Asymptotic order of the parallel volume difference

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Abstract

In this paper we continue the investigation of the asymptotic behavior of the parallel volume in Minkowski spaces as the distance tends to infinity that was started in [13]. We will show that the difference of the parallel volume of the convex hull of a body and the parallel volume of the body itself can at most have order r^{d-2} in dimension d. Then we will show that in the Euclidean case this difference can at most have order r^{d-3}. We will also examine the asymptotic behavior of the derivative of this difference as the distance tends to infinity. After this we will compute the derivative of $f_\mu(rK)$ in r, where f_μ is the Wills functional or a similar functional, K is a fixed body and rK is the Minkowski-product of r and K. Finally we will use these results to examine Brownian paths and Boolean models and derive new proofs for formulae for intrinsic volumes.

Keywords: Parallel volume · non-convex body · Wills functional

1 Introduction

The parallel volume of a body K at distance r is the volume of the set of all points, which have at most distance r from K, where by body we mean a non-empty compact set. Since Steiner [21] discovered in 1840 that the parallel volume of certain convex bodies is a polynomial (meanwhile it is known that this is true for all convex bodies), it has been studied intensively. Essential concepts of convex geometry, like intrinsic volumes, mixed volumes and support measures are usually defined with help of the parallel volume or relative or local versions of it. Moreover, the parallel volume has many applications, e.g. in stochastic geometry, geometric functional analysis or statistics. While in many of these applications the parallel volume of arbitrary bodies is of interest, it has been mainly investigated in the special case of convex bodies.

However, there are some important results for the parallel volume of non-convex bodies. The Brunn-Minkowski-inequality which gives an upper bound for the parallel volume, was proven by Lusternik [17] for arbitrary bodies. It implies the isoperimetric inequality and is closely related to many inequalities in various branches of mathematics and physics [6]. Kneser [16] and Sz.-Nagy [23] obtained inequalities saying that the parallel volume of a fixed body considered as function of the distance cannot be "too convex". Heveling, Hug and Last [7] showed that a planar body can only have polynomial parallel volume, if it is convex (see also [10]).

In [13] we have shown

$$\lim_{r\to\infty} V_2(\operatorname{conv} K + rB^2) - V_2(K + rB^2) = 0,$$

for an arbitrary body $K \subseteq \mathbb{R}^2$, where $K + L := \{x + y \mid x \in K, y \in L\}$ is the Minkowski-sum of two bodies $K, L \subseteq \mathbb{R}^d$, $rK := \{rx \mid x \in K\}$ is the Minkowski-product of a body $K \subseteq \mathbb{R}^d$ and a number $r \geq 0$, V_d denotes the d-dimensional Lebesgue measure and $\operatorname{conv} K$ denotes the convex hull of a body $K \subseteq \mathbb{R}^d$.
In the present paper we will show that the order of this convergence is $1/r$. Just like in [13] we will discuss extensions to Minkowski spaces, i.e. we will replace B^2 by another convex body B. We will show that the order is $1/r$, iff B contains a ball as summand (for the definition of a summand, see subsection 2.2). The result about the convergence order has an extension to higher dimensions, namely in dimension d the asymptotic order of this difference is r^{d-3}, provided that B has a ball a summand. We will also show that, if we do not assume that B has a ball as summand, this difference can have orders up to r^{d-2}, but not higher. In fact, these results hold for the expected value of parallel volume difference of random bodies, if certain integrability conditions are fulfilled. We will also discuss the case that the dimension (of the affine hull) of K is larger than that of B.
These results suggest that the asymptotic order of $\frac{d}{dr} V_d(\operatorname{conv} K + rB^d) - V_d(K + rB^d)$ is r^{d-4}. We will prove this in the case $d = 2$.
In [12] we introduced a large class of functionals f_μ that generalise the Wills functional. For a signed measure μ on the set $\mathcal{K}$ of all convex bodies fulfilling certain integrability assumptions we put

$$f_\mu : \mathcal{C} \to \mathbb{R}_0^+,\ K \mapsto \int_{\mathcal{K}} V_d(K + A)\, d\mu(A).$$

With help of the results about the asymptotic order of the parallel volume difference mentioned above, we can show that the first derivative of $\mathbb{R}_0^+ \to \mathbb{R}$, $r \mapsto f_\mu(rK)$ in $r = 0$ is $d \int_{\mathcal{K}} V(\operatorname{conv} K[1], A[d-1])\, d\mu(A)$, where $V(K[j], L[d-j])$ denotes the mixed volume (which will be introduced in subsection 2.3) of two convex bodies $K, L \subseteq \mathbb{R}^d$. We will also show that, if the second derivative exists, then it equals $d(d-1) \int_{\mathcal{K}} V(\operatorname{conv} K[2], A[d-2])\, d\mu(A)$ and give sufficient conditions for this second derivative to exist.
This paper is organized as follows. In Section 2 we collect mathematical tools, especially from geometry, that will be needed in later sections. In Section 3 we first show that the asymptotic order of $V_d(\operatorname{conv} K + rB) - V_d(K + rB)$ is r^{d-2} with the generalisations mentioned above. Then we will prove that in Minkowski spaces whose gauge bodies B have a ball as summand this difference has asymptotic order r^{d-3} and that this property characterizes Minkowski spaces whose gauge bodies B have a ball as summand in the planar case. Section 4 is devoted to the examination of derivative of the parallel volume difference. In Section 5 we will examine the derivatives of $\mathbb{R}_0^+ \to \mathbb{R}$, $r \mapsto f_\mu(rK)$. In Section 6 we present stochastic applications of the results of the previous sections. In particular, we examine the asymptotic behavior of the parallel volume of Brownian paths as the time tends to zero, prove results about the asymptotic behavior of the contact distribution of Boolean models, and give new proofs for formulae that express intrinsic volumes in terms of Gaussian random variables.

2 Preparation

In this section we collect tools, especially from geometry, that will be needed in later sections.

2.1 Analytical tools

We start with a version of the fundamental theorem of calculus that is stated in [2] as Corollary 6.3.7.

Theorem 1. *Let $F : [a, b] \to \mathbb{R}$ be a function. Then F is absolutly continuous, iff F is (Lebesgue-) a.e. differentiable and*

$$F(x) = F(a) + \int_a^x F'(t)\,dt, \quad x \in [a, b].$$

The following theorem is known as Rademacher's theorem (see e.g. [5, 3.1.6]).

Theorem 2. *Let $f : \mathbb{R}^d \to \mathbb{R}^n$ be a Lipschitz-continuous function. Then f is differentiable (Lebesgue-) almost everywhere.*

Def. 3. Let $G \subseteq \mathbb{R}^d$ be an open convex set and $f : G \to \mathbb{R}$ a convex function.

(i) A vector $\nu \in \mathbb{R}^d$ is called a subgradient of f in $x \in U$, if $f(y) \geq f(x) + \langle \nu, y - x\rangle$ for all $y \in G$.

(ii) A function $\theta : G \to \mathbb{R}^d$ is called a choice of subgradients of f, if $\theta(x)$ is a subgradient of f in x for every $x \in G$.

(iii) Let $x \in G$. If all choices of subgradients of f are differentiable in x with the same derivative, then f is said to be Alexandrov-twice-differentiable in x. The derivative of the choices of subgradients is called second derivative of f.

Theorem 4. *Let $G \subseteq \mathbb{R}^d$ be an open convex set and $f : G \to \mathbb{R}$ a convex function. Then f is Alexandrov-twice-differentiable in a.e. $x \in G$.*

Various proofs of this theorem are known. For a discussion, see [18, section 1.5, note 2].

2.2 Convex and Minkowski Geometry

In this subsection we first introduce tools from elementary convex geometry and then turn to Minkowski geometry. The parallel volume will be treated in the following two subsections.

A vector $u \in \mathbb{R}^d \setminus \{0\}$ is called *exterior normal vector* of a convex body $K \subseteq \mathbb{R}^d$ in a point $p \in K$, if

$$\langle x, u\rangle \leq \langle p, u\rangle, \quad x \in K.$$

For a convex body K the *support function* is defined by

$$h_K : \mathbb{R}^d \to \mathbb{R},\ u \mapsto \max\{\langle x, u\rangle \mid x \in K\}.$$

Corollary 1.7.3 in [18] says:

Theorem 5. *Let $K \subseteq \mathbb{R}^d$ be a convex body and $u \in \mathbb{R}^d \setminus \{0\}$. Then h_K is differentiable in u, iff there is a unique point $p \in \operatorname{bd} K$ with exterior normal vector u. In this case $\nabla h_K(u) = p$.*

A convex subset F of a convex body K is called *face* of K, if for all $x, y \in K$ with $\frac{x+y}{2} \in F$ we have $x, y \in F$. The *dimension* of a convex set is said to be the dimension of its affine hull. The *relative interior*, $\operatorname{relint} K$, of a convex set $K \subseteq \mathbb{R}^d$ is its interior w.r.t. its affine hull as surrounding topological space. By $\operatorname{bd} A$ we denote the boundary of a set $A \subseteq \mathbb{R}^d$.

Lemma 6. *Let $K \subseteq \mathbb{R}^d$ be a body and $x \in (\operatorname{bd} \operatorname{conv} K) \setminus K$. Then x is contained in the relative interior of a face of positive dimension of* $\operatorname{conv} K$.

Proof. According to [18, Theorem 2.1.2] the point x is contained in the relative interior of a face F of $\operatorname{conv} K$. So all we have to show is $F \neq \{x\}$. Since $(\operatorname{conv} K) \setminus F$ is convex, we cannot have $K \subseteq (\operatorname{conv} K) \setminus F$ by the definition of the convex hull. So $K \cap F \neq \emptyset$ and $x \in K$ implies $F \neq \{x\}$. □

A convex body $S \subseteq \mathbb{R}^d$ is called *summand* of a convex body $K \subseteq \mathbb{R}^d$, if for each point $p \in K$ there is a vector $t \in \mathbb{R}^d$ with

$$p \in t + S \subseteq K,$$

or, equivalently, if there is a convex body $M \subseteq \mathbb{R}^d$ such that $S + M = K$. For a more detailed introduction we refer to [18, sections 3.1 and 3.2].

Lemma 7. *Let K be a body with a summand RB^d, $R > 0$, $b \in \operatorname{bd} K$ a point with exterior unit normal ν and $t \in K$ another point. Then the following are equivalent:*

(i) $t = b - R\nu$

(ii) $b \in t + RB^d \subseteq K$

Proof. First we will show "$(ii) \Rightarrow (i)$". From (ii) we get $t + R\nu \in K$ and hence

$$\langle t, \nu \rangle + R = \langle t + R\nu, \nu \rangle \leq \langle b, \nu \rangle,$$

which implies $\langle b - t, \nu \rangle \geq R$. Since $\|b - t\| \leq R$, we conclude $b - t = R\nu$ and so obtain (i). Since there is a point t satisfying (ii), the converse statement must hold as well. □

A criterion for summands was given by Weil [25]. For this, recall that the essential infimum of a measurable function $f : \Omega \to \mathbb{R}$ on some measure space $(\Omega, \mathcal{A}, \mu)$ is

$$\operatorname{ess\,inf} f := \inf\{t \in \mathbb{R} \mid \mu(f^{-1}((-\infty, t))) > 0\}.$$

The support function $h_K : \mathbb{R}^d \to \mathbb{R}$ of a convex body $K \subseteq \mathbb{R}^d$ is convex and thus according to Theorem 4 a.e. Alexandrov-twice-differentiable. For $u \in S^{d-1}$ the set of subgradients of h_K in tu, $t > 0$, does not depend on t. Hence u is eigenvector with eigenvalue 0 of the Hessian matrix of h_K in u, if it exists. So the restriction of this Hessian matrix to $u^\perp$ maps to $u^\perp$. We denote its smallest eigenvalue by $R_1(K, u)$. Now [25, Theorem 1] yields the following:

Theorem 8. *A convex body $K \subseteq \mathbb{R}^d$ has a ball as summand, iff* $\operatorname{ess\,inf} R_1(K, \cdot) > 0$.

The *Hausdorff-metric* on $\mathcal{C}$ is defined by

$$d^H(K, L) := \min\{t \geq 0 \mid K \subseteq L + tB^d \text{ and } L \subseteq K + tB^d\}.$$

For further details we refer to [18, section 1.8].
We will now provide two lemmata dealing with the approximation w.r.t. the Hausdorff metric. The first one is known. Due to lack of reference, we will give its proof.

Lemma 9. *Let $K \in \mathcal{C}$. Then there is a sequence $(K_i)_{i\in\mathbb{N}^+}$ of finite subsets of K, converging to K in the Hausdorff metric.*

Proof. Let $i \in \mathbb{N}^+$ be fixed. Then

$$\{B_{1/i}(x) \mid x \in K\}$$

is an open cover of K, where $B_r(x) := \{y \in \mathbb{R}^d \mid \|y - x\| < r\}$ for $x \in \mathbb{R}^d$ and $r > 0$. Since K is compact, it has a finite subcover

$$\{B_{1/i}(x) \mid x \in K_i\}.$$

From $K_i \subseteq K$ and $K \subseteq K_i + \frac{1}{i}B^d$ we get $d^H(K, K_i) \leq \frac{1}{i}$.
Obviously, the sequence $(K_i)_{i\in\mathbb{N}^+}$ consists of finite subsets of K and converges to K. □

Lemma 10. *For each $\epsilon > 0$ there is a continuous map $T : \mathcal{K} \to \mathcal{K}$ with the following properties:*

(i) If K has a summand RB^d, $R > 0$, then the same holds for $T(K)$.

(ii) We have $d^H(K, T(K)) < \epsilon S$, if $K \subseteq SB^d$, $S > 0$.

(iii) For all $K \in \mathcal{K}$ the support function of $T(K)$ is infinitly differentiable on $\mathbb{R}^d \setminus \{0\}$.

Proof. Such a map T is given in [18, Theorem 3.3.1] and it is proven there that it is continuous and fulfils the properties (ii) and (iii). There it is also proven, that $T(K+L) = T(K) + T(L)$ holds for arbitrary $K, L \in \mathcal{K}$ and that $T(K) = K$ holds for balls K of arbitrary radius. But this implies (i). □

We let $B \subseteq \mathbb{R}^d$ be a convex body with $0 \in \operatorname{int} B$, the so-called *gauge body*. For a closed set $A \subseteq \mathbb{R}^d$ and $x \in \mathbb{R}^d$ we define the B-distance from x to A to be

$$d_B(A, x) := \min\{t \geq 0 \mid x \in A + tB\}.$$

For $x, y \in \mathbb{R}^d$ we put

$$d_B(y, x) := d_B(\{y\}, x).$$

Then it is easy to see that

$$d_B(x, y) = 0 \iff x = y, \qquad x, y \in \mathbb{R}^d, \tag{1}$$

$$d_B(x + \lambda u, x + \lambda v) = \lambda d_B(u, v), \qquad x, u, v \in \mathbb{R}^d,\ \lambda \in \mathbb{R}_0^+, \tag{2}$$

$$d_B(x, y) + d_B(y, z) \geq d_B(x, z), \qquad x, y, z \in \mathbb{R}^d, \tag{3}$$

and that $d_B : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}_0^+$ is continuous.
For a closed set $A \subseteq \mathbb{R}^d$ and $x \in \mathbb{R}^d$ we put

$$\Pi_B(A, x) := \{y \in A \mid d_B(y, x) = d_B(A, x)\}.$$

The *B-exoskeleton* $\operatorname{exo}_B(A)$ of A is the set of all points $x \in \mathbb{R}^d$, for which $\Pi_B(A, x)$ consists of more than one point. For $x \in \mathbb{R}^d \setminus \operatorname{exo}_B(A)$ we define the *B-metric projection* $p_B(A, x)$ of x onto A to be the unique point in $\Pi_B(A, x)$, and if moreover $x \notin A$, we put $u_B(A, x) := (x - p_B(A, x))/d_B(A, x)$. It is easy to see that $u_B(A, x) \in \operatorname{bd} B$. If the (Euclidean) exterior unit normal vector of B in $u_B(A, x)$ is determined uniquely, we call it $n_B(A, x)$. The *B-normal bundle* of A is

$$\mathcal{N}_B(A) := \{(p_B(A, x), u_B(A, x)) \mid x \in \mathbb{R}^d \setminus A \setminus \operatorname{exo}_B(A)\}.$$

Lemma 11. *Let $K, B \subseteq \mathbb{R}^d$ be convex bodies with $0 \in \operatorname{int} B$. Let $z \in \mathbb{R}^d \setminus K$ and put $r := d_B(K, z)$. Then we have:*

(i) $z \in \operatorname{bd}(K + rB)$

(ii) Every exterior normal vector of $K + rB$ in z is exterior normal vector of K in every point of $\Pi_B(K, z)$ and of B in every point of $\frac{1}{r}(z - \Pi_B(K, z))$.

(iii) Let $p \in \operatorname{bd} K$, $u \in \operatorname{bd} B$ and $s > 0$. Put $z := p + su$. If K in p and B in u have a common exterior normal vector ν, then we have $s = d_B(K, z)$ and $p \in \Pi_B(K, z)$.

Proof. (i) Since B and K are closed, we have $z \in K + rB$. If $z \in \operatorname{int}(K + rB)$, then there would be $\epsilon > 0$ with $z + \epsilon B \subseteq K + rB$. By the cancelation law for Minkowski sums (see e.g. [18, p. 41]), we get $z \in K + (r - \epsilon)B$ and thus $d_B(K, z) < r$, contradicting the assumption.
(ii) Let ν be an exterior normal vector of $K + rB$ in z, $p \in \Pi_B(K, z)$ and $u := \frac{1}{r}(z - p)$. Then for all $k \in K$ we have $k + ru \in K + rB$ and hence $\langle k + ru, \nu\rangle \leq \langle p + ru, \nu\rangle$, which implies $\langle k, \nu\rangle \leq \langle p, \nu\rangle$. So ν is exterior normal vector of K in p. The same way one can show that ν is exterior normal vector of B in u.
(iii) Clearly $d_B(K, z) \leq s$. Assume $d_B(K, z) < s$. Then there is $s' < s$, $b \in B$ and $k \in K$ with $z = k + s'b$. Due to $0 \in \operatorname{int} B$ we have $\langle u, \nu\rangle > 0$ and thus we get

$$\langle z, \nu\rangle = \langle k, \nu\rangle + s'\langle b, \nu\rangle < \langle p, \nu\rangle + s\langle u, \nu\rangle = \langle z, \nu\rangle,$$

which is a contradiction. Hence $s = d_B(K, z)$, which yields immediately $p \in \Pi_B(K, z)$. □

2.3 Mixed volumes and local versions

In this section we will introduce the mixed volumes, which are the analogues to the intrinsic volumes in Minkowski spaces, and certain measures that provide information about "where" the mixed volumes are.
For convex bodies $K, B \subseteq \mathbb{R}^d$ there are numbers $V(K[j], B[d - j])$, $j = 0, \ldots, d$, called *mixed volumes*, such that

$$V_d(K + rB) = \sum_{j=0}^{d} r^{d-j} \binom{d}{j} V(K[j], B[d - j]), \qquad r \geq 0.$$

It is easy to see that the mixed volumes fulfil

$$V(K[j], K[d-j]) = V_d(K), \quad j = 0, \dots, d,\ K \in \mathcal{K} \tag{4}$$

and that they are homogenous in the sense that

$$V(\lambda K[j], \mu B[d-j]) = \lambda^j \mu^{d-j} V(K[j], B[d-j]), \quad \lambda, \mu \in \mathbb{R}_0^+,\ j = 0, \dots, d,\ B, K \in \mathcal{K}. \tag{5}$$

It is well-known that they are monotone, i.e.

$$V(K[j], B[d-j]) \leq V(K'[j], B'[d-j]), \quad j = 0, \dots, d, \tag{6}$$

for $K, K', B, B' \in \mathcal{K}$ with $K \subseteq K'$ and $B \subseteq B'$.

Now assume that the gauge body B is strictly convex and satisfies $0 \in \operatorname{int} B$. It is known that for convex bodies $K \subseteq \mathbb{R}^d$ we have $\operatorname{exo}_B(K) = \emptyset$. Hence we can consider

$$\mu_r^B(K, \eta) := V_d\big(\{x \in (K + rB) \setminus K \mid (p_B(K, x), u_B(K, x)) \in \eta\}\big)$$

for $r \in \mathbb{R}_0^+$ and Borel-sets $\eta \subseteq \mathbb{R}^d \times \mathbb{R}^d$. There are measures $C_0^B(K, \cdot), \dots, C_{d-1}^B(K, \cdot)$, called *relative support measures*, on $\mathbb{R}^d \times \mathbb{R}^d$ with

$$\mu_r^B(K, \eta) = \sum_{j=0}^{d-1} r^{d-j} \kappa_{d-j} C_j^B(K, \eta)$$

for $r \in \mathbb{R}_0^+$ and Borel-sets $\eta \subseteq \mathbb{R}^d \times \mathbb{R}^d$ (see e.g. [9]). Their projections on the first component,

$$\Phi_j^B(K, \beta) := C_j^B(K, \beta \times \mathbb{R}^d), \quad j = 0, \dots, d-1,\ \beta \in \mathcal{B}(\mathbb{R}^d),$$

are called *relative curvature measures*. Their projections on the second component,

$$\Psi_j^B(K, \omega) := C_j^B(K, \mathbb{R}^d \times \omega), \quad j = 0, \dots, d-1,\ \omega \in \mathcal{B}(\mathbb{R}^d),$$

are called *relative area measures*.

The total masses of the measure defined above are, up to normalization, the mixed volumes. More precisely, for a strictly convex body $B \subseteq \mathbb{R}^d$ with $0 \in \operatorname{int} B$, a convex body $K \subseteq \mathbb{R}^d$ and $j \in \{0, \dots, d-1\}$ we have

$$\Phi_j^B(K, \mathbb{R}^d) = \Psi_j^B(K, \mathbb{R}^d) = C_j^B(K, \mathbb{R}^d \times \mathbb{R}^d) = \frac{\binom{d}{j}}{\kappa_{d-j}} V(K[j], B[d-j]). \tag{7}$$

The second part of this subsection is devoted to the examination of the mixed area measures. But before we come to their definition, we introduce the Hausdorff measure and the surface area measure.

Def. 12. Let $j \in \mathbb{R}_0^+$ and $A \subseteq \mathbb{R}^d$ an arbitrary subset. For $\delta > 0$ we put

$$\mathcal{H}_\delta^j(A) := \inf\Big\{\sum_{S \in \mathcal{G}} (\operatorname{diam} S)^j \mid \mathcal{G} \subseteq \mathbb{R}^d,\ \mathcal{G} \text{ is countable},\ A \subseteq \bigcup_{S \in \mathcal{G}} S,\ \operatorname{diam} S < \delta \text{ for all } S \in \mathcal{G}\Big\}.$$

The number

$$\mathcal{H}^j(A) := \kappa_j 2^{-j} \lim_{\delta \to 0} \mathcal{H}_\delta^j(A) \in \mathbb{R}_0^+ \cup \{\infty\}$$

is called *j-dimensional Hausdorff measure* of A.

The intuitive meaning of the Hausdorff measure is that $\mathcal{H}^j(A)$ gives the length, area, volume etc. of A, provided that A is j-dimensional. For a more detailed introduction of the Hausdorff measure, in particular for proofs of the facts that the restriction of the Hausdorff measure to the Borel-σ-algebra is really a measure and that the d-dimensional Hausdorff measure in $\mathbb{R}^d$ coincides with the Lebesgue measure, see e.g. [3].
For a convex body $K \subseteq \mathbb{R}^d$ we call the measure $S_{d-1}(K, \omega) := 2\Psi_{d-1}^{B^d}(K, \omega)$, $\omega \in \mathcal{B}(\mathbb{R}^d)$, which is concentrated on the unit sphere S^{d-1}, *surface area measure.* Its name derives from the fact (see e.g. [18, (4.2.24)]) that

$$S_{d-1}(K, \omega) = \mathcal{H}^{d-1}(\tau_K(\omega)), \tag{8}$$

where $\tau_K(\omega)$ is the set of all boundary points of K having an exterior unit normal vector in ω, the so-called *reverse sperical image.*
For convex bodies $K, B \subseteq \mathbb{R}^d$ the *mixed area measures* are defined to be measures $S(K[j], B[d-j-1], \cdot)$, $j = 0, \dots, d-1$, on $\mathbb{R}^d$ with

$$S_{d-1}(sK + rB, \omega) = \sum_{j=0}^{d-1} \binom{d-1}{j} s^j r^{d-j-1} S(K[j], B[d-j-1], \omega), \quad r, s \geq 0, \tag{9}$$

for any Borel-set $\omega \subseteq \mathbb{R}^d$. For further information on mixed area measure we refer to [18, section 5.1]. Seting $r = 1$ and $s = 0$ in (9), we obtain

$$S_{d-1}(B, \omega) = S(K[0], B[d-1], \omega), \quad \omega \in \mathcal{B}(\mathbb{R}^d). \tag{10}$$

The mixed area measures are related to the relative area measures. In order to make this relationship precise, we define the *reverse sperical image map* of a strictly convex body $L \subseteq \mathbb{R}^d$ to be the map $\tau_L : S^{d-1} \to \operatorname{bd} L$ that asigns to a vector $u \in S^{d-1}$ the point $p \in \operatorname{bd} L$ with exterior unit normal vector u. Since for a strictly convex body $L \subseteq \mathbb{R}^d$ the image of a set ω under the reverse sperical image map is its reverse sperical image, the use of the same symbol is no problem.

