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Valuation of Credit Derivatives

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Summary

In this work two main approaches for the evaluation of credit derivatives are analyzed: the copula based approach and the Markov Chain based approach. This work gives the opportunity to use the advantages and avoid disadvantages of both approaches. For example, modeling of contagion effects, i.e. modeling dependencies between counterparty defaults, is complicated under the copula approach. One remedy is to use Markov Chain, where it can be done directly.

The work consists of five chapters.

The first chapter of this work extends the model for the pricing of CDS contracts presented in the paper by Kraft and Steffensen (2007). In the widely used models for CDS pricing it is assumed that only borrower can default. In our model we assume that each of the counterparties involved in the contract may default. Calculated contract prices are compared with those calculated under usual assumptions. All results are summarized in the form of numerical examples and plots.

In the second chapter the copula and its main properties are described. The methods of constructing copulas as well as most common copulas families and its properties are introduced.

In the third chapter the method of constructing a copula for the existing Markov Chain is introduced. The cases with two and three counterparties are considered. Necessary relations between the transition intensities are derived to directly find some copula functions. The formulae for default dependencies like Spearman's rho and Kendall's tau for defined copulas are derived. Several numerical examples are presented in which the copulas are built for given Markov Chains.

The fourth chapter deals with the approximation of copulas if for a given Markov Chain a copula cannot be provided explicitly.

The fifth chapter concludes this thesis.

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Motivation

There are currently several approaches to price credit derivatives like CDO. One may start from a specification of dependent default intensities. A typical example is Duffie and Gârleanu (2001) or Kraft and Steffensen (2006). An alternative route is the structural approach, corresponding to a multivariate hitting time model, as illustrated by Hull et al. (2005). The previous approaches involve a calibration to marginal default distributions. On the other hand, the copula approach directly specifies the dependence structure, though in a somehow ad-hoc way. While the Gaussian copula model, introduced to the credit field by Li (2000) has become an industry standard, its theoretical foundations, such as credit spread dynamics may be questioned. For this purpose, copulas such as Clayton, Student t, double t, or Marshall-Olkin copulas have been proposed.

The factor approach is quite standard in credit risk modelling (see for instance Crouhy et al. (2000), Merino and Nyfeler (2002), Pykhtin and Dev (2002), Gordy (2003) and Frey and McNeil (2003)). In the case of homogeneous portfolios, it is often coupled with large sample approximation techniques. In such a framework, Gordy and Jones (2003) analyse the risks within CDO tranches. In order to deal with numerical issues, Gregory and Laurent (2003) and Laurent and Gregory (2005) have described a semi-analytical approach, based on factor models, for the pricing of basket credit derivatives and CDOs. This topic is also discussed by Andersen et al. (2003) and Hull and White (2004) among others. We will further rely on this factor approach, which also provides an easy to deal framework for model comparisons. Other contributions dedicated to comparing various copulas in the credit field are Das and Deng (2004) or the book by Cherubini et al. (2004).

Chapter 1

Pricing of Credit Default Swaps in a Markov Chain Framework

1.1 Introduction

In recent years, the market for credit derivatives has seen one of the largest growths of all the markets. Credit derivative give companies the possibility to trade and manage credit risk in the similar way as market risk. The purpose of handling these financial products is to manage and trade credit risk, i.e. the risk that a borrower may not be able to pay back a loan in time. Credit derivatives are being actively used not only for hedging purposes, but also as a way to improve the return on the capital. A bank might use credit derivatives to manage its portfolio of risk. Moreover with a credit derivative a bank can sell credit exposure and still keep a good relationship with an important client.

One of the most actively traded credit derivatives is a credit default swap (CDS). A CDS provides an insurance against the default (called a credit event) of a particular company (called the reference entity). The two mostly used approaches to model default events are the copula and intensity-based approaches. In this chapter we apply an intensity-based approach, where we model the defaults events by using Markov chains, for more details see Kraft and Steffensen (2007). Furthermore, this chapter explicitly models counterparty risk, i.e. in our model, each firm has a unique firm-specific counterparty structure that arises from its relation to other firms in the economy. By counterparty risk we mean the risk that the default of a firm's counterparty might affect its own default probability. As it was noted in Jarrow and Yu (2001), this approach has many benefits and describes the reality more precisely. Additionally, our model also captures contagion effects, i.e effects that the default probability of one firm is strongly affected by the default of another firm. For example, this effect can be observed, if a bank A holds a significant amount of firm's B debt, then default probability of the bank A can increase significantly after the default of the firm B.

In order to understand how a CDS works, let us consider an investor A who has bought a bond with maturity T-years from a company C. In order to protect himself from a default event on the bond, the investor A might want to buy a T-years CDS from a counterparty B on the specific bond that he has bought from the company C. Entering a CDS contract, the investor A has to pay a certain amount of money (called premium or fee payment) to the counterparty B at every predefined payment date until the maturity or until the credit event happens. The size of the payment to be paid to counterparty B by investor A is equals to the so-called CDS rate at the date when the CDS contract has been bought. Our main goal is to price a CDS contract, i.e. to determine the fair CDS rate.

1.2 The Model

As it was pointed out by Kraft and Steffensen (2007), Markov chains can be an important tool to model credit risk. A Markov chain is a stochastic process which is characterized by a finite number of states and transitions between this states. A transition from one state into another is associated with a credit event. In our case, such a credit event is a default of one of the firms in the market. In general case, not only the default events but also up- or downgrading in rating classes as well as bankruptcy events can be modeled with the help of Markov chains. Some states of the Markov chain may be absorbing states, i.e. once the process reaches one of these states it can not leave it any more.

The advantage of this approach is that in a Markov chain model the contagion effects can be easily modeled. If one of the firms defaults the default intensities of other firms will change to some new intensities. A more detailed description can be found in Kraft and Steffensen (2007), as well as in Jarrow and Yu (2001).

We are going to analyse a CDS contract using the framework settled up in Kraft and Steffensen (2007). For simplicity, we consider the case where the protection seller B promises to make a payment of one unit of money to the protection buyer A if a certain reference entity C defaults during the lifetime of the CDS contract. Usually such types of CDS contracts are called digital CDS contracts.

In our model we allow for the counterparty risk, i.e. in our case we allow that the counterparty B or the investor A fail to fulfil their contractual obligations.

In the model used by Kraft and Steffensen (2007), in the following the standard model, the protection buyer A continues to make his fee payments until his own default. The protection payment is made only if the protection seller B has not defaulted up to the default time of the reference entity C. Moreover, if investor A defaults before C defaults, protection seller B needs to fulfill his obligations as long as B is not in default and, independently of whether B is in default or not, investor A continues to make his fee payments until its own default, a default of reference entity C, or maturity whichever occurs first.

We improve the standard model. In our model, if counterparty B is defaulted before the company C or before the maturity time T, the contract will be immediately terminated. In this case, if the present value of the contract at the default time of the counterparty B is positive for counterparty B, then the counterparty B will receive some positive payment from the protection buyer A. Similar will occur, when the investor A defaults before the reference entity C defaults or before the maturity time T. In this case the contract will be immediately terminated and additionally, if the present value of the contract is positive for the protection buyer A, then the investor A will receive some positive payment from the counterparty B. Later in the paper, we will derive precise formulas for these payments.

To calculate CDS spreads, we need to evaluate the present value of the premium payments made by investor A to counterparty B, the so-called fee leg (also called premium leg) and the present value of the protection payment from the counterparty

B to the protection buyer A, the so-called protection leg (also called credit leg). Furthermore, in our model we need to evaluate the present value of the correction payments, which the investor A will pay, if the counterparty B defaults and which the counterparty B will pay, if the protection buyer A defaults. To do this we need the Corollary 3.1 from the paper by Kraft and Steffenson (2007):

Corollary 1.2.1 (Pricing via Transition Probabilities) *If the intensities and payments are constant, then*

- *The price of the continuous coupon payment at time t equals*

$$\sum_{k \in J} c^k \int_t^T P(t,s) q^{jk}(t,s) ds.$$

- *The price of the payments upon transition at time t equals*

$$\sum_{\nu \in J} \sum_{k \neq \nu} a^{\nu k} \lambda^{\nu k} \int_t^T P(t,s) q^{j\nu}(t,s) ds.$$

- *The price of the final payment at time t equals*

$$P(t,T) \sum_{k \in J} a^k q^{jk}(t,T).$$

There are two different types of CDS contracts that differ by the time when the protection payment is made. In the first case the counterparty B pays the protection payment to the investor A at the default time of the reference entity C and in the second case at the maturity of the CDS contract T, if the reference entity C has defaulted before the maturity time T. Both cases will be considered in this paper.

1.2.1 Protection Payment at Maturity

In this section we will derive the expression for the fee leg, the protection leg, and the correction payments for the case then CDS contract is settled at maturity. This mean that the protection payment is made by the protection seller B to the protection buyer A at maturity of the CDS contract T in a case of default event of

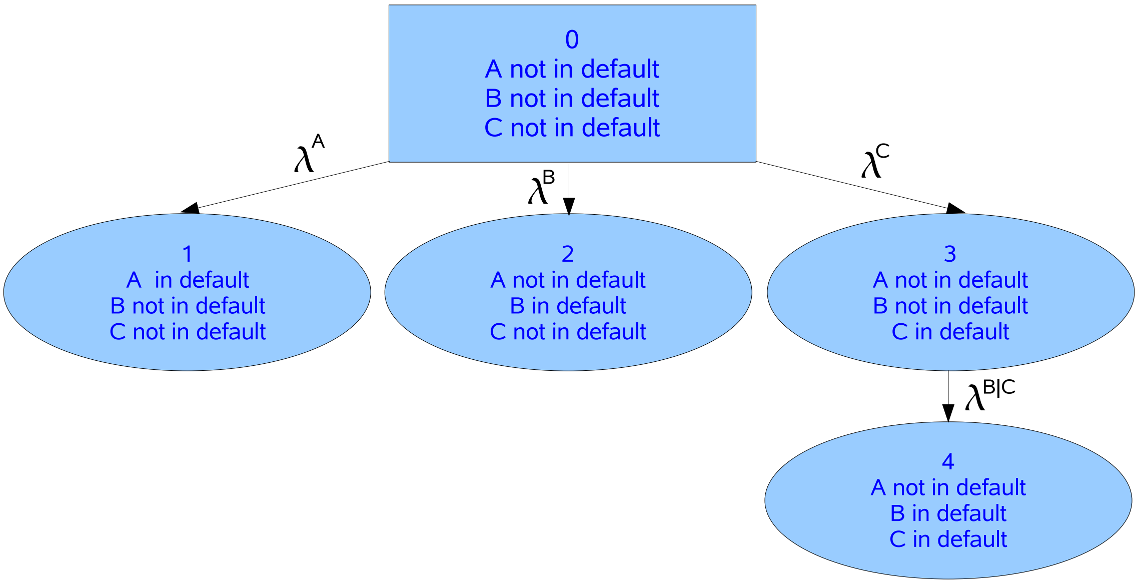

Figure 1.1: Markov chain describing the CDS contract settled at maturity

the reference entity C before the maturity time T. The corresponding Markov chain model is shown in the Figure 1.1.

In this case, the protection buyer A continuously makes fee payments to the protection seller B at payment dates until the reference entity C defaults or until the maturity T. On the other hand, the protection seller B makes only the protection payment to the investor A at the maturity T, if the reference entity C defaults before the end of the CDS contract. The investor A stops making premium payments if the counterparty B defaults or the investor A itself defaults. In both these cases the contract is terminated. State 0 is the initial state at the beginning of the CDS contract where all companies are not in default. State 1 describes the situation when investor A defaults. State 2 describes the case when counterparty B defaults. State 3 describes the default of the reference entity C. Since the protection payment is made at maturity, the counterparty B can also default before the maturity of the contract, but after the reference entity C has defaulted. In the Figure 1.1, this case is described by state 4. The λs in Figure 1.1 are the default intensities for the corresponding transitions between states.

Let $y(t)$ denote the fair CDS spread and assume that all intensities λ^{jk} are constant. The default intensities of A, B, and C given that no entity has already defaulted are denoted by $\lambda^A = \lambda^{01}$, $\lambda^B = \lambda^{02}$, and $\lambda^C = \lambda^{03}$. The default intensity of B given that C defaulted is denoted by $\lambda^{B|C} = \lambda^{34}$.

According to Kraft and Steffensen (2007), in the case of protection payments is made at maturity of the contract, the protection payment needs to be modeled as a contingent final payment. The fee leg can be modeled as a state-dependent coupon payment. By Corollary 3.1 of Kraft and Steffensen (2006), the value of the fee leg is given by

$$y(t)B^0_{fee}(t) = y(t) \sum_{k \in \mathcal{J}_F^M} \int_t^T P(t,s) q^{0k}(t,s) ds$$

with $P(t,s)$ – the price of zero-coupon bond, $q^{0k}(t,s)$ – is transition probability from state 0 to state k. Furthermore, $\mathcal{J}_F^M = \{0,3\}$ – the set of states where investor A pays fee payments. The value of the protection leg is given by

$$\bar{B}^0_{protection}(t) = q^{03}(t,T)P(t,T).$$

As we have said in the description of our model, if investor A defaults before the reference entity C the contract will be terminated and A will receive the following option: $max\{PV^A(\tau_A), 0\}$, with τ_A – the default time of B. The $PV^A(t) = \bar{B}^0_{protection}(t) - y(t_0)B^0_{fee}(t)$ is the present value of the CDS contract for the investor A and t_0 is the time of the beginning of the CDS contract. Following the Corollary 3.1 by Kraft and Steffenson (2006), this option is the payment upon transition. The value of this option is given by

$$\tilde{B}^0_{Aopt}(t) = \lambda^{01} \int_t^T max\{PV^A(s), 0\} P(t,s) q^{00}(t,s) ds.$$

On the other hand, if the counterparty B defaults before the reference entity C the contact will be also terminated and the counterpaty B receives an option $max\{PV^B(\tau_B), 0\}$, with τ_B – the default time of B. The $PV^B(t) = y(t_0)B^0_{fee}(t) - \bar{B}^0_{protection}(t)$ is the present value of the CDS contract for the protection seller B and t_0 is time of the beginning of the CDS contract. And again, following the Corollary 3.1 by Kraft and Steffenson (2007), this option is the payment upon transition. The value of this option is given by

$$\tilde{B}^0_{Bopt}(t) = \lambda^{02} \int_t^T max\{PV^B(s), 0\} P(t,s) q^{00}(t,s) ds.$$

As it was mentioned in Kraft and Steffensen (2007), the transition probabilities can be calculated in the following way

$$q^{0k} = \sum_{p(0,k)\in P(0,k)} \lambda^{p(0,k)} g^{p(0,k)(t,T)}$$

with $\lambda^{p(0,0)} = 1$, $p(0,k) = (0, p_1, ..., p_m, k)$ is the path from state 0 to state k, and $\lambda^{p(0,k)} = \lambda^{0p_1}\lambda^{p_1p_2}...\lambda^{p_mk}$. The function g is defined as follows

$$\begin{aligned} g^j(t,T) &= q^{jj}(t,T) = e^{-\lambda^{j*}(T-t)}, \\ g^{jk}(t,T) &= \frac{g^j(t,T) - g^k(t,T)}{\lambda^{k*} - \lambda^{j*}}, \end{aligned}$$

where $\lambda^{n*} = \sum_{i\in\mathcal{J}, i\neq n} \lambda^{ni}$ with $\mathcal{J}$ – set of all states.

The transition probabilities in our cases are defined as:

$$\begin{aligned} q^{00}(t,T) &= e^{-(\lambda^{01}+\lambda^{02}+\lambda^{03})(T-t)}, \\ q^{01}(t,T) &= \lambda^{01} g^{01} = \lambda^{01} \frac{e^{-\lambda^{0*}(T-t)} - e^{-\lambda^{1*}(T-t)}}{\lambda^{1*} - \lambda^{0*}} \\ &= \lambda^{01} \frac{1 - e^{-(\lambda^{01}+\lambda^{02}+\lambda^{03})(T-t)}}{\lambda^{01}+\lambda^{02}+\lambda^{03}}, \\ q^{02}(t,T) &= \lambda^{02} g^{02} = \lambda^{02} \frac{e^{-\lambda^{0*}(T-t)} - e^{-\lambda^{2*}(T-t)}}{\lambda^{2*} - \lambda^{0*}} \\ &= \lambda^{02} \frac{1 - e^{-(\lambda^{01}+\lambda^{02}+\lambda^{03})(T-t)}}{\lambda^{01}+\lambda^{02}+\lambda^{03}}, \\ q^{03}(t,T) &= \lambda^{03} g^{03} = \lambda^{03} \frac{e^{-\lambda^{0*}(T-t)} - e^{-\lambda^{3*}(T-t)}}{\lambda^{3*} - \lambda^{0*}} \\ &= \lambda^{03} \frac{e^{-(\lambda^{01}+\lambda^{02}+\lambda^{03})(T-t)} - e^{-\lambda^{34}(T-t)}}{\lambda^{34} - \lambda^{01} - \lambda^{02} - \lambda^{03}}. \end{aligned}$$

The value of a fair CDS spread for each time t can be found numerically by solving the following equation

$$y(t) B^0_{fee}(t) + \tilde{B}^0_{Bopt}(t) - \bar{B}^0_{protection}(t) - \tilde{B}^0_{Aopt}(t) = 0.$$

In the all earlier models for the CDS rate evaluation the equation for the fair CDS spread has looked like

$$y(t)B^0_{fee}(t) - \bar{B}^0_{protection}(t) = 0.$$

The terms $\tilde{B}^0_{Bopt}(t)$ and $\tilde{B}^0_{Aopt}(t)$ in our case are results of the options for the investor A and counterparty B after their defaults. In all earlier works the existence of these payments for investor and counterparty in the equation of the fair CDS rate was not considered!

For the constant risk-free interest rate our equation for the fair CDS spread can be numerically solved. Consider the numerical example of a CDS contract with 5 years to maturity. Assume that there is constant risk-free interest rate $r = 0.05$ in the market and starting date of the CDS contract is $t_0 = 0$. We compare three settings using three different scenarios:

(i) without counterparty risk and contagion effect,

(ii) with counterparty risk, but without contagion effect,

(iii) with counterparty risk and contagion effect.

In scenario (i), all intensities are set to be zero except $\lambda^C = 0.05$. It means that there are no counterparty risk and no contagion effect, because neither investor A nor counterparty B can default. In the case of scenario (ii), $\lambda^A = 0.01$, $\lambda^B = 0.01$ and $\lambda^C = 0.05$, as well as $\lambda^{B|C} = \lambda^B$. In this case there is counterparty risk, because the probabilities of A's and B's defaults are positive. But, there is no contagion effect because after the default of reference entity C the default probability of the counterparty B stays the same as before the default of C. In case (iii), we set $\lambda^A = 0.01$, $\lambda^B = 0.01$, $\lambda^C = 0.05$, and $\lambda^{B|C} = \lambda^B + 0.01 = 0.02$. In this case there is counterparty risk, because the probability of defaults of the investor and the counterparty are positive. Additionally, there is a contagion effect because default probability of the counterparty B given that the reference entity C has already defaulted is higher as before the default of C. The numerical results can be summarized in the Figure 1.2 where the values of fair CDS rates are given in basis points.

The blue curve corresponds to the case where no counterparty risk and no contagion effects are present. The red curve corresponds to the case where counterparty risk

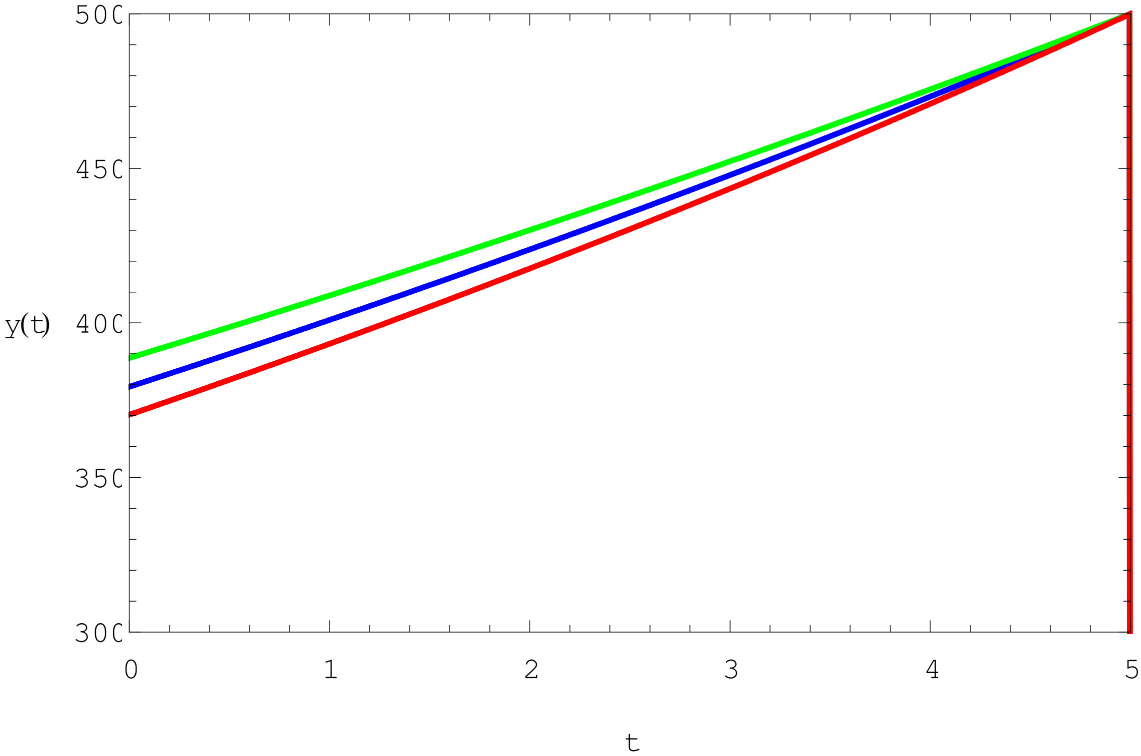

Figure 1.2: CDS rates depending on time. Scenario (i) – blue curve; (ii) – red curve;(iii) – green curve

exists. The green curve corresponds to the case where both counterparty risk and contagion effect are modeled. The value of CDS contracts at time $t = 0$ for λ-values from the example and for the case $\lambda^A = 0$ are given in the following table:

	(i)	(ii)	(iii)
y	389.4	379.3	370.2
$y_{\lambda^A=0}$	389.4	379.4	370.3

From this table we can see that if λ^A is zero, which means that A is default-free, the results are almost the same. Comparing the curves in the Figure 1.2, we can see that in the case without counterparty risk and contagion effect the value of CDS contract is the largest one. Furthermore, the effects of counterparty risk and contagion effect have together an impact of around 5%.

In Kraft and Steffensen (2007) it was assumed that if counterparty B defaults the investor A continues to make his fee payments. In the equation for the fair CDS spread the payments $\tilde{B}^0_{Bopt}(t)$ and $\tilde{B}^0_{Aopt}(t)$ were not involved. We have compared our results with the results obtained in Kraft and Steffensen (2006) for two different cases. In the first case, we assume that the probability that the investor A defaults is zero. In the second case, we assume that both default probabilities of the protection buyer A and the protection seller B are positive. In the first case, we get that CDS rates are almost the same for both compared models for the different parameter sets. It means that implementing such a complicated structure for the price of CDS contracts brings almost the same result as a standard model, described in Kraft and Steffensen (2007). So, in the case where the default probability of the investor A is zero, the standard model proposed by Kraft and Steffensen (2007) can safely be used.

On the other hand, if the investor A can default, the results differ from the standard model described in Kraft and Steffensen (2007). Let us consider the case where both, the protection buyer A and the protection buyer B, can default, in a more detail.

In the following table we compare fair CDS rate for our model and the model by Kraft and Steffenssen for the different given parameter sets at the time moment $t = 0$. In both models we used risk-free interest rate $r = 0.05$ and maturity $T = 5$ years.

	λ^A	λ^B	λ^C	$\lambda^{B\|C}$	y	y^{KS}	color
1	0.01	0.01	0.05	0.01	379.3	378.6	apricot
2	0.01	0.01	0.05	0.02	370.3	369.5	blue
3	0.05	0.01	0.05	0.01	378.9	375.6	red
4	0.05	0.01	0.05	0.02	369.9	366.3	cyan
5	0.01	0.05	0.05	0.01	341.0	340.0	black
6	0.01	0.05	0.05	0.06	334.3	331.8	pink
7	0.1	0.01	0.05	0.01	377.5	371.9	green
8	0.01	0.1	0.05	0.1	297.0	296.0	brawn

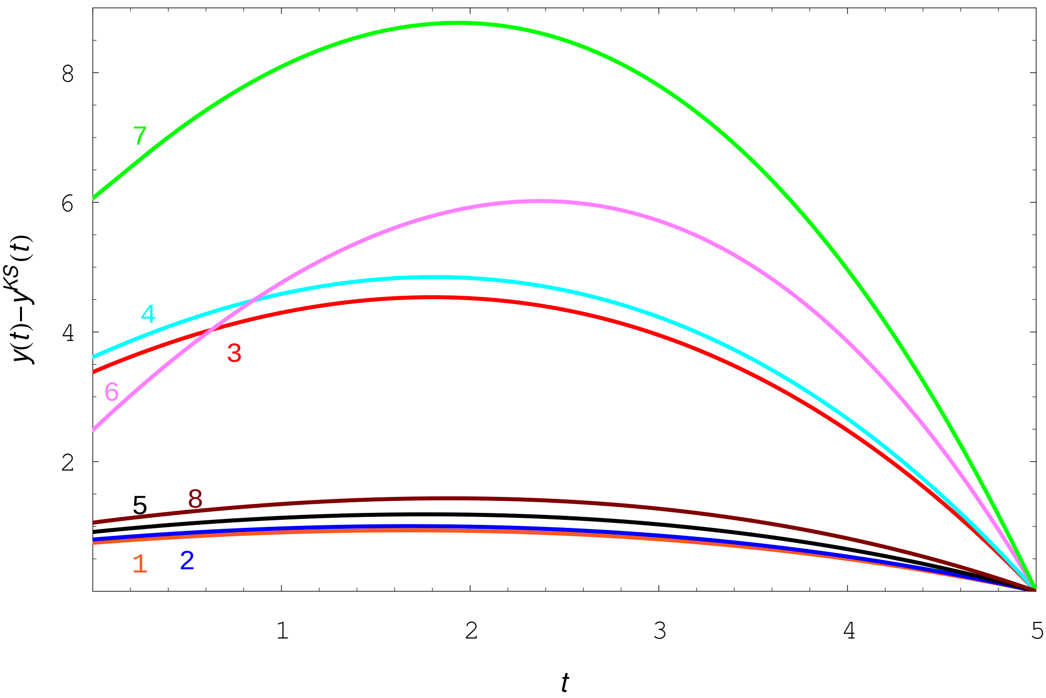

Figure 1.3: **Difference between two models**

From the table above we can see that for the more extreme cases with high probabilities of almost $5-10\%$ that the investor A defaults the difference between two models becomes more significant. If the probability of default of the counterparty B becomes bigger, the difference between the two models does not increase much. The existence of contagion effects makes the difference between the two models bigger. The Figure 1.3 summarizes the difference between CDS rates in our model y and CDS rates in Kraft-Steffensen model y^{KS} for the parameter sets given in the table. The colors of the corresponding curves are also given in the table.

1.2.2 Protection Payment at Default

In this section, we consider the case where the contact is settled upon default. It means that the protection seller B pays the protection payment immediately after the reference entity C defaults. The corresponding Markov chain model is shown in

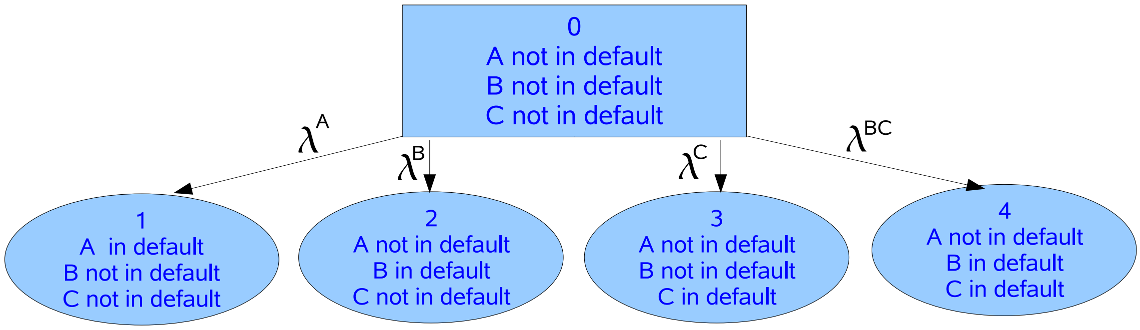

Figure 1.4: **Markov chain describing the CDS contract settled at default**

the Figure 1.4.

Same as before, the protection buyer A continuously makes fee payments to the protection seller B at payment dates until the reference entity C defaults, or until the maturity T. On the other hand, the protection seller B makes only a protection payment to the investor A at the time of default of the reference entity C. The investor A stops making premium payments, if the counterparty B defaults or the investor A itself defaults. In both these cases the contract is terminated. State 0 is the initial state at the beginning of the CDS contract where all companies are not in default. State 1 describes the situation when investor A defaults. State 2 describes the case when counterparty B defaults. State 3 describes the event of default of the reference entity C. In our case the protection payment is made at default time of the reference entity C. On the contrary to the Figure 1.1, the transition from state 3 to the state, where the counterparty B defaults after the reference entity C (state 4 in the Figure 1.1), is irrelevant for us in this case. In order to consider also the case that both parties, the protection seller B and the reference entity C, default simultaneously, we introduce an additional state 4, where we have the both market participants defaulted. In the previously described model (Figure 1.1) the case that both the protection seller B and the reference entity C default was also included. The λs in the Figure 1.4 are the default intensities for the corresponding transitions between states.

According to the Corollary 3.1 by Kraft and Steffensen (2007), the protection payments in this case needs to be modeled as payments upon transmission. The same as before the fee leg can be modeled as a state-dependent coupon payment.

Again, let $y(t)$ denote the fair CDS spread and assume that all intensities λ^{jk} are constant. The default intensities of A, B, and C are denoted by $\lambda^A = \lambda^{01}$, $\lambda^B = \lambda^{02}$, and $\lambda^C = \lambda^{03}$. The simultaneous default intensity of B and C is denoted by $\lambda^{BC} = \lambda^{04}$.

Then according to Kraft and Steffensen (2007), the value of the fee leg can be calculated as

$$y(t)B^0_{fee}(t) = y(t) \sum_{k \in \mathcal{J}^D_F} \int_t^T P(t,s)q^{0k}(t,s)ds$$

with $P(t,s)$ – the value of zero-coupon bond, $q^{0k}(t,s)$ – transition probability from state 0 to state k. Furthermore, $\mathcal{J}^D_F = \{0,3\}$ – the set of states where the investor A pays fee payments. The value of the protection leg is given by

$$\bar{B}^0_{protection}(t) = \lambda^{03} \int_t^T P(t,s)q^{00}(t,s)ds.$$

As we have said in the description of our model, if investor A defaults before the reference entity C the contract will be terminated and A will receive the following option: $max\{PV^A(\tau_A), 0\}$, with τ_A – the default time of investor A. The $PV^A(t) = \bar{B}^0_{protection}(t) - y(t_0)B^0_{fee}(t)$ is the present value of the CDS contract for the investor A and t_0 is the time of the beginning of the CDS contract. Following the Corollary 3.1 by Kraft and Steffenson (2007), this option is the payment upon transition. The value of this option is given by

$$\tilde{B}^0_{Aopt}(t) = \lambda^{01} \int_t^T max\{PV^A(s), 0\}P(t,s)q^{00}(t,s)ds.$$

On the other hand, if the counterparty B defaults before the reference entity C the contact will be also terminated and the counterparty B receives the option $max\{PV^B(\tau_B), 0\}$, with τ_B – the default time of B. The $PV^B(t) = y(t_0)B^0_{fee}(t) - \bar{B}^0_{protection}(t)$ is the present value of the CDS contract for the protection seller B and t_0 is time of the beginning of the CDS contract. And again, following the Corollary 3.1 by Kraft and Steffenson (2007), this option is the payment upon transition. The value of this option is given by

$$\begin{aligned}\tilde{B}^0_{Bopt}(t) &= \lambda^{02}\int_t^T max\{PV^B(s),0\}P(t,s)q^{00}(t,s)ds \\ &\quad + \lambda^{04}\int_t^T max\{PV^B(s),0\}P(t,s)q^{00}(t,s)ds \\ &= \lambda^{02}\int_t^T max\{PV^B(s),0\}P(t,s)q^{00}(t,s)ds.\end{aligned}$$

The second term in above equation vanishes, because if protection seller and reference entity default simultaneously, the present value of the CDS contract cannot be positive for B. After the default of C, investor A needs not to pay fee payments anymore, but the counterparty B should pay the protection payment at this moment, so the value of the CDS contract for the protection seller can be only negative in this moment.

According to the Corollary 3.1 in Kraft and Steffensen (2007), the correction payments in the default of the investor A or the counterparty B are defined as payments upon transition; their value equal to

$$\tilde{B}^0_{Bopt}(t) = \lambda^{02}\int_t^T max\{PV^B(s),0\}P(t,s)q^{00}(t,s)ds$$

with $PV^B(t) = y(t)B^0_{fee}(t) - \bar{B}^0_{protection}(t)$ and

$$\tilde{B}^0_{Aopt}(t) = \lambda^{01}\int_t^T max\{PV^A(s),0\}P(t,s)q^{00}(t,s)ds$$

with $PV^A(t) = \bar{B}^0_{protection}(t) - y(t)B^0_{fee}(t)$.

As previously, the CDS rate is given as a solution of the following equation:

$$y(t)B^0_{fee}(t) + \tilde{B}^0_{Bopt}(t) - \bar{B}^0_{protection}(t) - \tilde{B}^0_{Aopt}(t) = 0.$$

In the all earlier models for the CDS rate evaluation the equation for the fair CDS spread has looked like

$$y(t)B^0_{fee}(t) - \bar{B}^0_{protection}(t) = 0.$$

The terms $\tilde{B}^0_{Bopt}(t)$ and $\tilde{B}^0_{Aopt}(t)$ in our case are results of the options for the investor A and counterparty B after their defaults. In all earlier works the existence of these payments for investor and counterparty in the equation of the fair CDS rate was not considered!

Therefore, following Kraft and Steffensen (2007), we can find the transition probabilities

$$q^{0k} = \sum_{p(0,k)\in P(0,k)} \lambda^{p(0,k)} g^{p(0,k)(t,T)}$$

with $\lambda^{p(0,0)} = 1$, $p(0,k) = (0, p_1, ..., p_m, k)$ is the path from state 0 to state k, and $\lambda^{p(0,k)} = \lambda^{0p_1}\lambda^{p_1p_2}...\lambda^{p_mk}$. The function g is defined as follows

$$\begin{aligned}
g^j(t,T) &= q^{jj}(t,T) = e^{-\lambda^{j*}(T-t)}, \\
g^{jk}(t,T) &= \frac{g^j(t,T) - g^k(t,T)}{\lambda^{k*} - \lambda^{j*}},
\end{aligned}$$

where $\lambda^{n*} = \sum_{i\in\mathcal{J}, i\neq n} \lambda^{ni}$ with $\mathcal{J}$ – set of all states.

