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PF-MPC:
Particle Filter-Model Predictive Control

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Vorwort

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Darüber hinaus bietet die Reihe ein Forum für die Berichterstattung über die zahlreichen Kooperationsprojekte des Instituts mit Partnern aus Industrie und Wirtschaft.

Berichterstattung heißt hier Dokumentation des Transfers aktueller Ergebnisse aus mathematischer Forschungs- und Entwicklungsarbeit in industrielle Anwendungen und Softwareprodukte – und umgekehrt, denn Probleme der Praxis generieren neue interessante mathematische Fragestellungen.

Prof. Dr. Dieter Prätzel-Wolters
Institutsleiter

Kaiserslautern, im Juni 2001

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PF-MPC:
Particle Filter - Model Predictive Control

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Januar 2011

Abstract

In this article, a new model predictive control approach to nonlinear stochastic systems will be presented. The new approach is based on particle filters, which are usually used for estimating states or parameters. Here, two particle filters will be combined, the first one giving an estimate for the actual state based on the actual output of the system; the second one gives an estimate of a control input for the system. This is basically done by adopting the basic model predictive control strategies for the second particle filter. Later in this paper, this new approach is applied to a CSTR (continuous stirred-tank reactor) example and to the inverted pendulum.

1 Introduction

Model Predictive Control (MPC) is an established technique used for the control of mainly linear or linearized deterministic technical systems. Since nonlinear systems with intrinsic non-Gaussian disturbances are abundant especially in chemical and biological applications, there is an increasing need for the development of control strategies for nonlinear stochastic systems. With usual MPC, the control problem is formulated as an optimization problem. We show that, in a stochastic setting, the control problem can be reformulated as an estimation problem, in fact solutions are estimators based on smoothing densities of the stochastic system. It is generally impossible to compute analytical solutions to this problem in the case of nonlinear non-Gaussian systems. During the last 15 years, Sequential Monte Carlo methods (particle filters) have proved to be a powerful tool for the computation of approximate solutions. In this article, we want to show that by using two nested particle filters, one for state estimation (filtering), and one for horizon prediction, the MPC idea can be directly transferred to stochastic nonlinear systems. We will also show that complex control strategies can be easily incorporated into the procedure.

The article is structured as follows: After this introduction, we first describe the standard MPC approach for deterministic systems. Section 3 is devoted to stochastic state space models and Sequential Monte Carlo (SMC) methods. Our approach of applying two particle filters to the realization of MPC in this stochastic setting will be presented in section 4. We relate our approach to previous work in section 5, followed by the application of our method to two different stochastic systems in section 6. Conclusions and future work in section 7 are followed by an appendix where we present the proof of the theorem stating that our approach is indeed correspondent to classical MPC in the special case of deterministic systems.

2 Model Predictive Control

Model Predictive Control (MPC) is a class of predictive controllers which started developing in the late seventies. In contrast to the traditional control schemes,

such as PID-control, the input signal determined by an MPC-controller is not derived by only taking the actual system state into account. Instead, the MPC approach makes use of a process model which describes the system dynamics to predict the future behavior of the considered real system for a certain prediction horizon T_p. Based on this prediction the control inputs are determined in such a way that a given objective function will be minimized. Then the first input of the obtained control input sequence is applied to the system. At the next sample point, prediction and minimization are repeated for the same horizon, but are shifted one time step. An introduction to MPC can be found in [6].

2.1 Deterministic formulation of MPC

In this section, we recall the classical deterministic MPC approach. For discrete times $k = 0, 1, 2, \ldots$, the control system is given in state space representation by the difference equations

$$\begin{aligned} x_{k+1} &= f(x_k, u_k), \\ y_k &= g(x_k), \end{aligned} \tag{1}$$

with given initial state x_0 and subject to constraints given by

$$\begin{aligned} x_k &\in \mathcal{X} \subset \mathbb{R}^n \\ u_k &\in \mathcal{U} \subset \mathbb{R}^m. \end{aligned}$$

Here, x_k is the state, u_k the input and y_k the output of the system at time step k. Further $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$, $g : \mathbb{R}^n \to \mathbb{R}^q$ and $k \in \mathbb{N}$. Usually $\mathcal{X}$ and $\mathcal{U}$ are given by box constraints:

$$\begin{aligned} \mathcal{X} &= \{x \in \mathbb{R}^n \,|x_{min} \leq x \leq x_{max}\} \\ \mathcal{U} &= \{u \in \mathbb{R}^m \,|u_{min} \leq u \leq u_{max}\} \end{aligned}$$

where x_{min}, x_{max}, u_{min} and u_{max} are constant vectors.

The MPC strategy is to solve an open-loop optimal control problem at every time step k for a certain prediction horizon T_p. For a fixed time k, this optimal control problem can be formulated as an optimization problem with respect to a certain functional $J(x_k, \bar{u}_{k:(k+T_p)}, T_p)$, where we define $\bar{u}_{k:(k+T_p)} := \{\bar{u}_k, \ldots, \bar{u}_{k+T_p}\}$. The bar over the variables indicates that the control inputs $\bar{u}_j$, $j = k, k+1, \ldots, k+T_p$, are meant to be predicted at the given fixed time point k. It has to be distinguished from the real controls u_k for variable time k.

The optimal control problem is generally stated as

$$\min_{\bar{u}_{k:(k+T_p)}} J(x_k, \bar{u}_{k:(k+T_p)}, T_p) \tag{2}$$

subject to

$$\bar{x}_{j+1} = f(\bar{x}_j, \bar{u}_j) \tag{3}$$
$$\bar{y}_j = g(\bar{x}_j) \tag{4}$$

for $j \in \{k, \ldots, k+T_p\}$ with $\bar{x}_j \in \mathcal{X} \subset \mathbb{R}^n$, $\bar{u}_j \in U \subset \mathbb{R}^m$. The bar over the state variables $\bar{x}_j$ indicates again, that the respective states are meant to be predictions of the states at the fixed time k, to be distinguished from the actual states x_k when k varies. Usually, further constraints on the controls $\bar{u}_j$ are required, e.g. that the difference between successive controls $\Delta\bar{u}_j = \bar{u}_j - \bar{u}_{j-1}$ is small. In the ideal case, the initial value $\bar{x}_k$ is the actual state of the system at time k, $\bar{x}_k = x_k$. Usually, either only disturbed measurements of the states are available, or, if the state is not fully accessible, it has to be estimated via an observer based on the actual system output y_k.

The functional $J = J(x_k, \bar{u}_{k:(k+T_p)}, T_p)$ usually has the form

$$J = \sum_{j=k}^{k+T_p} \|\bar{u}_j - \bar{u}_{j-1}\|_Q^2 + \sum_{j=k+1}^{k+T_p} \|s_j - \hat{x}_j\|_R^2.$$

where the norms denote weighted Euclidean norms with weights given by matrices Q and R respectively. The first term realizes the mentioned contraints on $\Delta\bar{u}_j$ and penalizes too large values for these differences. The second term penalizes deviances of the system states from given setpoints s_j. These setpoints denote the trajectory of the system states which shall be preferrably attained through the control of the system.

Let $u^\star_{k:(k+T_p)}$ denote the optimal solution of the open-loop optimal control problem obtained at time k, i.e.

$$u^\star_{k:(k+T_p)} = \arg\min_{\bar{u}_{k:(k+T_p)}} J(x_k, \bar{u}_{k:(k+T_p)}, T_p).$$

Generally, only the first value $u^\star_k$ of this sequence will be used as control input for the time step k, and the rest of the sequence will be discarded. At the next time step $k+1$, when a new system output y_k is available, the optimization procedure will be repeated. In this way, one hopes to circumvent problems arising from differences between the real system and the model used for prediction, for instance due to linearization or time-discretization effects, or due to unmodelled disturbances or imperfect state estimates.

3 Particle Filter

We will now proceed from deterministic systems to stochastic systems. Since all real systems show stochasticity in some way, be it measurement noise or intrinsic disturbances, and since there is always some uncertainty left in modeling, it is natural to include this stochasticity into the models used for the purpose of control. While Gaussian noise is often straight away to be included especially into linear systems to which classical MPC can be applied, non-Gaussian disturbances and nonlinearity are often significant in chemical and biological applications. In these cases, classical MPC may fail due to the fact that it usually only works on linearized or approximated models. In this section, we therefore describe general stochastic state space models and the particle filter algorithm which can be used to compute approximate solutions to estimation problems occurring in these models. In the following section, we will then use the particle filters for the realization of an MPC approach with nonlinear, non-Gaussian stochastic state space models.

In general, the deterministic system given by eq. (1) can be augmented by stochastic components (variables) v_k and η_k, denoting state and observation noise, respectively. With difference equations, the stochastic system can be written as

$$\begin{aligned} x_{k+1} &= f(x_k, u_k, v_k), \\ y_k &= g(x_k, \eta_k). \end{aligned}$$

Since for all k, the values v_k and η_k are realizations of stochastic variables, as x_k and y_k are, too, we could alternatively define our system using the corresponding conditional probability densities

$$\begin{aligned} x_{k+1} &\sim a_k(\,.\,|x_k, u_k), \\ y_k &\sim b_k(\,.\,|x_k), \end{aligned}$$

with an additional initial probability density $x_0 \sim a_0(\,.\,)$ (the symbol $\sim$ means "sampled from"). In the following, we will use both descriptions interchangeably. The latter representation of the system is called stochastic general state space model. In the following paragraphs, we will give further details. To simplify further notation, we will omit the control u_k when writing the conditional densities a_k.

3.1 Definition of Stochastic General State Space model

Let $(\Omega, \mathcal{F}, P)$ be a probability space, and let $(\mathcal{X}, \mathcal{A}_\mathcal{X})$ and $(\mathcal{Y}, \mathcal{A}_\mathcal{Y})$ be measurable spaces, i.e. $\mathcal{A}_\mathcal{X}$ and $\mathcal{A}_\mathcal{Y}$ are σ-algebras on the sets $\mathcal{X}$ and $\mathcal{Y}$,

respectively. Furthermore, let a σ-finite reference measure $\mu_{\mathcal{X}}$ and $\mu_{\mathcal{Y}}$ be defined on each of these measurable spaces. Let $X_k : \Omega \longrightarrow \mathcal{X}$ denote a random variable on Ω for each $k \in \mathbb{N}$, and $Y_j : \Omega \longrightarrow \mathcal{Y}$ denote a random variable on Ω for each $j \in \mathbb{N}^*$.

The general state space model consists of an unobservable state process $(X_k)_{k\in\mathbb{N}}$ and an observation process $(Y_k)_{k\in\mathbb{N}^*}$. The state process is a Markov chain, i.e. a stochastic process with Markov property. This means that X_k conditioned on the past values is dependent only on the previous value, i.e.

$$P\left(X_k \in dx_k | X_{k-1} = x_{k-1}, \ldots, X_0 = x_0\right) = P\left(X_k \in dx_k | X_{k-1} = x_{k-1}\right).$$

Due to this property, the Markov process is completely described by the initial distribution $P(X_0 \in dx_0)$ and the state evolution for $X_{k-1} \to X_k$ (transition distribution)

$$P\left(X_k \in dx_k | X_{k-1} = x_{k-1}\right). \tag{5}$$

The observation (or measurement) Y_k at time k is conditionally independent of all other variables given X_k. The dependency between Y_k and X_k is given through the observation distribution

$$P\left(Y_k \in dy_k | X_k = x_k\right). \tag{6}$$

In the following, it is assumed that the initial distribution for X_0 and the transition distribution for $X_k|X_{k-1}$ are $\mu_{\mathcal{X}}$-continuous, and that the observation distribution for $Y_k|X_k$ is $\mu_{\mathcal{Y}}$-continuous. Then due to the Radon-Nikodym theorem the distributions in eqs. (5) and (6) can be expressed by densities a_k and b_k, respectively:

$$P\left(X_k \in dx_k | X_{k-1} = x_{k-1}\right) =: a_k(x_k|x_{k-1})\mu_{\mathcal{X}}(dx_k)$$

and

$$P\left(Y_k \in dy_k | X_k = x_k\right) =: b_k(y_k|x_k)\mu_{\mathcal{Y}}(dy_k).$$

The general state space model is then completely given by providing the initial state density $a_0(x_0)$, the state transition density $a_k(x_k|x_{k-1})$, and the observation density $b_k(y_k|x_k)$. Graphically the general state space model can be depicted as in Figure 1. Sometimes this model is referred to as *Hidden Markov Model* (HMM), but usually systems are only called HMM if the state space $\mathcal{X}$ is discrete and finite. We do not require these or other properties like linearity or Gaussianity.

