On Gyroscopic Stabilization

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CHAPTER 1

Introduction

We consider a system of the form

$$M\ddot{x} + D\dot{x} + Kx \;=\; 0\,, \quad x \in \mathbb{R}^n\,, \tag{1.1}$$

with a positive definite mass matrix M, a symmetric damping matrix D and a positive definite stiffness matrix K. If the equilibrium in (1.1) is unstable, a small disturbance is enough to set the system in motion again. The motion of the system sustains itself, an effect which is called *self-excitation* or *self-induced vibration*. The reason behind this effect is the presence of negative damping, which results for example from dry friction, see [1], [5], [18]. Another example is the Van der Pol Oscillator (see[13], p. 9), which due to a non-constant damping coefficient, locally has negative damping. This example will be further examined below.
Negative damping implies that the damping matrix D is indefinite or negative definite. Throughout our work, we assume D to be indefinite, and that system (1.1) possesses both stable and unstable modes and thus is unstable.
It is now the idea of *gyroscopic stabilization* to mix the modes of a system with indefinite damping such that the system is stabilized without introducing further dissipation. This is done by adding gyroscopic forces $G\dot{x}$ with a suitable skew-symmetric matrix G to the left-hand side (see e.g. [16]).

Definition 1.1. We call $G = -G^T \in \mathbb{R}^{n\times n}$ a *gyroscopic stabilizer* for the unstable system (1.1), if

$$M\ddot{x} + (D+G)\dot{x} + Kx \;=\; 0 \tag{1.2}$$

is asymptotically stable. In this case the system is *gyroscopically stabilizable*. We will even call G a gyroscopic stabilizer if

$$M\ddot{x} + (D+\gamma G)\dot{x} + Kx \;=\; 0$$

for some $\gamma \in \mathbb{R}$ is asymptotically stable, thus making the definition independent of the scaling of G.

As mentioned above, an example for the occurrence of negative damping is given by the Van der Pol Oscillator (see[13], p. 9),

$$\ddot{x} - \mu(1 - x^2)\dot{x} + x = 0 \tag{1.3}$$

with $\mu > 0$. For x close to zero, the damping coefficient in (1.3) is negative, thus self-excitation occurs. The trajectories of the system tend towards a limit cycle (see Figure 1.1), since the damping coefficient becomes positive when x is sufficiently far away from zero. It follows that here self-excitation is a local effect depending on x. We now

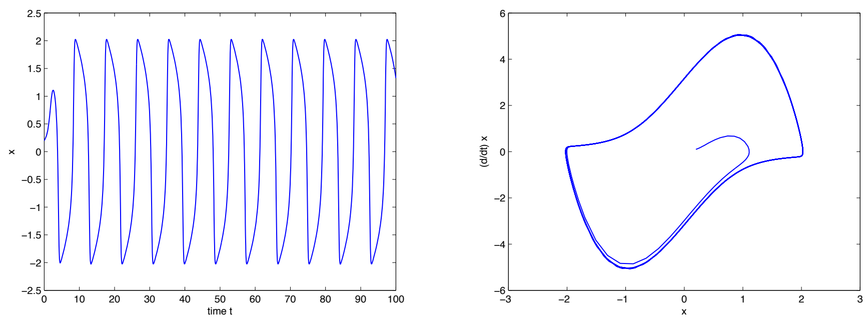

FIGURE 1.1. Solution of (1.3) with initial conditions $x(0) = 0.2$ and $\dot{x}(0) = 0.1$ plotted versus time on the left; phase portrait on the right

stabilize system (1.3) by coupling it via a conservative, gyroscopic term with two damped harmonic oscillators. We put $\mu = 3$ and consider the system

$$\begin{bmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{z} \end{bmatrix} + \begin{bmatrix} -3(1-x^2) & -g_3 & g_2 \\ g_3 & 2 & -g_1 \\ -g_2 & g_1 & 1.5 \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} + \begin{bmatrix} 1 & & \\ & \frac{1}{2} & \\ & & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$
$$= \ddot{v} + \big(D(v) + G\big)\dot{v} + Kv = 0\ . \tag{1.4}$$

For $g_1 = g_2 = g_3 = 0$ the above system is still decoupled. Thus x tends towards the limit cycle, while y and z will tend to zero.
By choosing $(g_1,\ g_2,\ g_3) = (4.5,\ 4.5,\ 4.5)$ and inspecting the eigenvalues, we see that the linearized system $\ddot{v} + \big(D(0) + G\big)\dot{v} + Kv = 0$ is asymptotically stable. According to ([20], §29, IX), it is implied that the equilibrium of the nonlinear system is at least locally asymptotically stable.

In Figure 1.2, the solution of the coupled nonlinear system (1.4) with initial conditions $(x(0)\ \dot{x}(0)\ y(0)\ \dot{y}(0)\ z(0)\ \dot{z}(0)) = \frac{1}{10}(2\ 1\ 2\ 1\ 2\ 1)$ is plotted versus time and in the phase space. In contrast to the non-coupled Van der Pol oscillator (1.3), the solution for z now tends to

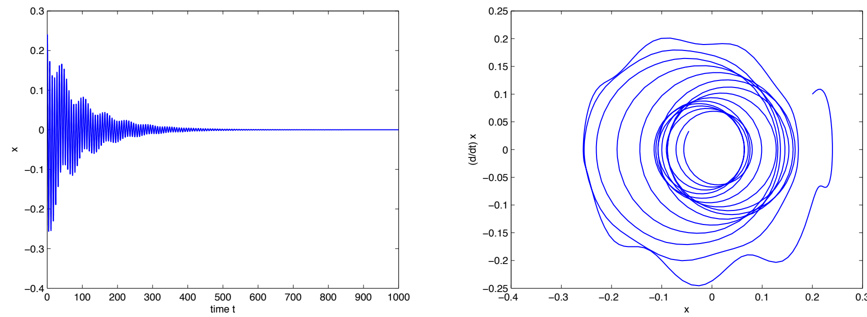

FIGURE 1.2. Solution for x in (1.4) with initial conditions $(x(0)\ \dot{x}(0)\ y(0)\ \dot{y}(0)\ z(0)\ \dot{z}(0)) = \frac{1}{10}(2\ 1\ 2\ 1\ 2\ 1)$ plottet versus time on the left; phase portrait of $(x(t), \dot{x}(t))$ with $t \in [0, 100]$ on the right

zero.

Figure 1.3 shows the solutions for y and z of (1.3). They behave very

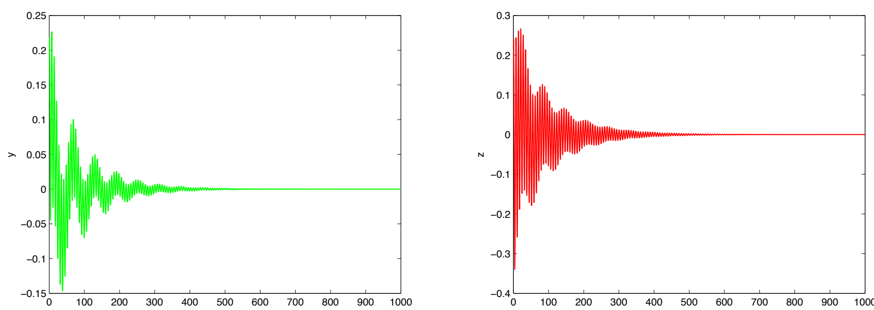

FIGURE 1.3. Solution for y and z in (1.4) with initial conditions $(x(0)\ \dot{x}(0)\ y(0)\ \dot{y}(0)\ z(0)\ \dot{z}(0)) = \frac{1}{10}(2\ 1\ 2\ 1\ 2\ 1)$ plottet versus time

similar to the solution for x, the gyroscopic term aligns the behavior of x, y, z.

This example demonstrates that a gyroscopic stabilization may even be applied for certain nonlinear systems.

The problem we will analyze in our work is: under which circumstances does a gyroscopic stabilizer exists, and how can it be constructed?

Let M be the unit matrix. A well-known necessary condition for gyroscopic stabilizability is that the traces of D and $K^{-1}D$ are both positive, see [14] or [15]. In the recent paper [15], the authors ask whether this condition is also sufficient. In the case $n = 2$ they give an affirmative answer. For $n > 2$ gyroscopic stabilizability so far has only

been shown under additional conditions.
We will use results on eigenvalue and eigenvector perturbation in order to derive a new sufficient condition for gyroscopic stabilizability. Our method is formulated as an inverse eigenvector problem, see chapter 2. We then show in chapter 3 how our method can be applied to systems of the form (1.1) in space dimension 2 and 3 and thereby show that the conditions $\operatorname{tr} D, \operatorname{tr} QDQ > 0$ are sufficient conditions for stabilizability. The two-dimensional case is known already and can be found in the literature, while the solution to the three-dimensional case so far was not known. An easily applicable construction method for a gyroscopic stabilizer G is shown, and we derive a visualization of the set of G in three dimensions for a given system of the form (1.1).
We also apply our construction method to space dimension 4 and 5. Actually, the construction in both cases is fairly similar. At least in dimension 4, we are again able to show the sufficiency of the conditions $\operatorname{tr} D, \operatorname{tr} QDQ > 0$ for gyroscopic stabilizability. Finally, we show how in some cases, the problem of constructing a gyroscopic stabilizer can be reduced to spaces of lower dimension.
Parts of chapters 2, 3 and 4 have already been published in a joint paper with Tobias Damm [7].

CHAPTER 2

Necessary and Sufficient Conditions

2.1. Necessary Condition

Since in the system

$$M\ddot{x} + (D+G)\dot{x} + Kx \;=\; 0\,, \quad x \in \mathbb{R}^n \tag{2.1}$$

we assume $M > 0$, there exists the unique positive definite square root of M, thus we may define $P > 0$ with $P^{-2} = M$. By multiplication from the left and with $z = P^{-1}x$, (2.1) becomes

$$\begin{aligned} PMPP^{-1}\ddot{x} + P(D+G)PP^{-1}\dot{x} + PKPP^{-1}x &= \\ PMP\ddot{z} + P(D+G)P\dot{z} + PKPz &= \\ I\ddot{z} + (\hat{D}+\hat{G})\dot{z} + \hat{K}z &= 0\,, \end{aligned}$$

where I denotes the unit matrix, $\hat{K} = \hat{K}^T > 0$, $\hat{G} = -\hat{G}^T$ and $\hat{D} = \hat{D}^T$. Thus, without loss generality, from now on we assume that in (2.1) we have $M = I$. Then, the second-order system can be written in first order form as

$$\frac{d}{dt}\begin{bmatrix} x \\ \dot{x} \end{bmatrix} = \begin{bmatrix} 0 & I \\ -K & -D-G \end{bmatrix}\begin{bmatrix} x \\ \dot{x} \end{bmatrix} = A_G \begin{bmatrix} x \\ \dot{x} \end{bmatrix}\,.$$

It is asymptotically stable if and only if $\sigma(A_G) \subset \mathbb{C}_-$, which implies $\operatorname{tr} A_G < 0$, i.e. $\operatorname{tr} D > 0$. Moreover, since $\sigma(A_G) \subset \mathbb{C}_-$ if and only if $\sigma(A_G^{-1}) \subset \mathbb{C}_-$, where

$$A_G^{-1} \;=\; \begin{bmatrix} -K^{-1}(D+G) & -K^{-1} \\ I & 0 \end{bmatrix}\,, \tag{2.2}$$

we conclude that also $0 < \operatorname{tr} K^{-1}(D+G) = \operatorname{tr} K^{-1}D$. These *necessary* criteria are well known (e.g. [14]), but their first appearance is difficult to track down. In the literature, [17] is often cited as the source, but actually, these conditions are not mentioned in the article.
We asked Prof. Dr. P. C. Müller about the origin of these criteria, which are well known to him for a long time, and he referred to Prof. Dr. K. Magnus. Müller was also so kind to provide us with his method to derive the conditions, which directly uses the characteristic polynomial p

of (1.2). Consider

$$p(\lambda) = \det(\lambda^2 I + \lambda(D+G) + K) = \lambda^m + \sum_{i=0}^{m-1} a_i \lambda^i \ ,$$

where $a_{m-1} = \operatorname{tr} D$. For asymptotic stability, it is necessary that $a_i > 0$ for all i, we immediately get the condition $\operatorname{tr} D > 0$.
Define q via

$$\begin{aligned} p(\lambda) &= \lambda^n \det K \det(\mu^2 I + \mu K^{-1}(D+G)) = \lambda^n \det K \ q(\mu) \\ &= \lambda^n \det K \ (\mu^m + \sum_{i=0}^{m-1} b_i \mu^i) \ , \end{aligned}$$

where $\mu = \frac{1}{\lambda}$. If p is asymptotically stable, then so is q, where q is actually the characteristic polynomial of (2.2). The condition $b_i > 0$ then implies $b_{m-1} = \operatorname{tr}(K^{-1}D) > 0$.

