

Testing for parameter stability in nonlinear autoregressive models*

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Abstract

In this paper we develop testing procedures for the detection of structural changes in nonlinear autoregressive processes. For the detection procedure we model the regression function by a single layer feedforward neural network. We show that CUSUM-type tests based on cumulative sums of estimated residuals, that have been intensively studied for linear regression, can be extended to this case. The limit distribution under the null hypothesis is obtained, which is needed to construct asymptotic tests. For a large class of alternatives it is shown that the tests have asymptotic power one. In this case, we obtain a consistent change-point estimator which is related to the test statistics.

Power and size are further investigated in a small simulation study with a particular emphasis on situations where the model is misspecified, i.e. the data is not generated by a neural network but some other regression function. As illustration, an application on the Nile data set as well as S&P log-returns is given.

Keywords: Change analysis, nonparametric regression, neural network, autoregressive process

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1 Introduction

The question of structural stability of models is very important in diverse areas of science such as economy, finance, geology, physics or quality control. In statistics the field of

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change-point analysis has a long tradition dating back to Page [27, 28], who introduced it in the context of quality control. It deals with the question whether the stochastic structure of an observed time series has changed at some unknown point in the sample. For a detailed discussion of the field of change-point analysis we refer to the book by Csörgő and Horváth [8].

During the past decades change-point problems have been attracting more and more interest and have been investigated in different ways. A classical scenario is a possible mean change in otherwise independent identically distributed random variables. Later on, this was extended to stability tests of the parameters of a regression function. Most of these tests are based on the variational or dynamical behavior of the partial sum process $\hat{S}_n(k)$ of the estimated residuals frequently by considering the weighted maxima. In this context, CUSUM based tests introduced by Brown et al. [4] are of particular importance. While, there is an abundant literature for CUSUM based tests for a change in the mean and a change in the parameter of a linear model (confer Csörgő and Horváth [8] and some of the references therein) in the nonlinear setting most of the results available are Kernel based approaches, see e.g. Müller [24] for change-point problems in nonparametric regression or Delgado and Hidalgo [9] for the specific problem of a nonparametric estimation of the location and size of the structural change. Little is done for nonlinear time series models, an exception is found in Andrews [3], where Wald, Lagrange Multiplier and Likelihood ratio tests based on generalized method of moments are designed for testing parameter stability and structural changes.

In this paper we focus on nonlinear autoregressive time series, where – for the change-point theory – we model the autoregression function by a neural network. Due to its universal approximation property, a large class of functions can be approximated by a neural network to any degree of accuracy (confer e.g. White [35] or Franke et al. [11]). Therefore, this setup is very general and able to model many real-life time series while – at the same time – being mathematical feasible and computationally easier to handle due to its parametric nature. This is also confirmed by our small simulation study where a special emphasis is given to autoregressive time series where the regression function is not a neural network (cf. Section 4). For the theoretic results we do not require the time series to be autoregressive with a neural network as regression function either, instead we consider general stationary and mixing time series.

We ask the question whether the underlying model remains constant over the observed period of time or whether some parameter change takes place leading to a nonstationary behavior of the observed time series.

For $\theta = (\nu_0, \dots, \nu_H, \boldsymbol{\alpha}_1, \dots, \boldsymbol{\alpha}_H, \beta_1, \dots, \beta_H)$, $\boldsymbol{\alpha}_j = (\alpha_{j1}, \dots, \alpha_{jp})$,

$$f(\mathbf{x}, \theta) = \nu_0 + \sum_{h=1}^H \nu_h \psi(\langle \boldsymbol{\alpha}_h, \mathbf{x} \rangle + \beta_h), \quad (1.1)$$

denotes a one layer feedforward neural network with H hidden neurons and \langle, \rangle is the classical scalar product on \mathbb{R}^p . In this paper we assume that ψ is twice continuously differentiable and belongs to the class of sigmoid activation functions that satisfy

$$\lim_{x \rightarrow -\infty} \psi(x) = 0, \quad \lim_{x \rightarrow \infty} \psi(x) = 1, \quad \psi(x) + \psi(-x) = 1. \quad (1.2)$$

A popular choice is the logistic function

$$\psi(x) = (1 + e^{-x})^{-1}.$$

The autoregressive time series model with a neural network as regression function is then given by

$$Z_t = f(Z_{t-1}, \theta_0) + \varepsilon_t, \quad (1.3)$$

where $\mathbb{Z}_{t-1} = (Z_{t-1}, \dots, Z_{t-p})$, θ_0 is fixed but unknown, ε_t is independent of $\mathcal{F}_{t-1} = \sigma\{Z_u, u \leq t-1\}$ the σ -algebra generated by the observations up to time $t-1$. Furthermore, $\{\varepsilon_t : 1 \leq t \leq n\}$ are independent identically distributed random errors having a positive variance and satisfying further conditions specified below.

Stockis et al. [33] use these time series as building blocks in a regime-switching model, the so called CHARME-models, in the context of financial time series. In their model the duration time in each regime is random and driven by a hidden Markov chain, while in classical change-point analysis the duration time is usually fixed and deterministic. Motivated by the CHARME time series this paper develops change-point tests in such a setup.

Description of the test procedure

Let us now consider a time series model with a change after an unknown time point $1 \leq k^* = k^*(n) \leq n$

$$X_t = \begin{cases} Z_t, & t \leq k^* \\ Y_{t-k^*}, & t > k^*, \end{cases} \quad (1.4)$$

where $\{Y_t\}$ is some time series which differs distributionally from $\{Z_t\}$. The unknown parameter k^* is called the change-point.

We are now interested in the testing problem

$$H_0 : k^* = n \quad \text{vs.} \quad H_1 : k^* < n.$$

Our testing procedures are based on various functionals of the partial sums of estimated residuals with respect to the model (1.3)

$$\hat{S}_n(k) = \sum_{t=p+1}^k \hat{\varepsilon}_t \quad (1.5)$$

$$= \sum_{t=p+1}^k \left(X_t - f(\mathbb{X}_{t-1}, \hat{\theta}_n) \right), \quad (1.6)$$

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where $\widehat{\theta}_n$ is the least-squares estimator of θ_0 (assuming the null hypothesis holds true). Precisely we minimize the nonlinear least squares (NLLS) with respect to θ

$$\begin{aligned} Q_n(\theta) &:= \sum_{t=p+1}^n (X_t - f(\mathbb{X}_{t-1}, \theta))^2 \\ &=: \sum_{t=p+1}^n q_t(\theta), \end{aligned} \tag{1.7}$$

thus we consider the nonlinear least squares estimator

$$\widehat{\theta}_n = \arg \min_{\theta \in K} Q_n(\theta) \tag{1.8}$$

for a suitable compact set K . The minimization is usually obtained by solving the nonlinear score function

$$\frac{\partial Q_n(\widehat{\theta}_n)}{\partial \theta} = 0,$$

which yields

$$\sum_{t=p+1}^n \widehat{\varepsilon}_t = 0. \tag{1.9}$$

In Section 2 the behavior of the estimator $\widehat{\theta}_n$ is investigated in a variety of situations including the correctly specified case without change (1.3) as well as possibly misspecified cases with and without change. Under appropriate assumptions $\widehat{\theta}_n$ is eventually in the interior of the compact set K (cf. Theorem 2.1), so that we can assume (1.9) for limit considerations.

In fact, if $\widehat{\theta}_n$ is not in the interior of K , we will reject the null hypothesis immediately since either a change occurred or the model (1.3) is not capable of modeling the observed time series sufficiently well.

Typically, test statistics are of the form

$$\begin{aligned} T_{n1} &= \max_{p < k < n} \left(\sqrt{\frac{n-p}{k(n-p-k)}} |\widehat{S}_n(k)| \right), \\ T_{n2}(q) &= \max_{p < k < n} \left(\frac{1}{\sqrt{n-p} q(\frac{k}{n-p})} |\widehat{S}_n(k)| \right), \\ T_{n3}(G) &= \max_{p+G < k \leq n} \frac{1}{\sqrt{G}} \left| \widehat{S}_n(k) - \widehat{S}_n(k-G) \right|, \\ \widetilde{T}_{n3}(G) &= \max_{p+G < k \leq n-p-G} \frac{1}{\sqrt{2G}} \left| \widehat{S}_n(k+G) - 2\widehat{S}_n(k) + \widehat{S}_n(k-G) \right|, \\ T_{n4}(r) &= \frac{1}{n-p} \sum_{k=p+1}^{n-1} \frac{1}{r(k/(n-p))} \left(\frac{1}{\sqrt{n-p}} \widehat{S}_n(k) \right)^2, \end{aligned} \tag{1.10}$$

where $q(\cdot)$ and $r(\cdot)$ are weight functions defined on $(0, 1)$ specified below and $G < n$.

Theorem 3.1 gives the null asymptotics for the above statistics, from which we can derive an asymptotic size α test. Theorem 3.2 gives some assumptions under which these tests have asymptotic power one in the correct as well as misspecified model. In this situation Corollary 3.1 shows how to obtain a consistent estimator for the change-point.

In some applications data is observed sequentially and a decision whether a change occurred or not has to be made online. An example are financial time series such a stock returns where adjustments of investment strategies should be made if a change occurred. The procedures discussed in this paper can also be extended to this situation, for details we refer to Kirch and Tadjuidje-Kamgaing [17].

The remainder of the paper is organized as follow: Statistical properties of the above NLLS-estimator under the null hypothesis as well as alternatives, the correctly specified as well as misspecified models are discussed in the next section.

Section 3 contains the asymptotic distribution of the proposed test statistics under the null hypothesis as well as some consistency results under alternatives. Results are derived taking possibly misspecified models into account. In addition, a consistent change-point estimator is obtained.

Section 4 illustrates the usefulness of the proposed procedures with a simulation study as well as an application to the Nile data set as well as S&P data.

The final sections contain the proofs.

2 Some Properties of Neural Network Estimators

In this section we discuss the behavior of the NLLS estimator (1.8) under the null hypothesis as well as alternatives including misspecified models. First, we consider the correctly specified null hypothesis, i.e.

$$X_t = Z_t = f(Z_{t-1}, \theta_0) + \varepsilon_t, \quad (2.1)$$

where $f(\mathbf{x}, \theta)$ is as in (1.1). Secondly, we consider the misspecified model, where we observe an arbitrary stationary process without change-point

$$X_t = Z_t \quad (2.2)$$

and finally the possibly misspecified change-point model

$$X_t = \begin{cases} Z_t, & t \leq k^*, \\ Y_{t-k^*}, & t > k^*, \end{cases} \quad (2.3)$$

where we assume:

A. 1. The change-point fulfills $k^* = \lfloor \lambda n \rfloor$ for some $0 < \lambda < 1$.

We now introduce some assumptions to derive consistency and asymptotic normality of the estimator in the above cases. For the case of a correctly specified null hypothesis, there is an abundant literature on this topic, however we present a proof that is different from the classical ones. Using a uniform law of large numbers for random variables defined on a separable Banach space (cf. Ranga Rao [31]), we need only the second derivative of the cost function, instead of the third as usual for the nonlinear parametric models (see e.g. Theorem 3.2.24 in Taniguchi and Kakizawa [34]). As a result the moment assumptions required here are weaker than for the classical proofs.

N. 1. Let $\{X_t : t \in \mathbb{Z}\}$ (as in (2.1)) be a stationary and ergodic process and $\mathbb{E}|\varepsilon_0|^\nu < \infty$, for some $\nu \geq 1$.

In the misspecified case we assume

MS. 1. Let $\{Z_t : t \in \mathbb{Z}\}$, $\{Y_t : t \in \mathbb{Z}\}$ be stationary and ergodic processes and $\mathbb{E}|Z_1|^\nu < \infty$ and $\mathbb{E}|Y_1|^\nu < \infty$ for some $\nu \geq 2$.