Since the state process $X_{0:k}$ is not observed directly (i.e. realizations of this process are not known) but solely through the observation process $Y_{1:k}$, the main task in a general state space model is to do inferences on the states $X_{0:k}$ given only realizations (observations, measurements) $y_{1:k}$ of the observation

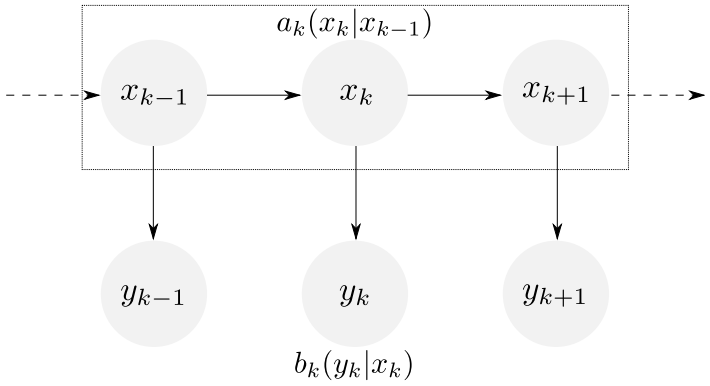


Figure 1 Diagram of the General State Space Model

process $Y_{1:k}$ (called state estimation or filtering). The basis for this is the computation of the joint smoothing distributions

$$P(X_{0:k} \in dx_{1:k}|Y_{1:k} = y_{1:k})$$

or, for practical reasons, the filtering distributions

$$P(X_k \in dx_k|Y_{1:k} = y_{1:k})$$

for each k. Once these distributions are known, the computation of point or interval estimates is possible. Nevertheless, the computation of joint smoothing and filtering distributions (or their densities) in general state space models is a major problem.

As in the majority of particle filter literature, given an arbitrary random variable X, we will now write $p(dx)$ for distributions $P(X \in dx)$, and if they have densities with respect to a reference measure μ, they will be denoted by $p(x)$. Except for the densities a_k and b_k, all other densities will just be denoted by p. Which density is meant will then be clear by its argument or by its context. For instance, the joint smoothing distribution $P(X_{0:k} \in dx_{0:k}|Y_{1:k} = y_{1:k})$ will just be written as

$$p(dx_{0:k}|y_{1:k}).$$

Furthermore, x will now denote the random variable X and its realization at the same time. From now on it will be assumed that $\mathcal{X} = \mathbb{R}^n$ and $\mathcal{Y} = \mathbb{R}^q$, the σ-algebras $\mathcal{A}_\mathcal{X}$ and $\mathcal{A}_\mathcal{Y}$ are the corresponding Borel σ-algebras, and also that the reference measures $\mu_\mathcal{X}$ and $\mu_\mathcal{Y}$ are the Lebesgue measures.

3.2 The Particle Filter algorithm

The particle filter is a sequential Monte Carlo method which allows the approximate computation of joint smoothing and filtering densities in general

state space models. Introductions to particle filters can also be found in [7], [2] and [11]. The basic idea is to approximate the targeted smoothing distribution $p(dx_{0:k}|y_{1:k})$ by a cloud of independent and identically distributed (i.i.d.) random samples $\{x_{0:k}^{(i)}, i = 1, \ldots, N_s\}$ with associated weights $\{w_k^{(i)}, i = 1, \ldots, N_s\}$, which satisfy $\sum_i w_k^{(i)} = 1$, so that the target distribution at time k can be approximated by

$$p(dx_{0:k}|y_{1:k}) \approx \sum_{i=1}^{N_s} w_k^{(i)} \delta_{x_{0:k}^{(i)}}(dx_{0:k})$$

where $\delta_{x_{0:k}^{(i)}}(dx_{0:k})$ is the delta-Dirac mass in $x_{0:k}^{(i)}$:

$$\delta_{x_{0:k}^{(i)}}(dx_{0:k}) = \begin{cases} 1 & \text{if } x_{0:k}^{(i)} \in dx_{0:k}, \\ 0 & \text{else.} \end{cases}$$

The tuples $(x_{0:k}^{(i)}, w_k^{(i)})$ are also referred to as particles. Due to degeneration problems one usually restricts the attention to the filter distribution $p(dx_k|y_{1:k})$, $k = 1, \ldots, N$, approximated by the particle cloud $(x_k^{(i)}, w_k^{(i)})_{i=1}^{N_s}$ in step k which can be interpreted as a representation of an empirical measure

$$\widehat{p}(dx_k) = \sum_{i=1}^{N_s} w_k^{(i)} \delta_{x_k^{(i)}}(dx_k).$$

For instance, it can be used to compute estimates of the mean, median, confidence intervals and quantiles of the filter distribution.

One particle filter algorithm is the SISR algorithm, which will be described in the following paragraph.

3.3 SISR algorithm

The SISR (Sequential Importance Sampling with Resampling) algorithm can be implemented with any choice of importance densities

$$q_0(x_0) \qquad \text{and} \qquad q_j(x_j|y_j, x_{j-1}), \quad j = 1, \ldots, k,$$

provided that their support is larger than the support of

$$a_0(x_0) \qquad \text{and} \qquad b_j(y_j|x_j)a_j(x_j|x_{j-1}), \quad j = 1, \ldots, k,$$

respectively. The SISR algorithm works on a set of N_s state samples $x_j^{(i)}$ and weights $w_j^{(i)}$, $i = 1, \ldots, N_s$, jointly called particles, where the sample size N_s

should be large. The state samples will be propagated according to the importance density q_j iteratively for $j = 1, \ldots, k$, and the weights will be updated using the system model and the measurements. To avoid a degeneration of the particle set, a resampling step is necessary. The degeneration can be measured in terms of the effective sample size (ESS) which can be estimated via

$$\widehat{N}_{\text{eff}} = \frac{N_s}{\sum_{i=1}^{N_s} \left(w_j^{(i)}\right)^2}.$$

If this estimate is below a threshold $N_T < N_s$, a resampling step will be done. Resampling (randomly) selects particles from the given particle set with probability given by the weights. This selection can be done in several ways. We use systematic resampling to compute the resampled particles $x_*^{(i)}$ and weights $w_*^{(i)}$:

- Divide the unit interval into N_s disjoint intervals

$$I_i = \left[\sum_{s=1}^{i-1} w_j^{(s)}, \sum_{s=1}^{i} w_j^{(s)}\right)$$

- Sample $\tilde{u}$ uniformly from $\mathcal{U}[0, 1]$ and set

$$u^{(l)} = \frac{l - 1 + \tilde{u}}{N_s}$$

- Set $x_*^{(l)} = x^{(i)}$ with i such that $u^{(l)} \in I_i$ and set $w_*^{(l)} = 1/N_s$ for $l = 1, \ldots, N_s$.

The complete SISR algorithm is then given by:

- Initialization:
  - Sample $x_0^{(i)}$ from $q_0(x_0)$ independently for $i = 1, \ldots, N_s$.
  - Compute the weights

$$w_0^{(i)} = \frac{a_0(x_0^{(i)})}{q_0(x_0^{(i)})}.$$

- Iteration over $j = 1, \ldots, k$:
  - If $\widehat{N}_{\text{eff}} < N_T$, resample.
  - Sample $x_j^{(i)}$ from $q_j(x_j | y_j, x_{j-1}^{(i)})$ independently for $i = 1, \ldots, N_s$.
  - Compute the unnormalized weights

$$\tilde{w}_j^{(i)} = w_{j-1}^{(i)} \frac{b_j(y_j | x_j^{(i)}) a_j(x_j | x_{j-1}^{(i)})}{q_j(x_j^{(i)} | y_j, x_{j-1}^{(i)})}$$

$$\text{for } i = 1, \ldots, N_s.$$

  - Compute the normalized weights

$$w_j^{(i)} = \frac{\tilde{w}_j^{(i)}}{\sum_{\iota=1}^{N_s} \tilde{w}_j^{(\iota)}} \qquad \text{for } i = 1, \ldots, N_s.$$

– Compute some statistical estimates (mean, median, confidence intervals etc.) based on the weighted particles $(x_j^{(i)}, w_j^{(i)})$, $i = 1, \ldots, N_s$.

3.4 Choice of importance distribution

The effectiveness of the algorithm is heavily dependent on the choice of the importance densities q_j. The optimal choice is

$$q_0(x_0) = a_0(x_0)$$

and

$$\begin{aligned} q_j(x_j|y_j, x_{j-1}) &= p(x_j|y_j, x_{j-1}) \\ &= \frac{b_j(y_j|x_j)a_j(x_j|x_{j-1})}{p(y_j|x_{j-1})} \end{aligned} \tag{7}$$

with unnormalized weights

$$\tilde{w}_j^{(i)} = w_{j-1}^{(i)} p(y_j|x_{j-1})$$

where

$$p(y_j|x_{j-1}) = \int b_j(y_j|x_j)a_j(x_j|x_{j-1})dx_j.$$

Unfortunately, this choice is usually not available for two reasons: sampling from $p(x_j|y_j, x_{j-1})$ is impossible and/or $p(y_j|x_{j-1})$ is not analytically computable.

In contrast, the easiest but not always good choice is to use the initial and transition densities of the model:

$$q_0(x_0) = a_0(x_0)$$

and

$$q_j(x_j|y_j, x_{j-1}) = a_j(x_j|x_{j-1}). \tag{8}$$

In this case, the unnormalized weights are just

$$\tilde{w}_j^{(i)} = w_{j-1}^{(i)} b_j(y_j|x_j) \qquad \text{for } i = 1, \ldots, N_s.$$

4 PF-MPC: Particle Filter - Model Predictive Control

The main idea behind our new approach is to use two distinct particle filters, one serving as state estimator in the standard way, and the other as predictor

for the control input of the model. This latter usage of the particle filter distinguishes our approach from others which use the particle filter algorithm; there, it is mostly used for state estimation only. In detail, our approach is the following:

Given a stochastic control system of the form

$$x_k = f(x_{k-1}, u_{k-1}, v_{k-1}),$$
$$y_k = g(x_k, \eta_k),$$

with initial condition $x_0 = \hat{x}_0 + v_{-1}$, where f and g can be nonlinear, and where the noise v_{k-1} and η_k need not be Gaussian, we may equivalently write this with corresponding state transition and observation densities (see beginning of section 3)

$$a_k(x_k|x_{k-1}, u_{k-1}),$$
$$b_k(y_k|x_k),$$

with initial density $a_0(x_0)$. During the run of our proposed MPC controller, when, at a fixed time k, the measurements y_k become available und the next control input $u_k^\star$ has to be determined, the first particle filter is applied with these densities as underlying model. This is done with the aim of estimating the filter distribution

$$p(dx_k|y_{1:k}, u_{k-1})$$

by approximating it with a cloud of particles (see section 3). From this particle cloud, a point estimate $\hat{x}_k$ for the actual state could be computed which could be used in the optimization procedure of the usual MPC approach, as it has been previously proposed. In contrast, we forward the complete particle cloud (together with their weights) to the second particle filter, which is initialized with these particles. The second particle filter works on different states and "observations", as will be explained in the following. At the fixed time k, each control sequence $\bar{u}_{k:k+T_p}$ for the horizon $j = k, k+1, \ldots, k+T_p$ can be seen as a realization of a stochastic process (Markov chain) $\bar{U}_{k:k+T_p}$ which is completely determined by its initial density $\bar{a}_{u,k}(\bar{u}_k)$ and the transition densities $\bar{a}_{u,j}(\bar{u}_j|\bar{u}_{j-1})$. These densities have to be provided by the user; as we will show later, they can be derived from the chosen control strategy and from potential constraints required for the controls. Given a realization $\bar{u}_{k:k+T_p}$ of the process $\bar{U}_{k:k+T_p}$, it is possible to successively sample realizations $\bar{x}_j$ given $\bar{x}_{j-1}$ and $\bar{u}_{j-1}$ for $j = k, \ldots, k+T_p$ from the state transition density a_j of our original system, once an initial value $\bar{x}_k$ is given. We note that the joint process $(\bar{X}_{k:k+T_p}, \bar{U}_{k:k+T_p})$ is also Markovian, the initial distribution being given by

$$\bar{a}_k(\bar{x}_k, \bar{u}_k)d\bar{x}_k d\bar{u}_k = \delta_{x_k}(d\bar{x}_k)\bar{a}_{u,k}(\bar{u}_k)d\bar{u}_k$$

and the transition density being given by

$$\bar{a}_j(\bar{x}_j, \bar{u}_j|\bar{x}_{j-1}, \bar{u}_{j-1}) = \bar{a}_{u,j}(\bar{u}_j|\bar{u}_{j-1})a_j(\bar{x}_j|\bar{x}_{j-1}, \bar{u}_{j-1}).$$