The transition probabilities in our cases are defined as:

$$\begin{aligned}
q^{00}(t,T) &= e^{-(\lambda^{01}+\lambda^{02}+\lambda^{03}+\lambda^{04})(T-t)}, \\
q^{01}(t,T) &= \lambda^{01}g^{01} = \lambda^{01}\frac{e^{-\lambda^{0*}(T-t)} - e^{-\lambda^{1*}(T-t)}}{\lambda^{1*} - \lambda^{0*}} \\
&= \lambda^{01}\frac{1 - e^{-(\lambda^{01}+\lambda^{02}+\lambda^{03}+\lambda^{04})(T-t)}}{\lambda^{01} + \lambda^{02} + \lambda^{03} + \lambda^{04}}, \\
q^{02}(t,T) &= \lambda^{02}g^{02} = \lambda^{02}\frac{e^{-\lambda^{0*}(T-t)} - e^{-\lambda^{2*}(T-t)}}{\lambda^{2*} - \lambda^{0*}} \\
&= \lambda^{02}\frac{1 - e^{-(\lambda^{01}+\lambda^{02}+\lambda^{03}+\lambda^{04})(T-t)}}{\lambda^{01} + \lambda^{02} + \lambda^{03} + \lambda^{04}}, \\
q^{03}(t,T) &= \lambda^{03}g^{03} = \lambda^{03}\frac{e^{-\lambda^{0*}(T-t)} - e^{-\lambda^{3*}(T-t)}}{\lambda^{3*} - \lambda^{0*}} \\
&= \lambda^{03}\frac{1 - e^{-(\lambda^{01}+\lambda^{02}+\lambda^{03}+\lambda^{04})(T-t)}}{\lambda^{01} + \lambda^{02} + \lambda^{03} + \lambda^{04}}, \\
q^{04}(t,T) &= \lambda^{04}g^{04} = \lambda^{04}\frac{e^{-\lambda^{0*}(T-t)} - e^{-\lambda^{4*}(T-t)}}{\lambda^{4*} - \lambda^{0*}} \\
&= \lambda^{04}\frac{1 - e^{-(\lambda^{01}+\lambda^{02}+\lambda^{03}+\lambda^{04})(T-t)}}{\lambda^{01} + \lambda^{02} + \lambda^{03} + \lambda^{04}}.
\end{aligned}$$

We apply our previous numerical example to this case where the CDS contract is settled upon default. In the same way as before, in scenario (i) $\lambda^C = 0.05$ and other intensities are set to be zero. It means that there are no counterparty risk and no contagion effect, because investor A and counterparty B are default-free. In case scenario (ii) the $\lambda^A = 0.01$, $\lambda^B = 0.01$, $\lambda^C = 0.05$, and $\lambda^{BC} = \lambda^B$. In this case there are counterparty risk, because the probabilities of A's and B's defaults are positive. But, there is no contagion effect because default intensity of reference entity C with the counterparty B is the same as default intensity of B. In case (iii) we set $\lambda^A = 0.01$, $\lambda^B = 0.01$, $\lambda^C = 0.05$, and $\lambda^{BC} = \lambda^B + 0.01 = 0.02$. In this case there are again the counterparty risk, because the probability of defaults of the investor and counterparty are positive. Additionally, there is a contagion effect because default probability of the counterparty B with the reference entity C is higher as the probabiity of default of counterparty B alone. The numerical results are summarized in the Figure 1.5 where the values of fair CDS rates are given in basis points.

The blue curve corresponds to the case where no counterparty risk and no contagion effects are present. The red curve corresponds to the case where counterparty risk is involved. The green curve corresponds to the case where both counterparty risk and contagion effect are present. The values at time $t = 0$ of the CDS contracts are given in the following table:

	(i)	(ii)	(iii)
y	444.7	456.0	455.7
$y_{\lambda^A=0}$	444.7	444.0	443.6

Comparing the curves in the Figure 1.5 we can see that in the case without counterparty risk and contagion effect the value of the fair CDS rate is the smallest one. Furthermore, we can see that if the investor A is default-free, the impact of the counterparty risk and the contagion effect is not as significant as in the previous case studied in the previous section. In general situation, where the probability of default of the investor A is positive, the impact of the counterparty risk is considerable; at the same time the impact of the contagion effect is almost negligible.

In the same way as before, in the following table we compare our results with results obtained in Kraft and Steffensen (2007) for different parameter sets. For all parameter sets we set maturity to $T = 5$ and risk-free interest rate to $r = 0.05$.

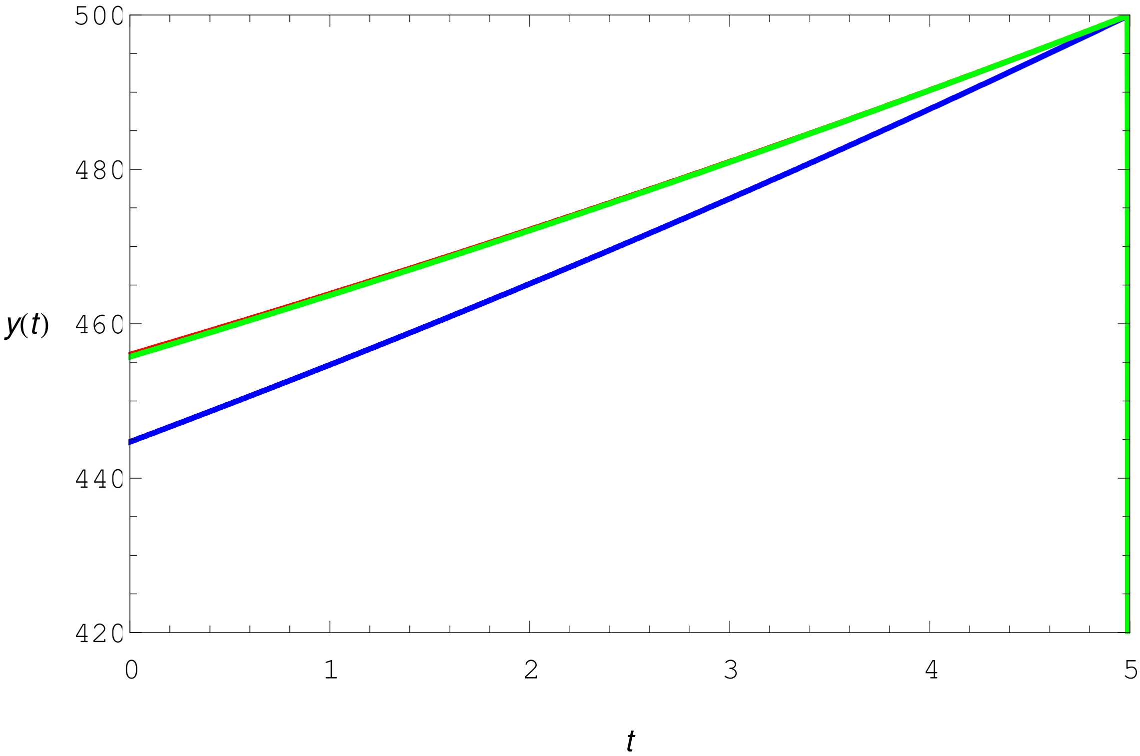

Figure 1.5: CDS rates depending on time. Scenario (i) – blue curve; (ii) – red curve; (iii) – green curve

	λ^A	λ^B	λ^C	$\lambda^{B\|C}$	y	y^{KS}	color
1	0.01	0.01	0.05	0.01	456.0	443.6	apricot
2	0.01	0.01	0.05	0.05	454.9	442.2	blue
3	0.05	0.01	0.05	0.01	501.2	442.2	red
4	0.05	0.01	0.05	0.05	500.9	439.2	cyan
5	0.02	0.1	0.05	0.05	465.0	438.9	black
6	0.02	0.1	0.05	0.1	564.1	437.3	pink
7	0.1	0.01	0.05	0.01	554.5	439.9	green
8	0.03	0.1	0.05	0.1	476.6	437.0	brawn

From the table we can see that there is a big difference between two models. For the more extreme cases with the high probabilities of almost $5-10\%$ that the investor A

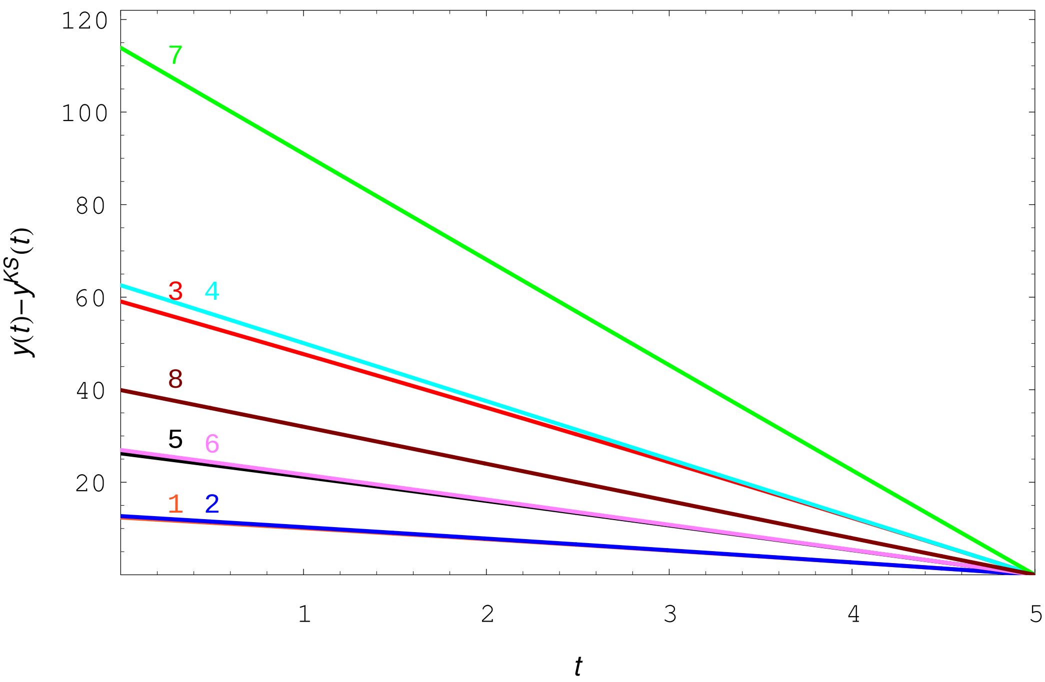

Figure 1.6: **Difference between two models**

defaults the difference between the two models becomes more than 100 basis points. In general, for all parameter sets the difference is more than 10 basis points. The Figure 1.6 summarizes the difference between CDS rates in our model y and CDS rates in Kraft-Steffensen model y^{KS} for the parameter sets given in the table. The colors of the corresponding curves are also given in the table.

Comparing cases 1 and 2 we again can see that the impact of the contagion effect on the CDS rates is only around 1 basis point even, if the probability of default of the counterparty B with the reference entity C is high, ($\lambda^{BC} = 0.05$). This means that influence of the contagion effects in this model is not very essential.

1.3 Conclusions

Our analysis confirms the fact that counterparty risk and contagion effects cannot be ignored for the calculation of fair CDS spreads. The same results were also obtained among others in Kraft and Steffensen (2007), and Yu (2005).

If the contact is settled at maturity and both λ^A and λ^B are relatively small (near $1-4\%$), the standard model can be used, because the difference between two models in this situation is negligible. The standard model can also be used if the probability of default of the investor A is low.

On the other hand, if the contract is settled at default, the difference between the two models becomes huge (in some cases more than 100 basis points). In this case, our model should be used to get correct fair spread.

Chapter 2

The Copula and its Properties

2.1 Mathematical and Statistical definition of Copulas

The copula concept is based on a separate statistical treatment of dependence and marginal behavior. The mathematical idea goes back to Sklar (1959), Sklar (1973) and Hoeffding (1940).

For a short introduction about the copula the reader should to refer to Genest and McKay (1986b), for an extensive review on that topic the reader should refer to Nelsen (1998), Joe (1997) and Schweizer (1991) or for their application in risk management to Rogge and Schönbucher (2003) or Embrechts, Lindskog and Mc Neil (2003). The first part of this section will be devoted to some notions in statistics; the interpretation of copula in the mathematical sense and then in term of random variable; and finally we will conclude this section with some important properties of copula functions.

Let us now first refresh some important statistical concepts. We will start by the notion of distribution and joint distribution function since they are at the cornerstone of the copula theory.

Definition 2.1.1 *A* ***distribution function*** *is a function F with domain $\mathbb{R}$ such that*

1. F is nondecreasing 2. $F(-\infty) = 0,\ F(+\infty) = 1.$

Definition 2.1.2 *A **joint (bivariate) distribution function** is a function H with domain $\mathbb{R}^2$ such that*

1. H is 2-increasing

2. $H(x, -\infty) = H(-\infty, y) = 0,\ H(+\infty, +\infty) = 1.$

In the last definition, the first condition simply stipulates that for every $x_1 \leq x_2$ and $y_1 \leq y_2$, $H(x_1, y_1) \leq H(x_2, y_2)$. While the second condition states that H should be bounded.

Definition 2.1.3 *Let S_1 and S_2 be nonempty subsets of $\bar{\mathbb{R}}$ and let H be a function such that $DomH = S_1 \times S_2$. Let $B = [x_1, x_2] \times [y_1, y_2]$ be a rectangle all of whose vertices are in $DomH$. Then the $H-$**volume of** B is given by*

$$V_H(B) = H(x_2, y_2) - H(x_2, y_1) - H(x_1, y_2) + H(x_1, y_1).$$

Note that $V_H(B)$ is also the $H-$mass 0f the rectangle $B = [x_1, x_2] \times [y_1, y_2]$.

Let us now move to the main topic of this section which is the definition of the notion of copula (in the bivariate case for the tractability of notation).

Definition 2.1.4 *A $2-$**copula** is a function $C : [0, 1]^2 \to [0, 1]$ satisfying:*

(i) Boundary conditions

$$C(0, x_2) = C(x_1, 0) = 0 \qquad \textit{for all} \qquad x_1, x_2 \in [0, 1] = I$$

and

$$C(x_1, 1) = x_1 \qquad \textit{and} \qquad C(1, x_2) = x_2 \qquad \textit{for all} \qquad x_1, x_2 \in [0, 1].$$

(ii) Monotonicity conditions

$$C(x_1, y_1) + C(x_2, y_2) - C(x_1, y_2) - C(x_2, y_1) \geq 0$$

for all $x_1, x_2, y_1, y_2 \in [0, 1]$ satisfying $x_1 \leq y_1$ and $x_2 \leq y_2$.

The last condition implies the continuity of C. While once again, the first part of the definition states that the 2-copula is a bounded function; the latter ensures the volume engendered by the rectangle $[x_1, x_2] \times [y_1, y_2]$ (or the 2-copula) is never negative. The copula can then be interpreted as a joint distribution. For the more pragmatic inclined; since the 2-copula can be seen as a volume in I, the form of the copula is the shape of a skewed continuous surface on the unit square which vertices belong to the unit cube.

Copulas are of interest because they link joint distributions to the one-dimensional marginal distributions. Sklar proved relation between copula and joint and margin distribution functions.

Theorem 2.1 (Sklar) *Let X_1, X_2 be a random variables with distribution functions F_1 and F_2, respectively, and joint distribution function F_{12}. Then there exists a copula C such that for all $x, y \in \overline{\mathbb{R}}$*

$$F_{12}(x, y) = C(F_1(x), F_2(y)). \tag{2.1}$$

If F_1 and F_2 are continuous, then C is unique; otherwise, C is uniquely determined on $RanF_1 \times RanF_2$. Conversely, if C is a copula, F_1 and F_2 are distribution functions, then function F_{12} defined by (2.1) is a joint distribution function with margins F_1 and F_2.

This Theorem first appeared in Sklar (1959). The name ”copula” was chosen to emphasize the manner in which a copula ”couples” a joint distribution function to its univariate margins. Copulas thus capture all of the information concerning the dependance structure of random variables irrespective of their distributions and so provide a natural framework for many investigations.

One example is to set F_1 and F_2 to be exponential margins and H to be the Gaussian link function with a given covariance matrix, R and call this bivariate distribution Z. As explained in Schönbucher (2003) in order to construct this copula one has to sample a vector of observation X from a multivariate Gaussian distribution with covariance matrix R. Then transform this vector X into a vector U by setting $u_i = \Phi(x_i)$; and finally compute the vector Y where $y_i = ln(u_i)$. Then we have that y_i follows the distribution Z.

By the Theorem 2.1, if the random variables X_1 and X_2 are continuous with joint distribution function F_{12}, then the copula is unique determined by (2.1). If, however, the random variables are not continuous, the copula C is not unique; in this case, the values of the copula are uniquely determined at points (x_1, x_2), where x_k is in range of F_k, $k = 1, 2$, and a copula C for which the expression above holds can be obtained by interpolating the values at these points in any manner consistent with the defining properties of a copula. Interpolation which is linear in each place ("bilinear interpolation") works, and we adopt the convention that bilinear interpolation is always used to fill in values at other points. With this convention we can refer to the copula of X_1 and X_2.

Now we can define $m-$dimensional copula function:

Definition 2.1.5 *For $m \geq 3$ an $m-$**copula** is a function $C : [0,1]^m \to [0,1]$ satisfying:*

(i) Boundary conditions

(a) $C(x_1, ..., x_{i-1}, 0, x_{i+1}, ..., x_m) = 0$ for all i and for all $x_1, ..., x_m \in [0,1]$;

(b) The function

$$(x_1, ..., x_{i-1}, x_{i+1}, ..., x_m) \to C(x_1, ..., x_{i-1}, 1, x_{i+1}, ..., x_m)$$

is an $(m-1)-$copula for all i.

(ii) Monotonicity conditions

$$\sum_{V \in R} sgn(V) C(V) \geq 0$$

for all rectangles R of the form $R = \prod_{i=1}^{m} [x_i, y_i]$, $x_i \leq y_i$.

Here, the sum is over all vertices $V = (\varepsilon_1, ..., \varepsilon_m)$ of the rectangle, where $\varepsilon_i = x_i$ or y_i, and

$$sgn(V) = \begin{cases} -1, & \text{if the number of } x_i\text{'s among the coordinates } V \text{ is odd,} \\ 1, & \text{otherwise} \end{cases}$$

Again, these conditions imply the continuity of C.

Theorem 2.2 (Sklar) *Let $X_1, ..., X_m$ be a random variables with distribution functions $F_1, ..., F_m$ respectively, and joint distribution function F. Then there exists an $m-$copula C such that*

$$F(x_1, ..., x_m) = C(F_1(x_1), ..., F_m(x_m)). \tag{2.2}$$

If $F_1, ..., F_m$ are all continuous, C is unique. Otherwise C is uniquely determined on $RanF_1 \times RanF_2 \times ... \times RanF_m$.

The Sklar's Theorems imply that for continuous multivariate distribution functions the univariate marginals and the dependence structure (encoded into the copula) can be separated in a unique way.

By the Theorem 2.2 if the random variables are all continuous, the $m-$copula in (2.2) is uniquely determined; otherwise it is uniquely determined at points $(x_1, ..., x_m)$, where x_k is in the range of F_k, $k = 1, ..., m$, and as before can be obtained at other points by interpolation. Here $m-$linear interpolation works, and we adopt the convention that it is always used. For discuss of this issues see Sklar (1956), Sklar (1973), Schweizer and Sklar (1974) and Schweizer and Sklar (1983).

Definition 2.1.6 *Let F be a distribution function. Then a* ***quasi-inverse*** *of F is any function $F^{(-1)}$ with domain I such that*

(i) if t is in $RanF$, then $F^{(-1)}(t)$ is any number x in $\overline{\mathbb{R}}$ such that $F(x) = t$ i.e., for all $t \in RanF$

$$F(F^{(-1)}(t)) = t$$

(ii) if t is not in $RanF$, then

$$F^{(-1)}(t) = inf\{x|F(x) \geq t\} = sup\{x|F(x) \leq t\}.$$

If F is strictly increasing, then it has but a single quasi-inverse, which is of course the ordinary inverse, for which we use a customary notation F^{-1}.

Corollary 2.1.1 *Let F, C, $F_1, ..., F_m$ be as in previous theorem, and let $F_1^{(-1)}$, ..., $F_m^{(-1)}$ be quasi-inverses of $F_1, ..., F_m$ respectively. Then for any $u \in I^m$*

$$C(u_1, ...u_m) = F\left(F_1^{(-1)}(u_1), ..., F_n^{(-1)}(u_n)\right).$$

Note that for any n-dimensional copula each k-dimensional margin of the copula is a k-dimensional copula itself ($1 \leq k \leq n$). The set of copulas is convex in the sense that every convex linear combination of copulas is a copula itself (Nelsen (1999), Ex. 2.3, p. 12).

Corollary 2.1.2 *Let $X_1, X_2, .., X_n$ are continuous independent random variables for $n \geq 2$ with joint distribution function*

$$F(x_1, x_2, ..., x_n) = F_1(x_1) \times F_2(x_2) \times ... \times F_n(x_n).$$

Then the copula for these independent variables is called the product copula

$$\Pi(u) = u_1 u_2 ... u_n.$$

After introducing the product copula one might naturally think about the existence of some lower and upper bounds around the product copula according to the dependence structure between the two random variables X and Y . These bounds are referred as the Fréchét-Hoeffding bonds for joint distribution functions of random variables.

Theorem 2.3 (Fréchét-Hoeffding) *Let X and Y be random variables with respectively marginal distribution functions F and G; and with joint distribution H. Then for all, x and y in $\mathbb{R}$*

$$max(F(x) + G(y) - 1, 0) \geq H(x, y) \geq min(F(x), G(y))$$

and

1. *Y is a.s an increasing function of X iif $H(x, y) = min(F(x), G(y))$*

2. *Y is a.s an decreasing function of X iif $H(x, y) = max\{(F(x) + G(y) - 1, 0\}$*

Corollary 2.1.3 *If u and v are uniform random variable, for all u and v in I*

$$W(u,v) = max(u+v-1,0) \leq C(u,v) \leq min(u,v) = M(u,v)$$

Proof: Lets take (u,v) an arbitrary point from $DomC$. Since $C(u,v) \leq C(1,v) = v$ and $C(u,v) \leq C(u,1) = u$ follows that $C(u,v) \leq min(u,v)$. Furthermore $V_C([u,1] \times [v,1]) \geq 0$ implies $C(u,v) \geq u+v-1$, which when combined with $C(u,v) \geq 0$ yields $C(u,v) \geq max(u+v-1,0)$. □

More general, let $M(u) = min\{u_1,...,u_n\}$ and $W(u) = max\{u_1+u_2+...+u_n-n+1,0\}$. The function $M(u)$ is copula function for all $n \geq 2$, whereas $W(u)$ is a copula only for $n=2$, but not for $n>2$. The functions W and M are known as *Fréchét-Hoeffding bounds* since for any vector $u=(u_1,...,u_n)$

$$W(u) \leq C(u) \leq M(u).$$

Note that both bounds are (pointwise) sharp. If we now look at the shape of the surface defined by the copula, we can say that this shape is bounded by the two Fréchét-Hoeffding bounds which are functions in the unit cube. To conclude the first part of this section, we introduce the second most interesting property of the copula (after the Sklar Theorem): their invariance to strictly increasing transformations and predictable behavior for more general strictly monotone transformations.

Definition 2.1.7 *If C_1 and C_2 are copulas, C_1 is* ***smaller*** *than C_2 (written $C_1 \prec C_2$) if*

$$C_1(\mathbf{u}) \leq C_2(\mathbf{u}),$$

for all $\mathbf{u}$ in $[0,1]^n$.

Theorem 2.4 *Let X,Y be continuous random variable with copula $C_{X,Y}$, α and β be strictly increasing functions on $RanG \times RanF$, then $C_{\alpha(X),\beta(Y)} = C_{X,Y}$. So that $C_{X,Y}$ is invariant under strictly increasing transformation of X and Y.*

Proof: Let X and Y have distributions function F and G and let $\alpha(X)$ and $\beta(Y)$ have distribution function L and M. If we now set α as (a monotonic) increasing function, we have the following expressions for the transformation of marginal distribution of X,

$$L(x) = P\{\alpha(X) \leq x\} = P\left\{X \leq \alpha^{-1}(x)\right\} = F(\alpha^{-1}(x)).$$

When looking at the transformation of the copula, we have

$$\begin{aligned}
C_{\alpha(X),\beta(Y)}(L(X), M(Y)) &= P\left\{\alpha(X) \leq x, \beta(Y) \leq y\right\} \\
&= P\left\{X \leq \alpha^{-1}(x), Y \leq \beta^{-1}(y)\right\} \\
&= C_{X,Y}(\alpha^{-1}(x), \beta^{-1}(y)) \\
&= C_{X,Y}(L(X), M(Y)). \qquad \square
\end{aligned}$$

When monotone transformations (but not increasing) are applied to copulas, the previous results do not hold anymore but nonetheless we can still make the following statement about the copula behavior.

Theorem 2.5 *Let X, Y be continuous random variables with copula $C_{X,Y}$. α and β be strictly monotone functions on $RanG \times RanF$,*

1. *If α is strictly increasing and β is strictly decreasing then*

$$C_{\alpha(X),\beta(Y)}(u, v) = u - C_{X,Y}(u, 1 - v)$$

2. *If β is strictly increasing and α is strictly decreasing then*

$$C_{\alpha(X),\beta(Y)}(u, v) = v - C_{X,Y}(1 - u, v)$$

3. *If α and β are both strictly decreasing then*

$$C_{\alpha(X),\beta(Y)}(u, v) = u + v - 1 - C_{X,Y}(1 - u, 1 - v)$$

Proof: Let X, Y follow the distributions F and G while $\alpha(X)$ and $\beta(X)$ follows the distribution L and M, with α and β monotone function so that,

$$C_{\alpha(X),\beta(Y)}\left(L(X), M(Y)\right) = P\{\alpha(X) \leq x; \beta(Y) \leq y\}.$$

So, when α is a strictly decreasing function and β is a strictly increasing function. Using the property of probability function $P(A^C \cap B) = P(B) - P(A \cap B)$ we get:

$$\begin{aligned}
C_{\alpha(X),\beta(Y)} &= P\{\alpha(X) \leq x; \beta(Y) \leq y\} \\
&= P\{\beta(Y) \leq y\} - P\{X \leq \alpha^{-1}(x); \beta(Y) \leq y\}
\end{aligned}$$

$$\begin{aligned}
&= C_{\beta(Y)}(M(Y)) - C_{X,\beta(Y)}(F(\alpha^{-1}(x)), M(y)) \\
&= C_{\beta(Y)}(M(Y)) - C_{X,\beta(Y)}(1 - L(x), M(y)) \\
&= v - C_{X,\beta(Y)}(1 - u, v) \\
&= v - C_{X,Y}(1 - u, v).
\end{aligned}$$

When α and β are both strictly decreasing functions, using the property of probability function $P(A^C \cap B^C) = 1 - P(A \cup B) = 1 - P(A) - P(B) - P(A \cap B)$, we get:

$$\begin{aligned}
C_{\alpha(X),\beta(Y)} &= P\{\alpha(X) \leq x; \beta(Y) \leq y\} \\
&= P\{X \leq \alpha^{-1}(x); Y \geq \beta^{-1}(y)\} \\
&= 1 - F(\alpha^{-1}(x)) - G(\beta^{-1}(y)) - C_{X,Y}(\alpha^{-1}(x), \beta^{-1}(y)) \\
&= 1 - (1-u) - (1-v) - C_{X,Y}(1-u, 1-v) \\
&= 1 + u + v - C_{X,Y}(1-u, 1-v). \qquad \square
\end{aligned}$$

2.1.1 Survival Copulas

In many applications, the random variables of interest are present the lifetimes of individuals or objects in some population. The probability of an individual living or surviving beyond time x is given by survival function (or survivor function, or reliability function) $\bar{F}(x) = P(X > x) = 1 - F(x)$, where F denotes the distribution function of X. When dealing with lifetimes, the natural range of random variable is often $[0, \infty)$; however, we will use the term "survival function" for $P(X > x)$ even if there range is $\bar{\mathbb{R}}$.

For a pair (X, Y) of random variables with joint distribution function H, the *joint survival function* is given by $\bar{H}(x, y) = P(X > x, Y > y)$. The margins of $\bar{H}$ are the functions $\bar{H}(x, -\infty)$ and $\bar{H}(-\infty, y)$, which are the univariate survival functions $\bar{F}$ and $\bar{G}$, respectively. A natural question is the following: Is there a relationship between univariate and joint distribution functions, as embodied in Sklar's Theorem 2.1. To answer this question, suppose that the copula of X and Y is C. Then we have

$$\bar{H}(x, y) = P(X > x, Y > y) = P(X > x) - P(X > x, Y \leq y)$$

$$
\begin{aligned}
&= 1 - P(X \le x) - (P(Y \le y) - P(X \le x, Y \le y)) \\
&= 1 - F(x) - G(y) + H(x,y) \\
&= \bar{F}(x) + \bar{G}(y) - 1 + C\left(1 - \bar{F}(x), 1 - \bar{G}(y)\right),
\end{aligned}
$$

so that if we define a function $\hat{C}$ from $\mathbf{I}^2$ into $\mathbf{I}$ by

$$\hat{C}(u,v) = u + v - 1 + C(1-u, 1-v),$$

then we have

$$\bar{H}(x,y) = \hat{C}(\bar{F}(x), \bar{G}(y)).$$

$\hat{C}$ is a copula (by Ex. 2.6.1 and Th.2.4.4 part 3, Joe(1997)). We refer to $\hat{C}$ as the *survival copula* of X and Y. Secondly, notice that the $\hat{C}$ "couples" the joint distribution function to its univariate margins in a manner completely analogous to the way in which a copula connects the joint distribution function to its margins.

Remark 2.6 In the same way, one can get a survival Copula for three random variables X, Y and Z with corresponding marginal distribution functions $F(x)$, $G(y)$ and $R(z)$. A function $\hat{C}$ from $\mathbf{I}^3$ into $\mathbf{I}$ is defined by

$$
\begin{aligned}
\hat{C}(u,v,w) &= u + v + w - 2 + C(1-u, 1-v, 1) + C(1, 1-v, 1-w) \\
&+ C(1-u, 1, 1-w) - C(1-u, 1-v, 1-w). \qquad (2.3)
\end{aligned}
$$

Proof:

$$
\begin{aligned}
\bar{H}(x,y,z) &= P(X > x, Y > y, Z > z) \\
&= P(X > x, Y > y) - P(X > x, Y > y, Z \le z) \\
&= P(X > x) - P(X > x, Y \le y) - (P(X > x, Z \le z) \\
&\quad -P(X > x, Y \le y, Z \le z)) \\
&= 1 - P(X \le x) - (P(Y \le y) - P(X \le x, Y \le y)) \\
&\quad -(P(Z \le z) - P(X \le x, Z \le z)) \\
&\quad -(P(Y \le y, Z \le z) - P(X \le x, Y \le y, Z \le z)) \\
&= \bar{F}(x) + \bar{G}(y) + \bar{R}(z) - 2 + C\left(1 - \bar{F}(x), 1 - \bar{G}(y)\right) \\
&\quad +C\left(1 - \bar{F}(x), 1 - \bar{G}(y), 1\right) + C\left(1 - \bar{F}(x), 1, 1 - \bar{R}(z)\right) \\
&\quad +C\left(1, 1 - \bar{G}(y), 1 - \bar{R}(z)\right) - C\left(1 - \bar{F}(x), 1 - \bar{G}(y), 1 - \bar{R}(z)\right). \square
\end{aligned}
$$

2.2 Dependence Concept

We concluded the last section by the copula invariance property to strictly increasing functions. It is worthwhile to be noted that this property is not shared by the well know (multivariate) elliptical distribution such as the Gaussian and Student ones. Furthermore as noted by Embrechts, Lindskog and Mc Neil (2003) *because "(. . .) most random variables are not jointly elliptically distributed and using linear correlation as a measure of dependence in such situation might prove very misleading"*

Let us recall that for Normally distributed random variable, the independence between random variables is equivalent to a Pearson correlation coefficient equal to zero. But this equivalence does not hold if the random variables fail to verify the normality assumption. This citation and remark provides us with the scope of this second part of this section, namely, documenting dependence measures between random variables. For an overview of that topic the reader should refer to Kruksal (1958) or for specific applications to copula to Schweizer and Wolf (1981). Following the properties of copulas, the more interesting measures will be the ones which can be solely defined in term of copula. Let us first introduce the notion of linear correlation since it will be used in the next sections.

Definition 2.2.1 *Let X and Y follow, respectively, the distribution F and G and jointly follow the distribution function H; then linear correlation coefficient ρ, for X and Y is defined as*

$$\rho(X,Y) = \frac{1}{\sqrt{Var(X)}\sqrt{Var(Y)}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left[H(x,y) - F(x)G(y)\right] dxdy$$

or if we use the fact that $u = F(x)$ and $v = G(y)$,

$$\rho(X,Y) = \frac{1}{\sqrt{Var(X)}\sqrt{Var(Y)}} \int_{0}^{1}\int_{0}^{1} \left[C(u,v) - uv)\right] dF^{-1}(u)dG^{-1}(v).$$

We clearly see from the last equation that the (linear) correlation coefficient is function of the inverse of the marginal distribution. Since usually these marginal distributions are not invariant under monotone transformations other measure of dependence are more appropriate when studying the dependence relationship in copulas. This is also a reason why the Pearson correlation coefficient is only able to catch linear relationship between variables.

Let us now introduce the notion of *concordance* since this will be used in the definition of other measure of association (which are scale invariant) between random variables.

Definition 2.2.2 *Let $(\tilde{x};\tilde{y})^T$ and $(x,y)^T$ be two observations from a vector $(X,Y)^T$ of continuous random variables. $(\tilde{x};\tilde{y})^T$ and $(x,y)^T$ are **concordant** if $(x-\tilde{x})(y-\tilde{y}) > 0$. $(\tilde{x};\tilde{y})^T$ and $(x,y)^T$ are **discordant** if $(x-\tilde{x})(y-\tilde{y}) < 0$.*

We can now define a the general notion of a concordance function Q. The following theorem can be found in Nelsen (1999) p. 127.

