Real Earth Oriented Gravitational Potential Determination

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Contents

1 Introduction

Approximations on regular surfaces are becoming more and more an important issue in recent times. The obvious applications involve approximation of data from different environmental sources, such as meteorology, oceanography or pollution. Applications include representing functions on the Earth's surface, which model temperature, pressure, ozone, gravitational and magnetic forces, elastic deformation etc. at all points on the surface of the Earth, based on a discrete sample of values taken at arbitrary points. In this thesis we are concerned with the new approximation methods that allow representation of the Earth's gravitational potential on regular surfaces.

The Earth's gravity field is one of the most fundamental forces. Although invisible, gravity is a complex force of nature that has an immeasurable impact on our everyday lives. It is often assumed that the force of gravity on the Earth's surface has a constant value, and gravity is considered acting in straight downward direction, but in fact its value varies subtly from place to place and its direction known as the plumb line is actually slightly curved. If the Earth had a perfectly spherical shape and if the mass inside the Earth were distributed homogeneously or rotationally symmetric, these considerations would be true and the line along which Newton's apple fell would indeed be a straight one. The gravitational field obtained in this way would be perfectly spherically symmetric. In reality, however, the situation is much more complex. Gravitational force deviates from one place to the other from that of a homogeneous sphere, due to a number of factors, such as the rotation of the Earth, the topographic features (the position of mountains, valleys or ocean trenches) and variations in density of the Earth's interior. As a consequence the precise knowledge of the Earth's gravitational potential and equipotential surfaces is crucial for all sciences that contribute to the study of the Earth, such as seismology, topography, solid geophysics or oceanography. With the growing awareness with respect to environmental problems like pollution and climate changes, this problem becomes every day a more and more important issue. In former geophysical prospecting, which was dominated by seismic reflection surveying, gravity methods have mostly been used as complements when difficulties with seismic methods have arisen. The determination of the gravitational field of the Earth was done by methods of classical potential theory as a solution to an exterior boundary value problem. Moreover, the up–to–now unrealistic assumption of a global coverage of boundary data (e.g., gravity anomalies) was required in geodetic and geophysical applications. Nowadays, however, with the advent of satellite based techniques, like the Global Positioning system (GPS), SLR (satellite laser ranging), SRA (satellite radar altimetry) and recent satellite missions different types of boundary values have become available (e.g., the gradients of both gravitational potential and field) and also the geometric shape of the Earth became measurable with unimaginable precision. High precision and resolution of the gravity field, obtained with space-borne satellite techniques have changed the ordinary routine in future prospecting. From being a secondary prospecting tool, the gravity field or the geoid, i.e., the equipotential surfaces, computed from (scattered) terrestrial and/or satellite data, are more and more used to locate prospective regions as well as individual prospects. As a consequence, the gravity field and the geoid determination, from actually being a major interest in physical geodesy, gained a renewed popularity in lots of different positioning, mapping and exploration applications, some of which are listed bellow.

- **Height Measurements**: Geometric heights can be obtained fast and efficiently from space positioning systems like GPS, GLONASS, and GALILEO. In order to convert the geometric heights into leveled heights, the precise geoid has to be subtracted. For this reason, the exact knowledge of the geoid is fundamental.
- **Prospecting and Exploration**: Gravity anomalies caused by, e.g., oil, gas structures or geothermal reservoirs can be detected by analyzing the small (spatial) variations of the Earth's gravitational potential.
- **Satellite Orbits**: Gravitational uncertainties must be taken into account when calculating the exact orbit of the spacecraft.
- **Solid Earth Physics**: Together with seismic velocities the gravity field is one of the most important signals of the Earth's interior being measurable in the exterior. More explicitly, internal density signatures are reflected by gravitational field signatures. Thus, by measuring the gravity anomalies, we get valuable information on mass inhomogeneities of the continental and oceanic lithosphere, which are caused by tectonic processes.
- **Physical Oceanography**: Computation of ocean circulations and currents caused by winds, slopes in temperature or salinity also requires precise knowledge of the geoid, since their determination depends directly on geometric distance between the geoid and the sea surface topography, and its variation in time. The transport processes of polluted material can also be determined.
- **Earth System**: Today's satellite missions are important tools for the investigation of global environmental problems, such as global sea–level changes and the CO_2–question. In this context, the geoid serves as an almost static reference for rapidly changing processes.

From a mathematical point of view the modeling of the gravitational potential on and outside the surface of the real Earth seems to be very simple. Namely, the potential is given by Newton's Law of Gravitation

$$V = \int_{Earth} \frac{\rho(x)}{|x - \cdot|} dx,$$

where ρ denotes the density distribution inside the Earth, which means that knowing the density distribution inside the Earth, we can easily calculate the gravitational potential. Unfortunately, we have very poor information about the Earth's interior, and as a matter of fact, the density is known in sufficient quality only in some areas near the Earth's surface and thus Newton's law is not applicable directly. Another possibility for providing the solution of the problem would be to measure the gravity vector (gravity being the resultant of gravitational and centrifugal force of the Earth's rotation) at discrete locations all over the surface of the Earth. But certainly, acquiring such terrestrial data is not as simple as one might think. First of all, if we want to model the potential with reasonable accuracy, we need a very dense equidistribution of points on the Earth's surface, and several areas are difficult to access due to their topographic structures. Secondly, measurements of the gravity vector are only possible on continents with a reasonable compromise between accuracy and costs and one has to look for other techniques such as altimetry on oceanic surfaces.

Until recent times, three essential data sources have been available (terrestrial, airborne and satellite), all of them combined, e.g., in the Earth Geopotential Model EGM96 (which consists of spherical harmonic coefficients up to degree 360). The data types used were: (mean) gravity anomalies, potential values over marine areas by satellite radar altimetry and (pseudo) ranges from orbit analysis of high flying satellites (of altitudes more than 600 km). Terrestrial measurements are not everywhere available, and even in regions where they are available do not always possess sufficient accuracy. In contrast, satellite data yield easily a (nearly) global and rather 'dense' data coverage, but due to the fact that gravitational force is exponentially attenuated with increasing distance from the Earth's surface, they do not reflect all gravity anomalies. Thus for the global coverage, measurements from several satellites and airplanes had to be combined, likewise being of heterogeneous type and having variable accuracies. As consequence, the situation was far away from having a dense global coverage with gravity measurements, not to mention with the homogeneous quality, and so neither of these three data sources nor their combination was sufficient for the applications listed above.

The latest satellite mission launched March 2009, namely the ESA's Gravity field and steady-state Ocean Circulation Explorer (GOCE), has brought a new level of understanding of the gravity field of the Earth. The entire satellite is actually one extremely sensitive measuring device flying at just 250 km above the Earth's surface an mapping global variations in the gravity field with extreme detail and accuracy. GOCE together with its predecessors GRACE (launched March, 2002, at altitude of 500 km) and CHAMP (launched in July, 2000, at altitude of 450 km) have done unprecedented measurements of the gravity field. The result is a unique model of the 'geoid' which yields valuable information about the distribution and flow of mass within the Earth, changes due to surface and deep currents in the ocean, runoff and ground water storage on land masses, exchanges between ice sheets or glaciers and the oceans etc., but mainly with techniques based on spherical reference surfaces. Nevertheless, using tables of spherical harmonic coefficients, geodetic observables, i.e., linear functionals such as geoid undulation, gravity anomaly, radial derivative at Earth's surface or at the satellite height can be evaluated. The latest gravity model EGM2008 consists of spherical harmonic coefficients up to degree 2190, thus providing scientists from all over the world with an efficient and cost-effective way to map the Earth's gravity fields with greatest accuracy on global basis.

In the view of these satellite missions the classical definition of geodesy as a scientific discipline concerned with the measurement and determination of the figure of the Earth and it's gravitational field in its exterior can be more and more realized from the data availability. The theory of spherical harmonic splines and wavelets were developed, showing that spline functions can be viewed as canonical generalizations of the outer harmonics, having desirable properties such as interpolating, smoothing, and best approximation functions, while harmonic wavelets are giving possibility of multiscale analysis as constituting 'building blocks' in the approximation of the gravitational potential. Seen from the mathematical point of view, however, new developments and numerical models are necessary in order to reach the goals described above from the huge data sets. The technological progress and the increasing observational accuracy require adequate mathematical methods with the need of observing geophysically more realistic reference surfaces than sphere and ellipsoid. Also, in recent years reasonably accurate measurements of the surface of the Earth have become available, so today we are in a position to discuss various developments and generalization of mathematical methods for integrals over regular regions, such as for example the Newton integral. This situation offers great challenges in developing a new mathematical framework for the determination of the geoid. Today we are interested in non–spherical boundaries when solving potential theory problems, such as the real Earth's surface. In this thesis we are concerned with developing the real Earth oriented strategies and methods for the Earth's gravitational potential determination. For this purpose we introduce the reproducing kernel Hilbert space of Newton potentials on and outside given regular surface with reproducing kernel defined as a Newton integral over it's interior.
The outline of the thesis is as follows:
The second chapter introduces the basic notation, important results from the theory of spherical harmonics and some basic theorems from potential theory.
In the third chapter, we introduce regular surfaces, boundary value problems, as well as formulations of approximating solutions, with respect to a given regular surface. The fourth chapter gives a closer look to the Earth's gravitational potential, the Newton potentials and their characterization in the interior and the exterior space of the Earth, in

relation to density function. We also present the L^2-decomposition for regions in $\mathbb{R}^3$ in terms of distributions, as a main strategy to impose the Hilbert space structure on the space of potentials on and outside a given regular surface. The properties of the Newton potential operator are investigated in relation to the closed subspace of harmonic density functions.

After these preparations, in the fifth chapter we are able to construct the reproducing kernel Hilbert space of Newton potentials on and outside a regular surface. The spline formulation for the solution to interpolation problems, corresponding to a set of bounded linear functionals is given, and corresponding convergence theorems are proven.

The sixth chapter deals with the representation of the used kernel in the spherical case. We recapitulate the basic results from the spherical harmonic splines theory, corresponding Sobolev spaces and the spherical reproducing kernels, as much as we need to establish a relation for the kernel to spherical ones. Then we prove that the representation of the reproducing kernel for the spherical Earth, corresponds to the representations of kernels such as Abel-Poisson or the singularity kernel. We also investigate the existence of the closed expression of the kernel. However, at this point it remains to be unknown to us.

So, in Chapter 7, we are led to consider certain discretization methods for integrals over regions in $\mathbb{R}^3$, in connection to theory of the multidimensional Euler summation formula for the Laplace operator. We discretize the Newton integral over the real Earth (representing the spline function) and give a priori estimates for approximate integration when using this discretization method.

The last chapter summarizes our results and gives some directions for the future research.

2 Preparatory Material

In this chapter we introduce the mathematical background of the thesis. The first two sections explain the basic notation and spherical nomenclature. The third section gives an introduction to potential theory, boundary value problems, and harmonic functions. Spherical harmonics as the most important functions in geosciences are introduced in the fourth section, as well as inner and outer harmonics. The last section deals with material involving the fundamental solution to the Laplacian.

2.1 Basic Notations

The letters $\mathbb{N}, \mathbb{N}_0, \mathbb{Z}$ and $\mathbb{R}$, denote the set of positive integers, non-negative integers, integers and real numbers, respectively. $\mathbb{R}^3$ denotes the three-dimensional Euclidian space. Using the canonical orthonormal basis in $\mathbb{R}^3$

$$\varepsilon^1 = (1,0,0)^T, \varepsilon^2 = (0,1,0)^T, \varepsilon^3 = (0,0,1)^T,$$

each element $x \in \mathbb{R}^3$ can be represented in cartesian coordinates as follows:

$$x = \sum_{i=1}^{3} x_i \varepsilon^i. \tag{2.1}$$

The *inner (scalar)* and *vector product* of two elements $x, y \in \mathbb{R}^3$ are defined respectively, by

$$x \cdot y = x^T y = x_1 y_1 + x_2 y_2 + x_3 y_3 \tag{2.2}$$

and

$$x \wedge y = \begin{pmatrix} x_2 y_3 - x_3 y_2 \\ x_3 y_1 - x_1 y_3 \\ x_1 y_2 - x_2 y_1 \end{pmatrix}. \tag{2.3}$$

Let $D \subset \mathbb{R}^3$ be a region, i.e., an open and connected set in $\mathbb{R}^3$. $C^{(k)}(D)$ denotes the set of all k-times continuously differentiable scalar functions $F : D \to \mathbb{R}$, with $k \in \mathbb{N}$. Moreover, $C^{(0)}(D)$ and $C^{(\infty)}(D)$ denote the set of all continuous and infinitely often continuously differentiable scalar functions $F : D \to \mathbb{R}$, respectively. If D is compact, then $C(D)(= C^{(0)}(D))$ equipped with the norm

$$||F||_{C(D)} = \sup_{x \in D} |F(x)|$$

is a Banach space.
For a measurable $D \subset \mathbb{R}^3$ and $1 \leq p < \infty$, we denote by $\left(L^p(D), ||\cdot||_p\right)$ the Banach space of all Lebesgue integrable functions F, such that

$$||F||_{L^p(D)} = \left(\int_D |F(x)|^p dx \right)^{\frac{1}{p}} < \infty.$$

Setting $p = 2$ we get the Hilbert space $\left(L^2(D), ||\cdot||_{L^2(D)}\right)$, equipped with the scalar product

$$(F, G)_{L^2(D)} = \int_D F(x)G(x)dx.$$

For later use we define the notion of a reproducing kernel in a Hilbert space.

Definition 2.1.1. *Let $(\mathcal{H}, (\cdot, \cdot)_{\mathcal{H}})$ be a real Hilbert space of functions defined on a domain $D \subset \mathbb{R}^n, n \in \mathbb{N}$. A function K, defined on $D \times D$, is called a reproducing kernel of $\mathcal{H}$ if the following properties are satisfied:*

i) $K(x, \cdot) \in \mathcal{H}, K(\cdot, x) \in \mathcal{H}$ *for all* $x \in D$

ii) $(K(x, \cdot), F)_{\mathcal{H}} = (K(\cdot, x), F)_{\mathcal{H}} = F(x)$ *for all* $F \in \mathcal{H}$ *and all* $x \in D$.

Reproducing kernels play an important role in this thesis, since they give the characterization of the approximating functions used for the gravitational field of the Earth. Several important results from the theory of reproducing kernels are listed below without proof. For details the reader is referred to, e.g., [3], [9].

Theorem 2.1.2. *Let $\mathcal{H}$ be a real Hilbert space of functions defined on $D \subset \mathbb{R}^n$. Then $\mathcal{H}$ possesses a reproducing kernel if and only if for each $y \in D$ the evaluation functional $\mathcal{L}_y(F) = F(y)$ is bounded, i.e.,*

$$|\mathcal{L}_y(F)| \leq c_y ||F||_{\mathcal{H}} \tag{2.4}$$

holds for some constant c_y and for all $F \in \mathcal{H}$.

Theorem 2.1.3. *If $\mathcal{H}$ has a reproducing kernel then the kernel is unique.*

Theorem 2.1.4. *Let $D \subset \mathbb{R}^n$ be a non–empty set and $\mathcal{H}$ be a separable reproducing kernel Hilbert space of real–valued functions on D. Assume that $\{\phi_n\}_{n \in \mathbb{N}_0} \subset \mathcal{H}$ is a complete orthonormal system in $\mathcal{H}$. Then its reproducing kernel K has the representation*

$$K(x, y) = \sum_{n=0}^{\infty} \phi_n(x)\phi_n(y). \tag{2.5}$$

Theorem 2.1.5. *Let $D \subset \mathbb{R}^n$ be a non–empty set and $\mathcal{H}$ be a separable reproducing kernel Hilbert space of real–valued functions on D. Let K be its reproducing kernel, $\mathcal{L}$ be a bounded linear functional on $\mathcal{H}$ and let $L \in \mathcal{H}$ denote its representer according to the Riesz representation theorem, i.e., $\mathcal{L}F = (L, F)_{\mathcal{H}}$ for all $F \in \mathcal{H}$. Assume that $\{\phi_n\}_{n\in\mathbb{N}_0} \subset \mathcal{H}$ is a complete orthonormal system in $\mathcal{H}$. Then the representer L is explicitly given by*

$$L(x) = \mathcal{L}K(\cdot, x) = \sum_{n=0}^{\infty}(\mathcal{L}\phi_n)\phi_n(x), \quad x \in D. \tag{2.6}$$

Theorem 2.1.6. *Let $\mathcal{H}$ be a reproducing kernel Hilbert space and let $\{\mathcal{L}_{x_i}\}_{i\in\mathbb{N}}$ be a set of bounded linear functionals such that $\{\mathcal{L}_{x_i}K(\cdot,\cdot)\}_{i\in\mathbb{N}} \subset \mathcal{H}$ defines a complete sequence of functions. Then*

$$\overline{span\{\mathcal{L}_{x_i}K(\cdot,\cdot)\}_{i\in\mathbb{N}}}^{||\cdot||_{\mathcal{H}}} = \mathcal{H}. \tag{2.7}$$

Definition 2.1.7. *Let $\mathcal{H}$ be a real Hilbert space. A linear operator $P : \mathcal{H} \to \mathcal{H}$, $F \mapsto PF$, is called a projection operator (projector) onto Im(P), if it satisfies $P^2 = P$. If additionally $(PF, G)_{\mathcal{H}} = (F, PG)_{\mathcal{H}}$ for all $F, G \in \mathcal{H}$, the operator is called an orthogonal projection operator (orthogonal projector) onto Im(P).*

The following lemma gives a useful characterization of orthogonal projection operators.

Lemma 2.1.8. *Let $\mathcal{H}$ be a real Hilbert space and let $P : \mathcal{H} \to \mathcal{H}, F \mapsto PF$ be a projection operator. The following two statements are equivalent:*

(i) P is an orthogonal projection operator onto Im(P),

(ii) $(F, S)_{\mathcal{H}} = (PF, S)_{\mathcal{H}}$ for all $F \in \mathcal{H}$ and all $S \in$ Im(P).

2.2 Cartesian Nomenclature

Next we introduce some differential operators in $\mathbb{R}^3$ which are used throughout this work. As usual, the *gradient operator* is denoted by ∇ and the *Laplace operator* by Δ. Their representations in Cartesian coordinates in $\mathbb{R}^3$ are well known

$$\nabla = \sum_{i=1}^{3}\left(\frac{\partial}{\partial x_i}\right)\varepsilon^i, \tag{2.8}$$

$$\Delta = \nabla \cdot \nabla = \sum_{i=1}^{3}\left(\frac{\partial}{\partial x_i}\right)^2. \tag{2.9}$$

The gradient and the Laplace operator applied to functions $F \in C^{(1)}(D)$ and $G \in C^{(2)}(D)$ respectively, where $D \subset \mathbb{R}^3$, are defined by

$$\text{grad } F(x) = \nabla F(x) = \left(\frac{\partial F}{\partial x_1}, \frac{\partial F}{\partial x_2}, \frac{\partial F}{\partial x_3}\right)^T \tag{2.10}$$

and

$$\Delta G(x) = \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}\right) G(x). \tag{2.11}$$

Let $\alpha_1, \alpha_2, \alpha_3$ be given non-negative integers. Then $\alpha = (\alpha_1, \alpha_2, \alpha_3)^T$ is called a three-dimensional *multi-index*. For a given multi-index $\alpha = (\alpha_1, \alpha_2, \alpha_3)^T \in \mathbb{N}_0^3$ and a given 3-tuple $x = (x_1, x_2, x_3)^T \in \mathbb{R}^3$ we let

$$\alpha! = \alpha_1! \cdot \alpha_2! \cdot \alpha_3!, \tag{2.12}$$

$$[\alpha] = \alpha_1 + \alpha_2 + \alpha_3, \tag{2.13}$$

$$|\alpha| = \sqrt{\alpha_1^2 + \alpha_2^2 + \alpha_3^2}, \tag{2.14}$$

$$x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3}. \tag{2.15}$$

We say $\alpha = (\alpha_1, \alpha_2, \alpha_3)^T$ is a 3−dimensional *multi-index of degree* n if $[\alpha] = n$. By definition, we set

$$\nabla^\alpha = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \left(\frac{\partial}{\partial x_2}\right)^{\alpha_2} \left(\frac{\partial}{\partial x_3}\right)^{\alpha_3}. \tag{2.16}$$

2.3 Spherical Nomenclature

The Earth is an almost perfect sphere. Deviations from its spherical shape are less than 0.4% of its mean radius (6371 km). Although the non-spherical approach is becoming more and more important in modern geosciences, especially due to the modern satellite techniques, spherical tools still play an important role. Thus, before turning to non-spherical geometries, we will give a detailed description of spherical tools.
We denote the unit sphere in $\mathbb{R}^3$ by Ω, i.e.,

$$\Omega = \{x \in \mathbb{R}^3, |x| = 1\}.$$

Any element $x \in \mathbb{R}^3$, $x = (x_1, x_2, x_3)^T$ with $|x| \neq 0$, has a unique representation

$$x = r\xi, \quad r = |x|, \quad \xi = \frac{x}{|x|},$$

where $r = |x|$ is the distance from x to the origin and $\xi \in \Omega$, with $\xi = (\xi_1, \xi_2, \xi_3)^T$ is uniquely determined directional unit vector of x. We set Ω^{int} for the 'inner space' of Ω, while Ω^{ext} denotes the 'outer space' of Ω. It is well-known that the total surface $||\Omega||$ of Ω is equal to 4π, i.e.,

$$\|\Omega\| = \int_{\Omega} d\omega = 4\pi. \tag{2.17}$$

Any arbitrary point $\xi \in \Omega$ can be given in spherical coordinates

$$\begin{aligned}\xi = t\varepsilon^3 + \sqrt{1-t^2}(\cos\varphi\varepsilon^1 + \sin\varphi\varepsilon^2),\\ \varphi \in [0, 2\pi), t \in [-1, 1], t = \cos\vartheta, \vartheta \in [-\pi, \pi].\end{aligned} \tag{2.18}$$

Written out this means that $x \in \mathbb{R}^3$ is given by the polar coordinates

$$x(r, \varphi, t) = \begin{pmatrix} r\sqrt{1-t^2}\cos\varphi \\ r\sqrt{1-t^2}\sin\varphi \\ rt \end{pmatrix}, \tag{2.19}$$

where $r \geq 0$ is the distance to the origin, φ denotes the spherical longitude, ϑ the spherical latitude and t the polar distance.

We are interested in constructing the orthonormal triad on Ω. We consider the vector function

$$\Phi : [0, \infty) \times [0, 2\pi) \times [-1, 1],$$

defined by

$$\Phi(r, \varphi, t) = \begin{pmatrix} r\sqrt{1-t^2}\cos\varphi \\ r\sqrt{1-t^2}\sin\varphi \\ rt \end{pmatrix}.$$

Setting $r = 1$ we get the local coordinate system on the unit sphere Ω as before. Now, calculating the derivatives of Φ and setting $r = 1$, the corresponding set of orthonormal unit vectors in the directions r, ϕ and t is easily determined by

$$(\xi =)\ \varepsilon^r(\varphi, t) = \begin{pmatrix} \sqrt{1-t^2}\cos\varphi \\ \sqrt{1-t^2}\sin\varphi \\ t \end{pmatrix},$$

$$\varepsilon^{\varphi}(\varphi, t) = \begin{pmatrix} \sin\varphi \\ \cos\varphi \\ 0 \end{pmatrix}$$

and

$$\varepsilon^t(\varphi, t) = \begin{pmatrix} t\cos\varphi \\ -t\sin\varphi \\ \sqrt{1-t^2} \end{pmatrix}.$$

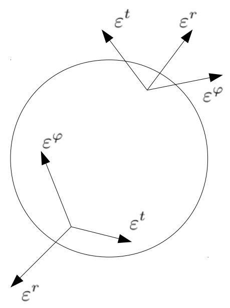

Figure 2.1: Moving local triad on the unit sphere

These vectors represent a moving orthonormal triad on the unit sphere Ω, where the vector ε^r is the radial vector, while the vectors $\varepsilon^\varphi, \varepsilon^t$ mark tangential directions.

We decompose the gradient and the Laplacian into a radial and angular part. The gradient ∇ in $\mathbb{R}^3$ is represented by

$$\nabla = \xi \frac{\partial}{\partial r} + \frac{1}{r}\nabla^*_\xi, \tag{2.20}$$

where ∇^* is the surface gradient on the unit sphere Ω. For the Laplace operator $\Delta = \nabla \cdot \nabla$ in $\mathbb{R}^3$ we have the representation

$$\Delta = \left(\frac{\partial}{\partial r}\right)^2 + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\Delta^*_\xi, \tag{2.21}$$

where Δ^* is the Beltrami operator on the unit sphere Ω. For explicit representations in polar coordinates see [38].

Next we will present some scalar function spaces for later use. In accordance to the previous notations $L^p(\Omega)$, $1 \le p < \infty$, denotes the space of all scalar functions $F : \Omega \to \mathbb{R}$ that are measurable and for which

$$\|F\|_{L^p(\Omega)} = \left(\int_\Omega |F(\eta)|^p d\omega(\eta)\right)^{1/p} < \infty. \tag{2.22}$$

As Ω is compact $L^p(\Omega) \subset L^q(\Omega)$, for $1 \le q \le p$. For $p = 2$ we get the space of all measurable and square-integrable functions on the sphere. $L^2(\Omega)$ is a Hilbert space with respect to the inner product given by

$$(F, G)_{L^2(\Omega)} = \int_\Omega F(\eta)G(\eta)d\omega(\eta). \tag{2.23}$$

$C^{(k)}(\Omega)$, $0 \le k \le \infty$, denotes the class of k-times continuously differentiable scalar functions $F : \Omega \to \mathbb{R}$. $C(\Omega) = C^{(0)}(\Omega)$, denoting the set of continuous scalar functions on the sphere, is a complete normed space equipped with the norm

$$\|F\|_{C(\Omega)} = \sup_{\xi \in \Omega} |F(\xi)|. \tag{2.24}$$

In connection with $(\cdot,\cdot)_{L^2(\Omega)}$, the space $C(\Omega)$ is a pre-Hilbert space. For each $F \in C(\Omega)$ we have the *norm estimate*

$$\|F\|_{L^2(\Omega)} \le \sqrt{4\pi}\|F\|_{C(\Omega)}. \tag{2.25}$$

$L^2(\Omega)$ is the completion of $C(\Omega)$ with respect to the norm $\|\cdot\|_{L^2(\Omega)}$, i.e.,

$$L^2(\Omega) = \overline{C(\Omega)}^{\|\cdot\|_{L^2(\Omega)}}. \tag{2.26}$$

Any function of the form $G_\xi : \Omega \to \mathbb{R}, \eta \mapsto G_\xi(\eta) = G(\xi \cdot \eta), \eta \in \Omega$, is called a ξ-zonal function on Ω. Zonal functions are constant on the set of all $\eta \in \Omega$ with $\xi \cdot \eta = h$, $h \in [-1,+1]$. The set of all ξ-zonal functions is isomorphic to the set of functions on the interval $[-1,+1]$. This gives rise to interpret $C[-1,+1]$ and $L^p[-1,+1]$ with norms defined correspondingly as subspaces of $C(\Omega)$ and $L^p(\Omega)$. More explicitly, we have

$$\|G\|_{L^p[-1,+1]} = \|G(\varepsilon^3\cdot)\|_{L^p(\Omega)} = \left(2\pi \int_{-1}^{+1} |G(t)|^p dt\right)^{1/p} \tag{2.27}$$

and

$$\|G\|_{C[-1,+1]} = \|G(\varepsilon^3\cdot)\|_{C(\Omega)} = \sup_{\xi \in \Omega} G(\varepsilon^3 \cdot \xi)| = \sup_{\xi \in \Omega} |G_{\varepsilon^3}(\xi)|. \tag{2.28}$$

Zonal functions (in the jargon of approximation theory, radial basis functions) show an important principle for many purposes on the sphere, namely rotational invariance. Using zonal functions, i.e., functions of axial symmetry, and varying their axes, many different approximation techniques can be constructed, for example multiscale approximation by spherical wavelets or approximation techniques such that the invariance of the corresponding pseudodifferential operator (with respect to Riemannian geometry of the sphere) is preserved. For more details the interested reader is referred to, e.g., [29], [38].

2.4 Spherical Harmonics

We next introduce the most commonly used harmonic functions for representing scalar functions on a spherical surface, namely the spherical harmonics. They form a complete orthonormal system in the Hilbert space $L^2(\Omega)$, and thus can be used for the construction of Fourier series in $L^2(\Omega)$. Spherical harmonics constitute the classical tool in gravitational field determination, they are used extensively in the gravitational and magnetic applications involving Laplace's equation. Important results of the theory of spherical harmonics are the addition theorem and the formula of Funk and Hecke. The addition theorem shows the relation between spherical harmonics and Legendre polynomials, i.e., zonal functions on the sphere. The connection between the addition theorem and the orthogonal invariance of the sphere is established by the Funk–Hecke formula. For more details the reader is referred to [14], [38].

Spherical harmonics are defined as restrictions of homogeneous harmonic polynomials. Given a homogeneous harmonic polynomial $H_n : \mathbb{R}^3 \to \mathbb{R}$ of degree n, the restriction $Y_n = H_n|_\Omega$ is called a *spherical harmonic of degree* n. The space of all spherical harmonics of degree n is denoted by $Harm_n(\Omega)$. This space is of dimension $2n+1$, i.e.,

$$d(Harm_n(\Omega)) = 2n + 1. \tag{2.29}$$

Spherical harmonics of different degrees are orthogonal in the sense of the $L^2(\Omega)$- inner product

$$(Y_n, Y_m)_{L^2(\Omega)} = \int_\Omega Y_n(\xi) Y_m(\xi) d\omega(\xi) = 0, \quad n \neq m. \tag{2.30}$$

Using the standard method of separation and observing the homogeneity we have $H_n(x) = r^n Y_n(\xi), x = r\xi, r = |x|, \xi \in \Omega$. From the identity

$$\frac{1}{r^2}\left(\frac{d}{dr}\right) r^2 \frac{d}{dr} r^n = n(n+1) r^{(n-2)} \tag{2.31}$$

it follows in connection with the harmonicity of H_n that

$$0 = \Delta_x H_n(x) = r^{(n-2)} n(n+1) Y_n(\xi) + r^{(n-2)} \Delta^*_\xi Y_n(\xi). \tag{2.32}$$

This means that any spherical harmonic $Y_n, n \in \mathbb{N}_0$, is an infinitely often differentiable eigenfunction of the Beltrami operator, corresponding to the eigenvalue $-n(n+1), n \in \mathbb{N}_0$. More explicitly,

$$\Delta^*_\xi Y_n(\xi) = (\Delta^*)^\wedge(n) Y_n(\xi), \quad \xi \in \Omega, \quad Y_n \in Harm_n(\Omega), \tag{2.33}$$

where the 'spherical symbol' $\{(\Delta^*)^\wedge(n)\}_{n\in\mathbb{N}_0}$ of the operator Δ^* is given by

$$(\Delta^*)^\wedge(n) = -n(n+1), n = 0, 1, ... \tag{2.34}$$

A special class of functions, which are members of the class of radial basis functions, are the Legendre polynomials. They can be defined via the Legendre operator

$$L_t = (d/dt)(1 - t^2)(d/dt),$$

which is the 'longitude-independent part' of the Beltrami operator. In doing so, the Legendre polynomial $P_n : [-1,+1] \to \mathbb{R}$ of degree n is the (uniquely defined) infinitely often differentiable eigenfunction of the Legendre operator L_t, corresponding to the eigenvalue $-n(n+1)$, i.e.,

$$L_t P_n(t) = -n(n+1)P_n(t), \quad t \in [-1,+1], \tag{2.35}$$

which satisfies $P_n(1) = 1$. It is well–known that the Legendre polynomials are orthogonal with respect to the $L^2([-1,+1])$–inner product, i.e.,

$$\int_{-1}^{+1} P_n(t)P_m(t)dt = \frac{2}{2n+1}\delta_{n,m}, \tag{2.36}$$

where $\delta_{n,m}$ is the Kronecker symbol. The Legendre polynomial P_n has the explicit representation

$$P_n(t) = \sum_{s=0}^{[n/2]} (-1)^s \frac{(2n-2s)!}{2^n(n-2s)!(n-s)!s!} t^{n-2s}, \quad t \in [-1,+1]. \tag{2.37}$$

Furthermore, we have

$$P_n(t) = \frac{1}{2^n n!}\left(\frac{d}{dt}\right)^n (t^2-1)^n, \quad t \in [-1,+1], \tag{2.38}$$

which is known as Rodrigues' formula. The Legendre polynomial P_n, seen as zonal function of the sphere $P_n(\xi\cdot)$, is the only spherical harmonic of degree n that is invariant with respect to orthogonal transformations which leave ξ fixed. The system $\{P_n\}_{n\in\mathbb{N}_0}$ of all Legendre polynomials is closed and complete in $L^2([-1,+1])$, with respect to $\|\cdot\|_{L^2[-1,+1]}$.
The Legendre transform $G \mapsto (LT)(G), G \in L^1[-1,+1]$ is defined by

$$(LT)(G)(n) = G^\wedge(n) = G^{\wedge_{L^2[-1,+1]}}(n) = (G, P_n)_{L^2[-1,+1]}. \tag{2.39}$$

The series

$$\sum_{n=0}^{\infty} \frac{2n+1}{4\pi} G^\wedge(n) P_n, \tag{2.40}$$

is called the Legendre expansion of G (with Legendre coefficients $G^\wedge(n), n = 0, 1, ...$). For all $G \in L^2[-1,+1]$ we have

$$\lim_{N\to\infty} \left\| G - \sum_{n=0}^{N} \frac{2n+1}{4\pi} G^\wedge(n) P_n \right\|_{L^2[-1,+1]} = 0. \tag{2.41}$$

This property in $L^2[-1,+1]$ is equivalent to the Parseval identity

$$(G,G)_{L^2[-1,+1]} = \sum_{n=0}^{\infty} \frac{2n+1}{4\pi} (G^\wedge(n))^2, \quad G \in L^2[-1,+1]. \tag{2.42}$$

By $\{Y_{n,k}\}_{k=1,...,2n+1}$ we denote a (maximal) complete orthonormal system in the space $Harm_n(\Omega)$ with respect to $(\cdot,\cdot)_{L^2(\Omega)}$. The following theorem, known as the addition theorem for spherical harmonics, relates functions on the unit sphere (spherical harmonics) of degree n to the univariate functions defined on the interval $[-1,+1]$ (Legendre polynomials).

Theorem 2.4.1. *(Addition Theorem) Let $\{Y_{n,k}\}_{k=1,...,2n+1}$ be an orthonormal system of spherical harmonics with respect to $(\cdot,\cdot)_{L^2(\Omega)}$ in $Harm_n(\Omega)$. Then*

$$\sum_{k=1}^{2n+1} Y_{n,k}(\xi)Y_{n,k}(\eta) = \frac{2n+1}{4\pi} P_n(\xi\cdot\eta), \quad \xi,\eta\in\Omega. \tag{2.43}$$

In connection with the Cauchy-Schwarz inequality we obtain as a consequence of the addition theorem

$$\sum_{k=1}^{2n+1} |Y_{n,k}(\xi)|^2 = \frac{2n+1}{4\pi} \tag{2.44}$$

and

$$|P_n(t)| \leq P_n(1) = 1, \quad t\in[-1,+1]. \tag{2.45}$$

It should be remarked that there exist infinitely many $L^2(\Omega)$-orthonormal systems in $Harm_n(\Omega)$ (for more details see, e.g., [38]). One example, usually used in the geosciences, is the system of fully normalized spherical harmonics in terms of Legendre functions (cf., e.g., [50]). Consider the $2n+1$ functions, expressed in polar coordinates (2.18)

$$Y_{n,k}(\xi) = c_{n,k} P_n^{|k|}(t)\cos(k\phi), \quad k=-n,...,0. \tag{2.46}$$

$$Y_{n,k}(\xi) = c_{n,k} P_n^{|k|}(t)\sin(k\phi), \quad k=1,...,n. \tag{2.47}$$

where the normalization coefficients $c_{n,k}$ are given by

$$c_{n,k} = \sqrt{(2-\delta_{0,k})\frac{2n+1}{4\pi}\frac{(n-|k|)!}{(n+|k|)!}}$$

and P_n^k denotes the associated Legendre functions of degree n and order k

$$P_n^k(t) = (1-t^2)^{k/2}\Big(\frac{d}{dt}\Big)^k P_n(t),$$

$k=0,...,n, t\in[-1,+1]$. For example the Earth Gravity Models EGM96 and EGM2008 are given as expansions in terms of fully–normalized spherical harmonics (see [63], [53]). From the numerical point of view it is important that there exist stable algorithms for the calculation of Legendre polynomials, Legendre functions and spherical harmonics (cf., e.g.,

[12]). These algorithms are based on three–term recurrence formulas. Representatives of such recurrence formulas are, for example for Legendre polynomials,

$$nP_n(t) + (2n-1)P_{n-1}(t) + (n-1)P_{n-2}(t) = 0, \tag{2.48}$$

and for their derivatives,

$$(n-k)P_n^{(k)}(t) - tP_n^{(k+1)}(t) + P_{n-1}^{(k+1)}(t) = 0, \tag{2.49}$$

where $P_0(t) = 1$ and $P_1(t) = t$. Equivalent formulas exist for Legendre functions $P_n^k(t)$, and thus for spherical harmonics. Based on these formulas, fast and stable algorithms can be derived for the evaluation of finite series of Legendre polynomials, Legendre functions and spherical harmonics (see, e.g., [11]).

For $t \in [-1, 1]$ and all $h \in (-1, 1)$

$$\sum_{n=0}^{\infty} P_n(t)h^n = \frac{1}{\sqrt{1 + h^2 - 2ht}}. \tag{2.50}$$

If $x = |x|\xi, y = |y|\eta$, where $\xi, \eta \in \Omega$, since

$$|x-y|^2 = (|x|^2 + |y|^2 - 2|x||y|(\xi \cdot \eta)) = |y|^2\big(\frac{|x|^2}{|y|^2} + 1 - 2\frac{|x|}{|y|}\xi \cdot \eta\big), \tag{2.51}$$

we have for the fundamental solution of the Laplace's equation the following expression

$$\frac{1}{|x-y|} = \frac{1}{|y|}\left(1 + \left(\frac{|x|}{|y|}\right)^2 - 2\frac{|x|}{|y|}\xi \cdot \eta\right)^{-1/2}. \tag{2.52}$$

Choosing $t = \xi \cdot \eta$ and $h = \frac{|x|}{|y|}$ we find from (2.50) for $|x| < |y|$ (or equivalently $\frac{|x|}{|y|} < 1$)

$$\frac{1}{|x-y|} = \frac{1}{|y|}\sum_{n=0}^{\infty}\big(\frac{|x|}{|y|}\big)^n P_n(\xi \cdot \eta). \tag{2.53}$$

For $0 \leq h < 1$ and $t \in [-1, 1]$ the following series representation can be derived from (2.50)

$$\sum_{n=0}^{\infty}(2n+1)P_n(t)h^n = \frac{1-h^2}{(1 + h^2 - 2ht)^{3/2}}. \tag{2.54}$$

For $F \in L^2(\Omega)$, the series

$$\sum_{n=0}^{\infty}\sum_{k=1}^{2n+1} F^{\wedge}(n,k)Y_{n,k} \tag{2.55}$$

is called the Fourier expansion (or spherical harmonic expansion) of F with Fourier (or spherical harmonic) coefficients given by

$$F^\wedge(n,k) = \int_\Omega F(\xi) Y_{n,k}(\xi) d\omega(\xi) \tag{2.56}$$

$n = 0,1,...; k = 1,...,2n+1$. For all $F \in L^2(\Omega)$ we have

$$\lim_{N\to\infty} \left\| F - \sum_{n=0}^{N} \sum_{k=1}^{2n+1} F^\wedge(n,k) Y_{n,k} \right\|_{L^2(\Omega)} = 0. \tag{2.57}$$

This property in $L^2(\Omega)$ is equivalent to the Parseval identity

$$(F,F)_{L^2(\Omega)} = \sum_{n=0}^{\infty} \sum_{k=1}^{2n+1} (F^\wedge(n,k))^2, \; F \in L^2(\Omega). \tag{2.58}$$

The recovery of a function $F \in L^2(\Omega)$ by its Fourier expansion (in the sense of $\|\cdot\|_{L^2(\Omega)}$) is equivalent to the following conditions:

(i) (*closure*) The system $\{Y_{n,k}\}_{\substack{n=0,1,...\\k=1,...,2n+1}}$ is closed in $L^2(\Omega)$, i.e., for any number $\varepsilon > 0$ and any function $F \in L^2(\Omega)$, there exist coefficients $d_{n,k}$, such that

$$\left\| F - \sum_{n=0}^{N} \sum_{k=1}^{2n+1} d_{n,k} Y_{n,k} \right\|_{L^2(\Omega)} \le \varepsilon, \tag{2.59}$$

(ii) (*completeness*) The system $\{Y_{n,k}\}_{\substack{n=0,1,...\\k=1,...,2n+1}}$ is complete in $L^2(\Omega)$, i.e., $F \in L^2(\Omega)$ with $F^\wedge(n,k) = 0$, for all n,k implies $F = 0$.

(iii) The system $\{Y_{n,k}\}_{\substack{n=0,1,...\\k=1,...,2n+1}}$, is a *Hilbert basis* of $L^2(\Omega)$, i.e.,

$$\overline{span_{\substack{n=0,1,...\\k=1,...,2n+1}} \{Y_{n,k}\}}^{\|\cdot\|_{L^2(\Omega)}} = L^2(\Omega). \tag{2.60}$$

The closure (and consequently the completeness) in $L^2(\Omega)$ states that spherical harmonics are able to represent square-integrable functions on the sphere within arbitrarily given accuracy in the $L^2(\Omega)$-topology. For $q \ge p \ge 0$ we denote by $Harm_{p,...,q}(\Omega)$ the space of all spherical harmonics of degrees n with $p \le n \le q$. Because of the orthogonality it is clear that $Harm_{p,...,q}(\Omega)$ may be written as orthogonal direct sum

$$Harm_{p,...,q}(\Omega) = \bigoplus_{n=p}^{q} Harm_n(\Omega). \tag{2.61}$$

$Harm_{p,...,q}(\Omega)$ is of dimension $\sum_{n=p}^{q}(2n+1)$. In particular, $Harm_{0,...,m}(\Omega)$ denotes the set of all spherical harmonics of degree $\leq m$, and we have

$$d(Harm_{0,...,m}(\Omega)) = \sum_{n=0}^{m}(2n+1) = (m+1)^2.$$

We denote $Harm_{0,...,\infty}(\Omega)$ simply as $Harm(\Omega)$.

For every $Y_n \in Harm_n(\Omega)$

$$\frac{2n+1}{4\pi}\int_{\Omega} P_n(\xi\cdot\eta)Y_n(\eta)d\omega(\eta) = Y_n(\xi), \quad \xi \in \Omega. \tag{2.62}$$

In other words, the kernel $K_{Harm_n(\Omega)}(\cdot,\cdot) : \Omega \times \Omega \to \mathbb{R}$ defined by

$$K_{Harm_n(\Omega)}(\xi,\eta) = \frac{2n+1}{4\pi}P_n(\xi\cdot\eta), \quad (\xi,\eta) \in \Omega \times \Omega, \tag{2.63}$$

represents the (uniquely determined) reproducing kernel in $Harm_n(\Omega)$.