N. 2. Assume that the parameter set $K \subseteq \mathbb{R}^{H(p+2)+1}$ is compact.

To derive theoretic results consider

$$\tilde{q}_t(\theta) = \begin{cases} (f(\mathbb{X}_{t-1}, \theta) - f(\mathbb{X}_{t-1}, \theta_0))^2 + 2\varepsilon_t (f(\mathbb{X}_{t-1}, \theta) - f(\mathbb{X}_{t-1}, \theta_0)), & X_t \text{ follows (2.1),} \\ (Z_t - f(\mathbb{Z}_{t-1}, \theta))^2, & X_t \text{ follows (2.2),} \\ (Z_t - f(\mathbb{Z}_{t-1}, \theta))^2 1_{\{t \leq \lfloor \lambda n \rfloor\}} + (Y_t - f(\mathbb{Y}_{t-1}, \theta))^2 1_{\{t > \lfloor \lambda n \rfloor\}}, & X_t \text{ follows (2.3),} \end{cases} \quad (2.4)$$

and $\tilde{Q}(\theta) = \sum_{t=p+1}^n \tilde{q}_t(\theta)$. Then, it holds $\hat{\theta}_n = \arg \min_{\theta} Q_n(\theta) = \arg \min_{\theta} \tilde{Q}_n(\theta)$. The different representation of $\hat{\theta}_n$ in case of model (1.3) has technical reasons as in the proofs only first moments are needed to obtain a uniform law of large numbers which in turn yields the consistency of the estimator. For the calculation of $\hat{\theta}_n$ in applications we use (1.8).

Furthermore, let

$$E(\theta) = \begin{cases} \mathbb{E}\tilde{q}_1(\theta), & \text{if no change occurs,} \\ \lambda \mathbb{E}\tilde{q}_1(\theta) + (1 - \lambda)\mathbb{E}\tilde{q}_n(\theta), & \text{if a change occurs,} \end{cases} \quad (2.5)$$

and

$$\tilde{\theta}_0 = \arg \min_{\theta} E(\theta) \quad (2.6)$$

which fulfills:

N. 3. $\tilde{\theta}_0$ is the unique minimizer of $E(\theta)$ and lies in the interior of K .

If the parameter space is chosen in such a way that the network is identifiable, then the true parameter θ_0 in the correctly specified model without change (2.1) is the unique minimizer of $E(\theta) = \mathbb{E}(f(\mathbb{X}_{t-1}, \theta_0) - f(\mathbb{X}_{t-1}, \theta))^2$, which shows that $\tilde{\theta}_0 = \theta_0$.

In order to get a better understanding of identifiability in neural networks, we state the following result of Hwang and Ding [15].

Lemma 2.1. *Assume that*

- (i) $f(\mathbf{x}, \tilde{\theta}_0)$ is not redundant (i.e. there exists no other networks with fewer hidden neurons ($H' < H$) that represent exactly the same relationship function).
- (ii) $f(\mathbf{x}, \tilde{\theta}_0)$ is irreducible, i.e., for all $i \neq 0, j \neq 0$,
 - a) $\nu_i \neq 0$
 - b) $\alpha_i \neq 0$
 - c) $(\alpha_i, \beta_i) \neq (\alpha_j, \beta_j)$ for all $i \neq j$.

If the activation function satisfies (1.2), then up to a family of permutations and transformations defined below, $f(\mathbf{x}, \tilde{\theta}_0)$ is identifiable unique.

To understand the transformation leading to non-identifiability, redefine $\theta = (\nu_0, \mu_1, \dots, \mu_H)$ with $\mu_h = (\nu_h, \beta_h, \alpha_h), h = 1, \dots, H$, then

1. a permutation of $\mu_i = (\nu_i, \beta_i, \alpha_i)$ and $\mu_j = (\nu_j, \beta_j, \alpha_j)$ still provide the same neural networks function, i.e. a permutation of the i -th neuron and j -th neuron will not change the value of the neural networks function.
2. Additionally, by using the relation in equation (1.2) one derives

$$\nu_i \psi(\langle \alpha_i, x \rangle + \beta_i) = \nu_i (1 - \psi(\langle -\alpha_i, x \rangle - \beta_i)).$$

Henceforth, we can easily verify that $(\nu_0, \mu_1, \dots, \mu_{i-1}, \mu_i, \mu_{i+1}, \dots, \mu_H)$ and $(\nu_0 + \nu_i, \mu_1, \dots, \mu_{i-1}, -\mu_i, \mu_{i+1}, \dots, \mu_H)$ yield the same neural network function.

In practice these transformations do not yield a problem, as in each segment of the parameter space $\tilde{\theta}_0$ is identifiably unique so that it is still guaranteed that $f(\mathbf{x}, \hat{\theta}_n)$ is close to $f(\mathbf{x}, \tilde{\theta}_0)$, which is all that matters for the below change-point test.

Theorem 2.1. *Let N.2 and N.3 hold. In the correctly specified case (2.1) let N.1 hold, in the misspecified cases (2.2) resp. (2.3) assume MS.1. Then $\hat{\theta}_n$ is strongly consistent for $\tilde{\theta}_0$, i.e.*

$$\hat{\theta}_n \rightarrow \tilde{\theta}_0 \quad a.s. \quad (n \rightarrow \infty).$$

In order to derive asymptotic normality we need some additional assumptions. Recall that a stationary process $\{T_t\}$ is called α - or strong mixing with mixing rate $\alpha(\cdot)$ if

$$\alpha(j) = \sup_{A \in \mathcal{F}_{-\infty}^0(T), B \in \mathcal{F}_j^\infty(T)} |P(A \cap B) - P(A)P(B)| \rightarrow 0 \text{ as } j \rightarrow \infty,$$

where $\mathcal{F}_{-\infty}^0(T)$ is the σ -algebra generated by T_0, T_{-1}, \dots and $\mathcal{F}_j^\infty(T)$ is the σ -algebra generated by T_j, T_{j+1}, \dots .

N. 4. The time series $X_t = f(\mathbb{X}_{t-1}, \theta_0) + \varepsilon_t$ is α -mixing with rate $\alpha(j) = O(n^{-c})$ for some $c > \nu/(2 - \nu)$, where $\nu > 2$ is such that $\mathbb{E}|\varepsilon_0|^\nu < \infty$.

This is a classical condition in nonlinear time series analysis and can be derived with little effort using the stability theory for Markov processes, see e.g. Meyn and Tweedie [23]. Indeed, this property is a consequence of the existence of a stationary solution that is geometric ergodic. In this situation the assumption on the rate is thus fulfilled since the time series is beta-mixing with an exponential rate.

In a more general setup and for the related CHARME-models this property has been proven by Stockis et al. [33].

Recently, different dependency concepts have been introduced to tackle some of the deficiencies of mixing conditions (cf. e.g. Doukhan and Louhichi [10] for the weak dependence approach). Our results remain true if these conditions ensure certain assumptions. For example to obtain the next theorem we need that $\frac{1}{\sqrt{n}}\nabla\tilde{Q}_n(\tilde{\theta}_0)$ fulfills a central limit theorem, for the results in the next section that it fulfills the law of iterated logarithm in addition to a strong invariance principle for $\zeta_t = Z_t - f(Z_{t-1}, \tilde{\theta}_0)$.

The advantage of using mixing conditions is twofold. First, for \mathbb{Z}_t mixing, processes of the form $g(\mathbb{Z}_t)$ for some measurable g are also strong mixing with the same rate (cf. the proof of Theorem 2.2). Secondly, all the results we need are available in the literature for mixing processes and follow for example from the invariance principle of Kuelbs and Philipp [21].

One example of a different dependency concept where a strong invariance principles holds is given by Wu [36] for processes of the form $Z_t = h(\varepsilon_t, \varepsilon_{t-1}, \dots)$, which also fits nicely into our context.

MS. 2. The time series $\{Z_t\}$ and $\{Y_t\}$ are independent and α -mixing with $\alpha(j) = O(n^{-c})$ for some $c > \nu/(2 - \nu)$, where $\nu > 2$ is such that $\mathbb{E}|Z_0|^\nu < \infty$ and $\mathbb{E}|Y_0|^\nu < \infty$.

Additionally, to achieve the asymptotic normality of the least-squares estimators we need to assume:

N. 5.

$$A = \nabla^2 E(\tilde{\theta}_0)$$

is positive definite, where ∇^2 denotes the Hesse matrix with respect to θ .

Finally, we can state asymptotic normality of the estimators.

Theorem 2.2. *Let N.2, N.3 and N.5 hold. Additionally, in the correctly specified case (2.1) let N.1 and N.4 and in the misspecified cases (2.2) resp. (2.3) let MS.1 and MS.2 hold. Then*

$$\sqrt{n}(\hat{\theta}_n - \tilde{\theta}_0) \xrightarrow{\mathcal{L}} \mathcal{N}(0, A^{-1}VA^{-1}),$$

where

$$V = \lim_{n \rightarrow \infty} \frac{1}{n} \mathbb{E} \nabla \tilde{Q}_n(\tilde{\theta}_0) (\nabla \tilde{Q}_n(\tilde{\theta}_0))^T$$

with $\nabla \tilde{Q}_n(\theta) = \frac{\partial \tilde{Q}_n(\theta)}{\partial \theta}$ is the gradient of $\tilde{Q}_n(\theta)$. The limit V exists but may be singular.

3 Consistency of the Change-Point Tests

In this section we derive the null asymptotics for the test statistics introduced in (1.10). Furthermore, we show that the corresponding tests have asymptotic power one for a large class of alternatives. In this case, we also obtain a consistent estimator for the change-point which is related to the test statistics above.

First, we need to introduce some more notation. We assume that the weight function q belongs to the class

$$Q_{0,1} = \{q : q \text{ is non-decreasing in a neighborhood of zero, non-increasing in a neighborhood of one and } \inf_{\eta \leq t \leq 1-\eta} q(t) > 0 \text{ for all } 0 < \eta < 1/2\}.$$

We need additionally that the following integral is finite for at least some $c > 0$

$$I^*(q, c) = \int_0^1 \frac{1}{t(1-t)} \exp \left\{ \frac{-cq^2(t)}{t(1-t)} \right\} dt.$$

A very important class of weight functions fulfilling these conditions are

$$q(t) = (t(1-t))^\gamma, \quad 0 \leq \gamma < 1/2,$$

where a γ close to $1/2$ rather detects early or late changes and a γ close to 0 detects changes in the middle. Note that for $\gamma = 1/2$ we obtain statistic T_{n1} (which is asymptotically independent from the statistics with $\gamma < 1/2$).

For details and further references confer Csörgő and Horváth [7], Chapter 4.

We assume that the weight function r fulfills for all $x \in (0, 1)$

$$r(x) > 0 \quad \text{and} \quad \int_0^1 \frac{t(1-t)}{r(t)} dt < \infty. \quad (3.1)$$

For more details and further references confer Csörgő and Horváth [8], Chapter 2.

Moreover define

$$\alpha(x) = \sqrt{2 \log x}, \quad \beta(x) = 2 \log x + \frac{1}{2} \log \log x - \frac{1}{2} \log \pi. \quad (3.2)$$

The key to the next theorem is a strong invariance principle for $\zeta_t = Z_t - f(Z_{t-1}, \tilde{\theta}_0)$, i.e. there exists a Wiener process $\{W(t)\}$ (possibly after enlarging the probability space) and $0 < \kappa < 1/2$, $\tau > 0$ such that

$$\sum_{i=1}^k (\zeta_i - \mathbb{E}\zeta_1) - \tau W(k) = O(k^{1/2-\kappa}) \quad a.s. \quad (3.3)$$

In case of a correctly specified model (1.3) we get $\zeta_t = \varepsilon_t$ is an i.i.d. sequence, which fulfills the above invariance principle with $\tau^2 = \text{var}(\varepsilon_1)$, $\kappa = (\nu - 2)/(2\nu)$. This is a

classical result by Komlós et al. [19, 20] and Major [22], which has subsequently been generalized to dependent random variables. For example, Kuelbs and Philipp [21] prove such a limit theorem for strong mixing sequences fulfilling MS.2. In particular (3.3) holds in the situation of the theorem below. Wu [36] proves the above invariance principle for certain processes of the type $Z_t = g(\varepsilon_t, \varepsilon_{t-1}, \dots)$, which also fits nicely into our context.

