University of Kaiserslautern
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Asymptotics for a thin elastic fiber in contact with a rigid body

Diploma-thesis

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I express the most deep gratitude to my family for supporting and accompanying me throughout my whole studies.

Herewith I declare that I did the presented work on my own using only results from the bibliography stated at the end and that these known results are clearly identified as such.

Kaiserslautern, March 3, 2011

Contents

Abstract

In this work a 3-dimensional contact elasticity problem for a thin fiber and a rigid foundation is studied. We describe the contact condition by a linear Robin-condition (by meaning of the penalized and linearized non-penetration and friction conditions). The dimension of the problem is reduced by an asymptotic approach. Scaling the Robin parameters s^{ε} as $s^{\varepsilon} = s\varepsilon^{\alpha}$, $\alpha = 0, 1, 2$ we obtain a recurrent chain of Neumann type boundary value problems which are considered only in the microscopic scale. The problem for the leading term $\boldsymbol{u}^0$ is a homogeneous Neumann problem, hence the leading term depends only on the slow variable. This motivates the choice of a multiplicative ansatz in the asymptotic expansion.

The theoretical results are illustrated with numerical examples performed with a commercial finite-element software-tool.

0.1 Notations

$\varepsilon, \delta -$ small parameters

Lower case greek and latin letters denote real numbers - for instance:

$\sigma_{ij} \in \mathbb{R}$

Bold lower case greek and latin letters denote real vectors - for instance:

$\boldsymbol{\sigma}_i = (\sigma_{ij})_{j=1,2,3} = (\sigma_{i1}, \sigma_{i2}, \sigma_{i3}) \in \mathbb{R}^3$

Bold upper case greek and latin letters denote real matrices - for instance:

$$\boldsymbol{\Sigma} = (\sigma_{ij})_{i,j=1,2,3} = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{pmatrix} \in \mathbb{R}^{3\times 3}$$

$\boldsymbol{e}_i = \boldsymbol{I}^i$ where $\boldsymbol{I}$ is the 3×3 identity matrix

$\tilde{\boldsymbol{A}} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}$ for a matrix $A = (a_{ij})_{i,j=1,2,3}$

$\tilde{\boldsymbol{x}} = (x_1, x_2)$ for a vector $\boldsymbol{x} \in \mathbb{R}^3$

$$\nabla(\cdot) = \left(\frac{\partial(\cdot)}{\partial x_1}, \frac{\partial(\cdot)}{\partial x_2}, \frac{\partial(\cdot)}{\partial x_3} \right)^T$$

$\boldsymbol{\Sigma}^i = (\sigma_{ij})_{j=1,2,3}$

$$\operatorname{div}(\boldsymbol{\Sigma}(\boldsymbol{u})) = \sum_{k=1}^{3} \frac{\partial}{\partial x_k} \boldsymbol{\Sigma}(\boldsymbol{u})^k$$

$$\nabla_{x_1 x_2}(\cdot) = \left(\frac{\partial(\cdot)}{\partial x_1}, \frac{\partial(\cdot)}{\partial x_2}, 0 \right)^T$$

$$\operatorname{div}(\boldsymbol{u}) = \frac{\partial(u_1)}{\partial x_1} + \frac{\partial(u_2)}{\partial x_2} + \frac{\partial(u_3)}{\partial x_3}$$

$$\operatorname{Div}(\boldsymbol{\Sigma}(\boldsymbol{u})) = \left(\sum_{j=1}^{3} \frac{\partial(\sigma(\boldsymbol{u})_{ij})}{\partial x_j} \right)_{i=1,2,3}$$

$$\Delta(\cdot) = \frac{\partial^2(\cdot)}{\partial x_1^2} + \frac{\partial^2(\cdot)}{\partial x_2^2} + \frac{\partial^2(\cdot)}{\partial x_3^2}$$

$$\Delta_{x_1 x_2}(\cdot) = \frac{\partial^2(\cdot)}{\partial x_1^2} + \frac{\partial^2(\cdot)}{\partial x_2^2}$$

$< 1 >= \mathbb{R}$

$y = \dfrac{x}{\varepsilon}$ – fast or microscopic variable

$H^1(\Omega) = \left(\overline{C^\infty(\overline{\Omega})}, ||||_{1,2}\right)$ for a Lipschitz domain Ω

$||u||_{1,2} = ||u||_{0,2} + ||\nabla u||_{0,2}$

$||u||_{0,2} = \sqrt{\int_\Omega |u|^2 dx}$

$\cdot : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ denotes the canonical scalar product in $\mathbb{R}^3$

BVP means boundary value problem

1 Introduction

1.1 Motivation

In this work a 3D model of a thin elastic fiber in contact with a rigid body is reduced to a 1D model using asymptotic methods. Asymptotic-or perturbation methods are used to reduce the complexity of a problem. These methods are widely applied in physics when a small parameter appears in the model. The small parameter might for instance appear as a geometric value, e.g. the thickness-or periodicity of a structure, or as a factor causing fast variation of a physical coefficient. In this work a small parameter appears as the relative thickness of a fiber, i.e ε represents the relation between the thickness and the length of the fiber. The consequence of such a relation is that the boundary value problem of elasticity contains two different scales, the longitudinal scale (macroscopic variable) of order $\mathcal{O}(1)$ and the cross-sectional scale (microscopic variable) of order $\mathcal{O}(\varepsilon)$. During the solving difficulties arise due to the fact that the solution lives on two different scales, this implies that a direct numerical computation is too expensive. To overcome these difficulties asymptotic approaches are used to separate the scales.

1.1.1 The Euler-Bernoulli equations

Under the Euler-Bernoulli hypotheses, that plane sections stay plane and normal to the longitudinal axis of the beam, 3D beams are modeled with the Euler-Bernoulli equations, which are one dimensional. As an example let us introduce a classical beam, the cantilever beam. A cantilever beam is a beam that is fixed at one end and free at the other end, see Figure 1.1 [1].

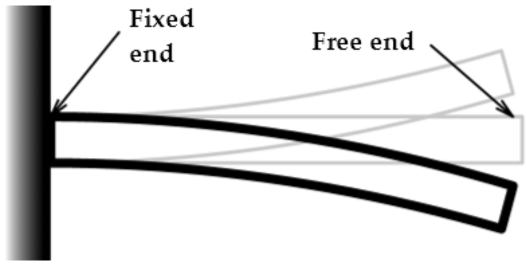


Figure 1.1: 2D cantilever beam

For this beam, the Euler-Bernoulli equations are the following

$$-EA\frac{\partial^2 u_3}{\partial x^2} = f_3(x) \qquad \text{in } (0,1) \tag{1.1}$$

$$EA\frac{\partial u_3}{\partial x} = 0 \qquad \text{on } 1 \tag{1.2}$$

$$u_3 = 0 \qquad \text{on } 0 \tag{1.3}$$

and for $i = 1, 2$

$$EI\frac{\partial^4 u_i}{\partial x^4} = f_i(x) \qquad \text{in } (0,1) \tag{1.4}$$

$$-EI\frac{\partial^3 u_i}{\partial x^3} = 0 \qquad \text{on } 1 \tag{1.5}$$

$$EI\frac{\partial^2 u_i}{\partial x^2} = 0 \qquad \text{on } 1 \tag{1.6}$$

$$\frac{\partial u_i}{\partial x} = 0 \qquad \text{on } 0 \tag{1.7}$$

$$u_i = 0 \qquad \text{on } 0 \tag{1.8}$$

$$\tag{1.9}$$

[1]Image taken form http://www.understandingcalculus.com/ with permission from the author

see [Kindmann, Frickel, 2002, p.22] and [Gross, Hauger, Schroder, Wall 2007, p.22-23,117-118]. Where $\boldsymbol{u} = (u_1, u_2, u_3)$ is the displacement vector with bending components u_i for $i = 1, 2$ and tension component u_3. The bending components of the displacement vector describe the lateral displacement of the beam and the tension component describes the longitudinal displacement. $\boldsymbol{f} = (f_1, f_2, f_3)^T$ is the body force. E is the Young modulus and can be expressed in terms of the Lame constants λ and μ as $E = \mu\frac{3\lambda+2\mu}{\lambda+\mu}$, $A = \int_\omega d(x_1, x_2)$ is the area of the cross-section ω, the constant product EA is the axial stiffness and the left hand side of (1.2) is the tension force. At this stage let us mention that the Young modulus of e material is indirectly related to its elasticity, the Young modulus of rubber is much smaller than the Young modulus of diamond. $I = \int_\omega x_i^2 d(x_1, x_2)$ is the area moment of inertia and the constant product EI is the bending stiffness. The left hand sides of (1.5) and (1.6) are the bending force and the bending moment respectively and the left hand side of (1.7) is the bending angle. The Dirichlet boundary conditions (1.3) and (1.8) describe a fixed end and are called geometric or kinematic boundary conditions while the Neumann boundary conditions (1.2), (1.5) and (1.6) describe a free end and are called dynamic boundary conditions.

In this work we replace the free end by an end that is in contact with a rigid body. We approximate a 3D contact elasticity problem by a mixed boundary value problem with Robin-type condition at the contact area and show that a dimension reduction leads to a replacement of the homogeneous Neumann boundary conditions (1.2) and (1.5) by homogeneous Robin-type conditions with constant coefficients in the 1D model. Which means that the (1.2) and (1.5) are replaced by $EA\frac{\partial u_3}{\partial x} = \gamma_t u_3$ and $EI\frac{\partial^3 u_i}{\partial x^3} = \gamma u_i$ respectively, where γ and γ_t are constants. This homogeneous Robin-type conditions with constant coefficients signify that the forces are proportional to the displacement.

Remark 1. *We remark that a homogeneous Robin-type boundary condition with constant coefficients looks as* $a\frac{\partial^k}{\partial x^k}\boldsymbol{u} + b\boldsymbol{u} = 0$. *In the following we refer to the term* $b\boldsymbol{u}$ *as the Robin-term and to* b *as the Robin-parameter.*

1.2 State of the art

In the last decades the one dimensional Euler-Bernoulli equations have been mathematically justified using asymptotics with regard to the beam thickness. This dimension reduction for thin elastic beams has been the subject of numerous works, see for instance [Panasenko, 2005] for isotropic homogeneous and heterogeneous beams, [Vodak, 2007] for isotropic homogeneous curved beams and [Alvarez-Dios, 1993] for for anisotropic ho-

mogeneous beams.
In the case of isotropic homogeneous thin elastic cylinders the classical Euler-Bernoulli equations (1.1)-(1.3) and (1.4)-(1.8) are obtained in [Panasenko, 2005] for a cantilever beam. In [Panasenko, 2005] we see that the dimension reduction of a 3D mixed Dirichlet/Neumann boundary value problem that models a 3D cantilever beam leads to a 1D mixed Dirichlet/Neumann boundary value problem modeling a 1D cantilever beam.
The aim of this work is to show that the dimension reduction of a 3D mixed Dirichlet/Neumann/Robin boundary value problem that approximates a 3D contact problem leads to a 1D Dirichlet/Robin boundary value problem approximating a 1D contact problem. Therefore the results of [Panasenko, 2005] for isotropic, homogeneous cantilever beams are extended to the case of a linearized contact of the elastic beam with a rigid body at an end.

1.2.1 Dimension reduction algorithm for a cantilever beam

We give a short overview of the dimension reduction algorithm for a cantilever beam performed in [Panasenko, 2005], see Figure 1.2. For this reason we list the main steps that are carried out in the algorithm.

1. State the 3D linear elasticity problem for a cantilever beam $\mathcal{P}_\varepsilon$. This problem has the form of a mixed Dirichlet/Neumann boundary value problem.

2. Assume that the solution $\boldsymbol{u}^\varepsilon$ is represented as a sum of 3 independent asymptotic expansions with respect to the beam thickness ε:

$$\boldsymbol{u}^\varepsilon = \boldsymbol{u}^\infty \quad , \text{ where } \quad \boldsymbol{u}^\infty = \boldsymbol{u}_B + \boldsymbol{u}_D + \boldsymbol{u}_N \tag{1.10}$$

   where $\boldsymbol{u}_B$ is responsible for the longitudinal interior, $\boldsymbol{u}_D$ is responsible for the Dirichlet boundary (fixed end) and $\boldsymbol{u}_N$ is responsible for the homogeneous Neumann boundary (free end).

3. For each expansion assume a multiplicative ansatz, i.e. every summand is a product of a function depending merely on the microscopic variable and derivatives of a function depending on the macroscopic variable only.

$$\boldsymbol{u}_B = \sum_{k=0}^{\infty} \varepsilon^k \boldsymbol{N}_k(\boldsymbol{y}) \frac{\partial^k}{\partial x_3^k} \boldsymbol{w}(x_3) \tag{1.11}$$

$$\boldsymbol{u}_D = \sum_{k=0}^{\infty} \varepsilon^k \boldsymbol{N}_k^D(\boldsymbol{y}) \frac{\partial^k}{\partial x_3^k} \boldsymbol{w}(x_3) \tag{1.12}$$

$$\boldsymbol{u}_N = \sum_{k=0}^{\infty} \varepsilon^k \boldsymbol{N}_k^N(\boldsymbol{y}) \frac{\partial^k}{\partial x_3^k} \boldsymbol{w}(x_3) \tag{1.13}$$

   Here x_3 is the macroscopic variable, in the case of a beam it is the longitudinal variable and $\boldsymbol{y} = \frac{\boldsymbol{x}}{\varepsilon}$ is the microscopic variable. Further $\boldsymbol{N}_k$, $\boldsymbol{N}_k^D$ and $\boldsymbol{N}_k^N$ are matrix functions with components in H^1.

4. Substitute $\boldsymbol{u}_B$, $\boldsymbol{u}_D$ and $\boldsymbol{u}_N$ separately into the 3D problem $\mathcal{P}_\varepsilon$. This leads to a recursive chain of auxiliary Neumann boundary value problems in the microscopic scale for $\boldsymbol{N}_k(\boldsymbol{y})$, $\boldsymbol{N}_k^D(\boldsymbol{y})$ and $\boldsymbol{N}_k^N(\boldsymbol{y})$ respectively.

5. To ensure that the solutions $\boldsymbol{N}_k(\boldsymbol{y})$, $\boldsymbol{N}_k^D(\boldsymbol{y})$ and $\boldsymbol{N}_N^N(\boldsymbol{y})$ exist, add constant matrices $\boldsymbol{H}_k$, $\boldsymbol{H}_k^D$ and $\boldsymbol{H}_k^N$ to each auxiliary Neumann boundary value problem and compute

them in order to obtain the desired solvability. These constant matrices are added to ensure that the sum of the right hand side of the Neumann problems is zero, see Section 7.

6. These constant matrices lead to a macroscopic ordinary differential equation $\mathcal{D}_\varepsilon(k)$ of order k. Via $\boldsymbol{H}_{k-1}$, $\boldsymbol{H}^D_{k-1}$ and $\boldsymbol{H}^N_{k-1}$ $\mathcal{D}_\varepsilon(k)$ contains data from the Neumann boundary value problems for $\boldsymbol{N}_{k-1}(\boldsymbol{y})$, $\boldsymbol{N}^D_{k-1}(\boldsymbol{y})$ and $\boldsymbol{N}^N_{k-1}(\boldsymbol{y})$.

7. Let ε tend to zero for $\mathcal{D}_\varepsilon(k)$, i.e. consider $\lim_{\varepsilon\to 0}\mathcal{D}_\varepsilon(k)$

The macroscopic problem $\lim_{\varepsilon\to 0}\mathcal{D}_\varepsilon(k)$ is called the limit problem. We highlight that the limit equation describes the effective mechanical properties of the 3D elasticity problem $\mathcal{P}_\varepsilon$ as ε tends to zero. Further we recall that the limit equations obtained in [Panasenko, 2005] for a cantilever beam are the Euler-Bernoulli equations (1.1)-(1.8). At this stage we notice that in [Panasenko, 2005] an averaged area $A = \frac{1}{|\omega|}\int_\omega d(x_1, x_2)(= 1)$ and an averaged area moment of inertia $I = \frac{1}{|\omega|}\int_\omega x_i^2 d(x_1, x_2)$ are obtained. The main mathematical tool used in the algorithm is the Fredholm-alternative. This Theorem is used to compute the constant matrices $\boldsymbol{H}_k$, $\boldsymbol{H}^D_k$ and $\boldsymbol{H}^N_k$ in step 5.

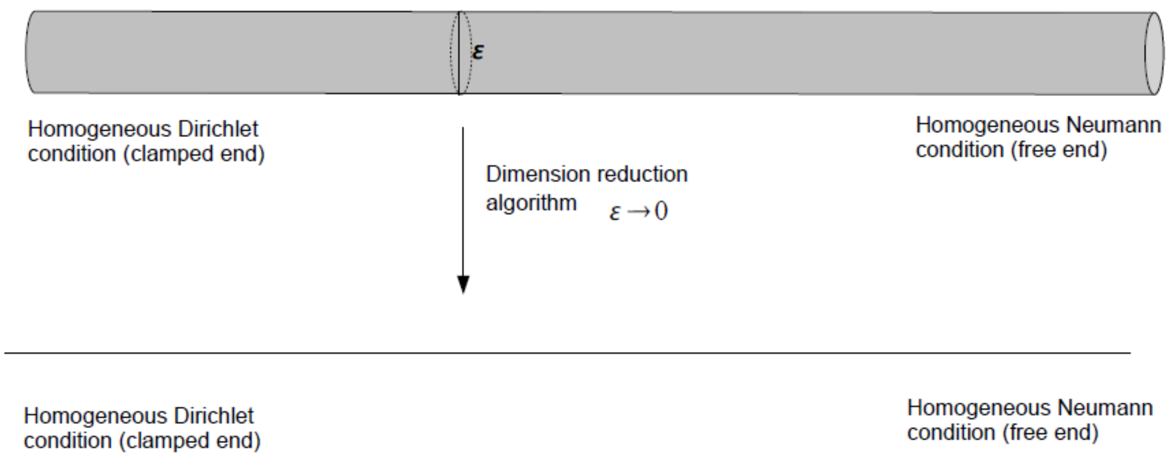


Figure 1.2: Dimension reduction of a 3D cantilever beam

1.3 Outline of the work

In the first part the contact between a thin fiber that is fixed at the left end and in contact with a rigid foundation on the right end is studied. Next a thin fiber that is fixed at both ends and in contact with a rigid fiber in its middle is considered. The dimension reduction of the 3D problems is performed with a slightly modified version of the procedure described in Subsubsection 1.2.1.

In this work the 3D elasticity contact problems are modeled with mixed Dirichlet/Neumann/Robin boundary value problems. The Dirichlet boundary conditions describe a clamped end, the homogeneous Neumann boundary conditions describe a free boundary and the Robin boundary conditions describe a contact surface. The ansatz is a general two scale expansion $\boldsymbol{u}^\infty = \sum_{k=0}^{\infty} \varepsilon^k \boldsymbol{u}^k(\frac{\boldsymbol{x}}{\varepsilon}, x_3)$, where x_3 is the longitudinal variable, and the multiplicative separation of scales is motivated by showing that the leading term $\boldsymbol{u}^0$ of the two scale expansion depends on the macroscopic scale (longitudinal variable) only. The leading term $\boldsymbol{u}^0$ serves as the function $\boldsymbol{w}$ in step (3) of the procedure in Subsubsection 1.2.1. Moreover no series is considered on the clamped end, the leading term is assumed to vanish at the fixed end. Additionally to the boundary layer corrector series responsible for the Neumann boundary condition introduced in step (3) in Subsubsection 1.2.1 a series is introduced at the contact area to compensate the influence of the Robin type boundary condition.

In Section 2 we emphasize and justify that the procedure described in Subsubsection 1.2.1 can only be executed when the recursive chain of auxiliary boundary value problems consists of Neumann boundary value problems. Thus another main aspect of this work is the scaling of the Robin-parameter. The Robin-parameters $s_i^\varepsilon = s_i \varepsilon^\alpha$ are scaled in such a way that a chain of Neumann type boundary value problems in the microscopic scale is obtained, in order to apply the algorithm described in [Panasenko, 2005]. This algorithm is based on the application of the Fredholm-alternative on the Neumann problems of the recursive chain. Therefore the scalings $\alpha = -1, 0, 1, 2$, are discussed. In Section 3 we conclude that a contact of two beams must be modeled at an end. Therefore we cut the beam at the contact area to generate two artificial ends. Finally one dimensional Robin-type limiting problems are obtained as ε tends to zero, see Figure 1.3. In the case of a hanging rod in contact with a rigid foundation at the lower end the complete ODE system is asymptotically derived. In the case of a beam in contact with a rigid beam in a part of the lateral boundary only the equilibrium-and force equation are constructed. The theoretical results for a thin fiber in contact with a rigid body at an end is illustrated numerically with a commercial finite-element software. We remark that the in this work we derive the limit problems by a formal asymptotic procedure, the convergence of the solutions is not discussed in this Diploma-thesis.

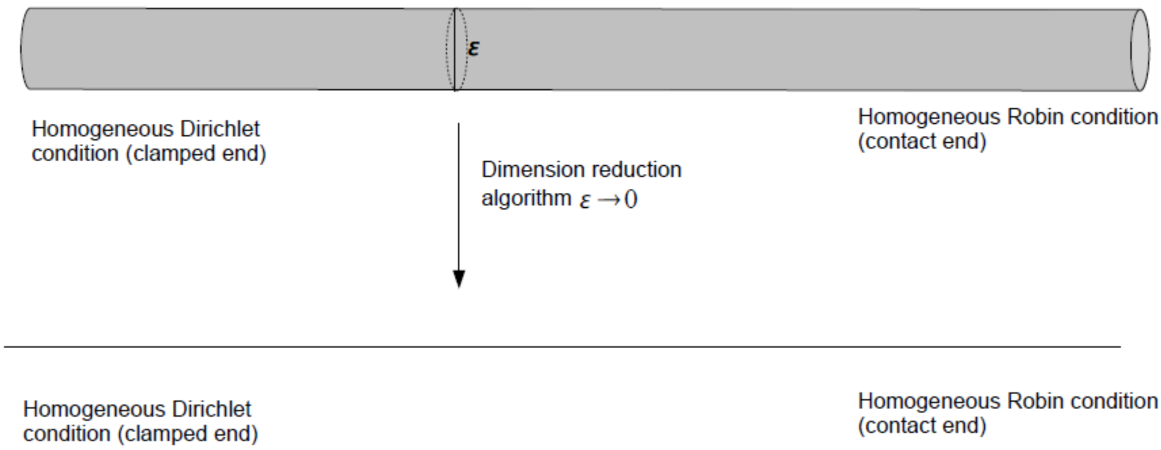


Figure 1.3: Dimension reduction of a 3D contact problem

2 3D contact problem for an elastic rod

In this subsection we consider a thin elastic rod that is fixed at one end and in contact with a rigid body on its other end, see Figure 2.1. We apply a slightly modified algorithm from [Panasenko, 2005] to construct the 1D limit problem as ε tends to zero.

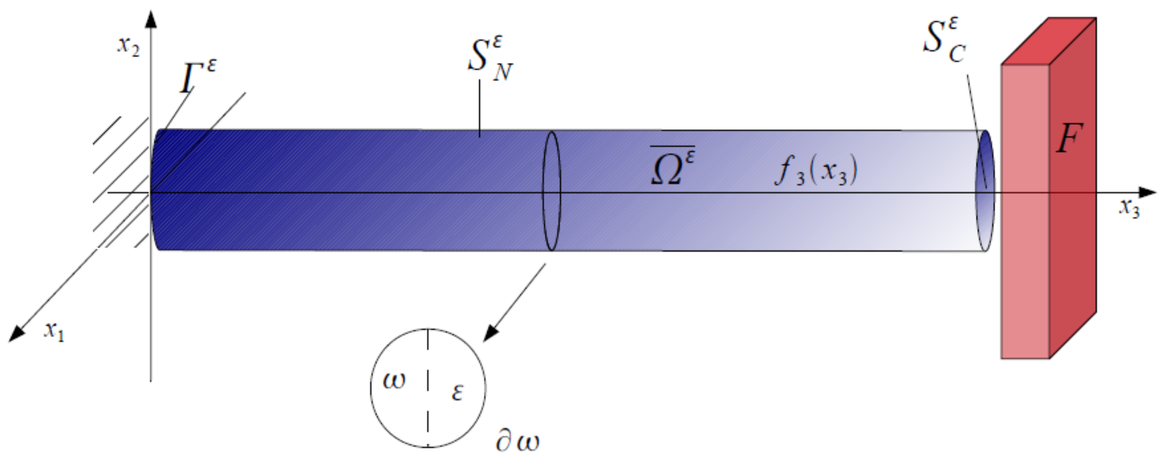


Figure 2.1: 3D rod in contact with a rigid foundation

2.1 Statement of the problem

We consider a 3D contact elasticity problem for a thin rod occupying the domain Ω^ε with a rigid foundation $\mathcal{F}$ at the contact surface S_C^ε, see Figure 2.1. The thin rod is clamped at its left end Γ^ε and the contact surface is located at its right end. On the lateral boundary of the fiber no traction is considered.

Assumptions 1. (Geometrical) *Let $\mathcal{O}(x_1, x_2, x_3)$ be a local coordinate system, in which the coordinate axes are identical with the principal axes of the body. W.L.O.G. the origin can be considered to be at an end of the fiber. The fiber ocuppies the set* $\Omega^\varepsilon = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : (x_1, x_2) \in \varepsilon\omega, x_3 \in (0, 1)\}$ *where the cross-section $\omega \subset \mathbb{R}^2$ is a symmetric domain with a smooth boundary. The lateral boundary of the fiber is denoted by* $S_N^\varepsilon = \left\{(x_1, x_2, x_3) \in \overline{\Omega}^\varepsilon : (\frac{x_1}{\varepsilon}, \frac{x_2}{\varepsilon}) \in \partial\omega, x_3 \in (0, 1)\right\}$*, the right end is given by* $S_C^\varepsilon = \left\{(x_1, x_2, x_3) \in \overline{\Omega}^\varepsilon : x_3 = 1\right\}$ *and the left end is set as* $\Gamma^\varepsilon = \left\{(x_1, x_2, x_3) \in \overline{\Omega}^\varepsilon : x_3 = 0\right\}$*. The diameter of the cross-section is given by the small parameter $0 < \varepsilon << 1$. The small parameter ε describes the relation between the thickness and the length of the fiber. The Robin-parameters s_i^ε are scaled as $s_i\varepsilon^\alpha$. We indicate that the choice of the coordinate system and the symmetry of ω implies that the product of inertia $\int_{\varepsilon\omega} x_i x_j d(x_1, x_2)$ for $i \neq j$ vanishes, since $\int_{\varepsilon\omega} x_i d(x_1, x_2) = 0$ and* $\int_{\varepsilon\omega} x_i x_j d(x_1, x_2) = \int_{\varepsilon\omega} x_j \left(\int_{\varepsilon\omega} x_i dx_i\right) dx_j$.

