Asymptotic Order of the Parallel Volume Difference

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Abstract

In this paper we investigate the asymptotic behaviour of the parallel volume of fixed non-convex bodies in Minkowski spaces as the distance r tends to infinity. We will show that the difference of the parallel volume of the convex hull of a body and the parallel volume of the body itself can at most have order r^{d-2} in a d-dimensional space. Then we will show that in Euclidean spaces this difference can at most have order r^{d-3}. These results have several applications, e.g. we will use them to compute the derivative of $f_\mu(rK)$ in $r = 0$, where f_μ is the Wills functional or a similar functional, K is a body and rK is the Minkowski-product of r and K. Finally we present applications concerning Brownian paths and Boolean models and derive new proofs for formulae for intrinsic volumes.

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1 Introduction

The parallel volume of a body $K \subseteq \mathbb{R}^d$ at distance $r \geq 0$ w.r.t. to a norm on $\mathbb{R}^d$ is the Lebesgue measure of the set of all points which have (w.r.t. the chosen norm) at most distance r from K, where by body we mean a non-empty compact subset of $\mathbb{R}^d$. Since Steiner [19] discovered in 1840 that the parallel volume w.r.t. the Euclidean norm of certain convex bodies is a polynomial (meanwhile it is known that this is true for all convex bodies and all norms), it has been studied intensively. Essential concepts of convex geometry, like intrinsic volumes, mixed volumes and support measures, are usually defined with help of the parallel volume or local versions of it. Moreover, the parallel volume has many applications, e.g. in stochastic geometry, geometric functional analysis or statistics. While in many of these applications the parallel volume of arbitrary bodies is of interest, it has been mainly investigated in the special case of convex bodies.

However, there are some important results for the parallel volume of non-convex bodies. The Brunn-Minkowski-inequality, which gives an upper bound for the parallel volume, was proven by Lusternik [16] for arbitrary bodies and arbitrary norms. It implies the isoperimetric inequality and is closely related to many other inequalities in various branches of mathematics and physics [4]. Kneser [15] and Sz.-Nagy [21] obtained inequalities saying that the Euclidean parallel volume, i.e. the parallel volume w.r.t. the

Euclidean norm, of a fixed body considered as function of the distance cannot be "too convex". Heveling, Hug and Last [6] showed that a planar body can only have polynomial Euclidean parallel volume if it is convex (see also [9]).

It is well known that the set of all points having at most distance r w.r.t. a certain norm from a body $K \subseteq \mathbb{R}^d$, which is called *parallel body* of K at distance r, equals $K+rB$, where $K+L := \{x+y \mid x \in K, y \in L\}$ is the Minkowski-sum of two bodies $K, L \subseteq \mathbb{R}^d$, $rK := \{rx \mid x \in K\}$ is the Minkowski-product of a body $K \subseteq \mathbb{R}^d$ and a number $r \geq 0$ and B is the so-called *gauge body* of the chosen norm, i.e. the set of all points whose norm is at most 1.

In [12] we have shown

$$\lim_{r\to\infty} V_2((\operatorname{conv} K) + rB) - V_2(K + rB) = 0, \tag{1}$$

for an arbitrary body $K \subseteq \mathbb{R}^2$ and convex bodies $B \subseteq \mathbb{R}^2$ fulfilling a certain smoothness assumption, where V_d denotes the d-dimensional Lebesgue measure and $\operatorname{conv} K$ denotes the convex hull of a body $K \subseteq \mathbb{R}^d$. In the present paper we will show that the order of the convergence in (1) is $1/r$ if B is a convex body which contains a (Euclidean) ball (of positive radius) as summand, where a convex body S is said to be a summand of a convex body B if there is a convex body K such that $B = K + S$. This result has an extension to higher dimensions, namely in dimension d the asymptotic order of $V_d((\operatorname{conv} K) + rB) - V_d(K + rB)$, which is called *parallel volume difference*, is (at most) r^{d-3}, provided that B is a convex body that has a ball as summand. The condition that B has a ball as summand is indeed necessary. In the planar case the fact that the order of $V_2((\operatorname{conv} K) + rB) - V_2(K + rB)$ is $1/r$ for any body $K \subseteq \mathbb{R}^2$ characterises gauge bodies B that have a ball as summand and still in higher dimension we will give an example, where K is a two-pointed set and B is a polytope, showing that the parallel volume difference can have order r^{d-2}. We will show that orders higher than r^{d-2} cannot occur. In fact, these results hold for the expected value of the parallel volume difference of random bodies if certain integrability conditions are fulfilled. We will also discuss the case that the dimension (of the affine hull) of K is larger than that of B.

Motivated by Vitale's work on the Wills functional (see [23] and the literature cited in there), we introduced in [11] a large class of functionals f_μ similar to the Wills functional. For a signed measure μ on $\mathcal{K}$ fulfilling certain integrability assumptions we put

$$f_\mu : \mathcal{C} \to \mathbb{R}_0^+,\ K \mapsto \int_{\mathcal{K}} V_d(K + A)\, d\mu(A),$$

where $\mathcal{C}$ denotes the set of all bodies, i.e. of all non-empty, compact subsets of $\mathbb{R}^d$, and $\mathcal{K}$ denotes the set of all convex bodies. As a consequence of the results about the asymptotic order of the parallel volume difference for large r mentioned above, the derivative of $\mathbb{R}_0^+ \to \mathbb{R},\ r \mapsto f_\mu(rK)$ in $r = 0$ is $d \int_{\mathcal{K}} V(\operatorname{conv} K[1], A[d-1])\, d\mu(A)$ for every body $K \subseteq \mathbb{R}^d$, where $V(L[j], L'[d-j])$ denotes the mixed volume (which will be introduced of page 5) of two convex bodies $L, L' \subseteq \mathbb{R}^d$. Moreover we will show that, if the second derivative exists, then it equals $d(d-1) \int_{\mathcal{K}} V(\operatorname{conv} K[2], A[d-2])\, d\mu(A)$ and we will give an sufficient condition for the second derivative to exist.

Since the contact distribution of a Boolean model is a functional of the expected parallel volume of its typical grain (see [18, section 2.4, 4.3 and 9.1]), we can use the results about the derivative of $f_\mu(rK)$ in order to derive a limit theorem about the contact distribution of a Boolean model as the intensity tends to zero including the asymptotic speed of this convergence.

As another application of the results about the derivatives of $f_\mu(rK)$, we will give new proofs for formulae that express the first and second intrinsic volume of the convex hull of a body as an expected value of certain geometric functionals of this body evaluated at a standard Gaussian random variable.

This paper is organized as follows. In Section 2 we collect mathematical tools, especially from geometry, that will be needed in later sections. In Section 3 we first show that the asymptotic order of $V_d(\operatorname{conv} K + rB) - V_d(K + rB)$ is r^{d-2} with the generalisations mentioned above. Then we will prove that in Minkowski spaces whose gauge bodies B have a ball as summand this difference has asymptotic order r^{d-3} and that this property characterizes Minkowski spaces whose gauge bodies B have a ball as summand in the planar case. In Section 4 we will differentiate $\mathbb{R}_0^+ \to \mathbb{R}$, $r \mapsto f_\mu(rK)$. In Section 5 we examine the asymptotic behaviour of the parallel volume of Brownian paths as the time tends to zero, present and prove the results about the contact distribution of Boolean models and prove the formulae for intrinsic volumes mentioned above.

2 Preparation

In this section we collect tools, especially from geometry, that will be needed in later sections.

We started with a weakened notion of twice differentiable, that is due to Alexandrov.

Def. 1. Let $G \subseteq \mathbb{R}^d$ be an open convex set and $f : G \to \mathbb{R}$ a convex function.

(i) A vector $\nu \in \mathbb{R}^d$ is called a subgradient of f in $x \in G$ if $f(y) \geq f(x) + \langle \nu, y - x \rangle$ for all $y \in G$.

(ii) A function $\theta : G \to \mathbb{R}^d$ is called a choice of subgradients of f if $\theta(x)$ is a subgradient of f in x for every $x \in G$.

(iii) Let $x \in G$. If all choices of subgradients of f are differentiable in x with the same derivative, then f is said to be Alexandrov-twice-differentiable in x. The derivative of the choices of subgradients is called second derivative of f.

Theorem 2. *Let $G \subseteq \mathbb{R}^d$ be an open convex set and $f : G \to \mathbb{R}$ a convex function. Then f is Alexandrov-twice-differentiable in a.e. $x \in G$.*

Various proofs of this theorem are known. For a discussion, see [17, section 1.5, note 2].

A vector $u \in \mathbb{R}^d \setminus \{0\}$ is called *exterior normal vector* of a convex set $K \subseteq \mathbb{R}^d$ in a point $p \in K$ if

$$\langle x, u \rangle \leq \langle p, u \rangle, \quad x \in K.$$

For a convex body K the *support function* is defined by

$$h_K : \mathbb{R}^d \to \mathbb{R},\ u \mapsto \max\{\langle x, u \rangle \mid x \in K\}.$$

Corollary 1.7.3 in [17] says:

Theorem 3. *Let $K \subseteq \mathbb{R}^d$ be a convex body and $u \in \mathbb{R}^d \setminus \{0\}$. Then h_K is differentiable in u iff there is a unique point $p \in K$ with exterior normal vector u. In this case $\nabla h_K(u) = p$.*

A convex subset F of a convex body K is called *face* of K if for all $x, y \in K$ with $\frac{x+y}{2} \in F$ we have $x, y \in F$. The *dimension* of a convex set is said to be the dimension of its affine hull. The *relative interior*, $\operatorname{relint} K$, of a convex set $K \subseteq \mathbb{R}^d$ is its interior w.r.t. its affine hull as surrounding topological space. By $\operatorname{bd} A$ we denote the boundary of a set $A \subseteq \mathbb{R}^d$.

Lemma 4. *Let $K \subseteq \mathbb{R}^d$ be a body and $x \in (\operatorname{bd} \operatorname{conv} K) \setminus K$. Then x is contained in the relative interior of a face of positive dimension of* $\operatorname{conv} K$.

Proof. According to [17, Theorem 2.1.2] the point x is contained in the relative interior of a face F of $\operatorname{conv} K$. So all we have to show is $F \neq \{x\}$. Since $(\operatorname{conv} K) \setminus F$ is convex, we cannot have $K \subseteq (\operatorname{conv} K) \setminus F$ by the definition of the convex hull. So $K \cap F \neq \emptyset$ and $x \notin K$ implies $F \neq \{x\}$. □

A convex body $S \subseteq \mathbb{R}^d$ is called *summand* of a convex body $K \subseteq \mathbb{R}^d$ if for each point $p \in K$ there is a vector $t \in \mathbb{R}^d$ with

$$p \in t + S \subseteq K,$$

or, equivalently, if there is a convex body $M \subseteq \mathbb{R}^d$ such that $S + M = K$. For a more detailed introduction we refer to [17, sections 3.1 and 3.2].

We let B^d denote the Euclidean unit ball.

Lemma 5. *Let K be a body with a summand RB^d, $R > 0$, $b \in \operatorname{bd} K$ a point with exterior unit normal ν and $t \in K$ another point. Then the following are equivalent:*

(i) $t = b - R\nu$

(ii) $b \in t + RB^d \subseteq K$

Proof. First we will show "*(ii)* ⇒ *(i)*". From (ii) we get $t + R\nu \in K$ and hence

$$\langle t, \nu \rangle + R = \langle t + R\nu, \nu \rangle \leq \langle b, \nu \rangle,$$

which implies $\langle b - t, \nu \rangle \geq R$. Since $\|b - t\| \leq R$, we conclude $b - t = R\nu$ and so obtain (i).

Since there is a point t satisfying (ii), the converse statement must hold as well. □

We let $B \subseteq \mathbb{R}^d$ be a convex body with $0 \in \operatorname{int} B$, called *gauge body* in the following. For a closed set $A \subseteq \mathbb{R}^d$ and $x \in \mathbb{R}^d$ we define the B-distance from x to A to be

$$d_B(A, x) := \min\{t \geq 0 \mid x \in A + tB\}.$$

For $x, y \in \mathbb{R}^d$ we put

$$d_B(y, x) := d_B(\{y\}, x).$$

Then it is easy to see that d_B is a metric that is induced by a norm and hence that $d_B : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}_0^+$ is continuous. Moreover, $B = \{y \in \mathbb{R}^d \mid d_B(y, 0) \leq 1\}$ and hence the terms "gauge body" introduced here and on page 2 coincide.

For a closed set $A \subseteq \mathbb{R}^d$ and $x \in \mathbb{R}^d$ we put

$$\Pi_B(A, x) := \{y \in A \mid d_B(y, x) = d_B(A, x)\}.$$

The *B-exoskeleton* $\mathrm{exo}_B(A)$ of A is the set of all points $x \in \mathbb{R}^d$, for which $\Pi_B(A, x)$ consists of more than one point. For $x \in \mathbb{R}^d \setminus \mathrm{exo}_B(A)$ we define the *B-metric projection* $p_B(A, x)$ of x onto A to be the unique point in $\Pi_B(A, x)$, and if moreover $x \notin A$, we put $u_B(A, x) := (x - p_B(A, x))/d_B(A, x)$. The *B-normal bundle* of A is

$$\mathcal{N}_B(A) := \{(p_B(A, x), u_B(A, x)) \mid x \in \mathbb{R}^d \setminus A \setminus \mathrm{exo}_B(A)\}.$$

Now we will introduce the mixed volumes, which are the analogues to the intrinsic volumes in Minkowski spaces, and certain measures that provide information about "where" the mixed volumes are.

