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Optimal Portfolios for Executive Stockholders

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"If you have a great manager,
you want to pay him very well."
Buffett, Warren (2004).

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Zusammenfassung

Wir entwickeln ein Modell zur Berechnung des in die eigenen Firmenaktien investierten Anteils und des Arbeitsaufwandes von leitenden Angestellten. Der leitende Angestellte - charakterisiert durch Risikoaversions- und Arbeitseffektivitätsparameter - investiert sein Vermögen ohne Einschränkungen in den Finanzmarkt einschließlich der Aktie der eigenen Firma, deren Wert er durch seinen Arbeitsaufwand beeinflussen kann. Die nutzenmaximierende Investitions- und Arbeitsaufwandsstrategie wird in geschlossener Form hergeleitet und mit einem Nutzenindifferenzargument die angemessene Entlohnung bestimmt. Der leitende Angestellte ist bei der Vertragserfüllung nicht eingeschränkt. Jedoch stellt die berechnete Arbeitsaufwandsstrategie einen Basisfall dar, der einen Einblick darin gibt, wie man die finanziellen Leistungen von leitenden Angestellten bemessen könnte und wie sie auf Einschränkungen bei der Vertragserfüllung reagieren könnten. Zudem betrachten wir ein hochqualifiziertes Individuum, das die Wahl zwischen zwei Karriereoptionen hat. Das Individuum kann zwischen einer mittleren Managementposition in einer großen Firma und einer leitenden Position in einer kleineren börsennotierten Gesellschaft wählen, in der es die Möglichkeit hat, den Wert der Aktie der eigenen Gesellschaft zu beeinflussen. Das Individuum investiert in den Finanzmarkt einschließlich der Aktie der kleineren börsennotierten Gesellschaft. Die nutzenmaximierende Konsum-, Investitions- und Arbeitsaufwandsstrategie wird in geschlossener Form hergeleitet. Es werden Bedingungen in Form eines Einkommensunterschiedes angegeben, bei denen das Individuum die Karriere in der kleineren börsennotierten Gesellschaft fortführt. Diese kann ein geringeres Einkommen anbieten. Das Einkommensdefizit wird durch die Möglichkeit, von einem gesteigerten Arbeitsaufwand durch Ankauf von eigenen Firmenaktien zu profitieren, kompensiert. Diese Ergebisse geben einen Einblick in das optimale Design von Verträgen.

Abstract

In this work, we develop a framework for analyzing an executive's own-company stockholding and work effort preferences. The executive, characterized by risk aversion and work effectiveness parameters, invests his personal wealth without constraint in the financial market, including the stock of his own company whose value he can directly influence with work effort. The executive's utility-maximizing personal investment and work effort strategy is derived in closed form for logarithmic and power utility and for exponential utility for the case of zero interest rates. Additionally, a utility indifference rationale is applied to determine his fair compensation. Being unconstrained by performance contracting, the executive's work effort strategy establishes a base case for theoretical or empirical assessment of the benefits or otherwise of constraining executives with performance contracting. Further, we consider a highly-qualified individual with respect to her choice between two distinct career paths. She can choose between a mid-level management position in a large company and an executive position within a smaller listed company with the possibility to directly affect the company's share price. She invests in the financial market including the share of the smaller listed company. The utility maximizing strategy from consumption, investment, and work effort is derived in closed form for logarithmic utility and power utility. Conditions for the individual to pursue her career with the smaller listed company are obtained. The participation constraint is formulated in terms of the salary differential between the two positions. The smaller listed company can offer less salary. The salary shortfall is offset by the possibility to benefit from her work effort by acquiring own-company shares. This gives insight into aspects of optimal contract design. Our framework is applicable to the pharmaceutical and financial industry, as well as the IT sector.

Introduction

The first two chapters of this work will deal with a topic with a quite actual background. Share-based payments are frequently used and discussed very controversially in recent time and they have meanwhile become a puplic interest. This topic has been widely discussed in the finance and economics theory in the so called 'principal-agent-problem', where the principal represents a share holder and the agent an executive. A natural question arising from that theory is how share-based payments, like for instance executive stock options, do increase the executive's incentive or work effort. Usually a 'constrained executive' is considered in the theory, since the common practice to incorporate executive stock options in the executive's compensation manipulates already the risk-taking in the own-company's stock. Our idea is to analyze an 'unconstrained executive' without any contraints on his compensation. This base case will then give us an insight how the agent can be influenced or even controlled by the principal.

Stemming from the agency theory fundamentals of Ross (1973), Jensen and Meckling (1976), Holmstrom (1979) and others, there has been much concern for the 'incentivization' link from equity-based executive compensation to corporate financial performance. The associated academic literature is extensive.[1] Counterpoint to past research, we consider the motivation for an executive with unconstrained (unincentivized) compensation to voluntarily link his personal wealth to his management success for the company. We develop a model framework that identifies the joint own-company stockholding and work effort strategy of a utility-maximizing executive. The executive's

[1] The summaries of Murphy (1999) and Core, Guay and Larcker (2003) are useful references.

compensation is assumed to be incorporated into his up-front total personal wealth, which he invests variously in a risk-free money market account, a diversified market portfolio, or his own company's stock. The executive is able to beneficially influence the value of his company via work effort; he gains utility from the increased value of his direct stockholding (within his overall personal portfolio), but loses utility for his work effort. The executive is characterized by a parameter of constant relative risk aversion (γ) or of constant absolute risk aversion (η), respectively, and two work effectiveness parameters (κ, representing inverse work productivity, and α, representing disutility stress).

A feature of our framework is that the executive's work effort, specified in terms of two control variables, non-systematic expected return and volatility (μ and σ), can be restated in terms of a single control variable, the non-systematic Sharpe ratio ($\lambda = (\mu - r)/\sigma$, where r is the risk-free rate of return). This reduces the dimensions of the problem and introduces a parametrization based on the well-known Sharpe ratio performance measure. The executive's optimal personal investment strategy ($\pi^\star$ and $\Pi^\star$, respectively) and work effort strategy ($\lambda^\star$) is then derived in closed form using stochastic control theory and the corresponding Hamilton-Jacobi-Bellman equations. Other technical papers similarly concerned with dynamic principal-agent models include Cadenillas, Cvitanic and Zapatero (2004), Korn and Kraft (2008) and Ou-Yang (2003), for example.

Our closed-form results demonstrate that an executive with superior work effectiveness (i.e. higher quality) will undertake more work effort for his company. But the extent to which any level of work effectiveness is put to use via work effort depends prominently on the executive's risk aversion; only if he has sufficiently low risk aversion to take on a substantial own-company stockholding he will have the incentive to apply substantial work effort. The results also provide guidance for identifying the executive's quality and risk aversion from demonstrated work effort. Or given identification of executive quality and risk aversion, the results indicate the own-company stockholding and work effort of an executive unconstrained by performance contracting, which establishes a base case for theoretical or empirical assessment of the benefits or otherwise of constraining the executive with performance contracting.

Freeing executives to self-incentivize may be a reasonable 'path of least resistance' in the light of some recent and not so recent research. For example, Dittmann and Maug (2007) were unable to rationalize observed executive compensation. Using a 'standard' principal-agent efficient contracting model, their analysis indicated that executives should not, in general, be compensated with options, and that it would commonly be optimal for executives to use private savings to purchase additional stock in their own companies. Bettis, Bizjak and Lemmon (2001) found that high-ranking corporate insiders use collars and swaps to cover a significant proportion of their own-company stockholdings, allowing them to unwind the constraint of equity-based compensation. Ross (2004) repudiated the folklore that giving options to agents makes them more willing to take risks (also see Carpenter (2000)). And Jensen and Murphy (1990) proposed that private political forces in the managerial labor market constrain pay-performance sensitivity, leading most CEOs to hold trivial fractions of their firms' stock. However, Hall and Liebman (1998) and Core and Larcker (2002), for example, found support for a link from equity-based executive compensation to corporate performance.

Whether subject to constrained or unconstrained (i.e. incentivized or unincentivized) compensation, an executive's actualized performance incentive will reflect a total personal wealth perspective. Ofek and Yermack (2000) found that once managers reach a certain own-company ownership level, they actively rebalance their personal portfolios when awarded equity compensation. And Garvey and Milbourn (2003) found that market risk has little effect on the use of stock-based pay for the average executive, suggesting that executives can undo any undesired market exposure from their incentive contracts by adjusting their personal portfolios. We thus maximize our risk averse executive's utility with respect to his total wealth investable across his own company's stock, a diversified market portfolio and a risk-free money market account. Our approach has parallels with Jin (2002), but uses a continuous-time setting with arguably a more intuitively appealing specification of work effort and its disutility. Also see Cvitanic (2008) for a more general continuous-time framework emphasizing incentive effects when the executive can hedge equity-based compensation. A natural future extension for our framework is to specify a constrained executive subject to an imposed own-company stockholding representative of performance contracting, and to contrast his work effort strategy with that of our unconstrained executive.

Chapter 3 will deal with another interesting applied problem which arises from observations that can be made at the employment market. That is, highly-qualified individuals have often the choice between different career paths.

We restate that the remuneration of managers should be linked to performance, see, e.g., Ross (1973), Jensen and Meckling (1976), Holmstrom (1979) and others, for the fundamentals of agency theory, and the summaries of Murphy (1999) and Core, Guay and Larcker (2003).

In contrast to past research, we investigate the motivation for an individual to voluntarily performance-link her private wealth by acquiring shares of the own-company. We consider a highly-qualified individual with respect to her choice between two distinct career paths. She can choose between a mid-level management position in a large company and an executive position in a smaller listed company with the possibility to directly affect the company's share price. The individual is assumed to be utility maximizing, deriving utility from terminal wealth and intertemporal consumption, and negative utility (disutility or cost) from work effort. The investment opportunities include the share of the smaller listed company and thus the individual can capitalize on her work effort by investing in own-company shares. Taking up the mid-level management position with the large company is the outside option in our setting. The outside option rules out the possibility to affect the share price of the smaller company. The individual is characterized by two time preference parameters (ρ, discount rate for utility from consumption, and $\tilde{\rho}$, discount rate for the disutility from work effort), and two work effectiveness parameters (κ, representing inverse work productivity, and α, representing disutility stress).

First, we analyze the individuals optimal control problem under the assumption that she takes up the offer from the smaller listed company. The optimal investment strategy ($\pi^\star$), consumption ($k^\star$), and work effort ($\lambda^\star$), respectively, are derived in closed form in the log-utility setting using stochastic control theory and the corresponding Hamilton-Jacobi-Bellman equations. We demonstrate that an executive with higher work effectiveness (quality) undertakes more work effort. Additionally, the broader constant relative risk aversion setting is explored. By imposing a sensible parameter restriction we are able to reduce the problem to a Riccati equation which we can solve in closed form. As second step, we identify conditions for the individual to work for the smaller listed company. The participation constraint is given in terms of the salary differential of the two job alternatives. In particular, we derive the minimal required salary $\delta^\star$ that needs to be offered by the smaller company to attract the individual and thereby characterize the participation constraint. In general, we find that a more talented individual requires a lower salary to be attracted to the smaller listed company. The salary shortfall is offset by the possibility to benefit from her work effort by acquiring shares of the company. This salary pattern can be observed in practice, e.g., in the pharmaceutical industry, the IT sector, and the financial industry.

The thesis is organized as follows. Chapter 1 establishes the framework of an unconstrained executive as described above with constant relative risk aversion. Section 1.1 introduces the notation and terminology, and as a first result the optimality problem is reformulated and simplified. In Section 1.2 the Hamilton-Jacobi-Bellman equations characterizing the utility maximization problem are derived, and closed-form solutions for the log-utility and the power-utility case are established. The results are illustrated and the executive's up-front fair compensation is given in Section 1.3.

Chapter 2 gives the framework of an unconstrained executive with constant absolute risk-aversion. Section 2.1 introduces the optimization problem, reformulates and simplifies it. In Section 2.2 the Hamilton-Jacobi-Bellman equation characterizing the utility maximization problem is derived, and a closed-form solution for the exponential-utility case assuming zero interest rates is established. The results are discussed and the executive's up-front fair compensation is given in Section 2.3.

In Chapter 3 we investigate the decision problem of the highly-qualified individual as stated above. Section 3.1 introduces the notation and terminology. In Section 3.2 the Hamilton-Jacobi-Bellman equations characterizing the utility and consumption maximization problem are derived, and closed-form solutions for the log-utility and the power-utility case are established. The participation constraint is derived for the case of logarithmic utility. The results are illustrated in Section 3.3.

Finally, we conclude and give an outlook for future research. A technical proof is moved into the Appendix.

Chapter 1

Own-Company Stockholding and Work Effort Preferences of an Unconstrained Executive with Constant Relative Risk Aversion

We now turn our focus to the framework of an unconstrained executive with constant relative risk aversion. This chapter follows the lines of Desmettre, Gould and Szimayer (2010).

1.1 Notation and Set-up

The financial market is defined on a filtered probability space $(\Omega, \mathcal{F}, P, (\mathcal{F}_t)_{t\geq 0})$ satisfying the usual hypothesis and large enough to support two independent standard Brownian motions, $W^P = (W_t^P)_{t\geq 0}$ and $W = (W_t)_{t\geq 0}$, where t indicates time. We consider a company executive that invests in the financial market. Specifically, the investment opportunities available to our executive are a risk-free money market account, a diversified market portfolio and his own company's stock. The risk-free money market account has the price process $B = (B_t)_{t\geq 0}$, with dynamics

$$\mathrm{d}B_t = r\, B_t\, \mathrm{d}t\,, \quad B_0 = 1\,, \tag{1.1}$$

where r is the instantaneous risk-free rate of return, hence $B_t = e^{r\,t}$. The price process of the market portfolio, $P = (P_t)_{t\geq 0}$, follows the stochastic differential equation (SDE)

$$\mathrm{d}P_t = P_t \left(\mu^P \,\mathrm{d}t + \sigma^P \,\mathrm{d}W_t^P\right), \quad P_0 \in \mathbb{R}^+ , \tag{1.2}$$

where μ^P and σ^P are respectively the expected return and volatility of the market portfolio. The company's stock price process, $S^{\mu,\sigma} = (S_t^{\mu,\sigma})_{t\geq 0}$, is a controlled diffusion with SDE

$$\mathrm{d}S_t^{\mu,\sigma} = S_t^{\mu,\sigma} \left(\mu_t \,\mathrm{d}t + \beta \left[\frac{\mathrm{d}P_t}{P_t} - r\mathrm{d}t \right] + \sigma_t \,\mathrm{d}W_t \right) , \; S_0^{\mu,\sigma} \in \mathbb{R}^+ , \tag{1.3}$$

where $\beta \in \mathbb{R}$ is the company's beta (i.e. the standardized covariance between the company's rate of return and that of the market portfolio, indicating systematic risk); μ is the company's expected rate of return for non-systematic risk (i.e. the expected return in excess of the beta-adjusted market portfolio's expected excess return); and σ is the company's non-systematic volatility. Both μ and σ are controlled by the executive. The company's stock price process and the market portfolio are dependent with the instantaneous correlation $\rho_t = \beta\,\sigma^P \,/\, \sqrt{\sigma_t^2 + (\beta\,\sigma^P)^2}$.

The executive influences the company's stock price dynamics by choice of the control strategy (μ,σ), which is specified to be associated with work effort. The control strategy can be conceptualized as deriving from the executive's corporate investment or financing strategy. For example, identifying and initiating positive net present value projects and optimal debt versus equity financing entails work effort that adds value and affects volatility. Value is added if μ is greater than r, indicating excess return compensation for non-systematic risk. To ensure sensible solutions we require $\mu \geq r$, which effectively bars the executive from destroying company value ($\mu < r$) and potentially profiting by shorting the company's stock.

The executive's instantaneous disutility of work effort at time t is represented by the disutility rate $c(t,V_t,\mu_t,\sigma_t)$ for control strategy (μ_t,σ_t), where V_t is the executive's wealth. We assume that the disutility rate $c : [0,T] \times \mathbb{R}^+ \times [r,\infty) \times \mathbb{R}^+ \to \mathbb{R}_0^+$ is a continuous and suitably differentiable function where T is the executive's time horizon.

The executive's starting wealth inclusive of his compensation, $V_0 > 0$, is invested in the financial market. Ongoing continuous-time portfolio adjustment is assumed to be free of short-selling constraints, and to be self-financing (i.e. no funds are added to or withdrawn from the executive's portfolio). The portfolio is allocated with fraction $\pi^P = (\pi_t^P)_{t\geq 0}$ invested in the market portfolio, fraction $\pi^S = (\pi_t^S)_{t\geq 0}$ in the company's stock, and the remainder in the risk-free account. For investment strategy $\pi = (\pi^P, \pi^S)$, and control strategy (μ,σ), we collect all controls in the vector process $u = (\pi^P, \pi^S, \mu, \sigma)$. The executive's wealth process, $V^u = (V_t^u)_{t\geq 0}$, is then

$$\mathrm{d}V_t^u = V_t^u \left((1 - \pi_t^P - \pi_t^S) \frac{\mathrm{d}B_t}{B_t} + \pi_t^P \frac{\mathrm{d}P_t}{P_t} + \pi_t^S \frac{\mathrm{d}S_t^{\mu,\sigma}}{S_t^{\mu,\sigma}} \right) , \; V_0^u \in \mathbb{R}^+ . \qquad (1.4)$$

This equation can be rewritten using the equations (1.1), (1.2) and (1.3) for the money market account, the market portfolio and the company's stock respectively as

$$\begin{aligned} \mathrm{d}V_t^u = V_t^u \Big(& \big[r + (\pi_t^P + \beta\, \pi_t^S)(\mu^P - r) + \pi_t^S(\mu_t - r) \big]\, \mathrm{d}t \\ & + \big[\pi_t^P + \beta\, \pi_t^S \big]\, \sigma^P \mathrm{d}W_t^P + \pi_t^S\, \sigma_t\, \mathrm{d}W_t \Big) , \quad V_0^u \in \mathbb{R}^+ . \end{aligned} \qquad (1.5)$$

The executive is assumed to maximize the expected terminal utility of his wealth for time horizon T, subject to some utility function which will be specified when deriving closed-form solutions.

Assuming that the control of the company's stock price behavior (μ,σ) is determined exogenously, the executive's *optimal investment decision* is then described by

$$\widehat{\Phi}(t, v) = \sup_{\pi \in \widehat{A}(t,v)} \mathbb{E}^{t,v}[U(V_T^\pi)] , \quad (t,v) \in [0,T] \times \mathbb{R}^+,$$

where $\widehat{A}(t,v)$ denotes the set of all admissible portfolio strategies π at time t corresponding to portfolio value (i.e. wealth) $v = V_t^\pi > 0$, U is a utility function, and $\mathbb{E}^{t,v}$ denotes the expectation conditional on t and v. See for example Korn and Korn (2001), where we note that in this classical set-up the wealth process $V^\pi = (V_t^\pi)_{t\geq 0} > 0$ does not depend on the exogenously given control strategy (μ,σ).

Definition 1.1.1

Let $0 \leq t \leq T$, t fixed. Further let (μ,σ) take values in $[r,\infty) \times (0,\infty)$. By $A(t,v)$ we denote the set of all admissible strategies $u = \left(\pi^P,\pi^S,\mu,\sigma\right)$ corresponding to portfolio value $v = V_t^u > 0$ at time t, which are $\{\mathcal{F}_s\,;\, t \leq s \leq T\}$-predictable processes, such that

(i) the company's stock price process

$$\mathrm{d}S_s^{\mu,\sigma} = S_s^{\mu,\sigma}\left(\mu_s\,\mathrm{d}s + \beta\left[\frac{\mathrm{d}P_s}{P_s} - r\mathrm{d}s\right] + \sigma_s\,\mathrm{d}W_s\right)\,, \quad S_t^{\mu,\sigma} \in \mathbb{R}^+\,,$$

has a unique non-negative solution and satisfies

$$\int_t^T (S_s^{\mu,\sigma})^2\left((\beta\sigma^P)^2 + (\sigma_s)^2\right)\,\mathrm{d}u \,<\, \infty \quad P-a.s.\,;$$

(ii) the wealth equation

$$\mathrm{d}V_s^u = V_s^u\left((1-\pi_s^P-\pi_s^S)\frac{\mathrm{d}B_s}{B_s} + \pi_s^P\frac{\mathrm{d}P_s}{P_s} + \pi_s^S\frac{\mathrm{d}S_s^{\mu,\sigma}}{S_s^{\mu,\sigma}}\right)\,, V_t^u \in \mathbb{R}^+\,,$$

has a unique non-negative solution and satisfies

$$\int_t^T (V_s^u)^2\left((\pi_s^P + \beta\,\pi_s^S)^2(\sigma^P)^2 + (\pi_s^S\sigma_s)^2\right)\,\mathrm{d}s \,<\, \infty \quad P-a.s.\,;$$

(iii) and the utility of wealth and the disutility of control satisfy

$$\mathbb{E}\left[U\left(V_T^u\right)^- + \int_t^T c(s,V_s^u,\mu_s,\sigma_s)\,\mathrm{d}s\right] \,<\, \infty\,.$$

The *optimal investment and control decision* is then the solution of

$$\Phi(t,v) = \sup_{u \in A(t,v)} \mathbb{E}^{t,v}\left[U\left(V_T^u\right) - \int_t^T c(s, V_s^u, \mu_s, \sigma_s)\,\mathrm{d}s\right], \tag{1.6}$$

where $(t,v) \in [0,T] \times \mathbb{R}^+$.

1.1.1 Restating the Set-up

A decomposition result for the optimal investment and control problem in (1.6) is derived. To do this we respecify the executive's control strategy in terms of a target non-systematic Sharpe ratio $\lambda = (\mu - r)/\sigma$; this supposes the executive makes investment or financing decisions with regard for their expected return to risk trade-off. Now the original four-dimensional maximization problem can be solved in two steps. The first step entails minimizing the disutility rate for the target non-systematic Sharpe ratio to obtain $c^\star(t,v,\lambda)$. The proof of Lemma 1.1.1 demonstrates that this is achievable given Assumption 1.1.1. For the second step, Theorem 1.1.3 shows that the optimal investment and control problem given by (1.6) can be restated and solved as a maximization problem over the reduced control vector $u' = (\pi^P, \pi^S, \lambda)$, with c replaced by $c^\star$.

The non-systematic expected return to risk trade-off represented by λ indicates the quality of the executive's control decision, which is associated with work effort and thereby disutility. Given λ, minimized disutility $c^\star$ is associated with the non-systematic volatility choice $\sigma^\star$ (see Lemma 1.1.1). That is, for a given level of control strategy quality represented by λ, $\sigma^\star$ is the non-systematic volatility associated with the 'easiest' control strategy from the executive's perspective.

Assumption 1.1.1 gives the conditions required for existence and uniqueness of $c^\star$ and $\sigma^\star$.

Assumption 1.1.1

The function $c : [0,T] \times \mathbb{R}^+ \times [r,\infty) \times \mathbb{R}^+ \to \mathbb{R}_0^+$, $(t,v,\mu,\sigma) \mapsto c(t,v,\mu,\sigma)$ *satisfies:*

(i) c *is continuous in* t *and* v*, and twice continuously differentiable in* μ *and* σ*;*

(ii) fix $(t,v,\lambda) \in [0,T] \times \mathbb{R}^+ \times \mathbb{R}_0^+$*, then*

$$\limsup_{\sigma \searrow 0} \quad \lambda \frac{\partial c}{\partial \mu}(t,v,r+\lambda\,\sigma,\sigma) + \frac{\partial c}{\partial \sigma}(t,v,r+\lambda\,\sigma,\sigma) \quad < \quad 0\,,$$

and

$$\sup_{\sigma > 0} \quad \lambda \frac{\partial c}{\partial \mu}(t,v,r+\lambda\,\sigma,\sigma) + \frac{\partial c}{\partial \sigma}(t,v,r+\lambda\,\sigma,\sigma) \quad > \quad 0\,;$$

(iii) it holds that

$$(\mu - r)^2 \frac{\partial^2 c}{\partial \mu^2} + 2\,\sigma\,(\mu - r)\,\frac{\partial^2 c}{\partial \mu\,\partial \sigma} + \sigma^2 \frac{\partial^2 c}{\partial \sigma^2} \quad > \quad 0\,;$$

(iv) for all (t,v)*:* $\inf_{\sigma>0} c(t,v,r,\sigma) = 0$.

In Assumption 1.1.1, (i) is a natural smoothness condition, (ii) and (iii) respectively ensure uniqueness and existence of the disutility $c^\star(t,v,\lambda)$ depending on the non-systematic Sharpe ratio λ, and (iv) is a natural norming condition that specifies a lower bound of zero disutility (i.e. zero work effort) when expected excess return is zero ($\mu = r$).

As an example, a disutility function that fulfills the conditions of Assumption 1.1.1 is

$$c(t,v,\mu,\sigma) = \kappa \left(\frac{\mu - r}{\sigma}\right)^\alpha + \nu\,(\sigma - \sigma_0)^2 = \kappa\lambda^\alpha + \nu\,(\sigma - \sigma_0)^2\,,$$

where $\mu \geq r$, $\sigma > 0$, $\kappa \geq 0$, $\nu > 0$, $\alpha > 0$; and $\sigma_0 > 0$ is the company's base-level non-systematic risk. Here c is proportional to λ depending on parameters κ and α (i.e. the executive's work effort is proportional to the quality of his control decision); and c increases with deviation of control choice σ from σ_0 depending on parameter ν (i.e. given λ, the executive's easiest control decision is to make investment or financing decisions that do not disrupt the company's base-level non-systematic volatility, which might be conceptualized as a preference for maintaining the status quo of the company's business model).

The following lemma establishes the first step of the decomposition result.

Lemma 1.1.1
Suppose Assumption 1.1.1 holds, then the minimization problem

$$\min_{\{\sigma>0:\mu=r+\lambda\,\sigma\}} c(t,v,\mu,\sigma)\,, \quad \textit{for } (t,v,\lambda) \in [0,T] \times \mathbb{R}^+ \times \mathbb{R}_0^+\,, \tag{1.7}$$

admits a unique solution $\sigma^\star(t,v,\lambda)$.

Proof. Fix $(t,v,\lambda) \in [0,T] \times \mathbb{R}^+ \times \mathbb{R}_0^+$ and define the function f by $f(\sigma) = c(t,v,r + \lambda\,\sigma, \sigma)$, for $\lambda \geq 0$. We need to show for f that a minimizing $\sigma^\star = \sigma^\star(t,v,\lambda)$ exists and is unique. Computing the first and second derivatives gives

$$f'(\sigma) = \lambda\,\frac{\partial c}{\partial \mu}(t,v,r + \lambda\,\sigma, \sigma) + \frac{\partial c}{\partial \sigma}(t,v,r + \lambda\,\sigma, \sigma)\,,$$

and

$$f''(\sigma) = \lambda^2\,\frac{\partial^2 c}{\partial \mu^2}(t,v,r+\lambda\,\sigma,\sigma)+2\,\lambda\,\frac{\partial^2 c}{\partial\sigma\partial\mu}(t,v,r+\lambda\,\sigma,\sigma)+\frac{\partial^2 c}{\partial\sigma^2}(t,v,r+\lambda\,\sigma,\sigma)\,.$$

By the differentiability assumption for c, f' is continuous and differentiable and f'' is continuous. Using elementary calculus rationale, the minimization problem $\min_{\sigma>0} f(\sigma)$ admits a unique solution if $f'(\sigma^\star) = 0$ has a solution and f is strictly convex.

For $f'(\sigma^\star) = 0$ to admit a solution that locally minimizes f, it is sufficient that f' starts below zero, $f'(0+) < 0$, and that f' takes on a positive value for some $\sigma > 0+$. This is given by Assumption 1.1.1 (ii). Moreover the condition f is strictly convex, $f'' > 0$, implies the solution is a unique global minimizer. Assumption 1.1.1 (iii) gives the strict convexity of f. □

Changing the parameters of the optimal investment and control problem in (1.6) from $u = (\pi^P, \pi^S, \mu, \sigma)$ to $u' = (\pi^P, \pi^S, \lambda)$, and replacing c by $c^\star$, requires adapting Def. 1.1.1 to the new setting. Before we present the new framework, observe that the company's stock price with respect to λ (and $\sigma^\star(\lambda)$) has the dynamics

$$\mathrm{d}S_t^\lambda = S_t^\lambda \left(\left[r + \lambda_t \sigma^\star(t, V_t^{u'}, \lambda_t) \right] \mathrm{d}t + \beta \left[\frac{\mathrm{d}P_t}{P_t} - r\mathrm{d}t \right] + \sigma^\star(t, V_t^{u'}, \lambda_t)\, \mathrm{d}W_t \right), \quad S_0^\lambda \in \mathbb{R}^+ . \tag{1.8}$$

Accordingly, the wealth equation will change to

$$\mathrm{d}V_t^{u'} = V_t^{u'} \left((1 - \pi_t^P - \pi_t^S) \frac{\mathrm{d}B_t}{B_t} + \pi_t^P \frac{\mathrm{d}P_t}{P_t} + \pi_t^S \frac{\mathrm{d}S_t^\lambda}{S_t^\lambda} \right), \quad V_0^{u'} \in \mathbb{R}^+ , \tag{1.9}$$

which implies the following rewritten representation using the equations (1.1), (1.2) and now (1.8) for the money market account, the market portfolio and the restated company's stock respectively:

$$\begin{aligned} \mathrm{d}V_t^{u'} = & V_t^{u'} \Big(\Big[r + (\pi_t^P + \beta\, \pi_t^S)(\mu^P - r) + \pi_t^S\, \lambda_t\, \sigma^\star(t, V_t^{u'}, \lambda_t) \Big]\, \mathrm{d}t \\ & + \left[\pi_t^P + \beta\, \pi_t^S \right] \sigma^P \mathrm{d}W_t^P + \pi_t^S\, \sigma^\star(t, V_t^{u'}, \lambda_t)\, \mathrm{d}W_t \Big), \ \ V_0^{u'} \in \mathbb{R}^+ . \end{aligned} \tag{1.10}$$

Further, we define the minimized disutility $c^\star$ corresponding to portfolio value $v > 0$ at time t via

$$c^\star(t, v, \lambda) := c(t, v, r + \lambda\, \sigma^\star(t, v, \lambda), \sigma^\star(t, v, \lambda)) = \min_{\{\sigma > 0 : \mu = r + \lambda\, \sigma\}} c(t, v, \mu, \sigma) . \tag{1.11}$$

For the stock price process S_t^λ defined in (1.8) we have to impose a technical condition similar to Def. 1.1.1 (i). The change of control from vector from $u = (\pi^P,\pi^S,\mu,\sigma)$ to $u' = (\pi^P,\pi^S,\lambda)$ is driven by the disutility function c, i.e. $\sigma^\star(t,v,\lambda)$ is determined by the form of c (see Lemma 1.1.1). The following assumption guarantees that the most cost efficient strategies are admissible.

Assumption 1.1.2

For a given control $u = (\pi^P,\pi^S,\mu,\sigma) \in A(t,v)$, the value process $V_t^{u'}$, $\lambda_t = (\mu_t - r)/\sigma_t$ and $\sigma^\star(t,V_t^{u'},\lambda_t)$ are determined in accordance with Lemma 1.1.1. The process S_t^λ defined in (1.8) *is assumed to satisfy*

$$\int_t^T (S_s^\lambda)^2 \left((\beta\sigma^P)^2 + (\sigma^\star(s,V_s^{u'},\lambda_s))^2\right) \mathrm{d}s \,<\, \infty \quad P-a.s..$$

Definition 1.1.2

Let $0 \le t \le T$, t fixed, and let further λ take values in $[0,\infty)$. Then we denote by $A'(t,v)$ the set of admissible strategies $u' = \left(\pi^P,\pi^S,\lambda\right)$ corresponding to portfolio value $v = V_t^{u'} > 0$ at time t, which are $\{\mathcal{F}_s\,;\, t \le s \le T\}$-predictable processes, such that

(i) the company's stock price process

$$\mathrm{d}S_s^\lambda = S_s^\lambda \left([r + \lambda_s\sigma_s^\star]\,\mathrm{d}t + \beta\left[\frac{\mathrm{d}P_s}{P_s} - r\mathrm{d}s\right] + \sigma_s^\star\,\mathrm{d}W_s \right), \quad S_t^\lambda \in \mathbb{R}^+,$$

has a unique non-negative solution and satisfies

$$\int_t^T (S_s^\lambda)^2 \left((\beta\sigma^P)^2 + (\sigma_s^\star)^2\right) \mathrm{d}s \,<\, \infty \quad P-a.s.;$$

(ii) the wealth equation

$$\mathrm{d}V_s^{u'} = V_s^{u'} \left((1 - \pi_s^P - \pi_s^S)\frac{\mathrm{d}B_s}{B_s} + \pi_s^P\frac{\mathrm{d}P_s}{P_s} + \pi_s^S\frac{\mathrm{d}S_s^\lambda}{S_s^\lambda} \right), \; V_t^{u'} \in \mathbb{R}^+,$$

has a unique non-negative solution and satisfies

$$\int_t^T \left(V_s^{u'}\right)^2 \left((\pi_s^P + \beta\,\pi_s^S)^2(\sigma^P)^2 + (\pi_s^S\sigma_s^\star)^2\right) \mathrm{d}s \,<\, \infty \quad P-a.s.;$$

(iii) and the utility of wealth and the minimized disutility of control satisfy

$$\mathbb{E}\left[U\left(V_T^{u'}\right)^- + \int_t^T c^\star(s, V_s^{u'}, \lambda_s)\,\mathrm{d}s\right] < \infty\,.$$

Theorem 1.1.3 (Correspondence Result)
Suppose (1.6) *admits a solution* Φ*, then this solution coincides with the value function of the optimal investment and control problem*

$$\Phi'(t,v) = \sup_{u' \in A'(t,v)} \mathbb{E}^{t,v}\left[U\left(V_T^{u'}\right) - \int_t^T c^\star\left(s,V_s^{u'},\lambda_s\right)\,\mathrm{d}s\right]\,,\ (t,v) \in [0,T] \times \mathbb{R}^+\,, \tag{1.12}$$

where $A'(t,v)$ *is given in Def. 1.1.2.*

Proof. Let

$$J(t,v;u) := \mathbb{E}^{t,v}\left[U(V_T^u) - \int_t^T c(s,V_s^u,\mu(s,V_s^u),\sigma(s,V_s^u))\,\mathrm{d}s\right]$$

and

$$J'(t,v;u') := \mathbb{E}^{t,v}\left[U(V_T^{u'}) - \int_t^T c^\star(s,V_s^{u'},\lambda(s,V_s^{u'}))\,\mathrm{d}s\right]\,.$$

The assertion is proven if we show that

$$\sup_{u \in A(t,v)} J(t,v;u) = \sup_{u' \in A'(t,v)} J'(t,v;u')\,, \tag{1.13}$$

i.e. the performance functionals J and J' admit the same value function $\Phi(t,v)$.

First, we are given a control vector $u = (\pi^P, \pi^S, \mu, \sigma) \in A(t,v)$ (and the resulting non-systematic Sharpe ratio $\lambda = (\mu - r)/\sigma$), and show that there exists a control vector $\tilde{u} = (\tilde{\pi}^P,\tilde{\pi}^S,\tilde{\lambda}) \in \tilde{A}(t,v)$ such that $J(t,v;u) \leq J'(t,v;\tilde{u})$. Note that replacing the controls μ and σ by λ and replacing the disutility c by $c^\star$ leads to two different systems of controlled SDEs describing the executive's utility-maximizing behavior. For control vector $\tilde{u} = (\tilde{\pi}^P, \tilde{\pi}^S, \tilde{\lambda})$, we write the dynamics of the resulting price processes as follows

$$\begin{aligned}
\mathrm{d}\tilde{B}_t &= \tilde{B}_t\, r\, \mathrm{d}t\ ,\ \mathrm{d}\tilde{P}_t = \tilde{P}_t \left(\left[r + \lambda^P \sigma^P \right] \mathrm{d}t + \sigma^P \mathrm{d}W_t^P \right)\ , \\
\mathrm{d}\tilde{S}_t &= \tilde{S}_t \left(\left[r + \tilde{\lambda}_t\, \sigma^\star(t, \tilde{V}_t^{\tilde{u}}, \tilde{\lambda}_t) \right] \mathrm{d}t + \beta \left[\frac{\mathrm{d}\tilde{P}_t}{\tilde{P}_t} - r\mathrm{d}t \right] + \sigma^\star(t, \tilde{V}_t^{\tilde{u}}, \tilde{\lambda}_t)\, \mathrm{d}W_t \right)\ , \\
\mathrm{d}\tilde{V}_t^{\tilde{u}} &= \tilde{V}_t^{\tilde{u}} \left(\left[r + \tilde{\pi}_t^P \lambda^P \sigma^P + \tilde{\pi}_t^S \left(\tilde{\lambda}_t\, \sigma^\star(t, \tilde{V}_t^{\tilde{u}}, \tilde{\lambda}_t) + \beta(\mu^P - r) \right) \right] \mathrm{d}t \right. \\
&\qquad \left. + \tilde{\pi}_t^P\, \sigma^P\, \mathrm{d}W_t^P + \tilde{\pi}_t^S \beta \sigma^P \mathrm{d}W_t^P + \tilde{\pi}_t^S\, \sigma^\star(t, \tilde{V}_t^{\tilde{u}}, \tilde{\lambda}_t)\, \mathrm{d}W_t \right)\ .
\end{aligned}$$

The system $(\tilde{B}, \tilde{P}, \tilde{S}, \tilde{V}^{\tilde{u}})$ is specified on the same probability space as the original system (B, P, S, V^u). We now choose the controls

$$\tilde{\lambda}_t := \lambda_t\ , \quad \tilde{\pi}_t^P := \pi_t^P + \pi_t^S\, \beta \left(1 - \frac{\sigma(t, V_t^u)}{\sigma^\star(t, \tilde{V}_t^{\tilde{u}}, \tilde{\lambda}_t)} \right)\ , \quad \tilde{\pi}_t^S := \pi_t^S \frac{\sigma(t, V_t^u)}{\sigma^\star(t, \tilde{V}_t^{\tilde{u}}, \tilde{\lambda}_t)}\ .$$

This yields that the integrands of the stochastic integrals (or, coefficients of the SDEs) defining $\mathrm{d}V^u$ and $\mathrm{d}\tilde{V}^{\tilde{u}}$ coincide almost-surely for each t. Noting that there exist continuous versions of the resulting processes V^u and $\tilde{V}^{\tilde{u}}$, we obtain uniformly on $[0, T]$

$$\tilde{V}^{\tilde{u}} = V^u\ , \quad \text{and} \quad \tilde{\lambda} = \lambda\ , \quad P - a.s.. \tag{1.14}$$

We remark here that by definition we also have $\tilde{B} = B$ and $\tilde{P} = P$. However, in general $\tilde{S} \neq S$. Continuing the proof, by $c^\star(t, v, \lambda) := c(t, v, r + \lambda\, \sigma^\star(t, v, \lambda), \sigma^\star(t, v, \lambda)) = \min_{\{\sigma > 0: \mu = r + \lambda\, \sigma\}} c(t, v, \mu, \sigma)$ and recalling that $\lambda = (\mu - r)/\sigma$ we have:

$$\begin{aligned}
J(t, v; u) &\leq \mathbb{E}^{t,v} \left[U(V_T^u) - \int_t^T c^\star(s, V_s^u, \lambda_s)\, \mathrm{d}s \right] \\
&\overset{(1.14)}{=} \mathbb{E}^{t,v} \left[U(\tilde{V}_T^{\tilde{u}}) - \int_t^T c^\star\left(s, \tilde{V}_s^{\tilde{u}}, \tilde{\lambda}_s \right)\, \mathrm{d}s \right] = J'(t, v; \tilde{u})\ .
\end{aligned}$$

To finish the first part of the proof, we have to ensure $\tilde{u} \in A'(t, v)$. This can be done by recalling that $u \in A(t, v)$ and checking conditions (i),(ii) and (iii) of Def. 1.1.2. First note that (i) is satisfied due to Assumption 1.1.2. To verify (ii) note that

$$\begin{aligned} \tilde{V}_s^{\tilde{u}} &= V_s^u\,, \quad P-a.s.\,, \\ \tilde{\pi}_s^P + \beta\,\tilde{\pi}_s^S &= \pi_s^P + \beta\,\pi_s^S\,, \quad P-a.s.\,, \\ \tilde{\pi}_s^S\,\sigma^\star(s,\tilde{V}_s^{\tilde{u}},\tilde{\lambda}_s) &= \pi_s^S\,\sigma_s\,, \quad P-a.s.\,, \end{aligned}$$

for $t \le s \le T$, and recall Def. 1.1.1 (ii). To verify (iii), check that $c_s^\star(s,\tilde{V}_s^{\tilde{u}},\tilde{\lambda}_s) \le c(s,V_s^u,\mu_s,\sigma_s)$, for $t \le s \le T$, and then recall Def. 1.1.1 (iii) to obtain an integrable upper bound.