Assumptions 2. (Physical) *We assume the fiber to be elastic, homogeneous and isotropic. Further in this work only infinitesimal displacements are considered. Additionally we assume the rigid body displacement matrix and the extended rigid body displacement matrix to be the identity matrix $\boldsymbol{I}$. This assumption is natural since only the first three columns influence the equilibrium equation and the tension and bending forces and the fourth column is responsible for torsion, see Remark 7.2 in Section 7 and [Panasenko, 2005, p.57,64 and 66,71-72]. Moreover we assume the Euler Bernoulli hypothesis for the thin fibers, i.e. we assume that the cross-section remains orthogonal to the middle line of the fiber.*

Remark 2. (Motivation of the Robin-type boundary condition) *The contact constraints in the weak formulation of the contact problem can be first penalized, then regularized and linearized. The resulting penalized contact condition in the weak formulation leads to its possible simplification in form of the Robin-type condition, see [Kikuchi, Oden, 1988, p.98-103].*

$$
\begin{aligned}
-\operatorname{Div}\boldsymbol{\Sigma}^\varepsilon &= \boldsymbol{f}(x_3) && \text{in } \Omega^\varepsilon \\
\boldsymbol{\Sigma}^\varepsilon\boldsymbol{\nu} &= \boldsymbol{S}^\varepsilon\boldsymbol{u}^\varepsilon && \text{on } S_C^\varepsilon \\
\boldsymbol{\Sigma}^\varepsilon\boldsymbol{\nu} &= 0 && \text{on } S_N^\varepsilon \\
\boldsymbol{u}^\varepsilon &= 0 && \text{on } \Gamma^\varepsilon
\end{aligned}
\tag{2.1}
$$

and componentwise, for $i = 1, 2, 3$

$$
\begin{aligned}
-\operatorname{div}\boldsymbol{\sigma}_i^\varepsilon &= f_i(x_3) && \text{in } \Omega^\varepsilon \\
\boldsymbol{\sigma}_i^\varepsilon\cdot\boldsymbol{\nu} &= s_i^\varepsilon u_i^\varepsilon && \text{on } S_C^\varepsilon \\
\boldsymbol{\sigma}_i^\varepsilon\cdot\boldsymbol{\nu} &= 0 && \text{on } S_N^\varepsilon \\
u_i^\varepsilon &= 0 && \text{on } \Gamma^\varepsilon
\end{aligned}
\tag{2.2}
$$

Where $\boldsymbol{u}$ is the displacement vector with bending components u_i, i=1,2, and tension component u_3, $\boldsymbol{S}^\varepsilon = \begin{pmatrix} s_1^\varepsilon & 0 & 0 \\ 0 & s_2^\varepsilon & 0 \\ 0 & 0 & s_3^\varepsilon \end{pmatrix}$ is the Robin-parameter matrix, $\boldsymbol{\nu} = (n_1, n_2, n_3)$ is the outward unit normal, $e(\boldsymbol{u}^\varepsilon)_{ij} = \frac{1}{2}\left(\frac{\partial u_j^\varepsilon}{\partial x_i} + \frac{\partial u_i^\varepsilon}{\partial x_j}\right)$ for $i, j = 1, 2, 3$ are the components of the strain tensor and $\boldsymbol{\Sigma}^\varepsilon = (\sigma_{ij}^\varepsilon)_{i,j=1,2,3}$ is the stress tensor with components $\sigma_{ij}^\varepsilon = 2\mu e\left(\boldsymbol{u}^\varepsilon\right)_{ij} + \lambda \operatorname{div}\left(\boldsymbol{u}^\varepsilon\right)\delta_{ij}$ with the Lame constants λ and μ. $\boldsymbol{f}(x_3)$ is the body force and is assumed to have only a tension component, i.e. $\boldsymbol{f}(x_3) = (0, 0, f_3(x_3))^T$.

The solution of (2.2) is sought as a solution in the weak sense. Therefore we seek for functions $u_i^\varepsilon \in V^\varepsilon = \{\varphi \in H^1(\Omega^\varepsilon) : \varphi|_\Gamma^\varepsilon = 0\}$ satisfying (2.3).

$$
\sum_{j=1}^{3}\int_{\Omega^\varepsilon}\sigma_{ij}^\varepsilon\frac{\partial}{\partial x_j}\varphi dx - \int_{S_C^\varepsilon} s_i u_i^\varepsilon \varphi ds = \int_{\Omega^\varepsilon} f_i^\varepsilon \varphi dx \quad \forall\varphi \in V^\varepsilon \tag{2.3}
$$

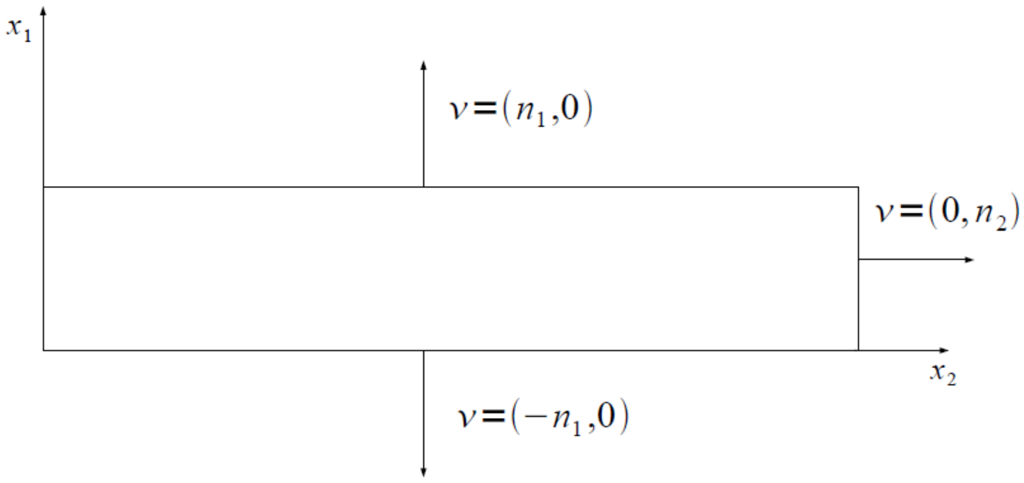
The existence and uniqueness of a solution of (2.3) are shown in

[Oleinik, Shamaev, Yosifian, 1992, p.317].

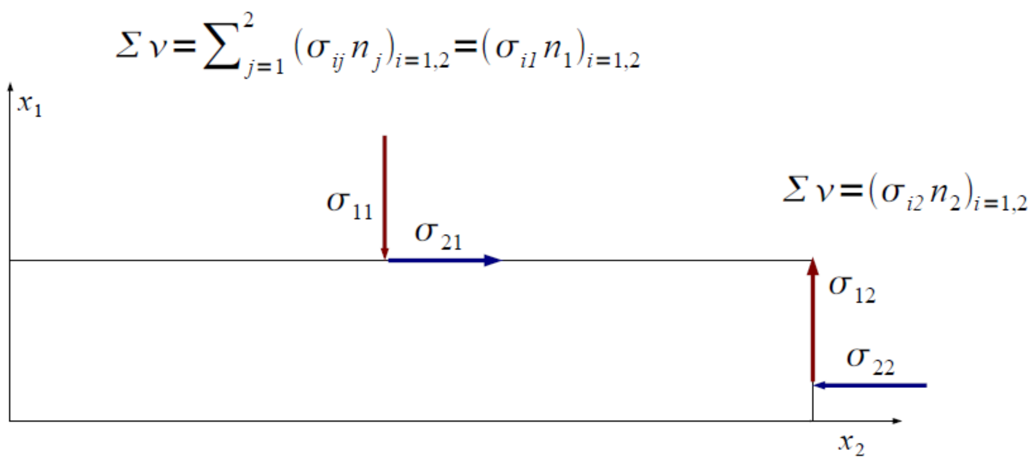
Remark 3. (Weak formulation) (2.3) *follows from* (2.2) *by multiplication by a test function* $\varphi \in V^\varepsilon$ *and partial integration, indeed*

$$\begin{aligned}
-\int_{\Omega^\varepsilon} \operatorname{div} \boldsymbol{\sigma}_i^\varepsilon \varphi dx &= -\sum_{j=1}^{3} \int_{\Omega^\varepsilon} \frac{\partial}{\partial x_j} \sigma_{ij}^\varepsilon \varphi dx \\
&= -\sum_{j=1}^{3} \left(\int_{\partial\Omega^\varepsilon} (\sigma_{ij}^\varepsilon n_j) \varphi ds - \int_{\Omega^\varepsilon} \sigma_{ij}^\varepsilon \frac{\partial}{\partial x_j} \varphi dx \right) \\
&= \sum_{j=1}^{3} \int_{\Omega^\varepsilon} \sigma_{ij}^\varepsilon \frac{\partial}{\partial x_j} \varphi dx - \left(\int_{S_C} (\boldsymbol{\sigma}_i^\varepsilon \cdot \boldsymbol{\nu}) \varphi ds + \int_{S_N} (\boldsymbol{\sigma}_i^\varepsilon \cdot \boldsymbol{\nu}) \varphi ds \right) \\
&= \sum_{j=1}^{3} \int_{\Omega^\varepsilon} \sigma_{ij}^\varepsilon \frac{\partial}{\partial x_j} \varphi dx - \int_{S_C} s_i u_i^\varepsilon \varphi ds \\
&= \int_{\Omega^\varepsilon} f_i^\varepsilon \varphi dx
\end{aligned}$$

Remark 4. (On traces) *The functions* $u \in H^1(\Omega^\varepsilon)$ *are understood as traces, i.e.* $u|_{\Omega^\varepsilon} = \mathrm{T}(u)$ *for the trace operator* T*, see [Dobrowolski, 2006, p.109].*



(a)



(b)

Figure 2.2: a) 2D rod with an outward unit normal , b) 2D draw with traction directions

2.2 Asymptotics

In this section asymptotics with respect to the fiber thickness ε are implemented for the dimension reduction. We perform a change of variables and substitute the displacement by a two scale asymptotic expansion with respect to the fiber thickness.
$x_i \mapsto y_i = \frac{x_i}{\varepsilon}$ for $i = 1, 2$, and $x_3 \mapsto (y_3 = \frac{x_3 - 1}{\varepsilon}, x_3)$.

Ansatz 1. (Two scale expansion)

$$\boldsymbol{u}^\varepsilon(x_1, x_2, x_3) = \boldsymbol{u}^\infty(y_1, y_2, y_3, x_3) \tag{2.4}$$

with

$$\boldsymbol{u}^\infty(y_1, y_2, y_3, x_3) = \sum_{i=0}^{\infty} \varepsilon^i \boldsymbol{u}^i(\boldsymbol{y}, x_3)$$

and

$$\boldsymbol{u}^i = 0 \textit{ for } i < 0 \tag{2.5}$$

Substituting (2.4) into (2.1) reads:

$$\begin{aligned}
-\operatorname{Div} \boldsymbol{\Sigma}^\infty &= \boldsymbol{f} && \text{in } \Omega \\
\boldsymbol{\Sigma}^\infty \boldsymbol{\nu} &= \boldsymbol{S}^\varepsilon \boldsymbol{u}^\infty && \text{on } S_C \\
\boldsymbol{\Sigma}^\infty \boldsymbol{\nu} &= 0 && \text{on } S_N \\
\boldsymbol{u}^\infty &= 0 && \text{on } \Gamma
\end{aligned} \tag{2.6}$$

where

$$\begin{aligned}
\Omega &= \left\{ (y_1, y_2, y_3, x_3) \in \mathbb{R}^4 : (y_1, y_2) \in \omega, y_3 \in (-\infty, 0), x_3 \in (0, 1) \right\} \\
S_N &= \left\{ (y_1, y_2, y_3, x_3) \in \overline{\Omega} : (y_1, y_2) \in \partial\omega, y_3 \in (-\infty, 0), x_3 \in (0, 1) \right\} \\
S_C &= \left\{ (y_1, y_2, y_3, x_3) \in \overline{\Omega} : y_3 = 0, x_3 = 1 \right\} \\
\Gamma &= \left\{ (y_1, y_2, y_3, x_3) \in \overline{\Omega} : y_3 = -\infty, x_3 = 0 \right\}
\end{aligned} \tag{2.7}$$

Theorem 1. (The limit problem) *The third component of the leading term of the solution of* (2.6) *solves* (2.8) *for* $\alpha = 0$ *and* (2.9) *for* $\alpha = 1$ *as* ε *tends to zero.*

$$\begin{aligned}
&-E\frac{\partial^2 u_3^0}{\partial x_3^2} = f_3(x_3) && x_3 \in (0,1) \text{ (equilibrium equation)} \\
&E\frac{\partial u_3^0}{\partial x_3} n_3 - s_3 u_3^0 = 0 && x_3 = 1 \text{ (force boundary condition)} \\
&u_3^0 = 0 && x_3 = 0 \text{ (displacement boundary condition)}
\end{aligned} \tag{2.8}$$

for $\alpha = 0$ *and*

$$\begin{aligned}
&-E\frac{\partial^2 u_3^0}{\partial x_3^2} = f_3(x_3) && x_3 \in (0,1) \text{ (equilibrium equation)} \\
&E\frac{\partial u_3^0}{\partial x_3} n_3 = 0 && x_3 = 1 \text{ (force boundary condition)} \\
&u_3^0 = 0 && x_3 = 0 \text{ (displacement boundary condition)}
\end{aligned} \tag{2.9}$$

for $\alpha = 1$.
Where solutions of (2.8) *and* (2.9) *are understood as functions* $u_3^0 \in V_0 = \{v \in H^1((0,1)) : v(0) = 0\}$ *satisfying*

$$\int_{(0,1)} E\frac{\partial u_3^0}{\partial x_3}\frac{\partial v}{\partial x_3} dx_3 - s_3 u_3^0(1)v(1) = \int_{(0,1)} f_3(x_3) v dx_3 \qquad \forall v \in V_0 \tag{2.10}$$

for $\alpha = 0$ *and*

$$\int_{(0,1)} E\frac{\partial u_3^0}{\partial x_3}\frac{\partial v}{\partial x_3} dx_3 = \int_{(0,1)} f_3(x_3) v dx_3 \qquad \forall v \in V_0$$

for $\alpha = 1$.

Where E *is the Young modulus, which can be expressed in terms of the Lame constants as* $E = E(\lambda, \mu) = \mu\frac{3\lambda+2\mu}{\lambda+\mu}$.

Theorem 1 is proven at the end of this section. The idea of the proof is based on the application of the Fredholm alternative for Neumann boundary value problems, which we also call their solvability condition. The limiting equations (2.8) and (2.9) follow from the solvability condition for u_3^2.

2.2.1 Sketch

To illustrate the procedure, we sketch the main steps of the proof.

1. We show that

$$\boldsymbol{u}^0 = \boldsymbol{u}^0(x_3)$$

2. Choose the main ansatz

$$\begin{aligned}\boldsymbol{u}^\infty(y_1, y_2, y_3, x_3) = &\sum_{k=0}^{\infty} \varepsilon^k \boldsymbol{N}_k(\tilde{\boldsymbol{y}}) \frac{\partial^k \boldsymbol{u}^0}{\partial x_3^k}(x_3) + \sum_{k=0}^{\infty} \varepsilon^k \boldsymbol{\Xi}_k(y_1, y_2, y_3) \frac{\partial^k \boldsymbol{u}^0}{\partial x_3^k}(x_3) \\ &+ \sum_{k=0}^{\infty} \varepsilon^k \sum_{s=1}^{\infty} {}^s\boldsymbol{\Theta}_{k-s(\alpha+1)}(y_1, y_2, y_3) \frac{\partial^{k-s(\alpha+1)} \boldsymbol{u}^0}{\partial x_3^{k-s(\alpha+1)}}(x_3) \end{aligned} \tag{2.11}$$

where the first and second sums correspond to $\boldsymbol{u}_B$ and $\boldsymbol{u}_N$ in the procedure mentioned in Subsubsection 1.2.1 and the third sum corresponds to the additional series mentioned in Subsection 1.3 responsible for the the Robin boundary condition.

3. Show that with (2.11) (2.6) implies

$$\begin{aligned} &\sum_{k=0}^{\infty} \varepsilon^{k-2} \boldsymbol{h}_k^3 \cdot \frac{\partial^k \boldsymbol{u}^0}{\partial x_3^k} = f_3(x_3) && x_3 \in (0,1) \\ &\sum_{k=0}^{\infty} \varepsilon^{k-1} \left({}^{\Xi}\boldsymbol{h}_k^3 \cdot \frac{\partial^k \boldsymbol{u}^0}{\partial x_3^k} + \sum_{s=1}^{\infty} {}^{s\Theta}\boldsymbol{h}_k^3 \cdot \frac{\partial^{k-s(\alpha+1)} \boldsymbol{u}^0}{\partial x_3^{k-s(\alpha+1)}} \right) = 0 && x_3 = 1 \end{aligned} \tag{2.12}$$

where $\boldsymbol{h}_k^3$, ${}^{\Xi}\boldsymbol{h}_k^3$ and ${}^{s\Theta}\boldsymbol{h}_k^3$ are constant vectors that follow from the solvability condition of boundary value problems for

$$\begin{aligned}\boldsymbol{u}^k(\boldsymbol{y}, x_3) = &\boldsymbol{N}_k(\tilde{\boldsymbol{y}}) \frac{\partial^k \boldsymbol{u}^0}{\partial x_3^k}(x_3) + \boldsymbol{\Xi}_k(y_1, y_2, y_3) \frac{\partial^k \boldsymbol{u}^0}{\partial x_3^k}(x_3) \\ &+ \sum_{s=1}^{\infty} {}^s\boldsymbol{\Theta}_{k-s(\alpha+1)}(y_1, y_2, y_3) \frac{\partial^{k-s(\alpha+1)} \boldsymbol{u}^0}{\partial x_3^{k-s(\alpha+1)}}(x_3)\end{aligned}$$

These constant solvability corrector vectors correspond to rows of the constant solvability corrector matrices mentioned in Subsubsection 1.2.1

4. Show that for $\varepsilon \to 0$ (2.12) yields

$$\boldsymbol{h}_2^3 \cdot \frac{\partial^2 \boldsymbol{u}^0}{\partial x_3^2} = f_3(x_3) \qquad x_3 \in (0,1) \tag{2.13}$$
$$^{\Xi}\boldsymbol{h}_1^3 \cdot \frac{\partial \boldsymbol{u}^0}{\partial x_3} + \sum_{s=1}^{\infty} {}^{s}\boldsymbol{\Theta}\boldsymbol{h}_1^3 \cdot \frac{\partial^{1-s(\alpha+1)}\boldsymbol{u}^0}{\partial x_3^{1-s(\alpha+1)}} = 0 \qquad x_3 = 1$$

5. Finally show that ${}^{\Xi}\boldsymbol{h}_1^2 = n_3 E\boldsymbol{e}_3$, ${}^{1}\boldsymbol{\Theta}\boldsymbol{h}_1^3 = s_3 \boldsymbol{N}^3_{1-(\alpha+1)}$, ${}^{s}\boldsymbol{\Theta}\boldsymbol{h}_1^3 = 0$ for $s \geq 2$ and $\boldsymbol{h}_2^3 = -E\boldsymbol{e}_3$

2.3 The algorithm

In this section the steps from the sketch of the previous subsection are carried out.

Proposition 1. (Dependence on the macroscopic scale of the leading term)

$$\boldsymbol{u}^0 = \boldsymbol{u}^0(x_3) \tag{2.14}$$

Remark 5. (Change of operators by the chain rule) *Applied on* $\boldsymbol{u}^i(y_1, y_2, y_3, x_3)$ *the differential operators in equation* (2.1) *change by the chain rule as follows:*

$$\begin{aligned}
\Delta &\mapsto \varepsilon^{-2}\left(\Delta_{\tilde{y}} + \frac{\partial^2}{\partial y_3^2}\right) + \varepsilon^{-1} 2 \frac{\partial}{\partial y_3}\frac{\partial}{\partial x_3} + \frac{\partial^2}{\partial x_3^2} \\
\nabla &\mapsto \varepsilon^{-1}\left(\nabla_{\tilde{y}} + \boldsymbol{e}_3 \frac{\partial}{\partial y_3}\right) + \boldsymbol{e}_3 \frac{\partial}{\partial x_3} \\
\operatorname{div} &\mapsto \varepsilon^{-1}\left(\operatorname{div}_{\tilde{y}} + \frac{\partial}{\partial y_3}\right) + \frac{\partial}{\partial x_3}
\end{aligned}$$

The appearance of the terms ε^k, $k = -2, -1, 0$ motivates the introduction of differential operators that have ε^k, $k = -2, -1, 0$ as coefficient. Before introducing these operators we set

$$\begin{aligned}
\tilde{e}(\tilde{\boldsymbol{u}})_{ij} &= \frac{1}{2}\left(\frac{\partial u_j}{\partial y_i} + \frac{\partial u_i}{\partial y_j}\right) && \text{for } i, j = 1, 2 \\
\tilde{\sigma}(\tilde{\boldsymbol{u}})_{ij} &= -2\mu \tilde{e}\left(\tilde{\boldsymbol{u}}\right)_{ij} - \lambda \operatorname{div}_{y_1, y_2}\left(\tilde{\boldsymbol{u}}\right) \delta_{ij} && \text{for } i, j = 1, 2
\end{aligned}$$

Definition 1. (Differential operators for different orders of ε)

$$
\begin{aligned}
L^{-1}(\boldsymbol{u}^k) &= -\sum_{i=1}^{2} \operatorname{div}_{\tilde{y}} \tilde{\boldsymbol{\sigma}}(\tilde{\boldsymbol{u}})_i \boldsymbol{e}_i - \mu \boldsymbol{e}_3 \Delta_{\tilde{y}} u_3^k - \mu \frac{\partial^2 \boldsymbol{u}^k}{\partial y_3^2} - (\lambda+\mu)\left(\nabla_{\tilde{y}} \frac{\partial u_3^k}{\partial y_3} + \boldsymbol{e}_3 \frac{\partial}{\partial y_3} \operatorname{div}_{\tilde{y}}(\tilde{\boldsymbol{u}}^k) + \boldsymbol{e}_3 \frac{\partial^2 u_3^k}{\partial y_3^2}\right) \\
L^{0}(\boldsymbol{u}^k) &= -(\lambda+\mu)\boldsymbol{e}_3 \frac{\partial}{\partial x_3} \operatorname{div}_{\tilde{y}} \boldsymbol{u}^k - (\lambda+\mu)\nabla_{\tilde{y}} \frac{\partial u_3^k}{\partial x_3} - 2\mu \frac{\partial}{\partial y_3}\frac{\partial}{\partial x_3}\boldsymbol{u}^k - 2(\lambda+\mu)\boldsymbol{e}_3 \frac{\partial}{\partial y_3}\frac{\partial u_3^k}{\partial x_3} \\
L^{1}(\boldsymbol{u}^k) &= -\mu \frac{\partial^2 \boldsymbol{u}^k}{\partial x_3^2} - (\lambda+\mu)\boldsymbol{e}_3 \frac{\partial^2 u_3^k}{\partial x_3^2} \\
G_{ij}^{-1}(\boldsymbol{u}^k) &= \tilde{\boldsymbol{\sigma}}(\tilde{\boldsymbol{u}}^k)_{ij} + \lambda \frac{\partial u_3^k}{\partial y_3}\delta_{ij} \\
G_{ij}^{0}(\boldsymbol{u}^k) &= \lambda \frac{\partial u_3^k}{\partial x_3}\delta_{ij} \\
G_{i3}^{-1}(\boldsymbol{u}^k) &= \mu \frac{\partial u_3^k}{\partial y_i} + \mu \frac{\partial u_i^k}{\partial y_3} \\
G_{i3}^{0}(\boldsymbol{u}^k) &= \mu \frac{\partial u_i^k}{\partial x_3} \\
G_{33}^{-1}(\boldsymbol{u}^k) &= \lambda \operatorname{div}_{\tilde{y}}(\tilde{\boldsymbol{u}}^k) + (\lambda+2\mu)\frac{\partial u_3^k}{\partial y_3} \\
G_{33}^{0}(\boldsymbol{u}^k) &= (\lambda+2\mu)\frac{\partial u_3^k}{\partial x_3}
\end{aligned}
$$

With definition 1 and equation (2.5) we have:

$$
\begin{aligned}
(-\operatorname{Div} \boldsymbol{\Sigma}^\infty)_i &= -\operatorname{div} \boldsymbol{\sigma}_i(\boldsymbol{u}^\infty) \\
&= \varepsilon^{-2} L^{-1}(\boldsymbol{u}^\infty)_i + \varepsilon^{-1} L^{0}(\boldsymbol{u}^\infty)_i + \varepsilon^{0} L^{1}(\boldsymbol{u}^\infty)_i \\
&= \sum_{k=0}^{\infty} \varepsilon^{k-2}\left(L^{-1}(\boldsymbol{u}^k)_i + L^{0}(\boldsymbol{u}^{k-1})_i + L^{1}(\boldsymbol{u}^{k-2})_i\right)
\end{aligned}
\tag{2.15}
$$

and

$$
\begin{aligned}
(\boldsymbol{\Sigma}^\infty \cdot \boldsymbol{\nu})_i &= \boldsymbol{\sigma}_i(\boldsymbol{u}^\infty)\cdot \boldsymbol{\nu} \\
&= \varepsilon^{-1} \sum_{k=1}^{3} G_{ij}^{-1}(\boldsymbol{u}^\infty) n_j + \varepsilon^{0} \sum_{k=1}^{3} G_{ij}^{0}(\boldsymbol{u}^\infty) n_j \\
&= \sum_{k=0}^{\infty} \varepsilon^{k-1}\left(\sum_{j=1}^{3}\left((G_{ij}^{-1}(\boldsymbol{u}^k)) + (G_{ij}^{0}(\boldsymbol{u}^{k-1}))\right) n_j\right)
\end{aligned}
$$

With this notation (2.6) takes the form:

$$\begin{aligned}
&\sum_{k=0}^{\infty} \varepsilon^{k-2}(L^{-1}(\boldsymbol{u}^k)_i + L^0(\boldsymbol{u}^{k-1})_i + L^1(\boldsymbol{u}^{k-2})_i) = f_i(x_3) && \text{in } \Omega \\
&\sum_{k=0}^{\infty} \varepsilon^{k-1}\left(\sum_{j=1}^{3}\left((G_{ij}^{-1}(\boldsymbol{u}^k)) + (G_{ij}^{0}(\boldsymbol{u}^{k-1}))\right) n_j\right) = s_i^{\varepsilon}\sum_{k=0}^{\infty}\varepsilon^{k-1}u_i^{k-1} && \text{on } S_C \\
&\sum_{k=0}^{\infty} \varepsilon^{k-1}\left(\sum_{j=1}^{3}\left((G_{ij}^{-1}(\boldsymbol{u}^k)) + (G_{ij}^{0}(\boldsymbol{u}^{k-1}))\right) n_j\right) = 0 && \text{on } S_N \\
&\sum_{k=0}^{\infty} \varepsilon^{k} u_i^k = 0 && \text{on } \Gamma
\end{aligned} \tag{2.16}$$

We notice that $L^{-1}(\boldsymbol{u}^k)_i = -\operatorname{div}_y(G_{ij}^{-1}(\boldsymbol{u}^k))$ for $i = 1, 2, 3$.

Definition 2.

$$\beta = \beta(\alpha) = \alpha + 1 \tag{2.17}$$

(2.16) induces the following chain of boundary value problems in divergence form in the microscopic scale $\boldsymbol{y}$. We point out that on the lateral boundary the outward unit normal $\boldsymbol{\nu}$ takes the form $\boldsymbol{\nu} = (n_1, n_2, 0)$ and at the right end $\boldsymbol{\nu}$ takes the form $\boldsymbol{\nu} = (0, 0, n_3)$, see Figure 2.2.

$$\begin{aligned}
&\mathbf{BVP}(\boldsymbol{u}^k) && \\
&-\operatorname{div}_y(G_{ij}^{-1}(\boldsymbol{u}^k)) = -L^0(\boldsymbol{u}^{k-1})_i - L^1(\boldsymbol{u}^{k-2})_i && \text{in } \omega \times (-\infty, 0) \\
&\sum_{j=1}^{2}(G_{ij}^{-1}(\boldsymbol{u}^k))n_j = -\sum_{j=1}^{2}(G_{ij}^{0}(\boldsymbol{u}^k))n_j && \text{on } \partial\omega \times (-\infty, 0) \\
&(G_{i3}^{-1}(\boldsymbol{u}^k))n_3 = -(G_{i3}^{0}(\boldsymbol{u}^k))n_3 + s_i u_i^{k-\beta} && \text{on } \omega \times \{0\}
\end{aligned} \tag{2.18}$$

Remark 6. (Scaling of the Robin-parameter) *At this stage we point out that for $\alpha = -1$ i.e.$\beta = 0$ we get a Robin-type condition on $y_3 = 0$ and for $\alpha \geq 0$ we get a Neumann condition. For our purpose, linking the lower dimensional right hand side of the boundary value problem with its solvability only Neumann boundary value problems are useful, see Remark 10 in Section 7. Hence in the following we consider only $\alpha = 0, 1$.*

Further we notice that in the microscale $\boldsymbol{y}$ there is no left end since the rod is of infinite length (see Figure 2.3) on the left hand side, hence there is no Dirichlet condition, since there is no clamped end.