The connection between spherical harmonics and radial basis functions is established by the Funk-Hecke formula

$$\int_{\Omega} G(\xi\cdot\eta)Y_n(\eta)d\omega(\eta) = G^{\wedge}(n)Y_n(\xi), \quad \xi \in \Omega, G \in L^1[-1,+1], \tag{2.64}$$

where the 'Legendre transform' of $G \in L^1[-1,+1]$ is given by

$$G^{\wedge}(n) = G^{\wedge_{L^2[-1,+1]}}(n) = 2\pi\int_{-1}^{+1} G(t)P_n(t)dt. \tag{2.65}$$

The Funk-Hecke formula founds the basis for the introduction of spherical convolutions (cf. [6], [7]) which, in turn, are fundamental for the theory of spherical singular integrals and spherical wavelets. Suppose that $F \in L^2(\Omega)$ and $G \in L^2[-1,+1]$. Then the function

$$G * F = \int_{\Omega} G(\cdot\eta)F(\eta)d\omega(\eta), \tag{2.66}$$

is called $L^2(\Omega)-$ spherical convolution of F against G. Two important properties of spherical convolutions should be listed

- If $F \in L^2(\Omega)$ and $G \in L^2[-1,+1]$, then $G * F$ is of class $L^2(\Omega)$, and we have
$$(G * F)^{\wedge}(n,k) = G^{\wedge}(n)F^{\wedge}(n,k), \quad n = 0,1,...; k = 1,...,2n+1. \tag{2.67}$$

- If $G_1, G_2 \in L^2[-1,+1]$, then the convolution of G_1 against G_2 is of class $C[-1,+1]$, and we have
$$(G_1(\cdot\eta) * G_2(\cdot\eta))^{\wedge}(n) = G_1^{\wedge}(n)G_2^{\wedge}(n). \tag{2.68}$$

2.5 Inner and Outer Harmonics

We consider a sphere $\Omega_R \subset \mathbb{R}^3$ around the origin with radius $R > 0$. Denote by Ω_R^{int} and Ω_R^{ext} the inner and the outer space of Ω_R, respectively. By virtue of the isomorphism $\Omega \ni \xi \mapsto R\xi \in \Omega_R$ we can assume functions $F : \Omega \to \mathbb{R}$ to be defined on Ω_R. With the surface measure $d\omega_R$ of Ω_R,

$$d\omega_R = R^2 d\omega, \tag{2.69}$$

we are able to introduce the $L^2(\Omega_R)$ - inner product $(\cdot,\cdot)_{L^2(\Omega_R)}$ and the associated norm $\|\cdot\|_{L^2(\Omega_R)}$, as usual. Obviously, an $L^2(\Omega)$ - orthonormal system of spherical harmonics forms an orthogonal system on Ω_R (with respect to $(\cdot,\cdot)_{L^2(\Omega_R)}$). More explicitly,

$$(Y_{n,k}, Y_{p,q})_{L^2(\Omega_R)} = \int_{\Omega_R} Y_{n,k}\left(\frac{x}{|x|}\right) Y_{p,q}\left(\frac{x}{|x|}\right) d\omega_R(x) = R^2 \delta_{n,p}\, \delta_{k,q}. \tag{2.70}$$

With the relationship $\xi \to R\xi$, the surface gradient $\nabla^{*;R}$ and the Beltrami operator $\Delta^{*;R}$ on Ω_R, respectively, have the representations

$$\nabla^{*;R} = (1/R)\nabla^* \tag{2.71}$$

and

$$\Delta^{*;R} = (1/R^2)\Delta^*. \tag{2.72}$$

The function spaces defined on Ω have their natural generalizations as spaces of functions defined on Ω_R. We have for example, $C(\Omega_R), L^p(\Omega_R)$, etc.
The system of spherical harmonics $\{Y_{n,k}^R\}_{\substack{n=0,1,\ldots\\k=1,\ldots,2n+1}}$, where

$$Y_{n,k}^R(x) = \frac{1}{R} Y_{n,k}\left(\frac{x}{|x|}\right), \quad x \in \Omega_R, \tag{2.73}$$

is orthonormal in $L^2(\Omega_R)$-sense, i.e.,

$$(Y_{n,k}^R, Y_{p,q}^R)_{L^2(\Omega_R)} = \int_{\Omega_R} Y_{n,k}^R(x) Y_{p,q}^R(x) d\omega_R(x) = \delta_{n,p}\, \delta_{k,q}. \tag{2.74}$$

Moreover,

$$C(\Omega_R) = \overline{span_{\substack{n=0,1,\ldots\\k=1,\ldots,2n+1}} \left(Y_{n,k}^R\right)}^{\|\cdot\|_{C(\Omega_R)}} \tag{2.75}$$

and

$$L^2(\Omega_R) = \overline{span_{\substack{n=0,1,\ldots\\k=1,\ldots,2n+1}} \left(Y_{n,k}^R\right)}^{\|\cdot\|_{L^2(\Omega_R)}}. \tag{2.76}$$

Next we introduce the inner (outer) harmonics as the solution of the interior (exterior) Dirichlet problem in Ω_R^{int} (Ω_R^{ext}) corresponding to the L^2–boundary values $Y_{n,k}^R$ on Ω_R.

The system $\{H^R_{n,k}\}_{\substack{n=0,1,\ldots\\k=1,\ldots,2n+1}}$ of *inner harmonics* of degree n and order k, defined by

$$H^R_{n,k}(x) = \left(\frac{|x|}{R}\right)^n Y^R_{n,k}(x), \quad x \in \mathbb{R}^3, \tag{2.77}$$

satisfies the following properties:

- $H^R_{n,k}$ is of class $C^{(\infty)}(\mathbb{R}^3)$,
- $\Delta H^R_{n,k}(x) = 0, \quad x \in \mathbb{R}^3$,
- $H^R_{n,k}|_{\Omega_R} = Y^R_{n,k}$,
- $\left(H^R_{n,k}, H^R_{p,q}\right)_{L^2(\Omega_R)} = \int_{\Omega_R} Y^R_{n,k}(x) Y^R_{p,q}(x) d\omega_R(x) = \delta_{n,p}\,\delta_{k,q}$.

The system $\{H^R_{-n-1,k}\}_{\substack{n=0,1,\ldots\\k=1,\ldots,2n+1}}$ of *outer harmonics* of degree n and order k, defined by

$$H^R_{-n-1,k}(x) = \left(\frac{R}{|x|}\right)^{n+1} Y^R_{n,k}(x), \quad x \in \mathbb{R}^3 \setminus \{0\}, \tag{2.78}$$

satisfies the following properties:

- $H^R_{-n-1,k}$ is of class $C^{(\infty)}(\mathbb{R}^3 \setminus \{0\})$,
- $\Delta H^R_{-n-1,k}(x) = 0, \quad x \in \mathbb{R}^3 \setminus \{0\}$,
- $H^R_{-n-1,k}$ is regular at infinity, i.e.,

$$\begin{aligned} \left|H^R_{-n-1,k}(x)\right| &= O\left(\tfrac{1}{|x|}\right), \qquad |x| \to \infty, \\ \left|\nabla H^R_{-n-1,k}(x)\right| &= O\left(\tfrac{1}{|x|^2}\right), \qquad |x| \to \infty, \end{aligned}$$

- $H^R_{-n-1,k}|_{\Omega_R} = Y^R_{n,k}$,
- $\left(H^R_{-n-1,k}, H^R_{-p-1,q}\right)_{L^2(\Omega_R)} = \delta_{n,p}\,\delta_{k,q}$.

In the case of $\Omega_R = \Omega$, i.e., $R = 1$, we have $H^R_{n,k}|_{R=1} = H^R_{-n-1,k}|_{R=1} = Y_{n,k}$ for all $n = 0, 1, \ldots; k = 1, \ldots., 2n+1$.

From the addition theorem of spherical harmonics (Theorem 2.4.1) it is easy to derive the following addition theorems for inner and outer harmonics

Theorem 2.5.1. *(Addition Theorems of Inner and Outer Harmonics).*
Let $\{H_{n,k}^R\}$ *and* $\{H_{-n-1,k}^R\}, k = 1, ..., 2n+1$ *be systems of inner and outer harmonics as defined before. Then the following identities hold true:*

$$\begin{aligned}\sum_{k=1}^{2n+1} & H_{n,k}^R(x) H_{n,k}^r(y) = \\ & \frac{2n+1}{4\pi} \left(\frac{|x||y|}{Rr}\right)^n P_n\left(\frac{x}{|x|} \cdot \frac{y}{|y|}\right), \quad x, y \in \mathbb{R}^3,\end{aligned} \tag{2.79}$$

$$\begin{aligned}\sum_{k=1}^{2n+1} & H_{-n-1,k}^R(x) H_{n,k}^r(y) = \\ & \frac{2n+1}{4\pi} \left(\frac{R}{|x|}\right)^{n+1} \left(\frac{|y|}{r}\right)^n P_n\left(\frac{x}{|x|} \cdot \frac{y}{|y|}\right), \quad x \in \mathbb{R}^3 \setminus \{0\}, y \in \mathbb{R}^3,\end{aligned} \tag{2.80}$$

and

$$\begin{aligned}\sum_{k=1}^{2n+1} & H_{-n-1,k}^R(x) H_{-n-1,k}^r(y) = \\ & \frac{2n+1}{4\pi} \left(\frac{Rr}{|x||y|}\right)^{n+1} P_n\left(\frac{x}{|x|} \cdot \frac{y}{|y|}\right), \quad x, y \in \mathbb{R}^3 \setminus \{0\}.\end{aligned} \tag{2.81}$$

In accordance with our above notation $Harm_n(\overline{\Omega_R^{int}})$ ($Harm_n(\overline{\Omega_R^{ext}})$) denotes the space of all inner (outer) harmonics of degree n. More explicitly we have

$$Harm_n(\overline{\Omega_R^{int}}) = span_{k=1,...,2n+1}(H_{n,k}^R) \tag{2.82}$$

and

$$Harm_n(\overline{\Omega_R^{ext}}) = span_{k=1,...,2n+1}(H_{-n-1,k}^R). \tag{2.83}$$

Thus we have:

$$d\left(Harm_n(\overline{\Omega_R^{int}})\right) = d\left(Harm_n(\overline{\Omega_R^{ext}})\right) = 2n+1. \tag{2.84}$$

For $m \geq 0$ we denote by $Harm_{0,...,m}(\Omega_R^{int})$ ($Harm_{0,...,m}(\Omega_R^{ext})$) the space of all inner (outer) spherical harmonics of degree $\leq m$. Because of the orthogonality it is clear that

$$Harm_{0,...,m}(\Omega_R^{int}) = \bigoplus_{n=0}^{m} Harm_n(\Omega_R^{int}), \tag{2.85}$$

$$d\left(Harm_{0,...,m}(\Omega_R^{int})\right) = \sum_{n=0}^{m} (2n+1) = (m+1)^2, \tag{2.86}$$

$$Harm_{0,\dots,m}(\Omega_R^{ext}) = \bigoplus_{n=0}^{m} Harm_n(\Omega_R^{ext}), \tag{2.87}$$

and

$$d\left(Harm_{0,\dots,m}(\Omega_R^{ext})\right) = \sum_{n=0}^{m}(2n+1) = (m+1)^2. \tag{2.88}$$

We denote the space $Harm_{0,\dots,\infty}(\Omega_R^{int})$ ($Harm_{0,\dots,\infty}(\Omega_R^{ext})$), accordingly defined, simply as $Harm(\Omega_R^{int})$ ($Harm(\Omega_R^{ext})$).
At the end of this section concerned with harmonics, we introduce the Kelvin transform in $\mathbb{R}^3$ with respect to the sphere Ω_R (cf., for example [51], [68]).

Theorem 2.5.2. *Let $G \subset \mathbb{R}^3$ be a region with $0 \notin G$. Consider a function F defined on G. Then the function*

$$\mathcal{K}(x) = \frac{R}{|x|} F\left(\frac{R^2}{|x|^2}x\right), \tag{2.89}$$

is called the Kelvin transform of F with respect to the sphere Ω_R. $\mathcal{K}$ is defined on the region G^*, arising from reflection of G with respect to Ω_R, i.e.,

$$G^* = \{x \in \mathbb{R}^3 | \frac{R^2}{|x|^2}x \in G\}. \tag{2.90}$$

If $F \in C^{(2)}(G)$, F is harmonic in G if and only if the Kelvin transform $\mathcal{K}$ is harmonic in G^*.

It should be noted that the outer harmonics are the Kelvin transforms of the inner harmonics, and vice versa.

2.6 Fundamental Solution to the Laplacian

The fundamental solution to the Laplacian plays a very important role in describing spline spaces for the Earth's gravitational potential approximation procedure. In this section we summarize some results from classical vector analysis and potential theory, which are of later use in this thesis.

2.6.1 Integral Theorems for Normal Regions in $\mathbb{R}^3$

A region $\mathcal{G} \in \mathbb{R}^3$ is called *normal*, if the *Gauss theorem*

$$\int_{\mathcal{G}} \nabla_x \cdot f(x)dx = \int_{\partial\mathcal{G}} f(x)\cdot\nu(x)d\omega(x), \tag{2.91}$$

is valid for its boundary $\partial\mathcal{G}$, where $f : \overline{\partial\mathcal{G}} \to \mathbb{R}^3$ is continuously differentiable vector field and ν denotes the outer (unit) normal field. Given a normal region $\mathcal{G} \in \mathbb{R}^3$, by letting $f = \nabla F$, $F \in C^{(2)}(\overline{\mathcal{G}})$, we obtain from the Gauss Theorem

$$\int_{\mathcal{G}} \Delta_x F(x)dx = \int_{\partial\mathcal{G}} \frac{\partial F}{\partial\nu}(x)d\omega(x), \tag{2.92}$$

where $\partial/\partial\nu$ denotes the derivative in the direction of the outer (unit normal) field ν. Consequently, for all functions $F \in C^{(1)}(\overline{\mathcal{G}}) \cap C^{(2)}(\mathcal{G})$ satisfying the Laplace equation $\Delta F = 0$ in $\mathcal{G}$, we have

$$\int_{\partial\mathcal{G}} \frac{\partial F}{\partial\nu}(x)d\omega(x) = 0. \tag{2.93}$$

Remark: By convention, $F \in C^{(1)}(\overline{\mathcal{G}}) \cap C^{(2)}(\mathcal{G})$ means that $F : \overline{\mathcal{G}} \to \mathbb{R}$ is continuously differentiable in $\overline{\mathcal{G}}$, such that $F|_{\mathcal{G}}$ is twice continuously differentiable.

For all vector fields $f = F\nabla G$, such that $F \in C^{(1)}(\overline{\mathcal{G}})$, $G \in C^{(2)}(\overline{\mathcal{G}})$, we get from the Gauss theorem

Theorem 2.6.1. *(First Green Theorem) Suppose that $\mathcal{G} \in \mathbb{R}^3$ is a normal region. For $F \in C^{(1)}(\overline{\mathcal{G}})$, $G \in C^{(2)}(\overline{\mathcal{G}})$ we have*

$$\int_{\mathcal{G}} \{F(x)\Delta_x G(x) + \nabla_x F(x) \cdot \nabla_x G(x)\}dx = \int_{\partial\mathcal{G}} F(x)\frac{\partial G}{\partial\nu}(x)d\omega(x). \tag{2.94}$$

Taking $f = F\nabla G - G\nabla F$ with $F, G \in C^{(2)}(\overline{\mathcal{G}})$, we obtain

Theorem 2.6.2. *(Second Green Theorem) Suppose that $\mathcal{G} \in \mathbb{R}^3$ is a normal region. For $F, G \in C^{(2)}(\overline{\mathcal{G}})$*

$$\int_{\mathcal{G}} \{F(x)\Delta_x G(x) - G(x)\Delta_x F(x)\}dx = \int_{\partial\mathcal{G}} \left\{F(x)\frac{\partial G}{\partial\nu}(x) - G(x)\frac{\partial F}{\partial\nu}(x)\right\} d\omega(x). \tag{2.95}$$

Let $y \in \mathcal{G}$ be fixed, where $\mathcal{G}$ is a region in $\mathbb{R}^3$. We are looking for a harmonic function U in $\mathcal{G} \setminus \{y\}$, such that

$$U(x) = F(|x-y|), x \in \mathcal{G} \setminus \{y\}, \tag{2.96}$$

i.e., U depends only on the mutual distance of x and y. From the identities

$$\frac{\partial}{\partial x_i} F(|x-y|) = F'(|x-y|)\frac{x_i - y_i}{|x-y|} \tag{2.97}$$

and

$$\left(\frac{\partial}{\partial x_i}\right)^2 F(|x-y|) = F''(|x-y|)\frac{(x_i-y_i)^2}{|x-y|^2} + F'(|x-y|)\left(\frac{1}{|x-y|} - \frac{(x_i-y_i)^2}{|x-y|^3}\right), \tag{2.98}$$

we easily obtain

$$\Delta_x F(|x-y|) = F''(|x-y|) + \frac{2}{|x-y|} F'(|x-y|) = 0. \tag{2.99}$$

In other words, $F(|x-y|)$ can be written in the form

$$F(|x-y|) = C_1 |x-y|^{-1} + C_2, \tag{2.100}$$

with some constants C_1, C_2. By convention, the function

$$x \mapsto F(|x-y|) = \frac{1}{4\pi} \frac{1}{|x-y|} \tag{2.101}$$

is called the *fundamental solution in* $\mathbb{R}^3$ *with respect to the Laplace operator* Δ. An important property of the fundamental solution of the Laplace operator is given by the following

Theorem 2.6.3. *For a continuous function* H *in the ball*

$$B_R(y) = \{x \in \mathbb{R}^3 : |x-y| < R\},$$

such that $R > r > 0$, *we have*

$$\lim_{\substack{r \to 0 \\ r > 0}} \int_{|x-y|=r} H(x) \frac{\partial}{\partial \nu_x} F(|x-y|) d\omega(x) = -H(y), \tag{2.102}$$

and

$$\lim_{\substack{r \to 0 \\ r > 0}} \int_{|x-y|=r} H(x) F(|x-y|) d\omega(x) = 0, \tag{2.103}$$

where the (unit) normal field ν *is directed to the exterior of* $B_R(y)$.

For proof reader is referred, e.g., to [51].

Next we want to apply the Second Green theorem (for a normal region $\mathcal{G}$ with continuously differentiable boundary $\partial\mathcal{G}$) especially to the functions

$$F : x \mapsto F(x) = 1, x \in \overline{\mathcal{G}}, \tag{2.104}$$

$$G : x \mapsto G(x) = F(|x-y|), x \in \overline{\mathcal{G}} \setminus \{y\}, \tag{2.105}$$

where $y \in \mathbb{R}^3$ is positioned in accordance with the following three cases:

Case $y \in \mathcal{G}$: For sufficiently small $\varepsilon > 0$ we obtain by integration by parts, i.e., the second Green theorem

$$\int_{\substack{x\in\mathcal{G}\\|x-y|\geq\varepsilon}} \underbrace{\Delta_x F(|x-y|)}_{=0}\,dx = \int_{x\in\partial\mathcal{G}} \frac{\partial}{\partial\nu_x}F(|x-y|)d\omega(x) + \int_{\substack{x\in\mathcal{G}\\|x-y|=\varepsilon}} \frac{\partial}{\partial\nu_x}F(|x-y|)d\omega(x). \tag{2.106}$$

In connection with Lemma 2.6.3 we therefore obtain by letting $\varepsilon \to 0$

$$\int_{\partial\mathcal{G}} \frac{\partial}{\partial\nu_x}F(|x-y|)d\omega(x) = 1. \tag{2.107}$$

Case $y \in \partial\mathcal{G}$: Again, by Green's theorem we obtain for $\varepsilon > 0$

$$-\int_{\substack{x\in\mathcal{G}\\|x-y|=\varepsilon}} \frac{\partial}{\partial\nu_x}F(|x-y|)d\omega(x) = \int_{x\in\partial\mathcal{G}} \frac{\partial}{\partial\nu_x}F(|x-y|)d\omega(x). \tag{2.108}$$

By letting $\varepsilon \to 0$ we now find in case of a continuously differentiable surface $\partial\mathcal{G}$

$$\int_{\partial\mathcal{G}} \frac{\partial}{\partial\nu_x}F(|x-y|)d\omega(x) = \frac{1}{2}. \tag{2.109}$$

Case $y \notin \overline{\mathcal{G}}$: The Second Green theorem now yields

$$\int_{\mathcal{G}} \underbrace{\Delta_x F(|x-y|)}_{=0}\,dx = \int_{\partial\mathcal{G}} \frac{\partial}{\partial\nu_x}F(|x-y|)d\omega(x). \tag{2.110}$$

Summarizing all results we obtain

Lemma 2.6.4. *Let $\mathcal{G} \subset \mathbb{R}^3$ be a normal region with continuously differentiable boundary $\partial\mathcal{G}$. Then*

$$\int_{\partial\mathcal{G}} \frac{\partial}{\partial\nu_x}F(|x-y|)d\omega(x) = \begin{cases} 1 & , \quad y \in \mathcal{G} \\ \frac{1}{2} & , \quad y \in \partial\mathcal{G} \\ 0 & , \quad y \notin \overline{\mathcal{G}} \end{cases}. \tag{2.111}$$

In other words, the integral is a measure for the 'solid angle' subtended by the boundary $\partial\mathcal{G}$ at the point $y \in \mathbb{R}^3$. Lemma 2.6.4 is a special case of the Third Green theorem in $\mathbb{R}^3$.

Theorem 2.6.5. *(Third Green Theorem)*

(i) Let $\mathcal{G}$ be a normal region with continuously differentiable boundary $\partial\mathcal{G}$. Suppose that $U : \overline{\mathcal{G}} \to \mathbb{R}$ is twice continuously differentiable, i.e., $U \in C^{(2)}(\overline{\mathcal{G}})$. Then we have

$$\begin{aligned} &\int_{\partial\mathcal{G}} \left\{ U(x)\frac{\partial}{\partial\nu_x}F(|x-y|) - F(|x-y|)\frac{\partial U}{\partial\nu}(x) \right\} d\omega(x) \\ &+ \int_{\mathcal{G}} F(|x-y|)\Delta U(x)dx = \begin{cases} U(y) & , \quad y \in \mathcal{G} \\ \frac{1}{2}U(y) & , \quad y \in \partial\mathcal{G} \\ 0 & , \quad y \notin \overline{\mathcal{G}} \end{cases}. \end{aligned} \tag{2.112}$$

(ii) Let $\mathcal{G}$ be a normal region. Suppose that $U : \overline{\mathcal{G}} \to \mathbb{R}$ is twice continuously differentiable, i.e., $U \in C^{(2)}(\overline{\mathcal{G}})$. Then we have

$$\begin{aligned}\int_{\partial\mathcal{G}} &\left\{ U(x) \frac{\partial}{\partial \nu_x} F(|x-y|) - F(|x-y|) \frac{\partial U}{\partial \nu}(x) \right\} d\omega(x) \\ &+ \int_{\mathcal{G}} F(|x-y|) \Delta U(x) dx = \alpha(y) U(y),\end{aligned} \tag{2.113}$$

where $\alpha(y), y \in \mathbb{R}^3$, is the solid angle at y, subtended by the surface $\partial\mathcal{G}$.

For proof the reader is referred to, e.g., [51], [68].

3 Real Earth Surface Based Methods

In what follows some results from potential theory will be given, generally following the course of the monograph [33]. We begin our considerations by introducing the concept of a regular surface.

3.1 Regular Surfaces

Definition 3.1.1. *A surface $\Sigma \subset \mathbb{R}^3$ is called a $C^{(k)}$-regular surface, if it satisfies the following properties:*

(i) *Σ divides $\mathbb{R}^3$ into the bounded region Σ^{int} (inner space) and unbounded region Σ^{ext} (outer space) defined by $\Sigma^{ext} = \mathbb{R}^3 \setminus \overline{\Sigma^{int}}$, $\overline{\Sigma^{int}} = \Sigma^{int} \cup \Sigma$.*

(ii) *Σ is a closed and compact surface free of double points.*

(iii) *The origin 0 is contained in Σ^{int}.*

(iv) *Σ is locally a $C^{(k)}$-surface (i.e., every point $x \in \Sigma$ has an open neighborhood $\mathcal{U} \subset \mathbb{R}^3$ such that $\Sigma \cap \mathcal{U}$ has a parametrization which is k-times continuously differentiable).*

Definition 3.1.2. *A $C^{(2)}$-regular surface is simply called a regular surface.*

Note that the $C^{(k)}$-regularity conditions on the surface Σ imply that the surface Σ has a k-times continuously differentiable *outer unit normal.* This outer unit normal (pointing by convention into the outer space Σ^{ext}) in the point $x \in \Sigma$ is denoted by $\nu(x)$. Georelevant regular surfaces are for example, the sphere, ellipsoid, (regular) Earth's surface.

Given a regular surface, there exist a positive constants α, β, such that

$$\alpha < \sigma^{inf} = \inf_{x \in \Sigma} |x| \leq \sup_{x \in \Sigma} |x| = \sigma^{sup} < \beta. \tag{3.1}$$

As usual, A^{int}, B^{int} (resp. A^{ext}, B^{ext}) denote the inner (resp. outer) space of the sphere A resp. B around the origin with radius α resp. β. A is a so–called *'Runge sphere'* for Σ^{ext}, and B is a so-called *'Runge sphere'* for Σ^{int}. Σ^{int}_{inf}, Σ^{int}_{sup} (resp. Σ^{ext}_{inf}, Σ^{ext}_{sup}) denote

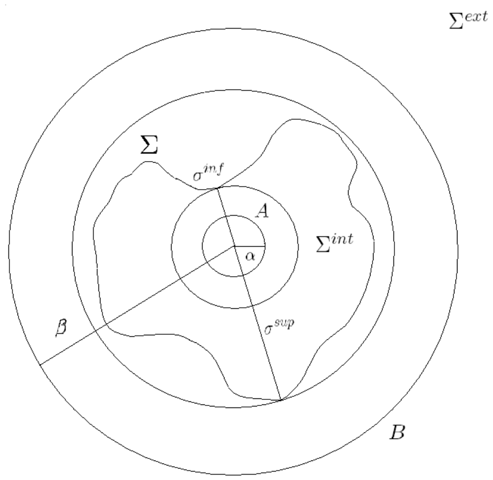

Figure 3.1: The geometric concept of a regular surface

the inner (resp. outer) space of the sphere Σ_{inf} (resp. Σ_{sup}) around the origin with radius σ^{inf} (resp. σ^{sup}).

Let Σ be a regular surface in the sense of Definition 3.1.2. The set

$$\Sigma(\tau) = \{x \in \mathbb{R}^3 | x = y + \tau\nu(y), y \in \Sigma\}, \tag{3.2}$$

generates a *parallel surface* which is exterior to Σ for $\tau > 0$ and interior for $\tau < 0$. It is well known from differential geometry (see, e.g., [60]), that if $|\tau|$ is sufficiently small, then the surface $\Sigma(\tau)$ is regular, and the normal to one parallel surface is a normal to the other.

Let Σ be a regular surface. Then the functions

$$(x, y) \mapsto \frac{|\nu(x) - \nu(y)|}{|x - y|}, \quad (x, y) \in \Sigma \times \Sigma, x \neq y, \tag{3.3}$$

and

$$(x, y) \mapsto \frac{|\nu(x) \cdot (x - y)|}{|x - y|^2}, \quad (x, y) \in \Sigma \times \Sigma, x \neq y, \tag{3.4}$$

are bounded. Hence, there exists a constant $M > 0$ such that

$$|\nu(x) - \nu(y)| \leq M|x - y| \tag{3.5}$$

and

$$|\nu(x) \cdot (x - y)| \leq M|x - y|^2, \tag{3.6}$$

for all $(x, y) \in \Sigma \times \Sigma$.

Definition 3.1.3. *A $C^{(k,\mu)}$-regular surface $\Sigma \subset \mathbb{R}^3$ with $\mu \in (0,1)$ is a locally $C^{(k)}$-regular surface, where every point $x \in \Sigma$ has a neighborhood $\mathcal{U} \subset \mathbb{R}^3$, such that $\Sigma \cap \mathcal{U}$ has a parametrization which is k-times μ- Hölder continuously differentiable.*

3.2 Function Spaces

Next we introduce some spaces of potentials that are of particular importance for gravitational field theory related to a regular surface Σ (such as the real Earth's surface).
For Σ being a regular surface we define two classes of potentials, namely

- $\mathrm{Pot}(\Sigma^{int})$ as the space of all functions U in $C^{(2)}(\Sigma^{int})$ satisfying the Laplace equation in the inner space Σ^{int} of Σ, i.e.,

$$\mathrm{Pot}(\Sigma^{int}) = \{U \in C^{(2)}(\Sigma^{int}) | \Delta U = 0\}. \tag{3.7}$$

- $\mathrm{Pot}(\Sigma^{ext})$ as the space of all functions U in $C^{(2)}(\Sigma^{ext})$ satisfying the Laplace equation in the outer space Σ^{ext} and being regular at infinity (that is, $|U(x)| = O(|x|^{-1})$, $|\nabla U(x)| = O(|x|^{-2})$ for $|x| \to \infty$ uniformly with respect to all directions $\xi = \frac{x}{|x|}$). In brief,

$$\mathrm{Pot}(\Sigma^{ext}) = \{U \in C^{(2)}(\Sigma^{ext}) | \Delta U = 0 \text{ and } U \text{ regular at infinity}\}. \tag{3.8}$$

For $k = 0, 1, ...$ we denote by $\mathrm{Pot}^{(k)}(\overline{\Sigma^{int}})$ the space of all functions $U \in C^{(k)}(\overline{\Sigma^{ext}})$ such that $U|_{\underline{\Sigma_{int}}}$ is of class $\mathrm{Pot}(\Sigma_{int})$. Analogously, $\mathrm{Pot}^{(k)}(\overline{\Sigma^{ext}})$ is the space of all functions $U \in C^{(k)}(\overline{\Sigma^{ext}})$ such that $U|_{\Sigma_{ext}}$ is of class $\mathrm{Pot}(\Sigma_{ext})$. In shorthand notation,

$$\mathrm{Pot}^{(k)}(\overline{\Sigma^{int}}) = \mathrm{Pot}(\Sigma^{int}) \cap C^{(k)}(\overline{\Sigma^{int}}) \tag{3.9}$$

and

$$\mathrm{Pot}^{(k)}(\overline{\Sigma^{ext}}) = \mathrm{Pot}(\Sigma^{ext}) \cap C^{(k)}(\overline{\Sigma^{ext}}). \tag{3.10}$$

Let U be of class $\mathrm{Pot}^{(0)}(\overline{\Sigma^{int}})$. Then the maximum/minimum principle of potential theory (see, e.g., [51], [68]) for the inner space Σ^{int} states

$$\sup_{x \in \overline{\Sigma^{int}}} |U(x)| \le \sup_{x \in \Sigma} |U(x)|. \tag{3.11}$$

Let U be of class $\mathrm{Pot}^{(0)}(\overline{\Sigma^{ext}})$. Then the maximum/minimum principle for the outer space Σ^{ext} gives

$$\sup_{x \in \overline{\Sigma^{ext}}} |U(x)| \le \sup_{x \in \Sigma} |U(x)|. \tag{3.12}$$

A function U possessing μ–Hölder continuous derivatives of k-th order is said to be of class $C^{(k,\mu)}$. We let

$$\mathrm{Pot}^{(k,\mu)}(\overline{\Sigma^{int}}) = \mathrm{Pot}(\Sigma^{int}) \cap C^{(k,\mu)}(\overline{\Sigma^{int}}) \tag{3.13}$$

and

$$\mathrm{Pot}^{(k,\mu)}(\overline{\Sigma^{ext}}) = \mathrm{Pot}(\Sigma^{ext}) \cap C^{(k,\mu)}(\overline{\Sigma^{ext}}). \tag{3.14}$$

Of particular importance for our considerations is the space $C^{(0,\mu)}(\Sigma)$ of all μ-Hölder continuous functions on Σ, $\mu \in (0,1)$. We discuss some properties of $C^{(0,\mu)}(\Sigma)$ in more detail. For $\mu' \leq \mu$ we have

$$C^{(0,\mu)}(\Sigma) \subset C^{(0,\mu')}(\Sigma). \tag{3.15}$$

$C^{(0,\mu)}(\Sigma)$ is a non-complete normed space with respect to the norm

$$||F||_{C^{(0)}(\Sigma)} = \sup_{x\in\Sigma} |F(x)|\,, \tag{3.16}$$

and a Banach space with respect to the norm

$$||F||_{C^{(0,\mu)}(\Sigma)} = \sup_{x\in\Sigma} |F(x)| + \sup_{\substack{x,y\in\Sigma \\ x\neq y}} \frac{|F(x)-F(y)|}{|x-y|^{\mu}}. \tag{3.17}$$

For $\mu' \leq \mu$ and $F \in C^{(0,\mu)}(\Sigma)$, there exists a positive constant $A = A(\mu',\mu)$

$$||F||_{C^{(0,\mu')}(\Sigma)} \leq A||F||_{C^{(0,\mu)}(\Sigma)}. \tag{3.18}$$

$C^{(0,\mu)}(\Sigma)$ is a non-complete normed space with respect to the norm $||\cdot||_{C^{(0,\mu')}(\Sigma)}$, for $\mu' \leq \mu$. For $F, H \in C^{(0,\mu)}(\Sigma)$ it is easy to verify that

$$||FH||_{C^{(0)}(\Sigma)} \leq ||F||_{C^{(0)}(\Sigma)}||H||_{C^{(0)}(\Sigma)}, \tag{3.19}$$

and

$$\begin{aligned} ||FH||_{C^{(0,\mu)}(\Sigma)} &\leq ||F||_{C^{(0,\mu)}(\Sigma)}||H||_{C^{(0)}(\Sigma)} + ||F||_{C^{(0)}(\Sigma)}||H||_{C^{(0,\mu)}(\Sigma)} \\ &\leq 2||F||_{C^{(0,\mu)}(\Sigma)}||H||_{C^{(0,\mu)}(\Sigma)}. \end{aligned} \tag{3.20}$$

In $C^{(0,\mu)}(\Sigma)$ we have the inner product

$$(F,H)_{L^2(\Sigma)} = \int_{\Sigma} F(x)H(x)d\omega(x), \tag{3.21}$$

where $d\omega$ denotes the surface element. The inner product $(\cdot,\cdot)_{L^2(\Sigma)}$ implies the norm

$$||F||_{L^2(\Sigma)} = ((F,F)_{L^2(\Sigma)})^{1/2}. \tag{3.22}$$

The space $(C^{(0,\mu)}(\Sigma), (\cdot,\cdot)_{L^2(\Sigma)})$ is a pre-Hilbert space. For every $F \in C^{(0,\mu)}(\Sigma)$ we have the norm estimate

$$||F||_{L^2(\Sigma)} \leq \sqrt{||\Sigma||} \cdot ||F||_{C^{(0)}(\Sigma)} \leq \sqrt{||\Sigma||} \cdot ||F||_{C^{(0,\mu)}(\Sigma)}, \tag{3.23}$$

where

$$||\Sigma|| = \int_\Sigma d\omega(x). \tag{3.24}$$

$L^2(\Sigma)$ is the completion of $C^{(0)}(\Sigma)$ and of $C^{(0,\mu)}(\Sigma)$ with respect to the norm $||\cdot||_{L^2(\Sigma)}$.

3.3 Boundary Value Problems

In accordance with previous notations we first introduce the classical boundary value problems of potential theory.

Interior Dirichlet Problem (IDP) Given $F \in C^{(0)}(\Sigma)$, find $U \in \mathrm{Pot}^{(0)}(\overline{\Sigma^{int}})$ such that $U^-_\Sigma = F$, where

$$U^-_\Sigma = \lim_{\substack{\tau\to 0\\ \tau>0}} U(x - \tau\nu(x)) = F(x), x \in \Sigma. \tag{3.25}$$

Exterior Dirichlet Problem (EDP) Given $F \in C^{(0)}(\Sigma)$, find $U \in \mathrm{Pot}^{(0)}(\overline{\Sigma^{ext}})$ such that $U^+_\Sigma = F$, where

$$U^+_\Sigma = \lim_{\substack{\tau\to 0\\ \tau>0}} U(x + \tau\nu(x)) = F(x), x \in \Sigma. \tag{3.26}$$

Interior Neumann Problem (INP) Given $F \in C^{(0)}(\Sigma)$, find $U \in \mathrm{Pot}^{(1)}(\overline{\Sigma^{int}})$ such that $\frac{\partial U^-}{\partial \nu_\Sigma} = F$, where

$$\frac{\partial U^-}{\partial \nu_\Sigma} = \lim_{\substack{\tau\to 0\\ \tau>0}} \nu(x) \cdot (\nabla U)(x - \tau\nu(x)) = F(x), x \in \Sigma. \tag{3.27}$$

Exterior Neumann Problem (ENP) Given $F \in C^{(0)}(\Sigma)$, find $U \in \mathrm{Pot}^{(1)}(\overline{\Sigma^{ext}})$ such that $\frac{\partial U^+}{\partial \nu_\Sigma} = F$, where

$$\frac{\partial U^+}{\partial \nu_\Sigma} = \lim_{\substack{\tau\to 0\\ \tau>0}} \nu(x) \cdot (\nabla U)(x + \tau\nu(x)) = F(x), x \in \Sigma. \tag{3.28}$$

Henceforth we restrict ourselves to the geoscientifically (more) relevant exterior boundary value problems.

Let D^+ (more accurately, D^+_Σ) denote the set

$$D^+ = \left\{ U^+_\Sigma \;\middle|\; U \in \mathrm{Pot}^{(0)}(\overline{\Sigma^{ext}}) \right\}. \tag{3.29}$$

The solution of (EDP) is always uniquely determined, hence, $D^+ = C^{(0)}(\Sigma)$. It can be formulated in terms of a potential of the form

$$U(x) = \int_\Sigma Q(y) \frac{\partial}{\partial \nu(y)} \frac{1}{|x-y|} d\omega(y) + \frac{1}{|x|} \int_\Sigma Q(y) d\omega(y), \quad Q \in C^{(0)}(\Sigma), \tag{3.30}$$

where Q satisfies the integral equation

$$F(x) = U^+_\Sigma(x) = 2\pi Q(x) + \frac{1}{|x|} \int_\Sigma Q(y) d\omega(y) + \int_\Sigma Q(y) \frac{\partial}{\partial \nu(y)} \frac{1}{|x-y|} d\omega(y). \tag{3.31}$$

It is shown ([33]) that

$$L^2(\Sigma) = \overline{D^+}^{||\cdot||_{L^2(\Sigma)}} = \overline{C^{(0)}(\Sigma)}^{||\cdot||_{L^2(\Sigma)}}. \tag{3.32}$$

Let N^+ (more accurately, N^+_Σ) denote the set

$$N^+ = \left\{ \frac{\partial U^+}{\partial \nu_\Sigma} \;\middle|\; U \in \mathrm{Pot}^{(1)}(\overline{\Sigma^{ext}}) \right\}. \tag{3.33}$$

The solution of (ENP) is always uniquely determined, hence, $N^+ = C^{(0)}(\Sigma)$. It can be formulated in terms of a potential of the form

$$U(x) = \int_\Sigma Q(y) \frac{1}{|x-y|} d\omega(y), \quad Q \in C^{(0)}(\Sigma), \tag{3.34}$$

where Q satisfies the integral equation

$$F(x) = \frac{\partial U^+}{\partial \nu_\Sigma}(x) = -2\pi Q(x) + \frac{\partial}{\partial \nu(x)} \int_\Sigma Q(y) \frac{1}{|x-y|} d\omega(y). \tag{3.35}$$

It is shown ([33]) that

$$L^2(\Sigma) = \overline{N^+}^{||\cdot||_{L^2(\Sigma)}}. \tag{3.36}$$

Of a special importance for our approach to gravitational potential determination from georelevant data on the real Earth's surface, is the *exterior oblique boundary–value problem*. In fact, it plays an important part in Earth sciences, particularly in geodetic and geophysical applications. For example, the determination of the gravitational field

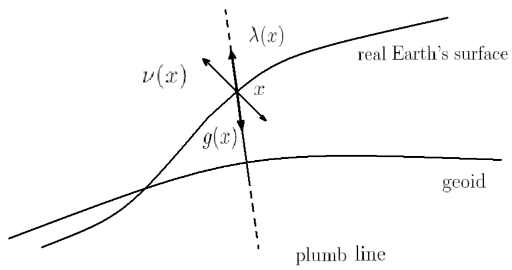

Figure 3.2: The gravity gradient

in the Earth's exterior using the (magnitudes of the) gravity gradients $g(x)(= -\lambda(x))$ (see Figure 3.2) as boundary values on the Earth's surface Σ, leads to an exterior oblique boundary–value problem, since the actual surface of the Earth does not coincide with the equipotential surface of the geoid.

Note that

$$g(x) \cdot (-\nu(x)) = (-\lambda(x)) \cdot (-\nu(x)) = \lambda(x) \cdot \nu(x), \quad x \in \Sigma. \tag{3.37}$$

Next we present the essential aspects of this problem.

Exterior Oblique Derivative Problem (EODP)

Let Σ be a $C^{(2,\mu)}$-regular surface with $\mu \in (0,1)$. Given $F \in C^{(0,\mu)}(\Sigma)$, find a function $U \in \mathrm{Pot}^{(1,\mu)}(\overline{\Sigma^{ext}})$, satisfying the boundary condition

$$\frac{\partial U^+}{\partial \lambda}(x) = \lim_{\substack{\tau \to 0 \\ \tau > 0}} \lambda(x) \cdot (\nabla U)(x + \tau\lambda(x)) = F(x), \quad x \in \Sigma, \tag{3.38}$$

where λ (more accurately, λ_Σ) is $C^{(1,\mu)}$-vector field on Σ with $|\lambda(x)| = 1$ satisfying

$$\inf_{x \in \Sigma} (\lambda(x) \cdot \nu(x)) > 0. \tag{3.39}$$

Remark: If the field λ coincides with the normal field ν on Σ, Equation (3.38) becomes the boundary condition of the classical exterior Neumann problem.