To conclude the proof we have to show that for a given control $\tilde{u} = (\tilde{\pi}^P,\tilde{\pi}^S,\tilde{\lambda}) \in A'(t,v)$ there is a corresponding control $u = (\pi^P,\pi^S,\mu,\sigma) \in A(t,v)$ s.t. $J'(t,v;\tilde{u}) \le J(t,v;u)$. To do so, set $\sigma_s = \sigma^\star(s,\tilde{V}_s^{\tilde{u}},\tilde{\lambda}_s)$, $\mu_s = r + \tilde{\lambda}_s\,\sigma_s$, as well as $\tilde{\pi}_s^P = \pi_s^P$ and $\tilde{\pi}_s^S = \pi_s^S$, for $t \le s \le T$, to obtain $J'(t,v;\tilde{u}) = J(t,v;u)$. Finally, $u \in A(t,v)$ is verified directly by checking Def. 1.1.1 using $\tilde{u} \in A'(t,v)$ and Def. 1.1.2. $\square$

1.2 Optimal Strategies

In this section we use stochastic control techniques to derive closed-form solutions to the investment and control decision problem in (1.12), for special choices of the utility and disutility functions. In particular we specify constant relative risk aversion. For the *relative* risk aversion parameter $\gamma > 0$, the utility function U is

$$U(v) = \begin{cases} \dfrac{v^{1-\gamma}}{1-\gamma}\,, & \text{for } \gamma > 0 \text{ and } \gamma \neq 1 \\ \log(v)\,, & \text{for } \gamma = 1\,, \end{cases} \tag{1.15}$$

and the disutility of control (i.e. work effort) $c^\star$ is

$$c^\star(t,v,\lambda) = \kappa\, v^{1-\gamma}\,\frac{\lambda^\alpha}{\alpha}\,, \quad \gamma > 0\,, \tag{1.16}$$

where $\kappa > 0$ and $\alpha > 2$ are the executive's work effectiveness parameters, respectively termed 'inverse work productivity' and 'disutility stress'. κ directly relates the executive's work effort disutility to the quality of his control decision as indicated by the non-systematic Sharpe ratio λ, and α indicates how rapidly his work effort disutility will rise for the sake of an improved λ. The requirement $\alpha > 2$ is a consequence of our set-up that ensures the executive's disutility grows with work effort, i.e. λ, at a rate that offsets (at some level of λ) the rate of his utility gain due to the flow-on from his work effort to the value of his own-company stockholding; this becomes evident with derivation of the solution to (1.12). A higher quality executive is able to achieve a given λ with lower disutility, and is able to improve λ with lower incremental disutility. That is, higher executive quality (i.e. higher work effectiveness) is implied by lower values of κ and α.

In (1.16), the scaling factor $v^{1-\gamma}$ relates the executive's disutility of work effort to his wealth (v) with a formulation based on the constant relative risk aversion formulation of the utility function in (1.15). Given a low (high) value of the relative risk aversion parameter, $0 < \gamma < 1$ ($\gamma > 1$), the executive's work effort disutility increases (decreases) with his wealth at a decreasing rate; and for $\gamma = 1$, work effort disutility is unrelated to wealth.

Remark 1.2.1

Our specification for the disutility of work effort is economically reasonable for the case $0 < \gamma < 1$. *For* $\gamma > 1$, *our specification produces decreasing disutility of work effort for an increasing level of wealth, keeping work effort constant. This is economically counter-intuitive.*

But it is very important for this utility/disutility set-up that the utility of wealth and the disutility of work effort are normed to take values on the same scale w.r.t. the wealth v.

A possible rationalization is to consider γ *to be positively related to the executive's work ethic, such that a high work ethic executive has comparatively* *low aversion to work effort at outset and will become further less averse to work effort if past effort or chance brings success as indicated by increased wealth. Whereas a low work ethic executive has comparatively high aversion to work effort and will become further more averse to work effort if his wealth increases. So in what follows we solve nevertheless our executive's investment and control problem for all values of* $\gamma > 0$.

For the remainder of the chapter we assume that the optimal investment and control problem (1.12) admits a value function $\Phi \in C^{1,2}$.

To guarantee that the candidates we will derive for the executive's optimal investment and control strategy (i.e. the choices for own-company stockholding, market portfolio holding and non-systematic Sharpe ratio) and value function are indeed optimal, we have to consider a more restrictive class of admissible strategies as follows.

Definition 1.2.1
Let $0 \leq t \leq T$, *t fixed, and let* λ *take values in* $[0,\infty)$. *Further choose* $\tilde{\epsilon} \in (0,\infty)$ *as close to zero as possible. Then by* $A'_\gamma(t,v)$ *we denote the set of admissible strategies* $u' = \left(\pi^P,\pi^S,\lambda\right) \in A'(t,v)$, *such that*

(i) for $\gamma > 0$ *and* $\gamma \neq 1$:

$$\int_t^T (\pi_s^P + \beta\pi_s^S)^{2+\tilde{\epsilon}}(\sigma^P)^{2+\tilde{\epsilon}} + \left(\pi_s^S\sigma_s^\star\right)^{2+\tilde{\epsilon}} \mathrm{d}s \leq C_1 < \infty, \textit{for some } C_1 \in \mathbb{R}_0^+; \tag{1.17}$$

$$\int_t^T \left|\pi_s^S\sigma_s^\star\lambda_s\right| \,\mathrm{d}u \leq C_2 < \infty\,, \ \textit{for some } C_2 \in \mathbb{R}_0^+\,; \tag{1.18}$$

(ii) for $\gamma = 1$:

$$\mathbb{E}\left[\int_t^T (\pi_s^P + \beta\,\pi_s^S)^2(\sigma^P)^2 + (\pi_s^S\sigma_s^\star)^2\,\mathrm{d}s\right] < \infty\,. \tag{1.19}$$

Restating the optimal investment and control problem:

$$\Phi(t, v) = \sup_{u' \in A'_\gamma(t,v)} \mathbb{E}^{t,v}\left[U(V_T^{u'}) - \int_t^T c^\star(s,V_s^{u'},\lambda_s)\,\mathrm{d}s\right], \tag{1.20}$$

where $(t,v) \in [0,T] \times \mathbb{R}^+$.

Remark 1.2.2

The results previously derived for $A'(t,v)$ remain valid for $A'_\gamma(t,v)$ since $A'_\gamma(t,v)$ is obviously a subset of $A'(t,v)$.

1.2.1 Hamilton-Jacobi-Bellman Equation

Having formulated the optimal investment and control decision problem with respect to the parameter set $u' = (\pi^P,\pi^S,\lambda)$ as given by (1.20), we can write down the corresponding Hamilton-Jacobi-Bellman equation (HJB); note that we formulate this equation with respect to a general utility function U and a general disutility function $c^\star$:

$$\begin{aligned} 0 &= \sup_{u' \in \mathbb{R}^2 \times [0,\infty)} \left[(L^{u'}\Phi)(t,v) - c^\star(t,v,\lambda)\right], \text{ for } (t,v) \in [0,T) \times \mathbb{R}^+, \\ U(v) &= \Phi(T,v), \quad \text{for } v \in \mathbb{R}^+, \end{aligned} \tag{1.21}$$

where the differential operator $L^{u'}$ is given by

$$\begin{aligned} (L^{u'}g)(t,v) = {} & \frac{\partial g}{\partial t}(t,v) + \frac{\partial g}{\partial v}(t,v)\, v\left(r + \pi^S\lambda\,\sigma^\star(t,v,\lambda) + \pi^S\beta[\mu^P - r]\right. \\ & \left. + \pi^P[\mu^P - r]\right) + \frac{1}{2}\frac{\partial^2 g}{\partial v^2}(t,v)\, v^2\left(\left[\pi^S\sigma^\star(t,v,\lambda)\right]^2 + \left[(\pi^P + \pi^S\beta)\sigma^P\right]^2\right). \end{aligned} \tag{1.22}$$

Potential maximizers $\pi^{P\star}$, $\pi^{S\star}$ and $\lambda^\star$ of the HJB (1.21) can be calculated by establishing the first order conditions:

$$\begin{aligned} \pi^{P\star}(t,v) &= -\frac{(\mu^P - r)}{v(\sigma^P)^2}\,\frac{\Phi_v(t,v)}{\Phi_{vv}(t,v)} - \beta\,\pi^{S\star}(t,v), \\ \pi^{S\star}(t,v) &= -\frac{\lambda^\star(t,v)}{v\sigma^\star(t,v,\lambda^\star(t,v))}\,\frac{\Phi_v(t,v)}{\Phi_{vv}(t,v)}, \end{aligned} \tag{1.23}$$

where $\lambda^\star$ is the solution of the implicit equation

$$\lambda \frac{\Phi_v^2(t,v)}{\Phi_{vv}(t,v)} + \frac{\partial c^\star}{\partial \lambda}(t,v,\lambda) = 0 \quad \text{for all } (t,v) \in [0,T] \times \mathbb{R}^+ , \tag{1.24}$$

where we have already used (1.23) to simplify the equation.

From (1.23), the executive's optimal wealth allocation to his own company's stock $\pi^{S^\star}$ depends on his optimal control decision for the stock price dynamics $\lambda^\star$. However, the executive's overall preference for investment exposure to systematic risk is independent of $\lambda^\star$. Therefore his optimal wealth allocation to the market portfolio $\pi^{P^\star}$ incorporates a deduction for the systematic risk exposure entailed by $\pi^{S^\star}$; because of this, $\pi^{P^\star}$ also depends on $\lambda^\star$ via $\pi^{S^\star}$ factored by the company's beta β.

Substituting the maximizers (1.23) in the HJB (1.21) yields:

$$\begin{aligned} 0 \;=\; & \Phi_t(t,v) + \Phi_v(t,v)\, v\, r - \frac{1}{2}(\lambda^\star(t,v))^2 \frac{\Phi_v^2(t,v)}{\Phi_{vv}(t,v)} - \frac{1}{2}(\lambda_P)^2 \frac{\Phi_v^2(t,v)}{\Phi_{vv}(t,v)} \\ & - c^\star(t,v,\lambda^\star(t,v)) , \end{aligned} \tag{1.25}$$

where $\lambda_P := \dfrac{\mu^P - r}{\sigma^P}$ is the Sharpe ratio of the market portfolio.
In the following section we solve (1.25) with choices (1.15) and (1.16) for the utility and disutility functions.

1.2.2 Closed-Form Solutions

Closed-form solutions are obtained for the optimal investment and control problem in (1.20) using the utility and disutility functions (1.15) and (1.16), first for the power-utility case ($\gamma > 0$ and $\gamma \neq 1$), and then for the log-utility case ($\gamma = 1$).

Theorem 1.2.1 (The power-utility case: $\gamma > 0$ and $\gamma \neq 1$)
The full solution of the maximization problem (1.20) can be summarized by

the strategy

$$\begin{aligned} \lambda^\star(t,v) &= \left(\frac{1}{\kappa\,\gamma}\,f(t)\right)^{\frac{1}{\alpha-2}}\,, \\ \pi^{P\star}(t,v) &= \frac{\mu^P - r}{\gamma\,(\sigma^P)^2} - \beta\,\pi^{S\star}(t,v)\,, \quad \pi^{S\star}(t,v) = \frac{\lambda^\star(t,v)}{\gamma\,\sigma^\star(t,v,\lambda^\star(t,v))}\,, \end{aligned} \tag{1.26}$$

and value function

$$\Phi(t,v) = \frac{v^{1-\gamma}}{1-\gamma}\,f(t)\,, \tag{1.27}$$

where

$$\begin{aligned} f(t) &= e^{(1-\gamma)\left(r+\frac{1}{2}\frac{\lambda_P^2}{\gamma}\right)(T-t)} \\ &\times \left(1 - \frac{(\alpha-2)\left(\frac{1}{\kappa\,\gamma}\right)^{\frac{2}{\alpha-2}}}{\alpha\,(2\,\gamma\,r + \lambda_P^2)}\left(e^{\frac{1-\gamma}{\alpha-2}\left(2\,r+\frac{\lambda_P^2}{\gamma}\right)(T-t)} - 1\right)\right)^{-\frac{\alpha-2}{2}}\,. \end{aligned} \tag{1.28}$$

Proof. First observe that a function F of the form $F(\lambda) = a\,\lambda^2 - b\,\lambda^\alpha$, $\lambda \geq 0$, for given constants $a, b > 0$ and $\alpha > 2$, has a unique maximizer $\lambda^\star$ and maximized value $F(\lambda^\star)$ given by

$$\lambda^\star = \left(\frac{2\,a}{\alpha\,b}\right)^{\frac{1}{\alpha-2}}\,, \quad \text{and} \quad F(\lambda^\star) = (\alpha-2)\,\alpha^{-\frac{\alpha}{\alpha-2}}\,2^{\frac{2}{\alpha-2}}\,a^{\frac{\alpha}{\alpha-2}}\,b^{-\frac{2}{\alpha-2}}\,. \tag{1.29}$$

Using this insight the first order condition for $\lambda^\star$ in (1.24) is now solved. Set

$$a = \frac{1}{2}\,\frac{\Phi_v^2}{-\Phi_{vv}}\,, \quad \text{and} \quad b = \frac{\kappa}{\alpha}\,v^{1-\gamma}\,,$$

then (1.29) gives

$$\lambda^\star = \left(\frac{1}{\kappa\,v^{1-\gamma}}\,\frac{\Phi_v^2}{-\Phi_{vv}}\right)^{\frac{1}{\alpha-2}}\,, \quad F(\lambda^\star) = \frac{\alpha-2}{2\,\alpha}\,\left(\kappa\,v^{1-\gamma}\right)^{-\frac{2}{\alpha-2}}\,\left(\frac{\Phi_v^2}{-\Phi_{vv}}\right)^{\frac{\alpha}{\alpha-2}}\,.$$

Now (1.25) reads

$$0 = \Phi_t + \Phi_v\,v\,r + \frac{1}{2}\,\frac{\Phi_v^2}{-\Phi_{vv}}\,\left(\frac{\mu^P - r}{\sigma^P}\right)^2 + \frac{\alpha-2}{2\,\alpha}\,\left(\kappa\,v^{1-\gamma}\right)^{-\frac{2}{\alpha-2}}\,\left(\frac{\Phi_v^2}{-\Phi_{vv}}\right)^{\frac{\alpha}{\alpha-2}}\,. \tag{1.30}$$

Using the separation ansatz $\Phi(t,v) = f(t)\,\frac{v^{1-\gamma}}{1-\gamma}$ with $f(T) = 1$ results in

$$\Phi_t = \dot{f}\,\frac{v^{1-\gamma}}{1-\gamma}\,, \quad \Phi_v = f\,v^{-\gamma}\,, \quad \Phi_{vv} = -\gamma\,f\,v^{-\gamma-1}\,, \quad \text{and} \quad \frac{\Phi_v^2}{-\Phi_{vv}} = \frac{f\,v^{1-\gamma}}{\gamma}\,.$$

Thus (1.30) becomes

$$\begin{aligned} 0 =& \dot{f}\,\frac{v^{1-\gamma}}{1-\gamma} + f\,v^{1-\gamma}\,r + \frac{1}{2}\,\frac{f\,v^{1-\gamma}}{\gamma}\,\left(\frac{\mu^P - r}{\sigma^P}\right)^2 \\ &+ \frac{\alpha-2}{2\,\alpha}\,\left(\kappa\,v^{1-\gamma}\right)^{-\frac{2}{\alpha-2}}\,\left(\frac{f\,v^{1-\gamma}}{\gamma}\right)^{\frac{\alpha}{\alpha-2}}\,. \end{aligned}$$

Dividing by $\frac{v^{1-\gamma}}{1-\gamma}$ and recalling $\lambda_P = (\mu^P - r)/\sigma^P$ gives

$$\dot{f} = f\,\left[-(1-\gamma)\,\left(r + \frac{1}{2}\,\frac{\lambda_P^2}{\gamma}\right)\right] + f^{\frac{\alpha}{\alpha-2}}\,\left[-(1-\gamma)\,\frac{\kappa}{2}\,\frac{\alpha-2}{\alpha}\,\left(\frac{1}{\kappa\,\gamma}\right)^{\frac{\alpha}{\alpha-2}}\right]\,. \tag{1.31}$$

This is a Bernoulli ordinary differential equation (ODE) of the form $\dot{f} = a_1\,f + a_\nu\,f^\nu$, with solution

$$f(t)^{1-\nu} = C\,e^{G(t)} + (1-\nu)\,e^{G(t)}\,\int_0^t e^{-G(s)}\,a_\nu\,\mathrm{d}s\,,$$

where $G(t) = (1-\nu)\,\int_0^t a_1(s)\,\mathrm{d}s$ and C is an arbitrary constant. In our setting we have $\nu = \frac{\alpha}{\alpha-2}$ and $(1-\nu) = \frac{-2}{\alpha-2}$ implying

$$a_1 = -(1-\gamma)\,\left(r + \frac{1}{2}\,\frac{\lambda_P^2}{\gamma}\right)\,, a_\nu = -(1-\gamma)\,\frac{\kappa}{2}\,\frac{\alpha-2}{\alpha}\,\left(\frac{1}{\kappa\,\gamma}\right)^{\frac{\alpha}{\alpha-2}}\,.$$

The formal solution $f(t)^{1-\nu}$ is explicitly calculated in three steps. First, compute

$$G(t) = -\frac{2\,a_1\,t}{\alpha-2}\,, \quad \text{and} \quad \int_0^t e^{-G(s)}\,a_\nu(s)\,\mathrm{d}s = \frac{\alpha-2}{2}\,\frac{a_\nu}{a_1}\,\left(e^{\frac{2\,a_1\,t}{\alpha-2}} - 1\right)\,,$$

then

$$f(t) = e^{a_1\,t}\,\left(C - \frac{a_\nu}{a_1}\,\left(e^{\frac{2\,a_1\,t}{\alpha-2}} - 1\right)\right)^{-\frac{\alpha-2}{2}}\,.$$

Finally, solve for C by using $f(T) = 1$ so that

$$C = e^{\frac{2\,a_1\,T}{\alpha-2}} + \frac{a_\nu}{a_1}\,\left(e^{\frac{2\,a_1\,T}{\alpha-2}} - 1\right)\,.$$

Note also that $f(0) = C^{-\frac{\alpha-2}{2}}$. Now

$$f(t) = e^{-a_1\,(T-t)} \left(1 - \frac{a_\nu}{a_1} \left(e^{-\frac{2\,a_1}{\alpha-2}\,(T-t)} - 1\right)\right)^{-\frac{\alpha-2}{2}} .$$

Substituting for a_1 and a_ν then yields the result for $f(t)$. Using $\frac{\Phi_v}{\Phi_{vv}} = -\frac{v}{\gamma}$ and the first order conditions in (1.23) we obtain the claimed optimal strategy $\lambda^\star$, $\pi^{P\star}$ and $\pi^{S\star}$. Finally note that our claimed optimal strategies are admissible, i.e. $u'^\star = (\pi^{S\star},\pi^{P\star},\lambda^\star) \in A'_\gamma(t,v)$. A sufficient condition for admissibility is that $\lambda^\star$, $\pi^{P\star}\,\sigma^P$, and $\pi^{S\star}\,\sigma^\star$ be uniformly bounded (see Def. 1.2.1); because these expressions are deterministic and continuous functions in u on $[t,T]$, they are hence uniformly bounded. □

Theorem 1.2.2 (The log-utility case: $\gamma = 1$)
The full solution of the maximization problem (1.20) can be summarized by the strategy

$$\begin{aligned} &\lambda^\star(t,v) = \kappa^{-\frac{1}{\alpha-2}}\,, \\ &\pi^{P\star}(t,v) = \frac{\mu^P - r}{(\sigma^P)^2} - \beta\,\pi^{S\star}(t,v)\,, \quad \pi^{S\star}(t,v) = \frac{\lambda^\star(t,v)}{\sigma^\star(t,v,\lambda^\star(t,v))}\,, \end{aligned} \tag{1.32}$$

and value function

$$\Phi(t,v) = \log(v) + \left[r + \frac{1}{2}\left(\frac{\mu^P - r}{\sigma^P}\right)^2 + \frac{\alpha-2}{2\,\alpha}\,\kappa^{-\frac{2}{\alpha-2}}\right](T-t)\,. \tag{1.33}$$

Proof. As in the power-utility case, first the implicit first order condition for $\lambda^\star$ in (1.24) is made explicit. This time set

$$a = \frac{1}{2}\,\frac{\Phi_v^2}{-\Phi_{vv}}\,, \quad \text{and} \quad b = \frac{\kappa}{\alpha}\,,$$

then (1.29) gives

$$\lambda^\star = \left(\frac{1}{\kappa}\,\frac{\Phi_v^2}{-\Phi_{vv}}\right)^{\frac{1}{\alpha-2}}\,, \quad \text{and} \quad F(\lambda^\star) = \frac{\alpha-2}{2\,\alpha}\,\kappa^{-\frac{2}{\alpha-2}}\left(\frac{\Phi_v^2}{-\Phi_{vv}}\right)^{\frac{\alpha}{\alpha-2}} .$$

The partial differential equation (PDE) for log-utility now reads

$$0 = \Phi_t + \Phi_v\,v\,r + \frac{1}{2}\,\frac{\Phi_v^2}{-\Phi_{vv}}\left(\frac{\mu^P - r}{\sigma^P}\right)^2 + \frac{\alpha-2}{2\,\alpha}\,\kappa^{-\frac{2}{\alpha-2}}\left(\frac{\Phi_v^2}{-\Phi_{vv}}\right)^{\frac{\alpha}{\alpha-2}} . \tag{1.34}$$

Using the ansatz $\Phi(t,v) = \log(v) + \varphi(T-t)$ results in

$$\Phi_t = -\varphi\,, \quad \Phi_v = \frac{1}{v}\,, \quad \Phi_{vv} = -\frac{1}{v^2}\,, \quad \text{and} \quad \Phi(T,v) = \log(v) = U(v)\,.$$

Then (1.34) reduces to

$$\varphi = r + \frac{1}{2}\left(\frac{\mu^P - r}{\sigma^P}\right)^2 + \frac{\alpha - 2}{2\,\alpha}\,\kappa^{-\frac{2}{\alpha-2}}\,.$$

Finally, noting $\Phi_v^2/\Phi_{vv} = -1$ and using the first order conditions in (1.23) establishes the claimed optimal strategy. Using identical rationale as in the proof of Theorem 1.2.1, we see that $u'^{\star} = (\pi^{S\star},\pi^{P\star},\lambda^{\star}) \in A_1'(t,v)$. Note that we also obtain the form of the optimal strategy by formally setting $\gamma = 1$ in Theorem 1.2.1. □

1.2.3 Verification Theorem

The solutions of the maximization problems given in Theorems 1.2.1 and 1.2.2 are candidates for the optimal investment and control choices for the problem in (1.20). In this section we verify that under sufficient assumptions these solutions are indeed optimal.

Theorem 1.2.3 (Verification Result)
Let $\kappa > 0$ and $\alpha > 2$. Assume the executive's utility and disutility functions are given by (1.15) and (1.16). Then the candidates given in (1.26) - (1.28) are the optimal investment and control strategy (i.e. own-company stockholding, market portfolio holding and non-systematic Sharpe ratio strategy) and value function of the optimal control problem (1.20) for the case $\gamma > 0$ and $\gamma \neq 1$; and the candidates given in (1.32) and (1.33) are the optimal investment and control strategy and value function of the optimal control problem (1.20) for the case $\gamma = 1$.

Proof. Define the performance functional of our optimal investment and control decision by

$$J(t,v;u') := \mathbb{E}^{t,v}\left[U\left(V_T^{u'}\right) - \int_t^T c^\star(s, V_s^{u'},\lambda_s)\,\mathrm{d}s\right],$$

where $(t,v) \in [0,T] \times \mathbb{R}^+$ and $u' = (\pi^P,\pi^S,\lambda) \in A'_\gamma(t,v)$. Recall the claimed optimal value function $\Phi \in C^{1,2}$, for $\gamma > 0$, and apply Ito's formula to obtain:

$$\begin{aligned}
&U\left(V_T^{u'}\right) - \int_t^T c^\star(s, V_s^{u'},\lambda_s)\,\mathrm{d}s = \Phi(T,V_T^{u'}) - \int_t^T \kappa\left(V_s^{u'}\right)^{1-\gamma}\frac{\lambda_s^\alpha}{\alpha}\,\mathrm{d}s = \Phi(t,v)\\
&+\int_t^T \Big(\Phi_t(s,V_s^{u'}) + \Phi_v(s,V_s^{u'})V_s^{u'}\left[r + \pi_s^S\,\lambda\,\sigma_s^\star + (\pi_s^P + \beta\,\pi_s^S)(\mu^P - r)\right]\\
&+1/2\,\Phi_{vv}(s,V_s^{u'})\left(V_s^{u'}\right)^2\left[((\pi_s^P + \beta\,\pi_s^S)\sigma^P)^2 + (\pi_s^S\sigma_s^\star)^2\right] - \kappa\left(V_s^{u'}\right)^{1-\gamma}\frac{\lambda_s^\alpha}{\alpha}\Big)\,\mathrm{d}s\\
&+\int_t^T \Phi_v(s,V_s^{u'})V_s^{u'}(\pi_s^P + \beta\,\pi_s^S)\sigma^P\,\mathrm{d}W_s^P + \int_t^T \Phi_v(s,V_s^{u'})V_s^{u'}\pi_s^S\sigma_s^\star\,\mathrm{d}W_s\,.
\end{aligned} \tag{1.35}$$

The remainder of the proof is divided into two parts. Part (a) establishes that the value function Φ coincides with the performance functional J evaluated at the claimed maximizers $u'^\star = (\pi^{S^\star},\pi^{P^\star},\lambda^\star)$, $\gamma > 0$. Part (b) shows the optimality of the candidate $u'^\star$, i.e.: $J(t,v;u') \le \Phi(t,v)$, for $u' \in A'_\gamma(t,v)$.

Part (a): We establish that $J(t,v;u'^\star) = \Phi(t,v)$. To do this we show that in the right hand side (RHS) of (1.35) the drift vanishes by the HJB (1.21) and that the local martingale component is a true martingale and hence disappears in expectation. And finally, it is verified that indeed $u'^\star \in A'_\gamma(t,v)$.

By construction, Φ with control $u'^\star$ satisfies the HJB-PDE in (1.21), that is,

$$\begin{aligned}
0 \;=\; &\Phi_t + \Phi_v\,v\,(r + \pi^{S\star}\,\lambda^\star\,\sigma^\star + (\pi^{P\star} + \beta\,\pi^{S\star})[\mu^P - r])\\
&+(1/2)\,\Phi_{vv}\,v^2([\pi^{S\star}\,\sigma^\star]^2 + [(\pi^{P\star} + \beta\,\pi^{S\star})\,\sigma^P]^2) - c^\star\,.
\end{aligned}$$

This eliminates the drift (Lebesgue integral) in (1.35) and we obtain

$$\begin{aligned}
&U\left(V_T^{u'\star}\right) - \int_t^T c^\star(s, V_s^{u'\star},\lambda_s^\star)\,\mathrm{d}s = \Phi(t,v)+\\
&\int_t^T \Phi_v(s,V_s^{u'\star})V_s^{u'\star}(\pi_s^{P\star} + \beta\,\pi_s^{S\star})\sigma^P\,\mathrm{d}W_s^P + \int_t^T \Phi_v(s,V_s^{u'\star})V_s^{u'\star}\pi_s^{S\star}\sigma_s^\star\,\mathrm{d}W_s\,.
\end{aligned}$$

For $J(t,v;u'^\star) = \Phi(t,v)$, it remains to prove that the local martingale component disappears in expectation. A sufficient condition is the square-integrability of the local martingale component

$$\mathbb{E}\left[\int_t^T \left(\Phi_v(s,V_s^{u'^\star})V_s^{u'^\star}\right)^2 \left([\pi_s^{P\star} + \beta\,\pi_s^{S\star}]^2(\sigma^P)^2 + [\pi_s^{S\star}\sigma_s^\star]^2\right)\,\mathrm{d}s\right] < \infty\,.$$

Using the explicit form of the candidates in (1.26), for $\gamma > 0$ and $\gamma \neq 1$, and in (1.32), for $\gamma = 1$, gives

$$\begin{aligned}\left(\Phi_v(s,V_s^{u'^\star})V_s^{u'^\star}\right)^2 &\left([\pi_s^{P\star} + \beta\,\pi_s^{S\star}]^2(\sigma^P)^2 + [\pi_s^{S\star}\sigma_s^\star]^2\right)\\ &= \frac{\left(V_s^{u'^\star}\right)^{2(1-\gamma)} f(s)^2}{\gamma^2}\left[\frac{(\mu^P - r)^2}{(\sigma^P)^2} + \left(\frac{1}{\kappa\gamma}f(s)\right)^{\frac{2}{\alpha-2}}\right],\end{aligned}$$

where we set $f = 1$, for $\gamma = 1$. The RHS is $\left(V_s^{u'^\star}\right)^{2(1-\gamma)}$ times a deterministic and continuous function on the compact set $[t,T]$. The deterministic part is uniformly bounded. Therefore, it is sufficient to focus on the stochastic component: $V^{u'^\star}$ satisfies

$$\mathrm{d}V_s^{u'^\star} = V_s^{u'^\star}\left[r\,\mathrm{d}s + \frac{\lambda_P^2}{\gamma}\,\mathrm{d}s + \frac{(\lambda^\star(s,V_s^{u'^\star}))^2}{\gamma}\,\mathrm{d}s + \frac{\lambda_P}{\gamma}\,\mathrm{d}W_s^P + \frac{\lambda^\star(s,V_s^{u'^\star})}{\gamma}\,\mathrm{d}W_s\right].$$

Recalling that $\lambda^\star(s,v)$ is a continuous function in s and does not depend on v, we see that $V_s^{u'^\star}$ follows a log-normal distribution with parameters being uniformly bounded, for all $s \in [t,T]$. Since all moments of a log-normally distributed random variable exist, it follows that the local martingale is a square-integrable martingale. This establishes $J(t,v;u'^\star) = \Phi(t,v)$. Finally, $u'^\star \in A'_\gamma(t,v)$ follows from the fact that $\pi^{P\star}$, $\pi^{S\star}\sigma^\star$, and $\lambda^\star$ are uniformly bounded on $[t,T]$, each $\gamma > 0$.

Part (b): Now we show the optimality, i.e. $J(t,v;u') \leq \Phi(t,v)$, for $u' \in A'_\gamma(t,v)$. As in (a), this is also based on the analysis of (1.35). The HJB (1.21) is applied to show that the drift component is bounded from above by zero. Then it is shown that the conditions in Def. 1.2.1 are sufficient for the local martingale component on the RHS of (1.35) to vanish in expectation.

By the HJB (1.21), Φ with arbitrary $u' = (\pi^P,\pi^S,\lambda) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}_0^+$ satisfies

$$\begin{aligned} 0 \;\geq\; & \Phi_t + \Phi_v\, v\,(r + \pi^S\,\lambda\,\sigma^\star + (\pi^P + \beta\,\pi^S)[\mu^P - r]) \\ & +(1/2)\,\Phi_{vv}\,v^2([\pi^S\,\sigma^\star]^2 + [(\pi^P + \beta\,\pi^S)\,\sigma^P]^2) - c^\star\,, \end{aligned}$$

for $(s,v) \in [t,T] \times \mathbb{R}^+$. This provides the point-wise upper bound zero for the drift in (1.35) and we obtain

$$\begin{aligned} & U\left(V_T^{u'}\right) - \int_t^T c^\star(s, V_s^{u'},\lambda_s)\,\mathrm{d}s \leq \Phi(t,v) + \\ & \underbrace{\int_t^T \Phi_v(s,V_s^{u'})V_s^{u'}(\pi_s^P + \beta\,\pi_s^S)\sigma^P\,\mathrm{d}W_s^P + \int_t^T \Phi_v(s,V_s^{u'})V_s^{u'}\pi_s^S\sigma^\star(s,V_s^{u'},\lambda_s)\,\mathrm{d}W_s}_{=:M_T^t}\;. \end{aligned} \tag{1.36}$$

We discuss two separate cases: (b1): $0 < \gamma < 1$ and $\gamma > 1$, and (b2): $\gamma = 1$.

Part (b1): $0 < \gamma < 1$ and $\gamma > 1$. Recall $\Phi_v(t,v) = f(t)\,v^{-\gamma}$ and calculate the quadratic variation of M^t

$$\begin{aligned} \langle M^t\rangle_T &= \int_t^T (V_s^{u'})^{2(1-\gamma)} f^2(s)\left([\pi_s^P + \beta\,\pi_s^S]^2(\sigma^P)^2 + [\sigma_s^\star\pi_s^S]^2\right)\,\mathrm{d}s \\ &\leq \frac{\epsilon^{\frac{\epsilon}{1+\epsilon}}}{1+\epsilon}\sup_{0\leq s\leq T} f(s)^2\Big[\int_t^T \left(V_s^{u'}\right)^{2\,(1-\gamma)\,\left(1+\frac{1}{\epsilon}\right)}\,\mathrm{d}s \\ &\qquad + \int_t^T \left([\pi_s^P + \beta\,\pi_s^S]^2(\sigma^P)^2 + [\sigma_s^\star\pi_s^S]^2\right)^{1+\epsilon}\,\mathrm{d}s\Big],\ \epsilon > 0\;, \end{aligned} \tag{1.37}$$

where the upper bound in the second line was achieved using inequality (A.1.2) given in Lemma 0.1.1 setting

$$x := (V_s^{u'})^{2(1-\gamma)} \text{ and } y := \left([\pi_s^P + \beta\,\pi_s^S]^2(\sigma^P)^2 + [\sigma_s^\star\pi_s^S]^2\right)\;.$$

We show that M^t is a martingale by deriving the integrability of the quadratic variation $\langle M^t\rangle_T$. First we use once more that f is a continuous function on the compact set $[0,T]$ and is uniformly bounded, and thus $\sup_{0\leq s\leq T} f(s)^2$ is finite. We are left to deal with the two expressions in the brackets of (1.37). The second expression is bounded in expectation by assumption, see (1.17) in Def. 1.2.1, setting $\epsilon = \frac{1}{2}\tilde{\epsilon}$. In what follows we establish that the first expression is finite by showing that

$$\mathbb{E}^{t,v}[(V_s^{\pi})^{\xi}] < \infty \quad \text{uniformly} , \tag{1.38}$$

with $\xi = 2\,(1-\gamma)\,\left(1+\frac{1}{\epsilon}\right)$, where $\xi > 0$ for $0 < \gamma < 1$ and $\xi < 0$ for $\gamma > 1$. Note further that $|\xi| < \infty$, since $\epsilon > 0$.