Considering now given setpoints s_j, we need a function which compares state realizations $\bar{x}_j$ with these setpoints s_j. In general, one could use any conditional density function $\bar{b}_j(s_j|\bar{x}_j)$ or even $\bar{b}_j(s_j|\bar{x}_j, \bar{u}_j)$ (depending also on the control value) which is maximal if $\bar{x}_j$ is optimal with respect to the setpoint s_j. Generally, in all cases where the term

$$\sum_{j=k+1}^{k+T_p} \|s_j - \bar{x}_j\|_R^2$$

appears in the functional J, a natural choice is the multivariate Gaussian density

$$\bar{b}_j(s_j|\bar{x}_j, \bar{u}_j) = \frac{1}{(2\pi)^{n/2}|R|^{-1/2}} \exp(\|s_j - \bar{x}_j\|_R^2)$$

for $j = k+1, \ldots, k+T_p$ (cf. Theorem 1). With this in mind, we have set up a general state space model with combined state $(\bar{x}_j, \bar{u}_j)$ and "observation" s_j given by the transition and observation densities

$$\bar{a}_j(\bar{x}_j, \bar{u}_j|\bar{x}_{j-1}, \bar{u}_{j-1}), \tag{9}$$

$$\bar{b}_j(s_j|\bar{x}_j, \bar{u}_j). \tag{10}$$

If we initialize this with the states from the particle cloud of the first filter, and additionally sample suitable first controls $\bar{u}_k$ (which we usually condition on the last applied control $u^\star_{k-1}$), then this allows us to use the Particle Filter algorithm to compute successively for each j an approximation to the "joint smoothing" distribution

$$p(d\bar{x}_{k:j}, d\bar{u}_{k:j}|s_{k+1:j})$$

which in turn leads to an approximation to the marginalisation

$$p(d\bar{u}_{k:j}|s_{k+1:j}) = \int_{\bar{x}_{k:j}} p(d\bar{x}_{k:j}, d\bar{u}_{k:j}|s_{k+1:j}).$$

Since the former distribution is given via a particle approximation, the latter marginalization is easily obtained by just discarding the values $\bar{x}_j$ in the particles. Finally, for $j = k+T_p$, we get an approximation to the distribution

$$p(d\bar{u}_{k:k+T_p}|s_{k+1:k+T_p}),$$

and marginalizing out all $\bar{u}_j$'s except the first one, we get an approximation to the distribution

$$p(d\bar{u}_k|s_{k+1:k+T_p}) = \int_{\bar{u}_{k+1:k+T_p}} p(d\bar{u}_{k:k+T_p}|s_{k+1:k+T_p}).$$

This last distribution can be interpreted in a Bayesian sense as the posterior distribution for the control u_k, given setpoints $s_{k+1:k+T_p}$, and is in this sense

optimal for given densities $\bar{a}_j$ and $\bar{b}_j$. In this way, the second particle filter estimates an approximation to the distribution of the next control u_k under the aspect of reaching predetermined setpoints $s_{k+1:k+T_p}$, as is the aim of common MPC. Then a point estimate $u_k^\star$ of the approximating particle distribution $\hat{p}_{N_s}(du_k|s_{k+1:k+T_p})$ can be computed, for instance the mode, mean, or median. The obtained point estimate $u_k^\star$ is then applied to the real system as control input until the next time step, in which the procedure is repeated. Thus the main MPC strategy is adopted. In figure 2 the control loop is sketched. Note

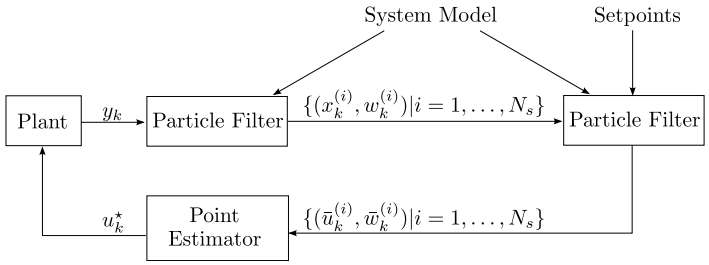


Figure 2 Sketch of the PF-MPC control loop at time k

that in the second particle filter, the setpoints s_j play the role of measurements in a standard particle filter.

4.1 Further implementation details

As mentioned above, the second particle filter, used for control prediction, has an expanded state $(\bar{x}_j, \bar{u}_j)$ and thus is applied to a different model than the first one. The general state space model at time k has the form

$$\bar{a}_j(\bar{x}_j, \bar{u}_j|\bar{x}_{j-1}, \bar{u}_{j-1}) \tag{11}$$

$$\bar{b}_j(s_j|\bar{x}_j, \bar{u}_j) \tag{12}$$

for $j \in \{k+1, \ldots, k+T_p\}$, see figure 3.

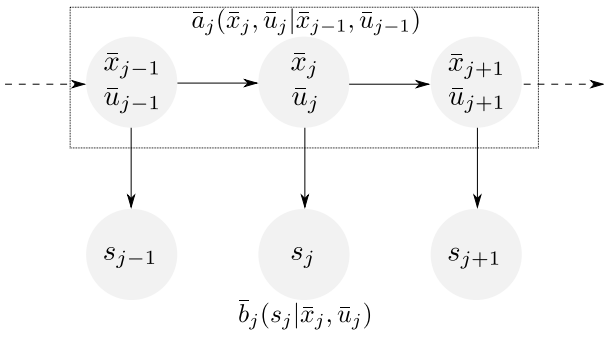


Figure 3 General State Space Model for the second Particle Filter

This can be seen as the general state space representation of a system

$$\bar{x}_j = f(\bar{x}_{j-1}, \bar{u}_{j-1}, \bar{v}_{j-1}) \tag{13}$$
$$\bar{u}_j = \bar{f}_u(\bar{x}_{j-1}, \bar{u}_{j-1}, \tilde{v}_{j-1}) \tag{14}$$
$$s_j = \bar{g}(\bar{x}_j, \bar{u}_j, \hat{v}_j) \tag{15}$$

for $j \in \{k+1, \ldots, k+T_p\}$ with $\bar{v}_{j-1}$, $\tilde{v}_{j-1}$ and $\hat{v}_j$ being random white noise following certain distributions. The initial conditions for this system as well as for the general state space representation are discussed later in this section. The functions f, $\bar{f}_u$ and $\bar{g}_u$ have to be chosen in the following way:

- The function f in eq. (13) is the same model, describing the state transitions of the real system, as it is used in the first particle filter. Also the noise $\bar{v}_{j-1}$ is sampled from the same distribution.
- The function $\bar{f}_u$ in eq. (14) defines the transition of the future control inputs. For example, if we choose

  $$\bar{f}_u(\bar{x}_{j-1}, \bar{u}_{j-1}, \tilde{v}_{j-1}) = \bar{u}_{j-1} + \tilde{v}_{j-1},$$

  then $\bar{u}_j$ lies in a neighborhood around its predecessor $\bar{u}_{j-1}$. If we choose the $\tilde{v}_{j-1}$'s as samples from a normal distribution $\mathcal{N}(0, \Sigma)$, the control input is constrained in a sense that the effort $\Delta\bar{u}_j$ is kept small depending on Σ. The effect on the control input $\bar{u}_j$ is similar to the effect of the term $\|\Delta\bar{u}\|_Q$ of a usual MPC objective function J. This link is further discussed in section 4.3. If one is not interested in keeping the control effort small one can set

  $$\bar{f}_u(\bar{x}_{j-1}, \bar{u}_{j-1}, \tilde{v}_{j-1}) = \tilde{v}_{j-1}$$

  and let the $\tilde{v}$'s for instance be normally or uniformly distributed. The control $\bar{u}_j$ is then independent from its predecessor $\bar{u}_{j-1}$. This makes constraining the input quite easy, which will be further discussed in section 4.2.
- The function $\bar{g}$ in eq. (15) defines the setpoint equation as part of the control strategy. The idea (and the actual meaning of this equation) is to define the constant s_j, the function $\bar{g}$ and the stochastic noise variable $\hat{V}_j$ in each step j in such a way that the constant s_j can be seen as a realization of the stochastic variable

  $$S_j = \bar{g}(\bar{X}_j, \bar{U}_j, \hat{V}_j).$$

  In the simplest case, s_j is a fixed value which denotes the desired state the system should attain, and $\bar{g}$ is defined as

  $$\bar{g}(\bar{X}_j, \bar{U}_j, \hat{V}_j) = \bar{X}_j + \hat{V}_j$$

  with normally distributed $\hat{V}_j$ with mean 0 (and some given variance σ^2). Since in this case $\hat{V}_j = S_j - \hat{X}_j$, this actually means that we expect the stochastic variable $S_j - \hat{X}_j$ to be normally distributed with mean 0, and each realization $s_j - x_j$ to be sampled from this distribution. Since s_j is fixed, this

means nothing else than that we expect the state x_j to be normally distributed with mean s_j (and variance σ^2). This is at least what we would like to attain in the controlled system. Nevertheless, the freedom in the choice of s_j, $\bar{g}$ and $\hat{V}_j$ allows much more complex control strategies which are moreover allowed to depend on the internal states x_j (via the function $\bar{g}$). An example of a complex control strategy can be found later in the pendulum example (section 6.2.1).

The initial states $\bar{x}_k^{(i)}$ and weights $\bar{w}_k^{(i)}$ are determined through the first particle filter in figure 2.

$$\bar{x}_k^{(i)} = x_k^{(i)}$$
$$\bar{w}_k^{(i)} = w_k^{(i)}$$

for all $i = 1, \ldots, N_s$. Initial controls $\bar{u}_k^{(i)}$ are obtained by using $u_{k-1}^\star$ and some distribution

$$\bar{u}_k^{(i)} \sim \bar{a}_{u,k}(d\bar{u}_k | u_{k-1}^\star)$$

or, equivalently, by a function

$$\bar{u}_k^{(i)} = \bar{f}_{u,k}(u_{k-1}^\star, \tilde{v}_{k-1}^{(i)}),$$

for example

$$\bar{u}_k^{(i)} = u_{k-1}^\star + \tilde{v}_{k-1}^{(i)},$$

where $\tilde{v}_{k-1}^{(i)}$ is some appropriate white noise (which may be identically distributed for all k).

Naturally, the particle filter is formulated for complete paths, like $\bar{u}_{k:k+l}^{(i)}$ for $l \in \{1, \ldots, T_p\}$. Usually one discards the paths and keeps only the last value $\bar{u}_{k+l}^{(i)}$. The reason for this is resampling, which is a necessary step in the particle filter algorithm to avoid weight degeneration. Nevertheless, due to this resampling step, the particle paths degenerate, and hence the paths are usually discarded and only the last value is kept. Here, the paths cannot be discarded completely, since the $\bar{u}_k^{(i)}$'s are needed for determining the control input. So, the values $\bar{u}_{k+1:k+l-1}^{(i)}$ are discarded and only $(\bar{u}_k^{(i)}, \bar{u}_{k+l}^{(i)})$ is kept, which then also undergoes resampling. Practically this can be done by introducing the following constant transition to (13)-(15):

$$\tilde{u}_j = \tilde{u}_{j-1} \qquad \text{with} \qquad \tilde{u}_k = \bar{u}_k. \tag{16}$$

The unavoidable degeneration of paths in the SISR algorithm also implies that the prediction horizon T_p should not be chosen too high.