For the statistics $T_{n2}(q)$ and $T_{n4}(r)$ it is sufficient but more technical if only a functional central limit theorem in addition to some Hájek-Rényi-type inequalities (for $q, r \neq 1$) holds (for some technical details we refer to Kirch [16], proof of Corollary 6.1).

We are now ready to state the asymptotic distribution under H_0 for the above statistics.

Theorem 3.1. *Let the null hypothesis hold, i.e. $X_t = Z_t$ and no change occurs. Furthermore assume N.2, N.3, MS.1 and MS.2 in addition to*

$$\widehat{\theta}_n - \widetilde{\theta}_0 = O_P(\sqrt{n}), \quad (3.4)$$

which holds e.g. under N.5 by Theorem 2.2.

a) Then we have for all $x \in \mathbb{R}$

$$P\left(\alpha(\log(n-p))\frac{T_{n1}}{\tau} - \beta(\log(n-p)) \leq x\right) \rightarrow \exp(-2e^{-x}) \quad \text{as } n \rightarrow \infty.$$

b) If $q \in Q_{0,1}$ and $I^*(q, c) < \infty$ for some $c > 0$, then

$$\frac{1}{\tau} T_{n2}(q) \xrightarrow{\mathcal{D}} \sup_{0 < t < 1} \frac{|B(t)|}{q(t)} \quad \text{as } n \rightarrow \infty.$$

c) If $G = G(n) \rightarrow \infty$, $\frac{G}{n} \rightarrow 0$ and $G^{-1}n^{1-2\kappa} \log n \rightarrow 0$ as $n \rightarrow \infty$, κ as in (3.3), then we have for all $x \in \mathbb{R}$

$$P\left(\alpha((n-p)/G)\frac{T_{n3}(G)}{\tau} - \beta((n-p)/G) \leq x\right) \rightarrow \exp(-2e^{-x}) \quad \text{as } n \rightarrow \infty.$$

d) If $G = G(n) \rightarrow \infty$, $\frac{G}{n} \rightarrow 0$ and $G^{-1}n^{1-2\kappa} \log n \rightarrow 0$ as $n \rightarrow \infty$, κ as in (3.3), then we have for all $x \in \mathbb{R}$

$$P\left(\alpha((n-p)/G)\frac{\widetilde{T}_{n3}(G)}{\tau} - \beta((n-p)/G) + \log(2/3) \leq x\right) \rightarrow \exp(-2e^{-x}) \quad \text{as } n \rightarrow \infty.$$

e) If r fulfills condition (3.1), then

$$\frac{1}{\tau^2} T_{n4}(r) \xrightarrow{\mathcal{D}} \int_0^1 \frac{B^2(t)}{r(t)} dt \quad \text{as } n \rightarrow \infty.$$

Here $\{B(t) : 0 \leq t \leq 1\}$ denotes a Brownian bridge. The assertions remain true if instead of the true long-run standard deviation $\tau > 0$ an estimator $\widehat{\tau}_n$ is used as long as for b) and e) $\widehat{\tau} - \tau = o_P(1)$, for a) $\widehat{\tau} - \tau = o_P((\log \log n)^{-1})$ and for c) and d) $\widehat{\tau} - \tau = o_P((\log n/G)^{-1})$.

The following lemma gives a variance estimator as is needed for the change-point tests above in case of a correctly specified model or put differently if the data at hand are well modeled by (1.3).

Lemma 3.1. *Let N.1 – N.5 hold. Then, under H_0 it holds for $2 \leq \nu < 4$ in N.1*

$$\begin{aligned} \widehat{\sigma}^2 &= \frac{1}{n - (H(p+2) + 1)} \sum_{j=p+1}^n \widehat{\varepsilon}_j^2 \\ &= \sigma^2 + o_P(n^{-(\nu-2)/\nu}) \end{aligned}$$

If $\nu \geq 4$, then we get the stronger rate $O_P(n^{-1/2})$.

Remark 3.1. In practical applications it is advisable to use the following adapted variance estimator, which – under appropriate assumptions – is also a consistent estimator under alternatives in the fully correctly specified situations, where $Y(\cdot)$ is also an autoregressive process based on a different neural network with an error-sequence with the same variance:

$$\begin{aligned} \widehat{\sigma}_{a,n}^2 &= \frac{\widehat{k}^*}{n} \frac{1}{\widehat{k}^* - (H(p+2) + 1)} \sum_{t=p+1}^{\widehat{k}^*} \widehat{\varepsilon}_t^2 \\ &\quad + \left(1 - \frac{\widehat{k}^*}{n}\right) \frac{1}{n - \widehat{k}^* - (H(p+2) + 1)} \sum_{t=\widehat{k}^*+1}^n \widehat{\varepsilon}_t^2 \end{aligned} \quad (3.5)$$

where \widehat{k}^* is as in Corollary 3.1

$$\begin{aligned} \widehat{\varepsilon}_t &= \begin{cases} X_t - f(\mathbb{X}_{t-1}, \widehat{\theta}_1), & t \leq \widehat{k}^*, \\ X_t - f(\mathbb{X}_{t-1}, \widehat{\theta}_2), & t > \widehat{k}^*, \end{cases} \\ \widehat{\theta}_1 &= \arg \min_{\theta} \sum_{t=p+1}^{\widehat{k}^*} (X_t - f(\mathbb{X}_{t-1}, \theta))^2, \quad \widehat{\theta}_2 = \arg \min_{\theta} \sum_{t=\widehat{k}^*+1}^n (X_t - f(\mathbb{X}_{t-1}, \theta))^2. \end{aligned}$$

Theorem 3.1 suggests to use an estimator for the long-run variance τ^2 instead of σ^2 in order to obtain asymptotically the correct size even under misspecification. To this end we propose to use a flattop-kernel estimator taking possible changes into account with the automatic bandwidth selection procedure by Politis [29]

$$\widehat{\tau}^2 = \widehat{\tau}^2(\Lambda_n) = \max \left(\widehat{R}(0) + 2 \sum_{k=1}^{\Lambda_n} w(k/\Lambda_n) \widehat{R}(k), \frac{1}{n} \widehat{\sigma}_{a,n}^2 \right), \quad (3.6)$$

where for $a > b$ we define $\sum_a^b = 0$, $\widehat{R}(k) = \frac{1}{n} \sum_{t=1}^{n-k} \widehat{\varepsilon}(t)\widehat{\varepsilon}(t+k)$ and

$$w(t) = \begin{cases} 1, & |t| \leq 1/2, \\ 2(1 - |t|), & 1/2 < t < 1, \\ 0, & t \geq 1. \end{cases}$$

The maximum on the right-hand side of the formula is needed to guarantee that the estimator is positive but remains scale invariant. In simulations we use the following automatic bandwidth selection procedure proposed by Politis [29]:

Automatic bandwidth selection procedure: Let $\widehat{\lambda}$ be the smallest positive integer such that $|\widehat{R}(\widehat{\lambda} + k)/\widehat{R}(0)| < 1.4\sqrt{\log n/n}$, for $k = 1, \dots, 3$. Then choose the bandwidth $\widehat{\Lambda}_n = 2\widehat{\lambda}$.

Hušková and Kirch [13] discuss the equivalent of this estimator in case of mean changes in dependent data. The problem is that the long-run variance τ^2 is much more difficult to estimate than σ^2 so that we do not obtain such good estimators for small sample sizes (cf. e.g. the simulation study in Hušková and Kirch [13]). On the other hand if the approximation of Z_t by a neural network based autoregressive process with parameter $\widetilde{\theta}_0$ is good enough, the two variances are almost equal, but the variance estimator (3.5) is much more accurate than the long-run variance estimator (3.6). In fact, in the simulations (cf. Section 4) it can be seen that using the variance estimator (3.5) for σ^2 results in a superior test performance than using (3.6) even in the misspecified situation.

Theorem 3.1 enables us to construct tests based on the above statistics with the correct asymptotic size. In the next theorem it is shown that those tests have asymptotic power one under a large class of alternatives.

Before we can state the theorem we first need to give some additional assumptions.

A. 2. (a) There exists $c > 0$ such that

$$P(\max[|\mathbb{E}Z_1 - \mathbb{E}f(Z_p, \theta)|_{\theta=\widehat{\theta}_n}|, |\mathbb{E}Y_1 - \mathbb{E}f(Y_p, \theta)|_{\theta=\widehat{\theta}_n}|] \geq c) \rightarrow 1.$$

(b) There exists $c > 0$ such that

$$P(|(\mathbb{E}Z_1 - \mathbb{E}f(Z_p, \theta)|_{\theta=\widehat{\theta}_n}) - (\mathbb{E}Y_1 - \mathbb{E}f(Y_p, \theta)|_{\theta=\widehat{\theta}_n})| \geq c) \rightarrow 1.$$

If the assumptions of Theorem 2.1 hold, A.2 a) is fulfilled if

$$|\mathbb{E}f(Z_p, \widetilde{\theta}_0) - \mathbb{E}Z_1| > 0.$$

Note that in this situation by the definition of $\widetilde{\theta}_0$ and N.3

$$|\mathbb{E}f(Z_p, \widetilde{\theta}_0) - \mathbb{E}Z_1| = \frac{1-\lambda}{\lambda} |\mathbb{E}Y_1 - \mathbb{E}f(Y_p, \theta)|.$$

If $\widetilde{\theta}_0$ does not exist under alternatives, the behavior of $\widehat{\theta}_n$ is arbitrary, but a) is still fulfilled if

$$\min_{\theta \in K} (\max(|\mathbb{E}f(X_p, \theta) - \mathbb{E}X_1|, |\mathbb{E}Y_1 - \mathbb{E}f(Y_p, \theta)|)) > 0. \quad (3.7)$$

Analogous expressions can be obtained for b).

Remark 3.2. Equation (3.7) essentially means that there exists no neural network $f(\mathbf{x}, \theta)$ with H hidden neurons for which $\mathbb{E}f(\mathbb{Z}_p, \theta) = \mathbb{E}Z_1$ as well as $\mathbb{E}f(\mathbb{Y}_1, \theta) = \mathbb{E}Y_1$. In the simple mean-change-model without neural networks (i.e. trivial neural networks where $H = 0$) both assumptions reduce to $\mathbb{E}Z_1 \neq \mathbb{E}Y_1$.

Assumptions like these are typical for change-point statistics that are based on estimated residuals and even occur in a simple linear regression situation (cf. e.g. Hušková and Koubkova [14]). Tests, which can detect general alternatives in the correctly specified model, can usually be obtained by using partial sums of vector-weighted estimated residuals, yet they are theoretically and computationally much more complicated and therefore will be investigated elsewhere. However, the approach discussed in this paper still yields reasonable results for a large class of changes as the simulation study in Section 4 shows and is computationally and theoretically feasible.

In Kirch and Tadjuidje-Kamgaing [18] we show that the tests have asymptotic power one even under certain local alternatives. The key tool is a uniform central limit theorem, which replaces the uniform law of large numbers (Theorem 5.1) in the below proof.

Theorem 3.2. *Assume that N.2 holds as well as MS.1, but here it is sufficient if the moment condition in MS.1 holds for $\nu \geq 1$. This includes the case of correct specification (1.4). Furthermore, the change-point fulfills Assumptions A.1.*

a) *Let Assumption A.2 (a) hold.*

(i) *It holds for all $c \in \mathbb{R}$*

$$P(\alpha(\log(n-p))T_{n1} - \beta(\log(n-p)) \geq c) \rightarrow 1.$$

(ii) *If $q \in Q_{0,1}$, then*

$$T_{n2}(q) \xrightarrow{P} \infty,$$

which means that $P(T_{n2}(q) \geq c) \rightarrow 1$ for all $c > 0$.