The solution of the wealth equation (1.10) starting at t with initial wealth v applying variation of constants is

$$V_s^{u'} = v\, e^{r(s-t)+\int_t^s (\pi_{\tilde{s}}^P+\beta\,\pi_{\tilde{s}}^S)\lambda^P\sigma^P+\pi_{\tilde{s}}^S\lambda_{\tilde{s}}\sigma_{\tilde{s}}^{\star}\,\mathrm{d}\tilde{s}}\; e^{L_s^t-\frac{1}{2}\left\langle L^t\right\rangle_s} ,$$

where $L_s^t = \int_t^s (\pi_{\tilde{s}}^P + \beta\,\pi_{\tilde{s}}^S)\sigma^P \mathrm{d}W_{\tilde{s}}^P + \int_t^s \pi_{\tilde{s}}^S\sigma_{\tilde{s}}^{\star}\mathrm{d}W_{\tilde{s}}$ and $\left\langle L^t\right\rangle_s = \int_t^s (\pi_{\tilde{s}}^P + \beta\,\pi_{\tilde{s}}^S)^2(\sigma^P)^2 + (\pi_{\tilde{s}}^S\sigma_{\tilde{s}}^{\star})^2\mathrm{d}\tilde{s}$. Then

$$(V_s^{u'})^{\xi} = v^{\xi}\; \underbrace{e^{\xi\,[r(s-t)+\int_t^s (\pi_{\tilde{s}}^P+\beta\,\pi_{\tilde{s}}^S)\lambda^P\sigma^P+\pi_{\tilde{s}}^S\lambda_{\tilde{s}}\sigma_{\tilde{s}}^{\star}\,\mathrm{d}\tilde{s}]}}_{=:R_s^t} \times \underbrace{e^{\xi\,L_s^t-\frac{1}{2}\,\xi\,\left\langle L^t\right\rangle_s}}_{=:Z_s^t}$$

Thus, condition (1.38) is for example fulfilled when

$$\mathbb{E}^{t,v}[(R_s^t)^2] < \infty \quad \text{and} \quad \mathbb{E}^{t,v}[(Z_s^t)^2] < \infty\,.$$

The square of R^t is given by

$$(R_s^t)^2 = e^{2\,\xi[r(s-t)+\int_t^s (\pi_{\tilde{s}}^P+\beta\,\pi_{\tilde{s}}^S)\lambda^P\sigma^P+\pi_{\tilde{s}}^S\lambda_{\tilde{s}}\sigma_{\tilde{s}}^{\star}\,\mathrm{d}\tilde{s}]} ,$$

which is uniformly bounded by a constant, see Def. 1.2.1, (1.17) and (1.18), and recalling that $\xi > 0$ for $0 < \gamma < 1$ and $\xi < 0$ for $\gamma > 1$ as well as $|\xi| < \infty$, since $\epsilon > 0$. This directly implies the square integrability of R^t.
The square of Z^t is given by

$$\begin{aligned}
(Z_s^t)^2 &= e^{2\,\xi\,L_s^t-\frac{1}{2}\,2\,\xi\,\left\langle L^t\right\rangle_s} \\
&= e^{2\,\xi\,L_s^t-(2\,\xi)^2\,\left\langle L^t\right\rangle_s} \times e^{(2\,\xi)^2\,\left\langle L^t\right\rangle_s-\frac{1}{2}\,2\,\xi\,\left\langle L^t\right\rangle_s} \\
&= e^{2\,\xi\,L_s^t-4\,\xi^2\,\left\langle L^t\right\rangle_s} \times e^{\xi\,(4\xi-1)\,\left\langle L^t\right\rangle_s} \\
&\leq \frac{1}{2}\left[\underbrace{e^{4\,\xi\,L_s^t-\frac{1}{2}\,16\,\xi^2\,\left\langle L^t\right\rangle_s}}_{=:\tilde{Z}_s^t}+\underbrace{e^{2\,\xi\,(4\xi-1)\,\left\langle L^t\right\rangle_s}}_{=:\tilde{R}_s^t}\right] ,
\end{aligned}$$

where the last line is again a straight forward upper bound.

The second factor $\tilde{R}^t$ is uniformly bounded by a constant by condition (1.17) of Def. 1.2.1. To finally obtain the square integrability of Z^t, it remains to prove that the first factor $\tilde{Z}^t_s = e^{4\,\xi\,L^t_s - \frac{1}{2}\,16\,\xi^2\,\langle L^t \rangle_s}$, $t \leq s \leq T$, is integrable. However, $\tilde{Z}^t$ is a strictly positive local martingale since it is the stochastic exponential of the local martingale $4\,\xi\,L^t$. The Novikov condition holds by (1.17), i.e. $\mathbb{E}^{t,v}(e^{\frac{1}{2}\,16\,\xi^2\,\langle L^t \rangle_T}) < \infty$, and hence $\tilde{Z}^t$ is a true martingale and $\mathbb{E}^{t,v}(\tilde{Z}^t_s) = 1$, $t \leq s \leq T$. In summary, the local martingale M^t is therefore a martingale vanishing in expectation in (1.36), and taking the conditional expectation of (1.36) gives the desired result

$$J(t,v;u') = \mathbb{E}^{t,v}\left[U(V_T^{u'}) - \int_t^T c^\star(s,V_s^{u'},\lambda_s)\,\mathrm{d}s\right] \leq \Phi(t,v)\,,\, u' \in A'_\gamma(t,v).$$

Part (b2): $\gamma = 1$. From $\Phi_v(t,v) = v^{-1}$ we obtain

$$M_T^t = \int_t^T (\pi_s^P + \beta\,\pi_s^S)\,\sigma^P\,\mathrm{d}W_s^P + \int_t^T \pi_s^S\,\sigma^\star(s,V_s^{u'},\lambda_s)\,\mathrm{d}W_s\,.$$

Def. 1.2.1 (ii) ensures the square-integrability. The local martingale M^t is therefore a martingale vanishing in expectation in (1.36), and $J(t,v;u') \leq \Phi(t,v)$, for $u' \in A'_1(t,v)$. □

1.3 Discussion and Implications of Results

Theorems 1.2.1, 1.2.2 and 1.2.3 indicate our unconstrained executive's maximized utility and associated optimal behavior in terms of personal portfolio selection and choice of work effort, subject to the constant relative risk aversion set-up. We now investigate the sensitivity of this optimal behavior to variation of the executive's risk aversion and work effectiveness characteristics. Additionally, we derive the fair compensation for the executive's work effort using a utility indifference approach (following the vein of, for example, Lambert, Larcker and Verrecchia (1991)).

The executive is characterized by the relative risk aversion coefficient ($\gamma > 0$) and the two work effectiveness parameters work productivity ($1/\kappa$, with $\kappa > 0$), and disutility stress ($\alpha > 2$). To produce results that have relativity to a base-level of work effort, as indicated by a base-level non-systematic Sharpe ratio control decision $\lambda_0 > 0$, the disutility $c^\star$ given by (1.16) is reparameterized by choosing

$$\tilde{\kappa} := \kappa \, (\lambda_0)^\alpha \,, \tag{1.39}$$

such that the CRRA utility/disutility set-up becomes

$$U(v) = \begin{cases} \dfrac{v^{1-\gamma}}{1-\gamma} \,, & \text{for } \gamma > 0 \text{ and } \gamma \neq 1 \\ \log(v) \,, & \text{for } \gamma = 1 \end{cases}$$

and

$$c^\star(t,v,\lambda) = \frac{\tilde{\kappa}}{\alpha} \, v^{1-\gamma} \left(\frac{\lambda}{\lambda_0} \right)^\alpha \,, \quad \text{for } \lambda \geq 0 \,, \quad \gamma > 0 \,.$$

In regard to the executive's optimal personal investment decisions $\pi^\star$, the optimal own-company stockholding $\pi^{S^\star}$ is a function of the optimal work effort choice, and the associated optimal volatility $\sigma^\star$ (see Lemma 1.1.1) which we do not explicitly specify. The optimal market portfolio allocation $\pi^{P^\star}$ considered in conjunction with the systematic risk exposure associated with $\pi^{S^\star}$ coincides with the results from classical utility maximization in the constant relative risk aversion setting, and is therefore of limited interest.

We now turn to the relationship between the executive's optimal work effort/control choice $\lambda^\star$, his characteristics $1/\tilde{\kappa}$ and α, and his utility indifference compensation, for log-utility and power-utility cases. It is worth reiterating that $\tilde{\kappa}$ ($1/\tilde{\kappa}$) directly (inversely) relates the executive's work effort disutility to the quality of his control decision as indicated by the non-systematic

Sharpe ratio λ, and α indicates how rapidly his work effort disutility will rise for the sake of an improved λ.

1.3.1 The Log-Utility Case

With assumption of log-utility ($\gamma = 1$), the executive's optimal choice of work effort for the new disutility parameterization is

$$\lambda^\star = \lambda_0^{\frac{\alpha}{\alpha-2}} \, (1/\tilde{\kappa})^{\frac{1}{\alpha-2}}$$

(see Theorem 1.2.2 for the optimal choice under the original parameterization). We assume work productivity satisfies

$$1/\tilde{\kappa} > \lambda_0^{-2}$$

to ensure optimal work effort is not less than the base-level, i.e. $\lambda^\star \geq \lambda_0 > 0$. Consequently, for $\lambda^\star = \lambda^\star(1/\tilde{\kappa},\alpha)$, the optimal work effort sensitivities to the work effectiveness parameters are

$$\frac{\partial \lambda^\star}{\partial(1/\tilde{\kappa})} = \frac{\tilde{\kappa}}{\alpha - 2}\, \lambda^\star > 0\,, \; \frac{\partial \lambda^\star}{\partial \alpha} = -\frac{\ln\left(\frac{1/\tilde{\kappa}}{\lambda_0^{-2}}\right)}{(\alpha-2)^2}\, \lambda^\star < 0, \text{ for } \alpha > 2 \text{ and } 1/\tilde{\kappa} > \lambda_0^{-2}.$$

That is, the executive's optimal work effort choice is positively related to his work productivity ($\partial\lambda^\star/\partial(1/\tilde{\kappa}) > 0$), and negatively related to his disutility stress ($\partial\lambda^\star/\partial\alpha < 0$). This result is illustrated by Figure 1.1, which graphs optimal work effort versus work productivity and disutility stress, with $\lambda_0 = 0.10$. Furthermore Figure 1.1 indicates that, for moderate and large values of disutility stress α, optimal work effort is mainly driven by work productivity $1/\tilde{\kappa}$; and optimal work effort is most sensitive to low values of work productivity close to the boundary value ($1/\tilde{\kappa} \gtrapprox \lambda_0^{-2} = 100$).

The limiting cases for work productivity are

$$\lim_{1/\tilde{\kappa} \searrow \lambda_0^{-2}} \lambda^\star(1/\tilde{\kappa}, \alpha) = \lambda_0 \quad \text{and} \quad \lim_{1/\tilde{\kappa} \nearrow \infty} \lambda^\star(1/\tilde{\kappa}, \alpha) = +\infty\,, \quad \text{for all } \alpha > 2\,,$$

indicating that the limit for deteriorating work productivity is base-level work effort λ_0, and ever increasing work productivity yields ever increasing work effort (to infinity).

Taking disutility stress to its limiting cases gives

$$\lim_{\alpha\searrow 2} \lambda^\star(1/\tilde{\kappa},\alpha) = +\infty \quad \text{and} \quad \lim_{\alpha\nearrow\infty} \lambda^\star(1/\tilde{\kappa},\alpha) = \lambda_0\,, \quad \text{for all } 1/\tilde{\kappa} > \lambda_0^{-2}\,,$$

indicating that the executive will deliver ever increasing work effort as disutility stress diminishes, and the totally stressed executive will deliver base-level work effort.

The value function specifying the executive's maximized utility can be written as the difference between the utility from his optimal personal investment decision and the disutility from his optimal work effort (see Theorem 1.2.2 for the value function under the original disutility parameterization):

$$\Phi(0{,}v) = \underbrace{\log(v) + \left[r + \frac{1}{2}\,(\lambda^P)^2 + \frac{1}{2}\,(\lambda^\star)^2\right] T}_{=\mathbb{E}^{0,v}[U(V_T^{u'\star})]} - \underbrace{\frac{1}{\alpha}\,(\lambda^\star)^2\,T}_{=\mathbb{E}^{0,v}\int_0^T c^\star(t,V_t^{u'\star},\lambda^\star(t,V_t^{u'\star}))\,\mathrm{d}t}\,. \tag{1.40}$$

We assume that the executive's fair compensation for the disutility of work effort is paid up-front with cash or marketable (unconstrained) securities of value Δv. Applying a utility indifference argument, the fair level of compensation satisfies

$$\Phi(0, v + \Delta v) = \Phi(0{,}v) + \mathbb{E}^{0,v}\left[\int_0^T c^\star(t, V_t^{u'\star}, \lambda^\star(t{,}V_t^{u'\star}))\,\mathrm{d}t\right], \tag{1.41}$$

which gives the following expression for the fair compensation rate:

Proposition 1.3.1
The fair compensation rate of the log-utility executive applying the indifference utility argument (1.41) *is given by*

$$\Delta v = v\left(e^{\frac{(\lambda^\star)^2\,T}{\alpha}} - 1\right) = v\left(e^{\frac{\lambda_0^2\,T}{\alpha}\left(\frac{1/\tilde{\kappa}}{\lambda_0^{-2}}\right)^{\frac{2}{\alpha-2}}} - 1\right).$$

Proof. Subtituting the representation (1.40) of the value function in (1.41) we see that the fair compensation rate satisfies

$$\log(v + \Delta\, v) = \log(v) + \mathbb{E}^{0,v}\left[\int_0^T c^\star(t, V_t^{u'^\star}, \lambda^\star(t, V_t^{u'^\star}))\,\mathrm{d}t\right],$$

which is equivalent to

$$\Delta v = v\left(e^{\mathbb{E}^{0,v}\left[\int_0^T c^\star(t,V_t^{u'^\star},\lambda^\star(t,V_t^{u'^\star}))\,\mathrm{d}t\right]} - 1\right).$$

Thus we are left with the calculation of the conditional expectation of the integrated minimized disutility rate:

$$\begin{aligned}\mathbb{E}^{0,v}\left[\int_0^T c^\star(t, V_t^{u'^\star}, \lambda^\star(t, V_t^{u'^\star}))\,\mathrm{d}t\right] &= \mathbb{E}^{0,v}\left[\int_0^T \frac{\kappa}{\alpha}\left(\kappa^{-\frac{1}{\alpha-2}}\right)^{\alpha}\mathrm{d}t\right]\\ &= \frac{\kappa^{-\frac{2}{\alpha-2}}}{\alpha}\,T = \frac{\lambda_0^{\frac{2\alpha}{\alpha-2}}\,\tilde{\kappa}^{-\frac{2}{\alpha-2}}}{\alpha}\,T,\end{aligned}$$

where we note that $(\lambda^*)^2 = \lambda_0^{\frac{2\alpha}{\alpha-2}}\,\tilde{\kappa}^{-\frac{2}{\alpha-2}}$. Plugging in this result finishes the proof. □

For $\Delta v = \Delta v(1/\tilde{\kappa},\alpha)$, the utility indifference compensation sensitivities to the work effectiveness parameters are

$$\frac{\partial \Delta v}{\partial(1/\tilde{\kappa})} = \frac{2\tilde{\kappa}}{\alpha-2}\,\frac{(\lambda^\star)^2 T}{\alpha}\,(\Delta v + v) \quad > \quad 0\,, \quad \text{for } \alpha > 2 \text{ and } 1/\tilde{\kappa} > \lambda_0^{-2}\,,$$

and

$$\frac{\partial \Delta v}{\partial \alpha} = -\left(\frac{1}{\alpha} + \frac{2\,\ln\left(\frac{1/\tilde{\kappa}}{\lambda_0^{-2}}\right)}{(\alpha-2)^2}\right)\frac{(\lambda^\star)^2 T}{\alpha}\,(\Delta v{+}v) \;<\; 0\,, \text{ for } \alpha > 2 \text{ and } 1/\tilde{\kappa} > \lambda_0^{-2},$$

indicating the sensible result that the executive's utility indifference compensation increases with work productivity and decreases with disutility stress. This result is illustrated by Figure 1.2, which graphs the executive's fair upfront compensation, based on the utility indifference rationale, versus work productivity and disutility stress, for the case where the executive's initial wealth is $v = \$5$ million, time horizon is $T = 10$ years, and base-level work effort is $\lambda_0 = 0.10$.

The limiting cases for work productivity are

$$\lim_{1/\tilde{\kappa}\searrow\lambda_0^{-2}} \Delta v(1/\tilde{\kappa},\alpha) = v\left(e^{\frac{\lambda_0^2 T}{\alpha}} - 1\right), \quad \text{and} \quad \lim_{1/\tilde{\kappa}\nearrow\infty} \Delta v(1/\tilde{\kappa},\alpha) = +\infty\,,$$

for all $\alpha > 2$, and the limiting cases for disutility stress are

$$\lim_{\alpha\searrow 2} \Delta v(1/\tilde{\kappa},\alpha) = +\infty\,, \quad \text{and} \quad \lim_{\alpha\nearrow\infty} \Delta v(1/\tilde{\kappa},\alpha) = 0\,, \quad \text{for all } 1/\tilde{\kappa} > \lambda_0^{-2}\,.$$

That is, with ever improving work effectiveness ($1/\tilde{\kappa} \nearrow \infty$ or $\alpha \searrow 2$), the executive's fair compensation is ever increasing (to infinity). And with ever diminishing work effectiveness ($1/\tilde{\kappa} \searrow \lambda_0^{-2}$ or $\alpha \nearrow \infty$), the executive's work effort decreases towards base-level (λ_0), for which the commensurate fair compensation is $v(e^{\lambda_0^2 T/\alpha} - 1)$; however, for the case where the executive becomes totally stressed ($\alpha \nearrow \infty$), the fair compensation limit is zero.

1.3.2 The Power-Utility Case

Now with assumption of power-utility, the executive's optimal choice of work effort is

$$\lambda^\star(t) = \lambda_0^{\frac{\alpha}{\alpha-2}} \left(\frac{1}{\tilde{\kappa}\,\gamma}\right)^{\frac{1}{\alpha-2}} f(t)^{\frac{1}{\alpha-2}}\,,$$

with

$$\begin{aligned} f(t) = \;& e^{(1-\gamma)\left(r+\frac{1}{2}\frac{\lambda_P^2}{\gamma}\right)(T-t)} \\ & \times \left(1 - \frac{(\alpha-2)\left(\frac{\lambda_0^\alpha}{\tilde{\kappa}\,\gamma}\right)^{\frac{2}{\alpha-2}}}{\alpha\,(2\,\gamma\,r + \lambda_P^2)} \left(e^{\frac{1-\gamma}{\alpha-2}\left(2\,r+\frac{\lambda_P^2}{\gamma}\right)(T-t)} - 1\right)\right)^{-\frac{\alpha-2}{2}}, \end{aligned}$$

(see Theorem 1.2.1 for the optimal choice under the original disutility parameterization). To ensure optimal work effort is not less than the base-level, we assume for the risk-free rate of return (in f)

$$r > -\lambda_P^2/(2\gamma)$$

(recalling that λ_P is the Sharpe ratio of the market portfolio), and for work productivity

$$1/\tilde{\kappa} > \begin{cases} \gamma\,\lambda_0^{-2}\,, & \text{for } \ 0 < \gamma < 1\,, \\ \gamma\,\lambda_0^{-2}\, f(0)^{-1}\,, & \text{for } \gamma > 1\,. \end{cases}$$

These conditions follow from the properties of the function f given by the equation above, where we keep in mind that $(1-\gamma) > 0$ for $0 < \gamma < 1$, $(1-\gamma) < 0$ for $\gamma > 1$ and that $f(T) = 1$ and from the fact that f is the solution of a Bernoulli ODE, which is decreasing in time for $0 < \gamma < 1$ and increasing in time for $\gamma > 1$, when we additionally fulfill that $r > -\lambda_P^2/(2\gamma)$ (therefore see equation (1.31) of Theorem 1.2.1). Also note that $f(0)$ reads

$$\begin{aligned} f(0) = & e^{(1-\gamma)\left(r+\frac{1}{2}\frac{\lambda_P^2}{\gamma}\right)T} \\ & \times \left(1 - \frac{(\alpha-2)\left(\frac{\lambda_0^\alpha}{\tilde{\kappa}\,\gamma}\right)^{\frac{2}{\alpha-2}}}{\alpha\,(2\,\gamma\,r + \lambda_P^2)}\left(e^{\frac{1-\gamma}{\alpha-2}\left(2\,r+\frac{\lambda_P^2}{\gamma}\right)T} - 1\right)\right)^{-\frac{\alpha-2}{2}}\,. \end{aligned}$$

As in the log-utility case we calculate the fair compensation rate of the executive:

Proposition 1.3.2
Applying the indifference utility argument (1.41), *the power-utility executive's utility indifference (fair) up-front compensation* Δv *is*

$$\begin{aligned} \Delta v = v \Bigg(& e^{\frac{1}{2\gamma}\int_0^T \lambda^\star(t)^2\,\mathrm{d}t} \\ & \times \left[1 - \frac{(\alpha-2)\left(\frac{\lambda_0^\alpha}{\tilde{\kappa}\,\gamma}\right)^{\frac{2}{\alpha-2}}}{\alpha\,(2\,\gamma\,r + \lambda_P^2)}\left(e^{\frac{1-\gamma}{\alpha-2}\left(2\,r+\frac{\lambda_P^2}{\gamma}\right)T} - 1\right)\right]^{\frac{(\alpha-2)}{2(1-\gamma)}} - 1\Bigg)\,. \end{aligned}$$

Proof. The solution presented for Δv is derived using the structural properties of the executive's optimal control vector $u'^{\star} = (\pi^{P^\star}, \pi^{S^\star}, \lambda^\star)$. An outside investor with knowledge of the work effort exercised by the executive (i.e. with knowledge of $\lambda^\star$) and his optimal investment strategies $\pi^{P^\star}$ and $\pi^{S^\star}$, will choose a control vector $\widehat{u'}^{\star} = (\widehat{\pi}^{P^\star}, \widehat{\pi}^{S^\star}, \widehat{\lambda}^\star)$ identical to the executive's control vector $u'^{\star}$. Denote $\widehat{\Phi}(0,v)$ to be the maximized utility of the outside investor, then it follows that

$$\widehat{\Phi}(0,v) = \Phi(0,v) + \mathbb{E}^{0,v}\left[\int_0^T c^\star(t, V_t^{u'^\star}, \lambda^\star(t,V_t^{u'^\star}))\,\mathrm{d}t\right].$$

Applying the utility indifference principle (1.41) we can then solve

$$\widehat{\Phi}(0,v) = \Phi(0,v+\Delta v) \tag{1.42}$$

to obtain Δv. So first, we have to calculate the value function $\widehat{\Phi}(0,v)$ of the outside investor. An outside investor with knowledge of the optimal control vector $u'^{\star}$ does not suffer from disutility and is characterized by the following Hamilton-Jacobi-Bellmann equation:

$$0 = \quad \widehat{\Phi}_t(t,v) + \widehat{\Phi}_v(t,v)\, v\, r - \frac{1}{2}(\lambda^\star(t))^2 \frac{\widehat{\Phi}_v^2(t,v)}{\widehat{\Phi}_{vv}(t,v)} - \frac{1}{2}(\lambda^P)^2 \frac{\widehat{\Phi}_v^2(t,v)}{\widehat{\Phi}_{vv}(t,v)},$$

where we have set $\lambda^\star(t,v) = \lambda^\star(t)$, since we already know from (1.26) that the optimal work effort does not depend on v.
Applying the ansatz $\widehat{\Phi}(t,v) = \widehat{f}(t)\,\frac{v^{1-\gamma}}{1-\gamma}$ with $\widehat{f}(T) = 1$ results in the ODE

$$\dot{\widehat{f}} = -(1-\gamma)\left[r + \frac{1}{2\gamma}(\lambda^\star(t))^2 + \frac{1}{2\gamma}\lambda_P^2\right]\widehat{f} \quad , \quad \widehat{f}(T) = 1\,,$$

which has the solution

$$\widehat{f}(t) = e^{(1-\gamma)\left[r(T-t) + \frac{\lambda_P^2}{2\gamma}(T-t) + \frac{1}{2\gamma}\int_t^T(\lambda^\star(s))^2\mathrm{d}s\right]}.$$

From (1.42) we then get that

$$\frac{(v)^{1-\gamma}}{1-\gamma}\,\widehat{f}(0) = \frac{(v+\Delta v)^{1-\gamma}}{1-\gamma}\,f(0) \quad \Leftrightarrow \quad \Delta v = v\left(\left[\frac{\widehat{f}(0)}{f(0)}\right]^{\frac{1}{1-\gamma}} - 1\right).$$

Plugging in the representations of $\widehat{f}$ and f, respectively, and simplifying gives the result. □

In contrast to the log-utility case, the sensitivities of the executive's optimal work effort $\lambda^\star$ and fair compensation Δv with respect to variations in his work effectiveness parameters cannot be shown with compact expressions. Instead we limit ourselves to graphical representations of the relationships, with additional consideration of the executive's risk aversion parameter γ.

Figure 1.3 displays optimal work effort over time for varying risk aversion (i.e. $\lambda^\star$ versus t and γ, for fixed values of $1/\tilde{\kappa}$, α and λ_0). The executive's disutility from work effort depends on his wealth v and risk aversion γ via the scaling factor $v^{1-\gamma}$, which is effectively a work aversion measure. For a given level of wealth, an executive with low risk aversion ($0 < \gamma < 1$) has higher work aversion than an executive with high risk aversion ($\gamma > 1$); furthermore, with increasing wealth, work aversion increases for a low risk aversion executive but decreases for a high risk aversion executive. If we suggest that a high risk aversion executive has high work ethic and a low risk aversion executive has low work ethic, our set-up assumes that: a high work ethic executive has comparatively low aversion to work effort and will become further less averse to work effort if past effort or chance brings success as indicated by increased wealth; and a low work ethic executive has comparatively high aversion to work effort and will become further more averse to work effort if his wealth increases. Nevertheless, a low risk aversion (i.e. low work ethic) executive is more willing to take on the risk associated with a larger own-company stockholding, and thus with more personal stake in his own company he always applies more work effort than a high risk aversion executive, ceteris paribus. These aspects are observable in Figure 1.3: a low risk aversion executive with $0 < \gamma < 1$ starts with a (comparatively) high level of work effort, which is expected to reduce over time (given that his wealth is expected to increase over time); whereas a high risk aversion executive with $\gamma > 1$ starts with a far lower level of work effort, which is expected to increase over time. Therefore observing the executive's work effort over time potentially reveals his risk aversion.

Figures 1.4 and 1.5 fix the executive's risk aversion at a relatively low level of $\gamma = 0.5$ and show optimal work effort over time for varying work effectiveness (i.e. respectively $\lambda^\star$ versus t and $1/\tilde{\kappa}$, and $\lambda^\star$ versus t and α). The executive's work effort increases with work effectiveness (but decreases over time given $\gamma = 0.5$). That is, work effort is positively related to work productivity $1/\tilde{\kappa}$, and negatively related to disutility stress α. The implication is that, for a given level of risk aversion, work effort distinguishes the work effectiveness (quality) of the executive.

The relationship between the executive's optimal work effort and his risk aversion and work effectiveness characteristics is reflected in his fair up-front compensation. Figures 1.6 and 1.7 show fair compensation versus pairings of risk aversion with each of work productivity and disutility stress (i.e. respectively Δv versus γ and $1/\tilde{\kappa}$, and Δv versus α and γ). Any combination of decreasing risk aversion, increasing work productivity, and decreasing disutility stress leads to higher work effort and commensurately higher fair compensation. The level of fair compensation is particularly prominently dependent on risk aversion: fair compensation sensitivity to work productivity and disutility stress is highest when risk aversion is low ($\gamma \approx 0.5$ or lower, see Figures 1.6 and 1.7). This result stems from the fact that, regardless of whether the executive has high work effectiveness or not, the company can only substantially benefit from the executive's quality if he has sufficiently low risk aversion to take on a substantial own-company stockholding and thereby have incentive to apply substantial work effort. Note that Figure 1.7 extends only to a minimum value of disutility stress $\alpha = 5$; not shown is that for lower disutility stress $\alpha \approx 4$ and below, fair compensation increases even more steeply.

1.4 Figures

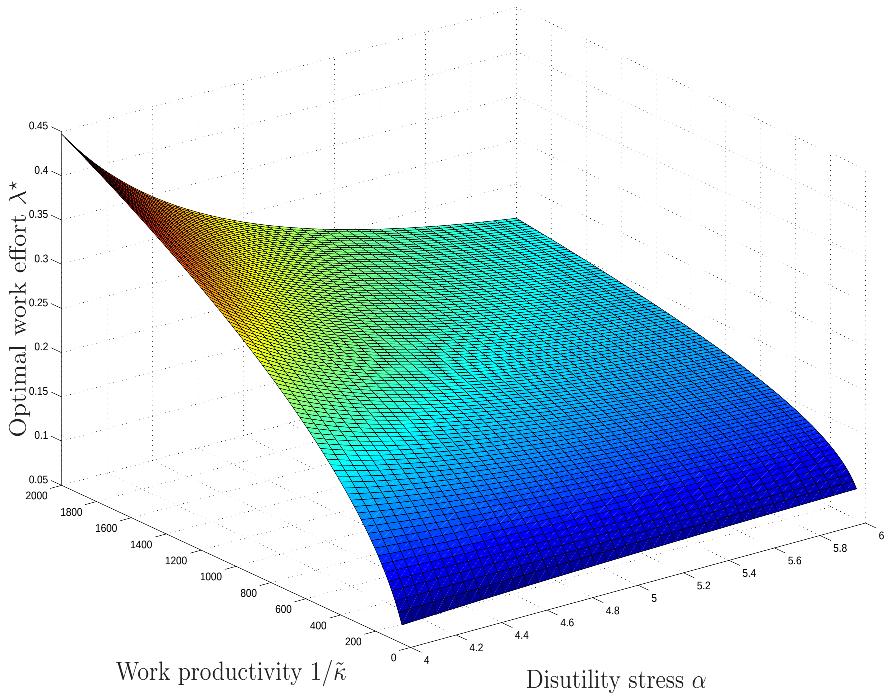

Figure 1.1: The log-utility executive's optimal work effort/control choice, in terms of optimal non-systematic Sharpe ratio $\lambda^\star$, w.r.t. his work effectiveness parameters, work productivity $1/\tilde{\kappa}$ and disutility stress α; given base-level work effort $\lambda_0 = 0.10$.

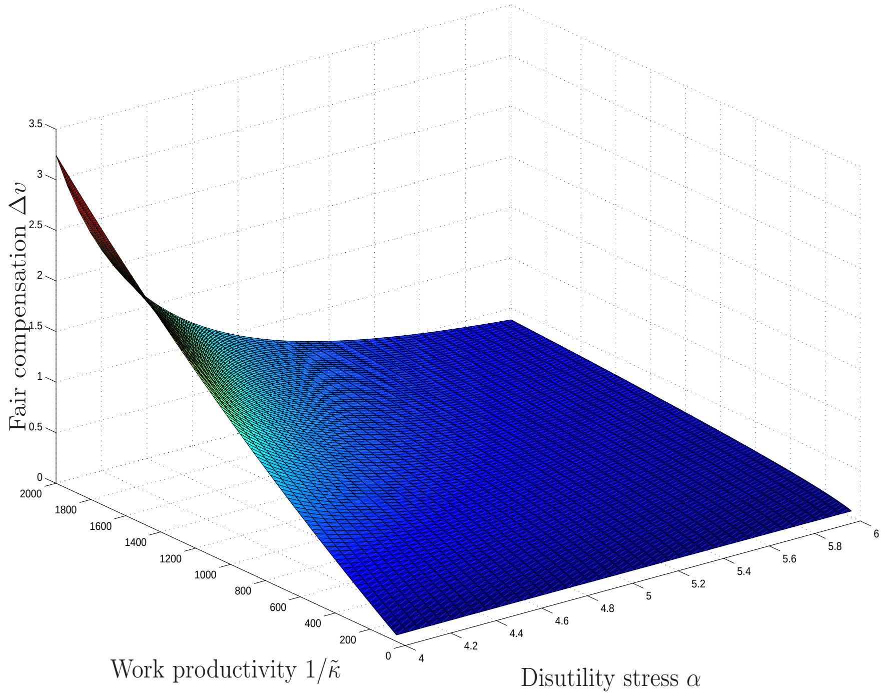

Figure 1.2: The log-utility executive's fair up-front compensation Δv, based on utility indifference, w.r.t. his work effectiveness parameters, work productivity $1/\tilde{\kappa}$ and disutility stress α; given initial wealth $v = \$5$ million, time horizon $T = 10$ years, and base-level work effort $\lambda_0 = 0.10$.

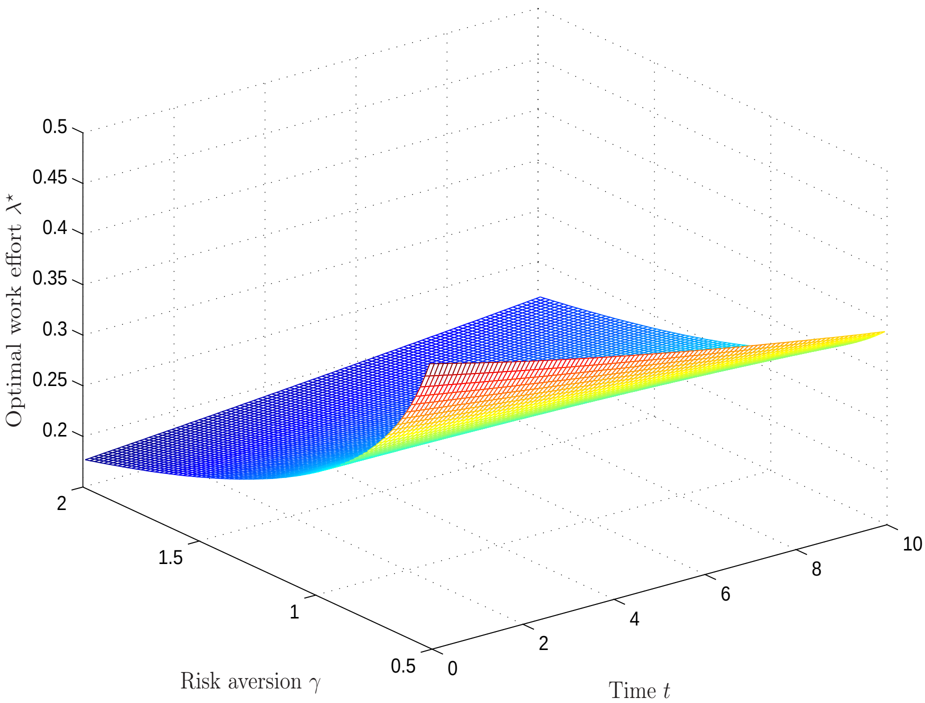

Figure 1.3: The power-utility executive's optimal work effort/control choice, in terms of optimal non-systematic Sharpe ratio $\lambda^\star$, w.r.t. time t, for varying risk-aversion γ; given work productivity $1/\tilde{\kappa} = 2000$, disutility stress $\alpha = 5$, and base-level work effort $\lambda_0 = 0.10$.

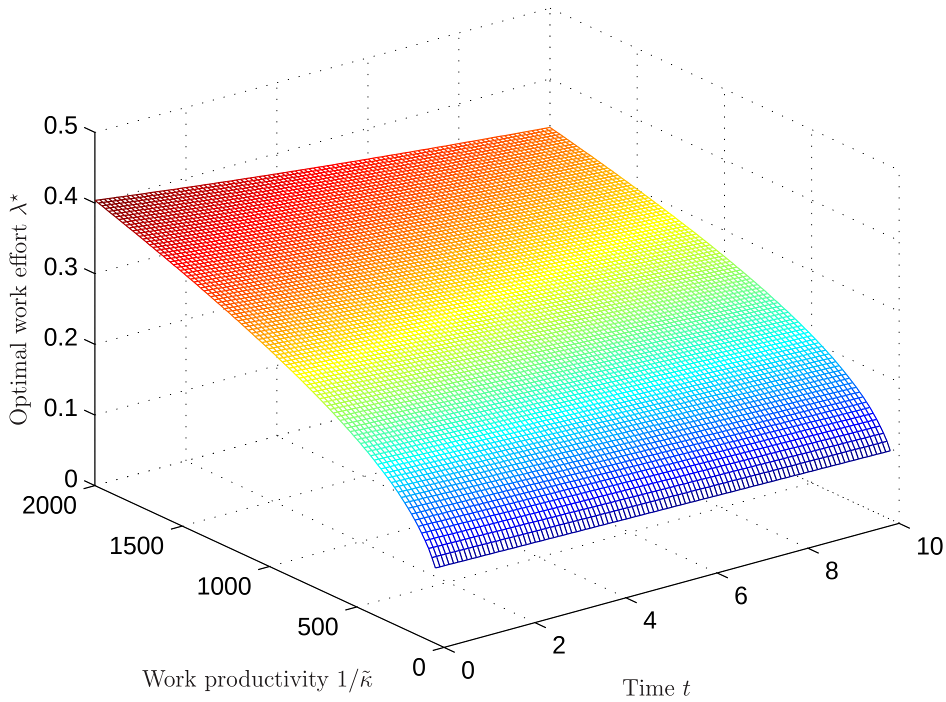

Figure 1.4: The power-utility executive's optimal work effort/control choice, in terms of optimal non-systematic Sharpe ratio $\lambda^\star$, w.r.t. time t, for varying work productivity $1/\tilde{\kappa}$; given risk aversion $\gamma = 0.5$, disutility stress $\alpha = 5$, and base-level work effort $\lambda_0 = 0.10$.

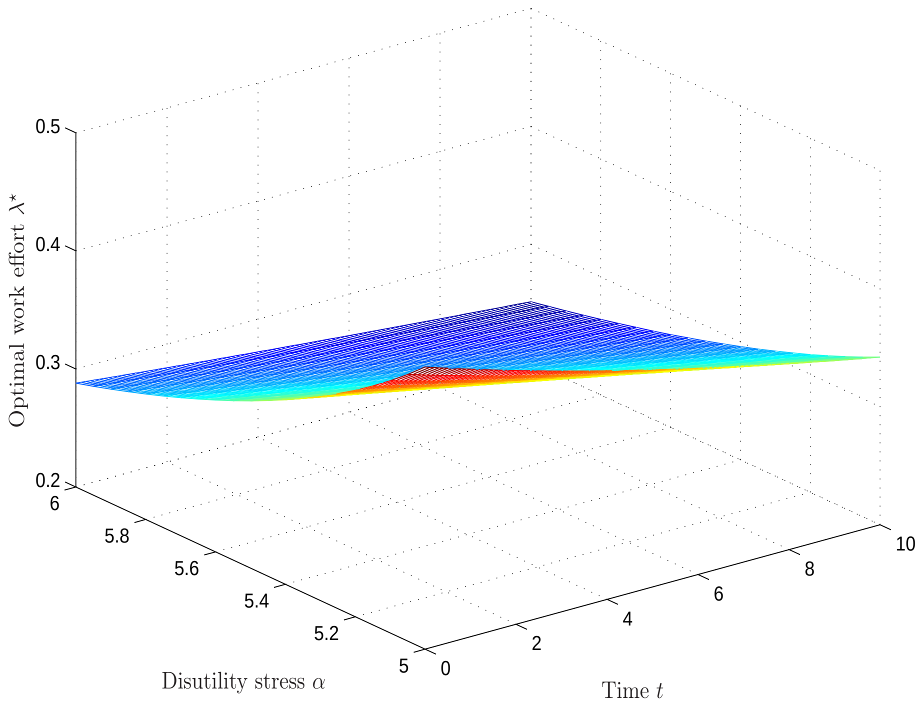

Figure 1.5: The power-utility executive's optimal work effort/control choice, in terms of optimal non-systematic Sharpe ratio $\lambda^{\star}$, w.r.t. time t, for varying disutility stress α; given risk aversion $\gamma = 0.5$, work productivity $1/\tilde{\kappa} = 2000$, and base-level work effort $\lambda_0 = 0.10$.

