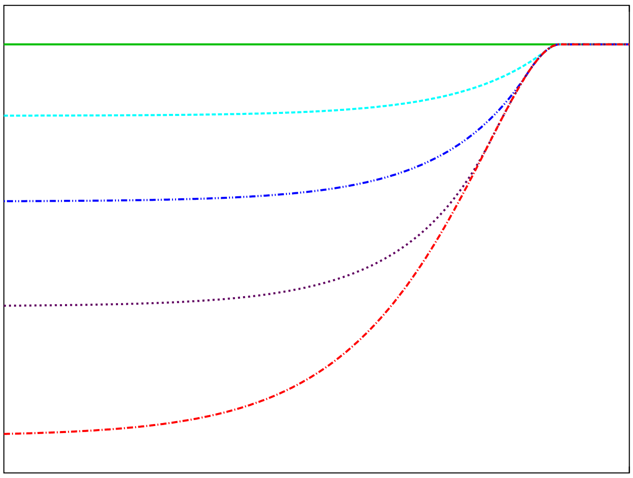
Optimal Investment for a Large Investor in a Regime-Switching Model

Michael Busch



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1. Gutachter: Prof. Dr. Ralf Korn
2. Gutachterin: Prof. Dr. Nicole Bäuerle

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Contents

Abbreviations and Symbols

Abbreviations

a.s.	almost surely
cf.	compare
e.g.	for example
i.e.	that is
resp.	respectively
w.r.t.	with respect to
lim	limit
max, min	maximum, minimum
sup, inf	supremum, infimum

Symbols

$v^\top$	transpose of vector v
A^{-1}	inverse of matrix A
x^+	$= \max\{x, 0\}$, the positive part of x
x^-	$= \max\{-x, 0\}$, the negative part of x
$x \wedge y$	$= \min\{x, y\}$
$x \vee y$	$= \max\{x, y\}$
$\|x\|$	absolute value of x
$\mathbb{N}$	natural numbers
$\mathbb{R}$	real numbers
$\mathbb{R}^+$	positive real numbers without 0
$\mathbb{R}^-$	negative real numbers without 0
$\mathbb{R}_0^+$	positive real numbers including 0
$\mathbb{R}_0^-$	negative real numbers including 0
$A \setminus B$	relative complement of set B in set A
1_A	indicator function of set A
$\emptyset$	empty set

$\exp(x)$	$= e^x$
$\ln(x)$	natural logarithm of x
$\mathrm{L}^1(\mathbb{P})$	space of $\mathbb{P}$-integrable processes
$\mathrm{L}^2(\mathbb{P})$	space of $\mathbb{P}$-square integrable processes
$C(M)$	space of continuous functions on M
$C^1(M)$	space of continuously differentiable functions on M
$C^2(M)$	space of twice continuously differentiable functions on M
$C^{1,2}(M \times N)$	space of functions that are continuously differentiable w.r.t. the first component and twice continuously differentiable w.r.t. the second component on $M \times N$
$f'(x)$	first-order derivative of f w.r.t. x
$f_t(t,x),\ f_x(t,x)$	first-order partial derivative of f w.r.t. t, resp. x
$f_{xx}(t,x)$	second-order partial derivative of f w.r.t. x
$x \nearrow x_0$	left-handed convergence of x towards x_0
$x \searrow x_0$	right-handed convergence of x towards x_0
$\mathbb{E}(X)$	expectation of the random variable X w.r.t. the probability measure $\mathbb{P}$
$\mathbb{E}(X\|Y)$	conditional expectation of the random variable X given Y
$\mathbb{E}^{\pi,c}(X)$	expectation of the random variable X w.r.t. the probability measure $\mathbb{P}^{\pi,c}$
$\mathcal{E}(X)$	stochastic exponential of the process X

1 Introduction

In the classical Merton investment problem of maximizing the expected utility from terminal wealth and intermediate consumption stock prices are independent of the investor who is optimizing his investment strategy. This is reasonable as long as the considered investor is small and thus does not influence the asset prices. However for an investor whose actions may affect the financial market the framework of the classical investment problem turns out to be inappropriate. Against this background various research was done on the field of including a relation between the investor and the financial market on which he is acting. Subsequently we present some different models.

In [Jarrow 1992] R. Jarrow discusses market manipulating trading strategies by large traders in a discrete time setting. In this context market manipulating strategies are defined as strategies that generate a positive real wealth without taking any risk. The financial market of the model by Jarrow consists of a riskless money market account and a risky stock where the relative stock price is an exogenously given function dependent on the large investor's actual and past holdings in the money market account and the stock, i.e.

$$\tfrac{P_t^1}{P_t^0} = G_t\big(\varphi_t^0, \varphi_{t-1}^0, \ldots, \varphi_0^0, \varphi_t^1, \varphi_{t-1}^1, \ldots, \varphi_0^1\big),\ t \in \{1, 2, \ldots, T\},\ P_0^0 = 1,\ P_0^1 = p_0^1,$$

where P^0, resp. P^1, is the price of the money market account, resp. the risky asset, and φ^i, $i = 0, 1$, denotes the corresponding holdings. Jarrow presents examples for the existence of market manipulating strategies under very general conditions. Further he provides a sufficient condition on the stock price process that excludes market manipulating strategies. The sufficient condition is that the stock price depends only on the large investor's actual holdings and is independent of his past portfolio compositions.

R. Jarrow extents his aforementioned model in [Jarrow 1994] via including a derivative security into the financial market. The relative stock price is now given by

$$\tfrac{P_t^1}{P_t^0} = G_t\big(\varphi_t^0, \varphi_t^1, \varphi_t^c\big),\ t \in \{1, 2, \ldots, T\},\ P_0^0 = 1,\ P_0^1 = p_0^1,$$

where φ^c denotes the number of derivatives in the large investor's portfolio. It turns out that the presence of the derivative security enables the existence of market manipulating strategies that would not have been possible if there was only the money market account and the stock. Corresponding to his results in [Jarrow 1992] Jarrow presents a sufficient condition that prevents these new market manipulating strategies. Finally a theory for the valuation of options in the discussed model is introduced. Hereby Jarrow verifies that the standard binomial option model still works, however, with random volatilities.

A continuous-time hedging problem of an investor whose portfolio strategy and wealth affect the riskless interest rate and the drift and volatility of the stock price process is dealt with in [Cvitanić, Ma 1996] by J. Cvitanić and J. Ma. In their paper the considered financial market consists of a riskless money market account and $\bar{n}$ risky assets with dynamics

$$\mathrm{d}P_t^0 = P_t^0 r_t^0(X_t, \varphi_t^0 P_t^0, \varphi_t^1 P_t^1, \ldots, \varphi_t^{\bar{n}} P_t^{\bar{n}})\mathrm{d}t,\ P_0^0 = 1,$$

$$\mathrm{d}P_t^n = \mu_t^n(P_t, X_t, \varphi_t^0 P_t^0, \varphi_t^1 P_t^1, \ldots, \varphi_t^{\bar{n}} P_t^{\bar{n}})\mathrm{d}t + \sum_{m=1}^{\bar{n}} \sigma_t^{n,m}(P_t, X_t, \varphi_t^0 P_t^0, \varphi_t^1 P_t^1, \ldots, \varphi_t^{\bar{n}} P_t^{\bar{n}})\mathrm{d}W_t^m,\ P_0^n = p_0^n,$$

where X denotes the investor's wealth process and $\varphi^n P^n$ equals the investor's portfolio process describing the amount of money invested in the n-th asset. Given the initial stock prices and a desired terminal wealth the investor is searching for the hedging portfolio process of an option that goes with the smallest initial endowment. It turns out that the problem corresponds to finding a solution of a forward-backward stochastic differential equation (FBSDE). Cvitanić and Ma provide conditions under which a solution to this FBSDE can be found.

D. Cuoco and J. Cvitanić investigate in [Cuoco, Cvitanić 1998] the continuous-time optimal investment problem of a large investor whose portfolio proportions impact on the instantaneous expected returns on the traded assets. The financial market consists of a riskless money market account and $\bar{n}$ risky assets with dynamics

$$\begin{aligned} \mathrm{d}P_t^0 &= P_t^0 r_t^0(\varphi_t^0 P_t^0, \varphi_t^1 P_t^1, \ldots, \varphi_t^{\bar{n}} P_t^{\bar{n}})\mathrm{d}t,\ P_0^0 = 1, \\ \mathrm{d}P_t^n &= P_t^n \left[\mu_t^n(\varphi_t^0 P_t^0, \varphi_t^1 P_t^1, \ldots, \varphi_t^{\bar{n}} P_t^{\bar{n}})\mathrm{d}t + \sum_{m=1}^{\bar{n}} \sigma_t^{n,m} \mathrm{d}W_t^m \right],\ P_0^n = p_0^n, \end{aligned}$$

where again $\varphi^n P^n$ denotes the amount of money invested in the n-th asset. Using martingale and duality methods they provide sufficient conditions for the existence of optimal strategies under general assumptions on the asset prices and the large investor's preferences. In specific examples of the investor's influence Cuoco and Cvitanić present explicit solutions for an investor with logarithmic utility.

In [Bank, Baum 2004] P. Bank and D. Baum consider a general, abstract continuous-time model for an illiquid financial market whose asset prices can be influenced by the trades of a large investor. The market they discuss consists of a riskless bank account and a risky asset whose dynamics are described by a family of continuous semimartingales that depend on the large investor's holdings in the asset, i.e.

$$P_t = P_t^{\varphi_t},\ t \in [0, T],$$

with the family $(P_t^{\varphi})_{t\in[0,T]}$, $\varphi \in \mathbb{R}$, and where φ denotes the investors holdings in the risky asset. As opposed to [Cvitanić, Ma 1996] and [Cuoco, Cvitanić 1998] where the investor was solely influencing the drift and volatility of the stock price, the model of Bank and Baum allows impacts on the stock price itself. In this model setting the authors prove the absence of arbitrage and investigate the problem of hedging attainable claims and the utility maximization problem using the Itô-Wentzell formula. It turns out that the large investor model inherits many properties of the underlying small investor model such as the attainability of claims, the determination of superreplication prices or the utility maximization.

In this thesis we provide a new approach to the field of large investor models. We study the optimal investment problem of a large investor in a jump-diffusion market which is in one of two states or regimes. The investor's portfolio proportions as well as his consumption rate affect the intensity of transitions between the different regimes. Hence the asset price dynamics are given by

$$\begin{aligned} \mathrm{d}P_t^0 &= P_t^0 r^{I_{t-}} \mathrm{d}t,\ P_0^0 = p_0^0, \\ \mathrm{d}P_t^n &= P_t^n \left[\mu_n^{I_{t-}} \mathrm{d}t + \sum_{m=1}^{\bar{m}} \sigma_{n,m}^{I_{t-}} \mathrm{d}W_t^m \right],\ P_0^n = p_0^n, \end{aligned}$$

where I is an $\{0,1\}$-valued state process with transition intensities $\vartheta^{i,1-i}(\pi, c)$, $i = 0, 1$, that depend on the investor's portfolio proportions π and consumption rate c. Thus the investor is 'large' in the sense that his investment decisions are interpreted by the market as *signals*: If, for instance, the large investor holds 25% of his wealth in a certain asset then the market may regard this as evidence for the corresponding asset to be priced incorrectly, and a regime shift becomes likely. More specifically, the large investor as modeled here may be the manager of a big mutual fund, a big insurance company or a sovereign wealth fund, or the executive of a company whose stocks are in his own portfolio. Typically, such investors have to disclose their portfolio allocations which impacts on market prices. But even if a large investor does not disclose his portfolio composition as it is the case of several hedge funds then the other market participants may speculate about the investor's strategy which finally could influence the asset prices. Since the investor's strategy only impacts on the regime shift intensities the asset prices do not necessarily react instantaneously. Hence as opposed to the aforementioned models where the investor has an immediate influence on the financial market in our model the influence is an indirect one.

Similar regime models of asset price dynamics have been used in the literature, albeit not in the context of large investors. In [Bäuerle, Rieder 2004] N. Bäuerle and U. Rieder study the optimal investment problem with Markov-modulated stock prices and observable drift. In their model the transition intensities are constants. They solve the problem of maximizing the expected utility from terminal by stochastic control methods for different kinds of utility functions. Besides CRRA utility for which they solve the investment problem explicitly they also consider a benchmark optimization problem. J. Sass and U. Haussmann investigate in [Sass, Haussmann 2004] the corresponding problem in the case of unobservable drift. They derive an explicit representation of the optimal trading strategy in terms of the unnormalized filter of the drift process. Further in [Diesinger, Kraft, Seifried 2009] P. Diesinger, H. Kraft and F. Seifried use a regime switching model to capture different states of liquidity.

Our model is a generalization of the two-states version of the Bäuerle-Rieder model. Hence as the Bäuerle-Rieder model it is suitable for long investment periods during which market conditions could change. The fact that the investor's influence enters the intensities of the transitions between the two states enables us to solve the investment problem of maximizing the expected utility from terminal wealth and intermediate consumption explicitly. We present the optimal investment strategy for a large investor with CRRA utility for three different kinds of strategy-dependent regime shift intensities – constant, step and affine intensity functions. In each case we derive the large investor's optimal strategy in explicit form only dependent on the solution of a system of coupled ODEs of which we show that it admits a unique global solution.

This thesis is organized as follows. In Section 2 we repeat the classical Merton investment problem of a small investor who does not influence the market. Further the Bäuerle-Rieder investment problem in which the market states follow a Markov chain with constant transition intensities is discussed.

Section 3 introduces the aforementioned investment problem of a large investor. Besides the mathematical framework and the HJB-system we present a verification theorem that is necessary to verify the optimality of the solutions to the investment problem that we derive later on.

The explicit derivation of the optimal investment strategy for a large investor with power utility is given in Section 4. For three kinds of intensity functions – constant, step and affine – we give the optimal solution and verify that the corresponding ODE-system admits a unique global solution. In case of the strategy-dependent intensity functions we distinguish three particular kinds of this dependency – portfolio-dependency, consumption-dependency and combined portfolio- and consumption-dependency. The corresponding results for an investor having logarithmic utility are shown in Section 5.

In the subsequent Section 6 we consider the special case of a market consisting of only two correlated stocks besides the money market account. We analyze the investor's optimal strategy when only the position in one of those two assets affects the market state whereas the position in the other asset is irrelevant for the regime switches.

Various comparisons of the derived investment problems are presented in Section 7. Besides the comparisons of the particular problems with each other we also dwell on the sensitivity of the solution concerning the parameters of the intensity functions. Finally we consider the loss the large investor had to face if he neglected his influence on the market.

In Section 8 we conclude this thesis.

2 Principles of Continuous-time Portfolio Optimization

In this section we recapitulate the classical Merton portfolio optimization problem (Subsection 2.1) and discuss the Bäuerle-Rieder investment problem (Subsection 2.2) which is the basis of our large investor model. For both models we describe the mathematical framework and formulate the optimal investment problem. This is followed by the derivation of the HJB-equation, resp. HJB-system, and the presentation of a verification theorem. Finally we provide the optimal solution of the investment problem. Due to its repetitive character this section does not contain the relevant proofs.

2.1 The Merton Investment Problem

This section describes an investment setting which nowadays is referred to as 'the Merton Problem' and summarizes the essence of the two fundamental papers [Merton 1969] and [Merton 1971] of Robert C. Merton.

2.1.1 Mathematical Framework

Informal Description. The financial market of the classical Merton problem consists of a locally riskless money market account P^0 and $\bar{n}$ risky assets P^n, $n = 1, \ldots, \bar{n}$. The asset prices are given by a diffusion model that is driven by an $\bar{m}$-dimensional Brownian motion W.

Asset Price Dynamics. The mathematical model of the asset price dynamics includes the filtered space $(\Omega, \mathfrak{F}, \mathfrak{F}(\cdot))$ with time horizon $[0, T]$ that is endowed with a reference probability measure $\mathbb{P}$ such that $\mathfrak{F}(\cdot)$ satisfies the usual conditions of right-continuity and completeness and $\mathfrak{F} = \mathfrak{F}(T)$. Further we assume that $(\Omega, \mathfrak{F}, \mathbb{P})$ carries an $\bar{m}$-dimensional $(\mathfrak{F}(\cdot), \mathbb{P})$-Wiener process W.

The **asset price dynamics** are then given by

$$\mathrm{d}P^0 = P^0 r \mathrm{d}t, \quad P^0(0) = p_0^0, \tag{2.1}$$

$$\mathrm{d}P^n = P^n \left[(r + \eta_n)\mathrm{d}t + \sum_{m=1}^{\bar{m}} \sigma_{n,m} \mathrm{d}W^m \right], \quad P^n(0) = p_0^n, \tag{2.2}$$

where the number $r \in [0, \infty)$, the vector $\eta \in \mathbb{R}^{\bar{n}}$ and the matrix $\sigma \in \mathbb{R}^{\bar{n} \times \bar{m}}$ are given parameters and $\sigma.\sigma^\top$ is positive definite.

Notice that in general the financial market as given above is incomplete.

Investor's Strategy. The investor is equipped with an initial wealth x_0 and specifies a **portfolio strategy** π and a **consumption rate** c. Then by (2.1) and (2.2) his **wealth** $X^{\pi,c}$ evolves according to the stochastic differential equation

$$\mathrm{d}X^{\pi,c} = X^{\pi,c} \left[(r + \pi^\top.\eta - c)\mathrm{d}t + \pi^\top.\sigma.\mathrm{d}W \right], \quad X^{\pi,c}(0) = x_0. \tag{2.3}$$

Let $\Pi \subseteq \mathbb{R}^{\bar{n}}$ be a given closed set. We denote by

$$\mathcal{A} \triangleq \{(\pi, c): \ (\pi, c) \ \text{ bounded, } \mathfrak{F}(\cdot)\text{-predictable, } \Pi \times \mathbb{R}_0^+\text{-valued and satisfies} \\ \mathbb{E}[\textstyle\int_0^T u_1(t, c(t)X^{\pi,c}(t))^- \mathrm{d}t + u_2(X^{\pi,c}(T))^-] < \infty\}$$

the class of admissible strategies where $u_1(t, \cdot)$ is a utility function for fixed $t \in [0, T]$ and u_2 is a utility function, too.

Definition 2.1 (Utility function). *A utility function is a strictly concave, strictly increasing, and continuously differentiable function $u: \ \mathbb{R}^+ \to \mathbb{R}$ satisfying*

$$u'(0) \triangleq \lim_{x \searrow 0} u'(x) = \infty, \quad u'(\infty) \triangleq \lim_{x \nearrow \infty} u'(x) = 0.$$

Remark 2.2. *The fact that the utility function u is strictly increasing implies that the utility is increasing with each additional unit of wealth. However the investor's risk aversion is reflected by the concavity of u, i.e. the marginal utility u' is decreasing which means that the benefit increase of an additional monetary unit is decreasing when the wealth x is increasing. Moreover the marginal utility at $x = 0$ given by $u'(0)$ is positive reflecting that few money is better than no money. Finally the utility function shows a so-called saturation effect as the marginal utility at $x = \infty$ given by $u'(\infty)$ vanishes.*

Example 1. *Typical utility functions are*

- *the power utility: $u(x) = \frac{1}{1-R}(x^{1-R} - 1)$, $R \in \mathbb{R}^+ \setminus \{1\}$,*
- *the logarithmic utility: $u(x) = \ln(x)$.*

*Both belong to the class of utility functions with **constant relative risk aversion** (CRRA). Besides the CRRA utilities there exist the utility functions with **constant absolute risk aversion** (CARA) of which the exponential utility function given by $u: \ \mathbb{R} \to \mathbb{R}$, $u(x) = -e^{-Rx}$, $R \in \mathbb{R}^+$ with $\lim_{x \searrow -\infty} u'(x) = \infty$ is a representative. In the following we focus on CRRA utilities.*

2.1.2 The Optimal Investment Problem

We suppose that the investor tries to maximize utility from terminal wealth as well as from intermediate consumption, and that his preferences are captured by a family of utility functions $u_1 \in \mathrm{C}^{1,2}([0, T] \times (0, \infty))$ and a utility function $u_2 \in \mathrm{C}^2(0, \infty)$. We always assume that $u_1(t, .)$, $t \in [0, T]$ fix, and u_2 are *polynomially bounded at* 0, i.e. that for some constants $K, \kappa, \delta > 0$ we have

$$\begin{aligned} |u_1(t, x)| &\le K(1 + \tfrac{1}{x})^\kappa \text{ for all } x \in (0, \delta) \text{ and } t \in [0, T] \text{ fixed,} \\ |u_2(x)| &\le K(1 + \tfrac{1}{x})^\kappa \text{ for all } x \in (0, \delta). \end{aligned} \tag{2.4}$$

Given the above dynamics, the investor's **optimal investment problem** is to

$$\text{maximize } \mathbb{E}\big[\textstyle\int_0^T u_1(t, c(t)X^{\pi,c}(t))\mathrm{d}t + u_2(X^{\pi,c}(T))\big] \text{ over } (\pi, c) \in \mathcal{A} \tag{$\mathrm{P_M}$}$$

given the initial wealth $X^{\pi,c}(0) = x_0$.

There exist two main approaches for solving the investment problem ($\mathrm{P_M}$). The first one uses dynamic programming methods and is called the **Stochastic Control Approach**. The second one which is based on the completeness of the financial market is called the **Martingale Approach**. In the subsequent section we present the solution of the investment problem ($\mathrm{P_M}$) following the **Stochastic Control Approach**.

2.1.3 Hamilton-Jacobi-Bellman Equation and the Verification Theorem

For solving the investment problem ($\mathrm{P_M}$) one defines the **value function** $v: [0,T] \times \mathbb{R}^+ \to \mathbb{R}$ by

$$v(t,x) \triangleq \sup_{(\pi,c)\in\mathcal{A}} \mathbb{E}\big[\int_t^T u_1(s, c(s)X^{\pi,c}(s))\mathrm{d}s + u_2(X^{\pi,c}(T)) \,|\, X^{\pi,c}(t) = x\big].$$

Note that $v(T,x) = u_2(x)$.

In the following we present a heuristic derivation of the so-called **Hamilton-Jacobi-Bellman equation** (HJB) which is a PDE for the value function v.

Assuming there exists an optimal strategy (π^*, c^*) we proceed as follows.

i) Definition of auxiliary strategies:

   For a given $(t,x) \in [0,T] \times \mathbb{R}^+$ and $\theta \in [t,T]$ we consider the following strategies on the interval $[t,T]$.

$$(\pi^1(s), c^1(s)) \triangleq (\pi^*(s), c^*(s)),$$
$$(\pi^2(s), c^2(s)) \triangleq \begin{cases} (\pi(s), c(s)) & \text{if } s \in [t,\theta], \\ (\pi^*(s), c^*(s)) & \text{if } s \in (\theta, T], \end{cases}$$

   for $s \in [t,T]$ and where (π, c) is an arbitrary admissible strategy.

ii) Calculation of the expected utilities:

   By assumption we get

$$\mathbb{E}\Big[\int_t^T u_1(s, c^1(s)X^{\pi^1,c^1}(s))\mathrm{d}s + u_2(X^{\pi^1,c^1}(T)) \,|\, X^{\pi^1,c^1}(t) = x\Big] = v(t,x).$$

   Further

$$\begin{aligned}
&\mathbb{E}\Big[\int_t^T u_1(s, c^2(s)X^{\pi^2,c^2}(s))\mathrm{d}s + u_2(X^{\pi^2,c^2}(T)) \,|\, X^{\pi^2,c^2}(t) = x\Big] \\
&\quad = \mathbb{E}\Big[\mathbb{E}\Big[\int_t^T u_1(s, c^2(s)X^{\pi^2,c^2}(s))\mathrm{d}s + u_2(X^{\pi^2,c^2}(T)) \,|\, X^{\pi^2,c^2}(\theta) = X^{\pi^2,c^2}(\theta)\Big] \,|\, X^{\pi^2,c^2}(t) = x\Big] \\
&\quad = \mathbb{E}\Big[\int_t^\theta u_1(s, c^2(s)X^{\pi^2,c^2}(s))\mathrm{d}s \\
&\qquad\quad + \mathbb{E}\Big[\int_\theta^T u_1(s, c^2(s)X^{\pi^2,c^2}(s))\mathrm{d}s + u_2(X^{\pi^2,c^2}(T)) \,|\, X^{\pi^2,c^2}(\theta) = X^{\pi^2,c^2}(\theta)\Big] \,|\, X^{\pi^2,c^2}(t) = x\Big] \\
&\quad = \mathbb{E}\Big[\int_t^\theta u_1(s, c(s)X^{\pi,c}(s))\mathrm{d}s \\
&\qquad\quad + \mathbb{E}\Big[\int_\theta^T u_1(s, c^*(s)X^{\pi^*,c^*}(s))\mathrm{d}s + u_2(X^{\pi^*,c^*}(T)) \,|\, X^{\pi^*,c^*}(\theta) = X^{\pi,c}(\theta)\Big] \,|\, X^{\pi,c}(t) = x\Big] \\
&\quad = \mathbb{E}\Big[\int_t^\theta u_1(s, c(s)X^{\pi,c}(s))\mathrm{d}s + v(\theta, X^{\pi,c}(\theta)) \,|\, X^{\pi,c}(t) = x\Big]
\end{aligned}$$

   for any admissible strategy (π, c).

iii) Taking the limit $\theta \searrow t$:

By definition of the two strategies we have

$$\mathbb{E}\Big[\int_t^\theta u_1(s, c(s)X^{\pi,c}(s))\mathrm{d}s + v(\theta, X^{\pi,c}(\theta)) \,|\, X^{\pi,c}(t) = x\Big] \le v(t,x), \tag{2.5}$$

where equality holds if $(\pi, c) = (\pi^*, c^*)$ is chosen.

Given that $v \in \mathrm{C}^{1,2}([0,T) \times (0,\infty)) \cap \mathrm{C}([0,T] \times (0,\infty))$ Itô's formula yields

$$\begin{aligned}\int_t^\theta u_1(s, c(s)X^{\pi,c}(s))\mathrm{d}s + v(\theta, X^{\pi,c}(\theta)) = v(t,x) + \int_t^\theta H(s, X^{\pi,c}(s), \pi(s), c(s))\mathrm{d}s \\ + \int_t^\theta \pi(s)^\top.\sigma X^{\pi,c}(s) v_x(s, X^{\pi,c}(s)).\mathrm{d}W(s)\end{aligned}$$

where we write $v_t \triangleq \frac{\partial v}{\partial t}$, $v_x \triangleq \frac{\partial v}{\partial x}$ and $v_{xx} \triangleq \frac{\partial^2 v}{\partial x^2}$ and

$$H(t,x,\pi,c) \triangleq u_1(t,cx) + v_t(t,x) + (r + \pi^\top.\eta - c)xv_x(t,x) + \tfrac{1}{2}\pi^\top.\sigma.\sigma^\top.\pi x^2 v_{xx}(t,x). \tag{2.6}$$

Since the local martingale term is in fact a martingale we can rewrite (2.5) as

$$\mathbb{E}\Big[\int_t^\theta H(s, X^{\pi,c}(s), \pi(s), c(s))\mathrm{d}s \,|\, X^{\pi,c}(t) = x\Big] \le 0, \tag{2.7}$$

Finally taking the limit $\theta \searrow t$ yields

$$H(t, X^{\pi,c}(t), \pi(t), c(t)) \le 0$$

or equivalently

$$u_1(t,cx) + v_t(t,x) + (r + \pi^\top.\eta - c)xv_x(t,x) + \tfrac{1}{2}\pi^\top.\sigma.\sigma^\top.\pi x^2 v_{xx}(t,x) \le 0$$

for any admissible strategy (π, c). As equality holds for $(\pi, c) = (\pi^*, c^*)$ we get the **HJB**

$$0 = \sup_{(\pi,c)\in\Pi\times\mathbb{R}_0^+} \big\{u_1(t,cx) + v_t(t,x) + (r + \pi^\top.\eta - c)xv_x(t,x) + \tfrac{1}{2}\pi^\top.\sigma.\sigma^\top.\pi x^2 v_{xx}(t,x)\big\} \tag{2.8}$$

for $(t,x) \in [0,T) \times (0,\infty)$ subject to the boundary condition

$$v(T,x) = u_2(x), \;\; x \in (0,\infty). \tag{2.9}$$

Remark 2.3. *Notice that the supremum in the HJB is taken over numbers and not over processes.*

Having obtained the HJB one can solve the investment problem $(\mathrm{P_M})$ by solving the corresponding HJB and afterwards verifying that the solution obtained in that way satisfies the assumptions made in the derivation of the HJB and moreover really is the solution of the investment problem $(\mathrm{P_M})$.

Solving the HJB usually works via choosing a certain type of utility function and then using a suitable separation ansatz for the value function v in order to transform the PDE into an ODE. Thereafter the maximizers of the HJB denoted by (π^*, c^*) are determined and inserted back into the HJB which thus becomes a classical ODE without the supremum to be taken. This ODE can then be solved using methods of the ODE-theory.

The evidence that the solution of the HJB is indeed the solution of the investment problem is then provided by the following verification theorem.

Theorem 2.4 (Verification Theorem)**.** *Suppose that* $v \in \mathrm{C}^{1,2}([0,T) \times (0,\infty)) \cap \mathrm{C}([0,T] \times (0,\infty))$ *is a solution to the HJB-system (2.8) subject to (2.9), and assume that* $|v(t,x)| \le K(1+\frac{1}{x})^\kappa$, $x \in (0,\delta)$, *and* $|v(t,x)| \le K(1+x)^\kappa$, $x > \frac{1}{\delta}$, *for some constants* $K, \kappa, \delta > 0$.

i) For any admissible strategy $(\pi, c) \in \mathcal{A}$ *and all* $t_0 \in [0,T]$, $x_0 \in (0,\infty)$, *we have*

$$\mathbb{E}\Big[\textstyle\int_{t_0}^T u_1(t, c(t)X^{\pi,c}(t))\mathrm{d}t + u_2(X^{\pi,c}(T)) \,|\, X^{\pi,c}(t_0) = x_0\Big] \le v(t_0, x_0).$$

ii) If there exists an admissible strategy (π^*, c^*) *such that*

$$(\pi^*(t,x), c^*(t,x)) \in \operatorname*{arg\,max}_{(\pi,c)\in\Pi\times\mathbb{R}_0^+} H(t,x,\pi,c) \text{ for } t \in [0,T),\ x \in (0,\infty)$$

where H *is given by (2.6) then it follows that*

$$v(t_0, x_0) = \sup_{(\pi,c)\in\mathcal{A}} \mathbb{E}\Big[\textstyle\int_{t_0}^T u_1(t, c(t)X^{\pi,c}(t))\mathrm{d}t + u_2(X^{\pi,c}(T)) \,|\, X^{\pi,c}(t_0) = x_0\Big]$$

for all $t_0 \in [0,T]$ *and* $x_0 \in (0,\infty)$, *and* (π^*, c^*) *is an optimal strategy for the investment problem* (P_M).

A detailed proof of a more general version of the Verification Theorem (Theorem 3.3) is given in Section 3.3.

2.1.4 Solution for CRRA Investors

In Section 4, especially Subsection 4.1, we solve the Merton investment problem for a CRRA investor in detail. At this point we just present the results.

Theorem 2.5 (Solution of the Merton investment problem with power utilities)**.** *Given the utility functions*

$$\begin{aligned} u_1(t,x) &= \varepsilon e^{-\delta t} \tfrac{1}{1-R}(x^{1-R} - 1),\ t \in [0,T],\ x \in (0,\infty),\ \varepsilon \in [0,\infty),\ \delta \in (0,\infty), \\ u_2(x) &= e^{-\delta T} \tfrac{1}{1-R}(x^{1-R} - 1),\ x \in (0,\infty), \end{aligned}$$

the optimal strategy (π^*, c^*) *of the Merton investment problem* (P_M) *is given by*

$$\pi^* = \tfrac{1}{R}(\sigma.\sigma^\top)^{-1}.\eta, \quad c^*(t) = \frac{\varepsilon^{\frac{1}{R}} e^{\frac{\delta-(1-R)\Psi}{R}(T-t)}}{1+\varepsilon^{\frac{1}{R}} \frac{R}{\delta-(1-R)\Psi}\left(e^{\frac{\delta-(1-R)\Psi}{R}(T-t)} - 1\right)}$$

for $t \in [0,T]$. *Further the value function reads*

$$\begin{aligned} v(t,x) = &\tfrac{1}{1-R} x^{1-R} e^{(1-R)\Psi(T-t)-\delta T}\left(1 + \varepsilon^{\frac{1}{R}} \tfrac{R}{\delta-(1-R)\Psi}\left(e^{\frac{\delta-(1-R)\Psi}{R}(T-t)} - 1\right)\right)^R \\ &- \tfrac{1}{1-R}\tfrac{1}{\delta} e^{-\delta t}(\varepsilon - (\varepsilon - \delta)e^{-\delta(T-t)}) \end{aligned}$$

for $(t,x) \in [0,T] \times (0,\infty)$ *where* Ψ *denotes the utility growth potential of the financial market given by*

$$\Psi \triangleq r + \tfrac{1}{2}\tfrac{1}{R}\eta^\top.(\sigma.\sigma^\top)^{-1}.\eta.$$

In case of $\varepsilon = 0$ *the optimal strategy* (π^*, c^*) *reads*

$$\pi^* = \tfrac{1}{R}(\sigma.\sigma^\top)^{-1}.\eta, \quad c^*(t) = 0$$

for $t \in [0,T]$ *and the value function is given by*

$$v(t,x) = e^{-\delta T}\tfrac{1}{1-R}(x^{1-R}e^{(1-R)\Psi(T-t)} - 1)$$

for $(t,x) \in [0,T] \times (0,\infty)$.

Theorem 2.6 (Solution of the Merton investment problem with logarithmic utilities)**.** *Given the utility functions*

$$\begin{aligned} u_1(t,x) &= \varepsilon e^{-\delta t}\ln(x),\ t \in [0,T],\ x \in (0,\infty),\ \varepsilon \in [0,\infty),\ \delta \in (0,\infty), \\ u_2(x) &= e^{-\delta T}\ln(x),\ x \in (0,\infty), \end{aligned}$$

the optimal strategy (π^*, c^*) *of the Merton investment problem* (P_M) *is given by*

$$\pi^* = (\sigma.\sigma^\top)^{-1}.\eta, \quad c^*(t) = \tfrac{\varepsilon\delta}{\varepsilon-(\varepsilon-\delta)e^{-\delta(T-t)}}$$

for $t \in [0,T]$. *Further the value function reads*

$$\begin{aligned} v(t,x) = {} & \tfrac{1}{\delta}e^{-\delta t}(\varepsilon - (\varepsilon-\delta)e^{-\delta(T-t)})\ln(x) + \tfrac{\varepsilon}{\delta}e^{-\delta t}(1 - e^{-\delta(T-t)})\big(\tfrac{\Psi}{\delta} + \ln(\varepsilon) - 1\big) - \varepsilon t e^{-\delta t} \\ & - (\varepsilon-\delta)e^{-\delta T}\big(\tfrac{\Psi}{\delta}(T-t) - T\big) - \tfrac{1}{\delta}e^{-\delta t}(\varepsilon - (\varepsilon-\delta)e^{-\delta(T-t)})\ln\big(\tfrac{1}{\delta}e^{-\delta t}(\varepsilon - (\varepsilon-\delta)e^{-\delta(T-t)})\big) \end{aligned}$$

for $(t,x) \in [0,T] \times (0,\infty)$.

In case of $\varepsilon = 0$ *the optimal strategy* (π^*, c^*) *reads*

$$\pi^* = (\sigma.\sigma^\top)^{-1}.\eta, \quad c^*(t) = 0$$

for $t \in [0,T]$ *and the value function is given by*

$$v(t,x) = e^{-\delta T}(\ln(x) + \Psi(T-t))$$

for $(t,x) \in [0,T] \times (0,\infty)$.

2.2 The Bäuerle-Rieder Investment Problem

In their paper [Bäuerle, Rieder 2004] N. Bäuerle and U. Rieder investigate the optimal investment problem in continuous-time with Markov-modulated asset prices and interest rates. They explicitly solve the corresponding problem of maximizing the expected utility from terminal wealth for different utility functions. Here we recapitulate their model using a slightly different notation and add intermediate consumption.

2.2.1 Mathematical Framework

Informal Description. Again, the financial market consists of a locally riskless money market account P^0 and $\bar{n}$ risky assets P^n, $n = 1, \dots, \bar{n}$. But as opposed to the Merton investment problem in which the financial market always stays in one state, in the Bäuerle-Rieder investment problem at each point of time, the market is in one of finitely many states. In each state the asset prices are given by a diffusion model that is driven by an $\bar{m}$-dimensional Brownian motion W with coefficients depending on the current state of the economy.

Asset Price Dynamics. The mathematical model of the asset price dynamics as described above is given as follows. We let $(\Omega, \mathfrak{F}, \mathfrak{F}(\cdot))$ be a filtered space with time horizon $[0, T]$ that is endowed with a reference probability measure $\mathbb{P}$ such that $\mathfrak{F}(\cdot)$ satisfies the usual conditions of right-continuity and completeness and $\mathfrak{F} = \mathfrak{F}(T)$. Further we assume that $(\Omega, \mathfrak{F}, \mathbb{P})$ carries an $\bar{m}$-dimensional $(\mathfrak{F}(\cdot), \mathbb{P})$-Wiener process W and $\bar{i}(\bar{i}-1)$ $(\mathfrak{F}(\cdot), \mathbb{P})$-Poisson processes $N^{i,j}$, $i, j \in \{0, \dots, \bar{i}-1\}$, $i \neq j$, with intensities $\vartheta^{i,j} \in (0, \infty)$, all of which are independent of each other.

The **state** of the market is described by the $\{0, \dots, \bar{i}-1\}$-valued process I that satisfies

$$\mathrm{d}I = \sum_{i=0}^{\bar{i}-1} \sum_{j=0, j\neq i}^{\bar{i}-1} (j-i) 1_{\{I_- = i\}} \mathrm{d}N^{i,j}, \quad I(0) = 0. \tag{2.10}$$

We denote by $\{\tau_k\}_{k \in \mathbb{N}_0}$ the corresponding sequence of jump times, i.e.

$$\tau_k \triangleq \inf \{t \in [\tau_{k-1}, T] \,:\, I(t) \neq I(\tau_{k-1})\} \text{ for } k \in \mathbb{N}, \text{ where } \tau_0 \triangleq 0.$$

Then the **asset price dynamics** are given by

$$\mathrm{d}P^0 = P^0 r^{I_-} \mathrm{d}t, \quad P^0(0) = p_0^0, \tag{2.11}$$

$$\begin{aligned} \mathrm{d}P^n &= P^n \left[(r^{I_-} + \eta_n^{I_-}) \mathrm{d}t + \sum_{m=1}^{\bar{m}} \sigma_{n,m}^{I_-} \mathrm{d}W^m \right] \text{ on } [\![\tau_{k-1}, \tau_k)\!), \\ P^n(0) &= p_0^n, \quad P^n(\tau_k) = P^n(\tau_k -), k \geq 1. \end{aligned} \tag{2.12}$$

Here for $i \in \{0, \dots, \bar{i}-1\}$ the number $r^i \in [0, \infty)$, the vector $\eta^i \in \mathbb{R}^{\bar{n}}$ and the matrix $\sigma^i \in \mathbb{R}^{\bar{n} \times \bar{m}}$ are given parameters and $\sigma^i.(\sigma^i)^\top$ is positive definite.

Notice that in general the financial market as given above is incomplete.

Investor's Strategy. The investor is equipped with an initial wealth x_0 and specifies for each state $i \in \{0, \dots, \bar{i}-1\}$ a **portfolio strategy** π^i and a **consumption rate** c^i which are applied when the economy is in state i. Then by (2.11) and (2.12) his **wealth** $X^{\pi,c}$ evolves according to the stochastic differential equation

$$\begin{aligned} \mathrm{d}X^{\pi,c} &= X^{\pi,c} \left[(r^{I_-} + (\pi^{I_-})^\top . \eta^{I_-} - c^{I_-}) \mathrm{d}t + (\pi^{I_-})^\top . \sigma^{I_-} . \mathrm{d}W \right] \text{ on } [\![\tau_{k-1}, \tau_k)\!), \\ X^{\pi,c}(0) &= x_0, \quad X^{\pi,c}(\tau_k) = X^{\pi,c}(\tau_k -), k \geq 1. \end{aligned} \tag{2.13}$$

We write $\pi \triangleq (\pi^0, \ldots, \pi^{\bar{i}-1})$, resp. $c \triangleq (c^0, \ldots, c^{\bar{i}-1})$, for brevity. Let $\Pi \subseteq \mathbb{R}^{\bar{n}}$ be a given closed set. We denote by

$$\begin{aligned}\mathcal{A} \triangleq \{(\pi, c) :\ (\pi^i, c^i)\ \text{ bounded, } \mathfrak{F}(\cdot)\text{-predictable, } \Pi \times \mathbb{R}_0^+\text{-valued and satisfies}\\ \mathbb{E}[\textstyle\int_0^T u_1(t, c^{I(t)}(t) X^{\pi,c}(t))^- \mathrm{d}t + u_2(X^{\pi,c}(T))^-] < \infty\}\end{aligned}$$

the class of admissible strategies where $u_1(t,.)$ is a utility function for fixed $t \in [0,T]$ and u_2 is a utility function, too.

2.2.2 The Optimal Investment Problem

We suppose that the investor tries to maximize utility from terminal wealth as well as from intermediate consumption, and that his preferences are captured by a family of utility functions $u_1 \in \mathrm{C}^{1,2}([0,T] \times (0,\infty))$ and a utility function $u_2 \in \mathrm{C}^2(0,\infty)$. We always assume that $u_1(t,.)$, $t \in [0,T]$ fix, and u_2 are *polynomially bounded at* 0, i.e. that for some constants $K, \kappa, \delta > 0$

$$\begin{aligned}|u_1(t,x)| &\le K(1 + \tfrac{1}{x})^\kappa \text{ for all } x \in (0,\delta) \text{ and } t \in [0,T] \text{ fixed,}\\ |u_2(x)| &\le K(1 + \tfrac{1}{x})^\kappa \text{ for all } x \in (0,\delta).\end{aligned} \tag{2.14}$$

Given the above dynamics, the investor's **optimal investment problem** is to

$$\text{maximize } \mathbb{E}\big[\textstyle\int_0^T u_1(t, c^{I(t)}(t) X^{\pi,c}(t))\mathrm{d}t + u_2(X^{\pi,c}(T))\big] \text{ over } (\pi, c) \in \mathcal{A} \tag{$\mathrm{P_{BR}}$}$$

given the initial wealth $X^{\pi,c}(0) = x_0$.

Again we follow the **Stochastic Control Approach**.

2.2.3 Hamilton-Jacobi-Bellman Equations and the Verification Theorem

Given the tuple $(v^0, \ldots, v^{\bar{i}-1})$ of value functions

$$v^i(t_0, x_0) = \sup_{(\pi,c)\in\mathcal{A}} \mathbb{E}[\textstyle\int_{t_0}^T u_1(t, c^{I(t)}(t) X^{\pi,c}(t))\mathrm{d}t + u_2(X^{\pi,c}(T)) \,|\, I(t_0) = i,\ X^{\pi,c}(t_0) = x_0]$$

for $i \in \{0, \ldots, \bar{i}-1\}$ the motivation for the Hamilton-Jacobi-Bellman system (**HJB-system**) works completely analogously to the motivation in Section 2.1.3 resulting in the HJB-system

$$\begin{aligned}0 = \sup_{(\pi,c)\in\Pi\times\mathbb{R}_0^+} \Big\{ & u_1(t,cx) + v_t^i(t,x) + (r^i + \pi^\top.\eta^i - c)x v_x^i(t,x) + \tfrac{1}{2}\pi^\top.\sigma^i.(\sigma^i)^\top.\pi x^2 v_{xx}^i(t,x)\\ &+ \sum_{j=0}^{\bar{i}-1} \vartheta^{i,j}\big[v^j(t,x) - v^i(t,x)\big]\Big\}\end{aligned} \tag{2.15}$$

for $(t,x) \in [0,T) \times (0,\infty)$ and $i \in \{0, \ldots, \bar{i}-1\}$, subject to the boundary conditions

$$v^i(T,x) = u_2(x),\ x \in (0,\infty) \text{ for } i \in \{0, \ldots, \bar{i}-1\}. \tag{2.16}$$

Notice that here the above HJB-system consists of coupled partial differential equations.

Remark 2.7. *The only difference between the HJB-system (2.15) of the Bäuerle-Rieder investment problem and the HJB (2.8) of the Merton investment problem is the switching term* $\sum_{j=0}^{\bar{\imath}-1} \vartheta^{i,j}\big[v^j(t,x) - v^i(t,x)\big]$ *which accommodates the possibility that the market could switch into another state. Notice that on small time intervals the probability of a regime switch from state i into state j is given by* $\mathbb{P}(I(t+\mathrm{d}t) = j \,|\, I(t) = i) = \vartheta^{i,j}\mathrm{d}t$. *Hence compared to the Merton HJB (2.8) one has to add the sum over the intensity-weighted differences of the value functions.*

Solving the HJB is done in analogy to the way described in Section 2.1.3 and the corresponding verification theorem is given as follows.

Theorem 2.8 (Verification Theorem)**.** *Suppose that* $(v^0,\dots,v^{\bar{\imath}-1})$ *with* $v^i \in \mathrm{C}^{1,2}([0,T)\times(0,\infty)) \cap \mathrm{C}([0,T]\times(0,\infty))$, $i \in \{0,\dots,\bar{\imath}-1\}$, *is a solution to the HJB-system (2.15) subject to (2.16), and assume that* $|v^i(t,x)| \le K(1+\frac{1}{x})^\kappa$, $x \in (0,\delta)$ *and* $|v^i(t,x)| \le K(1+x)^\kappa$, $x > \frac{1}{\delta}$ *for* $i \in \{0,\dots,\bar{\imath}-1\}$ *for some constants* $K,\kappa,\delta > 0$.

i) For any admissible strategy $(\pi,c) \in \mathcal{A}$ *and all* $t_0 \in [0,T]$, $x_0 \in (0,\infty)$, *we have*

$$\mathbb{E}\Big[\int_{t_0}^T u_1(t, c^{I(t)}(t)X^{\pi,c}(t))\mathrm{d}t + u_2(X^{\pi,c}(T)) \,|\, I(t_0) = i,\ X^{\pi,c}(t_0) = x_0\Big] \le v^i(t_0,x_0)$$

for $i \in \{0,\dots,\bar{\imath}-1\}$.

ii) If there exists an admissible strategy $(\pi^*,c^*) = ((\pi^{0,*},c^{0,*}),\dots,(\pi^{\bar{\imath}-1,*},c^{\bar{\imath}-1,*}))$ *such that*

$$(\pi^{i,*}(t,x), c^{i,*}(t,x)) \in \operatorname*{arg\,max}_{(\pi,c)\in\Pi\times\mathbb{R}_0^+} H^i(t,x,\pi,c) \text{ for } t \in [0,T),\ x \in (0,\infty)$$

where

$$\begin{aligned} H^i(t,x,\pi,c) \triangleq\ & u_1(t,cx) + v_t^i(t,x) + (r^i + \pi^\top.\eta^i - c)xv_x^i(t,x) + \tfrac{1}{2}\pi^\top.\sigma^i.(\sigma^i)^\top.\pi x^2 v_{xx}^i(t,x) \\ & + \sum_{j=0}^{\bar{\imath}-1} \vartheta^{i,j}\big[v^j(t,x) - v^i(t,x)\big], \end{aligned} \tag{2.17}$$

then it follows that

$$v^i(t_0,x_0) = \sup_{(\pi,c)\in\mathcal{A}} \mathbb{E}^{\pi,c}\Big[\int_{t_0}^T u_1(t, c^{I(t)}(t)X^{\pi,c}(t))\mathrm{d}t + u_2(X^{\pi,c}(T)) \,|\, I(t_0) = i,\ X^{\pi,c}(t_0) = x_0\Big]$$

for $i \in \{0,\dots,\bar{\imath}-1\}$ *and all* $t_0 \in [0,T]$ *and* $x_0 \in (0,\infty)$, *and* (π^*,c^*) *is an optimal strategy for the investment problem* (P_{BR}).

Again the proof is given in Section 3.3.

2.2.4 Solution for CRRA Investors

In Section 4, especially Subsection 4.1, we solve the Bäuerle-Rieder investment problem in a two state model for a CRRA investor in detail. At this point we just present the results.

Theorem 2.9 (Solution of the Bäuerle-Rieder investment problem with power utilities)**.** *Given the utility functions*

$$\begin{aligned} u_1(t,x) &= \varepsilon e^{-\delta t} \tfrac{1}{1-R}(x^{1-R}-1),\ t\in[0,T],\ x\in(0,\infty),\ \varepsilon\in[0,\infty),\ \delta\in(0,\infty),\\ u_2(x) &= e^{-\delta T}\tfrac{1}{1-R}(x^{1-R}-1),\ x\in(0,\infty), \end{aligned}$$

the optimal strategy (π^*, c^*) of the Bäuerle-Rieder investment problem (P_{BR}) is given by

$$\pi^{i,*} = \tfrac{1}{R}(\sigma^i.(\sigma^i)^\top)^{-1}.\eta^i, \quad c^{i,*}(t) = \big(\tfrac{\varepsilon\delta}{\varepsilon-(\varepsilon-\delta)e^{-\delta(T-t)}}\big)^{\frac{1}{R}} e^{-\frac{1-R}{R}g^i(t)}$$

for $t\in[0,T]$. Further the value function reads

$$v^i(t,x) = \tfrac{1}{1-R}\big(x^{1-R}e^{(1-R)g^i(t)}-1\big)\tfrac{1}{\delta}e^{-\delta t}(\varepsilon-(\varepsilon-\delta)e^{-\delta(T-t)})$$

for $(t,x)\in[0,T]\times(0,\infty)$ and $i\in\{0,\dots,\bar{i}-1\}$ where the function g^i is the solution of the following ODE-system

$$\begin{aligned} (g^i)'(t) = &-\Psi^i + \tfrac{\varepsilon\delta}{\varepsilon-(\varepsilon-\delta)e^{-\delta(T-t)}}\Big(1-\tfrac{R}{1-R}\big(\big(\tfrac{\varepsilon\delta}{\varepsilon-(\varepsilon-\delta)e^{-\delta(T-t)}}\big)^{\frac{1-R}{R}}e^{-\frac{1-R}{R}g^i(t)}-1\big)\Big)\\ &-\sum_{j=0}^{\bar{i}-1}\vartheta^{i,j}\tfrac{1}{1-R}\big(e^{(1-R)(g^j(t)-g^i(t))}-1\big) \end{aligned}$$

subject to the boundary conditions $g^i(T)=0$, $i\in\{0,\dots,\bar{i}-1\}$. Here Ψ^i denotes the utility growth potential of the financial market in state i which is given by

$$\Psi^i \triangleq r^i + \tfrac{1}{2}\tfrac{1}{R}(\eta^i)^\top.(\sigma^i.(\sigma^i)^\top)^{-1}.\eta^i.$$

In case of $\varepsilon=0$ the optimal strategy (π^*,c^*) reads

$$\pi^{i,*} = \tfrac{1}{R}(\sigma^i.(\sigma^i)^\top)^{-1}.\eta^i, \quad c^{i,*}(t)=0$$

for $t\in[0,T]$ and the value function is given by

$$v^i(t,x) = e^{-\delta T}\tfrac{1}{1-R}\big(x^{1-R}e^{(1-R)g^i(t)}-1\big)$$

for $(t,x)\in[0,T]\times(0,\infty)$ where g^i now satisfies

$$(g^i)'(t) = -\Psi^i - \sum_{j=0}^{\bar{i}-1}\vartheta^{i,j}\tfrac{1}{1-R}\big(e^{(1-R)(g^j(t)-g^i(t))}-1\big)$$

subject to the boundary conditions $g^i(T)=0$, $i\in\{0,\dots,\bar{i}-1\}$.

Theorem 2.10 (Solution of the Bäuerle-Rieder investment problem with logarithmic utilities)**.** *Given the utility functions*

$$u_1(t,x) = \varepsilon e^{-\delta t}\ln(x),\ t\in[0,T],\ x\in(0,\infty),\ \varepsilon\in[0,\infty),\ \delta\in(0,\infty),$$
$$u_2(x) = e^{-\delta T}\ln(x),\ x\in(0,\infty),$$

the optimal strategy (π^*, c^*) of the Bäuerle-Rieder investment problem (P_{BR}) is given by

$$\pi^{i,*} = (\sigma^i.(\sigma^i)^\top)^{-1}.\eta^i,\quad c^{i,*}(t) = \tfrac{\varepsilon\delta}{\varepsilon-(\varepsilon-\delta)e^{-\delta(T-t)}}$$

for $t\in[0,T]$. Further the value function reads

$$v^i(t,x) = (\ln(x)+g^i(t))\tfrac{1}{\delta}e^{-\delta t}(\varepsilon-(\varepsilon-\delta)e^{-\delta(T-t)})$$

for $(t,x)\in[0,T]\times(0,\infty)$ and $i\in\{0,\dots,\bar{i}-1\}$ where the function g^i is the solution of the following ODE-system

$$(g^i)'(t) = -\Psi^i + \tfrac{\varepsilon\delta}{\varepsilon-(\varepsilon-\delta)e^{-\delta(T-t)}}\Big(1-\ln\big(\tfrac{\varepsilon\delta}{\varepsilon-(\varepsilon-\delta)e^{-\delta(T-t)}}\big)+g^i(t)\Big) - \sum_{j=0}^{\bar{i}-1}\vartheta^{i,j}(g^j(t)-g^i(t))$$

subject to the boundary conditions $g^i(T)=0$, $i\in\{0,\dots,\bar{i}-1\}$.

In case of $\varepsilon=0$ the optimal strategy (π^*, c^*) reads

$$\pi^{i,*} = (\sigma^i.(\sigma^i)^\top)^{-1}.\eta^i,\quad c^{i,*}(t)=0$$

for $t\in[0,T]$ and the value function is given by

$$v^i(t,x) = e^{-\delta T}(\ln(x)+g^i(t))$$

for $(t,x)\in[0,T]\times(0,\infty)$ where g^i now satisfies

$$(g^i)'(t) = -\Psi^i - \sum_{j=0}^{\bar{i}-1}\vartheta^{i,j}(g^j(t)-g^i(t))$$

subject to the boundary conditions $g^i(T)=0$, $i\in\{0,\dots,\bar{i}-1\}$.

3 Continuous-time Portfolio Optimization for a Large Investor

In this section we introduce the investment problem for a large investor. In analogy to the previous section on the Merton and the Bäuerle-Rieder investment problem we begin with a description of the mathematical framework, followed by the formulation of the investment problem. Thereafter we give the HJB-system and present a verification theorem whose proof closes this section. A detailed derivation of the solution of the investment problem for particular utility functions is dealt with in the following Sections 4 and 5.

3.1 Mathematical Framework

Informal Description. The financial market consists of a locally riskless money market account P^0 and $\bar{n}$ risky assets P^n, $n = 1, \ldots, \bar{n}$. At each point of time, the market is either in the *normal* state $i = 0$ or in the *alerted* state $i = 1$. In normal and alerted times, asset prices are given by a jump-diffusion model that is driven by an $\bar{m}$-dimensional Brownian motion W and a $\bar{p}$-dimensional Poisson process, with coefficients depending on the current state of the economy. However, asset prices are also affected by the large investor's investment strategy π, where π_n is the fraction of wealth that he invests in asset n, as well as by his consumption rate c: The state of the market I jumps from i to $1 - i$ with intensity $\vartheta^{i,1-i}(\pi, c)$, where $\vartheta^{i,1-i}$ is a given **intensity function**; conversely, the large investor can observe regime shifts of the market. Thus the market takes the large investor's portfolio proportions and consumption rate as a *signal*. This additional dependence makes a non-standard specification of asset price dynamics necessary. In particular, on the one hand the investment strategy influences asset prices, while on the other hand it should be possible to use information on current and past prices in the portfolio decision.

Asset Price Dynamics. In the following, we construct a mathematical model of the asset price dynamics which have been described intuitively above. We let $(\Omega, \mathfrak{F}, \mathfrak{F}(\cdot))$ be a filtered space with time horizon $[0, T]$ that is endowed with a reference probability measure $\mathbb{P}$ such that $\mathfrak{F}(\cdot)$ satisfies the usual conditions of right-continuity and completeness and $\mathfrak{F} = \mathfrak{F}(T)$. Further we assume that $(\Omega, \mathfrak{F}, \mathbb{P})$ carries an $\bar{m}$-dimensional $(\mathfrak{F}(\cdot), \mathbb{P})$-Wiener process W, a $\bar{p}$-dimensional $(\mathfrak{F}(\cdot), \mathbb{P})$-Poisson process N with intensity $\lambda \in (0, \infty)^{\bar{p}}$, and two $(\mathfrak{F}(\cdot), \mathbb{P})$-Poisson processes $N^{0,1}$ and $N^{1,0}$ with intensity 1, all of which are independent of each other.

The **state** of the market is described by the $\{0, 1\}$-valued process I that satisfies

$$\mathrm{d}I = 1_{\{I_- = 0\}} \mathrm{d}N^{0,1} - 1_{\{I_- = 1\}} \mathrm{d}N^{1,0}, \quad I(0) = 0. \tag{3.1}$$

We denote by $\{\tau_k\}_{k \in \mathbb{N}_0}$ the corresponding sequence of jump times, i.e.

$$\tau_k \triangleq \inf \{t \in [\tau_{k-1}, T] \, : \, I(t) \neq I(\tau_{k-1})\} \text{ for } k \in \mathbb{N}, \text{ where } \tau_0 \triangleq 0.$$

Then the **asset price dynamics** are given by

$$\mathrm{d}P^0 = P^0_- r^{I-}\mathrm{d}t, \quad P^0(0) = p^0_0, \tag{3.2}$$

$$\begin{aligned}\mathrm{d}P^n &= P^n_-\left[(r^{I-} + \eta^{I-}_n)\mathrm{d}t + \sum_{m=1}^{\bar{m}} \sigma^{I-}_{n,m}\mathrm{d}W^m - \sum_{p=1}^{\bar{p}} \gamma^{I-}_{n,p}\mathrm{d}N^p\right] \text{ on } [\![\tau_{k-1}, \tau_k)\!), \\ P^n(0) &= p^n_0, \quad P^n(\tau_k) = [1 - \ell_n^{I(\tau_k-),1-I(\tau_k-)}]P^n(\tau_k-), k \geq 1.\end{aligned} \tag{3.3}$$

Here for $i = 0, 1$ the number $r^i \in [0, \infty)$, the vectors $\eta^i \in \mathbb{R}^{\bar{n}}$, $\ell^{i,1-i} \in (-\infty, 1)^{\bar{n}}$ and the matrices $\sigma^i \in \mathbb{R}^{\bar{n}\times\bar{m}}$ and $\gamma^i \in (-\infty, 1)^{\bar{n}\times\bar{p}}$ are given parameters and $\sigma^i.(\sigma^i)^\top$ is positive definite. Moreover $\ell^{i,1-i}$ models additional price jumps that occur on regime shifts.

In general the financial market as described above is incomplete as it was already the case in the Bäuerle-Rieder problem.

Large Investor. Turning to the large investor, we suppose that equipped with an initial wealth x_0 he specifies for each state $i = 0, 1$ a **portfolio strategy** π^i and a **consumption rate** c^i which are applied when the economy is in state i. Then by (3.2) and (3.3) his **wealth** $X^{\pi,c}$ evolves according to the stochastic differential equation

$$\begin{aligned}\mathrm{d}X^{\pi,c} &= X^{\pi,c}_-\left[(r^{I-} + (\pi^{I-})^\top.\eta^{I-} - c^{I-})\mathrm{d}t + (\pi^{I-})^\top.\sigma^{I-}.\mathrm{d}W - (\pi^{I-})^\top.\gamma^{I-}.\mathrm{d}N\right] \text{ on } [\![\tau_{k-1}, \tau_k)\!) \\ X^{\pi,c}(0) &= x_0, \quad X^{\pi,c}(\tau_k) = [1 - (\ell^{I(\tau_k-),1-I(\tau_k-)})^\top.\pi^{I(\tau_k-)}(\tau_k)]X^{\pi,c}(\tau_k-), k \geq 1.\end{aligned} \tag{3.4}$$

We write $\pi \triangleq (\pi^0, \pi^1)$, resp. $c \triangleq (c^0, c^1)$, for brevity. Our intuitive description requires that the large investor's portfolio choice and consumption affect the intensity of regime shifts. Let $\Pi \subseteq \mathbb{R}^{\bar{n}}$ be a given closed set. To avoid bankruptcy, we always choose $\Pi \subseteq \{\pi \in \mathbb{R}^{\bar{n}} : \pi_n \geq 0,\ n = 1, \ldots, \bar{n},\ \sum_{n=1}^{\bar{n}} \pi_n \leq 1\}$ as a subset of the unit simplex if $\gamma^0 \neq 0$, $\gamma^1 \neq 0$, $\ell^{0,1} \neq 0$ or $\ell^{1,0} \neq 0$.

Define the class of pre-admissible strategies by

$$\mathcal{A}_0 \triangleq \{(\pi, c) : (\pi^i, c^i) \text{ bounded, } \mathfrak{F}(\cdot)\text{-predictable and } \Pi \times \mathbb{R}^+_0\text{-valued for } i = 0, 1\}.$$

For each $(\pi, c) \in \mathcal{A}_0$ we construct a probability measure $\mathbb{P}^{\pi,c}$ on $(\Omega, \mathfrak{F})$ equivalent to $\mathbb{P}$ via the Girsanov transformation

$$\frac{\mathrm{d}\mathbb{P}^{\pi,c}}{\mathrm{d}\mathbb{P}} \triangleq \prod_{i=0,1} \exp\left\{\int_0^T [1 - \vartheta^{i,1-i}(\pi^i(t), c^i(t))]\mathrm{d}t\right\} \prod_{t\in[0,T],\, \Delta N^{i,1-i}(t)\neq 0} \vartheta^{i,1-i}(\pi^i(t), c^i(t)) \tag{3.5}$$

where the function $\vartheta^{i,1-i} : \mathbb{R}^{\bar{n}} \times \mathbb{R}^+_0 \to \mathbb{R}^+_0,\ (\pi, c) \longmapsto \vartheta^{i,1-i}(\pi, c)$ is deterministic and bounded on any closed subset of $\mathbb{R}^{\bar{n}} \times \mathbb{R}^+_0$. In order for this construction to be well-defined, we require

Lemma 3.1. *For any pair $(\pi, c) \in \mathcal{A}_0$ there exists a uniquely determined probability measure $\mathbb{P}^{\pi,c}$ on $\mathfrak{F} = \mathfrak{F}_T$ such that (3.5) is satisfied.*

Proof. We recall from equation (3.5) that $\frac{\mathrm{d}\mathbb{P}^{\pi,c}}{\mathrm{d}\mathbb{P}} = Z^{\pi,c}(T)$, where $Z^{\pi,c}$ is given by

$$Z^{\pi,c} \triangleq \prod_{i=0,1} \exp\left\{\int_0^\cdot [1 - \vartheta^{i,1-i}(\pi^i(t), c^i(t))]\mathrm{d}t\right\} \prod_{t\in[0,\cdot],\, \Delta N^{i,1-i}(t)\neq 0} \vartheta^{i,1-i}(\pi^i(t), c^i(t)). \tag{3.6}$$

To prove the assertion, we have to demonstrate that $Z^{\pi,c}$ is an $(\mathfrak{F}(\cdot),\mathbb{P})$-martingale. Note that $Z^{\pi,c}$ is the stochastic exponential of $\sum_{i=0,1}\int_0^\cdot[\vartheta^{i,1-i}(\pi^i(t),c^i(t))-1]\mathrm{d}\bar{N}^{i,1-i}(t)$, where $\bar{N}^{i,1-i}$ is given by

$$\bar{N}^{i,1-i}(t)\triangleq N^{i,1-i}(t)-t \text{ for } t\in[0,T].$$

Hence $Z^{\pi,c}$ is a local $(\mathfrak{F}(\cdot),\mathbb{P})$-martingale, and since

$$\sup_{t\in[0,T]}|Z^{\pi,c}(t)|\le e^{2T}\left[\max_{i=0,1,\,(\pi,c)\in\mathcal{A}_0}|\vartheta^{i,1-i}(\pi,c)|\right]^{N^{0,1}(T)+N^{1,0}(T)}\in\mathrm{L}^1(\mathbb{P}) \tag{3.7}$$

it follows that $Z^{\pi,c}$ is in fact a uniformly integrable $(\mathfrak{F}(\cdot),\mathbb{P})$-martingale. □

Note that since all measures $\mathbb{P}^{\pi,c}$, where $(\pi,c)\in\mathcal{A}_0$, are equivalent, the definition of the stochastic integral does not depend on (π,c). As a direct consequence of this construction we have the following result. We use the terminology of [Brémaud 1981].

Proposition 3.2. *Given a pair $(\pi,c)\in\mathcal{A}_0$, for $i=0,1$ the process $N^{i,1-i}$ is a counting process with $(\mathfrak{F}(\cdot),\mathbb{P}^{\pi,c})$-intensity $\vartheta^{i,1-i}(\pi^i,c^i)$. Moreover W is an $(\mathfrak{F}(\cdot),\mathbb{P}^{\pi,c})$-Wiener process and N is an $(\mathfrak{F}(\cdot),\mathbb{P}^{\pi,c})$-Poisson process, and*

$$[W,N]=[W,N^{i,1-i}]=[N^{i,1-i},N^{1-i,i}]=[N^{i,1-i},N]=0.$$

Proof. We recall from equation (3.6) that $\frac{\mathrm{d}\mathbb{P}^{\pi,c}}{\mathrm{d}\mathbb{P}}=Z^{\pi,c}(T)$. As $\mathbb{P}^{\pi,c}$ is equivalent to $\mathbb{P}$, quadratic covariation processes remain invariant and hence $[W,N]=[W,N^{i,1-i}]=[N^{i,1-i},N^{1-i,i}]=[N^{i,1-i},N]=0$. To show that W is an $(\mathfrak{F}(\cdot),\mathbb{P}^{\pi,c})$-Wiener process, it suffices by Lévy's characterization of Brownian motion to demonstrate that W is a local $(\mathfrak{F}(\cdot),\mathbb{P}^{\pi,c})$-martingale, i.e. that $Z^{\pi,c}W$ is a local $(\mathfrak{F}(\cdot),\mathbb{P})$-martingale. However, this is an immediate consequence of the product formula

$$\mathrm{d}(Z^{\pi,c}W)=Z^{\pi,c}_-\mathrm{d}W+W\mathrm{d}Z^{\pi,c}+\mathrm{d}Z^{\pi,c}\mathrm{d}W=Z^{\pi,c}_-\mathrm{d}W+W\mathrm{d}Z^{\pi,c}$$

because $[Z^{\pi,c},W]=0$. Next consider the counting processes $N^{i,1-i}$ and N. A direct computation via the product rule shows that with $\tilde{N}^{i,1-i}$ given by

$$\tilde{N}^{i,1-i}(t)\triangleq N^{i,1-i}(t)-\int_0^t\vartheta^{i,1-i}(\pi^i(s),c^i(s))\mathrm{d}s \text{ for } t\in[0,T],$$

we have

$$\mathrm{d}(Z^{\pi,c}\tilde{N}^{i,1-i})=\vartheta^{i,1-i}Z^{\pi,c}_-\mathrm{d}\bar{N}^{i,1-i}+\tilde{N}^{i,1-i}_-\mathrm{d}Z^{\pi,c}.$$

Thus $Z^{\pi,c}\tilde{N}^{i,1-i}$ is a local $(\mathfrak{F}(\cdot),\mathbb{P})$-martingale, so $\tilde{N}^{i,1-i}$ is a local $(\mathfrak{F}(\cdot),\mathbb{P}^{\pi,c})$-martingale, and $N^{i,1-i}$ has $(\mathfrak{F}(\cdot),\mathbb{P}^{\pi,c})$-intensity $\vartheta^{i,1-i}(\pi^i,c^i)$. A similar but simpler argument shows that N has $(\mathfrak{F}(\cdot),\mathbb{P}^{\pi,c})$-intensity λ, so N is an $(\mathfrak{F}(\cdot),\mathbb{P}^{\pi,c})$-Poisson process with intensity λ by a classical result of S. Watanabe, compare Theorem T5 in [Brémaud 1981]. □

We note from the proof of Lemma 3.1 that the compensated process $\tilde{N}^{i,1-i}$ is in fact a square integrable $(\mathfrak{F}(\cdot),\mathbb{P}^{\pi,c})$-martingale. Indeed, as $[\tilde{N}^{i,1-i}]=N^{i,1-i}$ we have

$$\mathbb{E}^{\pi,c}\left[[\tilde{N}^{i,1-i}]_T\right]=\mathbb{E}[Z^{\pi,c}_TN^{i,1-i}_T]<\infty$$

by virtue of the estimate (3.7).

Thus $\mathbb{P}^{\pi,c}$ describes the randomness in the asset market that the large investor observes if he chooses the portfolio strategy π and the consumption rate c. As required in our intuitive description, heuristically we then have

$$\mathbb{P}^{\pi,c}(I(t+\mathrm{d}t) \neq I(t) \,|\, \mathfrak{F}(t)) = \vartheta^{I(t),1-I(t)}(\pi^{I(t)}(t), c^{I(t)}(t))\mathrm{d}t.$$

We denote by

$$\mathcal{A} \triangleq \{(\pi, c) \in \mathcal{A}_0 : \; \mathbb{E}^{\pi,c}[\int_0^T u_1(t, c^{I(t)}(t)X^{\pi,c}(t))^- \mathrm{d}t + u_2(X^{\pi,c}(T))^-] < \infty\}$$

the class of admissible strategies where $u_1(t,\cdot)$ is a utility function for fixed $t \in [0,T]$ and u_2 is a utility function, too.

3.2 The Optimal Investment Problem

We suppose that the large investor tries to maximize utility from terminal wealth as well as from intermediate consumption, and that his preferences are captured by a family of utility functions $u_1 \in \mathrm{C}^{1,2}([0,T] \times (0,\infty))$ and a utility function $u_2 \in \mathrm{C}^2(0,\infty)$. We always assume that $u_1(t,\cdot)$, $t \in [0,T]$ fix, and u_2 are *polynomially bounded at* 0, i.e. that for some constants $K, \kappa, \delta > 0$

$$\begin{aligned} |u_1(t,x)| &\leq K(1 + \tfrac{1}{x})^\kappa \text{ for all } x \in (0,\delta) \text{ and } t \in [0,T] \text{ fixed}, \\ |u_2(x)| &\leq K(1 + \tfrac{1}{x})^\kappa \text{ for all } x \in (0,\delta). \end{aligned} \tag{3.8}$$

Given the above dynamics, the large investor's **optimal investment and consumption problem** is to

$$\text{maximize } \mathbb{E}^{\pi,c}\big[\int_0^T u_1(t, c^{I(t)}(t)X^{\pi,c}(t))\mathrm{d}t + u_2(X^{\pi,c}(T))\big] \text{ over } (\pi,c) \in \mathcal{A} \tag{P}$$

given the initial wealth $X^{\pi,c}(0) = x_0$.

Thus the investor tries to maximize expected utility from terminal wealth and from intermediate consumption, while he is aware of the fact that his investment strategy will affect asset prices in the sense that his portfolio proportions and his consumption rate trigger regime shifts in the market. This is reflected in the non-standard form of (P), where the expectation operator $\mathbb{E}^{\pi,c}$ depends on the investor's strategy (π, c). In the following section, we show how to solve the portfolio problem (P) with dynamic programming methods.

3.3 Hamilton-Jacobi-Bellman Equations and the Verification Theorem

A pair (v^0, v^1) of functions $v^0, v^1 \in \mathrm{C}^{1,2}([0,T) \times (0,\infty)) \cap \mathrm{C}([0,T] \times (0,\infty))$ is said to be a solution to the Hamilton-Jacobi-Bellman system, or more briefly the **HJB-system**, if it satisfies the following system of coupled partial differential equations:

$$\begin{aligned}0 = \sup_{(\pi,c)\in\Pi\times\mathbb{R}_0^+} \Big\{ & u_1(t,cx) + v_t^i(t,x) + (r^i + \pi^\top.\eta^i - c)xv_x^i(t,x) + \tfrac{1}{2}\pi^\top.\sigma^i.(\sigma^i)^\top.\pi x^2 v_{xx}^i(t,x) \\ & + \sum_{p=1}^{\bar{p}} \lambda_p \Big[v^i\big(t,[1-(\gamma_{\cdot,p}^i)^\top.\pi]x\big) - v^i(t,x)\Big] \\ & + \vartheta^{i,1-i}(\pi,c)\Big[v^{1-i}\big(t,[1-(\ell^{i,1-i})^\top.\pi]x\big) - v^i(t,x)\Big]\Big\}\end{aligned} \tag{3.9}$$

for $(t,x) \in [0,T) \times (0,\infty)$ and $i = 0,1$, subject to the boundary conditions

$$v^i(T,x) = u_2(x),\ \ x \in (0,\infty) \text{ for } i = 0,1. \tag{3.10}$$

Notice that the derivation of the HJB-system works completely analogously to the one in Section 2.1.3.

The following theorem is the main result of this section.

Theorem 3.3 (Verification Theorem)**.** *Suppose that* (v^0, v^1) *is a solution to the HJB-system (3.9) subject to (3.10), and assume that* $|v^i(t,x)| \le K(1+\frac{1}{x})^\kappa$, $x \in (0,\delta)$ *and* $|v^i(t,x)| \le K(1+x)^\kappa$, $x > \frac{1}{\delta}$ *for* $i = 0,1$ *for some constants* $K, \kappa, \delta > 0$.

i) For any admissible strategy $(\pi,c) \in \mathcal{A}$ *and all* $t_0 \in [0,T]$, $x_0 \in (0,\infty)$, *we have*

$$\mathbb{E}^{\pi,c}\Big[\textstyle\int_{t_0}^T u_1(t,c^{I(t)}(t)X^{\pi,c}(t))\mathrm{d}t + u_2(X^{\pi,c}(T))\,|\,I(t_0)=i,\ X^{\pi,c}(t_0)=x_0\Big] \le v^i(t_0,x_0) \text{ for } i=0,1.$$

ii) If there exists an admissible strategy $(\pi^*,c^*) = ((\pi^{0,*},c^{0,*}),(\pi^{1,*},c^{1,*}))$ *such that*

$$(\pi^{i,*}(t,x), c^{i,*}(t,x)) \in \underset{(\pi,c)\in\Pi\times\mathbb{R}_0^+}{\arg\max}\ H^i(t,x,\pi,c) \text{ for } t \in [0,T),\ x \in (0,\infty)$$

where

$$\begin{aligned}H^i(t,x,\pi,c) \triangleq\ & u_1(t,cx) + v_t^i(t,x) + (r^i + \pi^\top.\eta^i - c)xv_x^i(t,x) + \tfrac{1}{2}\pi^\top.\sigma^i.(\sigma^i)^\top.\pi x^2 v_{xx}^i(t,x) \\ & + \sum_{p=1}^{\bar{p}} \lambda_p \Big[v^i\big(t,[1-(\gamma_{\cdot,p}^i)^\top.\pi]x\big) - v^i(t,x)\Big] \\ & + \vartheta^{i,1-i}(\pi,c)\Big[v^{1-i}\big(t,[1-(\ell^{i,1-i})^\top.\pi]x\big) - v^i(t,x)\Big],\end{aligned} \tag{3.11}$$

then it follows that

$$v^i(t_0,x_0) = \sup_{(\pi,c)\in\mathcal{A}} \mathbb{E}^{\pi,c}\Big[\textstyle\int_{t_0}^T u_1(t,c^{I(t)}(t)X^{\pi,c}(t))\mathrm{d}t + u_2(X^{\pi,c}(T))\,|\,I(t_0)=i,\ X^{\pi,c}(t_0)=x_0\Big]$$

for $i = 0,1$ *and all* $t_0 \in [0,T]$ *and* $x_0 \in (0,\infty)$, *and* (π^*,c^*) *is an optimal strategy for the investment problem (P).*

Proof. Let an arbitrary admissible strategy $(\pi, c) = ((\pi^0, c^0), (\pi^1, c^1))$ be given, and suppose without loss of generality that $t_0 = 0$ and $I(0) = 0$. We denote by $\tilde{N}$ the compensated Poisson process associated to N, i.e.

$$\tilde{N}(t) \triangleq N(t) - \lambda t,\ \ t \in [0, T],$$

and similarly for $\tilde{N}^{i,1-i}$. Using Itô's formula we obtain for every stopping time θ with $\theta \le T$

$$\begin{aligned} &v^{I(\theta)}(\theta, X^{\pi,c}(\theta)) + \textstyle\int_0^\theta u_1(s-, c^{I(s-)}(s)X^{\pi,c}(s-))\mathrm{d}s \\ &\quad = v^{I(0)}(0, x_0) + \textstyle\int_0^\theta H^{I(s-)}(s-, X^{\pi,c}(s-), \pi^{I(s-)}(s), c^{I(s-)}(s))\mathrm{d}s \\ &\qquad + \textstyle\int_0^\theta \pi^{I(s-)}(s)^\top.\sigma^{I(s-)} X^{\pi,c}(s-) v_x^{I(s-)}(s-, X^{\pi,c}(s-)).\mathrm{d}W(s) \\ &\qquad + \sum_{p=1}^{\bar{p}} \textstyle\int_0^\theta \Big[v^{I(s-)}\left(s-, [1 - (\gamma_{\cdot,p}^{I(s-)})^\top.\pi^{I(s-)}(s)]X^{\pi,c}(s-)\right) - v^{I(s-)}(s-, X^{\pi,c}(s-))\Big]\mathrm{d}\tilde{N}^p(s) \\ &\qquad + \sum_{i=0}^{1} \textstyle\int_0^\theta 1_{\{I(s-)=i\}} \Big[v^{1-i}\left(s-, [1 - (\ell^{i,1-i})^\top.\pi^i(s)]X^{\pi,c}(s-)\right) - v^i(s-, X^{\pi,c}(s-))\Big]\mathrm{d}\tilde{N}^{i,1-1}(s). \end{aligned}$$

We need to show that the local martingale terms in fact represent martingales. For this purpose, we use a localization technique and set

$$O_q \triangleq \{x \in (0, \infty) \,:\, \tfrac{1}{q} < x < q\} \text{ and } Q_q \triangleq [0, T - \tfrac{1}{q}) \times O_q$$

for $q > \frac{1}{T}$. We denote by τ_q the exit time of $\{(t, X^{\pi,c}(t))\}$ from Q_q and let $\theta_q \triangleq T \wedge \tau_q$. As $X^{\pi,c}(\theta_q -) \in O_q$ and $\ell^{i,1-i} \in (-\infty, 1)^{\bar{n}}$, $\gamma^i \in (-\infty, 1)^{\bar{n}\times\bar{p}}$ for $i = 0, 1$, there exists an $r > q$ such that $X^{\pi,c}(\theta_q) \in O_r$. It follows that the integrands of the local martingale terms are bounded on $[\![0, \theta_q]\!]$, and hence

$$\begin{aligned} &\mathbb{E}^{\pi,c}\Big[v^{I(\theta_q)}(\theta_q, X^{\pi,c}(\theta_q)) + \textstyle\int_0^{\theta_q} u_1(s-, c^{I(s-)}(s)X^{\pi,c}(s-))\mathrm{d}s\Big] \\ &\quad = v^{I(0)}(0, x_0) + \mathbb{E}^{\pi,c}\Big[\textstyle\int_0^{\theta_q} H^{I(s-)}(s-, X^{\pi,c}(s-), \pi^{I(s-)}(s), c^{I(s-)}(s))\mathrm{d}s\Big]. \end{aligned}$$

Since v solves the HJB-system (3.9), we get

$$v^{I(0)}(0, x_0) \ge \mathbb{E}^{\pi,c}\Big[v^{I(\theta_q)}(\theta_q, X^{\pi,c}(\theta_q))\Big] + \mathbb{E}^{\pi,c}\Big[\textstyle\int_0^{\theta_q} u_1(s-, c(s)^{I(s-)}X^{\pi,c}(s-))\mathrm{d}s\Big],$$

for any $q > \frac{1}{T}$.

Since $\theta_q \to T$ as $q \to \infty$ a.s., we get together with the continuity of v^i for $i = 0, 1$ and the fact that $v^0(T, X^{\pi,c}(T)) = v^1(T, X^{\pi,c}(T)) = u_2(X^{\pi,c}(T))$ by (3.10) that

$$\lim_{q\to\infty} v^{I(\theta_q)}(\theta_q, X^{\pi,c}(\theta_q)) = v^{I(T)}(T, X^{\pi,c}(T)) \text{ a.s.}$$

Further the continuity of u_1 yields

$$\lim_{q\to\infty} \textstyle\int_0^{\theta_q} u_1(s-, c^{I(s-)}(s)X^{\pi,c}(s-))\mathrm{d}s = \int_0^T u_1(s-, c^{I(s-)}(s)X^{\pi,c}(s-))\mathrm{d}s \text{ a.s.}$$

The polynomial boundedness assumptions on v imply that $|v^i(t, x)| \le \bar{K} \max\{x^{-\bar{\kappa}}, x^{\bar{\kappa}}\}$ for $i = 0, 1$ and some constants $\bar{K}, \bar{\kappa} > 0$. For $\bar{v}^{i,k}(x) \triangleq \bar{K}x^k$ with $k \in \mathbb{R}$ Itô's formula yields

$$\mathrm{d}\bar{v}^{I(t),k}(X^{\pi,c}(t)) = \bar{v}^{I(t-),k}(X^{\pi,c}(t-))\left[A(t)\mathrm{d}t + B(t).\mathrm{d}W(t) + \sum_{p=1}^{\bar{p}} C^p(t)\mathrm{d}N^p(t) + \sum_{i=0}^{1} D^i(t)\mathrm{d}N^{i,1-i}(t)\right],$$

where

$$A(t) \triangleq k\big(r^{I(t-)} + (\pi^{I(t-)}(t))^\top.\eta^{I(t-)} - c^{I(t-)}(t)\big) + \tfrac{1}{2}k(k-1)(\pi^{I(t-)}(t))^\top.\sigma^{I(t-)}.(\sigma^{I(t-)})^\top.\pi^{I(t-)}(t),$$
$$B(t) \triangleq k\pi^{I(t-)}(t).\sigma^{I(t-)},\ C^p(t) \triangleq [1-(\gamma^{I(t-)}_{\cdot,p})^\top.\pi^{I(t-)}(t)]^k - 1,$$
$$D^i(t) \triangleq 1_{\{I(t-)=i\}}\Big[[1-(\ell^{i,1-i})^\top.\pi^i(t)]^k - 1\Big].$$

So $\bar{v}^{I(t),k}(X^{\pi,c}(t))$ can be written as a stochastic exponential, i.e.

$$\begin{aligned}\bar{v}^{I(t),k}(X^{\pi,c}(t)) &= \mathcal{E}_t\Big(\textstyle\int_0^\cdot A(s)\mathrm{d}s + \int_0^\cdot B(s).\mathrm{d}W(s) + \sum_{p=1}^{\bar{p}} \int_0^\cdot C^p(s)\mathrm{d}N^p(s) + \sum_{i=0}^{1} \int_0^\cdot D^i(s)\mathrm{d}N^{i,1-i}(s)\Big) \\ &= \exp\big(\textstyle\int_0^t A(s)\mathrm{d}s\big)\mathcal{E}_t\big(\int_0^\cdot B(s).\mathrm{d}W(s)\big)\prod_{p=1}^{\bar{p}} \prod_{\substack{s\in(0,t],\\ \Delta N^p(s)\neq 0}} (1+C^p(s)) \prod_{i=0}^{1} \prod_{\substack{s\in(0,t],\\ \Delta N^{i,1-i}(s)\neq 0}} (1+D^i(s)).\end{aligned}$$

By our assumptions on admissible strategies and the fact that $\ell^{i,1-i} \in (-\infty,1)^{\bar{n}}$ as well as $\gamma^i \in (-\infty,1)^{\bar{n}\times\bar{p}}$ for $i=0,1$ the integrands $A(t), B(t), C^p(t), D^0(t)$ and $D^1(t)$ are predictable and bounded; $C^p(t), D^0(t)$ and $D^1(t)$ only attain values in $(-1,\infty)$. Hence, let $\tilde{\kappa} > 0$ such that $|A(t)|$, $|B(t)|$, $|C^p(t)|$, $|D^0(t)|$, $|D^1(t)| \le \tilde{\kappa}$ for $t \in [0,T]$, $p=1,...,\bar{p}$. Then by Novikov's condition, $\mathcal{E}_t\big(\int_0^\cdot B(s).\mathrm{d}W(s)\big)$ is an $\mathrm{L}^2(\mathbb{P}^{\pi,c})$-martingale. Further, let

$$\xi_p = \tfrac{1}{2(\bar{p}+2)},\ p=1,...,\bar{p},\ \xi^{i,1-i} = \tfrac{1}{2(\bar{p}+2)},\ i=0,1,$$

i.e. $\frac{1}{2} + \sum_{p=1}^{\bar{p}} \xi_p + \sum_{i=0}^{1} \xi^{i,1-i} = 1$. Thus Hölder's and Doob's inequality yield

$$\begin{aligned}\mathbb{E}^{\pi,c}\left[\sup_{t\in[0,T]} \bar{v}^{I(t),k}(X^{\pi,c}(t))\right] &\le e^{\tilde{\kappa}T}\mathbb{E}^{\pi,c}\left[\sup_{t\in[0,T]} \mathcal{E}_t\big(\textstyle\int_0^\cdot B(s).\mathrm{d}W(s)\big)^2\right]^{\frac{1}{2}} \prod_{p=1}^{\bar{p}} \mathbb{E}^{\pi,c}\left[\sup_{t\in[0,T]} (1+\tilde{\kappa})^{\xi_p N^p(t)}\right]^{\frac{1}{\xi_p}} \\ &\quad \cdot \prod_{i=0}^{1} \mathbb{E}^{\pi,c}\left[\sup_{t\in[0,T]} (1+\tilde{\kappa})^{\xi^{i,1-i}N^{i,1-i}(t)}\right]^{\frac{1}{\xi^{i,1-i}}} \\ &\le 2e^{\tilde{\kappa}T}\mathbb{E}^{\pi,c}\left[\mathcal{E}_T\big(\textstyle\int_0^\cdot B(s).\mathrm{d}W(s)\big)^2\right]^{\frac{1}{2}} \prod_{p=1}^{\bar{p}} \mathbb{E}^{\pi,c}\left[(1+\tilde{\kappa})^{\xi_p N^p(T)}\right]^{\frac{1}{\xi_p}} \\ &\quad \cdot \prod_{i=0}^{1} \mathbb{E}^{\pi,c}\left[(1+\tilde{\kappa})^{\xi^{i,1-i}N^{i,1-i}(T)}\right]^{\frac{1}{\xi^{i,1-i}}}.\end{aligned}$$

Since N is an $(\mathfrak{F}(\cdot), \mathbb{P}^{\pi,c})$-Poisson process with intensity λ, we get

$$\mathbb{E}^{\pi,c}\left[(1+\tilde{\kappa})^{\xi_p N^p(T)}\right] = e^{((1+\tilde{\kappa})^{\xi_p}-1)T\lambda_p} < \infty,\ \text{for } p=1,...,\bar{p}.$$

Moreover,

$$\mathbb{E}^{\pi,c}\left[(1+\tilde{\kappa})^{\xi^{i,1-i}N^{i,1-i}(T)}\right] = \mathbb{E}\left[Z^{\pi,c}(T)(1+\tilde{\kappa})^{\xi^{i,1-i}N^{i,1-i}(T)}\right],$$

with $Z^{\pi,c}(T) = \frac{\mathrm{d}\mathbb{P}^{\pi,c}}{\mathrm{d}\mathbb{P}}$ as given in (3.5). Thus

$$\sup_{t\in[0,T]} \left| Z^{\pi,c}(t)(1+\tilde{\kappa})^{\xi^{i,1-i}N^{i,1-i}(t)} \right| \le e^{2T} \left[\max_{i=0,1,\,(\pi,c)\in\mathcal{A}} |\vartheta^{i,1-i}(\pi,c)| \right]^{N^{0,1}(T)+N^{1,0}(T)} (1+\tilde{\kappa})^{\xi^{i,1-i}N^{i,1-i}(T)}$$

where the right-hand side is in $\mathrm{L}^1(\mathbb{P})$ as $N^{i,1-i}$ is a $(\mathfrak{F}(\cdot),\mathbb{P})$-Poisson process.

Hence, $\mathbb{E}^{\pi,c}\left[(1+\tilde{\kappa})^{\xi^{i,1-i}N^{i,1-i}(T)}\right] < \infty$ and finally

$$\mathbb{E}^{\pi,c}\left[\sup_{t\in[0,T]} \bar{v}^{I(t),k}(X^{\pi,c}(t))\right] < \infty.$$

Since $|v^i(t,x)| \le \max\{\bar{v}^{i,-\bar{\kappa}}(x), \bar{v}^{i,\bar{\kappa}}(x)\}$ the family $\{v^{I(\theta_q)}(\theta_q, X^{\pi,c}(\theta_q))\}_{q>\frac{1}{T}}$ is uniformly integrable and we conclude that

$$\lim_{q\to\infty} \mathbb{E}^{\pi,c}[v^{I(\theta_q)}(\theta_q, X^{\pi,c}(\theta_q))] = \mathbb{E}^{\pi,c}[v^{I(T)}(T, X^{\pi,c}(T))].$$

It remains to show that

$$\lim_{q\to\infty} \mathbb{E}^{\pi,c}[\int_0^{\theta_q} u_1(s-, c^{I(s-)}(s)X^{\pi,c}(s-))\mathrm{d}s] = \mathbb{E}^{\pi,c}[\int_0^T u_1(s-, c^{I(s-)}(s)X^{\pi,c}(s-))\mathrm{d}s].$$

Therefore, we consider the positive and the negative part of u_1 separately.

Since $\int_0^{\theta_q} u_1(s-, c^{I(s-)}(s)X^{\pi,c}(s-))^+\mathrm{d}s \nearrow \int_0^T u_1(s-, c^{I(s-)}(s)X^{\pi,c}(s-))^+\mathrm{d}s$ as $q\to\infty$ a.s. the monotone convergence theorem yields

$$\lim_{q\to\infty} \mathbb{E}^{\pi,c}[\int_0^{\theta_q} u_1(s-, c^{I(s-)}(s)X^{\pi,c}(s-))^+\mathrm{d}s] = \mathbb{E}^{\pi,c}[\int_0^T u_1(s-, c^{I(s-)}(s)X^{\pi,c}(s-))^+\mathrm{d}s].$$

Further, since $\int_0^{\theta_q} u_1(s-, c^{I(s-)}(s)X^{\pi,c}(s-))^-\mathrm{d}s \nearrow \int_0^T u_1(s-, c^{I(s-)}(s)X^{\pi,c}(s-))^-\mathrm{d}s$ as $q\to\infty$ a.s. and $\mathbb{E}^{\pi,c}[\int_0^T u_1(s-, c^{I(s-)}(s)X^{\pi,c}(s-))^-\mathrm{d}s] < \infty$ for any admissible strategy, we get by the dominated convergence theorem that

$$\lim_{q\to\infty} \mathbb{E}^{\pi,c}[\int_0^{\theta_q} u_1(s-, c^{I(s-)}(s)X^{\pi,c}(s-))^-\mathrm{d}s] = \mathbb{E}^{\pi,c}[\int_0^T u_1(s-, c^{I(s-)}(s)X^{\pi,c}(s-))^-\mathrm{d}s].$$

Thus

$$\lim_{q\to\infty} \mathbb{E}^{\pi,c}[\int_0^{\theta_q} u_1(s-, c^{I(s-)}(s)X^{\pi,c}(s-))\mathrm{d}s] = \mathbb{E}^{\pi,c}[\int_0^T u_1(s-, c^{I(s-)}(s)X^{\pi,c}(s-))\mathrm{d}s]$$

is shown.

Hence, we have $\mathbb{E}^{\pi,c}[\int_0^T u_1(t, c^{I(t)}(t)X^{\pi,c}(t))\mathrm{d}t + u_2(X^{\pi,c}(T))] \le v^0(0,x_0)$ for any admissible strategy $(\pi,c)\in\mathcal{A}$, and part i) is established.

Part ii) now follows from the above argument if we note that the strategy (π^*, c^*) attains equality in the preceding arguments. This completes the proof. □

Remark 3.4. *The Verification Theorem ensures that in order to solve the investment problem (P) it suffices to determine the strategy that maximizes the right-hand side of the HJB-system (3.9) and to verify that the corresponding PDE-system admits a unique global solution that satisfies the aforementioned regularity conditions.*

4 Solution for CRRA Investors with Power Utility

In this section we derive the solution of the investment problem (P) for a large investor with power utility in a market without price jumps. First the particular HJB-system is specified. Thereafter we reduce this PDE-system to a simpler ODE-system via choosing a suitable ansatz for the value function v^i. Finally we present a detailed solution of the investment problem for three different kinds of intensity functions.

HJB-System. Throughout this section we suppose that there are no jumps in the asset price dynamics,

$$\gamma^i = 0 \text{ and } \ell^{i,1-i} = 0 \text{ for } i = 0, 1. \tag{NJ}$$

Moreover, we let

$$\Pi = \mathbb{R}^{\bar{n}}$$

and apply the following convention to extend the natural logarithm on $\mathbb{R}$

$$\ln(x) \triangleq -\infty \text{ for } x \in (-\infty, 0].$$

Under assumption (NJ) the HJB-system (3.9) simplifies to

$$\begin{aligned} 0 = \sup_{(\pi,c)\in\mathbb{R}^{\bar{n}}\times\mathbb{R}_0^+} \Big\{ & u_1(t, cx) + v_t^i(t, x) + (r^i + \pi^\top.\eta^i - c)xv_x^i(t, x) \\ & + \tfrac{1}{2}\pi^\top.\sigma^i.(\sigma^i)^\top.\pi x^2 v_{xx}^i(t, x) + \vartheta^{i,1-i}(\pi, c)[v^{1-i}(t, x) - v^i(t, x)] \Big\} \end{aligned} \tag{4.1}$$

for $(t, x) \in [0, T) \times (0, \infty)$ and $i = 0, 1$, subject to the boundary conditions

$$v^i(T, x) = u_2(x),\ x \in (0, \infty) \text{ for } i = 0, 1. \tag{4.2}$$

We assume that the investor's preferences are specified by CRRA utility functions of the form

$$\begin{aligned} u_1(t, x) &= \varepsilon e^{-\delta t} \tfrac{1}{1-R}(x^{1-R} - 1),\ t \in [0, T],\ x \in (0, \infty),\ \varepsilon \in [0, \infty), \\ u_2(x) &= e^{-\delta T} \tfrac{1}{1-R}(x^{1-R} - 1),\ x \in (0, \infty), \end{aligned} \tag{4.3}$$

where $\delta > 0$ is the utility discount factor that accommodates the chronological structure of consumption and terminal wealth and $R \in (0, \infty) \setminus \{1\}$ denotes the investor's relative risk aversion; see Section 5 for the case $R = 1$ of logarithmic utility. Choosing $\varepsilon = 1$ refers to the general problem of maximizing expected utility from terminal wealth and from intermediate consumption as well without preferring one to the other. The pure portfolio problem where the large investor solely gains utility from terminal wealth corresponds to $\varepsilon = 0$.

Remark 4.1. *Usually it would suffice to consider utility functions of the form*

$$\begin{aligned} u_1(t, x) &= \varepsilon e^{-\delta t} \tfrac{1}{1-R} x^{1-R},\ t \in [0, T],\ x \in (0, \infty),\ \varepsilon \in [0, \infty), \\ u_2(x) &= e^{-\delta T} \tfrac{1}{1-R} x^{1-R},\ x \in (0, \infty), \end{aligned}$$

since those just differ by constants from the utility functions (4.3). But the functions (4.3) have the advantage that those converge to the logarithmic utility functions as R tends to 1 which turns out to be quite convenient for the solution of the investment problem for an investor with logarithmic utility.

Notice that we take the supremum over $(\pi, c) \in \mathbb{R}^{\bar{n}} \times \mathbb{R}_0^+$ in the HJB-system (4.1), i.e. we allow c to be zero even though $u_1(t, cx)$ is not defined for $c = 0$ if $\varepsilon > 0$ since $\frac{1}{1-R}((cx)^{1-R} - 1)$ would involve a division by 0 in case of $R > 1$. If $\varepsilon = 0$ then $u_1(t, cx) = 0$ for all $c, x \in \mathbb{R}_0^+$ and the problem does not emerge. Further if $R < 1$ in case of $\varepsilon > 0$ then there is no problem either as then $0^{1-R} = 0$. Only if $\varepsilon > 0$ and $R > 1$ then $u_1(t, cx)$ is indefinite for $c = 0$. For notational convenience we maintain the maximization over $(\pi, c) \in \mathbb{R}^{\bar{n}} \times \mathbb{R}_0^+$ instead of discussing the two cases $(\pi, c) \in \mathbb{R}^{\bar{n}} \times \mathbb{R}^+$ $(\varepsilon > 0)$ and $(\pi, c) \in \mathbb{R}^{\bar{n}} \times \mathbb{R}_0^+$ $(\varepsilon = 0)$ separately. This is possible since we can overcome the problem of indefiniteness using the convention $0^{1-R} \triangleq \infty$ if $R > 1$ which implies $\frac{1}{1-R}((cx)^{1-R} - 1) = -\infty$. So as we are looking for the supremum in (4.1) it does not cause any trouble to take the supremum over $(\pi, c) \in \mathbb{R}^{\bar{n}} \times \mathbb{R}_0^+$ instead of $(\pi, c) \in \mathbb{R}^{\bar{n}} \times \mathbb{R}^+$ in case of $\varepsilon > 0$.

Finally, letting

$$\Psi^i \triangleq r^i + \frac{1}{2}\frac{1}{R}(\eta^i)^\top.(\sigma^i.(\sigma^i)^\top)^{-1}.\eta^i, \ i = 0, 1$$

denote the utility growth potentials of the markets in state 0 and 1, respectively, we assume throughout that the markets are labeled in such a way that

$$\Psi^0 > \Psi^1,$$

i.e. a Merton investor without market impact would prefer state 0 to state 1.

Reduced HJB-System. In order to solve the HJB-system (4.1), we conjecture

$$v^0(t, x) = \frac{1}{1-R} f(t)\big(\big(xe^{g(t)}\big)^{1-R} - 1\big), \quad v^1(t, x) = \frac{1}{1-R} f(t)\big(\big(xe^{g(t)-h(t)}\big)^{1-R} - 1\big) \tag{4.4}$$

for $(t, x) \in [0, T] \times (0, \infty)$ with C^1-functions f, g and h on $[0, T]$.

Remark 4.2 (Motivation for the ansatz)**.** *This particular ansatz mainly consists of two parts. The first one is the* time-dependent factor $f(t)$ *and the second one are the* time-dependent discount resp. accumulation factors $e^{g(t)-1_{\{i=1\}}h(t)}$.

The factor function f is necessary for the ansatz to yield a separation of the variables t and x since the model includes consumption. If the investor was not allowed to consume then one could choose $f(t) \triangleq 1$ and the ansatz would also work. To see the necessity of the factor function just set $f(t) = 1$ in equation (4.5) below. As long as $\varepsilon > 0$ there would remain just one term with x, i.e. the ansatz would fail.

Further the special form of the discount resp. accumulation factors is due to the major requirement in our model that the two market states are different concerning the large investor's point of view. This difference is accommodated by the function h. Since market state 1 is worse than state 0 one would expect that the wealth in state 1 is discounted at a higher rate, resp. accumulated at a smaller rate than in state 0. Aiming at h to be non-negative we therefore subtract h from g in the ansatz for the wealth function v^1. Later on it turns out that h is really non-negative (Lemma 4.3).

Inserting the ansatz (4.4) into (4.1), (4.2) and dividing the whole equation by the positive term $\big(xe^{g(t)-1_{\{i=1\}}h(t)}\big)^{1-R}$ yields

$$
\begin{aligned}
0 = \sup_{(\pi,c)\in\mathbb{R}^{\bar{n}}\times\mathbb{R}_0^+} \Big\{ & \tfrac{1}{1-R}\varepsilon e^{-\delta t}(c^{1-R} - x^{-(1-R)})e^{-(1-R)(g(t)-1_{\{i=1\}}h(t))} \\
& + \tfrac{1}{1-R} f'(t)(1 - x^{-(1-R)}e^{-(1-R)(g(t)-1_{\{i=1\}}h(t))}) \\
& + f(t)\Big(g'(t) - 1_{\{i=1\}}h'(t) + r^i + \pi^\top.\eta^i - \tfrac{1}{2}R\pi^\top.\sigma^i.(\sigma^i)^\top.\pi - c \\
& + \vartheta^{i,1-i}(\pi,c)\tfrac{1}{1-R}(e^{(-1)^{1-i}(1-R)h(t)} - 1)\Big)\Big\}
\end{aligned}
\tag{4.5}
$$

for $(t,x) \in [0,T) \times (0,\infty)$ and $i = 0,1$, subject to the boundary conditions

$$
f(T) = e^{-\delta T},\ g(T) = 0,\ h(T) = 0. \tag{4.6}
$$

Collecting the terms multiplying $x^{-(1-R)}$ we get the following ODE for the function f

$$
f'(t) = -\varepsilon e^{-\delta t},\ f(T) = e^{-\delta T}
$$

which has the solution

$$
f(t) = \tfrac{1}{\delta}e^{-\delta t}(\varepsilon - (\varepsilon - \delta)e^{-\delta(T-t)}). \tag{4.7}
$$

The remaining terms without x yield the **reduced HJB-system**

$$
\begin{aligned}
0 = \sup_{(\pi,c)\in\mathbb{R}^{\bar{n}}\times\mathbb{R}_0^+} \Big\{ & g'(t) - 1_{\{i=1\}}h'(t) + r^i + \pi^\top.\eta^i - \tfrac{1}{2}R\pi^\top.\sigma^i.(\sigma^i)^\top.\pi \\
& + \tfrac{\varepsilon\delta}{\varepsilon-(\varepsilon-\delta)e^{-\delta(T-t)}}\tfrac{1}{1-R}(e^{-(1-R)(g(t)-1_{\{i=1\}}h(t))}c^{1-R} - 1) - c \\
& + \vartheta^{i,1-i}(\pi,c)\tfrac{1}{1-R}(e^{(-1)^{1-i}(1-R)h(t)} - 1)\Big\}
\end{aligned}
\tag{4.8}
$$

for $t \in [0,T)$ and $i = 0,1$, subject to the boundary conditions

$$
g(T) = 0,\ h(T) = 0.
$$

The following lemma shows what we already expected when setting up the ansatz for the wealth function – the non-negativity of the function h.

Lemma 4.3 (Non-Negativity of h)**.** *If g and h are solutions of (4.8), then*

$$
h(t) \geq 0 \textit{ for every } t \in [0,T].
$$

Proof. The assertion follows from standard ODE arguments: If $h(t) = 0$ for some $t \in [0,T]$ then equation (4.8) implies

$$
\begin{aligned}
0 = & \sup_{(\pi,c)\in\mathbb{R}^{\bar{n}}\times\mathbb{R}_0^+} \Big\{ g'(t) + r^0 + \pi^\top.\eta^0 - \tfrac{1}{2}R\pi^\top.\sigma^0.(\sigma^0)^\top.\pi \\
& \qquad + \tfrac{\varepsilon\delta}{\varepsilon-(\varepsilon-\delta)e^{-\delta(T-t)}}\tfrac{1}{1-R}(e^{-(1-R)g(t)}c^{1-R} - 1) - c\Big\} \\
& - \sup_{(\pi,c)\in\mathbb{R}^{\bar{n}}\times\mathbb{R}_0^+} \Big\{ g'(t) - h'(t) + r^1 + \pi^\top.\eta^1 - \tfrac{1}{2}R\pi^\top.\sigma^1.(\sigma^1)^\top.\pi \\
& \qquad + \tfrac{\varepsilon\delta}{\varepsilon-(\varepsilon-\delta)e^{-\delta(T-t)}}\tfrac{1}{1-R}(e^{-(1-R)g(t)}c^{1-R} - 1) - c\Big\} \\
= & \ h'(t) + \Psi^0 - \Psi^1,
\end{aligned}
$$

and thus $h'(t) < 0$ as we required $\Psi^0 > \Psi^1$; since $h(T) = 0$, it follows that $h(t) \geq 0$ for all $t \in [0,T]$. □

Remark 4.4 (Interpretation of h)**.** *In order to understand the role of the function h we have to take a closer look on the ansatz for the value function $v^i(t,x)$ given by*

$$v^0(t,x) = \tfrac{1}{1-R} f(t)\big(\big(xe^{g(t)}\big)^{1-R} - 1\big), \quad v^1(t,x) = \tfrac{1}{1-R} f(t)\big(\big(xe^{g(t)-h(t)}\big)^{1-R} - 1\big).$$

Hence an investor being endowed with an initial wealth of x_0^0 when the market is in state 0 at time t would at the same time need a wealth of $x_0^1 = x_0^0 e^{h(t)}$ if the market was in state 1 in order to achieve the same expected utility from terminal wealth since $v^0(t,x_0^0) = v^1(t,x_0^0 e^{h(t)})$. Thus e^h can be interpreted as the investor's exchange ratio between the two market states. Since h is influenced by the investor's risk aversion R the notion the investor's exchange ratio *is reasonable as another investor having a different relative risk aversion would exhibit a different exchange ratio. Further, the non-negativity of h implies that x_0^1 is always bigger than or equal x_0^0 when requiring $v^0(t,x_0^0) = v^1(t,x_0^1)$. This mirrors the fact that from the investor's point of view state 1 is the adverse market state as compared to state 0. Therefore h, resp. e^h, is measuring the difference between the two market states perceived by the investor; the bigger h the bigger the perceived difference.*

In the following we present the solution of the investment problem (P) for three types of intensity functions. The first one representing the two-states version of the Bäuerle-Rieder investment problem are **constant intensity functions** which we deal with in Subsection 4.1. The second type corresponding to an *indirect* dependency on the investor's strategy are the **step intensity functions**. Those are discussed in Subsection 4.2. The third, *direct* type are the **affine intensity functions** presented in Subsection 4.3. In each subsection we dwell on different variants of the respective intensity functions and solve the related optimal investment problem.

4.1 Constant Intensity Functions

This subsection serves to transfer the two-states version of the Bäuerle-Rieder investment problem into the aforementioned nomenclature concerning the utility functions and the ansatz for the value function so that the optimal strategy and the related value function are comparable to the respective strategies and value functions of the investment problems with step, resp. affine intensity functions below.

We let the intensity functions $\vartheta^{i,1-i}$ be given as constants, i.e.

$$\vartheta^{i,1-i}(\pi,c) = C^i, \ (\pi,c) \in \mathbb{R}^{\bar{n}} \times \mathbb{R}_0^+ \tag{CI}$$

with $C^i \in \mathbb{R}_0^+$ for $i = 0,1$.

Inserting those constant intensity functions into the reduced HJB-system (4.8) the latter becomes

$$\begin{aligned} 0 = \sup_{(\pi,c)\in\mathbb{R}^{\bar{n}}\times\mathbb{R}_0^+} \Big\{ & g'(t) - 1_{\{i=1\}}h'(t) + r^i + \pi^\top.\eta^i - \tfrac{1}{2}R\pi^\top.\sigma^i.(\sigma^i)^\top.\pi \\ & + \tfrac{\varepsilon\delta}{\varepsilon-(\varepsilon-\delta)e^{-\delta(T-t)}} \tfrac{1}{1-R}\big(e^{-(1-R)(g(t)-1_{\{i=1\}}h(t))}c^{1-R} - 1\big) - c \\ & + C^i \tfrac{1}{1-R}\big(e^{(-1)^{1-i}(1-R)h(t)} - 1\big)\Big\} \end{aligned} \tag{4.9}$$

for $t \in [0,T)$ and $i = 0,1$, subject to the boundary conditions

$$g(T) = 0, \ h(T) = 0.$$

In the following we present the solution of the investment problem with constant regime shift intensities in general (Subsection 4.1.1) at first. Afterwards we dwell on the classical Merton investment problem with shift intensities $\vartheta^{i,1-i} = 0$ which we solve explicitly (Subsection 4.1.2).

4.1.1 Solution of the Investment Problem

In order to determine the maximizer in the HJB-system (4.9) we define functions $H^{\pi,i}: \mathbb{R}^{\bar{n}} \to \mathbb{R}$ and $H^{c,i}: [0,T] \times \mathbb{R} \times \mathbb{R}_0^+ \to \mathbb{R}$, $i = 0, 1$, given by

$$H^{\pi,i}(\pi) \triangleq r^i + \pi^\top.\eta^i - \tfrac{1}{2}R\pi^\top.\sigma^i.(\sigma^i)^\top.\pi,$$
$$H^{c,i}(t,x,c) \triangleq \tfrac{\varepsilon\delta}{\varepsilon-(\varepsilon-\delta)e^{-\delta(T-t)}}\tfrac{1}{1-R}(e^{-(1-R)x}c^{1-R} - 1) - c$$

where we use the already mentioned convention $0^{1-R} \triangleq \infty$ if $R > 1$. Notice that function $H^{c,i}$ is independent of t and x if $\varepsilon = 0$. Notice here that x is just a general variable independent of the investor's wealth.

Using those auxiliary functions the HJB-system (4.9) can be written as

$$0 = \sup_{(\pi,c)\in\mathbb{R}^{\bar{n}}\times\mathbb{R}_0^+} \Big\{ g'(t) - 1_{\{i=1\}}h'(t) + H^{\pi,i}(\pi) + H^{c,i}(t, g(t) - 1_{\{i=1\}}h(t), c) + C^i\tfrac{1}{1-R}(e^{(-1)^{1-i}(1-R)h(t)} - 1)\Big\} \tag{4.10}$$

for $t \in [0,T)$ and $i = 0, 1$, subject to the boundary conditions

$$g(T) = 0,\ h(T) = 0.$$

Writing the HJB-system in that way it is obvious that the maximization over $(\pi, c) \in \mathbb{R}^{\bar{n}} \times \mathbb{R}_0^+$ can be separated into two unrelated maximizations; one over $\pi \in \mathbb{R}^{\bar{n}}$ and one over $c \in \mathbb{R}_0^+$.

To find the supremum in (4.10) we present the maximizers of the functions $H^{\pi,i}$ and $H^{c,i}(t,x,\cdot)$ for arbitrary $(t,x) \in [0,T] \times \mathbb{R}$. This yields a family of maximizers dependent on (t,x). The maximizers of the HJB-system (4.10) are then obtained by replacing x by $g(t) - 1_{\{i=1\}}h(t)$.

The concavity of $H^{\pi,i}$ and $H^{c,i}(t,x,\cdot)$ and the first-order conditions imply

Lemma 4.5 (Maximizers of $H^{\pi,i}$ and $H^{c,i}(t,x,\cdot)$)**.** *For every $(t,x) \in [0,T] \times \mathbb{R}$ the maximizers*

$$\pi^{i,*} \triangleq \arg\max_{\pi\in\mathbb{R}^{\bar{n}}} H^{\pi,i}(\pi),\ \ c^{i,*}(t,x) \triangleq \arg\max_{c\in\mathbb{R}_0^+} H^{c,i}(t,x,c),\ \ i = 0, 1,$$

are given by the Merton strategy, i.e.

$$(\pi^{i,*}, c^{i,*}(t,x)) = (\pi^{i,M}, c^M(t,x))$$

where

$$\pi^{i,M} \triangleq \tfrac{1}{R}(\sigma^i.(\sigma^i)^\top)^{-1}.\eta^i, \quad c^M(t,x) \triangleq \big(\tfrac{\varepsilon\delta}{\varepsilon-(\varepsilon-\delta)e^{-\delta(T-t)}}\big)^{\frac{1}{R}}e^{-\frac{1-R}{R}x}.$$

Remark 4.6 (Merton consumption rate)**.** *In Lemma 4.5 we wrote that the maximizing consumption rate is* the *Merton consumption rate. However, one has to be careful with this formulation. In fact a unique Merton consumption rate – only determined by the parameters of the price dynamics r, η and σ, the investor's relative risk aversion R, the discount rate δ and the weighting factor ε – does not exist – except for the case of $\varepsilon = 0$ where $c^M(t,x) = 0$ for all $(t,x) \in [0,T] \times \mathbb{R}$ – whereas the Merton portfolio proportions π are unique. This comes from the fact that every consumption rate having the Merton form depends on the underlying functions g and h that are in turn determined by the whole model settings and not only by the parameters mentioned above. Thus the regime switching intensities are relevant, too. Hence we can only speak of consumption rates being of the* Merton type *having the form*

$$c^M(t,x) = \left(\tfrac{\varepsilon\delta}{\varepsilon-(\varepsilon-\delta)e^{-\delta(T-t)}}\right)^{\frac{1}{R}} e^{-\frac{1-R}{R}x}$$

for $(t,x) \in [0,T] \times \mathbb{R}$. If one was searching for a genuine representative of those Merton type consumption rates then a reasonable choice would be the optimal consumption rate in the classical Merton model with only one market state. This genuine model can be obtained via choosing $C^i = 0$ for $i = 0, 1$ which results in the following ODEs for h and g

$$h'(t) = -\left(\Psi^0 - \Psi^1\right) + \left(\tfrac{\varepsilon\delta}{\varepsilon-(\varepsilon-\delta)e^{-\delta(T-t)}}\right)^{\frac{1}{R}} e^{-\frac{1-R}{R}g(t)} \tfrac{R}{1-R}\left(e^{\frac{1-R}{R}h(t)} - 1\right), \tag{4.11}$$

$$g'(t) = -\Psi^0 + \tfrac{\varepsilon\delta}{\varepsilon-(\varepsilon-\delta)e^{-\delta(T-t)}}\Big(1 - \tfrac{R}{1-R}\Big(\Big(\tfrac{\varepsilon\delta}{\varepsilon-(\varepsilon-\delta)e^{-\delta(T-t)}}\Big)^{\frac{1-R}{R}} e^{-\frac{1-R}{R}g(t)} - 1\Big)\Big) \tag{4.12}$$

subject to the boundary conditions $h(T) = g(T) = 0$. Note that the two ODEs are uncoupled since ODE (4.12) for the function g is independent of the function h. Denoting by (g_{BSM}, h_{BSM}) the solution of the ODE-system (4.11), (4.12) the genuine Merton consumption rate is given by

$$c^{i,M}_{BSM}(t) \triangleq c^M(t, g_{BSM}(t) - 1_{\{i=1\}}h_{BSM}(t)) = \left(\tfrac{\varepsilon\delta}{\varepsilon-(\varepsilon-\delta)e^{-\delta(T-t)}}\right)^{\frac{1}{R}} e^{-\frac{1-R}{R}(g_{BSM}(t)-1_{\{i=1\}}h_{BSM}(t))}.$$

Later on in Subsection 4.1.2 we present the explicit solution of the Merton ODE-system (4.11), (4.12) and give the Merton consumption rate in explicit form.

If not otherwise stated we use $c^{i,M}(t)$ as shorthand notation for $c^M(t, g(t) - 1_{\{i=1\}}h(t))$ and simply call $c^{i,M}(t)$, $i = 0, 1$ the Merton consumption rate, *i.e.*

$$c^{i,M}(t) \triangleq c^M(t, g(t) - 1_{\{i=1\}}h(t)), \; i = 0, 1.$$

However one always should keep in mind the dependency of $c^{i,M}(t)$ on $g(t)$, resp. $g(t) - h(t)$. Only if $R = 1$ then the dependency on $g(t)$, resp. $g(t) - h(t)$ vanishes.