(iii) *If $G = G(n) \rightarrow \infty$, $\frac{\log n}{G} \rightarrow 0$ but $G/n \rightarrow 0$, then we have for all $c \in \mathbb{R}$*

$$P(\alpha((n-p)/G)T_{n3}(G) - \beta((n-p)/G) \geq c) \rightarrow 1.$$

(iv) *If r fulfills condition (3.1), then*

$$T_{n4}(r) \xrightarrow{P} \infty.$$

b) *Let Assumption A.2 (b) hold. If $G = G(n) \rightarrow \infty$, $\frac{\log n}{G} \rightarrow 0$ but $G/n \rightarrow 0$, then we have for all $c \in \mathbb{R}$*

$$P\left(\alpha((n-p)/G)\tilde{T}_{n3}(G) - \beta((n-p)/G) + \log(2/3) \geq c\right) \rightarrow 1.$$

We obtain asymptotic power one in case of an unknown variance, if the variance estimator is at least stochastically bounded under the alternative.

Based on the partial sum process $\{\hat{S}_n(k)\}$ we additionally obtain a consistent estimator for the change-point as the following corollary shows.

Corollary 3.1. *Let Assumptions N.2, N.3, MS.1 ($\nu \geq 1$ is sufficient) as well as A.1 and A.2 (a) hold. Then*

$$\frac{\hat{k}^*}{n} \xrightarrow{P} \lambda,$$

where

$$\hat{k}^* = \arg \max \left\{ \left| \hat{S}_n(k) \right| : 1 \leq k < n \right\}. \quad (3.8)$$

4 Simulation Study and Real Data Applications

In the previous sections we have shown that the derived tests have asymptotic level α and power one for a large class of alternatives.

In Section 4.1, we consider the behavior of the statistic if the data really is an autoregressive process generated by a neural network. As in reality this will generally not be fulfilled, the most pressing question is what happens under misspecification, which will be considered in Section 4.2. In this context the question arises how sensitive the procedure is with respect to the choice of the number of hidden neurons which is also considered there. Finally in Section 4.4, we would like to have a more detailed look at what kind of alternatives can be detected (cf. also Remark 3.2).

For illustrational purposes the methods will then be applied to two data sets which have been used frequently in the context of change-point detection, namely the Nile data set as well as S&P-returns in Section 4.5.

Because of limitations of space we restrict the simulations to the statistic $T_n := T_{n2}(q)$ with $q \equiv 1$, changes at $k^* = n/2$ and an autoregression of order 1. For the implementation of the test optimization algorithms to obtain the parameter $\hat{\theta}_n$ as in (1.8) (resp. for $\hat{\theta}_1, \hat{\theta}_2$ as in Remark 3.1) are needed. To this end the matlab algorithm *fminsearch* is used and the data set only included in the simulations if the variable *exitflag* of *fminsearch* indicates that the algorithm has converged. The idea behind this is that the test would only applied in situations where this is the case and the approximation by a neural network based process is reasonable.

4.1 Behavior of the Statistic under Model Specification

For sake of illustration, we consider the following correctly specified model including a change

$$X_t = \begin{cases} 0.5 + (1 + \exp(0.5(1 + \beta_1 X_{t-1})))^{-1} + \varepsilon_t, & t \leq n/2, \\ 0.5 - (1 + \exp(0.5(1 + \beta_2 X_{t-1})))^{-1} + \varepsilon_t, & t > n/2, \end{cases}$$

where ε_t is i.i.d. standard Gaussian random variables. The estimation is carried out with a neural network for which $H = 1$.

As illustration, Figure 4.1 shows one generated data set in addition to the corresponding CUSUM-Plot showing $1/\sqrt{\hat{\sigma}_{a,n}^2(n-p)}|\hat{S}_n(k)|$. Note that $T_n = \max_k 1/\sqrt{n-p}|\hat{S}_n(k)|$.

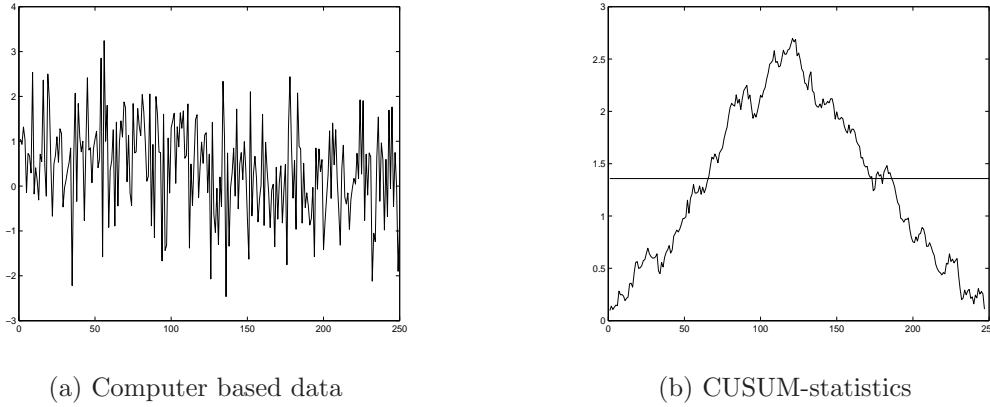


Figure 4.1: Neural Network based data and CUSUM statistic

Similar to the numerical study in the coming sections, we consider the sample sizes $n = 250, 500$ and base the empirical sizes and powers on 1000 replications (repeated experiments). For $\beta_1 = 0.7$ and $\beta_2 = -0.7$ the size 0.048 (0.053, 0.042) for variance estimators (3.5) ((3.6) and the true variance) are obtained for a length of $n = 250$ (respectively 0.046, 0.053, 0.046 for $n = 500$) and corresponding power is 0.999, (0.796, 0.999) for $n = 250$ (respectively 0.999, 0.997, 0.999 for $n = 500$).

More generally consider

$$X_t = \begin{cases} 0.5 + (1 + \exp(0.5(1 + 0.7X_{t-1})))^{-1} + \varepsilon_t, & t \leq n/2, \\ \mu_2 + \alpha_2 (1 + \exp(0.5(1 + \beta_2 X_{t-1})))^{-1} + \varepsilon_t, & t > n/2. \end{cases}$$

For various values of $(\mu_2, \alpha_2, \beta_2)$ the empirical size and power of the test statistic are summarized in Table 4.1. The results here are given for a sample size of $n = 250$ observations, EV stands for variance estimator (3.5), LV for (3.6) and KV for the true variance of the errors.

			(0.1, 1, 0.7)	(0.5, -1, 0.7)	(0.5,1,-0.7)	(0.5, -1, -0.7)
		size	power	power	power	power
H=1	EV	0.048	0.790	0.999	0.160	1
	LV	0.053	0.697	0.7960	0.146	0.6810
	KV	0.042	0.786	0.999	0.146	1

Table 4.1: Influence of the of the various parameters on the empirical size for a nominal level of 5% for the correctly specified models, $n = 250$.

Conclusions:

Using the estimator for the variance (3.5) yields comparable in fact even slightly better results than using the known variance which has been included as a benchmark value. Furthermore, the results are better especially in terms of power than using the long-run variance estimator. However, this is not surprising in the correctly specified model.

4.2 Behavior of the test statistics under misspecification

In this section we consider the more important misspecified situation, where the autoregression function is not truly given by a neural network but can only be approximated by it. In the simulations we consider linear autoregression functions $g(x) = a_0 + a_1x$ corresponding to a true AR(1)-process, as well as nonlinear autoregression functions

$$g(x) = \begin{cases} a_0 + a_1x, & x \leq c, \\ b_0 + b_1x, & x > c, \end{cases} \quad (\text{TAR}).$$

corresponding to a true TAR(1)-process (threshold autoregressive).

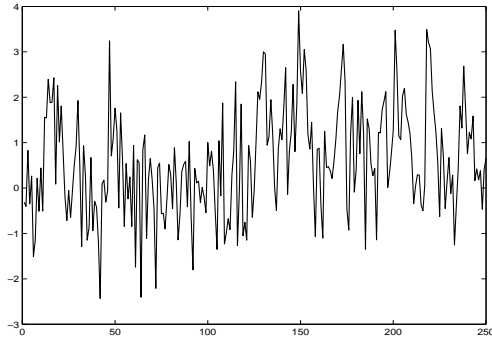
More precisely, we use standard normal errors and the following two AR as well as TAR parameters:

- AR 1: $g_0(x) = 0.3x, g_1(x) = 0.5 + 0.1x$
- AR 2: $g_0(x) = 0.3x, g_1(x) = 1 - 0.1x$
- TAR 1: $g_0(x) = 0.3x1_{\{x \geq 0\}} - 0.1x1_{\{x < 0\}}, g_1(x) = (0.5 + 0.5x)1_{\{x \geq 0\}} - 0.3x1_{\{x < 0\}}$
- TAR 2: $g_0(x) = 0.3x1_{\{x \geq 0\}} - 0.1x1_{\{x < 0\}}, g_1(x) = (1 - 0.1x)1_{\{x \geq 0\}} + (0.5 + 0.1x)1_{\{x < 0\}}$

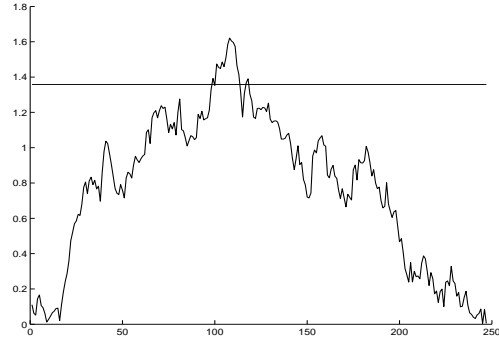
From the construction of the test statistic it is clear that the variance estimator plays a crucial role for the performance of the test both under the null hypothesis as well as under alternatives (cf. Remark 3.1). The empirical size and power (based on 1000 repeated experiments) for different scenarios are reported in Table 4.2. EV indicates that the estimator $\hat{\sigma}_{a,n}^2$ as given in 3.5 has been used, while for LV the long-run variance estimator (3.6) has been used. The true long-run variance is not known in the misspecified case so that it cannot be included here.

4 Simulation Study and Real Data Applications

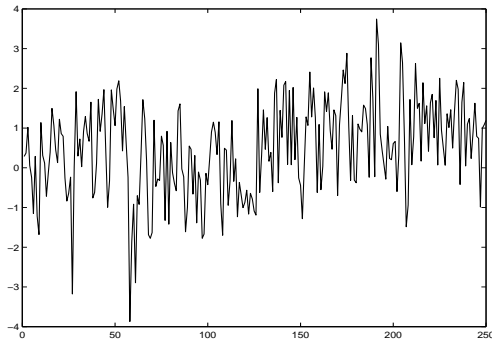
In Figure 4.2 one sample path for each four of the above misspecified scenarios is given in addition to a plot of $\frac{1}{\sqrt{\hat{\sigma}_{a,n}^2(n-p)}}|\hat{S}_n(k)|$ for this sample path, note that $T_n = \max_k 1/\sqrt{n-p}|\hat{S}_n(k)|$.