So altogether the setup for the second particle filter is the following: First initial controls $\bar{u}_k^{(i)}$ are created, then they are forwarded together with initial states

$(x_k^{(i)}, w_k^{(i)})$ to the second particle filter with underlying equations (13)-(15) and (16):

$$\begin{aligned}\bar{x}_j &= f(\bar{x}_{j-1}, \bar{u}_{j-1}, \bar{v}_{j-1})\\ \bar{u}_j &= \bar{f}_u(\bar{x}_{j-1}, \bar{u}_{j-1}, \tilde{v}_{j-1})\\ \tilde{u}_j &= \tilde{u}_{j-1}\\ s_j &= \bar{g}(\bar{x}_j, \bar{u}_j, \hat{v}_j)\end{aligned}$$

defining the following densities:

$$\begin{gathered}\bar{a}_j(\bar{x}_j, \bar{u}_j, \tilde{u}_j | \bar{x}_{j-1}, \bar{u}_{j-1}, \tilde{u}_{j-1})\\ \bar{b}_j(s_j | \bar{x}_j, \bar{u}_j)\end{gathered}$$

for $j \in \{k+1, \ldots, k+T_p\}$. The result after T_p prediction steps is then

$$\{(\bar{x}_{k+T_p}^{(i)}, \bar{u}_{k+T_p}^{(i)}, \tilde{u}_{k+T_p}^{(i)}, \bar{w}_{k+T_p}^{(i)}) | i = 1, \ldots, N_s\}.$$

These particles are then forwarded to a point estimator, where the control input $u_k^\star$ is obtained from

$$\{(\tilde{u}_{k+T_p}^{(i)}, \bar{w}_{k+T_p}^{(i)}) | i = 1, \ldots, N_s\}.$$

The mean, for instance, is obtained by

$$\sum_{i=1}^{N_s} \tilde{u}_{k+T_p}^{(i)} \bar{w}_{k+T_p}^{(i)}.$$

In figure 4, the PF-MPC control loop is presented in the particular case in which the transition density of the system is chosen as importance density (see the end of section 3). The grey numbers in figure 4 give the order of actions.

Overall, the control loop works as follows:

- Initialization:
  - Sample $x_0^{(i)}$ from $q_0(x_0)$ independently for $i = 1, \ldots, N_s$.
  - Compute the weights
$$w_0^{(i)} = \frac{a_0(x_0^{(i)})}{q_0(x_0^{(i)})}.$$
- Iteration over $k = 1, \ldots, N$:
  - If $\widehat{N}_{\text{eff}} < N_T$, resample.
  - Sample $x_k^{(i)}$ from $q_k(x_k | y_k, x_{k-1}^{(i)})$ independently for $i = 1, \ldots, N_s$.
  - Compute the unnormalized weights
$$\tilde{w}_k^{(i)} = w_{k-1}^{(i)} \frac{b_k(y_k | x_k^{(i)}) a_k(x_k | x_{k-1}^{(i)})}{q_k(x_k^{(i)} | y_k, x_{k-1}^{(i)})}$$
$$\text{for } i = 1, \ldots, N_s.$$

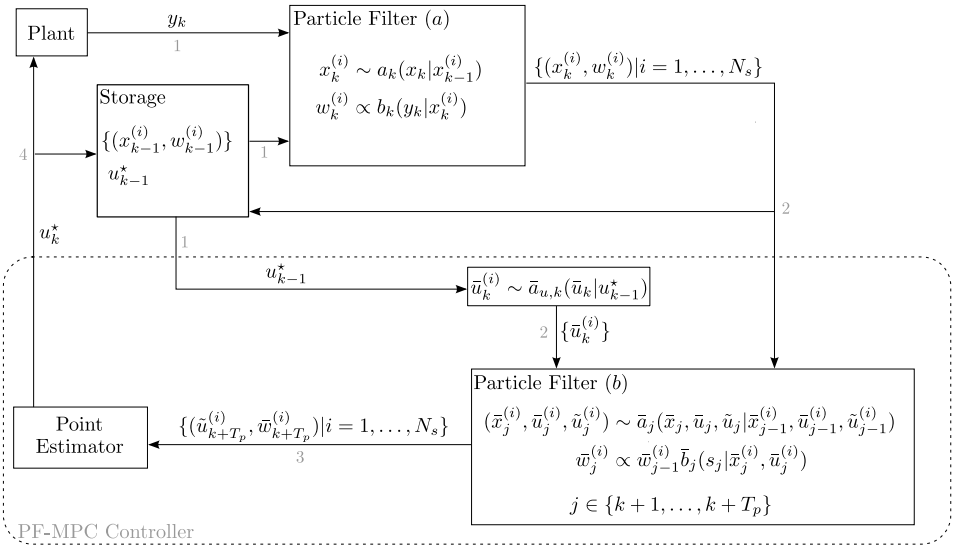


Figure 4 PF-MPC Control Loop

- Compute the normalized weights

$$w_k^{(i)} = \frac{\tilde{w}_k^{(i)}}{\sum_{\iota=1}^{N_s} \tilde{w}_k^{(\iota)}} \qquad \text{for } i = 1, \ldots, N_s.$$

- Control Unit — Initialization:
  - Sample $\bar{u}_k^{(i)}$ from $\bar{a}_{u,k}(\bar{u}_k|u_{k-1}^{\star})$ independently for $i = 1, \ldots, N_s$.
  - Set $\bar{x}_k^{(i)} = x_k^{(i)}$ and $\bar{w}_k^{(i)} = w_k^{(i)}$ for $i = 1, \ldots, N_s$.
- Control Unit — Iteration over $j = k+1, \ldots, k+T_p$:
  - If $\widehat{N}_{\text{eff}} < N_T$, resample.
  - Sample $\xi_j^{(i)} = (\bar{x}_j^{(i)}, \bar{u}_j^{(i)}, \tilde{u}_j^{(i)})$ from

$$q_j(\xi_j|s_j, \xi_{j-1}^{(i)})$$

    independently for $i = 1, \ldots, N_s$.
  - Compute the unnormalized weights

$$\tilde{w}_j^{(i)} = \bar{w}_{j-1}^{(i)} \frac{\bar{b}_j(s_j|\bar{x}_j^{(i)}, \bar{u}_j^{(i)}) \bar{a}_j(\xi_j|\xi_{j-1}^{(i)})}{q_j(\xi_j^{(i)}|s_j, \xi_{j-1}^{(i)})}$$

$$\text{for } i = 1, \ldots, N_s.$$

  - Compute the normalized weights

$$w_j^{(i)} = \frac{\tilde{w}_j^{(i)}}{\sum_{\iota=1}^{N_s} \tilde{w}_j^{(\iota)}} \qquad \text{for } i = 1, \ldots, N_s.$$

– Control Unit — Compute some statistical point estimate (mean, median, mode etc.) based on the weighted particles $(\tilde{u}^{(i)}_{k+T_p}, w^{(i)}_{k+T_p})$ to obtain $u^{\star}_k$

4.2 Constraints

4.2.1 Control input constraints

Apart from the possibility to require constraints for $\Delta\bar{u}_j$ as described above, it is easy to require constraints for the controls $\bar{u}_j$ directly. Without loss of generality, the control input is here assumed to be one-dimensional. One can distinguish between hard and soft constraints. Soft constraints mean that the input should lie in some specific region most of the time, but is allowed to leave the region sometimes. In our case, the constraints have to be formulated with the help of prior probabilities for $\bar{u}_j$. One example of a soft constraint is setting the input transition equation (14) equal to:

$$\bar{u}_j = \tilde{v}_{j-1} \qquad \text{with} \qquad \tilde{v}_{j-1} \sim \mathcal{N}(0, \sigma^2).$$

In this case the input lies around the origin and the variance σ^2 indicates the region where most of the inputs lie.

Hard constraints on the input mean, that the input has to lie in the interval $[a, b]$, with $a, b \in \mathbb{R}$. For this and the multi-dimensional case, the hard constraints are just box-constraints. These box-constraints can be realized in our case by setting equation (14) equal to:

$$\bar{u}_j = \tilde{v}_{j-1} \qquad \text{with} \qquad \tilde{v}_{j-1} \sim \mathcal{U}(a, b).$$

4.2.2 State/output constraints

Hard constraints have to be formulated in the system model or transition density, respectively. If this is not the case, constraints can be violated due to disturbances or the Euler-Maruyama discretization (see section 6). For that reason, only soft constraints are formulated here. These soft constraints are formulated as densities inside the observation density (15). There is no general way to do so, they have to be individually formulated for each case. Later, an example of how to handle constraints on the states will be given, when we discuss the control of the inverted pendulum, see section 6.2.2.

4.3 Link between classic MPC and PF-MPC

In this paragraph, we want to establish a link between classic MPC and the new developed particle filter approach in a particular case. We assume that our

system is deterministic, or that the expected value of the disturbance is taken as the nominal value of the system. This is the setup as in classical MPC. Since we still want to work in the proposed framework, our model is stochastic; this leads to delta distributions in the state transitions. In Theorem 1, it will be shown that PF-MPC, with this special setup, is doing approximately the same as classic MPC. But note that this is a theoretical result; the PF-MPC is applicable to a much broader class of systems than classical MPC, and in this particular case, the algorithm may be ineffective due to the delta distributions of the state transitions.

Theorem 1 *For a fixed time k, let the following system be given*

$$\bar{x}_j = f(\bar{x}_{j-1}, \bar{u}_{j-1}), \tag{17}$$
$$\bar{u}_j = \bar{u}_{j-1} + \tilde{v}_{j-1}, \tag{18}$$
$$s_j = \bar{x}_j + \hat{v}_j, \tag{19}$$

with $j \in \{k+1, \dots, k+T_p\}$, $T_p > k$, $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$, $\tilde{v}_{j-1} \sim \mathcal{N}(0, Q^{-1})$ and $\hat{v}_j \sim \mathcal{N}(0, R^{-1})$, where $Q \in \mathbb{R}^{m\times m}$ and $R \in \mathbb{R}^{n\times n}$ are symmetric positive definite matrices. Let initial $\bar{x}_k = \hat{x}_k$ and $\bar{u}_{k-1}$ be given and $\bar{u}_k = \bar{u}_{k-1} + \tilde{v}_{k-1}$ with $\tilde{v}_{k-1} \sim \mathcal{N}(0, Q^{-1})$. Then the mode of the distribution

$$p(\bar{u}_{k:k+T_p} | s_{k+1:k+T_p}) d\bar{u}_{k:k+T_p}$$

(i.e. the value where the maximum of the corresponding density is attained) is reached at the same point as the minimum of a usual MPC functional

$$J = \sum_{j=k}^{k+T_p} \|\bar{u}_j - \bar{u}_{j-1}\|_Q^2 + \sum_{i=k+1}^{k+T_p} \|s_i - \hat{x}_i\|_R^2.$$

Thus

$$\begin{aligned} &\arg \max_{\bar{u}_{k:k+T_p}} p(\bar{u}_{k:k+T_p} | s_{k+1:k+T_p}) \\ =&\arg \min_{\bar{u}_{k:k+T_p}} \sum_{j=k}^{k+T_p} \|\bar{u}_j - \bar{u}_{j-1}\|_Q^2 + \sum_{i=k+1}^{k+T_p} \|s_i - \hat{x}_i\|_R^2, \end{aligned}$$

where $\hat{x}_i = f(\hat{x}_{i-1}, \bar{u}_{i-1})$ is recursively defined, starting with $\hat{x}_k$.

The proof will be presented in section 8.