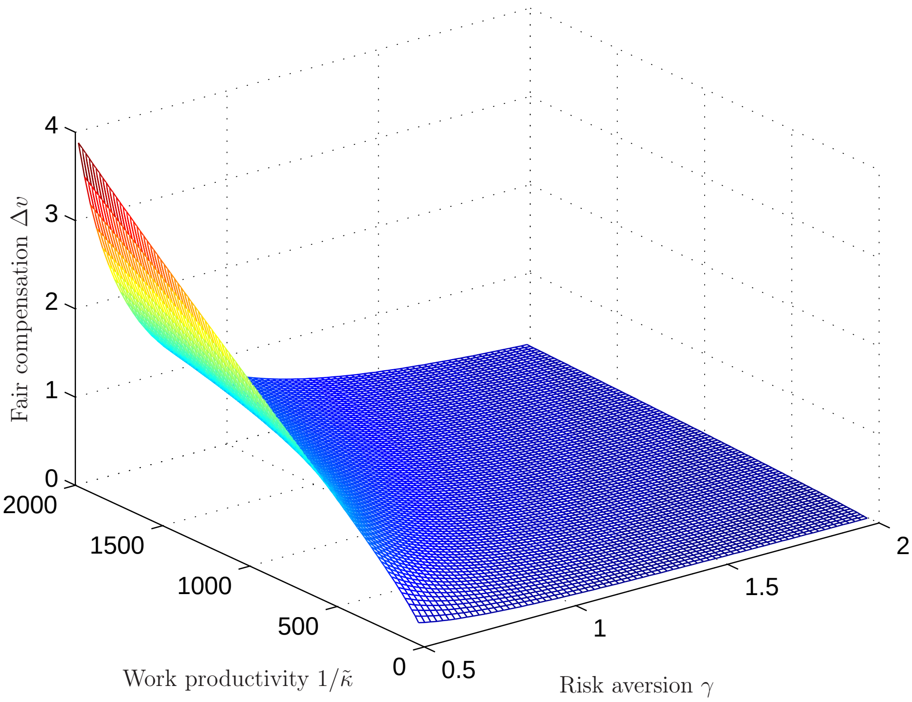

Figure 1.6: The power-utility executive's fair up-front compensation Δv, based on utility indifference, w.r.t. his work productivity $1/\tilde{\kappa}$ and risk aversion γ; given disutility stress $\alpha = 5$, initial wealth $v = \$5$ million, time horizon $T = 10$ years, and base-level work effort $\lambda_0 = 0.10$.

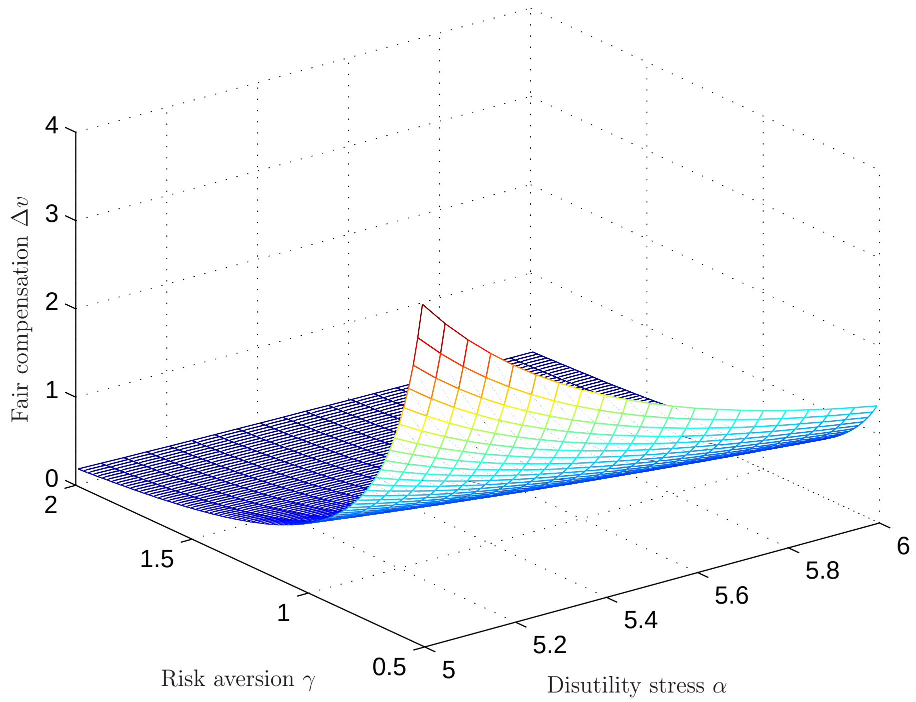

Figure 1.7: The power-utility executive's fair up-front compensation Δv, based on utility indifference, w.r.t. his risk aversion γ and disutility stress α; given work productivity $1/\tilde{\kappa} = 2000$, initial wealth $v = \$5$ million, time horizon $T = 10$ years, and base-level work effort $\lambda_0 = 0.10$.

Chapter 2

Own-Company Stockholding and Work Effort Preferences of an Unconstrained Executive with Constant Absolute Risk Aversion

The goal of this chapter is to solve our optimal investment and work effort decision (1.12) for a constant absolute risk aversion set-up. To do so, we use again the dynamic programmic method. It is widely supported in the literature (see for instance Korn (2007)) that for the case of an exponential utility set-up it is common to optimize the *absolute* value of the wealth invested into assets, i.e. the control of the pure investment problem is given by $\Pi := \pi\, v$, where v denotes the actual wealth and π the proportion invested into the risky asset(s). The reason for that is that when just optimizing the control π, the corresponding optimal strategies $\pi^\star$ are proportional to $1/v$ which may result in optimal strategies which tend to infinity, since the resulting wealth process w.r.t. the optimal strategies follows a generalized Brownian motion with drift. This process can become negative, and therefore in particular attain zero, which can cause optimal strategies that tend to infinity. To avoid these issues for our investment and work effort problem, we first reformulate it w.r.t. absolute investment strategies. So in what follows we will give the set-up along the lines of Chapter 1, then reformulate it, solve it for the case of exponential utility with zero interest rates and discuss the implications of these results.

2.1 The Set-up and its Reformulation

The mathematical set-up is kept the same as in Chapter 1 for the constant relative risk aversion case. The executive has the same underlying probability space, acts on the same financial market and suffers from a disutility rate depending on the control strategy (μ,σ). The only structural difference is that the executive invests now the absolute value $\Pi^P = \pi^P\, v$ into the market portfolio and the absolute value $\Pi^S = \pi^S\, v$ into the company's stock, respectively. We collect the absolute investment strategy $\Pi = (\Pi^P, \Pi^S)$ and the control strategy (μ,σ) now in the control vector process $u_{ab} = (\Pi^P, \Pi^S,\mu,\sigma)$. The executive's wealth process, $V^{u_{ab}} = (V_t^{u_{ab}})_{t\geq 0}$, for control vector u_{ab} is then

$$\mathrm{d}V_t^{u_{ab}} = \left(V_t^{u_{ab}} - \Pi_t^P - \Pi_t^S\right) \frac{\mathrm{d}B_t}{B_t} + \Pi_t^P \frac{\mathrm{d}P_t}{P_t} + \Pi_t^S \frac{\mathrm{d}S_t^{\mu,\sigma}}{S_t^{\mu,\sigma}}\,, \quad V_0^{u_{ab}} \in \mathbb{R}^+\,, \tag{2.1}$$

which can be rewritten using the equations (1.1), (1.2) and (1.3) for the money market account, the market portfolio and the company's stock respectively as

$$\begin{aligned} \mathrm{d}V_t^{u_{ab}} = &\left[r\, V_t^{u_{ab}} + (\Pi_t^P + \beta\, \Pi_t^S)(\mu^P - r) + \Pi_t^S(\mu_t - r)\right] \mathrm{d}t \\ &+ \left[\Pi_t^P + \beta\, \Pi_t^S\right] \sigma^P \mathrm{d}W_t^P + \Pi_t^S\, \sigma_t\, \mathrm{d}W_t\,, \quad V_0^{u_{ab}} \in \mathbb{R}^+\,. \end{aligned} \tag{2.2}$$

The optimal investment and control decision of the executive is then the solution of

$$\Phi(t,v) = \sup_{u_{ab}\in A(t,v)} \mathbb{E}^{t,v}\left[U\left(V_T^{u_{ab}}\right) - \int_t^T c^\star\left(s,V_s^{u_{ab}},\lambda_s\right)\, \mathrm{d}s\right]\,,\ (t,v) \in [0,T] \times \mathbb{R}^+\,, \tag{2.3}$$

where $A(t,v)$ is given in Def. 2.1.1.

Definition 2.1.1

Let $0 \leq t \leq T$, t fixed, and let further (μ,σ) take values in $[r,\infty) \times (0,\infty)$. Then we denote by $A(t,v)$ the set of admissible strategies $u_{ab} = \left(\Pi^P, \Pi^S, \mu, \sigma\right)$ corresponding to portfolio value $v = V_t^{u_{ab}} > 0$ at time t, which are $\{\mathcal{F}_s\,;\, t \leq s \leq T\}$-predictable processes, such that *(i) the company's stock price process*

$$\mathrm{d}S^{\mu,\sigma}_s = S^{\mu,\sigma}_s \left(\mu_s \, \mathrm{d}s + \beta \left[\frac{\mathrm{d}P_s}{P_s} - r\mathrm{d}s \right] + \sigma_s \, \mathrm{d}W_s \right) , \quad S^{\mu,\sigma}_t \in \mathbb{R}^+ ,$$

has a unique non-negative solution and satisfies

$$\int_t^T (S^{\mu,\sigma}_s)^2 \left((\beta\sigma^P)^2 + (\sigma_s)^2 \right) \mathrm{d}u \, < \, \infty \quad P - a.s. ;$$

(ii) the wealth equation

$$\mathrm{d}V^{u_{ab}}_s = \left(V^{u_{ab}}_s - \Pi^P_s - \Pi^S_s \right) \frac{\mathrm{d}B_s}{B_s} + \Pi^P_s \frac{\mathrm{d}P_s}{P_s} + \Pi^S_s \frac{\mathrm{d}S^{\mu,\sigma}_s}{S^{\mu,\sigma}_s} \, , V^{u_{ab}}_t \in \mathbb{R}^+ ,$$

has a unique non-negative solution and satisfies

$$\int_t^T \left((\Pi^P_s + \beta \, \Pi^S_s)^2 (\sigma^P)^2 + (\Pi^S_s \sigma_s)^2 \right) \mathrm{d}s \, < \, \infty \quad P - a.s. ;$$

(iii) and the utility of wealth and the disutility of control satisfy

$$\mathbb{E} \left[U \left(V^{u_{ab}}_T \right)^- + \int_t^T c(s, V^{u_{ab}}_s, \mu_s, \sigma_s) \, \mathrm{d}s \right] \, < \, \infty \, .$$

The reformulation of the optimization problem follows exactly the same steps as in the CRRA case. We restate that by Lemma 1.1.1 we know that the minimization problem

$$\min_{\{\sigma>0 : \mu = r + \lambda\,\sigma\}} c(t,v,\mu,\sigma) \, , \quad \text{for } (t,v,\lambda) \in [0,T] \times \mathbb{R}^+ \times \mathbb{R}^+_0 \, ,$$

admits a unique solution $\sigma^\star(t,v,\lambda)$. With this knowledge we are able to switch from the control vector $u_{ab} = \left(\Pi^P, \Pi^S, \mu, \sigma \right)$ to the reduced control vector $u'_{ab} = \left(\Pi^P, \Pi^S, \lambda \right)$. The corresponding restated equations of the company's stock price and the wealth equation, respectively, read

$$\mathrm{d}S^\lambda_t = S^\lambda_t \left(\left[r + \lambda_t \sigma^\star(t, V^{u'_{ab}}_t, \lambda_t) \right] \mathrm{d}t + \beta \left[\frac{\mathrm{d}P_t}{P_t} - r\mathrm{d}t \right] + \sigma^\star(t, V^{u'_{ab}}_t, \lambda_t) \, \mathrm{d}W_t \right) ,$$

$$S^\lambda_0 \in \mathbb{R}^+ , \tag{2.4}$$

and

$$\mathrm{d}V_t^{u'_{ab}} = (V_t^{u'_{ab}} - \Pi_t^P - \Pi_t^S)\,\frac{\mathrm{d}B_t}{B_t} + \Pi_t^P\frac{\mathrm{d}P_t}{P_t} + \Pi_t^S\frac{\mathrm{d}S_t^\lambda}{S_t^\lambda}\,, \quad V_0^{u'_{ab}} \in \mathbb{R}^+\,, \tag{2.5}$$

whereby the wealth equation admits the rewritten representation

$$\begin{aligned}\mathrm{d}V_t^{u'_{ab}} &= \Big[r\,V_t^{u'_{ab}} + (\Pi_t^P + \beta\,\Pi_t^S)(\mu^P - r) + \Pi_t^S\,\lambda_t\,\sigma^\star(t,V_t^{u'_{ab}},\lambda_t)\Big]\,\mathrm{d}t \\ &\quad + \big[\Pi_t^P + \beta\,\Pi_t^S\big]\,\sigma^P\mathrm{d}W_t^P + \Pi_t^S\,\sigma^\star(t,V_t^{u'_{ab}},\lambda_t)\,\mathrm{d}W_t\,, \quad V_0^{u'_{ab}} \in \mathbb{R}^+\,.\end{aligned} \tag{2.6}$$

Accordingly, we define the class of admissible strategies w.r.t. the reduced control vector u'_{ab}:

Definition 2.1.2

Let $0 \le t \le T$, t fixed, and let further λ take values in $[0,\infty)$. Then we denote by $A'(t,v)$ the set of admissible strategies $u'_{ab} = \big(\Pi^P,\Pi^S,\lambda\big)$ corresponding to portfolio value $v = V_t^{u'_{ab}} > 0$ at time t, which are $\{\mathcal{F}_s\,;\, t \le s \le T\}$-predictable processes, such that

(i) the company's stock price process

$$\mathrm{d}S_s^\lambda = S_s^\lambda\left([r + \lambda_s\sigma_s^\star]\,\mathrm{d}s + \beta\left[\frac{\mathrm{d}P_s}{P_s} - r\mathrm{d}s\right] + \sigma_s^\star\,\mathrm{d}W_s\right)\,, \quad S_t^\lambda \in \mathbb{R}^+\,,$$

has a unique non-negative solution and satisfies

$$\int_t^T (S_s^\lambda)^2\left((\beta\sigma^P)^2 + (\sigma_s^\star)^2\right)\,\mathrm{d}s \,<\, \infty \quad P - a.s.\,;$$

(ii) the wealth equation

$$\mathrm{d}V_s^{u'_{ab}} = \left(V_s^{u'_{ab}} - \Pi_s^P - \Pi_s^S\right)\,\frac{\mathrm{d}B_s}{B_s} + \Pi_s^P\frac{\mathrm{d}P_s}{P_s} + \Pi_s^S\frac{\mathrm{d}S_s^\lambda}{S_s^\lambda}\,,\; V_t^{u'_{ab}} \in \mathbb{R}^+\,,$$

has a unique non-negative solution and satisfies

$$\int_t^T \left((\Pi_s^P + \beta\,\Pi_s^S)^2(\sigma^P)^2 + (\Pi_s^S\sigma_s^\star)^2\right)\,\mathrm{d}s \,<\, \infty \quad P - a.s.\,;$$

(iii) and the utility of wealth and the minimized disutility of control satisfy

$$\mathbb{E}\left[U(V_T^{u'_{ab}})^- + \int_t^T c^\star(s, V_s^{u'_{ab}},\lambda_s)\,\mathrm{d}s\right] < \infty\,.$$

With the following analogon to Assumption 1.1.2, we can easily proof the correspondence theorem for our formulation with respect to the absolute investment strategies:

Assumption 2.1.1
For a given control $u_{ab} = (\Pi^P,\Pi^S,\mu,\sigma) \in A(t,v)$, the value process $V_t^{u'_{ab}}$, $\lambda_t = (\mu_t - r)/\sigma_t$ and $\sigma^\star(t,V_t^{u'_{ab}},\lambda_t)$ are determined in accordance with Lemma 1.1.1. The process S_t^λ defined in (2.4) is assumed to satisfy

$$\int_t^T (S_s^\lambda)^2 \left((\beta\sigma^P)^2 + (\sigma^\star(s,V_s^{u'_{ab}},\lambda_s))^2\right)\,\mathrm{d}s < \infty \quad P - a.s.\,.$$

Theorem 2.1.2 (Correspondence Result)
Suppose (2.3) admits a solution Φ, then this solution coincides with the value function of the optimal investment and control problem

$$\Phi'(t,v) = \sup_{u'_{ab}\in A'(t,v)} \mathbb{E}^{t,v}\left[U\left(V_T^{u'_{ab}}\right) - \int_t^T c^\star\left(s,V_s^{u'_{ab}},\lambda_s\right)\,\mathrm{d}s\right]\,,\ (t,v) \in [0,T]\times\mathbb{R}^+\,, \tag{2.7}$$

where $A'(t,v)$ is given in Def. 2.1.2.

Proof. Applying exactly the same steps as in Theorem 1.1.3, where we just substitute the control vectors $u = (\pi^P,\pi^S,\mu,\sigma)$ and $u' = (\pi^P,\pi^S,\lambda)$ from Chapter 1 by the control vectors $u_{ab} = (\Pi^P,\Pi^S,\mu,\sigma)$ and $u'_{ab} = (\Pi^P,\Pi^S,\lambda)$ together with Assumption 2.1.1 yields the result. □

For the remainder of the chapter we assume that the optimal investment and control problem (2.7) admits a value function $\Phi \in C^{1,2}$.

As in the power- and log-utility case, to guarantee that the candidates we will derive for the executive's optimal investment and control strategy as well as the value function are indeed optimal, we restrrict the class of admissible strategies now as follows:

Definition 2.1.3
Let $0 \le t \le T$, t fixed, and let λ take values in $[0,\infty)$. Further choose $\tilde{\epsilon} \in (0,\infty)$ as close to zero as possible. Then by $A'_\eta(t,v)$ we denote the set of admissible strategies $u'_{ab} \in A'(t,v)$, such that we have for $\eta > 0$:

$$\int_t^T (\Pi_s^P + \beta\,\Pi_s^S)^{2+\tilde{\epsilon}}(\sigma^P)^{2+\tilde{\epsilon}} + \left(\Pi_s^S \sigma_s^\star\right)^{2+\tilde{\epsilon}}\,\mathrm{d}s \le C_1 < \infty,\ \textit{for some } C_1 \in \mathbb{R}_0^+, \tag{2.8}$$

$$\int_t^T \Pi_s^S \sigma_s^\star \lambda_s\,\mathrm{d}u \ge C_2 > -\infty\,,\ \textit{for some } C_2 \in \mathbb{R}_0^+\,. \tag{2.9}$$

The optimal investment and control problem stated w.r.t. this more restrictive class is given by

$$\Phi(t,v) = \sup_{u'_{ab} \in A'_\eta(t,v)} \mathbb{E}^{t,v}\left[U(V_T^{u'_{ab}}) - \int_t^T c^\star(s,V_s^{u'_{ab}},\lambda_s)\,\mathrm{d}s\right], \tag{2.10}$$

where $(t,v) \in [0,T] \times \mathbb{R}^+$.

2.2 Optimal Strategies

In this section we give a closed-form solution for the optimal investment and control problem in (2.10) w.r.t. absolute investment strategies using an exponential utility/disutility set-up. Unfortunately this set-up is only solvable for the case of zero interest rates.

For the *absolute* risk aversion parameter $\eta > 0$, the utility function U is

$$U(v) = 1 - e^{-\eta v}\,, \tag{2.11}$$

and the disutility of control (i.e. work effort) $c^\star$ is

$$c^\star(t,v,\lambda) = \kappa\, e^{-\eta\, v}\, \frac{\lambda^\alpha}{\alpha}\,, \tag{2.12}$$

where $\kappa > 0$ and $\alpha > 2$ are again the executive's work effectiveness parameters, termed 'inverse work productivity' and 'disutility stress'. κ and α have the same properties and implications as mentioned in Section 1.2.

Now, in (2.12), the scaling factor $e^{-\eta\, v}$ relates the executive's disutility of work effort to his wealth (v) with a formulation based on the constant *absolute* risk aversion formulation of the utility function in (2.11). Given a positive value of the absolute risk aversion parameter, $\eta > 0$, the executive's disutility of work effort decreases with his wealth.

Remark 2.2.1

This specification for the disutility of work effort is economically counter-intuitive: For $\eta > 0$, our specification produces decreasing disutility of work effort for an increasing level of wealth, keeping work effort constant.

But it is also important for this exponential utility/disutility set-up that the utility of wealth and the disutility of work effort are normed to take values on the same scale w.r.t. the wealth v.

We refer again to the explanation of Chapter 1. We consider an executive with a rather high work ethic and we consider η to be positively related to the executive's work ethic. We repeat that our high work ethic executive has then a comparatively low aversion to work effort at outset and will become further less averse to work effort if past effort or chance brings success as indicated by increased wealth. In contrast to that a low work ethic executive would have a comparatively high aversion to work effort and will become further more averse to work effort if his wealth increases.

2.2.1 Hamilton-Jacobi-Bellman Equation

Having formulated the optimal investment and control decision problem with respect to the parameter set $u'_{ab} = (\Pi^P, \Pi^S, \lambda)$ as given by (2.7), we can write down the corresponding Hamilton-Jacobi-Bellman equation (HJB); again formulated with respect to a general utility function U and a general disutility function $c^\star$:

$$\begin{aligned} 0 &= \sup_{u'_{ab} \in \mathbb{R}^2 \times [0,\infty)} \Big[(L^{u'_{ab}} \Phi)(t,v) - c^\star(t,v,\lambda) \Big] \,, \text{ for } (t,v) \in [0,T) \times \mathbb{R}^+, \\ U(v) &= \Phi(T,v)\,, \quad \text{for } v \in \mathbb{R}^+\,, \end{aligned} \tag{2.13}$$

where the differential operator $L^{u'_{ab}}$ is given by

$$\begin{aligned} (L^{u'_{ab}} g)(t,v) = &\frac{\partial g}{\partial t}(t,v) + \frac{\partial g}{\partial v}(t,v) \Big(r\,v + \Pi^S \lambda\, \sigma^\star(t,v,\lambda) + (\Pi^P + \beta\, \Pi^S) \\ &\cdot [\mu^P - r] \Big) + \frac{1}{2} \frac{\partial^2 g}{\partial v^2}(t,v) \, \Big(\big[\Pi^S \sigma^\star(t,v,\lambda)\big]^2 + \big[(\Pi^P + \beta\, \Pi^S) \sigma^P\big]^2 \Big)\,. \end{aligned} \tag{2.14}$$

Potential maximizers $\Pi^{P^\star}$, $\Pi^{S^\star}$ and $\lambda^\star$ of the HJB (2.13) can be calculated by establishing the first order conditions:

$$\begin{aligned} \Pi^{P^\star}(t,v) &= -\frac{(\mu^P - r)}{(\sigma^P)^2} \frac{\Phi_v(t,v)}{\Phi_{vv}(t,v)} - \beta\, \Pi^{S^\star}(t,v) \quad , \\ \Pi^{S^\star}(t,v) &= -\frac{\lambda^\star(t,v)}{\sigma^\star(t,v,\lambda^\star(t,v))} \frac{\Phi_v(t,v)}{\Phi_{vv}(t,v)} \quad , \end{aligned} \tag{2.15}$$

where $\lambda^\star$ is the solution of the implicit equation

$$\lambda \frac{\Phi_v^2(t,v)}{\Phi_{vv}(t,v)} + \frac{\partial c^\star}{\partial \lambda}(t,v,\lambda) = 0 \qquad \text{for all } (t,v) \in [0,T] \times \mathbb{R}^+\,, \tag{2.16}$$

where we have already used (2.15) to simplify the equation.

Substitung the candidates (2.15) in the Hamilton-Jacobi-Bellman equation

(2.13) yields:

$$\begin{aligned} 0 =&\Phi_t(t,v) + \Phi_v(t,v)\,v\,r - \frac{1}{2}(\lambda^\star(t,v))^2 \frac{\Phi_v^2(t,v)}{\Phi_{vv}(t,v)} - \frac{1}{2}\lambda_P^2 \frac{\Phi_v^2(t,v)}{\Phi_{vv}(t,v)} \\ &- c^\star(t,v,\lambda)\,. \end{aligned} \tag{2.17}$$

In the following section we give the solution of this equation w.r.t. exponential utility and disutilty.

2.2.2 Closed-Form Solution

A closed-form solution is derived for the control problem (2.10) using the utility and disutility functions (2.11) and (2.12), for $\eta > 0$, where we have to restrict ouselves to the case of zero interest rates to obtain solvability of the problem.

Theorem 2.2.1 (The exponential-utility case: $\eta > 0$ with $r = 0$) *The full solution of the maximization problem (2.10) can be summarized by the strategy*

$$\begin{aligned} &\lambda^\star(t,v) = \left(\frac{1}{\kappa}\,f(t)\right)^{\frac{1}{\alpha-2}}\,, \\ &\Pi^{P\star}(t,v) = \frac{\mu^P}{\eta\,(\sigma^P)^2} - \beta\,\pi^{S\star}(t,v)\,, \quad \Pi^{S\star}(t,v) = \frac{\lambda^\star(t,v)}{\eta\,\sigma^\star(t,v,\lambda^\star(t,v))}\,, \end{aligned} \tag{2.18}$$

and value function

$$\Phi(t,v) = 1 - f(t)\,e^{-\eta\,v}\,, \tag{2.19}$$

where

$$\begin{aligned} f(t) =& e^{-\frac{1}{2}\left(\mu^P/\sigma^P\right)^2 (T-t)} \\ &\times \left(1 - \frac{(\alpha-2)\,\left(\sigma^P\right)^2\,\kappa^{-\frac{2}{\alpha-2}}}{\alpha\,\left(\mu^P\right)^2}\,\left(e^{-\frac{1}{\alpha-2}\left(\mu^P/\sigma^P\right)^2 (T-t)} - 1\right)\right)^{-\frac{\alpha-2}{2}}\,. \end{aligned} \tag{2.20}$$

Proof. First recall from Theorem 1.2.1 that a function F of the form $F(\lambda) = a\,\lambda^2 - b\,\lambda^\alpha$, $\lambda \geq 0$, for given constants $a, b > 0$ and $\alpha > 2$, has a unique maximizer $\lambda^\star$ and maximized value $F(\lambda^\star)$ given by

$$\lambda^\star = \left(\frac{2\,a}{\alpha\,b}\right)^{\frac{1}{\alpha-2}}, \quad \text{and} \quad F(\lambda^\star) = (\alpha-2)\,\alpha^{-\frac{\alpha}{\alpha-2}}\,2^{\frac{2}{\alpha-2}}\,a^{\frac{\alpha}{\alpha-2}}\,b^{-\frac{2}{\alpha-2}}\,. \quad (2.21)$$

Using this insight the first order condition for $\lambda^\star$ in (2.16) is now solved. Set

$$a = \frac{1}{2}\,\frac{\Phi_v^2}{-\Phi_{vv}}\,, \quad \text{and} \quad b = \frac{\kappa}{\alpha}\,e^{-\eta\,v}\,,$$

then (2.21) gives

$$\lambda^\star = \left(\frac{e^{\eta\,v}}{\kappa}\,\frac{\Phi_v^2}{-\Phi_{vv}}\right)^{\frac{1}{\alpha-2}}, \quad F(\lambda^\star) = \frac{\alpha-2}{2\,\alpha}\,\left(\frac{e^{\eta\,v}}{\kappa}\right)^{\frac{2}{\alpha-2}}\,\left(\frac{\Phi_v^2}{-\Phi_{vv}}\right)^{\frac{\alpha}{\alpha-2}}. \quad (2.22)$$

Now (2.17) reads

$$0 = \Phi_t + \Phi_v\,v\,r + \frac{1}{2}\,\frac{\Phi_v^2}{-\Phi_{vv}}\,\left(\lambda^P\right)^2 + \frac{\alpha-2}{2\,\alpha}\,\left(\frac{e^{\eta\,v}}{\kappa}\right)^{\frac{2}{\alpha-2}}\,\left(\frac{\Phi_v^2}{-\Phi_{vv}}\right)^{\frac{\alpha}{\alpha-2}}. \quad (2.23)$$

Using the separation ansatz $\Phi(t,v) = 1 - f(t)\,e^{-\eta\,v}$ with $f(T) = 1$ results in

$$\Phi_t = -\dot{f}\,e^{-\eta\,v}\,,\ \Phi_v = \eta\,f\,e^{-\eta\,v}\,,\ \Phi_{vv} = -\eta^2\,f\,e^{-\eta\,v}\,,\ \text{and}\ \frac{\Phi_v^2}{-\Phi_{vv}} = f(t)\,e^{-\eta\,v}\,.$$

Thus (2.23) becomes

$$0 = -\dot{f}\,e^{-\eta\,v} + \eta\,f\,e^{-\eta\,v}\,v\,r + \frac{1}{2}(\lambda^P)^2\,f\,e^{-\eta\,v} + \frac{\alpha-2}{2\,\alpha}\,\left(\frac{e^{\eta\,v}}{\kappa}\right)^{\frac{2}{\alpha-2}}\,\left(f\ e^{-\eta\,v}\right)^{\frac{\alpha}{\alpha-2}}.$$

Simplifying gives

$$-e^{-\eta\,v}\left\{\dot{f} - f\,\left[\eta\,v\,r + \frac{1}{2}\lambda_P^2\right] + f^{\frac{\alpha}{\alpha-2}}\,\left[-\frac{\alpha-2}{2\alpha}\,\kappa^{-\frac{2}{\alpha-2}}\right]\right\} = 0\,. \quad (2.24)$$

From (2.24), we see that the separation approach as given above only works if we have that $r = 0$. (2.24) with $r = 0$ is fullfilled when the expression in the brackets is equal to zero or we have that $\lim \eta \to \infty$, which is economically not reasonable. Thus we have to solve the Bernoulli ODE of the form

$$\dot{f} = f\,\frac{1}{2}\left(\frac{\mu^P}{\sigma^P}\right)^2 + f^{\frac{\alpha}{\alpha-2}}\,\frac{\alpha-2}{2\alpha}\,\kappa^{-\frac{2}{\alpha-2}}\,, \quad (2.25)$$

where we keep in mind that $(\lambda^P)^2 \stackrel{r=0}{=} (\mu^P/\sigma^P)^2$. The solution of this ODE is calculated as in the power-utility case. We repeat that a Bernoulli ODE of form $\dot{f} = a_1\, f + a_\nu\, f^\nu$, has the solution

$$f(t)^{1-\nu} = C\, e^{G(t)} + (1-\nu)\, e^{G(t)} \int_0^t e^{-G(s)}\, a_\nu\, \mathrm{d}s\,,$$

where $G(t) = (1-\nu) \int_0^t a_1(s)\, \mathrm{d}s$ and C is an arbitrary constant. In the exponential-utility setting we have $\nu = \frac{\alpha}{\alpha-2}$ and $(1-\nu) = \frac{-2}{\alpha-2}$ as well as

$$a_1 = \frac{1}{2}\left(\frac{\mu_P}{\sigma^P}\right)^2 \quad , \quad a_\nu = \frac{\alpha-2}{2\alpha}\,\kappa^{\frac{-\alpha}{\alpha-2}}\,.$$

The formal solution $f(t)^{1-\nu}$ is explicitly calculated in three steps. First, compute

$$G(t) = -\frac{2\,a_1\,t}{\alpha-2}\,, \quad \text{and} \quad \int_0^t e^{-G(s)}\, a_\nu(s)\, \mathrm{d}s = \frac{\alpha-2}{2}\,\frac{a_\nu}{a_1}\left(e^{\frac{2\,a_1\,t}{\alpha-2}} - 1\right)\,,$$

then

$$f(t) = e^{a_1\,t}\left(C - \frac{a_\nu}{a_1}\left(e^{\frac{2\,a_1\,t}{\alpha-2}} - 1\right)\right)^{-\frac{\alpha-2}{2}}\,.$$

Finally, solve for C by using $f(T) = 1$ so that

$$C = e^{\frac{2\,a_1\,T}{\alpha-2}} + \frac{a_\nu}{a_1}\left(e^{\frac{2\,a_1\,T}{\alpha-2}} - 1\right)\,.$$

Note also that $f(0) = C^{-\frac{\alpha-2}{2}}$. Now

$$f(t) = e^{-a_1\,(T-t)}\left(1 - \frac{a_\nu}{a_1}\left(e^{-\frac{2\,a_1}{\alpha-2}\,(T-t)} - 1\right)\right)^{-\frac{\alpha-2}{2}}\,.$$

Substituting for a_1 and a_ν then yields the result for $f(t)$. Using $\frac{\Phi_v}{\Phi_{vv}} = -\frac{1}{\eta}$ and the first order conditions in (2.15) and (2.22) we obtain the claimed optimal strategy $\lambda^\star$, $\Pi^{P\star}$ and $\Pi^{S\star}$. Finally note that our claimed optimal strategies are admissible, i.e. ${u'_{ab}}^\star = (\Pi^{S\star}, \Pi^{P\star}, \lambda^\star) \in A'_\eta(t,v)$. This is again shown as in the CRRA case. A sufficient condition for admissibility is that $\lambda^\star$, $\Pi^{P\star}\,\sigma^P$, and $\Pi^{S\star}\,\sigma^\star$ be uniformly bounded (see Def. 2.1.3); because these expressions are deterministic and continuous functions in u on $[t,T]$, they are hence uniformly bounded. □

Remark 2.2.2

Theorem 2.2.1 clearly indicates that the claimed optimal strategies are not proportional to $1/v$ *and we avoid the problem that they could tend to infinity.*

Remark 2.2.3

In a classical set-up without disutility the executive's optimal investment decision w.r.t. absolute investment strategies

$$\widehat{\Phi}(t,v) = \sup_{\Pi \in \widehat{A}(t,v)} \mathbb{E}^{t,v}[U(V_T^{\Pi})]\,, \quad (t,v) \in [0,T] \times \mathbb{R}^+,$$

where $\widehat{A}(t,v)$ *denotes the set of all admissible portfolio strategies* Π *at time* t *corresponding to portfolio value (i.e. wealth)* $v = V_t^{\Pi} > 0$, *can be solved for an exponential utility function as given in* (2.11) *using the ansatz*

$$\widehat{\Phi}(t,v) = e^{-\eta[f(t)\,v+g(t)]} \quad \textit{with} \quad f(T) = 1 \quad \textit{and} \quad g(T) = 0\,.$$

However in our set-up including a disutility of work effort, this technique does not work, since the additional term arising from the disutility in (2.23) *causes problems when applying the ansatz stated above in order to reduce equation* (2.23) *to ordinary differential equations w.r.t.* f *and* g.

2.2.3 Verification Theorem

The solutions of the maximization problem given in Theorem 2.2.1 are candidates for the optimal investment and control choices for the problem in (2.10) for the case $r = 0$. In this section we verify that under sufficient assumptions also these solutions are indeed optimal.

Theorem 2.2.2 (Verification Result for the Exponential Case)

Let $\kappa > 0$ *and* $\alpha > 2$*; further let* $r = 0$. *Assume the executive's utility and disutility functions are given by (2.11) and (2.12). Then the candidates given in (2.18) - (2.20) are the optimal investment and control strategy (i.e. own-company stockholding, market portfolio holding and non-systematic Sharpe* *ratio strategy) and value function of the optimal control problem (2.10) w.r.t. absolute investment strategies.*

Proof. Define the performance functional of our optimal investment and control decision by

$$J(t,v;u'_{ab}) := \mathbb{E}^{t,v}\left[U\left(V_T^{u'_{ab}}\right) - \int_t^T c^\star(s, V_s^{u'_{ab}},\lambda_s)\,\mathrm{d}s\right],$$

where $(t,v) \in [0,T] \times \mathbb{R}^+$ and $u'_{ab} = (\Pi^P,\Pi^S,\lambda) \in A'_\eta(t,v)$. First note that the wealth process (2.6) for $r = 0$ reads

$$\begin{aligned} \mathrm{d}V_t^{u'_{ab}} &= \left[(\Pi_t^P + \beta\,\Pi_t^S)\,\mu^P + \Pi_t^S\,\lambda_t\,\sigma_t^\star\right]\mathrm{d}t \\ &\quad + \left[\Pi_t^P + \beta\,\Pi_t^S\right]\sigma^P \mathrm{d}W_t^P + \Pi_t^S\,\sigma_t^\star\,\mathrm{d}W_t\,, \quad V_0^{u'_{ab}} \in \mathbb{R}^+\,. \end{aligned} \tag{2.26}$$

Recall the claimed optimal value function $\Phi \in C^{1,2}$ and apply Ito's formula for $\eta > 0$ to obtain:

$$\begin{aligned} &U\left(V_T^{u'_{ab}}\right) - \int_t^T c^\star(s, V_s^{u'_{ab}},\lambda_s)\,\mathrm{d}s = \Phi(T,V_T^{u'_{ab}}) - \int_t^T \kappa e^{-\eta V_s^{u'_{ab}}}\frac{\lambda_s^\alpha}{\alpha}\,\mathrm{d}s = \Phi(t,v) \\ &+ \int_t^T \Big(\Phi_t(s,V_s^{u'_{ab}}) + \Phi_v(s,V_s^{u'_{ab}})\left[\Pi_u^S\,\lambda_s\,\sigma_s^\star + (\Pi_s^P + \beta\,\Pi_s^S)\,\mu^P\right] \\ &+ 1/2\,\Phi_{vv}(s,V_s^{u'_{ab}})\left[((\Pi_s^P + \beta\,\Pi_s^S)\sigma^P)^2 + (\Pi_s^S\sigma_s^\star)^2\right] - \kappa e^{-\eta V_s^{u'_{ab}}}\frac{\lambda_s^\alpha}{\alpha}\Big)\,\mathrm{d}s \\ &+ \int_t^T \Phi_v(s,V_s^{u'_{ab}})\,(\Pi_s^P + \beta\,\Pi_s^S)\sigma^P\,\mathrm{d}W_s^P + \int_t^T \Phi_v(s,V_s^{u'_{ab}})\,\Pi_s^S\sigma_s^\star\,\mathrm{d}W_s\,. \end{aligned} \tag{2.27}$$

The remainder of the proof is divided into two parts. Part (a) establishes that the value function Φ coincides with the performance functional J evaluated at the claimed maximizers $u'^\star_{ab} = (\Pi^{P\star},\Pi^{S\star},\lambda^\star)$, $\eta > 0$. Part (b) shows the optimality of the candidate $u'^\star_{ab}$, i.e.: $J(t,v;u'_{ab}) \leq \Phi(t,v)$, for $u'_{ab} = (\Pi^P,\Pi^S,\lambda) \in A'_\eta(t,v)$.

Part (a): We establish that $J(t,v;u'^\star_{ab}) = \Phi(t,v)$. To do this we show that in the right hand side (RHS) of (2.27) the drift vanishes by the HJB (2.13) and that the local martingale component is a true martingale and hence disappears in expectation. And finally, it is verified that indeed $u'^{\star}_{ab} \in A'_\eta(t,v)$.

By construction, Φ with control $u'^{\star}_{ab}$ satisfies the HJB-PDE in (2.13), that is for $r = 0$,

$$\begin{aligned} 0 &= \Phi_t + \Phi_v \, (\Pi^{S\star}\, \lambda^\star\, \sigma^\star + (\Pi^{P\star} + \beta\, \Pi^{S\star})\, \mu^P) \\ &\quad +(1/2)\, \Phi_{vv}\, ([\Pi^{S\star}\, \sigma^\star]^2 + [(\Pi^{P\star} + \beta\, \Pi^{S\star})\, \sigma^P]^2) - c^\star . \end{aligned}$$

This eliminates the drift (Lebesgue integral) in (2.27) and we obtain

$$U\left(V_T^{u'^{\star}_{ab}}\right) - \int_t^T c^\star(s, V_s^{u'^{\star}_{ab}}, \lambda^\star_s)\, \mathrm{d}s = \Phi(t,v) +$$
$$\int_t^T \Phi_v(s, V_s^{u'^{\star}_{ab}})\, (\Pi_s^{P\star} + \beta\, \Pi_s^{S\star}) \sigma^P\, \mathrm{d}W_s^P + \int_t^T \Phi_v(s, V_s^{u'^{\star}_{ab}})\, \Pi_s^{S\star} \sigma_s^\star\, \mathrm{d}W_s\,.$$

For $J(t,v;u'^{\star}_{ab}) = \Phi(t,v)$, it remains to prove that the local martingale component disappears in expectation. A sufficient condition is the square-integrability of the local martingale component

$$\mathbb{E}\left[\int_t^T \left(\Phi_v(s, V_s^{u'^{\star}_{ab}})\right)^2 \left([\Pi_s^{P\star} + \beta\, \Pi_s^{S\star}]^2 (\sigma^P)^2 + [\Pi_s^{S\star}\, \sigma_s^\star]^2\right) \mathrm{d}s\right] < \infty\,.$$

Using the explicit form of the candidates in (2.18) and $\Phi_v = \eta\, f(t)\, e^{-\eta\, v}$, for $\eta > 0$, gives

$$\begin{aligned} &\left(\Phi_v(s, V_s^{u'^{\star}_{ab}})\right)^2 \left([\Pi_s^{P\star} + \beta\, \Pi_s^{S\star}]^2 (\sigma^P)^2 + [\Pi_s^{S\star}\, \sigma_s^\star]^2\right) \\ &\qquad = f(s)^2\, e^{-2\,\eta\, V_s^{u'^{\star}_{ab}}} \left[\left(\frac{\mu^P}{\sigma^P}\right)^2 + \left(\frac{1}{\kappa} f(s)\right)^{\frac{2}{\alpha-2}}\right] . \end{aligned}$$

The RHS is $e^{-2\,\eta\, V_s^{u'^{\star}_{ab}}}$ times a deterministic and continuous function on the compact set $[t,T]$. The deterministic part is uniformly bounded. Therefore, it is sufficient to focus on the stochastic component: $V^{u'^{\star}_{ab}}$ satisfies

$$\mathrm{d}V_s^{u'^{\star}_{ab}} = \left[\frac{(\mu^P)^2}{\eta\, (\sigma^P)^2} + \frac{(\lambda^\star(s, V_s^{u'^{\star}_{ab}}))^2}{\eta}\right] \mathrm{d}s + \frac{\mu^P}{\eta\, \sigma^P}\, \mathrm{d}W_s^P + \frac{\lambda^\star(s, V_s^{u'^{\star}_{ab}})}{\eta}\, \mathrm{d}W_s.$$

The solution of the above inhomogeneous wealth equation w.r.t. the optimal strategies ${u'}^{\star}_{ab}$ starting at t with initial wealth $V_t^{{u'}^{\star}_{ab}} = v$ applying variation of constants is

$$V_s^{{u'}^{\star}_{ab}} = v + \int_t^s \left(\frac{(\mu^P)^2}{\eta\,(\sigma^P)^2} + \frac{(\lambda^{\star}_{\tilde{s}})^2}{\eta} \right) \mathrm{d}\tilde{s} + \int_{\mathrm{t}}^{\mathrm{s}} \frac{\mu^{\mathrm{P}}}{\eta\,\sigma^{\mathrm{P}}}\,\mathrm{dW}^{\mathrm{P}}_{\tilde{\mathrm{s}}} + \int_{\mathrm{t}}^{\mathrm{s}} \frac{\lambda^{\star}_{\tilde{\mathrm{s}}}}{\eta}\,\mathrm{dW}_{\tilde{\mathrm{s}}}\,.$$

Recalling that $\lambda^{\star}(s,v)$ is a continuous function in s and does not depend on v, we see that $V_s^{{u'}^{\star}_{ab}}$ follows a normal distribution with mean

$$\mu_{V_s^{{u'}^{\star}_{ab}}} = v + \int_t^s \left(\frac{(\mu^P)^2}{\eta\,(\sigma^P)^2} + \frac{(\lambda^{\star}_{\tilde{s}})^2}{\eta} \right) \mathrm{d}\tilde{s}$$

and variance

$$\sigma^2_{V_s^{{u'}^{\star}_{ab}}} = \int_t^s \left(\frac{(\mu^P)^2}{\eta^2\,(\sigma^P)^2} + \frac{(\lambda^{\star}_{\tilde{s}})^2}{\eta^2} \right) \mathrm{d}\tilde{s}$$

being uniformly bounded, for all $s \in [t,T]$. Since all moments of a log-normally distributed random variable exist, it follows that the local martingale is a square-integrable martingale. This establishes $J(t,v;{u'}^{\star}_{ab}) = \Phi(t,v)$. Finally, ${u'}^{\star}_{ab} \in A'_{\eta}(t,v)$ follows from the fact that $\Pi^{P\star}$, $\Pi^{S\star}\,\sigma^{\star}$, and $\lambda^{\star}$ are uniformly bounded on $[t,T]$, for each $\eta > 0$.

Part (b): Now we show the optimality, i.e. $J(t,v;u'_{ab}) \leq \Phi(t,v)$, for $u'_{ab} \in A'_{\eta}(t,v)$. As in (a), this is also based on the analysis of (2.27). The HJB (2.13) is applied to show that the drift component is bounded from above by zero. Then it is shown that the conditions in Def. 2.1.3 are sufficient for the local martingale component on the RHS of (2.27) to vanish in expectation.

By the HJB (2.13), Φ with arbitrary $u'_{ab} = (\Pi^P,\Pi^S,\lambda) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^+_0$ with $r = 0$ satisfies

$$\begin{aligned} 0 \;\geq\; & \Phi_t + \Phi_v\,(\Pi^S\,\lambda\,\sigma^{\star} + (\Pi^P + \beta\,\Pi^S)\,\mu^P) \\ & + (1/2)\,\Phi_{vv}\,([\Pi^S\,\sigma^{\star}]^2 + [(\Pi^P + \beta\,\Pi^S)\,\sigma^P]^2) - c^{\star}\,, \end{aligned}$$

for $(s,v) \in [t,T] \times \mathbb{R}^+$. This provides the point-wise upper bound zero for the drift in (2.27) and we obtain

$$U\left(V_T^{u'_{ab}}\right) - \int_t^T c^\star(s, V_s^{u'_{ab}},\lambda_s)\,\mathrm{d}s \le \Phi(t,v) + \underbrace{\int_t^T \Phi_v(s,V_s^{u'_{ab}})\,(\Pi_s^P + \beta\,\Pi_s^S)\sigma^P\,\mathrm{d}W_s^P + \int_t^T \Phi_v(s,V_s^{u'_{ab}})\,\Pi_s^S\sigma^\star(s,V_s^{u'_{ab}},\lambda_s)\,\mathrm{d}W_s}_{=:M_T^t} . \tag{2.28}$$

Now recall $\Phi_v(t,v) = \eta\,f(t)\,e^{-\eta\,v}$ and calculate the quadratic variation of M^t:

$$\begin{aligned} \langle M^t\rangle_T &= \int_t^T \eta^2\,e^{-2\eta\,V_s^{u'_{ab}}} f^2(s)\left([\Pi_s + \beta\,\Pi_s^S]^2(\sigma^P)^2 + [\sigma_s^\star\Pi_s^S]^2\right)\,\mathrm{d}s \\ &\le \frac{\epsilon^{\frac{\epsilon}{1+\epsilon}}}{1+\epsilon}\eta^2 \sup_{0\le s\le T} f(s)^2\Big[\int_t^T e^{-2\,\eta\,V_s^{u'_{ab}}\,\left(1+\frac{1}{\epsilon}\right)}\,\mathrm{d}s \\ &\qquad + \int_t^T \left([\Pi_s + \beta\,\Pi_s^S]^2(\sigma^P)^2 + [\sigma_s^\star\Pi_s^S]^2\right)^{1+\epsilon}\,\mathrm{d}s\Big] \ , \ \epsilon > 0 \ , \end{aligned} \tag{2.29}$$

where the upper bound in the second line was achieved using inequality (A.1.2) given in Lemma 0.1.1 setting

$$x := e^{-2\,\eta\,V_s^{u'_{ab}}} \text{ and } y := \left([\Pi_s^P + \beta\,\Pi_s^S]^2(\sigma^P)^2 + [\sigma_s^\star\Pi_s^S]^2\right) \ .$$

We show that M^t is a martingale by deriving the integrability of the quadratic variation $\langle M^t\rangle_T$. First we use once more that f is a continuous function on the compact set $[0,T]$ and is uniformly bounded, and thus $\sup_{0\le s\le T} f(s)^2$ is finite. We are left to deal with the two expressions in the brackets of (2.29). The second expression is bounded in expectation by assumption, see (2.8) in Def. 2.1.3, setting $\epsilon = \frac{1}{2}\tilde{\epsilon}$. In what follows we establish that the first expression is finite by showing that

$$\mathbb{E}^{t,v}[e^{\xi\,V_s^{u'_{ab}}}] < \infty \quad \text{uniformly} \ , \tag{2.30}$$

with $\xi = -2\,\eta\,\left(1 + \frac{1}{\epsilon}\right)$, where $\xi < 0$ since $\eta > 0$ and we note that $|\xi| < \infty$ since $\epsilon > 0$.

The solution of the inhomogeneous wealth equation (2.26) starting at t with initial wealth $v = V_t^{u'_{ab}}$ applying variation of constants is

$$V_s^{u'_{ab}} = v + \int_t^s \left((\Pi_{\tilde{s}}^P + \beta\, \Pi_{\tilde{s}}^S)\, \mu^P + \Pi_{\tilde{s}}^S \lambda_{\tilde{s}} \sigma_{\tilde{s}}^\star \right) \mathrm{d}\tilde{s} \\ + \int_t^s (\Pi_{\tilde{s}}^P + \beta\, \Pi_{\tilde{s}}^S)\, \sigma^P \, \mathrm{d}W_{\tilde{s}}^P + \int_t^s \Pi_{\tilde{s}}^S\, \sigma_{\tilde{s}}^\star \, \mathrm{d}W_{\tilde{s}} \,.$$

Then

$$e^{\xi\, V_s^{u'_{ab}}} = e^{\xi\, v} \times \underbrace{e^{\xi \int_t^s \left((\Pi_{\tilde{s}}^P + \beta\, \Pi_{\tilde{s}}^S)\, \mu^P + \Pi_{\tilde{s}}^S \lambda_{\tilde{s}} \sigma_{\tilde{s}}^\star \right) \mathrm{d}\tilde{s}}}_{=:R_s^t} \\ \times \underbrace{e^{\xi \left(\int_t^s (\Pi_{\tilde{s}}^P + \beta\, \Pi_{\tilde{s}}^S)\, \sigma^P \, \mathrm{d}W_{\tilde{s}}^P + \int_t^s \Pi_{\tilde{s}}^S\, \sigma_{\tilde{s}}^\star \, \mathrm{d}W_{\tilde{s}} \right)}}_{=:Z_s^t} \,.$$

Thus, condition (2.30) is for example fulfilled when

$$\mathbb{E}^{t,v}[(R_s^t)^2] < \infty \quad \text{and} \quad \mathbb{E}^{t,v}[(Z_s^t)^2] < \infty \,.$$

The square of R_s^t is given by

$$(R_s^t)^2 = e^{2\,\xi \int_t^s \left((\Pi_{\tilde{s}}^P + \beta\, \Pi_{\tilde{s}}^S)\, \mu^P + \Pi_{\tilde{s}}^S \lambda_{\tilde{s}} \sigma_{\tilde{s}}^\star \right) \mathrm{d}\tilde{s}} \,,$$

which is uniformly bounded by a constant, see Def. 2.1.3, (2.8) and (2.9), and noting that $2\,\xi < 0$ for $\eta > 0$ and $|\xi| < \infty$ since $\epsilon > 0$. This directly implies the square integrability of R^t. With

$$L_s^t := \int_t^s (\Pi_{\tilde{s}}^P + \beta\, \Pi_{\tilde{s}}^S)\, \sigma^P \, \mathrm{d}W_{\tilde{s}}^P + \int_t^s \Pi_{\tilde{s}}^S\, \sigma_{\tilde{s}}^\star \, \mathrm{d}W_{\tilde{s}} \,,$$

and

$$\left\langle L^t \right\rangle_s := \int_t^s \left((\Pi_{\tilde{s}}^P + \beta\, \Pi_{\tilde{s}}^S)^2\, (\sigma^P)^2 + (\Pi_{\tilde{s}}^S\, \sigma_{\tilde{s}}^\star)^2 \right) \mathrm{d}\tilde{s} \,,$$

the square of Z_s^t is given by

$$(Z_s^t)^2 = e^{2\,\xi\, L_s^t} = e^{2\,\xi\, L_s^t - (2\,\xi)^2 \left\langle L^t \right\rangle_s} \times e^{+(2\,\xi)^2 \left\langle L^t \right\rangle_s} \\ = e^{2\,\xi\, L_s^t - 4\,\xi^2 \left\langle L^t \right\rangle_s} \times e^{4\,\xi^2 \left\langle L^t \right\rangle_s} \\ \leq \frac{1}{2} \Big[\underbrace{e^{4\,\xi\, L_s^t - \frac{1}{2}\, 16\, \xi^2 \left\langle L^t \right\rangle_s}}_{=:\tilde{Z}_s^t} + \underbrace{e^{8\,\xi^2\, \left\langle L^t \right\rangle_s}}_{=:\tilde{R}_s^t} \Big] \,,$$

where the last line is a straight forward upper bound.

The second factor $\tilde{R}^t$ is uniformly bounded by a constant by condition (2.8) of Def. 2.1.3, again noting that $|\xi| < \infty$ since $\epsilon > 0$. To finally obtain the square integrability of Z^t, it remains to prove that the first factor $\tilde{Z}^t_s = e^{4\,\xi\,L^t_s - \frac{1}{2}\,16\,\xi^2\,\langle L^t\rangle_s}$, $t \le s \le T$, is integrable. However, $\tilde{Z}^t$ is a strictly positive local martingale since it is the stochastic exponential of the local martingale $4\,\xi\,L^t$. The Novikov condition holds by (2.8), i.e. $\mathbb{E}^{t,v}(e^{\frac{1}{2}\,16\,\xi^2\langle L^t\rangle_T}) < \infty$, and hence $\tilde{Z}^t$ is a true martingale and $\mathbb{E}^{t,v}(\tilde{Z}^t_s) = 1$, $t \le s \le T$. In summary, the local martingale M^t is therefore a martingale vanishing in expectation in (2.28), and taking the conditional expectation of (2.28) gives the desired result

$$J(t,v;u'_{ab}) = \mathbb{E}^{t,v}\left[U(V_T^{u'_{ab}}) - \int_t^T c^\star(s,V_s^{u'_{ab}},\lambda_s)\,\mathrm{d}s\right] \le \Phi(t,v)\,, u'_{ab} \in A'_\eta(t,v).$$

And the proof is finished. □

2.3 Discussion and Implications of Results

Theorems 2.2.1 and 2.2.2 indicate our unconstrained executive's maximized utility and associated optimal behavior in terms of personal portfolio selection and choice of work effort, subject to the constant absolute risk aversion set-up. We proceed as in Chapter 1; we investigate the sensitivity of this optimal behavior to variation of the executive's risk aversion and work effectiveness characteristics. Again the utility indifference rationale (1.41) is applied to determine the fair compensation of an executive characterized by the exponential utility/disutility set-up.

Here, the executive is characterized by the constant absolute risk aversion coefficient ($\eta > 0$) and the two work effectiveness parameters work productivity ($1/\kappa$, with $\kappa > 0$), and disutility stress ($\alpha > 2$). We repeat that in order to produce results that have relativity to a base-level of work effort, as indicated by a base-level non-systematic Sharpe ratio control decision $\lambda_0 > 0$, the disutility $c^\star$ given by (2.12) is reparameterized by choosing

$$\tilde{\kappa} := \kappa \, (\lambda_0)^\alpha \, . \tag{2.31}$$

Then the CARA utility/disutility set-up becomes

$$U(v) = 1 - e^{-\eta \, v} \, , \quad \text{for } \eta > 0 \, ,$$

and

$$c^\star(t,v,\lambda) = \frac{\tilde{\kappa}}{\alpha} \, e^{-\eta \, v} \left(\frac{\lambda}{\lambda_0} \right)^\alpha \, , \quad \text{for } \lambda \geq 0 \, , \quad \eta > 0 \, .$$

Analogously to the constant relative risk aversion case, regarding the executive's optimal personal investment decisions $\Pi^\star$, the optimal own-company stockholding $\Pi^{S^\star}$ is a function of the optimal work effort choice, and the associated optimal volatility $\sigma^\star$ (see Lemma 1.1.1) which we do not explicitly specify. The optimal market portfolio allocation $\Pi^{P^\star}$ considered in conjunction with the systematic risk exposure associated with $\Pi^{S^\star}$ coincides with the results from classical utility maximization in the constant absolute risk aversion setting w.r.t. an exponnetial utility function, and is therefore of limited interest.

We insvestigate the relationship between the executive's optimal work effort/control choice $\lambda^\star$, his characteristics $1/\tilde{\kappa}$ and α, and his utility indifference compensation, for the exponential-utility case.

Note that the optimal work effort $\lambda^\star$ does not depend on the risk aversion parameter η in contrast to the CRRA set-up, which is caused by assuming $r = 0$ in equation (2.24) in the proof of Theorem 2.2.1 in order to make the separation approach work. So we have to limit the graphical representations to the behaviour w.r.t. the work effectiveness parameters and time.

After the reparametrization, the executive's optimal choice of work effort is

$$\lambda^\star(t) = \lambda_0^{\frac{\alpha}{\alpha-2}} \left(\frac{1}{\tilde{\kappa}}\right)^{\frac{1}{\alpha-2}} f(t)^{\frac{1}{\alpha-2}},$$

with

$$f(t) = e^{-\frac{1}{2}(\mu_P^2/\sigma^P)^2 (T-t)} \times \left(1 - \frac{(\alpha-2)(\sigma^P)^2 \left(\frac{\lambda_0^\alpha}{\tilde{\kappa}}\right)^{\frac{2}{\alpha-2}}}{2\alpha (\mu^P)^2} \left(e^{-\frac{1}{\alpha-2}(\mu^P/\sigma^P)^2 (T-t)} - 1\right)\right)^{-\frac{\alpha-2}{2}},$$

(see Theorem 2.2.1 for the optimal choice under the original disutility parameterization).

Again we want to ensure that the optimal work effort is not less than the base-level. In the exponential case we have to assume that the work productivity fulfills

$$1/\tilde{\kappa} > \lambda_0^{-2} f(0)^{-1}.$$

This condition follows from the properties of the function f given by the equation above, where we keep in mind that $\eta > 0$ and from the fact that f is the solution of a Bernoulli ODE, which is increasing in time for $\eta > 0$ (therefore see equation (2.25) in the proof of Theorem 2.2.1).

The following Proposition gives the fair compensation rate of the executive who is characterized by the CARA utility/disutility set-up.

Proposition 2.3.1
Using the indifference utility argument (1.41)*, the exponential-utility executive's utility indifference (fair) up-front compensation* Δv *is*

$$\Delta v = \frac{1}{2\eta} \int_0^T (\lambda^*(s))^2 \mathrm{d}s + \frac{\alpha-2}{\alpha\eta} \log\left(1 - \frac{(\alpha-2)(\sigma^P)^2 \left(\frac{\lambda_0^\alpha}{\tilde{\kappa}}\right)^{\frac{2}{\alpha-2}}}{2\alpha(\mu^P)^2} \left(e^{-\frac{1}{\alpha-2}(\mu^P/\sigma^P)^2 T} - 1\right)\right).$$

Proof. Applying the indifference utility argument (1.41) and the same argumentation as in the proof of Proposition 1.3.2, we obtain Δv by solving

$$\widehat{\Phi}(0,v) = \Phi(0,v + \Delta v)\,, \tag{2.32}$$

where $\widehat{\Phi}(0,v)$ denotes the maximized utility of an outside investor, who chooses his control vector $\widehat{u'}^{\star}_{ab} = (\widehat{\Pi}^{P^\star},\widehat{\Pi}^{S^\star},\widehat{\lambda}^\star)$ identical to the executive's control vector $u'^{\star}_{ab} = (\Pi^{P^\star},\Pi^{S^\star},\lambda^\star)$. An outside investor with knowledge of the optimal control vector $\widehat{u'}^{\star}$ does not suffer from disutility and is characterized by the following Hamilton-Jacobi-Bellmann equation (where we note that $r = 0$):

$$0 = \widehat{\Phi}_t(t,v) - \frac{1}{2}(\lambda^\star(t))^2\,\frac{\widehat{\Phi}^2_v(t,v)}{\widehat{\Phi}_{vv}(t,v)} - \frac{(\mu^P)^2}{2\,(\sigma^P)^2}\,\frac{\widehat{\Phi}^2_v(t,v)}{\widehat{\Phi}_{vv}(t,v)}\,,$$

where we have set $\lambda^\star(t,v) = \lambda^\star(t)$, since we already know from (2.18) that the optimal work effort does not depend on v.
Applying the ansatz $\widehat{\Phi}(t,v) = \widehat{f}(t)\,e^{-\eta v}$ with $\widehat{f}(T) = 1$ results in the ODE

$$\dot{\widehat{f}} = \left[\frac{1}{2}\,(\lambda^\star(t))^2 + \frac{(\mu^P)^2}{2\,(\sigma^P)^2}\right]\widehat{f} \quad , \quad \widehat{f}(T) = 1\,,$$

which has the solution

$$\widehat{f}(t) = e^{-\frac{(\mu^P)^2}{2\,(\sigma^P)^2}(T-t) - \frac{1}{2}\int_t^T(\lambda^\star(s))^2 \mathrm{d}s}\,.$$

From (2.32) we then get that

$$1 - \widehat{f}(0)\,e^{-\eta v} = 1 - f(0)\,e^{-\eta\,(v+\Delta v)} \quad \Leftrightarrow \quad \Delta v = -\frac{1}{\eta}\log\left(\frac{\widehat{f}(0)}{f(0)}\right).$$

Plugging in the representations of $\widehat{f}$ and f, respectively, and simplifying gives the result. □

Remark 2.3.1
The fair compensation Δv depends in contrast to the optimal work effort $\lambda^\star$ on the risk aversion η, which is caused by the representation of the value

function (compare equation (2.19)*). Further,* Δv *is independent of the initial wealth* v *of the executive. This is a consequence of the CARA set-up and the fact that already the optimal investment strategies* $\Pi^{P^\star}$ *and* $\Pi^{S^\star}$ *are independent of the actual wealth.*

As in the power-utility case, the sensitivities of the executive's optimal work effort $\lambda^\star$ and fair compensation Δv with respect to variations in his work effectiveness parameters cannot be shown with compact expressions. Instead we limit ourselves to graphical representations of the relationships, with additional consideration of the executive's parameter η of constant *absolute* risk aversion for the fair compensation.

Figures 2.1 and 2.2 show optimal work effort over time for varying work effectiveness parameters (i.e. respectively $\lambda^\star$ versus t and $1/\tilde{\kappa}$, and $\lambda^\star$ versus t and α). The executive's work effort increases with increasing work productivity and with increasing time (see Figure 2.1), and the executive's work effort increases with decreasing disutility stress and increasing time (see Figure 2.2), i.e. work effort is positively related to work productivity $1/\tilde{\kappa}$, and negatively related to disutility stress α. Given that the optimal work effort does not depend on the risk aversion η, it is an (at first sight) unexpected fact that the optimal work effort increases with time, since we know from the power-utility case that the optimal work effort of the power-utility executive does only increase in time for a rather high level of the relative risk aversion parameter γ. This can be interpreted as the exponential-utility executive with zero interest rates is in general of a risk-averse nature, which may stem from the fact that in a financial market with zero interest rates a loss in a risky asset would cause more damage to the executive than in an environment with higher interest rates, since the investment in the money market account will deliver no return when $r = 0$.

Figures 2.3 and 2.4 show the exponential utility executive's fair compensation versus pairings of risk aversion with each of work productivity and disutility stress (i.e. respectively Δv versus η and $1/\tilde{\kappa}$, and Δv versus α and η). Any combination of decreasing risk aversion, increasing work productivity, and decreasing disutility stress leads to higher work effort and commensurately higher fair compensation. We repeat that the fair compensation in the exponential utility/disutility set-up does not depend on the initial wealth v implying that the illustrations are true for any initial wealth of the executive. This is economically counter-intuitive and another drawback of this set-up. As in the CRRA set-up of Chapter 1, the level of fair compensation is particularly prominently dependent on risk aversion: fair compensation sensitivity to work productivity and disutility stress is highest when risk aversion is low ($\eta \approx 0.5$ or lower), which is emphasized by Figures 2.3 and 2.4. This can be considered as a confirmation of the correctness of the CRRA set-up of Chapter 1.

We summarize that the constant *absolute* risk aversion set-up of this chapter has many drawbacks compared to the constant *relative* risk aversion set-up of Chapter 1. The model itself has some limitations; we are not able to solve the optimal investment and control decision for a general $r \neq 0$ and the disutility of work effort produces decreasing disutility for increasing wealth for all values of $\eta > 0$. Further, the optimal work effort does not depend on the risk aversion η and the fair compensation rate is independent of the initial wealth of the executive. The constant *relative* risk aversion set-up is thus much more likely to produce reality-based results.

2.4 Figures

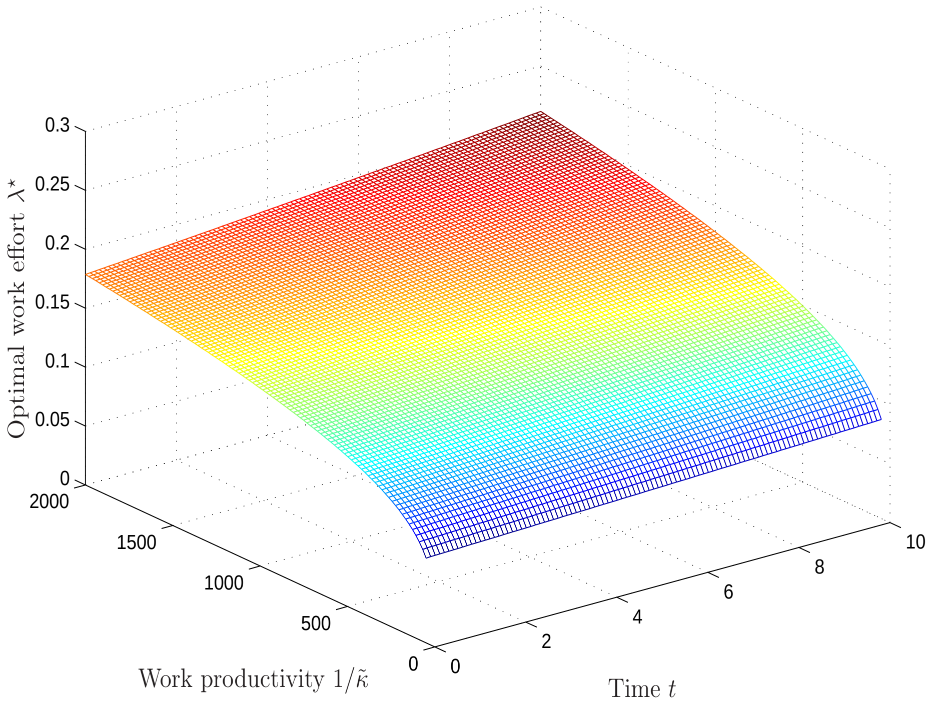

Figure 2.1: The exponential-utility executive's optimal work effort/control choice, in terms of optimal non-systematic Sharpe ratio $\lambda^\star$, w.r.t. time t, for varying work productivity $1/\tilde{\kappa}$; given disutility stress $\alpha = 5$, and base-level work effort $\lambda_0 = 0.10$.

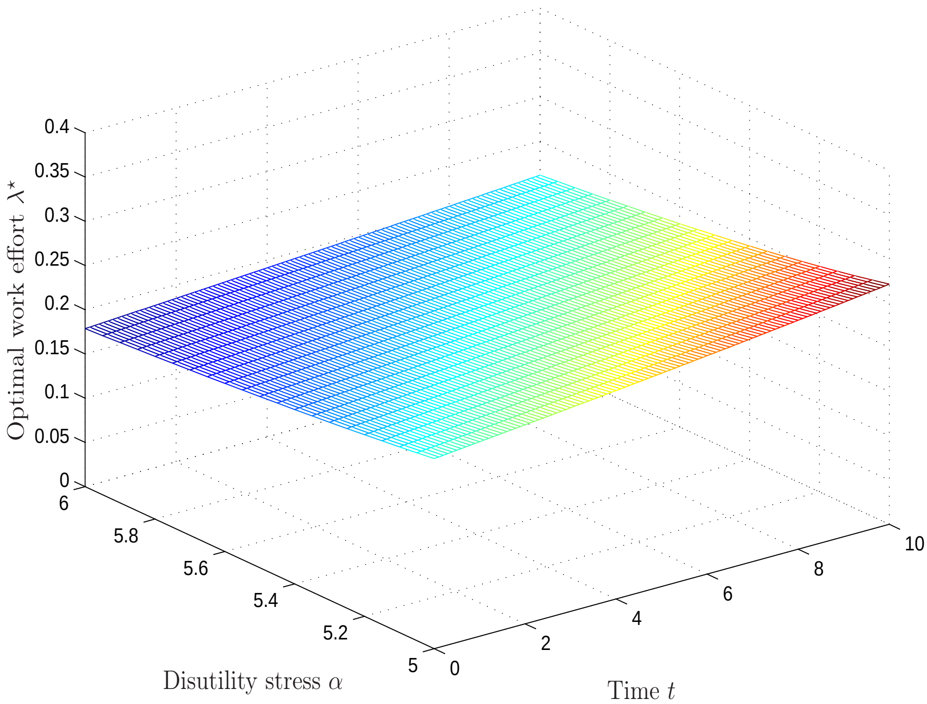

Figure 2.2: The exponential-utility executive's optimal work effort/control choice, in terms of optimal non-systematic Sharpe ratio $\lambda^\star$, w.r.t. time t, for varying disutility stress α; given work productivity $1/\tilde{\kappa} = 2000$, and base-level work effort $\lambda_0 = 0.10$.

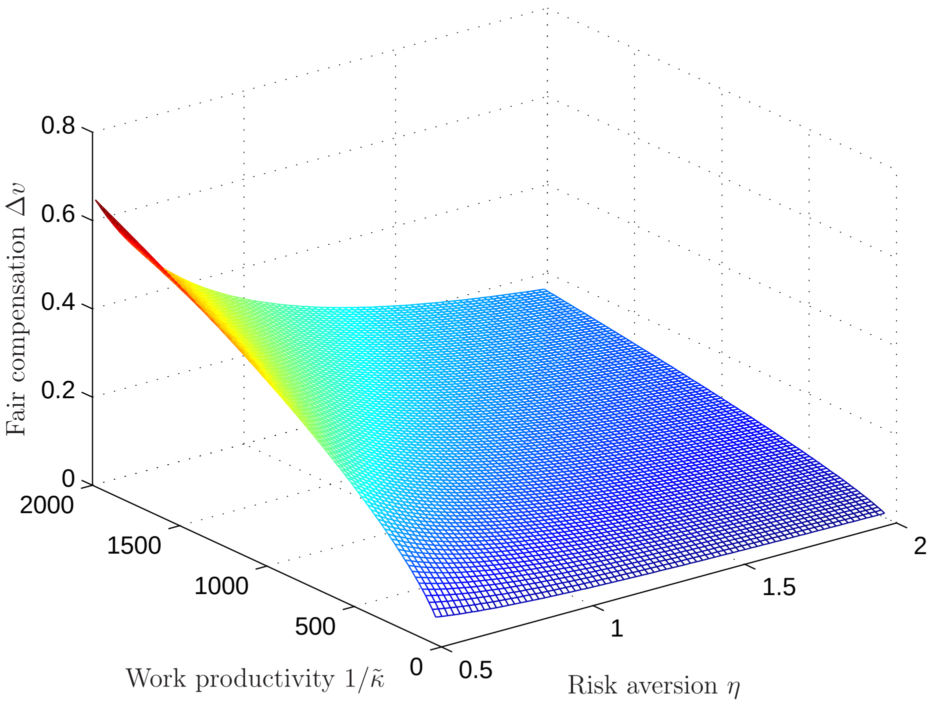

Figure 2.3: The exponential-utility executive's fair up-front compensation Δv, based on utility indifference, w.r.t. his work productivity $1/\tilde{\kappa}$ and risk aversion η; given disutility stress $\alpha = 5$, time horizon $T = 10$ years, and base-level work effort $\lambda_0 = 0.10$.

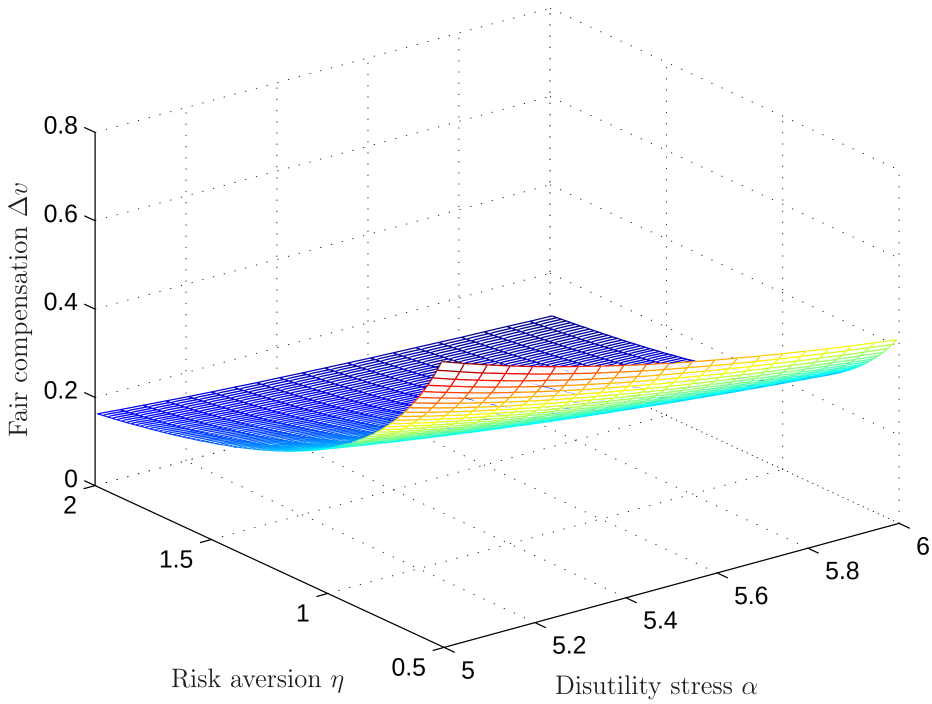

Figure 2.4: The exponential-utility executive's fair up-front compensation Δv, based on utility indifference, w.r.t. his risk aversion η and disutility stress α; given work productivity $1/\tilde{\kappa} = 2000$, time horizon $T = 10$ years, and base-level work effort $\lambda_0 = 0.10$.

Chapter 3

Work Effort, Consumption, and Portfolio Selection: When the Occupational Choice Matters

The framework of this chapter is a modification and extension of the framework in the past chapters. One shortcoming of that set-up is that the executive is subject to instanteneous disutility from work effort, but can benefit from the utility of the investment decision only at the end of the considered time horizon. This shortcoming is compensated by allowing the executive to consume continuously in time and to derive immediate utility from that. Another shortcoming of the past set-up is that the compensation of the executive is assumed to be included in the starting wealth. This shortcoming is resolved by assuming that the executive receives compensation at a deterministic and fix rate relative to his total personal wealth. Further we stick no longer that closely to the notion of an executive who is necessarily based in an ancestral company. We now consider a highly-qualified individual who has the choice between two distinct career paths at the beginning of the time horizon. She can choose between an executive position within a smaller listed company with the possibility to directly influence the own-company's performance and a mid-level management position in a large company without the ability to affect the own-company's performance. The latter possibility is referred to as outside option in the literature. In this chapter we follow the lines of Desmettre and Szimayer (2010).

3.1 Notation and Setup

We consider an individual endowed with given initial wealth. She manages here financial objectives by investing in the financial market and choosing her instantaneous consumption. The individual can also choose the level of work effort she applies.

3.1.1 Financial Market

First we specify the financial market. We are given a filtered probability space $(\Omega, \mathcal{F}, P, (\mathcal{F}_t)_{t\geq 0})$ satisfying the usual hypothesis and large enough to support two independent standard Brownian motions, $W^P = (W_t^P)_{t\geq 0}$ and $W = (W_t)_{t\geq 0}$. The investment opportunities available are a money market account, a diversified market portfolio, and shares of a small listed company making a job offer to the individual.

The risk-free money market account has the price process $B = (B_t)_{t\geq 0}$, with dynamics

$$\mathrm{d}B_t = r\, B_t\, \mathrm{d}t\,, \quad B_0 = 1\,, \tag{3.1}$$

where r is the instantaneous risk-free rate of return, hence $B_t = e^{r\,t}$.

The price process of the market portfolio, $P = (P_t)_{t\geq 0}$, follows the stochastic differential equation (SDE)

$$\mathrm{d}P_t = P_t\,(\mu_P\, \mathrm{d}t + \sigma_P\, \mathrm{d}W_t^P)\,, \quad P_0 \in \mathbb{R}^+\,, \tag{3.2}$$

where $\mu_P \in \mathbb{R}$ and $\sigma_P > 0$ are respectively the expected return rate and volatility of the market portfolio. The corresponding Sharpe ratio is then $\lambda_P = (\mu_P - r)/\sigma_P$.

The company's stock price process, $S^u = (S_t^u)_{t\geq 0}$, is a controlled diffusion with SDE

$$\mathrm{d}S_t^u = S_t^u \left([r + \lambda_t \, \sigma] \, \mathrm{d}t + \beta \left[\frac{\mathrm{d}P_t}{P_t} - r\mathrm{d}t \right] + \sigma \, \mathrm{d}W_t \right) , \quad S_0^u \in \mathbb{R}^+ , \tag{3.3}$$

where $\mu = r + \lambda \, \sigma$ is the company's expected return rate in excess of the beta-adjusted market portfolio's expected excess return rate (i.e. the expected return compensation for non-systematic risk), σ is the company's non-systematic volatility, and $\lambda = (\lambda_t)_{t \geq 0}$ is a control process collected in the control vector process u that will be specified below.

3.1.2 Controls and Wealth Process

The individual is endowed with the initial wealth $V_0 > 0$. She receives an instantaneous salary proportional to her current wealth at a relative rate δ. For an exogenously given time horizon, $T > 0$, the individual seeks to maximize her total utility by controlling the portfolio holdings, consumption, and work effort.

The portfolio is determined by a self-financing trading strategy given by the bivariate control process $\pi = (\pi^P, \pi^S)$, where $\pi^P = (\pi_t^P)_{t \geq 0}$ is the fraction of wealth invested in the market portfolio and $\pi^S = (\pi_t^S)_{t \geq 0}$ is the fraction of wealth invested in the company's stock. The remainder in the risk-free account, that is, the strategy is self-financing. The individual consumes instantaneously at the relative rate $k = (k_t)_{t \geq 0}$ proportional to the wealth V_t^π at time t, where $k_t \geq 0$, leading to a total consumption rate $k_t \, V_t^\pi$. Further, she influences the small company's stock price dynamics by choice of the control strategy $\lambda = (\lambda_t)_{t \geq 0}$, which is specified to be associated with work effort. The control strategy can be conceptualized as deriving from the individual's corporate investment. For example, identifying and initiating positive net present value projects. Value is added if $\mu = r + \lambda \, \sigma$ is greater than r, indicating excess return compensation for non-systematic risk. To ensure sensible solutions we require $\lambda \geq 0$, which effectively bars her from destroying company value ($\lambda < 0$) and potentially profiting by shorting the company's stock. All controls are collected in the vector process $u = (\pi^P,\pi^S,k,\lambda)$.

For a fixed salary rate δ, initial wealth $V_0 > 0$, and a control strategy u, the wealth process, $V^u = (V_t^u)_{t\geq 0}$, with starting value $V_0^u = V_0$ is given by

$$\mathrm{d}V_t^u = V_t^u \left(\left[1 - \pi_t^P - \pi_t^S\right] \frac{\mathrm{d}B_t}{B_t} + \pi_t^P \frac{\mathrm{d}P_t}{P_t} + \pi_t^S \frac{\mathrm{d}S_t^u}{S_t^u} + \delta \,\mathrm{d}t - k_t \,\mathrm{d}t \right), t \geq 0\,. \tag{3.4}$$

The above equation can be rewritten as follows

$$\begin{aligned} \mathrm{d}V_t^u = V_t^u \Big(& \left[r + \delta - k_t + (\pi_t^P + \beta\, \pi_t^S)\, \lambda_P\, \sigma_P + \pi_t^S\, \lambda_t\, \sigma\right] \mathrm{d}t \\ & + \left[\pi_t^P + \beta\, \pi_t^S\right] \sigma_P \,\mathrm{d}W_t^P + \pi_t^S\, \sigma \,\mathrm{d}W_t \Big), t \geq 0\,. \end{aligned} \tag{3.5}$$

3.1.3 Stochastic Control Problem

The individual is assumed to maximize the expected value of the terminal utility of her wealth for time horizon T, subject to some utility function U_1 and her consumption rate over the time period $[t,T]$, subject to some utility function U_2. The disutility for work effort is quantified by the cost function C. Both utility functions and the cost function will be specified when deriving closed-form solutions.

Assuming control of the company's stock price behavior λ is determined exogenously and comes at zero cost, the individual's *optimal investment and consumption decision* is then described by

$$\widehat{\Phi}(t,v) = \sup_{(\pi,k)\in\Pi(t,v)} \mathbb{E}^{t,v} \left[U_1(V_T^{(\pi,k)}) + \int_t^T U_2(s,V_s^{(\pi,k)},k_s)\,\mathrm{d}s \right], \tag{3.6}$$

for $(t,v) \in [0,T] \times \mathbb{R}^+$, where $\Pi(t,v)$ denotes the set of all admissible portfolio processes (π,k) at time t corresponding to portfolio value (i.e. wealth)

$v = V_t > 0$ (see for example Korn and Korn (2001)), U_1 and U_2 are utility functions, and $\mathbb{E}^{t,v}$ denotes the expectation conditional on t and v.

The *optimal investment and consumption control decision including work effort* is then the solution of

$$\Phi(t,v) = \sup_{u \in A(t,v)} \mathbb{E}^{t,v}\left[U_1(V_T^u) + \int_t^T U_2(s,V_s^u,k_s)\,\mathrm{d}s - \int_t^T C(s,V_s^u,\lambda_s)\,\mathrm{d}s \right], \tag{3.7}$$

for $(t,v) \in [0,T] \times \mathbb{R}^+$. The set of admissible strategies for the maximization $A(t,v)$ problem is made precise in the following definition.

Definition 3.1.1
Fix $(t,v) \in [0,T] \times \mathbb{R}^+$, then $u = \left(\pi^P,\pi^S,k,\lambda\right)$ is in the set of admissible strategies $A(t,v)$, if and only if u is an $\{\mathcal{F}_s\,;\, t \leq s \leq T\}$-predictable processes, such that

(i) the stock price equation

$$\mathrm{d}S_s^u = S_s^u \left([r + \lambda_s\,\sigma]\,\mathrm{d}s + \beta\left[\frac{\mathrm{d}P_s}{P_s} - r\mathrm{d}s\right] + \sigma\,\mathrm{d}W_s \right),$$

with initial condition $S_t^u \in \mathbb{R}^+$ admits a non-negative solution and

$$\int_t^T (S_s^u)^2\left(\sigma^2 + \beta^2\,\sigma_P^2\right)\,\mathrm{d}s \,<\, \infty \quad P-a.s.\,;$$

(ii) the wealth equation

$$\mathrm{d}V_s^u = V_s^u \left(\left[1 - \pi_s^P - \pi_s^S\right]\frac{\mathrm{d}B_s}{B_s} + \pi_s^P\frac{\mathrm{d}P_s}{P_s} + \pi_s^S\frac{\mathrm{d}S_s^u}{S_s^u} + \delta\,\mathrm{d}s - k_s\,\mathrm{d}s \right),$$

with initial condition $V_t^u = v$ has a unique non-negative solution and

$$\int_t^T (V_s^u)^2\left(([\pi_s^P + \beta\,\pi_s^S]\,\sigma_P)^2 + (\pi_s^S\sigma)^2\right)\,\mathrm{d}s \,<\, \infty \quad P-a.s.\,;$$

(iii) and the utility of wealth and consumption, and the disutility of control satisfy

$$\mathbb{E}\left[U_1(V_T^u)^- + \int_t^T U_2(s,V_s^u,k_s)^-\,\mathrm{d}s + \int_t^T C(s,V_s^u,\lambda_s)\,\mathrm{d}s \right] \,<\, \infty\,.$$

3.1.4 Outside Option

The individual can choose between two job offers at $t = 0$. As an alternative to taking on the executive position with the company with share price S^u, she can pursue her outside option and decide to work for a large company in a mid-management position paying a salary at rate $\widehat{\delta}$. In the latter case she cannot affect the stock price process any longer and hence $\widehat{\lambda} = 0$. The classical optimal investment and consumption decision applies.

Assume that portfolio process follows Eq. (3.5) where we set $\delta = \widehat{\delta}$ and $\lambda = \widehat{\lambda} = 0$. Then the optimal investment decision problem in Equation (3.6) determines the value of the outside option $\widehat{\Phi}(0,V_0)$ at time $t = 0$ for initial wealth $V_0 > 0$.

3.2 Optimal Strategies

In this section we use stochastic control techniques to derive closed-form solutions to the control problem in (3.7). Our main focus is placed on the log utility specification for utility from terminal wealth and consumption and disutility that is a power function of work effort applied. In addition, we also discuss the general constant relative risk aversion specification.

3.2.1 Hamilton-Jacobi-Bellman Equation

Having formulated the optimal investment and control decision problem including consumption with respect to the parameter set $u = (\pi^P,\pi^S,k,\lambda)$ as given by (3.7), we can write down the corresponding Hamilton-Jacobi-Bellman equation. Note that we formulate this equation with respect to a general utility functions U_1 and U_2 and a general cost function C. For

$(t,v) \in [0,T) \times \mathbb{R}^+$ we have

$$\frac{\partial \Phi}{\partial t}(t,v) + \sup_{u \in \mathcal{U}} \left[(L^u \Phi)(t,v) + U_2(t,v,k) - C(t,v,\lambda) \right] = 0\,, \tag{3.8}$$

with terminal condition $\Phi(T,v) = U_1(v)$, for $v \in \mathbb{R}^+$, where $\mathcal{U} = \mathbb{R}^2 \times [0,\infty)^2$ and the differential operator L^u is defined by

$$\begin{aligned}
&(L^u g)(t,v) = \\
&\frac{\partial g}{\partial v}(t,v)\, v \left(r + \delta + \pi^S(t,v)\, \lambda(t,v)\, \sigma + \left[\pi^P(t,v) + \beta\, \pi^S(t,v) \right] \lambda_P\, \sigma_P - k(t,v) \right) \\
&+ \frac{1}{2} \frac{\partial^2 g}{\partial v^2}(t,v)\, v^2 \left([\pi^S(t,v)\, \sigma]^2 + [\pi^P(t,v)\, \sigma_P + \beta\, \pi^S(t,v)\, \sigma_P]^2 \right)\ .
\end{aligned} \tag{3.9}$$

Potential maximizers $\pi^{P^\star}$, $\pi^{S^\star}$, $k^\star$ and $\lambda^\star$ of the HJB (3.8) can be calculated by establishing the first order conditions:

$$\begin{aligned}
\pi^{P^\star}(t,v) &= -\frac{\lambda_P}{v\, \sigma^P} \frac{\Phi_v(t,v)}{\Phi_{vv}(t,v)} - \beta\, \pi^{S^\star}(t,v)\,, \\
\pi^{S^\star}(t,v) &= -\frac{\lambda^\star(t,v)}{v\, \sigma} \frac{\Phi_v(t,v)}{\Phi_{vv}(t,v)}\,,
\end{aligned} \tag{3.10}$$

and $\lambda^\star$ is the solution of the implicit equation

$$\lambda \frac{\Phi_v^2(t,v)}{\Phi_{vv}(t,v)} + \frac{\partial C}{\partial \lambda}(t,v,\lambda) = 0\,, \quad \text{for all } (t,v) \in [0,T] \times \mathbb{R}^+\,, \tag{3.11}$$

where we have already used (3.10) to simplify the equation, and $k^\star$ is the solution of the equation

$$\frac{\partial U_2}{\partial k}(t,v,k) - v\, \Phi_v(t,v) = 0. \tag{3.12}$$

Substituting the maximizers (3.10) in the HJB (3.8) yields:

$$\begin{aligned}
\Phi_t(t,v) + \Phi_v(t,v)\, v\, (r + \delta - k^\star(t,v)) - \frac{1}{2}(\lambda^\star(t,v))^2 \frac{\Phi_v^2(t,v)}{\Phi_{vv}(t,v)} \\
- \frac{1}{2} \lambda_P^2 \frac{\Phi_v^2(t,v)}{\Phi_{vv}(t,v)} + U_2(t,k^\star(t,v)) - C(t,v,\lambda^\star(t,v)) = 0\,.
\end{aligned} \tag{3.13}$$

In the following we solve (3.13) with particular choices for the utility and disutility functions.

3.2.2 Closed-Form Solution for the Log-Utility Case

We specify the utility functions to be of log-utility type, belonging to the constant relative risk aversion class. The utility function of the final wealth U_1 is

$$U_1(v) = K\,\log(v)\,, \quad \text{for } v \in \mathbb{R}^+\,, \tag{3.14}$$

for a constant $K > 0$, the utility function of consumption U_2 is

$$U_2(t,k,v) = e^{-\rho\, t}\,\log(v\,k)\,, \quad \text{for } (t,v,k) \in [0,T] \times \mathbb{R}^+ \times \mathbb{R}_0^+\,, \tag{3.15}$$

where $\rho \in \mathbb{R}$ parametrizes the time preference, and the cost function of work effort C is

$$C(t,v,\lambda) = e^{-\tilde{\rho}\, t}\,\kappa\,\frac{\lambda^\alpha}{\alpha}\,, \quad \text{for } (t,v,\lambda) \in [0,T] \times \mathbb{R}^+ \times \mathbb{R}_0^+\,, \tag{3.16}$$

where $\kappa > 0$ and $\alpha > 2$ are the individual's work effectiveness parameters, respectively termed 'inverse work productivity' and 'disutility stress', and $\tilde{\rho} \in \mathbb{R}$ is a time preference parameter. The constant κ directly relates her work effort disutility to the quality of her control decision as indicated by the non-systematic Sharpe ratio λ, and α indicates how rapidly her work effort disutility will rise for the sake of an improved λ. The requirement $\alpha > 2$ is a consequence of our set-up that ensures the executive's disutility grows with work effort, i.e. λ, at a rate that offsets (at some level of λ) the rate of her utility gain due to the flow-on from her work effort to the value of her own-company stockholding; this becomes evident with derivation of the solution to (3.7). A higher quality individual is able to achieve a given λ with lower disutility, and is able to improve λ with lower incremental disutility. That is, higher individual quality (i.e. higher work effectiveness) is implied by lower values of κ and α.

For the remainder of the chapter we assume that the optimal investment and control problem (3.7) admits a value function $\Phi \in C^{1,2}$. To guarantee that the candidates we will derive for the executive's optimal investment and control strategy (i.e. the choices for own-company stockholding, market portfolio holding and non-systematic Sharpe ratio) and value function are indeed optimal, we have to consider a more restrictive class of admissible strategies as follows.

Definition 3.2.1

Fix $(t,v) \in [0,T] \times \mathbb{R}^+$. Then by $A'(t,v)$ we denote the set of admissible strategies $u \in A'(t,v)$, such that $u \in A(t,v)$ and

$$\mathbb{E}\left[\int_t^T \left(\pi_s^P + \beta\,\pi_s^S\right)^2 (\sigma_P)^2 + \left(\pi_s^S \sigma\right)^2 \,\mathrm{d}s\right] < \infty\,, \tag{3.17}$$

Restating the optimal investment and control problem:

$$\Phi(t,v) = \sup_{u \in A'(t,v)} \mathbb{E}^{t,v}\left[U_1(V_T^u) + \int_t^T U_2(s,V_s^u,k_s)\,\mathrm{d}s - \int_t^T C(s,V_s^u,\lambda_s)\,\mathrm{d}s\right], \tag{3.18}$$

for $(t,v) \in [0,T] \times \mathbb{R}^+$.

A closed-form solution is obtained for the optimal investment and control problem in (3.18) using the utility and disutility functions (3.14), (3.15) and (3.16).

Theorem 3.2.1

The full solution of the maximization problem (3.18) can be summarized by the strategy

$$\begin{aligned} \pi^{P\star}(t,v) &= \frac{\lambda_P}{\sigma_P} - \beta\,\pi^{S\star}(t,v)\,, \quad \pi^{S\star}(t,v) = \frac{\lambda^\star(t,v)}{\sigma}\,, \\ \lambda^\star(t,v) &= \left(\frac{e^{\tilde{\rho}t}}{\kappa}\,f(t)\right)^{\frac{1}{\alpha-2}}\,, \quad k^\star(t,v) = \frac{e^{-\rho t}}{f(t)}\,, \end{aligned} \tag{3.19}$$

and value function

$$\Phi(t,v) = f(t)\,\log(v) + g(t)\,, \tag{3.20}$$

with

$$f(t) = \begin{cases} K + \frac{e^{-\rho t} - e^{-\rho T}}{\rho}\,, & \textit{for } \rho \neq 0\,, \\ K + T - t\,, & \textit{for } \rho = 0\,, \end{cases} \tag{3.21}$$

and

$$g(t) = \left(r + \delta + \frac{1}{2}\,\lambda_P^2\right)\int_t^T f(s)\,\mathrm{d}s + \frac{\alpha-2}{2\,\alpha}\int_t^T \left(\frac{e^{\tilde{\rho}\,s}}{\kappa}\right)^{\frac{2}{\alpha-2}} f(s)^{\frac{\alpha}{\alpha-2}}\,\mathrm{d}s$$
$$- \int_t^T (1+\rho\,s)\,e^{-\rho\,s}\,\mathrm{d}s - \int_t^T e^{-\rho\,s}\,\log(f(s))\,\mathrm{d}s\,. \tag{3.22}$$

Proof. First observe that a function F of the form $F(\lambda) = a\,\lambda^2 - b\,\lambda^\alpha$, $\lambda \geq 0$, for given constants $a, b > 0$ and $\alpha > 2$, has a unique maximizer $\lambda^\star$ and maximized value $F(\lambda^\star)$ given by

$$\lambda^\star = \left(\frac{2\,a}{\alpha\,b}\right)^{\frac{1}{\alpha-2}}\,, \quad \text{and} \quad F(\lambda^\star) = (\alpha-2)\,\alpha^{-\frac{\alpha}{\alpha-2}}\,2^{\frac{2}{\alpha-2}}\,a^{\frac{\alpha}{\alpha-2}}\,b^{-\frac{2}{\alpha-2}}\,. \tag{3.23}$$

Using this insight, the first order condition for $\lambda^\star$ in (3.11) is now solved. Set

$$a = \frac{1}{2}\,\frac{\Phi_v^2}{-\Phi_{vv}}\,, \quad \text{and} \quad b = e^{-\tilde{\rho}\,t}\,\frac{\kappa}{\alpha}\,,$$

then (3.23) gives

$$\lambda^\star = \left(\frac{e^{\tilde{\rho}\,t}}{\kappa}\,\frac{\Phi_v^2}{-\Phi_{vv}}\right)^{\frac{1}{\alpha-2}}\,, \text{ and } F(\lambda^\star) = \frac{\alpha-2}{2\,\alpha}\left(\frac{e^{\tilde{\rho}\,t}}{\kappa}\right)^{\frac{2}{\alpha-2}}\left(\frac{\Phi_v^2}{-\Phi_{vv}}\right)^{\frac{\alpha}{\alpha-2}}\,. \tag{3.24}$$

Having specified the utility function U_2 of the consumption rate as $U_2(t,v,k) = e^{-\rho\,t}\,\log(v\,k)$, we can also solve the first order condition for the optimal consumption rate. Equation (3.12) then gives:

$$k^\star = \frac{e^{-\rho\,t}}{v\,\Phi_v}\,. \tag{3.25}$$

Substituting $\lambda^\star$ and $k^\star$ in equation (3.13) we get:

$$0 = \Phi_t + \Phi_v\,v\,(r+\delta) + \frac{1}{2}\,\lambda_P^2\,\frac{\Phi_v^2}{-\Phi_{vv}} + \frac{\alpha-2}{2\,\alpha}\left(\frac{e^{\tilde{\rho}\,t}}{\kappa}\right)^{\frac{2}{\alpha-2}}\left(\frac{\Phi_v^2}{-\Phi_{vv}}\right)^{\frac{\alpha}{\alpha-2}}$$
$$- e^{-\rho\,t} - \rho\,t\,e^{-\rho\,t} - e^{-\rho\,t}\,\log(\Phi_v)\,. \tag{3.26}$$

Using the ansatz $\Phi(t,v) = \log(v)\, f(t) + g(t)$ with $f(T) = K$ and $g(T) = 0$ results in

$$\Phi_t = \log(v)\dot{f}(t) + \dot{g}(t)\,, \quad \Phi_v = \frac{1}{v}\, f(t)\,, \quad \Phi_{vv} = -\frac{1}{v^2}\, f(t)\,, \text{and}$$
$$\Phi(T,v) = K\, \log(v) = U_1(v)\,.$$

Then (3.26) reduces to

$$\begin{aligned} 0 = \log(v)\dot{f}(t) + \dot{g}(t) + f(t)\left(r + \delta + \frac{1}{2}\,\lambda_P^2\right) + \frac{\alpha-2}{2\,\alpha}\left(\frac{e^{\tilde{\rho}\,t}}{\kappa}\right)^{\frac{2}{\alpha-2}} f(t)^{\frac{\alpha}{\alpha-2}} \\ - e^{-\rho\,t} - \rho\,t\,e^{-\rho\,t} + e^{-\rho\,t}\,\log(v) - e^{-\rho\,t}\,\log(f(t))\,. \end{aligned} \tag{3.27}$$

Taking the derivative of this equation w.r.t. v gives:

$$\frac{1}{v}\,\dot{f}(t) + \frac{1}{v}\,e^{-\rho\,t} = 0 \quad \Longleftrightarrow \quad \dot{f}(t) = -e^{-\rho\,t}\,.$$

Using the condition $f(T) = K$ we then get by integration

$$f(t) = \begin{cases} K + \frac{e^{-\rho\,t} - e^{-\rho\,T}}{\rho}\,, & \text{for } \rho \neq 0\,, \\ K + T - t\,, & \text{for } \rho = 0\,. \end{cases} \tag{3.28}$$

Following the derivation of f we can eliminate the $\log(v)$ in (3.27)

$$\begin{aligned} -\dot{g}(t) = f(t)\left(r + \delta + \frac{1}{2}\,\lambda_P^2\right) + \frac{\alpha-2}{2\,\alpha}\left(\frac{e^{\tilde{\rho}\,t}}{\kappa}\right)^{\frac{2}{\alpha-2}} f(t)^{\frac{\alpha}{\alpha-2}} \\ - e^{-\rho\,t} - \rho\,t\,e^{-\rho\,t} - e^{-\rho\,t}\,\log(f(t))\,, \quad \text{and} \;\; g(T) = 0\,. \end{aligned} \tag{3.29}$$

Equation (3.29) can now be solved by simple integration:

$$\begin{aligned} g(t) = \left(r + \delta + \frac{1}{2}\,\lambda_P^2\right)\int_t^T f(s)\,\mathrm{d}s + \frac{\alpha-2}{2\,\alpha}\int_t^T \left(\frac{e^{\tilde{\rho}\,s}}{\kappa}\right)^{\frac{2}{\alpha-2}} f(s)^{\frac{\alpha}{\alpha-2}}\,\mathrm{d}s \\ - \int_t^T (1 + \rho\,s)\,e^{-\rho\,s}\,\mathrm{d}s - \int_t^T e^{-\rho\,s}\,\log f(s)\,\mathrm{d}s\,, \end{aligned}$$

where $f(t)$ is given as above.

Combining the results for the functions f and g we then get the claimed result for the value function. Noting that $\Phi_v/\Phi_{vv} = -v$ and using the first order conditions in (3.10) establishes the claimed optimal strategies $\pi^{P^\star}$ and $\pi^{S^\star}$. Finally noting that $\Phi_v^2/\Phi_{vv} = -f(t)$ and using the solved first order condition (3.24), we get the desired result for the optimal sharpe ratio $\lambda^\star$ and plugging in $v\,\Phi_v = f$ in (3.25) we get the claimed result for the optimal consumption rate $k^\star$. The claimed optimal investment and control choices are deterministic and the optimal consumption rate are continuous on a compact support, so they are uniformly bounded implying $u^\star = (\pi^{S^\star},\pi^{P^\star},\lambda^\star,k^\star) \in A'(t,v)$. □

Remark 3.2.1
The expression for g in Theorem 3.2.1 can be partially calculated fairly explicitly. For $\rho \neq 0$ we obtain

$$\begin{aligned} g(t) = &\left(r+\delta+\frac{1}{2}\,\lambda_P^2\right)\left(K\,[T-t]+\frac{1}{\rho^2}\,\left[e^{-\rho\,t}-e^{-\rho\,T}(1+\rho[T-t])\right]\right)\\ &-\frac{1}{\rho}\,\left(e^{-\rho\,t}-e^{-\rho\,T}\right)-t\,e^{-\rho\,t}+T\,e^{-\rho\,T}+K\,\log(K)\\ &-\log\left(K+\frac{1}{\rho}\,\left[e^{-\rho\,t}-e^{-\rho\,T}\right]\right)\left(K+\frac{1}{\rho}\,\left[e^{-\rho\,t}-e^{-\rho\,T}\right]\right)\\ &+\frac{\alpha-2}{2\,\alpha}\int_t^T\left(\frac{e^{\tilde\rho\,s}}{\kappa}\right)^{\frac{2}{\alpha-2}}f(s)^{\frac{\alpha}{\alpha-2}}\,\mathrm{d}s\,. \end{aligned}$$

The integral in the last line can in general not be computed in closed form. However, it can be expressed as a hypergeometric function. For $\rho = 0$, the function g can be obtained by continuity in ρ, i.e. fix t and then compute the limit for $\rho \to 0$.

The solutions of the maximization problems given in Theorem 3.2.1 are candidates for the optimal investment and control choices as well as for the optimal consumption rate for the problem in (3.18). In the following theorem we verify that under sufficient assumptions these solutions are indeed optimal.

Theorem 3.2.2 (Verification)
Let $\kappa > 0$ and $\alpha > 2$. Assume the executive's utility function of wealth, the utility function of the consumption rate as well as the cost function are given

by (3.14), (3.15) and (3.16). Then the candidates given in (3.19) and (3.20) are the optimal investment and control strategy (i.e. own-company stockholding, market portfolio holding and non-systematic Sharpe ratio strategy), the optimal consumption rate and value function of the optimal control problem (3.18).

Proof. Define the performance functional of our optimal investment, consumption and control decision by

$$J'(t,v;u) := \mathbb{E}^{t,v}\left[U_1\left(V_T^u\right) + \int_t^T U_2(s,V_s^u,k_s)\,\mathrm{d}s - \int_t^T C(s,V_s^u,\lambda_s)\,\mathrm{d}s\right]. \tag{3.30}$$

Our candidates are optimal if we have

$$\begin{aligned} &J'(t,v;u^\star) = \Phi(t,v) \text{ with } u^\star = (\pi^{P\star},\pi^{S^\star},k^\star,\lambda^\star) \text{ and} \\ &J'(t,v;u) \leq \Phi(t,v)\,,\ \text{ for all } u = (\pi^P,\pi^S,k,\lambda)\ \in\ A_1'(t,v)\,. \end{aligned} \tag{3.31}$$

Let $u \in A_1'(t,v)$. Since $\Phi \in C^{1,2}$, we obtain by Ito's formula:

$$\begin{aligned} &\Phi(T,V_T^u) + \int_t^T e^{-\rho s}\,\log(V_s^u k_s)\,\mathrm{d}s - \int_t^T e^{-\tilde{\rho}\,s}\kappa\frac{\lambda_s^\alpha}{\alpha}\,\mathrm{d}s \\ &= \Phi(t,v) + \int_t^T \left(\Phi_t(s,V_s^u) + e^{-\rho s}\log(V_s^u k_s) - e^{-\tilde{\rho}\,s}\kappa\frac{\lambda_s^\alpha}{\alpha}\right)\mathrm{d}s \\ &+ \int_t^T \Phi_v(s,V_s^u)V_s^u\left(r + \delta - k_s + \left[\pi_s^P + \beta\pi_s^S\right]\lambda_P\sigma_P + \pi_s^S\lambda_s\sigma\right)\mathrm{d}s \\ &+ \frac{1}{2}\int_t^T \Phi_{vv}(s,V_s^u)\,(V_s^u)^2\left([(\pi_s^P + \beta\,\pi_s^S)\sigma_P]^2 + [\pi_s^S\sigma]^2\right)\mathrm{d}s \\ &+ \int_t^T \Phi_v(s,V_s^u)V_s^u\left(\pi_s^P + \beta\pi_s^S\right)\sigma_P\,\mathrm{d}W_s^P + \int_t^T \Phi_v(s,V_s^u)V_s^u\pi_s^S\sigma\,\mathrm{d}W_s. \end{aligned} \tag{3.32}$$

First, we investigate the optimal control $u^\star = (\pi^{P^\star},\pi^{S^\star},\lambda^\star,k^\star)$ given in (3.19). To show that the local martingale component in (3.32) vanishes in expectation we check the sufficient integrability condition

$$\mathbb{E}\left[\int_t^T \left(\Phi_v(s,V_s^{u^\star})V_s^{u^\star}\right)^2\left(\left[\pi_s^{P^\star} + \beta\,\pi_s^{S^\star}\right]^2\sigma_P^2 + \left[\pi_s^{S^\star}\right]^2\sigma^2\right)\mathrm{d}s\right] < \infty\,. \tag{3.33}$$

From (3.19) and (3.20) we obtain

$$\left(\Phi_v(s,V_s^{u^\star})V_s^{u^\star}\right)^2\left(\left[\pi_s^{P\star}+\beta\pi_s^{S\star}\right]^2\sigma_P^2+\left[\pi_s^{S\star}\right]^2\sigma^2\right)=(f(s))^2\left(\lambda_P^2+[\lambda^\star(s)]^2\right).$$

Now, f and $\lambda^\star$ are deterministic continuous functions on the compact $[0,T]$, and thus the above expression is uniformly bounded. Accordingly the expectation in (3.33) is finite, and the Wiener integrals in (3.32) vanish in expectation. Furthermore, Φ satisfies the HJB equation (3.8) implying

$$\begin{aligned}0=\,&\Phi_v(s,V_s^{u^\star})V_s^{u^\star}\left(r+\delta-k_s+\left[\pi_s^{P\star}+\beta\pi_s^{S\star}\right]\lambda_P\sigma_P+\pi_s^{S\star}\lambda_s^\star\sigma\right)\\&+\frac{1}{2}\,\Phi_{vv}(s,V_s^{u^\star})\left(V_s^{u^\star}\right)^2\left([(\pi_s^{P\star}+\beta\,\pi_s^{S\star})\sigma_P]^2+[\pi_s^{S\star}\sigma]^2\right)\\&+\Phi_t(s,V_s^{u^\star})+e^{-\rho s}\log(V_s^{u^\star}k_s^\star)-e^{-\tilde\rho s}\kappa\frac{(\lambda_s^\star)^\alpha}{\alpha},\quad\text{for } t\le s\le T\,.\end{aligned}$$

Then, using that $\Phi(T,v)=U_1(v)$ the expectation of (3.32) is:

$$\begin{aligned}\Phi(t,v)\;&=\;\mathbb{E}^{t,v}\left[\Phi(T,V_T^{u^\star})+\int_t^T e^{-\rho s}\log(V_s^{u^\star}k_s^\star)\,\mathrm{d}s-\int_t^T e^{-\tilde\rho s}\kappa\frac{(\lambda_s^\star)^\alpha}{\alpha}\,\mathrm{d}s\right]\\&=\;\mathbb{E}^{t,v}\left[U_1(V_T^{u^\star}))+\int_t^T U_2(s,V_s^{u^\star},k_s^\star)\,\mathrm{d}s-\int_t^T C(s,V_s^{u^\star},\lambda_s^\star)\,\mathrm{d}s\right]\\&=\;J'(t,v;u^\star)\,.\end{aligned}$$

Thus we have verified the first part of (3.31).

Next, fix $u\in A'(t,v)$. By the HJB equation (3.8), we have

$$\begin{aligned}0\;\ge\,&\Phi_t(s,V_s^u)+\Phi_v(s,V_s^u)V_s^u\left(r+\delta-k_s+\left[\pi_s^P+\beta\pi_s^S\right]\lambda_P\sigma_P+\pi_s^S\lambda_s\sigma\right)\\&+\frac{1}{2}\,\Phi_{vv}(s,V_s^u)\left(V_s^u\right)^2\left([(\pi_s^P+\beta\,\pi_s^S)\sigma_P]^2+[\pi_s^S\sigma]^2\right)\\&+e^{-\rho s}\log(V_s^uk_s)-e^{-\tilde\rho s}\kappa\frac{\lambda_s^\alpha}{\alpha},\quad\text{for } t\le s\le T\,.\end{aligned}$$

Substituting this in (3.32) and recalling that $\Phi_v(t,v)=\frac{1}{v}\,f(t)$ we get:

$$\begin{aligned}&\Phi(T,V_T^\pi)+\int_t^T e^{-\rho s}\log(V_s^uk_s)\mathrm{d}s-\int_t^T e^{-\tilde\rho s}\kappa\frac{\lambda_s^\alpha}{\alpha}\mathrm{d}s\\&\quad\le\Phi(t,v)+\int_t^T f(s)\,\left(\pi_s^P+\beta\,\pi_s^S\right)\sigma_P\,\mathrm{d}W_s^P+\int_t^T f(s)\,\pi_s^S\sigma\,\mathrm{d}W_s\,.\end{aligned}\tag{3.34}$$

Taking the expectation on both sides and keeping in mind that $\Phi(T,v) = U_1(v)$ then yields

$$\begin{aligned}
&J'(t,v;u)\\
&\quad= \mathbb{E}^{t,v}\left[U_1(V_T^u)) + \int_t^T U_2(s,V_s^u,k_s)\,\mathrm{d}s - \int_t^T C(s,V_s^u,\lambda_s)\,\mathrm{d}s\right]\\
&\quad= \mathbb{E}^{t,v}\left[\Phi(T,V_T^u) + \int_t^T e^{-\rho s}\log(V_s^u k_s)\mathrm{d}s - \int_t^T e^{-\tilde{\rho}s}\kappa\frac{\lambda_s^\alpha}{\alpha}\,\mathrm{d}s\right]\\
&\quad\leq \Phi(t,v) + \underbrace{\mathbb{E}^{t,v}\left[\int_t^T f(s)\,\left(\pi_s^P + \beta\,\pi_s^S\right)\sigma_P\,\mathrm{d}W_s^P + \int_t^T f(s)\,\pi_s^S\sigma\,\mathrm{d}W_s\right]}_{=0,\text{ by (3.17)}}.
\end{aligned}$$

The Wiener integral vanishes in expectation since the corresponding integrand is square integrable, since f is uniformly bounded and (3.17). □

3.2.3 Participation Constraint for the Log-Utility Case

The optimal strategies in Theorem 3.2.1 and Theorem 3.2.2 above apply in case the individual decides to work for the smaller listed company. However, she has the opportunity to take up an outside option, that is, working for a larger company in a mid-level management position. The outside option offers a contract that differs in the salary rate and foregoes the possibility of controlling the stock price of the smaller listed company. Next, we calculate the value of the outside option and derive the participation constraint.

The outside option pays a salary rate $\widehat{\delta}$. Taking on the position results in the loss of the ability to influence the stock price of the smaller listed company and therefore $\widehat{\lambda} = 0$. She can invest in the financial market. The classical optimal investment and consumption decision applies. For the remainder of this subsection we assume that the portfolio process follows Eq. (3.5) where we set $\delta = \widehat{\delta}$ and $\lambda = \widehat{\lambda} = 0$. Then the optimal investment decision problem in Equation (3.6) determines the value of the outside option $\widehat{\Phi}(0,V_0)$

at time $t = 0$ for initial wealth $V_0 > 0$. The solution $\widehat{\Phi}$ can be obtained as a simplification of the results in Theorem 3.2.1 and Theorem 3.2.2, i.e. $\widehat{\Phi}(t,v) = \widehat{f}(t)\,\log(v) + \widehat{g}(t)$ with

$$\widehat{f}(t) = \begin{cases} K + \frac{e^{-\rho t} - e^{-\rho T}}{\rho}, & \text{for } \rho \neq 0\,, \\ K + T - t\,, & \text{for } \rho = 0\,, \end{cases} \tag{3.35}$$

and

$$\begin{aligned} \widehat{g}(t) = {} & \left(r + \widehat{\delta} + \frac{1}{2}\,\lambda_P^2\right) \int_t^T \widehat{f}(s)\,\mathrm{d}s \\ & - \int_t^T (1 + \rho\, s)\, e^{-\rho\, s}\,\mathrm{d}s - \int_t^T e^{-\rho\, s}\,\log(\widehat{f}(s))\,\mathrm{d}s\,. \end{aligned} \tag{3.36}$$

Observe that $\widehat{f} = f$ and

$$g(t) - \widehat{g}(t) = (\delta - \widehat{\delta}) \int_t^T f(s)\,\mathrm{d}s + \frac{\alpha - 2}{2\,\alpha} \int_t^T \left(\frac{e^{\tilde{\rho}\, s}}{\kappa}\right)^{\frac{2}{\alpha-2}} f(s)^{\frac{\alpha}{\alpha-2}}\,\mathrm{d}s\,. \tag{3.37}$$

Based on the discussion above we can state the participation constraint.