Remark 13. From Theorem 5 we get that L is strictly convex, iff h_L is differentiable on $\mathbb{R}^d \setminus \{0\}$, and in this case we have $\nabla h_L(u) = \tau_L(\frac{u}{\|u\|})$ for all $u \in \mathbb{R}^d \setminus \{0\}$.

The relationship between relative area measures and mixed area measures is given by Theorem 2.14 in [8], which says:

Theorem 14. *Let $K, B \subseteq \mathbb{R}^d$ be two convex bodies such that $0 \in \operatorname{int} B$ and B is strictly convex. Then we have for $j \in \{0, \dots, d-1\}$ and Borel-sets $\gamma \subseteq \mathbb{R}^d$*

$$\Psi_j^B(K, \gamma) = \frac{\binom{d}{j}}{d\kappa_{d-j}} \int_{S^{d-1}} \mathbf{1}_\gamma(\nabla h_B(u)) h_B(u)\, S(K[j], B[d-j-1], du).$$

Now we will show that the 0-th relative curvature measure is concentrated on the extreme points. In the Euclidean special case this is well-known (see e.g. [18, (4.6.1)]). We remark that the set $\operatorname{ext} L$ of extreme points of a convex body $L \subseteq \mathbb{R}^d$ is a G_δ-set by [18, p. 66] and hence measurable.

Theorem 15. *Let $B \subseteq \mathbb{R}^d$ be a convex body with $0 \in \operatorname{int} B$, whose support function h_B is twice continuously differentiable on $\mathbb{R}^d \setminus \{0\}$. Let $L \subseteq \mathbb{R}^d$ be a convex body. Then the relative curvature measure $\Phi_0^B(L, \cdot)$ is concentrated on* $\operatorname{ext} L$.

Lemma 16. *Let $B, K \subseteq \mathbb{R}^d$ be two convex bodies such that $0 \in \operatorname{int} B$ and B is strictly convex. Then with $M := \max\{h_B(u) \mid u \in S^{d-1}\}$ we have for any Borel-set $\gamma \subseteq \operatorname{bd} B$*

$$\Psi_0^B(K, \gamma) \leq \frac{M}{d\kappa_d} \cdot \mathcal{H}^{d-1}(\gamma).$$

Proof. From Theorem 14, (10) and Remark 13 we get

$$\begin{aligned}
\Psi_0^B(K, \gamma) &= \frac{1}{d\kappa_d} \int_{S^{d-1}} \mathbf{1}_\gamma(\nabla h_B(u)) h_B(u) S_{d-1}(B, du) \\
&\leq \frac{1}{d\kappa_d} \int_{S^{d-1}} \mathbf{1}_\gamma(\tau_B(u)) M S_{d-1}(B, du) \\
&= \frac{M}{d\kappa_d} S_{d-1}(B, \{u \in S^{d-1} \mid \tau_B(u) \in \gamma\}).
\end{aligned}$$

Since $\tau_B : S^{d-1} \to \operatorname{bd} B$ is surjective, we derive from (8) that

$$S_{d-1}(B, \{u \in S^{d-1} \mid \tau_B(u) \in \gamma\}) = \mathcal{H}^{d-1}(\gamma),$$

which completes the proof of the lemma. □

Proof of Theorem 15. Since h_B is differentiable, B must be strictly convex by Remark 13. Hence $\Phi_0^B(L, \cdot)$ is defined. Let $\omega \subseteq S^{d-1}$ be the set of all (Euclidean) exterior unit normal vectors of L in points of $\operatorname{bd} L \setminus \operatorname{ext} L$. Since for every vector $u \in \omega$ there is more than one point in $\operatorname{bd} L$ having exterior normal vector u, [18, Theorem 2.2.9] implies $\mathcal{H}^{d-1}(\omega) = 0$. Let γ denote the set of all relative exterior normal vectors of L in points of $\operatorname{bd} L \setminus \operatorname{ext} L$. By [9, Lemma 2.1] we have $\gamma = \{\nabla h_B(u) \mid u \in \omega\}$. Since h_B is assumed to be twice continuously differentiable, ∇h_B is Lipschitz-continuous with Lipschitz-constant L, say. By Theorem 1 from [3, section 2.4] this implies

$$\mathcal{H}^{d-1}(\gamma) \leq L^{d-1} \cdot \mathcal{H}^{d-1}(\omega) = 0.$$

Because the relative support measure $C_0^B(L, \cdot)$ is concentrated on the relative normal bundel $\mathcal{N}_B(L)$, we get from Lemma 16 that

$$\begin{aligned}
\Phi_0^B(L, \operatorname{bd} L \setminus \operatorname{ext} L) &= C_0^B(L, (\operatorname{bd} L \setminus \operatorname{ext} L) \times \mathbb{R}^d) \\
&= C_0^B(L, (\operatorname{bd} L \setminus \operatorname{ext} L) \times \gamma) \\
&\leq \Psi_0^B(L, \gamma) \\
&\leq \frac{M}{d\kappa_d} \cdot \mathcal{H}^{d-1}(\gamma) \\
&= 0. \qquad \square
\end{aligned}$$

2.4 The parallel volume of arbitrary bodies

In the last subsection of this section we will collect results about parallel bodies and the parallel volume of arbitrary bodies.

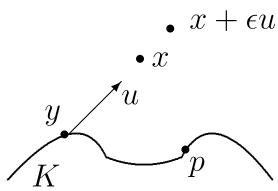
Lemma 17. *Let $B \subseteq \mathbb{R}^d$ be a convex body satisfying $RB^d \subseteq B$ for some $R > 0$. Let $K \subseteq \mathbb{R}^d$ be a body, $x \in \mathbb{R}^d$ and $r \in \mathbb{R}^+$.*

(i) If $x \in \operatorname{bd}(K + rB)$, then $d_B(K, x) = r$.

(ii) If $d_B(K, x) = r$ and $rR > \operatorname{diam} K$, then $x \in \operatorname{bd}(K + rB)$.

For an example, showing that the assumption $rR > \operatorname{diam} K$ cannot be dropped, see [10, p. 181].

Proof of Lemma 17. (i) From $x \in K + rB$ we get $d_B(K, x) \leq r$. If $d_B(K, x) < r$, then there would be $\rho < r$ such that $x \in K + \rho B$. Because of $0 \in \operatorname{int} B$ we would get $x \in \operatorname{int}(x + (r - \rho)B) \subseteq \operatorname{int}(K + rB)$.
(ii) From $d_B(K, x) = r$ we get $x \in K + rB$.



There are $y \in K$, $u \in B$ and $s \geq 0$ with $x = y + su$. Now $x + \epsilon u \notin K + rB$ for $\epsilon > 0$: Assume, there was $\epsilon > 0$ with $x + \epsilon u \in K + rB$. Then there was $p \in K$ with $x + \epsilon u - p \in rB$. From $\|y - p\| \leq \operatorname{diam} K < rR$ we get $y - p \in \operatorname{int} rRB^d \subseteq \operatorname{int} rB$. Due to the convexity of B and $y + (\epsilon + s)u - p \in rB$ we get

$$x - p = y + su - p = \frac{s}{\epsilon + s}(y + (\epsilon + s)u - p) + \frac{\epsilon}{\epsilon + s}(y - p) \in \operatorname{int} rB.$$

Hence there is $r_1 < r$ with $x - p \in r_1 B$ and thus $x \in K + r_1 B$. So $d_B(K, x) < r$, contradicting the assumption.
From $x \in K + rB$ and $x + \epsilon u \notin K + rB$ for $\epsilon > 0$ we get $x \in \operatorname{bd}(K + rB)$. $\square$

Theorem 18. *Let $K \subseteq \mathbb{R}^d$ be a body and $B \subseteq \mathbb{R}^d$ a convex body with $0 \in \operatorname{int} B$. Then for arbitrary numbers $r_2 > r_1 > 0$ we have*

$$V_d(K + r_2 B) - V_d(K + r_1 B) \leq \int_{r_1}^{r_2} \tfrac{d}{s}(V_d(K + sB) - V_d(K))\, ds.$$

Proof. Let $\epsilon > 0$. Because of $\frac{d}{dt}(t^d - 1)_{|t=1} = d$ there is $\delta > 0$ with $t^d - 1 \leq (d + \epsilon)(t - 1)$ for $t \in (1, 1 + \delta)$. Let $n \in \mathbb{N}^+$. Choose numbers $s^{(0)}, \ldots, s^{(m)}$, $m \in \mathbb{N}$, with $r_1 = s^{(0)} < \cdots < s^{(m)} = r_2$, satisfying $s^{(i)}/s^{(i-1)} < 1 + \delta$ and $s^{(i)} - s^{(i-1)} < \frac{1}{n}$ for all $i = 1, \ldots, m$. For fixed $i \in \{1, \ldots, m\}$ we conclude from [20, Theorem 4]

$$V_d(K + s^{(i)} B) - V_d(K) \leq \big(\frac{s^{(i)}}{s^{(i-1)}}\big)^d (V_d(K + s^{(i-1)} B) - V_d(K))$$

(the assumption that B is symmetric made in [20, Theorem 4] is not used its proof).

Hence

$$\begin{aligned}
V_d(K+s^{(i)}B)-&V_d(K+s^{(i-1)}B)\\
&= (V_d(K+s^{(i)}B)-V_d(K)) - (V_d(K+s^{(i-1)}B)-V_d(K))\\
&\le \big(\frac{s^{(i)}}{s^{(i-1)}}\big)^d (V_d(K+s^{(i-1)}B)-V_d(K)) - (V_d(K+s^{(i-1)}B)-V_d(K))\\
&= \left(\big(\frac{s^{(i)}}{s^{(i-1)}}\big)^d - 1\right)(V_d(K+s^{(i-1)}B)-V_d(K))\\
&\le (d+\epsilon)\frac{s^{(i)}-s^{(i-1)}}{s^{(i-1)}}(V_d(K+s^{(i-1)}B)-V_d(K))\\
&= (s^{(i)}-s^{(i-1)})\frac{d+\epsilon}{s^{(i-1)}}(V_d(K+s^{(i-1)}B)-V_d(K)).
\end{aligned}$$

Thus

$$\begin{aligned}
V_d(K+r_2B)-V_d(K+r_1B) &= \sum_{i=1}^m V_d(K+s^{(i)}B)-V_d(K+s^{(i-1)}B)\\
&\le \sum_{i=1}^m (s^{(i)}-s^{(i-1)})\frac{d+\epsilon}{s^{(i-1)}}(V_d(K+s^{(i-1)}B)-V_d(K))
\end{aligned}$$

Leting $n\to\infty$ we get from the Riemannian definition of the integral

$$V_d(K+r_2B)-V_d(K+r_1B) \le \int_{r_1}^{r_2} \frac{d+\epsilon}{s}(V_d(K+sB)-V_d(K))\,ds.$$

The integral exists, since $s\mapsto V_d(K+sB)$ is monotonically increasing and hence is continuous on $[r_1,r_2]$ except for at most countably many points, which implies that the integrand is continuous except for countably many points. Since $\epsilon>0$ was arbitrary, this shows the assertion. $\square$

We let $\mathcal{K}_0$ denote the set of all convex bodies with interior points.

Lemma 19. *For a fixed body $K\in\mathcal{C}$ the map*

$$\mathcal{K}_0\to\mathbb{R}_0^+,\ B\mapsto V_d(K+B)$$

is continuous.

Proof. Let $B\in\mathcal{K}_0$. Then B contains a ball of radius $R>0$, say, which has w.l.o.g. its center at the origin. Let $\epsilon>0$. Since

$$\mathbb{R}_0^+\to\mathbb{R}_0^+,\ r\mapsto V_d(K+rB)$$

is continuous according to Theorem 18, there is $\delta\in(0,1)$ with

$$V_d(K+B)-\epsilon < V_d(K+(1-\delta)B) < V_d(K+(1+\delta)B) < V_d(K+B)+\epsilon.$$

Let $\tilde{B}\in\mathcal{K}_0$ be a body whose Hausdorff distance from B is less than $R\delta$. Then

$$B\subseteq \tilde{B}+R\delta B^d\subseteq \tilde{B}+\delta B,$$

which implies $(1-\delta)B \subseteq \tilde{B}$ by the cancelation law for Minkowski sums, and

$$\tilde{B} \subseteq B + R\delta B^d \subseteq B + \delta B.$$

This however implies

$$V_d(K+B) - \epsilon < V_d(K+\tilde{B}) < V_d(K+B) + \epsilon. \quad \square$$

For a body $K \subseteq \mathbb{R}^d$ and a strictly convex body $B \subseteq \mathbb{R}^d$ with $0 \in \operatorname{int} B$ the function $d_B(K,\cdot)$ is Lipschitz continuous and hence differentiable a.e. by Theorem 2. It is easy to see, that the gradient $\nabla d_B(K,z)$ is never 0. Hence we can put

$$\nu_B(K,z) := \frac{\nabla d_B(K,z)}{\|\nabla d_B(K,z)\|}$$

for every point $z \in \mathbb{R}^d$, in which $d_B(K,\cdot)$ is differentiable.
Now [10, Proposition 2.8] is the following:

Theorem 20. *Let $K \subseteq \mathbb{R}^d$ be a body and $B \subseteq \mathbb{R}^d$ a strictly convex body with $0 \in \operatorname{int} B$. Then for any V_d-measurable function $f : \mathbb{R}^d \to \mathbb{R}_0^+$ we have*

$$\int_{\mathbb{R}^d \setminus K} f(x)\,dx = \int_0^r \int_{\operatorname{bd}(K+sB)} f(z) h_B(\nu_B(K,z))\, d\mathcal{H}^{d-1}(z)\, ds.$$

Now we want to show that $\nu_B(K,z)$ coincides with the vector $n_B(K,z)$ defined on page 6, provided $d_B(K,\cdot)$ is differentiable in z.

Lemma 21. *Let $K \subseteq \mathbb{R}^2$ be a body and $B \subseteq \mathbb{R}^2$ a smooth convex body with $0 \in \operatorname{int} B$. Then $\nabla d_B(K,z)$ is a positive multiple of $n_B(K,z)$ in all points $z \in \mathbb{R}^d \setminus K$, in which $d_B(K,\cdot)$ is differentiable.*

Proof. Let $z \in \mathbb{R}^2$ be a point, in which $d_B(K,\cdot)$ is differentiable. As shown in the proof of [10, Lemma 2.1], this implies $z \notin \operatorname{exo}_B(K)$. Because B is smooth, $n_B(K,z)$ is defined. Since $0 \in \operatorname{int} B$, there is $R > 0$ with $RB^2 \subseteq B$. Let $v \in S^1$ be a unit vector orthogonal to $n_B(K,z)$ and $\epsilon > 0$. Now consider the function

$$f : [-R,R] \to \mathbb{R},\, t \to -\max\{\langle x-z,\nu\rangle \mid x \in K+rB, \langle x-z,v\rangle = t\},$$

where $\nu := n_B(K,z)$. As B is smooth, we have $\lim_{t\to 0} f(t)/t = 0$. Put $\rho := d_B(K,z)$. From

$$z + \epsilon v \in K + \rho B + (f(\epsilon))B^d \subseteq K + (\rho + f(\epsilon)/R)B$$

we get $d_B(K, z+\epsilon v) \leq \rho + f(\epsilon)/R$. Hence the one-sided directional derivative of $d_B(K,\cdot)$ in z in direction v is not positive. Since, however, the same holds with v replaced by $-v$ and we have assumed that $d_B(K,\cdot)$ is differentiable in z, we get

$$\langle \nabla d_B(K,z), v\rangle = 0.$$

But this shows the assertion. $\square$

3 The main results

In this section we examine the asymptotic order of the parallel volume difference. First we show in Theorem 22 that this difference can at most have order r^{d-2} in arbitrary Minkowski spaces. In Theorem 26 we will show that in many Minkowski spaces, and in particular in the Euclidean space, it can at most have order r^{d-3}. The examples 24 and 29 as well as Corollary 30 show that our theorems are in a certain sense optimal.

We let ρ_B denote the radius of the largest ball contained in a convex body $B \subseteq \mathbb{R}^n$ and $\operatorname{diam} A$ the diameter of a subset $A \subseteq \mathbb{R}^n$. Moreover, we let $\hat{B}$ denote the affine hull of a body $B \subseteq \mathbb{R}^n$ and $B^\perp$ its affine-orthogonal compliment. For two bodies $K, B \subseteq \mathbb{R}^n$ we call

$$K_B := \bigcup_{x \in B^\perp} \operatorname{conv}(K \cap (\hat{B} + x))$$

the *B-convexification* of K.

For the notion of a random closed set, a random body, etc. we refer to [19].

We observe that the set $\mathcal{C}$ of non-empty compact sets and the set $\mathcal{K}$ of non-empty, convex compact sets are measurable by [19, Lemma 2.1.2 and Theorem 2.4.2]. As an easy consequence of [19, Theorem 12.3.2] maps that are continuous w.r.t. the Hausdorff-metric are measurable w.r.t. the Fell-Matheron-σ-algebra. The functions $\mathcal{K} \to \mathbb{R}$, $B \mapsto \rho_B$ and $\operatorname{diam} : \mathcal{C} \to \mathbb{R}$ are obviously continuous w.r.t. the Hausdorff metric and hence measurable. By [19, Theorem 12.3.5] the same holds for $\mathcal{C} \times \mathcal{C} \to \mathcal{C}$, $(K, L) \mapsto K + L$ and $\mathcal{C} \times \mathbb{R}_0^+ \to \mathcal{C}$, $(K, r) \mapsto rK$. The map $\mathcal{C} \times \mathcal{C} \to \mathcal{C}$, $(K, B) \mapsto K_B$ is shown to be measurable in [13, Lemma A.7]. The map $V_n : \mathcal{C} \to \mathbb{R}$ is upper semicontinuous by [19, Theorem 12.3.6] and hence measurable (see [19, p. 19]).

Theorem 22. *Let $1 < d \leq n$. Let $X \subseteq \mathbb{R}^n$ be a random body and $Y \subseteq \mathbb{R}^n$ a d-dimensional random convex body. Put $G := \max\{\operatorname{diam} X, 1\}$, $S := \max\{\operatorname{diam} Y, 1\}$ and $R := \rho_Y$. If*

$$c := d2^d \kappa_d \kappa_{n-d} \mathbb{E}\Big[\frac{S^{d-1} \cdot G^n}{R}\Big] < \infty,$$

then we have

$$\mathbb{E}[V_n(X_Y + rY) - V_n(X + rY)] < c \cdot r^{d-2}, \quad r \geq 1.$$

In the proof of this theorem, we need the function

$$\begin{aligned} w_B : \mathbb{R}_0^+ &\to \mathbb{R}_0^+, \\ r &\mapsto \min\{d_B(y, z) \mid y \in B^d,\, z \in \mathbb{R}^d,\, y \in \Pi_B(0y, z),\, d_B(0, z) = r\}, \end{aligned} \tag{11}$$

which is defined for convex bodies $B \subseteq \mathbb{R}^d$ with $0 \in \operatorname{int} B$. For a more detailed introduction see [13].

We first prove a lemma making the same statement as the theorem under additional assumptions and then argue that we can assume that these additional assumptions are fulfilled.

Lemma 23. *Let $K \subseteq \mathbb{R}^d, d > 1$, be a body and $B \subseteq \mathbb{R}^d$ be a convex body such that h_B is twice differentiable on $\mathbb{R}^d \setminus \{0\}$. Put $G := \max\{\operatorname{diam} K, 1\}$, $S := \max\{\operatorname{diam} B, 1\}$ and*

$$C' := d2^d \kappa_d \frac{S^{d-1} \cdot G^d}{\rho_B}.$$

Then we have for all $r \geq 1$

$$V_d(\operatorname{conv} K + rB) - V_d(K + rB) < C' \cdot r^{d-2}.$$

Proof. By the translation-invariance of the Lebesgue measure we may assume $\rho_B B^d \subseteq B$. Moreover, B is strictly convex, since we assumed h_B to be differentiable on $\mathbb{R}^d \setminus \{0\}$. We put $L := \operatorname{conv} K$. Then we get by [13, Lemma 3.4]

$$\begin{aligned}
V_d(L&+rB) - V_d(K + rB) \\
&= V_d((L + rB) \setminus (K + rB)) \\
&\leq V_d\big(\{x \in \mathbb{R}^d \mid p_B(L, x) \in L \setminus \operatorname{ext} L,\ d_B(L, x) \in (Gw_B(\tfrac{r}{G}), r]\} \cup (L \setminus (K + rB))\big) \\
&= \sum_{i=1}^{d} \kappa_i \Phi^B_{d-i}(L, L \setminus \operatorname{ext} L)\big(r^i - (Gw_B(\tfrac{r}{G}))^i\big) + V_d(L \setminus (K + rB)). \qquad (12)
\end{aligned}$$

Now we will derive an upper bound for the right-hand side in (12). The equation

$$r - w_B(r) \leq \frac{1}{\rho_B}, \quad r \in \mathbb{R}^+_0. \qquad (13)$$

is shown in [13, Remark 3.9] to be an easy corollary of the triangular inequality for d_B. Since obviously $w_B(s) \leq s$ for all $s \in \mathbb{R}^+$, we conclude

$$\begin{aligned}
r^i - (Gw_B(\tfrac{r}{G}))^i &= (r - Gw_B(\tfrac{r}{G})) \sum_{j=0}^{i-1} r^j (Gw_B(\tfrac{r}{G}))^{i-1-j} \\
&= G \cdot (\tfrac{r}{G} - w_B(\tfrac{r}{G})) \sum_{j=0}^{i-1} r^j (Gw_B(\tfrac{r}{G}))^{i-1-j} \\
&\leq \tfrac{G}{\rho_B} \sum_{j=0}^{i-1} r^j (G\tfrac{r}{G})^{i-1-j} \\
&= \tfrac{G}{\rho_B} \sum_{j=0}^{i-1} r^{i-1} \\
&= \tfrac{G}{\rho_B} i r^{i-1}.
\end{aligned}$$

By (7) we have

$$\kappa_i \Phi^B_{d-i}(L, L \setminus \operatorname{ext} L) \leq \binom{d}{i} V(L[d-i], B[i]).$$

Since L is contained in the circumsphere of K, whose radius is at most G, and B is contained in a ball of radius S, we get by (4) - (6)

$$V(L[d-i], B[i]) \leq V(GB^d[d-i], SB^d[i]) = G^{d-i} S^i \kappa_d.$$

Thus

$$\kappa_i \Phi^B_{d-i}(L, L \setminus \operatorname{ext} L) \leq \binom{d}{i} G^{d-i} S^i \kappa_d. \qquad (14)$$