Provided that both the boundary and the boundary values are of sufficient smoothness, the oblique derivative problem can be solved by well-known integral equation methods using the potential of a single layer.
For surfaces that fulfill the smoothness conditions, the uniqueness of a solution can be proven with the help of the extremum principle of Zaremba and Giraud (cf. [5], [58]) combined with the regularity condition at infinity imposed on the elements of $\mathrm{Pot}^{(1,\mu)}(\overline{\Sigma^{ext}})$.
The proof of the existence of a solution uses a single layer potential

$$U(x) = \int_{\Sigma} Q(y) \frac{1}{|x-y|} d\omega(y), \quad Q \in C^{(0,\mu)}(\Sigma). \tag{3.40}$$

Observing the discontinuity of the directional derivative of the single layer potential, from (3.38) we obtain for each $Q \in C^{(0,\mu)}(\Sigma)$ and all points $x \in \Sigma$

$$-2\pi Q(x)(\lambda(x) \cdot \nu(x)) + \int_{\Sigma}^{*} Q(y) \frac{1}{\partial\lambda(x)} \frac{1}{|x-y|} d\omega(y) = F(x). \tag{3.41}$$

The resulting integral equation (3.41) for $F \in C^{(0,\mu)}(\Sigma)$ is of singular type since the integral with the asterisk exists only in the sense of Cauchy's principal value. However, Miranda [58] has shown, for λ satisfying (3.39), all standard Fredholm theorems are still valid. As is well-known ([5]), the homogeneous integral equation corresponding to (3.41) has no solution other than $Q = 0$. Thus, the solution of the oblique derivative problem with boundary condition (3.38), exists and can be represented by a single layer potential (3.40). For more details the reader is referred to [18].
All in all, we are confronted with the following situation:

Let L^+ (more accurately L^+_Σ) denote the set

$$L^+ = \left\{ \frac{\partial U}{\partial \lambda_\Sigma} | U \in \mathrm{Pot}^{(1,\mu)}(\overline{\Sigma^{ext}}),\ \lambda \text{ is a } C^{(1,\mu)}\text{-unit vector field on } \Sigma \text{ , s.t. (3.39) is valid} \right\}. \tag{3.42}$$

The solution to (EODP) is always uniquely determined, hence, $L^+ = C^{(0,\mu)}(\Sigma)$.
It can be formulated in terms of a potential of the form

$$U(x) = \int_{\Sigma} Q(y) \frac{1}{|x-y|} d\omega(y), \quad Q \in C^{(0,\mu)}(\Sigma), \tag{3.43}$$

where Q satisfies the integral equation

$$F(x) = \frac{\partial U^+}{\partial \lambda}(x) = -2\pi Q(x)(\lambda(x) \cdot \nu(x)) + \int_{\Sigma}^{*} Q(y) \frac{\partial}{\partial\lambda(x)} \frac{1}{|x-y|} d\omega(y). \tag{3.44}$$

It shows that

$$L^2(\Sigma) = \overline{L^+}^{\|\cdot\|_{L^2(\Sigma)}}. \tag{3.45}$$

3.4 Regularity Theorems

In the context of the space $C^{(0)}(\Sigma)$ we are able to formulate regularity theorems. From maximum/minimum principle of potential theory we already know that

$$\sup_{x\in\overline{\Sigma^{ext}}} |U(x)| \leq \sup_{x\in\Sigma} \left|U_\Sigma^+(x)\right|, \tag{3.46}$$

holds for all $U \in \mathrm{Pot}^{(0)}(\overline{\Sigma^{ext}})$. Moreover, from the theory of integral equations, it follows that for $U \in \mathrm{Pot}^{(1)}(\overline{\Sigma^{ext}})$ there exists a constant C, such that

$$\sup_{x\in\overline{\Sigma^{ext}}} |U(x)| \leq C \sup_{x\in\Sigma} \left|\frac{\partial U^+}{\partial \nu_\Sigma}(x)\right|. \tag{3.47}$$

Moreover, regularity theorems can be verified in the $L^2(\Sigma)$ context.

Theorem 3.4.1. *Let U be of class $Pot^{(0)}(\overline{\Sigma^{ext}})$. Then there exists a constant $C(= C(k, K, \Sigma))$ such that*

$$\sup_{x\in K} \left|\nabla^{(k)}U(x)\right| \leq C \left(\int_\Sigma \left|U_\Sigma^+(x)\right|^2 d\omega(x)\right)^{1/2}, \tag{3.48}$$

for all $K \subset \Sigma^{ext}$ with $dist(\overline{K}, \Sigma) > 0$ and for all $k \in \mathbb{N}_0$.

Theorem 3.4.2. *Let U be of class $Pot^{(1)}(\overline{\Sigma^{ext}})$. Then there exists a constant $C(= C(k, K, \Sigma))$ such that*

$$\sup_{x\in K} \left|\nabla^{(k)}U(x)\right| \leq C \left(\int_\Sigma \left|\frac{\partial U^+}{\partial \nu_\Sigma}(x)\right|^2 d\omega(x)\right)^{1/2}, \tag{3.49}$$

for all $K \subset \Sigma^{ext}$ with $dist(\overline{K}, \Sigma) > 0$ and for all $k \in \mathbb{N}_0$.

As an immediate consequence of Theorem 3.4.1, Theorem 3.4.2 and the norm estimate (2.25) we obtain the following

Corollary 3.4.3. *Under the assumptions of Theorem 3.4.1 and Theorem 3.4.2, respectively, we have*

$$\sup_{x\in K} \left|\nabla^{(k)}U(x)\right| \leq \sqrt{||\Sigma||}\, C \sup_{x\in\Sigma} \left|U_\Sigma^+(x)\right| \tag{3.50}$$

and

$$\sup_{x\in K} \left|\nabla^{(k)}U(x)\right| \leq \sqrt{||\Sigma||}\, C \sup_{x\in\Sigma} \left|\frac{\partial U^+}{\partial \nu_\Sigma}(x)\right|. \tag{3.51}$$

Similarly, in the context of (exterior) oblique derivative problems, it was proven (see [17]), that for each vector field $U \in \mathrm{Pot}^{(1,\mu)}(\overline{\Sigma^{ext}})$ with $\mu \in (0,1)$, there exists a constant $C_\mu > 0$, with

$$\sup_{x\in\overline{\Sigma^{ext}}} |\nabla U(x)| \leq C_\mu \left\| \frac{\partial U}{\partial \lambda_\Sigma}(x) \right\|_{C^{(0,\mu)}(\Sigma)} . \tag{3.52}$$

Theorem 3.4.4. *Let $U \in Pot^{(1,\mu)}(\overline{\Sigma^{ext}})$ be the uniquely determined solution of the EODP corresponding to the boundary values* (3.38)*. Then*

$$\sup_{x\in K} \left|\nabla^{(k)} U(x)\right| \leq C \left(\int_\Sigma \left(\frac{\partial U^+}{\partial \lambda_\Sigma}(x) d\omega(x) \right) \right)^{1/2} , \tag{3.53}$$

holds for all $K \subset \Sigma^{ext}$, with $dist(\overline{K}, \Sigma) > 0$ and all $k \in \mathbb{N}_0$.

For proof the reader is referred to [33].

3.5 Locally Uniform Runge Approximation

In classical boundary-value problems of potential theory a result first motivated by C. Runge (1885) and later generalized by J.L. Walsh (1929) is of basic interest. For our geoscientifically relevant purpose of gravitational potential determination, it may be formulated as follows: Any function V satisfying Laplace's equation in Σ^{ext} and regular at infinity may be approximated by a function U, harmonic outside an arbitrarily given sphere inside $\overline{\Sigma^{int}}$, in the sense that for any given $\varepsilon > 0$, the inequality $|V(x) - U(x)| \leq \varepsilon$ holds for all points $x \in \mathbb{R}^3$ outside and on any closed surface completely surrounding the surface Σ in the outer space. The value ε may be arbitrarily small, and the surrounding surface may be arbitrarily close to the surface Σ.

In the terminology used in Sections 3.1 and 3.2, the Runge-Walsh theorem states that any function $V \in \mathrm{Pot}^{(0)}(\overline{\Sigma^{ext}})$ can be approximated in arbitrary accuracy in uniform sense by a potential U possessing a larger harmonicity domain. The domain of harmonicity of U may be chosen particularly to be the outer space of a 'Runge sphere' A, completely situated in the Earth's interior, i.e., we may choose $U \in \mathrm{Pot}^{(\infty)}(\overline{A^{ext}})$. Obviously, this is a pure existence theorem. Nothing is said about the approximation procedure and the structure of the approximation.

Assuming that the boundary Σ is a sphere around the origin, however, a constructive approximation of a potential in the outer space is available, e.g., by means of outer harmonics. More precisely, in a spherical approximation, a constructive version of the Runge-Walsh theorem can be established by finite truncations of Fourier expansions in terms of

outer harmonics, (see, e.g., [47]).
For every real number $\varepsilon > 0$, any function $F \in \mathrm{Pot}^{(0)}\left(\overline{A^{ext}}\right)$ and any compact set $K \subset A^{ext}$ with $\mathrm{dist}(K, A) > 0$, there exists an integer $N = N(\varepsilon)$ such that

$$\sup_{x \in K} \left| F(x) - \sum_{n=0}^{N} \sum_{j=1}^{2n+1} (F, H^{\alpha}_{-n-1,k})_{L^2(A)} H^{\alpha}_{-n-1,k}(x) \right| \leq \varepsilon. \tag{3.54}$$

As a matter of fact, the essential steps involved in the Fourier expansion method can be generalized to a non-spherical, i.e., regular boundary Σ. The main techniques for establishing these results are the jump relations and limit formulae and their formulations in the Hilbert space nomenclature of $(L^2(\Sigma), ||\cdot||_{L^2(\Sigma)})$ (see, e.g., [16], [33]). We restrict ourselves to the geophysically relevant exterior case.

Lemma 3.5.1. *Let Σ be a regular surface such that* (3.1) *holds true. Then the following statements are valid:*

(i) $(H^{\alpha}_{-n-1,k}|_{\Sigma})_{\substack{n=0,1,\ldots \\ k=1,\ldots,2n+1}}$ *is linearly independent,*

(ii) $\left(\frac{\partial H^{\alpha}_{-n-1,k}}{\partial \nu_{\Sigma}}\right)_{\substack{n=0,1,\ldots \\ k=1,\ldots,2n+1}}$ *is linearly independent.*

For the proof the reader is referred to [16], [33].

Theorem 3.5.2. *Let Σ be a regular surface such that* (3.1) *holds true. Then the following statements are valid:*

(i) $(H^{\alpha}_{-n-1,k}|_{\Sigma})_{\substack{n=0,1,\ldots \\ k=1,\ldots,2n+1}}$ *is complete in* $L^2(\Sigma) = \overline{D^+}^{||\cdot||_{L^2(\Sigma)}}$,

(ii) $\left(\frac{\partial H^{\alpha}_{-n-1,k}}{\partial \nu_{\Sigma}}\right)_{\substack{n=0,1,\ldots \\ k=1,\ldots,2n+1}}$ *is complete in* $L^2(\Sigma) = \overline{N^+}^{||\cdot||_{L^2(\Sigma)}}$.

For the proof the reader is referred to [16], [33].

From functional analysis we know that the properties of completeness and closure are equivalent in a Hilbert space such as $L^2(\Sigma)$. This leads us to the following corollary.

Corollary 3.5.3. *Under the assumptions of Theorem* 3.5.2 *the following statements are valid:*

(i) $(H^{\alpha}_{-n-1,k}|_{\Sigma})_{\substack{n=0,1,\ldots \\ k=1,\ldots,2n+1}}$ *is closed in* $L^2(\Sigma)$, *i.e., for given* $F \in L^2(\Sigma)$ *and arbitrary* $\varepsilon > 0$ *there exists a linear combination*

$$H_m = \sum_{n=0}^{m} \sum_{k=1}^{2n+1} a_{n,k} H^{\alpha}_{-n-1,k}|_{\Sigma}, \tag{3.55}$$

such that

$$||F - H_m||_{L^2(\Sigma)} \leq \varepsilon. \tag{3.56}$$

(ii) $\left(\frac{\partial H^{\alpha}_{-n-1,k}}{\partial \nu_\Sigma}\right)_{\substack{n=0,1,\ldots\\ k=1,\ldots,2n+1}}$ *is closed in* $L^2(\Sigma)$*, i.e., for given* $F \in L^2(\Sigma)$ *and arbitrary* $\varepsilon > 0$ *there exists a linear combination*

$$S_m = \sum_{n=0}^{m} \sum_{k=1}^{2n+1} a_{n,k} \frac{\partial H^{\alpha}_{-n-1,k}}{\partial \nu_\Sigma}, \tag{3.57}$$

such that

$$||F - S_m||_{L^2(\Sigma)} \leq \varepsilon. \tag{3.58}$$

Based on the results on outer harmonics developed in Section 3.5, a large number of ‘polynomial-based’ countable systems of potentials can be shown to have the L^2-closure property on Σ. The best known are ‘mass point representations’, i.e., fundamental solutions of the Laplace operator (see Section 2.6). Their L^2-closure is adequately described by using the concept of ‘fundamental systems’, which we recapitulate briefly for the case of regular surfaces.

Definition 3.5.4. *A sequence* $Y = (y_n)_{n=0,1,\ldots} \subset \Sigma^{int}$ *of points of the inner space* Σ^{int} *(with* $y_n \neq y_l$ *for* $n \neq l$*) is called a fundamental system in* Σ^{int} *if the following properties are satisfied*

(i) $dist(Y, \Sigma) > 0$*,*

(ii) for each $U \in Pot(\Sigma^{int})$ *the conditions* $U(y_n) = 0$ *for* $n = 0, 1, \ldots$ *imply* $U = 0$ *in* Σ^{int}*.*

Some examples of fundamental systems should be listed for the inner space Σ^{int}. $Y = (y_n)_{n=0,1\ldots}$ is, for example, a fundamental system in Σ^{int} if it is a dense set of points of one of the following subsets of Σ^{int}:

(i) region Ξ^{int} with $\overline{\Xi^{int}} \subset \Sigma^{int}$,

(ii) boundary $\partial\Xi^{int}$ of a region Ξ^{int} with $\overline{\Xi^{int}} \subset \Sigma^{int}$.

Theorem 3.5.5. *Let* Σ *be a regular surface such that* (3.1) *holds true. Then the following statements are valid:*

(i) For every fundamental system $Y = (y_n)_{n=0,1\ldots}$ *in* Σ^{int} *the system*

$$\left(x \mapsto |x - y_n|^{-1}, x \in \Sigma\right)_{n=0,1\ldots} \tag{3.59}$$

is closed in $L^2(\Sigma) = \overline{D^+}^{L^2(\Sigma)}$*.*

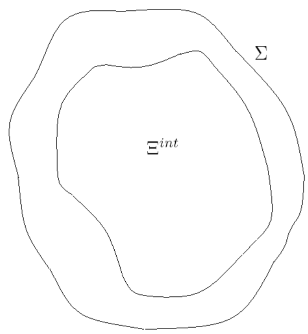

Figure 3.3: Region Ξ^{int} inside Σ

(ii) For every fundamental system $Y = (y_n)_{n=0,1\ldots}$ in Σ^{int} the system

$$\left(x \mapsto \frac{\partial}{\partial \nu(x)} |x - y_n|^{-1}, x \in \Sigma \right)_{n=0,1\ldots} \tag{3.60}$$

is closed in $L^2(\Sigma) = \overline{N^+}^{L^2(\Sigma)}$.

For proof the reader is referred to [16], [33]. Besides outer harmonics and mass poles there are other countable systems of potentials satisfying the properties of completeness and closure in $L^2(\Sigma)$. The systems generated by superposition (i.e., infinite clustering) of outer harmonics, turn out to be particularly suitable for numerical purposes (see e.g., [26], [27]).

Theorem 3.5.6. *Let Σ be a regular surface such that (3.1) is satisfied. Consider the kernel function*

$$\begin{aligned} K(x,y) &= \sum_{k=0}^{\infty} \sum_{l=1}^{2k+1} K^\wedge(k) H^\alpha_{k,l}(y) H^\alpha_{-k-1,l}(x) \\ &= \frac{\alpha}{|x|} \sum_{k=0}^{\infty} \frac{2k+1}{4\pi\alpha^2} K^\wedge(k) \left(\frac{|y|}{|x|} \right)^k P_k \left(\frac{x}{|x|} \cdot \frac{y}{|y|} \right), \end{aligned} \tag{3.61}$$

$y \in A^{int}$, $x \in \overline{A^{ext}}$. Let $Y = (y_n)_{n=0,1\ldots}$ be a fundamental system in Σ^{int}_{inf} with

$$\underline{\alpha} = \sup_{y \in Y} |y| < \alpha < \sigma^{inf} = \inf_{x \in \Sigma} |x|. \tag{3.62}$$

Suppose that

$$\sum_{k=0}^{\infty}(2k+1)\left|K^{\wedge}(k)\right|\left(\frac{\underline{\alpha}}{\alpha}\right)^{k}<\infty, \tag{3.63}$$

with $K^{\wedge}(k) \neq 0$ *for* $k = 0, 1, \ldots$. *Then the following statements are valid:*

(i) *The system*

$$(x \mapsto K(x, y_n), x \in \Sigma)_{n=0,1,\ldots} \tag{3.64}$$

is closed in $L^2(\Sigma) = \overline{D^+}^{L^2(\Sigma)}$.

(ii) *The system*

$$(x \mapsto \frac{\partial}{\partial\nu(x)} K(x, y_n), x \in \Sigma)_{n=0,1,\ldots} \tag{3.65}$$

is closed in $L^2(\Sigma) = \overline{N^+}^{L^2(\Sigma)}$.

Examples of kernel representations (3.61) are easily obtainable from known series expansions in terms of Legendre polynomials (see, e.g., [54]).

With the help of the Kelvin transform (2.89), analogous arguments hold for fundamental system in Σ^{ext}.
Applying the Kelvin transform with respect to the sphere A around origin with radius α, we are led to systems (see [32], [39])

$$\left(\overline{K}(x, \overline{y_n}) | x \in \overline{\Sigma^{ext}}\right)_{n=0,1,\ldots} \tag{3.66}$$

with

$$\begin{aligned}\overline{K}(x, \overline{y}) &= \sum_{k=0}^{\infty}\sum_{l=1}^{2k+1} K^{\wedge}(k) H^{\alpha}_{-k-1,l}(x) H^{\alpha}_{-k-1,l}(\overline{y}) \\ &= \sum_{k=0}^{\infty}\frac{2k+1}{4\pi\alpha^2}K^{\wedge}(k)\left(\frac{\alpha^2}{|x||\overline{y}|}\right)^{k+1} P_k\left(\frac{x}{|x|}\cdot\frac{\overline{y}}{|\overline{y}|}\right), \quad x \in \overline{\Sigma^{ext}}, \overline{y} \in \overline{Y} \subset \overline{\Sigma^{ext}_{inf}},\end{aligned} \tag{3.67}$$

where $\overline{Y} = (\overline{y}_n)_{n=0,1,\ldots}$ is the point system generated by Y by letting

$$\overline{y}_n = \frac{\alpha^2}{|y_n|^2} y_n, \quad n = 0, 1, \ldots \tag{3.68}$$

(thereby assuming that $0 \notin Y$). Note that our assumptions above imply the estimate

$$\sum_{k=0}^{\infty}(2k+1)\left|K^{\wedge}(k)\right|\left(\frac{\alpha}{\overline{\alpha}}\right)^{k}<\infty, \tag{3.69}$$

where $\overline{\alpha}$ is given by

$$\overline{\alpha} = \inf_{\overline{y} \in \overline{Y}} |\overline{y}| > \alpha. \tag{3.70}$$

Theorem 3.5.7. *Suppose that $\overline{Y} = (\overline{y}_n)_{n=0,1,...}$ is given as described above. Let $\overline{K}(x, \overline{y})$ be given by* (3.67) *with coefficients $K^\wedge(k) \neq 0$ for $k = 0, 1, ...$ satisfying* (3.69)*. Then the following properties hold true:*

(i) The system

$$(x \mapsto \overline{K}(x, \overline{y}_n), x \in \Sigma)_{n=0,1,...} \tag{3.71}$$

is closed in $L^2(\Sigma) = \overline{D^+}^{L^2(\Sigma)}$.

(ii) The system

$$(x \mapsto \frac{\partial}{\partial \nu(x)} \overline{K}(x, \overline{y}_n), x \in \Sigma)_{n=0,1,...} \tag{3.72}$$

is closed in $L^2(\Sigma) = \overline{N^+}^{L^2(\Sigma)}$.

Combining the L^2-closure (Theorem 3.5.2) for the system of outer harmonics and the regularity theorems (Theorem 3.4.1 and Theorem 3.4.2), we obtain the following

Theorem 3.5.8. *Let Σ be a regular surface satisfying the condition* (3.1)*.*
(EDP) For given $F \in C^{(0)}(\Sigma)$, let U be the potential of class $Pot^{(0)}(\overline{\Sigma^{ext}})$, with $U^+_\Sigma = F$. Then, for any given value $\varepsilon > 0$ and $K \subset \Sigma^{ext}$ with $dist(\overline{K}, \Sigma) > 0$, there exist an integer $m = m(\varepsilon)$ and a set of coefficients $a_{0,1}, ..., a_{m,1}, ..., a_{m,2m+1}$, such that

$$\left(\int_\Sigma \left| F(x) - \sum_{n=0}^{m} \sum_{k=1}^{2n+1} a_{n,k} H^\alpha_{-n-1,k}(x) \right|^2 d\omega(x) \right)^{1/2} \leq \varepsilon \tag{3.73}$$

and

$$\sup_{x \in K} \left| \left(\nabla^{(k)} \right) U(x) - \sum_{n=0}^{m} \sum_{k=1}^{2n+1} a_{n,k} \left(\nabla^{(k)} H^\alpha_{-n-1,k} \right)(x) \right| \leq C\varepsilon \tag{3.74}$$

hold for all $k \in \mathbb{N}_0$.
(ENP) For given $F \in C^{(0)}(\Sigma)$, let U satisfy $U \in Pot^{(1)}(\overline{\Sigma^{ext}}), \partial U^+ / \partial \nu_\Sigma = F$. Then, for any given value $\varepsilon > 0$ and $K \subset \Sigma^{ext}$ with $dist(\overline{K}, \Sigma) > 0$, there exist an integer $m = m(\varepsilon)$ and a set of coefficients $a_{0,1}, ..., a_{m,1}, ..., a_{m,2m+1}$, such that

$$\left(\int_\Sigma \left| F(x) - \sum_{n=0}^{m} \sum_{k=1}^{2n+1} a_{n,k} \frac{\partial H^\alpha_{-n-1,k}}{\partial \nu}(x) \right|^2 d\omega(x) \right)^{1/2} \leq \varepsilon \tag{3.75}$$

and

$$\sup_{x\in K}\left|\left(\nabla^{(k)}\right)U(x)-\sum_{n=0}^{m}\sum_{k=1}^{2n+1}a_{n,k}\left(\nabla^{(k)}H^{\alpha}_{-n-1,k}\right)(x)\right|\leq C\varepsilon \tag{3.76}$$

hold for all $k\in\mathbb{N}_0$.

In other words, locally uniform approximation is guaranteed in terms of outer harmonics, i.e., the L^2- approximation in terms of outer harmonics on Σ implies the uniform approximation (in ordinary sense) on each subset K with positive distance of $\overline{K}$ to Σ. The above theorems are non-constructive, since further information about the choice of m and the coefficients of the approximating linear combination is needed. In order to derive a constructive approximation theorem the system of potential values and normal derivatives, respectively, has to be orthonormalized on Σ. As a result, we obtain a '*(generalized) Fourier expansion*' *(orthogonal Fourier approximation)* that shows locally uniform approximation (see e.g., [16], [33]). The orthonormalization procedure can be performed (e.g., by the well-known Gram-Schmidt orthonormalization process) once and for all when the regular surface Σ is specified. Furthermore, locally uniform approximation by 'generalized Fourier expansions' can be obtained not only for the multipole system of outer harmonics, but also for the mass point and related kernel representations.
In what follows we summarize our generalized Fourier approach in

Theorem 3.5.9. *Let* Σ *be a regular surface such that* (3.1) *holds true.*

(EDP) Let $(D_n)_{n=0,1,\ldots}, D_n\in Pot^{(0)}(\overline{A^{ext}}), n=0,1,\ldots$ *be a Dirichlet basis in* $\overline{\Sigma^{ext}}$, *i.e., a sequence* $(D_n)_{n=0,1,\ldots}\subset Pot^{(0)}(\overline{A^{ext}})$ *satisfying the properties*

(i)

$$\overline{span_{n=0,1,\ldots}(D_n|_\Sigma)}^{\|\cdot\|_{L^2(\Sigma)}}=L^2(\Sigma), \tag{3.77}$$

(ii)

$$(D_n|_\Sigma, D_m|_\Sigma)_{L^2(\Sigma)}=\delta_{nm}. \tag{3.78}$$

If $F\in C^{(0)}(\Sigma)$, *then*

$$\lim_{m\to\infty}\left(\int_\Sigma\left|F(x)-\sum_{n=0}^{m}(F,D_n)_{L^2(\Sigma)}D_n(x)\right|^2 d\omega(x)\right)^{1/2}=0. \tag{3.79}$$

The potential $U\in Pot^{(0)}(\overline{\Sigma^{ext}}), U^+_\Sigma=F$, *can be represented in the form*

$$\lim_{m\to\infty}\sup_{x\in K}\left|U(x)-U^{(m)}(x)\right|=0, \tag{3.80}$$

with

$$U^{(m)} = \sum_{n=0}^{m} (F, D_n)_{L^2(\Sigma)} D_n, \tag{3.81}$$

for every $K \subset \Sigma^{ext}$ *with* $dist(\overline{K}, \Sigma) > 0$.

(ENP) Let $(N_n)_{n=0,1,\ldots}, N_n \in Pot^{(0)}(\overline{A^{ext}}), n = 0, 1, \ldots$ *be a Neumann basis in* $\overline{\Sigma^{ext}}$, *i.e., a sequence* $(N_n)_{n=0,1,\ldots} \subset Pot^{(0)}(\overline{A^{ext}})$ *satisfying the properties*

(i)

$$\overline{span_{n=0,1,\ldots}\left(\frac{\partial N_n}{\partial \nu_\Sigma}\right)}^{\|\cdot\|_{L^2(\Sigma)}} = L^2(\Sigma), \tag{3.82}$$

(ii)

$$\left(\frac{\partial N_n}{\partial \nu_\Sigma}, \frac{\partial N_m}{\partial \nu_\Sigma}\right)_{L^2(\Sigma)} = \delta_{nm}. \tag{3.83}$$

If $F \in C^{(0)}(\Sigma)$, *then*

$$\lim_{m\to\infty} \left(\int_\Sigma \left| F(x) - \sum_{n=0}^{m} \left(F, \frac{\partial N_n}{\partial \nu_\Sigma}\right)_{L^2(\Sigma)} \frac{\partial N_n}{\partial \nu_\Sigma}(x) \right|^2 d\omega(x) \right)^{1/2} = 0. \tag{3.84}$$

The potential $U \in Pot^{(1)}(\overline{\Sigma^{ext}})$, $\frac{\partial U^+}{\partial \nu_\Sigma} = F$, *can be represented in the form*

$$\lim_{m\to\infty} \sup_{x\in K} \left| U(x) - U^{(m)}(x) \right| = 0, \tag{3.85}$$

with

$$U^{(m)} = \sum_{n=0}^{m} \left(F, \frac{\partial N_n}{\partial \nu_\Sigma}\right)_{L^2(\Sigma)} N_n, \tag{3.86}$$

for every $K \subset \Sigma^{ext}$ *with* $dist(\overline{K}, \Sigma) > 0$.

Finally, having in mind the regularity theorems (Theorem 3.4.1 and Theorem 3.4.2) for the classical boundary value problem we get the following

Corollary 3.5.10. *Let* Σ *be a regular surface such that* (3.1) *is valid.*

(EDP) For given $F \in C^{(0)}(\Sigma)$, *let* U *satisfy* $U \in Pot^{(0)}(\overline{\Sigma^{ext}}), U^+_\Sigma = F$. *Furthermore,*

suppose that $(D_n)_{n=0,1,\ldots}$ *is a Dirichlet basis in* $\overline{\Sigma^{ext}}$*. Then*

$$\sup_{x\in K}\left|\left(\nabla^{(k)}U\right)(x)-\sum_{n=0}^{m}(F,D_n)_{L^2(\Sigma)}\left(\nabla^{(k)}D_n\right)(x)\right| \tag{3.87}$$
$$\leq C\left(||F||^2_{L^2(\Sigma)}-\sum_{n=0}^{m}(F,D_n)^2_{L^2(\Sigma)}\right)^{1/2},$$

holds for all $k\in\mathbb{N}_0$ *and all subsets* $K\subset\Sigma^{ext}$ *satisfying dist*$(\overline{K},\Sigma)>0$.

(ENP) For given $F\in C^{(0)}(\Sigma)$*, let* U *satisfy* $U\in Pot^{(1)}(\overline{\Sigma^{ext}})$, $\frac{\partial U^+}{\partial\nu_\Sigma}=F$*. Furthermore, suppose that* $(N_n)_{n=0,1,\ldots}$ *is a Neumann basis in* $\overline{\Sigma^{ext}}$*. Then*

$$\sup_{x\in K}\left|\left(\nabla^{(k)}U\right)(x)-\sum_{n=0}^{m}\left(F,\frac{\partial N_n}{\partial\nu_\Sigma}\right)_{L^2(\Sigma)}\left(\nabla^{(k)}N_n\right)(x)\right| \tag{3.88}$$
$$\leq C\left(||F||^2_{L^2(\Sigma)}-\sum_{n=0}^{m}\left(F,\frac{\partial N_n}{\partial\nu_\Sigma}\right)^2_{L^2(\Sigma)}\right)^{1/2},$$

holds for all $k\in\mathbb{N}_0$ *and all subsets* $K\subset\Sigma^{ext}$ *satisfying dist*$(\overline{K},\Sigma)>0$.

Next we turn over to the exterior oblique derivative problem. To this end we consider the pre-Hilbert space $\left(C^{(0,\mu)}(\Sigma),||\cdot||_{L^2(\Sigma)}\right)$. The following theorem gives the necessary closure condition for the case of the oblique derivative.

Theorem 3.5.11. *Let* $(D_n)_{n=0,1,\ldots}\subset Pot^{(0)}(\overline{A^{ext}})$ *be a Dirichlet basis in* $\overline{A^{ext}}$ *(see Theorem* 3.5.9*). Then the linear space*

$$span_{n=0,1,\ldots}\left(\frac{\partial D_n^+}{\partial\lambda_\Sigma}\right) \tag{3.89}$$

is dense in the pre-Hilbert space $\left(C^{(0,\mu)}(\Sigma),||\cdot||_{L^2(\Sigma)}\right)$.

Orthonormalizing the system $\left(\frac{\partial D_n^+}{\partial\lambda_\Sigma}\right)_{n=0,1,\ldots}$ we obtain the following systems:

(i) a closed and complete orthonormal system $D_n(\Sigma;\cdot)_{n=0,1,\ldots}$ in the pre-Hilbert space $\left(C^{(0,\mu)}(\Sigma),||\cdot||_{L^2(\Sigma)}\right)$,

(ii) corresponding solutions $\tilde{D}_n(\Sigma;\cdot)_{n=0,1,\ldots}$ to the EODPs $\tilde{D}_n(\Sigma;\cdot) \in \mathrm{Pot}^{(1,\mu)}(\overline{\Sigma^{ext}})$, $0 < \mu < 1$, such that

$$\frac{\partial \tilde{D}_n(\Sigma;\cdot)}{\partial \lambda_\Sigma} = D_n(\Sigma;\cdot). \tag{3.90}$$

Furthermore, for $U \in \mathrm{Pot}^{(1,\mu)}(\overline{\Sigma^{ext}})$, $F = \frac{\partial U^+}{\partial \lambda_\Sigma}$, the orthogonal (Fourier) expansion

$$\sum_{n=0}^{\infty} (F, D_n(\Sigma;\cdot))_{L^2(\Sigma)} \frac{\partial \tilde{D}_n(\Sigma;\cdot)}{\partial \lambda_\Sigma} \tag{3.91}$$

converges to F (in the sense of $||\cdot||_{L^2(\Sigma)}$). From the regularity theorem (Theorem 3.4.4) it follows that

$$U(x) = \sum_{n=0}^{\infty} (F, D_n(\Sigma;\cdot))_{L^2(\Sigma)} \tilde{D}_n(\Sigma;x), \quad x \in K, \tag{3.92}$$

holds uniformly on each subset K of Σ^{ext} with a positive distance of $\overline{K}$ to the boundary Σ.

3.6 Globally Uniform Approximation

The same results from the previous section remain valid when the regular surface Σ is replaced by any parallel surface $\Sigma(\tau)$ of distance $|\tau|$ to Σ (where $|\tau|$ is chosen sufficiently small). This fact can be exploited to verify the following closure properties (see [16]).

Theorem 3.6.1. *Let Σ be a regular surface satisfying (3.1). Then the following statements are true:*

(i) $\left(H^\alpha_{-n-1,k}|_\Sigma\right)$ *is closed in* $D^+ = C^{(0)}(\Sigma)$, *i.e.,*

$$D^+ = \overline{span\left(H^\alpha_{-n-1,k}|_\Sigma\right)}^{||\cdot||_{C^{(0)}(\Sigma)}} = C^{(0)}(\Sigma), \tag{3.93}$$

(ii) $\left(\frac{\partial H^\alpha_{-n-1,k}}{\partial \nu_\Sigma}|_\Sigma\right)$ *is closed in* $N^+ = C^{(0)}(\Sigma)$, *i.e.,*

$$N^+ = \overline{span\left(\partial H^\alpha_{-n-1,k}/\partial \nu_\Sigma\right)}^{||\cdot||_{C^{(0)}(\Sigma)}} = C^{(0)}(\Sigma). \tag{3.94}$$

Combining this results with the norm estimates (3.46) and (3.47) we get

Theorem 3.6.2. *Let Σ be a regular surface satisfying* (3.1)*. Then the following statements are valid:*

(EDP) For given $F \in D^+ = C^{(0)}(\Sigma)$, let U satisfy $U \in Pot^{(0)}(\overline{\Sigma^{ext}}), U_\Sigma^+ = F$. Then, to every $\varepsilon > 0$, there exist an integer $m = m(\varepsilon)$ and a finite set of a real numbers $a_{n,k}$ such that

$$\begin{aligned} &\sup_{x\in\overline{\Sigma^{ext}}} \left| U(x) - \sum_{n=0}^{m}\sum_{k=1}^{2n+1} a_{n,k} H^{\alpha}_{-n-1,k}(x) \right| \\ &\leq \sup_{x\in\Sigma} \left| F(x) - \sum_{n=0}^{m}\sum_{k=1}^{2n+1} a_{n,k} H^{\alpha}_{-n-1,k}(x) \right| \leq \varepsilon. \end{aligned} \tag{3.95}$$

(ENP) For given $F \in N^+ = C^{(0)}(\Sigma)$, let U satisfy $U \in Pot^{(1)}(\overline{\Sigma^{ext}})$, $\partial U^+/\partial \nu_\Sigma = F$. Then, to every $\varepsilon > 0$, there exist an integer $m = m(\varepsilon)$ and a finite set of a real numbers $a_{n,k}$ such that

$$\begin{aligned} &\sup_{x\in\overline{\Sigma^{ext}}} \left| U(x) - \sum_{n=0}^{m}\sum_{k=1}^{2n+1} a_{n,k} H^{\alpha}_{-n-1,k}(x) \right| \\ &\leq C \sup_{x\in\Sigma} \left| F(x) - \sum_{n=0}^{m}\sum_{k=1}^{2n+1} a_{n,k} \frac{\partial H^{\alpha}_{-n-1,k}}{\partial \nu_\Sigma}(x) \right| \leq \varepsilon. \end{aligned} \tag{3.96}$$

Unfortunately, a constructive procedure for determining the best approximation coefficients $a_{n,k}$ in the $||\cdot||_{C^{(0)}(\Sigma)}$-topology seems to be unknown. Therefore, harmonic splines are introduced on Hilbert subspaces of $\mathrm{Pot}^{(0)}(\overline{\Sigma^{ext}})$, so that the best approximations to solutions of boundary-value problems can be guaranteed on certain types of Sobolev-like subspaces of $\mathrm{Pot}^{(0)}(\overline{\Sigma^{ext}})$ (see Chapter 6).

Turning over to EODP we are confronted with the following (cf. [18])

Theorem 3.6.3. *Let Σ be a $C^{(2,\mu)}$-regular surface with $\mu \in (0,1)$ satisfying* (3.1)*. Then $\left(\frac{\partial H^{\alpha}_{-n-1,k}}{\partial \lambda_\Sigma}|_\Sigma\right)$ is closed in $C^{(0,\mu)}(\Sigma)$, i.e.,*

$$\overline{span\left(\partial H^{\alpha}_{-n-1,k}/\partial \lambda_\Sigma\right)}^{||\cdot||_{C^{(0,\mu)}(\Sigma)}} = C^{(0,\mu)}(\Sigma), \tag{3.97}$$

where $\mu \in (0,1)$ and the vector field λ on Σ is required to be $C^{(1,\mu)}$, to satisfy $|\lambda| = 1$ and to possess a non-vanishing normal part $\inf_{x\in\Sigma} \lambda(x)\cdot\nu(x) > 0$.

Theorem 3.6.4. *Let Σ be a $C^{(2,\mu)}$-regular surface with $\mu \in (0,1)$ satisfying* (3.1)*. Then the following statement is valid:*

(EODP) For given $F \in C^{(0,\mu)}(\Sigma)$ $\mu \in (0,1)$*, let* U *satisfy* $U \in Pot^{(1,\mu)}(\overline{\Sigma^{ext}})$*,* $\partial U^{+}/\partial\lambda_{\Sigma} = F$*. Then, to every* $\varepsilon > 0$*, there exists an integer* $m = m(\varepsilon)$ *and a finite set of real numbers* $a_{n,k}$ *such that*

$$\begin{aligned}&\sup_{x\in\overline{\Sigma^{ext}}}\left|U(x)-\sum_{n=0}^{m}\sum_{k=1}^{2n+1}a_{n,k}H^{\alpha}_{-n-1,k}(x)\right|\\&\leq C\left\|F(x)-\sum_{n=0}^{m}\sum_{k=1}^{2n+1}a_{n,k}\frac{\partial H^{\alpha}_{-n-1,k}}{\partial\lambda_{\Sigma}}(x)\right\|_{C^{(0,\mu)}(\Sigma)}\leq\varepsilon.\end{aligned}\tag{3.98}$$

More general results can be found in [17] and [43].

4 Real Earth Body Based Strategy

4.1 Geodetic Approach to Gravity and Gravitation

In this thesis we are interested in the determination of (the Earth's equipotential surfaces arising from) the Earth's gravitational potential from georelevant data. As already explained in the Introduction the exact knowledge of the Earth's gravitational potential, and subsequently the equipotential surfaces, especially the geoid, is of enormous importance in all sciences that contribute to the knowledge of Earth's processes.
In what follows we present some basic ingredients of the gravitational field theory (for more details the interested reader is referred to [50]).

The gravity acceleration (gravity) of the Earth is the resultant of the gravitational and

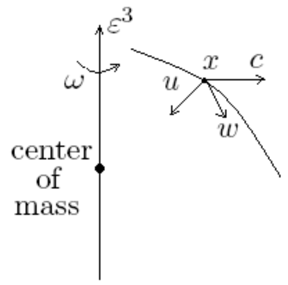

Figure 4.1: The gravitational and the centrifugal force

the centrifugal acceleration (Figure 4.1). The centrifugal force c arises as a result of the rotation of the Earth about it's axis (usually the ε^3-axis). Considering a rectangular coordinate system whose origin is at the Earth's center of gravity and z-axis coincides with the Earth's mean axis of rotation. The centrifugal force $\mathbf{c}$ on a unit mass is given by $\mathbf{c} = \omega^2 p$, where ω is the angular velocity of the Earth's rotation and $p = \sqrt{x^2 + y^2}$ is the distance from the axis of rotation. It can also be derived from a potential $C = \frac{1}{2}\omega^2(x^2 + y^2)$ so that $\mathbf{c} = \nabla C$. The gravity potential W of the Earth is the sum of the gravitational potential

V, and the centrifugal potential C, i.e.,

$$W = V + C \tag{4.1}$$

The gradient vector of W

$$\mathbf{g} = \nabla W = \left(\frac{\partial W}{\partial x}, \frac{\partial W}{\partial y}, \frac{\partial W}{\partial z}\right) \tag{4.2}$$

is called the *gravity vector* and we can write

$$\mathbf{g} = \nabla W = \nabla V + \nabla C. \tag{4.3}$$

The surfaces of constant gravity potential are called *equipotential surfaces* (in geodesy also known as geopotential surfaces). The *geoid* is an example of such a surface. Namely, it is a 'horizontal' or 'level' surface, which is everywhere perpendicular to the local direction of gravity. If there were no 'disturbations' in the ocean, it is where the sea surface would settle in equilibrium. The magnitude of the gravity vector, is the quantity $g = |\mathbf{g}|$ called the *gravity intensity* (or often just gravity in the narrower sense). It is measured in gals (1 gal = 1 cm sec^{-2}), the unit being named in honor of Galileo Galilei. The numerical value of g is about 978 gals at the equator, and 983 gals at the poles. The direction of the gravity vector is the direction of the *plumb line* or the vertical, and it is of basic significance for geodetic and astronomic measurements.

In Newtonian nomenclature, the gravitational potential V of the Earth generated by a mass-distribution F inside the Earth is given by the volume integral (Newton integral)

$$V(x) = G \int_{Earth} \frac{F(y)}{|x-y|} dy, \quad x \in \mathbb{R}^3, \tag{4.4}$$

where G is the gravitational constant ($G = 6.67422 \cdot 10^{-11} m^3/(\text{kg s}^2)$) and dy is the volume element. As is well known (see, e.g., [51]), the gravitational potential of the Earth corresponding to an integrable and bounded density function F, satisfies the Laplace equation in the outer space

$$\Delta V = 0, \tag{4.5}$$

and the Poisson equation in the interior space

$$\Delta V = -4\pi F. \tag{4.6}$$

The Newton integral (4.4) and its first derivatives are continuous everywhere on $\mathbb{R}^3$, i.e., $V \in C^{(1)}(\mathbb{R}^3)$. The second derivatives are analytic everywhere outside the real Earth surface, but they have a discontinuity when passing across the surface. Moreover, the gravitational potential V of the Earth, shows at infinity the following behavior:

(i) $|V(x)| = \mathcal{O}\left(\frac{1}{|x|}\right)$, $x \to \infty$,

(ii) $|\nabla V(x)| = \mathcal{O}\left(\frac{1}{|x|^2}\right)$ $x \to \infty$,

i.e., it is regular at infinity.

In the sequel we discuss the Newton integral in more detail.

4.2 Newton Potential

Let $\Sigma \subset \mathbb{R}^3$ be a regular surface. The Newtonian potential V can be expressed by the integral

$$V(x) = \int_{\Sigma^{int}} \frac{F(y)}{|x-y|} dy, \quad x \in \mathbb{R}^3, \tag{4.7}$$

where F is the density function. For this integral we have

Theorem 4.2.1. *Let $F : \overline{\Sigma^{int}} \to \mathbb{R}$ be of class $C^{(0)}(\overline{\Sigma^{int}})$. Then $V : \overline{\Sigma^{ext}} \to \mathbb{R}$ given by*

$$V(x) = \int_{\Sigma^{int}} F(y) \frac{1}{|x-y|} dy, \quad x \in \Sigma^{ext}, \tag{4.8}$$

satisfies the Laplace equation in Σ^{ext}, i.e.,

$$\Delta_x V(x) = \Delta_x \int_{\Sigma^{int}} F(y) \frac{1}{|x-y|} dy = \int_{\Sigma^{int}} F(y) \Delta_x \frac{1}{|x-y|} dy = 0. \tag{4.9}$$

Next, we are interested in showing that the Newton integral satisfies the Poisson equation in Σ^{int}, at least under some canonical conditions on the density function F. Our considerations essentially follow [33], [38].