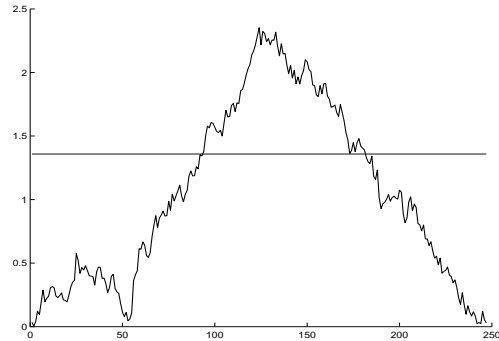
(a) AR 1: Sample Path



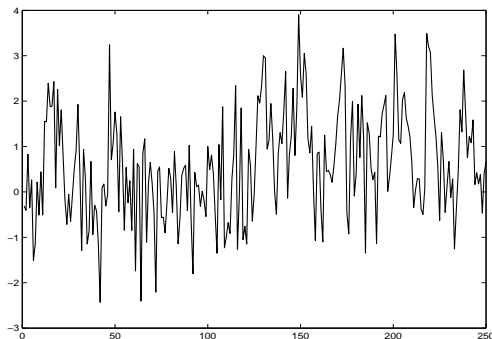
(b) AR 1: CUSUM Plot



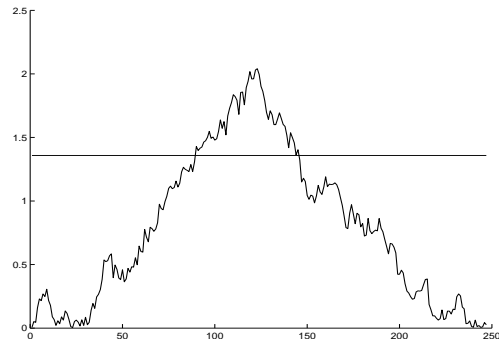
(c) AR 2: Sample Path



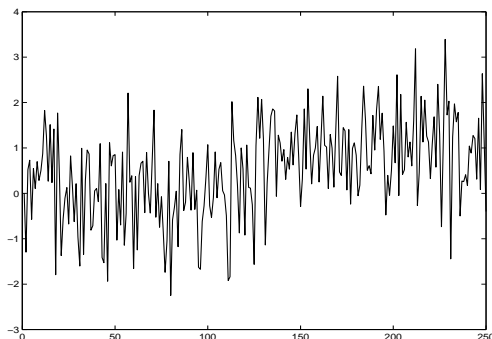
(d) AR 2: CUSUM Plot



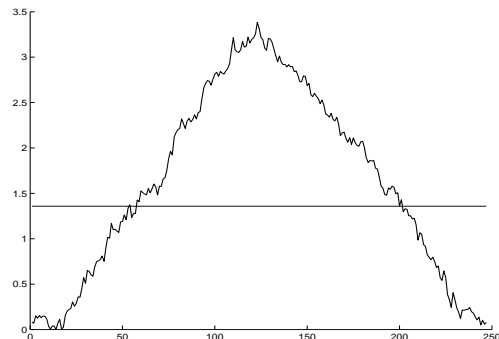
(e) TAR 1: Sample Path



(f) TAR 1: CUSUM Plot



(g) TAR 2: Sample Path



(h) TAR 2: CUSUM Plot

Figure 4.2: Sample Path and CUSUM plot for this sample path, $n = 250$

			size	power
AR 1	$n = 250$	EV	0.035	0.819
		LV	0.041	0.751
	$n = 500$	EV	0.044	0.996
		LV	0.043	0.976
AR 2	$n = 250$	EV	0.035	1
		LV	0.041	0.679
	$n = 500$	EV	0.044	1
		LV	0.043	0.556
TAR 1	$n = 250$	EV	0.035	0.933
		LV	0.043	0.847
	$n = 500$	EV	0.041	0.999
		LV	0.043	0.981
TAR 2	$n = 250$	EV	0.037	0.956
		LV	0.039	0.837
	$n = 500$	EV	0.041	0.999
		LV	0.043	0.932

Table 4.2: Empirical size and power for a nominal 5% level of the test for several scenarios.

Conclusions:

The test is conservative in all cases and has a good power even in the misspecified situations. Using the variance estimator (3.5) instead of the long-run variance estimator (3.6) results mainly in a smaller size but a larger power in our examples. These findings indicate that the errors of the approximating neural network autoregressive process are approximately independent so that the variance estimator (3.5) can be used and even yields superior results. Therefore, in the following the results are only given for this estimator.

4.3 Impact of Hidden Neurons:

Table 4.3 illustrates the influence of the number of hidden neurons on the power for the misspecified AR 1, AR 2, TAR 1 and TAR 2 models defined above. It can be seen that in all cases the power is good. For the AR 1 and AR 2 models the size is best for $H = 2$, while for the TAR 1 and TAR 2 models $H = 3$ delivers best results.

	AR 1		AR 2		TAR 1		TAR 2	
	size	power	size	power	size	power	size	power
H=2	0.046	0.989	0.046	1	0.040	0.998	0.040	1
H=3	0.042	0.994	0.042	1	0.041	0.999	0.041	1
H=4	0.034	0.991	0.034	1	0.028	1 0.999	0.028	1
H=5	0.033	0.994	0.033	1	0.035	1	0.035	1
H=6	0.032	0.993	0.032	1	0.041	0.999	0.041	0.998
H=10	0.036	0.987	0.036	1	0.024	0.998	0.024	1

Table 4.3: Influence of number of hidden neurons on empirical size and power for a nominal level of 5% for misspecified models, $n = 500$.

4.4 Power under Alternatives

Theorem 3.2 shows that certain alternatives are found with asymptotic power one. Remark 3.2 suggests that changes going along with a mean change will be more easily detectable, which is usually the case when statistics are based on estimated residuals. Therefore, we will have a closer look at this in simulations. Again, we consider the misspecified model, where the true process is generated by an AR(1)-process before the change-point and by a different AR(1)-process after the change-point.

Here, we consider the following four scenarios:

- AR 3: $g_0(x) = 1 + 0.5x$, $g_1(x) = 2$
- AR 4: $g_0(x) = 1 + \frac{2}{3}x$, $g_1(x) = 3$
- AR 5: $g_0(x) = 0.3x$, $g_1(x) = 0.9 - 0.8x$
- AR 6: $g_0(x) = 0.3x$, $g_1(x) = 1.5 - 0.5x$

For $H = 3$, the simulation results are summarized in Table 4.4.

	AR 3		AR 4		AR 5		AR 6	
	size	power	size	power	size	power	size	power
n=250	0.041	0.076	0.036	0.162	0.039	0.983	0.040	1
n=500	0.038	0.096	0.048	0.186	0.044	1.000	0.057	1

Table 4.4: Empirical size and power for a nominal level of 5% for misspecified models.

Note that $\mathbb{E}X_1 = \frac{a_0}{1-a_1}$ and $\mathbb{E}X_n = \frac{b_0}{1-b_1}$, where $g_0(x) = a_0 + a_1x$ and $g_1(x) = b_0 + b_1x$. This shows that in for AR 3 and 4 it holds $\mathbb{E}X_1 = \mathbb{E}X_n$, for AR 5 $|\mathbb{E}X_1 - \mathbb{E}X_n| = 0.5$ and for AR 6 $|\mathbb{E}X_1 - \mathbb{E}X_n| = 1$.

Conclusions:

As expected the power increases with increasing mean difference. However, in the cases

with no mean difference at all AR 1 and 2, we still get an unbiased test with a power that is indeed significantly higher than the level (which is additionally given for comparison). Interestingly, AR 4 has almost twice the power of model AR 3.

4.5 Real Data Applications

In this section we apply our testing procedure to two real data sets that have been frequently used in change-point analysis (cf. e.g. Wu [37], Zhang et al. [38] or the R-package `strucchange`). The first one is a hydrological data set namely the Nile river flow recorded at Aswan (1871-1970), the second are the S&P-stock returns.

The Nile river data as well as a plot of $\frac{1}{\sqrt{\sigma_{a,n}^2(n-p)}}|\widehat{S}_k|$ are given in Figure 4.3

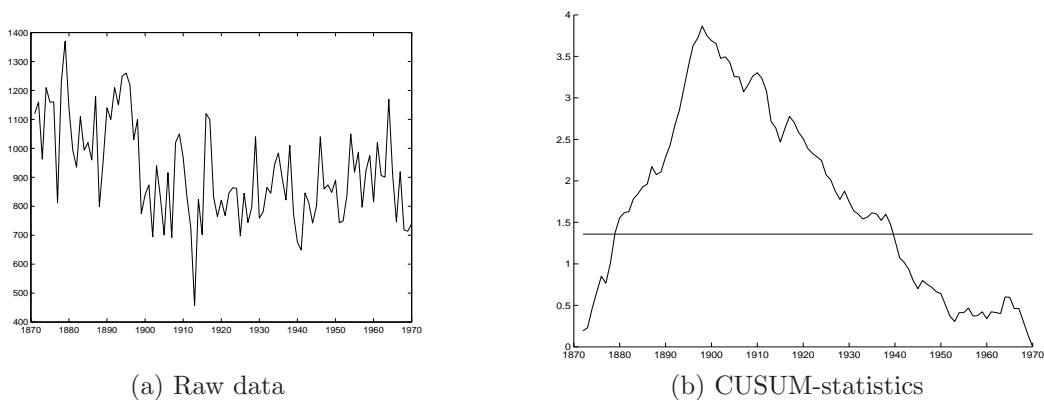


Figure 4.3: Nile river data and CUSUM statistic

The CUSUM plot shows that the null hypothesis of no change is clearly rejected and the change-point is estimated to have occurred in 1898 (cf. Corollary 3.1). This is in accordance to previous analyses of this data set. In fact, it was the year when the first Aswan dam was build.

An additional application of the test to the sub data sets before and after the estimated change-point did not yield any evidence of a second change-point.

The second data set we consider are the daily S&P log-returns from January 1992 to December 1999 (2022 observations) and July 1998 till June 2006 (2013 observations). The raw data sets, that consists of the daily index closing values, were downloaded from <http://finance.yahoo.com/>. S&P 500 is a world known stock index that is quoted at the New York stock exchange.

Instead of the log-returns r_t we apply the testing procedure to the log-transform of the squared returns $X_t = \log r_t^2$. A popular model for squared returns is given by the stochastic volatility model $r_t = \sigma_t Y_t$, where recently Stărică and Granger [32] have proposed that σ_t can be chosen piecewise constant while Y_t follow some weak autoregressive model (even i.i.d. noise is not such a bad assumption). The transformation

$\log r_t^2 = \log \sigma_t^2 + \log Y_t^2$ brings the data rather close to our assumed model and transforms changes in the volatility into mean changes which can be detected quite well by our procedure as the simulation study has shown. However, in extreme cases where the values of the returns are close to zero, the logarithm of the squared returns are too small to fit reasonably into the framework. Therefore, we use a slight modification of the *log*-transformation as introduced in Fuller [12], page 496.

$$X_t^* = \log(r_t^2 + \iota \hat{\sigma}_r^2) - \frac{\iota \hat{\sigma}_r^2}{r_t^2 + \iota \hat{\sigma}_r^2} \quad (4.1)$$

where ι is a small real number (we choose $\iota = 0.02$) and $\hat{\sigma}_r^2$ is the sample variance of the returns. Figures 4.4 and 4.5 illustrate the raw data for the two data sets, the log-returns as well as their subsequent squared log-transformation.