Using (12), (14) and Theorem 15, which says $\Phi_0^B(L, L\setminus \operatorname{ext} L) = 0$, we obtain

$$\begin{aligned}
&V_d(L+rB^d) - V_d(K+rB^d)\\
&\qquad\qquad \le \sum_{i=1}^{d} \kappa_i \Phi_{d-i}^B(L, L\setminus \operatorname{ext} L)\big(r^i - (Gw(\tfrac{r}{G}))^i\big) + V_d(L\setminus (K+rB))\\
&\qquad\qquad \le \sum_{i=1}^{d-1} \binom{d}{i} G^{d-i} S^i \kappa_d \cdot \tfrac{G}{\rho_B} i r^{i-1} + \kappa_d G^d\\
&\qquad\qquad \le \sum_{i=1}^{d-1} (d-1)\binom{d}{i} \kappa_d \frac{S^{d-1}G^d}{\rho_B} r^{d-2} + \kappa_d G^d r^{d-2}\\
&\qquad\qquad < \Big((d-1)2^d \kappa_d \frac{S^{d-1}G^d}{\rho_B} + \kappa_d G^d\Big) r^{d-2}\\
&\qquad\qquad < \Big(d2^d \kappa_d \frac{S^{d-1}\cdot G^d}{\rho_B}\Big) r^{d-2}. \qquad \square
\end{aligned}$$

Proof of Theorem 22. Let $r \ge 1$. We want to prove

$$\mathbb{E}[V_n(X_Y + rY) - V_n(X+rY)] < c \cdot r^{d-2}.$$

Put $X^x := X \cap (\hat{Y} + x)$ and $Z^x := \operatorname{conv} X^x$ for $x \in Y^\perp$.
By Lemma 10 there is a sequence $(Y_k)_{k\in\mathbb{N}}$ of random convex bodies lying a.s. in $\hat{Y}$ such that for all $k \in \mathbb{N}$ the support function of Y_k is twice continuously differentiable on $\hat{Y}\setminus\{0\}$, and that $\lim_{k\to\infty} Y_k = Y$ a.s. Due to Lemma 19 we have a.s. for all $x \in Y^\perp$, that

$$V_d(Z^x + rY) - V_d(X^x + rY) = \lim_{k\to\infty} V_d(Z^x + rY_k) - V_d(X^x + rY_k).$$

Putting

$$C' := d2^d \kappa_d \frac{S^{d-1}\cdot G^d}{R},$$

it follows from Lemma 23 that a.s. for all $x \in Y^\perp$ we have

$$V_d(Z^x + rY) - V_d(X^x + rY) = \lim_{k\to\infty} V_d(Z^x + rY_k) - V_d(X^x + rY_k) \le C' \cdot r^{d-2},$$

since diameter and ρ-number are continuous when considered as functionals $\mathcal{K}_0 \to \mathbb{R}_0^+$. Thus (we will comment on the measurability below)

$$\begin{aligned}
\mathbb{E}[V_n(X_Y + rY) - V_n(X+rY)] &= \mathbb{E}\left[\int_{X|Y^\perp} V_d(Z^x + rY) - V_d(X^x + rY)\, dx\right]\\
&\le \mathbb{E}\left[\int_{X|Y^\perp} C' \cdot r^{d-2}\, dx\right]\\
&= \mathbb{E}\left[V_{n-d}(X|Y^\perp) C'\right] \cdot r^{d-2}\\
&\le \mathbb{E}\left[\kappa_{n-d} G^{n-d} C'\right] \cdot r^{d-2}\\
&= c \cdot r^{d-2}.
\end{aligned}$$

It remains to show those expressions of the previous equation, whose measurability was not proven before the statement of Theorem 22, are measurable, too. The map $\mathcal{C}\times\mathcal{K}\times\mathbb{R}^n \to \mathcal{C}$, $(X,Y,x)\mapsto X^x$ is measurable by [13, Lemma A.1] and [19, Theorem 12.2.6]. The map $\operatorname{conv}:\mathcal{C}\to\mathcal{K}$ is measurable by [19, Theorem 12.3.5]. As an easy consequence of [19, Theorem 12.3.6] the Lebesgue measure V_d on the set of all bodies of $\mathbb{R}^n$, whose affine hull has at most dimension d, is upper semicontinuous and hence measurable. In the final version of [13] it will be shown that for measurable maps $f:\Omega\times\mathbb{R}^n\to\mathbb{R}$, where Ω denotes the probability space on which Y is defined, $\int_{Y^\perp} f(\omega,x)\,dx$ is a random variable. $\square$

Now we will show that the order r^{d-2} in Theorem 22 is optimal.

Example 24. Put $K:=\{-e_1,e_1\}$ and let $B:=\operatorname{conv}\{-e_1,e_1,\dots,-e_d,e_d\}$ be the unit ball of the L_1-norm. Because $V_{d-1}(\operatorname{conv}\{-e_2,e_2,\dots,-e_d,e_d\})=2^{d-1}/(d-1)!$, we have

$$\begin{aligned}
&V_d(\operatorname{conv}K+rB)-V_d(K+rB)\\
&\qquad=\int_{-1}^{1}V_{d-1}(\{(x_1,\dots,x_d)\in(\operatorname{conv}K+rB)\setminus(K+rB)\mid x_1=t\})\,dt\\
&\qquad=2\int_0^1 r^{d-1}2^{d-1}/(d-1)!-(r-t)^{d-1}2^{d-1}/(d-1)!\,dt\\
&\qquad=2^d\sum_{j=0}^{d-2}(-1)^{d-j}\frac{1}{j!(d-j)!}r^j.
\end{aligned}$$

While the order r^{d-2} is optimal under the assumptions of Theorem 26, we can show that under the additional assumption that the gauge body contains a ball as summand we have in fact order r^{d-3}. In particular, this is true in the Euclidean case.

For a convex body $B\subseteq\mathbb{R}^d$ we put

$$R(B):=\max\{\rho\in\mathbb{R}_0^+\mid \rho B^d \text{ is summand of } B\}.$$

The Blaschke selection theorem (see [18, Theorem 1.8.4]) and the continuity of Minkowski sums (see [19, Theorem 12.3.5]) imply that the maximum is attained. The following lemma tells use that the map $R:\mathcal{C}\to\mathbb{R}_0^+$ is measurable.

Lemma 25. *The map $R:\mathcal{C}\to\mathbb{R}_0^+$ is upper semicontinuous .*

Proof. Let $(K_i)_{i\in\mathbb{N}}$ be a sequence in $\mathcal{K}$ converging to $K\in\mathcal{K}$. Since $(R(K_i))_{i\in\mathbb{N}}$ is bounded, this sequence has a convergent subsequence, w.l.o.g. the sequence itself. Now there is a sequence $(\rho_i)_{i\in\mathbb{N}}$ converging to $\lim_{i\to\infty}R(K_i)$ and a sequence $(M_i)_{i\in\mathbb{N}}$ of convex bodies with $K_i=M_i+\rho_iB^d$ for all $i\in\mathbb{N}$. By the Blaschke selection theorem we can assume that $(M_i)_{i\in\mathbb{N}}$ converges to a convex body M. By the continuity of Minkowski sums we have $K=M+\lim_{i\to\infty}R(K_i)B^d$. Hence $R(K)\geq\lim_{i\to\infty}R(K_i)$. $\square$

Theorem 26. *Let $1<d\leq n$. Let $X\subseteq\mathbb{R}^n$ be a random body and $Y\subseteq\mathbb{R}^n$ a d-dimensional random convex body. Put $G:=\max\{\operatorname{diam}X,1\}$ and $S:=\max\{\operatorname{diam}Y,1\}$. If*

$$c:=d2^{d+2}\kappa_d\kappa_{n-d}\mathbb{E}\Big[\frac{S^d\cdot G^{n+1}}{R(Y)^3}\Big]<\infty,$$

then we have

$$\mathbb{E}[V_n(X_Y+rY)-V_n(X+rY)]<c\cdot r^{d-3},\quad r\geq1.$$

The main reason, why we obtain this sharper result now, is that we have instead of (13) the following lemma.

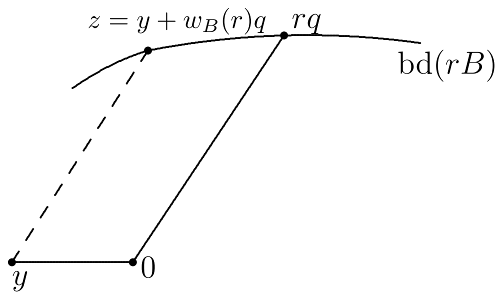
Lemma 27. *Let $B \subseteq \mathbb{R}^d$ be a convex body with interior points, which has a summand RB^d for some $R > 0$ and satisfies $B \subseteq SB^d$ for some $S > 0$. Assume that one largest ball contained in B has its center at the origin. Then with $C := \frac{4S}{R{\rho_B}^2}$ we have*

$$r - w_B(r) < \frac{C}{r}, \quad r \in \mathbb{R}_0^+.$$

Proof. If $r \leq \frac{4S}{R\rho_B}$, we conclude from (13)

$$r - w_B(r) \leq \frac{1}{\rho_B} \leq \frac{4S}{rR{\rho_B}^2}.$$

So let $r > \frac{4S}{R\rho_B}$ from now on. There are points $z \in \mathbb{R}^d$ and $y \in B^d$ with $y \in \Pi_B(0y, z)$, $d_B(0, z) = r$ and $d_B(y, z) = d_B(0y, z) = w_B(r)$. Put $q := (z - y)/d_B(y, z)$. Then $q \in \operatorname{bd} B$ by Lemma 11(ii). Moreover we let u denote the (Euclidean) exterior normal vector of B in q, which is determined uniquely, since RB^d is a summand of B, and let v denote an arbitrary unit vector perpendicular to u.



We will now proceed as follows: We will first show

$$r - w_B(r) = \frac{\langle rq - z, u\rangle}{\langle q, u\rangle}.$$

Then we will give an upper bound for $\langle rq - z, v\rangle$ in terms of $\langle rq - z, u\rangle$ and derive from the fact that B has a ball as summand an upper bound for $\langle rq - z, u\rangle$ in terms of $\langle rq - z, v\rangle$. Combining these two bounds will give the desired upper bound for $\langle rq - z, u\rangle$. So let $t \in [r, w_B(r)]$. In dimension $d \geq 3$ we cannot say anything about the angle between $0y$ and v, but by Lemma 11 there is a common exterior unit normal vector of B in q and of $0y$ in y and hence $\langle y, u\rangle = 0$. From $z = y + w_B(r)q$ we get on the one hand

$$\langle tq - z, u\rangle = \langle tq - w_B(r)q - y, u\rangle = (t - w_B(r))\langle q, u\rangle$$

and thus

$$t - w_B(r) = \frac{\langle tq - z, u\rangle}{\langle q, u\rangle} \tag{15}$$

and on the other hand

$$|\langle tq - z, v\rangle| = |(t - w_B(r))\langle q, v\rangle - \langle y, v\rangle| \leq (t - w_B(r))|\langle q, v\rangle| + 1.$$

From the last two equations we get

$$|\langle tq - z, v\rangle| \leq \frac{\langle tq - z, u\rangle}{\langle q, u\rangle}|\langle q, v\rangle| + 1.$$

Since $q \in B \subseteq SB^d$ we get $|\langle q, v\rangle| \leq S$ and since $\rho_B u \in \rho_B B^d \subseteq B$ and u is exterior normal vector of B in q we get $\langle q, u\rangle \geq \rho_B$. Thus

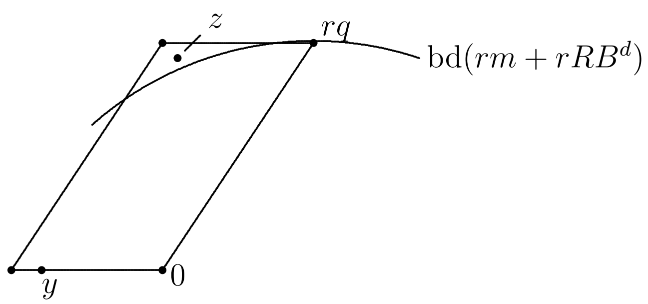
$$|\langle tq - z, v\rangle| \leq \frac{S}{\rho_B}\langle tq - z, u\rangle + 1. \tag{16}$$

We want to show

$$\langle tq - z, u\rangle < \frac{4}{Rr} \tag{17}$$

for all $t \in [w_B(r), r]$. Since this is true for $t = w_B(r)$ and the left-hand side of (17) is obviously continuous in t, it suffices to show that there is no $t \in [w_B(r), r]$ for which equality holds in (17). So assume, there is $t \in [w_B(r), r]$ for which equality holds in (17). By (16) we get

$$|\langle tq - z, v\rangle| \leq \frac{4S}{\rho_B Rr} + 1.$$



Now put $m := q - Ru$ and denote the unit vector in the direction $z - tm - \langle z - tm, u\rangle u$ by v_0. Since we have assumed that equality holds in (17), we get

$$\begin{aligned}
\|z - tm\|^2 &= \langle z - tm, u\rangle^2 + \langle z - tm, v_0\rangle^2 \\
&= \langle z - tq + tRu, u\rangle^2 + \langle z - tm, v_0\rangle^2 \\
&= (tR - \langle tq - z, u\rangle)^2 + \langle tq - z, v_0\rangle^2 \\
&\leq \left(tR - \frac{4}{Rr}\right)^2 + \left(\frac{4S}{\rho_B Rr} + 1\right)^2.
\end{aligned}$$

Due to

$$r > \frac{4S}{\rho_B R} \geq \max\{\tfrac{4}{\rho_B}, \tfrac{4}{R}\}$$

and (13) we get

$$\frac{t}{r} \geq \frac{r - \frac{1}{\rho_B}}{r} = 1 - \frac{1}{r\rho_B} \geq \frac{3}{4}$$

and thus

$$\Big(tR\Big)^2 - \Big(tR - \frac{4}{Rr}\Big)^2 = 8\frac{t}{r} - \Big(\frac{4}{Rr}\Big)^2 \geq 6 - 1 > 2^2 > \Big(\frac{4S}{\rho_B Rr} + 1\Big)^2.$$

Hence

$$\|z - tm\|^2 \leq \Big(tR - \frac{4}{Rr}\Big)^2 + \Big(\frac{4S}{\rho_B Rr} + 1\Big)^2 < \Big(tR\Big)^2.$$

So we have $z \in \operatorname{int}(tm + tRB^d)$. Since $m + RB^d \subseteq B$ by Lemma 7, we get $z \in \operatorname{int} tB$. Hence there is $t' < t$ with $z \in t'B$ and so $d_B(0, z) < t$, which contradicts $t \leq r = d_B(0, z)$. So we have proven inequality (17).

Now we use the equations (15) and (17) in the special case $t = r$ and we use again the inequality $\langle q, u\rangle \geq \rho_B$ we derived before (16) and get

$$r - w_B(r) = \frac{\langle rq - z, u\rangle}{\langle q, u\rangle} \leq \frac{\frac{4}{rR}}{\rho_B} = \frac{4}{rR\rho_B} < \frac{4S}{rR{\rho_B}^2}. \qquad \square$$

The following lemma is the counterpart to Lemma 23.

Lemma 28. *Let $K \subseteq \mathbb{R}^d, d > 1$, be a body and $B \subseteq \mathbb{R}^d$ be a convex body with a summand RB^d, $R > 0$, such that h_B is twice differentiable on $\mathbb{R}^d \setminus \{0\}$. Put $G := \max\{\operatorname{diam} K, 1\}$, $S := \max\{\operatorname{diam} B, 1\}$ and*

$$C' := d2^{d+2}\kappa_d \frac{S^d \cdot G^{d+1}}{R^3}.$$

Then we have for all $r \geq 1$

$$V_d(\operatorname{conv} K + rB) - V_d(K + rB) < C' \cdot r^{d-3}.$$

Proof. Just like in the proof of Lemma 23, B is strictly convex and we may assume $RB^d \subseteq B$. When we put again $L := \operatorname{conv} K$, (12) remains true. We obtain a better upper bound for $r^i - (Gw(\frac{r}{G}))^i$, $i = 1, \dots, d$, now. Lemma 27 says that

$$s - w_B(s) \leq \frac{C}{s},\ s \in \mathbb{R}^+,$$

where $C := \frac{4S}{R^3}$. Since obviously $w_B(s) \leq s$ for all $s \in \mathbb{R}^+$, we conclude

$$\begin{aligned}
r^i - (Gw_B(\tfrac{r}{G}))^i =& (r - Gw_B(\tfrac{r}{G})) \sum_{j=0}^{i-1} r^j (Gw_B(\tfrac{r}{G}))^{i-1-j} \\
=& G \cdot (\tfrac{r}{G} - w_B(\tfrac{r}{G})) \sum_{j=0}^{i-1} r^j (Gw_B(\tfrac{r}{G}))^{i-1-j} \\
\leq& G\tfrac{CG}{r} \sum_{j=0}^{i-1} r^j (G\tfrac{r}{G})^{i-1-j} \\
=& \tfrac{CG^2}{r} \sum_{j=0}^{i-1} r^{i-1} \\
=& CG^2 i r^{i-2}.
\end{aligned}$$

We will need a slightly different upper bound for $V_d(L\setminus(K+rB))$ now. Since $L\subseteq K+rB$, if $rR\geq G$, we have

$$V_d(L\setminus(K+rB))\leq V_d(L)\mathbf{1}_{\{rR<G\}}\leq \kappa_d G^d\frac{G}{rR}. \tag{18}$$

Using first (12) and then (14), which is still valid, too, (18) and Theorem 15, we obtain

$$\begin{aligned}
&V_d(L+rB^d)-V_d(K+rB^d)\\
&\qquad\leq \sum_{i=1}^{d}\kappa_i\Phi^B_{d-i}(L,L\setminus\operatorname{ext}L)\big(r^i-(Gw_B(\tfrac{r}{G}))^i\big)+V_d(L\setminus(K+rB))\\
&\qquad\leq \sum_{i=1}^{d-1}\binom{d}{i}G^{d-i}S^i\kappa_d\cdot CG^2ir^{i-2}+\kappa_dG^d\frac{G}{rR}\\
&\qquad\leq \sum_{i=1}^{d-1}(d-1)\binom{d}{i}\kappa_dS^{d-1}CG^{d+1}r^{d-3}+\kappa_d\frac{G^{d+1}}{R}r^{d-3}\\
&\qquad< \Big((d-1)2^d\kappa_dS^{d-1}CG^{d+1}+\kappa_d\frac{G^{d+1}}{R}\Big)r^{d-3}\\
&\qquad< \Big(d2^{d+2}\kappa_d\frac{S^d\cdot G^{d+1}}{R^3}\Big)r^{d-3}. \qquad\square
\end{aligned}$$

Now Theorem 26 can be proven the same way as Theorem 22.

In the rest of this section we will show that certain improvements of Theorem 26 are not possible. First we will show that the order r^{d-3} cannot be improved, even in the Euclidean deterministic case.

Example 29. Let $K:=\{v,-v\}$, where $v\in\mathbb{R}^d$ is a unit vector. Put

$$D(r):=\{x\in\mathbb{R}^d\mid |\langle x,v\rangle|\leq\tfrac{1}{2},\ \sqrt{r^2-\tfrac{1}{4}}<\|p_1(x)\|\leq r\},\quad r>\tfrac{1}{2},$$

where p_1 denotes the orthogonal projection from $\mathbb{R}^d$ onto the linear subspace perpendicular to v. Since $D_r\subseteq\operatorname{conv}K+rB^d$ and $D_r\cap(K+rB^d)=\emptyset$ we have

$$V_d(\operatorname{conv}K+rB^d)-V_d(K+rB^d)\geq V_d(D_r)=\kappa_{d-1}\big(r^{d-1}-\sqrt{r^2-\tfrac{1}{4}}^{\,d-1}\big).$$

A purely analytical computation shows that the latter expression is indeed of order r^{d-3}.

Now we will show that the assumption that B contains a ball as summand cannot be relaxed in the planar case.

Corollary 30. *Let $B\subseteq\mathbb{R}^2$ be a convex body. Then the following are equivalent:*

(i) There is a constant $c\in\mathbb{R}^+_0$ such that

$$V_2(\operatorname{conv}K+rB)-V_2(K+rB)<\frac{c}{r}$$

for all $r\in\mathbb{R}^+$ and every convex body $K\subseteq\mathbb{R}^2$ with $\operatorname{diam}K\leq 1$.

(ii) B has a summand RB^2, $R > 0$.

Proof. If (ii) is fulfilled, then by Theorem 26 there is a constant C such that

$$V_2(\operatorname{conv} K + rB) - V_2(K + rB) < \frac{C}{r}$$

for all $r \geq 1$ and every body $K \subseteq \mathbb{R}^2$ with $\operatorname{diam} K \leq 1$. For $r \in (0,1]$ and a body $K \subseteq \mathbb{R}^2$ with $\operatorname{diam} K \leq 1$ we have

$$V_2(\operatorname{conv} K + rB) - V_2(K + rB) < V_2(B^2 + B) \leq \frac{V_2(B^2 + B)}{r}.$$

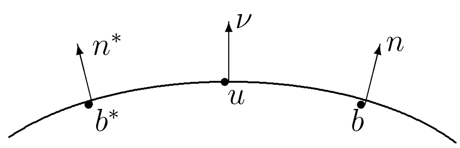
Hence we have

$$V_2(\operatorname{conv} K + rB) - V_2(K + rB) < \frac{\max\{C, V_2(B^2 + B)\}}{r}$$

for all $r \in \mathbb{R}^+$ and bodies $K \subseteq \mathbb{R}^2$ with $\operatorname{diam} K \leq 1$.
So now assume that (i) is fulfilled.
First assume that B has no interior points. Then B is contained in a line with unit normal vector τ, say. Let S be a segment of length 1 perpendicular to τ. Put $K := S \cup (S + \tau)$. Then

$$V_2(\operatorname{conv} K + rB) - V_2(K + rB) = 1 + rl,$$

where $l \geq 0$ denotes the length of B. Since this contradicts (i), B has interior points.
The support function h_B is convex and thus according to Theorem 4 a.e. Alexandrov-twice-differentiable. In particular, the second derivative h_B in ν in direction orthogonal to ν exists for a.e. $\nu \in S^1$. We call it $R(B, \nu)$. Now we will prove that there is a constant $\tilde{c} > 0$ such that $R(B, \nu) \geq \tilde{c}$ holds, whenever $R(B, \nu)$ exists. According to Theorem 8 this yields (ii).
So let $\nu \in S^1$ be a point, in which h_B is Alexandrov-twice-differentiable, and choose $\tau \in S^1$ perpendicular to ν. Let $\theta : \mathbb{R}^2 \to \mathbb{R}^2$ be a choice of subgradients of h_B. Then according to Theorem 5 the point $u := \theta(\nu)$ lies in $\operatorname{bd} B$ and has exterior normal vector ν.
There is $\epsilon > 0$ and $\xi \in \mathbb{R}^2$ with $B_\epsilon(\xi) \subseteq \operatorname{int} B$. Let $b_0 \in \operatorname{bd} B$ denote a point, which satisfies $0 < \langle b_0 - u, \tau \rangle \leq \frac{\epsilon}{2}$ and has an exterior unit normal vector n_0 with $\langle n_0, \nu \rangle > 0$. Let b_0^* and n_0^* be defined in the same way with $u - b_0^*$ instead of $b_0 - u$.



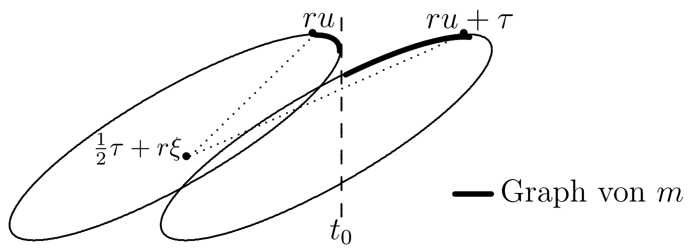
Let $n \in S^1$ be a vector with $\langle \nu, n \rangle > 0$ and $0 < \langle n - \nu, \tau \rangle \leq \min\{\langle n_0, \tau \rangle, -\langle n_0^*, \tau \rangle\}$. Put $n^* := 2\langle n, \nu \rangle \nu - n$. Now $b := \theta(n)$ and $b^* := \theta(n^*)$ are points in $\operatorname{bd} B$ with exterior normal vectors n resp. n^*. We put $K := \{0, \tau\}$ and $r := \frac{1}{2\langle b - b^*, \tau \rangle}$.
Now the point $\frac{1}{2}\tau + r\xi$ is both in $\operatorname{int}(rB)$ and in $\operatorname{int}(\tau + rB)$. Indeed,

$$\langle n_0^*, \tau \rangle \leq \langle n^*, \tau \rangle \leq \langle n, \tau \rangle \leq \langle n_0, \tau \rangle$$

and thus

$$\epsilon r \geq \frac{\langle b_0 - b_0^*, \tau\rangle}{2\langle b - b^*, \tau\rangle} \geq \tfrac{1}{2}.$$

So $\frac{1}{2}\tau + r\xi \in rB_\epsilon(\xi) \subseteq \mathrm{int}(rB)$ and $\frac{1}{2}\tau + r\xi \in \tau + rB_\epsilon(\xi) \subseteq \mathrm{int}(\tau + rB)$.