Remark 4.7. *So far the strategy $(\pi^{i,*}, c^{i,*})$ is only a candidate solution for the optimal investment problem. To verify the optimality we need to show that the related HJB-system has a global solution that satisfies the required regularity conditions stated in the Verification Theorem 3.3.*

Inserting the maximizing strategy $(\pi^{i,*}, c^{i,*}(t, g(t) - 1_{\{i=1\}}h(t)))$ into the reduced HJB-equation (4.10) the latter becomes a system of coupled backward ODEs

$$\begin{aligned} h'(t) = &- (\Psi^0 - \Psi^1) + \left(\tfrac{\varepsilon\delta}{\varepsilon-(\varepsilon-\delta)e^{-\delta(T-t)}}\right)^{\frac{1}{R}} e^{-\frac{1-R}{R}g(t)} \tfrac{R}{1-R}\left(e^{\frac{1-R}{R}h(t)} - 1\right) \\ &- C^0 \tfrac{1}{1-R}\left(e^{-(1-R)h(t)} - 1\right) + C^1 \tfrac{1}{1-R}\left(e^{(1-R)h(t)} - 1\right), \end{aligned} \tag{4.13}$$

$$\begin{aligned} g'(t) = &- \Psi^0 + \tfrac{\varepsilon\delta}{\varepsilon-(\varepsilon-\delta)e^{-\delta(T-t)}} \Big(1 - \tfrac{R}{1-R}\Big(\Big(\tfrac{\varepsilon\delta}{\varepsilon-(\varepsilon-\delta)e^{-\delta(T-t)}}\Big)^{\frac{1-R}{R}} e^{-\frac{1-R}{R}g(t)} - 1\Big)\Big) \\ &- C^0 \tfrac{1}{1-R}\left(e^{-(1-R)h(t)} - 1\right) \end{aligned} \tag{4.14}$$

subject to the boundary conditions

$$g(T) = 0, \; h(T) = 0. \tag{4.15}$$

Remark 4.8. *From an arithmetical point of view equation (4.14) could be further simplified, i.e. the term*

$$\tfrac{\varepsilon\delta}{\varepsilon-(\varepsilon-\delta)e^{-\delta(T-t)}} \Big(1 - \tfrac{R}{1-R}\Big(\Big(\tfrac{\varepsilon\delta}{\varepsilon-(\varepsilon-\delta)e^{-\delta(T-t)}}\Big)^{\frac{1-R}{R}} e^{-\frac{1-R}{R}g(t)} - 1\Big)\Big)$$

could be written as

$$\tfrac{1}{1-R} \tfrac{\varepsilon\delta}{\varepsilon-(\varepsilon-\delta)e^{-\delta(T-t)}} - \tfrac{R}{1-R}\Big(\tfrac{\varepsilon\delta}{\varepsilon-(\varepsilon-\delta)e^{-\delta(T-t)}}\Big)^{\frac{1}{R}} e^{-\frac{1-R}{R}g(t)},$$

too. But keeping the longer version makes it quite convenient to obtain the corresponding equations and formulas for the case of $R = 1$ using the convention

$$\tfrac{1}{1-R}\left(x^{(1-R)} - 1\right) \triangleq \ln(x), \; x > 0, \; \textit{for } R = 1,$$

as will be seen later on.

We now prove the existence of a solution of the two ODEs (4.13) and (4.14).

Lemma 4.9. *The ODE-system given by equation (4.13) and (4.14) subject to the boundary conditions (4.15) admits a unique global solution.*

Proof. The proof is structured as follows. We first show that the ODEs (4.13) and (4.14) have a unique local solution. Subsequently we verify that the right-hand sides of the ODEs for g, h and $g - h$ are suitably bounded implying that the local solution indeed is a global one.

- *Definition of auxiliary functions.* In order to achieve the above results we define the following auxiliary functions

$$\begin{aligned} \varrho : \; [0,T] \times \mathbb{R} \times \mathbb{R}_0^+ \to \mathbb{R}, \; & \varrho(t,x,y) \triangleq \Big(\tfrac{\varepsilon\delta}{\varepsilon-(\varepsilon-\delta)e^{-\delta(T-t)}}\Big)^{\frac{1}{R}} e^{-\frac{1-R}{R}x} \tfrac{R}{1-R}\left(e^{\frac{1-R}{R}y} - 1\right), \\ \varrho^i : \; [0,T] \times \mathbb{R} \to \mathbb{R}, \; & \varrho^i(t,x) \triangleq \tfrac{\varepsilon\delta}{\varepsilon-(\varepsilon-\delta)e^{-\delta(T-t)}} \Big(1 - \tfrac{R}{1-R}\Big(\Big(\tfrac{\varepsilon\delta}{\varepsilon-(\varepsilon-\delta)e^{-\delta(T-t)}}\Big)^{\frac{1-R}{R}} e^{-\frac{1-R}{R}x} - 1\Big)\Big), \\ \chi^i : \; \mathbb{R}_0^+ \to \mathbb{R}, \; & \chi^i(y) \triangleq (-1)^{1-i} C^i \tfrac{1}{1-R}\left(e^{(-1)^{1-i}(1-R)y} - 1\right) \end{aligned}$$

  and

$$\begin{aligned} F : \; [0,T] \times \mathbb{R} \times \mathbb{R}_0^+ \to \mathbb{R}, \; & F(t,x,y) \triangleq -(\Psi^0 - \Psi^1) + \varrho(t,x,y) + \chi^0(y) + \chi^1(y), \\ F^i : \; [0,T] \times \mathbb{R} \times \mathbb{R}_0^+ \to \mathbb{R}, \; & F^i(t,x,y) \triangleq -\Psi^i + \varrho^i(t,x) + (-1)^i \chi^i(y) \end{aligned}$$

$i = 0, 1$ so that h, g and $g - h$ satisfy

$$h'(t) = F(t, g(t), h(t)) \text{ and } g'(t) - 1_{\{i=1\}}h'(t) = F^i(t, g(t) - 1_{\{i=1\}}h(t), h(t)).$$

- *Existence of a unique local solution.* The functions ϱ and ϱ^i are obviously continuous in t and continuously differentiable in x and y. Moreover, χ^i is continuously differentiable in y, too. Therefore F and F^i are continuous in t and continuously differentiable in x and y which implies that they are locally Lipschitz continuous in x and y, too. Consequently the Theorem of Picard-Lindelöf ensures the existence of a unique local solution (g, h) to the ODEs (4.13) and (4.14). By Lemma 4.3 it follows moreover that every local solution (g, h) has the property $h \geq 0$.

- *Boundedness of the ODEs.* We show that $F(t, x, y)$ is bounded from below for every $(t, x, y) \in [0, T] \times \mathbb{R} \times \mathbb{R}_0^+$. Further we verify that $F^0(t, x, y)$ is bounded from below for every $(t, x, y) \in [0, T] \times \mathbb{R}_0^+ \times \mathbb{R}_0^+$ whereas $F^1(t, x, y)$ is bounded from above for every $(t, x, y) \in [0, T] \times \mathbb{R}_0^- \times \mathbb{R}_0^+$.

  In order to find the desired bounds we first need that the χ^i's are non-negative which is obvious since C^i and $(-1)^{1-i}\frac{1}{1-R}(e^{(-1)^{1-i}(1-R)y} - 1)$ both are non-negative as $y \in \mathbb{R}_0^+$ for $i = 0, 1$.

  Moreover $\varrho(t, x, y) \geq 0$ for every $(t, x, y) \in [0, T] \times \mathbb{R} \times \mathbb{R}_0^+$ whereas $\varrho^i(t, x)$ is monotonic increasing in x given $t \in [0, T]$. Further $\varrho^i(t, 0)$ is finite for every $t \in [0, T]$ since $0 < \delta \wedge \varepsilon \leq \frac{\varepsilon\delta}{\varepsilon - (\varepsilon - \delta)e^{-\delta(T-t)}} \leq \delta \vee \varepsilon < \infty$ $(\varepsilon > 0)$, resp. $\varrho^i(t, 0) = 0$ $(\varepsilon = 0)$. Hence we can define

$$\xi^0(T) \triangleq \min_{t \in [0,T]} \{\varrho^0(t, 0)\}, \ \xi^1(T) \triangleq \max_{t \in [0,T]} \{\varrho^1(t, 0)\}, \ T \in \mathbb{R}^+.$$

  The last results yield the desired bounds on F and F^i, i.e.

$$F(t, x, y) \geq -(\Psi^0 - \Psi^1) \text{ for every } (t, x, y) \in [0, T] \times \mathbb{R} \times \mathbb{R}_0^+, \qquad (*)$$
$$F^0(t, x, y) \geq -\Psi^0 + \xi^0(T) \text{ for every } (t, x, y) \in [0, T] \times \mathbb{R}_0^+ \times \mathbb{R}_0^+ \text{ and} \qquad (**)$$
$$F^1(t, x, y) \leq -\Psi^1 + \xi^1(T) \text{ for every } (t, x, y) \in [0, T] \times \mathbb{R}_0^- \times \mathbb{R}_0^+. \qquad (***)$$

- *Global solution.* Inequality $(*)$ implies that the local solution h of (4.13) satisfies

$$h(t) \leq (\Psi^0 - \Psi^1)(T - t) \text{ for every } t \in [0, T].$$

  This together with the non-negativity of h yields that the local solution $h(t)$ is linearly bounded in t. Therefore if there did not exist a global solution of the ODE-system this could only be caused by an explosion of the local solution g of (4.14) on the interval $[0, T]$.

  However, inequality $(**)$ implies

$$g(t) \leq (\Psi^0 - \xi^0(T))^+(T - t) \text{ for every } t \in [0, T]. \qquad (+)$$

  To see this we distinguish two cases; $\Psi^0 - \xi^0(T) \geq 0$ and $\Psi^0 - \xi^0(T) < 0$.

  - $\Psi^0 - \xi^0(T) \geq 0$. If $\Psi^0 - \xi^0(T) \geq 0$ then inequality $(**)$ shows that $g'(t) = F^0(t, g(t), h(t)) \geq -\Psi^0 + \xi^0(T)$ if $g(t) \geq 0$. Hence, $g(t) \leq (\Psi^0 - \xi^0(T))(T - t)$ for every $t \in [0, T]$.

  - $\Psi^0 - \xi^0(T) < 0$. If otherwise $\Psi^0 - \xi^0(T) < 0$ then $g(t) \leq 0$ for all $t \in [0, T]$ since for every $\hat{t}$ with $g(\hat{t}) = 0$ we have $g'(\hat{t}) = F^0(\hat{t}, g(\hat{t}), h(\hat{t})) = F^0(\hat{t}, 0, h(\hat{t})) \geq -\Psi^0 + \xi^0(T) > 0$ which together with $g(T) = 0$ implies the non-positivity of g.

Both cases combined then yield the desired result.

Further, inequality $(***)$ implies

$$g(t) - h(t) \geq -(\Psi^1 - \xi^1(T))^-(T-t) \text{ for every } t \in [0,T]. \qquad (++)$$

And again we distinguish two cases; $\Psi^1 - \xi^1(T) \leq 0$ and $\Psi^1 - \xi^1(T) > 0$.

- $\Psi^1 - \xi^1(T) \leq 0$. If $\Psi^1 - \xi^1(T) \leq 0$ then inequality $(***)$ shows that $g'(t) - h'(t) = F^1(t, g(t) - h(t), h(t)) \leq -\Psi^1 + \xi^1(T)$ if $g(t) - h(t) \leq 0$. Hence, $g(t) - h(t) \geq (\Psi^1 - \xi^1(T))(T-t)$ for every $t \in [0,T]$.
- $\Psi^1 - \xi^1(T) > 0$. If otherwise $\Psi^1 - \xi^1(T) > 0$ then $g(t) - h(t) \geq 0$ for all $t \in [0,T]$ since for every $\hat{t}$ with $g(\hat{t}) - h(\hat{t}) = 0$ we have $g'(\hat{t}) - h'(\hat{t}) = F^1(\hat{t}, g(\hat{t}) - h(\hat{t}), h(\hat{t})) = F^1(\hat{t}, 0, h(\hat{t})) \leq -\Psi^1 + \xi^1(T) < 0$ which together with $g(T) - h(T) = 0$ implies the non-negativity of $g - h$.

Both cases combined then yield the desired result.

Consequently, inequalities $(+)$ and $(++)$ together with the non-negativity of h imply that every local solution g satisfies

$$-(\Psi^1 - \xi^1(T))^-(T-t) \leq g(t) \leq (\Psi^0 - \xi^0(T))^+(T-t) \text{ for every } t \in [0,T].$$

Therefore, g cannot explode on $[0,T]$ which implies that the local solution (g,h) is indeed a global one.

This finishes the proof. □

Remark 4.10. *The proof of Lemma 4.9 turns out to be the archetype of the proofs of the existence of a unique global solution to the respective HJB-systems. The reason for this is that the respective ODEs only differ by the χ^i's whereas the ϱ's and ϱ^i's are always the same. In the following proofs concerning the existence of a unique global solution we therefore just present the respective χ^i's and prove that these are continuous in t, locally Lipschitz continuous in x and y and further non-negative. The rest of the proofs are completely analog to the proof of Lemma 4.9.*

The proof of Lemma 4.9 directly implies the following corollaries.

Corollary 4.11 (Time-dependent bounds on g and h)**.** *Let h and g be given by (4.13), (4.14) subject to the boundary conditions (4.15). Then*

$$-(\Psi^1 - \xi^1(T))^-(T-t) \leq g(t) \leq (\Psi^0 - \xi^0(T))^+(T-t) \ (\varepsilon > 0),$$
$$\textit{resp. } 0 \leq g(t) \leq \Psi^0(T-t) \ (\varepsilon = 0) \textit{ and}$$
$$0 \leq h(t) \leq (\Psi^0 - \Psi^1)(T-t)$$

for $t \in [0,T]$ where the $\xi^i(T)$'s are as defined in the proof of Lemma 4.9.

Corollary 4.12 (Time-independent bound on h)**.** *Let h and g be given by (4.13), (4.14) subject to the boundary conditions (4.15). Then*

$$0 \leq h(t) \leq \bar{h}$$

for $t \in [0,T]$ where $\bar{h}$ is the smallest positive root of $\bar{F}(y) \triangleq -(\Psi^0 - \Psi^1) + \chi^0(y) + \chi^1(y)$ if such a root exists; otherwise $\bar{h} = \infty$, i.e.

$$\bar{h} = \min\left\{y \in \mathbb{R}_0^+ \,:\, \bar{F}(y) = 0\right\},$$

with the convention $\min \emptyset \triangleq \infty$, where the χ^i's are as defined in the proof of Lemma 4.9.

Proof. The assertion follows from a simple ODE argument since

$$F(t,x,y) = -(\Psi^0 - \Psi^1) + \varrho(t,x,y) + \chi^0(y) + \chi^1(y) \geq \bar{F}(y)$$

for all $(t,x,y) \in [0,T] \times \mathbb{R} \times \mathbb{R}_0^+$ where we used the notation from the proof of Lemma 4.9. □

The boundary function on h as presented in Corollary 4.11 is a decreasing linear function that coincides with h at maturity T and has a slope given by $h'(T)$. Hence it is suitable for large values of t for which the deviation from h is not too large whereas for small values of t the non-linear behavior of h implies that the boundary function overshoots h by far. On the contrary the bound given in Corollary 4.12 is adequate for small values of t. Thus combining the two bounds on h yields

$$0 \leq h(t) \leq \min\{(\Psi^0 - \Psi^1)(T-t), \bar{h}\}$$

for $t \in [0,T]$.

We are now in the position to prove the optimality of the strategy that we found above.

Theorem 4.13 (Solution of the Investment Problem)**.** *Let condition (NJ) be satisfied, and let h and g be given by (4.13), (4.14) subject to the boundary conditions (4.15). Then the strategy*

$$(\pi^{i,*}, c^{i,*}(t)) \triangleq (\pi^{i,*}, c^{i,*}(t, g(t) - 1_{\{i=1\}}h(t))),\ t \in [0,T],\ i = 0,1$$

as given in Lemma 4.5 is optimal for the investment problem (P) with CRRA preferences and relative risk aversion R.

Proof. Since $(\pi^{i,*}, c^{i,*}(t))$ maximizes the reduced HJB-system (4.9) for each $t \in [0,T]$, optimality of the strategy $(\pi^{i,*}, c^{i,*})$ follows directly from the Verification Theorem 3.3. □

4.1.2 Solution of the Merton Investment Problem

Having generally discussed the solution of the investment problem (P) under constant intensity functions we now focus on a special variant of those, i.e. we consider the Merton investment problem in which the regime shift intensities are zero. This enables us to explicitly solve the ODE-system given by (4.13), (4.14).

Letting $C^i = 0$ the ODEs (4.13), (4.14) read

$$h'(t) = -(\Psi^0 - \Psi^1) + \left(\tfrac{\varepsilon\delta}{\varepsilon - (\varepsilon-\delta)e^{-\delta(T-t)}}\right)^{\frac{1}{R}} e^{-\frac{1-R}{R}g(t)} \tfrac{R}{1-R}\left(e^{\frac{1-R}{R}h(t)} - 1\right), \tag{4.16}$$

$$g'(t) = -\Psi^0 + \tfrac{\varepsilon\delta}{\varepsilon - (\varepsilon-\delta)e^{-\delta(T-t)}}\Big(1 - \tfrac{R}{1-R}\Big(\big(\tfrac{\varepsilon\delta}{\varepsilon - (\varepsilon-\delta)e^{-\delta(T-t)}}\big)^{\frac{1-R}{R}} e^{-\frac{1-R}{R}g(t)} - 1\Big)\Big) \tag{4.17}$$

subject to the boundary conditions

$$g(T)=0,\ h(T)=0. \tag{4.18}$$

On the basis of the procedure of solving the Merton problem with the conventional ansatz we use the following ansatz in order to solve the above ODE-system. We suppose

$$g(t)-1_{\{i=1\}}h(t)=\tfrac{R}{1-R}\ln(G^i(t))-\tfrac{1}{1-R}\ln(f(t)) \tag{4.19}$$

where the function G^i with $G^i(T)=e^{-\frac{1}{R}\delta T}$ is to be determined and f is given by (4.7). This ansatz implies that

$$e^{-\frac{1-R}{R}(g(t)-1_{\{i=1\}}h(t))}=\tfrac{(f(t))^{\frac{1}{R}}}{G^i(t)}. \tag{$*$}$$

Notice further that

$$\tfrac{\varepsilon\delta}{\varepsilon-(\varepsilon-\delta)e^{-\delta(T-t)}}=-\tfrac{f'(t)}{f(t)}. \tag{$**$}$$

Utilizing (4.19), ($*$) and ($**$) we can write the functions g and h, the Merton consumption rate $c^{i,M}$ and the value function v^i in terms of the functions G^i and f, i.e.

$$\begin{aligned}
g(t)&=\tfrac{R}{1-R}\ln(G^0(t))-\tfrac{1}{1-R}\ln(f(t)),\\
h(t)&=\tfrac{R}{1-R}\big(\ln(G^0(t))-\ln(G^1(t))\big),\\
c^{i,M}(t)&=\varepsilon^{\frac{1}{R}}e^{-\frac{1}{R}\delta t}(G^i(t))^{-1},\\
v^i(t,x)&=\tfrac{1}{1-R}x^{1-R}(G^i(t))^R-\tfrac{1}{1-R}f(t).
\end{aligned}$$

Further inserting the ansatz (4.19) into the system (4.16), (4.17) and taking ($*$) and ($**$) into account yields

$$\begin{aligned}
&g'(t)-1_{\{i=1\}}h'(t)=-\Psi^i+\tfrac{\varepsilon\delta}{\varepsilon-(\varepsilon-\delta)e^{-\delta(T-t)}}\Big(1-\tfrac{R}{1-R}\big(\big(\tfrac{\varepsilon\delta}{\varepsilon-(\varepsilon-\delta)e^{-\delta(T-t)}}\big)^{\frac{1-R}{R}}e^{-\frac{1-R}{R}(g(t)-1_{\{i=1\}}h(t))}-1\big)\Big)\\
\Leftrightarrow\ &\tfrac{R}{1-R}\tfrac{(G^i)'(t)}{G^i(t)}-\tfrac{1}{1-R}\tfrac{f'(t)}{f(t)}=-\Psi^i-\tfrac{f'(t)}{f(t)}\Big(1-\tfrac{R}{1-R}\big(\big(-\tfrac{f'(t)}{f(t)}\big)^{\frac{1-R}{R}}\tfrac{(f(t))^{\frac{1}{R}}}{G^i(t)}-1\big)\Big)\\
\Leftrightarrow\ &\tfrac{R}{1-R}\tfrac{(G^i)'(t)}{G^i(t)}-\tfrac{1}{1-R}\tfrac{f'(t)}{f(t)}=-\Psi^i-\tfrac{1}{1-R}\tfrac{f'(t)}{f(t)}-\tfrac{R}{1-R}\tfrac{(-f'(t))^{\frac{1}{R}}}{G^i(t)}\\
\Leftrightarrow\ &(G^i)'(t)=-\tfrac{1-R}{R}\Psi^iG^i(t)-(-f'(t))^{\frac{1}{R}}\\
\Leftrightarrow\ &(G^i)'(t)=-\tfrac{1-R}{R}\Psi^iG^i(t)-\varepsilon^{\frac{1}{R}}e^{-\frac{1}{R}\delta t}.
\end{aligned}$$

Thus in order to solve the Merton investment problem we just need to solve the following inhomogeneous ODE for G^i given by

$$(G^i)'(t)=-\tfrac{1-R}{R}\Psi^iG^i(t)-\varepsilon^{\frac{1}{R}}e^{-\frac{1}{R}\delta t}$$

subject to the boundary conditions $G^i(T)=e^{-\frac{1}{R}\delta T}$.

The variation of constants method yields that G^i can be written as

$$G^i(t)=G^{i,H}(t)\Big(G^i(T)+\int_t^T\tfrac{\varepsilon^{\frac{1}{R}}e^{-\frac{1}{R}\delta s}}{G^{i,H}(s)}\mathrm{d}s\Big)$$

where $G^{i,H}(t) = e^{\frac{1-R}{R}\Psi^i(T-t)}$ is the homogeneous solution of the aforementioned ODE. Calculating the integral we get that

$$G^i(t) = e^{\frac{1-R}{R}\Psi^i(T-t)-\frac{1}{R}\delta T}\Big(1 + \varepsilon^{\frac{1}{R}}\frac{R}{\delta-(1-R)\Psi^i}\Big(e^{\frac{\delta-(1-R)\Psi^i}{R}(T-t)} - 1\Big)\Big).$$

Inserting this solution into the above formulas depending on G^i yields explicit formulas for the functions g and h, the Merton consumption rate $c^{i,M}$ and the value function v^i at time t that are given in the following theorem.

Theorem 4.14 (Solution of the Merton Investment Problem). *Let condition (NJ) be satisfied, and let h and g be given by (4.16), (4.17) subject to the boundary conditions (4.18). Then the optimal strategy for the investment problem (P) with* CRRA *preferences and relative risk aversion R is given by*

$$(\pi^{i,*}, c^{i,*}(t)) = (\pi^{i,M}, c^{i,M}(t)),\ \ t \in [0,T]$$

where

$$\pi^{i,M} = \tfrac{1}{R}(\sigma^i.(\sigma^i)^\top)^{-1}.\eta^i, \quad c^{i,M}(t) = \frac{\varepsilon^{\frac{1}{R}}e^{\frac{\delta-(1-R)\Psi^i}{R}(T-t)}}{1+\varepsilon^{\frac{1}{R}}\frac{R}{\delta-(1-R)\Psi^i}\Big(e^{\frac{\delta-(1-R)\Psi^i}{R}(T-t)}-1\Big)}$$

for $i = 0, 1$. In particular g and h are given by

$$\begin{aligned}
g(t) &= \Psi^0(T-t) + \tfrac{R}{1-R}\ln\Big(1 + \varepsilon^{\frac{1}{R}}\tfrac{R}{\delta-(1-R)\Psi^0}\Big(e^{\frac{\delta-(1-R)\Psi^0}{R}(T-t)} - 1\Big)\Big) - \tfrac{1}{1-R}\ln\Big(\tfrac{\varepsilon}{\delta}(e^{\delta(T-t)} - 1) + 1\Big),\\
h(t) &= (\Psi^0 - \Psi^1)(T-t) + \tfrac{R}{1-R}\ln\Big(1 + \varepsilon^{\frac{1}{R}}\tfrac{R}{\delta-(1-R)\Psi^0}\Big(e^{\frac{\delta-(1-R)\Psi^0}{R}(T-t)} - 1\Big)\Big)\\
&\quad - \tfrac{R}{1-R}\ln\Big(1 + \varepsilon^{\frac{1}{R}}\tfrac{R}{\delta-(1-R)\Psi^1}\Big(e^{\frac{\delta-(1-R)\Psi^1}{R}(T-t)} - 1\Big)\Big)
\end{aligned}$$

so that the value functions read

$$\begin{aligned}
v^i(t,x) &= \tfrac{1}{1-R}x^{1-R}e^{(1-R)\Psi^i(T-t)-\delta T}\Big(1 + \varepsilon^{\frac{1}{R}}\tfrac{R}{\delta-(1-R)\Psi^i}\Big(e^{\frac{\delta-(1-R)\Psi^i}{R}(T-t)} - 1\Big)\Big)^R\\
&\quad - \tfrac{1}{1-R}\tfrac{1}{\delta}e^{-\delta t}(\varepsilon - (\varepsilon-\delta)e^{-\delta(T-t)})
\end{aligned}$$

for $i = 0, 1$.

If the investor was not allowed to consume, i.e. $\varepsilon = 0$, then the latter formulas simplify to

$$\pi^{i,M} = \tfrac{1}{R}(\sigma^i.(\sigma^i)^\top)^{-1}.\eta^i, \quad c^{i,M}(t) = 0$$

for $i = 0, 1$ and

$$\begin{aligned}
g(t) &= \Psi^0(T-t),\\
h(t) &= (\Psi^0 - \Psi^1)(T-t),\\
v^i(t,x) &= e^{-\delta T}\tfrac{1}{1-R}(x^{1-R}e^{(1-R)\Psi^i(T-t)} - 1)
\end{aligned}$$

for $i = 0, 1$.

Remark 4.15. *Remember the time-dependent upper bounds on g and h given in Corollary 4.11. The bound on h just coincides with the corresponding function h from the Merton investment problem given $\varepsilon = 0$. Concerning g this holds true in case of $\varepsilon = 0$.*

4.2 Step Intensity Functions

Having dealt with constant intensity functions we now come to a simple extension of those constant intensities, i.e. we consider piecewise constant intensity functions that exhibit at most one jump. Formally we let the intensity functions $\vartheta^{i,1-i}$ be given as step functions of the form

$$\vartheta^{i,1-i}(\pi,c) = C_1^i 1_{\{A^i+\pi^\top.B_\pi^i+B_c^i c\leq C^i\}} + C_2^i 1_{\{A^i+\pi^\top.B_\pi^i+B_c^i c>C^i\}},\ (\pi,c)\in\mathbb{R}^{\bar{n}}\times\mathbb{R}_0^+ \tag{SI}$$

with $A^i\in\mathbb{R}$, $B_\pi^i\in\mathbb{R}^{\bar{n}}$, $B_c^i\in\mathbb{R}$, $C^i\in\mathbb{R}$ and $C_j^i\in\mathbb{R}_0^+$ for $i=0,1$ and $j=1,2$ where $C_2^0>C_1^0$ and $C_1^1>C_2^1$. We omit the cases $C_1^i=C_2^i$, $i=0,1$, as this implies constant intensity functions that have already been discussed.

Such step intensity functions have the feature that the large investor's impact on the market is an indirect one. This is because his strategy (π,c) only determines whether the intensity is C_1^i or C_2^i, but not how large, resp. small, those two intensities are since these are predefined constants. In the following we give some explanations and interpretations of the parameters characterizing the step intensity functions.

Remark 4.16 (Interpretation of the intensity parameters)**.** *The step intensity functions can take on two different values, C_1^i and C_2^i, where C_1^i is advantageous and C_2^i is disadvantageous for the large investor. Be aware of the fact that the advantageous branch of $\vartheta^{i,1-i}$ is always closed whereas the disadvantageous branch is always open. This turns out to be necessary for the existence of an optimal strategy. The absolute value of the difference between those two values, $|C_1^i-C_2^i|$, mirrors the possible extent of the large investor's influence. Further, B_π^i and B_c^i determine the strength of that influence, resp. the sensitivity of the market concerning the strategy followed by the large investor – the bigger $|B_{\pi,n}^i|$, resp. $|B_c^i|$, the more sensitive the market. Here $B_{\pi,n}^i$ denotes the n-th element of B_π^i. Finally having specified the B^i's the critical barrier separating the advantageous from the disadvantageous strategies in terms of the resulting intensities is fixed via the parameters A^i and C^i.*

So far we only described when and to which extent the market is influenced by the large investor. The way in which the market reacts on his actions is determined by the sign of the B^i's. We first look at the portfolio parameter B_π^i. A positive $B_{\pi,n}^i$ corresponds to a market in which large positions in the n-th asset yield large ($i=0$), resp. small ($i=1$), transition intensities. So the other market participants disapprove the large investor's holdings in asset n in that his position could cause the market to turn into the adverse state (if $i=0$) or hinder an early jump back to the favorable state (if $i=1$).

In case of $B_{\pi,n}^i$ being negative large proportions in the n-th asset cause small ($i=0$), resp. large ($i=1$), transition intensities. Thus the large investor is accepted by the market, resp. the other market participants may think of him as having superior information about the n-th stock, such as a manager of a prosperous fund, or an executive of the company issuing the stock or even a person having insider information.

Having discussed the role of B_π^i we now go on describing the consumption parameter B_c^i. If $B_c^i>0$ then consuming at a high rate implies large ($i=0$), resp. small ($i=1$), regime shift intensities, i.e. the other market participants may interpret the large investor's high consumption rate as a bad signal for the future development, e.g. as a herald of a market crisis. Another example in which a positive value of B_c^i is reasonable is the large investor being the manager of a large mutual fund. In this context consumption can be interpreted as a reduction of the number of assets under management. A possible reason for such a reduction could be the absence of lucrative investment opportunities.

Finally, a negative B_c^i implies that consuming at a high rate yields small ($i = 0$), resp. large ($i = 1$), regime shift intensities. Thus the market somehow rewards the large investor if he consumes at a high rate. This could be the case if the investor's presence in the market is disapproved by the other market participants. A specialty arises if $\varepsilon = 0$, i.e. if the investor does not draw any utility from intermediate consumption in terms of the utility function. In this case a negative B_c^i may force the investor to consume just to achieve favorable transition intensities without generating any direct utility gain. In this context consumption could be interpreted as bribe and we will see later on that there exist parameter specifications under which the investor pays a bribe.

Inserting the step intensity functions into the reduced HJB-system (4.8) the latter further simplifies to

$$\begin{aligned} 0 = \sup_{(\pi,c)\in\mathbb{R}^{\bar{n}}\times\mathbb{R}_0^+} \Big\{ & g'(t) - 1_{\{i=1\}}h'(t) + r^i + \pi^\top.\eta^i - \tfrac{1}{2}R\pi^\top.\sigma^i.(\sigma^i)^\top.\pi \\ & + \tfrac{\varepsilon\delta}{\varepsilon-(\varepsilon-\delta)e^{-\delta(T-t)}}\tfrac{1}{1-R}\big(e^{-(1-R)(g(t)-1_{\{i=1\}}h(t))}c^{1-R} - 1\big) - c \\ & + \big(C_1^i 1_{\{A^i+\pi^\top.B_\pi^i+B_c^i c\le C^i\}} + C_2^i 1_{\{A^i+\pi^\top.B_\pi^i+B_c^i c> C^i\}}\big)\tfrac{1}{1-R}\big(e^{(-1)^{1-i}(1-R)h(t)} - 1\big)\Big\} \end{aligned} \tag{4.20}$$

for $t \in [0, T)$ and $i = 0, 1$, subject to the boundary conditions

$$g(T) = 0, \;\; h(T) = 0.$$

Subsequently, we present the solution of the investment problem (P) for three different variants of the step intensity functions. First, we discuss intensities that are influenced solely by the portfolio proportions π, the so called *portfolio-dependent intensities* (Subsection 4.2.1). Second, instead of the dependency on π we look at *consumption-dependent intensities* being functions just of the consumption rate c (Subsection 4.2.2). And finally the most general version – the *portfolio- and consumption-dependent intensities* – is regarded (Subsection 4.2.3).

We do not dwell on the simplest variant of the step intensity functions – the *portfolio- and consumption-independent intensities* – since those are just a special case of the constant intensity functions which have already been discussed in the previous section, i.e. setting $B_\pi^i = 0$ and $B_c^i = 0$ implies $\vartheta^{i,1-i}(\pi, c) = C_1^i 1_{\{A^i\le C^i\}} + C_2^i 1_{\{A^i>C^i\}}$ for $i = 0, 1$.

In each subsection we present the optimal strategies in closed form, i.e. we give explicit formulas for the optimal portfolio proportions and consumption rates that only depend on the solution of an ODE-system of which we show that it admits a unique global solution. Further we provide bounds on the solution of this ODE-system.

4.2.1 Portfolio-dependent Intensities

In this section we study the optimal investment problem when the large investor's portfolio proportions impact on the intensities of regime shifts whereas his consumption has no influence, i.e. we let

$$B_\pi^i \neq 0 \text{ and } B_c^i = 0 \text{ for } i = 0, 1, \tag{PD}$$

so that the intensities are now given by

$$\vartheta^{i,1-i}(\pi, c) = C_1^i 1_{\{A^i+\pi^\top.B_\pi^i\le C^i\}} + C_2^i 1_{\{A^i+\pi^\top.B_\pi^i> C^i\}}.$$

We denote by

$$\mathcal{F}^{\pi,i} \triangleq \left\{ \pi \in \mathbb{R}^{\bar{n}} \ : \ A^i + \pi^\top . B^i_\pi \leq C^i \right\}$$

the half space of all portfolio proportions that are favorable for the investor concerning the intensities of regime shifts. Letting

$$d^{\pi,i}(\pi) \triangleq A^i + \pi^\top . B^i_\pi - C^i$$

denote the 'distance' of the strategy π to the separating hyperplane, we have $\pi \in \mathcal{F}^{\pi,i}$ if and only if $d^{\pi,i}(\pi) \leq 0$.

In order to determine the maximizer in the HJB-system (4.20) we define functions $H^{\pi,i} : \mathbb{R}^+_0 \times \mathbb{R}^{\bar{n}} \to \mathbb{R}$ and $H^{c,i} : \ [0,T] \times \mathbb{R} \times \mathbb{R}^+_0 \to \mathbb{R}$, $i = 0, 1$, given by

$$\begin{aligned} H^{\pi,i}(y,\pi) &\triangleq r^i + \pi^\top . \eta^i - \tfrac{1}{2} R \pi^\top . \sigma^i . (\sigma^i)^\top . \pi + \left(C^i_1 1_{\{\pi \in \mathcal{F}^{\pi,i}\}} + C^i_2 1_{\{\pi \notin \mathcal{F}^{\pi,i}\}} \right) \tfrac{1}{1-R} (e^{(-1)^{1-i}(1-R)y} - 1), \\ H^{c,i}(t,x,c) &\triangleq \tfrac{\varepsilon\delta}{\varepsilon - (\varepsilon-\delta)e^{-\delta(T-t)}} \tfrac{1}{1-R} (e^{-(1-R)x} c^{1-R} - 1) - c \end{aligned}$$

where we use the already mentioned convention $0^{1-R} \triangleq \infty$ if $R > 1$. Notice that function $H^{c,i}$ is independent of t and x if $\varepsilon = 0$.

The HJB-system (4.20) reads

$$0 = \sup_{(\pi,c) \in \mathbb{R}^{\bar{n}} \times \mathbb{R}^+_0} \left\{ g'(t) - 1_{\{i=1\}} h'(t) + H^{\pi,i}(h(t),\pi) + H^{c,i}(t, g(t) - 1_{\{i=1\}} h(t), c) \right\} \tag{4.21}$$

for $t \in [0,T)$ and $i = 0, 1$, subject to the boundary conditions

$$g(T) = 0, \ \ h(T) = 0.$$

Writing the HJB-system in that way it is obvious that the maximization over $(\pi, c) \in \mathbb{R}^{\bar{n}} \times \mathbb{R}^+_0$ can be separated into two unrelated maximizations; one over $\pi \in \mathbb{R}^{\bar{n}}$ and one over $c \in \mathbb{R}^+_0$.

To find the supremum in (4.21) we present the maximizers of the functions $H^{\pi,i}(y,\cdot)$ and $H^{c,i}(t,x,\cdot)$ for arbitrary $(t,x,y) \in [0,T] \times \mathbb{R} \times \mathbb{R}^+_0$. This yields a family of maximizers dependent on (t,x,y). The maximizers of the HJB-system (4.21) are then obtained by replacing x and y by $g(t) - 1_{\{i=1\}} h(t)$ and $h(t)$.

Concerning the consumption rate the concavity of $H^{c,i}(t,x,\cdot)$ and the first-order condition imply

Lemma 4.17 (Maximizer of $H^{c,i}(t,x,\cdot)$)**.** *For every $(t,x) \in [0,T] \times \mathbb{R}$ the maximizer*

$$c^{i,*}(t,x) \triangleq \arg\max_{c \in \mathbb{R}^+_0} H^{c,i}(t,x,c), \ \ i = 0, 1,$$

is given by the Merton consumption rate, i.e.

$$c^{i,*}(t,x) = c^M(t,x).$$

Remark 4.18. *In case of $\varepsilon = 0$ the Merton consumption rate vanishes, i.e. $c^M(t,x) = 0$.*

In order to find the maximizing portfolio proportions we let $H_j^{\pi,i} : \mathbb{R}_0^+ \times \mathbb{R}^{\bar{n}} \to \mathbb{R}$, $i = 0, 1$ and $j = 1, 2$, be given by

$$H_j^{\pi,i}(y, \pi) \triangleq r^i + \pi^\top.\eta^i - \tfrac{1}{2} R \pi^\top.\sigma^i.(\sigma^i)^\top.\pi + C_j^i \tfrac{1}{1-R}(e^{(-1)^{1-i}(1-R)y} - 1)$$

such that $H^{\pi,i}(y, \pi) = H_1^{\pi,i}(y, \pi) 1_{\{\pi \in \mathcal{F}^{\pi,i}\}} + H_2^{\pi,i}(y, \pi) 1_{\{\pi \notin \mathcal{F}^{\pi,i}\}}$.

Since $H_j^{\pi,i}(y, \cdot)$ is a concave paraboloid for every $y \in \mathbb{R}_0^+$ and $j = 1, 2$, the two candidate solutions for the maximization of $H^{\pi,i}(y, \cdot)$ over π are $\pi^{i,M}$ and $\tilde{\pi}^{i,\mathrm{crit}}$ where

$$\tilde{\pi}^{i,\mathrm{crit}} \triangleq \underset{\left\{\pi \in \mathbb{R}^{\bar{n}} \,:\, A^i + \pi^\top.B_\pi^i = C^i\right\}}{\arg\max} H^{\pi,i}(y, \pi)$$

for $i = 0, 1$ and $y \in \mathbb{R}_0^+$. Using the Lagrange multiplier method we find that $\tilde{\pi}^{i,\mathrm{crit}}$ is given by

$$\tilde{\pi}^{i,\mathrm{crit}} = \tfrac{1}{R}(\sigma^i.(\sigma^i)^\top)^{-1}.\Big(\eta^i - B_\pi^i \tfrac{A^i + (\pi^{i,M})^\top.B_\pi^i - C^i}{\frac{1}{R}(B_\pi^i)^\top.(\sigma^i.(\sigma^i)^\top)^{-1}.B_\pi^i}\Big).$$

Having established the candidate solutions the following lemma presents the desired maximizers.

Lemma 4.19 (Maximizer of $H^{\pi,i}(y, \cdot)$)**.** *For every $y \in \mathbb{R}_0^+$ the maximizer*

$$\pi^{i,*}(y) \triangleq \underset{\pi \in \mathbb{R}^{\bar{n}}}{\arg\max}\, H^{\pi,i}(y, \pi), \;\; i = 0, 1,$$

is given by

$$\pi^{i,*}(y) = \begin{cases} \pi^{i,M} & \text{if } y < h^{i,\mathrm{crit}}, \\ \pi^{i,\mathrm{crit}} & \text{if } y \geq h^{i,\mathrm{crit}}, \end{cases}$$

where

$$h^{i,\mathrm{crit}} \triangleq (-1)^{1-i} \tfrac{1}{1-R} \ln\Big((1-R)\tfrac{\zeta^{i,\mathrm{crit}}}{C_2^i - C_1^i} + 1\Big)$$

with

$$\zeta^{i,\mathrm{crit}} \triangleq -\tfrac{1}{2} \tfrac{((A^i + (\pi^{i,M})^\top.B_\pi^i - C^i)^+)^2}{\frac{1}{R}(B_\pi^i)^\top.(\sigma^i.(\sigma^i)^\top)^{-1}.B_\pi^i}$$

and

$$\pi^{i,\mathrm{crit}} \triangleq \tfrac{1}{R}(\sigma^i.(\sigma^i)^\top)^{-1}.\Big(\eta^i - B_\pi^i \tfrac{(A^i + (\pi^{i,M})^\top.B_\pi^i - C^i)^+}{\frac{1}{R}(B_\pi^i)^\top.(\sigma^i.(\sigma^i)^\top)^{-1}.B_\pi^i}\Big).$$

Remark 4.20. *We use the following shorthand notations*

$$H_{1,\mathrm{crit}}^{\pi,i}(y) \triangleq H_1^{\pi,i}(y, \pi^{i,\mathrm{crit}}) \text{ and } H_{2,M}^{\pi,i}(y) \triangleq H_2^{\pi,i}(y, \pi^{i,M})$$

for $y \in \mathbb{R}_0^+$ and $i = 0, 1$.

Figure 4.1 shows the three typical shapes of the functions $H_1^{\pi,i}(y, \cdot)$, $H_2^{\pi,i}(y, \cdot)$ and $H^{\pi,i}(y, \cdot)$ that correspond to the different cases occurring in the following proof of Lemma 4.19.

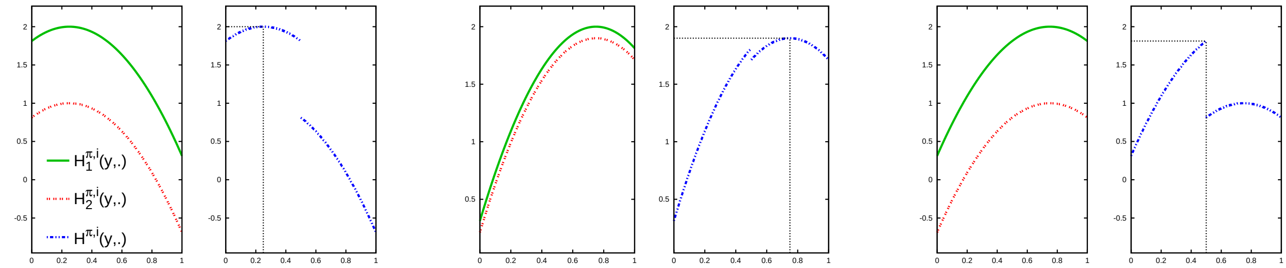

Figure 4.1: Typical shapes of $H_1^{\pi,i}(y,\cdot)$, $H_2^{\pi,i}(y,\cdot)$ and $H^{\pi,i}(y,\cdot)$ where $\mathfrak{F}^{\pi,i} = (-\infty, 0.5]$

Proof of Lemma 4.19. Let $y \in \mathbb{R}_0^+$ be given. A straight forward optimization yields that the two paraboloids $H_1^{\pi,i}(y,\cdot)$ and $H_2^{\pi,i}(y,\cdot)$ attain their maxima at $\pi^{i,M}$. Further since $C_1^0 < C_2^0$ and $C_1^1 > C_2^1$ we get that

$$H_1^{\pi,i}(y,\pi) > H_2^{\pi,i}(y,\pi) \text{ for all } \pi \in \mathbb{R}^{\bar{n}}. \tag{$*$}$$

In order to determine the maximizing portfolio strategy we distinguish the two cases $\pi^{i,M} \in \mathfrak{F}^{\pi,i}$ and $\pi^{i,M} \notin \mathfrak{F}^{\pi,i}$.

- $\pi^{i,M} \in \mathfrak{F}^{\pi,i}$. If $\pi^{i,M} \in \mathfrak{F}^{\pi,i}$ then by $(*)$ the maximizer of $H^{\pi,i}(y,\cdot)$ is given by the Merton strategy (cf. leftmost couple of plots in Figure 4.1), i.e.

$$\pi^{i,M} \in \mathfrak{F}^{\pi,i} \quad \Rightarrow \quad \pi^{i,*}(y) = \pi^{i,M}.$$

- $\pi^{i,M} \notin \mathfrak{F}^{\pi,i}$. If otherwise $\pi^{i,M} \notin \mathfrak{F}^{\pi,i}$ then $(*)$ implies that the maximizer of $H^{\pi,i}(y,\cdot)$ is either $\pi^{i,M}$ in case of $H_{2,M}^{\pi,i}(y) > H_{1,\text{crit}}^{\pi,i}(y)$ (cf. central couple of plots in Figure 4.1), or $\pi^{i,\text{crit}}$ in case of $H_{2,M}^{\pi,i}(y) \le H_{1,\text{crit}}^{\pi,i}(y)$ (cf. rightmost couple of plots in Figure 4.1). So we have to take a closer look on the condition $H_{2,M}^{\pi,i}(y) \le H_{1,\text{crit}}^{\pi,i}(y)$. Therefore with

$$\begin{aligned} H_{2,M}^{\pi,i}(y) \le H_{1,\text{crit}}^{\pi,i}(y) \quad &\Leftrightarrow \quad (C_2^i - C_1^i)\tfrac{1}{1-R}(e^{(-1)^{1-i}(1-R)y} - 1) \le -\tfrac{1}{2}\frac{((A^i + (\pi^{i,M})^\top.B_\pi^i - C^i)^+)^2}{\frac{1}{R}(B_\pi^i)^\top.(\sigma^i.(\sigma^i)^\top)^{-1}.B_\pi^i} \\ &\Leftrightarrow \quad (C_2^i - C_1^i)\tfrac{1}{1-R}(e^{(-1)^{1-i}(1-R)y} - 1) \le \zeta^{i,\text{crit}} \\ &\Leftrightarrow \quad y \ge (-1)^{1-i}\tfrac{1}{1-R}\ln\Big((1-R)\tfrac{\zeta^{i,\text{crit}}}{C_2^i - C_1^i} + 1\Big) \\ &\Leftrightarrow \quad y \ge h^{i,\text{crit}} \end{aligned}$$

we get

$$\pi^{i,M} \notin \mathfrak{F}^{\pi,i} \quad \Rightarrow \quad \pi^{i,*}(y) = \begin{cases} \pi^{i,M} & \text{if } y < h^{i,\text{crit}}, \\ \pi^{i,\text{crit}} & \text{if } y \ge h^{i,\text{crit}}. \end{cases}$$

Since $\pi^{i,M} \in \mathfrak{F}^{\pi,i}$ implies $h^{i,\text{crit}} = 0$ and $\pi^{i,\text{crit}} = \pi^{i,M}$ the last formula covers the case of $\pi^{i,M} \in \mathfrak{F}^{\pi,i}$, too. Hence the maximizing strategy in state i is given by

$$\pi^{i,*}(y) = \begin{cases} \pi^{i,M} & \text{if } y < h^{i,\text{crit}}, \\ \pi^{i,\text{crit}} & \text{if } y \ge h^{i,\text{crit}}. \end{cases}$$

□

Remark 4.21. *Notice that it is necessary for the existence of the maximizing strategy that the regime shift intensities are defined in such a way that they always attain the favorable value C_1^i for any strategy π that satisfies $A^i + \pi^\top.B_\pi^i = C^i$. Otherwise, the maximizing strategy would not exist if $H_{2,M}^{\pi,i}(y) \leq H_{1,\text{crit}}^{\pi,i}(y)$ in case of $\pi^{i,M} \notin \mathcal{F}^{\pi,i}$ (cf. rightmost couple of plots in Figure 4.1).*

Remark 4.22 (Interpretation)**.** *The maximizing strategy coincides with the Merton strategy when the latter belongs to the half space of favorable strategies. If this is not the case then the maximizing strategy is given by the Merton strategy as long as y is not bigger than $h^{i,\text{crit}}$. For all $y < h^{i,\text{crit}}$ it is not reasonable to deviate from the Merton strategy. Remember that y is the general variable that later on is to be replaced by h. Thus y represents the difference between the two market states as h does. Hence only if the difference overshoots the critical barrier $h^{i,\text{crit}}$ then it is reasonable to switch to the critical strategy $\pi^{i,\text{crit}}$.*

Remark 4.23 (Discontinuity of $\pi^{i,*}$)**.** *The maximizing strategy $\pi^{i,*}$ is in general discontinuous at the point $y = h^{i,\text{crit}}$. Only in the trivial case of $h^{i,\text{crit}} = 0$ the strategy does not jump at all.*

Remark 4.24. *So far the strategy $(\pi^{i,*}, c^{i,*})$ is only a candidate solution for the optimal investment problem. To verify the optimality we need to show that the related HJB-system has a global solution that satisfies the required regularity conditions stated in the Verification Theorem 3.3.*

Inserting $(\pi^{i,*}(h(t)), c^{i,*}(t, g(t) - 1_{\{i=1\}}h(t)))$ into the reduced HJB-equation (4.21) the latter becomes a system of coupled backward ODEs

$$\begin{aligned} h'(t) = &- (\Psi^0 - \Psi^1) + \left(\tfrac{\varepsilon\delta}{\varepsilon-(\varepsilon-\delta)e^{-\delta(T-t)}}\right)^{\frac{1}{R}} e^{-\frac{1-R}{R}g(t)} \tfrac{R}{1-R}\left(e^{\frac{1-R}{R}h(t)} - 1\right) \\ &- C_2^0 \tfrac{1}{1-R}\left(e^{-(1-R)h(t)} - 1\right) + C_2^1 \tfrac{1}{1-R}\left(e^{(1-R)h(t)} - 1\right) \\ &- \left[(C_1^0 - C_2^0)\tfrac{1}{1-R}\left(e^{-(1-R)h(t)} - 1\right) - \tfrac{1}{2}\tfrac{((A^0+(\pi^{0,M})^\top.B_\pi^0-C^0)^+)^2}{\frac{1}{R}(B_\pi^0)^\top.(\sigma^0.(\sigma^0)^\top)^{-1}.B_\pi^0}\right]^+ \\ &+ \left[(C_1^1 - C_2^1)\tfrac{1}{1-R}\left(e^{(1-R)h(t)} - 1\right) - \tfrac{1}{2}\tfrac{((A^1+(\pi^{1,M})^\top.B_\pi^1-C^1)^+)^2}{\frac{1}{R}(B_\pi^1)^\top.(\sigma^1.(\sigma^1)^\top)^{-1}.B_\pi^1}\right]^+, \end{aligned} \tag{4.22}$$

$$\begin{aligned} g'(t) = &- \Psi^0 + \tfrac{\varepsilon\delta}{\varepsilon-(\varepsilon-\delta)e^{-\delta(T-t)}}\left(1 - \tfrac{R}{1-R}\left(\left(\tfrac{\varepsilon\delta}{\varepsilon-(\varepsilon-\delta)e^{-\delta(T-t)}}\right)^{\frac{1-R}{R}} e^{-\frac{1-R}{R}g(t)} - 1\right)\right) \\ &- C_2^0 \tfrac{1}{1-R}\left(e^{-(1-R)h(t)} - 1\right) \\ &- \left[(C_1^0 - C_2^0)\tfrac{1}{1-R}\left(e^{-(1-R)h(t)} - 1\right) - \tfrac{1}{2}\tfrac{((A^0+(\pi^{0,M})^\top.B_\pi^0-C^0)^+)^2}{\frac{1}{R}(B_\pi^0)^\top.(\sigma^0.(\sigma^0)^\top)^{-1}.B_\pi^0}\right]^+ \end{aligned} \tag{4.23}$$

subject to the boundary conditions

$$g(T) = 0, \; h(T) = 0. \tag{4.24}$$

Remark 4.25. *From the proof of Lemma 4.19 we know that*

$$y \geq h^{i,\text{crit}} \quad \Leftrightarrow \quad (C_1^i - C_2^i)\tfrac{1}{1-R}\left(e^{(-1)^{1-i}(1-R)y} - 1\right) - \tfrac{1}{2}\tfrac{((A^i+(\pi^{i,M})^\top.B_\pi^i-C^i)^+)^2}{\frac{1}{R}(B_\pi^i)^\top.(\sigma^i.(\sigma^i)^\top)^{-1}.B_\pi^i} \geq 0.$$

Remark 4.26. *The aforementioned ODEs include the ODEs of the classical Merton problem (first rows) and the ODEs of the Bäuerle-Rieder problem with constant regime shift intensities C_2^i (first two rows).*

Lemma 4.27. *The ODE-system given by equation (4.22) and (4.23) subject to the boundary conditions (4.24) admits a unique global solution.*

Proof. The proof is essentially the same as the proof of Lemma 4.9. Only the χ^i's are different. Therefore we just present the actual χ^i's and verify that those satisfy the necessary continuity conditions and the non-negativity condition.

- *Definition of the χ^i's.* In the portfolio-dependent case the χ^i's are given by $\chi^i : R_0^+ \to \mathbb{R}$, $i = 0, 1$, with

$$\chi^i(y) \triangleq (-1)^{1-i}\Big(C_2^i \tfrac{1}{1-R}(e^{(-1)^{1-i}(1-R)y} - 1) + \Big[(C_1^i - C_2^i)\tfrac{1}{1-R}(e^{(-1)^{1-i}(1-R)y} - 1) + \zeta^{i,\text{crit}}\Big]^+\Big).$$

- *Continuity results on the χ^i's.* Being the composition of continuously differentiable functions and the $[\,\cdot\,]^+$-function χ^i is locally Lipschitz continuous.
- *Non-negativity of the χ^i's.* The non-negativity of χ^1 is obvious. Moreover the non-positivity of $\zeta^{0,\text{crit}}$ implies

$$(C_1^0 - C_2^0)\tfrac{1}{1-R}(e^{-(1-R)y} - 1) \geq \Big[(C_1^0 - C_2^0)\tfrac{1}{1-R}(e^{-(1-R)y} - 1) + \zeta^{0,\text{crit}}\Big]^+$$

  since we required $C_1^0 < C_2^0$. Hence the non-negativity of the χ^0 is proven as

$$\chi^0(y) \geq -C_1^0 \tfrac{1}{1-R}(e^{-(1-R)y} - 1) \geq 0 \text{ for every } y \in \mathbb{R}_0^+.$$

The remainder of the proof is identical to the proof of Lemma 4.9. □

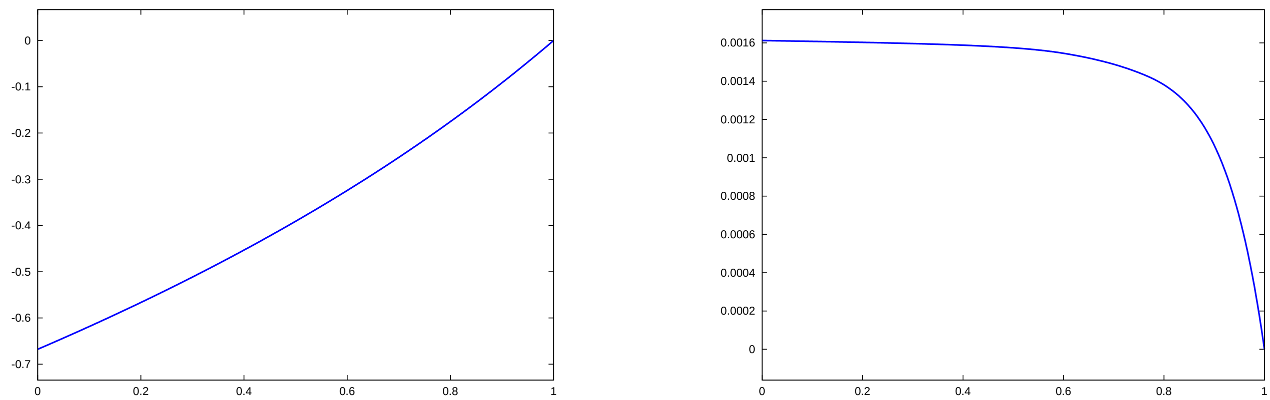

Figure 4.2: g (left) and h (right) as functions of t
($r^0 = r^1 = 0.03$, $\eta^0 = 0.1$, $\eta^1 = 0.02$, $\sigma^0 = 0.3$, $\sigma^1 = 0.6$, $\delta = 0.035$, $\varepsilon = 1$, $R = 3$, $T = 1$,
$A^0 = 17$, $B_\pi^0 = -5$, $C^0 = 14$, $C_1^0 = 5$, $C_2^0 = 10$, $A^1 = 5.5$, $B_\pi^1 = -4.1$, $C^1 = 5$, $C_1^1 = 5$, $C_2^1 = 1.25$)

Figure 4.2 shows an example for g and h. It can be seen that in this particular example h seems to be bounded from above, i.e. the difference of the two market states as perceived by the large investor cannot be arbitrarily large.

The proof of Lemma 4.27 directly implies the following two corollaries.

Corollary 4.28 (Time-dependent bounds on g and h)**.** *Let h and g be given by (4.22), (4.23) subject to the boundary conditions (4.24). Then*

$$-(\Psi^1 - \xi^1(T))^-(T-t) \le g(t) \le (\Psi^0 - \xi^0(T))^+(T-t) \ (\varepsilon > 0),$$
$$\textit{resp. } 0 \le g(t) \le \Psi^0(T-t) \ (\varepsilon = 0) \textit{ and}$$
$$0 \le h(t) \le (\Psi^0 - \Psi^1)(T-t)$$

for $t \in [0,T]$ where the $\xi^i(T)$'s are as defined in the proof of Lemma 4.9.

Corollary 4.29 (Time-independent bound on h)**.** *Let h and g be given by (4.22), (4.23) subject to the boundary conditions (4.24). Then*

$$0 \le h(t) \le \bar{h}$$

for $t \in [0,T]$ where $\bar{h}$ is the smallest positive root of $\bar{F}(y) \triangleq -(\Psi^0 - \Psi^1) + \chi^0(y) + \chi^1(y)$ if such a root exists; otherwise $\bar{h} = \infty$, i.e.

$$\bar{h} = \min\left\{y \in \mathbb{R}_0^+ \,:\, \bar{F}(y) = 0\right\},$$

with the convention $\min\emptyset \triangleq \infty$, where the χ^i's are as defined in the proof of Lemma 4.9.

Proof. The assertion follows from a simple ODE argument since

$$F(t,x,y) = -(\Psi^0 - \Psi^1) + \varrho(t,x,y) + \chi^0(y) + \chi^1(y) \ge \bar{F}(y)$$

for all $(t,x,y) \in [0,T] \times \mathbb{R} \times \mathbb{R}_0^+$ where we used the notation from the proof of Lemma 4.9. □

Figure 4.2 reveals that the boundary function on h as presented in Corollary 4.28 is suitable for large values of t whereas for small values of t the non-linear behavior of h implies that the boundary function overshoots h by far. The converse is true for the bound given in Corollary 4.29. Thus combining the two bounds on h yields

$$0 \le h(t) \le \min\{(\Psi^0 - \Psi^1)(T-t), \bar{h}\}$$

for $t \in [0,T]$.

The following theorem ensures that the strategy given in Lemmas 4.17 and 4.19 is indeed the optimal strategy for the optimal investment problem.

Theorem 4.30 (Solution of the Investment Problem)**.** *Let condition (NJ) be satisfied, and let h and g be given by (4.22), (4.23) subject to the boundary conditions (4.24). Then the strategy*

$$(\pi^{i,*}(t), c^{i,*}(t)) \triangleq (\pi^{i,*}(h(t)), c^{i,*}(t, g(t) - 1_{\{i=1\}}h(t))), \ t \in [0,T], \ i = 0, 1,$$

as given in Lemmas 4.17 and 4.19 is optimal for the investment problem (P) with CRRA preferences and relative risk aversion R.

Proof. Since $(\pi^{i,*}(t), c^{i,*}(t))$ maximizes the HJB-system (4.21) for each $t \in [0,T]$, optimality of the strategy $(\pi^{i,*}, c^{i,*})$ follows directly from the Verification Theorem 3.3. □

Remark 4.31. *Notice that the optimal consumption rate is of the Merton type but differs from the genuine Merton consumption rate as the underlying functions g and h are different from those in the Merton setting.*

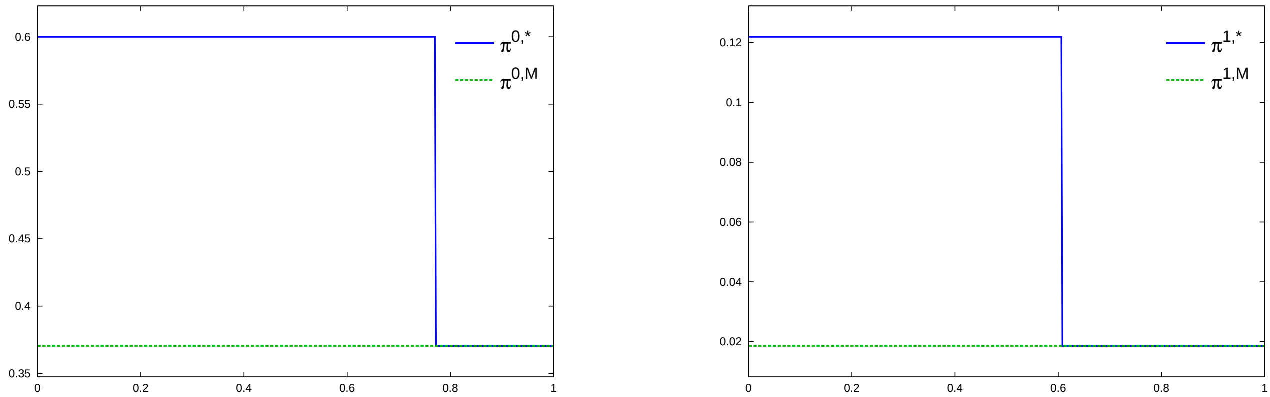

Figure 4.3: Optimal strategy vs. Merton strategy: $\pi^{i,*}$ and $\pi^{i,M}$ as functions of t
($r^0 = r^1 = 0.03$, $\eta^0 = 0.1$, $\eta^1 = 0.02$, $\sigma^0 = 0.3$, $\sigma^1 = 0.6$, $\delta = 0.035$, $\varepsilon = 1$, $R = 3$, $T = 1$,
$A^0 = 17$, $B^0_\pi = -5$, $C^0 = 14$, $C^0_1 = 5$, $C^0_2 = 10$, $A^1 = 5.5$, $B^1_\pi = -4.1$, $C^1 = 5$, $C^1_1 = 5$, $C^1_2 = 1.25$)

Figure 4.3 shows the optimal portfolio strategy and the Merton portfolio strategy in a setting where B^i_π is negative for $i = 0, 1$. Hence the large investor is forced to follow a portfolio strategy that temporarily exceeds the Merton proportion in order to generate advantageous regime shift intensities. In the example of Figure 4.3 the deviation is about 23% in state 0 whereas in state 1 it amounts to about 10%.

Further the strategy plotted in Figure 4.3 shows a typical feature of the optimal strategy for the large investor: Usually there exists a point in time, $t^i \in [0, T]$, such that $\pi^{i,*}(t) = \pi^{i,M}$ on $[t^i, T]$, $i = 0, 1$. This is due to the fact that with a decreasing time to maturity a regime shift becomes more and more improbable. Thus the only reason forcing the investor to deviate from the Merton strategy loses its strength when the time to maturity gets smaller. Hence the investor follows the Merton strategy as soon as the maturity is close enough.

The special form of the optimal portfolio strategy suggests a separation into the Merton strategy and an additional hedging component, i.e.

$$\pi^{i,*}(t) = \pi^{i,M} + \pi^{i,H}(t), \;\; i = 0, 1,$$

where the hedging component $\pi^{i,H}$ is given by

$$\pi^{i,H}(t) = -(\sigma^i.(\sigma^i)^\top)^{-1}.B^i_\pi \tfrac{(A^i + (\pi^{i,M})^\top.B^i_\pi - C^i)^+}{(B^i_\pi)^\top.(\sigma^i.(\sigma^i)^\top)^{-1}.B^i_\pi} 1_{\{h(t) \geq h^{i,\text{crit}}\}}.$$

This hedging component represents the necessary adjustment of the Merton strategy due to the investor's influence on the regime shift intensities.

Notice that $(\pi^{i,H}(t))^\top.B^i_\pi \leq 0$. The following lemma shows that $\pi^{i,H}$ benefits the regime shift intensities.

Lemma 4.32. *The optimal strategy* $(\pi^{i,*}, c^{i,*})$ *satisfies*

$$\vartheta^{0,1}(\pi^{0,*}(t), c^{0,*}(t)) \leq \vartheta^{0,1}(\pi^{0,M}, c^{0,M}(t)) \text{ and } \vartheta^{1,0}(\pi^{1,*}(t), c^{1,*}(t)) \geq \vartheta^{1,0}(\pi^{1,M}, c^{1,M}(t))$$

for every $t \in [0, T]$.

Proof. The assertion follows since

$$\begin{aligned}\vartheta^{0,1}(\pi^{0,*}(t),c^{0,*}(t)) &= C_1^0 1_{\{A^0+(\pi^{0,*}(t))^\top.B_\pi^0\le C^0\}} + C_2^0 1_{\{A^0+(\pi^{0,*}(t))^\top.B_\pi^0>C^0\}}\\ &= C_1^0 1_{\{A^0+(\pi^{0,M})^\top.B_\pi^0+(\pi^{0,H}(t))^\top.B_\pi^0\le C^0\}} + C_2^0 1_{\{A^0+(\pi^{0,M}(t))^\top.B_\pi^0(\pi^{0,H}(t))^\top.B_\pi^0>C^0\}}\\ &\le C_1^0 1_{\{A^0+(\pi^{0,M})^\top.B_\pi^0\le C^0\}} + C_2^0 1_{\{A^0+(\pi^{0,M}(t))^\top.B_\pi^0>C^0\}}\\ &= \vartheta^{0,1}(\pi^{0,M},c^{0,M}(t))\end{aligned}$$

and

$$\begin{aligned}\vartheta^{1,0}(\pi^{1,*}(t),c^{1,*}(t)) &= C_1^1 1_{\{A^1+(\pi^{1,*}(t))^\top.B_\pi^1\le C^1\}} + C_2^1 1_{\{A^1+(\pi^{1,*}(t))^\top.B_\pi^1>C^1\}}\\ &= C_1^1 1_{\{A^1+(\pi^{1,M})^\top.B_\pi^1+(\pi^{1,H}(t))^\top.B_\pi^1\le C^1\}} + C_2^1 1_{\{A^1+(\pi^{1,M}(t))^\top.B_\pi^1+(\pi^{1,H}(t))^\top.B_\pi^1>C^1\}}\\ &\ge C_1^1 1_{\{A^1+(\pi^{1,M})^\top.B_\pi^1\le C^1\}} + C_2^1 1_{\{A^1+(\pi^{1,M}(t))^\top.B_\pi^1>C^1\}}\\ &= \vartheta^{1,0}(\pi^{1,M},c^{1,M}(t))\end{aligned}$$

as $(\pi^{i,H}(t))^\top.B_\pi^i \le 0$. □

Remark 4.33. *Notice that the optimal strategy is a* compromise strategy*. At first glance the large investor is optimizing only the utility criterion of trading optimally in terms of generating the highest possible expected utility from intermediate consumption and final wealth. But as compared to the Merton, resp. the Bäuerle-Rieder model the investor also faces another criterion hidden in the utility criterion: The investor aims at investing in such a way that the regime shift intensities are as favorable as possible. We call this hidden criterion intensity criterion.*

Of course, the regime shift intensities implied by the choice of a certain strategy determine the performance of the investor's wealth and hence the intensity criterion is already covered by the utility criterion. But since the investor's influence on the regime shift intensities is the key point of our model we regard the intensity criterion as a second criterion the investor has to deal with when solving the optimal investment problem.

Thus the large investor faces a trade-off between optimizing the utility criterion on the one hand and optimizing the intensity criterion on the other hand. Pursuing only the utility goal would imply the Merton strategies to be optimal. However, solely aiming at favorable intensities would render any strategy in the favorable half space $\mathcal{F}^{\pi,i}$ *to be optimal.*

The optimal strategy that we derived here lies – as combination of the Merton strategy and the critical strategy which is part of $\mathcal{F}^{\pi,i}$ *– somewhere in between those two extreme strategies. Thus it is in general different from the Merton strategy but generates better regime shift intensities than those the Merton strategy would yield, i.e. it accommodates the aforementioned trade-off so that the notion* compromise strategy *is legitimate.*

4.2.2 Consumption-dependent Intensities

Instead of the portfolio proportions now the consumption rate determines the intensities of regime shifts, i.e. we let

$$B^i_\pi = 0 \text{ and } B^i_c \neq 0 \text{ for } i = 0, 1, \tag{CD}$$

so that the intensities are given by

$$\vartheta^{i,1-i}(\pi, c) = C^i_1 1_{\{A^i + B^i_c c \leq C^i\}} + C^i_2 1_{\{A^i + B^i_c c > C^i\}}.$$

We denote by

$$\mathcal{F}^{c,i} \triangleq \left\{ c \in \mathbb{R}^+_0 \,:\, A^i + B^i_c c \leq C^i \right\}$$

the half space of all consumption rates that are favorable for the investor concerning the intensities of regime shifts. In analogy to the previous subsection we let

$$d^{c,i}(c) \triangleq A^i + B^i_c c - C^i$$

denote the 'distance' of the strategy c to the separating hyperplane, i.e. $c \in \mathcal{F}^{c,i}$ if and only if $d^{c,i}(c) \leq 0$.

In order to determine the maximizer in the HJB-system (4.20) we define functions $H^{\pi,i} : \mathbb{R}^{\bar{n}} \to \mathbb{R}$ and $H^{c,i} : [0,T] \times \mathbb{R} \times \mathbb{R}^+_0 \times \mathbb{R}^+_0 \to \mathbb{R}$, $i = 0, 1$, given by

$$\begin{aligned} H^{\pi,i}(\pi) &\triangleq r^i + \pi^\top.\eta^i - \tfrac{1}{2} R \pi^\top.\sigma^i.(\sigma^i)^\top.\pi, \\ H^{c,i}(t,x,y,c) &\triangleq \tfrac{\varepsilon\delta}{\varepsilon - (\varepsilon - \delta) e^{-\delta(T-t)}} \tfrac{1}{1-R} (e^{-(1-R)x} c^{1-R} - 1) - c \\ &\quad + \left(C^i_1 1_{\{c \in \mathcal{F}^{c,i}\}} + C^i_2 1_{\{c \notin \mathcal{F}^{c,i}\}} \right) \tfrac{1}{1-R} (e^{(-1)^{1-i}(1-R)y} - 1) \end{aligned}$$

where we use the already mentioned convention $0^{1-R} \triangleq \infty$ if $R > 1$. Notice that the dependency of $H^{c,i}$ on t and x vanishes if $\varepsilon = 0$.

The HJB-system (4.20) is given by

$$0 = \sup_{(\pi,c) \in \mathbb{R}^{\bar{n}} \times \mathbb{R}^+_0} \left\{ g'(t) - 1_{\{i=1\}} h'(t) + H^{\pi,i}(\pi) + H^{c,i}(t, g(t) - 1_{\{i=1\}} h(t), h(t), c) \right\} \tag{4.25}$$

for $t \in [0,T)$ and $i = 0, 1$, subject to the boundary conditions

$$g(T) = 0, \;\; h(T) = 0.$$

As in the portfolio-dependent case taking the supremum over $(\pi, c) \in \mathbb{R}^{\bar{n}} \times \mathbb{R}^+_0$ can be separated into two unrelated maximizations; one over $\pi \in \mathbb{R}^{\bar{n}}$ and one over $c \in \mathbb{R}^+_0$.

To find the supremum in (4.25) we present the maximizers of the functions $H^{\pi,i}$ and $H^{c,i}(t,x,y,\cdot)$ for arbitrary $(t,x,y) \in [0,T] \times \mathbb{R} \times \mathbb{R}^+_0$. This yields a family of maximizers dependent on (t,x,y). The maximizers of the HJB-system (4.25) are then obtained by replacing x and y by $g(t) - 1_{\{i=1\}} h(t)$ and $h(t)$.

Concerning the portfolio proportions the concavity of $H^{\pi,i}$ and the first-order condition imply

Lemma 4.34 (Maximizer of $H^{\pi,i}$)**.** *The maximizer*

$$\pi^{i,*} \triangleq \operatorname*{arg\,max}_{\pi\in\mathbb{R}^{\bar{n}}} H^{\pi,i}(\pi),\ \ i=0,1,$$

is given by the Merton strategy, i.e.

$$\pi^{i,*} = \pi^{i,M}.$$

In order to determine the maximizing consumption rate we let $H_j^{c,i} : [0,T]\times\mathbb{R}\times\mathbb{R}_0^+\times\mathbb{R}_0^+ \to \mathbb{R}$, $i=0,1$ and $j=1,2$, given by

$$H_j^{c,i}(t,x,y,c) \triangleq \tfrac{\varepsilon\delta}{\varepsilon-(\varepsilon-\delta)e^{-\delta(T-t)}}\tfrac{1}{1-R}(e^{-(1-R)x}c^{1-R}-1) - c + C_j^i\tfrac{1}{1-R}(e^{(-1)^{1-i}(1-R)y}-1)$$

such that $H^{c,i}(t,x,y,c) = H_1^{c,i}(t,x,y,c)1_{\{c\in\mathcal{F}^{c,i}\}} + H_2^{c,i}(t,x,y,c)1_{\{c\notin\mathcal{F}^{c,i}\}}$. Again $H_j^{c,i}$ is *de facto* a function solely of y and c in case of $\varepsilon=0$.

If $\varepsilon>0$ then $H_j^{c,i}(t,x,y,\cdot)$ is concave for every $(t,x,y)\in[0,T]\times\mathbb{R}\times\mathbb{R}_0^+$ and $j=1,2$. Therefore the two candidate solutions for the maximization of $H^{c,i}(t,x,y,\cdot)$ over c in (4.25) are c^M and $\tilde{c}^{i,\mathrm{crit}}$ where

$$\tilde{c}^{i,\mathrm{crit}}(t) \triangleq \operatorname*{arg\,max}_{\left\{c\in\mathbb{R}_0^+ \,:\, A^i+B_c^ic=C^i\right\}} H^{c,i}(t,x,y,c).$$

If $\varepsilon=0$ then $H_j^{c,i}(t,x,y,\cdot)$ is linearly decreasing in c implying $c^M(t,x)=0$.

Note that $\tilde{c}^{i,\mathrm{crit}}$ only exists if $B_c^i>0$ and $A^i\le C^i$, resp. $B_c^i<0$ and $A^i\ge C^i$, where it is given by $-\frac{A^i-C^i}{B_c^i}$ for $i=0,1$. Otherwise the set $\left\{c\in\mathbb{R}_0^+ \,:\, A^i+B_c^ic=C^i\right\}$ would be empty.

Having established the candidate solutions the following lemma presents the desired maximizers.

Lemma 4.35 (Maximizer of $H^{c,i}(t,x,y,\cdot)$)**.** *For every $(t,x,y)\in[0,T]\times\mathbb{R}\times\mathbb{R}_0^+$ the maximizer*

$$c^{i,*}(t,x,y) \triangleq \operatorname*{arg\,max}_{c\in\mathbb{R}_0^+} H^{c,i}(t,x,y,c),\ \ i=0,1,$$

is given by

$$c^{i,*}(t,x,y) = \begin{cases} c^M(t,x) & \text{if } y < h^{i,\mathrm{crit}}(t,x),\\ c^{i,\mathrm{crit}}(t,x) & \text{if } y \ge h^{i,\mathrm{crit}}(t,x),\end{cases}$$

where

$$h^{i,\mathrm{crit}}(t,x) \triangleq (-1)^{1-i}\tfrac{1}{1-R}\ln\Big((1-R)\tfrac{\zeta^{i,\mathrm{crit}}(t,x)}{C_2^i-C_1^i}+1\Big)$$

with

$$\zeta^{i,\mathrm{crit}}(t,x) \triangleq \begin{cases} 0 & \text{if } B_c^i<0 \text{ and } A^i<C^i,\\ \Big[\tfrac{1}{1-R}\Big(\big(1-\tfrac{(d^{c,i}(c^M(t,x)))^+}{B_c^ic^M(t,x)}\big)^{1-R}-1\Big)+\tfrac{(d^{c,i}(c^M(t,x)))^+}{B_c^ic^M(t,x)}\Big]c^M(t,x) & \text{if } B_c^i<0 \text{ and } A^i\ge C^i,\\ & \text{or } B_c^i>0 \text{ and } A^i\le C^i,\\ -\infty & \text{if } B_c^i>0 \text{ and } A^i>C^i,\end{cases}$$

if $\varepsilon>0$, resp.

$$\zeta^{i,\mathrm{crit}}(t,x) \triangleq \begin{cases} 0 & \text{if } B_c^i<0 \text{ and } A^i<C^i,\\ \frac{(A^i-C^i)^+}{B_c^i} & \text{if } B_c^i<0 \text{ and } A^i\ge C^i,\\ & \text{or } B_c^i>0 \text{ and } A^i\le C^i,\\ -\infty & \text{if } B_c^i>0 \text{ and } A^i>C^i,\end{cases}$$

if $\varepsilon = 0$, *and*

$$c^{i,\mathrm{crit}}(t,x) \triangleq \begin{cases} c^M(t,x) & \text{if } B_c^i > 0 \text{ and } A^i > C^i, \text{ or } B_c^i < 0 \text{ and } A^i < C^i, \\ c^M(t,x) - \frac{(d^{c,i}(c^M(t,x)))^+}{B_c^i} & \text{if } B_c^i > 0 \text{ and } A^i \leq C^i, \text{ or } B_c^i < 0 \text{ and } A^i \geq C^i. \end{cases}$$

Remark 4.36. *Notice that in contrast to the portfolio-dependent setting the critical value $h^{i,\mathrm{crit}}$ is no longer constant. It is now a function of the Merton type consumption rate c^M. Only if $\varepsilon = 0$ then $\zeta^{i,\mathrm{crit}}$ and consequently $h^{i,\mathrm{crit}}$ are just constants independent of t and x. Further $\varepsilon = 0$ implies $c^M(t,x) = 0$ such that $c^{i,\mathrm{crit}}$ is constant, too. The maximizing consumption rate $c^{i,*}$ is therefore just a function of y.*

Remark 4.37. *We use the following shorthand notations*

$$H^{c,i}_{1,\mathrm{crit}}(t,x,y) \triangleq H_1^{c,i}(t,x,y,c^{i,\mathrm{crit}}(t,x)) \text{ and } H^{c,i}_{2,M}(t,x,y) \triangleq H_2^{c,i}(t,x,y,c^M(t,x))$$

for $(t,x,y) \in [0,T] \times \mathbb{R} \times \mathbb{R}_0^+$ and $i = 0,1$.

Remark 4.38. *The typical shapes of the functions $H^{c,i}(t,x,y,\cdot)$, $H_1^{c,i}(t,x,y,\cdot)$ and $H_2^{c,i}(t,x,y,\cdot)$ are comparable to those of the functions $H^{\pi,i}(y,\cdot)$, $H_1^{\pi,i}(y,\cdot)$ and $H_2^{\pi,i}(y,\cdot)$ as presented in Figure 4.1, i.e. $H_1^{c,i}(t,x,y,\cdot)$ and $H_2^{c,i}(t,x,y,\cdot)$ are also strictly concave and exhibit a unique maximum. Thus Figure 4.1 also serves as helpful illustration for the proof of Lemma 4.35.*

Proof of Lemma 4.35. Let $(t,x,y) \in [0,T] \times \mathbb{R} \times \mathbb{R}_0^+$ be given. A straight forward optimization yields that the two functions $H_1^{c,i}(t,x,y,\cdot)$ and $H_2^{c,i}(t,x,y,\cdot)$ attain their maxima at $c^M(t,x)$. Further, since $C_1^0 < C_2^0$ and $C_1^1 > C_2^1$ we get that

$$H_1^{c,i}(t,x,y,c) > H_2^{c,i}(t,x,y,c) \text{ for all } c \in \mathbb{R}_0^+. \qquad (*)$$

First we consider the trivial cases $B_c^i < 0$ and $A^i < C^i$, resp. $B_c^i > 0$ and $A^i > C^i$ in which the intensity functions are constant on the whole $\mathbb{R}^{\bar{n}} \times \mathbb{R}_0^+$.

- $B_c^i < 0$ *and* $A^i < C^i$. If $B_c^i < 0$ and $A^i < C^i$ then $c \in \mathcal{F}^{c,i}$ for all $c \in \mathbb{R}_0^+$. Therefore, $H^{c,i} = H_1^{c,i}$ and consequently the maximizer of $H^{c,i}(t,x,y,\cdot)$ is given by the Merton rate, i.e.

$$B_c^i < 0 \text{ and } A^i < C^i \quad \Rightarrow \quad c^{i,*}(t,x,y) = c^M(t,x).$$

- $B_c^i > 0$ *and* $A^i > C^i$. If $B_c^i > 0$ and $A^i > C^i$ then $c \notin \mathcal{F}^{c,i}$ for all $c \in \mathbb{R}_0^+$. Thus, $H^{c,i} = H_2^{c,i}$ and the maximizer of $H^{c,i}(t,x,y,\cdot)$ is again given by the Merton rate, i.e.

$$B_c^i > 0 \text{ and } A^i > C^i \quad \Rightarrow \quad c^{i,*}(t,x,y) = c^M(t,x).$$

- $B_c^i > 0$ *and* $A^i \leq C^i$, *resp.* $B_c^i < 0$ *and* $A^i \geq C^i$. In case of $B_c^i > 0$ and $A^i \leq C^i$, resp. $B_c^i < 0$ and $A^i \geq C^i$, the intensity functions exhibit a jump in $\mathbb{R}^{\bar{n}} \times \mathbb{R}_0^+$ and we need to distinguish the cases $c^M(t,x) \in \mathcal{F}^{c,i}$ and $c^M(t,x) \notin \mathcal{F}^{c,i}$.

  - $c^M(t,x) \in \mathcal{F}^{c,i}$. By $(*)$ it follows that the maximizer of $H^{c,i}(t,x,y,\cdot)$ is given by the Merton rate, i.e.

$$c^M(t,x) \in \mathcal{F}^{c,i} \quad \Rightarrow \quad c^{i,*}(t,x,y) = c^M(t,x).$$

- $c^M(t,x) \notin \mathcal{F}^{c,i}$. Otherwise, if $c^M(t,x) \notin \mathcal{F}^{c,i}$ then $(*)$ implies that the maximizer of $H^{c,i}(t,x,y,\cdot)$ is either $c^M(t,x)$ in case of $H^{c,i}_{2,M}(t,x,y) > H^{c,i}_{1,\mathrm{crit}}(t,x,y)$ or $c^{i,\mathrm{crit}}(t,x)$ in case of $H^{c,i}_{2,M}(t,x,y) \leq H^{c,i}_{1,\mathrm{crit}}(t,x,y)$. Hence with

$$
\begin{aligned}
& H^{c,i}_{2,M}(t,x,y) \leq H^{c,i}_{1,\mathrm{crit}}(t,x,y) \\
\Leftrightarrow\quad & (C^i_2 - C^i_1)\tfrac{1}{1-R}(e^{(-1)^{1-i}(1-R)y} - 1) \\
& \leq \begin{cases} \Big[\frac{1}{1-R}\Big(\big(1-\frac{(d^{c,i}(c^M(t,x)))^+}{B^i_c c^M(t,x)}\big)^{1-R} - 1\Big) + \frac{(d^{c,i}(c^M(t,x)))^+}{B^i_c c^M(t,x)}\Big] c^M(t,x) & \text{if } \varepsilon > 0, \\ \frac{(A^i - C^i)^+}{B^i_c} & \text{if } \varepsilon = 0, \end{cases} \\
\Leftrightarrow\quad & (C^i_2 - C^i_1)\tfrac{1}{1-R}(e^{(-1)^{1-i}(1-R)y} - 1) \leq \zeta^{i,\mathrm{crit}}(t,x) \\
\Leftrightarrow\quad & y \geq (-1)^{1-i}\tfrac{1}{1-R}\ln\Big((1-R)\tfrac{\zeta^{i,\mathrm{crit}}(t,x)}{C^i_2 - C^i_1} + 1\Big) \\
\Leftrightarrow\quad & y \geq h^{i,\mathrm{crit}}(t,x).
\end{aligned}
$$

  we get

$$
c^M(t,x) \notin \mathcal{F}^{c,i} \quad\Rightarrow\quad c^{i,*}(t,x,y) = \begin{cases} c^M(t,x) & \text{if } y < h^{i,\mathrm{crit}}(t,x), \\ c^{i,\mathrm{crit}}(t,x) & \text{if } y \geq h^{i,\mathrm{crit}}(t,x). \end{cases}
$$

  As $h^{i,\mathrm{crit}}(t,x) = 0$ if $c^M(t,x) \in \mathcal{F}^{c,i}$ the latter formula also covers the case $c^M(t,x) \in \mathcal{F}^{c,i}$.

Finally with $h^{i,\mathrm{crit}}(t,x)$ and $c^{i,\mathrm{crit}}(t,x)$ as defined for the trivial parameter specifications the maximizing strategy in state i is given by

$$
c^{i,*}(t,x,y) = \begin{cases} c^M(t,x) & \text{if } y < h^{i,\mathrm{crit}}(t,x), \\ c^{i,\mathrm{crit}}(t,x) & \text{if } y \geq h^{i,\mathrm{crit}}(t,x). \end{cases}
$$

Thus the lemma is proven. □

Remark 4.39 (Interpretation)**.** *The maximizing consumption rate coincides with the Merton consumption rate when the latter belongs to the half space of favorable consumption rates. If this is not the case then the maximizing consumption rate is given by the Merton consumption rate as long as y is not bigger than $h^{i,\mathrm{crit}}(t,x)$. For all $y < h^{i,\mathrm{crit}}(t,x)$ it is not reasonable to deviate from the Merton rate, i.e. the utility criterion dominates the intensity criterion as discussed in Remark 4.33. Only if y overshoots the critical barrier $h^{i,\mathrm{crit}}(t,x)$ then the maximizing consumption rate switches to the critical rate $c^{i,\mathrm{crit}}(t,x)$ since then the intensity criterion is more important than the utility criterion.*

Remark 4.40. *Even if $\varepsilon > 0$ it is possible that the maximizing consumption rate $c^{i,*}(t,x,y)$ is zero. This happens if $y \geq h^{i,\mathrm{crit}}(t,x)$ in case of $B^i_c > 0$ and $A^i = C^i$ with $R < 1$ because in this setting the critical consumption rate is given by $c^{i,\mathrm{crit}}(t,x) = 0$. Notice that $h^{i,\mathrm{crit}}(t,x)$ is finite if $B^i_c > 0$ and $A^i = C^i$ with $R < 1$ so that $y \geq h^{i,\mathrm{crit}}(t,x)$ is possible.*

Remark 4.41 (Discontinuity of $c^{i,*}$)**.** *The maximizing consumption rate $c^{i,*}$ exhibits jumps at all points $(\hat{t},\hat{x},\hat{y})$ in which $\hat{y} = h^{i,\mathrm{crit}}(\hat{t},\hat{x})$ and moreover $c^M(\hat{t},\hat{x}) \neq c^{i,\mathrm{crit}}(\hat{t},\hat{x})$.*

Remark 4.42 (Interpretation in case of $\varepsilon = 0$)**.** *Choosing $\varepsilon = 0$ represents a model in which the investor does not draw any utility from intermediate consumption at least concerning the direct impact of consumption on the utility functions. In the standard model without the investor's influence the optimal consumption rate consequently is 0. In our model this is different. When $y \geq h^{i,\text{crit}}$ in case of $B_c^i < 0$ and $A^i > C^i$ then the maximizing consumption rate is given by $c^{i,\text{crit}} = -\frac{A^i - C^i}{B_c^i} > 0$. Thus, the large investor consumes although this is not beneficial in terms of the utility from intermediate consumption. The reason for this behavior is that except for the case of $B_c^i < 0$ and $A^i > C^i$ a consumption rate of 0 yields the smallest ($i = 0$), resp. biggest ($i = 1$), possible intensity of a jump to the adverse ($i = 0$), resp. favorable ($i = 1$), market state. Only if $B_c^i < 0$ and $A^i > C^i$ then consuming at a rate of $-\frac{A^i - C^i}{B_c^i} > 0$ yields a smaller ($i = 0$), resp. bigger ($i = 1$), intensity than consuming at a rate of 0. Therefore, in order to reduce ($i = 0$), resp. augment ($i = 1$), the probability of a jump to the adverse ($i = 0$), resp. favorable ($i = 1$), market state, the large investor may consume.*

So far the above explanations are only relevant for the general maximizing consumption rate $c^{i,*}(t,x,y)$ and not necessarily for the optimal consumption rate. This is because it is not clear whether the situation $h(t) \geq h^{i,\text{crit}}$ – remember that the maximizing strategy of the HJB-system (4.25) is obtained by choosing $y = h(t)$ – could really occur. But later on we will provide an example showing that the aforementioned situation is really possible.

The following lemma provides a link between the maximizing consumption rate in case of $\varepsilon > 0$ and $\varepsilon = 0$.

Lemma 4.43 (Limiting behavior of $c^{i,\text{crit}}$, $h^{i,\text{crit}}$ and $c^{i,*}$ as ε tends to 0)**.** *Denote by $c^{i,\text{crit}}_{\varepsilon>0}$, $h^{i,\text{crit}}_{\varepsilon>0}$ and $c^{i,*}_{\varepsilon>0}$, resp. $c^{i,\text{crit}}_{\varepsilon=0}$, $h^{i,\text{crit}}_{\varepsilon=0}$ and $c^{i,*}_{\varepsilon=0}$, the critical consumption rate, the critical barrier and the maximizing consumption rate in case of $\varepsilon > 0$, resp. $\varepsilon = 0$. Then the following holds true for every $(t,x,y) \in [0,T] \times \mathbb{R} \times \mathbb{R}_0^+$.*

i) $\lim_{\varepsilon\to 0} c^{i,\text{crit}}_{\varepsilon>0}(t,x) = c^{i,\text{crit}}_{\varepsilon=0}(t,x)$.

ii) $\lim_{\varepsilon\to 0} h^{i,\text{crit}}_{\varepsilon>0}(t,x) = h^{i,\text{crit}}_{\varepsilon=0}(t,x)$ *unless $B_c^i > 0$ and $A^i = C^i$ in case of $R > 1$.*

iii) $\lim_{\varepsilon\to 0} c^{i,*}_{\varepsilon>0}(t,x,y) = c^{i,*}_{\varepsilon=0}(t,x,y)$.

Proof. Let $(t,x,y) \in [0,T] \times \mathbb{R} \times \mathbb{R}_0^+$ be fixed and notice that $\lim_{\varepsilon\to 0} c^M(t,x) = 0$ for every $(t,x) \in [0,T] \times \mathbb{R}$.

ad i) The assertion follows directly from $\lim_{\varepsilon\to 0} c^M(t,x) = 0$.

ad ii) If $B_c^i < 0$ and $A^i < C^i$, or $B_c^i > 0$ and $A^i > C^i$ then there is nothing to prove. So let $B_c^i < 0$ and $A^i \geq C^i$, or $B_c^i > 0$ and $A^i \leq C^i$ in the following and consider

$$
\begin{aligned}
\lim_{\varepsilon\to 0} \zeta^{i,\text{crit}}_{\varepsilon>0}(t,x) &= \lim_{\varepsilon\to 0}\Big(\Big[\tfrac{1}{1-R}\Big(\big(1 - \tfrac{(d^{c,i}(c^M(t,x)))^+}{B_c^i c^M(t,x)}\big)^{1-R} - 1\Big) + \tfrac{(d^{c,i}(c^M(t,x)))^+}{B_c^i c^M(t,x)}\Big] c^M(t,x)\Big) \\
&= \lim_{\varepsilon\to 0} \tfrac{(d^{c,i}(c^M(t,x)))^+}{B_c^i} + \lim_{\varepsilon\to 0} \tfrac{1}{1-R}\Big(\big(1 - \tfrac{(d^{c,i}(c^M(t,x)))^+}{B_c^i c^M(t,x)}\big)^{1-R} - 1\Big) c^M(t,x) \\
&= \tfrac{(A^i - C^i)^+}{B_c^i} + \lim_{\varepsilon\to 0} \tfrac{1}{1-R}\Big(\big(1 - \tfrac{(d^{c,i}(c^M(t,x)))^+}{B_c^i c^M(t,x)}\big)^{1-R} - 1\Big) c^M(t,x)
\end{aligned}
$$

We now show under which conditions $\lim_{\varepsilon\to 0}\frac{1}{1-R}\Big(\big(1-\frac{(d^{c,i}(c^M(t,x)))^+}{B_c^i c^M(t,x)}\big)^{1-R}-1\Big)c^M(t,x) = 0$ which implies $\lim_{\varepsilon\to 0}\zeta^{i,\text{crit}}_{\varepsilon>0}(t,x)=\zeta^{i,\text{crit}}_{\varepsilon=0}(t,x)$ and consequently also the assertion. To analyze the last limit we have to distinguish the following cases.

- $B_c^i<0$ *and* $A^i=C^i$. Here $(d^{c,i}(c^M(t,x)))^+=0$ which implies
$$\frac{1}{1-R}\Big(\big(1-\frac{(d^{c,i}(c^M(t,x)))^+}{B_c^i c^M(t,x)}\big)^{1-R}-1\Big)c^M(t,x)=\frac{1}{1-R}\Big(1^{1-R}-1\Big)c^M(t,x)=0.$$
- $B_c^i<0$ *and* $A^i>C^i$. Consider
$$\begin{aligned}&\lim_{\varepsilon\to 0}\frac{1}{1-R}\Big(\big(1-\frac{(d^{c,i}(c^M(t,x)))^+}{B_c^i c^M(t,x)}\big)^{1-R}-1\Big)c^M(t,x)\\ &=\lim_{\varepsilon\to 0}\frac{1}{1-R}\Big(\big(\frac{B_c^i c^M(t,x)-(d^{c,i}(c^M(t,x)))^+}{B_c^i}\big)^{1-R}(c^M(t,x))^{-(1-R)}-1\Big)c^M(t,x)\\ &=\lim_{\varepsilon\to 0}\frac{1}{1-R}\Big(\big(\frac{B_c^i c^M(t,x)-(d^{c,i}(c^M(t,x)))^+}{B_c^i}\big)^{1-R}(c^M(t,x))^{-(1-R)}c^M(t,x)-c^M(t,x)\Big)\\ &=\lim_{\varepsilon\to 0}\frac{1}{1-R}\big(\frac{B_c^i c^M(t,x)-(d^{c,i}(c^M(t,x)))^+}{B_c^i}\big)^{1-R}(c^M(t,x))^{R}-\lim_{\varepsilon\to 0}\frac{1}{1-R}c^M(t,x)\\ &=0\end{aligned}$$
since $\lim_{\varepsilon\to 0}\frac{B_c^i c^M(t,x)-(d^{c,i}(c^M(t,x)))^+}{B_c^i}=-\frac{(A^i-C^i)^+}{B_c^i}$ which is strictly positive as $B_c^i<0$ and $A^i>C^i$.
- $B_c^i>0$ *and* $A^i<C^i$. Consider now
$$\begin{aligned}&\lim_{\varepsilon\to 0}\frac{1}{1-R}\Big(\big(1-\frac{(d^{c,i}(c^M(t,x)))^+}{B_c^i c^M(t,x)}\big)^{1-R}-1\Big)c^M(t,x)\\ &=\lim_{\varepsilon\to 0}\frac{1}{1-R}\Big(\big(\frac{B_c^i c^M(t,x)-(d^{c,i}(c^M(t,x)))^+}{B_c^i c^M(t,x)}\big)^{1-R}-1\Big)c^M(t,x)\\ &=0\end{aligned}$$
since $\lim_{\varepsilon\to 0}\frac{B_c^i c^M(t,x)-(d^{c,i}(c^M(t,x)))^+}{B_c^i c^M(t,x)}=1$ if $B_c^i>0$ and $A^i<C^i$.
- $B_c^i>0$ *and* $A^i=C^i$. Notice that $(d^{c,i}(c^M(t,x)))^+=B_c^i c^M(t,x)$ if $B_c^i>0$ and $A^i=C^i$. Thus $1-\frac{(d^{c,i}(c^M(t,x)))^+}{B_c^i c^M(t,x)}=0$ and therefore
$$\frac{1}{1-R}\big(1-\frac{(d^{c,i}(c^M(t,x)))^+}{B_c^i c^M(t,x)}\big)^{1-R}=\begin{cases}0 & \text{if } R<1,\\ -\infty & \text{if } R>1.\end{cases}$$
Hence $\lim_{\varepsilon\to 0}\frac{1}{1-R}\Big(\big(1-\frac{(d^{c,i}(c^M(t,x)))^+}{B_c^i c^M(t,x)}\big)^{1-R}-1\Big)c^M(t,x)=0$ if $R<1$ whereas the limit is undefined if $R>1$.

ad iii) Assertions i) and ii) imply that $\lim_{\varepsilon\to 0}c^{i,*}_{\varepsilon>0}(t,x,y)=c^{i,*}_{\varepsilon=0}(t,x,y)$ unless $B_c^i>0$ and $A^i=C^i$ in case of $R>1$. Nevertheless the assertion is still true even if $B_c^i>0$, $A^i=C^i$ and $R>1$ because in that case $\lim_{\varepsilon\to 0}c^{i,\text{crit}}_{\varepsilon>0}(t,x)=c^{i,\text{crit}}_{\varepsilon=0}(t,x)=0=\lim_{\varepsilon\to 0}c^M(t,x)$, i.e. as ε tends to 0 the Merton as well as the critical consumption rate $c^{i,\text{crit}}_{\varepsilon>0}(t,x)$ converge towards 0 which is the optimal consumption rate in case of $\varepsilon=0$. Hence in the limit the distinction into the cases $y<h^{i,\text{crit}}_{\varepsilon>0}(t,x)$ and $y\geq h^{i,\text{crit}}_{\varepsilon>0}(t,x)$ is irrelevant and therefore $\lim_{\varepsilon\to 0}c^{i,*}_{\varepsilon>0}(t,x,y)=c^{i,*}_{\varepsilon=0}(t,x,y)$ holds true without any restriction. This finishes the proof. □

Remark 4.44. *So far the strategy $(\pi^{i,*}, c^{i,*})$ is only a candidate solution for the optimal investment problem. To verify the optimality we need to show that the related HJB-system has a global solution that satisfies the required regularity conditions stated in the Verification Theorem 3.3.*

Inserting $(\pi^{i,*}, c^{i,*}(t, g(t) - 1_{\{i=1\}}h(t), h(t)))$ into the reduced HJB-equation (4.25) the latter now becomes a system of coupled backward ODEs which for $\varepsilon > 0$ is given by

$$\begin{aligned}
h'(t) =& -(\Psi^0 - \Psi^1) + \big(\tfrac{\varepsilon\delta}{\varepsilon-(\varepsilon-\delta)e^{-\delta(T-t)}}\big)^{\frac{1}{R}} e^{-\frac{1-R}{R}g(t)} \tfrac{R}{1-R}(e^{\frac{1-R}{R}h(t)} - 1) \\
& - C_2^0 \tfrac{1}{1-R}(e^{-(1-R)h(t)} - 1) + C_2^1 \tfrac{1}{1-R}(e^{(1-R)h(t)} - 1) \\
& - \Big[(C_1^0 - C_2^0)\tfrac{1}{1-R}(e^{-(1-R)h(t)} - 1) + \Big[\tfrac{1}{1-R}\Big(\Big(1 - \tfrac{(A^0+B_c^0 c^M(t,g(t))-C^0)^+}{B_c^0 c^M(t,g(t))}\Big)^{1-R} - 1\Big) \\
& \qquad + \tfrac{(A^0+B_c^0 c^M(t,g(t))-C^0)^+}{B_c^0 c^M(t,g(t))}\Big] c^M(t, g(t))\Big] 1_{\{h(t)\geq h^{0,\mathrm{crit}}(t,g(t))\}} \\
& + \Big[(C_1^1 - C_2^1)\tfrac{1}{1-R}(e^{(1-R)h(t)} - 1) + \Big[\tfrac{1}{1-R}\Big(\Big(1 - \tfrac{(A^1+B_c^1 c^M(t,g(t)-h(t))-C^1)^+}{B_c^1 c^M(t,g(t)-h(t))}\Big)^{1-R} - 1\Big) \\
& \qquad + \tfrac{(A^1+B_c^1 c^M(t,g(t)-h(t))-C^1)^+}{B_c^1 c^M(t,g(t)-h(t))}\Big] c^M(t, g(t)-h(t))\Big] 1_{\{h(t)\geq h^{1,\mathrm{crit}}(t,g(t)-h(t))\}},
\end{aligned} \tag{4.26}$$

$$\begin{aligned}
g'(t) =& -\Psi^0 + \tfrac{\varepsilon\delta}{\varepsilon-(\varepsilon-\delta)e^{-\delta(T-t)}}\Big(1 - \tfrac{R}{1-R}\big(\big(\tfrac{\varepsilon\delta}{\varepsilon-(\varepsilon-\delta)e^{-\delta(T-t)}}\big)^{\frac{1-R}{R}} e^{-\frac{1-R}{R}g(t)} - 1\big)\Big) \\
& - C_2^0 \tfrac{1}{1-R}(e^{-(1-R)h(t)} - 1) \\
& - \Big[(C_1^0 - C_2^0)\tfrac{1}{1-R}(e^{-(1-R)h(t)} - 1) + \Big[\tfrac{1}{1-R}\Big(\Big(1 - \tfrac{(A^0+B_c^0 c^M(t,g(t))-C^0)^+}{B_c^0 c^M(t,g(t))}\Big)^{1-R} - 1\Big) \\
& \qquad + \tfrac{(A^0+B_c^0 c^M(t,g(t))-C^0)^+}{B_c^0 c^M(t,g(t))}\Big] c^M(t, g(t))\Big] 1_{\{h(t)\geq h^{0,\mathrm{crit}}(t,g(t))\}}
\end{aligned} \tag{4.27}$$

subject to the boundary conditions

$$g(T) = 0, \;\; h(T) = 0. \tag{4.28}$$

If $\varepsilon = 0$ then the ODE-system reads

$$\begin{aligned}
h'(t) =& -(\Psi^0 - \Psi^1) \\
& - C_2^0 \tfrac{1}{1-R}(e^{-(1-R)h(t)} - 1) + C_2^1 \tfrac{1}{1-R}(e^{(1-R)h(t)} - 1) \\
& - \Big((C_1^0 - C_2^0)\tfrac{1}{1-R}(e^{-(1-R)h(t)} - 1) + \tfrac{(A^0-C^0)^+}{B_c^0}\Big) 1_{\{h(t)\geq h^{0,\mathrm{crit}}\}} \\
& + \Big((C_1^1 - C_2^1)\tfrac{1}{1-R}(e^{(1-R)h(t)} - 1) + \tfrac{(A^1-C^1)^+}{B_c^1}\Big) 1_{\{h(t)\geq h^{1,\mathrm{crit}}\}},
\end{aligned} \tag{4.29}$$

$$\begin{aligned}
g'(t) =& -\Psi^0 \\
& - C_2^0 \tfrac{1}{1-R}(e^{-(1-R)h(t)} - 1) \\
& - \Big((C_1^0 - C_2^0)\tfrac{1}{1-R}(e^{-(1-R)h(t)} - 1) + \tfrac{(A^0-C^0)^+}{B_c^0}\Big) 1_{\{h(t)\geq h^{0,\mathrm{crit}}\}}
\end{aligned} \tag{4.30}$$

subject to the boundary conditions

$$g(T) = 0, \;\; h(T) = 0. \tag{4.31}$$

Remark 4.45. *The aforementioned ODEs include the ODEs of the classical Merton problem (first rows) and the ODEs of the Bäuerle-Rieder problem with constant regime shift intensities C_2^i (first two rows).*

Remark 4.46 (Implicit barrier on h)**.** *In ODE (4.26) we utilized the condition $h(t) \geq h^{i,\mathrm{crit}}(t, g(t) - 1_{\{i=1\}}h(t))$. In state 0 this is an explicit condition on the function h because $h^{0,\mathrm{crit}}(t, g(t))$ does not depend on $h(t)$. However in state 1 this condition can be either explicit or implicit depending on the parameters A^1, B_c^1 and C^1. For the trivial specifications $B_c^1 < 0$ and $A^1 < C^1$, resp. $B_c^1 > 0$ and $A^1 > C^1$, the barrier $h^{1,\mathrm{crit}}(t, g(t) - h(t))$ is given by 0, resp. ∞, implying that the condition $h(t) \geq h^{1,\mathrm{crit}}(t, g(t) - h(t))$ is explicit. But if $B_c^1 < 0$ and $A^1 \geq C^1$, resp. $B_c^1 > 0$ and $A^1 \leq C^1$, then $h^{1,\mathrm{crit}}(t, g(t)-h(t))$ is by itself a function of $h(t)$. Consequently, the condition $h(t) \geq h^{1,\mathrm{crit}}(t, g(t)-h(t))$ is implicit. Nevertheless we keep this implicit condition for notational convenience and because later on in the logarithmic case ($R = 1$) it turns out that $h^{i,\mathrm{crit}}(t, g(t) - 1_{\{i=1\}}h(t))$ is completely independent of the functions $g(t)$, resp. $g(t) - h(t)$, rendering the condition $h(t) \geq h^{i,\mathrm{crit}}(t, g(t) - 1_{\{i=1\}}h(t))$ explicit in both states and for all parameter specifications. So far the implicit condition $h(t) \geq h^{1,\mathrm{crit}}(t, g(t)-h(t))$ may be better understood when written in the equivalent way as $H_2^{c,1}(t, g(t) - h(t), h(t), c^M(t, g(t) - h(t))) \leq H_1^{c,1}(t, g(t) - h(t), h(t), c^{1,\mathrm{crit}}(t, g(t) - h(t)))$.*

Lemma 4.47. *The ODE-system given by equation (4.26) and (4.27) subject to the boundary conditions (4.28) ($\varepsilon > 0$), resp. (4.29) and (4.30) subject to the boundary conditions (4.31) ($\varepsilon = 0$), admits a unique global solution.*

Before proving the lemma we first provide some helpful technical results.

Lemma 4.48. *The following holds true for $R > 0$ and $z > -1$.*

i) $\frac{-\frac{R}{1-R}\left((1+z)^{-\frac{1-R}{R}}-1\right)}{(1+z)^{-\frac{1}{R}}} \geq z$ *($R \neq 1$), resp.* $\ln(1+z)(1+z) \geq z$ *($R = 1$).*

ii) $-\frac{R}{1-R}((1+z)^{-\frac{1-R}{R}} - 1) \leq z$ *($R \neq 1$), resp.* $\ln(1+z) \leq z$ *($R = 1$).*

iii) $\frac{1}{1-R}((1+z)^{1-R} - 1) \leq z$ *($R \neq 1$), resp.* $\ln((1+z)) \leq z$ *($R = 1$).*

Moreover, the assertions are even true in the limiting case $z \to -1$.

Proof. We prove the above assertions for $R \neq 1$. In case of $R = 1$ the proof remains valid using the convention $\frac{1}{1-R}(z^{1-R} - 1) \triangleq \ln(z)$ for $R = 1$. Let $z > -1$ and define the following auxiliary functions φ_1, φ_2, φ_3 and $\psi :\ (-1, \infty) \to \mathbb{R}$ given by

$$\varphi_1(z) \triangleq \frac{-\frac{R}{1-R}\left((1+z)^{-\frac{1-R}{R}}-1\right)}{(1+z)^{-\frac{1}{R}}},\ \varphi_2(z) \triangleq -\frac{R}{1-R}((1+z)^{-\frac{1-R}{R}} - 1),$$
$$\varphi_3(z) \triangleq \frac{1}{1-R}((1+z)^{1-R} - 1),\ \psi(z) \triangleq z.$$

ad i) Since $\varphi_1'(z) = \frac{1}{1-R}\left((1+z)^{\frac{1-R}{R}} - 1\right) + 1$ is strictly increasing, φ_1 is strictly convex. Further, ψ is the tangent of φ_1 at the point $z = 0$. Therefore $\varphi_1(z) \geq \psi(z)$ and the assertion is proven.

In order to prove that the assertion is still valid in the limiting case $z \to -1$ we have to distinguish the cases $R > 1$ and $R < 1$.

- $R > 1$. If $R > 1$ then the numerator of φ_1 converges to a finite limit as z tends to -1, i.e. $\lim_{z\to-1} -\frac{R}{1-R}\big((1+z)^{-\frac{1-R}{R}}-1\big) = \frac{R}{1-R}$ whereas the denominator diverges, i.e. $\lim_{z\to-1}(1+z)^{-\frac{1}{R}} = \infty$. Hence, $\lim_{z\to-1}\varphi_1(z) = 0 > -1$ and the assertion is proven.

- $R < 1$. If otherwise $R < 1$ then we first write the assertion in an equivalent way

$$\tilde{\varphi}_1(z) \triangleq \frac{\frac{R}{1-R}\big((1+z)^{-\frac{1-R}{R}}-1\big)}{(1+z)^{-\frac{1}{R}}} \leq -z.$$

  Now the numerator as well as the denominator of $\tilde{\varphi}_1$ diverge as z tends to -1, i.e. $\lim_{z\to-1}\frac{R}{1-R}\big((1+z)^{-\frac{1-R}{R}}-1\big) = \infty$ and $\lim_{z\to-1}(1+z)^{-\frac{1}{R}} = \infty$. Thus we can apply l'Hôpital's rule and get

$$\lim_{z\to-1}\tilde{\varphi}_1(z) = \lim_{z\to-1}\frac{\frac{\mathrm{d}}{\mathrm{d}z}\Big(\frac{R}{1-R}\big((1+z)^{-\frac{1-R}{R}}-1\big)\Big)}{\frac{\mathrm{d}}{\mathrm{d}z}\big((1+z)^{-\frac{1}{R}}\big)} = \lim_{z\to-1}R(1+z) = 0 < 1$$

  which is the desired result.

ad ii) Now $\varphi_2'(z) = (1+z)^{-\frac{1}{R}}$ is strictly decreasing implying that φ_2 is strictly concave. Once again, ψ is the tangent of φ_2 at the point $z = 0$ yielding the assertion $\varphi_2(z) \leq \psi(z)$.

Since $\lim_{z\to-1}\varphi_2(z) = -\infty < -1$ ($R \leq 1$), resp. $\lim_{z\to-1}\varphi_2(z) = \frac{R}{1-R} < -1$ ($R > 1$), the assertion is still true in the limiting case.

ad iii) Since $\varphi_3'(z) = (1+z)^{-R}$ is strictly decreasing, φ_3 is strictly concave. Moreover, ψ is the tangent of φ_3 at the point $z = 0$ and therefore the assertion $\varphi_3(z) \leq \psi(z)$ is valid.

As $\lim_{z\to-1}\varphi_3(z) = -\frac{1}{1-R} < -1$ ($R < 1$), resp. $\lim_{z\to-1}\varphi_3(z) = -\infty < -1$ ($R \geq 1$), the limiting case is proven, too. □

Now we can prove Lemma 4.47.

Proof of Lemma 4.47. The proof is essentially the same as the proof of Lemma 4.9. Only the χ^i's are different. Therefore we just present the actual χ^i's and verify that those satisfy the necessary continuity conditions and the non-negativity condition.

- *Definition of the χ^i's.* The χ^i's are given by $\chi^i: [0,T]\times\mathbb{R}\times\mathbb{R}_0^+ \to \mathbb{R}$, $i = 0, 1$, with

$$\begin{aligned}\chi^i(t,x,y) \triangleq (-1)^{1-i}\Big(&C_2^i\tfrac{1}{1-R}(e^{(-1)^{1-i}(1-R)y}-1)\\ &+\Big[(C_1^i - C_2^i)\tfrac{1}{1-R}(e^{(-1)^{1-i}(1-R)y}-1) + \zeta^{i,\mathrm{crit}}(t,x)\Big]1_{\{y\geq h^{i,\mathrm{crit}}(t,x)\}}\Big).\end{aligned}$$

- *Continuity results on the χ^i's.* We distinguish the three cases that already appeared in Lemma 4.35.

  - *$B_c^i < 0$ and $A^i < C^i$.* If $B_c^i < 0$ and $A^i < C^i$ then $\zeta^{i,\mathrm{crit}}(t,x) = 0$ and $h^{i,\mathrm{crit}}(t,x) = 0$. Hence

$$\begin{aligned}\chi^i(t,x,y) &= (-1)^{1-i}\Big(C_2^i\tfrac{1}{1-R}(e^{(-1)^{1-i}(1-R)y}-1) + (C_1^i - C_2^i)\tfrac{1}{1-R}(e^{(-1)^{1-i}(1-R)y}-1)\Big)\\ &= (-1)^{1-i}C_1^i\tfrac{1}{1-R}(e^{(-1)^{1-i}(1-R)y}-1).\end{aligned}$$

- $B_c^i > 0$ *and* $A^i > C^i$. If $B_c^i > 0$ and $A^i > C^i$ then $h^{i,\text{crit}}(t,x) = \infty$ for every $(t,x) \in [0,T] \times \mathbb{R}$ and

$$\chi^i(t,x,y) = (-1)^{1-i} C_2^i \tfrac{1}{1-R}(e^{(-1)^{1-i}(1-R)y} - 1).$$

- $B_c^i < 0$ *and* $A^i \geq C^i$, *or* $B_c^i > 0$ *and* $A^i \leq C^i$. If $B_c^i < 0$ and $A^i \geq C^i$, or $B_c^i > 0$ and $A^i \leq C^i$, then we know from the proof of Lemma 4.35 that $y \geq h^{i,\text{crit}}(t,x)$ is equivalent to

$$0 \leq (C_1^i - C_2^i)\tfrac{1}{1-R}(e^{(-1)^{1-i}(1-R)y} - 1) + \zeta^{i,\text{crit}}(t,x)$$

implying

$$\chi^i(t,x,y) = (-1)^{1-i}\Big(C_2^i\tfrac{1}{1-R}(e^{(-1)^{1-i}(1-R)y}-1) + \Big[(C_1^i - C_2^i)\tfrac{1}{1-R}(e^{(-1)^{1-i}(1-R)y}-1) + \zeta^{i,\text{crit}}(t,x)\Big]^+\Big).$$

In all three cases χ^i is continuous in t, and further, as a composition of continuously differentiable functions in x and y and the $[\,\cdot\,]^+$-function locally Lipschitz continuous in x and y.

- *Non-negativity of the* χ^i*'s.*

- $B_c^i < 0$ *and* $A^i < C^i$, *or* $B_c^i > 0$ *and* $A^i > C^i$. The χ^i's are trivially non-negative because $y \in \mathbb{R}_0^+$.
- $B_c^i < 0$ *and* $A^i \geq C^i$, *or* $B_c^i > 0$ *and* $A^i \leq C^i$. The non-negativity of χ^1 is obvious. In order to see that $\chi^0 \geq 0$ we need to show that $\zeta^{0,\text{crit}} \leq 0$. To prove this we distinguish the two cases $\varepsilon > 0$ and $\varepsilon = 0$.