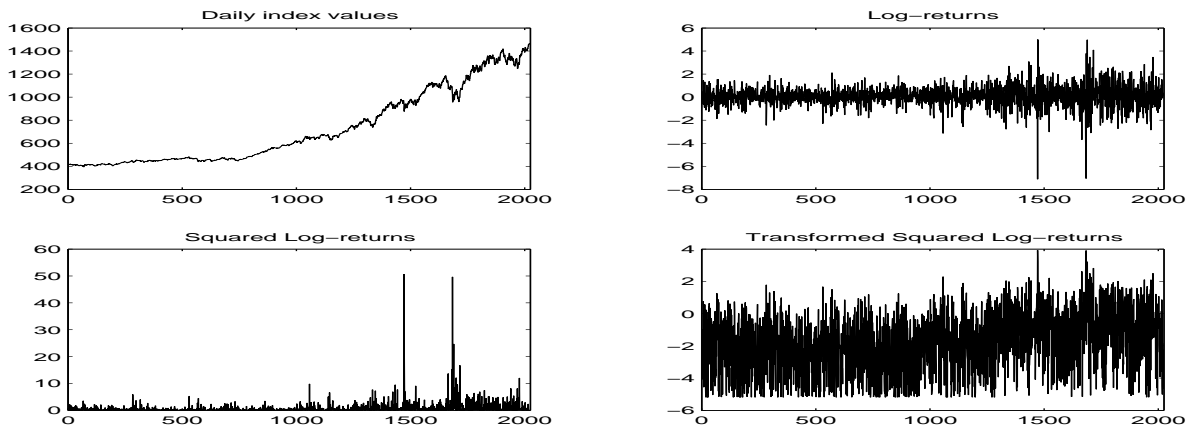


Figure 4.4: Daily S&P Values: January 1992- December 1999

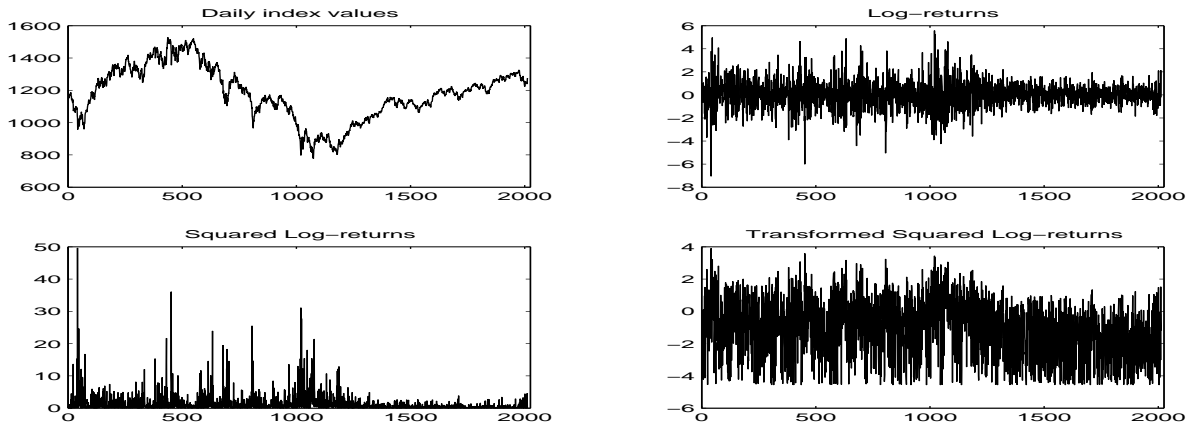


Figure 4.5: Daily S&P Values: July 1998- June 2006

The daily closing index values for both periods show the typical behavior of financial time series often referred to as 'stylized facts'. For example the returns exhibit some clustering and the squared returns consist of small positive values except for some local picks (large values). On the log-scale, the dynamic of the squared returns become more apparent.

Figure 4.6 shows the results of an application of our procedure to the log-transformed squared returns

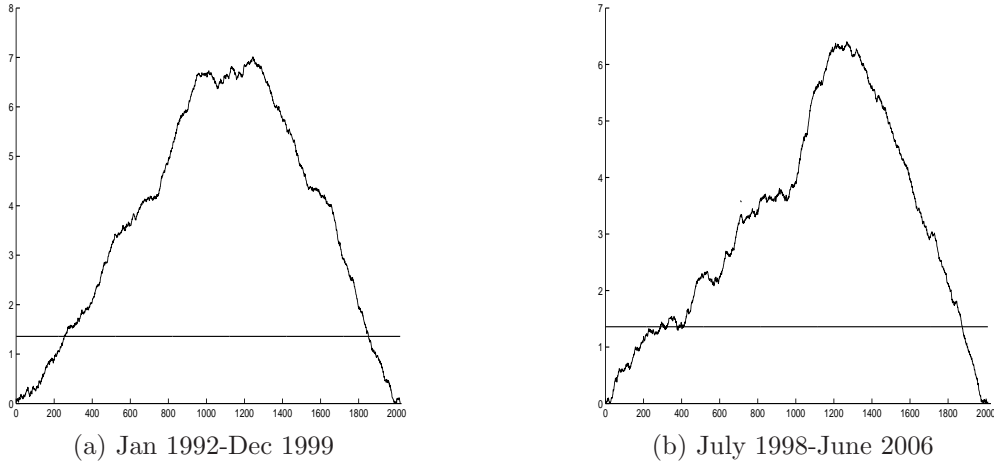


Figure 4.6: CUSUM Plots for S&P

The CUSUM plots show that the null hypothesis of no change are clearly rejected and the change-points are estimated to have occurred around the time points 1246(5th December, 1996) for the first data and 1269 (23rd July, 2003) for the second data set. Both data sets have been investigated by Zhang et al [38] in the context of discriminating change points from long memory behavior. However, their procedures are based on the squared returns rather than the log-transformed squared returns nevertheless they find potential change-points close to our estimated change-points.

When applying change-point tests to financial data sets one has to be careful as it is well known that change-point tests reject in the presence of long-range dependence, while at the same time time series with structural breaks yield long-memory effects (cf. Andreou et al. [2] as well as Mikosch and Stărică [1]).

5 Proofs of Section 2

We first state a uniform law of large numbers (ULLN) by Ranga Rao [31] for stationary and ergodic processes defined on a separable Banach space. As an application we obtain the uniform convergence of $Q_n(\theta)$ and its derivatives, which enables us to obtain the consistency of the NLLS estimator.

Theorem 5.1. *Let $\|\cdot\|$ be any norm on \mathbb{R}^d and $v_t(\theta)$ be a stationary ergodic random sequence with values in $\mathbb{C}(K, \mathbb{R}^d)$ satisfying*

$$\mathbb{E} \sup_{\theta \in K} \|v_1(\theta)\| < \infty,$$

then

$$\sup_{\theta \in K} \left\| \frac{1}{n} \sum_{t=1}^n v_t(\theta) - \mathbb{E}v_1(\theta) \right\| \longrightarrow 0 \quad a.s. \text{ as } n \longrightarrow \infty.$$

Proof. We refer to Theorem 6.5. in Ranga Rao [31]. ■

This enables us to prove the following uniform convergence theorem.

Proposition 5.1. *Let N.2 hold in addition to N.1 if X_t follows (2.1) respectively MS.1 in the misspecified cases (2.2) resp. (2.3).*

a) *Then it holds as $n \rightarrow \infty$*

$$\sup_{\theta \in K} \left| \frac{1}{n} \tilde{Q}_n(\theta) - E(\theta) \right| \rightarrow 0 \quad a.s.,$$

where $\tilde{Q}_n(\theta) = \sum_{t=p+1}^n \tilde{q}_t(\theta)$ with $\tilde{q}_t(\theta)$ as in (2.4), $E(\theta)$ as in (2.5).

b) *If $\{X_t\}$ follows (2.1) (correctly specified model without change) we additionally assume $\nu \geq 2$ in N.1. No additional assumption is needed in case of the misspecified model (2.2) resp. (2.3). Then as $n \rightarrow \infty$*

$$\sup_{\theta \in K} \left\| \frac{1}{n} \nabla^2 \tilde{Q}_n(\theta) - \nabla^2 E(\theta) \right\| \rightarrow 0 \quad a.s.,$$

where ∇^2 denotes the Hesse matrix with respect to θ .

A similar result can be formulated for the first partial derivative. However, we skip its presentation here as it is not needed for the proof of the main results in this paper.

Proof of Proposition 5.1. If X_t is stationary and ergodic (which is the case in the correctly specified as well as misspecified model without change), then

$$\tilde{q}_t(\theta) \quad \text{and} \quad \nabla^2 \tilde{q}_t(\theta)$$

are (pointwise for each θ) stationary and ergodic processes defined on $C(K, \mathbb{R})$ respectively $C(K, \mathbb{R}^{(H(p+2)+1) \times (H(p+2)+1)})$. Hence, preliminary conditions to make use of Theorem 5.1 are fulfilled. In case of a change at point $k^* = \lfloor \lambda n \rfloor$, this is still true under MS.1 for $\tilde{q}_{t,1}(\theta) = (Z_t - f(\mathbb{Z}_{t-1}, \theta))^2$ as well as $\tilde{q}_{t,2}(\theta) = (Y_t - f(\mathbb{Y}_{t-1}, \theta))^2$ and an analogous expression for b). It remains to show that $\mathbb{E} \sup_{\theta} \tilde{q}_1(\theta) < \infty$ and in case of a change that $\mathbb{E} \sup_{\theta} \tilde{q}_n(\theta) < \infty$ and corresponding expressions for b).

For a) this follows by $\sup_{\theta} |f(\mathbf{x}, \theta)| \leq D_1$ for some $D_1 > 0$ and Assumptions N.1 resp. MS.1. An application of Theorem 5.1 to $\{\tilde{q}_t : t = 1, \dots, \lfloor \lambda n \rfloor\}$ as well as $\{\tilde{q}_t : t = \lfloor \lambda n \rfloor + 1, \dots, n\}$ yields the assertion in case of a change-point at $k^* = \lfloor \lambda n \rfloor$.

For b) it follows by the existence of second moments of X_t, Z_t resp. Y_t because $\|\nabla^2 f(\mathbf{x}, \theta)\| \leq D_2 \max_{i=1, \dots, p} x_i^2$. This also shows that by the dominated convergence theorem one can exchange taking derivatives and expectations, showing that the expectation is indeed given by $\nabla^2 E(\theta)$. ■

The above proposition now enables us to prove Theorem 2.1.

Proof of Theorem 2.1. The assertion follows from Lemma 3.1 in Pötscher and Prucha [30] by Proposition 5.1 a) in addition to the identifiability uniqueness condition N.3. ■

Proof of Theorem 2.2. If X_t is α -mixing, then (X_t, \mathbb{X}_{t-1}) is α -mixing with the same rate as

$$\alpha_{(X_1, \mathbb{X}_0)}(k) \leq \alpha_X(k-p) \quad \text{for } k-p > 0.$$

$\nabla \tilde{q}_t(\theta)$ is a measurable function of (X_t, \mathbb{X}_{t-1}) by definition of $f(\mathbf{x}, \theta)$, hence by an application of N.4 resp. MS.2 is also α -mixing with the same rate as X_t – if no change occurs. If a change occurs this remains true for the processes before and after the change.

Since $\tilde{\theta}_0 = \arg \min_{\theta} \mathbb{E} \tilde{q}_1(\theta)$ if no change occurs we get by N.3 that

$$0 = \nabla \mathbb{E} \tilde{q}_1(\tilde{\theta}_0) = \mathbb{E} \nabla \tilde{q}_1(\tilde{\theta}_0),$$

where we can exchange the limits by the dominated convergence theorem because $\sup_{\theta} |\nabla f(\mathbf{x}, \theta)| \leq D \max(|x_1|, \dots, |x_p|)$ for some $D > 0$, hence by the mean value theorem a integrable majorant exists.

If no change occurs, by the strong invariance principle of Kuelbs and Philipp [21] for mixing sequences fulfilling N.4 resp. MS.2 the existence of V as well as the central limit theorem

$$\frac{1}{\sqrt{n}} \nabla \tilde{Q}_n(\tilde{\theta}_0) \xrightarrow{\mathcal{D}} N(0, V) \tag{5.1}$$

holds - for a different proof of such a central limit theorem in the univariate case we refer to Oodaira and Yoshihara [26].

Assertion (5.1) in case of a change follows by considering the central limit theorems before and after the change separately. By the independence of the process before and after the change the joint central limit theorem is obtained on noting that the joint expectation is zero due to the definition of $\tilde{\theta}_0$ and the joint variance is the sum of the variances due to the independence of the processes before and after the change.