Now every line of the form $\{x \in \mathbb{R}^2 \mid \langle x, \tau\rangle = t\}$ with $t \in [t_1, t_1 + 1]$, where $t_1 := r\langle u, \tau\rangle$, intersects at least one of the segments $[ru, \frac{1}{2}\tau + r\xi]$ or $[\frac{1}{2}\tau + r\xi, \tau + ru]$. Hence the sets $\{x \in K + rB \mid \langle x, \tau\rangle = t\}$, $t \in [t_1, t_1 + 1]$, are not empty and the function

$$m : [t_1, t_1 + 1] \to \mathbb{R},\ t \mapsto \max\{\langle x, \nu\rangle \mid x \in K + rB,\ \langle x, \tau\rangle = t\}$$

is defined. Now there is a number $t_0 \in [t_1, t_1 + 1]$ with

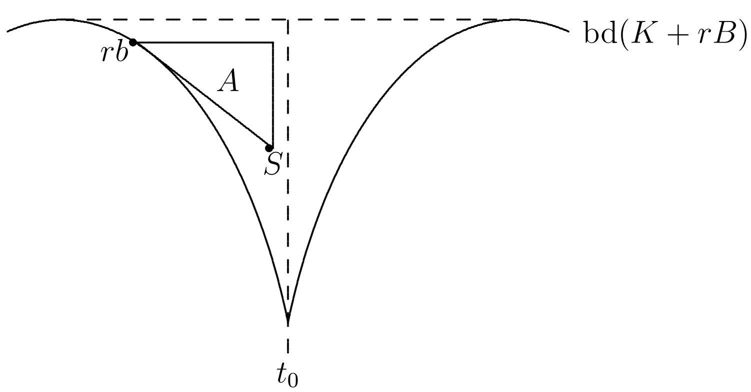
$$m(t) = \max\{\langle x, \nu\rangle \mid x \in rB,\ \langle x, \tau\rangle = t\}, \quad t < t_0,$$

and

$$m(t) = \max\{\langle x, \nu\rangle \mid x \in \tau + rB,\ \langle x, \tau\rangle = t\}, \quad t > t_0.$$

Since $\langle (\tau + rb^*) - rb, \tau\rangle = \frac{1}{2}$, we have either $t_0 - \langle rb, \tau\rangle \geq \frac{1}{4}$ or $\langle \tau + rb^*, \tau\rangle - t_0 \geq \frac{1}{4}$, w.l.o.g. the first.
Now we put

$$A := \{x \in \mathbb{R}^2 \mid \langle x - rb, \tau\rangle < \tfrac{1}{4},\ \langle x, n\rangle > \langle rb, n\rangle,\ \langle x, \nu\rangle \leq \langle rb, \nu\rangle\}.$$



It is easy to see that $A \subseteq \operatorname{conv} K + rB$, but $A \cap (K + rB) = \emptyset$. One cathetus of the rectangular triangle A has length $\frac{1}{4}$. In order to compute the length of the other cathetus, we let S denote the point with $\langle S, n\rangle = \langle rb, n\rangle$ und $\langle S, \tau\rangle = \langle rb, \tau\rangle + \frac{1}{4}$. Then

$$\begin{aligned} 0 &= \langle rb - S, n\rangle \\ &= \langle rb - S, \nu\rangle\langle n, \nu\rangle + \langle rb - S, \tau\rangle\langle n, \tau\rangle \\ &= \langle rb - S, \nu\rangle\langle n, \nu\rangle - \tfrac{1}{4}\langle n, \tau\rangle. \end{aligned}$$

Thus the length we looked for is

$$\langle rb, \nu\rangle - \langle S, \nu\rangle = \frac{\langle n, \tau\rangle}{4\langle n, \nu\rangle}.$$

Hence we get

$$\begin{aligned} \tfrac{1}{32}\langle n, \tau\rangle &\leq \tfrac{1}{2} \cdot \tfrac{1}{4} \cdot \frac{\langle n, \tau\rangle}{4\langle n, \nu\rangle} \\ &= V_2(A) \\ &\leq V_2(\operatorname{conv} K + rB) - V_2(K + rB) \\ &< \frac{c}{r} \\ &= \tfrac{c}{2}\langle b - b^*, \tau\rangle \\ &= \tfrac{c}{2}\langle \theta(n) - \theta(n^*), \tau\rangle. \end{aligned}$$

This means

$$\langle \theta(n) - \theta(n^*), \tau\rangle \geq \frac{1}{16c}\langle n, \tau\rangle = \frac{1}{32c}\langle n - n^*, \tau\rangle.$$

Since we assumed θ to be differentiable in ν and any vector in S^1 sufficiently close to ν could be chosen to be n, this implies

$$R(B, \nu) = \tfrac{\partial}{\partial\lambda}\langle \theta(\nu + \lambda\tau), \tau\rangle_{|\lambda=0} \geq \frac{1}{32c}.$$

Now Theorem 8 implies that B has a summand of the form RB^2, $R > 0$. $\square$

Conjecture 31. *There are convex bodies $B \subseteq \mathbb{R}^d$, $d \geq 3$, which contain no ball as summand, but for which there is a constant $c \in \mathbb{R}_0^+$ with*

$$V_d(\operatorname{conv} K + rB^d) - V_d(K + rB^d) < c \cdot r^{d-3} \tag{19}$$

for all $r \geq 1$ and for all bodies $K \subseteq \mathbb{R}^d$ with $\operatorname{diam} K \leq 1$.

Reason: Choose B to be a convex body, for which there are numbers $S > 0$ and $\alpha \in (1, 2)$ and a convex body $\tilde{B}$ with a ball as summand, such that

$$\{(x_1, \ldots, x_d) \in B \mid x_d \geq -S\} = \{(x_1, \ldots, x_d) \in \mathbb{R}^d \mid -S \leq x_d \leq -\|(x_1, \ldots, x_{d-1})\|^\alpha\}$$

and

$$\{(x_1, \ldots, x_d) \in B \mid x_d \leq -S\} = \{(x_1, \ldots, x_d) \in \tilde{B} \mid x_d \leq -S\}.$$

Now there is no $R > 0$ with $0 \in m + RB^d \subseteq B$ for any $m \in B$.
Geometric intuition tells us that it suffices to check (19) in the special case $K = \{-e_1, e_1\}$, where e_1 denotes the first unit vector. This is, however, an easy computation. $\square$

4 The derivative of the parallel volume difference

Theorem 26 suggests that under its assumptions, there is a constant $\tilde{c}$ with

$$\frac{d}{dr}\mathbb{E}[V_n(X+rY)-V_n(X_Y+rY)] < \tilde{c}\cdot r^{d-4}$$

for all sufficiently large r. However it does not imply this.
But we can prove this statement in the case, where the dimension d of Y is 2, under some additional regularity assumptions.

Theorem 32. *Let G, R and S be three $\mathbb{R}^+$-valued random variables with $R \le 1$ and $S, G \ge 1$ a.s. Let $X \subseteq \mathbb{R}^n$ be a random body whose diameter is a.s. less or equal to G and let $Y \subseteq \mathbb{R}^n$ be a random 2-dimensional body, which is a.s. strictly convex if considered as subset of $\hat{Y}$, which contains a.s. a ball of radius R as summand and which is a.s. subset of the ball SB^n. If*

$$c := 3200\kappa_{n-2}\mathbb{E}\left[\left(\frac{SG}{R^2}\right)^3 G^n S^2\right] < \infty,$$

then:

(i) For $b \ge a \ge 1$ we have

$$|\mathbb{E}[V_n(X+bY)-V_n(X_Y+bY)]-\mathbb{E}[V_n(X+aY)-V_n(X_Y+aY)]| < \frac{c}{a}-\frac{c}{b}.$$

(ii) The map $r \mapsto \mathbb{E}[V_n(X+rY)-V_n(X_Y+rY)]$ is differentiable for almost all $r \ge 1$ with

$$\Big|\frac{d}{dr}\mathbb{E}[V_n(X+rY)-V_n(X_Y+rY)]\Big| < c\cdot r^{-2}.$$

Again, we start by proving the theorem in the special case that $n = 2$ and all sets involved are deterministic.
Since the derivative of the parallel volume is an integral over the surface of the parallel body by Lemma 20, we will examine this boundary now. In the following we will always assume that the gauge body B fulfils the following condition:

(A) B is a strictly convex body, which has summand of the form RB^2, $R > 0$, and fulfils $RB^2 \subseteq B \subseteq SB^2$ for some $S > 0$.

Recall the definition of d_B, p_B, u_B and n_B on page 6.

Lemma 33. *Let $B, K \subseteq \mathbb{R}^2$ be two bodies such that B fulfils (A) and let $r > \frac{\operatorname{diam} K}{R}$ be a number. Let I be a closed convex set in the boundary of $L := \operatorname{conv} K$, whose endpoints lie in K. Let $u \in B$ denote the point, in which B has the same exterior unit normal vector ν as L has in the points of the relative interior of I. Let τ denote a unit vector orthogonal to ν and let j denote the length of I. Then the map*

$$f_r : I \to \mathbb{R}^+,\ y \mapsto \sup\{s > 0 \mid y + su \in K + rB\}$$

has the following properties:

(i) $\{z \in \operatorname{bd}(K+rB) \mid p_B(L,z) \in I\} = \{y + f_r(y)u \mid y \in I\}$

(ii) If $\sqrt{(2Rr)^2 - j^2}\,\langle \nu, u\rangle - j|\langle u, \tau\rangle| > 0$, *then* f_r *is Lipschitz continuous with Lipschitz constant*

$$\frac{j}{\sqrt{(2Rr)^2 - j^2}\,\langle \nu, u\rangle - j|\langle u, \tau\rangle|}.$$

(iii) For all $z \in \operatorname{bd}(K+rB)$ *with* $p_B(L,z) \in I$ *we have*

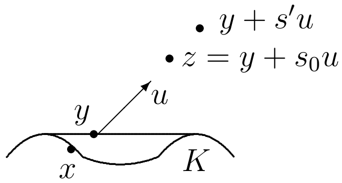
$$|h_B(n_B(K,z)) - h_B(\nu) + \langle u, \nu - n_B(K,z)\rangle| \le \frac{((1 + \frac{\|u\|}{4R})\operatorname{diam} K)^2}{2R} r^{-2},$$

if $n_B(K,z)$ *is defined.*

Proof. The definition of f_r makes sense, since

$$L \subseteq K + (\operatorname{diam} K)B^2 \subseteq \operatorname{int}(K+rB)$$

and thus the supremum is taken over a non-empty set.
(i) Let $y \in I$. Since we have $z := y + f_r(y)u \in K + rB$, but $z + \epsilon u \notin K + rB$ for all $\epsilon > 0$, we get $z \in \operatorname{bd}(K+rB)$. By Lemma 11(iii) we have $p_B(L,z) = y \in I$.
So now let $z \in \operatorname{bd}(K+rB)$ with $y := p_B(L,z) \in I$. Since u is the uniquely determined point from B with exterior normal vector ν, there is according to Lemma 11 a number $s_0 \in \mathbb{R}_0^+$ with $z = y + s_0 u$. Now we have on the one hand $y + s_0 u \in K + rB$. On the other hand we have $y + s'u \notin K + rB$ for all $s' > s_0$: Assume there is $s' > s_0$ with $y + s'u \in K + rB$. Then there is $x \in K$ with $y + s'u - x \in rB$.

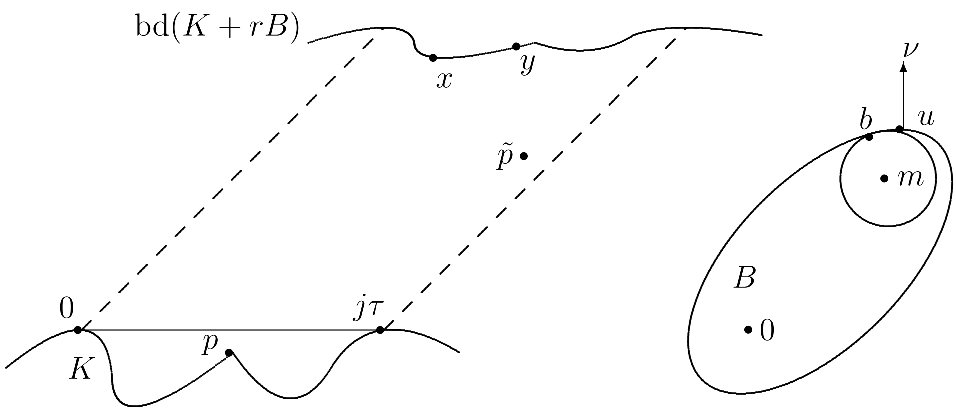


Moreover, $\|y - x\| \le \operatorname{diam} K < rR$, and hence $y - x \in \operatorname{int} rRB^2 \subseteq \operatorname{int} rB$. Since B is convex, we get

$$y + s_0 u - x = \frac{s_0}{s'}(y + s'u - x) + \frac{s' - s_0}{s'}(y - x) \in \operatorname{int} rB.$$

So $z \in \operatorname{int}(K+rB)$, contradicting the choice of z.
Thus $s_0 = \sup\{s > 0 \mid y + su \in K + rB\} = f_r(y)$ and therefore $z = y + f_r(y)u$.
(ii) We obey that τ is parallel to I, and introduce a (non-orthonormal) coordinate system by choosing the endpoint of I, from which the other lies in direction τ, as origin and identifying (x_1, x_2) with $x_1\tau + x_2 u$.
Let $x, y \in \operatorname{bd}(K+rB)$ with $p_B(L,x) \in I$ and $p_B(L,y) \in I$ and w.l.o.g. $y_2 \ge x_2$. Let $\epsilon > 0$. We assume $|x_1 - y_1| < \epsilon$ and $|\langle x - y, \tau\rangle| < \epsilon$. Moreover, let $p \in K$ and $b, m \in B$ denote points with $y = p + rb$ and $b \in m + RB^2 \subseteq B$ and put $\tilde{p} := p + rm$. Then from

$$\tilde{p} + rRB^2 = p + r(m + RB^2) \subseteq K + rB$$



we get $x \notin \operatorname{int}(\tilde p + rRB^2)$ and hence $\|x-\tilde p\| \geq Rr = \|y-\tilde p\|$. This gives

$$\begin{aligned}(x_1-\tilde p_1)^2 + 2(x_1-\tilde p_1)(x_2-\tilde p_2)\langle\tau,u\rangle + (x_2-\tilde p_2)^2\|u\|^2 \\ &= \|(x_1-\tilde p_1)\tau + (x_2-\tilde p_2)u\|^2 \\ &\geq \|(y_1-\tilde p_1)\tau + (y_2-\tilde p_2)u\|^2 \\ &= (y_1-\tilde p_1)^2 + 2(y_1-\tilde p_1)(y_2-\tilde p_2)\langle\tau,u\rangle + (y_2-\tilde p_2)^2\|u\|^2,\end{aligned}$$

and hence

$$\begin{aligned}[(x_1-\tilde p_1)^2 - (y_1-\tilde p_1)^2] + 2[(x_1-\tilde p_1)(x_2-\tilde p_2) - (y_1-\tilde p_1)(y_2-\tilde p_2)]\langle\tau,u\rangle \\ \geq [(y_2-\tilde p_2)^2 - (x_2-\tilde p_2)^2]\,\|u\|^2.\end{aligned}$$

Since

$$\begin{aligned}(x_1-\tilde p_1)(x_2-\tilde p_2) - (y_1-\tilde p_1)(y_2-\tilde p_2) \\ &= (x_1-\tilde p_1)(x_2-\tilde p_2) - [(y_1-x_1)+(x_1-\tilde p_1)](y_2-\tilde p_2) \\ &= (x_1-\tilde p_1)[(x_2-\tilde p_2)-(y_2-\tilde p_2)] - (y_1-x_1)(y_2-\tilde p_2) \\ &= (x_1-\tilde p_1)(x_2-y_2) - (y_1-x_1)(y_2-\tilde p_2),\end{aligned}$$

this is equivalent to

$$\begin{aligned}[x_1^2 - 2x_1\tilde p_1 - y_1^2 + 2y_1\tilde p_1] + 2[(x_1-\tilde p_1)(x_2-y_2) - (y_1-x_1)(y_2-\tilde p_2)]\langle\tau,u\rangle \\ &\geq [y_2^2 - 2y_2\tilde p_2 - x_2^2 + 2x_2\tilde p_2]\,\|u\|^2 \\ \Rightarrow (x_1-y_1)(x_1+y_1-2\tilde p_1) + 2[(x_1-y_1)(y_2-\tilde p_2)]\langle\tau,u\rangle \\ &\geq (y_2-x_2)(y_2+x_2-2\tilde p_2)\|u\|^2 - 2(x_1-\tilde p_1)(x_2-y_2)\langle\tau,u\rangle \\ \Rightarrow (x_1-y_1)[(x_1+y_1-2\tilde p_1) + 2(y_2-\tilde p_2)\langle\tau,u\rangle] \\ &\geq (y_2-x_2)[(y_2+x_2-2\tilde p_2)\|u\|^2 + 2(x_1-\tilde p_1)\langle\tau,u\rangle] \qquad (20)\end{aligned}$$

In order to find bounds for the second factor on either side of (20) we need the inequality

$$\langle x-\tilde p,\nu\rangle \geq \sqrt{(rR)^2 - (\tfrac{j}{2})^2}, \qquad (21)$$

that we will prove below. We obey, that this inequality holds also with x replaced by y. Since τ and ν are an orthonormal base, we have moreover (only with y, not with x):

$$\begin{aligned}|\langle y-\tilde{p},\tau\rangle| &= \sqrt{\|y-\tilde{p}\|^2-\langle y-\tilde{p},\nu\rangle^2} \\ &\leq \sqrt{(rR)^2-((rR)^2-(\tfrac{j}{2})^2)} \\ &= \tfrac{j}{2}.\end{aligned}$$

Thus we obtain

$$\begin{aligned}|(x_1+y_1-2\tilde{p}_1)+2(y_2-\tilde{p}_2)\langle\tau,u\rangle| &\leq |(2y_1-2\tilde{p}_1)\langle\tau,\tau\rangle+2(y_2-\tilde{p}_2)\langle u,\tau\rangle|+\epsilon \\ &= |\langle(2y_1-2\tilde{p}_1)\tau+2(y_2-\tilde{p}_2)u\,,\,\tau\rangle|+\epsilon \\ &= 2|\langle y-\tilde{p},\tau\rangle|+\epsilon \\ &\leq j+\epsilon\end{aligned}$$

and

$$\begin{aligned}&(y_2+x_2-2\tilde{p}_2)\|u\|^2+2(x_1-\tilde{p}_1)\langle\tau,u\rangle \\ &\qquad\geq \langle(y_2+x_2-2\tilde{p}_2)u+(y_1+x_1-2\tilde{p}_1)\tau\,,\,u\rangle-\epsilon|\langle\tau,u\rangle| \\ &\qquad= \langle y+x-2\tilde{p},u\rangle-\epsilon|\langle\tau,u\rangle| \\ &\qquad= \langle y+x-2\tilde{p},\tau\rangle\langle u,\tau\rangle+\langle y+x-2\tilde{p},\nu\rangle\langle u,\nu\rangle-\epsilon|\langle\tau,u\rangle| \\ &\qquad\geq \langle 2y-2\tilde{p},\tau\rangle\langle u,\tau\rangle-\epsilon|\langle u,\tau\rangle|+\langle y+x-2\tilde{p},\nu\rangle\langle u,\nu\rangle-\epsilon|\langle\tau,u\rangle| \\ &\qquad\geq -j|\langle u,\tau\rangle| \qquad\qquad +2\sqrt{(rR)^2-(\tfrac{j}{2})^2}\,\langle u,\nu\rangle-2\epsilon|\langle\tau,u\rangle| \\ &\qquad= \sqrt{(2rR)^2-j^2}\,\langle u,\nu\rangle-(j+2\epsilon)|\langle\tau,u\rangle|.\end{aligned}$$

With help of these inequalities and the assumption $y_2\geq x_2$ we get from (20)

$$\begin{aligned}|x_1-y_1|\cdot(j+\epsilon) &\geq |x_1-y_1|\cdot|(x_1+y_1-2\tilde{p}_1)+2(y_2-\tilde{p}_2)\langle\tau,u\rangle| \\ &\geq (x_1-y_1)[(x_1+y_1-2\tilde{p}_1)+2(y_2-\tilde{p}_2)\langle\tau,u\rangle] \\ &\geq (y_2-x_2)[(y_2+x_2-2\tilde{p}_2)\|u\|^2+2(x_1-\tilde{p}_1)\langle\tau,u\rangle] \\ &\geq (y_2-x_2)(\sqrt{(2rR)^2-j^2}\,\langle u,\nu\rangle-(j+2\epsilon)|\langle\tau,u\rangle|).\end{aligned}$$

Now we can drop the restrictions $|x_1-y_1|<\epsilon$ and $|\langle x-y,\tau\rangle|<\epsilon$: If e.g. $x_1<y_1$, then we let $x^{(0)},\ldots,x^{(n)}\in\mathrm{bd}(K+rB)$ denote points with $x=x^{(0)},y=x^{(n)}$, $x_1^{(0)}\leq\cdots\leq x_1^{(n)}$ such that $|x_1^{(j-1)}-x_1^{(j)}|<\epsilon$ and $|\langle x^{(j-1)}-x^{(j)},\tau\rangle|<\epsilon$ holds for $j=1,\ldots,n$. We get

$$\begin{aligned}&(y_2-x_2)\cdot(\sqrt{(2Rr)^2-j^2}\,\langle\nu,u\rangle-(j+2\epsilon)|\langle u,\tau\rangle|) \\ &\qquad= \sum_{j=1}^{n}(x_2^{(j)}-x_2^{(j-1)})\cdot(\sqrt{(2Rr)^2-j^2}\,\langle\nu,u\rangle-(j+2\epsilon)|\langle u,\tau\rangle|) \\ &\qquad\leq \sum_{j=1}^{n}|x_1^{(j-1)}-x_1^{(j)}|\cdot(j+\epsilon) \\ &\qquad= |x_1-y_1|\cdot(j+\epsilon).\end{aligned}$$

Since $\epsilon > 0$ was arbitrary and we assumed $\sqrt{(2rR)^2 - j^2}\langle u, \nu\rangle - j|\langle u, \tau\rangle| > 0$, we obtain

$$y_2 - x_2 \leq |x_1 - y_1| \frac{j}{\sqrt{(2rR)^2 - j^2}\langle u, \nu\rangle - j|\langle u, \tau\rangle|},$$

which shows the assertion, as $y_2 - x_2 \geq 0$.
It remains to show (21). We put $\bar{m} := u - R\nu$ and $A := \{0, j\tau\}$. The main part is the proof of

$$\langle x - r\bar{m}, \nu\rangle \geq \sqrt{(rR)^2 - (\tfrac{j}{2})^2}.$$

In order to show this inequality, we set $\rho_0 := d_B(L, x) \leq r$. Then $x \in I + \rho_0 u$, since $u = u_B(L, x)$. Because of $I \subseteq A + \frac{j}{2}B^2 \subseteq A + \frac{j}{2R}B$ we get

$$x \in I + \rho_0 B \subseteq A + (\rho_0 + \tfrac{j}{2R})B$$

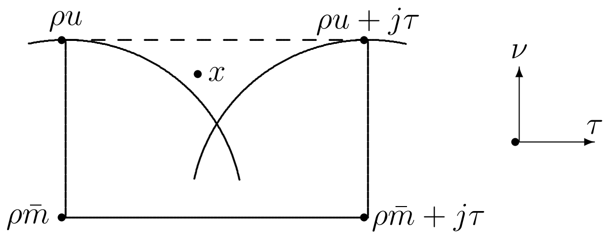
and thus

$$\rho_0 + \frac{j}{2R} \geq d_B(A, x) \geq d_B(K, x) = r,$$

since $0, j\tau \in K$. Hence $R(r - \rho_0) \leq \frac{j}{2}$ and with help of the assumption $r > \frac{\operatorname{diam} K}{R}$, we obtain

$$\rho_0 R \geq \frac{j}{2}. \tag{22}$$

Because B has a summand RB^2 and therefore $\bar{m} + RB^2 \subseteq B$ holds by Lemma 7, we conclude from $x \notin \operatorname{int}(K + rB)$, that $x \notin \operatorname{int}(A + \rho\bar{m} + \rho RB^2)$ holds for all $\rho \in (\rho_0, r)$. As $x \in I + \rho_0 u$ we obtain moreover $\langle x, \nu\rangle < \langle \rho u, \nu\rangle$ for all $\rho \in (\rho_0, r)$.