  - $\varepsilon > 0$. If $\varepsilon > 0$ then $\zeta^{0,\text{crit}}(t,x)$ is given by

$$\zeta^{0,\text{crit}}(t,x) = \Big[\tfrac{1}{1-R}\Big(\Big(1 - \tfrac{(d^{c,0}(c^M(t,x)))^+}{B_c^0 c^M(t,x)}\Big)^{1-R} - 1\Big) + \tfrac{(d^{c,0}(c^M(t,x)))^+}{B_c^0 c^M(t,x)}\Big] c^M(t,x).$$

  In case of $B_c^0 < 0$ and $A^0 \geq C^0$, or $B_c^0 > 0$ and $A^0 < C^0$ the non-positivity of $\zeta^{0,\text{crit}}(t,x)$ follows from Lemma 4.48, iii), with $z = -\frac{(d^{c,0}(c^M(t,x)))^+}{B_c^0 c^M(t,x)}$ where the lemma is applicable because

$$-\tfrac{(d^{c,0}(c^M(t,x)))^+}{B_c^0 c^M(t,x)} = -\tfrac{(d^{c,0}(c^M(t,x)))^+ - B_c^0 c^M(t,x)}{B_c^0 c^M(t,x)} - 1 = -\tfrac{\max\{A^0 - C^0, -B_c^0 c^M(t,x)\}}{B_c^0 c^M(t,x)} - 1 > -1.$$

  However, in case of $B_c^0 > 0$ and $A^0 = C^0$ which imply $(d^{c,0}(c^M(t,x)))^+ = B_c^0 c^M(t,x)$ using the convention $0^{1-R} = \infty$ for $R > 1$ we find

$$\zeta^{0,\text{crit}}(t,x) = \begin{cases} -\frac{R}{1-R} c^M(t,x) & \text{if } R < 1, \\ -\infty & \text{if } R > 1, \end{cases}$$

  which is non-positive, too.

  - $\varepsilon = 0$. If $\varepsilon = 0$ then

$$\zeta^{0,\text{crit}}(t,x) = \tfrac{(A^0 - C^0)^+}{B_c^0}.$$

  which is trivially non-positive.

The non-positivity of $\zeta^{0,\mathrm{crit}}$ implies

$$(C_1^0 - C_2^0)\tfrac{1}{1-R}(e^{-(1-R)y} - 1) \geq \left[(C_1^0 - C_2^0)\tfrac{1}{1-R}(e^{-(1-R)y} - 1) + \zeta^{0,\mathrm{crit}}(t,x)\right]^+$$

since we required $C_1^0 < C_2^0$. Hence

$$\chi^0(t,x,y) \geq -C_1^0\tfrac{1}{1-R}(e^{-(1-R)y} - 1) \geq 0 \text{ for every } (t,x,y) \in [0,T] \times \mathbb{R} \times \mathbb{R}_0^+.$$

The remainder of the proof is identical to the proof of Lemma 4.9. □

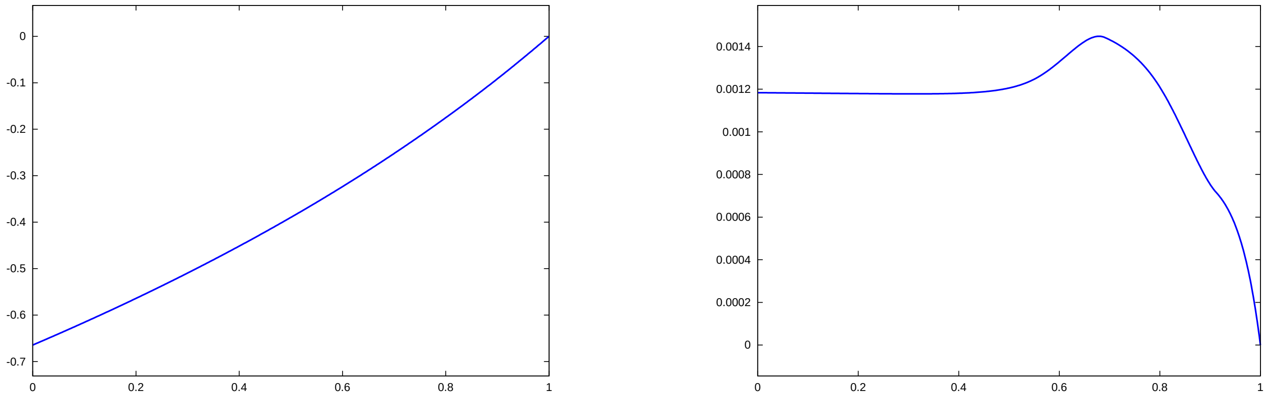

Figure 4.4: g (left) and h (right) as functions of t
($r^0 = r^1 = 0.03$, $\eta^0 = 0.1$, $\eta^1 = 0.06$, $\sigma^0 = 0.4$, $\sigma^1 = 0.7$, $\delta = 0.035$, $\varepsilon = 1$, $R = 1.5$, $T = 1$,
$A^0 = 17$, $B_c^0 = 3$, $C^0 = 19.5$, $C_1^0 = 10$, $C_2^0 = 20$, $A^1 = 5.5$, $B_c^1 = 5$, $C^1 = 9$, $C_1^1 = 5$, $C_2^1 = 1.25$)

Figure 4.4 shows an example for the functions g and h. It can be seen that the function h is not necessarily monotone, i.e. the difference between the two market states as perceived by the large investor can increase and decrease as well.

The proof of Lemma 4.47 directly implies the following corollaries.

Corollary 4.49 (Time-dependent bounds on g and h)**.** *Let h and g be given by (4.26), (4.27) subject to the boundary conditions (4.28) ($\varepsilon > 0$), resp. (4.29), (4.30) subject to the boundary conditions (4.31) ($\varepsilon = 0$). Then*

$$-(\Psi^1 - \xi^1(T))^-(T-t) \leq g(t) \leq (\Psi^0 - \xi^0(T))^+(T-t) \ (\varepsilon > 0),$$
$$\text{resp. } 0 \leq g(t) \leq \Psi^0(T-t) \ (\varepsilon = 0) \text{ and}$$
$$0 \leq h(t) \leq (\Psi^0 - \Psi^1)(T-t)$$

for $t \in [0,T]$ where the $\xi^i(T)$'s are as defined in the proof of Lemma 4.9.

Corollary 4.50 (Time-independent bound on h)**.** *Let h and g be given by (4.26), (4.27) subject to the boundary conditions (4.28) ($\varepsilon > 0$), resp. (4.29), (4.30) subject to the boundary conditions (4.31) ($\varepsilon = 0$). Then*

$$0 \leq h(t) \leq \bar{h}$$

for $t \in [0,T]$ where $\bar{h}$ is the smallest positive root of

$$\bar{F}(y) \triangleq \begin{cases} -(\Psi^0 - \Psi^1) - C_1^0 \frac{1}{1-R}(e^{-(1-R)y} - 1) + C_2^1 \frac{1}{1-R}(e^{(1-R)y} - 1) & \text{if } \varepsilon > 0, \\ -(\Psi^0 - \Psi^1) + \chi^0(t,x,y) + \chi^1(t,x,y) & \text{if } \varepsilon = 0 \end{cases}$$

if such a root exists; otherwise $\bar{h} = \infty$, i.e.

$$\bar{h} = \min\left\{y \in \mathbb{R}_0^+ \,:\, \bar{F}(y) = 0\right\},$$

with the convention $\min\emptyset \triangleq \infty$, where the χ^i's are as defined in the proof of Lemma 4.47.

Remark 4.51. *Notice that in case of $\varepsilon = 0$ the function $\bar{F}(y)$ is indeed a function solely in y although the χ^i's formally depend on t and x, too. However setting $\varepsilon = 0$ implies that $\zeta^{i,\text{crit}}$ and consequently $h^{i,\text{crit}}$ are independent of t and x. This yields that the χ^i's are independent of t and x, too.*

Proof of Corollary 4.50. The assertion follows from a simple ODE argument since

$$F(t,x,y) = -(\Psi^0 - \Psi^1) + \varrho(t,x,y) + \chi^0(t,x,y) + \chi^1(t,x,y) \geq \bar{F}(y)$$

for all $(t,x,y) \in [0,T] \times \mathbb{R} \times \mathbb{R}_0^+$ where we used the notation from the proof of Lemma 4.9. If $\varepsilon = 0$ then this is obviously true. In case of $\varepsilon > 0$ this is satisfied, too, since the proof of Lemma 4.47 showed that $\chi^0(t,x,y) \geq -C_1^0 \frac{1}{1-R}(e^{-(1-R)y} - 1)$ and $\chi^1(t,x,y) \geq C_2^1 \frac{1}{1-R}(e^{(1-R)y} - 1)$. □

The boundary function on h as presented in Corollary 4.49 is suitable for large values of t for which the deviation from h is not too large whereas for small values of t the non-linear behavior of h implies that the boundary function overshoots h by far. The converse is true for the bound given in Corollary 4.50. Thus combining the two bounds on h yields

$$0 \leq h(t) \leq \min\{(\Psi^0 - \Psi^1)(T-t), \bar{h}\}$$

for $t \in [0,T]$.

The following theorem ensures that the strategy given in Lemmas 4.35 and 4.34 is indeed the optimal strategy for the optimal investment problem.

Theorem 4.52 (Solution of the Investment Problem)**.** *Let condition (NJ) be satisfied, and let h and g be given by (4.26), (4.27) subject to the boundary conditions (4.28) ($\varepsilon > 0$), resp. (4.29), (4.30) subject to the boundary conditions (4.31) ($\varepsilon = 0$). Then the strategy*

$$(\pi^{i,*}, c^{i,*}(t)) \triangleq (\pi^{i,*}, c^{i,*}(t, g(t) - 1_{\{i=1\}}h(t), h(t))),\ t \in [0,T],\ i = 0,1,$$

as given in Lemmas 4.34 and 4.35 is optimal for the investment problem (P) with CRRA preferences and relative risk aversion R.

Proof. Since $(\pi^{i,*}, c^{i,*}(t))$ maximizes the reduced HJB-system (4.25) for each $t \in [0,T]$, optimality of the strategy $(\pi^{i,*}, c^{i,*})$ follows directly from the Verification Theorem 3.3. □

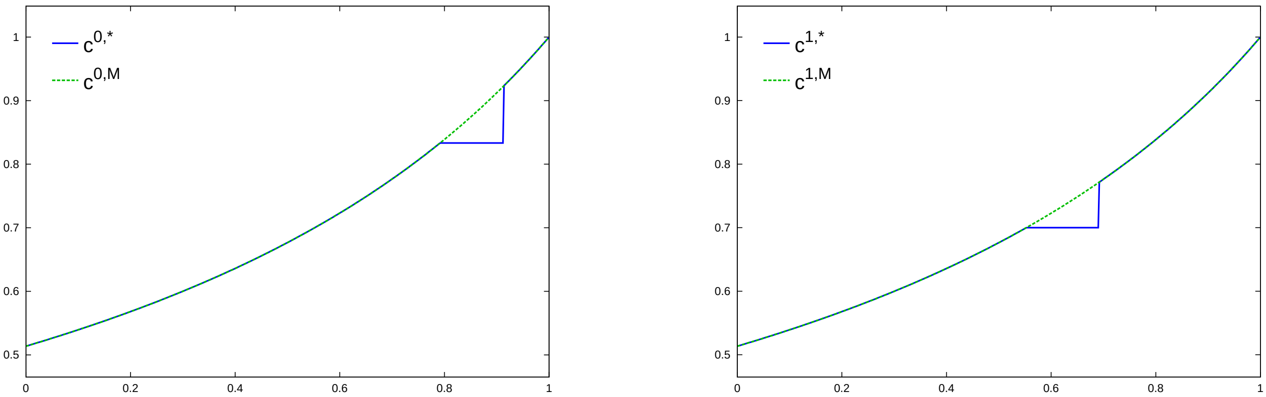

Figure 4.5: Optimal strategy vs. Merton strategy: $c^{i,*}$ and $c^{i,M}$ as functions of t
($r^0 = r^1 = 0.03$, $\eta^0 = 0.1$, $\eta^1 = 0.06$, $\sigma^0 = 0.4$, $\sigma^1 = 0.7$, $\delta = 0.035$, $\varepsilon = 1$, $R = 1.5$, $T = 1$, $A^0 = 17$, $B_c^0 = 3$, $C^0 = 19.5$, $C_1^0 = 10$, $C_2^0 = 20$, $A^1 = 5.5$, $B_c^1 = 5$, $C^1 = 9$, $C_1^1 = 5$, $C_2^1 = 1.25$)

Remark 4.53. *If not otherwise stated* $c^{i,\text{crit}}(t)$*, resp.* $h^{i,\text{crit}}(t)$*, is utilized as shorthand notation for* $c^{i,\text{crit}}(t, g(t) - 1_{\{i=1\}}h(t))$*, resp.* $h^{i,\text{crit}}(t, g(t) - 1_{\{i=1\}}h(t))$*, i.e.*

$$c^{i,\text{crit}}(t) \triangleq c^{i,\text{crit}}(t, g(t) - 1_{\{i=1\}}h(t)) \text{ and}$$
$$h^{i,\text{crit}}(t) \triangleq h^{i,\text{crit}}(t, g(t) - 1_{\{i=1\}}h(t)).$$

Figure 4.5 shows the optimal strategy and the Merton strategy in a setting where B_c^i is positive for $i = 0, 1$. Hence the large investor is forced to consume at a rate that is temporarily smaller than the Merton consumption rate in order to generate advantageous regime shift intensities. In both states the maximal deviation amounts to about 9%.

Since the Merton consumption rate is growing in time it is possible that the Merton rate is lying in the half space of favorable consumption rates first. Hence for small times t there is no need to deviate from the Merton rate. But as the latter is growing there exists a point at which it enters the half space of adverse strategies. From that point on it may be advantageous to deviate from the Merton consumption rate. Finally when the time to maturity gets smaller the probability of a regime shift becomes smaller and smaller and the investor turns back to the Merton rate as it was the case in the portfolio-dependent setting.

A particular example in which the investor is willing, resp. forced to consume although he does not draw any utility from this consumption in terms of the utility function is given in Figure 4.6. This consumption can be interpreted as the payment of a bribe. Notice that the large investor follows an extreme strategy. Until a certain point of time he pays a bribe at the maximal rate given by $c^{i,\text{crit}} = -\frac{A^i - C^i}{B_c^i}$ and then he cancels those payments completely. One would imagine that a continuous reduction of the bribe payment would be more reasonable. The reason for this extreme strategy switching is given by the special form of the intensity function. Being a step function it already includes this extreme character which is passed on to the optimal consumption rate, too. The optimal rate is either the Merton rate or the critical rate but nothing in between. In case of $\varepsilon > 0$ the time-dependency of the Merton consumption rate somehow tempers this extreme behavior. But for $\varepsilon = 0$ the Merton rate is no longer time-dependent and thus the extreme switching occurs straightly.

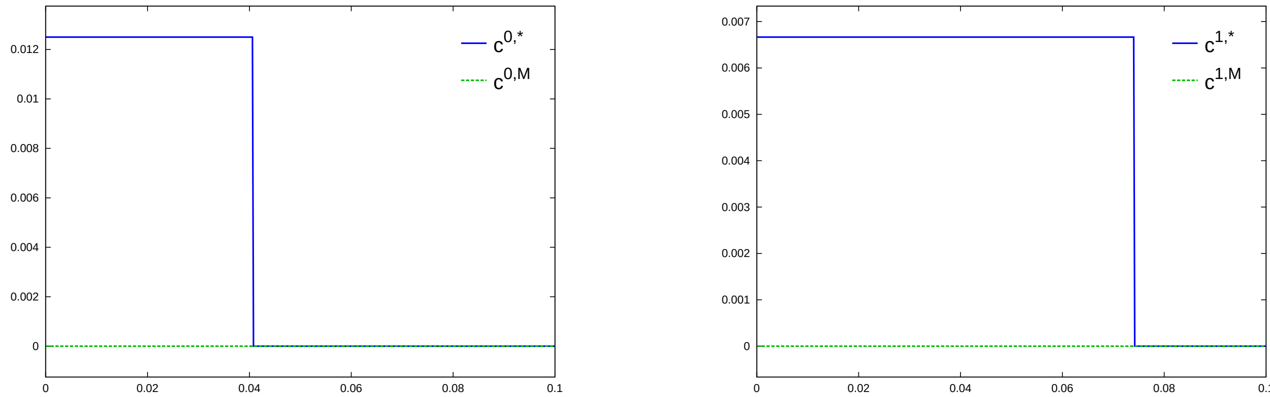

Figure 4.6: Optimal strategy vs. Merton strategy: $c^{i,*}$ and $c^{i,M}$ as functions of t
($r^0 = r^1 = 0.03$, $\eta^0 = 0.1$, $\eta^1 = 0.02$, $\sigma^0 = 0.4$, $\sigma^1 = 0.7$, $\delta = 0.035$, $\varepsilon = 0$, $R = 1.5$, $T = 0.1$, $A^0 = 11.75$, $B_c^0 = -20$, $C^0 = 11.5$, $C_1^0 = 2.5$, $C_2^0 = 25$, $A^1 = 5.4$, $B_c^1 = -15$, $C^1 = 5.3$, $C_1^1 = 20$, $C_2^1 = 2.5$)

As in the section on portfolio-dependent intensities the large investor's optimal consumption rate can be decomposed into the Merton consumption rate and an additional adjustment part which again accommodates the investor's influence on the market, i.e.

$$c^{i,*}(t) = c^{i,M}(t) + c^{i,A}(t), \;\; i = 0, 1$$

where the adjustment part $c^{i,A}$ is given by

$$c^{i,A}(t) = -\tfrac{(A^i + B_c^i c^{i,M}(t) - C^i)^+}{B_c^i} 1_{\{h(t) \geq h^{i,\mathrm{crit}}(t)\}}.$$

Notice that $B_c^i c^{i,A}(t) \leq 0$. Moreover $c^{i,A}(t) \leq 0$ if $B_c^i > 0$ and $c^{i,A}(t) \geq 0$ if $B_c^i < 0$. Thus a positive B_c^i forces the large investor to consume at a lower rate than the Merton one, whereas a negative B_c^i yields a higher consumption rate than the Merton one. As a consequence $c^{i,A}$ benefits the regime shift intensities which is shown in the following lemma.

Lemma 4.54. *The optimal strategy $(\pi^{i,*}, c^{i,*})$ satisfies*

$$\vartheta^{0,1}(\pi^{0,*}, c^{0,*}(t)) \leq \vartheta^{0,1}(\pi^{0,M}, c^{0,M}(t)) \text{ and } \vartheta^{1,0}(\pi^{1,*}, c^{1,*}(t)) \geq \vartheta^{1,0}(\pi^{1,M}, c^{1,M}(t))$$

for every $t \in [0, T]$.

Proof. The assertion follows since

$$\begin{aligned}
\vartheta^{0,1}(\pi^{0,*}, c^{0,*}(t)) &= C_1^0 1_{\{A^0 + B_c^0 c^{0,*}(t) \leq C^0\}} + C_2^0 1_{\{A^0 + B_c^0 c^{0,*}(t) > C^0\}} \\
&= C_1^0 1_{\{A^0 + B_c^0 c^{0,M}(t) + B_c^0 c^{0,A}(t) \leq C^0\}} + C_2^0 1_{\{A^0 + B_c^0 c^{0,M}(t) + B_c^0 c^{0,A}(t) > C^0\}} \\
&\leq C_1^0 1_{\{A^0 + B_c^0 c^{0,M}(t) \leq C^0\}} + C_2^0 1_{\{A^0 + B_c^0 c^{0,M}(t) > C^0\}} \\
&= \vartheta^{0,1}(\pi^{0,M}, c^{0,M}(t))
\end{aligned}$$

and

$$\begin{aligned}\vartheta^{1,0}(\pi^{1,*},c^{1,*}(t)) &= C_1^1 1_{\{A^1+B_c^1 c^{1,*}(t)\le C^1\}} + C_2^1 1_{\{A^1+B_c^1 c^{1,*}(t)>C^1\}}\\ &= C_1^1 1_{\{A^1+B_c^1 c^{1,M}(t)+B_c^1 c^{1,A}(t)\le C^1\}} + C_2^1 1_{\{A^1+B_c^1 c^{1,M}(t)+B_c^1 c^{1,A}(t)>C^1\}}\\ &\ge C_1^1 1_{\{A^1+B_c^1 c^{1,M}(t)\le C^1\}} + C_2^1 1_{\{A^1+B_c^1 c^{1,M}(t)>C^1\}}\\ &= \vartheta^{1,0}(\pi^{1,M},c^{1,M}(t))\end{aligned}$$

as $B_c^i c^{i,A}(t) \le 0$. □

Remark 4.55. *As in the last section the optimal consumption rate is a* compromise rate *composed of the Merton consumption rate and the critical consumption rate and thus generating better regime shift intensities than those the Merton rate would yield.*

4.2.3 Portfolio- and Consumption-dependent Intensities

Having discussed the optimal investment problem where the regime shifts are influenced either by the benchmark investor's portfolio proportions or by his consumption rate, we now consider the case where both – portfolio proportions and consumption rate – affect the shift intensities, i.e. we let

$$B_\pi^i \ne 0 \text{ and } B_c^i \ne 0 \text{ for } i = 0,1, \tag{PCD}$$

so that the intensities are given by

$$\vartheta^{i,1-i}(\pi,c) = C_1^i 1_{\{A^i+\pi^\top.B_\pi^i+B_c^i c\le C^i\}} + C_2^i 1_{\{A^i+\pi^\top.B_\pi^i+B_c^i c>C^i\}}.$$

We denote by

$$\mathfrak{F}^{\pi,c,i} \triangleq \left\{(\pi,c)\in\mathbb{R}^{\bar n}\times\mathbb{R}_0^+ \;:\; A^i+\pi^\top.B_\pi^i+B_c^i c\le C^i\right\}$$

the half space of all strategies that are favorable for the investor concerning the intensities of regime shifts. Letting

$$d^{\pi,c,i}(\pi,c)\triangleq A^i+\pi^\top.B_\pi^i+B_c^i c - C^i$$

denote the 'distance' of the strategy (π,c) to the separating hyperplane, we have $(\pi,c)\in\mathfrak{F}^{\pi,c,i}$ if and only if $d^{\pi,c,i}(\pi,c)\le 0$.

Further we let $H^{\pi,c,i}$, $H_j^{\pi,c,i} : [0,T]\times\mathbb{R}\times\mathbb{R}_0^+\times\mathbb{R}^{\bar n}\times\mathbb{R}_0^+\to\mathbb{R}$, $i=0,1$ and $j=1,2$, be given by

$$\begin{aligned}H^{\pi,c,i}(t,x,y,\pi,c) &\triangleq \tfrac{\varepsilon\delta}{\varepsilon-(\varepsilon-\delta)e^{-\delta(T-t)}}\tfrac{1}{1-R}(e^{-(1-R)x}c^{1-R}-1) - c + r^i + \pi^\top.\eta^i - \tfrac12 R\pi^\top.\sigma^i.(\sigma^i)^\top.\pi\\ &\quad + (C_1^i 1_{\{(\pi,c)\in\mathfrak{F}^{\pi,c,i}\}} + C_2^i 1_{\{(\pi,c)\notin\mathfrak{F}^{\pi,c,i}\}})\tfrac{1}{1-R}(e^{(-1)^{1-i}(1-R)y}-1),\\ H_j^{\pi,c,i}(t,x,y,\pi,c) &\triangleq \tfrac{\varepsilon\delta}{\varepsilon-(\varepsilon-\delta)e^{-\delta(T-t)}}\tfrac{1}{1-R}(e^{-(1-R)x}c^{1-R}-1) - c + r^i + \pi^\top.\eta^i - \tfrac12 R\pi^\top.\sigma^i.(\sigma^i)^\top.\pi\\ &\quad + C_j^i\tfrac{1}{1-R}(e^{(-1)^{1-i}(1-R)y}-1)\end{aligned}$$

such that $H^{\pi,c,i}(t,x,y,\pi,c) = H_1^{\pi,c,i}(t,x,y,\pi,c)1_{\{(\pi,c)\in\mathfrak{F}^{\pi,c,i}\}} + H_2^{\pi,c,i}(t,x,y,\pi,c)1_{\{(\pi,c)\notin\mathfrak{F}^{\pi,c,i}\}}$. Notice that the functions $H^{\pi,c,i}$ and $H_j^{\pi,c,i}$ are independent of t and x in case of $\varepsilon = 0$.

Using those auxiliary functions the HJB-system (4.20) is given by

$$0 = \sup_{(\pi,c)\in\mathbb{R}^{\bar n}\times\mathbb{R}_0^+} \Big\{ g'(t) - 1_{\{i=1\}}h'(t) + H^{\pi,c,i}(t, g(t) - 1_{\{i=1\}}h(t), h(t), \pi, c) \Big\} \tag{4.32}$$

for $t \in [0,T)$ and $i = 0,1$, subject to the boundary conditions

$$g(T) = 0, \;\; h(T) = 0.$$

To find the supremum in (4.32) we present the maximizers of $H^{\pi,c,i}(t,x,y,\cdot,\cdot)$ for arbitrary $(t,x,y) \in [0,T] \times \mathbb{R} \times \mathbb{R}_0^+$. This yields a family of maximizers dependent on (t,x,y). The maximizers of the HJB-system (4.32) are then obtained by replacing x and y by $g(t) - 1_{\{i=1\}}h(t)$ and $h(t)$.

If $\varepsilon > 0$ then $H_j^{\pi,c,i}(t,x,y,\cdot,\cdot)$ is concave for every $(t,x,y) \in [0,T]\times\mathbb{R}\times\mathbb{R}_0^+$ and $j = 1,2$. Therefore the two candidate solutions for the maximization of $H^{\pi,c,i}(t,x,y,\cdot,\cdot)$ over $(\pi,c) \in \mathbb{R}^{\bar n}\times\mathbb{R}_0^+$ are $(\pi^{i,M}, c^M)$ and $(\tilde\pi^{i,\text{crit}}, \tilde c^{i,\text{crit}})$ where

$$(\tilde\pi^{i,\text{crit}}(t,x), \tilde c^{i,\text{crit}}(t,x)) \triangleq \operatorname*{arg\,max}_{\{(\pi,c)\in\mathbb{R}^{\bar n}\times\mathbb{R}_0^+ \,:\, A^i+\pi^\top.B_\pi^i+B_c^i c=C^i\}} H^{\pi,c,i}(t,x,y,\pi,c).$$

The Lagrange multiplier method yields

$$\begin{aligned}\tilde\pi^{i,\text{crit}}(t,x) &= \tfrac{1}{R}(\sigma^i.(\sigma^i)^\top)^{-1}.\left(\eta^i + \tilde\lambda^{i,\text{crit}}(t,x)B_\pi^i\right),\\ \tilde c^{i,\text{crit}}(t,x) &= c^M(t,x)\left(1 - \tilde\lambda^{i,\text{crit}}(t,x)B_c^i\right)^{-\frac{1}{R}}\end{aligned}$$

for $i = 0,1$ and $(t,x) \in [0,T]\times\mathbb{R}$ where the Lagrange multiplier $\tilde\lambda^{i,\text{crit}}(t,x)$ is implicitly given via

$$\Lambda^{\pi,c,i}(t,x,\tilde\lambda^{i,\text{crit}}(t,x)) = -(A^i + (\pi^{i,M})^\top.B_\pi^i + B_c^i c^M(t,x) - C^i) \tag{4.33}$$

with $\Lambda^{\pi,c,i} : [0,T]\times\mathbb{R}\times(-\infty,\frac{1}{B_c^i}) \to \mathbb{R}$ $(B_c^i > 0)$, resp. $\Lambda^{\pi,c,i} : [0,T]\times\mathbb{R}\times(\frac{1}{B_c^i},\infty) \to \mathbb{R}$ $(B_c^i < 0)$ given by

$$\Lambda^{\pi,c,i}(t,z,\lambda) \triangleq \tfrac{1}{R}(B_\pi^i)^\top(\sigma^i.(\sigma^i)^\top)^{-1}.B_\pi^i\lambda + B_c^i\Big(\big(1 - B_c^i\lambda\big)^{-\frac{1}{R}} - 1\Big)c^M(t,z).$$

If $\varepsilon = 0$ then the function $H_j^{\pi,c,i}(t,x,y,\cdot,\cdot)$ is concave in π and linearly decreasing in c implying $c^M(t,x) = 0$. In order to determine $\tilde\pi^{i,\text{crit}}$ and $\tilde c^{i,\text{crit}}$ in case of $\varepsilon = 0$ we use the constraint $A^i + \pi^\top.B_\pi^i + B_c^i c = C^i$ to substitute $c = -\frac{A^i+\pi^\top.B_\pi^i-C^i}{B_c^i}$ and calculate $\tilde\pi^{i,\text{crit}}$ as

$$\tilde\pi^{i,\text{crit}} = \max_{\left\{\pi\in\mathbb{R}^{\bar n}:\, -\frac{A^i+\pi^\top.B_\pi^i-C^i}{B_c^i}\ge 0\right\}} H^{\pi,c,i}\big(t,x,y,\pi,-\tfrac{A^i+\pi^\top.B_\pi^i-C^i}{B_c^i}\big).$$

This yields the candidate

$$\bar\pi^{i,\text{crit}} = \tfrac{1}{R}(\sigma^i.(\sigma^i)^\top)^{-1}.\big(\eta^i + \tfrac{B_\pi^i}{B_c^i}\big).$$

But unfortunately the corresponding $\bar c^{i,\text{crit}}$ given by

$$\bar c^{i,\text{crit}} = -\frac{A^i+(\bar\pi^{i,\text{crit}})^\top.B_\pi^i-C^i}{B_c^i} = -\frac{A^i+(\pi^{i,M})^\top.B_\pi^i-C^i}{B_c^i} - \frac{\frac{1}{R}(B_\pi^i)^\top.(\sigma^i.(\sigma^i)^\top)^{-1}.B_\pi^i}{(B_c^i)^2}$$

may be negative whereas $\tilde{c}^{i,\mathrm{crit}} \geq 0$ is needed. It turns out that in order to get the optimal $\tilde{c}^{i,\mathrm{crit}}$ one just has to truncate $\bar{c}^{i,\mathrm{crit}}$ at 0, i.e.

$$\begin{aligned}\tilde{c}^{i,\mathrm{crit}} &= \Big(-\tfrac{A^i+(\pi^{i,M})^\top.B^i_\pi - C^i}{B^i_c} - \tfrac{\frac{1}{R}(B^i_\pi)^\top.(\sigma^i.(\sigma^i)^\top)^{-1}.B^i_\pi}{(B^i_c)^2}\Big)^+ \\ &= \tfrac{\frac{1}{R}(B^i_\pi)^\top.(\sigma^i.(\sigma^i)^\top)^{-1}.B^i_\pi}{(B^i_c)^2}\Big(\tfrac{A^i+(\pi^{i,M})^\top.B^i_\pi - C^i}{\frac{1}{R}(B^i_\pi)^\top.(\sigma^i.(\sigma^i)^\top)^{-1}.B^i_\pi}B^i_c + 1\Big)^-\end{aligned}$$

with corresponding

$$\tilde{\pi}^{i,\mathrm{crit}} = \tfrac{1}{R}(\sigma^i.(\sigma^i)^\top)^{-1}.\Big(\eta^i + B^i_\pi \tfrac{1}{B^i_c}\Big(1 - \Big(\tfrac{A^i+(\pi^{i,M})^\top.B^i_\pi - C^i}{\frac{1}{R}(B^i_\pi)^\top.(\sigma^i.(\sigma^i)^\top)^{-1}.B^i_\pi}B^i_c + 1\Big)^+\Big)\Big)$$

for $i = 0, 1$.

Before stating the optimal strategy we first present some properties of the function $\Lambda^{\pi,c,i}$ that we used to determine the Lagrange multipliers.

Lemma 4.56 (Properties of $\Lambda^{\pi,c,i}$). *The function $\Lambda^{\pi,c,i}$ given by*

$$\Lambda^{\pi,c,i} : [0,T] \times \mathbb{R} \times (-\infty, \tfrac{1}{B^i_c}) \to \mathbb{R}\ (B^i_c > 0), \ \textit{resp.}\ \Lambda^{\pi,c,i} : [0,T] \times \mathbb{R} \times (\tfrac{1}{B^i_c}, \infty) \to \mathbb{R}\ (B^i_c < 0)$$

with

$$\Lambda^{\pi,c,i}(t,z,\lambda) \triangleq \tfrac{1}{R}(B^i_\pi)^\top(\sigma^i.(\sigma^i)^\top)^{-1}.B^i_\pi\lambda + B^i_c\Big(\big(1 - B^i_c\lambda\big)^{-\frac{1}{R}} - 1\Big)c^M(t,z)$$

has the following properties for every given $(t,z) \in [0,T] \times \mathbb{R}$.

i) *$\Lambda^{\pi,c,i}$ is continuously differentiable in t, z and λ.*

ii) *If $\varepsilon > 0$ then $\Lambda^{\pi,c,i}(t,z,\cdot)$ is increasing and strictly convex ($B^i_c > 0$), resp. strictly concave ($B^i_c < 0$). If $\varepsilon = 0$ then $\Lambda^{\pi,c,i}(t,z,\cdot)$ is linearly increasing.*

iii) *$\Lambda^{\pi,c,i}(t,z,\lambda) < 0$ for $\lambda < 0$, $\Lambda^{\pi,c,i}(t,z,0) = 0$ and $\Lambda^{\pi,c,i}(t,z,\lambda) > 0$ for $\lambda > 0$.*

iv) *If $\varepsilon > 0$ then $\lim_{\lambda\to-\infty}\Lambda^{\pi,c,i}(t,z,\lambda) = -\infty$ and $\lim_{\lambda\to\frac{1}{B^i_c}}\Lambda^{\pi,c,i}(t,z,\lambda) = \infty$ ($B^i_c > 0$), resp. $\lim_{\lambda\to\frac{1}{B^i_c}}\Lambda^{\pi,c,i}(t,z,\lambda) = -\infty$ and $\lim_{\lambda\to\infty}\Lambda^{\pi,c,i}(t,z,\lambda) = \infty$ ($B^i_c < 0$).*

If otherwise $\varepsilon = 0$ then $\lim_{\lambda\to-\infty}\Lambda^{\pi,c,i}(t,z,\lambda) = -\infty$ and $\lim_{\lambda\to\infty}\Lambda^{\pi,c,i}(t,z,\lambda) = \infty$.

Proof. Let $(t,z) \in [0,T] \times \mathbb{R}$ be given.

ad i) Note that $c^M(t,z)$ is continuously differentiable in t and z which implies that $\Lambda^{\pi,c,i}$ is continuously differentiable in t and z, too. Further $\Lambda^{\pi,c,i}$ is obviously continuously differentiable in λ.

ad ii) Consider the derivative of $\Lambda^{\pi,c,i}$ with respect to λ which is given by

$$\Lambda^{\pi,c,i}_\lambda(t,z,\lambda) \triangleq \tfrac{\partial}{\partial\lambda}\Lambda^{\pi,c,i}(t,z,\lambda) = \tfrac{1}{R}(B^i_\pi)^\top(\sigma^i.(\sigma^i)^\top)^{-1}.B^i_\pi + \tfrac{1}{R}(B^i_c)^2\big(1 - B^i_c\lambda\big)^{-\frac{1+R}{R}}c^M(t,z).$$

If $\varepsilon > 0$ then $c^M(t,z) > 0$ and it is easy to see that $\Lambda^{\pi,c,i}_\lambda(t,z,\cdot)$ is non-negative and strictly increasing ($B^i_c > 0$), resp. strictly decreasing ($B^i_c < 0$), in λ. However, if $\varepsilon = 0$ then $c^M(t,z) = 0$ and $\Lambda^{\pi,c,i}_\lambda(t,z,\cdot)$ is a positive constant. This implies ii).

ad iii) Trivially $\Lambda^{\pi,c,i}(t,z,0)=0$ for any $(t,z)\in[0,T]\times\mathbb{R}$ which together with ii) implies iii).

ad iv) The limiting behavior of $\Lambda^{\pi,c,i}(t,z,\cdot)$ is clear when taking into account that $\Lambda^{c,i}(t,z,\cdot)$ has a pole at $\frac{1}{B_c^i}$ in case of $\varepsilon>0$. □

Lemma 4.56 directly implies

Corollary 4.57. *For each fixed $(t,z)\in[0,T]\times\mathbb{R}$ the function $\Lambda^{\pi,c,i}(t,z,\cdot)$ is bijective.*

Hence, there exists exactly one λ satisfying equation (4.33) and thus $\tilde{\lambda}^{i,\mathrm{crit}}(t,x)$ is well-defined. Moreover, $\tilde{c}^{i,\mathrm{crit}}(t,x)$ is well-defined as $1-\tilde{\lambda}^{i,\mathrm{crit}}(t,x)B_c^i>0$.

Having established the candidate solutions the following lemma presents the desired maximizers.

Lemma 4.58 (Maximizer of $H^{\pi,c,i}(t,x,y,\cdot,\cdot)$). *For every $(t,x,y)\in[0,T]\times\mathbb{R}\times\mathbb{R}_0^+$ the maximizer*

$$(\pi^{i,*}(t,x,y),c^{i,*}(t,x,y))\triangleq\underset{(\pi,c)\in\mathbb{R}^{\bar{n}}\times\mathbb{R}_0^+}{\arg\max}\ H^{\pi,c,i}(t,x,y,\pi,c),\ \ i=0,1,$$

is given by

$$(\pi^{i,*}(t,x,y),c^{i,*}(t,x,y))=\begin{cases}(\pi^{i,M},c^M(t,x)) & \text{if } y<h^{i,\mathrm{crit}}(t,x),\\ (\pi^{i,\mathrm{crit}}(t,x),c^{i,\mathrm{crit}}(t,x)) & \text{if } y\geq h^{i,\mathrm{crit}}(t,x),\end{cases}$$

where

$$h^{i,\mathrm{crit}}(t,x)\triangleq(-1)^{1-i}\tfrac{1}{1-R}\ln\Big((1-R)\tfrac{\zeta^{i,\mathrm{crit}}(t,x)}{C_2^i-C_1^i}+1\Big),$$

with

$$\zeta^{i,\mathrm{crit}}(t,x)\triangleq\begin{cases}-\frac{1}{2}\frac{1}{R}(B_\pi^i)^\top(\sigma^i.(\sigma^i)^\top)^{-1}.B_\pi^i(\lambda^{i,\mathrm{crit}}(t,x))^2\\ \quad+\Big[\frac{1}{1-R}\big((1-\lambda^{i,\mathrm{crit}}(t,x)B_c^i)^{-\frac{1-R}{R}}-1\big)-(1-\lambda^{i,\mathrm{crit}}(t,x)B_c^i)^{-\frac{1}{R}}+1\Big]c^M(t,x) & \text{if } \varepsilon>0,\\[2ex] -\frac{1}{2}\frac{\frac{1}{R}(B_\pi^i)^\top(\sigma^i.(\sigma^i)^\top)^{-1}.B_\pi^i}{(B_c^i)^2}\Big[\Big(1-\Big(\frac{(A^i+(\pi^{i,M})^\top.B_\pi^i-C^i)^+}{\frac{1}{R}(B_\pi^i)^\top.(\sigma^i.(\sigma^i)^\top)^{-1}.B_\pi^i}B_c^i+1\Big)^+\Big)^2\\ \qquad+2\Big(\frac{(A^i+(\pi^{i,M})^\top.B_\pi^i-C^i)^+}{\frac{1}{R}(B_\pi^i)^\top.(\sigma^i.(\sigma^i)^\top)^{-1}.B_\pi^i}B_c^i+1\Big)^-\Big] & \text{if } \varepsilon=0,\end{cases}$$

and where $\lambda^{i,\mathrm{crit}}(t,x)$ is uniquely determined by

$$\Lambda^{\pi,c,i}(t,x,\lambda^{i,\mathrm{crit}}(t,x))=-(A^i+(\pi^{i,M})^\top.B_\pi^i+B_c^ic^M(t,x)-C^i)^+. \tag{4.34}$$

Moreover

$$\pi^{i,\mathrm{crit}}(t,x)=\begin{cases}\frac{1}{R}(\sigma^i.(\sigma^i)^\top)^{-1}.\left(\eta^i+\lambda^{i,\mathrm{crit}}(t,x)B_\pi^i\right) & \text{if } \varepsilon>0,\\ \frac{1}{R}(\sigma^i.(\sigma^i)^\top)^{-1}.\Big(\eta^i+B_\pi^i\frac{1}{B_c^i}\Big(1-\Big(\frac{(A^i+(\pi^{i,M})^\top.B_\pi^i-C^i)^+}{\frac{1}{R}(B_\pi^i)^\top.(\sigma^i.(\sigma^i)^\top)^{-1}.B_\pi^i}B_c^i+1\Big)^+\Big)\Big) & \text{if } \varepsilon=0,\end{cases}$$

$$c^{i,\mathrm{crit}}(t,x)=\begin{cases}c^M(t,x)\left(1-\lambda^{i,\mathrm{crit}}(t,x)B_c^i\right)^{-\frac{1}{R}} & \text{if } \varepsilon>0,\\ \frac{\frac{1}{R}(B_\pi^i)^\top.(\sigma^i.(\sigma^i)^\top)^{-1}.B_\pi^i}{(B_c^i)^2}\Big(\frac{(A^i+(\pi^{i,M})^\top.B_\pi^i-C^i)^+}{\frac{1}{R}(B_\pi^i)^\top.(\sigma^i.(\sigma^i)^\top)^{-1}.B_\pi^i}B_c^i+1\Big)^- & \text{if } \varepsilon=0.\end{cases}$$

Remark 4.59. *If $\varepsilon = 0$ then $\zeta^{i,\mathrm{crit}}$ and consequently $h^{i,\mathrm{crit}}$ are just constants independent of t and x. Further $\varepsilon = 0$ implies that $\pi^{i,\mathrm{crit}}$ and $c^{i,\mathrm{crit}}$ are constants, too. The maximizing strategy $(\pi^{i,*}, c^{i,*})$ is therefore just a function of y.*

Remark 4.60. *We use the following shorthand notations*

$$H^{\pi,c,i}_{1,\mathrm{crit}}(t,x,y) \triangleq H^{\pi,c,i}_1(t,x,y,\pi^{i,\mathrm{crit}}(t,x), c^{i,\mathrm{crit}}(t,x)) \text{ and } H^{\pi,c,i}_{2,M}(t,x,y) \triangleq H^{\pi,c,i}_2(t,x,y,\pi^{i,M}, c^{M}(t,x))$$

for $(t,x,y) \in [0,T] \times \mathbb{R} \times \mathbb{R}^+_0$ and $i = 0,1$.

Proof of Lemma 4.58. Let $(t,x,y) \in [0,T] \times \mathbb{R} \times \mathbb{R}^+_0$ be given. A straight forward optimization yields that the two functions $H^{\pi,c,i}_1(t,x,y,\cdot,\cdot)$ and $H^{\pi,c,i}_2(t,x,y,\cdot,\cdot)$ attain their maxima at $(\pi^{i,M}, c^M(t,x))$. Further since $C^0_1 < C^0_2$ and $C^1_1 > C^1_2$ we get that

$$H^{\pi,c,i}_1(t,x,y,\pi,c) > H^{\pi,c,i}_2(t,x,y,\pi,c) \text{ for all } (\pi,c) \in \mathbb{R}^{\bar{n}} \times \mathbb{R}^+. \tag{$*$}$$

For notational convenience we identify the tuple (π, c) with ς and use the following shorthand notations throughout this proof.

$$\varsigma^{i,M}(t,x) \triangleq (\pi^{i,M}, c^M(t,x)),\ \varsigma^{i,\mathrm{crit}}(t,x) \triangleq (\pi^{i,\mathrm{crit}}(t,x), c^{i,\mathrm{crit}}(t,x)),\ \varsigma^{i,*}(t,x,y) \triangleq (\pi^{i,*}(t,x,y), c^{i,*}(t,x,y))$$

for $(t,x,y) \in [0,T] \times \mathbb{R} \times \mathbb{R}^+_0$, $i = 0,1$.

We distinguish the cases $\varsigma^{i,M}(t,x) \in \mathfrak{F}^{\pi,c,i}$ and $\varsigma^{i,M}(t,x) \notin \mathfrak{F}^{\pi,c,i}$.

- $\varsigma^{i,M}(t,x) \in \mathfrak{F}^{\pi,c,i}$. If $\varsigma^{i,M}(t,x) \in \mathfrak{F}^{\pi,c,i}$ then by $(*)$ the maximizer of $H^{\pi,c,i}(t,x,y,\cdot,\cdot)$ is given by $\varsigma^{i,M}(t,x)$, i.e.
$$\varsigma^{i,M}(t,x) \in \mathfrak{F}^{\pi,c,i} \quad \Rightarrow \quad \varsigma^{i,*}(t,x,y) = \varsigma^{i,M}(t,x).$$
- $\varsigma^{i,M}(t,x) \notin \mathfrak{F}^{\pi,c,i}$. If otherwise $\varsigma^{i,M}(t,x) \notin \mathfrak{F}^{\pi,c,i}$ then $(*)$ implies that the maximizer of $H^{\pi,c,i}(t,x,y,\cdot,\cdot)$ is either $\varsigma^{i,M}(t,x)$ in case of $H^{\pi,c,i}_{2,M}(t,x,y) > H^{\pi,c,i}_{1,\mathrm{crit}}(t,x,y)$ or $\varsigma^{i,\mathrm{crit}}(t,x)$ in case of $H^{\pi,c,i}_{2,M}(t,x,y) \le H^{\pi,c,i}_{1,\mathrm{crit}}(t,x,y)$. Hence with

$$\begin{aligned}
& H^{\pi,c,i}_{2,M}(t,x,y) \le H^{\pi,c,i}_{1,\mathrm{crit}}(t,x,y) \\
\Leftrightarrow\ & (C^i_2 - C^i_1)\tfrac{1}{1-R}\big(e^{(-1)^{1-i}(1-R)y} - 1\big) \\
& \le \begin{cases}
-\frac{1}{2}\frac{1}{R}(B^i_\pi)^\top.(\sigma^i.(\sigma^i)^\top)^{-1}.B^i_\pi(\lambda^{i,\mathrm{crit}}(t,x))^2 \\
\quad + \Big[\frac{1}{1-R}\Big(\big(1-\lambda^{i,\mathrm{crit}}(t,x)B^i_c\big)^{-\frac{1-R}{R}} - 1\Big) \\
\qquad - \Big(\big(1-\lambda^{i,\mathrm{crit}}(t,x)B^i_c\big)^{-\frac{1}{R}} - 1\Big)\Big]c^M(t,x) & \text{if } \varepsilon > 0, \\[2ex]
-\frac{1}{2}\frac{\frac{1}{R}(B^i_\pi)^\top(\sigma^i.(\sigma^i)^\top)^{-1}.B^i_\pi}{(B^i_c)^2} \\
\quad \cdot\Big[\Big(1-\Big(\frac{(A^i+(\pi^{i,M})^\top.B^i_\pi - C^i)^+}{\frac{1}{R}(B^i_\pi)^\top.(\sigma^i.(\sigma^i)^\top)^{-1}.B^i_\pi}B^i_c + 1\Big)^+\Big)^2 \\
\qquad + 2\Big(\frac{(A^i+(\pi^{i,M})^\top.B^i_\pi - C^i)^+}{\frac{1}{R}(B^i_\pi)^\top.(\sigma^i.(\sigma^i)^\top)^{-1}.B^i_\pi}B^i_c + 1\Big)^-\Big] & \text{if } \varepsilon = 0,
\end{cases} \\
\Leftrightarrow\ & (C^i_2 - C^i_1)\tfrac{1}{1-R}\big(e^{(-1)^{1-i}(1-R)y} - 1\big) \le \zeta^{i,\mathrm{crit}}(t,x) \\
\Leftrightarrow\ & y \ge (-1)^{1-i}\tfrac{1}{1-R}\ln\Big((1-R)\tfrac{\zeta^{i,\mathrm{crit}}(t,x)}{C^i_2 - C^i_1} + 1\Big) \\
\Leftrightarrow\ & y \ge h^{i,\mathrm{crit}}(t,x)
\end{aligned}$$

we get

$$\varsigma^{i,M}(t,x) \notin \mathcal{F}^{\pi,c,i} \quad \Rightarrow \quad \varsigma^{i,*}(t,x,y) = \begin{cases} \varsigma^{i,M}(t,x) & \text{if } y < h^{i,\mathrm{crit}}(t,x), \\ \varsigma^{i,\mathrm{crit}}(t,x) & \text{if } y \geq h^{i,\mathrm{crit}}(t,x). \end{cases}$$

Notice that $\varsigma^{i,M}(t,x) \in \mathcal{F}^{\pi,c,i}$ implies $\lambda^{i,\mathrm{crit}}(t,x) = 0$. Therefore $h^{i,\mathrm{crit}}(t,x) = 0$ and $\varsigma^{i,\mathrm{crit}}(t,x) = \varsigma^{i,M}(t,x)$ if $\varsigma^{i,M}(t,x) \in \mathcal{F}^{\pi,c,i}$ which yields that the latter formula also covers the case $\varsigma^{i,M}(t,x) \in \mathcal{F}^{\pi,c,i}$ and thus

$$\varsigma^{i,*}(t,x,y) = \begin{cases} \varsigma^{i,M}(t,x) & \text{if } y < h^{i,\mathrm{crit}}(t,x), \\ \varsigma^{i,\mathrm{crit}}(t,x) & \text{if } y \geq h^{i,\mathrm{crit}}(t,x). \end{cases}$$

Thus the lemma is proven. □

Remark 4.61. *If $\varepsilon = 0$ and $B^i_c > 0$ then the critical strategy is given by*

$$(\pi^{i,\mathrm{crit}}, c^{i,\mathrm{crit}}) = \Big(\tfrac{1}{R}(\sigma^i.(\sigma^i)^\top)^{-1}.\Big(\eta^i - B^i_\pi \tfrac{(A^i+(\pi^{i,M})^\top.B^i_\pi - C^i)^+}{\frac{1}{R}(B^i_\pi)^\top.(\sigma^i.(\sigma^i)^\top)^{-1}.B^i_\pi}\Big), 0\Big),$$

i.e. the consumption vanishes completely if $B^i_c > 0$.

Remark 4.62 (Interpretation)**.** *The maximizing strategy coincides with the Merton strategy when the latter belongs to the half space of favorable strategies. If this is not the case then the maximizing strategy is given by the Merton strategy as long as y is not bigger than $h^{i,\mathrm{crit}}(t,x)$. For all $y < h^{i,\mathrm{crit}}(t,x)$ it is not reasonable to deviate from the Merton strategy, i.e. the utility criterion dominates the intensity criterion as discussed in Remark 4.33. Only if y overshoots the critical barrier $h^{i,\mathrm{crit}}(t,x)$ then the maximizing strategy switches to the critical strategy $(\pi^{i,\mathrm{crit}}(t,x), c^{i,\mathrm{crit}}(t,x))$ since then the intensity criterion is more important than the utility criterion.*

Remark 4.63 (Discontinuity of $(\pi^{i,*}, c^{i,*})$)**.** *The maximizing strategy $(\pi^{i,*}, c^{i,*})$ exhibits jumps at all points $(\hat{t},\hat{x},\hat{y})$ in which $\hat{y} = h^{i,\mathrm{crit}}(\hat{t},\hat{x})$ and moreover $(\pi^{i,M}(\hat{t}), c^M(\hat{t},\hat{x})) \neq (\pi^{i,\mathrm{crit}}(\hat{t},\hat{x}), c^{i,\mathrm{crit}}(\hat{t},\hat{x}))$.*

Remark 4.64 (Interpretation in case of $\varepsilon = 0$)**.** *As already in the consumption-dependent case the maximizing consumption rate may be strictly positive even if $\varepsilon = 0$. This happens if $y \geq h^{i,\mathrm{crit}}$ and moreover $\frac{(A^i+(\pi^{i,M})^\top.B^i_\pi - C^i)^+}{\frac{1}{R}(B^i_\pi)^\top.(\sigma^i.(\sigma^i)^\top)^{-1}.B^i_\pi} B^i_c + 1 < 0$ which is only possible in case of $B^i_c < 0$. The maximizing consumption rate is then given by $c^{i,\mathrm{crit}}$ which is strictly positive. The reason for this behavior is the same as in the consumption-dependent setting. Consuming at the rate of $c^{i,\mathrm{crit}}$ further improves the regime shift intensities.*

So far the above explanations are relevant for the general maximizing strategy $(\pi^{i,*}(t,x,y), c^{i,*}(t,x,y))$ and not necessarily for the optimal strategy. This is because it is not clear whether the situation $h(t) \geq h^{i,\mathrm{crit}}$ – remember that the maximizing strategy of the HJB-system (4.32) is obtained by choosing $y = h(t)$ – could really occur. But later on we will provide an example showing that the aforementioned situation is really possible.

Lemma 4.65 (Bounds on $\lambda^{i,\mathrm{crit}}$)**.** *The function $\lambda^{i,\mathrm{crit}} : [0,T] \times \mathbb{R} \to \mathbb{R}$ with $\lambda^{i,\mathrm{crit}}(t,x)$ implicitly given by (4.34) satisfies*

$$\lambda^{i,\mathrm{crit}}(t,x) \in \begin{cases} \Big[-\frac{(A^i+(\pi^{i,M})^\top.B^i_\pi + B^i_c c^M(t,x) - C^i)^+}{\frac{1}{R}(B^i_\pi)^\top.(\sigma^i.(\sigma^i)^\top)^{-1}.B^i_\pi}, 0\Big] & \text{if } B^i_c > 0, \\ \Big[\frac{1}{B^i_c} \vee -\frac{(A^i+(\pi^{i,M})^\top.B^i_\pi + B^i_c c^M(t,x) - C^i)^+}{\frac{1}{R}(B^i_\pi)^\top.(\sigma^i.(\sigma^i)^\top)^{-1}.B^i_\pi}, 0\Big] & \text{if } B^i_c < 0 \end{cases}$$

for all $(t,x) \in [0,T] \times \mathbb{R}$. Thus, $\lambda^{i,\mathrm{crit}}(t,x)$ is especially non-positive.

Proof. Let $(t,x) \in [0,T] \times \mathbb{R}$ be given. We first verify the upper bound on $\lambda^{i,\mathrm{crit}}$ and then prove the lower one.

- *Upper bound.* Lemma 4.56, iii), directly implies that $\lambda^{i,\mathrm{crit}}(t,x) \le 0$, since $\Lambda^{\pi,c,i}(t,x,\lambda^{i,\mathrm{crit}}(t,x)) = -(d^{\pi,c,i}(\pi^{i,M}, c^M(t,x)))^+ \le 0$ by construction.
- *Lower bound.* Notice first that

$$\Lambda^{\pi,c,i}(t,x,\lambda) \le \tfrac{1}{R}(B^i_\pi)^\top(\sigma^i.(\sigma^i)^\top)^{-1}.B^i_\pi\lambda \triangleq \Lambda^{\pi,i}(\lambda) \qquad (*)$$

  for all $(t,x,\lambda) \in [0,T] \times \mathbb{R} \times (-\infty, 0]$ $(B^i_c > 0)$, resp. $(t,x,\lambda) \in [0,T] \times \mathbb{R} \times (\frac{1}{B^i_c}, 0]$ $(B^i_c < 0)$. Further $-\frac{(d^{\pi,c,i}(\pi^{i,M},c^M(t,x)))^+}{\frac{1}{R}(B^i_\pi)^\top.(\sigma^i.(\sigma^i)^\top)^{-1}.B^i_\pi}$ is the solution of $\Lambda^{\pi,i}(\lambda) = -(d^{\pi,c,i}(\pi^{i,M}, c^M(t,x)))^+$.

  If $B^i_c > 0$ then $\Lambda^{\pi,c,i}(t,x,\cdot)$ exhibits its pole at $\frac{1}{B^i_c} > 0$ so that $(*)$ implies that $\lambda^{i,\mathrm{crit}}(t,x) \ge -\frac{(d^{\pi,c,i}(\pi^{i,M},c^M(t,x)))^+}{\frac{1}{R}(B^i_\pi)^\top.(\sigma^i.(\sigma^i)^\top)^{-1}.B^i_\pi}$.

  However, if $B^i_c < 0$ then $\Lambda^{\pi,c,i}(t,x,\cdot)$ has its pole at $\frac{1}{B^i_c} < 0$. Consequently, $(*)$ yields that $\lambda^{i,\mathrm{crit}}(t,x) \ge \frac{1}{B^i_c} \vee -\frac{(d^{\pi,c,i}(\pi^{i,M},c^M(t,x)))^+}{\frac{1}{R}(B^i_\pi)^\top.(\sigma^i.(\sigma^i)^\top)^{-1}.B^i_\pi}$.

Thus the lemma is proven. □

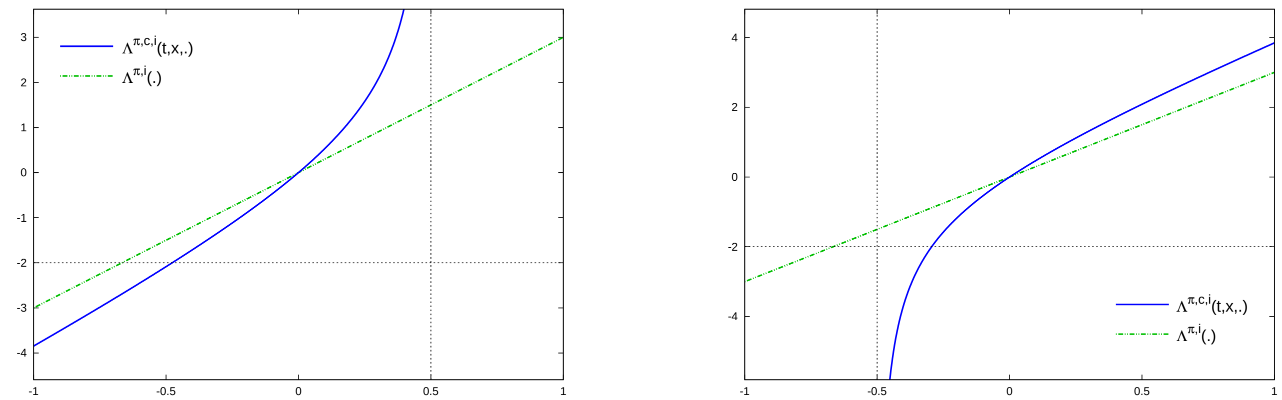

Figure 4.7: $\Lambda^{\pi,c,i}(t,x,\cdot)$ vs. $\Lambda^{\pi,i}(\cdot)$ with $B^i_c = 2$ (left) and $B^i_c = -2$ (right) as functions of λ
(Examplary $-(d^{\pi,c,i}(\pi^{i,M}, c^M(t,x)))^+ = -2$ is chosen)

In Figure 4.7 we draw the typical shapes of $\Lambda^{\pi,c,i}(t,x,\cdot)$ and $\Lambda^{\pi,i}(\cdot)$ to illustrate the arguments of the last proof.

We are now in the position to provide the link between the maximizing strategy in case of $\varepsilon > 0$ and $\varepsilon = 0$.

Lemma 4.66 (Limiting behavior of $(\pi^{i,\mathrm{crit}}, c^{i,\mathrm{crit}})$, $h^{i,\mathrm{crit}}$ and $(\pi^{i,*}, c^{i,*})$ as ε tends to 0)**.** *Denote by* $(\pi^{i,\mathrm{crit}}_{\varepsilon>0}, c^{i,\mathrm{crit}}_{\varepsilon>0})$, $h^{i,\mathrm{crit}}_{\varepsilon>0}$ *and* $(\pi^{i,*}_{\varepsilon>0}, c^{i,*}_{\varepsilon>0})$, *resp.* $(\pi^{i,\mathrm{crit}}_{\varepsilon=0}, c^{i,\mathrm{crit}}_{\varepsilon=0})$, $h^{i,\mathrm{crit}}_{\varepsilon=0}$ *and* $(\pi^{i,*}_{\varepsilon=0}, c^{i,*}_{\varepsilon=0})$, *the critical strategy, the critical barrier and the optimal strategy in case of* $\varepsilon > 0$, *resp.* $\varepsilon = 0$. *Then the following holds true for every* $(t,x,y) \in [0,T] \times \mathbb{R} \times \mathbb{R}^+_0$.

i) $\lim_{\varepsilon\to 0}(\pi^{i,\mathrm{crit}}_{\varepsilon>0}(t,x), c^{i,\mathrm{crit}}_{\varepsilon>0}(t,x)) = (\pi^{i,\mathrm{crit}}_{\varepsilon=0}(t,x), c^{i,\mathrm{crit}}_{\varepsilon=0}(t,x))$.

ii) $\lim_{\varepsilon\to 0} h^{i,\mathrm{crit}}_{\varepsilon>0}(t,x) = h^{i,\mathrm{crit}}_{\varepsilon=0}(t,x)$.

iii) $\lim_{\varepsilon\to 0}(\pi^{i,*}_{\varepsilon>0}(t,x,y), c^{i,*}_{\varepsilon>0}(t,x,y)) = (\pi^{i,*}_{\varepsilon=0}(t,x,y), c^{i,*}_{\varepsilon=0}(t,x,y))$.

Proof. Let $(t,x,y)\in[0,T]\times\mathbb{R}\times\mathbb{R}_0^+$ be given. Notice that for every $(t,x,\lambda)\in[0,T]\times\mathbb{R}\times(-\infty,\frac{1}{B_c^i})$ ($B_c^i>0$), resp. $(t,x,\lambda)\in[0,T]\times\mathbb{R}\times(\frac{1}{B_c^i},\infty)$ ($B_c^i<0$)

$$\begin{aligned}\lim_{\varepsilon\to 0}\Lambda^{\pi,c,i}_{\varepsilon>0}(t,x,\lambda) &= \lim_{\varepsilon\to 0}\Big(\tfrac{1}{R}(B_\pi^i)^\top(\sigma^i.(\sigma^i)^\top)^{-1}.B_\pi^i\lambda + B_c^i\Big(\big(1-B_c^i\lambda\big)^{-\frac{1}{R}}-1\Big)c^M(t,x)\Big)\\ &= \tfrac{1}{R}(B_\pi^i)^\top(\sigma^i.(\sigma^i)^\top)^{-1}.B_\pi^i\lambda\\ &\triangleq \Lambda^{\pi,c,i}_{\varepsilon=0}(\lambda)\end{aligned}$$

since $\lim_{\varepsilon\to 0}c^M(t,x)=0$. This convergence together with the results from Lemma 4.65 yield that

$$\begin{aligned}\lim_{\varepsilon\to 0}\lambda^{i,\mathrm{crit}}_{\varepsilon>0}(t,x) &= \begin{cases}-\frac{(d^{\pi,c,i}(\pi^{i,M},0))^+}{\frac{1}{R}(B_\pi^i)^\top.(\sigma^i.(\sigma^i)^\top)^{-1}.B_\pi^i} & \text{if } B_c^i>0,\\ \frac{1}{B_c^i}\vee-\frac{(d^{\pi,c,i}(\pi^{i,M},0))^+}{\frac{1}{R}(B_\pi^i)^\top.(\sigma^i.(\sigma^i)^\top)^{-1}.B_\pi^i} & \text{if } B_c^i<0,\end{cases}\\ &= \frac{1}{B_c^i}\Big(1-\Big(\frac{(d^{\pi,c,i}(\pi^{i,M},0))^+}{\frac{1}{R}(B_\pi^i)^\top.(\sigma^i.(\sigma^i)^\top)^{-1}.B_\pi^i}B_c^i+1\Big)^+\Big).\end{aligned}\tag{$*$}$$

Keeping this convergence result in mind we can now prove the assertions of the lemma.

ad i) The convergence of the critical portfolio strategy is a direct consequence of $(*)$, i.e.

$$\begin{aligned}\lim_{\varepsilon\to 0}\pi^{i,\mathrm{crit}}_{\varepsilon>0}(t,x) &= \lim_{\varepsilon\to 0}\tfrac{1}{R}(\sigma^i.(\sigma^i)^\top)^{-1}.\Big(\eta^i+\lambda^{i,\mathrm{crit}}_{\varepsilon>0}(t,x)B_\pi^i\Big)\\ &= \tfrac{1}{R}(\sigma^i.(\sigma^i)^\top)^{-1}.\Big(\eta^i+B_\pi^i\tfrac{1}{B_c^i}\Big(1-\Big(\frac{(d^{\pi,c,i}(\pi^{i,M},0))^+}{\frac{1}{R}(B_\pi^i)^\top.(\sigma^i.(\sigma^i)^\top)^{-1}.B_\pi^i}B_c^i+1\Big)^+\Big)\Big)\\ &= \pi^{i,\mathrm{crit}}_{\varepsilon=0}(t,x).\end{aligned}$$

To prove the limiting behavior of the critical consumption rate first note that

$$\Lambda^{\pi,c,i}_{\varepsilon>0}(t,x,\lambda^{i,\mathrm{crit}}_{\varepsilon>0}(t,x)) = -(d^{\pi,c,i}(\pi^{i,M},c^M(t,x)))^+$$

is equivalent to

$$c^M(t,x)\big(1-\lambda^{i,\mathrm{crit}}_{\varepsilon>0}(t,x)B_c^i\big)^{-\frac{1}{R}} = \frac{-(d^{\pi,c,i}(\pi^{i,M},c^M(t,x)))^+-\frac{1}{R}(B_\pi^i)^\top(\sigma^i.(\sigma^i)^\top)^{-1}.B_\pi^i\lambda^{i,\mathrm{crit}}_{\varepsilon>0}(t,x)}{B_c^i}+c^M(t,x)$$

as long as $\varepsilon>0$. Hence

$$\begin{aligned}\lim_{\varepsilon\to 0}c^{i,\mathrm{crit}}_{\varepsilon>0}(t,x) &= \lim_{\varepsilon\to 0}c^M(t,x)\big(1-\lambda^{i,\mathrm{crit}}_{\varepsilon>0}(t,x)B_c^i\big)^{-\frac{1}{R}}\\ &= \lim_{\varepsilon\to 0}\Big(\frac{-(d^{\pi,c,i}(\pi^{i,M},c^M(t,x)))^+-\frac{1}{R}(B_\pi^i)^\top(\sigma^i.(\sigma^i)^\top)^{-1}.B_\pi^i\lambda^{i,\mathrm{crit}}_{\varepsilon>0}(t,x)}{B_c^i}+c^M(t,x)\Big)\\ &= \lim_{\varepsilon\to 0}\frac{-(d^{\pi,c,i}(\pi^{i,M},c^M(t,x)))^+}{B_c^i}-\frac{\frac{1}{R}(B_\pi^i)^\top(\sigma^i.(\sigma^i)^\top)^{-1}.B_\pi^i}{B_c^i}\lim_{\varepsilon\to 0}\lambda^{i,\mathrm{crit}}_{\varepsilon>0}(t,x)+\lim_{\varepsilon\to 0}c^M(t,x)\end{aligned}$$

$$
\begin{aligned}
&= -\frac{(d^{\pi,c,i}(\pi^{i,M},0))^+}{B_c^i} - \frac{\frac{1}{R}(B_\pi^i)^\top(\sigma^i.(\sigma^i)^\top)^{-1}.B_\pi^i}{(B_c^i)^2}\Big(1-\Big(\frac{(d^{\pi,c,i}(\pi^{i,M},0))^+}{\frac{1}{R}(B_\pi^i)^\top.(\sigma^i.(\sigma^i)^\top)^{-1}.B_\pi^i}B_c^i+1\Big)^+\Big)\\
&= \frac{\frac{1}{R}(B_\pi^i)^\top.(\sigma^i.(\sigma^i)^\top)^{-1}.B_\pi^i}{(B_c^i)^2}\\
&\qquad\cdot\Big[-\frac{(d^{\pi,c,i}(\pi^{i,M},0))^+}{\frac{1}{R}(B_\pi^i)^\top.(\sigma^i.(\sigma^i)^\top)^{-1}.B_\pi^i}B_c^i - \Big(1-\Big(\frac{(d^{\pi,c,i}(\pi^{i,M},0))^+}{\frac{1}{R}(B_\pi^i)^\top.(\sigma^i.(\sigma^i)^\top)^{-1}.B_\pi^i}B_c^i+1\Big)^+\Big)\Big]\\
&= \frac{\frac{1}{R}(B_\pi^i)^\top.(\sigma^i.(\sigma^i)^\top)^{-1}.B_\pi^i}{(B_c^i)^2}\\
&\qquad\cdot\Big[-\Big(\frac{(d^{\pi,c,i}(\pi^{i,M},0))^+}{\frac{1}{R}(B_\pi^i)^\top.(\sigma^i.(\sigma^i)^\top)^{-1}.B_\pi^i}B_c^i+1\Big) + \Big(\frac{(d^{\pi,c,i}(\pi^{i,M},0))^+}{\frac{1}{R}(B_\pi^i)^\top.(\sigma^i.(\sigma^i)^\top)^{-1}.B_\pi^i}B_c^i+1\Big)^+\Big]\\
&= \frac{\frac{1}{R}(B_\pi^i)^\top.(\sigma^i.(\sigma^i)^\top)^{-1}.B_\pi^i}{(B_c^i)^2}\Big(\frac{(d^{\pi,c,i}(\pi^{i,M},0))^+}{\frac{1}{R}(B_\pi^i)^\top.(\sigma^i.(\sigma^i)^\top)^{-1}.B_\pi^i}B_c^i+1\Big)^-\\
&= c_{\varepsilon=0}^{i,\text{crit}}.
\end{aligned}
$$

ad ii) In order to verify the limiting behavior of $h_{\varepsilon>0}^{i,\text{crit}}$ we have to consider $\zeta_{\varepsilon>0}^{i,\text{crit}}$, i.e.

$$
\begin{aligned}
\lim_{\varepsilon\to 0}\zeta_{\varepsilon>0}^{i,\text{crit}}(t,x) &= \lim_{\varepsilon\to 0}\Big(-\frac{1}{2}\frac{1}{R}(B_\pi^i)^\top(\sigma^i.(\sigma^i)^\top)^{-1}.B_\pi^i(\lambda_{\varepsilon>0}^{i,\text{crit}}(t,x))^2\\
&\qquad + \Big[\frac{1}{1-R}\big((1-\lambda_{\varepsilon>0}^{i,\text{crit}}(t,x)B_c^i)^{-\frac{1-R}{R}}-1\big) - (1-\lambda_{\varepsilon>0}^{i,\text{crit}}(t,x)B_c^i)^{-\frac{1}{R}}+1\Big]c^M(t,x)\Big)\\
&= -\frac{1}{2}\frac{\frac{1}{R}(B_\pi^i)^\top(\sigma^i.(\sigma^i)^\top)^{-1}.B_\pi^i}{(B_c^i)^2}\Big(1-\Big(\frac{(d^{\pi,c,i}(\pi^{i,M},0))^+}{\frac{1}{R}(B_\pi^i)^\top.(\sigma^i.(\sigma^i)^\top)^{-1}.B_\pi^i}B_c^i+1\Big)^+\Big)^2\\
&\qquad + \lim_{\varepsilon\to 0}\Big(\Big[\frac{1}{1-R}\big((1-\lambda_{\varepsilon>0}^{i,\text{crit}}(t,x)B_c^i)^{-\frac{1-R}{R}}-1\big) - (1-\lambda_{\varepsilon>0}^{i,\text{crit}}(t,x)B_c^i)^{-\frac{1}{R}}+1\Big]c^M(t,x)\Big)\\
&= -\frac{1}{2}\frac{\frac{1}{R}(B_\pi^i)^\top(\sigma^i.(\sigma^i)^\top)^{-1}.B_\pi^i}{(B_c^i)^2}\Big(1-\Big(\frac{(d^{\pi,c,i}(\pi^{i,M},0))^+}{\frac{1}{R}(B_\pi^i)^\top.(\sigma^i.(\sigma^i)^\top)^{-1}.B_\pi^i}B_c^i+1\Big)^+\Big)^2\\
&\qquad + \lim_{\varepsilon\to 0}\Big(\frac{1}{1-R}\big((1-\lambda_{\varepsilon>0}^{i,\text{crit}}(t,x)B_c^i)^{-\frac{1-R}{R}}-1\big)c^M(t,x)\Big)\\
&\qquad - \lim_{\varepsilon\to 0}\Big((1-\lambda_{\varepsilon>0}^{i,\text{crit}}(t,x)B_c^i)^{-\frac{1}{R}}c^M(t,x)\Big) + \lim_{\varepsilon\to 0}c^M(t,x)\\
&= -\frac{1}{2}\frac{\frac{1}{R}(B_\pi^i)^\top(\sigma^i.(\sigma^i)^\top)^{-1}.B_\pi^i}{(B_c^i)^2}\Big(1-\Big(\frac{(d^{\pi,c,i}(\pi^{i,M},0))^+}{\frac{1}{R}(B_\pi^i)^\top.(\sigma^i.(\sigma^i)^\top)^{-1}.B_\pi^i}B_c^i+1\Big)^+\Big)^2\\
&\qquad + \lim_{\varepsilon\to 0}\Big(\frac{1}{1-R}\big((1-\lambda_{\varepsilon>0}^{i,\text{crit}}(t,x)B_c^i)^{-\frac{1-R}{R}}-1\big)c^M(t,x)\Big) - \lim_{\varepsilon\to 0}c_{\varepsilon>0}^{i,\text{crit}}(t,x)\\
&= -\frac{1}{2}\frac{\frac{1}{R}(B_\pi^i)^\top(\sigma^i.(\sigma^i)^\top)^{-1}.B_\pi^i}{(B_c^i)^2}\Big(1-\Big(\frac{(d^{\pi,c,i}(\pi^{i,M},0))^+}{\frac{1}{R}(B_\pi^i)^\top.(\sigma^i.(\sigma^i)^\top)^{-1}.B_\pi^i}B_c^i+1\Big)^+\Big)^2\\
&\qquad - \frac{\frac{1}{R}(B_\pi^i)^\top.(\sigma^i.(\sigma^i)^\top)^{-1}.B_\pi^i}{(B_c^i)^2}\Big(\frac{(d^{\pi,c,i}(\pi^{i,M},0))^+}{\frac{1}{R}(B_\pi^i)^\top.(\sigma^i.(\sigma^i)^\top)^{-1}.B_\pi^i}B_c^i+1\Big)^-\\
&\qquad + \lim_{\varepsilon\to 0}\Big(\frac{1}{1-R}\big((1-\lambda_{\varepsilon>0}^{i,\text{crit}}(t,x)B_c^i)^{-\frac{1-R}{R}}-1\big)c^M(t,x)\Big)\\
&= -\frac{1}{2}\frac{\frac{1}{R}(B_\pi^i)^\top(\sigma^i.(\sigma^i)^\top)^{-1}.B_\pi^i}{(B_c^i)^2}\\
&\qquad\cdot\Big[\Big(1-\Big(\frac{(d^{\pi,c,i}(\pi^{i,M},0))^+}{\frac{1}{R}(B_\pi^i)^\top.(\sigma^i.(\sigma^i)^\top)^{-1}.B_\pi^i}B_c^i+1\Big)^+\Big)^2 + 2\Big(\frac{(d^{\pi,c,i}(\pi^{i,M},0))^+}{\frac{1}{R}(B_\pi^i)^\top.(\sigma^i.(\sigma^i)^\top)^{-1}.B_\pi^i}B_c^i+1\Big)^-\Big]\\
&\qquad + \lim_{\varepsilon\to 0}\Big(\frac{1}{1-R}\big((1-\lambda_{\varepsilon>0}^{i,\text{crit}}(t,x)B_c^i)^{-\frac{1-R}{R}}-1\big)c^M(t,x)\Big)\\
&= \zeta_{\varepsilon=0}^{i,\text{crit}} + \lim_{\varepsilon\to 0}\Big(\frac{1}{1-R}\big((1-\lambda_{\varepsilon>0}^{i,\text{crit}}(t,x)B_c^i)^{-\frac{1-R}{R}}-1\big)c^M(t,x)\Big).
\end{aligned}
$$

We now show that $\lim_{\varepsilon\to 0}\Big(\frac{1}{1-R}\big((1-\lambda^{i,\mathrm{crit}}_{\varepsilon>0}(t,x)B^i_c)^{-\frac{1-R}{R}}-1\big)c^M(t,x)\Big)=0$ which then implies that $\lim_{\varepsilon\to 0}\zeta^{i,\mathrm{crit}}_{\varepsilon>0}(t,x)=\zeta^{i,\mathrm{crit}}_{\varepsilon=0}$ and consequently $\lim_{\varepsilon\to 0}h^{i,\mathrm{crit}}_{\varepsilon>0}(t,x)=h^{i,\mathrm{crit}}_{\varepsilon=0}$.

To prove the aforementioned limit we consider

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\Big(\tfrac{1}{1-R}\big((1-\lambda^{i,\mathrm{crit}}_{\varepsilon>0}(t,x)B^i_c\big)^{-\frac{1-R}{R}}-1\big)c^M(t,x)\Big)\\
&\quad=\lim_{\varepsilon\to 0}\Big(\tfrac{1}{1-R}c^M(t,x)\big(1-\lambda^{i,\mathrm{crit}}_{\varepsilon>0}(t,x)B^i_c\big)^{-\frac{1}{R}}(1-\lambda^{i,\mathrm{crit}}_{\varepsilon>0}(t,x)B^i_c)\Big)-\lim_{\varepsilon\to 0}\tfrac{1}{1-R}c^M(t,x)\\
&\quad=\lim_{\varepsilon\to 0}\Big(\tfrac{1}{1-R}c^{i,\mathrm{crit}}_{\varepsilon>0}(t,x)\big(1-\lambda^{i,\mathrm{crit}}_{\varepsilon>0}(t,x)B^i_c\big)\Big)\\
&\quad=\tfrac{1}{1-R}\lim_{\varepsilon\to 0}c^{i,\mathrm{crit}}_{\varepsilon>0}(t,x)\lim_{\varepsilon\to 0}\big(1-\lambda^{i,\mathrm{crit}}_{\varepsilon>0}(t,x)B^i_c\big)\\
&\quad=\tfrac{1}{1-R}\frac{\frac{1}{R}(B^i_\pi)^\top.(\sigma^i.(\sigma^i)^\top)^{-1}.B^i_\pi}{(B^i_c)^2}\Big(\frac{(d^{\pi,c,i}(\pi^{i,M},0))^+}{\frac{1}{R}(B^i_\pi)^\top.(\sigma^i.(\sigma^i)^\top)^{-1}.B^i_\pi}B^i_c+1\Big)^-\\
&\qquad\cdot\Big(1-\tfrac{1}{B^i_c}\Big(1-\Big(\frac{(d^{\pi,c,i}(\pi^{i,M},0))^+}{\frac{1}{R}(B^i_\pi)^\top.(\sigma^i.(\sigma^i)^\top)^{-1}.B^i_\pi}B^i_c+1\Big)^+\Big)B^i_c\Big)\\
&\quad=\tfrac{1}{1-R}\frac{\frac{1}{R}(B^i_\pi)^\top.(\sigma^i.(\sigma^i)^\top)^{-1}.B^i_\pi}{(B^i_c)^2}\Big(\frac{(d^{\pi,c,i}(\pi^{i,M},0))^+}{\frac{1}{R}(B^i_\pi)^\top.(\sigma^i.(\sigma^i)^\top)^{-1}.B^i_\pi}B^i_c+1\Big)^-\Big(\frac{(d^{\pi,c,i}(\pi^{i,M},0))^+}{\frac{1}{R}(B^i_\pi)^\top.(\sigma^i.(\sigma^i)^\top)^{-1}.B^i_\pi}B^i_c+1\Big)^+\\
&\quad=0
\end{aligned}$$

ad iii) Assertion iii) follows directly from i) and ii).

Thus the lemma is proven. □

Remark 4.67. *So far the strategy $(\pi^{i,*},c^{i,*})$ is only a candidate solution for the optimal investment problem. To verify the optimality we need to show that the related HJB-system has a global solution that satisfies the required regularity conditions stated in the Verification Theorem 3.3.*

Inserting $(\pi^{i,*}(t,g(t)-1_{\{i=1\}}h(t),h(t)),c^{i,*}(t,g(t)-1_{\{i=1\}}h(t),h(t)))$ into the HJB-equation (4.32) the latter now becomes a system of backward ODEs which for $\varepsilon>0$ is given by

$$\begin{aligned}
h'(t)=&-(\Psi^0-\Psi^1)+\big(\tfrac{\varepsilon\delta}{\varepsilon-(\varepsilon-\delta)e^{-\delta(T-t)}}\big)^{\frac{1}{R}}e^{-\frac{1-R}{R}g(t)}\tfrac{R}{1-R}(e^{\frac{1-R}{R}h(t)}-1)\\
&-C^0_2\tfrac{1}{1-R}(e^{-(1-R)h(t)}-1)+C^1_2\tfrac{1}{1-R}(e^{(1-R)h(t)}-1)\\
&-\Big[(C^0_1-C^0_2)\tfrac{1}{1-R}(e^{-(1-R)h(t)}-1)-\tfrac{1}{2}\tfrac{1}{R}(B^0_\pi)^\top.(\sigma^0.(\sigma^0)^\top)^{-1}.B^0_\pi(\lambda^{0,\mathrm{crit}}(t,g(t)))^2\\
&\quad+\Big[\tfrac{1}{1-R}\Big(\big(1-\lambda^{0,\mathrm{crit}}(t,g(t))B^0_c\big)^{-\frac{1-R}{R}}-1\Big)\\
&\qquad-\big(1-\lambda^{0,\mathrm{crit}}(t,g(t))B^0_c\big)^{-\frac{1}{R}}+1\Big]c^M(t,g(t))\Big]^+\\
&+\Big[(C^1_1-C^1_2)\tfrac{1}{1-R}(e^{(1-R)h(t)}-1)-\tfrac{1}{2}\tfrac{1}{R}(B^1_\pi)^\top.(\sigma^1.(\sigma^1)^\top)^{-1}.B^1_\pi(\lambda^{1,\mathrm{crit}}(t,g(t)-h(t)))^2\\
&\quad+\Big[\tfrac{1}{1-R}\Big(\big(1-\lambda^{1,\mathrm{crit}}(t,g(t)-h(t))B^1_c\big)^{-\frac{1-R}{R}}-1\Big)\\
&\qquad-\big(1-\lambda^{1,\mathrm{crit}}(t,g(t)-h(t))B^1_c\big)^{-\frac{1}{R}}+1\Big]c^M(t,g(t)-h(t))\Big]^+,
\end{aligned}\tag{4.35}$$

$$\begin{aligned}
g'(t) = &- \Psi^0 + \tfrac{\varepsilon\delta}{\varepsilon-(\varepsilon-\delta)e^{-\delta(T-t)}}\Big(1 - \tfrac{R}{1-R}\big(\big(\tfrac{\varepsilon\delta}{\varepsilon-(\varepsilon-\delta)e^{-\delta(T-t)}}\big)^{\frac{1-R}{R}} e^{-\frac{1-R}{R}g(t)} - 1\big)\Big) \\
&- C_2^0 \tfrac{1}{1-R}(e^{-(1-R)h(t)} - 1) \\
&- \Big[(C_1^0 - C_2^0)\tfrac{1}{1-R}(e^{-(1-R)h(t)} - 1) - \tfrac{1}{2}\tfrac{1}{R}(B_\pi^0)^\top.(\sigma^0.(\sigma^0)^\top)^{-1}.B_\pi^0(\lambda^{0,\text{crit}}(t,g(t)))^2 \\
&\quad + \Big[\tfrac{1}{1-R}\Big(\big(1-\lambda^{0,\text{crit}}(t,g(t))B_c^0\big)^{-\frac{1-R}{R}} - 1\Big) \\
&\qquad - \big(1-\lambda^{0,\text{crit}}(t,g(t))B_c^0\big)^{-\frac{1}{R}} + 1\Big]c^M(t,g(t))\Big]^+
\end{aligned} \tag{4.36}$$

subject to the boundary conditions

$$g(T) = 0, \;\; h(T) = 0. \tag{4.37}$$

If $\varepsilon = 0$ then the ODE-system reads

$$\begin{aligned}
h'(t) = &- (\Psi^0 - \Psi^1) \\
&- C_2^0 \tfrac{1}{1-R}(e^{-(1-R)h(t)} - 1) + C_2^1 \tfrac{1}{1-R}(e^{(1-R)h(t)} - 1) \\
&- \Big[(C_1^0 - C_2^0)\tfrac{1}{1-R}(e^{-(1-R)h(t)} - 1) - \tfrac{1}{2}\tfrac{\frac{1}{R}(B_\pi^0)^\top(\sigma^0.(\sigma^0)^\top)^{-1}.B_\pi^0}{(B_c^0)^2} \\
&\quad \cdot \Big[\Big(1 - \Big(\tfrac{(A^0+(\pi^{0,M})^\top.B_\pi^0-C^0)^+}{\frac{1}{R}(B_\pi^0)^\top.(\sigma^0.(\sigma^0)^\top)^{-1}.B_\pi^0}B_c^0 + 1\Big)^+\Big)^2 + 2\Big(\tfrac{(A^0+(\pi^{0,M})^\top.B_\pi^0-C^0)^+}{\frac{1}{R}(B_\pi^0)^\top.(\sigma^0.(\sigma^0)^\top)^{-1}.B_\pi^0}B_c^0 + 1\Big)^-\Big]\Big]^+ \\
&+ \Big[(C_1^1 - C_2^1)\tfrac{1}{1-R}(e^{(1-R)h(t)} - 1) - \tfrac{1}{2}\tfrac{\frac{1}{R}(B_\pi^1)^\top(\sigma^1.(\sigma^1)^\top)^{-1}.B_\pi^1}{(B_c^1)^2} \\
&\quad \cdot \Big[\Big(1 - \Big(\tfrac{(A^1+(\pi^{1,M})^\top.B_\pi^1-C^1)^+}{\frac{1}{R}(B_\pi^1)^\top.(\sigma^1.(\sigma^1)^\top)^{-1}.B_\pi^1}B_c^1 + 1\Big)^+\Big)^2 + 2\Big(\tfrac{(A^1+(\pi^{1,M})^\top.B_\pi^1-C^1)^+}{\frac{1}{R}(B_\pi^1)^\top.(\sigma^1.(\sigma^1)^\top)^{-1}.B_\pi^1}B_c^1 + 1\Big)^-\Big]\Big]^+,
\end{aligned} \tag{4.38}$$

$$\begin{aligned}
g'(t) = &- \Psi^0 \\
&- C_2^0 \tfrac{1}{1-R}(e^{-(1-R)h(t)} - 1) \\
&- \Big[(C_1^0 - C_2^0)\tfrac{1}{1-R}(e^{-(1-R)h(t)} - 1) - \tfrac{1}{2}\tfrac{\frac{1}{R}(B_\pi^0)^\top(\sigma^0.(\sigma^0)^\top)^{-1}.B_\pi^0}{(B_c^0)^2} \\
&\quad \cdot \Big[\Big(1 - \Big(\tfrac{(A^0+(\pi^{0,M})^\top.B_\pi^0-C^0)^+}{\frac{1}{R}(B_\pi^0)^\top.(\sigma^0.(\sigma^0)^\top)^{-1}.B_\pi^0}B_c^0 + 1\Big)^+\Big)^2 + 2\Big(\tfrac{(A^0+(\pi^{0,M})^\top.B_\pi^0-C^0)^+}{\frac{1}{R}(B_\pi^0)^\top.(\sigma^0.(\sigma^0)^\top)^{-1}.B_\pi^0}B_c^0 + 1\Big)^-\Big]\Big]^+
\end{aligned} \tag{4.39}$$

subject to the boundary conditions

$$g(T) = 0, \;\; h(T) = 0. \tag{4.40}$$

Remark 4.68. *From the proof of Lemma 4.58 we know that for* $(t,x,y) \in [0,T] \times \mathbb{R} \times \mathbb{R}_0^+$

$$\begin{aligned}
y \geq h^{i,\text{crit}}(t,x) \;\; \Leftrightarrow \;\; &(C_1^i - C_2^i)\tfrac{1}{1-R}(e^{(-1)^{1-i}(1-R)y} - 1) - \tfrac{1}{2}\tfrac{1}{R}(B_\pi^i)^\top.(\sigma^i.(\sigma^i)^\top)^{-1}.B_\pi^i(\lambda^{i,\text{crit}}(t,x))^2 \\
&+ \Big[\tfrac{1}{1-R}\Big(\big(1-\lambda^{i,\text{crit}}(t,x)B_c^i\big)^{-\frac{1-R}{R}} - 1\Big) - \big(1-\lambda^{i,\text{crit}}(t,x))B_c^i\big)^{-\frac{1}{R}} + 1\Big]c^M(t,x) \geq 0
\end{aligned}$$

if $\varepsilon > 0$ *and*

$$\begin{aligned}
y \geq h^{i,\text{crit}} \;\; \Leftrightarrow \;\; &(C_1^i - C_2^i)\tfrac{1}{1-R}(e^{(-1)^{1-i}(1-R)y} - 1) - \tfrac{1}{2}\tfrac{\frac{1}{R}(B_\pi^i)^\top(\sigma^i.(\sigma^i)^\top)^{-1}.B_\pi^i}{(B_c^i)^2} \\
&\cdot \Big[\Big(1 - \Big(\tfrac{(A^i+(\pi^{i,M})^\top.B_\pi^i-C^i)^+}{\frac{1}{R}(B_\pi^i)^\top.(\sigma^i.(\sigma^i)^\top)^{-1}.B_\pi^i}B_c^i + 1\Big)^+\Big)^2 + 2\Big(\tfrac{(A^i+(\pi^{i,M})^\top.B_\pi^i-C^i)^+}{\frac{1}{R}(B_\pi^i)^\top.(\sigma^i.(\sigma^i)^\top)^{-1}.B_\pi^i}B_c^i + 1\Big)^-\Big] \geq 0
\end{aligned}$$

if $\varepsilon = 0$.

Remark 4.69. *The aforementioned ODEs include the ODEs of the classical Merton problem (first rows) and the ODEs of the Bäuerle-Rieder problem with constant regime shift intensities C_2^i (first two rows).*

Remark 4.70. *Notice that the ODEs above resemble the ODEs from the portfolio-dependent model with $\varepsilon = 0$, namely the ODEs (4.22) and (4.23). The hidden difference lies in $\zeta^{i,\mathrm{crit}}$. Whereas $\zeta^{i,\mathrm{crit}}$ is given by*

$$\zeta_{\mathrm{PCD}}^{i,\mathrm{crit}} = -\frac{1}{2}\frac{\frac{1}{R}(B_\pi^i)^\top(\sigma^i.(\sigma^i)^\top)^{-1}.B_\pi^i}{(B_c^i)^2}\Bigg[\bigg(1-\Big(\frac{(A^i+(\pi^{i,M})^\top.B_\pi^i-C^i)^+}{\frac{1}{R}(B_\pi^i)^\top.(\sigma^i.(\sigma^i)^\top)^{-1}.B_\pi^i}B_c^i+1\Big)^+\bigg)^2+2\Big(\frac{(A^i+(\pi^{i,M})^\top.B_\pi^i-C^i)^+}{\frac{1}{R}(B_\pi^i)^\top.(\sigma^i.(\sigma^i)^\top)^{-1}.B_\pi^i}B_c^i+1\Big)^-\Bigg]$$

in case of portfolio- and consumption-dependent intensities it is given by

$$\zeta_{\mathrm{PD}}^{i,\mathrm{crit}} = -\frac{1}{2}\frac{((A^i+(\pi^{i,M})^\top.B_\pi^i-C^i)^+)^2}{\frac{1}{R}(B_\pi^i)^\top.(\sigma^i.(\sigma^i)^\top)^{-1}.B_\pi^i}$$

in case of portfolio-dependent intensities.

Therefore if $\varepsilon = 0$ and either $B_c^i > 0$, or $\frac{(A^i+(\pi^{i,M})^\top.B_\pi^i-C^i)^+}{\frac{1}{R}(B_\pi^i)^\top.(\sigma^i.(\sigma^i)^\top)^{-1}.B_\pi^i}B_c^i+1 \geq 0$ and $B_c^i < 0$ then $\zeta_{\mathrm{PCD}}^{i,\mathrm{crit}} = \zeta_{\mathrm{PD}}^{i,\mathrm{crit}}$ and both models coincide in that the maximizing strategies and the ODEs for the functions g and h are the same.

Lemma 4.71. *The ODE-system given by equation (4.35) and (4.36) subject to the boundary conditions (4.37) ($\varepsilon > 0$), resp. (4.38) and (4.39) subject to the boundary conditions (4.40) ($\varepsilon = 0$), admits a unique global solution.*

Before we can prove the lemma we need the following result on $\lambda^{i,\mathrm{crit}}$.

Lemma 4.72. *The function $\lambda^{i,\mathrm{crit}} : [0,T] \times \mathbb{R} \to \mathbb{R}_0^-$ where $\lambda^{i,\mathrm{crit}}(t,x)$ is implicitly given by the condition $\Lambda^{\pi,c,i}(t,x,\lambda^{i,\mathrm{crit}}(t,x)) = -(d^{\pi,c,i}(\pi^{i,M}, c^M(t,x)))^+$ is continuous in t and x and moreover locally Lipschitz continuous in x.*

Proof. In order to prove the assertion let

$$\bar{\Lambda}^{\pi,c,i}(t,x,\lambda) \triangleq \Lambda^{\pi,c,i}(t,x,\lambda) + (d^{\pi,c,i}(\pi^{i,M}, c^M(t,x)))^+$$

for $(t,x,\lambda) \in [0,T] \times \mathbb{R} \times (-\infty, \frac{1}{B_c^i})$ ($B_c^i > 0$), resp. $(t,x,\lambda) \in [0,T] \times \mathbb{R} \times (\frac{1}{B_c^i}, \infty)$ ($B_c^i < 0$).

- *Continuity in t and x.* As $\Lambda^{\pi,c,i}(t,x,\lambda)$ is continuous in t, x and λ which has been seen in Lemma 4.56 and as $d^{\pi,c,i}$ and $[\,\cdot\,]^+$ are continuous functions, too, $\bar{\Lambda}^{\pi,c,i}(t,x,\lambda)$ is also continuous in t, x and λ. Further $\lambda^{i,\mathrm{crit}}(t,x)$ is the unique root of $\bar{\Lambda}^{\pi,c,i}(t,x,\lambda)$.

  The continuity of $\lambda^{i,\mathrm{crit}}(t,x)$ in (t,x) is a consequence of the Theorem of the Maximum (Theorem 3.6 in [Stokey, Lucas, Prescott 1989]). Using the terminology of [Stokey, Lucas, Prescott 1989] we define correspondences Γ, $G : [0,T] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ and functions $\phi : [0,T] \times \mathbb{R} \times (-\infty, \frac{1}{B_c^i}) \to \mathbb{R}$ ($B_c^i > 0$), resp. $\phi : [0,T] \times \mathbb{R} \times (\frac{1}{B_c^i}, \infty) \to \mathbb{R}$ ($B_c^i < 0$), $\psi : [0,T] \times \mathbb{R} \to \mathbb{R}$, with $\mathcal{P}(\mathbb{R})$ the power set of $\mathbb{R}$, given by

$$\Gamma(t,x) \triangleq \begin{cases} \left[-\frac{(d^{\pi,c,i}(\pi^{i,M},c^M(t,x)))^+}{\frac{1}{R}(B^i_\pi)^\top.(\sigma^i.(\sigma^i)^\top)^{-1}.B^i_\pi}, 0\right] & \text{if } B^i_c > 0, \\ \left[\frac{1}{B^i_c} \vee -\frac{(d^{\pi,c,i}(\pi^{i,M},c^M(t,x)))^+}{\frac{1}{R}(B^i_\pi)^\top.(\sigma^i.(\sigma^i)^\top)^{-1}.B^i_\pi}, 0\right] & \text{if } B^i_c < 0, \end{cases}$$

$$\phi(t,x,\lambda) \triangleq -(\bar{\Lambda}^{\pi,c,i}(t,x,\lambda))^2, \ \psi(t,x) \triangleq \max_{\lambda\in\Gamma(t,x)} \phi(t,x,\lambda),$$

$$G(t,x) \triangleq \{\lambda \in \Gamma(t,x) \,:\, \phi(t,x,\lambda) = \psi(t,x)\}\,.$$

Notice that the correspondence Γ is compact-valued and continuous. Moreover, ϕ is continuous. The Theorem of the Maximum then implies that G is upper hemi-continuous. By construction, $\lambda^{i,\text{crit}}(t,x)$ is the unique maximizer of $\phi(t,x,\cdot)$. Therefore G is single-valued which implies that it is even continuous as given in [Stokey, Lucas, Prescott 1989]. As $\lambda^{i,\text{crit}} = G$, the continuity of $\lambda^{i,\text{crit}}$ is proven.

- *Local Lipschitz continuity in t and x.* Subsequently we verify that $\lambda^{i,\text{crit}}(t,x)$ is locally Lipschitz continuous in t and x. Note that for any $(\hat{t},\hat{x}) \in [0,T]\times\mathbb{R}$ satisfying $d^{\pi,c,i}(\pi^{i,M},c^M(\hat{t},\hat{x})) > 0$ the continuity of $d^{\pi,c,i}$ implies that there exists an open neighborhood of $(\hat{t},\hat{x})$ denoted by $N(\hat{t},\hat{x}) \subset [0,T]\times\mathbb{R}$ such that any $(t,x)\in N(\hat{t},\hat{x})$ satisfies $d^{\pi,c,i}(\pi^{i,M},c^M(t,x)) > 0$. Thus, $\bar{\Lambda}^{\pi,c,i}(t,x,\lambda) = \Lambda^{\pi,c,i}(t,x,\lambda) + d^{\pi,c,i}(\pi^{i,M},c^M(t,x))$ on $N(\hat{t},\hat{x})\times(-\infty,\frac{1}{B^i_c})$ ($B^i_c > 0$), resp. $N(\hat{t},\hat{x})\times(\frac{1}{B^i_c},\infty)$ ($B^i_c < 0$). Hence, $\bar{\Lambda}^{\pi,c,i}(t,x,\lambda)$ is even continuously differentiable on $N(\hat{t},\hat{x})\times(-\infty,\frac{1}{B^i_c})$ ($B^i_c > 0$), resp. $N(\hat{t},\hat{x})\times(\frac{1}{B^i_c},\infty)$ ($B^i_c < 0$) and we can consider the derivative of $\bar{\Lambda}^{\pi,c,i}$ with respect to λ, i.e.

$$\bar{\Lambda}^{\pi,c,i}_\lambda(\hat{t},\hat{x},\lambda^{i,\text{crit}}(\hat{t},\hat{x})) = \tfrac{1}{R}(B^i_\pi)^\top(\sigma^i.(\sigma^i)^\top)^{-1}.B^i_\pi + \tfrac{1}{R}(B^i_c)^2\big(1 - B^i_c\lambda^{i,\text{crit}}(\hat{t},\hat{x})\big)^{-\frac{1+R}{R}}c^M(\hat{t},\hat{x}) > 0$$

where $\bar{\Lambda}^{\pi,c,i}_\lambda(t,x,\lambda) \triangleq \frac{\partial}{\partial\lambda}\bar{\Lambda}^{\pi,c,i}(t,x,\lambda)$. The implicit function theorem then implies that there exists another open neighborhood $\tilde{N}(\hat{t},\hat{x}) \subset N(\hat{t},\hat{x})$ of $(\hat{t},\hat{x})$ such that $\lambda^{i,\text{crit}}$ is continuously differentiable on $\tilde{N}(\hat{t},\hat{x})$. In conclusion, for any $(\hat{t},\hat{x}) \in [0,T]\times\mathbb{R}$ satisfying $d^{\pi,c,i}(\pi^{i,M},c^M(\hat{t},\hat{x})) > 0$ there exists an open neighborhood $\tilde{N}(\hat{t},\hat{x})$ of $(\hat{t},\hat{x})$ on which $\lambda^{i,\text{crit}}$ is continuously differentiable.

  Analogously, we derive that for any $(\hat{t},\hat{x}) \in [0,T]\times\mathbb{R}$ satisfying $d^{\pi,c,i}(\pi^{i,M},c^M(\hat{t},\hat{x})) < 0$ there exists an open neighborhood $\tilde{N}(\hat{t},\hat{x})$ of $(\hat{t},\hat{x})$ on which $\lambda^{i,\text{crit}}$ is continuously differentiable, too.

  Consider now an arbitrary $(\hat{t},\hat{x}) \in [0,T]\times\mathbb{R}$. If $(\hat{t},\hat{x})$ satisfies $d^{\pi,c,i}(\pi^{i,M},c^M(\hat{t},\hat{x})) > 0$ then the above results imply that there exists an open neighborhood $N(\hat{t},\hat{x})$ of $(\hat{t},\hat{x})$ such that $\lambda^{i,\text{crit}}$ is continuously differentiable on $N(\hat{t},\hat{x})$. Consequently, $\lambda^{i,\text{crit}}$ is locally Lipschitz continuous on the set $\big\{(t,x)\in[0,T]\times\mathbb{R} \,:\, d^{\pi,c,i}(\pi^{i,M},c^M(t,x)) > 0\big\}$. Analogously, we find that $\lambda^{i,\text{crit}}$ is locally Lipschitz continuous on the set $\big\{(t,x)\in[0,T]\times\mathbb{R} \,:\, d^{\pi,c,i}(\pi^{i,M},c^M(t,x)) < 0\big\}$.

  However, if $(\hat{t},\hat{x})$ satisfies $d^{\pi,c,i}(\pi^{i,M},c^M(\hat{t},\hat{x})) = 0$ then consider an arbitrary open neighborhood $N(\hat{t},\hat{x})$ of $(\hat{t},\hat{x})$ and define $N^>(\hat{t},\hat{x}) \triangleq \big\{(t,x)\in N(\hat{t},\hat{x}) \,:\, d^{\pi,c,i}(\pi^{i,M},c^M(t,x)) > 0\big\}$. For all $(t,x) \in N^>(\hat{t},\hat{x})$ the derivative $\lambda^{i,\text{crit}}_x(t,x)$ exists and is finite. The same holds true for every $(t,x) \in N^<(\hat{t},\hat{x}) \triangleq \big\{(t,x)\in N(\hat{t},\hat{x}) \,:\, d^{\pi,c,i}(\pi^{i,M},c^M(t,x)) < 0\big\}$. To see this, note that

$$\bar{\Lambda}^{\pi,c,i}(t,x,\lambda^{i,\text{crit}}(t,x)) = 0$$

  for all $(t,x)\in[0,T]\times\mathbb{R}$. Hence

$$0 = \tfrac{\partial}{\partial x}\bar{\Lambda}^{\pi,c,i}(t,x,\lambda^{i,\text{crit}}(t,x)) = \bar{\Lambda}^{\pi,c,i}_x(t,x,\lambda^{i,\text{crit}}(t,x)) + \bar{\Lambda}^{\pi,c,i}_\lambda(t,x,\lambda^{i,\text{crit}}(t,x))\lambda^{i,\text{crit}}_x(t,x) \qquad (*)$$

where

$$\bar{\Lambda}_x^{\pi,c,i}(t,x,\lambda) = \begin{cases} B_c^i\big(1-\lambda B_c^i\big)^{-\frac{1}{R}} c_x^M(t,x) & \text{if } d^{\pi,c,i}(\pi^{i,M}, c^M(t,x)) > 0, \\ B_c^i\Big(\big(1-\lambda B_c^i\big)^{-\frac{1}{R}} - 1\Big) c_x^M(t,x) & \text{if } d^{\pi,c,i}(\pi^{i,M}, c^M(t,x)) < 0, \end{cases}$$

$$\bar{\Lambda}_\lambda^{\pi,c,i}(t,x,\lambda) = \tfrac{1}{R}(B_\pi^i)^\top(\sigma^i.(\sigma^i)^\top)^{-1}.B_\pi^i + \tfrac{1}{R}(B_c^i)^2\big(1-B_c^i\lambda\big)^{-\frac{1+R}{R}} c^M(t,x).$$

Since $\bar{\Lambda}_\lambda^{\pi,c,i}(t,x,\lambda) > 0$ we can transform $(*)$ into

$$\begin{aligned} \lambda_x^{i,\text{crit}}(t,x) &= -\frac{\bar{\Lambda}_x^{\pi,c,i}(t,x,\lambda^{i,\text{crit}}(t,x))}{\bar{\Lambda}_\lambda^{\pi,c,i}(t,x,\lambda^{i,\text{crit}}(t,x))} \\ &= \begin{cases} -\dfrac{B_c^i\big(1-\lambda^{i,\text{crit}}(t,x)B_c^i\big)^{-\frac{1}{R}} c_x^M(t,x)}{\frac{1}{R}(B_\pi^i)^\top(\sigma^i.(\sigma^i)^\top)^{-1}.B_\pi^i + \frac{1}{R}(B_c^i)^2\big(1-B_c^i\lambda^{i,\text{crit}}(t,x)\big)^{-\frac{1+R}{R}} c^M(t,x)} & \text{if } d^{\pi,c,i}(\pi^{i,M}, c^M(t,x)) > 0, \\ -\dfrac{B_c^i\big(1-\lambda^{i,\text{crit}}(t,x)B_c^i\big)^{-\frac{1}{R}} c_x^M(t,x) - B_c^i c_x^M(t,x)}{\frac{1}{R}(B_\pi^i)^\top(\sigma^i.(\sigma^i)^\top)^{-1}.B_\pi^i + \frac{1}{R}(B_c^i)^2\big(1-B_c^i\lambda^{i,\text{crit}}(t,x)\big)^{-\frac{1+R}{R}} c^M(t,x)} & \text{if } d^{\pi,c,i}(\pi^{i,M}, c^M(t,x)) < 0. \end{cases} \end{aligned}$$

Since in both cases the numerator is locally bounded and the denominator is strictly positive and bounded away from 0, $\lambda_x^{i,\text{crit}}(t,x)$ is locally bounded in either case. Hence we can define

$$\begin{aligned} \lambda_x^{i,\text{crit},>}(\hat{t},\hat{x}) &\triangleq \sup\left\{|\lambda_x^{i,\text{crit}}(t,x)| \,:\, (t,x) \in N^>(\hat{t},\hat{x})\right\}, \\ \lambda_x^{i,\text{crit},<}(\hat{t},\hat{x}) &\triangleq \sup\left\{|\lambda_x^{i,\text{crit}}(t,x)| \,:\, (t,x) \in N^<(\hat{t},\hat{x})\right\} \end{aligned}$$

which are both finite. Thus the Lipschitz constant is given by

$$L(\hat{t},\hat{x}) \triangleq \lambda_x^{i,\text{crit},>}(\hat{t},\hat{x}) \vee \lambda_x^{i,\text{crit},<}(\hat{t},\hat{x}).$$

The last results together with the continuity of $\lambda^{i,\text{crit}}$ imply that $\lambda^{i,\text{crit}}$ is locally Lipschitz continuous on $[0,T] \times \mathbb{R}$.

This finishes the proof. □

We now come to the proof of Lemma 4.71.

Proof of Lemma 4.71. The proof is essentially the same as the proof of Lemma 4.9. Only the χ^i's are different. Therefore we just present the actual χ^i's and verify that those satisfy the necessary continuity conditions and the non-negativity condition.

- *Definition of the χ^i's.* The χ^i's are given by $\chi^i : [0,T] \times \mathbb{R} \times \mathbb{R}_0^+ \to \mathbb{R}$, $i = 0, 1$, with

$$\chi^i(t,x,y) \triangleq (-1)^{1-i}\Big(C_2^i \tfrac{1}{1-R}(e^{(-1)^{1-i}(1-R)y} - 1) + \Big[(C_1^i - C_2^i)\tfrac{1}{1-R}(e^{(-1)^{1-i}(1-R)y} - 1) + \zeta^{i,\text{crit}}(t,x)\Big]^+\Big).$$

- *Continuity results on the χ^i's.* The χ^i's are continuous in t, and further, as compositions of continuously differentiable and locally Lipschitz continuous functions in x and y and the $[\,\cdot\,]^+$-function locally Lipschitz continuous in x and y.

- *Non-negativity of the χ^i's.* It is easy to see that $\chi^1 \geq 0$. The non-negativity of χ^0 follows from the non-positivity of $\zeta^{0,\mathrm{crit}}$. To see this we distinguish the two cases $\varepsilon > 0$ and $\varepsilon = 0$.

  - $\varepsilon > 0$. If $\varepsilon > 0$ then

$$\zeta^{0,\mathrm{crit}}(t,x) = -\frac{1}{2}\frac{1}{R}(B^0_\pi)^\top(\sigma^0.(\sigma^0)^\top)^{-1}.B^0_\pi(\lambda^{0,\mathrm{crit}}(t,x))^2 \\ + \Big[\tfrac{1}{1-R}\big((1-\lambda^{0,\mathrm{crit}}(t,x)B^0_c)^{-\frac{1-R}{R}} - 1\big) - (1-\lambda^{0,\mathrm{crit}}(t,x)B^0_c)^{-\frac{1}{R}} + 1\Big]c^M(t,x)$$

    whereof the first summand is obviously non-positive. Further the second summand is non-positive, too, since

$$\tfrac{1}{1-R}\Big(\big(1-\lambda^{0,\mathrm{crit}}(t,x)B^0_c\big)^{-\frac{1-R}{R}} - 1\Big) - \Big(\big(1-\lambda^{0,\mathrm{crit}}(t,x)B^0_c\big)^{-\frac{1}{R}} - 1\Big) \leq 0$$

    which follows from Lemma 4.48, iii), with $z = \big(1-\lambda^{0,\mathrm{crit}}(t,x)B^0_c\big)^{-\frac{1}{R}} - 1$. The lemma is applicable because $\big(1-\lambda^{0,\mathrm{crit}}(t,x)B^0_c\big)^{-\frac{1}{R}} > 0$.
  - $\varepsilon = 0$. If $\varepsilon = 0$ then

$$\zeta^{0,\mathrm{crit}}(t,x) = -\frac{1}{2}\frac{\frac{1}{R}(B^0_\pi)^\top(\sigma^0.(\sigma^0)^\top)^{-1}.B^0_\pi}{(B^0_c)^2}\Bigg[\bigg(1-\Big(\frac{(A^0+(\pi^{0,M})^\top.B^0_\pi-C^0)^+}{\frac{1}{R}(B^0_\pi)^\top.(\sigma^0.(\sigma^0)^\top)^{-1}.B^0_\pi}B^0_c+1\Big)^+\bigg)^2 \\ + 2\Big(\frac{(A^0+(\pi^{0,M})^\top.B^0_\pi-C^0)^+}{\frac{1}{R}(B^0_\pi)^\top.(\sigma^0.(\sigma^0)^\top)^{-1}.B^0_\pi}B^0_c+1\Big)^-\Bigg]$$

    which is trivially non-positive.

  The non-positivity of $\zeta^{i,\mathrm{crit}}$ implies that

$$(C^0_1 - C^0_2)\tfrac{1}{1-R}(e^{-(1-R)y} - 1) \geq \Big[(C^0_1 - C^0_2)\tfrac{1}{1-R}(e^{-(1-R)y} - 1) + \zeta^{0,\mathrm{crit}}(t,x)\Big]^+$$

  and hence

$$\chi^0(t,x,y) \geq -C^0_1\tfrac{1}{1-R}(e^{-(1-R)y} - 1) \geq 0 \text{ for every } (t,x,y) \in [0,T] \times \mathbb{R} \times \mathbb{R}^+_0.$$

The remainder of the proof is the same as the proof of Lemma 4.9. □

Examples for the functions g and h are given in Figure 4.8.

The proof of Lemma 4.71 directly implies the following corollaries.

Corollary 4.73 (Time-dependent bounds on g and h)**.** *Let h and g be given by (4.35), (4.36) subject to the boundary conditions (4.37) ($\varepsilon > 0$), resp. (4.38), (4.39) subject to the boundary conditions (4.40) ($\varepsilon = 0$). Then*

$$-(\Psi^1 - \xi^1(T))^-(T-t) \leq g(t) \leq (\Psi^0 - \xi^0(T))^+(T-t) \ (\varepsilon > 0), \\ \textit{resp. } 0 \leq g(t) \leq \Psi^0(T-t) \ (\varepsilon = 0) \textit{ and} \\ 0 \leq h(t) \leq (\Psi^0 - \Psi^1)(T-t)$$

for $t \in [0,T]$ where the $\xi^i(T)$'s are as defined in the proof of Lemma 4.9.

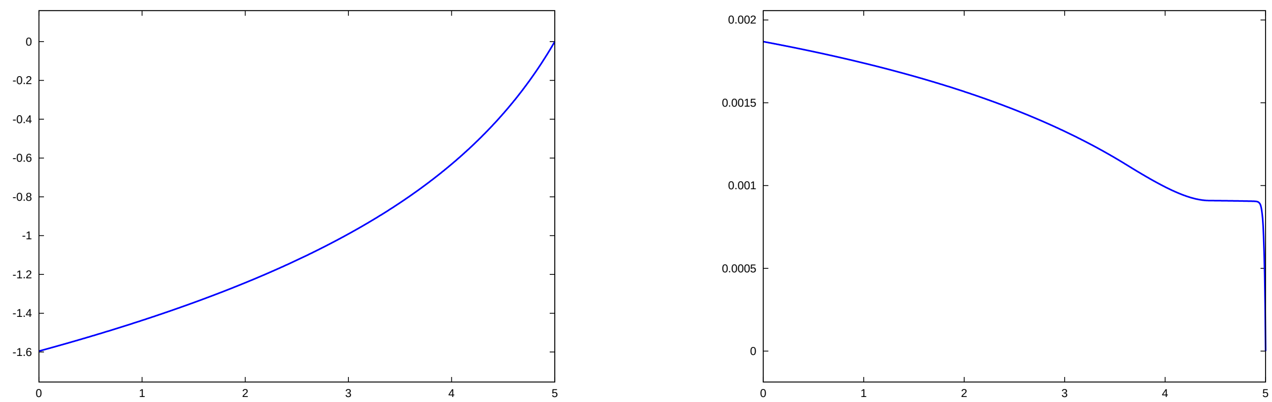

Figure 4.8: g (left) and h (right) as functions of t
($r^0 = r^1 = 0.03$, $\eta^0 = 0.1$, $\eta^1 = 0.06$, $\sigma^0 = 0.4$, $\sigma^1 = 0.7$, $\delta = 0.035$, $\varepsilon = 1$, $R = 0.5$, $T = 5$,
$A^0 = 27.5$, $B^0_\pi = 5$, $B^0_c = -15$, $C^0 = 27.5$, $C^0_1 = 10$, $C^0_2 = 50$,
$A^1 = 10$, $B^1_\pi = 5$, $B^1_c = -10$, $C^1 = 4.8$, $C^1_1 = 50$, $C^1_2 = 0.125$)

Corollary 4.74 (Time-independent bound on h)**.** *Let h and g be given by (4.35), (4.36) subject to the boundary conditions (4.37) ($\varepsilon > 0$), resp. (4.38), (4.39) subject to the boundary conditions (4.40) ($\varepsilon = 0$). Then*

$$0 \leq h(t) \leq \bar{h}$$

for $t \in [0, T]$ where $\bar{h}$ is the smallest positive root of

$$\bar{F}(y) \triangleq \begin{cases} -(\Psi^0 - \Psi^1) - C^0_1 \frac{1}{1-R}(e^{-(1-R)y} - 1) + C^1_2 \frac{1}{1-R}(e^{(1-R)y} - 1) & \text{if } \varepsilon > 0, \\ -(\Psi^0 - \Psi^1) + \chi^0(t,x,y) + \chi^1(t,x,y) & \text{if } \varepsilon = 0 \end{cases}$$

if such a root exists; otherwise $\bar{h} = \infty$, i.e.

$$\bar{h} = \min \left\{ y \in \mathbb{R}^+_0 \, : \, \bar{F}(y) = 0 \right\},$$

with the convention $\min \emptyset \triangleq \infty$, where the χ^i's are as defined in the proof of Lemma 4.71.

Remark 4.75. *Notice that in case of $\varepsilon = 0$ the function $\bar{F}(y)$ is indeed a function solely in y although the χ^i's formally depend on t and x, too. However setting $\varepsilon = 0$ implies that $\zeta^{i,\text{crit}}$ and consequently $h^{i,\text{crit}}$ are independent of t and x. This yields that the χ^i's are independent of t and x, too.*

Proof of Corollary 4.74. The assertion follows from a simple ODE argument since

$$F(t,x,y) = -(\Psi^0 - \Psi^1) + \varrho(t,x,y) + \chi^0(t,x,y) + \chi^1(t,x,y) \geq \bar{F}(y)$$

for all $(t,x,y) \in [0,T] \times \mathbb{R} \times \mathbb{R}^+_0$ where we used the notation from the proof of Lemma 4.9. If $\varepsilon = 0$ then this is obviously true. In case of $\varepsilon > 0$ this is satisfied, too, since the proof of Lemma 4.71 showed that $\chi^0(t,x,y) \geq -C^0_1 \frac{1}{1-R}(e^{-(1-R)y} - 1)$ and $\chi^1(t,x,y) \geq C^1_2 \frac{1}{1-R}(e^{(1-R)y} - 1)$. □

The boundary function on h as presented in Corollary 4.73 is suitable for large values of t for which the deviation from h is not too large whereas for small values of t the non-linear behavior of h implies that the boundary function overshoots h by far. The converse is true for the bound given in Corollary 4.74. Thus combining the two bounds on h yields

$$0 \le h(t) \le \min\{(\Psi^0 - \Psi^1)(T-t), \bar{h}\}$$

for $t \in [0,T]$.

The following theorem ensures that the strategy given in Lemma 4.58 is indeed the optimal strategy for the optimal investment problem.

Theorem 4.76 (Solution of the Investment Problem)**.** *Let condition (NJ) be satisfied, and let h and g be given by (4.35), (4.36) subject to the boundary conditions (4.37) ($\varepsilon > 0$), resp. (4.38), (4.39) subject to the boundary conditions (4.40) ($\varepsilon = 0$). Then the strategy*

$$(\pi^{i,*}(t), c^{i,*}(t)) \triangleq (\pi^{i,*}(t, g(t) - 1_{\{i=1\}}h(t), h(t)), c^{i,*}(t, g(t) - 1_{\{i=1\}}h(t), h(t))),\ t \in [0,T],\ i = 0,1,$$

as given in Lemma 4.58 is optimal for the investment problem (P) with CRRA *preferences and relative risk aversion R.*

Proof. Since $(\pi^{i,*}(t), c^{i,*}(t))$ maximizes the reduced HJB-system (4.20) for each $t \in [0,T]$, optimality of the strategy $(\pi^{i,*}, c^{i,*})$ follows directly from the Verification Theorem 3.3. □

Remark 4.77. *If not otherwise stated we use the following shorthand notations*

$$\pi^{i,\text{crit}}(t) \triangleq \pi^{i,\text{crit}}(t, g(t) - 1_{\{i=1\}}h(t)), \qquad c^{i,\text{crit}}(t) \triangleq c^{i,\text{crit}}(t, g(t) - 1_{\{i=1\}}h(t)),$$
$$h^{i,\text{crit}}(t) \triangleq h^{i,\text{crit}}(t, g(t) - 1_{\{i=1\}}h(t)), \qquad \lambda^{i,\text{crit}}(t) \triangleq \lambda^{i,\text{crit}}(t, g(t) - 1_{\{i=1\}}h(t)).$$

Figure 4.9 shows the optimal strategy and the Merton strategy in a setting where B^i_π is positive and B^i_c is negative for $i = 0,1$. Hence the large investor is forced to invest less than the Merton fractions and to consume at a rate that is greater than or equal the Merton consumption rate in order to generate advantageous regime shift intensities. As in the previous variants of the step intensity functions the investor turns back to the Merton strategy as soon as the time to maturity is suitably small. In the particular example of Figure 4.9 those times are $t \approx 3.6$ ($i = 0$), resp. $t \approx 4.4$ ($i = 1$).

As already in the consumption-dependent setting it is possible that the large investor consumes although $\varepsilon = 0$. Figure 4.10 provides an example for this situation.

The large investor's optimal portfolio strategy consists of the classical Merton strategy and an additional hedging component. Further the investor's optimal consumption rate can be decomposed into the Merton consumption rate and an adjustment part. The hedging component and the adjustment part result from the investor's influence on the market. So we can write

$$\pi^{i,*}(t) = \pi^{i,M} + \pi^{i,H}(t) \text{ and } c^{i,*}(t) = c^{i,M}(t) + c^{i,A}(t),\ i = 0,1,$$

where we have

$$\pi^{i,H}(t) = \begin{cases} \frac{1}{R}\lambda^{i,\text{crit}}(t)(\sigma^i.(\sigma^i)^\top)^{-1}.B^i_\pi 1_{\{h(t)\ge h^{i,\text{crit}}(t)\}} & \text{if } \varepsilon > 0, \\ \frac{1}{R}(\sigma^i.(\sigma^i)^\top)^{-1}.B^i_\pi \frac{1}{B^i_c}\Big(1 - \Big(\frac{(A^i + (\pi^{i,M})^\top.B^i_\pi - C^i)^+}{\frac{1}{R}(B^i_\pi)^\top.(\sigma^i.(\sigma^i)^\top)^{-1}.B^i_\pi}B^i_c + 1\Big)^+\Big)1_{\{h(t)\ge h^{i,\text{crit}}(t)\}} & \text{if } \varepsilon = 0, \end{cases}$$

$$c^{i,A}(t) = \begin{cases} \Big(\big(1 - \lambda^{i,\text{crit}}(t)B^i_c\big)^{-\frac{1}{R}} - 1\Big)c^{i,M}(t)1_{\{h(t)\ge h^{i,\text{crit}}(t)\}} & \text{if } \varepsilon > 0, \\ \frac{\frac{1}{R}(B^i_\pi)^\top.(\sigma^i.(\sigma^i)^\top)^{-1}.B^i_\pi}{(B^i_c)^2}\Big(\frac{(A^i + (\pi^{i,M})^\top.B^i_\pi - C^i)^+}{\frac{1}{R}(B^i_\pi)^\top.(\sigma^i.(\sigma^i)^\top)^{-1}.B^i_\pi}B^i_c + 1\Big)^- 1_{\{h(t)\ge h^{i,\text{crit}}(t)\}} & \text{if } \varepsilon = 0. \end{cases}$$

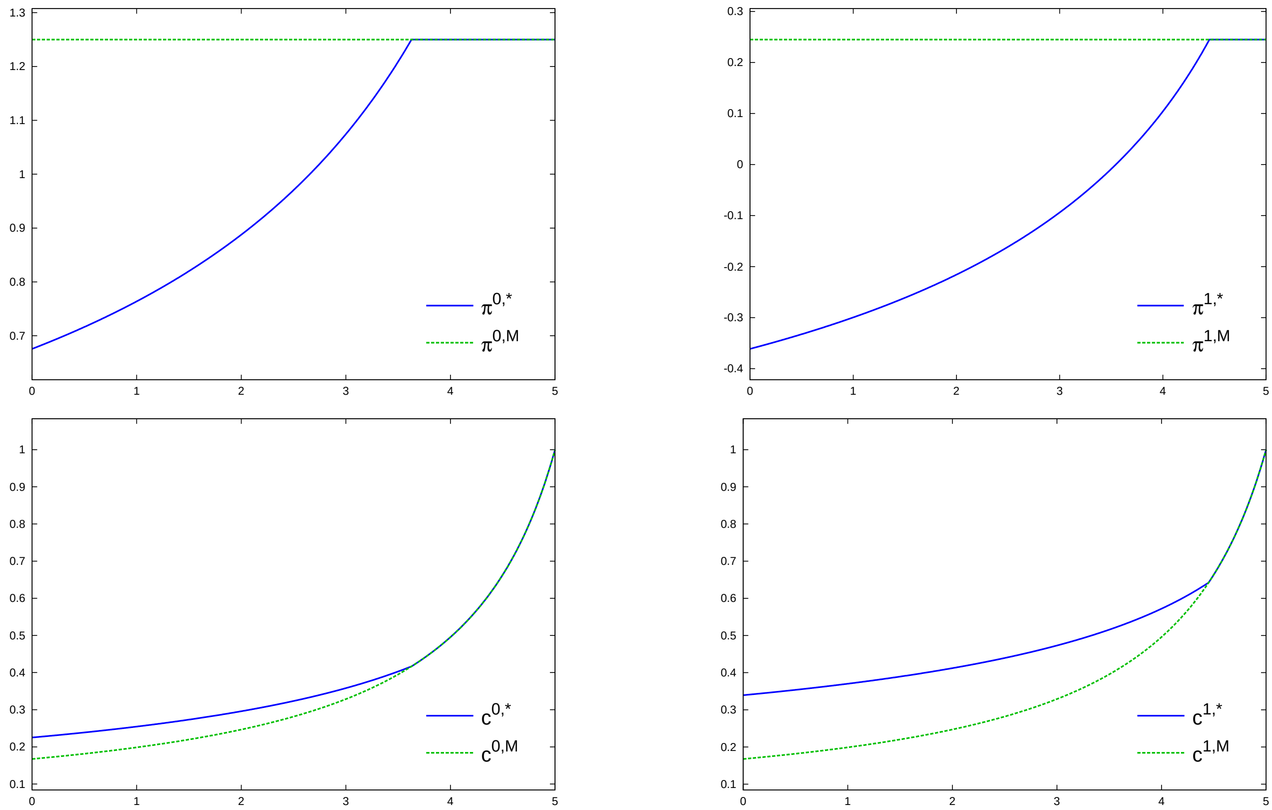

Figure 4.9: Optimal strategy vs. Merton strategy: $\pi^{i,*}$, $c^{i,*}$ and $\pi^{i,M}$, $c^{i,M}$ as functions of t
($r^0 = r^1 = 0.03$, $\eta^0 = 0.1$, $\eta^1 = 0.06$, $\sigma^0 = 0.4$, $\sigma^1 = 0.7$, $\delta = 0.035$, $\varepsilon = 1$, $R = 0.5$, $T = 5$,
$A^0 = 27.5$, $B^0_\pi = 5$, $B^0_c = -15$, $C^0 = 27.5$, $C^0_1 = 10$, $C^0_2 = 50$,
$A^1 = 10$, $B^1_\pi = 5$, $B^1_c = -10$, $C^1 = 4.8$, $C^1_1 = 50$, $C^1_2 = 0.125$)

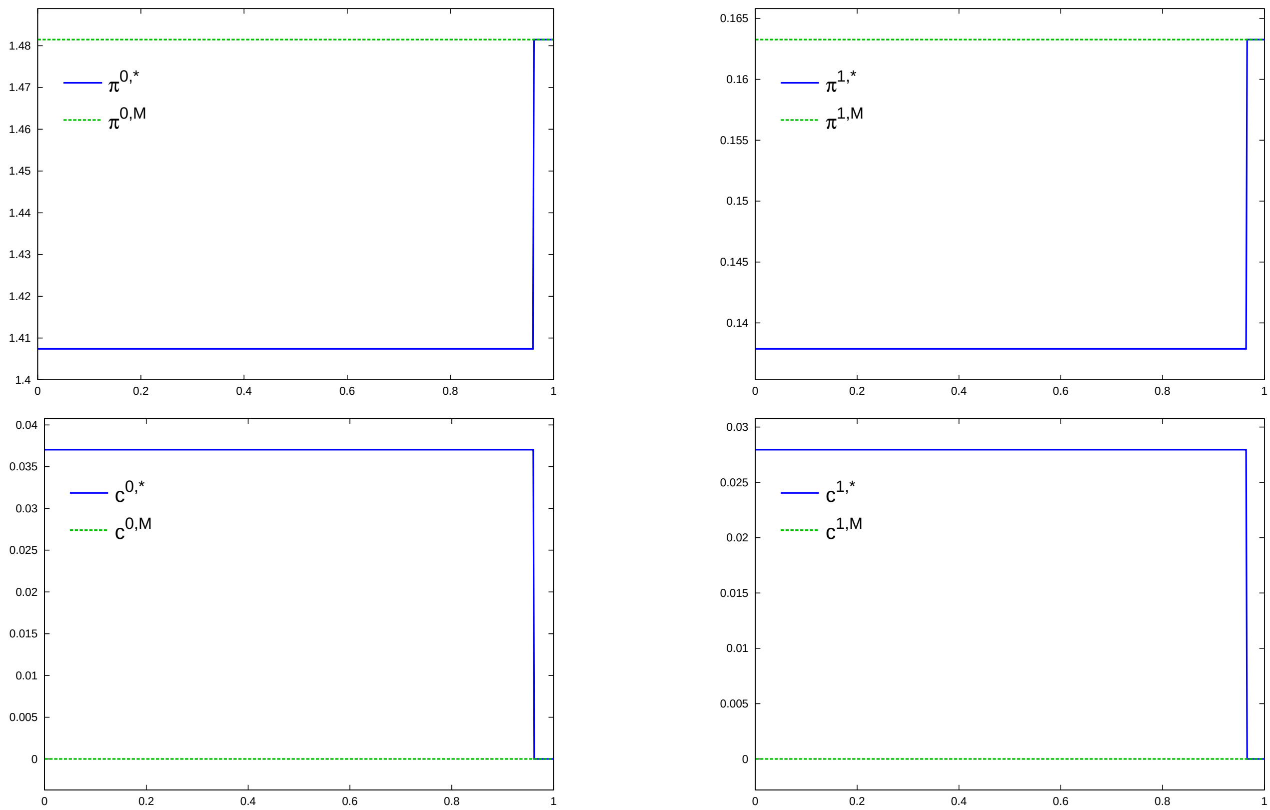

Figure 4.10: Optimal strategy vs. Merton strategy: $\pi^{i,*}$, $c^{i,*}$ and $\pi^{i,M}$, $c^{i,M}$ as functions of t
($r^0 = r^1 = 0.03$, $\eta^0 = 0.1$, $\eta^1 = 0.06$, $\sigma^0 = 0.3$, $\sigma^1 = 0.7$, $\delta = 0.035$, $\varepsilon = 0$, $R = 0.75$, $T = 1$,
$A^0 = 22.5$, $B^0_\pi = 0.5$, $B^0_c = -100$, $C^0 = 19.5$, $C^0_1 = 5$, $C^0_2 = 30$,
$A^1 = 11$, $B^1_\pi = 0.7$, $B^1_c = -75$, $C^1 = 9$, $C^1_1 = 25$, $C^1_2 = 5$)

Notice that $(\pi^{i,H}(t))^\top.B_\pi^i \le 0$. Moreover $B_c^i c^{i,A}(t) \le 0$. If $\varepsilon > 0$ this is true since $\lambda^{i,\text{crit}}(t) \le 0$. In case of $\varepsilon = 0$ this can be easily verified.

The following lemma shows that $(\pi^{i,H}, c^{i,A})$ benefits the regime shift intensities.

Lemma 4.78. *The optimal strategy* $(\pi^{i,*}, c^{i,*})$ *satisfies*

$$\vartheta^{0,1}(\pi^{0,*}(t), c^{0,*}(t)) \le \vartheta^{0,1}(\pi^{0,M}, c^{0,M}(t)) \text{ and } \vartheta^{1,0}(\pi^{1,*}(t), c^{1,*}(t)) \ge \vartheta^{1,0}(\pi^{1,M}, c^{1,M}(t))$$

for every $t \in [0,T]$.

Proof. The assertion follows since

$$\begin{aligned}
\vartheta^{0,1}(\pi^{0,*}(t), c^{0,*}(t)) &= C_1^0 1_{\{A^0+(\pi^{0,*}(t))^\top.B_\pi^0+B_c^0 c^{0,*}(t)\le C^0\}} + C_2^0 1_{\{A^0+(\pi^{0,*}(t))^\top.B_\pi^0+B_c^0 c^{0,*}(t)> C^0\}} \\
&= C_1^0 1_{\{A^0+(\pi^{0,M}(t))^\top.B_\pi^0+(\pi^{0,H}(t))^\top.B_\pi^0+B_c^0 c^{0,M}(t)+B_c^0 c^{0,A}(t)\le C^0\}} \\
&\quad + C_2^0 1_{\{A^0+(\pi^{0,M}(t))^\top.B_\pi^0+(\pi^{0,H}(t))^\top.B_\pi^0+B_c^0 c^{0,M}(t)+B_c^0 c^{0,A}(t)> C^0\}} \\
&\le C_1^0 1_{\{A^0+(\pi^{0,M}(t))^\top.B_\pi^0+B_c^0 c^{0,M}(t)\le C^0\}} + C_2^0 1_{\{A^0+(\pi^{0,M}(t))^\top.B_\pi^0+B_c^0 c^{0,M}(t)> C^0\}} \\
&= \vartheta^{0,1}(\pi^{0,M}, c^{0,M}(t))
\end{aligned}$$

and

$$\begin{aligned}
\vartheta^{1,0}(\pi^{1,*}(t), c^{1,*}(t)) &= C_1^1 1_{\{A^1+(\pi^{1,*}(t))^\top.B_\pi^1+B_c^1 c^{1,*}(t)\le C^1\}} + C_2^1 1_{\{A^1+(\pi^{1,*}(t))^\top.B_\pi^1+B_c^1 c^{1,*}(t)> C^1\}} \\
&= C_1^1 1_{\{A^1+(\pi^{1,M}(t))^\top.B_\pi^1+(\pi^{1,H}(t))^\top.B_\pi^1+B_c^1 c^{1,M}(t)+B_c^1 c^{1,A}(t)\le C^1\}} \\
&\quad + C_2^1 1_{\{A^1+(\pi^{1,M}(t))^\top.B_\pi^1+(\pi^{1,H}(t))^\top.B_\pi^1+B_c^1 c^{1,M}(t)+B_c^1 c^{1,A}(t)> C^1\}} \\
&\ge C_1^1 1_{\{A^1+(\pi^{1,M}(t))^\top.B_\pi^1+B_c^1 c^{1,M}(t)\le C^1\}} + C_2^1 1_{\{A^1+(\pi^{1,M}(t))^\top.B_\pi^1+B_c^1 c^{1,M}(t)> C^1\}} \\
&= \vartheta^{1,1}(\pi^{1,M}, c^{1,M}(t))
\end{aligned}$$

as $(\pi^{i,H}(t))^\top.B_\pi^i \le 0$ and $B_c^i c^{i,A}(t) \le 0$. □

Remark 4.79. *As in the last section the optimal strategy is a* compromise strategy *composed of the Merton strategy and the critical strategy and thus generating better regime shift intensities than those the Merton strategy would yield.*

4.3 Affine Intensity Functions

Having considered constant and piecewise constant intensity functions the next step is to discuss affine intensity functions. However since intensities have to be non-negative we cannot use pure affine functions. To overcome this problem of negativity we look at a class of continuous functions consisting of an affine and a constant part. For notational convenience we call those functions 'affine'. Thus in this subsection we let the intensity functions $\vartheta^{i,1-i}$ be given as functions of the form

$$\vartheta^{i,1-i}(\pi, c) = \max\{A^i + \pi^\top.B_\pi^i + B_c^i c, C^i\}, \ (\pi, c) \in \mathbb{R}^{\bar{n}} \times \mathbb{R}_0^+ \tag{AI}$$

with $A^i \in \mathbb{R}$, $B_\pi^i \in \mathbb{R}^{\bar{n}}$, $B_c^0 \in \mathbb{R}$, $B_c^1 \le 0$ and $C^i \ge 0$ for $i = 0, 1$. Positive values for B_c^1 are not allowed since this could cause an infinite consumption rate to be optimal in state 1 as will be seen later on.

Via step intensity functions the investor has only an indirect influence on the market whereas with those floored affine intensity functions the investor can directly determine the level of the regime shift intensities. The only restriction is that the intensities are not allowed to be smaller than a predefined, non-manipulable constant. As in the case of the step intensity functions we first give some explanations and possible interpretations of the parameters characterizing the 'affine' intensity functions.

Remark 4.80 (Interpretation of the intensity parameters). *The affine intensity functions can take on every value that is larger than or equal C^i. Thus, this minimal intensity can be regarded as being advantageous if $i = 0$, resp. disadvantageous if $i = 1$, for the large investor. Further, since $\vartheta^{i,1-i}$ is unbounded from above the possible extent of the large investor's influence is infinite. Again, B^i_π and B^i_c determine the strength of the influence, resp. the sensitivity of the market – the bigger $|B^i_{\pi,n}|$, resp. $|B^i_c|$, the more sensitive the market. Finally, having specified the minimal intensity and the B^i's the critical barrier separating the strategies that generate intensities bigger than C^i from those that cause the minimal intensity is fixed via the parameter A^i.*

As in the step intensity section the signs of $B^i_{\pi,n}$, resp. B^i_c, determine the way in which the market reacts on the large investor's presence. Thus, a positive $B^0_{\pi,n}$, resp. a negative $B^1_{\pi,n}$, corresponds to a market in which large positions in the n-th asset yield large ($i = 0$), resp. small ($i = 1$), transition intensities. So the other market participants disapprove the large investor's holdings in asset n in that his position could cause the market to turn into the adverse state (if $i = 0$) or hinder an early jump back to the favorable state (if $i = 1$).

In case of $B^0_{\pi,n}$ being negative, resp. $B^1_{\pi,n}$ being positive, large proportions in the n-th asset cause small ($i = 0$), resp. large ($i = 1$), transition intensities. Thus the large investor is accepted by the market, resp. the other market participants may think of him as having superior information about the n-th stock, such as a manager of a prosperous fund, or an executive of the company issuing the stock or even a person having insider information.

Having discussed the role of B^i_π we now go on describing the consumption parameter B^i_c. If $B^0_c > 0$, resp. $B^1_c < 0$, then consuming at a high rate implies large ($i = 0$), resp. small ($i = 1$), regime shift intensities, i.e. the other market participants may interpret the large investor's high consumption rate as a bad signal for the future development, e.g. as a herald of a market crisis. Another example in which a positive value of B^0_c, resp. a negative value of B^1_c, is reasonable is the large investor being the manager of a large mutual fund. In this context consumption can be interpreted as a reduction of the number of assets under management. A possible reason for such a reduction could be the absence of lucrative investment opportunities.

Finally, a negative B^0_c implies that consuming at a high rate yields small regime shift intensities. Thus the market somehow rewards the large investor's if he consumes at a high rate. This could be the case if the investor's presence in the market is disapproved by the other market participants. A specialty arises if $\varepsilon = 0$, i.e. if the investor does not draw any utility from intermediate consumption in terms of the utility function. In this case a negative B^0_c may force the investor to consume just to achieve favorable transition intensities without generating any direct utility gain. In this context consumption could be interpreted as bribe and we will see later on that there exist parameter specifications under which the investor pays a bribe.

Under the affine intensity functions the HJB-system (4.8) reads

$$0 = \sup_{(\pi,c)\in\mathbb{R}^{\bar{n}}\times\mathbb{R}_0^+} \Big\{ g'(t) - 1_{\{i=1\}}h'(t) + r^i + \pi^\top.\eta^i - \tfrac{1}{2}R\pi^\top.\sigma^i.(\sigma^i)^\top.\pi \\ + \tfrac{\varepsilon\delta}{\varepsilon-(\varepsilon-\delta)e^{-\delta(T-t)}}\tfrac{1}{1-R}(e^{-(1-R)(g(t)-1_{\{i=1\}}h(t))}c^{1-R} - 1) - c \\ + \max\{A^i + \pi^\top.B^i_\pi + B^i_c c, C^i\}\tfrac{1}{1-R}(e^{(-1)^{1-i}(1-R)h(t)} - 1)\Big\} \tag{4.41}$$

for $t \in [0,T)$ and $i = 0,1$, subject to the boundary conditions

$$g(T) = 0, \;\; h(T) = 0.$$

Subsequently, we present the solution of the investment problem (P) for three different variants of the affine intensity functions. First, we discuss intensities that are influenced solely by the portfolio proportions π, the so called *portfolio-dependent intensities* (Subsection 4.3.1). Second, instead of the dependency on π we look at *consumption-dependent intensities* being functions just of the consumption rate c (Subsection 4.3.2). And finally the most general version of *portfolio- and consumption-dependent intensities* is regarded (Subsection 4.3.3).

Again the simplest variant – the *portfolio- and consumption-independent intensities* – will not be discussed as it is a special form of the constant intensity functions where choosing $B^i_\pi = 0$ and $B^i_c = 0$ yields $\vartheta^{i,1-i}(\pi,c) = \max\{A^i, C^i\}$ for $i = 0,1$.

In each subsection we present the optimal strategies in closed form, i.e. we give explicit formulas for the optimal portfolio proportions and consumption rates that only depend on the solution of an ODE-system of which we show that it admits a unique global solution. Further we provide bounds on the solution of this ODE-system.

4.3.1 Portfolio-dependent Intensities

We now analyze the optimal investment problem when the intensities of regime shifts only depend on the investor's portfolio proportions π, i.e. we let

$$B^i_\pi \neq 0 \text{ and } B^i_c = 0 \text{ for } i = 0,1, \tag{PD}$$

so that the intensities are given by

$$\vartheta^{i,1-i}(\pi,c) = \max\{A^i + \pi^\top.B^i_\pi, C^i\}.$$

We denote by

$$\mathcal{I}^{\pi,0} \triangleq \Big\{\pi \in \mathbb{R}^{\bar{n}} \;:\; A^0 + \pi^\top.B^0_\pi > C^0\Big\},\; \mathcal{I}^{\pi,1} \triangleq \Big\{\pi \in \mathbb{R}^{\bar{n}} \;:\; A^1 + \pi^\top.B^1_\pi \geq C^1\Big\}$$

the half spaces of strategies that impact on the intensities of regime shifts. Thus

$$d^{\pi,i}(\pi) \triangleq A^i + \pi^\top.B^i_\pi - C^i$$

denotes the 'distance' of the strategy π to the separating hyperplane and we have $\pi \in \mathcal{I}^{\pi,0}$ if and only if $d^{\pi,0}(\pi) > 0$, resp. $\pi \in \mathcal{I}^{\pi,1}$ if and only if $d^{\pi,1}(\pi) \geq 0$.

At first sight it seems inconsistent that $\mathcal{I}^{\pi,1}$ includes the separating hyper plane whereas $\mathcal{I}^{\pi,0}$ does not. But this is actually in accordance with the setting in the section on step intensity functions. There the half space of favorable strategies included the separating hyper plane which turned out to be necessary for the existence of the solution to the optimal investment problem. Now, the analogon to the favorable half space of the previous section is in state 0 given by the subspace of all strategies that yield the smallest possible regime shift intensities. This subspace is just the complement of $\mathcal{I}^{\pi,0}$. On the contrary, in state 1 the complement of $\mathcal{I}^{\pi,1}$ contains all the unfavorable strategies since here small regime shift intensities are disadvantageous. This justifies the above definition of $\mathcal{I}^{\pi,0}$ and $\mathcal{I}^{\pi,1}$.

In order to determine the maximizer in the HJB-system (4.41) we define functions $H^{\pi,i} : \mathbb{R}_0^+ \times \mathbb{R}^{\bar{n}} \to \mathbb{R}$ and $H^{c,i} : [0,T] \times \mathbb{R} \times \mathbb{R}_0^+ \to \mathbb{R}$, $i = 0, 1$, given by

$$\begin{aligned} H^{\pi,i}(y,\pi) &\triangleq r^i + \pi^\top.\eta^i - \tfrac{1}{2} R\pi^\top.\sigma^i.(\sigma^i)^\top.\pi + \max\{A^i + \pi^\top.B_\pi^i, C^i\}\tfrac{1}{1-R}(e^{(-1)^{1-i}(1-R)y} - 1), \\ H^{c,i}(t,x,c) &\triangleq \tfrac{\varepsilon\delta}{\varepsilon-(\varepsilon-\delta)e^{-\delta(T-t)}}\tfrac{1}{1-R}(e^{-(1-R)x}c^{1-R} - 1) - c \end{aligned}$$

where we use the already mentioned convention $0^{1-R} \triangleq \infty$ if $R > 1$. Hence the HJB-system (4.41) reads

$$0 = \sup_{(\pi,c)\in\mathbb{R}^{\bar{n}}\times\mathbb{R}_0^+} \Big\{ g'(t) - 1_{\{i=1\}}h'(t) + H^{\pi,i}(h(t),\pi) + H^{c,i}(t, g(t) - 1_{\{i=1\}}h(t), c) \Big\} \tag{4.42}$$

for $t \in [0,T)$ and $i = 0, 1$, subject to the boundary conditions

$$g(T) = 0, \; h(T) = 0.$$

Writing the HJB-system in that way it is obvious that the maximization over $(\pi, c) \in \mathbb{R}^{\bar{n}} \times \mathbb{R}_0^+$ can be separated into two unrelated maximizations; one over $\pi \in \mathbb{R}^{\bar{n}}$ and one over $c \in \mathbb{R}_0^+$.

To find the supremum in (4.42) we present the maximizers of the functions $H^{\pi,i}(y,\cdot)$ and $H^{c,i}(t,x,\cdot)$ for arbitrary $(t,x,y) \in [0,T] \times \mathbb{R} \times \mathbb{R}_0^+$. This yields a family of maximizers dependent on (t,x,y). The maximizers of the HJB-system (4.42) are then obtained by replacing x and y by $g(t) - 1_{\{i=1\}}h(t)$ and $h(t)$.

Concerning the consumption rate the concavity of $H^{c,i}(t,x,\cdot)$ and the first-order condition imply

Lemma 4.81 (Maximizer of $H^{c,i}(t,x,\cdot)$)**.** *For every $(t,x) \in [0,T] \times \mathbb{R}$ the maximizer*

$$c^{i,*}(t,x) \triangleq \arg\max_{c\in\mathbb{R}_0^+} H^{c,i}(t,x,c), \; i = 0, 1,$$

is given by the Merton consumption rate, i.e.

$$c^{i,*}(t,x) = c^M(t,x).$$

Remark 4.82. *In case of $\varepsilon = 0$ the Merton consumption rate vanishes, i.e. $c^M(t,x) = 0$.*

In order to find the maximizing portfolio proportions we let $H_l^{\pi,i}$, $H_a^{\pi,i} : \mathbb{R}_0^+ \times \mathbb{R}^{\bar{n}} \to \mathbb{R}$, $i = 0, 1$ be given by

$$H_l^{\pi,i}(y,\pi) \triangleq r^i + \pi^\top.\eta^i - \tfrac{1}{2}R\pi^\top.\sigma^i.(\sigma^i)^\top.\pi + C^i \tfrac{1}{1-R}(e^{(-1)^{1-i}(1-R)y} - 1),$$
$$H_a^{\pi,i}(y,\pi) \triangleq r^i + \pi^\top.\eta^i - \tfrac{1}{2}R\pi^\top.\sigma^i.(\sigma^i)^\top.\pi + (A^i + \pi^\top.B_\pi^i)\tfrac{1}{1-R}(e^{(-1)^{1-i}(1-R)y} - 1)$$

such that $H^{\pi,i}(y,\pi) = H_l^{\pi,i}(y,\pi)1_{\{\pi\notin\mathfrak{I}^{\pi,i}\}} + H_a^{\pi,i}(y,\pi)1_{\{\pi\in\mathfrak{I}^{\pi,i}\}}$.

Since $H_l^{\pi,i}(y,\cdot)$ and $H_a^{\pi,i}(y,\cdot)$ are concave paraboloids for every $y \in \mathbb{R}_0^+$, the three candidate solutions for the maximization of $H^{\pi,i}(y,\cdot)$ over π are

$$\begin{aligned}
\pi^{i,M} &= \operatorname*{arg\,max}_{\pi\in\mathbb{R}^{\bar{n}}} H_l^{\pi,i}(y,\pi),\\
\tilde{\pi}^{i,*}(y) &\triangleq \operatorname*{arg\,max}_{\pi\in\mathbb{R}^{\bar{n}}} H_a^{\pi,i}(y,\pi),\\
\tilde{\pi}^{i,\mathrm{crit}} &\triangleq \operatorname*{arg\,max}_{\{\pi\in\mathbb{R}^{\bar{n}}\,:\,A^i+\pi^\top.B_\pi^i=C^i\}} H^{\pi,i}(y,\pi)
\end{aligned}$$

where

$$\begin{aligned}
\tilde{\pi}^{i,*}(y) &= \tfrac{1}{R}(\sigma^i.(\sigma^i)^\top)^{-1}.\big(\eta^i + B_\pi^i\tfrac{1}{1-R}(e^{(-1)^{1-i}(1-R)y} - 1)\big),\\
\tilde{\pi}^{i,\mathrm{crit}} &= \tfrac{1}{R}(\sigma^i.(\sigma^i)^\top)^{-1}.\big(\eta^i - B_\pi^i\tfrac{A^i+(\pi^{i,M})^\top.B_\pi^i-C^i}{\frac{1}{R}(B_\pi^i)^\top.(\sigma^i.(\sigma^i)^\top)^{-1}.B_\pi^i}\big)
\end{aligned}$$

for $y \in \mathbb{R}_0^+$, $i = 0, 1$.

Remark 4.83. *We use the following shorthand notations*

$$H_{l,M}^{\pi,i}(y) \triangleq H_l^{\pi,i}(y,\pi^{i,M}) \text{ and } H_{a,\sim}^{\pi,i}(y) \triangleq H_a^{\pi,i}(y,\tilde{\pi}^{i,*}(y))$$

for $y \in \mathbb{R}_0^+$ *and* $i = 0, 1$.

Before we present the maximizer of $H^{\pi,i}(y,\cdot)$ we first provide some useful results on the relation between $\pi^{i,M}$ and $\tilde{\pi}^{i,*}$ and the related function values $H_{l,M}^{\pi,i}$ and $H_{a,\sim}^{\pi,i}$.

Lemma 4.84. *For every* $y \in \mathbb{R}_0^+$ *the Merton strategy* $\pi^{i,M}$*, the candidate solution* $\tilde{\pi}^{i,*}$ *and their function values* $H_{l,M}^{\pi,i}$ *and* $H_{a,\sim}^{\pi,i}$ *are related as follows.*

i) $\pi^{0,M} \notin \mathfrak{I}^{\pi,0} \quad\Rightarrow\quad H_{l,M}^{\pi,0}(y) \le H_{a,\sim}^{\pi,0}(y) \quad\Rightarrow\quad \tilde{\pi}^{0,*}(y) \notin \mathfrak{I}^{\pi,0}$.

ii) $\pi^{1,M} \in \mathfrak{I}^{\pi,1} \quad\Rightarrow\quad H_{l,M}^{\pi,1}(y) \le H_{a,\sim}^{\pi,1}(y) \quad\Rightarrow\quad \tilde{\pi}^{1,*}(y) \in \mathfrak{I}^{\pi,1}$.

Proof. ad i) In order to prove the assertion we note that the following three equivalences hold true. Firstly, $\pi^{0,M} \notin \mathfrak{I}^{\pi,0}$ is by definition equivalent to

$$d^{\pi,0}(\pi^{0,M}) \le 0. \tag{$*$}$$

Secondly, some transformations yield that $H_{l,M}^{\pi,0}(y) \le H_{a,\sim}^{\pi,0}(y)$ is equivalent to

$$d^{\pi,0}(\pi^{0,M}) \le -\tfrac{1}{2}\tfrac{1}{R}(B_\pi^0)^\top.(\sigma^0.(\sigma^0)^\top)^{-1}.B_\pi^0\tfrac{1}{1-R}(e^{-(1-R)y} - 1). \tag{$**$}$$

And thirdly, $\tilde{\pi}^{0,*}(y) \notin \mathcal{I}^{\pi,0}$ is equivalent to

$$d^{\pi,0}(\pi^{0,M}) \leq -\tfrac{1}{R}(B^0_\pi)^\top.(\sigma^0.(\sigma^0)^\top)^{-1}.B^0_\pi \tfrac{1}{1-R}(e^{-(1-R)y} - 1). \qquad (***)$$

As $y \geq 0$ the right-hand side of $(**)$ is non-negative. Hence $(**)$ follows directly from $(*)$ which yields the first implication. The second implication follows from the fact that the right-hand side of $(**)$ is smaller than the right-hand side of $(***)$.

ad ii) The proof of assertion ii) works completely analogously since firstly, $\pi^{1,M} \in \mathcal{I}^{\pi,1}$ is by definition equivalent to

$$d^{\pi,1}(\pi^{1,M}) \geq 0,$$

secondly, $H^{\pi,1}_{l,M}(y) \leq H^{\pi,1}_{a,\sim}(y)$ is equivalent to

$$d^{\pi,1}(\pi^{1,M}) \geq -\tfrac{1}{2}\tfrac{1}{R}(B^1_\pi)^\top.(\sigma^1.(\sigma^1)^\top)^{-1}.B^1_\pi \tfrac{1}{1-R}(e^{(1-R)y} - 1),$$

and thirdly, $\tilde{\pi}^{1,*}(y) \in \mathcal{I}^{\pi,1}$ is equivalent to

$$d^{\pi,1}(\pi^{1,M}) \geq -\tfrac{1}{R}(B^1_\pi)^\top.(\sigma^1.(\sigma^1)^\top)^{-1}.B^1_\pi \tfrac{1}{1-R}(e^{(1-R)y} - 1).$$

□

Those technical results turn out to be quite helpful in the proof of the following lemma on the maximizer of $H^{\pi,i}(y,\cdot)$.

Lemma 4.85 (Maximizer of $H^{\pi,i}(y,\cdot)$)**.** *For every $y \in \mathbb{R}^+_0$ the maximizer*

$$\pi^{i,*}(y) \triangleq \arg\max_{\pi \in \mathbb{R}^{\bar{n}}} H^{\pi,i}(y,\pi), \;\; i = 0,1,$$

is given by

$$\pi^{0,*}(y) = \tfrac{1}{R}(\sigma^0.(\sigma^0)^\top)^{-1}.\big(\eta^0 + B^0_\pi \tfrac{1}{1-R}(e^{-(1-R)(y \wedge h^{0,\mathrm{crit}})} - 1)\big),$$

$$\pi^{1,*}(y) = \begin{cases} \pi^{1,M} & \text{if } y < h^{1,\mathrm{crit}}, \\ \tilde{\pi}^{1,*}(y) & \text{if } y \geq h^{1,\mathrm{crit}}, \end{cases}$$

where

$$h^{0,\mathrm{crit}} \triangleq -\tfrac{1}{1-R}\ln\Big(-(1-R)\tfrac{(A^0+(\pi^{0,M})^\top.B^0_\pi - C^0)^+}{\frac{1}{R}(B^0_\pi)^\top.(\sigma^0.(\sigma^0)^\top)^{-1}.B^0_\pi} + 1\Big),$$

$$h^{1,\mathrm{crit}} \triangleq \tfrac{1}{1-R}\ln\Big((1-R)2\tfrac{(A^1+(\pi^{1,M})^\top.B^1_\pi - C^1)^-}{\frac{1}{R}(B^1_\pi)^\top.(\sigma^1.(\sigma^1)^\top)^{-1}.B^1_\pi} + 1\Big)$$

and

$$\begin{aligned}\pi^{0,\mathrm{crit}} &\triangleq \tfrac{1}{R}(\sigma^0.(\sigma^0)^\top)^{-1}.\Big(\eta^0 + B^0_\pi \tfrac{1}{1-R}(e^{-(1-R)h^{0,\mathrm{crit}}} - 1)\Big) \\ &= \tfrac{1}{R}(\sigma^0.(\sigma^0)^\top)^{-1}.\Big(\eta^0 - B^0_\pi \tfrac{(A^0+(\pi^{0,M})^\top.B^0_\pi - C^0)^+}{\frac{1}{R}(B^0_\pi)^\top.(\sigma^0.(\sigma^0)^\top)^{-1}.B^0_\pi}\Big).\end{aligned}$$

The Figures 4.11 and 4.12 show the three, resp. four typical shapes of the functions $H^{\pi,i}_l(y,\cdot)$, $H^{\pi,i}_a(y,\cdot)$ and $H^{\pi,i}(y,\cdot)$ that correspond to the different cases occurring in the following proof of Lemma 4.85.

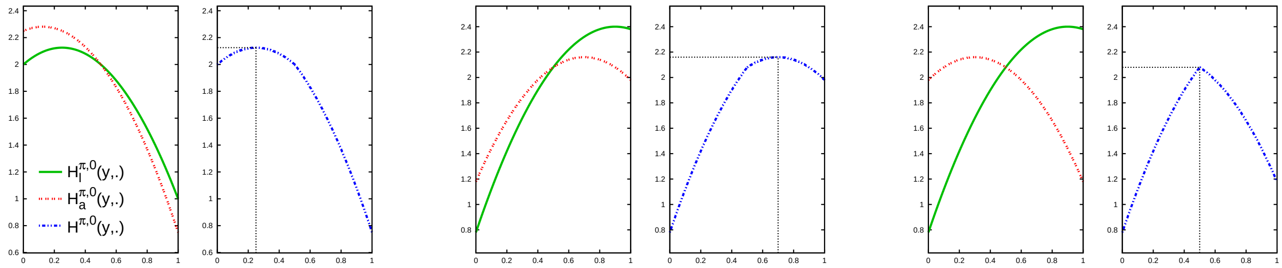

Figure 4.11: Typical shapes of $H_l^{\pi,0}(y,\cdot)$, $H_a^{\pi,0}(y,\cdot)$ and $H^{\pi,0}(y,\cdot)$ where $\mathcal{I}^{\pi,0} = (0.5, \infty)$

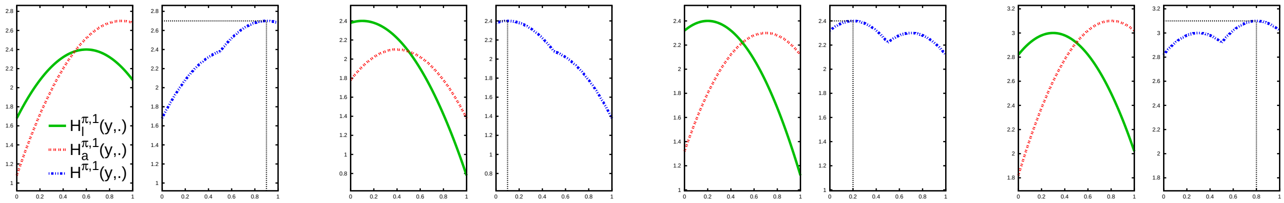

Figure 4.12: Typical shapes of $H_l^{\pi,1}(y,\cdot)$, $H_a^{\pi,1}(y,\cdot)$ and $H^{\pi,1}(y,\cdot)$ where $\mathcal{I}^{\pi,1} = [0.5, \infty)$

Proof of Lemma 4.85. The proof is structured as follows. First we deduce the maximizing strategy in state 0 and then continue with state 1. Let $y \in \mathbb{R}_0^+$ be given.

- *Maximizing strategy in state* 0. We first consider the case $\pi^{0,M} \notin \mathcal{I}^{\pi,0}$ and then $\pi^{0,M} \in \mathcal{I}^{\pi,0}$.

  - $\pi^{0,M} \notin \mathcal{I}^{\pi,0}$. If the Merton strategy $\pi^{0,M}$ is not part of the influencing half space $\mathcal{I}^{\pi,0}$ then Lemma 4.84, i), implies that $\tilde{\pi}^{0,*}(y) \notin \mathcal{I}^{\pi,0}$, either. Consequently $H^{\pi,0}(y,\cdot)$ exhibits only one maximum which is attained at the Merton strategy (cf. leftmost couple of plots in Figure 4.11), i.e.

$$\pi^{0,M} \notin \mathcal{I}^{\pi,0} \quad \Rightarrow \quad \pi^{0,*}(y) = \pi^{0,M}.$$

  - $\pi^{0,M} \in \mathcal{I}^{\pi,0}$. If the Merton strategy $\pi^{0,M}$ lies within $\mathcal{I}^{\pi,0}$ then $H^{\pi,0}(y,\cdot)$ has again just one maximum that is either the maximum of $H_a^{\pi,0}(y,\cdot)$ (cf. central couple of plots in Figure 4.11) or it is attained at the critical strategy $\pi^{0,\text{crit}}$ (cf. rightmost couple of plots in Figure 4.11). Notice that the maximizer of $H_a^{\pi,0}(y,\cdot)$ is given by $\tilde{\pi}^{0,*}(y)$. As long as $\tilde{\pi}^{0,*}(y)$ is lying in $\mathcal{I}^{\pi,0}$ it is even the maximizer of $H^{\pi,0}(y,\cdot)$. But as soon as $\tilde{\pi}^{0,*}(y)$ quits $\mathcal{I}^{\pi,0}$ the maximizer of $H^{\pi,0}(y,\cdot)$ is given by the critical strategy, i.e.

$$\pi^{0,M} \in \mathcal{I}^{\pi,0} \quad \Rightarrow \quad \pi^{0,*}(y) = \begin{cases} \tilde{\pi}^{0,*}(y) & \text{if } \tilde{\pi}^{0,*}(y) \in \mathcal{I}^{\pi,0}, \\ \pi^{0,\text{crit}} & \text{if } \tilde{\pi}^{0,*}(y) \notin \mathcal{I}^{\pi,0}. \end{cases}$$

  Combining the two cases we find

$$\pi^{0,*}(y) = \begin{cases} \pi^{0,M} & \text{if } \pi^{0,M} \notin \mathcal{I}^{\pi,0}, \\ \tilde{\pi}^{0,*}(y) & \text{if } \pi^{0,M} \in \mathcal{I}^{\pi,0} \text{ and } \tilde{\pi}^{0,*}(y) \in \mathcal{I}^{\pi,0}, \\ \pi^{0,\text{crit}} & \text{if } \pi^{0,M} \in \mathcal{I}^{\pi,0} \text{ and } \tilde{\pi}^{0,*}(y) \notin \mathcal{I}^{\pi,0}. \end{cases}$$

Hence, we need to analyze the condition $\tilde{\pi}^{0,*}(y) \in \mathfrak{I}^{\pi,0}$ in more detail.

$$\begin{aligned}
\tilde{\pi}^{0,*}(y) \in \mathfrak{I}^{\pi,0} \quad &\Leftrightarrow \quad d^{\pi,0}(\pi^{0,M}) > -\tfrac{1}{R}(B^0_\pi)^\top.(\sigma^0.(\sigma^0)^\top)^{-1}.B^0_\pi \tfrac{1}{1-R}(e^{-(1-R)y}-1) \\
&\Leftrightarrow \quad -\tfrac{1}{1-R}(e^{-(1-R)y}-1) < \frac{d^{\pi,0}(\pi^{0,M})}{\frac{1}{R}(B^0_\pi)^\top.(\sigma^0.(\sigma^0)^\top)^{-1}.B^0_\pi} \\
&\overset{(*)}{\Leftrightarrow} \quad -\tfrac{1}{1-R}(e^{-(1-R)y}-1) < \frac{(d^{\pi,0}(\pi^{0,M}))^+}{\frac{1}{R}(B^0_\pi)^\top.(\sigma^0.(\sigma^0)^\top)^{-1}.B^0_\pi} \\
&\Leftrightarrow \quad y < -\tfrac{1}{1-R}\ln\Big(-(1-R)\frac{(d^{\pi,0}(\pi^{0,M}))^+}{\frac{1}{R}(B^0_\pi)^\top.(\sigma^0.(\sigma^0)^\top)^{-1}.B^0_\pi} + 1\Big) \\
&\Leftrightarrow \quad y < h^{0,\mathrm{crit}}
\end{aligned}$$

where equivalence $(*)$ holds true since $-\frac{1}{1-R}(e^{-(1-R)y}-1)$ is non-negative as $y \geq 0$. Notice that by construction

$$\pi^{0,M} \notin \mathfrak{I}^{\pi,0} \quad \Leftrightarrow \quad h^{0,\mathrm{crit}} = 0.$$

Thus the maximizing strategy in state 0 is given by

$$\pi^{0,*}(y) = \begin{cases} \pi^{0,M} & \text{if } \pi^{0,M} \notin \mathfrak{I}^{\pi,0}, \\ \tilde{\pi}^{0,*}(y) & \text{if } \pi^{0,M} \in \mathfrak{I}^{\pi,0} \text{ and } y < h^{0,\mathrm{crit}}, \\ \pi^{0,\mathrm{crit}} & \text{if } \pi^{0,M} \in \mathfrak{I}^{\pi,0} \text{ and } y \geq h^{0,\mathrm{crit}} \end{cases}$$

which can equivalently be written as

$$\pi^{0,*}(y) = \tfrac{1}{R}(\sigma^0.(\sigma^0)^\top)^{-1}.\Big(\eta^0 + B^0_\pi \tfrac{1}{1-R}(e^{-(1-R)(y\wedge h^{0,\mathrm{crit}})}-1)\Big).$$

- *Maximizing strategy in state* 1. In state 1 we first consider the case $\pi^{1,M} \in \mathfrak{I}^{\pi,1}$ and then $\pi^{1,M} \notin \mathfrak{I}^{\pi,1}$.

  - $\pi^{1,M} \in \mathfrak{I}^{\pi,1}$. If the Merton strategy $\pi^{1,M}$ is lying in $\mathfrak{I}^{\pi,1}$ then by Lemma 4.84, ii), also $\tilde{\pi}^{1,*}(y) \in \mathfrak{I}^{\pi,1}$. Hence $H^{\pi,1}(y,\cdot)$ has a unique maximum at $\tilde{\pi}^{1,*}(y)$ (cf. first couple of plots in Figure 4.12), i.e.

  $$\pi^{1,M} \in \mathfrak{I}^{\pi,1} \quad \Rightarrow \quad \pi^{1,*}(y) = \tilde{\pi}^{1,*}(y).$$

  - $\pi^{1,M} \notin \mathfrak{I}^{\pi,1}$. However, if $\pi^{1,M} \notin \mathfrak{I}^{\pi,1}$ then $H^{\pi,1}(y,\cdot)$ may possess two local maxima, namely the maxima of $H^{\pi,1}_l(y,\cdot)$ and $H^{\pi,1}_a(y,\cdot)$ which are attained at $\pi^{1,M}$, resp. $\tilde{\pi}^{1,*}(y)$. As long as $\tilde{\pi}^{1,*}(y)$ does not lie within $\mathfrak{I}^{\pi,1}$, $H^{\pi,1}(y,\cdot)$ exhibits only one maximum that is achieved at $\pi^{1,M}$ (cf. second couple of plots in Figure 4.12). As soon as $\tilde{\pi}^{1,*}(y)$ enters $\mathfrak{I}^{\pi,1}$, $H^{\pi,1}(y,\cdot)$ exhibits the two local maxima mentioned above that have to be compared in order to find the global maximum (cf. third and fourth couple of plots in Figure 4.12). Hence we have

$$\pi^{1,M} \notin \mathfrak{I}^{\pi,1} \quad \Rightarrow \quad \pi^{1,*}(y) = \begin{cases} \pi^{1,M} & \text{if } \tilde{\pi}^{1,*}(y) \notin \mathfrak{I}^{\pi,1} \text{ or } [\tilde{\pi}^{1,*}(y) \in \mathfrak{I}^{\pi,1} \text{ and } H^{\pi,1}_{l,M}(y) > H^{\pi,1}_{a,\sim}(y)], \\ \tilde{\pi}^{1,*}(y) & \text{if } \tilde{\pi}^{1,*}(y) \in \mathfrak{I}^{\pi,1} \text{ and } H^{\pi,1}_{l,M}(y) \leq H^{\pi,1}_{a,\sim}(y). \end{cases}$$

Combining the two cases we can write the maximizing strategy in the following way.

$$\pi^{1,*}(y) = \begin{cases} \pi^{1,M} & \text{if } \pi^{1,M} \notin \mathfrak{I}^{\pi,1} \text{ and } \big[\tilde{\pi}^{1,*}(y) \notin \mathfrak{I}^{\pi,1} \text{ or } [\tilde{\pi}^{1,*}(y) \in \mathfrak{I}^{\pi,1} \text{ and } H^{\pi,1}_{l,M}(y) > H^{\pi,1}_{a,\sim}(y)]\big], \\ \tilde{\pi}^{1,*}(y) & \text{if } \pi^{1,M} \in \mathfrak{I}^{\pi,1} \text{ or } \big[\pi^{1,M} \notin \mathfrak{I}^{\pi,1} \text{ and } [\tilde{\pi}^{1,*}(y) \in \mathfrak{I}^{\pi,1} \text{ and } H^{\pi,1}_{l,M}(y) \leq H^{\pi,1}_{a,\sim}(y)]\big]. \end{cases}$$

We now consider the two conditions in more detail. Firstly

$$
\begin{aligned}
& \pi^{1,M} \notin \mathcal{I}^{\pi,1} \text{ and } \left[\tilde{\pi}^{1,*}(y) \notin \mathcal{I}^{\pi,1} \text{ or } [\tilde{\pi}^{1,*}(y) \in \mathcal{I}^{\pi,1} \text{ and } H^{\pi,1}_{l,M}(y) > H^{\pi,1}_{a,\sim}(y)]\right] \\
\Leftrightarrow \quad & \pi^{1,M} \notin \mathcal{I}^{\pi,1} \text{ and } [\tilde{\pi}^{1,*}(y) \notin \mathcal{I}^{\pi,1} \text{ or } H^{\pi,1}_{l,M}(y) > H^{\pi,1}_{a,\sim}(y)] \\
\overset{L.\ 4.84,\ ii)}{\Leftrightarrow} \quad & \pi^{1,M} \notin \mathcal{I}^{\pi,1} \text{ and } H^{\pi,1}_{l,M}(y) > H^{\pi,1}_{a,\sim}(y) \\
\overset{L.\ 4.84,\ ii)}{\Leftrightarrow} \quad & H^{\pi,1}_{l,M}(y) > H^{\pi,1}_{a,\sim}(y)
\end{aligned}
$$

Secondly

$$
\begin{aligned}
& \pi^{1,M} \in \mathcal{I}^{\pi,1} \text{ or } \left[\pi^{1,M} \notin \mathcal{I}^{\pi,1} \text{ and } [\tilde{\pi}^{1,*}(y) \in \mathcal{I}^{\pi,1} \text{ and } H^{\pi,1}_{l,M}(y) \le H^{\pi,1}_{a,\sim}(y)]\right] \\
\Leftrightarrow \quad & \pi^{1,M} \in \mathcal{I}^{\pi,1} \text{ or } [\tilde{\pi}^{1,*}(y) \in \mathcal{I}^{\pi,1} \text{ and } H^{\pi,1}_{l,M}(y) \le H^{\pi,1}_{a,\sim}(y)] \\
\overset{L.\ 4.84,\ ii)}{\Leftrightarrow} \quad & \pi^{1,M} \in \mathcal{I}^{\pi,1} \text{ or } H^{\pi,1}_{l,M}(y) \le H^{\pi,1}_{a,\sim}(y) \\
\overset{L.\ 4.84,\ ii)}{\Leftrightarrow} \quad & H^{\pi,1}_{l,M}(y) \le H^{\pi,1}_{a,\sim}(y)
\end{aligned}
$$

Hence we get

$$
\pi^{1,*}(y) = \begin{cases} \pi^{1,M} & \text{if } H^{\pi,1}_{l,M}(y) > H^{\pi,1}_{a,\sim}(y), \\ \tilde{\pi}^{1,*}(y) & \text{if } H^{\pi,1}_{l,M}(y) \le H^{\pi,1}_{a,\sim}(y). \end{cases}
$$

Thus, we need to take a closer look on the condition $H^{\pi,1}_{l,M}(y) \le H^{\pi,1}_{a,\sim}(y)$. Some straight forward calculations show

$$
\begin{aligned}
H^{\pi,1}_{l,M}(y) \le H^{\pi,1}_{a,\sim}(y) \quad \Leftrightarrow \quad & d^{\pi,1}(\pi^{1,M}) \ge -\tfrac{1}{2}\tfrac{1}{R}(B^1_\pi)^\top.(\sigma^1.(\sigma^1)^\top)^{-1}.B^1_\pi \tfrac{1}{1-R}(e^{(1-R)y} - 1) \\
\Leftrightarrow \quad & \tfrac{1}{1-R}(e^{(1-R)y} - 1) \ge -2\frac{d^{\pi,1}(\pi^{1,M})}{\frac{1}{R}(B^1_\pi)^\top.(\sigma^1.(\sigma^1)^\top)^{-1}.B^1_\pi} \\
\overset{(*)}{\Leftrightarrow} \quad & \tfrac{1}{1-R}(e^{(1-R)y} - 1) \ge 2\frac{(d^{\pi,1}(\pi^{1,M}))^-}{\frac{1}{R}(B^1_\pi)^\top.(\sigma^1.(\sigma^1)^\top)^{-1}.B^1_\pi} \\
\Leftrightarrow \quad & y \ge \tfrac{1}{1-R}\ln\Big((1-R)2\frac{(d^{\pi,1}(\pi^{1,M}))^-}{\frac{1}{R}(B^1_\pi)^\top.(\sigma^1.(\sigma^1)^\top)^{-1}.B^1_\pi} + 1\Big) \\
\Leftrightarrow \quad & y \ge h^{1,\text{crit}}
\end{aligned}
$$

where equivalence $(*)$ holds true since $\frac{1}{1-R}(e^{(1-R)y} - 1)$ is non-negative as $y \ge 0$. Notice that by construction

$$
\pi^{1,M} \in \mathcal{I}^{\pi,1} \quad \Leftrightarrow \quad h^{1,\text{crit}} = 0.
$$

Therefore the maximizing strategy in state 1 is given by

$$
\pi^{1,*}(y) = \begin{cases} \pi^{1,M} & \text{if } y < h^{1,\text{crit}}, \\ \tilde{\pi}^{1,*}(y) & \text{if } y \ge h^{1,\text{crit}}. \end{cases}
$$

Thus the lemma is proven. □

Remark 4.86 (Interpretation)**.** *Whenever the Merton strategy in state* 0 *is not part of the influencing half space then the maximizing strategy coincides with the Merton strategy. However, if the Merton strategy lies within the influencing half space then the maximizing strategy deviates from the Merton strategy.*

In state 1 *things are different. Here the maximizing strategy coincides with the Merton strategy if the Merton strategy is not lying within the influencing half space and* $y < h^{1,\text{crit}}$*, i.e. the utility criterion dominates the intensity criterion as discussed in Remark 4.33 as long as* $y < h^{1,\text{crit}}$*. Once* $y \geq h^{1,\text{crit}}$ *the situation changes and the intensity criterion is more important than the utility criterion, i.e. the maximizing strategy deviates from the Merton strategy.*

Remark 4.87 (Continuity of $\pi^{0,*}$ vs. discontinuity of $\pi^{1,*}$)**.** *The special form of the maximizing strategy in state* 0 *implies that* $\pi^{0,*}$ *is a continuous function in* y*.*

However in state 1 *the maximizing strategy may exhibit a discontinuity at* $y = h^{1,\text{crit}}$ *where* $\pi^{1,*}$ *jumps from* $\pi^{1,M}$ *to* $\tilde{\pi}^{1,*}(h^{1,\text{crit}})$*. Only in case of* $h^{1,\text{crit}} = 0$ *this discontinuity vanishes and* $\pi^{1,*}$ *is continuous.*

Remark 4.88. *So far the strategy* $(\pi^{i,*}, c^{i,*})$ *is only a candidate solution for the optimal investment problem. To verify the optimality we need to show that the related HJB-system has a global solution that satisfies the required regularity conditions stated in the Verification Theorem 3.3.*

Having established the maximizers of the HJB-system in general form inserting $(\pi^{i,*}(h(t), c^{i,*}(t, g(t) - 1_{\{i=1\}}h(t)))$ the reduced HJB-equation (4.42) now becomes a system of coupled backward ODEs

$$
\begin{aligned}
h'(t) = &-(\Psi^0 - \Psi^1) + \big(\tfrac{\varepsilon\delta}{\varepsilon-(\varepsilon-\delta)e^{-\delta(T-t)}}\big)^{\frac{1}{R}} e^{-\frac{1-R}{R}g(t)} \tfrac{R}{1-R}(e^{\frac{1-R}{R}h(t)} - 1) \\
&- C^0 \tfrac{1}{1-R}(e^{-(1-R)h(t)} - 1) + C^1 \tfrac{1}{1-R}(e^{(1-R)h(t)} - 1) \\
&- (A^0 + (\pi^{0,M})^\top . B^0_\pi - C^0)\tfrac{1}{1-R}(e^{-(1-R)(h(t)\wedge h^{0,\text{crit}})} - 1) \\
&- \tfrac{1}{2}\tfrac{1}{R}(B^0_\pi)^\top . (\sigma^0 . (\sigma^0)^\top)^{-1} . B^0_\pi \tfrac{1}{(1-R)^2}(e^{-(1-R)(h(t)\wedge h^{0,\text{crit}})} - 1)^2 \\
&+ (A^1 + (\pi^{1,M})^\top . B^1_\pi - C^1)\tfrac{1}{1-R}(e^{(1-R)(h(t)\vee h^{1,\text{crit}})} - 1) \\
&+ \tfrac{1}{2}\tfrac{1}{R}(B^1_\pi)^\top . (\sigma^1 . (\sigma^1)^\top)^{-1} . B^1_\pi \tfrac{1}{(1-R)^2}(e^{(1-R)(h(t)\vee h^{1,\text{crit}})} - 1)^2,
\end{aligned}
\tag{4.43}
$$

$$
\begin{aligned}
g'(t) = &-\Psi^0 + \tfrac{\varepsilon\delta}{\varepsilon-(\varepsilon-\delta)e^{-\delta(T-t)}}\Big(1 - \tfrac{R}{1-R}\big(\big(\tfrac{\varepsilon\delta}{\varepsilon-(\varepsilon-\delta)e^{-\delta(T-t)}}\big)^{\frac{1-R}{R}} e^{-\frac{1-R}{R}g(t)} - 1\big)\Big) \\
&- C^0 \tfrac{1}{1-R}(e^{-(1-R)h(t)} - 1) \\
&- (A^0 + (\pi^{0,M})^\top . B^0_\pi - C^0)\tfrac{1}{1-R}(e^{-(1-R)(h(t)\wedge h^{0,\text{crit}})} - 1) \\
&- \tfrac{1}{2}\tfrac{1}{R}(B^0_\pi)^\top . (\sigma^0 . (\sigma^0)^\top)^{-1} . B^0_\pi \tfrac{1}{(1-R)^2}(e^{-(1-R)(h(t)\wedge h^{0,\text{crit}})} - 1)^2
\end{aligned}
\tag{4.44}
$$

subject to the boundary conditions

$$
g(T) = 0, \;\; h(T) = 0. \tag{4.45}
$$

Remark 4.89. *From the proof of Lemma 4.85 we know that*

$$
y \geq h^{1,\text{crit}} \Leftrightarrow d^{\pi,1}(\pi^{1,M})\tfrac{1}{1-R}(e^{(1-R)y} - 1) + \tfrac{1}{2}\tfrac{1}{R}(B^1_\pi)^\top . (\sigma^1 . (\sigma^1)^\top)^{-1} . B^1_\pi \tfrac{1}{(1-R)^2}(e^{(1-R)y} - 1)^2 \geq 0.
$$

Remark 4.90. *The aforementioned ODEs include the ODEs of the classical Merton problem (first rows) and the ODEs of the Bäuerle-Rieder problem with constant regime shift intensities C^i (first two rows).*

The following lemma shows that the above ODE-system exhibits a unique global solution.

Lemma 4.91. *The ODE-system given by equation (4.43) and (4.44) subject to the boundary conditions (4.45) admits a unique global solution.*

Proof. The proof is essentially the same as the proof of Lemma 4.9. Only the χ^i's are different. Therefore we just present the actual χ^i's and verify that those satisfy the necessary continuity conditions and the non-negativity condition.

- *Definition of the χ^i's.* In the case of portfolio-dependent intensities the χ^i's are given by $\chi^i:\ \mathbb{R}_0^+ \to \mathbb{R}$, $i = 0, 1$, with

$$\begin{aligned}\chi^0(y) \triangleq & - C^0 \tfrac{1}{1-R}(e^{-(1-R)y} - 1) - d^{\pi,0}(\pi^{0,M}) \tfrac{1}{1-R}(e^{-(1-R)(y\wedge h^{0,\mathrm{crit}})} - 1) \\ & - \tfrac{1}{2}\tfrac{1}{R}(B^0_\pi)^\top.(\sigma^0.(\sigma^0)^\top)^{-1}.B^0_\pi \tfrac{1}{(1-R)^2}(e^{-(1-R)(y\wedge h^{0,\mathrm{crit}})} - 1)^2, \\ \chi^1(y) \triangleq & \, C^1 \tfrac{1}{1-R}(e^{(1-R)y} - 1) + d^{\pi,1}(\pi^{1,M}) \tfrac{1}{1-R}(e^{(1-R)(y\vee h^{1,\mathrm{crit}})} - 1) \\ & + \tfrac{1}{2}\tfrac{1}{R}(B^1_\pi)^\top.(\sigma^1.(\sigma^1)^\top)^{-1}.B^1_\pi \tfrac{1}{(1-R)^2}(e^{(1-R)(y\vee h^{1,\mathrm{crit}})} - 1)^2.\end{aligned}$$

- *Continuity results on the χ^i's.* Being compositions of continuously differentiable functions, the $\max\{\cdot,\cdot\}$- and the $\min\{\cdot,\cdot\}$-function the χ^i's are locally Lipschitz continuous.
- *Non-negativity of the χ^i's.* In order to verify that $\chi^0 \geq 0$ we distinguish the two cases that already appeared in the proof of Lemma 4.85; $\pi^{0,M} \notin \mathfrak{I}^{\pi,0}$ and $\pi^{0,M} \in \mathfrak{I}^{\pi,0}$.
  - $\pi^{0,M} \notin \mathfrak{I}^{\pi,0}$. If $\pi^{0,M} \notin \mathfrak{I}^{\pi,0}$ then $h^{0,\mathrm{crit}} = 0$ implying $\chi^0(y) = -C^0 \frac{1}{1-R}(e^{-(1-R)y} - 1)$ which is non-negative for every $y \in \mathbb{R}_0^+$.
  - $\pi^{0,M} \in \mathfrak{I}^{\pi,0}$. If $\pi^{0,M} \in \mathfrak{I}^{\pi,0}$ then $h^{0,\mathrm{crit}}$ is strictly positive and there are again two cases to be distinguished; $y < h^{0,\mathrm{crit}}$ and $y \geq h^{0,\mathrm{crit}}$.
    - $y < h^{0,\mathrm{crit}}$. In the proof of Lemma 4.85 we have seen that $y < h^{0,\mathrm{crit}}$ is equivalent to

    $$0 < - d^{\pi,0}(\pi^{0,M}) \tfrac{1}{1-R}(e^{-(1-R)y} - 1) - \tfrac{1}{R}(B^0_\pi)^\top.(\sigma^0.(\sigma^0)^\top)^{-1}.B^0_\pi \tfrac{1}{(1-R)^2}(e^{-(1-R)y} - 1)^2$$

    which obviously yields

    $$0 < - d^{\pi,0}(\pi^{0,M}) \tfrac{1}{1-R}(e^{-(1-R)y} - 1) - \tfrac{1}{2}\tfrac{1}{R}(B^0_\pi)^\top.(\sigma^0.(\sigma^0)^\top)^{-1}.B^0_\pi \tfrac{1}{(1-R)^2}(e^{-(1-R)y} - 1)^2,$$

    and thus $\chi^0(y) \geq 0$ for every $y \in \mathbb{R}_0^+$.
    - $y \geq h^{0,\mathrm{crit}}$. If otherwise $y \geq h^{0,\mathrm{crit}}$ then

    $$\chi^0(y) = -C^0 \tfrac{1}{1-R}(e^{-(1-R)y} - 1) + \tfrac{1}{2} \frac{((d^{\pi,0}(\pi^{0,M}))^+)^2}{\frac{1}{R}(B^0_\pi)^\top.(\sigma^0.(\sigma^0)^\top)^{-1}.B^0_\pi}$$

    which is positive for every $y \in \mathbb{R}_0^+$, too.

In order to prove that $\chi^1 \geq 0$ we now distinguish the two cases $\pi^{1,M} \in \mathcal{I}^{\pi,1}$ and $\pi^{1,M} \notin \mathcal{I}^{\pi,1}$.

- $\pi^{1,M} \in \mathcal{I}^{\pi,1}$. If $\pi^{1,M} \in \mathcal{I}^{\pi,1}$ then $h^{1,\text{crit}} = 0$ implying

$$\chi^1(y) = C^1 \tfrac{1}{1-R}(e^{(1-R)y} - 1) + d^{\pi,1}(\pi^{1,M}) \tfrac{1}{1-R}(e^{(1-R)y} - 1) + \tfrac{1}{2}\tfrac{1}{R}(B^1_\pi)^\top.(\sigma^1.(\sigma^1)^\top)^{-1}.B^1_\pi \tfrac{1}{(1-R)^2}(e^{(1-R)y} - 1)^2.$$

  Since $\pi^{1,M} \in \mathcal{I}^{\pi,1}$ also implies $d^{\pi,1}(\pi^{1,M}) \geq 0$ the non-negativity of χ^1 is proven.

- $\pi^{1,M} \notin \mathcal{I}^{\pi,1}$. If $\pi^{1,M} \notin \mathcal{I}^{\pi,1}$ then $h^{1,\text{crit}}$ is strictly positive and there are again two cases to be distinguished; $y < h^{1,\text{crit}}$ and $y \geq h^{1,\text{crit}}$.

  - $y < h^{1,\text{crit}}$. If $y < h^{1,\text{crit}}$ then

$$\begin{aligned}\chi^1(y) &= C^1 \tfrac{1}{1-R}(e^{(1-R)y} - 1) + d^{\pi,1}(\pi^{1,M}) \tfrac{1}{1-R}(e^{(1-R)h^{1,\text{crit}}} - 1) \\ &\quad + \tfrac{1}{2}\tfrac{1}{R}(B^1_\pi)^\top.(\sigma^1.(\sigma^1)^\top)^{-1}.B^1_\pi \tfrac{1}{(1-R)^2}(e^{(1-R)h^{1,\text{crit}}} - 1)^2 \\ &= C^1 \tfrac{1}{1-R}(e^{(1-R)y} - 1) + 2\frac{d^{\pi,1}(\pi^{1,M})(d^{\pi,1}(\pi^{1,M}))^-}{\frac{1}{R}(B^1_\pi)^\top.(\sigma^1.(\sigma^1)^\top)^{-1}.B^1_\pi} + 2\frac{((d^{\pi,1}(\pi^{1,M}))^-)^2}{\frac{1}{R}(B^1_\pi)^\top.(\sigma^1.(\sigma^1)^\top)^{-1}.B^1_\pi} \\ &= C^1 \tfrac{1}{1-R}(e^{(1-R)y} - 1)\end{aligned}$$

    which is non-negative as $y \geq 0$.

  - $y \geq h^{1,\text{crit}}$. If otherwise $y \geq h^{1,\text{crit}}$ then

$$\chi^1(y) = C^1 \tfrac{1}{1-R}(e^{(1-R)y} - 1) + d^{\pi,1}(\pi^{1,M}) \tfrac{1}{1-R}(e^{(1-R)y} - 1) + \tfrac{1}{2}\tfrac{1}{R}(B^1_\pi)^\top.(\sigma^1.(\sigma^1)^\top)^{-1}.B^1_\pi \tfrac{1}{(1-R)^2}(e^{(1-R)y} - 1)^2$$

    which is non-negative since we know from the proof of Lemma 4.85 that

$$y \geq h^{1,\text{crit}} \Leftrightarrow d^{\pi,1}(\pi^{1,M}) \tfrac{1}{1-R}(e^{(1-R)y} - 1) + \tfrac{1}{2}\tfrac{1}{R}(B^1_\pi)^\top.(\sigma^1.(\sigma^1)^\top)^{-1}.B^1_\pi \tfrac{1}{(1-R)^2}(e^{(1-R)y} - 1)^2 \geq 0.$$

The remainder of the proof is the same as the proof of Lemma 4.9. □

A particular example for the functions g and h is given in Figure 4.13.

We can even provide some bounds on the solution (g, h) as the proof of Lemma 4.91, resp. Lemma 4.9, directly implies the following corollaries.

Corollary 4.92 (Time-dependent bounds on g and h)**.** *Let h and g be given by (4.43), (4.44) subject to the boundary conditions (4.45). Then*

$$-(\Psi^1 - \xi^1(T))^-(T-t) \leq g(t) \leq (\Psi^0 - \xi^0(T))^+(T-t) \ (\varepsilon > 0),$$
$$\textit{resp. } 0 \leq g(t) \leq \Psi^0(T-t) \ (\varepsilon = 0) \textit{ and}$$
$$0 \leq h(t) \leq (\Psi^0 - \Psi^1)(T-t)$$

for $t \in [0, T]$ where the $\xi^i(T)$'s are as defined in the proof of Lemma 4.9.

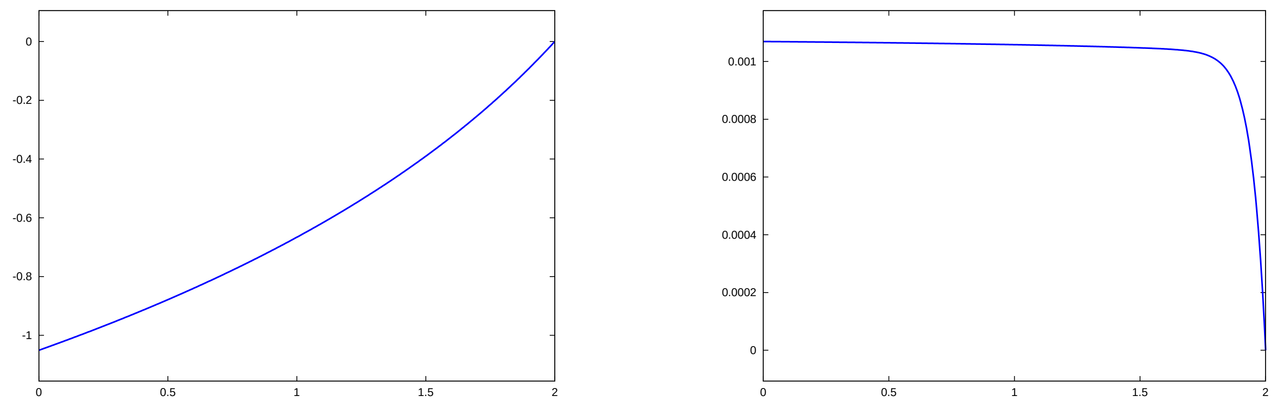

Figure 4.13: g (left) and h (right) as functions of t
($r^0 = r^1 = 0.03$, $\eta^0 = 0.09$, $\eta^1 = 0.01$, $\sigma^0 = 0.3$, $\sigma^1 = 0.5$, $\delta = 0.035$, $\varepsilon = 1$, $R = 2.5$, $T = 2$, $A^0 = 9$, $B^0_\pi = 5$, $C^0 = 10$, $A^1 = 5.5$, $B^1_\pi = -3$, $C^1 = 5.4$)

Corollary 4.93 (Time-independent bound on h)**.** *Let h and g be given by (4.43), (4.44) subject to the boundary conditions (4.45). Then*

$$0 \le h(t) \le \bar{h}$$

for $t \in [0, T]$ where $\bar{h}$ is the smallest positive root of $\bar{F}(y) \triangleq -(\Psi^0 - \Psi^1) + \chi^0(y) + \chi^1(y)$ if such a root exists; otherwise $\bar{h} = \infty$, i.e.

$$\bar{h} = \min \left\{ y \in \mathbb{R}_0^+ \,:\, \bar{F}(y) = 0 \right\},$$

with the convention $\min \emptyset \triangleq \infty$, where the χ^i's are as defined in the proof of Lemma 4.91.

Proof. The assertion follows from a simple ODE argument since

$$F(t, x, y) = -(\Psi^0 - \Psi^1) + \varrho(t, x, y) + \chi^0(y) + \chi^1(y) \ge \bar{F}(y)$$

for all $(t, x, y) \in [0, T] \times \mathbb{R} \times \mathbb{R}_0^+$ where we used the notation from the proof of Lemma 4.9. □

The boundary function on h as presented in Corollary 4.92 is suitable for large values of t for which the deviation from h is not too large whereas for small values of t the non-linear behavior of h implies that the boundary function overshoots h by far. The converse is true for the bound given in Corollary 4.93. Thus combining the two bounds on h yields

$$0 \le h(t) \le \min\{(\Psi^0 - \Psi^1)(T - t), \bar{h}\}$$

for $t \in [0, T]$.

The following theorem ensures that the strategy given in Lemmas 4.81 and 4.85 is indeed the optimal strategy for the optimal investment problem.

Theorem 4.94 (Solution of the Investment Problem)**.** *Let condition (NJ) be satisfied, and let h and g be given by (4.43), (4.44) subject to the boundary conditions (4.45). Then the strategy*

$$(\pi^{i,*}(t), c^{i,*}(t)) \triangleq (\pi^{i,*}(h(t)), c^{i,*}(t, g(t) - 1_{\{i=1\}} h(t))),\ t \in [0, T],\ i = 0, 1,$$

as given in Lemmas 4.81 and 4.85 is optimal for the investment problem (P) with CRRA preferences and relative risk aversion R.

Proof. Since $(\pi^{i,*}(t), c^{i,*}(t))$ maximizes the HJB-system (4.42) for each $t \in [0, T]$, optimality of the strategy $(\pi^{i,*}, c^{i,*})$ follows directly from the Verification Theorem 3.3. □

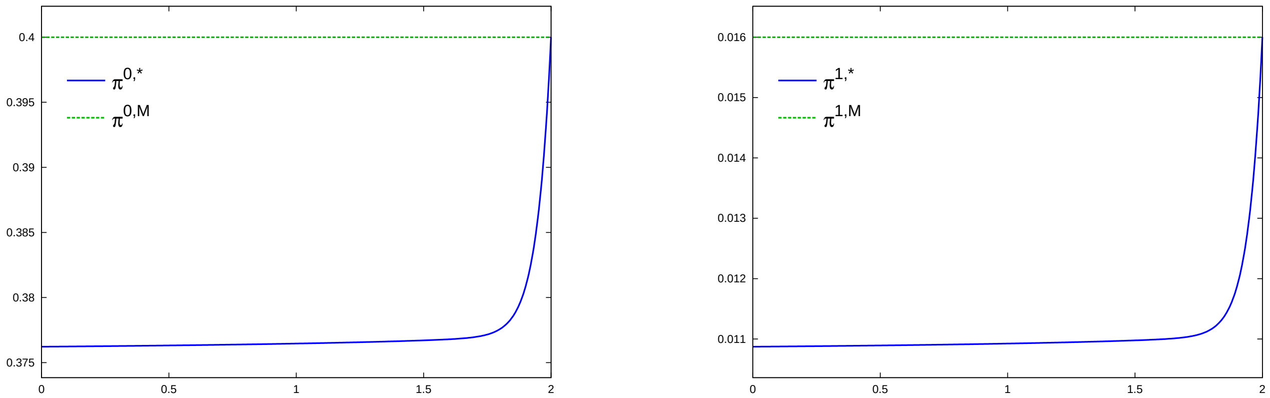

Figure 4.14: Optimal strategy vs. Merton strategy: $\pi^{i,*}$ and $\pi^{i,M}$ as functions of t
($r^0 = r^1 = 0.03$, $\eta^0 = 0.09$, $\eta^1 = 0.01$, $\sigma^0 = 0.3$, $\sigma^1 = 0.5$, $\delta = 0.035$, $\varepsilon = 1$, $R = 2.5$, $T = 2$,
$A^0 = 9$, $B^0_\pi = 5$, $C^0 = 10$, $A^1 = 5.5$, $B^1_\pi = -3$, $C^1 = 5.4$)

Figure 4.14 shows the optimal strategy and the Merton strategy in a setting where B^0_π is positive and B^1_π is negative. Hence the large investor is forced to follow a strategy with portfolio proportions that are smaller than the Merton ones in order to generate advantageous regime shift intensities. In state 0 the deviation is about 6% whereas in state 1 it amounts to about 31%.

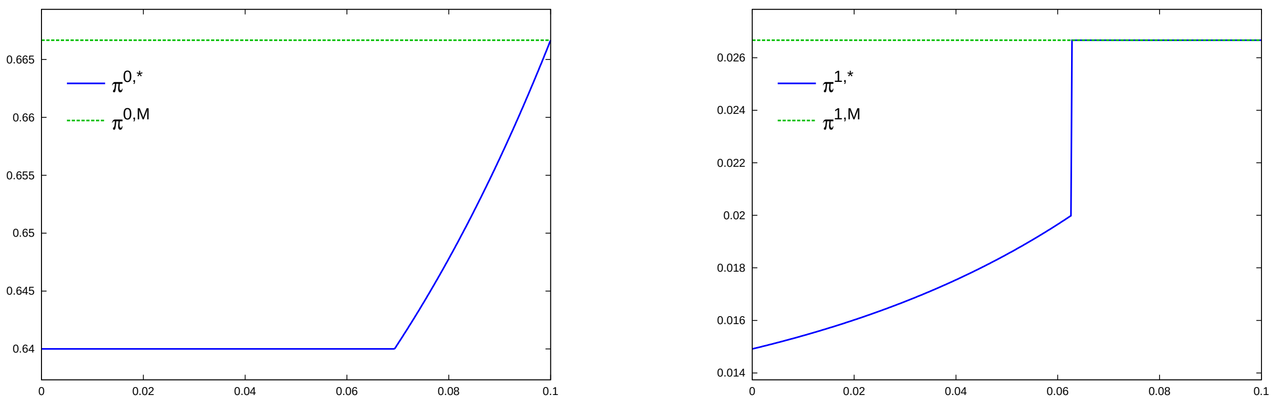

Figure 4.15: Optimal strategy vs. Merton strategy: $\pi^{i,*}$ and $\pi^{i,M}$ as functions of t
($r^0 = r^1 = 0.03$, $\eta^0 = 0.09$, $\eta^1 = 0.01$, $\sigma^0 = 0.3$, $\sigma^1 = 0.5$, $\delta = 0.035$, $\varepsilon = 1$, $R = 1.5$, $T = 0.1$,
$A^0 = 6.8$, $B^0_\pi = 5$, $C^0 = 10$, $A^1 = 5.47$, $B^1_\pi = -3$, $C^1 = 5.4$)

The special cases of $\pi^{0,*}$ reaching the critical strategy and $\pi^{1,*}$ jumping to the Merton strategy are shown in Figure 4.15.

Remark 4.95. *As one can see in Figures 4.14 and 4.15 the deviation of the optimal strategy from the Merton strategy usually shrinks continuously as t is growing; in $t = T$ it vanishes completely. This typical behavior is due to the fact that the chance, resp. the threat, of a possible jump to the favorable, resp. adverse, market state decreases as the time to maturity $T - t$ becomes smaller since the* *related probability decreases. Thus a relocation among the utility criterion and the intensity criterion as discussed in Remark 4.33 in favor of the utility criterion takes place. The utility aspect becomes more and more important and thus the optimal strategy approaches the Merton strategy. So far this is just a qualitative point of view.*

What is concretely meant by 'small' time to maturities and the aforementioned decreasing chance, resp. threat, i.e. the quantitative aspect of this behavior of the optimal strategy, highly depends on factors such as the level of the regime shift intensities, i.e. the intensities at the Merton strategies, the sensitivity of the market represented by B^i_π, the difference between the two market states measured by $\Psi^0 - \Psi^1$, the investor's risk aversion R and finally of course the time to maturity $T - t$. Moreover most of the aforementioned aspects depend on each other, e.g. the risk aversion enters the utility growth potential, the Merton strategy and thus the regime shift intensities at the Merton strategies. Hence the effects of the above factors on the optimal solution are far from being trivial.

As in the section on step intensity functions the special structure of the large investor's optimal portfolio strategy suggests a decomposition into the classical Merton strategy and an additional hedging component. This hedging component is due to the investor's influence on the market. So the optimal strategy can be written as

$$\pi^{i,*}(t) = \pi^{i,M} + \pi^{i,H}(t), \;\; i = 0, 1$$

where the hedging component $\pi^{i,H}$ is given by

$$\begin{aligned}
\pi^{0,H}(t) &\triangleq \tfrac{1}{R}(\sigma^0.(\sigma^0)^\top)^{-1}.B^0_\pi \tfrac{1}{1-R}(e^{-(1-R)h(t)\wedge h^{0,\mathrm{crit}}} - 1),\\
\pi^{1,H}(t) &\triangleq \tfrac{1}{R}(\sigma^1.(\sigma^1)^\top)^{-1}.B^1_\pi \tfrac{1}{1-R}(e^{(1-R)h(t)} - 1)1_{\{h(t)\geq h^{1,\mathrm{crit}}\}}.
\end{aligned}$$

Notice that $(\pi^{0,H}(t))^\top.B^0_\pi \leq 0$ whereas $(\pi^{1,H}(t))^\top.B^1_\pi \geq 0$. The following lemma shows that the investor achieves an intensity gain when using the optimal strategy $\pi^{i,*}$ instead of the Merton strategy $\pi^{i,M}$.

Lemma 4.96. *The optimal strategy $(\pi^{i,*}, c^{i,*})$ satisfies*

$$\vartheta^{0,1}(\pi^{0,*}(t), c^{0,*}(t)) \leq \vartheta^{0,1}(\pi^{0,M}, c^{0,M}(t)) \text{ and } \vartheta^{1,0}(\pi^{1,*}(t), c^{1,*}(t)) \geq \vartheta^{1,0}(\pi^{1,M}, c^{1,M}(t))$$

for every $t \in [0, T]$.

Proof. The assertion follows since

$$\begin{aligned}
\vartheta^{0,1}(\pi^{0,*}(t), c^{0,*}(t)) &= \max\{A^0 + (\pi^{0,*}(t))^\top.B^0_\pi, C^0\}\\
&= \max\{A^0 + (\pi^{0,M})^\top.B^0_\pi + (\pi^{0,H}(t))^\top.B^0_\pi, C^0\}\\
&\leq \max\{A^0 + (\pi^{0,M})^\top.B^0_\pi, C^0\}\\
&= \vartheta^{0,1}(\pi^{0,M}, c^{0,M}(t))
\end{aligned}$$

and

$$\begin{aligned}
\vartheta^{1,0}(\pi^{1,*}(t), c^{1,*}(t)) &= \max\{A^1 + (\pi^{1,*}(t))^\top.B^1_\pi, C^1\}\\
&= \max\{A^1 + (\pi^{1,M})^\top.B^1_\pi + (\pi^{1,H}(t))^\top.B^1_\pi, C^1\}\\
&\geq \max\{A^1 + (\pi^{1,M})^\top.B^1_\pi, C^1\}\\
&= \vartheta^{1,0}(\pi^{1,M}, c^{1,M}(t))
\end{aligned}$$

as $(\pi^{0,H}(t))^\top.B^0_\pi \leq 0$, resp. $(\pi^{1,H}(t))^\top.B^1_\pi \geq 0$. □

Remark 4.97. *Notice that the optimal strategy is a* compromise strategy*. The large investor faces the trade-off between trading optimally in terms of generating the highest possible expected utility from intermediate consumption and final wealth on the one hand and investing in such a way that the regime shift intensities are as favorable as possible on the other hand. Pursuing only the utility goal would imply the Merton strategies to be optimal. However, solely aiming at favorable intensities – small ones in state* 0 *and large ones in state* 1 *– would, in state* 0*, render any strategy not lying in the influencing half space* $\mathcal{I}^{\pi,i}$ *to be optimal, whereas in state* 1 *an optimal strategy would not exist since* $\vartheta^{1,0}$ *is unbounded from above.*

The optimal strategy that we derived here is a compromise strategy *in that it is in general different from the Merton strategy but generates better regime shift intensities than those the Merton strategy would yield, i.e. it accommodates the aforementioned trade-off.*

We have seen before that the optimal portfolio strategy in state 0 may be truncated at $\pi^{0,\mathrm{crit}}$. In the following we present a sufficient condition under which this truncation does not take place, i.e. $\pi^{i,*}(t) = \tilde{\pi}^{i,*}(t)$ for all $t \in [0,T]$ where $\tilde{\pi}^{i,*}(t) \triangleq \tilde{\pi}^{i,*}(h(t))$.

Proposition 4.98. *Let* $\pi^{i,M} \in \mathcal{I}^{\pi,i}$ *and suppose that* $A^0 + (\pi^{0,\circ})^\top.B^0_\pi \geq C^0$*, where*

$$\pi^{0,\circ} \triangleq \tfrac{1}{R}(\sigma^0.(\sigma^0)^\top)^{-1}.\left(\eta^0 - \left(\frac{1}{2}\frac{A^0+(\pi^{0,M})^\top.B^0_\pi-3C^0}{\frac{1}{R}(B^0_\pi)^\top.(\sigma^0.(\sigma^0)^\top)^{-1}.B^0_\pi} + \frac{\Psi^0-\Psi^1}{A^0+(\pi^{0,M})^\top.B^0_\pi-C^0}\right)B^0_\pi\right).$$

Then $\pi^{i,*}(t) = \tilde{\pi}^{i,*}(t)$ *for all* $t \in [0,T]$.

Proof. If $\pi^{1,M} \in \mathcal{I}^{\pi,1}$ then $h^{1,\mathrm{crit}} = 0$ implying $\pi^{1,*}(t) = \tilde{\pi}^{1,*}(t)$ for all $t \in [0,T]$. In state 0 the condition $\pi^{0,M} \in \mathcal{I}^{\pi,0}$ is necessary since otherwise $\pi^{0,*} = \pi^{0,M}$. Hence

$$h^{0,\mathrm{crit}} = -\frac{1}{1-R}\ln\Big(-(1-R)\frac{A^0+(\pi^{0,M})^\top.B^0_\pi-C^0}{\frac{1}{R}(B^0_\pi)^\top.(\sigma^0.(\sigma^0)^\top)^{-1}.B^0_\pi} + 1\Big).$$

The assertion of the proposition is equivalent to the condition

$$h(t) \leq h^{0,\mathrm{crit}} \text{ for all } t \in [0,T].$$

To prove this, we assume without loss of generality that $h^{0,\mathrm{crit}} < \infty$ and use a simple ODE argument. We let F as given in the proofs of the Lemmas 4.9 and 4.91. As $F(T, g(T), h(T)) = F(T,0,0) \leq 0$, it suffices to show that $F(t,x,h^{0,\mathrm{crit}}) \geq 0$ for every $(t,x) \in [0,T] \times \mathbb{R}$. Indeed, in this case it follows from the intermediate value theorem that for each $t \in [0,T]$ there exists some $\tilde{h}(t) \in [0, h^{0,\mathrm{crit}}]$ with $F(t,x,\tilde{h}(t)) = 0$. Thus $0 \leq h(t) \leq \max_{t\in[0,T]} \tilde{h}(t) \leq h^{0,\mathrm{crit}}$ for every $t \in [0,T]$.

To demonstrate under which conditions $F(t,x,h^{0,\mathrm{crit}}) \geq 0$ we write $F(t,x,h^{0,\mathrm{crit}}) = A + B$ with

$$A \triangleq \varrho(t,x,h^{0,\mathrm{crit}}) + \chi^1(h^{0,\mathrm{crit}}), \quad B \triangleq -(\Psi^0 - \Psi^1) + \chi^0(h^{0,\mathrm{crit}})$$

where we use the notation from the proofs of Lemmas 4.9 and 4.91.

The non-negativity of ϱ and χ^1 implies that $A \geq 0$. On the other hand,

$$\begin{aligned} B &= -(\Psi^0 - \Psi^1) - C^0 \tfrac{1}{1-R}(e^{-(1-R)h^{0,\text{crit}}} - 1) \\ &\quad - (A^0 + (\pi^{0,M})^\top . B^0_\pi - C^0) \tfrac{1}{1-R}(e^{-(1-R)h^{0,\text{crit}}} - 1) \\ &\quad - \tfrac{1}{2}\tfrac{1}{R}(B^0_\pi)^\top . (\sigma^0 . (\sigma^0)^\top)^{-1} . B^0_\pi \tfrac{1}{(1-R)^2}(e^{-(1-R)h^{0,\text{crit}}} - 1)^2 \\ &= -\Big[A^0 + \tfrac{1}{R}\Big(\eta^0 + \tfrac{1}{2}\tfrac{1}{1-R}(e^{-(1-R)h^{0,\text{crit}}} - 1)B^0_\pi\Big)^\top . (\sigma^0 . (\sigma^0)^\top)^{-1} . B^0_\pi \\ &\qquad + \frac{\Psi^0 - \Psi^1}{\frac{1}{1-R}(e^{-(1-R)h^{0,\text{crit}}} - 1)}\Big] \tfrac{1}{1-R}(e^{-(1-R)h^{0,\text{crit}}} - 1) \\ &= -(A^0 + (\tilde{\pi}^{0,\circ})^\top . B^0_\pi - C^0) \tfrac{1}{1-R}(e^{-(1-R)h^{0,\text{crit}}} - 1) \end{aligned} \tag{4.46}$$

where

$$\begin{aligned} \tilde{\pi}^{0,\circ} &\triangleq \tfrac{1}{R}(\sigma^0 . (\sigma^0)^\top)^{-1} . \left(\eta^0 + \left(\tfrac{1}{2}\tfrac{1}{1-R}(e^{-(1-R)h^{0,\text{crit}}} - 1) + \frac{\frac{\Psi^0 - \Psi^1}{\frac{1}{1-R}\left(e^{-(1-R)h^{0,\text{crit}}} - 1\right)} + C^0}{\frac{1}{R}(B^0_\pi)^\top . (\sigma^0 . (\sigma^0)^\top)^{-1} . B^0_\pi}\right) B^0_\pi\right) \\ &= \tfrac{1}{R}(\sigma^0 . (\sigma^0)^\top)^{-1} . \left(\eta^0 - \left(\tfrac{1}{2}\frac{A^0 + (\pi^{0,M})^\top . B^0_\pi - C^0}{\frac{1}{R}(B^0_\pi)^\top . (\sigma^0 . (\sigma^0)^\top)^{-1} . B^0_\pi} + \frac{\Psi^0 - \Psi^1}{A^0 + (\pi^{0,M})^\top . B^0_\pi - C^0} - \frac{C^0}{\frac{1}{R}(B^0_\pi)^\top . (\sigma^0 . (\sigma^0)^\top)^{-1} . B^0_\pi}\right) B^0_\pi\right) \\ &= \tfrac{1}{R}(\sigma^0 . (\sigma^0)^\top)^{-1} . \left(\eta^0 - \left(\tfrac{1}{2}\frac{A^0 + (\pi^{0,M})^\top . B^0_\pi - 3C^0}{\frac{1}{R}(B^0_\pi)^\top . (\sigma^0 . (\sigma^0)^\top)^{-1} . B^0_\pi} + \frac{\Psi^0 - \Psi^1}{A^0 + (\pi^{0,M})^\top . B^0_\pi - C^0}\right) B^0_\pi\right) \\ &= \pi^{0,\circ}. \end{aligned}$$

Since $A^0 + (\pi^{0,\circ})^\top . B^0_\pi \geq C^0$, we have $B \geq 0$ and the proof is complete. □

Observe that for an arbitrary parameter specification, the assumption of Proposition 4.98 is satisfied if A^0 is sufficiently large. This can be seen when writing

$$A^0 + (\pi^{0,\circ})^\top . B^0_\pi = \tfrac{1}{2}(A^0 + (\pi^{0,M})^\top . B^0_\pi + 3C^0) - \tfrac{1}{R}(B^0_\pi)^\top . (\sigma^0 . (\sigma^0)^\top)^{-1} . B^0_\pi \frac{\Psi^0 - \Psi^1}{A^0 + (\pi^{0,M})^\top . B^0_\pi - C^0}.$$

In a one-dimensional setting, we have a simple explicit criterion.

Corollary 4.99. *Let $\bar{n} = 1$ and $\pi^{i,M} \in \mathcal{I}^{\pi,i}$ for $i = 0, 1$. Then the assumption of Proposition 4.98 is fulfilled if*

$$A^0 \geq \begin{cases} C^0 + \sqrt{2\frac{1}{R}\frac{(B^0_\pi)^2}{(\sigma^0)^2}(r^0 - \Psi^1)^+} & \text{if } \pi^{0,M} B^0_\pi \geq 0, \\ -2\pi^{0,M} B^0_\pi + C^0 + \sqrt{2\frac{1}{R}\frac{(B^0_\pi)^2}{(\sigma^0)^2}(r^0 - \Psi^1)^+} & \text{if } \pi^{0,M} B^0_\pi < 0. \end{cases} \tag{4.47}$$

Proof. Recall that $A^0 + \pi^{0,\circ} B^0_\pi \geq C^0$ if and only if $B \geq 0$ in equation (4.46). When substituting $-\frac{1}{1-R}(e^{-(1-R)h^{0,\text{crit}}} - 1) = (A^0 + \pi^{0,M} B^0_\pi - C^0) R \frac{(\sigma^0)^2}{(B^0_\pi)^2}$ we find that $B \geq 0$ if and only if

$$(A^0)^2 + 2\pi^{0,M} B^0_\pi A^0 - (C^0)^2 - 2\tfrac{1}{R}\tfrac{(B^0_\pi)^2}{(\sigma^0)^2}(r^0 - \Psi^1) > 0.$$

Hence, with

$$C \triangleq (A^0)^2 + 2\pi^{0,M} B^0_\pi A^0 - (C^0)^2 - 2\tfrac{1}{R}\tfrac{(B^0_\pi)^2}{(\sigma^0)^2}(r^0 - \Psi^1)^+$$

we get that $A^0 + \pi^{0,\circ} B^0_\pi \geq C^0$ if $C \geq 0$. Regarding C as a quadratic polynomial in A^0, it is readily seen that $C \geq 0$ if

$$A^0 \geq -\pi^{0,M} B^0_\pi + \sqrt{(\pi^{0,M} B^0_\pi)^2 + (C^0)^2 + 2\frac{1}{R}\frac{(B^0_\pi)^2}{(\sigma^0)^2}(r^0 - \Psi^1)^+}.$$

In particular, we have $C \geq 0$ provided that

$$A^0 \geq -\pi^{0,M} B^0_\pi + \left|\pi^{0,M} B^0_\pi\right| + C^0 + \sqrt{2\frac{1}{R}\frac{(B^0_\pi)^2}{(\sigma^0)^2}(r^0 - \Psi^1)^+},$$

which yields the assertion. □

Remark 4.100 (Interpretation)**.** *Suppose that $r^0 \leq \Psi^1$. Then condition (4.47) in Corollary 4.99 simply means that the no-participation strategy $\pi^0 = 0$ ($\pi^0 = 2\pi^{0,M}$) satisfies*

$$A^0 + \pi^0 B^0_\pi \geq C^0$$

if $\pi^{0,M} B^0_\pi \geq 0$ ($\pi^{0,M} B^0_\pi < 0$).

4.3.2 Consumption-dependent Intensities

We now come to transition intensities solely dependent on the investor's consumption rate, i.e. we let

$$B^i_\pi = 0 \text{ and } B^i_c \neq 0 \text{ for } i = 0, 1 \tag{CD}$$

so that the intensities are given by

$$\vartheta^{i,1-i}(\pi, c) = \max\{A^i + B^i_c c, C^i\}.$$

We denote by

$$\mathcal{I}^{c,0} \triangleq \left\{c \in \mathbb{R}^+_0 \,:\, A^0 + B^0_c c > C^0\right\},\ \mathcal{I}^{c,1} \triangleq \left\{c \in \mathbb{R}^+_0 \,:\, A^1 + B^1_c c \geq C^1\right\}$$

the half spaces of consumption rates that impact on the intensities of regime shifts. Thus

$$d^{c,i}(c) \triangleq A^i + B^i_c c - C^i$$

denotes the 'distance' of the consumption rate c to the separating hyperplane and we have $c \in \mathcal{I}^{c,0}$ if and only if $d^{c,0}(c) > 0$, resp. $c \in \mathcal{I}^{c,1}$ if and only if $d^{c,1}(c) \geq 0$.

In order to determine the maximizer in the HJB-system (4.41) we define functions $H^{\pi,i} : \mathbb{R}^{\bar{n}} \to \mathbb{R}$ and $H^{c,i} : [0, T] \times \mathbb{R} \times \mathbb{R}^+_0 \times \mathbb{R}^+_0 \to \mathbb{R}$, $i = 0, 1$, given by

$$H^{\pi,i}(\pi) \triangleq r^i + \pi^\top.\eta^i - \tfrac{1}{2}R\pi^\top.\sigma^i.(\sigma^i)^\top.\pi,$$

$$H^{c,i}(t, x, y, c) \triangleq \frac{\varepsilon\delta}{\varepsilon-(\varepsilon-\delta)e^{-\delta(T-t)}}\frac{1}{1-R}(e^{-(1-R)x}c^{1-R} - 1) - c + \max\{A^i + B^i_c c, C^i\}\frac{1}{1-R}(e^{(-1)^{1-i}(1-R)y} - 1)$$

where we use the already mentioned convention $0^{1-R} \triangleq \infty$ if $R > 1$. Notice that function $H^{c,i}$ is independent of t and x in case of $\varepsilon = 0$.

The HJB-system (4.41) reads

$$0 = \sup_{(\pi,c)\in\mathbb{R}^{\bar{n}}\times\mathbb{R}_0^+} \Big\{ g'(t) - 1_{\{i=1\}}h'(t) + H^{\pi,i}(\pi) + H^{c,i}(t, g(t) - 1_{\{i=1\}}h(t), h(t), c) \Big\} \tag{4.48}$$

for $t \in [0,T)$ and $i = 0,1$, subject to the boundary conditions

$$g(T) = 0,\ \ h(T) = 0.$$

Writing the HJB-system in that way it is obvious that taking the supremum over $(\pi, c) \in \mathbb{R}^{\bar{n}} \times \mathbb{R}_0^+$ can be separated into two unrelated maximizations; one over $\pi \in \mathbb{R}^{\bar{n}}$ and one over $c \in \mathbb{R}_0^+$.

To find the supremum in (4.48) we present the maximizers of the functions $H^{\pi,i}$ and $H^{c,i}(t,x,y,\cdot)$ for arbitrary $(t,x,y) \in [0,T] \times \mathbb{R} \times \mathbb{R}_0^+$. This yields a family of maximizers dependent on (t,x,y). The maximizers of the HJB-system (4.48) are then obtained by replacing x and y by $g(t) - 1_{\{i=1\}}h(t)$ and $h(t)$.

Concerning the portfolio proportions the concavity of $H^{\pi,i}$ and the first-order condition imply

Lemma 4.101 (Maximizer of $H^{\pi,i}$)**.** *The maximizer*

$$\pi^{i,*} \triangleq \arg\max_{\pi\in\mathbb{R}^{\bar{n}}} H^{\pi,i}(\pi),\ \ i = 0,1,$$

is given by the Merton strategy, i.e.

$$\pi^{i,*} = \pi^{i,M}.$$

In order to determine the maximizing consumption rate we let $H_l^{c,i}$, $H_a^{c,i} :\ [0,T] \times \mathbb{R} \times \mathbb{R}_0^+ \times \mathbb{R}_0^+ \to \mathbb{R}$, $i = 0,1$, given by

$$H_l^{c,i}(t,x,y,c) \triangleq \tfrac{\varepsilon\delta}{\varepsilon-(\varepsilon-\delta)e^{-\delta(T-t)}} \tfrac{1}{1-R}(e^{-(1-R)x}c^{1-R} - 1) - c + C^i \tfrac{1}{1-R}(e^{(-1)^{1-i}(1-R)y} - 1),$$
$$H_a^{c,i}(t,x,y,c) \triangleq \tfrac{\varepsilon\delta}{\varepsilon-(\varepsilon-\delta)e^{-\delta(T-t)}} \tfrac{1}{1-R}(e^{-(1-R)x}c^{1-R} - 1) - c + (A^i + B_c^i c) \tfrac{1}{1-R}(e^{(-1)^{1-i}(1-R)y} - 1)$$

such that $H^{c,i}(t,x,y,c) = H_l^{c,i}(t,x,y,c)1_{\{c\notin\mathfrak{I}^{c,i}\}} + H_a^{c,i}(t,x,y,c)1_{\{c\in\mathfrak{I}^{c,i}\}}$.

If $\varepsilon > 0$ then $H_l^{c,i}(t,x,y,\cdot)$ and $H_a^{c,i}(t,x,y,\cdot)$ are concave for every $(t,x,y) \in [0,T] \times \mathbb{R} \times \mathbb{R}_0^+$. Therefore the three candidate solutions for the maximization in (4.48) are

$$c^M(t,x) = \arg\max_{c\in\mathbb{R}_0^+} H_l^{c,i}(t,x,y,c),$$
$$\tilde{c}^{i,*}(t,x,y) \triangleq \arg\max_{c\in\mathbb{R}_0^+} H_a^{c,i}(t,x,y,c),$$
$$\tilde{c}^{i,\mathrm{crit}} \triangleq \arg\max_{\{c\in\mathbb{R}_0^+ \,:\, A^i + B_c^i c = C^i\}} H^{c,i}(t,x,y,c)$$

where

$$\tilde{c}^{0,*}(t,x,y) = \begin{cases} \big(\tfrac{\varepsilon\delta}{\varepsilon-(\varepsilon-\delta)e^{-\delta(T-t)}}\big)^{\frac{1}{R}} e^{-\frac{1-R}{R}x} \big(1 - B_c^0 \tfrac{1}{1-R}(e^{-(1-R)y} - 1)\big)^{-\frac{1}{R}} & \text{if } y < h^{\mathrm{crit}}, \\ \infty & \text{if } y \geq h^{\mathrm{crit}}, \end{cases}$$
$$\tilde{c}^{1,*}(t,x,y) = \big(\tfrac{\varepsilon\delta}{\varepsilon-(\varepsilon-\delta)e^{-\delta(T-t)}}\big)^{\frac{1}{R}} e^{-\frac{1-R}{R}x} \big(1 - B_c^1 \tfrac{1}{1-R}(e^{(1-R)y} - 1)\big)^{-\frac{1}{R}}$$

for $(t, x, y) \in [0, T] \times \mathbb{R} \times \mathbb{R}_0^+$, $i = 0, 1$, with

$$h^{\text{crit}} \triangleq \begin{cases} \infty & \text{if } B_c^0 > 0, \\ -\frac{1}{1-R} \ln\left((1-R)\frac{1}{B_c^0} + 1\right) & \text{if } B_c^0 < 0, \end{cases}$$

so that $y < h^{\text{crit}}$ guarantees $1 - B_c^0 \frac{1}{1-R}(e^{-(1-R)y} - 1) > 0$. Moreover $\tilde{c}^{i,\text{crit}}$ only exists if $B_c^i > 0$ and $A^i \leq C^i$, resp. $B_c^i < 0$ and $A^i \geq C^i$, where it is given by $-\frac{A^i - C^i}{B_c^i}$, $i = 0, 1$, since otherwise the set $\left\{c \in \mathbb{R}_0^+ \, : \, A^i + B_c^i c = C^i\right\}$ is empty.

If $\varepsilon = 0$ then $H_l^{c,i}(t, x, y, \cdot)$ and $H_a^{c,i}(t, x, y, \cdot)$ are now linear functions in c implying

$$\tilde{c}^{0,*}(t, x, y) = \begin{cases} 0 & \text{if } y < h^{\text{crit}}, \\ \infty & \text{if } y \geq h^{\text{crit}}, \end{cases}$$

$$\tilde{c}^{1,*}(t, x, y) = 0$$

for $i = 0, 1$. Further as $H_l^{c,i}(t, x, y, \cdot)$ is even decreasing in c the Merton consumption rate vanishes, i.e. $c^M(t, x) = 0$.

Remark 4.102. *We use the following shorthand notations*

$$H_{l,M}^{c,i}(t, x, y) \triangleq H_l^{c,i}(t, x, y, c^M(t, x)) \text{ and } H_{a,\sim}^{c,i}(t, x, y) \triangleq H_a^{c,i}(t, x, y, \tilde{c}^{i,*}(t, x, y))$$

for $(t, x, y) \in [0, T] \times \mathbb{R} \times \mathbb{R}_0^+$ *and* $i = 0, 1$.

Before we present the maximizing consumption rate we introduce the auxiliary function $\Lambda^{c,i}$ which is the consumption-dependent analogon to the function $\Lambda^{\pi,c,i}$ some sections before.

Lemma 4.103 (Properties of $\Lambda^{c,i}$)**.** *The function* $\Lambda^{c,i}$ *given by*

$$\Lambda^{c,i} : [0, T] \times \mathbb{R} \times (-\infty, \tfrac{1}{B_c^i}) \to \mathbb{R} \ (B_c^i > 0), \ \text{resp. } \Lambda^{c,i} : [0, T] \times \mathbb{R} \times (\tfrac{1}{B_c^i}, \infty) \to \mathbb{R} \ (B_c^i < 0)$$

with

$$\Lambda^{c,i}(t, z, \lambda) \triangleq B_c^i\Big(\left(1 - B_c^i \lambda\right)^{-\frac{1}{R}} - 1\Big) c^M(t, z)$$

has the following properties for every given $(t, z) \in [0, T] \times \mathbb{R}$.

i) $\Lambda^{c,i}$ *is continuously differentiable in* t, z *and* λ.

ii) If $\varepsilon > 0$ *then* $\Lambda^{c,i}(t, z, \cdot)$ *is increasing and strictly convex (*$B_c^i > 0$*), resp. strictly concave (*$B_c^i < 0$*). If otherwise* $\varepsilon = 0$ *then* $\Lambda^{c,i}(t, z, \cdot) = 0$.

iii) $\Lambda^{c,i}(t, z, \lambda) \leq 0$ *for* $\lambda < 0$, $\Lambda^{c,i}(t, z, 0) = 0$ *and* $\Lambda^{c,i}(t, z, \lambda) \geq 0$ *for* $\lambda > 0$.

iv) If $\varepsilon > 0$ *then* $\lim_{\lambda \to -\infty} \Lambda^{c,i}(t, z, \lambda) = -B_c^i c^M(t, z)$ *and* $\lim_{\lambda \to \frac{1}{B_c^i}} \Lambda^{c,i}(t, z, \lambda) = \infty$ $(B_c^i > 0)$, *resp.* $\lim_{\lambda \to \frac{1}{B_c^i}} \Lambda^{c,i}(t, z, \lambda) = -\infty$ *and* $\lim_{\lambda \to \infty} \Lambda^{c,i}(t, z, \lambda) = -B_c^i c^M(t, z)$ $(B_c^i < 0)$.

Proof. Let $(t,z) \in [0,T] \times \mathbb{R}$ be given.

ad i) Since $c^M(t,z)$ is continuously differentiable in t and z it follows that $\Lambda^{c,i}$ is continuously differentiable in t and z, too. Further $\Lambda^{c,i}$ is obviously continuously differentiable in λ. Thus assertion i) is proven.

ad ii) The derivative of $\Lambda^{c,i}$ with respect to λ is given by

$$\Lambda^{c,i}_\lambda(t,z,\lambda) \triangleq \tfrac{\partial}{\partial\lambda}\Lambda^{c,i}(t,z,\lambda) = \tfrac{1}{R}(B^i_c)^2\big(1 - B^i_c\lambda\big)^{-\frac{1+R}{R}} c^M(t,z).$$

If $\varepsilon > 0$ then $c^M(t,z) > 0$ and it is easy to see that $\Lambda^{c,i}_\lambda(t,z,\cdot)$ is non-negative and strictly increasing ($B^i_c > 0$), resp. strictly decreasing ($B^i_c < 0$), in λ. If $\varepsilon = 0$ then $c^M(t,z) = 0$ and $\Lambda^{c,i}(t,z,\cdot) = 0$ holds trivially. This implies assertion ii).

ad iii) $\Lambda^{c,i}(t,z,0) = 0$ holds trivially for any $(t,z) \in [0,T] \times \mathbb{R}$ which together with assertion ii) yields assertion iii).

ad iv) The limiting behavior of $\Lambda^{c,i}(t,z,\cdot)$ is clear when taking into account that $\Lambda^{c,i}(t,z,\cdot)$ has a pole at $\frac{1}{B^i_c}$ in case of $\varepsilon > 0$. □

Lemma 4.103 directly implies

Corollary 4.104. *For each fixed $(t,z) \in [0,T] \times \mathbb{R}$ and $\varepsilon > 0$ the function*

$$\Lambda^{c,i}(t,z,\cdot) : (-\infty, \tfrac{1}{B^i_c}) \to (-B^i_c c^M(t,z), \infty)\ (B^i_c > 0),\ \textit{resp.}$$
$$\Lambda^{c,i}(t,z,\cdot) : (\tfrac{1}{B^i_c}, \infty) \to (-\infty, -B^i_c c^M(t,z))\ (B^i_c < 0)$$

is bijective.

Further we present the following results on the relation between c^M and $\tilde{c}^{i,*}$ and the related function values $H^{c,i}_{l,M}$ and $H^{c,i}_{a,\sim}$ that turn out to be quite helpful.

Lemma 4.105. *For every $(t,x,y) \in [0,T] \times \mathbb{R} \times \mathbb{R}^+_0$ the Merton consumption rate c^M, the candidate solution $\tilde{c}^{i,*}$ and their function values $H^{c,i}_{l,M}$ and $H^{c,i}_{a,\sim}$ are related as follows.*

i) $c^M(t,x) \notin \mathfrak{I}^{c,0} \quad \Rightarrow \quad H^{c,0}_{l,M}(t,x,y) \le H^{c,0}_{a,\sim}(t,x,y) \quad \Rightarrow \quad \tilde{c}^{0,*}(t,x,y) \notin \mathfrak{I}^{c,0}.$

ii) $c^M(t,x) \in \mathfrak{I}^{\pi,1} \quad \Rightarrow \quad H^{c,1}_{l,M}(t,x,y) \le H^{c,1}_{a,\sim}(t,x,y) \quad \Rightarrow \quad \tilde{c}^{1,*}(t,x,y) \in \mathfrak{I}^{c,1}.$

Proof. ad i) If $h^{\text{crit}} < \infty$ and $y \ge h^{\text{crit}}$ then $\tilde{c}^{0,*}(t,x,y) = \infty$ and therefore $H^{c,0}_{a,\sim}(t,x,y) = \infty$, too. Further $h^{\text{crit}} < \infty$ implies $B^0_c < 0$. Hence obviously $\tilde{c}^{0,*}(t,x,y) \notin \mathfrak{I}^{c,0}$ and there is nothing to prove.

If otherwise $y < h^{\text{crit}}$ then the following three equivalences hold true. Firstly, $c^M(t,x) \notin \mathfrak{I}^{c,0}$ is by definition equivalent to

$$d^{c,0}(c^M(t,x)) \le 0. \tag{$*$}$$

Secondly, some transformations yield that $H^{c,0}_{l,M}(t,x,y) \le H^{c,0}_{a,\sim}(t,x,y)$ is equivalent to

$$d^{c,0}(c^M(t,x)) \le \Big(\frac{-\frac{R}{1-R}\Big((1-B^0_c\frac{1}{1-R}(e^{-(1-R)y}-1))^{-\frac{1-R}{R}}-1\Big)}{\frac{1}{1-R}(e^{-(1-R)y}-1)} + B^0_c\Big)c^M(t,x). \tag{$**$}$$

And thirdly, $\tilde{c}^{0,*}(t,x,y) \notin \mathcal{I}^{c,0}$ is equivalent to

$$d^{c,0}(c^M(t,x)) \leq -B_c^0\big((1 - B_c^0 \tfrac{1}{1-R}(e^{-(1-R)y} - 1))^{-\frac{1}{R}} - 1\big)c^M(t,x). \qquad (***)$$

The right-hand side of $(**)$ is non-negative. This follows from the fact that $y \geq 0$ and from Lemma 4.48, ii), with $z = -B_c^0 \frac{1}{1-R}(e^{-(1-R)y} - 1)$. The lemma is applicable since $y < h^{\text{crit}}$ is equivalent to $-B_c^0 \frac{1}{1-R}(e^{-(1-R)y} - 1) > -1$. Hence $(**)$ follows directly from $(*)$ which yields the first implication. In order to prove the second implication we have to show that the right-hand side of $(**)$ is smaller than the right-hand side of $(***)$ which is equivalent to

$$0 \leq \left(\frac{-\frac{R}{1-R}\Big((1-B_c^0 \frac{1}{1-R}(e^{-(1-R)y}-1))^{-\frac{1-R}{R}} - 1\Big)}{(1-B_c^0 \frac{1}{1-R}(e^{-(1-R)y}-1))^{-\frac{1}{R}}} + B_c^0 \frac{1}{1-R}(e^{-(1-R)y} - 1) \right) c^M(t,x). \qquad (4.49)$$

This is trivially satisfied if $\varepsilon = 0$, since then $c^M(t,x) = 0$. If $\varepsilon > 0$ then the latter inequality follows from Lemma 4.48, i), again substituting $z = -B_c^0 \frac{1}{1-R}(e^{-(1-R)y} - 1)$.

ad ii) The proof of assertion ii) works completely analogously since firstly, $c^M(t,x) \in \mathcal{I}^{c,1}$ is by definition equivalent to

$$d^{c,1}(c^M(t,x)) \geq 0,$$

secondly, $H_{l,M}^{c,1}(t,x,y) \leq H_{a,\sim}^{c,1}(t,x,y)$ is equivalent to

$$d^{c,1}(c^M(t,x)) \geq \left(\frac{-\frac{R}{1-R}\Big((1-B_c^1 \frac{1}{1-R}(e^{(1-R)y}-1))^{-\frac{1-R}{R}} - 1\Big)}{\frac{1}{1-R}(e^{(1-R)y}-1)} + B_c^1 \right) c^M(t,x),$$

and thirdly, $\tilde{c}^{1,*}(t,x,y) \in \mathcal{I}^{c,1}$ is equivalent to

$$d^{c,1}(c^M(t,x)) \geq -B_c^1\big((1 - B_c^1 \tfrac{1}{1-R}(e^{(1-R)y} - 1))^{-\frac{1}{R}} - 1\big)c^M(t,x).$$

□

We are now able to give the maximizing consumption rate.

