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A new Wick formula for products of White Noise distributions and application to Feynman path integrands

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Inhaltsverzeichnis

Introduction

White Noise analysis is an important special case of the infinite dimensional calculus Gaussian analysis. The mathematical framework offers various generalizations of concepts known from finite-dimensional analysis, like differential operators and Fourier transform. Detailed information concerning these methods can be found in the monographs [32], [4], [34], [51], [61] and the articles [65], [42], [77], [74]. Within Gaussian analysis a various kind of problems can be represented and solved in mathematical rigorous way. These can be treated only in an infinite dimensional setting or in the framework of generalized functions. Such kind of problems arise in mathematical physics (like statistical mechanics, quantum field theory, quantum mechanics and polymer physics) and applied mathematics (Stochastic analysis, Dirichlet forms, stochastic partial differential equations or financial mathematics).

This work can be separated into two main parts:

- Further development of Gaussian analysis.
- Applications to path integrals.

Gaussian analysis and generalized functions

Over the last thirty years there has been an increasing interest in Gaussian and especially White Noise analysis, based on its rapid development in mathematical structure and applications in various domains. One underlying point for this sophisticated structure was the circle of ideas going under the heading characterization theorems. These results (see [41], [57], [66], [42], [19]) and their variations and refinements (see, e.g., [52], [59], [60], [78], [84] and references quoted there) were the starting point for a deep insight into the structure of spaces of smooth and generalized random variables over the white noise spaces or, more generally, Gaussian spaces. For detailed information we refer again to the books [34], [51] and [61].

The basic technical idea in the development of this theory is the use of dual pairs of spaces of test and generalized functionals. Of course, the usefulness of a particular test function space

depends on the application one has in mind. Hence, various dual pairs appear in the literature. Here construction, characterization and transformation concepts of a few kinds of dual pairs are presented in Chapter 2, where necessary preliminaries are given in Chapter 1. For applications the following spaces are of most interest in this thesis:

The Hida spaces:

We recall the construction of the nuclear triplet

$$(\mathcal{N}) \subset L^2(\mu) \subset (\mathcal{N})',$$

and give the construction of the second quantized space $(\mathcal{N})$ solely in terms of the topology of a nuclear space $\mathcal{N}$, independent of the particular representation as a projective limit, see Section 2.5.1.

The spaces of regular distributions:

In Section 2.6 spaces of regular test and generalized functions are discussed. These spaces are of various interest since in all terms the chaos expansion is given by Hilbert space kernels. Hence, also the distributions have an expansion in a series of n-fold stochastic integrals. In our case the triplet

$$\mathcal{G} \subset L^2(\mu) \subset \mathcal{G}',$$

is of most interest for applications. This triple was first introduced in [67] and later characterized via Bargman-Segal-spaces in [21].

In Chapter 3 and Chapter 4 we give further development to the inner structure of Gaussian spaces. Among them are differential calculus and products of generalized functions.

Differential calculus

In Section 3.1 differential operators of first and second order on spaces of generalized functions are discussed. Especially, Gâteaux derivatives and the so called Gross Laplacian are introduced and properties are listed. The ensuing Section 3.2 introduces linear operators based on differential operators in spaces of test and generalized functions. These are important for applications of the concept of generalized functions, e.g. for a mathematical rigorous definition of Feynman integrands. First we follow [34] and [82], see also [51] and [61], and define this operator on the test function space $(\mathcal{N})$. Later we prove in which case an extension to generalized functions is sensible. Let us denote that an important example where such extensions make sense is Donsker's delta. Moreover, we give a representation of such operators in terms of exponentials of differential operators. In detail the following operators are discussed:

Translation:

A translation in a direction of an element from the complexification of the underlying Hilbert space is defined first for test function spaces, see e.g. [34], [67]. In Section 3.2.1 we prove that a translation can be extended to a well-defined operator from the space of regular generalized functions $\mathcal{G}'$ into itself. In addition a representation of a translation in such a direction η is given by the exponential of the Gâteaux derivative in η, see Theorem 3.21.

Orthogonal projection:

Following [82] we consider a composition of a regular test functions with an orthogonal projection in a direction of an element from the complexification of the underlying Hilbert space as an operator from $\mathcal{G}$ into itself. In Corollary 3.19 it is shown that there exists no extension of such an orthogonal projection to spaces of generalized functions. Nevertheless, there exist regular generalized functions for which this composition is well-defined. Thus we decompose an orthogonal projection into an orthogonal projection (in the corresponding direction) of every kernel and an exponential of the second Gâteaux derivative w.r.t. the direction of the projection, see Theorem 3.21. Here the orthogonal projection of the kernels can be extended to a well-defined operator from $\mathcal{G}'$ into itself, see Lemma 3.16. Hence, a characterization of the domain of an orthogonal projection in the space of regular generalized functions $\mathcal{G}'$ is given via the domain of an exponential of the second Gâteaux derivative, see Theorem 3.20.

Complex scaling:

The so called scaling operator σ_z, $z \in \mathbb{C}$, discussed in Section 3.2.3, is of huge interest. Some properties of σ_z are collected and its domain and range is specified, first where it acts continuously (close to [34] and [82]). Moreover, we discuss how to extend domains of σ_z, since for applications to path integrals. Furthermore, a representation as a combination of the second quantization of $z\mathbb{1}$ (here $\mathbb{1}$ denotes the identity on the corresponding space) and an exponential of the Gross Laplacian is given in Theorem 3.32. Let us denote that the second quantization of $z\mathbb{1}$ can be extended to a continuous operator from $(S)'$ into itself. Thus close to the orthogonal projection the domain of the scaling operator can be characterized via an exponential of the Gross Laplacian.

Localized complex scaling:

In the special case of white noise spaces we can define a localized complex scaling operator $\sigma_{z,t_0,t}$, $z \in \mathbb{C}$, $-\infty < t_0 < t < \infty$, which is roughly speaking a restriction of the

scaling operator to a finite interval. In Theorem 3.42 its representation as a combination of the second quantization of $z\mathbb{1}$, a projection and an exponential of the Gross Laplacian restricted to the time-interval $[t_0, t]$ is shown. Furthermore, with help of the adjoint of the localized scaling operator a relation to the kinetic energy in a path integral combined with a Gaussian fall-off is given, see Theorem 3.40.

In addition we show that the underlying differential operators generate semigroups of linear operators on the corresponding underlying space of test functions.

Products of generalized functions

Since the test function spaces form algebras the inner or pointwise products are well-defined in any case. Furthermore, via the dual pairing the pointwise product of a test function and an element from the corresponding space of generalized functions is still a well-defined object in this distribution space. Thus one can define a pointwise product of generalized functions whenever the product of the corresponding generalized chaos decompositions defines an element of a suitable distribution space. In contrast to the pointwise product the so called Wick product is closed under the corresponding multiplication in several distribution spaces like Hida distributions and regular generalized functions, see e.g. [43] and [25], [19], [62],[81] for applications. In Theorem 4.9 it is shown in which case both products coincide under some projection properties. Note that these properties go back to the strongly independence defined in [3], see also [7] and [17].

For application products of Donsker's delta with generalized functions are of enormous interest. In [55] a formula for n-times product of various Donsker's deltas is achieved. This formula is used to construct products of Donsker's delta of Brownian motion and Brownian bridge at several times, see Section 4.3. In addition a relation between both via the Wick or strongly independent pointwise product is given, see Theorem 4.16. But for applications also products of arbitrary generalized functions with several types of Donsker's deltas are important. In Section 4.4 products of regular test functions with Donsker's delta are considered. These always exist since Donsker's delta is a regular distribution. In Theorem 4.24 it is shown that a representation of this product can always be presented by a translation combined with an orthogonal projection operator of the regular test function multiplied with Donsker's delta. The product therein is a Wick product, especially an independent pointwise product. In addition, we characterize the set of regular generalized functions which can be multiplied with Donsker's delta as a subset of $\mathcal{G}'$ for which the orthogonal projection exists, see Theorem 4.25 and Theorem 4.28. Thereby the orthogonal projection is mentioned in the direction of the function where Donsker's delta is located.

Applications to path integrals

In many branches of theoretical physics, e.g. quantum field theory and polymer physics (path) integrals are of particular interest. The initial idea of averages over paths has a mathematical meaning only for the solution of the heat equation. In this case, one can present the solution by a path integral, based on the Wiener measure. This is stated by the famous Feynman-Kac formula

$$E\left(\exp\left(\int_{t_0}^{t} V(x_0 + B_r)\, dr\right) f(x_0 + B_t)\right),$$

for suitable $f, V : \mathbb{R}^d \to \mathbb{R}$, $x_0 \in \mathbb{R}^d$ and $0 \le t_0 \le t \le T < \infty$, see e.g. [68]. Furthermore, for suitable potentials the heat kernel K_V is given by

$$\begin{aligned} K_V(x, t; x_0, t_0) = \frac{1}{\sqrt{2\pi(t - t_0)}} \exp\left(-\frac{1}{2(t - t_0)}(x_0 - x)^2\right) \\ \times E\left(\exp\left(\int_{t_0}^{t} V\left(x_0 - \frac{r - t_0}{t - t_0}(x_0 - x) + B_r - \frac{r - t_0}{t - t_0} B_t\right) dr\right)\right), \quad (1) \end{aligned}$$

for $0 < t_0 < t < T$, $x, x_0 \in \mathbb{R}^d$, see e.g. [31].

There have been a lot of attempts to write down solutions of complex scaled heat equations (like e.g. the Schrödinger equation) as a (path) integral in a mathematical rigorous way. The methods used in this context (e.g. analytic continuation, limits of finite dimensional approximation and Fourier transform) are always more involved and less direct than in the euclidean – i.e. Feynman-Kac – case. This is stated by the following fact: one may easily prove that there exists no reasonably well-behaved translation invariant measure on any infinite-dimensional Hilbert space. More detailed, for any translation invariant measure on a infinite dimensional Hilbert space such that all balls are measurable sets there must be many balls whose measure is either zero or ∞. Therefore, it is reasonable why the formal expression $D_\infty x$ used in some physical textbooks is problematic and misleading. But one may have hope that the ill defined 'measure' $D_\infty x$ combined with the kinetic energy term produces a well-defined complex measure with imaginary variance $\sigma^2 = i$, or that this combination can be represented as the limit of Gaussian measures. But this causes problems if we assume that cylinder functions are integrated in the obvious way, see [14]. In [6] it is shown that for any finite (complex or real) measure with N-dimensional densities

$$p_{t_N > \cdots > t_0}(x_N, \cdots, x_0) = \prod_{j=1}^{N} \frac{1}{\sqrt{2\pi i\gamma(t_j - t_{j-1})}} \exp\left(i\frac{(x_j - x_{j-1})^2}{2\gamma(t_j - t_{j-1})}\right)$$

one must have that $i\gamma \in \mathbb{R}^+$. Hence, there is no hope for measure theory to solve problems with path integration. Instead of giving a long list of publications with different approaches to path

integrals, we refer to [1] and the large number of references therein. We choose a white noise approach to construct a complex scaled heat kernel as the expectation of a generalized function for a new class of potentials. Furthermore, we construct Feynman integrands and give a relation between the general Gaussian ansatz for path integrals and the concept of the complex scaled heat kernel.

Complex scaling of the heat equation

Chapter 5 proposes a strategy to construct a solution of the complex scaled heat equation as the generalized expectation of a generalized function of white noise. This is inspired by [13], see also [6]. That means we construct a complex scaled Feynman-Kac-kernel with white noise methods for suitable potentials V by giving a meaning to

$$K(x,t \mid x_0,t_0) = \frac{1}{\sqrt{2\pi(t-t_0)z^2}} \exp\left(-\frac{1}{2(t-t_0)z^2}(x_0-x)^2\right) \times E\left(\exp\left(\frac{1}{z^2}\int_{t_0}^{t} V\left(x_0 + \frac{r-t_0}{t-t_0}(x-x_0) + zB_r - \frac{r-t_0}{t-t_0}B_t\right) dr\right)\right), \quad (2)$$

which is a scaled version of (1). This is done by inserting Donsker's delta in order to fix the final point $x \in \mathbb{R}^d$, and taking a generalized expectation, i.e.,

$$K_V(x,t;x_0,t_0) = E\left(\exp\left(\frac{1}{z^2}\int_{t_0}^{t} V(x+zB_r)\,dr\right)\sigma_z\delta(B_t-(x-x_0))\right), \quad (3)$$

whenever the integrand is a generalized function of white noise, e.g. a Hida distribution. The fundamental concept in proving this is a Wick product representation of the integrand

$$\exp\left(\frac{1}{z^2}\int_{t_0}^{t} V(x+zB_r)\,dr\right)\sigma_z\delta(B_t-(x-x_0)) = \exp\left(\frac{1}{z^2}\int_{t_0}^{t} V\left(x_0 + \frac{r-t_0}{t-t_0}(x-x_0) + z\left(B_r - \frac{r-t_0}{t-t_0}B_t\right)\right) dr\right) \diamond \sigma_z\delta(B_t-(x-x_0)),$$

whose generalized expectation coincides with (2). For analytic potentials these Wick product representation is shown in Section 5.3. The proof is based on a finite dimensional approximation close to the construction of classical Feynman-Kac-kernels, see e.g. [31]. In Section 5.4 we generalized this to potentials fulfilling some regularity conditions. Therein the knowledge from Section 4.4 is used.

Feynman integrands

The idea to realize Feynman integrals within the white noise framework was first mentioned in the work of Hida and Streit [35]. The basic concept therein goes back to Feynman's primal

construction of averages over paths, where the integral in white noise is understood as the dual pairing of a distribution with a test function. In this case, the Feynman integrand is an element of a suitable space of distributions which depends on the interacting potential. In the white noise framework the first attempt to include interaction with a potential was done in [40]. Khandekar and Streit constructed the Feynman integrand for a large class of potentials including singular ones. Basically they constructed a strong Dyson series converging in the space of Hida distributions. This causes various works where for different classes of potentials, Feynman integrals have been constructed in White Noise analysis. In Chapter 6 we give a short overview of these classes and a general ansatz for the Feynman integrand in white noise (for detailed constructions see e.g. [10], [40], [34], [20], [49], [5], [74], [9], [56], [27] and [26]). In most cases perturbation techniques was used, see [40], [34], [20], [49], [74], [9]. Nevertheless, in all cases the corresponding Feynman integrand exists as a Hida distribution or is in the larger space of Kondratiev distributions.

In Chapter 7 we give a new strategy to construct Feynman integrands in white noise based on the concept of complex scaling, see again [13], [6] and [30]. This is based on the representation of the kinetic energy factor combined with a Gaussian fall-off as a localized scaling operator and its adjoint. Within this concept the Feynman integrand for a new class of analytic potentials is constructed as generalized function of white noise, called Doss class. Let us remark that this class includes also non-perturbative accessible potentials. The techniques used here are analytic continuation, a relation between the localized scaling operator and normalized exponentials and the formula for products of regular generalized functions and Donsker's delta found in Theorem 4.25. Thus a relation to the complex scaled Feynman-Kac kernel is given, see Theorem 7.1. Among these calculations it is shown that not only the expectation of the Feynman integrand but also the T-transform of it has a physical meaning as a time-dependent propagator, see Theorem 7.16. This is proven without a formal integration by parts. Beyond that, we also construct a linear time-dependent complex scaled Feynman-Kac formula. Parts of this results are already published in [27] and [26]. Although the classes of potentials described in Chapter 6 might not lead us to regular generalized functions (in the worst case Kondratiev distributions), we are able to construct the corresponding Feynman integrands with help of the complex scaling ansatz as Hida or Kondratiev distributions, see Section 7.5. Furthermore we combine such a class (the Khandekar-Streit class) with the Doss class, see Theorem, 7.25. The underlying concepts are the perturbation techniques described in Chapter 6.

Chapter 1

Preliminaries

1.1 Some facts on nuclear triples

We start in considering a real separable Hilbert space $\mathcal{H}$ with inner product $(\cdot,\cdot)$ and norm $|\cdot|$. If $\mathcal{N}$ is a separable nuclear space (in the sense of Grothendieck) which is densely topologically embedded in $\mathcal{H}$ we can construct the Gel'fand triple

$$\mathcal{N} \subset \mathcal{H} \subset \mathcal{N}'.$$

By an extension of the inner product in $\mathcal{H}$

$$\langle \eta, \xi \rangle = (\eta, \xi), \quad f \in \mathcal{H},\ \xi \in \mathcal{N},$$

one can realize the dual pairing $\langle \cdot, \cdot \rangle$ of $\mathcal{N}'$ and $\mathcal{N}$. Instead of reproducing the abstract definition of nuclear spaces we give a complete (and convenient) characterization in terms of projective limits of countably Hilbert spaces (see e.g., [64] or [70] not only for the definition but also for the proof of the characterization).

Theorem 1.1. *The nuclear Frechet space $\mathcal{N}$ can be represented as*

$$\mathcal{N} = \bigcap_{p \in \mathbb{N}} \mathcal{H}_p,$$

where $\{\mathcal{H}_p, p \in \mathbb{N}\}$ is a family of Hilbert spaces such that for all $p_1, p_2 \in \mathbb{N}$ there exists $p \in \mathbb{N}$ such that the embeddings $\mathcal{H}_p \hookrightarrow \mathcal{H}_{p_1}$ and $\mathcal{H}_p \hookrightarrow \mathcal{H}_{p_2}$ are of Hilbert-Schmidt type. The topology of $\mathcal{N}$ is given by the projective limit topology, i.e., the coarsest topology on $\mathcal{N}$ such that the canonical embeddings $\mathcal{N} \hookrightarrow \mathcal{H}_p$ are continuous for all $p \in \mathbb{N}$.

We denote the Hilbertian norms on $\mathcal{H}_p$ by $|\cdot|_p$. Without loss of generality we always suppose that for all $p \in \mathbb{N}$ and for all $\xi \in \mathcal{N}$ the relation $|\xi| \leq |\xi|_p$ holds and that the system of norms is

ordered, i.e., $|\cdot|_p \leq |\cdot|_q$ for $p < q$. General duality theory tells us that the dual space $\mathcal{N}'$ can be written as

$$\mathcal{N}' = \bigcup_{p\in\mathbb{N}} \mathcal{H}_{-p},$$

provided with inductive limit topology by using the dual family of spaces $\{\mathcal{H}_{-p} := \mathcal{H}'_p,\ p \in \mathbb{N}\}$. Remember that the inductive limit topology (w.r.t. this family) is the finest topology on $\mathcal{N}'$ such that the embeddings $\mathcal{H}_{-p} \hookrightarrow \mathcal{N}'$ are continuous for a $p \in \mathbb{N}$. It is convenient to denote the norm on $\mathcal{H}_{-p}$ by $|\cdot|_{-p}$.

Moreover, we want to introduce the notion of tensor powers of a nuclear space. The simplest way to do this is to start from usual tensor powers $\mathcal{H}_p^{\otimes n}$, $n \in \mathbb{N}$, of Hilbert spaces. Since there is no danger of confusion, we will preserve the notation $|\cdot|_0$, $|\cdot|_p$, and $|\cdot|_{-p}$, for the norms on $\mathcal{H}^{\otimes n}$, $\mathcal{H}_p^{\otimes n}$ and $\mathcal{H}_{-p}^{\otimes n}$, respectively, for all $n \in \mathbb{N}$. Using the definition

$$\mathcal{N}^{\otimes n} := \operatorname{pr}\lim_{p\in\mathbb{N}} \mathcal{H}_p^{\otimes n},$$

one can prove (see again [64] or [70]) that $\mathcal{N}^{\otimes n}$ is a nuclear space which is called the n-th tensor power of $\mathcal{N}$. The dual space of $\mathcal{N}^{\otimes n}$ can be written as

$$(\mathcal{N}^{\otimes n})' = \operatorname{ind}\ \lim_{p\in\mathbb{N}} \mathcal{H}_{-p}^{\otimes n}.$$

All of the results quoted above hold also for complex spaces, especially for the complexified space $\mathcal{N}_{\mathbb{C}}$. By definition an element $\theta \in \mathcal{N}_{\mathbb{C}}$ decomposes into $\theta = \xi + i\eta$, where $\xi, \eta \in \mathcal{N}$. If we also introduce the corresponding complexified Hilbert spaces $\mathcal{H}_{p,\mathbb{C}}$ the inner product becomes

$$(\theta_1, \theta_2)_{\mathcal{H}_{p,\mathbb{C}}} = (\theta_1, \overline{\theta_2})_{\mathcal{H}_p} = (\xi_1, \xi_2)_{\mathcal{H}_p} + (\eta_1, \eta_2)_{\mathcal{H}_p} + i(\eta_1, \xi_2)_{\mathcal{H}_p} - i(\xi_1, \eta_2)_{\mathcal{H}_p},$$

for $\theta_1, \theta_2 \in \mathcal{H}_{p,\mathbb{C}}$, $\theta_1 = \xi_1 + i\eta_1$, $\theta_2 = \xi_2 + i\eta_2$, where $\xi_1, \xi_2, \eta_1, \eta_2 \in \mathcal{H}_p$. Thus, we have introduced a nuclear triple

$$\mathcal{N}_{\mathbb{C}}^{\otimes n} \subset \mathcal{H}_{p,\mathbb{C}}^{\otimes n} \subset \left(\mathcal{N}_{\mathbb{C}}^{\otimes n}\right)'.$$

We also want to introduce the (Boson or symmetric) Fock space $\Gamma(\mathcal{H})$ of $\mathcal{H}$ by

$$\Gamma(\mathcal{H}) = \bigoplus_{n=0}^{\infty} \mathcal{H}_{p,\mathbb{C}}^{\otimes n},$$

with the convention $\mathcal{H}_{p,\mathbb{C}}^{\otimes 0} := \mathbb{C}$ and the Hilbertian norm

$$\|\vec{\varphi}\|_{\Gamma(\mathcal{H})} = \sum_{n=0}^{\infty} \left|\varphi^{(n)}\right|_0^2, \qquad \vec{\varphi} = \{\varphi^{(n)},\ n \in \mathbb{N}\} \in \bigoplus_{n=0}^{\infty} \mathcal{H}_{p,\mathbb{C}}^{\otimes n}.$$

Example 1.2. *(i) If we consider a finite dimensional Hilbert space $\mathcal{H}$ of dimension $d \in \mathbb{N}$, then*

$$\mathcal{N} = \mathcal{H} = \mathcal{N}' \cong \mathbb{R}^d.$$

(ii) One can choose the real space of Schwartz test function, $S(\mathbb{R})$, of rapidly decreasing smooth functions as our nuclear space $\mathcal{N}$. Then the Hilbert space $\mathcal{H}$ can be taken as the space of real-valued square-integrable functions w.r.t. the Lebesque measure, $\mathcal{H} = L^2(\mathbb{R}, \mathbb{R})$. As one can see in later chapters this is the convenient choice of $\mathcal{N}$ and $\mathcal{H}$ in White Noise analysis. Usually one chooses the Hilbertian norms $\left\{ |\cdot|_p,\ p \in \mathbb{N} \right\}$ topologizing $S(\mathbb{R})$, which are given by

$$|g|_p := |H^p g|_0, \quad g \in S(\mathbb{R}).$$

Here H denotes the Hamiltonian of the Harmonic oscillator with ground state eigenvalue 2, written in a formula

$$Hg(t) := -\frac{\partial^2}{\partial t^2} g(t) + (t^2 + 1)g(t), \quad t \in \mathbb{R}.$$

Hence, in this case $\mathcal{H}_p$ is the completion of $S(\mathbb{R})$ w.r.t. $|\cdot|_p$.

1.2 Holomorphy on local convex spaces

In this section we collect some facts from the theory of holomorphic functions in locally convex topological vector spaces (over the complex field $\mathbb{C}$), see e.g. [11]. These topics are necessary tools to characterize the Gaussian spaces which we want to introduce in Chapter 2, (see Section 2.5 for the characterization). So let $\mathcal{E}$ be a locally convex topological vector space and $\mathcal{L}(\mathcal{E}^n)$ the space of n-linear mappings from $\mathcal{E}^n$ into $\mathbb{C}$. Moreover, we denote $\mathcal{L}_s(\mathcal{E}^n)$ to be the subspace of $\mathcal{L}(\mathcal{E}^n)$ of symmetric n-linear forms and $\mathcal{P}^n(\mathcal{E})$ the n-homogeneous polynomials on $\mathcal{E}$. Then there is a linear bijection

$$\mathcal{L}_s(\mathcal{E}^n) : A \leftrightarrow \hat{A} \in \mathcal{P}^n(\mathcal{E}).$$

Definition 1.3. *Let $\mathcal{U} \subset \mathcal{E}$ be open and G a function from $\mathcal{U}$ to $\mathbb{C}$. G is said to be Gâteaux-holomorphic, short G-holomorphic, if for all $\xi \in \mathcal{U}$ and for all $\theta \in \mathcal{E}$ the mapping from $\mathbb{C}$ to $\mathbb{C}$: $\lambda \mapsto G(\xi + \lambda\theta)$ is holomorphic in some neighborhood of zero in $\mathbb{C}$.*

If G is G-holomorphic then for every $\theta \in \mathcal{U}$ there exists a sequence of homogeneous polynomials $\frac{1}{n!} d^n \widehat{G(\theta)}$ such that

$$G(\xi + \theta) = \sum_{n=0}^{\infty} \frac{1}{n!} \widehat{d^n G}(\theta)(\xi), \tag{1.1}$$

for all ξ from some open set $\mathcal{V} \subset \mathcal{U}$. We call the function G holomorphic, if for all $\theta \in \mathcal{U}$ there exists an open neighborhood $\mathcal{V}$ of zero such that the sum in (1.1) converges uniformly on $\mathcal{V}$ (in ξ) to a continuous function. As an alternative to the denotation "holomorphic" one also says "entire" in different kind of literature. We say that G is holomorphic at ξ_0 if there is an open set $\mathcal{U}$ containing ξ_0 such that G is holomorphic on $\mathcal{U}$.

Proposition 1.4. *G is holomorphic if and only if it is G-holomorphic and locally bounded.*

The proof of this proposition can be found e.g., in [11]. Let us explicitly consider a function holomorphic at the point $0 \in \mathcal{E} = \mathcal{N}_{\mathbb{C}}$. Then, since we do not want to discern between different restrictions of one function, we consider germs of holomorphic functions. That means, we identify F and G if there exists an open neighborhood $\mathcal{U}$ with $0 \in \mathcal{U} \subset \mathcal{N}_{\mathbb{C}}$ such that $F(\xi) = G(\xi)$ for all $\xi \in \mathcal{U}$.

Definition 1.5. *We define* $\mathrm{Hol}_0(\mathcal{N}_{\mathbb{C}})$ *to be the algebra of germs of complex-valued functions on* $\mathcal{N}_{\mathbb{C}}$ *which are holomorphic at zero. It is equipped with the inductive limit topology given by the following family of norms*

$$n_{p,l,\infty}(G) = \sup_{|\xi|_p \leq 2^{-l}} |G(\xi)|, \quad p, l \in \mathbb{N}.$$

Corollary 1.6. *A function $G : \mathcal{N}_{\mathbb{C}} \to \mathbb{C}$ is an element of* $\mathrm{Hol}_0(\mathcal{N}_{\mathbb{C}})$ *if and only if there exist $p \in \mathbb{N}$, $\varepsilon > 0$ and $0 < C < \infty$ such that:*

(i) For all $\xi \in \mathcal{N}_{\mathbb{C}}$ with $|\xi|_p \leq \varepsilon$ and for all $\theta \in \mathcal{N}_{\mathbb{C}}$ the function

$$\mathbb{C} \ni \lambda \mapsto G(\xi + \lambda\theta) \in \mathbb{C},$$

is analytic at $0 \in \mathbb{C}$.

(ii) For all $\xi \in \mathcal{N}_{\mathbb{C}}$ with $|\xi|_p \leq \varepsilon$ one has that $|G(\xi)| \leq C$.

Chapter 2

Gaussian Analysis

This chapter is, with regard to contents, a repetition of the basic concepts of Gaussian Analysis. Among this are construction, characterization, transformations and examples for elements of several spaces of test and generalized functions. Hence, we point the reader familiar with Hida, Kondratiev and regular distributions to the next chapter.

2.1 Gaussian spaces

To introduce a probability measure on the vector space $\mathcal{N}'$ we consider the σ-algebra $C_\sigma(\mathcal{N}')$ generated by cylinder sets:

$$C^{\xi_1,\dots,\xi_n}_{F_1,\dots,F_n} := \left\{ x \in \mathcal{N}' \,\middle|\, \langle x, \xi_1\rangle \in F_1, \dots, \langle x, \xi_n\rangle \in F_n \right\}, \quad \xi_j \in \mathcal{N},\ F_j \in \mathcal{B}(\mathbb{R}),\ j = 1, \dots, n,\ n \in \mathbb{N},$$

where $\mathcal{B}(\mathbb{R})$ denotes the Borel σ-algebra on $\mathbb{R}$. By the characteristic function

$$\int_{\mathcal{N}'} \exp\left(i\langle x, \xi\rangle\right) d\mu(x) = \exp\Big(-\tfrac{1}{2}|\xi|_0^2\Big), \quad \xi \in \mathcal{N},$$

the canonical Gaussian measure on $(\mathcal{N}', C_\sigma(\mathcal{N}'))$ is given via Minlos' theorem, see e.g. [33], [4] and [34]. We consider the completion of the resultant probability space $(\mathcal{N}', C_\sigma(\mathcal{N}'), \mu)$ and denote the completion of the σ-algebra $C_\sigma(\mathcal{N}')$ w.r.t. μ by $C_{\sigma,\mu}(\mathcal{N}')$.

If a measurable function f defined on $\mathcal{N}'$ is integrable w.r.t. μ, that means $\int_{\mathcal{N}'} |f(x)|\, d\mu(x)$ is finite, we call the integral $\int_{\mathcal{N}'} f(x)\, d\mu(x)$ the expectation of f and denote it by $E_\mu(f)$. If the measure is fixed we write only E instead of E_μ. Furthermore, we define the space of integrable functions w.r.t. μ by $L^1(\mu) := L^1\left(\mathcal{N}', C_{\sigma,\mu}(\mathcal{N}'), \mu\right)$.

The space of complex-valued functions which are square-integrable w.r.t. our measure μ

$$L^2(\mu) := L^2\left(\mathcal{N}', C_{\sigma,\mu}(\mathcal{N}'), \mu\right),$$

is the central space in the setting of Gaussian Analysis. The inner product of it is given by

$$(f,g)_{L^2(\mu)} := \int_{\mathcal{N}'} \overline{f(x)} g(x) d\mu(x) \quad f,g \in L^2(\mu).$$

Example 2.1. *(i) For $\varphi^{(n)} \in N_{\mathbb{C}}^{\hat{\otimes} n}$, $n \in \mathbb{N}$, and $\varphi_0 \in \mathbb{C}$ we define the smooth Wick monomials of order n corresponding to the kernels $\varphi^{(n)}$ by*

$$I\left(\varphi^{(n)}\right)(x) := \left\langle :x^{\otimes n}:, \varphi^{(n)} \right\rangle, \quad x \in \mathcal{N}',\ n \in \mathbb{N}_0.$$

Here the maps

$$\mathcal{N}' \ni x \mapsto :x^{\otimes n}: \in \left(\mathcal{N}^{\hat{\otimes} n}\right)',$$

are the so called Wick powers of order n, $n \in \mathbb{N}$, and $\left\langle :x^{\otimes 0}:, \xi^{\otimes 0} \right\rangle := \xi^{\otimes 0} := 1$, see e.g. [4] or [34]. Note that the smooth Wick monomials of different order are orthogonal w.r.t. the inner product in $L^2(\mu)$.

(ii) By using an approximation we can construct Wick monomials $I\left(f^{(n)}\right)$ with kernels $f^{(n)} \in \mathcal{H}_{\mathbb{C}}^{\hat{\otimes} n}$. I.e., for any sequence $\left(\varphi_j^{(n)}\right)_{j\in\mathbb{N}} \subset \mathcal{N}_{\mathbb{C}}^{\hat{\otimes} n}$ converging to $f^{(n)}$ in $\mathcal{H}_{\mathbb{C}}^{\hat{\otimes} n}$ one has convergence of the corresponding Wick monomials $I\left(\varphi^{(n)}\right)$ to $I\left(f^{(n)}\right)$ in any $L^p(\mu)$, $p \geq 1$, see e.g. [4]. Therefore, we use $I\left(f^{(n)}\right) = \left\langle :x^{\otimes n}:, f^{(n)} \right\rangle$ as a formal notation for the measurable monomials introduced above. Again, we have an orthogonality property for Wick monomials associated to kernels $f^{(n)} \in \mathcal{H}_{\mathbb{C}}^{\hat{\otimes} n}$ and $g^{(m)} \in \mathcal{H}_{\mathbb{C}}^{\hat{\otimes} m}$, $n, m \in \mathbb{N}$:

$$\left(I\left(f^{(n)}\right), \left(g^{(m)}\right)\right)_{L^2(\mu)} = \int_{\mathcal{N}'} \overline{\left\langle :x^{\otimes n}:, f^{(n)} \right\rangle} \left\langle :x^{\otimes n}:, g^{(m)} \right\rangle d\mu(x) = \delta_{n,m} \left(\overline{f^{(n)}}, g^{(n)}\right)_{\mathcal{H}^{\hat{\otimes} n}}, \quad (2.1)$$

where $\delta_{n,m}$ denotes the Kronecker delta.

(iii) An important example for an element of $L^2(\mu)$ is the so called Wick exponential

$$:\exp(\langle x,\xi\rangle): := \frac{\exp(\langle x,\xi\rangle)}{E_\mu(\exp(\langle x,\xi\rangle))} = \exp(\langle x,\xi\rangle)\exp\left(-\frac{1}{2}|\xi|_0^2\right) = \sum_{n=0}^{\infty} \frac{1}{n!} \left\langle :x^{\otimes n}:, \xi^{\otimes n} \right\rangle, \quad (2.2)$$

for $x \in \mathcal{N}'$ and $\xi \in \mathcal{N}$.

In the following we give more special examples by choosing a concrete nuclear space $\mathcal{N}$ and a corresponding Hilbert space $\mathcal{H}$, in this case the white noise spaces.

Example 2.2. *(i) We consider the real Schwartz triple (or white noise triple)*

$$S(\mathbb{R}) \subset L^2(\mathbb{R},\mathbb{R}) \subset S'(\mathbb{R}).$$

In the sense of a $L^2(\mu)$-limit a version of Wiener's Brownian motion starting in zero at time t_0 is given by:

$$B_{t_0,t}(\omega) := \langle \omega, \mathbf{1}_{[t_0,t)} \rangle = \int_{t_0}^{t} \omega(s)\, ds, \quad 0 < t_0 < t < \infty, \tag{2.3}$$

where the later informal expression rigorously only makes sense for $\omega \in L^2(\mathbb{R})$. Here $\mathbf{1}_A$ denotes the indicator function of $A \subset \mathbb{R}$. Similar, for $0 < t_0 < t < \infty$ we can define a version of a Brownian bridge starting in zero at time t_0 and ending in zero at time t via

$$B^{0\to 0}_{t_0,t,s}(\omega) := \langle \omega, \mathbf{1}_{[t_0,s)} \rangle - \frac{s-t_0}{t-t_0} \langle \omega, \mathbf{1}_{[t_0,t)} \rangle, \quad t_0 \le s \le t. \tag{2.4}$$

(ii) To define d-parametric white noise, $d \in \mathbb{N}$, we consider the Gel'fand triple

$$S(\mathbb{R}^d) \subset L^2(\mathbb{R}^d, \mathbb{R}) \subset S'(\mathbb{R}^d).$$

Hence, a version of d-parameter Brownian sheet can be defined by

$$B(x_1, \dots, x_d)(\omega) := \langle \omega, \mathbb{1}_{[x_1 \wedge 0, 0 \vee x_1)} \cdot \ldots \cdot \mathbb{1}_{[x_d \wedge 0, 0 \vee x_d)} \rangle, \quad (x_1, \dots, x_d)^T \in \mathbb{R}^d,\ \omega \in S'(\mathbb{R}^d).$$

(iii) Following [78] one can also define vector-valued white noise. In this case one starts with the real separable Hilbert space of vector-valued square-integrable functions $L^2_d(\mathbb{R}) = L^2(\mathbb{R}, \mathbb{R}^d)$, $d \in \mathbb{N}$. Then $S_d(\mathbb{R})$, the space of vector-valued Schwartz test functions is chosen as its densely embedded nuclear subspace. Here the topology on $S_d(\mathbb{R})$ is given by the system of Hilbertian norms

$$|g|_p^2 = \sum_{j=1}^{d} |g_j|_p^2, \quad g = (g_1, \dots, g_d) \in S_d(\mathbb{R}),\ g_j \in S(\mathbb{R}),\ 1 \le j \le d,\ p \in \mathbb{N}_0.$$

The resultant vector-valued white noise triple is then given by

$$S_d(\mathbb{R}) \subset L^2_d(\mathbb{R}) \subset S'_d(\mathbb{R}).$$

Again, a version of (d-dimensional) Brownian motion is given by

$$B(t, \omega) := \langle \omega, \mathbb{1}_{[0,t)} \rangle := \Big(\langle \omega_j, \mathbb{1}_{[0,t)} \rangle \Big)_{j=1,\dots,d},\ 0 < t_0 < t < \infty,\ \omega = (\omega_1, \dots, \omega_d) \in S'_d(\mathbb{R}).$$

Note that on $S'_d(\mathbb{R})$ the canonical Gaussian measure μ_d can be determined via the characteristic function

$$C_d(g) := \exp\left(-\frac{1}{2} \int_{\mathbb{R}} \sum_{j=1}^{d} g_j^2(r)\, dr \right), \quad g \in S_d(\mathbb{R}).$$

Now we want to consider the space of smooth polynomials on $\mathcal{N}'$:

$$\mathcal{P}(\mathcal{N}') := \left\{ \varphi \;\middle|\; \varphi(x) = \sum_{n=0}^{N} \left\langle x^{\otimes n}, \tilde{\varphi}^{(n)} \right\rangle,\ \tilde{\varphi}^{(n)} \in \mathcal{N}_{\mathbb{C}}^{\hat{\otimes} n},\ x \in \mathcal{N}',\ N \in \mathbb{N} \right\}.$$

Of course, any $\varphi \in \mathcal{P}(\mathcal{N}')$ can be represented as a smooth Wick polynomial, i.e.,

$$\mathcal{P}(\mathcal{N}') = \left\{ \varphi \;\middle|\; \varphi(x) = \sum_{n=0}^{N} \left\langle :x^{\otimes n}:, \varphi^{(n)} \right\rangle,\ \varphi^{(n)} \in \mathcal{N}_{\mathbb{C}}^{\hat{\otimes} n},\ x \in \mathcal{N}',\ N \in \mathbb{N} \right\}.$$

Moreover, one can show that $\mathcal{P}(\mathcal{N}')$ is dense in $L^2(\mu)$. Consequently, for any $f \in L^2(\mu)$, there exists an Itô-Segal-Wiener chaos decomposition given by

$$f(x) = \sum_{n=0}^{\infty} \left\langle :x^{\otimes n}:, f^{(n)} \right\rangle, \quad f^{(n)} \in \mathcal{H}_{\mathbb{C}}^{\hat{\otimes} n},\ x \in \mathcal{N}'. \tag{2.5}$$

Hence, by (2.1) the norm of $f \in L^2(\mu)$ can be represented as follows

$$\|f\|^2_{L^2(\mu)} = \sum_{n=0}^{\infty} n! \left(\overline{f^{(n)}}, f^{(n)} \right).$$

We introduce a generalized Laplace transform on the space $L^2(\mu)$, called S-transform.

Definition 2.3. *Let $f \in L^2(\mu)$ and $\xi \in \mathcal{N}$. Then we define the S-transform of f in ξ as the inner product*

$$S f(\xi) := \left(\overline{f}, :\exp\left(\langle \cdot, \xi \rangle\right): \right)_{L^2(\mu)} = \exp\left(-\frac{1}{2} |\xi|_0^2 \right) \int_{\mathcal{N}'} f(x) \exp\left(\langle \cdot, \xi \rangle\right) d\mu(x).$$

By definition of the Wick monomials we obtain that

$$S f(\xi) = \sum_{n=0}^{\infty} \left(\xi^{\otimes n}, f^{(n)} \right), \quad \xi \in \mathcal{N},$$

where of course $f^{(n)}$ denotes the n-th kernel of the chaos decomposition of $\in L^2(\mu)$, see (2.5). Moreover, there exists an entire extension

$$S f(\theta) = \sum_{n=0}^{\infty} \left(\theta^{\otimes n}, f^{(n)} \right), \quad \theta \in \mathcal{N}_{\mathbb{C}}. \tag{2.6}$$

Remark 2.4. *For an explanation of the connection between S-transform and the Segal-Bargmann transform we refer to [42].*

Nevertheless, for practical purposes the distribution space $P'(\mathcal{N}')$ is in some sense too large. This in detailed is pointed out in e.g. [82].

2.2 Generalized functions

In this section we introduce a preliminary distribution theory in infinite dimensional Gaussian analysis. We first choose $P(\mathcal{N}')$ as our (minimal) test function space. The idea to use spaces of this type as adequate spaces of test functions is rather old, see [46]. Therein it is also discussed in which sense this space is minimal. As mentioned in the last section $P(\mathcal{N}')$ is densely embedded in $L^2(\mu)$. The space $P(\mathcal{N}')$ may be equipped with various different topologies, but there exists a natural one such that $P(\mathcal{N}')$ becomes a nuclear space, see e.g. [4]. The topology on $P(\mathcal{N}')$ is chosen such that it becomes isomorphic to the topological direct sum of tensor powers $\mathcal{N}_{\mathbb{C}}^{\hat{\otimes} n}$, see e.g. [70],

$$P(\mathcal{N}') \cong \bigotimes_{n=0}^{\infty} \mathcal{N}_{\mathbb{C}}^{\hat{\otimes} n}.$$

The isomorphism is given by

$$P(\mathcal{N}') \ni \varphi,\ \varphi(x) = \sum_{n=0}^{\infty} \langle :x^{\otimes n}:, \varphi^{(n)} \rangle,\ x \in \mathcal{N}' \longleftrightarrow \left\{ \varphi^{(n)} \,\middle|\, n \in \mathbb{N}_0 \right\} \in \bigotimes_{n=0}^{\infty} \mathcal{N}_{\mathbb{C}}^{\hat{\otimes} n},$$

where in this case only a finite number of $\varphi^{(n)}$ is non-zero. Instead of reproducing the full construction we describe the notion of convergence of sequences w.r.t. this topology on $P(\mathcal{N}')$. We consider

$$\varphi \in P(\mathcal{N}'), \qquad \varphi(x) = \sum_{n=0}^{N(\varphi)} \left\langle :x^{\otimes n}:, \varphi^{(n)} \right\rangle,$$

and a sequence $(\varphi_j)_{j\in\mathbb{N}}$ given by

$$\varphi_j \in P(\mathcal{N}'), \qquad \varphi_j(x) = \sum_{n=0}^{N(\varphi_j)} \left\langle :x^{\otimes n}:, \varphi_j^{(n)} \right\rangle.$$

Moreover, we define the mapping $p_n : P(\mathcal{N}') \to \mathcal{N}_{\mathbb{C}}^{\hat{\otimes} n}$ by $p_n\varphi := \varphi^{(n)}$. Then the sequence $(\varphi_j)_{j\in\mathbb{N}}$ of smooth polynomials converges to $\varphi \in P(\mathcal{N}')$ if and only if the set $\{N(\varphi_j),\ j \in \mathbb{N}\}$ is bounded and $p_n\varphi_j \to p_n\varphi$ as j goes to infinity in $\mathcal{N}_{\mathbb{C}}^{\hat{\otimes} n}$ for all $n \in \mathbb{N}$. Now we consider the triple

$$\mathcal{P}(\mathcal{N}') \subset L^2(\mu) \subset \mathcal{P}'(\mathcal{N}'),$$

where $\mathcal{P}'(\mathcal{N}')$ denotes the dual space of $\mathcal{P}(\mathcal{N}')$ w.r.t. $L^2(\mu)$. The (bilinear) dual pairing $\langle\!\langle \cdot, \cdot \rangle\!\rangle$ between $\mathcal{P}'(\mathcal{N}')$ and $\mathcal{P}(\mathcal{N}')$ is connected to the (sesquilinear) inner product on $L^2(\mu)$ by

$$\langle\!\langle f, \varphi \rangle\!\rangle = (\bar{f}, \varphi)_{L^2(\mu)}, \quad f \in L^2(\mu), \quad \varphi \in \mathcal{P}(\mathcal{N}'). \tag{2.7}$$

The concept of expectation can be extended from L^2-functions to distributions $\Phi \in \mathcal{P}'(\mathcal{N}')$ by

$$E_\mu(\Phi) := \langle\langle \Phi, \mathbb{1} \rangle\rangle, \quad \varphi \in \mathcal{P}(\mathcal{N}'),$$

since the constant function $\mathbb{1}$ is in $\mathcal{P}(\mathcal{N}')$. We are interested in providing a description of $\mathcal{P}'(\mathcal{N}')$ via a natural decomposition of $\Phi \in \mathcal{P}'(\mathcal{N}')$. So let $\Phi^{(n)} \in \mathcal{N}'^{\hat{\otimes} n}_{\mathbb{C}}$. Then there exists a distribution $I\left(\Phi^{(n)}\right)$ acting on polynomials $\varphi \in \mathcal{P}(\mathcal{N}')$ as

$$\left\langle\left\langle I\left(\Phi^{(n)}\right), \varphi\right\rangle\right\rangle = n! \left\langle \Phi^{(n)}, \varphi^{(n)} \right\rangle.$$

As a formal notation we use $I\left(\Phi^{(n)}\right) = \left\langle :\cdot^{\otimes n}:, \Phi^{(n)} \right\rangle$ for the distribution introduced above. Then any $\Phi \in \mathcal{P}'(\mathcal{N}')$ has the unique decomposition

$$\Phi(x) = \sum_{n=0}^{\infty} \left\langle : x^{\otimes n} :, \Phi^{(n)} \right\rangle, \; x \in \mathcal{N}',$$

where the sum above converges in $\mathcal{P}'(\mathcal{N}')$. Moreover, we have that

$$\langle\langle \Phi, \varphi \rangle\rangle = \sum_{n=0}^{\infty} n! \left\langle \Phi^{(n)}, \varphi^{(n)} \right\rangle, \quad \varphi \in \mathcal{P}(\mathcal{N}'),$$

see e.g. [45].

2.3 Spaces of test and generalized functions

In this section we consider projective and inductive limits of sequences of Hilbert spaces, the so called test and generalized functions. For doing this we define the following Hilbertian norm for a smooth Wick polynomial $\varphi(x) = \sum_{n=0}^{N} \left\langle : x^{\otimes n} :, \varphi^{(n)} \right\rangle$, $x \in \mathcal{N}'$, by

$$\|\varphi\|^2_{p,q,\beta} := \sum_{n=0}^{\infty} (n!)^{1+\beta} 2^{nq} \left|\varphi^{(n)}\right|^2_p,$$

for any $p, q \in \mathbb{Z}, \beta \in [-1, 1]$. Then for $p, q \in \mathbb{N}, \beta \in [0, 1]$ the completion of $\mathcal{P}(\mathcal{N}')$ w.r.t. $\|\cdot\|^2_{p,q,\beta}$ is a Hilbert space which is denoted by $(\mathcal{H}_p)^\beta_q$. Equivalently, we can define

$$(\mathcal{H}_p)^\beta_q := \left\{ f \in L^2(\mu) \;\middle|\; f(x) = \sum_{n=0}^{\infty} \left\langle : x^{\otimes n} :, f^{(n)} \right\rangle, \; x \in \mathcal{N}', \; \|f\|^2_{p,q,\beta} < \infty \right\}.$$

The space of test functions $(\mathcal{N})^\beta$ is then defined as the projective limit of the spaces $(\mathcal{H}_p)^\beta_q$, i.e.,

$$(\mathcal{N})^\beta := \bigcap_{p,q \geq 0} (\mathcal{H}_p)^\beta_q.$$

For a test function $\varphi \in (\mathcal{N})^\beta$ there exists a unique chaos decomposition

$$\varphi(x) = \sum_{n=0}^{N} \left\langle :x^{\otimes n}:, \varphi^{(n)} \right\rangle, \quad x \in \mathcal{N}',$$

since $(\mathcal{N})^\beta \subset L^2(\mu)$. Note that the kernels $\varphi^{(n)} \in (\mathcal{N})_{\mathbb{C}}^{\hat{\otimes} n}$, see e.g. [34]. Hence, for every $\varphi \in (\mathcal{N})^\beta$ there exists an unique (in $L^2(\mu)$) point-wise defined representative $\widetilde{\varphi}$ which is strongly continuous from $\mathcal{N}'$ into $\mathbb{C}$.

To consider generalized functions we denote the dual spaces of $(\mathcal{H}_p)_q^\beta$ and $(\mathcal{N})^\beta$ w.r.t. $L^2(\mu)$ by $(\mathcal{H}_{-p})_{-q}^{-\beta}$ and $(\mathcal{N})^{-\beta}$, respectively. We denote the corresponding bilinear dual pairing by $\langle\!\langle \cdot, \cdot \rangle\!\rangle$. This is connected to the sesquilinear inner product on $L^2(\mu)$ as described in (2.7). From general duality theory we can conclude that

$$(\mathcal{N})^{-\beta} = \bigcup_{p,q \geq 0} (\mathcal{H}_{-p})_{-q}^{-\beta}.$$

Hence, every distribution is of finite order, i.e., for every $\Phi \in (\mathcal{N})^{-\beta}$ there exists $p, q \in \mathbb{N}_0$ such that $\Phi \in (\mathcal{H}_{-p})_{-q}^{-\beta}$. Moreover, by definition the Hilbert space $(\mathcal{H}_{-p})_{-q}^{-\beta}$ can be described as follows:

$$(\mathcal{H}_{-p})_{-q}^{-\beta} := \left\{ \Phi \in \mathcal{P}'(\mathcal{N}') \;\middle|\; \Phi^{(n)} \in (\mathcal{N})_{\mathbb{C}}'^{\hat{\otimes} n},\ \|\Phi\|_{-p,-q,-\beta}^2 < \infty \right\}.$$

Remark 2.5. *If we consider the white noise triplet as in Example 2.2 (i) and the Hilbertian norms $|\cdot|_p$ as in Section 1.1 then the spaces $(\mathcal{N}) = (\mathcal{N})^0$ and $(\mathcal{N})' = (\mathcal{N})^{-0}$ coincide with the well-known spaces of Hida test functions and distributions, $(S(\mathbb{R}))$ and $(S(\mathbb{R}))'$, which are usually denoted by (S) and $(S)'$. Further information can be found in e.g. [44], [41], [48], [4], [66], [34] and [42].*

So we get the following chain of spaces

$$(\mathcal{N})^1 \subset (\mathcal{N})^\beta \subset (\mathcal{N}) \subset L^2(\mu) \subset (\mathcal{N})' \subset (\mathcal{N})^{-\beta} \subset (\mathcal{N})^{-1}.$$

The space $(\mathcal{N})^{-1}$ is called the space of Kondratiev distributions. This is (in some sense) the largest space of generalized functions which can be characterized with help of the S-transform (see Remark 2.8 below).

Example 2.6. *Let us again consider the white noise triplet introduced in Example 2.2. Then an example for an element of $(S)'$ is the so called Gaussian white noise process $\omega(t)$, $t \geq 0$, which is the derivative of Brownian motion in the weak sense. This process is defined by*

$$\omega(t) := \langle \omega, \delta_t \rangle, \quad t \geq 0,$$

where $\delta_t \in S'(\mathbb{R})$ denotes the Dirac delta function.

Remark 2.7. *By Kolmogorov's continuity theorem (see e.g. [79]) there exists a continuous version $\tilde{B}(t),\ t \geq 0$ of $B(t),\ t \geq 0$, called standard Brownian motion. In detail the existence of a continuous version of test functions is shown in [51]. Among this work we always consider continuous versions of a test function in calculations. In contrast to this we consider Brownian motion as defined in Example 2.2. But this does not matter since we use equality of random variables in mean square or in the sense of generalized functions.*

2.4 Transformations of generalized functions

In this section we want to extend the S-transform introduced in Definition 2.3 to generalized functions. Moreover, we want to define a generalized Fourier transform for such functions. Let us consider the Wick exponential as in (2.2):

$$: \exp(\langle x, \xi \rangle) : := \sum_{n=0}^{\infty} \frac{1}{n!} \langle : x^{\otimes n} :, \xi^{\otimes n} \rangle, \quad \text{for } x \in \mathcal{N}' \text{ and } \xi \in \mathcal{N}.$$

Calculating its p, q, β-norm yields

$$\left\| : \exp(\langle \cdot, \xi \rangle) : \right\|_{p,q,\beta}^2 := \sum_{n=0}^{\infty} (n!)^{1+\beta} 2^{nq} \left| \frac{1}{n!} \xi^{\otimes n} \right|_p^2 = \sum_{n=0}^{\infty} (n!)^{\beta-1} 2^{nq} |\xi|_p^{2n}. \tag{2.8}$$

The expression (2.8) is finite for all $\beta < 1$ and therefore $: \exp(\langle x, \xi \rangle) : \in (\mathcal{N})^{\beta}$. Hence, using the dual pairing between $(\mathcal{N})^{\beta}$ and $(\mathcal{N})^{-\beta}$, there exists a natural extension of the S-transform of a distribution $\Phi \in (\mathcal{N})^{-\beta},\ \beta < 1$. I.e., for $\Phi \in (\mathcal{N})^{-\beta}$ with kernels $\Phi^{(n)}$, $n \in \mathbb{N}$, we can define the S-transform by

$$S(\Phi)(\xi) := \langle\!\langle\, \Phi, : \exp(\langle \cdot, \xi \rangle) : \,\rangle\!\rangle = \sum_{n=0}^{\infty} \left\langle \Phi^{(n)}, \xi^{\otimes n} \right\rangle,$$

for all $\xi \in \mathcal{N}$. In the case when $\beta = 1$ the norm in (2.8) is finite if and only if $2^q |\xi|_p^2 < 1$. Hence, in contrast to the case when $\beta \in [0, 1)$ the Wick exponentials are not in the test function space $(\mathcal{N})^1$. They are only in the spaces $(\mathcal{H}_p)_q^1$ for which $|\xi|_p^2 < 2^{-q}$.

Nevertheless, we can define the S-transform of an element of $(\mathcal{N})^{-1}$, since every distribution is of finite order. I.e., for all $\Phi \in (\mathcal{N})^{-1}$ there exist some $p, q \in \mathbb{N}_0$ such that $\Phi \in (\mathcal{H}_{-p})_{-q}^{-1}$. Thus, for all $\xi \in \mathcal{N}$ with $2^q |\xi|_p^2 < 1$ the S-transform of Φ is a well-defined object given by

$$S(\Phi)(\xi) := \langle\!\langle\, \Phi, : \exp(\langle \cdot, \xi \rangle) : \,\rangle\!\rangle = \sum_{n=0}^{\infty} \left\langle \Phi^{(n)}, \xi^{\otimes n} \right\rangle. \tag{2.9}$$

Close to (2.6) this definition can be directly extended to complex vectors $\theta \in \mathcal{N}_{\mathbb{C}}$ with $|\theta|_p^2 < 2^{-q}$ by

$$S(\Phi)(\theta) := \langle\!\langle\, \Phi, : \exp(\langle \cdot, \theta \rangle) : \,\rangle\!\rangle = \sum_{n=0}^{\infty} \left\langle \Phi^{(n)}, \theta^{\otimes n} \right\rangle.$$

In this case the Wick exponentials have complex kernels. Therefore, for all $\Phi \in (\mathcal{H}_{-p})_{-q}^{-1}$ the S-transform is well-defined for all θ from the following neighborhood of zero:

$$\mathcal{U}_{p,q} := \left\{ \theta \in \mathcal{N} \,\middle|\, |\theta|_p^2 < 2^{-q} \right\}, \quad p, q \in \mathbb{N}_0.$$

Remark 2.8. *Now we can understand why $(\mathcal{N})^{-1}$ is somehow the largest space of generalized functions, since for $\beta > 1$ the space $(\mathcal{N})^{-\beta}$ con not be defined. In this case $\left\| : \exp(\langle \cdot, \xi \rangle) : \right\|_{p,q,\beta}^2 = \sum_{n=0}^{\infty} (n!)^{1+\beta} 2^{nq} \left| \frac{1}{n!} \xi^{\otimes n} \right|_p^2$ is not finite for any p, q if $\xi \neq 0$. That means the Wick exponentials are not longer in $(\mathcal{H}_p)_q^{\beta}$ and the S-transform is no longer a well-defined object.*

A transformation, which is later useful for applications, is the generalized Fourier transform, called T-transform. This can be defined by its relation to the S-transform:

$$(T\Phi)(\theta) := \exp\left(-\frac{1}{2}(\theta, \theta)\right)(S\Phi)(i\theta), \quad \Phi \in (\mathcal{N})^{-1},\ \theta \in \mathcal{U}, \tag{2.10}$$

where $\mathcal{U} \subset \mathcal{N}_{\mathbb{C}}$ is an open neighborhood of zero such that the S-transform is well-defined. One can calculate that the following definition for the T-transform is equivalent

$$(T\Phi)(\theta) := \langle\!\langle\, \Phi, \exp(i\langle \cdot, \theta \rangle) \,\rangle\!\rangle, \quad \theta \in \mathcal{U}.$$

Therefore, it is indeed a generalized Fourier transform.

2.5 Characterization of generalized functions

In this section the characterization of Hida and Kondratiev distributions via their S- and T-transforms are given. Note that both transformations map a generalized function to a holomorphic function, as defined in Section 1.2. Hence, we show now that the S- and T-transforms give isomorphisms between the spaces of generalized functions and suitable spaces of holomorphic functions.

2.5.1 Hida distributions

The characterization of Hida distributions by their S- and T-transforms was first discussed in [66] and then shown in [42]. But before we formulate the characterization theorem we have to define the appropriate space of holomorphic functions.

Definition 2.9. *A function $F : \mathcal{N} \to \mathbb{C}$ is called U-functional iff*

(i) F is "ray-analytic": that means for all $\xi, \theta \in \mathcal{N}$ the mapping

$$\mathbb{R} \ni y \mapsto F(\xi + y\theta) \in \mathbb{C}$$

has an analytic continuation to $\mathbb{C}$ as an entire function.

(ii) F is uniformly bounded of order 2, i.e., there exist some constants $0 < K, D < \infty$ and some $p \in \mathbb{N}$ such that for all $z \in \mathbb{C}$, $\xi \in \mathcal{N}$

$$|F(z\xi)| \leq K \exp\left(D|z|^2 |\xi|_p^2\right).$$

From Section 1.2 one can easily conclude that U-functionals have a holomorphic extension to entire functions on $\mathcal{N}_{\mathbb{C}}$.

Theorem 2.10. *The following statements are equivalent:*

(i) $F : \mathcal{N} \to \mathbb{C}$ is a U-functional.

(ii) F is the T-transform of a unique Hida distribution $\Phi \in (\mathcal{N})'$.

(iii) F is the S-transform of a unique Hida distribution $\tilde{\Phi} \in (\mathcal{N})'$.

Theorem 2.11. *Let $\{F_n\}_{n\in\mathbb{N}}$ denote a sequence of U-functionals with the following properties:*

(i) For all $\xi \in \mathcal{N}$, $\{F_n(\xi)\}_{n\in\mathbb{N}}$ is a Cauchy sequence.

(ii) There exist some constants $P, Q > 0$ and $p \in \mathbb{N}_0$ such that the relation

$$\left|F_n(z\xi)\right| \leq P \exp\left(Q|z|^2 \, |\xi|_p^2\right)$$

holds for all $n \in \mathbb{N}$, $\xi \in \mathcal{N}$ and $z \in \mathbb{C}$. Here F_n denotes the entire analytic extension.

Then there exists a unique $\Phi \in (\mathcal{N})'$ such that $\left(T^{-1}F_n\right)_{n\in\mathbb{N}}$ converges strongly to Φ.

This theorem is also valid for the S-transform.

Theorem 2.12. *Let $(\Lambda, \mathcal{A}, \nu)$ denote a measure space and $\lambda \mapsto \phi(\lambda)$ a mapping from Λ to $(\mathcal{N})'$. Let $F(\lambda)$ denote the T-transform of $\Phi(\lambda)$ with the following conditions for all $\lambda \in \Lambda$:*

(i) $\lambda \mapsto F(\lambda, \xi)$ is a measurable function for all $\xi \in \mathcal{N}$.

(ii) There exists a $p \in \mathbb{N}_0$ such that for all $\xi \in \mathcal{N}$ and $z \in \mathbb{C}$ the relation

$$\left|F(\lambda, z\xi)\right| \leq P(\lambda) \exp\left((Q(\lambda)|z|^2|\xi|_p^2\right),$$

holds for some $Q \in L^\infty(\Lambda, \nu)$ and $P \in L^1(\Lambda, \nu)$.