The theorem shows that the PF-MPC approach does approximately the same as classic MPC, in the special case where the system from Theorem 1 is chosen and the mode is taken as point estimator — keeping in mind that the second particle filter computes just an approximation of the distribution

$$p(\bar{u}_{k:k+T_p} | s_{k+1:k+T_p}) d\bar{u}_{k:k+T_p}.$$

5 Relation to existing methods

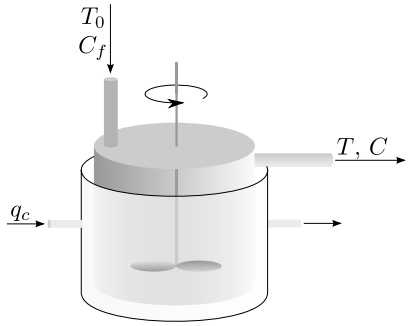
In the majority of articles which use the particle filter algorithm for Model Predictive Control, the particle filter is exclusively used as state estimator by computing a point estimate from the particle cloud. This point estimate is then used in the usual MPC optimization procedure. Our first particle filter serves the same purpose of state estimation, but in contrast we use the complete particle cloud for the subsequent steps of determining the control input. In the overview article [1] the authors present methods where the optimal control input is either found by enumeration (in the case of only finitely many possible controls) or by gradient methods (in the continuous case). A more recent article is [5], but also here, SMC methods are used only for state estimation. Another approach with particle methods was taken by [4], the idea there is to approximate the original stochastic control problem by a deterministic one, thus avoiding the use of particle filters at all. The optimization problem is instead solved with Mixed-Integer Linear Programming techniques. The approach which possibly comes closest to our method is described in the recent article [15] (which came to our knowledge only after the submission of this article). Similar to our method, they, too, use a particle-based approach in the control prediction/optimization step. In our notation, at a given time k, they sample a complete control path $\bar{u}_{k:k+T_p}$ for each particle, and given this path, a state path $x_{k:k+T_p}$. This path is then used to compute incremental weights via a cost function. In contrast to our approach, the path $\bar{u}_{k:k+T_p}$ is changed completely(!) and the procedure repeated iteratively. The Monte Carlo method used there is a variant of Simulated Annealing and, since the paths are always sampled completely in each iteration, principally not sequential. In contrast, our approach is fully sequential because incremental weights are computed for each j after sampling of $\bar{u}_j$ and $\bar{x}_j$, and resampling is applied if necessary. Also, in contrast to Simulated Annealing as a direct optimization method, our approach computes some kind of posterior distribution, and optimization is done more indirectly. We expect a truly sequential method, as ours is, to be superior to a method where complete paths have to be resampled. Nevertheless, this remains to be shown in future experiments.

6 Examples

In this section, the PF-MPC control approach is applied to two nonlinear disturbed systems, namely to a *continuous stirred-tank reactor* (CSTR) and to the inverted pendulum. The PF-MPC controller is implemented in **R**, a language and environment for statistical computing and graphics [18]. All computations were made on a Thinkpad T60 with a 2GHz Core2Duo processor and 2GB of RAM.

Generally, the differential equations used for describing the CSTR and the inverted pendulum are deterministic. In order to be able to introduce disturbances we want to transform them into *stochastic differential equations* (SDE). For discretizing these SDEs, the *Euler-Maruyama method*, which is similar to the well known Euler method for ordinary differential equations, will be used.

6.1 Application to the CSTR



C	Current concentration
C_f	Feed concentration
T	Current temperature
T_0	Feed temperature
q_c	Coolant stream temperature

Figure 5 Sketch of the CSTR

A continuous stirred-tank reactor (CSTR) as it is sketched in figure 5 is a chemical reactor. The reactor is fed with some chemical A with concentration C_f and temperature T_0. Inside the reactor, a first-order exothermic and irreversible chemical reaction $A \rightarrow B$ takes place. It is assumed that the interior of the reactor is perfectly mixed. This means that the temperature T and the concentration C is the same at each point in the reactor. The reactor is cooled by a coolant stream q_c which flows around the reactor. The aim is to keep the concentration C of the product at a certain level. This is achieved by manipulating the coolant stream, which is thus the control input of the system.

Only the temperature T of the system will be measured (disturbed by some measurement noise), the concentration C itself, which has to be controlled, is not accessible directly. The CSTR can be modeled by the following differential equations:

$$\begin{aligned}\dot{C} &= \frac{q}{V}(C_f - C) - k_0 C e^{\frac{-E}{RT}} \\ &=: f_1(C,T)\end{aligned} \tag{20}$$

$$\begin{aligned}\dot{T} &= \frac{q}{V}(T_0 - T) - \frac{k_0 \Delta H}{\rho c_p} C e^{\frac{-E}{RT}} \\ &\quad + \frac{\rho_c c_{pc} q_c}{\rho c_p V}\left(1 - e^{\frac{-hA}{q_c \rho_c c_{pc}}}\right)(T_{C_0} - T) \\ &=: f_2(C,T)\end{aligned} \tag{21}$$

where the parameters are set as shown in table 1. This specific example was

Parameter	**Description**	**Unit**	**Value**
C_f	Feed concentration	mol/l	1
T_0	Feed temperature	K	350
V	Reactor volume	l	100
q	Process flow rate	$1/\text{min}$	100
T_{C_0}	Inlet coolant temperature	K	350
hA	Heat transfer term	K cal/min	7×10^5
k_0	Reaction rate constant	$1/\text{min}$	7.2×10^{10}
E/R	Activation energy term	K	10^4
ΔH	Heat of reaction	cal/mol	-2×10^5
ρ, ρ_c	Liquid densities	g/l	1000
c_p, c_{pc}	Specific heats	K cal/g	1

Table 1 Parameters of the CSTR

taken from [5], which in turn had been taken from [3]. The CSTR will be simulated and controlled using the PF-MPC approach. For that reason, we transform the ordinary differential equations (20) and (21) to the following SDE by adding stochastic terms:

$$dX(t) = f(X(t))dt + F(X(t))dW(t)$$

with

$$\begin{aligned}X(t) &= (C(t), T(t)), \\ f(X(t)) &= (f_1(C(t), T(t)), f_2(C(t), T(t))), \\ F(X(t)) &= \begin{pmatrix}\eta_C & 0 \\ 0 & \eta_T\end{pmatrix}\end{aligned}$$

where $dW = (dW_C(t), dW_T(t))$ is a 2-dimensional Wiener process. The entries η_C and η_T are both positive real numbers and their particular choices for the different simulation runs will be described later in the corresponding paragraphs.

Discretization of this SDE with the Euler-Maruyama scheme at a sampling rate of $\delta = 0.05$min and time instances $t_k = k\delta$ for $k = 0, 1, 2, \ldots$ leads to the transition densities denoted by a_k in section 4. We will use these for simulation runs of the CSTR, and for the simultaneous application of our PF-MPC controller to the simulated reactor. During the simulation, measurements of the temperature T at each sampling point will also be generated. These simulated measurements will be additionally disturbed by an artificial measurement error $w \sim \mathcal{N}(0, \hat{\eta}_T^2)$ (realizing the observation density b_k). From these measurements, the filter distribution of the concentration C and temperature T will be estimated with the first particle filter. Then a control input candidate will be determined by the PF-MPC controller, realized by the second particle filter, so that the concentration C follows the setpoint $s_{C_1} = 0.1$ for $t \in [0, 150 \cdot \delta)$. At $t = 150 \cdot \delta$, we will change the setpoint to $s_{C_2} = 0.12$ for $t \in [150 \cdot \delta, 300 \cdot \delta]$. We realize the setpoint distribution $\hat{b}_j$ of the second particle filter by the following equation:

$$s_C(j) = C(j) + \hat{v}_C(j)$$

with $\hat{v}_C(j) \sim \mathcal{N}(0, \hat{\eta}_{\text{set}}^2)$.

The distribution of the control q_c (denoted by $\bar{a}_{u,j}$ in section 4) is chosen as

$$\bar{q}_c(j) = \bar{q}_c(j-1) + \tilde{v}(j-1)$$

with

$$\tilde{v}(j-1) \sim \mathcal{N}(0, \hat{\eta}_{\text{pred}}^2),$$

thus obtaining a small control effort. In the following, results of simulations with different setups will be presented.

6.1.1 Results of the CSTR simulations

In the first simulation run, very small disturbances (given by the parameters η_C and η_T) for the states C and T, respectively, as well as a small measurement noise (given by $\hat{\eta}_T$) will be used to figure out if the PF-MPC control loop is able to handle the CSTR. With this model, it is possible to use the optimal importance distribution eq. (7) in the SISR algorithm. The following parameters will be used:

$$\begin{array}{lll} T_p = 20 & N_s = 1000 & \\ \eta_C = 0.000001 & \eta_T = 0.000001 & \\ \hat{\eta}_T = 0.5 & \hat{\eta}_{\text{pred}} = 0.4 & \hat{\eta}_{\text{set}} = 0.005 \end{array}$$

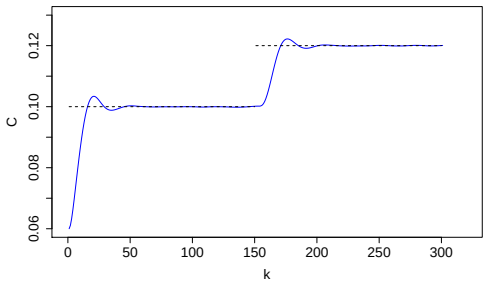


Figure 6 State C over k

An example of a simulated trajectory of the controlled state C can be seen in figure 6. Note that this trajectory is not accessible in the real system, and that it has not been used in the controller. In this example, the PF-MPC control loop works very well and is able to handle the slightly disturbed CSTR. In figure 7, the control input computed by our algorithm in this run is shown. In the figures

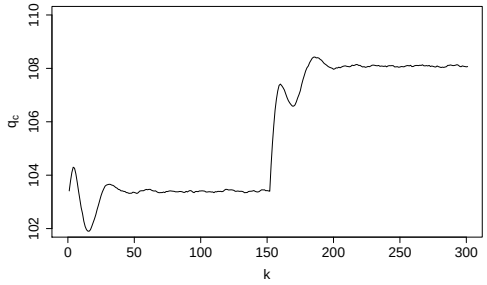


Figure 7 Control q_c over k

8-10, we plot the prediction horizons for the state C and the control candidate $\tilde{q}_c$ at certain fixed time steps k, illustrating the results of the second particle filter (the x-axis represents values $j = k, \ldots, k + T_p$). In these figures, a solid line depicts the mean estimate. Additionally, vertical stripes sketch the particle distribution at each j, where darker tones mean higher density, and lighter tones mean lower density. We will adhere to this indication for the following figures concerning the prediction horizons. In figure 8, one can already see that at time $k = 0$, the PF-MPC controller predicts that the concentration C will reach its predetermined setpoint $s_{C_1} = 0.1$. At $k = 100$, where the concentration C already follows s_{C_1}, the prediction horizon for C is to be seen in figure 9. It can be observed from this figure that the predicted concentration will stay stable at the setpoint s_{C_1} for the whole prediction horizon $j = k, \ldots, k + T_p$. At $k = 150$ where the setpoint change $s_{C_1} \rightarrow s_{C_2}$ takes place, one can see in figure 10 that C will reach s_{C_2}. The average computing time the control loop needs for one step with the parameters chosen above is approximately 0.12 seconds. Since the model of the CSTR is sampled with $\delta = 0.05$min which equals 3sec, the