Now we will show that x lies for all $\rho \in (\rho_0, r]$ in the rectangle plotted in the figure. Precisely this means that the inequality $\langle x, \nu\rangle < \langle \rho u, \nu\rangle$, which was already mentioned, holds and the three scalar products $\langle x - \rho\bar{m}, \tau\rangle$, $\langle \rho\bar{m} + j\tau - x, \tau\rangle$ and $\langle x - \rho\bar{m}, \nu\rangle$ are positive. Assume the contrary. Since these terms are obviously positive for $\rho = \rho_0$ and continuous in ρ, there must be a minimal value $\rho_1 \in (\rho_0, r]$, for which one of the scalar products is not positive. Clearly, for $\rho = \rho_1$ none of the scalar products is negative. Now the first one cannot be 0, since

$$0 \leq \langle x - \rho_1\bar{m}, \nu\rangle < \langle \rho_1 u - \rho_1\bar{m}, \nu\rangle = \rho_1 R$$

and $\langle x - \rho_1\bar{m}, \tau\rangle = 0$ would give $x \in \operatorname{int}(A + \rho_1\bar{m} + \rho_1 RB^2)$. For an analogue reason the second one cannot be zero either. However, the third one must be positive, too.

Namely, $0 \le \langle x-\rho_1\bar{m},\tau\rangle$ and $0 \le \langle \rho_1\bar{m}+j\tau-x,\tau\rangle$ yield $|\langle x-\rho_1\bar{m},\tau\rangle| \le \frac{j}{2}$ or $|\langle \rho_1\bar{m}+j\tau-x,\tau\rangle| \le \frac{j}{2}$. Since $\frac{j}{2} < \rho_1 R$ holds by (22), from $\langle x-\rho_1\bar{m},\nu\rangle = 0$, we would get $x \in \operatorname{int}(A+\rho_1\bar{m}+\rho_1 RB^2)$. Hence the three expression are also positive for $\rho = r$ and thus either $|\langle x-r\bar{m},\tau\rangle| \le \frac{j}{2}$ or $|\langle r\bar{m}+j\tau-x,\tau\rangle| \le \frac{j}{2}$ holds, w.l.o.g. the first one. As $x \notin \operatorname{int}(A+r\bar{m}+rRB^2)$, we conclude

$$\langle x-r\bar{m},\nu\rangle = \sqrt{\|x-r\bar{m}\|^2-\langle x-r\bar{m},\tau\rangle^2} \ge \sqrt{(rR)^2-(\tfrac{j}{2})^2}. \tag{23}$$

Moreover $m+R\nu \in B$, which gives $\langle \bar{m}+R\nu,\nu\rangle = \langle u,\nu\rangle \ge \langle m+R\nu,\nu\rangle$ and thus $\langle \bar{m},\nu\rangle \ge \langle m,\nu\rangle$. Because of $\langle p,\nu\rangle \le 0$ we obtain

$$\begin{aligned}\langle x-\tilde{p},\nu\rangle &= \langle x-p-rm,\nu\rangle\\ &\ge \langle x-r\bar{m},\nu\rangle\\ &\ge \sqrt{(rR)^2-(\tfrac{j}{2})^2},\end{aligned}$$

which completes the proof of (21).

(iii) Again, we introduce a coordinate system such that the endpoint of I from which the other one is in direction τ is the origin. Let $z \in \operatorname{bd}(K+rB)$ be a point with $p_B(L,z) \in I$, for which $n_B(K,z)$, and hence $p_B(K,z)$ and $u_B(K,z)$ are defined. We put $b := u_B(K,z)$ and show

$$\|b-u\| \le ((1+\tfrac{\|u\|}{4R})\operatorname{diam} K)\cdot r^{-1}. \tag{24}$$

On the one hand $z = p+rb$ holds, where $p := p_B(K,z)$, and on the other $z = y+\rho u$ with $y := p_B(L,z) \in I$ and $\rho := d_B(L,z) \le r$. This gives

$$r\cdot\|b-u\| = \|(z-\rho u)-(z-rb)-(r-\rho)u\| = \|y-p-(r-\rho)u\|. \tag{25}$$

In order to derive an upper bound for $r-\rho$, we put again $\bar{m} := u-R\nu$. Because of (23) and $\langle y,\nu\rangle = 0$ we get

$$\begin{aligned}\sqrt{(rR)^2-(\tfrac{j}{2})^2} &\le \langle z-r\bar{m},\nu\rangle\\ &= \rho\langle u,\nu\rangle - r\langle \bar{m},\nu\rangle\\ &= \rho\langle u,\nu\rangle - r\langle u-R\nu,\nu\rangle\\ &= -(r-\rho)\langle u,\nu\rangle + r\langle R\nu,\nu\rangle\\ &\le -(r-\rho)R + rR\\ &= \rho R,\end{aligned}$$

where we used for the second inequality, that $r-\rho \ge 0$ and $\langle u,\nu\rangle \ge \langle R\nu,\nu\rangle = R$, since $R\nu \in RB^2 \subseteq B$.

This yields $(rR)^2-(\frac{j}{2})^2 \le (\rho R)^2$ and thus

$$(r-\rho)rR^2 \le (r-\rho)(r+\rho)R^2 = (rR)^2-(\rho R)^2 \le (\tfrac{j}{2})^2.$$

By the assumption $rR \ge \operatorname{diam} K$ and $j \le \operatorname{diam} K$ we get

$$(r-\rho)(\operatorname{diam} K)R \le \frac{j\cdot\operatorname{diam} K}{4}$$

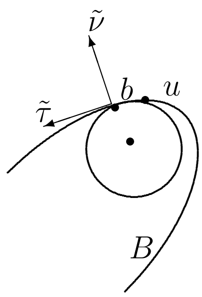
and hence

$$r - \rho \leq \frac{j}{4R}.$$

Using (25) we get

$$\begin{aligned} r \cdot \|b - u\| &= \|y - p - (r-\rho)u\| \\ &\leq \|y - p\| + (r-\rho)\|u\| \\ &\leq \operatorname{diam} K + \frac{j}{4R}\|u\| \\ &\leq \left(1 + \frac{\|u\|}{4R}\right) \cdot \operatorname{diam} K. \end{aligned}$$

Now we have shown (24).
Put $\tilde{\nu} := n_B(K, z)$ and let $\tilde{\tau}$ denote a unit vector perpendicular to $\tilde{\nu}$. The ball of radius R with center at $b - R\tilde{\nu}$ is contained in B according to Lemma 7 and thus u is not in the interior of this ball.



Hence

$$\begin{aligned} R^2 &\leq \|(b - R\tilde{\nu}) - u\|^2 \\ &= \langle (b - R\tilde{\nu}) - u, \tilde{\nu}\rangle^2 + \langle (b - R\tilde{\nu}) - u, \tilde{\tau}\rangle^2 \\ &= (\langle b - u, \tilde{\nu}\rangle - R)^2 + \langle b - u, \tilde{\tau}\rangle^2 \\ &= \langle b - u, \tilde{\nu}\rangle^2 - 2R\langle b - u, \tilde{\nu}\rangle + R^2 + \langle b - u, \tilde{\tau}\rangle^2. \end{aligned}$$

With inequality (24) this gives

$$\begin{aligned} 2R\langle b - u, \tilde{\nu}\rangle &\leq \langle b - u, \tilde{\nu}\rangle^2 + \langle b - u, \tilde{\tau}\rangle^2 \\ &= \|b - u\|^2 \\ &\leq \left(\left(1 + \tfrac{\|u\|}{4R}\right) \cdot \operatorname{diam} K \cdot r^{-1}\right)^2, \end{aligned}$$

which is equivalent to

$$\langle b - u, \tilde{\nu}\rangle \leq \frac{(1 + \frac{\|u\|}{4R})^2 (\operatorname{diam} K)^2}{2R} \cdot r^{-2}.$$

Since we have $\langle b - u, \tilde{\nu}\rangle \geq 0$ by the choice of $\tilde{\nu}$ and

$$h_B(\tilde{\nu}) - h_B(\nu) + \langle u, \nu - \tilde{\nu}\rangle \;=\; \langle b, \tilde{\nu}\rangle - \langle u, \nu\rangle + \langle u, \nu - \tilde{\nu}\rangle \;=\; \langle b - u, \tilde{\nu}\rangle,$$

we obtain

$$0 \le h_B(\tilde{\nu}) - h_B(\nu) + \langle u, \nu - \tilde{\nu} \rangle \le \frac{(1 + \frac{\|u\|}{4R})^2 (\operatorname{diam} K)^2}{2R} \cdot r^{-2}. \qquad \square$$

According to Lemma 33 a particular set is a graph of a Lipschitz function with respect to a not orthonormal coordinate system. The following lemma implies that this set is also a graph of a Lipschitz function with respect to an orthonormal coordinate system.

Lemma 34. *Let F be a set of the form $\{x\tau + f(x)u \mid x \in I\}$, where $I = [0, j]$ is an interval, τ is a unit vector, u is a vector linearly independent of τ and $f : I \to \mathbb{R}$ is a Lipschitz function with Lipschitz constant L, satisfying $f(0) = f(j)$. If $L\langle \tau, u \rangle < 1$, then F has a representation of the form $\{x\tau + \tilde{f}(x)\nu \mid x \in \tilde{I}\}$, where ν is the unit vector in direction $u - \langle u, \tau \rangle \tau$, this means ν is perpendicular to τ, $\tilde{I}$ is an interval of length j and $\tilde{f} : \tilde{I} \to \mathbb{R}$ is a Lipschitz function with Lipschitz constant*

$$\frac{L\langle \nu, u \rangle}{1 - L\langle \tau, u \rangle}.$$

Proof. Let $z_1, z_2 \in F$, $z_1 \neq z_2$. Then there are $x_1, x_2 \in I$ with $z_i = x_i\tau + f(x_i)u$, $i = 1, 2$, and w.l.o.g. $x_1 < x_2$. Now we have

$$\begin{aligned} \langle z_2 - z_1, \tau \rangle &= \langle x_2\tau + f(x_2)u - x_1\tau - f(x_1)u, \tau \rangle \\ &= x_2 - x_1 + (f(x_2) - f(x_1))\langle u, \tau \rangle \\ &\ge (x_2 - x_1)(1 - L\langle u, \tau \rangle) \qquad (26) \\ &> 0. \end{aligned}$$

Hence each line perpendicular to τ intersects F in at most one point and thus there is a function $\tilde{f} : \tilde{I} \to \mathbb{R}$ with domain $\tilde{I} := \{\langle z, \tau \rangle \mid z \in F\}$ for which $F = \{x\tau + \tilde{f}(x)\nu \mid x \in \tilde{I}\}$ holds. So every point $z \in F$ can be represented in the form $z = x\tau + \tilde{f}(x)\nu$ with some $x \in \tilde{I}$. This gives $\langle z, \tau \rangle = x$ and $\langle z, \nu \rangle = \tilde{f}(x)$. Hence

$$\langle z, \nu \rangle = \tilde{f}(\langle z, \tau \rangle). \qquad (27)$$

Since f is continuous, F is connected and thus $\tilde{I}$ must be an interval. Moreover, the length of $\tilde{I}$ is

$$\begin{aligned} \max\{\langle z, \tau \rangle | z \in F\} - \min\{\langle z, \tau \rangle | z \in F\} &= \langle j\tau + f(j)u, \tau \rangle - \langle 0 + f(0)u, \tau \rangle \\ &= j + (f(j) - f(0))\langle u, \tau \rangle \\ &= j. \end{aligned}$$

For the proof of the Lipschitz constant let $z_1, z_2 \in F$ and x_1, x_2 as above. Then

$$\begin{aligned} |\langle z_2 - z_1, \nu \rangle| &= |\langle x_2\tau + f(x_2)u - x_1\tau - f(x_1)u, \nu \rangle| \\ &= |f(x_2) - f(x_1)|\langle u, \nu \rangle \\ &\le (x_2 - x_1)L\langle u, \nu \rangle. \end{aligned}$$

This gives, together with equality (27) and inequality (26)

$$\begin{aligned}|\tilde{f}(\langle z_2,\tau\rangle)-\tilde{f}(\langle z_1,\tau\rangle)| &= |\langle z_2,\nu\rangle-\langle z_1,\nu\rangle| \\ &\leq (x_2-x_1)L\langle u,\nu\rangle \\ &\leq \frac{L\langle u,\nu\rangle}{1-L\langle u,\tau\rangle}(\langle z_2,\tau\rangle-\langle z_1,\tau\rangle). \qquad \square\end{aligned}$$

Our aim is, as mentioned above, to determine the “size” of the boundary of $K+rB$. More precisely, we mean by size the Hausdorff measure defined in Definition 12 in the Euclidean case and a certain integral taken w.r.t. the Hausdorff measure in the general case. We will see that only these parts of the boundary are important, of which we have shown in Lemma 33 that they are graphs of Lipschitz functions. Hence the following lemma is of interest for us.

Lemma 35. *Let I be an interval and $f: I \to \mathbb{R}$ a Lipschitz continuous function with Lipschitz constant L. Let $F := \{(y, f(y)) \mid y \in I\} \subseteq \mathbb{R}^2$ be the graph of f and $g: F \to \mathbb{R}_0^+$ a measurable function. Then*

$$\int_I g(y,f(y))\,dy \leq \int_F g(z)\ d\mathcal{H}^1(z) \leq \sqrt{1+L^2}\int_I g(y,f(y))\,dy.$$

Proof. We will conclude this lemma from the area formula [5, Theorem 3.2.3 (2)]. Since f is Lipschitz continuous, there is a Lipschitz continuous function $\tilde{f}: \mathbb{R} \to \mathbb{R}^2$ with

$$\tilde{f}(y) = \begin{pmatrix} y \\ f(y) \end{pmatrix},\ y \in I.$$

The area formula, applied to

$$u: \mathbb{R} \to \mathbb{R},\ y \mapsto \begin{cases} g(y,f(y)), & \text{if } y \in I, \\ 0, & \text{if } y \notin I, \end{cases}$$

gives

$$\int_I g(y,f(y))\|\tilde{f}'(y)\|\,dy = \int_{\mathbb{R}^2} \sum_{y\in\tilde{f}^{-1}(\{z\})} u(y)\,d\mathcal{H}^1(z).$$

For $z \in \mathbb{R}^2$ we have

$$\sum_{y\in\tilde{f}^{-1}(\{z\})} u(y) = \sum_{y\in\tilde{f}^{-1}(\{z\})\cap I} u(y) + \sum_{y\in\tilde{f}^{-1}(\{z\})\setminus I} u(y) = g(z)\mathbf{1}_{\{z\in F\}} + 0.$$

Hence

$$\int_I g(y,f(y))\|\tilde{f}'(y)\|\,dy = \int_F g(z)\,d\mathcal{H}^1(z). \tag{28}$$

Since f is Lipschitz continuous with Lipschitz constant L, we have $|f'(y)| < L$ for all $y \in I$, in which f is differentiable, and hence

$$1 \leq \|\tilde{f}'(y)\| \leq \sqrt{1+L^2}.$$

Together with (28) this implies the assertion. $\square$

Lemma 36. *Assume that $B \subseteq \mathbb{R}^2$ fulfils condition (A) and let R and S be the numbers from (A). Let $K \subseteq \mathbb{R}^2$ be a body and put $L := \operatorname{conv} K$. Then for almost all $r > 2R^{-2} S \operatorname{diam} K$ we have*

$$\left(-(\tfrac{S}{R} + 4) \frac{S(2\pi \operatorname{diam} K)^3}{R^2} \right) \cdot r^{-2} \\ \leq \int_{\operatorname{bd}(K+rB)} h_B(n_B(K,z))\, d\mathcal{H}^1(z) - \int_{\operatorname{bd}(L+rB)} h_B(n_B(L,z))\, d\mathcal{H}^1(z) \\ \leq \left((10 + \tfrac{S}{R}) \frac{S(2\pi \operatorname{diam} K)^3}{R^2} \right) \cdot r^{-2}.$$

Remark 37. Just like Lemma 36 the inequality

$$0 \leq \mathcal{H}^1(\operatorname{bd}(K + rB)) - \mathcal{H}^1(\operatorname{bd}(L + rB)) \leq \frac{2(4\pi \cdot \operatorname{diam} K)^3}{(Rr)^2}$$

can be proven.

Proof of Lemma 36. Since $d_B(K, \cdot)$ is Lipschitz-continuous, it is differentiable a.e. by Theorem 2. Hence Theorem 20 shows that

$$\mathcal{H}^1(\{x \in \operatorname{bd}(K + rB) \mid \ d_B(K, \cdot) \text{ is not differentiable in } x\}) = 0 \tag{29}$$

holds for almost all $r > 0$. Let $r > 2R^{-2} S \operatorname{diam} K$ be a number satisfying (29). Obviously

$$\int_{\operatorname{bd}(K+rB)} h_B(n_B(K,z))\, d\mathcal{H}^1(z) - \int_{\operatorname{bd}(L+rB)} h_B(n_B(L,z))\, d\mathcal{H}^1(z) \\ = \int_{\operatorname{bd}(K+rB)\setminus\operatorname{bd}(L+rB)} h_B(n_B(K,z))\, d\mathcal{H}^1(z) \\ - \int_{\operatorname{bd}(L+rB)\setminus\operatorname{bd}(K+rB)} h_B(n_B(L,z))\, d\mathcal{H}^1(z) \\ + \int_{\operatorname{bd}(K+rB)\cap\operatorname{bd}(L+rB)} h_B(n_B(K,z)) - h_B(n_B(L,z))\, d\mathcal{H}^1(z). \tag{30}$$

Let $z \in \operatorname{bd}(K+rB) \cap \operatorname{bd}(L+rB)$ and put $p := p_B(L, z)$. Then for any point $k \in K \setminus \{p\} \subseteq L \setminus \{p\}$, Lemma 17 implies $d_B(k, z) > d_B(L, z) = r = d_B(K, z)$, which yields $p_B(K, z) = p$ and $n_B(K, z) = n_B(L, z)$. As $z \in \operatorname{bd}(K+rB) \cap \operatorname{bd}(L+rB)$ was arbitrary, the last integral in (30) is 0.
According to Lemma 17 for any point

$$z \in \big(\operatorname{bd}(L + rB) \setminus \operatorname{bd}(K + rB)\big) \cup \big(\operatorname{bd}(K + rB) \setminus \operatorname{bd}(L + rB)\big)$$

we have $d(K, z) > d(L, z)$ and hence $p_B(L, z) \in (\operatorname{bd} L) \setminus K$. By Lemma 6 every point of $(\operatorname{bd} L) \setminus K$ is contained in the relative interior of an edge of L and by [18, Theorem 2.2.5] there are (at most) countably many edges of L. For every edge J of L the set $((\operatorname{bd} L) \setminus K) \cap J = J \setminus K$ is open in J. Since any open subset of $\mathbb{R}$ is the union of countably many pairwise disjoint open intervals, $((\operatorname{bd} L) \setminus K) \cap J$ is the union of countably many pairwise disjoint sets that are open in J and convex. Hence there is a countable set $\mathcal{I}$ of pairwise disjoint 1-dimensional convex and relatively open sets, such that $(\mathrm{bd}\, L) \setminus K = \bigcup_{I \in \mathcal{I}} I$. The length of $I \in \mathcal{I}$ will be denoted by l_I, the Euclidean outer unit normal vector of L in the points of I by ν_I and the point in $\mathrm{bd}\, B$ with exterior normal vector ν_I by u_I. So $\mathrm{bd}(L + rB) \setminus \mathrm{bd}(K + rB)$ is the union of all sets $I + ru_I$, $I \in \mathcal{I}$, by Lemma 11 and $\mathrm{bd}(K + rB) \setminus \mathrm{bd}(L + rB)$ is the union of all sets

$$F_r^I := \{y + f_r^I(y)u_I \mid y \in I\}, \quad I \in \mathcal{I},$$

where f_r^I, $I \in \mathcal{I}$, denote the functions from Lemma 33. Hence we get

$$\int_{\mathrm{bd}(L+rB)\setminus \mathrm{bd}(K+rB)} h_B(n_B(L, z))\, d\mathcal{H}^1(z) = \sum_{I\in\mathcal{I}} l_I h_B(\nu_I)$$

and

$$\int_{\mathrm{bd}(K+rB)\setminus \mathrm{bd}(L+rB)} h_B(n_B(K, z))\, d\mathcal{H}^1(z) = \sum_{I\in\mathcal{I}} \int_{F_r^I} h_B(n_B(K, z))\, d\mathcal{H}^1(z).$$

This gives

$$\begin{aligned}
&\int_{\mathrm{bd}(K+rB)} h_B(n_B(K, z))\, d\mathcal{H}^1(z) - \int_{\mathrm{bd}(L+rB)} h_B(n_B(L, z))\, d\mathcal{H}^1(z) \\
&= \int_{\mathrm{bd}(K+rB)\setminus \mathrm{bd}(L+rB)} h_B(n_B(K, z))\, d\mathcal{H}^1(z) - \int_{\mathrm{bd}(L+rB)\setminus \mathrm{bd}(K+rB)} h_B(n_B(L, z))\, d\mathcal{H}^1(z) \\
&= \sum_{I\in\mathcal{I}} \int_{F_r^I} h_B(n_B(K, z))\, d\mathcal{H}^1(z) - l_I h_B(\nu_I). \qquad (31)
\end{aligned}$$

Now we will show that the condition $\sqrt{(2Rr)^2 - l_I^2}\, \langle \nu_I, u_I\rangle - l_I|\langle \tau_I, u_I\rangle| > 0$ from Lemma 33(ii) is satisfied, where τ_I is a unit vector orthogonal to ν_I. We have $|\langle \tau_I, u_I\rangle| \leq \|\tau_I\| \cdot \|u_I\| \leq S$, because $u_I \in B$. Since $R\nu_I \in RB^d \subseteq B$ and ν_I is exterior normal vector of B in u_I, we get $\langle \nu_I, u_I\rangle \geq \langle \nu_I, R\nu_I\rangle = R$. From the assumption $r > 2R^{-2}S \,\mathrm{diam}\, K$ we get $\mathrm{diam}\, K < Rr$ and $S \cdot \mathrm{diam}\, K < \frac{1}{2}R^2 r$ and thus

$$\begin{aligned}
\sqrt{(2Rr)^2 - l_I^2}\langle \nu_I, u_I\rangle - l_I|\langle \tau_I, u_I\rangle| &\geq R\sqrt{4(Rr)^2 - (\mathrm{diam}\, K)^2} - S \cdot \mathrm{diam}\, K \\
&\geq R\sqrt{3}(Rr) - S \cdot \mathrm{diam}\, K \\
&> R^2 r \\
&> 0.
\end{aligned}$$

So Lemma 33(ii) implies that f_r^I is Lipschitz continuous with Lipschitz constant

$$L := \frac{l_I}{\sqrt{(2Rr)^2 - l_I^2}\, \langle \nu_I, u_I\rangle - l_I|\langle \tau_I, u_I\rangle|}.$$

In order to check the assumptions of Lemma 34, we notice

$$|L\langle \tau_I, u_I\rangle| \leq \frac{\mathrm{diam}\, K}{R^2 r} S \leq \frac{1}{2}.$$

Further we let p_I and $p_I + l_I\tau_I$ denote the endpoints of I. Then $p_I + ru_I \in K + rB$, and considering scalar products with ν_I one gets $p_I + su_I \notin K + rB$ for all $s > r$.