Then Φ is Bochner integrable and

$$\int_\Lambda \Phi(\lambda)\, d\nu(\lambda) \in (\mathcal{N})'.$$

Moreover, for $\varphi \in (\mathcal{N})$ we have that

$$\left\langle\!\!\left\langle \int_\Lambda \Phi(\lambda)\, d\nu(\lambda), \varphi \right\rangle\!\!\right\rangle = \int_\Lambda \langle\!\langle \Phi(\lambda), \varphi \rangle\!\rangle\, d\nu(\lambda).$$

This allows us to interchange T-transform and integration, i.e., for $\xi \in \mathcal{N}$

$$T\left(\int_\Lambda \Phi(\lambda)\, d\nu(\lambda)\right)(\xi) = \int_\Lambda T(\Phi(\lambda))(\xi)\, d\nu(\lambda).$$

Again, the same theorem holds for the S-transform. For the proofs see e.g. [34].

Example 2.13 (Normalized exponentials)**.** *For later application one would like to give a meaning to the formal expression*

$$\Phi := \exp\left(\frac{1}{2}(1 - z^{-2})\langle x, x\rangle\right), \quad z \in \mathbb{C} \setminus \{0\}, \quad x \in \mathcal{N}'.$$

Using finite dimensional approximations to calculate its S-transform, we see that the sequence factorizes in a convergent sequence of U-functionals and a divergent pre-factor. So instead of constructing the ill defined expression Φ, we consider its multiplicative renormalization (see [34, Sec. 3] for more details) $J_z = \Phi/E(\Phi)$. So the divergent pre-factor cancels in each step of approximation. For J_z we also use the suggestive notation of normalized exponential

$$J_z = \mathrm{Nexp}\left(\frac{1}{2}(1 - z^{-2})\langle x, x\rangle\right).$$

The resulting S-transform is given by

$$S J_z(\xi) = \exp\left(-\frac{1}{2}(1 - z^2)(\xi, \xi)\right), \quad \xi \in \mathcal{H}_\mathbb{C}.$$

The right hand side is obviously a U-functional and thus by characterization $J_z \in (\mathcal{N})'$.

To define the kinetic energy factor in path integrals in addition we consider the following informal expression on the white noise space, i.e., $\mathcal{N} = S_d(\mathbb{R})$ and $\mathcal{H} = L^2_d(\mathbb{R})$, $0 \leq d \leq \infty$,

$$\exp\left(\frac{1}{2}(1 - z^{-2}) \int_{t_0}^t \omega(r)^2\, dr\right), \quad \omega \in S'_d(\mathbb{R}),$$

where z is a complex constant and $-\infty < t_0 < t < \infty$. Again, we define the normalized exponential

$$J_{z,t_0,t}(\omega) := \mathrm{Nexp}\left(\frac{1}{2}(1-z^{-2})\int_{t_0}^{t}\omega(r)^2\,dr\right), \quad \omega \in S'_d(\mathbb{R}),$$

as a distribution via the following S -transform

$$S\left(\mathrm{Nexp}\left(\frac{1}{2}(1-z^{-2})\int_{t_0}^{t}\omega(r)^2\,dr\right)\right)(g) = \exp\left(-\frac{1}{2}(1-z^{2})\sum_{j=0}^{d}\int_{t_0}^{t} g_j^2(r)\,dr\right),$$

for $g = (g_1, \ldots, g_d) \in S_d(\mathbb{R})$.

Example 2.14 (Donsker's delta)**.** *In order to 'pin' Brownian motion at a point $a \in \mathbb{R}$ we want to consider the formal composition of the Dirac delta distribution with a Brownian motion in one dimension (B_t), $0 < t < \infty$, i.e., $\delta(B(t) - a)$. This can be given a precise meaning as a Hida distribution, see e.g.* [50]*. More general, using the Fourier-transform representation of Dirac delta, informally*

$$\delta(\langle\cdot,\eta\rangle - a) = \lim_{n\to\infty}\varphi_n =: \lim_{n\to\infty}\int_{-n}^{n}\exp(i\lambda(\langle\cdot,\eta\rangle - a))\,d\lambda,$$

for some $\eta \in \mathcal{H}$, $\eta \neq 0$. Of course, this is only sensible for choices of $\mathcal{N}$ for which $\langle\cdot,\eta\rangle$, is not vector-valued. Thus, applying Theorem 2.11 to the sequence $(\varphi_n)_{n\in\mathbb{N}}$ leads us to a well-defined Hida distribution. Its T-transform and S-transform are given by

$$T(\delta(\langle\cdot,\eta\rangle - a))(\xi) = \frac{1}{\sqrt{2\pi(\eta,\eta)}}\exp\left(-\frac{1}{2(\eta,\eta)}(i(\xi,\eta)-a)^2 - \frac{1}{2}(\xi,\xi)\right)$$

and

$$S(\delta(\langle\cdot,\eta\rangle - a))(\xi) = \frac{1}{\sqrt{2\pi(\eta,\eta)}}\exp\left(-\frac{1}{2(\eta,\eta)}((\xi,\eta)-a)^2\right),$$

for all $\xi \in \mathcal{N}$. Moreover, with (2.9) *and the generating function for the Hermite polynomials, see e.g. [80], it follows that*

$$\begin{aligned} S(\delta(\langle\cdot,\eta\rangle - a))(\xi) &= \frac{1}{\sqrt{2\pi(\eta,\eta)}}\exp\left(-\frac{1}{2(\eta,\eta)}((\xi,\eta)-a)^2\right) \\ &= \frac{1}{\sqrt{2\pi(\eta,\eta)}}\exp\left(-\frac{a^2}{2(\eta,\eta)}\right)\sum_{n=0}^{\infty}\frac{1}{n!}H_n\left(\frac{a}{\sqrt{2\langle\eta,\eta\rangle}}\right)(2(\eta,\eta))^{-n/2}\langle\xi^{\otimes n},\eta^{\otimes n}\rangle. \end{aligned}$$

Hence, its chaos decomposition is given by

$$\delta(\langle\cdot,\eta\rangle - a) = \sum_{n=0}^{\infty}\left\langle :\cdot^{\otimes n}:, f^{(n)}\right\rangle,$$

where the kernels $f^{(n)}$ are given by

$$f^{(n)} = \frac{1}{\sqrt{2\pi(\eta,\eta)}n!} \exp\left(-\frac{a^2}{2(\eta,\eta)}\right) H_n\left(\frac{a}{\sqrt{2(\eta,\eta)}}\right) (2(\eta,\eta))^{-n/2} \eta^{\otimes n}. \tag{2.11}$$

For a detailed construction and proofs we refer to [54].

Example 2.15. *Let us again consider the vector-valued white noise spaces, see Example 2.2 (iii). In Example 2.14 the formal composition of the Dirac delta distribution with a d-dimensional Brownian motion (B_t), $0 < t < \infty$, i.e., $\delta(B(t) - a)$, $a = (a_1, \ldots, a_d) \in \mathbb{R}^d$ is not included. But for applications also the meaning of*

$$\delta^d(\langle \omega, \eta \rangle - a), \quad \omega \in S'_d(\mathbb{R}),$$

is also of interest, for $\eta \in L^2(\mathbb{R})$. Here δ^d denotes the Dirac delta function in $S'(\mathbb{R}^d)$ and $\langle \omega, \eta \rangle = (\langle \omega_j, \eta \rangle)_{j=1,\ldots,d}$, $\omega = (\omega_1, \ldots, \omega_j) \in S'_d(\mathbb{R})$. Using the characteristic function μ_d, see Example 2.2 (iii), the S-transform is given by

$$S\left(\delta(\langle \cdot, \eta \rangle - a)\right)(g) = \frac{1}{(2\pi(\eta,\eta))^{\frac{d}{2}}} \exp\left(-\frac{1}{2(\eta,\eta)}((g,\eta) - a)^2\right),$$

for all $g = (g_1, \ldots, g_d) \in S_d(\mathbb{R})$. Here $(g, \eta) = ((g_j, \eta))_{j=1,\ldots,d}$.

2.5.2 Kondratiev distributions

Now we are interested in recalling the characterization theorems for Kondratiev distributions, for details and proofs see [42]. Here the corresponding suitable space of holomorphic functions is given by $Hol_0(\mathcal{N}_{\mathbb{C}})$, see Section 1.2 for the definition.

Theorem 2.16. *Let $\mathcal{U} \subseteq \mathcal{N}_{\mathbb{C}}$ be open and $F : \mathcal{U} \to \mathbb{C}$ be holomorphic at zero, then there exists a unique $\Phi \in (\mathcal{N})^{-1}$ such that $T\Phi = F$. Conversely, let $\Phi \in (\mathcal{N})^{-1}$ then $T\Phi$ is holomorphic at zero. The correspondence between F and Φ is a bijection if we identify holomorphic functions, which coincide on an open neighborhood of zero.*

As a consequence of the characterization we have also a criterion for sequences and integrals with respect to an additional parameter.

Theorem 2.17. *Let $(\Phi_n)_{n\in\mathbb{N}}$ be a sequence in $(\mathcal{N})^{-1}$, such that there exists a set $\mathcal{U}_{p,q} \subset \mathcal{N}_{\mathbb{C}}$, $p, q \in \mathbb{N}_0$, so that the following statements hold:*

(i) All $T\Phi_n$ are holomorphic on $\mathcal{U}_{p,q}$.

(ii) There exists $0 < C < \infty$ *such that* $|T\Phi_n(\theta)| \leq C$ *for all* $\theta \in \mathcal{U}_{p,q}$ *and all* $n \in \mathbb{N}$.

(iii) $(T\Phi_n(\theta))_{n\in\mathbb{N}}$ *is a Cauchy sequence in* $\mathbb{C}$ *for all* $\theta \in \mathcal{U}_{p,q}$.

Then $(\Phi_n)_{n\in\mathbb{N}}$ *converges strongly in* $(\mathcal{N})^{-1}$.

Theorem 2.18. *Let* $(\Lambda, \mathcal{A}, \nu)$ *be a measure space and* $\lambda \mapsto \Phi_\lambda$ *a mapping from* Λ *to* $(\mathcal{N})^{-1}$. *We assume that there exists a set* $\mathcal{U}_{p,q} \subset \mathcal{N}_{\mathbb{C}}$, $p, q \in \mathbb{N}_0$, *such that:*

(i) $T\Phi_\lambda$, *is holomorphic on* $\mathcal{U}_{p,q}$ *for every* $\lambda \in \Lambda$.

(ii) the mapping $\lambda \mapsto T\Phi_\lambda(\theta)$ *is measurable for every* $\theta \in \mathcal{U}_{p,q}$.

(iii) there exists a function $C \in L^1(\Lambda, \nu)$ *such that*

$$|T\Phi_\lambda(\theta)| \leq C(\lambda),$$

for all $\theta \in \mathcal{U}_{p,q}$ *and for* ν*-almost all* $\lambda \in \Lambda$.

Then there exist $p', q' \in \mathbb{N}_0$, *which only depend on* p, q, *such that* Φ_λ *is Bochner integrable. In particular,*

$$\int_\Lambda \Phi_\lambda \mathrm{d}\nu(\lambda) \in (\mathcal{N})^{-1}$$

and $T\left[\int_\Lambda \Phi_\lambda \mathrm{d}\nu(\lambda)\right]$ *is holomorphic on* $\mathcal{U}_{p',q'}$. *We may interchange dual pairing and integration*

$$\left\langle\!\!\left\langle \int_\Lambda \Phi_\lambda \mathrm{d}\nu(\lambda), \varphi \right\rangle\!\!\right\rangle = \int_\Lambda \left\langle\!\!\left\langle \Phi_\lambda, \varphi \right\rangle\!\!\right\rangle \mathrm{d}\nu(\lambda), \quad \varphi \in (\mathcal{N})^1.$$

Close to Section 2.5.1 Theorem 2.16, Theorem 2.17 and Theorem 2.18, are also valid for the S-transform.

Example 2.19 (Local times)**.** *As in Example 2.2 (iii) we consider d-dimensional white noise. We are interested in the local time, which is intuitively a measure for the mean time a Brownian particle spends at a given point* $a \in \mathbb{R}^d$. *Informally this is stated by Tanaka's formula*

$$L(r, a) = \frac{1}{r}\int_0^r \delta^d(B(t) - a)\, dt, \quad r \in \mathbb{R}^+, \tag{2.12}$$

where $\delta(B(t) - a)$ *is Donsker's delta, see Example 2.15. Note that the local time does not exist as a Bochner integral in any* $(\mathcal{N})^{-\beta}$, $\beta < 1$, *because for any* $a \neq 0$ *there exists some* $g \in S_d(\mathbb{R})$ *such that* $\int_0^t g(s)ds = a$. *But then (use Example 2.15 and* (2.10)*)*

$$S\delta^d(B(t) - a)(g) = \left(\frac{1}{2\pi t}\right)^{d/2},$$

which is clearly not integrable for $d \geq 2$. Thus,

$$S\,L(r,a)(g) = \frac{1}{r}\int_0^r S\,\delta^d(B(t)-a)(g)\,dt$$

cannot be defined for all $g \in S_d(\mathbb{R})$, i.e., it cannot be an entire function. This illustrates the advantage of using the large space $(S)^{-1}$. For $a \in \mathbb{R}^d$, $a \neq 0$, we choose q such that $2^{-q} < \frac{1}{2}|a|$ and consider

$$\mathcal{U}_{1,q} = \{h \in S_{d,\mathbb{C}}(\mathbb{R}) \mid 2^q|h|_1 < 1\}.$$

Then by an easy calculation (again use Example 2.14 and (2.10)*) we obtain that for all $g \in \mathcal{U}_{1,q}$ the relation*

$$|S\,\delta^d(B(t)-a)(g)| \leq C(t) := \left(\frac{1}{2\pi t}\right)^{d/2} \exp\left(-\frac{a^2}{2t} + |a|^2\right)$$

holds. Since $C(t)$ is integrable w.r.t. the Lebesgue measure on the interval $[0,r]$ it follows that $L(r,a) \in (\mathcal{N})^{-1}$ by Theorem 2.18. Details of this proof and also more useful examples for elements of $(\mathcal{N})^{-1}$ can be found in [43].

2.6 Regular generalized functions

As a special class of generalized functions we introduce the regular test and generalized functions and recall their characterization via the Bargmann-Segal spaces, in this section. Note that this type of test and generalized functions was first introduced in [67] and later characterized in [21]. An important application resort are the studies of stochastic (partial) differential equations, see e.g. [22], [23], [3] or [7].

Before we introduce spaces of regular distributions we define the meaning of being regular for a generalized function $\Phi \in \mathcal{P}(\mathcal{N}')$. Therefore, we introduce the space of smooth polynomials of order $m \in \mathbb{N}_0$:

$$\mathcal{P}_m(\mathcal{N}') := \left\{\varphi \in \mathcal{P}(\mathcal{N}') \;\middle|\; \sum_{n=0}^{m} \left\langle :x^{\otimes n}:, \varphi^{(n)}\right\rangle,\ \varphi^{(n)} \in \mathcal{N}_{\mathbb{C}}^{\hat{\otimes} n},\ x \in \mathcal{N}'\right\}.$$

Definition 2.20. *A generalized function $\Phi \in \mathcal{P}'(\mathcal{N}')$ is called regular generalized function if for all $m \in \mathbb{N}_0$ there exists some $f_m \in L^2(\mu)$ such that for all $\varphi \in \mathcal{P}_m(\mathcal{N}')$ one has that*

$$\langle\!\langle \Phi, \varphi \rangle\!\rangle = \int_{\mathcal{N}'} f_m(x)\varphi(x)\,d\mu(x).$$

The subset of regular generalized functions from $\mathcal{P}'(\mathcal{N}')$ is denoted by $\mathcal{P}'_{reg}(\mathcal{N}')$.

The next theorem gives a characterization of regular generalized functions by the kernels in their generalized chaos decomposition. The proof can be found in e.g. [19].

Theorem 2.21. *A generalized function Φ is in $\mathcal{P}'_{reg}(\mathcal{N}')$ if and only if for all $n \in \mathbb{N}_0$ the kernel $\Phi^{(n)}$ is in $\mathcal{H}_{\mathbb{C}}^{\hat{\otimes} n}$.*

Close to Section 2.3 for $q \in \mathbb{Z}$ and $\beta \in [0, 1]$ we consider the norms

$$\|\varphi\|_{q,\beta}^2 := \sum_{n=0}^{\infty} (n!)^{1+\beta} 2^{nq} \left|\varphi^{(n)}\right|_0^2,$$

for smooth Wick polynomials $\varphi = \sum_{n=0}^{N} I_n\left(\varphi^{(n)}\right)$. Then we define the Hilbert spaces $\mathcal{G}_q^\beta$ as the completion of $\mathcal{P}(\mu)$ w.r.t. $\|\cdot\|_{0,q,\beta}$. Equivalently, we can define

$$\mathcal{G}_q^\beta := \left\{ f \in L^2(\mu) \,\middle|\, f = \sum_{n=0}^{\infty} \left\langle :x^{\otimes n}:, f^{(n)} \right\rangle,\ \|f\|_{0,q,\beta}^2 < \infty \right\}.$$

The space of regular test functions $\mathcal{G}^\beta$ is then given by the projective limit of the spaces $\mathcal{G}_q^\beta$:

$$\mathcal{G}^\beta = \bigcap_{q \le 0} \mathcal{G}_q^\beta.$$

We denote the dual spaces of $\mathcal{G}_q^\beta$ and $\mathcal{G}^\beta$ w.r.t. $L^2(\mu)$ by $\mathcal{G}_{-q}^{-\beta}$ and $\mathcal{G}^{-\beta}$, respectively. Again, from general duality theory we know that

$$\mathcal{G}^{-\beta} = \bigcup_{q \le 0} \mathcal{G}_{-q}^{-\beta}.$$

Finally we obtain the chain of spaces

$$\mathcal{G}^1 \subset \mathcal{G}^\beta \subset \mathcal{G} \subset L^2(\mu) \subset \mathcal{G}' \subset \mathcal{G}^{-\beta} \subset \mathcal{G}^{-1}.$$

The spaces $\mathcal{G}^0$ and $\mathcal{G}^{-0}$ have been introduced in [67] and are denoted by $\mathcal{G}$ and $\mathcal{G}'$, respectively.

Again, the bilinear dual pairing $\langle\!\langle \cdot, \cdot \rangle\!\rangle$ between $\mathcal{G}^1$ and $\mathcal{G}^{-1}$ is connected to the sesquilinear inner product on $L^2(\mu)$ by

$$\langle\!\langle f, \varphi \rangle\!\rangle = (\overline{f}, \varphi)_{L^2(\mu)}, \quad f \in L^2(\mu), \varphi \in \mathcal{G}^1.$$

Since the constant function 1 is in $\mathcal{G}^1$ we may extend the concept of expectation from integrable functions to distributions $\Phi \in \mathcal{G}^{-1}$:

$$E_\mu(\Phi) := \langle\!\langle \Phi, 1 \rangle\!\rangle.$$

It is not hard to see that $\mathcal{G}^{\pm\beta}_{\pm q}$ is a Hilbert space which can be described as follows

$$\mathcal{G}^{\pm\beta}_{\pm q} = \left\{ \Phi = \sum_{n=0}^{\infty} I_n(\Phi^{(n)}) \;\middle|\; \Phi^{(n)} \in \mathcal{H}^{\hat{\otimes} n},\ \|\Phi\|^2_{\pm q, \pm\beta} < \infty \right\}.$$

This description of $\mathcal{G}^{\pm\beta}_{\pm q}$ (and therefore also of $\mathcal{G}^{\pm\beta}$) shows that its elements have the property that the Wick monomials in their (generalized) chaos decomposition are square-integrable functions. This is the characteristic feature of so called regular generalized functions.

2.6.1 Characterization of the space $\mathcal{G}'$ via the Bargmann-Segal space

To understand the basic concept of the characterization of $\mathcal{G}$ and $\mathcal{G}'$ via the Bargmann-Segal space, we first have to introduce a Gaussian measure on the complexification of $\mathcal{N}'$, which is denoted by $\mathcal{N}'_{\mathbb{C}}$. Therefore, we define the measure $\mu_{\frac{1}{2}}$ as the Gaussian measure with characteristic function $C_{\frac{1}{2}}(\xi) := \exp\left(-\frac{1}{4}(\xi,\xi)\right)$, $\xi \in \mathcal{N}$. The covariance operator corresponding to $\mu_{\frac{1}{2}}$ is then given by $\frac{1}{2}\mathbb{1}$, where $\mathbb{1}$ denotes the identity operator on $\mathcal{H}$. Following e.g. [41] or [21], we are now able to define a Gaussian measure on the measurable space $\left(\mathcal{N}'_{\mathbb{C}}, \mathcal{C}_\sigma(\mathcal{N}'_{\mathbb{C}})\right)$.

Definition 2.22. *Let $z = x + iy \in \mathcal{N}'_{\mathbb{C}}$, $x, y \in \mathcal{N}'$, then we define the measure*

$$d\nu(z) := d\mu_{\frac{1}{2}}(x) \times d\mu_{\frac{1}{2}}(y)$$

on the measurable space $\left(\mathcal{N}'_{\mathbb{C}}, \mathcal{C}_\sigma(\mathcal{N}'_{\mathbb{C}})\right)$.

Remark 2.23. *Now close to Section 2.1 we can consider the space of square-integrable complex valued functions $L^2(\nu) := L^2\left(\mathcal{N}'_{\mathbb{C}}, \mathcal{C}_\sigma(\mathcal{N}'_{\mathbb{C}}), \nu\right)$. In addition one can construct smooth holomorphic orthogonal monomials close to* (2.5).

Let us denote the set $\mathbb{P}$ to be the set of all orthogonal projections on $\mathcal{H}$ such that for any $P \in \mathbb{P}$ one has that $\mathcal{R}(P) \subset \mathcal{N}$ and $\dim \mathcal{R}(P) < \infty$, where $\mathcal{R}(P)$ denotes the range of the operator P.

Definition 2.24. *A function $H : \mathcal{H}_{\mathbb{C}} \to \mathbb{C}$ is in the Bargmann-Segal space $E_2(\nu)$ if it satisfies the following conditions:*

(i) H is entire on $\mathcal{H}_{\mathbb{C}}$, and

(ii)

$$\sup_{P \in \mathbb{P}} \int_{\mathcal{N}'_{\mathbb{C}}} |H(Pz)|^2 \, d\nu(z) < \infty.$$

Several properties, also for families of Bargmann-Segal spaces, can be found in [21]. Therein also the following characterization theorems are shown.

Corollary 2.25. *A test function φ is an element of $\mathcal{G}$ if and only if for any $\lambda > 0$ the function $(S\varphi)(\lambda\cdot) \in E^2(\nu)$.*

Theorem 2.26. *The generalized function Φ is an element of $\mathcal{G}'$ if and only if there exists $\varepsilon > 0$ such that the measurable extension of $S\Phi(\varepsilon\cdot)$ exists and is an element of $E^2(\nu)$.*

Using the series expansion of $H \in E^2(\nu)$ noted in Remark 2.23 together with the summability property of its kernels given by the square-integrability of H, one can restrict H to holomorphic functions on $\mathcal{H}_{\mathbb{C}}$. This mapping can be used to construct, see [21], a natural isomorphism between $E^2(\nu)$ and the in the literature well-known Bargmann-Segal space on a Hilbert space, see [71], [2], [72], and [73]. In applications of the characterization theorem this isomorphism turned out to be useful, see [21].

Example 2.27. *Donsker's delta $\delta(\langle\cdot,\eta\rangle - a)$ is in $\mathcal{G}'$, for all $\eta \in \mathcal{H}$ and $a \in \mathbb{R}$.*

Proof. The statement can be shown in two different ways. On the one side, see Example 2.14, we know that $\delta(\langle\cdot,\eta\rangle - a) \in (\mathcal{N})'$. Thus, one only has to verify that its $\|\cdot\|_{-q,0}$-norm is finite for some $0 < q < \infty$. For this proof we refer to [82].

On the other side following Theorem 2.26 one has to check whether there exists some $\varepsilon > 0$ such that $S\left(\delta(\langle\cdot,\eta\rangle - a)\right)(\varepsilon\cdot) \in E^2(\nu)$. In the white noise case, i.e., $\mathcal{N} = S(\mathbb{R})$ and $\mathcal{H} = L^2(\mathbb{R})$, this is already proven in [21] for a composition of the Dirac delta distribution with a Brownian motion. Let $\Phi := \delta(\langle\cdot,\eta\rangle - a)$, $\eta \in \mathcal{H}$ and $a \in \mathbb{R}$. Then, obviously $S(\Phi)(\varepsilon\cdot)$ is entire on $\mathcal{H}_{\mathbb{C}}$, see Example 2.14. Moreover

$$\begin{aligned}
&\int_{\mathcal{N}'_{\mathbb{C}}} |S(\Phi)(\varepsilon z)|^2 \, d\nu(z) \\
&\qquad = \frac{1}{2\pi(\eta,\eta)} \int_{\mathcal{N}'_{\mathbb{C}}} \exp\left(-\frac{1}{2(\eta,\eta)}((\varepsilon z,\eta) - a)^2 - \frac{1}{2(\eta,\eta)}(\overline{(\varepsilon z,\eta)} - a)^2\right) d\nu(z) \\
&\qquad\qquad = \frac{1}{2\pi(\eta,\eta)} \int_{\mathbb{C}} \exp\left(-\frac{\varepsilon^2}{2}(u^2 + \overline{u}^2)\right) d\gamma(u),
\end{aligned}$$

where the density

$$d\gamma(u) = \frac{1}{\pi} \exp\left(-(x^2 + y^2)\right) dx\, dy, \quad u = x + iy,\ x, y \in \mathbb{R}.$$

Therefore

$$\int_{\mathcal{N}'_{\mathbb{C}}} |S(\Phi)(\varepsilon z)|^2 \, d\nu(z) = \frac{1}{2\pi^2(\eta,\eta)} \int_{\mathbb{R}} \int_{\mathbb{R}} \exp\left((3\varepsilon^2 - 1)x^2 + (\varepsilon^2 - 1)y^2\right) dx\, dy,$$

is finite for all $0 < \varepsilon < \frac{1}{\sqrt{3}}$. Thus, since

$$\int_{\mathcal{N}'_{\mathbb{C}}} |S(\Phi)(\varepsilon P z)|^2 \, d\nu(z) \leq \int_{\mathcal{N}'_{\mathbb{C}}} |S(\Phi)(\varepsilon z)|^2 \, d\nu(z) < \infty,$$

we get that

$$\sup_{P \in \mathbb{P}} \int_{\mathcal{N}'_{\mathbb{C}}} |S(\Phi)(\varepsilon P z)|^2 \, d\nu(z) \leq \int_{\mathcal{N}'_{\mathbb{C}}} |S(\Phi)(\varepsilon z)|^2 \, d\nu(z) < \infty.$$

Hence, $S\Phi(\varepsilon \cdot) \in E^2(\nu)$ for all $0 < \varepsilon < \frac{1}{\sqrt{3}}$ and therefore $\Phi \in \mathcal{G}'$ by Theorem 2.26. □

Analogously to the proof of Example 2.27 one can show that the d-dimensional Donsker's delta, see Example 2.15 is in a regular distribution.

Remark 2.28. *Similarly to Theorem 2.26 one can prove a characterization of $\mathcal{G}^{-1}$, whereas the Bargmann-Segal space is replaced by an infinite dimensional Hardy space.*

2.6.2 Independence of regular generalized functions

A priori it is not clear how to define independence for regular generalized functions, since they are not pointwisely defined. We first give a definition of independent random variables in $\mathcal{G}'$ according to [7], see also [3] and [17]. That means as in Example 2.2 (i) we consider $\mathcal{N} = S(\mathbb{R})$ and $\mathcal{H} = L^2(\mathbb{R}; dx)$. Let I be an interval in $\mathbb{R}$, and denote by $\mathcal{F}_I$ the σ-algebra generated by the random variables $\langle \cdot, \mathbb{1}_{[t_0,t)} \rangle - \langle \cdot, \mathbb{1}_{[t_0,s)} \rangle$, $t_0, s, t \in I$, $t_0 \leq s \leq t$.

Definition 2.29. *We call $\Phi \in \mathcal{G}'$ $\mathcal{F}_I$-measurable, if for all $g \in S(\mathbb{R})$,*

$$S(\Phi)(g) = S(\Phi)(\mathbb{1}_I g).$$

Remark 2.30. *(i) Recall that $f \in L^2(\mu)$ is $\mathcal{F}_I$-measurable if and only if $Sf(g) = Sf(\mathbb{1}_I g)$ for all $g \in S(\mathbb{R})$.*

(ii) Note that the S-transform of an element in $\mathcal{G}'$ has a continuous extension from $S(\mathbb{R})$ to $L^2(\mathbb{R})$, hence for $\Phi \in \mathcal{G}'$ we have that $S(\Phi)(\mathbb{1}_I g)$ is well-defined for all intervals $I \subset \mathbb{R}$ and all $g \in S(\mathbb{R})$.

Definition 2.31. *Two generalized random variables Φ, $\Psi \in \mathcal{G}'$ are called independent if there exist intervals I, $J \subset \mathbb{R}$ whose intersection has Lebesgue measure zero, and Φ is $\mathcal{F}_I$-measurable, and Ψ is $\mathcal{F}_J$-measurable.*

Of course, close to a multi-dimensional Brownian motion (see Example 2.2 (iii)) we can define independence on the space $S_d(\mathbb{R})$, $d \geq 0$, analogously to the above considerations.

Chapter 3

Differential calculus and related operations in Gaussian spaces

In Section 3.1 we discuss differential operators of first and second order on spaces of generalized functions. Especially, Gâteaux derivatives and the Gross Laplacian are introduced and properties are listed. The ensuing Section 3.2 introduced linear operators based on differential operators on spaces of test and generalized functions. These are important for applications of the concept of generalized functions, e.g. for a mathematical rigorous definition of Feynman integrands.

3.1 Differential operators in Gaussian spaces

In this section we consider several differential operators on $(\mathcal{N})^\beta$, $0 \leq \beta \leq 1$. Instead of giving a long list of comprehensive references we refer to the books [61], [34] and [51]. We start in defining Gâteaux derivatives on the test function space $(\mathcal{N})^\beta$, $0 \leq \beta \leq 1$, close to Definition 1.3.

Definition 3.1. *Let $\varphi \in (\mathcal{N})^\beta$, $0 \leq \beta \leq 1$, and $x, y \in \mathcal{N}'$. We consider the function $\lambda \to \varphi(x + \lambda y)$ on $\mathbb{R}$. If this function is differentiable at $\lambda = 0$, we say that φ is Gâteaux differentiable at x in direction y and denote*

$$D_y\varphi(x) = \frac{\partial}{\partial\lambda}\varphi(x + \lambda y)\Big|_{\lambda=0}.$$

If φ is Gâteaux differentiable at x in direction y for all $x \in \mathcal{N}'$ we simply call it Gâteaux differentiable in direction y.

Lemma 3.2. *For $\varphi, \psi \in (\mathcal{N})^\beta$, $0 \leq \beta \leq 1$, and $y \in \mathcal{N}'$ the Gâteaux derivative has the following properties:*

(i) The chaos expansion of $D_y\varphi$ is given by

$$D_y\varphi(x) = \sum_{n=0}^{\infty} n\langle : x^{\otimes(n-1)} : \hat{\otimes} y, \varphi^{(n)}\rangle, \quad x \in \mathcal{N}',$$

where $\varphi^{(n)}$ again denotes the n-th kernel of φ.

(ii) A product rule is given by:

$$D_y\psi\varphi = (D_y\psi)\varphi + \psi(D_y\varphi).$$

(iii) The S-transform of $D_y\varphi$ in $\xi \in \mathcal{N}$ is given by

$$S(D_y\varphi)(\xi) = \frac{\partial}{\partial\lambda} S(\varphi)(\xi + \lambda y)\Big|_{\lambda=0}.$$

Lemma 3.3. *For $\eta \in \mathcal{H}$ the differential operator D_η is continuous form $\mathcal{G}^\beta$, $0 \le \beta \le 1$, into itself. Moreover, it extends continuously from $\mathcal{G}^{-\beta}$, $0 \le \beta < 1$, into itself and the statements (i)-(iii) in Lemma 3.2 also holds.*

The proof can be done analogously to e.g. [51]. Therein it is shown that for $\eta \in \mathcal{N}$ the differential operator D_η extends continuously from $(\mathcal{N})^{-\beta}$, $0 \le \beta < 1$, into itself.

Definition 3.4. *Let $\varphi \in (\mathcal{N})^\beta$, $0 \le \beta < 1$, with kernels $\varphi^{(n)}$, $n \in \mathbb{N}$. Then for any $x \in \mathcal{N}'$ we define the Gross Laplacian of φ by*

$$\Delta_G\varphi(x) = \sum_{n=0}^{\infty}(n+2)(n+1)\Big\langle : x^{\otimes n} :, \langle \mathrm{Tr}, \varphi^{(n+2)}\rangle\Big\rangle.$$

Here $\mathrm{Tr} \in \mathcal{N}'^{\otimes 2}$ *denotes the trace kernel defined by*

$$\langle \mathrm{Tr}, \xi \otimes \eta\rangle = (\xi, \eta), \quad \xi, \eta \in \mathcal{N}. \tag{3.1}$$

Theorem 3.5. *The Gross Laplacian is a continuous operator from $(\mathcal{N})^\beta$, $0 \le \beta \le 1$, into itself.*

Let us consider the test function spaces w.r.t. the Schwartz triplet, see Example 2.2, $(S)^\beta$, $0 \le \beta < 1$. The Gâteaux derivative in direction δ_t, $t \in \mathbb{R}$, is denoted by $D_{\delta_t} =: \partial_t$. For $h \in L^2(\mathbb{R})$ and $\varphi \in (S)$ we get that

$$D_h\varphi(x) = \int_{\mathbb{R}} h(t)\partial_t\varphi(x)dt, \quad x \in (S)^\beta.$$

Theorem 3.6. *A relation of the Gross Laplacian and ∂_t, $t \in \mathbb{R}$, on $(S)^\beta$, $0 \le \beta < 1$, is given as follows:*

$$\Delta_G = \int_{\mathbb{R}} \partial_t^2 dt.$$

For the proofs and more properties of this differential operators we refer to [51] and [34].

3.2 Operators on Gaussian spaces related to differential operators

In this section we introduce special kind of linear operators on spaces of test and generalized functions. In detail we discuss translations, orthogonal projections and scaling operators. First we follow e.g. [34] and [82] and define this operators on spaces of test functions $(\mathcal{N})$ or $\mathcal{G}$. Later we show in which case extensions to generalized functions make sense. Moreover, we give a representation of such operators in terms of differential operators.

3.2.1 Translation operator

We start in considering the translation operator, which is sometimes also called shift operator.

Definition 3.7. *We define the translation operator for $\eta \in \mathcal{N}'$ by*

$$\begin{aligned} \tau_\eta : (\mathcal{N}) &\to (\mathcal{N}) \\ \varphi &\mapsto \varphi(\cdot + \eta). \end{aligned}$$

Close to [34] we define

$$: (x+\eta)^{\otimes n} : := \sum_{k=0}^{n} \binom{n}{k} : x^{\otimes(n-k)} : \otimes \eta^{\otimes k}$$

and for $\varphi \in (\mathcal{N})$ with kernels $\varphi^{(n)}$, $n \in \mathbb{N}$, we have that

$$\tau_\eta \varphi = \varphi(x+\eta) := \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \binom{n+k}{n} \left\langle : x^{\otimes k} :, \left(\eta^{\otimes k}, \varphi^{(n+k)}\right)_{\mathcal{H}^{\otimes k}} \right\rangle, \tag{3.2}$$

whenever the series converges for arbitrary $\eta \in \mathcal{N}'$. It was shown in [51] that τ_η is a continuous mapping from $(\mathcal{N})^\beta$ into itself.

Theorem 3.8. *Let $\eta \in \mathcal{H}_\mathbb{C}$ and $\varphi \in \mathcal{G}$ then $\tau_\eta \varphi \in \mathcal{G}$.*

This Theorem was shown in [67] and [82] with help of (3.2) but can be much easier realized using the characterization of the test function space $\mathcal{G}$, see Corollary 2.25.

Proof. Let $H \in E^2(\nu)$ then, by the theorem of Cameron-Martin, $H(\cdot - \eta/\lambda)$ is also in $E^2(\nu)$ for all $\eta \in \mathcal{H}_\mathbb{C}$ and all $\lambda > 0$. So for $\varphi \in \mathcal{G}$, $S(\tau_\eta \varphi)(\lambda \cdot) = S(\varphi)(\lambda(\cdot - \eta/\lambda)) \in E^2(\nu)$ since $S(\varphi)(\lambda \cdot) \in E^2(\nu)$ by Corollary 2.25. Hence again by Corollary 2.25 we get that $\tau_\eta \varphi \in \mathcal{G}$. □

Remark 3.9. *We extend the translation operator to a generalized function whenever the series representation* (3.2) *leads us to a well-defined object in the same space of generalized functions. But since not only the generalized function but also the operation τ_η, $\eta \in \mathcal{H}_\mathbb{C}$, has no pointwise sense we restrict ourself first to regular generalized functions. This is a suggestive restriction, since all regular distributions have square-integrable kernels. Hence, for all $k, n \in \mathbb{N}_0$ we have that*

$$\left\langle :x^{\otimes k}:, \left(\eta^{\otimes k}, \varphi^{(n+k)}\right)_{\mathcal{H}^{\otimes k}} \right\rangle$$

is also regular for all $\eta \in \mathcal{H}_\mathbb{C}$. Hence in case of convergence of (3.2) *again we might obtain a regular generalized function.*

Theorem 3.10. *The mapping τ_η, $\eta \in \mathcal{H}_\mathbb{C}$, can be extended to a mapping $\tilde{\tau}_\eta : \mathcal{G}' \to \mathcal{G}'$.*

Proof. Let $\varphi, \psi \in \mathcal{G}$ and $\eta \in \mathcal{H}_\mathbb{C}$, then $\tau_\eta\varphi$ and $\tau_{-\eta}\psi$ are in $\mathcal{G}$ by Theorem 3.8. Moreover, we have that

$$S((\tau_\eta\varphi)\psi)(0) = S(\varphi(\cdot + \eta)\psi)(0) = S(\varphi\psi(\cdot - \eta))(\eta) = S(\varphi\tau_{-\eta}\psi)(\eta).$$

Hence we get that

$$\begin{aligned}\langle\!\langle \tau_\eta\varphi, \psi \rangle\!\rangle = S(\varphi(\cdot + \eta)\psi)(0) = S(\varphi\psi(\cdot - \eta))(\eta) &= \langle\!\langle \varphi\tau_{-\eta}\psi, :\exp(\langle\cdot, \eta\rangle): \rangle\!\rangle \\ &= \langle\!\langle :\exp(\langle\cdot, \eta\rangle): \varphi, \tau_{-\eta}\psi \rangle\!\rangle = \langle\!\langle \tau^\dagger_{-\eta}(:\exp(\langle\cdot, \eta\rangle): \varphi), \psi \rangle\!\rangle.\end{aligned}$$

Therefore, $\tau_\eta = \tau^\dagger_{-\eta}(:\exp(\langle\cdot, \eta\rangle): \varphi)$ for all $\varphi \in \mathcal{G}$. Here $\tau^\dagger_{-\eta}$ denotes the adjoint of $\tau_{-\eta}$. But since $\tau_{-\eta}$ is continuous from $\mathcal{G}$ into itself also $\tau^\dagger_{-\eta}$ is continuous from $\mathcal{G}'$ into itself. Hence by continuity of the multiplication operator $:\exp(\langle\cdot, \eta\rangle):$ we get that $\tau^\dagger_{-\eta} :\exp(\langle\cdot, \eta\rangle):$ is continuous from $\mathcal{G}'$ into itself. Let $(\varphi_n)_{n\in\mathbb{N}} \subset \mathcal{G}$ be a sequence with limit $\Phi \in \mathcal{G}'$. Moreover let $\alpha > 0$ such that $\tau^\dagger_{-\eta}(:\exp(\langle\cdot, \eta\rangle): \Phi) \in \mathcal{G}_{-\alpha}$. Then for all $\psi \in \mathcal{G}$ we obtain

$$\left|\langle\!\langle \tau_\eta\varphi_n, \psi \rangle\!\rangle\right| = \left|\langle\!\langle \tau^\dagger_{-\eta}(:\exp(\langle\cdot, \eta\rangle): \varphi_n), \psi \rangle\!\rangle\right| \leq \left\|\tau^\dagger_{-\eta}(:\exp(\langle\cdot, \eta\rangle): \varphi_n)\right\|_{-\alpha,0}\left\|\psi\right\|_{\alpha,0}.$$

Hence $\lim_{n\to\infty}\langle\!\langle \tau_\eta\varphi_n, \psi \rangle\!\rangle$ exists for all $\psi \in \mathcal{G}$ and is equal to $\langle\!\langle \tau^\dagger_{-\eta}(:\exp(\langle\cdot, \eta\rangle): \Phi), \psi \rangle\!\rangle$. Thus we can extend τ_η to a continuous linear operator $\tilde{\tau}_\eta := \tau^\dagger_{-\eta} :\exp(\langle\cdot, \eta\rangle):$ from $\mathcal{G}'$ into $\mathcal{G}'$. □

Remark 3.11. *One can show Theorem 3.10 for other spaces of test and generalized functions.*

(i) For $\mathcal{G}^{-\beta}$, $0 < \beta < 1$, the proof is close to the proof of Theorem 3.10. If $\beta = 1$, of course $\tau_\eta\Phi$ does not exist for all $\eta \in \mathcal{H}_\mathbb{C}$ and $\Phi \in \mathcal{G}^{-1}$, since η might not be in the subset of $\mathcal{H}_\mathbb{C}$ where the S-transform of Φ is holomorphic. From now on we write τ_η instead of $\tilde{\tau}_\eta$, since all extensions coincide on their common domain.

(ii) If one considers a distribution space $(\mathcal{N})^{-\beta}$, $0 \leq \beta < 1$, Theorem 3.10 only holds for $\eta \in \mathcal{N}_{\mathbb{C}}$. This was shown in [51, Theorem 10.22]. As in the case of regular generalized functions for $\beta = 1$ a generalization of this theorem is not possible for arbitrary $\eta \in \mathcal{N}_{\mathbb{C}}$.

Representation by differential operators

Following e.g. [51] or [67] the translation operator can be represented in terms of differential operators.

Theorem 3.12. *For any $\eta \in \mathcal{N}'_{\mathbb{C}}$ the translation operator on $(\mathcal{N})$ can be represented by*

$$\tau_\eta = \exp(D_\eta) := \sum_{k=0}^{\infty} \frac{1}{k!} D_\eta^k.$$

Moreover, for $\varphi \in (\mathcal{N})$ the S-transform of $\tau_\eta \varphi$ is given by

$$S(\tau_\eta \varphi)(\xi) = S(\exp(D_\eta)\varphi)(\xi) = \exp(D_\eta) S(\varphi)(\xi) = \sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{\partial}{\partial t}\right)^k S(\varphi)(\xi + t\eta)\Big|_{t=0}. \tag{3.3}$$

Proof. Let $\eta \in \mathcal{N}'_{\mathbb{C}}$ and $\varphi = \sum_{n=0}^{\infty} \langle :\cdot^{\otimes n}:, \varphi^{(n)} \rangle \in (\mathcal{N})$, then $\tau_\eta \varphi \in (\mathcal{N})$. Moreover, its S-transform in $\xi \in \mathcal{N}$ is given by

$$S(\tau_\eta \varphi)(\xi) = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \binom{n+k}{n} \left(\xi^{\otimes n}, \left(\eta^{\otimes k}, \varphi^{(n+k)}\right)_{\mathcal{H}^{\otimes k}}\right).$$

But

$$\begin{aligned}
\exp(D_\eta) S(\varphi)(\xi) &= \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{n=0}^{\infty} D_\eta^k \left(\xi^{\otimes n}, \varphi^{(n)}\right) \\
&= \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} \frac{1}{k!} \left(\frac{\partial}{\partial t}\right)^k \left((\xi + t\eta)^{\otimes n}, \varphi^{(n)}\right)\Big|_{t=0} = \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} \frac{1}{k!} \left(\frac{\partial}{\partial t}\right)^k \sum_{j=0}^{n} \binom{n}{j} t^j \left(\xi^{\otimes(n-j)} \otimes \eta^{\otimes j}, \varphi^{(n)}\right)\Big|_{t=0} \\
&= \sum_{k=0}^{\infty} \sum_{n=k}^{\infty} \binom{n}{k} \left(\xi^{\otimes(n-k)} \otimes \eta^{\otimes k}, \varphi^{(n)}\right) = \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} \binom{n+k}{k} \left(\xi^{\otimes n}, \left(\eta^{\otimes k}, \varphi^{(n+k)}\right)_{\mathcal{H}^{\otimes k}}\right).
\end{aligned}$$

If φ has only a finite number of non-vanishing kernels then the summations can be interchanged by Lemma 3.2 (iii). Let $\varphi \in (\mathcal{N})$ be an arbitrary test function with kernels $\varphi^{(n)}$, $n \in \mathbb{N}$. Then the sequence $\varphi_M \in (\mathcal{N})$ given by $\sum_{n=0}^{M} \langle :\cdot^{\otimes n}:, \varphi^{(n)} \rangle$ converges in $(\mathcal{N})$ to φ. Moreover by continuity of τ_η we get that $\tau_\eta \varphi_M$ converges to $\tau_\eta \varphi$ in $(\mathcal{N})$. Therefore (3.3) holds whenever

$$\sum_{k=0}^{\infty} \left\| \sum_{n=0}^{\infty} \binom{n+k}{k} \left\langle\!\left\langle :\cdot^{\otimes n}:, \left(\eta^{\otimes k}, \varphi^{(n+k)}\right)_{\mathcal{H}^{\otimes k}} \right\rangle\!\right\rangle \right\|_{p,q,0}^2 < \infty.$$

Let $p, q \in \mathbb{N}$. Then there exists some $s \in \mathbb{N}$ such that $\eta \in \mathcal{H}_{-p-s}$. With $(n+k)! < n!k!2^{n+k}$ and $(n+k)!\left|\varphi^{(n+k)}\right|^2_{p+s} \leq 2^{-\alpha(n+k)}\|\varphi\|^2_{p+s,\alpha}$, for all $\alpha \in \mathbb{N}$, we get that

$$\begin{aligned}
&\sum_{k=0}^{\infty}\left\|\sum_{n=0}^{\infty}\binom{n+k}{k}\left\langle\!\left\langle :\cdot^{\otimes n}:, \left(\eta^{\otimes k}, \varphi^{(n+k)}\right)_{\mathcal{H}^{\otimes k}}\right\rangle\!\right\rangle\right\|^2_{p,q,0} \\
&\quad \leq \sum_{k=0}^{\infty}\sum_{n=0}^{\infty} n!2^{qn}\binom{n+k}{k}^2 |\eta|^{2k}_{-p-s}\left|\varphi^{(n+k)}\right|^2_{p+s} \leq \sum_{k=0}^{\infty}\sum_{n=0}^{\infty} 2^{qn}2^{-\alpha(n+k)}\frac{(n+k)!}{n!(k!)^2}|\eta|^{2k}_{-p-s}\|\varphi\|^2_{p+s,\alpha} \\
&\quad \leq \sum_{k=0}^{\infty} 2^{-\alpha k}\sum_{n=0}^{\infty} 2^{qn}2^{-\alpha n}\frac{2^{n+k}}{k!}|\eta|^{2k}_{-p-s}\|\varphi\|^2_{p+s,\alpha} \leq \|\varphi\|^2_{p+s,\alpha}\sum_{k=0}^{\infty}\frac{2^{-(\alpha-1)k}}{k!}|\eta|^{2k}_{-p-s}\sum_{n=0}^{\infty} 2^{(q+1-\alpha)n} < \infty,
\end{aligned}$$

for all $p, q \in \mathbb{N}$ and a suitable choice of $\alpha \in \mathbb{N}$. □

Corollary 3.13. *Let $\eta \in \mathcal{H}_{\mathbb{C}}$ and $\Phi \in \mathcal{G}'$ then $\tau_\eta\Phi \in \mathcal{G}'$ and its S-transform is given by*

$$S(\tau_\eta\Phi) = \exp(D_\eta)S(\Phi) = \sum_{k=0}^{\infty}\frac{1}{k!}\left(\frac{\partial}{\partial t}\right)^k S(\Phi)(\cdot + t\eta)\Big|_{t=0}. \tag{3.4}$$

Remark 3.14. *One can show that* (3.4) *holds for $\eta \in \mathcal{N}_{\mathbb{C}}$ and $\Phi \in (\mathcal{N})^{-\beta}$. Hence the translation operator τ_η, $\eta \in \mathcal{N}'_{\mathbb{C}}$, can be applied to $\Phi \in (\mathcal{N})^{-\beta}$, $0 \leq \beta < 1$, whenever*

$$\sum_{k=0}^{\infty}\frac{1}{k!}\left(\frac{\partial}{\partial t}\right)^k S(\Phi)(\cdot + t\eta_n)\Big|_{t=0}$$

defines a Cauchy sequence in the sense of Theorem 2.11 for a sequence $(\eta_n)_{n\in\mathbb{N}} \subset \mathcal{N}_{\mathbb{C}}$ converging to η in $\mathcal{N}'_{\mathbb{C}}$.

3.2.2 Orthogonal projection

In this section we are interested in a realization of a composition of an orthogonal projection on $\mathcal{N}'$ with a test or generalized function.

Definition 3.15. *Let $\eta \in \mathcal{N}$, $|\eta|_0 = 1$, then we define the projection on the orthogonal complement (orthogonal projection) of the subspace spanned by η by:*

$$P_{\perp,\eta} : \mathcal{N}' \to \mathcal{N}', \qquad x \mapsto P_{\perp,\eta}x = x - \langle x, \eta\rangle\eta.$$

Moreover, we define the orthogonal projection on $(\mathcal{N}^{\otimes n})'$ by

$$P^{\otimes n}_{\perp,\eta} : (\mathcal{N}^{\otimes n})' \to (\mathcal{N}^{\otimes n})', \qquad \Phi^{(n)} \mapsto P^{\otimes n}_{\perp,\eta}\Phi^{(n)}.$$

In addition for $0 \leq \beta < 1$ we denote the orthogonal projection in the kernels by

$$\tilde{P}_\eta : \mathcal{G}^\beta \to \mathcal{G}^\beta, \qquad \varphi = \sum_{n=0}^{\infty}\left\langle :\cdot^{\otimes n}: \varphi^{(n)}\right\rangle \mapsto \sum_{n=0}^{\infty}\left\langle :\cdot^{\otimes n}:, P^{\otimes n}_{\perp,\eta}\varphi^{(n)}\right\rangle.$$

Let us remark that $\left(\xi^{\otimes n}, P^{\otimes n}_{\perp,\eta}\varphi^{(n)}\right) = \left(P^{\otimes n}_{\perp,\eta}\xi^{\otimes n}, \varphi^{(n)}\right)$, for all kernels $\varphi^{(n)} \in \mathcal{H}^{\otimes n}$ and all $\eta \in \mathcal{N}_{\mathbb{C}}$. Thus, we define a projection $\tilde{P}_\eta$ in direction $\eta \in \mathcal{H}_{\mathbb{C}}$ by

$$\begin{aligned} \tilde{P}_\eta : \{S(\Phi),\ \Phi \in \mathcal{G}'\} &\to \{S(\Phi),\ \Phi \in \mathcal{G}'\} \\ F &\mapsto \tilde{P}_\eta F =: F \circ P_{\perp,\eta}. \end{aligned} \tag{3.5}$$

Of course, $\tilde{P}_\eta$ is not well-defined on $\mathrm{Hol}_0(\mathcal{N}_{\mathbb{C}})$ in general. Nevertheless, we show that on the subspace $\{S(\Phi),\ \Phi \in \mathcal{G}^{-\beta}\}$, $0 \le \beta < 1$, the definition of $\tilde{P}_\eta$ given in (3.5) is sensible for all $\eta \in \mathcal{H}_{\mathbb{C}}$, $|\eta|_0 = 1$. In addition we show that the definition of the projection in the kernels $\tilde{P}_\eta$ (see Definition 3.15) can be extended to generalized functions. Moreover, its S-transform coincides with the orthogonal projection $\tilde{P}_\eta$ on $\{S(\Phi),\ \Phi \in \mathcal{G}^{-\beta}\} \subset \mathrm{Hol}_0(\mathcal{N}_{\mathbb{C}})$.

Lemma 3.16. *Let $\Phi \in \mathcal{G}^{-\beta}$, $0 \le \beta < 1$, with kernels $\Phi^{(n)}$, $n \in \mathbb{N}$, and $\eta \in \mathcal{H}_{\mathbb{C}}$, $|\eta|_0 = 1$. Then $\tilde{P}_\eta\,\Phi$ is also in $\mathcal{G}^{-\beta}$ and its S-transform is given by*

$$S\left(\tilde{P}_\eta\Phi\right)(\xi) = \tilde{P}_\eta S(\Phi)(\xi),$$

for all $\xi \in \mathcal{N}$.

Proof. Let $\Phi \in \mathcal{G}^{-\beta}$, $0 \le \beta < 1$, with kernels $\Phi^{(n)}$, $n \in \mathbb{N}$, and $\eta \in \mathcal{H}_{\mathbb{C}}$. Then since $\Phi^{(n)} \in \mathcal{H}^{\hat{\otimes} n}$ and $\eta \in \mathcal{H}_{\mathbb{C}}$ we get that $P^{\otimes n}_{\perp,\eta}\Phi^{(n)}$ is also in $\mathcal{H}^{\hat{\otimes} n}$. For showing this we first note that

$$\left|P^{\otimes n}_{\perp,\eta}\left(\eta^{\otimes 2k}, \Phi^{(n+2k)}\right)_{\mathcal{H}^{\otimes 2k}}\right|_0 \le |\eta|_0^{2k}\left|\Phi^{(n+2k)}\right|_0 \le \left|\Phi^{(n+2k)}\right|_0, \tag{3.6}$$

for all $k, n \in \mathbb{N}$, since $|\eta|_0 = 1$. Then

$$\begin{aligned} \left\|\tilde{P}_\eta\Phi\right\|^2_{-q,-\beta} &= \left\|\sum_{n=0}^{\infty}\left\langle :\cdot^{\otimes n}:, P^{\otimes n}_{\perp,\eta}\Phi^{(n)}\right\rangle\right\|^2_{-q,-\beta} \\ &= \sum_{n=0}^{\infty}(n!)^{1-\beta}2^{-nq}\left|P^{\otimes n}_{\perp,\eta}\Phi^{(n)}\right|_0^2 \le \sum_{n=0}^{\infty}(n!)^{1-\beta}2^{-nq}\left|\Phi^{(n)}\right|_0^2 < \infty. \end{aligned}$$

Moreover, for $\xi \in \mathcal{N}$ we have that

$$\begin{aligned} S\left(\tilde{P}_\eta\Phi\right)(\xi) = S\left(\sum_{n=0}^{\infty}\left\langle :\cdot^{\otimes n}:, P^{\otimes n}_{\perp,\eta}\Phi^{(n)}\right\rangle\right)(\xi) &= \sum_{n=0}^{\infty}\left\langle P^{\otimes n}_{\perp,\eta}\Phi^{(n)}, \xi^{\otimes n}\right\rangle \\ &= \sum_{n=0}^{\infty}\left\langle \Phi^{(n)}, P^{\otimes n}_{\perp,\eta}\xi^{\otimes n}\right\rangle = S(\Phi)(\xi - (\xi,\eta)\eta) = \tilde{P}_\eta S(\Phi)(\xi). \end{aligned}$$

Thus $\tilde{P}_\eta$ is given by

$$\tilde{P}_\eta : \mathcal{G}^{-\beta} \to \mathcal{G}^{-\beta}, \qquad \Phi \mapsto S^{-1}\left(\tilde{P}_\eta S(\Phi)\right).$$

□

Remark 3.17. *Note that for $\eta \in \mathcal{N}$ the projection $\tilde{P}_\eta : \mathcal{G}^{-\beta} \to \mathcal{G}^{-\beta}$ coincides with the generalized conditional expectation with respect to the filtration induced by the set $\mathcal{M}_\eta := \mathcal{N} \setminus \operatorname{span}\{\eta\}$, see e.g. [3] or [21] for more details.*

Theorem 3.18. *Let $\eta \in \mathcal{H}$, $|\eta|_0 = 1$, then there exists a linear mapping*

$$\begin{aligned} P_\eta : \mathcal{G}^\beta &\to \mathcal{G}^\beta \\ \varphi &\mapsto P_\eta \varphi =: \varphi \circ P_{\perp,\eta}. \end{aligned}$$

Moreover, the chaos decomposition of $P_\eta \varphi$ is given by

$$\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(n+2k)!}{k!n!} \left(-\frac{1}{2}\right)^k \left\langle :\cdot^{\otimes n}:, P_{\perp,\eta}^{\otimes n}\left(\eta^{\otimes 2k}, \varphi^{(n+2k)}\right)_{\mathcal{H}^{\otimes 2k}} \right\rangle, \tag{3.7}$$

where $\varphi^{(n)}$ denotes the kernels of φ.

Proof for $\beta = 0$. Let $\eta \in \mathcal{H}$, $|\eta|_0 = 1$, and $\varphi = \sum_{n=0}^{\infty} \left\langle :\cdot^{\otimes n}:, \varphi^{(n)} \right\rangle \in \mathcal{G}$ where only a finite number of kernels $\varphi^{(n)}$, $n \in \mathbb{N}$, are non-vanishing. Then for $x \in \mathcal{N}'$ we get that

$$\begin{aligned} P_\eta \varphi(x) = P_\eta \sum_{n=0}^{\infty} \left\langle :x^{\otimes n}:, \varphi^{(n)} \right\rangle &= \sum_{n=0}^{\infty} \left\langle :(P_\eta x)^{\otimes n}:, \varphi^{(n)} \right\rangle \\ &= \sum_{n=0}^{\infty} \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{n!}{k!(n-2k)!} \left(-\frac{1}{2}\right)^k \left\langle \left(:x^{\otimes(n-2k)}: \circ P_{\perp,\eta}^{\otimes(n-2k)}\right) \otimes \eta^{\otimes 2k}, \varphi^{(n)} \right\rangle \\ &= \sum_{l=0}^{\infty} \sum_{k=0}^{\infty} \frac{(l+2k)!}{k!l!} \left(-\frac{1}{2}\right)^k \left\langle :x^{\otimes l}:, P_{\perp,\eta}^{\otimes n}\left(\eta^{\otimes 2k}, \varphi^{(l+2k)}\right)_{\mathcal{H}^{\otimes 2k}} \right\rangle. \end{aligned} \tag{3.8}$$

Of course, convergence and exchanging of the summations do not cause problems since both sums are finite, thus $P_\eta \varphi \in \mathcal{G}$. Now we observe whether the kernels $\hat{\varphi}^{(n)}$, $n \in \mathbb{N}$, given by

$$\hat{\varphi}^{(n)} := \sum_{k=0}^{\infty} \frac{(n+2k)!}{k!n!} \left(-\frac{1}{2}\right)^k P_{\perp,\eta}^{\otimes n}\left(\eta^{\otimes 2k}, \varphi^{(n+2k)}\right)_{\mathcal{H}^{\otimes 2k}}$$

are still well-defined if we do not restrict ourself to finite linear combinations. Then we obtain by (3.6) that

$$\begin{aligned} \left|\hat{\varphi}^{(n)}\right|_0 \le \sum_{k=0}^{\infty} \frac{(n+2k)!}{k!n!} \left(\frac{1}{2}\right)^k \left|\varphi^{(n+2k)}\right|_0 &\le \frac{1}{n!} \sum_{k=0}^{\infty} \frac{(n+2k)!}{k!} \left(\frac{1}{2}\right)^k \left|\varphi^{(n+2k)}\right|_0 \\ &\le \|\varphi\|_{q,0} \frac{1}{n!} \left(\sum_{k=0}^{\infty} \frac{(n+2k)!}{(k!)^2} \left(\frac{1}{2}\right)^{2k} 2^{-q(n+2k)} \right)^{\frac{1}{2}}, \end{aligned}$$

while using Schwartz inequality and the fact that $(n+2k)!\left|\varphi^{(n+2k)}\right|_0^2 \leq 2^{-q(n+2k)}\|\varphi\|_{q,0}^2$, for all $n, q \in \mathbb{N}$. Thus with $(n+2k)! \leq 2^{n+2k} n!(2k)!$ and $(2k)! < (2^k k!)^2$ it follows that

$$\left|\hat{\varphi}^{(n)}\right|_0 \leq \|\varphi\|_{q,0} \frac{1}{\sqrt{n!}} 2^{n/2(1-q)} \left(\sum_{k=0}^{\infty} 2^{-(q-1)(2k)} \right)^{\frac{1}{2}}$$
$$\leq \|\varphi\|_{q,0} \frac{1}{\sqrt{n!}} 2^{n/2(1-q)} \left(1 - 2^{-2q}\right)^{-\frac{1}{2}} =: \|\varphi\|_{q,0} C_n < \infty, \quad (3.9)$$

if q is large enough. Therefore, for every $\alpha \in \mathbb{N}$ one gets that

$$\|P_\eta \varphi\|_{\alpha,0}^2 = \sum_{n=0}^{\infty} n! 2^{\alpha n} \left|\hat{\varphi}^{(n)}\right|_0^2 \leq \sum_{n=0}^{\infty} 2^{\alpha n} \|\varphi\|_{q,0}^2 2^{n(1-q)} \left(1 - 2^{-2q}\right)^{-1}$$
$$\leq \|\varphi\|_{q,0}^2 \left(1 - 2^{-2q}\right)^{-1} \sum_{n=0}^{\infty} 2^{\alpha n} 2^{n(1-q)} \leq \|\varphi\|_{q,0}^2 \left(1 - 2^{-2q}\right)^{-1} \left(1 - 2^{-2(q-\alpha)}\right)^{-1/2} < \infty,$$

again if q is large enough. □

Corollary 3.19. *There exists no extension of the projection operator P_η, $\eta \in \mathcal{H}$, from $\mathcal{G}'$ into $\mathcal{G}'$.*

Proof. If we assume that there exists a continuous extension $P_\eta : \mathcal{G}' \to \mathcal{G}'$ then its adjoint w.r.t. the inner product in $\mathcal{G}'$, $P_\eta^\dagger$, is continuous from $\mathcal{G}$ into itself. But for $\xi, \theta, \eta \in \mathcal{H}$, one has that

$$\exp\left((P_\eta \xi, \theta) - \frac{1}{2}(\xi, \eta)^2\right) = S\left(P_\eta : \exp\left(\langle \cdot, \xi \rangle\right) :\right)(\theta) = \langle\langle P_\eta : \exp\left(\langle \cdot, \xi \rangle\right) :, : \exp\left(\langle \cdot, \theta \rangle\right) : \rangle\rangle$$
$$= \langle\langle : \exp\left(\langle \cdot, \xi \rangle\right) :, P_\eta^\dagger : \exp\left(\langle \cdot, \theta \rangle\right) : \rangle\rangle = S\left(P_\eta^\dagger : \exp\left(\langle \cdot, \theta \rangle\right) :\right)(\xi).$$

Thus for $\theta = 0$ one gets that

$$S\left(P_\eta^\dagger 1\right)(\xi) = S\left(P_\eta^\dagger : \exp\left(\langle \cdot, 0 \rangle\right) :\right)(\xi) = \exp\left(-\frac{1}{2}(\xi, \eta)^2\right) = \sqrt{2\pi} S\left(\delta(\langle \cdot, \eta \rangle)\right)(\xi).$$

Thus $\Phi, P_\eta^\dagger$ is no continuous map from $\mathcal{G}$ into itself. □

Close to this proof one can show that an extension to other spaces of generalized functions is not possible.