2.2. An Inverse Eigenvector Problem

To analyze *sufficiency* of the conditions $\operatorname{tr} D, \operatorname{tr} K^{-1}D > 0$ we first reformulate the gyroscopic stabilization problem as an inverse eigenvector problem. According to the spectral mapping theorem [12, Prop. A.1.16], the matrix A_G is asymptotically stable if and only if the matrix

$$-A_G - A_G^{-1} \quad = \quad \begin{bmatrix} K^{-1}(D+G) & K^{-1} - I \\ K - I & D+G \end{bmatrix}$$

is positive stable (i.e. has all eigenvalues in $\mathbb{C}_+$). Let Q denote the positive definite square root of K^{-1}. A similarity transformation with $T = \begin{bmatrix} Q & 0 \\ 0 & I \end{bmatrix}$ brings $-(A_G + A_G^{-1})$ to the form

$$\begin{aligned} -T^{-1}(A_G + A_G^{-1})T \quad &= \quad \begin{bmatrix} Q(D+G)Q & Q - Q^{-1} \\ Q^{-1} - Q & D+G \end{bmatrix} \\ &= \quad \begin{bmatrix} QGQ & 0 \\ 0 & G \end{bmatrix} + \begin{bmatrix} QDQ & Q - Q^{-1} \\ Q^{-1} - Q & D \end{bmatrix} . \end{aligned} \tag{2.3}$$

By a perturbation argument we can formulate a stabilizability criterion as an inverse eigenvector problem.

Proposition 2.1. *Let $\tau > 0$ and $D_\tau = D - \tau P$ for some positive definite P.*
For system (1.2) to be gyroscopically stabilizable it is sufficient that there exists a skew-symmetric matrix $G = -G^T$ with the following properties:

(a) *Both G and QGQ only have simple eigenvalues.*

(b) *If v is an eigenvector of G then $v^* D_\tau v \geq 0$.*
(c) *If w is an eigenvector of QGQ then $w^* Q D_\tau Q w \geq 0$.*

Proof. Instead of (2.3) consider the matrix

$$M_\varepsilon = \begin{bmatrix} QGQ & 0 \\ 0 & G \end{bmatrix} + \varepsilon \begin{bmatrix} QDQ & Q - Q^{-1} \\ Q^{-1} - Q & D \end{bmatrix} .$$

We show that for small $\varepsilon > 0$ this matrix is positive stable, which implies that $\varepsilon^{-1} G$ is a gyroscopic stabilizer.
Note that all eigenvalues of M_ε are perturbations of the imaginary eigenvalues of M_0. We will show that for each eigenvalue $\lambda_0 \in \sigma(M_0) = \sigma(G) \cup \sigma(QGQ) \subset i\,\mathbb{R}$ of multiplicity k the perturbed matrix M_ε has k eigenvalues with positive real part in a neighbourhood of λ_0.

(i) Assume $\lambda_0 \in \sigma(G) \setminus \sigma(QGQ)$. Then G has an eigenvector $v \in \mathbb{C}^n$, so that $\|v\| = 1$ and $Gv = \lambda_0 v$. Condition (a) implies that λ_0 is a simple eigenvalue of M_0. A unit eigenvector of M_0 is given by $v_0 = [0, v]^T$. For small $\varepsilon > 0$ a standard perturbation result (e.g. [19, Thm. IV 2.3]) gives that M_ε has a simple eigenvalue $\lambda_\epsilon = \lambda_0 + \epsilon v^* D v + \mathcal{O}(\epsilon^2)$.
Since $v^* D v > 0$ by (b), we have $\lambda_\varepsilon \in \mathbb{C}_+$.

(ii) Assume $\lambda_0 \in \sigma(QGQ) \setminus \sigma(G)$. For a corresponding unit eigenvector $w_0 = [w, 0]^T$ of M_0, an analogous argument as in the first case shows that M_ϵ has a simple eigenvalue $\lambda_\epsilon = \lambda_0 + \epsilon w^* Q D Q w + \mathcal{O}(\epsilon^2) \in \mathbb{C}_+$.

(iii) Assume $\lambda_0 \in \sigma(QGQ) \cap \sigma(G)$. Then λ_0 is a double eigenvalue of M_0. The corresponding two-dimensional invariant subspace is spanned by vectors v_0 and w_0 as in the first two cases. For small $\varepsilon \geq 0$ the perturbed matrix M_ε also has a two-dimensional invariant subspace, which depends smoothly on ε and coincides with $\mathrm{Span}\{v_0, w_0\}$ for $\varepsilon = 0$. The restriction of M_ε to this subspace has the representation (e.g. [19, Thm. V 2.8])

$$\begin{aligned} & [v_0, w_0]^* M_\varepsilon [v_0, w_0] + \mathcal{O}(\varepsilon^2) \\ = \; & \lambda_0 I + \begin{bmatrix} v^* D v & v^*(Q^{-1} - Q) w \\ w^*(Q - Q^{-1}) v & w^* Q D Q w \end{bmatrix} + \mathcal{O}(\varepsilon^2) . \end{aligned}$$

The 2×2-matrix in the previous term is positive stable, since it has positive trace and positive determinant. Thus M_ε has two positive stable eigenvalues (counting multiplicity) in a neighbourhood of λ_0.

□

If $\frac{\operatorname{tr} D}{n} \leq \frac{\operatorname{tr} QDQ}{\operatorname{tr} Q^2}$ we consider $\tau = \frac{\operatorname{tr} D}{n}$, such that we have

$$\begin{aligned} \operatorname{tr} D_\tau &= 0 \\ \operatorname{tr} QD_\tau Q &\geq 0 \, . \end{aligned}$$

Otherwise, if $\frac{\operatorname{tr} D}{n} > \frac{\operatorname{tr} QDQ}{\operatorname{tr} Q^2}$, we consider the system defined via A_G^{-1} in (2.2) as

$$\frac{d}{dt}\begin{bmatrix} \dot{y} \\ y \end{bmatrix} = \begin{bmatrix} -K^{-1}(D+G) & -K^{-1} \\ I & 0 \end{bmatrix}\begin{bmatrix} \dot{y} \\ y \end{bmatrix},$$

or equivalently

$$0 = \ddot{y} + Q^2(D+G)\dot{y} + Q^2 y \, . \tag{2.4}$$

We put $z = Q^{-1}y$ and multiply from the left with Q^{-1}, then (2.4) becomes

$$0 = \ddot{z} + Q(D+G)Q\dot{z} + Q^2 z \, .$$

We set $\tilde{D} = QDQ$, $\tilde{Q} = Q^{-1}$ and $\tilde{G} = QGQ$, i. e.

$$0 = \ddot{z} + (\tilde{D} + \tilde{G})\dot{z} + \tilde{Q}^{-2} z \, . \tag{2.5}$$

System (2.5) is stable if and only if the original system (1.1) is stable. Note that from $\frac{\operatorname{tr} D}{n} > \frac{\operatorname{tr} QDQ}{\operatorname{tr} Q^2}$ it follows that $\operatorname{tr} \tilde{Q}\tilde{D}\tilde{Q} > n\frac{\operatorname{tr} \tilde{D}}{\operatorname{tr} \tilde{Q}^{-2}}$.
For system (2.5) we get with $\tilde{\tau} = \frac{\operatorname{tr} \tilde{D}}{\operatorname{tr} \tilde{Q}^{-2}}$ and $P = \tilde{Q}^{-2}$:

$$\begin{aligned} \operatorname{tr}(\tilde{D} - \tfrac{\operatorname{tr} \tilde{D}}{\operatorname{tr} \tilde{Q}^{-2}}\tilde{Q}^{-2}) &= 0 \, , \\ \operatorname{tr}(\tilde{Q}(\tilde{D} - \tfrac{\operatorname{tr} \tilde{D}}{\operatorname{tr} \tilde{Q}^{-2}}\tilde{Q}^{-2})\tilde{Q}) &= \operatorname{tr}(\tilde{Q}\tilde{D}\tilde{Q}) - \operatorname{tr} \tilde{D}\frac{\operatorname{tr} I}{\operatorname{tr} \tilde{Q}^{-2}} \\ &= \operatorname{tr}(\tilde{Q}\tilde{D}\tilde{Q}) - \operatorname{tr} \tilde{D}\frac{n}{\operatorname{tr} \tilde{Q}^{-2}} \geq 0 \, . \end{aligned}$$

Also, if $\tilde{G}$ is such that $\tilde{v}_i^* \tilde{D}_\tau \tilde{v}_i = 0$ for all eigenvectors $\tilde{v}_i$ of $\tilde{G}$ and $\tilde{w}_i^* \tilde{Q}\tilde{D}_\tau \tilde{Q}\tilde{w}_i \geq 0$ for all eigenvectors $\tilde{w}_i$ of $\tilde{Q}\tilde{D}\tilde{Q}$, then for the skew-symmetric $G = Q^{-1}\tilde{G}Q^{-1}$ we have $v_i^* D v_i \geq 0$ for all eigenvectors $v_i = \tilde{w}_i$ of G and $w_i^* QDQ w_i \geq 0$ for all eigenvectors $w_i = \tilde{v}_i$ of QDQ. Hence from now on we always consider $\operatorname{tr} D_\tau = 0$ and $\operatorname{tr} QD_\tau Q \geq 0$; for simplicity, we write again D, Q instead of $\tilde{D}, \tilde{Q}$.
Throughout the following chapters, we will show that a solution to the following problem exists.

Problem 2.2. *For symmetric matrices $D, Q \in \mathbb{R}^{n\times n}$ satisfying $Q > 0$, $\operatorname{tr} D = 0$ and $\operatorname{tr} QDQ \geq 0$ find $G = -G^T$ such that*

(a) *both G and QGQ only have simple eigenvalues,*
(b) *if v is an eigenvector of G then $v^* D v = 0$,*

(c) *if w is an eigenvector of QGQ then $w^*QDQw \geq 0$.*

Remark 2.3. Consider the characteristic polynomial p of the system $\ddot{x} + (D+G)\dot{x} + Kx = 0$. We have

$$\begin{aligned} p(\lambda) &= \det\left(\lambda^2 I + \lambda(D+G) + K\right) \\ &= \det\left(\lambda^2 I + \lambda(D+G) + K\right)^T \\ &= \det\left(\lambda^2 I + \lambda(D-G) + K\right) . \end{aligned}$$

Thus it follows that G is a gyroscopic stabilizer if and only if $-G$ is a gyroscopic stabilizer.

2.3. Traces And Indefinite Scalar Products

The conditions $v^*Dv = 0, w^*QDQw \geq 0$ from problem 2.2 can be considered in context of indefinite scalar products. Thus, we will in short give some background on the subject and introduce some notation that we will use in the subsequent chapters.

As before, by $\operatorname{tr} A$ we denote the trace of a square matrix. We will extend this notion now and define the trace of a matrix on a subspace. It is well-known that $\operatorname{tr} BC = \operatorname{tr} CB$ if the product BC is a square matrix. Hence, if $U = [u_1, \dots, u_n] \in \mathbb{C}^{n\times n}$ is unitary then

$$\operatorname{tr} A = \operatorname{tr} AUU^* = \operatorname{tr} U^*AU = \sum_{j=1}^{n} u_j^* A u_j \ .$$

More generally, if the matrix $U = [u_1, \dots, u_k] \in \mathbb{C}^{n\times k}$ has orthonormal columns, we write $\mathcal{U} = \operatorname{Span}\{u_1, \dots, u_k\}$ and $P_{\mathcal{U}} = UU^*$ for the orthogonal projection onto $\mathcal{U}$. Then $P_{\mathcal{U}}AP_{\mathcal{U}}$ is the projection of A to $\mathcal{U}$ and

$$\operatorname{tr}_{\mathcal{U}} A := \operatorname{tr}(P_{\mathcal{U}}AP_{\mathcal{U}}) = \operatorname{tr}(U^*AU) = \sum_{j=1}^{k} u_j^* A u_j$$

is the trace of the projected matrix. It is important that $\operatorname{tr}_{\mathcal{U}}$ depends continuously on $\mathcal{U}$, or, equivalently, on the orthogonal projector $P_{\mathcal{U}}$.

A matrix $D = D^T \in \mathbb{R}^{n\times n}$ is positive definite, if $\operatorname{tr}_{\mathcal{U}} D > 0$ for all non-zero subspaces $\mathcal{U} \subset \mathbb{C}^n$. If D is indefinite, there exists a vector $u \in \mathbb{R}^n$ with $u^T Du = 0$.