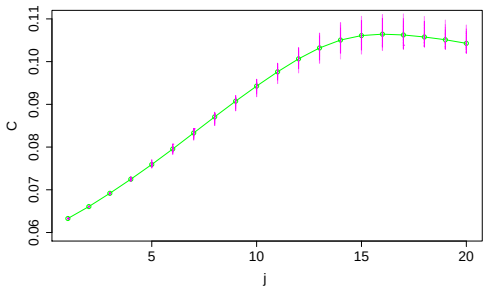


Figure 8 Prediction of C at $k = 0$

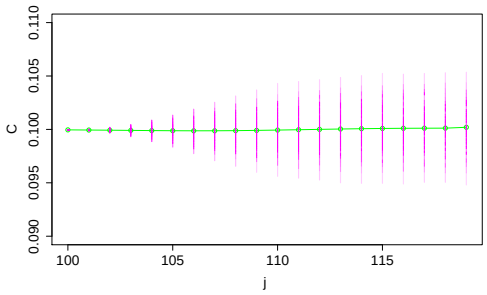


Figure 9 Prediction of C at $k = 100$

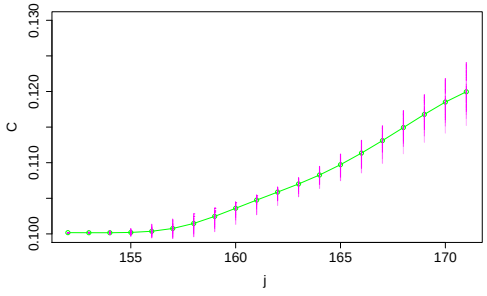


Figure 10 Prediction of C at $k = 150$

computation for one step is around 25 times faster than the real-time process. Thus, for the first parameter constellation, the PF-MPC controller worked very well in the simulations and would be applicable to the real system. In the next simulation run, the controller will be tested on a simulated system with increased state and measurement noise, i.e. increased values for η_C, η_T, and $\hat{\eta}_T$. The following parameters will be used:

$$\begin{array}{lll} T_p = 20 & N_s = 1000 & \\ \eta_C = 0.001 & \eta_T = 0.1 & \\ \hat{\eta}_T = 2.5 & \hat{\eta}_{\text{pred}} = 0.4 & \hat{\eta}_{\text{set}} = 0.005 \end{array}$$

With these parameters, the simulated trajectory of the state C over time k obtained during a simulation run with applied PF-MPC controller can be found in figure 11. Here, too, one observes that the PF-MPC control loop works very

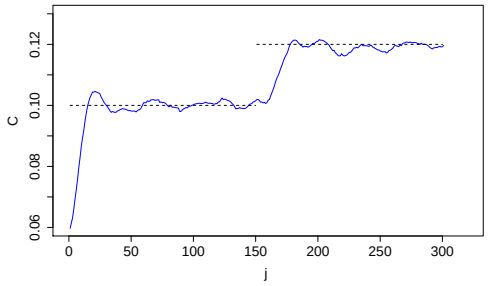


Figure 11 State C over k

well and that it can handle the CSTR with larger transition and observation errors. As expected, due to the higher disturbances, the concentration C shows higher variation around its setpoint than in the previous case. The predictions are not plotted and discussed here, because they are very similar to the ones already discussed in the case where the disturbances were small.

6.2 Application to the inverted pendulum

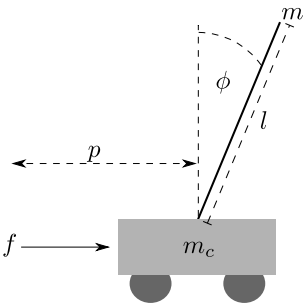
The following parameters are assumed to hold for the inverted pendulum, which is sketched in figure 12:

$$\begin{array}{ll} l = 0.5\text{m} & m = 0.3\text{kg} \\ m_c = 3\text{kg} & g = 9.81 \frac{\text{m}}{\text{s}^2} \end{array}$$

The behavior of the inverted pendulum can be modeled by the following two coupled differential equations:

$$\ddot{\phi} = \frac{g \sin\phi + \cos\phi \frac{-f - ml\dot{\phi}^2 \sin\phi}{m_c + m}}{l\left(\frac{4}{3} - \frac{m\cos^2\phi}{m_c + m}\right)},$$

$$\ddot{p} = \frac{f + ml(\dot{\phi}^2 \sin\phi - \ddot{\phi}\cos\phi)}{m_c + m}.$$



p	Position of the cart
ϕ	Deflection of the pendulum
l	Length of the pendulum
m	Mass of the pendulum
m_c	Mass of the cart
g	Gravitational acceleration
f	Force/control input

Figure 12 Sketch of the pendulum

By adding process noise, a stochastic differential equation can be formulated:

$$dX(t) = h(X(t)) + H(X(t))dW(t) \tag{22}$$

with

$$X(t) = \begin{bmatrix} \phi_1(t) \\ \phi_2(t) \\ p_1(t) \\ p_2(t) \end{bmatrix} = \begin{bmatrix} \phi(t) \\ \dot{\phi}(t) \\ p(t) \\ \dot{p}(t) \end{bmatrix},$$

$$h(X(t)) = \begin{bmatrix} \phi_2(t) \\ \frac{g \sin\phi_1(t) + \cos\phi_1(t) \frac{-f(t) - ml\phi_2^2 \sin\phi_1(t)}{m_c+m}}{l\left(\frac{4}{3} - \frac{m\cos^2\phi_1(t)}{m_c+m}\right)} \\ p_2(t) \\ \frac{f(t) + ml(\phi_2(t)^2 \sin\phi_1(t) - \dot{\phi}_2(t)\cos\phi_1(t))}{m_c+m} \end{bmatrix},$$

$$H(X(t)) = \begin{bmatrix} 0 & 0 \\ \eta_{\phi_2} & 0 \\ 0 & 0 \\ 0 & \eta_{p_2} \end{bmatrix}$$

where $W(t)$ is a two-dimensional Wiener process. The entries η_{ϕ_2} and η_{p_2} are constant positive real numbers. The SDE (22) is again discretized with the Euler-Maruyama method and a sampling rate of $\delta = 0.05$ seconds. The initial value of the SDE (22) is

$$X_0 = \begin{bmatrix} \pi & 0 & 0 & 0 \end{bmatrix}^T,$$

which means that the cart is standing at the origin with pendulum down.

6.2.1 Controlling deflection and velocity of the inverted pendulum

We will now discuss how to control the inverted pendulum and how additional constraints can be applied. The goal of controlling the pendulum is to stabilize the pendulum in the upright position while the cart should stay in a predefined area. We will first discuss the setpoint conditions and afterwards how to implement the necessary constraints for the cart. Possibly, the easiest way is to use a setpoint for the deflection of the pendulum only, and set it to 0, which corresponds to the upright position the pendulum should be stabilized in (note that one has to cope with the fact that $\phi = \phi_1$ is cyclic modulo 2π, thus $0 \equiv 2\pi k$ for all $k \in \mathbb{Z}$). Using this idea works fine and delivers good results, but we want to show that the PF-MPC controller allows the application of more complex control strategies. In the following, we want to show how those can be easily implemented using more complex setpoint equations.

The idea for a possible control strategy can be derived from the following deliberations: Consider the pendulum when it slightly deviates from its swung-up position ($\phi_1 > 0$) with zero velocity ($\phi_2 = 0$). Clearly, the pendulum then drops down and begins swinging. Simulation results are shown in figure 13, where the resulting velocity and the angle of the pendulum are plotted against each other. As this figure shows, the velocity of the pendulum is nearly 0 every time it reaches the upright position ($\phi_1 \equiv 0 \mod 2\pi$). We conclude that,

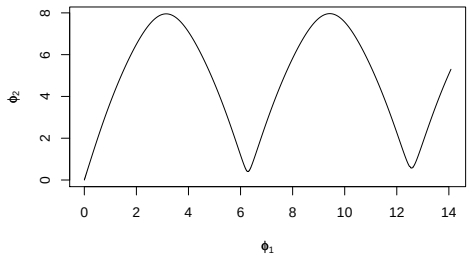


Figure 13 Swinging Pendulum

for each deflection angle, the corresponding velocity obtained in this way can be seen as nearly optimal for reaching the upright position with velocity 0 at this point. Since the behaviour of the system is similar when time is reversed, the same is true if we reverse time by multiplying the velocity by -1. The idea is now to first approximate the function giving the optimal velocity for each deviation angle on the interval $[0, 2\pi]$, and to use it to build a setpoint-determinator function for the velocity of the pendulum dependent on its deflection. So the aim of this function is, given a deflection ϕ_1, to return the needed velocity the pendulum should have in ϕ_1 to reach the upright position smoothly. Once the pendulum has swung up, we must avoid any slight positive deviation from the upright position ($0 < \phi_1 < \beta$ for some small angle β) causing

the controller to swing the pendulum again around in direction $\phi_1 = 2\pi$. Rather, the setpoint for the velocity should be set to the opposite direction. Note also that we have to reverse angles and velocities when the actual velocity of the pendulum is negative ($\phi_2 < 0$).

We thus obtain the right setpoint equation within three steps: We first apply quadratic approximation, using as interpolation points the angles $\phi_1 = 0$ and $\phi_1 = 2\pi$ (upright position) with velocity equal to zero, and the simulated value of the velocity at the opposite angle $\phi_1 = \pi$ in order to obtain an approximated function f_1. In our case, the result was

$$f_1(\phi_1) = -0.806314\phi_1^2 + 5.06622\phi_1.$$

Using f_1, we construct

$$f_2(\phi_1) = \begin{cases} -f_1(\phi_1), & \text{if } \phi_1 \in [0, \beta] \\ f_1(\phi_1), & \text{if } \phi_1 \in (\beta, 2\pi) \end{cases}$$

and finally the setpoint determination function

$$f_3(\phi_1, \phi_2) = \begin{cases} f_2(\phi_1), & \text{if } \phi_2 > 0, \\ -f_2(-\phi_1), & \text{if } \phi_2 < 0. \end{cases}$$

The constant β gives the angle until which the controller should try to stabilize the already swung up pendulum. One should keep in mind that ϕ_1 is 2π cyclic. The function f_3 gives approximately the right velocity for all angles for the pendulum to reach the upright position with zero velocity there. Due to the construction of f_3 the pendulum will also be balanced in the fixed point, as long as it does not drop beyond the angles β or $-\beta$. The function f_3 with $\beta = 0.4$ is plotted in figure 14. The setpoint equation used in the PF-MPC controller (15)

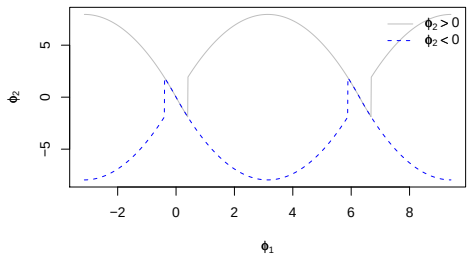


Figure 14 Setpoint determination function f_3

then equals for all times j

$$s_{\phi_2}(j) = f_3(\phi_1(j), \phi_2(j)) - \phi_2(j) + \hat{v}_{\phi_2}(j), \tag{23}$$

with $\hat{v}_{\phi_2}(j) \sim \mathcal{N}(0, \hat{\eta}^2_{\phi_2})$ and $s_{\phi_2}(j)$ set to 0. The corresponding density $\hat{b}$ is then proportional to

$$e^{\frac{-(f_3(\phi_1,\phi_2)-\phi_2)^2}{2\hat{\eta}^2_{\phi_2}}}.$$

6.2.2 Controlling the inverted pendulum with constraints on the cart position

Until now only the angle or the velocity of the pendulum were considered under the aspect of stabilizing the pendulum. But it is necessary as well to control (constrain) the movement of the cart on which the pendulum is fixed, because being uncontrolled, the cart would possibly move in one direction and never stop. As already mentioned in section 4.2, the constraints have to be formulated in densities. The easiest way of constraining the cart is to introduce an additional setpoint equation for the cart position, determining the region which the cart should remain in:

$$s_{p_1}(j) = p_1(j) + \hat{v}_{p_1}(j). \tag{24}$$

So, if the cart should stay around the origin, s_{p_1} should be chosen to be zero. The variance $\hat{\eta}^2_{p_1}$ of the added Gaussian noise $\hat{v}_{p_1} \sim \mathcal{N}(0, \hat{\eta}^2_{p_1})$ together with $\hat{\eta}^2_{\phi_2}$ determine the width of the region the cart should not leave. Alternatively or additionally, one could constrain the cart velocity in a certain way with another setpoint equality:

$$s_{p_2}(j) = p_2(j) + \hat{v}_{p_2}(j) \tag{25}$$

with $s_{p_2}(j) = 0$ for all j and

$$\hat{v}_{p_2}(j) \sim \mathcal{N}(0, \sigma^2_{p_2}(j)).$$

The variance $\sigma^2_{p_2}(j)$ is not constant but dependent on the actual velocity of the cart. It is chosen such that it is large if the cart is inside a given interval $[p_{1\,\min}, p_{1\,\max}]$ and small if it is outside and the velocity has the wrong direction. That means, if the position of the cart is left of $p_{1\,\min}$ and the velocity is positive, the variance is large because the cart is on the way back, but the variance is small if the velocity is negative because otherwise the cart would veer away further from $[p_{1\,\min}, p_{1\,\max}]$. Small variance $\sigma_{\min p_2}$ means that the cart slows down and large variance $\sigma_{\max p_2}$ means that the cart is allowed to move. The values of these variances have to be tuned by the user, and are dependent on the model and particle filter parameters. One way of formulating the variance $\sigma^2_{p_2}(j)$ mathematically is:

$$\sigma^2_{p_2}(j) = \begin{cases} \sigma_{\max p_2} & \begin{cases} \text{if } p_1(j) > p_{1\,\min} \text{ and } p_2(j) < 0 \\ \text{or } p_1(j) < p_{1\,\max} \text{ and } p_2(j) > 0 \end{cases} \\ \sigma_{\min p_2} & \begin{cases} \text{if } p_1(j) \leq p_{1\,\min} \text{ and } p_2(j) < 0 \\ \text{or } p_1(j) \geq p_{1\,\max} \text{ and } p_2(j) > 0 \end{cases} \end{cases} \tag{26}$$

(see figure 15).