Theorem 3.2.3
Let $\widehat{\delta}$ be the salary rate of the outside option. Then the value of the outside option is the solution $\widehat{\Phi}$ to optimal investment and consumption problem in (3.6) with dynamics (3.5) where we set $\delta = \widehat{\delta}$ and $\lambda = \widehat{\lambda} = 0$. The participation constraint for the individual is

$$\delta \;\geq\; \widehat{\delta} - \frac{(\alpha - 2)}{2\,\alpha}\, \frac{\int_0^T \left(\frac{e^{\tilde{\rho}\, t}}{\kappa}\right)^{\frac{2}{\alpha-2}} f(t)^{\frac{\alpha}{\alpha-2}}\,\mathrm{d}t}{\int_0^T f(t)\,\mathrm{d}t}\,, \tag{3.38}$$

where f is given in Theorem 3.2.1.

Proof. The value function is of the form $\widehat{\Phi} = \widehat{f}(t)\,\log(v) + \widehat{g}(t)$ with $\widehat{f} = f$ and $\widehat{g} - g$ given in (3.37). Then we have of course $\Phi(t,v) - \widehat{\Phi}(t,v) = g(t) - \widehat{g}(t)$ and the participation constraint $\Phi(0,V_0) \geq \widehat{\Phi}(0,V_0)$ follows as stated in (3.38). $\square$

Remark 3.2.2

Using the representation of the function f

$$f(t) = \begin{cases} K + \frac{e^{-\rho\, t} - e^{-\rho\, T}}{\rho}\,, & \textit{for } \rho \neq 0\,, \\ K + T - t\,, & \textit{for } \rho = 0\,, \end{cases}$$

we can rewrite the particicpation constraint as

$$\delta \geq \begin{cases} \widehat{\delta} - \dfrac{(\alpha-2)}{2\,\alpha}\, \dfrac{\int_0^T \left(\frac{e^{\tilde{\rho}\, t}}{\kappa}\right)^{\frac{2}{\alpha-2}} f(t)^{\frac{\alpha}{\alpha-2}}\, \mathrm{d}t}{K\,T + \frac{1}{\rho^2}\left[1 - e^{-\rho T}(1+\rho T)\right]}\,, & \textit{for } \rho \neq 0\,, \\ \widehat{\delta} - \dfrac{(\alpha-2)}{2\,\alpha}\, \dfrac{\int_0^T \left(\frac{e^{\tilde{\rho}\, t}}{\kappa}\right)^{\frac{2}{\alpha-2}} f(t)^{\frac{\alpha}{\alpha-2}}\, \mathrm{d}t}{K\,T + \frac{1}{2}T^2}\,, & \textit{for } \rho = 0\,. \end{cases}$$

3.2.4 Closed-Form Solution for the Power-Utility Case

In this subsection, we derive a closed-form solution for the case of power utility. In particular, we specify a constant relative risk aversion utility-disutility set-up. For the relative risk aversion parameter $\gamma > 1$, the utility function of the final wealth U_1 is

$$U_1(v) = \frac{v^{1-\gamma}}{1-\gamma}\,, \quad \text{for } \gamma > 1\,, \tag{3.39}$$

the utility function of the consumption U_2 is

$$U_2(k,v) = \frac{(v\,k)^{1-\gamma}}{1-\gamma}\,, \quad \text{for } \gamma > 1\,, \tag{3.40}$$

and the disutility of control (i.e. work effort) C is

$$C(v,\lambda) = \kappa\, v^{1-\gamma}\, \frac{\lambda^\alpha}{\alpha}\,, \quad \text{for } \gamma > 1\,, \tag{3.41}$$

where $\kappa > 0$ and $\alpha > 2$ are as in the log-utility part.

Compared to the log-utility setup we have made the simplifying assumption that utility from consumption in (3.40) and the cost from work effort in (3.41) are not depending on time, see (3.15) and (3.16) for time preferences in the log-utility setup. This assumption enables us to obtain a tractable formulation of the problem. However, we require a further structural assumption linking the cost function parameter α to the relative risk aversion γ. The following condition is assumed to hold:

$$\alpha = 2\gamma + 2\,. \tag{3.42}$$

Condition (3.42) enables us to reduce an ODE of inhomogeneous Bernoulli type that appears when solving the HJB equation to an ODE of Riccati type, which we are able to solve in closed-form. This restriction is however not counterintuitive. A more risk averse individual is implicitly assumed to be more sensitive towards work. When focusing on the optimal work effort $\lambda^\star$ as a main result we can rely on the results of the first chapter. The results w.r.t. this related framework (although without consumption and salary) indicate that $\lambda^\star$ decreases with increasing risk aversion as well as with increasing disutility stress; compare Figures 1.3 and 1.5. So by relating those two parameters via (3.42) we do not change the qualitative behavior of the optimal work effort.

Analogously to the log-utility case, to guarantee indeed the optimality of the candidates we will derive for the executive's optimal investment and control strategy and value function, we consider again a more restrictive class of admissible strategies as follows.

Definition 3.2.2

Fix $(t,v) \in [0,T] \times \mathbb{R}^+$. Further choose $\tilde{\epsilon} \in (0,\infty)$ as close to zero as possible. Then for $\gamma > 1$, we denote by $A'_\gamma(t,v)$ the set of admissible strategies $u \in A'_\gamma(t,v)$, such that $u \in A_\gamma(t,v)$ and

$$\int_t^T (\pi_s^P + \beta\,\pi_s^S)^{2+\tilde{\epsilon}}(\sigma^P)^{2+\tilde{\epsilon}} + \left(\pi_s^S\sigma\right)^{2+\tilde{\epsilon}}\,\mathrm{d}s \le C_1 < \infty\,, \quad \textit{for some } C_1 \in \mathbb{R}_0^+\,, \tag{3.43}$$

$$\int_t^T \pi_s^S\sigma\lambda_s\,\mathrm{d}s \ge C_2 > -\infty\,, \quad \textit{for some } C_2 \in \mathbb{R}_0^+\,, \tag{3.44}$$

$$\int_t^T k_s\,\mathrm{d}s \le C_3 < \infty\,, \quad \textit{for some } C_3 \in \mathbb{R}_0^+\,. \tag{3.45}$$

Theorem 3.2.4 (The power-utility case: $\gamma > 1$)
Suppose that the relative risk aversion parameter γ and the disutility stress parameter α are connected via the relation (3.42)*, then the full solution of the maximization problem (3.18) can be summarized by the strategy*

$$\begin{aligned} \pi^{P\star}(t,v) &= \frac{\mu^P - r}{\gamma\,(\sigma^P)^2} - \beta\,\pi^{S\star}(t,v)\,, \quad \pi^{S\star}(t,v) = \frac{\lambda^\star(t,v)}{\gamma\,\sigma^\star(t,v,\lambda^\star(t,v))}\,, \\ \lambda^\star(t,v) &= \left(\frac{1}{\kappa\,\gamma}\,f(t)\right)^{\frac{1}{2\gamma}}\,, \quad k^\star(t,v) = (f(t))^{-\frac{1}{\gamma}}\,, \end{aligned} \tag{3.46}$$

and value function

$$\Phi(t,v) = \frac{v^{1-\gamma}}{1-\gamma}\,f(t)\,, \tag{3.47}$$

where

$$f(t) = \left(\frac{2(1-g_P)\sqrt{C_0}}{2\sqrt{C_0}e^{-2\sqrt{C_0}(T-t)} + (1-g_P)\left(e^{-2\sqrt{C_0}(T-t)} - 1\right)} + g_P\right)^{-\gamma}\,, \tag{3.48}$$

with

$$C_0 = \frac{(\gamma-1)^2}{4\gamma^2}\left(r + \delta + \frac{1}{2}\frac{\lambda_P^2}{\gamma}\right)^2 - \frac{\kappa}{2}\frac{(1-\gamma)}{(1+\gamma)}\left(\frac{1}{\kappa\gamma}\right)^{\frac{\gamma+1}{\gamma}}\,, \tag{3.49}$$

and

$$g_P = -\frac{1-\gamma}{2\gamma}\left(r + \delta + \frac{1}{2}\frac{\lambda_P^2}{\gamma}\right) + \sqrt{C_0}\,. \tag{3.50}$$

Proof. First observe that a function F of the form $F(\lambda) = a\,\lambda^2 - b\,\lambda^\alpha$, $\lambda \ge 0$, for given constants $a, b > 0$ and $\alpha > 2$, has a unique maximizer $\lambda^\star$

and maximized value $F(\lambda^\star)$ given by

$$\lambda^\star = \left(\frac{2\,a}{\alpha\,b}\right)^{\frac{1}{\alpha-2}}\,, \quad \text{and} \quad F(\lambda^\star) = (\alpha-2)\,\alpha^{-\frac{\alpha}{\alpha-2}}\,2^{\frac{2}{\alpha-2}}\,a^{\frac{\alpha}{\alpha-2}}\,b^{-\frac{2}{\alpha-2}}\,. \tag{3.51}$$

Using this insight the first order condition for $\lambda^\star$ in (3.11) is now solved. Set

$$a = \frac{1}{2}\,\frac{\Phi_v^2}{-\Phi_{vv}}\,, \quad \text{and} \quad b = \frac{\kappa}{\alpha}\,v^{1-\gamma}\,,$$

then (3.51) gives

$$\lambda^\star = \left(\frac{1}{\kappa\,v^{1-\gamma}}\,\frac{\Phi_v^2}{-\Phi_{vv}}\right)^{\frac{1}{\alpha-2}}\,, \quad F(\lambda^\star) = \frac{\alpha-2}{2\,\alpha}\,\left(\kappa\,v^{1-\gamma}\right)^{-\frac{2}{\alpha-2}}\,\left(\frac{\Phi_v^2}{-\Phi_{vv}}\right)^{\frac{\alpha}{\alpha-2}}\,.$$

Having specified the utility function as $U_2(t,k_t) = \frac{(v\,k)^{1-\gamma}}{1-\gamma}$, the first order condition (3.12) for the optimal consumption rate becomes:

$$k^\star = \frac{1}{v}\,\left(\Phi_v\right)^{-\frac{1}{\gamma}}\,.$$

Substituting $\lambda^\star$ and $k^\star$ in (3.13) then yields:

$$\begin{aligned} 0 \;&= \Phi_t + \Phi_v\,v\,(r+\delta) + \frac{1}{2}\,\frac{\Phi_v^2}{-\Phi_{vv}}\,\left(\lambda^P\right)^2 \\ &\quad + \frac{\alpha-2}{2\,\alpha}\,\left(\kappa\,v^{1-\gamma}\right)^{-\frac{2}{\alpha-2}}\,\left(\frac{\Phi_v^2}{-\Phi_{vv}}\right)^{\frac{\alpha}{\alpha-2}} + \frac{\gamma}{1-\gamma}\,\left(\Phi_v\right)^{\frac{\gamma-1}{\gamma}}\,. \end{aligned} \tag{3.52}$$

Using the separation ansatz $\Phi(t,v) = f(t)\,\frac{v^{1-\gamma}}{1-\gamma}$ results in

$$\Phi_t = \dot{f}\,\frac{v^{1-\gamma}}{1-\gamma}\,, \quad \Phi_v = f\,v^{-\gamma}\,, \quad \Phi_{vv} = -\gamma\,f\,v^{-\gamma-1}\,, \quad \text{and} \quad f(T) = 1\,. \tag{3.53}$$

Thus (3.52) becomes

$$\begin{aligned} 0 = \dot{f}\,\frac{v^{1-\gamma}}{1-\gamma} + f\,v^{1-\gamma}\,(r+\delta) + \frac{1}{2}\,\frac{f\,v^{1-\gamma}}{\gamma}\,\left(\lambda^P\right)^2 \\ + \frac{\alpha-2}{2\,\alpha}\,\left(\kappa\,v^{1-\gamma}\right)^{-\frac{2}{\alpha-2}}\,\left(\frac{f\,v^{1-\gamma}}{\gamma}\right)^{\frac{\alpha}{\alpha-2}} + \frac{\gamma}{1-\gamma}v^{1-\gamma}f^{\frac{\gamma-1}{\gamma}}\,. \end{aligned}$$

Dividing by $\frac{v^{1-\gamma}}{1-\gamma}$ and then defining

$$\begin{aligned} a_1 =&(1-\gamma)\,\left(r+\delta+\frac{1}{2}\,\frac{\lambda_P^2}{\gamma}\right)\,,\, a_n = (1-\gamma)\,\frac{\kappa}{2}\,\frac{\alpha-2}{\alpha}\,\left(\frac{1}{\kappa\,\gamma}\right)^{\frac{\alpha}{\alpha-2}}\,, \\ a_m =&\gamma\,, \quad n = \frac{\alpha}{\alpha-2}\,, \quad \text{and} \quad m = \frac{\gamma-1}{\gamma}\,. \end{aligned} \tag{3.54}$$

results in an ordinary differential equation of the form

$$\dot{f} + a_1\, f + a_n\, f^n + a_m\, f^m = 0\,. \tag{3.55}$$

The ansatz $g = f^{1-n}$ yields $\dot{g} = \dfrac{1-n}{f^n}\,\dot{f}$ and thus

$$\dot{g} + a_1\,(1-n)\,g + a_m\,(1-n)\,g^{\frac{m-n}{1-n}} = -a_n\,(1-n) \quad , \quad g(T) = 1\,.$$

Using (3.42), i.e. $\alpha = 2 + 2\,\gamma$, and plugging in the coefficients in (3.54) we obtain the following ODE of Riccati type

$$\dot{g} - \frac{1-\gamma}{\gamma}\left(r + \delta + \frac{1}{2}\frac{\lambda_P^2}{\gamma}\right) g - g^2 = \frac{\kappa}{2}\,\frac{1-\gamma}{1+\gamma}\left(\frac{1}{\kappa\,\gamma}\right)^{\frac{\gamma+1}{\gamma}} . \tag{3.56}$$

This ODE can be solved if we know a particular solution g_P, since then we can reduce this ODE by using the standard ansatz

$$h = 1/(g - g_P)$$

to the following linear form:

$$\dot{h} + \left[2g_P + \frac{\gamma-1}{\gamma}\left(r + \delta + \frac{1}{2}\frac{\lambda_P^2}{\gamma}\right)\right] h + 1 = 0\,, \qquad h(T) = \frac{1}{1-g_P}\,.$$

This equation can now be solved by variation of constants. A nonnegative particular solution of (3.56) is

$$\begin{aligned} g_P = &-\frac{1-\gamma}{2\gamma}\left(r + \delta + \frac{1}{2}\frac{\lambda_P^2}{\gamma}\right) \\ &+ \sqrt{\frac{(\gamma-1)^2}{4\,\gamma^2}\left(r + \delta + \frac{1}{2}\frac{\lambda_P^2}{\gamma}\right)^2 - \frac{\kappa\,(1-\gamma)}{2\,(1+\gamma)}\left(\frac{1}{\kappa\gamma}\right)^{\frac{\gamma+1}{\gamma}}}\,, \end{aligned}$$

which means that we have to solve the following inhomogeneous linear ODE

$$\dot{h} + \left[2\sqrt{\frac{(\gamma-1)^2}{4\gamma^2}\left(r + \delta + \frac{1}{2}\frac{\lambda_P^2}{\gamma}\right)^2 - \frac{\kappa\,(1-\gamma)}{2\,(1+\gamma)}\left(\frac{1}{\kappa\gamma}\right)^{\frac{\gamma+1}{\gamma}}}\;\right] h + 1 = 0\,. \tag{3.57}$$

Now applying variation of constants and using that $h(T) = 1/(1-g_P)$, the solution of this ODE is

$$h(t) = \frac{1}{1-g_P}\, e^{2\sqrt{C_0}\,(T-t)} + \frac{1}{2\sqrt{C_0}} \left(e^{2\sqrt{C_0}\,(T-t)} - 1\right), \tag{3.58}$$

where

$$C_0 = \frac{(\gamma-1)^2}{4\gamma^2}\left(r+\delta+\frac{1}{2}\frac{\lambda_P^2}{\gamma}\right)^2 - \frac{\kappa\,(1-\gamma)}{2\,(1+\gamma)}\left(\frac{1}{\kappa\gamma}\right)^{\frac{\gamma+1}{\gamma}}.$$

Transforming the result back to the function f we get

$$f(t) = \left(g_P + \frac{2(1-g_P)\sqrt{C_0}}{2\sqrt{C_0}e^{2\sqrt{C_0}(T-t)} + (1-g_P)\left(e^{2\sqrt{C_0}(T-t)}-1\right)}\right)^{-\gamma}. \tag{3.59}$$

Using the representations (3.53) we get

$$\lambda^\star(t,v) = \left(\frac{1}{\kappa\, v^{1-\gamma}}\,\frac{\Phi_v^2}{-\Phi_{vv}}\right)^{\frac{1}{\alpha-2}} = \left(\frac{1}{\kappa\gamma} f(t)\right)^{\frac{1}{\alpha-2}} = \left(\frac{1}{\kappa\gamma} f(t)\right)^{\frac{1}{2\gamma}},$$

and

$$\pi^{P^\star}(t,v) = -\frac{(\mu^P - r)}{v(\sigma^P)^2}\,\frac{\Phi_v(t,v)}{\Phi_{vv}(t,v)} - \beta\,\pi^{S^\star}(t,v) = \frac{\mu^P - r}{\gamma\,(\sigma^P)^2} - \beta\,\pi^{S^\star}(t,v),$$

$$\pi^{S^\star}(t,v) = -\frac{\lambda^\star(t,v)}{v\sigma^\star(t,v,\lambda^\star(t,v))}\,\frac{\Phi_v(t,v)}{\Phi_{vv}(t,v)} = \frac{\lambda^\star(t,v)}{\gamma\,\sigma^\star(t,v,\lambda^\star(t,v))},$$

as well as

$$k^\star(t,v) = \frac{1}{v}\,(\phi_v(t,v))^{-\frac{1}{\gamma}} = \frac{1}{v}\,\left(f(t)\,v^{-\gamma}\right)^{-\frac{1}{\gamma}} = (f(t))^{-\frac{1}{\gamma}}.$$

And the proof is finished. □

Remark 3.2.3

Establishing the solution is based on the function f in (3.55)*. The transformation $g = f^{-1/\gamma}$ is applied and requires f to be nonnegative. Accordingly, the function g satisfies the Riccati ODE in* (3.56) *and lives also on $\mathbb{R}^+$. As a solution strategy we identifying a particular solution g_P. This works for $\gamma > 1$, since then $g_P > 0$, i.e. the particular solution is in the region where g is specified on. However, the solution strategy breaks down for $0 < \gamma < 1$.*

Then we would have $g_P < 0$ and this candidate is not an admissible solution. This explains why we cannot provide a solution for the case $0 < \gamma < 1$, at least, with our methods at hand.

Again, we need to show that the candidates derived in Theorem (3.2.4) are indeed optimal. This is done in the following verification theorem.

Theorem 3.2.5 (Verification Result for the Case $\gamma > 1$)
Let $\kappa > 0$ and $\alpha > 2$ and $\alpha = 2\gamma + 2$. Assume the utility function of wealth, the utility function of the consumption rate and the disutility function are given by (3.39), (3.40) and (3.41), respectively. Then the candidates given via (3.46) - (3.50) are the optimal investment and control strategy (i.e. own-company stockholding, market portfolio holding and non-systematic Sharpe ratio strategy), the optimal consumption rate and value function of the optimal control problem (3.18) for the case $\gamma > 1$.

Proof. Define the performance functional of our optimal investment, consumption and control decision again by (3.30). Our candidates are optimal if we have

$$J'(t,v;u^\star) = \Phi(t,v) \text{ with } u^\star = (\pi^{P^\star},\pi^{S^\star},k^\star,\lambda^\star) \text{ and}$$
$$J'(t,v;u) \le \Phi(t,v)\,, \text{ for all } u = (\pi^P,\pi^S,k,\lambda) \in A'_\gamma(t,v)\,.$$

Let $u \in A'_\gamma(t,v)$. Since $\Phi \in C^{1,2}$, we obtain by Ito's formula:

$$\begin{aligned}
\Phi(T,V^u_T) - \int_t^T \kappa (V^u_s)^{1-\gamma}\frac{\lambda_s^\alpha}{\alpha}\,\mathrm{d}s + \int_t^T \frac{(V^u_s\,k)^{1-\gamma}}{1-\gamma}\,\mathrm{d}s = \Phi(t,v)\, + \\
\int_t^T \Big\{ \Phi_t(s,V^u_s) + \Phi_v(s,V^u_s)V^u_s\big[r + \pi^S_s\,\lambda_s\,\sigma + (\pi^P_s + \beta\,\pi^S_s)\lambda^P\sigma^P + \delta - k_s\big] \\
+ 1/2\Phi_{vv}(s,V^u_s)\,(V^u_s)^2\,\big[(\pi^P_s + \beta\,\pi^S_s)^2(\sigma^P)^2 + (\pi^S_s\sigma)^2\big] \\
- \kappa (V^u_s)^{1-\gamma}\frac{\lambda_s^\alpha}{\alpha}\mathrm{d}s + \frac{(V^u_s\,k)^{1-\gamma}}{1-\gamma}\Big\} \\
+ \int_t^T \Phi_v(s,V^u_s)V^u_s(\pi^P_s + \beta\,\pi^S_s)\sigma^P\,\mathrm{d}W^P_s + \int_t^T \Phi_v(s,V^u_s)V^u_s\pi^S_s\sigma\,\mathrm{d}W_s\,. \quad (3.60)
\end{aligned}$$

For the optimality candidates given in (3.46), the local martingale component in (3.60) disappears. A sufficient condition to verify this is the square integrability condition

$$\mathbb{E}\left[\int_t^T \left(\Phi_v(s,V_s^{u^\star})V_s^{u^\star}\right)^2 \left([\pi_s^{P^\star} + \beta\,\pi_s^{S^\star}]^2(\sigma^P)^2 + [\pi_s^{S^\star}\sigma]^2\right)\right] \mathrm{d}s \,<\, \infty\,. \qquad (*)$$

Now substituting the candidates from (3.46) - (3.50) yields

$$\begin{aligned}\left(\Phi_v(s,V_s^{u^\star})V_s^{u^\star}\right)^2 &\left([\pi_s^{P^\star} + \beta\,\pi_s^{S^\star}]^2(\sigma^P)^2 + [\pi_s^{S^\star}\sigma]^2\right) \\ &= \frac{\left(V_s^{u^\star}\right)^{2(1-\gamma)} f(s)^2}{\gamma^2}\left[(\lambda^P)^2 + \left(\frac{1}{\kappa\gamma}f(s)\right)^{\frac{1}{\gamma}}\right]\,. \qquad (**)\end{aligned}$$

The RHS of $(**)$ is $\left(V_s^{u^\star}\right)^{2(1-\gamma)}$ times a deterministic and continuous function on the compact set $[0,T]$. The deterministic part is uniformly bounded. Therefore it is sufficient to focus on the stochastic component: $V_s^{u^\star}$ satisfies the wealth equation

$$\begin{aligned}\mathrm{d}V_t^{u^\star} = V_t^{u^\star}\Bigg[r\,\mathrm{d}t + \frac{\lambda_P^2}{\gamma}\,\mathrm{d}t + \frac{(\lambda^\star(t,V_t^{u^\star}))^2}{\gamma}\,\mathrm{d}t \,-\, (f(t))^{-\frac{1}{\gamma}}\,\mathrm{d}t + \delta \mathrm{d}t \\ + \frac{\lambda_P}{\gamma}\,\mathrm{d}W_t^P + \frac{\lambda^\star(t,V_t^{u^\star})}{\gamma}\,\mathrm{d}W_t\Bigg]\,,\end{aligned}$$

for which we have substituted the optimality candidates (3.46) in the original wealth equation. Recalling that $\lambda^\star(t,v)$ is a deterministic function in t and further does not depend on v and that $f(t)$ is a deterministic function as well, we see that $V_t^{u^\star}$ follows a log-normal distribution for all $t \geq 0$ with parameters being uniformly bounded for all $t \in [0,T]$. Since all moments of a log-normally distributed random variable exist, we have proven $(*)$. Furthermore Φ satisfies the HJB equation (3.8), i.e. for $u = u^\star = (\pi^{P^\star},\pi^{S^\star},k^\star,\lambda^\star)$, the choice (3.41) of the disutility function and the choice (3.40) of the consumpion rate we have:

$$\begin{aligned}&\Phi_t(s,V_s^{u^\star}) \,+\, \Phi_v(s,V_s^{u^\star})V_s^{u^\star}\left[r + \pi_s^{S^\star}\,\lambda_s^\star\,\sigma + \left(\pi_s^{P^\star} + \beta\,\pi_s^{S^\star}\right)\lambda^P\sigma^P + \delta - k_s^\star\right] \\ &+ 1/2\,\Phi_{vv}(s,V_s^{u^\star})\left(V_s^{u^\star}\right)^2\left[(\pi_s^{P^\star} + \beta\,\pi_s^{S^\star})^2(\sigma^P)^2 + (\pi_s^{S^\star}\sigma)^2\right] \\ &- \kappa\left(V_s^{u^\star}\right)^{1-\gamma}\frac{(\lambda_s^\star)^\alpha}{\alpha} + \frac{\left(V_s^{u^\star}k^\star\right)^{1-\gamma}}{1-\gamma} = 0\,.\end{aligned}$$

Then, for $u = u^\star$, the expectation of equation (3.60) using $\Phi(T,v) = v^{1-\gamma}/(1-\gamma)$ is:

$$\mathbb{E}^{t,v}\left[\frac{\left(V_T^{u^\star}\right)^{1-\gamma}}{1-\gamma}\right] - \mathbb{E}^{t,v}\left[\int_t^T \kappa\left(V_s^{u^\star}\right)^{1-\gamma}\frac{(\lambda_s^\star)^\alpha}{\alpha}\mathrm{d}s\right] + \mathbb{E}^{t,v}\left[\int_t^T \frac{\left(V_s^{u^\star}k^\star\right)^{1-\gamma}}{1-\gamma}\,\mathrm{d}s\right]$$
$$= J'(t,v;u^\star) = \Phi(t,v)\,.$$

The optimality of our candidates is finally shown if we have for all $u \in A'_\gamma(t,v)$:

$$\mathbb{E}^{t,v}\left[\frac{(V_T^u)^{1-\gamma}}{1-\gamma}\right] - \mathbb{E}^{t,v}\left[\int_t^T \kappa\,(V_s^u)^{1-\gamma}\frac{(\lambda_s)^\alpha}{\alpha}\,\mathrm{d}s\right] + \mathbb{E}^{t,v}\left[\int_t^T \frac{(V_s^u k)^{1-\gamma}}{1-\gamma}\,\mathrm{d}s\right]$$
$$= J'(t,v;u) \le \Phi(t,v)\,. \tag{3.61}$$

Also, since Φ satisfies the HJB equation (3.8), we get for all $u \in A'_\gamma(t,v)$:

$$\begin{aligned}\Phi_t(s,V_s^u) \;+\; &\Phi_v(s,V_s^u)V_s^u\left[r + \pi_s^S\,\lambda_s\,\sigma + (\pi_s^P + \beta\,\pi_s^P)\lambda^P\sigma^P + \delta - k_s\right]\\ &+ 1/2\,\Phi_{vv}(s,V_s^u)\,(V_s^u)^2\left[(\pi_s^P + \beta\,\pi_s^S)^2(\sigma^P)^2 + (\pi_s^{S\star}\sigma)^2\right]\\ &- \kappa\,(V_s^u)^{1-\gamma}\frac{(\lambda_s)^\alpha}{\alpha} + \frac{(V_s^u k)^{1-\gamma}}{1-\gamma} \le 0\,.\end{aligned}$$

Substituting this in (3.60) , recalling that $\Phi_v(t,v) = f(t)\,v^{-\gamma}$, we get:

$$\begin{aligned}\Phi(T,V_T^u) - \int_t^T \kappa\,(V_s^u)^{1-\gamma}\frac{\lambda_s^\alpha}{\alpha}\,\mathrm{d}s + \int_t^T \frac{(V_s^u k)^{1-\gamma}}{1-\gamma}\,\mathrm{d}s \;\le \Phi(t,v)\\ + \underbrace{\int_t^T (V_s^u)^{1-\gamma}f(s)\left(\pi_s^P + \beta\,\pi_s^S\right)\sigma^P\,\mathrm{d}W_s^P + \int_t^T (V_s^u)^{1-\gamma}f(s)\pi_s^S\sigma\,\mathrm{d}W_s}_{=:M_T^t}\,.\end{aligned} \tag{3.62}$$

To verify equation (3.61), we impose conditions under which the local martingale M^t is a martingale. Recall $\Phi_v(t,v) = f(t)\,v^{-\gamma}$ and calculate the quadratic variation of M^t

$$\begin{aligned}\langle M^t\rangle_T &= \int_t^T (V_s^u)^{2(1-\gamma)} f^2(s)\left([\pi_s^P+\beta\,\pi_s^S]^2(\sigma^P)^2+[\sigma\pi_s^S]^2\right)\,\mathrm{d}s\\ &\le \frac{\epsilon^{\frac{\epsilon}{1+\epsilon}}}{1+\epsilon}\sup_{0\le s\le T} f(s)^2\Big(\int_t^T (V_s^u)^{2\,(1-\gamma)\,\left(1+\frac{1}{\epsilon}\right)}\,\mathrm{d}s \\ &\qquad + \int_t^T \left([\pi_s^P+\beta\,\pi_s^S]^2(\sigma^P)^2+[\sigma\pi_s^S]^2\right)^{1+\epsilon}\,\mathrm{d}s\Big)\ ,\ \epsilon>0\ ,\end{aligned} \tag{3.63}$$

where the upper bound in the second line was achieved using inequality (A.1.2) given in Lemma 0.1.1 setting

$$x := (V_s^u)^{2(1-\gamma)} \text{ and } y := \left([\pi_s^P+\beta\,\pi_s^S]^2(\sigma^P)^2+[\sigma_s^\star\pi_s^S]^2\right)\ .$$

We show that M^t is a martingale by deriving the integrability of the quadratic variation $\langle M^t\rangle_T$. First we use that f is a continuous function on the compact set $[0,T]$ and is uniformly bounded, and thus $\sup_{0\le s\le T} f(s)^2$ is finite. We are left to deal with the two expressions in the brackets of (3.63). The second expression is bounded in expectation by assumption, see (3.43) in Def. 3.2.2, setting $\epsilon=\frac{1}{2}\tilde{\epsilon}$. In what follows we establish that that the first expression is finite by showing that

$$\mathbb{E}^{t,v}[(V_s^u)^\xi] < \infty \quad \text{uniformly}\ , \tag{3.64}$$

with $\xi = 4(1-\gamma)\left(1+\frac{1}{\epsilon}\right) < 0$ for $\gamma>1$ and $|\xi|<\infty$ since $\epsilon>0$.

Applying variation of constants, the solution of the wealth equation (3.5) starting at t with initial wealth $v = V_t^u$ is

$$V_s^u = v\, e^{(r+\delta)(s-t)+\int_t^s\left((\pi_{\tilde{s}}^P+\beta\,\pi_{\tilde{s}}^S)\lambda^P\sigma^P+\pi_{\tilde{s}}^S\lambda_{\tilde{s}}\sigma-k_{\tilde{s}}\right)\mathrm{d}\tilde{s}}\, e^{L_s^t-\frac{1}{2}\left\langle L^t\right\rangle_s}\,,$$

where $L_s^t = \int_t^s(\pi_{\tilde{s}}^P+\beta\,\pi_{\tilde{s}}^S)\sigma^P\mathrm{d}W_{\tilde{s}}^P + \int_0^t \pi_{\tilde{s}}^S\sigma\,\mathrm{d}W_{\tilde{s}}$ and $\left\langle L^t\right\rangle_s = \int_t^s(\pi_{\tilde{s}}^P + \beta\,\pi_{\tilde{s}}^S)^2(\sigma^P)^2+(\pi_{\tilde{s}}^S\sigma)^2\mathrm{d}\tilde{s}$.
Using this we have

$$(V_s^u)^\xi = v^\xi\, \underbrace{e^{\xi L_s^t-\frac{1}{2}\,\xi\left\langle L^t\right\rangle_s}}_{=:Z_s^t} \times \underbrace{e^{\xi\left[(r+\delta)(s-t)+\int_t^s\left((\pi_{\tilde{s}}^P+\beta\,\pi_{\tilde{s}}^S)\lambda^P\sigma^P+\pi_{\tilde{s}}^S\lambda_{\tilde{s}}\sigma-k_{\tilde{s}}\right)\mathrm{d}\tilde{s}\right]}}_{=:R_s^t}\ .$$

Thus, condition (3.64) is for example fulfilled when

$$\mathbb{E}^{t,v}[(R_s^t)^2] < \infty \quad \text{and} \quad \mathbb{E}^{t,v}[(Z_s^t)^2] < \infty\,.$$

The square of R^t is given by

$$(R_s^t)^2 = e^{2\,\xi[(r+\delta)(s-t)+\int_t^s(\pi^P_{\tilde{s}}+\beta\,\pi^S_{\tilde{s}})\lambda^P\sigma^P+\pi^S_{\tilde{s}}\lambda_{\tilde{s}}\sigma-k_{\tilde{s}}\,\mathrm{d}\tilde{s}]}\,,$$

which is uniformly bounded by a constant, see Def. 3.2.2, (3.43), (3.44) and (3.45) and recalling that $\xi < 0$ for $\gamma > 1$ as well as $|\xi| < \infty$ since $\epsilon > 0$, and keeping in mind that $k_t \geq 0$, $t \leq s \leq T$, by assumption. This directly implies the square integrability of R^t. The square of Z^t is given by

$$\begin{aligned}
(Z_s^t)^2 &= e^{2\,\xi\,L_s^t-\frac{1}{2}\,2\,\xi\,\langle L^t\rangle_s} \\
&= e^{2\,\xi\,L_s^t-(2\,\xi)^2\,\langle L^t\rangle_s} \times e^{(2\,\xi)^2\,\langle L^t\rangle_s-\frac{1}{2}\,2\,\xi\,\langle L^t\rangle_s} \\
&= e^{2\,\xi\,L_s^t-4\,\xi^2\,\langle L^t\rangle_s} \times e^{\xi\,(4\xi-1)\,\langle L^t\rangle_s} \\
&\leq \frac{1}{2}\Big[\underbrace{e^{4\,\xi\,L_s^t-\frac{1}{2}\,16\,\xi^2\,\langle L^t\rangle_s}}_{=:\tilde{Z}_s^t} + \underbrace{e^{2\,\xi\,(4\xi-1)\,\langle L^t\rangle_s}}_{=:\tilde{R}_s^t}\Big]\,,
\end{aligned}$$

where the last line is a straight forward upper bound. The second factor $\tilde{R}^t$ is uniformly bounded by a constant by condition (3.43) of Def. 3.2.2. To finally obtain the square integrability of Z^t, it remains to prove that the first factor $\tilde{Z}_s^t = e^{4\,\xi\,L_s^t-\frac{1}{2}\,16\,\xi^2\,\langle L^t\rangle_s}$, $t \leq s \leq T$, is integrable. However, $\tilde{Z}^t$ is a strictly positive local martingale since it is the stochastic exponential of the local martingale $4\,\xi\,L^t$. The Novikov condition holds by (3.43), i.e. $\mathbb{E}^{t,v}(e^{\frac{1}{2}\,16\,\xi^2\langle L^t\rangle_T}) < \infty$, and hence $\tilde{Z}^t$ is a true martingale and $\mathbb{E}^{t,v}(\tilde{Z}_s^t) = 1$, $t \leq s \leq T$. In summary, the local martingale M^t is therefore a martingale vanishing in expectation in (3.62), implying (3.61) for $u \in \mathcal{A}'_\gamma(t,v)$. □

3.3 Discussion and Implications of Results

The previous section established results on the optimal behavior of the individual and derived the participation constraint, i.e. conditions for the her to accept the offer by the smaller listed company. In the following we discuss the results by investigating the sensitivities of the optimal strategies and the participation constraint when varying model parameters.

3.3.1 Optimal Work Effort

Theorems 3.2.1 and 3.2.2 indicate the individual's maximized utility and associated optimal behavior in terms of personal portfolio selection, consumption and work effort decision, given that she accepts to job offer by the smaller listed company, all subject to the log utility set-up. We now investigate the sensitivity of the optimal work effort to variations of the work effectiveness characteristics and the time preferences. Note that the portfolio selection and consumption are in line with standard results in the log utility setup and are here of limited interest.

The individual is characterized by the work effectiveness parameters work productivity ($1/\kappa$, with $\kappa > 0$), and disutility stress ($\alpha > 2$) and the time preferences of consumption from work effort ($\rho \in \mathbb{R}$) and disutility ($\tilde{\rho} \in \mathbb{R}$), respectively. To produce results that have relativity to a base-level of work effort, as indicated by a base-level non-systematic Sharpe ratio control decision $\lambda_0 > 0$, the disutility C given by (3.16) is reparametrized to

$$C(t,v,\lambda) = e^{-\tilde{\rho}\,t}\,\frac{\tilde{\kappa}}{\alpha}\left(\frac{\lambda}{\lambda_0}\right)^{\alpha}, \quad \text{for } \lambda \geq 0\,, \quad \gamma > 0,$$

and the utility of wealth U_1 and the utility of consumption U_2 remain unchanged.