Now $f_r^I(p_I) = r$ follows immediately from the definition of f_r^I and the same way we get $f_r^I(p_I + l_I\tau_I) = r$. Thus

$$f_r^I(p_I) = f_r^I(p_I + l_I\tau_I). \tag{32}$$

Hence by Lemma 34 F_r^I has a representation of the form $\{q_I + x\tau_I + \tilde{f}_r^I(x)\nu_I \mid x \in \operatorname{int}\tilde{I}\}$, where $q_I \in \mathbb{R}^2$ is a point, $\tilde{I} := [0, l_I]$ and $\tilde{f}_r^I : \tilde{I} \to \mathbb{R}$ is a Lipschitz continuous function with Lipschitz constant

$$\tilde{L} := \frac{L\langle \nu_I, u_I\rangle}{1 - L\langle \tau_I, u_I\rangle}.$$

Now

$$\begin{aligned}
\tilde{L} &\leq L \cdot \frac{\langle \nu_I, u_I\rangle}{1 - \frac{1}{2}} \\
&= \frac{l_I}{\sqrt{(2Rr)^2 - l_I^2}\,\langle \nu_I, u_I\rangle - l_I|\langle \tau_I, u_I\rangle|}\frac{\langle \nu_I, u_I\rangle}{\frac{1}{2}} \\
&= \frac{2l_I}{\sqrt{(2Rr)^2 - l_I^2} - l_I\frac{|\langle \tau_I, u_I\rangle|}{\langle \nu_I, u_I\rangle}} \\
&\leq \frac{2l_I}{\sqrt{(2Rr)^2 - (\operatorname{diam} K)^2} - \operatorname{diam} K\frac{S}{R}} \\
&\leq \frac{2l_I}{\sqrt{3}(Rr) - \operatorname{diam} K\frac{S}{R}} \\
&\leq \frac{2l_I}{Rr}.
\end{aligned}$$

So

$$\sqrt{1 + \tilde{L}^2} < \sqrt{1 + \tilde{L}^2 + \tfrac{1}{4}\tilde{L}^4} = 1 + \tfrac{1}{2}\tilde{L}^2 \leq 1 + 2\Big(\frac{l_I}{Rr}\Big)^2 \tag{33}$$

and

$$\tilde{L} \leq \frac{2l_I}{Rr} \leq \frac{R}{S} \leq 1. \tag{34}$$

According to Lemma 33(iii)

$$|h_B(n_B(K, z)) - h_B(\nu_I) + \langle u_I, \nu_I - n_B(K, z)\rangle| \leq c_I r^{-2}$$

holds for all $z \in F_r^I \setminus \operatorname{exo}_B(K)$, where

$$c_I := \frac{((1 + \frac{\|u_I\|}{4R})\operatorname{diam} K)^2}{2R}.$$

We put

$$g_I(x) := q_I + x\tau_I + \tilde{f}_r^I(x)\nu_I,\ x \in \tilde{I}. \tag{35}$$

Then $F_r^I = \{g_I(x) \mid x \in \operatorname{int} \tilde{I}\}$ and from Lemma 35 and (33) we get

$$\begin{aligned}
&\int_{F_r^I} h_B(n_B(K,z))\, d\mathcal{H}^1(z) - l_I h_B(\nu_I) \\
&\quad \le \sqrt{1+\tilde{L}^2} \cdot \int_{\tilde{I}} h_B(n_B(K, g_I(x)))\, dx - l_I h_B(\nu_I) \\
&\quad \le \big(1 + 2\big(\frac{l_I}{Rr}\big)^2\big) \cdot \int_{\tilde{I}} \big(h_B(\nu_I) - \langle u_I, \nu_I - n_B(K, g_I(x))\rangle + c_I r^{-2}\big)\ dx - l_I h_B(\nu_I) \\
&\quad = \big(1 + 2\big(\frac{l_I}{Rr}\big)^2\big) \cdot l_I h_B(\nu_I) - l_I h_B(\nu_I) + (1 + 2(\frac{l_I}{Rr})^2) \cdot l_I c_I r^{-2} \\
&\qquad\qquad - \big(1 + 2\big(\frac{l_I}{Rr}\big)^2\big) \cdot \int_{\tilde{I}} \langle u_I, \nu_I - n_B(K, g_I(x))\rangle\, dx \\
&\quad = 2\big(\frac{l_I}{Rr}\big)^2 \cdot l_I h_B(\nu_I) + \big(1 + 2\big(\frac{l_I}{Rr}\big)^2\big) \cdot l_I c_I r^{-2} \\
&\qquad\qquad - \big(1 + 2\big(\frac{l_I}{Rr}\big)^2\big) \cdot \int_{\tilde{I}} \langle u_I, \nu_I - n_B(K, g_I(x))\rangle\, dx. \qquad (36)
\end{aligned}$$

Now we are going to find an upper bound for the absolute value of the last integral. Since for $x, x' \in I$ we have $\|g_I(x) - g_I(x')\| \in [|x - x'|, \sqrt{1+\tilde{L}^2}|x - x'|]$, g_I is differentiable for a.a. $x \in \tilde{I}$ with

$$\|g_I'(x)\| \in \big[1, \sqrt{1+\tilde{L}^2}\ \big].$$

Hence for a.a. $x \in \tilde{I}$ we have

$$\big|\langle \nu_I, g_I'(x)\rangle \big(1 - \frac{1}{\|g_I'(x)\|}\big)\big| \le \tilde{L}\big(1 - \frac{1}{\sqrt{1+\tilde{L}^2}}\big) \qquad (37)$$

and

$$\big|\big\langle \tau_I, \frac{g_I'(x)}{\|g_I'(x)\|}\big\rangle\big| = \frac{1}{\|g_I'(x)\|} \ge \frac{1}{\sqrt{1+\tilde{L}^2}}. \qquad (38)$$

Since $d_B(K, g_I(\cdot))$ is constant, $g_I'(x)$ is perpendicular to $\nabla d_B(K,z)_{|z=g_I(x)}$ and hence by Lemma 21 to $n_B(K, g_I(x))$ for all $x \in \tilde{I}$, for which $d_B(K,\cdot)$ is differentiable in $g_I(x)$. The orthonormal bases $(n_B(K, g_I(x)), g_I'(x)/\|g_I'(x)\|)$ and (ν_I, τ_I) have the same orientation. Further we choose a vector v such that $(\frac{u_I}{\|u_I\|}, v)$ is a another orthonormal base with the same orientation. Then

$$\begin{aligned}
&\Big|\int_{\tilde{I}} \big\langle \frac{u_I}{\|u_I\|}, \nu_I - n_B(K, g_I(x))\big\rangle\, dx\Big| \\
&\qquad = \Big|\int_{\tilde{I}} \big\langle v, \tau_I - \frac{g_I'(x)}{\|g_I'(x)\|}\big\rangle\, dx\Big| \\
&\qquad = \Big|\langle \tau_I, v\rangle \int_{\tilde{I}} \big\langle \tau_I, \tau_I - \frac{g_I'(x)}{\|g_I'(x)\|}\big\rangle\, dx + \langle \nu_I, v\rangle \int_{\tilde{I}} \big\langle \nu_I, \tau_I - \frac{g_I'(x)}{\|g_I'(x)\|}\big\rangle\, dx\Big| \\
&\qquad \le \Big|\int_{\tilde{I}} \big\langle \tau_I, \tau_I - \frac{g_I'(x)}{\|g_I'(x)\|}\big\rangle\, dx\Big| + \Big|\int_{\tilde{I}} \langle \nu_I, -g_I'(x)\rangle\, dx\Big| \\
&\qquad\qquad + \Big|\int_{\tilde{I}} \big\langle \nu_I, g_I'(x)\big(1 - \frac{1}{\|g_I'(x)\|}\big)\big\rangle\, dx\Big| \\
&\qquad \le \int_{\tilde{I}} 1 - \frac{1}{\sqrt{1+\tilde{L}^2}}\, dx + \Big|\int_{\tilde{I}} \tfrac{d}{dx} \langle \nu_I, -g_I(x)\rangle\, dx\Big| + \int_{\tilde{I}} \tilde{L}\big(1 - \frac{1}{\sqrt{1+\tilde{L}^2}}\big)\, dx,
\end{aligned}$$

where the last inequality is due to (38) and (37). Since $x \mapsto \langle \nu_I, -g_I(x) \rangle$ is Lipschitz continuous and hence absolutely continuous, the fundamental theorem of calculus (Theorem 1) and (32) give

$$\int_{\tilde{I}} \tfrac{d}{dx} \langle \nu_I, -g_I(x) \rangle \, dx = \lim_{x \to l_I} \langle \nu_I, -g_I(x) \rangle - \lim_{x \to 0} \langle \nu_I, -g_I(x) \rangle = 0.$$

Hence the above sum equals

$$\begin{aligned}
& l_I\Big(1 - \frac{1}{\sqrt{1+\tilde{L}^2}}\Big) + 0 + l_I \tilde{L}\big(1 - \frac{1}{\sqrt{1+\tilde{L}^2}}\big) \\
& \leq l_I(1+\tilde{L})\big(\sqrt{1+\tilde{L}^2} - 1\big) \\
& \leq 2 l_I \big(1 + 2\big(\frac{l_I}{Rr}\big)^2 - 1\big) \\
& = \frac{4 l_I^3}{(Rr)^2},
\end{aligned}$$

where the last inequality is due to (34) and (33).
Hence (36) gives

$$\begin{aligned}
\int_{F_r^I} & h_B(n_B(K,z)) \, d\mathcal{H}^1(z) - l_I h_B(\nu_I) \\
& \leq 2\big(\frac{l_I}{Rr}\big)^2 \cdot l_I h_B(\nu_I) + \big(1 + 2\big(\frac{l_I}{Rr}\big)^2\big) \cdot l_I c_I r^{-2} \\
& \qquad + \big(1 + 2\big(\frac{l_I}{Rr}\big)^2\big) \cdot S \Big| \int_{\tilde{I}} \big\langle \frac{u_I}{\|u_I\|}, \nu_I - n_B(K, g_I(x)) \big\rangle \, dx \Big| \\
& \leq 2\big(\frac{l_I}{Rr}\big)^2 \cdot l_I h_B(\nu_I) + \big(1 + 2\big(\frac{l_I}{Rr}\big)^2\big) \cdot l_I \frac{((1 + \frac{\|u_I\|}{4R}) \operatorname{diam} K)^2}{2R} r^{-2} \\
& \qquad + \big(1 + 2\big(\frac{l_I}{Rr}\big)^2\big) \cdot S \frac{4 l_I^3}{(Rr)^2}.
\end{aligned}$$

So we get from (31)

$$\begin{aligned}
\int_{\operatorname{bd}(K+rB)} & h_B(n_B(K,z)) \, d\mathcal{H}^1(z) - \int_{\operatorname{bd}(L+rB)} h_B(n_B(L,z)) \, d\mathcal{H}^1(z) \\
& = \sum_{I \in \mathcal{I}} \int_{F_r^I} h_B(n_B(K,z)) \, d\mathcal{H}^1(z) - l_I h_B(\nu_I) \\
& \leq \sum_{I \in \mathcal{I}} 2\big(\frac{l_I}{Rr}\big)^2 \cdot l_I h_B(\nu_I) + \big(1 + 2\big(\frac{l_I}{Rr}\big)^2\big) \cdot l_I \frac{\big((1 + \frac{\|u_I\|}{4R}) \operatorname{diam} K\big)^2}{2R} r^{-2} \\
& \qquad + \big(1 + 2\big(\frac{l_I}{Rr}\big)^2\big) \cdot \frac{4 l_I^3 S}{(Rr)^2}.
\end{aligned}$$

Since from the assumption $r > \frac{2S \operatorname{diam} K}{R^2}$ one can conclude $\frac{l_I}{Rr} < \frac{1}{2}$ and hence $1+2(\frac{l_I}{Rr})^2 < 2$, this expression is smaller than

$$\begin{aligned}
&\sum_{I\in\mathcal{I}} 2\frac{l_I^3}{(Rr)^2}S + 2l_I\frac{(\frac{5S}{4R}\operatorname{diam} K)^2}{2R}\cdot r^{-2} + 8\frac{l_I^3 S}{(Rr)^2}\\
&\leq \left(10S\frac{\sum_{I\in\mathcal{I}} l_I^3}{R^2} + 2\frac{S^2(\operatorname{diam} K)^2\sum_{I\in\mathcal{I}} l_I}{R^3}\right)\cdot r^{-2}\\
&\leq \left(10S\frac{(\sum_{I\in\mathcal{I}} l_I)^3}{R^2} + 2\frac{S^2(\operatorname{diam} K)^2\sum_{I\in\mathcal{I}} l_I}{R^3}\right)\cdot r^{-2}\\
&\leq \left(10\frac{S(2\pi\operatorname{diam} K)^3}{R^2} + 2\frac{S^2(\operatorname{diam} K)^2\cdot 2\pi\operatorname{diam} K}{R^3}\right)\cdot r^{-2}\\
&\leq \left((10+\tfrac{S}{R})\frac{S(2\pi\operatorname{diam} K)^3}{R^2}\right)\cdot r^{-2},
\end{aligned}$$

where we obtain the inequality $\sum_{I\in\mathcal{I}} l_I \leq 2\pi\cdot\operatorname{diam} K$ as follows: The sets $I\in\mathcal{I}$ are pairwise disjoint subsets of $\operatorname{bd}\operatorname{conv} K$ and hence $\sum_{I\in\mathcal{I}} l_I \leq \mathcal{H}^1(\operatorname{bd}\operatorname{conv} K)$. By [18, p. 210] we get $\sum_{I\in\mathcal{I}} l_I \leq 2V_1(\operatorname{conv} K)$. Since $\operatorname{conv} K$ is contained in a ball of radius $\operatorname{diam} K$, the monotonicity of the intrinsic volumes (6) gives $\sum_{I\in\mathcal{I}} l_I \leq 2V_1(\operatorname{conv} K) \leq 2\pi\cdot\operatorname{diam} K$. Now the second inequality of the assertion is proven.

The proof of the first inequality is similar to that of the second one. We will only present the computations that are different.

Lemma 35 implies for $I\in\mathcal{I}$ that

$$\begin{aligned}
&\int_{F_r^I} h_B(n_B(K,z))\,d\mathcal{H}^1(z) - l_I h_B(\nu_I)\\
&\qquad\geq \int_{\tilde{I}} h_B(n_B(K,g_I(x)))\,dx - l_I h_B(\nu_I)\\
&\qquad\geq \int_{\tilde{I}} h_B(\nu_I) - \langle u_I, \nu_I - n_B(K,g_I(x))\rangle - c_I r^{-2}\,dx - l_I h_B(\nu_I)\\
&\qquad\geq -l_I c_I r^{-2} - \int_{\tilde{I}}\langle u_I, \nu_I - n_B(K,g_I(x))\rangle\,dx\\
&\qquad\geq -l_I\frac{((1+\frac{\|u_I\|}{4R})\operatorname{diam} K)^2}{2R}r^{-2} - \frac{4l_I^3 S}{(Rr)^2}
\end{aligned}$$

and hence

$$\begin{aligned}
&\int_{\mathrm{bd}(K+rB)} h_B(n_B(K,z))\,d\mathcal{H}^1(z) - \int_{\mathrm{bd}(L+rB)} h_B(n_B(L,z))\,d\mathcal{H}^1(z) \\
&\qquad = \sum_{I\in\mathcal{I}} \int_{F_r^I} h_B(n_B(K,z))\,d\mathcal{H}^1(z) - l_I h_B(\nu_I) \\
&\qquad \geq -\sum_{I\in\mathcal{I}} \left(l_I \frac{((1+\frac{\|u_I\|}{4R})\operatorname{diam} K)^2}{2R} r^{-2} + \frac{4 l_I^3 S}{(Rr)^2} \right) \\
&\qquad \geq -\left(\sum\nolimits_{I\in\mathcal{I}} l_I\right) \frac{((\frac{5S}{4R})\operatorname{diam} K)^2}{2R} r^{-2} - \frac{4S\cdot \sum_{I\in\mathcal{I}} l_I^3}{(Rr)^2} \\
&\qquad \geq -(2\pi\cdot\operatorname{diam} K) \frac{((\frac{5S}{4R})\operatorname{diam} K)^2}{2R} r^{-2} - \frac{4S\cdot(2\pi\cdot\operatorname{diam} K)^3}{(Rr)^2} \\
&\qquad \geq \Big[-(\tfrac{S}{R}+4) \frac{S(2\pi\cdot\operatorname{diam} K)^3}{R^2} \Big] \cdot r^{-2}
\end{aligned}$$

□

Now we obtain the following special case of Theorem 32(i):

Lemma 38. *Let $K \subseteq \mathbb{R}^2$ be a body and $L := \operatorname{conv} K$. Then for all $b \geq a \geq 2R^{-2}S \operatorname{diam} K$ we have*

$$\begin{aligned}
&|(V_2(K+bB) - V_2(L+bB)) - (V_2(K+aB) - V_2(L+aB))| \\
&\qquad\qquad \leq \int_a^b \left((10+\tfrac{S}{R}) \frac{S(2\pi \operatorname{diam} K)^3}{R^2} \right) \cdot s^{-2}\,ds. \quad (39)
\end{aligned}$$

Proof. By Theorem 20 and Lemma 21 we have

$$\begin{aligned}
&(V_2(K+bB) - V_2(L+bB)) - (V_2(K+aB) - V_2(L+aB)) \\
&\qquad = \int_a^b \left(\int_{\mathrm{bd}(K+sB)} h_B(n_B(K,z))\,d\mathcal{H}^1(z) - \int_{\mathrm{bd}(L+sB)} h_B(n_B(L,z))\,d\mathcal{H}^1(z) \right) ds
\end{aligned}$$

for all $b \geq a \geq 0$. Due to Lemma 36 this implies the assertion. □

Proof of Theorem 32. (i) Like in the proof of Theorem 22 we put $X^x := X \cap (x+\hat{Y})$ and $Z^x := \operatorname{conv} X^x$ for $x \in Y^\perp$ and we let $X|Y^\perp$ denote the image of X under the orthogonal projection onto $Y^\perp$.
Let $b > a \geq 1$. For fixed $x \in Y^\perp$ we can identify $\hat{Y}$ and $x+\hat{Y}$ at the same time with $\mathbb{R}^2$ in such a way that $RB^2 \subseteq Y$.
In each of the cases that $2R^{-2}S \operatorname{diam} X^x$ is greater than b, is in $[a,b]$, or is less than a, Lemma 38 and Theorem 18 imply

$$\begin{aligned}
&(V_2(X^x+bY) - V_2(Z^x+bY)) - (V_2(X^x+aY) - V_2(Z^x+aY)) \\
&\qquad \leq \int_a^b \mathbf{1}_{\{s>2R^{-2}S\operatorname{diam} X^x\}} \left((10+\tfrac{S}{R}) \frac{S(2\pi\operatorname{diam} X^x)^3}{R^2} \right) \cdot s^{-2}\,ds \\
&\qquad\quad + \int_a^b \mathbf{1}_{\{s\leq 2R^{-2}S\operatorname{diam} X^x\}} \tfrac{2}{s} (V_2(X^x+sY) + V_2(Z^x+sY))\,ds.
\end{aligned}$$

Now Fubini's Theorem gives

$$
\begin{aligned}
&\mathbb{E}[V_n(X+bY)-V_n(X_Y+bY)]-\mathbb{E}[V_n(X+aY)-V_n(X_Y+aY)]\\
&\quad=\mathbb{E}\int_{X|Y^\perp}(V_2(X^x+bY)-V_2(Z^x+bY))-(V_2(X^x+aY)-V_2(Z^x+aY))\,dx\\
&\quad\le\mathbb{E}\int_{X|Y^\perp}\left(\int_a^b \mathbf{1}_{\{s>2R^{-2}S\operatorname{diam}X^x\}}\left((10+\tfrac{S}{R})\frac{S(2\pi\operatorname{diam}X^x)^3}{R^2}\right)\cdot s^{-2}\,ds\right.\\
&\qquad\qquad\left.+\int_a^b \mathbf{1}_{\{s\le 2R^{-2}S\operatorname{diam}X^x\}}\tfrac{2}{s}(V_2(X^x+sY)+V_2(Z^x+sY))\,ds\right)dx\\
&\quad\le\mathbb{E}\,\kappa_{n-2}G^{n-2}\left(\int_a^b\left((10+\tfrac{S}{R})\frac{S(2\pi G)^3}{R^2}\right)\cdot s^{-2}\,ds\right.\\
&\qquad\qquad\left.+\int_a^b\frac{(2R^{-2}SG)^3}{s^3}\tfrac{2}{s}2\pi(G+sS)^2\,ds\right)\\
&\quad\le\int_a^b\mathbb{E}\left[\kappa_{n-2}G^{n-2}\cdot\left((11\tfrac{S}{R})\frac{S(2\pi G)^3}{R^2}\cdot s^{-2}+\frac{(2SG)^3}{R^6s^4}4\pi(2GsS)^2\right)\right]ds\\
&\quad\le 3200\kappa_{n-2}\int_a^b\mathbb{E}\left[\left(\frac{SG}{R^2}\right)^3G^nS^2\right]\cdot s^{-2}\,ds\\
&\quad=\frac{c}{a}-\frac{c}{b}.
\end{aligned}
$$

(ii) Since $r\mapsto\mathbb{E}[V_n(X+rY)-V_n(X_Y+rY)]$ is Lipschitz continuous on $[1,\infty)$ by part (i), we conclude, e.g. using Theorem 1, that this function is differentiable a.e. In those points, where the derivative exists, we get by an elementary computation from part (i) that its absolute value is less than c/r^2. □

5 Weighted parallel volumes and differentiability

In this section we apply the theorems from Sections 3 and 4 to the functions that map a non-negative real number r onto the real number $f_\mu(rK)$, where the functional f_μ is a certain generalisation of the Wills functional. In Theorem 39 we will show that such a function is infinitely differentiable in $r>0$, if f_μ fulfils strong regularity assumptions, e.g. if f_μ is the Wills functional. Then we will compute in Theorem 40 and Theorem 41 under weaker regularity assumptions the first derivative in $r=0$ and, if it exists, also the second derivative. The third and longest part of this section will be giving sufficient conditions for the existence of this second derivative. In Corollary 44 we give the derivatives from Theorem 40 and Theorem 41 in the special case, where f_μ is the Wills functional.
The results of this section answer a question of R.A. Vitale, who asked what the geometric meaning of the derivatives in Corollary 44 is.
A *signed measure* is a measure that may take negative values. For a precise introduction, see e.g. [2]. Here we always assume that it has finite total variation. The variation measure of a signed measure μ will be denoted by $|\mu|$.
The Wills functional is defined by

$$W:\mathcal{C}\to\mathbb{R},\ K\mapsto\mathbb{E}\,V_d(K+\Lambda B^d),\tag{40}$$

where Λ is an $\mathbb{R}_0^+$-valued random variable with distribution function $1 - e^{-\pi t^2}$. If K is convex, then the Wills functional of K equals the sum of its intrinsic volumes.
For a convex body $B \subseteq \mathbb{R}^d$ and a signed measure ρ on $\mathbb{R}_0^+$ with finite d-th moment we call

$$\mathcal{C} \to \mathbb{R},\, K \mapsto \int_{\mathbb{R}_0^+} V_d(K + \lambda B)\, d\rho(\lambda)$$

ρ-weighted B-parallel volume.
Finally, for a signed measure μ on $\mathcal{K}$ satisfying

$$\int_{\mathcal{K}} V_d(K + A)\, d|\mu|(A) < \infty, \quad K \in \mathcal{C}, \tag{41}$$

we put

$$f_\mu : \mathcal{C} \to \mathbb{R},\, K \mapsto \int_{\mathcal{K}} V_d(K + A)\, d\mu(A).$$

For further information on these functionals, see [12].

Theorem 39. *Let $B \subseteq \mathbb{R}^d$ be a convex body and $X \subseteq \mathbb{R}^d$ a random body satisfying $\mathbb{E}\, V_d(\operatorname{conv} X + xB) < \infty$ for all $x > 0$. Let ρ be a signed measure on $\mathbb{R}_0^+$, which is absolutely continuous and has density $f(\lambda) = \sum_{i=1}^n y_i e^{P_i(\lambda)}$ w.r.t. the Lebesgue measure, where the y_i are real numbers and the P_i are on $\mathbb{R}^+$ strictly monotonically decreasing polynomials for $i = 1, \ldots, n$, $n \in \mathbb{N}$. Then the map $r \mapsto \mathbb{E} \int V_d(rX + \lambda B)\, d\rho(\lambda)$ is infinitely differentiable in $r > 0$.*
In particular, this is true for $r \mapsto \mathbb{E}\, W(rX)$.