Theorem 3.20. *The projection operator P_η, $\eta \in \mathcal{H}$, $|\eta|_0 = 1$, can be applied to $\Phi \in \mathcal{G}^{-\beta}$, $0 \leq \beta < 1$, with kernels $\Phi^{(n)}$, $n \in \mathbb{N}$, if and only if*

$$\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(n+2k)!}{k!n!} \left(-\frac{1}{2}\right)^k \left\langle : \cdot^{\otimes n} :, \left(\eta^{\otimes 2k}, \Phi^{(n+2k)}\right)_{\mathcal{H}^{\otimes 2k}} \right\rangle,$$

is also an element of $\mathcal{G}^{-\beta}$.

The proof follows directly from Lemma 3.16 and (3.7).

Representation by differential operators

Close to the translation operator one can also represent the projection operator by differential operators combined with the orthogonal projection $\tilde{P}_\eta$, defined as in the proof of Lemma 3.16.

Theorem 3.21. *For any $\eta \in \mathcal{H}$, $|\eta|_0 = 1$, the projection operator on $\mathcal{G}^\beta$ can be represented by*

$$P_\eta = \tilde{P}_\eta \exp\Big(-\frac{1}{2}D_\eta^2\Big) := \tilde{P}_\eta \sum_{k=0}^{\infty} \frac{1}{k!}\Big(-\frac{1}{2}\Big)^k D_\eta^{2k}.$$

Moreover, for $\varphi \in \mathcal{G}^\beta$ the S-transform of $P_\eta\varphi$ is given by

$$S(P_\eta\varphi)(\xi) = \tilde{P}_\eta S\left(\exp\Big(-\frac{1}{2}D_\eta^2\Big)\varphi\right)(\xi) = \tilde{P}_\eta \exp\Big(-\frac{1}{2}D_\eta^2\Big)S(\varphi)(\xi).$$

Proof for $\beta = 0$. Let $\eta \in \mathcal{H}$ and $\varphi = \sum_{n=0}^{\infty} \langle :x^{\otimes n}:, \varphi^{(n)}\rangle \in \mathcal{G}$, then by Theorem 3.18 it follows that $P_\eta\varphi$ is also in $\mathcal{G}$. Moreover, its chaos decomposition is given by

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{(n+2k)!}{k!n!}\left(-\frac{1}{2}\right)^k \left\langle :\cdot^{\otimes n}:, P_{\perp,\eta}^{\otimes n}\left(\eta^{\otimes 2k}, \varphi^{(n+2k)}\right)_{\mathcal{H}^{\otimes 2k}}\right\rangle$$
$$= \tilde{P}_\eta \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{(n+2k)!}{k!n!}\left(-\frac{1}{2}\right)^k \left\langle :\cdot^{\otimes n}:, \left(\eta^{\otimes 2k}, \varphi^{(n+2k)}\right)_{\mathcal{H}^{\otimes 2k}}\right\rangle.$$

Hence by Lemma 3.16 one only has to show that

$$\exp\Big(-\frac{1}{2}D_\eta^2\Big)\varphi = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{(n+2k)!}{k!n!}\left(-\frac{1}{2}\right)^k \left\langle :\cdot^{\otimes n}:, \left(\eta^{\otimes 2k}, \varphi^{(n+2k)}\right)_{\mathcal{H}^{\otimes 2k}}\right\rangle.$$

Due to (3.8), using Lemma 3.2 (i) and the fact that $D_\eta^{2k}\left\langle :\cdot^{\otimes n}:, \varphi^{(n)}\right\rangle = 0$ for all $0 < n < 2k$, we get that

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{(n+2k)!}{k!n!}\left(-\frac{1}{2}\right)^k \left\langle :\cdot^{\otimes n}:, \left(\eta^{\otimes 2k}, \varphi^{(n+2k)}\right)_{\mathcal{H}^{\otimes 2k}}\right\rangle = \sum_{n=0}^{\infty} \exp\Big(-\frac{1}{2}D_\eta^2\Big)\left\langle :\cdot^{\otimes n}:, \varphi^{(n)}\right\rangle.$$

But then for $\xi \in \mathcal{N}$ one has that

$$\exp\Big(-\frac{1}{2}D_\eta^2\Big)S(\varphi)(\xi) = \sum_{k=0}^{\infty}\frac{1}{k!}\left(-\frac{1}{2}\right)^k \sum_{n=0}^{\infty} D_\eta^{2k}\left\langle \varphi^{(n)}, \xi^{\otimes n}\right\rangle$$
$$= \sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{1}{k!}\left(-\frac{1}{2}\right)^k \left(\frac{\partial}{\partial t}\right)^{2k}\left\langle \varphi^{(n)}, (\xi + t\eta)^{\otimes n}\right\rangle\Big|_{t=0}$$
$$= \sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{1}{k!}\left(-\frac{1}{2}\right)^k \left(\frac{\partial}{\partial t}\right)^{2k}\sum_{j=0}^{n}\binom{n}{j}t^j\left\langle \varphi^{(n)}, \xi^{\otimes(n-j)}\otimes\eta^{\otimes j}\right\rangle\Big|_{t=0}$$

$$= \sum_{k=0}^{\infty}\sum_{n=2k}^{\infty}\frac{1}{k!}\binom{n}{2k}(2k)!\left(-\frac{1}{2}\right)^k\left\langle\varphi^{(n)},\xi^{\otimes(n-2k)}\otimes\eta^{\otimes 2k}\right\rangle$$

$$= \sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{(n+2k)!}{k!n!}\left(-\frac{1}{2}\right)^k\left\langle\left(\eta^{\otimes 2k},\varphi^{(n+2k)}\right)_{\mathcal{H}^{\otimes(2k)}},\xi^{\otimes n}\right\rangle.$$

Thus one only has to check whether the sums can be interchanged. But this can easily be shown for regular test function where only a finite number of kernels are non-vanishing, using Lemma 3.2 (iii). Now let $\varphi\in\mathcal{G}$ with kernels $\varphi^{(n)}$, $n\in\mathbb{N}$, then the sequences given by $\varphi_M := \sum_{n=0}^{M}\langle :\cdot^{\otimes n}:,\varphi^{(n)}\rangle$ and $P_\eta\varphi_M$, $M\in\mathbb{N}$, converge to φ and $P_\eta\varphi$ in $\mathcal{G}$ as M goes to infinity by the norm estimate in the proof of Theorem 3.18. Thus we can interchange the sums if

$$\sum_{k=0}^{\infty}\left\|\sum_{n=0}^{\infty}\frac{(n+2k)!}{k!n!}\left(-\frac{1}{2}\right)^k\left\langle :\cdot^{\otimes n}:,\left(\eta^{\otimes 2k},\varphi^{(n+2k)}\right)_{\mathcal{H}^{\otimes 2k}}\right\rangle\right\|_{\alpha,0}^2<\infty,$$

for all $\alpha\in\mathbb{N}$. Following [82] there exists some $q\in\mathbb{N}$ such that

$$\left\|\sum_{n=0}^{\infty}\frac{(n+2k)!}{k!n!}\left(-\frac{1}{2}\right)^k\left\langle :\cdot^{\otimes n}:,\left(\eta^{\otimes 2k},\varphi^{(n+2k)}\right)_{\mathcal{H}^{\otimes 2k}}\right\rangle\right\|_{\alpha,0}^2 \le \sum_{n=0}^{\infty}n!2^{\alpha n}\left(\frac{(n+2k)!}{k!n!}\right)^2\left(\frac{1}{2}\right)^{2k}\left|\varphi^{(n+2k)}\right|_0^2$$

$$\le \|\varphi\|_{q,0}^2 2^{-kq}\sum_{n=0}^{\infty}\binom{n+2k}{2k}2^{(\alpha-q)n} = \|\varphi\|_{q,0}^2(1-2^{\alpha-q})^{-1}(2^q-2^\alpha)^{-2k}.$$

Hence

$$\sum_{k=0}^{\infty}\left\|\sum_{n=0}^{\infty}\frac{(n+2k)!}{k!n!}\left(-\frac{1}{2}\right)^k\left\langle :\cdot^{\otimes n}:,\left(\eta^{\otimes 2k},\varphi^{(n+2k)}\right)_{\mathcal{H}^{\otimes 2k}}\right\rangle\right\|_{\alpha,0}^2 \le \|\varphi\|_{q,0}^2(1-2^{\alpha-q})^{-1}\sum_{k=0}^{\infty}(2^q-2^\alpha)^{-2k},$$

which converges for a suitable choice of q. □

Corollary 3.22. *The projection operator P_η, $\eta\in\mathcal{H}$, $|\eta|_0=1$, can be applied to $\Phi\in\mathcal{G}'$, whenever*

$$\sum_{k=0}^{\infty}\frac{1}{k!}\left(-\frac{1}{2}\right)^k\left(\frac{\partial}{\partial t}\right)^{2k}S(\Phi)(\cdot+t\eta)\Big|_{t=0} \tag{3.10}$$

is in the Bargmann-Segal space $E^2(\nu)$.

Proof. Let $\Phi\in\mathcal{G}'$ such that $P_\eta\Phi\in\mathcal{G}'$, with kernels $\Phi^{(n)}$ and $\hat{\Phi}^{(n)}$ in $\mathcal{H}^{\hat{\otimes}n}$, respectively. Here

$$\hat{\Phi}^{(n)} = \sum_{k=0}^{\infty}\frac{(n+2k)!}{k!n!}\left(-\frac{1}{2}\right)^k P_{\perp,\eta}^{\otimes n}\left(\eta^{\otimes 2k},\Phi^{(n+2k)}\right)_{\mathcal{H}^{\otimes 2k}}. \tag{3.11}$$

We define φ_M, $M\in\mathbb{N}$, by $\varphi_M := \sum_{n=0}^{M}\langle :\cdot^{\otimes n}:,\Phi^{(n)}\rangle$. Then $\varphi_M\in\mathcal{G}$ for all $M\in\mathbb{N}$ since the sum is finite. Moreover, the sequence $(\varphi_M)_{M\in\mathbb{N}}$ converges to Φ in $\mathcal{G}'$. Thus there exists some $\alpha\in\mathbb{N}$

such that $\|\Phi - \varphi_M\|_{-\alpha,0}$ converge to zero as M goes to infinity. In addition $P_\eta\varphi_M \in \mathcal{G}$ and the sequence of kernels $\left(\hat{\varphi}_M^{(n)}\right)_{M\in\mathbb{N}}$ converges to $\hat{\Phi}^{(n)}$ for all $n \in \mathbb{N}$. Let us assume that there exists some $q \in \mathbb{N}$, $\alpha < q$, and some constant $0 < C < \infty$ such that $\|P_\eta\Phi - P_\eta\varphi_M\|_{-q,0} \leq C$ for all $M \in \mathbb{N}$. Then $\|P_\eta\Phi - P_\eta\varphi_M\|_{-(q+1),0} \leq \sum_{n=0}^{\infty} 2^{-n}C$ for all $M \in \mathbb{N}$. Thus by Lebesgue dominated convergence $\left(P_\eta\varphi_M\right)_{M\in\mathbb{N}}$ converges to $P_\eta\Phi$ w.r.t. the norm $\|\cdot\|_{-(q+1),0}$ and therefore in $\mathcal{G}'$.

It is left to show that there exists some $q \in \mathbb{N}$, $\alpha < q$, and some constant $0 < C < \infty$ such that $\|P_\eta\Phi - P_\eta\varphi_M\|_{-q,0} \leq C$ for all $M \in \mathbb{N}$. Let us denote that the generalized function given by its kernels

$$\sum_{k=j}^{\infty} \frac{(n+2k)!}{k!n!}\left(-\frac{1}{2}\right)^k P_{\perp,\eta}^{\otimes n}\left(\eta^{\otimes 2k}, \Phi^{(n+2k)}\right)_{\mathcal{H}^{\otimes 2k}}, \quad n \in \mathbb{N},$$

is also in $\mathcal{G}_{-\alpha}$. This causes by the fact that these kernels have to vanish in $\mathcal{H}^{\otimes n}$ as j goes to infinity, since $\hat{\Phi}^{(n)} \in \mathcal{H}^{\otimes n}$. Thus there exists some $N \in \mathbb{N}$ such that for all $j > N$ there exists some $\alpha(j) \leq \alpha$ such that

$$\left|\sum_{k=j}^{\infty} \frac{(n+2k)!}{k!n!}\left(-\frac{1}{2}\right)^k P_{\perp,\eta}^{\otimes n}\left(\eta^{\otimes 2k}, \Phi^{(n+2k)}\right)_{\mathcal{H}^{\otimes 2k}}\right|^2 < \frac{2^{\alpha(j)}}{n!}, \tag{3.12}$$

for all $n \in \mathbb{N}$. Moreover, for all $\varepsilon > 0$ there exists an $N' \in \mathbb{N}$ such that for all $l > N'$ we get that

$$\sum_{n=l}^{\infty} n!2^{-\alpha n}\left|\hat{\Phi}^{(n)} - \varphi^{(n)}\right|_0^2 < \varepsilon. \tag{3.13}$$

Here $\varphi^{(n)}$ denote the n-th kernel of an arbitrary regular test function φ. Thus let $M \in \mathbb{N}$ such that $\frac{M}{2} > \max\{N, N'\}$, then by (3.12) and (3.13) we obtain

$$\begin{aligned}
\left\|P_\eta\Phi - P_\eta\varphi_M\right\|_{-\alpha-1,0} &= \sum_{n=0}^{\infty} n!2^{-(\alpha+1)n}\left|\hat{\Phi}^{(n)} - \hat{\varphi}_M^{(n)}\right|_0^2 \\
&= \sum_{n=0}^{\frac{M}{2}} n!2^{-(\alpha+1)n}\left|\hat{\Phi}^{(n)} - \hat{\varphi}_M^{(n)}\right|_0^2 + \sum_{n=\frac{M}{2}}^{\infty} n!2^{-(\alpha+1)n}\left|\hat{\Phi}^{(n)} - \hat{\varphi}_M^{(n)}\right|_0^2 \\
&\leq \sum_{n=0}^{\frac{M}{2}} n!2^{-(\alpha+1)n}\left|\sum_{k=\frac{M}{4}}^{\infty} \frac{(n+2k)!}{k!n!}\left(-\frac{1}{2}\right)^k P_{\perp,\eta}^{\otimes n}(\eta^{\otimes 2k}, \Phi^{(n+2k)})_{\mathcal{H}^{\otimes 2k}}\right|^2 + \varepsilon \leq \sum_{n=0}^{\infty} 2^{-n} + \varepsilon < \infty.
\end{aligned}$$

Therefore $P_\eta\varphi_M$ converges to $P_\eta\Phi$ in $\mathcal{G}'$, i.e., $S(P_\eta\Phi) \in E^2(\nu)$. Using Theorem 3.21 we get that

$$\begin{aligned}
S(P_\eta\Phi)(\xi) &= \lim_{M\to\infty} S(P_\eta\varphi_M)(\xi) = \lim_{M\to\infty} \tilde{P}_\eta S\left(\exp\left(-\frac{1}{2}D_\eta^2\right)\varphi_M\right)(\xi) \\
&= \tilde{P}_\eta \lim_{M\to\infty} \exp\left(-\frac{1}{2}D_\eta^2\right) S(\varphi_M)(\xi) = \tilde{P}_\eta \exp\left(-\frac{1}{2}D_\eta^2\right) \lim_{M\to\infty} S(\varphi_M)(\xi) = \tilde{P}_\eta \exp\left(-\frac{1}{2}D_\eta^2\right) S(\Phi)(\xi),
\end{aligned}$$

for all $\xi \in \mathcal{N}$. Since by Lemma 3.16 $\tilde{P}_\eta : \mathcal{G}' \to \mathcal{G}'$, also $\exp\left(-\frac{1}{2}D_\eta^2\right) S(\Phi) \in E^2(\nu)$.

Vise versa, let (3.10) be an element of $E^2(\nu)$, then there exists an element $\Psi \in \mathcal{G}'$ whose S-transform is given by (3.10). Hence by Lemma 3.16 also $\tilde{P}_\eta \Psi \in \mathcal{G}'$. Using the above defined approximation $\varphi_M,\ M \in \mathbb{N}$, it is easy to see that $\left(\exp\left(-\frac{1}{2}D_\eta^2\right)\varphi_M\right)_{M\in\mathbb{N}}$ converge to Ψ. Moreover using (2.9) the kernels of $\tilde{P}_\eta \Psi$ are given by $\hat{\Phi}^{(n)}$ as in (3.11). Thus by definition of P_η we get that $\tilde{P}_\eta \Psi = P_\eta \Phi \in \mathcal{G}'$. □

Example 3.23. *Let $\eta, \theta \in \mathcal{H}$, $|\eta|_0 = 1$, and $a \in \mathbb{C}$ then $\delta(\langle\cdot,\theta\rangle - a) \in \mathcal{G}'$. Moreover $P_\eta \delta(\langle\cdot,\theta\rangle - a) \in \mathcal{G}'$ if and only if $\theta \neq \eta$ and $(\eta,\theta) \neq 1$.*

Proof. Let $\theta \in \mathcal{H},\ \theta \neq 0$, and $a \in \mathbb{R}$. Then $\delta(\langle\cdot,\theta\rangle - a) \in \mathcal{G}'$ as shown in Example 2.27. Thus with (2.11) we get for $\eta \in \mathcal{H}$, $|\eta|_0 = 1$, that

$$\begin{aligned}
\exp\left(-\frac{1}{2}D_\eta^2\right) S\left(\delta(\langle\cdot,\theta\rangle - a)\right)(\xi) &= \sum_{k=0}^{\infty} \frac{1}{k!}\left(-\frac{1}{2}\right)^k D_\eta^{2k} S\left(\delta(\langle\cdot,\theta\rangle - a)\right)(\xi) \\
&= \sum_{k=0}^{\infty} \frac{1}{k!}\left(-\frac{1}{2}\right)^k \left(\frac{\partial}{\partial t}\right)^{2k} \exp\left(-\frac{t^2(\eta,\theta)^2 + 2t(\eta,\theta)((\xi,\theta)-a)}{2(\theta,\theta)}\right)\Bigg|_{t=0} S\left(\delta(\langle\cdot,\theta\rangle - a)\right)(\xi) \\
&= S\left(\delta(\langle\cdot,\theta\rangle - a)\right)(\xi) \sum_{k=0}^{\infty} \frac{1}{k!}\left(-\frac{1}{2}\right)^k H_{2k}\left(\frac{(\xi,\theta)-a}{\sqrt{2(\theta,\theta)}}\right)(2(\theta,\theta))^{-k}(\eta,\theta)^{2k} \\
&= S\left(\delta(\langle\cdot,\theta\rangle - a)\right)(\xi) \sum_{k=0}^{\infty} L_k^{-(\frac{1}{2})}\left(\frac{((\xi,\theta)-a)^2}{2(\theta,\theta)}\right)(\theta,\theta)^{-k}(\eta,\theta)^{2k},
\end{aligned}$$

where $L_k^{-(\frac{1}{2})}$ denotes th k-th Laguerre polynomial with index $-\frac{1}{2}$. Here we use the relation

$$H_{2k}(x) = (-1)^k 2^{2k} k! L_k^{-(\frac{1}{2})}(x^2), \quad x \in \mathbb{R}.$$

Using the generating function of the Laguerre polynomials (see e.g. [80]) one gets that

$$\exp\left(-\frac{1}{2}D_\eta^2\right) S\left(\delta(\langle\cdot,\theta\rangle - a)\right)(\xi) = S\left(\delta(\langle\cdot,\theta\rangle - a)\right)(\xi) \frac{1}{1-(\eta,\theta)} \exp\left(-\left(\frac{(\xi,\theta)-a}{(\theta,\theta)(1-(\eta,\theta))}\right)^2\right).$$

Obviously, this is a U-functional if $\eta \neq \theta$ and $(\eta,\theta) \neq 1$. If one considers a sequence $(\theta_n)_{n\in\mathbb{N}} \subset \mathcal{H}$ converging to θ, where $\theta = \eta$ or $(\theta,\eta) = 1$. Then

$$\lim_{n\to\infty} \exp\left(-\frac{1}{2}D_\eta^2\right) S\left(\delta(\langle\cdot,\theta_n\rangle - a)\right)(\xi) = \delta((\xi,\theta) - a)$$

which is indeed not a U-functional. □

3.2.3 Scaling operator

Another import operator which is also at first only defined on spaces of test functions is the so called 'scaling operator'.

Definition 3.24. *Let φ be the continuous version of an element of $(\mathcal{N})^\beta$ for $0 \leq \beta \leq 1$. Then for $z \in \mathbb{C}$ we define the scaling of φ by*

$$(\sigma_z \varphi)(x) = \varphi(zx), \qquad x \in \mathcal{N}'.$$

Theorem 3.25. *Let $0 \leq \beta \leq 1$ then the scaling operator has the following properties:*

(i) For all $z \in \mathbb{C}$ the mapping $\varphi \mapsto \sigma_z \varphi$ is continuous from $(\mathcal{N})^\beta$ into itself.

(ii) For $\varphi, \psi \in (\mathcal{N})^\beta$ the equation

$$\sigma_z(\varphi\psi) = (\sigma_z\varphi)(\sigma_z\psi),$$

holds.

A representation of $\sigma_z\varphi$ via its chaos decomposition is given by

$$\sigma_z \varphi = \sum_{n=0}^{\infty} \left\langle :x^{\otimes n}:, \varphi_z^{(n)} \right\rangle, \tag{3.14}$$

with kernels

$$\varphi_z^{(n)} = z^n \sum_{k=0}^{\infty} \frac{(n+2k)!}{k!n!} \left(\frac{z^2-1}{2} \right)^k \mathrm{tr}^k \varphi^{(n+2k)}.$$

Here $\mathrm{tr}^k\varphi^{(n+2k)}$ is defined by

$$\mathrm{tr}^k \varphi^{(n+2k)} := \left(\mathrm{Tr}^{\otimes k}, \varphi^{(n+2k)} \right)_{\mathcal{H}^{\otimes 2k}} \in \mathcal{N}^{\hat{\otimes} n},$$

where the trace kernel Tr is given by (3.1). Since we consider a continuous mapping from $(\mathcal{N})^\beta$ into itself one can define the adjoint scaling operator $\sigma_z^\dagger : (\mathcal{N})^{-\beta} \to (\mathcal{N})^{-\beta}$ by

$$\left\langle\!\left\langle \sigma_z^\dagger \Phi, \psi \right\rangle\!\right\rangle = \left\langle\!\left\langle \Phi, \sigma_z \psi \right\rangle\!\right\rangle,$$

for all $\Phi \in (\mathcal{N})^{-\beta}$ and all $\psi \in (\mathcal{N})^\beta$.

Lemma 3.26. *For all $z \in \mathbb{C}$ and $\Phi \in (\mathcal{N})^{-\beta}$ we have that*

$$\sigma_z^\dagger \Phi = J_z \diamond \Gamma_z \Phi,$$

where Γ_z is defined by $S(\Gamma_z\Psi)(\xi) = S(\Psi)(z\xi)$, for all $\Psi \in (\mathcal{N})^{-\beta}$ and $\xi \in \mathcal{N}$, and J_z is defined as in Example 2.13. In particular

$$\sigma_z^\dagger \mathbb{1} = J_z.$$

Remark 3.27. *Note that close to Lemma 3.16 one can show that Γ_z is a continuous map from $(\mathcal{N})^{-\beta}$ into itself for all $z \in \mathbb{C} \setminus \{0\}$. Moreover, Γ_z is an extension of $\Gamma(z\mathbb{1})$, where Γ denotes the usual second quantization, see e.g. [75].*

Lemma 3.28. *Let $\varphi \in (\mathcal{N})^{\beta}$ then*

$$J_z\varphi = \sigma_z^{\dagger}(\sigma_z\varphi).$$

Proof. Let $\varphi, \psi \in (\mathcal{N})^{\beta}$ then

$$\left\langle\!\left\langle J_z\varphi, \psi \right\rangle\!\right\rangle = \left\langle\!\left\langle \sigma_z^{\dagger}\mathbb{1}, \varphi\psi \right\rangle\!\right\rangle = \left\langle\!\left\langle \sigma_z\varphi, \sigma_z\psi \right\rangle\!\right\rangle = \left\langle\!\left\langle \sigma_z^{\dagger}(\sigma_z\varphi), \psi \right\rangle\!\right\rangle .$$

□

Remark 3.29. *Since $(\mathcal{N})^{\beta}$ is an algebra for $0 \leq \beta \leq 1$ a product of a generalized function $\Phi \in (\mathcal{N})^{-\beta}$ and a test function $\varphi \in (\mathcal{N})^{\beta}$ can be realized as*

$$\left\langle\!\left\langle \Phi\varphi, \psi \right\rangle\!\right\rangle = \left\langle\!\left\langle \Phi, \varphi\psi \right\rangle\!\right\rangle, \quad \psi \in (\mathcal{N})^{\beta}.$$

Theorem 3.30. *Let $(\varphi_n)_{n\in\mathbb{N}}$ be a sequence of test functionals in $(\mathcal{N})^{\beta}$. Then for $z \in \mathbb{C}$ the following statements are equivalent:*

(i) The sequence $(J_z\varphi_n)_{n\in\mathbb{N}}$ converges in $(\mathcal{N})^{-\beta}$ to an element Ψ.

(ii) The sequence $(\sigma_z\varphi_n)_{n\in\mathbb{N}}$ converges in $(\mathcal{N})^{-\beta}$.

(iii) The sequence $\left(E(\psi\sigma_z\varphi_n)\right)_{n\in\mathbb{N}}$ converges for all $\psi \in (\mathcal{N})^{\beta}$.

Hence the action of Ψ is given by

$$\left\langle\!\left\langle \Psi, \psi \right\rangle\!\right\rangle = \lim_{n\to\infty} E(\sigma_z(\varphi_n\psi))$$

if one of the conditions (i) to (iii) holds.

Remark 3.31. *Following Theorem 3.30 and Lemma 3.28 it is natural to define the product of J_z with $\Phi \in (\mathcal{N})^{-\beta}$ by*

$$J_z\Phi := \sigma_z^{\dagger} \lim_{n\to\infty} (\sigma_z\varphi_n),$$

where $(\varphi_n)_{n\in\mathbb{N}} \subset (\mathcal{N})^{\beta}$ converges in $(\mathcal{N})^{-\beta}$ to Φ, whenever $\lim_{n\to\infty} \sigma_z\varphi_n$ is a well-defined object in $(\mathcal{N})^{-\beta}$.

Representation by differential operators

Again we are interested in a representation by differential operators.

Theorem 3.32. *For any $z \in \mathbb{C}$, $z \neq 0$, we can represent the scaling operator on $(\mathcal{N})^\beta$, $0 \leq \beta < 1$, by*

$$\sigma_z = \Gamma_z \exp\left(\frac{z^2-1}{2}\Delta_G\right) := \Gamma_z \sum_{k=0}^{\infty} \frac{1}{k!}\left(\frac{z^2-1}{2}\right)^k \Delta_G^k,$$

where Δ_G denotes the Gross Laplacian and Γ_z is defined as in Lemma 3.26. Moreover, for $\varphi \in (\mathcal{N})^\beta$ the S-transform of $\sigma_z\varphi$ is given by

$$S(\sigma_z\varphi)(\xi) = \Gamma_z S\left(\exp\left(\frac{z^2-1}{2}\Delta_G\right)\varphi\right)(\xi).$$

The proof is similar to the proof of Theorem 3.21.

Complex scaling of Donsker's delta

Let us consider the S-transform of Donsker's delta $\delta(\langle\cdot,\eta\rangle - a)$, as in Example 2.14, for $a \in \mathbb{R}$ and $\eta \in \mathcal{N}$:

$$F_{\eta,a}(\xi) := S\left(\delta(\langle\cdot,\eta\rangle - a)\right)(\xi) = \frac{1}{\sqrt{2\pi(\eta,\eta)}} \exp\left(-\frac{1}{2(\eta,\eta)}(\langle\xi,\eta\rangle - a)^2\right).$$

Clearly, $F_{h,a}(f)$ is analytic in the parameter a. Thus it is possible to extend it to complex a and the resulting expression is still a U-functional. Analogously, we can extend $F_{\eta,a}(f)$ first continuously to $\eta \in \mathcal{H}$ and obviously, it has an analytic continuation to $\eta \in \mathcal{H}_\mathbb{C}$ (the complexification of $\mathcal{H}$), similar to Definition 2.9. One only has to be careful with the square root, hence we exclude $\eta \in \mathcal{H}_\mathbb{C}$ with $(\eta,\eta) < 0$ (note that due to the bilinear extension of $(\cdot,\cdot)$ to $\mathcal{H}_\mathbb{C}$, negative values are possible). In this case $F_{\eta,a}(f)$ is again a well-defined U-functional.

Theorem 3.33. *Let $a \in \mathbb{C}$ and $\eta \in \mathcal{H}_\mathbb{C}$ with $\arg(\eta,\eta) \neq \pi$. Then*

$$\delta(\langle\cdot,\eta\rangle - a) = \sum_{n=0}^{\infty} \left\langle :x^{\otimes n}:, f^{(n)}\right\rangle,$$

is in $\mathcal{G}'$, i.e., for all $q \in \mathbb{N}$ one has that $\|\delta(\langle\cdot,\eta\rangle - a)\|_{q,0}$ is finite. Note that the kernels $f^{(n)}$ are given as in (2.11).

The proof can be done analogously to Example 2.27. Let us remark that a construction of a complex scaled version of Donsker's delta was achieved in [54] with help of the following

approximating sequence: For $a \in \mathbb{C}$, $|\alpha| < \frac{\pi}{4}$, $\eta \in \mathcal{N}$ and $z \in S_\alpha := \left\{z \in \mathbb{C} \mid \arg z \in (-\frac{\pi}{4} + \alpha, \frac{\pi}{4} + \alpha)\right\}$ and define

$$\begin{aligned} \varphi_{n,z} : \mathcal{N}' &\to \mathbb{C} \\ x &\mapsto \int_{\gamma_{\alpha,n}} \exp(i\nu(z\langle x, \eta\rangle - a))\, d\nu, \end{aligned} \tag{3.15}$$

where $\gamma_{\alpha,n} := \{\exp(i\alpha)s, s \in \mathbb{R}, |s| < n\}$.

Lemma 3.34. *δ is homogeneous of degree -1 in $z \in S_\alpha$:*

$$\sigma_z \delta(\langle \cdot, \eta\rangle - a) = \frac{1}{z}\delta\Big(\langle \cdot, \eta\rangle - \frac{1}{z}a\Big).$$

For the proof and a detailed construction of complex scaled Donsker's deltas we refer to [54]. Again the same statements can be shown for the vector-valued case, see Example 2.15.

3.2.4 Localized scaling operator

Let us again consider the vector-valued white noise case $\mathcal{N} = S_d(\mathbb{R})$ and $\mathcal{H} = L^2_d(\mathbb{R})$, $d \geq 1$. For application we are also interested in a representation in terms of differential operators of localized normalized exponentials

$$J_{z,t_0,t} := \mathrm{Nexp}\left(\frac{1}{2}(1 - z^{-2}) \int_{t_0}^{t} \omega^2(s)\, ds\right),$$

where $-\infty \leq t_0 \leq t < \infty$, see Example 2.13 for its rigorous definition as a Hida distibution.

Definition 3.35. *For $\varphi \in (S_d)^\beta$, $0 \leq \beta \leq 1$, we define $\sigma_{z,t_0,t}\varphi$ via its chaos decomposition, which is given by*

$$\sigma_{z,t_0,t}\varphi = \sum_{n=0}^{\infty} \left\langle :\omega^{\otimes n}:, \varphi^{(n)}_{z,t_0,t}\right\rangle, \tag{3.16}$$

with kernels

$$\varphi^{(n)}_{z,t_0,t} = z^n \sum_{k=0}^{\infty} \frac{(n+2k)!}{k!n!} \left(\frac{z^2 - 1}{2}\right)^k \mathrm{tr}^k_{t_0,t}\varphi^{(n+2k)}.$$

Here $\mathrm{tr}^k_{t_0,t}\varphi^{(n+2k)}$ is defined by

$$\mathrm{tr}^k_{t_0,t}\varphi^{(n+2k)} := \left(\mathrm{Tr}^{\otimes k}_{t_0,t}, \varphi^{(n+2k)}\right)_{L^2_d(\mathbb{R})^{\otimes 2k}} \in S_d(\mathbb{R})^{\hat{\otimes} n},$$

where the localized trace kernel $\mathrm{Tr}_{t_0,t} \in S'_d(\mathbb{R})^{\otimes 2}$ is defined by

$$\langle \mathrm{Tr}_{t_0,t}, f \otimes g\rangle = \sum_{j=1}^{d} \int_{t_0}^{t} f_j(s) g_j(s) ds, \quad f = (f_1, \ldots, f_d),\ g = (g_1, \ldots, g_d) \in S_d(\mathbb{R}). \tag{3.17}$$

Theorem 3.36. *Let $0 \leq \beta \leq 1$ then the localized scaling operator has the following properties:*

(i) For all $z \in \mathbb{C}$ and all $-\infty < t_0 < t < \infty$ the mapping $\varphi \mapsto \sigma_{z,t_0,t}\varphi$ is continuous from $(S_d)^\beta$ into itself.

(ii) For $\varphi, \psi \in (S_d)^\beta$ the equation

$$\sigma_{z,t_0,t}(\varphi\psi) = (\sigma_{z,t_0,t}\varphi)(\sigma_{z,t_0,t}\psi),$$

holds.

The proof follows directly by Theorem 3.25 and the fact that $|\mathrm{Tr}_{t_0,t}|_{-p} \leq |\mathrm{Tr}|_{-p}$. We denote that the application of the operator $\sigma_{z,t_0,t}$ to a generalized function $\Phi \in (S_d)^{-\beta}$ is sensible whenever (3.16) is well-defined in $(S_d)^{-\beta}$. Since we consider a continuous mapping from $(S_d)^\beta$ into itself one can define the adjoint scaling operator $\sigma^\dagger_{z,t_0,t} : (S_d)^{-\beta} \to (S_d)^{-\beta}$ by

$$\left\langle\!\left\langle \sigma^\dagger_{z,t_0,t}\Phi, \psi \right\rangle\!\right\rangle = \left\langle\!\left\langle \Phi, \sigma_{z,t_0,t}\psi \right\rangle\!\right\rangle,$$

for all $\Phi \in (S_d)^{-\beta}$ and all $\psi \in (S_d)^\beta$.

Lemma 3.37. *For all $z \in \mathbb{C}$, $-\infty < t_0 < t < \infty$ and $\Phi \in (S_d)^{-\beta}$ we have that*

$$\sigma^\dagger_{z,t_0,t}\Phi = J_{z,t_0,t} \diamond \Gamma_z\Phi.$$

In particular

$$\sigma^\dagger_{z,t_0,t}\mathbb{1} = J_{z,t_0,t}.$$

Proof. Let $z \in \mathbb{C}$, $-\infty < t_0 < t < \infty$, $\Phi \in (S_d)^{-\beta}$ and $g \in S_d(\mathbb{R})$. Using

$$\begin{aligned}
\sigma_{z,t_0,t} : \exp(\langle\omega, g\rangle) &:= \sum_{n=0}^{\infty} \left\langle : \omega^{\otimes n} :, z^n \sum_{k=0}^{\infty} \frac{(n+2k)!}{k!n!} \left(\frac{z^2-1}{2}\right)^k \mathrm{tr}^k_{t_0,t} \frac{1}{(n+2k)!} g^{\otimes(n+2k)} \right\rangle \\
&= \sum_{n=0}^{\infty} \left\langle : \omega^{\otimes n} :, z^n \sum_{k=0}^{\infty} \frac{1}{k!n!} \left(\frac{z^2-1}{2}\right)^k \left(\int_{t_0}^t g^2(s)ds\right)^k g^{\otimes n} \right\rangle \\
&= \sum_{n=0}^{\infty} \left\langle : \omega^{\otimes n} :, z^n \frac{1}{n!} g^{\otimes n} \right\rangle \exp\left(\frac{z^2-1}{2} \int_{t_0}^t g^2(s)ds\right) \\
&=: \exp(\langle\omega, zg\rangle) : \exp\left(\frac{z^2-1}{2} \int_{t_0}^t g^2(s)ds\right), \qquad (3.18)
\end{aligned}$$

we get that

$$\begin{aligned}
S\left(\sigma^\dagger_{z,t_0,t}\Phi\right)(g) &= \left\langle\!\left\langle \sigma^\dagger_{z,t_0,t}\Phi, : \exp(\langle\cdot, g\rangle) : \right\rangle\!\right\rangle = \left\langle\!\left\langle \Phi, \sigma_{z,t_0,t} : \exp(\langle\cdot, g\rangle) : \right\rangle\!\right\rangle \\
&= \left\langle\!\left\langle \Phi, : \exp(\langle\cdot, zg\rangle) : \right\rangle\!\right\rangle \exp\left(\frac{z^2-1}{2} \int_{t_0}^t g^2(s)ds\right) = S\left(\Gamma_z\Phi\right)(g) S\left(J_{z,t_0,t}\right)(g).
\end{aligned}$$

□

Remark 3.38. *(i) Similar to* (3.18) *one can calculate that*

$$\sigma_{z,t_0,t}\exp(i\langle\cdot,g\rangle) = \exp(iz\langle\cdot,g\rangle)\exp\left(\frac{z^2-1}{2}\int_{[t_0,t]^c} g^2(s)ds\right)$$
$$= \sigma_z\exp(i\langle\cdot,g\rangle)\exp\left(\frac{z^2-1}{2}\int_{[t_0,t]^c} g^2(s)ds\right),$$

for all $z \in \mathbb{C}$, $-\infty < t_0 < t < \infty$ and $g \in S_d(\mathbb{R})$. Here $[t_0,t]^c$ denotes the complement of the time interval $[t_0,t]$.

(ii) Using (3.18) *we can show close to Corollary 3.19 that there does not exists an extension $\tilde{\sigma}_z : (S_d)' \to (S_d)'$ or other distribution spaces.*

Lemma 3.39. *Let $\varphi \in (S_d)^\beta$ then*

$$J_{z,t_0,t}\varphi = \sigma^\dagger_{z,t_0,t}(\sigma_{z,t_0,t}\varphi).$$

The proof can be done in analogy to Lemma 3.28.

Theorem 3.40. *Let $(\varphi_n)_{n\in\mathbb{N}}$ be a sequence of test functionals in $(S_d)^\beta$. Then for $z \in \mathbb{C}$ and $-\infty < t_0 < t < \infty$ the following statements are equivalent:*

(i) The sequence $(J_{z,t_0,t}\varphi_n)_{n\in\mathbb{N}}$ converges in $(S_d)^{-\beta}$ to an element Ψ.

(ii) The sequence $(\sigma_{z,t_0,t}\varphi_n)_{n\in\mathbb{N}}$ converges in $(S_d)^{-\beta}$.

(iii) The sequence $\left(E(\psi\sigma_{z,t_0,t}\varphi_n)\right)_{n\in\mathbb{N}}$ converges for all $\psi \in (S_d)^\beta$.

Hence the action of Ψ is given by

$$\left\langle\!\left\langle \Psi,\psi \right\rangle\!\right\rangle = \lim_{n\to\infty} E(\sigma_{z,t_0,t}(\varphi_n\psi))$$

if one of the conditions (i) to (iii) holds.

Remark 3.41. *(i) Close to Remark 3.31 it is natural to define the product of $J_{z,t_0,t}$ with $\Phi \in (S_d)^{-\beta}$ by*

$$J_{z,t_0,t}\Phi := \sigma^\dagger_{z,t_0,t}\lim_{n\to\infty}(\sigma_{z,t_0,t}\varphi_n),$$

where $(\varphi_n)_{n\in\mathbb{N}} \subset (S_d)^\beta$ converges in $(S_d)^{-\beta}$ to Φ, whenever $\lim_{n\to\infty}\sigma_{z,t_0,t}\varphi_n$ is a well-defined object in $(S_d)^{-\beta}$.

(ii) If the sequence $(\varphi_n)_{n\in\mathbb{N}} \subset (S_d)$ *as in (i) with kernels* $\varphi_n^{(k)}$, $k \in \mathbb{N}$, *fulfills*

$$\mathrm{Tr}^l_{t_0,t}\varphi_n^{\otimes(k+2l)} = \mathrm{Tr}^l\varphi_n^{\otimes(k+2l)},$$

for all $k, l \in \mathbb{N}$, *then*

$$\sigma_{z,t_0,t}\Phi = \sigma_z\Phi \quad \text{but} \quad J_{z,t_0,t}\Phi \neq J_z\Phi.$$

For example Donsker's delta in Brownian motion $\delta(\langle\cdot, 1\!\!1_{[t_0,t)}\rangle - a)$, $a \in \mathbb{R}$, *with approximation sequence as in* (3.15) *fulfills this assumption.*

Representation by differential operators

Close to Theorem 3.32 there exists a representation of the localized scaling operator by differential operators.

Theorem 3.42. *For any* $z \in \mathbb{C}$, $z \neq 0$, *and* $-\infty < t_0 < t < \infty$ *the localized scaling operator on* $(S_d)^\beta$, $0 \leq \beta < 1$, *is given by*

$$\sigma_{z,t_0,t} = \Gamma_z \exp\left(\frac{z^2-1}{2}D_{1_{[t_0,t)}}\right) := \Gamma_z \sum_{k=0}^{\infty} \frac{1}{k!}\left(\frac{z^2-1}{2}\right)^k \Delta^k_{1_{[t_0,t)}},$$

where $\Delta_{1_{[t_0,t)}}$ *is defined by*

$$\Delta_{1_{[t_0,t)}}\varphi(x) = \sum_{n=0}^{\infty}(n+2)(n+1)\Big\langle :x^{\otimes n}:, \langle \mathrm{Tr}_{t_0,t}, \varphi^{(n+2)}\rangle\Big\rangle, \quad \varphi \in (\mathcal{N}),$$

and Γ_z *is as in Lemma 3.26. Moreover, for* $\varphi \in (S_d)^\beta$ *the* S*-transform of* $\sigma_z\varphi$ *is given by*

$$S(\sigma_{z,t_0,t}\varphi)(g) = \Gamma_z S\left(\exp\left(-\frac{1}{2}\Delta_{1_{[t_0,t)}}\right)\varphi\right)(g) = \Gamma_z S\left(\sum_{k=0}^{\infty}\frac{1}{k!}\left(\frac{z^2-1}{2}\right)^k \Delta^k_{1_{[t_0,t)}}\varphi\right)(g),$$

for $g \in S_d(\mathbb{R})$.

Again the proof is close to the proof of Theorem 3.21.

Remark 3.43. *Note* $\Delta_{1_{[t_0,t)}}$, $-\infty < t_0 < t < \infty$, *can be understood as a composition of the Gross Laplacian with the orthogonal projection* $P_{1_{[t_0,t)^c}}$, *where* $1\!\!1_{[t_0,t)^c}$ *denotes the indicator function on the complement of the interval* $[t_0, t)$.

3.3 Semigroups on generalized functions

The theory of semigroups is of huge interest in several parts of mathematics where one is interested in solving partial differential equations. Instead of giving a complete list of references to this voluminous topic we refer to the classics [36], [39] and [63]. In this section we consider the underlying differential operators of the scaling and the projection operator as semigroups. The basic concept and also the idea of the proofs go back to [51, Section 11].

Definition 3.44. *A family $T(t)$, $t \geq 0$, of bounded operators on a Banach space $(B, \|\cdot\|_B)$ is called semigroup if the following properties are fulfilled:*

(i) $T(0) = Id$;

(ii) $T(s+t) = T(s)T(t)$ *for all* $s, t \geq 0$;

$T(t)$ is called quasi-bounded if there exist constants $\omega \geq 0$ and $M \geq 1$ such that

(iii) $\|T(t)\|_B \leq Me^{\omega t}$ *for all* $t \geq 0$.

If $M = 1$ and $\omega = 0$ then $T(t)$ is called semigroup of contractions. A semigroup $T(t)$, $t \geq 0$, is said to be C_0 if

(iv) $\lim_{t\downarrow 0} T(t)\varphi = \varphi$ *for every* $\varphi \in B$.

Definition 3.45. *The operator A defined on*

$$D(A) := \left\{\varphi \in H \,\middle|\, \lim_{t\downarrow 0} \frac{\varphi - T(t)\varphi}{t} \text{ exists}\right\}$$

and

$$A\varphi := \lim_{t\downarrow 0} \frac{\varphi - T(t)\varphi}{t}$$

is called the infinitesimal generator (short: generator) of the semigroup $T(t)$, $t \geq 0$.

Theorem 3.46. *Let $\eta \in \mathcal{H}$ with $|\eta|_0 = 1$. Then $\left(\exp\left(-\frac{1}{2}tD_\eta^2\right)\right)_{0\leq t\leq 1}$ is a strongly continuous semigroup on the space of regular test functions $\mathcal{G}$ with generator $-\frac{1}{2}D_\eta^2$, w.r.t. the family of norms $\|\cdot\|_{q,0}$, for all $q \in \mathbb{N}$.*

Without loss of generality we consider semigroups on the time interval $[0, 1]$.

Proof. Let $\eta \in \mathcal{H}$ with $|\eta|_0 = 1$, then we know by Theorem 3.18 that $\exp\left(-\frac{1}{2}D_\eta^2\right)$ leaves the space $\mathcal{G}$ invariant. Thus for all $t \in [0,1]$ we get that

$$T(t)\varphi := \exp\left(-t\frac{1}{2}D_\eta^2\right)\varphi = \sum_{n=0}^{\infty}\left\langle :\cdot^{\otimes n}, \hat{\varphi}_t^{(n)}\right\rangle$$
$$= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(n+2k)!}{k!n!}\left(-\frac{t}{2}\right)^k\left\langle :\cdot^{\otimes n}:, \left(\eta^{\otimes 2k}, \varphi^{(n+2k)}\right)_{\mathcal{H}^{\otimes 2k}}\right\rangle,$$

is in $\mathcal{G}$. This causes by the fact that for all $n \in \mathbb{N}$ there exists a $q \in \mathbb{N}$ such that

$$\left|\hat{\varphi}_t^{(n)}\right| \leq C_n\|\varphi\|_{q,0},$$

where $0 < C_n < \infty$ is given as in the proof of Theorem 3.18, see (3.9). Thus $T(t)_{0\leq t\leq 1}$ is well-defined and obviously $T(0)\varphi = \varphi$ for all $\varphi \in \mathcal{G}$. Moreover, using the Cauchy product of series one can show that $T(t+s)\varphi = T(t)T(s)\varphi$ for all $\varphi \in \mathcal{G}$ and all $0 < s < t$, with $s + t < 1$.

For showing strong continuity of the semigroup let $t \in [0,1]$ and $\varphi \in \mathcal{G}$. Then

$$T(t)\varphi - \varphi = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(n+2k)!}{k!n!}\left(-\frac{t}{2}\right)^k\left\langle :\cdot^{\otimes n}:, \left(\eta^{\otimes 2k}, \varphi^{(n+2k)}\right)_{\mathcal{H}^{\otimes 2k}}\right\rangle - \sum_{n=0}^{\infty}\left\langle :\cdot^{\otimes n}, \varphi^{(n)}\right\rangle$$
$$= t\sum_{n=0}^{\infty}\sum_{k=1}^{\infty}\frac{(n+2k)!}{k!n!}\left(-\frac{1}{2}\right)^k t^{k-1}\left\langle :\cdot^{\otimes n}:, \left(\eta^{\otimes 2k}, \varphi^{(n+2k)}\right)_{\mathcal{H}^{\otimes 2k}}\right\rangle,$$

and therefore

$$\|T(t)\varphi - \varphi\|_{q,0}^2 \leq t^2\sum_{n=0}^{\infty}C_n^2\|\varphi\|_{q,0}^2,$$

which converges to zero as t goes to zero. Moreover, this implies that

$$\frac{T(t)\varphi - \varphi}{t} + \frac{1}{2}D_\eta^2\varphi = t\sum_{n=0}^{\infty}\sum_{k=2}^{\infty}\frac{(n+2k)!}{k!n!}\left(-\frac{1}{2}\right)^k t^{k-2}\left\langle :\cdot^{\otimes n}:, \left(\eta^{\otimes 2k}, \varphi^{(n+2k)}\right)_{\mathcal{H}^{\otimes 2k}}\right\rangle.$$

Thus

$$\lim_{t\downarrow 0}\frac{T(t)\varphi - \varphi}{t} = -\frac{1}{2}D_\eta^2\varphi, \qquad \varphi \in \mathcal{G},$$

since also

$$\left\|\frac{T(t)\varphi - \varphi}{t} + \frac{1}{2}D_\eta^2\varphi\right\|_{q,0}^2 \leq t^2\sum_{n=0}^{\infty}C_n\|\varphi\|_{q,0}^2.$$

Hence $T(t)_{t\in[0,1]}$ is a continuous (quasi-bounded) semigroup with generator $\frac{1}{2}D_\eta^2$. □

Theorem 3.47. *Let $z \in \mathbb{C}$ such that $\left|\frac{z^2-1}{2}\right| < 1$. Then $\left(\exp\left(-\frac{z^2-1}{2}t\Delta_G\right)\right)_{0\le t\le 1}$ is a strongly continuous semigroup on the space of Hida test functions $(\mathcal{N})$ with generator $-\frac{z^2-1}{2}\Delta_G$ w.r.t. the family of norms $\|\cdot\|_{p,q,\alpha}$, $p, q, \alpha \in \mathbb{N}$.*

The proof can be done similar to the proof of Theorem 3.46, see also [51, Section 11] for the proof for $z = -1$.

Remark 3.48. *One can guess that the differential operator $\left(\exp\left(\frac{z^2-1}{2}\Delta_{t_0,t}\right)\right)_{-\infty<t_0<t<\infty}$ corresponding to the localized scaling operator $\sigma_{z,t_0,t}$ defines a two-parametric group on (S_d), $d \ge 1$, w.r.t. the family of norms $\|\cdot\|_{p,q,\alpha}$, $p, q, \alpha \in \mathbb{N}$. Here the propagator property is naturally given by the fact that*

$$\mathrm{tr}^k_{t_0,t} = \sum_{j=0}^{k} \binom{k}{j} \mathrm{tr}^j_{t_0,r} \mathrm{tr}^{k-j}_{r,t},$$

for all $-\infty < t_0 < r < t < \infty$.

Chapter 4

Products of generalized functions

For applications of Gaussian analysis certain products of generalized functions are of interest. In this section we observe pointwise products and Wick products in different spaces of test and generalized functions. Moreover, we show in which case both products coincide. At least we consider special products of regular generalized functions with Donsker's delta.

4.1 Pointwise products of generalized functions

Since $(\mathcal{N})$ is an algebra the pointwise product of $\varphi, \psi \in (\mathcal{N})$ is defined by the algebraic product of their chaos decompositions, see (2.5). Therein the product of two kernels $\varphi^{(n)} \in \mathcal{N}_{\mathbb{C}}^{\hat{\otimes} n}$ and $\psi^{(m)} \in \mathcal{N}_{\mathbb{C}}^{\hat{\otimes} m}$, $n, m \in \mathbb{N}$, is then given by

$$\left\langle :x^{\otimes n}:, \varphi^{(n)} \right\rangle \left\langle :x^{\otimes m}:, \psi^{(m)} \right\rangle = \sum_{k=0}^{n \wedge m} k! \binom{n}{k} \binom{m}{k} \left\langle :x^{\otimes (n+m-2k)}:, \varphi^{(n)} \hat{\otimes}_k \psi^{(m)} \right\rangle, \quad x \in \mathcal{N}', \tag{4.1}$$

where $\hat{\otimes}_k$ denotes the k-times symmetric tensor product, see e.g. [68]. Hence the chaos decomposition of $\varphi\psi$ is given by

$$\varphi\psi := \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \sum_{k=0}^{n \wedge m} k! \binom{n}{k} \binom{m}{k} \left\langle :\cdot^{\otimes (n+m-2k)}:, \varphi^{(n)} \hat{\otimes}_k \psi^{(m)} \right\rangle.$$

Proposition 4.1. *The space $\mathcal{G}^\beta$, $0 \leq \beta \leq 1$, is closed under pointwise multiplication. More precisely, multiplication is a separately continuous bilinear map from $\mathcal{G}^\beta \times \mathcal{G}^\beta$ into $\mathcal{G}^\beta$.*

For the proof see [19] and also [67] for the case $\beta = 1$.

Having in mind the generalized chaos decomposition of elements from $(\mathcal{N})'$ by extending (4.1) we can define the formal product of $\Phi, \Psi \in (\mathcal{N})'$ with kernels $\Phi^{(n)}$, $n \in \mathbb{N}$, and $\Psi^{(m)}$, $m \in \mathbb{N}$,

respectively, by

$$\Phi\Psi := \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{k=0}^{n\wedge m} k!\binom{n}{k}\binom{m}{k}\left\langle :\cdot^{\otimes(n+m-2k)}:, \Phi^{(n)}\hat{\otimes}_k\Psi^{(m)}\right\rangle, \tag{4.2}$$

whenever the sum on the right hand side is in a suitable distribution space.

Remark 4.2. *Following the definition of the pointwise product in* (4.2) *for* $\Phi, \Psi \in (\mathcal{N})'$ *and* $z \in \mathbb{C}$ *one obviously gets the following property of the scaling operator:*

$$\sigma_z(\Phi\Psi) = (\sigma_z\Phi)(\sigma_z\Psi),$$

whenever the right hand side and the pointwise product $\Phi\Psi$ *exists in* $(\mathcal{N})'$*. Of course, this also holds for the localized scaling operator.*

4.2 The Wick product

In contrast to the pointwise product it has been shown that $(\mathcal{N})^{-1}$ (and other distribution spaces) is closed under so called Wick multiplication, see [43] and [62],[81], [19], [25] for applications.

Definition 4.3. *Let* $\Phi, \Psi \in (\mathcal{N})^{-1}$*. Then we define the Wick product* $\Psi \diamond \Phi$ *by*

$$\Psi \diamond \Phi := S^{-1}\left(S\Phi \cdot S\Psi\right).$$

The wick product is a well-defined object since $\mathrm{Hol}_0(\mathcal{N}_{\mathbb{C}})$ is an algebra. Thus by the characterization Theorem, see Theorem 2.16, there exists an element $\Psi \diamond \Phi \in (\mathcal{N})^{-1}$ such that $S\left(\Psi \diamond \Phi\right) = S(\Psi) \cdot S(\Phi)$.

Remark 4.4. *One can also define the Wick product via its chaos decomposition. Let* $\Phi, \Psi \in (\mathcal{N})^{-1}$ *with kernel* $\Phi^{(k)}$ *and* $\Psi^{(l)}, k, l \in \mathbb{N}_0$*, respectively. Then the kernels of* $\Xi := \Psi \diamond \Phi$ *are given by*

$$\Xi^{(n)} := \sum_{k+l=n} \Phi^{(k)}\hat{\otimes}\Psi^{(l)},$$

for all $n \in \mathbb{N}$*.*

Inductively, one can define the n-th wick power of a white noise distribution Φ by

$$\Phi^{\diamond n} := S^{-1}\left((S\Phi)^n\right)$$

in $(\mathcal{N})^{-1}$. Moreover one can also define Wick polynomials by taking finite linear combinations of Wick powers, i.e., $\sum_{n=1}^{N} a_n\Phi^{\diamond n}$ for some constants $a_n \in \mathbb{C}$, $n = 1, \ldots, N$, for some $N \in \mathbb{N}$. It is even possible to define Wick analytic functions in $(\mathcal{N})^{-1}$ under very general assumptions.

Theorem 4.5. *Let $F : \mathbb{C} \to \mathbb{C}$ be analytic in a neighborhood of the point $z_0 = E(\Phi)$, $\Phi \in (\mathcal{N})^{-1}$. Then $F^{\diamond}(\Phi)$ defined by $S(F^{\diamond}(\Phi)) = F(S\Phi)$ exists in $(\mathcal{N})^{-1}$.*

For the proof see e.g. [19].

Example 4.6. *An equation of the form*

$$\Psi \diamond X = \Phi, \qquad \text{for some } X \in (\mathcal{N})^{-1}$$

can be solved if $E_\mu(\Phi) = S\Phi(0) \neq 0$. That implies $(S\Phi)^{-1} \in \mathrm{Hol}_0(\mathcal{N}_{\mathbb{C}})$. Thus $\phi^{\diamond(-1)} = S^{-1}((S\Phi)^{-1}) \in (\mathcal{N})^{-1}$. Then $X := \phi^{\diamond(-1)} \diamond \Psi$ is the solution in $(\mathcal{N})^{-1}$.

For more instructive examples we refer the reader to [43].
Note that the wick product leaves the space $\mathcal{G}^{-1}$ invariant. This implies the possibility to introduce more mathematical structures on the space of regular distributions, like conditional expectations or martingales, see e.g. [25] or [19].
In this thesis we are interested in the question in which cases pointwise and Wick product coincides. First we follow [3], see also [7], in answering this question for independent regular generalized functions from $\mathcal{G}'$.

Lemma 4.7. *The pointwise product of random variables has a well-defined extension to pairs Ψ, Φ of independent generalized random variables (here independence is mentioned in the sense of Definition 2.31, i.e., we consider the white noise spaces as in Example 2.2 (i)) in $\mathcal{G}'$ so that $\Psi \cdot \Phi \in \mathcal{G}'$. Moreover, the formula*

$$\Psi \cdot \Phi = \Psi \diamond \Phi,$$

holds.

For the proof see [3]. Following (4.2) the formal product of $\Psi\Phi$ of elements $\Psi,\ \Phi \in \mathcal{G}^{-\beta}$, $0 \leq \beta \leq 1$, has square-integrable wick monomials in its chaos decomposition. Hence the pointwise product extends to elements from $\mathcal{G}'$ whenever the formal series of the corresponding chaos decompositions converges in $\mathcal{G}'$. Now we are interested in extending this kind of independence not only to other spaces of regular distributions, but also without measureability conditions w.r.t. some filtrations as in Lemma 4.7, i.e., as in Definition 2.31.