The individual's optimal work effort for the new disutility parametrization is

$$\lambda^{\star}(t,v) = \lambda_0^{\frac{\alpha}{\alpha-2}}\left(\frac{e^{\tilde{\rho}\,t}}{\tilde{\kappa}}f(t)\right)^{\frac{1}{\alpha-2}} \tag{3.65}$$

(see Theorem 3.2.1 for the optimal choice under the original parametrization). Assuming that the inverse work productivity satisfies

$$1/\tilde{\kappa} > \lambda_0^{-2}\, e^{|\tilde{\rho}|\,T}/K\,, \tag{3.66}$$

we guarantee that the optimal work effort $\lambda^\star$ is not less than the base level λ_0, i.e. $\lambda^\star \geq \lambda_0 > 0$. If not stated otherwise, the default values for the parameters are $\alpha = 5$, $1/\tilde{\kappa} = 1000$, $r = 0.05$, $\lambda_P = 0.20$, $\lambda_0 = 0.10$, $\rho = 0.10$, $\tilde{\rho} = -0.10$, $K = 1$, $\widehat{\delta} = 0.20$, and $T = 10$.

The individual's optimal work effort choice is positively related to her work productivity and negatively related to her disutility stress. This result is illustrated by Figures 3.1 and 3.2, which graph the optimal work effort $\lambda^\star$ versus time t and work productivity $1/\tilde{\kappa}$ and, time t and disutility stress α, respectively. Both figures indicate that the individual's optimal work effort is negatively related to time, i.e. $\lambda^\star$ is decreasing over time. The individual spends in general more work effort at the beginning of the time horizon. Note that the monotonicity of the optimal work effort depends on the sign of ρ, see discussion of Figure 3.4 below.

Figure 3.3 shows the optimal work effort choice $\lambda^\star$ w.r.t. the time preference of consumption ρ and time t. The figure indicates that with increasing time the optimal work effort decreases as already observed above. This implies that the individual is more productive at the beginning of her career path. The optimal work effort is also decreasing for increasing time preference of consumption ρ. An individual which has a higher consumption preference will deliver a lower work effort, especially at the beginning of the time horizon.

Figure 3.4 graphs the optimal work effort choice $\lambda^\star$ w.r.t. the time preference of disutility $\tilde{\rho}$ and time t. The optimal work effort is positively related to the time preference of work related disutility $\tilde{\rho}$, i.e. with increasing value of $\tilde{\rho}$ the individual is becoming more productive and delivers a higher level of the optimal work effort indicating a reasonable behavior: The higher the cost for spending work effort the lower is the optimal work effort. Note that positive values of $\tilde{\rho}$ are associated with work effort becoming cheaper over time. For this parameter set, we first observe over time an increase of work effort and then a decrease at the end of the time horizon. However, typically we expect $\tilde{\rho}$ to be negative, i.e., work effort becomes more expensive with the passing of time.

3.3.2 Participation Constraint

The participation constraint is given in Theorem 3.2.3. Denote $\delta^\star$ the minimal salary rate such that the participation constraint holds, i.e. $\delta^\star = \inf\{\delta \in \mathbb{R} : \delta \text{ satisfies (3.38)}\}$. Taking account of the reparametrization gives

$$\delta^\star = \begin{cases} \widehat{\delta} - \dfrac{(\alpha-2)}{2\,\alpha}\,\lambda_0^{\frac{2\alpha}{\alpha-2}}\,\dfrac{\int_0^T \left(\frac{e^{\tilde{\rho}\,t}}{\tilde{\kappa}}\right)^{\frac{2}{\alpha-2}} f(t)^{\frac{\alpha}{\alpha-2}}\,\mathrm{d}t}{K\,T + \frac{1}{\rho^2}\left[1 - e^{-\rho T}(1+\rho T)\right]}\,, & \text{for } \rho \neq 0\,, \\ \widehat{\delta} - \dfrac{(\alpha-2)}{2\,\alpha}\,\lambda_0^{\frac{2\alpha}{\alpha-2}}\,\dfrac{\int_0^T \left(\frac{e^{\tilde{\rho}\,t}}{\tilde{\kappa}}\right)^{\frac{2}{\alpha-2}} f(t)^{\frac{\alpha}{\alpha-2}}\,\mathrm{d}t}{K\,T + \frac{1}{2}\,T^2}\,, & \text{for } \rho = 0\,. \end{cases} \tag{3.67}$$

Now, $\alpha > 2$ by assumption and $f > 0$ by Theorem 3.2.1. And the minimal salary rate of the smaller listed company that is satisfying the participation constraint is always below the salary rate of the outside option, i.e. $\delta^\star < \widehat{\delta}$. The salary rate discount can be explained by the fact that the smaller company is offering in return for the reduced salary the possibility to affect the share price by work effort and thereby to increase the utility derived from the individual's investment. In the following we investigate the minimal required salary rate $\delta^\star$ depending on the individual's parameters (work productivity $1/\tilde{\kappa}$, disutility stress α, time preference of consumption ρ and time preference for work effort $\tilde{\rho}$) to characterize individuals that are attracted by an offer of the smaller listed company.

Figure 3.5 displays the minimal required salary rate $\delta^\star$ w.r.t. disutility stress α and work productivity $1/\tilde{\kappa}$. The minimal required salary rate is decreasing with increasing work productivity and increasing with increasing disutility stress. This means that a more productive individual is willing to accept a lower salary rate because she can compensate the loss of utility by the ability to improve the unsystematic Sharpe ratio λ. On the other hand, an individual with a higher disutility stress requires a higher salary rate to accept the contract from the smaller listed company.

The effect of the time preferences is shown in Figure 3.6. The required minimal salary rate $\delta^{\star}$ is graphed against the time preference of consumption ρ and the time preference of disutility from work effort $\tilde{\rho}$, respectively. Increasing the time preference parameter for consumption increases the minimal required salary rate. In contrast, the required minimal salary rate decreases with increasing time preference of disutility. This is attributed to the average disutility from work effort being lower for a higher value of $\tilde{\rho}$. The individual will deliver a higher work effort (see also Figure 3.4), and therefore accept a lower salary since she gains more utility from an improved unsystematic Sharpe ratio.

We summarize that the offered salary rate δ can act as a selection device for the smaller listed company. Under the assumption that potential job candidates have an identical outside option, the group of individuals satisfying a more restrictive participation constraint is in general more talented, i.e. the individuals exhibit a lower disutility stress α, a higher productivity $1/\tilde{\kappa}$, a lower time preference for consumption ρ, and a higher time preference for disutility from work effort $\tilde{\rho}$. Viewing the holdings in the own-company shares ($\pi^{S\star}(t) = \lambda^{\star}(t)/\sigma$) as a way of voluntarily linking the pay to performance, our results reflect common practice in executive remuneration. A more talented manager is in general attracted by a lower fixed salary component and a higher performance linked salary component.

3.3.3 Work Effort/Consumption Allocation

Figures 3.1 - 3.4 indicate that the optimal Sharpe ratio is decreasing with increasing time. The optimal consumption rate shows the opposite behaviour: It increases with increasing time and reaches its maximum at the end of the time horizon. This behaviour is shown in Figure 3.7, which graphs the optimal consumption rate $k^\star$ w.r.t. the time preference ρ and time t for fixed time horizon $T = 5$ years. This qualitative behaviour can be conceptualized mathematically more rigorously: The time dynamics of the optimal work effort and the optimal consumption weighted with an appropriate factor dependent on the corresponding time preferences ρ and $\tilde{\rho}$ are anti-proprotional:

$$(\alpha - 2) \log \left(\lambda^\star e^{-\frac{\tilde{\rho}}{(\alpha-2)} t} \right) + \log \left(k^\star e^{\rho t} \right) = \log \frac{\lambda_0^\alpha}{\tilde{\kappa}} .$$

This can be interpreted as a time budget being distributed between the work and the consumption of the individual as the weighted sum on the log scale sums up to a constant. If we now assume that the time allocated for the work effort and the time allocated for the consumption are suitable weighted functions on the log scale for both, we can even normalize the work effort-consumption allocation such that it sums up to 1, i.e. assuming that

$$\widehat{t(\lambda^\star)} = \frac{(\alpha - 2)}{\log(1+T)} \log \left(\lambda^\star e^{-\frac{\tilde{\rho}}{(\alpha-2)} t} \lambda_0^{-\frac{\alpha}{\alpha-2}} \tilde{\kappa}^{\frac{1}{\alpha-2}} \right) , \qquad (3.68)$$

and

$$\widehat{t(k^\star)} = \frac{1}{\log(1+T)} \left(\log(k^\star e^{\rho t}) + \log(1+T) \right) , \qquad (3.69)$$

yields

$$\widehat{t(\lambda^\star)} + \widehat{t(k^\star)} = 1 .$$

Figure 3.8 displays the work effort-consumption allocation for fixed work productivity $1/\tilde{\kappa} = 1000$, disutility stress $\alpha = 5$, base-level work effort $\lambda_0 = 0.10$, time preference of disutility $\tilde{\rho} = -0.10$, and varying time preferences of consumption, in particular, $\rho = 0.0$ (LUHS), $\rho = 0.05$ (RUHS), $\rho = 0.10$ (LBHS), and $\rho = 0.15$ (RBHS), respectively. It shows that the allocation sums constantly up to 1 and further that time spent at work is decreasing with the passing of time and conversely, time spent for consumption increases with the passing of time. The individual starts with a relatively high work effort and towards the end of the investment horizon the consumption increases to a level of 100%. We note that the starting level of work effort is decreasing with increasing values of the time preference of consumption ρ. This indicates that an individual with a higher time preference of consumption will consume more at the beginning of the time period than an individual with a lower time preference of consumption and vice versa. Note that for the limit $\rho \to 0$, equations (3.19) as well as (3.68) and (3.69), respectively, imply that $\widehat{t(\lambda^\star)} = 1$ and $\widehat{t(k^\star)} = 0$ at time $t = 0$, which means that a highly qualified individual with a zero time preference of consumption will spend her full time budget for work effort in the beginning of her employment. This result is shown in the left upper hand side of the figure.

3.4 Figures

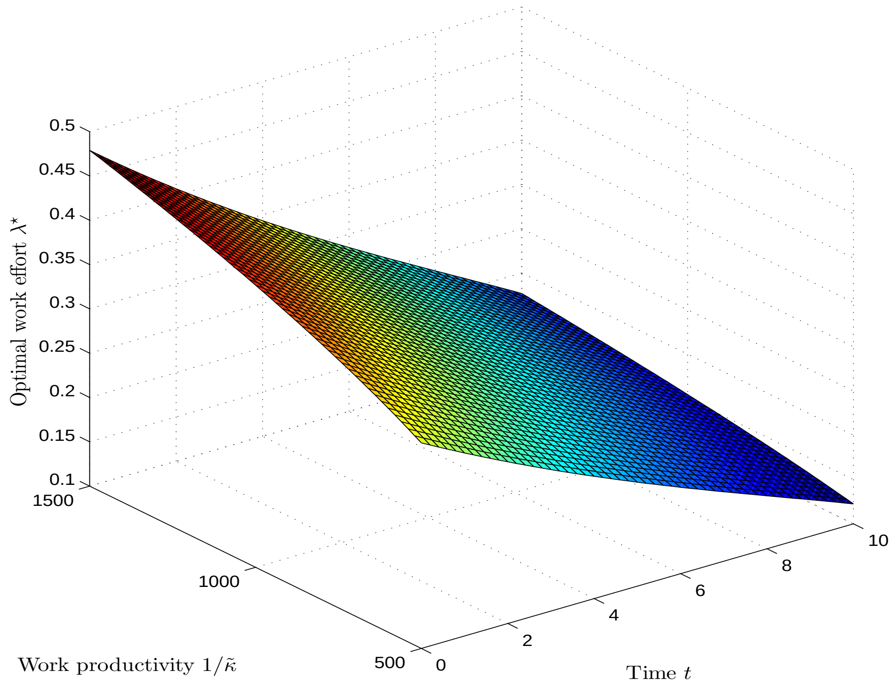

Figure 3.1: Optimal work effort $\lambda^\star$ w.r.t. work productivity $1/\tilde{\kappa}$ and time t for fixed disutility stress $\alpha = 5$, time preferences $\rho = 0.10$ and $\tilde{\rho} = -0.10$, $K = 1$, base-level work effort $\lambda_0 = 0.10$ and time horizon $T = 10$ years.

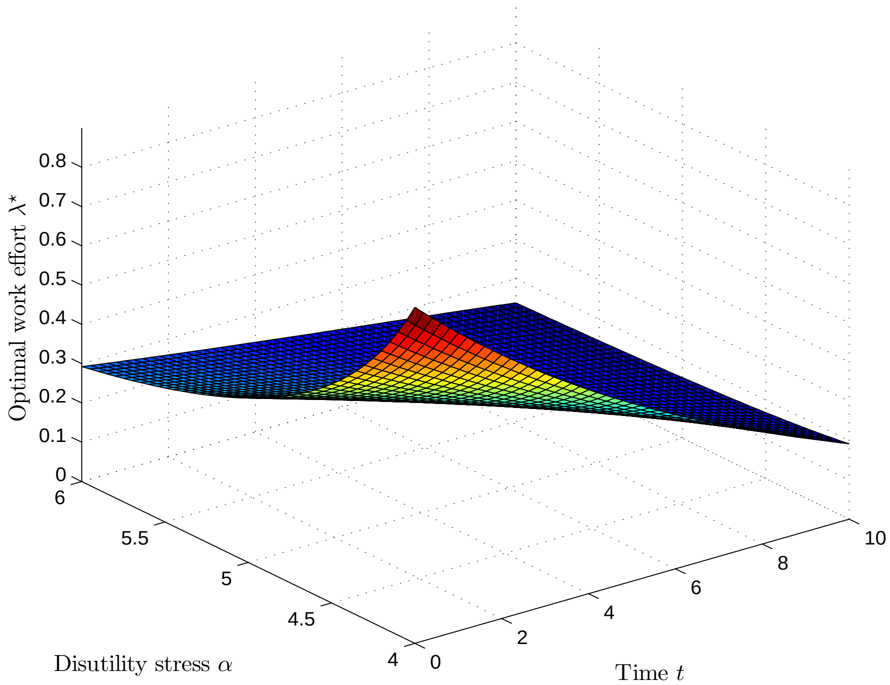

Figure 3.2: Optimal work effort $\lambda^\star$ w.r.t. disutility stress α and time t for fixed work productivity $1/\tilde{\kappa} = 1000$, time preferences $\rho = 0.10$ and $\tilde{\rho} = -0.10$, $K = 1$, base-level work effort $\lambda_0 = 0.10$ and time horizon $T = 10$ years.

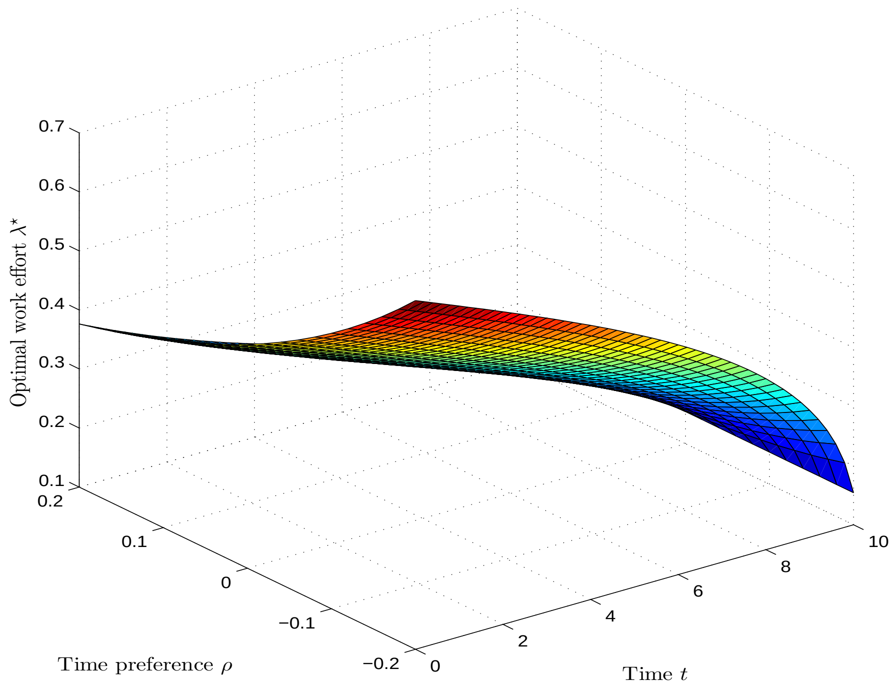

Figure 3.3: Optimal work effort $\lambda^\star$ w.r.t. the time preference of consumption ρ and time t for fixed work productivity $1/\tilde{\kappa} = 1000$, $\alpha = 5$, time preference $\tilde{\rho} = -0.10$, $K = 1$, base-level work effort $\lambda_0 = 0.10$ and time horizon $T = 10$ years.

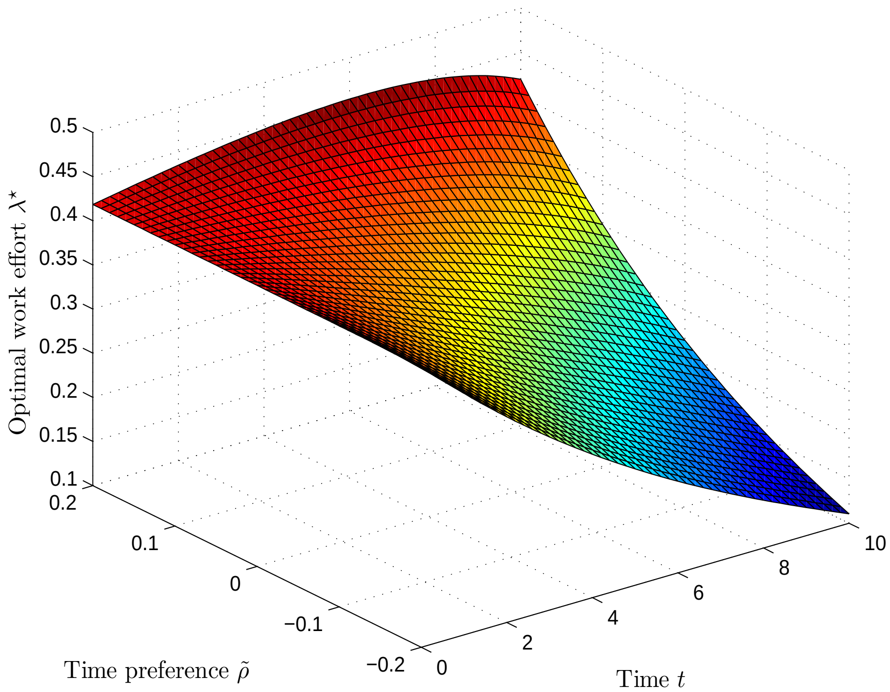

Figure 3.4: Optimal work effort $\lambda^\star$ w.r.t. the time preference of disutility $\tilde{\rho}$ and time t for fixed work productivity $1/\tilde{\kappa} = 1000$, disutility stress $\alpha = 5$, time preference $\rho = 0.10$, $K = 1$, base-level work effort $\lambda_0 = 0.10$ and time horizon $T = 10$ years.

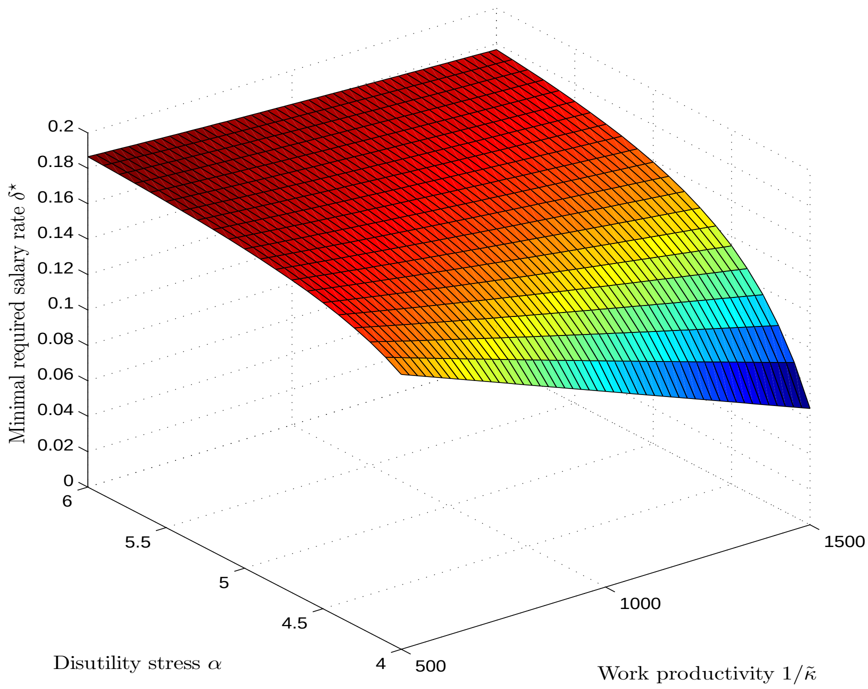

Figure 3.5: Minimal required salary rate $\delta^\star$ w.r.t. disutility stress α and work productivity $1/\tilde{\kappa}$ for fixed time preferences $\rho = 0.10$ and $\tilde{\rho} = -0.10$, $K = 1$, base-level work effort $\lambda_0 = 0.10$, outside salary rate $\widehat{\delta} = 0.2$, and time horizon $T = 10$ years.

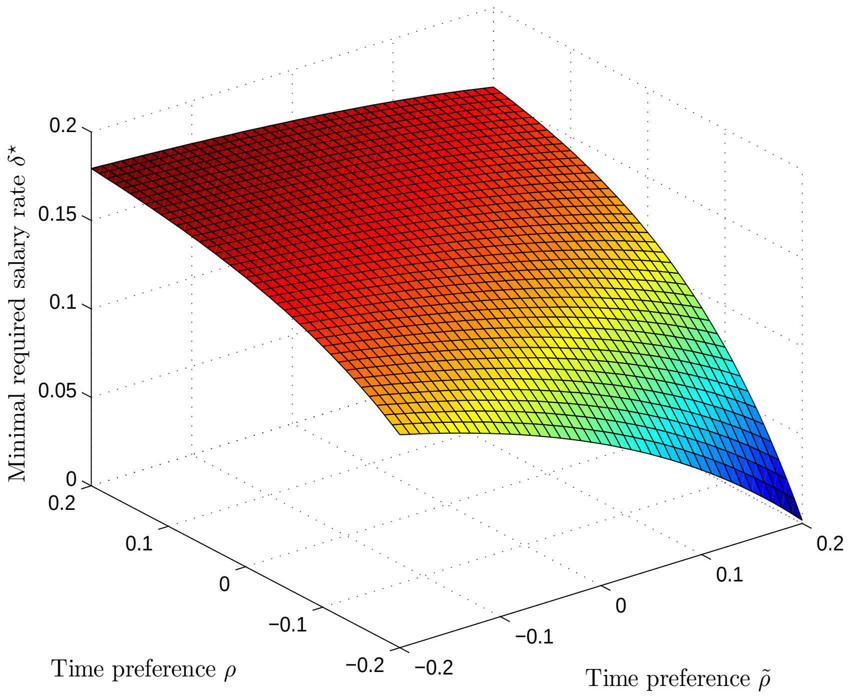

Figure 3.6: Minimal required salary rate $\delta^\star$ w.r.t. the time preferences ρ and $\tilde{\rho}$ for fixed work productivity $1/\tilde{\kappa} = 1000$, disutility stress $\alpha = 5$, $K = 1$, base-level work effort $\lambda_0 = 0.10$ and time horizon $T = 10$ years.

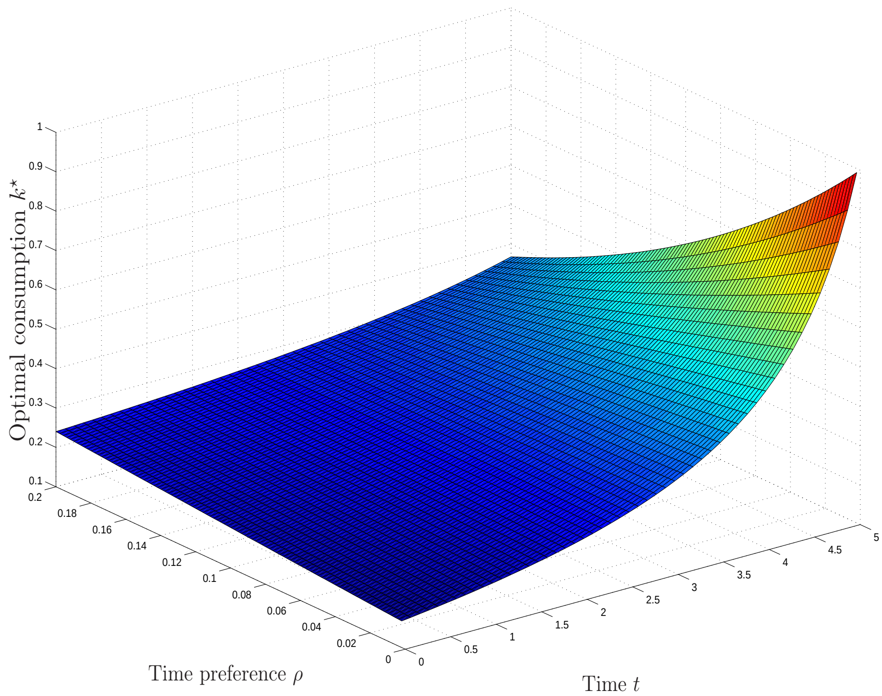

Figure 3.7: The log-utility individual's optimal consumption rate $k^\star$ w.r.t. the time preference ρ of consumption and time t for given time horizon $T = 5$ years.

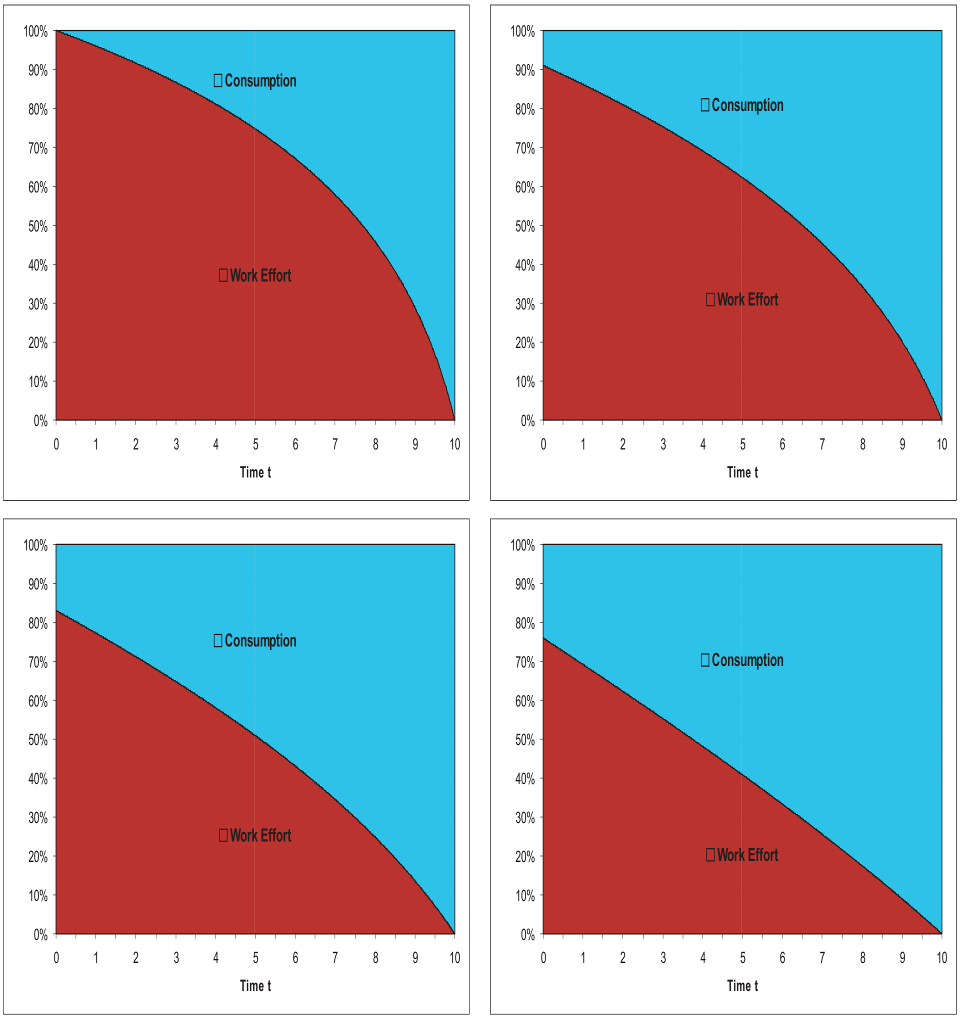

Figure 3.8: The log-utility individual's effort-consumption evaluation over time t for fixed work productivity $1/\tilde{\kappa} = 1000$, disutility stress $\alpha = 5$, base level work effort $\lambda_0 = 0.10$, time preference of disutility $\tilde{\rho} = -0.10$, and time preferences of consumption $\rho = 0.0$ (LUHS), $\rho = 0.05$ (RUHS), $\rho = 0.10$ (LBHS), and $\rho = 0.15$ (RBHS), respectively.

Conclusion and Outlook for Future Research

In Chapter 1, we establish a model framework that gives insight into an unconstrained executive's own-company stockholding and work effort preferences examined in a constant relative risk aversion set-up. Further, an indifference utility rationale is applied to determine the executive's up-front fair compensation. The executive's optimal work effort choice $\lambda^\star$ and fair compensation Δv depend sensibly on his characteristics, risk aversion γ, work productivity $1/\kappa$, and disutility stress α. The executive's risk aversion is indicated by his work effort over time; and for a given level of risk aversion, the executive's work effectiveness quality (where higher quality is associated with higher work productivity and/or lower disutility stress) is distinguished by his work effort at a point in time. For empirical purposes, work effort might be observed with an empirical non-systematic Sharpe ratio or some other company performance measure.

We demonstrate that an executive with higher work effectiveness (quality) undertakes more work effort, which is associated with higher fair (utility indifference) compensation. Thus the executive is rewarded twice for his quality. First he receives higher compensation as a direct reward; and second he benefits from his work effort via his own-company stockholding, which can be considered an indirect reward.

The extent to which the company benefits from the executive's work effectiveness depends prominently on his risk aversion. Only if he has sufficiently low risk aversion to take on a substantial own-company stockholding, he will have the incentive to apply substantial work effort for the benefit of the company. Consequently the executive's fair compensation is negatively related to his risk aversion.

Given identification of executive risk aversion and quality, our framework indicates the own-company stockholding and work effort of an unconstrained executive. This establishes a base case for theoretical or empirical assessment of the benefits or otherwise of constraining the executive with performance contracting.

A future extension of this framework is to allow the executive to invest in executive stock options with the company's stock price process as underlying. The aim is then to calculate optimal option portfolios for the executive applying the techniques to determine optimal option portfolios given in Korn and Trautmann (1999) and using the analytic representation of the value of executive stock options shown in Cvitanić, Wiener and Zapatero (2008). Optimal strategies will then be compared and contrasted to the strategies of an executive who cannot invest in options. This set-up reflects more the compensation structure of 'constrained' executives who are incentivized by the agent by including executive stock options in their compensation.

In Chapter 2, we establish the framework of Chapter 1 given that the executive is characterized by a constant absolute risk aversion set-up. Results are only obtained for the case of zero interest rates due to the limitations of this set-up. The results from Chapter 1 are confirmed to a large extent. The executive's optimal work effort choice $\lambda^\star$ depends on the work effectiveness parameters work productivity $1/\kappa$, and disutility stress α, but not on his risk aversion η as a consequence of only being able to solve this set-up for zero interest rates. However, the fair compensation Δv depends on the full parameter set and shows an analogous behaviour as in the constant relative risk aversion set-up. Again, an executive with higher work effectiveness (quality) undertakes more work effort, which results in a higher fair (utility indifference) compensation. Further the fair compensation is once more negatively related to the risk aversion of the executive. We summarize that the constant relative risk aversion set-up described and solved in Chapter 1 is widely confirmed by the constant absolute risk aversion set-up and is much more likely to produce reality-based results.

In Chapter 3, we establish a model framework that gives insight into an individual's occupational choice when she can choose between two different positions. She is offered an executive position in a smaller listed company where she can affect the company's share price by work effort. Alternatively, she can take up a mid-level management position with a larger company but then forgoes the possibility to affect the other company's share price. We identify conditions for the individual to work for the smaller listed company where the participation constraint is given in terms of the salary differential of the two job alternatives. In particular, we derive the minimal required salary $\delta^\star$ that needs to be offered by the smaller company to attract the individual and thereby characterize the participation constraint. In general, we find that a more talented individual requires a lower salary to be attracted to the smaller listed company. This salary pattern can be observed in practice, e.g., in the pharmaceutical industry, the IT sector, and the financial industry.

Given that the participation constraint holds, we give explicit solutions for the individual's utility maximizing behavior in terms of the investment strategy ($\pi = (\pi^P,\pi^S)$), consumption (k), and work effort (λ). Overall, our results depend sensibly on her characteristics, work productivity $1/\kappa$, disutility stress α, time preference of consumption ρ, and time preference of work effort $\tilde{\rho}$. We demonstrate that a highly-qualified individual with higher work effectiveness (quality) undertakes more work effort, which is associated with a lower minimal required salary $\delta^\star$. The main analysis is performed in the log-utility setting. However, we also explore the broader setup of constant relative risk aversion and derive a closed-form solution for the case when the risk aversion parameter γ is bigger than 1.

A future development of this work is to extend the semi-static game between the individual and the smaller listed company to a stochastic differential game. The aim of the company is then to maximize share holder value. The additional control available to the company is the quantity of share-based payments granted to the individual that affect her holdings in the company's shares. The stochastic differential game can then be investigated for equilibria. This setup is likely to provide more insight into the design of optimal share-based payments.

As a concluding remark of this work we can come back to the citation in the very beginning of this thesis. The results of this thesis indicate that a more productive executive (or highly-qualified individual, respectively) is characterized by a higher optimal work effort. This higher optimal work effort increases the value of the own-company's stock itself. These implications are reflected in the salary pattern of the model for the unconstrained executive; a more productive executive receives a higher fair up-front compensation. In contrast, the highly-qualified individual with the possibility of an outside option will accept a lower salary rate when she can directly influence the own-company's shares. Taking these effects into account, Warren Buffet was quite right in 2004, when he said that "If you have a great manager, you want to pay him very well.".

Appendix

A Generalized Inequality

The goal of this appendix is to prove a generalized version of the well-known inequality

$$x\,y \leq \frac{1}{2}\left(x^2 + y^2\right) \quad \text{for all } x,y \in \mathbb{R}\,. \tag{A.1.1}$$

The following lemma is very useful in order to derive the weakest possible integrability conditions in the Definitions 1.2.1, 2.1.3 and 3.2.2, which are needed in the proofs of the Verification Theorems 1.2.3, 2.2.2 and 3.2.5.

Lemma 0.1.1
Let $\epsilon \in (0,\infty)$. Then it holds for all $x,y \in \mathbb{R}$

$$x\,y \leq |x\,y| \leq \frac{\epsilon^{\frac{\epsilon}{1+\epsilon}}}{1+\epsilon}\left(|x|^{1+\frac{1}{\epsilon}} + |y|^{1+\epsilon}\right)\,. \tag{A.1.2}$$

Proof. W.l.o.g. assume that $x,y \in \mathbb{R}_0^+$. Further let $\alpha > 1$ and $\beta > 1$. First we will prove that

$$c_\alpha\left(y^\alpha + x^\beta\right) \geq x\,y\,,$$

for a suitable choice of $c_\alpha \in \mathbb{R}^+$ and $\beta = \beta(\alpha)$. This identity can be rewritten as

$$y^\alpha - c\,x\,y + x^\beta \geq 0\,, \text{ where } c = c_\alpha^{-1}\,.$$

Fix $x \in \mathbb{R}_0^+$. The function $f_x : [0,\infty) \subset \mathbb{R}_0^+ \to \mathbb{R}$ defined as

$$f_x(y) := y^\alpha - c\,x\,y + x^\beta$$

is strictly convex for $\alpha > 1$, hence the zero of $\partial f_x / \partial y$ is the unique minimum of f_x, which is given by

$$y_0 = \left(\frac{c\,x}{\alpha}\right)^{\frac{1}{\alpha-1}} .$$

Plugging in this minimum gives

$$\begin{aligned} & \left(\frac{c\,x}{\alpha}\right)^{\frac{\alpha}{\alpha-1}} - c\,x \left(\frac{c\,x}{\alpha}\right)^{\frac{1}{\alpha-1}} + x^{\beta} \geq 0 \\ \Leftrightarrow \quad & (c\,x)^{\frac{\alpha}{\alpha-1}} \left(\alpha^{-\frac{\alpha}{\alpha-1}} - \alpha^{-\frac{1}{\alpha-1}}\right) + x^{\beta} \geq 0 \\ \Leftrightarrow \quad & c^{\frac{\alpha}{\alpha-1}}\, \alpha^{-\frac{\alpha}{\alpha-1}}\, (1-\alpha) + x^{\beta - \frac{\alpha}{\alpha-1}} \geq 0 \end{aligned}$$

Choosing $\beta = \beta(\alpha) = \frac{\alpha}{\alpha-1}$ (note $\beta > 1$ since $\alpha > 1$) yields

$$\begin{aligned} & c^{\frac{\alpha}{\alpha-1}}\, \alpha^{-\frac{\alpha}{\alpha-1}}\, (1-\alpha) \geq -1 \\ \Leftrightarrow \quad & c \leq \alpha\,(\alpha-1)^{-\frac{\alpha-1}{\alpha}} , \end{aligned}$$

which implies that

$$c_\alpha = \frac{1}{c} \geq \frac{(\alpha-1)^{\frac{\alpha-1}{\alpha}}}{\alpha} .$$

Putting all this together, we have shown that for the choice $\beta = \frac{\alpha}{\alpha-1}$ and with $c_\alpha = (\alpha-1)^{\frac{\alpha-1}{\alpha}}/\alpha$ the following inequality holds

$$x\,y \leq \frac{(\alpha-1)^{\frac{\alpha-1}{\alpha}}}{\alpha} \left(y^{\alpha} + x^{\frac{\alpha}{\alpha-1}}\right) .$$

Now choosing $\alpha = (1+\epsilon)$ (note $\alpha = (1+\epsilon) > 1$, since $\epsilon \in (0,\infty)$) yields the claimed result of the lemma. □

Remark 0.1.1

Choosing $\epsilon = 1$ *yields the well-known inequality* (A.1.1) *as a special case of* (A.1.2).

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