Proof. Fubini's theorem is valid for signed measures as well, but now its integrability assumptions have to be fulfilled w.r.t. the variation measures. Since $\mathbb{E}\, V_d(\operatorname{conv} X + xB) < \infty$ holds for all $x > 0$ and the form of f implies that all moments of ρ exist, we can apply Fubini's theorem and get

$$\begin{aligned}
\mathbb{E} \int_{\mathbb{R}_0^+} V_d(rX + \lambda B)\, d\rho(\lambda) &= r^d \int_{\mathbb{R}_0^+} \mathbb{E}\, V_d(X + \tfrac{\lambda}{r} B)\, d\rho(\lambda) \\
&= r^d \int_0^\infty \mathbb{E}\, V_d(X + \tfrac{\lambda}{r} B) f(\lambda)\, d\lambda \\
&= r^d \int_0^\infty \mathbb{E}\, V_d(X + xB) f(rx) r\, dx.
\end{aligned}$$

The integrand of the last integral is obviously infinitely differentiable for any $x \in \mathbb{R}_0^+$. The standard theorems for switching integral and differential hold for signed measures as well, where the integrability assumptions have to be fulfilled w.r.t. the variation measure. In order to check these integrability assumptions, observe that for $k \in \mathbb{N}$ there are numbers $c^k_{\alpha\beta;i}$, $\alpha, \beta \in \mathbb{N}$, $i \in \{1, \ldots, n\}$, of which all but finitely many are 0, such that

$$\tfrac{\partial^k}{\partial r^k} f(rx) r = \sum_{\alpha,\beta,i} y_i c^k_{\alpha\beta;i} r^\alpha x^\beta e^{P_i(rx)}.$$

Choose $R_0 \in (0, r)$ and $R_1 > r$ and put

$$h : \mathbb{R}_0^+ \to \mathbb{R},\, x \mapsto \mathbb{E} \sum_{i,\alpha,\beta} V_d(X + xB) |y_i c^k_{\alpha\beta;i}| R_1^\alpha x^\beta e^{P_i(R_0 x)}.$$

Since h is integrable (w.r.t. the Lebesgue measure on $\mathbb{R}_0^+$) and for all $s \in (R_0, R_1)$ and all $x \geq 0$ we have

$$h(x) \geq |\mathbb{E}\, V_d(X + xB) \sum_{\alpha,\beta,i} y_i c^k_{\alpha\beta;i} s^\alpha x^\beta e^{P_i(sx)}|,$$

an easy induction shows that $r \mapsto \mathbb{E} \int V_d(rX + \lambda B)\, d\rho(\lambda)$ is k times differentiable for any $k \in \mathbb{N}^+$. □

For a convex body $B \subseteq \mathbb{R}^d$ we let $S(B)$ denote the minimum of the radius of the circumsphere of B and 1 and we recall

$$R(B) := \sup\{\rho \in \mathbb{R}_0^+ \mid \rho B^d \text{ is summand of } B\}.$$

Theorem 40. *Let $X \subseteq \mathbb{R}^d$, $d > 1$, be a random body with $\mathbb{E}\,(\operatorname{diam} X)^d < \infty$ and μ a signed measure on $\mathcal{K}$, which is concentrated on the set of all convex bodies having interior points, and fulfils $\int_{\mathcal{K}} S(A)^{d-1}\, d|\mu|(A) < \infty$ and (41). Then $r \mapsto \mathbb{E} f_\mu(rX)$ is differentiable in $r = 0$ with*

$$\tfrac{d}{dr}\mathbb{E} f_\mu(rX)_{|r=0} = d \int_{\mathcal{K}} \mathbb{E} V(\operatorname{conv} X[1], A[d-1])\, d\mu(A). \tag{42}$$

The mixed volume is continuous as shown in the proof of [18, Theorem 5.1.6] and hence measurable.

Proof. We first show this theorem in the special case, where μ is the Dirac measure in a convex body B which has interior points and X is deterministic. Although this is an easy consequence of [15, Corollary 2(2)], we find it convenient to give a proof using the same methods as the proof of Theorem 41 below. By Theorem 22 for the map

$$\Delta : \mathbb{R}_0^+ \to \mathbb{R}_0^+,\ s \mapsto V_d(\operatorname{conv} X + sB) - V_d(X + sB),$$

there is a constant $c \in \mathbb{R}_0^+$ with $\Delta(s) < c \cdot s^{d-2}$ for $s \geq 1$. Hence

$$\begin{aligned} V_d(rX + B) &= r^d \cdot V_d(X + \tfrac{1}{r}B) \\ &= r^d \cdot \left(\sum_{j=0}^{d} \binom{d}{j} (\tfrac{1}{r})^{d-j} V(\operatorname{conv} X[j], B[d-j]) - \Delta(\tfrac{1}{r}) \right) \\ &= \sum_{j=0}^{d} \binom{d}{j} r^j V(\operatorname{conv} X[j], B[d-j]) - r^d \cdot \Delta(\tfrac{1}{r}). \end{aligned} \tag{43}$$

Since $0 \leq r^d \cdot \Delta(\frac{1}{r}) \leq c \cdot r^2$ for $r \leq 1$, we conclude

$$\tfrac{d}{dr}\, V_d(rX + B) = dV(\operatorname{conv} X[1], B[d-1]).$$

The integrability assumption, needed to generalize the statement from this special case to the general case, are fulfilled, since we have assumed $\int_{\mathcal{K}} S(A)^{d-1}\, d|\mu|(A) < \infty$ and $\mathbb{E}\,(\operatorname{diam} X)^d < \infty$ and an easy computation shows that for $r < 1$ we have

$$\frac{V_d(rX + B) - V_d(B)}{r} \leq \sum_{j=1}^{d} \binom{d}{j} (\operatorname{diam} X)^j (\operatorname{diam} B)^{d-j} \kappa_d. \quad \square$$

Theorem 41. *Let $X \subseteq \mathbb{R}^d$, $d > 1$, be a random body with $\mathbb{E}\,(\operatorname{diam} X)^{d+1} < \infty$ and μ a signed measure on $\mathcal{K}$, which fulfils the integrability assumptions (41) and*

$$\int_{\mathcal{K}} \frac{S(A)^d}{R(A)^3}\, d|\mu|(A) < \infty.$$

(i) Then

$$\mathbb{E} f_\mu(rX) = \sum_{j=0}^{2} \binom{d}{j} r^j \int_{\mathcal{K}} \mathbb{E}V(\operatorname{conv} X[j], A[d-j])\, d\mu(A) + O(r^3)$$

as $r \to 0$.

(ii) If the second derivative exists, then

$$\tfrac{d^2}{dr^2}\mathbb{E} f_\mu(rX)_{|r=0} = d(d-1) \int_{\mathcal{K}} \mathbb{E}V(\operatorname{conv} X[2], A[d-2])\, d\mu(A). \tag{44}$$

Proof. (i) Put $Z := \operatorname{conv} X$. By Theorem 26 for each convex body B, which contains a ball as summand, there is a map $\Delta_B : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ with

$$\mathbb{E}\, V_d(X + sB) = \mathbb{E}V_d(Z + sB) - \Delta_B(s)$$

such that

$$0 \le \Delta_B(s) < c \cdot \frac{S(B)^d}{R(B)^3} s^{d-3} \tag{45}$$

holds for all $s > 1$ with a constant $c \in \mathbb{R}_0^+$ that is independent of B, but depends on the distribution of X. Just like in (43) we get

$$\mathbb{E} f_\mu(rX) = \sum_{j=0}^{d} \binom{d}{j} r^j \int_{\mathcal{K}} \mathbb{E}V(Z[j], A[d-j])\, d\mu(A) - \int_{\mathcal{K}} r^d \Delta_A(\tfrac{1}{r})\, d\mu(A). \tag{46}$$

Moreover, for $r < 1$ we have

$$\left| \int_{\mathcal{K}} r^d \Delta_A(\tfrac{1}{r})\, d\mu(A) \right| \le \int_{\mathcal{K}} r^d \cdot c \cdot \frac{S(A)^d}{R(A)^3} (\tfrac{1}{r})^{d-3}\, d|\mu|(A) \le r^3 \cdot c \cdot \int_{\mathcal{K}} \frac{S(A)^d}{R(A)^3}\, d|\mu|(A). \tag{47}$$

So (i) is shown.
(ii) Assume, $\mathbb{E} f_\mu(rX)$ is twice differentiable with

$$\tfrac{d^2}{dr^2}\mathbb{E} f_\mu(rX)_{|r=0} \neq d(d-1) \int_{\mathcal{K}} \mathbb{E}V(\operatorname{conv} X[2], A[d-2])\, d\mu(A). \tag{48}$$

Then (46) yields that $h(r) := \int_{\mathcal{K}} r^d \Delta_A(\frac{1}{r})\, d\mu(A)$ is twice differentiable, too, and $h''(0) \neq 0$, w.l.o.g. $h''(0) > 0$. Hence for each $\gamma \in \mathbb{R}^+$ there is $\epsilon \in (0, 1)$ with $h'(r) > \gamma r^2$ for all $r \in (0, \epsilon)$. Putting $\gamma := 3c \int_{\mathcal{K}} S(A)^d / R(A)^3\, d|\mu|(A)$ and integrating this over $(0, \epsilon)$, we get $h(\epsilon) > \epsilon^3 c \int_{\mathcal{K}} S(A)^d / R(A)^3\, d|\mu|(A)$, which contradicts (47). □

We will now provide sufficient conditions for the existence of the second derivative in (44). First we will show that this second derivative always exists in the planar case, if we modify the notion of the derivative appropriately.
We put $R'(B) := \min\{R(B), 1\}$ for $B \in \mathcal{K}$.

Theorem 42. *Let $X \subseteq \mathbb{R}^2$ be a random body with $\mathbb{E}\,(\operatorname{diam} X)^5 < \infty$ and μ a signed measure on $\mathcal{K}$, which is concentrated on the set of all strictly convex bodies and fulfils the integrability conditions (41) and*

$$\int_{\mathcal{K}} \frac{S(A)^5}{R'(A)^6}\, d|\mu|(A) < \infty.$$

Then $h_1(r) := \frac{d}{dr}\mathbb{E}f_\mu(rX)$ is defined for a.a. $r \in \mathbb{R}^+$ and we have

$$\lim_{r\to 0} \frac{h_1(r) - h_1(0)}{r} = 2\,\mathbb{E}V_2(\operatorname{conv} X) \cdot \mu(\mathcal{K}).$$

Proof. We put $Z := \operatorname{conv} X$. For a strictly convex body $B \subseteq \mathbb{R}^2$ we put $\Delta_B(s) := \mathbb{E}[V_2(Z + sB) - V_2(X + sB)]$ for all $s > 0$. If B has moreover a ball as summand, then Theorem 32 implies that there is a constant $\tilde{c} \in \mathbb{R}_0^+$, which is independent of B but depends on the distribution of X, such that

$$|\Delta_B(b) - \Delta_B(a)| < \tilde{c} \cdot \frac{S(B)^5}{R'(B)^6}\left(\tfrac{1}{a} - \tfrac{1}{b}\right) \tag{49}$$

for all $b > a \geq 1$.
Now we will get bounds for the derivative of the right-hand side of (46). For $s < r < 1$ and a strictly convex body $B \subseteq \mathbb{R}^2$ with a ball as summand we have according to (45) and (49)

$$\begin{aligned}
|r^2\Delta_B(\tfrac{1}{r}) - s^2\Delta_B(\tfrac{1}{s})| &\leq |r^2\Delta_B(\tfrac{1}{r}) - r^2\Delta_B(\tfrac{1}{s})| + |r^2\Delta_B(\tfrac{1}{s}) - s^2\Delta_B(\tfrac{1}{s})| \\
&\leq r^2\tilde{c} \cdot \frac{S(B)^5}{R'(B)^6}\left(\frac{1}{1/r} - \frac{1}{1/s}\right) + (r^2 - s^2) \cdot c\frac{S(B)^2}{R(B)^3} \cdot (\tfrac{1}{r})^{-1} \\
&\leq r^2\tilde{c} \cdot \frac{S(B)^5}{R'(B)^6}(r - s) + (r^2 - s^2) \cdot c\frac{S(B)^5}{R'(B)^6} \cdot r \\
&\leq (2r^3 - rs^2 - r^2 s) \cdot \max\{c, \tilde{c}\} \cdot \frac{S(B)^5}{R'(B)^6}.
\end{aligned}$$

Hence

$$|\tfrac{d}{dr}\mathbb{E}\int_{\mathcal{K}} r^2\Delta_A(\tfrac{1}{r})\, d\mu(A)| \leq 3r^2 \cdot \max\{c, \tilde{c}\} \cdot \int_{\mathcal{K}} \frac{S(A)^5}{R'(A)^6}\, d\mu(A).$$

Together with (42), (46) and Fubini's theorem this gives

$$\begin{aligned}
&\lim_{r\to 0}\frac{h_1(r) - h_1(0)}{r} \\
&\quad = \lim_{r\to 0}\frac{\frac{d}{dr}\mathbb{E}\int_{\mathcal{K}} V_2(rX + A)\, d\mu(A) - 2\int_{\mathcal{K}} \mathbb{E}V(Z[1], A[1])\, d\mu(A)}{r} \\
&\quad = \lim_{r\to 0}\frac{\sum_{j=1}^{2}\binom{2}{j} j r^{j-1}\int_{\mathcal{K}} \mathbb{E}V(Z[j], A[2-j])\, d\mu(A) - 2\int_{\mathcal{K}} \mathbb{E}V(Z[1], A[1])\, d\mu(A)}{r} \\
&\qquad\qquad - \lim_{r\to 0}\frac{\frac{d}{dr}\int_{\mathcal{K}} r^2\Delta_A(\frac{1}{r})\, d\mu(A)}{r} \\
&\quad = 2\int_{\mathcal{K}} \mathbb{E}V(Z[2], A[0])\, d\mu(A) - 0 \\
&\quad = 2\,\mathbb{E}V_2(Z) \cdot \mu(\mathcal{K}). \qquad \square
\end{aligned}$$

Now we will show that the second derivative in (44) exists, if f_μ is a weighted parallel volume fulfilling some regularity assumptions.

Theorem 43. *Let $X \subseteq \mathbb{R}^d$, $d > 1$, be a random body with $\mathbb{E}(\operatorname{diam} X)^{d+1} < \infty$ and $B \subseteq \mathbb{R}^d$ a convex body with a summand RB^d, $R > 0$. Let ρ be a signed measure on $\mathbb{R}_0^+$, which has finite d-th moment and is absolutely continouos w.r.t. the Lebesgue measure and has a differentiable density g. Assume, there is a constant $A > 0$ such that for all $x \in \mathbb{R}^+$ we have*

$$|g(x)| \leq \frac{A}{x^{d-1}}, \qquad |g(x)| \leq A, \qquad |g'(x)| \leq \frac{A}{x^d}, \qquad |g'(x)| \leq \frac{A}{x}.$$

Then the map $r \mapsto \mathbb{E} \int V_d(rX + \lambda B)\, d\rho(\lambda)$ is twice differentiable in $r = 0$ with

$$\tfrac{d^2}{dr^2}\mathbb{E} \int V_d(rX + \lambda B)\, d\rho(\lambda) = d(d-1)\mu_{d-2}\mathbb{E}V(\operatorname{conv} X[2], B[d-2]),$$

where μ_{d-2} is the $(d-2)$-th moment of ρ.

Proof. We have to distinguish cases w.r.t. the dimension. First we consider the case $d \geq 3$. We put again $Z := \operatorname{conv} X$ and $\Delta_A(s) := \mathbb{E}[V_d(Z + sA) - V_d(X + sA)]$ for $s > 0$ and convex bodies A. For $s, \lambda \in \mathbb{R}^+$ we have

$$\Delta_{\lambda B}(s) = \mathbb{E}[V_d(Z + s\lambda B) - V_d(X + s\lambda B)] = \Delta_B(s\lambda).$$

By Theorem 26 there is a constant c with $\Delta(s) < c \cdot s^{d-3}$ for all $s \geq 1$, where $\Delta := \Delta_B$. Hence there is a constant c_1 with $\Delta(s) < c \cdot s^{d-3} + c_1$ for all $s \geq 0$.
We will compute the derivative of $h(r) := \int_{\mathbb{R}_0^+} r^d \Delta(\frac{\lambda}{r})\, d\rho(\lambda)$ in a point $r \geq 0$. We have

$$h(r) = \int_0^\infty r^d \Delta(\tfrac{\lambda}{r}) g(\lambda)\, d\lambda = \int_0^\infty \Delta(x) g(rx) r^{d+1}\, dx.$$

Now we will check the integrability conditions needed in order to differentiate this integral pointwise. Let $R_1 > 1$. We abbreviate $a \wedge b := \min\{a, b\}$. For any $r \in [0, R_1]$ we have

$$\begin{aligned}
&|\tfrac{d}{dr}\, \Delta(x) g(rx) r^{d+1}| \\
&\quad = \ |\Delta(x) g'(rx) x r^{d+1} + (d+1)\Delta(x) g(rx) r^d| \\
&\quad \leq \ (cx^{d-3} + c_1)\Big[\frac{A}{(rx)^d} \wedge \frac{A}{rx}\Big] x r^{d+1} \ + \ (d+1)(cx^{d-3} + c_1)\Big[\frac{A}{(rx)^{d-1}} \wedge A\Big] r^d \\
&\quad = \ (1 + d + 1)(cx^{d-3} + c_1)\Big[\frac{A}{(rx)^{d-1}} \wedge A\Big] r^d \\
&\quad \leq \ (d+2)[(cx^{-2} + c_1 x^{1-d}) A R_1 \wedge (cx^{d-3} + c_1) A R_1^d].
\end{aligned}$$

Moreover,

$$\int_0^\infty [(cx^{-2} + c_1 x^{1-d}) A R_1 \wedge (cx^{d-3} + c_1) A R_1^d]\, dx < \infty,$$

since the integrand is of order x^0 for $x \to 0$ and of order x^{-2} for $x \to \infty$. Observe the difference between the situation here and the situation in Theorem 39: In Theorem 39 we wanted to switch differential and integral in $\frac{d}{dr} \int_0^\infty \mathbb{E}\, V_d(X + xB) g(rx) r^{d+1}\, dx$. However, the integrability assumption was only fufilled for $r > 0$ and not for $r = 0$. Now we have

replaced $\mathbb{E}\, V_d(X + xB)$ by the smaller value $\Delta(x)$ and whence the integrability condition is now fulfilled for $r = 0$, too.
For $r \in [0, 1]$ we have further

$$
\begin{aligned}
|h'(r)| &= \left| \int_0^\infty \tfrac{d}{dr}\, \Delta(x) g(rx) r^{d+1}\, dx \right| \\
&= \left| \int_0^\infty \Delta(x)(g'(rx) x r^{d+1} + (d+1) g(rx) r^d)\, dx \right| \\
&\leq \int_0^\infty r^{d-1} \Delta(\tfrac{\lambda}{r}) \left(|g'(\lambda)|\lambda + (d+1)|g(\lambda)| \right)\, d\lambda \\
&\leq \int_0^\infty r^{d-1} \Delta(\tfrac{\lambda}{r}) \left(\Big[\frac{A}{\lambda^d} \wedge \frac{A}{\lambda}\Big]\lambda + (d+1)\Big[\frac{A}{\lambda^{d-1}} \wedge A\Big] \right)\, d\lambda \\
&= \int_0^\infty r^{d-1} \Delta(\tfrac{\lambda}{r})(d+2)\Big[\frac{A}{\lambda^{d-1}} \wedge A\Big]\, d\lambda \\
&\leq \int_0^\infty r^{d-1} (c(\tfrac{\lambda}{r})^{d-3} + c_1)(d+2)\Big[\frac{A}{\lambda^{d-1}} \wedge A\Big]\, d\lambda \\
&\leq r^2 \int_0^\infty (c\lambda^{d-3} + c_1)(d+2)\Big[\frac{A}{\lambda^{d-1}} \wedge A\Big]\, d\lambda.
\end{aligned}
$$

Since the integrand in the last line is of order λ^0 for $\lambda \to 0$ and of order λ^{-2} for $\lambda \to \infty$, the integral is finite. Hence $h''(0) = 0$ and by (46) this shows the statement.

Now we examine the case $d = 2$. Let Δ, c and h be defined as above. Then $\Delta(s) < cs^{-1}$ holds for all $s > 0$. Since Δ is bounded on compact intervals, there is $c' \in \mathbb{R}_0^+$ such that $\Delta(s) < c'$ for all $s > 0$.
Again we can compute the derivative of h by pointwise differentiation. However, we have to find a new way of checking the integrability conditions. Let $R_1 > 1$ and $r \in [0, R_1]$. Then

$$
\begin{aligned}
|\tfrac{d}{dr}\, \Delta(x) g(rx) r^3| &= |\Delta(x) g'(rx) x r^3 + 3\Delta(x) g(rx) r^2| \\
&\leq [cx^{-1} \wedge c'] \cdot \Big[\frac{A}{(rx)^2} \wedge \frac{A}{rx}\Big] x r^3 \ + \ 3[cx^{-1} \wedge c'] \cdot \Big[\frac{A}{rx} \wedge A\Big] r^2 \\
&= (1+3)[cx^{-1} \wedge c'] \cdot \Big[\frac{A}{rx} \wedge A\Big] r^2 \\
&\leq 4A[cx^{-1} \wedge c'] \cdot \Big[\frac{r}{x} \wedge r^2\Big] \\
&\leq 4A[cx^{-1} \wedge c'] \cdot \Big[\frac{R_1}{x} \wedge R_1^2\Big].
\end{aligned}
$$

Further

$$
\int_0^\infty 4A[cx^{-1} \wedge c'] \cdot \Big[\frac{R_1}{x} \wedge R_1^2\Big]\, dx \leq 4AR_1^2 \int_0^\infty \Big[\frac{c}{x^2} \wedge c'\Big]\, dx < \infty.
$$

Hence we can change differential and integral and for $r \in [0, 1]$ we obtain in a similar way as above

$$\begin{aligned}
|h'(r)| &\leq \int_0^\infty r\cdot\Delta(\tfrac{\lambda}{r})\left(\Big[\frac{A}{\lambda^2}\wedge\frac{A}{\lambda}\Big]\lambda+3\Big[\frac{A}{\lambda}\wedge A\Big]\right)d\lambda\\
&\leq \int_0^\infty r\cdot\Big[c(\tfrac{\lambda}{r})^{-1}\wedge c_1\Big]\cdot 4\Big[\frac{A}{\lambda}\wedge A\Big]\,d\lambda\\
&\leq 4r\int_0^\infty\Big[\frac{crA}{\lambda^2}\wedge c_1A\Big]\,d\lambda\\
&= 4r\int_0^{\sqrt{\frac{crA}{c_1A}}} c_1A\,d\lambda+4r\int_{\sqrt{\frac{crA}{c_1A}}}^\infty\frac{crA}{\lambda^2}\,d\lambda\\
&= 4r\sqrt{\frac{cr}{c_1}}\cdot c_1A+4r\frac{crA}{\sqrt{cr/c_1}}\\
&= 8r\sqrt{r}\cdot A\sqrt{c\cdot c_1}.
\end{aligned}$$

Just like in the first case, this shows the assertion. □

Now we will reformulate the theorems of this section in the special case, where the functional f_μ is the Wills functional.