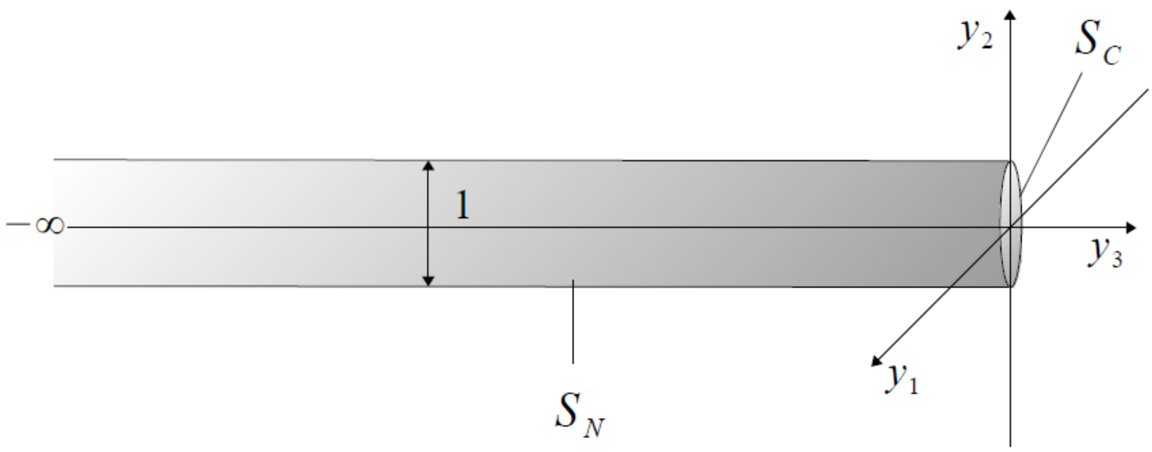


Figure 2.3: 3D rod in the microscopic scale

Further we notice that in the microscale $\boldsymbol{y}$ there is no left end since the rod is of infinite length (see Figure 2.3) on the left hand side, hence there is no Dirichlet condition, since there is no clamped end.

Proof. (of Proposition 5)

By (2.5) and (2.18) we obtain a homogeneous Neumann problem for the leading order $\boldsymbol{u}^0$:

$$\begin{aligned}
&\mathbf{BVP}(\boldsymbol{u}^0) \\
&-\operatorname{div}_y(G_{ij}^{-1}(\boldsymbol{u}^0)) = 0 && \text{in } \omega \times (-\infty, 0) \\
&\sum_{j=1}^{2} (G_{ij}^{-1}(\boldsymbol{u}^0)) n_j = 0 && \text{on } \partial\omega \times (-\infty, 0) \\
&(G_{i3}^{-1}(\boldsymbol{u}^0)) n_3 = 0 && \text{on } \omega \times \{0\}
\end{aligned}$$

From the fact that homogeneous Neumann problems admit only constant solutions, see [Evans, 1998, p.346], we deduce:

$$\boldsymbol{u}^0 = \boldsymbol{u}^0(x_3) \tag{2.19}$$

□

The fact that the leading term of the expansion depends only on the macroscopic scale motivates a multiplicative separation of scales in the asymptotic expansion. Therefore we choose an ansatz like in [Panasenko, 2005, p.23] and add the boundary layer correctors $\varepsilon^\beta \boldsymbol{u}^0(x_3) + \mathcal{O}(\varepsilon^{\beta+1}) = \sum_{k=0}^{\infty} \varepsilon^k \sum_{s=1}^{\infty} {}^s\boldsymbol{\Theta}_{k-s\beta}(y_1, y_2, y_3) \frac{\partial^{k-s\beta}\boldsymbol{u}^0}{\partial x_3^{k-s\beta}}(x_3)$ at the contact area.

Ansatz 2. (Main ansatz)

$$\begin{aligned}\boldsymbol{u}^\infty(y_1, y_2, y_3, x_3) = &\sum_{k=0}^{\infty} \varepsilon^k \boldsymbol{N}_k(\tilde{\boldsymbol{y}}) \frac{\partial^k \boldsymbol{u}^0}{\partial x_3^k} + \sum_{k=0}^{\infty} \varepsilon^k \boldsymbol{\Xi}_k(y_1, y_2, y_3) \frac{\partial^k \boldsymbol{u}^0}{\partial x_3^k}(x_3) \\ &+ \sum_{k=0}^{\infty} \varepsilon^k \sum_{s=1}^{\infty} {}^s\boldsymbol{\Theta}_{k-s\beta}(y_1, y_2, y_3) \frac{\partial^{k-s\beta}\boldsymbol{u}^0}{\partial x_3^{k-s\beta}}(x_3)\end{aligned} \tag{2.20}$$

Where $\operatorname{supp}(\boldsymbol{\Xi}_k, {}^s\boldsymbol{\Theta}_{k-s\beta}) \subset \omega \times \mathcal{B}_\delta(0)$ *is compact and*

$$\boldsymbol{N}_0, {}^s\boldsymbol{\Theta}_0 = \boldsymbol{I}, \boldsymbol{N}_k, {}^s\boldsymbol{\Theta}_k = 0 \text{ for } k < 0, \boldsymbol{\Xi}_k = 0 \text{ for } k \leq 0 \tag{2.21}$$

and

$$\boldsymbol{u}^0 \in V_0^3 = \left\{\boldsymbol{v} \in H^1((0,1))^3 : \boldsymbol{v}(0) = 0\right\} \tag{2.22}$$

We notice that no boundary layer corrector is introduced for the Dirichlet boundary, instead the leading term is chosen from an appropriate Sobolev space.

Definition 3.

$$\boldsymbol{v}^\infty := \sum_{k=0}^{\infty} \varepsilon^k \boldsymbol{N}_k(\tilde{\boldsymbol{y}}) \frac{\partial^k \boldsymbol{u}^0}{\partial x_3^k} \tag{2.23}$$

$$\boldsymbol{z}^\infty := \sum_{k=0}^{\infty} \varepsilon^k \boldsymbol{\Xi}_k(y_1, y_2, y_3) \frac{\partial^k \boldsymbol{u}^0}{\partial x_3^k}(x_3) \tag{2.24}$$

$${}^s\boldsymbol{w}^\infty := \sum_{k=0}^{\infty} \varepsilon^k \sum_{s=1}^{\infty} {}^s\boldsymbol{\Theta}_{k-s\beta}(y_1, y_2, y_3) \frac{\partial^{k-s\beta}\boldsymbol{u}^0}{\partial x_3^{k-s\beta}}(x_3) \tag{2.25}$$

Then the series (2.20) writes as:

$$\boldsymbol{u}^\infty = \boldsymbol{v}^\infty + \boldsymbol{z}^\infty + {}^s\boldsymbol{w}^\infty$$

Proposition 2. (Equation of infinite order for the leading term) (2.16) *implies:*

$$\begin{aligned}
&\sum_{k=0}^{\infty} \varepsilon^{k-2} \boldsymbol{h}_k^i \cdot \frac{\partial^k \boldsymbol{u}^0}{\partial x_3^k} = f_i(x_3) && x_3 \in (0,1) \qquad (2.26)\\
&\sum_{k=0}^{\infty} \varepsilon^{k-1} \left({}^{\Xi}\boldsymbol{h}_k^i \cdot \frac{\partial^k \boldsymbol{u}^0}{\partial x_3^k} + \sum_{s=1}^{\infty} {}^{s\Theta}\boldsymbol{h}_k^i \cdot \frac{\partial^{k-s\beta} \boldsymbol{u}^0}{\partial x_3^{k-s\beta}} \right) = 0 && x_3 = 1
\end{aligned}$$

where $\boldsymbol{h}_k^3$*,* ${}^{\Xi}\boldsymbol{h}_k^3$ *and* ${}^{s\Theta}\boldsymbol{h}_k^3$ *are constant vectors that follow from the solvability condition of boundary value problems for* $\boldsymbol{u}^k$*.*

To be able to separate appropriately the problems we split the operators in Definition 1.

Definition 4. (Differential operators in $\tilde{\boldsymbol{y}}$ and x_3)

$$\begin{aligned}
\tilde{L}^{-1}(\boldsymbol{u}^k) &= -\sum_{i=1}^{2} \operatorname{div}_{\tilde{y}} \tilde{\boldsymbol{\sigma}}(\tilde{\boldsymbol{u}}^k)_i \boldsymbol{e}_i - \mu \boldsymbol{e}_3 \Delta_{\tilde{y}} u_3^k \\
\tilde{L}^{0}(\boldsymbol{u}^k) &= -(\lambda+\mu)\boldsymbol{e}_3 \frac{\partial}{\partial x_3} \operatorname{div}_{\tilde{y}} \boldsymbol{u}^k - (\lambda+\mu)\nabla_{\tilde{y}} \frac{\partial u_3^k}{\partial x_3} \\
\tilde{L}^{1}(\boldsymbol{u}^k) &= L^1(\boldsymbol{u}^k) \\
\tilde{G}_{ij}^{-1}(\boldsymbol{u}^k) &= \tilde{\boldsymbol{\sigma}}(\tilde{\boldsymbol{u}}^k)_{ij} \\
\tilde{G}_{ij}^{0}(\boldsymbol{u}^k) &= G_{ij}^0(\boldsymbol{u}^k) \\
\tilde{G}_{i3}^{-1}(\boldsymbol{u}^k) &= \mu \frac{\partial u_3^k}{\partial y_i} \\
\tilde{G}_{i3}^{0}(\boldsymbol{u}^k) &= G_{i3}^0(\boldsymbol{u}^k) \\
\tilde{G}_{33}^{-1}(\boldsymbol{u}^k) &= \lambda \operatorname{div}_{\tilde{y}}(\tilde{\boldsymbol{u}}^k) \\
\tilde{G}_{33}^{0}(\boldsymbol{u}^k) &= G_{33}^0(\boldsymbol{u}^k)
\end{aligned}$$

Substituting (2.20) into (2.18) and with the linearity of the operators in Definitions 1, 6 we obtain the chains of boundary value problems stated below.

$$\begin{aligned}
&\mathbf{BVP}(\boldsymbol{v}^k) && (2.27)\\
&-\operatorname{div}_y(\tilde{G}_{ij}^{-1}(\boldsymbol{v}^k)) = -\tilde{L}^0(\boldsymbol{v}^{k-1})_i - \tilde{L}^1(\boldsymbol{v}^{k-2})_i && \text{in } \omega\\
&\sum_{j=1}^{2}(\tilde{G}_{ij}^{-1}(\boldsymbol{v}^k))n_j = -\sum_{j=1}^{2}(\tilde{G}_{ij}^{0}(\boldsymbol{v}^{k-1}))n_j && \text{on } \partial\omega
\end{aligned}$$

and

$$\begin{aligned}
&\mathbf{BVP}(\boldsymbol{z}^k) && (2.28)\\
&-\operatorname{div}_y(G_{ij}^{-1}(\boldsymbol{z}^k)) = -L^0(\boldsymbol{z}^{k-1})_i - L^1(\boldsymbol{z}^{k-2})_i && \text{in } \omega\times(-\infty,0)\\
&\sum_{j=1}^{2}(G_{ij}^{-1}(\boldsymbol{z}^k))n_j = -\sum_{j=1}^{2}(G_{ij}^{0}(\boldsymbol{z}^{k-1}))n_j && \text{on } \partial\omega\times(-\infty,0)\\
&G_{i3}^{-1}(\boldsymbol{z}^k)n_3 = -G_{i3}^{0}(\boldsymbol{z}^{k-1})n_3 - G_{i3}^{-1}(\boldsymbol{v}^k)n_3 - G_{i3}^{0}(\boldsymbol{v}^{k-1})n_3 && \text{on } \omega\times\{0\}
\end{aligned}$$

and

$$\begin{aligned}
&\mathbf{BVP}({}^s\boldsymbol{w}^k) && (2.29)\\
&-\operatorname{div}_y(G_{ij}^{-1}({}^s\boldsymbol{w}^k)) = -L^0({}^s\boldsymbol{w}^{k-1})_i - L^1({}^s\boldsymbol{w}^{k-2})_i && \text{in } \omega\times(-\infty,0)\\
&\sum_{j=1}^{2}(G_{ij}^{-1}({}^s\boldsymbol{w}^k))n_j = -\sum_{j=1}^{2}(G_{ij}^{0}({}^s\boldsymbol{w}^{k-1}))n_j && \text{on } \partial\omega\times(-\infty,0)\\
&G_{i3}^{-1}({}^s\boldsymbol{w}^k)n_3 = -G_{i3}^{0}({}^s\boldsymbol{w}^{k-1})n_3 + s_i u_i^{k-\beta} && \text{on } \omega\times\{0\}
\end{aligned}$$

As [Panasenko, 2005, 59-60] we separate (2.27) (2.28) and (2.29) from x_3-derivatives, therefore defining differential operators independently of x_3-derivatives is useful. For a matrix $\boldsymbol{A}_k \in \{\boldsymbol{N}_k, \boldsymbol{\Xi}_k, {}^s\boldsymbol{\Theta}_{k-s\beta}\}$ we define the operators below.

Definition 5. (Differential operators in the micro scale)

$$
\begin{aligned}
L_y^{-1}(\boldsymbol{A}_k)_i &= -\operatorname{div}_{\boldsymbol{y}} \boldsymbol{\sigma}(\boldsymbol{A}_k)_i && \textit{for } i = 1,2,3 \\
L_y^0(\boldsymbol{A}_k)_i &= -(\lambda+\mu)\frac{\partial}{\partial y_i}\boldsymbol{A}_k^3 - 2\mu\frac{\partial}{\partial y_3}\boldsymbol{A}_k^i && \textit{for } i = 1,2 \\
L_y^0(\boldsymbol{A}_k)_3 &= -(\lambda+\mu)\operatorname{div}_{\tilde{y}} \widetilde{\boldsymbol{A}}_k - 2\mu\frac{\partial}{\partial y_3}\boldsymbol{A}_k^3 - 2(\lambda+\mu)\frac{\partial}{\partial y_3}\boldsymbol{A}_k^3 \\
L_y^1(\boldsymbol{A}_k)_i &= -\mu\boldsymbol{A}_k^i && \textit{for } i = 1,2 \\
L_y^1(\boldsymbol{A}_k)_3 &= -(\lambda+2\mu)\boldsymbol{A}_k^3 \\
G_{yij}^{-1}(\boldsymbol{A}_k) &= \boldsymbol{\sigma}(\boldsymbol{A}_k)_{ij} && \textit{for } i = 1,2 \\
G_{yij}^{0}(\boldsymbol{A}_k) &= \lambda\boldsymbol{A}_k^3\delta_{ij} && \textit{for } i = 1,2 \\
G_{yi3}^{-1}(\boldsymbol{A}_k) &= \mu\frac{\boldsymbol{A}_k^3}{\partial y_i} + \mu\frac{\partial\boldsymbol{A}_k^i}{\partial y_3} && \textit{for } i = 1,2 \\
G_{yi3}^{0}(\boldsymbol{A}_k) &= \mu\boldsymbol{A}_k^i && \textit{for } i = 1,2 \\
G_{y33}^{-1}(\boldsymbol{A}_k) &= \lambda\operatorname{div}_{\boldsymbol{y}}(\boldsymbol{A}_k) + 2\mu\frac{\partial\boldsymbol{A}_k^3}{\partial y_3} \\
G_{y33}^{0}(\boldsymbol{A}_k) &= (\lambda+2\mu)\boldsymbol{A}_k^3
\end{aligned}
$$

Analogously $\tilde{L}_y^k(\cdot)_i$, $\tilde{G}_{yij}^k(\cdot)$ are defined.

Definition 6. (Differential operators in $\tilde{\boldsymbol{y}}$ only)

$$
\begin{aligned}
\tilde{L}_y^{-1}(\boldsymbol{A}_k)_i &= -\operatorname{div}_{\tilde{\boldsymbol{y}}} \boldsymbol{\sigma}(\boldsymbol{A}_k)_i && \textit{for } i = 1,2,3 \\
\tilde{L}_y^0(\boldsymbol{A}_k)_i &= -(\lambda+\mu)\frac{\partial}{\partial y_i}\boldsymbol{A}_k^3 && \textit{for } i = 1,2 \\
\tilde{L}_y^0(\boldsymbol{A}_k)_3 &= -(\lambda+\mu)\operatorname{div}_{\tilde{y}} \widetilde{\boldsymbol{A}}_k \\
\tilde{L}_y^1(\boldsymbol{A}_k)_i &= -\mu\boldsymbol{A}_k^i && \textit{for } i = 1,2 \\
\tilde{L}_y^1(\boldsymbol{A}_k)_3 &= -(\lambda+2\mu)\boldsymbol{A}_k^3 \\
\tilde{G}_{yij}^{-1}(\boldsymbol{A}_k) &= \tilde{\boldsymbol{\sigma}}(\boldsymbol{A}_k)_{ij} && \textit{for } i = 1,2 \\
\tilde{G}_{yij}^{0}(\boldsymbol{A}_k) &= \lambda\boldsymbol{A}_k^3\delta_{ij} && \textit{for } i = 1,2 \\
\tilde{G}_{yi3}^{-1}(\boldsymbol{A}_k) &= \mu\frac{\boldsymbol{A}_k^3}{\partial y_i} && \textit{for } i = 1,2 \\
\tilde{G}_{yi3}^{0}(\boldsymbol{A}_k) &= \mu\boldsymbol{A}_k^i && \textit{for } i = 1,2 \\
\tilde{G}_{y33}^{-1}(\boldsymbol{A}_k) &= \lambda\operatorname{div}_{\tilde{\boldsymbol{y}}}(\boldsymbol{A}_k) \\
\tilde{G}_{y33}^{0}(\boldsymbol{A}_k) &= (\lambda+2\mu)\boldsymbol{A}_k^3
\end{aligned}
$$

By the multiplicative ansatz we have $\boldsymbol{v}^k = \boldsymbol{N}_k(\tilde{y})\frac{\partial^k \boldsymbol{u}^0}{\partial x_3^k}$. This multiplicative separation of scales permits us to decouple (2.27) from x_3-derivatives as

$$\begin{aligned}
&\mathbf{BVP}(\boldsymbol{N}_k \frac{\partial^k \boldsymbol{u}^0}{\partial x_3^k}) && (2.30)\\
&-\operatorname{div}_y(\tilde{G}^{-1}_{yij}(\boldsymbol{N}_k))\frac{\partial^k \boldsymbol{u}^0}{\partial x_3^k} = -\tilde{L}^0_y(\boldsymbol{N}_{k-1})_i \frac{\partial^k \boldsymbol{u}^0}{\partial x_3^k} - \tilde{L}^1_y(\boldsymbol{N}_{k-2})_i \frac{\partial^k \boldsymbol{u}^0}{\partial x_3^k} && \text{in } \omega\\
&\sum_{j=1}^{2}(\tilde{G}^{-1}_{yij}(\boldsymbol{N}_k))n_j \frac{\partial^k \boldsymbol{u}^0}{\partial x_3^k} = -\sum_{j=1}^{2}(\tilde{G}^{0}_{yij}(\boldsymbol{N}_{k-1}))_i n_j \frac{\partial^k \boldsymbol{u}^0}{\partial x_3^k} && \text{on } \partial\omega
\end{aligned}$$

hence

$$\begin{aligned}
&\mathbf{BVP}(\boldsymbol{N}_k) && (2.31)\\
&-\operatorname{div}_y(\tilde{G}^{-1}_{yij}(\boldsymbol{N}_k)) = -\tilde{L}^0_y(\boldsymbol{N}_{k-1})_i - \tilde{L}^1_y(\boldsymbol{N}_{k-2})_i && \text{in } \omega\\
&\sum_{j=1}^{2}(\tilde{G}^{-1}_{yij}(\boldsymbol{N}_k))n_j = -\sum_{j=1}^{2}(\tilde{G}^{0}_{yij}(\boldsymbol{N}_{k-1}))_i n_j && \text{on } \partial\omega
\end{aligned}$$

Analogously for (2.28) and (2.29) we get

$$\begin{aligned}
&\mathbf{BVP}(\boldsymbol{\Xi}_k) && (2.32)\\
&-\operatorname{div}_y(G^{-1}_{yij}(\boldsymbol{\Xi}_k)) = -L^0_y(\boldsymbol{\Xi}_{k-1})_i - L^1_y(\boldsymbol{\Xi}_{k-2})_i && \text{in } \omega\times(-\infty,0)\\
&\sum_{j=1}^{2}(G^{-1}_{yij}(\boldsymbol{\Xi}_k))n_j = -\sum_{j=1}^{2}(G^{0}_{yij}(\boldsymbol{\Xi}_{k-1}))n_j && \text{on } \partial\omega\times(-\infty,0)\\
&G^{-1}_{yi3}(\boldsymbol{\Xi}_k)n_3 = -G^0_{i3}(\boldsymbol{\Xi}_{k-1})n_3 - G^{-1}_{yi3}(\boldsymbol{N}_k)n_3 - G^0_{yi3}(\boldsymbol{N}_{k-1})n_3 && \text{on } \omega\times\{0\}
\end{aligned}$$

and

$$\begin{aligned}
&\mathbf{BVP}({}^1\boldsymbol{\Theta}_{k-\beta}) && (2.33)\\
&-\operatorname{div}_y(G^{-1}_{yij}({}^1\boldsymbol{\Theta}_{k-\beta}) = -L^0_y({}^1\boldsymbol{\Theta}_{k-1-\beta})_i - L^1_y({}^1\boldsymbol{\Theta}_{k-2-\beta})_i && \text{in } \omega\times(-\infty,0)\\
&\sum_{j=1}^{2}(G^{-1}_{yij}({}^1\boldsymbol{\Theta}_{k-\beta})n_j = -\sum_{j=1}^{2}(G^{0}_{yij}({}^1\boldsymbol{\Theta}_{k-1-\beta})n_j && \text{on } \partial\omega\times(-\infty,0)\\
&G^{-1}_{yi3}({}^1\boldsymbol{\Theta}_{k-\beta})n_3 = -G^0_{yi3}({}^1\boldsymbol{\Theta}_{k-1-\beta})n_3 + s_i(\boldsymbol{N}^i_{k-\beta} + \boldsymbol{\Xi}^i_{k-\beta}) && \text{on } \omega\times\{0\}
\end{aligned}$$

and

$$
\begin{aligned}
&\mathbf{BVP}({}^s\mathbf{\Theta}_{k-s\beta}) && (2.34)\\
&-\operatorname{div}_y(G^{-1}_{yij}({}^s\mathbf{\Theta}_{k-\beta}) = -L^0_y({}^s\mathbf{\Theta}_{k-1-s\beta})_i - L^1_y({}^s\mathbf{\Theta}_{k-2-s\beta})_i && \text{in } \omega\times(-\infty,0)\\
&\sum_{j=1}^{2}(G^{-1}_{yij}({}^s\mathbf{\Theta}_{k-s\beta})n_j = -\sum_{j=1}^{2}(G^0_{yij}({}^s\mathbf{\Theta}_{k-1-s\beta})n_j && \text{on } \partial\omega\times(-\infty,0)\\
&G^{-1}_{yi3}({}^s\mathbf{\Theta}_{k-s\beta})n_3 = -G^0_{yi3}({}^s\mathbf{\Theta}_{k-1-s\beta})n_3 + s_i({}^{s-1}\mathbf{\Theta}_{k-s\beta}) && \text{on } \omega\times\{0\}
\end{aligned}
$$

To make sure that (2.31), (2.32), (2.33) and (2.34) are solvable we proceed as [Panasenko, 2005, p.60], therefore we need the following definition.