Lemma 4.106 (Maximizer of $H^{c,i}(t,x,y,\cdot)$)**.** *For every $(t,x,y) \in [0,T] \times \mathbb{R} \times \mathbb{R}_0^+$ let*

$$c^{i,*}(t,x,y) \triangleq \operatorname*{arg\,max}_{c \in \mathbb{R}_0^+} H^{c,i}(t,x,y,c), \;\; i = 0,1.$$

If $\varepsilon > 0$ then the maximizer $c^{i,*}$ is given by

$$c^{0,*}(t,x,y) = \big(\tfrac{\varepsilon\delta}{\varepsilon-(\varepsilon-\delta)e^{-\delta(T-t)}}\big)^{\frac{1}{R}} e^{-\frac{1-R}{R}x}\Big(1 - B_c^0 \tfrac{1}{1-R}(e^{-(1-R)(y \wedge h^{0,\text{crit}}(t,x))} - 1)\Big)^{-\frac{1}{R}},$$
$$c^{1,*}(t,x,y) = \begin{cases} c^M(t,x) & \text{if } H_{l,M}^{c,1}(t,x,y) > H_{a,\sim}^{c,1}(t,x,y), \\ \tilde{c}^{1,*}(t,x,y) & \text{if } H_{l,M}^{c,1}(t,x,y) \leq H_{a,\sim}^{c,1}(t,x,y), \end{cases}$$

with

$$h^{0,\text{crit}}(t,x) \triangleq -\tfrac{1}{1-R}\ln((1-R)\lambda^{0,\text{crit}}(t,x) + 1),$$

where

$$\lambda^{0,\mathrm{crit}}(t,x) = \begin{cases} 0 & \text{if } B_c^0 < 0 \text{ and } A^0 \le C^0, \\ \frac{1}{B_c^0}\Big(1 - \big(1 - \frac{(A^0 + B_c^0 c^M(t,x) - C^0)^+}{B_c^0 c^M(t,x)}\big)^{-R}\Big) & \text{if } B_c^0 > 0 \text{ and } A^0 < C^0, \\ & \text{or } B_c^0 < 0 \text{ and } A^0 > C^0, \\ -\infty & \text{if } B_c^0 > 0 \text{ and } A^0 \ge C^0 \end{cases}$$

is the unique solution of

$$\Lambda^{c,0}(t,x,\lambda^{0,\mathrm{crit}}(t,x)) = -(A^0 + B_c^0 c^M(t,x) - C^0)^+ \tag{4.50}$$

if it exists. Otherwise, $\lambda^{0,\mathrm{crit}}(t,x) = -\infty$*. Moreover*

$$c^{0,\mathrm{crit}}(t,x) \triangleq \begin{cases} c^M(t,x) & \text{if } B_c^0 > 0 \text{ and } A^0 \ge C^0, \text{ or } B_c^0 < 0 \text{ and } A^0 \le C^0, \\ c^M(t,x) - \frac{(A^0 + B_c^0 c^M(t,x) - C^0)^+}{B_c^0} & \text{if } B_c^0 > 0 \text{ and } A^0 < C^0, \text{ or } B_c^0 < 0 \text{ and } A^0 > C^0. \end{cases}$$

If $\varepsilon = 0$ *then the maximizer* $c^{i,*}$ *is given by*

$$c^{0,*}(t,x,y) = \begin{cases} 0 & \text{if } y < h^{0,\mathrm{crit}}, \\ c^{0,\mathrm{crit}} & \text{if } y \ge h^{0,\mathrm{crit}}, \end{cases}$$
$$c^{1,*}(t,x,y) = 0$$

where

$$h^{0,\mathrm{crit}} \triangleq -\tfrac{1}{1-R}\ln((1-R)\lambda^{0,\mathrm{crit}} + 1)$$

with

$$\lambda^{0,\mathrm{crit}} = \begin{cases} 0 & \text{if } B_c^0 < 0 \text{ and } A^0 \le C^0, \text{ or } B_c^0 > 0 \text{ and } A^0 < C^0, \\ \frac{1}{B_c^0} & \text{if } B_c^0 < 0 \text{ and } A^0 > C^0, \\ -\infty & \text{if } B_c^0 > 0 \text{ and } A^0 \ge C^0, \end{cases}$$

and

$$c^{0,\mathrm{crit}} \triangleq \begin{cases} 0 & \text{if } B_c^0 > 0 \text{ and } A^0 \ge C^0, \text{ or } B_c^0 < 0 \text{ and } A^0 \le C^0, \\ -\frac{(A^0 - C^0)^+}{B_c^0} & \text{if } B_c^0 > 0 \text{ and } A^0 < C^0, \text{ or } B_c^0 < 0 \text{ and } A^0 > C^0. \end{cases}$$

Remark 4.107. *In case of* $\varepsilon = 0$ *the maximizing consumption rate* $c^{i,*}$ *is just a function of* y*, i.e. the dependency on* t *and* x *vanishes.*

Remark 4.108. *The typical shapes of the functions* $H^{c,i}(t,x,y,\cdot)$*,* $H_l^{c,i}(t,x,y,\cdot)$ *and* $H_a^{c,i}(t,x,y,\cdot)$ *are comparable to those of the functions* $H^{\pi,i}(y,\cdot)$*,* $H_l^{\pi,i}(y,\cdot)$ *and* $H_a^{\pi,i}(y,\cdot)$ *as presented in Figures 4.11, 4.12, i.e.* $H_l^{c,i}(t,x,y,\cdot)$ *and* $H_a^{c,i}(t,x,y,\cdot)$ *are also strictly concave and exhibit a unique maximum. Thus Figures 4.11, 4.12 also serve as helpful illustrations for the proof of Lemma 4.106.*

Proof of Lemma 4.106. The proof is structured as follows. First we deduce the maximizing consumption rate in case of $\varepsilon > 0$ and then continue with the case $\varepsilon = 0$. Let $(t,x,y) \in [0,T] \times \mathbb{R} \times \mathbb{R}_0^+$ be given.

- *Maximizing consumption rate in state* 0 *($\varepsilon > 0$).* Let $\varepsilon > 0$. We first consider the trivial cases $B_c^0 < 0$ and $A^0 \leq C^0$, resp. $B_c^0 > 0$ and $A^0 \geq C^0$, and then go on with the non-trivial cases $B_c^0 < 0$ and $A^0 > C^0$, resp. $B_c^0 > 0$ and $A^0 < C^0$.

  - $B_c^0 < 0$ *and* $A^0 \leq C^0$. In case of $B_c^0 < 0$ and $A^0 \leq C^0$ the intensity function $\vartheta^{0,1}$ is constant on the whole $\mathbb{R}^{\bar{n}} \times \mathbb{R}_0^+$ and $H^{c,0} = H_l^{c,0}$ on $[0,T] \times \mathbb{R} \times \mathbb{R}_0^+ \times \mathbb{R}_0^+$. Thus $H^{c,0}(t,x,y,\cdot)$ exhibits only one maximum that lies at the Merton consumption rate, i.e.

$$B_c^0 < 0 \text{ and } A^0 \leq C^0 \quad \Rightarrow \quad c^{0,*}(t,x,y) = c^M(t,x).$$

  - $B_c^0 > 0$ *and* $A^0 \geq C^0$. If $B_c^0 > 0$ and $A^0 \geq C^0$ then $\vartheta^{0,1}$ is linear on $\mathbb{R}^{\bar{n}} \times \mathbb{R}_0^+$ and $H^{c,0} = H_a^{c,0}$ on $[0,T] \times \mathbb{R} \times \mathbb{R}_0^+ \times \mathbb{R}_0^+$. Again $H^{c,0}(t,x,y,\cdot)$ has only one maximum that now is attained at $\tilde{c}^{0,*}(t,x,y)$, i.e.

$$B_c^0 > 0 \text{ and } A^0 \geq C^0 \quad \Rightarrow \quad c^{0,*}(t,x,y) = \tilde{c}^{0,*}(t,x,y).$$

  - $B_c^0 < 0$ *and* $A^0 > C^0$*, resp.* $B_c^0 > 0$ *and* $A^0 < C^0$. Now $H^{c,0}$ is piecewise given by $H_l^{c,0}$, resp. $H_a^{c,0}$, and we distinguish the two cases $c^M(t,x) \notin \mathfrak{I}^{c,0}$, resp. $c^M(t,x) \in \mathfrak{I}^{c,0}$.

    - $c^M(t,x) \notin \mathfrak{I}^{c,0}$. If $c^M(t,x) \notin \mathfrak{I}^{c,0}$ then by Lemma 4.105, i), also $\tilde{c}^{0,*}(t,x,y) \notin \mathfrak{I}^{c,0}$. This implies that $H^{c,0}(t,x,y,\cdot)$ exhibits only one maximum, namely at $c^M(t,x)$, i.e.

$$c^M(t,x) \notin \mathfrak{I}^{c,0} \quad \Rightarrow \quad c^{0,*}(t,x,y) = c^M(t,x).$$

    - $c^M(t,x) \in \mathfrak{I}^{c,0}$. If $c^M(t,x) \in \mathfrak{I}^{c,0}$ then $H^{c,0}(t,x,y,\cdot)$ possesses again just one maximum that is either the maximum of $H_a^{c,0}(t,x,y,\cdot)$ or it is attained at the critical consumption rate $c^{0,\text{crit}}(t,x)$. The maximizer of $H_a^{c,0}(t,x,y,\cdot)$ is given by $\tilde{c}^{0,*}(t,x,y)$. As long as $\tilde{c}^{0,*}(t,x,y)$ is in the influencing half space it is even the maximizer of $H^{c,0}(t,x,y,\cdot)$. But as soon as $\tilde{c}^{0,*}(t,x,y)$ quits $\mathfrak{I}^{c,0}$ then the critical consumption rate is given by the maximizer of $H^{c,0}(t,x,y,\cdot)$, i.e.

$$c^M(t,x) \in \mathfrak{I}^{c,0} \quad \Rightarrow \quad c^{0,*}(t,x,y) = \begin{cases} \tilde{c}^{0,*}(t,x,y) & \text{if } \tilde{c}^{0,*}(t,x,y) \in \mathfrak{I}^{c,0}, \\ c^{0,\text{crit}}(t,x) & \text{if } \tilde{c}^{0,*}(t,x,y) \notin \mathfrak{I}^{c,0}. \end{cases}$$

  Combining the two cases we get

$$c^{0,*}(t,x,y) = \begin{cases} c^M(t,x) & \text{if } c^M(t,x) \notin \mathfrak{I}^{c,0}, \\ \tilde{c}^{0,*}(t,x,y) & \text{if } c^M(t,x) \in \mathfrak{I}^{c,0} \text{ and } \tilde{c}^{0,*}(t,x,y) \in \mathfrak{I}^{c,0}, \\ c^{0,\text{crit}}(t,x) & \text{if } c^M(t,x) \in \mathfrak{I}^{c,0} \text{ and } \tilde{c}^{0,*}(t,x,y) \notin \mathfrak{I}^{c,0}. \end{cases}$$

  We now have to analyze the conditions under which $\tilde{c}^{0,*}(t,x,y) \in \mathfrak{I}^{c,0}$ in more detail.

$$
\begin{aligned}
& \tilde{c}^{0,*}(t,x,y) \in \mathcal{I}^{c,0} \\
\Leftrightarrow\quad & d^{c,0}(c^M(t,x)) > B_c^0 c^M(t,x)\big(1-(1-B_c^0\tfrac{1}{1-R}(e^{-(1-R)y}-1))^{-\frac{1}{R}}\big) \text{ and } y < h^{\text{crit}} \\
\overset{(*)}{\Leftrightarrow}\quad & (d^{c,0}(c^M(t,x)))^+ > B_c^0 c^M(t,x)\big(1-(1-B_c^0\tfrac{1}{1-R}(e^{-(1-R)y}-1))^{-\frac{1}{R}}\big) \text{ and } y < h^{\text{crit}} \\
\Leftrightarrow\quad & -\tfrac{1}{1-R}(e^{-(1-R)y}-1) < \tfrac{1}{B_c^0}\Big(\Big(1-\tfrac{(d^{c,0}(c^M(t,x)))^+}{B_c^0 c^M(t,x)}\Big)^{-R}-1\Big) \text{ and } y < h^{\text{crit}} \\
\Leftrightarrow\quad & y < -\tfrac{1}{1-R}\ln\Big(-(1-R)\tfrac{1}{B_c^0}\Big(\Big(1-\tfrac{(d^{c,0}(c^M(t,x)))^+}{B_c^0 c^M(t,x)}\Big)^{-R}-1\Big)+1\Big) \text{ and } y < h^{\text{crit}} \\
\Leftrightarrow\quad & y < \Big(-\tfrac{1}{1-R}\ln\Big(-(1-R)\tfrac{1}{B_c^0}\Big(\Big(1-\tfrac{(d^{c,0}(c^M(t,x)))^+}{B_c^0 c^M(t,x)}\Big)^{-R}-1\Big)+1\Big)\wedge h^{\text{crit}}\Big) \\
\Leftrightarrow\quad & y < -\tfrac{1}{1-R}\ln\Big(-(1-R)\tfrac{1}{B_c^0}\Big(\Big(1-\tfrac{(d^{c,0}(c^M(t,x)))^+}{B_c^0 c^M(t,x)}\Big)^{-R}-1\Big)+1\Big) \\
\Leftrightarrow\quad & y < h^{0,\text{crit}}(t,x)
\end{aligned}
$$

where equivalence $(*)$ holds true since the right-hand side of the left inequality is positive. This is true since $0 < (1-B_c^0\frac{1}{1-R}(e^{-(1-R)y}-1))^{-\frac{1}{R}} \leq 1$ ($B_c^0 > 0$), resp. $(1-B_c^0\frac{1}{1-R}(e^{-(1-R)y}-1))^{-\frac{1}{R}} \geq 1$ ($B_c^0 < 0$), as $y \geq 0$. Further, the expression $\Big(1-\frac{(d^{c,0}(c^M(t,x)))^+}{B_c^0 c^M(t,x)}\Big)^{-R}$ is well-defined as we discuss the parameter specifications $B_c^0 < 0$ and $A^0 > C^0$, resp. $B_c^0 > 0$ and $A^0 < C^0$, for which $1-\frac{(d^{c,0}(c^M(t,x)))^+}{B_c^0 c^M(t,x)} > 0$. Finally h^{crit} can be neglected due to the fact that $\Big(1-\frac{(A^0+B_c^0 c^M(t,x)-C^0)^+}{B_c^0 c^M(t,x)}\Big)^{-R} > 0$. Notice further that

$$c^M(t,x) \notin \mathcal{I}^{c,0} \quad \Leftrightarrow \quad h^{0,\text{crit}}(t,x) = 0.$$

Hence

$$c^{0,*}(t,x,y) = \begin{cases} c^M(t,x) & \text{if } c^M(t,x) \notin \mathcal{I}^{c,0}, \\ \tilde{c}^{0,*}(t,x,y) & \text{if } c^M(t,x) \in \mathcal{I}^{c,0} \text{ and } y < h^{0,\text{crit}}(t,x), \\ c^{0,\text{crit}}(t,x) & \text{if } c^M(t,x) \in \mathcal{I}^{c,0} \text{ and } y \geq h^{0,\text{crit}}(t,x), \end{cases}$$

or equivalently

$$c^{0,*}(t,x,y) = \big(\tfrac{\varepsilon\delta}{\varepsilon-(\varepsilon-\delta)e^{-\delta(T-t)}}\big)^{\frac{1}{R}} e^{-\frac{1-R}{R}x}\big(1-B_c^0\tfrac{1}{1-R}(e^{-(1-R)(y\wedge h^{0,\text{crit}}(t,x))}-1)\big)^{-\frac{1}{R}}.$$

Thus

$$
\begin{aligned}
& B_c^0 < 0 \text{ and } A^0 > C^0, \text{ resp. } B_c^0 > 0 \text{ and } A^0 < C^0 \\
\Rightarrow\quad & c^{0,*}(t,x,y) = \big(\tfrac{\varepsilon\delta}{\varepsilon-(\varepsilon-\delta)e^{-\delta(T-t)}}\big)^{\frac{1}{R}} e^{-\frac{1-R}{R}x}\big(1-B_c^0\tfrac{1}{1-R}(e^{-(1-R)(y\wedge h^{0,\text{crit}}(t,x))}-1)\big)^{-\frac{1}{R}}.
\end{aligned}
$$

With $h^{0,\text{crit}}(t,x)$ as defined in the lemma the above formula is also valid in the aforementioned trivial parameter cases.

- *Maximizing consumption rate in state* 1 *($\varepsilon > 0$)*. Again we begin with the trivial case $A^1 < C^1$ and then continue with the non-trivial case $A^1 \geq C^1$.

  - $A^1 < C^1$. In this case the intensity function $\vartheta^{1,0}$ is constant on the whole $\mathbb{R}^{\bar{n}} \times \mathbb{R}_0^+$ since $B_c^1 < 0$. Further $H^{c,1} = H_l^{c,1}$ on $[0,T] \times \mathbb{R} \times \mathbb{R}_0^+ \times \mathbb{R}_0^+$. Thus $H^{c,1}(t,x,y,\cdot)$ exhibits its unique maximum at the Merton rate, i.e.

$$A^1 < C^1 \quad \Rightarrow \quad c^{1,*}(t,x,y) = c^M(t,x).$$

  - $A^1 \geq C^1$. Here we first consider $c^M(t,x) \in \mathcal{I}^{c,1}$ and then $c^M(t,x) \notin \mathcal{I}^{c,1}$.

    - $c^M(t,x) \in \mathcal{I}^{c,1}$. If $c^M(t,x) \in \mathcal{I}^{c,1}$ then by Lemma 4.105, ii), also $\tilde{c}^{1,*}(t,x,y) \in \mathcal{I}^{c,1}$. Hence $H^{c,1}(t,x,y,\cdot)$ exhibits its unique maximum at $\tilde{c}^{1,*}(t,x,y)$, i.e.

$$c^M(t,x) \in \mathcal{I}^{c,1} \quad \Rightarrow \quad c^{1,*}(t,x,y) = \tilde{c}^{1,*}(t,x,y).$$

    - $c^M(t,x) \notin \mathcal{I}^{c,1}$. If $c^M(t,x) \notin \mathcal{I}^{c,1}$ then $H^{c,1}(t,x,y,\cdot)$ may possess two local maxima; the maximum of $H_l^{c,1}(t,x,y,\cdot)$ and the one of $H_a^{c,1}(t,x,y,\cdot)$, given by $c^M(t,x)$, resp. $\tilde{c}^{1,*}(t,x,y)$. As long as $\tilde{c}^{1,*}(t,x,y)$ does not lie in the influencing half space then $H^{c,1}(t,x,y,\cdot)$ has only one maximum, namely at $c^M(t,x)$. But as soon as $\tilde{c}^{1,*}(t,x,y)$ enters the influencing half space then $H^{c,1}(t,x,y,\cdot)$ exhibits the two local maxima mentioned above which have to be compared in order to find the global maximum, i.e.

$$c^M(t,x) \notin \mathcal{I}^{c,1} \;\Rightarrow\; c^{1,*}(t,x,y) = \begin{cases} c^M(t,x) & \text{if } \tilde{c}^{1,*}(t,x,y) \notin \mathcal{I}^{c,1} \text{ or } [\tilde{c}^{1,*}(t,x,y) \in \mathcal{I}^{c,1} \\ & \text{and } H_{l,M}^{c,1}(t,x,y) > H_{a,\sim}^{c,1}(t,x,y)], \\ \tilde{c}^{1,*}(t,x,y) & \text{if } \tilde{c}^{1,*}(t,x,y) \in \mathcal{I}^{c,1} \\ & \text{and } H_{l,M}^{c,1}(t,x,y) \leq H_{a,\sim}^{c,1}(t,x,y). \end{cases}$$

  Combining the two cases we arrive at

$$c^{1,*}(t,x,y) = \begin{cases} c^M(t,x) & \text{if } c^M(t,x) \notin \mathcal{I}^{c,1} \text{ and } \big[\tilde{c}^{1,*}(t,x,y) \notin \mathcal{I}^{c,1} \\ & \text{or } [\tilde{c}^{1,*}(t,x,y) \in \mathcal{I}^{c,1} \text{ and } H_{l,M}^{c,1}(t,x,y) > H_{a,\sim}^{c,1}(t,x,y)]\big], \\ \tilde{c}^{1,*}(t,x,y) & \text{if } c^M(t,x) \in \mathcal{I}^{c,1} \text{ or } \big[c^M(t,x) \notin \mathcal{I}^{c,1} \\ & \text{and } [\tilde{c}^{1,*}(t,x,y) \in \mathcal{I}^{c,1} \text{ and } H_{l,M}^{c,1}(t,x,y) \leq H_{a,\sim}^{c,1}(t,x,y)]\big]. \end{cases}$$

  To simplify this we have a look at the two conditions. Firstly

$$\begin{aligned} & c^M(t,x) \notin \mathcal{I}^{c,1} \text{ and } \big[\tilde{c}^{1,*}(t,x,y) \notin \mathcal{I}^{c,1} \\ & \quad \text{or } [\tilde{c}^{1,*}(t,x,y) \in \mathcal{I}^{c,1} \text{ and } H_{l,M}^{c,1}(t,x,y) > H_{a,\sim}^{c,1}(t,x,y)]\big] \\ \Leftrightarrow \quad & c^M(t,x) \notin \mathcal{I}^{c,1} \text{ and } [\tilde{c}^{1,*}(t,x,y) \notin \mathcal{I}^{c,1} \text{ or } H_{l,M}^{c,1}(t,x,y) > H_{a,\sim}^{c,1}(t,x,y)] \\ \overset{L.\ 4.105,\ ii)}{\Leftrightarrow} \quad & c^M(t,x) \notin \mathcal{I}^{c,1} \text{ and } H_{l,M}^{c,1}(t,x,y) > H_{a,\sim}^{c,1}(t,x,y) \\ \overset{L.\ 4.105,\ ii)}{\Leftrightarrow} \quad & H_{l,M}^{c,1}(t,x,y) > H_{a,\sim}^{c,1}(t,x,y) \end{aligned}$$

Secondly

$$\begin{aligned} & c^M(t,x) \in \mathcal{J}^{c,1} \text{ or } \big[c^M(t,x) \notin \mathcal{J}^{c,1} \\ & \quad \text{and } [\tilde{c}^{1,*}(t,x,y) \in \mathcal{J}^{c,1} \text{ and } H^{c,1}_{l,M}(t,x,y) \le H^{c,1}_{a,\sim}(t,x,y)]\big] \\ \Leftrightarrow \quad & c^M(t,x) \in \mathcal{J}^{c,1} \text{ or } [\tilde{c}^{1,*}(t,x,y) \in \mathcal{J}^{c,1} \text{ and } H^{c,1}_{l,M}(t,x,y) \le H^{c,1}_{a,\sim}(t,x,y)] \\ \overset{L.\ 4.105,\ ii)}{\Leftrightarrow} \quad & c^M(t,x) \in \mathcal{J}^{c,1} \text{ or } H^{c,1}_{l,M}(t,x,y) \le H^{c,1}_{a,\sim}(t,x,y) \\ \overset{L.\ 4.105,\ ii)}{\Leftrightarrow} \quad & H^{c,1}_{l,M}(t,x,y) \le H^{c,1}_{a,\sim}(t,x,y) \end{aligned}$$

Hence we get

$$c^{1,*}(t,x,y) = \begin{cases} c^M(t,x) & \text{if } H^{c,1}_{l,M}(t,x,y) > H^{c,1}_{a,\sim}(t,x,y), \\ \tilde{c}^{1,*}(t,x,y) & \text{if } H^{c,1}_{l,M}(t,x,y) \le H^{c,1}_{a,\sim}(t,x,y). \end{cases}$$

Some simple transformations show

$$\begin{aligned} & H^{c,1}_{l,M}(t,x,y) \le H^{c,1}_{a,\sim}(t,x,y) \\ \Leftrightarrow \quad & \Big(-\tfrac{R}{1-R}\Big(\big(1 - B^1_c \tfrac{1}{1-R}(e^{(1-R)y}-1)\big)^{-\frac{1-R}{R}} - 1\Big) + B^1_c \tfrac{1}{1-R}(e^{(1-R)y}-1)\Big) c^M(t,x) \\ & \quad \le d^{c,1}(c^M(t,x)) \tfrac{1}{1-R}(e^{(1-R)y}-1) \\ \overset{(*)}{\Leftrightarrow} \quad & \Big(-\tfrac{R}{1-R}\Big(\big(1 - B^1_c \tfrac{1}{1-R}(e^{(1-R)y}-1)\big)^{-\frac{1-R}{R}} - 1\Big) + B^1_c \tfrac{1}{1-R}(e^{(1-R)y}-1)\Big) c^M(t,x) \\ & \quad \le -(d^{c,1}(c^M(t,x)))^{-} \tfrac{1}{1-R}(e^{(1-R)y}-1) \end{aligned}$$

where equivalence $(*)$ is satisfied because the left-hand side of the inequality is non-positive. Unfortunately, we cannot derive an explicit condition on y; not even in the logarithmic case $R=1$. Therefore, we stick to the condition $H^{c,1}_{l,M}(t,x,y) \le H^{c,1}_{a,\sim}(t,x,y)$. However $H^{c,1}_{l,M}(t,x,y) \le H^{c,1}_{a,\sim}(t,x,y)$ is trivially satisfied if $c^M(t,x) \in \mathcal{J}^{c,1}$, i.e.

$$c^M(t,x) \in \mathcal{J}^{c,1} \quad \Rightarrow \quad H^{c,1}_{l,M}(t,x,y) \le H^{c,1}_{a,\sim}(t,x,y).$$

Hence we get

$$A^1 \ge C^1 \quad \Rightarrow \quad c^{1,*}(t,x,y) = \begin{cases} c^M(t,x) & \text{if } H^{c,1}_{l,M}(t,x,y) > H^{c,1}_{a,\sim}(t,x,y), \\ \tilde{c}^{1,*}(t,x,y) & \text{if } H^{c,1}_{l,M}(t,x,y) \le H^{c,1}_{a,\sim}(t,x,y). \end{cases}$$

Notice that the trivial parameter specification $A^1 < C^1$ in which $c^{1,*}(t,x,y) = c^M(t,x)$ is covered, too, since then

$$\begin{aligned} & H^{c,1}_{l,M}(t,x,y) \le H^{c,1}_{a,\sim}(t,x,y) \\ \Leftrightarrow \quad & -\tfrac{R}{1-R}\Big(\big(1 - B^1_c \tfrac{1}{1-R}(e^{(1-R)y}-1)\big)^{-\frac{1-R}{R}} - 1\Big) c^M(t,x) \le (A^1 - C^1)\tfrac{1}{1-R}(e^{(1-R)y}-1) \end{aligned}$$

where the left-hand side is now positive. Hence, if $A^1 < C^1$ then $H^{c,1}_{l,M}(t,x,y) \le H^{c,1}_{a,\sim}(t,x,y)$ is not valid as $(A^1 - C^1)\frac{1}{1-R}(e^{(1-R)y}-1) \le 0$, except for $y=0$.

We now proceed with the case $\varepsilon = 0$ in which the Merton consumption rate vanishes, i.e. $c^M(t,x) = 0$ for all $(t,x) \in [0,T] \times \mathbb{R}$.

- *Maximizing consumption rate in state 0 ($\varepsilon = 0$).* Let now $\varepsilon = 0$. Again we first look at the trivial parameter specifications $B_c^0 < 0$ and $A^0 \le C^0$, resp. $B_c^0 > 0$ and $A^0 \ge C^0$.

  - *$B_c^0 < 0$ and $A^0 \le C^0$.* If $B_c^0 < 0$ and $A^0 \le C^0$ then none of the admissible consumption rates $c \in \mathbb{R}_0^+$ lie in $\mathcal{I}^{c,0}$, i.e. $H^{c,0} = H_l^{c,0}$ on $[0,T] \times \mathbb{R} \times \mathbb{R}_0^+ \times \mathbb{R}_0^+$ and thus the Merton rate maximizes $H^{c,0}(t,x,y,\cdot)$, i.e.
$$B_c^0 < 0 \text{ and } A^0 \le C^0 \quad \Rightarrow \quad c^{0,*}(t,x,y) = 0.$$
  - *$B_c^0 > 0$ and $A^0 \ge C^0$.* If $B_c^0 > 0$ and $A^0 \ge C^0$ then the intensity function $\vartheta^{0,1}$ is linear on $\mathbb{R}^{\bar{n}} \times \mathbb{R}_0^+$ and $H^{c,0} = H_a^{c,0}$ on $[0,T] \times \mathbb{R} \times \mathbb{R}_0^+ \times \mathbb{R}_0^+$. Consequently the maximizer is given by $\tilde{c}^{0,*}(t,x,y) = 0$ since $\tilde{c}^{0,*}(t,x,y) = 0$ for all $(t,x,y) \in [0,T] \times \mathbb{R} \times \mathbb{R}_0^+$ if $B_c^0 > 0$, i.e.
$$B_c^0 > 0 \text{ and } A^0 \ge C^0 \quad \Rightarrow \quad c^{0,*}(t,x,y) = 0.$$
  - *$B_c^0 < 0$ and $A^0 > C^0$, resp. $B_c^0 > 0$ and $A^0 < C^0$.* As $c^M(t,x) = 0$ we distinguish the cases $0 \notin \mathcal{I}^{c,0}$ and $0 \in \mathcal{I}^{c,0}$.

    - $0 \notin \mathcal{I}^{c,0}$. If $0 \notin \mathcal{I}^{c,0}$ then by Lemma 4.105, i), also $\tilde{c}^{0,*}(t,x,y) \notin \mathcal{I}^{c,0}$. Thus $H^{c,0}(t,x,y,\cdot)$ exhibits its unique maximum at $c^M(t,x) = 0$, i.e.
$$0 \notin \mathcal{I}^{c,0} \quad \Rightarrow \quad c^{0,*}(t,x,y) = 0.$$
    - $0 \in \mathcal{I}^{c,0}$. If $0 \in \mathcal{I}^{c,0}$ then $c^{0,*}(t,x,y) = \tilde{c}^{0,*}(t,x,y)$ as long as $\tilde{c}^{0,*}(t,x,y) \in \mathcal{I}^{c,0}$ and $c^{0,*}(t,x,y) = c^{0,\text{crit}}$ as soon as $\tilde{c}^{0,*}(t,x,y) \notin \mathcal{I}^{c,0}$, i.e.
$$0 \in \mathcal{I}^{c,0} \quad \Rightarrow \quad c^{0,*}(t,x,y) = \begin{cases} \tilde{c}^{0,*}(t,x,y) & \text{if } \tilde{c}^{0,*}(t,x,y) \in \mathcal{I}^{c,0}, \\ c^{0,\text{crit}} & \text{if } \tilde{c}^{0,*}(t,x,y) \notin \mathcal{I}^{c,0}. \end{cases}$$

  Combining the two cases we get
$$c^{0,*}(t,x,y) = \begin{cases} 0 & \text{if } 0 \notin \mathcal{I}^{c,0}, \\ \tilde{c}^{0,*}(t,x,y) & \text{if } 0 \in \mathcal{I}^{c,0} \text{ and } \tilde{c}^{0,*}(t,x,y) \in \mathcal{I}^{c,0}, \\ c^{0,\text{crit}} & \text{if } 0 \in \mathcal{I}^{c,0} \text{ and } \tilde{c}^{0,*}(t,x,y) \notin \mathcal{I}^{c,0}. \end{cases}$$

  Therefore, we have to analyze the conditions under which $\tilde{c}^{0,*}(t,x,y) \in \mathcal{I}^{c,0}$ in more detail. Since
$$\tilde{c}^{0,*}(t,x,y) \triangleq \begin{cases} 0 & \text{if } y < h^{\text{crit}}, \\ \infty & \text{if } y \ge h^{\text{crit}}, \end{cases}$$
  it follows that
$$\tilde{c}^{0,*}(t,x,y) \in \mathcal{I}^{c,0} \quad \Leftrightarrow \quad A^0 - C^0 > 0 \text{ and } y < h^{\text{crit}}.$$
  Hence with $h^{0,\text{crit}}$ and $c^{0,\text{crit}}$ we get
$$B_c^0 < 0 \text{ and } A^0 > C^0, \text{ resp. } B_c^0 > 0 \text{ and } A^0 < C^0 \quad \Rightarrow \quad c^{0,*}(t,x,y) = \begin{cases} 0 & \text{if } y < h^{0,\text{crit}}, \\ c^{0,\text{crit}} & \text{if } y \ge h^{0,\text{crit}}. \end{cases}$$

The formula above also covers the trivial parameter specifications.

- *Maximizing consumption rate in state* 1 *($\varepsilon = 0$).* The maximizing consumption rate in state 1 is given by $c^{1,*}(t,x,y) = 0$. If $A^1 < C^1$ then $\vartheta^{1,0}$ is constant on the whole $\mathbb{R}^{\bar{n}} \times \mathbb{R}_0^+$ and $H^{c,1} = H_l^{c,1}$ on $[0,T] \times \mathbb{R} \times \mathbb{R}_0^+ \times \mathbb{R}_0^+$ implying $c^{1,*}(t,x,y) = c^M(t,x) = 0$. Further if $A^1 \geq C^1$ then $0 \in \mathfrak{I}^{c,1}$ which in analogy to the case of $\varepsilon > 0$ yields that $c^{1,*}(t,x,y) = \tilde{c}^{1,*}(t,x,y) = 0$.

Thus the lemma is proven. □

Remark 4.109 (Interpretation)**.** *Whenever the Merton consumption rate in state* 0 *is not part of the influencing half space then the maximizing consumption rate coincides with the Merton rate. However, if the Merton consumption lies within the influencing half space then the maximizing consumption rate may deviate from the Merton rate.*

In state 1 *things are different. If $\varepsilon > 0$ then the maximizing consumption rate coincides with the Merton consumption rate if the Merton rate is not lying within the influencing half space and either $\tilde{c}^{1,*} \notin \mathfrak{I}^{c,1}$ or $\tilde{c}^{1,*} \in \mathfrak{I}^{c,1}$ but $H_{l,M}^{c,1}(t,x,y) > H_{a,\sim}^{c,1}(t,x,y)$. Otherwise the maximizing consumption rate deviates from the Merton rate. In case of $\varepsilon = 0$ the maximizing consumption rate equals the Merton rate.*

Remark 4.110 (Continuity of $c^{0,*}$ vs. discontinuity of $c^{1,*}$ ($\varepsilon > 0$))**.** *The special form of the maximizing strategy in state* 0 *together with the continuity of $\lambda^{0,\mathrm{crit}}$ imply that $c^{0,*}$ is a continuous function in t, x and y.*

However in state 1 *the maximizing strategy exhibits a discontinuity at all $(\hat{t}, \hat{x}, \hat{y})$ satisfying $H_{l,M}^{c,1}(\hat{t}, \hat{x}, \hat{y}) = H_{a,\sim}^{c,1}(\hat{t}, \hat{x}, \hat{y})$ and $c^M(\hat{t}, \hat{x}) \neq \tilde{c}^{1,*}(\hat{t}, \hat{x}, \hat{y})$ where $c^{1,*}$ jumps from $c^M(\hat{t}, \hat{x})$ to $\tilde{c}^{1,*}(\hat{t}, \hat{x}, \hat{y})$.*

Remark 4.111 (Discontinuity of $c^{0,*}$ vs. continuity of $c^{1,*}$ ($\varepsilon = 0$))**.** *If $\varepsilon = 0$ then $c^{0,*}$ may be discontinuous in y. Namely if $B_c^0 < 0$ and $A^0 > C^0$ then $c^{0,*}$ exhibits a jump at $y = h^{0,\mathrm{crit}}$ where it jumps from* 0 *to $c^{0,\mathrm{crit}}$*

In state 1 *the maximizing strategy $c^{1,*}$ is obviously continuous in y if $\varepsilon = 0$.*

Remark 4.112 (Negativity of B_c^1)**.** *We do not allow B_c^1 to take on positive values for the following reason. If B_c^1 was positive then similarly to state* 0 *the maximizer of $H_a^{c,1}(t,x,y,\cdot)$ would be ∞ as soon as $y \geq \frac{1}{1-R} \ln((1-R)\frac{1}{B_c^1} + 1)$. In state* 0 *this does not cause any problem as before the maximizer $c^{0,*}$ becomes ∞ it hits the critical value $c^{0,\mathrm{crit}}$ and does not overshoot it. This is reasonable as from $c^{0,\mathrm{crit}}$ onwards the intensity of a jump to state* 1 *will not get any smaller so that there is no reason for the investor to further deviate from the Merton rate. But in state* 1 *things are different. If $B_c^1 > 0$ then the intensity for a jump to the better state* 0 *could get infinitely large as $\vartheta^{1,0}$ would be unbounded from above and there would not be any potential barrier for the maximizer $c^{1,*}$. Hence whenever the market was in state* 1 *and y was large enough then it would be optimal for the investor to consume at an infinitely large rate for an infinitesimal short time thus guaranteeing the jump back to state* 0*. But this strategy would not be admissible. Therefore, $B_c^1 > 0$ has to be prohibited.*

Remark 4.113 (Interpretation in case of $\varepsilon = 0$)**.** *Choosing $\varepsilon = 0$ represents a model in which the investor does not draw any utility from intermediate consumption at least concerning the direct impact of consumption on the utility functions. In the standard model without the investor's influence the optimal consumption rate consequently is* 0*. In our model this is different. In the special case of $B_c^0 < 0$ and $A^0 > C^0$ it is possible that the maximizing consumption rate is strictly positive. This happens if $y \geq h^{0,\mathrm{crit}} = -\frac{1}{1-R} \ln\big((1-R)\frac{1}{B_c^0} + 1\big)$. Thus, the large investor consumes although this is not beneficial in terms of the utility from intermediate consumption. The reason for this behavior is that except for* *the case of $B_c^0 < 0$ and $A^0 > C^0$ a consumption rate of* 0 *yields the smallest possible intensity of a jump to the adverse market state. Only if $B_c^0 < 0$ and $A^0 > C^0$ then consuming at a rate of $-\frac{A^0 - C^0}{B_c^0}$ yields a smaller intensity than consuming at a rate of* 0*. Therefore, in order to reduce the probability of a jump to the adverse market state, the large investor may consume.*

So far the above explanations are relevant for the general maximizing strategy $c^{0,*}(t,x,y)$ and not necessarily for the optimal strategy. This is because it is not clear whether the situation $h(t) \geq h^{0,\text{crit}}$ – remember that the maximizing strategy of the HJB-system (4.48) is obtained by choosing $y = h(t)$ – could really occur. But later on we will provide an example showing that the aforementioned situation is really possible.

The following lemma provides a link between the maximizing consumption rate in case of $\varepsilon > 0$ and $\varepsilon = 0$.

Lemma 4.114 (Limiting behavior of $c^{0,\text{crit}}$, $h^{0,\text{crit}}$ and $c^{i,*}$ as ε tends to 0)**.** *Denote by $c_{\varepsilon>0}^{0,\text{crit}}$, $h_{\varepsilon>0}^{0,\text{crit}}$ and $c_{\varepsilon>0}^{i,*}$, resp. $c_{\varepsilon=0}^{0,\text{crit}}$, $h_{\varepsilon=0}^{0,\text{crit}}$ and $c_{\varepsilon=0}^{i,*}$, the critical consumption rate, the critical barrier and the maximizing consumption rate in case of $\varepsilon > 0$, resp. $\varepsilon = 0$. Then the following holds true for every $(t,x,y) \in [0,T] \times \mathbb{R} \times \mathbb{R}_0^+$.*

i) $\lim_{\varepsilon \to 0} c_{\varepsilon>0}^{0,\text{crit}}(t,x) = c_{\varepsilon=0}^{0,\text{crit}}$.

ii) $\lim_{\varepsilon \to 0} h_{\varepsilon>0}^{0,\text{crit}}(t,x) = h_{\varepsilon=0}^{0,\text{crit}}$.

iii) $\lim_{\varepsilon \to 0} c_{\varepsilon>0}^{i,*}(t,x,y) = c_{\varepsilon=0}^{i,*}(t,x,y)$.

Proof. Let $(t,x,y) \in [0,T] \times \mathbb{R} \times \mathbb{R}_0^+$ be fixed and notice that

$$\lim_{\varepsilon \to 0} c^M(t,x) = 0 \text{ for every } (t,x) \in [0,T] \times \mathbb{R}. \qquad (*)$$

ad i) The assertion is a direct consequence of $(*)$.

ad ii) Again $(*)$ implies $\lim_{\varepsilon \to 0} \lambda_{\varepsilon>0}^{0,\text{crit}}(t,x) = \lambda_{\varepsilon=0}^{0,\text{crit}}$ which yields the assertion.

ad iii) The assertions i) and ii) together with $(*)$ yield $\lim_{\varepsilon \to 0} c_{\varepsilon>0}^{0,*}(t,x,y) = c_{\varepsilon=0}^{0,*}(t,x,y)$. Moreover since $\lim_{\varepsilon \to 0} \tilde{c}_{\varepsilon>0}^{1,*}(t,x,y) = 0 = \tilde{c}_{\varepsilon=0}^{1,*}(t,x,y)$ we also get $\lim_{\varepsilon \to 0} c_{\varepsilon>0}^{1,*}(t,x,y) = c_{\varepsilon=0}^{1,*}(t,x,y)$. □

Remark 4.115. *So far the strategy $(\pi^{i,*}, c^{i,*})$ is only a candidate solution for the optimal investment problem. To verify the optimality we need to show that the related HJB-system has a global solution that satisfies the required regularity conditions stated in the Verification Theorem 3.3.*

Having established the maximizers of the HJB-system in general form inserting $(\pi^{i,*}, c^{i,*}(t, g(t) - 1_{\{i=1\}}h(t), h(t)))$ the reduced HJB-equation (4.48) now becomes a system of coupled backward ODEs which for $\varepsilon > 0$ is given by

$$
\begin{aligned}
h'(t) =& -(\Psi^0 - \Psi^1) + \big(\tfrac{\varepsilon\delta}{\varepsilon-(\varepsilon-\delta)e^{-\delta(T-t)}}\big)^{\frac{1}{R}} e^{-\frac{1-R}{R}g(t)} \tfrac{R}{1-R}\big(e^{\frac{1-R}{R}h(t)} - 1\big) \\
&- C^0 \tfrac{1}{1-R}(e^{-(1-R)h(t)} - 1) + C^1 \tfrac{1}{1-R}(e^{(1-R)h(t)} - 1) \\
&- \Big[(A^0 + B_c^0 c^M(t, g(t)) - C^0)\tfrac{1}{1-R}(e^{-(1-R)(h(t)\wedge h^{0,\mathrm{crit}}(t,g(t)))} - 1) \\
&\qquad + \Big(\tfrac{R}{1-R}\big((1 - B_c^0 \tfrac{1}{1-R}(e^{-(1-R)(h(t)\wedge h^{0,\mathrm{crit}}(t,g(t)))} - 1))^{-\frac{1-R}{R}} - 1\big) \\
&\qquad\qquad - B_c^0 \tfrac{1}{1-R}(e^{-(1-R)(h(t)\wedge h^{0,\mathrm{crit}}(t,g(t)))} - 1)\Big) c^M(t, g(t))\Big] \\
&+ \Big[(A^1 + B_c^1 c^M(t, g(t) - h(t)) - C^1)\tfrac{1}{1-R}(e^{(1-R)h(t)} - 1) \\
&\qquad + \Big(\tfrac{R}{1-R}\big((1 - B_c^1 \tfrac{1}{1-R}(e^{(1-R)h(t)} - 1))^{-\frac{1-R}{R}} - 1\big) \\
&\qquad\qquad - B_c^1 \tfrac{1}{1-R}(e^{(1-R)h(t)} - 1)\Big) c^M(t, g(t) - h(t))\Big]^+,
\end{aligned}
\tag{4.51}
$$

$$
\begin{aligned}
g'(t) =& -\Psi^0 + \tfrac{\varepsilon\delta}{\varepsilon-(\varepsilon-\delta)e^{-\delta(T-t)}}\Big(1 - \tfrac{R}{1-R}\big(\big(\tfrac{\varepsilon\delta}{\varepsilon-(\varepsilon-\delta)e^{-\delta(T-t)}}\big)^{\frac{1-R}{R}} e^{-\frac{1-R}{R}g(t)} - 1\big)\Big) \\
&- C^0 \tfrac{1}{1-R}(e^{-(1-R)h(t)} - 1) \\
&- \Big[(A^0 + B_c^0 c^M(t, g(t)) - C^0)\tfrac{1}{1-R}(e^{-(1-R)(h(t)\wedge h^{0,\mathrm{crit}}(t,g(t)))} - 1) \\
&\qquad + \Big(\tfrac{R}{1-R}\big((1 - B_c^0 \tfrac{1}{1-R}(e^{-(1-R)(h(t)\wedge h^{0,\mathrm{crit}}(t,g(t)))} - 1))^{-\frac{1-R}{R}} - 1\big) \\
&\qquad\qquad - B_c^0 \tfrac{1}{1-R}(e^{-(1-R)(h(t)\wedge h^{0,\mathrm{crit}}(t,g(t)))} - 1)\Big) c^M(t, g(t))\Big]
\end{aligned}
\tag{4.52}
$$

subject to the boundary conditions

$$g(T) = 0, \;\; h(T) = 0. \tag{4.53}$$

If $\varepsilon = 0$ then the ODE-system reads

$$
\begin{aligned}
h'(t) =& -(\Psi^0 - \Psi^1) \\
&- C^0 \tfrac{1}{1-R}(e^{-(1-R)h(t)} - 1) + C^1 \tfrac{1}{1-R}(e^{(1-R)h(t)} - 1) \\
&- (A^0 - C^0)\tfrac{1}{1-R}(e^{-(1-R)(h(t)\wedge h^{0,\mathrm{crit}})} - 1) \\
&+ (A^1 - C^1)^+ \tfrac{1}{1-R}(e^{(1-R)h(t)} - 1),
\end{aligned}
\tag{4.54}
$$

$$
\begin{aligned}
g'(t) =& -\Psi^0 \\
&- C^0 \tfrac{1}{1-R}(e^{-(1-R)h(t)} - 1) \\
&- (A^0 - C^0)\tfrac{1}{1-R}(e^{-(1-R)(h(t)\wedge h^{0,\mathrm{crit}})} - 1)
\end{aligned}
\tag{4.55}
$$

subject to the boundary conditions

$$g(T) = 0, \;\; h(T) = 0. \tag{4.56}$$

Remark 4.116. *From the proof of Lemma 4.106 we know that for* $(t,x,y) \in [0,T] \times \mathbb{R} \times \mathbb{R}_0^+$

$$\begin{aligned} & H_{l,M}^{c,1}(t,x,y) \leq H_{a,\sim}^{c,1}(t,x,y) \\ \Leftrightarrow \quad & \Big(-\tfrac{R}{1-R}\Big(\big(1 - B_c^1 \tfrac{1}{1-R}(e^{(1-R)y} - 1)\big)^{-\frac{1-R}{R}} - 1\Big) + B_c^1 \tfrac{1}{1-R}(e^{(1-R)y} - 1)\Big) c^M(t,x) \\ & \leq d^{c,1}(c^M(t,x)) \tfrac{1}{1-R}(e^{(1-R)y} - 1). \end{aligned}$$

Remark 4.117. *The aforementioned ODEs include the ODEs of the classical Merton problem (first rows) and the ODEs of the Bäuerle-Rieder problem with constant regime shift intensities* C^i *(first two rows).*

The following lemma shows that the above ODE-system exhibits a unique global solution.

Lemma 4.118. *The ODE-system given by equation (4.51) and (4.52) subject to the boundary conditions (4.53) ($\varepsilon > 0$), resp. (4.54) and (4.55) subject to the boundary conditions (4.56) ($\varepsilon = 0$), admits a unique global solution.*

Proof of Lemma 4.118. The proof is essentially the same as the proof of Lemma 4.9. Only the χ^i's are different. Therefore we just present the actual χ^i's and verify that those satisfy the necessary continuity conditions and the non-negativity condition. However in the consumption-dependent case we have to distinguish the cases $\varepsilon > 0$ and $\varepsilon = 0$. So we first give the χ^i's and their properties for $\varepsilon > 0$ and then for $\varepsilon = 0$.

- *Definition of the χ^i's ($\varepsilon > 0$).* In case of $\varepsilon > 0$ the χ^i's are given by $\chi^i : \ [0,T] \times \mathbb{R} \times \mathbb{R}_0^+ \to \mathbb{R}$, $i = 0,1$, with

$$\begin{aligned} \chi^0(t,x,y) \triangleq & - C^0 \tfrac{1}{1-R}(e^{-(1-R)y} - 1) - \Big[d^{c,0}(c^M(t,x)) \tfrac{1}{1-R}(e^{-(1-R)(y \wedge h^{0,\mathrm{crit}}(t,x))} - 1) \\ & + \Big(\tfrac{R}{1-R}\big((1 - B_c^0 \tfrac{1}{1-R}(e^{-(1-R)(y \wedge h^{0,\mathrm{crit}}(t,x))} - 1))^{-\frac{1-R}{R}} - 1\big) \\ & \qquad - B_c^0 \tfrac{1}{1-R}(e^{-(1-R)(y \wedge h^{0,\mathrm{crit}}(t,x))} - 1)\Big) c^M(t,x)\Big], \\ \chi^1(t,x,y) \triangleq & \ C^1 \tfrac{1}{1-R}(e^{(1-R)y} - 1) + \Big[d^{c,1}(c^M(t,x)) \tfrac{1}{1-R}(e^{(1-R)y} - 1) \\ & + \Big(\tfrac{R}{1-R}\big((1 - B_c^1 \tfrac{1}{1-R}(e^{(1-R)y} - 1))^{-\frac{1-R}{R}} - 1\big) - B_c^1 \tfrac{1}{1-R}(e^{(1-R)y} - 1)\Big) c^M(t,x)\Big]^+. \end{aligned}$$

- *Continuity results on the χ^i's ($\varepsilon > 0$).* The χ^i's are continuous in t and further, as compositions of continuously differentiable functions in x and y, the $[\,\cdot\,]^+$- and the $\min\{\cdot,\cdot\}$-function locally Lipschitz continuous in x and y.

- *Non-negativity of the χ^i's ($\varepsilon > 0$).* Obviously, $\chi^1(t,x,y) \geq 0$ for every $(t,x,y) \in [0,T] \times \mathbb{R} \times \mathbb{R}_0^+$. To see that $\chi^0(t,x,y) \geq 0$ for every $(t,x,y) \in [0,T] \times \mathbb{R} \times \mathbb{R}_0^+$ we distinguish the three cases that already appeared in the proof of Lemma 4.106.

  - $B_c^0 < 0$ *and* $A^0 \leq C^0$. If $B_c^0 < 0$ and $A^0 \leq C^0$ then $h^{0,\mathrm{crit}}(t,x) = 0$ for every $(t,x) \in [0,T] \times \mathbb{R}$ and hence $\chi^0(t,x,y) = -C^0 \frac{1}{1-R}(e^{-(1-R)y} - 1)$ which is non-negative as $y \geq 0$.

- $B_c^0 > 0$ *and* $A^0 \geq C^0$. If $B_c^0 > 0$ and $A^0 \geq C^0$ then $h^{0,\text{crit}}(t,x) = \infty$ for every $(t,x) \in [0,T] \times \mathbb{R}$ and hence

$$\begin{aligned}\chi^0(t,x,y) = &- C^0 \tfrac{1}{1-R}(e^{-(1-R)y} - 1) - d^{c,0}(c^M(t,x)) \tfrac{1}{1-R}(e^{-(1-R)y} - 1) \\ &- \Big(\tfrac{R}{1-R}\big((1 - B_c^0 \tfrac{1}{1-R}(e^{-(1-R)y} - 1))^{-\frac{1-R}{R}} - 1\big) - B_c^0 \tfrac{1}{1-R}(e^{-(1-R)y} - 1)\Big) c^M(t,x).\end{aligned}$$

Further, $B_c^0 > 0$ and $A^0 \geq C^0$ implies that $\tilde{c}^{0,*}(t,x,y) \in \mathcal{I}^{c,0}$ which by Lemma 4.105, i), yields that $H_{l,M}^{c,0}(t,x,y) > H_{a,\sim}^{c,0}(t,x,y)$. But this is equivalent to

$$\begin{aligned}0 < &- d^{c,0}(c^M(t,x)) \tfrac{1}{1-R}(e^{-(1-R)y} - 1) \\ &- \Big(\tfrac{R}{1-R}\big((1 - B_c^0 \tfrac{1}{1-R}(e^{-(1-R)y} - 1))^{-\frac{1-R}{R}} - 1\big) - B_c^0 \tfrac{1}{1-R}(e^{-(1-R)y} - 1)\Big) c^M(t,x),\end{aligned}$$

so that $\chi^0(t,x,y) > 0$.

- $B_c^0 < 0$ *and* $A^0 > C^0$, *resp.* $B_c^0 > 0$ *and* $A^0 < C^0$. We distinguish the cases $c^M(t,x) \notin \mathcal{I}^{c,0}$ and $c^M(t,x) \in \mathcal{I}^{c,0}$.

  - $c^M(t,x) \notin \mathcal{I}^{c,0}$. If $c^M(t,x) \notin \mathcal{I}^{c,0}$ then $h^{0,\text{crit}}(t,x) = 0$ and therefore we get $\chi^0(t,x,y) = -C^0 \frac{1}{1-R}(e^{-(1-R)y} - 1) \geq 0$ as $y \geq 0$.
  - $c^M(t,x) \in \mathcal{I}^{c,0}$. If $c^M(t,x) \in \mathcal{I}^{c,0}$ then $h^{0,\text{crit}}(t,x)$ is strictly positive and we have to distinguish the cases $y < h^{0,\text{crit}}(t,x)$ and $y \geq h^{0,\text{crit}}(t,x)$.

    - $y < h^{0,\text{crit}}(t,x)$. If $y < h^{0,\text{crit}}(t,x)$ then

$$\begin{aligned}\chi^0(t,x,y) = &- C^0 \tfrac{1}{1-R}(e^{-(1-R)y} - 1) - d^{c,0}(c^M(t,x)) \tfrac{1}{1-R}(e^{-(1-R)y} - 1) \\ &- \Big(\tfrac{R}{1-R}\big((1 - B_c^0 \tfrac{1}{1-R}(e^{-(1-R)y} - 1))^{-\frac{1-R}{R}} - 1\big) - B_c^0 \tfrac{1}{1-R}(e^{-(1-R)y} - 1)\Big) c^M(t,x).\end{aligned}$$

    From the proof of Lemma 4.106 we know that $y < h^{0,\text{crit}}(t,x)$ is equivalent to $\tilde{c}^{0,*}(t,x,y) \in \mathcal{I}^{c,0}$ which by Lemma 4.105, i), implies that $H_{l,M}^{c,0}(t,x,y) > H_{a,\sim}^{c,0}(t,x,y)$. Hence it follows that $\chi^0(t,x,y) > 0$.

    - $y \geq h^{0,\text{crit}}(t,x)$. If $y \geq h^{0,\text{crit}}(t,x)$ then

$$\begin{aligned}\chi^0(t,x,y) = &- C^0 \tfrac{1}{1-R}(e^{-(1-R)y} - 1) - d^{c,0}(c^M(t,x)) \tfrac{1}{1-R}(e^{-(1-R)h^{0,\text{crit}}(t,x)} - 1) \\ &- \Big(\tfrac{R}{1-R}\big((1 - B_c^0 \tfrac{1}{1-R}(e^{-(1-R)h^{0,\text{crit}}(t,x)} - 1))^{-\frac{1-R}{R}} - 1\big) \\ &\qquad - B_c^0 \tfrac{1}{1-R}(e^{-(1-R)h^{0,\text{crit}}(t,x)} - 1)\Big) c^M(t,x) \\ = &- C^0 \tfrac{1}{1-R}(e^{-(1-R)y} - 1) - (d^{c,0}(c^M(t,x)))^+ \tfrac{1}{1-R}(e^{-(1-R)h^{0,\text{crit}}(t,x)} - 1) \\ &- \Big(\tfrac{R}{1-R}\big((1 - B_c^0 \tfrac{1}{1-R}(e^{-(1-R)h^{0,\text{crit}}(t,x)} - 1))^{-\frac{1-R}{R}} - 1\big) \\ &\qquad - B_c^0 \tfrac{1}{1-R}(e^{-(1-R)h^{0,\text{crit}}(t,x)} - 1)\Big) c^M(t,x) \\ = &- C^0 \tfrac{1}{1-R}(e^{-(1-R)y} - 1) - \Big(\tfrac{R}{1-R}\big((1 - B_c^0 \tfrac{1}{1-R}(e^{-(1-R)h^{0,\text{crit}}(t,x)} - 1))^{-\frac{1-R}{R}} - 1\big) \\ &- B_c^0 \tfrac{1}{1-R}(e^{-(1-R)h^{0,\text{crit}}(t,x)} - 1)(1 - B_c^0 \tfrac{1}{1-R}(e^{-(1-R)h^{0,\text{crit}}(t,x)} - 1))^{-\frac{1}{R}}\Big) c^M(t,x),\end{aligned}$$

since $d^{c,0}(c^M(t,x)) > 0$ as $c^M(t,x) \in \mathcal{J}^{c,0}$, and $\lambda^{0,\text{crit}}(t,x) = \frac{1}{1-R}(e^{-(1-R)h^{0,\text{crit}}(t,x)} - 1)$ is the unique solution of equation (4.50). Moreover part i) of Lemma 4.48 yields with $z = -B_c^0 \frac{1}{1-R}(e^{-(1-R)h^{0,\text{crit}}(t,x)} - 1)$ that

$$\begin{aligned}0 \leq -\Big(&\tfrac{R}{1-R}\big((1 - B_c^0 \tfrac{1}{1-R}(e^{-(1-R)h^{0,\text{crit}}(t,x)} - 1))^{-\frac{1-R}{R}} - 1\big)\\ &- B_c^0 \tfrac{1}{1-R}(e^{-(1-R)h^{0,\text{crit}}(t,x)} - 1)(1 - B_c^0 \tfrac{1}{1-R}(e^{-(1-R)h^{0,\text{crit}}(t,x)} - 1))^{-\frac{1}{R}}\Big),\end{aligned}$$

thus implying $\chi^0(t,x,y) \geq 0$. Here Lemma 4.48, i), is applicable since

$$\begin{aligned}-B_c^0 \tfrac{1}{1-R}(e^{-(1-R)h^{0,\text{crit}}(t,x)} - 1) &= -B_c^0 \lambda^{0,\text{crit}}(t,x)\\ &= -\Big(1 - \big(1 - \tfrac{(A^0 + B_c^0 c^M(t,x) - C^0)^+}{B_c^0 c^M(t,x)}\big)^{-R}\Big)\\ &> -1\end{aligned}$$

because $\big(1 - \frac{(A^0 + B_c^0 c^M(t,x) - C^0)^+}{B_c^0 c^M(t,x)}\big) > 0$ as we discuss the case $B_c^0 < 0$ and $A^0 > C^0$, resp. $B_c^0 > 0$ and $A^0 < C^0$.

Subsequently we let $\varepsilon = 0$.

- *Definition of the χ^i's ($\varepsilon = 0$).* If $\varepsilon = 0$ then the χ^i's are given by $\chi^i : \mathbb{R}_0^+ \to \mathbb{R}$, $i = 0, 1$, with

$$\begin{aligned}\chi^0(y) &\triangleq -C^0 \tfrac{1}{1-R}(e^{-(1-R)y} - 1) - (A^0 - C^0)\tfrac{1}{1-R}(e^{-(1-R)(y \wedge h^{0,\text{crit}})} - 1),\\ \chi^1(y) &\triangleq C^1 \tfrac{1}{1-R}(e^{(1-R)y} - 1) + (A^1 - C^1)^+ \tfrac{1}{1-R}(e^{(1-R)y} - 1).\end{aligned}$$

- *Continuity results on the χ^i's ($\varepsilon = 0$).* Being compositions of continuously differentiable functions, the $[\,\cdot\,]^+$- and the $\min\{\cdot,\cdot\}$-function the χ^i's are locally Lipschitz continuous.
- *Non-negativity of the χ^i's ($\varepsilon = 0$).* The non-negativity of χ^1 is obvious. To verify the non-negativity of χ^0 note that $h^{0,\text{crit}} = 0$ if $A^0 < C^0$.

The remainder of the proof is the same as the proof of Lemma 4.9. □

Figure 4.16 provides an example for the functions g and h.

The proof of Lemma 4.118 directly implies the following corollaries.

Corollary 4.119 (Time-dependent bounds on g and h)**.** *Let h and g be given by (4.51), (4.52) subject to the boundary conditions (4.53) ($\varepsilon > 0$), resp. (4.54), (4.55) subject to the boundary conditions (4.56) ($\varepsilon = 0$). Then*

$$\begin{gathered}-(\Psi^1 - \xi^1(T))^-(T-t) \leq g(t) \leq (\Psi^0 - \xi^0(T))^+(T-t)\ (\varepsilon > 0),\\ \textit{resp. } 0 \leq g(t) \leq \Psi^0(T-t)\ (\varepsilon = 0) \textit{ and}\\ 0 \leq h(t) \leq (\Psi^0 - \Psi^1)(T-t)\end{gathered}$$

for $t \in [0,T]$ where the $\xi^i(T)$'s are as defined in the proof of Lemma 4.9.

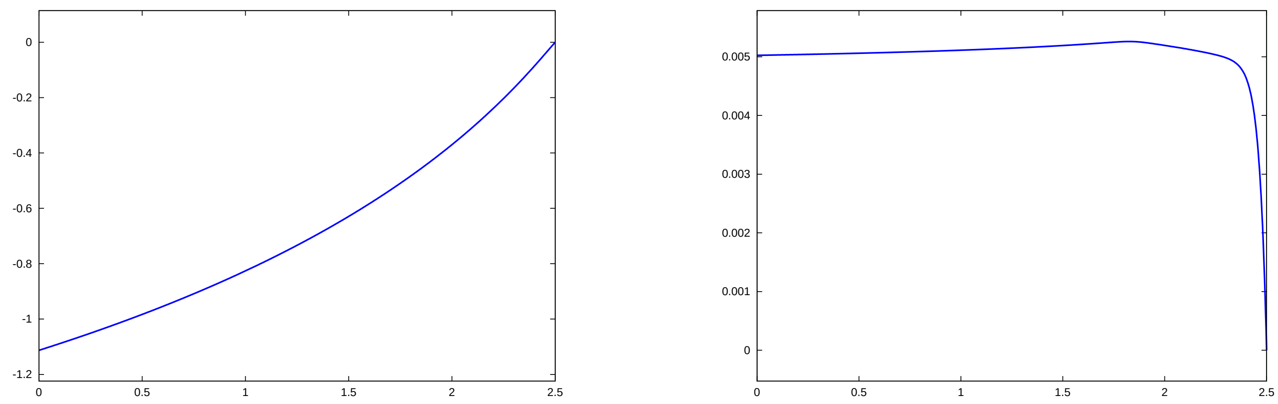

Figure 4.16: g (left) and h (right) as functions of t
($r^0 = r^1 = 0.03$, $\eta^0 = 0.09$, $\eta^1 = 0.01$, $\sigma^0 = 0.2$, $\sigma^1 = 0.6$, $\delta = 0.035$, $\varepsilon = 1$, $R = 0.75$, $T = 2.5$,
$A^0 = 9$, $B_c^0 = 10$, $C^0 = 15$, $A^1 = 13$, $B_c^1 = -5$, $C^1 = 7.5$)

Corollary 4.120 (Time-independent bound on h)**.** *Let h and g be given by (4.51), (4.52) subject to the boundary conditions (4.53) ($\varepsilon > 0$), resp. (4.54), (4.55) subject to the boundary conditions (4.56) ($\varepsilon = 0$). Then*

$$0 \leq h(t) \leq \bar{h}$$

for $t \in [0, T]$ where $\bar{h}$ is the smallest positive root of

$$\bar{F}(y) \triangleq \begin{cases} -(\Psi^0 - \Psi^1) - C^0 \frac{1}{1-R}(e^{-(1-R)y} - 1) + C^1 \frac{1}{1-R}(e^{(1-R)y} - 1) & \text{if } \varepsilon > 0, \\ -(\Psi^0 - \Psi^1) + \chi^0(y) + \chi^1(y) & \text{if } \varepsilon = 0 \end{cases}$$

if such a root exists; otherwise $\bar{h} = \infty$, i.e.

$$\bar{h} = \min \left\{ y \in \mathbb{R}_0^+ \,:\, \bar{F}(y) = 0 \right\},$$

with the convention $\min \emptyset \triangleq \infty$, where the χ^i's are as defined in the proof of Lemma 4.118.

Proof. The assertion follows from a simple ODE argument since

$$F(t, x, y) = -(\Psi^0 - \Psi^1) + \varrho(t, x, y) + \chi^0(t, x, y) + \chi^1(t, x, y) \geq \bar{F}(y)$$

for all $(t, x, y) \in [0, T] \times \mathbb{R} \times \mathbb{R}_0^+$ where we used the notation from the proof of Lemma 4.9. If $\varepsilon = 0$ then this is obviously true. In case of $\varepsilon > 0$ this is satisfied, too, since the proof of Lemma 4.118 showed that $\chi^0(t, x, y) \geq -C^0 \frac{1}{1-R}(e^{-(1-R)y} - 1)$ and $\chi^1(t, x, y) \geq C^1 \frac{1}{1-R}(e^{(1-R)y} - 1)$. □

The boundary function on h as presented in Corollary 4.119 is suitable for large values of t for which the deviation from h is not too large whereas for small values of t the non-linear behavior of h implies that the boundary function overshoots h by far. The converse is true for the bound given in Corollary 4.120. Thus combining the two bounds on h yields

$$0 \leq h(t) \leq \min\{(\Psi^0 - \Psi^1)(T - t), \bar{h}\}$$

for $t \in [0, T]$.

Since the above ODE-system is solvable we can now verify that the strategy given above is indeed the optimal strategy that solves the investment problem.

Theorem 4.121 (Solution of the Investment Problem)**.** *Let condition (NJ) be satisfied, and let h and g be given by (4.51), (4.52) subject to the boundary conditions (4.53) ($\varepsilon > 0$), resp. (4.54), (4.55) subject to the boundary conditions (4.56) ($\varepsilon = 0$). Then the strategy*

$$(\pi^{i,*}, c^{i,*}(t)) \triangleq (\pi^{i,*}, c^{i,*}(t, g(t) - 1_{\{i=1\}}h(t), h(t))),\ t \in [0,T],\ i = 0, 1,$$

as given in Lemmas 4.101 and 4.106 is optimal for the investment problem (P) with CRRA *preferences and relative risk aversion R.*

Proof. Since $(\pi^{i,*}, c^{i,*}(t))$ maximizes the reduced HJB-system (4.48) for each $t \in [0,T]$, optimality of the strategy $(\pi^{i,*}, c^{i,*})$ follows directly from the Verification Theorem 3.3. □

Remark 4.122. *If not otherwise stated we use the following shorthand notations*

$$\begin{aligned}
c^{i,\mathrm{crit}}(t) &\triangleq c^{i,\mathrm{crit}}(t, g(t) - 1_{\{i=1\}}h(t)),\\
h^{i,\mathrm{crit}}(t) &\triangleq h^{i,\mathrm{crit}}(t, g(t) - 1_{\{i=1\}}h(t)),\\
H^{c,i}_{l,M}(t) &\triangleq H^{c,i}_{l}(t, g(t) - 1_{\{i=1\}}h(t), h(t), c^{i,M}(t)),\\
H^{c,i}_{a,\sim}(t) &\triangleq H^{c,i}_{a}(t, g(t) - 1_{\{i=1\}}h(t), h(t), \tilde{c}^{i,*}(t))
\end{aligned}$$

for $t \in [0,T]$, $i = 0, 1$.

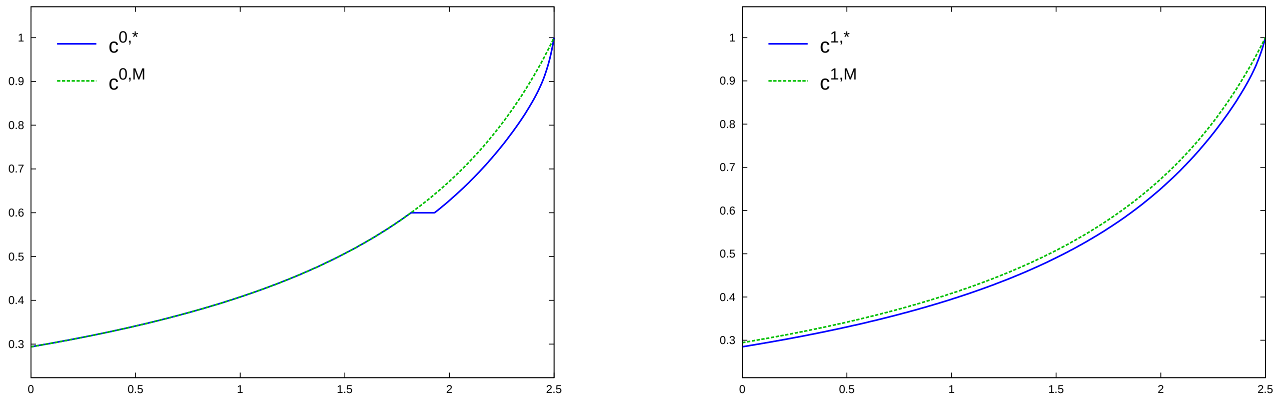

Figure 4.17: Optimal strategy vs. Merton strategy: $c^{i,*}$ and $c^{i,M}$ as functions of t
($r^0 = r^1 = 0.03$, $\eta^0 = 0.09$, $\eta^1 = 0.01$, $\sigma^0 = 0.2$, $\sigma^1 = 0.6$, $\delta = 0.035$, $\varepsilon = 1$, $R = 0.75$, $T = 2.5$,
$A^0 = 9$, $B^0_c = 10$, $C^0 = 15$, $A^1 = 13$, $B^1_c = -5$, $C^1 = 7.5$)

Figure 4.17 shows the optimal strategy and the Merton strategy in a setting where B^0_c is positive and B^1_c is negative. Hence the large investor is forced to consume at a rate that is temporarily smaller than the Merton consumption rate in order to generate advantageous regime shift intensities. In state 0 the maximal deviation amounts to about 6% whereas in state 1 it is at most 3.5%.

As in the portfolio-dependent case the optimal consumption rate converges continuously to the Merton consumption rate as the time to maturity decreases. The reason for this behavior is the same as in the portfolio-dependent setting. Further due to the time dependency of the Merton consumption rate there is no need for the investor to deviate from the Merton rate until $t \approx 1.8$ in state 0. From this time on the Merton rate enters the half space of influencing consumption rates which makes it necessary to deviate from the Merton rate.

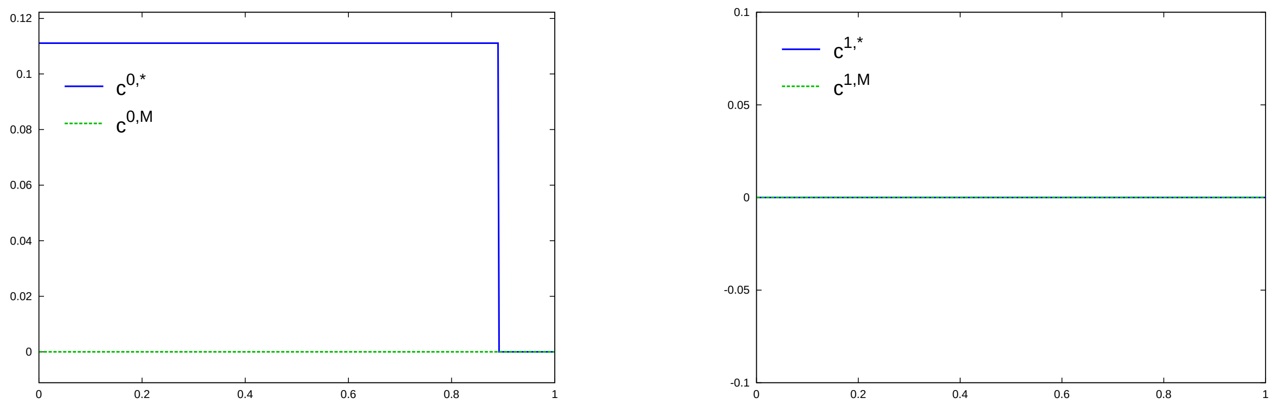

Figure 4.18: Optimal strategy vs. Merton strategy: $c^{i,*}$ and $c^{i,M}$ as functions of t
($r^0 = r^1 = 0.03$, $\eta^0 = 0.1$, $\eta^1 = 0.01$, $\sigma^0 = 0.2$, $\sigma^1 = 0.6$, $\delta = 0.035$, $\varepsilon = 0$, $R = 0.25$, $T = 1$, $A^0 = 20$, $B^0_c = -45$, $C^0 = 15$, $A^1 = 0$, $B^1_c = 0$, $C^1 = 0$)

Figure 4.18 displays the case where the large investor is forced to consume although he draws no utility from intermediate consumption. In the example corresponding to Figure 4.18 the market will stay in state 1 as soon as it enters it. So in order to avoid a regime shift into state 1 the large investor consumes at a rate of about 11% until $t \approx 0.9$. Afterwards he stops comsuming until $t = T$.

Again the optimal consumption rate shows an extreme behavior if $\varepsilon = 0$. Until $t \approx 0.9$ the investor consumes at the critical rate and then immediately stops consuming. One would expect the strategy in the case of $\varepsilon = 0$ to be similarly continuous as in the case of $\varepsilon > 0$. The reason for the extreme behavior can be found in the HJB-system (4.48). There the only term including c is given by $-c + \max\{A^i + B^i_c c, C^i\}\frac{1}{1-R}(e^{(-1)^{1-i}(1-R)h(t)} - 1)$ where $-c$ is the usual consumption impact and $\max\{A^i + B^i_c c, C^i\}\frac{1}{1-R}(e^{(-1)^{1-i}(1-R)h(t)} - 1)$ goes back to the consumption-dependency of the regime shift intensity. Those two components are of the same magnitude, i.e. they are both linear in c. Usually – without the investor's influence – the maximizing consumption rate is 0 as the linear function $H^{c,i}(t, g(t) - 1_{\{i=1\}}h(t), h(t), \cdot)$ is usually decreasing. Due to the investor's impact on the market the slope of $H^{c,i}(t, g(t) - 1_{\{i=1\}}h(t), h(t), \cdot)$ becomes positive when $h(t)$ is large enough. When this occurs the optimal consumption rate immediately jumps to the critical rate $c^{0,\text{crit}}$. Thus the extreme behavior is due to the affine form of the investor's influence.

As in the section on step intensity functions the special structure of the large investor's optimal consumption rate suggests a decomposition into the Merton consumption rate and an additional adjustment component. This adjustment component results from the investor's influence on the market. So the optimal consumption rate can be written as

$$c^{i,*}(t) = c^{i,M}(t) + c^{i,A}(t) \text{ for } i = 0, 1$$

where we have

$$c^{0,A}(t) = \begin{cases} \left(\left(1 - B_c^0 \frac{1}{1-R}(e^{-(1-R)(h(t)\wedge h^{0,\mathrm{crit}}(t))} - 1)\right)^{-\frac{1}{R}} - 1\right) c^{0,M}(t) & \text{if } \varepsilon > 0, \\ -\frac{(A^0-C^0)^+}{B_c^0} 1_{\{h(t)\geq h^{0,\mathrm{crit}}\}} & \text{if } \varepsilon = 0, \end{cases}$$

$$c^{1,A}(t) = \begin{cases} \left(\left(1 - B_c^1 \frac{1}{1-R}(e^{(1-R)h(t)} - 1)\right)^{-\frac{1}{R}} - 1\right) c^{1,M}(t) 1_{\{H_{l,M}^{c,1}(t)\leq H_{a,\sim}^{c,1}(t)\}} & \text{if } \varepsilon > 0, \\ 0 & \text{if } \varepsilon = 0. \end{cases}$$

Note that $c^{0,A}(t) \leq 0$ if $B_c^0 > 0$ and $c^{0,A}(t) \geq 0$ if $B_c^0 < 0$, whereas $c^{1,A}(t) \geq 0$ if $B_c^1 > 0$ and $c^{1,A}(t) \leq 0$ if $B_c^1 < 0$. Thus, a positive value of B_c^0, resp. a negative value of B_c^1, corresponds to a market in which the large investor is forced to consume less than the Merton consumption in order to reduce the probability of a jump to the adverse market state, resp. enlarge the probability of a jump to the favorable market state. A possible interpretation for this market behavior could be that the other market participants consider the large investor's consumption as a negative signal. In case of B_c^0 being negative the large investor now has to consume at a higher rate than the Merton one to avoid the adverse market state. So the other market participants somehow disapprove his presence in the market and reward high consumption rates. Thus $c^{i,A}$ benefits the regime shift intensities which is shown in the following lemma.

Lemma 4.123. *The optimal strategy* $(\pi^{i,*}, c^{i,*})$ *satisfies*

$$\vartheta^{0,1}(\pi^{0,*}, c^{0,*}(t)) \leq \vartheta^{0,1}(\pi^{0,M}, c^{0,M}(t)) \text{ and } \vartheta^{1,0}(\pi^{1,*}, c^{1,*}(t)) \geq \vartheta^{1,0}(\pi^{1,M}, c^{1,M}(t))$$

for every $t \in [0, T]$.

Proof. The assertion follows since

$$\begin{aligned} \vartheta^{0,1}(\pi^{0,*}, c^{0,*}(t)) &= \max\{A^0 + B_c^0 c^{0,*}(t), C^0\} \\ &= \max\{A^0 + B_c^0 c^{0,M}(t) + B_c^0 c^{0,A}(t), C^0\} \\ &\leq \max\{A^0 + B_c^0 c^{0,M}(t), C^0\} \\ &= \vartheta^{0,1}(\pi^{0,M}, c^{0,M}(t)) \end{aligned}$$

and

$$\begin{aligned} \vartheta^{1,0}(\pi^{1,*}, c^{1,*}(t)) &= \max\{A^1 + B_c^1 c^{1,*}(t), C^1\} \\ &= \max\{A^1 + B_c^1 c^{1,M}(t) + B_c^1 c^{1,A}(t), C^1\} \\ &\geq \max\{A^1 + B_c^1 c^{1,M}(t), C^1\} \\ &= \vartheta^{1,0}(\pi^{1,M}, c^{1,M}(t)) \end{aligned}$$

as $B_c^0 c^{0,A}(t) \leq 0$, resp. $B_c^1 c^{1,A}(t) \geq 0$. □

Remark 4.124. *As in the last section the optimal consumption rate that we derived here is a* compromise rate *in that it is in general different from the Merton rate but generates better regime shift intensities than those the Merton rate would yield, i.e. it accommodates the aforementioned trade-off.*

We have seen before that in case of $\varepsilon = 0$ the optimal consumption rate in state 0 may jump from 0 to the critical consumption rate $c^{0,\text{crit}} = -\frac{(A^0-C^0)^+}{B_c^0}$ which is strictly positive if $B_c^0 < 0$ and $A^0 > C^0$. For all other settings of the parameters B_c^0, A^0 and C^0 the optimal consumption rate is just 0. Subsequently we will present a sufficient condition under which this jump in case of $B_c^0 < 0$ and $A^0 > C^0$ does not occur, i.e. $c^{0,*}(t) = 0$ for all $t \in [0,T]$.

Proposition 4.125. *Let $\varepsilon = 0$ and $B_c^0 < 0$ and $A^0 > C^0$. Further suppose that $-(\Psi^0 - \Psi^1) - \frac{A^0}{B_c^0} \geq 0$ or equivalently $A^0 + B_c^0 c^{0,\circ} \geq C^0$, where*

$$c^{0,\circ} \triangleq (\Psi^0 - \Psi^1) + \tfrac{C^0}{B_c^0}.$$

Then $c^{0,*}(t) = 0$ for all $t \in [0,T]$.

Proof. Let $\varepsilon = 0$. Since $B_c^0 < 0$ and $A^0 > C^0$ we get $h^{0,\text{crit}} = -\frac{1}{1-R}\ln\Big((1-R)\frac{1}{B_c^0} + 1\Big)$. The assertion of the proposition is equivalent to the condition

$$h(t) \leq h^{0,\text{crit}} \text{ for all } t \in [0,T].$$

To prove this, we use a simple ODE argument. We let F as given in the proofs of the Lemmas 4.9 and 4.118. As $F(T, g(T), h(T)) = F(T,0,0) \leq 0$, it suffices to show that $F(t,x,h^{0,\text{crit}}) \geq 0$ for every $(t,x) \in [0,T] \times \mathbb{R}$. Indeed, in this case it follows from the intermediate value theorem that for each $t \in [0,T]$ there exists some $\tilde{h}(t) \in [0, h^{0,\text{crit}}]$ with $F(t,x,\tilde{h}(t)) = 0$. Thus $0 \leq h(t) \leq \max_{t\in[0,T]} \tilde{h}(t) \leq h^{0,\text{crit}}$ for every $t \in [0,T]$.

To demonstrate under which conditions $F(t,x,h^{0,\text{crit}}) \geq 0$ we write $F(t,x,h^{0,\text{crit}}) = A + B$ with

$$A \triangleq \chi^1(h^{0,\text{crit}}), \quad B \triangleq -(\Psi^0 - \Psi^1) + \chi^0(h^{0,\text{crit}})$$

where we use the notation from the proofs of Lemmas 4.9 and 4.118.

The non-negativity of χ^1 implies that $A \geq 0$. On the other hand,

$$\begin{aligned} B &= -(\Psi^0 - \Psi^1) - C^0\tfrac{1}{1-R}(e^{-(1-R)h^{0,\text{crit}}} - 1) - (A^0 - C^0)\tfrac{1}{1-R}(e^{-(1-R)h^{0,\text{crit}}} - 1) \\ &= -(\Psi^0 - \Psi^1) - \tfrac{A^0}{B_c^0}. \end{aligned}$$

Since $-(\Psi^0 - \Psi^1) - \frac{A^0}{B_c^0} \geq 0$, we have $B \geq 0$ and the proof is complete. □

Remark 4.126. *Notice that $c^{0,\circ}$ is not necessarily non-negative. Thus $c^{0,\circ}$ cannot be interpreted as a consumption rate.*

Observe that the assumption of Proposition 4.125 is satisfied if A^0, resp. $|B_c^0|$ is sufficiently large.

4.3.3 Portfolio- and Consumption-dependent Intensities

Having discussed the optimal investment problem where the regime shifts are influenced either by the benchmark investor's portfolio proportions or by his consumption rate, we now consider the case where both affect the shift intensities, i.e.

$$B_\pi^i \neq 0 \text{ and } B_c^i \neq 0 \text{ for } i = 0,1, \tag{PCD}$$

so that the intensities are given by

$$\vartheta^{i,1-i}(\pi, c) = \max\{A^i + \pi^\top.B^i_\pi + B^i_c c, C^i\}.$$

We denote by

$$\mathcal{I}^{\pi,c,0} \triangleq \left\{(\pi, c) \in \mathbb{R}^{\bar{n}} \times \mathbb{R}^+_0 \ : \ A^0 + \pi^\top.B^0_\pi + B^0_c c > C^0\right\},$$
$$\mathcal{I}^{\pi,c,1} \triangleq \left\{(\pi, c) \in \mathbb{R}^{\bar{n}} \times \mathbb{R}^+_0 \ : \ A^1 + \pi^\top.B^1_\pi + B^1_c c \geq C^1\right\}$$

the half spaces of strategies that impact on the intensities of regime shifts. Moreover,

$$d^{\pi,c,i}(\pi, c) \triangleq A^i + \pi^\top.B^i_\pi + B^i_c c - C^i$$

denotes the 'distance' of the strategy (π, c) to the separating hyperplane and we have $(\pi, c) \in \mathcal{I}^{\pi,c,0}$ if and only if $d^{\pi,c,0}(\pi, c) > 0$, resp. $(\pi, c) \in \mathcal{I}^{\pi,c,1}$ if and only if $d^{\pi,c,1}(\pi, c) \geq 0$.

In order to determine the maximizer in the HJB-system (4.41) we define the function $H^{\pi,c,i} : [0, T] \times \mathbb{R} \times \mathbb{R}^+_0 \times \mathbb{R}^{\bar{n}} \times \mathbb{R}^+_0 \to \mathbb{R}$ given by

$$\begin{aligned} H^{\pi,c,i}(t, x, y, \pi, c) \triangleq \; & \tfrac{\varepsilon\delta}{\varepsilon-(\varepsilon-\delta)e^{-\delta(T-t)}} \tfrac{1}{1-R}(e^{-(1-R)x}c^{1-R} - 1) - c + r^i + \pi^\top.\eta^i - \tfrac{1}{2}R\pi^\top.\sigma^i.(\sigma^i)^\top.\pi \\ & + \max\{A^i + \pi^\top.B^i_\pi + B^i_c c, C^i\}\tfrac{1}{1-R}(e^{(-1)^{1-i}(1-R)y} - 1). \end{aligned}$$

Hence the HJB-system (4.41) reads

$$0 = \sup_{(\pi,c)\in\mathbb{R}^{\bar{n}}\times\mathbb{R}^+_0} \left\{g'(t) - 1_{\{i=1\}}h'(t) + H^{\pi,c,i}(t, g(t) - 1_{\{i=1\}}h(t), h(t), \pi, c)\right\} \tag{4.57}$$

for $t \in [0, T)$ and $i = 0, 1$, subject to the boundary conditions

$$g(T) = 0, \ h(T) = 0.$$

To find the supremum in (4.57) we present the maximizer of $H^{\pi,c,i}(t, x, y, \cdot, \cdot)$ for arbitrary $(t, x, y) \in [0, T] \times \mathbb{R} \times \mathbb{R}^+_0$. This yields a family of maximizers dependent on (t, x, y). The maximizers of the HJB-system (4.57) are then obtained by replacing x and y by $g(t) - 1_{\{i=1\}}h(t)$ and $h(t)$.

Let therefore $H^{\pi,c,i}_l$, $H^{\pi,c,i}_a : [0, T] \times \mathbb{R} \times \mathbb{R}^+_0 \times \mathbb{R}^{\bar{n}} \times \mathbb{R}^+_0 \to \mathbb{R}$ given by

$$\begin{aligned} H^{\pi,c,i}_l(t, x, y, \pi, c) \triangleq \; & \tfrac{\varepsilon\delta}{\varepsilon-(\varepsilon-\delta)e^{-\delta(T-t)}} \tfrac{1}{1-R}(e^{-(1-R)x}c^{1-R} - 1) - c + r^i + \pi^\top.\eta^i - \tfrac{1}{2}R\pi^\top.\sigma^i.(\sigma^i)^\top.\pi \\ & + C^i\tfrac{1}{1-R}(e^{(-1)^{1-i}(1-R)y} - 1), \\ H^{\pi,c,i}_a(t, x, y, \pi, c) \triangleq \; & \tfrac{\varepsilon\delta}{\varepsilon-(\varepsilon-\delta)e^{-\delta(T-t)}} \tfrac{1}{1-R}(e^{-(1-R)x}c^{1-R} - 1) - c + r^i + \pi^\top.\eta^i - \tfrac{1}{2}R\pi^\top.\sigma^i.(\sigma^i)^\top.\pi \\ & + (A^i + \pi^\top.B^i_\pi + B^i_c c)\tfrac{1}{1-R}(e^{(-1)^{1-i}(1-R)y} - 1) \end{aligned}$$

such that $H^{\pi,c,i}(t, x, y, \pi, c) = H^{\pi,c,i}_l(t, x, y, \pi, c)1_{\{(\pi,c)\notin\mathcal{I}^{\pi,c,i}\}} + H^{\pi,c,i}_a(t, x, y, \pi, c)1_{\{(\pi,c)\in\mathcal{I}^{\pi,c,i}\}}$.

If $\varepsilon > 0$ then $H_l^{\pi,c,i}(t,x,y,\cdot,\cdot)$ and $H_a^{\pi,c,i}(t,x,y,\cdot,\cdot)$ are concave for every $(t,x,y) \in [0,T] \times \mathbb{R} \times \mathbb{R}_0^+$. Therefore the three candidate solutions for the maximization in (4.57) are

$$\begin{aligned}
(\pi^{i,M}, c^M(t,x)) &= \underset{(\pi,c)\in\mathbb{R}^{\bar{n}}\times\mathbb{R}_0^+}{\arg\max}\; H_l^{\pi,c,i}(t,x,y,\pi,c),\\
(\tilde{\pi}^{i,*}(t,x,y), \tilde{c}^{i,*}(t,x,y)) &= \underset{(\pi,c)\in\mathbb{R}^{\bar{n}}\times\mathbb{R}_0^+}{\arg\max}\; H_a^{\pi,c,i}(t,x,y,\pi,c),\\
(\tilde{\pi}^{i,\text{crit}}(t,x), \tilde{c}^{i,\text{crit}}(t,x)) &\triangleq \underset{\left\{(\pi,c)\in\mathbb{R}^{\bar{n}}\times\mathbb{R}_0^+ \,:\, A^i+\pi^\top.B_\pi^i+B_c^i c=C^i\right\}}{\arg\max}\; H^{\pi,c,i}(t,x,y,\pi,c)
\end{aligned}$$

where

$$\begin{aligned}
\tilde{\pi}^{i,*}(t,x,y) &= \tfrac{1}{R}(\sigma^i.(\sigma^i)^\top)^{-1}.\big(\eta^i + B_\pi^i \tfrac{1}{1-R}(e^{(-1)^{1-i}(1-R)y} - 1)\big),\\
\tilde{c}^{0,*}(t,x,y) &= \begin{cases} \big(\tfrac{\varepsilon\delta}{\varepsilon-(\varepsilon-\delta)e^{-\delta(T-t)}}\big)^{\frac{1}{R}} e^{-\frac{1-R}{R}x}\big(1 - B_c^0 \tfrac{1}{1-R}(e^{-(1-R)y} - 1)\big)^{-\frac{1}{R}} & \text{if } y < h^{\text{crit}},\\ \infty & \text{if } y \geq h^{\text{crit}}, \end{cases}\\
\tilde{c}^{1,*}(t,x,y) &= \big(\tfrac{\varepsilon\delta}{\varepsilon-(\varepsilon-\delta)e^{-\delta(T-t)}}\big)^{\frac{1}{R}} e^{-\frac{1-R}{R}x}\big(1 - B_c^1 \tfrac{1}{1-R}(e^{(1-R)y} - 1)\big)^{-\frac{1}{R}},\\
\tilde{\pi}^{i,\text{crit}}(t,x) &= \tfrac{1}{R}(\sigma^i.(\sigma^i)^\top)^{-1}.\big(\eta^i + \tilde{\lambda}^{i,\text{crit}}(t,x)B_\pi^i\big),\\
\tilde{c}^{i,\text{crit}}(t,x) &= \big(\tfrac{\varepsilon\delta}{\varepsilon-(\varepsilon-\delta)e^{-\delta(T-t)}}\big)^{\frac{1}{R}} e^{-\frac{1-R}{R}x}\big(1 - \tilde{\lambda}^{i,\text{crit}}(t,x)B_c^i\big)^{-\frac{1}{R}}
\end{aligned}$$

for $(t,x,y) \in [0,T] \times \mathbb{R} \times \mathbb{R}_0^+$, $i = 0, 1$, with

$$h^{\text{crit}} \triangleq \begin{cases} \infty & \text{if } B_c^0 > 0,\\ -\frac{1}{1-R}\ln\big((1-R)\frac{1}{B_c^0} + 1\big) & \text{if } B_c^0 < 0 \end{cases}$$

and $\tilde{\lambda}^{i,\text{crit}}(t,x)$ implicitly given via

$$\Lambda^{\pi,c,i}(t,x,\tilde{\lambda}^{i,\text{crit}}(t,x)) = -(A^i + (\pi^{i,M})^\top.B_\pi^i + B_c^i c^M(t,x) - C^i).$$

If $\varepsilon = 0$ then $H_l^{\pi,c,i}(t,x,y,\cdot,\cdot)$ and $H_a^{\pi,c,i}(t,x,y,\cdot,\cdot)$ are concave in π and linear in c for every $(t,x,y) \in [0,T] \times \mathbb{R} \times \mathbb{R}_0^+$ implying

$$\begin{aligned}
\tilde{\pi}^{i,*}(y) &= \tfrac{1}{R}(\sigma^i.(\sigma^i)^\top)^{-1}.\big(\eta^i + B_\pi^i \tfrac{1}{1-R}(e^{(-1)^{1-i}(1-R)y} - 1)\big),\\
\tilde{c}^{0,*}(y) &= \begin{cases} 0 & \text{if } y < h^{\text{crit}},\\ \infty & \text{if } y \geq h^{\text{crit}}, \end{cases}\\
\tilde{c}^{1,*}(y) &= 0,\\
\tilde{\pi}^{i,\text{crit}} &= \tfrac{1}{R}(\sigma^i.(\sigma^i)^\top)^{-1}.\Big(\eta^i + B_\pi^i \tfrac{1}{B_c^i}\Big(1 - \Big(\tfrac{A^i+(\pi^{i,M})^\top.B_\pi^i-C^i}{\frac{1}{R}(B_\pi^i)^\top.(\sigma^i.(\sigma^i)^\top)^{-1}.B_\pi^i}B_c^i + 1\Big)^+\Big)\Big),\\
\tilde{c}^{i,\text{crit}} &= \tfrac{\frac{1}{R}(B_\pi^i)^\top.(\sigma^i.(\sigma^i)^\top)^{-1}.B_\pi^i}{(B_c^i)^2}\Big(\tfrac{A^i+(\pi^{i,M})^\top.B_\pi^i-C^i}{\frac{1}{R}(B_\pi^i)^\top.(\sigma^i.(\sigma^i)^\top)^{-1}.B_\pi^i}B_c^i + 1\Big)^-
\end{aligned}$$

for $(t,x,y) \in [0,T] \times \mathbb{R} \times \mathbb{R}_0^+$, $i = 0, 1$. Further $H_l^{\pi,c,i}(t,x,y,\cdot,\cdot)$ is even decreasing in c yielding $c^M(t,x) = 0$.

Remark 4.127. *We use the following shorthand notations*

$$H^{\pi,c,i}_{l,M}(t,x,y) \triangleq H^{\pi,c,i}_{l}(t,x,y,\pi^{i,M},c^M(t,x)),$$
$$H^{\pi,c,i}_{a,\sim}(t,x,y) \triangleq H^{\pi,c,i}_{a}(t,x,y,\tilde{\pi}^{i,*}(t,x,y),\tilde{c}^{i,*}(t,x,y))$$

for $(t,x,y) \in [0,T]\times\mathbb{R}\times\mathbb{R}^+_0$ *and* $i=0,1$.

Before we present the maximizer of $H^{\pi,c,i}(t,x,y,\cdot,\cdot)$ we first need to provide the following helpful results on the relation between $(\pi^{i,M},c^M)$ and $(\tilde{\pi}^{i,*},\tilde{c}^{i,*})$ and the related function values $H^{\pi,c,i}_{l,M}$ and $H^{\pi,c,i}_{a,\sim}$.

Lemma 4.128. *For every* $(t,x,y)\in[0,T]\times\mathbb{R}\times\mathbb{R}^+_0$ *the Merton strategy* $(\pi^{i,M},c^M)$*, the candidate solution* $(\tilde{\pi}^{i,*},\tilde{c}^{i,*})$ *and their function values* $H^{\pi,c,i}_{l,M}$ *and* $H^{\pi,c,i}_{a,\sim}$ *are related as follows.*

i) $(\pi^{0,M},c^M(t,x)) \notin \mathcal{I}^{\pi,c,0} \;\Rightarrow\; H^{\pi,c,0}_{l,M}(t,x,y) \le H^{\pi,c,0}_{a,\sim}(t,x,y) \;\Rightarrow\; (\tilde{\pi}^{0,*}(t,x,y),\tilde{c}^{0,*}(t,x,y)) \notin \mathcal{I}^{\pi,c,0}.$

ii) $(\pi^{1,M},c^M(t,x)) \in \mathcal{I}^{\pi,c,1} \;\Rightarrow\; H^{\pi,c,1}_{l,M}(t,x,y) \le H^{\pi,c,1}_{a,\sim}(t,x,y) \;\Rightarrow\; (\tilde{\pi}^{1,*}(t,x,y),\tilde{c}^{1,*}(t,x,y)) \in \mathcal{I}^{\pi,c,1}.$

Proof. ad i) If $h^{\text{crit}}<\infty$ and $y\ge h^{\text{crit}}$ then $\tilde{c}^{0,*}(t,x,y)=\infty$ whereas every component of $\tilde{\pi}^{0,*}(t,x,y)$ is finite and therefore $H^{\pi,c,0}_{a,\sim}(t,x,y)=\infty$, too. Further $h^{\text{crit}}<\infty$ implies $B^0_c<0$. Hence obviously $(\tilde{\pi}^{0,*}(t,x,y),\tilde{c}^{0,*}(t,x,y))\notin\mathcal{I}^{\pi,c,0}$ and there is nothing to prove.

If otherwise $y<h^{\text{crit}}$ then the following three equivalences hold true. Firstly, $(\pi^{0,M},c^M(t,x))\notin\mathcal{I}^{\pi,c,0}$ is by definition equivalent to

$$d^{\pi,c,0}(\pi^{0,M},c^M(t,x)) \le 0. \tag{$*$}$$

Secondly, some transformations yield that $H^{\pi,c,0}_{l,M}(t,x,y)\le H^{\pi,c,0}_{a,\sim}(t,x,y)$ is equivalent to

$$\begin{aligned} d^{\pi,c,0}(\pi^{0,M},c^M(t,x)) \le &-\tfrac{1}{2}\tfrac{1}{R}(B^0_\pi)^\top.(\sigma^0.(\sigma^0)^\top)^{-1}.B^0_\pi\tfrac{1}{1-R}(e^{-(1-R)y}-1)\\ &+\left(\frac{-\frac{R}{1-R}\left((1-B^0_c\frac{1}{1-R}(e^{-(1-R)y}-1))^{-\frac{1-R}{R}}-1\right)}{\frac{1}{1-R}(e^{-(1-R)y}-1)}+B^0_c\right)c^M(t,x). \end{aligned} \tag{$**$}$$

And thirdly, $(\tilde{\pi}^{0,*}(t,x,y),\tilde{c}^{0,*}(t,x,y))\notin\mathcal{I}^{\pi,c,0}$ is equivalent to

$$\begin{aligned} d^{\pi,c,0}(\pi^{0,M},c^M(t,x)) \le &-\tfrac{1}{R}(B^0_\pi)^\top.(\sigma^0.(\sigma^0)^\top)^{-1}.B^0_\pi\tfrac{1}{1-R}(e^{-(1-R)y}-1)\\ &-B^0_c\big((1-B^0_c\tfrac{1}{1-R}(e^{-(1-R)y}-1))^{-\frac{1}{R}}-1\big)c^M(t,x) \end{aligned} \tag{$*{*}*$}$$

The right-hand side of $(**)$ is non-negative. This follows from the fact that $y\ge 0$ and from Lemma 4.48, ii), with $z=-B^0_c\frac{1}{1-R}(e^{-(1-R)y}-1)$. The lemma is applicable since $y<h^{\text{crit}}$ is equivalent to $-B^0_c\frac{1}{1-R}(e^{-(1-R)y}-1)>-1$. Hence $(**)$ follows directly from $(*)$ which yields the first implication. In order to prove the second implication we have to show that the right-hand side of $(**)$ is smaller than the right-hand side of $(*{*}*)$ which is equivalent to

$$\begin{aligned} 0\le &-\tfrac{1}{2}\tfrac{1}{R}(B^0_\pi)^\top.(\sigma^0.(\sigma^0)^\top)^{-1}.B^0_\pi\tfrac{1}{1-R}(e^{-(1-R)y}-1)\\ &+\left(\frac{-\frac{R}{1-R}\left((1-B^0_c\frac{1}{1-R}(e^{-(1-R)y}-1))^{-\frac{1-R}{R}}-1\right)}{\frac{1}{1-R}(e^{-(1-R)y}-1)}+B^0_c(1-B^0_c\tfrac{1}{1-R}(e^{-(1-R)y}-1))^{-\frac{1}{R}}\right)c^M(t,x). \end{aligned}$$

This is trivially satisfied if $\varepsilon = 0$, since then $c^M(t,x) = 0$. If $\varepsilon > 0$ then the latter inequality follows from Lemma 4.48, i), again substituting $z = -B_c^0 \frac{1}{1-R}(e^{-(1-R)y} - 1)$.

ad ii) The proof of assertion ii) works completely analogously since firstly, $(\pi^{1,M}, c^M(t,x)) \in \mathcal{I}^{\pi,c,1}$ is by definition equivalent to

$$d^{\pi,c,1}(\pi^{1,M}, c^M(t,x)) \geq 0,$$

secondly, $H^{\pi,c,1}_{l,M}(t,x,y) \leq H^{\pi,c,1}_{a,\sim}(t,x,y)$ is equivalent to

$$\begin{aligned} d^{\pi,c,1}(\pi^{1,M}, c^M(t,x)) \geq &- \tfrac{1}{2}\tfrac{1}{R}(B_\pi^1)^\top.(\sigma^1.(\sigma^1)^\top)^{-1}.B_\pi^1 \tfrac{1}{1-R}(e^{(1-R)y} - 1) \\ &+ \Big(\frac{-\frac{R}{1-R}\big((1-B_c^1\frac{1}{1-R}(e^{(1-R)y}-1))^{-\frac{1-R}{R}}-1\big)}{\frac{1}{1-R}(e^{(1-R)y}-1)} + B_c^1\Big)c^M(t,x) \end{aligned}$$

and thirdly, $(\tilde{\pi}^{1,*}(y), \tilde{c}^{1,*}(y)) \in \mathcal{I}^{\pi,c,1}$ is equivalent to

$$\begin{aligned} d^{\pi,c,1}(\pi^{1,M}, c^M(t,x)) \geq &- \tfrac{1}{R}(B_\pi^1)^\top.(\sigma^1.(\sigma^1)^\top)^{-1}.B_\pi^1 \tfrac{1}{1-R}(e^{(1-R)y} - 1) \\ &- B_c^1\big((1 - B_c^1\tfrac{1}{1-R}(e^{(1-R)y} - 1))^{-\frac{1}{R}} - 1\big)c^M(t,x). \end{aligned}$$

□

We now present the maximizing strategy.

Lemma 4.129 (Maximizer of $H^{\pi,c,i}(t,x,y,\cdot,\cdot)$)**.** *For every $(t,x,y) \in [0,T] \times \mathbb{R} \times \mathbb{R}_0^+$ let*

$$(\pi^{i,*}(t,x,y), c^{i,*}(t,x,y)) \triangleq \underset{(\pi,c)\in\mathbb{R}^{\bar{n}}\times\mathbb{R}_0^+}{\arg\max}\ H^{\pi,c,i}(t,x,y,\pi,c),\ \ i = 0,1.$$

If $\varepsilon > 0$ then the maximizer $(\pi^{i,*}, c^{i,*})$ is given by

$$\begin{aligned} \pi^{0,*}(t,x,y) &= \tfrac{1}{R}(\sigma^0.(\sigma^0)^\top)^{-1}.\big(\eta^0 + B_\pi^0\tfrac{1}{1-R}(e^{-(1-R)(y\wedge h^{0,\mathrm{crit}}(t,x))} - 1)\big), \\ c^{0,*}(t,x,y) &= \big(\tfrac{\varepsilon\delta}{\varepsilon-(\varepsilon-\delta)e^{-\delta(T-t)}}\big)^{\frac{1}{R}} e^{-\frac{1-R}{R}x}\big(1 - B_c^0\tfrac{1}{1-R}(e^{-(1-R)(y\wedge h^{0,\mathrm{crit}}(t,x))} - 1)\big)^{-\frac{1}{R}}, \\ (\pi^{1,*}(t,x,y), c^{1,*}(t,x,y)) &= \begin{cases} (\pi^{1,M}, c^M(t,x)) & \text{if } H^{\pi,c,1}_{l,M}(t,x,y) > H^{\pi,c,1}_{a,\sim}(t,x,y), \\ (\tilde{\pi}^{1,*}(t,x,y), \tilde{c}^{1,*}(t,x,y)) & \text{if } H^{\pi,c,1}_{l,M}(t,x,y) \leq H^{\pi,c,1}_{a,\sim}(t,x,y), \end{cases} \end{aligned}$$

with

$$h^{0,\mathrm{crit}}(t,x) \triangleq -\tfrac{1}{1-R}\ln((1-R)\lambda^{0,\mathrm{crit}}(t,x) + 1),$$

where $\lambda^{0,\mathrm{crit}}(t,x)$ is implicitly given via

$$\Lambda^{\pi,c,0}(t,x,\lambda^{0,\mathrm{crit}}(t,x)) = -(A^0 + (\pi^{0,M})^\top.B_\pi^0 + B_c^0 c^M(t,x) - C^0)^+. \tag{4.58}$$

Moreover

$$\begin{aligned} \pi^{0,\mathrm{crit}}(t,x) &\triangleq \tfrac{1}{R}(\sigma^0.(\sigma^0)^\top)^{-1}.\big(\eta^0 + B_\pi^0\tfrac{1}{1-R}(e^{-(1-R)h^{0,\mathrm{crit}}(t,x)} - 1)\big), \\ c^{0,\mathrm{crit}}(t,x) &\triangleq c^M(t,x)\big(1 - B_c^0\tfrac{1}{1-R}(e^{-(1-R)h^{0,\mathrm{crit}}(t,x)} - 1)\big)^{-\frac{1}{R}}. \end{aligned}$$

If $\varepsilon = 0$ *then the maximizer* $(\pi^{i,*}, c^{i,*})$ *is given by*

$$\pi^{0,*}(t,x,y) = \tfrac{1}{R}(\sigma^0.(\sigma^0)^\top)^{-1}.\big(\eta^0 + B^0_\pi \tfrac{1}{1-R}(e^{-(1-R)(y\wedge h^{0,\mathrm{crit}})} - 1)\big),$$

$$c^{0,*}(t,x,y) = \begin{cases} 0 & \text{if } y < h^{0,\mathrm{crit}}, \\ c^{0,\mathrm{crit}} & \text{if } y \geq h^{0,\mathrm{crit}}, \end{cases}$$

$$(\pi^{1,*}(t,x,y), c^{1,*}(t,x,y)) = \begin{cases} (\pi^{1,M}, 0) & \text{if } y < h^{1,\mathrm{crit}}, \\ (\tilde{\pi}^{1,*}(t,x,y), 0) & \text{if } y \geq h^{1,\mathrm{crit}}, \end{cases}$$

where

$$h^{0,\mathrm{crit}} \triangleq -\tfrac{1}{1-R}\ln\Big((1-R)\tfrac{1}{B^0_c}\Big(1 - \Big(\tfrac{(A^0+(\pi^{0,M})^\top.B^0_\pi - C^0)^+}{\frac{1}{R}(B^0_\pi)^\top.(\sigma^0.(\sigma^0)^\top)^{-1}.B^0_\pi} B^0_c + 1\Big)^+\Big) + 1\Big),$$

$$h^{1,\mathrm{crit}} \triangleq \tfrac{1}{1-R}\ln\Big((1-R)2\tfrac{(A^1+(\pi^{1,M})^\top.B^1_\pi - C^1)^-}{\frac{1}{R}(B^1_\pi)^\top.(\sigma^1.(\sigma^1)^\top)^{-1}.B^1_\pi} + 1\Big)$$

and

$$\pi^{0,\mathrm{crit}} = \tfrac{1}{R}(\sigma^0.(\sigma^0)^\top)^{-1}.\Big(\eta^0 + B^0_\pi \tfrac{1}{B^0_c}\Big(1 - \Big(\tfrac{(A^0+(\pi^{0,M})^\top.B^0_\pi - C^0)^+}{\frac{1}{R}(B^0_\pi)^\top.(\sigma^0.(\sigma^0)^\top)^{-1}.B^0_\pi} B^0_c + 1\Big)^+\Big)\Big),$$

$$c^{0,\mathrm{crit}} = \frac{\frac{1}{R}(B^0_\pi)^\top.(\sigma^0.(\sigma^0)^\top)^{-1}.B^0_\pi}{(B^0_c)^2}\Big(\tfrac{(A^0+(\pi^{0,M})^\top.B^0_\pi - C^0)^+}{\frac{1}{R}(B^0_\pi)^\top.(\sigma^0.(\sigma^0)^\top)^{-1}.B^0_\pi} B^0_c + 1\Big)^-.$$

Remark 4.130. *In case of* $\varepsilon = 0$ *the maximizing strategy* $(\pi^{i,*}, c^{i,*})$ *is just a function of* y*, i.e. the dependency on* t *and* x *vanishes.*

Proof of Lemma 4.129. We first verify the maximizing strategies in case of $\varepsilon > 0$ and then continue with the case $\varepsilon = 0$. Let $(t,x,y) \in [0,T] \times \mathbb{R} \times \mathbb{R}^+_0$. For notational convenience we identify the tuple (π, c) with ς and use the following shorthand notations throughout this proof.

$$\varsigma^{i,M}(t,x) \triangleq (\pi^{i,M}, c^M(t,x)), \qquad \varsigma^{i,\mathrm{crit}}(t,x) \triangleq (\pi^{i,\mathrm{crit}}(t,x), c^{i,\mathrm{crit}}(t,x)),$$

$$\tilde{\varsigma}^{i,*}(t,x,y) \triangleq (\tilde{\pi}^{i,*}(t,x,y), \tilde{c}^{i,*}(t,x,y)), \qquad \varsigma^{i,*}(t,x,y) \triangleq (\pi^{i,*}(t,x,y), c^{i,*}(t,x,y))$$

for $(t,x,y) \in [0,T] \times \mathbb{R} \times \mathbb{R}^+_0$, $i = 0, 1$.

- *Maximizing strategy in state* 0 *(*$\varepsilon > 0$*).* Let $\varepsilon > 0$. We distinguish the cases $\varsigma^{0,M}(t,x) \notin \mathfrak{I}^{\pi,c,0}$ and $\varsigma^{0,M}(t,x) \in \mathfrak{I}^{\pi,c,0}$.

  - $\varsigma^{0,M}(t,x) \notin \mathfrak{I}^{\pi,c,0}$*.* If the Merton strategy $\varsigma^{0,M}(t,x)$ does not lie in the influencing half space $\mathfrak{I}^{\pi,c,0}$ then Lemma 4.128, i), implies that $\tilde{\varsigma}^{0,*}(t,x,y)$ is not lying in $\mathfrak{I}^{\pi,c,0}$ either. Consequently, $H^{\pi,c,0}(t,x,y,\cdot,\cdot)$ exhibits only one maximum which is attained at the Merton strategy, i.e.

  $$\varsigma^{0,M}(t,x) \notin \mathfrak{I}^{\pi,c,0} \quad \Rightarrow \quad \varsigma^{0,*}(t,x,y) = \varsigma^{0,M}(t,x).$$

  - $\varsigma^{0,M}(t,x) \in \mathfrak{I}^{\pi,c,0}$*.* If the Merton strategy is part of $\mathfrak{I}^{\pi,c,0}$ then $H^{\pi,c,0}(t,x,y,\cdot,\cdot)$ has again just one maximum that is either the maximum of $H^{\pi,c,0}_a(t,x,y,\cdot,\cdot)$ or it is attained at the critical strategy $\varsigma^{0,\mathrm{crit}}(t,x)$. The maximum of $H^{\pi,c,0}_a(t,x,y,\cdot,\cdot)$ is given by $\tilde{\varsigma}^{0,*}(t,x,y)$. As long as $\tilde{\varsigma}^{0,*}(t,x,y)$ is in the influencing half space it is even the maximizer of $H^{\pi,c,0}(t,x,y,\cdot,\cdot)$. But as soon as $\tilde{\varsigma}^{0,*}(t,x,y)$ quits $\mathfrak{I}^{\pi,c,0}$ then the critical strategy is given by the maximizer of $H^{\pi,c,0}(t,x,y,\cdot,\cdot)$, i.e.

  $$\varsigma^{0,M}(t,x) \in \mathfrak{I}^{\pi,c,0} \quad \Rightarrow \quad \varsigma^{0,*}(t,x,y) = \begin{cases} \tilde{\varsigma}^{0,*}(t,x,y) & \text{if } \tilde{\varsigma}^{0,*}(t,x,y) \in \mathfrak{I}^{\pi,c,0}, \\ \varsigma^{0,\mathrm{crit}}(t,x) & \text{if } \tilde{\varsigma}^{0,*}(t,x,y) \notin \mathfrak{I}^{\pi,c,0}. \end{cases}$$

Combining the two cases we get

$$\varsigma^{0,*}(t,x,y) = \begin{cases} \varsigma^{0,M}(t,x) & \text{if } \varsigma^{0,M}(t,x) \notin \mathcal{I}^{\pi,c,0}, \\ \tilde{\varsigma}^{0,*}(t,x,y) & \text{if } \varsigma^{0,M}(t,x) \in \mathcal{I}^{\pi,c,0} \text{ and } \tilde{\varsigma}^{0,*}(t,x,y) \in \mathcal{I}^{\pi,c,0}, \\ \varsigma^{0,\text{crit}}(t,x) & \text{if } \varsigma^{0,M}(t,x) \in \mathcal{I}^{\pi,c,0} \text{ and } \tilde{\varsigma}^{0,*}(t,x,y) \notin \mathcal{I}^{\pi,c,0}. \end{cases}$$

We now have to analyze the condition $\tilde{\varsigma}^{0,*}(t,x,y) \in \mathcal{I}^{\pi,c,0}$ in more detail.

$$\begin{aligned} \tilde{\varsigma}^{0,*}(t,x,y) \in \mathcal{I}^{\pi,c,0} \;\Leftrightarrow\;& d^{\pi,c,0}(\varsigma^{0,M}(t,x)) \\ & > -\tfrac{1}{R}(B_\pi^0)^\top.(\sigma^0.(\sigma^0)^\top)^{-1}.B_\pi^0 \tfrac{1}{1-R}(e^{-(1-R)y}-1) \\ & \quad - B_c^0 c^M(t,x)\big((1-B_c^0\tfrac{1}{1-R}(e^{-(1-R)y}-1))^{-\frac{1}{R}}-1\big) \text{ and } y < h^{\text{crit}} \\ \overset{(*)}{\Leftrightarrow}\;& (d^{\pi,c,0}(\varsigma^{0,M}(t,x)))^+ \\ & > -\tfrac{1}{R}(B_\pi^0)^\top.(\sigma^0.(\sigma^0)^\top)^{-1}.B_\pi^0 \tfrac{1}{1-R}(e^{-(1-R)y}-1) \\ & \quad - B_c^0 c^{0,M}(t)\big((1-B_c^0\tfrac{1}{1-R}(e^{-(1-R)y}-1))^{-\frac{1}{R}}-1\big) \text{ and } y < h^{\text{crit}} \\ \Leftrightarrow\;& -(d^{\pi,c,0}(\varsigma^{0,M}(t,x)))^+ < \Lambda^{\pi,c,0}\big(t,x,\tfrac{1}{1-R}(e^{-(1-R)y}-1)\big) \text{ and } y < h^{\text{crit}} \\ \Leftrightarrow\;& \tfrac{1}{1-R}(e^{-(1-R)y}-1) > \lambda^{0,\text{crit}}(t,x) \text{ and } y < h^{\text{crit}} \\ \Leftrightarrow\;& y < h^{0,\text{crit}}(t,x) \text{ and } y < h^{\text{crit}} \\ \Leftrightarrow\;& y < h^{0,\text{crit}}(t,x) \wedge h^{\text{crit}} \\ \Leftrightarrow\;& y < h^{0,\text{crit}}(t,x) \end{aligned}$$

where equivalence $(*)$ holds true since the right-hand side of the left inequality is positive. This is true since $0 < (1 - B_c^0 \frac{1}{1-R}(e^{-(1-R)y}-1))^{-\frac{1}{R}} \le 1$ ($B_c^0 > 0$), resp. $(1 - B_c^0 \frac{1}{1-R}(e^{-(1-R)y}-1))^{-\frac{1}{R}} \ge 1$ ($B_c^0 < 0$), as $y \ge 0$. The last equivalence follows from the fact that in case of $B_c^0 < 0$ we have that $h^{0,\text{crit}}(t,x) < h^{\text{crit}}$ since $\lambda^{0,\text{crit}}(t,x) > \frac{1}{B_c^0}$. Notice that

$$\varsigma^{0,M}(t,x) \notin \mathcal{I}^{\pi,c,0} \quad \Leftrightarrow \quad h^{0,\text{crit}}(t,x) = 0.$$

Hence

$$\varsigma^{0,*}(t,x,y) = \begin{cases} \varsigma^{0,M}(t,x) & \text{if } \varsigma^{0,M}(t,x) \notin \mathcal{I}^{\pi,c,0}, \\ \tilde{\varsigma}^{0,*}(t,x,y) & \text{if } \varsigma^{0,M}(t,x) \in \mathcal{I}^{\pi,c,0} \text{ and } y < h^{0,\text{crit}}(t,x), \\ \varsigma^{0,\text{crit}}(t,x) & \text{if } \varsigma^{0,M}(t,x) \in \mathcal{I}^{\pi,c,0} \text{ and } y \ge h^{0,\text{crit}}(t,x), \end{cases}$$

or equivalently

$$\begin{aligned} \pi^{0,*}(t,x,y) &= \tfrac{1}{R}(\sigma^0.(\sigma^0)^\top)^{-1}.\big(\eta^0 + B_\pi^0 \tfrac{1}{1-R}(e^{-(1-R)(y\wedge h^{0,\text{crit}}(t,x))}-1)\big), \\ c^{0,*}(t,x,y) &= \big(\tfrac{\varepsilon\delta}{\varepsilon-(\varepsilon-\delta)e^{-\delta(T-t)}}\big)^{\frac{1}{R}} e^{-\frac{1-R}{R}x}\big(1 - B_c^0 \tfrac{1}{1-R}(e^{-(1-R)(y\wedge h^{0,\text{crit}}(t,x))}-1)\big)^{-\frac{1}{R}}. \end{aligned}$$

We now go on with the determination of the maximizing strategy in state 1.