Corollary 44. *Let $X\subseteq\mathbb{R}^d$, $d\geq 2$, be a random body with $\mathbb{E}(\operatorname{diam}X)^{d+1}<\infty$. Then $r\mapsto\mathbb{E}\,W(rX)$ is twice differentiable in $r=0$ and we have*

$$\tfrac{d}{dr}\mathbb{E}\,W(rX)_{|r=0}=\mathbb{E}\,V_1(\operatorname{conv}X)$$

and

$$\tfrac{d^2}{dr^2}\mathbb{E}\,W(rX)_{|r=0}=2\cdot\mathbb{E}\,V_2(\operatorname{conv}X).$$

Proof. A straight-forward computation shows that for the random variable Λ from the definition (40) of W we have

$$\kappa_{d-1}\mathbb{E}\,\Lambda^{d-1}=\kappa_{d-2}\mathbb{E}\,\Lambda^{d-2}=1.$$

Now Theorem 40 yields

$$\begin{aligned}
\tfrac{d}{dr}\mathbb{E}\,W(rX)_{|r=0} &= \binom{d}{1}\mathbb{E}\,\Lambda^{d-1}\cdot\mathbb{E}\,V(\operatorname{conv}X[1],B^d[d-1])\\
&= \kappa_{d-1}\mathbb{E}\,\Lambda^{d-1}\cdot\mathbb{E}\,V_1(\operatorname{conv}X)\\
&= \mathbb{E}\,V_1(\operatorname{conv}X).
\end{aligned}$$

We will now show that the Lebesgue density $g(x)=2\pi x\cdot e^{-\pi x^2}$ of Λ fulfils the assumptions of Theorem 43. We have $g'(x)=2\pi\cdot e^{-\pi x^2}-4\pi^2x^2\cdot e^{-\pi x^2}$ and hence $|g'(x)|\leq(2\pi+4\pi^2x^2)\cdot e^{-\pi x^2}$. Moreover,

$$\lim_{x\to 0}2\pi x\cdot e^{-\pi x^2}=0$$
$$\lim_{x\to\infty}2\pi x\cdot e^{-\pi x^2}\cdot x^{d-1}=0$$
$$\lim_{x\to 0}(2\pi+4\pi^2x^2)\cdot e^{-\pi x^2}=2\pi$$
$$\lim_{x\to\infty}(2\pi+4\pi^2x^2)\cdot e^{-\pi x^2}\cdot x^d=0$$

Hence we can apply Theorem 43 and the second assertion ensues just like the first. $\square$

Considering Example 29 it should not be to surprising that formulas analogue to the formulas presented in this section for the third derivative do not hold. However, we find it worthy to present an example explicitely showing that even $\frac{d^3}{dr^3}W(rK)_{|r=0} = 6V_3(\operatorname{conv} K)$ does not hold in general.

Example 45. Let $K \subseteq \mathbb{R}^3$ be a body, whose parallel volume is a polynomial, $V_3(K + sB^3) = \sum_{i=0}^{3} c_i s^i$, say, and for which $V_3(K) \neq V_3(\operatorname{conv} K)$. Such bodies exist, as shown in [7, section 4]. Let Λ denote again the random variable with distribution function $1 - e^{-\pi t^2}$, $t \geq 0$. Then

$$\begin{aligned} W(rK) &= r^3\, \mathbb{E}\, V_3\big(K + \frac{\Lambda}{r} B^3\big) \\ &= r^3 \mathbb{E} \sum_{i=0}^{3} c_i \big(\frac{\Lambda}{r}\big)^i \\ &= \sum_{i=0}^{3} c_i \mathbb{E}\Lambda^i\, r^{3-i}. \end{aligned}$$

Thus

$$\tfrac{d^3}{dr^3} W(rK)_{|r=0} = \tfrac{d^3}{dr^3} \sum_{i=0}^{3} c_i \mathbb{E}\Lambda^i\, r^{3-i}{}_{|r=0} = 6c_0 = 6V_3(K + 0B^3) = 6V_3(K).$$

6 Stochastic applications

In this section we will apply the results from the previous sections to Wiener sausages, Boolean models and Gaussian random variables. In Corollary 46 we obtain a third-order expansion of the expected volume of a Wiener sausage as the time tends to zero. In Theorem 47 we show that the failure rate of the contact distribution of a Boolean model does not change too much, if each grain is replaced by its convex hull and Theorem 49 is a limit theorem including the asymptotic speed of the convergence about the contact distribution of a Boolean model as the intensity tends to zero. Then we will present some statements that finally lead to the formulae in Theorem 55 that express the first and second intrinsic volume of the convex hull of a body as an expected value of certain geometric functionals of this body evaluated at a standard Gaussian random variable.

The parallel body of a Brownian path is called Wiener sausage. While there are many papers dealing with the asymptotic behavior of the volume of the Wiener sausage as the time tends to infinity (see [11] and the literature cited therein), [14] seems to be the only one dealing with its asymptotics as the time tends to 0. There it was shown that

$$\mathbb{E}\, V_d(S_t + rB^d) = \kappa_d r^d + \frac{d\sqrt{2}\kappa_d}{\sqrt{\pi}} r^{d-1}\sqrt{t} + o(\sqrt{t}) = \frac{\pi^{d/2}}{\Gamma(\frac{d}{2}+1)} r^d + \frac{2\sqrt{2}\pi^{(d-1)/2}}{\Gamma(\frac{d}{2})} r^{d-1}\sqrt{t} + o(\sqrt{t}),$$

where $S_t \subseteq \mathbb{R}^d$ denotes a Brownian path up to time t. Now putting together Theorem 41 and [14, Prop. 1.4] we obtain:

Corollary 46. *For $r \geq 0$ we have as $t \to 0$*

$$\begin{aligned}\mathbb{E}V_d(S_t + rB^d) &= \kappa_d r^d + \frac{d\sqrt{2}\kappa_d}{\sqrt{\pi}} r^{d-1}\sqrt{t} + \frac{(d-1)\kappa_{d-2}\pi}{2} r^{d-2} t + O(t^{3/2}) \\ &= \frac{\pi^{d/2}}{\Gamma(\frac{d}{2}+1)} r^d + \frac{2\sqrt{2}\pi^{(d-1)/2}}{\Gamma(\frac{d}{2})} r^{d-1}\sqrt{t} + \frac{(d-1)\pi^{d/2}}{2\Gamma(\frac{d}{2})} r^{d-2} t + O(t^{3/2}).\end{aligned}$$

Now we turn to the contact distributions of Boolean models. For the introduction of the notions of a Boolean model and the contact distribution, see [19, sections 4.3 and 2.4]. Here we consider only stationary Boolean models and assume that their grain distributions are defined on the set $\mathcal{C}_0$ of centered bodies (see [19, section 4.1]). The contact distribution of a stationary Boolean model Z in $\mathbb{R}^d$ with intensity γ and grain distribution $\mathbb{Q}$ is

$$H_B^Z(r) = 1 - \exp\Big(-\gamma \int_{\mathcal{C}_0} V_d(A + rB^*) - V_d(A)\, d\mathbb{Q}(A)\Big), \quad r \geq 0, \tag{50}$$

(see [19, Theorem 9.1.1]), where $B^* := \{-x \mid x \in B\}$ for $B \subseteq \mathbb{R}^d$.
The *failure rate* of a reell-valued, absolutely continuous distribution with distribution function F and density f is

$$\lambda : \mathbb{R} \to \mathbb{R}_0^+,\ t \mapsto \begin{cases} \frac{f(t)}{1-F(t)} & \text{if } F(t) < 1, \\ 0 & \text{if } F(t) = 1. \end{cases}$$

Theorem 47. *Let $B \subseteq \mathbb{R}^2$ be a convex body with a ball as summand and $0 \in \operatorname{int} B$. Further let Z be a stationary Boolean model in $\mathbb{R}^2$ induced by the marked point process $\sum \delta_{(X_i,Z_i)}$, that is $Z = \bigcup(X_i + Z_i)$. Assume that the typical grain Z_0 of Z fulfils $\mathbb{E}(\operatorname{diam} Z_0)^5 < \infty$. Then $\bar{Z} := \bigcup \operatorname{conv}(X_i + Z_i)$ is a stationary Boolean model, too. Let λ and $\bar{\lambda}$ denote the failure rates of the contact distributions of Z and $\bar{Z}$ w.r.t. B. Then there is a constant $C > 0$ such that for a.a. sufficiently large r we have*

$$|\lambda(r) - \bar{\lambda}(r)| < C \cdot r^{-2}.$$

Theorem 47 is an immediate consequence of Theorem 32(ii) and Lemma 48.

Lemma 48. *Let Z be a stationary Boolean model in $\mathbb{R}^d$ with intensity γ and grain distribution $\mathbb{Q}$ and let $B \subseteq \mathbb{R}^d$ be a convex body with $0 \in \operatorname{int} B$. Then the failure rate of the contact distribution of Z w.r.t. B is for a.a. $r \geq 0$*

$$\lambda(r) = \gamma \cdot \tfrac{\partial}{\partial t} \int_{\mathcal{C}_0} V_d(A + tB^*)\, d\mathbb{Q}(A)_{|t=r}.$$

Proof. First we show that any map $f : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ with

$$f(r) = \exp\Big(-\gamma \int_{\mathcal{C}_0} V_d(A + rB^*) - V_d(A)\, d\mathbb{Q}(A)\Big) \cdot \gamma \cdot \tfrac{\partial}{\partial t} \int_{\mathcal{C}_0} V_d(A + tB^*)\, d\mathbb{Q}(A)_{|t=r}$$

for any $r \geq 0$, in which the derivative exists, is a density of the contact distribution. Let $R_2 > R_1 \geq 0$. Let $r_1, r_2 \in [R_1, R_2]$ such that $r_1 \leq r_2$. Then by Theorem 18 we have

$$\begin{aligned}\int_{\mathcal{C}_0} V_d(A + r_2B^*) - V_d(A + r_1B^*)\, d\mathbb{Q}(A) &\leq \int_{\mathcal{C}_0}\int_{r_1}^{r_2} \tfrac{d}{s}(V_d(A + sB^*) - V_d(A))\, ds\, d\mathbb{Q}(A) \\ &\leq (r_2 - r_1) \cdot \frac{d}{R_1} \int_{\mathcal{C}_0} V_d(A + R_2B^*) - V_d(A)\, d\mathbb{Q}(A).\end{aligned}$$

So $t \mapsto \int_{\mathcal{C}_0} V_d(A + tB^*) \, d\mathbb{Q}(A)$ is Lipschitz continuous on $[R_1, R_2]$. Since $t \mapsto e^{-\gamma t}$ is Lipschitz continuous on $\mathbb{R}_0^+$, H_B^Z is Lipschitz continuous on $[R_1, R_2]$. Because R_1 and R_2 were arbitrary, Theorem 1 implies that f is a density of H_B^Z.
Now the assertion is immediate from the definition of the failure rate. □

In the next theorem we will examine the asymptotic behavior of the contact distribution of a Boolean model as the intensity tends to zero.

Theorem 49. *Let $Z^{(r)}$, $r \in \mathbb{R}^+$, be stationary Boolean models in $\mathbb{R}^d$ with intensity r^d and a typical grain Z_0 that is independent of r and fulfils $\mathbb{E}(\operatorname{diam} Z_0)^{d+1} < \infty$. Let $B \subseteq \mathbb{R}^d$ be a convex body with $0 \in \operatorname{int} B$. Let D be an $\mathbb{R}_0^+$-valued random variable with distribution function $1 - \exp(-t^d V_d(B))$. Then we have*

$$r \cdot d_B(Z^{(r)}, 0) \overset{r \to 0}{\longrightarrow} D$$

in distribution. More precisely for $t \geq 0$ we have

$$\lim_{r \to 0} \frac{\mathbb{P}(r \cdot d_B(Z^{(r)}, 0) \leq t) - (1 - e^{-t^d V_d(B)})}{r} = e^{-t^d V_d(B)} t^{d-1} \mathbb{E} V(\operatorname{conv} Z_0[1], B^*[d-1]).$$

Proof. For a convex body $A \subseteq \mathbb{R}^d$ we let δ_A denote the Dirac measure on $\mathcal{K}$ in A. From (50) and Theorem 40 we get

$$\begin{aligned}
&\lim_{r \to 0} \frac{\mathbb{P}(r \cdot d_B(Z^{(r)}, 0) \leq t) - (1 - e^{-t^d V_d(B)})}{r} \\
&= \lim_{r \to 0} \frac{(1 - e^{-r^d \mathbb{E} V_d(Z_0 + \frac{t}{r} B^*)}) - (1 - e^{-t^d V_d(B)})}{r} \\
&= \lim_{r \to 0} \frac{e^{-V_d(tB^*)} - e^{-\mathbb{E} V_d(rZ_0 + tB^*)}}{r} \\
&= -\tfrac{d}{dr} \, e^{-\mathbb{E} V_d(rZ_0 + tB^*)}{}_{|r=0} \\
&= -e^{-V_d(tB^*)} \cdot \left(-\tfrac{d}{dr} \, \mathbb{E} \int_{\mathcal{K}} V_d(rZ_0 + A) \, d\delta_{tB*}(A)_{|r=0} \right) \\
&= e^{-t^d V_d(B)} \cdot \left(\mathbb{E} \int_{\mathcal{K}} V(\operatorname{conv} Z_0[1], A[d-1]) \, d\delta_{tB^*}(A) \right) \\
&= e^{-t^d V_d(B)} t^{d-1} \mathbb{E} V(\operatorname{conv} Z_0[1], B^*[d-1]). \qquad \square
\end{aligned}$$

In the third part of this section we give a new proof for formulae that use Gaussian random variables in order to compute the first and second intrinsic volume the convex hull of a body. We do this by finding expressions for the first and second derivative of $W(rK)$ involving Gaussian random variables and comparing these expressions to the ones from Corollary 44.
Vitale [24] derived the following representation of the Wills functional.

Theorem 50. *Let Z be a standard-normal distributed random vector in $\mathbb{R}^d$ and $K \subseteq \mathbb{R}^d$ compact. Then*

$$W(K) = \mathbb{E} \exp(\max\{\langle a, Z \rangle - \tfrac{\|a\|^2}{2} \mid \tfrac{a}{\sqrt{2\pi}} \in K\}).$$

Now we let $A \subseteq \mathbb{R}^d$ denote a fixed finite set. For $r \in \mathbb{R}_0^+$ and $z \in \mathbb{R}^d$ we let $a_z^r \in \sqrt{2\pi}A$ denote a point that satisfies

$$\langle a_z^r, z\rangle - \tfrac{r}{2}\|a_z^r\|^2 = \max\{\langle a, z\rangle - \tfrac{r}{2}\|a\|^2 \mid \tfrac{a}{\sqrt{2\pi}} \in A\}$$

in such a way that $\mathbb{R}^d \to \mathbb{R}^d$, $z \mapsto a_z^r$ is measurable. We have to argue that such a choice is possible. We abbreviate $f_z(a) := \langle a, z\rangle - \frac{r}{2}\|a\|^2$ and $\tilde{A} := \{a \in \mathbb{R}^d \mid \frac{a}{\sqrt{2\pi}} \in A\}$. Now

$$\beta : \mathbb{R}^d \mapsto \mathcal{F}, z \mapsto \operatorname{argmax}_{a\in\tilde{A}} f_z(a) := \{a \in \tilde{A} \mid f_z(a) = \max\{f_z(b) \mid b \in \tilde{A}\}\}$$

is upper semicontinuous. Since the lower tangent point (see [19, p. 110]) can be shown to be measurable by semicontinuity arguments, it is possible to select one point from $\beta(Z)$ in a measurable way.
From now on Z is a standard-normal distributed random vector in $\mathbb{R}^d$.

Theorem 51. *With these denominations we have for $r \geq 0$*

$$\tfrac{\partial}{\partial r} W(rA) = \mathbb{E}\exp\left(\langle ra_Z^r, Z\rangle - \tfrac{1}{2}\|ra_Z^r\|^2\right) \cdot (\langle a_Z^r, Z\rangle - r\|a_Z^r\|^2) \tag{51}$$

and

$$\frac{\partial^2}{\partial r^2} W(rA) = \mathbb{E}\ \exp\left(\langle ra_Z^r, Z\rangle - \tfrac{1}{2}\|ra_Z^r\|^2\right) \cdot [(\langle a_Z^r, Z\rangle - r\|a_Z^r\|^2)^2 - \|a_Z^r\|^2]. \tag{52}$$

Proof. Let $r \in \mathbb{R}_0^+$. Then for two different points $a, a' \in \sqrt{2\pi}A$ we have

$$\langle a, Z\rangle - \tfrac{r}{2}\|a\|^2 \neq \langle a', Z\rangle - \tfrac{r}{2}\|a'\|^2$$

a.s. and hence a_Z^r is determined uniquely a.s. If a_Z^r is determined uniquely, then there is neighbourhood of r such that for all s from this neighbourhood $a_Z^r = a_Z^s$ holds. Thus

$$\tfrac{\partial}{\partial r} \exp\left(\langle ra_Z^r, Z\rangle - \tfrac{1}{2}\|ra_Z^r\|^2\right) = \exp\left(\langle ra_Z^r, Z\rangle - \tfrac{1}{2}\|ra_Z^r\|^2\right) \cdot (\langle a_Z^r, Z\rangle - r\|a_Z^r\|^2). \tag{53}$$

By Theorem 50 we have

$$\begin{aligned}
\tfrac{\partial}{\partial r} W(rA) &= \tfrac{\partial}{\partial r}\mathbb{E}\ \exp(\max\{\langle ra, Z\rangle - \tfrac{\|ra\|^2}{2} \mid \tfrac{a}{\sqrt{2\pi}} \in K\}) \\
&= \tfrac{\partial}{\partial r}\mathbb{E}\ \exp\left(\langle ra_Z^r, Z\rangle - \tfrac{1}{2}\|ra_Z^r\|^2\right) \qquad (54)
\end{aligned}$$

If we can switch differential and expected value in the last expression, equation (53) will yield the assertion (51). In order to check the integrability assumptions, we choose $R > 0$ with $A \subseteq RB^d$. For $s \in [0, r+1]$ we put $I := [\min\{r, s\}, \max\{r, s\}]$. Since the map $t \mapsto \exp(\langle ta_Z^t, Z\rangle - \frac{1}{2}\|ta_Z^t\|^2)$ is a.s. continuous and piecewise differentiable, we obtain a.s.

$$\begin{aligned}
&|\exp(\langle sa_Z^s, Z\rangle - \tfrac{1}{2}\|sa_Z^s\|^2) - \exp(\langle ra_Z^r, Z\rangle - \tfrac{1}{2}\|ra_Z^r\|^2)| \\
&\qquad \leq |s-r| \cdot \max\{|\tfrac{d}{dt}\exp(\langle ta_Z^t, Z\rangle - \tfrac{1}{2}\|ta_Z^t\|^2)| \mid t \in I\} \\
&\qquad \leq |s-r| \cdot \max\{\exp(\langle ta_Z^t, Z\rangle - \tfrac{1}{2}\|ta_Z^t\|^2) \cdot |\langle a_Z^t, Z\rangle - t\|a_Z^t\|^2| \mid t \in I\} \\
&\qquad \leq |s-r| \cdot \exp((r+1)R\|Z\|) \cdot (R\|Z\| + (r+1)R^2).
\end{aligned}$$

The random variable on the right hand side has finite expected value, which completes the proof of (51).
The proof of (52) is analogue to the proof of (51). The only difference is that in the place, where Theorem 50 was used, now equation (51) has to be used. □

For a body $K \subseteq \mathbb{R}^d$ and a vector $u \in \mathbb{R}^d$ we put

$$h_K(u) := \max\{\langle k, u\rangle \mid k \in K\} = h_{\operatorname{conv} K}(u)$$

and we choose a point $H(K;u) \in K$ satisfying

$$\langle H(K;u), u\rangle = h_K(u).$$

If K is convex, then $H(K;u)$ is an arbitrary point of the support set $H_K(u)$. No matter whether K is convex or not, $H(K;Z)$ is determined uniquely a.s.

Corollary 52. *With the denominations introduced above and right now we have*

(i) $\frac{\partial}{\partial r} W(rA)_{|r=0} = \sqrt{2\pi} \cdot \mathbb{E}\, h_A(Z)$

(ii) $\frac{\partial^2}{\partial r^2} W(rA)_{|r=0} = 2\pi \cdot \mathbb{E}\,[h_A(Z)^2 - \|H(A;Z)\|^2]$.

Proof. From the definition of a_z^r we get

$$\langle a_Z^0, Z\rangle = \max\{\langle a, Z\rangle \mid a \in \sqrt{2\pi}A\} = \sqrt{2\pi} \cdot h_A(Z)$$

and hence, because of $a_Z^0 \in \sqrt{2\pi}A$,

$$a_Z^0 = \sqrt{2\pi} \cdot H(A;Z).$$

So Theorem 51 yields the assertion. □

Comparing the Corollaries 44 and 52 we obtain the following corollary.

Corollary 53. *With the denominations introduced above we have*

(i) $V_1(\operatorname{conv} A) = \sqrt{2\pi} \cdot \mathbb{E}\, h_A(Z)$

(ii) $V_2(\operatorname{conv} A) = \pi \cdot \mathbb{E}\,[h_A(Z)^2 - \|H(A;Z)\|^2]$.

In order to generalize Corollary 53 from finite sets to compact sets, we need continuity arguments. According to [19, Theorem 12.3.5] the map $\operatorname{conv} : \mathcal{C} \to \mathcal{K}$ is continuous and the intrinsic volumes and $K \mapsto h_K(u)$, $u \in S^{d-1}$, are continuous according to [18, p. 210 resp. Lemma 1.8.10], if considered as functions $\mathcal{K} \to \mathbb{R}$.

Lemma 54. *Let $u \in \mathbb{R}^d$ and $K \in \mathcal{C}$ such that $H(K;u)$ is determined uniquely and let $(K_i)_{i\in\mathbb{N}}$ be a sequence converging to K. Then for any choice of $H(K_i;u)$ we have*

$$\lim_{i\to\infty} H(K_i;u) = H(K;u).$$

Proof. It suffices to show that any subsequence $(K_{m(i)})_{i\in\mathbb{N}}$ of $(K_i)_{i\in\mathbb{N}}$ contains a subsequence $(K_{m(r(i))})_{i\in\mathbb{N}}$ such that

$$\lim_{i\to\infty} H(K_{m(r(i))}; u) = H(K; u).$$

So let $(K_{m(i)})_{i\in\mathbb{N}}$ be a subsequence of $(K_i)_{i\in\mathbb{N}}$. Since $(K_{m(i)})_{i\in\mathbb{N}}$ converges, there is $R \in \mathbb{R}^+$ with $K_{m(i)} \subseteq RB^d$ for all $i \in \mathbb{N}$ and, in particular,

$$H(K_{m(i)}; u) \in RB^d, \;\; i \in \mathbb{N}.$$

Hence this sequence has a convergent subsequence $(H(K_{m(r(i))}; u))_{i\in\mathbb{N}}$. Now

$$\langle \lim_{i\to\infty} H(K_{m(r(i))}; u), u\rangle = \lim_{i\to\infty} \langle H(K_{m(r(i))}; u), u\rangle = \lim_{i\to\infty} h_{K_{m(r(i))}}(u) = h_K(u).$$

Because of $\lim_{i\to\infty} H(K_{m(r(i))}; u) \in K$ we get

$$\lim_{i\to\infty} H(K_{m(r(i))}; u) = H(K; u). \qquad \square$$

Theorem 55. *Let $K \subseteq \mathbb{R}^d$ be a body and Z a standard-normal distributed random vector in $\mathbb{R}^d$. Then*

(i) $V_1(\operatorname{conv} K) = \sqrt{2\pi} \cdot \mathbb{E}\, h_K(Z)$

(ii) $V_2(\operatorname{conv} K) = \pi \cdot \mathbb{E}\,[h_K(Z)^2 - \|H(K; Z)\|^2]$.

Proof. We prove only the second statement, since the first one ensues the same way, only slightly easier. According to Lemma 9 there is a sequence $(A_i)_{i\in\mathbb{N}}$ of finite subsets of K converging to K. Now Corollary 53, the continuity statements before and in Lemma 54, and the dominated convergence theorem, which can be applied since $A_i \subseteq K$ holds for all $i \in \mathbb{N}$, give

$$\begin{aligned} V_2(\operatorname{conv} K) &= \lim_{i\to\infty} V_2(\operatorname{conv} A_i) \\ &= \lim_{i\to\infty} \pi \cdot \mathbb{E}\,[h_{A_i}(Z)^2 - \|H(A_i; Z)\|^2] \\ &= \pi \cdot \mathbb{E} \lim_{i\to\infty} [h_{A_i}(Z)^2 - \|H(A_i; Z)\|^2] \\ &= \pi \cdot \mathbb{E}\,[h_K(Z)^2 - \|H(K; Z)\|^2]. \qquad \square \end{aligned}$$

Theorem 55 is not essentially new. The first statement is a special case of Proposition 14 in [22], whose proof is based on the stochastic independence of $\|Z\|$ and $\frac{Z}{\|Z\|}$ and the projection formula from integral geometry. The second statement is new, but part (ii) of Corollary 53 can be derived from [1, (3.10.1)] by using that the covariance of $\langle a, Z\rangle$ is $\|a\|^2$ for $a \in \mathbb{R}^d$.

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