By a Taylor expansion there exists θ_n^* such that $\|\theta_n^* - \tilde{\theta}_0\| \leq \|\hat{\theta}_n - \tilde{\theta}_0\|$ with

$$\begin{aligned} 0 &= \nabla \tilde{Q}_n(\hat{\theta}_n) \\ &= \nabla \tilde{Q}_n(\tilde{\theta}_0) + (\hat{\theta}_n - \tilde{\theta}_0) \nabla^2 \tilde{Q}_n(\theta_n^*). \end{aligned}$$

Hence,

$$\nabla \tilde{Q}_n(\theta_0) = -(\hat{\theta}_n - \tilde{\theta}_0) \nabla^2 \tilde{Q}_n(\theta_n^*)$$

By Theorem 2.1 $\theta_n^* \xrightarrow{a.s.} \tilde{\theta}_0$. Since $\tilde{\theta}_0$ is an interior point of K we obtain that θ_n^* is (a.s.) an interior point of K for n large enough. Thus Proposition 5.1 together with the Dominated Convergence Theorem implies as $\sup_{\theta} |\nabla^2 f(\mathbf{x}, \theta)| \leq D_3 \max(x_1^2, \dots, x_p^2)$

$$\frac{1}{n} \nabla^2 \tilde{Q}_n(\theta_n^*) \rightarrow \nabla^2 E(\tilde{\theta}_0) \quad a.s.,$$

which is positive definite by Assumption N.5. This yields the assertion by (5.1). ■

6 Proofs of Section 3

To prove Theorem 3.1 the following lemma plays the crucial role. It allows to replace the estimated errors by the centered errors, for which the assertions of Theorem 3.1 are well-known.

Lemma 6.1. *Assume that N.2– N.4 and (3.4) hold in addition to MS.1 and MS.2 in the no change situation, i.e. $\lambda = 1$. Let $\zeta_t = Z_t - f(\mathbb{X}_{t-1}, \tilde{\theta}_0)$. Then it holds*

$$\begin{aligned} a) \quad & \max_{p < k \leq n} \sqrt{\frac{n-p}{k(n-p-k)}} \left| \sum_{t=p+1}^k (\hat{\varepsilon}_t - (\zeta_t - \bar{\zeta}_{n-p})) \right| = O_P \left(\sqrt{\frac{\log \log n}{n}} \right), \\ b) \quad & \max_{p+G < k \leq n} \frac{1}{\sqrt{G}} \left| \sum_{t=k-G+1}^k (\hat{\varepsilon}_t - (\zeta_t - \bar{\zeta}_{n-p})) \right| = O_P \left(\sqrt{\frac{\log \log n}{G}} \right), \end{aligned}$$

where $\bar{\zeta}_n = 1/(n-p) \sum_{t=p+1}^n \zeta_t$. In the correctly specified model without change (2.1) it holds $\zeta_t = \varepsilon_t$.

Proof. By Theorem 2.1 it holds *a.s.* that $\hat{\theta}_n \in K$ for n large enough, which implies (1.9). Hence,

$$\begin{aligned} \sum_{t=p+1}^k (\hat{\varepsilon}_t - (\zeta_t - \bar{\zeta}_n)) &= \sum_{t=p+1}^k (\hat{\varepsilon}_t - \zeta_t) - \frac{k}{n-p} \sum_{t=p+1}^n (\hat{\varepsilon}_t - \zeta_t) \\ &= \sum_{t=p+1}^k (f(\mathbb{X}_{t-1}, \theta_0) - f(\mathbb{X}_{t-1}, \hat{\theta}_n)) - \frac{k}{n-p} \sum_{t=p+1}^n (f(\mathbb{X}_{t-1}, \theta_0) - f(\mathbb{X}_{t-1}, \hat{\theta}_n)) \quad (6.1) \end{aligned}$$

A Taylor expansion of f yields

$$\begin{aligned} f(\mathbb{X}_{t-1}, \hat{\theta}_n) - f(\mathbb{X}_{t-1}, \tilde{\theta}_0) &= \nabla f(\mathbb{X}_{t-1}, \tilde{\theta}_0)^T (\hat{\theta}_n - \tilde{\theta}_0) \\ &\quad + \frac{1}{2} (\hat{\theta}_n - \tilde{\theta}_0)^T \nabla^2 f(\mathbb{X}_{t-1}, \xi) (\hat{\theta}_n - \tilde{\theta}_0), \quad (6.2) \end{aligned}$$

where $\nabla f(\mathbb{X}_{t-1}, \theta)$ is the gradient with respect to θ and $\nabla^2 f(\mathbb{X}_{t-1}, \theta)$ is the Hessian matrix, $\tilde{\theta}_0 < \xi < \hat{\theta}_n$ elementwise. Furthermore the Hessian matrix is by Assumption N.2 uniformly bounded by $O(1) \max_{1 \leq i \leq p} \max_{1 \leq j \leq p} |X_{t-i} X_{t-j}|$ similarly as in the proof of Theorem 2.2. The uniform ergodic theorem 5.1 yields

$$\sup_{p < k \leq n} \frac{1}{k} \sum_{t=p+1}^k \|\nabla^2 f(\mathbb{X}_{t-1}, \xi)\|_\infty = O_P(1),$$

where $\|(\alpha_{i,j})\|_\infty = \max_{i,j} |\alpha_{i,j}|$. Together with (6.2) this yields uniformly in k

$$\begin{aligned} & \sum_{t=p+1}^k (f(\mathbb{X}_{t-1}, \hat{\theta}_n) - f(\mathbb{X}_{t-1}, \tilde{\theta}_0)) \\ &= \sum_{t=p+1}^k \nabla f(\mathbb{X}_{t-1}, \tilde{\theta}_0)^T (\hat{\theta}_n - \tilde{\theta}_0) + O_P \left(k \|\hat{\theta}_n - \tilde{\theta}_0\|^2 \right). \quad (6.3) \end{aligned}$$

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Assumptions N.1 and N.2 show that $\mathbb{E}|\nabla f(\mathbb{X}_{t-1}, \tilde{\theta}_0)|^\nu < \infty$, since $\|\nabla f(\mathbb{X}_{t-1}, \tilde{\theta}_0)\|_\infty = O(\max_{1 \leq j \leq k} |X_{t-j}|)$, where $\|(a_i)\|_\infty = \max_i |a_i|$. As in the proof of Theorem 2.2 Assumption N.4 shows that $\nabla f(\mathbb{X}_{t-1}, \tilde{\theta}_0)$ fulfills the assumptions in MS.2. Hence by Kuelbs and Philipp [21] an invariance principle analogous to (3.3) holds, which implies the following law of iterated logarithm

$$\sum_{t=p+1}^k (\nabla f(\mathbb{X}_{t-1}, \theta_0)^T - \mathbb{E}\nabla f(\mathbb{X}_{t-1}, \theta_0)^T) = O(\sqrt{k \log \log k}) \quad a.s. \quad (6.4)$$

A different proof of a law of iterated logarithm for mixing sequences but in the univariate situation is given by Oodaira and Yoshihara [25]. Together with Theorem 2.2 (6.1), (6.3) and (6.4) yield

$$\max_{p < k \leq n} \frac{1}{\sqrt{k}} \left| \sum_{t=p+1}^k (\hat{\varepsilon}_t - (\zeta_t - \bar{\zeta}_n)) \right| = O_P \left(\sqrt{\frac{\log \log n}{n}} \right).$$

Similar arguments yield the assertion if we replace $1/\sqrt{k}$ by $1/\sqrt{n-p-k}$ since by (6.1)

$$\sum_{t=p+1}^k (\hat{\varepsilon}_t - (\zeta_t - \bar{\zeta}_n)) = \sum_{t=k+1}^n (\hat{\varepsilon}_t - (\zeta_t - \bar{\zeta}_n)).$$

This proves assertion a). Similar arguments yield that

$$\sum_{t=k-G+1}^k (\hat{\varepsilon}_t - (\zeta_t - \bar{\zeta}_{n-p})) = O_P \left(\sqrt{\log \log n} \right)$$

uniformly in k , thus assertion b) follows. ■

Proof of Theorem 3.1. Lemma 6.1 shows that we can replace $\hat{\varepsilon}_i$ in the statistics by $\zeta_i - \bar{\zeta}_{n-p}$. Concerning T_{n2} note that

$$\frac{t(1-t)}{q(t)} = O(1),$$

by Csörgő and Horváth [7] (Chapter 4, Corollary 1.2). By the invariance principle (3.3) one can replace ζ_i by i.i.d. normal random variables with mean 0 and variance τ^2 by standard arguments from change-point analysis which are sketched below. The results then follow by classical results where $\hat{\varepsilon}(t)$ in the statistics is replaced by $\xi(t) - \bar{\xi}_{n-p}$ and $\{\zeta_i\}$ i.i.d. $N(0, \tau^2)$. For the proofs and further references we refer to the book by Csörgő and Horváth [8]. The result for d) can be found in Chen [5].

W.l.o.g. let $\mathbb{E}\zeta_1 = 0$. The invariance principle (3.3) implies a law of iterated logarithm from which we can deduce

$$\max_{1 \leq k \leq \log n} \sqrt{\frac{n}{k(n-k)}} \left| \sum_{t=p+1}^k (\zeta_t - \bar{\zeta}_{n-p}) \right| = o_P \left(\frac{\beta(\log n)}{\alpha(\log n)} \right)$$

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and an analogous expression for $k \geq n - \log n$ as well as for $\sum_{t=p+1}^k (\xi_t - \bar{\xi}_{n-p})$. This shows that

$$\begin{aligned} & P \left(\alpha(\log n) \max_{1 \leq k \leq n} \sqrt{\frac{n}{k(n-k)}} \left| \sum_{t=p+1}^k (\zeta_t - \bar{\zeta}_{n-p}) \right| - \beta(\log n) \leq y \right) \\ &= P \left(\alpha(\log n) \max_{\log n \leq k \leq n - \log n} \sqrt{\frac{n}{k(n-k)}} \left| \sum_{t=p+1}^k (\zeta_t - \bar{\zeta}_{n-p}) \right| - \beta(\log n) \leq y \right) + o(1). \end{aligned}$$

Another application of the invariance principle shows that

$$\max_{\log n \leq k \leq n - \log n} \sqrt{\frac{n}{k(n-k)}} \left| \sum_{t=p+1}^k (\zeta_t - \bar{\zeta}_{n-p}) - \sum_{t=p+1}^k (\xi_t - \bar{\xi}_{n-p}) \right| = o_P(1)$$

finishing the proof for T_{n1} . Noting that by Csörgő and Horváth [7] (Chapter 4, Corollary 4) $\lim_{c_n \rightarrow 0} \sup_{t \leq c_n} \frac{\sqrt{t}}{q(t)} = 0$ the results for the other statistics follow similarly. ■

Proof of Lemma 3.1. First let us recall, from Theorem 2.2, that

$$\hat{\theta}_n - \theta_0 = O_p \left(\frac{1}{\sqrt{n}} \right). \quad (6.5)$$

Then, we rewrite

$$\begin{aligned} \sum_{t=p+1}^n \hat{\varepsilon}_t^2 &= \sum_{t=p+1}^n \varepsilon_t^2 - 2 \sum_{t=p+1}^n \varepsilon_t (f(\mathbb{X}_{t-1}, \hat{\theta}_n) - f(\mathbb{X}_{t-1}, \theta_0)) \\ &\quad + \sum_{t=p+1}^n (f(\mathbb{X}_{t-1}, \hat{\theta}_n) - f(\mathbb{X}_{t-1}, \theta_0))^2. \end{aligned} \quad (6.6)$$

From Marcinkiewicz-Zygmund (cf. e.g. Chow and Teicher [6], Theorem 5.2.2) we have for $2 \leq \nu < 4$

$$\frac{1}{n - (H(p+2) + 1)} \sum_{t=p+1}^n \varepsilon_t^2 = \sigma^2 + o_P(n^{-(\nu-2)/\nu})$$

respectively $O_P(n^{-1/2})$ by the central limit theorem if at least four moments exist. Analogously to (6.3) using only a first-order Taylor expansion we get by (6.5):

$$\begin{aligned} \frac{1}{n - (H(p+2) + 1)} \sum_{t=p+1}^n (f(\mathbb{X}_{t-1}, \hat{\theta}_n) - f(\mathbb{X}_{t-1}, \theta_0))^2 &= O_P \left(\left\| \hat{\theta}_n - \theta_0 \right\|^2 \right) \\ &= O_P \left(\frac{1}{n} \right). \end{aligned}$$