Theorem 2.7 *Let $(X,Y)^T$ and $(\tilde{X},\tilde{Y})^T$ be independent vectors of continuous random variables with joint distribution function H and $\tilde{H}$, respectively, with common margins F and G. Let C, $\tilde{C}$ denote the copulas of $(X,Y)^T$ and $(\tilde{X},\tilde{Y})^T$, respectively, so that $H(x,y) = C\left(F(x),G(y)\right)$ and $\tilde{H}(x,y) = \tilde{C}\left(F(x),G(y)\right)$. Let Q denote the difference between the probability of concordance and discordance of $(X,Y)^T$ and $(\tilde{X},\tilde{Y})^T$ respectively,*

$$Q(H,\tilde{H}) = P\{(\tilde{X}-X)(\tilde{Y}-Y) > 0\} - P\{(\tilde{X}-X)(\tilde{Y}-Y) < 0\}$$

then

$$Q(H,\tilde{H}) = 4\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\tilde{H}(x,y)dH(x,y) - 1$$

or if $F(x) = u$ and $G(y) = v$,

$$Q(C,\tilde{C}) = 4\int_{0}^{1}\int_{0}^{1}\tilde{C}(u,v)dC(u,v) - 1.$$

Proof: If we denote P_c the probability of concordance, then $1-P_c$ is the probability of discordance and $Q = 2P_c - 1$. So that,

$$\begin{aligned} Q(C,\tilde{C}) &= 2P\{(\tilde{X}-X)(\tilde{Y}-Y)\} - 1 \\ &= 2E\left[P\{(\tilde{X}-X)(\tilde{Y}-Y)|\tilde{X}=\tilde{x},\tilde{Y}=\tilde{y}\}\right] - 1 \end{aligned}$$

$$= \quad 2E\left[2\tilde{H}(x,y) - F(x) - G(y) + 1\right] - 1 = 4E\left[H(x,y)\right] - 1.$$

Where we have used in the third line,

$$P\{\tilde{X} \geq x, \tilde{Y} \geq y\} \quad = \quad \tilde{H}(x,y) - F(x) - G(y) + 1$$

since $\tilde{X}$ and $\tilde{Y}$ are independent with $F(x)$, $G(y)$ uniform random variables and the definition in term of copula holds by the Sklar theorem. □

Corollary 2.2.1 *Let C, $\tilde{C}$, and Q be as given in Theorem 2.7. Then*

1. *Q is symmetric in its arguments: $Q(C, \tilde{C}) = Q(\tilde{C}, C)$*
2. *Q is nondecreasing in each argument: if $C \prec C'$, then $Q(C, \tilde{C}) \leq Q(C', \tilde{C})$*
3. *Copulas can be replaced by survival copulas in Q, i.e. $Q(C, \tilde{C}) = Q(\hat{C}, \hat{\tilde{C}})$*

According to Scarsini (1984), a set of desirable properties for a concordance measure would include those following.

Definition 2.2.3 *A numeric measure κ of association between two continuous random variable X and Y, whose copula is C, is a **measure of concordance** if it satisfies the following properties:*

1. *κ is defined for every pairs X, Y*
2. *$-1 \leq \kappa \leq 1$ and $\kappa_{-X,X} = -1$*
3. *$\kappa_{X,Y} = \kappa_{Y,X}$*
4. *If X and Y are independent $\kappa_{X,Y} = 0$*
5. *$\kappa_{-X,Y} = \kappa_{X,-Y} = -\kappa_{X,Y}$*
6. *If C_1 and C_2 are copulas such that $V(C_1) < V(C_2)$ then $\kappa_{C_1} < \kappa_{C_2}$*
7. *If $\{(X_n, Y_n)\}$ is a sequence of continuous random variables with copulas C_n and if $\{C_n\}$ converge point-wise to C, then $\lim\limits_{n \to +\infty} \kappa_{C_n} = \kappa_C$*

A consequence from the last Definition, the Corollary 2.1.1 and Theorem 2.3 is stated in the following theorem

Theorem 2.8 *Let κ be a measure of concordance for continuous random variable X and Y*

1. *If Y is almost surely an increasing function of X then $\kappa_{X,Y} = \kappa_M = 1$*
2. *If Y is almost surely a decreasing function of X then $\kappa_{X,Y} = \kappa_W = -1$*
3. *If α and β are almost surely strictly monotone functions, respectively, on $RanX$ and $RanY$, then $\kappa_{\alpha(X),\beta(Y)} = \kappa_{X,Y}$*

The most direct measure of association following the Theorem 2.1 is the Kendall's τ which is the difference between the probability of concordance and discordance as previously defined.

Definition 2.2.4 *Let X and Y follow jointly a bivariate distribution H. The **Kendall's** τ for X and Y is then defined as*

$$\tau(X,Y) = 4\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} H(x,y)dH(x,y) - 1$$

or if we use the fact that $F(x) = u$ and $G(y) = v$,

$$\tau(X,Y) = 4\int_0^1\int_0^1 C(u,v)dC(u,v) - 1.$$

Note that the integral above is the expected value of the random variable $C(U,V)$, where U, $V \sim U(0,1)$ with joint distribution function C, i.e. $\tau(X,Y) = 4\mathbb{E}\left(C(U,V)\right) - 1$.

To evaluate the Kendall's τ one can also use the following theorem, a proof of which can be found in Nelsen (1999, p.131).

Theorem 2.9 *Let C be a Copula such that the product $(\partial C/\partial u)(\partial C/\partial v)$ is integrable on $[0,1]^2$. Then*

$$\int\int_{[0,1]^2} C(u,v)dC(u,v) = \frac{1}{2} - \int\int_{[0,1]^2} \frac{\partial}{\partial u}C(u,v)\frac{\partial}{\partial v}C(u,v)dudv. \tag{2.4}$$

We now move to another measure of association between random variable, the Spearman's ρ_s.

Definition 2.2.5 *Let $(X,Y)^T$, $(\tilde{X},\tilde{Y})^T$ and $(X',Y')^T$ be independent copies, the* ***Spearman's*** *ρ_s for a random vector $(X,Y)^T$ is then defined as*

$$\rho_s(X,Y) = 3\Big(P\{(X-\tilde{X})(Y-Y')>0\} - P\{(X-\tilde{X})(Y-Y')<0\}\Big)$$

$$\rho_s(X,Y) = 12\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}[H(x,y)-F(x)G(y)]dF(x)dG(y)$$

or if we use $F(x)=u$ and $G(y)=v$,

$$\rho_s(X,Y) = 12\int_0^1\int_0^1[C(u,v)-uv]dudv.$$

Using Theorem 2.7 and the first part of Corollary 2.2.1 we obtain the following result.

Theorem 2.10 *Let $(X,Y)^T$ be a vector of continuous random variables with copula C. Then Spearman's rho for $(X,Y)^T$ is given by*

$$\rho_S(X,Y) = 12\int_0^1\int_0^1 uvdC(u,v) - 3 = 12\int_0^1\int_0^1 C(u,v)dudv - 3. \tag{2.5}$$

Hence, if $X \sim F$ and $Y \sim G$, we let $U=F(X)$ and $V=G(Y)$, then

$$\begin{aligned}\rho_S(X,Y) &= 12\int_0^1\int_0^1 uvdC(u,v) - 3 = 12\mathbb{E}(UV) - 3\\ &= \frac{\mathbb{E}(UV)-1/4}{1/12} = \frac{Cov(U,V)}{\sqrt{Var(U)}\sqrt{Var(V)}}\\ &= \rho\left(F(X),G(Y)\right).\end{aligned}$$

Following the last two definitions about the Spearman's ρ_s and Kendall's τ we have the following theorem.

Theorem 2.11 *If X and Y are continuous random variables whose copula is $C_{X,Y}$, then Spearman's ρ as in definition 2.2.5 and Kendall's τ as defined in Definition 2.2.4 satisfy the measure of concordance definition and Theorem 2.8 for a measure of concordance.*

Proof: For the definition if the measure of concordance: the first condition is satisfied by the definition of a probability; the second, by the Fréchét-Hoeffding bounds; the third, by the exchangeability of copula (i.e. $C(u, v) = C(v, u)$); the fourth, by definition of the product copula; the fifth by the exchangeability and the Fréchét-Hoeffding bounds. For the sixth and seventh the reader should refer to Nelsen (1998), pg 137. $\square$

For the Theorem 2.8: If we take two random variables X, Y and fix $X = Y$ so that their copula will be the upper Fréchét-Hoeffding bound and $\tau(X, Y) = 4\int_0^1 x dx - 1$. We can use the same argument for the condition 2 and setting $X = -Y$. The third condition holds by the copula invariance to strictly increasing function property.

Another interesting measure of association is the one related to the tail dependence. Broadly speaking with this measure we try so see how random extreme events from different marginal distribution happen together. Such measure has an implicit interpretation in finance: the probability that two firms default together, the probability two stocks crash together, etc...

Definition 2.2.6 *Let $(X, Y)^T$ be a vector of continuous random variables with marginals, respectively, F and G. The **coefficient of upper and lower dependence** (if they do exist) are respectively defined as*

$$\lim_{u \nearrow 1} P\{Y > G^{-1}(u) | X > F^{-1}(u)\} = \lambda_U,$$

$$\lim_{u \searrow 0} P\{Y \leq G^{-1}(u) | X \leq F^{-1}(u)\} = \lambda_L.$$

The random variables are then said to have upper tail dependence if $\lambda_U \in (0, 1]$ and lower tail dependence if $\lambda_L \in (0, 1]$. If $\lambda_U = 0$ or $\lambda_L = 0$, X and Y are said to be asymptotically independent in the upper tail or in the lower tail correspondingly.

By using the Bayes Theorem (and the property of probability function $P(A^C \cap B^C) = 1 - P(A \cup B) = 1 - P(A) - P(B) - P(A \cap B)$), the previous expressions can be redefined in term of copula.

Corollary 2.2.2 *Let $u \in I$ and C be a 2-copula, then coefficients of upper and lower dependance can be, respectively re-expressed as*

$$\lim_{u \nearrow 1} \frac{C(u,u) - 2u + 1}{1-u} = \lambda_U,$$

$$\lim_{u \searrow 0} \frac{C(u,u)}{u} = \lambda_L.$$

Consider a pair of $U(0,1)$ random variables (U,V) with copula C. First note that $P\{V \leq v|U=u\} = \partial C(u,v)/\partial u$ and $P\{V > v|U=u\} = 1 - \partial C(u,v)/\partial u$, and similarly when conditioning on V. Then

$$\begin{aligned} \lambda_U &= \lim_{u \nearrow 1} \frac{C(u,u) - 2u + 1}{1-u} = -\lim_{u \nearrow 1} \frac{dC(u,u) - 2}{du} \\ &= 2 - \lim_{u \nearrow 1} \left(\frac{\partial}{\partial s} C(s,t)|_{s=t=u} + \frac{\partial}{\partial t} C(s,t)|_{s=t=u} \right) \\ &= \lim_{u \nearrow 1} \left(P\{V > v|U=u\} + P\{U > u|V=v\} \right). \end{aligned} \tag{2.6}$$

Furthermore, if C is an exchangeable copula, i.e. $C(u,v) = C(v,u)$, then expression for λ_U simplifies to

$$\lambda_U = 2 \lim_{u \nearrow 1} P\{V > v|U=u\}.$$

An alternative formula for λ_L can be defined in a similar way:

$$\begin{aligned} \lambda_L &= \lim_{u \searrow 0} \frac{C(u,u)}{u} = \lim_{u \searrow 0} \frac{dC(u,u)}{du} \\ &= \lim_{u \searrow 0} \left(\frac{\partial}{\partial s} C(s,t)|_{s=t=u} + \frac{\partial}{\partial t} C(s,t)|_{s=t=u} \right) \\ &= \lim_{u \searrow 0} \left(P\{V < v|U=u\} + P\{U < u|V=v\} \right). \end{aligned} \tag{2.7}$$

Furthermore if C is an exchangeable copula, then expression for λ_L simplifies to

$$\lambda_L = 2 \lim_{u \nearrow 1} P\{V < v|U=u\}.$$

Since the coefficient of upper and lower dependence belong to the unit interval I and can be expressed in term of copula, they agree with the notion of numeric measure of association as described by Scarsini (1984).

Remark 2.12 Hence it follows that

$$\lambda_U = \lim_{u \nearrow 1} \frac{C(u,u) - 2u + 1}{1-u} = \lim_{u \nearrow 1} \frac{\hat{C}(1-u, 1-u)}{1-u} = \lim_{u \searrow 0} \frac{\hat{C}(u,u)}{u},$$

so the coefficient of upper tail dependence of C is the coefficient of lower tail dependence of $\hat{C}$. Similarly the coefficient of lower tail dependence of C is the coefficient of upper tail dependence of $\hat{C}$.

Quite interestingly as noted by Schönbucher (2003) if random variables have tail dependence it means that there should be some singularities in the volume defined by the shape of their joint distribution. Namely for the lower dependence case, as $u \to 0$, the joint distribution probability mass or the volume described by the rectangle $[0,u] \times [0,u]$ tend to zero at speed λ_L (and not u^2).

2.3 Methods of constructing copulas

2.3.1 The Inversion Method

If the joint distribution function with continuous margins is given, we can easily find a copula by inverting the margin distribution functions:

$$C(s,t) = H\left(F^{(-1)}(s), G^{(-1)}(t)\right) \text{ for } s,\ t \in [0,1]. \tag{2.8}$$

Of course, this can be done equally as well using survival functions (recall that $\hat{C}$ is a copula):

$$\hat{C}(s,t) = \bar{H}\left(\bar{F}^{(-1)}(s), \bar{G}^{(-1)}(t)\right), \text{ for } s,\ t \in [0,1].$$

where $\bar{F}^{(-1)}$ denotes a quasi-inverse of $\bar{F}$, defined analogously to $F^{(-1)}$ in Definition 2.1.6; or equivalently $\bar{F}^{(-1)}(t) = F^{(-1)}(1-t)$.

If H admits a density h and we denote by f and g the densities of marginals F and G, it follows from (2.8), that C has a density c given by

$$c(s,t) = \frac{h\left(F^{(-1)}(s), G^{(-1)}(t)\right)}{f\left(F^{(-1)}(s)\right) g\left(G^{(-1)}(t)\right)} \text{ for } s,\ t \in [0,1]. \tag{2.9}$$

Thus, to compute the copula of a given joint probability distribution function, we proceed as follows:

(a) Compute the joint distribution function H and the marginal distribution functions F and G.

(b) Find the inverse marginal distribution functions $F^{(-1)}$ and $G^{(-1)}$.

(c) Use (2.8) to compute the copula C or use (2.9) to obtain the copula density c.

2.4 Copulas families

While having defined the properties of copulas and the notion of measure of association in the last two sections, this section will be devoted to an overview of some of copula functions in term of their form and properties.

2.4.1 Elliptical Copulas

The first two copula functions presented in this sub-section come from the class of elliptical distributions. Broadly speaking when an elliptical copula (or a joint elliptical distribution) is seen from above, the contour lines of this distribution have elliptical shapes. These copulas have the radial symmetry property and their main advantage is the ease of sampling from them while, on the other hand, they do not have a simple closed form.

Definition 2.4.1 *Let Φ denotes the standard univariate normal distribution function and let Φ_R the standard normal multivariate distribution with covariance matrix R. The **Gaussian copula** is then defined as*

$$C_{\Phi_R}(u_1, u_2, ..., u_n; R) = \Phi_R(\Phi^{-1}(u_1), \Phi^{-1}(u_2), ..., \Phi^{-1}(u_n)).$$

To sample from this copula, let suppose we have derived the correlation matrix ρ from the covariance matrix R. We then compute, A, the Cholesky decomposition of ρ, then sample a vector X of n independent random variables from a standard Gaussian distribution. We set $Y = AX$ and transform the components of the vector Y into uniform random variables by setting $u_i = \Phi(yi)$. So $U \sim C_{\Phi_R}$.

Definition 2.4.2 *Let t_ν denotes the standard univariate Student distribution function with ν degrees of freedom and let $t_{\nu,R}$ be the multivariate Student distribution with covariance matrix R and ν degrees of freedom. The Student copula is then defined as*

$$C_{t_{\nu,R}}(u_1, u_2, ..., u_n; R) = t_{\nu,R}\left(t_\nu^{-1}(u_1), t_\nu^{-1}(u_2), ..., t_\nu^{-1}(u_n)\right).$$

It is also easy to sample from this copula by using the definition of a Student random variable as the ratio of a standard Gaussian random variable and the square root of Chi Square random variable divided by its degree of freedom. Suppose we have derived the correlation matrix ρ from the covariance matrix R. We then compute, A, the Cholesky decomposition of ρ; then sample a vector X of n independent random variables from a standard Gaussian distribution and set $Y = AX$. We then sample a vector Z of n independent Chi square distributed random variables with ν degrees of freedom. We set $W = Y\sqrt{\nu/Z}$ and denote $u_i = t_{\nu,R}(wi)$. So $U \sim C_{t_{\nu,R}}$.

In order to compute the measure of dependence for these elliptical copulas we rely on the following theorem.

Theorem 2.13 *Let X be a vector of random variables which follows an elliptical distribution and have covariance matrix R, then*

$$\tau(X_i, X_j) = \frac{2}{\pi} arcsin(R_{ij}),$$

$$\rho(X_i, X_j) = \frac{6}{\pi} arcsin\left(\frac{R_{ij}}{2}\right).$$

Proof: Kruksal (1958) pg 827.

Once again, let us remark that this measure, for the elliptical distributions, rely on the linear coefficient and so fails to take into account non linear relationship between random variables.

Let us now move to the tail dependence for this class of elliptical copulas. As previously stated since these copulas do not have a simple closed form, the computation of the tail dependence is not as straightforward as the Archimedean copulas (which will be covered in the next section). The derivation which follows rely on the results from Embrechts, Lindskog and McNeil (2003).

If we use the definition of the coefficient of tail dependance as defined in the previous subsection and apply to it the L'Hopital rule and remark that $P\{V \leq v|U = u\} = \partial C(u,v)/\partial u$ (and $P\{V > v|U = u\} = 1 - \partial C(u,v)/\partial u$). We then have that,

$$\lambda_U = -\lim_{u \nearrow 1} \left[-2 + \left.\frac{\partial C(u,v)}{\partial u}\right|_{u=v} + \left.\frac{\partial C(u,v)}{\partial v}\right|_{v=u} \right],$$

$$\lambda_L = -\lim_{u \searrow 0} \left[P\{V \leq v|U = u\} - P\{U \leq u|V = vu\} \right].$$

Since the copulas are exchangeable (i.e $C(u,v) = C(v,u)$), we have:

$$\lambda_U = -2 \lim_{u \nearrow 1} \left[P\{V \leq v|U = u\} \right].$$

If we now define $x = F^{-1}(u)$ and $y = F^{-1}(v)$ where $x, y \in \mathbf{R}$ with F and G the marginal distribution of X and Y . We can now rewrite the previous limit as

$$\begin{aligned} \lambda_U &= -2 \lim_{x \to +\infty} \left[P\{F^{-1}(V) \geq x|F^{-1}(U) = x\} \right] \\ &= -2 \lim_{x \to +\infty} P\{X \geq x|Y = x\}. \end{aligned}$$

- If $F = \Phi$, the standard Gaussian distribution, and by using the fact that for bivariate standard Gaussian distribution $Y|X = x \sim N(\rho x; 1 - \rho^2)$. We can rewrite the previous expression as

$$\begin{aligned} \lambda_U &= -2 \lim_{x \to +\infty} \left[1 - \Phi\left(\frac{x - \rho x}{\sqrt{1 - \rho^2}} \right) \right] \\ &= -2 \lim_{x \to +\infty} \left[1 - \Phi\left(\frac{x\sqrt{1 - \rho}}{\sqrt{1 + \rho}} \right) \right]. \end{aligned}$$

  When $\rho < 1$ the Gaussian copula has no upper tail dependence. By the radial symmetry this argument also holds for the lower tail dependence coefficient.

- If $F = t_\nu$, the Student distribution with ν degrees of freedom and by using the fact (see Demarta and McNeil (2004) or Galiani (2001) for a formal proof) that

$P\{X > x|X = x\} = 1 - t_{\nu+1}\left[\left(\frac{\nu+x^2}{\nu+1}\right)^{-1/2}\frac{x-rx}{\sqrt{1-r^2}}\right]$. When computing two times the limits of this last expression we find the coefficient of upper dependence:

$$\lambda_U = 2 - 2t_{\nu+1}\left[\frac{((\nu+1)(1-r))^{1/2}}{\sqrt{1-r}}\right].$$

This last expression shows that the tail dependence parameter is function of the degree of freedom and the linear correlation. Let us remark that even with a correlation coefficient equal to zero there is still some tail dependence. While when the degrees of freedom tends to the infinity, the behavior of tail dependence in the Student distribution tend to the behavior of the tail dependence in the Gaussian distribution (i.e. is equal to zero when $\rho < 1$).

2.4.2 Marshall Olkin Copula

The next class of copulas presented in this section was first introduced by Marshall and Olkin(1967a,b) and is particularly relevant when modeling the joint distribution of objects lifetime when these lifetime are related to each others. By example the lifetime of light bulbs (from a same brand) or the lifetime of some bonds from companies in a same business sector, etc. To be more precise, the Marshall-Olkin of copula aim to build multivariate distribution of marginally distributed exponential random variables. The dependence between these exponential random variables, by example in the bivariate case, is created by taking into account that at any time t during the object lifetime, either one object will die or the two together.

In this framework it is understood that the lifetime of an object follows a stopped $Poisson(\lambda)$ process (i.e. this object will die at the time of the first jump in the Poisson process). In the bivariate case (we have objects 1 and 2, with lifetime X and Y), at each time t (before the death of a component) 3 types of events (and their complement) can happen: either only component 1 die (let denote the time when this event happens E_1) or only component 2 die (E_2) or component 1 and 2 both die together (E_{12}). Let us remark that with this definition the lifetime of component 1, X is $min(E_1, E_{12})$ and the lifetime of component 2 is $min(E_2, E_{12})$.

To model these three events, three independent Poisson processes with parameter λ_1, λ_2 and λ_{12} are used. The survival probability, F for the object 1 at time x (in the bivariate case) is

$$\overline{F}_1(x) = P\{E_1 > x\}P\{E_{12} > x\} = exp[-(\lambda_1 + \lambda_{12})x] = exp[-\lambda_1^* x],$$

with $\lambda_1^* = \lambda_1 + \lambda_{12}$. In the bivariate example, the survival function $\overline{H}$ is defined as

$$\begin{aligned} \overline{H}(x, y) &= P\{E_1 > x\}P\{E_2 > y\}P\{E_{12} > max(x, y)\} \\ &= exp\left[-\lambda_1 x - \lambda_2 y - \lambda_{12} max(x, y)\right] \end{aligned}$$

In order now to express the joint survival distribution H, in term of its survival copula, C, we need to work on the last equation. By first noticing that $max(x, y) = x + y - min(x, y)$, we can rewrite the last equation,

$$\begin{aligned} \overline{H}(x, y) &= exp\left[-(\lambda_1 + \lambda_{12})x - (\lambda_2 + \lambda_{12})y + \lambda_{12} min(x, y)\right] \\ &= \overline{F}_1(x)\overline{F}_2(y) min\{exp(\lambda_{12}x), exp(\lambda_{12}y)\} \end{aligned}$$

and by setting $\overline{F}_1(x) = u,\ \overline{F}_2(y) = v,\ \alpha_1 = \frac{\lambda_{12}}{\lambda_{12}+\lambda_1},\ \alpha_2 = \frac{\lambda_{12}}{\lambda_{12}+\lambda_2}$. So that

$$exp(\lambda_{12}x) = u^{-\alpha_1}, \quad exp(\lambda_{12}y) = v^{-\alpha_2}.$$

Now substituting in the definition of H, the survival copula, C, for X and Y is

$$C(u, v) = uv \cdot min(u^{-\alpha_1}, v^{-\alpha_2}) = min\left(vu^{-\alpha_1}, uv^{-\alpha_2}\right)$$

This last computation provides us the form of the Marshall-Olkin copula

$$C^{MO}_{\alpha_1,\alpha_2}(u, v) = min\{vu^{1-\alpha_1}, uv^{1-\alpha_2}\} = \begin{cases} vu^{1-\alpha_1}, & u^{\alpha_2} > v^{\alpha_1}, \\ uv^{1-\alpha_2}, & u^{\alpha_2} < v^{\alpha_1}. \end{cases} \tag{2.10}$$

With $\alpha_1, \alpha_2 \in (0; 1)$. The Fréchét-Hoeffding bounds in this case are defined as follows:

$$C_{\alpha_1,0} = C_{0,\alpha_2} = C_\Pi, \quad C_{1,1} = M.$$

While we have the following representation for the Spearman's ρ and Kendall's τ (for a formal proof see Nelsen (1999))

$$\rho_s\left(C^{MO}_{\alpha_1,\alpha_2}\right) = \frac{3\alpha_1\alpha_2}{2\alpha_1 + 2\alpha_2 - \alpha_1\alpha_2}; \quad \tau\left(C^{MO}_{\alpha_1,\alpha_2}\right) = \frac{\alpha_1\alpha_2}{\alpha_1 + \alpha_2 - \alpha_1\alpha_2}.$$

The coefficient of upper dependence can easily be computed by once again using its definition in term of copula and applying to it the LHopital rule,

$$\lambda_U^{C^{MO}_{\alpha_1,\alpha_2}} = \lim_{u \searrow 1} \frac{C^{MO}_{\alpha_1,\alpha_2}(u,u)}{1-u} = \begin{cases} \alpha_2, & \alpha_1 > \alpha_2, \\ \alpha_1, & \alpha_1 < \alpha_2. \end{cases} \tag{2.11}$$

The coefficient of upper dependence is

$$\lambda_U^{C^{MO}_{\alpha_1,\alpha_2}} = min\{\alpha_1, \alpha_2\}.$$

While it was easy to sample from elliptical copulas, for this last copula this becomes more tedious. It is mainly due to the large number of random variable needed to sample: $2^n - 1$ uniform random variables in order to later build n dependent exponential random variables. Once the uniform random variables are sampled, one should find an ordering and group them together into subsets. By example if $n = 3$ then $i = 7$, one could set the three first random variables be the intensities of the lifetime of the three object; then set the three following be the intensities of the event at which two objects will die together and the last random variable the intensity for the time at which all the three objects will die together. After these subsets are defined one need to use them to compute the n different (correlated) intensities, λ_i^* , of the n objects lifetime. The marginal distribution can now readily be computed by using the intensities, λ_i^*. The n variate from the Marshall-Olkin copula , $(\nu_i, ..., \nu_i)$, are then found by applying the same transformations on the intensities and find the parameters α_i's as described in the bivariate case above. The n-Marshall-Olkin copula will now have n parameters.

2.4.3 Archimedean Copulas

The class of elliptical copulas is appealing for their ease of sampling while on the other hand we have seen that the measure of dependence computations are quite involved and this class of copulas has restricted properties. It is then natural to try to find other class of copulas which could have some desired properties in term of the measure of dependence and which could easily be computed. The goal of this section is to present such kind of copula. For an overview on this specific topic one should refer to Genest and McKay (1986b).

We have seen previously that the product copula was easily computed since this copula is the product of uniform random variables. If we take the log transformation of this copula, we have

$$\log\left(C_{\Pi}\right) = \log(u) + \log(v).$$

If we are now interested to build a copula not for independent uniform random variables but uniform variables that might be related (either through a linear or non linear relationship) we could apply a simple parametric transformation to the product copula in such a way that the parameters in this transformation would produce the desired dependence structure between these uniform random variables. In that case we would have

$$\varphi(C_{\Pi}(u,v)) = \varphi(u) + \varphi(v).$$

If we now solve the previous equation for $C(u,v)$ we have constructed the following Archimedean copula

$$C(u,v) = \varphi^{[-1]}\left(\varphi(u) + \varphi(v)\right),$$

where φ is called the *generator* of this Archimedean Copula and $\varphi^{[-1]}$ is its *pseudo-inverse.*

Let us now more formally introduce the notion of pseudo-inverse of the previously defined generator function by imposing some condition on it so that the Archimedean copula follows the definition of copula.

Definition 2.4.3 *Let φ be a continuous, strictly decreasing function from I to $[0;+\infty]$ such that $\varphi(1) = 0$. The* ***pseudo inverse*** *of φ is a function, $\varphi^{[-1]}$, with* **Dom**$\varphi^{[-1]} = [0;+\infty]$ *and* ***Ran***$\varphi^{[-1]} = I$ *given by*

$$\varphi^{[-1]}(u) = \begin{cases} u, & 0 \geq u \geq \varphi(u), \\ 0, & \varphi(u) \geq u \geq +\infty. \end{cases} \tag{2.12}$$

Let us remark that if $\varphi(0) = +\infty$ then $\varphi^{-1} = \varphi^{[-1]}$ and in the case φ is called a *strict generator.*

Theorem 2.14 *Let φ be a continuous strictly decreasing function from $[0;1]$ to $[0;+\infty]$ such that $\varphi(1) = 0$ and let $\varphi^{[-1]}$ be the pseudo-inverse of φ. Let C be the* *function from* $[0;1]^2$ *to* $[0;1]$ *given by*

$$C(u,v) = \varphi^{[-1]}\left(\varphi(u) + \varphi(v)\right). \tag{2.13}$$

Then C *is a copula if and only if* φ *is convex.*

Proof: Nelsen (1999) p.91. □

There are as many families of Archimedean copulas as there are function φ which verifies the previous definition.

Example 2.4.1 *Let* $\varphi(t) = (-\ln t)^\theta$*, where* $\theta \geq 1$*. Clearly* $\varphi(t)$ *is continuous and* $\varphi(1) = 0$*.* $\varphi'(t) = -\frac{\theta}{t}(-\ln t)^{\theta-1}$*, so* φ *is a strictly decreasing function from* $[0,1]$ *to* $[0,\infty]$*.* $\varphi''(t) > 0$ *on [0, 1], so* φ *is convex. Moreover* $\varphi(0) = \infty$*, so* φ *is a strict generator. From (2.13) we get*

$$C_\theta(u,v) = \varphi^{[-1]}\left(\varphi(u) + \varphi(v)\right) = exp\left\{-\left[(-\ln u)^\theta + (-\ln v)^\theta\right]^{1/\theta}\right\}.$$

Furthermore $C_1 = \Pi$ *and* $\lim_{\theta\to\infty} C_\theta = M$ *(recall that* $\Pi(u,v) = uv$ *and* $M(u,v) = \min(u,v)$*). This copula family is called* ***Gumbel*** *family.*

Example 2.4.2 *Let* $\varphi(t) = \left(t^{-\theta} - 1\right)/\theta$*, where* $\theta \in [-1,\infty)\backslash\{0\}$*. This gives the* ***Clayton*** *family*

$$C_\theta(u,v) = \max\left(\left[u^{-\theta} + v^{-\theta} - 1\right]^{-1/\theta}, 0\right). \tag{2.14}$$

For $\theta > 0$ *the copulas are strict and the copula expression simplifies to*

$$C_\theta(u,v) = \left[u^{-\theta} + v^{-\theta} - 1\right]^{-1/\theta}. \tag{2.15}$$

The Clayton family has lower tail dependence for $\theta > 0$*, and* $C_1 = W$*,* $\lim_{\theta\to 0} C_\theta = \Pi$ *and* $\lim_{\theta\to\infty} C_\theta = M$

Example 2.4.3 *Let* $\varphi(t) = -\ln\frac{e^{-\theta t}-1}{e^{-\theta}-1}$*, where* $\theta \in \mathbb{R}\backslash\{0\}$*. This gives the* ***Frank*** *family*

$$C_\theta(u,v) = -\frac{1}{\theta}\ln\left(1 + \frac{\left(e^{-\theta u} - 1\right)\left(e^{-\theta v} - 1\right)}{e^{-\theta} - 1}\right). \tag{2.16}$$

The Frank copulas are strict Archimedean copulas. Furthermore $\lim_{\theta\to-\infty} C_\theta = W$, $\lim_{\theta\to 0} C_\theta = \Pi$ *and* $\lim_{\theta\to\infty} C_\theta = M$. *Members of the Frank family are the only Archimedean copulas which satisfy the equation* $C(u,v) = \hat{C}(u,v)$ *for so-called radial symmetry, see Frank (1979) for details.*

Example 2.4.4 *Let* $\varphi(t) = 1-t$ *for* t *in* $[0,1]$. *Then* $\varphi^{[-1]}(t) = 1-t$ *for* t *in [0, 1], and* 0 *for* $t > 1$*; i.e.,* $\varphi^{[-1]}(t) = max(1-t, 0)$. *Since* $C(u,v) = max(u+v-1, 0) =:$ $W(u,v)$*, we see that the bivariate Fréchét-Hoeffding lower bound* W *is Archimedean.*

Recall that Kendall's tau for a copula C can be expressed as a double integral of C. This double integral is in most cases not straightforward to evaluate. However for an Archimedean copula, Kendall's tau can be expressed as an (one-dimensional) integral of the generator and its derivative, as shown in the following theorem from Genest and MacKay (1986a).

Theorem 2.15 *Let* X *and* Y *be random variables with an Archimedean copula* C *generated by* φ. *Kendall's tau of* X *and* Y *is given by*

$$\tau(C) = 1 + 4\int_0^1 \frac{\varphi(t)}{\varphi'(t)}dt. \tag{2.17}$$

For Proof see Embrechts *at al.* (2003).

For Archimedean copulas, tail dependence can be expressed in terms of the generators.

Theorem 2.16 *Let* φ *be a strict generator such that* φ^{-1} *belongs to the class of Laplace transforms of strictly positive random variables. If* $(\varphi^{-1})'(0)$ *is finite, then*

$$C(u,v) = \varphi^{-1}(\varphi(u) + \varphi(v))$$

does not have upper tail dependence. If C *has upper tail dependence, then* $(\varphi^{-1})'(0) = -\infty$ *and the coefficient of upper tail dependence is given by*

$$\lambda_U = 2 - 2\lim_{s\searrow 0}\left[\left(\varphi^{-1}\right)'(2s)/\left(\varphi^{-1}\right)'(s)\right].$$

The coefficient of lower tail dependence is equal to

$$\lambda_L = 2\lim_{s\searrow 0}\left[\left(\varphi^{-1}\right)'(2s)/\left(\varphi^{-1}\right)'(s)\right].$$

The proof of this theorem can be found in Joe (1997), p. 103.

Chapter 3

From Markov Chain to Copula

3.1 Introduction

In this chapter we will assume that default times of some counterparties are described by Markov Chain. We will introduce the procedure of finding the marginal and joint distribution functions of these default times. Then we will derive the method of finding the inverse functions of the marginal distribution functions. As soon as joint distribution function and inverse marginal distribution functions are defined one can easily find the copula function by the procedure described in the section 2.3.1. We will close this chapter with different examples and graphical illustrations for the cases of two and three firms.

3.2 Marginal and Joint Distributions Functions of Default Times

Using the procedure described by Leung and Kwok (2006) the marginal and joint distribution functions of default times can be easily found.

3.2.1 Marginal Distribution

Remember that we use the framework defined by Kraft and Steffenson (2007). Once the conditional transition density matrix $Q(t,s|Y)$ has been defined, it can be used to derive the marginal distribution of $\tau_i, i = 1, 2, ..., N$. The marginal distribution function of the default time τ_i of obligor i is defined by

$$F_i(t_i) = P\left(\tau_i \leq t_i\right), \qquad i = 1, 2, ..., N.$$

Let $\mu_Y(y)$ be the probability measure which gives the law of Y. To obtain $F_i(t_i)$, we sum over all states j with default of the ith obligor of all transition probabilities moving from state 0 (none of the obligors defaults) to state j, and subsequently integrate over the distribution of $\mu_Y(y)$. This gives

$$F_i(t_i) = \int \sum_{j \in J_i} q_{0j}^{|y}(0, t_i)\mu_Y(y),$$

where J_i consist of the states in which obligor i has defaulted.