Theorem 4.2.2. *Let $F : \overline{\Sigma^{int}} \to \mathbb{R}$ be of class $C^{(0)}(\overline{\Sigma^{int}})$. Then $V : \overline{\Sigma^{int}} \to \mathbb{R}$ given by*

$$V(x) = \int_{\Sigma^{int}} F(y) \frac{1}{|x-y|} dy, \quad x \in \overline{\Sigma^{int}} \tag{4.10}$$

is of class $C^{(1)}(\overline{\Sigma^{int}})$, and we have

$$\nabla_x V(x) = \int_{\Sigma^{int}} F(y) \nabla_x \frac{1}{|x-y|} dy = -\int_{\Sigma^{int}} F(y) \frac{x-y}{|x-y|^3} dy, \quad x \in \overline{\Sigma^{int}}. \tag{4.11}$$

Proof: We replace the fundamental solution of potential theory

$$S : (x, y) \mapsto S(|x-y|),\ x \neq y,$$

given by

$$S(|x-y|) = \frac{1}{|x-y|}, \tag{4.12}$$

by a 'regularization' of the form

$$S_\delta(|x-y|) = \begin{cases} \frac{1}{2\delta}(3 - \frac{1}{\delta^2}|x-y|^2), & |x-y| \leq \delta \\ \frac{1}{|x-y|}, & |x-y| > \delta, \end{cases} \tag{4.13}$$

where $\delta > 0$. In other words, by letting $r = |x-y|$, we replace $S(r) = \frac{1}{r}$, by

$$S_\delta(r) = \begin{cases} \frac{1}{2\delta}(3 - \frac{1}{\delta^2}r^2), & r \leq \delta \\ \frac{1}{r}, & r > \delta, \end{cases}. \tag{4.14}$$

It can be easily seen that S_δ is continuously differentiable for all for all $r \geq 0$, and moreover $S(r) = S_\delta(r)$ for all $r > \delta$.
We set

$$V(x) = \int_{\Sigma^{int}} F(y) S(|x-y|) dy \tag{4.15}$$

and

$$V_{S_\delta}(x) = \int_{\Sigma^{int}} F(y) S_\delta(|x-y|) dy. \tag{4.16}$$

The integrands of V and V_{S_δ} differ only in the ball around the point x with radius δ. Moreover, the function $F : \overline{\Sigma^{int}} \to \mathbb{R}$ is supposed to be continuous. Hence, it is uniformly bounded on $\overline{\Sigma^{int}}$ by

$$||F||_{C(\overline{\Sigma^{int}})} = \sup_{x \in \overline{\Sigma^{int}}} |F(x)|. \tag{4.17}$$

This shows us that

$$|V(x) - V_{S_\delta}(x)| = O\left(\int_{|x-y| \leq \delta} (S(|x-y|) - S_\delta(|x-y|)) dy \right) = O(\delta^2). \tag{4.18}$$

Therefore, V is of class $C^{(0)}(\overline{\Sigma^{int}})$, as a limit of a uniformly convergent sequence of continuous functions on $\overline{\Sigma^{int}}$. Furthermore we set

$$v = \int_{\Sigma^{int}} F(y) \nabla_x S(|x-y|) dy \tag{4.19}$$

and

$$v_{S_\delta}(x) = \int_{\Sigma^{int}} F(y)\nabla_x S_\delta(|x-y|)dy. \tag{4.20}$$

Note that

$$\left|\nabla_x \frac{1}{|x-y|}\right| = \left|-\frac{x-y}{|x-y|^3}\right| = \frac{1}{|x-y|^2}, \tag{4.21}$$

so the integrals v and v_{S_δ} exist for all $x \in \overline{\Sigma^{int}}$. It is not difficult to see that

$$\sup_{x\in\overline{\Sigma^{int}}} |v(x) - v_{S_\delta}(x)| = \sup_{x\in\overline{\Sigma^{int}}} |\nabla V(x) - \nabla V_{S_\delta}(x)| = O(\delta). \tag{4.22}$$

Consequently, v is a continuous vector field on $\overline{\Sigma^{int}}$. Moreover, as the relation (4.22) holds uniformly on $\overline{\Sigma^{int}}$, we obtain

$$v(x) = \nabla V(x) = \int_{\Sigma^{int}} F(y)\nabla_x S(|x-y|)dy. \tag{4.23}$$

This is the desired result.
Next we come to the Poisson equation under the assumption of Hölder continuity of the function F on $\overline{\Sigma^{int}}$.

Theorem 4.2.3. *Let $F : \overline{\Sigma^{int}} \to \mathbb{R}$ be Hölder continuous with exponent $\alpha \in (0,1]$ on $\overline{\Sigma^{int}}$. Then the potential $V : \overline{\Sigma^{int}} \to \mathbb{R}$*

$$V(x) = \int_{\Sigma^{int}} F(y)\frac{1}{|x-y|}dy, \quad x \in \overline{\Sigma^{int}} \tag{4.24}$$

satisfies the Poisson differential equation

$$\Delta_x V(x) = -4\pi F(x), \tag{4.25}$$

for all $x \in \overline{\Sigma^{int}}$.

Proof: We introduce the function

$$H_\delta(|x-y|) = \begin{cases} \frac{1}{2\delta^3}(5 - \frac{3}{\delta^2}|x-y|^2), & |x-y| \le \delta \\ \\ \frac{1}{|x-y|^3}, & |x-y| > \delta, \end{cases} \tag{4.26}$$

Setting $r = |x-y|$, we obtain

$$H_\delta(r) = \begin{cases} \frac{1}{2\delta^3}(5 - \frac{3}{\delta^2}r^2), & r \le \delta \\ \\ \frac{1}{r^3}, & r > \delta, \end{cases} \tag{4.27}$$

thus H_δ is continuously differentiable for all $r \geq 0$. Moreover, by already known arguments, it can be shown (cf. Theorem 4.2.2) that the vector field

$$-\int_{\Sigma^{int}} F(y)(x-y)H_\delta(|x-y|)dy. \tag{4.28}$$

converges uniformly on $\overline{\Sigma^{int}}$ to the limit field

$$\nabla V(x) = -\int_{\Sigma^{int}} F(y)\frac{x-y}{|x-y|^3}dy. \tag{4.29}$$

Now for all $x \in \mathbb{R}^3$ with $|x-y| \leq \delta$, a simple calculation yields

$$\nabla_x \cdot ((x-y)H_\delta(|x-y|)) = \frac{15}{2}\left(\frac{1}{\delta^3} - \frac{|x-y|^2}{\delta^5}\right). \tag{4.30}$$

Furthermore,

$$\int_{|x-y|\leq\delta} \nabla_x \cdot ((x-y)H_\delta(|x-y|)) = 4\pi. \tag{4.31}$$

Hence it is not hard to verify that

$$\begin{aligned}
&-\nabla_x \cdot \int_{\Sigma^{int}} F(y)(x-y)H_\delta(|x-y|)dy \\
&= -\int_{|x-y|\leq\delta} F(y)\nabla_x \cdot ((x-y)H_\delta(|x-y|))dy \\
&= -4\pi F(x) \\
&+ \int_{|x-y|\leq\delta} (F(x)-F(y))\nabla_x \cdot ((x-y)H_\delta(|x-y|))dy.
\end{aligned} \tag{4.32}$$

The Hölder continuity of F then assures the estimate

$$\sup_{x\in\overline{\Sigma^{int}}} \left| -\nabla_x \cdot \int_{\Sigma^{int}} F(y)(x-y)H_\delta(|x-y|)dy + 4\pi F(x)\right| = O(\delta^\alpha). \tag{4.33}$$

In a analogous way, we are able to show that the first partial derivatives of (4.31) uniformly converge to continuous limit fields. This shows that ∇V is differentiable in $\overline{\Sigma^{int}}$, and we have

$$\Delta_x \int_{\Sigma^{int}} \frac{F(y)}{|x-y|}dy = -4\pi F(x), \tag{4.34}$$

as required.

4.3 L^2-decomposition for Regions in $\mathbb{R}^3$

Note that from now on the Earth surface will be understood as a regular surface Σ, and it's interior and exterior space, respectively is denoted by Σ^{int} and Σ^{ext}.
We are interested in introducing an appropriate Hilbert space structure on the space of potentials in $\overline{\Sigma^{ext}}$, generated by the Newton integral (4.7). In order to accomplish this, some characterization of the density function $F : \Sigma^{int} \to \mathbb{R}$ is required. It will be shown that associating the density function to the class of harmonic functions in $L^2(\Sigma^{int})$ is closely related to the Hilbert space structure we are interested in. In the following, some necessary definitions and theorems will be first introduced. For more details the reader is referred to [1] or [69].

4.3.1 Distributionally Harmonic Functions in $L^2(\Sigma^{int})$

Definition 4.3.1. *Let $D \subset \mathbb{R}^3$ be an open set. We define*

$$C_0^{(k)}(D) = \left\{ F \in C^{(k)}(D) |\ supp(F) \ \text{ is compact in } \ D \right\}, \tag{4.35}$$

for $k \in \mathbb{N} \cup \{\infty\}$, where

$$supp(F) = \overline{\{x \in D | F(x) \neq 0\}} \tag{4.36}$$

is the support of F.

Theorem 4.3.2. *The space $C_0^{(\infty)}(D)$ is dense in $L^p(D)$, $1 \leq p < \infty$.*

Theorem 4.3.3. *The space $C_0^{(0)}(D)$ is dense in $L^p(D)$, $1 \leq p < \infty$.*

For the proof the reader is referred, e.g., to [1].

Definition 4.3.4. *Let D be an open set. By $\mathcal{D}(D)$ we denote the space $C_0^{(\infty)}(D)$ equipped with the following topology:*

The sequence $\{\varphi_k\}_{k\in\mathbb{N}} \subset \mathcal{D}(D)$ is called convergent to 0 iff

(i) there exists a bounded set $E \subset D$ such that every φ_k vanishes outside E.

(ii) the sequence $\{\nabla^\alpha \varphi_k\}_{k\in\mathbb{N}}$ is convergent to 0 with respect to $C^{(0)}(D)$ for every multi-index α.

Elements of $\mathcal{D}(D)$ are called test functions.

Definition 4.3.5. *Let $D \subset \mathbb{R}^3$ be an open set. Elements of the dual space $\mathcal{D}^*(D)$, i.e., continuous linear functionals*

$$\mathcal{F} : \mathcal{D}(D) \to \mathbb{R}, \tag{4.37}$$

are called distributions. Scalar multiplication and addition are defined for functionals and are used for distributions in the same way ([69]).

In Hilbert spaces like $L^2(D)$ (D measurable) continuous linear functionals on the Hilbert space can be represented by a scalar product with a given element that only depends on the functional. This is a result of Riesz's representation theorem. But on $\mathcal{D}(D)$ this is not always possible.

Definition 4.3.6. *Let $D \subset \mathbb{R}^3$ be a given open and measurable set and let $\mathcal{F} \in \mathcal{D}^*(D)$ be a given distribution. If there exists a function $F : D \to \mathbb{R}$ which is locally integrable, i.e., F is Lebesgue integrable on every compact subset of D, such that*

$$\mathcal{F}\varphi = \int_D F(x)\varphi(x)dx \tag{4.38}$$

for all test functions $\varphi \in \mathcal{D}(D)$, then $\mathcal{F}$ is called a regular distribution. ([69])

Theorem 4.3.7. *Let $D \subset \mathbb{R}^3$ be a given open and measurable set. If $\mathcal{F} \in \mathcal{D}^*(D)$ is a regular distribution then the corresponding function F is uniquely determined with the exception of a set of Lebesgue measure zero.*

Hence regular distributions are usually identified with their corresponding functions in the sense of an L^p- space.

Definition 4.3.8. *Let $D \subset \mathbb{R}^3$ be an open set. A sequence $\{\mathcal{F}_k\}_{k\in\mathbb{N}}$ in $\mathcal{D}^*(D)$ is called convergent to $\mathcal{F} \in \mathcal{D}^*(D)$ iff*

$$\lim_{k\to\infty} (\mathcal{F}_k\varphi) = \mathcal{F}\varphi \ \text{ for all } \ \varphi \in \mathcal{D}(D). \tag{4.39}$$

Derivatives of distributions can also be defined.

Definition 4.3.9. *Let $D \subset \mathbb{R}^3$ be an open set and $\mathcal{F} \in \mathcal{D}^*(D)$ be a given distribution. If there exists a distribution $\mathcal{K} \in \mathcal{D}^*(D)$ such that*

$$\mathcal{K}\varphi = (-1)^{|\alpha|}\mathcal{F}(\nabla^\alpha\varphi), \tag{4.40}$$

for every $\varphi \in \mathcal{D}(D)$, then we write

$$\nabla^\alpha\mathcal{F} = \mathcal{K}. \tag{4.41}$$

We are particularly interested in one special case.

Definition 4.3.10. *Let $D \subset \mathbb{R}^3$ be an open and measurable set. A functional $\mathcal{F} \in \mathcal{D}^*(D)$ is called a distributionally harmonic functional or harmonic distribution iff*

$$\Delta \mathcal{F} = 0, \tag{4.42}$$

where Δ is the Laplace operator. We denote the set of all regular harmonic $L^2(D)$–distributions in $\mathcal{D}^*(D)$ by $Harm(D)$, i.e.,

$$Harm(D) = \{F \in L^2(D) | \Delta F = 0\}. \tag{4.43}$$

With Definition 4.3.10 we apparently obtain a generalization of the harmonic functions. It is easy to prove the following

Theorem 4.3.11. *Let $D \subset \mathbb{R}^3$ be a given open and measurable set. Then the space $Harm(D)$ is a closed subspace of $L^2(D)$ (concerning the functions that correspond to the regular distributions).*

Proof: Let $F, G \in Harm(D)$ be given and $r \in \mathbb{R}$. Then

$$\int_D F(x)\Delta\varphi(x)dx = \int_D G(x)\Delta\varphi(x)dx = 0, \tag{4.44}$$

for all $\varphi \in \mathcal{D}(D)$. Hence

$$\int_D (F(x) + G(x))\, \Delta\varphi(x)dx = 0 \tag{4.45}$$

and

$$\int_D (rF(x))\, \Delta\varphi(x)dx = 0, \tag{4.46}$$

for all $\varphi \in \mathcal{D}(D)$. Thus $Harm(D)$ is a linear subspace of $L^2(D)$.

Let $\{F_n\}_{n\in\mathbb{N}} \subset Harm(D)$ be a given arbitrary sequence with

$$\lim_{n\to\infty} F_n = F \in L^2(D). \tag{4.47}$$

Then

$$\int_D F_n(x)\Delta\varphi(x)dx = 0, \tag{4.48}$$

for all $\varphi \in \mathcal{D}(D)$ and all $n \in \mathbb{N}$. As strong convergence implies weak convergence we get

$$\int_D \lim_{n\to\infty} F_n(x)\Delta\varphi(x)dx = \lim_{n\to\infty} \int_D F_n(x)\Delta\varphi(x)dx = 0, \tag{4.49}$$

for all $\varphi \in \mathcal{D}(D)$. Hence $F \in Harm(D)$. This is the desired result.

Later we will see that the set $Harm(D)$ plays a significant role in this thesis, especially in introducing a reproducing kernel Hilbert spaces which we use for representing the gravitational potential.
A fundamental property of Hilbert spaces is the following (see [49])

Theorem 4.3.12. *Let $\mathcal{H}$ be a Hilbert space and $G \subset \mathcal{H}$ be a closed linear subspace. Then there exists a decomposition*

$$\mathcal{H} = G \oplus G^{\perp}, \tag{4.50}$$

i.e., for every $x \in \mathcal{H}$, there exist unique vectors $y \in G$, and $z \in G^{\perp}$, such that

$$x = y + z.$$

Moreover, there exist projection operators

$$P_1 : \mathcal{H} \to G \ \text{ and } \ P_2 : \mathcal{H} \to G^{\perp}, \tag{4.51}$$

such that the decomposition is given by

$$x = P_1 x + P_2 x. \tag{4.52}$$

From Theorem 4.3.11 and Theorem 4.3.12 we thus have the following

Corollary 4.3.13. $L^2(\Sigma^{int}) = Harm(\Sigma^{int}) \oplus (Harm(\Sigma^{int}))^{\perp}$.

4.3.2 Volume Potentials to Harmonic Density Distributions

Obviously,

$$V(x) = \int_{\Sigma^{int}} \frac{F(y)}{|x-y|} dy, \quad x \in \overline{\Sigma^{ext}}, F \in L^2(\Sigma^{int}) \tag{4.53}$$

defines a linear operator

$$\mathscr{P} : L^2(\Sigma^{int}) \to \mathscr{P}(L^2(\Sigma^{int})) \tag{4.54}$$

$$\mathscr{P} : F \mapsto V \tag{4.55}$$

such that for every density function $F \in L^2(\Sigma^{int})$,

$$\mathscr{P}F = \int_{\Sigma^{int}} \frac{F(y)}{|\cdot - y|} dy. \tag{4.56}$$

We denote by $\mathscr{H}$ the space $\mathscr{P}(L^2(\Sigma^{int}))$ of potentials in $\overline{\Sigma^{ext}}$ representing the images of the density functions from $L^2(\Sigma^{int})$ under the Newton operator $\mathscr{P}$, i.e., we say that a function V is an element in $\mathscr{H}$, if we can write V in the form (4.53).
By the definition of $\mathscr{H}$ the operator $\mathscr{P}$ is surjective, but is not one to one: to a given potential $P \in \mathscr{H}$ there corresponds an infinite set of density functions $F \in L^2(\Sigma^{int})$ that generate the same potential. This results from the fact that none of the contributions to a given density function $F \in L^2(\Sigma^{int})$, coming from the nullspace of the operator $\mathscr{P}$ has an actual contribution to the generated potential in the free space, i.e., for every arbitrary density function there exists an infinite-dimensional set of different density functions which generate exactly the same potential. Obviously we have to consider the nullspace $N(\mathscr{P})$ of the integral operator (4.56). This space consists of all density functions $F \in L^2(\Sigma^{int})$, that generate potentials which are zero in $\overline{\Sigma^{ext}}$, i.e.,

$$N(\mathscr{P}) = \{F \in L^2(\Sigma^{int}) \mid \mathscr{P}F = 0\}. \tag{4.57}$$

It is shown (see, e.g., [70]) that the space $N(\mathscr{P})$ consists of precisely those functions which are orthogonal to harmonic functions on Σ^{int}, i.e., we can state the following

Theorem 4.3.14. *For the nullspace $N(\mathscr{P})$ of the Newton's gravitational potential operator $\mathscr{P}$ the following statement is valid*

$$N(\mathscr{P})^{\perp} = Harm(\Sigma^{int}). \tag{4.58}$$

From Corollary 4.3.13 it then easily follows that

$$L^2(\Sigma^{int}) = N(\mathscr{P}) \oplus Harm(\Sigma^{int}). \tag{4.59}$$

Let P_1 and P_2 be the orthogonal projections of $L^2(\Sigma^{int})$ on $N(\mathscr{P})$ and $Harm(\Sigma^{int})$ respectively. Then every element $F \in L^2(\Sigma^{int})$ can be written as

$$F = P_1F + P_2F. \tag{4.60}$$

Moreover applying the linear operator $\mathscr{P}$ in (4.60) yields

$$\begin{aligned} \mathscr{P}F &= \mathscr{P}P_1F + \mathscr{P}P_2F \\ &= \mathscr{P}P_2F. \end{aligned} \tag{4.61}$$

Having in mind that

$$\|F\|_{L^2(\Sigma^{int})} = \|P_1F\|_{L^2(\Sigma^{int})} + \|P_2F\|_{L^2(\Sigma^{int})}, \tag{4.62}$$

(4.61) means that the harmonic density function P_2F is that element of $L^2(\Sigma^{int})$, which has the smallest norm among all density functions in $L^2(\Sigma^{int})$, that generate in $\mathscr{H}$ the same potential as F. Thus we can state

Corollary 4.3.15. *For every potential $P \in \mathscr{H}$, there exists a unique harmonic density function $F \in Harm(\Sigma^{int})$, such that $\mathscr{P}F = \mathscr{P}P_2F = P$.*

5 Real Earth Body Based Methods

In gravitational field theory, the relation between the object function, i.e., the gravitational potential V and the data is non-linear. However, it may supposed to be linear if we go over from the gravitational potential to a (suitably defined) anomalous gravitational potential (see, e.g., [50]). Mathematically, the handling of the anomalous gravitational potential is equivalent to restricting the gravitational field theory to a linear relation of the object function to the data. In consequence, this assumption is supposed to hold true for the remaining part of this thesis. To be more concrete, the actual problem of gravitational theory today is the determining of a harmonic function, regular at infinity, to certain linear functionals, for example, discrete boundary data on the Earth's surface or discrete satellite data from space. In consequence, gravitational field theory canonically leads to interpolation based on a specific linear functionals, usually functional values or derivatives in certain (discretely given) points.
The problem of interpolating the gravitational potential outside the surface of the Earth, from heterogeneous, i.e., terrestrial, airborne, and spaceborne data demands us to give these potentials a mathematical structure by which both quality and computability of the approximation can be attacked. In the conventional geodetic approach due to [42], [52], it was proposed, that the class of approximating functions should conveniently be structured as a Hilbert space with reproducing kernel. There are several reasons for using this topological structure:

i) In accordance with the fact that the Laplace operator is a linear differential operator, the gravitational potential can be obtained by superposition of certain potential functions.

ii) By introducing the norm in a reproducing Hilbert space, it is easily possible to specify the class of approximating functions and to control the accuracy of the approximation.

iii) All linear (observation) functionals of terrestrial, airborne as well as spaceborne origin can be identified with elements of the dual space of this Hilbert space.

iv) Reproducing kernel functions turn out to have extremely desirable properties as interpolating, smoothing, and best approximation functions.

The problem of interpolation using reproducing kernels becomes inextricably involved with the problem of choosing a specific norm (see [20], [52]). This exposes the strength and the weakness of the method of interpolation. Given a norm in a Hilbert space we can calculate the reproducing kernel (if it exists), which again delivers the interpolating function under minimum norm assumptions.

For computational reasons, the spline interpolation of the Earth's gravitational potential is usually done in a spherical framework (see e.g., [19], [24], [25]). In this work, however, the intention is to propose a spline approach for the real Earth. Essential tools are a reproducing property involving the Newton potential and the decomposition of $L^2(\Sigma^{int})$ (as explained in the previous chapter).

5.1 Reproducing Kernel Hilbert Space

In the previous chapter we have already introduced the space $\mathscr{H} = \mathscr{P}(L^2(\Sigma^{int}))$ of potentials in $\overline{\Sigma^{ext}}$ representing the images of the density functions from $L^2(\Sigma^{int})$ under the Newton operator

$$\mathscr{P}F = \int_{\Sigma^{int}} \frac{F(y)}{|\cdot - y|} dy. \tag{5.1}$$

In this section we will present a way to impose a Hilbert space structure on the space $\mathscr{H}$. Using the decomposition (4.59) of the Hilbert space of density functions $L^2(\Sigma^{int})$, this can be easily done by restricting the Newton potential operator $\mathscr{P}$ to the closed subspace in $L^2(\Sigma^{int})$ of harmonic density functions on Σ^{int}. Indeed discarding the nonharmonic density contributions to the potentials in $\mathscr{H}$, it is clear that the operator

$$\widetilde{\mathscr{P}} = \mathscr{P}|_{Harm(\Sigma^{int})} : Harm(\Sigma^{int}) \to \mathscr{H}, \tag{5.2}$$

is bijective, since for every potential $P \in \mathscr{H}$, there exists a unique harmonic density function $F \in Harm(\Sigma^{int})$, such that $P = \widetilde{\mathscr{P}}F$ (see Corollary 4.3.15). This enables us to define the norm in $\mathscr{H}$ in the following way: for every $P \in \mathscr{H}$ let

$$\|P\|_{\mathscr{H}} = \|\widetilde{\mathscr{P}}F\|_{\mathscr{H}} = \|F\|_{L^2(\Sigma^{int})}, \tag{5.3}$$

where F is the unique harmonic density function $F \in Harm(\Sigma^{int})$ such that $\mathscr{P}F = \widetilde{\mathscr{P}}F = P$. Moreover, we are able to define a scalar product in $\mathscr{H}$ by

$$(\widetilde{\mathscr{P}}F, \widetilde{\mathscr{P}}G)_{\mathscr{H}} = (F, G)_{L^2(\Sigma^{int})}, \quad \text{for} \quad F, G \in Harm(\Sigma^{int}). \tag{5.4}$$

By this we have imposed a Hilbert space structure on the space of potentials $\mathscr{H}$ by using the isometric operator $\widetilde{\mathscr{P}}$ between $L^2(\Sigma^{int})$ and $\mathscr{H}$.

5.1.1 The Reproducing Kernel

In the following we will prove that $\mathscr{H}$ is a reproducing kernel Hilbert space, i.e., a Hilbert space equipped with a reproducing kernel. Considering the kernel function $k : \overline{\Sigma^{ext}} \times \Sigma^{int} \to \mathbb{R}$

$$k(x,y) = \frac{1}{|x-y|}, \tag{5.5}$$

it is clear that for a fixed $x \in \overline{\Sigma^{ext}}$, $k(x,\cdot)$ is an element of $Harm(\Sigma^{int})$ (it is an element of $C^{(\infty)}(\Sigma^{int})$ and harmonic in Σ^{int}). Thus, from (5.1) and our previous considerations, it is clear from Hilbert space theory (cf., e.g., [55]) that at the point $x \in \overline{\Sigma^{ext}}$ we can represent a given potential $P \in \mathscr{H}$ as

$$P(x) = \widetilde{\mathscr{P}}F(x) = (F, k(x,\cdot))_{L^2(\Sigma^{int})}, \tag{5.6}$$

for some $F \in Harm(\Sigma^{int})$. It is remarkable that not only for $x \in \Sigma^{ext}$, but also for all points $x \in \overline{\Sigma^{ext}}$ the functional $\mathcal{L}_x(P) = P(x)$ is a bounded functional on $\mathscr{H}$.
Indeed, from the representation

$$P(x) = (F, k(x,\cdot))_{L^2(\Sigma^{int})}, \tag{5.7}$$

and Cauchy–Schwarz inequality we get

$$|P(x)|^2 \leq ||F||^2_{L^2(\Sigma^{int})} ||k(x,\cdot)||^2_{L^2(\Sigma^{int})}. \tag{5.8}$$

Then from (5.3), it follows that

$$|P(x)|^2 \leq C(x) \cdot ||P||^2_{\mathscr{H}}, \quad \text{for every} \quad P \in \mathscr{H}, \tag{5.9}$$

where $C(x) = ||k(x,\cdot)||^2_{L^2(\Sigma^{int})}$ for a fixed $x \in \overline{\Sigma^{ext}}$.

Thus the necessary and sufficient condition for a Hilbert space to possess a reproducing kernel (see, e.g., [3], [9]) is fulfilled (see Theorem 2.1.2).

Next we want to find the explicit expression of the reproducing kernel

$$\mathscr{K}(x,y) : \overline{\Sigma^{ext}} \times \overline{\Sigma^{ext}} \to \mathbb{R} \tag{5.10}$$

for the space $\mathscr{H}$. Obviously, for every $P \in \mathscr{H}$ it must satisfy the property

$$P(x) = (P, \mathscr{K}(x,\cdot))_{\mathscr{H}}, \quad x \in \overline{\Sigma^{ext}}. \tag{5.11}$$

Thus, from (5.4) and (5.6), for fixed $x \in \overline{\Sigma^{ext}}$ and for $F \in Harm(\Sigma^{int})$, such that $\widetilde{\mathscr{P}}F = P$ we get

$$P(x) = (F, k(x,\cdot))_{L^2(\Sigma^{int})} = (\widetilde{\mathscr{P}}F, \widetilde{\mathscr{P}}k(x,\cdot))_{\mathscr{H}} = (P, \widetilde{\mathscr{P}}k(x,\cdot))_{\mathscr{H}}. \tag{5.12}$$

But this means that

$$\mathscr{K}(x,\cdot) = \widetilde{\mathscr{P}}k(x,\cdot), \tag{5.13}$$

i.e., we have the following expression for the reproducing kernel of the space of potentials $\mathscr{H}$

$$\mathscr{K}(x,\cdot) = \int_{\Sigma^{int}} \frac{dz}{|x-z||\cdot - z|} = (k(x,\cdot), k(\cdot,\cdot))_{L^2(\Sigma^{int})}, \quad x \in \overline{\Sigma^{ext}}, \tag{5.14}$$

i.e., we can formulate the following

Theorem 5.1.1. *The space $\mathscr{H}$ of Newton integrals* (4.4) *in $\overline{\Sigma^{ext}}$, corresponding to harmonic density functions, is a reproducing kernel Hilbert space with the reproducing kernel*

$$\mathscr{K}(x,\cdot) = \int_{\Sigma^{int}} \frac{dz}{|x-z||\cdot - z|}, \quad x \in \overline{\Sigma^{ext}}. \tag{5.15}$$

The reproducing kernel (5.15) is of great importance for our later considerations, so we will do a closer examination of it.

Remark: Equation (5.13) clearly states that for a fixed $x \in \overline{\Sigma^{ext}}$, the reproducing kernel $\mathscr{K}(x,\cdot)$ is a Newtonian potential corresponding to the harmonic density function $\frac{1}{|x-\cdot|}$ from $L^2(\Sigma^{int})$. Moreover, for a fixed $x \in \overline{\Sigma^{ext}}$, the potential $\mathscr{K}(x,\cdot)$ is an element of the space $\mathrm{Pot}^{(0)}(\overline{\Sigma^{ext}})$. Indeed, for a fixed $x \in \overline{\Sigma^{ext}}$, the density $\frac{1}{|x-\cdot|}$ is an element of $L^1(\Sigma^{int})$. This fact assures (see [64]) that $\mathscr{K}(x,\cdot)$ satisfies the Laplace equation in Σ^{ext}, i.e.,

$$\mathscr{K}(x,\cdot) \in \mathrm{Pot}(\Sigma^{ext}). \tag{5.16}$$

Moreover, the potentials corresponding to densities in $L^2(\Sigma^{int})$ are elements in $C^{(0)}(\mathbb{R}^3)$ (see [64]). Altogether we have

$$\mathscr{K}(x,\cdot) \in \mathrm{Pot}(\Sigma^{ext}) \cap C^{(0)}(\overline{\Sigma^{ext}}), \quad x \in \overline{\Sigma^{ext}}, \tag{5.17}$$

i.e.,

$$\mathscr{K}(x,\cdot) \in \mathrm{Pot}^{(0)}(\overline{\Sigma^{ext}}), \quad x \in \overline{\Sigma^{ext}}. \tag{5.18}$$

As already mentioned this is an extraordinary fact, since it means that now in interpolation methods we will be able to use potentials of the same nature as the Earth's gravitational potential, instead of using outer harmonic expressions (as in the spherically harmonic case (cf. [24]) of harmonic splines), which are of class $\mathrm{Pot}^{(\infty)}(\overline{A^{ext}})$ (see Chapter 6). The serious problem may be the fact that, for given $x, y \in \overline{\Sigma^{ext}}$, it is impossible to find a closed expression for this kernel in terms of elementary functions (as in the spherically harmonic case [24]) and from obvious reasons of expensiveness and time consumption, in the case of practical implementation this is a very unfortunate situation. Nevertheless, the reproducing kernel is available in integral form for any geophisically relevant geometry (like ellipsoid, geoid, actual Earth's surface). In addition for practically interesting functionals $\mathcal{L}$ on $\mathscr{H}$, certain discretization methods for $\mathcal{L}\mathcal{L}\mathscr{K}(\cdot,\cdot)$ can be found based on the calculus of the Euler summation formula. This will be the subject of the last chapter.

5.2 Discrete Boundary Value Problems

Contrary to the classical boundary value problems (see Chapter 3), where the solution process requires the continuous knowledge of the boundary function as a whole, in the case of the discrete boundary value problems, values of the boundary functions are known only in a set of discrete points on the boundary. In accordance to our considerations in the previous section, we consider the following discrete boundary value problems which are adequate in gravitational theory for a general geometry.

Discrete Exterior Dirichlet Problem (DEDP)

Let Σ be a regular surface. Let $\{x_1, ..., x_N\}$ be a discrete set of N points on Σ and let $\alpha_i = U(x_i), i = 1, ..., N$, be a given data set corresponding to a function $U \in C^{(0)}(\Sigma)$. Find an approximation $U_N \in \mathrm{Pot}^{(0)}(\overline{\Sigma^{ext}})$ to the potential $U : \overline{\Sigma^{ext}} \to \mathbb{R}$, $U \in \mathrm{Pot}^{(0)}(\overline{\Sigma^{ext}})$ such that

$$U_N(x_i) = U(x_i) = \alpha_i, i = 1, ..., N.$$

Discrete Exterior Oblique Derivative Problem (DEODP)

Let Σ be a $C^{(2,\mu)}$-regular surface, $\mu \in (0, 1)$. Let λ be a unit $C^{(1,\mu)}$-vector field on Σ, such that

$$\inf_{x\in\Sigma}(\lambda(x) \cdot \nu(x)) > 0, \tag{5.19}$$

where $\nu(x)$ denotes the outer unit normal vector field on Σ. Let $\{x_1, ..., x_N\}$ be a discrete set of N points on Σ and let $\alpha_i = \frac{\partial U}{\partial \lambda_\Sigma}(x_i), i = 1, ..., N$, be a given data set corresponding to a function $U \in C^{(1,\mu)}(\Sigma)$. Find an approximation $U_N \in \mathrm{Pot}^{(1,\mu)}(\overline{\Sigma^{ext}})$ to the potential $U \in \mathrm{Pot}^{(1,\mu)}(\overline{\Sigma^{ext}})$, such that

$$\frac{\partial U_N}{\partial \lambda_\Sigma}(x_i) = \frac{\partial U}{\partial \lambda_\Sigma}(x_i) = \alpha_i, i = 1, ..., N.$$

5.3 Bounded Linear Functionals

We first consider the Dirichlet functional. From (5.9), we easily get the following

Lemma 5.3.1. *For each $x \in \overline{\Sigma^{ext}}$, the linear functional $\mathcal{D}_x$ defined by*

$$\mathcal{D}_x : P \mapsto \mathcal{D}_x P = P(x), \quad P \in \mathscr{H}, \tag{5.20}$$

is bounded on $\mathscr{H}$, i.e.,

$$|\mathcal{D}_x P| = |P(x)| \leq C(x, \Sigma) ||P||_{\mathscr{H}}, \tag{5.21}$$

where $C(x, \Sigma) = ||k(x, \cdot)||_{L^2(\Sigma^{int})}$. Moreover, for each point $x \in \overline{\Sigma^{ext}}$,

$$y \mapsto \mathscr{K}(x, y), \quad y \in \overline{\Sigma^{ext}}, \tag{5.22}$$

is an element of $\mathscr{H}$, and for all $P \in \mathscr{H}$, we have (see Section 5.1.1)

$$\mathcal{D}_x P = P(x) = (P, \mathscr{K}(x, \cdot))_{\mathscr{H}}. \tag{5.23}$$

In other words, Lemma 5.3.1 states that the Dirichlet functional is bounded on the space of potentials $\mathscr{H}$.

Next we consider the linear functional of the oblique derivative N_x for the potentials in $\mathscr{H}$.

$$\mathcal{N}_x : P \mapsto \mathcal{N}_x P = \frac{\partial P}{\partial \lambda_\Sigma}(x), \quad P \in \mathscr{H}, x \in \Sigma, \tag{5.24}$$

where λ is a unit $C^{(1,\mu)}$-vector field on Σ, satisfying the conditions (5.19). Clearly, within the oblique framework, Σ is considered to be a $C^{(2,\mu)}$-regular surface, with $\mu \in (0, 1)$. Let x be an arbitrary point on Σ, P be a given potential in $\mathscr{H}$, and suppose that $\varepsilon > 0$ is arbitrary. For given unit $C^{(1,\mu)}$-vector field λ on Σ, satisfying the conditions (5.19), we investigate the limit of $\varepsilon^{-1} (P(x) - P(x + \varepsilon\lambda(x)))$, as ε tends to 0. The reproducing property together with the Cauchy inequality yield

$$\varepsilon^{-1} |P(x) - P(x + \varepsilon\lambda(x))| \leq \varepsilon^{-1} ||P||_{\mathscr{H}} ||\mathscr{K}(x, \cdot) - \mathscr{K}(x + \varepsilon\lambda(x), \cdot)||_{\mathscr{H}}. \tag{5.25}$$

For the potential $\mathscr{K}(x, \cdot) - \mathscr{K}(x + \varepsilon\lambda(x), \cdot)$ we find the following representation

$$\int_{\Sigma^{int}} \left(\frac{1}{|x - z|} - \frac{1}{|x + \varepsilon\lambda(x) - z|} \right) \frac{1}{|\cdot - z|} dz. \tag{5.26}$$

From our definition of the norm in $\mathscr{H}$ and the triangle inequality we easily find

$$\begin{aligned} & ||\mathscr{K}(x,\cdot) - \mathscr{K}(x+\varepsilon\lambda(x),\cdot)||^2_{\mathscr{H}} \\ = & \left\| \frac{1}{|x-z|} - \frac{1}{|x+\varepsilon\lambda(x)-z|} \right\|^2_{L^2(\Sigma^{int})} \\ = & \int_{\Sigma^{int}} \left| \frac{1}{|x-z|} - \frac{1}{|x+\varepsilon\lambda(x)-z|} \right|^2 dz \\ = & \int_{\Sigma^{int}} \frac{\Big| |x+\varepsilon\lambda(x)-z| - |x-z| \Big|^2}{|x-z|^2|x+\varepsilon\lambda(x)-z|^2} dz \\ \leq & \int_{\Sigma^{int}} \frac{|\varepsilon\lambda(x)|^2}{|x-z|^2|x+\varepsilon\lambda(x)-z|^2} dz \end{aligned} \tag{5.27}$$

.

Having in mind that λ is a unit vector field on Σ, i.e., $|\lambda(x)| = 1, x \in \Sigma$ we have

$$\varepsilon^{-2}||\mathscr{K}(x,\cdot) - \mathscr{K}(x+\varepsilon\lambda(x),\cdot)||^2_{\mathscr{H}} \leq \int_{\Sigma^{int}} \frac{dz}{|x-z|^2|x+\varepsilon\lambda(x)-z|^2}. \tag{5.28}$$

Obviously, as ε tends to zero, the last integral is divergent for $x \in \Sigma$ and we are not able to guarantee in canonical way the boundedness of the linear functional for the oblique derivative.

Instead of considering Newton integrals on Σ^{int} our concept now is to replace Σ^{int} by Σ^{int}_τ, i.e., in the case of the functional of the oblique derivative we replace the regular surface Σ, by an inner parallel surface Σ_τ at distance $|\tau|$ to Σ

$$\Sigma_\tau = \{x \in \mathbb{R}^3 | x = y - \tau\nu(y), \tau > 0, y \in \Sigma\}, \tag{5.29}$$

where $\nu(y)$ as usual denotes outer unit normal (pointing into the outer space Σ^{ext}) at the point $y \in \Sigma$. In doing so we have to choose a sufficiently small $|\tau|$, in order to preserve the regularity of the parallel surface Σ_τ (see Section 3.1). Then, for the regular surface Σ_τ, in analogous way as in the Section 5.1, we start from the image space $\mathscr{H}_\tau = \mathscr{P}_\tau(L^2(\Sigma^{int}_\tau))$ of potentials in Σ^{ext}_τ, generated by the Newton potential operator

$$\mathscr{P}_\tau F = \int_{\Sigma^{int}_\tau} \frac{F(y)}{|\cdot - y|} dy. \tag{5.30}$$

Using the decomposition of the space of density functions $L^2(\Sigma^{int}_\tau)$), we restrict the Newton potential operator to the closed subspace $Harm(\Sigma^{int}_\tau)$ of harmonic density functions in $L^2(\Sigma^{int}_\tau)$. The isometric operator

$$\widetilde{\mathscr{P}_\tau} = \mathscr{P}_\tau|_{Harm(\Sigma^{int}_\tau)} : Harm(\Sigma^{int}_\tau) \to \mathscr{H}_\tau \tag{5.31}$$

between $L^2(\Sigma_\tau^{int})$ and $\mathscr{H}_\tau$ imposes a Hilbert space structure on $\mathscr{H}_\tau$. The scalar product in $\mathscr{H}_\tau$ is defined by

$$(\widetilde{\mathscr{P}}_\tau F, \widetilde{\mathscr{P}}_\tau G)_{\mathscr{H}_\tau} = (F, G)_{L^2(\Sigma_\tau^{int})}, \tag{5.32}$$

for $F, G \in Harm(\Sigma_\tau^{int})$. By analogous arguments as in Section 5.1, it clearly follows that the space $\mathscr{H}_\tau$ possesses a uniquely determined reproducing kernel $\mathscr{K}_\tau$ related to Σ_τ

$$\mathscr{K}_\tau(x,y) = \int_{\Sigma_\tau^{int}} \frac{dz}{|x-z||y-z|}, x, y \in \overline{\Sigma_\tau^{ext}}. \tag{5.33}$$

Of course, it is an element of the class $\mathrm{Pot}^{(0)}(\overline{\Sigma_\tau^{ext}})$.

Now, considering a potential P from $\mathscr{H}_\tau$ and the reproducing kernel $\mathscr{K}_\tau$, in analogous way as described above, we get the following modified version of the Equation (5.28)

$$\varepsilon^{-2}||\mathscr{K}_\tau(x,\cdot) - \mathscr{K}_\tau(x+\varepsilon\lambda(x),\cdot)||^2_{\mathscr{H}} \le \int_{\Sigma_\tau^{int}} \frac{dz}{|x-z|^2|x+\varepsilon\lambda(x)-z|^2}, \tag{5.34}$$

only now we have the following estimate

$$\varepsilon^{-2}||\mathscr{K}_\tau(x,\cdot) - \mathscr{K}_\tau(x+\varepsilon\lambda(x),\cdot)||^2_{\mathscr{H}_\tau} \le C(x, \Sigma_\tau), \tag{5.35}$$

where

$$C(x,\Sigma_\tau) = \int_{\Sigma_\tau^{int}} \frac{dz}{|x-z|^4}. \tag{5.36}$$

The last integral is convergent for all $x \in \overline{\Sigma^{ext}}$. Then we get for the Equation (5.25) in the case of $\mathscr{H}_\tau$

$$\begin{aligned} &\varepsilon^{-1}\left|P(x) - P(x+\varepsilon\lambda(x))\right| \\ &\le \varepsilon^{-1}||P||_{\mathscr{H}_\tau}||\mathscr{K}_\tau(x,\cdot) - \mathscr{K}_\tau(x+\varepsilon\lambda(x),\cdot)||_{\mathscr{H}_\tau} \\ &\le ||P||_{\mathscr{H}_\tau}\sqrt{C(x,\Sigma_\tau)}. \end{aligned} \tag{5.37}$$

This shows that the functional $\mathcal{N}_x$ of the oblique derivative for the points x on the surface Σ is bounded, but with respect to the $\mathscr{H}_\tau$-topology. Thus, we can state the following

Lemma 5.3.2. *Let x be a point of the regular surface Σ. Then the function*

$$y \mapsto \lambda_\Sigma(y) \cdot \nabla \mathscr{K}_\tau(x,y), \quad y \in \overline{\Sigma^{ext}}, \tag{5.38}$$

is the representer of the linear functional

$$\mathcal{N}_x : P \mapsto \mathcal{N}_x P = \frac{\partial P}{\partial \lambda_\Sigma}(x), \quad P \in \mathscr{H}_\tau, \tag{5.39}$$

i.e.,

$$\mathcal{N}_x P = (P, \lambda_\Sigma \cdot \nabla \mathscr{K}_\tau(x,\cdot))_{\mathscr{H}_\tau}, \quad \textit{for all} \quad P \in \mathscr{H}_\tau, x \in \overline{\Sigma^{ext}}, \tag{5.40}$$

where

$$\mathscr{K}_\tau(x,y) = \int_{\Sigma_\tau^{int}} \frac{dz}{|x-z||y-z|}, \quad x, y \in \overline{\Sigma_\tau^{ext}}. \tag{5.41}$$

5.4 Solution to the Interpolation Problems

In principle, in the Dirichlet case, if we have the measurements $\alpha_1, ..., \alpha_N$ of a scalar field U at points $x_1, ..., x_N$ on the surface Σ, we want to find the harmonic function that fits these data, and which has also the minimum norm in the space that is used. Interpolation of the gravitational potential field in terms of reproducing kernels immediately leads to a spline formulation, i.e., a minimum norm formulation. The potentials are considered being elements of a certain Hilbert space possessing a reproducing kernel, while the measured values at the points $x_1, ..., x_N$ are assumed to be linearly independent bounded functionals $\mathcal{L}_1, ..., \mathcal{L}_N$ of the gravitational potential U. It is known that if the Hilbert space possesses a reproducing kernel, we can find the solution as a linear combination of the representers $\mathcal{L}\mathscr{K}(x_i, \cdot)$ of the functionals $\mathcal{L}_i$, i.e., the solution is exactly the projection of U on the N-dimensional linear subspace of the used Hilbert space, spanned by the linearly independent representers $\mathcal{L}\mathscr{K}(x_i, \cdot), i = 1, ..., N$.
Having in mind that the reproducing kernel we have constructed, is a volume integral over the surface Σ (i.e., it carries the information of the regular geometry Σ), the interpolating splines corresponding to given bounded functionals, also gives the interpolating potential a characterization which is strongly connected to the body Σ^{int}, i.e., the geometry of Σ is reflected in the representation of the interpolating splines.

5.4.1 Solution to DEDP

In Section 5.3 we have proven that the Dirichlet functional of the gravitational potential for points on the surface Σ, is bounded on the reproducing kernel Hilbert space $\mathscr{H}$ as defined in Section 5.1. Let $\{\alpha_1, ..., \alpha_N\}$ be a given data set of Dirichlet functionals for the unknown potential U, corresponding to the discrete set $X_N = \{x_1, ..., x_N\}$ of pairwise disjoint points on Σ, i.e., for $i = 1, ..., N$

$$\mathcal{D}_i U = U(x_i) = \alpha_i. \tag{5.42}$$

Our aim is to find the smoothest $\mathscr{H}$ – interpolant corresponding to data set $\{\alpha_1, ..., \alpha_N\}$, where by 'smoothest' we mean that the norm is minimized in $\mathscr{H}$. In other words, the problem is to find a function $S^U_{\mathcal{D}_1,...,\mathcal{D}_N}$ in the set

$$\mathcal{I}^U_{\mathcal{D}_1,...,\mathcal{D}_N} = \{P \in \mathscr{H} \quad | \quad \mathcal{D}_i P = \alpha_i, \quad i = 1, ..., N\}, \tag{5.43}$$

such that

$$\left\| S^U_{\mathcal{D}_1,...,\mathcal{D}_N} \right\|_{\mathscr{H}} = \inf_{P \in \mathcal{I}^U_{\mathcal{D}_1,...,\mathcal{D}_N}} \|P\|_{\mathscr{H}} . \tag{5.44}$$

The corresponding representer of the functional $\mathcal{D}_i$ can be written as

$$\mathcal{D}_i\mathscr{K}(\cdot,\cdot) = \mathscr{K}(x_i,\cdot), \tag{5.45}$$

where $\mathscr{K}$ is the reproducing kernel of $\mathscr{H}$. Then, for a given set $\{\mathcal{D}_1, ..., D_N\}$ of N Dirichlet functionals on $\mathscr{H}$, corresponding to the set $X_N = \{x_1, ..., x_N\}$ of points on Σ, we have the set of representers

$$\{\mathcal{D}_1\mathscr{K}(\cdot,\cdot), ..., \mathcal{D}_N\mathscr{K}(\cdot,\cdot)\}. \tag{5.46}$$

The reproducing property of $\mathscr{K}$ yields, for $i = 1, ..., N$, and $P \in \mathscr{H}$

$$\mathcal{D}_iP = (\mathcal{D}_i\mathscr{K}(\cdot,\cdot), P)_{\mathscr{H}}. \tag{5.47}$$

Having in mind that the reproducing kernel is given as a Newton integral (5.15), so are the representers of the functionals $\mathcal{D}_i$, i.e.,

$$\mathcal{D}_i\mathscr{K}(\cdot,\cdot) = \int_{\Sigma^{int}} \frac{dz}{|x_i - z||\cdot - z|}. \tag{5.48}$$

In order to present the solution method, we first introduce the following

Definition 5.4.1. *A system X_N of points $x_1, ..., x_N$ on the surface Σ is called fundamental system on Σ, if the corresponding representers $\mathcal{L}_1\mathscr{K}(\cdot,\cdot), ..., \mathcal{L}_N\mathscr{K}(\cdot,\cdot)$ of a given linear functional $\mathcal{L}$, are linearly independent.*

The interpolating spline function is defined as follows

Definition 5.4.2. *Let $X_N = \{x_1, ..., x_N\}$ be a given fundamental system of points on Σ and let $\{\mathcal{D}_1, ..., \mathcal{D}_N\}$ be the set of the corresponding bounded linear Dirichlet functionals.Then, any function of the form*

$$S(x) = \sum_{i=1}^{N} a_i\mathcal{D}_i\mathscr{K}(\cdot,x) = \sum_{i=1}^{N} a_i \int_{\Sigma^{int}} \frac{dz}{|x_i - z||x - z|}, \quad x \in \overline{\Sigma^{ext}}, \tag{5.49}$$

with arbitrarily given (real) coefficients $a_1, ..., a_N$ is called a $\mathscr{H}$-spline relative to $\{\mathcal{D}_1, ..., \mathcal{D}_N\}$.

Obviously the space

$$S_{\mathscr{H}}(\mathcal{D}_1, ..., \mathcal{D}_N) = span\{\mathcal{D}_1\mathscr{K}(\cdot,\cdot), ..., \mathcal{D}_N\mathscr{K}(\cdot,\cdot)\}, \tag{5.50}$$

of all $\mathscr{H}$-splines relative to $\{\mathcal{D}_1, ..., \mathcal{D}_N\}$, is an N-dimensional subspace of $\mathscr{H}$.
As an immediate consequence of the reproducing property (5.47), viz. the $\mathscr{H}$-spline formula we get the following

Lemma 5.4.3. *Let S be a function of class $S_{\mathscr{H}}(\mathcal{D}_1,...,\mathcal{D}_N)$. Then for each $P \in \mathscr{H}$, the following identity is valid*

$$(S,P)_{\mathscr{H}} = \sum_{i=1}^{N} a_i \mathcal{D}_i P. \tag{5.51}$$

Now the problem of determining the smoothest function in the set (5.43) of all $\mathscr{H}$-interpolants is related to a system of linear equations which needs to be solved to obtain the spline coefficients. Indeed, the application of the linear functionals $\mathcal{D}_1,...,\mathcal{D}_N$ to the $\mathscr{H}$-spline of the form (5.49), yields N linear equations in the coefficients $a_1^N,...,a_N^N$

$$\sum_{j=1}^{N} a_j^N \mathcal{D}_i \mathcal{D}_j \mathscr{K}(\cdot,\cdot) = \mathcal{D}_i U, \quad i = 1,...,N. \tag{5.52}$$

The elements of the coefficients matrix

$$\left(\mathcal{D}_i \mathcal{D}_j \mathscr{K}(\cdot,\cdot)\right)_{i,j=1,...,N}, \tag{5.53}$$

are given by

$$\mathcal{D}_i \mathcal{D}_j \mathscr{K}(\cdot,\cdot) = \int_{\Sigma^{int}} \frac{1}{|x_i - z||x_j - z|} dz. \tag{5.54}$$

Since the coefficient matrix as Gram matrix of the N linearly independent functions $\mathcal{D}_1\mathscr{K}(\cdot,\cdot),...,\mathcal{D}_N\mathscr{K}(\cdot,\cdot)$ is non-singular, the linear system (5.52) is uniquely solvable. Together with the set of linear bounded functionals and the reproducing kernel Hilbert space $\mathscr{H}$, the coefficients $a_1^N,...,a_N^N$ define the unique interpolating spline we are looking for. Thus we can state

Lemma 5.4.4. *(Uniqueness of interpolation) For given $U \in \mathscr{H}$ there exist a unique element in $S_{\mathscr{H}}(\mathcal{D}_1,...,\mathcal{D}_N) \cap \mathcal{I}^U_{\mathcal{D}_1,...,\mathcal{D}_N}$. We denote this element by $S^U_{\mathcal{D}_1,...,\mathcal{D}_N}$.*

Moreover, we have the following

Lemma 5.4.5. *The interpolating $\mathscr{H}$-spline $S^U_{\mathcal{D}_1,...,\mathcal{D}_N}$ of U (relative to $\{\mathcal{D}_1,...,\mathcal{D}_N\}$) is the $\mathscr{H}$-orthogonal projection of U onto the space $S_{\mathscr{H}}(\mathcal{D}_1,...,\mathcal{D}_N)$.*

Proof: Due to the uniqueness of the solution of the interpolation problem, for all $V \in S_{\mathscr{H}}(\mathcal{D}_1,...,\mathcal{D}_N)$ we have $S^V_{\mathcal{D}_1,...,\mathcal{D}_N} = V$. Consequently, the $\mathscr{H}$- spline interpolation operator is a projector onto the spline space $S_{\mathscr{H}}(\mathcal{D}_1,...,\mathcal{D}_N)$. We now use Lemma 2.1.8 to prove that this projector is even an orthogonal projector.
Let U in $\mathscr{H}$ be arbitrary. Then, by construction

$$(U, \mathcal{D}_i\mathscr{K}(\cdot,\cdot))_{\mathscr{H}} = \mathcal{D}_i U = \mathcal{D}_i S^U_{\mathcal{D}_1,...,\mathcal{D}_N} = (S^U_{\mathcal{D}_1,...,\mathcal{D}_N}, \mathcal{D}_i\mathscr{K}(\cdot,\cdot))_{\mathscr{H}} \tag{5.55}$$

is valid for $i = 1, ..., N$. Hence, for all $S = \sum_{i=1}^{N} b_i \mathcal{D}_i \mathscr{K}(\cdot,\cdot) \in S_{\mathscr{H}}(\mathcal{D}_1, ..., \mathcal{D}_N)$, the identity

$$(U, S)_{\mathscr{H}} = (S^U_{\mathcal{D}_1,...,\mathcal{D}_N}, S)_{\mathscr{H}} \tag{5.56}$$

is also valid. According to Lemma 2.1.8, the $\mathscr{H}$-spline interpolation operator is an orthogonal projector.

The upcoming lemmata give several properties, namely the minimum norm properties which also justify the use of the name 'spline' for such interpolants.

Lemma 5.4.6. *(First minimum property) If $P \in \mathcal{I}^U_{\mathcal{D}_1,...,\mathcal{D}_N}$, then*

$$||P||^2_{\mathscr{H}} = ||S^U_{\mathcal{D}_1,...,\mathcal{D}_N}||^2_{\mathscr{H}} + ||S^U_{\mathcal{D}_1,...,\mathcal{D}_N} - P||^2_{\mathscr{H}}. \tag{5.57}$$

Lemma 5.4.7. *(Second minimum property) If $S \in S_{\mathscr{H}}(\mathcal{D}_1, ..., \mathcal{D}_N)$ and $P \in \mathcal{I}^U_{\mathcal{D}_1,...,\mathcal{D}_N}$, then*

$$||S - P||^2_{\mathscr{H}} = ||S^U_{\mathcal{D}_1,...,\mathcal{D}_N} - P||^2_{\mathscr{H}} + ||S - S^U_{\mathcal{D}_1,...,\mathcal{D}_N}||^2_{\mathscr{H}}. \tag{5.58}$$

Proof: Since the solution of the normal equations is nothing else than the orthogonal projection onto the N–dimensional subspace $S_{\mathscr{H}}(\mathcal{D}_1, ..., \mathcal{D}_N)$, the remainder function $S^U_{\mathcal{D}_1,...,\mathcal{D}_N} - P$ is orthogonal to each element in $S_{\mathscr{H}}(\mathcal{D}_1, ..., \mathcal{D}_N)$. This is likewise true for every $P \in \mathcal{I}^U_{\mathcal{D}_1,...,\mathcal{D}_N}$ and in particular, we obtain

$$(S^U_{\mathcal{D}_1,...,\mathcal{D}_N} - P, S^U_{\mathcal{D}_1,...,\mathcal{D}_N})_{\mathscr{H}} = 0. \tag{5.59}$$

Then it follows that

$$\begin{aligned}(P, P)_{\mathscr{H}} &= (S^U_{\mathcal{D}_1,...,\mathcal{D}_N} - (S^U_{\mathcal{D}_1,...,\mathcal{D}_N} - P), S^U_{\mathcal{D}_1,...,\mathcal{D}_N} - (S^U_{\mathcal{D}_1,...,\mathcal{D}_N} - P))_{\mathscr{H}} \\ &= (S^U_{\mathcal{D}_1,...,\mathcal{D}_N}, S^U_{\mathcal{D}_1,...,\mathcal{D}_N})_{\mathscr{H}} + (S^U_{\mathcal{D}_1,...,\mathcal{D}_N} - P, S^U_{\mathcal{D}_1,...,\mathcal{D}_N} - P)_{\mathscr{H}},\end{aligned} \tag{5.60}$$

which is, in fact equation (5.57). Similar arguments are valid for the equation (5.58). Equation (5.57) states that in fact, the spline $S^U_{\mathcal{D}_{x_1},...,\mathcal{D}_{x_N}}$ solves the minimum norm interpolation problem (5.44), i.e. that $S^U_{\mathcal{D}_{x_1},...,\mathcal{D}_{x_N}}$ is the interpolant in $\mathscr{H}$, with the smallest norm in the $\mathscr{H}$–topology, while equation (5.58) states that $S^U_{\mathcal{D}_{x_1},...,\mathcal{D}_{x_N}}$ is closest to U among all possible splines in $S_{\mathscr{H}}(\mathcal{D}_{x_1}, ..., \mathcal{D}_{x_N})$, also to be understood in the $\mathscr{H}$–topology.

Summarizing our results we finally find

Theorem 5.4.8. *The interpolation problem*

$$||S^U_{\mathcal{D}_1,...,\mathcal{D}_N}||_{\mathscr{H}} = \inf_{P \in \mathcal{I}^U_{\mathcal{D}_1,...,\mathcal{D}_N}} ||P||_{\mathscr{H}}, \tag{5.61}$$

is well-posed in the sense that its solution exists, is unique, and depends continuously on the data $\alpha_1, ..., \alpha_N$. The uniquely determined solution $S^U_{\mathcal{D}_1,...,\mathcal{D}_N}$ is given in the explicit form

$$S^U_{\mathcal{D}_1,...,\mathcal{D}_N}(x) = \sum_{i=1}^{N} a_i^N \int_{\Sigma^{int}} \frac{1}{|x_i - z||x - z|} dz, \quad x \in \overline{\Sigma^{ext}}, \tag{5.62}$$

where the coefficients $a_1^N, ..., a_N^N$ satisfy the linear equations

$$\sum_{i=1}^{N} a_i^N \int_{\Sigma^{int}} \frac{1}{|x_i - z||x_j - z|} dz = \alpha_j, \quad j = 1, ..., N. \tag{5.63}$$

Remark: It should be noted that the requirement for the linear independence of the given bounded linear functionals is not necessary from the theoretical point of view, but essential for numerical computations. It guarantees that the $\mathscr{H}$-spline coefficients are uniquely determined, i.e., that the linear equation system (5.52) is uniquely solvable. Without linear independence of the functionals, the dimension of the spline space $S_{\mathscr{H}}(\mathcal{D}_1, ..., \mathcal{D}_N)$ is smaller than N, and the coefficients of the interpolating $\mathscr{H}$-spline $S^U_{\mathcal{D}_{x_1},...,\mathcal{D}_{x_N}}$ of $U \in \mathscr{H}$ relative to $\{\mathcal{D}_1, ..., \mathcal{D}_N\}$ are no longer uniquely determined. Nevertheless, the interpolating $\mathscr{H}$-spline is the uniquely determined orthogonal projection of U onto the spline space $S_{\mathscr{H}}(\mathcal{D}_1, ..., \mathcal{D}_N)$ and all the spline properties are still valid.

It is an important question whether the interpolating spline to a certain function on Σ, converges to this function as the number N of bounded linear functionals tends to infinity. Such a convergence property justifies the approximation by the given interpolating spline. Note that the linear independence of the given functionals is not required also for the upcoming convergence results for $\mathscr{H}$-splines.

We start our considerations with the following

Theorem 5.4.9. *Let $X = \{x_i\}_{i\in\mathbb{N}} \subset \Sigma$ be a countable dense set of points on Σ. Then*

$$\overline{span\{\mathscr{K}(x_i, \cdot) | i \in \mathbb{N}\}}^{||\cdot||_{\mathscr{H}}} = \mathscr{H}. \tag{5.64}$$

Proof: Our purpose is to show that the properties $P \in \mathscr{H}$ and $(\mathscr{K}(x_i, \cdot), P)_{\mathscr{H}} = 0$ for all $i \in \mathbb{N}$, imply that $P = 0$. Then the statement of the Theorem 5.4.9, follows immediately from Theorem 2.1.6. Clearly, the condition $(\mathscr{K}(x_i, \cdot), P)_{\mathscr{H}} = 0$ is equivalent to $P(x_i) = 0$ for all $i \in \mathbb{N}$. Now, according to our construction, P is continuous function on Σ. Hence it follows that if $P(x) \neq 0$ for some $x \in \Sigma$, then P would be different from zero in a whole neighborhood of X on Σ. But this is impossible because of the density of X on Σ. This proves the theorem.

Next we state the obvious generalization of Theorem 5.4.9 by means of Dirichlet functionals on $\mathscr{H}$.

Theorem 5.4.10. *Let $X = \{x_i\}_{i\in\mathbb{N}} \subset \Sigma$ be a countable dense set of points on Σ. Let $\{\mathcal{D}_i\}_{i\in\mathbb{N}}$ be the set of bounded linear Dirichlet functionals on $\mathscr{H}$, corresponding to the set X. Then*

$$\overline{span_{\mathcal{D}_i \in X}\{\mathcal{D}_i\mathscr{K}(\cdot,\cdot) | i \in \mathbb{N}\}}^{||\cdot||_{\mathscr{H}}} = \mathscr{H}. \tag{5.65}$$

Now, Theorem 5.4.10 enables us to prove the following convrgence theorem for the solution of DEDP.

Theorem 5.4.11. *Let the assumptions and the notation be the same as in Theorem* 5.4.10. *Let $U \in \mathscr{H}$ be arbitrary function and let $S^U_{\mathcal{D}_1,...,\mathcal{D}_N}$ be the interpolating $\mathscr{H}$–spline relative to $\{\mathcal{D}_1,...,\mathcal{D}_N\}$. Then the following convergence results hold true:*

(i) $\lim_{N\to\infty} ||U - S^U_{\mathcal{D}_1,...,\mathcal{D}_N}||_{\mathscr{H}} = 0$

(ii) $\lim_{N\to\infty} ||U - S^U_{\mathcal{D}_1,...,\mathcal{D}_N}||_{C(\overline{\Sigma^{ext}})} = 0$

Proof: The spline interpolation operator $U \mapsto S^U_{\mathcal{D}_1,...,\mathcal{D}_N}$ is the orthogonal projection onto $span\{\mathcal{D}_1\mathscr{K}(\cdot,\cdot),...,\mathcal{D}_N\mathscr{K}(\cdot,\cdot)\}$. Due to Theorem 5.4.10, the Gram Schmidt orthonormalization process can be used to obtain successively a complete orthonormal system $\{\Phi_n\}_n \in \mathbb{N}$ in $\mathscr{H}$, such that $span\{\mathcal{D}_1\mathscr{K}(\cdot,\cdot),...,\mathcal{D}_N\mathscr{K}(\cdot,\cdot)\} = span\{\Phi_1,...,\Phi_{N'}\}$, where $N' = N'(N) \leq N$, for all $N \in \mathbb{N}$. Then

$$S^U_{\mathcal{D}_1,...,\mathcal{D}_N} = \sum_{n=1}^{N'} (U,\Phi_n)_{\mathscr{H}}\Phi_n, \tag{5.66}$$

and this is the truncated Fourier series of U, with respect to the orthonormal system $\{\Phi_n\}_n \in \mathbb{N}$, and it converges in $\mathscr{H}$–sense to U for $N \to \infty$. This proves (i).
In order to prove (ii), we realize that for arbitrary $x \in \overline{\Sigma^{ext}}$, reproducing property of $\mathscr{H}$ and Cauchy inequality, yields

$$\begin{aligned} |U(x) - S^U_{\mathcal{D}_1,...,\mathcal{D}_N}(x)| &= |(U - S^U_{\mathcal{D}_1,...,\mathcal{D}_N}, \mathcal{D}_x\mathscr{K}(\cdot,\cdot))_{\mathscr{H}}| \\ &\leq ||U - S^U_{\mathcal{D}_1,...,\mathcal{D}_N}||_{\mathscr{H}} ||\mathcal{D}_x\mathscr{K}(\cdot,\cdot)||_{\mathscr{H}}. \end{aligned} \tag{5.67}$$

From the representation (5.45) of the reproducing kernel $\mathcal{D}_x\mathscr{K}(\cdot,\cdot)$, and our definition of the norm in $\mathscr{H}$, we find for $x \in \overline{\Sigma^{ext}}$ the following estimate for the norm of the representer $\mathcal{D}_x\mathscr{K}(\cdot,\cdot)$ in $\mathscr{H}$

$$||\mathcal{D}_x\mathscr{K}(\cdot,\cdot)||_{\mathscr{H}} \leq \left\| \frac{1}{|x-z|} \right\|_{L^2(\Sigma^{int})} = \left(\int_{\Sigma^{int}} \frac{1}{|x-z|^2} dz \right)^{1/2}, \tag{5.68}$$

and we know that the last integral is convergent. Then, from (5.67), for arbitrary $x \in \overline{\Sigma^{ext}}$ we have

$$|U(x) - S^U_{\mathcal{D}_1,...,\mathcal{D}_N}(x)| \leq C(x,\Sigma)||U - S^U_{\mathcal{D}_1,...,\mathcal{D}_N}||_{\mathscr{H}}, \tag{5.69}$$

where

$$C(x,\Sigma) = \left(\int_{\Sigma^{int}} \frac{1}{|x-z|^2} dz\right)^{1/2}. \tag{5.70}$$

Equation (5.69) together with (i) shows the validity of (ii).

This proves the Theorem 5.4.11.

5.4.2 Solution to DEODP

The solution to DEODP will be constructed in the similar way, but with a different solution space. Obviously, from our discussion in Section 5.3, it is clear that we will seek the solution to DEODP in the space $\mathscr{H}_\tau$ (since the functional of the oblique derivative is bounded on $\mathscr{H}_\tau$). Let $\{\alpha_1, ..., \alpha_N\}$ be a given data set of functionals of the oblique derivative for the unknown potential U, corresponding to the discrete set $X_N = \{x_1, ..., x_N\}$ of points on Σ, i.e., for $i = 1, ..., N$

$$\mathcal{N}_i U = \frac{\partial U}{\partial \lambda_\Sigma}(x_i) = \alpha_i. \tag{5.71}$$

Our aim is to find the smoothest function $S^U_{\mathcal{N}_1,...,\mathcal{N}_N} \in \mathscr{H}_\tau$ from the set of all $\mathscr{H}_\tau$– interpolants, where the norm is minimized in $\mathcal{H}_\tau$, i.e., the problem is to find a function S_N in the set

$$\mathcal{I}^U_{\mathcal{N}_N,...,\mathcal{N}_N} = \{P \in \mathcal{H}_\tau \quad | \quad \mathcal{N}_i P = \alpha_i, \quad i = 1, ..., N\}, \tag{5.72}$$

such that

$$\left\|S^U_{\mathcal{N}_1,...,\mathcal{N}_N}\right\|_{\mathscr{H}_\tau} = \inf_{P \in \mathcal{I}^U_{\mathcal{N}_1,...,\mathcal{N}_N}} \|P\|_{\mathscr{H}_\tau}. \tag{5.73}$$

As previously we denote the representer of the functional $\mathcal{N}_i$ is by $\mathcal{N}_i \mathscr{K}_\tau(\cdot,\cdot)$, where $\mathscr{K}_\tau$ is the reproducing kernel of $\mathscr{H}_\tau$, defined by (5.33), i.e.,

$$\mathcal{N}_i \mathscr{K}_\tau(\cdot,\cdot) = \mathcal{N}\mathscr{K}_\tau(x_i,\cdot) = \frac{\partial \mathscr{K}_\tau(x_i,\cdot)}{\partial \lambda_\Sigma(x_i)}. \tag{5.74}$$

Again considering the Newton integral representation of the reproducing kernel, we write

$$\mathcal{N}_i \mathscr{K}_\tau(\cdot,\cdot) = \frac{\partial}{\partial \lambda_\Sigma(x_i)} \int_{\Sigma^{int}_\tau} \frac{dz}{|x_i - z||\cdot - z|} = \int_{\Sigma^{int}_\tau} \frac{-\lambda(x_i)(x_i - z)}{|x_i - z|^3 |\cdot - z|} dz \tag{5.75}$$

For the set $\{\mathcal{N}_1, ..., \mathcal{N}_N\}$ of N linearly independent bounded linear functionals on $\mathscr{H}_\tau$, corresponding to the set $X_N = \{x_1, ..., x_N\}$ of points on Σ, and the set of representers $\{\mathcal{N}_1\mathscr{K}_\tau(\cdot,\cdot), ..., \mathcal{N}_N\mathscr{K}_\tau(\cdot,\cdot)\}$ such that for $i = 1, ..., N$

$$\mathcal{N}_i P = (\mathcal{N}_i\mathscr{K}_\tau(\cdot,\cdot), P)_{\mathscr{H}_\tau}, \quad P \in \mathscr{H}_\tau, \tag{5.76}$$

the $\mathscr{H}_\tau$-spline relative to $\{\mathcal{N}_1, ..., \mathcal{N}_N\}$ is given by

$$S(x) = \sum_{i=1}^{N} a_i \mathcal{N}_i \mathscr{K}_\tau(\cdot, x) = \sum_{i=1}^{N} a_i \int_{\Sigma_\tau^{int}} \frac{-\lambda(x_i)(x_i - z)}{|x_i - z|^3 |x - z|} dz, \quad x \in \overline{\Sigma^{ext}}. \tag{5.77}$$

In analogous way the same results as in previous subsection are valid also for the $\mathscr{H}_\tau$-spline, i.e., we have the following

Lemma 5.4.12. *Let S be a function of class $S_{\mathscr{H}_\tau}(\mathcal{N}_1, ..., \mathcal{N}_N)$. Then for each $P \in \mathscr{H}_\tau$, the following is valid*

$$(S, P)_{\mathscr{H}_\tau} = \sum_{i=1}^{N} a_i \mathcal{N}_i P. \tag{5.78}$$

Lemma 5.4.13. *(Uniqueness of interpolation) For given $U \in \mathscr{H}_\tau$ there exist a unique element in $S_{\mathscr{H}}(\mathcal{N}_1, ..., \mathcal{N}_N) \cap \mathcal{I}^U_{\mathcal{N}_1,...,\mathcal{N}_N}$. This element is denoted by $S^U_{\mathcal{N}_1,...,\mathcal{N}_N}$.*

Proof: Application of the linear functionals $\mathcal{N}_1, ..., \mathcal{N}_N$ to the $\mathscr{H}_\tau$-spline of the form (5.77), yields N linear equations in the coefficients $a_1^N, ..., a_N^N$

$$\sum_{j=1}^{N} a_j^N \mathcal{N}_i \mathcal{N}_j \mathscr{K}_\tau(\cdot,\cdot) = \mathcal{N}_i U, \quad i = 1, ..., N, \tag{5.79}$$

where the elements of the coefficients matrix

$$\left(\mathcal{N}_i \mathcal{N}_j \mathscr{K}_\tau(\cdot,\cdot)\right)_{i,j=1,...,N}, \tag{5.80}$$

are given by

$$\mathcal{N}_i \mathcal{N}_j \mathscr{K}_\tau(\cdot,\cdot) = \int_{\Sigma_\tau^{int}} \frac{\lambda(x_i)\lambda(x_j)(x_i - z)(x_j - z)}{|x_i - z|^3 |x_j - z|^3} dz. \tag{5.81}$$

The same argumentation as before yields that the linear system (5.79) is uniquely solvable.

Lemma 5.4.14. *The interpolating $\mathscr{H}$-spline $S^U_{\mathcal{N}_1,...,\mathcal{N}_N}$ of U (relative to $\{\mathcal{N}_1, ..., \mathcal{N}_N\}$) is the $\mathscr{H}$-orthogonal projection of U onto the space $S_{\mathscr{H}}(\mathcal{N}_1, ..., \mathcal{N}_N)$.*

Minimum norm properties for the interpolating spline are also valid.

Lemma 5.4.15. *(First minimum property) If* $P \in \mathcal{I}^U_{\mathcal{N}_1,...,\mathcal{N}_N}$*, then*

$$||P||^2_{\mathscr{H}_\tau} = ||S^U_{\mathcal{N}_1,...,\mathcal{N}_N}||^2_{\mathscr{H}_\tau} + ||S^U_{\mathcal{N}_1,...,\mathcal{N}_N} - P||^2_{\mathscr{H}_\tau}. \tag{5.82}$$

Lemma 5.4.16. *(Second minimum property) If* $S \in S_{\mathscr{H}_\tau}(\mathcal{N}_1, ..., \mathcal{N}_N)$ *and* $P \in \mathcal{I}^U_{\mathcal{N}_1,...,\mathcal{N}_N}$*, then*

$$||S - P||^2_{\mathscr{H}_\tau} = ||S^U_{\mathcal{N}_1,...,\mathcal{N}_N} - P||^2_{\mathscr{H}_\tau} + ||S - S^U_{\mathcal{N}_1,...,\mathcal{N}_N}||^2_{\mathscr{H}_\tau}. \tag{5.83}$$

Altogether we have

Theorem 5.4.17. *The interpolation problem*

$$||S^U_{\mathcal{N}_1,...,\mathcal{N}_N}||_{\mathscr{H}_\tau} = \inf_{P \in \mathcal{I}^U_{\mathcal{N}_1,...,\mathcal{N}_N}} ||P||_{\mathscr{H}_\tau}, \tag{5.84}$$

is well-posed in the sense that its solution exists, is unique, and depends continuously on the data $\alpha_1, ..., \alpha_N$*. The uniquely determined solution* $S^U_{\mathcal{N}_1,...,\mathcal{N}_N}$ *is given in the explicit form*

$$S^U_{\mathcal{N}_1,...,\mathcal{N}_N}(x) = \sum_{i=1}^N a_i^N \int_{\Sigma^{int}_\tau} \frac{-\lambda(x_i)(x_i - z)}{|x_i - z|^3 |x - z|} dz, \quad x \in \overline{\Sigma^{ext}}, \tag{5.85}$$

where the coefficients $a_1^N, ..., a_N^N$ *satisfy the linear equations*

$$\sum_{i=1}^N a_i^N \int_{\Sigma^{int}_\tau} \frac{\lambda(x_i)\lambda(x_j)(x_i - z)(x_j - z)}{|x_i - z|^3 |x_j - z|^3} dz = \alpha_j, \quad j = 1, ..., N. \tag{5.86}$$

As remarked earlier, the linear independence for the set $\mathcal{N}_1, ..., \mathcal{N}_N$ of bounded linear functionals is not necessary. All results are also valid if $\mathcal{N}_1, ..., \mathcal{N}_N$ are linearly dependent. In this case uniqueness of the $\mathscr{H}$- spline is still guaranteed, because the spline interpolation operator is still an orthogonal projector. However, the system of linear equations (5.79) is no longer uniquely solvable.

An obvious analogue to Theorem 5.4.9 is also valid for the space $\mathscr{H}_\tau$ and its reproducing kernel, i.e., we have

Theorem 5.4.18. *Let* $X = \{x_i\}_{i \in \mathbb{N}} \subset \Sigma$ *be a countable dense set of points on* Σ*. Then*

$$\overline{span\{\mathscr{K}_\tau(x_i, \cdot) | i \in \mathbb{N}\}}^{||\cdot||_{\mathscr{H}_\tau}} = \mathscr{H}_\tau. \tag{5.87}$$

Generalizing the Theorem 5.4.18 for the functionals of the oblique derivative we get

Theorem 5.4.19. *Let $X = \{x_i\}_{i\in\mathbb{N}} \subset \Sigma$ be a countable dense set of points on Σ. Let $\{\mathcal{N}_i\}_{i\in\mathbb{N}}$ be the set of bounded linear oblique derivative functionals on $\mathscr{H}$, corresponding to the set X. Then*

$$\overline{span_{\mathcal{N}_i\in X}\{\mathcal{N}_i\mathscr{K}_\tau(\cdot,\cdot)|i\in\mathbb{N}\}}^{||\cdot||_{\mathscr{H}_\tau}} = \mathscr{H}_\tau. \tag{5.88}$$

Proof: Our purpose is to show that the properties $P \in \mathscr{H}_\tau$ and $(\mathcal{N}_i\mathscr{K}_\tau(\cdot,\cdot), P)_{\mathscr{H}_\tau} = 0$ for all $i \in \mathbb{N}$, imply that $P = 0$. Then the statement of the theorem, follows immediately from Theorem 2.1.6. According to our construction, P is harmonic function in Σ_τ^{ext}. Clearly, the condition $(\mathcal{N}_i\mathscr{K}_\tau(\cdot,\cdot), P)_{\mathscr{H}_\tau} = 0$ is equivalent to $\frac{\partial P}{\partial\lambda_\Sigma}(x_i) = 0$ for all $i \in \mathbb{N}$. Due to harmonicity of P in Σ_τ^{ext}, this means that $\frac{\partial P}{\partial\lambda_\Sigma}(x) = 0$ for all $x \in \Sigma \subset \Sigma_\tau^{ext}$, i.e., $P(x) = C$ for $x \in \Sigma$. Moreover, the potentials from $\mathscr{H}_\tau$ are regular at infinity. This yields $C = 0$, and thus $P = 0$ as required.
We are now able to give the following theorem for the convergence of the interpolating spline $S^U_{\mathcal{N}_{x_1},\ldots,\mathcal{N}_{x_N}}$:

Theorem 5.4.20. *Let the assumptions and the notation be the same as in Theorem* 5.4.19. *Let $U \in \mathscr{H}$ be arbitrary function and let $S^U_{\mathcal{N}_1,\ldots,\mathcal{N}_N}$ be the interpolating $\mathscr{H}$–spline relative to $\{\mathcal{N}_1,\ldots,\mathcal{N}_N\}$. Then the following convergence results hold true:*

(i) $\lim_{N\to\infty} ||U - S^U_{\mathcal{N}_1,\ldots,\mathcal{N}_N}||_{\mathscr{H}_\tau} = 0$

(ii) $\lim_{N\to\infty} ||U - S^U_{\mathcal{N}_1,\ldots,\mathcal{N}_N}||_{C(\overline{\Sigma^{ext}})} = 0$

Proof: The proof of (i) follows by analogous arguments as in Theorem 5.4.11.
In order to prove (ii), we realize that for arbitrary $x \in \overline{\Sigma^{ext}} \subset \Sigma_\tau^{ext}$, reproducing property of $\mathscr{H}_\tau$ and Cauchy inequality, yields

$$\begin{aligned} |U(x) - S^U_{\mathcal{N}_1,\ldots,\mathcal{N}_N}(x)| &= |(U - S^U_{\mathcal{N}_1,\ldots,\mathcal{N}_N}, \mathcal{N}_x\mathscr{K}_\tau(\cdot,\cdot))| \\ &\le ||U - S^U_{\mathcal{N}_1,\ldots,\mathcal{N}_N}||_{\mathscr{H}_\tau} ||\mathcal{N}_x\mathscr{K}_\tau(\cdot,\cdot)||_{\mathscr{H}_\tau}. \end{aligned} \tag{5.89}$$

From the representation (5.75) of the reproducing kernel $\mathcal{N}_x\mathscr{K}_\tau(\cdot,\cdot)$, and our definition of the norm in $\mathscr{H}_\tau$, we find for $x \in \Sigma \subset \Sigma_\tau^{ext}$ the following estimate for the norm of the representer $\mathcal{N}_x\mathscr{K}_\tau(\cdot,\cdot)$ in $\mathscr{H}_\tau$

$$||\mathcal{N}_x\mathscr{K}_\tau(\cdot,\cdot)||_{\mathscr{H}_\tau} \le \left\|\frac{\lambda(x)(x-z)}{|x-z|^3}\right\|_{L^2(\Sigma_\tau^{int})} = \int_{\Sigma_\tau^{int}} \left|\frac{\lambda(x)(x-z)}{|x-z|^3}\right|^2 dz \le \int_{\Sigma_\tau^{int}} \frac{dz}{|x-z|^4}, \tag{5.90}$$

where the last integral is convergent for $x \in \Sigma$. Thus, from (5.89) for arbitrary $x \in \overline{\Sigma^{ext}}$ we have

$$|U(x) - S^U_{\mathcal{N}_1,\ldots,\mathcal{N}_N}(x)| \le C(x,\Sigma_\tau)||U - S^U_{\mathcal{N}_1,\ldots,\mathcal{N}_N}||_{\mathscr{H}_\tau}, \tag{5.91}$$

where

$$C(x, \Sigma_\tau) = \int_{\Sigma_\tau^{int}} \frac{dz}{|x-z|^4}. \tag{5.92}$$

Equation (5.91) together with (i) shows the validity of (ii).

6 Spherical Earth Geometry

In the previous chapter we have constructed a reproducing kernel Hilbert space of potentials outside the regular surface Σ under consideration, as a space of approximating functions to the gravitational potential outside Σ. We have shown that the reproducing kernel is a potential of class $\mathrm{Pot}^{(0)}(\overline{\Sigma^{ext}})$, represented by a Newton integral over the body Σ^{int}.

The advantage of this reproducing kernel is twofold: First, it shows the same properties as the gravitational potential, and second it carries the information of the regular surface Σ under consideration. This is a great step forward, especially having in mind the conventional Runge approach, where the reproducing kernels used in gravitational potential determination, are of class $\mathrm{Pot}^{(\infty)}(\overline{A^{ext}})$. They are generated as sequences of outer harmonics corresponding to a Runge sphere A (see Section 3.1), i.e., they possess a larger harmonicity domain than the gravitational potential. Obviously, the reproducing kernel $\mathscr{K}$ is a 'natural' choice in the interpolation processes. It is certainly more appropriate, than the spherically oriented kernels with 'Runge sphere' inside Σ^{int}. Choosing Σ especially as a sphere this rises a question whether $\mathscr{K}$ is a kind of generalization to these types of kernels known from spherically harmonic spline theory (see [20], [24])? In spherical nomenclature we are able to show that $\mathscr{K}$ can indeed be considered a generalized version of a spherically harmonic kernel function, and this justifies its importance even more. The theory of spherical harmonic interpolation is well–known (see [20], [24], [31], [33], [41], [44], [48]). However, in order to explain these properties we must in short present some basic elements of this theory.

6.1 Spherically Based Runge Approach

The spherically harmonic spline theory is using the theoretical background of outer harmonics under a Sobolev space formalism, to solve the problem of gravitational potential determination discretely. The approximating spherically harmonic potentials are assumed to be functions in a Sobolev-like Hilbert space equipped with a reproducing kernel. All elements of this space are harmonic functions outside a 'Runge sphere' inside Σ^{int}, i.e., they are harmonic in the $\overline{A^{ext}}$. The Runge–Walsh approximation property then yields that every function harmonic outside the regular surface Σ, may be approximated arbitrarily well (in uniform sense) by elements of this space. Sobolev space considerations are of great importance in spherical harmonic based approximation methods for gravitational potential determination. The Sobolev space formalism has proven conceptually suitable to verify the boundedness of the linear functionals representing the gravitational observables. Working with the gravitational potential in the exterior of a regular surface Σ we have the advantage to deal with a function which contains outer harmonics contributions of any degree. In other words, the Sobolev space to constitute the reference class for the gravitational potential determination contains all spaces $Harm_n(\overline{A^{ext}}), n \in \mathbb{N}_0$. Consequently, not only low degrees are involved (as, for example, in truncated Fourier series), but also high degree contributions are present. Thus, the spline theory seems to be much more of advantage if high frequency approximations are searched for.

6.1.1 The Hilbert Spaces $\mathcal{H}\left(\{A_n\}; \overline{A^{ext}}\right)$

In the following we introduce the (Sobolev-like) Hilbert spaces $\mathcal{H}(\{A_n\}; \overline{A^{ext}})$ of harmonic functions which serve as reference spaces for spherically harmonic spline theory. As already mentioned, from the mathematical point of view, functions in $\mathcal{H}(\{A_n\}; \overline{A^{ext}})$ can be seen as series expansions in terms of outer harmonics with certain assumptions on the growth of the coefficients.

Let $\mathcal{A} = \{\{A_n\}_{n\in\mathbb{N}_0} | \, A_n \in \mathbb{R}^+ \text{ for all } n \in \mathbb{N}_0\}$ denote the set of all sequences of positive real numbers. Given a sequence $\{A_n\}_{n\in\mathbb{N}_0} \in \mathcal{A}$, we consider the linear space $\mathcal{E} = \mathcal{E}(\{A_n\}; \overline{A^{ext}})$, $\mathcal{E} \subset \mathrm{Pot}^{(\infty)}(\overline{A^{ext}})$ of all potentials F of the form

$$F = \sum_{n=0}^{\infty} \sum_{j=1}^{2n+1} F^{\wedge}(n,j) H^{\alpha}_{-n-1,j} \tag{6.1}$$

whose Fourier coefficients (with respect to $L^2(A)$)

$$F^{\wedge}(n,j) = F^{\wedge_{L^2(A)}}(n,j) = \int_A F(x) H^{\alpha}_{-n-1,j} d\omega_\alpha(x) \tag{6.2}$$

satisfy

$$\sum_{n=0}^{\infty} \sum_{j=1}^{2n+1} A_n^2 (F^{\wedge}(n,j))^2 < \infty \tag{6.3}$$

The last sum is imposed as a norm for $\mathcal{E}$

$$||F||_{\mathcal{H}(\{A_n\};\overline{A^{ext}})} = \left(\sum_{n=0}^{\infty} \sum_{j=1}^{2n+1} A_n^2 (F^{\wedge}(n,j))^2 \right)^{1/2} \tag{6.4}$$

Definition 6.1.1. *The Sobolev space* $\mathcal{H}\left(\{A_n\};\overline{A^{ext}}\right)$ *is defined by*

$$\mathcal{H}(\{A_n\};\overline{A^{ext}}) = \overline{\mathcal{E}(\{A_n\};\overline{A^{ext}})}^{||\cdot||_{\mathcal{H}(\{A_n\};\overline{A^{ext}})}}. \tag{6.5}$$

It is a Hilbert space equipped with the inner product

$$(F,G)_{\mathcal{H}(\{A_n\};\overline{A^{ext}})} = \sum_{n=0}^{\infty}\sum_{j=1}^{2n+1} A_n^2 F^{\wedge}(n,j)G^{\wedge}(n,j) \tag{6.6}$$

for $F, G \in \mathcal{H}(\{A_n\};\overline{A^{ext}})$, where $F^{\wedge}(n,j)$ and $G^{\wedge}(n,j)$ are Fourier coefficients of F and G with respect to $L^2(A)$. Every element F of the space $\mathcal{H}(\{A_n\};\overline{A^{ext}})$ is uniquely determined by its Fourier coefficients $F^{\wedge}(n,j)$ that satisfy

$$||F||^2_{\mathcal{H}(\{A_n\};\overline{A^{ext}})} = \left(\sum_{n=0}^{\infty}\sum_{j=1}^{2n+1} A_n^2 (F^{\wedge}(n,j))^2\right) < \infty, \tag{6.7}$$

and F can be formally represented by the expansion

$$F = \sum_{n=0}^{\infty}\sum_{j=1}^{2n+1} F^{\wedge}(n,j) H^{\alpha}_{-n-1,j}, \tag{6.8}$$

which has to be understood in 'distributional sense' (at least on A). Condition (6.7) determines the maximal possible growth behavior of the Fourier coefficients. It follows directly from the definition of $\mathcal{H}(\{A_n\};\overline{A^{ext}})$ that the set $\{A_n^{-1}H^{\alpha}_{-n-1,k}\}_{n\in\mathbb{N}_0,k=1,\dots,2n+1}$ is a complete orthonormal system in $\mathcal{H}(\{A_n\};\overline{A^{ext}})$.

Remark: In particular, we let

$$\mathcal{H}_s(\overline{A^{ext}}) = \mathcal{H}(\{(n+1/2)^s\};\overline{A^{ext}}), s \in \mathbb{R}. \tag{6.9}$$

Especially for $s = 0$ we have

$$\mathcal{H}_0(\overline{A^{ext}}) = \mathcal{H}(\{1\};\overline{A^{ext}}). \tag{6.10}$$

The space $\mathcal{H}_0(\overline{A^{ext}})$ may be understood as the space of all harmonic functions in $\overline{A^{ext}}$, regular at infinity, corresponding to $L^2(A)$–restrictions. Its norm $||\cdot||_{\mathcal{H}_0(\overline{A^{ext}})}$ can be understood as the $L^2(A)$–norm. Loosely spoken, the topology of $\mathcal{H}_0(\overline{A^{ext}})$ is led back to the topology of $L^2(A) = \mathcal{H}_0(\overline{A^{ext}})|_A$ and $\mathcal{H}_0(\overline{A^{ext}})$ forms the harmonic continuations of $L^2(A)$-functions.
According to our construction, the space $\mathrm{Pot}^{\infty}(\overline{A^{ext}})$ is a dense subspace of $\mathcal{H}_s(\overline{A^{ext}})$ for

each s. Moreover, if $t < s$, then $||F||_{\mathcal{H}_t(\overline{A^{ext}})} \leq ||F||_{\mathcal{H}_s(\overline{A^{ext}})}$.

When we associate to a potential $F \in \mathrm{Pot}^{\infty}(\overline{A^{ext}})$ the series (6.8), it is of fundamental importance to know if the series converges uniformly on $\overline{A^{ext}}$. The answer is provided by an analogue of the Sobolev lemma. In order to present this lemma, we first introduce the concept of summable sequences.

Definition 6.1.2. *A sequence $\{A_n\}_{n\in\mathbb{N}_0} \in \mathcal{A}$ is called summable if it satisfies the summability condition*

$$\sum_{n=0}^{\infty} \frac{2n+1}{4\pi} \frac{1}{A_n^2} < \infty. \tag{6.11}$$

Lemma 6.1.3. *(Sobolev Lemma) If a sequence $\{A_n\}_{n\in\mathbb{N}_0} \in \mathcal{A}$ is summable, then each $F \in \mathcal{H}(\{A_n\}; \overline{A^{ext}})$ corresponds to a potential of class $Pot^{(0)}(\overline{A^{ext}})$.*

Proof: For a sufficiently large integer N and all $x \in \overline{A^{ext}}$ we find in connection to Cauchy-Schwarz inequality and the addition theorem for outer harmonics

$$\begin{aligned} \left| \sum_{n=0}^{N} \sum_{k=1}^{2n+1} F^{\wedge}(n,k) H^{\alpha}_{-n-1,k} \right|^2 &= \left| \sum_{n=0}^{N} \sum_{k=1}^{2n+1} A_n^{-1} H^{\alpha}_{-n-1,k} A_n F^{\wedge}(n,k) \right|^2 \\ \leq \left(\sum_{n=0}^{N} \sum_{k=1}^{2n+1} |A_n|^{-2} (H^{\alpha}_{-n-1,k})^2 \right) &\times \left(\sum_{n=0}^{N} \sum_{k=1}^{2n+1} |A_n|^2 (F^{\wedge}(n,k))^2 \right) \\ \leq \left(\sum_{n=0}^{\infty} \frac{2n+1}{4\pi\alpha^2} |A_n|^{-2} \right) &||F||^2_{\mathcal{H}(\{A_n\}; \overline{A^{ext}})} < \infty \end{aligned} \tag{6.12}$$

This shows that the series (6.8) converges uniformly to the function $F \in \mathcal{H}(\{A_n\}; \overline{A^{ext}})$. Now, as the limit function of the uniformly convergent series of continuous functions is also a continuous function, we get the statement of Lemma 6.1.3.
The Sobolev lemma gives us the right to call the spaces $\mathcal{H}(\{A_n\}; \overline{A^{ext}})$ created via summable sequences, Sobolev spaces.

Considering the differentiability of potentials in the Sobolev spaces, we present the following

Lemma 6.1.4. *If $F \in \mathcal{H}_s(\overline{A^{ext}})$ with $s > k+1$, then F corresponds to a function of class $Pot^{(k)}(\overline{A^{ext}})$.*

Moreover, the following lemma is valid (see e.g., [31], [38]):

Lemma 6.1.5. *Suppose that F is of class $\mathcal{H}_s(\overline{A^{ext}})$, $s > [l]+1$, for a given multi-index l. Then*

$$\sup_{x\in\overline{A^{ext}}}\left|(\nabla^l F)(x) - \sum_{n=0}^{N}\sum_{k=1}^{2n+1} F^\wedge(n,k)\nabla^l H^\alpha_{-n-1,k}\right| \le CN^{[l]+1-s}||F||_{\mathcal{H}_s(\overline{A^{ext}})}, \tag{6.13}$$

holds for all positive integers N, and C is a positive constant independent of F.

Theorem 6.1.6. *Let $\{A_n\}_{n\in\mathbb{N}_0} \in \mathcal{A}$ be a summable sequence. Then $\mathcal{H}(\{A_n\};\overline{A^{ext}})$ is a reproducing kernel Hilbert space with the reproducing kernel given by*

$$\begin{aligned} K_{\mathcal{H}(\{A_n\};\overline{A^{ext}})}(x,y) &= \sum_{n=0}^{\infty}\sum_{j=1}^{2n+1}\frac{1}{A_n}H^\alpha_{-n-1,j}(x)\frac{1}{A_n}H^\alpha_{-n-1,j}(y) \\ &= \sum_{n=0}^{\infty}\frac{1}{A_n^2}\frac{2n+1}{4\pi\alpha^2}\left(\frac{\alpha^2}{|x||y|}\right)^{n+1} P_n\Big(\frac{x}{|x|}\cdot\frac{y}{|y|}\Big) \end{aligned} \tag{6.14}$$

where $x, y \in \overline{A^{ext}}$.

Proof: From the Sobolev lemma it follows that $\mathcal{H}(\{A_n\};\overline{A^{ext}})$ is a subset of $\mathrm{Pot}^{(0)}(\overline{A^{ext}})$. A necessary and sufficient condition that $\mathcal{H}(\{A_n\};\overline{A^{ext}})$ possesses a reproducing kernel is that, for all $x \in \overline{A^{ext}}$, the evaluation functional (i.e., Dirichlet functional) $\mathcal{D}_x : \mathcal{H}(\{A_n\};\overline{A^{ext}}) \to \mathbb{R}$ given by $\mathcal{D}_x F = F(x)$, $x \in \overline{A^{ext}}$ is bounded. This can be easily derived from (6.8) and (6.12). Namely we have

$$|\mathcal{D}_x F| = |F(x)| \le \left(\sum_{n=0}^{\infty}\frac{1}{A_n^2}\frac{2n+1}{4\pi\alpha^2}\right)^{1/2} ||F||_{\mathcal{H}(\overline{A^{ext}})}. \tag{6.15}$$

The statement of the theorem then follows directly from Theorem 2.1.4, with

$$\begin{aligned} K_{\mathcal{H}(\{A_n\};\overline{A^{ext}})}(x,x) &= \sum_{n=0}^{\infty}\frac{1}{A_n^2}\frac{2n+1}{4\pi\alpha^2}\left(\frac{\alpha^2}{|x|^2}\right)^{n+1} \\ &\le \frac{1}{4\pi\alpha^2}\sum_{n=0}^{\infty}(2n+1)\frac{1}{A_n^2} < \infty \end{aligned} \tag{6.16}$$

The following theorem imposes the necessary conditions on the sequence $\{A_n\}_{n\in\mathbb{N}_0} \in \mathcal{A}$, for the boundedness of the functionals of the oblique derivative on $\mathcal{H}(\{A_n\};\overline{A^{ext}})$.

Theorem 6.1.7. *Let $\{A_n\}_{n\in\mathbb{N}_0} \in \mathcal{A}$ be a summable sequence for which the following condition holds: There exists a constant $C > 0$ and a parameter $s > 2$, such that*

$$A_n \ge C\,(n+1/2)^s \tag{6.17}$$

for all but finitely many $n \in \mathbb{N}_0$.
Then the first directional derivative $\mathcal{N}_x : \mathcal{H}(\{A_n\}; \overline{A^{ext}}) \to \mathbb{R}$, defined by

$$\mathcal{N}_x F = \frac{\partial F}{\partial \lambda_x}(x), \tag{6.18}$$

in the point $x \in \overline{A^{ext}}$ and direction λ, $|\lambda| = 1$, where $\lambda(x) \cdot \nu(x) > 0$ in case $x \in A$ (ν is the unit normal field on A), is a well-defined bounded linear functional with the representer

$$\mathcal{N} K_{\mathcal{H}(\{A_n\};\overline{A^{ext}})}(\cdot, x) = \sum_{n=0}^{\infty} \sum_{j=1}^{2n+1} \frac{1}{A_n} \mathcal{N}_x H^{\alpha}_{-n-1,j}(x) \frac{1}{A_n} H^{\alpha}_{-n-1,j}(\cdot), x \in \overline{A^{ext}}, \tag{6.19}$$

which is a continuous function on $\overline{A^{ext}}$.

Note that condition (6.17) can be dropped if we restrict ourselves to first directional derivatives $\mathcal{N}_x$ in points $x \in A^{ext}$. For details and for the proof of Theorem 6.1.7, the reader is referred to [48].

Reproducing kernel representations may be used to act as basis system in reproducing Sobolev spaces.

Theorem 6.1.8. *Let $\{A_n\}_{n\in\mathbb{N}_0} \in \mathcal{A}$ be a summable sequence. Assume that $X = \{x_i\}_{i\in\mathbb{N}}$ is a countable dense set of points on a regular surface $\Xi \subset \overline{A^{ext}}$ (for example, Runge sphere A, 'spherical Earth's surface' Ω_R, actual Earth's surface Σ). Then*

$$\overline{\mathrm{span}_{x_i\in X} K_{\mathcal{H}(\{A_n\};\overline{A^{ext}})}(x_i, \cdot)}^{\|\cdot\|_{\mathcal{H}(\{A_n\};\overline{A^{ext}})}} = \mathcal{H}(\{A_n\}; \overline{A^{ext}}),$$

For proof the reader is referred to [31].

6.1.2 Spherically Harmonic $\mathcal{H}(\{A_n\}; \overline{A^{ext}})$–Splines

The Sobolev spaces of harmonic functions $\mathcal{H}(\{A_n\}; \overline{A^{ext}})$ allow the definition of harmonic splines (see [20], [27] for the original papers or the text books [31], [38]). These splines are introduced with respect to a set of linear bounded functionals which provide interpolation conditions. The choise of the solution space $\mathcal{H}(\{A_n\}; \overline{A^{ext}})$, i.e., the corresponding sequence $\{A_n\}_{n\in\mathbb{N}_0} \in \mathcal{A}$, is dictated by the specifics of the functional under consideration. Since we are interested in the interpolation with respect to the functional of the oblique derivative, from now on we consider in accordance to Theorem 6.1.7, that the Sobolev-like space $\mathcal{H}(\{A_n\}; \overline{A^{ext}})$ corresponds to a summable sequence $\{A_n\}_{n\in\mathbb{N}_0} \in \mathcal{A}$, such that condition (6.17) is valid for all but finitely many $n \in \mathbb{N}_0$. It should be noted that when solving the satellite problems the spline method can be used even for the case when no summability condition is imposed on the sequence $\{A_n\}_{n\in\mathbb{N}_0} \in \mathcal{A}$ (cf. [48]). However, in such cases an additional requirement must be imposed on the sequence $\{A_n\}_{n\in\mathbb{N}_0} \in \mathcal{A}$, i.e., it must be required that $A_n \geq 1$ for all but finitely many $n \in \mathbb{N}$.

Definition 6.1.9. *Let $\{\mathcal{L}_1, ...\mathcal{L}_N\}$ be a set of N linearly independent bounded linear functionals on the Sobolev-type Hilbert space $\mathcal{H}(\{A_n\}; \overline{A^{ext}})$. Then any function S of the form*

$$S(x) = \sum_{i=1}^{N} a_i \mathcal{L}_i K_{\mathcal{H}(\{A_n\};\overline{A^{ext}})}(\cdot, x), x \in \overline{A^{ext}}, \tag{6.20}$$

with a set of real numbers $\{a_1, ..., a_N\} \subset \mathbb{R}$ is called a $\mathcal{H}(\{A_n\}; \overline{A^{ext}})$-spline relative to $\{\mathcal{L}_1, ...\mathcal{L}_N\}$.

The function space of all $\mathcal{H}(\{A_n\}; \overline{A^{ext}})$–splines relative to $\{\mathcal{L}_1, ...\mathcal{L}_N\}$ is denoted by $S_{\mathcal{H}(\{A_n\};\overline{A^{ext}})}(\mathcal{L}_1, ...\mathcal{L}_N)$.

$\mathcal{H}(\{A_n\}; \overline{A^{ext}})$–spline interpolation problem

Let $F \in \mathcal{H}(\{A_n\}; \overline{A^{ext}})$, and let $\{\mathcal{L}_1, ...\mathcal{L}_N\}$ be a set of N linearly independent bounded linear functionals on the Hilbert space $\mathcal{H}(\{A_n\}; \overline{A^{ext}})$. As usual we denote the representer of $\mathcal{L}_i$, by $\mathcal{L}_i K_{\mathcal{H}(\{A_n\};\overline{A^{ext}})}(\cdot, \cdot)$, $i = 1, ..., N$. The space of all interpolating functions in $\mathcal{H}(\{A_n\}; \overline{A^{ext}})$ for F relative to $\mathcal{L}_1, ..., \mathcal{L}_N$ is denoted by

$$\mathcal{I}_{\mathcal{L}_1,...,\mathcal{L}_N} = \{G \in \mathcal{H}(\{A_n\}; \overline{A^{ext}}) \quad | \quad \mathcal{L}_i G = \mathcal{L}_i F, i = 1, ..., N\}.$$

The $\mathcal{H}(\{A_n\}; \overline{A^{ext}})$–spline interpolation problem is to determine a function

$$S^F_{\mathcal{H}(\{A_n\};\overline{A^{ext}})}(\mathcal{L}_1, ...\mathcal{L}_N) \in S_{\mathcal{H}(\{A_n\};\overline{A^{ext}})}(\mathcal{L}_1, ...\mathcal{L}_N) \cap \mathcal{I}_{\mathcal{L}_1,...,\mathcal{L}_N},$$

i.e., to determine a spline $S^F_{\mathcal{H}(\{A_n\};\overline{A^{ext}})}(\mathcal{L}_1, ...\mathcal{L}_N)$ which fulfills the interpolation conditions

$$\mathcal{L}_i S^F_{\mathcal{H}(\{A_n\};\overline{A^{ext}})}(\mathcal{L}_1, ...\mathcal{L}_N) = \mathcal{L}_i F,$$

for all $i = 1, ..., N$.
In analogous way as in the Section 5.4, the solution to the interpolation problem corresponding to $\mathcal{H}(\{A_n\}; \overline{A^{ext}})$-splines relative to a finite set of linear bounded functionals, relates the interpolation conditions to a system of linear equations which needs to be solved to obtain the spline coefficients. Together with the set of linear bounded functionals and the Sobolev space $\mathcal{H}(\{A_n\}; \overline{A^{ext}})$ (or the corresponding representers) these coefficients define the interpolating spline. For this spline the analogous minimum norm properties are valid as for the splines defined in Section 5.4.

Theorem 6.1.10. *Let $F \in \mathcal{H}(\{A_n\};\overline{A^{ext}})$ and let $\{\mathcal{L}_1, ...\mathcal{L}_N\}$ be a set of N linearly independent bounded linear functionals on the Hilbert space $\mathcal{H}(\{A_n\};\overline{A^{ext}})$. Then the $\mathcal{H}(\{A_n\};\overline{A^{ext}})$–spline interpolation problem relative to $\{\mathcal{L}_1, ...\mathcal{L}_N\}$ is uniquely solvable and its solution has the representation*

$$S^F_{\mathcal{H}(\{A_n\};\overline{A^{ext}})}(\mathcal{L}_1, ...\mathcal{L}_N) = \sum_{i=1}^{N} a_i^N \mathcal{L}_i K_{\mathcal{H}(\{A_n\};\overline{A^{ext}})}(\cdot, x), \tag{6.21}$$

with $\mathcal{H}(\{A_n\};\overline{A^{ext}})$–spline coefficients $a_i^N, i = 1, ..., N$. These coefficients are uniquely given by the following system of linear equations

$$\sum_{i=1}^{N} a_i^N \mathcal{L}_i \mathcal{L}_j K_{\mathcal{H}(\{A_n\};\overline{A^{ext}})}(\cdot, \cdot) = \mathcal{L}_j F, \quad j = 1, ..., N. \tag{6.22}$$

The interpolating spline has the following properties

(i)

$$||S^F_{\mathcal{H}(\{A_n\};\overline{A^{ext}})}(\mathcal{L}_1, ...\mathcal{L}_N)||_{\mathcal{H}(\{A_n\};\overline{A^{ext}})} \leq ||F||_{\mathcal{H}(\{A_n\};\overline{A^{ext}})}. \tag{6.23}$$

(ii) (first minimum property)

$$\begin{aligned} ||G||^2_{\mathcal{H}(\{A_n\};\overline{A^{ext}})} &= \left\| S^F_{\mathcal{H}(\{A_n\};\overline{A^{ext}})}(\mathcal{L}_1, ...\mathcal{L}_N) \right\|^2_{\mathcal{H}(\{A_n\};\overline{A^{ext}})} \\ &\quad + \left\| G - S^F_{\mathcal{H}(\{A_n\};\overline{A^{ext}})}(\mathcal{L}_1, ...\mathcal{L}_N) \right\|^2_{\mathcal{H}(\{A_n\};\overline{A^{ext}})}, \end{aligned} \tag{6.24}$$

for all $G \in \mathcal{I}_{\mathcal{L}_1,...,\mathcal{L}_N}$, i.e., $S^F_{\mathcal{H}(\{A_n\};\overline{A^{ext}})}(\mathcal{L}_1, ...\mathcal{L}_N)$ is the interpolating function of F in $\mathcal{H}(\{A_n\};\overline{A^{ext}})$ with minimal norm.

(iii) (second minimum property)

$$\begin{aligned} ||S - G||^2_{\mathcal{H}(\{A_n\};\overline{A^{ext}})} &= \left\| S^F_{\mathcal{H}(\{A_n\};\overline{A^{ext}})}(\mathcal{L}_1, ...\mathcal{L}_N) - G \right\|^2_{\mathcal{H}(\{A_n\};\overline{A^{ext}})} \\ &\quad + \left\| S - S^F_{\mathcal{H}(\{A_n\};\overline{A^{ext}})}(\mathcal{L}_1, ...\mathcal{L}_N) \right\|^2_{\mathcal{H}(\{A_n\};\overline{A^{ext}})}, \end{aligned} \tag{6.25}$$

for all $G \in \mathcal{I}_{\mathcal{L}_1,...,\mathcal{L}_N}$ and $S \in S_{\mathcal{H}(\{A_n\};\overline{A^{ext}})}(\mathcal{L}_1, ...\mathcal{L}_N)$.

The proof can be found in [31] or [38].

The link between the boundary value problems on regular surfaces and harmonic $\mathcal{H}(\{A_n\};\overline{A^{ext}})$–splines with respect to the Runge sphere $A \subset \Sigma^{int}$, is provided by the Runge–Walsh theorem

Theorem 6.1.11. *Let Σ be a $C^{(k)}$- regular surface, $k \geq 2$, with A being a Runge sphere in Σ^{int} (see Section 3.1). Then the following holds true:*

$$\overline{span\{H^{\alpha}_{-n-1,j}|_{\overline{\Sigma^{ext}}} : n \in \mathbb{N}_0, k = 1, ..., 2n+1\}}^{||\cdot||_{C(\overline{\Sigma^{ext}})}} = Pot^{(0)}(\overline{\Sigma^{ext}}). \tag{6.26}$$

See [16] and [26] for the proof.

An immediate consequence of Theorem 6.1.11 is the following corollary that allows us to use functions of a reproducing kernel Hilbert space $\mathcal{H}(\{A_n\}; \overline{A^{ext}})$ for approximation.

Corollary 6.1.12. *Let Σ be a $C^{(k)}$- regular surface, $k \geq 2$, with a A being a Runge sphere in Σ^{int}. Let $\{A_n\}_{n\in\mathbb{N}_0} \in \mathcal{A}$ be a summable sequence. Then*

$$\overline{\mathcal{H}(\{A_n\}; \overline{A^{ext}})|_{\overline{\Sigma^{ext}}}}^{||\cdot||_{C(\overline{\Sigma^{ext}})}} = Pot^{(0)}(\overline{\Sigma^{ext}}). \tag{6.27}$$

A proof of Corollary 6.1.12 can be found e.g., in [48].

The convergence of the interpolating spline $S^F_{\mathcal{H}(\{A_n\};\overline{A^{ext}})}(\mathcal{L}_1, ...\mathcal{L}_N)$ to $F \in \mathcal{H}(\{A_n\}; \overline{A^{ext}})$ as $N \to \infty$, is guaranteed by the following

Theorem 6.1.13. *Let $\{A_n\}_{n\in\mathbb{N}_0} \in \mathcal{A}$ be a summable sequence such that condition (6.17) is valid for all but finitely many $n \in \mathbb{N}_0$. Let $\Sigma \subset \overline{A^{ext}}$ be a $C^{(2,\mu)}$–regular surface and let $X = \{x_1, x_2, ...\} \subset \Sigma$ be a dense point set on the surface Σ. Assume the set of bounded linear functionals $\{\mathcal{L}_1, ...\mathcal{L}_N\}$ on $\mathcal{H}(\{A_n\}; \overline{A^{ext}})$ consists of either*

(i) evaluation functionals $\mathcal{D}_i : \mathcal{H}(\{A_n\}; \overline{A^{ext}}) \to \mathbb{R}$, $F \mapsto \mathcal{D}_i F = F(x_i)$

(ii) oblique derivatives $\mathcal{N}_i : \mathcal{H}(\{A_n\}; \overline{A^{ext}}) \to \mathbb{R}$, $F \mapsto \mathcal{N}_i F = \frac{\partial F}{\partial\lambda(x_i)}(x_i)$, where λ denotes a unit $C^{1,\mu}$–vector field in $\mathbb{R}^3$, that satisfies $\lambda(x) \cdot \nu(x) > 0$ for all $x \in \Sigma$, with $\nu(x)$ being the outer unit normal vector in x and $x_i \in X \subset \Sigma$.

Let $F \in \mathcal{H}(\{A_n\}; \overline{A^{ext}})$ be arbitrary and let $S^F_{\mathcal{H}(\{A_n\};\overline{A^{ext}})}(\mathcal{L}_1, ...\mathcal{L}_N)$ be the interpolating $\mathcal{H}(\{A_n\}; \overline{A^{ext}})$–spline of F relative to $\mathcal{L}_1, ...\mathcal{L}_N$. Then

$$\lim_{N\to\infty} ||F - S^F_{\mathcal{H}(\{A_n\};\overline{A^{ext}})}(\mathcal{L}_1, ...\mathcal{L}_N)||_{\mathcal{H}(\{A_n\};\overline{A^{ext}})} = 0, \tag{6.28}$$

$$\lim_{N\to\infty} ||F - S^F_{\mathcal{H}(\{A_n\};\overline{A^{ext}})}(\mathcal{L}_1, ...\mathcal{L}_N)||_{C(\overline{\Omega^{ext}_r})} = 0 \quad \textit{for every} \quad r \geq \alpha, \tag{6.29}$$

and

$$\lim_{N\to\infty} ||F - S^F_{\mathcal{H}(\{A_n\};\overline{A^{ext}})}(\mathcal{L}_1, ...\mathcal{L}_N)||_{C(\overline{\Sigma^{ext}})} = 0. \tag{6.30}$$

Furthermore, the following statement hold true due to summability of the sequence $\{A_n\}_{n\in\mathbb{N}_0}$

$$\overline{span\{\mathcal{L}_i K_{\mathcal{H}(\{A_n\};\overline{A^{ext}})}(\cdot,\cdot)|_{\overline{\Omega_r^{ext}}} | i\in\mathbb{N}\}}^{||\cdot||_{C(\overline{\Omega_r^{ext}})}} = Pot^{(0)}(\overline{\Omega_r^{ext}}) \quad \textit{for every} \quad r\geq\alpha, \tag{6.31}$$

and

$$\overline{span\{\mathcal{L}_i K_{\mathcal{H}(\{A_n\};\overline{A^{ext}})}(\cdot,\cdot)|_{\overline{\Sigma^{ext}}} | i\in\mathbb{N}\}}^{||\cdot||_{C(\overline{\Sigma^{ext}})}} = Pot^{(0)}(\overline{\Sigma^{ext}}). \tag{6.32}$$

For the proof the reader is referred to [48]. For more detailed stability and convergence estimates we recommend [27], [38] or [31].

Theorem 6.1.13 yields the following convergence theorems for the treatment of the boundary value problems with spherically harmonic splines, i.e., splines related to a Runge sphere A.

Theorem 6.1.14. *Let the assumptions and the notation of Theorem* 6.1.13 *be valid. Suppose that u is an element of the class*

$$\mathcal{D}_\Sigma\mathcal{H}(\{A_n\};\overline{A^{ext}}) = \{\mathcal{D}_\Sigma U | \mathcal{D}_\Sigma : x\mapsto \mathcal{D}_x U, U\in\mathcal{H}(\{A_n\};\overline{A^{ext}}), x\in\Sigma\}. \tag{6.33}$$

Let $\{X_N\}$ *be a sequence of fundamental systems* $X_N=\{x_1,...,x_N\}$ *such that* $\Theta_N\to 0$ *as* $N\to\infty$. *Then the solution of the exterior boundary value problem*

$$U\in\mathcal{H}(\{A_n\};\overline{A^{ext}}), \quad \mathcal{D}_\Sigma U = u, \tag{6.34}$$

can be approximated in the sense that, to every $\varepsilon>0$, *there exist an integer* $N=N(\varepsilon)$ *and a harmonic spline* U_N *such that*

$$\sup_{x\in\overline{\Sigma^{ext}}} |U(x)-U_N(x)| \leq \varepsilon. \tag{6.35}$$

Theorem 6.1.15. *Let the assumptions and the notation of Theorem* 6.1.13 *be valid. Suppose that u is an element of the class*

$$\mathcal{N}_\Sigma\mathcal{H}(\{A_n\};\overline{A^{ext}}) = \{\mathcal{N}_\Sigma U | \mathcal{N}_\Sigma : x\mapsto \mathcal{N}_x U, U\in\mathcal{H}(\{A_n\};\overline{A^{ext}}), x\in\Sigma\}. \tag{6.36}$$

Let $\{X_N\}$ *be a sequence of fundamental systems* $X_N=\{x_1,...,x_N\}$ *such that* $\Theta_N\to 0$ *as* $N\to\infty$. *Then the solution of the exterior boundary value problem*

$$U\in\mathcal{H}(\{A_n\};\overline{A^{ext}}), \quad \mathcal{N}_\Sigma U = u, \tag{6.37}$$

can be approximated in the sense that, to every $\varepsilon>0$, *there exist an integer* $N=N(\varepsilon)$ *and a harmonic spline* U_N *such that*

$$\sup_{x\in\overline{\Sigma^{ext}}} |U(x)-U_N(x)| \leq \varepsilon. \tag{6.38}$$

To establish the solution of the boundary value problems in its classical form we combine the last theorems with an extension of Helly's theorem

Theorem 6.1.16. *Let M be a dense and convex subset in a normed, not necessarily complete linear space $(X, ||\cdot||)$. Let $\mathcal{L}_1, ..., \mathcal{L}_N$ be N bounded linear functionals on X. For any $F \in X$ and $\varepsilon > 0$, there exists a $G \in M$ such that*

(i) $||F - G||_X < \varepsilon$

(ii) $\mathcal{L}_i F = \mathcal{L}_i G$ *for all* $i = 1, ..., N$.

The general version of the theorem is proven in [72].

Having in mind the density arguments from Corollary 6.1.12 the extension of Helly's theorem yields the existence of the approximating $\mathcal{H}(\{A_n\}; \overline{A^{ext}})$–spline to a given function $U \in \mathrm{Pot}^{(0)}(\overline{\Sigma^{ext}})$, in the sense that, to every $\varepsilon > 0$, there exist an integer $N = N(\varepsilon)$ and a harmonic spline U_N such that

$$\sup_{x \in \overline{\Sigma^{ext}}} |U(x) - U_N(x)| \leq \varepsilon. \tag{6.39}$$

The last result shows that best approximations to solutions of boundary-value problems can be guaranteed on certain types of Sobolev-like subspaces of $\mathrm{Pot}^{(0)}(\overline{\Sigma^{ext}})$ by using harmonic $\mathcal{H}(\{A_n\}; \overline{A^{ext}})$–splines.

6.2 Spherical Representation of the Kernel $\mathscr{K}$

Next we will investigate the connection between the reproducing kernel $\mathscr{K}$ we have introduced in Chapter 5 and the spherically oriented kernels. For this purpose we will consider the special case $\Sigma = A$, i.e., we calculate the kernel $\mathscr{K}$ for the Runge sphere $A = \Omega_\alpha$.
In this case we get for the reproducing kernel $\mathscr{K}(x, \cdot)$ in $\mathscr{H}$

$$\mathscr{K}(x, y) = \int_{A^{int}} \frac{dz}{|x - z||y - z|}. \tag{6.40}$$

Now using the known expansions in spherical harmonics for fundamental solutions (of the Laplace's equation) appearing in the integral (see Section 2.5) we can write

$$\frac{1}{|x - z|} = \sum_{n=0}^{\infty} \frac{|z|^n}{|x|^{n+1}} P_n\left(\frac{x}{|x|} \cdot \frac{z}{|z|}\right). \tag{6.41}$$

and

$$\frac{1}{|y-z|} = \sum_{m=0}^{\infty} \frac{|z|^m}{|y|^{m+1}} P_m\left(\frac{y}{|y|} \cdot \frac{z}{|z|}\right). \tag{6.42}$$

Substituting this expressions in (6.40) we get

$$\begin{aligned} \mathscr{K}(x,y) &= \int_{A^{int}} \frac{1}{|x-z|} \frac{1}{|y-z|} dz \\ = \int_0^\alpha \int_{\Omega_r} \sum_{n=0}^{\infty} \frac{|z|^n}{|x|^{n+1}} P_n\left(\frac{x}{|x|} \cdot \frac{z}{|z|}\right) &\cdot \sum_{m=0}^{\infty} \frac{|z|^m}{|y|^{m+1}} P_m\left(\frac{y}{|y|} \cdot \frac{z}{|z|}\right) d\omega_r\left(\frac{z}{|z|}\right) dr \end{aligned} \tag{6.43}$$

Using the addition theorem (2.43) for spherical harmonics the last expression can be written as

$$\begin{aligned} &\int_0^\alpha \int_{\Omega_r} \sum_{n=0}^{\infty} \frac{r^n}{|x|^{n+1}} \sum_{j=-n}^{n} \frac{4\pi}{2n+1} Y_{n,j}\left(\frac{x}{|x|}\right) Y_{n,j}\left(\frac{z}{|z|}\right) \\ &\quad \cdot \sum_{m=0}^{\infty} \frac{r^m}{|y|^{m+1}} \sum_{k=-m}^{m} \frac{4\pi}{2m+1} Y_{m,k}\left(\frac{y}{|y|}\right) Y_{m,k}\left(\frac{z}{|z|}\right) d\omega_r\left(\frac{z}{|z|}\right) dr \\ &\overset{(2.70)}{=} \int_0^\alpha \sum_{n=0}^{\infty} \frac{r^{2n}}{|x|^{n+1}|y|^{n+1}} \sum_{j=-n}^{n} \left(\frac{4\pi}{2n+1}\right)^2 Y_{n,j}\left(\frac{x}{|x|}\right) Y_{n,j}\left(\frac{y}{|y|}\right) r^2 dr \\ &\overset{(2.43)}{=} \int_0^\alpha \sum_{n=0}^{\infty} \frac{r^{2n+2}}{|x|^{n+1}|y|^{n+1}} \left(\frac{4\pi}{2n+1}\right)^2 \frac{2n+1}{4\pi} P_n\left(\frac{x}{|x|} \cdot \frac{y}{|y|}\right) dr \\ &= \sum_{n=0}^{\infty} \frac{4\pi}{2n+1} \cdot \frac{1}{|x|^{n+1}|y|^{n+1}} P_n\left(\frac{x}{|x|} \cdot \frac{y}{|y|}\right) \int_0^\alpha r^{2n+2} dr \\ &= \sum_{n=0}^{\infty} \frac{4\pi}{2n+1} \frac{1}{|x|^{n+1}|y|^{n+1}} P_n\left(\frac{x}{|x|} \cdot \frac{y}{|y|}\right) \frac{\alpha^{2n+3}}{2n+3} \\ &= \sum_{n=0}^{\infty} \frac{4\pi\alpha}{(2n+1)(2n+3)} \cdot \left(\frac{\alpha^2}{|x||y|}\right)^{n+1} P_n\left(\frac{x}{|x|} \cdot \frac{y}{|y|}\right) \\ &= \sum_{n=0}^{\infty} \frac{(4\pi)^2\alpha^3}{(2n+1)^2(2n+3)} \cdot \frac{2n+1}{4\pi\alpha^2} \left(\frac{\alpha^2}{|x||y|}\right)^{n+1} P_n\left(\frac{x}{|x|} \cdot \frac{y}{|y|}\right) \end{aligned} \tag{6.44}$$

Altogether we have

$$\mathscr{K}(x,y) = \sum_{n=0}^{\infty} \frac{(4\pi)^2\alpha^3}{(2n+1)^2(2n+3)} \cdot \frac{2n+1}{4\pi\alpha^2} \left(\frac{\alpha^2}{|x||y|}\right)^{n+1} P_n\left(\frac{x}{|x|} \cdot \frac{y}{|y|}\right). \tag{6.45}$$

This means that in case of $\Sigma = A$, the kernel (5.15) corresponds to the type of kernels defined by (6.14) and the reproducing kernel Hilbert space $\mathscr{H}$ (Section 5.1) we have used as a solution space (Section 5.4) corresponds to the space $\mathcal{H}\Big(\{A_n\}; \overline{A^{ext}}\Big)$, where A_n is the summable sequence

$$A_n = 4\pi(2n+1)(2n+3)^{1/2}\alpha^{-3/2}, \tag{6.46}$$

i.e.,

$$\mathscr{H} = \mathcal{H}\Big(\{4\pi(2n+1)(2n+3)^{1/2}\alpha^{-3/2}\}; \overline{A^{ext}}\Big). \tag{6.47}$$

This shows us that we can consider the kernel (5.15) as a generalized version to the spherically oriented kernels. Following figures represent the reproducing kernel $\mathscr{K}$, calculated for $x, y \in \Omega$ and different values of α using the Clenshaw algorithm.

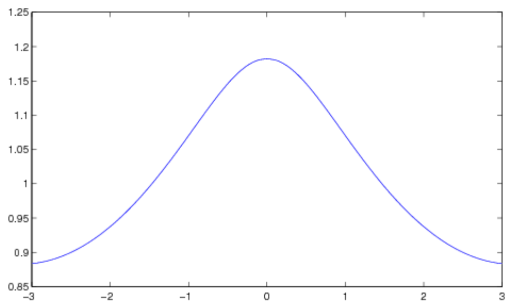

Figure 6.1: Kernel $\mathscr{K}$ on Ω with $\alpha = 0.7$.

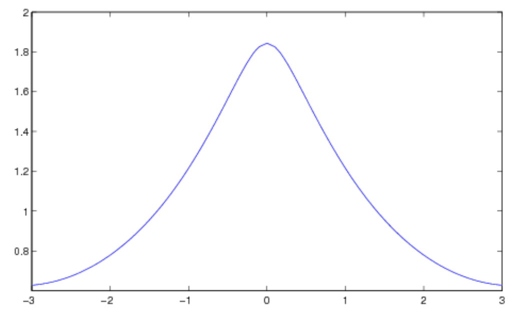

Figure 6.2: Kernel $\mathscr{K}$ on Ω with $\alpha = 0.9$.

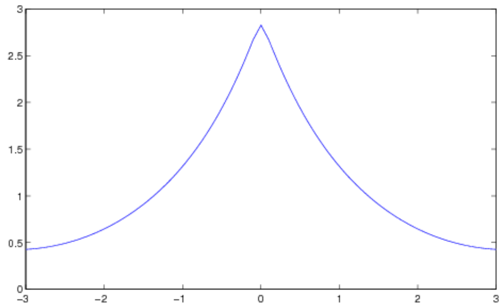

Figure 6.3: Kernel $\mathscr{K}$ on Ω with $\alpha = 0.99$.

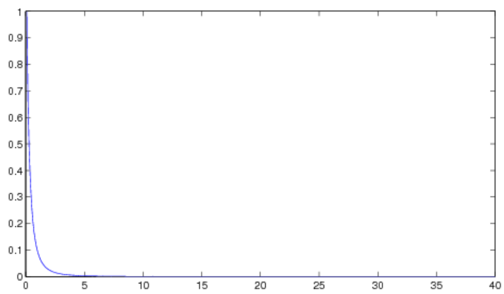

Figure 6.4: Coefficients A_n^{-1} of $\mathscr{K}$ with $\alpha = 0.7$.

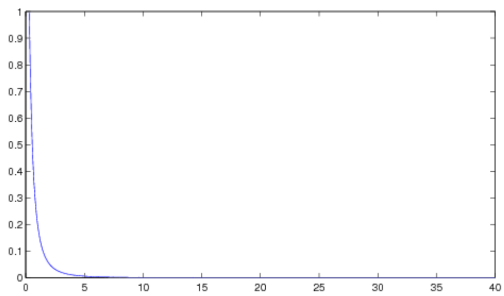

Figure 6.5: Coefficients A_n^{-1} of $\mathscr{K}$ with $\alpha = 0.9$.

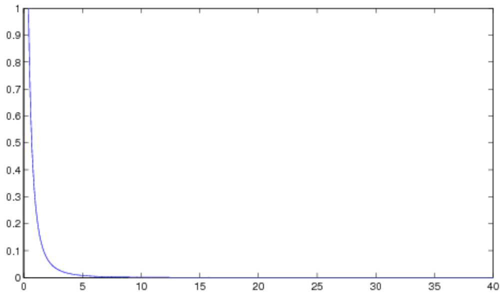

Figure 6.6: Coefficients A_n^{-1} of $\mathscr{K}$ with $\alpha = 0.99$.

6.3 Explicit Representations of the Kernel $\mathscr{K}$

It is obvious that for computational reasons of special importance are the reproducing kernels which have closed expressions. For some special classes of summable sequences $\{A_n\}_{n\in\mathbb{N}_0}$ we can find closed representations of the reproducing kernel as an elementary function by the use of the addition theorem (2.43) as well as (2.50) or (2.54), respectively. In the following we present the most often used spherically oriented kernels.

(i) Kernels of Abel–Poisson type: $A_n = h^{-n/2}$ for $h \in (0,1)$

$$\begin{aligned} &K_{\mathcal{H}\left(\{h^{-n/2}\};\overline{A^{ext}}\right)}(x,y) \\ &= \sum_{n=0}^{\infty} h^n \frac{2n+1}{4\pi\alpha^2}\left(\frac{\alpha^2}{|x||y|}\right)^{n+1} P_n\left(\frac{x}{|x|}\cdot\frac{y}{|y|}\right) \\ &= \frac{1}{4\pi}\frac{|x|^2|y|^2 - h^2\alpha^4}{\left(|x|^2|y|^2 + h^2\alpha^4 - 2h\alpha^2(x\cdot y)\right)^{3/2}} \end{aligned} \tag{6.48}$$

with $x, y \in \overline{A^{ext}}$.

(ii) Kernels of Singularity type: $A_n = \left(n+\frac{1}{2}\right)h^{-n/2}$ for $h \in (0,1)$

$$\begin{aligned} &K_{\mathcal{H}\left(\{\left(n+\frac{1}{2}\right)h^{-n/2}\};\overline{A^{ext}}\right)}(x,y) \\ &= \sum_{n=0}^{\infty} \frac{h^n}{\left(n+\frac{1}{2}\right)}\frac{2n+1}{4\pi\alpha^2}\left(\frac{\alpha^2}{|x||y|}\right)^{n+1} P_n\left(\frac{x}{|x|}\cdot\frac{y}{|y|}\right) \end{aligned}$$

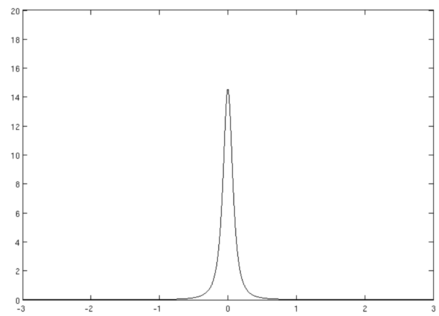

Figure 6.7: Kernel of Abel-Poisson type on Ω with $h = 0.9$

$$= \frac{1}{2\pi} \frac{1}{\big(|x|^2|y|^2 + h^2\alpha^4 - 2h\alpha^2(x \cdot y)\big)^{1/2}}$$

with $x, y \in \overline{A^{ext}}$.

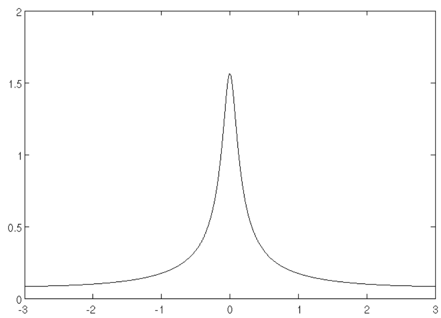

Figure 6.8: Singularity Kernel on Ω with $h = 0.9$

(iii) Kernels of Logarithmic Type: $A_n = \big(n+1\big)^{1/2}\big(2n+1\big)^{1/2} h^{-n/2}$ for $h \in (0,1)$

$$\begin{aligned} &K_{\mathcal{H}\left(\{\left(n+1\right)^{1/2}\left(2n+1\right)^{1/2} h^{-n/2}\};\overline{A^{ext}}\right)}(x,y) \\ &= \sum_{n=0}^{\infty} \frac{h^n}{\big(n+1\big)\big(2n+1\big)} \frac{2n+1}{4\pi R^2} \left(\frac{\alpha^2}{|x||y|}\right)^{n+1} P_n\left(\frac{x}{|x|} \cdot \frac{y}{|y|}\right) \\ &= \frac{1}{4\pi h\alpha^2} \log\left(1 + \frac{2h\alpha^2}{\big(|x|^2|y|^2 + h^2\alpha^4 - 2h\alpha^2(x \cdot y)\big)^{1/2} + |x||y| - h\alpha^2}\right) \end{aligned}$$

with $x, y \in \overline{A^{ext}}$.

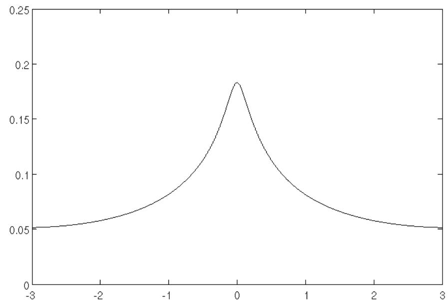

Figure 6.9: Logarithmic Kernel on Ω with $h = 0.9$

Next we will do an investigation on the existence of a closed expression for the kernel $\mathscr{K}$. We have

$$\mathscr{K}(x,y) = \frac{4\pi\alpha^3}{|x||y|}\sum_{n=0}^{\infty}\left(\frac{\alpha^2}{|x||y|}\right)^n \frac{1}{(2n+1)(2n+3)} P_n\left(\frac{x}{|x|}\cdot\frac{y}{|y|}\right). \tag{6.49}$$

Writing $h_1 = \frac{\alpha^2}{|x||y|} = \left(\frac{\alpha}{\sqrt{|x||y|}}\right) = h^2$, and using partial fraction we get

$$\begin{aligned}\mathscr{K}(x,y) &= \frac{4\pi\alpha^3}{|x||y|}\sum_{n=0}^{\infty} h^{2n}\frac{1}{(2n+1)(2n+3)}P_n\left(\frac{x}{|x|}\cdot\frac{y}{|y|}\right)\\ &= \frac{2\pi\alpha^3}{|x||y|}\left(\underbrace{\sum_{n=0}^{\infty} h_1^n \frac{1}{2n+1}P_n\left(\frac{x}{|x|}\cdot\frac{y}{|y|}\right)}_{=S_1} - \underbrace{\sum_{n=0}^{\infty} h_1^n\frac{1}{2n+3}P_n\left(\frac{x}{|x|}\cdot\frac{y}{|y|}\right)}_{=S_2}\right).\end{aligned} \tag{6.50}$$

For the sum S_1 we get from (2.50) for $t = \frac{x}{|x|}\cdot\frac{y}{|y|}$

$$\sum_{n=0}^{\infty} h_1^n P_n(t) = \sum_{n=0}^{\infty}(h^2)^n P_n(t) = \frac{1}{\sqrt{1+h^4-2th^2}}. \tag{6.51}$$

Integrating both sides with respect to h, we get

$$\sum_{n=0}^{\infty}\frac{h^{2n+1}}{2n+1}P_n(t) = \int \frac{1}{\sqrt{1+h^4-2th^2}}dh = \mathcal{F}(h,t) = \mathcal{F}(\sqrt{h_1},t), \tag{6.52}$$

where

$$\mathcal{F}(h,t) = \frac{-\left(i\sqrt{\frac{h^2}{\sqrt{t^2-1}-t}+1}\sqrt{1-\frac{h^2}{\sqrt{t^2-1}+t}}F\left(i\sinh^{-1}\left(\sqrt{\frac{1}{\sqrt{t^2-1}-t}}\cdot h\right)\Big|\frac{t-\sqrt{t^2-1}}{t+\sqrt{t^2-1}}\right)\right)}{\sqrt{\frac{1}{\sqrt{t^2-1}-t}}\sqrt{2th^2+h^4+1}} \tag{6.53}$$

and $F(h,t)$ is an elliptic integral of I kind.

In conclusion we have for the sum S_1

$$S_1 = \sum_{n=0}^{\infty} \frac{h_1^n}{2n+1} P_n(t) = \frac{1}{\sqrt{h_1}} \mathcal{F}(\sqrt{h_1}, t). \tag{6.54}$$

In a similar way we calculate the sum S_2. Equation (2.50) yields for $t = \frac{x}{|x|} \cdot \frac{y}{|y|}$ and for the sum

$$\sum_{n=0}^{\infty} h^{2n+2} P_n(t) = h^2 \sum_{n=0}^{\infty} (h^2)^n P_n(t) = \frac{h^2}{\sqrt{1+h^4-2th^2}}. \tag{6.55}$$

Again integrating both sides with respect to h, we get

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{h^{2n+3}}{2n+3} P_n(t) &= \int \frac{h^2}{\sqrt{1+h^4-2th^2}} dh \\ = h^3 \sum_{n=0}^{\infty} \frac{h^{2n}}{2n+3} P_n(t) &= \mathcal{G}(h,t) = \mathcal{G}(\sqrt{h_1}, t), \end{aligned} \tag{6.56}$$

where $\mathcal{G}(h,t)$ is defined via

$$\begin{aligned} &\frac{E\left(i\sinh^{-1}\left(\sqrt{\frac{1}{\sqrt{t^2-1}-t}}\cdot h\right)\Big|\frac{t-\sqrt{t^2-1}}{t+\sqrt{t^2-1}}\right) - F\left(i\sinh^{-1}\left(\sqrt{\frac{1}{\sqrt{t^2-1}-t}}\cdot h\right)\Big|\frac{t-\sqrt{t^2-1}}{t+\sqrt{t^2-1}}\right)}{\sqrt{\frac{1}{\sqrt{t^2-1}-t}}\sqrt{-2th^2+h^4+1}} \\ &\times(\sqrt{t^2-1}+t)\sqrt{\frac{h^2}{\sqrt{t^2-1}-t}+1}\sqrt{1-\frac{h^2}{\sqrt{t^2-1}+t}} \end{aligned} \tag{6.57}$$

and $F(h,t)$ and $E(h,t)$ are elliptic integrals of I and II kind respectively. In conclusion we have for the sum S_2

$$S_2 = \sum_{n=0}^{\infty} \frac{h_1^n}{2n+3} P_n(t) = \frac{1}{h_1^{3/2}} \mathcal{G}(\sqrt{h_1}, t). \tag{6.58}$$

Finally for the reproducing kernel (6.45) we have

$$\mathscr{K}(x,y) = \int_{A^{int}} \frac{1}{|x-z|} \frac{1}{|y-z|} dz = \frac{2\pi\alpha^3}{|x||y|} \left(\frac{1}{\sqrt{h_1}} \mathcal{F}(\sqrt{h_1}, t) + \frac{1}{h_1^{3/2}} \mathcal{G}(\sqrt{h_1}, t) \right). \tag{6.59}$$

For elliptic integrals of I and II kind is known that there exist closed expression only in the case $t = -1$ or $t = 1$. For other values of t, namely for which we are interested, the integral must be calculated numerically. This means that the closed expression for this kernel does not exist even in the case of spherical boundary. Thus, in order to use this kernel for practical purposes, we must consider certain numerical integration methods for its integral expression. This is the topic of the last chapter, where we present a discretization method for the approximate integration of the kernel.

6.4 Numerical Illustration

In the following we present an example of a spherically harmonic spline application for the oblique derivative functionals with randomly distributed points on the Earth's surface. For more details on this procedures the reader is referred to [44].
For a test scenario, 100000 points on the Earth's surface are considered given by the free Terrainbase model, Figure 6.10 (see e.g., [46] for more details). The resulting potential is depicted in Figure 6.11. For the oblique derivatives the directions provided by gradients of the gravitational potential are used, i.e., the gradient is applied to the EGM96 model. A spherically harmonic $\mathcal{H}$–spline for the oblique derivative functionals (with $\alpha = 6.3781363 \cdot 10^6$) is then computed with randomly distributed points on the Earth's surface. These points are uniformly distributed on a sphere, with different distances from the origin. The computation of the spline of singularity type with $h = 0.965$ yields the results displayed in Figure 6.12, while the Figure 6.13 shows the absolute error between the spline and the original potential V.

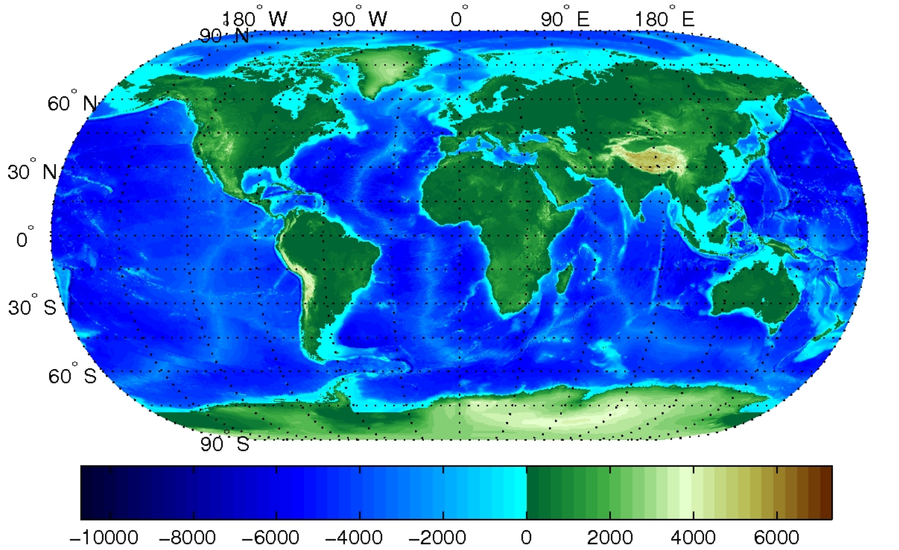

Figure 6.10: The Terrainbase model (heights in m).

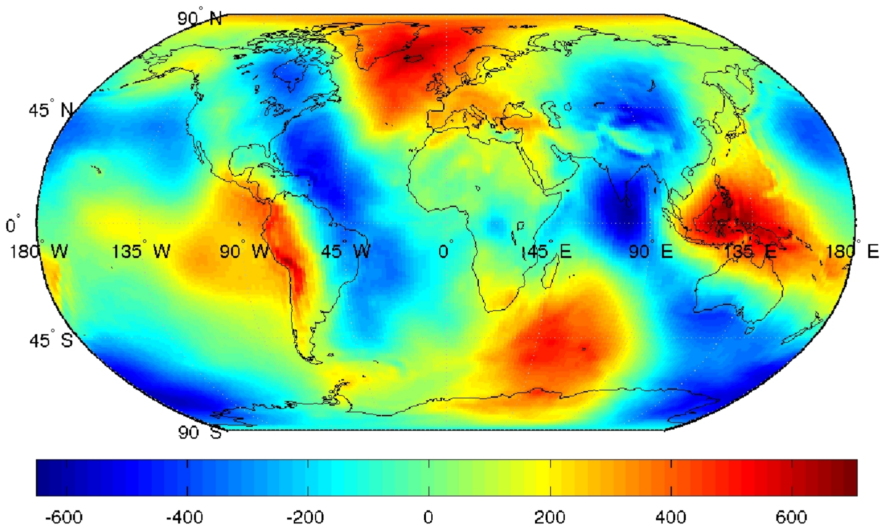

Figure 6.11: The EGM96 gravitational potential.

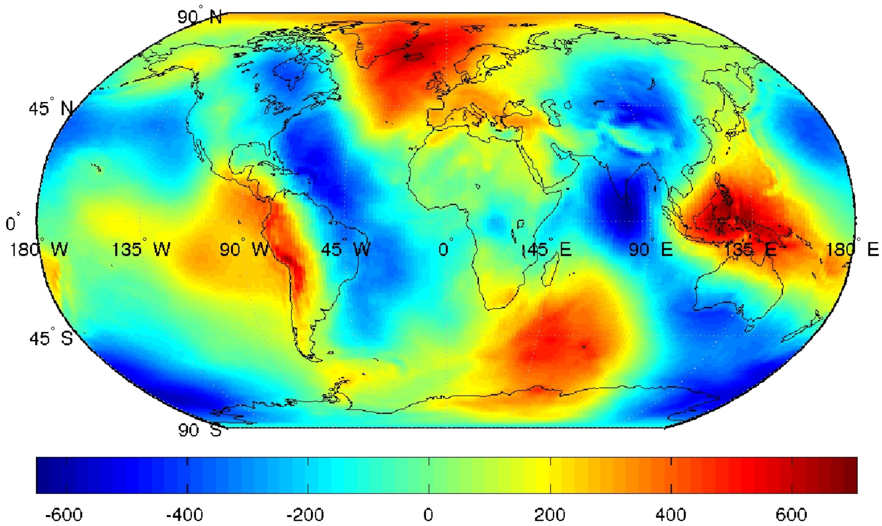

Figure 6.12: Spline with the singularity kernel for oblique derivative functionals with randomly distributed points on the Earth's surface

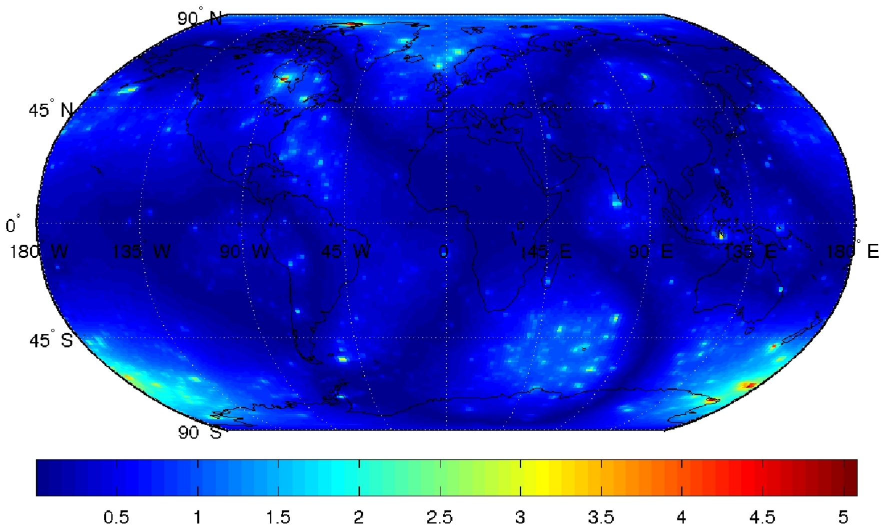

Figure 6.13: Absolute error.

7 Integral Discretization Based on Euler Summation

As already mentioned, of great importance for practical implementation of the spline method is the existence of the closed representation for the reproducing kernel under consideration. However, in Section 6.3, we did not succeed in finding such representation for given $x, y \in \overline{\Sigma^{ext}}$, even for the case of a sphere. Going over to a regular surfaces as the real Earth surface this task becomes even more complicated. Thus, when using this kernel in approximation processes, one loses the convenience offered by reproducing kernels with closed representations in elementary form, like the Abel-Poisson or the singularity kernel. At the end, we must cope with the integral expression of the reproducing kernel, i.e., with the Newton integral over the inner space of a regular surface. In doing so, we were led to consider certain discretization methods, concerning regions in $\mathbb{R}^3$. The basic tool for our purposes is the multidimensional Euler summation formula. A certain discretization of the inner space of a regular surface, yields a numerical integration procedure for the calculation of the reproducing kernel. Moreover, the a priori estimate of the error term is given, under a certain constrain of the density function.

7.1 Threedimensional Lattices

Let g_1, g_2, g_3 be arbitrary linearly independent vectors in Euclidean space $\mathbb{R}^3$.

Definition 7.1.1. *The set* Λ *of all points*

$$g = n_1 g_1 + n_2 g_2 + n_3 g_3 \tag{7.1}$$

$(n_i \in \mathbb{Z}, i = 1, 2, 3)$ *is called a lattice in* $\mathbb{R}^3$ *with basis* g_1, g_2, g_3.

Clearly, the vectors $\varepsilon^1, \varepsilon^2, \varepsilon^3$ form a lattice basis of $\mathbb{Z}^3$. Trivially, a lattice basis $\{g_1, g_2, g_3\}$ is related to the canonical basis $\{\varepsilon^1, \varepsilon^2, \varepsilon^3\}$ in $\mathbb{R}^3$ via the formula

$$g_i = \sum_{j=1}^{3} (g_i \cdot \varepsilon^j) \varepsilon^j. \tag{7.2}$$

Each lattice vector $g \in \Lambda$, described by

$$g = \sum_{k=1}^{3} n_k g_k, \tag{7.3}$$

is determined with respect to the basis $\{\varepsilon^1, \varepsilon^2, \varepsilon^3\}$ by the equation

$$g = \sum_{k=1}^{3} n_k \sum_{j=1}^{3} (g_k \cdot \varepsilon^j) \varepsilon^j = \sum_{j=1}^{3} \left(\sum_{k=1}^{3} (g_k \cdot \varepsilon^j) n_k \right) \varepsilon^j. \tag{7.4}$$

In other words, the components of g with respect to the basis $\{\varepsilon^1, \varepsilon^2, \varepsilon^3\}$ are obtainable by a simple multiplication of the matrix $(g_k \cdot \varepsilon^j)_{j,k=1,2,3}$ with the vector $(n_k)_{k=1,2,3}$ of integer values.

Definition 7.1.2. *The half-open parallelotope $\mathcal{F}$ consisting of the points $x \in \mathbb{R}^3$ with*

$$x = t_1 g_1 + t_2 g_2 + t_3 g_3, \quad -\frac{1}{2} \leq t_i < \frac{1}{2}, \quad i = 1, 2, 3. \tag{7.5}$$

$\mathcal{F}$ is called the fundamental cell of the lattice Λ.

From Linear Algebra (see, e.g., [9]) it is well known that the volume of $\mathcal{F}$ is equal to the quantity

$$||\mathcal{F}|| = \int_{\mathcal{F}} dV = \sqrt{det(g_i \cdot g_j)_{i,j=1,2,3}} \tag{7.6}$$

(dx is the volume element). Since the vectors g_1, g_2, g_3 are assumed to be linearly independent, there exists a system of vectors h_1, h_2, h_3 in $\mathbb{R}^3$ such that

$$h_j \cdot g_i = \delta_{ij} = \begin{cases} 0 & , \quad i \neq j \\ 1 & , \quad i = j \end{cases} \tag{7.7}$$

(δ_{ij} is the Kronecker symbol).

Definition 7.1.3. *The lattice with basis h_1, h_2, h_3 is called the inverse (or dual) lattice Λ^{-1} to Λ.*

The inverse lattice Λ^{-1} consists of all vectors $h \in \mathbb{R}^3$ such that the inner product $h \cdot g$ is an integer for all $g \in \Lambda$. Obviously,

$$\Lambda = (\Lambda^{-1})^{-1}. \tag{7.8}$$

Moreover, for the fundamental cell $\mathcal{F}^{-1}$ of the inverse lattice Λ^{-1} we have

$$||\mathcal{F}^{-1}|| = ||\mathcal{F}||^{-1}. \tag{7.9}$$

Example:
Let $\Lambda = \tau\mathbb{Z}^3$ be the lattice generated by the basis $\tau\varepsilon^1, \tau\varepsilon^2, \tau\varepsilon^3$, where τ is a positive number and $\varepsilon^1, \varepsilon^2, \varepsilon^3$ form the canonical orthonormal basis in $\mathbb{R}^3$. Then the volume of the fundamental cell is $||\mathcal{F}|| = \tau^3$. Generating vectors of the inverse lattice Λ^{-1} are $\tau^{-1}\varepsilon^1, \tau^{-1}\varepsilon^2, \tau^{-1}\varepsilon^3$. The volume of the fundamental cell of the inverse lattice is given by $||\mathcal{F}^{-1}|| = \tau^{-3} = ||\mathcal{F}||^{-1}$

7.1.1 Λ-periodic Functions

The functions $\Phi_h, \;\; h \in \Lambda^{-1}$, defined by

$$\Phi_h(x) = \frac{1}{\sqrt{||\mathcal{F}||}} e(h \cdot x) = \frac{1}{\sqrt{||\mathcal{F}||}} e^{2\pi i (h \cdot x)}, \;\; x \in \mathbb{R}^3, \tag{7.10}$$

are Λ-*periodic*, i.e.,

$$\Phi_h(x+g) = \Phi_h(x) \tag{7.11}$$

for all $x \in \mathbb{R}^3$ and all $g \in \Lambda$. An elementary calculation yields

$$\Delta_x \Phi_h(x) + \Delta^\wedge(h)\Phi_h(x) = 0, \quad \Delta^\wedge(h) = 4\pi^2 h^2, \quad h \in \Lambda^{-1}. \tag{7.12}$$

The space of all $F \in C^{(m)}(\mathbb{R}^3)$ that are Λ-periodic is denoted by $C_\Lambda^{(m)}(\mathbb{R}^3), 0 \le m \le \infty$. $L_\Lambda^p(\mathbb{R}^3), 1 \le p < \infty$ is the space of all $F : \mathbb{R}^3 \to \mathbb{C}$ that are Λ-periodic and are Lebesgue-measurable on $\mathcal{F}$ with

$$||F||_{L_\Lambda^p(\mathbb{R}^3)} = \left(\int_{\mathcal{F}} |F(x)|^p dx \right)^{\frac{1}{p}} < \infty. \tag{7.13}$$

Clearly, $C_\Lambda^{(0)}(\mathbb{R}^3) \subset L_\Lambda^p(\mathbb{R}^3)$. As is well-known, $L_\Lambda^2(\mathbb{R}^3)$ is the completion of $C_\Lambda^{(0)}(\mathbb{R}^3)$ with respect to the norm $||\cdot||_{L_\Lambda^2(\mathbb{R}^3)}$, i.e.,

$$L_\Lambda^2(\mathbb{R}^3) = \overline{C_\Lambda^{(0)}(\mathbb{R}^3)}^{||\cdot||_{L_\Lambda^2(\mathbb{R}^3)}}. \tag{7.14}$$

An easy calculation shows

$$\int_{\mathcal{F}} \Phi_h(x)\overline{\Phi_{h'}(x)}dx = \delta_{hh'} = \begin{cases} 0 & , \quad h = h' \\ 1 & , \quad h \ne h'. \end{cases} \tag{7.15}$$

In other words, the system $\{\Phi_h\}_{h\in\Lambda^{-1}}$ is orthonormal with respect to the $L_\Lambda^2(\mathbb{R}^3)$-inner product, i.e.,

$$(\Phi_h, \Phi_{h'})_{L_\Lambda^2(\mathbb{R}^3)} = \delta_{hh'}. \tag{7.16}$$

We shall say that λ is an eigenvalue of the lattice Λ with respect to the operator Δ, if there is a non-trivial solution U of the differential equation $(\Delta + \lambda)U = 0$ satisfying the 'boundary condition of periodicity' $U(x + g) = U(x)$ for all $g \in \Lambda$. The function U then is called eigenfunction of the lattice Λ with respect to the eigenvalue λ and the operator Δ. Since the solutions of $(\Delta + \lambda)U = 0$ are analytic, application of the standard multi-dimensional Fourier theory (see for example [71]) shows that the functions Φ_h are the only eigenfunctions. Furthermore, the scalars

$$\Delta^\wedge(h)(= \Delta^\wedge_\Lambda(h)) = 4\pi^2 h^2, \quad h \in \Lambda^{-1} \tag{7.17}$$

are the eigenvalues of Δ with respect to lattice Λ. The set of all eigenvalues of Δ with respect to Λ is the *spectrum* $\mathrm{Spect}_\Delta(\Lambda)$

$$\mathrm{Spect}_\Delta(\Lambda) = \{\Delta^\wedge(h) | \Delta^\wedge(h) = 4\pi^2 h^2, h \in \Lambda^{-1}\}. \tag{7.18}$$

The system $\{\Phi_h\}_{h\in\Lambda^{-1}}$ is closed and complete in the pre-Hilbert space $(C^{(0)}_\Lambda(\mathbb{R}^3); || \cdot ||_{L^2_\Lambda(\mathbb{R}^3)})$ as well as in the Hilbert space $L^2_\Lambda(\mathbb{R}^3)$. A fundamental result in Fourier analysis is that each $F \in L^2_\Lambda(\mathbb{R}^3)$ can be represented by its Fourier series in the sense

$$\lim_{N\to\infty} ||F - \sum_{\substack{|h|\le N \\ h\in\Lambda^{-1}}} F^\wedge_\Lambda(h)\Phi_h||_{L^2_\Lambda(\mathbb{R}^3)} = 0, \tag{7.19}$$

where the Fourier coefficients $F^\wedge_\Lambda(h)$ of F are given by

$$F^\wedge_\Lambda(h) = \int_{\mathcal{F}} F(x)\overline{\Phi_h(x)}dx, \quad h \in \Lambda^{-1}. \tag{7.20}$$

The Parseval identity then tells us that, for each $F \in L^2_\Lambda(\mathbb{R}^3)$,

$$\int_{\mathcal{F}} |F(x)|^2 dx = \sum_{h\in\Lambda^{-1}} |F^\wedge_\Lambda(h)|^2. \tag{7.21}$$

A useful corollary of (7.19) (see [67]) is that any $F \in L^1_\Lambda(\mathbb{R}^3)$ with

$$\sum_{h\in\Lambda^{-1}} |F^\wedge_\Lambda(h)| < \infty, \tag{7.22}$$

can be modified on a set of measure zero so that it is in $C^{(0)}_\Lambda(\mathbb{R}^3)$ and equals its Fourier series, i.e.,

$$F(x) = \sum_{h\in\Lambda^{-1}} F^\wedge_\Lambda(h)\Phi_h(x), \quad \text{for all} \quad x \in \mathcal{F}. \tag{7.23}$$

Suppose that $F \in C^{(k)}_\Lambda(\mathbb{R}^3)$ with $k > \frac{3}{2}$. Then

$$\int_{\mathcal{F}} \nabla^\alpha F(x)\overline{\Phi_h(x)}dV(x) = (-2\pi i h)^\alpha F^\wedge_\Lambda(h), \tag{7.24}$$

whenever $F \in C_\Lambda^{(k)}(\mathbb{R}^3)$ and $\alpha = (\alpha_1, \alpha_2, \alpha_3)^T$ is chosen such that $[\alpha] = \alpha_1 + \alpha_2 + \alpha_3 \leq k$. Since $\nabla^\alpha F$ is continuous in $\mathcal{F}$ it must belong to $L^2_\Lambda(\mathbb{R}^3)$. In other words,

$$\sum_{[\alpha]=k} \left(\sum_{h\in\Lambda^{-1}} |F_\Lambda^\wedge(h)|^2((2\pi h)^\alpha)^2 \right) < \infty. \tag{7.25}$$

Moreover, there exists a constant C (dependent on $3k$) such that

$$C|h|^{2k} \leq \sum_{[\alpha]=k} ((2\pi h)^\alpha)^2. \tag{7.26}$$

From the Cauchy-Schwarz inequality we therefore obtain for all $N > 0$, and for all $x \in \mathbb{R}^3$

$$\begin{aligned} \left| \sum_{\substack{|h|\leq N\\ h\in\Lambda^{-1}}} F_\Lambda^\wedge(h)\Phi_h(x) \right| &\leq \sum_{\substack{|h|\leq N\\ h\in\Lambda^{-1}}} |F_\Lambda^\wedge(h)| \\ &\leq \sum_{\substack{|h|\leq N\\ h\in\Lambda^{-1}}} |F_\Lambda^\wedge(h)| \left(\sum_{[\alpha]=k} ((2\pi h)^\alpha)^2 \right)^{1/2} C^{-1/2}|h|^{-k} \\ &\leq \left(\sum_{\substack{|h|\leq N\\ h\in\Lambda^{-1}}} |F_\Lambda^\wedge(h)|^2 \sum_{[\alpha]=k} ((2\pi h)^\alpha)^2 \right)^{1/2} \times \left(\sum_{\substack{|h|\leq N\\ h\in\Lambda^{-1}}} |h|^{-2k} \right)^{1/2} C^{-1/2} \end{aligned} \tag{7.27}$$

If $k > \frac{3}{2}$, the sum $\sum_{|h|\leq N} |h|^{-2k}$ is finite, hence, the last expression must also be finite. This leads us to the following statement:

Theorem 7.1.4. *If $F \in C_\Lambda^{(k)}(\mathbb{R}^3)$ with $k > \frac{3}{2}$, then*

$$\sum_{h\in\Lambda^{-1}} |F_\Lambda^\wedge(h)| < \infty, \tag{7.28}$$

where $F_\Lambda^\wedge(h)$ are the Fourier coefficients of F.

This means that we have the pointwise convergence of the Fourier series in this case, i.e., when the assumptions of the Theorem 7.1.4 are fulfilled, we have

$$F(x) = \sum_{\substack{|h|\leq N\\ h\in\Lambda^{-1}}} F_\Lambda^\wedge(h)\Phi_h(x). \tag{7.29}$$

7.2 Euler Green Function

In the literature there are known various formulas for numerical integration using cubature sums, such as Euler summation formula, but they are mostly presented for the one–dimensional case and related to parallelepipeds in iterated one-dimensional way. Since our integration must be done over the inner space $\Sigma^{int} \subset \mathbb{R}^3$ of a regular surface Σ, in the following we present a generalization of these formulas, however, in specifically three-dimensional way. The idea is the same, namely to put the lattice inside the regular surface Σ and then to approximate the integral via the sum of the functional values at the lattice points (see Figure 7.1). The essential idea is based on the interpretation of the Bernoulli function occurring in the classical (one–dimensional) Euler summation formula, as Green's function $G(\Delta;\Lambda;\cdot)$ with respect to the Laplace operator Δ corresponding to 'boundary conditions of periodicity' to a lattice Λ under consideration. In other words, the Green function, acts as the connecting tool to convert a differential equation involving the Laplace operator corresponding to (periodic boundary conditions) into an associated integral equation, i.e., the Euler summation formula. For broader and more detailed approach the interested reader is referred to [40].

7.2.1 Defining Properties

In order to guarantee the existence of the Green function $G(\Delta;\Lambda;\cdot)$, we imbed this function in a more general framework involving Helmholtz operators $\Delta + \lambda$ (see e.g., [40]). Seen from the point of view of mathematical physics, the Λ-Euler (Green) function is the Green function with respect to the operator $\Delta + \lambda$, $\lambda \in \mathbb{R}$, in $\mathbb{R}^3$ and the 'boundary condition' of periodicity with regard to the lattice Λ.
If there is no confusion likely to arise we simply use the notation $G(\Delta + \lambda; x)$ instead of $G(\Delta + \lambda;\Lambda; x)$.

Definition 7.2.1. *$G(\Delta+\lambda;\cdot) : \mathbb{R}^3\setminus\Lambda \to \mathbb{R}$, $\lambda \in \mathbb{R}$ fixed, is called Λ-Euler (Green) function with respect to the operator $\Delta + \lambda$ if it has the following properties:*

(i) (periodicity) For all $x \in \mathbb{R}^3 \setminus \Lambda$ and $g \in \Lambda$

$$G(\Delta + \lambda; x + g) = G(\Delta + \lambda; x) \tag{7.30}$$

holds true.

(ii) (differential equation) $G(\Delta + \lambda;\cdot)$ is twice continuously differentiable for $x \notin \Lambda$ with

$$(\Delta + \lambda)G(\Delta + \lambda; x) = 0, \tag{7.31}$$

if $\lambda \notin \mathrm{Spect}_\Delta(\Lambda)$, *and*

$$(\Delta+\lambda)G(\Delta+\lambda;x) = -\frac{1}{\sqrt{||\mathcal{F}||}} \sum_{\substack{\Delta^\wedge(h)+\lambda=0 \\ h\in\Lambda^{-1}}} \Phi_h(x), \tag{7.32}$$

if $\lambda \in \mathrm{Spect}_\Delta(\Lambda)$ *(note that the summation on the right hand side is to be taken over all* $h \in \Lambda^{-1}$ *satisfying* $(\Delta+\lambda)^\wedge(h) = -\lambda + \Delta^\wedge(h) = -\lambda + 4\pi^2 h^2 = 0$, *i.e.,* $\lambda = 4\pi^2 h^2$*).*

(iii) (characteristic singularity) In the neighborhood of the origin

$$G(\Delta+\lambda;x) + F(|x|) = O(1), \tag{7.33}$$

and

$$\nabla_x G(\Delta+\lambda;x) + \nabla_x F(|x|) = O(|x^{-1}|), \tag{7.34}$$

where the $F : x \mapsto F(|x|) = 1/|x|, x \neq 0$, *is the fundamental solution in* $\mathbb{R}^3$ *with respect to* Δ *(see (2.101)).*

(iv) (normalization) For all $h \in \Lambda^{-1}$ *with* $(\Delta+\lambda)^\wedge(h) = 0$

$$\int_{\mathcal{F}} G(\Delta+\lambda;x)\overline{\Phi_h(x)}dx = 0. \tag{7.35}$$

7.2.2 Uniqueness of the Euler (Green) Function

Next we deal with the essential results of the theory of Λ-Euler (Green) functions with respect to the Helmholtz operator $\Delta+\lambda, \lambda \in \mathbb{R}$. All these results can be derived directly by means of partial integration, i.e., by Green's second theorem observing the 'boundary condition' of Λ-periodicity and the particular construction of the fundamental solution F of the Laplace operator Δ in $\mathbb{R}^3$. Let Λ be n arbitrary lattice in $\mathbb{R}^3$. By application of Green's theorem we obtain for every (sufficiently small) $\varepsilon > 0$ and all lattice points $h \in \Lambda^{-1}$ with $(\Delta+\lambda)^\wedge(h) \neq 0$

$$\begin{aligned}
&\int_{\substack{x\in\mathcal{F}\\ |x|\geq\varepsilon}} \left((\Delta+\lambda)G(\Delta+\lambda;x)\overline{\Phi_h(x)} - G(\Delta+\lambda;x)(\Delta+\lambda)\overline{\Phi_h(x)} \right) d(x) \\
&= \int_{x\in\partial\mathcal{F}} \left(\left(\frac{\partial G(\Delta+\lambda;x)}{\partial\nu} \right)\overline{\Phi_h(x)} - G(\Delta+\lambda;x)\left(\frac{\partial\overline{\Phi_h(x)}}{\partial\nu} \right) \right) d\omega(x) \\
&+ \int_{\substack{x\in\mathcal{F}\\ |x|=\varepsilon}} \left(\left(\frac{\partial G(\Delta+\lambda;x)}{\partial\nu} \right)\overline{\Phi_h(x)} - G(\Delta+\lambda;x)\left(\frac{\partial\overline{\Phi_h(x)}}{\partial\nu} \right) \right) d\omega(x),
\end{aligned} \tag{7.36}$$

where ν is the outward directed (unit) normal field. Because of the Λ-periodicity of both the Λ-Euler (Green) function and the functions $\Phi_h, h \in \Lambda^{-1}$, the integral over the boundary $\partial\mathcal{F}$ of the fundamental cell vanishes in (7.36). Moreover, observing the differential equation (Condition (ii)) of the Λ-Euler (Green) function we get

$$\begin{aligned} &-(\lambda - \Delta^\wedge(h)) \int_{\substack{x\in\mathcal{F}\\|x|=\varepsilon}} G(\Delta+\lambda;x)\overline{\Phi_h(x)}dx \\ &= \int_{\substack{x\in\mathcal{F}\\|x|=\varepsilon}} \left(\left(\frac{\partial G(\Delta+\lambda;x)}{\partial\nu}\right)\overline{\Phi_h(x)} - G(\Delta+\lambda;x)\left(\frac{\partial\overline{\Phi_h(x)}}{\partial\nu}\right)\right)d\omega(x). \end{aligned} \tag{7.37}$$

Letting $\varepsilon \to 0$ we obtain, in connection with Lemma 2.6.3, the identity

$$(\lambda - \Delta^\wedge(h)) \int_{\mathcal{F}} G(\Delta+\lambda;x)\overline{\Phi_h(x)}dx = \frac{1}{\sqrt{||\mathcal{F}||}}, \quad \Delta^\wedge(h) \neq \lambda. \tag{7.38}$$

Consequently, for all $h \in \Lambda^{-1}$ with $\Delta^\wedge(h) \neq \lambda$, i.e., $4\pi^2h^2 \neq \lambda$, the Fourier coefficients of $G(\Delta+\lambda;\cdot)$ read as follows

$$\int_{\mathcal{F}} G(\Delta+\lambda;x)\overline{\Phi_h(x)}dx = -\frac{1}{\sqrt{||\mathcal{F}||}}\frac{1}{(\Delta+\lambda)^\wedge(h)}. \tag{7.39}$$

In addition, the normalization condition (iv) tells us that for all $h \in \Lambda^{-1}$ with $\Delta^\wedge(h) = 4\pi^2h^2 = \lambda$

$$\int_{\mathcal{F}} G(\Delta+\lambda;x)\overline{\Phi_h(x)}dx = 0 \tag{7.40}$$

Thus combining (7.38) and (7.40) we find the following result.

Lemma 7.2.2. *For $h \in \Lambda^{-1}$*

$$\int_{\mathcal{F}} G(\Delta+\lambda;x)\overline{\Phi_h(x)}dx = \begin{cases} 0 & ,\lambda = 4\pi^2h^2 \\ -\frac{1}{\sqrt{||\mathcal{F}||}}\frac{1}{\lambda-4\pi^2h^2} & ,\lambda \neq 4\pi^2h^2 \end{cases} \tag{7.41}$$

From Lemma 7.2.2 we are immediately able to verify the uniqueness of the Λ-Euler (Green) function $G(\Delta+\lambda;\cdot)$ by virtue of the completeness of the system $\Phi_h, h \in \Lambda^{-1}$ in the Hilbert space $L^2_\Lambda(\mathbb{R}^3)$. We formulate this result in next theorem.

Theorem 7.2.3. *There exists one and only one Λ-Euler (Green) function $G(\Delta+\lambda;\cdot)$ satisfying the constituting conditions (i)-(iv) listed under Definition* 7.2.1.

Proof: The difference of two Λ-Euler (Green) functions with respect to the Helmholtz operator $\Delta + \lambda$ permits only vanishing Fourier coefficients. Moreover, the difference is a continuously differentiable function on $\mathbb{R}^3$. Then the completeness of the system $\Phi_h, h \in \Lambda^{-1}$ tells us that the difference vanishes in $\mathbb{R}^3$. Thus the two Λ-Euler (Green) functions are identical.

7.2.3 Existence of the Euler (Green) Function

As we have seen, the Λ-Euler (Green) function $G(\Delta + \lambda; \cdot)$ is uniquely determined by its defining properties. Unfortunately, in $\mathbb{R}^3$, the bilinear expansion of the function $G(\Delta+\lambda; \cdot)$ does not converge absolutely and uniformly. In [40] however, the existence of the function $G(\Delta+\lambda; \cdot)$ is guaranteed by following Hilbert's classical approach to the theory of Green's functions. Namely, after giving an explicit representation of the Λ-Euler (Green) function to the operator $\Lambda - 1$, the Fredholm theory of linear integral equations corresponding to (weak) singular kernels is used to deduce the full theory for the Λ-Euler (Green) function in $\mathbb{R}^q$. Here we present only some important results from this theory, i.e., the results that are relevant for this thesis.

The following identity compares the Green function to an operator $\Delta + \lambda$ with the Green function to the operator $\Delta + \lambda^*$, where $\lambda^* \notin \mathrm{Spect}_\Delta(\Lambda)$.

Lemma 7.2.4. *Under the assumption that $\lambda \neq \lambda^*, \lambda^* \notin \mathrm{Spect}_\Delta(\Lambda)$ the identity*

$$\begin{aligned} G(\Delta + \lambda; x) = G(\Delta + \lambda^*; x) + \frac{1}{||\mathcal{F}||} \sum_{\substack{(\Delta+\lambda)^\wedge(h)=0 \\ h\in\Lambda^{-1}}} \frac{1}{\lambda - \lambda^*} \Phi_h(x) \\ +(\lambda - \lambda^*) \int_{\mathcal{F}} G(\Delta + \lambda^*; x-y) G(\Delta + \lambda; y) dy, \end{aligned} \tag{7.42}$$

holds true for all $x \in \mathcal{F} \setminus \{0\}$, where the sum on the right hand side only occurs if λ is an eigenvalue, i.e., $\lambda \in \mathrm{Spect}_\Delta(\Lambda)$.

It can be shown that the Λ-lattice function with respect to the operator $\Delta - 1$ is expressible by a series expansion in terms of Kelvin function $K_0(3; \cdot)$ of dimension 3.

Lemma 7.2.5. *For $q = 3$ and $\lambda^* = -1$*

$$G(\Delta - 1; x) = -\frac{1}{8\pi} \sum_{g\in\Lambda} K_0(3; |x+g|), \tag{7.43}$$

where

$$K_0(3; r) = \frac{e^{-r}}{r}. \tag{7.44}$$

The Poisson's differential equation of potential theory (Theorem 4.2.3) admits the following transfer to the theory of the Λ-Euler (Green) function.

Lemma 7.2.6. *Assume that $F : y \mapsto F(y)$, $y \in \mathcal{F}$, is bounded, Λ-periodic function that satisfies a Lipschitz-condition in the neighborhood of a point $x \in \mathcal{F}$. Then the function*

$$U(x) = \int_{\mathcal{F}} G(\Delta + \lambda; x-y) F(y) dy \tag{7.45}$$

is continuously differentiable in $\mathbb{R}^3$ *and twice continuously differentiable in* x*, such that*

$$(\Delta_x + \lambda)U(x) = F(x) - \sum_{\substack{(\Delta+\lambda)^\wedge(h)=0 \\ h\in\Lambda}} \int_{\mathcal{F}} F(y)\overline{\Phi_h(y)}dy\Phi_h(x), \tag{7.46}$$

where the sum on the right side of (7.46) *only occurs if* $\lambda \in \mathrm{Spect}_\Delta(\Lambda)$.

After these preparatory material, the solvability of the linear (Fredholm) integral equation

$$\begin{aligned} H(\lambda;x) = G(\Delta-1;x) + \frac{1}{\sqrt{\|\mathcal{F}\|}} \sum_{\substack{(\Delta+\lambda)^\wedge(h)=0 \\ h\in\Lambda^{-1}}} \frac{1}{\lambda+1}\Phi_h(x) \\ -(\lambda+1)\int_{\mathcal{F}} G(\Delta-1;x-y)H(\lambda;y)dy, \end{aligned} \tag{7.47}$$

motivated by Lemma 7.2.4, can be handled in accordance with the well-known Fredholm alternative (see [8]). In other words, the Fredholm theory of (weakly singular) integral equations justifies the existence of all $G(\Delta+\lambda;\cdot)$. Thus we can state

Theorem 7.2.7. *(Existence of the Euler (Green) Function* $G(\Delta+\lambda;\cdot)$*)*

(ia) If λ *is not an eigenvalue, i.e.,* $\lambda \notin \mathrm{Spect}_\Delta(\Lambda)$*, then the integral equation (*7.47*) possesses a unique solution* $H(\lambda;\cdot)$*.*

(ib) If λ *is an eigenvalue, i.e.,* $\lambda \in \mathrm{Spect}_\Delta(\Lambda)$*, then the integral equation (*7.47*) possesses a unique solution* $H(\lambda;\cdot)$ *under the condition*

$$\int_{\mathcal{F}} H(\lambda;x)\Phi_h(x)dx = 0, \tag{7.48}$$

for all $h \in \Lambda^{-1}$ *with* $(\Delta+\lambda)^\wedge(h) = 0$.

(ii) $H(\lambda;\cdot)$*, as specified by (i) satisfies all defining conditions of the function* $G(\Delta+\lambda;\cdot)$*, hence, in light of the uniqueness theorem (Theorem* 7.2.3*),*

$$H(\lambda;\cdot) = G(\Delta+\lambda;\cdot). \tag{7.49}$$

7.3 Euler Summation Formula for the Inner Space of a Regular surface

Next we generalize the Euler summation formula to the three-dimensional case. In fact, we give its formulation for the (iterated) Laplace operator Δ^m, arbitrary lattices $\Lambda \in \mathbb{R}^3$ and the inner space of a regular surface Σ (cf. Section 2.6.1). The essential tool is the Euler (Green) function with respect to the Laplace operator and its constituting properties (as introduced in the previous section).

7.3.1 Euler Summation Formulas for the Laplace Operator

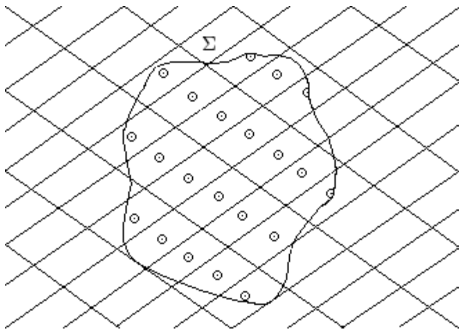

Figure 7.1: Multidimensional Lattice

Let $\Lambda \subset \mathbb{R}^3$ be an arbitrary lattice. Let $\Sigma^{int} \subset \mathbb{R}^3$ is the inner space of the regular surface Σ. Let F be a function of class $C^{(2)}(\overline{\Sigma^{int}})$. Then, for every (sufficiently small) $\varepsilon > 0$, the second Green theorem gives

$$\begin{aligned}\int_{\substack{x\in\overline{\Sigma^{int}}\\ x\notin\mathbb{B}_\epsilon+\Lambda}} &\Big(F(x)(\Delta)G(\Delta;x) - G(\Delta;x)(\Delta)F(x)\Big)dx\\ &= \int_{\substack{x\in\Sigma\\ x\notin\mathbb{B}_\epsilon+\Lambda}} \left(F(x)\frac{\partial}{\partial\nu}G(\Delta;x) - G(\Delta;x)\frac{\partial F}{\partial\nu}(x)\right)d\omega(x)\\ + \sum_{\substack{g\in\overline{\Sigma^{int}}\\ g\in\Lambda}} &\int_{\substack{|x-g|=\varepsilon\\ x\in\overline{\Sigma^{int}}}} \left(F(x)\frac{\partial}{\partial\nu}G(\Delta;x) - G(\Delta;x)\frac{\partial F}{\partial\nu}(x)\right)d\omega(x),\end{aligned} \tag{7.50}$$

where ν is the outer (unit) normal field, and $\mathbb{B}_\epsilon = \{x \in \mathbb{R}^3 | |x| \leq \varepsilon\}$. Observing the differential equation (Condition (ii) of Definition 7.2.1) we get

$$\int_{\substack{x\in\overline{\Sigma^{int}}\\ x\notin\mathbb{B}_\epsilon+\Lambda}} F(x)\Delta G(\Delta;x)dx = -\frac{1}{||\mathcal{F}||}\int_{\substack{x\in\overline{\Sigma^{int}}\\ x\notin\mathbb{B}_\epsilon+\Lambda}} F(x)dx. \tag{7.51}$$

Hence, on passing to the limit $\varepsilon \to 0$ and observing the characteristic singularity of the Λ-Euler (Green) function (Condition (iii) of Definition 7.2.1) we obtain in connection with Lemma 2.6.3 the following

Theorem 7.3.1. *(Lattice Point Generated Euler Summation Formula for the Laplace Operator Δ) Let Λ be an arbitrary lattice in $\mathbb{R}^3$. Let $\Sigma^{int} \subset \mathbb{R}^3$ be the inner space of the regular surface Σ. Let F be twice continuously differentiable on $\overline{\Sigma^{int}}$. Then we have*

$$\begin{aligned}\sum_{\substack{g\in\overline{\Sigma^{int}}\\ g\in\Lambda}}{}' F(g) = \frac{1}{||\mathcal{F}||}\int_{\Sigma^{int}} F(x)dx \\ + \int_{\Sigma^{int}} G(\Delta;x)\Delta F(x)dx \\ + \int_{\Sigma}\left(F(x)\frac{\partial}{\partial\nu}G(\Delta;x) - G(\Delta;x)\frac{\partial F}{\partial\nu}(x)\right)d\omega(x),\end{aligned} \tag{7.52}$$

where $\frac{\partial}{\partial\nu}$ denotes in the direction of the outer normal ν and

$$\sum_{\substack{g\in\overline{\Sigma^{int}}\\ g\in\Lambda}}{}' F(g) = \sum_{\substack{g\in\overline{\Sigma^{int}}\\ g\in\Lambda}} F(g) - \frac{1}{2}\sum_{\substack{g\in\Sigma\\ g\in\Lambda}} F(g). \tag{7.53}$$

This formula provides a comparison between the integral over the inner space Σ^{int} of the regular surface Σ and the (weighted) sum over all functional values of the twice continuously differentiable function F in lattice points $g \in \overline{\Sigma^{int}}$ under explicit knowledge of the remainder term in integral form. Moreover, this formula is an immediate generalization to the three-dimensional case of the one-dimensional Euler summation formula, where $G(\Delta;\cdot)$ takes the role of the Bernoulli polynomial of degree 2.

7.3.2 Euler Summation Formulas for the Iterated Laplace Operators

The Euler summation formula (Theorem 7.3.1) can be extended by use of higher order derivatives. For that purpose we introduce Λ-Euler (Green) functions to the iterated operator $\Delta, m \in \mathbb{N}$.

Definition 7.3.2. *The function $G(\Delta^m;\cdot), m \in \mathbb{N}$, defined by*

$$G(\Delta^1;x) = G(\Delta;x) \tag{7.54}$$

$$G(\Delta^m;x) = \int_{\mathcal{F}} G(\Delta^{m-1};z)G(\Delta;x-z)dz, \tag{7.55}$$

$m = 2, 3, ...$ is called Λ-Euler (Green) function with respect to the operator Δ^m.

Obviously, for all $x \notin \Lambda$ and $g \in \Lambda$

$$G(\Delta^m;x+g) = G(\Delta^m;x) \tag{7.56}$$

is satisfied, i.e., $G(\Delta^m;\cdot)$ is Λ-periodic. In analogy to techniques of potential theory it can be proved that

$$G(\Delta^m;x) = O(|x|^{2m-3}), \text{ for } 2m \le 3, \tag{7.57}$$

while for $2m > 3$, $G(\Delta^m;x)$ is continuous in $\mathbb{R}^3$. Moreover, the differential equation

$$\Delta G(\Delta^m;x) = G(\Delta^{m-1};x), \ x \notin \Lambda, \tag{7.58}$$

$m = 2, 3, ...,$, represents a recursion relation relating the Λ-Euler (Green) function with respect to the operator Δ^m, to the Λ-Euler (Green) function with respect to the operator Δ^{m-1}. The bilinear expansion of $G(\Delta^m;x)$ in terms of eigenfunctions, which is equivalent to the (formal) Fourier (orthogonal) expansion, reads for $m = 2, 3, ...$

$$\frac{1}{\sqrt{||\mathcal{F}||}} \sum_{\substack{\Delta^\wedge(h)\neq 0 \\ h\in\Lambda^{-1}}} \frac{\Phi_h(x)}{(-\Delta^\wedge(h))^m}, \tag{7.59}$$

where

$$\Delta^\wedge(h) = 4\pi^2 h^2. \tag{7.60}$$

Note, for $m > 3/2$ it follows that there exists a constant $C > 0$ such that

$$\left| \sum_{\substack{\Delta^\wedge(h)\neq 0 \\ h\in\Lambda^{-1}}} \frac{\Phi_h(x)}{(-\Delta^\wedge(h))^m} \right| = \left| \frac{1}{\sqrt{||\mathcal{F}||}} \sum_{\substack{4\pi^2h^2\neq 0 \\ h\in\Lambda^{-1}}} \frac{e^{2\pi i h\cdot x}}{(-4\pi^2h^2)^m} \right| \le C \sum_{h\in\Lambda^{-1}} \frac{1}{(1+h^2)^m} < \infty. \tag{7.61}$$

Altogether we are able to formulate

Lemma 7.3.3. *For $m > 3/2$, the Λ- Euler (Green) function $G(\Delta^m;\cdot)$ is continuous in $\mathbb{R}^3$, and we have for all $x, y \in \mathbb{R}^3$*

$$G(\Delta^m;x-y) = \frac{1}{\sqrt{||\mathcal{F}||}} \sum_{\substack{\Delta^\wedge(h)\neq 0 \\ h\in\Lambda^{-1}}} \frac{\Phi_h(x)\overline{\Phi_h(y)}}{(-\Delta^\wedge(h))^m}. \tag{7.62}$$

Provided that the function $F : \overline{\Sigma^{int}} \to \mathbb{R}, \Sigma^{int} \subset \mathbb{R}^3$, is of class $C^{(2m)}(\overline{\Sigma^{int}}), m \in \mathbb{N}$, we get from the second Green theorem by use of the differential equation (7.58) for the Λ-Euler

(Green) function with respect to Δ^m

$$\begin{aligned}
&\int_{\substack{x\in\overline{\Sigma^{int}}\\ x\notin\mathbb{B}_\epsilon+\Lambda}} G(\Delta^{k+1};x)(\Delta^{k+1}F(x))dx - \int_{\substack{x\in\overline{\Sigma^{int}}\\ x\notin\mathbb{B}_\epsilon+\Lambda}} (\Delta G(\Delta^{k+1};x))\Delta^k F(x)dx \\
&\qquad = \int_{\substack{x\in\Sigma\\ x\notin\mathbb{B}_\epsilon+\Lambda}} G(\Delta^{k+1};x)\left(\frac{\partial}{\partial\nu}\Delta^k F(x)\right)d\omega(x) \\
&\qquad\quad - \int_{\substack{x\in\Sigma\\ x\notin\mathbb{B}_\epsilon+\Lambda}} \left(\frac{\partial}{\partial\nu}G(\Delta^{k+1};x)\right)\Delta^k F(x)d\omega(x) \\
&\qquad\quad + \sum_{\substack{g\in\overline{\Sigma^{int}}\\ g\in\Lambda}} \int_{\substack{|x-g|=\varepsilon\\ x\in\overline{\Sigma^{int}}}} G(\Delta^{k+1};x)\left(\frac{\partial}{\partial\nu}\Delta^k F(x)\right)d\omega(x) \\
&\qquad\quad - \sum_{\substack{g\in\overline{\Sigma^{int}}\\ g\in\Lambda}} \int_{\substack{|x-g|=\varepsilon\\ x\in\overline{\Sigma^{int}}}} \left(\frac{\partial}{\partial\nu}G(\Delta^{k+1};x)\right)\Delta^k F(x)d\omega(x),
\end{aligned} \tag{7.63}$$

for every (sufficiently small) $\varepsilon > 0$ and $k \in \mathbb{N}$ with $k \in [1, m-1]$. From classical potential theory (see, e.g., [51]), we know that the integrals over all hyperspheres tend to 0 as $\varepsilon \to 0$. This leads us to the recursion formula

$$\begin{aligned}
&\int_{\Sigma^{int}} G(\Delta^{k+1};x)\Delta^{k+1}F(x)dx \\
&= \int_{\Sigma^{int}} G(\Delta^k;x)\Delta^k F(x)dx \\
&\quad + \int_\Sigma G(\Delta^{k+1};x)\left(\frac{\partial}{\partial\nu}\Delta^k F(x)\right)d\omega(x) \\
&\quad - \int_\Sigma \left(\frac{\partial}{\partial\nu}G(\Delta^{k+1};x)\right)\Delta^k F(x)d\omega(x).
\end{aligned} \tag{7.64}$$

Consequently we find for $F \in C^{(2m)}(\overline{\Sigma^{int}})$

$$\begin{aligned}
&\int_{\Sigma^{int}} G(\Delta;x)\Delta F(x)dx \\
&= \int_{\Sigma^{int}} G(\Delta^m;x)\Delta^m F(x)dx \\
&\quad + \sum_{k=1}^{m-1} \int_\Sigma \left(\frac{\partial}{\partial\nu}G(\Delta^{k+1};x)\right)\Delta^k F(x)d\omega(x) \\
&\quad - \sum_{k=1}^{m-1} \int_\Sigma G(\Delta^{k+1};x)\left(\frac{\partial}{\partial\nu}\Delta^k F(x)\right)d\omega(x).
\end{aligned} \tag{7.65}$$

In connection with the Euler summation formula (Theorem 7.3.1) we therefore obtain the *extended Euler summation formula*, i.e., the Euler summation formula with respect to the operator $\Delta^m, m \in \mathbb{N}$ (see [22]).

Theorem 7.3.4. *(Euler Summation Formula for the Operator Δ^m) Let $\Sigma^{int} \subset \mathbb{R}^3$ be the inner space of the regular surface Σ. Suppose that F is of class $C^{(2m)}(\overline{\Sigma^{int}})$. Then we have*

$$\begin{aligned} \sum_{\substack{g \in \overline{\Sigma^{int}} \\ g \in \Lambda}}{}' F(g) = & \frac{1}{||\mathcal{F}||} \int_{\Sigma^{int}} F(x) dx \\ & + \int_{\Sigma^{int}} G(\Delta^m; x) \Delta^m F(x) dx \\ & + \sum_{k=1}^{m-1} \int_{\Sigma} \left(\frac{\partial}{\partial \nu} G(\Delta^{k+1}; x) \right) \Delta^k F(x) d\omega(x) \\ & - \sum_{k=1}^{m-1} \int_{\Sigma} G(\Delta^{k+1}; x) \left(\frac{\partial}{\partial \nu} \Delta^k F(x) \right) d\omega(x). \end{aligned} \tag{7.66}$$

Replacing the lattice Λ by a shifted lattice $\Lambda + \{x\}$ we obtain from the extended Euler summation formula (Theorem 7.3.4) the following result.

Theorem 7.3.5. *Let Λ be an arbitrary lattice in $\mathbb{R}^3$. Let $\Sigma^{int} \subset \mathbb{R}^3$ be the inner space of the regular surface Σ. Suppose that F is a member of class $C^{(2m)}(\overline{\Sigma^{int}}), m \in \mathbb{N}$. Then, for every $x \in \mathbb{R}^3$,*

$$\begin{aligned} \sum_{\substack{g+x \in \overline{\Sigma^{int}} \\ g \in \Lambda}}{}' F(g+x) = & \frac{1}{||\mathcal{F}||} \int_{\Sigma^{int}} F(y) dy \\ & + \int_{\Sigma^{int}} G(\Delta^m; x-y) \Delta_y^m F(y) dy \\ & + \sum_{k=0}^{m-1} \int_{\Sigma} \left(\frac{\partial}{\partial \nu_y} G(\Delta^{k+1}; x-y) \right) (\Delta_y^k F(y)) d\omega(y) \\ & - \sum_{k=0}^{m-1} \int_{\Sigma} G(\Delta^{k+1}; x-y) \left(\frac{\partial}{\partial \nu_y} \Delta_y^k F(y) \right) d\omega(y), \end{aligned} \tag{7.67}$$

where

$$\sum_{\substack{g+x \in \overline{\Sigma^{int}} \\ g \in \Lambda}}{}' F(g+x) = \sum_{\substack{g+x \in \overline{\Sigma^{int}} \\ g \in \Lambda}} \alpha(g+x) F(g+x). \tag{7.68}$$

7.4 Discretization of Integrals over the Inner Space of a Regular Surface

Let $\Sigma^{int} \subset \mathbb{R}^3$ be the inner space of the regular surface Σ, $\Lambda \in \mathbb{R}^3$ be an arbitrary lattice and let F be of class $C^{(2m)}(\overline{\Sigma^{int}})$. Starting from extended Euler summation formula for the iterated Laplacian Δ^m we find for $x = 0$

$$\begin{aligned}\sum_{\substack{g\in\overline{\Sigma^{int}}\\ g\in\Lambda}}{}' F(g) - \frac{1}{||\mathcal{F}||}\int_{\Sigma^{int}} F(y)dy - \int_{\Sigma} P_L\left[F(y), G^{(m)}(\Delta;\Lambda;y)\right] d\omega(y) \\ = \int_{\Sigma^{int}} G(\Delta^m;\Lambda;y)(\Delta^m F(y))dy,\end{aligned} \tag{7.69}$$

where

$$\begin{aligned}&\int_{\Sigma} P_L\left[F(y), G^{(m)}(\Delta;\Lambda;y)\right] d\omega(y) \\ &= \sum_{r=0}^{m-1}\int_{\Sigma}\left(G(\Delta^{r+1};\Lambda;y)(\frac{\partial}{\partial\nu}\Delta^r F(y))d\omega(x) - (\frac{\partial}{\partial\nu}G(\Delta^{r+1};\Lambda;y))\Delta^r F(y)\right) d\omega(y).\end{aligned} \tag{7.70}$$

This formula gives a comparison between the integral over Σ^{int} and the sum of functional values of the integrand at the lattice points inside the region.
The following theorem gives an a priori estimate for the error term.

Theorem 7.4.1. *Let $\Sigma^{int} \subset \mathbb{R}^3$ be the inner space of the regular surface Σ and let F be a function of class $C^{(2m)}(\overline{\Sigma^{int}})$, where $m > 3/2$. Suppose that there exists a constant C, such that*

$$\int_{\Sigma^{int}} |\Delta^m F(x)|\, dx \le C. \tag{7.71}$$

Then

$$\begin{aligned}\left|\int_{\Sigma^{int}} F(x)dx - ||\mathcal{F}|| \sum_{\substack{g\in\overline{\Sigma^{int}}\\ g\in\Lambda}}{}' F(g) + ||\mathcal{F}|| \int_{\Sigma} P_L\left[F(x), G^{(m)}(\Delta;\Lambda;x)\right] d\omega(x)\right| \\ \le ||\mathcal{F}|| C(2\pi)^{-2m}\zeta(2m;\Lambda^{-1}),\end{aligned} \tag{7.72}$$

where

$$\zeta(s,\Lambda) = \sum_{|g|\neq 0} |g|^{-s}, \tag{7.73}$$

is the (Epstein) zeta function.

Proof: For $m > 3/2$ we have

$$\begin{aligned}
&\left| \int_{\Sigma^{int}} F(x)dx - ||\mathcal{F}|| \sum_{\substack{g\in\overline{\Sigma^{int}} \\ g\in\Lambda}}{}' F(g) + ||\mathcal{F}|| \int_{\Sigma} P_L \left[F(x), G^{(m)}(\Delta;\Lambda;x) \right] d\omega(x) \right| \\
&\quad \leq ||\mathcal{F}|| \int_{\Sigma^{int}} |G(\Delta^m;\Lambda;x)||\Delta^m F(x)|dx \\
&\quad \leq ||\mathcal{F}|| \sup_{x\in\overline{\Sigma^{int}}} |G(\Delta^m;\Lambda;x)| \cdot C \qquad (7.74) \\
&\quad \leq ||\mathcal{F}|| \sup_{x\in\overline{\Sigma^{int}}} \left| \sum_{\substack{|h|\neq 0 \\ h\in\Lambda^{-1}}} \frac{\Phi_h(x)}{(-4\pi^2 h^2)^m} \right| \cdot C \leq ||\mathcal{F}|| \cdot C \sum_{\substack{|h|\neq 0 \\ h\in\Lambda^{-1}}} \frac{1}{(4\pi^2)^m |h|^{2m}} \\
&\qquad = ||\mathcal{F}|| \cdot C (2\pi)^{-2m} \zeta(2m;\Lambda^{-1}).
\end{aligned}$$

For our purposes of numerical integration over the inner space Σ^{int} of a regular surface Σ, we use the 'τ-dilated lattice' $\tau\mathbb{Z}^3$ ($\tau > 0$) and $m > 3/2$. We then have

$$\begin{aligned}
\int_{\Sigma^{int}} F(x)dx &= \tau^3 \sum_{\substack{g\in\overline{\Sigma^{int}} \\ g\in\tau\mathbb{Z}^3}}{}' F(g) + \tau^3 \int_{\Sigma^{int}} G(\Delta^m;\tau\mathbb{Z}^3;x)\Delta^m F(x)dx \\
&\quad + \tau^3 \sum_{k=0}^{m-1} \int_{\Sigma} (\frac{\partial}{\partial\nu} G(\Delta^{k+1};\tau\mathbb{Z}^3;x))\Delta^k F(x)d\omega(x), \qquad (7.75) \\
&\quad - \tau^3 \sum_{k=0}^{m-1} \int_{\Sigma} G(\Delta^{k+1};\tau\mathbb{Z}^3;x)(\frac{\partial}{\partial\nu}\Delta^k F(x))d\omega(x).
\end{aligned}$$

Theorem 7.4.2. *Given a lattice $\Lambda \subset \mathbb{R}^3$, $\tau > 0$ and $m > 3/2$, the following identity is valid:*

$$G(\Delta^m;\tau\mathbb{Z}^3;x) = \tau^{2m-3} G(\Delta^m;\mathbb{Z}^3;\tau^{-1}x). \qquad (7.76)$$

Proof: Using Lemma 7.3.3 we have for $m > 3/2$ and $\tau > 0$

$$\begin{aligned} G(\Delta^m;\tau\mathbb{Z}^3;x) &= \frac{1}{\tau^3}\sum_{\substack{h\neq 0\\ h\in\tau^{-1}\mathbb{Z}^3}} \frac{1}{(-4\pi^2h^2)^m} e^{2\pi i h\cdot x} \\ &= \frac{1}{\tau^3}\sum_{\substack{l\neq 0\\ l\in\mathbb{Z}^3}} \frac{1}{(-4\pi^2(\frac{l}{\tau})^2)^m} e^{2\pi i \frac{l}{\tau}\cdot x} \\ &= \frac{1}{\tau^3}\sum_{\substack{l\neq 0\\ l\in\mathbb{Z}^3}} \frac{\tau^{2m}}{(-4\pi^2l^2)^m} e^{2\pi i l \frac{x}{\tau}} \\ &= \tau^{2m-3}\sum_{\substack{l\neq 0\\ l\in\mathbb{Z}^3}} \frac{1}{(-4\pi^2l^2)^m} e^{2\pi i l(\tau^{-1}\cdot x)} \\ &= \tau^{2m-3} G(\Delta^m;\mathbb{Z}^3;\tau^{-1}x). \end{aligned} \tag{7.77}$$

This proves Theorem 7.4.2.

This leads us to the following summation formula for the 'τ-dilated lattice'

$$\begin{aligned} \int_{\Sigma^{int}} F(x)dx = \tau^3 & \sum_{\substack{g\in\overline{\Sigma^{int}}\\ g\in\tau\mathbb{Z}^3}}{}' F(g) \\ & +\tau^{2m}\int_{\Sigma^{int}} G(\Delta^m;\mathbb{Z}^3;\tau^{-1}x)\Delta^m F(x)dx \\ & +\sum_{k=0}^{m-1}\tau^{2k+2}\int_{\Sigma}(\frac{\partial}{\partial\nu}G(\Delta^{k+1};\mathbb{Z}^3;\tau^{-1}x))\Delta^k F(x)d\omega(x) \\ & -\sum_{k=0}^{m-1}\tau^{2k+2}\int_{\Sigma}G(\Delta^{k+1};\mathbb{Z}^3;\tau^{-1}x)(\frac{\partial}{\partial\nu}\Delta^k F(x))d\omega(x). \end{aligned} \tag{7.78}$$

For the a priori estimate of the error term in connection to Theorem 7.4.1, we then have the following

Theorem 7.4.3. *Let $\Sigma^{int} \subset \mathbb{R}^3$ be the inner space of a regular surface Σ and let F be a function of class $C^{(2m)}(\overline{\Sigma^{int}})$, where $m > 3/2$. Suppose that there exists a constant C, such that*

$$\int_{\Sigma^{int}} |\Delta^m F(x)|\, dx \leq C. \tag{7.79}$$

Then

$$\left| \int_{\Sigma^{int}} F(x)dx - \tau^3 \sum_{\substack{g \in \overline{\Sigma^{int}} \\ g \in \Lambda}}{}' F(g) + \tau^3 \int_{\Sigma} P_L \left[F(x), G^{(m)}(\Delta; \mathbb{Z}^3; \tau^{-1}x) \right] d\omega(x) \right| \tag{7.80}$$
$$\leq \tau^{2m} C (2\pi)^{-2m} \zeta(2m; \mathbb{Z}^3),$$

where

$$\zeta(s, \mathbb{Z}^3) = \sum_{|g| \neq 0} |g|^{-s}, \tag{7.81}$$

is the (Epstein) zeta function.

Obviously, the remainder term gives the accuracy of the numerical procedure in the sense that the denser the lattice, the better approximation to the volume integral under consideration.

7.5 Discretization of the Spline Kernel $\mathscr{K}$

Next we give the estimate for the remainder term in the case of the spline integral. For more general approach in higher dimensions the reader is referred to [40] and the references therein.

The essential tool in the spline formulation for the gravitational potential determination, related to the inner space of a regular surface Σ (as presented in Chapter 5) was the reproducing kernel given in the form

$$\mathscr{K}(x, y) = \int_{\Sigma^{int}} \frac{dz}{|x - z||y - z|}. \tag{7.82}$$

For points in Σ^{ext} our purpose is to apply Theorem 7.4.1 with $m = 2$. Denote by G, the integrand in (7.82), i.e.,

$$G(z) = \frac{1}{|x - z||y - z|}, z \in \overline{\Sigma^{int}} \tag{7.83}$$

For $x, y, \in \Sigma^{ext}$, the function G is an element in $C^{(\infty)}(\overline{\Sigma^{int}})$ and we have chosen the regular surface Σ (see Section 4.3), so the assumptions of the Theorem 7.4.1 are fulfilled.
We need to prove the existence of a constant C, such that

$$\int_{\Sigma^{int}} \left| \Delta_z^2 G(z) \right| dz \leq C. \tag{7.84}$$

Using the formula

$$\Delta(H \cdot K) = H\Delta K + 2\nabla H \cdot \nabla K + K\Delta H, \qquad \text{for} \qquad H, K \in C^{(2)}(\mathbb{R}^3), \tag{7.85}$$

and the harmonicity of $\frac{1}{|x-z|}$ and $\frac{1}{|y-z|}$ in Σ^{int} we get for the Laplacian $\Delta_z G(z)$ the following

$$\begin{aligned}
\Delta_z G(z) &= \nabla_z \cdot \nabla_z G(z) = \nabla_z \cdot \nabla_z \left(\frac{1}{|x-z|} \frac{1}{|y-z|} \right) \\
&= \nabla_z \cdot \left(\frac{1}{|y-z|} \nabla_z \frac{1}{|x-z|} + \frac{1}{|x-z|} \nabla_z \frac{1}{|y-z|} \right) \\
&= \nabla_z \cdot \left(\frac{1}{|y-z|} \nabla_z \frac{1}{|x-z|} \right) + \nabla_z \cdot \left(\frac{1}{|x-z|} \nabla_z \frac{1}{|y-z|} \right) \\
&= \frac{1}{|y-z|} \Delta_z \frac{1}{|x-z|} + \frac{1}{|x-z|} \Delta_z \frac{1}{|y-z|} + 2\nabla_z \frac{1}{|x-z|} \nabla_z \frac{1}{|y-z|} \\
&= 2\nabla_z \frac{1}{|x-z|} \nabla_z \frac{1}{|y-z|} \\
&= \frac{2(x-z)(y-z)}{|x-z|^3 |y-z|^3}.
\end{aligned} \tag{7.86}$$

An elementary calculation yields

$$\begin{aligned}
\Delta_z^2 G(z) &= 2 \sum_{i=1}^{3} \frac{\partial^2}{\partial z_i^2} \frac{1}{|x-z|} \frac{\partial^2}{\partial z_i^2} \frac{1}{|y-z|} \\
&\quad + 2 \sum_{i=1}^{3} \sum_{\substack{j=1 \\ j \neq i}}^{3} \frac{\partial^2}{\partial z_i \partial z_j} \frac{1}{|x-z|} \frac{\partial^2}{\partial z_i \partial z_j} \frac{1}{|y-z|} \\
&\quad + \sum_{i=1}^{3} \frac{\partial}{\partial z_i} \frac{1}{|x-z|} \frac{\partial}{\partial z_i} \Delta_z \frac{1}{|y-z|} \\
&\quad + \sum_{i=1}^{3} \frac{\partial}{\partial z_i} \Delta_z \frac{1}{|x-z|} \frac{\partial}{\partial z_i} \frac{1}{|y-z|}.
\end{aligned} \tag{7.87}$$

For the first summand in (7.87) we find the following estimate

$$\begin{aligned}
&\left| \frac{\partial^2}{\partial z_i^2} \frac{1}{|x-z|} \frac{\partial^2}{\partial z_i^2} \frac{1}{|y-z|} \right| \\
&= \left| \left(\frac{-1}{|x-z|^3} + \frac{3(x_i - z_i)^2}{|x-z|^5} \right) \left(\frac{-1}{|y-z|^3} + \frac{3(y_i - z_i)^2}{|y-z|^5} \right) \right|
\end{aligned}$$

$$\begin{aligned}
&\leq \frac{1}{|x-z|^3|y-z|^3} + \frac{|-3(x_i-z_i)^2|}{|x-z|^5|y-z|^3} + \frac{|-3(y_i-z_i)^2|}{|y-z|^5|x-z|^3} + \frac{|3(x_i-z_i)^2(y_i-z_i)^2|}{|x-z|^5|y-z|^5} \\
&\leq \frac{1}{|x-z|^3|y-z|^3} + \frac{3|x-z|^2}{|x-z|^5|y-z|^3} + \frac{3|y-z|^2}{|y-z|^5|x-z|^3} + \frac{3|x-z|^2|y-z|^2}{|x-z|^5|y-z|^5} \\
&\leq \frac{10}{|x-z|^3|y-z|^3} \quad i = 1,2,3.
\end{aligned} \tag{7.88}$$

In similar way find the estimates for the other summands in (7.87)

$$\begin{aligned}
&\left|\frac{\partial^2}{\partial z_i \partial z_j}\frac{1}{|x-z|}\frac{\partial^2}{\partial z_i \partial z_j}\frac{1}{|y-z|}\right| \\
&= \frac{|9(x_i-z_i)(x_j-z_j)(y_i-z_i)(y_j-z_j)|}{|x-z|^5|y-z|^5} \\
&\leq 9\frac{|x-z|^2|y-z|^2}{|x-z|^5|y-z|^5} \\
&\leq \frac{9}{|x-z|^3|y-z|^3}, \quad i,j = 1,2,3, i \neq j.
\end{aligned} \tag{7.89}$$

For the last two summands we have

$$\begin{aligned}
&\left|\frac{\partial}{\partial z_i}\frac{1}{|x-z|}\Delta_z\frac{1}{|y-z|}\right| \\
&\leq \frac{9|x-z||y-z|}{|x-z|^3|y-z|^5} + \frac{45|x-z||y-z|^3}{|x-z|^3|y-z|^7} + \frac{6|x-z||y-z|}{|x-z|^3|y-z|^5} \\
&\leq \frac{60}{|x-z|^2|y-z|^4}. \quad i = 1,2,3.
\end{aligned} \tag{7.90}$$

Similarly

$$\begin{aligned}
&\left|\frac{\partial}{\partial z_i}\Delta_z\frac{1}{|x-z|}\frac{\partial}{\partial z_i}\frac{1}{|y-z|}\right| \\
&\leq \frac{9|x-z||y-z|}{|x-z|^5|y-z|^3} + \frac{45|x-z|^3|y-z|}{|x-z|^7|y-z|^3} + \frac{6|x-z||y-z|}{|x-z|^5|y-z|^3} \\
&\leq \frac{60}{|x-z|^4|y-z|^2}, \quad i = 1,2,3.
\end{aligned} \tag{7.91}$$

Moreover, since Σ is a regular surface (cf. Definition 3.1), we have for points $p \in \Sigma^{ext}$ and $z \in \overline{\Sigma^{int}}$

$$|p| > \alpha, |z| < \beta, \tag{7.92}$$

where α and β are as in (3.1).
Now using the triangle inequality, we find for for points $p \in \Sigma^{ext}$ and $z \in \overline{\Sigma^{int}}$

$$\frac{1}{|p-z|} \leq \frac{1}{||p|-|z||} \leq \frac{1}{|\alpha-\beta|}. \tag{7.93}$$

Thus we can state the following

Theorem 7.5.1. *Given the points $x, y \in \Sigma^{ext}$ and the function*

$$z \mapsto G(z) = \frac{1}{|x-z||y-z|}, z \in \overline{\Sigma^{int}}, \tag{7.94}$$

the following estimate is valid:

$$\int_{\Sigma^{int}} \left|\Delta_z^2 G(z)\right| dz \leq 528\ C, \tag{7.95}$$

where

$$C = C(\Sigma) = \frac{||\Sigma^{int}||}{|\alpha-\beta|^6}, \tag{7.96}$$

where α, β are chosen as in (3.1)*.*

Proof: Indeed, from (7.87) we find

$$\begin{aligned} \int_{\Sigma^{int}} \left|\Delta_z^2 G(z)\right| dz \leq\ & 2 \int_{\Sigma^{int}} \sum_{i=1}^{3} \left| \frac{\partial^2}{\partial z_i^2} \frac{1}{|x-z|} \frac{\partial^2}{\partial z_i^2} \frac{1}{|y-z|} \right| dz \\ & + 2 \int_{\Sigma^{int}} \sum_{i=1}^{3} \sum_{\substack{j=1 \\ j\neq i}}^{3} \left| \frac{\partial^2}{\partial z_i \partial z_j} \frac{1}{|x-z|} \frac{\partial^2}{\partial z_i \partial z_j} \frac{1}{|y-z|} \right| dz \\ & + \int_{\Sigma^{int}} \sum_{i=1}^{3} \left| \frac{\partial}{\partial z_i} \frac{1}{|x-z|} \frac{\partial}{\partial z_i} \Delta_z \frac{1}{|y-z|} \right| dz \\ & + \int_{\Sigma^{int}} \sum_{i=1}^{3} \left| \frac{\partial}{\partial z_i} \Delta_z \frac{1}{|x-z|} \frac{\partial}{\partial z_i} \frac{1}{|y-z|} \right| dz. \end{aligned} \tag{7.97}$$

Having in mind the estimates (7.88), (7.89), (7.90) and (7.91) we then find

$$\begin{aligned}\int_{\Sigma^{int}} \left|\Delta_z^2 G(z)\right| dz \le\ & 2\cdot 3 \int_{\Sigma^{int}} \frac{10}{|x-z|^3|y-z|^3} dz \\ &+ 2\cdot 3\cdot 2 \int_{\Sigma^{int}} \frac{9}{|x-z|^3|y-z|^3} dz \\ &+ 3 \int_{\Sigma^{int}} \frac{60}{|x-z|^2|y-z|^4} dz \\ &+ 3 \int_{\Sigma^{int}} \frac{60}{|x-z|^4|y-z|^2} dz \\ &= 168 \int_{\Sigma^{int}} \frac{dz}{|x-z|^3|y-z|^3} dz \\ &+ 180 \int_{\Sigma^{int}} \frac{dz}{|x-z|^2|y-z|^4} \\ &+ 180 \int_{\Sigma^{int}} \frac{dz}{|x-z|^4|y-z|^2}.\end{aligned}$$

Finally, using the estimate (7.93), we find for given $x, y \in \Sigma^{ext}$

$$\int_{\Sigma^{int}} \left|\Delta_z^2 G(z)\right| dz \le 528\ \ C(\Sigma), \tag{7.98}$$

with C defined as in (7.96). This proves Theorem 7.5.1.
Now we can state in connection to Theorem 7.4.1 the following theorem concerning the discretization of the interpolating kernel $\mathscr{K}$

Theorem 7.5.2. *Let Σ be a regular surface and let x, y be a given points in Σ^{ext}. Then the integral*

$$\mathscr{K}(x,y) = \int_{\Sigma^{int}} \frac{dz}{|x-z||y-z|}, \tag{7.99}$$

can be approximated by the expression

$$\sum_{\substack{g\in\overline{\Sigma^{int}}\\ g\in\mathbb{Z}^3}}{}' G(g) + \int_\Sigma P_L\left[G(z), G^{(2)}(\Delta;\mathbb{Z}^3;z)\right] d\omega(z), \tag{7.100}$$

where G is defined in (7.94), such that

$$\begin{aligned}&\left|\int_{\Sigma^{int}} G(z)dz - \sum_{\substack{g\in\overline{\Sigma^{int}}\\ g\in\mathbb{Z}^3}}{}' G(g) + \int_\Sigma P_L\left[G(z), G^{(2)}(\Delta;\mathbb{Z}^3;z)\right] d\omega(z)\right| \\ &\le (2\pi)^{-4}\zeta(4;\mathbb{Z}^3) C_1(\Sigma),\end{aligned} \tag{7.101}$$

where

$$C_1 = 528C(\Sigma), \tag{7.102}$$

with C *defined as in* (7.96) *and*

$$\zeta(s, \mathbb{Z}^3) = \sum_{|g| \neq 0} |g|^{-s}, \tag{7.103}$$

is the (Epstein) zeta function.

Going over to the 'τ-dilated lattice' $\tau\mathbb{Z}^3$, $\tau > 0$ in connection to Theorem 7.4.3 we find

Theorem 7.5.3. *For the approximation procedure the following a priori error is valid*

$$\begin{aligned} &\left| \int_{\Sigma^{int}} G(z)dz - \tau^3 \sum_{\substack{g \in \overline{\Sigma^{int}} \\ g \in \mathbb{Z}^3}}{}' G(g) + \tau^3 \int_{\Sigma} P_L \left[G(z), G^{(2)}(\Delta; \mathbb{Z}^3; \tau^{-1}z) \right] d\omega(z) \right| \\ &\qquad \leq \tau^4 (2\pi)^{-4} \zeta(4; \mathbb{Z}^3) C_1(\Sigma), \end{aligned} \tag{7.104}$$

where

$$C_1 = 528C(\Sigma), \tag{7.105}$$

with C *defined as in* (7.96) *and*

$$\zeta(s, \mathbb{Z}^3) = \sum_{|g| \neq 0} |g|^{-s}, \tag{7.106}$$

is the (Epstein) zeta function.

The summation formulas presented here can be used in solving the Dirichlet and the oblique derivative boundary problem corresponding to data sets for points in Σ^{ext}. Thus, combined with a suitable regularization methods, they can also be applied in solving the SST or SGG satellite problems. Moreover, in the case of the oblique derivative we can even consider point sets in $\overline{\Sigma^{ext}}$, i.e., the data sets of points on Σ can be taken into consideration. Remember that in this case the reproducing kernel is defined with respect to the inner parallel surface, so for the points on Σ, we have the necessary smoothness conditions for the integrand.

8 Summary and Outlook

In this thesis we are trying to make the first modest steps to overcome the problems of spherical harmonic theory and to push the boundaries of today's geomathematical approaches, by finding a suitable way to determine the gravitational potential, with respect to the actual Earth's surface. Replacing the sphere by the regular surface and overcoming the main problems in spherical approach, will benefit to the development of new perspectives in all application fields. We have managed to make the first steps towards the approximations of functions on regular surfaces. The solutions to boundary problems of potential theory are given with respect to the real Earth surface. We have constructed the Hilbert space of potentials on and outside a regular surface, with reproducing kernel defined as Newton integral over the it's inner space. Thus, proposed spline formulation reflects the specific geometry of a given regular surface. Moreover, it is shown that the spline function, i.e., minimum norm interpolant, has the same domain of harmonicity as the gravitational potential, i.e., it is harmonic outside, and continuous on the Earth surface. This is a step forward in comparison to spherical harmonic spline formulation, where solution was given as a superposition of reproducing kernels that are harmonic functions down to a Runge sphere, i.e., the gravitational potential on and outside the Earth surface, was approximated using functions with larger harmonicity domain. Moreover, in the case of spherical Earth, it turns out that the reproducing kernel we have used, possesses the representation analogous to spherical harmonic kernels. This means that the reproducing kernel we have constructed, can be considered to be some kind of generalization to spherical oriented kernels.We have given a constructive way for gravitational potential determination and also convergence theorems were proven for interpolating functions to real geometries. However, the closed expression for this kernel seems to be unknown, even for the spherical case. Thus, considering a practical implementation, we proposed certain discretization methods for integrals over the inner space of a regular surface, involving multidimensional Euler summation formula. Also we managed to give a priori estimates for approximate integration of the interpolating Newton integral. The upcoming research in the field of Metaharmonic Lattice Point Theory (Geomathematics Group, TU Kaiserslautern) raises a great hope for further practical implementation of this research.

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