Definition 7. (Solvability correctors)

$$
\begin{aligned}
\boldsymbol{h}^i_k(\tilde{\boldsymbol{y}}) &= \tilde{L}^{-1}_y(\boldsymbol{N}_k)_i + \tilde{L}^0_y(\boldsymbol{N}_{k-1})_i + \tilde{L}^1_y(\boldsymbol{N}_{k-2})_i\\
{}^{\Xi}\boldsymbol{h}^i_k(\tilde{\boldsymbol{y}},0) &= \Big(G^{-1}_{yi3}(\mathbf{\Xi}_k+\boldsymbol{N}_k)|_{y_3=0} + G^0_{yi3}(\mathbf{\Xi}_{k-1}+\boldsymbol{N}_{k-1})|_{y_3=0}\Big)\, n_3\\
{}^{1\mathbf{\Theta}}\boldsymbol{h}^i_k(\tilde{\boldsymbol{y}}) &= \sum_{j=1}^{2}\Big(G^{-1}_{yij}({}^1\mathbf{\Theta}_{k-1-\beta})|_{y_3=0} + G^0_{yij}({}^1\mathbf{\Theta}_{k-2-\beta})|_{y_3=0}\Big)\, n_j - s_i(\boldsymbol{N}^{k-\beta}_i + \mathbf{\Xi}^i_{k-\beta})|_{y_3=0}\\
{}^{s\mathbf{\Theta}}\boldsymbol{h}^i_k(\tilde{\boldsymbol{y}},0) &= \sum_{j=1}^{2}\Big(G^{-1}_{yij}({}^s\mathbf{\Theta}_{k-1-\beta})|_{y_3=0} + G^0_{yij}({}^s\mathbf{\Theta}_{k-2-\beta})|_{y_3=0}\Big)\, n_j - s_i({}^{s-1}\mathbf{\Theta}^i_{k-s\beta})|_{y_3=0}
\end{aligned}
$$

Assumptions 3. (Constant solvability correctors) *We assume* $\boldsymbol{h}^i_k, {}^{\Xi}\boldsymbol{h}^i_k, {}^{1\mathbf{\Theta}}\boldsymbol{h}^i_k, {}^{s\mathbf{\Theta}}\boldsymbol{h}^i_k$ *to be constant, see [Panasenko, 2005, p.24,60,70].*

Proof. (of Proposition 2)

With Definition 7 and Assumption 3, (2.31), (2.32), (2.33) and (2.34) read:

$$
\begin{aligned}
&\mathbf{BVP}(\boldsymbol{N}_k) && (2.35)\\
&-\operatorname{div}_y(\tilde{G}^{-1}_{yij}(\boldsymbol{N}_k)) = -\tilde{L}^0_y(\boldsymbol{N}_k)_i - \tilde{L}^1_y(\boldsymbol{N}_{k-1})_i + \boldsymbol{h}^i_k && \text{in } \omega\\
&\sum_{j=1}^{2}(\tilde{G}^{-1}_{yij}(\boldsymbol{N}_k))n_j = -\sum_{j=1}^{2}(\tilde{G}^0_{yij}(\boldsymbol{N}_{k-1}))_i n_j && \text{on } \partial\omega
\end{aligned}
$$

and

$$\textbf{BVP}(\boldsymbol{\Xi}_k) \tag{2.36}$$

$$\begin{aligned}
&-\operatorname{div}_y(G^{-1}_{yij}(\boldsymbol{\Xi}_k)) = -L^0_y(\boldsymbol{\Xi}_{k-1})_i - L^1_y(\boldsymbol{\Xi}_{k-2})_i && \text{in } \omega\times(-\infty,0)\\
&\sum_{j=1}^{2}(G^{-1}_{yij}(\boldsymbol{\Xi}_k))n_j = -\sum_{j=1}^{2}(G^{0}_{yij}(\boldsymbol{\Xi}_{k-1}))n_j && \text{on } \partial\omega\times(-\infty,0)\\
&G^{-1}_{yi3}(\boldsymbol{\Xi}_k)n_3 = -G^0_{i3}(\boldsymbol{\Xi}_{k-1})n_3 - G^{-1}_{yi3}(\boldsymbol{N}_k)n_3 - G^0_{yi3}(\boldsymbol{N}_{k-1})n_3 + {}^{\Xi}\boldsymbol{h}^i_k && \text{on } \omega\times\{0\}
\end{aligned}$$

and

$$\textbf{BVP}({}^1\boldsymbol{\Theta}_{k-\beta}) \tag{2.37}$$

$$\begin{aligned}
&-\operatorname{div}_y(G^{-1}_{yij}({}^1\boldsymbol{\Theta}_{k-\beta}) = -L^0_y({}^1\boldsymbol{\Theta}_{k-1-\beta})_i - L^1_y({}^1\boldsymbol{\Theta}_{k-2-\beta})_i && \text{in } \omega\times(-\infty,0)\\
&\sum_{j=1}^{2}(G^{-1}_{yij}({}^1\boldsymbol{\Theta}_{k-\beta})n_j = -\sum_{j=1}^{2}(G^{0}_{yij}({}^1\boldsymbol{\Theta}_{k-1-\beta})n_j && \text{on } \partial\omega\times(-\infty,0)\\
&G^{-1}_{yi3}({}^1\boldsymbol{\Theta}_{k-\beta})n_3 = -G^0_{yi3}({}^1\boldsymbol{\Theta}_{k-1-\beta})n_3 + s_i(\boldsymbol{N}^i_{k-\beta} + \boldsymbol{\Xi}^i_{k-\beta}) + {}^{1\boldsymbol{\Theta}}\boldsymbol{h}^i_k && \text{on } \omega\times\{0\}
\end{aligned}$$

and

$$\textbf{BVP}({}^s\boldsymbol{\Theta}_{k-s\beta}) \tag{2.38}$$

$$\begin{aligned}
&-\operatorname{div}_y(G^{-1}_{yij}({}^s\boldsymbol{\Theta}_{k-\beta}) = -L^0_y({}^s\boldsymbol{\Theta}_{k-1-s\beta})_i - L^1_y({}^s\boldsymbol{\Theta}_{k-2-s\beta})_i && \text{in } \omega\times(-\infty,0)\\
&\sum_{j=1}^{2}(G^{-1}_{yij}({}^s\boldsymbol{\Theta}_{k-s\beta})n_j = -\sum_{j=1}^{2}(G^{0}_{yij}({}^s\boldsymbol{\Theta}_{k-1-s\beta})n_j && \text{on } \partial\omega\times(-\infty,0)\\
&G^{-1}_{yi3}({}^s\boldsymbol{\Theta}_{k-s\beta})n_3 = -G^0_{yi3}({}^s\boldsymbol{\Theta}_{k-1-s\beta})n_3 + s_i({}^{s-1}\boldsymbol{\Theta}_{k-s\beta}) + {}^{s\boldsymbol{\Theta}}\boldsymbol{h}^i_k && \text{on } \omega\times\{0\}
\end{aligned}$$

And $\boldsymbol{h}^i_k, {}^{\Xi}\boldsymbol{h}^i_k, {}^{1\boldsymbol{\Theta}}\boldsymbol{h}^i_k, {}^{s\boldsymbol{\Theta}}\boldsymbol{h}^i_k$ are computed via the Fredholm-alternative, see Remark 10 in Section 7, i.e. the solvability condition for (2.35), (2.36), (2.37) and (2.38) in the weak sense, hence

$$\boldsymbol{h}_k^i = -\left(\frac{1}{|\omega|}\int_\omega -\tilde{L}_y^0(\boldsymbol{N}_k)_i - \tilde{L}_y^1(\boldsymbol{N}_{k-2})_i - \operatorname{div}_{\tilde{y}} \tilde{G}_{yij}^0(\widetilde{\boldsymbol{N}}_{k-1}) d\tilde{y}\right) \tag{2.39}$$

$$\begin{aligned} {}^{\Xi}\boldsymbol{h}_k^i = &-\left(\frac{1}{|\omega|}\int_{\omega\times(-\infty,0)} -L_y^0(\boldsymbol{\Xi}_{k-1})_i - L_y^1(\boldsymbol{\Xi}_{k-2})_i - \operatorname{div}_{\tilde{y}} G_{yij}^0(\boldsymbol{\Xi}_{k-1}) dy\right) \\ &-\left(\frac{1}{|\omega|}\int_\omega -G_{i3}^0(\boldsymbol{\Xi}_{k-1})|_{y_3=0} n_3 - G_{i3}^{-1}(\boldsymbol{N}_k) n_3 - G_{i3}^0(\boldsymbol{N}_{k-1}) n_3 d\tilde{y}\right) \end{aligned} \tag{2.40}$$

$$\begin{aligned} {}^{1\boldsymbol{\Theta}}\boldsymbol{h}_k^i = &-\left(\frac{1}{|\omega|}\int_{\omega\times(-\infty,0)} -L_y^0({}^1\boldsymbol{\Theta}_{k-1-\beta})_i - L_y^1({}^1\boldsymbol{\Theta}_{k-2-\beta})_i - \operatorname{div}_{\tilde{y}} G_{ij}^0({}^1\boldsymbol{\Theta}_{k-1-\beta}) dy\right) \\ &-\left(\frac{1}{|\omega|}\int_\omega -G_{i3}^0({}^1\boldsymbol{\Theta}_{k-1-\beta})|_{y_3=0} n_3 + s_i(\boldsymbol{N}_{k-\beta}^i + \boldsymbol{\Xi}_{k-\beta}^i|_{y_3=0}) d\tilde{y}\right) \end{aligned} \tag{2.41}$$

$$\begin{aligned} {}^{s\boldsymbol{\Theta}}\boldsymbol{h}_k^i = &-\left(\frac{1}{|\omega|}\int_{\omega\times(-\infty,0)} -L_y^0({}^s\boldsymbol{\Theta}_{k-1-s\beta})_i - L_y^1({}^s\boldsymbol{\Theta}_{k-2-s\beta})_i - \operatorname{div}_{\tilde{y}} G_{yij}^0({}^s\boldsymbol{\Theta}_{k-1-s\beta}) dy\right) \\ &-\left(\frac{1}{|\omega|}\int_\omega -G_{yi3}^0({}^s\boldsymbol{\Theta}_{k-1-s\beta})|_{y_3=0} n_3 + s_i({}^{s-1}\boldsymbol{\Theta}_{k-1-s\beta}^i|_{y_3=0}) d\tilde{y}\right) \end{aligned} \tag{2.42}$$

Finally in Ω we obtain

$$\begin{aligned} &\sum_{k=0}^\infty \varepsilon^{k-2}(L^{-1}(\boldsymbol{u}^k)_i + L^0(\boldsymbol{u}^{k-1})_i + L^1(\boldsymbol{u}^{k-2})_i) \\ &= \sum_{k=0}^\infty \varepsilon^{k-2}((L_y^{-1}(\boldsymbol{N}^k)_i + L_y^0(\boldsymbol{N}_{k-1})_i + L_y^1(\boldsymbol{N}_{k-2})_i) + L_y^{-1}(\boldsymbol{\Xi}_k)_i + L_y^0(\boldsymbol{\Xi}_{k-1})_i + L_y^1(\boldsymbol{\Xi}_{k-2})_i))\frac{\partial^k \boldsymbol{u}^0}{\partial x_3^k} \\ &+ \sum_{k=0}^\infty \varepsilon^{k-2}\sum_{s=1}^\infty((L_y^{-1}({}^s\boldsymbol{\Theta}_{k-s})_i + L_y^0({}^s\boldsymbol{\Theta}_{k-1-s})_i + L_y^1({}^s\boldsymbol{\Theta}_{k-2-s})_i))\frac{\partial^{k-s}\boldsymbol{u}^0}{\partial x_3^{k-s}} \\ &= \sum_{k=0}^\infty \varepsilon^{k-2}\boldsymbol{h}_k^i \cdot \frac{\partial^k \boldsymbol{u}^0}{\partial x_3^k} \end{aligned}$$

and on S_C we get

$$
\begin{aligned}
&\sum_{k=0}^{\infty}\varepsilon^{k-1}\left(\sum_{j=1}^{3}(G_{ij}^{-1}(\boldsymbol{u}^k)+G_{ij}^{0}(\boldsymbol{u}^{k-1}))n_j - s_i u_i^{k-1}\right)\\
&=\sum_{k=0}^{\infty}\varepsilon^{k-1}\left(\sum_{j=1}^{3}((G_{yij}^{-1}(\boldsymbol{N}_k)+G_{yij}^{0}(\boldsymbol{N}_{k-1})+G_{yij}^{-1}(\boldsymbol{\Xi}_k)+G_{yij}^{0}({}^1\boldsymbol{\Xi}_k))n_j)\frac{\partial^k\boldsymbol{u}^0}{\partial x_3^k}\right)\\
&=\sum_{k=0}^{\infty}\varepsilon^{k-1}\left(\sum_{j=1}^{3}(G_{yij}^{-1}({}^1\boldsymbol{\Theta}_{k-1})+G_{yij}^{0}({}^1\boldsymbol{\Theta}_{k-2}))n_j - s_i(\boldsymbol{N}_{k-1}^i+\boldsymbol{\Xi}_{k-1}))\frac{\partial^{k-1}\boldsymbol{u}^0}{\partial x_3^{k-1}}\right)\\
&=\sum_{k=0}^{\infty}\sum_{s=2}^{\infty}\varepsilon^{k-1}\left(\sum_{j=1}^{3}(G_{yij}^{-1}({}^s\boldsymbol{\Theta}_{k-s})+G_{yij}^{0}({}^s\boldsymbol{\Theta}_{k-1-s}))n_j - s_i({}^{s-1}\boldsymbol{\Theta}_{k-s})\frac{\partial^{k-s}\boldsymbol{u}^0}{\partial x_3^{k-s}}\right)\\
&=\sum_{k=0}^{\infty}\varepsilon^{k-1}\left({}^{\Xi}\boldsymbol{h}_k^i\cdot\frac{\partial^k\boldsymbol{u}^0}{\partial x_3^k}+\sum_{s=1}^{\infty}{}^{s\boldsymbol{\Theta}}\boldsymbol{h}_k^i\cdot\frac{\partial^{k-s\beta}\boldsymbol{u}^0}{\partial x_3^{k-s\beta}}\right)
\end{aligned}
$$

□

Lemma 1. $\boldsymbol{h}_0^i, \boldsymbol{h}_1^i, {}^{\Xi}\boldsymbol{h}_0^i, {}^{1\boldsymbol{\Theta}}\boldsymbol{h}_0^i, {}^{s\boldsymbol{\Theta}}\boldsymbol{h}_0^i = 0$ *for* $i = 1, 2, 3$

Proof. By (2.39), (2.40), (2.41), (2.42) and (2.21) the Lemma is proven. □

Proposition 3. (Limiting 1D equation) *For* $\varepsilon \to 0$ (2.26) *reads for i=3:*

$$
\begin{aligned}
&\boldsymbol{h}_2^3\cdot\frac{\partial^2\boldsymbol{u}^0}{\partial x_3^2} = f_3(x_3) && x_3\in(0,1)\\
&{}^{\Xi}\boldsymbol{h}_1^3\cdot\frac{\partial\boldsymbol{u}^0}{\partial x_3}+\sum_{s=1}^{\infty}{}^{s\boldsymbol{\Theta}}\boldsymbol{h}_1^3\cdot\frac{\partial^{1-s\beta}\boldsymbol{u}^0}{\partial x_3^{1-s\beta}} = 0 && x_3 = 1
\end{aligned}
\tag{2.43}
$$

Proof. With Lemma 1 we get:

$$
\begin{aligned}
&\lim_{\varepsilon\to 0}\sum_{k=0}^{\infty}\varepsilon^{k-2}\boldsymbol{h}_k^3\cdot\frac{\partial^k\boldsymbol{u}^0}{\partial x_3^k} = \boldsymbol{h}_2^3\cdot\frac{\partial^2\boldsymbol{u}^0}{\partial x_3^2}\\
&\lim_{\varepsilon\to 0}\sum_{k=0}^{\infty}\varepsilon^{k-1}\left({}^{\Xi}\boldsymbol{h}_k^3\cdot\frac{\partial^k\boldsymbol{u}^0}{\partial x_3^k}+\sum_{s=1}^{\infty}{}^{s\boldsymbol{\Theta}}\boldsymbol{h}_k^3\cdot\frac{\partial^{k-s\beta}\boldsymbol{u}^0}{\partial x_3^{k-s\beta}}\right) = {}^{\Xi}\boldsymbol{h}_1^3\cdot\frac{\partial\boldsymbol{u}^0}{\partial x_3}+\sum_{s=1}^{\infty}{}^{s\boldsymbol{\Theta}}\boldsymbol{h}_1^3\cdot\frac{\partial^{1-s\beta}\boldsymbol{u}^0}{\partial x_3^{1-s\beta}}
\end{aligned}
$$

□

(2.43) is the limit equation for the tension of the fiber with contact at the right end. At this stage we emphasize that this 1D Robin-type problem for the leading term u_3^0 includes data from the problems for $\boldsymbol{u}^1$ and $\boldsymbol{u}^2$ contained in the terms $\boldsymbol{h}_2^3$, ${}^{\Xi}\boldsymbol{h}_1^3$ and ${}^{s\boldsymbol{\Theta}}\boldsymbol{h}_1^3$.

Lemma 2.

$$\boldsymbol{h}_2^3 = -E\boldsymbol{e}_3 \tag{2.44}$$

From (2.39) we know:

$$\boldsymbol{h}_2^3 = -\frac{1}{|\omega|}\int_\omega \lambda\,\mathrm{div}_{\tilde{y}}(\widetilde{\boldsymbol{N}}_1) + (\lambda+2\mu)\boldsymbol{N}_0^3 d\tilde{y} \tag{2.45}$$

To compute $\boldsymbol{h}_2^3$ we need to obtain $\widetilde{\boldsymbol{N}}_1$. For that reason we consider the chain for $\boldsymbol{N}_k$ and remark that (2.35) reads componentwise:

For $i = 1, 2$:

$$\begin{aligned} -\,\mathrm{div}_{\tilde{y}}(\boldsymbol{\sigma}(\tilde{\boldsymbol{N}}_k))_i &= \mu\boldsymbol{N}_{k-2}^i + (\lambda+\mu)(\frac{\partial}{\partial y_i}\boldsymbol{N}_{k-1}^3) + \boldsymbol{h}_k^i && \text{in } \omega \\ \boldsymbol{\sigma}(\tilde{\boldsymbol{N}}_k)_i \cdot \tilde{\boldsymbol{\nu}} &= -n_i\lambda\boldsymbol{N}_{k-1}^3 && \text{on } \partial\omega \end{aligned} \tag{2.46}$$

We notice that on $\partial\omega$ $\boldsymbol{\nu} = (n_1, n_2, 0)$ and that $\tilde{\boldsymbol{N}}_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$.
For $k = 1$ $i = 1, 2$ (2.46) reads:

$$\begin{aligned} -\,\mathrm{div}(\boldsymbol{\sigma}(\tilde{\boldsymbol{N}}_1))_i &= 0 && \text{in } \omega \\ \boldsymbol{\sigma}(\tilde{\boldsymbol{N}}_1)_i \cdot \tilde{\boldsymbol{\nu}} &= -n_i\lambda\boldsymbol{e}_3 && \text{on } \partial\omega \end{aligned} \tag{2.47}$$

Next we proceed with (2.47). (2.47) is solved analytically by $\boldsymbol{N}_1^i = (0, 0, -\frac{\lambda}{2(\lambda+\mu)}y_i)$for $i = 1, 2$. Indeed, we introduce some new operators in order to get clear problems for the components of $\widetilde{\boldsymbol{N}}_k^i$.

Definition 8. (Decoupled differential operators for the rows of $\widetilde{\boldsymbol{N}}_k^i$)

$$\tilde{B}_i^1(\boldsymbol{N}_k^1) = n_1\mu\left(\frac{\partial(\boldsymbol{N}_k^1)}{\partial y_i} + \frac{\partial(\boldsymbol{N}_k^1)}{\partial y_1}\delta_{i1}\right) + \lambda n_1\frac{\partial(\boldsymbol{N}_k^1)}{\partial y_1}\delta_{i1} + n_2\mu\frac{\partial(\boldsymbol{N}_k^1)}{\partial y_2}\delta_{i1} + \lambda n_2\frac{\partial(\boldsymbol{N}_k^1)}{\partial y_1}\delta_{i2}$$
$$\tilde{B}_i^2(\boldsymbol{N}_k^2) = n_2\mu\left(\frac{\partial(\boldsymbol{N}_k^2)}{\partial y_i} + \frac{\partial(\boldsymbol{N}_k^2)}{\partial y_2}\delta_{i2}\right) + \lambda n_2\frac{\partial(\boldsymbol{N}_k^2)}{\partial y_2}\delta_{i2} + n_1\mu\frac{\partial(\boldsymbol{N}_k^2)}{\partial y_1}\delta_{i2} + \lambda n_1\frac{\partial(\boldsymbol{N}_k^2)}{\partial y_2}\delta_{i1}$$
$$\tilde{C}_i^1(\boldsymbol{N}_k^1) = -\mu\left(\frac{\partial^2(\boldsymbol{N}_k^1)}{\partial y_1\partial y_i} + \frac{\partial^2(\boldsymbol{N}_k^1)}{\partial y_1^2}\delta_{i1}\right) - \lambda\frac{\partial^2(\boldsymbol{N}_k^1)}{\partial y_1^2}\delta_{i1} - \mu\frac{\partial^2(\boldsymbol{N}_k^1)}{\partial y_2^2}\delta_{i1} - \lambda\frac{\partial^2(\boldsymbol{N}_k^1)}{\partial y_2\partial y_1}\delta_{i2}$$
$$\tilde{C}_i^2(\boldsymbol{N}_k^2) = -\mu\left(\frac{\partial^2(\boldsymbol{N}_k^2)}{\partial y_2\partial y_i} + \frac{\partial^2(\boldsymbol{N}_k^2)}{\partial y_2^2}\delta_{i2}\right) - \lambda\frac{\partial^2(\boldsymbol{N}_k^2)}{\partial y_2^2}\delta_{i2} - \mu\frac{\partial^2(\boldsymbol{N}_k^2)}{\partial y_1^2}\delta_{i2} - \lambda\frac{\partial^2(\boldsymbol{N}_k^2)}{\partial y_1\partial y_2}\delta_{i1}$$

In terms of these operators (2.47) reads:
For $i = 1, 2$:

$$\begin{aligned} \tilde{C}_i^1(n_1^{11}) + \tilde{C}_i^2(n_1^{21}) &= 0 && \text{in } \omega \\ \tilde{B}_i^1(n_1^{11}) + \tilde{B}_i^2(n_1^{21}) &= 0 && \text{on } \partial\omega \end{aligned} \tag{2.48}$$

and

$$\begin{aligned} \tilde{C}_i^1(n_1^{12}) + \tilde{C}_i^2(n_1^{22}) &= 0 && \text{in } \omega \\ \tilde{B}_i^1(n_1^{12}) + \tilde{B}_i^2(n_1^{22}) &= 0 && \text{on } \partial\omega \end{aligned} \tag{2.49}$$

and

$$\begin{aligned} \tilde{C}_i^1(n_1^{13}) + \tilde{C}_i^2(n_1^{23}) &= 0 && \text{in } \omega \\ \tilde{B}_i^1(n_1^{13}) + \tilde{B}_i^2(n_1^{23}) &= -n_i\lambda && \text{on } \partial\omega \end{aligned} \tag{2.50}$$

(2.48), (2.49) and (2.50) are solved analytically by

$$n_1^{i3} = -\frac{\lambda}{2(\lambda+\mu)}y_i, \quad n_1^{i1} = n_1^{i2} = 0 \text{ for } i = 1, 2 \tag{2.51}$$

see [Panasenko, 2005, p.95]. Indeed, (2.48) and (2.49) are homogeneous problems solved trivially by the zero function and $\tilde{C}_i^1(n_1^{13}) + \tilde{C}_i^2(n_1^{23}) = 0$ is clear since $\tilde{C}_i^1$ and $\tilde{C}_i^2$

are second-order differential operators and $n_1^{i3} = \mathcal{O}(y_i)$, i=1,2. It is left to show that $\tilde{B}_i^1(n_1^{13}) + \tilde{B}_i^2(n_1^{23}) = -n_i\lambda$. W.O.L.G. lets assume $i = 1$.

$$\begin{aligned}
\tilde{B}_i^1(n_1^{13}) + \tilde{B}_i^2(n_1^{23}) &= (2\mu + \lambda)n_1 \frac{\partial n_1^{13}}{\partial y_1} + \lambda n_1 \frac{\partial n_1^{23}}{\partial y_2} \\
&= -\frac{(2\mu + \lambda)\lambda + \lambda^2}{2(\lambda + \mu)} n_1 \\
&= -\lambda n_1
\end{aligned}$$

Proof. (of Lemma 2) With (2.45) and (2.51) we get

$$\begin{aligned}
\boldsymbol{h}_2^3 &= -\frac{1}{|\omega|}\int_\omega \lambda \sum_{k=1}^{2} \frac{\partial}{\partial y_k}(\boldsymbol{N}_1^k) + (\lambda+2\mu)\boldsymbol{e}_3 d\tilde{y} \\
&= -\frac{1}{|\omega|}\int_\omega -\frac{\lambda^2}{2(\lambda+\mu)} \sum_{k=1}^{2} \frac{\partial}{\partial y_k}(0,0,y_k) + (\lambda+2\mu)\boldsymbol{e}_3 d\tilde{y} \\
&= -\frac{1}{|\omega|}\int_\omega -\frac{\lambda^2}{\lambda+\mu}\boldsymbol{e}_3 + (\lambda+2\mu)\boldsymbol{e}_3 d\tilde{y} \\
&= -E\boldsymbol{e}_3
\end{aligned}$$

Which is a well known result, see [Panasenko, 2005, p.91]. □

Lemma 3. *${}^{\Xi}\boldsymbol{h}_1^2 = n_3 E\boldsymbol{e}_3$, ${}^{1}\boldsymbol{\Theta}\boldsymbol{h}_1^3 = s_3\boldsymbol{N}_{1-\beta}^3$ and ${}^{s}\boldsymbol{\Theta}\boldsymbol{h}_1^3 = 0$ for $s \geq 2$.*

Proof. (2.40) reads:

$$ {}^{\Xi}\boldsymbol{h}_1^2 = G_{33}^{-1}(\boldsymbol{N}_1)n_3 + G_{33}^{0}(\boldsymbol{N}_0)n_3 \tag{2.52}$$

Hence

$$ {}^{\Xi}\boldsymbol{h}_1^3 = \lambda \sum_{k=1}^{2} \frac{\partial}{\partial y_k}(\boldsymbol{N}_1^k)n_3 + (\lambda+2\mu)n_3\boldsymbol{e}_3 = n_3\boldsymbol{e}_3 E$$

which is a well known result, see for instance [Panasenko, 2005, p.91].

By (2.41) and (2.42) we have ${}^{1}\boldsymbol{\Theta}\boldsymbol{h}_1^3 = s_3\boldsymbol{N}_{1-\beta}^3$ and ${}^{s}\boldsymbol{\Theta}\boldsymbol{h}_1^3 = 0$ for $s \geq 2$.

□

Proof. (of Theorem 1) With $\boldsymbol{h}_2^3, {}^{\Xi}\boldsymbol{h}_1^3, {}^{1}\boldsymbol{\Theta}\boldsymbol{h}_1^3, {}^{s}\boldsymbol{\Theta}\boldsymbol{h}_1^3$ and $\boldsymbol{h}_0^i = \boldsymbol{h}_1^i$ computed and $n_3 = 1$ (2.43) yields:

$$\begin{aligned}
-E\frac{\partial^2 u_3^0}{\partial x_3^2} &= f_3(x_3) && x_3 \in (0,1) \\
E\frac{\partial u_3^0}{\partial x_3} - s_3 u_3^0 &= 0 && x_3 = 1
\end{aligned} \tag{2.53}$$

for $\beta = 1$ and

$$-E\frac{\partial^2 u_3^0}{\partial x_3^2} = f_3(x_3) \qquad x_3 \in (0,1) \tag{2.54}$$
$$E\frac{\partial u_3^0}{\partial x_3} = 0 \qquad x_3 = 1$$

for $\beta = 2$.

Since by ansatz 2 $u_3^0 \in V_0$, (2.53) and (2.54) yield the following variational formulations

$$\int_{(0,1)} E\frac{\partial u_3^0}{\partial x_3}\frac{\partial v}{\partial x_3}dx_3 - s_3 u_3^0(1)v(1) = \int_{(0,1)} f_3(x_3)v dx_3 \qquad \forall v \in V_0$$

for $\alpha = 0$ and

$$\int_{(0,1)} E\frac{\partial u_3^0}{\partial x_3}\frac{\partial v}{\partial x_3}dx_3 = \int_{(0,1)} f_3(x_3)v dx_3 \qquad \forall v \in V_0$$

for $\alpha = 1$.

□

Proposition 4. (Existence of a strong solution to the limit-problem) (2.53) *and* (2.54) *have a strong solution. We show that the corresponding homogeneous problems of the second order ordinary boundary value problems* (2.53) *and* (2.54) *have only the trivial solution, then by [Denk,2007/08, Theorem 6.8, p.64]* (2.53) *and* (2.54) *are solvable in the strong sense.*

Proof. The strong homogeneous equation of (2.54) is

$$-E\frac{\partial^2 u_3^0}{\partial x_3^2} = f_3(x_3) \qquad x_3 \in (0,1) \tag{2.55}$$
$$E\frac{\partial u_3^0}{\partial x_3} - s_3 u_3^0 = 0 \qquad x_3 = 1 \tag{2.56}$$
$$u_3^0 = 0 \qquad x_3 = 0 \tag{2.57}$$

Lets proceed with the ansatz: $u_3^0(x_3) = ax_3 + b$ with $a, b \in \mathbb{R}$. Then (2.57) implies $b = 0$. And (2.56) implies $E\frac{\partial u_3^0}{\partial x_3} - s_3 u_3^0 = Ea - s_3(a+b) = Ea(1-s_3) = 0$ hence $a = 0$. For (2.54) the only difference is the equation $E\frac{\partial u_3^0}{\partial x_3} = 0$, hence again $a = 0$. □

Corollary 1. (Existence of a weak solution) *Since every strong solution is also a weak* *solution, see [Braess, 1997, p.27],* (2.53) *and* (2.54) *are solvable in the weak sense.*

3 3D contact problem for an elastic beam

In this section notations, physical assumptions and results from the previous sections are used. In the following we use the symbol ˆ to identify terms that differ from terms of the previous section, the displacement for instance is denoted by $\hat{\boldsymbol{u}}$. Further the same dimension reduction procedure is implemented. The idea of the section is to analyze the contact of an elastic beam with a rigid one, see Figure 3.1. Further in this section we accentuate that the contact must be modeled at an end of the beam. Therefore the beam is cut in two halves at the contact area. By [Panasenko, 2005, p.78] we know that the results obtained for one half apply to the case of the junction of the two halves when additionally the transmission conditions $[\boldsymbol{u}^\varepsilon] = 0$ and $[\boldsymbol{\Sigma}^\varepsilon \boldsymbol{u}^\varepsilon] = 0$ are imposed on the junction area, where $[\cdot]$ denotes the jump between the two halves, see [Panasenko, 2005, p.58 and 73]. Therefore W.L.O.G. only the left half of the original beam is considered.