Definition 4.8. *We call two regular generalized functions $\Phi, \Psi \in \mathcal{G}^{-\beta}$, $0 \leq \beta < 1$, projective independent w.r.t. $\eta \in \mathcal{H}$ if for all $\xi \in \mathcal{N}$ one has that*

$$S(\Phi)(\xi) = S(\Phi)((\xi, \eta)\eta) \quad \text{and} \quad S(\Psi)(\xi) = S(\Psi)(\xi - (\xi, \eta)\eta).$$

Theorem 4.9. *If $\Phi, \Psi \in \mathcal{G}^{-\beta}$, $0 \leq \beta < 1$ are projective independent w.r.t. $\eta \in \mathcal{H}$ then the Wick product coincides with the pointwise product.*

Proof. Let $\Phi, \Psi \in \mathcal{G}^{-\beta}$, $0 \leq \beta < 1$, $\eta \in \mathcal{H}$ and $\xi \in \mathcal{N}$. Then the S-transforms of Φ and Ψ naturally extends to element from $\mathcal{H}$. I.e., we have on the one side that

$$S(\Phi)(\xi) = \sum_{n=0}^{\infty} \left(\xi^{\otimes n}, \Phi^{(n)}\right) = S(\Phi)((\xi,\eta)\eta) = \sum_{n=0}^{\infty} (\xi,\eta)^n \left(\eta^{\otimes n}, \Phi^{(n)}\right) = \sum_{n=0}^{\infty} \left(\xi^{\otimes n}, \left(\eta^{\otimes n}, \Phi^{(n)}\right)\eta^{\otimes n}\right),$$

where $\Phi^{(n)} \in \mathcal{H}_{\mathbb{C}}^{\hat{\otimes} n}$ denotes the n-th kernel of the chaos decomposition of Φ. On the other side we get that

$$S(\Psi)(\xi) = \sum_{m=0}^{\infty} \left(\xi^{\otimes m}, \Psi^{(m)}\right) = S(\Psi)(\xi - (\xi,\eta)\eta) = \sum_{m=0}^{\infty} \left((\xi - (\xi,\eta)\eta)^{\otimes m}, \Psi^{(m)}\right)$$
$$= \sum_{m=0}^{\infty} \left(\xi^{\otimes m}, P_{\eta}^{\otimes m}\Psi^{(m)}\right),$$

where $\Psi^{(m)} \in \mathcal{H}_{\mathbb{C}}^{\hat{\otimes} m}$ denotes the m-th kernel of the chaos decomposition of Ψ. Hence for all $n, m \in \mathbb{N}_0$ we have that $\Phi^{(n)}$ and $\Psi^{(m)}$ coincides with $\left(\eta^{\otimes n}, \Phi^{(n)}\right)\eta^{\otimes n}$ and $\Psi^{(m)} - \left(\eta^{\otimes m}, \Psi^{(m)}\right)\eta^{\otimes m}$ in $\mathcal{H}_{\mathbb{C}}^{\hat{\otimes} n}$ and $\mathcal{H}_{\mathbb{C}}^{\hat{\otimes} m}$, respectively.

Now let $n, m, k \in \mathbb{N}$ such that $k \leq m \wedge n$, then we have that

$$\Phi^{(n)} \hat{\otimes}_k \Psi^{(m)} = \left(\left(\eta^{\otimes n}, \Phi^{(n)}\right)\eta^{\otimes n}\right) \hat{\otimes}_k P_{\eta}^{\otimes m}\Psi^{(m)} = 0,$$

since $\left(\eta^{\otimes n}, \Phi^{(n)}\right)\eta^{\otimes n}$ and $\Psi^{(m)} - \left(\eta^{\otimes m}, \Psi^{(m)}\right)\eta^{\otimes m}$ are orthogonal. Hence we get that

$$\Psi \diamond \Phi = \sum_{n=0}^{\infty} \sum_{k+l=n} \left\langle :\cdot^{\otimes n}:, \Phi^{(k)} \hat{\otimes} \Psi^{(l)} \right\rangle = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \sum_{k=0}^{n \wedge m} k! \binom{n}{k}\binom{m}{k} \left\langle :\cdot^{\otimes(n+m-2k)}:, \Phi^{(n)} \hat{\otimes}_k \Psi^{(m)} \right\rangle = \Psi\Phi.$$

□

Remark 4.10. *For $\eta \in \mathcal{N}$ Theorem 4.9 can be extended to other distribution spaces. In addition, $\Phi, \Psi \in (\mathcal{N})^{-\beta}$, $0 \leq \beta < 1$, can be projective independent w.r.t. $\eta \in \mathcal{H}$ if for all $\xi \in \mathcal{N}$ one has that*

$$S(\Phi)(\xi) = S(\Phi)((\xi,\eta)\eta) \quad \textit{and} \quad S(\Psi)(\xi) = S(\Psi)(\xi - (\xi,\eta)\eta).$$

Of course, this condition is now stronger since it is not clear whether the S-transforms above are defined for elements of $\mathcal{H}$. An idea of the proof is given as follows: Since $\Phi, \Psi \in (\mathcal{N})^{-\beta}$, $0 \leq \beta < 1$, there exist some sequences $(\varphi_n)_{n\in\mathbb{N}}, (\psi_n)_{n\in\mathbb{N}} \subset (\mathcal{N})^{\beta}$, with kernels $\varphi_n^{(m)}$, $\psi_n^{(m)}$, $m \in \mathbb{N}$, converging to Φ and Ψ in $(\mathcal{N})^{-\beta}$. Of course, this sequences might not be projective independent.

Thus we define the sequences $(F_n)_{n\in\mathbb{N}}, (G_n)_{n\in\mathbb{N}} \subset \mathcal{G}^\beta$ *by its kernels* $F_n^{(m)} = \left(\eta^{\otimes m}, \varphi_n^{(m)}\right)\eta^{\otimes m}$ *and* $G_n^{(m)} = P_\eta^{\otimes m}\psi_n^{(m)}$ *for all* $n, m \in \mathbb{N}$ *which also converge to* Φ *and* Ψ*, respectively. Thus*

$$S(\Phi)(\xi)S(\Psi)(\xi) = S(\Phi)((\xi,\eta)\eta)S(\Psi)(\xi - (\xi,\eta)\eta) = \lim_{n\to\infty} S(\varphi_n)((\xi,\eta)\eta)S(\psi_n)(\xi - (\xi,\eta)\eta)$$
$$= \lim_{n\to\infty} S(F_n)(\xi)S(G_n)(\xi) = \lim_{n\to\infty} S(F_n)(\xi)S(G_n)(\xi) = \lim_{n\to\infty} S(F_nG_n)(\xi) = S(\Phi\Psi)(\xi),$$

for all $\xi \in \mathcal{N}$.

4.3 Products of Donsker's deltas

As shown in Theorem 3.33 Donsker's delta is a regular distribution. Hence we are not sure whether products of it are well-defined objects in $\mathcal{G}'$ or $(\mathcal{N})'$. For example this is not true for a product of any Donsker's delta with itself. In this Section we only consider the white noise case $\mathcal{N} = S(\mathbb{R})$ and $\mathcal{H} = L^2(\mathbb{R})$ as in Example 2.2 (i).

Theorem 4.11. *One can define an n-times product of Donsker's deltas by*

$$\Phi := \prod_{j=1}^n \sigma_z\delta(\langle\cdot, h_j\rangle - a_j) := \frac{1}{(2\pi)^n}\prod_{j=1}^n \int_\gamma \exp\left(i\lambda_j(z\langle\cdot, h_j\rangle - a_j)\right) \times_{j=1}^n d\lambda_j, \tag{4.3}$$

in the sense of Bochner, see Theorem 2.12. Here $\gamma = \left\{e^{-i\alpha}x \mid x \in \mathbb{R}\right\}$, $z \in S_\alpha := \left\{z \in \mathbb{C} \mid \arg z \in (-\frac{\pi}{4}+\alpha, \frac{\pi}{4}+\alpha)\right\}$, $|\alpha| < \pi/4$*, and* h_j *linear independent elements of* $L^2(\mathbb{R})$ *and* $a_j \in \mathbb{C}$, $j = 1, \ldots, n$. *Then* Φ *is an element of* $(S)'$ *and for* $g \in S(\mathbb{R})$ *its* S*-transform is given by*

$$S\Phi(g) = \frac{1}{\sqrt{(2\pi z^2)^n \det M}} \exp\left(-\frac{1}{2}\left((h,g) - \frac{1}{z}a\right)M^{-1}\left((h,g) - \frac{1}{z}a\right)^T\right), \tag{4.4}$$

where M *denotes the Gram matrix to* $h_1, \ldots, h_n$ *defined by* $M := (h_k, h_l)_{k,l=1,\ldots,n}$, $(h,g) := ((h_1,g), \ldots, (h_n,g))$ *and* $a := (a_1, \ldots, a_n)$.

Again for the proof see e.g. [55].

4.3.1 Products of Donsker's deltas of Brownian motion and Brownian bridge

In the following we only consider products for $z \in \overline{S_0}$, where $\overline{S_0}$ denotes the closure of S_0, see Theorem 4.11 for its definition.

Lemma 4.12. *Let* $0 < t < \infty$, $z \in \overline{S_0}$, $n \in \mathbb{N}$, $x_k \in \mathbb{C}$, *and* $t_k := t\frac{k}{n}$, $1 \le k \le n$. *Then the Gram matrix, defined as in Theorem 4.11, corresponding to Brownian motion at discrete times* t_k, $1 \le k \le n$, *is given by* $M_{1,n} = (\mathbb{1}_{[0,t_k)}, \mathbb{1}_{[0,t_l)})_{k,l=1,\dots,n} = \frac{t}{n}\min(k,l)_{k,l=1,\dots,n}$, *see Example 2.2 (i),* (2.3). *So for*

$$\Phi := \prod_{k=1}^{n} \sigma_z\delta(\langle\cdot, \mathbb{1}_{[0,t_k)}\rangle - x_k) \in (S)' \tag{4.5}$$

one has that

$$S(\Phi)(g) = \frac{1}{\sqrt{(2\pi z^2 \frac{t}{n})^n}} \times \exp\left(-\frac{n}{2t}\left(\left((\mathbb{1}_{[0,t_1)}, g) - \frac{x_1}{z}\right)^2 + \sum_{k=1}^{n-1}\left((\mathbb{1}_{[0,t_k)}, g) - \frac{x_k}{z} - \left((\mathbb{1}_{[0,t_{k+1})}, g) - \frac{x_{k+1}}{z}\right)\right)^2\right)\right), \tag{4.6}$$

for all $g \in S(\mathbb{R})$.

Proof. By induction one can show that for all $n \in \mathbb{N}$ the inverse of $M_{1,n}$ is given by

$$M_{1,n}^{-1} = \frac{n}{t}\begin{pmatrix} 2 & -1 & 0 & . & . & . & . & . & 0 \\ -1 & 2 & -1 & 0 & . & . & . & . & 0 \\ 0 & -1 & 2 & -1 & 0 & . & . & . & 0 \\ . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . \\ . & . & . & . & 0 & -1 & 2 & -1 & 0 \\ . & . & . & . & . & 0 & -1 & 2 & -1 \\ 0 & . & . & . & . & . & 0 & -1 & 1 \end{pmatrix}$$

By Gauss algorithm (here add the $k+1$-th row to the k-th row $k = n-1, \dots, 1$) one can show that $\det M_{1,n}^{-1} = \left(\frac{n}{t}\right)^n$, for all $n \in \mathbb{N}$. Hence Theorem 4.11 yields (4.6). □

Lemma 4.13. *Let* $0 < t < \infty$, $z \in \overline{S_0}$, $n \in \mathbb{N}$, $y_k \in \mathbb{C}$. *Then we define* $t_k := t\frac{k}{n}$ *for* $1 \le k \le n$ *and* $h_k := \mathbb{1}_{[0,t_k)} - \frac{k}{n}\mathbb{1}_{[0,t)}$ *for* $1 \le k \le n-1$. *The Gram matrix corresponding to Brownian bridge at discrete times* t_k, $1 \le k \le n-1$, *is given by* $M_{2,n-1} = (h_k, h_l)_{k,l=1,\dots,n} = \left(\left(\mathbb{1}_{[0,t_k)} - \frac{k}{n}\mathbb{1}_{[0,t)}, \mathbb{1}_{[0,t_l)} - \frac{l}{n}\mathbb{1}_{[0,t)}\right)\right)_{k,l=1,\dots,n} = \frac{t}{n}\left(\min(k,l) - \frac{kl}{n}\right)_{k,l=1,\dots,n}$, *see 2.2 (i),* (2.4). *Hence for*

$$\Psi := \prod_{k=1}^{n-1} \sigma_z\delta(\langle\cdot, h_k\rangle - y_k) \in (S)' \tag{4.7}$$

one has that

$$S\left(\Psi\right)(g) = \frac{1}{\sqrt{\frac{1}{n}(2\pi z^2 \frac{t}{n})^{n-1}}}$$
$$\times \exp\left(-\frac{n}{2t}\left(\left((h_1, g) - \frac{y_1}{z}\right)^2 + \sum_{k=1}^{n-2}\left((h_k, g) - \frac{y_k}{z} - \left((h_{k+1}, g) - \frac{y_{k+1}}{z}\right)\right)^2 + \left((h_{n-1}, g) - \frac{y_{n-1}}{z}\right)^2\right)\right), \tag{4.8}$$

for all $g \in S(\mathbb{R})$.

Proof. One can easy calculate that

$$M_{2,n-1} = M_{2,n-1}M_{1,n-1}^{-1}M_{1,n-1} = \begin{pmatrix} 1 & 0 & . & . & . & 0 \\ 0 & 1 & 0 & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & 0 & 1 & 0 \\ -\frac{1}{n} & -\frac{2}{n} & . & . & . -\frac{n-2}{n} & 1-\frac{n-1}{n} \end{pmatrix} M_{1,n-1},$$

therefore

$$M_{2,n-1}^{-1} = M_{1,n-1}^{-1}\begin{pmatrix} 1 & 0 & . & . & . & 0 \\ 0 & 1 & 0 & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & 0 & 1 & 0 \\ -\frac{\frac{1}{n}}{1-\frac{n-1}{n}} & -\frac{\frac{2}{n}}{1-\frac{n-1}{n}} & . & . & . -\frac{\frac{n-2}{n}}{1-\frac{n-1}{n}} & \frac{1}{1-\frac{n-1}{n}} \end{pmatrix}$$

$$= \frac{n}{t}\begin{pmatrix} 2 & -1 & 0 & . & . & . & . & . & 0 \\ -1 & 2 & -1 & 0 & . & . & . & . & 0 \\ 0 & -1 & 2 & -1 & 0 & . & . & . & 0 \\ . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . \\ . & . & . & . & 0 & -1 & 2 & -1 & 0 \\ . & . & . & . & . & 0 & -1 & 2 & -1 \\ 0 & . & . & . & . & . & 0 & -1 & 2 \end{pmatrix}.$$

Since $\det M_{1,n-1}^{-1} = \left(\frac{n}{t}\right)^{n-1}$, see in the proof of Lemma 4.12, the determinant of $M_{2,n-1}^{-1}$ is given by $\det M_{2,n-1}^{-1} = n\left(\frac{n}{t}\right)^{n-1}$. Now again by Theorem 4.11 we obtain (4.8). □

Remark 4.14. *Let us denote that $\frac{t}{n}M_{2,n-1}$ is a discretization of the negative Laplace operator.*

In the next Theorem we consider a relation between a product of Donsker's deltas for Brownian motion and a product of Donsker's deltas for Brownian bridge. In the corresponding proof we need the following Lemma.

Lemma 4.15. *Let $n \in \mathbb{N}$, and $a_1, \dots, a_{n-1}, b \in \mathbb{C}$. Then*

$$\left(a_1 - \frac{1}{n}b\right)^2 + \sum_{k=1}^{n-2}\left(a_k + \frac{1}{n}b - a_{k+1}\right)^2 + \left(a_{n-1} - \frac{n-1}{n}b\right)^2$$
$$= a_1^2 + \sum_{k=1}^{n-2}(a_k - a_{k+1})^2 + (a_{n-1} - b)^2 - \frac{1}{n}b^2.$$

Proof. Let $n \in \mathbb{N}$, and $a_1, \dots, a_{n-1}, b \in \mathbb{C}$. Because of the binomial theorem the following equation holds:

$$\begin{aligned}
&\left(a_1 - \frac{1}{n}b\right)^2 + \sum_{k=1}^{n-2}\left(a_k + \frac{1}{n}b - a_{k+1}\right)^2 + \left(a_{n-1} - \frac{n-1}{n}b\right)^2 \\
&= a_1^2 + \sum_{k=1}^{n-3}(a_k - a_{k+1})^2 + \frac{n-2}{n^2}b^2 - \frac{2}{n}ba_{n-2} + \left(a_{n-2} + \frac{1}{n}b - a_{n-1}\right)^2 + \left(a_{n-1} - \frac{n-1}{n}b\right)^2 \\
&= a_1^2 + \sum_{k=1}^{n-2}(a_k - a_{k+1})^2 + \frac{n-1}{n^2}b^2 - 2\frac{1}{n}ba_{n-1} + a_{n-1}^2 - 2\frac{n-1}{n}ba_{n-1} + \left(\frac{n-1}{n}b\right)^2 \\
&= a_1^2 + \sum_{k=1}^{n-2}(a_k - a_{k+1})^2 + (a_{n-1} - b)^2 + \frac{n-1-n^2+(n-1)^2}{n^2}b^2 \\
&= a_1^2 + \sum_{k=1}^{n-2}(a_k - a_{k+1})^2 + (a_{n-1} - b)^2 - \frac{1}{n}b^2.
\end{aligned}$$

□

Theorem 4.16. *Let $0 < t < \infty$, $z \in \overline{S_0}$, $n \in \mathbb{N}$ and $x_k \in \mathbb{R}$, $k = 1, \dots, n$. We define $t_k := t\frac{k}{n}$, $k = 1, \dots, n$, $y_k := x_k - \frac{k}{n}x_n$, $k = 1, \dots, n-1$, and $h_k := \mathbb{1}_{[0,t_k)} - \frac{k}{n}\mathbb{1}_{[0,t)}$, $1 \le k \le n-1$. Then for Φ and Ψ as in* (4.5) *and* (4.7)*, respectively, one has that*

$$S(\Phi)(g) = S(\Psi)(g)\frac{1}{z}S(\delta(\langle\cdot, \mathbb{1}_{[0,t)}\rangle - x_n/z))(g) = S(\Psi)(g)\frac{1}{\sqrt{2\pi z t}}\exp\left(\frac{1}{2t}(g, \mathbb{1}_{[0,t)}) - x_n/z\right)^2,$$

for $g \in S(\mathbb{R})$. I.e.,

$$\Phi = \Psi \diamond \sigma_z\delta(\langle\cdot, \mathbb{1}_{[0,t)}\rangle - x).$$

Proof. From Lemma 4.12 we know that

$$S(\Phi)(g) = \frac{1}{\sqrt{(2\pi z^2 \frac{t}{n})^n}} \exp\Bigg(-\frac{n}{2t}\Bigg(\Big((\mathbb{1}_{[0,t_1)}, g) - \frac{x_1}{z} \Big)^2 + \sum_{k=1}^{n-1} \Big((\mathbb{1}_{[0,t_k)}, g) - \frac{x_{k+1}}{z} - \Big((\mathbb{1}_{[0,t_{k+1})}, g) - \frac{x_{k+1}}{z} \Big) \Big)^2 \Bigg) \Bigg),$$

for all $f \in S(\mathbb{R})$. Using Lemma 4.15 and Lemma 4.13 one gets that

$$\begin{aligned} S(\Phi)(g) &= \frac{1}{\sqrt{(2\pi z^2 \frac{t}{n})^n}} \times \exp\Bigg(-\frac{t}{2n}\Bigg(\Big((h_1, g) - \frac{y_1}{z} \Big)^2 \\ &\quad + \sum_{k=1}^{n-2} \Big((h_k, g) - \frac{y_k}{z} - \Big((h_{k+1}, g) - \frac{y_{k+1}}{z} \Big) \Big)^2 + \Big((h_{n-1}, g) - \frac{y_{n-1}}{z} \Big)^2 \Bigg) \Bigg) \\ &= \frac{\sqrt{\frac{1}{n}(2\pi \frac{t}{n})^{n-1}}}{\sqrt{(2\pi \frac{t}{n})^n}} \exp\Big(-\frac{1}{2t} \left((g, \mathbb{1}_{[0,t)}) - x_n/z \right) \Big) S(\Psi)(g), \end{aligned}$$

for all $g \in S(\mathbb{R})$. Hence

$$S(\Phi)(g) = \frac{1}{\sqrt{2\pi z^2 t}} \exp\Big(-\frac{1}{2t} \left((g, \mathbb{1}_{[0,t)}) - x_n/z \right) \Big) S(\Psi)(g), \quad g \in S(\mathbb{R}).$$

□

Corollary 4.17. *The Wick product in Theorem 4.16 is a pointwise product.*

Proof. By Lemma 4.7 one only has to show that the regular generalized functions Ψ and the scaled Donsker's delta $\sigma_z \delta(\langle \cdot, \mathbb{1}_{[0,t)} \rangle - (x - x_0))$ are independent. This is true since for all $g \in S(\mathbb{R})$ on the one side

$$S\sigma_z \delta(\langle \cdot, \mathbb{1}_{[0,t)} \rangle - (x - x_0))(g) = S\sigma_z \delta(\langle \cdot, \mathbb{1}_{[0,t)} \rangle - (x - x_0))(\mathbb{1}_{[0,t)} g)$$

and on the other side

$$S(\Psi)(g) = S(\Psi)(g - \langle g, \mathbb{1}_{[0,t)} \rangle \mathbb{1}_{[0,t)}) = S(\Psi)\left(\mathbb{1}_{[0,t)^c} (g - \langle g, \mathbb{1}_{[0,t)} \rangle \mathbb{1}_{[0,t)}) \right) = S(\Psi)(\mathbb{1}_{[0,t)^c} g).$$

Here we used that for $h_k := \mathbb{1}_{[0,t_k)} - \frac{k}{n} \mathbb{1}_{[0,t)}$, $1 \le k \le n-1$, we have that

$$\begin{aligned} (h_k, g - \langle g, \mathbb{1}_{[0,t)} \rangle \mathbb{1}_{[0,t)}) &= (h_k, g) - (h_k, \langle g, \mathbb{1}_{[0,t)} \rangle \mathbb{1}_{[0,t)}) \\ &= (h_k, g) - (\mathbb{1}_{[0,t_k)} \langle g, \mathbb{1}_{[0,t)} \rangle \mathbb{1}_{[0,t)}) - \left(\frac{k}{n} \mathbb{1}_{[0,t)}, \langle g, \mathbb{1}_{[0,t)} \rangle \mathbb{1}_{[0,t)} \right) = (h_k, g). \end{aligned} \tag{4.9}$$

□

4.3.2 Finitely based Hida distributions in terms of products of Donsker's deltas

Following [82] we can use Theorem 4.11 to extend the scaling operator. Note that we still consider the white noise case $\mathcal{N} = S(\mathbb{R})$ and $\mathcal{H} = L^2(\mathbb{R})$. Let $h_j \in L^2(\mathbb{R})$, $1 \leq j \leq n$ be linear independent, such that $M = ((h_k, h_l))_{k,l=1,\dots,n}$ is positive definite, and $G : \mathbb{R}^n \to \mathbb{C}$ such that $G \in L^p(\nu_M)$ for some $p > 1$. Here ν_M denotes the measure on $\mathbb{R}^n$ having density $\exp\left(-\frac{1}{2}\sum_{k,j=1}^n x_k M^{-1}_{k,j} x_j\right)$ w.r.t. the Lebesgue measure on $\mathbb{R}^n$. In this case one can define $\phi \in L^p(\mu)$ by

$$\phi(\cdot) := G\left(\langle \cdot, h_1\rangle, \dots, \langle \cdot, h_n\rangle\right).$$

Since ϕ only depends on a finite number of 'coordinates' $\langle \cdot, h_k\rangle$, $k \in \mathbb{N}$, we call it a finitely based Hida distribution. The definition goes back to [47]. Therein only smooth $h_k \in S(\mathbb{R})$, $1 \leq j \leq n$ are considered.

Lemma 4.18. *If $G \in L^p(\nu_M)$ the following relation holds*

$$G\left(\langle \cdot, h_1\rangle, \dots, \langle \cdot, h_n\rangle\right) = \int_{\mathbb{R}^n} G(x_1, \dots, x_n) \prod_{j=1}^n \delta\left(\langle \cdot, h_j\rangle - x_j\right) dx_1 \dots dx_n,$$

where the integral in $(S)'$ is in the sense of Bochner as in Theorem 2.12.

The proof is postponed because the existence of the Bochner integral will follow from the more general discussion in the next theorem. Then the equality follows from a comparison on the dense set of exponential functions. Now it is natural to try the following extension of σ_z :

$$\sigma_z \phi = \int_{\mathbb{R}^n} G(x_1, \dots, x_n) \prod_{j=1}^n \sigma_z \delta\left(\langle \cdot, h_j\rangle - x_j\right) dx_1 \dots dx_n$$

whenever the right hand side is a well-defined Bochner integral in $(S)'$. For such a realization stronger conditions on G are necessary.

Theorem 4.19. *Let $z \in S_0$ (i.e., $\Re \frac{1}{z^2} > 0$) and let*

$$G \in L^p_{\mathbb{C}}\left(\nu_{z,M,\varepsilon}\right),$$

for some $p > 1$ and $0 < \varepsilon < \infty$. Here $\nu_{z,M,\varepsilon}$ is a measure on $\mathbb{R}^n$ having density

$$\exp\left(-\frac{1}{2}\left(\Re\left(\frac{1}{z^2}\right) - \varepsilon\right) \sum_{k,j=1}^n x_k M^{-1}_{k,j} x_j\right),$$

w.r.t. the Lebesgue measure on $\mathbb{R}^n$*. Again* $h_j \in L^2(\mathbb{R})$*,* $1 \le j \le n$*, are linear independent, such that* $M = ((h_k, h_l))_{k,l=1,\dots,n}$ *is positive definite. Then*

$$\sigma_z\phi = \int_{\mathbb{R}^n} G(x_1,\dots,x_n) \prod_{j=1}^n \sigma_z\delta\Big(\langle\cdot, h_j\rangle - x_j\Big)\, dx_1 \dots dx_n$$

is a well-defined Bochner integral in $(S)'$*.*

Proof. From Theorem 4.11 we can estimate

$$\begin{aligned}\left|S\left(\prod_{j=1}^n \sigma_z\delta\Big(\langle\cdot, h_j\rangle - x_j\Big)\right)(g)\right| &\\ \le \frac{1}{\sqrt{(2\pi|z|^2)^n \det M}} \exp\Bigg(-\frac{1}{2}\left(\Re\frac{1}{z^2} - \varepsilon\right)\sum_{k,j=1}^n x_k M^{-1}_{k,j} x_j &\\ + \frac{1}{2}\left(\frac{1}{2\varepsilon} + 1\right)|z|^2 |g|_0^2 \sum_{k,j=1}^n |h_k|_0 M^{-1}_{k,j} |h_j|_0\Bigg),&\end{aligned}$$

for all $g \in S(\mathbb{R})$. Here the term linear in $(x_1 \dots, x_n)$ is estimated by using a general estimation for positive quadratic forms; for all $\varepsilon > 0$

$$\begin{aligned}\frac{1}{|z|}\left|\sum_{k,j=1}^n (g, h_k) M^{-1}_{k,j} x_j\right| &\le \varepsilon \sum_{k,j=1}^n x_k M^{-1}_{k,j} x_j + \frac{1}{4\varepsilon|z|^2}\sum_{k,j=1}^n (g, h_k) M^{-1}_{k,j} (g, h_j)\\ &\le \varepsilon \sum_{k,j=1}^n x_k M^{-1}_{k,j} x_j + \frac{1}{4\varepsilon|z|^2}|g|_0^2 \sum_{k,j=1}^n |h_k|_0 M^{-1}_{k,j} |h_j|_0.\end{aligned}$$

Now we choose $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ and $\varepsilon > 0$ such that $q\varepsilon < \frac{1}{2}\Re\frac{1}{z^2}$. Then

$$\begin{aligned}&\int_{\mathbb{R}^n} |G(x_1,\dots,x_n)| \exp\left(-\frac{1}{2}\left(\Re\frac{1}{z^2} - \varepsilon\right)\sum_{k,j=1}^n x_k M^{-1}_{k,j} x_j\right) d^n x\\ &\le \left(\int_{\mathbb{R}^n} |G(x_1,\dots,x_n)|^p \exp\left(-\frac{1}{2}\left(\Re\frac{1}{z^2} - \varepsilon\right)\sum_{k,j=1}^n x_k M^{-1}_{k,j} x_j\right) dx_1 \dots dx_n\right)^{\frac{1}{p}}\\ &\quad\times\left(\int_{\mathbb{R}^n} \exp\left(-\frac{1}{2}\left(\Re\frac{1}{z^2} - q\varepsilon\right)\sum_{k,j=1}^n x_k M^{-1}_{k,j} x_j\right) dx_1 \dots dx_n\right)^{\frac{1}{q}}\\ &= \left(\int_{\mathbb{R}^n} |G(x_1,\dots,x_n)|^p d\nu_{z,M,\varepsilon}(x_1 \dots x_n)\right)^{\frac{1}{p}}\\ &\qquad\left(\int_{\mathbb{R}^n} \exp\left(-\frac{1}{2}\left(\Re\frac{1}{z^2} - q\varepsilon\right)\sum_{k,j=1}^n x_k M^{-1}_{k,j} x_j\right) dx_1 \dots dx_n\right)^{\frac{1}{q}}\end{aligned}$$

is finite because of our assumptions. Hence by Theorem 2.12 the statement holds. □

Theorem 4.20. *Let $z \in S_\alpha$, $|\alpha| < \frac{\pi}{4}$, and let*

$$G \in L^p_{\mathbb{C}}\left(\nu_{|z,M,\varepsilon|}\right),$$

for all $1 \leq p < \infty$ and $0 < \varepsilon < \infty$. Here $\nu_{|z,M,\varepsilon|}$ is a measure on $\mathbb{R}^n$ having density

$$\exp\left(\frac{1}{2}\left(\left|\Re\left(\frac{1}{z^2}\right)\right| + \varepsilon\right)\sum_{k,j=1}^{n} x_k M^{-1}_{k,j} x_j\right),$$

w.r.t. the Lebesgue measure on $\mathbb{R}^n$. Again $h_j \in L^2(\mathbb{R})$, $1 \leq j \leq n$, are linear independent, such that $M = ((h_k, h_l))_{k,l=1,\ldots,n}$ is positive definite. Then

$$\sigma_z\phi = \int_{\mathbb{R}^n} G(x_1, \ldots, x_n) \prod_{j=1}^{n} \sigma_z\delta\left(\langle\cdot, h_j\rangle - x_j\right) dx_1 \ldots dx_n$$

is a well-defined Bochner integral in $(S)'$.

The proof follows directly by the fact that our assumption implies that

$$\int_{\mathbb{R}^n} |G(x_1, \ldots, x_n)| \left|S\left(\prod_{j=1}^{n} \sigma_z\delta\left(\langle\cdot, h_j\rangle - x_j\right)\right)(g)\right| dx_1 \ldots dx_n < \infty.$$

Again let $z \in S_\alpha$ and $h_j \in L^2(\mathbb{R})$, $1 \leq j \leq n$ be linear independent, such that $M = ((h_k, h_l))_{k,l=1,\ldots,n}$ is positive definite, and $G : \mathbb{R}^n \to C$ in $L^p_{\mathbb{C}}\left(\nu_{|z,M,\varepsilon|}\right)$ for all $1 \leq p < \infty$ such that in addition $G(z\cdot) \in L^p_{\mathbb{C}}(\nu_M)$. Then

$$\phi := G\left(\langle\cdot, h_1\rangle, \cdots, \langle\cdot, h_n\rangle\right) \in L^p(\mu).$$

and

$$\sigma_z\phi = G\left(z\langle\cdot, h_1\rangle, \cdots, z\langle\cdot, h_n\rangle\right) \in L^p(\mu).$$

Thus on the one side one gets by Lemma 4.18 that

$$\sigma_z\phi = \int_{\mathbb{R}^n} G(zx_1, \ldots, zx_n) \prod_{j=1}^{n} \delta\left(\langle\cdot, h_j\rangle - x_j\right) dx_1 \ldots dx_n$$

and on the other side by Theorem 4.20 that

$$\sigma_z\phi = \int_{\mathbb{R}^n} G(x_1, \ldots, x_n) \prod_{j=1}^{n} \sigma_z\delta\left(\langle\cdot, h_j\rangle - x_j\right) dx_1 \ldots dx_n.$$

Since $\sigma_z\phi$ should be well-defined object the following equation must be true:

$$\begin{aligned}\sigma_z\phi &= \int_{\mathbb{R}^n} G(zx_1,\dots,zx_n)\prod_{j=1}^n \delta\left(\langle\cdot,h_j\rangle - x_j\right)dx_1\dots dx_n\\ &= \frac{1}{z}\int_{z\mathbb{R}^n} G(y_1,\dots,y_n)\prod_{j=1}^n \delta\left(\langle\cdot,h_j\rangle - y_j/z\right)dy_1\dots dy_n\\ &= \int_{z\mathbb{R}^n} G(y_1,\dots,y_n)\prod_{j=1}^n \sigma_z\delta\left(\langle\cdot,h_j\rangle - y_j\right)dy_1\dots dy_n\\ &= \int_{\mathbb{R}^n} G(x_1,\dots,x_n)\prod_{j=1}^n \sigma_z\delta\left(\langle\cdot,h_j\rangle - x_j\right)dx_1\dots dx_n.\end{aligned}$$

Of course, equality is here mentioned in distribution sense, i.e., in $(S)'$.

Example 4.21. *Let H_n, $n \in \mathbb{N}$, be a Hermite function, then for $x_0 \in \mathbb{R}$ and $0 < t < \infty$ the following equation holds:*

$$\begin{aligned}\sigma_{\sqrt{i}}H_n\left(x_0 + \langle\cdot, \mathbb{1}_{[0,t)}\rangle\right) &= \int_{\mathbb{R}} H_n(y)\sigma_{\sqrt{i}}\delta\left(\langle\cdot, \mathbb{1}_{[0,t)}\rangle - (y - x_0)\right)dy\\ &= \frac{1}{i}\int_{\mathbb{R}} H_n(y)\delta\left(\langle\cdot, \mathbb{1}_{[0,t)}\rangle - \frac{y - x_0}{\sqrt{i}}\right)dy.\end{aligned}$$

Remark 4.22. *Instead of integration with respect to $G(x)\cdot d^n x$ we can use complex measures ν on $\mathbb{R}^n$ to define the more general distribution*

$$\int_{\mathbb{R}^n}\prod_{j=1}^n \sigma_z\delta\left(\langle\cdot,h_j\rangle - x_j\right)d\nu(x_1,\dots x_n).$$

4.4 Products of Donsker's deltas with regular generalized functions

Following Theorem 3.8 and Theorem 3.18 one can apply a composition of shifting and projection operator to $\varphi \in \mathcal{G}$ and $\tau_{a\eta}P_\eta\varphi = \varphi\left(\cdot + (a - \langle\cdot,\eta\rangle)\eta\right) \in \mathcal{G}$. In this section we are interested in a representation of products of Donsker's delta (first as in Example 2.14) with regular generalized functions with help of this composition.

Lemma 4.23. *Let $\varphi \in \mathcal{G}$, $\eta \in \mathcal{H}$, $|\eta|_0 = 1$, and $a \in \mathbb{C}$, then*

$$\langle\!\langle\delta\left(\langle\cdot,\eta\rangle - a\right),\varphi\rangle\!\rangle = \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}a^2}E\left(\varphi\left(\cdot + (a - \langle\cdot,\eta\rangle)\eta\right)\right) = \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}a^2}E\left(\tau_{a\eta}P_\eta\varphi\right).$$

This can be easy shown first for Wick exponentials. Then by continuity of $\tau_{a\eta}$ and P_η on $\mathcal{G}$, the statement follows for arbitrary regular test functions.

Theorem 4.24. *Let $\varphi \in \mathcal{G}$, $\eta \in \mathcal{H}$, $|\eta|_0 = 1$, and $a \in \mathbb{C}$. Then*

$$\delta\left(\langle\cdot,\eta\rangle - a\right)\varphi(\cdot) \in \mathcal{G}'$$

and

$$\delta\left(\langle\cdot,\eta\rangle - a\right)\varphi(\cdot) = \delta\left(\langle\cdot,\eta\rangle - a\right) \diamond \varphi\left(\cdot + (a - \langle\cdot,\eta\rangle)\eta\right) = \delta\left(\langle\cdot,\eta\rangle - a\right)\tau_{a\eta}P_\eta\varphi,$$

where the product is an projective independent product in the sense of Definition 4.8.

Proof. Let $\varphi \in \mathcal{G}$, $\eta \in \mathcal{H}$, $|\eta|_0 = 1$, and $a \in \mathbb{C}$, then $\delta\left(\langle\cdot,\eta\rangle - a\right)\varphi(\cdot) \in \mathcal{G}'$, since $\delta\left(\langle\cdot,\eta\rangle - a\right) \in \mathcal{G}'$, see Theorem 3.33. Moreover, we know by Theorem 3.10 and Theorem 3.18 that $\tau_{a\eta}P_\eta\varphi = \varphi\left(\cdot + (a - \langle\cdot,\eta\rangle)\eta\right) \in \mathcal{G}$. Hence for all $\xi \in \mathcal{N}$ we get with Lemma 4.23 that

$$\begin{aligned}
S\Big(\delta\left(\langle\cdot,\eta\rangle - a\right)\varphi(\cdot)\Big)(\xi) &= \Big\langle\!\Big\langle \delta\left(\langle\cdot,\eta\rangle - a\right)\varphi, : \exp\left(\langle\cdot,\xi\rangle\right) : \Big\rangle\!\Big\rangle \\
&= \Big\langle\!\Big\langle \delta\left(\langle\cdot,\eta\rangle - a\right), \varphi\exp\left(\langle\cdot,\xi\rangle\right) \Big\rangle\!\Big\rangle \exp\left(-\tfrac{1}{2}|\xi|_0^2\right) \\
&= \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}a^2}E\Big(\varphi\left(\cdot + (a - \langle\cdot,\eta\rangle)\eta\right)\exp\left(\langle\cdot + (a - \langle\cdot,\eta\rangle)\eta,\xi\rangle\right)\Big)\exp\left(-\tfrac{1}{2}|\xi|_0^2\right) \\
&= \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}\left((\eta,\xi) - a\right)^2\right)\exp\left(-\tfrac{1}{2}\left(|\xi|_0^2 - (\eta,\xi)^2\right)\right) \\
&\quad \times E\Big(\varphi\left(\cdot + (a - \langle\cdot,\eta\rangle)\eta\right)\exp\left(\cdot - \langle\cdot,\eta\rangle\eta,\xi\rangle\right)\Big) \\
&= S\Big(\delta\left(\langle\cdot,\eta\rangle - a\right)\Big)(\xi)S\Big(\varphi\left(\cdot + (a - \langle\cdot,\eta\rangle)\eta\right)\Big)(\xi - (\eta,\xi)\eta) \\
&= S\Big(\delta\left(\langle\cdot,\eta\rangle - a\right)\Big)(\xi)S\Big(\tau_{a\eta}P_\eta\varphi\Big)(\xi - (\eta,\xi)\eta).
\end{aligned}$$

Since $\tau_{a\eta}P_\eta\varphi \in \mathcal{G}$, we get with Definition 2.3 that

$$\begin{aligned}
S\Big(\varphi\left(\cdot + (a - \langle\cdot,\eta\rangle)\eta\right)\Big)(\xi - (\eta,\xi)\eta) &= E\Big(\varphi\left(\cdot + \xi - (\eta,\xi)\eta + (a - \langle\cdot + \xi - (\eta,\xi)\eta,\eta\rangle)\eta\right)\Big) \\
&= E\Big(\varphi\left(\cdot + \xi + (a - \langle\cdot + \xi,\eta\rangle)\eta\right)\Big) = S\Big(\varphi\left(\cdot + (a - \langle\cdot,\eta\rangle)\eta\right)\Big)(\xi).
\end{aligned}$$

Hence since

$$S\Big(\delta\left(\langle\cdot,\eta\rangle - a\right)\Big)(\xi) = S\Big(\delta\left(\langle\cdot,\eta\rangle - a\right)\Big)((\xi,\eta)\eta)$$

we get by Theorem 4.9 that

$$\delta\left(\langle\cdot,\eta\rangle - a\right)\varphi(\cdot) = \delta\left(\langle\cdot,\eta\rangle - a\right) \diamond \varphi\left(\cdot + (a - \langle\cdot,\eta\rangle)\eta\right) = \delta\left(\langle\cdot,\eta\rangle - a\right)\varphi\left(\cdot + (a - \langle\cdot,\eta\rangle)\eta\right).$$

□

Theorem 4.25. *Let $\Phi \in \mathcal{G}'$, such that $P_\eta\Phi = \Phi(\cdot - \langle\cdot,\eta\rangle\eta)$ is also in $\mathcal{G}'$, $\eta \in \mathcal{H}$, $|\eta|_0 = 1$, and $a \in \mathbb{C}$. Then*

$$\delta(\langle\cdot,\eta\rangle - a)\,\Phi(\cdot) \in \mathcal{G}'$$

and

$$\begin{aligned}\delta(\langle\cdot,\eta\rangle - a)\,\Phi(\cdot) &= \delta(\langle\cdot,\eta\rangle - a)\ \diamond\ \Phi(\cdot + (a - \langle\cdot,\eta\rangle)\eta)\\ &= \delta(\langle\cdot,\eta\rangle - a)\ \diamond\ \tau_{a\eta}P_\eta\Phi = \delta(\langle\cdot,\eta\rangle - a)\,\tau_{a\eta}P_\eta\Phi,\end{aligned}$$

where the product is an projective independent product in the sense of Definition 4.8.

Proof. Let $\Phi \in \mathcal{G}'$, with kernels $\Phi^{(n)}$, $n \in \mathbb{N}$, such that $P_\eta\Phi = \Phi(\cdot - \langle\cdot,\eta\rangle\eta) \in \mathcal{G}'$. We define the sequence $(\varphi_M)_{M\in\mathbb{N}} \subset \mathcal{G}$ by

$$\varphi_M = \sum_{n=0}^{\infty}\left\langle :\cdot^{\otimes n}:, \varphi^{(n)}\right\rangle = \sum_{n=0}^{M}\left\langle :\cdot^{\otimes n}:, \Phi^{(n)}\right\rangle,$$

as in the proof of Corollary 3.22. Therein it is shown that $(\varphi_M)_{M\in\mathbb{N}}$ converges to Φ in $\mathcal{G}'$ and also $\left(P_\eta\varphi_M\right)_{M\in\mathbb{N}}$ converges to $P_\eta\Phi$ in $\mathcal{G}'$. Since $\tau_{a\eta}$ is continuous on $\mathcal{G}'$ (see Theorem 3.10) we get that $\left(\tau_{a\eta}P_\eta\varphi_M\right)_{M\in\mathbb{N}}$ also converges in $\mathcal{G}'$ to $\tau_{a\eta}P_\eta\Phi$. Hence by Theorem 4.24 and continuity of the Wick product we get that

$$\begin{aligned}\lim_{M\to\infty} S\Big(\delta(\langle\cdot,\eta\rangle - a)\,\varphi_M(\cdot)\Big)(\xi) &= \lim_{M\to\infty} S\Big(\delta(\langle\cdot,\eta\rangle - a)\Big)(\xi)S\Big(\tau_{a\eta}P_\eta\varphi_M\Big)(\xi)\\ &= S\Big(\delta(\langle\cdot,\eta\rangle - a)\Big)(\xi)S\Big(\tau_{a\eta}P_\eta\Phi\Big)(\xi),\end{aligned}$$

for all $\xi \in \mathcal{N}$. Thus since

$$S\Big(\tau_{a\eta}P_\eta\Phi\Big)(\xi) = \lim_{M\to\infty} S\Big(\tau_{a\eta}P_\eta\varphi_M\Big)(\xi) = \lim_{M\to\infty} S\Big(\tau_{a\eta}P_\eta\varphi_M\Big)(\xi - (\xi,\eta)\eta) = S\Big(\tau_{a\eta}P_\eta\Phi\Big)(\xi - (\xi,\eta)\eta),$$

we get by Theorem 4.9 that

$$\begin{aligned}\delta(\langle\cdot,\eta\rangle - a)\,\Phi(\cdot) &= \delta(\langle\cdot,\eta\rangle - a)\ \diamond\ \Phi(\cdot + (a - \langle\cdot,\eta\rangle)\eta)\\ &= \delta(\langle\cdot,\eta\rangle - a)\ \diamond\ \tau_{a\eta}P_\eta\Phi = \delta(\langle\cdot,\eta\rangle - a)\,\tau_{a\eta}P_\eta\Phi.\end{aligned}$$

□

Definition 4.26. *Let $\eta \in \mathcal{H}$, $|\eta|_0 = 1$, and $a \in \mathbb{C}$. Then we define the linear subspace of $\mathcal{G}'$, $C_{\eta,a}$, by*

$$C_{\eta,a} := \{\Phi \in \mathcal{G}' \mid \Phi\delta(\langle\cdot,\eta\rangle - a) \in \mathcal{G}'\}.$$

Corollary 4.27. *Let $\eta \in \mathcal{H}$, $|\eta|_0 = 1$, and $a \in \mathbb{C}$. Then*

$$C_{\eta,a} = \left\{\Phi \in \mathcal{G}' \mid \Phi(\cdot - \langle \cdot, \eta\rangle\eta) = P_\eta \Phi \in \mathcal{G}'\right\}.$$

Proof. By Theorem 4.25 we know that

$$\left\{\Phi \in \mathcal{G}' \mid \Phi(\cdot - \langle \cdot, \eta\rangle\eta) = P_\eta \Phi \in \mathcal{G}'\right\} \subset C_{\eta,a}.$$

So let $\Phi \in C_{\eta,a}$, and assume that $\Phi(\cdot - \langle \cdot, \eta\rangle\eta) = P_\eta\Phi \notin \mathcal{G}'$. Then since $\tau_{a\eta}$ is continuous in $\mathcal{G}'$, see Theorem 3.10, it follows that $\tau_{a\eta}P_\eta\Phi \notin \mathcal{G}'$. Now let $(\varphi_M)_{M\in\mathbb{N}} \subset \mathcal{G}$ be defined as in the proof of Corollary 3.22 and Theorem 4.25. Then $\tau_{a\eta}P_\eta\varphi_M \in \mathcal{G}$ for all $M \in \mathbb{N}$ and by definition $\varphi_M \in C_{\eta,a}$. But

$$\lim_{M\to\infty} \tau_{a\eta}P_\eta\varphi_M \notin \mathcal{G}'.$$

Since $(\varphi_M)_{M\in\mathbb{N}}$ converges to Φ in $\mathcal{G}'$ we get that

$$\Phi\delta\left(\langle \cdot, \eta\rangle - a\right) = \lim_{M\to\infty} \varphi_M\delta\left(\langle \cdot, \eta\rangle - a\right) = \lim_{M\to\infty} \varphi_M(\cdot - \langle \cdot, \eta\rangle\eta + a\eta) \diamond \delta\left(\langle \cdot, \eta\rangle - a\right),$$

which is a contradiction to the continuity of the Wick product in $\mathcal{G}'$, see Section 4.2. □

Theorem 4.28. *Let $\eta \in \mathcal{H}$, $|\eta|_0 = 1$, and $a \in \mathbb{C}$. Then*

$$C_{\eta,a} = \{\Phi \in \mathcal{G}' \mid \Phi\delta\left(\langle \cdot, \eta\rangle - a\right) \in \mathcal{G}'\} = \left\{\Phi \in \mathcal{G}' \;\middle|\; \exp\left(-\frac{1}{2}D_\eta^2\right)\Phi \in \mathcal{G}'\right\}.$$

The proof follows directly by Corollary 3.22.

Remark 4.29. *Note that Corollary 3.22 also implies that the operator $\tau_{a\eta}P_\eta$ is continuous on $C_{\eta,a}$.*

Remark 4.30. *Of course we are interested in an extension of $C_{\eta,a}$, $\eta \in \mathcal{H}$, $|\eta|_0 = 1$, $a \in \mathbb{C}$, which includes elements of $(\mathcal{N})^{-1} \setminus \mathcal{G}'$. This causes more problems since $\tau_{a\eta}$ is not defined on $(\mathcal{N})^{-1} \setminus \mathcal{G}'$.*

(i) Let $(\Phi_n)_{n\in\mathbb{N}} \subset C_{\eta,a}$ be a sequence which converges to $\Phi \in (\mathcal{N})^{-1}$,$\eta \in \mathcal{H}$, $|\eta|_0 = 1$, $a \in \mathbb{C}$. If $(\tau_{a\eta}P_\eta\Phi_n)_{n\in\mathbb{N}}$ converges in $(\mathcal{N})^{-1}$ then we define $\tau_{a\eta}P_\eta\Phi := \lim_{n\to\infty} \tau_{a\eta}P_\eta\Phi_n$. In that case $\lim_{n\to\infty} \tau_{a\eta}P_\eta\Phi_n$ and $\delta(\langle \cdot, \eta\rangle - a)$ are projective independent (see Remark 4.10), since

$$\begin{aligned} S\left(\lim_{n\to\infty} \tau_{a\eta}P_\eta\Phi_n\right)(\xi) &= \lim_{n\to\infty} S\left(\tau_{a\eta}P_\eta\Phi_n\right)(\xi) \\ &= \lim_{n\to\infty} S\left(\tau_{a\eta}P_\eta\Phi_n\right)(\xi - (\xi,\eta)\eta) = S\left(\lim_{n\to\infty} \tau_{a\eta}P_\eta\Phi_n\right)(\xi - (\xi,\eta)\eta), \end{aligned}$$

for all $\xi \in U$. Here $U \subset \mathcal{N}_\mathbb{C}$ open, denotes the subset where $S(\Phi)$ is holomorphic at zero, see Theorem 2.16.

(ii) Let $\Phi(\lambda) \subset C_{\eta,a}$, $\lambda \in \Omega$, and $\Phi := \int_\Omega \Phi(\lambda)\, d\lambda \in (\mathcal{N})^{-1}$, in the sense of Theorem 2.18. If $\int_\Omega \tau_{a\eta} P_\eta \Phi(\lambda)\, d\lambda \in (\mathcal{N})^{-1}$ then we define $\tau_{a\eta} P_\eta \Phi := \int_\Omega \tau_{a\eta} P_\eta \Phi(\lambda)\, d\lambda$. Again projective independence is naturally given.

Example 4.31. *Let us again consider the white noise case $\mathcal{N} = S(\mathbb{R})$ and $\mathcal{H} = L^2(\mathbb{R})$ and an n-times product of Donsker's deltas. Then for $-\infty < t_0 < t_1 < t_n < \infty$ and $x_0, \dots, x_n \in \mathbb{R}$ we get that*

$$\prod_{j=1}^{n} \delta(\langle \cdot, \mathbb{1}_{[t_0,t_j)} \rangle - (x_j - x_0)) = \prod_{j=1}^{n} \delta(\langle \cdot, \mathbb{1}_{[t_{j-1},t_j)} \rangle - (x_j - x_{j-1})), \tag{4.10}$$

where the product on the right hand side is an n-times independent product.

Proof. Let $n = 2$, $-\infty < t_0 < t_1 < t_2 < \infty$ and $x_0, x_1, x_2 \in \mathbb{R}$, then, since $\mathbb{1}_{[t_0,t_1)}$ and $\mathbb{1}_{[t_0,t_2)}$ are linear independent

$$\begin{aligned}
&\delta(\langle \cdot, \mathbb{1}_{[t_0,t_1)} \rangle - (x_1 - x_0))\delta(\langle \cdot, \mathbb{1}_{[t_0,t_2)} \rangle - (x_2 - x_0)) \\
&\qquad = \frac{1}{\sqrt{t_1 - t_0}} \delta\left(\frac{1}{\sqrt{t_1 - t_0}} \langle \cdot, \mathbb{1}_{[t_0,t_1)} \rangle - \frac{x_1 - x_0}{\sqrt{t_1 - t_0}} \right) \delta(\langle \cdot, \mathbb{1}_{[t_0,t_2)} \rangle - (x_2 - x_0))
\end{aligned}$$

exists by Theorem 4.11, where the equality is given by Lemma 3.34. Moreover, by Theorem 4.28 and Example 3.23 we get that $\delta(\langle \cdot, \mathbb{1}_{[t_0,t_2)} \rangle - (x_2 - x_0)) \in C_{\frac{1}{\sqrt{t_1-t_0}} \mathbf{1}_{[t_0,t_1)}, \frac{x_1-x_0}{\sqrt{t_1-t_0}}}$. Therefore

$$\begin{aligned}
&\delta(\langle \cdot, (t_1 - t_0)^{-1/2} \mathbb{1}_{[t_0,t_1)} \rangle - (t_1 - t_0)^{-1/2}(x_1 - x_0))\delta(\langle \cdot, \mathbb{1}_{[t_0,t_2)} \rangle - (x_2 - x_0)) \\
&\quad = \delta(\langle \cdot, \mathbb{1}_{[t_0,t_1)} \rangle - (x_1 - x_0)) \diamond \tau_{(t_1-t_0)(x_1-x_0)\mathbf{1}_{[t_0,t_1)}} P_{(t_1-t_0)^{-1/2}\mathbf{1}_{[t_0,t_1)}} \delta(\langle \cdot, \mathbb{1}_{[t_0,t_2)} \rangle - (x_2 - x_0)) \\
&\qquad = \delta(\langle \cdot, \mathbb{1}_{[t_0,t_1)} \rangle - (x_1 - x_0)) \\
&\diamond \delta(\langle \cdot - \langle \cdot, (t_1 - t_0)^{-1/2} \mathbb{1}_{[t_0,t_1)} \rangle (t_1 - t_0)^{-1/2} \mathbb{1}_{[t_0,t_1)} + (t_1 - t_0)(x_1 - x_0) \mathbb{1}_{[t_0,t_1)}, \mathbb{1}_{[t_0,t_2)} \rangle - (x_2 - x_0)) \\
&\quad = \delta(\langle \cdot, \mathbb{1}_{[t_0,t_1)} \rangle - (x_1 - x_0)) \diamond \delta(-\langle \cdot, \mathbb{1}_{[t_0,t_1)} \rangle + \langle \cdot, \mathbb{1}_{[t_0,t_2)} \rangle - (x_2 - x_0) + (x_1 - x_0)) \\
&\qquad = \delta(\langle \cdot, \mathbb{1}_{[t_0,t_1)} \rangle - (x_1 - x_0))\delta(\langle \cdot, \mathbb{1}_{[t_1,t_2)} \rangle - (x_2 - x_1)).
\end{aligned}$$

By induction (4.10) holds. □

Remark 4.32. *For $\eta \in \mathcal{H}$, $\eta \neq 0$ and $a \in \mathbb{C}$ due to the homogeneity of Donsker's delta of degree -1, see Lemma 3.34, the set $C_{\eta,a}$ is given by $C_{\frac{1}{|\eta|_0}\eta, \frac{1}{|\eta|_0}a}$.*

Remark 4.33. *In analogy to this Section one can show in the vector-valued case that a product of d-dimensional Donsker's delta, see Example 2.15, with $\Phi \in \mathcal{G}'$ is given as follows*

$$\Phi \delta^d (\langle \omega, \eta \rangle - a) = \tau_{a\eta} P_{\vec{\eta}} \Phi \diamond \delta^d (\langle \omega, \eta \rangle - a),$$

where $a\eta = (a_1\eta, \ldots, a_d\eta), \vec{\eta} = (\eta, \ldots, \eta) \in L^2_d(\mathbb{R})$ *for* $\eta \in L^2(\mathbb{R})$ *and* $a = (a_1, \ldots, a_d) \in \mathbb{R}^d$. *This causes by the informal relation*

$$\delta^d\left(\langle \omega, \eta \rangle - a\right) = \prod_{j=1}^{d} \delta\left(\langle \omega_j, \eta \rangle - a_j\right), \quad \omega = (\omega_1, \ldots, \omega_d) \in S'_d(\mathbb{R}),$$

for $\eta \in L^2(\mathbb{R})$ *and* $a = (a_1, \ldots, a_d) \in \mathbb{R}^d$.

Chapter 5

Wick product representation for the integrand of the complex scaled Feynman-Kac-kernel

In this chapter we construct integral kernels to complex scaled heat equations. More precisely, these kernels are given by the generalized expectations of Hida distributions. That means that we also construct the integrands corresponding to these kernels. For analytic potentials these integrands are given via a Wick product representation, see Section 5.3. The proof is based on a finite dimensional approximation close to the construction of classical Feynman-Kac-kernels, see e.g. [31]. In Section 5.4 we generalized this to potentials fulfilling some regularity conditions. Therein the knowledge from Section 4.4 is used.

5.1 Complex-scaled heat equation

It is well-known from Stochastic Analysis (see e.g. [31], [38] or [76]) that a solution to the heat equation

$$\begin{cases} \frac{\partial}{\partial t}\psi(t,x) = \frac{1}{2}\Delta\psi(t,x) + V(x)\psi(t,x) \\ \qquad \psi(0,x) = f(x), \quad 0 \le t \le T < \infty,\ x \in \mathbb{R}^d, \end{cases} \tag{5.1}$$

is stated by the famous Feynman-Kac-formula (here Δ denotes the Laplace operator on $\mathbb{R}^d$). I.e., for a suitable potential $V : \mathcal{O} \to \mathbb{C}$, $\mathcal{O} \subset \mathbb{R}^d$ open, $d \ge 1$, and initial function $f : \mathbb{R}^d \to \mathbb{C}$ the unique solution to the heat equation (5.1) is given by

$$\psi(t,x) = E\left(\exp\left(\int_0^t V(x+B_r)\,dr\right) f(x+B_t)\right), \quad t \in [0,T],\ x \in \mathcal{O}, \tag{5.2}$$

where E denotes the expectation w.r.t. a Brownian motion B starting at 0. Moreover for $t \in (t_0, T]$ and $x, x_0 \in \mathbb{R}^d$ the heat kernel $K_V : \mathbb{R}^d \times \mathbb{R}^d \times (0, T] \to \mathbb{R}$ is given by

$$K_V(x, t; x_0, t_0) = \frac{1}{\sqrt{2\pi(t - t_0)}} \exp\left(-\frac{1}{2(t - t_0)}(x_0 - x)^2\right) \\ E\left(\exp\left(\int_{t_0}^{t} V\left(x_0 - \frac{r - t_0}{t - t_0}(x_0 - x) + B_r - \frac{r - t_0}{t - t_0} B_t\right) dr\right)\right), \quad (5.3)$$

for suitable potentials, see e.g. [31]. Observe that in the integral we are dealing with a Brownian bridge (starting at time t_0 in x_0 and ending at time t in x). In this section we consider time-independent complex scaled heat equations. That means for all $0 \le t \le T$, $x \in \mathcal{O}$, $\mathcal{O} \subset \mathbb{R}^d$ open, $d \ge 1$, and $z \in \mathbb{C}$ the following partial differential equation is of interest

$$\begin{cases} \frac{\partial}{\partial t}\psi(t, x) = -z^2 \frac{1}{2}\Delta\psi(t, x) + \frac{1}{z^2} V(x)\psi(t, x) \\ \psi(0, x) = f(x), \end{cases} \quad (5.4)$$

again for suitable potentials V and initial functions f.

5.2 Complex scaled Feynman-Kac-formula

Upon this section we construct a solution of a similar form to (5.2) for the complex scaled heat equation (5.4) with scaling $z \in \mathbb{C}$. This section goes back to [13], see also [6].

Assumption 5.1. *For $\mathcal{O} \subset \mathbb{R}^d$ open, where $\mathbb{R}^d \setminus \mathcal{O}$ is a set of Lebesgue measure zero, we define the set $\mathcal{D}_z \subset \mathbb{C}$ by*

$$\mathcal{D}_z := \left\{x + zy \,\middle|\, x \in \mathcal{O} \text{ and } y \in \mathbb{R}^d\right\} \quad (5.5)$$

and consider analytic functions $V : \mathcal{D}_z \to \mathbb{C}$ and $f : D_z \to \mathbb{C}$.

Assumption 5.2. *For $0 \le t \le T < \infty$ and $x \in \mathcal{D}_z$ we assume that the expressions*

$$E\left(\sup_{u \le t}\left|\exp\left(\frac{1}{z^2}\int_0^u V\big(x + zB_r\big)\, dr\right) f\big(x + zB_u\big)\right|\right)$$

and

$$E\left(\sup_{u \le t}\left|\left(\frac{i}{2}\frac{\partial^2}{\partial x^2} - iV(x)\right)\exp\left(\frac{1}{z^2}\int_0^u V\big(x + zB_r\big)\, dr\right) f\big(x + zB_u\big)\right|\right)$$

are finite. Again E denotes the expectation w.r.t. a Brownian motion B starting at 0.