6.2.3 Results of the inverted pendulum simulations

For our simulation run, we will use the setpoint equation (23) and the additional setpoint equation (25). The model and PF-MPC controller will be initialized with

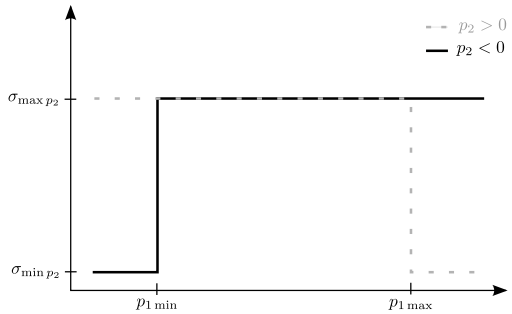


Figure 15 Variance of p_2

the following parameters

$$\begin{array}{lll} \eta_{\phi_2} = 0.1 & T_p = 26 & N_s = 500 \\ \eta_{p_2} = 0.1 & \hat{\eta}_{meas} = 0.01 & \hat{\eta}_{\phi_2} = 0.5 \\ p_{1\,\min} = -0.4 & p_{1\,\max} = 0.4 & \\ \sigma_{\min p_2} = 0.001 & \sigma_{\max p_2} = 5 & \end{array}$$

In this case, it is not possible to use the optimal importance distribution for the particle filters. Instead, we used the state transitions, see eq. (8). We again simulate the pendulum and apply the PF-MPC controller to the simulation. The simulated trajectory of the cart together with the pendulum is plotted in figure 16. The color changes from black to brighter tones as time increases. The trajectories are also plotted separately and can be found in figures 17 and 18. The figures show clearly that the controller stabilizes the pendulum while satisfying the constraints. Figure 18 shows that the cart remains in the desired

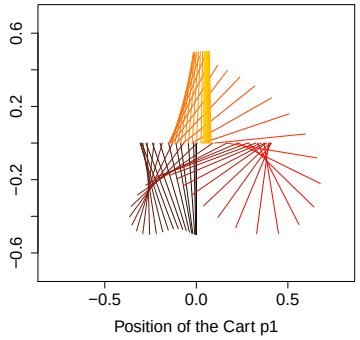


Figure 16 Moving pendulum over time

interval $[-0.4, 0.4]$. If one looks at the fixed time step $k = 25$, one observes in the corresponding figure 19 that even the controller predictions for the cart position remain in the desired interval $[-0.4, 0.4]$. In figure 20, we see the trajectory of the pendulum velocity depicted. The light-colored curve in this figure shows the corresponding setpoints for the velocities, which are determined by the setpoint determinator function f_3. Summarizing, we can say that PF-MPC works very well even if the setpoint equation is more complex and

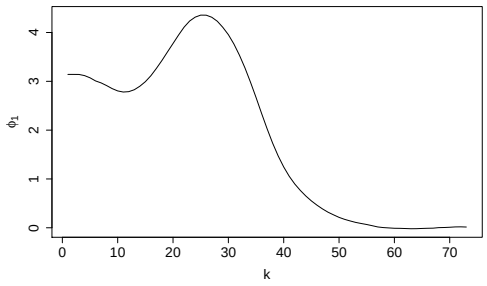


Figure 17 Deflection ϕ_1 of the pendulum

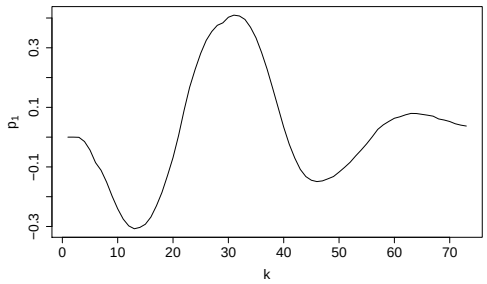


Figure 18 Position of the cart p_1

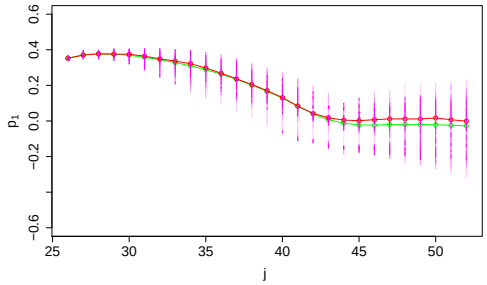


Figure 19 Prediction of p_1 at $k = 25$

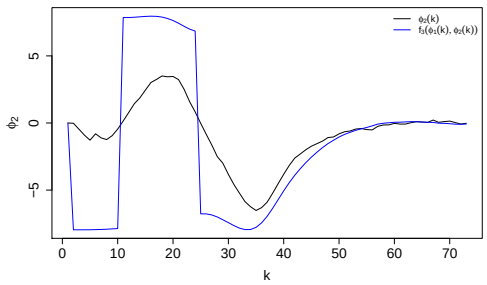


Figure 20 Pendulum velocity ϕ_2 and corresponding setpoint

if we have additional constraints on states or controls. One drawback with the parameter setting we applied in our simulation run is that the pendulum is not controllable in real time, since it is approximately 0.01 seconds to slow in each step. By reducing samples and prediction horizon and by increasing sampling time, however, the controller is also able to handle the pendulum in real time.

7 Summary and conclusion

In this article, a new model predictive control technique has been presented. It appeared that this new approach, which is based on the particle filter algorithm, performs very well. It is capable of handling nonlinear disturbed systems in real time, as it was seen by the CSTR and by the inverted pendulum simulations.

The advantage of the PF-MPC loop is that the current state estimations are given in distributions and not, as in usual control theory, as single point estimates. This provides much more information for the control unit. Another advantage is that complicated input constraints can be handled very easily, soft as well as hard constraints. For state and output constraints so far only soft constraints have been considered, which the controller also handled well. In addition to this, the controller is not restricted to Gaussian noise. It should be noted that this approach is supported by the well-developed Monte Carlo theory, and it is therefore less heuristic than many other approaches.

Clearly there are also some drawbacks. The most serious drawback is that it is hard to state anything about the stability of the PF-MPC loop, as is the case for most nonlinear control schemes. One also cannot say anything about the amount of particles needed for estimating the states and controlling the system. In some cases, one has to use so many particles to obtain good results that the computational effort is too high to handle the system in real time. In the pendulum example, the controller was capable of real time handling (results not shown), but the results with more samples looked much better. Another drawback is that in general the choice of the system transition densities as importance densities is not always a good choice. In this case, the particle filter may possibly show a bad performance or even fail. This drawback can be compensated by using better importance densities. Their development and further investigation can be seen as a future goal.

Also, future work should investigate possibilities for the application of this newly

developed PF-MPC controller on a real system, for the implementation of better discretization schemes (replacing the Euler-Maruyama scheme), and for the usage of better point estimates for the control inputs (replacing the mean value). It is worth taking a look at hard output constraints, too. An idea for this could be to reject drawn samples which are in contradiction to constraints.

Overall, the proposed controller has great potential, in particular when applied to nonlinear disturbed systems.

8 Proof of Theorem 1

W.l.o.g. let $k = 0$. First the following proportionality will be shown:

$$p(\bar{u}_{0:T_p}|s_{1:T_p}) \propto \prod_{j=0}^{T_p} p(\bar{u}_j|\bar{u}_{j-1}) \prod_{i=1}^{T_p} p(s_i|\hat{x}_i).$$

Therefore, we first consider the posterior density

$$p(\bar{x}_{0:T_p}, \bar{u}_{0:T_p}|s_{1:T_p})$$

of the joint states $(\bar{x}_{0:T_p}, \bar{u}_{0:T_p})$. For this posterior density it holds that

$$\begin{aligned} &p(\bar{x}_{0:T_p}, \bar{u}_{0:T_p}|s_{1:T_p}) \\ &\propto p(s_{T_p}|\bar{x}_{T_p}, \bar{u}_{T_p}) p(\bar{x}_{T_p}, \bar{u}_{T_p}|\bar{x}_{T_p-1}, \bar{u}_{T_p-1}) \\ &\quad \cdot p(\bar{x}_{0:T_p-1}, \bar{u}_{0:T_p-1}|s_{1:T_p-1}). \end{aligned} \tag{27}$$

Since s_{T_p} is conditionally independent from $\bar{u}_{T_p}$ given $\bar{x}_{T_p}$, it holds that

$$p(s_{T_p}|\bar{x}_{T_p}, \bar{u}_{T_p}) = p(s_{T_p}|\bar{x}_{T_p}). \tag{28}$$

Further, $\bar{x}_{T_p}$ is conditionally independent from $\bar{u}_{T_p}$ given $\bar{x}_{T_p-1}$, and $\bar{u}_{T_p}$ is conditionally independent from $\bar{x}_{T_p-1}$ given $\bar{u}_{T_p-1}$. Thus the following holds for the transition density of the joint state:

$$\begin{aligned} &p(\bar{x}_{T_p}, \bar{u}_{T_p}|\bar{x}_{T_p-1}, \bar{u}_{T_p-1}) \\ &= p(\bar{x}_{T_p}|\bar{u}_{T_p}, \bar{x}_{T_p-1}, \bar{u}_{T_p-1}) p(\bar{u}_{T_p}|\bar{x}_{T_p-1}, \bar{u}_{T_p-1}) \\ &= p(\bar{x}_{T_p}|\bar{x}_{T_p-1}, \bar{u}_{T_p-1}) p(\bar{u}_{T_p}|\bar{u}_{T_p-1}). \end{aligned} \tag{29}$$

Plugging (28) and (29) into (27), one obtains

$$\begin{aligned}
&p(\bar{x}_{0:T_p}, \bar{u}_{0:T_p}|s_{1:T_p}) \\
&\propto p(s_{T_p}|\bar{x}_{T_p})p(\bar{x}_{T_p}|\bar{x}_{T_p-1}, \bar{u}_{T_p-1})p(\bar{u}_{T_p}|\bar{u}_{T_p-1}) \\
&\quad \cdot p(\bar{x}_{0:T_p-1}, \bar{u}_{0:T_p-1}|s_{1:T_p-1}).
\end{aligned}$$

From this, one recursively gets the following decomposition

$$\begin{aligned}
&p(\bar{x}_{0:T_p}, \bar{u}_{0:T_p}|s_{1:T_p}) \\
&\propto p(\bar{x}_0)p(\bar{u}_0|\bar{u}_{-1}) \prod_{i=1}^{T_p} p(s_i|\bar{x}_i)p(\bar{x}_i|\bar{x}_{i-1}, \bar{u}_{i-1})p(\bar{u}_i|\bar{u}_{i-1}).
\end{aligned}$$