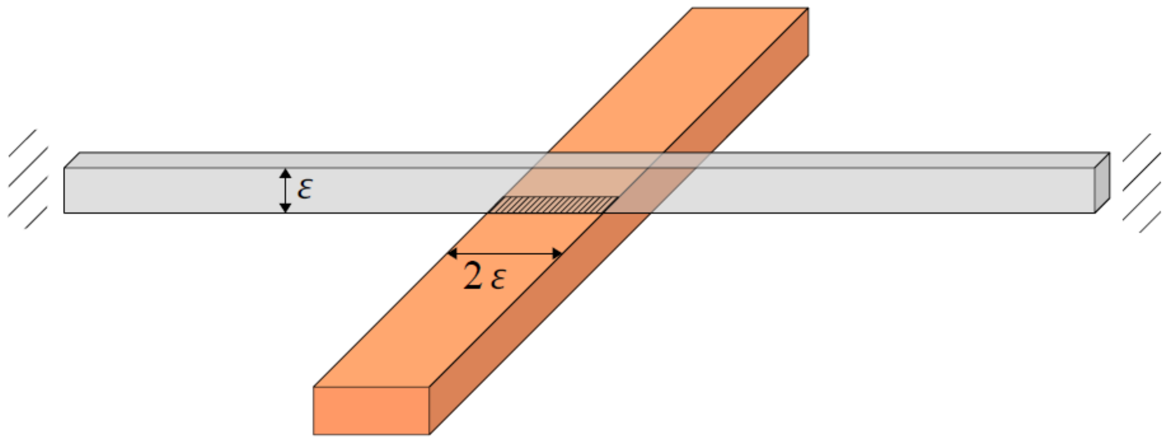


Figure 3.1: Elastic fiber in contact with a rigid one

3.1 Statement of the problem

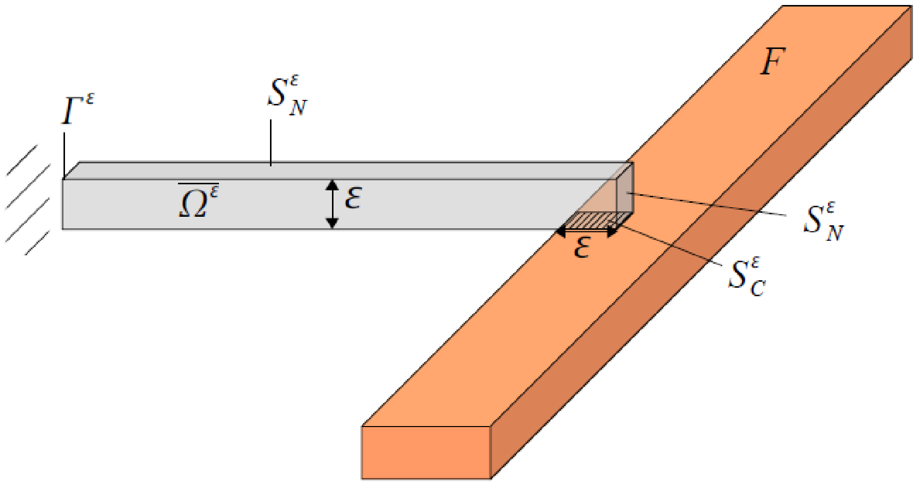
We consider a 3D contact elasticity problem for a thin rod occupying the domain $\hat{\Omega}^\varepsilon$ with a rigid fiber $\mathcal{F}$ at the contact surface $\hat{S}^\varepsilon_C$, see Figure 3.2. The thin rod is clamped at its left end Γ^ε and the contact surface is located at an ε surrounding on a section of its lateral boundary at its right end. On the remaining part of the boundary no traction is considered.

Assumptions 4. (Geometrical)
Let the fiber be given by $\hat{\Omega}^\varepsilon = \{(x_1,x_2,x_3) \in \mathbb{R}^3 : (x_1,x_2) \in \varepsilon\omega, x_3 \in (-1,0)\}$ *where the cross-section* $\omega \subset \mathbb{R}^2$ *is a symmetric domain with smooth boundary. The free boundary of the fiber is denoted by*
$\hat{S}^\varepsilon_N = A^\varepsilon \cup B^\varepsilon \cup C^\varepsilon$ *with* $A^\varepsilon = \left\{(x_1,x_2,x_3) \in \overline{\hat{\Omega}}^\varepsilon : (\frac{x_1}{\varepsilon}, \frac{x_2}{\varepsilon}) \in \partial\omega, x_3 \in (-1,-\varepsilon)\right\}$, $B^\varepsilon = \left\{(x_1,x_2,x_3) \in \overline{\hat{\Omega}}^\varepsilon : (x_1,x_2) \in \varepsilon\omega, x_3 = 0\right\}$ *and*
$C^\varepsilon = \left\{(x_1,x_2,x_3) \in \overline{\hat{\Omega}}^\varepsilon : (\frac{x_1}{\varepsilon}, \frac{x_2}{\varepsilon}) \in \partial\omega_N, x_3 \in [-\varepsilon,0]\right\}$ *the left end is given by* $\Gamma^\varepsilon = \left\{(x_1,x_2,x_3) \in \overline{\hat{\Omega}}^\varepsilon : x_3 = -1\right\}$. *The contact area is denoted as*
$\hat{S}^\varepsilon_C = \left\{(x_1,x_2,x_3) \in \overline{\hat{\Omega}}^\varepsilon : (\frac{x_1}{\varepsilon}, \frac{x_2}{\varepsilon}) \in \partial\omega_R, x_3 \in [-\varepsilon,0]\right\}$. *The diameter of the cross-section is given by the small parameter* $0 < \varepsilon << 1$. *The robin parameters* s_i^ε *are scaled as* $s_i\varepsilon^\alpha$ *for* $i = 1,2$ *and* $s_3^\varepsilon = s_3^{\alpha-2}$. *The body force* $\hat{\boldsymbol{f}}^\varepsilon$ *is scaled as* $\hat{\boldsymbol{f}}^\varepsilon(x_3) = (\varepsilon^\alpha \hat{f}_1(x_3), \varepsilon^\alpha \hat{f}_2(x_3), \varepsilon^{\alpha-2}\hat{f}_3(x_3))^T$.

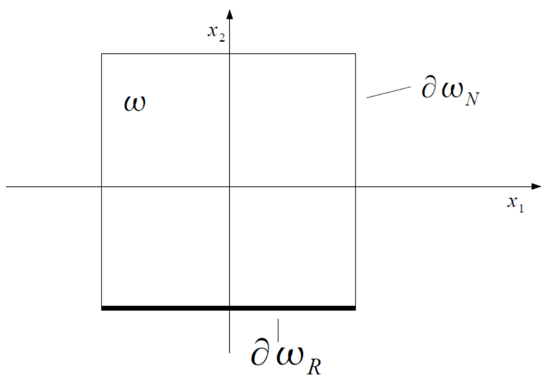
$$\begin{aligned} -\operatorname{Div} \hat{\boldsymbol{\Sigma}}^\varepsilon &= \hat{\boldsymbol{f}}^\varepsilon && \text{in } \in \hat{\Omega}^\varepsilon \\ \hat{\boldsymbol{\Sigma}}^\varepsilon \boldsymbol{\nu} &= \boldsymbol{S}^\varepsilon \hat{\boldsymbol{u}}^\varepsilon && \text{on } \hat{S}^\varepsilon_C \\ \hat{\boldsymbol{\Sigma}}^\varepsilon \boldsymbol{\nu} &= 0 && \text{on } \hat{S}^\varepsilon_N \\ \hat{\boldsymbol{u}}^\varepsilon &= 0 && \text{on } \Gamma^\varepsilon \end{aligned} \tag{3.1}$$

As in the previous section $\boldsymbol{S}^\varepsilon = \begin{pmatrix} \varepsilon^\alpha s_1 & 0 & 0 \\ 0 & \varepsilon^\alpha s_2 & 0 \\ 0 & 0 & \varepsilon^{\alpha-2} s_3 \end{pmatrix}$ is the robin parameter matrix, $\boldsymbol{\nu} = (n_1,n_2,n_3)$ is the outward unit normal, $e(\hat{\boldsymbol{u}}^\varepsilon)_{ij} = \frac{1}{2}\left(\frac{\partial \hat{u}^\varepsilon_j}{\partial x_i} + \frac{\partial \hat{u}^\varepsilon_i}{\partial x_j}\right)$ for $i,j = 1,2,3$ are the components of the strain tensor and $\hat{\boldsymbol{\Sigma}}^\varepsilon = (\hat{\sigma}^\varepsilon_{ij})_{i,j=1,2,3}$ is the stress tensor with components $\hat{\sigma}^\varepsilon_{ij} = 2\mu e\left(\hat{\boldsymbol{u}}^\varepsilon\right)_{ij} + \lambda \operatorname{div}\left(\hat{\boldsymbol{u}}^\varepsilon\right)\delta_{ij}$ with the Lame constants λ and μ.

We recall that as in the previous section the solution of (3.1) is sought as a solution in the weak sense. Therefore we seek for functions $\hat{u}^\varepsilon_i \in V^\varepsilon = \left\{v \in H^1(\hat{\Omega}^\varepsilon) : v|_\Gamma = 0\right\}$



(a)



(b)

Figure 3.2: a) Elastic rod in contact with rigid rod , b) Zoomed cross-section at the contact area

satisfying (3.2).

$$\sum_{j=1}^{3}\int_{\hat{\Omega}^\varepsilon}\hat{\sigma}_{ij}^\varepsilon\frac{\partial}{\partial x_j}v dx-\int_{\hat{S}_C}s_i\hat{u}_i^\varepsilon v dx=\int_{\hat{\Omega}^\varepsilon}\hat{f}_i^\varepsilon v dx\qquad\forall v\in V^\varepsilon \tag{3.2}$$

3.2 Asymptotics

We perform a change of variables and substitute the displacement by a two scale asymptotic expansion with respect to the fiber thickness.
$x_i \mapsto y_i = \frac{x_i}{\varepsilon}$ for $i = 1, 2$, and $x_3 \mapsto (y_3 = \frac{x_3}{\varepsilon}, x_3)$.

Ansatz 3. (Two scale expansion)

$$\hat{\boldsymbol{u}}^\varepsilon(x_1,x_2,x_3)=\hat{\boldsymbol{u}}^\infty(y_1,y_2,y_3,x_3) \tag{3.3}$$

with

$$\hat{\boldsymbol{u}}^\infty(y_1,y_2,y_3,x_3)=\sum_{i=0}^{\infty}\varepsilon^i\hat{\boldsymbol{u}}^i(\boldsymbol{y},x_3)$$

Substituting (3.3) into (3.1) reads:

$$\begin{aligned}
-\operatorname{Div}\hat{\boldsymbol{\Sigma}}^\infty &= \hat{\boldsymbol{f}}^\varepsilon && \text{in } \hat{\Omega}\\
\hat{\boldsymbol{\Sigma}}^\infty\boldsymbol{\nu} &= \boldsymbol{S}^\varepsilon\hat{\boldsymbol{u}}^\infty && \text{on } \hat{S}_C\\
\hat{\boldsymbol{\Sigma}}^\infty\boldsymbol{\nu} &= 0 && \text{on } \hat{S}_N\\
\hat{\boldsymbol{u}}^\infty &= 0 && \text{on } \Gamma
\end{aligned} \tag{3.4}$$

where

$$\begin{aligned}
\hat{\Omega} &= \left\{(y_1, y_2, y_3, x_4) \in \mathbb{R}^4 : (y_1, y_2) \in \omega, y_3 \in (-\infty, 0), x_3 \in (-1, 0)\right\} \\
\hat{S}_C &= \Big\{(y_1, y_2, y_3, x_3) \in \overline{\hat{\Omega}} : (y_1, y_2) \in \partial\omega, y_3 \in [-1, 0], x_3 \in [-\varepsilon, 0]\Big\} \\
\hat{S}_N &= A \cup B \cup C \\
C &= \Big\{(y_1, y_2, y_3, x_3) \in \overline{\hat{\Omega}} : (y_1, y_2) \in \partial\omega_N, y_3 \in [-1, 0] x_3 \in [-\varepsilon, 0]\Big\} \\
B &= \Big\{(y_1, y_2, y_3, x_3) \in \overline{\hat{\Omega}} : (y_1, y_2) \in \omega, y_3 = 0, x_3 = 0\Big\} \\
A &= \Big\{(y_1, y_2, y_3, x_3) \in \overline{\hat{\Omega}} : (y_1, y_2) \in \partial\omega, y_3 \in (-\infty, 1), x_3 \in (-1, -\varepsilon)\Big\} \\
\Gamma &= \Big\{(y_1, y_2, y_3, x_3) \in \overline{\hat{\Omega}} : y_3 = -\infty, x_3 = -1\Big\}
\end{aligned}$$

Theorem 2. (The limit problem) *As ε tends to zero the limit problem for the leading term of the solution of* (3.4) *has the following equilibrium equation, and force-and displacement boundary conditions for $\beta = 3$, i.e. $\alpha = 2$.*
For i=1,2

$$\begin{aligned} EI\frac{\partial^4 \hat{u}_i^0}{\partial x_3^4} &= \hat{f}_i(x_3) && x_3 \in (-1,0) \text{ (equilibrium equation)} \\ -EI\frac{\partial^3 \hat{u}_i^0}{\partial x_3^3} - s_i\frac{|\partial\omega_R|}{|\omega|}\hat{u}_i^0 &= 0 && x_3 = 0 \text{ (force boundary condition)} \\ \hat{u}_i^0 &= 0 && x_3 = -1 \text{ (displacement boundary condition)} \end{aligned} \tag{3.5}$$

and for $i = 3$

$$\begin{aligned} -E\frac{\partial^2 \hat{u}_3^0}{\partial x_3^2} &= \hat{f}_3(x_3) && x_3 \in (-1,0) \text{ (equilibrium equation)} \\ E\frac{\partial \hat{u}_3^0}{\partial x_3} - s_3\frac{|\partial\omega_R|}{|\omega|}\hat{u}_3^0 &= 0 && x_3 = 0 \text{ (force boundary condition)} \\ \hat{u}_3^0 &= 0 && x_3 = -1 \text{ (displacement boundary condition)} \end{aligned} \tag{3.6}$$

Where I is the averaged area moment of inertia $I = \frac{1}{|\omega|}\int_\omega y_i^2 d\tilde{y}$. The solution of (3.6) *is understood a function*
$\hat{u}_3^0 \in V_0 = \{v \in H^1((-1,0)) : v(-1) = 0\}$ *satisfying the weak problem of* (3.6)*, see Equation* (2.10) *in Section 2. The solution u_i^0 for $i = 1,2$ of* (3.5) *is understood as a weak solution of the weak formulation of the complete ODE system of* (3.5) *containing also bending-angle and moment:*

$$\int_{(-1,0)} EI\frac{\partial^2 u_i^0}{\partial x_3^2}\frac{\partial^2 v}{\partial x_3^2}dx_3 - s_i\frac{|\partial\omega_R|}{|\omega|}u_i^0(0)v(0) - EI\frac{\partial^2 u_i^0}{\partial x_3^2}\frac{\partial v}{\partial x_3}|_{-1}^0 = \int_{(-1,0)} f_i(x_3)v dx_3 \ \forall v \in V_0$$

for $i = 1,2$, see Remark 7.

This theorem is proven at the end of this section.

Remark 7. (On the bending-angle and moment) *The equations for the bending angle $\frac{\partial u_i}{\partial x_i}$ and the bending moment $EI\frac{\partial^2 u_i}{\partial x_i^2}$ on the fixed- and contact end respectively are still to be asymptotically constructed. This is a subject of the continuation of this work. The deduction of these two boundary conditions requires taking into account the matrix of rigid displacements and its extension, see [Panasenko, 2005, p.90-92]. In [Panasenko, 2005, p.69] the bending angle at fixed end is derived. In this Diploma -thesis for the reason of lack of time the matrix of rigid displacements and its extension are not considered.*

Therefore we only construct asymptotically the equilibrium- and force equations, which do not depend on the neglected matrices, see Subsection 7.2.

Proposition 5. (Dependence on the macroscopic scale of the leading term)

$$\hat{\boldsymbol{u}}^0 = \hat{\boldsymbol{u}}^0(x_3) \tag{3.7}$$

Proof. With defintion 1 we have:

$$\begin{aligned} -\left(\operatorname{Div}\hat{\boldsymbol{\Sigma}}^{\infty}\right)_i &= \sum_{k=0}^{\infty} \varepsilon^{k-2}\left(L^{-1}(\hat{\boldsymbol{u}}^k)_i + L^0(\hat{\boldsymbol{u}}^{k-1})_i + L^1(\hat{\boldsymbol{u}}^{k-2})_i\right) \\ (\hat{\boldsymbol{\Sigma}}^{\infty}\cdot\boldsymbol{\nu})_i &= \sum_{k=0}^{\infty} \varepsilon^{k-1}\left(\sum_{j=1}^{3}\left((G_{ij}^{-1}(\hat{\boldsymbol{u}}^k)) + (G_{ij}^{0}(\hat{\boldsymbol{u}}^{k-1}))\right) n_j\right) \end{aligned} \tag{3.8}$$

With this notation (3.4) takes the form: For $i = 1, 2$

$$\begin{aligned} &\sum_{k=0}^{\infty} \varepsilon^{k-2}\left(L^{-1}(\hat{\boldsymbol{u}}^k) + L^0(\hat{\boldsymbol{u}}^{k-1})_i + L^1(\hat{\boldsymbol{u}}^{k-2})_i\right) = \varepsilon^{\alpha}\hat{f}_i && \text{in } \hat{\Omega} \\ &\sum_{k=0}^{\infty} \varepsilon^{k-1}\left(\sum_{j=1}^{3}\left((G_{ij}^{-1}(\hat{\boldsymbol{u}}^k)) + (G_{ij}^{0}(\hat{\boldsymbol{u}}^{k-1}))\right) n_j\right) = s_i\varepsilon^{\alpha}\sum_{k=0}^{\infty}\varepsilon^{k-1}\hat{u}_i^{k-1} && \text{on } \hat{S}_C \\ &\sum_{k=0}^{\infty} \varepsilon^{k-1}\left(\sum_{j=1}^{3}\left((G_{ij}^{-1}(\hat{\boldsymbol{u}}^k))_i + (G_{ij}^{0}(\hat{\boldsymbol{u}}^{k-1}))_i\right) n_j\right) = 0 && \text{on } \hat{S}_N \\ &\sum_{k=0}^{\infty} \varepsilon^{k}\hat{u}_i^k = 0 && \text{on } \Gamma \end{aligned} \tag{3.9}$$

and for $i = 3$

$$\begin{aligned} &\sum_{k=0}^{\infty} \varepsilon^{k-2}\left(L^{-1}(\hat{\boldsymbol{u}}^k)_3 + L^0(\hat{\boldsymbol{u}}^{k-1}) + L^1(\hat{\boldsymbol{u}}^{k-2})_3\right) = \varepsilon^{\alpha-2}\hat{f}_3 && \text{in } \hat{\Omega} \\ &\sum_{k=0}^{\infty} \varepsilon^{k-1}\left(\sum_{j=1}^{3}\left((G_{3j}^{-1}(\hat{\boldsymbol{u}}^k)) + (G_{3j}^{0}(\hat{\boldsymbol{u}}^{k-1}))\right) n_j\right) = s_3^{\alpha-2}\hat{u}_3^{\infty} && \text{on } \hat{S}_C \\ &\sum_{k=0}^{\infty} \varepsilon^{k-1}\left(\sum_{j=1}^{3}\left((G_{3j}^{-1}(\hat{\boldsymbol{u}}^k)) + (G_{3j}^{0}(\hat{\boldsymbol{u}}^{k-1}))\right) n_j\right) = 0 && \text{on } \hat{S}_N \\ &\hat{u}_3^{\infty} = 0 && \text{on } \Gamma \end{aligned} \tag{3.10}$$

(3.9) and (3.10) induce the following chains in the microscopic scale.

For $i = 1, 2$

$$\begin{aligned}
&\mathbf{BVP}(\hat{\boldsymbol{u}}^k) && \qquad (3.11)\\
&-\operatorname{div}_y(G^{-1}_{ij}(\hat{\boldsymbol{u}}^k)) = -L^0(\hat{\boldsymbol{u}}^{k-1})_i - L^1(\hat{\boldsymbol{u}}^{k-2})_i && \text{in } \omega \times (-\infty, 0)\\
&\sum_{j=1}^{2}(G^{-1}_{ij}(\hat{\boldsymbol{u}}^k))n_j = -\sum_{j=1}^{2}(G^0_{ij}(\hat{\boldsymbol{u}}^{k-1}))n_j && \text{on } \partial\omega \times (-\infty, -1)\\
&\sum_{j=1}^{2}(G^{-1}_{ij}(\hat{\boldsymbol{u}}^k))n_j = -\sum_{j=1}^{2}(G^0_{ij}(\hat{\boldsymbol{u}}^{k-1}))n_j && \text{on } \partial\omega_N \times [-1, 0]\\
&\sum_{j=1}^{2}(G^{-1}_{ij}(\hat{\boldsymbol{u}}^k))n_j = -\sum_{j=1}^{2}(G^0_{ij}(\hat{\boldsymbol{u}}^{k-1}))n_j + s_i\hat{u}_i^{k-\beta} && \text{on } \partial\omega_R \times [-1, 0]\\
&(G^{-1}_{i3}(\hat{\boldsymbol{u}}^k))n_3 = -(G^0_{i3}(\hat{\boldsymbol{u}}^{k-1}))n_3 && \text{on } \omega \times \{0\}
\end{aligned}$$

and for $i = 3$

$$\begin{aligned}
&\mathbf{BVP}(\hat{\boldsymbol{u}}^k) && \qquad (3.12)\\
&-\operatorname{div}_y(G^{-1}_{3j}(\hat{\boldsymbol{u}}^k)) = -L^0(\hat{\boldsymbol{u}}^{k-1})_3 - L^1(\hat{\boldsymbol{u}}^{k-2})_3 && \text{in } \omega \times (-\infty, 0)\\
&\sum_{j=1}^{2}(G^{-1}_{3j}(\hat{\boldsymbol{u}}^k))n_j = -\sum_{j=1}^{2}(G^0_{3j}(\hat{\boldsymbol{u}}^{k-1}))n_j && \text{on } \partial\omega \times (-\infty, -1)\\
&\sum_{j=1}^{2}(G^{-1}_{3j}(\hat{\boldsymbol{u}}^k))n_j = -\sum_{j=1}^{2}(G^0_{3j}(\hat{\boldsymbol{u}}^{k-1}))n_j && \text{on } \partial\omega_N \times [-1, 0]\\
&\sum_{j=1}^{2}(G^{-1}_{3j}(\hat{\boldsymbol{u}}^k))n_j = -\sum_{j=1}^{2}(G^0_{3j}(\hat{\boldsymbol{u}}^{k-1}))n_j + s_3\hat{u}_3^{k-(\beta-2)} && \text{on } \partial\omega_R \times [-1, 0]\\
&(G^{-1}_{33}(\hat{\boldsymbol{u}}^k))n_3 = -(G^0_{33}(\hat{\boldsymbol{u}}^{k-1}))n_3 && \text{on } \omega \times \{0\}
\end{aligned}$$

As in Remark 6 in Section 2 we point out that for $\beta = 0$ and $\beta = 2$ we get a Robin-type problem for the first two components and the third component of the solution respectively. Hence we set $\beta = 3$ in order to obtain a homogeneous Neumann-type problem.

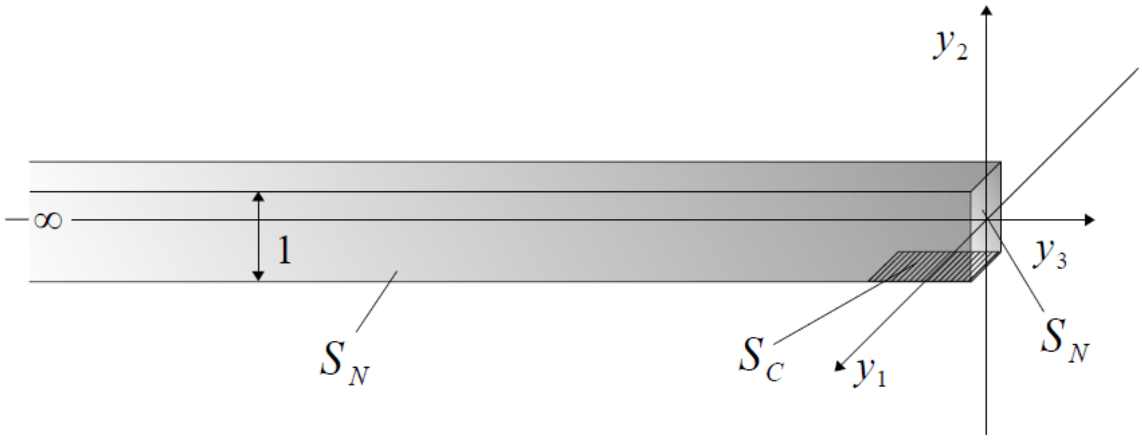


Figure 3.3: 3D rod in the microscale

$$
\begin{aligned}
&\mathbf{BVP}(\hat{\boldsymbol{u}}^0) && && (3.13)\\
&-\operatorname{div}_y(G_{ij}^{-1}(\hat{\boldsymbol{u}}^0)) = 0 && \text{in } \omega\times(-\infty,0) \\
&\sum_{j=1}^{2}(G_{ij}^{-1}(\hat{\boldsymbol{u}}^0))n_j = 0 && \text{on } \partial\omega\times(-\infty,0) \\
&(G_{i3}^{-1}(\hat{\boldsymbol{u}}^0))n_3 = 0 && \text{on } \omega\times\{0\}
\end{aligned}
$$

From the fact that homogeneous Neumann problems admit only constant solutions, see [Evans, 1998, p.346], we deduce:

$$\hat{\boldsymbol{u}}^0 = \hat{\boldsymbol{u}}^0(x_3) \tag{3.14}$$

□

We notice that the scaling $\beta = 3$, i.e. $\alpha = 2$ implies that the body force bending components $\hat{f}_i^\varepsilon$ are scaled as $\hat{f}_i^\varepsilon = \varepsilon^2 \hat{f}_i$ which is often done in the context of the asymptotic dimension reduction of elastic beams, see for instance [Vodak, 2007, p.50] and [Palencia, Hubert, 1999, p.376].

Ansatz 4. (Main ansatz)

$$\hat{\boldsymbol{u}}^{\infty}(y_1,y_2,y_3,x_3) = \sum_{k=0}^{\infty}\varepsilon^k \boldsymbol{N}_k(\tilde{\boldsymbol{y}})\frac{\partial^k \hat{\boldsymbol{u}}^0}{\partial x_3^k} + \sum_{k=0}^{\infty}\varepsilon^k \boldsymbol{\Xi}_k(y_1,y_2,y_3)\frac{\partial^k \hat{\boldsymbol{u}}^0}{\partial x_3^k}(x_3)$$
$$+\sum_{k=0}^{\infty}\varepsilon^k \sum_{s=1}^{\infty} {}^s\hat{\boldsymbol{\Theta}}_{k-s3}(y_1,y_2,y_3)\frac{\partial^{k-s3}\hat{\boldsymbol{u}}^0}{\partial x_3^{k-s3}}(x_3) \tag{3.15}$$

Where $\operatorname{supp}(\boldsymbol{\Xi}_k) \subset \omega \times \mathcal{B}_{\delta}(0)$ and $\operatorname{supp}({}^s\hat{\boldsymbol{\Theta}}_{k-s3}) \subset \omega \times \mathcal{B}_{1+\delta}(0)$ are compact and

$$\boldsymbol{N}_0, {}^s\hat{\boldsymbol{\Theta}}_0 = \boldsymbol{I}, \boldsymbol{N}_k, {}^s\hat{\boldsymbol{\Theta}}_k = 0 \text{ for } k<0, \boldsymbol{\Xi}_k = 0 \text{ for } k \le 0 \tag{3.16}$$

and

$$\hat{\boldsymbol{u}}^0 \in V_0^3 = \left\{\boldsymbol{v} \in H^1((0,1))^3 : \boldsymbol{v}(0) = 0\right\} \tag{3.17}$$

Similar as in the previous section we have
$\varepsilon^3\hat{\boldsymbol{u}}^0 + \mathcal{O}(\varepsilon^4) = \sum_{k=0}^{\infty}\varepsilon^k \sum_{s=1}^{\infty} {}^s\hat{\boldsymbol{\Theta}}_{k-s3}(y_1,y_2,y_3)\frac{\partial^{k-s3}\hat{\boldsymbol{u}}^0}{\partial x_3^{k-s3}}(x_3)$

Proposition 6. (Equation of infinite order for the leading term)
(3.9) *and* (3.10) *imply*

$$\begin{aligned}
&\sum_{k=0}^{\infty}\varepsilon^{k-4}\boldsymbol{h}_k^i \cdot \frac{\partial^k\hat{\boldsymbol{u}}^0}{\partial x_3^k} = \hat{f}_i(x_3) && x_3 \in (-1,0)\\
&\sum_{k=0}^{\infty}\varepsilon^{k-3}\left({}^{\Xi}\boldsymbol{h}_k^i \cdot \frac{\partial^k\hat{\boldsymbol{u}}^0}{\partial x_3^k} + \sum_{s=1}^{\infty} {}^{s\hat{\Theta}}\boldsymbol{h}_k^i \cdot \frac{\partial^{k-s3}\hat{\boldsymbol{u}}^0}{\partial x_3^{k-s3}}\right) = 0 && x_3 = 0
\end{aligned} \tag{3.18}$$

and

$$\begin{aligned}
&\sum_{k=0}^{\infty}\varepsilon^{k-2}\boldsymbol{h}_k^3 \cdot \frac{\partial^k\hat{\boldsymbol{u}}^0}{\partial x_3^k} = \hat{f}_3(x_3) && x_3 \in (-1,0)\\
&\sum_{k=0}^{\infty}\varepsilon^{k-1}\left({}^{\Xi}\boldsymbol{h}_k^3 \cdot \frac{\partial^k\hat{\boldsymbol{u}}^0}{\partial x_3^k} + \sum_{s=1}^{\infty} {}^{s\hat{\Theta}}\boldsymbol{h}_k^3 \cdot \frac{\partial^{k-s3}\hat{\boldsymbol{u}}^0}{\partial x_3^{k-s3}}\right) = 0 && x_3 = 0
\end{aligned} \tag{3.19}$$

Where $\boldsymbol{h}_k^i, {}^{\Xi}\boldsymbol{h}_k^i, {}^{1\hat{\Theta}}\boldsymbol{h}_k^i, {}^{s\hat{\Theta}}\boldsymbol{h}_k^i$ *follow from the solvability condition for* $\hat{\boldsymbol{u}}^k$*.*

Proof. Analogously as in the previous section with ansatz 3.15 we obtain the following chain of boundary value problems in the microscale decoupled from x_3-derivatives.

For i=1,2,3

$$
\begin{aligned}
&\mathbf{BVP}(\boldsymbol{N}_k) && (3.20)\\
&-\operatorname{div}_y(\tilde{G}^{-1}_{yij}(\boldsymbol{N}_k)) = -\tilde{L}^0_y(\boldsymbol{N}_{k-1})_i - \tilde{L}^1_y(\boldsymbol{N}_{k-2})_i + \boldsymbol{h}^i_k && \text{in } \omega\\
&\sum_{j=1}^{2}(\tilde{G}^{-1}_{yij}(\boldsymbol{N}_k))n_j = -\sum_{j=1}^{2}(\tilde{G}^{0}_{yij}(\boldsymbol{N}_{k-1}))n_j && \text{on } \partial\omega
\end{aligned}
$$

and

$$
\begin{aligned}
&\mathbf{BVP}(\boldsymbol{\Xi}_k) && (3.21)\\
&-\operatorname{div}_y(G^{-1}_{yij}(\boldsymbol{\Xi}_k)) = -L^0_y(\boldsymbol{\Xi}_{k-1})_i - L^1_y(\boldsymbol{\Xi}_{k-2})_i && \text{in } \omega\times(-\infty,0)\\
&\sum_{j=1}^{2}(G^{-1}_{yij}(\boldsymbol{\Xi}_k))n_j = -\sum_{j=1}^{2}(G^{0}_{yij}(\boldsymbol{\Xi}_{k-1}))n_j && \text{on } \partial\omega\times(-\infty,0)\\
&G^{-1}_{yi3}(\boldsymbol{\Xi}_k)n_3 = -G^0_{i3}(\boldsymbol{\Xi}_{k-1})n_3 - G^{-1}_{yi3}(\boldsymbol{N}_k)n_3 - G^0_{yi3}(\boldsymbol{N}_{k-1})n_3 + {}^{\Xi}\boldsymbol{h}^i_k && \text{on } \omega\times\{0\}
\end{aligned}
$$

and for $i = 1, 2$

$$
\begin{aligned}
&\mathbf{BVP}({}^1\hat{\boldsymbol{\Theta}}_{k-3}) && (3.22)\\
&-\operatorname{div}_y(G^{-1}_{yij}({}^1\hat{\boldsymbol{\Theta}}_{k-3}) = -L^0_y({}^1\hat{\boldsymbol{\Theta}}_{k-1-3})_i - L^1_y({}^1\hat{\boldsymbol{\Theta}}_{k-2-3})_i && \text{in } \omega\times(-\infty,0)\\
&\sum_{j=1}^{2}(G^{-1}_{yij}({}^1\hat{\boldsymbol{\Theta}}_{k-3})n_j = -\sum_{j=1}^{2}(G^{0}_{yij}({}^1\hat{\boldsymbol{\Theta}}_{k-1-3})n_j && \text{on } \partial\omega\times(-\infty,-1)\\
&\sum_{j=1}^{2}(G^{-1}_{yij}({}^1\hat{\boldsymbol{\Theta}}_{k-3})_i n_j = -\sum_{j=1}^{2}(G^{0}_{yij}({}^1\hat{\boldsymbol{\Theta}}_{k-1-3})_i n_j && \text{on } \partial\omega_N\times[-1,0]\\
&\sum_{j=1}^{2}(G^{-1}_{yij}({}^1\hat{\boldsymbol{\Theta}}_{k-3})n_j = -\sum_{j=1}^{2}(G^{0}_{yij}({}^1\hat{\boldsymbol{\Theta}}_{k-1-3})n_j + s_i(\boldsymbol{N}^i_{k-3} + \boldsymbol{\Xi}^i_{k-3}) && \text{on } \partial\omega_R\times[-1,0]\\
&G^{-1}_{yi3}({}^1\hat{\boldsymbol{\Theta}}_{k-3})n_3 = -G^0_{yi3}({}^1\hat{\boldsymbol{\Theta}}_{k-1-3})n_3 + {}^{1\hat{\boldsymbol{\Theta}}}\boldsymbol{h}^i_k && \text{on } \omega\times\{0\}
\end{aligned}
$$

and

$$\begin{aligned}
&\mathbf{BVP}({}^s\hat{\boldsymbol{\Theta}}_{k-s3}) && (3.23)\\
&-\operatorname{div}_y(G^{-1}_{yij}({}^s\hat{\boldsymbol{\Theta}}_{k-s3}) = -L^0_y({}^s\hat{\boldsymbol{\Theta}}_{k-1-3})_i - L^1_y({}^s\hat{\boldsymbol{\Theta}}_{k-2-s3})_i && \text{in } \omega\times(-\infty,0)\\
&\sum_{j=1}^{2}(G^{-1}_{yij}({}^s\hat{\boldsymbol{\Theta}}_{k-s3})n_j = -\sum_{j=1}^{2}(G^{0}_{yij}({}^s\hat{\boldsymbol{\Theta}}_{k-1-s3})_i n_j && \text{on } \partial\omega\times(-\infty,-1)\\
&\sum_{j=1}^{2}(G^{-1}_{yij}({}^s\hat{\boldsymbol{\Theta}}_{k-s3})n_j = -\sum_{j=1}^{2}(G^{0}_{yij}({}^s\hat{\boldsymbol{\Theta}}_{k-1-s3})n_j && \text{on } \partial\omega_N\times[-1,0]\\
&\sum_{j=1}^{2}(G^{-1}_{yij}({}^s\hat{\boldsymbol{\Theta}}_{k-s3})n_j = -\sum_{j=1}^{2}(G^{0}_{yij}({}^s\hat{\boldsymbol{\Theta}}_{k-1-s3})n_j + s_i({}^{s-1}\hat{\boldsymbol{\Theta}}_{k-s3}) && \text{on } \partial\omega_R\times[-1,0]\\
&G^{-1}_{yi3}({}^s\hat{\boldsymbol{\Theta}}_{k-s3})n_3 = -G^{0}_{yi3}({}^s\hat{\boldsymbol{\Theta}}_{k-1-s3})n_3 + {}^{s\hat{\boldsymbol{\Theta}}}\boldsymbol{h}^i_k && \text{on } \omega\times\{0\}
\end{aligned}$$

and for $i = 3$

$$\begin{aligned}
&\mathbf{BVP}({}^1\hat{\boldsymbol{\Theta}}_{k-1}) && (3.24)\\
&-\operatorname{div}_y(G^{-1}_{y3j}({}^1\hat{\boldsymbol{\Theta}}_{k-1}) = -L^0_y({}^1\hat{\boldsymbol{\Theta}}_{k-2})_3 - L^1_y({}^1\hat{\boldsymbol{\Theta}}_{k-3})_3 && \text{in } \omega\times(-\infty,0)\\
&\sum_{j=1}^{2}(G^{-1}_{y3j}({}^1\hat{\boldsymbol{\Theta}}_{k-1})n_j = -\sum_{j=1}^{2}(G^{0}_{y3j}({}^1\hat{\boldsymbol{\Theta}}_{k-2})n_j && \text{on } \partial\omega\times(-\infty,-1)\\
&\sum_{j=1}^{2}(G^{-1}_{y3j}({}^1\hat{\boldsymbol{\Theta}}_{k-1})n_j = -\sum_{j=1}^{2}(G^{0}_{y3j}({}^1\hat{\boldsymbol{\Theta}}_{k-2})n_j && \text{on } \partial\omega_N\times[-1,0]\\
&\sum_{j=1}^{2}(G^{-1}_{y3j}({}^1\hat{\boldsymbol{\Theta}}_{k-1})n_j = -\sum_{j=1}^{2}(G^{0}_{y3j}({}^1\hat{\boldsymbol{\Theta}}_{k-2})n_j + s_3(\boldsymbol{N}^i_{k-1} + \boldsymbol{\Xi}^3_{k-1}) && \text{on } \partial\omega_R\times[-1,0]\\
&G^{-1}_{y33}({}^1\hat{\boldsymbol{\Theta}}_{k-1})n_3 = -G^{0}_{y33}({}^1\hat{\boldsymbol{\Theta}}_{k-2})n_3 + {}^{1\hat{\boldsymbol{\Theta}}}\boldsymbol{h}^3_k && \text{on } \omega\times\{0\}
\end{aligned}$$

and

$$\mathbf{BVP}({}^s\hat{\boldsymbol{\Theta}}_{k-s}) \tag{3.25}$$

$$\begin{aligned}
&-\operatorname{div}_y(G^{-1}_{y3j}({}^s\hat{\boldsymbol{\Theta}}_{k-s}) = -L^0_y({}^s\hat{\boldsymbol{\Theta}}_{k-1-s})_3 - L^1_y({}^s\hat{\boldsymbol{\Theta}}_{k-2-s})_3 && \text{in } \omega\times(-\infty,0)\\
&\sum_{j=1}^{2}(G^{-1}_{y3j}({}^s\hat{\boldsymbol{\Theta}}_{k-s})n_j = -\sum_{j=1}^{2}(G^{0}_{y3j}({}^s\hat{\boldsymbol{\Theta}}_{k-1-s})n_j && \text{on } \partial\omega\times(-\infty,-1)\\
&\sum_{j=1}^{2}(G^{-1}_{y3j}({}^s\hat{\boldsymbol{\Theta}}_{k-s})n_j = -\sum_{j=1}^{2}(G^{0}_{y3j}({}^s\hat{\boldsymbol{\Theta}}_{k-1-s})n_j && \text{on } \partial\omega_N\times[-1,0]\\
&\sum_{j=1}^{2}(G^{-1}_{y3j}({}^s\hat{\boldsymbol{\Theta}}_{k-s})n_j = -\sum_{j=1}^{2}(G^{0}_{y3j}({}^s\hat{\boldsymbol{\Theta}}_{k-1-s})n_j + s_3({}^{s-1}\hat{\boldsymbol{\Theta}}_{k-s}) && \text{on } \partial\omega_R\times[-1,0]\\
&G^{-1}_{y33}({}^s\hat{\boldsymbol{\Theta}}_{k-s})n_3 = -G^{0}_{y33}({}^s\hat{\boldsymbol{\Theta}}_{k-1-s})n_3 + {}^{s\hat{\boldsymbol{\Theta}}}\boldsymbol{h}^3_k && \text{on } \omega\times\{0\}
\end{aligned}$$

And $\boldsymbol{h}^i_k$, ${}^{\Xi}\boldsymbol{h}^i_k$, ${}^{1\hat{\boldsymbol{\Theta}}}\boldsymbol{h}^i_k$, ${}^{s\hat{\boldsymbol{\Theta}}}\boldsymbol{h}^i_k$ are computed via the Fredholm-alternative, i.e. the solvability condition for (3.20), (3.21), (3.24) and (3.25) in the weak sence, hence
For $i = 1, 2, 3$

$$\boldsymbol{h}^i_k = -\left(\frac{1}{|\omega|}\int_\omega -\tilde{L}^0_y(\boldsymbol{N}_{k-1})_i - \tilde{L}^1_y(\boldsymbol{N}_{k-2})_i - \operatorname{div}_{\tilde{y}}\tilde{G}^0_{yij}(\widetilde{\boldsymbol{N}}_{k-1})d\tilde{y}\right) \tag{3.26}$$

$$\begin{aligned}
{}^{\Xi}\boldsymbol{h}^i_k = &-\left(\frac{1}{|\omega|}\int_{\omega\times(-\infty,0)} -L^0_y(\boldsymbol{\Xi}_{k-1})_i - L^1_y(\boldsymbol{\Xi}_{k-2})_i - \operatorname{div}_{\tilde{y}} G^0_{yij}(\boldsymbol{\Xi}_{k-1})dy\right) \\
&-\left(\frac{1}{|\omega|}\int_\omega -G^0_{i3}(\boldsymbol{\Xi}_{k-1})|_{y_3=0}n_3 - G^{-1}_{i3}(\boldsymbol{N}_k)n_3 - G^0_{i3}(\boldsymbol{N}_{k-1})n_3 d\tilde{y}\right)
\end{aligned} \tag{3.27}$$

and for $i = 1, 2$

$$\begin{aligned}
{}^{1\hat{\boldsymbol{\Theta}}}\boldsymbol{h}^i_k = &-\left(\frac{1}{|\omega|}\int_{\omega\times(-\infty,0)} -L^0_y({}^1\hat{\boldsymbol{\Theta}}_{k-4})_i - L^1_y({}^1\hat{\boldsymbol{\Theta}}_{k-5})_i - \operatorname{div}_{\tilde{y}} G^0_{ij}({}^1\hat{\boldsymbol{\Theta}}_{k-4})dy\right) \\
&-\left(\frac{1}{|\omega|}\int_\omega -G^0_{i3}({}^1\hat{\boldsymbol{\Theta}}_{k-4})|_{y_3=0}n_3 d\tilde{y} + s_i\frac{1}{|\omega|}\int_{\partial\omega_R\times(-1,0)}(\boldsymbol{N}^i_{k-3} + \boldsymbol{\Xi}^i_{k-3})ds dy_3\right)
\end{aligned} \tag{3.28}$$

$$\begin{aligned}
{}^{s\hat{\boldsymbol{\Theta}}}\boldsymbol{h}^i_k = &-\left(\frac{1}{|\omega|}\int_{\omega\times(-\infty,0)} -L^0_y({}^s\hat{\boldsymbol{\Theta}}_{k-1-s3})_i - L^1_y({}^s\hat{\boldsymbol{\Theta}}_{k-2-s3})_i - \operatorname{div}_{\tilde{y}} G^0_{yij}({}^s\hat{\boldsymbol{\Theta}}_{k-1-s3})dy\right) \\
&-\left(\frac{1}{|\omega|}\int_\omega -G^0_{yi3}({}^s\hat{\boldsymbol{\Theta}}_{k-1-s3})|_{y_3=0}n_3 d\tilde{y} + s_i\frac{1}{|\omega|}\int_{\partial\omega_R\times(-1,0)}({}^{s-1}\hat{\boldsymbol{\Theta}}^i_{k-1-s3})ds dy_3\right)
\end{aligned} \tag{3.29}$$

and for $i = 3$

$$
\begin{aligned}
{}^{1}\hat{\mathbf{\Theta}}\boldsymbol{h}_k^3 = &-\left(\frac{1}{|\omega|}\int_{\omega\times(-\infty,0)} -L_y^0({}^{1}\hat{\mathbf{\Theta}}_{k-2})_3 - L_y^1({}^{1}\hat{\mathbf{\Theta}}_{k-3})_3 - \mathrm{div}_{\tilde{y}}\, G_{3j}^0({}^{1}\hat{\mathbf{\Theta}}_{k-2})dy\right) \qquad (3.30)\\
&-\left(\frac{1}{|\omega|}\int_{\omega} -G_{33}^0({}^{1}\hat{\mathbf{\Theta}}_{k-2})|_{y_3=0}n_3 d\tilde{y} + s_3\frac{1}{|\omega|}\int_{\partial\omega_R\times(-1,0)}(\boldsymbol{N}_{k-1}^3 + \mathbf{\Xi}_{k-1}^3)dsdy_3\right)\\
{}^{s}\hat{\mathbf{\Theta}}\boldsymbol{h}_k^3 = &-\left(\frac{1}{|\omega|}\int_{\omega\times(-\infty,0)} -L_y^0({}^{s}\hat{\mathbf{\Theta}}_{k-1-s})_3 - L_y^1({}^{s}\hat{\mathbf{\Theta}}_{k-2-s})_3 - \mathrm{div}_{\tilde{y}}\, G_{y3j}^0({}^{s}\hat{\mathbf{\Theta}}_{k-1-s})dy\right) \qquad (3.31)\\
&-\left(\frac{1}{|\omega|}\int_{\omega} -G_{y33}^0({}^{s}\hat{\mathbf{\Theta}}_{k-1-s})|_{y_3=0}n_3 d\tilde{y} + s_3\frac{1}{|\omega|}\int_{\partial\omega_R\times(-1,0)}({}^{s-1}\hat{\mathbf{\Theta}}_{k-1-s}^3)dsdy_3\right)
\end{aligned}
$$

Finaly we get in $\hat{\Omega}$

$$
\begin{aligned}
&\sum_{k=0}^{\infty}\varepsilon^{k-2}(L^{-1}(\hat{\boldsymbol{u}}^k) + L^0(\hat{\boldsymbol{u}}^{k-1})_i + L^1(\hat{\boldsymbol{u}}^{k-2})_i) = \varepsilon^2\hat{f}_i(x_3)\\
&\Rightarrow\\
&\sum_{k=0}^{\infty}\varepsilon^{k-4}(L^{-1}(\hat{\boldsymbol{u}}^k) + L^0(\hat{\boldsymbol{u}}^{k-1})_i + L^1(\hat{\boldsymbol{u}}^{k-2})_i) = \hat{f}_i(x_3)
\end{aligned}
$$

and

$$
\begin{aligned}
&\sum_{k=0}^{\infty}\varepsilon^{k-4}(L^{-1}(\hat{\boldsymbol{u}}^k)_i + L^0(\hat{\boldsymbol{u}}^{k-1})_i + L^1(\hat{\boldsymbol{u}}^{k-2})_i)\\
&= \sum_{k=0}^{\infty}\varepsilon^{k-4}((L_y^{-1}(\boldsymbol{N}^k)_i + L_y^0(\boldsymbol{N}_{k-1})_i + L_y^1(\boldsymbol{N}_{k-2})_i) + L_y^{-1}(\mathbf{\Xi}_k)_i + L_y^0(\mathbf{\Xi}_{k-1})_i + L_y^1(\mathbf{\Xi}_{k-2})_i))\frac{\partial^k\hat{\boldsymbol{u}}^0}{\partial x_3^k}\\
&+ \sum_{k=0}^{\infty}\varepsilon^{k-4}\sum_{s=1}^{\infty}((L_y^{-1}({}^{s}\hat{\mathbf{\Theta}}_{k-s})_i + L_y^0({}^{s}\hat{\mathbf{\Theta}}_{k-1-s})_i + L_y^1({}^{s}\hat{\mathbf{\Theta}}_{k-2-s})_i))\frac{\partial^{k-s}\hat{\boldsymbol{u}}^0}{\partial x_3^{k-s}}\\
&= \sum_{k=0}^{\infty}\varepsilon^{k-4}\boldsymbol{h}_k^i\cdot\frac{\partial^k\hat{\boldsymbol{u}}^0}{\partial x_3^k}
\end{aligned}
$$

Further in $\hat{S}_C$ we get

$$\sum_{k=0}^{\infty}\varepsilon^{k-1}\left(\sum_{j=1}^{3}\Big((G_{ij}^{-1}(\hat{\boldsymbol{u}}^{k}))+(G_{ij}^{0}(\hat{\boldsymbol{u}}^{k-1}))\Big)\,n_j\right)=s_i\varepsilon^2\sum_{k=0}^{\infty}\varepsilon^{k}\hat{u}_i^{k}$$

$$\Rightarrow$$

$$\sum_{k=0}^{\infty}\varepsilon^{k-3}\left(\sum_{j=1}^{3}\Big((G_{ij}^{-1}(\hat{\boldsymbol{u}}^{k}))+(G_{ij}^{0}(\hat{\boldsymbol{u}}^{k-1}))\Big)\,n_j\right)=s_i\sum_{k=0}^{\infty}\varepsilon^{k-3}\hat{u}_i^{k-3}$$

$$\Rightarrow$$

$$\sum_{k=0}^{\infty}\varepsilon^{k-3}\left(\sum_{j=1}^{3}(G_{ij}^{-1}(\hat{\boldsymbol{u}}^{k})+G_{ij}^{0}(\hat{\boldsymbol{u}}^{k-1}))n_j-s_i\hat{u}_i^{k-3}\right)=0$$

and

$$\begin{aligned}
&\sum_{k=0}^{\infty}\varepsilon^{k-3}\left(\sum_{j=1}^{3}(G_{ij}^{-1}(\hat{\boldsymbol{u}}^{k})+G_{ij}^{0}(\hat{\boldsymbol{u}}^{k-1}))n_j-s_i\hat{u}_i^{k-3}\right)\\
&=\sum_{k=0}^{\infty}\varepsilon^{k-3}\left(\sum_{j=1}^{3}((G_{yij}^{-1}(\boldsymbol{N}_k)+G_{yij}^{0}(\boldsymbol{N}_{k-1})+G_{yij}^{-1}(\boldsymbol{\Xi}_k)+G_{yij}^{0}({}^{1}\boldsymbol{\Xi}_k))n_j)\frac{\partial^k\hat{\boldsymbol{u}}^0}{\partial x_3^k}\right)\\
&=\sum_{k=0}^{\infty}\varepsilon^{k-3}\left(\sum_{j=1}^{3}(G_{yij}^{-1}({}^{1}\hat{\boldsymbol{\Theta}}_{k-3})+G_{yij}^{0}({}^{1}\hat{\boldsymbol{\Theta}}_{k-4}))n_j-s_i(\boldsymbol{N}_{k-3}^{i}+\boldsymbol{\Xi}_{k-3}))\frac{\partial^{k-3}\hat{\boldsymbol{u}}^0}{\partial x_3^{k-3}}\right)\\
&=\sum_{k=0}^{\infty}\sum_{s=2}^{\infty}\varepsilon^{k-3}\left(\sum_{j=1}^{3}(G_{yij}^{-1}({}^{s}\hat{\boldsymbol{\Theta}}_{k-s3})+G_{yij}^{0}({}^{s}\hat{\boldsymbol{\Theta}}_{k-1-s3}))n_j-s_i({}^{s-1}\hat{\boldsymbol{\Theta}}_{k-s3})\frac{\partial^{k-s3}\hat{\boldsymbol{u}}^0}{\partial x_3^{k-s3}}\right)\\
&=\sum_{k=0}^{\infty}\varepsilon^{k-3}\left({}^{\Xi}\boldsymbol{h}_k^i\cdot\frac{\partial^k\hat{\boldsymbol{u}}^0}{\partial x_3^k}+\sum_{s=1}^{\infty}{}^{s\hat{\Theta}}\boldsymbol{h}_k^i\cdot\frac{\partial^{k-s3}\hat{\boldsymbol{u}}^0}{\partial x_3^{k-s3}}\right)
\end{aligned}$$

(3.36) follows directly from Proposition (2). □

Remark 8. (Contact on the end) *We highlight that the terms* $-G_{i3}^{-1}(\boldsymbol{N}_k)n_3-G_{i3}^{0}(\boldsymbol{N}_{k-1})n_3$ *in* (3.21) *lead to the bending force in the limit equation, see* (3.27)*, Lemma 8 and [Panasenko, 2005, p.70-71]. Further we underline that these terms vanish in the longitudinal interior of the fiber, since* $n_3=0$ *in the longitudinal interior of the cylinder, see Figures 2.2, 2.2. Therefore the contact shall be located on an end. Mechanically this is natural since in the Euler-Bernoulli equations the forces are dynamic boundary conditions.*

Lemma 4. $\boldsymbol{h}_k^i=0$ *for* $k=0,1,2,3$ *and* $i=1,2$ *and* $\boldsymbol{h}_0^3=0$*.*

Proof. From Lemma 1 we know that $\boldsymbol{h}_k^i=0$ for $k=0,1$ and $i=1,2,3$. From (3.26) we know:

$$\boldsymbol{h}_2^i = -\left(\frac{1}{|\omega|}\int_\omega -\tilde{L}_y^0(\boldsymbol{N}_1)_i - \tilde{L}_y^1(\boldsymbol{N}_0)_i - \operatorname{div}_{\tilde{y}} \tilde{G}_{yij}^0(\widetilde{\boldsymbol{N}}_1)d\tilde{y}\right) = -\left(\frac{1}{|\omega|}\int_\omega \mu\frac{\partial \boldsymbol{N}_1^3}{\partial y_i} + \mu \boldsymbol{N}_0^i d\tilde{y}\right)$$

(3.20) yields componentwise:

$$\begin{aligned} -\mu\Delta_{y_1y_2}\boldsymbol{N}_1^3 &= 0 && \text{in } \omega \\ \mu\frac{\partial \boldsymbol{N}_k^3}{\partial \tilde{\boldsymbol{\nu}}} &= -\mu\tilde{\boldsymbol{N}}_0\boldsymbol{\nu} && \text{on } \partial\omega \end{aligned}$$

which is solved by $\boldsymbol{N}_1^3 = (-y_1, -y_2, 0)$, see [Panasenko, 2005, p.96], indeed:

$$\frac{\partial N_k^3}{\partial \tilde{\boldsymbol{\nu}}} = \sum_{j=1}^2 \frac{\partial}{\partial y_j}(-y_1, -y_2, 0)n_j = (-n_1, -n_2, 0) = -\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}(n_1, n_2, 0)^T = -\tilde{\boldsymbol{N}}_0\boldsymbol{\nu}$$

hence

$$\begin{aligned} \boldsymbol{h}_2^i &= -\left(\frac{1}{|\omega|}\int_\omega \mu\frac{\partial \boldsymbol{N}_1^3}{\partial y_i} + \mu\boldsymbol{N}_0^i d\tilde{y}\right) \\ &= -\left(\frac{1}{|\omega|}\int_\omega \mu\frac{\partial}{\partial y_i}(-y_1, -y_2, 0) + \mu\boldsymbol{e}_i d\tilde{y}\right) \\ &= -\left(\frac{1}{|\omega|}\int_\omega -\mu\boldsymbol{e}_i + \mu\boldsymbol{e}_i d\tilde{y}\right) = 0 \end{aligned}$$