Theorem 5.3. *Let $0 \leq t \leq T < \infty$ and $V : \mathcal{D}_z \to \mathbb{C}$ and $f : \mathbb{C} \to \mathbb{C}$ such that Assumption 5.1 and Assumption 5.2 are fulfilled. Then the unique solution $\psi : [0,T] \times \mathcal{D}_z \to \mathbb{C}$ to the complex scaled heat equation* (5.4) *is given by*

$$\psi(t,x) = E\left(\exp\left(\frac{1}{z^2}\int_0^t V\Big(x + zB_r\Big)\,dr\right) f\Big(x + zB_t\Big)\right). \tag{5.6}$$

For more details and the proof see [13]. Note that (5.6) is holomorphic on $\mathcal{D}_z$. This statement follows by the theorems of Fubini and Morera.

Corollary 5.4. *Let us consider the case of the free motion, i.e., $V \equiv 0$. We assume that $f : \mathcal{D}_z \to \mathbb{C}$ is an analytic function, such that $E\Big[f\Big(y + zB_t\Big)\Big]$, $y \in \mathcal{D}_z$, $0 \leq t \leq T$, exists and is uniformly bounded on* $[0,T]$. *Moreover, let*

$$\omega \mapsto \sup_{s\in[0,T]} \Big|\Delta f\Big(y + zB_s(\omega)\Big)\Big|$$

be integrable, then

$$z\frac{\partial}{\partial t}E\Big[f\Big(x + zB_t\Big)\Big] = -z^2\frac{1}{2}\Delta E\Big[f\Big(x + zB_t\Big)\Big],$$

for $x \in \mathcal{O}$, $0 \leq t \leq T$.

Of course for $z \in \mathbb{R}$ it is enough to claim square-integrability of f instead of analyticity (see e.g. [68]).

Remark 5.5. *We denote $H(\mathcal{D}_z)$ to be the set of holomorphic functions from $\mathcal{D}_z$ to $\mathbb{C}$. In the sense of Fröhlich (see [16]) by the solution* (5.6) *of the complex scaled heat equation an unbounded semigroup from a subset $D_z(t)$ of $H(\mathcal{D}_z)$ into itself is defined.*

5.3 Construction of the complex scaled heat kernel

Among this section we will only consider the one dimensional case, $d = 1$. In the euclidean case, a solution to the heat equation is stated by the Feynman-Kac formula, see (5.2), and the corresponding heat kernel is given as in (5.3). In white noise one can construct the integral kernel by inserting Donsker's delta in order to fix the final point $x \in \mathbb{R}$, and taking the generalized expectation, i.e.,

$$K_V(x,t;x_0,t_0) = E\left(\exp\left(\int_{t_0}^t V\left(x_0 + \langle\cdot, \mathbb{1}_{[t_0,r)}\rangle\right)\,dr\right)\delta(\langle\cdot, \mathbb{1}_{[t_0,t)}\rangle - (x - x_0))\right),$$

where the integrand is e.g. a Hida distribution. We are not only interested in a mathematical meaning of the informal expression inside the expectation on the right hand side, we also want

to answer the question for which potentials we can generalize this kernel to a complex scaled situation. In formulas this means for suitable potentials V, $z \in \mathbb{C}$, we are interested in the product

$$\exp\left(\frac{1}{z^2}\int_{t_0}^{t} V\left(x + z\left\langle \cdot, \mathbb{1}_{[t_0,r)}\right\rangle\right) dr\right)\sigma_z\delta(\langle \cdot, \mathbb{1}_{[t_0,t)}\rangle - (x - x_0)), \tag{5.7}$$

as a generalized function of white noise. In addition we show that

$$\begin{aligned}\exp&\left(\frac{1}{z^2}\int_{t_0}^{t} V\left(x + z\left\langle \cdot, \mathbb{1}_{[t_0,r)}\right\rangle\right) dr\right)\sigma_z\delta(\langle \cdot, \mathbb{1}_{[t_0,t)}\rangle - (x - x_0))\\ &= \exp\left(\frac{1}{z^2}\int_{t_0}^{t} V\left(x_0 + \frac{r - t_0}{t - t_0}(x - x_0) + z\left(\left\langle \cdot, \mathbb{1}_{[t_0,r)}\right\rangle - \frac{r - t_0}{t - t_0}\left\langle \cdot, \mathbb{1}_{[0,t)}\right\rangle\right)\right) dr\right)\\ &\qquad\qquad \diamond\, \sigma_z\delta\left(\langle \cdot, \mathbb{1}_{[t_0,t)}\rangle - (x - x_0)\right). \end{aligned}\tag{5.8}$$

Taking its generalized expectation we obtain

$$\begin{aligned}K_V(x, t; x_0, t_0) = &\frac{1}{\sqrt{2\pi(t - t_0)z^2}}\exp\left(-\frac{1}{2(t - t_0)z^2}(x_0 - x)^2\right)\\ &\times E\left(\exp\left(\frac{1}{z^2}\int_{t_0}^{t} V\left(x_0 + \frac{r - t_0}{t - t_0}(x - x_0) + z\left(\left\langle \cdot, \mathbb{1}_{[t_0,r)}\right\rangle - \frac{r - t_0}{t - t_0}\left\langle \cdot, \mathbb{1}_{[t_0,t)}\right\rangle\right)\right) dr\right)\right),\end{aligned}$$

which is a scaled version of (5.3). Of course this is only possible if $\sigma_z\delta(\langle \cdot, \mathbb{1}_{[t_0,t)}\rangle - (x - x_0))$ is a Hida distribution. Hence, we only consider $z \in \overline{S_0}$, see Theorem 4.11 for its definition. Again, the potential V should be defined on the subset $\mathcal{D}_z$, see (5.5), of the complex plain such that $V(x + zy)$ is well-defined for $x, y \in \mathbb{R}$.

Assumption 5.6. *Let $z \in \overline{S_0}$ and $0 < T < \infty$. We assume that the potential $V : \mathcal{D}_z \to \mathbb{C}$ is analytic and that there exist a constant $0 < A < \infty$, a locally bounded function $B : \mathcal{O} \to \mathbb{R}$ and some $0 < \varepsilon < \frac{1}{4T}$ such that for all $x_0 \in \mathcal{O}$ and $y \in \mathbb{R}$ one has that*

$$\left|\exp\left(\frac{1}{z^2}V(x)\right)\right| \le A\exp\left(\varepsilon x^2\right) \quad \textit{and} \quad \left|\exp\left(\frac{1}{z^2}V\left(x_0 + zy\right)\right)\right| \le B(x_0)\exp\left(\varepsilon y^2\right).$$

To get along with this class one should have in mind the following (also non-perturbative) potentials.

Example 5.7. *Let $z = 1$ and $p : \mathbb{R} \to \mathbb{R}$ a polynomial. Then the potential*

$$\begin{aligned}V : \mathbb{R} &\to \mathbb{R}\\ x &\mapsto -\exp(p(x))\end{aligned}$$

fulfills Assumption 5.6.

Example 5.8. *For $n \in \mathbb{N}_0$ and $z = \sqrt{i}$, we have a look at the potential*

$$\begin{aligned} V : \mathbb{C} &\to \mathbb{C} \\ x &\mapsto (-1)^{n+1} a_{4n+2} x^{4n+2} + \sum_{j=1}^{4n+1} a_j x^j, \end{aligned}$$

for $a_0, \dots, a_{4n+1} \in \mathbb{C}$ and $a_{4n+2} > 0$. Using $\frac{1}{z^2} z^{4n+2} = \frac{1}{i}(\sqrt{i})^{4n+2} = -1$ one can easily show that Assumption 5.6 is fulfilled.

Example 5.9. *Consider $O = \mathbb{R} \setminus \{b\}$, $b \in \mathbb{R}$, and $z = \sqrt{i}$. Then the potentials*

(i)

$$\begin{aligned} V : \mathcal{D}_{\sqrt{i}} &\to \mathbb{C} \\ x &\mapsto \frac{a}{|x-b|^n}, \end{aligned}$$

where $n \in \mathbb{N}$, $a \in \mathbb{C}$ and $b \in \mathbb{R}$,

(ii)

$$\begin{aligned} V : \mathcal{D}_{\sqrt{i}} &\to \mathbb{C} \\ x &\mapsto \frac{a}{(x-b)^n}, \end{aligned}$$

for $a \in \mathbb{C}$, $b \in \mathbb{R}$ and $n \in \mathbb{N}$,

fulfill Assumption 5.6. This causes by the natural representation of V, as in (i), as an entire function

$$\begin{aligned} V : \mathcal{D}_{\sqrt{i}} &\to \mathbb{C} \\ x &\mapsto \exp\Big(\log(a) - \frac{n}{2}\log\Big((x-b)^2\Big)\Big), \end{aligned}$$

which can be estimate by

$$\Big|V\Big(x + \sqrt{i}y\Big)\Big| = |a| \,\Big|\exp\Big(-\frac{n}{2}\log\Big((x-b+\sqrt{i}y)^2\Big)\Big| \le |a| \,\exp\Big(-\frac{n}{2}\log\Big(\frac{(x-b)^2}{2}\Big)\Big), \tag{5.9}$$

for all $x \in O$ and $y \in \mathbb{R}$. For the proof of this formula see, [13].

Note that this examples also fulfill Assumption 5.2. Similar examples for potentials are considered in [13] and [26]. In the following we prove the existence of (5.8) as a Hida distribution via an approximation by finitely based Hida distributions.

5.3.1 Approximation by finitely based Hida distribution

In this section we give a meaning to (5.8) for potentials $V : \mathcal{D}_z \to \mathbb{C}$ fulfilling Assumption 5.6. W.l.o.g. we consider the case $t_0 = 0$. Then for $0 < t < T < \infty$ and $n \in \mathbb{N}$, we define the decomposition of the time interval $[0, t]$, given by $t_k := t\frac{k}{n}$, $k = 1, \dots, n$. Then by Assumption 5.6 the Riemann approximation,

$$\phi_n := \exp\left(\frac{1}{z^2}\frac{t}{n}\sum_{k=1}^{n-1} V\Big(x_0 + z\,\langle\cdot, \mathbb{1}_{[0,t_k)}\rangle\Big)\right), \tag{5.10}$$

is a well-defined $L^2(\mu)$-function for all $z \in \overline{S_0}$.

Proposition 5.10. *The product of a complex scaled Donsker's delta with the Riemann approximation defined as in* (5.10) *can be defined as an operation in* $(S)'$*. Hence,*

$$\Phi_n := \exp\left(\frac{1}{z^2}\frac{t}{n}\sum_{k=1}^{n-1} V\Big(x_0 + z\,\langle\cdot, \mathbb{1}_{[0,t_k)}\rangle\Big)\right)\sigma_z\delta(\langle\cdot, \mathbb{1}_{[0,t)}\rangle - (x - x_0)), \tag{5.11}$$

is a Hida distribution for all $x \in \mathbb{R}$, $0 < t < T < \infty$ *and* $n \in \mathbb{N}$.

Proof. We define

$$\begin{aligned} G : \mathbb{R}^{n-1} &\to \mathbb{C} \\ y = (y_1, \dots, y_{n-1}) &\mapsto \exp\left(\frac{1}{z^2}\frac{t}{n}\sum_{k=1}^{n-1} V\Big(x_0 + y_k\Big)\right). \end{aligned} \tag{5.12}$$

Then by Assumption 5.6 the functions $y \mapsto G(y)$ and $y \mapsto G(zy)$ are in $L^2_{\mathbb{C}}\left(\mathbb{R}^n, \nu_{M^{-1}_{1,n-1}}\right)$, where $M_{1,n-1}$ is defined as in the proof of Lemma 4.12. Therefore, by Lemma 4.18 the function ϕ_n, defined as in (5.10), can be represented as

$$\phi_n = \int_{\mathbb{R}^{n-1}} G(zy) \prod_{k=1}^{n-1} \delta\left(\langle\cdot, \mathbb{1}_{[0,t_k)}\rangle - y_k\right) d^{n-1}y,$$

for all $n \in \mathbb{N}$. From Lemma 3.34 we know that $\sigma_z\delta$ is homogeneous of degree -1, so we get that for all $y = (y_1, \dots, y_{n-1}) \in \mathbb{R}^{n-1}$ the function

$$\begin{aligned} \Phi_{n,y} := \prod_{k=1}^{n-1} \delta\left(\langle\cdot, \mathbb{1}_{[0,t_k)}\rangle - y_k\right) \sigma_z\delta(\langle\cdot, \mathbb{1}_{[0,t)}\rangle - (x - x_0)) \\ = \frac{1}{z}\prod_{k=1}^{n-1} \delta\left(\langle\cdot, \mathbb{1}_{[0,t_k)}\rangle - y_k\right) \delta\left(\langle\cdot, \mathbb{1}_{[0,t)}\rangle - \frac{(x - x_0)}{z}\right), \end{aligned}$$

is a Hida distribution for all $n \in \mathbb{N}$. Moreover, as shown in Lemma 4.12 its S-transform evaluated at $g \in S(\mathbb{R})$ is given by

$$S\left(\Phi_{n,y}\right)(g) = \frac{1}{z\sqrt{(2\pi\frac{t}{n})^n}} \exp\Bigg(- \frac{n}{2t}\Bigg(((\mathbb{1}_{[0,t_1)}, g) - y_1)^2 + \sum_{k=1}^{n-1} ((\mathbb{1}_{[0,t_k)}, g) - y_k - ((\mathbb{1}_{[0,t_{k+1})}, g) - y_{k+1}))^2 \Bigg)\Bigg),$$

where $y_n := \frac{x-x_0}{z}$. Since $y \mapsto G(zy) \in L^2_{\mathbb{C}}\left(\mathbb{R}^{n-1}, \nu_{M^{-1}_{1,n-1}}\right)$ (again $M_{1,n-1}$ defined as in the proof of Lemma 4.12) the function $\mathbb{R}^{n-1} \ni y = (y_1, \ldots, y_{n-1}) \mapsto G(zy)\, S\Phi_{n,y}(g)$, $g \in S(\mathbb{R})$, fulfills the assumptions of Theorem 2.12. Hence, $\Phi_n := \int_{\mathbb{R}^{n-1}} G(zy)\, \Phi_{n,y}\, dy$ is a Hida distribution and its S-transform is given by

$$S(\Phi_n)(g) = \int_{\mathbb{R}^{n-1}} G(zy)\, S\Phi_{n,y}(g)\, dy,$$

for all $g \in S(\mathbb{R})$. □

Note that one can not show, e.g. by using Theorem 2.11, that the sequence $(\Phi_n)_{n\in\mathbb{N}}$ (defined by (5.11)) converges in $(S)'$, since its S-transform can not be estimated independent of n.

Proposition 5.11. *Let $h_k := \mathbb{1}_{[0,t_k)} - \frac{k}{n}\mathbb{1}_{[0,t)}$, $1 \le k \le n-1$. Moreover, let $x_0, x \in O$ such that $x_0 + \frac{r}{t}(x - x_0) \in O$, for all $0 \le r \le t$, and Φ_n, $n \in \mathbb{N}$, be as in Proposition 5.10. Then*

$$\Psi_n := \exp\left(\frac{1}{z^2}\frac{t}{n}\sum_{k=1}^{n-1} V\left(x_0 + \frac{k}{n}(x - x_0) + z\langle\cdot, h_k\rangle\right)\right) \in L^2(\mu), \qquad n \in \mathbb{N},$$

and

$$\Phi_n = \Psi_n \diamond \sigma_z\delta\left(\langle\omega, \mathbb{1}_{[0,t)}\rangle - (x - x_0)\right) \in (S)'.$$

Proof. As shown in the proof of Proposition 5.10 Φ_n is a Hida distribution for all $n \in \mathbb{N}$. Its S-transform at $g \in S(\mathbb{R})$ is given by

$$S(\Phi_n)(g) = \int_{\mathbb{R}^{n-1}} G(zy)\, S\Phi_{n,y}(g)\, dy,$$

where G and $\Phi_{n,y}$, $y \in \mathbb{R}^{n-1}$, are defined as in the proof of Proposition 5.10. By Theorem 4.16 one gets that

$$S(\Phi_n)(g) = \int_{\mathbb{R}^{n-1}} G(zy)\, S\Psi_{n,y}(g)\, dy\; S\left(\sigma_z\delta\left(\langle\omega, \mathbb{1}_{[0,t)}\rangle - (x - x_0)\right)\right)(g),$$

where $\Psi_{n,y} := \prod_{k=1}^{n-1} \delta\left(\langle \cdot, h_k\rangle - y_k - \frac{k}{n}\frac{(x-x_0)}{z}\right)$. Hence it is left to show that

$$S(\Psi_n)(g) = \int_{\mathbb{R}^{n-1}} G(zy)\, S\Psi_{n,y}(g)\, dy, \tag{5.13}$$

which can be done by the following integral transformations:

$$\begin{aligned}\int_{\mathbb{R}^{n-1}} G(zy)\, S\Psi_{n,y}(g)\, dy &= \int_{\mathbb{R}^{n-1}} G(zy)\, S\left(\prod_{k=1}^{n-1} \delta\left(\langle \cdot, h_k\rangle - \left(y_k - \frac{k}{n}\frac{(x-x_0)}{z}\right)\right)\right)(g)\, dy \\ &= \frac{1}{z}\int_{\gamma_{1,n-1}} G(u)\, S\left(\prod_{k=1}^{n-1} \delta\left(\langle \cdot, h_k\rangle - \frac{u_k - \frac{k}{n}(x-x_0)}{z}\right)\right)(g)\, du \\ &= \frac{1}{z}\int_{\gamma_{2,n-1}} G(u + w(x-x_0))\, S\left(\prod_{k=1}^{n-1} \delta\left(\langle \cdot, h_k\rangle - \frac{u_k}{z}\right)\right)(g)\, du,\end{aligned}$$

where $\gamma_{1,n-1} := \left\{u \in \mathbb{C}^{n-1} \middle| \, u := zy,\ y \in \mathbb{R}^{n-1}\right\}$,
$\gamma_{2,n-1} := \left\{u \in \mathbb{C}^{n-1} \middle| \, u_k := z(y_k - \frac{k}{n}(x-x_0)),\ y \in \mathbb{R}^{n-1}\right\}$ and $w \in \mathbb{R}^{n-1}$ is given by $w_k = \frac{k}{n}$, $k = 1, \ldots, n-1$. Again by an integral transformation

$$\begin{aligned}\int_{\mathbb{R}^{n-1}} G(zy)\, S\Psi_{n,y}(g)\, dy &= \int_{\gamma_{3,n-1}} G(zy + w(x-x_0))\, S\left(\prod_{k=1}^{n-1} \delta\left(\langle \cdot, h_k\rangle - y_k\right)\right)(g)\, dy \\ &= \int_{\mathbb{R}^{n-1}} G(zy + w(x-x_0))\, S\left(\prod_{k=1}^{n-1} \delta\left(\langle \cdot, h_k\rangle - y_k\right)\right)(g)\, dy,\end{aligned}$$

where $\gamma_{3,n-1} := \left\{y \in \mathbb{R}^{n-1} \middle| \, y_k := x_k - \frac{k}{n}(x-x_0),\ y \in \mathbb{R}^{n-1}\right\}$. Since we assumed that $x_0 + \frac{k}{n}(x-x_0) \in \mathcal{O}$ it follows by Assumption 5.6 that

$$y = (y_1, \ldots, y_{n-1}) \mapsto \exp\left(\frac{1}{z^2}\frac{t}{n}\sum_{k=0}^{n-1} V\left(x_0 + zy_k + \frac{k}{n}(x-x_0)\right)\right) \in L^2\left(\mathbb{R}^{n-1}, \nu_{M_{2,n-1}^{-1}}\right),$$

where $M_{2,n-1}$ is defined as in the proof of Lemma 4.13. Hence again by Lemma 4.18 we get that

$$S(\Psi_n)(g) = \int_{\mathbb{R}^{n-1}} G(zy + w(x-x_0))\, S\left(\prod_{k=1}^{n-1} \delta\left(\langle \cdot, h_k\rangle - y_k\right)\right)(g)\, dy,$$

and therefore (5.13) is true. □

Proposition 5.12. *Let ϕ_n and Ψ_n, $n \in \mathbb{N}$, be defined as in* (5.10) *and Theorem 5.11, respectively. Then $\phi_n, \Psi_n \in L^2(\mu)$ for all $n \in \mathbb{N}$. Moreover, the sequences $(\Psi_n)_{n\in\mathbb{N}}$ and $(\phi_n)_{n\in\mathbb{N}}$ converge in $L^2(\mu)$ to*

$$\phi := \exp\left(\frac{1}{z^2}\int_0^t V\left(x_0 + z\left\langle \cdot, \mathbb{1}_{[0,r)}\right\rangle\right) dr\right),$$

and

$$\Psi := \exp\left(\frac{1}{z^2}\int_0^t V\left(x_0 - \frac{r}{t}(x-x_0) + z\left\langle \cdot, \mathbb{1}_{[0,r)} - \frac{r}{t}\mathbb{1}_{[0,t)}\right\rangle\right) dr\right),$$

respectively.

Using Assumption 5.6 the proof follows directly by Lebesgue dominated convergence.

Theorem 5.13. *Let* $0 \le t \le T < \infty$, $t_k := t\frac{k}{n}$ *for* $1 \le k \le n$, $n \in \mathbb{N}$, $x, x_0 \in O$ *such that* $x_0 + \frac{r}{t}(x - x_0) \in O$, *for all* $0 \le r \le t$, *and* Φ_n *as in Proposition 5.10. Then the sequence of Hida distributions* $(\Phi_n)_{n\in\mathbb{N}}$ *converges in* $(S)'$, *and it is natural to identify the limit object with* (5.7), *i.e.,*

$$\Phi := \exp\left(\frac{1}{z^2}\int_0^t V\Big(x_0 + z\left\langle \cdot, \mathbb{1}_{[0,r)}\right\rangle\Big)\, dr\right)\sigma_z\delta\left(\left\langle \cdot, \mathbb{1}_{[0,t)}\right\rangle - (x - x_0)\right) := \lim_{n\to\infty}\Phi_n. \tag{5.14}$$

Moreover, relation (5.8) *holds, which implies that the* S*-transform of* Φ *is given by*

$$S\Phi(g) = S\left(\sigma_z\delta\left(\left\langle \cdot, \mathbb{1}_{[0,t)}\right\rangle - (x - x_0)\right)\right)(g) S\Psi(g), \quad g \in S(\mathbb{R}),$$

where Ψ *is defined as in Proposition 5.12.*

Proof. The proof follows directly by the proofs of Proposition 5.10, Proposition 5.12 and Theorem 2.11. □

Corollary 5.14. *The Wick product in* (5.8) *and Theorem 5.13 is a pointwise product.*

Proof. Example 3.23 and Proposition 5.12 implies that $\delta\left(\left\langle \cdot, \mathbb{1}_{[0,t)}\right\rangle - (x - x_0)\right)$ and Ψ (defined as in Proposition 5.12) are regular generalized functions. Hence, by Lemma 4.7 one only has to show that the $L^2(\mu)$-function Ψ and the scaled Donsker's delta $\sigma_z\delta\left(\left\langle \cdot, \mathbb{1}_{[0,t)}\right\rangle - (x - x_0)\right)$ are independent. This is true since for all $g \in S(\mathbb{R})$ on the one side

$$S\sigma_z\delta\left(\left\langle \cdot, \mathbb{1}_{[0,t)}\right\rangle - (x - x_0)\right)(g) = S\sigma_z\delta\left(\left\langle \cdot, \mathbb{1}_{[0,t)}\right\rangle - (x - x_0)\right)(\mathbb{1}_{[0,t)}g)$$

and on the other side

$$S(\Psi)(g) = S(\Psi)(g - (g, \mathbb{1}_{[0,t)})\mathbb{1}_{[0,t)}) = S(\Psi)\left(\mathbb{1}_{[0,t)^c}(g - (g, \mathbb{1}_{[0,t)})\mathbb{1}_{[0,t)})\right) = S(\Psi)(\mathbb{1}_{[0,t)^c}g). \tag{5.15}$$

Here the first equality of (5.15) can be shown by using (2.3) close to (4.9). □

Lemma 5.15. *Let* $n \in \mathbb{N}$, $0 < t \le T < \infty$, $t_k = t\frac{k}{n}$, $k = 1, \dots, n-1$ *and* $y \in \mathbb{R}$ *then the expression*

$$\Xi_y := \int_{\mathbb{R}^{n-1}} \prod_{k=1}^{n-1} \delta\left(\left\langle \cdot, \mathbb{1}_{[0,t_k)}\right\rangle - y_k\right)\delta\left(\left\langle \cdot, \mathbb{1}_{[0,t)}\right\rangle - y\right)\, dy_1 \dots dy_{n-1}$$

is a Hida distribution for all $y \in \mathbb{R}$ and the function $y \mapsto \Xi_y$ is Bochner integrable. Furthermore, its S-transform is given by

$$S\left(\Xi_y\right)(g) = \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{1}{2t}\left((\mathbb{1}_{[0,t)}, g) - y_n\right)^2\right),$$

for all $g \in S(\mathbb{R})$.

Proof. Let $n \in \mathbb{N}$, $0 \le t \le T$, $t_k = t\frac{k}{n}$, $k = 1, \dots, n-1$ and $y_1, \dots, y_n \in \mathbb{R}$. Then

$$\prod_{k=1}^{n} \delta\left(\langle \cdot, \mathbb{1}_{[0,t_k)}\rangle - y_k\right)$$

is a Hida distribution (see Lemma 4.12) and its S-transform is given by

$$\begin{aligned} S(\Phi)(g) = &\frac{1}{\sqrt{(2\pi\frac{t}{n})^n}} \\ &\times \exp\left(-\frac{n}{2t}\left(\left((\mathbb{1}_{[0,t_1)}, g) - y_1\right)^2 + \sum_{k=1}^{n-1}\left((\mathbb{1}_{[0,t_k)}, g) - y_k - \left((\mathbb{1}_{[0,t_{k+1})}, g) - y_{k+1}\right)\right)^2\right)\right), \end{aligned}$$

for all $g \in S(\mathbb{R})$, which is obviously integrable w.r.t. $y_1, \dots, y_{n-1}$ (use Theorem 2.12). Integrating w.r.t. the first variable yields:

$$\begin{aligned} \int_{\mathbb{R}^{n-1}} S(\Phi)(g)\, dy_1 \dots dy_{n-1} = &\frac{1}{\sqrt{(2\pi\frac{t}{n})^n}} \int_{\mathbb{R}^{n-1}} \exp\left(-\frac{n}{2t}\left((\mathbb{1}_{[0,t_1)}, g) - y_1\right)^2\right) \\ &\times \exp\left(-\frac{n}{2t}\left(\sum_{k=1}^{n-1}\left((\mathbb{1}_{[0,t_k)}, g) - y_k - \left((\mathbb{1}_{[0,t_{k+1})}, g) - y_{k+1}\right)\right)^2\right)\right) dy_1 \dots dy_{n-1} \\ &= \frac{1}{\sqrt{(2\pi\frac{t}{n})^n}} \int_{\mathbb{R}^{n-1}} \exp\left(-\frac{n}{2t}\left({y_1}^2 + \sum_{k=1}^{n-2}(y_k - y_{k+1})^2\right)\right) \\ &\quad \times \exp\left(-\frac{n}{2t}\left(y_{n-1} - \left((\mathbb{1}_{[0,t_n)}, g) - y_n\right)\right)^2\right) dy_1 \dots dy_{n-1} \\ = &\frac{1}{\sqrt{(2\pi\frac{t}{n})^n}} \sqrt{\pi}\left(\frac{n}{t}\right)^{-1/2} \int_{\mathbb{R}^{n-2}} \exp\left(\frac{1}{4}\frac{n}{t}y_2^2\right) \exp\left(-\frac{n}{2t}\left(\sum_{k=2}^{n-2}(y_k - y_{k+1})^2\right)\right) \\ &\quad \times \exp\left(-\frac{n}{2t}\left(y_{n-1} - \left((\mathbb{1}_{[0,t_n)}, g) - y_n\right)\right)^2\right) dy_2 \dots dy_{n-1}. \end{aligned}$$

Here we use that for $a, b \in \mathbb{R}$, $a > 0$, one has that

$$\int_{\mathbb{R}} \exp\left(-ax^2 - bx\right) dx = \sqrt{\frac{\pi}{a}} \exp\left(\frac{b^2}{4a}\right), \tag{5.16}$$

see e.g. [18]. Inductively one can show that the prefactor a_k in (5.16) for an integration w.r.t. y_k, $k = 2, \dots, n-1$, is given by

$$a_k = \frac{n}{t}\left(1 - \frac{1}{4a_{k-1}}\right) = \frac{n}{t}\frac{k+1}{2k}.$$

Hence, an integration w.r.t. $y_2, \dots, y_{n-1}$ yields

$$\begin{aligned}\int_{\mathbb{R}^{n-1}} S\,(\Phi)\,(g)\,dy_1 \dots dy_{n-1} &= \frac{1}{\sqrt{(2\pi)^n}}\left(\frac{n}{t}\right)^{n/2}\sqrt{\pi^{n-1}}\left(\frac{n}{t}\prod_{k=1}^{n-2}\frac{n}{t}\frac{k+2}{(k+1)2}\right)^{-1/2}\\ &\quad \exp\left(\frac{n}{t}\frac{((\mathbb{1}_{[0,t_n)}, g) - y_n)^2}{4\frac{n-1+1}{(n-1)2}}\right)\exp\left(-\frac{n}{t}\frac{1}{2}((\mathbb{1}_{[0,t_n)}, g) - y_n)^2\right)\\ &= \frac{1}{\sqrt{4\pi t}}\left(n\prod_{k=1}^{n-2}\frac{k+1}{k+2}\right)^{1/2}\exp\left(-\frac{1}{2t}((\mathbb{1}_{[0,t_n)}, g) - y_n)^2\right)\\ &= \frac{1}{\sqrt{2\pi t}}\exp\left(-\frac{1}{2t}((\mathbb{1}_{[0,t_n)}, g) - y_n)^2\right).\end{aligned}$$

□

Remark 5.16. *For sure, Lemma 5.15 just describes the free heat kernel.*

5.3.2 The solution of the complex scaled heat equation simulated by the free heat kernel

As shown in Section 5.2 the unique solution to (5.4) is given by

$$E\left(\exp\left(\frac{1}{z^2}\int_0^t V\left(x_0 + z\langle\cdot, \mathbb{1}_{[0,r)}\rangle\right)dr\right) f\left(x_0 + z\langle\cdot, \mathbb{1}_{[0,t)}\rangle\right)\right),$$

for all $V : \mathcal{D}_z \to \mathbb{C}$ and all $f : \mathbb{C} \to \mathbb{C}$ for which Assumption 5.2 holds, $x_0 \in O$ and $0 < t \le T < \infty$. Let us assume that Assumption 5.6 also holds. Then, close to Theorem 5.13 we can show that

$$\begin{aligned}&\exp\left(\frac{1}{z^2}\int_{t_0}^t V\left(x_0 + z\langle\cdot, \mathbb{1}_{[t_0,r)}\rangle\right)\,dr\right)\delta\left(\langle\cdot, \mathbb{1}_{[t_0,t)}\rangle - y\right)\\ &\quad = \exp\left(\frac{1}{z^2}\int_{t_0}^t V\left(x_0 + z\frac{r-t_0}{t-t_0}y + z\left\langle\cdot, \mathbb{1}_{[t_0,r)} - \frac{r-t_0}{t-t_0}\mathbb{1}_{[t_0,t)}\right\rangle\right)\,dr\right)\delta(\langle\cdot, \mathbb{1}_{[t_0,t)}\rangle - y), \quad (5.17)\end{aligned}$$

for all $x_0 \in O$, $y \in \mathbb{R}$, $0 \le t_0 < t \le T$.

Theorem 5.17. *Let $V : \mathcal{D}_z \to \mathbb{C}$ and all $f : \mathbb{C} \to \mathbb{C}$ fulfill Assumption 5.1 and Assumption 5.6. Moreover, we assume that $|f(x_0 + zy)| \le B_1(x_0)\exp(\varepsilon_1 y^2)$, where $B_1 : \mathbb{R} \to \mathbb{R}$ locally bounded*

and $0 < \varepsilon_1 < \frac{1}{2T}$. *Then for* $x_0 \in O$, $y \in \mathbb{R}$ *and* $0 \le t_0 < t \le T$ *one has the following scaled integral representation:*

$$\begin{aligned} &S\left(\exp\left(\frac{1}{z^2}\int_{t_0}^{t} V\left(x_0 + z\langle\cdot, \mathbb{1}_{[t_0,r)}\rangle\right)\,dr\right) f\left(x_0 + z\langle\cdot, \mathbb{1}_{[t_0,t)}\rangle\right)\right)(g) \\ &\quad = \int_{\mathbb{R}} f(x+zy) S\left(\exp\left(\frac{1}{z^2}\int_{t_0}^{t} V\left(x_0 + z\frac{r-t_0}{t-t_0}y + z\left\langle\cdot, \mathbb{1}_{[t_0,r)} - \frac{r-t_0}{t-t_0}\mathbb{1}_{[t_0,t)}\right\rangle\right)\,dr\right)\right)(g) \\ &\qquad\qquad \times S\left(\delta(\langle\cdot, \mathbb{1}_{[t_0,t)}\rangle - y)\right)(g)\,dy, \quad (5.18) \end{aligned}$$

for all $g \in S(\mathbb{R})$.

Proof. Let $x_0 \in O$, $y \in \mathbb{R}$ and $0 \le t_0 < t \le T$. Then since we assumed that $|f(x_0 + zy)| \le B_1(x_0)\exp(\varepsilon_1 y^2)$, for some locally bounded function $c_2 : \mathbb{R} \to \mathbb{R}$ and $0 < \varepsilon < \frac{1}{2T}$, we obtain that $f\left(x_0 + z\langle\cdot, \mathbb{1}_{[t_0,t)}\rangle\right) \in L^2(\mu)$. Furthermore, by Assumption 5.1 we know that

$$\exp\left(\frac{1}{z^2}\int_{t_0}^{t} V\left(x_0 + z\langle\cdot, \mathbb{1}_{[t_0,r)}\rangle\right)\,dr\right) f\left(x_0 + z\langle\cdot, \mathbb{1}_{[t_0,t)}\rangle\right)$$

is also in $L^2(\mu)$. Thus using Lemma 4.18 and (5.17) we obtain that

$$\begin{aligned} &S\left(\exp\left(\frac{1}{z^2}\int_{t_0}^{t} V\left(x_0 + z\langle\cdot, \mathbb{1}_{[t_0,r)}\rangle\right)\,dr\right) f\left(x_0 + z\langle\cdot, \mathbb{1}_{[t_0,t)}\rangle\right)\right)(g) \\ &\quad = \int_{\mathbb{R}} f(x+zy) S\left(\exp\left(\frac{1}{z^2}\int_{t_0}^{t} V\left(x_0 + z\langle\cdot, \mathbb{1}_{[t_0,r)}\rangle\right)\,dr\right)\delta(\langle\cdot, \mathbb{1}_{[t_0,t)}\rangle - y\right)(g)\,dy \\ &\quad = \int_{\mathbb{R}} f(x+zy) S\left(\exp\left(\frac{1}{z^2}\int_{t_0}^{t} V\left(x_0 + z\frac{r-t_0}{t-t_0}y + z\left\langle\cdot, \mathbb{1}_{[t_0,r)} - \frac{r-t_0}{t-t_0}\mathbb{1}_{[t_0,t)}\right\rangle\right)\,dr\right)\right)(g) \\ &\qquad\qquad \times S\left(\delta(\langle\cdot, \mathbb{1}_{[t_0,t)}\rangle - y)\right)(g)\,dy, \end{aligned}$$

for all $g \in S(\mathbb{R})$. □

Remark 5.18. *(i) Indeed* (5.6) *and* (5.18) *provide us with a solution to the complex scaled heat equation* (5.4) *with help of a real Brownian bridge, as in the non-scaled case. That means one can simulate it by a solution of the classical (i.e. non-scaled) heat equation for a complex potential, see Lemma 5.15. In contrast to this the complex scaled heat kernel described in* (5.8) *is based on a complex scaled Brownian bridge.*

(ii) Examples for such pairs of potentials and initial functions are e.g. potentials as in Example 5.8 and Example 5.9 together with Hermite functions or polynomials.

(iii) There is no hope to extend the set of possible initial functions f *in* (5.18)*, since they must be analytic. Also an approximation of* f *by a sequence of analytic functions* $(f_n)_{n\in\mathbb{N}}$

fulfilling Assumption 5.2 and Assumption 5.6 (of course together with a suitable potential) does not help. This is stated by the fact that convergence of (5.18) *implies that* $(f_n)_{n\in\mathbb{N}}$ *converges locally uniformly. Otherwise, there is no natural indication for interchanging limit and integral. But a sequence of analytic functions converging locally uniformly to a function implies that this function is also analytic. Therefore, the function* f *is analytic. Thus,* f *itself is a possible initial function for* (5.18) *if the integral exists.*

5.3.3 Verifying the scaled heat equation

For showing that the expectation of (5.7) is the integral kernel of the complex scaled heat equation further integrability and differentiability conditions on the potential V are necessary. Note that these conditions are only used to provide the existence and locally boundedness of the first and second derivative w.r.t. $x \in \mathcal{D}_z$. I.e., the derivative exists and expectation and derivation can be interchanged.

Assumption 5.19. *Let* $z \in \overline{S_0}$, $0 < T < \infty$ *and* $V : \mathcal{D}_z \to \mathbb{C}$ *such that Assumption 5.6 is fulfilled. Then we require that there exist a locally bounded function* $C : \mathcal{O} \times \mathcal{O} \to \mathbb{R}$ *and some* $0 < \varepsilon < \frac{1}{8T}$ *such that for all* $x_0, x_1 \in \mathcal{O}$ *and* $y \in \mathbb{R}$ *one has that*

$$\left| V(x_0 + zy) \exp\left(\frac{1}{z^2} V(x_1 + zy) \right) \right| \leq C(x_0, x_1) \exp\left(\varepsilon y^2\right)$$

and

$$\left| V'(x_0 + zy) \exp\left(\frac{1}{z^2} V(x_1 + zy) \right) \right| \leq C(x_0, x_1) \exp\left(\varepsilon y^2\right).$$

Theorem 5.20. *Let* $z \in \overline{S_0}$, $0 < T < \infty$ *and* $V : \mathcal{D}_z \to \mathbb{C}$ *such that Assumption 5.19 is fulfilled, then*

$$K(x, t; x_0, t_0) := E\left(\exp\left(\frac{1}{z^2} \int_{t_0}^{t} V\Big(x_0 + z\langle \cdot, \mathbb{1}_{[t_0,r)} \rangle\Big) dr \right) \sigma_z \delta\left(\langle \cdot, \mathbb{1}_{[t_0,t)} \rangle - (x - x_0) \right) \right) \tag{5.19}$$

solves the scaled heat equation for all $x, x_0 \in \mathcal{O}$, $0 < t_0 < t < T$, *such that* $x_0 + \frac{r-t_0}{t-t_0}(x - x_0) \in \mathcal{O}$, $t_0 \leq r \leq t$, *i.e.,*

$$\left(\frac{\partial}{\partial t} - z^2 \frac{1}{2} \Delta - \frac{1}{z^2} V(x) \right) K(x, t; x_0, t_0) = 0, \tag{5.20}$$

with initial condition

$$\lim_{t \searrow t_0} K(x, t; x_0, t_0) = \delta(x - x_0).$$

Remark 5.21. *Note that for* $z = \sqrt{i}$ *we solve the Schrödinger equation.*

Proof of Theorem 5.20. Let $x, x_0 \in O$, $0 < t_0 < t < T$, such that $x_0 + \frac{r-t_0}{t-t_0}(x - x_0) \in O$, for all $t_0 \leq r \leq t$, and $V : D_z \to \mathbb{C}$ such that Assumption 5.19 is fulfilled, then we know from Theorem 5.13 that (5.19) is given by

$$K(x,t;x_0,t_0) = E\left(\exp\left(\frac{1}{z^2}\int_{t_0}^{t} V\Big(x_0 + \frac{r-t_0}{t-t_0}(x-x_0) + z\langle\cdot, \mathbb{1}_{[t_0,r)}\rangle - \frac{r-t_0}{t-t_0} z\langle\cdot, \mathbb{1}_{[t_0,t)}\rangle\Big)\,dr\right)\right)$$
$$\times E\left(\sigma_z\delta\left(\langle\cdot, \mathbb{1}_{[t_0,t)}\rangle - (x-x_0)\right)\right).$$

Hence, the time derivative of $K(x,t;x_0,t_0)$ is given by

$$\begin{aligned}\frac{\partial}{\partial t}K(x,t;x_0,t_0) &= \left(\frac{\partial}{\partial t}E\left(\sigma_z\delta\left(\langle\cdot, \mathbb{1}_{[t_0,t)}\rangle - (x-x_0)\right)\right)\right)\\ &\quad\times E\left(\exp\left(\frac{1}{z^2}\int_{t_0}^{t} V\Big(x_0 + \frac{r-t_0}{t-t_0}(x-x_0) + z\langle\cdot, \mathbb{1}_{[t_0,r)}\rangle - \frac{r-t_0}{t-t_0} z\langle\cdot, \mathbb{1}_{[t_0,t)}\rangle\Big)\,dr\right)\right)\\ &\qquad + E\left(\sigma_z\delta\left(\langle\cdot, \mathbb{1}_{[t_0,t)}\rangle - (x-x_0)\right)\right)\\ &\quad\times \frac{\partial}{\partial t}E\left(\exp\left(\frac{1}{z^2}\int_{t_0}^{t} V\Big(x_0 + \frac{r-t_0}{t-t_0}(x-x_0) + z\langle\cdot, \mathbb{1}_{[t_0,r)}\rangle - \frac{r-t_0}{t-t_0} z\langle\cdot, \mathbb{1}_{[t_0,t)}\rangle\Big)\,dr\right)\right). \qquad (5.21)\end{aligned}$$

From the free case (see e.g. [74]) we know that

$$\frac{\partial}{\partial t}E\left(\sigma_z\delta\left(\langle\cdot, \mathbb{1}_{[t_0,t)}\rangle - (x-x_0)\right)\right) = z^2\frac{1}{2}\Delta E\left(\sigma_z\delta\left(\langle\cdot, \mathbb{1}_{[t_0,t)}\rangle - (x-x_0)\right)\right).$$

From now on we denote $B_{t_0,t} := \langle\cdot, \mathbb{1}_{[t_0,t)}\rangle$ to get shorter equations. To consider the time derivative of the second part we first have a look at the difference quotient from the right side:

$$\begin{aligned}&\frac{\partial}{\partial t}^{+} E\left(\exp\left(\frac{1}{z^2}\int_{t_0}^{t} V\Big(x_0 + \frac{r-t_0}{t-t_0}(x-x_0) + zB_{t_0,r} - \frac{r-t_0}{t-t_0} zB_{t_0,t}\Big)\,dr\right)\right)\\ &= \lim_{h\searrow 0}\frac{1}{h}\Bigg(E\left(\exp\left(\frac{1}{z^2}\int_{t_0}^{t+h} V\Big(x_0 + \frac{r-t_0}{t+h-t_0}(x-x_0) + zB_{t_0,r} - \frac{r-t_0}{t+h-t_0} zB_{t_0,t+h}\Big)\,dr\right)\right)\\ &\qquad - E\left(\exp\left(\frac{1}{z^2}\int_{t_0}^{t} V\Big(x_0 + \frac{r-t_0}{t-t_0}(x-x_0) + zB_{t_0,r} - \frac{r-t_0}{t-t_0} zB_{t_0,t}\Big)\,dr\right)\right)\Bigg)\end{aligned}$$

Let $h > 0$ such that $t + h \leq T$ then we get by adding zero that

$$\begin{aligned}&\frac{1}{h}\Bigg[E\left(\exp\left(\frac{1}{z^2}\int_{t_0}^{t+h} V\left(x_0 + \frac{r-t_0}{t-t_0+h}(x-x_0) + zB_{t_0,r} - \frac{r-t_0}{t-t_0+h} zB_{t_0,t+h}\right)\,dr\right)\right)\\ &\qquad - E\left(\exp\left(\frac{1}{z^2}\int_{t_0}^{t} V\left(x_0 + \frac{r-t_0}{t-t_0}(x-x_0) + zB_{t_0,r} - \frac{r-t_0}{t-t_0} zB_{t_0,t}\right)\,dr\right)\right)\Bigg]\\ &= \frac{1}{h}\Bigg[E\left(\exp\left(\frac{1}{z^2}\int_{t_0}^{t+h} V\left(x_0 + \frac{r-t_0}{t-t_0+h}(x-x_0) + zB_{t_0,r} - \frac{r-t_0}{t-t_0+h} zB_{t_0,t+h}\right)\,dr\right)\right)\end{aligned}$$

$$
\begin{aligned}
&- E\left(\exp\left(\frac{1}{z^2}\int_{t_0}^{t} V\left(x_0 + \frac{r-t_0}{t-t_0+h}(x-x_0) + zB_{t_0,r} - \frac{r-t_0}{t-t_0+h} zB_{t_0,t+h}\right) dr\right)\right)\\
&+ E\left(\exp\left(\frac{1}{z^2}\int_{t_0}^{t} V\left(x_0 + \frac{r-t_0}{t-t_0+h}(x-x_0) + zB_{t_0,r} - \frac{r-t_0}{t-t_0+h} zB_{t_0,t+h}\right) dr\right)\right)\\
&\quad - E\left(\exp\left(\frac{1}{z^2}\int_{t_0}^{t} V\left(x_0 + \frac{r-t_0}{t-t_0}(x-x_0) + zB_{t_0,r} - \frac{r-t_0}{t-t_0} zB_{t_0,t+h}\right) dr\right)\right)\\
&\quad + E\left(\exp\left(\frac{1}{z^2}\int_{t_0}^{t} V\left(x_0 + \frac{r-t_0}{t-t_0}(x-x_0) + zB_{t_0,r} - \frac{r-t_0}{t-t_0} zB_{t_0,t+h}\right) dr\right)\right)\\
&\qquad - E\left(\exp\left(\frac{1}{z^2}\int_{t_0}^{t} V\left(x_0 + \frac{r-t_0}{t-t_0}(x-x_0) + zB_{t_0,r} - \frac{r-t_0}{t-t_0} zB_{t_0,t}\right) dr\right)\right)\Bigg]. \qquad (5.22)
\end{aligned}
$$

Now we consider it as three difference quotients, separately. For the last two summands of (5.22) we get that

$$
\begin{aligned}
&\lim_{h\searrow 0}\frac{1}{h}\Bigg[E\left(\exp\left(\frac{1}{z^2}\int_{t_0}^{t} V\left(x_0 + \frac{r-t_0}{t-t_0}(x-x_0) + zB_{t_0,r} - \frac{r-t_0}{t-t_0} zB_{t_0,t+h}\right) dr\right)\right)\\
&\qquad - E\left(\exp\left(\frac{1}{z^2}\int_{t_0}^{t} V\left(x_0 + \frac{r-t_0}{t-t_0}(x-x_0) + zB_{t_0,r} - \frac{r-t_0}{t-t_0} zB_{t_0,t}\right) dr\right)\right)\Bigg]\\
&= \frac{\partial}{\partial u}^{+} E\left(\exp\left(\frac{1}{z^2}\int_{t_0}^{t} V\left(x_0 + \frac{r-t_0}{t-t_0}(x-x_0) + zB_{t_0,r} - \frac{r-t_0}{t-t_0} zB_{t_0,u}\right) dr\right)\right)\Bigg|_{u=t}\\
&\qquad = \frac{\partial}{\partial u}\int_{S'(\mathbb{R})} \exp\left(\frac{1}{z^2}\int_{t_0}^{t} V\Bigg(x_0 + \frac{r-t_0}{t-t_0}(x-x_0) + zB_{t_0,r}(\omega)\right.\\
&\qquad\qquad \left. - \frac{r-t_0}{t-t_0} zB_{t_0,t}(\omega) - \frac{r-t_0}{t-t_0} zB_{t,u}(\omega)\Bigg) dr\right) d\mu(\omega)\Bigg|_{u=t}\\
&\qquad = \frac{\partial}{\partial u}\int_{S'(\mathbb{R})}\int_{S'(\mathbb{R})} \exp\left(\frac{1}{z^2}\int_{t_0}^{t} V\Bigg(x_0 + \frac{r-t_0}{t-t_0}(x-x_0) + zB_{t_0,r}(\omega_1)\right.\\
&\qquad\qquad \left. - \frac{r-t_0}{t-t_0} zB_{t_0,t}(\omega_1) - \frac{r-t_0}{t-t_0} zB_{t,u}(\omega_2)\Bigg) dr\right) d\mu(\omega_2) d\mu(\omega_1)\Bigg|_{u=t}.
\end{aligned}
$$

Here we use the fact that $B_{t_0,r}$ and $B_{t,t+h}$ are independent for all $t_0 \le r \le t$. Therefore, we get by Corollary 5.4 and since V is analytic that

$$
\begin{aligned}
&\frac{\partial}{\partial u}\int_{S'(\mathbb{R})} \exp\left(\frac{1}{z^2}\int_{t_0}^{t} V\Bigg(x_0 + \frac{r-t_0}{t-t_0}(x-x_0) + zB_{t_0,r}(\omega_1)\right.\\
&\qquad\qquad \left. - \frac{r-t_0}{t-t_0} zB_{t_0,t}(\omega_1) - \frac{r-t_0}{t-t_0} zB_{t,u}(\omega_2)\Bigg) dr\right) d\mu(\omega_2)\Bigg|_{u=t}\\
&= z^2\frac{1}{2}\frac{\partial^2}{\partial y^2}\int_{S'(\mathbb{R})} \exp\left(\frac{1}{z^2}\int_{t_0}^{t} V\left(x_0 + zB_{t_0,r}(\omega_1) - \frac{r-t_0}{t-t_0}(y + zB_{t,u}(\omega_2))\right) dr\right) d\mu(\omega_2)\Bigg|_{y=c(x,x_0,\omega_1)}\\
&\qquad = z^2\frac{1}{2}\frac{\partial^2}{\partial x^2}\int_{S'(\mathbb{R})} \exp\left(\frac{1}{z^2}\int_{t_0}^{t} V\Bigg(x_0 + \frac{r-t_0}{t-t_0}(x-x_0) + zB_{t_0,r}(\omega_1)\right.
\end{aligned}
$$

$$- \frac{r-t_0}{t-t_0} zB_{t_0,t}(\omega_1) - \frac{r-t_0}{t-t_0} zB_{t,u}(\omega_2)\Bigg) dr\Bigg) d\mu(\omega_2)$$

for μ- almost every $\omega_1 \in S'(\mathbb{R})$, where $c(x, x_0, \omega_1) = -x + x_0 + zB_{t_0,t}(\omega_1)$. Hence ,

$$\frac{\partial}{\partial u}^{+} E\left(\exp\left(\frac{1}{z^2}\int_{t_0}^{t} V\left(x_0 + \frac{r-t_0}{t-t_0}(x-x_0) + zB_{t_0,r} - \frac{r-t_0}{t-t_0} zB_{t_0,u}\right) dr\right)\right)\Bigg|_{u=t}$$
$$= z^2 \frac{1}{2}\Delta E\left(\exp\left(\frac{1}{z^2}\int_{0}^{t-t_0} V\left(x_0 + \frac{r-t_0}{t-t_0}(x-x_0) + zB_{t_0,r} - \frac{r-t_0}{t-t_0} zB_{t_0,t}\right) dr\right)\right).$$

For the first two summands of (5.22) we get that

$$\frac{1}{h}\Bigg[E\left(\exp\left(\frac{1}{z^2}\int_{t_0}^{t+h} V\left(x_0 + \frac{r-t_0}{t-t_0+h}(x-x_0) + zB_{t_0,r} - \frac{r-t_0}{t-t_0+h} zB_{t_0,t+h}\right) dr\right)\right)$$
$$- E\left(\exp\left(\frac{1}{z^2}\int_{t_0}^{t} V\left(x_0 + \frac{r-t_0}{t-t_0+h}(x-x_0) + zB_{t_0,r} - \frac{r-t_0}{t-t_0+h} zB_{t_0,t+h}\right) dr\right)\right)\Bigg]$$
$$= \frac{1}{h}\Bigg[E\Bigg(\exp\left(\frac{1}{z^2}\int_{t_0}^{t} V\left(x_0 + \frac{r-t_0}{t-t_0+h}(x-x_0) + zB_{0,r} - \frac{r-t_0}{t-t_0+h} zB_{t_0,t+h}\right) dr\right)$$
$$\left(\exp\left(\frac{1}{z^2}\int_{t}^{t+h} V\left(x_0 + \frac{r-t_0}{t-t_0+h}(x-x_0) + zB_{t_0,r} - \frac{r-t_0}{t-t_0+h} zB_{t_0,t+h}\right) dr\right) - 1\right)\Bigg)\Bigg].$$

Hence, by Assumption 5.19 and the fact that $u \mapsto B_{t,u}$ is continuous on $[t, T]$ the limit is given by

$$\lim_{h\searrow 0}\frac{1}{h}\Bigg[E\left(\exp\left(\frac{1}{z^2}\int_{t_0}^{t+h} V\left(x_0 + \frac{r-t_0}{t-t_0+h}(x-x_0) + zB_{t_0,r} - \frac{r-t_0}{t-t_0+h} zB_{t_0,t+h}\right) dr\right)\right)$$
$$- E\left(\exp\left(\frac{1}{z^2}\int_{t_0}^{t} V\left(x_0 + \frac{r-t_0}{t-t_0+h}(x-x_0) + zB_{t_0,r} - \frac{r-t_0}{t-t_0+h} zB_{t_0,t+h}\right) dr\right)\right)\Bigg]$$
$$= \frac{1}{z^2}V(x)E\left(\exp\left(\frac{1}{z^2}\int_{t_0}^{t} V\left(x_0 + \frac{r-t_0}{t-t_0}(x-x_0) + zB_{t_0,r} - \frac{r-t_0}{t-t_0} zB_{t_0,t}\right) dr\right)\right).$$

For the remaining two summands of (5.22) it is easy to see with Assumption 5.19 and the mean value theorem that there exist some $s \in (t, t+h)$ such that

$$\frac{1}{h}\Bigg[E\left(\exp\left(\frac{1}{z^2}\int_{t_0}^{t} V\left(x_0 + \frac{r-t_0}{t-t_0+h}(x-x_0) + zB_{t_0,r} - \frac{r-t_0}{t-t_0+h} zB_{t_0,t+h}\right) dr\right)\right)$$
$$- E\left(\exp\left(\frac{1}{z^2}\int_{t_0}^{t} V\left(x_0 + \frac{r-t_0}{t-t_0}(x-x_0) + zB_{t_0,r} - \frac{r-t_0}{t-t_0} zB_{t_0,t+h}\right) dr\right)\right)\Bigg]$$
$$= \frac{\partial}{\partial u}E\left(\exp\left(\frac{1}{z^2}\int_{t_0}^{t} V\left(x_0 + \frac{r-t_0}{u-t_0}(x-x_0) + zB_{t_0,r} - \frac{r-t_0}{u-t_0} zB_{t_0,t+h}\right) dr\right)\right)\Bigg|_{u=s}$$
$$= E\left(\frac{\partial}{\partial u}\left(\frac{1}{z^2}\int_{t_0}^{t} V\left(x_0 + \frac{r-t_0}{u-t_0}(x-x_0) + zB_{t_0,r} - \frac{r-t_0}{u-t_0} zB_{t_0,t+h}\right) dr\right)\right)\Bigg|_{u=s}$$

$$\times \exp\left(\frac{1}{z^2}\int_{t_0}^{t} V\left(x_0 + \frac{r-t_0}{s-t_0}(x-x_0) + zB_{t_0,r} - \frac{r-t_0}{s-t_0} zB_{t_0,t+h}\right) dr\right)\Bigg)$$

$$= E\Bigg(\frac{1}{z^2}\int_{t_0}^{t} V'\left(x_0 + \frac{r-t_0}{s-t_0}(x-x_0) + zB_{t_0,r} - \frac{r-t_0}{s-t_0} zB_{t_0,t+h}\right)\frac{-(r-t_0)}{(s-t_0)^2}(x - x_0 + zB_{t_0,t+h})\, dr$$

$$\times \exp\left(\frac{1}{z^2}\int_{t_0}^{t} V\left(x_0 + \frac{r-t_0}{s-t_0}(x-x_0) + zB_{t_0,r} - \frac{r-t_0}{s-t_0} zB_{t_0,t+h}\right) dr\right)\Bigg),$$

and therefore the limit is given by

$$\lim_{h\searrow 0}\frac{1}{h}\Bigg[E\left(\exp\left(\frac{1}{z^2}\int_{t_0}^{t} V\left(x_0 + \frac{r-t_0}{t-t_0+h}(x-x_0) + zB_{t_0,r} - \frac{r-t_0}{t-t_0+h} zB_{t_0,t+h}\right) dr\right)\right)$$

$$- E\left(\exp\left(\frac{1}{z^2}\int_{t_0}^{t} V\left(x_0 + \frac{r-t_0}{t-t_0}(x-x_0) + zB_{t_0,r} - \frac{r-t_0}{t-t_0} zB_{t_0,t+h}\right) dr\right)\right)\Bigg]$$

$$= E\Bigg(\frac{1}{z^2}\int_{t_0}^{t} V'\left(x_0 + \frac{r-t_0}{t-t_0}(x-x_0) + zB_{t_0,r} - \frac{r-t_0}{(t-t_0)} zB_{t_0,t}\right)\frac{-(r-t_0)}{(t-t_0)^2}(x - x_0 + zB_{t_0,t})\, dr$$

$$\times \exp\left(\frac{1}{z^2}\int_{t_0}^{t} V\left(x_0 + \frac{r-t_0}{t-t_0}(x-x_0) + zB_{t_0,r} - \frac{r-t_0}{t-t_0} zB_{t_0,t}\right) dr\right)\Bigg).$$

On the one side, similar as in the proof of Corollary 4.17 and Corollary 5.14 one can show that the random variables $B_{t_0,t}$ and

$$\frac{1}{z^2}\int_{t_0}^{t} V'\left(x_0 + \frac{r-t_0}{t-t_0}(x-x_0) + zB_{t_0,r} - \frac{r-t_0}{t-t_0} zB_{t_0,t}\right)$$

$$\times \exp\left(\frac{1}{z^2}\int_{t_0}^{t} V\left(x_0 + \frac{r-t_0}{t-t_0}(x-x_0) + zB_{t_0,r} - \frac{r-t_0}{t-t_0} zB_{t_0,t}\right) dr\right)$$

are independent. Hence, since $E(B_{t_0,t}) = 0$ we get that

$$E\Bigg(zB_{t_0,t}\frac{1}{z^2}\int_{t_0}^{t} V'\left(x_0 + \frac{r-t_0}{t-t_0}(x-x_0) + zB_{t_0,r} - \frac{r-t_0}{t-t_0} zB_{t_0,t}\right)\frac{-(r-t_0)}{(t-t_0)^2}\, dr$$

$$\times \exp\left(\frac{1}{z^2}\int_{t_0}^{t} V\left(x_0 + \frac{r-t_0}{t-t_0}(x-x_0) + zB_{t_0,r} - \frac{r-t_0}{t-t_0} zB_{t_0,t}\right) dr\right)\Bigg) = 0.$$

On the other side we get the cross terms by

$$E\left(\sigma_z \delta\left(\langle \cdot, \mathbb{1}_{[t_0,t)}\rangle - (x - x_0)\right)\right)$$

$$\times E\Bigg(\frac{1}{z^2}\int_{t_0}^{t} V'\left(x_0 + \frac{r-t_0}{t-t_0}(x-x_0) + zB_{t_0,r} - \frac{r-t_0}{t-t_0} zB_{t_0,t}\right)\frac{-(r-t_0)}{(t-t_0)^2}(x - x_0)\, dr$$

$$\times \exp\left(\frac{1}{z^2}\int_{t_0}^{t} V\left(x_0 + \frac{r-t_0}{t-t_0}(x-x_0) + zB_{t_0,r} - \frac{r-t_0}{t-t_0} zB_{t_0,t}\right) dr\right)\Bigg)$$

$$= (-1)\frac{(x-x_0)}{t-t_0} E\left(\sigma_z \delta\left(\langle \cdot, \mathbb{1}_{[t_0,t)}\rangle - (x - x_0)\right)\right)$$

$$E\left(\frac{\partial}{\partial x}\exp\left(\frac{1}{z^2}\int_{t_0}^{t} V\left(x_0+\frac{r-t_0}{t-t_0}(x-x_0)+zB_{t_0,r}-\frac{r-t_0}{t-t_0}zB_{t_0,t}\right)dr\right)\right)$$
$$= z^2\frac{\partial}{\partial x}E\left(\sigma_z\delta\left(\langle\cdot,\mathbb{1}_{[t_0,t)}\rangle-(x-x_0)\right)\right)$$
$$\times\frac{\partial}{\partial x}E\left(\exp\left(\frac{1}{z^2}\int_{t_0}^{t} V\left(x_0+\frac{r-t_0}{t-t_0}(x-x_0)+zB_{t_0,r}-\frac{r-t_0}{t-t_0}zB_{t_0,t}\right)dr\right)\right).$$

Summing up again yields

$$\frac{\partial}{\partial t}^{+}K(x,t;x_0,t_0) = -z^2\frac{1}{2}\Delta K(x,t;x_0,t_0)+\frac{1}{z^2}V(x)K(x,t;x_0,t_0).$$

Analogously one can show that (5.20) is also true for $\frac{\partial}{\partial t}^{-}K(x,t;x_0,t_0)$. Note that

$$\lim_{t\searrow t_0}K(x,t;x_0,t_0)=\delta(x-x_0),$$

since

$$\lim_{t\searrow t_0}E\left(\exp\left(\frac{1}{z^2}\int_{t_0}^{t} V\left(x_0+\frac{r-t_0}{t-t_0}(x-x_0)+zB_{t_0,r}-\frac{r-t_0}{t-t_0}zB_{t_0,t}\right)dr\right)\right)=1$$

and

$$E\left(\sigma_z\delta\left(\langle\cdot,\mathbb{1}_{[t_0,t)}\rangle-(x-x_0)\right)\right)$$

describes a Dirac sequence in $t-t_0$ (see its T-transform in Example 2.14). □

5.3.4 The complex scaled heat kernel as an Integral operator

In this section we are interested in the solution of (5.4) to more general initial states f, e.g. $f\in L^2(\mathbb{R})\cap C_0^\infty(\Omega)$, where $C_0^\infty(\Omega)$ denotes the set of arbitrary often differentiable functions with compact support in $\Omega\subset\mathcal{O}$. We consider f as a function defined on $\mathbb{R}$ by extending it on $\mathbb{R}\setminus\Omega$ by zero.