The *mixed volumes* $V(K[j], B[d-j])$, $j = 0, \ldots, d$, of convex bodies $K, B \subseteq \mathbb{R}^d$ are the uniquely determined numbers such that

$$V_d(K + rB) = \sum_{j=0}^{d} r^{d-j} \binom{d}{j} V(K[j], B[d-j]), \qquad r \geq 0.$$

For further information on mixed volumes, see e.g. [17, Section 5.1].

Now assume that B is a strictly convex body satisfying $0 \in \operatorname{int} B$. It is known that for convex bodies $K \subseteq \mathbb{R}^d$ we have $\mathrm{exo}_B(K) = \emptyset$. Hence we can consider

$$\mu_r^B(K, \eta) := V_d\big(\{x \in (K + rB) \setminus K \mid (p_B(K, x), u_B(K, x)) \in \eta\}\big)$$

for $r \in \mathbb{R}_0^+$ and Borel-sets $\eta \subseteq \mathbb{R}^d \times \mathbb{R}^d$. There are uniquely determined measures $C_j^B(K, \cdot)$, $j = 0, \ldots, d-1$, called *relative support measures*, on $\mathbb{R}^d \times \mathbb{R}^d$ with

$$\mu_r^B(K, \eta) = \sum_{j=0}^{d-1} r^{d-j} \kappa_{d-j} C_j^B(K, \eta)$$

for $r \in \mathbb{R}_0^+$ and Borel-sets $\eta \subseteq \mathbb{R}^d \times \mathbb{R}^d$ (see e.g. [8]). Their projections on the first component,

$$\Phi_j^B(K, \beta) := C_j^B(K, \beta \times \mathbb{R}^d), \quad j = 0, \ldots, d-1,\ \beta \in \mathcal{B}(\mathbb{R}^d),$$

are called *relative curvature measures*. Their projections on the second component,

$$\Psi_j^B(K, \omega) := C_j^B(K, \mathbb{R}^d \times \omega), \quad j = 0, \ldots, d-1,\ \omega \in \mathcal{B}(\mathbb{R}^d),$$

are called *relative area measures*.

The total masses of the measure defined above are, up to normalization, the mixed volumes. More precisely, for a strictly convex body $B \subseteq \mathbb{R}^d$ with $0 \in \operatorname{int} B$, a convex body $K \subseteq \mathbb{R}^d$ and $j \in \{0, \ldots, d-1\}$ we have

$$\Phi_j^B(K, \mathbb{R}^d) = \Psi_j^B(K, \mathbb{R}^d) = C_j^B(K, \mathbb{R}^d \times \mathbb{R}^d) = \frac{\binom{d}{j}}{\kappa_{d-j}} V(K[j], B[d-j]). \tag{2}$$

The Hausdorff measure $\mathcal{H}^j(A)$ of a subset $A \subseteq \mathbb{R}^d$ is the length, area, volume etc. of A, provided that A is j-dimensional. For a precise introduction see e.g. [3].

For a convex body $K \subseteq \mathbb{R}^d$ the measure $S_{d-1}(K, \omega) := 2\Psi_{d-1}^{B^d}(K, \omega)$, $\omega \in \mathcal{B}(\mathbb{R}^d)$, which is concentrated on the unit sphere S^{d-1}, is called *surface area measure*. Its name derives from the fact (see e.g. [17, (4.2.24)]) that

$$S_{d-1}(K, \omega) = \mathcal{H}^{d-1}(\tau_K(\omega)), \quad \omega \in S^{d-1}, \tag{3}$$

where $\tau_K(\omega)$ is the set of all boundary points of K having an exterior unit normal vector in ω, the so-called *reverse spherical image* of ω.

For convex bodies $K, B \subseteq \mathbb{R}^d$ the *mixed area measures* are defined to be the measures $S(K[j], B[d-j-1], \cdot)$, $j = 0, \dots, d-1$, on $\mathbb{R}^d$ with

$$S_{d-1}(sK + rB, \omega) = \sum_{j=0}^{d-1} \binom{d-1}{j} s^j r^{d-j-1} S(K[j], B[d-j-1], \omega), \quad r, s \geq 0, \tag{4}$$

for any Borel-set $\omega \subseteq \mathbb{R}^d$. For further information on mixed area measures we refer to [17, section 5.1].

The mixed area measures are related to the relative area measures. In order to make this relationship precise, we define the *reverse spherical image map* of a strictly convex body $L \subseteq \mathbb{R}^d$ to be the map $\tau_L : S^{d-1} \to \operatorname{bd} L$ that assigns to a vector $u \in S^{d-1}$ the point $p \in \operatorname{bd} L$ with exterior unit normal vector u. Since for a strictly convex body $L \subseteq \mathbb{R}^d$ the image of a set ω under the reverse spherical image map is its reverse spherical image, the use of the same symbol is no problem.

Remark 6. From Theorem 3 we get that L is strictly convex iff h_L is differentiable on $\mathbb{R}^d \setminus \{0\}$, and in this case we have $\nabla h_L(u) = \tau_L(\frac{u}{\|u\|})$ for all $u \in \mathbb{R}^d \setminus \{0\}$.

The relationship between relative area measures and mixed area measures is given by Theorem 2.14 in [7], which says:

Theorem 7. *Let $K, B \subseteq \mathbb{R}^d$ be two convex bodies such that $0 \in \operatorname{int} B$ and B is strictly convex. Then we have for $j \in \{0, \dots, d-1\}$ and Borel-sets $\gamma \subseteq \mathbb{R}^d$*

$$\Psi_j^B(K, \gamma) = \frac{\binom{d}{j}}{d\kappa_{d-j}} \int_{S^{d-1}} \mathbf{1}_\gamma(\nabla h_B(u)) h_B(u)\, S(K[j], B[d-j-1], du).$$

A point $x \in \mathbb{R}^d$ is called an *extreme point* of a convex body $L \subseteq \mathbb{R}^d$ if $\{x\}$ is a face of L. Now we will show that the 0-th relative curvature measure of a convex body is concentrated on the set of its extreme points. For the Euclidean curvature measure this is well-known (see e.g. [17, (4.6.1)]). We remark that the set $\operatorname{ext} L$ of extreme points of a convex body $L \subseteq \mathbb{R}^d$ is an intersection of countably many open sets by [17, p. 66] and hence measurable.

Theorem 8. *Let $B \subseteq \mathbb{R}^d$ be a convex body with $0 \in \operatorname{int} B$, whose support function h_B is twice continuously differentiable on $\mathbb{R}^d \setminus \{0\}$. Let $L \subseteq \mathbb{R}^d$ be a convex body. Then the relative curvature measure $\Phi_0^B(L, \cdot)$ is concentrated on $\operatorname{ext} L$.*

Lemma 9. *Let $B, K \subseteq \mathbb{R}^d$ be two convex bodies such that $0 \in \operatorname{int} B$ and B is strictly convex. Then with $M := \max\{h_B(u) \mid u \in S^{d-1}\}$ we have for any Borel-set $\gamma \subseteq \operatorname{bd} B$*

$$\Psi_0^B(K, \gamma) \leq \frac{M}{d\kappa_d} \cdot \mathcal{H}^{d-1}(\gamma).$$

Proof. From Theorem 7 and Remark 6 we get, since $S(K[0], B[d-1], \cdot) = S_{d-1}(B, \cdot)$,

$$\begin{aligned}
\Psi_0^B(K, \gamma) &= \frac{1}{d\kappa_d} \int_{S^{d-1}} \mathbf{1}_\gamma(\nabla h_B(u)) h_B(u) S_{d-1}(B, du) \\
&\leq \frac{1}{d\kappa_d} \int_{S^{d-1}} \mathbf{1}_\gamma(\tau_B(u)) M S_{d-1}(B, du) \\
&= \frac{M}{d\kappa_d} S_{d-1}(B, \{u \in S^{d-1} \mid \tau_B(u) \in \gamma\}).
\end{aligned}$$

Since $\tau_B : S^{d-1} \to \operatorname{bd} B$ is surjective, we derive from (3) that

$$S_{d-1}(B, \{u \in S^{d-1} \mid \tau_B(u) \in \gamma\}) = \mathcal{H}^{d-1}(\gamma),$$

which completes the proof of the lemma. □

Proof of Theorem 8. Since h_B is differentiable, B must be strictly convex by Remark 6. Hence $\Phi_0^B(L, \cdot)$ is defined. Let $\omega \subseteq S^{d-1}$ be the set of all (Euclidean) exterior unit normal vectors of L in points of $\operatorname{bd} L \setminus \operatorname{ext} L$. Since for every vector $u \in \omega$ there is more than one point in $\operatorname{bd} L$ having exterior normal vector u, [17, Theorem 2.2.9] implies $\mathcal{H}^{d-1}(\omega) = 0$. Let γ denote the set of all relative exterior normal vectors of L in points of $\operatorname{bd} L \setminus \operatorname{ext} L$. By [8, Lemma 2.1] we have $\gamma = \{\nabla h_B(u) \mid u \in \omega\}$. Since h_B is assumed to be twice continuously differentiable, ∇h_B is Lipschitz-continuous with Lipschitz-constant L, say. By Theorem 1 from [3, section 2.4] this implies

$$\mathcal{H}^{d-1}(\gamma) \leq L^{d-1} \cdot \mathcal{H}^{d-1}(\omega) = 0.$$

Because the relative support measure $C_0^B(L, \cdot)$ is concentrated on the relative normal bundel $\mathcal{N}_B(L)$, we get from Lemma 9 that

$$\begin{aligned}
\Phi_0^B(L, \operatorname{bd} L \setminus \operatorname{ext} L) &= C_0^B(L, (\operatorname{bd} L \setminus \operatorname{ext} L) \times \mathbb{R}^d) \\
&= C_0^B(L, (\operatorname{bd} L \setminus \operatorname{ext} L) \times \gamma) \\
&\leq \Psi_0^B(L, \gamma) \\
&\leq \frac{M}{d\kappa_d} \cdot \mathcal{H}^{d-1}(\gamma) \\
&= 0. \qquad \square
\end{aligned}$$

We finish this section by a continuity result.

For this, we define the *Hausdorff-metric* on $\mathcal{C}$ by

$$d^H(K, L) := \min\{t \geq 0 \mid K \subseteq L + tB^d \text{ and } L \subseteq K + tB^d\}.$$

For further details we refer to [17, section 1.8].

We let $\mathcal{K}_0$ denote the set of all convex bodies with interior points.

Lemma 10. *For a fixed body $K \in \mathcal{C}$ the map*

$$\mathcal{K}_0 \to \mathbb{R}_0^+,\ B \mapsto V_d(K+B)$$

is continuous w.r.t. the Hausdorff-metric.

Proof. Let $B \in \mathcal{K}_0$. Then B contains a ball of radius $R > 0$, say, which has w.l.o.g. its center at the origin. Let $\epsilon > 0$. Since

$$\mathbb{R}_0^+ \to \mathbb{R}_0^+,\ r \mapsto V_d(K+rB)$$

is continuous according to [5, Lemma 3], there is $\delta \in (0,1)$ with

$$V_d(K+B) - \epsilon < V_d(K+(1-\delta)B) < V_d(K+(1+\delta)B) < V_d(K+B) + \epsilon.$$

Let $\tilde{B} \in \mathcal{K}_0$ be a body whose Hausdorff distance from B is less than $R\delta$. Then

$$B \subseteq \tilde{B} + R\delta B^d \subseteq \tilde{B} + \delta B,$$

which implies $(1-\delta)B \subseteq \tilde{B}$ by the cancelation law for Minkowski sums (see [17, p. 41], and

$$\tilde{B} \subseteq B + R\delta B^d \subseteq B + \delta B.$$

This, however, implies

$$V_d(K+B) - \epsilon < V_d(K+\tilde{B}) < V_d(K+B) + \epsilon. \quad \square$$

3 The main results

In this section we examine the asymptotic order of the parallel volume difference. First we show in Theorem 11 that this difference can at most have order r^{d-2} in a d-dimensional Minkowski spaces. In Theorem 15 we will show that in many Minkowski spaces, and in particular in the Euclidean space, it can at most have order r^{d-3}. The examples 13 and 18 as well as Corollary 19 show that our theorems are in a certain sense optimal.

We will always assume $d > 1$ in the remainder of the paper, since the parallel volume difference is obviously zero for all large enough r as soon as $d = 1$.

We let ρ_B denote the radius of the largest ball contained in a convex body $B \subseteq \mathbb{R}^n$ and $\operatorname{diam} A$ the diameter of a subset $A \subseteq \mathbb{R}^n$. Moreover, we let $\hat{B}$ denote the affine hull of a body $B \subseteq \mathbb{R}^n$ and $B^\perp$ the orthogonal complement of $\hat{B}$. For two bodies $K, B \subseteq \mathbb{R}^n$ we call

$$K_B := \bigcup_{x \in B^\perp} \operatorname{conv}(K \cap (\hat{B} + x))$$

the B-*convexification* of K.

A random closed set is a measurable map from some probability space to the space of all closed subsets of $\mathbb{R}^n$ equipped with the Borel-σ-algebra of the Fell-Matheron-topology (see e.g. [18] for details). A random (convex) body is a random closed set that a.s. takes values in the set of all (convex) bodies in $\mathbb{R}^n$.

We observe that the set $\mathcal{C}$ of non-empty compact sets and the set $\mathcal{K}$ of non-empty, convex compact sets are measurable by [18, Lemma 2.1.2 and Theorem 2.4.2]. The functions $\mathcal{K} \to \mathbb{R}$, $B \mapsto \rho_B$ and $\operatorname{diam} : \mathcal{C} \to \mathbb{R}$ are obviously continuous w.r.t. the Hausdorff-metric (defined on page 7) and hence measurable due to [18, Theorem 12.3.2]. By [18, Theorem 12.3.5] the same holds for $\mathcal{C} \times \mathcal{C} \to \mathcal{C}$, $(K, L) \mapsto K + L$ and $\mathcal{C} \times \mathbb{R}_0^+ \to \mathcal{C}$, $(K, r) \mapsto rK$. The map $\mathcal{C} \times \mathcal{C} \to \mathcal{C}$, $(K, B) \mapsto K_B$ is shown to be measurable in [12, Lemma A.7]. The map $V_n : \mathcal{C} \to \mathbb{R}$ is upper semicontinuous by [18, Theorem 12.3.6] and hence measurable (see [18, p. 19]).

Theorem 11. *Let $1 < d \leq n$. Let $X \subseteq \mathbb{R}^n$ be a random body and $Y \subseteq \mathbb{R}^n$ a random convex body that is d-dimensional a.s. Put $G := \max\{\operatorname{diam} X, 1\}$, $S := \max\{\operatorname{diam} Y, 1\}$ and $R := \rho_Y$. If*

$$c := d2^d \kappa_d \kappa_{n-d} \mathbb{E}\Big[\frac{S^{d-1} \cdot G^n}{R}\Big] < \infty,$$

then we have

$$\mathbb{E}[V_n(X_Y + rY) - V_n(X + rY)] < c \cdot r^{d-2}, \quad r \geq 1.$$

We first prove a lemma making the same statement as Theorem 11 under additional assumptions, in particular $n = d$. For this we need the function

$$\begin{aligned} &w_B : \mathbb{R}_0^+ \to \mathbb{R}_0^+, \\ &\qquad r \mapsto \min\{d_B(y, z) \mid y \in B^d,\, z \in \mathbb{R}^d,\, y \in \Pi_B(0y, z),\, d_B(0, z) = r\}, \end{aligned} \tag{5}$$

which is defined for convex bodies $B \subseteq \mathbb{R}^d$ with $0 \in \operatorname{int} B$, where $xy := \{\lambda x + (1 - \lambda)y \mid \lambda \in \mathbb{R}\}$ is the line through x and y if $x \neq y$ and the singleton $\{x\}$ otherwise. For a more detailed introduction see [12].

Lemma 12. *Let $K \subseteq \mathbb{R}^d, d > 1$, be a body and $B \subseteq \mathbb{R}^d$ be a convex body such that h_B is twice differentiable on $\mathbb{R}^d \setminus \{0\}$. Put $G := \max\{\operatorname{diam} K, 1\}$, $S := \max\{\operatorname{diam} B, 1\}$ and*

$$C' := d2^d \kappa_d \frac{S^{d-1} \cdot G^d}{\rho_B}.$$

Then we have for all $r \geq 1$

$$V_d(\operatorname{conv} K + rB) - V_d(K + rB) < C' \cdot r^{d-2}.$$

Proof. By the translation-invariance of the Lebesgue measure we may assume $\rho_B B^d \subseteq B$. Moreover, B is strictly convex, since we assumed h_B to be differentiable on $\mathbb{R}^d \setminus \{0\}$.

We put $L := \operatorname{conv} K$. Then we get by [12, Lemma 3.4]

$$\begin{aligned} V_d(L+&rB) - V_d(K + rB) \\ &= V_d((L + rB) \setminus (K + rB)) \\ &\leq V_d\big(\{x \in \mathbb{R}^d \setminus L \mid p_B(L, x) \in L \setminus \operatorname{ext} L,\ d_B(L, x) \in (Gw_B(\tfrac{r}{G}), r]\} \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \cup (L \setminus (K + rB))\big) \\ &= \sum_{i=1}^{d} \kappa_i \Phi_{d-i}^B(L, L \setminus \operatorname{ext} L)\big(r^i - (Gw_B(\tfrac{r}{G}))^i\big) + V_d(L \setminus (K + rB)). \end{aligned} \tag{6}$$

Now we will derive an upper bound for the right-hand side in (6). The equation

$$r - w_B(r) \le \frac{1}{\rho_B}, \quad r \in \mathbb{R}_0^+. \tag{7}$$

is shown in [12, formula (8)] to be an easy corollary of the triangular inequality for d_B. Since obviously $w_B(s) \le s$ for all $s \in \mathbb{R}^+$, we conclude

$$\begin{aligned} r^i - (Gw_B(\tfrac{r}{G}))^i &= (r - Gw_B(\tfrac{r}{G})) \sum_{j=0}^{i-1} r^j (Gw_B(\tfrac{r}{G}))^{i-1-j} \\ &= G \cdot (\tfrac{r}{G} - w_B(\tfrac{r}{G})) \sum_{j=0}^{i-1} r^j (Gw_B(\tfrac{r}{G}))^{i-1-j} \\ &\le \tfrac{G}{\rho_B} \sum_{j=0}^{i-1} r^j (G\tfrac{r}{G})^{i-1-j} \\ &= \tfrac{G}{\rho_B} \sum_{j=0}^{i-1} r^{i-1} \\ &= \tfrac{G}{\rho_B} i r^{i-1}. \end{aligned}$$

By (2) we have

$$\kappa_i \Phi_{d-i}^B(L, L \setminus \operatorname{ext} L) \le \binom{d}{i} V(L[d-i], B[i]).$$

Since L is contained in the circumsphere of K, whose radius is at most G, and B is contained in a ball of radius S, we get by elementary properties of mixed volumes ([17, p. 277 and (5.1.24)])

$$V(L[d-i], B[i]) \le V(GB^d[d-i], SB^d[i]) = G^{d-i} S^i \kappa_d.$$

Thus

$$\kappa_i \Phi_{d-i}^B(L, L \setminus \operatorname{ext} L) \le \binom{d}{i} G^{d-i} S^i \kappa_d. \tag{8}$$

Using (6), (8) and Theorem 8, which says $\Phi_0^B(L, L \setminus \operatorname{ext} L) = 0$, we obtain

$$\begin{aligned} & V_d(L + rB^d) - V_d(K + rB^d) \\ & \quad \le \sum_{i=1}^{d} \kappa_i \Phi_{d-i}^B(L, L \setminus \operatorname{ext} L)\big(r^i - (Gw_B(\tfrac{r}{G}))^i\big) + V_d(L \setminus (K + rB)) \\ & \quad \le \sum_{i=1}^{d-1} \binom{d}{i} G^{d-i} S^i \kappa_d \cdot \tfrac{G}{\rho_B} i r^{i-1} + \kappa_d G^d \\ & \quad \le \sum_{i=1}^{d-1} (d-1) \binom{d}{i} \kappa_d \frac{S^{d-1} G^d}{\rho_B} r^{d-2} + \kappa_d G^d r^{d-2} \\ & \quad < \Big((d-1) 2^d \kappa_d \frac{S^{d-1} G^d}{\rho_B} + \kappa_d G^d\Big) r^{d-2} \\ & \quad < \Big(d 2^d \kappa_d \frac{S^{d-1} \cdot G^d}{\rho_B}\Big) r^{d-2}. \qquad \square \end{aligned}$$

Proof of Theorem 11. Let $r \geq 1$. We want to prove

$$\mathbb{E}[V_n(X_Y + rY) - V_n(X + rY)] < c \cdot r^{d-2}.$$

Put $X^x := X \cap (\hat{Y} + x)$ and $Z^x := \operatorname{conv} X^x$ for $x \in Y^\perp$.

As an easy consequence of [17, Theorem 3.3.1] there is a sequence $(Y_k)_{k \in \mathbb{N}}$ of random convex bodies lying a.s. in $\hat{Y}$ such that for all $k \in \mathbb{N}$ the support function of Y_k is twice continuously differentiable on $\hat{Y} \setminus \{0\}$, and that $\lim_{k\to\infty} Y_k = Y$ a.s. Due to Lemma 10 we have a.s. for all $x \in Y^\perp$, that

$$V_d(Z^x + rY) - V_d(X^x + rY) = \lim_{k\to\infty} V_d(Z^x + rY_k) - V_d(X^x + rY_k),$$

where $V_d(K)$ denotes for a set K whose affine hull is d-dimensional the value the Lebesgue measure on $\hat{K}$ assigns to K. Putting

$$C' := d2^d \kappa_d \frac{S^{d-1} \cdot G^d}{R},$$

it follows from Lemma 12 that a.s. for all $x \in Y^\perp$ we have

$$V_d(Z^x + rY) - V_d(X^x + rY) = \lim_{k\to\infty} V_d(Z^x + rY_k) - V_d(X^x + rY_k) \leq C' \cdot r^{d-2},$$

since diameter and ρ-number are continuous when considered as functionals $\mathcal{K}_0 \to \mathbb{R}_0^+$. Thus (we will comment on the measurability below)

$$\begin{aligned}
&\mathbb{E}[V_n(X_Y + rY) - V_n(X + rY)] \\
&\quad = \mathbb{E}\Big[\int_{X|Y^\perp} V_d(Z^x + rY) - V_d(X^x + rY)\,dx\Big] \\
&\quad \leq \mathbb{E}\Big[\int_{X|Y^\perp} C' \cdot r^{d-2}\,dx\Big] \\
&\quad = \mathbb{E}\big[V_{n-d}(X|Y^\perp)C'\big] \cdot r^{d-2} \\
&\quad \leq \mathbb{E}\big[\kappa_{n-d} G^{n-d} C'\big] \cdot r^{d-2} \\
&\quad = c \cdot r^{d-2}.
\end{aligned}$$

It remains to show those expressions of the previous equation, whose measurability was not proven before the statement of Theorem 11, are measurable, too. The map $\mathcal{C} \times \mathcal{K} \times \mathbb{R}^n \to \mathcal{C}$, $(X, Y, x) \mapsto X^x$ is measurable by [12, Lemma A.1] and [18, Theorem 12.2.6]. The map $\operatorname{conv} : \mathcal{C} \to \mathcal{K}$ is measurable by [18, Theorem 12.3.5]. As an easy consequence of [18, Theorem 12.3.6] the map $K \mapsto V_d(K)$ on the set of all bodies of $\mathbb{R}^n$, whose affine hull has dimension d, is upper semicontinuous and hence measurable. In the final version of [12] it will be shown that for measurable maps $f : \Omega \times \mathbb{R}^n \to \mathbb{R}$, where Ω denotes the probability space on which Y is defined, $\int_{Y^\perp} f(\omega, x)\,dx$ is a random variable. □

Now we will show that the order r^{d-2} in Theorem 11 is optimal.

Example 13. Put $K := \{-e_1, e_1\}$ and let $B := \operatorname{conv}\{-e_1, e_1, \ldots, -e_d, e_d\}$ be the unit ball of the L_1-norm. Because $V_{d-1}(\operatorname{conv}\{-e_2, e_2, \ldots, -e_d, e_d\}) = 2^{d-1}/(d-1)!$, we have

$$\begin{aligned} &V_d(\operatorname{conv} K + rB) - V_d(K + rB) \\ &\quad = \int_{-1}^{1} V_{d-1}(\{(x_1, \ldots, x_d) \in (\operatorname{conv} K + rB) \setminus (K + rB) \mid x_1 = t\})\, dt \\ &\quad = 2\int_{0}^{1} r^{d-1}2^{d-1}/(d-1)! - (r-t)^{d-1}2^{d-1}/(d-1)!\, dt \\ &\quad = 2^d \sum_{j=0}^{d-2} (-1)^{d-j} \frac{1}{j!(d-j)!} r^j. \end{aligned}$$

While the order r^{d-2} is optimal under the assumptions of Theorem 11, we can show that under an additional assumption on the smoothness of the gauge body Y we have in fact order r^{d-3}.

For a convex body $B \subseteq \mathbb{R}^d$ we put

$$R(B) := \max\{\rho \in \mathbb{R}_0^+ \mid \rho B^d \text{ is summand of } B\}.$$

The Blaschke selection theorem (see [17, Theorem 1.8.4]) and the continuity of Minkowski sums (see [18, Theorem 12.3.5]) imply that the maximum is attained. The following lemma tells us that the map $R : \mathcal{C} \to \mathbb{R}_0^+$ is measurable.

Lemma 14. *The map $R : \mathcal{C} \to \mathbb{R}_0^+$ is upper semicontinuous .*

Proof. Let $(K_i)_{i\in\mathbb{N}}$ be a sequence in $\mathcal{K}$ converging to $K \in \mathcal{K}$. Since $(R(K_i))_{i\in\mathbb{N}}$ is bounded, this sequence has a convergent subsequence, w.l.o.g. the sequence itself. Now there is a sequence $(\rho_i)_{i\in\mathbb{N}}$ in $\mathbb{R}_0^+$ converging to $\lim_{i\to\infty} R(K_i)$ and a sequence $(M_i)_{i\in\mathbb{N}}$ of convex bodies with $K_i = M_i + \rho_i B^d$ for all $i \in \mathbb{N}$. By the Blaschke selection theorem we can assume that $(M_i)_{i\in\mathbb{N}}$ converges to a convex body M. By the continuity of Minkowski sums we have $K = M + \lim_{i\to\infty} R(K_i)B^d$. Hence $R(K) \geq \lim_{i\to\infty} R(K_i)$. $\square$

Theorem 15. *Let $X \subseteq \mathbb{R}^n$ be a random body and $Y \subseteq \mathbb{R}^n$ a d-dimensional random convex body, $1 < d \leq n$. Put $G := \max\{\operatorname{diam} X, 1\}$ and $S := \max\{\operatorname{diam} Y, 1\}$. If*

$$c := d2^{d+2}\kappa_d\kappa_{n-d}\mathbb{E}\Big[\frac{S^d \cdot G^{n+1}}{R(Y)^3}\Big] < \infty,$$

then we have

$$\mathbb{E}[V_n(X_Y + rY) - V_n(X + rY)] < c \cdot r^{d-3}, \quad r \geq 1.$$

The main reason, why we obtain this sharper result now, is that we have instead of (7) the following lemma.