3.2.2 Joint Distribution

The joint distribution of the default times is defined as

$$F(t_1, t_2, ..., t_N) = P\left(\tau_1 \leq t_1, ..., \tau_N \leq t_N\right).$$

To express $F\left(t_1, t_2, ..., t_N\right)$ in terms of $q^{ij|Y}(t_k, t_{k+1})$, we consider the decomposition of the event $\{\tau_1 \leq t_1, ..., \tau_N \leq t_N\}$ into the union of the following mutually exclusive sub-events. Without loss of generality, we assume $t_1 \leq t_2 \leq ... \leq t_N$. The first sub-event is the default of all obligors within $[0, t_1]$, whose probability is given by $q^{1J|Y}(0, t_1)$, with state J in which all obligors have defaulted. The second sub-event corresponds to the default of all obligors within $(0, t_2]$, while obligor 1 but not all obligors have defaulted by t_1. Similarly, in the third sub-event, all obligors have defaulted by t_3. However, obligor 1 must default within $(0, t_1]$, obligor 2 must default within $(0, t_2]$ while not all obligors have defaulted by t_2. In the last sub-event, obligor k must default within $(0, t_k]$, $k = 1, 2, ..., N-1$, while not all obligors have defaulted by t_{N-1}. In addition to the above requirements, we also require that once an obligor has defaulted, it remains in the default state forever. Assuming $t_1 \leq t_2 \leq ... \leq t_N$, the joint distribution function can be expressed as

$$\begin{aligned} F(t_1, t_2, ..., t_N) &= \int \Big[q^{0J|y}(0, t_1) + \sum_{j_1 \in J_1} q^{0j_1|y}(0, t_1) q^{j_1 J|y}(t_1, t_2) \\ &+ \sum_{j_1 \in J_1, j_2 \in J_2} q^{0j_1|y}(0, t_1) q^{j_1 j_2|y}(t_1, t_2) q^{j_2 J|y}(t_2, t_3) + ... + \\ &+ \sum_{j_1 \in J_1, ..., j_N \in J_N} q^{0j_1|y}(0, t_1) q^{j_1 j_2|y}(t_1, t_2) \cdots q^{j_{N-1} J|y}(t_{N-1}, t_N) \Big] d\mu_Y(y), \end{aligned}$$

where J_i, $i = 1, ..., N-1$ is the set of states, in which first i obligors has defaulted within $(0, t_i]$ but not all of the obligors have defaulted by t_i.

Proof: We will use the method of mathematical induction. Let $N = 2$ and $t_1 \leq t_2$, then

$$\begin{aligned} F(t_1, t_2) &= P(\tau_1 \leq t_1, \tau_2 \leq t_2) = E\left[1_{\{\tau_1 \leq t_1\}} 1_{\{\tau_2 \leq t_2\}}\right] \\ &= E\left[1_{\{\tau_1 \leq t_1\}} E\left[1_{\{\tau_2 \leq t_2\}} | \mathcal{F}_{t_1}\right]\right] = E\left[1_{\{\tau_1 \leq t_1\}} E\left[1_{\{\tau_2 \leq t_1\}} + 1_{\{t_1 < \tau_2 \leq t_2\}} | \mathcal{F}_{t_1}\right]\right] \\ &= E\left[1_{\{\tau_1 \leq t_1\}} \left\{1_{\{\tau_2 \leq t_1\}} + E\left[1_{\{t_1 < \tau_2 \leq t_2\}} | \mathcal{F}_{t_1}\right]\right\}\right] \\ &= E\left[1_{\{\tau_1 \leq t_1\}} 1_{\{\tau_2 \leq t_1\}}\right] + E\left[1_{\{\tau_1 \leq t_1\}} E\left[1_{\{t_1 < \tau_2 \leq t_2\}} | \mathcal{F}_{t_1}\right]\right] \\ &= E\left[1_{\{\tau_1 \leq t_1, \tau_2 \leq t_1\}}\right] + E\left[1_{\{\tau_1 \leq t_1\}}\right] E\left[1_{\{t_1 < \tau_2 \leq t_2\}} | \mathcal{F}_{t_1}\right] \\ &= P(\tau_1 \leq t_1, \tau_2 \leq t_1) + P(\tau_1 \leq t_1) P(t_1 < \tau_2 \leq t_2) \\ &= \int \left[q^{0J|y}(0, t_1) + \sum_{j_1 \in J_1} q^{0j_1|y}(0, t_1) q^{j_1 J|y}(t_1, t_2) \right] d\mu_Y(y). \end{aligned}$$

Now assume that for $N-1$ the formula is true. Assume that $t_1 \leq t_2 \leq ... \leq t_N$ and check the formula for N obligors:

$$\begin{aligned} F(t_1, t_2, ..., t_N) &= Pr[\tau_1 \leq t_1, \tau_2 \leq t_2, ..., \tau_N \leq t_N] \\ &= E\left[1_{\{\tau_1 \leq t_1\}} 1_{\{\tau_2 \leq t_2\}} ... 1_{\{\tau_N \leq t_N\}}\right] \\ &= E\left[1_{\{\tau_1 \leq t_1\}} ... 1_{\{\tau_{N-1} \leq t_{N-1}\}} E\left[1_{\{\tau_N \leq t_N\}} | \mathcal{F}_{t_{N-1}}\right]\right] \\ &= E\left[1_{\{\tau_1 \leq t_1\}} ... 1_{\{\tau_{N-1} \leq t_{N-1}\}} E\left[1_{\{\tau_N \leq t_{N-1}\}} + 1_{\{t_{N-1} < \tau_N \leq t_N\}} | \mathcal{F}_{t_{N-1}}\right]\right] \\ &= E\left[1_{\{\tau_1 \leq t_1\}} ... 1_{\{\tau_N \leq t_{N-1}\}}\right] \\ &\quad + E\left[1_{\{\tau_1 \leq t_1\}} ... 1_{\{\tau_{N-1} \leq t_{N-1}\}} E\left[1_{\{t_{N-1} < \tau_N \leq t_N\}} | \mathcal{F}_{t_{N-1}}\right]\right] \\ &= E\left[1_{\{\tau_1 \leq t_1, ..., \tau_N \leq t_{N-1}\}}\right] \end{aligned}$$

$$
\begin{aligned}
& +E\left[1_{\{\tau_1\le t_1\}}...1_{\{\tau_{N-1}\le t_{N-1}\}}\right]E\left[1_{\{t_{N-1}<\tau_N\le t_N\}}|\mathcal{F}_{t_{N-1}}\right] \\
= \; & P\left(\tau_1\le t_1,...,\tau_N\le t_{N-1}\right) \\
& +P\left(\tau_1\le t_1,...,\tau_{N-1}\le t_{N-1}\right)P\left(t_{N-1}<\tau_N\le t_N\right),
\end{aligned}
$$

where the last summand coincide with the last summand of our initial formula. Now we can easily show that

$$
\begin{aligned}
P\big(\tau_1\le t_1, \;\; ... \;\; \tau_N\le t_{N-1}\big) = & \int\Big[q^{0J|y}(0,t_1)+\sum_{j_1\in J_1}q^{0j_1|y}(0,t_1)q^{j_1J|y}(t_1,t_2) \\
& +\sum_{j_1\in J_1,j_2\in J_2}q^{0j_1|y}(0,t_1)q^{j_1j_2|y}(t_1,t_2)q^{j_2J|y}(t_2,t_3)+...+ \\
& +\sum_{j_1\in J_1,...,j_{N-1}\in J_{N-1}}q^{0j_1|y}(0,t_1)q^{j_1j_2|y}(t_1,t_2)\cdots q^{j_{N-2}J|y}(t_{N-2},t_{N-1})\Big]d\mu_Y(y).
\end{aligned}
$$

Using the same procedure as before

$$
\begin{aligned}
P\big(\tau_1\le t_1, \;\; ... \;\; \tau_N\le t_{N-1}\big) &= E\left[1_{\{\tau_1\le t_1\}}...1_{\{\tau_N\le t_{N-1}\}}\right] \\
&= E\left[1_{\{\tau_1\le t_1\}}...1_{\{\tau_{N-2}\le t_{N-2}\}}E\left[1_{\{\tau_{N-1}\le t_{N-1}\}}1_{\{\tau_N\le t_{N-1}\}}|\mathcal{F}_{t_{N-2}}\right]\right] \\
&= E\Big[1_{\{\tau_1\le t_1\}}...1_{\{\tau_{N-2}\le t_{N-2}\}} \\
&\quad \times E\left[(1_{\{\tau_{N-1}\le t_{N-2}\}}+1_{\{t_{N-2}<\tau_{N-1}\le t_{N-1}\}})1_{\{\tau_N\le t_{N-1}\}}|\mathcal{F}_{t_{N-2}}\right]\Big] \\
&= E\left[1_{\{\tau_1\le t_1\}}...1_{\{\tau_{N-2}\le t_{N-2}\}}1_{\{\tau_{N-1}\le t_{N-2}\}}E\left[1_{\{\tau_N\le t_{N-1}\}}|\mathcal{F}_{t_{N-2}}\right]\right] \\
&\quad +E\left[1_{\{\tau_1\le t_1\}}...1_{\{\tau_{N-2}\le t_{N-2}\}}E\left[1_{\{t_{N-2}<\tau_{N-1}\le t_{N-1}\}}1_{\{\tau_N\le t_{N-1}\}}|\mathcal{F}_{t_{N-2}}\right]\right] \\
&= E\Big[1_{\{\tau_1\le t_1\}}...1_{\{\tau_{N-2}\le t_{N-2}\}}1_{\{\tau_{N-1}\le t_{N-2}\}} \\
&\quad \times E\left[(1_{\{\tau_N\le t_{N-2}\}}+1_{\{t_{N-2}<\tau_N\le t_{N-1}\}})|\mathcal{F}_{t_{N-2}}\right]\Big] \\
&\quad +E\Big[1_{\{\tau_1\le t_1\}}...1_{\{\tau_{N-2}\le t_{N-2}\}} \\
&\quad \times E\left[1_{\{t_{N-2}<\tau_{N-1}\le t_{N-1}\}}(1_{\{\tau_N\le t_{N-2}\}}+1_{\{t_{N-2}<\tau_N\le t_{N-1}\}})|\mathcal{F}_{t_{N-2}}\right]\Big] \\
&= E\left[1_{\{\tau_1\le t_1\}}...1_{\{\tau_{N-2}\le t_{N-2}\}}1_{\{\tau_{N-1}\le t_{N-2}\}}E\left[1_{\{\tau_N\le t_{N-2}\}}|\mathcal{F}_{t_{N-2}}\right]\right] \\
&\quad +E\left[1_{\{\tau_1\le t_1\}}...1_{\{\tau_{N-2}\le t_{N-2}\}}1_{\{\tau_{N-1}\le t_{N-2}\}}E\left[1_{\{t_{N-2}<\tau_N\le t_{N-1}\}}|\mathcal{F}_{t_{N-2}}\right]\right] \\
&\quad +E\left[1_{\{\tau_1\le t_1\}}...1_{\{\tau_{N-2}\le t_{N-2}\}}E\left[1_{\{t_{N-2}<\tau_{N-1}\le t_{N-1}\}}1_{\{t_{N-2}<\tau_N\le t_{N-1}\}}|\mathcal{F}_{t_{N-2}}\right]\right] \\
&\quad +E\left[1_{\{\tau_1\le t_1\}}...1_{\{\tau_{N-2}\le t_{N-2}\}}E\left[1_{\{t_{N-2}<\tau_{N-1}\le t_{N-1}\}}1_{\{\tau_N\le t_{N-2}\}}|\mathcal{F}_{t_{N-2}}\right]\right] \\
&= E\left[1_{\{\tau_1\le t_1\}}...1_{\{\tau_{N-2}\le t_{N-2}\}}1_{\{\tau_{N-1}\le t_{N-2}\}}1_{\{\tau_N\le t_{N-2}\}}\right]
\end{aligned}
$$

$$
\begin{aligned}
&+E\left[1_{\{\tau_1\le t_1\}}...1_{\{\tau_{N-2}\le t_{N-2}\}}1_{\{\tau_{N-1}\le t_{N-2}\}}E\left[1_{\{t_{N-2}<\tau_N\le t_{N-1}\}}|\mathcal{F}_{t_{N-2}}\right]\right]\\
&+E\left[1_{\{\tau_1\le t_1\}}...1_{\{\tau_{N-2}\le t_{N-2}\}}E\left[1_{\{t_{N-2}<\tau_{N-1}\le t_{N-1}\}}1_{\{t_{N-2}<\tau_N\le t_{N-1}\}}|\mathcal{F}_{t_{N-2}}\right]\right]\\
&+E\left[1_{\{\tau_1\le t_1\}}...1_{\{\tau_{N-2}\le t_{N-2}\}}1_{\{\tau_N\le t_{N-2}\}}E\left[1_{\{t_{N-2}<\tau_{N-1}\le t_{N-1}\}}|\mathcal{F}_{t_{N-2}}\right]\right]\\
= \; & E\left[1_{\{\tau_1\le t_1\}}...1_{\{\tau_{N-2}\le t_{N-2}\}}1_{\{\tau_{N-1}\le t_{N-2}\}}1_{\{\tau_N\le t_{N-2}\}}\right]\\
&+\int\Big[\sum_{j_1\in J_1,...,j_{N-1}\in J_{N-1}} q^{0j_1|y}(0,t_1)q^{j_1j_2|y}(t_1,t_2)\times\cdots\times\\
&\times q^{j_{N-2}J|y}(t_{N-2},t_{N-1})\Big]d\mu_Y(y).
\end{aligned}
$$

Doing the same procedure with first summand, we obtain our formula. □

3.3 Calculating Transitions Probabilities

From now on we will consider the constant intensities λ. Then according to Kraft and Steffenson (2007) the transition probabilities can be found in a following way

$$q^{0k}(t,T)=\sum_{p(0,k)\in P(0,k)}\lambda^{p(0,k)}g^{p(0,k)(t,T)}$$

with $\lambda^{p(0,0)}=1$, $p(0,k)=(0,p_1,...,p_m,k)$ is the path from state 0 to state k, and $\lambda^{p(0,k)}=\lambda^{0p_1}\lambda^{p_1p_2}...\lambda^{p_mk}$. The function g is defined as follows

$$
\begin{aligned}
g^j(t,T) &= q^{jj}(t,T)=e^{-\lambda^{j*}(T-t)},\\
g^{jk}(t,T) &= \frac{g^j(t,T)-g^k(t,T)}{\lambda^{k*}-\lambda^{j*}},\\
&\dots\\
g^{i_1\dots i_mjkk}(t,T) &= \frac{g^{i_1\dots i_mj}(t,T)-g^{i_1\dots i_mk}(t,T)}{\lambda^{k*}-\lambda^{j*}},
\end{aligned}
$$

where $\lambda^{n*}=\sum_{i\in\mathcal{J},i\neq n}\lambda^{ni}$ with $\mathcal{J}$ – set of all states. Moreover, we assume $\lambda^{k*}\neq\lambda^{j*}$ for all our applications.

3.4 Case of Two Firms

In the previous sections we have described how can be found the distribution functions of default times. Now we can use the inversion method described in the section

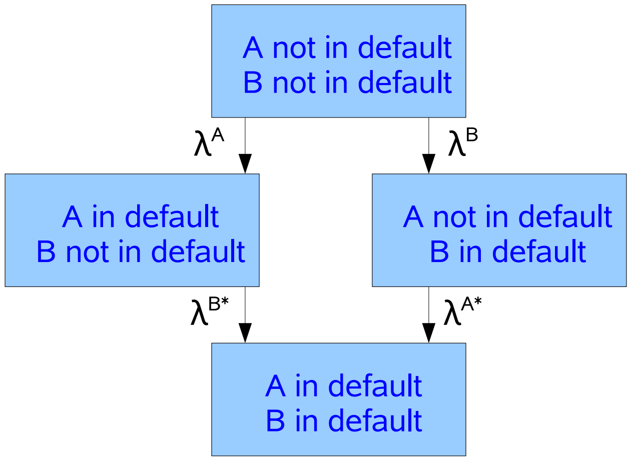

Figure 3.1: **Markov chain describing two firms case**

2.3.1 to find corresponding copula for any given Markov Chain. To do this we need to construct the corresponding distribution functions of default times, then to find inverse functions of marginal distribution functions and then substitute them into joint distribution function. So let us consider all this steps in details . Assume that we have Markov process, which describes the case of two firms described by the Figure 3.1. Let defaults times of firms A and B be τ_A and τ_B accordingly. We can find joint and marginal distribution function of these default times: $F_A(t) = P[\tau_A \leq t]$, $F_B(t) = P[\tau_B \leq t]$ and $F_{A,B}(t,s) = P[\tau_A \leq t, \tau_B \leq s]$.

Assume that transition intensities $\lambda^A = \lambda^{01}$, $\lambda^B = \lambda^{02}$, $\lambda^{A*} = \lambda^{23}$, $\lambda^{B*} = \lambda^{13}$ are constant and are given. The transition probabilities according to the section 3.3 are defined as:

$$\begin{aligned}
q^{00}(t,T) &= e^{-(\lambda^{01}+\lambda^{02})(T-t)} = e^{-\lambda^{0*}(T-t)}, \\
q^{01}(t,T) &= \lambda^{01} g^{01} = \lambda^{01} \frac{e^{-\lambda^{0*}(T-t)} - e^{-\lambda^{1*}(T-t)}}{\lambda^{1*} - \lambda^{0*}} \\
&= \lambda^{01} \frac{e^{-\lambda^{13}(T-t)} - e^{-(\lambda^{01}+\lambda^{02})(T-t)}}{\lambda^{01} + \lambda^{02} - \lambda^{13}}, \\
q^{02}(t,T) &= \lambda^{02} g^{02} = \lambda^{02} \frac{e^{-\lambda^{0*}(T-t)} - e^{-\lambda^{2*}(T-t)}}{\lambda^{2*} - \lambda^{0*}}
\end{aligned}$$

$$
\begin{aligned}
&= \lambda^{02}\frac{e^{-\lambda^{23}(T-t)} - e^{-(\lambda^{01}+\lambda^{02})(T-t)}}{\lambda^{01}+\lambda^{02}-\lambda^{23}},\\
q^{03}(t,T) &= \lambda^{01}\lambda^{13}g^{013}(t,T) + \lambda^{02}\lambda^{23}g^{023}(t,T),\\
q^{13}(t,T) &= \lambda^{13}g^{13} = \lambda^{13}\frac{e^{-\lambda^{1*}(T-t)} - e^{-\lambda^{3*}(T-t)}}{\lambda^{3*}-\lambda^{1*}}\\
&= \lambda^{13}\frac{1-e^{-\lambda^{13}(T-t)}}{\lambda^{13}},\\
q^{23}(t,T) &= \lambda^{23}g^{23} = \lambda^{23}\frac{e^{-\lambda^{2*}(T-t)} - e^{-\lambda^{3*}(T-t)}}{\lambda^{3*}-\lambda^{2*}}\\
&= \lambda^{23}\frac{1-e^{-\lambda^{23}(T-t)}}{\lambda^{23}}.
\end{aligned}
$$

The marginal distribution functions according to the section 3.2.1 are:

$$
\begin{aligned}
F_A(t) &= q^{03}(0,t) + q^{01}(0,t) = 1 - q^{02}(0,t) - q^{00}(0,t)\\
&= 1 - e^{-\lambda^{0*}t} + \frac{\lambda^{02}}{\lambda^{0*}-\lambda^{23}}\left(e^{-\lambda^{0*}t} - e^{-\lambda^{23}t}\right), \qquad (3.1)
\end{aligned}
$$

$$
\begin{aligned}
F_B(s) &= q^{03}(0,s) + q^{02}(0,s) = 1 - q^{01}(0,s) - q^{00}(0,t)\\
&= 1 - e^{-\lambda^{0*}t} + \frac{\lambda^{01}}{\lambda^{0*}-\lambda^{13}}\left(e^{-\lambda^{0*}t} - e^{-\lambda^{13}t}\right), \qquad (3.2)
\end{aligned}
$$

where $\lambda^{0*} = \lambda^{01}+\lambda^{02}$. In the previous equations we have use that $q^{00}(0,t)+q^{01}(0,t)+q^{02}(0,t)+q^{03}(0,t) = 1$. The joint distribution function in case $t \leq s$ according to the section 3.2.2 is:

$$
\begin{aligned}
F_{A,B}(t,s) &= q^{03}(0,t) + q^{01}(0,t)q^{13}(t,s)\\
&= 1 - q^{00}(0,t) - q^{01}(0,t) - q^{02}(0,t) + q^{01}(0,t)q^{13}(t,s) =\\
&= 1 - e^{-\lambda^{0*}t} - \frac{\lambda^{02}}{\lambda^{0*}-\lambda^{23}}e^{-\lambda^{23}t}\left[1 - e^{-\lambda^{0*}t}\right]\\
&\quad -\frac{\lambda^{01}}{\lambda^{0*}-\lambda^{13}}e^{-\lambda^{13}s}\left[1 - e^{(\lambda^{13}-\lambda^{0*})t}\right],
\end{aligned}
$$

and

$$
\begin{aligned}
F_{A,B}(t,s) &= q^{03}(0,s) + q^{02}(0,s)q^{23}(s,t)\\
&= 1 - q^{00}(0,s) - q^{01}(0,s) - q^{02}(0,s) + q^{02}(0,s)q^{23}(s,t)\\
&= 1 - e^{-\lambda^{0*}s} - \frac{\lambda^{01}}{\lambda^{0*}-\lambda^{13}}e^{-\lambda^{13}s}\left[1 - e^{-\lambda^{0*}s}\right]\\
&\quad -\frac{\lambda^{02}}{\lambda^{0*}-\lambda^{23}}e^{-\lambda^{23}t}\left[1 - e^{(\lambda^{23}-\lambda^{0*})s}\right]
\end{aligned}
$$

for case $t > s$.

3.4.1 Computation of Inverse Marginal Distributions

According to the section 2.3.1 we need to find inverse functions of marginal distribution functions of default time for each firm involved in our model. Consider marginal distribution function $F_A(t)$. It means, given $u \in [0,1]$ we have to solve equation $u = F_A(t)$ to find $t = F_A^{-1}(u)$. Substituting $\sigma(t) = \sigma = e^{-\lambda^{0*}t}$ in the equation (3.1) we obtain

$$F_A(t) = \hat{F}_A(\sigma(t)) \quad = \quad \hat{F}_A(\sigma) = 1 - P(\sigma) = u \tag{3.3}$$

with

$$P(\sigma) \triangleq (1-\alpha)\,\sigma + \alpha\sigma^{\frac{\lambda^{23}}{\lambda^{0*}}}, \quad \text{where } \alpha \triangleq \frac{\lambda^{02}}{\lambda^{0*} - \lambda^{23}}. \tag{3.4}$$

Similarly, to compute $F_B^{-1}(s)$ we have to solve equation

$$1 - v = Q(\tau) \tag{3.5}$$

with

$$Q(\tau) \triangleq (1-\beta)\,\tau + \beta\tau^{\frac{\lambda^{13}}{\lambda^{0*}}}, \quad \text{where } \beta \triangleq \frac{\lambda^{01}}{\lambda^{0*} - \lambda^{13}} \text{ and } \tau(s) = \tau = e^{-\lambda^{0*}s}.$$

Remark 3.1 The distribution functions $F_A(t)$ and $F_B(s)$ are strictly increasing functions of t and s correspondingly, after substitution the new functions $\hat{F}_A(\sigma(t)) = F_A(t)$ and $\hat{F}_B(\tau(s)) = F_B(s)$ are decreasing functions of their arguments. This means that the inverse functions $\hat{F}_A^{-1}(u)$ and $\hat{F}_B^{-1}(v)$ will be also decreasing functions of their arguments.

We can easily solve equations (3.3) and (3.5) if we let

$$\lambda^{23} = m\lambda^{0*} \text{ and } \lambda^{13} = n\lambda^{0*} \text{ for some } m,\ n = 0, 1, 2, 3, \cdots \tag{3.6}$$

In this case equations (3.3) and (3.5) we be (degenerate) polynomial equations of degrees m and n.

Note that we do not assume $m = n$. In particular, it is possible to solve explicitly if $m,\ n \in \{0, 1, 2, 3, 4\}$.

Another case which can be solved explicitly is obtained by choosing $n = \frac{p}{q}$, $m = \frac{k}{l}$ for $p,\ q,\ k,\ l = 1, 2, 3, \cdots$ For instance, choosing $n = 0.5$ leads to quadratic equation.

As soon as we've found inverse functions $\hat{F}_A^{-1}(u)$ and $\hat{F}_B^{-1}(v)$, the quasi inverse functions of the marginal distribution functions can be found as

$$F_A^{-1}(u) = -\frac{\ln\left(\hat{F}_A^{-1}(u)\right)}{\lambda^{0*}} \text{ and } F_B^{-1}(v) = -\frac{\ln\left(\hat{F}_B^{-1}(v)\right)}{\lambda^{0*}}.$$

As we've seen the marginal distribution functions after the substitution are polynomial equations of some natural degree. Also if in the equation (3.6) $n,\ m$ are rational we can reduce it to some polynomial equation of natural degree. So, to find the inverse function we just need to solve some polynomial equation of natural degree. So, assume that $\lambda^{23} = n\lambda^{0*}$ and consider equation (3.3). We get

$$1 - u = (1 - \alpha)\sigma + \alpha\sigma^n.$$

We should remark, that $\alpha < 0$ for all $n > 1$ and $\alpha > 0$ for all $n < 1$. In the following we will find the inverse functions for different values of n. Remember that a polynomial equation of power n can have up to n solutions $\sigma_i(u)$.

Remark 3.2 Remember that $\sigma = e^{-\lambda^{0*}t}$ with t from zero to ∞. So, we are interested at the one of the solutions, which satisfies $\sigma(0) = 1$ and $\sigma(1) = 0$. As soon as we have found this solution of the polynomial equation $\sigma(u) \triangleq \hat{F}_A^{-1}(u)$, the inverse function F_A^{-1} can be easily found as

$$F_A^{-1}(u) = -\frac{\ln(\sigma(u))}{\lambda^{0*}} = -\frac{\ln\left(\hat{F}_A^{-1}(u)\right)}{\lambda^{0*}}.$$

In the following we will refer to this solution as the right solution.

Let us consider different solutions of the polynomial equations of the form of (3.3) dependent on the value of n.

Case 1: $n = 2$: $\lambda^{23} = 2\lambda^{0*}$

In this case we get quadratic equation

$$1 - u = (1 - \alpha)\sigma + \alpha\sigma^2. \tag{3.7}$$

This equation have two solutions

$$\sigma_{1,2}(u) = \frac{-1+\alpha \pm \sqrt{1+2\alpha+\alpha^2-4u\alpha}}{2\alpha}.$$

Consider our solutions for $u = 0$ and $u = 1$:

$$\begin{aligned}
\sigma_1(0) &= \frac{-1+\alpha-\sqrt{1+2\alpha+\alpha^2}}{2\alpha} = \frac{-1+\alpha-(1+\alpha)}{2\alpha} = \frac{-1}{\alpha} \\
\sigma_1(1) &= \frac{-1+\alpha-\sqrt{1+2\alpha+\alpha^2-4\alpha}}{2\alpha} = \frac{-1+\alpha-(1-\alpha)}{2\alpha} = \frac{-1+\alpha}{\alpha} \\
\sigma_2(0) &= \frac{-1+\alpha+\sqrt{1+2\alpha+\alpha^2}}{2\alpha} = \frac{-1+\alpha+(1+\alpha)}{2\alpha} = 1 \\
\sigma_2(1) &= \frac{-1+\alpha+\sqrt{1+2\alpha+\alpha^2-4\alpha}}{2\alpha} = \frac{-1+\alpha+(1-\alpha)}{2\alpha} = 0
\end{aligned}$$

The right solution will be

$$\sigma(u) = \sigma_2(u) = \frac{-1+\alpha+\sqrt{1+2\alpha+\alpha^2-4u\alpha}}{2\alpha}. \tag{3.8}$$

We should remark, that in this case it was easy to find the right solution. In the following we will only write out all the possible solutions. Only after the specification of the parameters one can find the right solution.

Case 2: $n = 3$**:** $\lambda^{23} = 3\lambda^{0*}$

In this case the equation (3.3) will be incomplete cubic equation (for details see http://eqworld.ipmnet.ru/ru/solutions/ae/ae-toc1.htm)

$$1 - u = (1-\alpha)\sigma + \alpha\sigma^3$$

or equivalently

$$\sigma^3 + p\sigma + q = 0 \tag{3.9}$$

with $p = \frac{1-\alpha}{\alpha}$ and $q = -\frac{1-u}{\alpha}$.

The roots of the incomplete cubic equation (3.9) are given by

$$\sigma_1(u) = A(u) + B(u)$$

and

$$\sigma_{2,3}(u) = -\frac{1}{2}\left(A(u) + B(u)\right) \pm i\frac{\sqrt{3}}{2}\left(A(u) - B(u)\right),$$

where

$$A(u) = \left(-\frac{q}{2} + \sqrt{D(u)}\right)^{1/3}, \quad B(u) = \left(-\frac{q}{2\alpha} - \sqrt{D(u)}\right)^{1/3}$$

and

$$D(u) = \left(\frac{p}{3p}\right)^3 + \left(\frac{q}{2}\right)^2, \quad i^2 = -1$$

with $A(u)$ and $B(u)$ being any of the values of the respective cubic roots such that $A(u)B(u) = -\frac{1-\alpha}{3\alpha}$. Using the this property, we can write the roots as

$$\sigma_1(u) = A(u) - \frac{1-\alpha}{3A(u)\alpha}$$

and

$$\sigma_{2,3}(u) = -\frac{1 \mp \sqrt{3}i}{2}A(u) + \frac{1 \pm \sqrt{3}i}{2}\frac{1-\alpha}{3A(u)\alpha},$$

If we substitute $A(u)$ in the previous equations we get

$$\begin{aligned}
\sigma_1(u) &= -\frac{2^{1/3}(1-\alpha)}{\left(27\alpha^2 - 27\alpha^2 u + \sqrt{108(1-\alpha)^3\alpha^3 + (27\alpha^2 - 27\alpha^2 u)^2}\right)^{1/3}} \\
&\quad + \frac{\left(27\alpha^2 - 27\alpha^2 u + \sqrt{108(1-\alpha)^3\alpha^3 + (27\alpha^2 - 27\alpha^2 u)^2}\right)^{1/3}}{3 2^{1/3}\alpha}, \\
\sigma_2(u) &= \frac{(1+i\sqrt{3})(1-\alpha)}{2^{2/3}\left(27\alpha^2 - 27\alpha^2 u + \sqrt{108(1-\alpha)^3\alpha^3 + (27\alpha^2 - 27\alpha^2 u)^2}\right)^{1/3}} \\
&\quad - \frac{(1-i\sqrt{3})\left(27\alpha^2 - 27\alpha^2 u + \sqrt{108(1-\alpha)^3\alpha^3 + (27\alpha^2 - 27\alpha^2 u)^2}\right)^{1/3}}{6 2^{1/3}\alpha}, \\
\sigma_3(u) &= \frac{(1-i\sqrt{3})(1-\alpha)}{2^{2/3}\left(27\alpha^2 - 27\alpha^2 u + \sqrt{108(1-\alpha)^3\alpha^3 + (27\alpha^2 - 27\alpha^2 u)^2}\right)^{1/3}} \\
&\quad - \frac{(1+i\sqrt{3})\left(27\alpha^2 - 27\alpha^2 u + \sqrt{108(1-\alpha)^3\alpha^3 + (27\alpha^2 - 27\alpha^2 u)^2}\right)^{1/3}}{6 2^{1/3}\alpha}.
\end{aligned} \tag{3.10}$$

The right solution according to the remark 3.2 can be easily found after specifying the parameter α.

Case 3: $n = 4$**:** $\lambda^{23} = 4\lambda^{0*}$

In this the equation (3.3) will be incomplete quartic equation (see http://eqworld.ipmnet.ru/en/solutions/ae/ae0108.pdf)

$$1 - u = (1-\alpha)\sigma + \alpha\sigma^4 \tag{3.11}$$

or equivalently

$$\sigma^4 + q\sigma + r = 0 \tag{3.12}$$

with $r = -\frac{u-1}{\alpha}$ and $q = \frac{1-\alpha}{\alpha}$.

Decartes–Euler solution of the incomplete quartic equation are given by

$$\begin{aligned}
\sigma_1(u) &= \frac{1}{2}\left(\sqrt{z_1(u)} + \sqrt{z_2(u)} + \sqrt{z_3(u)}\right), \\
\sigma_2(u) &= \frac{1}{2}\left(\sqrt{z_1(u)} - \sqrt{z_2(u)} - \sqrt{z_3(u)}\right), \\
\sigma_3(u) &= \frac{1}{2}\left(-\sqrt{z_1(u)} + \sqrt{z_2(u)} - \sqrt{z_3(u)}\right), \\
\sigma_4(u) &= \frac{1}{2}\left(-\sqrt{z_1(u)} - \sqrt{z_2(u)} + \sqrt{z_3(u)}\right),
\end{aligned} \tag{3.13}$$

where $z_1(u)$, $z_2(u)$, $z_3(u)$ are roots of the cubic equation

$$z^3 - 4rz - q^2 = 0, \tag{3.14}$$

which is called the cubic resolvent of equation (3.12). The signs of the roots in (3.13) are chosen so that

$$\sqrt{z_1(u)}\sqrt{z_2(u)}\sqrt{z_3(u)} = -\frac{1-\alpha}{\alpha}.$$

The roots of (3.14) can be found by same procedure as in case 2.

Again, given the parameter α one can find the right solution (see remark 3.2) of the equation (3.12) and so the inverse function of the marginal distribution function.

Case 4: $n = \frac{1}{2}$**:** $\lambda^{23} = \frac{1}{2}\lambda^{0*}$

In this case we get quadratic equation

$$1 - u = (1-\alpha)\sigma + \alpha\sigma^{\frac{1}{2}} \tag{3.15}$$

or equivalently

$$(1-\alpha)^2\sigma^2 - \left(2(1-\alpha)(1-u)+\alpha^2\right)\sigma + (1-u)^2 = 0.$$

This equation have two solutions

$$\sigma_{1,2}(u) = \frac{-2(1-\alpha)(1-u)+\alpha^2 \mp \sqrt{4(1-\alpha)(1-u)+\alpha^2}}{2(1-\alpha)^2}.$$

Same as in case 1 the right solution is:

$$\sigma(u) = \frac{-2(1-\alpha)(1-u)+\alpha^2 + \sqrt{4(1-\alpha)(1-u)+\alpha^2}}{2(1-\alpha)^2}.$$

There are also possible to find the solutions of the equation (3.15) by substituting $\sigma^{\frac{1}{2}} = \tau$. In this case equation (3.15) will be reduced to the equation (3.7).

Case 5: $n = \frac{1}{3}$**:** $\lambda^{23} = \frac{1}{3}\lambda^{0*}$

This case will be done same as case 2. In this case we get cubic equation:

$$1 - u = (1-\alpha)\sigma + \alpha\sigma^{\frac{1}{3}} \tag{3.16}$$

or equivalently

$$(1-u-(1-\alpha)\sigma)^3 = \alpha^3\sigma.$$

Opposed to case 2 we have here complete cubic equation. The roots of complete cubic equation

$$a\sigma^3 + b\sigma^2 + c\sigma + d = 0 \tag{3.17}$$

are:

$$\sigma_k = y_k - \frac{b}{3a} \quad k = 1, 2, 3,$$

where the y_k are roots of the incomplete cubic equation (3.9) with coefficients

$$p = -\frac{1}{3}\left(\frac{b}{a}\right)^2 + \frac{c}{a} \quad q = \frac{2}{27}\left(\frac{b}{a}\right)^3 - \frac{bc}{3a^2} + \frac{d}{a}.$$

For details see http://eqworld.ipmnet.ru/ru/solutions/ae/ae-toc1.htm . The solutions y_k can be found in the same way as in case 2.

One can also solve the equation (3.16) by substituting $\sigma^{1/3} = \tau$. The we get exactly equation (3.9) with $p = \frac{\alpha}{1-\alpha}$ and $q = -\frac{1-u}{1-\alpha}$.