Theorem 5.22. *Let $V:D_z\to\mathbb{C}$ such that Assumption 5.6 is fulfilled and $f\in L^2(\mathbb{R})\cap C_0^\infty(\Omega)$. Here $\Omega\subset\mathcal{O}$ connected and open with the property that for all $x_0\in\mathcal{O}$ and all $y\in\Omega$ it follows that $x_0+\frac{r-t_0}{t-t_0}(y-x_0)\in\mathcal{O}$, for all $0\le t_0\le r\le t<T<\infty$. Then*

$$\int_{\mathbb{R}} f(y)S\left(\sigma_z\delta\left(\langle\cdot,\mathbb{1}_{[t_0,t)}\rangle-y\right)\right)(g)$$
$$\times S\left(\exp\left(\frac{1}{z^2}\int_{t_0}^{t} V\Big(x_0+\frac{r-t_0}{t-t_0}(y-x_0)+z\langle\cdot,\mathbb{1}_{[t_0,r)}\rangle-\frac{r-t_0}{t-t_0}z\langle\cdot,\mathbb{1}_{[t_0,t)}\rangle\Big)dr\right)\right)(g)\,dy, \quad (5.23)$$

exists for all $g\in S(\mathbb{R})$. In addition, if $g=0$ then (5.23) solves the complex scaled heat equation (5.4).

Proof. Let $V : \mathcal{D}_z \to \mathbb{C}$ such that Assumption 5.6 is fulfilled, $f \in L^2(\mathbb{R}) \cap C_0^\infty(\Omega)$, $x_0 \in \mathcal{O}$ and $0 \le t_0 < t < T < \infty$. Since $\Omega \subset \mathcal{O}$ fulfills the property that for all $x_0 \in \mathcal{O}$ and all $y \in \Omega$ it follows that $x_0 + \frac{r-t_0}{t-t_0}(y - x_0) \in \mathcal{O}$, for all $t_0 < r < t < T < \infty$, we get by Theorem 5.13 that

$$\exp\left(\frac{1}{z^2}\int_{t_0}^{t} V\Big(x_0 + \frac{r-t_0}{t-t_0}(y-x_0) + z\langle\cdot, \mathbb{1}_{[t_0,r)}\rangle - \frac{r-t_0}{t-t_0} z\langle\cdot, \mathbb{1}_{[t_0,t)}\rangle\Big)\,dr\right) \diamond \sigma_z\delta\left(\langle\cdot, \mathbb{1}_{[t_0,t)}\rangle - y\right)$$

is a well-defined Hida distribution. By continuity of the potential and continuity of the function $y \mapsto S\left(\sigma_z\delta\left(\langle\cdot, \mathbb{1}_{[t_0,t)}\rangle - y\right)\right)(g)$, $g \in S(\mathbb{R})$, also

$$\begin{aligned} y \mapsto\ & S\left(\sigma_z\delta\left(\langle\cdot, \mathbb{1}_{[t_0,t)}\rangle - y\right)\right)(g) \\ & \times S\left(\exp\left(\frac{1}{z^2}\int_{t_0}^{t} V\Big(x_0 + \frac{r-t_0}{t-t_0}(y-x_0) + z\langle\cdot, \mathbb{1}_{[t_0,r)}\rangle - \frac{r-t_0}{t-t_0} z\langle\cdot, \mathbb{1}_{[t_0,t)}\rangle\Big)dr\right)\right)(g) \end{aligned}$$

is continuous for all $g \in S(\mathbb{R})$ and therefore locally bounded. Since $f \in L^2(\mathbb{R}) \cap C_0^\infty(\Omega)$ we get that (5.23) exists for all $g \in S(\mathbb{R})$. The fact that the complex scaled heat equation is solved by (5.23) for $g = 0$, follows directly by Theorem 5.20 and Lebesgue dominated convergence. □

Remark 5.23. *Theorem 5.22 implies that the S-transform, evaluated at $g \in S(\mathbb{R})$, of the complex scaled heat kernel is in the distribution space $D'(\Omega)$, see [12] or [69] for its definition and characteristic properties.*

5.4 Wick product representation of the generalized scaled heat kernel

Of course, one is not only interested in analytic potentials. In this section we extend Theorem 5.13 to potentials where the exponential part of the potentials, see ϕ in Proposition 5.12, are regular generalized functions of white noise. To consider also higher space dimensions for $\mathcal{O} \subset \mathbb{R}^d$, such that $\mathbb{R}^d \setminus \mathcal{O}$ is a set of Lebesgue measure zero, and $z \in \mathbb{C} \setminus \{0\}$ we define the set $\mathcal{D}_z \subset \mathbb{C}$ by

$$\mathcal{D}_{z,d} := \left\{x + zy \,\middle|\, x \in \mathcal{O} \text{ and } y \in \mathbb{R}^d\right\}, \tag{5.24}$$

in analogy to (5.5).

Theorem 5.24. *Let $0 < T < \infty$, $z \in \overline{S_0}$ and $V : \mathcal{D}_{z,d} \subset \mathbb{R}^d \to \mathbb{C}$ be a potential such that*

$$\phi := \exp\left(\frac{1}{z^2}\int_{t_0}^{t} V\left(x_0 + \langle\cdot, \mathbb{1}_{[t_0,r)}\rangle\right)\,dr\right)$$

and

$$\sigma_z\phi = \exp\left(\frac{1}{z^2}\int_{t_0}^{t} V\left(x_0 + z\langle\cdot, \mathbb{1}_{[t_0,r)}\rangle\right)\,dr\right)$$

are well-defined objects in $C_{\frac{1}{\sqrt{t-t_0}}\vec{\mathbb{1}}_{[t_0,t)},\frac{x-x_0}{\sqrt{t-t_0}}}$ *and* $C_{\frac{1}{\sqrt{t-t_0}}\vec{\mathbb{1}}_{[t_0,t)},\frac{x-x_0}{z\sqrt{t-t_0}}}$ *(see Definition 4.26), respectively, for all* $0 \le t_0 < t \le T$ *and* $x_0, x \in O$*, such that* $x_0 + \frac{r-t_0}{t-t_0}x \in O$*,* $t_0 \le r \le t$*. Then*

$$\Phi := \left(\exp\left(\frac{1}{z^2}\int_{t_0}^{t} V\left(x_0 + \langle\cdot, \mathbb{1}_{[t_0,r)}\rangle\right)\,dr\right)\delta^d(\langle\cdot, \mathbb{1}_{[t_0,t)}\rangle - (x - x_0))\right)$$

is a well-defined Hida distribution. Furthermore, $\sigma_z\Phi$ *is a Hida distribution and can be represent as follows*

$$\begin{aligned}\sigma_z\Phi &= \sigma_z\tau_{\frac{x-x_0}{(t-t_0)}\mathbb{1}_{[t_0,t)}}P_{\frac{1}{\sqrt{t-t_0}}\vec{\mathbb{1}}_{[t_0,t)}}\exp\left(\frac{1}{z^2}\int_{t_0}^{t} V\left(x_0 + z\langle\cdot, \mathbb{1}_{[t_0,r)}\rangle\right)\,dr\right)\diamond\sigma_z\delta^d(\langle\cdot, \mathbb{1}_{[t_0,t)}\rangle - (x - x_0))\\ &= \tau_{\frac{x-x_0}{z(t-t_0)}\mathbb{1}_{[t_0,t)}}P_{\frac{1}{\sqrt{t-t_0}}\vec{\mathbb{1}}_{[t_0,t)}}\sigma_z\exp\left(\frac{1}{z^2}\int_{t_0}^{t} V\left(x_0 + z\langle\cdot, \mathbb{1}_{[t_0,r)}\rangle\right)\,dr\right)\diamond\sigma_z\delta^d(\langle\cdot, \mathbb{1}_{[t_0,t)}\rangle - (x - x_0)), \quad (5.25)\end{aligned}$$

where the Wick product inside is a projective independent pointwise product in the sense of Definition 4.8. Furthermore, if V *is continuous then*

$$\sigma_z\phi = \sigma_z\exp\left(\frac{1}{z^2}\int_{t_0}^{t} V\left(x_0 + \frac{r-t_0}{t-t_0}(x-x_0) + \left(\langle\cdot, \mathbb{1}_{[t_0,r)}\rangle - \frac{r-t_0}{t-t_0}\langle\cdot, \mathbb{1}_{[t_0,t)}\rangle\right)\right)dr\right)$$
$$\diamond\sigma_z\delta^d(\langle\cdot, \mathbb{1}_{[t_0,t)}\rangle - (x - x_0)).$$

Here only the one dimensional case is proven. The d-dimensional follows by a componentwise consideration of the path.

Proof. Let $V : \mathcal{D}_z \subset \mathbb{R} \to \mathbb{C}$ be a potential $0 \le t_0 \le t \le T$ and $x, x_0 \in O$ such that ϕ and $\sigma_z\phi$ are in $C_{\frac{1}{\sqrt{t-t_0}}1_{[t_0,t)},\frac{x-x_0}{\sqrt{t-t_0}}}$ and $C_{\frac{1}{\sqrt{t-t_0}}1_{[t_0,t)},\frac{x-x_0}{z\sqrt{t-t_0}}}$, respectively. Then by Theorem 4.25 we get that the generalized function

$$\begin{aligned}\Phi &:= \exp\left(\frac{1}{z^2}\int_{t_0}^{t} V(x_0 + \langle\cdot, \mathbb{1}_{[t_0,r)}\rangle)\,dr\right)\delta(\langle\cdot, \mathbb{1}_{[t_0,t)}\rangle + x_0 - x)\\ &= \sigma_z\left(\tau_{\frac{x-x_0}{(t-t_0)}\mathbb{1}_{[t_0,t)}}P_{\frac{1}{\sqrt{t-t_0}}\mathbb{1}_{[t_0,t)}}\exp\left(\frac{1}{z^2}\int_{t_0}^{t} V\left(x_0 + \langle\cdot, \mathbb{1}_{[t_0,r)}\rangle\right)\,dr\right)\diamond\delta(\langle\cdot, \mathbb{1}_{[t_0,t)}\rangle - (x - x_0))\right) \in \mathcal{G}'.\end{aligned}$$

Applying once more Theorem 4.25 we get that

$$\begin{aligned}\Psi &:= \sigma_z\exp\left(\frac{1}{z^2}\int_{t_0}^{t} V(x_0 + \langle\cdot, \mathbb{1}_{[t_0,r)}\rangle)\,dr\right)\sigma_z\delta(\langle\cdot, \mathbb{1}_{[t_0,t)}\rangle + x_0 - x)\\ &= \tau_{\frac{x-x_0}{z(t-t_0)}\mathbb{1}_{[t_0,t)}}P_{\frac{1}{\sqrt{t-t_0}}\mathbb{1}_{[t_0,t)}}\sigma_z\exp\left(\frac{1}{z^2}\int_{t_0}^{t} V\left(x_0 + z\langle\cdot, \mathbb{1}_{[t_0,r)}\rangle\right)\,dr\right)\diamond\sigma_z\delta(\langle\cdot, \mathbb{1}_{[t_0,t)}\rangle - (x - x_0)) \in \mathcal{G}'.\end{aligned}$$

Of course, in both formulas above the Wick products are independent pointwise products. Moreover, see Remark 4.2, it follows that $\sigma_z\Phi = \Psi$. Let us assume that V is continuous. Then, since

$$\tau_{\frac{x-x_0}{(t-t_0)}\mathbf{1}_{[t_0,t)}} P_{\frac{1}{\sqrt{t-t_0}}\mathbf{1}_{[t_0,t)}} \left(x_0 + \langle \cdot, \mathbb{1}_{[t_0,r)}\rangle\right) = x_0 + \frac{r-t_0}{t-t_0}(x-x_0) + \langle \cdot, \mathbb{1}_{[t_0,r)}\rangle - \frac{r-t_0}{t-t_0}\langle \cdot, \mathbb{1}_{[t_0,t)}\rangle$$

and

$$\tau_{\frac{x-x_0}{z(t-t_0)}\mathbf{1}_{[t_0,t)}} P_{\frac{1}{\sqrt{t-t_0}}\mathbf{1}_{[t_0,t)}} \left(x_0 + z\langle \cdot, \mathbb{1}_{[t_0,r)}\rangle\right) = x_0 + \frac{r-t_0}{t-t_0}(x-x_0) + z\langle \cdot, \mathbb{1}_{[t_0,r)}\rangle - z\frac{r-t_0}{t-t_0}\langle \cdot, \mathbb{1}_{[t_0,t)}\rangle$$

we get by Remark 4.29 and the continuity of V, exp and the integral that

$$\begin{aligned}\Psi = \sigma_z \exp\left(\frac{1}{z^2}\int_{t_0}^{t} V\left(x_0 + \frac{r-t_0}{t-t_0}(x-x_0) + \left(\langle \cdot, \mathbb{1}_{[t_0,r)}\rangle - \frac{r-t_0}{t-t_0}\langle \cdot, \mathbb{1}_{[t_0,t)}\rangle\right)\right)dr\right)\\ \times \sigma_z\delta\left(\langle \cdot, \mathbb{1}_{[t_0,t)}\rangle - (x-x_0)\right)\\ = \sigma_z\left(\exp\left(\frac{1}{z^2}\int_{t_0}^{t} V\left(x_0 + \frac{r-t_0}{t-t_0}(x-x_0) + \left(\langle \cdot, \mathbb{1}_{[t_0,r)}\rangle - \frac{r-t_0}{t-t_0}\langle \cdot, \mathbb{1}_{[t_0,t)}\rangle\right)\right)dr\right)\right.\\ \left.\times \delta\left(\langle \cdot, \mathbb{1}_{[t_0,t)}\rangle - (x-x_0)\right)\right) = \sigma_z\Phi.\end{aligned}$$

□

Since without further integrability and differentiability conditions we can not verify the corresponding scaled heat equation we call the objects constructed in Theorem 5.24 generalized complex scaled heat kernels.

Remark 5.25. *Assume that*

$$\exp\left(\frac{1}{z^2}\int_{t_0}^{t} V\left(x_0 + z\langle \cdot, \mathbb{1}_{[t_0,r)}\rangle\right)dr\right)$$

and

$$\exp\left(\frac{1}{z^2}\int_{t_0}^{t} V\left(x_0 + \frac{r-t_0}{t-t_0}(x-x_0) + \left(\langle \cdot, \mathbb{1}_{[t_0,r)}\rangle - \frac{r-t_0}{t-t_0}\langle \cdot, \mathbb{1}_{[t_0,t)}\rangle\right)\right)dr\right)$$

and also the scaling, σ_z, $z \in \overline{S_0}$, of both are just well-defined in $(S_d)'$ or $(S_d)^{-1}$. Then it is still possible to define the generalized complex scaled heat kernel and its integrand by an approximation following Remark 4.30.

Chapter 6

Feynman Path integration in the framework of White Noise Analysis

The idea of realizing Feynman integrands in white noise framework goes back to [35]. In this chapter e describe the huge class of potentials for which the Feynman integrand is constructed as a generalized function of white noise. Therefore, we consider the subclasses in separate sections. For details and proofs we refer to [34], [10], [40], [20], [49], [5], [74], [9], [56] and [27].

6.1 How to realize Feynman integrals in White Noise analysis?

As proposed by Feynman, see [15], quantum mechanical transition amplitudes may be thought of as kind of averaging over fluctuating paths $x(r),\ t_0 < r < t, -\infty < t_0 < t < \infty$, with oscillatory weight functions given in terms of the classical action

$$S(x) = \int_{t_0}^{t} L(x(r), \dot{x}(r), r)\, dr.$$

Typically the Lagrangian L is of the form

$$L(x, \dot{x}, \cdot) = \frac{1}{2} m\dot{x}^2 - V(x, m\dot{x}, \cdot),$$

for a particle of mass m moving in the force field of a potential V. Informally the Feynman path integral is then expressed by

$$K(t, x; t_0, x_0) = \mathrm{N} \int \exp\left(\frac{i}{\hbar} S(x)\right) \prod_{t_0<r<t} dx(r), \quad \hbar = \frac{h}{2\pi}.$$

Here h is Planck's constant and the integral is thought of as being over all paths $x(r)$, $t_0 < r < t$, with $x(t_0) = x_0$ and $x(t) = x$. The quantum mechanical propagator $K(t, x; t_0, x_0)$ represents the quantum mechanical transposition amplitude for a particle to be found at position $x \in \mathbb{R}^d$ at time time t given that the particle was at position $x_0 \in \mathbb{R}^d$ at an earlier time t_0. From the mathematical point of view $K(t, x; t_0, x_0)$ is nothing else than the integral kernel of the corresponding evolution system $U(t, t_0)$. In detail

$$U(t, t_0)\Psi(x) = \int_{\mathbb{R}^d} K(t, x; t_0, u)\Psi(u)\, du, \quad U(t_0, t_0)\Psi = \Psi,$$

for a normalizable state Ψ, where the evolution system $U(t, t_0)$ solves the Schrödinger equation

$$i\hbar\partial_t\Big(U(t, t_0)\Psi\Big) = H(t)\Big(U(t, t_0)\Psi\Big), \quad U(t_0, t_0)\Psi = \Psi, \tag{6.1}$$

corresponding to the Hamiltonian

$$H(t) = -\frac{\hbar^2}{2m}\Delta + V(q, \nabla, t),$$

for a potential V.

As described in Chapter 5, see (5.2), in the euclidean case, i.e., $t \mapsto -it$, the Feynman-Kac formula gives an explicit integral representation for the heat kernel.

Here we use white noise techniques to realize Feynman integrals. Our approach is related to the Feynman-Kac formula, because in both frameworks a Gaussian measure is taken as reference measure. In contrast, however, we work in physical time and have as integrand a generalized function instead of an integrable function. That means for a potential V we have to find $I_V \in (S_d)'$ such that $E(I_V) = K_V(t, x; t_0, x_0)$, $x, x_0 \in \mathbb{R}^d$, $-\infty < t_0 < t < \infty$. A general Gaussian ansatz is of the form

$$I_V = \mathrm{Nexp}\left(\frac{i}{2}\int_{t_0}^t \dot{x}(r)^2\, dr + \frac{1}{2}\int_{t_0}^t \dot{x}(r)^2\, dr\right)\mathrm{Nexp}\left(-i\int_{t_0}^t V(x(r), \dot{x}(r), r)\, dr\right)\cdot\delta(x(t)-x), \tag{6.2}$$

where Nexp is a normalized exponential, $\frac{i}{2}\int_{t_0}^t \dot{x}(r)^2\, dr$ corresponds to the kinetic energy, the term $\frac{1}{2}\int_{t_0}^t \dot{x}(r)^2\, dr$ is used to compensate the Gaussian fall off between t_0 and t and $-i\int_{t_0}^t V(x(r), \dot{x}(r), r)\, dr$ takes the potential energy into account.

6.2 The free Feynman integrand

We follow [35], see also [10] or [74], in viewing the Feynman integral as a weighted average over Brownian paths in $\mathbb{R}^d$ which are modeled for $x_0 \in \mathbb{R}^d$ and $t \in \mathbb{R}$ by

$$x(t) := x_0 + \sqrt{\frac{\hbar}{m}}\langle\omega, \mathbb{1}_{[t_0,t)}\rangle, \quad \omega \in (S_d)', \tag{6.3}$$

as in Example 2.2 (iii). For simplification, we set $\hbar = m = 1$. So the Feynman integrand corresponding to the free motion is

$$I_0 = \mathrm{Nexp}\Big(\frac{i+1}{2}\int_{t_0}^{t}\omega^2(r)\,dr\Big)\,\delta^d(x(t)-x), \quad -\infty < t_0 < t < \infty,\ x \in \mathbb{R}^d,$$

where the normalized exponential is defined as in Example 2.13. To fix the starting point, we use Donsker's delta $\delta^d(x(t)-x_0)$, which plays the role of an initial distribution, see Example 2.15 for its definition. The T-transform of I_0 at $g \in S_d(\mathbb{R})$ is given by

$$(TI_0)(g) = \frac{1}{\sqrt{(2\pi i|\Lambda|)^d}}\exp\Big(-\frac{i}{2}|g_\Lambda|_0^2 - \frac{1}{2}|g_{\Lambda^c}|_0^2 + \frac{i}{2|\Lambda|}\Big(\int_{t_0}^{t} g(r)\,dr + x - x_0\Big)^2\Big), \tag{6.4}$$

where $\Lambda = [t_0, t]$, $x = x(t) \in \mathbb{R}^d$, g_Λ denotes the restriction of g to Λ and g_{Λ^c} is the restriction of g to the complement of Λ. So the Feynman integral $E(I_0) = (TI_0)(0)$ is the free particle propagator

$$\frac{1}{\sqrt{(2\pi i|\Lambda|)^d}}\ \exp\Big(\frac{i}{2|\Lambda|}(x-x_0)^2\Big).$$

With a formal integration by parts we get

$$(TI_0)(g) = \exp\Big(-\frac{1}{2}|g_{\Lambda^c}|_0^2 + ig(t)\cdot x - ig(t_0)\cdot x_0\Big)\,E\Big(I_0\ \exp\Big(-i\int_{t_0}^{t}\dot g(r)\cdot x(r)\,dr\Big)\Big).$$

Then the term

$$\exp\Big(-i\int_{t_0}^{t}\dot g(r)\cdot x(r)\,dr\Big)$$

would be arise from a time-dependent potential $W_{\dot g}(x,t) = \dot g(t)\cdot x$, $x \in \mathbb{R}^d$, $g \in S_d(\mathbb{R})$ and $-\infty < t_0 < t < \infty$. So it is natural to check that

$$(TI_0)(g) = \exp\Big(-\frac{1}{2}|g_{\Lambda^c}|_0^2 + ig(t)\cdot x - ig(t_0)\cdot x_0\Big)K_0^{(\dot g)}(x,t;x_0,t_0), \tag{6.5}$$

where

$$K_0^{(\dot g)}(x,t;x_0,t_0) := \frac{1}{\sqrt{(2\pi i|\Lambda|)^d}}\ \exp(ig(t_0)\cdot x_0 - ig(t)\cdot x)$$
$$\times\ \exp\Big(-\frac{i}{2}|g_\Lambda|_0^2 + \frac{i}{2|\Lambda|}\Big(\int_{t_0}^{t} g(r)\,dr + x - x_0\Big)^2\Big)$$

denotes the Green's function corresponding to the potential $W_{\dot g}$, that means $K_0^{(\dot g)}$ obeys the Schrödinger equation

$$\Big(\frac{\partial}{\partial t} - z^2\frac{1}{2}\Delta - \dot g(t)\cdot x\Big)K_0^{(\dot g)}(x,t;x_0,t_0) = 0,$$

with initial condition

$$\lim_{t \searrow t_0} K_0^{(\dot{g})}(x, t; x_0, t_0) = \delta(x - x_0)\,.$$

More generally, one calculates

$$T\Big(I_0 \prod_{j=1}^{n} \delta^d(x(t_j) - x_j)\Big)(g) = \exp\Big(-\frac{1}{2}|g_{\Lambda^c}|_0^2 + ig(t)\cdot x - ig(t_0)\cdot x_0\Big) \times \prod_{j=1}^{n+1} K_0^{(\dot{g})}(x_j, t_j; x_{j-1}, t_{j-1}), \quad (6.6)$$

where $t_0 < t_1 < \ldots < t_{n+1} = t$, $x_j \in \mathbb{R}^d$, $1 \le j \le n+1$ and $x_{n+1} = x$.

6.3 The perturbed Feynman integrand

In order to pass from the free motion to more general situations two steps have to be shown. First of all we have to justify the pointwise multiplication of I_0 with the interaction term. That means one has to give a mathematical meaning to the heuristic expression

$$I_V = I_0 \exp\Big(-i \int_{t_0}^{t} V(x(r), r)dr\Big),$$

where the time states $-\infty < t_0 < t \le T < \infty$ and the path x are defined as in Section 6.2 and $V : \mathbb{R}^d \to \mathbb{C}$, $d \in \mathbb{N}$, is a suitable potential. Secondly it has to be shown that $E(I_V)$ solves the Schrödinger equation for the Hamiltonian $-\frac{1}{2}\Delta + V$. Depending on the potential structure we only give the integrand I_V or its T-transform and refer to literature for the proofs and also for the verification of the Schrödinger equation. Let us remark that we are not interested in a complete list of examples. Here only classes are considered for which the Feynman integrand can be constructed also with help of the complex scaling ansatz, see Chapter 7.

6.3.1 The Khandekar-Streit class

In [40] the authors construct the Feynman integrands corresponding to potentials which are finite signed Borel measures with compact support. This work was generalized in [54] to a bigger class of potentials by allowing time-dependent potentials. Moreover, the assumption that V must have a compact support was replaced by a Gaussian fall-off condition. The underlying idea is a power series expansion of $\exp\big(-i \int_{t_0}^{t} V(x(r), r)\, dr\big)$ while using that $V(x(r), r) = \int_{\mathbb{R}} V(x, r)\delta(x(r) - x)dx$, for a path $x(r)$, $t_0 < r < t$, as in (6.3). I.e.,

$$I_0 \exp\Big(-i \int_{t_0}^{t} V(x(r))\, dr\Big) = \sum_{n=0}^{\infty} (-i)^n \int_{\mathbb{R}^n} \int_{\Lambda_n} \prod_{j=1}^{n} V(x_j, r_j) I_0 \delta(x(r_j) - x_j)\, dr_j dx_j,$$

where $\Lambda_n = \{(r_1, \ldots, r_n) \mid t_0 = r_0 < r_1 < \ldots < r_n < t\}$. For considering singular potentials, V is no longer taken to be a function, but a measure ν. Thus, one consider a finite signed Borel measure on $\mathbb{R} \times \Lambda$, $\Lambda = [T_0, T]$, for some $-\infty < T_0 < T < \infty$. The marginal measures, denoted by ν_x and ν_t, are given by

$$\nu_x(A \in \mathcal{B}(\mathbb{R})) = \nu(A \times \Lambda)$$

and

$$\nu_t(B \in \mathcal{B}(\Lambda)) = \nu(\mathbb{R} \times B).$$

Definition 6.1. *Let $\nu = \nu_+ - \nu_-$ be a finite Borel measure on $\mathbb{R} \times \Lambda$ where the marginal measures $|\nu|_x = (\nu_+ + \nu_-)_x$ and ν_t satisfy:*

(i) There exists $0 < R < \infty$ such that for all $R < r < \infty$ there exists some $0 < \beta < \infty$ such that

$$|\nu|_x(\{x : |x| > r\}) < \exp(-\beta r^2);$$

(ii) $|\nu|_t$ has L^∞ density.

The space of "potentials" given by such measures is called Khandekar-Streit-class.

For a measure ν as in Definition 6.1, $T_0 \leq t_0 < t \leq T$ and $x, x_0 \in \mathbb{R}^d$ the Feynman integrand

$$I_V = I_0 + \sum_{n=1}^{\infty} (-i)^n \int_{\mathbb{R}^n} \int_{\Lambda_n} \left(I_0 \prod_{j=1}^{n} \delta(x(r_j) - x_j) \right) \nu(dr_j dx_j),$$

exists as an element of $(S_d)'$, see again [54] and also [40]. Therein, its T-transform is given via Theorem 2.11 and Theorem 2.12 by

$$T(I_V)(g) = I_0 + \sum_{n=1}^{\infty} (-i)^n \int_{\mathbb{R}^n} \int_{\Lambda_n} T\left(I_0 \prod_{j=1}^{n} \delta(x(r_j) - x_j) \right)(g)\, \nu(dr_j dx_j),$$

for all $g \in S_d(\mathbb{R})$.

6.3.2 The harmonic oscillator

To define the Feynman integrand of the harmonic oscillator (here for space dimension d = 1),

$$I_h = I_0 \exp\left(-i \int_{t_0}^{t} U(x(r))\, dr \right), \quad U(x) = \frac{1}{2} k^2 x^2,$$

with $0 < k|\Lambda| < \frac{\pi}{2}$, $\Lambda = [t_0, t]$, $|\Lambda| = |t - t_0|$, $-\infty < t_0 < t < \infty$, several solutions are possible. On the one side again one can consider the exponential as a perturbation series. On the other side the T-transform can be directly calculated while using the quadratic structure of the potential, see [10]. The T-transform of I_h is given by

$$\begin{aligned} TI_h(g) = & \sqrt{\frac{k}{2\pi i \sin k|\Lambda|}} \exp\left(\frac{i}{2}|g|_0^2\right) \\ & \times \exp\left(\frac{ik}{2\sin k|\Lambda|}\left(\left(x_0^2 + x^2\right)\cos k|\Lambda| - 2x_0 x + 2x\int_{t_0}^{t} g(r)\cos k(r - t_0)dr\right.\right. \\ & \left.\left. - 2x_0\int_{t_0}^{t} g(r)\cos k(t - r)\,dr + 2\int_{t_0}^{t}\int_{t_0}^{r_1} g(r_1)g(r_2)\cos k(t - r_1)\cos k(r_2 - t_0)dr_2 dr_1\right)\right). \end{aligned}$$

TI_h is easily seen to be a U-functional, for all $x_0, x \in \mathbb{R}$. Thus, $I_h \in (S)'$.

Remark 6.2. *In [8] the authors constructed the Feynman integrand corresponding to potentials $V = U + V_1$, where V_1 is in the Khandekar-Streit class. Furthermore, in [24] a generalization to several quadratic actions is given.*

6.3.3 The Albeverio-Høegh-Krohn class

In [82] the Feynman integrand corresponding to Fresnel integrable potentials was constructed as Kondratiev distributions.

Definition 6.3. *Let m be a complex measure on the Borel sets of $\mathbb{R}^d$, $d \geq 1$. A complex-valued function V is called Fresnel integrable (see e.g. [1]) if*

$$V(x) = \int_{\mathbb{R}^d} e^{i\alpha\cdot x} dm(\alpha).$$

We call V to be an element from the Albeverio-Høegh-Krohn class if

$$\int_{\mathbb{R}^d} e^{C|\alpha|}\, d|m|(\alpha) < \infty, \quad \textit{for all } \ 0 < C < \infty.$$

Close to Section 6.3.1 and Section 6.3.2 one gives a meaning to the Feynman integrand corresponding to a potential as in Definition 6.3 by an expansion of the exponential into a perturbation series. Then

$$I_V := \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int_{[t_0,t]^n} \int_{\mathbb{R}^{dn}} I_0 \cdot \prod_{j=1}^{n} e^{i\alpha_j\cdot x(r_j)} \prod_{j=1}^{n} dm(\alpha_j)\, dr_j,$$

for $-\infty < t_0 < t < \infty$ and $x_0, x \in \mathbb{R}^d$, where the series converges in the strong topology of $(\mathcal{S}_d)^{-1}$. Therefore, the T-transform can be expressed by

$$TI_V(g) = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int_{[t_0,t]^n} \int_{\mathbb{R}^{dn}} T\left(I_0 \cdot \prod_{j=1}^{n} e^{i\alpha_j \cdot x(r_j)}\right)(g) \prod_{j=1}^{n} dm(\alpha_j)\, dr_j,$$

for all g in a neighborhood of zero

$$U_{p,q} := \left\{ g \in S_{d,\mathbb{C}} \,\middle|\, 2^q |\theta|_p < 1 \right\},$$

for some $p, q \in \mathbb{N}_0$.

6.3.4 The Westerkamp-Kuna-Streit class

Inspired by the Albeverio-Høegh-Krohn class in [49] the Feynman integrand for the following class of potentials is constructed:

Definition 6.4. *Let m be a complex measure on the Borel sets of $\mathbb{R}^d$, $d \geq 1$, fulfilling the following condition*

$$\int_{\mathbb{R}^d} e^{C|\alpha|}\, d|m|(\alpha) < \infty, \quad \textit{for all } 0 < C < \infty. \tag{6.7}$$

We define a potential V on $\mathbb{R}^d$ by

$$V(x) = \int_{\mathbb{R}^d} e^{\alpha \cdot x} dm(\alpha).$$

The linear space of potentials of this form is called Westerkamp-Kuna-class.

Remark 6.5. *A consequence of the condition above* (6.7) *is that the measure m is finite. By Lebesgue's dominated convergence theorem we obtain that the potentials are restrictions to the real line of entire functions. In particular they are locally bounded and without singularities. However, they are in general unbounded at $\pm\infty$.*

Example 6.6. *(i) Every finite measure with compact support fulfills condition* (6.7).

(ii) The simplest example is the Dirac measure in one dimension $m(\alpha) := g\, \delta_a(\alpha)$ for $a > 0$ and $g \in \mathbb{R}$. The associated potential is $V(x) = g\, e^{ax}$. Obviously all polynomials of exponential functions of the above kind are also in our class, too, e.g. $\sinh(ax)$, $\cosh(ax)$.

(iii) If we choose a Gaussian density, we get potentials of the form $V(x) = ge^{bx^2}$ with $b, x \in \mathbb{R}$.

(iv) Further entire functions of arbitrary high order of growth are inside of our class. More explicitly, the measures $m(\alpha) := \Theta(\alpha)\exp\left(-k\alpha^{1+b}\right)$ *with* $b,\ k > 0$ *and* $x \in \mathbb{R}$ *fulfill the condition* (6.7)*. The corresponding potentials are entire functions of order* $1 + \frac{1}{b}$*, see [58] Lemma 7.2.1.*

The Feynman integrand for a potential V as in Definition 6.4 is then given by

$$I := \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int_{[t_0,t]^n} \int_{\mathbb{R}^{dn}} I_0 \cdot \prod_{j=1}^{n} e^{\alpha_j \cdot x(r_j)} \prod_{j=1}^{n} dm(\alpha_j)\, dr_j,$$

for $-\infty < t_0 < t < \infty$ and $x_0, x \in \mathbb{R}^d$, where the series again converges in the strong topology of $(S_d)^{-1}$. Therefore, the T-transform can be expressed by

$$TI_V(g) = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int_{[t_0,t]^n} \int_{\mathbb{R}^{dn}} T\left(I_0 \cdot \prod_{j=1}^{n} e^{\alpha_j \cdot x(r_j)}\right)(g) \prod_{j=1}^{n} dm(\alpha_j)\, dr_j,$$

for all g in a neighborhood of zero

$$U_{p,q} := \left\{ g \in S_{d,\mathbb{C}} \,\middle|\, 2^q |\theta|_p < 1 \right\},$$

for some $p, q \in \mathbb{N}_0$.

Let us remark that in contrast to (6.3) in this construction the path runs backward in time, i.e.,

$$x(r) = x - \sqrt{\frac{\hbar}{m}} \langle \omega, \mathbb{1}_{(r,t]} \rangle, \quad \omega \in (S_d)'.$$

Here again $0 \le t_0 < t \le T < \infty$ and $x \in \mathbb{R}^d$.

Remark 6.7. *One can also consider the Feynman integrand for potentials which are combinations, especially sums, of potentials from the Westerkamp-Kuna-Streit class and the Khandekar-Streit class, see [9].*

6.4 The T-transform of the Feynman integrand as a time-dependent propagator

As shown in (6.5) there exists a natural connection between the T-transform evaluated at $g \in S_d(\mathbb{R})$ of the free Feynman integrand and the propagator to the time dependent potential $x \mapsto \dot{g}(t) \cdot x$. Furthermore, in all cases considered in Section 6.3 the T-transform also provides us with a solution to the time-dependent Schrödinger equation. That means

$$\left(i\frac{\partial}{\partial t} + \frac{1}{2}\triangle_d - V(x) - x\dot{g}(t)\right) K_V^{(\dot{g})}(x, t; x_0, t_0) = 0,$$

with initial condition

$$\lim_{t \searrow t_0} K_V^{(\dot{g})}(x, t; x_0, t_0) = \delta(x - x_0)\,,$$

for all $x, x_0 \in \mathbb{R}^d$, $-\infty < t_0 < t < \infty$, where the connection between propagator and T-transform is given by

$$K_V^{(\dot{g})}(x, t; x_0, t_0) = T I_V(g) \cdot \exp\left(\frac{i}{2} \int_{[t_0,t]^c} g^2(s)\, \mathrm{d}s + i x_0 \cdot g(t_0) - i x \cdot g(t)\right). \tag{6.8}$$

Remark 6.8. *(i) Note that in every case of potentials* (6.8) *was obtained by an informal integration by parts. We give this informal step a rigorous meaning for a new class of potentials in Section 7.3.2.*

(ii) In [27] it is shown that (6.8) *holds under the natural assumption that the kernels of I_V, $I_V^{(n)}$, have support in $[t_0, t]^n$ for all $n \in \mathbb{N}$.*

Chapter 7

Feynman integrals and complex scaling

In this Chapter we use the concept of complex scaling to construct Feynman integrands as generalized functions of white noise. We not only appropriate this method for analytic potentials as discussed in Chapter 5, but also to classes of potentials discussed in Chapter 6. In addition, we give a rigorous derivation for the relation between the time-dependent propagator and the T-transform of the integrand. Among this a time-dependent complex scaled Feynman-Kac formula is derived. Parts of these results are already published in [27] and [26].

7.1 General strategy of complex scaling

In Example 2.13 it has been shown that the part of the kinetic energy combined with the compensating term of the Gaussian fall-off of the white noise measure yields a well-defined Hida distribution. More general this is given by

$$J_{z,t_0,t} := \mathrm{Nexp}\left(\frac{1}{2}(1-z^{-2})\int_{t_0}^{t}\omega(r)^2\,dr\right),$$

for $-\infty < t_0 < t < \infty$ and $z \in \mathbb{C}$, $z \neq 0$. It is discussed in Section 3.2.3 that for $\phi, \psi \in (S_d)$ and $z \in \mathbb{C}$ the following relations hold:

$$J_{z,t_0,t}\varphi = \sigma^{\dagger}_{z,t_0,t}(\sigma_{z,t_0,t}\varphi)$$

and

$$J_{z,t_0,t}\varphi\psi := \sigma^{\dagger}_{z,t_0,t}(\sigma_{z,t_0,t}\varphi\sigma_{z,t_0,t}\psi). \tag{7.1}$$

Thus, applying Theorem 3.40 and Remark 3.41 one gets that if $\sigma_{z,t_0,t}\Phi \in (S_d)'$ also the product $J_{z,t_0,t}\Phi \in (S_d)'$. Hence, for a path $x(r),\ t_0 < r < t$, as in (6.3) one achieves

$$J_{z,t_0,t}\delta^d(x(t)-x) = \sigma^{\dagger}_{z,t_0,t}(\sigma_{z,t_0,t}\delta^d(x(t)-x)),$$

for all $0 < t_0 < t < \infty$ and all $x, x_0 \in \mathbb{R}^d$. Moreover, using Remark 3.38 (i) and Remark 3.41 (ii) its T-transform is given by

$$\begin{aligned}
T\left(J_{z,t_0,t}\delta^d(x(t)-x)\right)(g) &= \left\langle\!\left\langle \sigma^\dagger_{z,t_0,t}\sigma_z\delta^d(x(t)-x), \exp\left(i\langle\cdot, g\rangle\right)\right\rangle\!\right\rangle \\
&= \left\langle\!\left\langle \sigma_z\delta^d(x(t)-x), \exp\left(iz\langle\cdot, g\rangle\right)\right\rangle\!\right\rangle \exp\left(\frac{z^2-1}{2}\int_{\Lambda^c} g^2(r)\,dr\right) \\
&= \left\langle\!\left\langle \sigma_z\delta^d(x(t)-x), \sigma_z\exp\left(i\langle\cdot, g\rangle\right)\right\rangle\!\right\rangle \exp\left(\frac{z^2-1}{2}|g_{\Lambda^c}|_0^2\right) \\
= T\left(\sigma_z^\dagger\sigma_z\delta^d(x(t)-x)\right)(g)\exp\left(\frac{z^2-1}{2}|g_{\Lambda^c}|_0^2\right) &= T\left(\sigma_z\delta^d(x(t)-x)\right)(zg)\exp\left(\frac{z^2-1}{2}|g_{\Lambda^c}|_0^2\right) \\
= \frac{1}{(2\pi z^2(t-t_0))^{\frac{d}{2}}}\exp\left(-\frac{1}{2(t-t_0)}\left(iz\int_{t_0}^t g(r)dr + \frac{(x-x_0)}{z}\right)^2 - \frac{z^2}{2}\int_{\mathbb{R}} g^2(r)\,dr\right)\exp\left(\frac{z^2-1}{2}|g_{\Lambda^c}|_0^2\right) \\
= \frac{1}{(2\pi z^2(t-t_0))^{\frac{d}{2}}}\exp\left(-\frac{1}{2(t-t_0)}\left(iz\int_{t_0}^t g(r)\,dr + \frac{(x-x_0)}{z}\right)^2\right) \\
\times\exp\left(-\frac{z^2}{2}|g_\Lambda|_0^2 - \frac{1}{2}|g_{\Lambda^c}|_0^2\right),
\end{aligned}$$

for all $g \in S_d(\mathbb{R})$, see Example 2.14 for the T-transform of Donsker's delta. Here $\Lambda = [t_0, t]$, g_Λ denotes the restriction of g to Λ and g_{Λ^c} is the restriction of g to the complement of Λ. Comparing this with (6.4) by uniqueness of the T-transform we get that

$$I_0 = J_{\sqrt{i},t_0,t}\delta^d(x(r)-x) = \sigma^\dagger_{\sqrt{i},t_0,t}(\sigma_{\sqrt{i},t_0,t}\delta^d(x(r)-x)) = \sigma^\dagger_{\sqrt{i},t_0,t}(\sigma_{\sqrt{i}}\delta^d(x(r)-x)).$$

Here the last equality follows by Remark 3.41 (ii). Thus, the Feynman integrand corresponding to an arbitrary potential V, which only depends on the path, can be informal written as

$$\begin{aligned}
I_V &= \mathrm{N}\exp\left(\frac{i+1}{2}\int_{t_0}^t \omega^2\,dr\right)\exp\left(-i\int_{t_0}^t V(x_0 + \langle\omega, \mathbb{1}_{[t_0,r)}\rangle)\,dr\right)\delta^d(\langle\omega, \mathbb{1}_{[t_0,t)}\rangle + x_0 - x) \\
&= \sigma^\dagger_{\sqrt{i},t_0,t}\left(\sigma_{\sqrt{i}}\left(\exp\left(-i\int_{t_0}^t V(x_0 + \langle\omega, \mathbb{1}_{[t_0,r)}\rangle)\,dr\right)\sigma_{\sqrt{i}}\delta^d(\langle\omega, \mathbb{1}_{[t_0,t)}\rangle + x_0 - x)\right)\right),
\end{aligned}$$

for $x, x_0 \in \mathbb{R}^d$ and $0 < t_0 < t < \infty$. Here equality is mentioned in $(S_d)'$ as an informal application of Theorem 3.40, Remark 3.41 and Remark 4.2. But in which case (7.1) can be extended to generalized function? Since $\sigma^\dagger_{\sqrt{i},t_0,t} : (S_d)^{-1} \to (S_d)^{-1}$ one has only to verify that

$$\sigma_{\sqrt{i}}\left(\exp\left(-i\int_{t_0}^t V\left(x_0 + \langle\omega, \mathbb{1}_{[t_0,r)}\rangle\right)\,dr\right)\delta^d(\langle\omega, \mathbb{1}_{[t_0,t)}\rangle + x_0 - x)\right)$$

is a generalized function of white noise. Thus, in contrast to the common white noise ansatz for constructing Feynman integrands we do not want to give a meaning to the informal product

$$I_0\exp\left(-i\int_{t_0}^t V(x_0 + \langle\omega, \mathbb{1}_{[t_0,r)}\rangle)\,dr\right).$$

In the complex scaling ansatz we give a meaning to the product

$$\exp\left(-i\int_{t_0}^{t} V(x_0+\langle\omega,\mathbb{1}_{[t_0,r)}\rangle)\,dr\right)\delta^d(\langle\omega,\mathbb{1}_{[t_0,t)}\rangle+x_0-x)$$

with help of the attainment from Section 4.4 and following Section 3.2.3 to decide whether we can apply the scaling operator $\sigma_{\sqrt{i}}$.

Theorem 7.1. *Let $-\infty < T_0 \le t_0 < t \le T < \infty$ and $V : \mathcal{D} \subset \mathbb{C}^d \to \mathbb{C}$ be a potential such that*

$$\exp\left(-i\int_{t_0}^{t} V(x(r))\,dr\right) \quad \text{and} \quad \sigma_{\sqrt{i}}\exp\left(-i\int_{t_0}^{t} V(x(r))\,dr\right) \tag{7.2}$$

are well-defined objects in $C_{\frac{1}{\sqrt{t-t_0}}\vec{\mathbb{1}}_{[t_0,t)},\frac{x-x_0}{\sqrt{t-t_0}}}$ and $C_{\frac{1}{\sqrt{t-t_0}}\vec{\mathbb{1}}_{[t_0,t)},\frac{x-x_0}{\sqrt{i(t-t_0)}}}$, respectively, for all $x_0, x \in \mathcal{D}\cap\mathbb{R}$, see Definition 4.26. Then

$$I_V = J_{\sqrt{i},t_0,t}\exp\left(-i\int_{t_0}^{t} V(x(r))\,dr\right)\delta^d(x(t)-x)$$

is a well-defined Hida distribution. Equivalently, one can write

$$I_V = \sigma^{\dagger}_{\sqrt{i},t_0,t}\left(\sigma_{\sqrt{i}}\left(\tau_{\frac{x-x_0}{(t-t_0)}}\mathbb{1}_{[t_0,t)}P_{\frac{1}{\sqrt{t-t_0}}\vec{\mathbb{1}}_{[t_0,t)}}\exp\left(-i\int_{t_0}^{t} V(x(r))\,dr\right)\right)\diamond\sigma_{\sqrt{i}}\delta^d(x(t)-x)\right), \tag{7.3}$$

where the Wick product inside is a pointwise independent product (in the sense of Definition 4.8).

The proof for the one dimensional case is given by Theorem 5.24. The d-dimensional case follows by a componentwise consideration of the path.

Thus, the Gaussian ansatz for the Feynman integrand as described in (6.2) is a Hida distribution for all potentials V fulfilling the assumptions of Theorem 7.1. Nevertheless, it is not clear whether it solves the Schrödinger equation.

7.2 The Doss class

For $\mathcal{O} \subset \mathbb{R}^d$ open, $d \ge 1$, where $\mathbb{R}\setminus\mathcal{O}$ is a set of Lebesgue measure zero, we consider the set $\mathcal{D}_{\sqrt{i},d} \subset \mathbb{C}^d$ as in (5.24). Analogously to Assumption 5.6 and Assumption 5.19, we give conditions on analytic potentials defined on $\mathcal{D}_{\sqrt{i},d}$, such that the corresponding Feynman integrands are Hida distributions and their generalized expectation solves the Schrödinger equation.

Assumption 7.2. *Let $0 < T < \infty$. We assume that the potential $V : \mathcal{D}_{\sqrt{i},d} \to \mathbb{C}$ is analytic and that there exist a constant $0 < A < \infty$, a locally bounded functions $B : \mathcal{O} \to \mathbb{R}$ and some $\varepsilon < \frac{1}{8T}$ such that for all $x_0 \in \mathcal{O}$ and $y \in \mathbb{R}^d$ one has that*

$$\left|\exp\left(-iV(x)\right)\right| \le A\exp\left(\varepsilon x^2\right) \quad \text{and} \quad \left|\exp\left(-iV\left(x_0+\sqrt{i}y\right)\right)\right| \le B(x_0)\exp\left(\varepsilon y^2\right).$$

Assumption 7.3. *Let $0 < T < \infty$ and $V : \mathcal{D}_{\sqrt{i},d} \to \mathbb{C}$ such that Assumption 5.6 is fulfilled. Then we require that there exist a locally bounded function $C : \mathcal{O} \times \mathcal{O} \to \mathbb{R}$ and some $0 < \varepsilon < \frac{1}{8T}$ such that for all $x_0, x_1 \in \mathcal{O}$ and $y \in \mathbb{R}^d$ one has that*

$$\left| V\left(x_0 + \sqrt{i}y\right) \exp\left(-iV\left(x_1 + \sqrt{i}y\right)\right) \right| \leq C(x_0, x_1) \exp\left(\varepsilon y^2\right)$$

and

$$\sum_{j=1}^{d} \left| \frac{\partial}{\partial z_j} V\left(x_0 + \sqrt{i}y\right) \exp\left(-iV\left(x_1 + \sqrt{i}y\right)\right) \right| \leq C(x_0, x_1) \exp\left(\varepsilon y^2\right).$$

Here $\frac{\partial}{\partial z_j}$ denotes the derivative of $z = (z_1, \ldots, z_d) \mapsto V(z)$, w.r.t. z_j, $j = 1, \ldots, d$.

Definition 7.4. *Let $\mathcal{D}_{\sqrt{i},d}$ be defined as in (5.24) then we consider analytic potentials $V : \mathcal{D}_{\sqrt{i},d} \to \mathbb{C}$ for which Assumption 7.2 and Assumption 7.3 are fulfilled. We call the linear space of such potentials Doss-class.*

Example 7.5. *Examples for elements of the Doss class can be derived close to Example 5.9:*

(i) **Polynomial potentials**
For $n, d \in \mathbb{N}_0$ and $z = \sqrt{i}$, we have a look at the potential

$$\begin{aligned} V : \mathbb{C}^d &\to \mathbb{C} \\ x &\mapsto (-1)^{n+1}(a_{4n+2}, x)_{\text{euk}}^{4n+2} + \sum_{j=1}^{4n+1} (a_j, x)_{\text{euk}}^{j}, \end{aligned}$$

for $a_0 \in \mathbb{C}$, $a_1, \ldots, a_{4n+1} \in \mathbb{C}^d$ and $a_{4n+1} = (a_{4n+1,1}, \ldots, a_{4n+1,d}) \in \mathbb{R}^d$, $a_{4n+2,k} > 0$, $k = 1, \ldots, d$. Here $(\cdot, \cdot)_{\text{euk}}$ denotes the euklidian scalar product on $\mathbb{C}^d$.

(ii) **The harmonic oscillator**
For $n = 0$ in (i) we can also consider potentials given by some constants $a_0 \in \mathbb{C}$, , $a_1 \in \mathbb{C}^d$ and $a_2 = (a_{2,1}, \ldots, a_{2,d}) \in \mathbb{R}^d$ such that $\sum_{k=1}^{d} \left|a_{2,k}\right| < \frac{1}{8T}$. Remind that T is the upper bound for the observed times.

(iii) **Non-perturbative accessible potenials**

(a)

$$\begin{aligned} V : \mathcal{D}_{\sqrt{i}} &\to \mathbb{C} \\ x &\mapsto \frac{a}{|(x, b)_{\text{euk}} - c|^n}, \end{aligned}$$

where $n \in \mathbb{N}$, $a \in \mathbb{C}$, $b \in \mathbb{R}^d$, and $c \in \mathbb{R}$.

(b)

$$\begin{aligned} V : \mathcal{D}_{\sqrt{i}} &\to \mathbb{C} \\ x &\mapsto \frac{a}{((x,b)_{\text{euk}} - c)^n}, \end{aligned}$$

again $n \in \mathbb{N}$, $a \in \mathbb{C}$, $b \in \mathbb{R}^d$ and $c \in \mathbb{R}$. Let us remark that close to Example 5.9 the natural expression as an analytic function of the first potential is given by

$$\begin{aligned} V : \mathcal{D}_{\sqrt{i}} &\to \mathbb{C} \\ x &\mapsto \exp\Big(\log(a) - \frac{n}{2}\log\Big(((x,b)_{\text{euk}} - c)^2\Big)\Big), \end{aligned}$$

which can be estimated in analogy to (5.9).

Theorem 7.6. *Let V be defined as in Definition 7.4, $0 \le t_0 < t \le T$ and $x, x_0 \in \mathcal{O}$ such that $x_0 + \frac{r-t_0}{t-t_0}(x - x_0) \in \mathcal{O}$ for all $t_0 < r < t$. Then the corresponding Feynman integrand exists as a Hida distribution and is given by* (7.3).

Proof. Let V be defined as in Definition 7.4, $0 \le t_0 < t \le T$ and $x, x_0 \in \mathcal{O}$ such that $x_0 + \frac{r-t_0}{t-t_0}(x - x_0) \in \mathcal{O}$. Then the proof follows directly by Theorem 7.1 if

$$\Phi := \exp\left(-i\int_{t_0}^{t} V(x(r))\,dr\right)$$

and $\sigma_{\sqrt{i}}\Phi$ are well-defined objects in $C_{\frac{1}{\sqrt{t-t_0}}\vec{1}_{[t_0,t)}, \frac{x-x_0}{\sqrt{t-t_0}}}$ and $C_{\frac{1}{\sqrt{t-t_0}}\vec{1}_{[t_0,t)}, \frac{x-x_0}{\sqrt{i(t-t_0)}}}$, respectively. By Assumption 7.2 it is easy to show that

$$\begin{aligned} \Psi &:= \exp\left(\frac{1}{z^2}\int_{t_0}^{t} V\left(x_0 + \frac{r-t_0}{t-t_0}(x-x_0) + \langle\cdot, 1\!\!1_{[t_0,r)}\rangle - \frac{r-t_0}{t-t_0}\langle\cdot, 1\!\!1_{[t_0,t)}\rangle\right)dr\right) \\ &= \exp\left(\frac{1}{z^2}\int_{t_0}^{t} V\left(x_0 + \left\langle\cdot - \frac{1}{t-t_0}\langle\cdot, 1\!\!1_{[t_0,t)}\rangle 1\!\!1_{[t_0,t)} + (x-x_0)\frac{1}{t-t_0}(1\!\!1_{[t_0,t)}, 1\!\!1_{[t_0,r)}), 1\!\!1_{[t_0,r)}\right\rangle\right)dr\right) \end{aligned}$$

and $\sigma_{\sqrt{i}}\Psi$ are in $L^2(\mu) \subset \mathcal{G}'$. Thus, by considering the chaos decomposition of Ψ and $\sigma_{\sqrt{i}}\Psi$ we get that that $\tau_{\frac{x-x_0}{(t-t_0)}1_{[t_0,t)}} P_{\frac{1}{\sqrt{t-t_0}}\vec{1}_{[t_0,t)}}\Phi = \Psi$ and $\tau_{\frac{x-x_0}{\sqrt{i}(t-t_0)}1_{[t_0,t)}} P_{\frac{1}{\sqrt{t-t_0}}\vec{1}_{[t_0,t)}}\sigma_{\sqrt{i}}\Phi = \sigma_{\sqrt{i}}\Psi$. Hence, by definition Φ and $\sigma_{\sqrt{i}}\Phi$ are in $C_{\frac{1}{\sqrt{t-t_0}}\vec{1}_{[t_0,t)}, \frac{x-x_0}{\sqrt{t-t_0}}}$ and $C_{\frac{1}{\sqrt{t-t_0}}\vec{1}_{[t_0,t)}, \frac{x-x_0}{\sqrt{i(t-t_0)}}}$, respectively. □

Theorem 7.7. *Let V be defined as in Definition 7.4 then*

$$\begin{aligned} K_V(x,t;x_0,t_0) &= E(I_V) \\ &= E\left(\sigma_{\sqrt{i}}\left(\exp\left(-i\int_{t_0}^{t} V\Big(x_0 + \sqrt{i}\,\langle\cdot, 1\!\!1_{[t_0,r)}\rangle\Big)dr\right)\sigma_{\sqrt{i}}\delta\left(\langle\cdot, 1\!\!1_{[t_0,t)}\rangle - (x-x_0)\right)\right) \end{aligned}$$

solves the Schrödinger equation for all $x, x_0 \in O$ *such that* $x_0 + \frac{r-t_0}{t-t_0}(x - x_0)$ *and* $0 \leq t_0 < r < t \leq T$

$$\left(i\frac{\partial}{\partial t} + \frac{1}{2}\Delta - V(x)\right) K_V(x, t; x_0, t_0) = 0, \tag{7.4}$$

with initial condition

$$\lim_{t \searrow t_0} K_V(x, t; x_0, t_0) = \delta(x - x_0).$$

For $d = 1$ the proof follows directly by Theorem 5.20 and the fact that for a Hida distribution Φ one has that $E\left(\sigma^{\dagger}_{\sqrt{i}}\Phi\right) = E(\Phi)$. For $d \geq 1$ the proof can be done similar.

7.2.1 Independence of the time direction of the path

Following Feynman's primal idea of average over path it is natural to work on the path

$$x_1(r) = x_0 + \langle \omega, \mathbb{1}_{[t_0,r)} \rangle,$$

$x_0 \in \mathbb{R}^d$, $0 \leq t_0 < r < t \leq T < \infty$, as in Section 6.2. Nevertheless, in [49] and [8] one describes the Feynman integrand with help of paths

$$x_2(r) := x - \langle \omega, \mathbb{1}_{(r,t]} \rangle,$$

$x \in \mathbb{R}^d$, $0 \leq t_0 < r < t \leq T < \infty$, running backward in time. In the following theorem we show that both models coincide in the complex scaling ansatz.

Theorem 7.8. *Let* $0 < T < \infty$*,* V *as in Definition 7.4 and* $x, x_0 \in O$ *such that* $x_0 + \frac{r-t_0}{t-t_0}(x - x_0)$ *for all* $0 \leq t_0 < r < t \leq T$*. Then both path models for the Feynman integrand coincide. I.e.,*

$$\begin{aligned} S(I_V)(g) &= S\left(J_{\sqrt{i},t_0,t} \exp\left(-i\int_{t_0}^{t} V(x_1(r))\,dr\right) \cdot \delta(x_1(t) - x)\right)(g) \\ &= S\left(J_{\sqrt{i},t_0,t} \exp\left(-i\int_{t_0}^{t} V(x_2(r))\,dr\right) \cdot \delta(x_2(t_0) - x_0)\right)(g) \end{aligned} \tag{7.5}$$

for all $g \in S_d(\mathbb{R})$.

Again, we only proof the one dimensional case.