Then, the marginalized density $p(\bar{u}_{0:T_p}|s_{1:T_p})$ is given by

$$\begin{aligned}
&p(\bar{u}_{0:T_p}|s_{1:T_p}) \\
&= \int p(\bar{x}_{0:T_p}, \bar{u}_{0:T_p}|s_{1:T_p}) d\bar{x}_{0:T_p} \\
&\propto \int p(\bar{x}_0)p(\bar{u}_0|\bar{u}_{-1}) \\
&\qquad \cdot \prod_{i=1}^{T_p} p(s_i|\bar{x}_i)p(\bar{x}_i|\bar{x}_{i-1}, \bar{u}_{i-1})p(\bar{u}_i|\bar{u}_{i-1}) d\bar{x}_{0:T_p} \\
&= \prod_{j=0}^{T_p} p(\bar{u}_j|\bar{u}_{j-1}) \int p(\bar{x}_0) \prod_{i=1}^{T_p} p(s_i|\bar{x}_i)p(\bar{x}_i|\bar{x}_{i-1}, \bar{u}_{i-1}) d\bar{x}_{0:T_p}.
\end{aligned}$$

Now, we collect all components containing $\bar{x}_0$ and split the integral in the following way:

$$\begin{aligned}
p(\bar{u}_{0:T_p}|s_{1:T_p}) \propto \prod_{j=0}^{T_p} p(\bar{u}_j|\bar{u}_{j-1}) \iint p(\bar{x}_0)p(\bar{x}_1|\bar{x}_0, \bar{u}_0) d\bar{x}_0 \\
\cdot p(s_1|\bar{x}_1) \prod_{i=2}^{T_p} p(s_i|\bar{x}_i)p(\bar{x}_i|\bar{x}_{i-1}, \bar{u}_{i-1}) d\bar{x}_{1:T_p}.
\end{aligned} \tag{30}$$

Since the transition of $\bar{x}_j$ is deterministic and $\hat{x}_0$ is known, the following holds for the integral over $\bar{x}_0$:

$$\begin{aligned}
\int p(\bar{x}_0)p(\bar{x}_1|\bar{x}_0, \bar{u}_0) d\bar{x}_0 &= \int p(\bar{x}_1|\bar{x}_0, \bar{u}_0) \delta_{\hat{x}_0}(d\bar{x}_0) \\
&= p(\bar{x}_1|\hat{x}_0, \bar{u}_0),
\end{aligned}$$

and therefore (30) is equal to

$$p(\bar{u}_{0:T_p}|s_{1:T_p})$$

$$\propto \prod_{j=0}^{T_p} p(\bar{u}_j|\bar{u}_{j-1}) \int p(\bar{x}_1|\hat{x}_0,\bar{u}_0)p(s_1|\bar{x}_1) \cdot \prod_{i=2}^{T_p} p(s_i|\bar{x}_i)p(\bar{x}_i|\bar{x}_{i-1},\bar{u}_{i-1})d\bar{x}_{1:T_p}.$$

The components containing $\bar{x}_1$ are now collected and the integral is split again

$$\begin{aligned} &p(\bar{u}_{0:T_p}|s_{1:T_p}) \\ &\propto \prod_{j=0}^{T_p} p(\bar{u}_j|\bar{u}_{j-1}) \iint p(s_1|\bar{x}_1)p(\bar{x}_1|\hat{x}_0,\bar{u}_0)p(\bar{x}_2|\bar{x}_1,\bar{u}_1)d\bar{x}_1 \\ &\cdot p(s_2|\bar{x}_2) \prod_{i=3}^{T_p} p(s_i|\bar{x}_i)p(\bar{x}_i|\bar{x}_{i-1},\bar{u}_{i-1})d\bar{x}_{2:T_p}. \end{aligned}$$

The integral over $\bar{x}_1$

$$\int p(s_1|\bar{x}_1)p(\bar{x}_1|\hat{x}_0,\bar{u}_0)p(\bar{x}_2|\bar{x}_1,\bar{u}_1)d\bar{x}_1$$

is equal to

$$\begin{aligned} &\int p(s_1|\bar{x}_1)p(\bar{x}_2|\bar{x}_1,\bar{u}_1)\delta_{\underbrace{f(\hat{x}_0,\bar{u}_0)}_{=\hat{x}_1}}(d\bar{x}_1) \\ &= p(s_1|\hat{x}_1)p(\bar{x}_2|\hat{x}_1,\bar{u}_1) \end{aligned}$$

and thus (30) equals

$$\begin{aligned} &p(\bar{u}_{0:T_p}|s_{1:T_p}) \\ &\propto \prod_{j=0}^{T_p} p(\bar{u}_j|\bar{u}_{j-1}) \int p(s_1|\hat{x}_1)p(\bar{x}_2|\hat{x}_1,\bar{u}_1)p(s_2|\bar{x}_2) \\ &\cdot \prod_{i=3}^{T_p} p(s_i|\bar{x}_i)p(\bar{x}_i|\bar{x}_{i-1},\bar{u}_{i-1})d\bar{x}_{2:T_p}. \end{aligned}$$

The former steps are repeated recursively for $\bar{x}_2$ up to $\bar{x}_{T_p}$. This finally results in

$$p(\bar{u}_{0:T_p}|s_{1:T_p}) \propto \prod_{j=0}^{T_p} p(\bar{u}_j|\bar{u}_{j-1}) \prod_{i=1}^{T_p} p(s_i|\hat{x}_i).$$

Since $\tilde{v}_{j-1}$ and $\hat{v}_{j-1}$ are Gaussian, the following holds:

$$p(\bar{u}_{0:T_p}|s_{1:T_p})$$

$$
\begin{aligned}
&\propto \prod_{j=0}^{T_p} p(\bar{u}_j|\bar{u}_{j-1}) \prod_{i=1}^{T_p} p(s_i|\hat{x}_i) \\
&\propto \prod_{j=0}^{T_p} \exp\left(-\frac{1}{2}(\bar{u}_j - \bar{u}_{j-1})^T Q(\bar{u}_j - \bar{u}_{j-1})\right) \\
&\quad \cdot \prod_{i=1}^{T_p} \exp\left(-\frac{1}{2}(s_i - \hat{x}_i)^T R(s_i - \hat{x}_i)\right) \\
&= \prod_{j=0}^{T_p} \exp\left(-\frac{1}{2}\|\bar{u}_j - \bar{u}_{j-1}\|_Q^2\right) \prod_{i=1}^{T_p} \exp\left(-\frac{1}{2}\|s_i - \hat{x}_i\|_R^2\right) \\
&= \exp\left(-\frac{1}{2}\sum_{j=0}^{T_p} \|\bar{u}_j - \bar{u}_{j-1}\|_Q^2\right) \exp\left(-\frac{1}{2}\sum_{i=1}^{T_p} \|s_i - \hat{x}_i\|_R^2\right) \\
&= \exp\left(-\frac{1}{2}\left(\sum_{j=0}^{T_p} \|\bar{u}_j - \bar{u}_{j-1}\|_Q^2 + \sum_{i=1}^{T_p} \|s_i - \hat{x}_i\|_R^2\right)\right).
\end{aligned}
$$

Thus the following holds:

$$
\begin{aligned}
&\arg\max_{\bar{u}_{0:T_p}} p(\bar{u}_{0:T_p}|s_{1:T_p}) = \\
&\arg\max_{\bar{u}_{0:T_p}} \exp\left(-\frac{1}{2}\left(\sum_{j=0}^{T_p} \|\bar{u}_j - \bar{u}_{j-1}\|_Q^2 + \sum_{i=1}^{T_p} \|s_i - \hat{x}_i\|_R^2\right)\right).
\end{aligned}
$$

The maximum of a strictly monotone function is reached at the same point as the maximum of its argument, hence

$$
\begin{aligned}
&\arg\max_{\bar{u}_{0:T_p}} \exp\left(-\frac{1}{2}\left(\sum_{j=0}^{T_p} \|\bar{u}_j - \bar{u}_{j-1}\|_Q^2 + \sum_{i=1}^{T_p} \|s_i - \hat{x}_i\|_R^2\right)\right) \\
&= \arg\max_{\bar{u}_{0:T_p}} -\frac{1}{2}\left(\sum_{j=0}^{T_p} \|\bar{u}_j - \bar{u}_{j-1}\|_Q^2 + \sum_{i=1}^{T_p} \|s_i - \hat{x}_i\|_R^2\right) \\
&= \arg\min_{\bar{u}_{0:T_p}} \left(\sum_{j=0}^{T_p} \|\bar{u}_j - \bar{u}_{j-1}\|_Q^2 + \sum_{i=1}^{T_p} \|s_i - \hat{x}_i\|_R^2\right).
\end{aligned}
$$

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References

[1] C. Andrieu, A. Doucet, S.S. Singh, and V.B. Tadić. Particle methods for change detection, system identification, and control. *Proceedings of the IEEE*, 92(3):423–438, March 2004.

[2] M.S. Arulampalam, S. Maskell, N. Gordon, and T. Clapp. A tutorial on particle filters for online nonlinear/non-Gaussian Bayesian tracking. *Signal Processing, IEEE Transactions on*, 50(2):174–188, Feb 2002.

[3] S.I. Biagiola and J.A. Solsona. Use of state estimation for inferential nonlinear MPC: a case study. *Chem. Eng. J*, 106:13–24, 2005.

[4] L. Blackmore. Probabilistic particle control approach to optimal, robust predictive control. In *Proceedings of the AIAA Guidance, Navigation and Control Conference*, 2006.

[5] S.K. Botchu and S. Ungarala. Nonlinear model predictive control based on sequential Monte Carlo state estimation. *8th International IFAC Symposium on Dynamics and Control of Processs Systems, Preprints*, 3:31–36, Jun 2007.

[6] E. F. Camacho and C. Bordons. *Model predictive control. With a foreword by M. J. Grimble and M. A. Johnson. 2nd ed.* Advanced Textbooks in Control and Signal Processing. London: Springer. xxii, 435 p.5 , 2004.

[7] O. Cappe, S.J. Godsill, and E. Moulines. An overview of existing methods and recent advances in sequential Monte Carlo. *Proceedings of the IEEE*, 95(5):899–924, May 2007.

[8] H. Chen and F. Allgöwer. A quasi-infinite horizon nonlinear model predictive control scheme with guaranteed stability. *Automatica*, 34(10):1205–1218, 1998.

[9] Zhe Chen. Bayesian filtering: From Kalman filters to particle filters, and beyond. Technical report, McMaster University, 2003.

[10] Arnaud Doucet, Nando De Freitas, and Neil Gordon. *Sequential Monte Carlo methods in practice*. Springer, New York ; London, 2001.

[11] Arnaud Doucet, Simon Godsill, and Christophe Andrieu. On sequential Monte Carlo sampling methods for Bayesian filtering. *Statistics and Computing*, 10:197–208, 2000.

[12] Hans Driessen and Yvo Boers. MAP estimation in particle filter tracking. *Target Tracking and Data Fusion: Algorithms and Applications, 2008 IET Seminar on*, pages 41–45, April 2008.

[13] Jeroen Hol, Thomas Schön, and Fredrik Gustafsson. On resampling algorithms for particle filters. Technical report, Department of Electrical Engineering, Linköping University, January 2007.

[14] L. Imsland, R. Findeisen, E. Bullinger, F. Allgöwer, and B.A. Foss. A note on stability, robustness and performance of output feedback nonlinear model predictive control. *Journal of Process Control*, 13(7):633–644, October 2003.

[15] N. Kantas, J.M. Maciejowski, and A. Lecchini-Visintini. Sequential monte carlo for model predictive control. In *L. Magni et al. (Eds.): Nonlinear Model Predictive Control, LNCIS 384*, pages 263–273. Springer-Verlag, Berlin Heidelberg, 2009.

[16] P.E. Kloeden and E. Platen. *Numerical Solution of Stochastic Differential Equations*. Springer, Berlin, 1999.

[17] A. Kong, J.S. Liu, and W.H. Wong. Sequential imputations and Bayesian missing data problems. *J. Am. Statist. Ass*, 89(425):278–288, 1994.

[18] R Development Core Team. *R: A Language and Environment for Statistical Computing*. R Foundation for Statistical Computing, Vienna, Austria, 2009. ISBN 3-900051-07-0.

[19] D. Stahl. PF-MPC: Particle Filter - Model Predictive Control. Master's thesis, TU Kaiserslautern, 2009.

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