Again, given the parameter α one can define the right solution and so the inverse distribution function (see remark 3.2).

Case 6: $n = \frac{1}{4}$**:** $\lambda^{23} = \frac{1}{4}\lambda^{0*}$

This case will be done same as case 3. In this case we get quatic equation:

$$1 - u = (1-\alpha)\sigma + \alpha\sigma^{\frac{1}{4}} \tag{3.18}$$

or equivalently

$$1 - u = (1-\alpha)\tau^4 + \alpha\tau,$$

with $\tau = \sigma^{\frac{1}{4}}$.

We got equation (3.12) with $q = \frac{\alpha}{1-\alpha}$ and $r = -\frac{1-u}{1-\alpha}$.

Case 7: $n = \frac{2}{3}$**:** $\lambda^{23} = \frac{2}{3}\lambda^{0*}$

This case will be done same as case 3. In this case we get quatic equation:

$$1 - u = (1-\alpha)\sigma + \alpha\sigma^{\frac{3}{2}} \tag{3.19}$$

or equivalently

$$1 - u = (1-\alpha)\tau^2 + \alpha\tau^3.$$

We got equation (3.17) with $a = \alpha$, $b = 1 - \alpha$, $c = 0$ and $d = u - 1$.

3.4.2 Numerical Examples

In the following we will consider some numerical examples, where we can easily calculate copula for our model. The problem for copula calculation is that each marginal distribution function is the sum of exponential functions with different arguments. So, in general, it is not easy to find inverse function of marginal distribution function.

The easiest case is the case, where our firms are independent. For our model its means that there no contagion effects. This case will be our first example. The next examples will introduce the cases, where the intensities are dependent in such a way, that inverse functions for distribution functions can be analytically found (see previous section).

Example 3.4.1 (Case 1: Without Contagion Effect) *For the case, where the contagion effect is absent, the τ_A and τ_B are independent. It is mean that $\lambda^{13} = \lambda^{02}$ and $\lambda^{23} = \lambda^{01}$. In this case*

$$F_A(t) = 1 - e^{-t\lambda_{01}}$$

and

$$F_B(s) = 1 - e^{-s\lambda_{02}}.$$

The joint distribution for such intensities is

$$F_{A,B}(t,s) = (1 - e^{t\lambda_{01}})(1 - e^{s\lambda_{02}}) = F_A(t)F_B(s).$$

This means that our copula in this case is only a product copula, i.e.

$$C(u,v) = uv.$$

In the Figure 3.2 we can see the copula and the contour plot of it.

The Kendall's τ is equal to 0, while the Spearmans's ρ is also equal to zero. The upper and lower tail dependance measures are also zero.

To simplify our mathematical calculations let us at the begin rewrite the marginal and joint distribution functions in terms of $\alpha \triangleq \frac{\lambda^{02}}{\lambda^{0*} - \lambda^{23}}$, $\beta \triangleq \frac{\lambda^{01}}{\lambda^{0*} - \lambda^{13}}$, $m \triangleq \frac{\lambda^{23}}{\lambda^{0*}}$ and $n = \frac{\lambda^{13}}{\lambda^{0*}}$.

We get

$$F_A(\sigma(t)) = \hat{F}_A(\sigma) = 1 - (1-\alpha)\,\sigma - \alpha\sigma^m, \tag{3.20}$$

$$F_B(\tau(s)) = \hat{F}_B(\tau) = 1 - (1-\beta)\,\tau - \beta\tau^n, \tag{3.21}$$

with $\sigma(t) = \sigma = e^{-\lambda^{0*}t}$, $\tau(s) = \tau = e^{-\lambda^{0*}s}$. For the joint distribution function we have two functions: $F^1_{A,B}(t,s)$ for $t < s$ (or equivalently $\sigma(t) > \tau(s)$) and $F^2_{A,B}(t,s)$

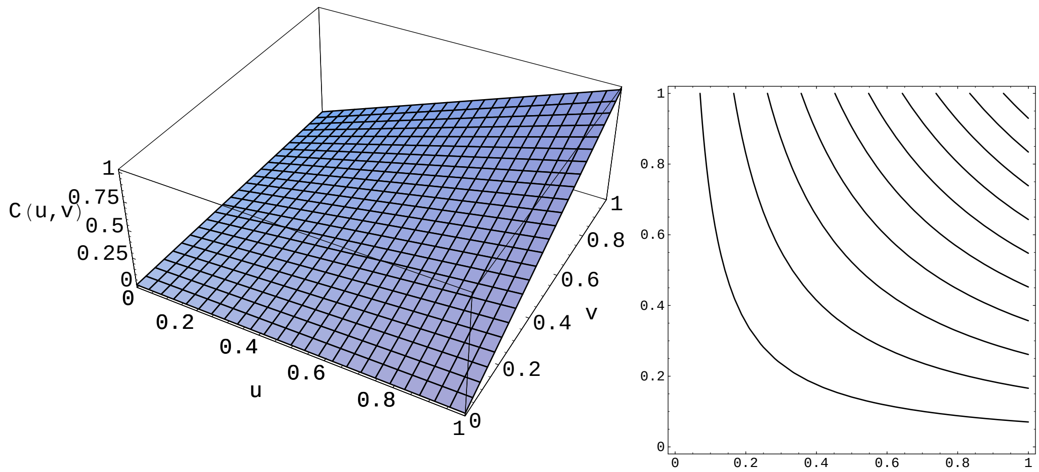

Figure 3.2: **Copula for the case of two independent firms**

for $t \geq s$ (or equivalently $\sigma(t) \leq \tau(s)$):

$$F^1_{A,B}(\sigma(t),\tau(s)) = \hat{F}^1_{A,B}(\sigma,\tau) = 1 - \beta\tau^n + \sigma\left(-1+\beta\left(\frac{\tau}{\sigma}\right)^n + \alpha\right) - \alpha\sigma^m \quad (3.22)$$

$$F^2_{A,B}(\sigma(t),\tau(s)) = \hat{F}^2_{A,B}(\sigma,\tau) = 1 - \beta\tau^n + \tau\left(-1+\alpha\left(\frac{\sigma}{\tau}\right)^m + \beta\right) - \alpha\sigma^m \quad (3.23)$$

To define the dependance coefficients we can use function

$$H(x,y) = \begin{cases} \hat{F}^1_{A,B}(x,y), & \text{for } x > y, \\ \hat{F}^2_{A,B}(x,y), & \text{for } x \leq y \end{cases} \quad (3.24)$$

with $x, y \in [0,1]$.

Then according to the section 2.2 we can define:

$$\tau_K(A,B) = 1 - 4\int\int_{[0,1]^2} \frac{\partial}{\partial x}H(x,y)\frac{\partial}{\partial y}H(x,y)dxdy,$$

$$\rho_s(X,Y) = 12\int_0^1\int_0^1 [H(x,y) - \hat{F}_A(x)\hat{F}_B(y)]d\hat{F}_A(x)d\hat{F}_B(y).$$

Remember that $H(x,y)$ consist of two parts its means that

$$\int\int_{[0,1]^2} H(x,y)dxdy = \int_0^1\int_0^x \hat{F}^2_{A,B}(x,y)dydx + \int_0^1\int_0^y \hat{F}^1_{A,B}(x,y)dxdy.$$

Such decomposition holds also for the derivatives of $H(x,y)$, i.e

$$
\begin{aligned}
\tau_K(A,B) &= 1-4\left[\int_0^1\int_0^x \frac{\partial}{\partial x}\hat{F}^2_{A,B}(x,y)\frac{\partial}{\partial y}\hat{F}^2_{A,B}(x,y)dydx\right.\\
&\quad\left.+\int_0^1\int_0^y \frac{\partial}{\partial x}\hat{F}^1_{A,B}(x,y)\frac{\partial}{\partial y}\hat{F}^1_{A,B}(x,y)dxdy\right],\\
\rho_s(X,Y) &= 12\int_0^1\int_0^x\left[\hat{F}^2_{A,B}(x,y)-\hat{F}_A(x)\hat{F}_B(y)\right]d\hat{F}_B(y)d\hat{F}_A(x)\\
&\quad+12\int_0^1\int_0^y\left[\hat{F}^1_{A,B}(x,y)-\hat{F}_A(x)\hat{F}_B(y)\right]d\hat{F}_A(x)d\hat{F}_B(y).
\end{aligned}
$$

Substituting equations (3.20), (3.21) in the previous equation we get

$$
\begin{aligned}
\tau_K(A,B) &= 1-\frac{2\alpha(m-1)(n+1)+\alpha^2(m-1)^2(n+1)}{(n+1)(m+1)}\\
&\quad-\frac{(m+1)\left(\beta(2+\beta(n-1))(n-1)+2(n+1)\right)}{(n+1)(m+1)},\\
\rho_s(X,Y) &= \frac{1}{(1+m)(2+m)(1+n)(2+n)(1+m+n)}\\
&\quad\times\Big((1+m)(2+m)(1+m+n)\left(4\beta(-1+n)+2\beta^2(-1+n)^2+(1+n)(2+n)\right)\\
&\quad+2\alpha^2(-1+m)^2(1+n)(\beta(2+2m-n)(-1+n)+(2+n)(1+m+n))\\
&\quad-2\alpha(-1+m)\Big(-2(1+n)(2+n)(1+m+n)\\
&\quad+\beta^2(1+m)(-1+n)^2(m-2(1+n))\\
&\quad+2\beta(-1+n)\left(2n(1+n)+m^2(2+n)+m(2+n(2+n))\right)\Big)\Big).
\end{aligned}
$$

For detailed calculations see appendix B. Now we will proceed with numerical examples.

Case 2: With contagion effect $\lambda^{13}=\lambda^{23}=2(\lambda^{01}+\lambda^{02})$

In this case the default intensity of each firm increases in two times after the default of another firm. For our model it means that there is a contagion effect, because the probability of default growth after the default of counterparty.

For this case the margin distribution functions of defaults times τ_A and τ_B according to the section 3.2.1 are

$$
F_A(t)=\frac{(e^{-t(\lambda^{01}+\lambda^{02})}-1)((-\lambda^{01}+(e^{-t(\lambda^{01}+\lambda^{02})}-1)\lambda^{02})}{\lambda^{01}+\lambda^{02}}
$$

and

$$F_B(t) = \frac{(e^{-t(\lambda^{01}+\lambda^{02})} - 1)((\lambda^{01}(e^{-t(\lambda^{01}+\lambda^{02})} - 1) - \lambda^{02})}{\lambda^{01} + \lambda^{02}}$$

respectively. The joint distribution function for $t \leq s$ according to the section 3.2.2 is

$$\begin{aligned} F_{A,B}(t,s) &= \frac{e^{-3t(\lambda^{01}+\lambda^{02})}(e^{t(\lambda^{01}+\lambda^{02})} - 1)}{\lambda^{01} + \lambda^{02}} \\ &\times \left(-\lambda^{01} e^{(3t-2s)(\lambda^{01}+\lambda^{02})} - \lambda^{02} e^{t(\lambda^{01}+\lambda^{02})} + (\lambda^{01} + \lambda^{02}) e^{2t(\lambda^{01}+\lambda^{02})}\right) \end{aligned}$$

and the joint distribution function for the case $t > s$ according to the section 3.2.2 is

$$\begin{aligned} F_{A,B}(t,s) &= \frac{e^{-2t(\lambda^{01}+\lambda^{02})}(e^{t(\lambda^{01}+\lambda^{02})} - 1)}{\lambda^{01} + \lambda^{02}} \\ &\times \left(-\lambda^{02} - \lambda^{01} e^{2(t-s)(\lambda^{01}+\lambda^{02})} + (\lambda^{01} + \lambda^{02}) e^{(2t-s)(\lambda^{01}+\lambda^{02})}\right). \end{aligned}$$

The marginal and joint distribution functions in terms of α, β, τ and σ are described by equations (3.20)-(3.23). In our case $n = m = 2$,

$$\alpha = \frac{-\lambda^{02}}{\lambda^{0*}} \qquad \text{and} \qquad \beta = \frac{-\lambda^{01}}{\lambda^{0*}}.$$

The inverse marginal distribution functions can be found according to the equation (3.8). So we get the inverse function for distribution function of the firm's A default time as

$$\begin{aligned} \hat{F}_A^{-1}(u) &= \frac{-1 + \alpha + \sqrt{1 + 2\alpha + \alpha^2 - 4u\alpha}}{2\alpha} \\ &= \frac{\lambda^{01} + 2\lambda^{02} - \sqrt{(\lambda^{01})^2 + 4u\lambda^{01}\lambda^{02} + 4u(\lambda^{02})^2}}{2\lambda^{02}} \end{aligned} \tag{3.25}$$

and inverse function for distribution function of the firm's B default time as

$$\begin{aligned} \hat{F}_B^{-1}(v) &= \frac{-1 + \beta + \sqrt{1 + 2\beta + \beta^2 - 4v\beta}}{2\beta} \\ &= \frac{2\lambda^{01} + \lambda^{02} - \sqrt{4v(\lambda^{01})^2 + 4v\lambda^{01}\lambda^{02} + (\lambda^{02})^2}}{2\lambda^{01}}. \end{aligned} \tag{3.26}$$

As soon as we've found inverse functions $\hat{F}_A^{-1}(u)$ and $\hat{F}_B^{-1}(v)$, the inverse functions of the marginal distribution functions can be found as

$$F_A^{-1}(u) = -\frac{\ln\left(\hat{F}_A^{-1}(u)\right)}{\lambda^{0*}} \text{ and } F_B^{-1}(v) = -\frac{\ln\left(\hat{F}_B^{-1}(v)\right)}{\lambda^{0*}}.$$

Now we can find copula just by using the procedure described in section 2.3.1:

$$C(u,v) = F_{A,B}\left(F_A^{-1}(u), F_B^{-1}(v)\right) = \hat{F}_{A,B}\left(\hat{F}_A^{-1}(u), \hat{F}_B^{-1}(v)\right).$$

Or equivalently, we will get the copula function by substituting inverse functions (3.25) and (3.26) in the equations (3.22) and (3.23). For the case $\hat{F}_A^{-1}(u) \geq \hat{F}_B^{-1}(v)$ we get:

$$\begin{aligned} C(u,v) &= u + \frac{\lambda^{01} + 2u\lambda^{02} - \sqrt{(\lambda^{01})^2 + 4u\lambda^{02}(\lambda^{01}+\lambda^{02})}}{2\lambda^{01}} \\ &\times \frac{\left(2\lambda^{01} + \lambda^{02} - \sqrt{(\lambda^{02})^2 + 4v\lambda^{01}(\lambda^{02}+\lambda^{01})}\right)^2}{\left(\lambda^{01} + 2\lambda^{02} - \sqrt{(\lambda^{01})^2 + 4u\lambda^{02}(\lambda^{01}+\lambda^{02})}\right)^2}. \end{aligned} \tag{3.27}$$

Doing the same procedure with the joint distribution function (3.23) for the case $t > s$ we will find the second part of copula-function for the case $\hat{F}_A^{-1}(u) < \hat{F}_B^{-1}(v)$:

$$\begin{aligned} C(u,v) &= v + \frac{2v\lambda^{01} + \lambda^{02} - \sqrt{(\lambda^{02})^2 + 4v\lambda^{01}(\lambda^{01}+\lambda^{02})}}{2\lambda^{02}} \\ &\times \frac{\left(\lambda^{01} + 2\lambda^{02} - \sqrt{(\lambda^{01})^2 + 4u\lambda^{02}\left(\lambda^{02}+\lambda^{02}\right)}\right)^2}{\left(2\lambda^{01} + \lambda^{02} - \sqrt{(\lambda^{02})^2 + 4v\lambda^{01}(\lambda^{01}+\lambda^{02})}\right)^2}. \end{aligned} \tag{3.28}$$

For illustration of our result consider the following numerical examples:

Example 3.4.2 *Let $\lambda^{01} = \lambda^{02} = 0.01$ and $\lambda^{23} = \lambda^{13} = 0.04$. In this case according to the equations (3.27) and (3.28) the copula function is*

$$C(u,v) = v + \frac{\left(1 + 2v - \sqrt{1+8v}\right)\left(3 - \sqrt{1+8u}\right)^2}{2\left(3 - \sqrt{1+8v}\right)^2}$$

for $u \geq v$, and

$$C(u,v) = u + \frac{\left(1 + 2u - \sqrt{1+8u}\right)\left(3 - \sqrt{1+8v}\right)^2}{2\left(3 - \sqrt{1+8u}\right)^2}$$

for $u < v$. The Figure 3.3 show this copula and the contour plot of it. The copula in this case is symmetric about $u = v$, because the $\lambda^{01} = \lambda^{02}$.

The Kendall's τ in this case is equal to $\frac{1}{2}$, while the Spearmans's ρ is equal to $\frac{27}{40}$. The upper and lower tail dependance measures are corresponding $\lambda_U = \frac{2}{3}$ and $\lambda_L = 0$.

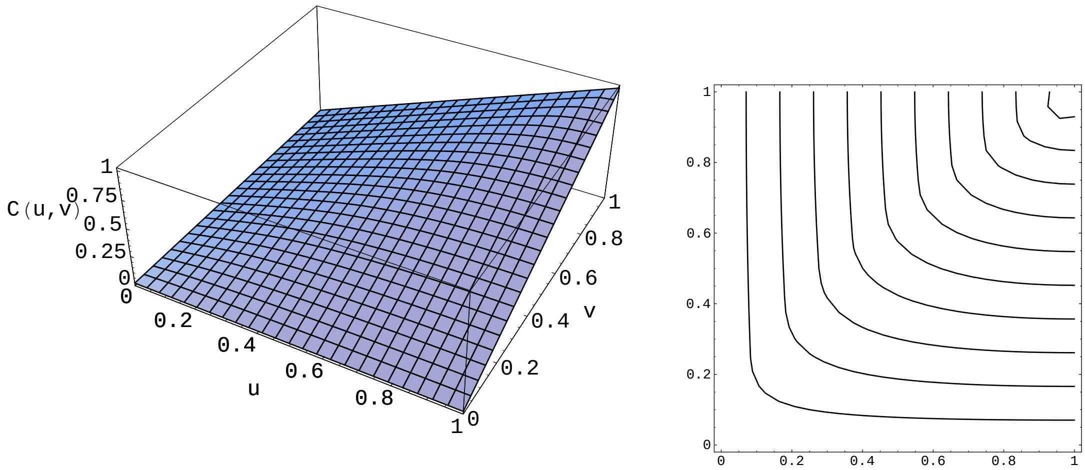

Figure 3.3: **Copula for the case** $\lambda^{01} = \lambda^{02} = 0.01$**,** $\lambda^{23} = \lambda^{13} = 0.04$

Example 3.4.3 *Let* $\lambda^{01} = 1/1500$*,* $\lambda^{02} = 1/100$ *and* $\lambda^{23} = \lambda^{13} = 32/1500$ *Again according to the equations (3.27) and (3.28) the copula function for this case is*

$$C(u,v) \;=\; v + \frac{\left(2v + 15 - \sqrt{225+64v}\right)\left(31 - \sqrt{1+960u}\right)^2}{2\left(17 - \sqrt{225+64v}\right)^2}$$

for $\sqrt{1+960v} \leq 15\sqrt{225+64u} - 224$*, and*

$$C(u,v) \;=\; u + \frac{\left(30u + 1 - \sqrt{1+960u}\right)\left(\sqrt{225+64v} - 17\right)^2}{2\left(\sqrt{1+960u} - 31\right)^2}$$

for $\sqrt{1+960v} > 15\sqrt{225+64u} - 224$*.*

The Figure 3.4 show this copula and the contour plot of it.

The Kendall's τ *in this case is equal to* $\frac{241}{384}$*, while the Spearmans's* ρ *is equal to* $1223/1536$*. The upper and lower tail dependence measures are correspondingly* $\lambda_U = \frac{258}{289}$ *and* $\lambda_L = 0$*.*

Case 3: $\lambda^{13} = \lambda^{23} = n(\lambda^{01} + \lambda^{02}),\ n \in \mathbb{N}$

Now we can extend previous case and assume, that $\lambda^{13} = \lambda^{23} = n(\lambda^{01} + \lambda^{02})$, $n \in \mathbb{N}$. Following the same procedure as in the previous case, we can easily find the copula

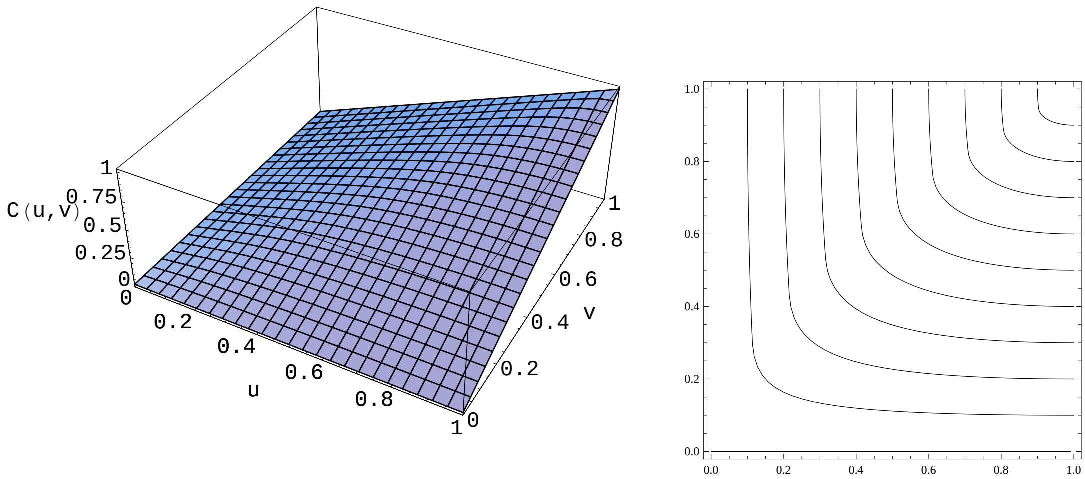

Figure 3.4: **Copula for the case** $\lambda^{01} = 1/100$**,** $\lambda^{02} = 1/1500$**,** $\lambda^{23} = \lambda^{13} = 8/375$

function for this case. So accordingly to the section 3.2.1, the marginal distribution function are:

$$F_A(t) = 1 - e^{-t(\lambda^{01}+\lambda^{02})} - \frac{\lambda^{01} e^{-nt(\lambda^{01}+\lambda^{02})}(e^{-(n-1)t(\lambda^{01}+\lambda^{02})} - 1)}{(n-1)(\lambda^{01}+\lambda^{02})}$$

and

$$F_B(t) = 1 - e^{-t(\lambda^{01}+\lambda^{02})} - \frac{\lambda^{02} e^{-nt(\lambda^{01}+\lambda^{02})}(e^{-(n-1)t(\lambda^{01}+\lambda^{02})} - 1)}{(n-1)(\lambda^{01}+\lambda^{02})}.$$

According to the section 3.2.2 the joint distribution function for $t \leq s$ is:

$$\begin{aligned} F_{A,B}(t,s) &= \frac{e^{-nt(\lambda^{01}+\lambda^{02})}}{(n-1)(\lambda^{01}+\lambda^{02})} \\ &\quad \times \Big(e^{n(t-s)(\lambda^{01}+\lambda^{02})}\lambda^{01} - e^{((2n-1)t-ns)(\lambda^{01}+\lambda^{02})}\lambda^{01} + \lambda^{02} \\ &\quad + e^{nt(\lambda^{01}+\lambda^{02})}(n-1)(\lambda^{01}+\lambda^{02}) + e^{(n-1)t(\lambda^{01}+\lambda^{02})}(\lambda^{01} - n(\lambda^{01}+\lambda^{02}))\Big) \end{aligned}$$

and

$$\begin{aligned} F_{A,B}(t,s) &= \frac{e^{-nt(\lambda^{01}+\lambda^{02})}}{(n-1)(\lambda^{01}+\lambda^{02})} \\ &\quad \times \Big(e^{n(t-s)(\lambda^{01}+\lambda^{02})}\lambda^{01} - e^{(n_1)s(\lambda^{01}+\lambda^{02})}\lambda^{02} + \lambda^{02} \end{aligned}$$

$$+e^{nt(\lambda^{01}+\lambda^{02})}(n-1)(\lambda^{01}+\lambda^{02})+e^{(nt-s)(\lambda^{01}+\lambda^{02})}(\lambda^{02}-n(\lambda^{01}+\lambda^{02}))\Big)$$

for $t > s$.

Again we can rewrite marginal and joint distribution functions in terms of α, β, τ and σ. In our case $m = n$, this means that

$$\alpha = \frac{\lambda^{02}}{(1-n)\lambda^{0*}} \qquad \text{and} \qquad \beta = \frac{\lambda^{01}}{(1-m)\lambda^{0*}}.$$

According to the equations (3.20) and (3.21) the marginal distribution functions are

$$\begin{aligned} F_A(\sigma(t)) &= \hat{F}_A(\sigma) = \frac{(n-n\sigma+\sigma-1)\lambda^{01}+(n-1+\sigma^n-n\sigma)\lambda^{02}}{(n-1)(\lambda^{01}+\lambda^{02})} \\ &= 1-(1-\alpha)\sigma-\alpha\sigma^n \end{aligned}$$

and

$$\begin{aligned} F_B(\tau(s)) &= \hat{F}_B(\tau) = \frac{(n-n\tau+\tau-1)\lambda^{02}+(n-1+\tau^n-n\tau)\lambda^{01}}{(n-1)(\lambda^{01}+\lambda^{02})} \\ &= 1-(1-\beta)\tau-\beta\tau^n. \end{aligned}$$

The joint distribution function for $\sigma(t) \leq \tau(s)$ according to the equation (3.22 is

$$\begin{aligned} F^1_{A,B}(\sigma(t),\tau(s)) &= \hat{F}^1_{A,B}(\sigma,\tau) \\ &= -\frac{((\sigma-1)n+1-\sigma-\tau^n+\sigma^{n-1}y^n)\lambda^{01}-(n-1-n\sigma+\sigma^n)\lambda^{02}}{(\lambda^{01}+\lambda^{02})(n-1)} \\ &= 1-\beta\tau^n+\sigma\left(-1+\beta\left(\frac{\tau}{\sigma}\right)^n+\alpha\right)-\alpha\sigma^n \end{aligned}$$

and according to the equation (3.23)

$$\begin{aligned} F^2_{A,B}(\sigma(t),\tau(s)) &= \hat{F}^2_{A,B}(\sigma,\tau) \\ &= \frac{(n-1-n\tau+\tau^n)\lambda^{01}-(1+n(\tau-1)-\tau+\sigma^n(\tau^{1-n}-1))\lambda^{02}}{(\lambda^{01}+\lambda^{02})(n-1)} \\ &= 1-\beta\tau^n+\tau\left(-1+\alpha\left(\frac{\sigma}{\tau}\right)^n+\beta\right)-\alpha\sigma^n \end{aligned}$$

for $\sigma(t) > \tau(s)$.

The inverse marginal distribution functions can be found according to the section 3.4.1. Then we have to choose the solutions in such way, that time variables t and s

are positive. This implies that the solution of this equations $\sigma(u)$ and $\tau(v)$ are from 0 to 1.

Unfortunately, it is impossible to write down a general solutions of n-th power equation. So let us consider two numerical examples for odd and even n.

Example 3.4.4 $\mathbf{n = 2k + 1}$ *In this example we will consider the case, where $k = 1$. This implies $n = 3$. According to the section 3.4.1 the inverse marginal distribution functions can be found by solving the equations (3.3) for $\hat{F}_A(\sigma)$ and $\hat{F}_B(\tau)$, They both are cubic equations. Each of this equations has three solutions, they are defined by equations (3.10). We have to choose one of them, such that time variables t and s are from 0 to ∞, or equivalently σ and τ from 0 to 1.*

To choose the right solution we need to specify other parameters. So let $\lambda^{01} = 0.001$ and $\lambda^{02} = 0.002$. The right solutions are:

$$\sigma(u) = \hat{F}_A^{-1}(u) = \frac{48 + A^2(u)}{6A(u)}$$

and

$$\tau(v) = \hat{F}_B^{-1}(v) = \frac{84 + B^2(v)}{6B(v)}$$

with

$$A(u) = \sqrt[3]{12}(27 - 27u - \sqrt{-39 - 1458u - 729u^2})^{\frac{1}{3}}$$

and

$$B(v) = 2\sqrt[3]{3}(27 - 27v - \sqrt{-300 - 1458v - 729v^2})^{\frac{1}{3}},$$

and $u, v \in [0; 1]$.

So, now we have to substitute the inverse functions of marginal distribution function into the joint distribution function to get the copula function. So we get

$$C(u, v) = 1 + \frac{512}{3A^3(u)} + \frac{A^3(u)}{648} + \frac{(2034 + 60A^2(u) + A^4(u))(84 + B^2(v))^3}{1296(48 + A^2(u))^2 B^3(v)}$$

for $\sigma(u) \leq \tau(v)$, and

$$C(u, v) = 1 + \frac{1372}{3B^3(v)} + \frac{B^3(v)}{1296} + \frac{(7056 + 132B^2(v) + B^4(v))(48 + A^2(u))^3}{648(84 + B^2(v))^2 A^3(u)}$$

for $\sigma(u) > \tau(v)$ with $A(u)$ and $B(v)$ defined above.

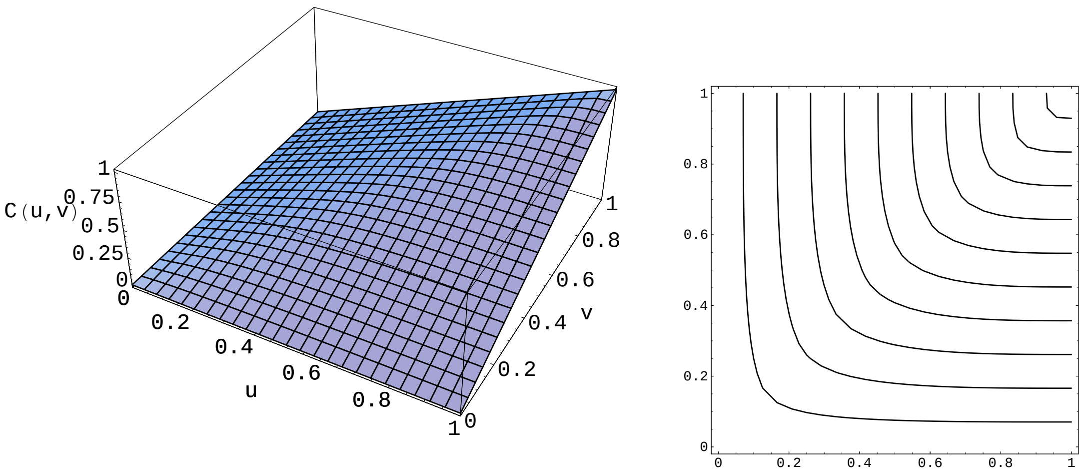

Figure 3.5: **Copula for the case** $\lambda^{01} = 0.001$**,** $\lambda^{02} = 0.002$**,** $\lambda^{23} = \lambda^{13} = 0.009$

The Figure 3.5 show this copula and contour plot of it graphically.

The Kendall's τ in this case is equal to $\frac{23}{36}$, while the Spearmans's ρ is equal to $\frac{57}{70}$.

Example 3.4.5 $n = 2k$

To not involve very complicated calculations we take $k = 2$ and consequently $n = 4$. According to the section 3.4.1 the inverse marginal distribution functions can be found by solving the equations (3.3) for $\hat{F}_A(\sigma)$ and $\hat{F}_B(\tau)$. They both are quatic equations. Each of these equations has four solutions, they are defined by equations (3.13). We have to choose one of them, such that time variables t and s are from 0 to ∞, or equivalently σ and τ from 0 to 1.

To specify the right solutions let $\lambda^{01} = 0.001$ and $\lambda^{02} = 0.002$.

The solutions we need are:

$$\hat{F}_A^{-1}(u) = \sigma(u) = \frac{1}{4}\left(\sqrt{2}\sqrt{A(u)} - 2\sqrt{-\frac{1}{2}A(u) + \frac{11\sqrt{2}}{\sqrt{A(u)}}}\right)$$

and

$$\hat{F}_B^{-1}(v) = \tau(v) = \frac{1}{2}\left(\sqrt{B(v)} - \sqrt{-B(v) + \frac{20}{\sqrt{B(v)}}}\right)$$

with

$$A(u) = \frac{24 - 24u + \left(121 - \sqrt{817 + 41472u - 41472u^2 + 13824u^3}\right)^{\frac{2}{3}}}{\left(121 - \sqrt{817 + 41472u - 41472u^2 + 13824u^3}\right)^{\frac{1}{3}}}$$

and

$$B(v) = \frac{12 - 12v + \left(50 - 2\sqrt{193 + 1296u - 1296u^2 + 432u^3}\right)^{\frac{2}{3}}}{\left(50 - 2\sqrt{193 + 1296u - 1296u^2 + 432u^3}\right)^{\frac{1}{3}}}.$$

Then the copula for this case will be

$$\begin{aligned} C(u,v) &= 1 + \frac{\sqrt{\frac{22\sqrt{2}}{\sqrt{A(u)}} - A(u)} - \sqrt{A(u)}}{2\sqrt{2}} \\ &\quad + \frac{1}{144A(u)}\left(-242 + 18A(u)\sqrt{\frac{44\sqrt{2}}{\sqrt{A(u)}} - 2A(u)} - 18\sqrt{2}A^{\frac{3}{2}}(u) + A^3(u)\right) \\ &\quad \times \left(-2 + \frac{4\left(\sqrt{\frac{20}{\sqrt{B(v)}} - B(v)} - \sqrt{B(v)}\right)^4}{\left(\sqrt{\frac{22\sqrt{2}}{\sqrt{A(u)}} - A(u)} - \sqrt{A(u)}\right)^4}\right) \end{aligned}$$

for $\sigma(u) \leq \tau(v)$ *, and*

$$\begin{aligned} C(u,v) &= 1 + \frac{\sqrt{\frac{20}{\sqrt{B(v)}} - B(v)} - \sqrt{B(v)}}{2} \\ &\quad + \frac{1}{36B(v)}\left(-100 + 18B(v)\sqrt{\frac{20}{\sqrt{B(v)}} - B(v)} - 18B^{\frac{3}{2}}(v) + B^3(v)\right) \\ &\quad \times \left(-1 + \frac{\left(\sqrt{\frac{22\sqrt{2}}{\sqrt{A(u)}} - A(u)} - \sqrt{A(u)}\right)^4}{4\left(\sqrt{\frac{20}{\sqrt{B(v)}} - B(v)} - \sqrt{B(v)}\right)^4}\right) \end{aligned}$$

for $\sigma(u) > \tau(v)$ *with* $A(u)$ *and* $B(v)$ *defined above.*

The Figure 3.6 show this copula and contour plot of it.

The Kendall's τ *in this case is equal to* $\frac{32}{45}$ *, while the Spearmans's* ρ *is equal to* 1064/1215.

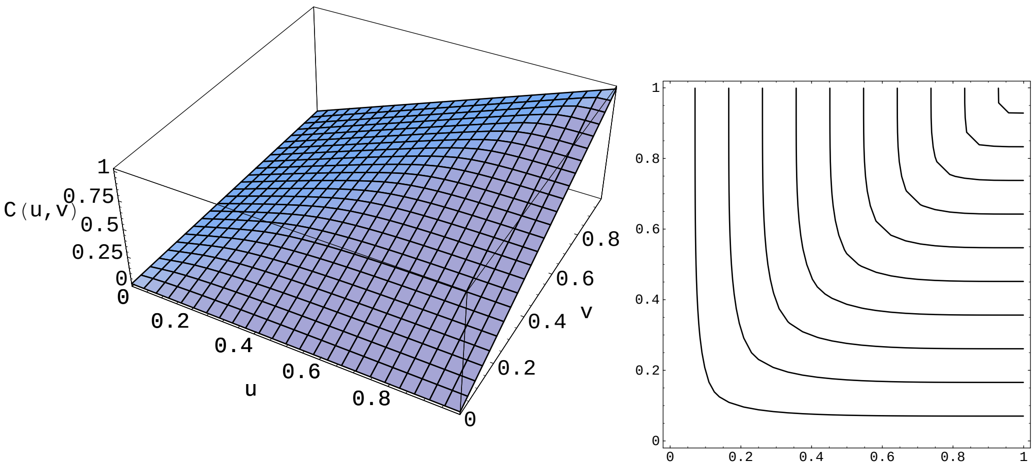

Figure 3.6: **Copula for the case** $\lambda^{01} = 0.001$**,** $\lambda^{02} = 0.002$**,** $\lambda^{23} = \lambda^{13} = 0.012$

Case 4: $\lambda^{13} = n(\lambda^{01} + \lambda^{02})$**,** $\lambda^{23} = m(\lambda^{01} + \lambda^{02})$**,** $n, m \in \mathbb{N}$

Now we can generalize case 3. Let's take $\lambda^{13} = n(\lambda^{01} + \lambda^{02})$ and $\lambda^{23} = m(\lambda^{01} + \lambda^{02})$ with $n, m \in \mathbb{N}$.

Then the margin distribution functions according to the chapter 3.2.1 are:

$$F_A(t) = 1 - e^{-t(\lambda^{01}+\lambda^{02})} - \frac{\lambda^{01} e^{-mt(\lambda^{01}+\lambda^{02})}(e^{-(m-1)t(\lambda^{01}+\lambda^{02})} - 1)}{(m-1)(\lambda^{01}+\lambda^{02})}$$

and

$$F_B(t) = 1 - e^{-t(\lambda^{01}+\lambda^{02})} - \frac{\lambda^{02} e^{-nt(\lambda^{01}+\lambda^{02})}(e^{-(n-1)t(\lambda^{01}+\lambda^{02})} - 1)}{(n-1)(\lambda^{01}+\lambda^{02})}.$$

The joint distribution function according to the 3.2.2 for $t < s$ is:

$$\begin{aligned} F_{A,B}(t,s) &= 1 + \frac{\lambda^{01}(e^{-ns(\lambda^{01}+\lambda^{02})} - e^{-(n(s-t)+t)(\lambda^{01}+\lambda^{02})})}{(n-1)(\lambda^{01}+\lambda^{02})} \\ &\quad + \frac{\lambda^{02} e^{-mt(\lambda^{01}+\lambda^{02})}}{(m-1)(\lambda^{01}+\lambda^{02})} - e^{-t(\lambda^{01}+\lambda^{02})}\left(1 + \frac{\lambda^{02}}{(m-1)(\lambda^{01}+\lambda^{02})}\right) \end{aligned}$$

and

$$F_{A,B}(t,s) = 1 - e^{-s(\lambda^{01}+\lambda^{02})} - \frac{\lambda^{02} e^{-mt(\lambda^{01}+\lambda^{02})}\left(-1 + e^{(m-1)s(\lambda^{01}+\lambda^{02})}\right)}{(m-1)(\lambda^{01}+\lambda^{02})}$$

$$-\frac{\left(e^{-s(\lambda^{01}+\lambda^{02})} - e^{-ns(\lambda^{01}+\lambda^{02})}\right)\lambda^{01}}{(n-1)(\lambda^{01}+\lambda^{02})}$$

for $t \geq s$.

We can rewrite marginal and joint distribution functions in terms of α, β, τ and σ using equations (3.20)-(3.23). In our case

$$\alpha = \frac{\lambda^{02}}{(1-m)\lambda^{0*}} \qquad \text{and} \qquad \beta = \frac{-\lambda^{01}}{(1-n)\lambda^{0*}}.$$

Then the marginal distribution functions are

$$\begin{aligned} F_A(\sigma(t)) &= \hat{F}_A(\sigma) = \frac{(m - mx + x - 1)\lambda^{01} + (m - 1 + x^m - mx)\lambda^{02}}{(m-1)(\lambda^{01}+\lambda^{02})} \\ &= 1 - (1-\alpha)\,\sigma - \alpha\sigma^m \end{aligned}$$

and

$$\begin{aligned} F_B(\tau(s)) &= \hat{F}_B(\tau) = \frac{(n - ny + y - 1)\lambda^{02} + (n - 1 + y^n - ny)\lambda^{01}}{(n-1)(\lambda^{01}+\lambda^{02})} \\ &= 1 - (1-\beta)\,\tau - \beta\tau^n \end{aligned}$$

with $\sigma(t) = \sigma = e^{-\lambda^{0*}t}$, $\tau(s) = \tau = e^{-\lambda^{0*}s}$.

The joint distribution function for $\sigma(t) \leq \tau(s)$ is

$$\begin{aligned} F^1_{A,B}(\sigma(t),\tau(s)) &= \hat{F}^1_{A,B}(\sigma,\tau) \\ &= -1 + \frac{y^n(1-x^{1-n})\lambda^{01}}{(\lambda^{01}+\lambda^{02})(n-1)} + \frac{\lambda^{02}x^m}{(m-1)(\lambda^{01}+\lambda^{02})} \\ &\quad -x\left(1 + \frac{\lambda^{02}}{(m-1)(\lambda^{01}+\lambda^{02})}\right) \\ &= 1 - \beta\tau^n + \tau\left(-1 + \alpha\left(\frac{\sigma}{\tau}\right)^m + \beta\right) - \alpha\sigma^m \end{aligned}$$

and

$$\begin{aligned} F^2_{A,B}(\sigma(t),\tau(s)) &= \hat{F}^2_{A,B}(\sigma,\tau) \\ &= 1 + \frac{x^m(1-y^{1-m})\lambda^{02}}{(\lambda^{01}+\lambda^{02})(m-1)} + \frac{\lambda^{01}y^n}{(n-1)(\lambda^{01}+\lambda^{02})} \\ &\quad -y\left(1 + \frac{\lambda^{01}}{(n-1)(\lambda^{01}+\lambda^{02})}\right) \\ &= 1 - \beta\tau^n + \sigma\left(-1 + \beta\left(\frac{\tau}{\sigma}\right)^n + \alpha\right) - \alpha\sigma^m \end{aligned}$$

for $\sigma(t) > \tau(s)$.

The inverse marginal distribution functions can be found according to the section 3.4.1. Then we have to choose the solutions in such way, that time variables t and s are positive. This implies that the solution of this equations $\sigma(u)$ and $\tau(v)$ are from 0 to 1.

Consider a numerical example:

Example 3.4.6 *Let $n = 2$, $m = 4$, $\lambda^{01} = 0.001$, and $\lambda^{02} = 0.0015$.*

Following all previous steps: Find inverse functions of marginal distribution functions, substituting them into the joint distribution function, we get the copula function. The inverse functions of marginal distribution functions according to the equations (3.8) for $\hat{F}_B(\sigma)$ and equations (3.13) for $\hat{F}_A(\tau)$ are:

$$\hat{F}_A^{-1}(u) = \sigma(u) = -\frac{1}{2}\sqrt{-A(u) + \frac{12}{A(u)}} + \frac{1}{2}\sqrt{A(u)},$$

with

$$A(u) = B(u) + \frac{20(1-u)}{3B(u)}$$

and

$$B(u) = \frac{\left(972 - \sqrt{944784 - 4(60-60u)^3}\right)^{1/3}}{(54)^{1/3}},$$

and

$$\hat{F}_B^{-1}(v) = \tau(v) = \frac{1}{4}\left(7 - \sqrt{9+40v}\right).$$

The copula function for this case is:

$$\begin{aligned} C(u,v) \;=\; & \frac{1}{120A(u) - 20\sqrt{\frac{12}{\sqrt{A(u)}} - A(u)}A(u)^2} \big(216 - 6A(u)^3 \\ & + \sqrt{\frac{12}{\sqrt{A(u)}} - A(u)}A(u)^4 \\ & + A(u)\left(294 - 36\sqrt{\frac{12}{\sqrt{A(u)}} - A(u)} + 120v - 42\sqrt{9+40v}\right) \\ & + \sqrt{\frac{12}{\sqrt{A(u)}} - A(u)}A(u)^{3/2}\left(29 + 20v - 7\sqrt{9+40v}\right) \end{aligned}$$

$$+ A(u)^2\left(-29-49\sqrt{\frac{12}{\sqrt{A(u)}}-A(u)}+\left(-20-20\sqrt{\frac{12}{\sqrt{A(u)}}-A(u)}\right)v\right.$$
$$\left.\left.+ 7\sqrt{9+40v}+7\sqrt{\frac{12}{\sqrt{A(u)}}-A(u)}\sqrt{9+40v}\right)\right)$$

for $\sigma(u) \le \tau(v)$ *, and*

$$\begin{aligned} C(u,v) &= \frac{1}{\left(-7+\sqrt{9+40v}\right)^4}8\left(\frac{117}{16}\left(\sqrt{\frac{12}{\sqrt{A(u)}}-A(u)}-\sqrt{A(u)}\right)^4\right. \\ &+ \frac{39}{2}\left(\sqrt{\frac{12}{\sqrt{A(u)}}-A(u)}-\sqrt{A(u)}\right)^4 v+\frac{5}{2}\left(\sqrt{\frac{12}{\sqrt{A(u)}}-A(u)}-\sqrt{A(u)}\right)^4 v^2 \\ &- \frac{39}{16}\left(\sqrt{\frac{12}{\sqrt{A(u)}}-A(u)}-\sqrt{A(u)}\right)^4\sqrt{9+40v} \\ &\left.- \frac{7}{4}\left(\sqrt{\frac{12}{\sqrt{A(u)}}-A(u)}-\sqrt{A(u)}\right)^4 v\sqrt{9+40v}+\frac{1}{8}v\left(-7+\sqrt{9+40v}\right)^4\right) \end{aligned}$$

for $\sigma(u) > \tau(v)$ *with* $A(u)$ *defined above. The Figure 3.7 show this copula and contour plot of it.*

The Kendall's τ *in this case is equal to* $232/375$*, while the Spearmans's* ρ *is equal to* $10364/13125$*. The upper and lower tail dependance measures are correspondingly* $\lambda_U = 0$ *and* $\lambda_L = 0.6666667$*.*

Case 5: $\lambda^{13} = \frac{1}{n}(\lambda^{01}+\lambda^{02})$**,** $\lambda^{23} = \frac{1}{m}(\lambda^{01}+\lambda^{02})$**,** $n,\ m \in \mathbb{N}$

In the same way as in the previous cases one can also find the copula function for the rational relations like $\lambda^{13} = \frac{1}{n}(\lambda^{01}+\lambda^{02})$, $\lambda^{23} = \frac{1}{m}(\lambda^{01}+\lambda^{02})$, $n,\ m \in \mathbb{N}$.

The marginal distribution functions according to the chapter 3.2.1 are:

$$F_A(t) = 1 + e^{-t(\lambda^{01}+\lambda^{02})}\frac{\lambda^{01}-m\lambda^{01}+\lambda^{02}-m\lambda^{02}e^{\frac{(m-1)t(\lambda^{01}+\lambda^{02})}{m}}}{(m-1)(\lambda^{01}+\lambda^{02})}$$

and

$$F_B(t) = 1 + e^{-t(\lambda^{01}+\lambda^{02})}\frac{\lambda^{01}-n\lambda^{02}+\lambda^{02}-n\lambda^{02}e^{\frac{(n-1)t(\lambda^{01}+\lambda^{02})}{n}}}{(n-1)(\lambda^{01}+\lambda^{02})}.$$

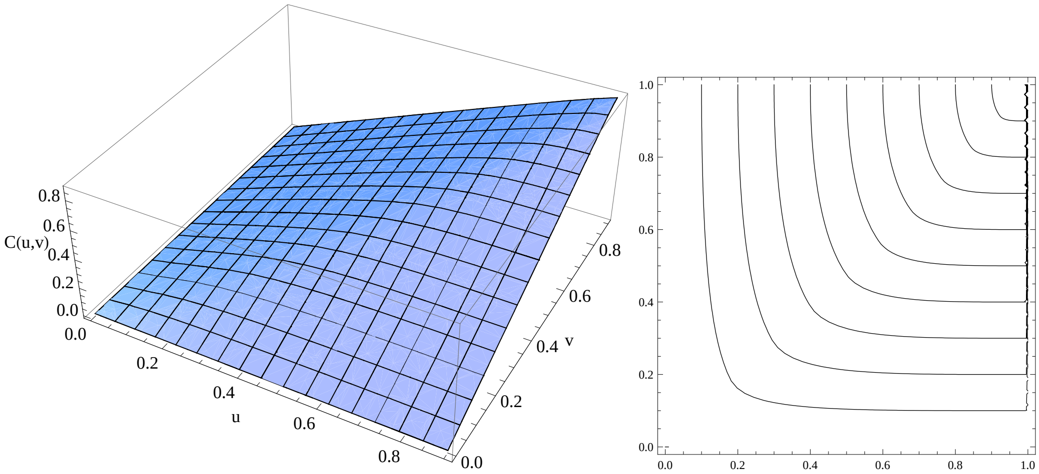

Figure 3.7: Copula for the case $\lambda^{01} = 0.001$, $\lambda^{02} = 0.0015$, $\lambda^{23} = 0.005$ and $\lambda^{13} = 0.01$

The joint distribution function according to the 3.2.2 for $t \le s$ is:

$$\begin{aligned} F_{A,B}(t,s) &= 1 - \frac{e^{-\frac{s(\lambda^{01}+\lambda^{02})}{n}} n\lambda^{01}}{(-1+n)(\lambda^{01}+\lambda^{02})} + \frac{e^{-\frac{(s+(-1+n)t)(\lambda^{01}+\lambda^{02})}{n}} n\lambda^{01}}{(-1+n)(\lambda^{01}+\lambda^{02})} \\ &\quad - \frac{e^{-\frac{t(\lambda^{01}+\lambda^{02})}{m}} m\lambda^{02}}{(-1+m)(\lambda^{01}+\lambda^{02})} + e^{-t(\lambda^{01}+\lambda^{02})}\left(-1 + \frac{m\lambda^{02}}{(-1+m)(\lambda^{01}+\lambda^{02})}\right) \end{aligned}$$

and

$$\begin{aligned} F_{A,B}(t,s) &= 1 - e^{-s(\lambda^{01}+\lambda^{02})} - \frac{e^{-s(\lambda^{01}+\lambda^{02})}\left(-1 + e^{\frac{(-1+n)s(\lambda^{01}+\lambda^{02})}{n}}\right) n\lambda^{01}}{(-1+n)(\lambda^{01}+\lambda^{02})} \\ &\quad - \frac{e^{-s(\lambda^{01}+\lambda^{02})}\left(-1 + e^{\frac{(-1+m)s(\lambda^{01}+\lambda^{02})}{m}}\right) m\lambda^{02}}{(-1+m)(\lambda^{01}+\lambda^{02})} \\ &\quad + \frac{\left(-e^{-s(\lambda^{01}+\lambda^{02})} + e^{-\frac{s(\lambda^{01}+\lambda^{02})}{n}}\right)\left(1 - e^{\frac{(s-t)(\lambda^{01}+\lambda^{02})}{m}}\right) n\lambda^{02}}{(-1+n)(\lambda^{01}+\lambda^{02})} \end{aligned}$$

for $t > s$.

Same as before to find the copula for this case we need to find a inverse functions of marginal distribution functions to use the procedure of finding the copula function described in the chapter 2.3.1. Following the same procedure as before, we can using the substitution

$$t = \frac{-m\ln(\sigma)}{\lambda^{01} + \lambda^{02}}$$

and

$$s = \frac{-n\ln(\tau)}{\lambda^{01} + \lambda^{02}}.$$

Then the marginal distribution functions will be

$$\begin{aligned} F_A(\sigma(t)) &= \hat{F}_A(\sigma) = 1 - \sigma^m + \frac{m\left(-\sigma + \sigma^m\right)\lambda^{02}}{(-1+m)(\lambda^{01} + \lambda^{02})} \\ &= 1 - \sigma^m(1-\alpha) - \alpha\sigma \end{aligned}$$

and

$$\begin{aligned} F_B(\tau(s)) &= \hat{F}_B(\tau) = 1 - \tau^n + \frac{n\left(-\tau + \tau^n\right)\lambda^{01}}{(-1+n)(\lambda^{01} + \lambda^{02})} \\ &= 1 - \tau^n(1-\beta) - \beta\tau \end{aligned}$$

with $\alpha = \frac{m\lambda^{02}}{(m-1)(\lambda^{01}+\lambda^{02})}$ and $\beta = \frac{n\lambda^{01}}{(n-1)(\lambda^{01}+\lambda^{02})}$.

The joint distribution function for $\sigma \leq \tau$ will be

$$\begin{aligned} F_{A,B}(\sigma(t), \tau(s)) = \hat{F}_{A,B}(\sigma, \tau) &= \frac{-(-1+m)\left(1 - \sigma^m + n\left(-1 + \sigma^m + \tau - \sigma^{\frac{m(-1+n)}{n}}\tau\right)\right)\lambda^{01}}{(-1+m)(-1+n)(\lambda^{01} + \lambda^{02})} \\ &+ \frac{(-1+n)\left(-1 + m - m\sigma + \sigma^m\right)\lambda^{02}}{(-1+m)(-1+n)(\lambda^{01} + \lambda^{02})} \end{aligned}$$

and

$$\begin{aligned} F_{A,B}(\sigma(t), \tau(s)) = \hat{F}_{A,B}(\sigma, \tau) &= 1 - \tau^n + \frac{n\left(-\tau + \tau^n\right)\lambda^{01}}{(-1+n)(\lambda^{01} + \lambda^{02})} \\ &+ \frac{m\left(\tau^n - \tau^{\frac{n}{m}}\right)\lambda^{02}}{(-1+m)(\lambda^{01} + \lambda^{02})} - \frac{n\tau^{-\frac{n}{m}}\left(-\tau + \tau^n\right)\left(-\sigma + \tau^{\frac{n}{m}}\right)\lambda^{02}}{(-1+n)(\lambda^{01} + \lambda^{02})} \end{aligned}$$

for $\sigma > \tau$.

Same as before to find inverse functions we again have to solve the following equations:

$$\hat{F}_A(\sigma) = u \qquad \text{and} \qquad \hat{F}_B(\tau) = v. \tag{3.29}$$

Remember, that in this case $\sigma = e^{-\frac{\lambda^{0*}t}{m}}$ and $\tau = e^{-\frac{\lambda^{0*}s}{n}}$.

Consider a numerical example:

Example 3.4.7 *Let $n = 2$, $m = 3$, $\lambda^{01} = 1/100$, and $\lambda^{02} = 1/200$. So we get $\lambda^{13} = 3/400$ and $\lambda^{23} = 1/200$ So, in this case the marginal distribution function $\hat{F}_A(\sigma)$ and $\hat{F}_B(\tau)$ are cubic and quadratic equations and have consequently three or two solutions correspondingly.*

To find inverse functions of marginal distribution functions we can use equations (3.16) and (3.7). Substituting them into the joint distribution function, we get the copula function.

The needed solutions of equations for inverse functions are:

$$\hat{F}_A^{-1}(u) = \sigma(u) = -\frac{1}{\sqrt[3]{3}A(u)} + \frac{A(u)}{\sqrt[3]{9}}$$

and

$$\hat{F}_B^{-1}(v) = \tau(v) = 2 - \sqrt{1+3v}$$

with $A(u) = \left(9 - 9u + \sqrt{3}\sqrt{28 - 54u + 27u^2}\right)^{1/3}$.

The copula function for this case is:

$$\begin{aligned} C(u,v) &= 2 - u + \frac{1}{3^{1/3}A(u)} - \frac{A(u)}{3^{2/3}} + \frac{\left(3^{2/3} - 3^{1/3}A^2(u)\right)^3}{27A^3(u)} \\ &-\frac{4}{81}\left(9\left(\frac{A^3(u) - 3^{1/3}A(u)}{A^2(u)}\right)^{3/2} + \frac{\left(3^{2/3} - 3^{1/3}A^2(u)\right)^3}{A^3(u)}\right) \\ &+\frac{4}{81}\left(9\left(\frac{A^3(u) - 3^{1/3}A(u)}{A^2(u)}\right)^{3/2} + \frac{\left(3^{2/3} - 3^{1/3}A^2(u)\right)^3}{A^3(u)}\right) \\ &\times\left(1 - \frac{3\sqrt{3}\left(2 - \sqrt{1+3v}\right)}{\left(\frac{-3^{2/3}+3^{1/3}A^2(u)}{A(u)}\right)^{3/2}}\right) \end{aligned}$$

for $\sigma(u) \leq \tau(v)$, *and*

$$\begin{aligned} C(u,v) &= 3^{1/3}\left(3^{1/3} - A^2(u)\right) \\ &\times\frac{\left(-5 + 4\sqrt{1+3v} + \left(2 - \sqrt{1+3v}\right)^{2/3}\right)}{6A(u)\left(2 - \sqrt{1+3v}\right)^{2/3}} \\ &+\frac{3v\left(-3^{2/3} + 3^{1/3}A^2(u) + 2A(u)\left(2 - \sqrt{1+3v}\right)^{2/3}\right)}{6A(u)\left(2 - \sqrt{1+3v}\right)^{2/3}} \end{aligned}$$

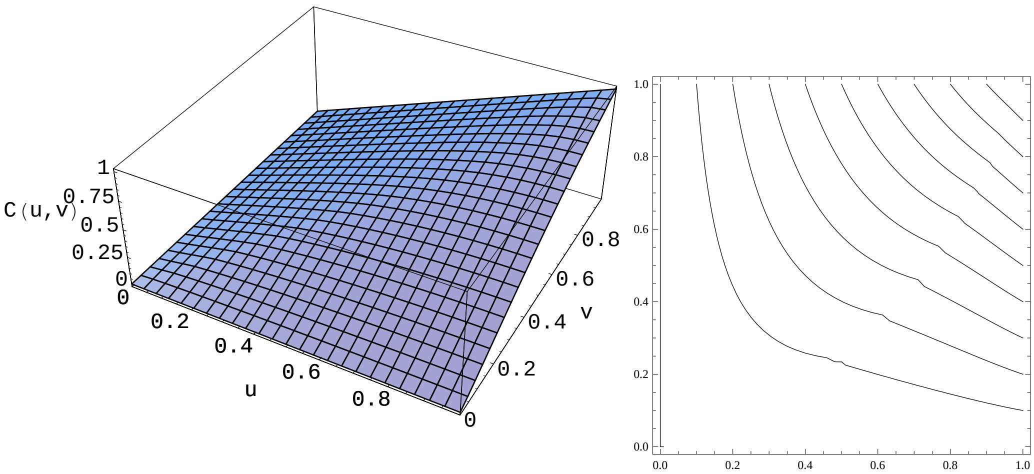

Figure 3.8: Copula for the case $\lambda^{01} = 1/100$, $\lambda^{02} = 1/200$, $\lambda^{23} = 1/200$ and $\lambda^{13} = 3/400$

for $\sigma(u) > \tau(v)$.

The Figure 3.8 show this copula and contour plot of it.

The Kendall's τ *in this case is equal to* $-1/108$, *while the Spearmans's* ρ *is equal to* $-13/594$. *The upper and lower tail dependence measures are corresponding* $\lambda_L = 0$ *and* $\lambda_U = 0$.

Case 6: $\lambda^{13} = p(\lambda^{01} + \lambda^{02})$**,** $\lambda^{23} = q(\lambda^{01} + \lambda^{02})$**,** $p,\ q \in \mathbb{Q}$ **(rational numbers)**

Now we can generate all the previous cases and find the copula function for the rational numbers p and q, where $p = \frac{n}{m}$ and $q = \frac{k}{l}$ with $k,\ l,\ n,\ m \in \mathbb{N}$.

The marginal distribution functions according to the chapter 3.2.1 are:

$$F_A(t) = 1 - e^{-t(\lambda^{01}+\lambda^{02})} - \frac{\left(-e^{-t(\lambda^{01}+\lambda^{02})} + e^{-\frac{kt(\lambda^{01}+\lambda^{02})}{l}}\right)\lambda^{02}}{\lambda^{01} + \lambda^{02} - \frac{k(\lambda^{01}+\lambda^{02})}{l}}$$

and

$$F_B(t) = 1 - e^{-t(\lambda^{01}+\lambda^{02})} - \frac{\left(-e^{-t(\lambda^{01}+\lambda^{02})} + e^{-\frac{nt(\lambda^{01}+\lambda^{02})}{m}}\right) m\lambda^{01}}{(m-n)(\lambda^{01}+\lambda^{02})}.$$

The joint distribution function according to the 3.2.2 for $t \leq s$ is:

$$\begin{aligned}
F_{A,B}(t,s) &= 1 - \frac{e^{-\frac{ns(\lambda^{01}+\lambda^{02})}{m}} m\lambda^{01}}{(m-n)(\lambda^{01}+\lambda^{02})} + \frac{e^{-\frac{(n(s-t)+mt)(\lambda^{01}+\lambda^{02})}{m}} m\lambda^{01}}{(m-n)(\lambda^{01}+\lambda^{02})} \\
&\quad + \frac{e^{-\frac{kt(\lambda^{01}+\lambda^{02})}{l}} l\lambda^{02}}{(k-l)(\lambda^{01}+\lambda^{02})} + e^{-t(\lambda^{01}+\lambda^{02})} \left(-1 + \frac{l\lambda^{02}}{(-k+l)(\lambda^{01}+\lambda^{02})}\right)
\end{aligned}$$

and

$$\begin{aligned}
F_{A,B}(t,s) &= 1 - e^{-s(\lambda^{01}+\lambda^{02})} - \frac{\left(-e^{-s(\lambda^{01}+\lambda^{02})} + e^{-\frac{ns(\lambda^{01}+\lambda^{02})}{m}}\right) m\lambda^{01}}{(m-n)(\lambda^{01}+\lambda^{02})} \\
&\quad + \frac{\left(-e^{-s(\lambda^{01}+\lambda^{02})} + e^{-\frac{ns(\lambda^{01}+\lambda^{02})}{m}}\right)\left(1 - e^{\frac{k(s-t)(\lambda^{01}+\lambda^{02})}{l}}\right) m\lambda^{02}}{(m-n)(\lambda^{01}+\lambda^{02})} \\
&\quad - \frac{\left(-e^{-s(\lambda^{01}+\lambda^{02})} + e^{-\frac{ks(\lambda^{01}+\lambda^{02})}{l}}\right)\lambda^{02}}{\lambda^{01}+\lambda^{02} - \frac{k(\lambda^{01}+\lambda^{02})}{l}}
\end{aligned}$$

for $t > s$.

Same as before to find the copula for this case we need to find a inverse functions of marginal distribution functions to use the procedure of finding the copula function described in the chapter 2.3.1. Following the same procedure as before, we can using the substitution

$$t = \frac{-l\ln(\sigma)}{(\lambda^{01}+\lambda^{02})}$$

and

$$s = \frac{-m\ln(\tau)}{(\lambda^{01}+\lambda^{02})}.$$

Then the marginal distribution functions will be

$$\begin{aligned}
F_A(\sigma(t)) = \hat{F}_A(\sigma) &= 1 - \sigma^l + \frac{l\left(\sigma^k - \sigma^l\right)\lambda^{02}}{(k-l)(\lambda^{01}+\lambda^{02})} \\
&= 1 - \sigma^l + \alpha l\left(\sigma^k - \sigma^l\right)
\end{aligned}$$

and

$$\begin{aligned}
F_B(\tau(s)) = \hat{F}_B(\tau) &= 1 - \tau^m + \frac{m\left(\tau^m - \tau^n\right)\lambda^{01}}{(m-n)(\lambda^{01}+\lambda^{02})} \\
&= 1 - \tau^m + \beta m\left(\tau^m - \tau^n\right)
\end{aligned}$$

with $\alpha = \frac{\lambda^{02}}{(k-l)(\lambda^{01}+\lambda^{02})}$ and $\beta = \frac{\lambda^{01}}{(m-n)(\lambda^{01}+\lambda^{02})}$.

The joint distribution function for $\sigma \leq \tau$ will be

$$F_{A,B}(\sigma(t),\tau(s)) = \hat{F}_{A,B}(\sigma,\tau) \;=\; 1 + \frac{-\frac{m\left(1-\sigma^{l-\frac{ln}{m}}\right)\tau^n\lambda^{01}}{m-n} + \frac{l\sigma^k\lambda^{02}+\sigma^l(l\lambda^{01}-k(\lambda^{01}+\lambda^{02}))}{k-l}}{\lambda^{01}+\lambda^{02}}$$

and

$$\begin{aligned} F_{A,B}(\sigma(t),\tau(s)) = \hat{F}_{A,B}(\sigma,\tau) \;&=\; 1-\tau^m + \frac{m\left(\tau^m-\tau^n\right)\lambda^{01}}{(m-n)(\lambda^{01}+\lambda^{02})} + \frac{l\left(-\tau^m+\tau^{\frac{km}{l}}\right)\lambda^{02}}{(k-l)(\lambda^{01}+\lambda^{02})} \\ &\quad + \frac{m\tau^{-\frac{km}{l}}\left(x^k-\tau^{\frac{km}{l}}\right)\left(\tau^m-\tau^n\right)\lambda^{02}}{(m-n)(\lambda^{01}+\lambda^{02})} \end{aligned}$$

for $\sigma > \tau$. Again the marginal distribution functions are polynomial. To find inverse functions we have to solve the next equations:

$$\hat{F}_A(\sigma) = u \qquad \text{and} \qquad \hat{F}_B(\tau) = v. \tag{3.30}$$

Consider a numerical example:

Example 3.4.8 *Let $n = 1$, $m = 2$, $k = 3$, $l = 4$, $\lambda^{01} = 1/100$, and $\lambda^{02} = 1/300$. So we get $\lambda^{13} = 1/150$ and $\lambda^{23} = 1/100$.*

So now we follow all previous steps: Find inverse functions of marginal distribution functions, substituting them into the joint distribution function, we get the copula function. For our parameters we have two polynomial equations of degree four and two. To solve them we can use results of section 3.4.1.

The needed solutions of equations for inverse functions are:

$$\hat{F}_A^{-1}(u) = x(u) = (1-u)^{1/3},$$

and

$$\hat{F}_B^{-1}(v) = y(v) = \frac{1}{2}\left(3-\sqrt{1+8v}\right).$$

The copula function for this case is (should be recalculated):

$$C(u,v) \;=\; \frac{1}{4}\left(4u-3\left(-1+(1-u)^{2/3}\right)\left(-3+\sqrt{1+8v}\right)\right)$$

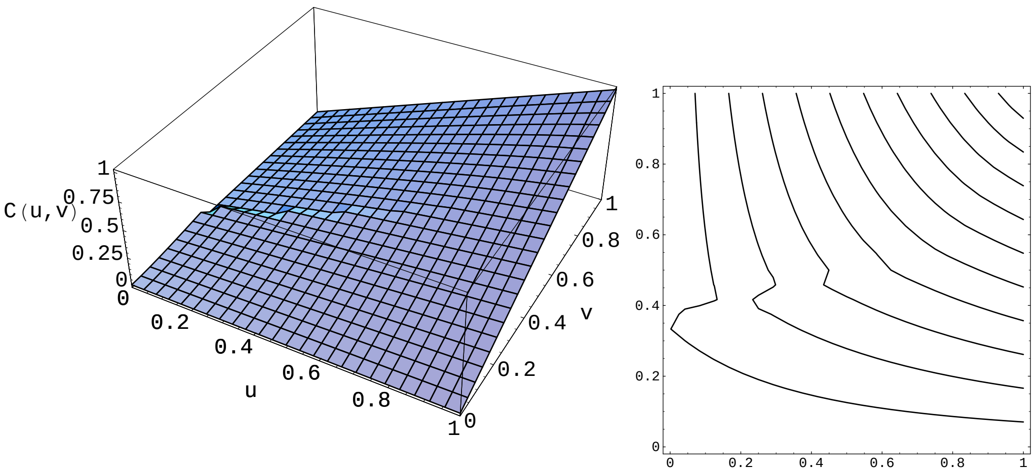

Figure 3.9: Copula for the case $\lambda^{01} = 1/100$, and $\lambda^{02} = 1/300$, $\lambda^{13} = 1/150$ and $\lambda^{23} = 1/100$

for $\sigma(u) \leq \tau(v)$ *, and*

$$C(u,v) \quad = \quad -1 + v + \frac{\sqrt{3 - \sqrt{1+8v}}}{\sqrt{2}} + \frac{1}{2}u\left(2 - \sqrt{6 - 2\sqrt{1+8v}}\right)$$

for $\sigma(u) > \tau(v)$ *with* $A(u)$ *defined above. The Figure 3.9 show this copula and contour plot of it.*

The Kendall's τ *in this case is equal to* $-7/24$*, while the Spearmans's* ρ *is equal to* $-87/154$*. The upper and lower tail dependence measures are corresponding* $\lambda_U = 0$ *and* $\lambda_L = 0$*.*

Case 7: General Case

In the general case, it is not always possible to find inverse functions of marginal distribution function analytically. Numerical solutions are feasible for the given parameter set.

The marginal distribution function are sum of exponential functions with different arguments. So, it may be reasonable in the general case consider some approximation of the exponent functions, for example Taylor expansion. We will try to find an approximation of the copula in a general case in the chapter 4.

The other way to find a copula function for a given Markov Chain is an approximation of transition intensities by the rational numbers. As soon as we found some rational numbers p and q such that equations

$$\lambda^{23} = p\left(\lambda^{01} + \lambda^{02}\right) \text{ and } \lambda^{13} = q\left(\lambda^{01} + \lambda^{02}\right) \text{ for some } p,\ q \in \mathbb{Q}\setminus\{1\}$$

hold, we can use the method described above to find copula function for a given Markov Chain.

3.5 Case of Three Firms

Assume that we have Markov process, which describes the case of three firms as in the Figure 3.10. Assume that defaults times of firms A, B and C are τ_A, τ_B and τ_C accordingly. We can find joint and marginal distributions of this defaults times according to the sections 3.2.2 and 3.2.1: $F_A(t) = P(\tau_A \le t)$, $F_B(t) = P(\tau_B \le t)$, $F_C(t) = P(\tau_C \le t)$ and $F_{A,B,C}(t,s,z) = P(\tau_A \le t, \tau_B \le s, \tau_C \le z)$.

Assume that after default of some firm the default intensities of others firms will be changed it means

$$\lambda_t^A = a_{10} + a_{12}\mathbf{1}_{\{\tau_B \le t\}} + a_{13}\mathbf{1}_{\{\tau_C \le t\}} + a_{14}\mathbf{1}_{\{\tau_B \le t, \tau_C \le t\}},$$

$$\lambda_t^B = a_{20} + a_{21}\mathbf{1}_{\{\tau_A \le t\}} + a_{23}\mathbf{1}_{\{\tau_C \le t\}} + a_{24}\mathbf{1}_{\{\tau_A \le t, \tau_C \le t\}},$$

$$\lambda_t^C = a_{30} + a_{31}\mathbf{1}_{\{\tau_A \le t\}} + a_{32}\mathbf{1}_{\{\tau_B \le t\}} + a_{14}\mathbf{1}_{\{\tau_A \le t, \tau_B \le t\}}.$$

Here a_{i0} is original default intensity of firm i; a_{ij} is the default intensity increment of firm i after default of firm j, $i = 1,2,3,\ j = 1,2,3$; a_{i4} is increment of default intensity of firm i after default of two others firms.

For the case described by the figure 3.10 it means that for example $\lambda^{01} = a_{10}$, $\lambda^{14} = a_{10} + a_{12}$, $\lambda^{35} = a_{10} + a_{13}$ – default of A after C, $\lambda^{57} = a_{30} + a_{31} + a_{32} + a_{34}$ – default of C after A and B, and so on.

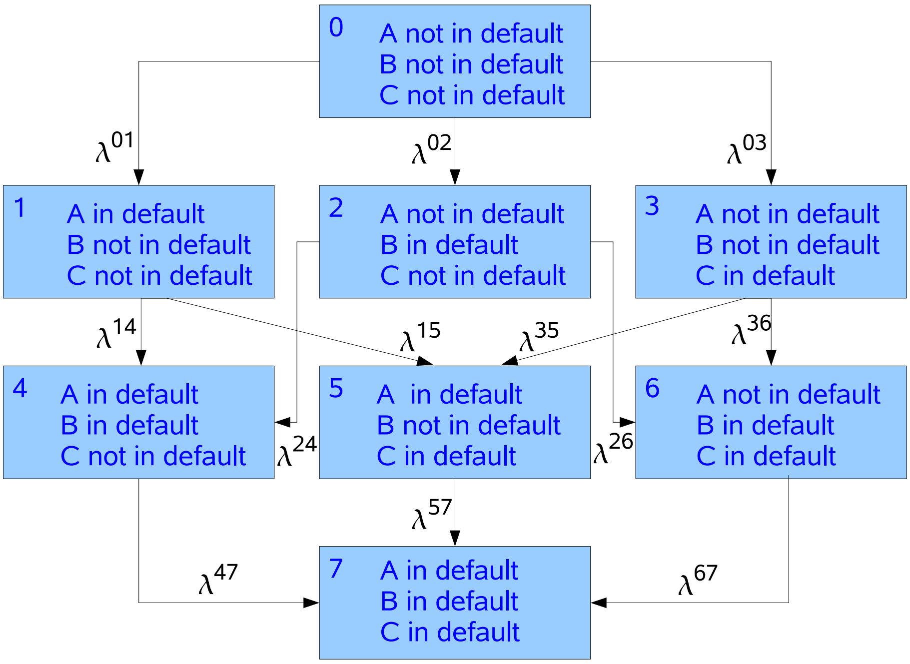

Figure 3.10: **Markov chain describing three firms case**

Same as for the case of two firms we can use the inversion method described in the section 2.3.1 to find a copula for this case. According to the section 3.2.1 the marginal distribution functions are defined as

$$\begin{aligned}
F_A(t) &= q^{01}(0,t) + q^{04}(0,t) + q^{05}(0,t) + q^{07}(0,t) \\
&= 1 - q^{02}(0,t) - q^{00}(0,t) - q^{03}(0,t) - q^{06}(0,t), \\
F_B(s) &= q^{02}(0,s) + q^{04}(0,s) + q^{06}(0,s) + q^{07}(0,s) \\
&= 1 - q^{01}(0,s) - q^{00}(0,t) - q^{03}(0,s) - q^{05}(0,s), \\
F_C(z) &= q^{03}(0,z) + q^{05}(0,z) + q^{06}(0,z) + q^{07}(0,z) \\
&= 1 - q^{01}(0,s) - q^{00}(0,t) - q^{02}(0,s) - q^{04}(0,s),
\end{aligned}$$

the second equation part holds because of

$$\begin{aligned}
q^{00}(0,t) &+ q^{01}(0,t) + q^{02}(0,t) + q^{03}(0,t) \\
&+ q^{04}(0,t) + q^{05}(0,t) + q^{06}(0,t) + q^{07}(0,t) = 1.
\end{aligned}$$

The joint distribution function according to the chapter 3.2.2 is:

$$F_{A,B,C}(t,s,z) = q^{07}(0,t) + q^{01}(0,t)q^{17}(t,s) + q^{04}(0,t)q^{47}(t,s) +$$

$$q^{05}(0,t)q^{57}(t,s) + q^{01}(0,t)q^{14}(t,s)q^{47}(s,z) + q^{04}(0,t)q^{44}(t,s)q^{47}(s,z),$$

for $t \leq s \leq z$,

$$F_{A,B,C}(t,s,z) = q^{07}(0,t) + q^{01}(0,t)q^{17}(t,z) + q^{04}(0,t)q^{47}(t,z) +$$

$$q^{05}(0,t)q^{57}(t,z) + q^{01}(0,t)q^{15}(t,z)q^{57}(z,s) + q^{05}(0,t)q^{55}(t,z)q^{47}(z,s),$$

for $t \leq z \leq s$,

$$F_{A,B,C}(t,s,z) = q^{07}(0,s) + q^{02}(0,s)q^{27}(s,t) + q^{04}(0,s)q^{47}(s,t) +$$

$$q^{06}(0,s)q^{67}(s,t) + q^{02}(0,s)q^{24}(s,t)q^{47}(t,z) + q^{04}(0,s)q^{44}(s,t)q^{47}(t,z),$$

for $s \leq t \leq z$,

$$F_{A,B,C}(t,s,z) = q^{07}(0,s) + q^{02}(0,s)q^{27}(s,z) + q^{04}(0,s)q^{47}(s,z) +$$

$$q^{06}(0,s)q^{67}(s,z) + q^{02}(0,s)q^{26}(s,z)q^{67}(z,t) + q^{06}(0,s)q^{66}(s,z)q^{67}(z,t),$$

for $s \leq z \leq t$,

$$F_{A,B,C}(t,s,z) = q^{07}(0,z) + q^{03}(0,z)q^{37}(z,t) + q^{05}(0,z)q^{57}(z,t) +$$

$$q^{06}(0,z)q^{67}(z,t) + q^{03}(0,z)q^{35}(z,t)q^{57}(t,s) + q^{05}(0,z)q^{55}(z,t)q^{57}(t,s),$$

for $z \leq t \leq s$ and

$$F_{A,B,C}(t,s,z) = q^{07}(0,z) + q^{03}(0,z)q^{37}(z,s) + q^{05}(0,z)q^{57}(z,s) +$$

$$q^{06}(0,z)q^{67}(z,s) + q^{03}(0,z)q^{36}(z,s)q^{67}(s,t) + q^{06}(0,z)q^{66}(z,s)q^{67}(s,t),$$

for $z \leq s \leq t$.

The transition probabilities according to Kraft and Steffensen (2007) are defined as

$$q^{ii}(t,T) = e^{-\lambda^{i*}(T-t)}, \qquad i \in \{0,1,...,7\},$$

i.e.

$$
\begin{aligned}
q^{00}(t,T) &= e^{-(\lambda^{01}+\lambda^{02}+\lambda^{03})(T-t)}, \\
q^{01}(t,T) &= \lambda^{01} g^{01} = \lambda^{01} \frac{e^{-\lambda^{0*}(T-t)} - e^{-\lambda^{1*}(T-t)}}{\lambda^{1*} - \lambda^{0*}} \\
&= \lambda^{01} \frac{e^{-(\lambda^{14}+\lambda^{15})(T-t)} - e^{-(\lambda^{01}+\lambda^{02}+\lambda^{03})(T-t)}}{\lambda^{01}+\lambda^{02}+\lambda^{03}-\lambda^{14}-\lambda^{15}}, \\
q^{02}(t,T) &= \lambda^{02} g^{02} = \lambda^{02} \frac{e^{-\lambda^{0*}(T-t)} - e^{-\lambda^{2*}(T-t)}}{\lambda^{2*} - \lambda^{0*}} \\
&= \lambda^{02} \frac{e^{-(\lambda^{24}+\lambda^{26})(T-t)} - e^{-(\lambda^{01}+\lambda^{02}+\lambda^{03})(T-t)}}{\lambda^{01}+\lambda^{02}+\lambda^{03}-\lambda^{24}-\lambda^{26}}, \\
q^{03}(t,T) &= \lambda^{03} g^{03} = \lambda^{03} \frac{e^{-\lambda^{0*}(T-t)} - e^{-\lambda^{3*}(T-t)}}{\lambda^{3*} - \lambda^{0*}} \\
&= \lambda^{03} \frac{e^{-(\lambda^{35}+\lambda^{36})(T-t)} - e^{-(\lambda^{01}+\lambda^{02}+\lambda^{03})(T-t)}}{\lambda^{01}+\lambda^{02}+\lambda^{03}-\lambda^{36}-\lambda^{35}},
\end{aligned}
$$

and so on. For simplification we use next notation: $\lambda^{i*} = \sum_{j \neq i} \lambda^{ij}$.

After substitution of expressions for transition probabilities into distribution functions we get

$$F_A(t) = 1 - C_A^0 e^{-\lambda^{0*}t} + C_A^2 e^{-\lambda^{2*}t} + C_A^3 e^{-\lambda^{3*}t} + C_A^6 e^{-\lambda^{67}t},$$

$$F_B(s) = 1 - C_B^0 e^{-\lambda^{0*}t} + C_B^1 e^{-\lambda^{1*}t} + C_B^3 e^{-\lambda^{3*}t} + C_B^5 e^{-\lambda^{57}t},$$

$$F_C(z) = 1 - C_C^0 e^{-\lambda^{0*}t} + C_C^1 e^{-\lambda^{1*}t} + C_C^2 e^{-\lambda^{2*}t} + C_C^4 e^{-\lambda^{47}t}$$

with C_A^0, C_B^0, C_C^0, C_A^2, C_B^1, C_C^1, C_A^3, C_B^3, C_C^2, C_A^6, C_B^5, C_C^4 are constants dependent on transition intensities. The values of this constants are presented in Appendix.

3.5.1 Case 1: Independence.

In this case as before independence means that the transition intensities stay the same after defaults of other firms. In the notations of a_{ij} it means that $a_{i0} \neq 0$, $i = 1, ..., 4$ and $a_{ij} = 0$ for $j \neq 0$. In this case the distribution functions will be

$$F_A(t) = 1 - e^{-\lambda^{01}t},$$

$$F_B(s) = 1 - e^{-\lambda^{02}s},$$

$$F_C(z) = 1 - e^{-\lambda^{03} z}.$$

The joint distribution in this case is defined as

$$F_{A,B,C}(t,s,z) = \left(1 - e^{-\lambda^{01} t}\right)\left(1 - e^{-\lambda^{02} s}\right)\left(1 - e^{-\lambda^{03} z}\right)$$

for $\forall u, v, r \in [0,1]$ and this mean that the copula in this case will be

$$C_{A,B,C}(u,v,r) = uvr.$$

3.5.2 Case 2: Part Dependence

In this case we assume change of transition intensities after default of first firm. But we suppose that after default of second firm the intensity will stay the same, i.e. $a_{i0} \neq 0$, $i = 1, ..., 3$ and $a_{ij} = a$ for $i > 1$ and $j \neq 4$, and $a_{ij} = -a$ for $i > 1$ and $j = 4$. Then the marginal distribution functions according to the section 3.2.1 are:

$$F_A(t) = 1 - \frac{ae^{-\lambda^{0*} t} - (\lambda^{02} + \lambda^{03})e^{-(a+\lambda^{01})t}}{a - \lambda^{02} - \lambda^{03}},$$

$$F_B(s) = 1 - \frac{ae^{-\lambda^{0*} s} - (\lambda^{01} + \lambda^{03})e^{-(a+\lambda^{02})s}}{a - \lambda^{01} - \lambda^{03}},$$

$$F_C(z) = 1 - \frac{ae^{-\lambda^{0*} z} - (\lambda^{01} + \lambda^{02})e^{-(a+\lambda^{03})z}}{a - \lambda^{01} - \lambda^{02}}.$$

The same as before to find a copula function for this case,we will use the inversion method described in the section 2.3.1: As the first step we have to find the inverse functions of marginal distribution function. To do this we will use the same approach as for the case of two firms: we will use substitution

$$\begin{aligned} t &= -\frac{\ln(\sigma)}{\lambda^{0*}}, \\ s &= -\frac{\ln(\tau)}{\lambda^{0*}}, \\ z &= -\frac{\ln(\theta)}{\lambda^{0*}}. \end{aligned} \tag{3.31}$$

Then we get new distribution functions:

$$
\begin{aligned}
F_A(\sigma(t)) = \hat{F}_A(\sigma) &= 1 - \frac{a\sigma - (\lambda^{02} + \lambda^{03})\sigma^{\frac{a+\lambda^{01}}{\lambda^{0*}}}}{a - \lambda^{02} - \lambda^{03}}, \\
F_B(\tau(s)) = \hat{F}_B(\tau) &= 1 - \frac{a\tau - (\lambda^{01} + \lambda^{03})\tau^{\frac{a+\lambda^{02}}{\lambda^{0*}}}}{a - \lambda^{01} - \lambda^{03}}, \\
F_C(\theta(z)) = \hat{F}_C(\theta) &= 1 - \frac{a\theta - (\lambda^{01} + \lambda^{02})\theta^{\frac{a+\lambda^{03}}{\lambda^{0*}}}}{a - \lambda^{01} - \lambda^{02}}.
\end{aligned}
$$

To find inverse functions for marginal distribution functions we have to consider some more additional cases. As in the case of two firms we will use the fact that the distribution function can be introduced as a polynomial equation for same defined parameters. So, assuming that

$$
\begin{aligned}
a + \lambda^{01} &= k\lambda^{0*}, \\
a + \lambda^{02} &= l\lambda^{0*}, \\
a + \lambda^{03} &= m\lambda^{0*},
\end{aligned}
$$

and solving this equation with respect to the parameter a, we get

$$
\begin{aligned}
\lambda^{01} &= -\frac{-a - 2ak + al + am}{k + l + m - 1}, \\
\lambda^{02} &= -\frac{-a + ak - 2al + am}{k + l + m - 1}, \\
\lambda^{03} &= -\frac{-a + ak + al - 2am}{k + l + m - 1}.
\end{aligned}
$$

After this procedure the marginal distribution functions will be:

$$
\begin{aligned}
F_A(\sigma(t)) &= \hat{F}_A(\sigma) = 1 - \frac{(m + l + k - 1)\sigma - (2 + l + m - 2k)\sigma^k}{3(k - 1)}, \\
F_B(\tau(s)) &= \hat{F}_B(\tau) = 1 - \frac{(m + l + k - 1)\tau - (2 + k + m - 2l)\tau^l}{3(l - 1)}, \\
F_C(\theta(z)) &= \hat{F}_C(\theta) = 1 - \frac{(m + l + k - 1)\theta - (2 + k + m - 2m)\theta^m}{3(m - 1)}.
\end{aligned}
$$

Now to find the inverse functions of marginal distribution function we have to solve polynomial equations of degree k, l and m correspondingly for $F_A(\sigma) = u$, $F_B(\tau) = v$

and $F_C(\theta) = r$ for different parameters k, m, and l. To do this we can use section 3.4.1. Then from σ, τ, θ we can find the values of outgoing parameters t, s and z. Substituting them in the joint distribution function introduced at the section 3.5 we will get copula.

For example for the case of $k = m = l = \frac{1}{2}$ we get the copula:

$$\begin{aligned} C(u,v,r) &= -\frac{1}{9t^3(u)}\Big(s^2(v)\left(5t^2(u) + t^5(u) - z^2(r) + t^3(u)(z^6(r) - 6)\right) \\ &\quad + t^2(u)(t(u) - 1)\left(-5z^2(r) + t^2(u)(z^2(r) - 6) + t(u)(9 + z^2(r)\right)\Big) \end{aligned}$$

for $u \le v$, $u \le r$, and

$$\begin{aligned} C(u,v,r) &= \frac{1}{9s^3(v)}\Big(-15s^4(v) + t^2(u)z^2(r) - s^5(v)\left(t^2(u) + z^2(r) - 6\right) \\ &\quad -5s^2(v)\left(t^2(u) + z^2(r)\right) + s^3(v)\left(9 + 6z^2(r) - t^2(u)\left(z^2 8(r) - 6\right)\right)\Big) \end{aligned}$$

for $u > v$ and $u > r$, and

$$\begin{aligned} C(u,v,r) &= -\frac{1}{9z^3(r)}\Big(z^2(r)(z(r) - 1) \times \left(9z(r) - 6z^2(r) + t^2(u)(z^2(r) + z(r) - 5)\right) \\ &\quad + s^2(v)\left(t^2(u)(z^3(r) - 1) + z^2(r)\left(5 - 6z(r) + z^3(r)\right)\right)\Big) \end{aligned}$$

for $r > u$ and $r > v$ with

$$t(u) = \frac{5 - \sqrt{1 + 24u}}{4}, \qquad s(v) = \frac{5 - \sqrt{1 + 24v}}{4} \qquad \text{and}$$

$$z(r) = \frac{5 - \sqrt{1 + 34r}}{4}.$$

And graphically we can see the copulas $C_{AB}(u,v)$, $C_{AC}(u,r)$ and $C_{BC}(v,r)$ for the cases $r = 1$, $v = 1$, $u = 1$, correspondingly and $a = 0.01$ on the Figure 3.11. Of course this case can be considered for more complicated λ's and relations between them. The same approaches as for the case of two firms can be used.

3.5.3 Case 3: Full dependence

According to the section 3.5 the marginal distribution functions for the case of three firms are:

$$F_A(t) = 1 - C_A^0 e^{-\lambda^{0*} t} + C_A^2 e^{-\lambda^{2*} t} + C_A^3 e^{-\lambda^{3*} t} + C_A^6 e^{-\lambda^{67} t},$$

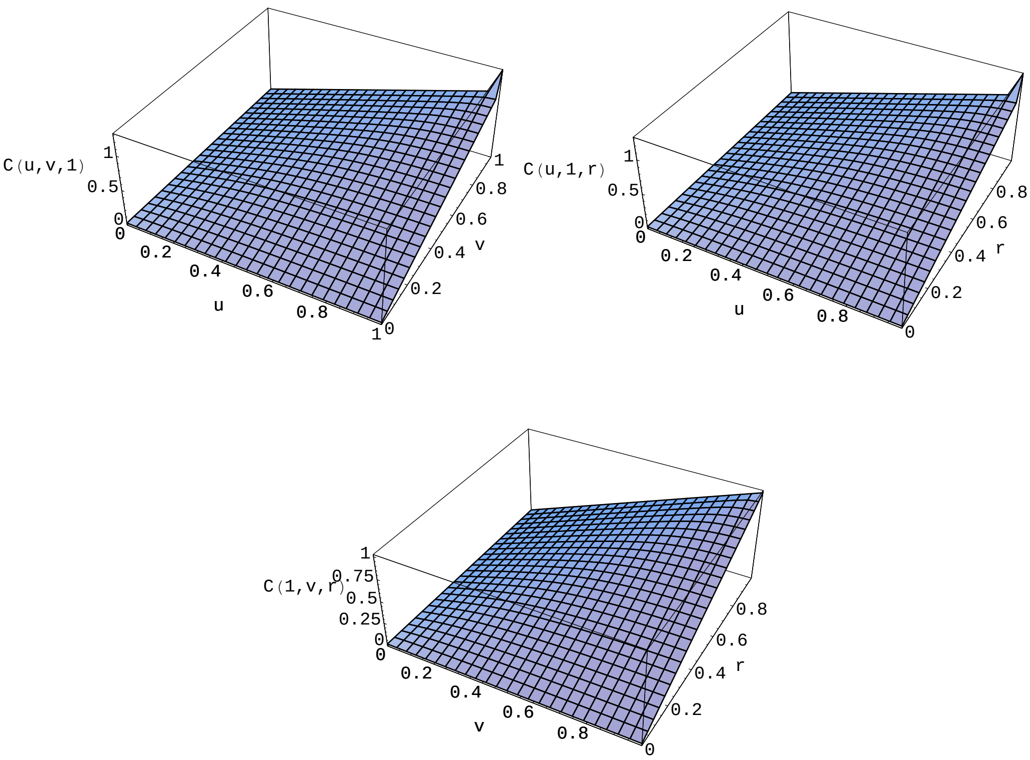

Figure 3.11: Copula describing three firms part dependance case: $u = 1$, $v = 1$, $r = 1$; $k = m = l = \frac{1}{2}$ and $a = 0.01$

$$F_B(s) = 1 - C_B^0 e^{-\lambda^{0*}t} + C_B^1 e^{-\lambda^{1*}t} + C_B^3 e^{-\lambda^{3*}t} + C_B^5 e^{-\lambda^{57}t},$$
$$F_C(z) = 1 - C_C^0 e^{-\lambda^{0*}t} + C_C^1 e^{-\lambda^{1*}t} + C_C^2 e^{-\lambda^{2*}t} + C_C^4 e^{-\lambda^{47}t}$$

with all constants defined in Appendix.

To find the copula function we need to find inverse functions of marginal distribution functions. To simplify this procedure we again will find some relations between λ^{i*}, $i = 1, ..., 6$.

So consider transition probabilities λ^{i*} which satisfy:

$$\begin{aligned} \frac{\lambda^{0*}}{\lambda^{1*}} &= k, & \tfrac{\lambda^{0*}}{\lambda^{2*}} &= l, & \frac{\lambda^{0*}}{\lambda^{3*}} &= m, \\ \frac{\lambda^{0*}}{\lambda^{4*}} &= n, & \tfrac{\lambda^{0*}}{\lambda^{5*}} &= p, & \frac{\lambda^{0*}}{\lambda^{6*}} &= q \end{aligned} \tag{3.32}$$

with $k, l, m, n, p, q \in \mathbb{Q}\backslash\{1\}$.

Let us fix some a_{ij} in the expressions of λ^{ij} and solve this equations. We should remember, that we have 12 unknown a_{ij}. Given equations (3.32) we need to fix a half of them. We get:

$$\begin{aligned} a_{14} &= -\frac{(a_{10}+a_{20}+a_{30})(q-m)}{mq} + a_{20} + a_{23} - a_{12}, \\ a_{24} &= -\frac{(a_{10}+a_{20}+a_{30})(p-k)}{kp} + a_{30} + a_{31} - a_{32}, \\ a_{34} &= -\frac{(a_{10}+a_{20}+a_{30})(n-l)}{ln} + a_{10} + a_{12} - a_{31}, \\ a_{13} &= \frac{a_{10}+a_{20}+a_{30}}{m} - a_{20} - a_{10} - a_{23}, \\ a_{21} &= \frac{a_{10}+a_{20}+a_{30}}{k} - a_{20} - a_{30} - a_{31}, \\ a_{32} &= \frac{a_{10}+a_{20}+a_{30}}{l} - a_{10} - a_{12} - a_{30}. \end{aligned}$$

It is also possible to fix other six a_{ij} and then solve equation (3.32) for the rest of a_{ij}. In this case we will get the similar equations for the marginal distribution functions.

We can use the same substitution as before in the marginal distribution functions:

$$\begin{aligned} t &= -\frac{\ln(\sigma)}{\lambda^{0*}}, \\ s &= -\frac{\ln(\tau)}{\lambda^{0*}}, \\ z &= -\frac{\ln(\theta)}{\lambda^{0*}}. \end{aligned} \tag{3.33}$$

As result the sum of exponential functions with different parameters will be transformed into the polynomial functions. So, we get new distribution functions:

$$\begin{aligned} F_A(\sigma(t)) &= \hat{F}_A(\sigma) = 1 - C_A^0\sigma + C_A^2\sigma^{\frac{1}{l}} + C_A^3\sigma^{\frac{1}{m}} + C_A^6\sigma^{\frac{1}{q}}, \\ F_B(\tau(s)) &= \hat{F}_B(\tau) = 1 - C_B^0\tau + C_B^1\tau^{\frac{1}{k}} + C_B^3\tau^{\frac{1}{m}} + C_B^5\tau^{\frac{1}{p}}, \\ F_C(\theta(z)) &= \hat{F}_C(\theta) = 1 - C_C^0\theta^{+}C_C^1\theta^{\frac{1}{k}} + C_C^2\theta^{\frac{1}{l}} + C_C^4\theta^{\frac{1}{n}}. \end{aligned}$$

New distribution functions are polynomial functions. Now choosing different values of $a_{10},\ a_{20},\ a_{30},\ a_{12},\ a_{23},\ a_{31}$, and $k,\ l,\ m,\ n,\ p,\ q \in \mathbb{Q}\backslash\{1\}$ we can easily find a inverse functions of the distribution functions according to the section 3.4.1 and then find a copula function.

Consider an numerical example:

Example 3.5.1 *Let $a_{10} = 0.01$, $a_{20} = 0.02$, $a_{30} = 0.015$, $a_{12} = 0.002$, $a_{23} = 0.005$, $a_{31} = 0.003$, $k = \frac{1}{3}$, $l = \frac{1}{3}$,$m = \frac{1}{3}$, $n = \frac{1}{2} = p = q$. Then we the parameters of transition probabilities will be equal:*

$$\begin{aligned} a_{14} = -\frac{11}{500}, &\qquad a_{24} = -\frac{4}{125}, \\ a_{34} = -\frac{9}{250}, &\qquad a_{13} \ = \ \frac{1}{10}, \\ a_{21} \ = \ \frac{97}{1000}, &\qquad a_{32} \ = \ \frac{27}{250}. \end{aligned}$$

The distribution functions for this parameters after substitution (3.33) will be:

$$\begin{aligned} \hat{F}_A(\sigma) &= -\frac{(\sigma-1)(45+7\sigma(2\sigma-7))}{45}, \\ \hat{F}_B(\tau) &= -\frac{(\tau-1)(270+\tau(47\tau-197))}{270}, \\ \hat{F}_C(\theta) &= \frac{135-227\theta+94\theta^2-\theta^3}{135} \end{aligned}$$

with $\sigma,\ \tau,\ \theta \in [0,1]$

To find inverse functions we have to solve the following equations: $F_A(\sigma) = u$, $F_B(\tau) = v$ and $F_C(\theta) = r$. Again, for solving these equation we can use section 3.4.1. As inverse function of the marginal distribution function we have to choose one of the solutions, which results in positive time, i.e. σ, τ and θ are from 0 to 1.

The suitable solutions are

$$\begin{aligned} \hat{F_A}^{-1}(u) = \sigma(u) &= \frac{3}{2} - \frac{1}{2\sqrt[3]{21}A(u)} - \frac{A(u)}{2\sqrt[3]{(21)^2}}, \\ \hat{F_B}^{-1}(v) = \tau(v) &= \frac{1}{141}\left(244 - \frac{6311}{B(v)} + B(v)\right), \\ \hat{F_C}^{-1}(z) = \theta(r) &= \frac{47}{3} - \frac{14498}{12\sqrt[3]{2}D(r)} - \frac{D(r)}{3\sqrt[3]{4}}, \end{aligned}$$

where

$$
\begin{aligned}
A(u) &= \left(189 + 5670u - 2\sqrt{105}\sqrt{85 + 5103u + 76545u^2}\right)^{1/3}, \\
B(v) &= \left(-1521413 - 8051805v + 423\sqrt{15}\sqrt{956080 + 9128478v + 24155415v^2}\right)^{1/3}, \\
D(r) &= \left(-322916 + 3645r - 27\sqrt{5}\sqrt{-27690 - 645832r + 3645r^2}\right)^{1/3}
\end{aligned}
$$

with inverse functions of the marginal distribution functions are

$$
\begin{aligned}
F_A^{(-1)}(u) &= e^{-\lambda^{0*}\sigma(u)}, \\
F_B^{(-1)}(v) &= e^{-\lambda^{0*}\tau(v)}, \\
F_C^{(-1)}(r) &= e^{-\lambda^{0*}\theta(r)}.
\end{aligned}
$$

Substituting the inverse functions of marginal distribution functions in the joint distribution function we get the copula for this case:

$$
\begin{aligned}
C(u,v,r) &= -\frac{1}{270\tau^2(v)}\Big((\tau(v)-1)\Big(-197\tau^3(v) + 47\tau^4(v) + 164\sigma^2(u)\theta(r) \\
&\quad -44\theta^3(r) + \tau^2(v)\left(270 + 25\sigma^2(u) + 78\theta^2(r)\right) \\
&\quad +\tau(v)\left(-22\theta^2(r)\left(5 + 2\theta(r)\right) + \sigma^2(u)\left(164\theta(r) - 353\right)\right)\Big)\Big)
\end{aligned}
$$

for $\sigma(u) \le \theta(r) \le \tau(v)$,

$$
\begin{aligned}
C(u,v,r) &= -\frac{1}{270\theta^2(r)}\Big((\theta(r)-1)\Big(\left(65\tau^3(v) + 25\tau(v)\sigma^2(u)\right)(\theta(r)+1) \\
&\quad +4\tau^2(v)\theta(r)(3\theta(r)-58) \\
&\quad +2\theta(r)\left(\sigma^2(u)\left(82\theta(r)-107\right) + \theta(r)\left(135 - 92\theta(r) + 2\theta^2(r)\right)\right)\Big)\Big)
\end{aligned}
$$

for $\sigma(u) \le \tau(v) \le \theta(r)$,

$$
\begin{aligned}
C(u,v,r) &= -\frac{1}{270\theta^2(r)}\Big((\theta(r)-1)\Big(65\sigma^3(u)(\theta(r)+1) + 4\sigma^2(u)\theta(r)(3\theta(r)-58) \\
&\quad +2\theta^2(r)\left(135 - 92\theta(r) + 2\theta^2(r)\right) \\
&\quad +\tau^2(v)\left(25\sigma(u)(\theta(r)+1) + 2\theta(r)(82\theta(r)-107)\right)\Big)\Big)
\end{aligned}
$$

for $\tau(v) \le \sigma(u) \le \theta(r)$,

$$
C(u,v,r) = -\frac{1}{135\sigma^2(u)}\Big((\sigma(u)-1)\Big(-147\sigma^3(u) + 42\sigma^4(u) + 24\theta^3(r)
$$

$$+2\sigma(u)\theta^2(r)(12\theta(r)-43)+\sigma^2(u)(8\theta^2(r)+135)$$
$$+\tau^2(v)\left(55\sigma^2(u)+6\theta(r)+\sigma(u)(6\theta(r)-67)\right)\Big)\Big)$$

for $y(v) \leq \theta(r) \leq \sigma(u)$,

$$\begin{aligned} C(u,v,r) &= -\frac{1}{270\tau^2(v)}\Big((\tau(v)-1)\Big(-197\tau^3(v)+47\tau^4(v) \\ &\quad +8\sigma(u)(13\sigma^2(u)+2\theta^2(r))+\tau^2(v)(270+25\sigma^2(u)+78\theta^2(r)) \\ &\quad +\tau(v)\left(-353\sigma^2(u)+104\sigma^3(u)-110\theta^2(r)+16\sigma(u)\theta^2(r)\right)\Big)\Big) \end{aligned}$$

for $\theta(r) \leq \sigma(u) \leq \tau(v)$, *and*

$$\begin{aligned} C(u,v,r) &= -\frac{1}{135\sigma^2(u)}\Big((\sigma(u)-1)\Big(-147\sigma^3(u)+42\sigma^4(u) \\ &\quad +(39\tau(v)\theta^2(r)-9\tau^3(v))(\sigma+1) \\ &\quad -86\sigma(u)\theta^2(r)+\sigma^2(u)(8\theta^2(r)+135)+\tau^2(v)\left(55\sigma^2(u)-67\right)\Big)\Big) \end{aligned}$$

for $\theta(r) \leq \tau(v) \leq \sigma(u)$.

Graphically we can see the copulas $C_{AB}(u,v)$, $C_{AC}(u,r)$ *and* $C_{BC}(v,r)$ *for the cases* $r=1$, $v=1$ *and* $u=1$ *correspondingly, and* $a=0.01$ *on the Figure 3.12.*

3.6 Conclusions

In this chapter we have shown that it is possible to find a copula function for a given Markov Chain. We have considered two cases with two and three firms. One can also extend this approach for the case with more than three firms.

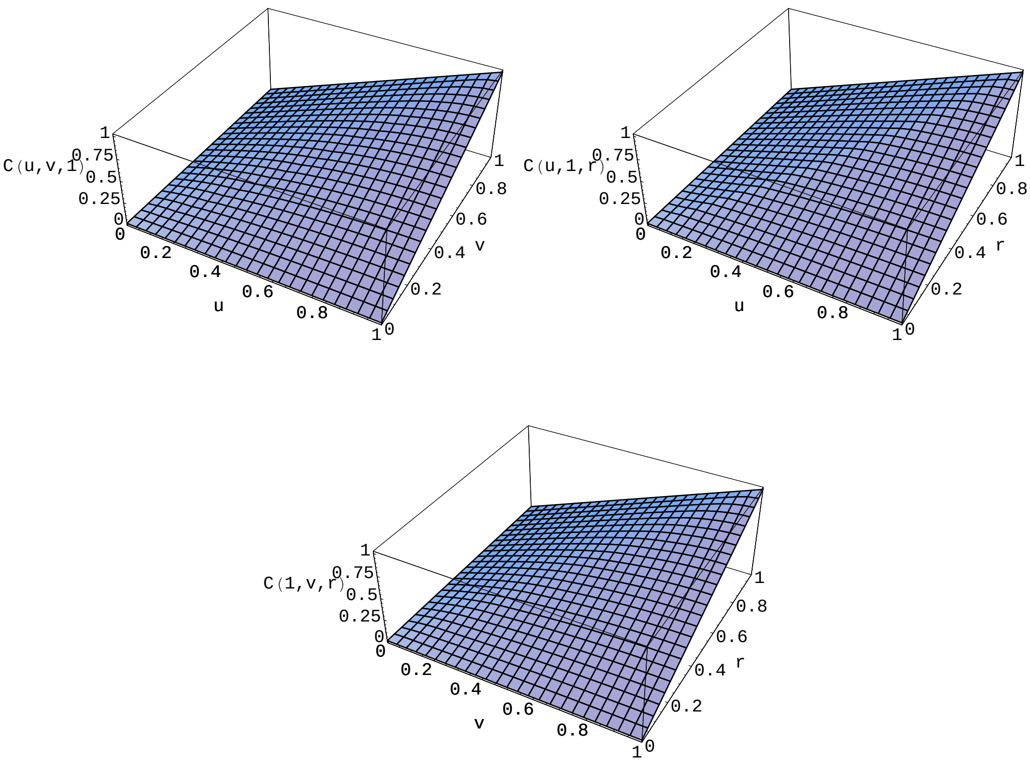

Figure 3.12: Copula describing three firms full dependance case: $a_{10} = 0.01$, $a_{20} = 0.02$, $a_{30} = 0.015$, $a_{12} = 0.002$, $a_{23} = 0.005$, $a_{31} = 0.003$, $k = \frac{1}{3}$, $l = \frac{1}{3}$, $m = \frac{1}{3}$, $n = \frac{1}{2} = p = q$

Chapter 4

Approximation of Copula

4.1 Introduction

As we said in chapter 3 it is not always possible to find quasi-inverse functions of marginal distribution function in general case. In this section we will build the approximation of the marginal distribution function.

The marginal distribution function are sum of exponential functions with different arguments. So may be reasonable in the general case consider some approximation of the exponent functions, for example Taylor expansion. The problem is to estimate how good is this approximation. If we assume that the maximal value of the time will be 10 years and the coefficients λ's are $k * 0.001$ with $k \leq 10$ then we can use the fact that

$$e^{ax} \approx 1 + ax + \frac{(ax)^2}{2}.$$

In this case we will get an error of the rate $(ax)^3 = 0.001$. Figure 4.1 show how it looks graphically.