If G is skew-symmetric then $\operatorname{tr} G = 0$. Moreover, if $P = QQ^T$ is positive definite then also $\operatorname{tr} PG = \operatorname{tr} Q^TGQ = 0$.

Note that the eigenvalues of a skew-symmetric matrix G are either zero or complex conjugate pairs of purely imaginary numbers and a set of eigenvectors of a skew-symmetric matrix can be chosen as an orthonormal basis of $\mathbb{C}^n$. The complex eigenvectors can be chosen as conjugate pairs. If v, w is any pair of normalized complex conjugate orthogonal vectors, we have

$$\begin{aligned} v^*Dv = w^*Dw &= \frac{1}{2}\operatorname{tr}_{\operatorname{Span}\{v,w\}} D \\ &= \frac{1}{2}\operatorname{tr}_{\operatorname{Span}\{b_1,b_2\}} D = \frac{1}{2}(b_1^*Db_1 + b_2^*Db_2) \end{aligned} \tag{2.6}$$

for any orthonormal basis $\{b_1, b_2\}$ of $\operatorname{Span}\{v, w\}$ and any matrix D of suitable size, a fact we use freqently.

Definition 2.4. An indefinite scalar product $\langle\, .\, ,\, .\, \rangle_D$ induced by an indefinite matrix D in the vector space $\mathcal{V} = \mathbb{C}^n$ is a bilinear form with

(i) $\langle x, y\rangle_D = \langle y, x\rangle_D^*$ for all $x, y \in \mathcal{V}$,
(ii) $\langle ax + by, z\rangle_D = a\langle x, z\rangle_D + b\langle y, z\rangle_D$ for all $x, y, z \in \mathcal{V}$ and $a, b \in \mathbb{C}$.

If D is singular and $x \in \ker(D)$, then we have $\langle x, y\rangle_D = 0$ for all $y \in \mathbb{C}^n$. In [11], a brief introduction on indefinite scalar products and Krein spaces can be found.

Definition 2.5. A vector $x \in \mathcal{V}$ is said to be

· *D-positive* if $\langle x, x\rangle_D > 0$,
· *D-neutral* or *D-isotropic* if $\langle x, x\rangle_D = 0$,
· *D-negative* if $\langle x, x\rangle_D < 0$.

Accordingly, we call a subspace $\mathcal{U} \subset \mathcal{V}$ D-positive if $\langle x, x\rangle_D > 0$ for all $x \in \mathcal{U}$; a D-negative subspace is defined analogously. In addition, we call a subspace *indefinite* if it contains both positive and negative vectors.

For a finite-dimensional subspace $\mathcal{U} = \operatorname{Span}\{u_1, \dots, u_m\} \subseteq \mathcal{V}$, we define the gramian

$$R_{\mathcal{U}} = \begin{pmatrix} \langle u_1, u_1\rangle_D & \dots & \langle u_1, u_m\rangle_D \\ \vdots & \ddots & \vdots \\ \langle u_m, u_1\rangle_D & \dots & \langle u_m, u_m\rangle_D \end{pmatrix}.$$

Since D is hermitian, the gramian $R_{\mathcal{U}}$ is hermitian as well.

Lemma 2.6. *[11] Let $u_1, \dots, u_m \in \mathcal{V}$ be linearly independent. Then $\mathcal{U} = \operatorname{Span}\{u_1, \dots, u_m\}$ is a positive subspace of $\mathcal{V}$ if and only if $R_{\mathcal{U}}$ is positive definite.*

Proof. Since $u_1, \dots, u_m$ are linearly independent, any $x \in \mathcal{U}$ has a unique representation $x = [u_1, \dots, u_m]y$ with $y \in \mathbb{C}^m$. Since $\langle x, x\rangle_D = x^*Dx = y^*[u_1, \dots, u_m]^*D[u_1, \dots, u_m]y > 0$, it follows that $R_{\mathcal{U}}$ is positive definite if and only if $\mathcal{U}$ is positive. □

The idea of the proof for indefinite and negative definite subspaces is similar. Note that an indefinite real subspace contains at least one nonzero real isotropic vector, a fact that we will use extensively.
We now recall a well-known fact about quadrics (see for example Prop. 14.3.1 in [3], [9]).

Proposition 2.7. *Let $D \in \mathbb{R}^{n \times n}$ with $n \geq 3$ be a symmetric matrix. Then the set $\mathcal{M} = \{x \in \mathbb{R}^n \mid x^* D x = 0\}$ is path-connected.
Furthermore, the set $\mathcal{M}_1 = \{x \in \mathbb{R}^n \mid \|x\| = 1\ ,\ x^* D x = 0\}$ consists of at most two path-connected components.*

Let $x, y \in \mathbb{R}^n$, $n \geq 3$, with $\|x\| = \|y\| = 1$ be given. Then by proposition 2.7 it follows that with some arbitrary $a, b \in \mathbb{R}^n$, $a < b$, there exists a continuous mapping $z : [a, b] \to \mathbb{R}^n$ with the properties $\|z(t)\| = 1$ for all t, $z(a) = x$ and either $z(b) = y$ or $z(b) = -y$.

For further use we define the numerical range of a symmetric matrix.

Definition 2.8. Let $D \in \mathbb{R}^{n \times n}$ be symmetric. Then the *numerical range* of D is defined as

$$\operatorname{nr} D := \{x^* D x \mid x \in \mathbb{R}^n\ ,\ \|x\| = 1\}\ .$$

2.4. A Special Case

Here we present a special case for which the existence of a gyroscopic stabilizer is already known, see for example [15]. We apply Problem 2.2 and show how to actually construct a gyroscopic stabilizer G.

Proposition 2.9. *([15]) Let $\operatorname{tr} D = 0$ and $Q = cI$ with $c > 0$ be a multiple of the unit matrix. Then there exists a gyroscopic stabilizer.*

Proof. We write down the proof only for odd $n = 2m + 1$, the proof for even n is analogous.
Since $Q = cI$, conditions (b) and (c) from problem 2.2 are identical. Thus, we just need to make sure that (b) is satisfied.
Given a symmetric matrix D with $\operatorname{tr} D = 0$, it is known ([8], [10]), that there exists a real orthogonal matrix $U = [u_1, \ldots, u_n]$ such that the diagonal elements of $U^* D U$ are all identically zero. Define

$$\begin{aligned} G &= U \operatorname{diag}(0,\ \rho_1 G_0,\ \ldots,\ \rho_m G_0) U^*\ , \\ G_0 &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\ , \end{aligned}$$

where $\rho_i \neq 0$ for all i and $\rho_i \neq \rho_j$ for $i \neq j$. Then u_1 is an eigenvector for the eigenvalue zero of G and we have $u_1^* D u_1 = 0$ by construction.

Now consider the pair of eigenvalues $\pm i\rho_i$ and let x, y be a pair of orthonormal eigenvectors for $\pm i\rho_i$. Then we have

$$\mathrm{Span}\{x, y\} = \mathrm{Span}\{u_{2i}, u_{2i+1}\} \ .$$

By construction,

$$\mathrm{tr}_{\mathrm{Span}\{u_{2i}, u_{2i+1}\}} D = u_{2i}^* D u_{2i} + u_{2i+1}^* D u_{2i+1} = 0 \ ,$$

thus it follows that with (2.6) that $x^* D x = y^* D y = 0$. □

CHAPTER 3

Space Dimensions Two and Three

3.1. Existence of G in Space Dimension Two

The existence of a gyroscopic stabilizer in space dimension two is well known ([2]). Here we demonstrate that also our newly derived sufficient condition implies the existence of a gyroscopic stabilizer.

PROPOSITION 3.1. *Let $D, Q \in \mathbb{R}^{2\times 2}$ with $Q > 0$ and $\operatorname{tr} D = 0$. Then*

$$G = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

solves problem 2.2.

PROOF. Let v_1, v_2 be a pair of normalized eigenvectors of G. By (2.6), we have $v_1^* D v_1 = v_2^* D v_2 = \frac{1}{2} \operatorname{tr} D = 0$. Analogously, let w_1, w_2 be a pair of normalized eigenvectors of G, then $w_1^* QDQ w_1 = w_2^* QDQ w_2 = \frac{1}{2} \operatorname{tr} QDQ > 0$. $\square$

Since there is basically only one gyroscopic stabilizer G, the more interesting question is whether we can actually compute the coefficient γ, such that with $\operatorname{tr} D, \operatorname{tr} QDQ > 0$, the system $\ddot{x} + (D + \gamma G)\dot{x} + Kx = 0$ is stable. We use the approach from [2, p. 196 ff], where the Hurwitz criterion [12, Cor. 3.4.71] is applied. The result can also be found in [14].

LEMMA 3.2. *[12, Cor. 3.4.71], [14] Let $D, Q \in \mathbb{R}^{2\times 2}$ be given with $\operatorname{tr} D, \operatorname{tr} QDQ > 0$ and let $\ddot{x} + D\dot{x} + Kx = 0$ with $K = Q^{-2}$ be unstable. Then $\ddot{x} + (D + \gamma G)\dot{x} + Kx = 0$ with G as in proposition 3.1 is stable if and only if*

$$\gamma^2 > -\operatorname{tr} K - \det D + \frac{\operatorname{tr} D \det K}{\operatorname{tr} K^{-1} D} + \frac{\operatorname{tr} K^{-1} D}{\operatorname{tr} D} \ .$$

Proof. Let $p(\lambda) = \det(\lambda^2 I + \lambda(D+\gamma G) + Q^{-2})$. Direct computation shows that

$$\begin{aligned} p(\lambda) &= \lambda^4 + \operatorname{tr} D\, \lambda^3 + (\operatorname{tr} K + \det D + \gamma^2)\lambda^2 + \operatorname{tr}(K^{-1}D)\lambda + \det K \\ &= \sum_{i=0}^{4} a_i \lambda^i \, . \end{aligned}$$

For stability, it is necessary and sufficient that all principal minors of the associated Hurwitz matrix $M(p)$ are positive. Here we have

$$M(p) = \begin{pmatrix} a_3 & a_1 & 0 \\ a_4 & a_2 & a_0 \\ 0 & a_3 & a_1 \end{pmatrix} .$$

With the criterion of Liénard-Chipart [12, Th. 3.4.73] it is even necessary and sufficient for stability that $a_i > 0$ for all i and the principal minors of odd order are positive, which in this case gives us the condition $\det M(p) = a_3(a_1 a_2 - a_0 a_3) - a_1^2 a_4 > 0$.
By asumption, $a_4, a_3, a_1, a_0 > 0$, thus if $\det M(p) > 0$ then $a_2 > 0$. Here, we get

$$\begin{aligned} \det M(p) &= \operatorname{tr} D\big(\operatorname{tr} K^{-1}D \det K(\operatorname{tr} K + \gamma^2 + \det D) - \det K \operatorname{tr} D\big) \\ &\quad -(\operatorname{tr} K^{-1} D \det K)^2 > 0 \, . \end{aligned}$$

Solving for γ^2 yields

$$\gamma^2 > -\operatorname{tr} K - \det D + \frac{\operatorname{tr} D \det K}{\operatorname{tr} K^{-1} D} + \frac{\operatorname{tr} K^{-1} D}{\operatorname{tr} D} \, .$$

□

Note that if $\ddot{x} + D\dot{x} + Kx = 0$ is already stable, it follows that adding a gyroscopic term in $\mathbb{R}^2$ never destabilizes the system, since G only plays a role in a_2 and $M(p)$, and both terms become larger by adding G. We can even conclude that in $\mathbb{R}^2$, the conditions of Proposition 2.1 are not only sufficient, but also necessary for gyroscopic stabilizability.

3.2. Existence of G in Space Dimension Three

Proposition 3.3. *Let $D, Q \in \mathbb{R}^{3\times 3}$ with $Q > 0$ and $\operatorname{tr} D = 0$. Choose $\omega \in \mathbb{R}\backslash\{0\}$ and an orthonormal basis $\{u_1, u_2, u_3\}$ of $\mathbb{R}^3$ so that $u_1^* D u_1 = 0$. Then*

$$G = \big[u_1, u_2, u_3\big] \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \omega \\ 0 & -\omega & 0 \end{pmatrix} \big[u_1, u_2, u_3\big]^T \tag{3.1}$$

solves Problem 2.2.

Proof. By construction, $v_1 = u_1$ is an eigenvector for the eigenvalue 0 of G, and we have $v_1^* D v_1 = 0$. Since G is skew-symmetric, the eigenvectors are orthogonal and the eigenvectors v_2, v_3 for the imaginary eigenvalues can be assumed to be complex conjugate and normalized. From (2.6) it follows that it suffices to show that $\operatorname{tr}_{\operatorname{Span}\{v_2,v_3\}} D \geq 0$. From $\operatorname{Span}\{v_2, v_3\} = \operatorname{Span}\{u_2, u_3\}$ one has

$$\begin{aligned} \operatorname{tr}_{\operatorname{Span}\{v_2,v_3\}} D &= \operatorname{tr}_{\operatorname{Span}\{u_2,u_3\}} D \\ &= \operatorname{tr} D - \operatorname{tr}_{\operatorname{Span}\{u_1\}} D = 0 \, . \end{aligned}$$

Since $QGQQ^{-1}v_1 = 0$, it follows that $w_1 = \frac{Q^{-1}v_1}{\|Q^{-1}v_1\|}$ is an eigenvector for the eigenvalue 0 of QGQ. Since $w_1^* QDQ w_1 = 0$, we have $\operatorname{tr}_{\operatorname{Span} w_1} D = 0$. Let w_2, w_3 denote the other eigenvectors of QDQ. We use (2.6) again and get

$$\begin{aligned} \operatorname{tr}_{\operatorname{Span}\{w_2,w_3\}} QDQ &= \operatorname{tr} QDQ - \operatorname{tr}_{w_1} QDQ \\ &= \operatorname{tr} QDQ \quad \geq 0 \, , \end{aligned}$$

which completes the proof. $\square$

As seen in the previous section, in $\mathbb{R}^2$, any gyroscopic stabilizer satisfies the conditions provided by Proposition 2.1, basically there is just one. In $\mathbb{R}^3$, the situation is already different, there exist gyroscopic stabilizers whose eigenvectors do not satisfy the mentioned conditions, as the following example shows. But still, as shown with proposition

Example 3.4. Consider the system

$$0 = \ddot{x} + \begin{pmatrix} -4 & -2 & 3 \\ -2 & 8 & -12 \\ 3 & -12 & 4 \end{pmatrix} \dot{x} + \begin{pmatrix} 5 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{pmatrix}^{-2} x \, .$$

We have $\operatorname{tr} D = 8$ and $\operatorname{tr} QDQ = 8$, and the system is unstable, since the maximum of the real parts of the eigenvalues is 6.5803. Now, there exists a gyroscopic stabilizer

$$G = \begin{pmatrix} 0 & -1 & -11 \\ 1 & 0 & -26 \\ 11 & 26 & 0 \end{pmatrix} ,$$

such that the largest real part of the eigenvalues of $\ddot{x} + (D+G)\dot{x} + Q^{-2}x = 0$ is about -0.000032. But for the eigenvector $v = (26,\ -11,\ 1)$ with eigenvalue zero, we have $v^* D v = -168 < 0$. Thus, the matrix G does not result from our construction in Proposition 3.3. This shows that in $\mathbb{R}^3$, not every gyroscopic stabilizer satisfies the conditions given by Problem 2.2.

3.3. The Set of Gyroscopic Stabilizers in $\mathbb{R}^3$

We first investigate the solutions of the *shifted* problem 2.2, which themselves are a subset of the gyroscopic stabilizers that satisfy the conditions of Proposition 2.1. Thus, at first we consider pairs D, Q such that $\operatorname{tr} D = 0, \operatorname{tr} QDQ \geq 0$.
The construction of G in Proposition 3.3 actually just depends on finding an isotropic vector with respect to D. Let

$$G = \begin{pmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{pmatrix} \tag{3.2}$$

be a skew symmetric matrix. Then $g = (a, b, c)^T$ is an eigenvector of G for the eigenvalue zero. This shows that the specific choice of $\{u_2, u_3\}$ in Proposition 3.3 does not play any role in the construction of G. The actual form of G is already completely determined just by u_1, but the question of how to scale G remains open.
The question of how to scale G with a suitable γ such that the system actually becomes stable is still open, we just know that γ exists. Therefore, here we consider g to be normalized, and instead of speaking about the matrix G, we now just talk about the associated vector g.
For a given symmetric D with $\operatorname{tr} D = 0$, we define the sets

$$\begin{aligned} S_0 &= \{g \mid g^* D g = 0,\ \|g\| = 1\}\ , \\ S_+ &= \{g \mid g^* D g > 0,\ \|g\| = 1\}\ , \\ S_- &= \{g \mid g^* D g < 0,\ \|g\| = 1\}\ . \end{aligned}$$

Throughout our work, a quadric $\{g \mid g^* D g = 0\}$, where D is symmetric, will be referred to as the *cone* generated by D. Clearly $S_0 \subset \{g \mid g^* D g = 0\}$. If $g \in S_0$, then the associated G satisfies the conditions of problem 2.2. The shape of the sets S_0, S_+, S_- is determined by the eigenvalue structure of D. Let $v_-(D),\ v_0(D),\ v_+(D)$ denote the number of eigenvalues of A counting algebraic multiplicities that are, respectively, negative, zero, and positive. Define the inertia of D as $\operatorname{In}(D) = (v_-(D)\ v_0(D)\ v_+(D))$.
We assume D to be nonzero, then from $\operatorname{tr} D = 0$ it follows that $\operatorname{In}(D)$ takes either the form $(2, 0, 1)$ or $(1, 1, 1)$ or $(1, 0, 2)$.
The set S_0 is the intersection of the cone defined by $\{x \mid x^* D x = 0\}$ and the unit sphere, and according to [9], the set $\{x \mid x^* D x = 0\}$ is path-connected. The intersection with the unit sphere consists of one connected component in the case that $\operatorname{In}(D) = (1, 0, 1)$, or two connected components if either $\operatorname{In}(D) = (2, 0, 1)$ or $\operatorname{In}(D) = (1, 0, 2)$.
Figure 3.1 shows the cone which is determined by the isotropic vectors

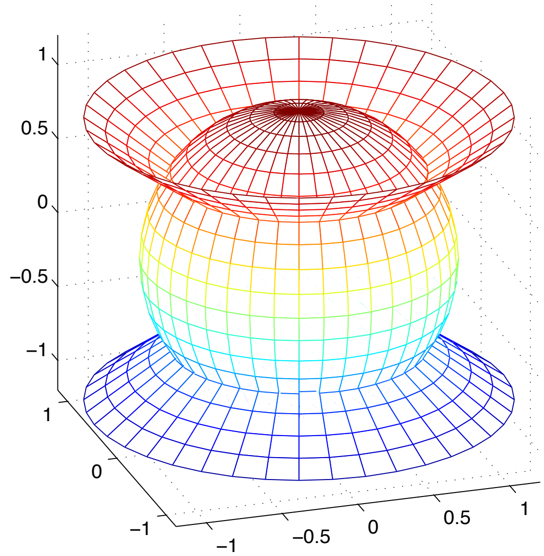

FIGURE 3.1. Intersection of unit sphere with cone S_0

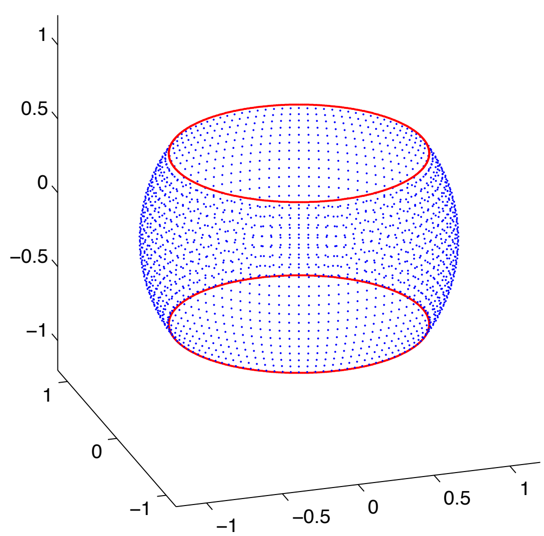

FIGURE 3.2. Sets S_+ (blue) and S_0 (red)

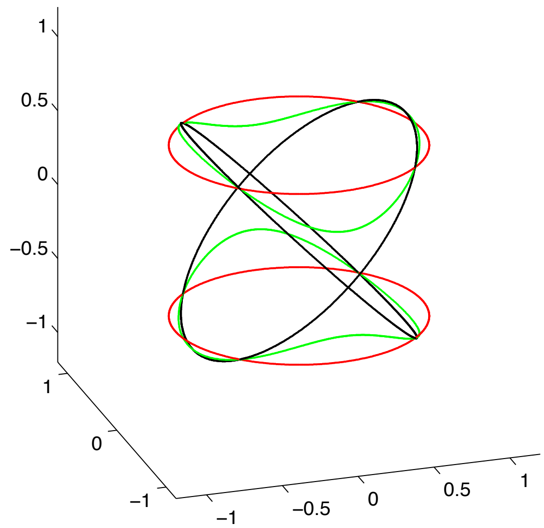

FIGURE 3.3. Set S_0 for different D

of the matrix $D = \operatorname{diag}(1\ 1\ -2)$. Exactly one eigenvalue is negative and two are positive, thus each of the sets S_- and S_0 splits into two connected components, while S_+ consists of exactly one connected component. Figure 3.2 shows the set S_+ visualized by blue dots, and the set S_0 in red; the latter is the intersection line of the cone in Figure 3.1 with the unit sphere.

In Figure 3.3, several possible shapes for S_0 depending on the given D can be seen. The red set is for $D = \operatorname{diag}(1\ 1\ -2)$, the green set for $D = \operatorname{diag}(6\ 1\ -7)$ and the black set for $D = \operatorname{diag}(1\ 0\ -1)$. While the first two sets in green and red are given by intersections of actual cones with the unit sphere, the black set arises from the degenerate case, where S_0 consists of exactly one connected component.

We now come to the original problem and consider pairs D, Q with $\operatorname{tr} D > 0, \operatorname{tr} QDQ > 0$, as in Proposition 2.1. Consider the pair D, Q with

$$D = \begin{pmatrix} 1.3 & 0 & 0 \\ 0 & 1.3 & 0 \\ 0 & 0 & -1.7 \end{pmatrix}, \quad Q = \begin{pmatrix} 7 & 3 & 0 \\ 3 & 4 & 2 \\ 0 & 2 & 6 \end{pmatrix}.$$

While in the situation $\operatorname{tr} D = 0$ it was enough to identify the g with $g^* D g = 0$ in order to find the gyroscopic stabilizers, we now have

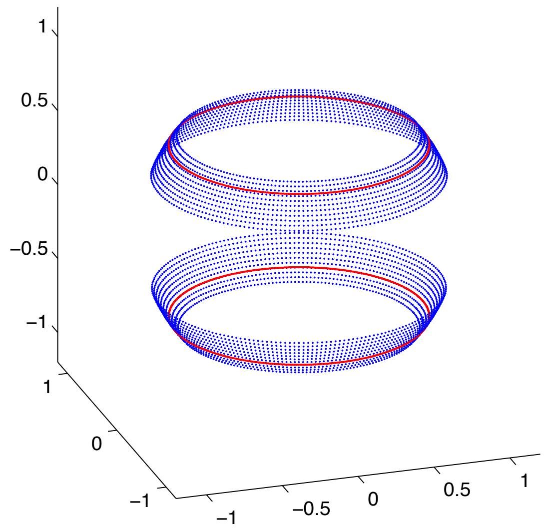

FIGURE 3.4. x with $0 < x^*Dx < \mathrm{tr}(D)$

to consider the conditions $v^*Dv > 0$ for all eigenvectors of G and $w^*QDQw > 0$ for all eigenvectors of QGQ separately.

Consider a g such that $0 < g^*Dg < \mathrm{tr}(D)$. Let v_2, v_3 be a normalized pair of eigenvectors for the complex eigenvalues of the associated G. Then since $v_2^*Dv_2 = v_3^*Dv_3$ and $g^*Dg + v_2^*Dv_2 + v_3^*Dv_3 = \mathrm{tr}(D)$ it follows that $0 < v_2^*Dv_2,\ v_3^*Dv_3 < \mathrm{tr}\, D$. Figure 3.4 shows in blue the set of g such that the conditions on the eigenvectors of the associated G are satisfied. The boundaries of the blue set are determined by the cones defined by $x^*Dx = 0$ and $x^*(D - \mathrm{tr}(D)I)x = 0$.

The two red lines show those g for which $g^*(D - \frac{\mathrm{tr}\, D}{3}I)g = 0$, thus they are the solutions to the *shifted* problem as formulated in 2.2 and coincide with the set S_0 as in Figure 3.2. By construction, the red set is completely contained in the blue set.

Now consider the eigenvectors of QGQ. As in Proposition 3.3, we know that $w_1 = \frac{Q^{-1}g}{\|Q^{-1}g\|}$ is an eigenvector for the eigenvalue zero of QGQ, and we denote normalized eigenvectors for the complex eigenvalues with w_2, w_3. Let $0 < w_1^*QDQw_1 = \frac{g^*Dg}{\|Q^{-1}g\|^2} < \mathrm{tr}(QDQ)$. With $w_2^*QDQw_2 = w_3^*QDQw_3$ and

$$w_1^*QDQw_1 + w_2^*QDQw_2 + w_3^*QDQw_3 = \mathrm{tr}(QDQ)$$

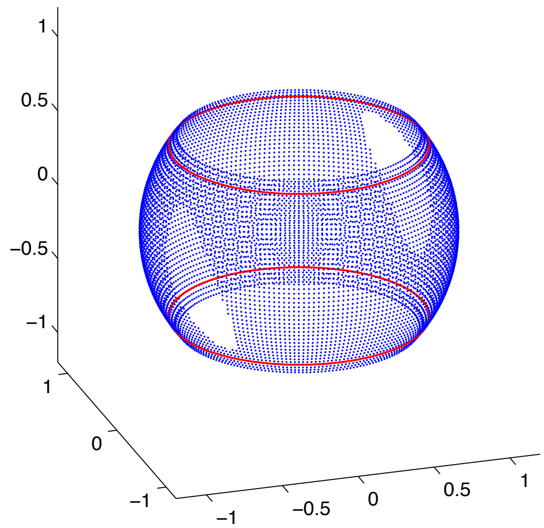

FIGURE 3.5. x with $0 < \frac{x^* Dx}{\|Q^{-1}x\|^2} < \mathrm{tr}(QDQ)$

it follows that

$$0 < w_2^* QDQw_2, w_3^* QDQw_3 < \mathrm{tr}\, QDQ\ ,$$

satisfying the eigenvector conditions on QGQ. In Figure 3.5, the set of the g such that the eigenvector conditions for QGQ are satisfied, is shown in blue. The boundaries of the blue set are determined by the two cones defined by $x^* Dx = 0$ and $x^*(QDQ - \mathrm{tr}(QDQ)I)x = 0$. The red set is the same as in the previous figures and again completely contained in the blue set by construction.

Since the eigenvector conditions on G and QGQ need to be satisfied simultaneously, the set of the gyroscopic stabilizers is given by the intersection of the sets in Figures 3.4 and 3.5. This intersection is the blue set in Figure 3.6, and it is non-empty, as can be seen by the fact that the red line, consisting of those g for which $g^*(D - \frac{\mathrm{tr}\, D}{3} I)g = 0$, is contained in the intersection. Altogether, in Figure 3.6 all g such that the conditions in Proposition 2.1 are satisfied, are plotted in blue. This blue set is bounded by exactly three cones given by $g^* Dg = 0$, $x^*(D - \mathrm{tr}(D)I)x = 0$ and $x^*(QDQ - \mathrm{tr}(QDQ)I)x = 0$.
But Proposition 2.1 just provided a sufficient condition for the existence of a gyroscopic stabilizer G (or g), thus one can assume that actually there are more gyroscopic stabilizers than we constructed so far.

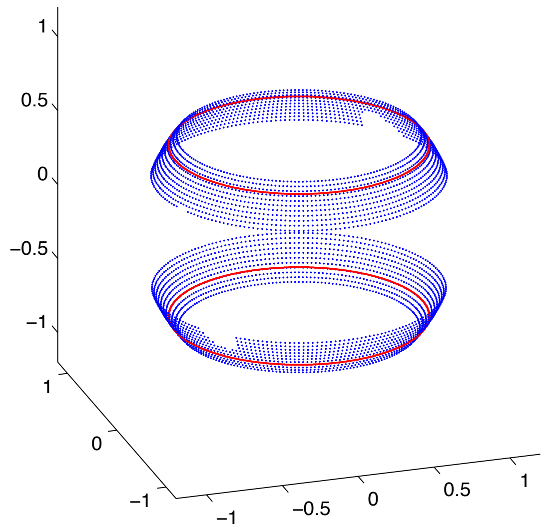

FIGURE 3.6. x with $0 < x^*Dx < \operatorname{tr}(D)$ and $0 < \frac{x^*Dx}{\|Q^{-1}x\|^2} < \operatorname{tr}(QDQ)$

Figure 3.7 shows again in blue the g that result from Proposition 2.1 and in red the g such that our sufficient conditions are violated but still the associated G is a gyroscopic stabilizer. The red g were found using Matlab: we directly computed the eigenvalues of the matrix

$$M = \begin{bmatrix} 0 & I \\ -K & -D - \gamma G \end{bmatrix}$$

with coefficients γ in the range from zero to 2000; in the case the matrix was stable, the associated g was plotted.

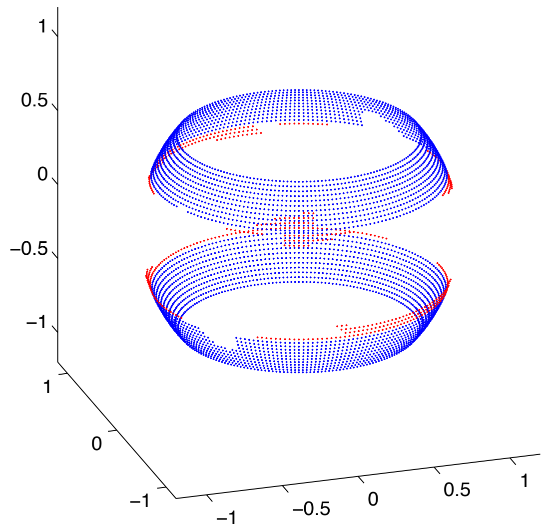

FIGURE 3.7. g resulting from Proposition 2.1 (blue), other stabilizing g (red)

CHAPTER 4

Space Dimension Four

In the three-dimensional case, we exploited the fact that the skew-symmetric matrices $G, QGQ \in \mathbb{R}^{3\times 3}$ both have a zero eigenvalue, and the corresponding eigenvectors are related via multiplication with Q^{-1}. Now we construct $G \in \mathbb{R}^{4\times 4}$ with a double eigenvalue zero, allowing us to identify spaces containing eigenvectors of QGQ. Then we use a perturbation argument to move the zero eigenvalues along $i\,\mathbb{R}$.

4.1. Perturbation of Eigenvalues and Eigenvectors

Proposition 4.1. *For some $\delta \in \mathbb{R}$ and an orthogonal matrix $Z = [z_1, z_2, z_3, z_4] \in \mathbb{R}^{4\times 4}$ let*

$$G_\delta = Z \begin{pmatrix} 0 & \delta & 0 & 0 \\ -\delta & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} Z^T . \tag{4.1}$$

If for some $\tau > 0$ we have

(i) $\operatorname{tr}_{\operatorname{Span}\{z_1,z_2\}} D_\tau \geq 0$ *and* $\operatorname{tr}_{\operatorname{Span}\{z_3,z_4\}} D_\tau \geq 0$,

(ii) $\operatorname{tr}_{Q^{-1}\operatorname{Span}\{z_1,z_2\}} QD_\tau Q \geq 0$ *and* $\operatorname{tr}_{Q\operatorname{Span}\{z_3,z_4\}} QD_\tau Q \geq 0$,

then there exists $\delta \neq 0$ so that (a), (b), and (c) in Proposition 2.1 hold for G_δ and $\tau/2$.

The idea behind the above statement is as follows. The matrix G_0 is constructed such that after a small perturbation δ of the double zero eigenvalue of G, the conditions (a), (b) and (c) in Proposition 2.1 are satisfied, which is ensured by the assumptions (i) and (ii).
Then, by changing the value of δ by a tiny amount, the space associated with the zero eigenvalues of $QG_\delta Q$ is changed a bit as well, but still the conditions posed on the eigenvectors of $QG_\delta Q$ remain satisfied as long as the change in δ remains sufficiently small.

Proof. By continuity of eigenvalues, it is clear that (a) in Proposition 2.1 holds for small $|\delta| \neq 0$.
Using (2.6) and the structure of G_δ we conclude that assumption (b) is equivalent to $\operatorname{tr}_{\operatorname{Span}\{z_1,z_2\}} D_\tau \geq 0$ and $\operatorname{tr}_{\operatorname{Span}\{z_3,z_4\}} D_\tau \geq 0$ for all

$\delta \in \mathbb{R}\setminus\{0\}$.
To verify (c), note that $QG_\delta Q$ has two conjugate pairs of imaginary eigenvalues, which we denote by $\pm\lambda_\delta$ and $\pm\mu_\delta$. These depend continuously on δ (where $\lambda_0 = 0$). The same is true (e.g. [19]) for the invariant subspaces

$$\begin{aligned}\mathcal{V}_\lambda(\delta) &:= \operatorname{Ker}\left((QG_\delta Q)^2 + |\lambda_\delta|^2 I\right) ,\\ \mathcal{V}_\mu(\delta) &:= \operatorname{Ker}\left((QG_\delta Q)^2 + |\mu_\delta|^2 I\right) .\end{aligned}$$

By assumption for $\delta = 0$ and $\eta = \tau$ we have with $D = D_\tau + \tau P$:

$$\begin{aligned}\operatorname{tr}_{\mathcal{V}_\lambda(\delta)} QDQ &\geq \eta \operatorname{tr}_{\mathcal{V}_\lambda(\delta)} QPQ > 0 ,\\ \operatorname{tr}_{\mathcal{V}_\mu(\delta)} QDQ &\geq \eta \operatorname{tr}_{\mathcal{V}_\mu(\delta)} QPQ > 0 .\end{aligned}$$

By continuity, the same holds for $\eta = \tau/2$ and sufficiently small δ. Together with (2.6) this completes the proof. $\square$

Thus, we can relax the conditions in Problem 2.2 slightly and reformulate it, such that it relates to two-dimensional spaces containing pairs of complex conjugate eigenvectors instead of relating directly to eigenvectors.

Problem 4.2. *For symmetric matrices $D, Q \in \mathbb{R}^{4\times 4}$ satisfying $Q > 0$, $\operatorname{tr} D = 0$ and $\operatorname{tr} QDQ \geq 0$, find G_0 as in (4.1) such that $\operatorname{tr}_{\operatorname{Ker} G_0} D = 0$ and $\operatorname{tr} QDQ \geq \operatorname{tr}_{\operatorname{Ker} QG_0Q} QDQ \geq 0$.*

Let $\{u_1, u_2, u_3, u_4\}$ denote an orthonormal set of eigenvectors of Q with corresponding eigenvalues $\lambda_k > 0$. We consider the numbers $u_i^* D u_i$. Since $\operatorname{tr} D = 0$ we either have $u_i^* D u_i = 0$ for all i or some of these numbers are positive and some are negative. In the following propositions, we make a complete distinction between all possible cases.

4.2. Construction Of G Via Eigenvectors Of Q

Proposition 4.3. *Assume that for some ordering of the u_i the spaces $\mathcal{U}_{12} = \operatorname{Span}\{u_1, u_2\}$ and $\mathcal{U}_{34} = \operatorname{Span}\{u_3, u_4\}$ are both D-indefinite or D-singular. Then there exists a skew-symmetric G_0 solving Problem 4.2.*

The idea of the following proof is to construct a two-dimensional space via identifying a basis consisting of D-neutral vectors. Note that a D-neutral basis does not imply that each vector in the space is D-neutral. The fact that $\mathcal{U}_{12}$ and $\mathcal{U}_{34}$ are orthogonal and remain orthogonal under multiplication with Q^{-1} is used.

Proof. By our assumptions on $\mathcal{U}_{12}$ and $\mathcal{U}_{34}$, there exist normalized vectors $z_1 \in \mathcal{U}_{12}$ and $z_2 \in \mathcal{U}_{34}$ with $\langle z_1, z_1\rangle_D = \langle z_2, z_2\rangle_D = 0$. Let $Z = \left[z_1, z_2, z_3, z_4\right] \in \mathbb{R}^{4\times 4}$ be orthogonal and define G_0 as in (4.1).

Then $\mathrm{Span}\{z_1, z_2\} = \mathrm{Ker}\, G_0$ and $\mathrm{tr}_{\mathrm{Ker}\, G_0} D = 0$.
Again by construction, $\{Q^{-1}z_1, Q^{-1}z_2\}$ is an orthogonal basis of the space $\mathrm{Ker}\, QG_0Q$ and

$$\langle Q^{-1}z_1, Q^{-1}z_1\rangle_{QDQ} = \langle Q^{-1}z_2, Q^{-1}z_2\rangle_{QDQ} = \langle z_1, z_1\rangle_D = \langle z_2, z_2\rangle_D = 0 \; .$$

Hence $\mathrm{tr}_{\mathrm{Ker}\, QG_0Q}\, QDQ = 0$, i.e. G_0 solves Problem 4.2. □

Note, that the assumptions of Prop. 4.3 may only fail, if three of the numbers $u_i^* D u_i$ are positive and one is negative, or vice versa.

Proposition 4.4. *Assume that $u_i^* D u_i < 0$ for exactly one fixed $i \in \{1, 2, 3, 4\}$ and $u_k^* D u_k > 0$ for all $k \neq i$.*
Assume further that

$$K_{mn} = \begin{pmatrix} u_m^* D u_m & u_m^* D u_n \\ u_n^* D u_m & u_n^* D u_n \end{pmatrix}$$

be nonnegative definite for any choice of distinct $m, n \in \{1, 2, 3, 4\} \backslash \{i\}$. Then there exists a skew-symmetric G_0 solving Problem 4.2.

Remark 4.5. We can fix j, m, n arbitrarily provided that we have $\{i, j, m, n\} = \{1, 2, 3, 4\}$. Thus we may assume $\lambda_j, \lambda_m, \lambda_n$ to be ordered arbitrarily. For any such choice,

$$K_{ij} = \begin{pmatrix} u_i^* D u_i & u_i^* D u_j \\ u_j^* D u_i & u_j^* D u_j \end{pmatrix}$$

necessarily is indefinite. If K_{mn} was not nonnegative definite, then it would be indefinite or singular, and we could apply Prop. 4.3.

Proof. (of Proposition 4.4) We denote the eigenvalues of K_{ij} by μ_i, μ_j and those of K_{mn} by μ_m and μ_n, where in accordance with our assumptions $\mu_i < 0$, $\mu_j > 0$, and $\mu_n \geq \mu_m \geq 0$. Since $\mathrm{tr}\, D = \mu_i + \mu_j + \mu_m + \mu_n = 0$, we have $\mu_i \leq -\mu_m - \mu_n$.
Thus $[\mu_j, \mu_i] \supset [-\mu_m, -\mu_n]$, i.e. (e.g. [4, Ex. I.2.9])

$$\{x_1^* K_{ij} x_1 \mid \|x_1\| = 1\} \;\; \supset \;\; \{-x_2^* K_{mn} x_2 \mid \|x_2\| = 1\} \; .$$

Thus, for each normalized $z_2 \in \mathrm{Span}\{u_m, u_n\}$ there is a normalized $z_1 = z_1(z_2) \in \mathrm{Span}\{u_i, u_j\}$ so that

$$z_1^* D z_1 = -z_2^* D z_2 \; . \tag{4.2}$$

We can choose $z_1 = f(\alpha) = \cos(\alpha)\tilde{u}_i + \sin(\alpha)\tilde{u}_j$ with $\alpha \in [0, \pi/2]$, where $\tilde{u}_i, \tilde{u}_j$ are orthonormal and $[\tilde{u}_i\, \tilde{u}_j]^* D [\tilde{u}_i\, \tilde{u}_j] = \mathrm{diag}(\mu_i\, \mu_j)$. Then the mapping $g : \alpha \mapsto f(\alpha)^* D f(\alpha)$ is continuous and strictly monotonically increasing and therefore continuously invertible. Since the mapping $z_2 \mapsto z_2^* D z_2$ is also continuous, we can assume the mapping $z_2 \mapsto z_1(z_2) = z_1(g^{-1}(z_2^* D z_2))$ to be continuous.

We now consider three different cases.
(i) Assume $\lambda_i = \max_k \lambda_k$. Since

$$\begin{aligned} 0 \ \leq \ & \operatorname{tr} QDQ = \sum_{k=1}^{4} u_k^* QDQ u_k \\ = \ & \sum_{k=1}^{4} \lambda_k^2 u_k^* D u_k \leq \lambda_i^2 \sum_{k=1}^{4} u_k^* D u_k = 0 \ , \end{aligned}$$

it follows that $\lambda_i = \lambda_k$ for all k, i.e. $Q = \lambda_k I$. But this case is solved by Proposition 2.9.
(ii) Let $\min_k \lambda_k < \lambda_i < \max_k \lambda_k$ and assume without loss of generality that $\lambda_m \leq \lambda_i, \lambda_j \leq \lambda_n$. Then

$$\begin{aligned} \lambda_m^{-1} = \|Q^{-1} u_m\| \ & \geq \ \|Q^{-1} z_1(u_m)\| \ , \\ \lambda_n^{-1} = \|Q^{-1} u_n\| \ & \leq \ \|Q^{-1} z_1(u_n)\|. \end{aligned}$$

By the mean value theorem there exists a normalized $z_2 = \cos(\beta) u_m + \sin(\beta) u_n \in \operatorname{Span}\{u_m, u_n\}$ so that

$$\|Q^{-1} z_2\| \ = \ \|Q^{-1} z_1(z_2)\| \ . \tag{4.3}$$

We extend $z_1 = z_1(z_2)$ and z_2 to an orthogonal matrix $Z = [z_1, \ldots, z_4]$ and define G_0 as in (4.1).
Then $\operatorname{Span}\{z_1, z_2\} = \operatorname{Ker} G_0$ and $\operatorname{tr}_{\operatorname{Ker} G_0} D = 0$. Moreover, we have that $\{Q^{-1} z_1, Q^{-1} z_2\}$ is an orthogonal basis of $\operatorname{Ker} QG_0Q$ and (using (4.2) and (4.3)) we have

$$\operatorname{tr}_{\operatorname{Ker} QG_0Q} QDQ \ = \ \frac{z_1^* D z_1}{\|Q^{-1} z_1\|^2} + \frac{z_2^* D z_2}{\|Q^{-1} z_2\|^2} = 0 \ .$$

Hence G_0 solves Problem 4.2.
(iii) Let $\lambda_i = \min_k \lambda_k$ and assume $\lambda_m \geq \lambda_j$. Let $z_2 = u_m$ and $z_1 = z_1(z_2) \in \operatorname{Span}\{u_i, u_j\}$. Then

$$\lambda_i^{-2} \geq \|Q^{-1} z_1\|^2 \geq \lambda_j^{-2} \geq \lambda_m^{-2} = \|Q^{-1} z_2\|^2 \ .$$

With G_0 again as in (4.1), we have $\operatorname{tr}_{\operatorname{Ker} G_0} D = 0$ and

$$\operatorname{tr}_{\operatorname{Ker} QG_0Q} QDQ = \frac{z_1^* D z_1}{\|Q^{-1} z_1\|^2} + \frac{z_2^* D z_2}{\|Q^{-1} z_2\|^2} \geq 0 \ ,$$

because $-z_1^* D z_1 = z_2^* D z_2 = u_m^* D u_m$. On the other hand

$$\begin{aligned} \operatorname{tr}_{\operatorname{Ker} QG_0Q} QDQ \ & = \ \tfrac{z_1^* D z_1}{\|Q^{-1} z_1\|^2} + \tfrac{z_2^* D z_2}{\|Q^{-1} z_2\|^2} \\ & \leq \ u_m^* D u_m (\lambda_m^2 - \lambda_i^2) \\ & \leq \ \sum_{k=1}^{4} \lambda_i^2 u_k^* D u_k = \operatorname{tr} QDQ \ . \end{aligned}$$

Again G_0 solves Problem 4.2. □

Finally we consider the case where three of the numbers $u_i^* D u_i$ are negative and one is positive.

Proposition 4.6. *Assume that $u_i^* D u_i > 0$ for exactly one fixed $i \in \{1,2,3,4\}$ and $u_k^* D u_k < 0$ for all $k \neq i$. Assume further that*

$$K_{mn} = \begin{pmatrix} u_m^* D u_m & u_m^* D u_n \\ u_n^* D u_m & u_n^* D u_n \end{pmatrix}$$

be nonpositive definite for any choice of distinct $m, n \in \{1,2,3,4\} \backslash \{i\}$. Then there is a skew-symmetric G_0 solving Problem 4.2.

The proof is essentially the same as in Proposition 4.4.

Proof. We denote the eigenvalues of K_{ij} by μ_i, μ_j and those of K_{mn} by μ_m and μ_n, where in accordance with our assumptions $\mu_i > 0$, $\mu_j < 0$, and $\mu_n \leq \mu_m \leq 0$. Since $\operatorname{tr} D = \mu_i + \mu_j + \mu_m + \mu_n = 0$, we have $\mu_i \geq -\mu_m - \mu_n$.
Thus $[\mu_j, \mu_i] \supset [-\mu_m, -\mu_n]$, i.e. (e.g. [4, Ex. I.2.9])

$$\{x_1^* K_{ij} x_1 \mid \|x_1\| = 1\} \quad \supset \quad \{-x_2^* K_{mn} x_2 \mid \|x_2\| = 1\} \ .$$

Thus, for each normalized $z_2 \in \operatorname{Span}\{u_m, u_n\}$ there is a normalized $z_1 = z_1(z_2) \in \operatorname{Span}\{u_i, u_j\}$ so that

$$z_1^* D z_1 = -z_2^* D z_2 \ . \tag{4.4}$$

We can choose $z_1 = f(\alpha) = \cos(\alpha)\tilde{u}_i + \sin(\alpha)\tilde{u}_j$ with $\alpha \in [0, \pi/2]$, where $\tilde{u}_i, \tilde{u}_j$ are orthonormal and $[\tilde{u}_i\, \tilde{u}_j]^* D [\tilde{u}_i\, \tilde{u}_j] = \operatorname{diag}(\mu_i\, \mu_j)$, where $\mu_i > 0 > \mu_j$. Then the mapping $g : \alpha \mapsto f(\alpha)^* D f(\alpha)$ is continuous and strictly monotonically decreasing on $[0, \pi/2]$ and therefore continuously invertible. Since the mapping $z_2 \mapsto z_2^* D z_2$ is also continuous, we can assume the mapping $z_2 \mapsto z_1(z_2) = z_1(g^{-1}(z_2^* D z_2))$ to be continuous. We now consider three different cases.
(i) Assume $\lambda_i = \min_k \lambda_k$. Since

$$\begin{aligned} 0 \quad \leq \quad & \operatorname{tr} QDQ = \sum_{k=1}^{4} u_k^* QDQ u_k \\ = \quad & \sum_{k=1}^{4} \lambda_k^2 u_k^* D u_k \leq \lambda_i^2 \sum_{k=1}^{4} u_k^* D u_k = 0 \ , \end{aligned}$$

it follows that $\lambda_i = \lambda_k$ for all k, i.e. $Q = \lambda_i I$. But this case is solved by Proposition 2.9.

(ii) Let $\min_k \lambda_k < \lambda_i < \max_k \lambda_k$ and assume without loss of generality that $\lambda_m \leq \lambda_i, \lambda_j \leq \lambda_n$. Then

$$\begin{aligned} \lambda_m^{-1} = \|Q^{-1}u_m\| &\geq \|Q^{-1}z_1(u_m)\| , \\ \lambda_n^{-1} = \|Q^{-1}u_n\| &\leq \|Q^{-1}z_1(u_n)\|. \end{aligned}$$

By the mean value theorem there exists a normalized $z_2 = \cos(\beta)u_m + \sin(\beta)u_n \in \operatorname{Span}\{u_m, u_n\}$ so that

$$\|Q^{-1}z_2\| = \|Q^{-1}z_1(z_2)\| . \tag{4.5}$$

We extend $z_1 = z_1(z_2)$ and z_2 to an orthogonal matrix $Z = [z_1, \dots, z_4]$ and define G_0 as in (4.1).
Then we have $\operatorname{Span}\{z_1, z_2\} = \operatorname{Ker} G_0$ and $\operatorname{tr}_{\operatorname{Ker} G_0} D = 0$. Moreover, $\{Q^{-1}z_1, Q^{-1}z_2\}$ is an orthogonal basis of $\operatorname{Ker} QG_0Q$ and (using (4.4) and (4.5)) we have

$$\operatorname{tr}_{\operatorname{Ker} QG_0Q} QDQ = \frac{z_1^* D z_1}{\|Q^{-1}z_1\|^2} + \frac{z_2^* D z_2}{\|Q^{-1}z_2\|^2} = 0 .$$

Hence G_0 solves Problem 4.2.
(iii) Let $\lambda_i = \max_k \lambda_k$ and assume $\lambda_m \leq \lambda_j$.
Let $z_2 = u_m$ and $z_1 = z_1(z_2) \in \operatorname{Span}\{u_i, u_j\}$. Then

$$\lambda_i^{-2} \leq \|Q^{-1}z_1\|^2 \leq \lambda_j^{-2} \leq \lambda_m^{-2} = \|Q^{-1}z_2\|^2 .$$

With G_0 again as in (4.1), we have $\operatorname{tr}_{\operatorname{Ker} G_0} D = 0$ and

$$\operatorname{tr}_{\operatorname{Ker} QG_0Q} QDQ = \frac{z_1^* D z_1}{\|Q^{-1}z_1\|^2} + \frac{z_2^* D z_2}{\|Q^{-1}z_2\|^2} \geq 0 ,$$

because $-z_1^* D z_1 = z_2^* D z_2 = u_m^* D u_m$. On the other hand

$$\begin{aligned} \operatorname{tr}_{\operatorname{Ker} QG_0Q} QDQ &= \tfrac{z_1^* D z_1}{\|Q^{-1}z_1\|^2} + \tfrac{z_2^* D z_2}{\|Q^{-1}z_2\|^2} \\ &\leq u_m^* D u_m (\lambda_m^2 - \lambda_i^2) \\ &= \lambda_m^2 u_m^* D u_m - \lambda_i^2 u_m^* D u_m \\ &= \lambda_m^2 u_m^* D u_m + \lambda_i^2 (u_i^* D u_i + u_j^* D u_j + u_n^* D u_n) \\ &\leq \sum_{k=1}^4 \lambda_i^2 u_k^* D u_k = \operatorname{tr} QDQ . \end{aligned}$$

Again G_0 solves Problem 4.2. □

Since all appearing cases have been considered, we have the following result.

Theorem 4.7. *Let D and $Q > 0$ be in $\mathbb{R}^{4\times 4}$ with $\operatorname{tr} D, \operatorname{tr} QDQ > 0$. Then there exists a gyroscopic stabilizer.*

4.3. Examples

Here, we provide examples for some of the cases discussed in the previous section and show how to do an explicit construction. Since Q is positive definite and therefore diagonalizable, for simplicity we directly take examples where Q is diagonal. All computations are performed in Matlab.

Example 4.8. We construct an example as in Proposition 4.3. Consider the system $\ddot{x} + D\dot{x} + Q^{-2}x = 0$, where

$$D = \begin{pmatrix} 1 & -3 & 0 & -2 \\ -3 & -2 & -1 & -3 \\ 0 & -1 & -3 & 3 \\ -2 & -3 & 3 & 8 \end{pmatrix}, \quad Q = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix}.$$

The system is unstable since there is an eigenvalue with positive real part 4.7740. We have $\frac{\operatorname{tr} D}{n} = 1 < \frac{94}{30} = \frac{\operatorname{tr} QDQ}{\operatorname{tr} Q^2}$, thus we take $\tau = 1$ and $P = I$ according to Proposition 2.1 and get

$$D_\tau = \begin{pmatrix} 0 & -3 & 0 & -2 \\ -3 & -3 & -1 & -3 \\ 0 & -1 & -4 & 3 \\ -2 & -3 & 3 & 7 \end{pmatrix}.$$

Now $\operatorname{Span}\{e_1, e_2\}$ and $\operatorname{Span}\{e_3, e_4\}$ are both either D-indefinite or singular, thus in each of them there exists a D-isotropic vector.
In $\operatorname{Span}\{e_1, e_2\}$, a normalized isotropic vector can be taken as $z_1 = e_1$. We diagonalize $K_{34} = [e_3\ e_4]^* D [e_3\ e_4] = U_{34} V_{34} U_{34}^*$ and get

$$U_{34} = \begin{pmatrix} -0.9690 & 0.2471 \\ 0.2471 & 0.9690 \end{pmatrix}, \quad V_{34} = \begin{pmatrix} -4.7650 & 0 \\ 0 & 7.7650 \end{pmatrix},$$

thus an isotropic vector is given by

$$\tilde{z}_2 = U \begin{pmatrix} \sqrt{7.7650} \\ -\sqrt{4.7650} \end{pmatrix}, \quad z_2 = [e_3\ e_4] \frac{\tilde{z}_2}{\|\tilde{z}_2\|} = [e_3\ e_4] \begin{pmatrix} 0 \\ 0 \\ -0.9152 \\ -0.4030 \end{pmatrix}.$$

We complete z_1, z_2 to an orthonormal basis and according to Proposition 4.1 we put

$$G_\delta = Z \begin{pmatrix} 0 & \delta & 0 & 0 \\ -\delta & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} Z^T,$$

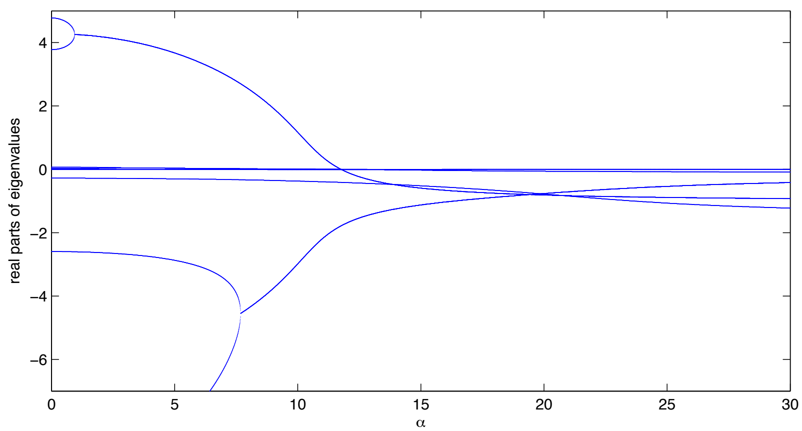

FIGURE 4.1. Plot of the real parts of the eigenvalues of $\ddot{x} + (D + \alpha G_{0.0001})\dot{x} + Kx = 0$ with α on the horizontal axis

where the matrix Z is chosen as

$$Z = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -0.9152 & 0 & -0.4030 \\ 0 & -0.4030 & 0 & 0.9152 \end{pmatrix}.$$

According to our construction, we just know that αG_δ stabilizes our system for *small* δ and *large* α. We put $\delta = 0.0001$ and plot the real parts of the eigenvalues of the system versus α. Via Matlab we find that the system is stable for all $\alpha_\delta > 11.7564$ as can be seen in Figure 4.1. While our construction is independent of the scaling or norm of Q, it should we noted that the actual α_δ does depend on it.
Figure 4.2 shows the dependance of α on δ: the green area signifies the parameter combinations (α, δ), for which the system is unstable, the blue area signifies stability. Our construction just predicted that small δ and large α yield stability, but the actual stability domain (the blue area) appears to be a lot larger.
Also interesting is the fact that the stability domain as computed via MATLAB is not symmetric around the horizontal axis, which means that replacing δ by $-\delta$ might alter the stability properties of the system, a fact which cannot be deduced from our construction. Note that $G_\delta \neq -G_{-\delta}$, since only in one eigenvalue block the sign is changed.

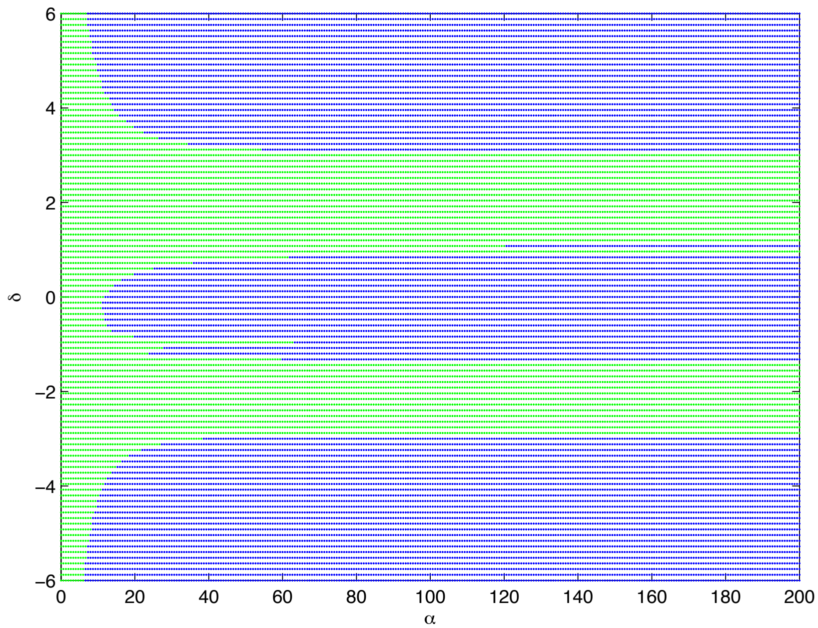

FIGURE 4.2. The blue area indicates parameter combinations (α, δ) that result in stability of $\ddot{x} + (D + \alpha G_\delta)\dot{x} + Kx = 0$, the green area indicates instability

EXAMPLE 4.9. Here we will do a construction as in Proposition 4.4 (iii). Consider the system $\ddot{x} + D\dot{x} + Q^{-2}x = 0$, where

$$D = \begin{pmatrix} -13 & -1 & 2 & 2 \\ -1 & 9 & 2 & 5 \\ 2 & 2 & 2 & 1 \\ 2 & 5 & 1 & 6 \end{pmatrix}, \quad Q = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix}.$$

The largest real part of the eigenvalues of the system is 13.5055, so the system is unstable. With $\frac{\operatorname{tr} D}{n} = 1 < \frac{137}{30} = \frac{\operatorname{tr} QDQ}{\operatorname{tr} Q^2}$ we put $\tau = 1$ and $P = I$ according to Proposition 2.1 and get

$$D_\tau = \begin{pmatrix} -14 & -1 & 2 & 2 \\ -1 & 8 & 2 & 5 \\ 2 & 2 & 1 & 1 \\ 2 & 5 & 1 & 5 \end{pmatrix} : .$$

The space $\operatorname{Span}\{e_1, e_j\}$ is D-indefinite for any $j = 2, \ldots, 4$ while for any $p, q \neq 1$ the space $\operatorname{Span}\{e_p, e_q\}$ is D-positive definite. In particular, $e_1^* D e_1$ is negative and λ_1 is the smallest eigenvalue of Q, thus we are

in the case (iii) of Proposition 4.4.
We fix, according to the construction in the proposition, $i = 1$ and $j = 2$ and put $z_2 = e_4$. A normalized $z_1 = z_1(z_2)$ such that $z_1^* D z_1 + z_2^* D z_2 = 0$ can be constructed as follows. Consider a normalized vector $y \in \mathbb{R}^{2\times 2}$ such that

$$y^* K_{12} y = -z_2^* D z_2 \ .$$

Then

$$y^*(K_{12} + z_2^* D z_2 I) y = 0 \ ,$$

so as in the example before we need an isotropic vector of the matrix $T = K_{12} + z_2^* D z_2 I$. We diagonalize $T = UVU^*$ and obtain

$$U = \begin{pmatrix} -0.9990 & -0.0453 \\ -0.0453 & 0.9990 \end{pmatrix} , \quad V = \begin{pmatrix} -9.0454 & \\ & 13.0454 \end{pmatrix} ,$$

thus an isotropic vector of T is given by

$$\tilde{z}_1 = U \begin{pmatrix} \sqrt{13.0454} \\ -\sqrt{9.0454} \end{pmatrix} , \ z_1 = [e_1 \ e_2] \frac{\tilde{z}_1}{\|\tilde{z}_1\|} = [e_1 \ e_2] \begin{pmatrix} -0.7387 \\ -0.6741 \\ 0 \\ 0 \end{pmatrix} .$$

In fact, we have $z_2^* D z_2 = 5 = -z_1^* D z_1$. We complete z_1, z_2 to an orthonormal basis and according to Proposition 4.1 we define

$$\begin{aligned} Z &= \begin{pmatrix} 0 & -0.7387 & 0 & 0.6741 \\ 0 & -0.6741 & 0 & -0.7387 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} , \\ G_\delta &= Z \begin{pmatrix} 0 & \delta & 0 & 0 \\ -\delta & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} Z^T . \end{aligned}$$

With $\delta = 0.0001$, the system $\ddot{x} + (D + \alpha G_\delta)\dot{x} + Q^{-2}x = 0$ is stable for $\alpha \geq 69752$. Note that in this case, the coefficient α needed to stabilize the system is a lot larger than in the previous example, the eigenvalues of the system $\ddot{x} + (D + \alpha G_{0.0001})\dot{x} + Kx = 0$ are less sensitive with respect to the parameter α as can be seen in Figure 4.3. One of the reasons of this effect is, as already mentioned, the scaling of Q.
Figure 4.4 shows again the dependance of α on δ, where the green area signifies the parameter combinations (α, δ), for which the system is unstable, the blue area signifies stability. It can be seen that the higher α gets, the smaller the absolute value of δ can be chosen, such

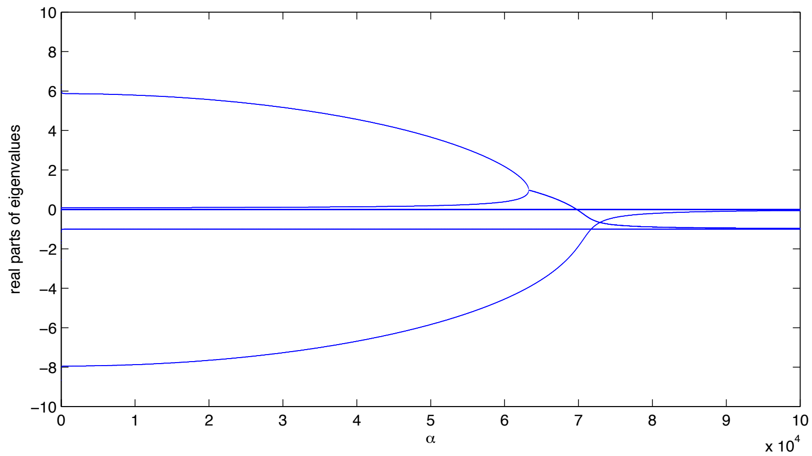

FIGURE 4.3. Plot of the real parts of the eigenvalues of $\ddot{x} + (D + \alpha G_{0.0001})\dot{x} + Kx = 0$ with α on the horizontal axis

that the pair α, δ yields stability. In contrast to the previous example, the value $\delta = 0$ does not seem to be admissible.

EXAMPLE 4.10. Here, we construct a stabilizer for the case of Proposition 4.4 (ii). The main difficulty is the construction of the mapping $z_2 \mapsto z_1(z_2)$. Consider the system $\ddot{x} + D\dot{x} + Q^{-2}x = 0$, where

$$D = \begin{pmatrix} 9 & -1 & 2 & 5 \\ -1 & -13 & 2 & 2 \\ 2 & 2 & 2 & 1 \\ 5 & 2 & 1 & 6 \end{pmatrix}, \quad Q = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix}.$$

The largest real part of the eigenvalues of the system is 13.5055, the system is unstable. With $\frac{\operatorname{tr} D}{n} = 1 < \frac{137}{30} = \frac{\operatorname{tr} QDQ}{\operatorname{tr} Q^2}$ we put $\tau = 1$ and $P = I$ in Proposition 2.1 and get

$$D_\tau = \begin{pmatrix} 8 & -1 & 2 & 5 \\ -1 & -14 & 2 & 2 \\ 2 & 2 & 1 & 1 \\ 5 & 2 & 1 & 5 \end{pmatrix}.$$

Any space $\operatorname{Span}\{e_2, e_j\}$ with $j \neq 2$ is D-indefinite, while the spaces $\operatorname{Span}\{e_1, e_3\}$, $\operatorname{Span}\{e_1, e_4\}$, $\operatorname{Span}\{e_3, e_4\}$ are each D-positive definite, thus we are in the situation of Proposition 4.4 (ii).

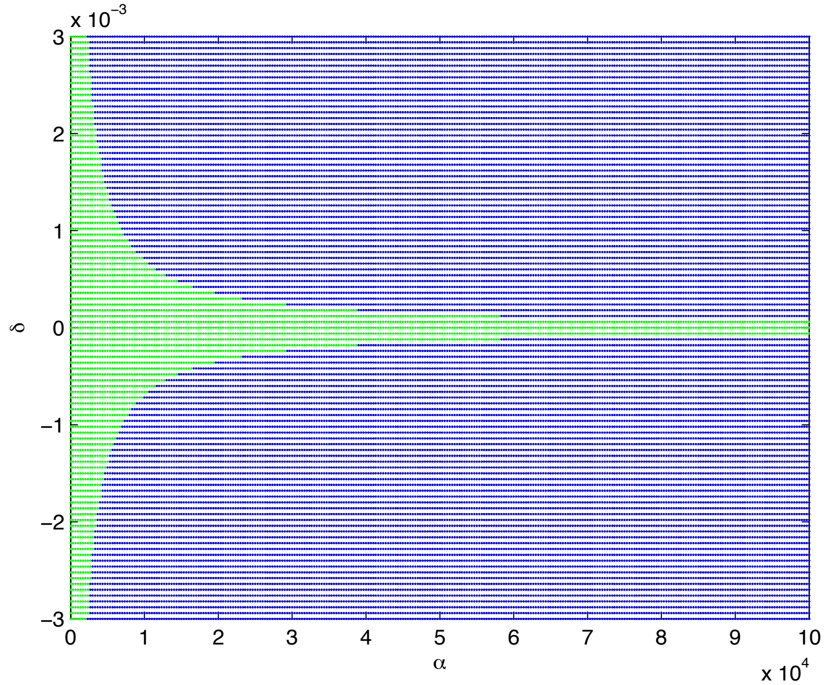

FIGURE 4.4. The blue area indicates parameter combinations (α, δ) that result in stability of $\ddot{x} + (D + \alpha G_\delta)\dot{x} + Kx = 0$, the green area indicates instability

We put $(i, j, m, n) = (2, 3, 1, 4)$ according to Proposition 4.4, and

$$\begin{aligned} K_{ij} &= \big[e_i\ e_j\big]^* D \big[e_i\ e_j\big] = \begin{pmatrix} -14 & 2 \\ 2 & 1 \end{pmatrix} , \\ K_{mn} &= \big[e_m\ e_n\big]^* D \big[e_m\ e_n\big] = \begin{pmatrix} 8 & 5 \\ 5 & 5 \end{pmatrix} . \end{aligned}$$

Let the eigenvalues of K_{ij} be μ_i, μ_j with $\mu_i \le 0 \le \mu_j$ and those of K_{mn} be $\mu_m, \mu_n > 0$. Clearly there exists a pair of orthonormal vectors $x_i, x_j \in \operatorname{Span}\{u_i, u_j\}$ and $x_m, x_n \in \operatorname{Span}\{u_m, u_n\}$ such that with $X_{ij} = [x_i, x_j]$ and $X_{mn} = [x_m, x_n]$ we have

$$\begin{aligned} \operatorname{diag}(\mu_i, \mu_j) &= \operatorname{diag}(-14.2621 \quad 1.2621) = X_{ij}^T D X_{ij} , \\ \operatorname{diag}(\mu_m, \mu_n) &= \operatorname{diag}(1.2798 \quad 11.7202) = X_{mn}^T D X_{mn} . \end{aligned}$$

Since K_{ij} is indefinite, we can assume without loss of generality

$$\mu_i \le u_i^* D u_i < 0 \le u_j^* D u_j \le \mu_j .$$

From $\operatorname{tr} D = \sum_{k=1}^4 u_k^* D u_k = \sum_{k=1}^4 \mu_k = 0$ we conclude

$$\begin{aligned} \mu_i \le -(\mu_n + \mu_m) \;&\le\; 0 \le \mu_j \;, \\ 0 \;&\le\; \mu_m, \mu_n \;. \end{aligned} \tag{4.6}$$

Therefore, for each normalized $z_2 \in \operatorname{Span}\{u_m, u_n\}$ there exists a normalized $z_1 \in \operatorname{Span}\{u_i, u_j\}$ such that

$$z_1^* D z_1 = -z_2^* D z_2 \;.$$

Let $z_2(\beta) = \cos(\beta) x_m + \sin(\beta) x_n$ for $\beta \in [0, 2\pi)$ and define $z_1(\alpha_\beta) = \cos(\alpha_\beta) x_i + \sin(\alpha_\beta) x_j$. Now we construct a continuous mapping $\beta \mapsto \alpha_\beta$ that satisfies

$$z_1(\alpha_\beta)^* D z_1(\alpha_\beta) = -z_2(\beta)^* D z_2(\beta) \tag{4.7}$$

for all β. We write (4.7) as

$$0 \;=\; \cos^2(\alpha_\beta)\mu_i + \sin^2(\alpha_\beta)\mu_j + \cos^2(\beta)\mu_m + \sin^2(\beta)\mu_n \;.$$

With $\sin^2(\alpha_\beta) = 1 - \cos^2(\alpha_\beta)$ we get

$$0 = \cos^2(\alpha_\beta)\big(\mu_i - \mu_j\big) + \mu_j + \cos^2(\beta)\mu_m + \sin^2(\beta)\mu_n \;.$$

and finally

$$\cos^2(\alpha_\beta) = \frac{\mu_j + \cos^2(\beta)\mu_m + \sin^2(\beta)\mu_n}{\mu_j - \mu_i} \;,$$

where the denominator is nonzero by construction. Also, the real square root of the above fraction exists and is less or equal 1, since from (4.6) it follows that

$$\begin{aligned} \frac{\mu_j + \cos^2(\beta)\mu_m + \sin^2(\beta)\mu_n}{\mu_j - \mu_i} \;&\le\; \frac{\mu_j + \cos^2(\beta)\mu_m + \sin^2(\beta)\mu_n}{\mu_j + \mu_m + \mu_n} \\ &\le\; \frac{\mu_j + \mu_m + \mu_n}{\mu_j + \mu_m + \mu_n} \le 1 \;, \end{aligned}$$

and we also have

$$\frac{\mu_j + \cos^2(\beta)\mu_m + \sin^2(\beta)\mu_n}{\mu_j - \mu_i} \ge 0 \;,$$

since both numerator and denominator are nonnegative. Therefore,

$$\alpha(\beta) = \arccos\big(\frac{\mu_j + \cos^2(\beta)\mu_m + \sin^2(\beta)\mu_n}{\mu_j - \mu_i}\big)^{\frac{1}{2}}$$

is well defined and satisfies (4.7). In our case, it takes the form

$$\alpha(\beta) = \arccos\big(\frac{1.2621 + \cos^2(\beta)1.2798 + \sin^2(\beta)11.7202}{1.2621 + 14.2621}\big)^{\frac{1}{2}} \;.$$

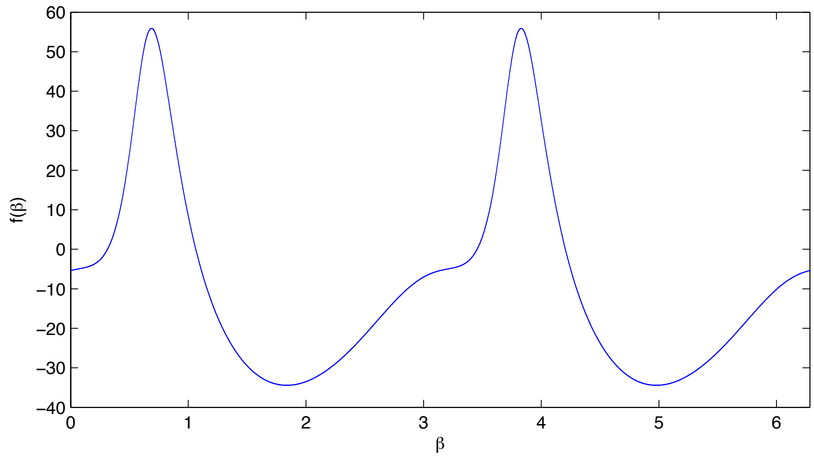

FIGURE 4.5. Plot of the function f on $[0\ 2\pi)$

With

$$\begin{aligned} z_1(\alpha_\beta) &= \cos(\alpha_\beta)x_i + \sin(\alpha_\beta)x_j \\ z_2(\beta) &= \cos(\beta)x_m + \sin(\beta)x_n \end{aligned}$$

we define the mapping

$$f : [0, 2\pi) \to \mathbb{R}\ , f(\beta) = \frac{z_1(\alpha_\beta)^* D z_1(\alpha_\beta)}{\|Q^{-1}z_1(\alpha_\beta)\|^2} + \frac{z_2(\beta)^* D z_2(\beta)}{\|Q^{-1}z_2(\beta)\|^2}\ ,$$

which now according to Proposition 4.4 has at least one zero. Figure 4.5 shows that actually there are several zeros we can choose from, we take $\beta = 3.4547$, complete $z_1(\alpha_\beta), z_2(\beta)$ to an orthonormal basis which we write into the columns of

$$Z = \begin{pmatrix} 0 & -0.3208 & -0.9471 & 0 \\ 0.5872 & 0 & 0 & 0.8095 \\ 0.8095 & 0 & 0 & -0.5872 \\ 0 & 0.9471 & -0.3208 & 0 \end{pmatrix}\ ,$$

and define as in the previous examples

$$G_\delta = