be the bijection constructed above. Suppose that G/H is cyclic. We have
$$M_\chi = \{\chi\lambda \mid \lambda \in \mathrm{Irr}(G \mid 1_H)\}$$
by Theorem 5.3. By Theorem 15.3 we know that $\hat{f}$ that is compatible with multiplication by linear characters arising from $\mathrm{Irr}(G/H)$, therefore $|M_\chi| = |M_{\hat{f}(\chi)}|$ and $|M_\chi| = |I_G(\eta) : H|$ by Theorem 5.2. Since G/H is cyclic this number uniquely determines $I_G(\eta)$. The action of G on $N_\eta = N_\chi$ is equivalent to the action on the left cosets of $I_G(\eta)$, therefore we can find a bijective mapping of N_χ onto $N_{\hat{f}(\chi)}$ respecting the conjugation action of G, that is, diagonal automorphisms of H. Every character in $\mathrm{Irr}_{p'}(\mathbf{B}_{\mathrm{sc}}^F)$ lies in some N_χ for $\chi \in \mathrm{Irr}_{p'}(\mathbf{B}_u^F)$ thus we can define a bijective mapping
$$f : \mathrm{Irr}_{p'}(\mathbf{B}_{\mathrm{sc}}^F) \longrightarrow \mathrm{Irr}_{p'}(\mathbf{G}_{\mathrm{sc}}^F)$$
in this way. By construction it is compatible with the action of diagonal automorphisms and the other properties follow directly from those of $\hat{f}$.

We are left with the case when G/H is not cylic, e.g., type $\mathsf{D}_{\mathrm{even}}$. We can again apply Theorem 5.3 since all $\eta \in \mathrm{Irr}_{p'}(H)$ extend to their respective inertia groups by Proposition 11.3 on the side of $\mathbf{B}_u^F$ and by [CE04, Theorem 15.11] for $\mathbf{G}_u^F$. Unfortunately the number $|I_G(\eta) : H|$ does not determine $I_G(\eta)$ uniquely in this case. Since $\ker d = H$ and H acts trivial on N_χ the action of $d(G)$ on the cosets of $d(I)$ is also equivalent to the action of G on N_χ. Since $d(I)$ always contains $d(t_{0^{(1)}}) = (\mu^2, 1)$ and $d(t_{0^{(2)}}) = (1, \mu^2)$ it is of index at most 4 in $d(G)$. If it is of index 4 or 1 the action of G on N_χ is uniquely determined. So it remains to consider the case where $d(I)$ is of index 2 in $d(G)$. There are three possible non–equivalent actions: one of $d(t_{n-1}) = (\mu, 1)$, $d(t_n) = (1, \mu)$ and $d(t_1) = (\mu, \mu)$ acts trivially, i.e., is contained in $d(I)$ and the other two have the same non–trivial action. For fixed $k, k' \in \{1, 2\}$, $k \neq k'$ consider the group $H^* := \ker \det^{(k)}$. We have $H \lhd H^* \lhd G$ and G/H^* and H^*/H are cyclic. We define $M_\chi^* := \{\chi\lambda \mid \lambda \in \mathrm{Irr}(G/H^*)\}$ and $N_\chi^* \subseteq \mathrm{Irr}_{p'}(H^*)$ as the

set of characters of H^* below χ, let $I^* := I_G(\eta^*)$ be the inertia group of $\eta^* \in N_\chi^*$ in G. We proceed as in the cyclic case: $d^* := \det^{(k')}$ provides an isomorphism of G/H^* with $\mathbb{F}_q^\times$ and $|d^*(I^*)| = |M_\chi^*|$. There are only two possible cases

    1. if $|M_\chi^*| = q - 1$ then $|\operatorname{Irr}(H^* \mid \eta)| = \frac{q-1}{2}$ and

    2. if $|M_\chi^*| = \frac{q-1}{2}$ then $|\operatorname{Irr}(H^* \mid \eta)| = 2(q - 1)$.

In the first case we must have $|H^* : I_{H^*}(\eta)| = 2$ and there is an element $h_k \in H^*$ that doesn't stabilize η, otherwise we set $h_k := 1$. We repeat the construction interchanging the roles of k and k'. We can now distinguish the different actions of G on $d(I)$:

    1. h_1 non–trivial, h_2 trivial: $d(h_1)d(I) = d(t_{n-1})d(I) = d(t_1)d(I)$ and $d(t_n) \in d(I)$,

    2. h_2 non–trivial, h_1 trivial: $d(h_2)d(I) = d(t_n)d(I) = d(t_1)d(I)$ and $d(t_{n-1}) \in d(I)$,

    3. h_1 and h_2 non–trivial: $d(t_n)d(I) = d(t_{n-1})d(I)$ and $d(t_1) \in d(I)$.

Thus the action is decided by the three numbers $|M_\chi|$ and $|M_\chi^*|$ for $k = 1, 2$ and the proposed bijection exists. $\qquad\square$

We have described the action of all automorphisms within sets $N_\chi \subseteq \operatorname{Irr}_{p'}(\mathbf{B}_{sc}^F)$ in Section 11 and computed the stabilizers in the group of automorphisms of all characters in $\operatorname{Irr}_{p'}(\mathbf{B}_{sc}^F)$. Unfortunately we know very little about the action of the (non–diagonal) automorphisms on $N_\chi \subseteq \operatorname{Irr}_{p'}(\mathbf{G}_{sc}^F)$, so we cannot refine the somewhat arbitrary choice of the bijection on the sets N_χ at this point. Also see Conjecture 16.1 in the next section.

Example 15.5. If $\mathbf{Z}(\mathbf{G}_{sc}^F)$ is trivial, the maps $\hat{f}$ and f conincide. If the group of automorphisms of $\mathbf{G}_{sc}^F$ is cyclic (i.e. the order of Γ is prime to the order of F_p and $\mathbf{G}_{sc}$ is not of type D_4) then our results about $\hat{f}$ show that $\mathbf{G}_{sc}^F$ is "good" (see the proof of [Bru09a, Theorem 5]).

16. Outlook

There are two main problems that need to be solved to show that $\mathbf{G}_{\mathrm{sc}}^F$ is good in the sense of [IMN07] with our method.

- The description of the action of non–diagonal automorphisms within the sets $N_\eta \subseteq \mathrm{Irr}_{p'}(\mathbf{G}_{\mathrm{sc}}^F)$.
- The bijection also needs to preserve extension properties in the group of automorphisms.

Both problems are beyond the scope of this work. For the first problem our description of the action of automorphisms on $N_\eta \subseteq \mathrm{Irr}_{p'}(\mathbf{B}_{\mathrm{sc}}^F)$ in Section 11 gives some hints. In particular we show (Proposition 11.13) that for any graph or field automorphism that stabilizes N_η there always exists at least one character of N_η that is fixed by it. Therefore the action of field and graph automorphisms on the sets N_η can be solely described by their action on the conjugating elements of $\mathbf{G}_{\mathrm{u}}^F$, that is to say:

Conjecture 16.1. *Let N_η denote the orbit of $\eta \in \mathrm{Irr}_{p'}(\mathbf{G}_{\mathrm{sc}}^F)$ under the action of the group of diagonal automorphisms D and $A := \mathrm{Stab}_{\mathrm{Out}(\mathbf{G}_{\mathrm{sc}}^F)}(N_\eta)/D$. Then the action of A on N_η is equivalent to the conjugation action of A on $D/I_D(\eta)$.*

Note that the action of A on $D/I_D(\eta)$ is well defined regardless of truth or falsity of the conjecture. If D is cylic then $I_D(\eta) \triangleleft A$. In type $\mathsf{D}_{\mathrm{even}}$ an automorphism that acts non–trivially on D can only be contained in $A = \mathrm{Stab}_{\mathrm{Out}(\mathbf{G}_{\mathrm{sc}}^F)}(N_\eta)$ if it stabilizes $I_D(\eta)$, thus $I_D(\eta) \triangleleft A$ here as well.

For the extension problem we can make a few interesting observations, that show, that the extension properties pose a non–trivial problem.

Proposition 16.2. *("The common case") Let $\chi \in \mathrm{Irr}_{p'}(\mathbf{G}_{\mathrm{u}}^F)$. Suppose M_χ, and thus N_χ, are stabilized by a field automorphism σ and that the reduced label $(a_0, \ldots, a_n)$ of χ has a non–zero λ^1–position (see the Proof of Theorem 11.8 for the definition of "reduced label" and Example 11.9 for a definition of λ^1–position). Then M_χ contains a character that is stabilized by σ.*

Proof. Since the λ^1–position i is non–zero, we find another label $(a'_0, \ldots, a'_n)$ in M_χ with $a'_i = 1$ by multiplying with a_i^{-1} in this position. Now σ stabilizes M_χ and fixes $a'_i = 1$ by Proposition 9.4. All other labels in M_χ have a different value at the i–th position, since i is a λ^1–position and thus all possible values are taken. $\square$

We call this "the common case", since there exists a λ^1–position in the reduced labels for every type, and so at least $(q-1)$ of every q characters are in such a set. In this

case N_χ only contains a single character by Proposition 11.6. However, a set M_χ that is stabilized by a field automorphism, needs not always contain an invariant character.

Example 16.3. Consider $\mathbf{G}_{\mathrm{u}}$ of type B_2 with Frobenius map $F = F_9$ and $\chi \in \mathrm{Irr}_{p'}(\mathbf{G}_{\mathrm{u}}^{F})$ with label $(\mu, 0, 0)$ where μ is a generator of $\mathbb{F}_9^\times$. Then M_χ contains the characters with labels

$$(\mu, 0, 0), \ (\mu^3, 0, 0), \ (\mu^5, 0, 0) , (\mu^7, 0, 0),$$

none of which are fixed by F_3, but M_χ is stabilized by F_3.

There are even examples of $\chi \in \mathrm{Irr}_{p'}(\mathbf{G}_{\mathrm{u}}^{F} \,\big|\, 1_{\mathbf{Z}(\mathbf{G}_{\mathrm{sc}}^{F})})$ above the trivial character of the center and field automorphisms σ, such that M_χ does not contain a σ–invariant character even though the whole set M_χ is stabilized by σ.

Example 16.4. Consider $\mathbf{G}_{\mathrm{u}}$ of type A_8 with Frobenius map $F = F_{7^3}$ and field automorphism $\sigma = F_7$. Let χ be the character of $\mathrm{Irr}_{p'}(\mathbf{G}_{\mathrm{u}}^{F})$ with label $(1, 0, 0, \mu, 0, 0, 0, 0, 0)$ where μ is a primitive 9–th root of unity. Then M_χ (of size $(7^3 - 1)/3$) is stabilized by σ, but not a single character is fixed. The set $N_{f^{-1}(\chi)} \subseteq \mathrm{Irr}_{p'}(\mathbf{B}_{\mathrm{sc}}^{F})$ contains three characters each of which is fixed by σ (see Section 11).

IV. Appendix

17. List of Cartan matrices and their inverses

We use the same numbering as [Hum78], the Cartan matrices C can be found in [Hum78, Section 11.4], their inverses are given in Table 1 of [Hum78, Section 13.2] as the coefficients of fundamental weights. Note that Carter [Car72] and Humphreys [Hum78] use transposed notation. We have chosen to align with Humphreys.

17.1. Type A_n

Dynkin diagram
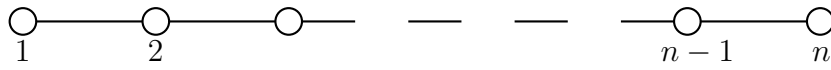
Cartan matrix

$$
\begin{pmatrix}
2 & -1 & 0 & & . & . & . & & 0 \\
-1 & 2 & -1 & 0 & & . & . & . & & 0 \\
0 & -1 & 2 & -1 & 0 & . & . & . & & 0 \\
. & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & -1 & 2 & -1 & 0 \\
0 & 0 & 0 & 0 & . & . & . & -1 & 2 & -1 \\
0 & 0 & 0 & 0 & . & . & . & 0 & -1 & 2
\end{pmatrix}
$$

i–th row of the inverse

$$
\frac{1}{n+1} \big(1(n-i+1) \quad 2(n-i+1) \quad . \quad . \quad i(n-i+1)| \quad i(n-i) \quad . \quad . \quad i \cdot 2 \quad i \big)
$$

17.2. **Type** B_n

Dynkin diagram
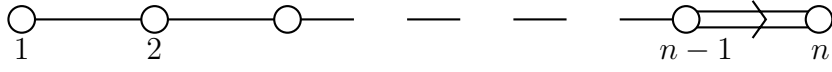
Cartan matrix

$$\begin{pmatrix} 2 & -1 & 0 & & . & . & . & & 0 \\ -1 & 2 & -1 & 0 & . & . & . & & 0 \\ 0 & -1 & 2 & -1 & 0 & . & . & . & 0 \\ . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & . & . & . & -1 & 2 & -2 \\ 0 & 0 & 0 & 0 & . & . & . & 0 & -1 & 2 \end{pmatrix}$$

i–th row of the inverse

$$\begin{cases} \left(1 \quad 2 \quad . \quad . \quad i| \quad i \quad . \quad . \quad i \right) & i \le n - 1 \\ \frac{1}{2} \left(1 \quad 2 \quad . \quad . \quad . \quad . \quad . \quad n \right) & i = n \end{cases}$$

17.3. **Type** C_n

Dynkin diagram
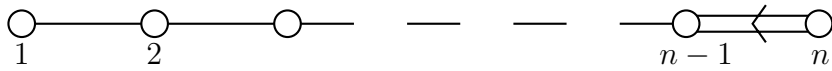
Cartan matrix

$$\begin{pmatrix} 2 & -1 & 0 & & . & . & . & & 0 \\ -1 & 2 & -1 & 0 & . & . & . & & 0 \\ 0 & -1 & 2 & -1 & 0 & . & . & . & 0 \\ . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & . & . & . & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & . & . & . & 0 & -2 & 2 \end{pmatrix}$$

i–th row of the inverse

$$\begin{pmatrix} 1 & 2 & . & . & i| & i & . & . & i & \frac{1}{2}i \end{pmatrix}$$

17.4. Type D_n

Dynkin diagram
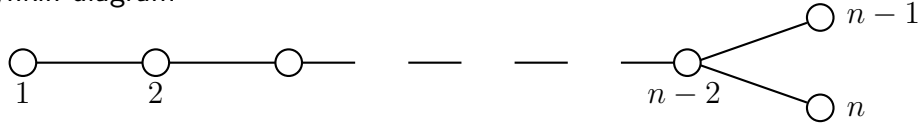
Cartan matrix

$$\begin{pmatrix}
2 & -1 & 0 & & & . & . & . & & 0 \\
-1 & 2 & -1 & 0 & & . & . & . & & 0 \\
0 & -1 & 2 & -1 & 0 & . & . & . & & 0 \\
. & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & -1 & 2 & -1 & 0 & 0 \\
. & . & . & . & . & . & -1 & 2 & -1 & -1 \\
0 & 0 & 0 & 0 & . & . & 0 & -1 & 2 & 0 \\
0 & 0 & 0 & 0 & . & . & 0 & -1 & 0 & 2
\end{pmatrix}$$

i–th row of the inverse

$$\begin{cases}
\begin{pmatrix} 1 & 2 & . & . & i| & i & . & i & \frac{i}{2} & \frac{i}{2} \end{pmatrix} & i \leq n-2 \\[2mm]
\begin{pmatrix} \frac{1}{2} & \frac{2}{2} & . & . & . & \frac{n-2}{2} & \frac{n}{4} & \frac{n-2}{4} \end{pmatrix} & i = n-1 \\[2mm]
\begin{pmatrix} \frac{1}{2} & \frac{2}{2} & . & . & . & \frac{n-2}{2} & \frac{n-2}{4} & \frac{n}{4} \end{pmatrix} & i = n
\end{cases}$$

17.5. Type E_6

Dynkin diagram
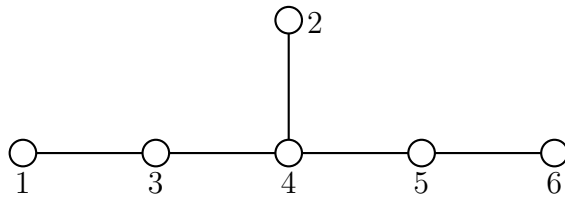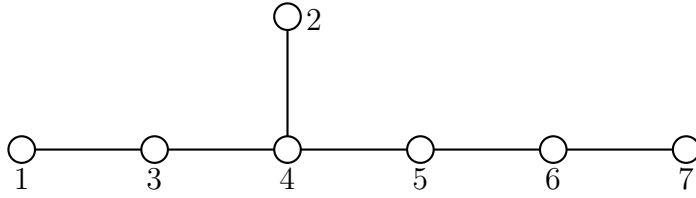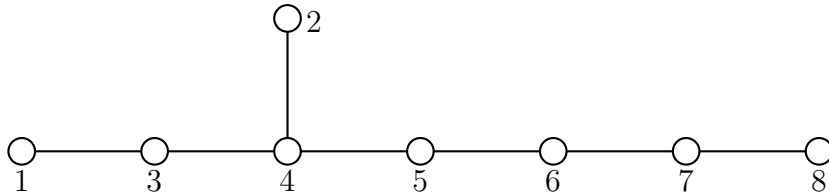
Cartan matrix and inverse