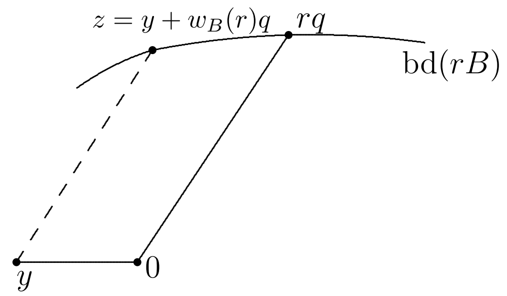
Lemma 16. *Let $B \subseteq \mathbb{R}^d$ be a convex body with interior points, which has a summand RB^d for some $R > 0$ and satisfies $B \subseteq SB^d$ for some $S > 0$. Assume that one largest ball contained in B has its center at the origin. Then, with $C := \frac{4S}{R{\rho_B}^2}$, we have*

$$r - w_B(r) < \frac{C}{r}, \quad r \in \mathbb{R}_0^+.$$

Proof. If $r \leq \frac{4S}{R\rho_B}$, we conclude from (7)

$$r - w_B(r) \leq \frac{1}{\rho_B} \leq \frac{4S}{rR{\rho_B}^2}.$$

So let $r > \frac{4S}{R\rho_B}$ from now on. By (5) there are points $z \in \mathbb{R}^d$ and $y \in B^d$ with $y \in \Pi_B(0y, z)$, $d_B(0, z) = r$ and $d_B(y, z) = d_B(0y, z) = w_B(r)$. Put $q := (z - y)/d_B(y, z)$. Then $q \in \operatorname{bd} B$ due to [12, Lemma 2.1]. Moreover, we let u denote the (Euclidean) exterior normal vector of B in q, which is determined uniquely, since RB^d is a summand of B, and let v denote an arbitrary unit vector perpendicular to u.



Let $t \in [w_B(r), r]$. By [12, Lemma 2.1] there is a common exterior unit normal vector of B in q and of $0y$ in y and hence $\langle y, u\rangle = 0$. From $z = y + w_B(r)q$ we get on the one hand

$$\langle tq - z, u\rangle = \langle tq - w_B(r)q - y, u\rangle = (t - w_B(r))\langle q, u\rangle$$

and thus

$$t - w_B(r) = \frac{\langle tq - z, u\rangle}{\langle q, u\rangle} \tag{9}$$

and on the other hand

$$|\langle tq - z, v\rangle| = |(t - w_B(r))\langle q, v\rangle - \langle y, v\rangle| \leq (t - w_B(r))|\langle q, v\rangle| + 1.$$

From the last two equations we get

$$|\langle tq - z, v\rangle| \leq \frac{\langle tq - z, u\rangle}{\langle q, u\rangle}|\langle q, v\rangle| + 1.$$

Since $q \in B \subseteq SB^d$ we get $|\langle q, v\rangle| \leq S$ and since $\rho_B u \in \rho_B B^d \subseteq B$ and u is exterior normal vector of B in q we get $\langle q, u\rangle \geq \rho_B$. Thus

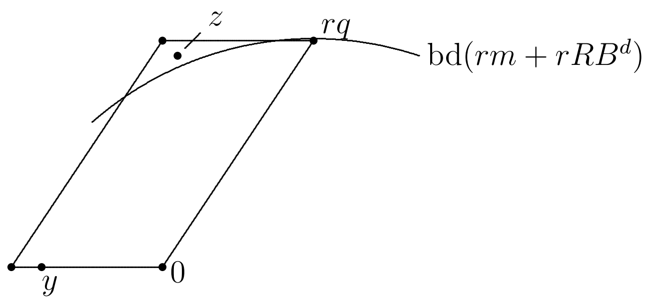
$$|\langle tq - z, v\rangle| \leq \frac{S}{\rho_B}\langle tq - z, u\rangle + 1. \tag{10}$$

We want to show

$$\langle tq - z, u\rangle < \frac{4}{Rr} \tag{11}$$

for all $t \in [w_B(r), r]$. Since this is true for $t = w_B(r)$ and the left-hand side of (11) is obviously continuous in t, it suffices to show that there is no $t \in [w_B(r), r]$ for which equality holds in (11). So assume, there is $t \in [w_B(r), r]$ for which equality holds in (11). By (10) we get

$$|\langle tq - z, v\rangle| \le \frac{4S}{\rho_B R r} + 1.$$



Now put $m := q - Ru$ and denote the unit vector in the direction $z - tm - \langle z - tm, u\rangle u$ by v_0. Since we have assumed that equality holds in (11), we get

$$\begin{aligned} \|z - tm\|^2 &= \langle z - tm, u\rangle^2 + \langle z - tm, v_0\rangle^2 \\ &= \langle z - tq + tRu, u\rangle^2 + \langle z - tq + tRu, v_0\rangle^2 \\ &= (tR - \langle tq - z, u\rangle)^2 + \langle tq - z, v_0\rangle^2 \\ &\le \left(tR - \frac{4}{Rr}\right)^2 + \left(\frac{4S}{\rho_B R r} + 1\right)^2. \end{aligned}$$

Due to

$$r > \frac{4S}{\rho_B R} \ge \max\{\tfrac{4}{\rho_B}, \tfrac{4}{R}\}$$

and (7) we get

$$\frac{t}{r} \ge \frac{r - \frac{1}{\rho_B}}{r} = 1 - \frac{1}{r\rho_B} \ge \frac{3}{4}$$

and thus

$$\left(tR\right)^2 - \left(tR - \frac{4}{Rr}\right)^2 = 8\frac{t}{r} - \left(\frac{4}{Rr}\right)^2 \ge 6 - 1 > 2^2 > \left(\frac{4S}{\rho_B R r} + 1\right)^2.$$

Hence

$$\|z - tm\|^2 \le \left(tR - \frac{4}{Rr}\right)^2 + \left(\frac{4S}{\rho_B R r} + 1\right)^2 < \left(tR\right)^2.$$

So we have $z \in \operatorname{int}(tm + tRB^d)$. Since $m + RB^d \subseteq B$ by Lemma 5, we get $z \in \operatorname{int} tB$. Hence there is $t' < t$ with $z \in t'B$ and so $d_B(0, z) < t$, which contradicts $t \le r = d_B(0, z)$. So we have proven inequality (11).

Now we use the equations (9) and (11) in the special case $t = r$ and we use again the inequality $\langle q, u\rangle \ge \rho_B$ we derived before (10) and get

$$r - w_B(r) = \frac{\langle rq - z, u\rangle}{\langle q, u\rangle} < \frac{\frac{4}{rR}}{\rho_B} = \frac{4}{rR\rho_B} \le \frac{4S}{rR\rho_B{}^2}. \qquad \Box$$

The following lemma is the counterpart to Lemma 12.

Lemma 17. *Let $K \subseteq \mathbb{R}^d$ be a body and $B \subseteq \mathbb{R}^d$ be a convex body with a summand $RB^d, R > 0$, such that h_B is twice differentiable on $\mathbb{R}^d \setminus \{0\}$. Put $G := \max\{\operatorname{diam} K, 1\}$, $S := \max\{\operatorname{diam} B, 1\}$ and*

$$C' := d2^{d+2}\kappa_d \frac{S^d \cdot G^{d+1}}{R^3}.$$

Then we have for all $r \geq 1$

$$V_d(\operatorname{conv} K + rB) - V_d(K + rB) < C' \cdot r^{d-3}.$$

Proof. Just like in the proof of Lemma 12, B is strictly convex and we may assume $RB^d \subseteq B$. When we put again $L := \operatorname{conv} K$, (6) remains true. Lemma 16 says that

$$s - w_B(s) \leq \frac{C}{s},\ s \in \mathbb{R}^+,$$

where $C := \frac{4S}{R^3}$. Since obviously $w_B(s) \leq s$ for all $s \in \mathbb{R}^+$, we conclude

$$\begin{aligned} r^i - (Gw_B(\tfrac{r}{G}))^i =& (r - Gw_B(\tfrac{r}{G})) \sum_{j=0}^{i-1} r^j (Gw_B(\tfrac{r}{G}))^{i-1-j} \\ =& G \cdot (\tfrac{r}{G} - w_B(\tfrac{r}{G})) \sum_{j=0}^{i-1} r^j (Gw_B(\tfrac{r}{G}))^{i-1-j} \\ \leq& G\tfrac{CG}{r} \sum_{j=0}^{i-1} r^j (G\tfrac{r}{G})^{i-1-j} \\ =& \tfrac{CG^2}{r} \sum_{j=0}^{i-1} r^{i-1} \\ =& CG^2 i r^{i-2}. \end{aligned}$$

Since $L \subseteq K + rB$ if $rR \geq G$, we have

$$V_d(L \setminus (K + rB)) \leq V_d(L)\mathbf{1}_{\{rR<G\}} \leq \kappa_d G^d \frac{G}{rR}. \tag{12}$$

Using first (6) and then (8), which is still valid, too, (12) and Theorem 8, we obtain

$$\begin{aligned} V_d(L+&rB^d) - V_d(K + rB^d) \\ &\leq \sum_{i=1}^{d} \kappa_i \Phi^B_{d-i}(L, L \setminus \operatorname{ext} L)\big(r^i - (Gw_B(\tfrac{r}{G}))^i\big) + V_d(L \setminus (K + rB)) \\ &\leq \sum_{i=1}^{d-1} \binom{d}{i} G^{d-i} S^i \kappa_d \cdot CG^2 i r^{i-2} + \kappa_d G^d \frac{G}{rR} \\ &\leq \sum_{i=1}^{d-1} (d-1)\binom{d}{i} \kappa_d S^{d-1} C G^{d+1} r^{d-3} + \kappa_d \frac{G^{d+1}}{R} r^{d-3} \\ &< \Big((d-1)2^d \kappa_d S^{d-1} C G^{d+1} + \kappa_d \frac{G^{d+1}}{R}\Big) r^{d-3} \\ &< \Big(d2^{d+2}\kappa_d \frac{S^d \cdot G^{d+1}}{R^3}\Big) r^{d-3}. \qquad \square \end{aligned}$$

Now Theorem 15 can be proven the same way as Theorem 11.

In the rest of this section we will show that certain improvements of Theorem 15 are not possible. First we will show that the order r^{d-3} cannot be improved, even when all sets involved are deterministic and the gauge body Y is the Euclidean unit ball.

Example 18. Let $K := \{v, -v\}$, where $v \in \mathbb{R}^d$ is a unit vector. Put

$$D(r) := \{x \in \mathbb{R}^d \mid |\langle x, v\rangle| \leq \tfrac{1}{2}, \sqrt{r^2 - \tfrac{1}{4}} < \|p_1(x)\| \leq r\}, \quad r > \tfrac{1}{2},$$

where p_1 denotes the orthogonal projection from $\mathbb{R}^d$ onto the linear subspace perpendicular to v. Since $D_r \subseteq \operatorname{conv} K + rB^d$ and $D_r \cap (K + rB^d) = \emptyset$ we have

$$V_d(\operatorname{conv} K + rB^d) - V_d(K + rB^d) \geq V_d(D_r) = \kappa_{d-1}\big(r^{d-1} - \sqrt{r^2 - \tfrac{1}{4}}^{\,d-1}\big).$$

A purely analytical computation shows that the latter expression is indeed of order r^{d-3}.

Now we will show that the assumption that B contains a ball as summand cannot be relaxed in the planar case.

Corollary 19. *Let $B \subseteq \mathbb{R}^2$ be a convex body. Then the following are equivalent:*

(i) There is a constant $c \in \mathbb{R}_0^+$ such that

$$V_2(\operatorname{conv} K + rB) - V_2(K + rB) < \frac{c}{r}$$

for all $r \in \mathbb{R}^+$ and every convex body $K \subseteq \mathbb{R}^2$ with $\operatorname{diam} K \leq 1$.

(ii) B has a summand RB^2, $R > 0$.

Proof. If (ii) is fulfilled, then by Theorem 15 there is a constant C such that

$$V_2(\operatorname{conv} K + rB) - V_2(K + rB) < \frac{C}{r}$$

for all $r \geq 1$ and every body $K \subseteq \mathbb{R}^2$ with $\operatorname{diam} K \leq 1$. For $r \in (0, 1]$ and a body $K \subseteq \mathbb{R}^2$ with $\operatorname{diam} K \leq 1$ we have

$$V_2(\operatorname{conv} K + rB) - V_2(K + rB) < V_2(B^2 + B) \leq \frac{V_2(B^2 + B)}{r}.$$

Hence we have

$$V_2(\operatorname{conv} K + rB) - V_2(K + rB) < \frac{\max\{C, V_2(B^2 + B)\}}{r}$$

for all $r \in \mathbb{R}^+$ and bodies $K \subseteq \mathbb{R}^2$ with $\operatorname{diam} K \leq 1$.