Proof. Let V be defined as in Definition 7.4, $x, x_0 \in O$ such that $x_0 + \frac{r-t_0}{t-t_0}(x - x_0)$ is also in O for all $0 \leq t_0 < r < t \leq T < \infty$. Then analogous to Theorem 7.6 one can show that

$$\exp\left(-i\int_{t_0}^{t} V(x_2(r))\,dr\right) \quad \text{and} \quad \sigma_{\sqrt{i}}\exp\left(-i\int_{t_0}^{t} V(x_2(r))\,dr\right)$$

are well-defined objects in $C_{\frac{1}{\sqrt{t-t_0}}\vec{\mathbb{1}}_{(t_0,t]},\frac{x-x_0}{\sqrt{t-t_0}}}$ and $C_{\frac{1}{\sqrt{t-t_0}}\vec{\mathbb{1}}_{(t_0,t]},\frac{x-x_0}{\sqrt{i(t-t_0)}}}$, respectively. Therefore, we get the following equality in $(S)'$:

$$\begin{aligned}
&J_{\sqrt{i},t_0,t}\exp\left(-i\int_{t_0}^{t} V(x_2(r))\,dr\right)\cdot\delta(x_2(t)-x_0)\\
&\qquad\qquad = \sigma^{\dagger}_{\sqrt{i},t_0,t}\left(\sigma_{\sqrt{i}}\left(\exp\left(-i\int_{t_0}^{t} V(x_2(r))\,dr\right)\cdot\delta(x_2(t)-x_0)\right)\right)\\
&= \sigma^{\dagger}_{\sqrt{i},t_0,t}\left(\exp\left(-i\int_{t_0}^{t} V\left(x+\frac{t-r}{t-t_0}(x_0-x)+\sqrt{i}\,\langle\cdot,\mathbb{1}_{(r,t]}\rangle-\frac{t-r}{t-t_0}\sqrt{i}\,\langle\cdot,\mathbb{1}_{(t_0,t]}\rangle\right)dr\right)\right.\\
&\qquad\qquad\qquad\qquad \left.\diamond\sigma_{\sqrt{i}}\delta(x_2(t_0)-x_0)\right).
\end{aligned}$$

Moreover

$$\begin{aligned}
&x+\frac{t-r}{t-t_0}(x_0-x)+\sqrt{i}\,\langle\cdot,\mathbb{1}_{(r,t]}\rangle-\frac{t-r}{t-t_0}\sqrt{i}\,\langle\cdot,\mathbb{1}_{(t_0,t]}\rangle\\
&\qquad = x+\frac{t+t_0-t_0-r}{t-t_0}(x_0-x)+\sqrt{i}\,\langle\cdot,\mathbb{1}_{[r,t)}\rangle-\frac{t+t_0-t_0-r}{t-t_0}\sqrt{i}\,\langle\cdot,\mathbb{1}_{[t_0,t)}\rangle\\
&\qquad\qquad = x_0+\frac{r-t_0}{t-t_0}(x-x_0)+\sqrt{i}\,\langle\cdot,\mathbb{1}_{[t_0,r)}\rangle-\frac{r-t_0}{t-t_0}\sqrt{i}\,\langle\cdot,\mathbb{1}_{[t_0,t)}\rangle,
\end{aligned}$$

for all $t_0 \le r \le t$. Here the first equality holds on $(S)'$ by uniqueness of the S-transform. Hence, since

$$S\left(\sigma_{\sqrt{i}}\delta(x_2(t_0)-x_0)\right)(g) = S\left(\sigma_{\sqrt{i}}\delta(x_1(t_0)-x_0)\right)(g), \quad g\in S(\mathbb{R}),$$

equation (7.5) holds. □

7.3 Solutions to time-dependent Schrödinger equations

As in the free case, see (6.5), we are interested in a mathematical meaning of the T-transform of Feynman integrands for potentials from the Doss class as time-dependent propagator.

7.3.1 The time-dependent complex scaled Feynman-Kac formula

In the following we consider a generalization of Section 5.1 to the case where the potential depends linear on the time. However, we only consider the Schrödinger equation in one space dimension, i.e., $d = 1$:

$$\begin{cases} i\frac{\partial}{\partial t}(U(t,t_0)f)(x) = (H(t)U(t,t_0)f)(x) \\ \qquad (U(t_0,t_0)f)(x) = f(x), \end{cases} \qquad x\in O, \quad 0\le t_0 < t \le T, \tag{7.6}$$

where $H(t) := -\frac{1}{2}\Delta + V(x) + \dot{g}(t)x$ for $g \in C^1(\mathbb{R}, \mathbb{C})$ and $0 \leq T < \infty$. Here $C^1(\mathbb{R}, \mathbb{C})$ denotes the set of continuous differentiable functions from $\mathbb{R}$ to $\mathbb{C}$. These results are already published in [26]. Let us remark that if the potential V is time-dependent a solution is given by two-parameter operator semigroup, see e.g. [68], [63] or [83]).

Assumption 7.9. *Again for $\mathcal{O} \subset \mathbb{R}$ open, where $\mathbb{R} \setminus \mathcal{O}$ is a set of Lebesgue measure zero, we define the set $\mathcal{D}_{\sqrt{i}} \subset \mathbb{C}$ as in (5.5) and consider analytic functions $V : \mathcal{D}_{\sqrt{i}} \to \mathbb{C}$ and $f : \mathbb{C} \to \mathbb{C}$. We require that for all $0 \leq t_0 < t \leq T$ and $V_{\dot{g}}(x,t) := V(x) + \dot{g}(t)x$ the expectation*

$$E\left(\sup_{u \leq t}\left|\exp\left(-i\int_0^{u-t_0} V_{\dot{g}}\big(u-r, z+\sqrt{i}B_r\big)\,dr\right) f\big(z+\sqrt{i}B_{u-t_0}\big)\right|\right)$$

is finite.

Assumption 7.10. *Let $V : \mathcal{D}_{\sqrt{i}} \to \mathbb{C}$ and $f : \mathbb{C} \to \mathbb{C}$ such that Assumption 7.9 is fulfilled and $V_{\dot{g}},\ g \in C^1(\mathbb{R}, \mathbb{C})$, defined as in Assumption 7.9.*

(i) *For all $u, v, s, l \in [0, T]$ and all $z \in \mathcal{D}_{\sqrt{i}}$ we require that*

$$\begin{aligned} &E^1\Bigg[\Bigg|\exp\left(-i\int_0^u V_{\dot{g}}\big(v-r, z+\sqrt{i}B_r^1\big)\,dr\right) \\ &\times E^2\Bigg[\exp\left(-i\int_0^s V_{\dot{g}}\big(l-r, z+\sqrt{i}B_u^1+\sqrt{i}B_r^2\big)\,dr\right) f\big(z+\sqrt{i}B_u^1+\sqrt{i}B_s^2\big)\Bigg]\Bigg|\Bigg] < \infty. \end{aligned} \tag{7.7}$$

(ii) *For all $z \in \mathcal{D}_{\sqrt{i}}$, $0 \leq t_0 \leq t \leq T$ and some $0 < \varepsilon \leq T$ the functions*

$$\begin{aligned} \omega \mapsto \sup_{0 \leq h \leq \varepsilon}\Bigg|&\left(V_{\dot{g}}\big(t, z+\sqrt{i}B_h(\omega)\big) + \int_0^h \frac{\partial}{\partial t} V_{\dot{g}}\big(t+h-r, z+\sqrt{i}B_r(\omega)\big)\,dr\right) \\ &\times \exp\left(-i\int_0^h V_{\dot{g}}\big(t+h-r, z+\sqrt{i}B_r(\omega)\big)\,dr\right) f\big(z+\sqrt{i}B_h(\omega)\big)\Bigg| \end{aligned} \tag{7.8}$$

and

$$\begin{aligned} \omega \mapsto \sup_{h \in [0,T]}\Bigg|&\Delta E^2\Bigg[\exp\left(-i\int_0^{t-t_0} V_{\dot{g}}\big(t-r, z+\sqrt{i}B_h^1(\omega)+\sqrt{i}B_r^2\big)\,dr\right) \\ &\times f\big(z+\sqrt{i}B_h^1(\omega)+\sqrt{i}B_{t-t_0}^2\big)\Bigg]\Bigg| \end{aligned} \tag{7.9}$$

are integrable.

Here B^1 and B^2 are Brownian motions starting at 0 with corresponding expectations E^1 and E^2, respectively. Moreover, Δ denotes $\frac{\partial^2}{\partial z^2}$ and $\frac{\partial}{\partial t}$ the derivative w.r.t. the first variable.

As in Remark 5.5 we denote $H(\mathcal{D}_{\sqrt{i}})$ to be the set of holomorphic functions from $\mathcal{D}_{\sqrt{i}}$ to $\mathbb{C}$. In the following we show that the operator $U(t,t_0) : D(t,t_0) \subset H(\mathcal{D}_{\sqrt{i}}) \to H(\mathcal{D}_{\sqrt{i}})$, $0 \le t_0 \le t \le T$, given by

$$U(t,t_0)f(z) := E\Big[\exp\Big(-i\int_0^{t-t_0} V_{\dot{g}}\big(t-r, z+\sqrt{i}B_r\big)\,dr\Big) f\big(z+\sqrt{i}B_{t-t_0}\big)\Big], \quad z \in \mathcal{D}_{\sqrt{i}}, \tag{7.10}$$

provides us with a solution to (7.6). Here by $D(t,t_0)$ we denote the set of functions $f \in H(\mathcal{D}_{\sqrt{i}})$ such that the expectation in (7.10) is a well-defined object in $H(\mathcal{D}_{\sqrt{i}})$.

Lemma 7.11. *Let V and f fulfill the Assumptions 7.9 and 7.10 then the operator $U(t,t_0)$, $0 \le t_0 \le t \le T$, as in* (7.10)*, maps from $D(t,t_0)$ to $H(\mathcal{D}_{\sqrt{i}})$. Moreover, $U(u,t_0)f \in D(t,u)$ and one gets that*

$$U(t,t_0)f(z) = U(t,u)(U(u,t_0)f)(z),$$

for all $0 \le t_0 \le u \le t \le T$ and $z \in \mathcal{D}_{\sqrt{i}}$.

Proof. The property that $U(t,t_0)$, $0 \le t_0 \le t \le T$, as in (7.10), maps from $D(t,t_0)$ to $H(\mathcal{D}_{\sqrt{i}})$ follows by using Morera and Assumption 7.9. The fact that $U(u,t_0)f \in D(t,u)$ follows by Assumption 7.10 (i). Let $0 \le t_0 \le u \le t \le T$ and $z \in \mathcal{D}_{\sqrt{i}}$, then one gets with the Markov property and the time-reversibility of Brownian motion that

$$\begin{aligned}
U(t,t_0)f(z) &= E\Big[\exp\Big(-i\int_0^{t-t_0} V_{\dot{g}}\big(t-r, z+\sqrt{i}B_r\big)\,dr\Big) f\big(z+\sqrt{i}B_{t-t_0}\big)\Big] \\
&= E\Big[\exp\Big(-i\int_0^{t-u} V_{\dot{g}}\big(t-r, z+\sqrt{i}B_r\big)\,dr\Big) \\
&\quad\times \exp\Big(-i\int_{t-u}^{t-u+u-t_0} V_{\dot{g}}\big(t-r, z+\sqrt{i}B_r\big)\,dr\Big) f\big(z+\sqrt{i}B_{t-t_0}\big)\Big] \\
&= E\Big[\exp\Big(-i\int_0^{t-u} V_{\dot{g}}\big(t-r, z+\sqrt{i}B_r\big)\,dr\Big) \\
&\quad\times \exp\Big(-i\int_0^{u-t_0} V_{\dot{g}}\big(u-r+t-u, z+\sqrt{i}B_{r+t-u}\big)\,dr\Big) f\big(z+\sqrt{i}B_{t-u+u-t_0}\big)\Big] \\
&= E^1\Big[\exp\Big(-i\int_0^{t-u} V_{\dot{g}}\big(t-r, z+\sqrt{i}B^1_r\big)\,dr\Big) \\
&\quad\times E^2\Big[\exp\Big(-i\int_0^{u-t_0} V_{\dot{g}}\big(u-r, z+\sqrt{i}B^1_{t-u}+\sqrt{i}B^2_r\big)\,dr\Big) f\big(z+\sqrt{i}B^1_{t-u}+\sqrt{i}B^2_{u-t_0}\big)\Big]\Big] \\
&= U(t,u)(U(u,t_0)f)(z).
\end{aligned} \tag{7.11}$$

□

One can show that by $U(t,t_0)$, $0 \le t_0 \le t \le T$, a pointwise-defined (unbounded) evolution system is given.

Theorem 7.12. *Let $0 < T < \infty$, V, $V_{\dot{g}}$, $g \in C^1(\mathbb{R},\mathbb{C})$, defined as in Assumption 7.9, and f such that Assumption 7.9 and 7.10 are fulfilled. Then $U(t,t_0)f(x)$, $0 \le t_0 < t \le T$, $x \in \mathcal{O}$, given in* (7.10) *solves the Schrödinger equation* (7.6).

Proof. Let $0 \le t_0 < t \le T$, $x \in \mathcal{O}$ and $g \in C^1(\mathbb{R},\mathbb{C})$. If we have a look at the difference quotient from the right side, we get with Lemma 7.11 that

$$\begin{aligned}\frac{\partial}{\partial t}^{+} U(t,t_0)f(x) &= \lim_{h\searrow 0} \frac{U(t+h,t_0)-U(t,t_0)}{h} f(x)\\ &= \lim_{h\searrow 0} \frac{U(t+h,t)-U(t,t)}{h} U(t,t_0)f(x).\end{aligned}$$

Hence, it is left to show that

$$\lim_{h\searrow 0} \frac{U(t+h,t)k(x)-U(t,t)k(x)}{h} = H(t)k(x),$$

for $k = U(t,t_0)f$. Note that

$$\begin{aligned}&\lim_{h\searrow 0}\frac{1}{h}E\Big[\exp\Big(-i\int_0^{t+h-t} V_{\dot{g}}\big(t+h-r,x+\sqrt{i}B_r\big)\,dr\Big)k\big(x+\sqrt{i}B_h\big)-k\big(x+\sqrt{i}B_0\big)\Big]\\ &\quad= \lim_{h\searrow 0}E\Big[\frac{1}{h}\exp\Big(-i\int_0^{h} V_{\dot{g}}\big(t+h-r,x+\sqrt{i}B_r\big)\,dr\Big)k\big(x+\sqrt{i}B_h\big)-\frac{1}{h}k\big(x+\sqrt{i}B_h\big)\Big]\\ &\qquad+\lim_{h\searrow 0}E\Big[\frac{1}{h}k\big(x+\sqrt{i}B_h\big)-\frac{1}{h}k\big(x+\sqrt{i}B_0\big)\Big]. \qquad (7.12)\end{aligned}$$

The integrand of the first summand yields

$$\lim_{h\searrow 0}\frac{1}{h}\Big(\exp\Big(-i\int_0^{h} V_{\dot{g}}\big(t+h-r,x+\sqrt{i}B_r\big)\,dr\Big)-1\Big)k\big(x+\sqrt{i}B_h\big) \qquad (7.13)$$

$$= -iV_{\dot{g}}\big(t,x+\sqrt{i}B_0\big)k\big(x+\sqrt{i}B_0\big) = -iV_{\dot{g}}(t,x)k(x). \qquad (7.14)$$

Hence, by Assumption 7.10 (ii), the mean value theorem and Lebesgue dominated convergence

$$\lim_{h\searrow 0}E\Big[\frac{1}{h}\Big(\exp\Big(-i\int_0^{h} V_{\dot{g}}\big(t+h-r,x+\sqrt{i}B_r\big)\,dr\Big)-1\Big)k\big(x+\sqrt{i}B_h)\Big)\Big] = -iV_{\dot{g}}(t,x)k(x).$$

Moreover, we know by Remark 5.4 and Assumption 7.10 (ii) that $E\big[k(x+\sqrt{i}B_t)\big]$ solves the free Schrödinger equation, hence

$$\lim_{h\searrow 0}\frac{1}{h}E\big[k\big(x+\sqrt{i}B_h\big)-k\big(x+\sqrt{i}B_0\big)\big] = -i\frac{1}{2}\Delta k(x).$$

Similar with

$$\begin{aligned}\frac{\partial}{\partial t}^{-} U(t,t_0)f(x) &= \lim_{h\searrow 0}\frac{U(t-h,t_0)-U(t,t_0)}{h}f(x)\\ &= \lim_{h\searrow 0}\frac{U(t-h,t-h)-U(t,t-h)}{h}U(t-h,t_0)f(x)\end{aligned}$$

one can show the same for the difference quotient from the left side. □

Remark 7.13. *It is shown in [26] that the examples considered in Example 5.9 fulfill Assumption 7.9 and Assumption 7.10.*

Remark 7.14. *One can also consider the so called backward Schrödinger equation:*

$$\begin{cases} i\frac{\partial}{\partial t_0}(Y(t,t_0)f)(x) = (H(t)Y(t,t_0)f)(x) \\ \qquad (Y(t,t)f)(x) = f(x), \end{cases} \qquad x\in O,\ 0\le t_0\le t\le T, \tag{7.15}$$

where $H(t) := -\frac{1}{2}\Delta + V(x) + \dot{g}(t)x$ *for* $g \in C^1(\mathbb{R},\mathbb{C})$ *and* $0 \le T < \infty$. *In many cases when one considered the Heat equation, a solution to the backward equation is given by the so called backward propagator, which is the adjoint of the forward propagator (i.e., the propagator corresponding to a time-dependent Heat equation close to* (7.6)*), see e.g. [37], [53], [28] or [29]. Hence, we call* (7.15) *the dual problem of* (7.6). *In the non-euclidean case we are only able to construct a pointwise solution to both problems, which might not be integrable, hence we can not check whether we still have a relation between both solutions. Nevertheless, under the suitable integrability and differentiability assumptions (similar to Assumption 7.9 and Assumption 7.10) a solution to* (7.15) *can be constructed. Analogously to* (7.10) *the operator* $Y(t,t_0) : D(t,t_0) \subset H(\mathcal{D}_{\sqrt{i}}) \to H(\mathcal{D}_{\sqrt{i}})$, $0 \le t_0 \le t \le T$, *given by*

$$Y(t,t_0)f(z) := E\left[\exp\left(-i\int_0^{t-t_0} V_{\dot{g}}\big(r+t_0, z+\sqrt{i}B_r\big)\,dr\right)f\big(z+\sqrt{i}B_{t-t_0}\big)\right], \quad z\in\mathcal{D}_{\sqrt{i}}, \tag{7.16}$$

provides us with a solution to (7.15). *Here by* $D(t,t_0)$ *we denote the set of functions* $f \in H(\mathcal{D}_{\sqrt{i}})$ *such that the expectation in* (7.10) *is a well-defined object in* $H(\mathcal{D}_{\sqrt{i}})$.

7.3.2 T-transform of the Feynman integrand as time-dependent propagator

Close to Section 6.4 we are interested in a relation between the T-transform of the Feynman integrand corresponding to a potential from the Doss class V and the solution to the time-dependent Schrödinger equation corresponding to the potential $x \mapsto V(x) + \dot{g}(t)x$, $0 \le t \le T$. Again, we only consider the one dimensional case.

Lemma 7.15. *Let $g \in S(\mathbb{R})$ and $0 \leq t_0 < t \leq T < \infty$, then*

$$\langle \cdot, g\mathbb{1}_{[t_0,t)}\rangle = -\int_{t_0}^{t} \dot{g}(r)\langle \cdot, \mathbb{1}_{[t_0,r)}\rangle\, dr + g(t)\langle \cdot, \mathbb{1}_{[t_0,t)}\rangle$$

in $\mathcal{G}$.

Proof. By the characterization theorem, see Theorem 2.10, it suffices to show equality of the S-transforms. Let $g \in S(\mathbb{R})$ and $0 \leq t_0 < t \leq T < \infty$. Then one can easy calculate that

$$S(\langle \cdot, g\mathbb{1}_{[t_0,t)}\rangle)(h) = \int_{t_0}^{t} h(r)g(r)\, dr$$

and

$$S\left(-\int_{t_0}^{t} \dot{g}(r)\langle \omega, \mathbb{1}_{[t_0,r)}\rangle\, dr + g(t)\langle \cdot, \mathbb{1}_{[t_0,t)}\rangle\right)(h) = -\int_{t_0}^{t} \dot{g}(r)\int_{t_0}^{r} h(s)\, ds\, dr + g(t)\int_{t_0}^{t} h(r)\, dr,$$

for all $h \in S(\mathbb{R})$. Using integration by parts we get that both S-transforms coincide. Since $\langle \cdot, g\mathbb{1}_{[t_0,t)}\rangle \in \mathcal{G}$ the statement holds in $\mathcal{G}$. □

Theorem 7.16. *Let V be as in Defintion 7.4, $0 \leq t_0 < t \leq T < \infty$, $x, x_0 \in \mathbb{R}$ such that $x_0 + \frac{r-t_0}{t-t_0}(x - x_0) \in \mathcal{O}$ for all $t_0 < r < t$. Then for all $g \in S(\mathbb{R})$ we get that*

$$\begin{aligned}
&\exp\left(-ixg(t) + ix_0 g(t_0) + \frac{1}{2}|g_{\Lambda^c}|_0^2\right) TI_V(g) \\
&\qquad = E\Bigg(\sigma_{\sqrt{i}} \exp\left(-i\int_{t_0}^{t} V(x_0 + \langle \cdot, \mathbb{1}_{[t_0,r)}\rangle) + \dot{g}(r)(x_0 + \langle \cdot, \mathbb{1}_{[t_0,r)}\rangle)\right) dr\Bigg) \\
&\qquad\qquad \times \sigma_{\sqrt{i}}\delta(\langle \cdot, \mathbb{1}_{[t_0,t)}\rangle + x_0 - x)\Bigg) =: K_V^{(\dot{g})}(x, t; x_0, t_0),
\end{aligned}$$

where $\Lambda = [t_0, t]$ and g_{Λ^c} is the restriction of g to the complement of Λ.

Proof. Let V be as in Definition 7.4, $x, x_0 \in \mathbb{R}$ such that $x_0 + \frac{r-t_0}{t-t_0}(x - x_0) \in \mathcal{O}$ for all $t_0 < r < t$. Then we know by Theorem 7.6 that

$$I_V = \sigma^{\dagger}_{\sqrt{i}}\left(\sigma_{\sqrt{i}} \exp\left(-i\int_{t_0}^{t} V\left(x_0 + \langle \omega, \mathbb{1}_{[t_0,r)}\rangle\right)\, dr\right) \cdot \delta(\langle \omega, \mathbb{1}_{[t_0,t)}\rangle + x_0 - x)\right) \in (S)'.$$

Moreover, using Remark 3.38 (i) its T-transform in $g \in S(\mathbb{R})$ is given by

$$\begin{aligned}
TI_V(g) &= \langle\!\langle I_V, \exp(i\langle \cdot, g\rangle)\rangle\!\rangle \\
&= \left\langle\!\!\left\langle \sigma_{\sqrt{i},t_0,t} \exp\left(-i\int_{t_0}^{t} V(x_0 + \langle \cdot, \mathbb{1}_{[t_0,r)}\rangle)\, dr\right) \cdot \delta(\langle \cdot, \mathbb{1}_{[t_0,t)}\rangle + x_0 - x), \exp(i\sqrt{i}\langle \cdot, g\rangle)\right\rangle\!\!\right\rangle
\end{aligned}$$

$$\times \exp\left(\frac{i-1}{2}\int_{[t_0,t]^c} g^2(r)\,dr\right)$$
$$= \langle\!\langle I_V, \exp(i\langle\cdot, g\mathbb{1}_{[t_0,t)}\rangle)\rangle\!\rangle E\left(\exp(i\sqrt{i}\langle\cdot, g\mathbb{1}_{[t_0,t)^c}\rangle)\right)$$
$$= TI_V\left(g\mathbb{1}_{[t_0,t)}\right)\exp\left(-\frac{1}{2}|g_{\Lambda^c}|_0^2\right), \quad (7.17)$$

since the Hida distributions $\exp(i\sqrt{i}\langle\cdot, g\mathbb{1}_{[t_0,t)^c}\rangle)$ and

$$\exp\left(-i\int_{t_0}^t V(x_0+\sqrt{i}\langle\cdot, \mathbb{1}_{[t_0,r)}\rangle)\,dr\right)\exp(i\sqrt{i}\langle\cdot, g\mathbb{1}_{[t_0,t)}\rangle)\cdot\sigma_{\sqrt{i}}\delta(\langle\cdot, \mathbb{1}_{[t_0,t)}\rangle + x_0 - x)$$

are independent in the sense of Definition 4.8. Applying Lemma 7.15 one gets that

$$TI_V\left(g\mathbb{1}_{[t_0,t)}\right) = E\Bigg(\exp\left(-i\int_{t_0}^t V(x_0+\sqrt{i}\langle\cdot, \mathbb{1}_{[t_0,r)}\rangle)\,dr\right)$$
$$\times\exp\left(-i\sqrt{i}\int_{t_0}^t \dot{g}(r)\langle\cdot, \mathbb{1}_{[t_0,r)}\rangle dr + i\sqrt{i}g(t)\langle\cdot, \mathbb{1}_{[t_0,t)}\rangle\right)\cdot\sigma_{\sqrt{i}}\delta(\langle\cdot, \mathbb{1}_{[t_0,t)}\rangle + x_0 - x)\Bigg).$$

Since

$$\exp\left(-i\int_{t_0}^t V(x_0+\sqrt{i}\langle\cdot, \mathbb{1}_{[t_0,r)}\rangle)\,dr\right) \in C_{\frac{1}{\sqrt{t-t_0}}\mathbb{1}_{[t_0,t)}, \frac{x-x_0}{\sqrt{i(t-t_0)}}}$$

and

$$\exp\left(-i\sqrt{i}\int_{t_0}^t \dot{g}(r)\langle\cdot, \mathbb{1}_{[t_0,r)}\rangle dr + i\sqrt{i}g(t)\langle\cdot, \mathbb{1}_{[t_0,t)}\rangle\right) \in \mathcal{G}$$

we get that the product of both is also in $C_{\frac{1}{\sqrt{t-t_0}}\mathbb{1}_{[t_0,t)}, \frac{x-x_0}{\sqrt{i(t-t_0)}}}$, where

$$P_{\frac{1}{\sqrt{t-t_0}}\mathbb{1}_{[t_0,t)}}\tau_{\frac{x-x_0}{\sqrt{i}(t-t_0)}\mathbb{1}_{[t_0,t)}}\Bigg[\exp\left(-i\int_{t_0}^t V(x_0+\sqrt{i}\langle\cdot, \mathbb{1}_{[t_0,r)}\rangle)\,dr\right)$$
$$\times\exp\left(-i\sqrt{i}\int_{t_0}^t \dot{g}(r)\langle\cdot, \mathbb{1}_{[t_0,r)}\rangle dr + i\sqrt{i}g(t)\langle\cdot, \mathbb{1}_{[t_0,t)}\rangle\right)\Bigg]$$
$$= \exp\left(-i\int_{t_0}^t V\left(x_0 + \frac{s-t_0}{t-t_0}(x-x_0) + \sqrt{i}\left(\langle\cdot, \mathbb{1}_{[t_0,r)}\rangle - \frac{r-t_0}{t-t_0}\langle\cdot, \mathbb{1}_{[t_0,t)}\rangle\right)\right)dr\right)$$
$$\times\tau_{\frac{x-x_0}{\sqrt{i}(t-t_0)}\mathbb{1}_{[t_0,t)}}P_{\frac{1}{\sqrt{t-t_0}}\mathbb{1}_{[t_0,t)}}\exp\left(-i\sqrt{i}\int_{t_0}^t \dot{g}(r)\langle\cdot, \mathbb{1}_{[t_0,r)}\rangle dr + i\sqrt{i}g(t)\langle\cdot, \mathbb{1}_{[t_0,t)}\rangle\right).$$

Therefore, with Theorem 4.25 it follows that

$$= E\Bigg(\exp\left(-i\int_{t_0}^t V\left(x_0 + \frac{s-t_0}{t-t_0}(x-x_0) + \sqrt{i}\left(\langle\cdot, \mathbb{1}_{[t_0,r)}\rangle - \frac{r-t_0}{t-t_0}\langle\cdot, \mathbb{1}_{[t_0,t)}\rangle\right)\right)dr\right)$$
$$\tau_{\frac{x-x_0}{\sqrt{i}(t-t_0)}\mathbb{1}_{[t_0,t)}}P_{\frac{1}{\sqrt{t-t_0}}\mathbb{1}_{[t_0,t)}}\exp\left(-i\sqrt{i}\int_{t_0}^t \dot{g}(r)\langle\cdot, \mathbb{1}_{[t_0,r)}\rangle dr + i\sqrt{i}g(t)\langle\cdot, \mathbb{1}_{[t_0,t)}\rangle\right)\Bigg)$$

$$\times E\left(\sigma_{\sqrt{i}}\delta(\langle\cdot, \mathbb{1}_{[t_0,t)}\rangle + x_0 - x)\right).$$

One has on the one side that

$$\tau_{\frac{x-x_0}{\sqrt{i}(t-t_0)}\mathbf{1}_{[t_0,t)}} P_{\frac{1}{\sqrt{t-t_0}}\mathbf{1}_{[t_0,t)}} \exp\left(i\sqrt{i}g(t)\langle\cdot, \mathbb{1}_{[t_0,t)}\rangle\right) = \exp(ig(t)(x-x_0)),$$

and on the other side that

$$\tau_{\frac{x-x_0}{\sqrt{i}(t-t_0)}\mathbf{1}_{[t_0,t)}} P_{\frac{1}{\sqrt{t-t_0}}\mathbf{1}_{[t_0,t)}} \exp\left(-i\sqrt{i}\int_{t_0}^{t}\dot{g}(r)\langle\cdot, \mathbb{1}_{[t_0,r)}\rangle dr\right)$$
$$= \exp\left(-i\int_{t_0}^{t}\dot{g}(r)\left(\frac{x-x_0}{(t-t_0)} + \sqrt{i}\left(\langle\cdot, \mathbb{1}_{[t_0,r)}\rangle - \frac{r-t_0}{t-t_0}\langle\cdot, \mathbb{1}_{[t_0,t)}\rangle\right)\right)dr\right).$$

Hence, together one gets that

$$\tau_{\frac{x-x_0}{\sqrt{i}(t-t_0)}\mathbf{1}_{[t_0,t)}} P_{\frac{1}{\sqrt{t-t_0}}\mathbf{1}_{[t_0,t)}} \exp\left(-i\sqrt{i}\int_{t_0}^{t}\dot{g}(r)\langle\cdot, \mathbb{1}_{[t_0,r)}\rangle\, dr + i\sqrt{i}g(t)\langle\cdot, \mathbb{1}_{[t_0,t)}\rangle\right)$$
$$= -i\int_{t_0}^{t}\dot{g}(r)\left(x_0 + \frac{r-t_0}{(t-t_0)}(x-x_0) + \sqrt{i}\left(\langle\cdot, \mathbb{1}_{[t_0,r)}\rangle - \frac{r-t_0}{t-t_0}\langle\cdot, \mathbb{1}_{[t_0,t)}\rangle\right)\right)dr$$
$$+ ig(t)(x-x_0) + ix_0\int_{t_0}^{t}\dot{g}(r)\,dr$$
$$= -i\int_{t_0}^{t}\dot{g}(r)\left(x_0 + \frac{r-t_0}{(t-t_0)}(x-x_0) + \sqrt{i}\left(\langle\cdot, \mathbb{1}_{[t_0,r)}\rangle - \frac{r-t_0}{t-t_0}\langle\cdot, \mathbb{1}_{[t_0,t)}\rangle\right)\right)dr$$
$$+ ig(t)x - ig(t_0)x_0.$$

Inserting this in (7.17) it follows that

$$TI_V(g) = \exp\left(-\frac{1}{2}|g_{\Lambda^c}|_0^2\right)\exp(ig(t)x - ig(t_0)x_0)$$
$$\times E\Bigg(\exp\Bigg(-i\int_{t_0}^{t}V\left(x_0 + \frac{r-t_0}{(t-t_0)}(x-x_0) + \sqrt{i}\left(\langle\cdot, \mathbb{1}_{[t_0,r)}\rangle - \frac{r-t_0}{t-t_0}\langle\cdot, \mathbb{1}_{[t_0,t)}\rangle\right)\right)$$
$$+ \dot{g}(r)\left(x_0 + \frac{r-t_0}{(t-t_0)}(x-x_0) + \sqrt{i}\left(\langle\cdot, \mathbb{1}_{[t_0,r)}\rangle - \frac{r-t_0}{t-t_0}\langle\cdot, \mathbb{1}_{[t_0,t)}\rangle\right)\right)dr\Bigg)\Bigg)$$
$$\times E\left(\sigma_{\sqrt{i}}\delta(\langle\cdot, \mathbb{1}_{[t_0,t)}\rangle + x_0 - x)\right).$$

Again, by Theorem 4.25 we obtain

$$\exp\left(-ixg(t) + ix_0g(t_0) + \frac{1}{2}|g_{\Lambda^c}|_0^2\right)TI_V(g)$$
$$= E\Bigg(\sigma_{\sqrt{i}}\exp\left(-i\int_{t_0}^{t}V(x_0 + \langle\cdot, \mathbb{1}_{[t_0,r)}\rangle) + \dot{g}(r)(x_0 + \langle\cdot, \mathbb{1}_{[t_0,r)}\rangle)\,dr\right)$$
$$\times\sigma_{\sqrt{i}}\delta(\langle\cdot, \mathbb{1}_{[t_0,t)}\rangle + x_0 - x)\Bigg).$$

□

Close to (6.6) we can decompose this expectation in fractional states and times.

Theorem 7.17. *Let V be defined as in Definition 7.4, $t_0 < t_1 < \ldots < t_{n+1} = t$ and $x_j \in \mathcal{O}$ such that there exists a convex set $\mathcal{A} \subset \mathcal{O}$ with $x_j \in \mathcal{A}$ for all $0 \leq j \leq n+1$. Then*

$$T\Big(I_V \prod_{j=1}^{n} \delta^d(x(t_j) - x_j)\Big)(g) = \exp\Big(-\frac{1}{2}|g_{\Lambda^c}|_0^2 + ig(t)\cdot x - ig(t_0)\cdot x_0\Big) \times \prod_{j=1}^{n+1} K_V^{(\dot{g})}(x_j, t_j; x_{j-1}, t_{j-1}),$$

for all $g \in S_d(\mathbb{R})$.

Proof. The proof follows by induction. Here we only proof the case $n = 1$ and $d = 1$, the induction step and higher dimensions can be treated analogously. Let V be defined as in Definition 7.4, $t_0 < t_1 < t_2 = t$ and $x_j \in \mathcal{O}$ such that there exists a convex set $\mathcal{A} \subset \mathcal{O}$ with $x_j \in \mathcal{A}$ for all $0 \leq j \leq 2$. Then

$$I_V\delta(x(t_1) - x_1) = \sigma^{\dagger}_{\sqrt{i},t_0,t}\sigma_{\sqrt{i}}\Big(\exp\Big(-i\int_{t_0}^{t} V(x_0 + \langle\cdot, \mathbb{1}_{[t_0,r)}\rangle)\,dr\Big)\delta(x(t) - x)\delta(x(t_1) - x_1)\Big)$$

is a well-defined Hida distribution. This follows by (4.10), Assumption 7.2 and the following equality:

$$\begin{aligned}
&\sigma_{\sqrt{i},t_0,t}\Big(\exp\Big(-i\int_{t_0}^{t} V(x_0 + \langle\cdot, \mathbb{1}_{[t_0,r)}\rangle)\,dr\Big)\delta(x(t) - x)\delta(x(t_1) - x_1)\Big)\\
&\quad= \exp\Big(-i\int_{t_0}^{t_1} V(x_0 + \sqrt{i}\langle\cdot, \mathbb{1}_{[t_0,r)}\rangle)\,dr\Big)\exp\Big(-i\int_{t_1}^{t_2} V(x_0 + \sqrt{i}\langle\cdot, \mathbb{1}_{[t_0,r)}\rangle)\,dr\Big)\\
&\qquad\times \sigma_{\sqrt{i}}\delta(\langle\cdot, \mathbb{1}_{[t_0,t_1)}\rangle - (x_1 - x_0))\sigma_{\sqrt{i}}\delta(\langle\cdot, \mathbb{1}_{[t_1,t_2)}\rangle - (x_2 - x_1))\\
&\quad= \tau_{\frac{x_1-x_0}{\sqrt{i}(t_1-t_0)}\mathbb{1}_{[t_0,t_1)}} P_{\frac{1}{\sqrt{t_1-t_0}}\mathbb{1}_{[t_0,t_1)}}\Big(\exp\Big(-i\int_{t_0}^{t_1} V(x_0 + \sqrt{i}\langle\cdot, \mathbb{1}_{[t_0,r)}\rangle)\,dr\Big)\\
&\qquad\times \exp\Big(-i\int_{t_1}^{t_2} V(x_0 + \sqrt{i}\langle\cdot, \mathbb{1}_{[t_0,r)}\rangle)\,dr\Big)\Big)\\
&\qquad\times \sigma_{\sqrt{i}}\delta(\langle\cdot, \mathbb{1}_{[t_1,t_2)}\rangle - (x_2 - x_1)) \diamond \sigma_{\sqrt{i}}\delta(\langle\cdot, \mathbb{1}_{[t_0,t_1)}\rangle - (x_1 - x_0))\\
&= \exp\Big(-i\int_{t_0}^{t_1} V(x_0 + \frac{r-t_0}{t_1-t_0}(x_1 - x_0) + \sqrt{i}\langle\cdot, \mathbb{1}_{[t_0,r)}\rangle - \sqrt{i}\frac{r-t_0}{t_1-t_0}(x_1 - x_0)\langle\cdot, \mathbb{1}_{[t_0,t_1)}\rangle)\,dr\Big)\\
&\quad\times \exp\Big(-i\int_{t_1}^{t_2} V(x_0 + \frac{t_1-t_0}{t_1-t_0}(x_1 - x_0) + \sqrt{i}\langle\cdot, \mathbb{1}_{[t_0,r)}\rangle - \sqrt{i}\frac{t_1-t_0}{t_1-t_0}\langle\cdot, \mathbb{1}_{[t_0,t_1)}\rangle)\,dr\Big)\\
&\qquad\times \sigma_{\sqrt{i}}\delta(\langle\cdot, \mathbb{1}_{[t_1,t_2)}\rangle - (x_2 - x_1)) \diamond \sigma_{\sqrt{i}}\delta(\langle\cdot, \mathbb{1}_{[t_0,t_1)}\rangle - (x_1 - x_0))\\
&= \exp\Big(-i\int_{t_0}^{t_1} V(x_0 + \frac{r-t_0}{t_1-t_0}(x_1 - x_0) + \sqrt{i}\langle\cdot, \mathbb{1}_{[t_0,r)}\rangle - \sqrt{i}\frac{r-t_0}{t_1-t_0}(x_1 - x_0)\langle\cdot, \mathbb{1}_{[t_0,t_1)}\rangle)\,dr\Big)
\end{aligned}$$

$$\times\, \tau_{\frac{x_2-x_1}{\sqrt{i}(t_2-t_1)}\mathbf{1}_{[t_1,t_2)}} P_{\frac{1}{\sqrt{t_2-t_1}}\mathbf{1}_{[t_1,t_2)}} \exp\left(-i \int_{t_1}^{t_2} V(x_1 + \sqrt{i}\langle \cdot, \mathbb{1}_{[t_1,r)}\rangle)\, dr \right)$$

$$\diamond\, \sigma_{\sqrt{i}}\delta(\langle \cdot, \mathbb{1}_{[t_1,t_2)}\rangle - (x_2 - x_1)) \diamond \sigma_{\sqrt{i}}\delta(\langle \cdot, \mathbb{1}_{[t_0,t_1)}\rangle - (x_1 - x_0))$$

$$= \exp\left(-i \int_{t_0}^{t_1} V(x_0 + \frac{r-t_0}{t_1-t_0}(x_1 - x_0) + \sqrt{i}\langle \cdot, \mathbb{1}_{[t_0,r)}\rangle - \sqrt{i}\frac{r-t_0}{t_1-t_0}\langle \cdot, \mathbb{1}_{[t_0,t_1)}\rangle)\, dr \right)$$

$$\diamond\, \exp\left(-i \int_{t_1}^{t_2} V(x_1 + \frac{r-t_1}{t_2-t_1}(x_2 - x_1) + \sqrt{i}\langle \cdot, \mathbb{1}_{[t_1,r)}\rangle - \sqrt{i}\frac{r-t_1}{t_2-t_1}\langle \cdot, \mathbb{1}_{[t_1,t_2)}\rangle)\, dr \right)$$

$$\diamond\, \sigma_{\sqrt{i}}\delta(\langle \cdot, \mathbb{1}_{[t_1,t_2)}\rangle - (x_2 - x_1)) \diamond \sigma_{\sqrt{i}}\delta(\langle \cdot, \mathbb{1}_{[t_0,t_1)}\rangle - (x_1 - x_0)).$$

Remark that every projection and translation in the equation above is well-defined since $\mathcal{A}$ is convex and by Assumption 7.2. Moreover, we use that $\tau_{\frac{x_2-x_1}{\sqrt{i}(t_2-t_1)}\mathbf{1}_{[t_1,t_2)}} P_{\frac{1}{\sqrt{t_2-t_1}}\mathbf{1}_{[t_1,t_2)}}$ applied to the exponential which only depends on the time interval $[t_0, t_1)$ is again this exponential due to the disjointness of the time intervals $[t_0, t_1)$ and $[t_1, t_2)$. Note that in the equation above again every Wick product can be understood as a projective independent pointwise product. Now close to the proof of Theorem 7.16 one can show that

$$T(I_V\delta(x(t_1) - x_1))(g) = \exp\left(-\frac{1}{2}|g_{\Lambda^c}|_0^2 + ig(t)\cdot x - ig(t_0)\cdot x_0 \right)$$
$$\times\, K_V^{(\dot{g})}(x_1, t_1; x_0, t_0) K_V^{(\dot{g})}(x_2, t; x_1, t_1).$$

□

7.4 Verifying the time-dependent Schrödinger equation

In this section we show that $K_V^{(\dot{g})}(x, t; x_0, t_0)$, defined as in Theorem 7.16, solves the time-dependent Schrödinger equation.

Theorem 7.18. *Let $0 < T < \infty$, $g \in S_d(\mathbb{R})$ and $V : \mathcal{D}_{\sqrt{i},d} \to \mathbb{C}$ as in Definition 7.4, then $K_V^{(\dot{g})}(x, t; x_0, t_0)$, see Theorem 7.16 for its definition, solves the Schrödinger equation for all $x, x_0 \in O$, $0 \le t_0 < t \le T$, such that $x_0 + \frac{r-t_0}{t-t_0}(x - x_0) \in O$, for all $t_0 < r < t$, i.e.,*

$$\left(i\frac{\partial}{\partial t} - \frac{1}{2}\Delta - V_{\dot{g}}(t, x) \right) K_V^{(\dot{g})}(x, t; x_0, t_0) = 0, \tag{7.18}$$

with initial condition

$$\lim_{t \searrow t_0} K_V^{(\dot{g})}(x, t; x_0, t_0) = \delta(x - x_0).$$

Note that the proof is nearly identical to proof of Theorem 5.20. Again we only consider the one-dimensional case.

Proof. Let $x, x_0 \in \mathcal{O}$, $0 \leq t_0 < t \leq T$, such that $x_0 + \frac{r-t_0}{t-t_0}(x - x_0) \in \mathcal{O}$, for all $t_0 < r < t$, and $V : \mathcal{D}_{\sqrt{i}} \to \mathbb{C}$ such that Assumption 7.3 is fulfilled, then we know from Theorem 7.16 that $K_V^{(\dot{g})}(x, t; x_0, t_0)$ is given by

$$\begin{aligned} &K_V^{(\dot{g})}(x, t; x_0, t_0) \\ &\quad = E\left(\exp\left(-i\int_{t_0}^{t} V_{\dot{g}}\Big(r, x_0 + \frac{r-t_0}{t-t_0}(x-x_0) + \sqrt{i}\langle\cdot, \mathbb{1}_{[t_0,r)}\rangle - \frac{r-t_0}{t-t_0}\sqrt{i}\langle\cdot, \mathbb{1}_{[t_0,t)}\rangle\Big)dr\right)\right) \\ &\qquad\qquad \times E\left(\sigma_{\sqrt{i}}\delta\left(\langle\cdot, \mathbb{1}_{[t_0,t)}\rangle - (x-x_0)\right)\right). \end{aligned}$$

Hence, the time derivative of $K(x, t; x_0, t_0)$ is given by

$$\begin{aligned} \frac{\partial}{\partial t}K_V^{(\dot{g})}(x, t; x_0, t_0) &= \left(\frac{\partial}{\partial t}E\left(\sigma_{\sqrt{i}}\delta\left(\langle\cdot, \mathbb{1}_{[t_0,t)}\rangle - (x-x_0)\right)\right)\right) \\ &\times E\left(\exp\left(-i\int_{t_0}^{t} V_{\dot{g}}\Big(r, x_0 + \frac{r-t_0}{t-t_0}(x-x_0) + \sqrt{i}\langle\cdot, \mathbb{1}_{[t_0,r)}\rangle - \frac{r-t_0}{t-t_0}\sqrt{i}\langle\cdot, \mathbb{1}_{[t_0,t)}\rangle\Big)dr\right)\right) \\ &\qquad + E\left(\sigma_{\sqrt{i}}\delta\left(\langle\cdot, \mathbb{1}_{[t_0,t)}\rangle - (x-x_0)\right)\right) \\ &\times \frac{\partial}{\partial t}E\left(\exp\left(-i\int_{t_0}^{t} V_{\dot{g}}\Big(r, x_0 + \frac{r-t_0}{t-t_0}(x-x_0) + \sqrt{i}\langle\cdot, \mathbb{1}_{[t_0,r)}\rangle - \frac{r-t_0}{t-t_0}\sqrt{i}\langle\cdot, \mathbb{1}_{[t_0,t)}\rangle\Big)dr\right)\right). \end{aligned} \tag{7.19}$$

From the free case (see e.g. [74]) we know that

$$\frac{\partial}{\partial t}E\left(\sigma_{\sqrt{i}}\delta\left(\langle\cdot, \mathbb{1}_{[t_0,t)}\rangle - (x-x_0)\right)\right) = -i\frac{1}{2}\Delta E\left(\sigma_{\sqrt{i}}\delta\left(\langle\cdot, \mathbb{1}_{[t_0,t)}\rangle - (x-x_0)\right)\right).$$

To get shorter equations from now on we denote $B_{t_0,t} := \langle\cdot, \mathbb{1}_{[t_0,t)}\rangle$. To consider the time derivative of the second part we first have a look at the difference quotient from the right side:

$$\begin{aligned} &\frac{\partial}{\partial t}^{+} E\left(\exp\left(-i\int_{t_0}^{t} V_{\dot{g}}\Big(r, x_0 + \frac{r-t_0}{t-t_0}(x-x_0) + \sqrt{i}B_{t_0,r} - \frac{r-t_0}{t-t_0}\sqrt{i}B_{t_0,t}\Big)dr\right)\right) \\ &= \lim_{h\searrow 0}\frac{1}{h}\Bigg(E\left(\exp\left(-i\int_{t_0}^{t+h} V_{\dot{g}}\Big(r, x_0 + \frac{r-t_0}{t+h-t_0}(x-x_0) + \sqrt{i}B_{t_0,r} - \frac{r-t_0}{t+h-t_0}\sqrt{i}B_{t_0,t+h}\Big)dr\right)\right) \\ &\qquad - E\left(\exp\left(-i\int_{t_0}^{t} V_{\dot{g}}\Big(r, x_0 + \frac{r-t_0}{t-t_0}(x-x_0) + \sqrt{i}B_{t_0,r} - \frac{r-t_0}{t-t_0}\sqrt{i}B_{t_0,t}\Big)dr\right)\right)\Bigg) \end{aligned}$$

Let $h > 0$ such that $t + h \leq T$ then by adding zero we get that

$$\begin{aligned} &\frac{1}{h}\Bigg[E\left(\exp\left(-i\int_{t_0}^{t+h} V_{\dot{g}}\left(r, x_0 + \frac{r-t_0}{t-t_0+h}(x-x_0) + \sqrt{i}B_{t_0,r} - \frac{r-t_0}{t-t_0+h}\sqrt{i}B_{t_0,t+h}\right)dr\right)\right) \\ &\qquad - E\left(\exp\left(-i\int_{t_0}^{t} V_{\dot{g}}\left(r, x_0 + \frac{r-t_0}{t-t_0}(x-x_0) + \sqrt{i}B_{t_0,r} - \frac{r-t_0}{t-t_0}\sqrt{i}B_{t_0,t}\right)dr\right)\right)\Bigg] \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{h}\Bigg[E\left(\exp\left(-i\int_{t_0}^{t+h} V_{\dot g}\left(r, x_0 + \frac{r-t_0}{t-t_0+h}(x-x_0) + \sqrt{i}B_{t_0,r} - \frac{r-t_0}{t-t_0+h}\sqrt{i}B_{t_0,t+h}\right)dr\right)\right)\\
&\quad - E\left(\exp\left(-i\int_{t_0}^{t} V_{\dot g}\left(r, x_0 + \frac{r-t_0}{t-t_0+h}(x-x_0) + \sqrt{i}B_{t_0,r} - \frac{r-t_0}{t-t_0+h}\sqrt{i}B_{t_0,t+h}\right)dr\right)\right)\\
&\quad + E\left(\exp\left(-i\int_{t_0}^{t} V_{\dot g}\left(r, x_0 + \frac{r-t_0}{t-t_0+h}(x-x_0) + \sqrt{i}B_{t_0,r} - \frac{r-t_0}{t-t_0+h}\sqrt{i}B_{t_0,t+h}\right)dr\right)\right)\\
&\quad - E\left(\exp\left(-i\int_{t_0}^{t} V_{\dot g}\left(r, x_0 + \frac{r-t_0}{t-t_0}(x-x_0) + \sqrt{i}B_{t_0,r} - \frac{r-t_0}{t-t_0}\sqrt{i}B_{t_0,t+h}\right)dr\right)\right)\\
&\quad + E\left(\exp\left(-i\int_{t_0}^{t} V_{\dot g}\left(r, x_0 + \frac{r-t_0}{t-t_0}(x-x_0) + \sqrt{i}B_{t_0,r} - \frac{r-t_0}{t-t_0}\sqrt{i}B_{t_0,t+h}\right)dr\right)\right)\\
&\quad - E\left(\exp\left(-i\int_{t_0}^{t} V_{\dot g}\left(r, x_0 + \frac{r-t_0}{t-t_0}(x-x_0) + \sqrt{i}B_{t_0,r} - \frac{r-t_0}{t-t_0}\sqrt{i}B_{t_0,t}\right)dr\right)\right)\Bigg]. \qquad (7.20)
\end{aligned}
$$

Now we consider it as three difference quotients, separately. For the last two summands of (7.20) we get that

$$
\begin{aligned}
&\lim_{h\searrow 0}\frac{1}{h}\Bigg[E\left(\exp\left(-i\int_{t_0}^{t} V_{\dot g}\left(r, x_0 + \frac{r-t_0}{t-t_0}(x-x_0) + \sqrt{i}B_{t_0,r} - \frac{r-t_0}{t-t_0}\sqrt{i}B_{t_0,t+h}\right)dr\right)\right)\\
&\qquad - E\left(\exp\left(-i\int_{t_0}^{t} V_{\dot g}\left(r, x_0 + \frac{r-t_0}{t-t_0}(x-x_0) + \sqrt{i}B_{t_0,r} - \frac{r-t_0}{t-t_0}\sqrt{i}B_{t_0,t}\right)dr\right)\right)\Bigg]\\
&= \frac{\partial}{\partial u}^{+} E\left(\exp\left(-i\int_{t_0}^{t} V_{\dot g}\left(r, x_0 + \frac{r-t_0}{t-t_0}(x-x_0) + \sqrt{i}B_{t_0,r} - \frac{r-t_0}{t-t_0}\sqrt{i}B_{t_0,u}\right)dr\right)\right)\Bigg|_{u=t}\\
&\qquad = \frac{\partial}{\partial u}\int_{S'(\mathbb{R})} \exp\Bigg(-i\int_{t_0}^{t} V_{\dot g}\Bigg(r, x_0 + \frac{r-t_0}{t-t_0}(x-x_0) + \sqrt{i}B_{t_0,r}(\omega)\\
&\qquad\qquad - \frac{r-t_0}{t-t_0}\sqrt{i}B_{t_0,t}(\omega) - \frac{r-t_0}{t-t_0}\sqrt{i}B_{t,u}(\omega)\Bigg)dr\Bigg)d\mu(\omega)\Bigg|_{u=t}\\
&\qquad = \frac{\partial}{\partial u}\int_{S'(\mathbb{R})}\int_{S'(\mathbb{R})} \exp\Bigg(-i\int_{t_0}^{t} V_{\dot g}\Bigg(r, x_0 + \frac{r-t_0}{t-t_0}(x-x_0) + \sqrt{i}B_{t_0,r}(\omega_1)\\
&\qquad\qquad - \frac{r-t_0}{t-t_0}\sqrt{i}B_{t_0,t}(\omega_1) - \frac{r-t_0}{t-t_0}\sqrt{i}B_{t,u}(\omega_2)\Bigg)dr\Bigg)d\mu(\omega_2)d\mu(\omega_1)\Bigg|_{u=t}.
\end{aligned}
$$

Here we use the fact that $B_{t_0,r}$ and $B_{t,t+h}$ are independent for all $r \le t$. Therefore, we get by Corollary 5.4 and since $V_{\dot g}$ is analytic that

$$
\begin{aligned}
&\frac{\partial}{\partial u}\int_{S'(\mathbb{R})} \exp\Bigg(-i\int_{t_0}^{t} V_{\dot g}\Bigg(r, x_0 + \frac{r-t_0}{t-t_0}(x-x_0) + \sqrt{i}B_{t_0,r}(\omega_1)\\
&\qquad - \frac{r-t_0}{t-t_0}\sqrt{i}B_{t_0,t}(\omega_1) - \frac{r-t_0}{t-t_0}\sqrt{i}B_{t,u}(\omega_2)\Bigg)dr\Bigg)d\mu(\omega_2)\Bigg|_{u=t}\\
&\qquad = -i\frac{1}{2}\frac{\partial^2}{\partial y^2}\int_{S'(\mathbb{R})} \exp\Bigg(-i\int_{t_0}^{t} V_{\dot g}\Bigg(r, x_0 + \sqrt{i}B_{t_0,r}(\omega_1)
\end{aligned}
$$

$$
\begin{aligned}
&\quad - \frac{r-t_0}{t-t_0}\left(y + \sqrt{i}B_{t,u}(\omega_2)\right)\Bigg)dr\Bigg)d\mu(\omega_2)\Bigg|_{y=(-x+x_0)+\sqrt{i}B_{t_0,t}(\omega_1)} \\
&= -i\frac{1}{2}\frac{\partial^2}{\partial x^2}\int_{S'(\mathbb{R})}\exp\Bigg(-i\int_{t_0}^{t}V_{\dot{g}}\Bigg(r, x_0 + \frac{r-t_0}{t-t_0}(x-x_0) + \sqrt{i}B_{t_0,r}(\omega_1) \\
&\qquad - \frac{r-t_0}{t-t_0}\sqrt{i}B_{t_0,t}(\omega_1) - \frac{r-t_0}{t-t_0}\sqrt{i}B_{t,u}(\omega_2)\Bigg)dr\Bigg)d\mu(\omega_2)
\end{aligned}
$$

for μ- almost every $\omega_1 \in S'(\mathbb{R})$. Hence

$$
\begin{aligned}
&\frac{\partial}{\partial u}^{+} E\left(\exp\left(-i\int_{t_0}^{t}V_{\dot{g}}\left(r, x_0 + \frac{r-t_0}{t-t_0}(x-x_0) + \sqrt{i}B_{t_0,r} - \frac{r-t_0}{t-t_0}\sqrt{i}B_{t_0,u}\right)dr\right)\right)\Bigg|_{u=t} \\
&\quad = -i\frac{1}{2}\Delta E\left(\exp\left(-i\int_{0}^{t-t_0}V_{\dot{g}}\left(r, x_0 + \frac{r-t_0}{t-t_0}(x-x_0) + \sqrt{i}B_{t_0,r} - \frac{r-t_0}{t-t_0}\sqrt{i}B_{t_0,t}\right)dr\right)\right).
\end{aligned}
$$

For the first two summands of (7.20) we get that

$$
\begin{aligned}
&\frac{1}{h}\Bigg[E\left(\exp\left(-i\int_{t_0}^{t+h}V_{\dot{g}}\left(r, x_0 + \frac{r-t_0}{t-t_0+h}(x-x_0) + \sqrt{i}B_{t_0,r} - \frac{r-t_0}{t-t_0+h}\sqrt{i}B_{t_0,t+h}\right)dr\right)\right) \\
&\quad - E\left(\exp\left(-i\int_{t_0}^{t}V_{\dot{g}}\left(r, x_0 + \frac{r-t_0}{t-t_0+h}(x-x_0) + \sqrt{i}B_{t_0,r} - \frac{r-t_0}{t-t_0+h}\sqrt{i}B_{t_0,t+h}\right)dr\right)\right)\Bigg] \\
&= \frac{1}{h}\Bigg[E\Bigg(\exp\left(-i\int_{t_0}^{t}V_{\dot{g}}\left(r, x_0 + \frac{r-t_0}{t-t_0+h}(x-x_0) + \sqrt{i}B_{0,r} - \frac{r-t_0}{t-t_0+h}\sqrt{i}B_{t_0,t+h}\right)dr\right) \\
&\quad \left(\exp\left(-i\int_{t}^{t+h}V_{\dot{g}}\left(r, x_0 + \frac{r-t_0}{t-t_0+h}(x-x_0) + \sqrt{i}B_{t_0,r} - \frac{r-t_0}{t-t_0+h}\sqrt{i}B_{t_0,t+h}\right)dr\right) - 1\right)\Bigg)\Bigg].
\end{aligned}
$$

One can easy show that if V fulfills Assumption 7.3 then this also holds for $V_{\dot{g}}(t,\cdot)$ uniformly on $[0,T]$. Due to this and the fact that $u \mapsto B_{t,u}$ is continuous on $[t,T]$ the limit is given by

$$
\begin{aligned}
&\lim_{h\searrow 0}\frac{1}{h}\Bigg[E\left(\exp\left(-i\int_{t_0}^{t+h}V_{\dot{g}}\left(r, x_0 + \frac{r-t_0}{t-t_0+h}(x-x_0) + \sqrt{i}B_{t_0,r} - \frac{r-t_0}{t-t_0+h}\sqrt{i}B_{t_0,t+h}\right)dr\right)\right) \\
&\quad - E\left(\exp\left(-i\int_{t_0}^{t}V_{\dot{g}}\left(r, x_0 + \frac{r-t_0}{t-t_0+h}(x-x_0) + \sqrt{i}B_{t_0,r} - \frac{r-t_0}{t-t_0+h}\sqrt{i}B_{t_0,t+h}\right)dr\right)\right)\Bigg] \\
&\quad = -iV_{\dot{g}}(t,x)E\left(\exp\left(-i\int_{t_0}^{t}V_{\dot{g}}\left(r, x_0 + \frac{r-t_0}{t-t_0}(x-x_0) + \sqrt{i}B_{t_0,r} - \frac{r-t_0}{t-t_0}\sqrt{i}B_{t_0,t}\right)dr\right)\right).
\end{aligned}
$$

For the remaining two summands of (7.20) it is easy to see with Assumption 5.19 and the mean value theorem that there exists some $s \in (t, t+h)$ such that

$$
\begin{aligned}
&\frac{1}{h}\Bigg[E\left(\exp\left(-i\int_{t_0}^{t}V_{\dot{g}}\left(r, x_0 + \frac{r-t_0}{t-t_0+h}(x-x_0) + \sqrt{i}B_{t_0,r} - \frac{r-t_0}{t-t_0+h}\sqrt{i}B_{t_0,t+h}\right)dr\right)\right) \\
&\quad - E\left(\exp\left(-i\int_{t_0}^{t}V_{\dot{g}}\left(r, x_0 + \frac{r-t_0}{t-t_0}(x-x_0) + \sqrt{i}B_{t_0,r} - \frac{r-t_0}{t-t_0}\sqrt{i}B_{t_0,t+h}\right)dr\right)\right)\Bigg]
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{\partial}{\partial u} E\left(\exp\left(-i\int_{t_0}^{t} V_{\dot g}\left(r, x_0 + \frac{r-t_0}{u-t_0}(x-x_0) + \sqrt{i}B_{t_0,r} - \frac{r-t_0}{u-t_0}\sqrt{i}B_{t_0,t+h}\right) dr\right)\right)\Bigg|_{u=s} \\
&= E\left(\frac{\partial}{\partial u}\left(-i\int_{t_0}^{t} V_{\dot g}\left(r, x_0 + \frac{r-t_0}{u-t_0}(x-x_0) + \sqrt{i}B_{t_0,r} - \frac{r-t_0}{u-t_0}\sqrt{i}B_{t_0,t+h}\right) dr\right)\Bigg|_{u=s}\right. \\
&\qquad \left.\times \exp\left(-i\int_{t_0}^{t} V_{\dot g}\left(r, x_0 + \frac{r-t_0}{s-t_0}(x-x_0) + \sqrt{i}B_{t_0,r} - \frac{r-t_0}{s-t_0}\sqrt{i}B_{t_0,t+h}\right) dr\right)\right) \\
&= E\left(-i\int_{t_0}^{t} V'_{\dot g}\left(r, x_0 + \frac{r-t_0}{s-t_0}(x-x_0) + \sqrt{i}B_{t_0,r} - \frac{r-t_0}{s-t_0}\sqrt{i}B_{t_0,t+h}\right)\frac{-(r-t_0)}{(s-t_0)^2}(x-x_0+\sqrt{i}B_{t_0,t+h})\, dr\right. \\
&\qquad \left.\times \exp\left(-i\int_{t_0}^{t} V_{\dot g}\left(r, x_0 + \frac{r-t_0}{s-t_0}(x-x_0) + \sqrt{i}B_{t_0,r} - \frac{r-t_0}{s-t_0}\sqrt{i}B_{t_0,t+h}\right) dr\right)\right),
\end{aligned}
$$

and therefore the limit is given by

$$
\begin{aligned}
&\lim_{h\searrow 0}\frac{1}{h}\Bigg[E\left(\exp\left(-i\int_{t_0}^{t} V_{\dot g}\left(r, x_0 + \frac{r-t_0}{t-t_0+h}(x-x_0) + \sqrt{i}B_{t_0,r} - \frac{r-t_0}{t-t_0+h}\sqrt{i}B_{t_0,t+h}\right) dr\right)\right) \\
&\qquad - E\left(\exp\left(-i\int_{t_0}^{t} V_{\dot g}\left(r, x_0 + \frac{r-t_0}{t-t_0}(x-x_0) + \sqrt{i}B_{t_0,r} - \frac{r-t_0}{t-t_0}\sqrt{i}B_{t_0,t+h}\right) dr\right)\right)\Bigg] \\
&= E\left(-i\int_{t_0}^{t} V'_{\dot g}\left(r, x_0 + \frac{r-t_0}{t-t_0}(x-x_0) + \sqrt{i}B_{t_0,r} - \frac{r-t_0}{(t-t_0)}\sqrt{i}B_{t_0,t}\right)\frac{-(r-t_0)}{(t-t_0)^2}(x-x_0+\sqrt{i}B_{t_0,t})\, dr\right. \\
&\qquad \left.\times \exp\left(-i\int_{t_0}^{t} V_{\dot g}\left(r, x_0 + \frac{r-t_0}{t-t_0}(x-x_0) + \sqrt{i}B_{t_0,r} - \frac{r-t_0}{t-t_0}\sqrt{i}B_{t_0,t}\right) dr\right)\right).
\end{aligned}
$$

On the one side, using Lemma 4.7 one can show that the random variables $B_{t_0,t}$ and

$$
\begin{aligned}
&-i\int_{t_0}^{t} V'_{\dot g}\left(r, x_0 + \frac{r-t_0}{t-t_0}(x-x_0) + \sqrt{i}B_{t_0,r} - \frac{r-t_0}{t-t_0}\sqrt{i}B_{t_0,t}\right) \\
&\qquad \times \exp\left(-i\int_{t_0}^{t} V_{\dot g}\left(r, x_0 + \frac{r-t_0}{t-t_0}(x-x_0) + \sqrt{i}B_{t_0,r} - \frac{r-t_0}{t-t_0}\sqrt{i}B_{t_0,t}\right) dr\right)
\end{aligned}
$$

are independent. Hence, since $E(B_{t_0,t}) = 0$ we get that

$$
\begin{aligned}
&E\left(\sqrt{i}B_{t_0,t}\left(-i\int_{t_0}^{t} V'_{\dot g}\left(r, x_0 + \frac{r-t_0}{t-t_0}(x-x_0) + \sqrt{i}B_{t_0,r} - \frac{r-t_0}{t-t_0}\sqrt{i}B_{t_0,t}\right)\frac{-(r-t_0)}{(t-t_0)^2}\, dr\right)\right. \\
&\qquad \left.\times \exp\left(-i\int_{t_0}^{t} V_{\dot g}\left(r, x_0 + \frac{r-t_0}{t-t_0}(x-x_0) + \sqrt{i}B_{t_0,r} - \frac{r-t_0}{t-t_0}\sqrt{i}B_{t_0,t}\right) dr\right)\right) = 0.
\end{aligned}
$$

On the other side we get the cross terms by

$$
\begin{aligned}
&E\left(\sigma_{\sqrt{i}}\delta\left(\langle\cdot, \mathbb{1}_{[t_0,t)}\rangle - (x-x_0)\right)\right) \\
&\quad \times E\left(-i\int_{t_0}^{t} V'_{\dot g}\left(r, x_0 + \frac{r-t_0}{t-t_0}(x-x_0) + \sqrt{i}B_{t_0,r} - \frac{r-t_0}{t-t_0}\sqrt{i}B_{t_0,t}\right)\frac{-(r-t_0)}{(t-t_0)^2}(x-x_0)\, dr\right.
\end{aligned}
$$

$$\times \exp\left(-i\int_{t_0}^{t} V_{\dot g}\left(r, x_0 + \frac{r-t_0}{t-t_0}(x-x_0) + \sqrt{i}B_{t_0,r} - \frac{r-t_0}{t-t_0}\sqrt{i}B_{t_0,t}\right)dr\right)\right)$$
$$= (-1)\frac{(x-x_0)}{t-t_0}E\left(\sigma_{\sqrt{i}}\delta\left(\langle\cdot, \mathbb{1}_{[t_0,t)}\rangle - (x-x_0)\right)\right)$$
$$E\left(\frac{\partial}{\partial x}\exp\left(-i\int_{t_0}^{t} V_{\dot g}\left(r, x_0 + \frac{r-t_0}{t-t_0}(x-x_0) + \sqrt{i}B_{t_0,r} - \frac{r-t_0}{t-t_0}\sqrt{i}B_{t_0,t}\right)dr\right)\right)$$
$$= -i\frac{\partial}{\partial x}E\left(\sigma_{\sqrt{i}}\delta\left(\langle\cdot, \mathbb{1}_{[t_0,t)}\rangle - (x-x_0)\right)\right)$$
$$\times \frac{\partial}{\partial x}E\left(\exp\left(-i\int_{t_0}^{t} V_{\dot g}\left(r, x_0 + \frac{r-t_0}{t-t_0}(x-x_0) + \sqrt{i}B_{t_0,r} - \frac{r-t_0}{t-t_0}\sqrt{i}B_{t_0,t}\right)dr\right)\right).$$

Summing up again yields

$$i\frac{\partial}{\partial t}^{+} K_V^{(\dot g)}(x,t;x_0,t_0) = -\frac{1}{2}\Delta K_V^{(\dot g)}(x,t;x_0,t_0) + V_{\dot g}(t,x)K_V^{(\dot g)}(x,t;x_0,t_0).$$

Analogously one can show that (7.18) is also true for $\frac{\partial}{\partial t}^{-} K_V^{(\dot g)}(x,t;x_0,t_0)$. Again

$$\lim_{t\searrow t_0} K_V^{(\dot g)}(x,t;x_0,t_0) = \delta(x-x_0).$$

□

Remark 7.19. *In white noise one would arise the kernel to* (7.18) *by inserting Donsker's delta as an initial distribution for the solutions found in Theorem 7.12. In detail, close to Theorem 5.13 the kernel is given by*

$$E\left[\exp\left(-i\int_0^{t-t_0} V_{\dot g}\left(t-r, z+\sqrt{i}\langle\omega, \mathbb{1}_{[0,r)}\rangle\right)dr\right)\delta\left(x-x_0+\sqrt{i}\langle\omega, \mathbb{1}_{[0,t-t_0)}\rangle\right)\right]. \tag{7.21}$$

where $V_{\dot g}(t,x) := V(x) + \dot g(t)x$. Comparing this with $K_V^{(\dot g)}(x,t;x_0,t_0)$ for the first time, it seems to be a contradiction. But using suitable variable transformation, similar to Theorem 7.8, one can show that both expectations coincide.