- *Maximizing strategy in state* 1 *($\varepsilon > 0$).* Here we first consider $\varsigma^{1,M}(t,x) \in \mathcal{I}^{\pi,c,1}$ and then $\varsigma^{1,M}(t,x) \notin \mathcal{I}^{\pi,c,1}$.

  - $\varsigma^{1,M}(t,x) \in \mathcal{I}^{\pi,c,1}$. If the Merton strategy $\varsigma^{1,M}(t,x)$ lies within $\mathcal{I}^{\pi,c,1}$ then Lemma 4.128, ii), implies that $\tilde{\varsigma}^{1,*}(t,x,y)$ lies in $\mathcal{I}^{\pi,c,1}$, too. Hence $H^{\pi,c,1}(t,x,y,\cdot,\cdot)$ exhibits its unique maximum at $\varsigma^{1,*}(t,x,y) = \tilde{\varsigma}^{1,*}(t,x,y)$, i.e.

$$\varsigma^{1,M}(t,x) \in \mathcal{I}^{\pi,c,1} \quad \Rightarrow \quad \varsigma^{1,*}(t,x,y) = \tilde{\varsigma}^{1,*}(t,x,y).$$

  - $\varsigma^{1,M}(t,x) \notin \mathcal{I}^{\pi,c,1}$. If $\varsigma^{1,M}(t,x)$ is not in $\mathcal{I}^{\pi,c,1}$ then $H^{\pi,c,1}(t,x,y,\cdot,\cdot)$ may possess two local maxima; the maximum of $H_l^{\pi,c,1}(t,x,y,\cdot,\cdot)$ and the maximum of $H_a^{\pi,c,1}(t,x,y,\cdot,\cdot)$, given by $\varsigma^{1,M}(t,x)$, resp. $\tilde{\varsigma}^{1,*}(t,x,y)$. As long as $\tilde{\varsigma}^{1,*}(t,x,y)$ does not lie in the influencing half space then $H^{\pi,c,1}(t,x,y,\cdot,\cdot)$ has only one maximum, namely at $\varsigma^{1,*}(t,x,y) = \varsigma^{1,M}(t,x)$. But as soon as $\tilde{\varsigma}^{1,*}(t,x,y)$ enters the influencing half space then $H^{\pi,c,1}(t,x,y,\cdot,\cdot)$ exhibits the two local maxima mentioned above which have to be compared in order to find the global one, i.e.

$$\varsigma^{1,M}(t,x) \notin \mathcal{I}^{\pi,c,1} \;\Rightarrow\; \varsigma^{1,*}(t,x,y) = \begin{cases} \varsigma^{1,M}(t,x) & \begin{array}{l}\text{if } \tilde{\varsigma}^{1,*}(t,x,y) \notin \mathcal{I}^{\pi,c,1} \text{ or } [\tilde{\varsigma}^{1,*}(t,x,y) \in \mathcal{I}^{\pi,c,1} \\ \text{and } H_{l,M}^{\pi,c,1}(t,x,y) > H_{a,\sim}^{\pi,c,1}(t,x,y)],\end{array} \\ \tilde{\varsigma}^{1,*}(t,x,y) & \begin{array}{l}\text{if } \tilde{\varsigma}^{1,*}(t,x,y) \in \mathcal{I}^{\pi,c,1} \\ \text{and } H_{l,M}^{\pi,c,1}(t,x,y) \le H_{a,\sim}^{\pi,c,1}(t,x,y).\end{array} \end{cases}$$

Combining the two cases we get

$$\varsigma^{1,*}(t,x,y) = \begin{cases} \varsigma^{1,M}(t,x) & \begin{array}{l}\text{if } \varsigma^{1,M}(t,x) \notin \mathcal{I}^{\pi,c,1} \text{ and } \big[\tilde{\varsigma}^{1,*}(t,x,y) \notin \mathcal{I}^{\pi,c,1} \\ \text{or } [\tilde{\varsigma}^{1,*}(t,x,y) \in \mathcal{I}^{\pi,c,1} \text{ and } H_{l,M}^{\pi,c,1}(t,x,y) > H_{a,\sim}^{\pi,c,1}(t,x,y)]\big],\end{array} \\ \tilde{\varsigma}^{1,*}(t,x,y) & \begin{array}{l}\text{if } \varsigma^{1,M}(t,x) \in \mathcal{I}^{\pi,c,1} \text{ or } \big[\varsigma^{1,M}(t,x) \notin \mathcal{I}^{\pi,c,1} \\ \text{and } [\tilde{\varsigma}^{1,*}(t,x,y) \in \mathcal{I}^{\pi,c,1} \text{ and } H_{l,M}^{\pi,c,1}(t,x,y) \le H_{a,\sim}^{\pi,c,1}(t,x,y)]\big].\end{array} \end{cases}$$

We have a look at the two conditions. Firstly

$$\begin{aligned} & \varsigma^{1,M}(t,x) \notin \mathcal{I}^{\pi,c,1} \\ & \text{and } \big[\tilde{\varsigma}^{1,*}(t,x,y) \notin \mathcal{I}^{\pi,c,1} \text{ or } [\tilde{\varsigma}^{1,*}(t,x,y) \in \mathcal{I}^{\pi,c,1} \text{ and } H_{l,M}^{\pi,c,1}(t,x,y) > H_{a,\sim}^{\pi,c,1}(t,x,y)]\big] \\ \Leftrightarrow\quad & \varsigma^{1,M}(t,x) \notin \mathcal{I}^{\pi,c,1} \text{ and } [\tilde{\varsigma}^{1,*}(t,x,y) \notin \mathcal{I}^{\pi,c,1} \text{ or } H_{l,M}^{\pi,c,1}(t,x,y) > H_{a,\sim}^{\pi,c,1}(t,x,y)] \\ \overset{L.\ 4.128,\ ii)}{\Leftrightarrow}\quad & \varsigma^{1,M}(t,x) \notin \mathcal{I}^{\pi,c,1} \text{ and } H_{l,M}^{\pi,c,1}(t,x,y) > H_{a,\sim}^{\pi,c,1}(t,x,y) \\ \overset{L.\ 4.128,\ ii)}{\Leftrightarrow}\quad & H_{l,M}^{\pi,c,1}(t,x,y) > H_{a,\sim}^{\pi,c,1}(t,x,y) \end{aligned}$$

Secondly

$$\begin{aligned} & \varsigma^{1,M}(t,x) \in \mathcal{I}^{\pi,c,1} \\ & \text{or } \big[\varsigma^{1,M}(t,x) \notin \mathcal{I}^{\pi,c,1} \text{ and } [\tilde{\varsigma}^{1,*}(t,x,y) \in \mathcal{I}^{\pi,c,1} \text{ and } H_{l,M}^{\pi,c,1}(t,x,y) \le H_{a,\sim}^{\pi,c,1}(t,x,y)]\big] \\ \Leftrightarrow\quad & \varsigma^{1,M}(t,x) \in \mathcal{I}^{\pi,c,1} \text{ or } [\tilde{\varsigma}^{1,*}(t,x,y) \in \mathcal{I}^{\pi,c,1} \text{ and } H_{l,M}^{\pi,c,1}(t,x,y) \le H_{a,\sim}^{\pi,c,1}(t,x,y)] \\ \overset{L.\ 4.128,\ ii)}{\Leftrightarrow}\quad & \varsigma^{1,M}(t,x) \in \mathcal{I}^{\pi,c,1} \text{ or } H_{l,M}^{\pi,c,1}(t,x,y) \le H_{a,\sim}^{\pi,c,1}(t,x,y) \\ \overset{L.\ 4.128,\ ii)}{\Leftrightarrow}\quad & H_{l,M}^{\pi,c,1}(t,x,y) \le H_{a,\sim}^{\pi,c,1}(t,x,y) \end{aligned}$$

Hence we get

$$\varsigma^{1,*}(t,x,y) = \begin{cases} \varsigma^{1,M}(t,x) & \text{if } H^{\pi,c,1}_{l,M}(t,x,y) > H^{\pi,c,1}_{a,\sim}(t,x,y), \\ \tilde{\varsigma}^{1,*}(t,x,y) & \text{if } H^{\pi,c,1}_{l,M}(t,x,y) \le H^{\pi,c,1}_{a,\sim}(t,x,y). \end{cases}$$

Some simple transformations show

$$\begin{aligned} & H^{\pi,c,1}_{l,M}(t,x,y) \le H^{\pi,c,1}_{a,\sim}(t,x,y) \\ \Leftrightarrow\quad & \Big(-\tfrac{R}{1-R}\Big(\big(1 - B^1_c \tfrac{1}{1-R}(e^{(1-R)y}-1)\big)^{-\frac{1-R}{R}} - 1\Big) + B^1_c \tfrac{1}{1-R}(e^{(1-R)y}-1)\Big)c^M(t,x) \\ & - \tfrac{1}{2}\tfrac{1}{R}(B^1_\pi)^\top.(\sigma^1.(\sigma^1)^\top)^{-1}.B^1_\pi \tfrac{1}{(1-R)^2}(e^{(1-R)y}-1)^2 \\ & \quad \le d^{\pi,c,1}(\varsigma^{1,M}(t,x))\tfrac{1}{1-R}(e^{(1-R)y}-1) \\ \overset{(*)}{\Leftrightarrow}\quad & \Big(-\tfrac{R}{1-R}\Big(\big(1 - B^1_c \tfrac{1}{1-R}(e^{(1-R)y}-1)\big)^{-\frac{1-R}{R}} - 1\Big) + B^1_c \tfrac{1}{1-R}(e^{(1-R)y}-1)\Big)c^M(t,x) \\ & - \tfrac{1}{2}\tfrac{1}{R}(B^1_\pi)^\top.(\sigma^1.(\sigma^1)^\top)^{-1}.B^1_\pi \tfrac{1}{(1-R)^2}(e^{(1-R)y}-1)^2 \\ & \quad \le -(d^{\pi,c,1}(\varsigma^{1,M}(t,x)))^- \tfrac{1}{1-R}(e^{(1-R)y}-1) \end{aligned}$$

where equivalence $(*)$ is satisfied because the left-hand side of the inequality is non-positive. Unfortunately, we cannot derive an explicit condition on y; not even in the logarithmic case $R = 1$. Therefore, we stick to the condition $H^{\pi,c,1}_{l,M}(t,x,y) \le H^{\pi,c,1}_{a,\sim}(t,x,y)$ which is trivially satisfied if $\varsigma^{1,M}(t,x) \in \mathcal{I}^{\pi,c,1}$, i.e.

$$\varsigma^{1,M}(t,x) \in \mathcal{I}^{\pi,c,1} \quad\Rightarrow\quad H^{\pi,c,1}_{l,M}(t,x,y) \le H^{\pi,c,1}_{a,\sim}(t,x,y).$$

We now proceed with the case of $\varepsilon = 0$ which works quite analogously to the one of $\varepsilon > 0$. As a special feature the Merton consumption rate vanishes, i.e. $c^M(t,x) = 0$ for all $(t,x) \in [0,T] \times \mathbb{R}$.

- *Maximizing strategy in state* 0 *($\varepsilon = 0$)*. The arguments are the same as in the case of $\varepsilon > 0$. But now

$$\begin{aligned} \tilde{\varsigma}^{0,*}(t,x,y) \in \mathcal{I}^{\pi,c,0} \;&\Leftrightarrow\; d^{\pi,c,0}(\varsigma^{0,M}(t,x)) > -\tfrac{1}{R}(B^0_\pi)^\top.(\sigma^0.(\sigma^0)^\top)^{-1}.B^0_\pi \tfrac{1}{1-R}(e^{-(1-R)y}-1) \text{ and } y < h^{\text{crit}} \\ &\Leftrightarrow\; \frac{d^{\pi,c,0}(\varsigma^{0,M}(t,x))}{\frac{1}{R}(B^0_\pi)^\top.(\sigma^0.(\sigma^0)^\top)^{-1}.B^0_\pi} > -\tfrac{1}{1-R}(e^{-(1-R)y}-1) \text{ and } y < h^{\text{crit}} \\ &\Leftrightarrow\; \frac{(d^{\pi,c,0}(\varsigma^{0,M}(t,x)))^+}{\frac{1}{R}(B^0_\pi)^\top.(\sigma^0.(\sigma^0)^\top)^{-1}.B^0_\pi} > -\tfrac{1}{1-R}(e^{-(1-R)y}-1) \text{ and } y < h^{\text{crit}} \\ &\Leftrightarrow\; y < -\tfrac{1}{1-R}\ln\Big(-(1-R)\frac{(d^{\pi,c,0}(\varsigma^{0,M}(t,x)))^+}{\frac{1}{R}(B^0_\pi)^\top.(\sigma^0.(\sigma^0)^\top)^{-1}.B^0_\pi} + 1\Big) \text{ and } y < h^{\text{crit}} \\ &\Leftrightarrow\; y < \Big(-\tfrac{1}{1-R}\ln\Big(-(1-R)\frac{(d^{\pi,c,0}(\varsigma^{0,M}(t,x)))^+}{\frac{1}{R}(B^0_\pi)^\top.(\sigma^0.(\sigma^0)^\top)^{-1}.B^0_\pi} + 1\Big)\Big) \wedge h^{\text{crit}} \\ &\Leftrightarrow\; y < -\tfrac{1}{1-R}\ln\Big((1-R)\tfrac{1}{B^0_c}\Big(1 - \Big(\frac{(d^{\pi,c,0}(\varsigma^{0,M}(t,x)))^+}{\frac{1}{R}(B^0_\pi)^\top.(\sigma^0.(\sigma^0)^\top)^{-1}.B^0_\pi}B^0_c + 1\Big)^+\Big) + 1\Big) \\ &\Leftrightarrow\; y < h^{0,\text{crit}} \end{aligned}$$

Finally, the maximizing strategy in state 0 is given by

$$\begin{aligned} \pi^{0,*}(t,x,y) &= \tfrac{1}{R}(\sigma^0.(\sigma^0)^\top)^{-1}.\Big(\eta^0 + B^0_\pi \tfrac{1}{1-R}(e^{-(1-R)(y\wedge h^{0,\text{crit}})}-1)\Big), \\ c^{0,*}(t,x,y) &= \begin{cases} 0 & \text{if } y < h^{0,\text{crit}}, \\ c^{0,\text{crit}} & \text{if } y \ge h^{0,\text{crit}}. \end{cases} \end{aligned}$$

- *Maximizing strategy in state* 1 *($\varepsilon = 0$).* The analysis of state 1 is again similar to the one in case of $\varepsilon > 0$. But here we have the advantage that the condition $H^{\pi,c,1}_{l,M}(t,x,y) \leq H^{\pi,c,1}_{a,\sim}(t,x,y)$ can be transformed into an explicit condition on y, i.e.

$$\begin{aligned}
H^{\pi,c,1}_{l,M}(t,x,y) \leq H^{\pi,c,1}_{a,\sim}(t,x,y) \;&\Leftrightarrow\; d^{\pi,c,1}(\varsigma^{1,M}(t,x)) \geq -\tfrac{1}{2}\tfrac{1}{R}(B^1_\pi)^\top.(\sigma^1.(\sigma^1)^\top)^{-1}.B^1_\pi \tfrac{1}{1-R}(e^{(1-R)y}-1)\\
&\Leftrightarrow\; -2\frac{d^{\pi,c,1}(\varsigma^{1,M}(t,x))}{\frac{1}{R}(B^1_\pi)^\top.(\sigma^1.(\sigma^1)^\top)^{-1}.B^1_\pi} \leq \tfrac{1}{1-R}(e^{(1-R)y}-1)\\
&\Leftrightarrow\; 2\frac{(d^{\pi,c,1}(\varsigma^{1,M}(t,x)))^-}{\frac{1}{R}(B^1_\pi)^\top.(\sigma^1.(\sigma^1)^\top)^{-1}.B^1_\pi} \leq \tfrac{1}{1-R}(e^{(1-R)y}-1)\\
&\Leftrightarrow\; y \geq \tfrac{1}{1-R}\ln\Big((1-R)2\frac{(d^{\pi,c,1}(\varsigma^{1,M}(t,x)))^-}{\frac{1}{R}(B^1_\pi)^\top.(\sigma^1.(\sigma^1)^\top)^{-1}.B^1_\pi}+1\Big)\\
&\Leftrightarrow\; y \geq h^{1,\text{crit}}
\end{aligned}$$

  After all, the maximizing strategy in state 1 is given by

$$(\pi^{1,*}(t,x,y), c^{1,*}(t,x,y)) = \begin{cases}(\pi^{1,M},0) & \text{if } y < h^{1,\text{crit}},\\ (\tilde{\pi}^{1,*}(t,x,y),0) & \text{if } y \geq h^{1,\text{crit}}.\end{cases}$$

Thus the lemma is proven. □

Remark 4.131 (Interpretation)**.** *Whenever the Merton strategy in state* 0 *is not part of the influencing half space then the maximizing strategy coincides with the Merton strategy. However, if the Merton strategy lies within the influencing half space then the maximizing strategy deviates from the Merton strategy.*

In state 1 *things are different. Here the maximizing strategy coincides with the Merton strategy if the Merton strategy is not lying within the influencing half space and either* $(\tilde{\pi}^{1,*}, \tilde{c}^{1,*}) \notin \mathcal{I}^{\pi,c,1}$ *or* $(\tilde{\pi}^{1,*}, \tilde{c}^{1,*}) \in \mathcal{I}^{\pi,c,1}$ *but* $H^{\pi,c,1}_{l,M}(t,x,y) > H^{\pi,c,1}_{a,\sim}(t,x,y)$*. Otherwise the maximizing strategy deviates from the Merton strategy.*

Remark 4.132 (Continuity of $(\pi^{0,*}, c^{0,*})$ vs. discontinuity of $(\pi^{1,*}, c^{1,*})$ $(\varepsilon > 0)$)**.** *The special form of the maximizing strategy together with the continuity of* $\lambda^{0,\text{crit}}$ *(Lemma 4.72) imply that* $(\pi^{0,*}, c^{0,*})$ *is continuous in* t, x *and* y.

But in state 1 *the maximizing strategy exhibits a discontinuity at all* $(\hat{t},\hat{x},\hat{y})$ *satisfying* $H^{\pi,c,1}_{l,M}(\hat{t},\hat{x},\hat{y}) = H^{\pi,c,1}_{a,\sim}(\hat{t},\hat{x},\hat{y})$ *and* $(\pi^{1,M}, c^M(\hat{t},\hat{x})) \neq (\tilde{\pi}^{1,*}(\hat{t},\hat{x},\hat{y}), \tilde{c}^{1,*}(\hat{t},\hat{x},\hat{y}))$ *where* $c^{1,*}$ *jumps from* $(\pi^{1,M}, c^M(\hat{t},\hat{x}))$ *to* $(\tilde{\pi}^{1,*}(\hat{t},\hat{x},\hat{y}), \tilde{c}^{1,*}(\hat{t},\hat{x},\hat{y}))$.

Remark 4.133 (Continuity of $\pi^{0,*}$ vs. discontinuity of $c^{0,*}$ $(\varepsilon = 0)$)**.** *If* $\varepsilon = 0$ *then* $\pi^{0,*}$ *is obviously continuous in* y *whereas* $c^{0,*}$ *may jump from* 0 *to* $c^{0,\text{crit}}$ *at* $y = h^{0,\text{crit}}$.

Remark 4.134 (Discontinuity of $\pi^{1,*}$ vs. continuity of $c^{1,*}$ $(\varepsilon = 0)$)**.** *If* $\varepsilon = 0$ *then* $\pi^{1,*}$ *may jump at* $y = h^{1,\text{crit}}$ *from* $\pi^{1,M}$ *to* $\tilde{\pi}^{1,*}(h^{1,\text{crit}})$ *whereas* $c^{1,*}$ *is trivially continuous in* y.

Remark 4.135 (Negativity of B^1_c)**.** *As in the consumption-dependent case we do not allow* B^1_c *to take on positive values. The reason for this is the same as in the consumption-dependent setting. Once* $y \geq \frac{1}{1-R}\ln((1-R)\frac{1}{B^1_c}+1)$ *the maximizing consumption rate would be* ∞*. Hence whenever the market was in state* 1 *and* y *was large enough then it would be optimal for the investor to consume at an infinitely large rate for an infinitesimal short time thus guaranteeing the jump back to state* 0*. But this strategy would not be admissible. Therefore,* $B^1_c > 0$ *has to be prohibited.*

Remark 4.136 (Interpretation in case of $\varepsilon = 0$)**.** *Even in the portfolio- and consumption dependent setting the large investor may consume although $\varepsilon = 0$. In case of $B_c^0 < 0$ and $y \geq h^{0,\mathrm{crit}}$ the optimal consumption may be strictly positive, namely if $\frac{(A^0+(\pi^{0,M})^\top.B_\pi^0-C^0)^+}{\frac{1}{R}(B_\pi^0)^\top.(\sigma^0.(\sigma^0)^\top)^{-1}.B_\pi^0}B_c^0 + 1 < 0$ which implies that $c^{0,\mathrm{crit}} > 0$.*

So far the above explanations are relevant for the general maximizing strategy $(\pi^{0,*}(t,x,y), c^{0,*}(t,x,y))$ and not necessarily for the optimal strategy. This is because it is not clear whether the situation $h(t) \geq h^{0,\mathrm{crit}}$ – remember that the maximizing strategy of the HJB-system (4.57) is obtained by choosing $y = h(t)$ – could really occur. But later on we will provide an example showing that the aforementioned situation is really possible.

The following lemma provides a link between the maximizing consumption rate in case of $\varepsilon > 0$ and $\varepsilon = 0$.

Lemma 4.137 (Limiting behavior of $(\pi^{i,\mathrm{crit}}, c^{i,\mathrm{crit}})$, $h^{i,\mathrm{crit}}$ and $(\pi^{i,*}, c^{i,*})$ as ε tends to 0)**.** *Denote by $(\pi_{\varepsilon>0}^{i,\mathrm{crit}}, c_{\varepsilon>0}^{i,\mathrm{crit}})$, $h_{\varepsilon>0}^{i,\mathrm{crit}}$ and $(\pi_{\varepsilon>0}^{i,*}, c_{\varepsilon>0}^{i,*})$, resp. $(\pi_{\varepsilon=0}^{i,\mathrm{crit}}, c_{\varepsilon=0}^{i,\mathrm{crit}})$, $h_{\varepsilon=0}^{i,\mathrm{crit}}$ and $(\pi_{\varepsilon=0}^{i,*}, c_{\varepsilon=0}^{i,*})$, the critical strategy, the critical barrier and the maximizing strategy in case of $\varepsilon > 0$, resp. $\varepsilon = 0$. Then the following holds true for every $(t,x,y) \in [0,T] \times \mathbb{R} \times \mathbb{R}_0^+$.*

i) $\lim_{\varepsilon\to 0}(\pi_{\varepsilon>0}^{0,\mathrm{crit}}(t,x), c_{\varepsilon>0}^{0,\mathrm{crit}}(t,x)) = (\pi_{\varepsilon=0}^{0,\mathrm{crit}}, c_{\varepsilon=0}^{0,\mathrm{crit}})$.

ii) $\lim_{\varepsilon\to 0} h_{\varepsilon>0}^{0,\mathrm{crit}}(t,x) = h_{\varepsilon=0}^{0,\mathrm{crit}}$ *and* $\lim_{\varepsilon\to 0} H_{l,M,\varepsilon>0}^{\pi,c,1}(t,x,y) = H_{l,M,\varepsilon=0}^{\pi,c,1}(y)$, *resp.* $\lim_{\varepsilon\to 0} H_{a,\sim,\varepsilon>0}^{\pi,c,1}(t,x,y) = H_{a,\sim,\varepsilon=0}^{\pi,c,1}(y)$.

iii) $\lim_{\varepsilon\to 0}(\pi_{\varepsilon>0}^{i,*}(t,x,y), c_{\varepsilon>0}^{i,*}(t,x,y)) = (\pi_{\varepsilon=0}^{i,*}(t,x,y), c_{\varepsilon=0}^{i,*}(t,x,y))$.

Proof. Let $(t,x,y) \in [0,T]\times\mathbb{R}\times\mathbb{R}_0^+$ be given and notice that for every $(t,x,\lambda) \in [0,T]\times\mathbb{R}\times(-\infty, \frac{1}{B_c^0})$ $(B_c^0 > 0)$, resp. $(t,x,\lambda) \in [0,T] \times \mathbb{R} \times (\frac{1}{B_c^0}, \infty)$ $(B_c^0 < 0)$

$$\begin{aligned}\lim_{\varepsilon\to 0}\Lambda_{\varepsilon>0}^{\pi,c,0}(t,x,\lambda) &= \lim_{\varepsilon\to 0}\Big(\tfrac{1}{R}(B_\pi^0)^\top(\sigma^0.(\sigma^0)^\top)^{-1}.B_\pi^0\lambda + B_c^0\Big(\big(1-B_c^0\lambda\big)^{-\frac{1}{R}}-1\Big)c^M(t,x)\Big)\\ &= \tfrac{1}{R}(B_\pi^0)^\top(\sigma^0.(\sigma^0)^\top)^{-1}.B_\pi^0\lambda\\ &\triangleq \Lambda_{\varepsilon=0}^{\pi,c,0}(\lambda)\end{aligned}$$

since $\lim_{\varepsilon\to 0} c^M(t,x) = 0$. This convergence together with the results from Lemma 4.65 yield that

$$\begin{aligned}\lim_{\varepsilon\to 0}\lambda_{\varepsilon>0}^{0,\mathrm{crit}}(t,x) &= \begin{cases} -\frac{(d^{\pi,c,0}(\pi^{0,M},0))^+}{\frac{1}{R}(B_\pi^0)^\top.(\sigma^0.(\sigma^0)^\top)^{-1}.B_\pi^0} & \text{if } \frac{(d^{\pi,c,0}(\pi^{0,M},0))^+}{\frac{1}{R}(B_\pi^0)^\top.(\sigma^0.(\sigma^0)^\top)^{-1}.B_\pi^0}B_c^0 + 1 > 0,\\ \frac{1}{B_c^0} & \text{if } \frac{(d^{\pi,c,0}(\pi^{0,M},0))^+}{\frac{1}{R}(B_\pi^0)^\top.(\sigma^0.(\sigma^0)^\top)^{-1}.B_\pi^0}B_c^0 + 1 \leq 0,\end{cases}\\ &= \frac{1}{B_c^0}\Big(1-\Big(\frac{(d^{\pi,c,0}(\pi^{0,M},0))^+}{\frac{1}{R}(B_\pi^0)^\top.(\sigma^0.(\sigma^0)^\top)^{-1}.B_\pi^0}B_c^0+1\Big)^+\Big).\end{aligned}$$

Now we can prove the assertions of the lemma.

ad i) Concerning the critical portfolio strategy we get

$$\begin{aligned}\lim_{\varepsilon\to 0}\pi^{0,\text{crit}}_{\varepsilon>0}(t,x) &= \lim_{\varepsilon\to 0}\tfrac{1}{R}(\sigma^0.(\sigma^0)^\top)^{-1}.\Big(\eta^0+\lambda^{0,\text{crit}}_{\varepsilon>0}(t,x)B^0_\pi\Big)\\ &= \tfrac{1}{R}(\sigma^0.(\sigma^0)^\top)^{-1}.\Big(\eta^0+B^0_\pi\tfrac{1}{B^0_c}\Big(1-\Big(\tfrac{(d^{\pi,c,0}(\pi^{0,M},0))^+}{\frac{1}{R}(B^0_\pi)^\top.(\sigma^0.(\sigma^0)^\top)^{-1}.B^0_\pi}B^0_c+1\Big)^+\Big)\Big)\\ &= \pi^{0,\text{crit}}_{\varepsilon=0}.\end{aligned}$$

To prove the limiting behavior of the critical consumption rate note that

$$c^M(t,x)\big(1-\lambda^{0,\text{crit}}_{\varepsilon>0}(t,x)B^0_c\big)^{-\frac{1}{R}} = \frac{-(d^{\pi,c,0}(\pi^{0,M},c^M(t,x)))^+-\frac{1}{R}(B^0_\pi)^\top(\sigma^0.(\sigma^0)^\top)^{-1}.B^0_\pi\lambda^{0,\text{crit}}_{\varepsilon>0}(t,x)}{B^0_c}+c^M(t,x)$$

by definition as long as $\varepsilon>0$. Hence

$$\begin{aligned}\lim_{\varepsilon\to 0}c^{0,\text{crit}}_{\varepsilon>0}(t,x) &= \lim_{\varepsilon\to 0}c^M(t,x)\big(1-\lambda^{0,\text{crit}}_{\varepsilon>0}(t,x)B^0_c\big)^{-\frac{1}{R}}\\ &= \lim_{\varepsilon\to 0}\Big(\tfrac{-(d^{\pi,c,0}(\pi^{0,M},c^M(t,x)))^+-\frac{1}{R}(B^0_\pi)^\top(\sigma^0.(\sigma^0)^\top)^{-1}.B^0_\pi\lambda^{0,\text{crit}}_{\varepsilon>0}(t,x)}{B^0_c}+c^M(t,x)\Big)\\ &= -\tfrac{(d^{\pi,c,0}(\pi^{0,M},0))^+}{B^0_c}-\tfrac{\frac{1}{R}(B^0_\pi)^\top(\sigma^0.(\sigma^0)^\top)^{-1}.B^0_\pi}{(B^0_c)^2}\Big(1-\Big(\tfrac{(d^{\pi,c,0}(\pi^{0,M},0))^+}{\frac{1}{R}(B^0_\pi)^\top.(\sigma^0.(\sigma^0)^\top)^{-1}.B^0_\pi}B^0_c+1\Big)^+\Big)\\ &= \tfrac{\frac{1}{R}(B^0_\pi)^\top.(\sigma^0.(\sigma^0)^\top)^{-1}.B^0_\pi}{(B^0_c)^2}\Big(\Big(\tfrac{(d^{\pi,c,0}(\pi^{0,M},0))^+}{\frac{1}{R}(B^0_\pi)^\top.(\sigma^0.(\sigma^0)^\top)^{-1}.B^0_\pi}B^0_c+1\Big)^+-\Big(\tfrac{(d^{\pi,c,0}(\pi^{0,M},0))^+}{\frac{1}{R}(B^0_\pi)^\top.(\sigma^0.(\sigma^0)^\top)^{-1}.B^0_\pi}B^0_c+1\Big)\Big)\\ &= \tfrac{\frac{1}{R}(B^0_\pi)^\top.(\sigma^0.(\sigma^0)^\top)^{-1}.B^0_\pi}{(B^0_c)^2}\Big(\tfrac{(d^{\pi,c,0}(\pi^{0,M},0))^+}{\frac{1}{R}(B^0_\pi)^\top.(\sigma^0.(\sigma^0)^\top)^{-1}.B^0_\pi}B^0_c+1\Big)^-\\ &= c^{0,\text{crit}}_{\varepsilon=0}\end{aligned}$$

ad ii) The limiting behavior of $h^{0,\text{crit}}_{\varepsilon>0}$ is obvious since

$$\begin{aligned}\lim_{\varepsilon\to 0}h^{0,\text{crit}}_{\varepsilon>0}(t,x) &= \lim_{\varepsilon\to 0}-\tfrac{1}{1-R}\ln((1-R)\lambda^{0,\text{crit}}_{\varepsilon>0}(t,x)+1)\\ &= -\tfrac{1}{1-R}\ln\Big((1-R)\tfrac{1}{B^0_c}\Big(1-\Big(\tfrac{(d^{\pi,c,0}(\pi^{0,M},0))^+}{\frac{1}{R}(B^0_\pi)^\top.(\sigma^0.(\sigma^0)^\top)^{-1}.B^0_\pi}B^0_c+1\Big)^+\Big)+1\Big)\\ &= -\tfrac{1}{1-R}\ln\Big(-(1-R)\tfrac{(d^{\pi,c,0}(\pi^{0,M},0))^+}{\frac{1}{R}(B^0_\pi)^\top.(\sigma^0.(\sigma^0)^\top)^{-1}.B^0_\pi}+1\Big)\wedge h^{\text{crit}}\\ &= h^{0,\text{crit}}_{\varepsilon=0}.\end{aligned}$$

Further

$$\begin{aligned}\lim_{\varepsilon\to 0}H^{\pi,c,1}_{l,M,\varepsilon>0}(t,x,y) &= \lim_{\varepsilon\to 0}\Big(-\tfrac{R}{1-R}c^M(t,x)-\tfrac{1}{1-R}\tfrac{\varepsilon\delta}{\varepsilon-(\varepsilon-\delta)e^{-\delta(T-t)}}+\Psi^1+C^1\tfrac{1}{1-R}(e^{(1-R)y}-1)\Big)\\ &= \Psi^1+C^1\tfrac{1}{1-R}(e^{(1-R)y}-1)\\ &= H^{\pi,c,1}_{l,M,\varepsilon=0}(y)\end{aligned}$$

and

$$\begin{aligned}\lim_{\varepsilon\to 0} H^{\pi,c,1}_{a,\sim,\varepsilon>0}(t,x,y) &= \lim_{\varepsilon\to 0}\Big(-\tfrac{R}{1-R}c^M(t,x)\big(1-B^1_c\tfrac{1}{1-R}(e^{(1-R)y}-1)\big)-\tfrac{1}{1-R}\tfrac{\varepsilon\delta}{\varepsilon-(\varepsilon-\delta)e^{-\delta(T-t)}}+\Psi^1\\ &\qquad -\tfrac{1}{2}\tfrac{1}{R}(B^1_\pi)^\top.(\sigma^1.(\sigma^1)^\top)^{-1}.B^1_\pi\tfrac{1}{(1-R)^2}(e^{(1-R)y}-1)^2\\ &\qquad +(A^1+(\pi^{1,M})^\top.B^1_\pi)\tfrac{1}{1-R}(e^{(1-R)y}-1)\Big)\\ &= \Psi^1-\tfrac{1}{2}\tfrac{1}{R}(B^1_\pi)^\top.(\sigma^1.(\sigma^1)^\top)^{-1}.B^1_\pi\tfrac{1}{(1-R)^2}(e^{(1-R)y}-1)^2\\ &\quad +(A^1+(\pi^{1,M})^\top.B^1_\pi)\tfrac{1}{1-R}(e^{(1-R)y}-1)\\ &= H^{\pi,c,1}_{a,\sim,\varepsilon=0}(y).\end{aligned}$$

ad iii) The limiting behavior of the maximizing strategy is a consequence of i) and ii). In state 0 this is obvious. For state 1 notice further that $H^{\pi,c,1}_{l,M,\varepsilon=0}(y) > H^{\pi,c,1}_{a,\sim,\varepsilon=0}(y)$ is equivalent to $y < h^{1,\mathrm{crit}}_{\varepsilon=0}$. This finishes the proof. □

Remark 4.138. *So far the strategy $(\pi^{i,*}, c^{i,*})$ is only a candidate solution for the optimal investment problem. To verify the optimality we need to show that the related HJB-system has a global solution that satisfies the required regularity conditions stated in the Verification Theorem 3.3.*

Having established the strategy that maximizes the HJB-system in general form inserting $(\pi^{i,*}(t, g(t) - 1_{\{i=1\}}h(t), h(t)), c^{i,*}(t, g(t) - 1_{\{i=1\}}h(t), h(t)))$ the reduced HJB-equation (4.57) now becomes a system of coupled backward ODEs which for $\varepsilon > 0$ is given by

$$\begin{aligned} h'(t) = &-(\Psi^0-\Psi^1)+\big(\tfrac{\varepsilon\delta}{\varepsilon-(\varepsilon-\delta)e^{-\delta(T-t)}}\big)^{\frac{1}{R}}e^{-\frac{1-R}{R}g(t)}\tfrac{R}{1-R}\big(e^{\frac{1-R}{R}h(t)}-1\big)\\ &-C^0\tfrac{1}{1-R}(e^{-(1-R)h(t)}-1)+C^1\tfrac{1}{1-R}(e^{(1-R)h(t)}-1)\\ &-\Big[(A^0+(\pi^{0,M})^\top.B^0_\pi+B^0_c c^M(t,g(t))-C^0)\tfrac{1}{1-R}(e^{-(1-R)(h(t)\wedge h^{0,\mathrm{crit}}(t,g(t)))}-1)\\ &\qquad+\tfrac{1}{2}\tfrac{1}{R}(B^0_\pi)^\top.(\sigma^0.(\sigma^0)^\top)^{-1}.B^0_\pi\tfrac{1}{(1-R)^2}(e^{-(1-R)(h(t)\wedge h^{0,\mathrm{crit}}(t,g(t)))}-1)^2\\ &\qquad+\Big(\tfrac{R}{1-R}\big((1-B^0_c\tfrac{1}{1-R}(e^{-(1-R)(h(t)\wedge h^{0,\mathrm{crit}}(t,g(t)))}-1))^{-\frac{1-R}{R}}-1\big)\\ &\qquad\quad-B^0_c\tfrac{1}{1-R}(e^{-(1-R)(h(t)\wedge h^{0,\mathrm{crit}}(t,g(t)))}-1)\Big)c^M(t,g(t))\Big]\\ &+\Big[(A^1+(\pi^{1,M})^\top.B^1_\pi+B^1_c c^M(t,g(t)-h(t))-C^1)\tfrac{1}{1-R}(e^{(1-R)h(t)}-1)\\ &\qquad+\tfrac{1}{2}\tfrac{1}{R}(B^1_\pi)^\top.(\sigma^1.(\sigma^1)^\top)^{-1}.B^1_\pi\tfrac{1}{(1-R)^2}(e^{(1-R)h(t)}-1)^2\\ &\qquad+\Big(\tfrac{R}{1-R}\big((1-B^1_c\tfrac{1}{1-R}(e^{(1-R)h(t)}-1))^{-\frac{1-R}{R}}-1\big)\\ &\qquad\quad-B^1_c\tfrac{1}{1-R}(e^{(1-R)h(t)}-1)\Big)c^M(t,g(t)-h(t))\Big]^+,\end{aligned} \tag{4.59}$$

$$\begin{aligned} g'(t) = &- \Psi^0 + \tfrac{\varepsilon\delta}{\varepsilon-(\varepsilon-\delta)e^{-\delta(T-t)}}\Big(1 - \tfrac{R}{1-R}\big(\big(\tfrac{\varepsilon\delta}{\varepsilon-(\varepsilon-\delta)e^{-\delta(T-t)}}\big)^{\frac{1-R}{R}} e^{-\frac{1-R}{R}g(t)} - 1\big)\Big) \\ &- C^0 \tfrac{1}{1-R}(e^{-(1-R)h(t)} - 1) \\ &- \Big[(A^0 + (\pi^{0,M})^\top.B^0_\pi + B^0_c c^M(t,g(t)) - C^0)\tfrac{1}{1-R}(e^{-(1-R)(h(t)\wedge h^{0,\mathrm{crit}}(t,g(t)))} - 1) \\ &\quad + \tfrac{1}{2}\tfrac{1}{R}(B^0_\pi)^\top.(\sigma^0.(\sigma^0)^\top)^{-1}.B^0_\pi \tfrac{1}{(1-R)^2}(e^{-(1-R)(h(t)\wedge h^{0,\mathrm{crit}}(t,g(t)))} - 1)^2 \\ &\quad + \Big(\tfrac{R}{1-R}\big((1 - B^0_c \tfrac{1}{1-R}(e^{-(1-R)(h(t)\wedge h^{0,\mathrm{crit}}(t,g(t)))} - 1))^{-\frac{1-R}{R}} - 1\big) \\ &\qquad - B^0_c \tfrac{1}{1-R}(e^{-(1-R)(h(t)\wedge h^{0,\mathrm{crit}}(t,g(t)))} - 1)\Big)c^M(t,g(t))\Big] \end{aligned} \tag{4.60}$$

subject to the boundary conditions

$$g(T) = 0, \ h(T) = 0. \tag{4.61}$$

If $\varepsilon = 0$ then the ODE-system reads

$$\begin{aligned} h'(t) = &- (\Psi^0 - \Psi^1) \\ &- C^0 \tfrac{1}{1-R}(e^{-(1-R)h(t)} - 1) + C^1 \tfrac{1}{1-R}(e^{(1-R)h(t)} - 1) \\ &- (A^0 + (\pi^{0,M})^\top.B^0_\pi - C^0)\tfrac{1}{1-R}(e^{-(1-R)(h(t)\wedge h^{0,\mathrm{crit}})} - 1) \\ &- \tfrac{1}{2}\tfrac{1}{R}(B^0_\pi)^\top.(\sigma^0.(\sigma^0)^\top)^{-1}.B^0_\pi \tfrac{1}{(1-R)^2}(e^{-(1-R)(h(t)\wedge h^{0,\mathrm{crit}})} - 1)^2 \\ &+ (A^1 + (\pi^{1,M})^\top.B^1_\pi - C^1)\tfrac{1}{1-R}(e^{(1-R)(h(t)\vee h^{1,\mathrm{crit}})} - 1) \\ &+ \tfrac{1}{2}\tfrac{1}{R}(B^1_\pi)^\top.(\sigma^1.(\sigma^1)^\top)^{-1}.B^1_\pi \tfrac{1}{(1-R)^2}(e^{(1-R)(h(t)\vee h^{1,\mathrm{crit}})} - 1)^2, \end{aligned} \tag{4.62}$$

$$\begin{aligned} g'(t) = &- \Psi^0 \\ &- C^0 \tfrac{1}{1-R}(e^{-(1-R)h(t)} - 1) \\ &- (A^0 + (\pi^{0,M})^\top.B^0_\pi - C^0)\tfrac{1}{1-R}(e^{-(1-R)(h(t)\wedge h^{0,\mathrm{crit}})} - 1) \\ &- \tfrac{1}{2}\tfrac{1}{R}(B^0_\pi)^\top.(\sigma^0.(\sigma^0)^\top)^{-1}.B^0_\pi \tfrac{1}{(1-R)^2}(e^{-(1-R)(h(t)\wedge h^{0,\mathrm{crit}})} - 1)^2 \end{aligned} \tag{4.63}$$

subject to the boundary conditions

$$g(T) = 0, \ h(T) = 0. \tag{4.64}$$

Remark 4.139. *The aforementioned ODEs include the ODEs of the classical Merton problem (first rows) and the ODEs of the Bäuerle-Rieder problem with constant regime shift intensities C^i (first two rows).*

Remark 4.140. *Notice that the ODEs above resemble the ODEs from the portfolio-dependent model with $\varepsilon = 0$, namely the ODEs (4.43) and (4.44). The hidden difference lies in $h^{0,\mathrm{crit}}$. Whereas $h^{0,\mathrm{crit}}$ is given by*

$$h^{0,\mathrm{crit}}_{\mathrm{PCD}} = -\tfrac{1}{1-R}\ln\Big((1-R)\tfrac{1}{B^0_c}\Big(1 - \Big(\tfrac{(A^0+(\pi^{0,M})^\top.B^0_\pi-C^0)^+}{\frac{1}{R}(B^0_\pi)^\top.(\sigma^0.(\sigma^0)^\top)^{-1}.B^0_\pi}B^0_c + 1\Big)^+\Big) + 1\Big)$$

in case of portfolio- and consumption-dependent intensities it is given by

$$h^{0,\mathrm{crit}}_{\mathrm{PD}} = -\tfrac{1}{1-R}\ln\Big(-(1-R)\tfrac{(A^0+(\pi^{0,M})^\top.B^0_\pi-C^0)^+}{\frac{1}{R}(B^0_\pi)^\top.(\sigma^0.(\sigma^0)^\top)^{-1}.B^0_\pi} + 1\Big)$$

in case of portfolio-dependent intensities.

Therefore if $\varepsilon = 0$ *and* $\frac{(A^0+(\pi^{0,M})^\top.B^0_\pi - C^0)^+}{\frac{1}{R}(B^0_\pi)^\top.(\sigma^0.(\sigma^0)^\top)^{-1}.B^0_\pi} B^0_c + 1 \geq 0$ *then* $h^{0,\text{crit}}_{\text{PCD}} = h^{0,\text{crit}}_{\text{PD}}$ *and both models coincide in that the the optimal strategies and the underlying ODEs for the functions* g *and* h *are the same. Notice that* $\frac{(A^0+(\pi^{0,M})^\top.B^0_\pi - C^0)^+}{\frac{1}{R}(B^0_\pi)^\top.(\sigma^0.(\sigma^0)^\top)^{-1}.B^0_\pi} B^0_c + 1 \geq 0$ *is trivially satisfied if* $B^0_c > 0$.

Lemma 4.141. *The ODE-system given by equation (4.59) and (4.60) subject to the boundary conditions (4.61)* ($\varepsilon > 0$)*, resp. (4.62) and (4.63) subject to the boundary conditions (4.64)* ($\varepsilon = 0$)*, admits a unique global solution.*

Proof. As in the consumption-dependent case we have to distinguish the cases $\varepsilon > 0$ and $\varepsilon = 0$.

- *Definition of the* χ^i*'s* ($\varepsilon > 0$). Let $\varepsilon > 0$. Then the χ^i's are given by $\chi^i: \ [0,T] \times \mathbb{R} \times \mathbb{R}^+_0$, $i = 0, 1$, with

$$\begin{aligned}
\chi^0(t,x,y) \triangleq &- C^0 \tfrac{1}{1-R}(e^{-(1-R)y} - 1) - d^{\pi,c,0}(\pi^{0,M}, c^M(t,x)) \tfrac{1}{1-R}(e^{-(1-R)(y \wedge h^{0,\text{crit}}(t,x))} - 1) \\
&- \tfrac{1}{2}\tfrac{1}{R}(B^0_\pi)^\top.(\sigma^0.(\sigma^0)^\top)^{-1}.B^0_\pi \tfrac{1}{(1-R)^2}(e^{-(1-R)(y \wedge h^{0,\text{crit}}(t,x))} - 1)^2 \\
&- \Big(\tfrac{R}{1-R}\big((1 - B^0_c \tfrac{1}{1-R}(e^{-(1-R)(y \wedge h^{0,\text{crit}}(t,x))} - 1))^{-\frac{1-R}{R}} - 1\big) \\
&\qquad - B^0_c \tfrac{1}{1-R}(e^{-(1-R)(y \wedge h^{0,\text{crit}}(t,x))} - 1)\Big) c^M(t,x), \\
\chi^1(t,x,y) \triangleq &\ C^1 \tfrac{1}{1-R}(e^{(1-R)y} - 1) + \Big[d^{\pi,c,1}(\pi^{1,M}, c^M(t,x)) \tfrac{1}{1-R}(e^{(1-R)y} - 1) \\
&+ \tfrac{1}{2}\tfrac{1}{R}(B^1_\pi)^\top.(\sigma^1.(\sigma^1)^\top)^{-1}.B^1_\pi \tfrac{1}{(1-R)^2}(e^{(1-R)y} - 1)^2 \\
&+ \Big(\tfrac{R}{1-R}\big((1 - B^1_c \tfrac{1}{1-R}(e^{(1-R)y} - 1))^{-\frac{1-R}{R}} - 1\big) - B^1_c \tfrac{1}{1-R}(e^{(1-R)y} - 1)\Big) c^M(t,x)\Big]^+.
\end{aligned}$$

- *Continuity results on the* χ^i*'s* ($\varepsilon > 0$). The χ^i's are continuous in t and further, as compositions of continuously differentiable and locally Lipschitz continuous functions in x and y, the $[\,\cdot\,]^+$- and the $\min\{\cdot,\cdot\}$-function locally Lipschitz continuous in x and y.

- *Non-negativity of the* χ^i*'s* ($\varepsilon > 0$). The non-negativity of χ^1 is obvious. In order to see that $\chi^0 \geq 0$ we distinguish the two cases that already appeared in the proof of Lemma 4.129; $(\pi^{0,M}, c^M(t,x)) \notin \mathcal{I}^{\pi,c,0}$ and $(\pi^{0,M}, c^M(t,x)) \in \mathcal{I}^{\pi,c,0}$.

  - $(\pi^{0,M}, c^M(t,x)) \notin \mathcal{I}^{\pi,c,0}$. If $(\pi^{0,M}, c^M(t,x)) \notin \mathcal{I}^{\pi,c,0}$ then $h^{0,\text{crit}}(t,x) = 0$ implying $\chi^0(t,x,y) = -C^0 \frac{1}{1-R}(e^{-(1-R)y} - 1)$ which is non-negative as $y \geq 0$.

  - $(\pi^{0,M}, c^M(t,x)) \in \mathcal{I}^{\pi,c,0}$. If otherwise $(\pi^{0,M}, c^M(t,x)) \in \mathcal{I}^{\pi,c,0}$ then $h^{0,\text{crit}}(t,x)$ is strictly positive and there are again two cases to be distinguished; $y < h^{0,\text{crit}}(t,x)$ and $y \geq h^{0,\text{crit}}(t,x)$.

    - $y < h^{0,\text{crit}}(t,x)$. We know from the proof of Lemma 4.129 that $y < h^{0,\text{crit}}(t,x)$ is equivalent to $(\tilde{\pi}^{0,*}(t,x,y), \tilde{c}^{0,*}(t,x,y)) \in \mathcal{I}^{\pi,c,0}$ which, further, by Lemma 4.128, i), implies that $H^{\pi,c,0}_l(t,x,y) > H^{\pi,c,0}_a(t,x,y)$. Some simple calculations show that $H^{\pi,c,0}_{l,M}(t,x,y) > H^{\pi,c,0}_{a,\sim}(t,x,y)$

is equivalent to

$$\begin{aligned}
0 <& - d^{\pi,c,0}(\pi^{0,M}, c^M(t,x))\tfrac{1}{1-R}(e^{-(1-R)y} - 1) \\
&- \tfrac{1}{2}\tfrac{1}{R}(B_\pi^0)^\top.(\sigma^0.(\sigma^0)^\top)^{-1}.B_\pi^0\tfrac{1}{(1-R)^2}(e^{-(1-R)y} - 1)^2 \\
&- \Big(\tfrac{R}{1-R}\big((1 - B_c^0\tfrac{1}{1-R}(e^{-(1-R)y} - 1))^{-\frac{1-R}{R}} - 1\big) - B_c^0\tfrac{1}{1-R}(e^{-(1-R)y} - 1)\Big)c^M(t,x)
\end{aligned}$$

which implies $\chi^0(t,x,y) > 0$.

- $y \geq h^{0,\mathrm{crit}}(t,x)$. If otherwise $y \geq h^{0,\mathrm{crit}}(t,x)$ then

$$\begin{aligned}
\chi^0(t,x,y) =& - C^0\tfrac{1}{1-R}(e^{-(1-R)y} - 1) - d^{\pi,c,0}(\pi^{0,M}, c^M(t,x))\tfrac{1}{1-R}(e^{-(1-R)h^{0,\mathrm{crit}}(t,x)} - 1) \\
&- \tfrac{1}{2}\tfrac{1}{R}(B_\pi^0)^\top.(\sigma^0.(\sigma^0)^\top)^{-1}.B_\pi^0\tfrac{1}{(1-R)^2}(e^{-(1-R)h^{0,\mathrm{crit}}(t,x)} - 1)^2 \\
&- \Big(\tfrac{R}{1-R}\big((1 - B_c^0\tfrac{1}{1-R}(e^{-(1-R)h^{0,\mathrm{crit}}(t,x)} - 1))^{-\frac{1-R}{R}} - 1\big) \\
&\quad - B_c^0\tfrac{1}{1-R}(e^{-(1-R)h^{0,\mathrm{crit}}(t,x)} - 1)\Big)c^M(t,x) \\
=& - C^0\tfrac{1}{1-R}(e^{-(1-R)y} - 1) - (d^{\pi,c,0}(\pi^{0,M}, c^M(t,x)))^+\tfrac{1}{1-R}(e^{-(1-R)h^{0,\mathrm{crit}}(t,x)} - 1) \\
&- \tfrac{1}{2}\tfrac{1}{R}(B_\pi^0)^\top.(\sigma^0.(\sigma^0)^\top)^{-1}.B_\pi^0\tfrac{1}{(1-R)^2}(e^{-(1-R)h^{0,\mathrm{crit}}(t,x)} - 1)^2 \\
&- \Big(\tfrac{R}{1-R}\big((1 - B_c^0\tfrac{1}{1-R}(e^{-(1-R)h^{0,\mathrm{crit}}(t,x)} - 1))^{-\frac{1-R}{R}} - 1\big) \\
&\quad - B_c^0\tfrac{1}{1-R}(e^{-(1-R)h^{0,\mathrm{crit}}(t,x)} - 1)\Big)c^M(t,x) \\
=& - C^0\tfrac{1}{1-R}(e^{-(1-R)y} - 1) + \tfrac{1}{2}\tfrac{1}{R}(B_\pi^0)^\top.(\sigma^0.(\sigma^0)^\top)^{-1}.B_\pi^0\tfrac{1}{(1-R)^2}(e^{-(1-R)h^{0,\mathrm{crit}}(t,x)} - 1)^2 \\
&- \Big(\tfrac{R}{1-R}\big((1 - B_c^0\tfrac{1}{1-R}(e^{-(1-R)h^{0,\mathrm{crit}}(t,x)} - 1))^{-\frac{1-R}{R}} - 1\big) \\
&\quad - B_c^0\tfrac{1}{1-R}(e^{-(1-R)h^{0,\mathrm{crit}}(t,x)} - 1)(1 - B_c^0\tfrac{1}{1-R}(e^{-(1-R)h^{0,\mathrm{crit}}(t,x)} - 1))^{-\frac{1}{R}}\Big)c^M(t,x),
\end{aligned}$$

since $\lambda^{0,\mathrm{crit}}(t,x) = \frac{1}{1-R}(e^{-(1-R)h^{0,\mathrm{crit}}(t,x)} - 1)$ is the unique solution of equation (4.58). Further, Lemma 4.48, i), yields with $z = -B_c^0\frac{1}{1-R}(e^{-(1-R)h^{0,\mathrm{crit}}(t,x)} - 1)$ that

$$\begin{aligned}
0 \leq& - \Big(\tfrac{R}{1-R}\big((1 - B_c^0\tfrac{1}{1-R}(e^{-(1-R)h^{0,\mathrm{crit}}(t,x)} - 1))^{-\frac{1-R}{R}} - 1\big) \\
&\quad - B_c^0\tfrac{1}{1-R}(e^{-(1-R)h^{0,\mathrm{crit}}(t,x)} - 1)(1 - B_c^0\tfrac{1}{1-R}(e^{-(1-R)h^{0,\mathrm{crit}}(t,x)} - 1))^{-\frac{1}{R}}\Big),
\end{aligned}$$

where the lemma is applicable since

$$-B_c^0\tfrac{1}{1-R}(e^{-(1-R)h^{0,\mathrm{crit}}(t,x)} - 1) = -B_c^0\lambda^{0,\mathrm{crit}}(t,x) > -1.$$

If $B_c^0 > 0$ this is trivially true since $\lambda^{0,\mathrm{crit}}(t,x) \leq 0$. If otherwise $B_c^0 < 0$ then $\lambda^{0,\mathrm{crit}}(t,x) > \frac{1}{B_c^0}$ implying the desired result.

Thus after all $\chi^0(t,x,y) \geq 0$.

We now come to the case $\varepsilon = 0$.

- *Definition of the χ^i's ($\varepsilon = 0$).* Let $\varepsilon = 0$. Then the χ^i's are given by $\chi^i: R_0^+ \to \mathbb{R}$, $i = 0, 1$, with

$$\begin{aligned}\chi^0(y) \triangleq &- C^0 \tfrac{1}{1-R}(e^{-(1-R)y} - 1) - d^{\pi,c,0}(\pi^{0,M}, 0) \tfrac{1}{1-R}(e^{-(1-R)(y \wedge h^{0,\text{crit}})} - 1) \\ &- \tfrac{1}{2}\tfrac{1}{R}(B^0_\pi)^\top.(\sigma^0.(\sigma^0)^\top)^{-1}.B^0_\pi \tfrac{1}{(1-R)^2}(e^{-(1-R)(y \wedge h^{0,\text{crit}})} - 1)^2, \\ \chi^1(y) \triangleq &\, C^1 \tfrac{1}{1-R}(e^{(1-R)y} - 1) + d^{\pi,c,1}(\pi^{1,M}, 0) \tfrac{1}{1-R}(e^{(1-R)(y \vee h^{1,\text{crit}})} - 1) \\ &+ \tfrac{1}{2}\tfrac{1}{R}(B^1_\pi)^\top.(\sigma^1.(\sigma^1)^\top)^{-1}.B^1_\pi \tfrac{1}{(1-R)^2}(e^{(1-R)(y \vee h^{1,\text{crit}})} - 1)^2.\end{aligned}$$

- *Continuity results on the χ^i's ($\varepsilon = 0$).* The functions χ^i's are as compositions of continuously differentiable functions, the $\min\{\cdot,\cdot\}$- and the $\max\{\cdot,\cdot\}$-function locally Lipschitz continuous.

- *Non-negativity of χ^0 ($\varepsilon = 0$).* In order to prove the non-negativity of $\chi^0 \geq 0$ we distinguish the cases $(\pi^{0,M}, 0) \notin \mathcal{I}^{\pi,c,0}$ and $(\pi^{0,M}, 0) \in \mathcal{I}^{\pi,c,0}$.

  - $(\pi^{0,M}, 0) \notin \mathcal{I}^{\pi,c,0}$. If $(\pi^{0,M}, 0) \notin \mathcal{I}^{\pi,c,0}$ then $h^{0,\text{crit}} = 0$ implying $\chi^0(y) = -C^0 \frac{1}{1-R}(e^{-(1-R)y} - 1) \geq 0$ as $y \geq 0$.

  - $(\pi^{0,M}, 0) \in \mathcal{I}^{\pi,c,0}$. If otherwise $(\pi^{0,M}, 0) \in \mathcal{I}^{\pi,c,0}$ then there are again two cases to be distinguished; $y < h^{0,\text{crit}}$ and $y \geq h^{0,\text{crit}}$.

    - $y < h^{0,\text{crit}}$. If $y < h^{0,\text{crit}}$ then $(\tilde{\pi}^{0,*}(t,x,y), 0) \in \mathcal{I}^{\pi,c,0}$ which by Lemma 4.128, i), implies that $H^{\pi,c,0}_{l,M}(t,x,y) > H^{\pi,c,0}_{a,\sim}(t,x,y)$. As $H^{\pi,c,0}_{l,M}(t,x,y) > H^{\pi,c,0}_{a,\sim}(t,x,y)$ is equivalent to

$$0 < - d^{\pi,c,0}(\pi^{0,M}, 0) \tfrac{1}{1-R}(e^{-(1-R)y} - 1) - \tfrac{1}{2}\tfrac{1}{R}(B^0_\pi)^\top.(\sigma^0.(\sigma^0)^\top)^{-1}.B^0_\pi \tfrac{1}{(1-R)^2}(e^{-(1-R)y} - 1)^2$$

    it follows that $\chi^0(y) > 0$ as $y \geq 0$.

    - $y \geq h^{0,\text{crit}}$. If $y \geq h^{0,\text{crit}}$ then

$$\begin{aligned}\chi^0(y) = &- C^0 \tfrac{1}{1-R}(e^{-(1-R)y} - 1) - d^{\pi,c,0}(\pi^{0,M}, 0) \tfrac{1}{1-R}(e^{-(1-R)h^{0,\text{crit}}} - 1) \\ &- \tfrac{1}{2}\tfrac{1}{R}(B^0_\pi)^\top.(\sigma^0.(\sigma^0)^\top)^{-1}.B^0_\pi \tfrac{1}{(1-R)^2}(e^{-(1-R)h^{0,\text{crit}}} - 1)^2 \\ = &- C^0 \tfrac{1}{1-R}(e^{-(1-R)y} - 1) - (d^{\pi,c,0}(\pi^{0,M}, 0))^+ \tfrac{1}{1-R}(e^{-(1-R)h^{0,\text{crit}}} - 1) \\ &- \tfrac{1}{2}\tfrac{1}{R}(B^0_\pi)^\top.(\sigma^0.(\sigma^0)^\top)^{-1}.B^0_\pi \tfrac{1}{(1-R)^2}(e^{-(1-R)h^{0,\text{crit}}} - 1)^2\end{aligned}$$

    where

$$\tfrac{1}{1-R}(e^{-(1-R)h^{0,\text{crit}}} - 1) = \begin{cases} \dfrac{(d^{\pi,c,0}(\pi^{0,M},0))^+}{\frac{1}{R}(B^0_\pi)^\top.(\sigma^0.(\sigma^0)^\top)^{-1}.B^0_\pi} & \text{if } B^0_c > 0, \\ \frac{1}{B^0_c}\Big(1 - \Big(\frac{(d^{\pi,c,0}(\pi^{0,M},0))^+}{\frac{1}{R}(B^0_\pi)^\top.(\sigma^0.(\sigma^0)^\top)^{-1}.B^0_\pi} B^0_c + 1\Big)^+\Big) & \text{if } B^0_c < 0. \end{cases}$$

    Thus, if $B^0_c > 0$ then

$$\chi^0(y) = -C^0 \tfrac{1}{1-R}(e^{-(1-R)y} - 1) + \tfrac{1}{2} \frac{((d^{\pi,c,0}(\pi^{0,M},0))^+)^2}{\frac{1}{R}(B^0_\pi)^\top.(\sigma^0.(\sigma^0)^\top)^{-1}.B^0_\pi} \geq 0.$$

If otherwise $B^0_c < 0$ then

$$\begin{aligned}
\chi^0(y) =& - C^0 \tfrac{1}{1-R}(e^{-(1-R)y} - 1) - (d^{\pi,c,0}(\pi^{0,M},0))^+ \tfrac{1}{B^0_c}\Big(1 - \Big(\tfrac{(d^{\pi,c,0}(\pi^{0,M},0))^+}{\frac{1}{R}(B^0_\pi)^\top.(\sigma^0.(\sigma^0)^\top)^{-1}.B^0_\pi} B^0_c + 1\Big)^+\Big) \\
&- \tfrac{1}{2}\tfrac{1}{R}(B^0_\pi)^\top.(\sigma^0.(\sigma^0)^\top)^{-1}.B^0_\pi \tfrac{1}{(B^0_c)^2}\Big(1 - \Big(\tfrac{(d^{\pi,c,0}(\pi^{0,M},0))^+}{\frac{1}{R}(B^0_\pi)^\top.(\sigma^0.(\sigma^0)^\top)^{-1}.B^0_\pi} B^0_c + 1\Big)^+\Big)^2 \\
=& - C^0 \tfrac{1}{1-R}(e^{-(1-R)y} - 1) - \tfrac{\frac{1}{R}(B^0_\pi)^\top.(\sigma^0.(\sigma^0)^\top)^{-1}.B^0_\pi}{(B^0_c)^2}\Big(1 - \Big(\tfrac{(d^{\pi,c,0}(\pi^{0,M},0))^+}{\frac{1}{R}(B^0_\pi)^\top.(\sigma^0.(\sigma^0)^\top)^{-1}.B^0_\pi} B^0_c + 1\Big)^+\Big) \\
&\cdot \Big(\tfrac{(d^{\pi,c,0}(\pi^{0,M},0))^+}{\frac{1}{R}(B^0_\pi)^\top.(\sigma^0.(\sigma^0)^\top)^{-1}.B^0_\pi} B^0_c + \tfrac{1}{2}\Big(1 - \Big(\tfrac{(d^{\pi,c,0}(\pi^{0,M},0))^+}{\frac{1}{R}(B^0_\pi)^\top.(\sigma^0.(\sigma^0)^\top)^{-1}.B^0_\pi} B^0_c + 1\Big)^+\Big)\Big) \\
=& - C^0 \tfrac{1}{1-R}(e^{-(1-R)y} - 1) + \tfrac{1}{2}\tfrac{\frac{1}{R}(B^0_\pi)^\top.(\sigma^0.(\sigma^0)^\top)^{-1}.B^0_\pi}{(B^0_c)^2} \\
&\cdot \Big(\Big(1 - \Big(\tfrac{(d^{\pi,c,0}(\pi^{0,M},0))^+}{\frac{1}{R}(B^0_\pi)^\top.(\sigma^0.(\sigma^0)^\top)^{-1}.B^0_\pi} B^0_c + 1\Big)^+\Big)^2 + 2\Big(\tfrac{(d^{\pi,c,0}(\pi^{0,M},0))^+}{\frac{1}{R}(B^0_\pi)^\top.(\sigma^0.(\sigma^0)^\top)^{-1}.B^0_\pi} B^0_c + 1\Big)^-\Big).
\end{aligned}$$

Thus after all $\chi^0(y) \geq 0$ as $y \geq 0$.

- *Non-negativity of χ^1 ($\varepsilon = 0$).* Here we have to distinguish the cases $(\pi^{1,M},0) \in \mathcal{I}^{\pi,c,1}$ and $(\pi^{1,M},0) \notin \mathcal{I}^{\pi,c,1}$.
  - $(\pi^{1,M},0) \in \mathcal{I}^{\pi,c,1}$. If $(\pi^{1,M},0) \in \mathcal{I}^{\pi,c,1}$ then $h^{1,\text{crit}} = 0$ and $\chi^1(y) \geq 0$ holds trivially since $(\pi^{1,M},0) \in \mathcal{I}^{\pi,c,1}$ is equivalent to $d^{\pi,c,1}(\pi^{1,M},0) \geq 0$.
  - $(\pi^{1,M},0) \notin \mathcal{I}^{\pi,c,1}$. If $(\pi^{1,M},0) \notin \mathcal{I}^{\pi,c,1}$ then $h^{1,\text{crit}}$ is strictly positive and we need to distinguish the cases $y < h^{1,\text{crit}}$ and $y \geq h^{1,\text{crit}}$.
    - $y < h^{1,\text{crit}}$. If $y < h^{1,\text{crit}}$ then $\chi^1(y) = C^1 \frac{1}{1-R}(e^{(1-R)y} - 1) \geq 0$ as $y \geq 0$ by the definition of $h^{1,\text{crit}}$.
    - $y \geq h^{1,\text{crit}}$. The proof of Lemma 4.129 showed that $y \geq h^{1,\text{crit}}$ implies $H^{\pi,c,1}_{l,M}(t,x,y) \leq H^{\pi,c,1}_{a,\sim}(t,x,y)$ which is equivalent to

$$0 \leq d^{\pi,c,1}(\pi^{1,M},0)\tfrac{1}{1-R}(e^{(1-R)y} - 1) + \tfrac{1}{2}\tfrac{1}{R}(B^1_\pi)^\top.(\sigma^1.(\sigma^1)^\top)^{-1}.B^1_\pi \tfrac{1}{(1-R)^2}(e^{(1-R)y} - 1)^2$$

and thus $\chi^1(y) \geq 0$ as $y \geq 0$.

□

Figure 4.19 shows an example for the functions g and h.

Corollary 4.142 (Time-dependent bounds on g and h)**.** *Let h and g be given by (4.59), (4.60) subject to the boundary conditions (4.61) ($\varepsilon > 0$), resp. (4.62), (4.63) subject to the boundary conditions (4.64) ($\varepsilon = 0$). Then*

$$-(\Psi^1 - \xi^1(T))^-(T-t) \leq g(t) \leq (\Psi^0 - \xi^0(T))^+(T-t) \ (\varepsilon > 0),$$
$$\textit{resp. } 0 \leq g(t) \leq \Psi^0(T-t) \ (\varepsilon = 0) \textit{ and}$$
$$0 \leq h(t) \leq (\Psi^0 - \Psi^1)(T-t)$$

for $t \in [0,T]$ where the $\xi^i(T)$'s are as defined in the proof of Lemma 4.9.

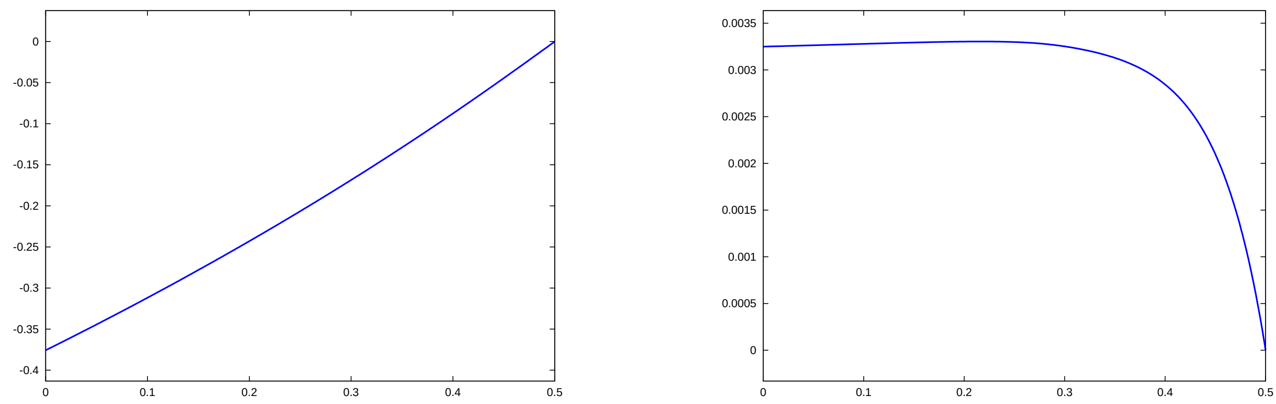

Figure 4.19: g (left) and h (right) as functions of t
($r^0 = r^1 = 0.035$, $\eta^0 = 0.09$, $\eta^1 = 0.01$, $\sigma^0 = 0.2$, $\sigma^1 = 0.5$, $\delta = 0.04$, $\varepsilon = 1$, $R = 1.5$, $T = 0.5$, $A^0 = 4$, $B^0_\pi = -1.5$, $B^0_c = 10$, $C^0 = 10$, $A^1 = 13.5$, $B^1_\pi = 0.75$, $B^1_c = -5$, $C^1 = 8$)

Corollary 4.143 (Time-independent bound on h)**.** *Let h and g be given by (4.59), (4.60) subject to the boundary conditions (4.61) ($\varepsilon > 0$), resp. (4.62), (4.63) subject to the boundary conditions (4.64) ($\varepsilon = 0$). Then*

$$0 \leq h(t) \leq \bar{h}$$

for $t \in [0, T]$ where $\bar{h}$ is the smallest positive root of

$$\bar{F}(y) \triangleq \begin{cases} -(\Psi^0 - \Psi^1) - C^0 \frac{1}{1-R}(e^{-(1-R)y} - 1) + C^1 \frac{1}{1-R}(e^{(1-R)y} - 1) & \text{if } \varepsilon > 0, \\ -(\Psi^0 - \Psi^1) + \chi^0(y) + \chi^1(y) & \text{if } \varepsilon = 0 \end{cases}$$

if such a root exists; otherwise $\bar{h} = \infty$, i.e.

$$\bar{h} = \min \left\{ y \in \mathbb{R}^+_0 \,:\, \bar{F}(y) = 0 \right\},$$

with the convention $\min \emptyset \triangleq \infty$, where the χ^i's are as defined in the proof of Lemma 4.141.

Proof. The assertion follows from a simple ODE argument since

$$F(t, x, y) = -(\Psi^0 - \Psi^1) + \varrho(t, x, y) + \chi^0(t, x, y) + \chi^1(t, x, y) \geq \bar{F}(y)$$

for all $(t, x, y) \in [0, T] \times \mathbb{R} \times \mathbb{R}^+_0$ where we used the notation from the proof of Lemma 4.9. If $\varepsilon = 0$ then this is obviously true. In case of $\varepsilon > 0$ this is satisfied, too, since the proof of Lemma 4.141 showed that $\chi^0(t, x, y) \geq -C^0 \frac{1}{1-R}(e^{-(1-R)y} - 1)$ and $\chi^1(t, x, y) \geq C^1 \frac{1}{1-R}(e^{(1-R)y} - 1)$. □

The boundary function on h as presented in Corollary 4.142 is suitable for large values of t for which the deviation from h is not too large whereas for small values of t the non-linear behavior of h implies that the boundary function overshoots h by far. The converse is true for the bound given in Corollary 4.143. Thus combining the two bounds on h yields

$$0 \leq h(t) \leq \min\{(\Psi^0 - \Psi^1)(T - t), \bar{h}\}$$

for $t \in [0, T]$.

Since the above ODE-system is solvable we can now verify that the strategy given above is indeed the optimal strategy that solves the investment problem.

Theorem 4.144 (Solution of the Investment Problem)**.** *Let condition (NJ) be satisfied, and let h and g be given by (4.59), (4.60) subject to the boundary conditions (4.61) ($\varepsilon > 0$), resp. (4.62), (4.63) subject to the boundary conditions (4.64) ($\varepsilon = 0$). Then the strategy*

$$(\pi^{i,*}(t), c^{i,*}(t)) \triangleq (\pi^{i,*}(t, g(t) - 1_{\{i=1\}}h(t), h(t)), c^{i,*}(t, g(t) - 1_{\{i=1\}}h(t), h(t))),\ t \in [0,T],\ i = 0, 1,$$

as given in Lemma 4.129 is optimal for the investment problem (P) with CRRA *preferences and relative risk aversion R.*

Proof. Since $(\pi^{i,*}(t), c^{i,*}(t))$ maximizes the reduced HJB-system (4.57) for each $t \in [0,T]$, optimality of the strategy $(\pi^{i,*}, c^{i,*})$ follows directly from the Verification Theorem 3.3. □

Remark 4.145. *If not otherwise stated we use the following shorthand notations*

$$\begin{aligned}
\pi^{i,\text{crit}}(t) &\triangleq \pi^{i,\text{crit}}(t, g(t) - 1_{\{i=1\}}h(t)),\\
c^{i,\text{crit}}(t) &\triangleq c^{i,\text{crit}}(t, g(t) - 1_{\{i=1\}}h(t)),\\
h^{i,\text{crit}}(t) &\triangleq h^{i,\text{crit}}(t, g(t) - 1_{\{i=1\}}h(t)),\\
H^{\pi,c,i}_{l,M}(t) &\triangleq H^{\pi,c,i}_{l}(t, g(t) - 1_{\{i=1\}}h(t), h(t), \pi^{i,M}, c^{i,M}(t)),\\
H^{\pi,c,i}_{a,\sim}(t) &\triangleq H^{\pi,c,i}_{a}(t, g(t) - 1_{\{i=1\}}h(t), h(t), \tilde{\pi}^{i,*}(t), \tilde{c}^{i,*}(t)).
\end{aligned}$$

Figure 4.20 shows the optimal strategy and the Merton strategy in a setting where B^0_π is negative, resp. B^1_π is positive, and B^0_c is positive, resp. B^1_c is negative. Hence the large investor is forced to invest more than the Merton fractions and to consume at a rate that is smaller than or equal the Merton consumption rate in order to generate advantageous regime shift intensities.

Once again the deviation from the Merton strategy decreases continuously towards 0 as the time to maturity gets smaller.

A particular example in which the large investor consumes although $\varepsilon = 0$ is given in Figure 4.21. Notice that the optimal portfolio strategy in state 0 converges continuously towards the Merton strategy as the time to maturity decreases whereas the optimal consumption rate shows the already discussed extreme behavior.

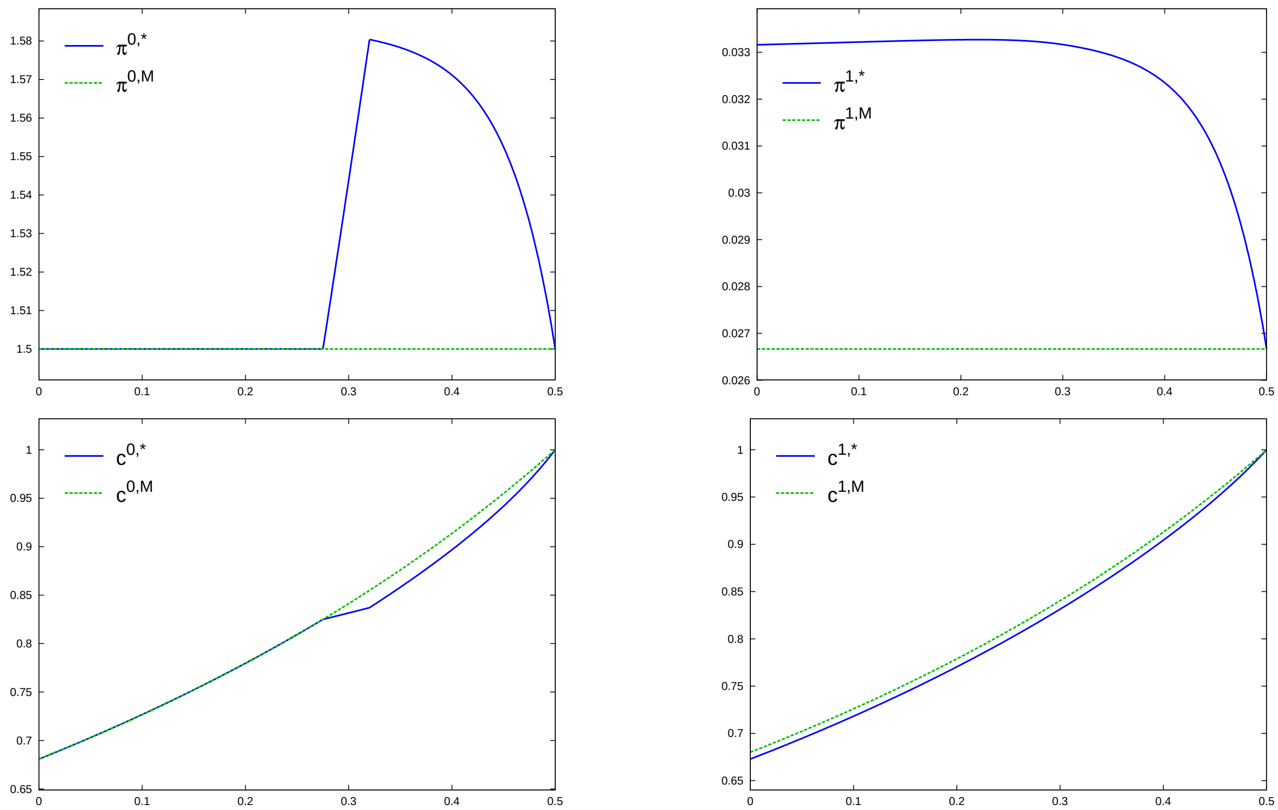

Figure 4.20: Optimal strategy vs. Merton strategy: $\pi^{i,*}$, $c^{i,*}$ and $\pi^{i,M}$, $c^{i,M}$ as functions of t
($r^0 = r^1 = 0.035$, $\eta^0 = 0.09$, $\eta^1 = 0.01$, $\sigma^0 = 0.2$, $\sigma^1 = 0.5$, $\delta = 0.04$, $\varepsilon = 1$, $R = 1.5$, $T = 0.5$,
$A^0 = 4$, $B^0_\pi = -1.5$, $B^0_c = 10$, $C^0 = 10$, $A^1 = 13.5$, $B^1_\pi = 0.75$, $B^1_c = -5$, $C^1 = 8$)

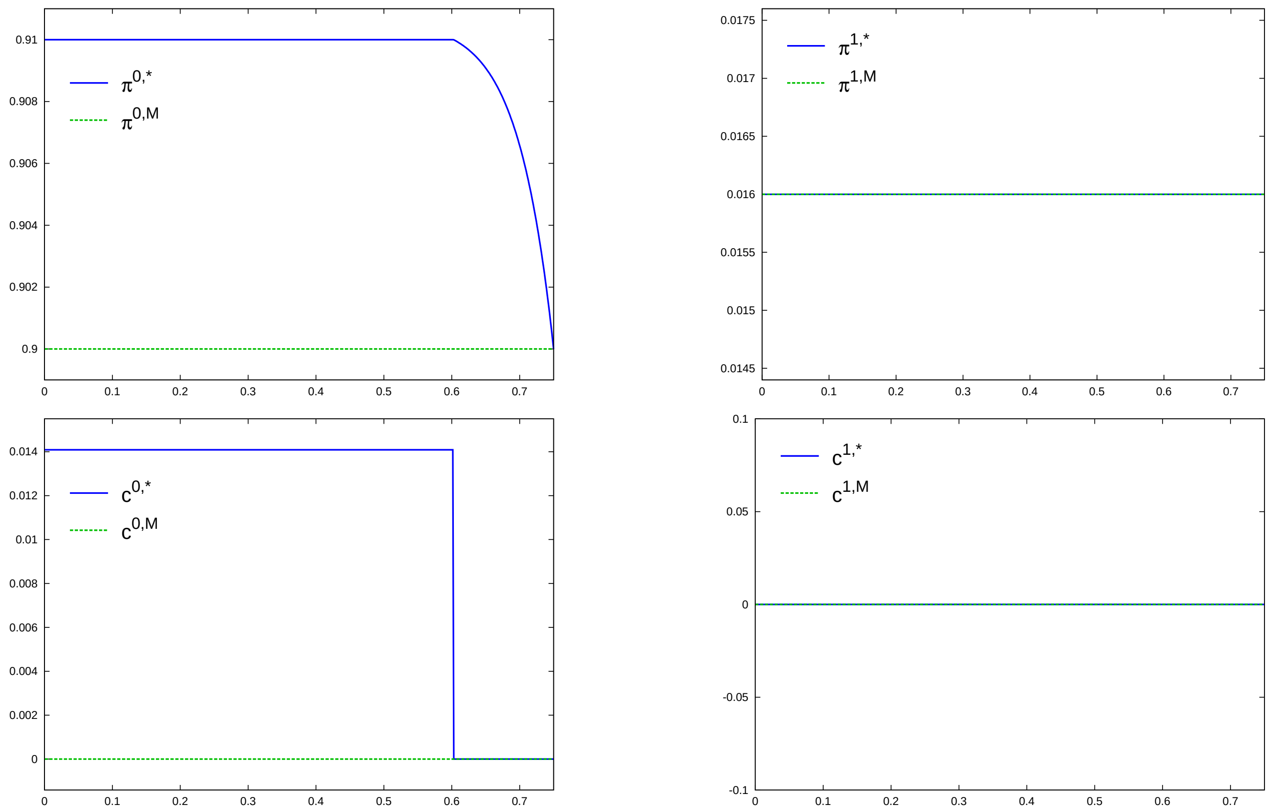

Figure 4.21: Optimal strategy vs. Merton strategy: $\pi^{i,*}$, $c^{i,*}$ and $\pi^{i,M}$, $c^{i,M}$ as functions of t
($r^0 = r^1 = 0.035,\ \eta^0 = 0.09,\ \eta^1 = 0.01,\ \sigma^0 = 0.2,\ \sigma^1 = 0.5,\ \delta = 0.04,\ \varepsilon = 0,\ R = 2.5,\ T = 0.75,$
$A^0 = 17.5,\ B^0_\pi = -0.5,\ B^0_c = -500,\ C^0 = 10,\ A^1 = -1,\ B^1_\pi = 0.25,\ B^1_c = 0,\ C^1 = 2$)

The large investor's optimal portfolio strategy consists of the classical Merton strategy and an additional hedging component. Further the investor's optimal consumption rate can be decomposed into the Merton consumption rate and an adjustment part. The hedging component and the adjustment part result from the investor's influence on the market. So we can write

$$(\pi^{i,*}(t), c^{i,*}(t)) = (\pi^{i,M} + \pi^{i,H}(t), c^{i,M}(t) + c^{i,A}(t)) \text{ for } i = 0, 1$$

where

$$\pi^{0,H}(t) = \tfrac{1}{R}(\sigma^0.(\sigma^0)^\top)^{-1}.B^0_\pi \tfrac{1}{1-R}(e^{-(1-R)(h(t)\wedge h^{0,\mathrm{crit}}(t))} - 1),$$

$$c^{0,A}(t) = \begin{cases} \Big(\big(1 - B^0_c \tfrac{1}{1-R}(e^{-(1-R)(h(t)\wedge h^{0,\mathrm{crit}}(t))} - 1)\big)^{-\frac{1}{R}} - 1\Big) c^{0,M}(t) & \text{if } \varepsilon > 0, \\ \frac{\frac{1}{R}(B^0_\pi)^\top.(\sigma^0.(\sigma^0)^\top)^{-1}.B^0_\pi}{(B^0_c)^2} \Big(\frac{(A^0 + (\pi^{0,M})^\top.B^0_\pi - C^0)^+}{\frac{1}{R}(B^0_\pi)^\top.(\sigma^0.(\sigma^0)^\top)^{-1}.B^0_\pi} B^0_c + 1\Big)^- 1_{\{h(t)\geq h^{0,\mathrm{crit}}\}} & \text{if } \varepsilon = 0, \end{cases}$$

$$\pi^{1,H}(t) = \begin{cases} \frac{1}{R}(\sigma^1.(\sigma^1)^\top)^{-1}.B^1_\pi \frac{1}{1-R}(e^{(1-R)h(t)} - 1) 1_{\{H^{\pi,c,1}_{l,M}(t) \leq H^{\pi,c,1}_{a,\sim}(t)\}} & \text{if } \varepsilon > 0, \\ \frac{1}{R}(\sigma^1.(\sigma^1)^\top)^{-1}.B^1_\pi \frac{1}{1-R}(e^{(1-R)h(t)} - 1) 1_{\{h(t)\geq h^{1,\mathrm{crit}}\}} & \text{if } \varepsilon = 0, \end{cases}$$

$$c^{1,A}(t) = \begin{cases} \Big(\big(1 - B^1_c \tfrac{1}{1-R}(e^{(1-R)h(t)} - 1)\big)^{-\frac{1}{R}} - 1\Big) c^{1,M}(t) 1_{\{H^{\pi,c,1}_{l,M}(t) \leq H^{\pi,c,1}_{a,\sim}(t)\}} & \text{if } \varepsilon > 0, \\ 0 & \text{if } \varepsilon = 0. \end{cases}$$

Note that $(\pi^{0,H}(t))^\top.B^0_\pi \leq 0$, resp. $(\pi^{1,H}(t))^\top.B^1_\pi \geq 0$. Moreover $B^0_c c^{0,A}(t) \leq 0$, resp. $B^1_c c^{1,A}(t) \geq 0$.

The following lemma shows that $(\pi^{i,H}, c^{i,A})$ benefits the regime shift intensities.

Lemma 4.146. *The optimal strategy* $(\pi^{i,*}, c^{i,*})$ *satisfies*

$$\vartheta^{0,1}(\pi^{0,*}(t), c^{0,*}(t)) \leq \vartheta^{0,1}(\pi^{0,M}, c^{0,M}(t)) \text{ and } \vartheta^{1,0}(\pi^{1,*}(t), c^{1,*}(t)) \geq \vartheta^{1,0}(\pi^{1,M}, c^{1,M}(t))$$

for every $t \in [0, T]$.

Proof. The assertion follows since

$$\begin{aligned} \vartheta^{0,1}(\pi^{0,*}(t), c^{0,*}(t)) &= \max\{A^0 + (\pi^{0,*}(t))^\top.B^0_\pi + B^0_c c^{0,*}(t), C^0\} \\ &= \max\{A^0 + (\pi^{0,M}(t))^\top.B^0_\pi + (\pi^{0,H}(t))^\top.B^0_\pi + B^0_c c^{0,M}(t) + B^0_c c^{0,A}(t), C^0\} \\ &\leq \max\{A^0 + (\pi^{0,M}(t))^\top.B^0_\pi + B^0_c c^{0,M}(t), C^0\} \\ &= \vartheta^{0,1}(\pi^{0,M}, c^{0,M}(t)) \end{aligned}$$

and

$$\begin{aligned} \vartheta^{1,0}(\pi^{1,*}(t), c^{1,*}(t)) &= \max\{A^1 + (\pi^{1,*}(t))^\top.B^1_\pi + B^1_c c^{1,*}(t), C^1\} \\ &= \max\{A^1 + (\pi^{1,M}(t))^\top.B^1_\pi + (\pi^{1,H}(t))^\top.B^1_\pi + B^1_c c^{1,M}(t) + B^1_c c^{1,A}(t), C^1\} \\ &\geq \max\{A^1 + (\pi^{1,M}(t))^\top.B^1_\pi + B^1_c c^{1,M}(t), C^1\} \\ &= \vartheta^{1,0}(\pi^{1,M}, c^{1,M}(t)) \end{aligned}$$

as $(\pi^{0,H}(t))^\top.B^0_\pi \leq 0$, resp. $(\pi^{1,H}(t))^\top.B^1_\pi \geq 0$, and $B^0_c c^{0,A}(t) \leq 0$, resp. $B^1_c c^{1,A}(t) \geq 0$ □

Remark 4.147. *As in the last section the optimal strategy that we derived here is a* compromise strategy *in that it is in general different from the Merton strategy but generates better regime shift intensities than those the Merton strategy would yield, i.e. it accommodates the aforementioned trade-off.*

We have seen before that in case of $\varepsilon = 0$ the optimal portfolio strategy in state 0 may be truncated at $\pi^{0,\text{crit}}$ and that further the optimal consumption rate may jump from 0 to the critical consumption rate $c^{0,\text{crit}}$. In the following we present a sufficient condition under which this truncation and the jump do not take place, i.e. $(\pi^{i,*}(t), c^{i,*}(t)) = (\tilde{\pi}^{i,*}(t), 0)$ for all $t \in [0,T]$ where $\tilde{\pi}^{i,*}(t) \triangleq \tilde{\pi}^{i,*}(t, g(t) - 1_{\{i=1\}} h(t), h(t))$.

Proposition 4.148. *Let $\varepsilon = 0$ and $(\pi^{i,M}, 0) \in \mathcal{I}^{\pi,c,i}$. Further suppose that $A^0 + (\pi^{0,\circ})^\top . B^0_\pi + B^0_c c^{0,\circ} \geq C^0$, where*

$$\pi^{0,\circ} \triangleq \tfrac{1}{R}(\sigma^0.(\sigma^0)^\top)^{-1}. \left(\eta^0 + \tfrac{1}{2}\tfrac{1}{B^0_c}\left(1 + \frac{(B^0_c)^2}{\frac{1}{R}(B^0_\pi)^\top.(\sigma^0.(\sigma^0)^\top)^{-1}.B^0_\pi} \frac{\Psi^0 - \Psi^1}{1 - \left(\frac{A^0 + (\pi^{0,M})^\top.B^0_\pi - C^0}{\frac{1}{R}(B^0_\pi)^\top.(\sigma^0.(\sigma^0)^\top)^{-1}.B^0_\pi} B^0_c + 1\right)^+}\right) B^0_\pi\right),$$

$$c^{0,\circ} \triangleq \frac{C^0}{B^0_c} + \frac{1}{2}\frac{\frac{1}{R}(B^0_\pi)^\top.(\sigma^0.(\sigma^0)^\top)^{-1}.B^0_\pi}{(B^0_c)^2} \left(\frac{(B^0_c)^2}{\frac{1}{R}(B^0_\pi)^\top.(\sigma^0.(\sigma^0)^\top)^{-1}.B^0_\pi} \frac{\Psi^0 - \Psi^1}{1 - \left(\frac{A^0 + (\pi^{0,M})^\top.B^0_\pi - C^0}{\frac{1}{R}(B^0_\pi)^\top.(\sigma^0.(\sigma^0)^\top)^{-1}.B^0_\pi} B^0_c + 1\right)^+} - \left(\frac{A^0 + (\pi^{0,M})^\top.B^0_\pi - C^0}{\frac{1}{R}(B^0_\pi)^\top.(\sigma^0.(\sigma^0)^\top)^{-1}.B^0_\pi} B^0_c + 1\right)^+\right).$$

Then $(\pi^{i,*}(t), c^{i,*}(t)) = (\tilde{\pi}^{i,*}(t), 0)$ for all $t \in [0,T]$.

Proof. Let $\varepsilon = 0$. If $(\pi^{1,M}, 0) \in \mathcal{I}^{\pi,c,1}$ then $h^{1,\text{crit}} = 0$ implying $(\pi^{1,*}(t), c^{1,*}(t)) = (\tilde{\pi}^{1,*}(t), 0)$ for all $t \in [0,T]$. In state 0 the condition $(\pi^{0,M}, 0) \in \mathcal{I}^{\pi,c,0}$ is necessary since otherwise $(\pi^{0,*}, 0) = (\pi^{0,M}, 0)$. Hence

$$h^{0,\text{crit}} = -\tfrac{1}{1-R} \ln\Big((1-R)\tfrac{1}{B^0_c}\Big(1 - \Big(\frac{A^0 + (\pi^{0,M})^\top.B^0_\pi - C^0}{\frac{1}{R}(B^0_\pi)^\top.(\sigma^0.(\sigma^0)^\top)^{-1}.B^0_\pi} B^0_c + 1\Big)^+\Big) + 1\Big).$$

The assertion of the proposition is equivalent to the condition

$$h(t) \leq h^{0,\text{crit}} \text{ for all } t \in [0,T].$$

To prove this, we assume without loss of generality that $h^{0,\text{crit}} < \infty$ and use a simple ODE argument. We let F as given in the proofs of the Lemmas 4.9 and 4.141. As $F(T, g(T), h(T)) = F(T,0,0) \leq 0$, it suffices to show that $F(t,x,h^{0,\text{crit}}) \geq 0$ for every $(t,x) \in [0,T] \times \mathbb{R}$. Indeed, in this case it follows from the intermediate value theorem that for each $t \in [0,T]$ there exists some $\tilde{h}(t) \in [0, h^{0,\text{crit}}]$ with $F(t,x,\tilde{h}(t)) = 0$. Thus $0 \leq h(t) \leq \max_{t\in[0,T]} \tilde{h}(t) \leq h^{0,\text{crit}}$ for every $t \in [0,T]$.

To demonstrate under which conditions $F(t,x,h^{0,\text{crit}}) \geq 0$ we write $F(t,x,h^{0,\text{crit}}) = A + B$ with

$$A \triangleq \chi^1(h^{0,\text{crit}}), \quad B \triangleq -(\Psi^0 - \Psi^1) + \chi^0(h^{0,\text{crit}})$$

where we use the notation from the proofs of Lemmas 4.9 and 4.141.

The non-negativity of χ^1 implies that $A \geq 0$. On the other hand,

$$\begin{aligned}
B &= -(\Psi^0 - \Psi^1) - C^0 \tfrac{1}{1-R}(e^{-(1-R)h^{0,\text{crit}}} - 1) \\
&\quad - (A^0 + (\pi^{0,M})^\top . B_\pi^0 - C^0) \tfrac{1}{1-R}(e^{-(1-R)h^{0,\text{crit}}} - 1) \\
&\quad - \tfrac{1}{2}\tfrac{1}{R}(B_\pi^0)^\top . (\sigma^0.(\sigma^0)^\top)^{-1}.B_\pi^0 \tfrac{1}{(1-R)^2}(e^{-(1-R)h^{0,\text{crit}}} - 1)^2 \\
&= -\Big[A^0 + (\pi^{0,M})^\top . B_\pi^0 + \tfrac{1}{2}\tfrac{1}{R}(B_\pi^0)^\top . (\sigma^0.(\sigma^0)^\top)^{-1}.B_\pi^0 \tfrac{1}{1-R}(e^{-(1-R)h^{0,\text{crit}}} - 1) \\
&\qquad + \frac{\Psi^0 - \Psi^1}{\frac{1}{1-R}(e^{-(1-R)h^{0,\text{crit}}} - 1)}\Big] \tfrac{1}{1-R}(e^{-(1-R)h^{0,\text{crit}}} - 1) \\
&= -\Bigg[A^0 + (\pi^{0,M})^\top . B_\pi^0 + \tfrac{1}{2}\tfrac{1}{R}(B_\pi^0)^\top . (\sigma^0.(\sigma^0)^\top)^{-1}.B_\pi^0 \tfrac{1}{B_c^0}\left(1 - \left(\frac{A^0 + (\pi^{0,M})^\top . B_\pi^0 - C^0}{\frac{1}{R}(B_\pi^0)^\top . (\sigma^0.(\sigma^0)^\top)^{-1}.B_\pi^0} B_c^0 + 1\right)^+\right) \\
&\qquad + \frac{\Psi^0 - \Psi^1}{\frac{1}{1-R}(e^{-(1-R)h^{0,\text{crit}}} - 1)}\Bigg] \tfrac{1}{1-R}(e^{-(1-R)h^{0,\text{crit}}} - 1) \\
&= -\Bigg[A^0 + \tfrac{1}{R}\left(\eta^0 + \tfrac{1}{2}\left(\tfrac{1}{B_c^0} + \frac{1}{\frac{1}{R}(B_\pi^0)^\top . (\sigma^0.(\sigma^0)^\top)^{-1}.B_\pi^0} \frac{\Psi^0 - \Psi^1}{\frac{1}{1-R}(e^{-(1-R)h^{0,\text{crit}}} - 1)}\right) B_\pi^0\right)^\top . (\sigma^0.(\sigma^0)^\top)^{-1}.B_\pi^0 \\
&\qquad + \tfrac{1}{2}\left(\tfrac{1}{B_c^0}\frac{\Psi^0 - \Psi^1}{\frac{1}{1-R}(e^{-(1-R)h^{0,\text{crit}}} - 1)} - \frac{\frac{1}{R}(B_\pi^0)^\top . (\sigma^0.(\sigma^0)^\top)^{-1}.B_\pi^0}{(B_c^0)^2}\left(\frac{A^0 + (\pi^{0,M})^\top . B_\pi^0 - C^0}{\frac{1}{R}(B_\pi^0)^\top . (\sigma^0.(\sigma^0)^\top)^{-1}.B_\pi^0} B_c^0 + 1\right)^+\right) B_c^0\Bigg] \\
&\quad \cdot \tfrac{1}{1-R}(e^{-(1-R)h^{0,\text{crit}}} - 1) \\
&= -(A^0 + (\tilde\pi^{0,\circ})^\top . B_\pi^0 + B_c^0 \tilde c^{0,\circ} - C^0)\tfrac{1}{1-R}(e^{-(1-R)h^{0,\text{crit}}} - 1)
\end{aligned}$$

where

$$\begin{aligned}
\tilde\pi^{0,\circ} &\triangleq \tfrac{1}{R}(\sigma^0.(\sigma^0)^\top)^{-1}\left(\eta^0 + \tfrac{1}{2}\left(\tfrac{1}{B_c^0} + \frac{1}{\frac{1}{R}(B_\pi^0)^\top . (\sigma^0.(\sigma^0)^\top)^{-1}.B_\pi^0}\frac{\Psi^0 - \Psi^1}{\frac{1}{1-R}(e^{-(1-R)h^{0,\text{crit}}} - 1)}\right) B_\pi^0\right) \\
&= \tfrac{1}{R}(\sigma^0.(\sigma^0)^\top)^{-1}\left(\eta^0 + \tfrac{1}{2}\left(\tfrac{1}{B_c^0} + \frac{B_c^0}{\frac{1}{R}(B_\pi^0)^\top . (\sigma^0.(\sigma^0)^\top)^{-1}.B_\pi^0}\frac{\Psi^0 - \Psi^1}{1 - \left(\frac{A^0 + (\pi^{0,M})^\top . B_\pi^0 - C^0}{\frac{1}{R}(B_\pi^0)^\top . (\sigma^0.(\sigma^0)^\top)^{-1}.B_\pi^0} B_c^0 + 1\right)^+}\right) B_\pi^0\right) \\
&= \pi^{0,\circ}
\end{aligned}$$

and

$$\begin{aligned}
\tilde c^{0,\circ} &\triangleq \tfrac{C^0}{B_c^0} + \tfrac{1}{2}\left(\tfrac{1}{B_c^0}\frac{\Psi^0 - \Psi^1}{\frac{1}{1-R}(e^{-(1-R)h^{0,\text{crit}}} - 1)} - \frac{\frac{1}{R}(B_\pi^0)^\top . (\sigma^0.(\sigma^0)^\top)^{-1}.B_\pi^0}{(B_c^0)^2}\left(\frac{A^0 + (\pi^{0,M})^\top . B_\pi^0 - C^0}{\frac{1}{R}(B_\pi^0)^\top . (\sigma^0.(\sigma^0)^\top)^{-1}.B_\pi^0} B_c^0 + 1\right)^+\right) \\
&= \tfrac{C^0}{B_c^0} + \tfrac{1}{2}\Bigg(\frac{\Psi^0 - \Psi^1}{1 - \left(\frac{A^0 + (\pi^{0,M})^\top . B_\pi^0 - C^0}{\frac{1}{R}(B_\pi^0)^\top . (\sigma^0.(\sigma^0)^\top)^{-1}.B_\pi^0} B_c^0 + 1\right)^+} \\
&\qquad - \frac{\frac{1}{R}(B_\pi^0)^\top . (\sigma^0.(\sigma^0)^\top)^{-1}.B_\pi^0}{(B_c^0)^2}\left(\frac{A^0 + (\pi^{0,M})^\top . B_\pi^0 - C^0}{\frac{1}{R}(B_\pi^0)^\top . (\sigma^0.(\sigma^0)^\top)^{-1}.B_\pi^0} B_c^0 + 1\right)^+\Bigg) \\
&= c^{0,\circ}.
\end{aligned}$$

Since $A^0 + (\pi^{0,\circ})^\top . B_\pi^0 + B_c^0 c^{0,\circ} \geq C^0$, we have $B \geq 0$ and the proof is complete. □

Remark 4.149. *Notice that* $c^{0,\circ}$ *is not necessarily non-negative. Thus* $c^{0,\circ}$ *cannot be interpreted as a consumption rate.*

5 Solution for CRRA Investors with Logarithmic Utility

In this section we present the solution of the investment problem (P) for a large investor with logarithmic utility. As in the last section on power utility functions we assume that condition (NJ) is satisfied and that $\Pi = \mathbb{R}^{\bar{n}}$. For logarithmic preferences, the results from the preceding section remain valid *mutatis mutandis*. More specifically, the utility functions now have the form

$$u_1(t,x) = \varepsilon e^{-\delta t}\ln(x),\ u_2(x) = e^{-\delta T}\ln(x)$$

for $\varepsilon \in [0,\infty)$ and $(t,x) \in [0,T]\times(0,\infty)$. Further the ansatz comparable to the one from the section on power utilities is given by

$$v^0(t,x) = f(t)(\ln(x)+g(t)),\ v^1(t,x) = f(t)(\ln(x)+g(t)-h(t))$$

for $(t,x) \in [0,T]\times(0,\infty)$ with $f(T) = e^{-\delta T}$, $g(T) = h(T) = 0$. Using the convention that

$$\tfrac{1}{1-R}(x^{(1-R)}-1) \triangleq \ln(x),\ x>0, \quad \tfrac{1}{1-R}\ln((1-R)x+1) \triangleq x \text{ for } R=1$$

the results from the last section can be transferred. It turns out that the function f is again given by (4.7) and thus is independent of the investor's relative risk aversion R. Moreover the **reduced HJB-system** is now given by

$$\begin{aligned} 0 = \sup_{(\pi,c)\in\mathbb{R}^{\bar{n}}\times\mathbb{R}_0^+} \Big\{ & g'(t) - 1_{\{i=1\}}h'(t) + r^i + \pi^\top.\eta^i - \tfrac{1}{2}\pi^\top.\sigma^i.(\sigma^i)^\top.\pi \\ & + \tfrac{\varepsilon\delta}{\varepsilon-(\varepsilon-\delta)e^{-\delta(T-t)}}\big(\ln(c) - (g(t) - 1_{\{i=1\}}h(t))\big) - c + \vartheta^{i,1-i}(\pi,c)(-1)^{1-i}h(t)\Big\} \end{aligned} \tag{5.1}$$

for $t \in [0,T)$ and $i = 0,1$, subject to the boundary conditions

$$g(T) = 0,\ h(T) = 0,$$

implying that Lemma 4.3 still holds true.

The logarithmic setting with risk aversion $R = 1$ has two nice features as will be seen in the following. On the one hand the Merton consumption rate turns out to be independent of the functions g and h which implies that it is the same in both market states, i.e.

$$c^{i,M}(t) = \tfrac{\varepsilon\delta}{\varepsilon-(\varepsilon-\delta)e^{-\delta(T-t)}},\ i = 0,1.$$

On the other hand the ODEs for g and h decouple from each other as the ODEs for h are independent of the function g.

In the following subsections we give the optimal strategies and the corresponding ODEs for the previously discussed types of intensity functions without proving optimality, resp. existence of global solutions. Using the above conventions the corresponding proofs from the previous section on power utility functions remain valid for the logarithmic utility function, too. Only in case of the Merton investment problem we derive the solution of the corresponding ODE-system in an explicit way since the derivation is not comparable to the one in the setting with power utilities.

5.1 Constant Intensity Functions

5.1.1 Solution of the Investment Problem

The log-utility versions of the functions $H^{\pi,i}$ and $H^{c,i}$, $i=0,1$, are given by $H^{\pi,i}: \mathbb{R}^{\bar{n}} \to \mathbb{R}$ and $H^{c,i}: [0,T]\times\mathbb{R}\times\mathbb{R}_0^+ \to \mathbb{R}$, $i=0,1$, with

$$\begin{aligned} H^{\pi,i}(\pi) &\triangleq r^i + \pi^\top.\eta^i - \tfrac{1}{2}\pi^\top.\sigma^i.(\sigma^i)^\top.\pi, \\ H^{c,i}(t,x,c) &\triangleq \tfrac{\varepsilon\delta}{\varepsilon-(\varepsilon-\delta)e^{-\delta(T-t)}}(\ln(c)-x)-c. \end{aligned}$$

The special form of $H^{c,i}$ yields that the maximizing consumption rate is going to be independent of x.

Lemma 5.1 (Maximizers of $H^{\pi,i}$ and $H^{c,i}(t,x,\cdot)$)**.** *For every $(t,x)\in[0,T]\times\mathbb{R}$ the maximizers*

$$\pi^{i,*} \triangleq \arg\max_{\pi\in\mathbb{R}^{\bar{n}}} H^{\pi,i}(\pi),\ \ c^{i,*}(t) \triangleq \arg\max_{c\in\mathbb{R}_0^+} H^{c,i}(t,x,c),\ \ i=0,1,$$

are given by the Merton strategy, i.e.

$$(\pi^{i,*}, c^{i,*}(t)) = (\pi^{i,M}, c^M(t))$$

where

$$\pi^{i,M} \triangleq (\sigma^i.(\sigma^i)^\top)^{-1}.\eta^i, \quad c^M(t) \triangleq \tfrac{\varepsilon\delta}{\varepsilon-(\varepsilon-\delta)e^{-\delta(T-t)}}.$$

The corresponding system of backward ODEs is given by

$$\begin{aligned} h'(t) =& -(\Psi^0-\Psi^1) + \tfrac{\varepsilon\delta}{\varepsilon-(\varepsilon-\delta)e^{-\delta(T-t)}}h(t) \\ &+ (C^0+C^1)h(t), \end{aligned} \tag{5.2}$$

$$\begin{aligned} g'(t) =& -\Psi^0 + \tfrac{\varepsilon\delta}{\varepsilon-(\varepsilon-\delta)e^{-\delta(T-t)}}\Big(1-\ln\big(\tfrac{\varepsilon\delta}{\varepsilon-(\varepsilon-\delta)e^{-\delta(T-t)}}\big)+g(t)\Big) \\ &+ C^0h(t) \end{aligned} \tag{5.3}$$

subject to the boundary conditions

$$g(T)=0,\ \ h(T)=0. \tag{5.4}$$

In the case without consumption, i.e. $\varepsilon=0$, h and g can even be determined explicitly, i.e.

$$h(t) = \begin{cases} \frac{\Psi^0-\Psi^1}{C^0+C^1}(1-e^{-(C^0+C^1)(T-t)}) & \text{if } C^0+C^1>0, \\ (\Psi^0-\Psi^1)(T-t) & \text{if } C^0+C^1=0, \end{cases}$$

$$g(t) = \begin{cases} \frac{C^0\Psi^1+C^1\Psi^0}{C^0+C^1}(T-t) + \frac{C^0(\Psi^0-\Psi^1)}{(C^0+C^1)^2}(1-e^{-(C^0+C^1)(T-t)}) & \text{if } C^0+C^1>0, \\ \Psi^0(T-t) & \text{if } C^0+C^1=0, \end{cases}$$

Remark 5.2 (Asymptotic behavior)**.** *In case of $\varepsilon=0$ and $C^0+C^1>0$ the functions h and g show the following asymptotic behavior. For $t\to-\infty$ the function h is converging towards $h^{\text{stat}} \triangleq \frac{\Psi^0-\Psi^1}{C^0+C^1}$ whereas g is tending to $-\infty$ with an asymptotic slope of $-\frac{C^0\Psi^1+C^1\Psi^0}{C^0+C^1}$.*

Theorem 5.3 (Solution of the Investment Problem)**.** *Let condition (NJ) be satisfied, and let h and g be given by (5.2), (5.3) subject to the boundary conditions (5.4). Then the strategy*

$$(\pi^{i,*}(t), c^{i,*}(t)) \triangleq (\pi^{i,*}, c^{i,*}(t)),\ t \in [0,T],\ i = 0,1,$$

as given in Lemma 5.1 is optimal for the investment problem (P) with CRRA *preferences and relative risk aversion $R = 1$.*

5.1.2 Solution of the Merton Investment Problem

In the Merton setting the ODEs (5.2), (5.3) read

$$h'(t) = -\left(\Psi^0 - \Psi^1\right) + \tfrac{\varepsilon\delta}{\varepsilon-(\varepsilon-\delta)e^{-\delta(T-t)}} h(t), \tag{5.5}$$

$$g'(t) = -\Psi^0 + \tfrac{\varepsilon\delta}{\varepsilon-(\varepsilon-\delta)e^{-\delta(T-t)}}\Big(1 - \ln\big(\tfrac{\varepsilon\delta}{\varepsilon-(\varepsilon-\delta)e^{-\delta(T-t)}}\big) + g(t)\Big) \tag{5.6}$$

subject to the boundary conditions

$$g(T) = 0,\ h(T) = 0. \tag{5.7}$$

On the basis of the procedure of solving the Merton problem with the conventional ansatz we use the following ansatz in order to solve the above ODE-system, i.e.

$$g(t) - 1_{\{i=1\}} h(t) = \tfrac{G^i(t)}{f(t)} \tag{5.8}$$

where the function G^i with $G^i(T) = 0$ is to be determined and f is given by (4.7).

Remark 5.4. *Note that the function G^i from ansatz (5.8) is not comparable with the function G^i from ansatz (4.19) in the power utility case.*

Notice that

$$\tfrac{\varepsilon\delta}{\varepsilon-(\varepsilon-\delta)e^{-\delta(T-t)}} = -\tfrac{f'(t)}{f(t)}. \tag{$*$}$$

Utilizing (5.8) and ($*$) we can write the functions g, h and the value function v^i in terms of the functions G^i and f, i.e.

$$\begin{aligned} g(t) &= \tfrac{G^0(t)}{f(t)}, \\ h(t) &= \tfrac{G^0(t)-G^1(t)}{f(t)}, \\ v^i(t,x) &= f(t)\ln(x) + G^i(t). \end{aligned}$$

Further inserting the ansatz (5.8) into the system (5.5), (5.6) and taking ($*$) into account yields

$$\begin{aligned} & g'(t) - 1_{\{i=1\}} h'(t) = -\Psi^i + \tfrac{\varepsilon\delta}{\varepsilon-(\varepsilon-\delta)e^{-\delta(T-t)}}\Big(1 - \ln\big(\tfrac{\varepsilon\delta}{\varepsilon-(\varepsilon-\delta)e^{-\delta(T-t)}}\big) + g(t) - 1_{\{i=1\}} h(t)\Big) \\ \Leftrightarrow\ & \tfrac{(G^i)'(t)f(t) - G^i(t)f'(t)}{(f(t))^2} = -\Psi^i - \tfrac{f'(t)}{f(t)}\Big(1 - \ln\big(-\tfrac{f'(t)}{f(t)}\big) + \tfrac{G^i(t)}{f(t)}\Big) \\ \Leftrightarrow\ & \tfrac{(G^i)'(t)}{f(t)} - \tfrac{f'(t)}{f(t)}\tfrac{G^i(t)}{f(t)} = -\Psi^i - \tfrac{f'(t)}{f(t)}\Big(1 - \ln\big(-\tfrac{f'(t)}{f(t)}\big) + \tfrac{G^i(t)}{f(t)}\Big) \\ \Leftrightarrow\ & (G^i)'(t) = -\Psi^i f(t) - f'(t)\Big(1 - \ln\big(-\tfrac{f'(t)}{f(t)}\big)\Big). \end{aligned}$$

Integrating the whole equation we get

$$G^i(t) = \tfrac{\varepsilon}{\delta}e^{-\delta t}(1-e^{-\delta(T-t)})\big(\tfrac{\Psi^i}{\delta}+\ln(\varepsilon)-1\big) - (\varepsilon-\delta)e^{-\delta T}\big(\tfrac{\Psi^i}{\delta}(T-t)-T\big) - \varepsilon t e^{-\delta t}$$
$$- \tfrac{1}{\delta}e^{-\delta t}(\varepsilon-(\varepsilon-\delta)e^{-\delta(T-t)})\ln\big(\tfrac{1}{\delta}e^{-\delta t}(\varepsilon-(\varepsilon-\delta)e^{-\delta(T-t)})\big).$$

Inserting this into the above formulas depending on G^i yields explicit formulas for the functions g, h and the value function v^i that are given in the following theorem.