Further

$$\begin{aligned}
\boldsymbol{h}_3^i &= -\left(\frac{1}{|\omega|}\int_\omega -\tilde{L}_y^0(\boldsymbol{N}_2)_i - \tilde{L}_y^1(\boldsymbol{N}_1)_i - \operatorname{div}_{\tilde{y}} \tilde{G}_{yij}^0(\widetilde{\boldsymbol{N}}_2)d\tilde{y}\right)\\
&= -\left(\frac{1}{|\omega|}\int_\omega \mu\frac{\partial \boldsymbol{N}_2^3}{\partial y_i} + \mu \boldsymbol{N}_1^i d\tilde{y}\right)\\
&= -\sum_{j=1}^2\left(\frac{1}{|\omega|}\int_\omega (\mu\frac{\partial \boldsymbol{N}_2^3}{\partial y_j} + \mu \boldsymbol{N}_1^j)\delta_{ij} d\tilde{y}\right)\\
&= -\sum_{j=1}^2\left(\frac{1}{|\omega|}\int_\omega (\mu\frac{\partial \boldsymbol{N}_2^3}{\partial y_j} + \mu \boldsymbol{N}_1^j)\frac{\partial y_i}{\partial y_j} d\tilde{y}\right) \qquad (3.32)\\
&= \sum_{j=1}^2\left(\frac{1}{|\omega|}\int_\omega \frac{\partial}{\partial y_j}(\mu\frac{\partial \boldsymbol{N}_2^3}{\partial y_j} + \mu \boldsymbol{N}_1^j) y_i d\tilde{y}\right) \qquad (3.33)\\
&= \frac{1}{|\omega|}\int_\omega (\mu\Delta_{\tilde{y}}\boldsymbol{N}_2^3 + \mu \operatorname{div}_{\tilde{y}} \widetilde{\boldsymbol{N}}_1) y_i d\tilde{y}\\
&= \frac{1}{|\omega|}\int_\omega (-(\lambda+2\mu)\boldsymbol{N}_0^3 - \lambda \operatorname{div}_{\tilde{y}} \widetilde{\boldsymbol{N}}_1 - \boldsymbol{h}_2^3) y_i d\tilde{y}\\
&= \frac{1}{|\omega|}\int_\omega (-(\lambda+2\mu)\boldsymbol{e}_3 + \frac{\lambda^2}{\lambda+\mu}\boldsymbol{e}_3 + E\boldsymbol{e}_3) y_i d\tilde{y}\\
&= \frac{1}{|\omega|}\int_\omega (-E\boldsymbol{e}_3 + E\boldsymbol{e}_3) y_i d\tilde{y}\\
&= 0
\end{aligned}$$

where the step from (3.32) to (3.33) follows from the definition of the weak derivative. □

Lemma 5. *$^{\Xi}\boldsymbol{h}_k^i = 0$ and $^{s\hat{\Theta}}\boldsymbol{h}_k^i = 0$ for $k = 0, 1, 2$, $i = 1, 2$ and $s \geq 1$. Additionally $^{\Xi}\boldsymbol{h}_0^3 = 0$ and $^{s\hat{\Theta}}\boldsymbol{h}_k^3 = 0$ for $k = 0, 1$ and $s \geq 1$.*

Proof. From Lemma 1 we know $^{\Xi}\boldsymbol{h}_k^i = 0$ for $k = 0, 1$, $i = 1, 2$ and $^{\Xi}\boldsymbol{h}_0^3 = 0$ and $^{s\hat{\Theta}}\boldsymbol{h}_0^3 = 0$. (3.27) yields

$$\begin{aligned}
^{\Xi}\boldsymbol{h}_2^i = &-\left(\frac{1}{|\omega|}\int_{\omega\times(-\infty,0)} -L_y^0(\boldsymbol{\Xi}_1)_i - \operatorname{div}_{\tilde{y}} G_{yij}^0(\boldsymbol{\Xi}_1)dy\right)\\
&-\left(\frac{1}{|\omega|}\int_\omega -G_{i3}^0(\boldsymbol{\Xi}_1)|_{y_3=0}n_3 - G_{i3}^{-1}(\boldsymbol{N}_k)n_3 - G_{i3}^0(\boldsymbol{N}_1)n_3 d\tilde{y}\right)
\end{aligned} \qquad (3.34)$$

From (3.21) we get the problem for $\boldsymbol{\Xi}_1$

$$\begin{aligned}
&\mathbf{BVP}(\boldsymbol{\Xi}_1) && \\
&-\operatorname{div}_y(G^{-1}_{yij}(\boldsymbol{\Xi}_1)) = 0 && \text{in } \omega \times (-\infty, 0) \\
&\sum_{j=1}^{2}(G^{-1}_{yij}(\boldsymbol{\Xi}_1))n_j = 0 && \text{on } \partial\omega \times (-\infty, 0) \\
&G^{-1}_{yi3}(\boldsymbol{\Xi}_1)n_3 = -G^{-1}_{yi3}(\boldsymbol{N}_1)n_3 - G^{0}_{yi3}(\boldsymbol{N}_0)n_3 && \text{on } \omega \times \{0\}
\end{aligned}$$

Since $G^{-1}_{yi3}(\boldsymbol{N}_1) + G^{0}_{yi3}(\boldsymbol{N}_0) = \mu\frac{\partial}{\partial y_i}(-y_1, -y_2, 0) + \mu\boldsymbol{e}_i = -\mu\boldsymbol{e}_i + \boldsymbol{e}_i = 0$ we obtain that $\boldsymbol{\Xi}_1 = 0$ solves $\mathbf{BVP}(\boldsymbol{\Xi}_1)$. Hence

$$\begin{aligned}
{}^{\Xi}\boldsymbol{h}^i_2 =& \frac{1}{|\omega|}\int_\omega G^{-1}_{i3}(\boldsymbol{N}_2)n_3 G^0_{i3}(\boldsymbol{N}_1)n_3 d\tilde{y} \\
=& \left(\frac{1}{|\omega|}\int_\omega \mu\frac{\partial \boldsymbol{N}^3_2}{\partial y_i} + \mu\boldsymbol{N}^i_1 d\tilde{y}\right) \\
=& -\boldsymbol{h}^i_3 \\
=& 0
\end{aligned}$$

hence ${}^{\Xi}\boldsymbol{h}^i_2 = 0$. Further form (3.29) and (3.31) we get directly ${}^{s\hat{\boldsymbol{\Theta}}}\boldsymbol{h}^i_k = 0$ for $k = 0, 1, 2$, $i = 1, 2$ and $s \geq 1$. □

Lemma 6. ${}^{s\hat{\boldsymbol{\Theta}}}\boldsymbol{h}^i_3 = 0$ *for* $i = 1, 2$ *and* ${}^{s\hat{\boldsymbol{\Theta}}}\boldsymbol{h}^3_1 = 0$ *for* $s \geq 2$.

Proof. Lemma 1 and (3.31) proof the Lemma. □

Proposition 7. (Limiting 1D equation)
(3.35) *and* (3.36) *yield for* $\varepsilon \to 0$ *:*
For $i = 1, 2$

$$\begin{aligned}
&\boldsymbol{h}^i_4 \cdot \frac{\partial^4 \hat{\boldsymbol{u}}^0}{\partial x_3^4} = \hat{f}_i(x_3) && x_3 \in (-1, 0) \\
&{}^{\Xi}\boldsymbol{h}^i_3 \cdot \frac{\partial^3 \hat{\boldsymbol{u}}^0}{\partial x_3^3} + {}^{1\hat{\boldsymbol{\Theta}}}\boldsymbol{h}^i_3 \cdot \hat{\boldsymbol{u}}^0 = 0 && x_3 = 0
\end{aligned} \tag{3.35}$$

and for $i = 3$

$$\begin{aligned}
&\boldsymbol{h}^3_k \cdot \frac{\partial^2 \hat{\boldsymbol{u}}^0}{\partial x_3^2} = \hat{f}_3(x_3) && x_3 \in (-1, 0) \\
&{}^{\Xi}\boldsymbol{h}^3_1 \cdot \frac{\partial \hat{\boldsymbol{u}}^0}{\partial x_3} + {}^{1\hat{\boldsymbol{\Theta}}}\boldsymbol{h}^3_k \cdot \hat{\boldsymbol{u}}^0 = 0 && x_3 = 0
\end{aligned} \tag{3.36}$$

Proof. With Lemmata 4, (5), and (6) we have

$$\lim_{\varepsilon\to 0}\sum_{k=0}^{\infty}\varepsilon^{k-4}\boldsymbol{h}_k^i\cdot\frac{\partial^k\hat{\boldsymbol{u}}^0}{\partial x_3^k}=\boldsymbol{h}_4^i\cdot\frac{\partial^4\hat{\boldsymbol{u}}^0}{\partial x_3^4}$$

$$\lim_{\varepsilon\to 0}\sum_{k=0}^{\infty}\varepsilon^{k-3}\left({}^{\Xi}\boldsymbol{h}_k^i\cdot\frac{\partial^k\hat{\boldsymbol{u}}^0}{\partial x_3^k}+\sum_{s=1}^{\infty}{}^{s}\hat{\boldsymbol{\Theta}}\boldsymbol{h}_k^i\cdot\frac{\partial^{k-s3}\hat{\boldsymbol{u}}^0}{\partial x_3^{k-s3}}\right)={}^{\Xi}\boldsymbol{h}_3^i\cdot\frac{\partial^3\hat{\boldsymbol{u}}^0}{\partial x_3^3}+{}^{1}\hat{\boldsymbol{\Theta}}\boldsymbol{h}_3^i\cdot\hat{\boldsymbol{u}}^0$$

$$\lim_{\varepsilon\to 0}\sum_{k=0}^{\infty}\varepsilon^{k-2}\boldsymbol{h}_k^3\cdot\frac{\partial^k\hat{\boldsymbol{u}}^0}{\partial x_3^k}=\boldsymbol{h}_k^2\cdot\frac{\partial^2\hat{\boldsymbol{u}}^0}{\partial x_3^2}$$

$$\lim_{\varepsilon\to 0}\sum_{k=0}^{\infty}\varepsilon^{k-1}\left({}^{\Xi}\boldsymbol{h}_k^3\cdot\frac{\partial^k\hat{\boldsymbol{u}}^0}{\partial x_3^k}+\sum_{s=1}^{\infty}{}^{s}\hat{\boldsymbol{\Theta}}\boldsymbol{h}_k^3\cdot\frac{\partial^{k-s3}\hat{\boldsymbol{u}}^0}{\partial x_3^{k-s3}}\right)={}^{\Xi}\boldsymbol{h}_1^3\cdot\frac{\partial\hat{\boldsymbol{u}}^0}{\partial x_3}+{}^{1}\hat{\boldsymbol{\Theta}}\boldsymbol{h}_k^3\cdot\hat{\boldsymbol{u}}^0$$

□

Lemma 7. $\boldsymbol{h}_4^i = EI\boldsymbol{e}_i$ *for* $i = 1, 2$ *and* $\boldsymbol{h}_2^3 = -E\boldsymbol{e}_3$.

Proof.

$$\boldsymbol{h}_4^i=-\left(\frac{1}{|\omega|}\int_\omega -\tilde{L}_y^0(\boldsymbol{N}_3)_i-\tilde{L}_y^1(\boldsymbol{N}_2)_i-\operatorname{div}_{\tilde{y}}\tilde{G}_{yij}^0(\widetilde{\boldsymbol{N}}_2)d\tilde{y}\right)=-\left(\frac{1}{|\omega|}\int_\omega \mu\frac{\partial\boldsymbol{N}_3^3}{\partial y_i}+\mu\boldsymbol{N}_2^i d\tilde{y}\right)$$

Where $\boldsymbol{N}_2^i$ for $i=1,2$ with $n_2^{12}=n_2^{21}=\frac{\lambda}{2(\lambda+\mu)}y_1y_2$, $n_2^{11}=\frac{\lambda}{4(\lambda+\mu)}(y_1^2-y_2^2)$, $n_2^{22}=\frac{\lambda}{4(\lambda+\mu)}(y_2^2-y_1^2)$ and $n_2^{i3}=0$ solve
For $i=1,2$:

$$\begin{aligned} -\operatorname{div}(\hat{\boldsymbol{\sigma}}(\tilde{\boldsymbol{N}}_2))_i &= -\lambda\boldsymbol{e}_i && \text{in } \omega \\ \hat{\boldsymbol{\sigma}}(\tilde{\boldsymbol{N}}_2)_i\cdot\tilde{\boldsymbol{\nu}} &= n_i\lambda(y_1,y_2,0) && \text{on } \partial\omega \end{aligned} \tag{3.37}$$

, see [Panasenko, 2005, 96]. Indeed, in terms of operators of definition 8 (3.37) reads

$$\begin{aligned} \tilde{C}_i^1(n_2^{11})+\tilde{C}_i^2(n_2^{21}) &= -\lambda && \text{in } \omega \\ \tilde{B}_i^1(n_2^{11})+\tilde{B}_i^2(n_2^{21}) &= n_i\lambda y_i && \text{on } \partial\omega \end{aligned}$$

W.L.O.G. lets assume i=1, hence

$$\begin{aligned}
\tilde{C}_1^1(n_2^{11}) + \tilde{C}_1^2(n_2^{21}) &= -(2\mu+\lambda)\frac{\partial^2}{\partial y_1^2}n_2^{11} - (\lambda+\mu)\frac{\partial^2}{\partial y_1 \partial y_2}n_2^{21} - \mu\frac{\partial^2}{\partial y_2^2}n_2^{11} \\
&= -\frac{(\lambda+2\mu)\lambda}{4(\lambda+\mu)}2 - \frac{(\lambda+\mu)\lambda}{2(\lambda+\mu)} - \frac{\mu\lambda}{4(\lambda+\mu)}(-2) \\
&= \frac{-2(\mu\lambda+\lambda^2)}{2(\mu+\lambda)} = -\lambda
\end{aligned}$$

and

$$\begin{aligned}
\tilde{B}_1^1(n_2^{11}) + \tilde{B}_1^2(n_2^{21}) &= (2\mu+\lambda)n_1\frac{\partial}{\partial y_1}n_2^{11} + \mu n_2\frac{\partial}{\partial y_2}n_2^{11} + \mu n_2\frac{\partial}{\partial y_1}n_2^{21} + \lambda n_1\frac{\partial}{\partial y_2}n_2^{21} \\
&= \frac{(\lambda+2\mu)\lambda 2n_1}{4(\lambda+\mu)}y_1 + \frac{n_2\mu(-\lambda 2)}{4(\mu+\lambda)}y_2 + \frac{n_2\mu\lambda}{2(\mu+\lambda)}y_2 + \frac{n_1\mu\lambda}{2(\mu+\lambda)}y_1 \\
&= n_1\lambda y_1\frac{2\mu+2\lambda}{2(\lambda+\mu)} = n_1\lambda y_1
\end{aligned}$$

hence

$$\begin{aligned}
\boldsymbol{h}_4^i &= -\left(\frac{1}{|\omega|}\int_\omega \mu\frac{\partial \boldsymbol{N}_3^3}{\partial y_i} + \mu\boldsymbol{N}_2^i d\tilde{y}\right) \\
&= -\sum_{j=1}^{2}\left(\frac{1}{|\omega|}\int_\omega (\mu\frac{\partial \boldsymbol{N}_3^3}{\partial y_j} + \mu\boldsymbol{N}_2^j)\delta_{ij} d\tilde{y}\right) \\
&= -\sum_{j=1}^{2}\left(\frac{1}{|\omega|}\int_\omega (\mu\frac{\partial \boldsymbol{N}_3^3}{\partial y_j} + \mu\boldsymbol{N}_2^j)\frac{\partial y_i}{\partial y_j} d\tilde{y}\right) \\
&= \sum_{j=1}^{2}\left(\frac{1}{|\omega|}\int_\omega \frac{\partial}{\partial y_j}(\mu\frac{\partial \boldsymbol{N}_3^3}{\partial y_j} + \mu\boldsymbol{N}_2^j) y_i d\tilde{y}\right) \\
&= \left(\frac{1}{|\omega|}\int_\omega (\mu\Delta_{\tilde{y}}\boldsymbol{N}_3^3 + \mu\,\mathrm{div}_{\tilde{y}}\,\widetilde{\boldsymbol{N}}_2) y_i d\tilde{y}\right) && (3.38) \\
&= \frac{1}{|\omega|}\int_\omega (-(\lambda+2\mu)\boldsymbol{N}_1^3 - \lambda\,\mathrm{div}_{\tilde{y}}\,\widetilde{\boldsymbol{N}}_2 - \boldsymbol{h}_3^3) y_i d\tilde{y} && (3.39) \\
&= \frac{1}{|\omega|}\int_\omega ((\lambda+2\mu)(y_1,y_2,0) - \frac{\lambda^2}{\lambda+\mu}(y_1,y_2,0)) y_i d\tilde{y} \\
&= E\frac{1}{|\omega|}\int_\omega y_i^2 d\tilde{y}\boldsymbol{e}_i \\
&= EI\boldsymbol{e}_i
\end{aligned}$$

We notice that the step from (3.38) to (3.39) follows from the fact that $\boldsymbol{N}_3^3$ solves (3.20) which reads as (3.40). For $i = 3$:

$$-\mu\Delta_{y_1y_2}\boldsymbol{N}_3^3 = (\lambda+2\mu)N_1^3 + (\lambda+\mu)\operatorname{div}_{y_1y_2}(\tilde{\boldsymbol{N}}_2) + \boldsymbol{h}_3^3 \qquad \text{in } \omega \tag{3.40}$$
$$\mu\frac{\partial \boldsymbol{N}_k^3}{\partial\tilde{\boldsymbol{\nu}}} = -\mu\tilde{\boldsymbol{N}}_2\boldsymbol{\nu} \qquad \text{on } \partial\omega$$

We recall that by the symmetry of the cross-section ω, see Assumption 1, we have $\int_\omega y_i d\tilde{y} = 0$, in particular $\int_\omega y_i y_j d\tilde{y} = \int_\omega y_i^2 d\tilde{y}\delta_{ij}$ and $\boldsymbol{h}_3^3\int_\omega y_i d\tilde{y} = 0$. Additionally Lemma 2 yields $\boldsymbol{h}_2^3 = -E\boldsymbol{e}_3$. □

Remark 9. (Uniqueness of $\boldsymbol{N}_k$) *At this stage let us mention that in [Panasenko, 2005] n_2^{11} and n_2^{11} take the form:*

$$n_2^{11} = \frac{\lambda}{4(\lambda+\mu)}\left((y_1^2 - y_2^2) - \frac{1}{|\omega|}\int_\omega (y_1^2 - y_2^2)d\tilde{\boldsymbol{y}}\right)$$
$$n_2^{22} = \frac{\lambda}{4(\lambda+\mu)}\left((y_2^2 - y_1^2) - \frac{1}{|\omega|}\int_\omega (y_2^2 - y_1^2)\tilde{\boldsymbol{y}}\right)$$

The constants $-\frac{1}{|\omega|}\int_\omega(y_1^2 - y_2^2)d\tilde{\boldsymbol{y}}$ and $-\frac{1}{|\omega|}\int_\omega(y_2^2 - y_1^2)d\tilde{\boldsymbol{y}}$ are added to guarantee that $\int_\omega n_2^{ii}d\tilde{\boldsymbol{y}} = 0$ for $i = 1, 2$. This property is required to fulfill the requirements of Lemma 2.2.1 in [Panasenko, 2005, p.40] which is an extension of the Lax-Milgram Theorem (see Section 7) to Neumann boundary value problems. The vanishing integral $\int_\omega \boldsymbol{N}_k d\tilde{\boldsymbol{y}} = 0$ is required for the uniqueness of the solution, see also [Braess, 1997, p.44]. In this work however we refer to [Dobrowolski, 2006, p.221] for the existence of the solution and don't require that property. Since for the dimension reduction algorithm we need derivatives of n_2^{ii} the constants vanish and hence do not influence the procedure, see the proof of Lemma 7.

Lemma 8. ${}^{\Xi}\boldsymbol{h}_3^i = -EIn_3\boldsymbol{e}_i$ *for* $i = 1, 2$ *and* ${}^{\Xi}\boldsymbol{h}_1^3 = En_3\boldsymbol{e}_3$.

Proof. (3.30) yields

$$\begin{aligned}{}^{\Xi}\boldsymbol{h}_3^i = &-\left(\frac{1}{|\omega|}\int_{\omega\times(-\infty,0)} -L_y^0(\boldsymbol{\Xi}_2)_i - \operatorname{div}_{\tilde{y}} G_{yij}^0(\boldsymbol{\Xi}_2)dy\right)\\ &-\left(\frac{1}{|\omega|}\int_\omega -G_{i3}^0(\boldsymbol{\Xi}_2)|_{y_3=0}n_3 - G_{i3}^{-1}(\boldsymbol{N}_3)n_3 - G_{i3}^0(\boldsymbol{N}_2)n_3 d\tilde{y}\right)\\ =&\frac{1}{|\omega|}\int_{\omega\times(-\infty,0)} L_y^0(\boldsymbol{\Xi}_2)_i + \operatorname{div}_y G_{yij}^0(\boldsymbol{\Xi}_2)dy + \frac{1}{|\omega|}\int_\omega G_{i3}^{-1}(\boldsymbol{N}_3)n_3 + G_{i3}^0(\boldsymbol{N}_2)n_3 d\tilde{y}\end{aligned}$$

Where

$$\frac{1}{|\omega|}\int_{\omega} G_{i3}^{-1}(\boldsymbol{N}_3)n_3 + G_{i3}^{0}(\boldsymbol{N}_2)n_3 d\tilde{y} = -\boldsymbol{e}_i EI$$
$$\frac{1}{|\omega|}\int_{\omega\times(-\infty,0)} L_y^0(\boldsymbol{\Xi}_2)_i + \operatorname{div}_y G_{yij}^{0}(\boldsymbol{\Xi}_2)dy = 0$$

Indeed, from (3.21) we get the problem for $\boldsymbol{\Xi}_2$ for $i = 1,2,3$

$$\begin{aligned}
&\mathbf{BVP}(\boldsymbol{\Xi}_2) && \quad (3.41)\\
&-\operatorname{div}_y(G_{yij}^{-1}(\boldsymbol{\Xi}_2)) = 0 && \text{in } \omega\times(-\infty,0)\\
&\sum_{j=1}^{2}(G_{yij}^{-1}(\boldsymbol{\Xi}_2))n_j = 0 && \text{on } \partial\omega\times(-\infty,0)\\
&G_{yi3}^{-1}(\boldsymbol{\Xi}_2)n_3 = -G_{yi3}^{-1}(\boldsymbol{N}_2)n_3 - G_{yi3}^{0}(\boldsymbol{N}_1)n_3 && \text{on } \omega\times\{0\}
\end{aligned}$$

hence

$$\begin{aligned}
&\frac{1}{|\omega|}\int_{\omega\times(-\infty,0)} L_y^0(\boldsymbol{\Xi}_2)_i + \operatorname{div}_y G_{yij}^{0}(\boldsymbol{\Xi}_2)dy\\
=&\frac{1}{|\omega|}\int_{\omega\times(-\infty,0)} -(\lambda+\mu)\frac{\partial}{\partial y_i}\boldsymbol{\Xi}_2^3 - 2\mu\frac{\partial}{\partial y_3}\boldsymbol{\Xi}_2^i + \lambda\frac{\partial}{\partial y_i}\boldsymbol{\Xi}_2^3 + \mu\frac{\partial}{\partial y_3}\boldsymbol{\Xi}_2^i dy\\
=&\frac{1}{|\omega|}\int_{\omega\times(-\infty,0)} -\mu\frac{\partial}{\partial y_i}\boldsymbol{\Xi}_2^3 - \mu\frac{\partial}{\partial y_3}\boldsymbol{\Xi}_2^i dy\\
=&-\frac{1}{|\omega|}\int_{\omega\times(-\infty,0)} \mu\frac{\partial}{\partial y_i}\boldsymbol{\Xi}_2^3 + \mu\frac{\partial}{\partial y_3}\boldsymbol{\Xi}_2^i dy\\
=&-\frac{1}{|\omega|}\int_{\omega\times(-\infty,0)} G_{yi3}^{-1}(\boldsymbol{\Xi}_2)dy\\
=&-\sum_{j=1}^{3}\frac{1}{|\omega|}\int_{\omega\times(-\infty,0)} G_{yj3}^{-1}(\boldsymbol{\Xi}_2)\frac{\partial y_i}{\partial y_j}dy\\
=&\sum_{j=1}^{3}\frac{1}{|\omega|}\int_{\omega\times(-\infty,0)} \frac{\partial}{\partial y_j}G_{yj3}^{-1}(\boldsymbol{\Xi}_2)y_i dy\\
=&\frac{1}{|\omega|}\int_{\omega\times(-\infty,0)} \operatorname{div}_y G_{yj3}^{-1}(\boldsymbol{\Xi}_2)y_i dy\\
=&0
\end{aligned}$$

Where the last equation follows from the symmetry $G^{-1}_{yj3} = G^{-1}_{y3j}$ and (3.41). Further

$$\frac{1}{|\omega|}\int_\omega G^{-1}_{i3}(\boldsymbol{N}_3)n_3 + G^0_{i3}(\boldsymbol{N}_2)n_3 d\tilde{y} = (\frac{1}{|\omega|}\int_\omega \mu\frac{\partial \boldsymbol{N}^3_3}{\partial y_i} + \mu\boldsymbol{N}^i_2 d\tilde{y})n_3 = -\boldsymbol{h}^i_4 n_3 = -\boldsymbol{e}_i EIn_3$$

Further with Lemma 3 we get ${}^{\Xi}\boldsymbol{h}^i_3 = -EIn_3\boldsymbol{e}_i$. Which are well known results, see [Panasenko, 2005, p.91] □

Lemma 9. ${}^1\hat{\boldsymbol{\Theta}}\boldsymbol{h}^i_3 = -\boldsymbol{e}_i\frac{|\partial\omega_R|}{|\omega|}s_i$ *for* $i = 1,2$ *and* ${}^1\hat{\boldsymbol{\Theta}}\boldsymbol{h}^3_1 = -\boldsymbol{e}_3\frac{|\partial\omega_R|}{|\omega|}s_3$

Proof. From (3.28) and (3.30) we get

$$\begin{aligned}{}^1\hat{\boldsymbol{\Theta}}\boldsymbol{h}^i_3 &= -s_i\frac{1}{|\omega|}\int_{\partial\omega_R\times(-1,0)}(\boldsymbol{N}^i_0 + \boldsymbol{\Xi}^i_0)dsdy_3 = -\boldsymbol{e}_i\frac{|\partial\omega_R|}{|\omega|}s_i\\ {}^1\hat{\boldsymbol{\Theta}}\boldsymbol{h}^3_1 &= -s_3\frac{1}{|\omega|}\int_{\partial\omega_R\times(-1,0)}(\boldsymbol{N}^3_0 + \boldsymbol{\Xi}^3_0)dsdy_3 = -\boldsymbol{e}_3\frac{|\partial\omega_R|}{|\omega|}s_3\end{aligned}$$