So now assume that (i) is fulfilled.

First assume that B has no interior points. Then B is contained in a line with unit normal vector τ, say. Let S be a segment of length 1 perpendicular to τ. Put $K := S \cup (S + \tau)$. Then

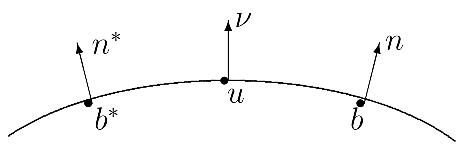
$$V_2(\operatorname{conv} K + rB) - V_2(K + rB) = 1 + rl,$$

where $l \geq 0$ denotes the length of B. Since this contradicts (i), B has interior points.

Since the support function h_B is convex, it is a.e. Alexandrov-twice-differentiable according to Theorem 2. In particular, the second derivative h_B in ν in direction orthogonal to ν exists for a.e. $\nu \in S^1$. We call it $R(B,\nu)$. According to a theorem of Weil ([24, Theorem 1]) is suffices to show that there is a constant $\tilde{c} > 0$ such that $R(B,\nu) \geq \tilde{c}$ holds, whenever $R(B,\nu)$ exists, in order to prove (ii).

So let $\nu \in S^1$ be a point in which h_B is Alexandrov-twice-differentiable and choose $\tau \in S^1$ perpendicular to ν. Let $\theta : \mathbb{R}^2 \to \mathbb{R}^2$ be a choice of subgradients of h_B. Then according to Theorem 3 the point $u := \theta(\nu)$ lies in $\operatorname{bd} B$ and has exterior normal vector ν.

There is $\epsilon > 0$ and $\xi \in \mathbb{R}^2$ with $B_\epsilon(\xi) \subseteq \operatorname{int} B$. Let $b_0 \in \operatorname{bd} B$ denote a point, which satisfies $0 < \langle b_0 - u, \tau\rangle \leq \frac{\epsilon}{2}$ and has an exterior unit normal vector n_0 with $\langle n_0, \nu\rangle > 0$. Let b_0^* and n_0^* be defined in the same way with $u - b_0^*$ instead of $b_0 - u$.



Let $n \in S^1$ be a vector with $0 < \langle n - \nu, \tau\rangle \leq \min\{\langle n_0, \tau\rangle, -\langle n_0^*, \tau\rangle\}$ and $\langle \nu, n\rangle > 0$. Put $n^* := 2\langle n, \nu\rangle\nu - n$. Now $b := \theta(n)$ and $b^* := \theta(n^*)$ are points in $\operatorname{bd} B$ with exterior normal vectors n resp. n^*. We put $K := \{0, \tau\}$ and $r := \frac{1}{2\langle b - b^*, \tau\rangle}$.

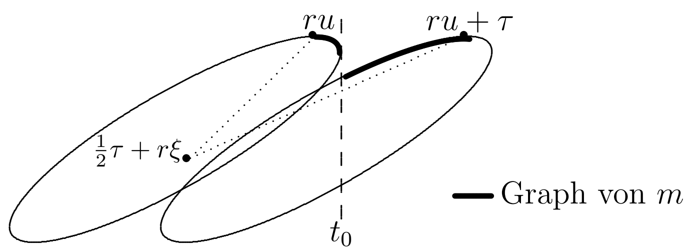
Now the point $\frac{1}{2}\tau + r\xi$ is both in $\operatorname{int}(rB)$ and in $\operatorname{int}(\tau + rB)$. Indeed,

$$\langle n_0^*, \tau\rangle \leq \langle n^*, \tau\rangle \leq \langle n, \tau\rangle \leq \langle n_0, \tau\rangle$$

and thus

$$\epsilon r \geq \frac{\langle b_0 - b_0^*, \tau\rangle}{2\langle b - b^*, \tau\rangle} \geq \tfrac{1}{2}.$$

So $\frac{1}{2}\tau + r\xi \in rB_\epsilon(\xi) \subseteq \operatorname{int}(rB)$ and $\frac{1}{2}\tau + r\xi \in \tau + rB_\epsilon(\xi) \subseteq \operatorname{int}(\tau + rB)$.



Now every line of the form $\{x \in \mathbb{R}^2 \mid \langle x, \tau\rangle = t\}$ with $t \in [t_1, t_1+1]$, where $t_1 := r\langle u, \tau\rangle$, intersects at least one of the segments $[ru, \frac{1}{2}\tau + r\xi]$ or $[\frac{1}{2}\tau + r\xi, \tau + ru]$. Hence the sets $\{x \in K + rB \mid \langle x, \tau\rangle = t\}$, $t \in [t_1, t_1 + 1]$, are not empty and the function

$$m : [t_1, t_1 + 1] \to \mathbb{R},\ t \mapsto \max\{\langle x, \nu\rangle \mid x \in K + rB,\ \langle x, \tau\rangle = t\}$$

is defined. Now there is a number $t_0 \in [t_1, t_1 + 1]$ with

$$m(t) = \max\{\langle x, \nu\rangle \mid x \in rB,\ \langle x, \tau\rangle = t\}, \quad t < t_0,$$

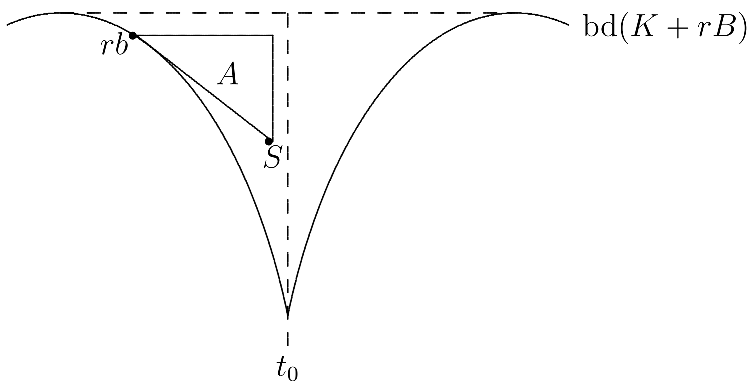
and

$$m(t) = \max\{\langle x, \nu\rangle \mid x \in \tau + rB,\ \langle x, \tau\rangle = t\}, \quad t > t_0.$$

Since $\langle (\tau + rb^*) - rb, \tau\rangle = \frac{1}{2}$, we have either $t_0 - \langle rb, \tau\rangle \geq \frac{1}{4}$ or $\langle \tau + rb^*, \tau\rangle - t_0 \geq \frac{1}{4}$, w.l.o.g. the first.

Now we put

$$A := \{x \in \mathbb{R}^2 \mid \langle x - rb, \tau\rangle < \tfrac{1}{4},\ \langle x, n\rangle > \langle rb, n\rangle,\ \langle x, \nu\rangle \leq \langle rb, \nu\rangle\}.$$



It is easy to see that $A \subseteq \operatorname{conv} K + rB$, but $A \cap (K + rB) = \emptyset$. One cathetus of the rectangular triangle A has length $\frac{1}{4}$. In order to compute the length of the other cathetus, we let S denote the point with $\langle S, n\rangle = \langle rb, n\rangle$ und $\langle S, \tau\rangle = \langle rb, \tau\rangle + \frac{1}{4}$. Then

$$\begin{aligned} 0 &= \langle rb - S, n\rangle \\ &= \langle rb - S, \nu\rangle\langle n, \nu\rangle + \langle rb - S, \tau\rangle\langle n, \tau\rangle \\ &= \langle rb - S, \nu\rangle\langle n, \nu\rangle - \tfrac{1}{4}\langle n, \tau\rangle. \end{aligned}$$

Thus the length we looked for is

$$\langle rb, \nu\rangle - \langle S, \nu\rangle = \frac{\langle n, \tau\rangle}{4\langle n, \nu\rangle}.$$

Hence we get

$$\begin{aligned} \tfrac{1}{32}\langle n, \tau\rangle &\leq \tfrac{1}{2} \cdot \tfrac{1}{4} \cdot \frac{\langle n, \tau\rangle}{4\langle n, \nu\rangle} \\ &= V_2(A) \\ &\leq V_2(\operatorname{conv} K + rB) - V_2(K + rB) \\ &< \frac{c}{r} \\ &= \tfrac{c}{2}\langle b - b^*, \tau\rangle \\ &= \tfrac{c}{2}\langle \theta(n) - \theta(n^*), \tau\rangle. \end{aligned}$$

This means

$$\langle \theta(n) - \theta(n^*), \tau\rangle \geq \frac{1}{16c}\langle n, \tau\rangle = \frac{1}{32c}\langle n - n^*, \tau\rangle.$$

Since we assumed θ to be differentiable in ν and any vector in S^1 sufficiently close to ν could be chosen to be n, this implies

$$R(B,\nu) = \tfrac{\partial}{\partial\lambda}\langle\theta(\nu+\lambda\tau),\tau\rangle_{|\lambda=0} \geq \frac{1}{32c}.$$

Now [24, Theorem 1] implies that B has a summand of the form RB^2, $R > 0$. □

Conjecture 20. *There are convex bodies $B \subseteq \mathbb{R}^d$, $d \geq 3$, which contain no ball as summand, but for which there is a constant $c \in \mathbb{R}_0^+$ with*

$$V_d(\operatorname{conv} K + rB^d) - V_d(K + rB^d) < c \cdot r^{d-3} \tag{13}$$

for all $r \geq 1$ and for all bodies $K \subseteq \mathbb{R}^d$ with $\operatorname{diam} K \leq 1$*.*

Reason: Choose B to be a convex body, for which there are numbers $S > 0$ and $\alpha \in (1,2)$ and a convex body $\tilde{B}$ with a ball as summand, such that

$$\begin{aligned}\{(x_1,\dots,x_d) \in B\ |x_d \geq -S\}\\ = \{(x_1,\dots,x_d) \in \mathbb{R}^d \mid -S \leq x_d \leq -\|(x_1,\dots,x_{d-1})\|^\alpha\}\end{aligned}$$

and

$$\{(x_1,\dots,x_d) \in B \mid x_d \leq -S\} = \{(x_1,\dots,x_d) \in \tilde{B} \mid x_d \leq -S\}.$$

Now there is no $R > 0$ with $0 \in m + RB^d \subseteq B$ for any $m \in B$.

Geometric intuition tells us that it suffices to check (13) in the special case $K = \{-\frac{1}{2}e_1, \frac{1}{2}e_1\}$, where e_1 denotes the first unit vector. This is, however, an easy computation. □

4 Weighted parallel volumes and differentiability

In this section we apply the theorems from Section 3 to the functions that map a non-negative real number r onto the real number $f_\mu(rK)$, where the functional f_μ is a certain generalisation of the Wills functional and $K \subseteq \mathbb{R}^d$ is a body. In Theorem 21 we will show that such a function is infinitely differentiable in $r > 0$ if f_μ fulfils strong regularity assumptions, e.g. if f_μ is the Wills functional. Then we will compute in Theorem 22 and Theorem 23 under weaker regularity assumptions the first derivative in $r = 0$ and, if it exists, also the second derivative. Sufficient conditions for the existence of this second derivative will be given in 24. In Corollary 25 we give the derivatives from Theorem 22 and Theorem 23 in the special case, where f_μ is the Wills functional.

The results of this section answer a question of R.A. Vitale, who asked what the geometric meaning of the derivatives in Corollary 25 is.

A *signed measure* is a measure that may take negative values. For a precise introduction, see e.g. [2]. Here we always assume that it has finite total variation. The variation measure of a signed measure μ will be denoted by $|\mu|$.

The Wills functional is defined by

$$W : \mathcal{C} \to \mathbb{R},\ K \mapsto \mathbb{E}\, V_d(K + \Lambda B^d), \tag{14}$$

where Λ is an $\mathbb{R}_0^+$-valued random variable with distribution function $1 - e^{-\pi t^2}$. If K is convex, then the Wills functional of K equals the sum of its intrinsic volumes.

More generally, for a convex body $B \subseteq \mathbb{R}^d$ and a signed measure ρ on $\mathbb{R}_0^+$ with finite d-th moment we call

$$\mathcal{C} \to \mathbb{R},\ K \mapsto \int_{\mathbb{R}_0^+} V_d(K + \lambda B)\, d\rho(\lambda)$$

ρ-weighted B-parallel volume.

Finally, for a signed measure μ on $\mathcal{K}$ satisfying

$$\int_{\mathcal{K}} V_d(K + A)\, d|\mu|(A) < \infty, \quad K \in \mathcal{C}, \tag{15}$$

we put

$$f_\mu : \mathcal{C} \to \mathbb{R},\ K \mapsto \int_{\mathcal{K}} V_d(K + A)\, d\mu(A).$$

For further information on these functionals, see [11].

Theorem 21. *Let $B \subseteq \mathbb{R}^d$ be a convex body and $X \subseteq \mathbb{R}^d$ a random body satisfying $\mathbb{E}\, V_d(\operatorname{conv} X + xB) < \infty$ for all $x > 0$. Let ρ be a signed measure on $\mathbb{R}_0^+$ which is absolutely continuous and has density $f(\lambda) = \sum_{i=1}^n y_i e^{P_i(\lambda)}$ w.r.t. the Lebesgue measure, where the y_i are real numbers and the P_i are on $\mathbb{R}^+$ strictly monotonically decreasing polynomials for $i = 1, \dots, n$, $n \in \mathbb{N}$. Then the map $r \mapsto \mathbb{E} \int V_d(rX + \lambda B)\, d\rho(\lambda)$ is infinitely differentiable in $r > 0$.*

In particular, this is true for $r \mapsto \mathbb{E}\, W(rX)$.

Proof. Fubini's theorem is valid for signed measures as well, but now its integrability assumptions have to be fulfilled w.r.t. the variation measures. Since $\mathbb{E}\, V_d(\operatorname{conv} X + xB) < \infty$ holds for all $x > 0$ and the form of f implies that all moments of ρ exist, we can apply Fubini's theorem and get

$$\begin{aligned}
\mathbb{E} \int_{\mathbb{R}_0^+} V_d(rX + \lambda B)\, d\rho(\lambda) &= r^d \int_{\mathbb{R}_0^+} \mathbb{E}\, V_d(X + \tfrac{\lambda}{r} B)\, d\rho(\lambda) \\
&= r^d \int_0^\infty \mathbb{E}\, V_d(X + \tfrac{\lambda}{r} B) f(\lambda)\, d\lambda \\
&= r^d \int_0^\infty \mathbb{E}\, V_d(X + xB) f(rx) r\, dx.
\end{aligned}$$

The integrand of the last integral is obviously infinitely differentiable for any $x \in \mathbb{R}_0^+$. The standard theorems for switching integral and differential hold for signed measures as well, where the integrability assumptions have to be fulfilled w.r.t. the variation measure.

In order to check these integrability assumptions, observe that for $k \in \mathbb{N}$ there are numbers $c^k_{\alpha\beta;i}$, $\alpha, \beta \in \mathbb{N}$, $i \in \{1, \dots, n\}$, of which all but finitely many are 0, such that

$$\tfrac{\partial^k}{\partial r^k} f(rx) r = \sum_{\alpha,\beta,i} y_i c^k_{\alpha\beta;i} r^\alpha x^\beta e^{P_i(rx)}.$$

Choose $R_0 \in (0, r)$ and $R_1 > r$ and put

$$h : \mathbb{R}_0^+ \to \mathbb{R},\ x \mapsto \mathbb{E} \sum_{i,\alpha,\beta} V_d(X + xB) |y_i c^k_{\alpha\beta;i}| R_1^\alpha x^\beta e^{P_i(R_0 x)}.$$

Since h is integrable (w.r.t. the Lebesgue measure on $\mathbb{R}_0^+$) and for all $s \in (R_0, R_1)$ and all $x \geq 0$ we have

$$h(x) \geq |\mathbb{E}\, V_d(X + xB) \sum_{\alpha,\beta,i} y_i c^k_{\alpha\beta;i} s^\alpha x^\beta e^{P_i(sx)}|,$$

an easy induction shows that $r \mapsto \mathbb{E} \int V_d(rX + \lambda B)\, d\rho(\lambda)$ is k times differentiable for any $k \in \mathbb{N}^+$. □

For a convex body $B \subseteq \mathbb{R}^d$ we let $S(B)$ denote the maximum of the radius of the circumsphere of B and 1 and we recall

$$R(B) := \sup\{\rho \in \mathbb{R}_0^+ \mid \rho B^d \text{ is summand of } B\}.$$

Theorem 22. *Let $X \subseteq \mathbb{R}^d$ be a random body with $\mathbb{E}\,(\operatorname{diam} X)^d < \infty$ and μ a signed measure on $\mathcal{K}$, which is concentrated on the set of all convex bodies having interior points and fulfils $\int_{\mathcal{K}} S(A)^{d-1}\, d|\mu|(A) < \infty$ and (15). Then $r \mapsto \mathbb{E} f_\mu(rX)$ is differentiable in $r = 0$ with*

$$\tfrac{d}{dr}\mathbb{E} f_\mu(rX)_{|r=0} = d \int_{\mathcal{K}} \mathbb{E} V(\operatorname{conv} X[1], A[d-1])\, d\mu(A). \tag{16}$$

The mixed volume is continuous as shown in the proof of [17, Theorem 5.1.6] and hence measurable; the functional $f_\mu : \mathcal{C} \to \mathbb{R}$ is measurable by the considerations made before Theorem 11 and Fubini's Theorem.

Proof. We first show this theorem in the special case, where μ is the Dirac measure in a convex body B which has interior points and X is deterministic. Although this is an easy consequence of [14, Corollary 2(2)], we find it convenient to give a proof using the same methods as the proof of Theorem 23 below. By Theorem 11 for the map

$$\Delta : \mathbb{R}_0^+ \to \mathbb{R}_0^+,\ s \mapsto V_d(\operatorname{conv} X + sB) - V_d(X + sB),$$

there is a constant $c \in \mathbb{R}_0^+$ with $\Delta(s) < c \cdot s^{d-2}$ for $s \geq 1$. Hence

$$\begin{aligned} V_d(rX + B) &= r^d \cdot V_d(X + \tfrac{1}{r}B) \\ &= r^d \Big(\sum_{j=0}^{d} \binom{d}{j} (\tfrac{1}{r})^{d-j} V(\operatorname{conv} X[j], B[d-j]) - \Delta(\tfrac{1}{r}) \Big) \\ &= \sum_{j=0}^{d} \binom{d}{j} r^j V(\operatorname{conv} X[j], B[d-j]) - r^d \cdot \Delta(\tfrac{1}{r}). \end{aligned} \tag{17}$$

Since $0 \leq r^d \cdot \Delta(\frac{1}{r}) \leq c \cdot r^2$ for $r \leq 1$, we conclude

$$\tfrac{d}{dr} V_d(rX + B) = dV(\operatorname{conv} X[1], B[d-1]).$$

The integrability assumption that are needed to generalize the statement from this special case to the general case are fulfilled, since we have assumed $\mathbb{E}\,(\operatorname{diam} X)^d < \infty$ and $\int_{\mathcal{K}} S(A)^{d-1}\, d|\mu|(A) < \infty$ and an easy computation shows that for $r < 1$ we have

$$\frac{V_d(rX + B) - V_d(B)}{r} \leq \sum_{j=1}^{d} \binom{d}{j} (\operatorname{diam} X)^j (\operatorname{diam} B)^{d-j} \kappa_d. \qquad \square$$

Theorem 23. *Let $X \subseteq \mathbb{R}^d$ be a random body with $\mathbb{E}\,(\operatorname{diam} X)^{d+1} < \infty$ and μ a signed measure on $\mathcal{K}$ which fulfils the integrability assumptions (15) and*

$$\int_{\mathcal{K}} \frac{S(A)^d}{R(A)^3}\, d|\mu|(A) < \infty.$$

(i) Then

$$\mathbb{E} f_\mu(rX) = \sum_{j=0}^{2} \binom{d}{j} r^j \int_{\mathcal{K}} \mathbb{E} V(\operatorname{conv} X[j], A[d-j])\, d\mu(A) + O(r^3)$$

as $r \to 0$.

(ii) If the second derivative exists, then

$$\tfrac{d^2}{dr^2} \mathbb{E} f_\mu(rX)_{|r=0} = d(d-1) \int_{\mathcal{K}} \mathbb{E} V(\operatorname{conv} X[2], A[d-2])\, d\mu(A). \tag{18}$$

Proof. (i) Put $Z := \operatorname{conv} X$. By Theorem 15 for each convex body B which contains a ball as summand there is a map $\Delta_B : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ with

$$\mathbb{E}\, V_d(X + sB) = \mathbb{E} V_d(Z + sB) - \Delta_B(s)$$

such that

$$0 \le \Delta_B(s) < c \cdot \frac{S(B)^d}{R(B)^3} s^{d-3} \tag{19}$$

holds for all $s > 1$ with a constant $c \in \mathbb{R}_0^+$ that is independent of B, but depends on the distribution of X. Just like in (17) we get

$$\mathbb{E} f_\mu(rX) = \sum_{j=0}^{d} \binom{d}{j} r^j \int_{\mathcal{K}} \mathbb{E} V(Z[j], A[d-j])\, d\mu(A) - \int_{\mathcal{K}} r^d \Delta_A(\tfrac{1}{r})\, d\mu(A). \tag{20}$$

Moreover, for $r < 1$ we have

$$\begin{aligned} \left| \int_{\mathcal{K}} r^d \Delta_A(\tfrac{1}{r})\, d\mu(A) \right| &\le \int_{\mathcal{K}} r^d \cdot c \cdot \frac{S(A)^d}{R(A)^3} (\tfrac{1}{r})^{d-3}\, d|\mu|(A) \\ &\le r^3 \cdot c \cdot \int_{\mathcal{K}} \frac{S(A)^d}{R(A)^3}\, d|\mu|(A). \end{aligned} \tag{21}$$

So (i) is shown.

(ii) Assume, $\mathbb{E} f_\mu(rX)$ is twice differentiable with

$$\tfrac{d^2}{dr^2} \mathbb{E} f_\mu(rX)_{|r=0} \ne d(d-1) \int_{\mathcal{K}} \mathbb{E} V(\operatorname{conv} X[2], A[d-2])\, d\mu(A). \tag{22}$$

Then (20) yields that $h(r) := \int_{\mathcal{K}} r^d \Delta_A(\frac{1}{r})\, d\mu(A)$ is twice differentiable, too, and $h''(0) \ne 0$, w.l.o.g. $h''(0) > 0$. Hence for each $\gamma \in \mathbb{R}^+$ there is $\epsilon \in (0,1)$ with $h'(r) > \gamma r^2$ for all $r \in (0, \epsilon)$. Putting $\gamma := 3c \int_{\mathcal{K}} S(A)^d / R(A)^3\, d|\mu|(A)$ and integrating this over $(0, \epsilon)$, we get $h(\epsilon) > \epsilon^3 c \int_{\mathcal{K}} S(A)^d / R(A)^3\, d|\mu|(A)$, which contradicts (21). □

Now we will show that the second derivative in (18) exists if f_μ is a weighted parallel volume fulfilling some regularity assumptions.

Theorem 24. *Let $X \subseteq \mathbb{R}^d$ be a random body with $\mathbb{E}(\operatorname{diam} X)^{d+1} < \infty$ and $B \subseteq \mathbb{R}^d$ a convex body with a summand RB^d, $R > 0$. Let ρ be a signed measure on $\mathbb{R}_0^+$, which has finite d-th moment and is absolutely continouos w.r.t. the Lebesgue measure and has a differentiable density g. Assume, there is a constant $A > 0$ such that for all $x \in \mathbb{R}^+$ we have*

$$|g(x)| \leq \frac{A}{x^{d-1}}, \qquad |g(x)| \leq A, \qquad |g'(x)| \leq \frac{A}{x^d}, \qquad |g'(x)| \leq \frac{A}{x}.$$

Then the map $r \mapsto \mathbb{E} \int V_d(rX + \lambda B)\, d\rho(\lambda)$ is twice differentiable in $r = 0$ with

$$\tfrac{d^2}{dr^2}\mathbb{E} \int V_d(rX + \lambda B)\, d\rho(\lambda) = d(d-1)\mu_{d-2}\mathbb{E}V(\operatorname{conv} X[2], B[d-2]),$$

where μ_{d-2} is the $(d-2)$-th moment of ρ.

Proof. We have to distinguish cases w.r.t. the dimension. First we consider the case $d \geq 3$.

We put again $Z := \operatorname{conv} X$ and $\Delta_A(s) := \mathbb{E}[V_d(Z + sA) - V_d(X + sA)]$ for $s > 0$ and convex bodies A. For $s, \lambda \in \mathbb{R}^+$ we have

$$\Delta_{\lambda B}(s) = \mathbb{E}[V_d(Z + s\lambda B) - V_d(X + s\lambda B)] = \Delta_B(s\lambda).$$

By Theorem 15 there is a constant c with $\Delta(s) < c \cdot s^{d-3}$ for all $s \geq 1$, where $\Delta := \Delta_B$. Hence there is a constant c_1 with $\Delta(s) < c \cdot s^{d-3} + c_1$ for all $s \geq 0$.

We will compute the derivative of $h(r) := \int_{\mathbb{R}_0^+} r^d \Delta(\frac{\lambda}{r})\, d\rho(\lambda)$ in a point $r \geq 0$. We have

$$h(r) = \int_0^\infty r^d \Delta(\tfrac{\lambda}{r}) g(\lambda)\, d\lambda = \int_0^\infty \Delta(x) g(rx) r^{d+1}\, dx.$$

Now we will check the integrability conditions needed in order to differentiate this integral pointwise. Let $R_1 > 1$. We abbreviate $a \wedge b := \min\{a, b\}$. For any $r \in [0, R_1]$ we have

$$\begin{aligned}
&|\tfrac{d}{dr}\, \Delta(x) g(rx) r^{d+1}| \\
&\quad= |\Delta(x) g'(rx) x r^{d+1} + (d+1)\Delta(x) g(rx) r^d| \\
&\quad\leq (cx^{d-3} + c_1)\Big[\frac{A}{(rx)^d} \wedge \frac{A}{rx}\Big] x r^{d+1} \\
&\qquad\qquad + (d+1)(cx^{d-3} + c_1)\Big[\frac{A}{(rx)^{d-1}} \wedge A\Big] r^d \\
&\quad= (1 + d + 1)(cx^{d-3} + c_1)\Big[\frac{A}{(rx)^{d-1}} \wedge A\Big] r^d \\
&\quad\leq (d+2)[(cx^{-2} + c_1 x^{1-d}) A R_1 \wedge (cx^{d-3} + c_1) A R_1^d].
\end{aligned}$$

Moreover,

$$\int_0^\infty [(cx^{-2} + c_1 x^{1-d}) A R_1 \wedge (cx^{d-3} + c_1) A R_1^d]\, dx < \infty,$$

since the integrand is of order x^0 for $x \to 0$ and of order x^{-2} for $x \to \infty$. Observe the difference between the situation here and the situation in Theorem 21: In Theorem 21 we wanted to switch differential and integral in $\frac{d}{dr} \int_0^\infty \mathbb{E}\, V_d(X + xB) g(rx) r^{d+1}\, dx$. However, the integrability assumption was only fulfilled for $r > 0$ and not for $r = 0$. Now we have replaced $\mathbb{E}\, V_d(X + xB)$ by the smaller value $\Delta(x)$ and whence the integrability condition is now fulfilled for $r = 0$, too.

For $r \in [0, 1]$ we have further

$$\begin{aligned}
|h'(r)| &= \left| \int_0^\infty \tfrac{d}{dr}\, \Delta(x) g(rx) r^{d+1}\, dx \right| \\
&= \left| \int_0^\infty \Delta(x)(g'(rx)xr^{d+1} + (d+1)g(rx)r^d)\, dx \right| \\
&\leq \int_0^\infty r^{d-1}\Delta(\tfrac{\lambda}{r})\, (|g'(\lambda)|\lambda + (d+1)|g(\lambda)|\,)\; d\lambda \\
&\leq \int_0^\infty r^{d-1}\Delta(\tfrac{\lambda}{r}) \left(\Big[\frac{A}{\lambda^d} \wedge \frac{A}{\lambda}\Big]\lambda + (d+1)\Big[\frac{A}{\lambda^{d-1}} \wedge A\Big] \right)\, d\lambda \\
&= \int_0^\infty r^{d-1}\Delta(\tfrac{\lambda}{r})(d+2)\Big[\frac{A}{\lambda^{d-1}} \wedge A\Big]\, d\lambda \\
&\leq \int_0^\infty r^{d-1}(c(\tfrac{\lambda}{r})^{d-3} + c_1)(d+2)\Big[\frac{A}{\lambda^{d-1}} \wedge A\Big]\, d\lambda \\
&\leq r^2 \int_0^\infty (c\lambda^{d-3} + c_1)(d+2)\Big[\frac{A}{\lambda^{d-1}} \wedge A\Big]\, d\lambda.
\end{aligned}$$

Since the integrand in the last line is of order λ^0 for $\lambda \to 0$ and of order λ^{-2} for $\lambda \to \infty$, the integral is finite. Hence $h''(0) = 0$ and by formula (20) this shows the statement.

Now we examine the case $d = 2$. Let Δ, c and h be defined as above. Then $\Delta(s) < cs^{-1}$ holds w.l.o.g. for all $s > 0$. Since Δ is bounded on compact intervals, there is $c' \in \mathbb{R}_0^+$ such that $\Delta(s) < c'$ for all $s > 0$.

Again we can compute the derivative of h by pointwise differentiation. However, we have to find a new way of checking the integrability conditions. Let $R_1 > 1$ and $r \in [0, R_1]$. Then

$$\begin{aligned}
|\tfrac{d}{dr}\, \Delta(x) g(rx) r^3| &= |\Delta(x)g'(rx)xr^3 + 3\Delta(x)g(rx)r^2| \\
&\leq [cx^{-1} \wedge c'] \cdot \Big[\frac{A}{(rx)^2} \wedge \frac{A}{rx}\Big] xr^3 \\
&\qquad + 3[cx^{-1} \wedge c'] \cdot \Big[\frac{A}{rx} \wedge A\Big] r^2 \\
&= (1+3)[cx^{-1} \wedge c'] \cdot \Big[\frac{A}{rx} \wedge A\Big] r^2 \\
&\leq 4A[cx^{-1} \wedge c'] \cdot \Big[\frac{r}{x} \wedge r^2\Big] \\
&\leq 4A[cx^{-1} \wedge c'] \cdot \Big[\frac{R_1}{x} \wedge R_1^2\Big].
\end{aligned}$$

Further

$$\int_0^\infty 4A[cx^{-1} \wedge c'] \cdot \Big[\frac{R_1}{x} \wedge R_1^2\Big]\, dx \leq 4AR_1^2 \int_0^\infty \Big[\frac{c}{x^2} \wedge c'\Big]\, dx < \infty.$$

Hence we can change differential and integral and for $r \in [0, 1]$ we obtain in a similar way as above

$$\begin{aligned}
|h'(r)| &\leq \int_0^\infty r \cdot \Delta(\tfrac{\lambda}{r}) \left(\Big[\frac{A}{\lambda^2} \wedge \frac{A}{\lambda}\Big]\lambda + 3\Big[\frac{A}{\lambda} \wedge A\Big] \right) d\lambda \\
&\leq \int_0^\infty r \cdot \Big[c(\tfrac{\lambda}{r})^{-1} \wedge c_1\Big] \cdot 4\Big[\frac{A}{\lambda} \wedge A\Big]\, d\lambda \\
&\leq 4r \int_0^\infty \Big[\frac{crA}{\lambda^2} \wedge c_1 A\Big]\, d\lambda \\
&= 4r \int_0^{\sqrt{\frac{crA}{c_1 A}}} c_1 A \, d\lambda + 4r \int_{\sqrt{\frac{crA}{c_1 A}}}^\infty \frac{crA}{\lambda^2}\, d\lambda \\
&= 4r\sqrt{\frac{cr}{c_1}} \cdot c_1 A + 4r \frac{crA}{\sqrt{cr/c_1}} \\
&= 8r\sqrt{r} \cdot A\sqrt{c \cdot c_1}.
\end{aligned}$$

Just like in the first case, this shows the assertion. □

Now we will reformulate the theorems of this section in the special case, where the functional f_μ is the Wills functional.

Corollary 25. *Let $X \subseteq \mathbb{R}^d$ be a random body with $\mathbb{E}(\operatorname{diam} X)^{d+1} < \infty$. Then $r \mapsto \mathbb{E}\, W(rX)$ is twice differentiable in $r = 0$ and we have*

$$\tfrac{d}{dr}\mathbb{E}\, W(rX)_{|r=0} = \mathbb{E}\, V_1(\operatorname{conv} X)$$

and

$$\tfrac{d^2}{dr^2}\mathbb{E}\, W(rX)_{|r=0} = 2 \cdot \mathbb{E}\, V_2(\operatorname{conv} X).$$

Proof. A straight-forward computation shows that for the random variable Λ from the definition (14) of W we have

$$\kappa_{d-1}\mathbb{E}\,\Lambda^{d-1} = \kappa_{d-2}\mathbb{E}\,\Lambda^{d-2} = 1.$$

Now Theorem 22 yields

$$\begin{aligned}
\tfrac{d}{dr}\mathbb{E}\, W(rX)_{|r=0} &= \binom{d}{1}\mathbb{E}\,\Lambda^{d-1} \cdot \mathbb{E}\, V(\operatorname{conv} X[1], B^d[d-1]) \\
&= \kappa_{d-1}\mathbb{E}\,\Lambda^{d-1} \cdot \mathbb{E}\, V_1(\operatorname{conv} X) \\
&= \mathbb{E}\, V_1(\operatorname{conv} X).
\end{aligned}$$

We will now show that the Lebesgue density $g(x) = 2\pi x \cdot e^{-\pi x^2}$ of Λ fulfils the assumptions of Theorem 24. We have $g'(x) = 2\pi \cdot e^{-\pi x^2} - 4\pi^2 x^2 \cdot e^{-\pi x^2}$ and hence $|g'(x)| \leq (2\pi + 4\pi^2 x^2) \cdot e^{-\pi x^2}$. Moreover,

$$\lim_{x\to 0} 2\pi x \cdot e^{-\pi x^2} = 0$$

$$\lim_{x\to \infty} 2\pi x \cdot e^{-\pi x^2} \cdot x^{d-1} = 0$$

$$\lim_{x\to 0} (2\pi + 4\pi^2 x^2) \cdot e^{-\pi x^2} = 2\pi$$

$$\lim_{x\to \infty} (2\pi + 4\pi^2 x^2) \cdot e^{-\pi x^2} \cdot x^d = 0$$

Hence we can apply Theorem 24 and the second assertion ensues just like the first. □

Considering Example 18 it should not be too surprising that formulas analogue to the formulas presented in this section for the third derivative do not hold. However, we find it worthy to present an example explicitly showing that even $\frac{d^3}{dr^3}W(rK)_{|r=0} = 6V_3(\operatorname{conv} K)$ does not hold in general.

Example 26. Let $K \subseteq \mathbb{R}^3$ be a body, whose parallel volume is a polynomial, $V_3(K + sB^3) = \sum_{i=0}^{3} c_i s^i$, say, and for which $V_3(K) \neq V_3(\operatorname{conv} K)$. Such bodies exist, as shown in [6, section 4]. Let Λ denote again the random variable with distribution function $1 - e^{-\pi t^2}$, $t \geq 0$. Then

$$\begin{aligned} W(rK) &= r^3\, \mathbb{E}\, V_3\big(K + \frac{\Lambda}{r}B^3\big) \\ &= r^3 \mathbb{E} \sum_{i=0}^{3} c_i \big(\frac{\Lambda}{r}\big)^i \\ &= \sum_{i=0}^{3} c_i \mathbb{E}\Lambda^i\, r^{3-i}. \end{aligned}$$

Thus

$$\tfrac{d^3}{dr^3} W(rK)_{|r=0} = \tfrac{d^3}{dr^3} \sum_{i=0}^{3} c_i \mathbb{E}\Lambda^i\, r^{3-i}{}_{|r=0} = 6c_0 = 6V_3(K + 0B^3) = 6V_3(K).$$

5 Stochastic applications

In this section we will apply the results from the previous sections to Wiener sausages (Corollary 27), Boolean models (Theorem 28) and Gaussian random variables (Theorem 34).

The parallel body of a Brownian path is called Wiener sausage. While there are many papers dealing with the asymptotic behaviour of the volume of the Wiener sausage as the time tends to infinity (see [10] and the literature cited therein), [13] seems to be the only one dealing with its asymptotics as the time tends to 0. There it was shown that

$$\begin{aligned} \mathbb{E}\, V_d(S_t + rB^d) &= \kappa_d r^d + \frac{d\sqrt{2}\kappa_d}{\sqrt{\pi}} r^{d-1}\sqrt{t} + o(\sqrt{t}) \\ &= \frac{\pi^{d/2}}{\Gamma(\frac{d}{2}+1)} r^d + \frac{2\sqrt{2}\pi^{(d-1)/2}}{\Gamma(\frac{d}{2})} r^{d-1}\sqrt{t} + o(\sqrt{t}), \end{aligned}$$

where $S_t \subseteq \mathbb{R}^d$ denotes a Brownian path up to time t. Putting together Theorem 23 and [13, Prop. 1.4] we obtain:

Corollary 27. *For $r \geq 0$ we have as $t \to 0$*

$$\begin{aligned}\mathbb{E}V_d(S_t + rB^d) &= \kappa_d r^d + \frac{d\sqrt{2}\kappa_d}{\sqrt{\pi}} r^{d-1}\sqrt{t} + \frac{(d-1)\kappa_{d-2}\pi}{2} r^{d-2} t + O(t^{3/2}) \\ &= \frac{\pi^{d/2}}{\Gamma(\frac{d}{2}+1)} r^d + \frac{2\sqrt{2}\pi^{(d-1)/2}}{\Gamma(\frac{d}{2})} r^{d-1}\sqrt{t} + \frac{(d-1)\pi^{d/2}}{2\Gamma(\frac{d}{2})} r^{d-2} t \\ &\qquad + O(t^{3/2}).\end{aligned}$$

Now we turn to the contact distributions of Boolean models. For the introduction of the notions of a Boolean model and the contact distribution, see [18, sections 4.3 and 2.4]. Here we consider only stationary Boolean models and assume that their grain distributions are defined on the set $\mathcal{C}_0$ of centered bodies (see [18, section 4.1]). The contact distribution of a stationary Boolean model Z in $\mathbb{R}^d$ with intensity γ and grain distribution $\mathbb{Q}$ is

$$H_B^Z(r) = 1 - \exp\Big(-\gamma \int_{\mathcal{C}_0} V_d(A + rB^*) - V_d(A)\, d\mathbb{Q}(A)\Big), \quad r \geq 0, \tag{23}$$

(see [18, Theorem 9.1.1]), where $B^* := \{-x \mid x \in B\}$ for $B \subseteq \mathbb{R}^d$.

Theorem 28. *Let $Z^{(r)}$, $r \in \mathbb{R}^+$, be stationary Boolean models in $\mathbb{R}^d$ with intensity r^d and a typical grain Z_0 that fulfils $\mathbb{E}(\operatorname{diam} Z_0)^{d+1} < \infty$ and is independent of r and . Let $B \subseteq \mathbb{R}^d$ be a convex body with $0 \in \operatorname{int} B$. Let D be an $\mathbb{R}_0^+$-valued random variable with distribution function $1 - \exp(-t^d V_d(B))$. Then we have*

$$r \cdot d_B(Z^{(r)}, 0) \overset{r\to 0}{\longrightarrow} D$$

in distribution. More precisely, for $t \geq 0$ we have

$$\begin{aligned}\lim_{r\to 0} & \frac{\mathbb{P}(r \cdot d_B(Z^{(r)}, 0) \leq t) - (1 - e^{-t^d V_d(B)})}{r} \\ &\qquad = e^{-t^d V_d(B)} t^{d-1} \mathbb{E}V(\operatorname{conv} Z_0[1], B^*[d-1]).\end{aligned}$$

Proof. For a convex body $A \subseteq \mathbb{R}^d$ we let δ_A denote the Dirac measure on $\mathcal{K}$ in A. From (23) and Theorem 22 we get

$$\begin{aligned}\lim_{r\to 0} & \frac{\mathbb{P}(r \cdot d_B(Z^{(r)}, 0) \leq t) - (1 - e^{-t^d V_d(B)})}{r} \\ &= \lim_{r\to 0} \frac{(1 - e^{-r^d \mathbb{E}V_d(Z_0 + \frac{t}{r}B^*)}) - (1 - e^{-t^d V_d(B)})}{r} \\ &= \lim_{r\to 0} \frac{e^{-V_d(tB^*)} - e^{-\mathbb{E}V_d(rZ_0 + tB^*)}}{r} \\ &= -\tfrac{d}{dr}\, e^{-\mathbb{E}V_d(rZ_0 + tB^*)}{}_{|r=0} \\ &= -e^{-V_d(tB^*)} \cdot \Big(-\tfrac{d}{dr}\, \mathbb{E} \int_{\mathcal{K}} V_d(rZ_0 + A)\, d\delta_{tB*}(A)_{|r=0} \Big) \\ &= e^{-t^d V_d(B)} \cdot \Big(\mathbb{E} \int_{\mathcal{K}} V(\operatorname{conv} Z_0[1], A[d-1])\, d\delta_{tB^*}(A) \Big) \\ &= e^{-t^d V_d(B)} t^{d-1} \mathbb{E}V(\operatorname{conv} Z_0[1], B^*[d-1]). \qquad \square\end{aligned}$$

In the third part of this section we give a new proof for formulae that use Gaussian random variables in order to compute the first and second intrinsic volume of the convex hull of a body. We do this by finding expressions for the first and second derivative of $W(rK)$ involving Gaussian random variables and comparing these expressions to the ones from Corollary 25.

Vitale [22] derived the following representation of the Wills functional.

Theorem 29. *Let Z be a standard-normal distributed random vector in $\mathbb{R}^d$ and $K \subseteq \mathbb{R}^d$ a body. Then*

$$W(K) = \mathbb{E}\, \exp(\max\{\langle a, Z\rangle - \tfrac{\|a\|^2}{2} \mid \tfrac{a}{\sqrt{2\pi}} \in K\}).$$

Now we let $A \subseteq \mathbb{R}^d$ denote a fixed finite set. For $r \in \mathbb{R}_0^+$ and $z \in \mathbb{R}^d$ we let $a_z^r \in \sqrt{2\pi}A$ denote a point that satisfies

$$\langle a_z^r, z\rangle - \tfrac{r}{2}\|a_z^r\|^2 = \max\{\langle a, z\rangle - \tfrac{r}{2}\|a\|^2 \mid \tfrac{a}{\sqrt{2\pi}} \in A\}$$

in such a way that $\mathbb{R}^d \to \mathbb{R}^d$, $z \mapsto a_z^r$ is measurable. In order to show that such a choice is possible, we abbreviate $f_z(a) := \langle a, z\rangle - \frac{r}{2}\|a\|^2$ and $\tilde{A} := \{a \in \mathbb{R}^d \mid \frac{a}{\sqrt{2\pi}} \in A\}$. Now

$$\beta : \mathbb{R}^d \mapsto \mathcal{F}, z \mapsto \operatorname{argmax}_{a\in\tilde{A}} f_z(a) := \{a \in \tilde{A} \mid f_z(a) = \max\{f_z(b) \mid b \in \tilde{A}\}\}$$

is upper semicontinuous. Since the lower tangent point (see [18, p. 110]) can be shown to be measurable by semicontinuity arguments, it is possible to select one point from $\beta(z)$ in a measurable way.

From now on Z is a standard-normal distributed random vector in $\mathbb{R}^d$.

Theorem 30. *With these denominations we have for $r \geq 0$*

$$\frac{\partial}{\partial r} W(rA) = \mathbb{E} \exp\left(\langle ra_Z^r, Z\rangle - \tfrac{1}{2}\|ra_Z^r\|^2\right) \cdot (\langle a_Z^r, Z\rangle - r\|a_Z^r\|^2) \tag{24}$$

and

$$\frac{\partial^2}{\partial r^2} W(rA) = \mathbb{E}\, \exp\left(\langle ra_Z^r, Z\rangle - \tfrac{1}{2}\|ra_Z^r\|^2\right) \cdot [(\langle a_Z^r, Z\rangle - r\|a_Z^r\|^2)^2 - \|a_Z^r\|^2]. \tag{25}$$

Proof. Let $r \in \mathbb{R}_0^+$. Then for two different points $a, a' \in \sqrt{2\pi}A$ we have

$$\langle a, Z\rangle - \tfrac{r}{2}\|a\|^2 \neq \langle a', Z\rangle - \tfrac{r}{2}\|a'\|^2$$

a.s. and hence a_Z^r is determined uniquely a.s. If a_Z^r is determined uniquely, then there is neighbourhood of r such that for all s from this neighbourhood $a_Z^r = a_Z^s$ holds. Thus

$$\frac{\partial}{\partial r} \exp\left(\langle ra_Z^r, Z\rangle - \tfrac{1}{2}\|ra_Z^r\|^2\right) = \exp\left(\langle ra_Z^r, Z\rangle - \tfrac{1}{2}\|ra_Z^r\|^2\right) \cdot (\langle a_Z^r, Z\rangle - r\|a_Z^r\|^2). \tag{26}$$

By Theorem 29 we have

$$\begin{aligned} \frac{\partial}{\partial r} W(rA) &= \frac{\partial}{\partial r}\mathbb{E}\, \exp(\max\{\langle ra, Z\rangle - \tfrac{\|ra\|^2}{2} \mid \tfrac{a}{\sqrt{2\pi}} \in K\}) \\ &= \frac{\partial}{\partial r}\mathbb{E}\, \exp\left(\langle ra_Z^r, Z\rangle - \tfrac{1}{2}\|ra_Z^r\|^2\right) \end{aligned} \tag{27}$$

If we can switch differential and expected value in the last expression, equation (26) will yield the assertion (24). In order to check the integrability assumptions, we choose $R > 0$ with $A \subseteq RB^d$. For $s \in [0, r+1]$ we put $I := [\min\{r, s\}, \max\{r, s\}]$. Since the map $t \mapsto \exp(\langle ta_Z^t, Z\rangle - \frac{1}{2}\|ta_Z^t\|^2)$ is a.s. continuous and piecewise differentiable, we obtain a.s.

$$
\begin{aligned}
&|\exp(\langle sa_Z^s, Z\rangle - \tfrac{1}{2}\|sa_Z^s\|^2) - \exp(\langle ra_Z^r, Z\rangle - \tfrac{1}{2}\|ra_Z^r\|^2)| \\
&\quad \le |s-r| \cdot \max\{|\tfrac{d}{dt}\exp(\langle ta_Z^t, Z\rangle - \tfrac{1}{2}\|ta_Z^t\|^2)| \mid t \in I\} \\
&\quad \le |s-r| \cdot \max\{\exp(\langle ta_Z^t, Z\rangle - \tfrac{1}{2}\|ta_Z^t\|^2) \cdot |\langle a_Z^t, Z\rangle - t\|a_Z^t\|^2| \mid t \in I\} \\
&\quad \le |s-r| \cdot \exp((r+1)R\|Z\|) \cdot (R\|Z\| + (r+1)R^2).
\end{aligned}
$$

The random variable on the right hand side has finite expected value, which completes the proof of (24).

The proof of (25) is analogue to the proof of (24). The only difference is that in the place, where Theorem 29 was used, now equation (24) has to be used. □

For a body $K \subseteq \mathbb{R}^d$ and a vector $u \in \mathbb{R}^d$ we put

$$h_K(u) := \max\{\langle k, u\rangle \mid k \in K\} = h_{\operatorname{conv} K}(u)$$

and we choose a point $H(K; u) \in K$ satisfying

$$\langle H(K; u), u\rangle = h_K(u).$$

Then $H(K; Z)$ is determined uniquely a.s.

Corollary 31. *With the denominations introduced above and right now we have*

(i) $\frac{\partial}{\partial r} W(rA)_{|r=0} = \sqrt{2\pi} \cdot \mathbb{E}\, h_A(Z)$

(ii) $\frac{\partial^2}{\partial r^2} W(rA)_{|r=0} = 2\pi \cdot \mathbb{E}\,[h_A(Z)^2 - \|H(A; Z)\|^2]$.

Proof. From the definition of a_z^r we get

$$\langle a_Z^0, Z\rangle = \max\{\langle a, Z\rangle \mid a \in \sqrt{2\pi} A\} = \sqrt{2\pi} \cdot h_A(Z)$$

and hence, because of $a_Z^0 \in \sqrt{2\pi} A$,

$$a_Z^0 = \sqrt{2\pi} \cdot H(A; Z).$$

So Theorem 30 yields the assertion. □

Comparing the Corollaries 25 and 31 we obtain the following corollary.

Corollary 32. *With the denominations introduced above we have*

(i) $V_1(\operatorname{conv} A) = \sqrt{2\pi} \cdot \mathbb{E}\, h_A(Z)$

(ii) $V_2(\operatorname{conv} A) = \pi \cdot \mathbb{E}\,[h_A(Z)^2 - \|H(A; Z)\|^2]$.

In order to generalize Corollary 32 from finite sets to compact sets, we need continuity arguments. According to [18, Theorem 12.3.5] the map $\operatorname{conv} : \mathcal{C} \to \mathcal{K}$ is continuous (w.r.t. the Hausdorff metric defined on page 7) and the intrinsic volumes and $K \mapsto h_K(u)$, $u \in S^{d-1}$, are continuous according to [17, p. 210 resp. Lemma 1.8.10] if considered as functions $\mathcal{K} \to \mathbb{R}$.

Lemma 33. *Let $u \in \mathbb{R}^d$ and $K \in \mathcal{C}$ such that $H(K;u)$ is determined uniquely and let $(K_i)_{i\in\mathbb{N}}$ be a sequence converging to K. Then for any choice of $H(K_i;u)$ we have*

$$\lim_{i\to\infty} H(K_i;u) = H(K;u).$$

Proof. It suffices to show that any subsequence $(K_{m(i)})_{i\in\mathbb{N}}$ of $(K_i)_{i\in\mathbb{N}}$ contains a subsequence $(K_{m(r(i))})_{i\in\mathbb{N}}$ such that

$$\lim_{i\to\infty} H(K_{m(r(i))};u) = H(K;u).$$

So let $(K_{m(i)})_{i\in\mathbb{N}}$ be a subsequence of $(K_i)_{i\in\mathbb{N}}$. Since $(K_{m(i)})_{i\in\mathbb{N}}$ converges, there is $R \in \mathbb{R}^+$ with $K_{m(i)} \subseteq RB^d$ for all $i \in \mathbb{N}$ and, in particular,

$$H(K_{m(i)};u) \in RB^d, \ i \in \mathbb{N}.$$

Hence this sequence has a convergent subsequence $(H(K_{m(r(i))};u))_{i\in\mathbb{N}}$. Now

$$\langle \lim_{i\to\infty} H(K_{m(r(i))};u), u\rangle = \lim_{i\to\infty} \langle H(K_{m(r(i))};u), u\rangle = \lim_{i\to\infty} h_{K_{m(r(i))}}(u) = h_K(u).$$

Because of $\lim_{i\to\infty} H(K_{m(r(i))};u) \in K$ we get

$$\lim_{i\to\infty} H(K_{m(r(i))};u) = H(K;u). \qquad \square$$

Theorem 34. *Let $K \subseteq \mathbb{R}^d$ be a body and Z a standard-normal distributed random vector in $\mathbb{R}^d$. Then*

(i) $V_1(\operatorname{conv} K) = \sqrt{2\pi} \cdot \mathbb{E}\, h_K(Z)$

(ii) $V_2(\operatorname{conv} K) = \pi \cdot \mathbb{E}\,[h_K(Z)^2 - \|H(K;Z)\|^2]$.

Proof. We prove only the second statement, since the first one ensues the same way, only slightly easier. It is well-known that there is a sequence $(A_i)_{i\in\mathbb{N}}$ of finite subsets of K converging to K. Now Corollary 32, the continuity statements before and in Lemma 33, and the dominated convergence theorem, which can be applied since $A_i \subseteq K$ holds for all $i \in \mathbb{N}$, give

$$\begin{aligned} V_2(\operatorname{conv} K) &= \lim_{i\to\infty} V_2(\operatorname{conv} A_i) \\ &= \lim_{i\to\infty} \pi \cdot \mathbb{E}\,[h_{A_i}(Z)^2 - \|H(A_i;Z)\|^2] \\ &= \pi \cdot \mathbb{E} \lim_{i\to\infty} [h_{A_i}(Z)^2 - \|H(A_i;Z)\|^2] \\ &= \pi \cdot \mathbb{E}\,[h_K(Z)^2 - \|H(K;Z)\|^2]. \qquad \square \end{aligned}$$

Theorem 34 is not essentially new. The first statement is a special case of Proposition 14 in [20], whose proof is based on the stochastic independence of $\|Z\|$ and $\frac{Z}{\|Z\|}$ and the projection formula from integral geometry. The second statement is new, but part (ii) of Corollary 32 can be derived from [1, (3.10.1)] by using that the covariance of $\langle a, Z\rangle$ is $\|a\|^2$ for $a \in \mathbb{R}^d$.

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