Theorem 5.5 (Solution of the Merton Investment Problem)**.** *Let condition (NJ) be satisfied, and let h and g be given by (5.5), (5.6) subject to the boundary conditions (5.7). Then the optimal strategy for the investment problem (P) with* CRRA *preferences and relative risk aversion $R=1$ is given by*

$$(\pi^{i,*}, c^{i,*}(t)) = (\pi^{i,M}, c^{i,M}(t)),\ \ t\in[0,T]$$

where

$$\pi^{i,M} = (\sigma^i.(\sigma^i)^\top)^{-1}.\eta^i, \quad c^{i,M}(t) = \tfrac{\varepsilon\delta}{\varepsilon-(\varepsilon-\delta)e^{-\delta(T-t)}}$$

for $i=0,1$. In particular g and h are given by

$$g(t) = \Psi^0(T-t) + \tfrac{\varepsilon(1-e^{-\delta(T-t)})(\frac{\Psi^0}{\delta}+\ln(\varepsilon)-1)+(\delta(\varepsilon-\delta)e^{-\delta(T-t)}-\varepsilon\Psi^0)(T-t)}{\varepsilon-(\varepsilon-\delta)e^{-\delta(T-t)}} - \ln\big(\tfrac{1}{\delta}(\varepsilon-(\varepsilon-\delta)e^{-\delta(T-t)})\big),$$
$$h(t) = (\Psi^0-\Psi^1)(T-t) + \varepsilon\tfrac{\frac{1}{\delta}(1-e^{-\delta(T-t)})-(T-t)}{\varepsilon-(\varepsilon-\delta)e^{-\delta(T-t)}}(\Psi^0-\Psi^1)$$

so that the value functions read

$$v^i(t,x) = \tfrac{1}{\delta}e^{-\delta t}(\varepsilon-(\varepsilon-\delta)e^{-\delta(T-t)})\ln(x) + \tfrac{\varepsilon}{\delta}e^{-\delta t}(1-e^{-\delta(T-t)})\big(\tfrac{\Psi^i}{\delta}+\ln(\varepsilon)-1\big) - \varepsilon t e^{-\delta t}$$
$$-(\varepsilon-\delta)e^{-\delta T}\big(\tfrac{\Psi^i}{\delta}(T-t)-T\big) - \tfrac{1}{\delta}e^{-\delta t}(\varepsilon-(\varepsilon-\delta)e^{-\delta(T-t)})\ln\big(\tfrac{1}{\delta}e^{-\delta t}(\varepsilon-(\varepsilon-\delta)e^{-\delta(T-t)})\big)$$

for $i=0,1$.

If the investor was not allowed to consume, i.e. $\varepsilon=0$, then the latter formulas simplify to

$$\pi^{i,M} = (\sigma^i.(\sigma^i)^\top)^{-1}.\eta^i, \quad c^{i,M}(t) = 0$$

for $i=0,1$ and

$$g(t) = \Psi^0(T-t),$$
$$h(t) = (\Psi^0-\Psi^1)(T-t),$$
$$v^i(t,x) = e^{-\delta T}(\ln(x)+\Psi^i(T-t))$$

for $i=0,1$.

Remark 5.6 (Asymptotic behavior)**.** *In case of $\varepsilon>0$ the functions h and g show the following asymptotic behavior. For $t\to-\infty$ the function h is converging towards $h^{\text{stat}} \triangleq \frac{\Psi^0-\Psi^1}{\delta}$ whereas g is tending to $g^{\text{stat}} \triangleq \frac{\Psi^0}{\delta}+\ln(\delta)-1$.*

5.2 Step Intensity Functions

5.2.1 Portfolio-dependent Intensities

The log-utility versions of the functions $H^{\pi,i}$ and $H^{c,i}$, $i = 0, 1$, are given by $H^{\pi,i} : \mathbb{R}_0^+ \times \mathbb{R}^{\bar{n}} \to \mathbb{R}$ and $H^{c,i} :\ [0, T] \times \mathbb{R} \times \mathbb{R}_0^+ \to \mathbb{R}$, $i = 0, 1$, with

$$\begin{aligned} H^{\pi,i}(y, \pi) &\triangleq r^i + \pi^\top.\eta^i - \tfrac{1}{2}\pi^\top.\sigma^i.(\sigma^i)^\top.\pi + (-1)^{1-i}\big(C_1^i 1_{\{\pi\in\mathcal{F}^{\pi,i}\}} + C_2^i 1_{\{\pi\notin\mathcal{F}^{\pi,i}\}}\big)y, \\ H^{c,i}(t, x, c) &\triangleq \tfrac{\varepsilon\delta}{\varepsilon-(\varepsilon-\delta)e^{-\delta(T-t)}}(\ln(c) - x) - c. \end{aligned}$$

Lemma 5.7 (Maximizer of $H^{c,i}(t, x, \cdot)$)**.** *For every $(t, x) \in [0, T] \times \mathbb{R}$ the maximizer*

$$c^{i,*}(t) \triangleq \arg\max_{c\in\mathbb{R}_0^+} H^{c,i}(t, x, c),\ \ i = 0, 1,$$

is given by the Merton consumption rate, i.e.

$$c^{i,*}(t) = c^M(t).$$

Lemma 5.8 (Maximizer of $H^{\pi,i}(y, \cdot)$)**.** *For every $y \in \mathbb{R}_0^+$ the maximizer*

$$\pi^{i,*}(y) \triangleq \arg\max_{\pi\in\mathbb{R}^{\bar{n}}} H^{\pi,i}(y, \pi),\ \ i = 0, 1,$$

is given by

$$\pi^{i,*}(y) = \begin{cases} \pi^{i,M} & \text{if } y < h^{i,\text{crit}}, \\ \pi^{i,\text{crit}} & \text{if } y \geq h^{i,\text{crit}}, \end{cases}$$

where

$$h^{i,\text{crit}} \triangleq (-1)^{1-i}\tfrac{\zeta^{i,\text{crit}}}{C_2^i - C_1^i}$$

with

$$\zeta^{i,\text{crit}} \triangleq -\tfrac{1}{2}\tfrac{((A^i+(\pi^{i,M})^\top.B_\pi^i-C^i)^+)^2}{(B_\pi^i)^\top.(\sigma^i.(\sigma^i)^\top)^{-1}.B_\pi^i}$$

and

$$\pi^{i,\text{crit}} \triangleq (\sigma^i.(\sigma^i)^\top)^{-1}.\Big(\eta^i - B_\pi^i\tfrac{(A^i+(\pi^{i,M})^\top.B_\pi^i-C^i)^+}{(B_\pi^i)^\top.(\sigma^i.(\sigma^i)^\top)^{-1}.B_\pi^i}\Big).$$

Inserting the maximizing portfolio proportions and consumption rates into the reduced HJB-equation

(5.1) the latter now becomes a system of backward ODEs

$$\begin{aligned} h'(t) = & -(\Psi^0 - \Psi^1) + \tfrac{\varepsilon\delta}{\varepsilon-(\varepsilon-\delta)e^{-\delta(T-t)}} h(t) \\ & + (C_2^0 + C_2^1) h(t) \\ & - \Big[-(C_1^0 - C_2^0) h(t) - \tfrac{1}{2} \tfrac{((A^0+(\pi^{0,M})^\top.B_\pi^0 - C^0)^+)^2}{(B_\pi^0)^\top.(\sigma^0.(\sigma^0)^\top)^{-1}.B_\pi^0} \Big]^+ \\ & + \Big[(C_1^1 - C_2^1) h(t) - \tfrac{1}{2} \tfrac{((A^1+(\pi^{1,M})^\top.B_\pi^1 - C^1)^+)^2}{(B_\pi^1)^\top.(\sigma^1.(\sigma^1)^\top)^{-1}.B_\pi^1} \Big]^+, \end{aligned} \tag{5.9}$$

$$\begin{aligned} g'(t) = & -\Psi^0 + \tfrac{\varepsilon\delta}{\varepsilon-(\varepsilon-\delta)e^{-\delta(T-t)}} \Big(1 - \ln\big(\tfrac{\varepsilon\delta}{\varepsilon-(\varepsilon-\delta)e^{-\delta(T-t)}}\big) + g(t) \Big) \\ & + C_2^0 h(t) \\ & - \Big[-(C_1^0 - C_2^0) h(t) - \tfrac{1}{2} \tfrac{((A^0+(\pi^{0,M})^\top.B_\pi^0 - C^0)^+)^2}{(B_\pi^0)^\top.(\sigma^0.(\sigma^0)^\top)^{-1}.B_\pi^0} \Big]^+ \end{aligned} \tag{5.10}$$

subject to the boundary conditions

$$g(T) = 0, \;\; h(T) = 0. \tag{5.11}$$

Theorem 5.9 (Solution of the Investment Problem)**.** *Let condition (NJ) be satisfied, and let h and g be given by (5.9), (5.10) subject to the boundary conditions (5.11). Then the strategy*

$$(\pi^{i,*}(t), c^{i,*}(t)) \triangleq (\pi^{i,*}(h(t)), c^{i,*}(t)), \;\; t \in [0,T], \;\; i = 0,1,$$

as given in Lemmas 5.7 and 5.8 is optimal for the investment problem (P) with CRRA *preferences and relative risk aversion $R = 1$.*

5.2.2 Consumption-dependent Intensities

The log-utility versions of the functions $H^{\pi,i}$ and $H^{c,i}$, $i = 0,1$, are given by $H^{\pi,i} : \mathbb{R}^{\bar{n}} \to \mathbb{R}$ and $H^{c,i} : [0,T] \times \mathbb{R} \times \mathbb{R}_0^+ \times \mathbb{R}_0^+ \to \mathbb{R}$, $i = 0,1$, with

$$\begin{aligned} H^{\pi,i}(\pi) &\triangleq r^i + \pi^\top.\eta^i - \tfrac{1}{2}\pi^\top.\sigma^i.(\sigma^i)^\top.\pi, \\ H^{c,i}(t,x,y,c) &\triangleq \tfrac{\varepsilon\delta}{\varepsilon-(\varepsilon-\delta)e^{-\delta(T-t)}}(\ln(c) - x) - c + (-1)^{1-i}\big(C_1^i 1_{\{c\in\mathcal{F}^{c,i}\}} + C_2^i 1_{\{c\notin\mathcal{F}^{c,i}\}}\big)y. \end{aligned}$$

In case of $\varepsilon = 0$ the function $H^{c,i}$ is *de facto* a function solely of y and c.

Lemma 5.10 (Maximizer of $H^{\pi,i}$)**.** *The maximizer*

$$\pi^{i,*} \triangleq \operatorname*{arg\,max}_{\pi\in\mathbb{R}^{\bar{n}}} H^{\pi,i}(\pi), \;\; i = 0,1,$$

is given by the Merton strategy, i.e.

$$\pi^{i,*} = \pi^{i,M}.$$

Lemma 5.11 (Maximizer of $H^{c,i}(t,x,y,\cdot)$)**.** *For every $(t,x,y) \in [0,T] \times \mathbb{R} \times \mathbb{R}_0^+$ the maximizer*

$$c^{i,*}(t,y) \triangleq \operatorname*{arg\,max}_{c\in\mathbb{R}_0^+} H^{c,i}(t,x,y,c), \;\; i = 0,1,$$

is given by

$$c^{i,*}(t,y) = \begin{cases} c^M(t) & \text{if } y < h^{i,\text{crit}}(t), \\ c^{i,\text{crit}}(t) & \text{if } y \geq h^{i,\text{crit}}(t), \end{cases}$$

where

$$h^{i,\text{crit}}(t) \triangleq (-1)^{1-i} \frac{\zeta^{i,\text{crit}}(t)}{C_2^i - C_1^i}$$

with

$$\zeta^{i,\text{crit}}(t) \triangleq \begin{cases} 0 & \text{if } B_c^i < 0 \text{ and } A^i < C^i, \\ \Big[\ln\Big(1 - \frac{(A^i + B_c^i c^M(t) - C^i)^+}{B_c^i c^M(t)}\Big) + \frac{(A^i + B_c^i c^M(t) - C^i)^+}{B_c^i c^M(t)}\Big] c^M(t) & \text{if } B_c^i < 0 \text{ and } A^i \geq C^i, \\ & \text{or } B_c^i > 0 \text{ and } A^i \leq C^i, \\ -\infty & \text{if } B_c^i > 0 \text{ and } A^i > C^i, \end{cases}$$

if $\varepsilon > 0$*, resp.*

$$\zeta^{i,\text{crit}}(t) \triangleq \begin{cases} 0 & \text{if } B_c^i < 0 \text{ and } A^i < C^i, \\ \frac{(A^i - C^i)^+}{B_c^i} & \text{if } B_c^i < 0 \text{ and } A^i \geq C^i, \\ & \text{or } B_c^i > 0 \text{ and } A^i \leq C^i, \\ -\infty & \text{if } B_c^i > 0 \text{ and } A^i > C^i, \end{cases}$$

if $\varepsilon = 0$*, and*

$$c^{i,\text{crit}}(t) \triangleq \begin{cases} c^M(t) & \text{if } B_c^i > 0 \text{ and } A^i > C^i, \text{ or } B_c^i < 0 \text{ and } A^i < C^i, \\ c^M(t) - \frac{(A^i + B_c^i c^M(t) - C^i)^+}{B_c^i} & \text{if } B_c^i > 0 \text{ and } A^i \leq C^i, \text{ or } B_c^i < 0 \text{ and } A^i \geq C^i. \end{cases}$$

Remark 5.12. *Notice that in contrast to the portfolio-dependent setting the critical value* $h^{i,\text{crit}}$ *is no longer constant. It is now a function of the Merton type consumption rate* c^M*. Only if* $\varepsilon = 0$ *then* $\zeta^{i,\text{crit}}$ *and consequently* $h^{i,\text{crit}}$ *are just constants independent of* t*. Further* $\varepsilon = 0$ *implies* $c^M(t) = 0$ *such that* $c^{i,\text{crit}}$ *is constant, too. The maximizing consumption rate* $c^{i,*}$ *is therefore just a function of* y*.*

Inserting the maximizing portfolio proportions and consumption rates into the reduced HJB-equation (5.1) the latter now becomes a system of backward ODEs which for $\varepsilon > 0$ is given by

$$\begin{aligned} h'(t) = &- (\Psi^0 - \Psi^1) + \frac{\varepsilon\delta}{\varepsilon - (\varepsilon - \delta)e^{-\delta(T-t)}} h(t) \\ &+ (C_2^0 + C_2^1) h(t) \\ &- \Big[-(C_1^0 - C_2^0) h(t) \\ &\quad + \Big(\ln\Big(1 - \frac{(A^0 + B_c^0 c^M(t) - C^0)^+}{B_c^0 c^M(t)}\Big) + \frac{(A^0 + B_c^0 c^M(t) - C^0)^+}{B_c^0 c^M(t)}\Big) c^M(t)\Big] 1_{\{h(t) \geq h^{0,\text{crit}}(t)\}} \\ &+ \Big[(C_1^1 - C_2^1) h(t) \\ &\quad + \Big(\ln\Big(1 - \frac{(A^1 + B_c^1 c^M(t) - C^1)^+}{B_c^1 c^M(t)}\Big) + \frac{(A^1 + B_c^1 c^M(t) - C^1)^+}{B_c^1 c^M(t)}\Big) c^M(t)\Big] 1_{\{h(t) \geq h^{1,\text{crit}}(t)\}}, \end{aligned} \tag{5.12}$$

$$\begin{aligned} g'(t) = &- \Psi^0 + \frac{\varepsilon\delta}{\varepsilon - (\varepsilon - \delta)e^{-\delta(T-t)}} \Big(1 - \ln\big(\frac{\varepsilon\delta}{\varepsilon - (\varepsilon - \delta)e^{-\delta(T-t)}}\big) + g(t)\Big) \\ &+ C_2^0 h(t) \\ &- \Big[-(C_1^0 - C_2^0) h(t) \\ &\quad + \Big(\ln\Big(1 - \frac{(A^0 + B_c^0 c^M(t) - C^0)^+}{B_c^0 c^M(t)}\Big) + \frac{(A^0 + B_c^0 c^M(t) - C^0)^+}{B_c^0 c^M(t)}\Big) c^M(t)\Big] 1_{\{h(t) \geq h^{0,\text{crit}}(t)\}} \end{aligned} \tag{5.13}$$

subject to the boundary conditions

$$g(T) = 0, \;\; h(T) = 0. \tag{5.14}$$

If $\varepsilon = 0$ then the ODE-system reads

$$\begin{aligned} h'(t) = &- (\Psi^0 - \Psi^1) \\ &+ (C_2^0 + C_2^1)h(t) \\ &- \Big(-(C_1^0 - C_2^0)h(t) + \tfrac{(A^0-C^0)^+}{B_c^0}\Big)1_{\{h(t)\geq h^{0,\mathrm{crit}}\}} \\ &+ \Big((C_1^1 - C_2^1)h(t) + \tfrac{(A^1-C^1)^+}{B_c^1}\Big)1_{\{h(t)\geq h^{1,\mathrm{crit}}\}}, \end{aligned} \tag{5.15}$$

$$\begin{aligned} g'(t) = &- \Psi^0 \\ &+ C_2^0 h(t) \\ &- \Big(-(C_1^0 - C_2^0)h(t) + \tfrac{(A^0-C^0)^+}{B_c^0}\Big)1_{\{h(t)\geq h^{0,\mathrm{crit}}\}} \end{aligned} \tag{5.16}$$

subject to the boundary conditions

$$g(T) = 0, \;\; h(T) = 0. \tag{5.17}$$

Remark 5.13. *Note that the barrier $h^{i,\mathrm{crit}}(t)$ is now independent of $g(t) - 1_{\{i=1\}}h(t)$. Therefore the condition $h(t) \geq h^{i,\mathrm{crit}}(t)$ is explicit for $i = 0, 1$. Remember that for $R \neq 1$ the condition $h(t) \geq h^{1,\mathrm{crit}}(t, x)$ was partly implicit.*

Theorem 5.14 (Solution of the Investment Problem)**.** *Let condition (NJ) be satisfied, and let h and g be given by (5.12), (5.13) subject to the boundary conditions (5.14) ($\varepsilon > 0$), resp. (5.15), (5.16) subject to the boundary conditions (5.17) ($\varepsilon = 0$). Then the strategy*

$$(\pi^{i,*}(t), c^{i,*}(t)) \triangleq (\pi^{i,*}, c^{i,*}(t, h(t))), \;\; t \in [0, T], \;\; i = 0, 1,$$

as given in Lemmas 5.10 and 5.11 is optimal for the investment problem (P) with CRRA *preferences and relative risk aversion $R = 1$.*

5.2.3 Portfolio- and Consumption-dependent Intensities

The log-utility version of the function $H^{\pi,c,i}$, $i = 0, 1$, is given by $H^{\pi,c,i} : [0, T] \times \mathbb{R} \times \mathbb{R}_0^+ \times \mathbb{R}^{\bar{n}} \times \mathbb{R}_0^+ \to \mathbb{R}$, $i = 0, 1$, with

$$\begin{aligned} H^{\pi,c,i}(t, x, y, \pi, c) \triangleq &\tfrac{\varepsilon\delta}{\varepsilon-(\varepsilon-\delta)e^{-\delta(T-t)}}(\ln(c) - x) - c + r^i + \pi^\top.\eta^i - \tfrac{1}{2}\pi^\top.\sigma^i.(\sigma^i)^\top.\pi \\ &+ (-1)^{1-i}(C_1^i 1_{\{(\pi,c)\in\mathcal{F}^{\pi,c,i}\}} + C_2^i 1_{\{(\pi,c)\notin\mathcal{F}^{\pi,c,i}\}})y. \end{aligned}$$

If $\varepsilon = 0$ then $H^{\pi,c,i}$ solely depends on y, π and c.

Lemma 5.15 (Maximizer of $H^{\pi,c,i}(t, x, y, \cdot, \cdot)$)**.** *For every $(t, x, y) \in [0, T] \times \mathbb{R} \times \mathbb{R}_0^+$ the maximizer*

$$(\pi^{i,*}(t, y), c^{i,*}(t, y)) \triangleq \underset{(\pi,c)\in\mathbb{R}^{\bar{n}}\times\mathbb{R}_0^+}{\arg\max} \; H^{\pi,c,i}(t, x, y, \pi, c), \;\; i = 0, 1,$$

is given by

$$(\pi^{i,*}(t,y), c^{i,*}(t,y)) = \begin{cases} (\pi^{i,M}, c^M(t)) & \text{if } y < h^{i,\text{crit}}(t), \\ (\pi^{i,\text{crit}}(t), c^{i,\text{crit}}(t)) & \text{if } y \geq h^{i,\text{crit}}(t), \end{cases}$$

with

$$h^{i,\text{crit}}(t) \triangleq (-1)^{1-i} \tfrac{\zeta^{i,\text{crit}}(t)}{C_2^i - C_1^i},$$

with

$$\zeta^{i,\text{crit}}(t) \triangleq \begin{cases} -\frac{1}{2}(B_\pi^i)^\top(\sigma^i.(\sigma^i)^\top)^{-1}.B_\pi^i(\lambda^{i,\text{crit}}(t))^2 \\ + \Big[-\ln\big(1-\lambda^{i,\text{crit}}(t)B_c^i\big) - \frac{1}{1-\lambda^{i,\text{crit}}(t)B_c^i} + 1\Big]c^M(t) & \text{if } \varepsilon > 0, \\ \\ -\frac{1}{2}\frac{(B_\pi^i)^\top(\sigma^i.(\sigma^i)^\top)^{-1}.B_\pi^i}{(B_c^i)^2}\Big[\Big(1-\Big(\frac{(A^i+(\pi^{i,M})^\top.B_\pi^i-C^i)^+}{(B_\pi^i)^\top.(\sigma^i.(\sigma^i)^\top)^{-1}.B_\pi^i}B_c^i+1\Big)^+\Big)^2 \\ \qquad + 2\Big(\frac{(A^i+(\pi^{i,M})^\top.B_\pi^i-C^i)^+}{(B_\pi^i)^\top.(\sigma^i.(\sigma^i)^\top)^{-1}.B_\pi^i}B_c^i+1\Big)^-\Big] & \text{if } \varepsilon = 0, \end{cases}$$

and where $\lambda^{i,\text{crit}}(t)$ *is uniquely determined by*

$$\Lambda^{\pi,c,i}(t, \lambda^{i,\text{crit}}(t)) = -(A^i + (\pi^{i,M})^\top.B_\pi^i + B_c^i c^M(t) - C^i)^+.$$

Moreover

$$\pi^{i,\text{crit}}(t) = \begin{cases} (\sigma^i.(\sigma^i)^\top)^{-1}.\big(\eta^i + \lambda^{i,\text{crit}}(t)B_\pi^i\big) & \text{if } \varepsilon > 0, \\ (\sigma^i.(\sigma^i)^\top)^{-1}.\Big(\eta^i + B_\pi^i\frac{1}{B_c^i}\Big(1-\Big(\frac{(A^i+(\pi^{i,M})^\top.B_\pi^i-C^i)^+}{(B_\pi^i)^\top.(\sigma^i.(\sigma^i)^\top)^{-1}.B_\pi^i}B_c^i+1\Big)^+\Big)\Big) & \text{if } \varepsilon = 0, \end{cases}$$

$$c^{i,\text{crit}}(t) = \begin{cases} c^M(t)\big(1-\lambda^{i,\text{crit}}(t)B_c^i\big)^{-1} & \text{if } \varepsilon > 0, \\ \frac{(B_\pi^i)^\top.(\sigma^i.(\sigma^i)^\top)^{-1}.B_\pi^i}{(B_c^i)^2}\Big(\frac{(A^i+(\pi^{i,M})^\top.B_\pi^i-C^i)^+}{(B_\pi^i)^\top.(\sigma^i.(\sigma^i)^\top)^{-1}.B_\pi^i}B_c^i+1\Big)^- & \text{if } \varepsilon = 0. \end{cases}$$

Here the log-utility version of the function $\Lambda^{\pi,c,i}$ given by

$$\Lambda^{\pi,c,i} : [0,T] \times (-\infty, \tfrac{1}{B_c^i}) \to \mathbb{R}\ (B_c^i > 0),\ \text{resp. } \Lambda^{\pi,c,i} : [0,T] \times (\tfrac{1}{B_c^i}, \infty) \to \mathbb{R}\ (B_c^i < 0)$$

with

$$\Lambda^{\pi,c,i}(t,\lambda) \triangleq (B_\pi^i)^\top(\sigma^i.(\sigma^i)^\top)^{-1}.B_\pi^i\lambda + B_c^i\Big(\big(1-B_c^i\lambda\big)^{-1} - 1\Big)c^M(t).$$

Remark 5.16. *If* $\varepsilon = 0$ *then* $\zeta^{i,\text{crit}}$ *and consequently* $h^{i,\text{crit}}$ *are just constants independent of* t*. Further* $\varepsilon = 0$ *implies that* $\pi^{i,\text{crit}}$ *and* $c^{i,\text{crit}}$ *are constants, too. The maximizing strategy* $(\pi^{i,*}, c^{i,*})$ *is therefore just a function of* y*.*

Inserting the maximizing portfolio proportions and consumption rates into the reduced HJB-equation

(5.1) the latter now becomes a system of backward ODEs which for $\varepsilon > 0$ is given by

$$\begin{aligned} h'(t) =& -(\Psi^0 - \Psi^1) + \tfrac{\varepsilon\delta}{\varepsilon-(\varepsilon-\delta)e^{-\delta(T-t)}} h(t) \\ &+ (C_2^0 + C_2^1) h(t) \\ &- \Big[-(C_1^0 - C_2^0) h(t) - \tfrac{1}{2} (B_\pi^0)^\top . (\sigma^0 . (\sigma^0)^\top)^{-1} . B_\pi^0 (\lambda^{0,\mathrm{crit}}(t))^2 \\ &\qquad + \Big(-\ln\big(1 - \lambda^{0,\mathrm{crit}}(t) B_c^0\big) - \big(1 - \lambda^{0,\mathrm{crit}}(t) B_c^0\big)^{-1} + 1 \Big) c^M(t) \Big]^+ \\ &+ \Big[(C_1^1 - C_2^1) h(t) - \tfrac{1}{2} (B_\pi^1)^\top . (\sigma^1 . (\sigma^1)^\top)^{-1} . B_\pi^1 (\lambda^{1,\mathrm{crit}}(t))^2 \\ &\qquad + \Big(-\ln\big(1 - \lambda^{1,\mathrm{crit}}(t) B_c^1\big) - \big(1 - \lambda^{1,\mathrm{crit}}(t) B_c^1\big)^{-1} + 1 \Big) c^M(t) \Big]^+, \end{aligned} \tag{5.18}$$

$$\begin{aligned} g'(t) =& -\Psi^0 + \tfrac{\varepsilon\delta}{\varepsilon-(\varepsilon-\delta)e^{-\delta(T-t)}} \Big(1 - \ln\big(\tfrac{\varepsilon\delta}{\varepsilon-(\varepsilon-\delta)e^{-\delta(T-t)}}\big) + g(t) \Big) \\ &+ C_2^0 h(t) \\ &- \Big[-(C_1^0 - C_2^0) h(t) - \tfrac{1}{2} (B_\pi^0)^\top . (\sigma^0 . (\sigma^0)^\top)^{-1} . B_\pi^0 (\lambda^{0,\mathrm{crit}}(t))^2 \\ &\qquad + \Big(-\ln\big(1 - \lambda^{0,\mathrm{crit}}(t) B_c^0\big) - \big(1 - \lambda^{0,\mathrm{crit}}(t) B_c^0\big)^{-1} + 1 \Big) c^M(t) \Big]^+ \end{aligned} \tag{5.19}$$

subject to the boundary conditions

$$g(T) = 0, \;\; h(T) = 0. \tag{5.20}$$

If $\varepsilon = 0$ then the ODE-system reads

$$\begin{aligned} h'(t) =& -(\Psi^0 - \Psi^1) \\ &+ (C_2^0 + C_2^1) h(t) \\ &- \Big[-(C_1^0 - C_2^0) h(t) - \tfrac{1}{2} \tfrac{(B_\pi^0)^\top (\sigma^0 . (\sigma^0)^\top)^{-1} . B_\pi^0}{(B_c^0)^2} \\ &\quad \cdot \Big[\Big(1 - \Big(\tfrac{(A^0 + (\pi^{0,M})^\top . B_\pi^0 - C^0)^+}{(B_\pi^0)^\top . (\sigma^0 . (\sigma^0)^\top)^{-1} . B_\pi^0} B_c^0 + 1 \Big)^+ \Big)^2 + 2 \Big(\tfrac{(A^0 + (\pi^{0,M})^\top . B_\pi^0 - C^0)^+}{(B_\pi^0)^\top . (\sigma^0 . (\sigma^0)^\top)^{-1} . B_\pi^0} B_c^0 + 1 \Big)^- \Big] \Big]^+ \\ &+ \Big[(C_1^1 - C_2^1) h(t) - \tfrac{1}{2} \tfrac{(B_\pi^1)^\top (\sigma^1 . (\sigma^1)^\top)^{-1} . B_\pi^1}{(B_c^1)^2} \\ &\quad \cdot \Big[\Big(1 - \Big(\tfrac{(A^1 + (\pi^{1,M})^\top . B_\pi^1 - C^1)^+}{(B_\pi^1)^\top . (\sigma^1 . (\sigma^1)^\top)^{-1} . B_\pi^1} B_c^1 + 1 \Big)^+ \Big)^2 + 2 \Big(\tfrac{(A^1 + (\pi^{1,M})^\top . B_\pi^1 - C^1)^+}{(B_\pi^1)^\top . (\sigma^1 . (\sigma^1)^\top)^{-1} . B_\pi^1} B_c^1 + 1 \Big)^- \Big] \Big]^+, \end{aligned} \tag{5.21}$$

$$\begin{aligned} g'(t) =& -\Psi^0 \\ &+ C_2^0 h(t) \\ &- \Big[-(C_1^0 - C_2^0) h(t) - \tfrac{1}{2} \tfrac{(B_\pi^0)^\top (\sigma^0 . (\sigma^0)^\top)^{-1} . B_\pi^0}{(B_c^0)^2} \\ &\quad \cdot \Big[\Big(1 - \Big(\tfrac{(A^0 + (\pi^{0,M})^\top . B_\pi^0 - C^0)^+}{(B_\pi^0)^\top . (\sigma^0 . (\sigma^0)^\top)^{-1} . B_\pi^0} B_c^0 + 1 \Big)^+ \Big)^2 + 2 \Big(\tfrac{(A^0 + (\pi^{0,M})^\top . B_\pi^0 - C^0)^+}{(B_\pi^0)^\top . (\sigma^0 . (\sigma^0)^\top)^{-1} . B_\pi^0} B_c^0 + 1 \Big)^- \Big] \Big]^+ \end{aligned} \tag{5.22}$$

subject to the boundary conditions

$$g(T) = 0, \;\; h(T) = 0. \tag{5.23}$$

Theorem 5.17 (Solution of the Investment Problem)**.** *Let condition (NJ) be satisfied, and let h and g be given by (5.18), (5.19) subject to the boundary conditions (5.20) ($\varepsilon > 0$), resp. (5.21), (5.22) subject to the boundary conditions (5.23) ($\varepsilon = 0$). Then the strategy*

$$(\pi^{i,*}(t), c^{i,*}(t)) \triangleq (\pi^{i,*}(t, h(t)), c^{i,*}(t, h(t))), \ t \in [0,T], \ i = 0,1,$$

as given in Lemma 5.15 is optimal for the investment problem (P) with CRRA *preferences and relative risk aversion $R = 1$.*

5.3 Affine Intensity Functions

5.3.1 Portfolio-dependent Intensities

The log-utility versions of the functions $H^{\pi,i}$, $H_l^{\pi,i}$, $H_a^{\pi,i}$ and $H^{c,i}$, $i = 0,1$, are given by $H^{\pi,i}, H_l^{\pi,i}, H_a^{\pi,i}$: $\mathbb{R}_0^+ \times \mathbb{R}^{\bar{n}} \to \mathbb{R}$ and $H^{c,i} : [0,T] \times \mathbb{R} \times \mathbb{R}_0^+ \to \mathbb{R}$ for $i = 0,1$ with

$$\begin{aligned}
H^{\pi,i}(y,\pi) &\triangleq r^i + \pi^\top.\eta^i - \tfrac{1}{2}\pi^\top.\sigma^i.(\sigma^i)^\top.\pi + (-1)^{1-i}\max\{A^i + \pi^\top.B_\pi^i, C^i\}y,\\
H_l^{\pi,i}(y,\pi) &\triangleq r^i + \pi^\top.\eta^i - \tfrac{1}{2}\pi^\top.\sigma^i.(\sigma^i)^\top.\pi + (-1)^{1-i}C^i y,\\
H_a^{\pi,i}(y,\pi) &\triangleq r^i + \pi^\top.\eta^i - \tfrac{1}{2}\pi^\top.\sigma^i.(\sigma^i)^\top.\pi + (-1)^{1-i}(A^i + \pi^\top.B_\pi^i)y,\\
H^{c,i}(t,x,c) &\triangleq \tfrac{\varepsilon\delta}{\varepsilon-(\varepsilon-\delta)e^{-\delta(T-t)}}(\ln(c) - x) - c.
\end{aligned}$$

Further the candidate solution for the portfolio proportions reads

$$\tilde{\pi}^{i,*}(y) \triangleq (\sigma^i.(\sigma^i)^\top)^{-1}.\big(\eta^i + (-1)^{1-i}B_\pi^i y\big).$$

Lemma 5.18 (Maximizer of $H^{c,i}(t,x,\cdot)$)**.** *For every $(t,x) \in [0,T] \times \mathbb{R}$ the maximizer*

$$c^{i,*}(t) \triangleq \arg\max_{c \in \mathbb{R}_0^+} H^{c,i}(t,x,c), \ i = 0,1,$$

is given by the Merton consumption rate, i.e.

$$c^{i,*}(t) = c^M(t).$$

Lemma 5.19 (Maximizer of $H^{\pi,i}(y,\cdot)$)**.** *For every $y \in \mathbb{R}_0^+$ the maximizer*

$$\pi^{i,*}(y) \triangleq \arg\max_{\pi \in \mathbb{R}^{\bar{n}}} H^{\pi,i}(y,\pi), \ i = 0,1,$$

is given by

$$\begin{aligned}
\pi^{0,*}(y) &= (\sigma^0.(\sigma^0)^\top)^{-1}.\big(\eta^0 - B_\pi^0(y \wedge h^{0,\text{crit}})\big),\\
\pi^{1,*}(t) &= \begin{cases} \pi^{1,M} & \text{if } y < h^{1,\text{crit}},\\ \tilde{\pi}^{1,*}(y) & \text{if } y \geq h^{1,\text{crit}}, \end{cases}
\end{aligned}$$

where

$$\begin{aligned}
h^{0,\text{crit}} &\triangleq \tfrac{(A^0 + (\pi^{0,M})^\top.B_\pi^0 - C^0)^+}{(B_\pi^0)^\top.(\sigma^0.(\sigma^0)^\top)^{-1}.B_\pi^0},\\
h^{1,\text{crit}} &\triangleq 2\tfrac{(A^1 + (\pi^{1,M})^\top.B_\pi^1 - C^1)^-}{(B_\pi^1)^\top.(\sigma^1.(\sigma^1)^\top)^{-1}.B_\pi^1}
\end{aligned}$$

and

$$\pi^{0,\text{crit}} \triangleq (\sigma^0.(\sigma^0)^\top)^{-1}.\big(\eta^0 - B_\pi^0\tfrac{(A^0 + (\pi^{0,M})^\top.B_\pi^0 - C^0)^+}{(B_\pi^0)^\top.(\sigma^0.(\sigma^0)^\top)^{-1}.B_\pi^0}\big).$$

The reduced HJB-equation (5.1) now becomes a system of backward ODEs

$$\begin{aligned} h'(t) = &-(\Psi^0-\Psi^1)+\tfrac{\varepsilon\delta}{\varepsilon-(\varepsilon-\delta)e^{-\delta(T-t)}}h(t)\\ &+(C^0+C^1)h(t)\\ &+(A^0+(\pi^{0,M})^\top.B^0_\pi-C^0)(h(t)\wedge h^{0,\mathrm{crit}})-\tfrac12(B^0_\pi)^\top.(\sigma^0.(\sigma^0)^\top)^{-1}.B^0_\pi(h(t)\wedge h^{0,\mathrm{crit}})^2\\ &+(A^1+(\pi^{1,M})^\top.B^1_\pi-C^1)(h(t)\vee h^{1,\mathrm{crit}})+\tfrac12(B^1_\pi)^\top.(\sigma^1.(\sigma^1)^\top)^{-1}.B^1_\pi(h(t)\vee h^{1,\mathrm{crit}})^2, \end{aligned} \tag{5.24}$$

$$\begin{aligned} g'(t) = &-\Psi^0+\tfrac{\varepsilon\delta}{\varepsilon-(\varepsilon-\delta)e^{-\delta(T-t)}}\Big(1-\ln\big(\tfrac{\varepsilon\delta}{\varepsilon-(\varepsilon-\delta)e^{-\delta(T-t)}}\big)+g(t)\Big)\\ &+C^0h(t)\\ &+(A^0+(\pi^{0,M})^\top.B^0_\pi-C^0)(h(t)\wedge h^{0,\mathrm{crit}})-\tfrac12(B^0_\pi)^\top.(\sigma^0.(\sigma^0)^\top)^{-1}.B^0_\pi(h(t)\wedge h^{0,\mathrm{crit}})^2 \end{aligned} \tag{5.25}$$

subject to the boundary conditions

$$g(T)=0,\ \ h(T)=0. \tag{5.26}$$

Theorem 5.20 (Solution of the Investment Problem)**.** *Let condition (NJ) be satisfied, and let h and g be given by (5.24), (5.25) subject to the boundary conditions (5.26). Then the strategy*

$$(\pi^{i,*}(t),c^{i,*}(t))\triangleq(\pi^{i,*}(h(t)),c^{i,*}(t)),\ t\in[0,T],\ i=0,1,$$

as given in Lemmas 5.18 and 5.19 is optimal for the investment problem (P) with CRRA *preferences and relative risk aversion $R=1$.*

For simplicity, we focus on the case when $\pi^{i,M}\in\mathfrak{I}^{\pi,i}$ and $A^0+(\pi^{0,\circ})^\top.B^0_\pi\geq C^0$ in the following. Then equation (5.24) is a backward Riccati ODE,

$$h'(t)=\alpha_0+\alpha_1(t)h(t)+\alpha_2h(t)^2,\ t\in[0,T]$$

with boundary condition $h(T)=0$, where

$$\begin{aligned} \alpha_0 &\triangleq -(\Psi^0-\Psi^1),\\ \alpha_1(t) &\triangleq A^0+(\pi^{0,M})^\top.B^0_\pi+A^1+(\pi^{1,M})^\top.B^1_\pi+\tfrac{\varepsilon\delta}{\varepsilon-(\varepsilon-\delta)e^{-\delta(T-t)}},\\ \alpha_2 &\triangleq -\tfrac12\Big((B^0_\pi)^\top.(\sigma^0.(\sigma^0)^\top)^{-1}.B^0_\pi-(B^1_\pi)^\top.(\sigma^1.(\sigma^1)^\top)^{-1}.B^1_\pi\Big). \end{aligned}$$

In the case without utility from intermediate consumption, i.e. $\varepsilon=0$, h can even be determined explicitly since α_1 gets constant, i.e.

$$\alpha_1=A^0+(\pi^{0,M})^\top.B^0_\pi+A^1+(\pi^{1,M})^\top.B^1_\pi.$$

Moreover, since $\pi^{i,M}\in\mathfrak{I}^{\pi,i}$, which is equivalent to $A^0+(\pi^{0,M})^\top.B^0_\pi>C^0$, resp. $A^1+(\pi^{1,M})^\top.B^1_\pi\geq C^1$, we get that $\alpha_1>0$. Hence

$$h(t)=\begin{cases}\frac{\alpha_1}{2\alpha_2}\Big(D\frac{1+\frac{1-D}{1+D}e^{-\alpha_1D(T-t)}}{1-\frac{1-D}{1+D}e^{-\alpha_1D(T-t)}}-1\Big) & \text{if } \alpha_2\neq0 \text{ and } D>0,\\ \frac{\alpha_1}{2\alpha_2}\big(\frac{2}{\alpha_1(T-t)+2}-1\big), & \text{if } \alpha_2\neq0 \text{ and } D=0,\\ \frac{\alpha_0}{\alpha_1}(e^{-\alpha_1(T-t)}-1) & \text{if } \alpha_2=0\end{cases}$$

for $t\in[0,T]$, where $D\triangleq\sqrt{1-4\frac{\alpha_0\alpha_2}{\alpha_1^2}}$.

Remark 5.21 (Asymptotic behavior)**.** *For $t \to -\infty$ the function h is converging towards*

$$h^{\text{stat}} = \begin{cases} \frac{\alpha_1}{2\alpha_2}(D-1) & \text{for } \alpha_2 \neq 0, \\ -\frac{\alpha_0}{\alpha_1} & \text{for } \alpha_2 = 0. \end{cases}$$

Remark 5.22. *It is not possible that $D < 0$ since in that case h would exhibit periodic poles which would contradict the boundedness of h that has been proved in the section on power utility functions.*

5.3.2 Consumption-dependent Intensities

The log-utility versions of the functions $H^{\pi,i}, H^{c,i}, H_l^{c,i}$ and $H_a^{c,i}$, $i = 0, 1$, are given by $H^{\pi,i} : \mathbb{R}^{\bar{n}} \to \mathbb{R}$ and $H^{c,i}, H_l^{c,i}, H_a^{c,i} : [0,T] \times \mathbb{R} \times \mathbb{R}_0^+ \times \mathbb{R}_0^+ \to \mathbb{R}$ for $i = 0, 1$ with

$$\begin{aligned} H^{\pi,i}(\pi) &\triangleq r^i + \pi^\top.\eta^i - \tfrac{1}{2}\pi^\top.\sigma^i.(\sigma^i)^\top.\pi, \\ H^{c,i}(t,x,y,c) &\triangleq \tfrac{\varepsilon\delta}{\varepsilon-(\varepsilon-\delta)e^{-\delta(T-t)}}(\ln(c)-x) - c + (-1)^{1-i}\max\{A^i + B_c^i c, C^i\}y, \\ H_l^{c,i}(t,x,y,c) &\triangleq \tfrac{\varepsilon\delta}{\varepsilon-(\varepsilon-\delta)e^{-\delta(T-t)}}(\ln(c)-x) - c + (-1)^{1-i}C^i y, \\ H_a^{c,i}(t,x,y,c) &\triangleq \tfrac{\varepsilon\delta}{\varepsilon-(\varepsilon-\delta)e^{-\delta(T-t)}}(\ln(c)-x) - c + (-1)^{1-i}(A^i + B_c^i c)y. \end{aligned}$$

Notice that the functions $H^{c,i}$, $H_l^{c,i}$ and $H_a^{c,i}$ are independent of t and x if $\varepsilon = 0$.

Moreover the candidate solution for the consumption rate reads

$$\begin{aligned} \tilde{c}^{0,*}(t,y) &= \begin{cases} \frac{\varepsilon\delta}{\varepsilon-(\varepsilon-\delta)e^{-\delta(T-t)}}\left(1 + B_c^0 y\right)^{-1} & \text{if } y < h^{\text{crit}}, \\ \infty & \text{if } y \geq h^{\text{crit}}, \end{cases} \\ \tilde{c}^{1,*}(t,y) &= \frac{\varepsilon\delta}{\varepsilon-(\varepsilon-\delta)e^{-\delta(T-t)}}\left(1 - B_c^1 y\right)^{-1} \end{aligned}$$

for $(t,y) \in [0,T] \times \mathbb{R}_0^+$ with

$$h^{\text{crit}} \triangleq \begin{cases} \infty & \text{if } B_c^0 > 0, \\ -\frac{1}{B_c^0} & \text{if } B_c^0 < 0. \end{cases}$$

Lemma 5.23 (Maximizer of $H^{\pi,i}$)**.** *The maximizer*

$$\pi^{i,*} \triangleq \arg\max_{\pi \in \mathbb{R}^{\bar{n}}} H^{\pi,i}(\pi), \ \ i = 0, 1,$$

is given by the Merton strategy, i.e.

$$\pi^{i,*} = \pi^{i,M}.$$

Lemma 5.24 (Maximizer of $H^{c,i}(t,x,y,\cdot)$)**.** *For every $(t,x,y) \in [0,T] \times \mathbb{R} \times \mathbb{R}_0^+$ let*

$$c^{i,*}(t,y) \triangleq \arg\max_{c \in \mathbb{R}_0^+} H^{c,i}(t,x,y,c), \ \ i = 0, 1.$$

If $\varepsilon > 0$ then the maximizer $c^{i,*}$ is given by

$$\begin{aligned} c^{0,*}(t,y) &= \frac{\varepsilon\delta}{\varepsilon-(\varepsilon-\delta)e^{-\delta(T-t)}}\left(1 + B_c^0(y \wedge h^{0,\text{crit}}(t))\right)^{-1} \\ c^{1,*}(t,y) &= \begin{cases} c^M(t) & \text{if } H_{l,M}^{c,1}(t,x,y) > H_{a,\sim}^{c,1}(t,x,y), \\ \tilde{c}^{1,*}(t,y) & \text{if } H_{l,M}^{c,1}(t,x,y) \leq H_{a,\sim}^{c,1}(t,x,y), \end{cases} \end{aligned}$$

with

$$h^{0,\mathrm{crit}}(t) \triangleq -\lambda^{0,\mathrm{crit}}(t),$$

and where

$$\lambda^{0,\mathrm{crit}}(t) = \begin{cases} 0 & \text{if } B_c^0 < 0 \text{ and } A^0 \le C^0, \\ \frac{1}{B_c^0}\Big(1 - \big(1 - \frac{(A^0 + B_c^0 c^M(t) - C^0)^+}{B_c^0 c^M(t)}\big)^{-1}\Big) & \text{if } B_c^0 > 0 \text{ and } A^0 < C^0, \\ & \text{or } B_c^0 < 0 \text{ and } A^0 > C^0, \\ -\infty & \text{if } B_c^0 > 0 \text{ and } A^0 \ge C^0 \end{cases}$$

is the unique solution of

$$\Lambda^{c,0}(t, \lambda^{0,\mathrm{crit}}(t)) = -(A^0 + B_c^0 c^M(t) - C^0)^+ \tag{5.27}$$

if it exists. Otherwise, $\lambda^{0,\mathrm{crit}}(t) = -\infty$. *Moreover*

$$c^{0,\mathrm{crit}}(t) \triangleq \begin{cases} c^M(t) & \text{if } B_c^0 > 0 \text{ and } A^0 \ge C^0, \text{ or } B_c^0 < 0 \text{ and } A^0 \le C^0, \\ c^M(t) - \frac{(A^0 + B_c^0 c^M(t) - C^0)^+}{B_c^0} & \text{if } B_c^0 > 0 \text{ and } A^0 < C^0, \text{ or } B_c^0 < 0 \text{ and } A^0 > C^0. \end{cases}$$

If $\varepsilon = 0$ *then the maximizer* $c^{i,*}$ *is given by*

$$c^{0,*}(t,y) = \begin{cases} 0 & \text{if } y < h^{0,\mathrm{crit}}, \\ c^{0,\mathrm{crit}} & \text{if } y \ge h^{0,\mathrm{crit}}, \end{cases}$$
$$c^{1,*}(t,y) = 0$$

where

$$h^{0,\mathrm{crit}} \triangleq -\lambda^{0,\mathrm{crit}}$$

with

$$\lambda^{0,\mathrm{crit}} = \begin{cases} 0 & \text{if } B_c^0 < 0 \text{ and } A^0 \le C^0 \text{ or } B_c^0 > 0 \text{ and } A^0 < C^0, \\ \frac{1}{B_c^0} & \text{if } B_c^0 < 0 \text{ and } A^0 > C^0, \\ -\infty & \text{if } B_c^0 > 0 \text{ and } A^0 \ge C^0, \end{cases}$$

and

$$c^{0,\mathrm{crit}} \triangleq \begin{cases} 0 & \text{if } B_c^0 > 0 \text{ and } A^0 \ge C^0, \text{ or } B_c^0 < 0 \text{ and } A^0 \le C^0, \\ -\frac{(A^0 - C^0)^+}{B_c^0} & \text{if } B_c^0 > 0 \text{ and } A^0 < C^0, \text{ or } B_c^0 < 0 \text{ and } A^0 > C^0. \end{cases}$$

Here the log-utility version of the function $\Lambda^{c,i}$ given by

$$\Lambda^{c,i} : [0,T] \times (-\infty, \tfrac{1}{B_c^i}) \to \mathbb{R} \ (B_c^i > 0), \text{ resp. } \Lambda^{c,i} : [0,T] \times (\tfrac{1}{B_c^i}, \infty) \to \mathbb{R} \ (B_c^i < 0)$$

with

$$\Lambda^{c,i}(t,\lambda) \triangleq B_c^i\Big(\big(1 - B_c^i \lambda\big)^{-1} - 1\Big)c^M(t).$$

Remark 5.25. *In case of* $\varepsilon > 0$ *the optimal consumption rate in state 1,* $c^{1,*}$*, is independent of* x *although it depends on the comparison of* $H^{c,1}_{l,M}(t,x,y)$ *and* $H^{c,1}_{a,\sim}(t,x,y)$ *which both depend on* x*. However the special structure of* $H^{c,1}_l$*, resp.* $H^{c,1}_a$*, in case of* $R = 1$ *is the reason for this – at first sight – paradoxical relation; namely*

$$H^{c,1}_{l,M}(t,x,y) \le H^{c,1}_{a,\sim}(t,x,y) \quad \Leftrightarrow \quad \big(\ln\big(1 - B_c^1 y\big) + B_c^1 y\big)c^M(t) \le -(d^{c,1}(c^M(t)))^- y$$

which is obviously independent of x.

If $\varepsilon = 0$ then the maximizer $c^{i,*}$ is solely a function of y.

The reduced HJB-equation (5.1) now becomes a system of backward ODEs which for $\varepsilon > 0$ is given by

$$\begin{aligned} h'(t) = &- (\Psi^0 - \Psi^1) + \tfrac{\varepsilon\delta}{\varepsilon-(\varepsilon-\delta)e^{-\delta(T-t)}} h(t) \\ &+ (C^0 + C^1)h(t) \\ &+ (A^0 + B_c^0 c^M(t) - C^0)(h(t) \wedge h^{0,\text{crit}}(t)) \\ &- \Big(-\ln\big(1 + B_c^0(h(t) \wedge h^{0,\text{crit}}(t))\big) + B_c^0(h(t) \wedge h^{0,\text{crit}}(t))\Big)c^M(t) \\ &+ \Big[(A^1 + B_c^1 c^M(t) - C^1)h(t) + \Big(-\ln\big(1 - B_c^1 h(t)\big) - B_c^1 h(t)\Big)c^M(t)\Big]^+, \end{aligned} \tag{5.28}$$

$$\begin{aligned} g'(t) = &- \Psi^0 + \tfrac{\varepsilon\delta}{\varepsilon-(\varepsilon-\delta)e^{-\delta(T-t)}}\Big(1 - \ln\big(\tfrac{\varepsilon\delta}{\varepsilon-(\varepsilon-\delta)e^{-\delta(T-t)}}\big) + g(t)\Big) \\ &+ C^0 h(t) \\ &+ (A^0 + B_c^0 c^M(t) - C^0)(h(t) \wedge h^{0,\text{crit}}(t)) \\ &- \Big(-\ln\big(1 + B_c^0(h(t) \wedge h^{0,\text{crit}}(t))\big) + B_c^0(h(t) \wedge h^{0,\text{crit}}(t))\Big)c^M(t) \end{aligned} \tag{5.29}$$

subject to the boundary conditions

$$g(T) = 0,\ h(T) = 0. \tag{5.30}$$

If $\varepsilon = 0$ then the ODE-system reads

$$\begin{aligned} h'(t) = &- (\Psi^0 - \Psi^1) \\ &+ (C^0 + C^1)h(t) \\ &+ (A^0 - C^0)(h(t) \wedge h^{0,\text{crit}}) + (A^1 - C^1)^+ h(t), \end{aligned} \tag{5.31}$$

$$\begin{aligned} g'(t) = &- \Psi^0 \\ &+ C^0 h(t) \\ &+ (A^0 - C^0)(h(t) \wedge h^{0,\text{crit}}) \end{aligned} \tag{5.32}$$

subject to the boundary conditions

$$g(T) = 0,\ h(T) = 0. \tag{5.33}$$

Theorem 5.26 (Solution of the Investment Problem)**.** *Let condition (NJ) be satisfied, and let h and g be given by (5.28), (5.29) subject to the boundary conditions (5.30) ($\varepsilon > 0$), resp. (5.31), (5.32) subject to the boundary conditions (5.33) ($\varepsilon = 0$). Then the strategy*

$$(\pi^{i,*}(t), c^{i,*}(t)) \triangleq (\pi^{i,*}, c^{i,*}(t, h(t))),\ t \in [0, T],\ i = 0, 1,$$

as given in Lemmas 5.23 and 5.24 is optimal for the investment problem (P) with CRRA *preferences and relative risk aversion $R = 1$.*

Before going on with the portfolio- and consumption-dependent intensity functions we consider the special parameter setting in which the investor is forced to consume although $\varepsilon = 0$, i.e. $B_c^0 < 0$ and $A^0 > C^0$. In the case with logarithmic utilities we can present a stronger version of Proposition 4.125 in which we gave a sufficient condition that guaranteed that the optimal consumption rate will not jump. Remember that in the proof of Proposition 4.125 it was necessary to show that $F(t, x, h^{0,\text{crit}}) \geq 0$ for all $(t, x) \in [0, T] \times \mathbb{R}$. We wrote $F(t, x, h^{0,\text{crit}}) = A + B$ with $A = \chi^1(h^{0,\text{crit}})$ and $B = -(\Psi^0 - \Psi^1) + \chi^0(h^{0,\text{crit}})$ and subsequently only discussed B as A was trivially non-negative. Hence the resulting estimate was not a sharp one. Taking advantage of the simpler structure of the χ^i's in the logarithmic setting we now include A into our considerations. With

$$\begin{aligned} A + B &= -(\Psi^0 - \Psi^1) + (C^0 + C^1)h^{0,\text{crit}} + (A^0 - C^0)h^{0,\text{crit}} + (A^1 - C^1)^+ h^{0,\text{crit}} \\ &= -(\Psi^0 - \Psi^1) - \tfrac{A^0 + (A^1 - C^1)^+ + C^1}{B_c^0} \end{aligned}$$

the stronger version of Proposition 4.125 reads

Proposition 5.27. *Let $\varepsilon = 0$ and $B_c^0 < 0$ and $A^0 > C^0$. Further suppose that $-(\Psi^0 - \Psi^1) - \frac{A^0 + (A^1 - C^1)^+ + C^1}{B_c^0} \geq 0$ or equivalently $A^0 + B_c^0 c^{o,\circ} \geq C^0$, where*

$$c^{0,\circ} \triangleq (\Psi^0 - \Psi^1) + \tfrac{C^0 + (A^1 - C^1)^+ + C^1}{B_c^0}.$$

Then $c^{0,*}(t) = 0$ for all $t \in [0, T]$.

With this stronger result we are in the position to solve ODE (5.31) explicitly in case of $B_c^0 < 0$ and $A^0 > C^0$. First we determine the time $t^{0,\text{crit}}$ at which h reaches the barrier $h^{0,\text{crit}} = -\frac{1}{B_c^0}$ – from a backward perspective. For this purpose we solve the ODE (5.31) where we omit $h^{0,\text{crit}}$, resp. set it to ∞, i.e. we consider

$$(\tilde{h}^{(1)})'(t) = -\,(\Psi^0 - \Psi^1) + (A^0 + (A^1 - C^1)^+ + C^1)\tilde{h}^{(1)}(t)$$

subject to the boundary condition $\tilde{h}^{(1)}(T) = 0$ which has the solution

$$\tilde{h}^{(1)}(t) = \tfrac{\Psi^0 - \Psi^1}{A^0 + (A^1 - C^1)^+ + C^1}\Big(1 - e^{-(A^0 + (A^1 - C^1)^+ + C^1)(T-t)}\Big).$$

Notice that $A^0 + (A^1 - C^1)^+ + C^1 > 0$ since we assumed $A^0 > C^0 \geq 0$. Thus $\tilde{h}^{(1)}$ is decreasing in t. Equating $\tilde{h}^{(1)}(t) = -\frac{1}{B_c^0}$ then yields

$$t^{0,\text{crit}} = T + \tfrac{1}{A^0 + (A^1 - C^1)^+ + C^1} \ln\Big(\tfrac{1}{B_c^0}\tfrac{A^0 + (A^1 - C^1)^+ + C^1}{\Psi^0 - \Psi^1} + 1\Big).$$

Since $\frac{1}{B_c^0}\frac{A^0 + (A^1 - C^1)^+ + C^1}{\Psi^0 - \Psi^1} < 0$ it follows that $t^{0,\text{crit}} < T$. Notice further that $\frac{1}{B_c^0}\frac{A^0 + (A^1 - C^1)^+ + C^1}{\Psi^0 - \Psi^1} + 1$ is possibly non-positive which implies that potentially $t^{0,\text{crit}} = -\infty$, i.e. h would never hit $h^{0,\text{crit}}$. This is particularly the case if $A^0 + B_c^0 c^{0,\circ} \geq 0$ which is equivalent to $\frac{1}{B_c^0}\frac{A^0 + (A^1 - C^1)^+ + C^1}{\Psi^0 - \Psi^1} + 1 \leq 0$.

If $t^{0,\text{crit}} = -\infty$ then the solution of ODE (5.31) is given by $\tilde{h}^{(1)}$, i.e. $h = \tilde{h}^{(1)}$. In case of $t^{0,\text{crit}} > -\infty$ we have to consider the behavior of h from $t^{0,\text{crit}}$ onwards – remember the backward perspective. On this interval h satisfies the following ODE which is obtained by replacing the term $h(t) \wedge h^{0,\text{crit}}$ by $h^{0,\text{crit}}$ in ODE (5.31), i.e. the ODE reads

$$(\tilde{h}^{(2)})'(t) = -\left(\Psi^0 - \Psi^1\right) - \tfrac{A^0 - C^0}{B_c^0} + (C^0 + (A^1 - C^1)^+ + C^1)\tilde{h}^{(2)}(t)$$

for $t \in (-\infty, t^{0,\text{crit}}]$ subject to the boundary condition $\tilde{h}^{(2)}(t^{0,\text{crit}}) = h^{0,\text{crit}} = -\frac{1}{B_c^0}$ which has the solution

$$\tilde{h}^{(2)}(t) = -\tfrac{1}{B_c^0} e^{-(C^0+(A^1-C^1)^+ + C^1)(t^{0,\text{crit}}-t)} + \tfrac{\Psi^0 - \Psi^1 + \frac{A^0-C^0}{B_c^0}}{C^0+(A^1-C^1)^+ + C^1}\Big(1 - e^{-(C^0+(A^1-C^1)^+ + C^1)(t^{0,\text{crit}}-t)}\Big).$$

The derivative of $\tilde{h}^{(2)}$ is consequently given by

$$(\tilde{h}^{(2)})'(t) = -\Big(\Psi^0 - \Psi^1 + \tfrac{A^0+(A^1-C^1)^+ + C^1}{B_c^0}\Big) e^{-(C^0+(A^1-C^1)^+ + C^1)(t^{0,\text{crit}}-t)}.$$

This is negative since we assumed that $t^{0,\text{crit}} > -\infty$ which is equivalent to $\Psi^0 - \Psi^1 + \frac{A^0+(A^1-C^1)^+ + C^1}{B_c^0} > 0$. Thus $\tilde{h}^{(2)}$ is also decreasing in t, i.e. except for $t = t^{0,\text{crit}}$ the function $\tilde{h}^{(2)}$ will never hit $h^{0,\text{crit}}$ again.

Combining the two cases $t^{0,\text{crit}} = -\infty$ and $t^{0,\text{crit}} > -\infty$ the explicit solution of ODE (5.31) in case of $B_c^0 < 0$ and $A^0 > C^0$ is given by

$$h(t) = \begin{cases} \tilde{h}^{(2)}(t) & \text{if } t \in (-\infty, t^{0,\text{crit}}], \\ \tilde{h}^{(1)}(t) & \text{if } t \in (t^{0,\text{crit}}, T] \end{cases}$$

for $t \in [0, T]$.

By integration the solution for the function g turns out to be

$$g(t) = \begin{cases} \tilde{g}^{(2)}(t) + \tilde{g}^{(1)}(t^{0,\text{crit}}) & \text{if } t \in (-\infty, t^{0,\text{crit}}], \\ \tilde{g}^{(1)}(t) & \text{if } t \in (t^{0,\text{crit}}, T] \end{cases}$$

for $t \in [0, T]$ where

$$\tilde{g}^{(1)}(t) \triangleq \Big(\Psi^0 - A^0 \tfrac{\Psi^0 - \Psi^1}{A^0+(A^1-C^1)^+ + C^1}\Big)(T-t) + A^0 \tfrac{\Psi^0 - \Psi^1}{(A^0+(A^1-C^1)^+ + C^1)^2}\Big(1 - e^{-(A^0+(A^1-C^1)^+ + C^1)(T-t)}\Big),$$

$$\begin{aligned} \tilde{g}^{(2)}(t) \triangleq{}& \Big(\Psi^0 + \tfrac{A^0-C^0}{B_c^0} - C^0 \tfrac{\Psi^0 - \Psi^1 + \frac{A^0-C^0}{B_c^0}}{C^0+(A^1-C^1)^+ + C^1}\Big)(t^{0,\text{crit}} - t) \\ &+ C^0 \tfrac{1}{C^0+(A^1-C^1)^+ + C^1}\left(\tfrac{1}{B_c^0} + \tfrac{\Psi^0 - \Psi^1 + \frac{A^0-C^0}{B_c^0}}{C^0+(A^1-C^1)^+ + C^1}\right)\Big(1 - e^{-(C^0+(A^1-C^1)^+ + C^1)(t^{0,\text{crit}}-t)}\Big). \end{aligned}$$

Remark 5.28 (Asymptotic behavior)**.** *Given $\varepsilon = 0$ in case of $B_c^0 < 0$ and $A^0 > C^0$ the functions h and g show the following asymptotic behavior. For $t \to -\infty$ the function h is converging towards h^{stat} where*

$$h^{\text{stat}} \triangleq \begin{cases} \frac{\Psi^0 - \Psi^1}{A^0+(A^1-C^1)^+ + C^1} & \textit{if } t^{0,\text{crit}} = -\infty, \\ \frac{\Psi^0 - \Psi^1 + \frac{A^0-C^0}{B_c^0}}{C^0+(A^1-C^1)^+ + C^1} & \textit{if } t^{0,\text{crit}} > -\infty. \end{cases}$$

Furthermore g exhibits the asymptotic slope

$$\lim_{t \to -\infty} g'(t) = \begin{cases} -\Psi^0 + A^0 \frac{\Psi^0 - \Psi^1}{A^0+(A^1-C^1)^+ + C^1} & \textit{if } t^{0,\text{crit}} = -\infty, \\ -\Psi^0 - \frac{A^0-C^0}{B_c^0} + C^0 \frac{\Psi^0 - \Psi^1 + \frac{A^0-C^0}{B_c^0}}{C^0+(A^1-C^1)^+ + C^1} & \textit{if } t^{0,\text{crit}} > -\infty. \end{cases}$$

Finally the corresponding optimal consumption rate in state 0 is given by

$$c^{0,*}(t) = \begin{cases} c^{0,\text{crit}} & \text{if } t \in (-\infty, t^{0,\text{crit}}], \\ 0 & \text{if } t \in (t^{0,\text{crit}}, T] \end{cases}$$

for $t \in [0,T]$.

5.3.3 Portfolio- and Consumption-dependent Intensities

The log-utility versions of the functions $H^{\pi,c,i}$, $H_l^{\pi,c,i}$ and $H_a^{\pi,c,i}$, $i = 0,1$, are given by $H^{\pi,c,i}, H_l^{\pi,c,i}, H_a^{\pi,c,i}$: $[0,T] \times \mathbb{R} \times \mathbb{R}_0^+ \times \mathbb{R}^{\bar{n}} \times \mathbb{R}_0^+ \to \mathbb{R}$ for $i = 0,1$ with

$$\begin{aligned} H^{\pi,c,i}(t,x,y,\pi,c) &\triangleq \tfrac{\varepsilon\delta}{\varepsilon-(\varepsilon-\delta)e^{-\delta(T-t)}}(\ln(c) - x) - c + r^i + \pi^\top.\eta^i - \tfrac{1}{2}\pi^\top.\sigma^i.(\sigma^i)^\top.\pi \\ &\quad + (-1)^{1-i}\max\{A^i + \pi^\top.B_\pi^i + B_c^i c, C^i\}y, \\ H_l^{\pi,c,i}(t,x,y,\pi,c) &\triangleq \tfrac{\varepsilon\delta}{\varepsilon-(\varepsilon-\delta)e^{-\delta(T-t)}}(\ln(c) - x) - c + r^i + \pi^\top.\eta^i - \tfrac{1}{2}\pi^\top.\sigma^i.(\sigma^i)^\top.\pi + (-1)^{1-i}C^i y, \\ H_a^{\pi,c,i}(t,x,y,\pi,c) &\triangleq \tfrac{\varepsilon\delta}{\varepsilon-(\varepsilon-\delta)e^{-\delta(T-t)}}(\ln(c) - x) - c + r^i + \pi^\top.\eta^i - \tfrac{1}{2}\pi^\top.\sigma^i.(\sigma^i)^\top.\pi \\ &\quad + (-1)^{1-i}(A^i + \pi^\top.B_\pi^i + B_c^i c)y. \end{aligned}$$

Notice that the functions $H^{\pi,c,i}$, $H_l^{\pi,c,i}$ and $H_a^{\pi,c,i}$ are independent of t and x in case of $\varepsilon = 0$.

Further the candidate solution reads

$$\begin{aligned} \tilde{\pi}^{i,*}(t,y) &= (\sigma^i.(\sigma^i)^\top)^{-1}.\big(\eta^i + (-1)^{1-i}B_\pi^i y\big), \\ \tilde{c}^{0,*}(t,y) &= \begin{cases} \frac{\varepsilon\delta}{\varepsilon-(\varepsilon-\delta)e^{-\delta(T-t)}}\big(1 + B_c^0 y\big)^{-1} & \text{if } y < h^{\text{crit}}, \\ \infty & \text{if } y \geq h^{\text{crit}}, \end{cases} \\ \tilde{c}^{1,*}(t,y) &= \tfrac{\varepsilon\delta}{\varepsilon-(\varepsilon-\delta)e^{-\delta(T-t)}}\big(1 - B_c^1 y\big)^{-1} \end{aligned}$$

for $(t,y) \in [0,T] \times \mathbb{R}_0^+$, $i = 0,1$.

Lemma 5.29 (Maximizer of $H^{\pi,c,i}(t,x,y,\cdot,\cdot)$)**.** *For every $(t,x,y) \in [0,T] \times \mathbb{R} \times \mathbb{R}_0^+$ let*

$$(\pi^{i,*}(t,y), c^{i,*}(t,y)) \triangleq \operatorname*{arg\,max}_{(\pi,c)\in\mathbb{R}^{\bar{n}}\times\mathbb{R}_0^+} H^{\pi,c,i}(t,x,y,\pi,c), \;\; i = 0,1.$$

If $\varepsilon > 0$ then the maximizer $(\pi^{i,*}, c^{i,*})$ is given by

$$\begin{aligned} \pi^{0,*}(t,y) &= (\sigma^0.(\sigma^0)^\top)^{-1}.\big(\eta^0 - B_\pi^0(y \wedge h^{0,\text{crit}}(t))\big), \\ c^{0,*}(t,y) &= \tfrac{\varepsilon\delta}{\varepsilon-(\varepsilon-\delta)e^{-\delta(T-t)}}\big(1 + B_c^0(y \wedge h^{0,\text{crit}}(t))\big)^{-1}, \\ (\pi^{1,*}(t,y), c^{1,*}(t,y)) &= \begin{cases} (\pi^{1,M}, c^M(t)) & \text{if } H_{l,M}^{\pi,c,1}(t,x,y) > H_{a,\sim}^{\pi,c,1}(t,,x,y), \\ (\tilde{\pi}^{1,*}(t,y), \tilde{c}^{1,*}(t,y)) & \text{if } H_{l,M}^{\pi,c,1}(t,x,y) \leq H_{a,\sim}^{\pi,c,1}(t,x,y) \end{cases} \end{aligned}$$

where

$$h^{0,\text{crit}}(t) \triangleq -\lambda^{0,\text{crit}}(t),$$

and where $\lambda^{0,\mathrm{crit}}(t)$ *is implicitly given via*

$$\Lambda^{\pi,c,0}(t,\lambda^{0,\mathrm{crit}}(t)) = -(A^0 + (\pi^{0,M})^\top.B^0_\pi + B^0_c c^M(t) - C^0)^+.$$

Moreover

$$\begin{aligned}\pi^{0,\mathrm{crit}}(t) &\triangleq (\sigma^0.(\sigma^0)^\top)^{-1}.\big(\eta^0 - B^0_\pi h^{0,\mathrm{crit}}(t)\big),\\ c^{0,\mathrm{crit}}(t) &\triangleq \tfrac{\varepsilon\delta}{\varepsilon-(\varepsilon-\delta)e^{-\delta(T-t)}}\big(1 + B^0_c h^{0,\mathrm{crit}}(t)\big)^{-1}.\end{aligned}$$

If $\varepsilon = 0$ *then the maximizer* $(\pi^{i,*}, c^{i,*})$ *is given by*

$$\begin{aligned}\pi^{0,*}(t,y) &= (\sigma^0.(\sigma^0)^\top)^{-1}.\big(\eta^0 - B^0_\pi(y\wedge h^{0,\mathrm{crit}})\big),\\ c^{0,*}(t,y) &= \begin{cases} 0 & \textit{if } y < h^{0,\mathrm{crit}},\\ c^{0,\mathrm{crit}} & \textit{if } y \ge h^{0,\mathrm{crit}},\end{cases}\\ (\pi^{1,*}(t,y), c^{1,*}(t,y)) &= \begin{cases} (\pi^{1,M}, 0) & \textit{if } y < h^{1,\mathrm{crit}},\\ (\tilde\pi^{1,*}(t,y), 0) & \textit{if } y \ge h^{1,\mathrm{crit}},\end{cases}\end{aligned}$$

where

$$\begin{aligned}h^{0,\mathrm{crit}} &\triangleq \tfrac{(A^0+(\pi^{0,M})^\top.B^0_\pi - C^0)^+}{(B^0_\pi)^\top.(\sigma^0.(\sigma^0)^\top)^{-1}.B^0_\pi} \wedge h^{\mathrm{crit}},\\ h^{1,\mathrm{crit}} &\triangleq 2\tfrac{(A^1+(\pi^{1,M})^\top.B^1_\pi - C^1)^-}{(B^1_\pi)^\top.(\sigma^1.(\sigma^1)^\top)^{-1}.B^1_\pi},\\ h_{\mathrm{crit}} &\triangleq \begin{cases}\infty & \textit{if } B^0_c > 0,\\ -\tfrac{1}{B^0_c} & \textit{if } B^0_c < 0\end{cases}\end{aligned}$$

and

$$\begin{aligned}\pi^{0,\mathrm{crit}} &= (\sigma^0.(\sigma^0)^\top)^{-1}.\Big(\eta^0 + B^0_\pi \tfrac{1}{B^0_c}\Big(1 - \Big(\tfrac{(A^0+(\pi^{0,M})^\top.B^0_\pi - C^0)^+}{(B^0_\pi)^\top.(\sigma^0.(\sigma^0)^\top)^{-1}.B^0_\pi}B^0_c + 1\Big)^+\Big)\Big)\\ c^{0,\mathrm{crit}} &= \tfrac{(B^0_\pi)^\top.(\sigma^0.(\sigma^0)^\top)^{-1}.B^0_\pi}{(B^0_c)^2}\Big(\tfrac{(A^0+(\pi^{0,M})^\top.B^0_\pi - C^0)^+}{(B^0_\pi)^\top.(\sigma^0.(\sigma^0)^\top)^{-1}.B^0_\pi}B^0_c + 1\Big)^-.\end{aligned}$$

Remark 5.30. *As in the consumption-dependent case* $(\pi^{1,*}, c^{1,*})$ *depends only on* t *and* y *if* $\varepsilon > 0$ *since*

$$\begin{aligned}& H^{\pi,c,1}_{l,M}(t,x,y) \le H^{\pi,c,1}_{a,\sim}(t,x,y)\\ \Leftrightarrow\quad & \Big(\ln\big(1 - B^1_c y\big) + B^1_c y\Big)c^M(t) - \tfrac12 (B^1_\pi)^\top.(\sigma^1.(\sigma^1)^\top)^{-1}.B^1_\pi y^2 \le -(d^{\pi,c,1}(\pi^{1,M}, c^M(t)))^- y\end{aligned}$$

is independent of x.

If $\varepsilon = 0$ *then the maximizer* $(\pi^{i,*}, c^{i,*})$ *is solely a function of* y.

Inserting the maximizing strategies the reduced HJB-equation (5.1) now becomes a system of backward

ODEs which for $\varepsilon > 0$ is given by

$$\begin{aligned} h'(t) = &-(\Psi^0 - \Psi^1) + \tfrac{\varepsilon\delta}{\varepsilon-(\varepsilon-\delta)e^{-\delta(T-t)}} h(t) \\ &+ (C^0 + C^1)h(t) \\ &+ (A^0 + (\pi^{0,M})^\top . B^0_\pi + B^0_c c^M(t) - C^0)(h(t) \wedge h^{0,\mathrm{crit}}(t)) \\ &- \tfrac{1}{2}(B^0_\pi)^\top . (\sigma^0 . (\sigma^0)^\top)^{-1} . B^0_\pi (h(t) \wedge h^{0,\mathrm{crit}}(t))^2 \\ &- \Big(-\ln\big(1 + B^0_c(h(t) \wedge h^{0,\mathrm{crit}}(t))\big) + B^0_c(h(t) \wedge h^{0,\mathrm{crit}}(t))\Big) c^M(t) \\ &+ \Big[(A^1 + (\pi^{1,M})^\top . B^1_\pi + B^1_c c^M(t) - C^1)h(t) \\ &\quad + \tfrac{1}{2}(B^1_\pi)^\top . (\sigma^1 . (\sigma^1)^\top)^{-1} . B^1_\pi (h(t))^2 + \Big(-\ln\big(1 - B^1_c h(t)\big) - B^1_c h(t)\Big) c^M(t) \Big]^+, \end{aligned} \tag{5.34}$$

$$\begin{aligned} g'(t) = &-\Psi^0 + \tfrac{\varepsilon\delta}{\varepsilon-(\varepsilon-\delta)e^{-\delta(T-t)}} \Big(1 - \ln\big(\tfrac{\varepsilon\delta}{\varepsilon-(\varepsilon-\delta)e^{-\delta(T-t)}}\big) + g(t)\Big) \\ &+ C^0 h(t) \\ &+ (A^0 + (\pi^{0,M})^\top . B^0_\pi + B^0_c c^M(t) - C^0)(h(t) \wedge h^{0,\mathrm{crit}}(t)) \\ &- \tfrac{1}{2}(B^0_\pi)^\top . (\sigma^0 . (\sigma^0)^\top)^{-1} . B^0_\pi (h(t) \wedge h^{0,\mathrm{crit}}(t))^2 \\ &- \Big(-\ln\big(1 + B^0_c(h(t) \wedge h^{0,\mathrm{crit}}(t))\big) + B^0_c(h(t) \wedge h^{0,\mathrm{crit}}(t))\Big) c^M(t) \end{aligned} \tag{5.35}$$

subject to the boundary conditions

$$g(T) = 0, \;\; h(T) = 0. \tag{5.36}$$

If $\varepsilon = 0$ then the ODE-system reads

$$\begin{aligned} h'(t) = &-(\Psi^0 - \Psi^1) \\ &+ (C^0 + C^1)h(t) \\ &+ (A^0 + (\pi^{0,M})^\top . B^0_\pi - C^0)(h(t) \wedge h^{0,\mathrm{crit}}) - \tfrac{1}{2}(B^0_\pi)^\top . (\sigma^0 . (\sigma^0)^\top)^{-1} . B^0_\pi (h(t) \wedge h^{0,\mathrm{crit}})^2 \\ &+ (A^1 + (\pi^{1,M})^\top . B^1_\pi - C^1)(h(t) \vee h^{1,\mathrm{crit}}) + \tfrac{1}{2}(B^1_\pi)^\top . (\sigma^1 . (\sigma^1)^\top)^{-1} . B^1_\pi (h(t) \vee h^{1,\mathrm{crit}})^2, \end{aligned} \tag{5.37}$$

$$\begin{aligned} g'(t) = &-\Psi^0 \\ &+ C^0 h(t) \\ &+ (A^0 + (\pi^{0,M})^\top . B^0_\pi - C^0)(h(t) \wedge h^{0,\mathrm{crit}}) - \tfrac{1}{2}(B^0_\pi)^\top . (\sigma^0 . (\sigma^0)^\top)^{-1} . B^0_\pi (h(t) \wedge h^{0,\mathrm{crit}})^2 \end{aligned} \tag{5.38}$$

subject to the boundary conditions

$$g(T) = 0, \;\; h(T) = 0. \tag{5.39}$$

Theorem 5.31 (Solution of the Investment Problem)**.** *Let condition (NJ) be satisfied, and let h and g be given by (5.34), (5.35) subject to the boundary conditions (5.36) ($\varepsilon > 0$), resp. (5.37), (5.38) subject to the boundary conditions (5.39) ($\varepsilon = 0$). Then the strategy*

$$(\pi^{i,*}(t), c^{i,*}(t)) \triangleq (\pi^{i,*}(t, h(t)), c^{i,*}(t, h(t))), \;\; t \in [0, T], \;\; i = 0, 1,$$

as given in Lemma 5.29 is optimal for the investment problem (P) with CRRA preferences and relative risk aversion $R = 1$.

If $\varepsilon = 0$ then focusing on the case $(\pi^{i,M}, 0) \in \mathfrak{I}^{\pi,c,i}$ and $A^0 + (\pi^{0,\circ})^\top.B^0_\pi + B^0_c c^{0,\circ} \geq C^0$ equation (5.37) becomes a backward Riccati ODE,

$$h'(t) = \alpha_0 + \alpha_1 h(t) + \alpha_2 h(t)^2,\ t \in [0,T]$$

with boundary condition $h(T) = 0$, where

$$\begin{aligned}
\alpha_0 &\triangleq -(\Psi^0 - \Psi^1),\\
\alpha_1 &\triangleq A^0 + (\pi^{0,M})^\top.B^0_\pi + A^1 + (\pi^{1,M})^\top.B^1_\pi,\\
\alpha_2 &\triangleq -\tfrac{1}{2}\Big((B^0_\pi)^\top.(\sigma^0.(\sigma^0)^\top)^{-1}.B^0_\pi - (B^1_\pi)^\top.(\sigma^1.(\sigma^1)^\top)^{-1}.B^1_\pi\Big)
\end{aligned}$$

so that h can even be determined explicitly.

Moreover, since $\pi^{i,M} \in \mathfrak{I}^{\pi,i}$, which is equivalent to $A^0 + (\pi^{0,M})^\top.B^0_\pi > C^0$, resp. $A^1 + (\pi^{1,M})^\top.B^1_\pi \geq C^1$, we get that $\alpha_1 > 0$. Hence

$$h(t) = \begin{cases} \frac{\alpha_1}{2\alpha_2}\Big(D\frac{1+\frac{1-D}{1+D}e^{-\alpha_1 D(T-t)}}{1-\frac{1-D}{1+D}e^{-\alpha_1 D(T-t)}} - 1\Big) & \text{if } \alpha_2 \neq 0 \text{ and } D > 0,\\ \frac{\alpha_1}{2\alpha_2}(\frac{2}{\alpha_1(T-t)+2} - 1), & \text{if } \alpha_2 \neq 0 \text{ and } D = 0,\\ \frac{\alpha_0}{\alpha_1}(e^{-\alpha_1(T-t)} - 1) & \text{if } \alpha_2 = 0 \end{cases}$$

for $t \in [0,T]$, where $D \triangleq \sqrt{1 - 4\frac{\alpha_0\alpha_2}{\alpha_1^2}}$.

Remark 5.32 (Asymptotic behavior)**.** *For $t \to -\infty$ the function h is converging towards*

$$h^{\text{stat}} = \begin{cases} \frac{\alpha_1}{2\alpha_2}(D-1) & \text{for } \alpha_2 \neq 0,\\ -\frac{\alpha_0}{\alpha_1} & \text{for } \alpha_2 = 0. \end{cases}$$

Remark 5.33. *It is not possible that $D < 0$ since in that case h would exhibit periodic poles which would contradict the boundedness of h that has been proved in the section on power utility functions.*

Remark 5.34. *Notice that this explicit solution coincides with the explicit solution that has been found in the portfolio-dependent setting in case of $\pi^{i,M} \in \mathfrak{I}^{\pi,i}$ and $A^0 + (\pi^{0,\circ})^\top.B^0_\pi \geq C^0$ (where $\pi^{0,\circ}$ is given as in the portfolio-dependent setting) when $\varepsilon = 0$.*

6 A Special Case: Two Correlated Assets

In this section we take a closer look on a special variant of our model: We analyze the optimal investment strategy of a large investor acting in a market with only two risky assets besides the riskless money market account. Those two assets shall be driven by two correlated Wiener processes and, moreover, only the investor's holdings in one of those two assets shall impact on the regime shift intensities. This can be interpreted as two assets from the same market segment but only the large investor's actions concerning one of these stocks are monitored by the other market participants whereas in the other stock he can act as if he was a small investor.

Asset Price Dynamics. The asset price dynamics are

$$\begin{aligned}
\mathrm{d}P^0 &= P^0 r^{I-}\mathrm{d}t, \quad P^0(0) = p_0^0,\\
\mathrm{d}P^1 &= P^1\left[(r^{I-} + \eta_1^{I-})\mathrm{d}t + \sigma_1^{I-}\mathrm{d}W^1\right], \quad P^1(0) = p_0^1,\\
\mathrm{d}P^2 &= P^2\left[(r^{I-} + \eta_2^{I-})\mathrm{d}t + \rho^{I-}\sigma_2^{I-}\mathrm{d}W^1 + \sqrt{1-(\rho^{I-})^2}\sigma_2^{I-}\mathrm{d}W^2\right], \quad P^2(0) = p_0^2,
\end{aligned}$$

where $\sigma_j^i \neq 0$, $j = 1, 2$, and $\rho^i \in (-1, 1)$ is the correlation coefficient between the two Wiener processes in state $i = 0, 1$. Thus the volatility matrix is given by

$$\sigma^i = \begin{pmatrix} \sigma_1^i & 0 \\ \rho^i\sigma_2^i & \sqrt{1-(\rho^i)^2}\sigma_2^i \end{pmatrix}$$

implying

$$(\sigma^i.(\sigma^i)^\top)^{-1} = \begin{pmatrix} \frac{1}{1-(\rho^i)^2}\frac{1}{(\sigma_1^i)^2} & -\frac{\rho^i}{1-(\rho^i)^2}\frac{1}{\sigma_1^i\sigma_2^i} \\ -\frac{\rho^i}{1-(\rho^i)^2}\frac{1}{\sigma_1^i\sigma_2^i} & \frac{1}{1-(\rho^i)^2}\frac{1}{(\sigma_2^i)^2} \end{pmatrix}.$$

Focusing on the portfolio-dependent variants of the aforementioned models with step, resp. affine intensity functions we analyze the impact of the existence of a second, correlated asset on the optimal portfolio strategy.

6.1 Step Intensity Functions

Since only the investor's portfolio proportions in the first asset shall impact on the regime shift intensities we let $B_\pi^i \triangleq (B_{\pi,1}^i, 0)^\top$ so that the intensity function is given by

$$\vartheta^{i,1-i}(\pi, c) = C_1^i 1_{\{A^i+\pi_1 B_{\pi,1}^i \leq C^i\}} + C_2^i 1_{\{A^i+\pi_1 B_{\pi,1}^i > C^i\}}, \ (\pi, c) \in \mathbb{R}^2 \times \mathbb{R}_0^+.$$

Lemma 6.1 (Maximizer of $H^{\pi,i}(y,\cdot)$). *For every $y \in \mathbb{R}_0^+$ the maximizer*

$$\pi^{i,*}(y) \triangleq \operatorname*{arg\,max}_{\pi\in\mathbb{R}^2} H^{\pi,i}(y, \pi), \ i = 0, 1,$$

is given by

$$\pi^{i,*}(y) = \begin{cases} \pi^{i,M} & \text{if } y < h^{i,\text{crit}}, \\ \pi^{i,\text{crit}} & \text{if } y \geq h^{i,\text{crit}}, \end{cases}$$

where

$$h^{i,\mathrm{crit}} \triangleq (-1)^{1-i}\frac{1}{1-R}\ln\Big((1-R)\frac{\zeta^{i,\mathrm{crit}}}{C_2^i-C_1^i}+1\Big)$$

with

$$\zeta^{i,\mathrm{crit}} \triangleq -\frac{1}{2}\frac{((A^i+\pi_1^{i,M}B_{\pi,1}^i-C^i)^+)^2}{\frac{1}{R}\frac{1}{1-(\rho^i)^2}\frac{1}{(\sigma_1^i)^2}(B_{\pi,1}^i)^2}$$

and

$$\pi_1^{i,M} = \frac{1}{R}\Big(\frac{1}{1-(\rho^i)^2}\frac{\eta_1^i}{(\sigma_1^i)^2}-\frac{\rho^i}{1-(\rho^i)^2}\frac{\eta_2^i}{\sigma_1^i\sigma_2^i}\Big), \qquad \pi_1^{i,\mathrm{crit}} = \pi_1^{i,M}-\frac{(A^i+\pi_1^{i,M}B_{\pi,1}^i-C^i)^+}{B_{\pi,1}^i},$$

$$\pi_2^{i,M} = \frac{1}{R}\Big(\frac{1}{1-(\rho^i)^2}\frac{\eta_2^i}{(\sigma_2^i)^2}-\frac{\rho^i}{1-(\rho^i)^2}\frac{\eta_1^i}{\sigma_1^i\sigma_2^i}\Big), \qquad \pi_2^{i,\mathrm{crit}} = \pi_2^{i,M}+\rho^i\frac{\sigma_1^i}{\sigma_2^i}\frac{(A^i+\pi_1^{i,M}B_{\pi,1}^i-C^i)^+}{B_{\pi,1}^i}.$$

Further the ODEs for the functions g and h are given by

$$\begin{aligned} h'(t) =& -(\Psi^0-\Psi^1)+\Big(\frac{\varepsilon\delta}{\varepsilon-(\varepsilon-\delta)e^{-\delta(T-t)}}\Big)^{\frac{1}{R}}e^{-\frac{1-R}{R}g(t)}\frac{R}{1-R}(e^{\frac{1-R}{R}h(t)}-1)\\ & -C_2^0\frac{1}{1-R}(e^{-(1-R)h(t)}-1)+C_2^1\frac{1}{1-R}(e^{(1-R)h(t)}-1)\\ & -\Big[(C_1^0-C_2^0)\frac{1}{1-R}(e^{-(1-R)h(t)}-1)-\frac{1}{2}\frac{((A^0+\pi_1^{0,M}B_{\pi,1}^0-C^0)^+)^2}{\frac{1}{R}\frac{1}{1-(\rho^0)^2}\frac{1}{(\sigma_1^0)^2}(B_{\pi,1}^0)^2}\Big]^+\\ & +\Big[(C_1^1-C_2^1)\frac{1}{1-R}(e^{(1-R)h(t)}-1)-\frac{1}{2}\frac{((A^1+\pi_1^{1,M}B_{\pi,1}^1-C^1)^+)^2}{\frac{1}{R}\frac{1}{1-(\rho^1)^2}\frac{1}{(\sigma_1^1)^2}(B_{\pi,1}^1)^2}\Big]^+, \end{aligned} \tag{6.1}$$

$$\begin{aligned} g'(t) =& -\Psi^0+\frac{\varepsilon\delta}{\varepsilon-(\varepsilon-\delta)e^{-\delta(T-t)}}\Big(1-\frac{R}{1-R}\Big(\Big(\frac{\varepsilon\delta}{\varepsilon-(\varepsilon-\delta)e^{-\delta(T-t)}}\Big)^{\frac{1-R}{R}}e^{-\frac{1-R}{R}g(t)}-1\Big)\Big)\\ & -C_2^0\frac{1}{1-R}(e^{-(1-R)h(t)}-1)\\ & -\Big[(C_1^0-C_2^0)\frac{1}{1-R}(e^{-(1-R)h(t)}-1)-\frac{1}{2}\frac{((A^0+\pi_1^{0,M}B_{\pi,1}^0-C^0)^+)^2}{\frac{1}{R}\frac{1}{1-(\rho^0)^2}\frac{1}{(\sigma_1^0)^2}(B_{\pi,1}^0)^2}\Big]^+ \end{aligned} \tag{6.2}$$

subject to the boundary condition $h(T)=g(T)=0$ where

$$\Psi^i = r^i+\frac{1}{R}\frac{1}{1+\rho^i}\Big(\frac{1}{2}\frac{1}{1-\rho^i}\big(\frac{\eta_1^i}{\sigma_1^i}-\frac{\eta_2^i}{\sigma_2^i}\big)^2+\frac{\eta_1^i\eta_2^i}{\sigma_1^i\sigma_2^i}\Big).$$

Dependency of $\pi^{i,*}$, g and h on ρ^i. Concerning the impact of the correlation coefficient on the optimal strategy we observe that the optimal portfolio proportion in the second asset may deviate from the Merton proportion if $\rho^i \neq 0$, i.e. the large investor may deviate from the Merton strategy even in the second asset although the portfolio proportions in this asset do not affect the regime shift intensities. Hence this deviation is solely due to compensational reasons. Whereas the investor faces the trade-off between the utility criterion and the intensity criterion in the first asset, he can neglect the intensity criterion concerning his proportions in the second asset.

Thus the large investor can use his shares in the second, correlated asset in order to rearrange the relation of the utility criterion and the intensity criterion in a favorable manner. Roughly speaking, by $B_{\pi,2}^i = 0$ the investor obtains a degree of freedom that he uses to further optimize his portfolio.

However the existence of a second, correlated asset does not influence the critical portfolio strategy $\pi_1^{i,\text{crit}}$ in the first asset at least in the non-trivial case of $\pi^{i,M} \notin \mathcal{F}^{\pi,i}$. Given that the optimal strategy jumps it always jumps to the critical strategy $\pi_1^{i,\text{crit}}$ which is given by $\pi_1^{i,\text{crit}} = -\frac{A^i - C^i}{B^i_{\pi,1}}$ in the non-trivial case of $\pi^{i,M} \notin \mathcal{F}^{\pi,i}$, resp. $\pi_1^{i,\text{crit}} = \pi_1^{i,M}$ in the trivial case $\pi^{i,M} \in \mathcal{F}^{\pi,i}$ where no jump occurs.

Unfortunately this is the only observable, unambiguous influence of the correlation coefficient. The size of the deviation of the optimal strategy from the Merton strategy in the second asset depends on ρ^i in a non-monotonic way. This is due to the fact that the Merton proportions by themselves depend on the correlation coefficient ρ^i in a non-monotonic way. Thus the optimal strategy and the functions g and h do not monotonically depend on ρ^i.

Exemplary we take a look at $\zeta^{i,\text{crit}}$. Varying ρ^i causes two effects. Firstly the denominator of the fraction in $\zeta^{i,\text{crit}}$ is growing in $|\rho^i|$ with limit ∞. Thus, if the numerator did not change a bigger $|\rho^i|$ would imply a bigger $\zeta^{i,\text{crit}}$ – remember the factor $-\frac{1}{2}$ – and thus a smaller $h^{i,\text{crit}}$. But secondly the numerator also depends on ρ^i via the Merton proportion $\pi_1^{i,M}$. Varying the correlation coefficient changes the position of the Merton strategy relative to the separating hyperplane between the half spaces of favorable and adverse strategies. However, whereas the influence of ρ^i on the denominator is clear, independent of the parameter setting, its impact on the numerator is ambiguous.

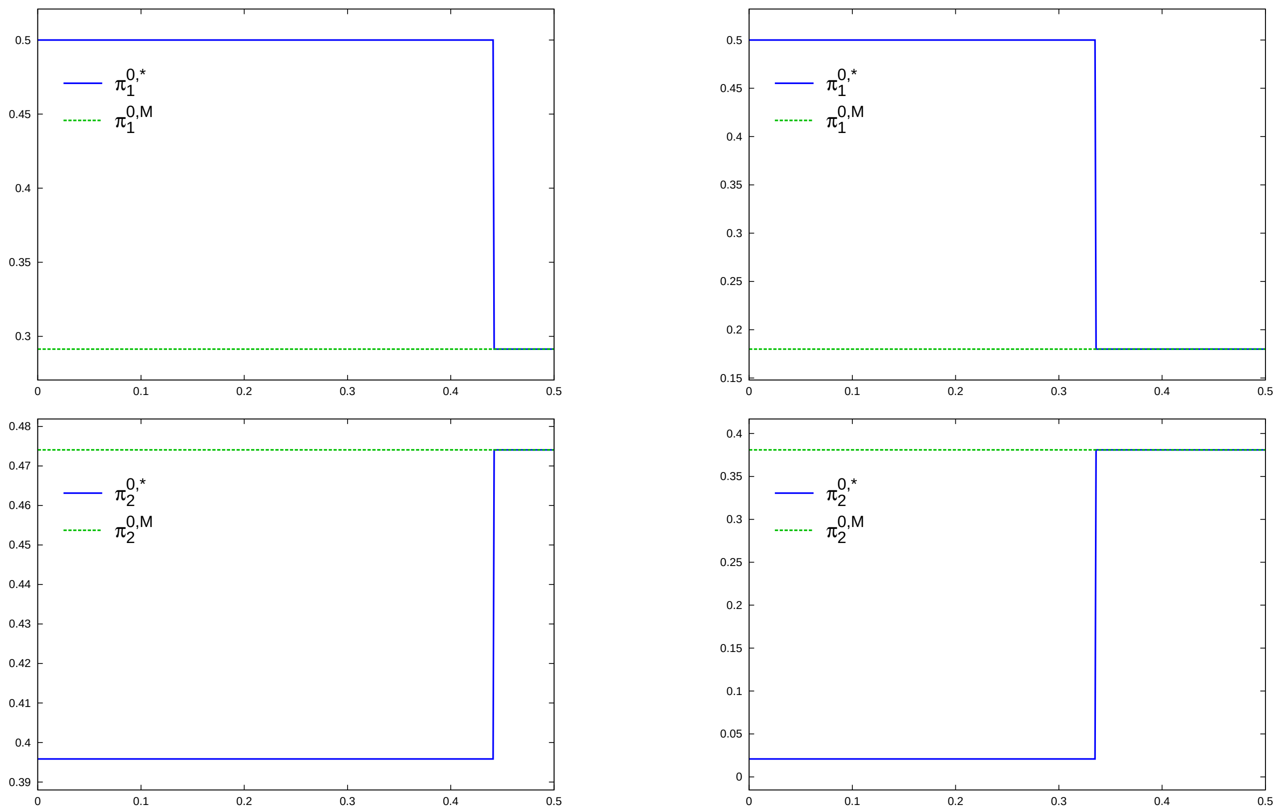

Figure 6.1: $\pi_j^{0,*}$ and $\pi_j^{0,M}$ as functions of t for $\rho = 0.25$ (left) and $\rho = 0.75$ (right)
($r^0 = r^1 = 0.03$, $\eta^0 = (0.1, 0.07)^\top$, $\eta^1 = (0.02, 0.05)^\top$, $\sigma_1^0 = 0.3$, $\sigma_2^0 = 0.2$, $\sigma_1^1 = 0.6$, $\sigma_2^1 = 0.4$,
$\delta = 0.035$, $\varepsilon = 1$, $R = 3$, $T = 0.5$,
$A^0 = 16.5$, $B_\pi^0 = (-5, 0)^\top$, $C^0 = 14$, $C_1^0 = 5$, $C_2^0 = 10$, $A^1 = 5.1$, $B_\pi^1 = (-4, 0)^\top$, $C^1 = 5$, $C_1^1 = 5$, $C_2^1 = 1.25$)

Figure 6.1 shows the optimal strategy for the first asset (upper plots) and the second asset (lower plots) in state 0 for a correlation coefficient given by $\rho = 0.25$ (left plots) and $\rho = 0.75$ (right plots). In case of $\rho = 0.75$ the deviation of the optimal strategy from the Merton strategy in the second asset amounts to the significant size of about 93%.

6.2 Affine Intensity Functions

Again let $B^i_\pi \triangleq (B^i_{\pi,1}, 0)^\top$ so that the intensity function is given by

$$\vartheta^{i,1-i}(\pi, c) = \max\{A^i + \pi_1 B^i_{\pi,1}, C^i\},\ (\pi, c) \in \mathbb{R}^2 \times \mathbb{R}^+_0.$$

Lemma 6.2 (Maximizer of $H^{\pi,i}(y,\cdot)$)**.** *For every $y \in \mathbb{R}^+_0$ the maximizer*

$$\pi^{i,*}(y) \triangleq \underset{\pi \in \mathbb{R}^2}{\arg\max}\, H^{\pi,i}(y, \pi),\ i = 0, 1,$$

is given by

$$\pi^{0,*}(y) = \begin{pmatrix} \pi_1^{0,M} + \frac{1}{R}\frac{1}{1-(\rho^0)^2}\frac{1}{(\sigma_1^0)^2} B^0_{\pi,1}\frac{1}{1-R}\left(e^{-(1-R)(y\wedge h^{0,crit})} - 1\right) \\ \pi_2^{0,M} - \frac{1}{R}\frac{\rho^0}{1-(\rho^0)^2}\frac{1}{\sigma_1^0\sigma_2^0} B^0_{\pi,1}\frac{1}{1-R}\left(e^{-(1-R)(y\wedge h^{0,crit})} - 1\right) \end{pmatrix},$$

$$\pi^{1,*}(y) = \begin{cases} \begin{pmatrix} \pi_1^{1,M} \\ \pi_2^{1,M} \end{pmatrix} & \text{if } y < h^{1,\mathrm{crit}}, \\ \begin{pmatrix} \pi_1^{1,M} + \frac{1}{R}\frac{1}{1-(\rho^1)^2}\frac{1}{(\sigma_1^1)^2} B^1_{\pi,1}\frac{1}{1-R}\left(e^{(1-R)y} - 1\right) \\ \pi_2^{1,M} - \frac{1}{R}\frac{\rho^1}{1-(\rho^1)^2}\frac{1}{\sigma_1^1\sigma_2^1} B^1_{\pi,1}\frac{1}{1-R}\left(e^{(1-R)y} - 1\right) \end{pmatrix} & \text{if } y \geq h^{1,\mathrm{crit}}, \end{cases}$$

where

$$h^{0,\mathrm{crit}} \triangleq -\frac{1}{1-R}\ln\Bigg(-(1-R)\frac{(A^0+\pi_1^{0,M}B^0_{\pi,1}-C^0)^+}{\frac{1}{R}\frac{1}{1-(\rho^0)^2}\frac{1}{(\sigma_1^0)^2}(B^0_{\pi,1})^2} + 1\Bigg),$$
$$h^{1,\mathrm{crit}} \triangleq \frac{1}{1-R}\ln\Bigg((1-R)2\frac{(A^1+\pi_1^{1,M}B^1_{\pi,1}-C^1)^-}{\frac{1}{R}\frac{1}{1-(\rho^1)^2}\frac{1}{(\sigma_1^1)^2}(B^1_{\pi,1})^2} + 1\Bigg),$$

and

$$\pi_1^{i,M} = \frac{1}{R}\left(\frac{1}{1-(\rho^i)^2}\frac{\eta_1^i}{(\sigma_1^i)^2} - \frac{\rho^i}{1-(\rho^i)^2}\frac{\eta_2^i}{\sigma_1^i\sigma_2^i}\right), \qquad \pi_1^{0,\mathrm{crit}} = \pi_1^{0,M} - \frac{(A^0+\pi_1^{0,M}B^0_{\pi,1}-C^0)^+}{B^0_{\pi,1}},$$
$$\pi_2^{i,M} = \frac{1}{R}\left(\frac{1}{1-(\rho^i)^2}\frac{\eta_2^i}{(\sigma_2^i)^2} - \frac{\rho^i}{1-(\rho^i)^2}\frac{\eta_1^i}{\sigma_1^i\sigma_2^i}\right), \qquad \pi_2^{0,\mathrm{crit}} = \pi_2^{0,M} + \rho^0\frac{\sigma_1^0}{\sigma_2^0}\frac{(A^0+\pi_1^{0,M}B^0_{\pi,1}-C^0)^+}{B^0_{\pi,1}}.$$

The ODEs for the functions g and h are given by

$$
\begin{aligned}
h'(t) = &- (\Psi^0 - \Psi^1) + \big(\tfrac{\varepsilon\delta}{\varepsilon-(\varepsilon-\delta)e^{-\delta(T-t)}}\big)^{\frac{1}{R}} e^{-\frac{1-R}{R}g(t)} \tfrac{R}{1-R}(e^{\frac{1-R}{R}h(t)} - 1) \\
&- C^0 \tfrac{1}{1-R}(e^{-(1-R)h(t)} - 1) + C^1 \tfrac{1}{1-R}(e^{(1-R)h(t)} - 1) \\
&- (A^0 + \pi_1^{0,M} B^0_{\pi,1} - C^0) \tfrac{1}{1-R}(e^{-(1-R)(h(t)\wedge h^{0,\text{crit}})} - 1) \\
&- \tfrac{1}{2}\tfrac{1}{R}\tfrac{1}{1-(\rho^0)^2}\tfrac{1}{(\sigma_1^0)^2}(B^0_{\pi,1})^2 \tfrac{1}{(1-R)^2}(e^{-(1-R)(h(t)\wedge h^{0,\text{crit}})} - 1)^2 \\
&+ (A^1 + \pi_1^{1,M} B^1_{\pi,1} - C^1) \tfrac{1}{1-R}(e^{(1-R)(h(t)\vee h^{1,\text{crit}})} - 1) \\
&+ \tfrac{1}{2}\tfrac{1}{R}\tfrac{1}{1-(\rho^1)^2}\tfrac{1}{(\sigma_1^1)^2}(B^1_{\pi,1})^2 \tfrac{1}{(1-R)^2}(e^{(1-R)(h(t)\vee h^{1,\text{crit}})} - 1)^2,
\end{aligned}
\tag{6.3}
$$

$$
\begin{aligned}
g'(t) = &- \Psi^0 + \tfrac{\varepsilon\delta}{\varepsilon-(\varepsilon-\delta)e^{-\delta(T-t)}}\Big(1 - \tfrac{R}{1-R}\big(\big(\tfrac{\varepsilon\delta}{\varepsilon-(\varepsilon-\delta)e^{-\delta(T-t)}}\big)^{\frac{1-R}{R}} e^{-\frac{1-R}{R}g(t)} - 1\big)\Big) \\
&- C^0 \tfrac{1}{1-R}(e^{-(1-R)h(t)} - 1) \\
&- (A^0 + \pi_1^{0,M} B^0_{\pi,1} - C^0) \tfrac{1}{1-R}(e^{-(1-R)(h(t)\wedge h^{0,\text{crit}})} - 1) \\
&- \tfrac{1}{2}\tfrac{1}{R}\tfrac{1}{1-(\rho^0)^2}\tfrac{1}{(\sigma_1^0)^2}(B^0_{\pi,1})^2 \tfrac{1}{(1-R)^2}(e^{-(1-R)(h(t)\wedge h^{0,\text{crit}})} - 1)^2
\end{aligned}
\tag{6.4}
$$

subject to the boundary condition $h(T) = g(T) = 0$.

Dependency of $\pi^{i,*}$, g and h on ρ^i. As already in the previous section on step intensity functions the optimal portfolio proportion in the second asset deviates from the Merton proportion although the strategy in the second asset does not affect the regime shift intensities. Again the large investor can compensate parts of the deviation from the Merton strategy in the first asset with an adequate deviation from the Merton strategy in the second asset. In contrast to the step intensity case the size of the deviation is influenced by the existence of a second, correlated asset to a greater extend than in the step intensity case. Whereas the critical strategy $\pi_1^{0,\text{crit}}$ remains unaffected as in the last section the candidate solution $\tilde{\pi}^{i,*}$ now changes directly when adding a second, correlated asset.

Again the impact of the correlation coefficient on the optimal strategy and the functions g and h is not monotonic since the Merton strategy depends in a non-monotonic way on ρ^i.

Figure 6.2 shows the optimal strategy for the first asset (upper plots) and the second asset (lower plots) in state 0 for a correlation coefficient given by $\rho = 0.25$ (left plots) and $\rho = 0.75$ (right plots). Notice that for $\rho = 0.75$ the relative deviations from the Merton strategy are higher (+20% in stock 1, -11% in stock 2) than in case of $\rho = 0.25$ (+8% in stock 1, -2% in stock 2). This suggests that the large investor makes use of a higher correlation in that he can increase the deviation from Merton proportion in the first stock yielding better regime shift intensities while compensating this larger deviation with a suitable position in the second asset.

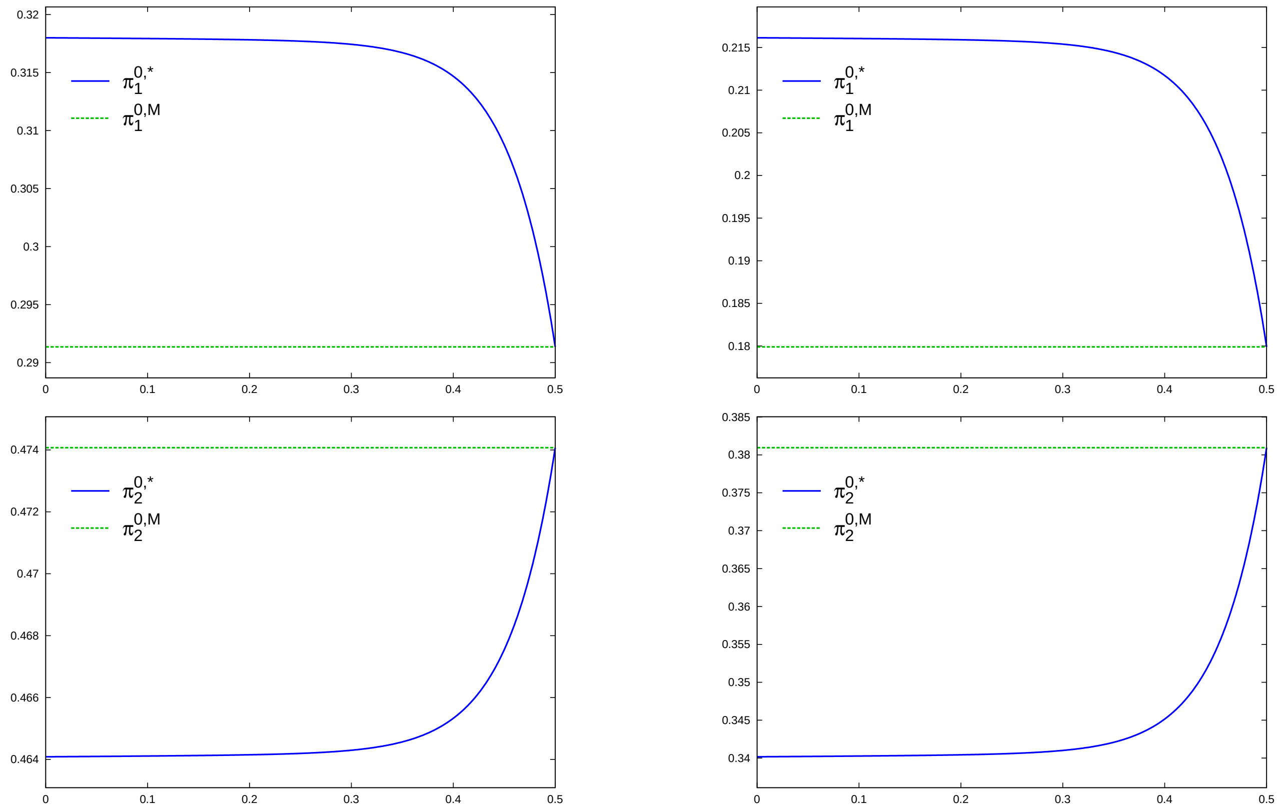

Figure 6.2: $\pi_j^{0,*}$ and $\pi_j^{0,M}$ as functions of t for $\rho = 0.25$ (left) and $\rho = 0.75$ (right) ($r^0 = r^1 = 0.03$, $\eta^0 = (0.1, 0.07)^\top$, $\eta^1 = (0.02, 0.05)^\top$, $\sigma_1^0 = 0.3$, $\sigma_2^0 = 0.2$, $\sigma_1^1 = 0.6$, $\sigma_2^1 = 0.4$, $\delta = 0.035$, $\varepsilon = 1$, $R = 3$, $T = 0.5$, $A^0 = 16.5$, $B_\pi^0 = (-5, 0)^\top$, $C^0 = 14$, $A^1 = 5.5$, $B_\pi^1 = (4, 0)^\top$, $C^1 = 0$)

7 Model Comparisons

In this section we discuss comparisons of the aforementioned variants of the investment problems with each other, with the classical Merton investment problem and the Bäuerle-Rieder investment problem.

When comparing two investment problems, resp. models A and B we denote by $x^i_{\mathrm{A}}(t)$, resp. $x^i_{\mathrm{B}}(t)$, the wealth that is required in state i at time t to generate the maximal expected utility from terminal wealth and intermediate consumption given by $v^i_{\mathrm{A}}(t, x^i_{\mathrm{A}})$, resp. $v^i_{\mathrm{B}}(t, x^i_{\mathrm{B}})$, in model A, resp. model B. In order to compare investment problem A with investment problem B we determine the wealth that would be necessary in model B to obtain the same maximal expected utility from terminal wealth and intermediate consumption as in model A equipped with $x^i_{\mathrm{A}}(t)$ at time t. Thus we are searching $x^i_{\mathrm{B}}(t)$ such that $v^i_{\mathrm{A}}(t, x^i_{\mathrm{A}}) = v^i_{\mathrm{B}}(t, x^i_{\mathrm{B}})$ given $x^i_{\mathrm{A}}(t)$. Therefore we consider

$$\begin{aligned}
& v^i_{\mathrm{A}}(t, x^i_{\mathrm{A}}(t)) = v^i_{\mathrm{B}}(t, x^i_{\mathrm{B}}(t)) \\
\Leftrightarrow\quad & \tfrac{1}{1-R} f_{\mathrm{A}}(t)\Big(\big(x^i_{\mathrm{A}}(t) e^{g_{\mathrm{A}}(t) - 1_{\{i=1\}} h_{\mathrm{A}}(t)}\big)^{1-R} - 1\Big) = \tfrac{1}{1-R} f_{\mathrm{B}}(t)\Big(\big(x^i_{\mathrm{B}}(t) e^{g_{\mathrm{B}}(t) - 1_{\{i=1\}} h_{\mathrm{B}}(t)}\big)^{1-R} - 1\Big) \\
\Leftrightarrow\quad & x^i_{\mathrm{A}}(t) e^{g_{\mathrm{A}}(t) - 1_{\{i=1\}} h_{\mathrm{A}}(t)} = x^i_{\mathrm{B}}(t) e^{g_{\mathrm{B}}(t) - 1_{\{i=1\}} h_{\mathrm{B}}(t)} \\
\Leftrightarrow\quad & \tfrac{x^i_{\mathrm{A}}(t) - x^i_{\mathrm{B}}(t)}{x^i_{\mathrm{A}}(t)} = 1 - e^{g_{\mathrm{A}}(t) - 1_{\{i=1\}} h_{\mathrm{A}}(t) - (g_{\mathrm{B}}(t) - 1_{\{i=1\}} h_{\mathrm{B}}(t))}
\end{aligned}$$

Notice that $f_{\mathrm{A}} = f_{\mathrm{B}}$ since the function $f \neq 0$ is independent of the choice of the intensity function and thus independent of the models that we are discussing here.

The **wealth ratio** defined by

$$w^i_{\mathrm{A,B}}(t) \triangleq 1 - e^{g_{\mathrm{A}}(t) - 1_{\{i=1\}} h_{\mathrm{A}}(t) - (g_{\mathrm{B}}(t) - 1_{\{i=1\}} h_{\mathrm{B}}(t))}$$

attains values in $(-\infty, 1]$ and satisfies $w^i_{\mathrm{A,B}}(T) = 0$. It measures the profitability, resp. unprofitability of model B relative to model A when starting at time $t \in [0, T]$ in state i. As t converges towards T the wealth ratio converges towards 0. This is because the difference between two models vanishes when the time to maturity converges towards 0.

If $w^i_{\mathrm{A,B}}(t) > 0$ then it would suffice to begin with an initial wealth of $x^i_{\mathrm{B}}(t) = (1 - w^i_{\mathrm{A,B}}(t)) x^i_{\mathrm{A}}(t) < x^i_{\mathrm{A}}(t)$ at time t in model B in order to achieve the same utility results as in model A, i.e. model B would be more profitable than model A. If otherwise $w^i_{\mathrm{A,B}}(t) < 0$ then the converse is true; the investor needed an initial wealth of $x^i_{\mathrm{B}}(t) = (1 - w^i_{\mathrm{A,B}}(t)) x^i_{\mathrm{A}}(t) > x^i_{\mathrm{A}}(t)$ at time t in model B in order to achieve the same utility results as in model A, i.e. model B would be less profitable than model A.

Hence, if the investor could choose between investment problem A and investment problem B then the wealth ratio $w^i_{\mathrm{A,B}}(t)$ measures the price as percentage of the initial wealth in investment problem A the investor would have to pay, resp. would receive, for choosing investment problem B. Those comparisons seem to be quite theoretical and without any practical use as usually the investor is not in the position to choose the market model in which he acts. However we will see later on that such comparisons are well suited for analyzing interesting properties of the particular model such as the parameter sensitivity.

At first glance, the concept of the wealth ratio can only be used to compare the investment problems that we considered in Section 4 (Subsection 7.1), e.g. portfolio-dependent models and consumption-dependent models. Further we can utilize the concept of the wealth ratio to analyze the impact of parameter changes within a particular model (Subsection 7.2). But moreover the wealth ratio is even suitable to determine the loss a large investor would have to face if he neglected his influence on the market (Subsection 7.3).

7.1 Comparison of the Different Investment Problems

At first we compare the investment problem for a large investor with the Merton investment problem (Subsection 7.1.1). Thereafter in Subsection 7.1.2 comparisons of similar models with different types of intensity functions are made, e.g. we compare the portfolio-dependent, step intensity model with the portfolio-dependent, affine intensity model. Finally we take a look at the difference between models having the same type of intensity function (Subsection 7.1.3), e.g. the portfolio-dependent, affine intensity model and the consumption-dependent, affine intensity model. The comparisons with the Bäuerle-Rieder investment problem are discussed in Subsection 7.3.

7.1.1 Comparison with the Merton Investment Problem

When comparing the investment problem for a large investor with the Merton investment problem one first has to determine which particular Merton investment problem to choose. It is reasonable to use the same model parameters as in the large investor problem that should be compared with except for the market parameters r, η and σ. Every choice of those three parameters implies a particular utility growth potential Ψ_{M}. Given the large investor problem with market parameters r^i, η^i and σ^i and corresponding utility growth potentials Ψ^i, where $\Psi^0 > \Psi^1$, we can differentiate three possible cases; $\Psi_{\mathrm{M}} \leq \Psi^1$, $\Psi^1 < \Psi_{\mathrm{M}} < \Psi^0$ and $\Psi_{\mathrm{M}} \geq \Psi^0$.