$$
\begin{pmatrix}
2 & 0 & -1 & 0 & 0 & 0 \\
0 & 2 & 0 & -1 & 0 & 0 \\
-1 & 0 & 2 & -1 & 0 & 0 \\
0 & -1 & -1 & 2 & -1 & 0 \\
0 & 0 & 0 & -1 & 2 & -1 \\
0 & 0 & 0 & 0 & -1 & 2
\end{pmatrix}, \quad
\frac{1}{3}
\begin{pmatrix}
4 & 3 & 5 & 6 & 4 & 2 \\
3 & 6 & 6 & 9 & 6 & 3 \\
5 & 6 & 10 & 12 & 8 & 4 \\
6 & 9 & 12 & 18 & 12 & 6 \\
4 & 6 & 8 & 12 & 10 & 5 \\
2 & 3 & 4 & 6 & 5 & 4
\end{pmatrix}
$$

17.6. **Type** E_7

Dynkin diagram

Cartan matrix and inverse

$$
\begin{pmatrix}
2 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 2 & 0 & -1 & 0 & 0 & 0 \\
-1 & 0 & 2 & -1 & 0 & 0 & 0 \\
0 & -1 & -1 & 2 & -1 & 0 & 0 \\
0 & 0 & 0 & -1 & 2 & -1 & 0 \\
0 & 0 & 0 & 0 & -1 & 2 & -1 \\
0 & 0 & 0 & 0 & 0 & -1 & 2
\end{pmatrix}, \quad
\frac{1}{2}
\begin{pmatrix}
4 & 4 & 6 & 8 & 6 & 4 & 2 \\
4 & 7 & 8 & 12 & 9 & 6 & 3 \\
6 & 8 & 12 & 16 & 12 & 8 & 4 \\
8 & 12 & 16 & 24 & 18 & 12 & 6 \\
6 & 9 & 12 & 18 & 15 & 10 & 5 \\
4 & 6 & 8 & 12 & 10 & 8 & 4 \\
2 & 3 & 4 & 6 & 5 & 4 & 3
\end{pmatrix}
$$

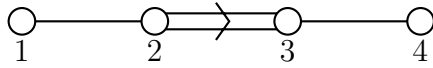## 17.7. **Type** E_8

Dynkin diagram

Cartan matrix and inverse

$$
\begin{pmatrix}
2 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 2 & 0 & -1 & 0 & 0 & 0 & 0 \\
-1 & 0 & 2 & -1 & 0 & 0 & 0 & 0 \\
0 & -1 & -1 & 2 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 2
\end{pmatrix},
\begin{pmatrix}
4 & 5 & 7 & 10 & 8 & 6 & 4 & 2 \\
5 & 8 & 10 & 15 & 12 & 9 & 6 & 3 \\
7 & 10 & 14 & 20 & 16 & 12 & 8 & 4 \\
10 & 15 & 20 & 30 & 24 & 18 & 12 & 6 \\
8 & 12 & 16 & 24 & 20 & 15 & 10 & 5 \\
6 & 9 & 12 & 18 & 15 & 12 & 8 & 4 \\
4 & 6 & 8 & 12 & 10 & 8 & 6 & 3 \\
2 & 3 & 4 & 6 & 5 & 4 & 3 & 2
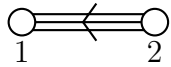\end{pmatrix}
$$

17.8. Type F_4

Dynkin diagram

Cartan matrix and inverse

$$
\begin{pmatrix}
2 & -1 & 0 & 0 \\
-1 & 2 & -2 & 0 \\
0 & -1 & 2 & -1 \\
0 & 0 & -1 & 2
\end{pmatrix},
\begin{pmatrix}
2 & 3 & 4 & 2 \\
3 & 6 & 8 & 4 \\
2 & 4 & 6 & 3 \\
1 & 2 & 3 & 2
\end{pmatrix}
$$

17.9. Type G_2

Dynkin diagram

Cartan matrix and inverse

$$
\begin{pmatrix}
2 & -1 \\
-3 & 2
\end{pmatrix},
\begin{pmatrix}
2 & 1 \\
3 & 2
\end{pmatrix}
$$

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