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Concerning the mixed term the Cauchy-Schwartz inequality yields

$$\frac{1}{n - (H(p+2) + 1)} \sum_{t=p+1}^n \varepsilon_t (f(\mathbb{X}_{t-1}, \widehat{\theta}_n) - f(\mathbb{X}_{t-1}, \theta_0)) = O_P(n^{-1/2}),$$

which finishes the proof by (6.6). ■

Proof of Theorem 3.2. We can assume w.l.o.g. that $\widehat{\theta}_n$ is in the interior of K since otherwise we reject the null hypothesis right away. Hence, by (1.9) it holds

$$\widehat{S}_n(k^*) = \sum_{j=p+1}^{k^*} (X_j - f(\mathbb{X}_{j-1}, \widehat{\theta}_n)) = - \sum_{j=k^*+1}^n (X_j - f(\mathbb{X}_{j-1}, \widehat{\theta}_n)). \quad (6.7)$$

This yields on the one hand

$$\begin{aligned} \widehat{S}_n(k^*) &= \sum_{i=p+1}^{k^*} (Z_i - f(\mathbb{Z}_{i-1}, \widehat{\theta}_n)) \\ &= k^* (\mathbb{E}Z_1 - \mathbb{E}f(\mathbb{Z}_p, \theta)|_{\theta=\widehat{\theta}_n}) + \sum_{i=p+1}^{k^*} (Z_i - \mathbb{E}Z_1) \\ &\quad + O\left(\sup_{\theta \in K} \left| \sum_{i=p+1}^{k^*} (f(\mathbb{Z}_{i-1}, \theta) - \mathbb{E}f(\mathbb{Z}_p, \theta)) \right|\right) \\ &= \lambda n (\mathbb{E}Z_1 - \mathbb{E}f(\mathbb{Z}_p, \theta)|_{\theta=\widehat{\theta}_n}) + o_P(n) \end{aligned}$$

since by Theorem 5.1 a uniform law of large numbers holds (note that $\sup_x \sup_{\theta \in K} |f(\mathbf{x}, \theta)| = O(1)$).

On the other hand we obtain similarly (for $\mathbb{Y}_j = (Y_j, \dots, Y_{j-p})$) using (6.7)

$$\widehat{S}_n(k^*) = -(1 - \lambda)n (\mathbb{E}Y_1 - \mathbb{E}f(\mathbb{Y}_p, \theta)|_{\theta=\widehat{\theta}_n}) + o_P(n),$$

since $f(\mathbb{Y}_j, \theta)$ is stationary and ergodic by Assumption *MS.1*. Together we obtain by Assumption A.2 (a)

$$\left| \widehat{S}_n(k^*) \right| \geq nC \min(\lambda, 1 - \lambda) + o_P(n) \quad (6.8)$$

for some constant $C > 0$, which implies immediately that

$$(\log \log n)^{-1/2} T_{n1} \xrightarrow{P} \infty,$$

which implies in return assertion a) (i). Furthermore, due to Assumption A.1 and since $q \in Q_{0,1}$ equation (6.8) implies

$$T_{n2}(q) \xrightarrow{P} \infty,$$

hence a) (ii). Concerning a) (iii) similar arguments yield

$$\begin{aligned} \left| \hat{S}_n(k^*) - \hat{S}_n(k^* - G) \right| &= G \left| \mathbb{E}Z_1 - \mathbb{E}f(\mathbb{Z}_p, \theta) \Big|_{\theta=\hat{\theta}_n} \right| + o_P(G) \\ \left| \hat{S}_n(k^* + G) - \hat{S}_n(k^*) \right| &= G \left| \mathbb{E}Y_1 - \mathbb{E}f(\mathbb{Y}_p, \theta) \Big|_{\theta=\hat{\theta}_n} \right| + o_P(G), \end{aligned}$$

hence by Assumption A.2 (a)

$$T_{n3}(G) \geq \sqrt{G}C + o_P(\sqrt{G})$$

for some $C > 0$, which in turn yields $(\log(n/G))^{-1/2}T_{n3}(G) \xrightarrow{P} \infty$ and hence the assertion. Concerning a) (iv) we need again a slight variation of the argument, namely it holds uniformly in k

$$\left| \hat{S}_n(k) \right| = \begin{cases} k \left| \mathbb{E}Z_1 - \mathbb{E}f(\mathbb{Z}_p, \theta) \Big|_{\theta=\hat{\theta}_n} \right| + o_P(k), & n/\log n < k \leq k^*, \\ (n-k) \left| \mathbb{E}Y_1 - \mathbb{E}f(\mathbb{Y}_p, \theta) \Big|_{\theta=\hat{\theta}_n} \right| + o_P(n-k), & k^* < k < n - n/\log n. \end{cases}$$

Since by (3.1)

$$\frac{1}{n} \sum_{k^* \geq k \geq \frac{n}{\log n}} \frac{k^2/n}{r(k/(n-p))} \geq \frac{n}{\log n} \left(\int_0^\lambda \frac{t}{r(t)} dt + o(1) \right)$$

and an analogous expression for $k^* \leq j \leq n - n/\log n$, we obtain by Assumption A.2 (a)

$$T_{n4}(r) \geq o_P\left(\frac{n}{\log n}\right) + C \frac{n}{\log n}$$

for some $C > 0$ and hence the assertion.

Similarly to $T_{n3}(G)$ we obtain for b) that

$$\begin{aligned} \tilde{T}_{n3}(G) &\geq \frac{1}{\sqrt{2G}} \left| \hat{S}_n(k^* + G) - 2\hat{S}_n(k^*) + \hat{S}_n(k^* - G) \right| \\ &\geq \sqrt{G/2} \left| (\mathbb{E}Z_1 - \mathbb{E}f(\mathbb{Z}_p, \theta) \Big|_{\theta=\hat{\theta}_n}) - (\mathbb{E}Y_1 - \mathbb{E}f(\mathbb{Y}_p, \theta) \Big|_{\theta=\hat{\theta}_n}) \right| + o_P(\sqrt{G}), \end{aligned}$$

and hence the assertion. ■

Proof of Corollary 3.1. The proof of Theorem 3.2 yields that

$$\sup_{t \in [0,1]} \left| \frac{1}{n} \left| \hat{S}_n(\lfloor nt \rfloor) \right| - L_n(t) \right| = o_P(1),$$

where

$$L_n(t) = \begin{cases} t \left| \mathbb{E}Z_1 - \mathbb{E}f(\mathbb{Z}_p, \theta) \Big|_{\theta=\hat{\theta}_n} \right|, & t < \lambda, \\ \max\left(\lambda \left| \mathbb{E}Z_1 - \mathbb{E}f(\mathbb{Z}_p, \theta) \Big|_{\theta=\hat{\theta}_n} \right|, (1-\lambda) \left| \mathbb{E}Y_1 - \mathbb{E}f(\mathbb{Y}_p, \theta) \Big|_{\theta=\hat{\theta}_n} \right|\right), & t = \lambda, \\ (1-t) \left| \mathbb{E}Y_1 - \mathbb{E}f(\mathbb{Y}_p, \theta) \Big|_{\theta=\hat{\theta}_n} \right|, & t > \lambda. \end{cases}$$

$L_n(t)$ has a unique maximum in $t = \lambda$ for all n , is equicontinuous (with respect to n) for all $t \neq \lambda$ (since $f(x, \theta)$ is bounded). Let $\xi_n := \max(|\mathbb{E}f(\mathbb{Z}_p, \theta)|_{\theta=\hat{\theta}_n} - \mathbb{E}Z_1|, |\mathbb{E}Y_1 - \mathbb{E}f(\mathbb{Y}_p, \theta)|_{\theta=\hat{\theta}_n}|)$ with $P(\xi_n \geq c) \rightarrow 1$ by Assumption A.2 (a). Since $\xi \geq c - (c - \xi)_+$ and $P((c - \xi)_+ \geq \varepsilon) \leq P(\xi < c) = 1 - P(\xi \geq c) \rightarrow 0$, it holds $\xi_n \geq c + o_P(1)$. We can now show that

$$\inf_n (L_n(\lambda) - L_n(t)) > c(t, \lambda) + o_P(1), \quad c(t, \lambda) > 0 \quad \text{for any } t \neq \lambda$$

We prove this assertion for $t < \lambda$, the assertion for $t > \lambda$ is analogous. First, consider the case when $\hat{\theta}_n$ is such that $|\mathbb{E}f(\mathbb{Z}_p, \theta)|_{\theta=\hat{\theta}_n} - \mathbb{E}Z_1| \geq \min(\frac{1-\lambda}{\lambda}, 1) \xi_n$. In this case it can easily be seen that

$$L_n(\lambda) - L_n(t) \geq (\lambda - t) \xi_n \min\left(\frac{1-\lambda}{\lambda}, 1\right) \geq c(\lambda - t) \min\left(\frac{1-\lambda}{\lambda}, 1\right) + o_P(1).$$

Otherwise it holds $|\mathbb{E}f(\mathbb{Z}_p, \theta) - \mathbb{E}Z_1| < \min(\frac{1-\lambda}{\lambda}, 1) c$, hence by definition of ξ_n that $|\mathbb{E}Y_1 - \mathbb{E}f(\mathbb{Y}_p, \theta)|_{\theta=\hat{\theta}_n}| \geq c$, which yields

$$L_n(\lambda) - L_n(t) \geq \xi_n \left[1 - \lambda - t \min\left(\frac{1-\lambda}{\lambda}, 1\right)\right] \geq (\lambda - t) c \min\left(\frac{1-\lambda}{\lambda}, 1\right) + o_P(1),$$

proving the assertion.

From this we can conclude the proof. We only give the argument for real sequences $s_n(t)$ and $l_n(t)$ – the assertion for random sequences $\frac{1}{n} \left| \tilde{S}_n(\lfloor nt \rfloor) \right|$ and $L_n(t)$ then follows via the subsequence principle. This variation of the standard proof is necessary since we do not know anything about the limit of $L_n(t)$ (not even if it exists): Suppose $\hat{\lambda}_n = \arg \max(s_n(t)) \not\rightarrow \lambda$. Because $[0, 1]$ is compact there exists a subsequence $\hat{\lambda}_{\alpha(n)}$ and $t_1 \neq \lambda$ with $\hat{\lambda}_{\alpha(n)} \rightarrow t_1$, hence

$$|s_{\alpha(n)}(\hat{\lambda}_{\alpha(n)}) - y_{\alpha(n)}(t_1)| \leq \max_t |s_{\alpha(n)}(t) - y_{\alpha(n)}(t)| + |y_{\alpha(n)}(\hat{\lambda}_{\alpha(n)}) - y_{\alpha(n)}(t_1)| \rightarrow 0,$$

since by assumption $\sup_t |s_{\alpha(n)}(t) - l_n(t)| \rightarrow 0$ and by the equicontinuity of $l_n, n \in \mathbb{N}$. Since $\inf_n (l_n(\lambda) - l_n(t_1)) > c(t_1, \lambda) > 0$ we conclude

$$y_{\alpha(n)}(\lambda) - s_{\alpha(n)}(\hat{\lambda}_{\alpha(n)}) = y_{\alpha(n)}(\lambda) - y_{\alpha(n)}(t_1) + y_{\alpha(n)}(t_1) - s_{\alpha(n)}(\hat{\lambda}_{\alpha(n)}) > c(t_1, \lambda) + o(1),$$

but this is a contradiction since by assumption it holds

$$|s_n(\hat{\lambda}_n) - l_n(\lambda)| = \left| \sup_t s_n(t) - \sup_t l_n(t) \right| \leq \sup_t |s_n(t) - l_n(t)| \rightarrow 0.$$

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