In case of $\Psi_{\mathrm{M}} \leq \Psi^1$ it is clear that the large investor should prefer the large investor problem to the Merton investment problem. If $\Psi_{\mathrm{M}} < \Psi^1$ then the market of the Merton investment problem is less profitable than both market states of the large investor problem. However if $\Psi_{\mathrm{M}} = \Psi^1$ then the Merton market attains the same profitability as the market in the adverse state 1 – in terms of the utility growth potential. But the large investor model involves the possibility that the market setting could improve. Thus it is to be preferred to the Merton model (cf. Figure 7.1). On the contrary, given $\Psi_{\mathrm{M}} \geq \Psi^0$, the large investor would choose the Merton problem instead of the large investor problem for the analog reason (cf. Figure 7.4).

However, when $\Psi^1 < \Psi_{\mathrm{M}} < \Psi^0$ then it is *a priori* not clear whichever model is preferable. Examples for this case are given in Figures 7.2 and 7.3. In the example of Figure 7.2 Ψ_{M} is closer to Ψ^1 than to Ψ^0, whereas in Figure 7.3 the opposite holds true. Notice that although $\Psi_{\mathrm{M}} < \Psi^0$ in Figure 7.3 the Merton investment problem turns out to be favorable when compared with the large investor investment problem. This could be explained by the fact that the market setting in the Merton model does not change, i.e. there is no risk concerning the market parameters, whereas in the large investor model the market situation could worsen – a risk the investor has to face. Apparently in the example of Figure 7.3 the large investor would rather accept a lesser profitable market – in terms of the utility growth potential – than choosing a market that could be better but although worse than the Merton market, i.e. the related risk is too high.

But even if the parameters r^i, η^i and σ^i, resp. r_{M}, η_{M} and σ_{M}, and thus the utility growth potentials Ψ^i, resp. Ψ_{M} did not vary one could set up two different regime shift intensities so that the Merton investment problem is not preferable to the large investor problem in the first setting (cf. Figure 7.5), whereas the converse is true in the second setting (cf. Figure 7.6).

Notice moreover that the wealth ratio shown in the left picture of Figure 7.5 is neither non-negative nor non-positive, i.e. it changes the sign while t is growing. Thus the large investor's rating of the two investment problems may change during time.

One main reason for the uncertainty of the large investor's preferences solely based on the utility growth potentials and for the time-dependency of the model rating is the fact that the profitability of the market in the large investor problem is described not only by the two utility growth potentials Ψ^0 and Ψ^1 but also by the regime switching intensities that link the two market states. Since those intensities are moreover dependent of the optimal strategy the comparison with a chosen Merton investment problem becomes quite complex.

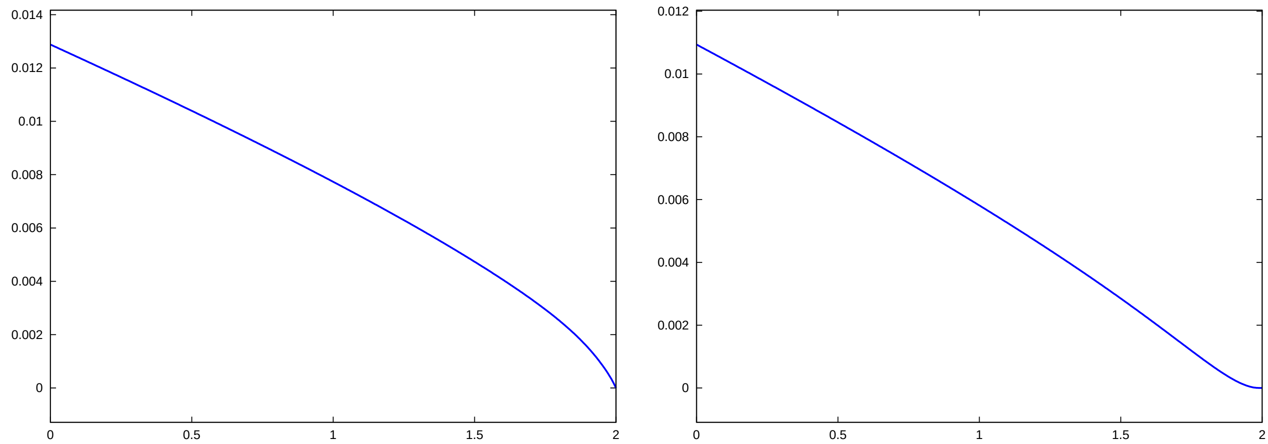

Figure 7.1: Wealth ratios $w^0_{\mathrm{M,PD_{steps}}}$ (left), $w^1_{\mathrm{M,PD_{steps}}}$ (right) as function of t in case of $\Psi_{\mathrm{M}} \leq \Psi^1$
($r^0 = r^1 = r_{\mathrm{M}} = 0.03$, $\eta^0 = 0.1$, $\eta^1 = 0.02$, $\eta_{\mathrm{M}} = 0.02$, $\sigma^0 = 0.3$, $\sigma^1 = 0.6$, $\sigma_{\mathrm{M}} = 0.6$,
i.e. $\Psi^0 = 0.05\bar{2}$, $\Psi^1 = 0.030\bar{2}$, $\Psi_{\mathrm{M}} = 0.030\bar{2}$, and $\delta = 0.035$, $\varepsilon = 1$, $R = 2.5$, $T = 2$,
$A^0 = 17$, $B^0_\pi = -5$, $C^0 = 14$, $C^0_1 = 5$, $C^0_2 = 10$, $A^1 = 5.5$, $B^1_\pi = -7$, $C^1 = 5$, $C^1_1 = 5$, $C^1_2 = 1.25$)

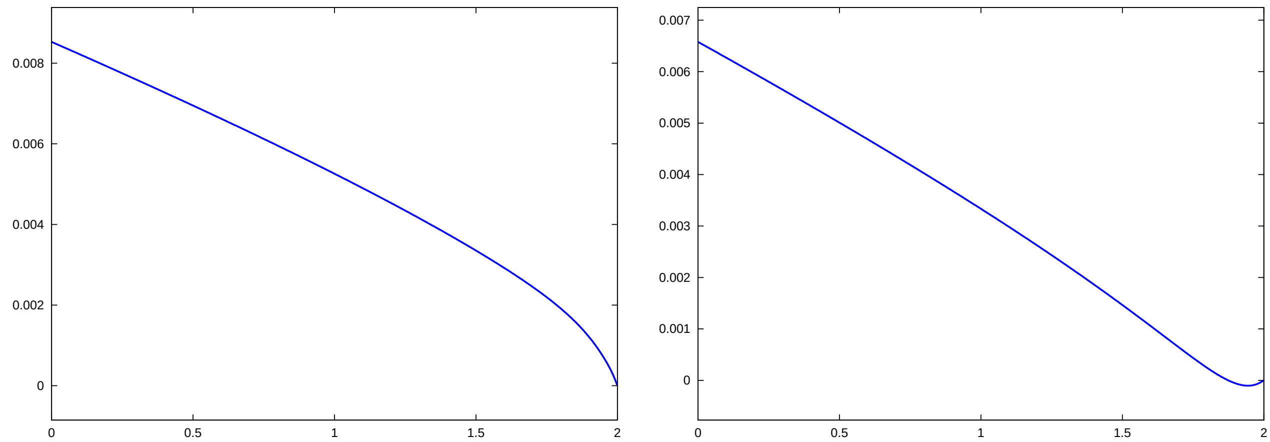

Figure 7.2: Wealth ratios $w^0_{\mathrm{M,PD_{steps}}}$ (left), $w^1_{\mathrm{M,PD_{steps}}}$ (right) as function of t in case of $\Psi^1 < \Psi_{\mathrm{M}} < \Psi^0$
($r^0 = r^1 = r_{\mathrm{M}} = 0.03,\ \eta^0 = 0.1,\ \eta^1 = 0.02,\ \eta_{\mathrm{M}} = 0.06,\ \sigma^0 = 0.3,\ \sigma^1 = 0.6,\ \sigma_{\mathrm{M}} = 0.45,$
i.e. $\Psi^0 = 0.05\bar{2},\ \Psi^1 = 0.030\bar{2},\ \Psi_{\mathrm{M}} = 0.033\bar{5}$ and $\delta = 0.035,\ \varepsilon = 1,\ R = 2.5,\ T = 2,$
$A^0 = 17,\ B^0_\pi = -5,\ C^0 = 14,\ C^0_1 = 5,\ C^0_2 = 10,\ A^1 = 5.5,\ B^1_\pi = -7,\ C^1 = 5,\ C^1_1 = 5,\ C^1_2 = 1.25$)

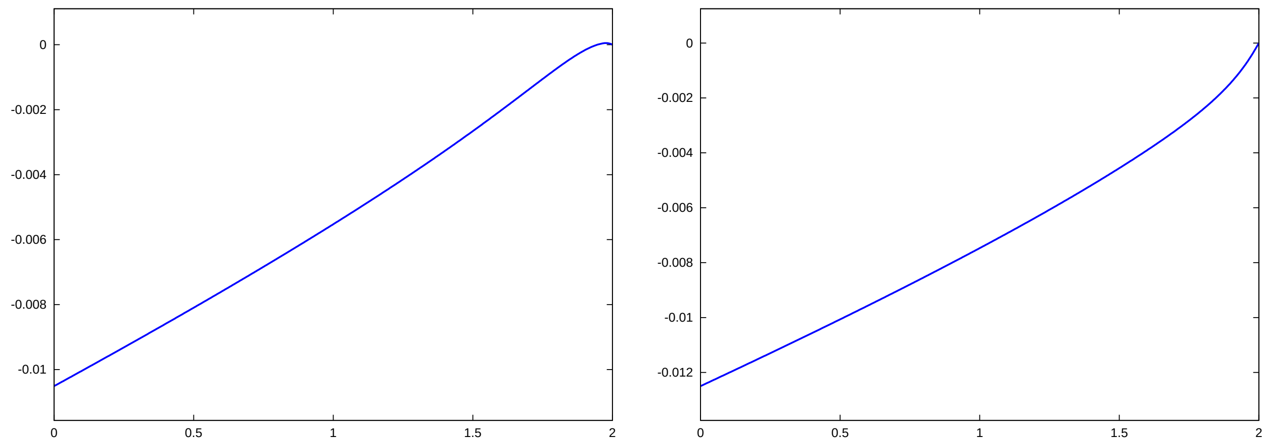

Figure 7.3: Wealth ratios $w^0_{\mathrm{M,PD_{steps}}}$ (left), $w^1_{\mathrm{M,PD_{steps}}}$ (right) as function of t in case of $\Psi^1 < \Psi_{\mathrm{M}} < \Psi^0$
($r^0 = r^1 = r_{\mathrm{M}} = 0.03,\ \eta^0 = 0.1,\ \eta^1 = 0.02,\ \eta_{\mathrm{M}} = 0.09,\ \sigma^0 = 0.3,\ \sigma^1 = 0.6,\ \sigma_{\mathrm{M}} = 0.3,$
i.e. $\Psi^0 = 0.05\bar{2},\ \Psi^1 = 0.030\bar{2},\ \Psi_{\mathrm{M}} = 0.048$ and $\delta = 0.035,\ \varepsilon = 1,\ R = 2.5,\ T = 2,$
$A^0 = 17,\ B^0_\pi = -5,\ C^0 = 14,\ C^0_1 = 5,\ C^0_2 = 10,\ A^1 = 5.5,\ B^1_\pi = -7,\ C^1 = 5,\ C^1_1 = 5,\ C^1_2 = 1.25$)

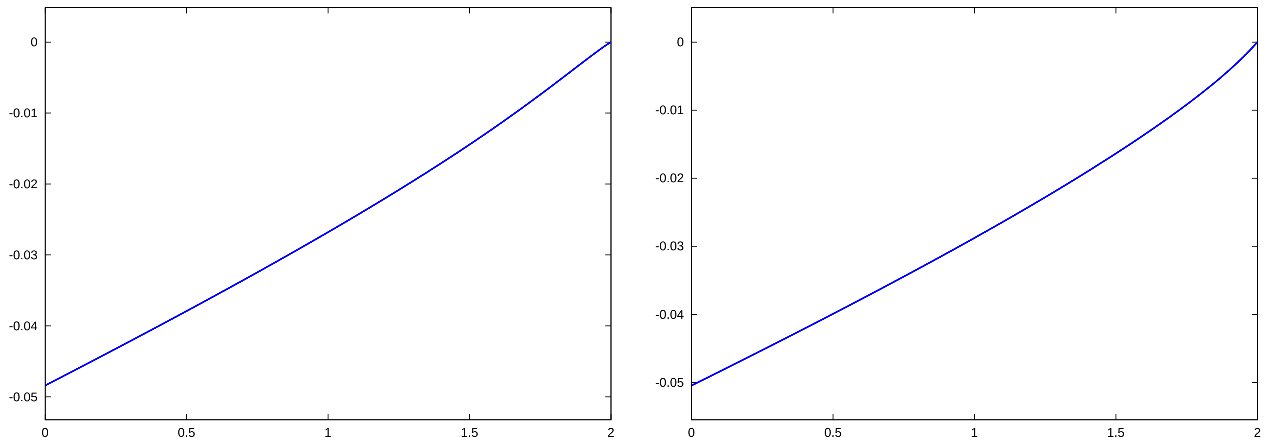

Figure 7.4: Wealth ratios $w^0_{\mathrm{M,PD_{steps}}}$ (left), $w^1_{\mathrm{M,PD_{steps}}}$ (right) as function of t in case of $\Psi_{\mathrm{M}} \geq \Psi^0$
($r^0 = r^1 = r_{\mathrm{M}} = 0.03$, $\eta^0 = 0.1$, $\eta^1 = 0.02$, $\eta_{\mathrm{M}} = 0.12$, $\sigma^0 = 0.3$, $\sigma^1 = 0.6$, $\sigma_{\mathrm{M}} = 0.25$,
i.e. $\Psi^0 = 0.05\bar{2}$, $\Psi^1 = 0.030\bar{2}$, $\Psi_{\mathrm{M}} = 0.07608$ and $\delta = 0.035$, $\varepsilon = 1$, $R = 2.5$, $T = 2$,
$A^0 = 17$, $B^0_\pi = -5$, $C^0 = 14$, $C^0_1 = 5$, $C^0_2 = 10$, $A^1 = 5.5$, $B^1_\pi = -7$, $C^1 = 5$, $C^1_1 = 5$, $C^1_2 = 1.25$)

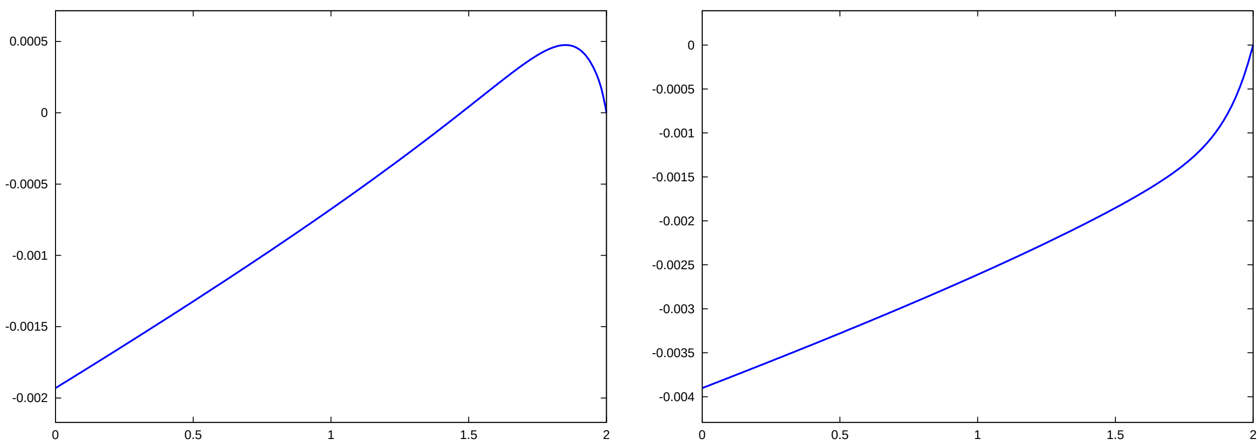

Figure 7.5: Wealth ratios $w^0_{\mathrm{M,PD_{steps}}}$ (left), $w^1_{\mathrm{M,PD_{steps}}}$ (right) as function of t in case of $\Psi^1 < \Psi_{\mathrm{M}} < \Psi^0$
($r^0 = r^1 = r_{\mathrm{M}} = 0.03$, $\eta^0 = 0.1$, $\eta^1 = 0.02$, $\eta_{\mathrm{M}} = 0.06$, $\sigma^0 = 0.3$, $\sigma^1 = 0.6$, $\sigma_{\mathrm{M}} = 0.25$,
i.e. $\Psi^0 = 0.05\bar{2}$, $\Psi^1 = 0.030\bar{2}$, $\Psi_{\mathrm{M}} = 0.04152$ and $\delta = 0.035$, $\varepsilon = 1$, $R = 2.5$, $T = 2$,
$A^0 = 17$, $B^0_\pi = -5$, $C^0 = 14$, $C^0_1 = 5$, $C^0_2 = 10$, $A^1 = 5.5$, $B^1_\pi = -7$, $C^1 = 5$, $C^1_1 = 5$, $C^1_2 = 1.25$)

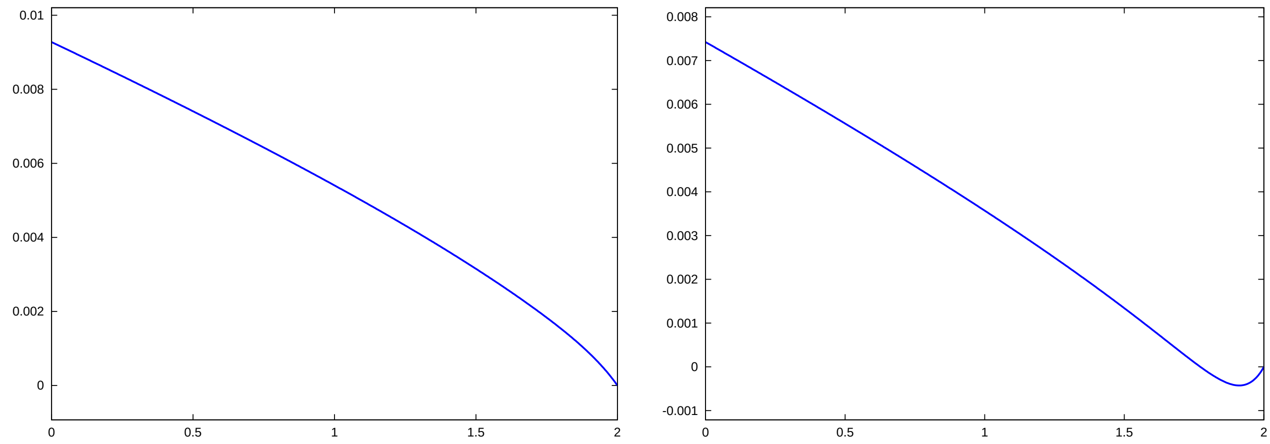

Figure 7.6: Wealth ratios $w^0_{\mathrm{M,PD_{steps}}}$ (left), $w^1_{\mathrm{M,PD_{steps}}}$ (right) as function of t in case of $\Psi^1 < \Psi_\mathrm{M} < \Psi^0$ ($r^0 = r^1 = r_\mathrm{M} = 0.03$, $\eta^0 = 0.1$, $\eta^1 = 0.02$, $\eta_\mathrm{M} = 0.06$, $\sigma^0 = 0.3$, $\sigma^1 = 0.6$, $\sigma_\mathrm{M} = 0.25$, i.e. $\Psi^0 = 0.05\bar{2}$, $\Psi^1 = 0.030\bar{2}$, $\Psi_\mathrm{M} = 0.04152$ and $\delta = 0.035$, $\varepsilon = 1$, $R = 2.5$, $T = 2$, $A^0 = 17$, $B^0_\pi = -5$, $C^0 = 14$, $C^0_1 = 1$, $C^0_2 = 2$, $A^1 = 5.5$, $B^1_\pi = -7$, $C^1 = 5$, $C^1_1 = 10$, $C^1_2 = 9$)

7.1.2 Comparison of Similar Investment Problems

In the following we compare the models with step intensity functions with the ones with affine intensity functions for the same kind of the investor's influence, i.e. we consider the wealth ratios $w^i_{\mathrm{PD_{affine},PD_{steps}}}$, $w^i_{\mathrm{CD_{affine},CD_{steps}}}$ and $w^i_{\mathrm{PCD_{affine},PCD_{steps}}}$. In order to make the models comparable we have to adjust the particular intensity functions in a suitable way (cf. Figure 7.7). Let therefore

$$\begin{aligned}
\vartheta^{i,1-i}_{\mathrm{steps}}(\pi,c) &= C^i_{1,\mathrm{steps}} 1_{\{A^i_{\mathrm{steps}}+\pi^\top.B^i_{\pi,\mathrm{steps}}+B^i_{c,\mathrm{steps}}c\le C^i_{\mathrm{steps}}\}} + C^i_{2,\mathrm{steps}} 1_{\{A^i_{\mathrm{steps}}+\pi^\top.B^i_{\pi,\mathrm{steps}}+B^i_{c,\mathrm{steps}}c> C^i_{\mathrm{steps}}\}},\\
\vartheta^{i,1-i}_{\mathrm{affine}}(\pi,c) &= \max\{A^i_{\mathrm{affine}} + \pi^\top.B^i_{\pi,\mathrm{affine}} + B^i_{c,\mathrm{affine}}c, C^i_{\mathrm{affine}}\}.
\end{aligned}$$

Setting

$$\begin{array}{lllll}
A^0_{\mathrm{steps}} = A^0 & B^0_{\pi,\mathrm{steps}} = B^0_\pi & B^0_{c,\mathrm{steps}} = B^0_c & C^0_{\mathrm{steps}} = C^0 & C^0_{1,\mathrm{steps}} = C^0\\
A^0_{\mathrm{affine}} = A^0 & B^0_{\pi,\mathrm{affine}} = B^0_\pi & B^0_{c,\mathrm{affine}} = B^0_c & C^0_{\mathrm{affine}} = C^0 & C^1_{1,\mathrm{steps}} = C^1_1\\
A^1_{\mathrm{steps}} = -A^1 & B^1_{\pi,\mathrm{steps}} = -B^1_\pi & B^1_{c,\mathrm{steps}} = -B^1_c & C^1_{\mathrm{steps}} = -C^1 & C^0_{2,\mathrm{steps}} = C^0_2\\
A^1_{\mathrm{affine}} = A^1 & B^1_{\pi,\mathrm{affine}} = B^1_\pi & B^1_{c,\mathrm{affine}} = B^1_c & C^1_{\mathrm{affine}} = C^1 & C^1_{2,\mathrm{steps}} = C^1
\end{array}$$

for given parameters A^i, B^i_π, B^i_c, C^i, $C^0_2 < C^0$ and $C^1_1 > C^1$ the comparable intensity functions read

$$\begin{aligned}
\vartheta^{0,1}_{\mathrm{steps}}(\pi,c) &= C^0 1_{\{A^0+\pi^\top.B^0_\pi+B^0_c c\le C^0\}} + C^0_2 1_{\{A^0+\pi^\top.B^0_\pi+B^0_c c> C^0\}},\\
\vartheta^{0,1}_{\mathrm{affine}}(\pi,c) &= \max\{A^0 + \pi^\top.B^0_\pi + B^0_c c, C^0\},\\
\vartheta^{1,0}_{\mathrm{steps}}(\pi,c) &= C^1_1 1_{\{-A^1-\pi^\top.B^1_\pi-B^1_c c\le -C^1\}} + C^1 1_{\{-A^1-\pi^\top.B^1_\pi-B^1_c c> -C^1\}},\\
\vartheta^{1,0}_{\mathrm{affine}}(\pi,c) &= \max\{A^1 + \pi^\top.B^1_\pi + B^1_c c, C^1\}.
\end{aligned}$$

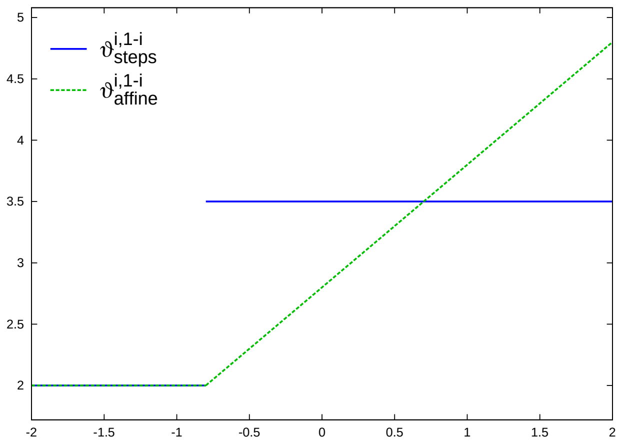

Figure 7.7: Comparable intensity functions $\vartheta^{i,1-i}_{\text{steps}}$ and $\vartheta^{i,1-i}_{\text{affine}}$

Figure 7.7 illustrates that it is *a priori* not clear which kind of intensity function is favorable for the large investor. This depends highly on the position of the Merton strategy relative to two neuralgic 'points' – the one at which $\vartheta^{i,1-i}_{\text{affine}}$ exhibits its kink and the one at which $\vartheta^{i,1-i}_{\text{steps}}$ and the affine branch of $\vartheta^{i,1-i}_{\text{affine}}$ coincide. The time- and intensity-dependency of the Merton consumption rate even worsen this problem concerning the comparability.

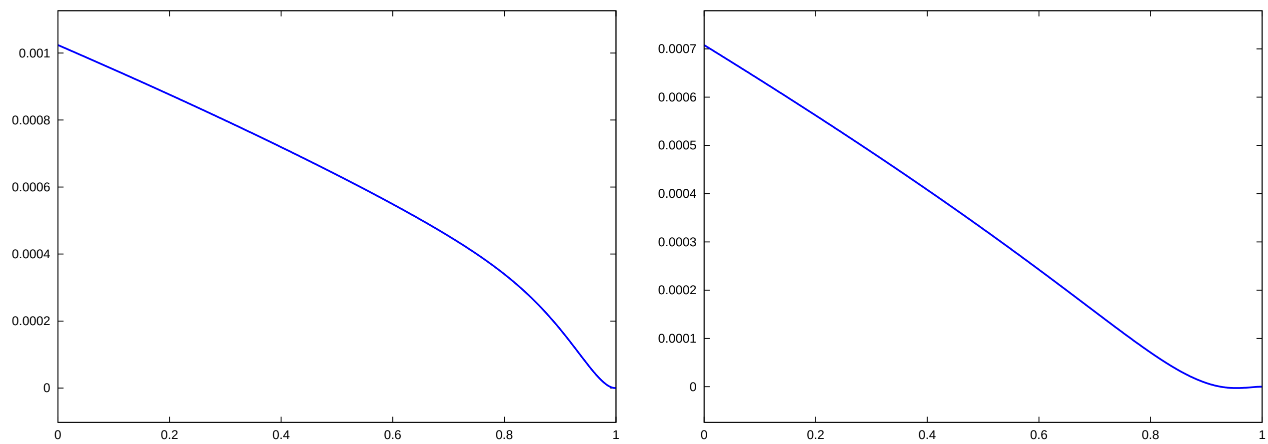

Figure 7.8: Wealth ratio $w^i_{\text{PD}_{\text{affine}},\text{PD}_{\text{steps}}}$ in state 0 (left) and state 1 (right) as function of t
($r^0 = r^1 = 0.03$, $\eta^0 = 0.1$, $\eta^1 = 0.02$, $\sigma^0 = 0.3$, $\sigma^1 = 0.6$, $\delta = 0.035$, $\varepsilon = 1$, $R = 3$, $T = 1$,
$A^0 = 17$, $B^0_\pi = -5$, $C^0 = 5$, $C^0_2 = 10$, $A^1 = 5.5$, $B^1_\pi = 4$, $C^1 = 1.25$, $C^1_1 = 5$)

The Figures 7.8, 7.9 and 7.10 show the wealth ratios of interest in different settings with comparable intensities. Neither the step intensity functions nor the affine intensity functions are generally favorable for the large investor. In the example of Figure 7.8 the step intensity functions are preferable whereas in the example of Figure 7.9 the converse is true. Notice further that $w^0_{\text{PCD}_{\text{affine}},\text{PCD}_{\text{steps}}}$ as shown in left picture of Figure 7.10 is positive until about $t \approx 0.85$. Thereafter it is negative. This reflects the aforementioned problem concerning the comparability of the two models with step, resp. affine intensities.

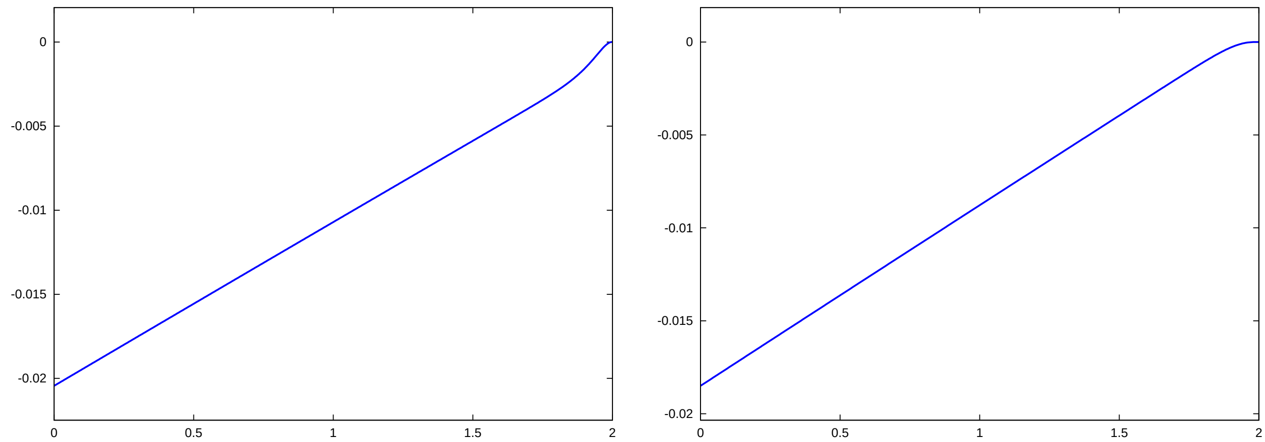

Figure 7.9: Wealth ratio $w^i_{\text{CD}_{\text{affine}},\text{CD}_{\text{steps}}}$ in state 0 (left) and state 1 (right) as function of t
($r^0 = r^1 = 0.03$, $\eta^0 = 0.1$, $\eta^1 = 0.01$, $\sigma^0 = 0.2$, $\sigma^1 = 0.7$, $\delta = 0.035$, $\varepsilon = 0$, $R = 0.5$, $T = 2$,
$A^0 = 16$, $B^0_c = 20$, $C^0 = 10$, $C^0_2 = 20$, $A^1 = 5$, $B^1_c = -10$, $C^1 = 2.5$, $C^1_1 = 5$)

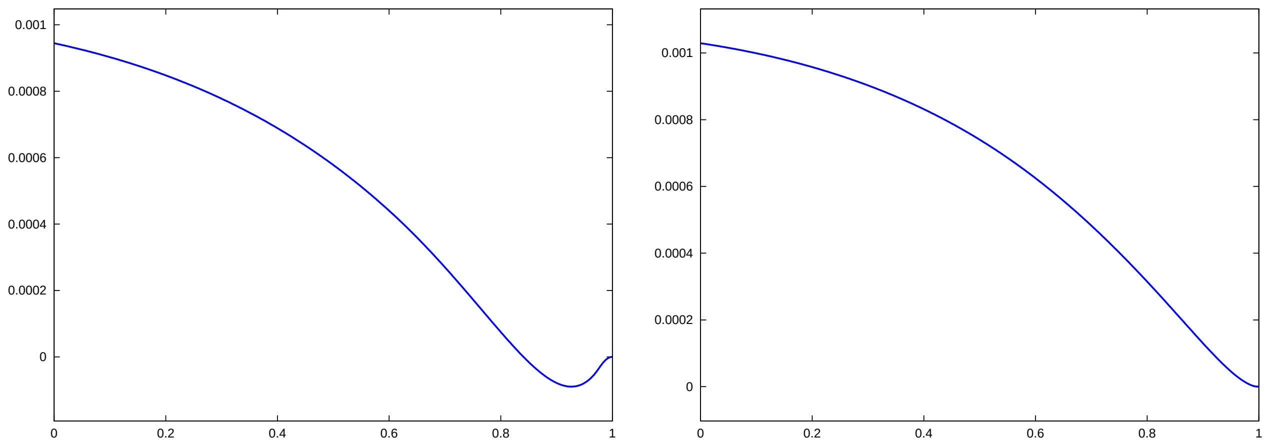

Figure 7.10: Wealth ratio $w^i_{\text{PCD}_{\text{affine}},\text{PCD}_{\text{steps}}}$ in state 0 (left) and state 1 (right) as function of t
($r^0 = r^1 = 0.03$, $\eta^0 = 0.1$, $\eta^1 = 0.06$, $\sigma^0 = 0.4$, $\sigma^1 = 0.7$, $\delta = 0.035$, $\varepsilon = 1$, $R = 1.5$, $T = 1$,
$A^0 = 10$, $B^0_\pi = -5$, $B^0_c = 3$, $C^0 = 10$, $C^0_2 = 20$, $A^1 = 6$, $B^1_\pi = 7$, $B^1_c = -5$, $C^1 = 1.25$, $C^1_1 = 5$)

7.1.3 Comparison of Investment Problems with Identical Intensity Type

In this subsection we dwell on comparisons of investment problem having the same type of intensity function but different dependencies on the large investor, i.e. we compare the solely portfolio-dependent, the solely consumption-dependent and the combined portfolio- and consumption-dependent models with each other. The dependency on the investor's strategy is characterized by the parameters A^i, B^i_π, B^i_c, C^i. For a given quadruple $(A^i, B^i_\pi, B^i_c, C^i)$ we compare the portfolio-dependent variant with $(A^i, B^i_\pi, 0, C^i)$, the consumption-dependent variant with $(A^i, 0, B^i_c, C^i)$ and the portfolio- and consumption-dependent variant with $(A^i, B^i_\pi, B^i_c, C^i)$ of the investment problem with step, resp. affine intensity functions.

In the example of Figure 7.11 the addition of the consumption-dependency to the portfolio-dependency is not profitable for the large investor. However this effect shrinks when the time to maturity decreases. Figure 7.12 shows an example in which the inclusion of the portfolio-dependency to the consumption-dependency would be favorable for the investor. Note here the non-monotonic behavior of the wealth ratios. The addition of the portfolio-dependency is most profitable near $t \approx 1$. Finally an example in which the consumption-dependent model is worse than the portfolio-dependent model is given in Figure 7.13. Again as in Figure 7.11 for smaller times t the consumption-dependent model is worse than for larger t.

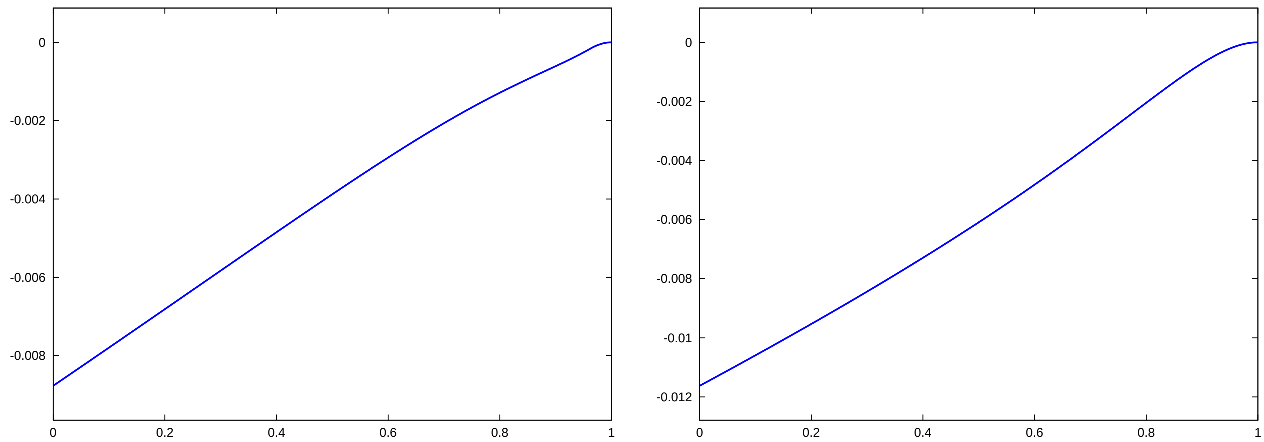

Figure 7.11: Wealth ratio $w^i_{\mathrm{PD_{steps}},\mathrm{PCD_{steps}}}$ in state 0 (left) and state 1 (right) as function of t
($r^0 = r^1 = 0.03$, $\eta^0 = 0.1$, $\eta^1 = 0.02$, $\sigma^0 = 0.3$, $\sigma^1 = 0.6$, $\delta = 0.035$, $\varepsilon = 1$, $R = 1$, $T = 1$,
$A^0 = 17$, $B^0_\pi = -5$, $B^0_c = 5$, $C^0 = 14$, $C^0_1 = 5$, $C^0_2 = 10$, $A^1 = 5.5$, $B^1_\pi = -7$, $B^1_c = 7.5$, $C^1 = 5$, $C^1_1 = 5$, $C^1_2 = 1.25$)

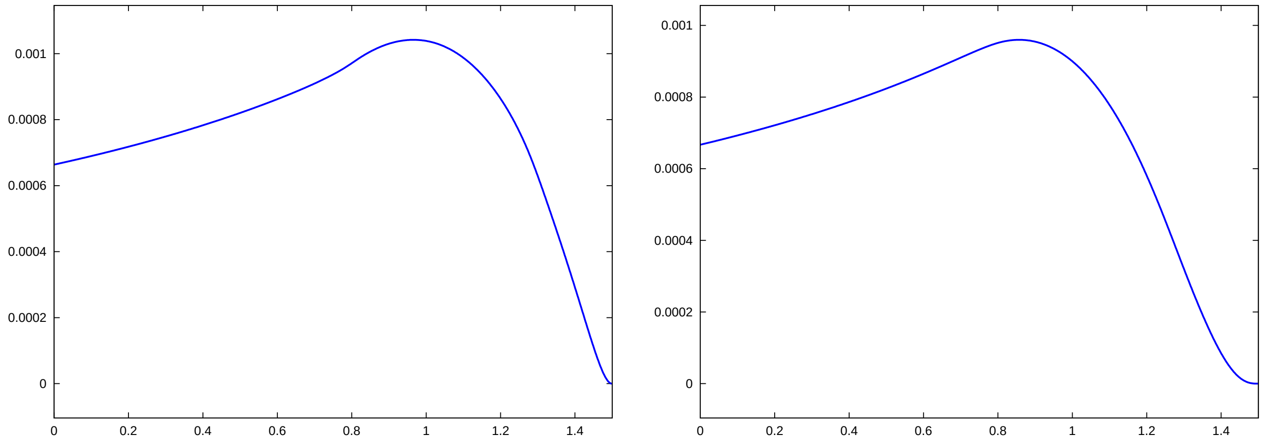

Figure 7.12: Wealth ratio $w^i_{\mathrm{CD_{affine}},\mathrm{PCD_{affine}}}$ in state 0 (left) and state 1 (right) as function of t
($r^0 = r^1 = 0.035$, $\eta^0 = 0.09$, $\eta^1 = 0.01$, $\sigma^0 = 0.2$, $\sigma^1 = 0.5$, $\delta = 0.04$, $\varepsilon = 1$, $R = 1.5$, $T = 1.5$,
$A^0 = 4$, $B^0_\pi = -1.5$, $B^0_c = 10$, $C^0 = 10$, $A^1 = 13.5$, $B^1_\pi = 0.75$, $B^1_c = -5$, $C^1 = 8$)

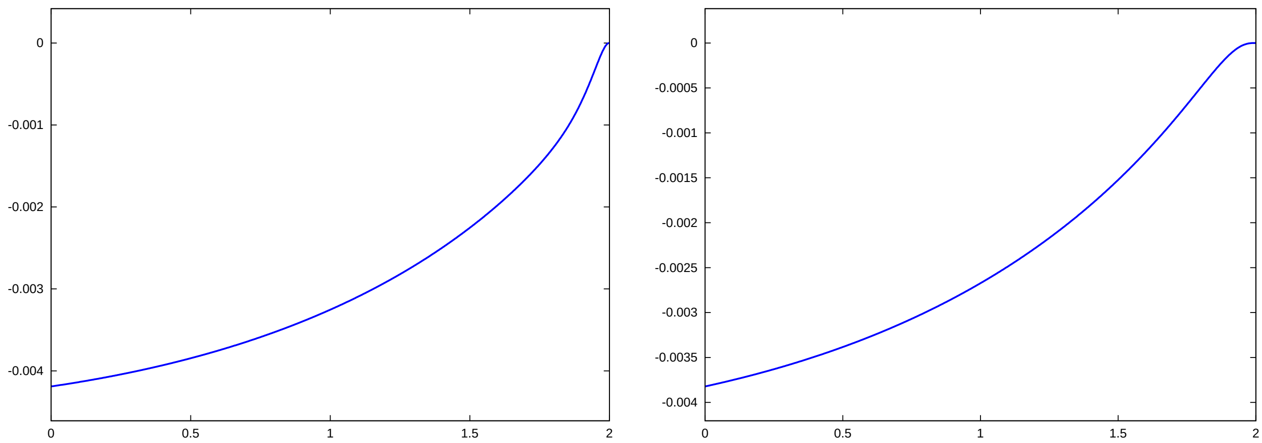

Figure 7.13: Wealth ratio $w^i_{\mathrm{PD_{affine}},\mathrm{CD_{affine}}}$ in state 0 (left) and state 1 (right) as function of t
($r^0 = r^1 = 0.03$, $\eta^0 = 0.09$, $\eta^1 = 0.01$, $\sigma^0 = 0.3$, $\sigma^1 = 0.5$, $\delta = 0.035$, $\varepsilon = 1$, $R = 1.5$, $T = 2$,
$A^0 = 6.8$, $B^0_\pi = 5$, $B^0_c = 20$, $C^0 = 10$, $A^1 = 5.47$, $B^1_\pi = -3$, $B^1_c = -5$, $C^1 = 5.4$)

7.2 Parameter Sensitivity

We now investigate how changes of the parameters of the intensity function $\vartheta^{i,1-i}$ impact on the optimal strategy and the value function. At first glance one would think of *ceteris paribus* analyses, i.e. varying one parameter while keeping the others fix. However depending on the particular version of the intensity functions it turns out to be quite suitable to vary several parameters at once. The reason for this is that solely changing the parameter A^i, B^i_π, B^i_c or C^i may impact on the position of the Merton strategy relative to the separating hyperplane between $\mathcal{F}^i$ and $\mathbb{R}^{\bar{n}} \times \mathbb{R}^+_0 \setminus \mathcal{F}^i$ in the model with step intensity functions, resp. $\mathcal{I}^i$ and $\mathbb{R}^{\bar{n}} \times \mathbb{R}^+_0 \setminus \mathcal{I}^i$ in the model with affine intensity functions, measured by $d^{\pi,c,i}$. But regarding the Merton strategy as a reference strategy it may be preferable to analyze also those variations of the intensity functions that keep the position of the Merton strategy fix. We call those variations *Merton invariant intensity variations.*

As we have already seen in Remark 4.6 the Merton type consumption rate is time-dependent except for the case of $\varepsilon = 0$ and depends moreover on the regime shift intensities unless $R = 1$. Within our setting of intensity functions with constant parameters the aforementioned Merton invariant intensity variations thus only work either in the models with solely portfolio-dependent intensities denoted by (PD) or in the models with consumption-dependent intensities in case of $\varepsilon = 0$ denoted by $(\mathrm{CD}_{\varepsilon=0})$ and $(\mathrm{PCD}_{\varepsilon=0})$. In the following we discuss two typical Merton invariant intensity variations – the *intensity shift* and the *rotation around the Merton strategy.*

In order to change the constant A^i in a Merton invariant way we have to vary C^i by the same amount. We call such a simultaneous change of the pair (A^i, C^i) an *intensity shift.* Notice that this kind of variation is only suitable for the affine intensity functions since in case of the step intensity functions it would not cause any difference if we took the pair $(A^i + C, C^i + C)$ instead of the pair (A^i, C^i) where $C \in [-C^i, \infty)$ is a constant. To achieve the corresponding *intensity shift* in the setting with step intensity functions one has to vary C^i_1 and C^i_2 by the same amount.

Further when analyzing the sensitivity of the optimal strategy on the parameter B^i_π we have to change the parameter A^i in such a way that the position of the Merton strategy does not change. Denote by $(\hat{A}^i, \hat{B}^i_\pi)$ the pair of the reference parameters. When changing $\hat{B}^i_\pi$ into $\tilde{B}^i_\pi$ we would have to choose $\tilde{A}^i = \hat{A}^i + (\pi^{i,M})^\top.(\hat{B}^i_\pi - \tilde{B}^i_\pi)$ such that $\tilde{A}^i + (\pi^{i,M})^\top.\tilde{B}^i_\pi = \hat{A}^i + (\pi^{i,M})^\top.\hat{B}^i_\pi$. We call this variation of the pair (A^i, B^i_π) a *rotation around the Merton strategy.* Concerning changes of B^i_c in the models $(\mathrm{CD}_{\varepsilon=0})$ and $(\mathrm{PCD}_{\varepsilon=0})$ the parameter A^i does not need to be adjusted since $c^{i,M}(t) = 0$. Thus a rotation around the Merton strategy consists just of a variation of the parameter B^i_c.

7.2.1 Step Intensity Functions

We first consider some *ceteris paribus intensity variations* and then go on with the *Merton invariant intensity variations.*

Ceteris paribus intensity variations. Figure 7.14 shows the wealth ratios for the comparisons of the model (PD_1) with the models (PD_k), $k = 1, \dots, 5$, where $A^0_1 = 16.5$, $A^0_2 = 16.75$, $A^0_3 = 17$, $A^0_4 = 17.25$ and $A^0_5 = 17.5$. In this particular setting an increasing A^0 means that the Merton strategy moves deeper in the complementary half space $\mathbb{R}^{\bar{n}} \setminus \mathcal{F}^{\pi,0}$. Thus the bigger A^0 the more the large investor has to deviate from the Merton strategy in order to achieve favorable regime shift intensities. Therefore big values of A^0 are disadvantageous for the large investor.

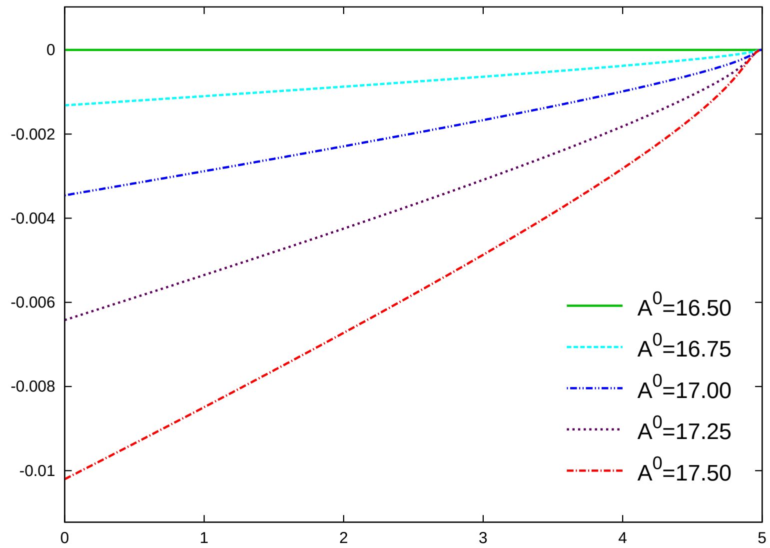

Figure 7.14: Wealth ratio $w^0_{\mathrm{PD}_1,\mathrm{PD}_k}$ for different values of A^0 as function of t
($r^0 = r^1 = 0.03$, $\eta^0 = 0.1$, $\eta^1 = 0.02$, $\sigma^0 = 0.3$, $\sigma^1 = 0.6$, $\delta = 0.035$, $\varepsilon = 1$, $R = 2.5$, $T = 5$, $A^0_1 = 16.5$, $B^0_\pi = -5$, $C^0 = 14$, $C^0_1 = 5$, $C^0_2 = 10$, $A^1 = 5.5$, $B^1_\pi = -7$, $C^1 = 5$, $C^1_1 = 5$, $C^1_2 = 1.25$)

Figure 7.15 shows the wealth ratios for the comparisons of the model (CD_1) with the models (CD_k), $k = 1, \ldots, 5$, where $C^1_{1,1} = 5$, $C^1_{1,2} = 10$, $C^1_{1,3} = 15$, $C^1_{1,4} = 20$ and $C^1_{1,5} = 25$. The augmentation of the intensity parameter C^1_1 comes along with an increase of the possible extent of the large investor's influence, i.e. the chances for a jump from the adverse market state to the favorable one grow. Hence the large investor benefits from a bigger C^1_1.

Merton invariant intensity variations. Figure 7.16 shows the consequences of an intensity shift in state 0. The wealth ratios are decreasing as the intensity level is increasing, i.e. the large investor benefits from small intensity levels which is the intuitive result.

The impact of a rotation around the Merton strategy is shown in Figure 7.17. It displays the wealth ratios for the comparisons of the model (PD_1) with the models (PD_k), $k = 1, \ldots, 5$, where $(A^0_1, B^0_{\pi,1}) = (17, -5.5)$, $(A^0_2, B^0_{\pi,2}) = (16.91, -5.25)$, $(A^0_3, B^0_{\pi,3}) = (16.82, -5)$, $(A^0_4, B^0_{\pi,4}) = (16.72, -4.75)$ and $(A^0_5, B^0_{\pi,5}) = (16.63, -4)$. Notice that here $B^0_{\pi,k}$ does not stand for the k-th component of B^0_π. In this context we denote by $B^0_{\pi,k}$ the value of B^0_π in the k-th setting.

Figure 7.18 shows the particular wealth ratios for the models with $(A^0_1, B^0_{\pi,1}) = (17, 12.5)$, $(A^0_2, B^0_{\pi,2}) = (15.44, 13.75)$, $(A^0_3, B^0_{\pi,3}) = (13.88, 15)$, $(A^0_4, B^0_{\pi,4}) = (12.31, 16.25)$ and $(A^0_5, B^0_{\pi,5}) = (10.75, 15)$.

The Figures 7.17 and 7.18 suggest that for every time t the wealth ratio $w^0_{\mathrm{PD}_1,\mathrm{PD}_k}(t)$ is decreasing as $|B^0_\pi|$ is decreasing as long as the sign of B^0_π does not change, i.e. the investor appears to benefit from a sensitive market. At first glance this seems counterintuitive – one would expect that the investor profits from a lesser sensitive market. However the large investor can use the sensitivity of the market

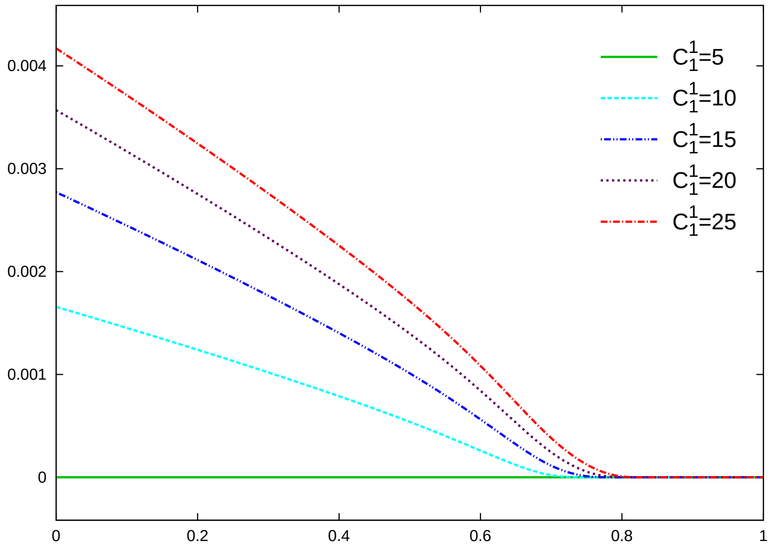

Figure 7.15: Wealth ratio $w^0_{\mathrm{CD}_1,\mathrm{CD}_k}$ for different values of C^1_1 as function of t
($r^0 = r^1 = 0.03$, $\eta^0 = 0.1$, $\eta^1 = 0.06$, $\sigma^0 = 0.4$, $\sigma^1 = 0.7$, $\delta = 0.035$, $\varepsilon = 1$, $R = 1.5$, $T = 1$,
$A^0 = 17$, $B^0_c = 3$, $C^0 = 19.5$, $C^0_1 = 10$, $C^0_2 = 20$, $A^1 = 5.5$, $B^1_c = 5$, $C^1 = 9$, $C^1_{1,1} = 5$, $C^1_2 = 1.25$)

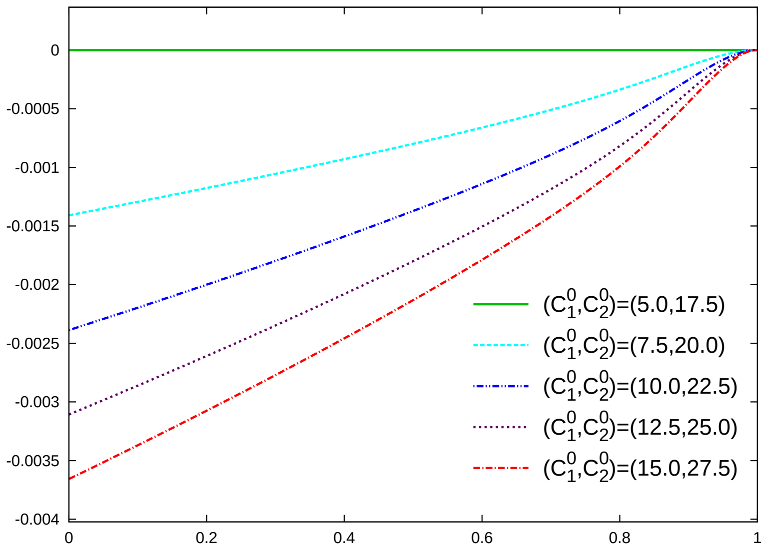

Figure 7.16: Wealth ratio $w^0_{\mathrm{PCD}_1,\mathrm{PCD}_k}$ for different values of C^0_1 and C^0_2 as function of t
($r^0 = r^1 = 0.03$, $\eta^0 = 0.1$, $\eta^1 = 0.06$, $\sigma^0 = 0.4$, $\sigma^1 = 0.7$, $\delta = 0.035$, $\varepsilon = 1$, $R = 1.5$, $T = 1$,
$A^0 = 17$, $B^0_\pi = -5$, $B^0_c = 3$, $C^0 = 19.5$, $C^0_{1,1} = 5$, $C^0_{2,1} = 17.5$, $A^1 = 5.5$, $B^1_\pi = -7$, $B^1_c = 5$, $C^1 = 9$, $C^1_1 = 5$, $C^1_2 = 2.5$)

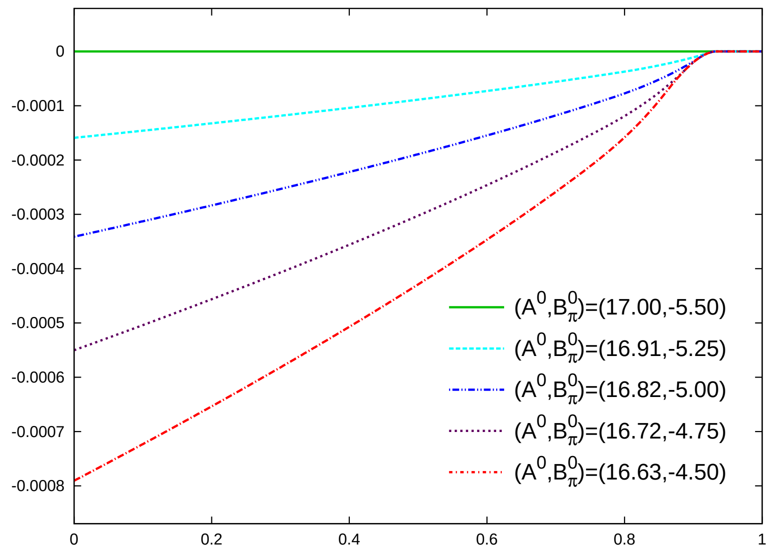

Figure 7.17: Wealth ratio $w^0_{\mathrm{PD}_1,\mathrm{PD}_k}$ for different values of (A^0, B^0_π) as function of t
$(r^0 = r^1 = 0.03,\ \eta^0 = 0.1,\ \eta^1 = 0.02,\ \sigma^0 = 0.3,\ \sigma^1 = 0.6,\ \delta = 0.035,\ \varepsilon = 1,\ R = 3,\ T = 1,$
$A^0_1 = 17,\ B^0_{\pi,1} = -5.5,\ C^0 = 14,\ C^0_1 = 5,\ C^0_2 = 10,\ A^1 = 5.5,\ B^1_\pi = -4.1,\ C^1 = 5,\ C^1_1 = 5,\ C^1_2 = 1.25)$

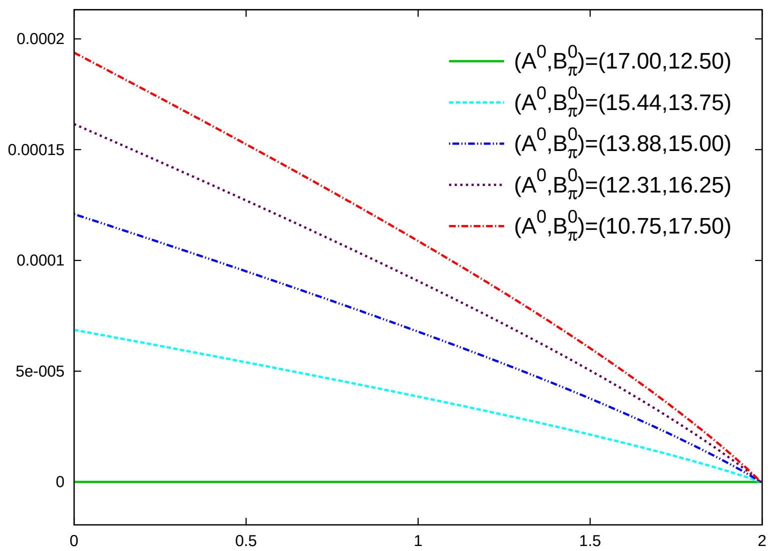

Figure 7.18: Wealth ratio $w^0_{\mathrm{PD}_1,\mathrm{PD}_k}$ for different values of (A^0, B^0_π) as function of t
$(r^0 = r^1 = 0.03,\ \eta^0 = 0.1,\ \eta^1 = 0.02,\ \sigma^0 = 0.4,\ \sigma^1 = 0.7,\ \delta = 0.035,\ \varepsilon = 1,\ R = 0.5,\ T = 2,$
$A^0_1 = 17,\ B^0_{\pi,1} = 12.5,\ C^0 = 31.5,\ C^0_1 = 2.5,\ C^0_2 = 12.5,\ A^1 = 5.5,\ B^1_\pi = 7,\ C^1 = 14,\ C^1_1 = 10,\ C^1_2 = 2.5)$

to influence the regime shift intensities in a favorable way. From this point of view a large sensitivity goes along with the fact that in order to achieve the same regime shift intensity the investor does not need to deviate as far from the Merton strategy as he had to if the sensitivity was smaller.

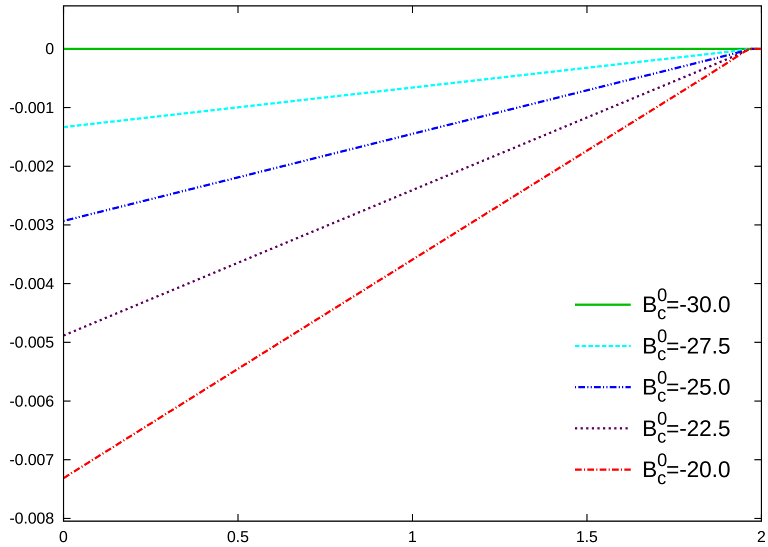

Figure 7.19: Wealth ratio $w^0_{\mathrm{CD}_1,\mathrm{CD}_k}$ for different values of B^0_c as function of t
($r^0 = r^1 = 0.03$, $\eta^0 = 0.1$, $\eta^1 = 0.02$, $\sigma^0 = 0.4$, $\sigma^1 = 0.7$, $\delta = 0.035$, $\varepsilon = 0$, $R = 1.5$, $T = 2$,
$A^0 = 11.75$, $B^0_{c,1} = -30$, $C^0 = 11.5$, $C^0_1 = 2.5$, $C^0_2 = 25$, $A^1 = 5.4$, $B^1_c = -15$, $C^1 = 5.3$, $C^1_1 = 20$, $C^1_2 = 2.5$)

Figure 7.19 suggests that a larger sensitivity of the market concerning the large investor's consumption is favorable for the large investor as it was the case in the portfolio-dependent model. The figure displays the wealth ratios for the comparisons of the model ($\mathrm{CD}_{\varepsilon=0,1}$) with the models ($\mathrm{CD}_{\varepsilon=0,k}$), $k = 1, \dots, 5$, where $B^0_{c,1} = -30$, $B^0_{c,2} = -27.5$, $B^0_{c,3} = -25$, $B^0_{c,4} = -22.5$ and $B^0_{c,5} = -20$.

7.2.2 Affine Intensity Functions

As in the case of step intensity functions we first consider some *ceteris paribus intensity variations* and then go on with the *Merton invariant intensity variations*.

Ceteris paribus intensity variations. Figure 7.20 shows that increasing values of A^i imply that the intensities $\vartheta^{i,1-i}$ increase and moreover that the half spaces of strategies that impact on the regime shift intensities $\mathcal{I}^i$ are getting larger. Hence bigger values of A^0 are disadvantageous than smaller ones, whereas the opposite is true for A^1.

Figure 7.21 shows the wealth ratios for the comparisons of the model (CD_1) with the models (CD_k), $k = 1, \dots, 5$, where $A^1_1 = 10$, $A^1_2 = 11.25$, $A^1_3 = 12.5$, $A^1_4 = 13.75$ and $A^1_5 = 15$. As we have already seen in Figure 7.20 the large investor benefits from bigger values of A^1.

Figure 7.22 shows the wealth ratios for the comparisons of the model (PCD_1) with the models (PCD_k), $k = 1, \dots, 5$, where $C^1_1 = 2.5$, $C^1_2 = 3.75$, $C^1_3 = 5$, $C^1_4 = 6.25$ and $C^1_5 = 7.5$. The large investor profits

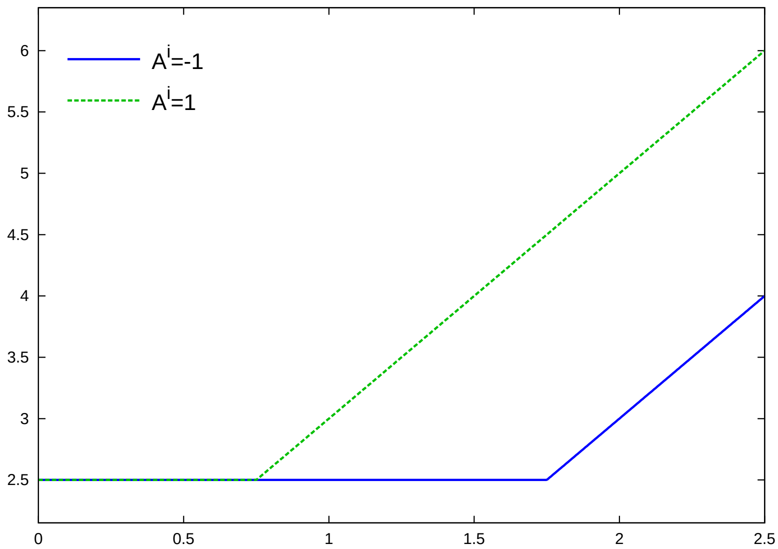

Figure 7.20: Intensity $\vartheta^{i,1-i}$ for different values of A^i

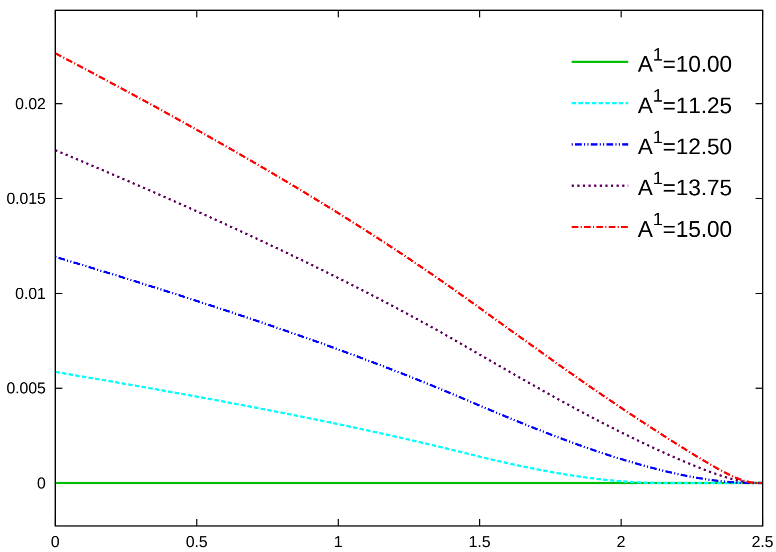

Figure 7.21: Wealth ratio $w^0_{\mathrm{CD}_1,\mathrm{CD}_k}$ for different values of A^1 as function of t ($r^0 = r^1 = 0.03$, $\eta^0 = 0.09$, $\eta^1 = 0.01$, $\sigma^0 = 0.2$, $\sigma^1 = 0.6$, $\delta = 0.035$, $\varepsilon = 1$, $R = 0.75$, $T = 2.5$, $A^0 = 9$, $B^0_c = 10$, $C^0 = 15$, $A^1_1 = 10$, $B^1_c = -5$, $C^1 = 7.5$)

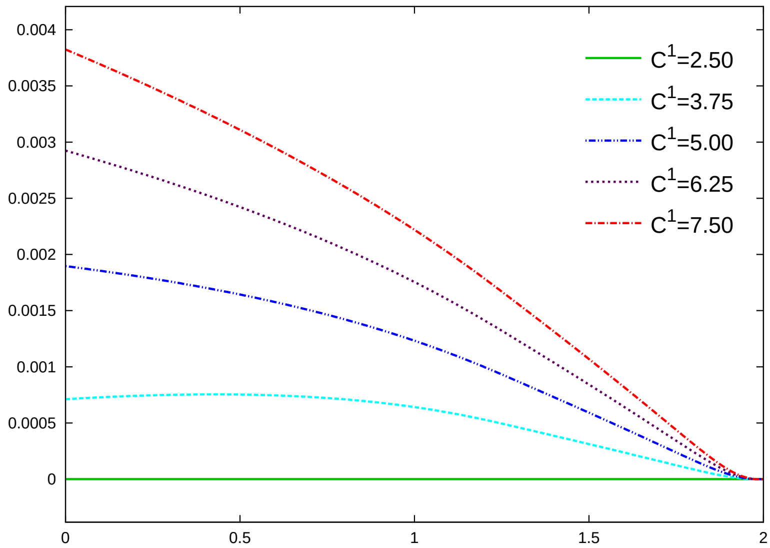

Figure 7.22: Wealth ratio $w^0_{\mathrm{PCD}_1,\mathrm{PCD}_k}$ for different values of C^1 as function of t ($r^0 = r^1 = 0.03$, $\eta^0 = 0.09$, $\eta^1 = 0.01$, $\sigma^0 = 0.3$, $\sigma^1 = 0.5$, $\delta = 0.035$, $\varepsilon = 1$, $R = 2.5$, $T = 2$, $A^0 = 5$, $B^0_\pi = 5$, $B^0_c = 10$, $C^0 = 10$, $A^1 = 5.5$, $B^1_\pi = -3$, $B^1_c = -5$, $C^1_1 = 2.5$)

from the growing minimal regime shift intensity in market state 1.

Merton invariant intensity variations. Figure 7.23 shows the consequences of an intensity shift in state 0. It displays the wealth ratios for the comparisons of the model (PD_1) with the models (PD_k), $k = 1, \ldots, 5$, where $(A^0_1, C^0_1) = (5, 10)$, $(A^0_2, C^0_2) = (7.5, 12.5)$, $(A^0_3, C^0_3) = (10, 15)$, $(A^0_4, C^0_4) = (12.5, 17.5)$ and $(A^0_5, C^0_5) = (15, 20)$. The wealth ratios are decreasing as the intensity level is increasing, i.e. the large investor benefits from small intensity levels which is the intuitive result.

The rotation around the Merton strategy is dealt with in Figures 7.24 to 7.26. As already in the step intensity case the Figures 7.24 and 7.25 suggest that bigger absolute values of the parameter B^i_π are advantageous given the sign of B^i_π does not change. The same holds true for the consumption parameter B^i_c (cf. Figure 7.26).

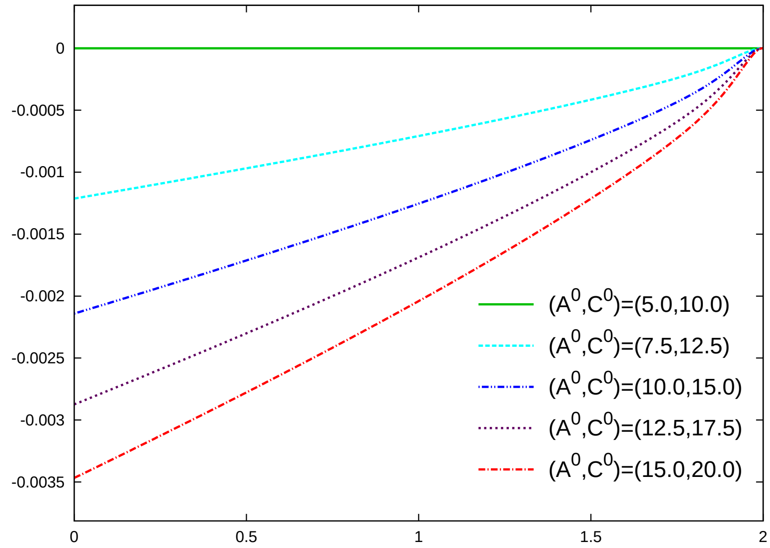

Figure 7.23: Wealth ratio $w^0_{\mathrm{PD}_1,\mathrm{PD}_k}$ for different values of (A^0, C^0) as function of t
($r^0 = r^1 = 0.03$, $\eta^0 = 0.09$, $\eta^1 = 0.01$, $\sigma^0 = 0.3$, $\sigma^1 = 0.5$, $\delta = 0.035$, $\varepsilon = 1$, $R = 2.5$, $T = 2$,
$A^0_1 = 5$, $B^0_\pi = 5$, $C^0_1 = 10$, $A^1 = 5.5$, $B^1_\pi = -3$, $C^1 = 5.4$)

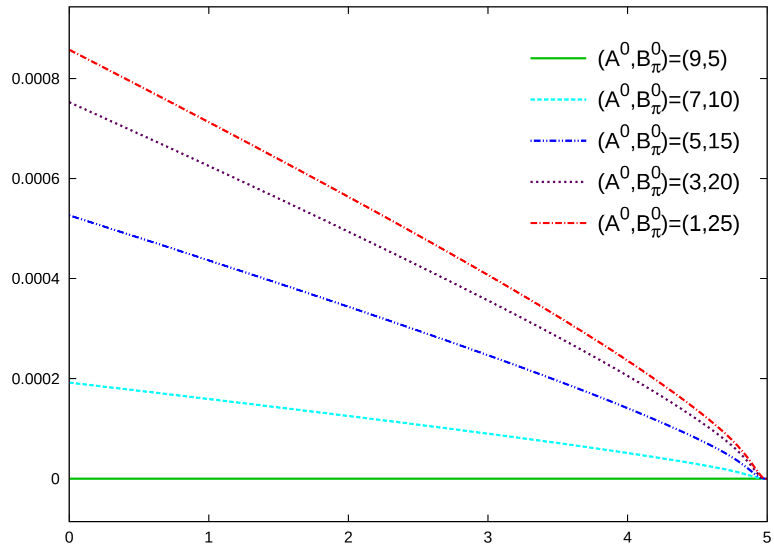

Figure 7.24: Wealth ratio $w^0_{\mathrm{PD}_1,\mathrm{PD}_k}$ for different values of (A^0, B^0_π) as function of t
($r^0 = r^1 = 0.03$, $\eta^0 = 0.09$, $\eta^1 = 0.01$, $\sigma^0 = 0.3$, $\sigma^1 = 0.5$, $\delta = 0.035$, $\varepsilon = 1$, $R = 2.5$, $T = 5$,
$A^0_1 = 9$, $B^0_{\pi,1} = 5$, $C^0 = 10$, $A^1 = 5.5$, $B^1_\pi = -3$, $C^1 = 5.4$)

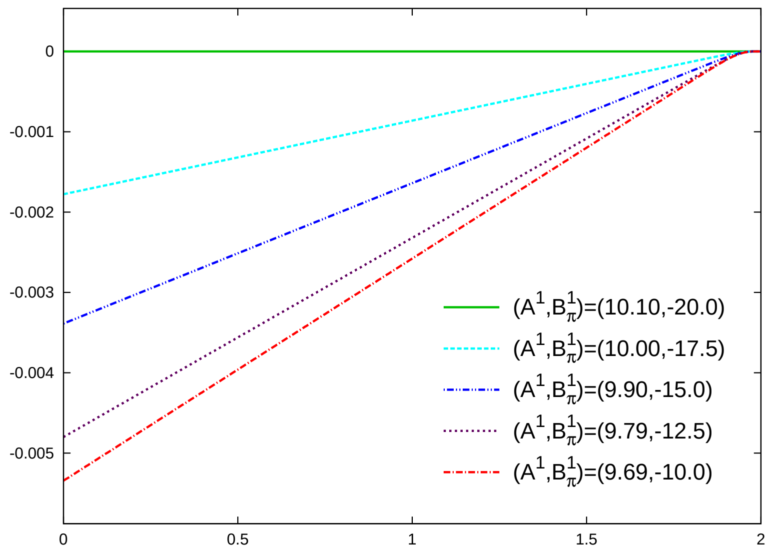

Figure 7.25: Wealth ratio $w^0_{\mathrm{PD}_1,\mathrm{PD}_k}$ for different values of (A^1, B^1_π) as function of t ($r^0 = r^1 = 0.03$, $\eta^0 = 0.1$, $\eta^1 = 0.01$, $\sigma^0 = 0.3$, $\sigma^1 = 0.7$, $\delta = 0.035$, $\varepsilon = 0$, $R = 0.5$, $T = 2$, $A^0 = -7$, $B^0_\pi = 20$, $C^0 = 30$, $A^1_1 = 10.1$, $B^1_{\pi,1} = -20$, $C^1 = 10$)

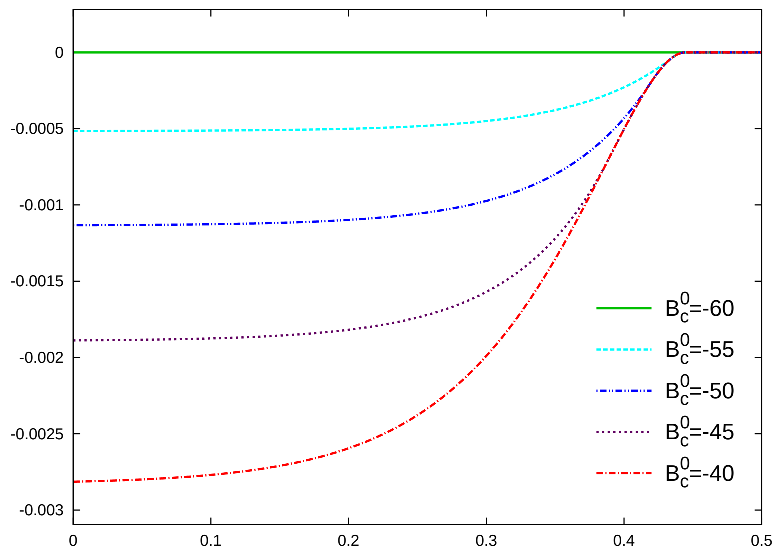

Figure 7.26: Wealth ratio $w^0_{\mathrm{CD}_1,\mathrm{CD}_k}$ for different values of B^0_c as function of t ($r^0 = r^1 = 0.03$, $\eta^0 = 0.1$, $\eta^1 = 0.01$, $\sigma^0 = 0.2$, $\sigma^1 = 0.6$, $\delta = 0.035$, $\varepsilon = 0$, $R = 0.25$, $T = 0.5$, $A^0 = 20$, $B^0_{c,1} = -40$, $C^0 = 15$, $A^1 = 0$, $B^1_c = 0$, $C^1 = 0$)

7.3 Price of Misconception

In order to determine the loss the large investor would have to face if he neglected his influence on the market we first have to discuss what it means that the investor ignores his influence. One would think that the investor would follow the Merton strategy instead of the optimal strategy. But whereas the Merton portfolio proportions are unique – independent of the model – the consumption rate of Merton type depends on the particular model, resp. on the regime shift intensities; except for a logarithmic investor with $R = 1$, as we have already seen in Remark 4.6. Hence it is reasonable to assume that the investor follows a strategy of the Merton type. However it remains to decide which particular type among the different Merton types should be used. An expedient choice would be the Merton type consumption rate resulting from the Bäuerle-Rieder model with regime shift intensities given by $\vartheta_{\mathrm{BR}}^{i,1-i} = \vartheta^{i,1-i}(\pi^{i,M}, c^{i,M}(t))$, $i = 0, 1$.

However, as long as $B_c^i \neq 0$ and $\varepsilon \neq 0$ the inclusion of the Merton type consumption rate causes some difficulties. First of all we have only discussed the Bäuerle-Rieder model with constant regime shift intensities. Only if $B_c^i = 0$ or $\varepsilon = 0$ then $\vartheta^{i,1-i}(\pi^{i,M}, c^{i,M}(t))$ is time-independent. Further the inclusion of the Merton consumption rate would cause a kind of circular argument since we needed the Merton consumption rate which is an output of the Bäuerle-Rieder model in order to determine the regime shift intensities which are an input of the Bäuerle-Rieder model. Again, only if $B_c^i = 0$ or $\varepsilon = 0$ then $\vartheta^{i,1-i}(\pi^{i,M}, c^{i,M}(t))$ is consumption-independent; except for the trivial case $\varepsilon = 0$ in which $c^{i,M}(t) = 0$. For the last two reasons we concentrate on the implications of the large investor's misconception in the portfolio-dependent model (PD) and in the models with dependency on consumption in case of $\varepsilon = 0$ ($\mathrm{CD}_{\varepsilon=0}$) and ($\mathrm{PCD}_{\varepsilon=0}$). Thus the particular regime shift intensities in the relevant Bäuerle-Rieder model are given by $\vartheta_{\mathrm{BR}}^{i,1-i} = \vartheta^{i,1-i}(\pi^{i,M}, 0)$.

Having specified the suitable regime shift intensities for the Bäuerle-Rieder model we obtain the optimal strategy $(\pi_{\mathrm{BR}}^{i,*}, c_{\mathrm{BR}}^{i,*}) = (\pi^{i,M}, c_{\mathrm{BR}}^{i,M})$. In order to determine the expected utility from terminal wealth and intermediate consumption a large investor would achieve when he neglected his influence on the market we have to solve the HJB-system over the set of admissible strategies that are given just by $(\pi_{\mathrm{BR}}^{i,*}, c_{\mathrm{BR}}^{i,*}(t))$. It turns out that the value function of the Bäuerle-Rieder model with intensities given by $\vartheta^{i,1-i}(\pi^{i,M}, 0)$ equals the expected utility from terminal wealth and intermediate consumption of the large investor in the models (PD), ($\mathrm{CD}_{\varepsilon=0}$) or ($\mathrm{PCD}_{\varepsilon=0}$) when following the Merton type strategy $(\pi_{\mathrm{BR}}^{i,*}, c_{\mathrm{BR}}^{i,*})$. The relevant wealth ratios for the analysis of the investor's misconception in the particular models are given by $w_{\mathrm{BR,PD}}^i$, $w_{\mathrm{BR,CD}_{\varepsilon=0}}^i$ and $w_{\mathrm{BR,PCD}_{\varepsilon=0}}^i$.

In the following we illustrate the implications of the large investor's misconception in the case of step intensity functions (Subsection 7.3.1) and affine intensity functions (Subsection 7.3.2) with several examples.

7.3.1 Step Intensity Functions

Figures 7.27 to 7.32 show the wealth ratios, resp. the prices of the investor's misconception in the model with step intensity functions. We present the wealth ratios for the three model variants mentioned above, i.e. (PD), ($\mathrm{CD}_{\varepsilon=0}$) and ($\mathrm{PCD}_{\varepsilon=0}$); each with two different investment horizons. In the discussed examples the loss due to the misconception attains rather significant values of percentage order. Moreover the loss gets bigger when the investment horizon T becomes larger. Hence the investor's influence on the market should not be neglected when setting up his investment strategy.

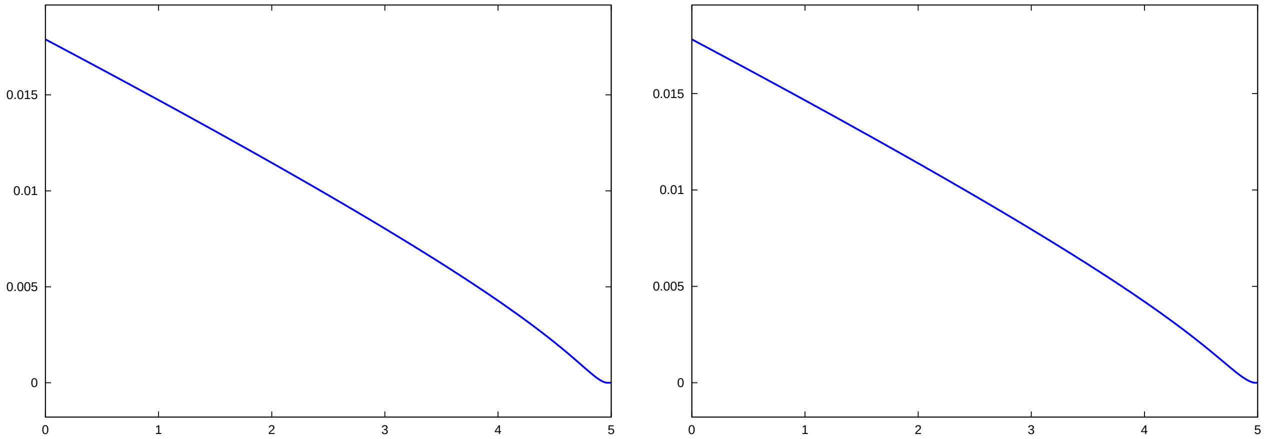

Figure 7.27: Wealth ratio $w^i_{\text{BR,PD}}$ in state 0 (left) and state 1 (right) as function of t
($r^0 = r^1 = 0.03$, $\eta^0 = 0.1$, $\eta^1 = 0.02$, $\sigma^0 = 0.3$, $\sigma^1 = 0.6$, $\delta = 0.035$, $\varepsilon = 1$, $R = 2.5$, $T = 5$,
$A^0 = 17$, $B^0_\pi = -5$, $C^0 = 14$, $C^0_1 = 5$, $C^0_2 = 10$, $A^1 = 5.5$, $B^1_\pi = -7$, $C^1 = 5$, $C^1_1 = 5$, $C^1_2 = 1.25$)

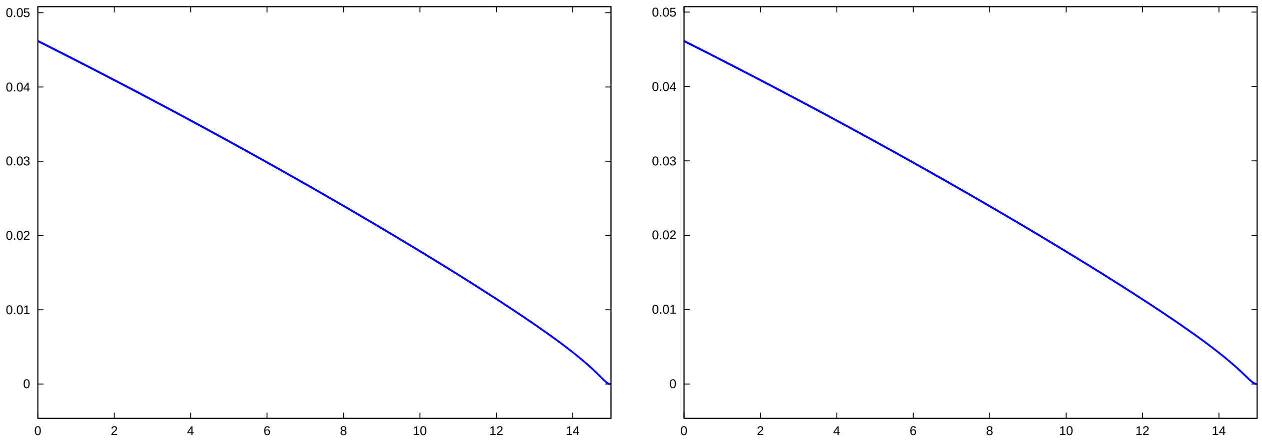

Figure 7.28: Wealth ratio $w^i_{\text{BR,PD}}$ in state 0 (left) and state 1 (right) as function of t
($r^0 = r^1 = 0.03$, $\eta^0 = 0.1$, $\eta^1 = 0.02$, $\sigma^0 = 0.3$, $\sigma^1 = 0.6$, $\delta = 0.035$, $\varepsilon = 1$, $R = 2.5$, $T = 15$,
$A^0 = 17$, $B^0_\pi = -5$, $C^0 = 14$, $C^0_1 = 5$, $C^0_2 = 10$, $A^1 = 5.5$, $B^1_\pi = -7$, $C^1 = 5$, $C^1_1 = 5$, $C^1_2 = 1.25$)

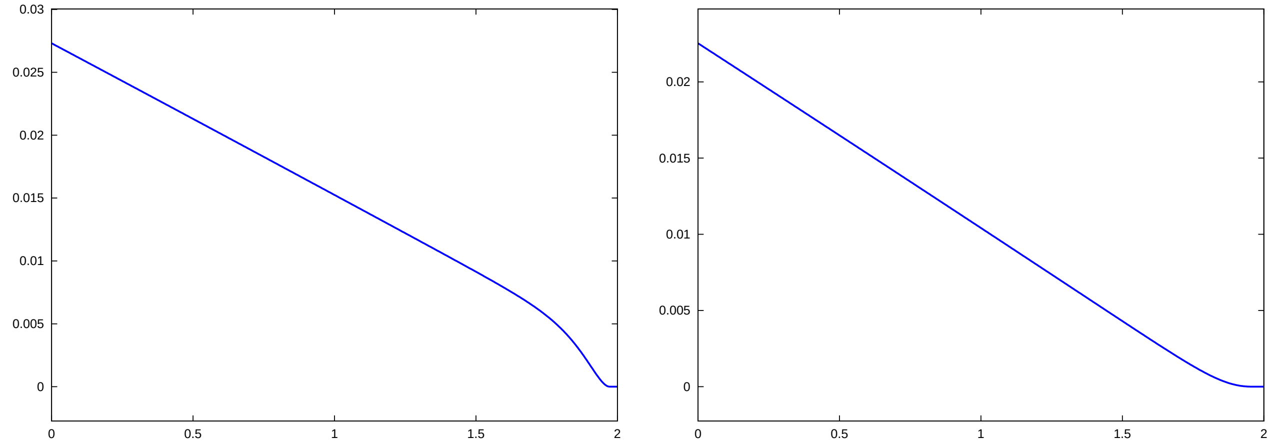

Figure 7.29: Wealth ratio $w^i_{\mathrm{BR,CD}_{\varepsilon=0}}$ in state 0 (left) and state 1 (right) as function of t
($r^0 = r^1 = 0.03$, $\eta^0 = 0.1$, $\eta^1 = 0.01$, $\sigma^0 = 0.2$, $\sigma^1 = 0.7$, $\delta = 0.035$, $\varepsilon = 0$, $R = 0.5$, $T = 2$,
$A^0 = 16$, $B^0_c = -20$, $C^0 = 15$, $C^0_1 = 10$, $C^0_2 = 20$, $A^1 = 5$, $B^1_c = -10$, $C^1 = 3$, $C^1_1 = 5$, $C^1_2 = 2.5$)

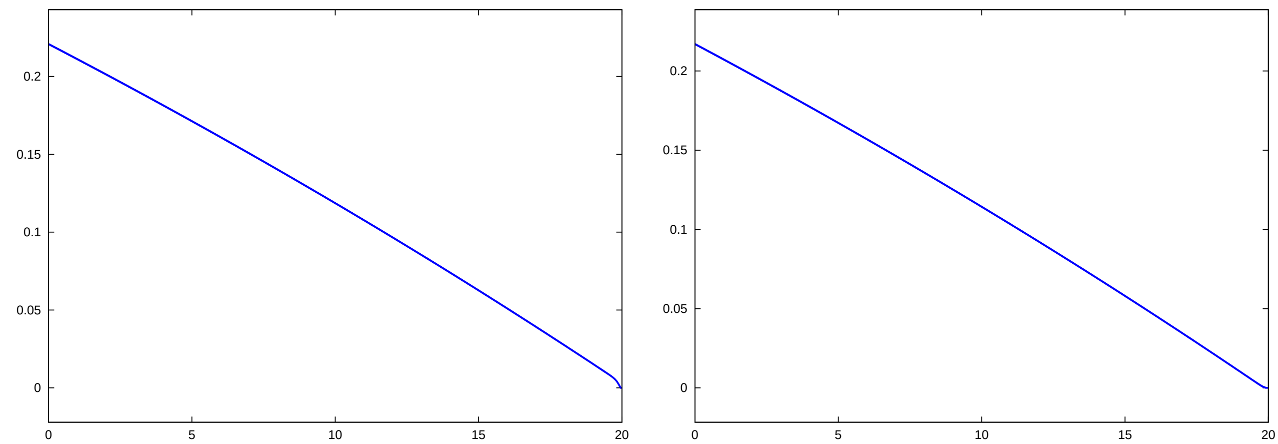

Figure 7.30: Wealth ratio $w^i_{\mathrm{BR,CD}_{\varepsilon=0}}$ in state 0 (left) and state 1 (right) as function of t
($r^0 = r^1 = 0.03$, $\eta^0 = 0.1$, $\eta^1 = 0.01$, $\sigma^0 = 0.2$, $\sigma^1 = 0.7$, $\delta = 0.035$, $\varepsilon = 0$, $R = 0.5$, $T = 20$,
$A^0 = 16$, $B^0_c = -20$, $C^0 = 15$, $C^0_1 = 10$, $C^0_2 = 20$, $A^1 = 5$, $B^1_c = -10$, $C^1 = 3$, $C^1_1 = 5$, $C^1_2 = 2.5$)

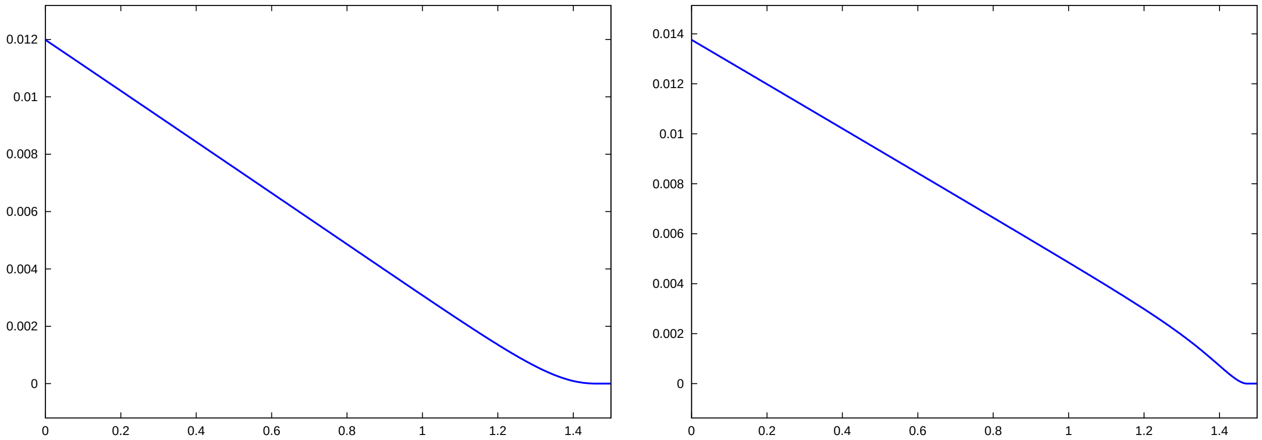

Figure 7.31: Wealth ratio $w^i_{\mathrm{BR,PCD}_{\varepsilon=0}}$ in state 0 (left) and state 1 (right) as function of t
($r^0 = r^1 = 0.03$, $\eta^0 = 0.1$, $\eta^1 = 0.06$, $\sigma^0 = 0.3$, $\sigma^1 = 0.7$, $\delta = 0.035$, $\varepsilon = 0$, $R = 1.5$, $T = 1.5$,
$A^0 = 22.5$, $B^0_\pi = -5$, $B^0_c = 10$, $C^0 = 19.5$, $C^0_1 = 5$, $C^0_2 = 30$, $A^1 = 11$, $B^1_\pi = -7$, $B^1_c = 5$, $C^1 = 9$, $C^1_1 = 25$, $C^1_2 = 5$)

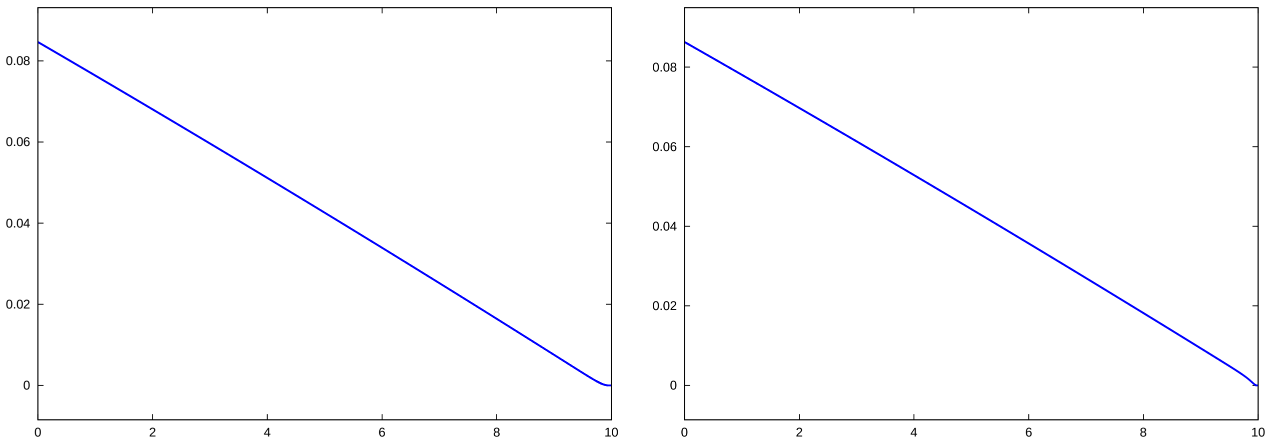

Figure 7.32: Wealth ratio $w^i_{\mathrm{BR,PCD}_{\varepsilon=0}}$ in state 0 (left) and state 1 (right) as function of t
($r^0 = r^1 = 0.03$, $\eta^0 = 0.1$, $\eta^1 = 0.06$, $\sigma^0 = 0.3$, $\sigma^1 = 0.7$, $\delta = 0.035$, $\varepsilon = 0$, $R = 1.5$, $T = 10$,
$A^0 = 22.5$, $B^0_\pi = -5$, $B^0_c = 10$, $C^0 = 19.5$, $C^0_1 = 5$, $C^0_2 = 30$, $A^1 = 11$, $B^1_\pi = -7$, $B^1_c = 5$, $C^1 = 9$, $C^1_1 = 25$, $C^1_2 = 5$)

7.3.2 Affine Intensity Functions

Examples for the price of the investor's misconception in the affine intensity setting are given in the Figures 7.33 to 7.38. The wealth ratios for the three model variants mentioned above, i.e. (PD), ($CD_{\varepsilon=0}$) and ($PCD_{\varepsilon=0}$) – each with two different investment horizons – are presented. As in the step intensity case the loss due to the misconception attains rather significant values of percentage order that increase with growing investment horizon T which again implies that the investor's influence on the market should not be neglected.

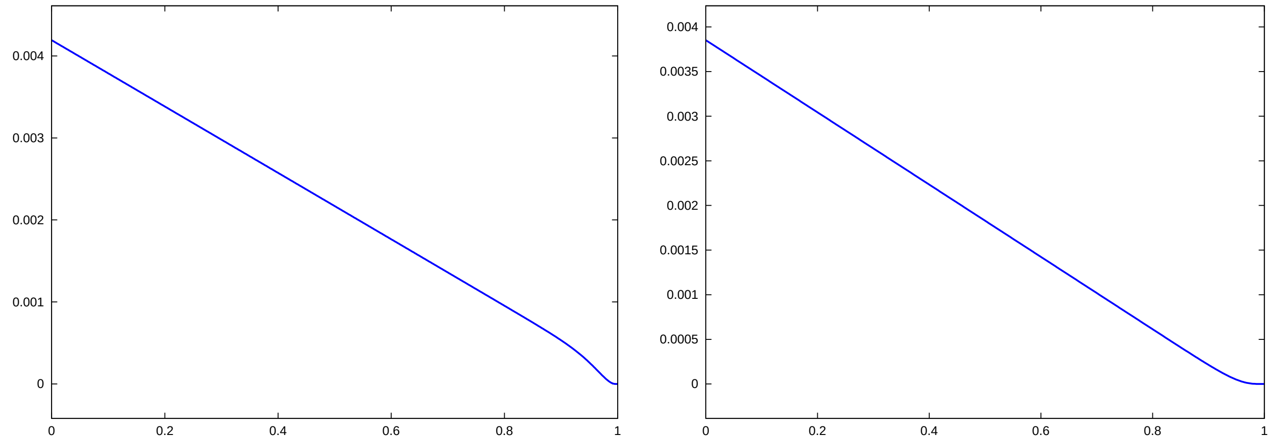

Figure 7.33: Wealth ratio $w^i_{\mathrm{BR,PD}}$ in state 0 (left) and state 1 (right) as function of t
($r^0 = r^1 = 0.03$, $\eta^0 = 0.1$, $\eta^1 = 0.01$, $\sigma^0 = 0.3$, $\sigma^1 = 0.7$, $\delta = 0.035$, $\varepsilon = 0$, $R = 0.5$, $T = 1$, $A^0 = -7$, $B^0_\pi = 20$, $C^0 = 30$, $A^1 = 10.1$, $B^1_\pi = -10$, $C^1 = 10$)

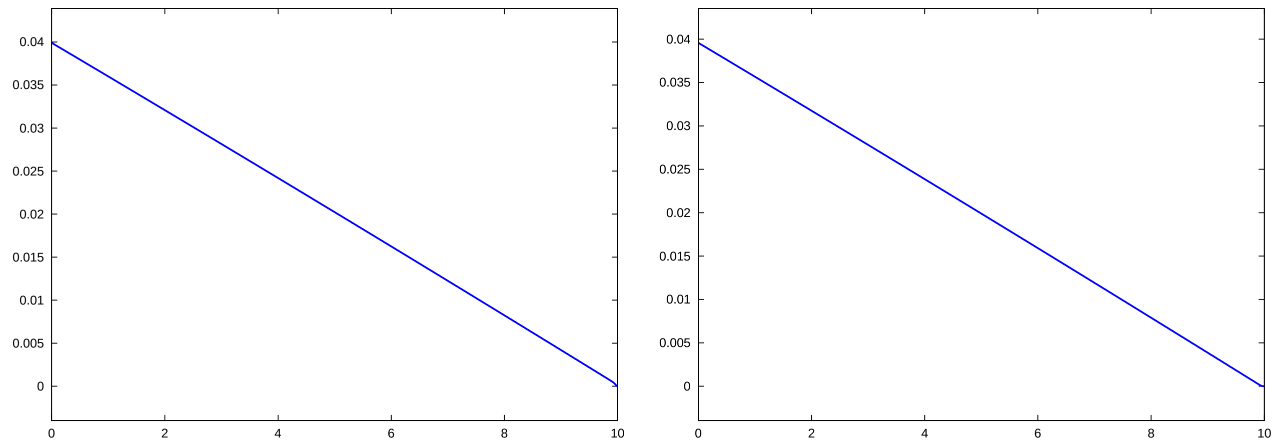

Figure 7.34: Wealth ratio $w^i_{\mathrm{BR,PD}}$ in state 0 (left) and state 1 (right) as function of t
($r^0 = r^1 = 0.03$, $\eta^0 = 0.1$, $\eta^1 = 0.01$, $\sigma^0 = 0.3$, $\sigma^1 = 0.7$, $\delta = 0.035$, $\varepsilon = 0$, $R = 0.5$, $T = 10$, $A^0 = -7$, $B^0_\pi = 20$, $C^0 = 30$, $A^1 = 10.1$, $B^1_\pi = -10$, $C^1 = 10$)

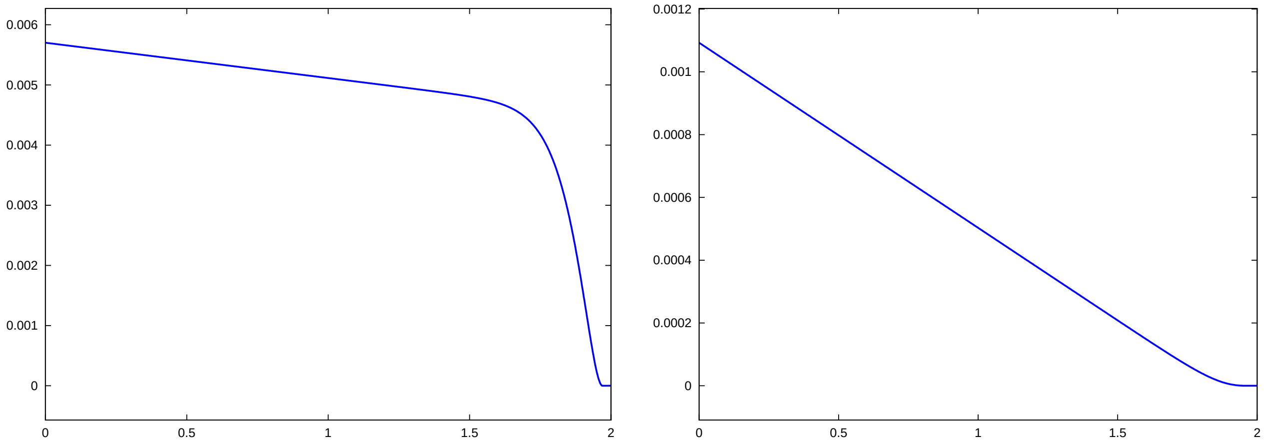

Figure 7.35: Wealth ratio $w^i_{\mathrm{BR,CD}_{\varepsilon=0}}$ in state 0 (left) and state 1 (right) as function of t
($r^0 = r^1 = 0.03,\ \eta^0 = 0.1,\ \eta^1 = 0.01,\ \sigma^0 = 0.2,\ \sigma^1 = 0.7,\ \delta = 0.035,\ \varepsilon = 0,\ R = 0.25,\ T = 2,$
$A^0 = 20,\ B^0_c = -90,\ C^0 = 15,\ A^1 = 0,\ B^1_c = 0,\ C^1 = 0.125$)

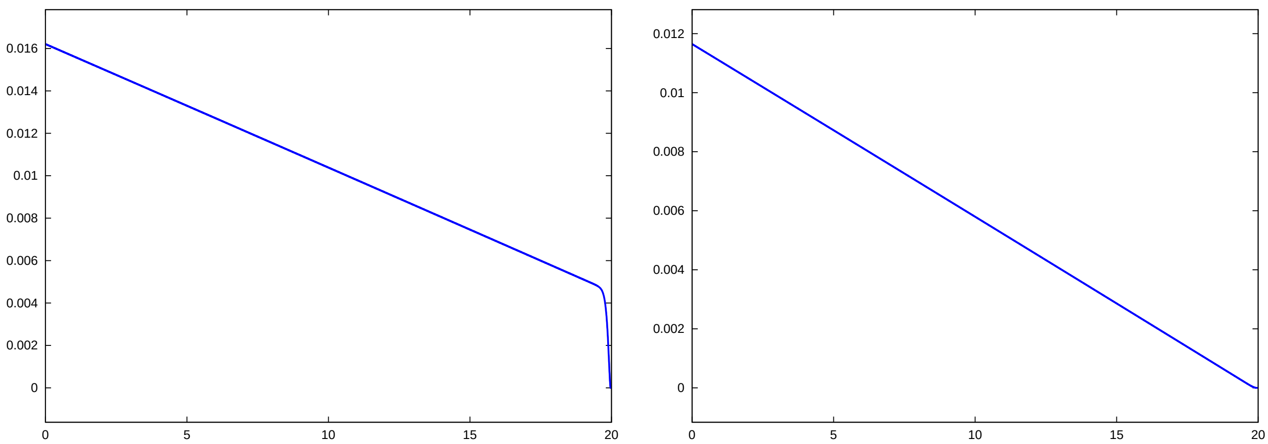

Figure 7.36: Wealth ratio $w^i_{\mathrm{BR,CD}_{\varepsilon=0}}$ in state 0 (left) and state 1 (right) as function of t
($r^0 = r^1 = 0.03,\ \eta^0 = 0.1,\ \eta^1 = 0.01,\ \sigma^0 = 0.2,\ \sigma^1 = 0.7,\ \delta = 0.035,\ \varepsilon = 0,\ R = 0.25,\ T = 20,$
$A^0 = 20,\ B^0_c = -90,\ C^0 = 15,\ A^1 = 0,\ B^1_c = 0,\ C^1 = 0.125$)

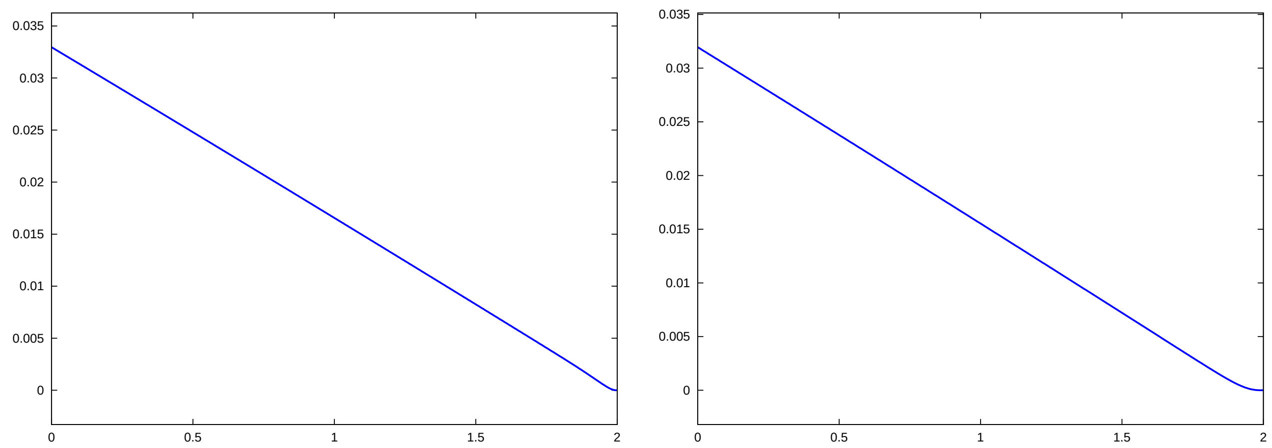

Figure 7.37: Wealth ratio $w^i_{\text{BR},\text{PCD}_{\varepsilon=0}}$ in state 0 (left) and state 1 (right) as function of t
($r^0 = r^1 = 0.035$, $\eta^0 = 0.09$, $\eta^1 = 0.01$, $\sigma^0 = 0.2$, $\sigma^1 = 0.5$, $\delta = 0.04$, $\varepsilon = 0$, $R = 1.5$, $T = 2$, $A^0 = 100$, $B^0_\pi = -50$, $B^0_c = -1000$, $C^0 = 10$, $A^1 = 8$, $B^1_\pi = 25$, $B^1_c = 0$, $C^1 = 8$)

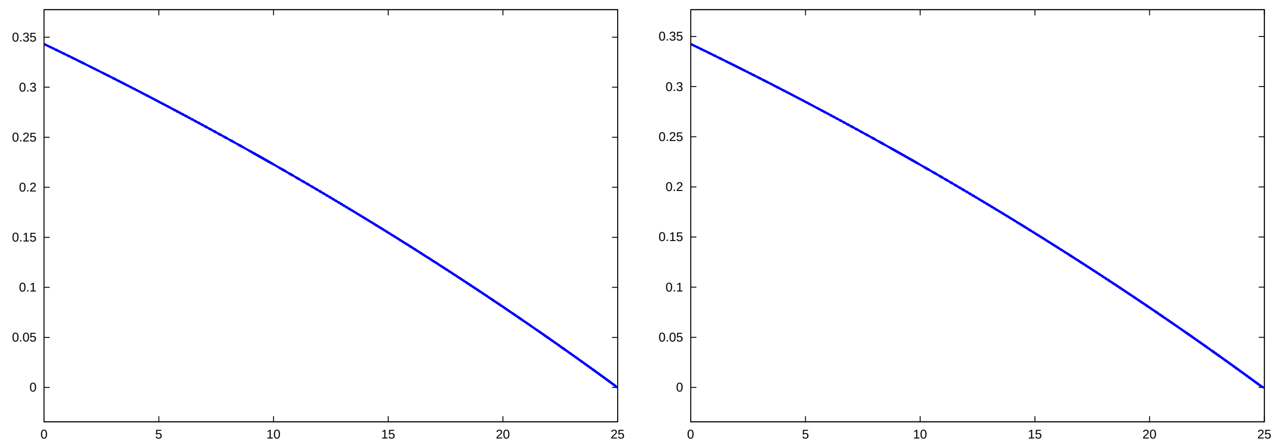

Figure 7.38: Wealth ratio $w^i_{\text{BR},\text{PCD}_{\varepsilon=0}}$ in state 0 (left) and state 1 (right) as function of t
($r^0 = r^1 = 0.035$, $\eta^0 = 0.09$, $\eta^1 = 0.01$, $\sigma^0 = 0.2$, $\sigma^1 = 0.5$, $\delta = 0.04$, $\varepsilon = 0$, $R = 1.5$, $T = 25$, $A^0 = 100$, $B^0_\pi = -50$, $B^0_c = -1000$, $C^0 = 10$, $A^1 = 8$, $B^1_\pi = 25$, $B^1_c = 0$, $C^1 = 8$)

8 Conclusion

In this thesis we set up a continuous-time investment problem for a large investor whose investment strategy influences the financial market. In order to include this influence in the model we described the financial market as a regime model where the transition intensities are functions of the investor's strategy. After formulating the mathematical framework and the optimal investment problem of maximizing the expected utility from terminal wealth as well as from intermediate consumption we proved a general verification theorem that covers jumps in the stock prices and a general class of intensity functions. Thereafter we focused on a model with continuous asset prices and investigated three different kinds of intensity functions – constant, step and affine – that enabled us to solve the investment problem explicitly for an investor with CRRA utility. We derived the solution following the stochastic control approach for three variants of the aforementioned strategy-dependency – portfolio-dependency, consumption-dependency and combined portfolio- and consumption-dependency. It turned out that whereas the optimal portfolio strategy in the Merton problem and the Bäuerle-Rieder problem was constant, in our model with portfolio-dependency it becomes a time-dependent function thus accommodating the impact on the market state. As in the Merton and the Bäuerle-Rieder problem the optimal consumption rate is a time-dependent function but deviates from the Merton-type optimal consumption rate in the cases with consumption-dependent intensities.

The sections on the solutions for investors with power and logarithmic utility were followed by a short discussion of the special case of two correlated stocks. This showed that in case of correlated assets deviations from the Merton strategy are possible although no direct intensity impact is given. Hence deviations may occur for compensational reasons, too. Thereafter we investigated the differences of the presented models. We found out that the investor's preferences concerning the particular models vary over time and may even change sign, i.e. during the investment period it is possible that the firstly preferred model becomes disadvantageous after some time. The subsection on the parameter sensitivity revealed that the large investor seems to benefit from a sensitive market which at first glance is quite counterintuitive but turns out to be the logical consequence. In a sensitive market the large investor does not have to deviate far from the Merton strategy in order to influence the market in his favor. Finally we addressed the question what would be the loss the investor had to face if he neglected his market impact. Here it turned out that – depending of course on the particular parameter setting – the loss may be remarkably high.

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Scientific Career

21/03/1983	born in Kirchheimbolanden
08/1989 – 07/1993	Primary school: GHS Flomborn
08/1993 – 03/2002	Grammar school: Gymnasium am Römerkastell Alzey
03/2002	Abitur
10/2002 – 07/2007	study of Management Mathematics ("Wirtschaftsmathematik") at University of Kaiserslautern
10/2004	Vordiplom in Management Mathematics at University of Kaiserslautern
11/2005 – 03/2006	student assistant at Fraunhofer Institut für Techno- und Wirtschaftsmathematik
05/2006 – 09/2006	internship, DekaBank
10/2006 – 12/2006	visiting student at University of Oxford
02/2007 – 07/2007	student assistant at Fraunhofer Institut für Techno- und Wirtschaftsmathematik
07/2007	Diplom in Management Mathematics (Dipl.-Math. oec.) at University of Kaiserslautern, specialization in mathematical finance
since 07/2007	PhD student of Prof. Dr. Ralf Korn
09/2007 – 10/2007	internship, HypoVereinsbank

Wissenschaftlicher Werdegang

21/03/1983	geboren in Kirchheimbolanden
08/1989 – 07/1993	Grund- und Hauptschule Flomborn
08/1993 – 03/2002	Gymnasium am Römerkastell Alzey
03/2002	Abitur
10/2002 – 07/2007	Studium der Wirtschaftsmathematik an der TU Kaiserslautern
10/2004	Vordiplom in Wirtschaftsmathematik an der TU Kaiserslautern
11/2005 – 03/2006	wissenschaftliche Hilfskraft am Fraunhofer Institut für Techno- und Wirtschaftsmathematik
05/2006 – 09/2006	Praktikum bei der DekaBank
10/2006 – 12/2006	Gaststudent an der University of Oxford
02/2007 – 07/2007	wissenschaftliche Hilfskraft am Fraunhofer Institut für Techno- und Wirtschaftsmathematik
07/2007	Diplom in Wirtschaftsmathematik (Dipl.-Math. oec.) an der TU Kaiserslautern, Vertiefungsrichtung Finanzmathematik
seit 07/2007	Doktorand bei Prof. Dr. Ralf Korn
09/2007 – 10/2007	Praktikum bei der HypoVereinsbank