So in the following sections we will try to find an approximation of a copula function for a Markov Chain.

To see how good is our approximation we will summarize this section with numerical example.

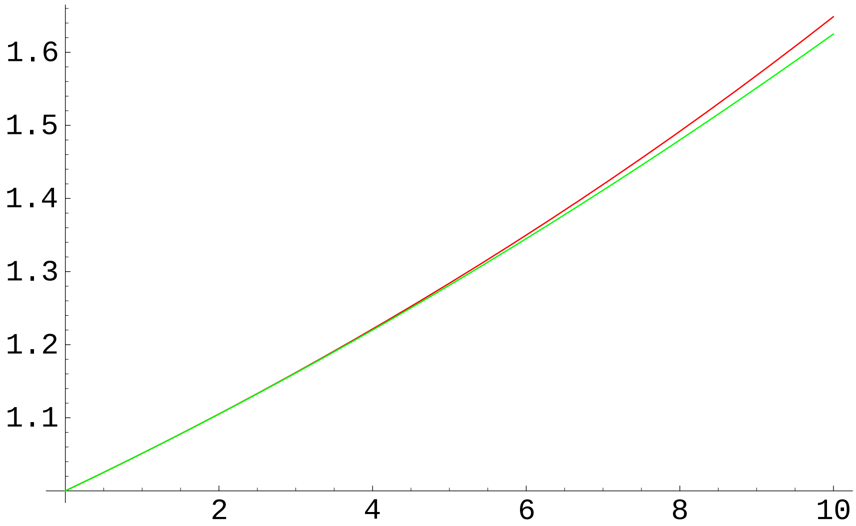

Figure 4.1: Comparison of Taylor expansion of second degree and exponential function, $a = 0.05$

4.2 Numerical Example: Case of Two Firms

According to the section 3.4 (under the assumptions of the chapter 3) the marginal and joint distribution functions in general case are:

$$
\begin{aligned}
F_A(t) &= 1 - q^{02}(0,t) - q^{00}(0,t) \\
&= 1 - \frac{e^{-(\lambda^{01}+\lambda^{02})t}\left(\lambda^{23} - \lambda^{01} - \lambda^{02} e^{-(\lambda^{23}-\lambda^{01}+\lambda^{02})t}\right)}{\lambda^{23} - \lambda^{01} - \lambda^{02}}, && (4.1)\\
F_B(t) &= 1 - q^{01}(0,t) - q^{00}(0,t) \\
&= 1 - \frac{e^{-(\lambda^{01}+\lambda^{02})t}\left(\lambda^{13} - \lambda^{02} - \lambda^{01} e^{-(\lambda^{13}-\lambda^{01}+\lambda^{02})t}\right)}{\lambda^{13} - \lambda^{01} - \lambda^{02}}, && (4.2)\\
F_{A,B}(t,s) &= 1 - q^{01}(0,t) - q^{02}(0,t) - q^{00}(0,t) + q^{01}(0,t) q^{13}(t,s) \\
&= 1 - \frac{(\lambda^{23} - \lambda^{01})\, e^{-(\lambda^{01}+\lambda^{02})t} - \lambda^{02} e^{-\lambda^{23} t}}{\lambda^{23} - \lambda^{01} - \lambda^{02}} && (4.3)\\
&- \lambda^{01} \frac{e^{-\lambda^{13} t} - e^{-\left(\lambda^{02}+\lambda^{01}\right)t}}{\lambda^{01} + \lambda^{02} - \lambda^{13}} e^{-\lambda^{13}(s-t)}, \text{ for } t \leq s, \\
F_{A,B}(t,s) &= 1 - \frac{(\lambda^{13} - \lambda^{02})\, e^{-(\lambda^{01}+\lambda^{02})t} - \lambda^{01} e^{-\lambda^{13} t}}{\lambda^{13} - \lambda^{01} - \lambda^{02}} && (4.4)
\end{aligned}
$$

$$- \quad \lambda^{02} \frac{e^{-\lambda^{23}t} - e^{-\left(\lambda^{02}+\lambda^{01}\right)t}}{\lambda^{01} + \lambda^{02} - \lambda^{23}} e^{-\lambda^{23}(s-t)}, \text{ for } t > s.$$

To find copula for this joint and marginal distributions, we can use the procedure described in the section 2.3.1:

$$C(u, v) = F_{A,B}(F_A^{(-1)}(u), F_B^{(-1)}(v)),$$

where $F_A^{(-1)}(u),\ F_B^{(-1)}(v))$ are quasi-inverse functions of $F_A(t),\ F_B(s)$. So, we need to find inverse functions of marginal distribution functions or equivalently to solve following equations

$$F_A(t) = u, \tag{4.5}$$

$$F_B(s) = v. \tag{4.6}$$

Using the approximation of the exponential function we will build some approximation of quasi-inverse functions and so we will get a approximation of a copula function. So, substituting $e^{ax} \approx 1 + ax + \frac{(ax)^2}{2}$ in the marginal distribution functions we get:

$$\begin{aligned}
F_A(t) &= 1 - \frac{e^{-(\lambda^{01}+\lambda^{02})t}\left(\lambda^{23} - \lambda^{01} - \lambda^{02} e^{-(\lambda^{23}-\lambda^{01}+\lambda^{02})t}\right)}{\lambda^{23} - \lambda^{01} - \lambda^{02}} \\
&\approx t\left(-\lambda^{01}(-1 + t(\lambda^{01} + \lambda^{02})) + t\lambda^{02}\lambda^{23}\right) \\
&= \tilde{F}_A(t), \\
F_B(t) &= 1 - \frac{e^{-(\lambda^{01}+\lambda^{02})t}\left(\lambda^{13} - \lambda^{02} - \lambda^{01} e^{-(\lambda^{13}-\lambda^{01}+\lambda^{02})t}\right)}{\lambda^{13} - \lambda^{01} - \lambda^{02}} \\
&\approx t\left(-\lambda^{02}(-1 + t(\lambda^{01} + \lambda^{02})) + t\lambda^{01}\lambda^{13}\right) \\
&= \tilde{F}_B(t).
\end{aligned}$$

The approximated distribution functions are quadratic equations, so we can easily solve equations (4.5) - (4.6) and accordingly find quasi-inverse functions of the approximated distribution functions. Each of this equations has two solutions. We have to choose one of them, such that solutions have values in interval $[0; 1]$. The solutions of equation (4.5) are

$$t_1 = \frac{-\lambda^{01} - \sqrt{(\lambda^{01})^2 - 4u(\lambda^{01})^2 - 4u\lambda^{01}\lambda^{02} + 4u\lambda^{02}\lambda^{23}}}{2\left(-(\lambda^{01})^2 - \lambda^{01}\lambda^{02} + \lambda^{02}\lambda^{23}\right)},$$

$$t_2 = \frac{-\lambda^{01} + \sqrt{(\lambda^{01})^2 - 4u(\lambda^{01})^2 - 4u\lambda^{01}\lambda^{02} + 4u\lambda^{02}\lambda^{23}}}{2\left(-(\lambda^{01})^2 - \lambda^{01}\lambda^{02} + \lambda^{02}\lambda^{23}\right)}.$$

The solutions of equation (4.6) are

$$s_1 = \frac{-\lambda^{02} - \sqrt{(\lambda^{021})^2 - 4v(\lambda^{02})^2 - 4v\lambda^{01}\lambda^{02} + 4v\lambda^{02}\lambda^{13}}}{2\left(-(\lambda^{02})^2 - \lambda^{01}\lambda^{02} + \lambda^{01}\lambda^{13}\right)},$$
$$s_2 = \frac{-\lambda^{02} + \sqrt{(\lambda^{021})^2 - 4v(\lambda^{02})^2 - 4v\lambda^{01}\lambda^{02} + 4v\lambda^{02}\lambda^{13}}}{2\left(-(\lambda^{02})^2 - \lambda^{01}\lambda^{02} + \lambda^{01}\lambda^{13}\right)}.$$

The quasi-inverse functions of marginal distribution functions are

$$\tilde{F}_A^{(-1)}(u) = \frac{-\lambda^{01} + \sqrt{(\lambda^{01})^2 - 4u(\lambda^{01})^2 - 4u\lambda^{01}\lambda^{02} + 4u\lambda^{02}\lambda^{23}}}{2\left(-(\lambda^{01})^2 - \lambda^{01}\lambda^{02} + \lambda^{02}\lambda^{23}\right)}, \quad (4.7)$$
$$\tilde{F}_B^{(-1)}(v) = \frac{-\lambda^{02} + \sqrt{(\lambda^{021})^2 - 4v(\lambda^{02})^2 - 4v\lambda^{01}\lambda^{02} + 4v\lambda^{02}\lambda^{13}}}{2\left(-(\lambda^{02})^2 - \lambda^{01}\lambda^{02} + \lambda^{01}\lambda^{13}\right)}. \quad (4.8)$$

Now substituting them into expression for the joint distribution function we get a corresponding approximation of a copula function

$$\begin{aligned}\tilde{C}(u,v) &= F_{A,B}(\tilde{F}_A^{(-1)}(u), \tilde{F}_B^{(-1)}(v)) \\ &= \begin{cases} 1 & - q^{01}\left(0, \tilde{F}_A^{(-1)}(u)\right) - q^{02}\left(0, \tilde{F}_A^{(-1)}(u)\right) - \ q^{00}\left(0, \tilde{F}_A^{(-1)}(u)\right) \\ & + q^{01}\left(0, \tilde{F}_A^{(-1)}(u)\right) q^{13}\left(\tilde{F}_A^{(-1)}(u), \tilde{F}_B^{(-1)}(v)\right), \ v \le u, \\ 1 & - q^{01}\left(0, \tilde{F}_B^{(-1)}(u)\right) - q^{02}\left(0, \tilde{F}_B^{(-1)}(u)\right) - \ q^{00}\left(0, \tilde{F}_B^{(-1)}(u)\right) \\ & + q^{02}\left(0, \tilde{F}_B^{(-1)}(u)\right) q^{23}\left(\tilde{F}_B^{(-1)}(u), \tilde{F}_A^{(-1)}(v)\right), \ u > v. \end{cases}\end{aligned}$$

Now we can estimate how gut is our approximation. Consider the case, where $\lambda^{01} + \lambda^{02} = a$ and $\lambda^{13} = \lambda^{23} = 2(\lambda^{01} + \lambda^{02}) = 2a$.

In this case, the approximations of distributions functions will be:

$$\tilde{F}_A(t) = at(1 + at) = \tilde{F}_B(t).$$

The quasi-inverse functions of corresponding distribution functions are:

$$\tilde{F}_A^{(-1)}(u) = \frac{-1 + \sqrt{1 + 4u}}{2a},$$

$$\tilde{F}_B^{(-1)}(v) = \frac{-1+\sqrt{1+4v}}{2a}.$$

The approximation of the copula function will be: For the case $u \leq v$:

$$\begin{aligned}\tilde{C}_1(u,v) &= \frac{1}{2}\Big(2+e^{2(1-\sqrt{1+4u})}-3e^{1-\sqrt{1+4u}} \\ &+ e^{2(1-\sqrt{1+4v})}-e^{1+\sqrt{1+4u}-2\sqrt{1+4v}}\Big)\end{aligned}$$

and for the case $u > v$:

$$\begin{aligned}\tilde{C}_2(u,v) &= \frac{1}{2}\Big(2+e^{2(1-\sqrt{1+4v})}-3e^{1-\sqrt{1+4v}} \\ &+ e^{2(1-\sqrt{1+4u})}-e^{1+\sqrt{1+4v}-2\sqrt{1+4u}}\Big).\end{aligned}$$

From section 3.4 we have exact expression for this copula function: For the case $u \leq v$:

$$\begin{aligned}C_1(u,v) &= 1+\frac{\lambda^{01}(\lambda^{01}+2u\lambda^{02}-\sqrt{(\lambda^{01})^2+4u\lambda^{01}\lambda^{02}+4u(\lambda^{02})^2})}{2(\lambda^{02})^2} \\ &- \frac{\lambda^{01}+2\lambda^{02}-\sqrt{(\lambda^{01})^2+4u\lambda^{01}\lambda^{02}+4u(\lambda^{02})^2}}{2\lambda^{02}} \\ &+ \frac{(\lambda^{01}+2u\lambda^{02}-\sqrt{(\lambda^{01})^2+4u\lambda^{01}\lambda^{02}+4u(\lambda^{02})^2})}{2\lambda^{01}} \\ &\times \frac{(2\lambda^{01}+\lambda^{02}-\sqrt{(\lambda^{02})^2+4v\lambda^{01}(\lambda^{02}+\lambda^{01})})^2}{(\lambda^{01}+2\lambda^{02}-\sqrt{(\lambda^{01})^2+4u\lambda^{01}\lambda^{02}+4u(\lambda^{02})^2})^2} \\ &= \frac{(-3+\sqrt{1+8v})^2\left(1+2u-\sqrt{1+8u}+\frac{2u(-3+\sqrt{1+8u})12}{(-3+\sqrt{1+8v})^2}\right)^2}{2(-3+\sqrt{1+8u})^2},\end{aligned}$$

and for the case $u > v$:

$$\begin{aligned}C_2(u,v) &= 1-\frac{2\lambda^{01}+\lambda^{02}-\sqrt{(\lambda^{02})^2+4v\lambda^{01}(\lambda^{01}+(\lambda^{02})}}{2\lambda^{01}} \\ &+ \frac{2v\lambda^{01}+\lambda^{02}-\sqrt{(\lambda^{02})^2+4v\lambda^{01}(\lambda^{01}+\lambda^{02})}}{2\lambda^{01}} \\ &+ \frac{(2v\lambda^{01}+\lambda^{02}-\sqrt{(\lambda^{02})^2+4v\lambda^{01}(\lambda^{01}+\lambda^{02})})}{2\lambda^{02}} \\ &\times \frac{(\lambda^{01}+2\lambda^{02}-\sqrt{(\lambda^{01})^2+4u\lambda^{01}\lambda^{02}+4u(\lambda^{02})^2})^2}{(2\lambda^{01}+\lambda^{02}-\sqrt{(\lambda^{02})^2+4v\lambda^{01}(\lambda^{01}+\lambda^{02})})^2}\end{aligned}$$

$$= \frac{(-3+\sqrt{1+8u})^2\left(1+2v-\sqrt{1+8v}+\frac{2v(-3+\sqrt{1+8v})12}{(-3+\sqrt{1+8u})^2}\right)^2}{2(-3+\sqrt{1+8v})^2}.$$

Now we can estimate the difference between two copulas as quadratic error:

$$\begin{aligned}\epsilon &= \int_0^1\int_0^1 |C(u,v)-\tilde{C}(u,v)|dudv \\ &= \int_0^1\int_0^v |C_1(u,v)-\tilde{C}_1(u,v)|dudv + \int_0^1\int_0^u |C_1(u,v)-\tilde{C}_1(u,v)|dvdu \\ &= 0.0956078.\end{aligned}$$

With the same procedure one can get an approximation of the copula with Taylor Expansion of the exponential function to the third degree, i.e. $e^{ax} \approx 1+ax+\frac{(ax)^2}{2}+\frac{(ax)^2}{6}$.

4.3 Conclusions

One can see, that in general case, when direct calculation are not possible, it still possible to find some approximation of the copula function for each Markov Chain. Obviously, the quality of the approximation is dependent of the degree of Taylor approximation function: higher degree will result in more precise estimation, but will lead to more complicated mathematical equations. One can also use the other approximation of the exponential function to get an approximation of copula function in general case. In the case of more than two firms one can use the same procedure. It will involve much more complicated mathematical statements, but still possible.

Chapter 5

Conclusions and Further Investigations

The first chapter of this work was devoted to the some interesting aspects of credit derivatives pricing: default of each contract counterparty. It was shown that it is an important point, because in some cases the influence of such assumption lead to the significant price differences.

The main result of this work was connection of two main approaches of credit derivatives pricing. We have found a way to connect Markov Chain Model with the Copula based approach. We have also shown that in general case it is not easy to find copula function for given Markov Chain. For this case also possible way to find an approximation of copula function was introduced.

In this work we were concentrated only on the Markov Chain framework of Kraft and Steffensen (2007). It is also possible to consider some other Markov Chain models or some general intensity-based models.

The possible way from Copula to Markov Chain can be also investigated.

It is also possible some further investigations on the copula approximation and moreover one can try to see the impact of different connected Models on the price of credit derivative.

Appendix A

Coefficients of the Marginal Distribution Functions

The coefficients of the marginal distribution functions are:

$$C_A^0 = -1 + \frac{\lambda^{02}}{\lambda^{0*} - \lambda^{2*}} + \frac{\lambda^{03}}{\lambda^{0*} - \lambda^{3*}} + \frac{\lambda^{02}\lambda^{26}}{\lambda^{6*} - \lambda^{2*}}\Big(\frac{1}{\lambda^{0*} - \lambda^{2*}} - \frac{1}{\lambda^{0*} - \lambda^{6*}}\Big) +$$
$$\frac{\lambda^{03}\lambda^{36}}{\lambda^{6*} - \lambda^{3*}}\Big(\frac{1}{\lambda^{0*} - \lambda^{3*}} - \frac{1}{\lambda^{0*} - \lambda^{6*}}\Big);$$

$$C_A^2 = \frac{\lambda^{02}(\lambda^{67} - \lambda^{24})}{(\lambda^{0*} - \lambda^{2*})(\lambda^{2*} - \lambda^{67})};$$

$$C_A^3 = \frac{\lambda^{03}(\lambda^{67} - \lambda^{35})}{(\lambda^{0*} - \lambda^{3*})(\lambda^{3*} - \lambda^{67})};$$

$$C_A^6 = \frac{\frac{\lambda^{02}\lambda^{26}}{\lambda^{2*} - \lambda^{67}} + \frac{\lambda^{03}\lambda^{36}}{\lambda^{3*} - \lambda^{67}}}{\lambda^{0*} - \lambda^{67}};$$

$$C_B^0 = -1 + \frac{\lambda^{01}}{\lambda^{0*} - \lambda^{1*}} + \frac{\lambda^{03}}{\lambda^{0*} - \lambda^{3*}} + \frac{\lambda^{01}\lambda^{15}}{\lambda^{5*} - \lambda^{1*}}\Big(\frac{1}{\lambda^{0*} - \lambda^{1*}} - \frac{1}{\lambda^{0*} - \lambda^{5*}}\Big) +$$
$$\frac{\lambda^{03}\lambda^{35}}{\lambda^{5*} - \lambda^{3*}}\Big(\frac{1}{\lambda^{0*} - \lambda^{3*}} - \frac{1}{\lambda^{0*} - \lambda^{5*}}\Big);$$

$$C_B^1 = \frac{\lambda^{01}}{\lambda^{0*} - \lambda^{1*}}\Big(1 + \frac{\lambda^{15}}{\lambda^{5*} - \lambda^{1*}}\Big);$$

$$C_B^3 = \frac{\lambda^{03}}{\lambda^{0*} - \lambda^{3*}}\Big(1 + \frac{\lambda^{13}}{\lambda^{5*} - \lambda^{3*}}\Big);$$

$$C_B^5 = \frac{1}{\lambda^{0*} - \lambda^{5*}}\Big(\frac{\lambda^{01}\lambda^{15}}{\lambda^{5*} - \lambda^{1*}} + \frac{\lambda^{03}\lambda^{35}}{\lambda^{5*} - \lambda^{3*}}\Big);$$

$$C_C^0 = -1 + \frac{\lambda^{01}}{\lambda^{0*} - \lambda^{1*}} + \frac{\lambda^{02}}{\lambda^{0*} - \lambda^{2*}} + \frac{\lambda^{01}\lambda^{14}}{\lambda^{4*} - \lambda^{1*}}\Big(\frac{1}{\lambda^{0*} - \lambda^{1*}} - \frac{1}{\lambda^{0*} - \lambda^{4*}}\Big) + \frac{\lambda^{02}\lambda^{24}}{\lambda^{4*} - \lambda^{2*}}\Big(\frac{1}{\lambda^{0*} - \lambda^{2*}} - \frac{1}{\lambda^{0*} - \lambda^{4*}}\Big);$$

$$C_C^1 = \frac{\lambda^{01}}{\lambda^{0*} - \lambda^{1*}}\Big(1 + \frac{\lambda^{14}}{\lambda^{4*} - \lambda^{1*}}\Big);$$

$$C_C^2 = \frac{\lambda^{02}}{\lambda^{0*} - \lambda^{2*}}\Big(1 + \frac{\lambda^{24}}{\lambda^{4*} - \lambda^{2*}}\Big);$$

$$C_B^4 = \frac{1}{\lambda^{0*} - \lambda^{4*}}\Big(\frac{\lambda^{01}\lambda^{14}}{\lambda^{4*} - \lambda^{1*}} + \frac{\lambda^{02}\lambda^{24}}{\lambda^{4*} - \lambda^{2*}}\Big);$$

Appendix B

Calculation of Kendall's tau and Spearman's rho

To obtain Kendall's tau and Spearman's rho a little manipulation is needed. Let

$$n = \frac{\lambda_{13}}{\lambda_{01}+\lambda_{02}}, \qquad m = \frac{\lambda_{23}}{\lambda_{01}+\lambda_{02}},$$

$$\alpha = \frac{\lambda_{02}}{\lambda_{01}+\lambda_{02}}, \qquad \beta = \frac{\lambda_{01}}{\lambda_{01}+\lambda_{02}},$$

$$x = \sigma = e^{-\lambda^{0*}t}, \qquad y = \sigma = e^{-\lambda^{0*}s}.$$

by the equations (3.1)-(3.3) we get:

$$\begin{aligned}\hat{F}_A(x) &= 1-(1-\alpha)\,x-\alpha x^m,\\ \hat{F}_B(y) &= 1-(1-\beta)\,y-\beta y^n,\end{aligned}$$

and

$$\begin{aligned}\hat{F}^1_{A,B}(x(t),y(s)) &= 1-\beta y^n + x\left(-1+\beta\left(\frac{y}{x}\right)^n+\alpha\right)-\alpha x^m\\ \hat{F}^2_{A,B}(x(t),y(s)) &= 1-\beta y^n + y\left(-1+\alpha\left(\frac{x}{y}\right)^m+\beta\right)-\alpha x^m\end{aligned}$$

Our task is now to calculate

$$\tau(A,B) = 1-4\int_0^1\int_0^x \frac{\partial}{\partial x}\hat{F}^2_{A,B}(x,y)\frac{\partial}{\partial y}\hat{F}^2_{A,B}(x,y)dydx$$

$$+\int_0^1\int_0^y \frac{\partial}{\partial x}\hat{F}^1_{A,B}(x,y)\frac{\partial}{\partial y}\hat{F}^1_{A,B}(x,y)dxdy,$$

$$\begin{aligned}\rho_s(X,Y) &= 12\int_0^1\int_0^x \left[\hat{F}^2_{A,B}(x,y)-\hat{F}_A(x)\hat{F}_B(y)\right]d\hat{F}_B(y)d\hat{F}_A(x)\\ &\quad +12\int_0^1\int_0^y \left[\hat{F}^1_{A,B}(x,y)-\hat{F}_A(x)\hat{F}_B(y)\right]d\hat{F}_A(x)d\hat{F}_B(y).\end{aligned}$$

Our aim is to calculate all derivatives of transformed joint and marginal distribution functions. So, we get $\hat{F}^1_{A,B}(x,y)$, $\hat{F}^2_{A,B}(x,y)$, $\hat{F}_A(x)$, $\hat{F}_B(y)$:

$$\begin{aligned}\frac{\partial}{\partial x}\hat{F}_A(x) &= \alpha-1-m\alpha x^{m-1},\\ \frac{\partial}{\partial y}\hat{F}_B(y) &= \beta-1-n\alpha y^{n-1},\\ \frac{\partial}{\partial x}\hat{F}^1_{A,B}(x,y) &= \alpha-1-m\alpha x^{m-1}-(n-1)\beta\left(\frac{y}{x}\right)^n,\\ \frac{\partial}{\partial y}\hat{F}^1_{A,B}(x,y) &= -n\beta y^{n-1}+n\beta\left(\frac{y}{x}\right)^{n-1},\\ \frac{\partial}{\partial x}\hat{F}^2_{A,B}(x,y) &= -m\alpha x^{m-1}+m\alpha\left(\frac{x}{y}\right)^{m-1},\\ \frac{\partial}{\partial y}\hat{F}^2_{A,B}(x,y) &= \beta-1-n\beta y^{n-1}-(m-1)\alpha\left(\frac{x}{y}\right)^m.\end{aligned}$$

Firstly we proceed with Kendall's tau. Using previous results we obtain

$$\begin{aligned}&\int_0^1\int_0^x \frac{\partial}{\partial x}\hat{F}^2_{A,B}(x,y)\frac{\partial}{\partial y}\hat{F}^2_{A,B}(x,y)dydx\\ &=\int_0^1\int_0^x\left(\alpha-1-m\alpha x^{m-1}-(n-1)\beta\left(\frac{y}{x}\right)^n\right)\left(-n\beta y^{n-1}+n\beta\left(\frac{y}{x}\right)^{n-1}\right)dydx\\ &=\int_0^1\frac{\beta x\left((2-2\alpha+\beta(n-1))x+2\alpha m x^m\right)(x^n-x)}{2x^2}dx\\ &=\frac{1}{2}\beta\left(-1+\frac{3\beta}{2}-\alpha+\frac{2\alpha}{m+1}-\frac{\beta n}{2}-\frac{2(\alpha+\beta-1)}{n+1}+\frac{2\alpha m}{m+n}\right),\end{aligned}$$

and

$$\begin{aligned}&\int_0^1\int_0^y \frac{\partial}{\partial x}\hat{F}^1_{A,B}(x,y)\frac{\partial}{\partial y}\hat{F}^1_{A,B}(x,y)dxdy\\ &=\int_0^1\int_0^y\left(-m\alpha x^{m-1}+m\alpha\left(\frac{x}{y}\right)^{m-1}\right)\left(\beta-1-n\beta y^{n-1}-(m-1)\alpha\left(\frac{x}{y}\right)^m\right)dxdy\end{aligned}$$

$$= \int_0^1 \frac{\alpha y \left((2-2\beta+\alpha(m-1))y + 2\beta n y^n\right)(y^m - y)}{2y^2} dy$$
$$= \frac{1}{2}\alpha\left(-1 + \frac{3\alpha}{2} - \beta + \frac{2\beta}{n+1} - \frac{\alpha m}{2} - \frac{2(\alpha+\beta-1)}{m+1} + \frac{2\beta n}{m+n}\right).$$

Adding both equations we obtain Kendall's tau is

$$\begin{aligned}\tau(A,B) &= 1 - \frac{2\alpha(m-1)(n+1) + \alpha^2(m-1)^2(n+1)}{(n+1)(m+1)} \\ &\quad + \frac{(m+1)\left(\beta(2+\beta(n-1))(n-1) + 2(n+1)\right)}{(n+1)(m+1)}.\end{aligned}$$

Now we consider Spearman's rho. It contains also two parts.

$$\begin{aligned}\rho_s(X,Y) &= 12\int_0^1\int_0^x \left[\hat{F}^2_{A,B}(x,y) - \hat{F}_A(x)\hat{F}_B(y)\right] d\hat{F}_B(y)d\hat{F}_A(x) \\ &\quad + 12\int_0^1\int_0^y \left[\hat{F}^1_{A,B}(x,y) - \hat{F}_A(x)\hat{F}_B(y)\right] d\hat{F}_A(x)d\hat{F}_B(y).\end{aligned}$$

Let us calculate each of these parts and then add them. Firstly, consider each of these parts before the integration and simplify them:

$$\left(\hat{F}^2_{A,B}(x,y) - \hat{F}_A(x)\hat{F}_B(y)\right)\frac{\partial}{\partial x}\hat{F}_A(x)\frac{\partial}{\partial y}\hat{F}_B(y)$$
$$= -\frac{1}{xy}\left((-1+\alpha)x - \alpha m x^m\right)\left((-1+\beta)y - \beta n y^n\right)$$
$$\times\left((-1+\beta)\left(1+(-1+\alpha)x - \alpha x^m\right)y + \beta\left(x - \alpha x + \alpha x^m\right)y^n - \beta x\left(\frac{y}{x}\right)^n\right)$$

and

$$\left(\hat{F}^1_{A,B}(x,y) - \hat{F}_A(x)\hat{F}_B(y)\right)\frac{\partial}{\partial x}\hat{F}_A(x)\frac{\partial}{\partial y}\hat{F}_B(y)$$
$$= -\frac{1}{xy}\left((-1+\alpha)x - \alpha m x^m\right)\left((-1+\beta)y - \beta n y^n\right)$$
$$\times\left(-\alpha\left(\frac{x}{y}\right)^m y + (-1+\alpha)x\left(1+(-1+\beta)y - \beta y^n\right) + a x^m\left(y - \beta y + \beta y^n\right)\right).$$

Now we can proceed with integration of the simplified parts

$$\int_0^1\int_0^x \left(\hat{F}^2_{A,B}(x,y) - \hat{F}_A(x)\hat{F}_B(y)\right)\frac{\partial}{\partial x}\hat{F}_A(x)\frac{\partial}{\partial y}\hat{F}_B(y)dydx$$

$$
\begin{aligned}
&= -\int_0^1 \int_0^x \frac{1}{xy} \left((-1+\alpha)x - \alpha m x^m\right)\left((-1+\beta)y - \beta n y^n\right) \\
&\quad \times \left((-1+\beta)\left(1+(-1+\alpha)x - \alpha x^m\right)y + \beta\left(x - \alpha x + \alpha x^m\right)y^n - \beta x \left(\frac{y}{x}\right)^n\right) dydx \\
&= -\int_0^1 \frac{1}{2(1+n)x}\left((-1+\alpha)x - \alpha m x^m\right)\Big((-1+\beta)(-1+\beta(-1+n)-n)x^2 \\
&\quad +(-1+\alpha)(-1+\beta)^2(1+n)x^3 - \alpha(-1+\beta)^2(1+n)x^{2+m} \\
&\quad -\beta(\beta(-1+n)-2n)x^{1+n} - 2(-1+\alpha)(-1+\beta)\beta(1+n)x^{2+n} \\
&\quad +2\alpha(-1+\beta)\beta(1+n)x^{1+m+n} + (-1+\alpha)\beta^2(1+n)x^{1+2n} \\
&\quad -\alpha\beta^2(1+n)x^{m+2n}\Big)dx \\
&= \frac{1}{24}\Bigg(\frac{(-1+b)^2}{(1+m)(2+m)(3+m)} \\
&\quad \times \left((1+m)(2+m)(3+m) + 2\alpha(-1+m)(1+m)(6+m) + 3\alpha^2\left(2-3m+m^3\right)\right) \\
&\quad + \frac{-4b(8-5b+\alpha(4+(-10+3\alpha)b)) - 2(-1+\alpha)b(-8+(5+3\alpha)b)m}{(2+m)(1+n)} \\
&\quad + \frac{12(-1+\alpha)b(-4+3b)}{2+n} + \frac{24(-1+\alpha)^2(-1+b)b}{3+n} \\
&\quad - \frac{6\alpha^2 b^2 m}{m+n} - \frac{12\alpha b(-2(1+m)+b(2+m))}{1+m+n} \\
&\quad - \frac{24(-1+\alpha)\alpha(-1+b)b(1+m)}{2+m+n} + \frac{24\alpha^2(-1+b)bm}{1+2m+n} + \frac{12(-1+\alpha)\alpha b^2}{1+m+2n}\Bigg)
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_0^1 \int_0^y \left(\hat{F}^1_{A,B}(x,y) - \hat{F}_A(x)\hat{F}_B(y)\right)\frac{\partial}{\partial x}\hat{F}_A(x)\frac{\partial}{\partial y}\hat{F}_B(y)dxdy \\
&= -\int_0^1 \int_0^y \frac{1}{xy}\left((-1+\alpha)x - \alpha m x^m\right)\left((-1+\beta)y - \beta n y^n\right) \\
&\quad \times \left(-\alpha\left(\frac{x}{y}\right)^m y + (-1+\alpha)x\left(1+(-1+\beta)y - \beta y^n\right) + a x^m\left(y - \beta y + \beta y^n\right)\right) dydx \\
&= -\int_0^1 \frac{1}{2(1+m)y}\left((-1+\beta)y - \beta n y^n\right)\Big((-1+\alpha)(-1+\alpha(-1+m)-m)y^2 \\
&\quad +(-1+\alpha)^2(-1+\beta)(1+m)y^3 - \alpha(\alpha(-1+m)-2m)y^{1+m} \\
&\quad -2(-1+\alpha)\alpha(-1+\beta)(1+m)y^{2+m} + \alpha^2(-1+\beta)(1+m)y^{1+2m} \\
&\quad -(-1+\alpha)^2\beta(1+m)y^{2+n} + 2(-1+\alpha)\alpha\beta(1+m)y^{1+m+n} - \alpha^2\beta(1+m)y^{2m+n}\Big)dy
\end{aligned}
$$

$$
\begin{aligned}
&= -\frac{1}{2(1+m)}\Bigg(\frac{1}{2}a^2(-1+b)^2 + \frac{1}{3}(-1+a)(-1+b)(-1+a(-1+m)-m) \\
&\quad + \frac{1}{4}(-1+a)^2(-1+b)^2(1+m) - \frac{a(-1+b)(a(-1+m)-2m)}{2+m} \\
&\quad - \frac{2(-1+a)a(-1+b)^2(1+m)}{3+m} - \frac{(-1+a)b(-1+a(-1+m)-m)n}{2+n} \\
&\quad - \frac{(-1+a)^2(-1+b)b(1+m)}{3+n} \\
&\quad - \frac{(-1+a)^2(-1+b)b(1+m)n}{3+n} + \frac{a^2b^2(1+m)n}{2(m+n)} \\
&\quad + \frac{ab(a(-1+m)-2m)n}{1+m+n} + \frac{2(-1+a)a(-1+b)b(1+m)}{2+m+n} \\
&\quad + \frac{2(-1+a)a(-1+b)b(1+m)n}{2+m+n} - \frac{a^2(-1+b)b(1+m)}{1+2m+n} \\
&\quad - \frac{a^2(-1+b)b(1+m)n}{1+2m+n} + \frac{(-1+a)^2b^2(1+m)n}{2+2n} \\
&\quad - \frac{2(-1+a)ab^2(1+m)n}{1+m+2n}\Bigg).
\end{aligned}
$$

Finally, to obtain Spearman's rho we just need to add both parts. So, we get:

$$
\begin{aligned}
\rho_s(X,Y) &= \frac{1}{(1+m)(2+m)(1+n)(2+n)(1+m+n)} \\
&\quad \times\Big((1+m)(2+m)(1+m+n)\left(4\beta(-1+n)+2\beta^2(-1+n)^2+(1+n)(2+n)\right) \\
&\quad +2\alpha^2(-1+m)^2(1+n)(\beta(2+2m-n)(-1+n)+(2+n)(1+m+n)) \\
&\quad -2\alpha(-1+m)\Big(-2(1+n)(2+n)(1+m+n) \\
&\quad +\beta^2(1+m)(-1+n)^2(m-2(1+n)) \\
&\quad +2\beta(-1+n)\left(2n(1+n)+m^2(2+n)+m(2+n(2+n))\right)\Big)\Big).
\end{aligned}
$$

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