7.5 Complex scaling for Feynman integrands corresponding to non-smooth or rapidly growing potentials

In the following we are interested in representation of the Feynman integrand for potentials V for which

$$\sigma_{\sqrt{i}}\exp\left(-i\int_{t_0}^{t} V\left(x_0 + \left\langle\cdot, \mathbb{1}_{[t_0,r)} - \frac{t-r}{t-t_0}\mathbb{1}_{[t_0,r)}\right\rangle\right)dr\right)$$

is not a regular generalized function, and Theorem 4.25 and therefore Theorem 7.1 can not be applied. In detail, we consider the Khandekar-Streit class and the Westerkamp-Kuna-Streit class using the series expansion of their exponentials as described in Section 6.3.

7.5.1 The Khandekar-Streit class

Close to Section 6.3.1 we consider a finite signed Borel measure ν, as in Definition 6.1. For getting shorter equations we only consider the one dimensional case $d = 1$ and time-independent measures. As shown in [54], for a path as in (6.3), $-\infty < t_0 < t < \infty$ and $x_0, x \in \mathbb{R}$, the corresponding Feynman integrand is given by

$$I_V = \sum_{n=0}^{\infty}(-i)^n \int_{\mathbb{R}^n}\int_{\Lambda_n}\left(I_0 \prod_{j=1}^{n}\delta(x(r_j)-x_j)\right)dr_j\, d\nu(x_j),$$

where $\Lambda_n = \{(r_1,\ldots,r_n) \mid t_0 = r_0 < r_1 < \ldots < r_n < t\}$. Thus, informally we can write

$$\begin{aligned} I_V &= \sum_{n=0}^{\infty}(-i)^n \int_{\mathbb{R}^n}\int_{\Lambda_n}\sigma^{\dagger}_{\sqrt{i},t_0,t}\sigma_{\sqrt{i}}\Bigg(\delta(\langle\omega, \mathbb{1}_{[t_0,t)}\rangle-(x-x_0)) \\ &\qquad\qquad \times\prod_{j=1}^{n}\delta(\langle\cdot, \mathbb{1}_{[t_0,r_j)}\rangle-(x_j-x_0))\Bigg)dr_j\, d\nu(x_j) \\ &= \sigma^{\dagger}_{\sqrt{i},t_0,t}\sigma_{\sqrt{i}}\Bigg(\sum_{n=0}^{\infty}(-i)^n\int_{\mathbb{R}^n}\int_{\Lambda_n}\delta(\langle\omega, \mathbb{1}_{[t_0,t)}\rangle-(x-x_0)) \\ &\qquad\qquad \times\prod_{j=1}^{n}\delta(\langle\cdot, \mathbb{1}_{[t_0,r_j)}\rangle-(x_j-x_0))dr_j\, d\nu(x_j)\Bigg). \end{aligned} \tag{7.22}$$

For a mathematical meaning of (7.22) roughly speaking every expression inside must be a Hida distribution. Let us remark that $\sigma^{\dagger}_{\sqrt{i},t_0,t}$ is a continuous operator form $(S)'$ into itself. Thus, whenever

$$\begin{aligned} &\sum_{n=0}^{\infty}(-i)^n\int_{\mathbb{R}^n}\int_{\Lambda_n}\sigma_{\sqrt{i}}\left(\delta(\langle\cdot, \mathbb{1}_{[t_0,t)}\rangle-(x-x_0))\prod_{j=1}^{n}\delta(\langle\cdot, \mathbb{1}_{[t_0,r_j)}\rangle-(x_j-x_0))\right)dr_j\,\nu(dx_j) \\ &\quad= \sigma_{\sqrt{i}}\left(\sum_{n=0}^{\infty}(-i)^n\int_{\mathbb{R}^n}\int_{\Lambda_n}\delta(\langle\cdot, \mathbb{1}_{[t_0,t)}\rangle-(x-x_0))\prod_{j=1}^{n}\delta(\langle\cdot, \mathbb{1}_{[t_0,r_j)}\rangle-(x_j-x_0))dr_j\,\nu(dx_j)\right) \end{aligned}$$

in $(S)'$ as limits of finite sums (7.22) holds by Theorem 2.12 and Theorem 2.11.

Theorem 7.20. *Let ν be a time-independent finite signed Borel measure as in Definition 6.1, then the corresponding Feynman integrand is given by*

$$\begin{aligned} I_V = \sigma^{\dagger}_{\sqrt{i},t_0,t}\Bigg(&\sigma_{\sqrt{i}}\tau_{\frac{x-x_0}{(t-t_0)}\mathbb{1}_{[t_0,t)}}P_{\frac{1}{\sqrt{t-t_0}}\mathbb{1}_{[t_0,t)}}\left(\sum_{n=0}^{\infty}(-i)^n\int_{\mathbb{R}^n}\int_{\Lambda_n}\left(\prod_{j=1}^{n}\delta(\langle\cdot, \mathbb{1}_{[t_0,r_j)}\rangle-(x_j-x_0))\right)dr_j d\nu(x_j)\right) \\ &\diamond\,\sigma_{\sqrt{i}}\delta(\langle\omega, \mathbb{1}_{[t_0,t)}\rangle-(x-x_0))\Bigg), \end{aligned} \tag{7.23}$$

for all $x, x_0 \in \mathbb{R}$ and all $-\infty < t_0 < t < \infty$. Here the Wick product is an independent pointwise product.

Proof. Let ν be a time-independent finite signed Borel measure ν as in Definition 6.1. I.e., ν is a measure on $\mathbb{R}$ fulfilling property (i) of Definition 6.1. In order to show (7.23) one has to verify that the operator $\sigma_{\sqrt{i}}\tau_{\frac{x-x_0}{(t-t_0)}\mathbb{1}_{[t_0,t)}}P_{\frac{1}{\sqrt{t-t_0}}\mathbb{1}_{[t_0,t)}}$ can be applied to

$$\sum_{n=0}^{\infty}(-i)^n\int_{\mathbb{R}^n}\int_{\Lambda_n}\prod_{j=1}^{n}\delta(x_0+\langle\cdot,\mathbb{1}_{[t_0,r_j)}\rangle-x_j)\,dr_j\,d\nu(x_j),$$

for all $x_0, x\in\mathbb{R}$ and $-\infty<t_0<t<\infty$. Using the notation $r_0=t_0$, we get by using Theorem 4.25, see also Example 4.31, that

$$\Phi_n:=\prod_{j=1}^{n}\delta(x_0+\langle\cdot,\mathbb{1}_{[t_0,r_j)}\rangle-x_j)=\prod_{j=1}^{n}\delta\left(\langle\cdot,\mathbb{1}_{[r_{j-1},r_j)}\rangle-(x_j-x_{j-1})\right)$$

and

$$\Psi_n:=\sigma_{\sqrt{i}}\Phi_n=\prod_{j=1}^{n}\sigma_{\sqrt{i}}\delta\left(\langle\cdot,\mathbb{1}_{[r_{j-1},r_j)}\rangle-(x_j-x_{j-1})\right)$$

are regular generalized functions which are in $C_{\frac{1}{\sqrt{t-t_0}}\mathbb{1}_{[t_0,t)},\frac{x-x_0}{\sqrt{t-t_0}}}$ and $C_{\frac{1}{\sqrt{t-t_0}}\mathbb{1}_{[t_0,t)},\frac{x-x_0}{\sqrt{i(t-t_0)}}}$, respectively. Note that both products contain of projective independent random variables. Thus, their S-transforms are given by

$$S\Phi_n(zg)=\prod_{j=1}^{n}\frac{1}{\sqrt{2\pi(r_j-r_{j-1})}}\exp\left(-\frac{1}{2(r_j-r_{j-1})}\left(\left\langle zg,\mathbb{1}_{[r_{j-1},r_j)}\right\rangle-(x_j-x_{j-1})\right)^2\right)$$

and

$$S\Psi_n(zg)=\prod_{j=1}^{n}\frac{1}{i\sqrt{2\pi((r_j-r_{j-1})}}\exp\left(-\frac{1}{i2(r_j-r_{j-1})}\left(\sqrt{i}\left\langle zg,\mathbb{1}_{[r_{j-1},r_j)}\right\rangle-(x_j-x_{j-1})\right)^2\right),$$

for all $g\in S(\mathbb{R})$ and all $z\in\mathbb{C}$. Both can be estimated by

$$\prod_{j=1}^{n}\frac{1}{\sqrt{2\pi(r_j-r_{j-1})}}\exp\left(\left(\frac{1}{2}+\frac{1}{\varepsilon}\right)|z|^2\sup_{s\in[t_0,t]}|g(s)|^2\right)\exp\left(\varepsilon\sup_{0\le j\le n}|x_j|^2\right),\tag{7.24}$$

for some $\varepsilon>0$. Here we use the fact that $2ab\le\frac{1}{\varepsilon}a^2+\varepsilon b^2$, for all $0<a,b<\infty$. Let us remark that the estimation above only deviates from the estimation in the original proof of the construction of the Feynman integrand for this class (see [40] and [54]) since we consider the S- and not the T-transform.

To calculate the S-transforms of $\tau_{\frac{x-x_0}{(t-t_0)}\mathbb{1}_{[t_0,t)}}P_{\frac{1}{\sqrt{t-t_0}}\mathbb{1}_{[t_0,t)}}\Phi_n$ the following relation is useful:

$$\tau_{\frac{x-x_0}{(t-t_0)}\mathbb{1}_{[t_0,t)}}P_{\frac{1}{\sqrt{t-t_0}}\mathbb{1}_{[t_0,t)}}\Phi_n\diamond\delta\left(\langle\cdot,\mathbb{1}_{[t_0,t)}\rangle-(x-x_0)\right)$$

$$= \prod_{j=1}^{n+1} \delta(x_0 + \langle \cdot, \mathbb{1}_{[t_0,r_j)} \rangle - x_j) = \prod_{j=1}^{n+1} \delta\Big(\langle \cdot, \mathbb{1}_{[r_{j-1},r_j)} \rangle - (x_j - x_{j-1})\Big),$$

where we use the notation $t_{n+1} = t$ and $x_{n+1} = x$. Thus it follows that

$$\begin{aligned} &S\left(\tau_{\frac{x-x_0}{\sqrt{i}(t-t_0)}\mathbb{1}_{[t_0,t)}} P_{\frac{1}{\sqrt{t-t_0}}\mathbb{1}_{[t_0,t)}} \Phi_n\right)(zg) \\ &\quad = \prod_{j=1}^{n+1} S\left(\delta\left(\langle \cdot, \mathbb{1}_{[r_{j-1},r_j)} \rangle - (x_j - x_{j-1})\right)\right)(zg)\left(S\left(\delta\left(\langle \cdot, \mathbb{1}_{[t_0,t)} - (x - x_0)\rangle\right)\right)(zg)\right)^{-1} \\ &\quad = \prod_{j=1}^{n+1} \frac{1}{\sqrt{2\pi(r_j - r_{j-1})}} \exp\left(-\frac{1}{2(r_j - r_{j-1})}\left(\left\langle zg, \mathbb{1}_{[r_{j-1},r_j)}\right\rangle - (x_j - x_{j-1})\right)^2\right) \\ &\qquad\qquad \times \sqrt{2\pi(t-t_0)} \exp\left(+\frac{1}{2(t-t_0)}\left(\langle zg, \mathbb{1}_{[t_0,t)}\rangle - (x - x_0)\right)^2\right), \end{aligned} \tag{7.25}$$

for all $g \in S(\mathbb{R})$ and $z \in \mathbb{C}$. Using Lemma 3.34 we obtain

$$\tau_{\frac{x-x_0}{\sqrt{i}(t-t_0)}\mathbb{1}_{[t_0,t)}} P_{\frac{1}{\sqrt{t-t_0}}\mathbb{1}_{[t_0,t)}} \Psi_n = \tau_{\frac{x-x_0}{\sqrt{i}(t-t_0)}\mathbb{1}_{[t_0,t)}} P_{\frac{1}{\sqrt{t-t_0}}\mathbb{1}_{[t_0,t)}} \sigma_{\sqrt{i}} \Phi_n = \sigma_{\sqrt{i}} \tau_{\frac{x-x_0}{(t-t_0)}\mathbb{1}_{[t_0,t)}} P_{\frac{1}{\sqrt{t-t_0}}\mathbb{1}_{[t_0,t)}} \Phi_n.$$

Therefore, we get that

$$\begin{aligned} &S\left(\tau_{\frac{x-x_0}{\sqrt{i}(t-t_0)}\mathbb{1}_{[t_0,t)}} P_{\frac{1}{\sqrt{t-t_0}}\mathbb{1}_{[t_0,t)}} \Psi_n\right)(zg) \\ &\quad = \prod_{j=1}^{n+1} \frac{1}{i\sqrt{2\pi(r_j - r_{j-1})}} \exp\left(-\frac{1}{i2(r_j - r_{j-1})}\left(\left\langle zg, \mathbb{1}_{[r_{j-1},r_j)}\right\rangle - (x_j - x_{j-1})\right)^2\right) \\ &\qquad\qquad \times \sqrt{2\pi(t-t_0)} \exp\left(-\frac{i}{2(t-t_0)}\left(\langle zg, \mathbb{1}_{[t_0,t)}\rangle - (x - x_0)\right)^2\right), \end{aligned} \tag{7.26}$$

for all $g \in S(\mathbb{R})$ and $z \in \mathbb{C}$. One can calculate that (7.25) and (7.26) are bounded by

$$\begin{aligned} &\prod_{j=1}^{n+1} \frac{1}{\sqrt{2\pi(r_j - r_{j-1})}} \exp\left(\left(\frac{1}{2} + \frac{1}{\varepsilon}\right)|z|^2 \sup_{s\in[t_0,t]} |g(s)|^2\right) \exp\left(\varepsilon \sup_{0\le j\le n+1} |x_j|^2\right) \\ &\qquad\qquad \times A(t_0,t,x,x_0) \exp\left(B|z|^2 \sup_{s\in[t_0,t]} |g(s)|^2\right), \end{aligned} \tag{7.27}$$

where $0 < A(t_0,t,x,x_0), B < \infty$ are chosen such that

$$\sqrt{2\pi(t-t_0)} \exp\left(\frac{1}{2(t-t_0)}\left|\langle zg, \mathbb{1}_{[t_0,t)}\rangle - (x - x_0)\right|^2\right) \le A(t_0,t,x,x_0) \exp\left(B|z|^2 \sup_{s\in[t_0,t]} |g(s)|^2\right),$$

for all $g \in S(\mathbb{R})$ and $z \in \mathbb{C}$. Using (7.24), (7.27) and the formula

$$\int_{\Lambda_n} \prod_{j=1}^{n+1} \frac{1}{\left(\sqrt{2\pi(r_j - r_{j-1})}\right)^\alpha} dr_j = \frac{\Gamma(1-\alpha)^{n+1}}{(2\pi)^\alpha} \frac{|t-t_0|^{n(1-\alpha)-\alpha}}{\Gamma((n+1)(1-\alpha))}, \quad 0 < \alpha < 1,$$

we get for $\Xi \in \left\{\Phi_n,\ \Psi_n,\ \tau_{\frac{x-x_0}{(t-t_0)}\mathbf{1}_{[t_0,t)}} P_{\frac{1}{\sqrt{t-t_0}}\mathbf{1}_{[t_0,t)}}\Phi_n,\ \tau_{\frac{x-x_0}{\sqrt{i}(t-t_0)}\mathbf{1}_{[t_0,t)}} P_{\frac{1}{\sqrt{t-t_0}}\mathbf{1}_{[t_0,t)}}\Psi_n\right\}$ that

$$\begin{aligned}
&\int_{\mathbb{R}^n}\int_{\Lambda_n} |S\,\Xi(zg)|\,dr_j\,d\nu(x_j)\\
&\qquad\le A(t_0,t,x,x_0)\exp\left(B|z|^2\sup_{s\in[t_0,t]}|g(s)|^2\right)\frac{\Gamma(1/2)^{n+1}}{\sqrt{2\pi}}\,\frac{|t-t_0|^{(n-1)/2}}{\Gamma\left((n+1)/2\right)}\\
&\quad\times\int_{\mathbb{R}^n}\exp\left(\left(\frac{1}{2}+\frac{1}{\varepsilon}\right)|z|^2\sup_{s\in[T_0,T]}|g(s)|^2\right)\exp\left(\left(\frac{1}{2}+\varepsilon\right)\sup_{0\le j\le n+1}|x_j|^2\right)\prod_{j=1}^{n}d\nu(x_j)\\
&\qquad\le A(t_0,t,x,x_0)\exp\left(B|z|^2\sup_{s\in[t_0,t]}|g(s)|^2\right)\frac{\Gamma(1/2)^{n+1}}{\sqrt{2\pi}}\,\frac{|t-t_0|^{(n-1)/2}}{\Gamma\left((n+1)/2\right)}\\
&\qquad\qquad\times\exp\left(\varepsilon|x_0|^2\right)\exp\left(\varepsilon|x|^2\right)Q^n\exp\left(\left(\frac{1}{2}+\frac{1}{\varepsilon}\right)|z|^2\sup_{s\in[t_0,t]}|g(s)|^2\right),
\end{aligned}$$

for all $g\in S(\mathbb{R})$ and $z\in\mathbb{C}$. Here Q (due to Assumption (i) of Definition 6.1) is given by

$$Q:=\int_{\mathbb{R}}\exp\left(\varepsilon|y|^2\right)d|\nu|(y)<\infty.$$

By Theorem 2.12 $\widehat{\Phi}_n := \int_{\mathbb{R}^n}\int_{\Lambda_n}\Phi_n\,dr_j\,d\nu(x_j)$ and $\widehat{\Psi}_n := \int_{\mathbb{R}^n}\int_{\Lambda_n}\Psi_n\,dr_j\,d\nu(x_j)$ are Hida distribution with the property that $\sigma_{\sqrt{i}}\widehat{\Phi}_n = \widehat{\Psi}_n$. In the same way we obtain that

$$\int_{\mathbb{R}^n}\int_{\Lambda_n}\tau_{\frac{x-x_0}{(t-t_0)}\mathbf{1}_{[t_0,t)}}P_{\frac{1}{\sqrt{t-t_0}}\mathbf{1}_{[t_0,t)}}\Phi_n\,dr_j\,d\nu(x_j)$$

and

$$\int_{\mathbb{R}^n}\int_{\Lambda_n}\tau_{\frac{x-x_0}{\sqrt{i}(t-t_0)}\mathbf{1}_{[t_0,t)}}P_{\frac{1}{\sqrt{t-t_0}}\mathbf{1}_{[t_0,t)}}\Psi_n\,dr_j\,d\nu(x_j)$$

are Hida distributions. By Remark 3.31, Remark 4.30 and Remark 4.32 we get that the equation

$$\begin{aligned}
&\sigma_{\sqrt{i}}\tau_{\frac{x-x_0}{(t-t_0)}\mathbf{1}_{[t_0,t)}}P_{\frac{1}{\sqrt{t-t_0}}\mathbf{1}_{[t_0,t)}}\widehat{\Phi}_n=\sigma_{\sqrt{i}}\int_{\mathbb{R}^n}\int_{\Lambda_n}\tau_{\frac{x-x_0}{(t-t_0)}\mathbf{1}_{[t_0,t)}}P_{\frac{1}{\sqrt{t-t_0}}\mathbf{1}_{[t_0,t)}}\Phi_n\,dr_j\,d\nu(x_j)\\
&=\tau_{\frac{x-x_0}{\sqrt{i}(t-t_0)}\mathbf{1}_{[t_0,t)}}P_{\frac{1}{\sqrt{t-t_0}}\mathbf{1}_{[t_0,t)}}\int_{\mathbb{R}^n}\int_{\Lambda_n}\sigma_{\sqrt{i}}\Phi_n\,dr_j\,d\nu(x_j)=\int_{\mathbb{R}^n}\int_{\Lambda_n}\tau_{\frac{x-x_0}{\sqrt{i}(t-t_0)}\mathbf{1}_{[t_0,t)}}P_{\frac{1}{\sqrt{t-t_0}}\mathbf{1}_{[t_0,t)}}\Psi_n\,dr_j\,d\nu(x_j)\\
&\qquad=\tau_{\frac{x-x_0}{\sqrt{i}(t-t_0)}\mathbf{1}_{[t_0,t)}}P_{\frac{1}{\sqrt{t-t_0}}\mathbf{1}_{[t_0,t)}}\widehat{\Psi}_n.
\end{aligned}$$

holds in $(S)'$. Moreover,

$$\sum_{n=0}^{\infty}\frac{\Gamma(1/2)^{n+1}}{\sqrt{2\pi}}\,\frac{|t-t_0|^{(n-1)/2}}{\Gamma\left((n+1)/2\right)}\exp\left(\varepsilon|x_0|^2\right)\exp\left(\varepsilon|x|^2\right)Q^n<\infty.$$

Thus, using Theorem 2.11 $\sum_{n=0}^{\infty}\widehat{\Phi}_n$ and $\sum_{n=0}^{\infty}\widehat{\Psi}_n$ are Hida distribution with the property that $\sigma_{\sqrt{i}}\sum_{n=0}^{\infty}\widehat{\Phi}_n = \sum_{n=0}^{\infty}\widehat{\Psi}_n$. Furthermore, also

$$\sum_{n=0}^{\infty}\tau_{\frac{x-x_0}{(t-t_0)}\mathbb{1}_{[t_0,t)}}P_{\frac{1}{\sqrt{t-t_0}}\mathbb{1}_{[t_0,t)}}\widehat{\Phi}_n \quad \text{and} \quad \sum_{n=0}^{\infty}\tau_{\frac{x-x_0}{\sqrt{i}(t-t_0)}\mathbb{1}_{[t_0,t)}}P_{\frac{1}{\sqrt{t-t_0}}\mathbb{1}_{[t_0,t)}}\widehat{\Psi}_n$$

are Hida distributions. Again, by Remark 3.31, Remark 4.30 and Remark 4.32 it follows that

$$\begin{aligned}
\sigma_{\sqrt{i}}\sum_{n=0}^{\infty}\widehat{\Phi}_n\diamond\sigma_{\sqrt{i}}\delta(\langle\cdot,\mathbb{1}_{[t_0,t)}\rangle+x_0-x) &= \sigma_{\sqrt{i}}\tau_{\frac{x-x_0}{(t-t_0)}\mathbb{1}_{[t_0,t)}}P_{\frac{1}{\sqrt{t-t_0}}\mathbb{1}_{[t_0,t)}}\sum_{n=0}^{\infty}\widehat{\Phi}_n\diamond\sigma_{\sqrt{i}}\delta(\langle\cdot,\mathbb{1}_{[t_0,t)}\rangle+x_0-x)\\
&= \sum_{n=0}^{\infty}\sigma_{\sqrt{i}}\tau_{\frac{x-x_0}{(t-t_0)}\mathbb{1}_{[t_0,t)}}P_{\frac{1}{\sqrt{t-t_0}}\mathbb{1}_{[t_0,t)}}\widehat{\Phi}_n\diamond\sigma_{\sqrt{i}}\delta(\langle\cdot,\mathbb{1}_{[t_0,t)}\rangle+x_0-x)\\
&= \sum_{n=0}^{\infty}\tau_{\frac{x-x_0}{\sqrt{i}(t-t_0)}\mathbb{1}_{[t_0,t)}}P_{\frac{1}{\sqrt{t-t_0}}\mathbb{1}_{[t_0,t)}}\widehat{\Psi}_n\diamond\sigma_{\sqrt{i}}\delta(\langle\cdot,\mathbb{1}_{[t_0,t)}\rangle+x_0-x)\\
&= \sum_{n=0}^{\infty}\widehat{\Psi}_n\sigma_{\sqrt{i}}\delta(\langle\cdot,\mathbb{1}_{[t_0,t)}\rangle+x_0-x). \qquad (7.28)
\end{aligned}$$

□

7.5.2 The Westerkamp-Kuna-Streit class

In this section we consider the Feynman integrands of potentials as in Definition 6.4 within the complex scaling ansatz. Again, only the one dimensional case is treated. Remember that such a potential V on $\mathbb{R}$ is given by

$$V(x) = \int_{\mathbb{R}} e^{\alpha\cdot x}\,dm(\alpha), \qquad (7.29)$$

where m is a complex measure on the Borel sets on $\mathbb{R}$, fulfilling (6.7). Close to Section 6.3.4 one has to consider the series expansion of the following objects:

$$\exp\left(-i\int_{t_0}^{t}V(x_0+\langle\cdot,\mathbb{1}_{[t_0,r)}\rangle)\,dr\right), \quad \tau_{\frac{x-x_0}{(t-t_0)}\mathbb{1}_{[t_0,t)}}P_{\frac{1}{\sqrt{t-t_0}}\mathbb{1}_{[t_0,t)}}\exp\left(-i\int_{t_0}^{t}V(x_0+\langle\cdot,\mathbb{1}_{[t_0,r)}\rangle)\,dr\right) \qquad (7.30)$$

and

$$\begin{aligned}
&\sigma_{\sqrt{i}}\exp\left(-i\int_{t_0}^{t}V(x_0+\langle\cdot,\mathbb{1}_{[t_0,r)}\rangle)\,dr\right),\\
&\tau_{\frac{x-x_0}{\sqrt{i}(t-t_0)}\mathbb{1}_{[t_0,t)}}P_{\frac{1}{\sqrt{t-t_0}}\mathbb{1}_{[t_0,t)}}\sigma_{\sqrt{i}}\exp\left(-i\int_{t_0}^{t}V(x_0+\langle\cdot,\mathbb{1}_{[t_0,r)}\rangle)\,dr\right), \qquad (7.31)
\end{aligned}$$

and show existence of them in $(S)^{-1}$.

Theorem 7.21. *For a potential $V : \mathbb{R} \to \mathbb{C}$ as in Definition 6.4 the Feynman integrand is given by*

$$I_V = \sigma^{\dagger}_{\sqrt{i},t_0,t}\Bigg(\sigma_{\sqrt{i}}\tau_{\frac{x-x_0}{(t-t_0)}1_{[t_0,t)}}P_{\frac{1}{\sqrt{t-t_0}}1_{[t_0,t)}}\left(\sum_{n=0}^{\infty}\frac{(-i)^n}{n!}\int_{[t_0,t]^n}\int_{\mathbb{R}^n} I_0\prod_{j=1}^{n}e^{\alpha_j\cdot x(r_j)}\prod_{j=1}^{n}dm(\alpha_j)\,dr_j\right)$$
$$\diamond\,\sigma_{\sqrt{i}}\delta(\langle\cdot,\mathbb{1}_{[t_0,t)}\rangle + x_0 - x)\Bigg),$$

for all $x, x_0 \in \mathbb{R}$ and all $-\infty < t_0 < t < \infty$. Again the Wick product coincides with the independent pointwise product.

Proof. Let V be a potential as in Definition 6.4 and m the corresponding measure. We will only show existence of (7.31) in $(S)^{-1}$. Existence of (7.30) in $(S)^{-1}$ can be treated analogously. Following Theorem 3.30 and Remark 3.31 the series expansions of the objects in (7.31) are given by

$$\begin{aligned}\sigma_{\sqrt{i}}\exp\left(-i\int_{t_0}^{t}V(x_0+\langle\cdot,\mathbb{1}_{[t_0,r)}\rangle)\,dr\right) &= \sigma_{\sqrt{i}}\sum_{n=0}^{\infty}\frac{(-i)^n}{n!}\int_{[t_0,t]^n}\int_{\mathbb{R}^n}\prod_{j=1}^{n}e^{\alpha_j\cdot x(r_j)}\prod_{j=1}^{n}dm(\alpha_j)\,dr_j\\ &= \sum_{n=0}^{\infty}\frac{(-i)^n}{n!}\int_{[t_0,t]^n}\int_{\mathbb{R}^n}\prod_{j=1}^{n}\exp\left(\alpha_j\cdot\left(x_0+\sqrt{i}\langle\cdot,\mathbb{1}_{[t_0,r_j)}\rangle\right)\right)\prod_{j=1}^{n}dm(\alpha_j)\,dr_j.\end{aligned}$$

and

$$\begin{aligned}&\sigma_{\sqrt{i}}\exp\left(-i\int_{t_0}^{t}V\left(x_0-\frac{r-t_0}{t-t_0}(x_0-x)+\left\langle\cdot,\mathbb{1}_{[t_0,r)}-\frac{r-t_0}{t-t_0}\mathbb{1}_{[t_0,r)}\right\rangle\right)dr\right)\\ &=\sum_{n=0}^{\infty}\frac{(-i)^n}{n!}\int_{[t_0,t]^n}\int_{\mathbb{R}^n}\prod_{j=1}^{n}\exp\left(\alpha_j\cdot\left(x_0-\frac{r_j-t_0}{t-t_0}(x_0-x)+\sqrt{i}\left\langle\cdot,\mathbb{1}_{[t_0,r_j)}-\frac{r_j-t_0}{t-t_0}\mathbb{1}_{[t_0,t)}\right\rangle\right)\right)\\ &\qquad\qquad\prod_{j=1}^{n}dm(\alpha_j)\,dr_j.\end{aligned}$$

We first consider the products of exponentials inside the integrals for arbitrary $\alpha_j \in \mathbb{R}$, and $t_0 < r_1 \ldots < r_n < t$, $j = 1, \ldots, n$. Let us denote them by φ_n and ψ_n, respectively. Obviously, both are regular generalized test functions and therefore in $C_{\frac{1}{\sqrt{t-t_0}}1_{[t_0,t)},\frac{x-x_0}{\sqrt{i(t-t_0)}}}$. Thus

$$\begin{aligned}S(\varphi_n)(g) &= S\left(\prod_{j=1}^{n}\exp\left(\alpha_j\cdot\left(x_0+\sqrt{i}\langle\cdot,\mathbb{1}_{[t_0,r_j)}\rangle\right)\right)\right)(g)\\ &= \exp\left(\sum_{j=1}^{n}\alpha_j\cdot x_0\right)\prod_{j=1}^{n}\exp\left(\frac{1}{2}\left(g+\sqrt{i}\alpha_j\mathbb{1}_{[t_0,r_j)},g+\sqrt{i}\alpha_j\mathbb{1}_{[t_0,r_j)}\right)\right)\end{aligned}$$

$$= \exp\left(\sum_{j=1}^{n} \alpha_j \cdot x_0\right) \exp\left(\frac{1}{2}(g,g)\right) \prod_{j=1}^{n} \exp\left(\frac{1}{2} i\alpha_j^2 \left(\mathbb{1}_{[t_0,r_j)}, \mathbb{1}_{[t_0,r_j)}\right)\right) \exp\left(\sqrt{i}\alpha_j \left(g, \mathbb{1}_{[t_0,r_j)}\right)\right)$$

and

$$S(\psi_n)(g) = S\left(\prod_{j=1}^{n} \exp\left(\alpha_j \cdot \left(x_0 - \frac{r_j - t_0}{t - t_0}(x_0 - x) + \sqrt{i}\left\langle \cdot, \mathbb{1}_{[t_0,r_j)} - \frac{r_j - t_0}{t - t_0}\mathbb{1}_{[t_0,t)}\right\rangle\right)\right)\right)(g)$$

$$= \exp\left(\sum_{j=1}^{n} \alpha_j \cdot \left(x_0 - \frac{r_j - t_0}{t - t_0}(x_0 - x)\right)\right) \exp\left(\frac{1}{2}(g,g)\right) \prod_{j=1}^{n} \exp\left(\sqrt{i}\alpha_j \left(g, \mathbb{1}_{[t_0,r_j)} - \frac{r_j - t_0}{t - t_0}\mathbb{1}_{[t_0,t)}\right)\right)$$

$$\times \exp\left(\frac{1}{2} i\alpha_j^2 \left(\mathbb{1}_{[t_0,r_j)} - \frac{r_j - t_0}{t - t_0}\mathbb{1}_{[t_0,t)}, \mathbb{1}_{[t_0,r_j)} - \frac{r_j - t_0}{t - t_0}\mathbb{1}_{[t_0,t)}\right)\right),$$

for all $g \in S(\mathbb{R})$. Obviously, both can be extended to $h \in S_{\mathbb{C}}(\mathbb{R})$. Using the Cauchy-Schwarz inequality the modulus of $S(\varphi_n)(h)$ and $S(\psi_n)(h)$, $h \in S_{\mathbb{C}}(\mathbb{R})$, can be estimated by

$$C_n(\alpha_1, \ldots, \alpha_n, h) := \exp\left(\sum_{j=1}^{n} |\alpha_j| \cdot (2|x_0| + |x|)\right) \exp\left(\frac{1}{2}|h|_0^2\right) \prod_{j=1}^{n} \exp\left(2\sqrt{t - t_0}|\alpha_j||h|_0\right).$$

Moreover

$$\int_{[t_0,t]^n} \int_{\mathbb{R}^n} C_n(\alpha_1, \ldots, \alpha_n, h) \prod_{j=1}^{n} dm(\alpha_j)\, dr_j$$

$$\leq \exp\left(\frac{1}{2}|h|_0^2\right)(t - t_0)^n \left(\int_{\mathbb{R}} \exp\left(|\alpha| \cdot (2|x_0| + |x|)\right) \exp\left(2\sqrt{t - t_0}|\alpha||h|_0\right) d|m|(\alpha)\right)^n < \infty,$$

by the condition on the measure (6.7). Since $S(\varphi_n)(h)$ and $S(\psi_n)(h)$ are measurable and entire in $h \in S_{\mathbb{C}}(\mathbb{R})$, due to Theorem 2.18, there exists an open neighborhood of zero U independent of n and

$$\Phi_n := \int_{[t_0,t]^n} \int_{\mathbb{R}^n} \prod_{j=1}^{n} e^{\alpha_j(x_0 + \sqrt{i}\langle \cdot, \mathbb{1}_{[t_0,r_j)}\rangle)} \prod_{j=1}^{n} dm(\alpha_j)\, dr_j$$

and

$$\Psi_n := \int_{[t_0,t]^n} \int_{\mathbb{R}^n} \prod_{j=1}^{n} e^{\left(\alpha_j \cdot \left(x_0 - \frac{r_j - t_0}{t - t_0}(x_0 - x) + \sqrt{i}\langle \cdot, \mathbb{1}_{[t_0,r_j)} - \frac{r_j - t_0}{t - t_0}\mathbb{1}_{[t_0,t)}\rangle\right)\right)} \prod_{j=1}^{n} dm(\alpha_j)\, dr_j$$

are Kondratiev distributions. By Remark 4.30 we obtain $\tau_{\frac{x - x_0}{\sqrt{i}(t - t_0)}\mathbb{1}_{[t_0,t)}} P_{\frac{1}{\sqrt{t - t_0}}\mathbb{1}_{[t_0,t)}} \Phi_n = \Psi_n$. Moreover, $S\left(\sum_{n=0}^{M} \Phi_n\right)(h)$ and $S\left(\sum_{n=0}^{M} \Psi_n\right)(h)$ can be estimated by

$$\exp\left(\frac{1}{2}|h|_0^2\right) \exp\left((t - t_0) \int_{\mathbb{R}} \exp\left(|\alpha| \cdot (2|x_0| + |x|)\right) \exp\left(2\sqrt{t - t_0}|\alpha||h|_0\right) d|m|(\alpha)\right) < \infty,$$

for all $h \in U$. Hence, by Theorem 2.17 also $\sum_{n=0}^{\infty} \Phi_n$ and $\sum_{n=0}^{\infty} \Psi_n$ are Kondratiev distributions with

$\tau_{\frac{x-x_0}{\sqrt{i(t-t_0)}}\mathbf{1}_{[t_0,t)}} P_{\frac{1}{\sqrt{t-t_0}}\mathbf{1}_{[t_0,t)}} \sum_{n=0}^{\infty} \Phi_n = \sum_{n=0}^{\infty} \Psi_n$. Now close to (7.28) the statement holds. □

Remark 7.22. *(i) One can show the statements of Theorem 7.8, Theorem 7.16 and Theorem 7.17 not only for the Khandekar-Streit-class but also for the Westerkamp-Kuna-class. This can be done first for the approximating sequences $(\widehat{\Phi}_n)_{n\in\mathbb{N}}$ and $(\Phi_n)_{n\in\mathbb{N}}$, respectively (see the proof of Theorem 7.20 and Theorem 7.21).*

(ii) For the solutions of the corresponding Schrödinger equations we refer to [54] and [49].

(iii) Note that similar to Theorem 7.20 and Theorem 7.21 it should be possible to construct a representation of the Feynman integrand for non-smooth and rapidly growing potentials (see Remark 6.7 and [9] for its detailed construction) via scaling, translation and orthogonal projection.

7.6 The Feynman integrand for the Khandekar-Streit class combined with the Doss class

As considered in several articles, see e.g. [8] or [9], one is interested in combinations of different classes of potentials for which Feynman integrands exists as generalized functions of white noise. In this section we consider the classical Khandekar-Streit class combined with the Doss class. I.e., we construct the Feynman integrand corresponding to a potential of the form $V = V_1 + V_2$, where $V_1 : \mathbb{C} \to \mathbb{C}$ is as in Definition 7.4 and V_2 is given as in [40]. In detail,

$$V_2(x) = \int_{\mathbb{R}} \delta(x-y)\, dm(y), \tag{7.32}$$

where $dm(y) := v_2(y)dy$ is a finite Borel measure of compact support, K. For simplicity we assume that $v_2 : \mathbb{R} \to \mathbb{R}$ to be a continuous function with compact support. To consider such potentials V the following lemmata are useful.

Lemma 7.23. *Let $k : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ be a measurable function and B a real-valued Brownian motion, then*

$$E\Big[k(\|B\|_{\sup,T})\Big] \leq 2\left(\frac{2}{\pi T}\right)^{1/2} \int_0^{\infty} k(u) e^{-\frac{u^2}{2T}}\, du,$$

where $\|\cdot\|_{\sup,T}$ denotes the supremum norm over $[0,T]$.

For the proof see [13, Sec.1, Lem.1].

Lemma 7.24. *Let $V : \mathbb{C} \to \mathbb{C}$ be a potential as in Definition 7.4, $0 \le t_0 = r_0 < r_1 < \ldots < r_n < r_{n+1} = t$ and $x_j \in \mathbb{R}$, $j = 1, \ldots, n$ for $n \in \mathbb{N}$. Then there exists some constants $0 < C < \infty$ such that*

$$\Bigg| S\Bigg(\exp\Bigg(- i \sum_{j=1}^{n+1} \int_{r_{j-1}}^{t_j} V\Bigg(x_{j-1} + \frac{r - r_{j-1}}{r_j - r_{j-1}}(x_j - x_{j-1}) + \sqrt{i}\langle \cdot, \mathbb{1}_{[r_{j-1}, r)} \rangle - \sqrt{i}\frac{r - r_{j-1}}{r_j - r_{j-1}} \langle \cdot, \mathbb{1}_{[r_{j-1}, r_j)} \rangle \Bigg) dr \Bigg) \Bigg)(zg) \Bigg| \le \sup_{y \in K_n} B(y) C \exp\left(\frac{1}{2} |z|^2 |g|_0^2 \right) \quad (7.33)$$

for all $z \in \mathbb{C}$ and all $g \in S(\mathbb{R})$. Here $B : \mathbb{R} \to \mathbb{R}$ is defined as in Assumption 7.2 and K_n denotes the convex hull of $\{x_1, \ldots, x_{n+1}\}$.

Proof. Let $V : \mathbb{C} \to \mathbb{C}$ be a potential as in Definition 7.4, $0 \le t_0 = r_0 < r_1 < \ldots < r_n < r_{n+1} = t$ and $x_j \in \mathbb{R}$, $j = 1, \ldots, n$ for $n \in \mathbb{N}$. Then we get by Assumption 7.2 that the integrand of (7.33) is in $L^2(\mu)$. Let us denote this integrand by Ψ_n. We can approximate Ψ_n with help of the decompositions of the time intervals $[r_{j-1}, r_j]$ given by $t_{k,j} := \frac{(r_j - r_{j-1})k}{m}$ for $m \in \mathbb{N}$. I.e.,

$$\Psi_{n,m} := \exp\Bigg(- i \sum_{j=1}^{n+1} \frac{r_j - r_{j-1}}{m} \sum_{k=0}^{m} V\Bigg(x_{j-1} + \frac{t_{k,j} - r_{j-1}}{r_j - r_{j-1}}(x_j - x_{j-1}) + \sqrt{i}\langle \cdot, \mathbb{1}_{[r_{j-1}, t_{k,j})} \rangle - \sqrt{i}\frac{t_{k,j} - r_{j-1}}{r_j - r_{j-1}} \langle \cdot, \mathbb{1}_{[r_{j-1}, r_j)} \rangle \Bigg) \Bigg),$$

converges to Ψ_n as m goes to infinity in $L^2(\mu)$. Thus, it is enough to estimate the $L^2(\mu)$-norm of $\Psi_{n,m}$ and give an estimate for it independent of n and m. Let us define the set

$$K_{n,m} := \left\{ x_{j-1} + \frac{t_{k,j} - r_{j-1}}{r_j - r_{j-1}}(x_j - x_{j-1}) \,|\, j = 1, \ldots, n \,;\, k = 1, \ldots, m \right\} \subset \mathbb{R}.$$

Then we get by Assumption 7.2 that

$$\Bigg| \exp\Bigg(- i \sum_{j=1}^{n+1} \frac{r_j - r_{j-1}}{m} \sum_{k=0}^{m} V\Bigg(x_{j-1} + \frac{t_{k,j} - r_{j-1}}{r_j - r_{j-1}}(x_j - x_{j-1}) + \sqrt{i}\langle \omega, \mathbb{1}_{[r_{j-1}, t_{k,j})} \rangle - \sqrt{i}\frac{t_{k,j} - r_{j-1}}{r_j - r_{j-1}} \langle \omega, \mathbb{1}_{[r_{j-1}, r_j)} \rangle \Bigg) \Bigg) \Bigg|^2$$
$$\le \max_{y \in K_{n,m}} \Bigg| \exp\Bigg(- 2i \sum_{j=1}^{n+1} \frac{r_j - r_{j-1}}{m} \sum_{k=0}^{m} V\Bigg(y + \sqrt{i}\langle \omega, \mathbb{1}_{[r_{j-1}, t_{k,j})} \rangle - \sqrt{i}\frac{t_{k,j} - r_{j-1}}{r_j - r_{j-1}} \langle \omega, \mathbb{1}_{[r_{j-1}, r_j)} \rangle \Bigg) \Bigg) \Bigg|$$

$$\leq \max_{y\in K_{n,m}} \max_{1\leq k\leq m} \max_{1\leq j\leq n} \left| \exp\left(-2i \sum_{j=1}^{n+1} \frac{r_j - r_{j-1}}{m} \sum_{k=0}^{m} V\left(y + \sqrt{i}\langle \omega, \mathbb{1}_{[r_{j-1},t_{k,j})}\rangle - \sqrt{i}\frac{t_{k,j} - r_{j-1}}{r_j - r_{j-1}}\langle \omega, \mathbb{1}_{[r_{j-1},r_j)}\rangle\right)\right)\right|$$

$$\leq \max_{y\in K_{n,m}} B(y) \max_{1\leq k\leq m} \max_{1\leq j\leq n} \exp\left(\varepsilon \left|\langle \omega, \mathbb{1}_{[r_{j-1},t_{k,j})}\rangle - \frac{t_{k,j} - r_{j-1}}{r_j - r_{j-1}}\langle \omega, \mathbb{1}_{[r_{j-1},r_j)}\rangle\right|^2\right)$$

$$\leq \max_{y\in K_{n,m}} B(y) \exp\left(4\varepsilon \|\langle \omega, \mathbb{1}_{t_0,t)}\rangle\|^2_{\sup,T}\right).$$

Of course, the equation above first only holds for $\omega \in L^2(\mathbb{R})$. Thus by Lemma 7.23 and Assumption 7.2 we get that there exists some $0 < \varepsilon < \frac{1}{8T}$ such that

$$\int_{S'(\mathbb{R})} \left|\Psi_{n,m}\right|^2 d\mu(\omega) \leq \max_{y\in K_{n,m}} B(y)\, 2\left(\frac{2}{\pi T}\right)^{1/2} \int_0^\infty \exp\left(4\varepsilon u^2 - \frac{u^2}{2T}\right) du =: C < \infty.$$

As calculated in (2.8) we know that

$$\left\|: \exp\left(\langle \cdot, zg\rangle\right) :\right\|_{0,0} = \exp\left(\frac{1}{2}|z|^2 |g|_0^2\right),$$

for all $z \in \mathbb{C}$ and $g \in S(\mathbb{R})$. Hence (7.24) holds. □

Theorem 7.25. *Let $V = V_1 + V_2$, where $V_1 : \mathbb{C} \to \mathbb{C}$ is in the Doss class (see Definition 7.4d V_2 is defined as in* (7.32)*. Then the corresponding Feynman integrand is given as a Hida distribution by*

$$I_V = \sum_{n=0}^{\infty} (-i)^n \int_{\mathbb{R}^n} \int_{\Lambda_n} I_{V_1} \delta(x(r_j) - x_j) \prod_{j=1}^{n} v_2(x_j)\, dr_j\, dx_j, \tag{7.34}$$

for all $x, x_0 \in \mathbb{R}$ and all $0 \leq t_0 < t \leq T < \infty$.

Let us remark that we only show existence of I_V and do not prove a representation of it by scaling translation and projection, since this can be shown analogously to Theorem 7.20.

Proof. By continuity of $\sigma^{\dagger}_{z,t_0,t}$ on $(S)'$ it is enough to show that

$$\Phi := \sum_{n=0}^{\infty} (-i)^n \int_{\mathbb{R}^n} \int_{\Lambda_n} \sigma_{\sqrt{i}}\left(\exp\left(-i \int_{t_0}^{t} V(x(r))\, dr\right) \delta(x(t) - x) \prod_{j=1}^{n} \delta(x(r_j) - x_j) v_2(x_j)\right) dr_j\, dx_j,$$

is a Hida distribution. First, we ensure that every integrand of the formula above is a Hida distribution. Let $0 \leq t_0 = r_0 < r_1 < \ldots < r_n < r_{n+1} = t < T$ and $x_j \in \mathbb{R}$, $j = 1, \ldots, n$ for $n \in \mathbb{N}$. Then we know by Theorem 7.17 that

$$\Phi_n := \sigma_{\sqrt{i}}\left(\exp\left(-i \int_{t_0}^{t} V(x(r))\, dr\right) \delta(x(t) - x) \prod_{j=1}^{n} \delta(x(r_j) - x_j)\, v_2(x_j)\right)$$

$$= \prod_{j=1}^{n+1} \Big(\sigma_{\sqrt{i}}\delta(\langle\cdot, \mathbb{1}_{[r_{j-1},r_j)}\rangle - (x_j - x_{j-1}))\Big) v_2(x_j)$$

$$\times \exp\left(-i\int_{r_{j-1}}^{t_j} V\Big(x_{j-1} + \frac{r - r_{j-1}}{r_j - r_{j-1}}(x_j - x_{j-1}) + \sqrt{i}\langle\cdot, \mathbb{1}_{[r_{j-1},r)}\rangle - \sqrt{i}\frac{r - r_{j-1}}{r_j - r_{j-1}}\langle\cdot, \mathbb{1}_{[r_{j-1},r_j)}\rangle\Big)\, dr\right)$$

is in $\mathcal{G}'$, where every product inside is independent. Thus

$$S(\Phi_n)(g) = \prod_{j=1}^{n+1} S\left(\sigma_{\sqrt{i}}\delta(\langle\cdot, \mathbb{1}_{[r_{j-1},r_j)}\rangle - (x_j - x_{j-1}))\right)(g) v_2(x_j)$$

$$\times S\left(\exp\left(-i\int_{r_{j-1}}^{t_j} V\Big(x_{j-1} + \frac{r - r_{j-1}}{r_j - r_{j-1}}(x_j - x_{j-1}) + \sqrt{i}\left\langle\cdot, \mathbb{1}_{[r_{j-1},r)} - \frac{r - r_{j-1}}{r_j - r_{j-1}}\mathbb{1}_{[r_{j-1},r_j)}\right\rangle\Big)\, dr\right)\right)(g),$$

for all $g \in S(\mathbb{R})$. On the one side we know by Lemma 7.24 that there exists some constant $0 < C < \infty$ such that

$$S\left(\exp\left(-i\int_{r_{j-1}}^{t_j} V\Big(x_{j-1} + \frac{r - r_{j-1}}{r_j - r_{j-1}}(x_j - x_{j-1}) + \langle\cdot, \mathbb{1}_{[r_{j-1},r)}\rangle - \frac{r - r_{j-1}}{r_j - r_{j-1}}\langle\cdot, \mathbb{1}_{[r_{j-1},r_j)}\rangle\Big)\, dr\right)\right)(zg)$$

$$\leq \sup_{y\in K_n} B(y)\, C \exp\left(\frac{1}{2}|z|^2|g|_0^2\right) < \sup_{y\in K} B(y)\, C \exp\left(\frac{1}{2}|z|^2|g|_0^2\right),$$

for all $z \in \mathbb{C}$ and all $g \in S(\mathbb{R})$. On the other side as shown in the proof of Theorem 7.20 that there exists some $0 < \varepsilon < \infty$ such that

$$\left|\prod_{j=1}^{n+1} S\left(\sigma_{\sqrt{i}}\delta(\langle\cdot, \mathbb{1}_{[r_{j-1},r_j)}\rangle - (x_j - x_{j-1}))\right)(zg)\right|$$

$$\leq \prod_{j=1}^{n+1} \frac{1}{\sqrt{2\pi(r_j - r_{j-1})}} \exp\left(\left(\frac{1}{2} + \frac{1}{\varepsilon}\right)|z|^2 \sup_{s\in[t_0,t]} |g(s)|^2\right) \exp\left(\varepsilon \sup_{0\leq j\leq n+1} |x_j|^2\right),$$

for all $z \in \mathbb{C}$ and all $g \in S(\mathbb{R})$. Using the formula

$$\int_{\Lambda_n} \prod_{j=1}^{n+1} \frac{1}{\left(\sqrt{2\pi(r_j - r_{j-1})}\right)^{\alpha}}\, dr_j = \frac{\Gamma(1-\alpha)^{n+1}}{(2\pi)^{\alpha}} \frac{|t - t_0|^{n(1-\alpha)-\alpha}}{\Gamma((n+1)(1-\alpha))}, \quad 0 < \alpha < 1,$$

and the fact that there exists some constant $0 < Q < \infty$ such that

$$\int_{\mathbb{R}^n} \prod_{j=1}^{n+1} \exp\left(\varepsilon \sup_{0\leq j\leq n+1} |x_j|^2\right) |v_2|(x_j)\, dx_j \leq Q^n \exp\left(\varepsilon|x_0|^2\right) \exp\left(\varepsilon|x|^2\right) < \infty,$$

we get that

$$\left|\int_{\mathbb{R}^n}\int_{\Lambda_n} S(\Phi_n)(zg) v_2(x_j)\, dr_j\, d(x_j)\right|$$

$$
\begin{aligned}
&\leq \sup_{y\in K} B(y) C \exp\left(\frac{1}{2}|z|^2|g|_0^2\right) \frac{\Gamma(1/2)^{n+1}}{\sqrt{2\pi}} \frac{|t-t_0|^{(n-1)/2}}{\Gamma\left((n+1)/2\right)} \\
&\times \exp\left(\left(\frac{1}{2}+\frac{1}{\varepsilon}\right)|z|^2 \sup_{s\in[t_0,t]} |g(s)|^2\right) \exp\left(\varepsilon|x_0|^2\right) \exp\left(\varepsilon|x|^2\right) Q^n \\
&=: C_n \exp\left(\left(\frac{1}{2}+\frac{1}{\varepsilon}\right)|z|^2 \sup_{s\in[t_0,t]} |g(s)|^2\right) \exp\left(\frac{1}{2}|z|^2|g|_0^2\right) \exp\left(\varepsilon|x_0|^2\right) \exp\left(\varepsilon|x|^2\right),
\end{aligned}
$$

for all $z \in \mathbb{C}$ and all $g \in S(\mathbb{R})$. Thus, by Theorem 2.12 we get that $\int_{\mathbb{R}^n} \int_{\Lambda_n} \Phi_n \nu_2(x_j)\, dr_j\, d(x_j)$ is a Hida distribution. Furthermore, since $\sum_{n=0}^{\infty} C_n < \infty$ we get by Theorem 2.11 that $\Phi = \sum_{n=0}^{\infty} \int_{\mathbb{R}^n} \int_{\Lambda_n} \Phi_n \nu_2(x_j)\, dr_j\, d(x_j)$ is also a Hida distribution. □

Remark 7.26. *(i) One can also considered arbitrary $V_1 : \mathcal{D}_{\sqrt{i}} \to \mathbb{C}$. But in this case one has to care of the fact that the convex hull of the set $\{x_0, x\} \cup K$ must be contained in the open set O, see Definition 7.4. Remind that $K \subset \mathbb{R}$ is the compact support of the potential V_2, see* (7.32)*, and $x, x_0 \in O$.*

(ii) In general it seems not to be possible to extend Theorem 7.25 to measures ν_2 which have no compact support, but a Gaussian-fall off, as in Definition 6.1. This is caused by the fact that the function $B : \mathbb{R} \to \mathbb{R}$ might of higher growth. For example for the potential $x \mapsto x^6$, see Example 5.9 (i), the function B is of growth $x \mapsto \exp(x^5)$.

(iii) It is still left to show that I_V defined as in Theorem 7.25 solves the corresponding Schrödinger equation. But this can be done similar to [54, Theorem 3.4] while using Theorem 7.18.

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