□

Proof. (of Theorem 2) With Proposition 7, Lemmata 7, 8 and 9 and $n_3 = 1$ we obtain: For i=1,2

$$\begin{aligned} &EI\frac{\partial^4\hat{u}^0_i}{\partial x^4_3} = \hat{f}_i(x_3) && x_3\in(-1,0)\\ -&EI\frac{\partial^3\hat{u}^0_i}{\partial x^3_3} - s_i\frac{|\partial\omega_R|}{|\omega|}\hat{u}^0_i = 0 && x_3 = 0\end{aligned} \tag{3.42}$$

and for $i = 3$

$$\begin{aligned} -&E\frac{\partial^2\hat{u}^0_3}{\partial x^2_3} = \hat{f}_3(x_3) && x_3\in(-1,0)\\ &E\frac{\partial\hat{u}^0_3}{\partial x_3} - s_3\frac{|\partial\omega_R|}{|\omega|}\hat{u}^0_3 = 0 && x_3 = 0\end{aligned} \tag{3.43}$$

Since by ansatz 4 $\hat{u}^0_i \in V_0$, for $i = 1,2,3$ (3.42) and (3.43) yield the following variational formulation.

$$\int_{(-1,0)} EI\frac{\partial^2\hat{u}^0_i}{\partial x^2_3}\frac{\partial^2 v}{\partial x^2_3}dx_3 - s_i\frac{|\partial\omega_R|}{|\omega|}\hat{u}^0_i(0)v(0) - EI\frac{\partial^2\hat{u}^0_i}{\partial x^2_3}\frac{\partial v}{\partial x_3}|^0_{-1} = \int_{(-1,0)}\hat{f}_i(x_3)vdx_3\ \forall v\in V_0$$

for $i = 1,2$

$$\int_{(-1,0)} E\frac{\partial\hat{u}^0_3}{\partial x_3}\frac{\partial v}{\partial x_3}dx_3 - s_3\frac{|\partial\omega_R|}{|\omega|}\hat{u}^0_3(0)v(0) = \int_{(-1,0)}\hat{f}_3(x_3)vdx_3\ \forall v\in V_0$$

for $i = 3$. □

Proposition 8. (Existence of a strong solution to the limit-problem) (3.42) *and* (3.43) *have a strong solution. We show that the corresponding homogeneous problems of the fourth order ordinary boundary value problems* (3.42) *and* (3.43) *have only the trivial solution, then by [Denk,2007/08, Theorem 6.8, p.64]* (3.42) *and* (3.43) *are solvable in the strong sense.*

Proof. W.L.O.G we consider the interval $(0,1)$, where 0 is the fixed-and 1 is the contact end. For (3.43) we refer to 4. For (3.43) we assume the angle and the moment to vanish at the fixed and contact end respectively, i.e. $\frac{\partial u_i(0)}{x_3} = 0$ and $\frac{\partial^2 u_i(1)}{\partial x_3^2} = 0$. Further set $\gamma = s_i \frac{|\partial \omega_R|}{|\omega|}$ for $i = 1, 2$. Under these assumptions the strong homogeneous equation of (3.42) is

$$EI \frac{\partial^4 u_i}{\partial x_3^4} = 0 \qquad \text{in } (0,1) \tag{3.44}$$

$$-EI \frac{\partial^3 u_i}{\partial x_3^3} - \gamma u_i(x_3) = 0 \qquad \text{on } 1 \tag{3.45}$$

$$EI \frac{\partial^2 u_i}{\partial x_3^2} = 0 \qquad \text{on } 1 \tag{3.46}$$

$$\frac{\partial u_i}{\partial x_3} = 0 \qquad \text{on } 0 \tag{3.47}$$

$$u_i = 0 \qquad \text{on } 0 \tag{3.48}$$

Lets proceed with the ansatz: $u_i^0(x_3) = ax_3^3 + bx^2 + cx + d$ with $a, b, c, d \in \mathbb{R}$. Then (3.48) implies $d = 0$ and (3.47) implies $c = 0$. Further

$$\text{(3.46) implies } 6a + 2b = 0 \text{ hence } a = -\frac{b}{3} \tag{3.49}$$

$$\text{(3.45) implies } 6a - \gamma(a + b) = 0 \text{ hence } a = -\frac{\gamma b}{6 - \gamma} \tag{3.50}$$

since γ varies with ω, (3.49) and (3.50) imply $a = 0 = b$

□

Corollary 2. (Existence of a weak solution) *Since every strong solution is also a weak solution, see [Braess, 1997, p.27],* (3.42) *and* (3.43) *are solvable in the weak sense.*

4 Numerical examples

In this Section we illustrate the results from Section 2 numerically. First we consider a hanging rod clamped from above, under gravity force, and compute the longitudinal displacement along the central line of the rod, once with the help of a 3D-finite-element software[2] and once analytically by solving the limiting 1D ODE obtained in Section 2. Then we compare the displacement along the central line of the 3D and 1D solution. In the next step we fix the length and vary the thickness of the rod in the 3D example. We recognize that the error between the 1D and the 3D solution decreases as the thickness of the rod decreases. Further we also execute the same numerical experiment for a hanging rod in contact with a rigid foundation at its lower end. We found out that the absolute error between a solution of the corresponding 1D ODE and the 3D-finite-element computation is of size 10^{-6} and 10^{-5} respectively.

4.1 Comparison of the 3D- and the 1D solutions for a rod under gravity force

We start comparing the 1D solution of (2.53) with the displacement along the center line of the 3D solution of a hanging rod, a rod that is fixed at the upper end and free at its lower end, see Figure 4.1.

Assumptions 5. (Experimental details) *We model a rod by a thin rectangular parallelepiped o We let the cross-section* $\omega_a = (-\frac{a}{2}, \frac{a}{2})^2$, *for* $a = 1, 10, 100$ *of the parallelepiped be quadratic, see Figure 4.3, and we let its length be* $1000mm$ *and set the relative relation between the thickness and the length* $\varepsilon_a = \frac{a}{1000}$. *The body force* $f_3(x_3) = 7.86 \cdot 10^{-2} \frac{gmm}{s^2}$ *is constant and induced by the gravity. The longitudinal displacement along the center line of the rod of the 3D-finite-element computation is compared to the longitudinal displacement obtained by solving the 1D mixed Robin/Neumann/Dirichlet ODE which was derived in this work, see Figure 4.4. We choose steel as the material of the rod*

[2]Commercial finite-element software-tool ANSYS 12.1

with Young modulus $E = 2 \cdot 10^5 \frac{gmm}{s^2}$ *and density (1D)* $\rho = 7.86 \cdot 10^{-3} \frac{g}{mm}$. *The interval considered is* $[-L, 0]$ *with* $L = 1000mm$.

We notice that an analytical solution of (2.53) is

$$u_3^0(x_3) = -\frac{f_3(x_3)}{2EL}\left(x_3^2 + \frac{s_3L^2 - 2EL}{E - s_3L}x_3\right) \tag{4.1}$$

Further we recall that for $s_3 = 0$ (2.53) is the equation describing a hanging rod, see for instance [Panasenko, 2005, p.61]. And for $s_3 = 0$ (4.1) reads $u_3^0(x_3) = -\frac{f_3(x_3)}{2EL}\left(x_3^2 - 2Lx_3\right)$, which is a well known solution for the Euler-Bernoulli equation for a hanging rod, see for instance [Gross, Hauger, Schroder, Wall 2007, p.23].

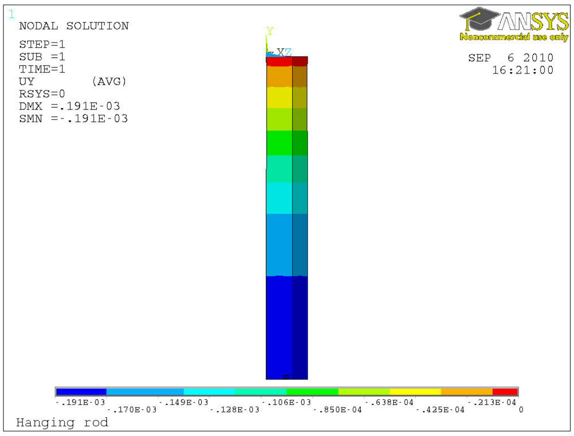
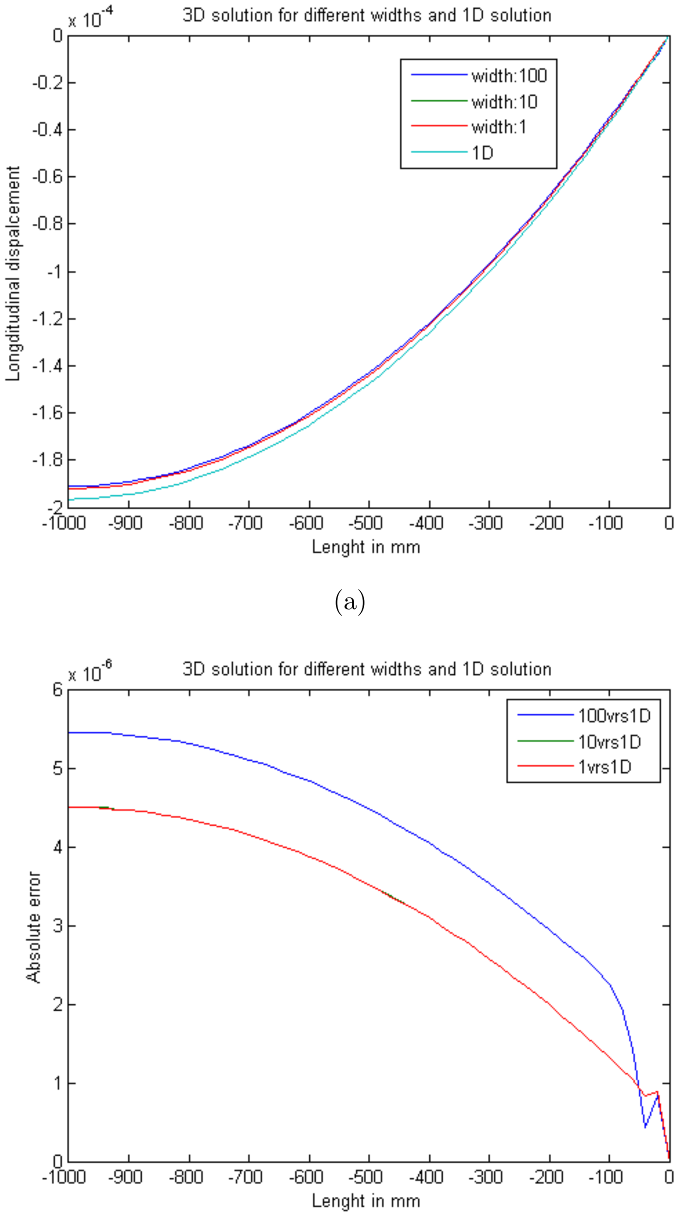


Figure 4.1: Rod with relative thickness $\varepsilon_{100} = 0,1$



(a)

(b)

Figure 4.2: a)Numerical- vrs analytical approximation , b) absolute error

Width (in mm)	**Norm of absolute error**
100	$3.0563 \cdot 10^{-5}$
10	$2.4250 \cdot 10^{-5}$
1	$2.4246 \cdot 10^{-5}$

Table 4.1: Euclidean norm of absolute value of error for different thickness of the rod (free lower end)

4.2 Comparison of the 3D- and the 1D solutions for a rod under gravity force in contact with a rigid foundation at its lower end

In this subsection we proceed as in the subsection before and compare the displacement along the center line of a rod computed once by a 3D-finite-element simulation and once by solving (2.53) analytically. In this subsection the rod is in contact with a rigid foundation at its lower end, see Figure 4.3. The Robin-parameter s_3 in (4.1) is set as $s_3 = 5500$ (empirical choice).

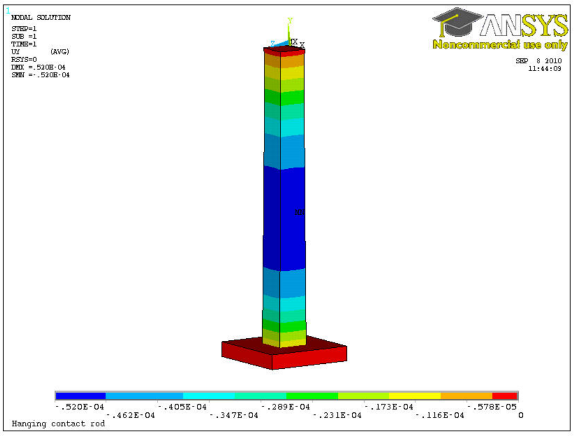
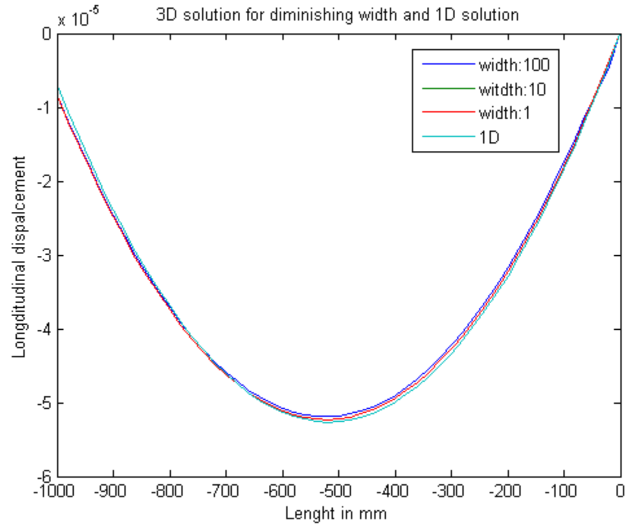
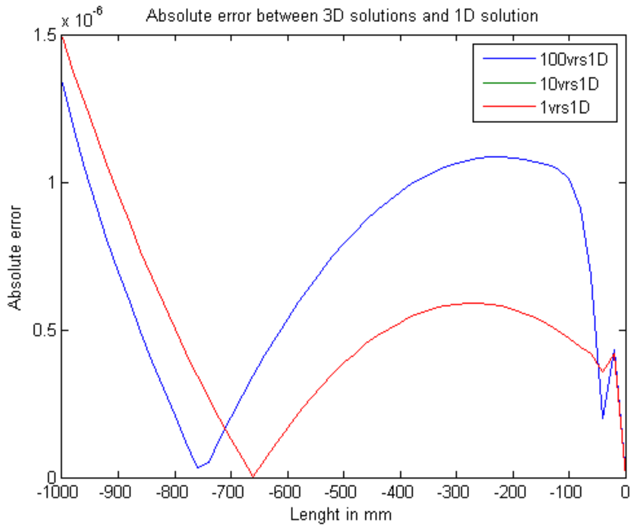


Figure 4.3: Rod with relative thickness $\varepsilon_{100} = 0,1$

Additionally we remark that the red and the green graphs in Figures 4.2 and 4.4 overlap. At this stage we mention that an improved meshing in the 3D finite-element simulation might reduce the error between the different 3D solutions for different widths.



(a)



(b)

Figure 4.4: a)3D solution for different widths and 1D solution , b) absolute error between the 3D solutions and the 1D solution

Width (in mm)	**Norm of absolute error**
100	$5.7340 \cdot 10^{-6}$
10	$4.3537 \cdot 10^{-6}$
1	$4.3536 \cdot 10^{-6}$

Table 4.2: Euclidean norm of absolute error for different thickness of the rod (in contact)

We see that for diminishing thickness of the rod the numerical solution to the contact problem approaches the analytical solution of (2.53).

We see that a solution of the ODE derived asymptotically in Section 2 approaches the corresponding 3D solution up to an absolute error of size 10^{-6}, see Figure 4.4, while in the case of a hanging rod without contact the size of the absolute error between the analytic solution of the Euler-Bernoulli equation and the corresponding 3D solution is of size 10^{-5} , see Figure 4.2. In both cases the norm of the absolute error gets smaller as the thickness of the rod gets smaller.

5 Conclusions

In this work two different contact problems where considered. One for a hanging rod in contact with a rigid body at the lower end and one for a beam that is fixed at one end and in contact with a rigid body on a part of the lateral boundary at other end. For both problems 3D linear elasticity contact problems are established, where the contact is described by a Robin-type boundary condition. For the first contact problem we construct asymptotically the complete 1D limit problem, for the thickness of the rod tending to zero. We recall that for that asymptotic dimension reduction we use a slightly modified algorithm presented in [Panasenko, 2005] for contact less rods. For the second contact problem only the equilibrium- and force equations are asymptotically deduced. Moreover the results obtained for the dimension reduction of the problem modeling a hanging rod in contact with a rigid body at its lower end are illustrated numerically. The numerical results show that for diminishing thickness of the rod the error between the numerical and analytical solution also diminishes.

We conclude that the dimension reduction of a contact-elasticity problem with Robin-type boundary condition at the contact area effectively leads to an mixed ODE with Robin-type boundary at the contact area. Moreover we showed that the error between the analytical solution of this ODE and the corresponding 3D solution behaves similar as the error between the solution of the Euler-Bernoulli ODE and the 3D solution corresponding to a contact less hanging rod.

6 Outlook

The Fredholm-alternative permits analyzing the existence of solutions of Neumann boundary value problems. This is sufficient for the solution of the inner Neumann problems for the functions $\boldsymbol{N}_k$. Yet the boundary layer corrector functions $\boldsymbol{\Xi}$ and ${}^s\boldsymbol{\Theta}$ shall not only exist but stabilize exponentially to zero when leaving the boundary on which they are concentrated. For the corrector function $\boldsymbol{\Xi}$ concentrated at one end in the case of a free end this property has been shown in [Panasenko, 2005]. The theory that permits analyzing solutions in unbounded domains on exponential decay at infinity are the Saint Venants principle and Theorems of Phragmen-Lindelof type, see [Oleinik, Shamaev, Yosifian, 1992]. In a continuation of this work, the exponential decay of the boundary layer corrector functions $\boldsymbol{\Xi}$ and ${}^s\boldsymbol{\Theta}$ will be studied. In this work it is shown that the bending and tension forces are proportional to the displacement on the contact area. A subject of further investigation is to show that the bending moment is proportional to the bending angle, in order to complete the limiting equation in Section 3. These results are to be illustrated numerically too. Moreover the convergence of the solution and the error estimate are to be analyzed.

7 Appendix

7.1 The Fredholm-alternative

The Fredholm-alternative is functional-analytical tool that permits linking the right hand sides of Neumann boundary value problems with the solvability of the problems themselves.
In the dimension reduction of boundary value problems for elastic structures, for instance beams or plates it is widely used. Since the Fredholm-alternative permits linking the right hand sides of Neumann boundary value problems with the solvability of the problem itself new equations for the right hansides are obtained. This property is of interest in the dimension reduction since the right hand sides are often of a lower dimension.

Theorem 3. (Fredholm-alternative) *Let* X, Y *be Banach spaces and* $\mathrm{A} \in \mathcal{L}(X, Y)$ *a Fredholm operator with the property that* $dim\mathcal{N}(\mathrm{A}) = codim\mathcal{R}(\mathrm{A})$.

$$\begin{aligned}
&\textit{Then:} \\
&\textit{Either } \mathrm{A} \textit{ is bijective} \qquad (7.1) \\
&\ \textit{or else} \\
&\mathrm{A} \textit{ is not bijective and } \mathrm{A}\, x = f \textit{ is solvable if and only if} \\
&f \perp \mathcal{N}(\mathrm{A}^*) \qquad (7.2)
\end{aligned}$$

For a simple Neumann problem for instance the application of the Fredholm alternative looks as follows.
Let the following Neumann boundary problem be given.

$$\begin{aligned}
-\Delta u &= f && \text{in } \Omega \qquad (7.3) \\
\nabla u \cdot n &= g && \text{on } \partial\Omega
\end{aligned}$$

Then the variational formulation reads:

$$\begin{aligned}
&\text{Find } u \in H^{1,2}(\Omega) \text{ s.t.} \\
&\int_\Omega \nabla u \nabla v dx = \int_\Omega f v dx + \int_{\partial\Omega} g v ds \quad \forall v \in H^{1,2}(\Omega)
\end{aligned} \tag{7.4}$$

We set $a(u,v) = \int_\Omega \nabla u \nabla v dx$ and $F(v) = \int_\Omega f v dx + \int_{\partial\Omega} g v ds$. Then (7.4) reads:

$$\begin{aligned}
&\text{Find } u \in H^1(\Omega) \text{ s.t.} \\
&a(u,v) = F(v) \quad \forall v \in H^1(\Omega)
\end{aligned} \tag{7.5}$$

The operator A associated to $a(\cdot,\cdot)$ by $a(u,v) = (\mathrm{A}\, u, v)$ is a Fredholm operator satisfying the hypotheses of Theorem (3), see [Dobrowolski, 2006]. Hence the Fredholm alternative is valid. We remark that $\mathcal{N}(\mathrm{A}^*) =< 1 >$ since:

$$\begin{aligned}
&v \in \mathcal{N}(\mathrm{A}^*) \Rightarrow (u, \mathrm{A}^*\, v) = 0 \quad \forall u \in H^1(\Omega) \\
&\Rightarrow (\mathrm{A}\, v, v) = 0 \Rightarrow \|\nabla v\|_2 = 0 \\
&\Rightarrow v = const\ dx\text{−almost everywhere}
\end{aligned}$$

Since $\mathrm{A}\, u = 0$ for $u = const$ A is not injective and hence not bijective, therefore by the Fredholm alternative (7.5) is solvable if and only if:

$$\mathrm{A}\, u \perp \mathcal{N}(\mathrm{A}^*)$$

Where

$$\mathrm{A}\, u \perp < 1 > \Leftrightarrow (\mathrm{A}\, u, 1) = 0 \Leftrightarrow F(1) = \int_\Omega f dx + \int_{\partial\Omega} g ds = 0$$

Consequently (7.5) is solvable if and only if:

$$\int_\Omega f dx + \int_{\partial\Omega} g ds = 0 \tag{7.6}$$

We remark, that the necessity of (7.6) for the solvability of (7.3) in the strong sense can be shown applying the Gauss formula.

Remark 10. (The Robin-type boundary value problem) *We point out that neither Dirichlet nor Robin-type boundary value problems require the solvability condition* (7.6)*. This is due to the fact that the Lax-Milgram Theorem* (4) *holds for Dirichlet and Robin-type boundary value problems fulfilling the theorems hypotheses. For a Robin-type boundary value problem (of linear elasticity) the hypotheses of the Lax-Milgram theorem are fulfilled, see for instance [Oleinik, Shamaev, Yosifian, 1992, p.317] or [Steinbach, 2003, p.75]. Hence for Robin-type linear elasticity problems the operator* A_R *associated to the bilinear form* $a_R(\cdot,\cdot)$ *by* $a_R(u,v) = (\mathrm{A}_R\, u, v)$ *is bijective. Therefore the alternative* (7.1) *holds, which imposes no restriction on the right hand sides.*

Theorem 4. ([Alt, 1999, p.147](Lax-Milgram theorem)) *Let* X *be a Hilbert space over the field* $\mathbb{K}$ *and* $a : X \times X \to \mathbb{K}$ *a sesquilinear form such that for all* $x, y \in X$*:*

$$\text{(i) } |a(x,y)| \leq C_0 ||x||_X ||y||_X \qquad \text{(continuous)}$$
$$\text{(ii) } |a(x,y)| \geq c_0 ||x||_X^2 \qquad \text{(coercive)}$$

for some $0 \leq c_0 \leq C_0 < \infty$*. Then there exists a unique bijective mapping* $A \in L(X)$ *such that*

$$a(y,x) = (y, Ax)_X \;\; \forall x, y \in X$$

Furthermore,

$$||A|| \leq C_0 \text{ and } ||A^{-1}|| \leq \frac{1}{c_0}$$

7.2 The matrix of rigid displacements

In this Subsection we justify that in this work the matrix of rigid displacements and its extension were neglected. We refer to results from [Panasenko, 2005] and use the variables of that work. Therefore we notice that in [Panasenko, 2005] x_1 is the longitudinal variable and x_2 and x_3 are cross-sectional coordinates. The matrix of rigid displacements referred to in Assumption (2) and its extension have the the form

$$\boldsymbol{\Phi} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -ay_3 \\ 0 & 0 & 1 & ay_2 \end{pmatrix}, \quad \tilde{\boldsymbol{\Phi}} = \begin{pmatrix} 1 & 0 & 0 & 0 & -y_2 & -y_3 \\ 0 & 1 & 0 & -ay_3 & y_1 & 0 \\ 0 & 0 & 1 & ay_2 & 0 & y_1 \end{pmatrix} \tag{7.7}$$

where $a = \left(\frac{1}{|\omega|}\int_\omega (y_2^2 + y_3^2) d\tilde{\boldsymbol{y}}\right)^{-\frac{1}{2}}$
see [Panasenko, 2005, p.56,66]. In [Panasenko, 2005] these matrices influence the dimension reduction algorithm. To show that influence we first notice that the constant vectors introduced in Definition (7) induce the matrices

$$\boldsymbol{H}_k = (\boldsymbol{h}_k^i)_{i=1,2,3}, \quad {}^{\Xi}\boldsymbol{H}_k = ({}^{\Xi}\boldsymbol{h}_k^i)_{i=1,2,3}$$

The corresponding matrices in [Panasenko, 2005] that are responsible for tension and bending equilibrium and force equations have the form

$$\boldsymbol{H}_2^B = \begin{pmatrix} -E & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & M \end{pmatrix}, \qquad \boldsymbol{H}_4^B = \begin{pmatrix} C_1 & 0 & 0 & 0 \\ 0 & -EI & 0 & 0 \\ 0 & 0 & -EI & 0 \\ 0 & 0 & 0 & C_4 \end{pmatrix} \tag{7.8}$$

$$\boldsymbol{H}_1^N = -\begin{pmatrix} -E & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & M \end{pmatrix}, \qquad \boldsymbol{H}_3^N = -\begin{pmatrix} C_1 & 0 & 0 & 0 \\ 0 & -EI & 0 & 0 \\ 0 & 0 & -EI & 0 \\ 0 & 0 & 0 & C_4 \end{pmatrix} \tag{7.9}$$

We remark that only the first 4 rows were taken, since the other rows do have no influence on the equilibrium and force equations. $\boldsymbol{\Phi}$ and $\tilde{\boldsymbol{\Phi}}$ are matrix coefficients of these matrices and the solvability correctors in [Panasenko, 2005] read as $\boldsymbol{\Phi}\boldsymbol{H}_2^B$, $\boldsymbol{\Phi}\boldsymbol{H}_4^B$, $\tilde{\boldsymbol{\Phi}}\boldsymbol{H}_1^N$ and $\tilde{\boldsymbol{\Phi}}\boldsymbol{H}_3^N$. Let us point out, that only the first three rows of these solvability correctors influence equilibrium-and force equations. Further we notice that the fourth column has only an influence on the torsion, see [Panasenko, 2005, p.64/71-72].

Lemma 10. (The matrices of rigid displacements have no influence on the equilibrium and force equations) *Let* $\boldsymbol{\Phi}$, $\tilde{\boldsymbol{\Phi}}$, $\boldsymbol{H}_2^B$, $\boldsymbol{H}_4^B$, $\boldsymbol{H}_1^N$ *and* $\boldsymbol{H}_3^N$ *be given as above. Then*

$$(\boldsymbol{\Phi}\boldsymbol{H}_2^B)^i = (\boldsymbol{H}_2^B)^i, \qquad (\boldsymbol{\Phi}\boldsymbol{H}_4^B)^i = (\boldsymbol{H}_4^B)^i \tag{7.10}$$

$$(\tilde{\boldsymbol{\Phi}}\boldsymbol{H}_1^N)^i = (\boldsymbol{H}_1^N)^i, \qquad (\tilde{\boldsymbol{\Phi}}\boldsymbol{H}_3^N)^i = (\boldsymbol{H}_3^N)^i \tag{7.11}$$

for $i = 1, 2, 3$

Proof. A straight forward multiplication of the sparse matrices (7.7), (7.8) and (7.9) proofs the Lemma. □

Therefore it is justified to neglect the matrices $\boldsymbol{\Phi}$ and $\tilde{\boldsymbol{\Phi}}$ in this work, since in this work only equilibrium equations and force equations are derived.

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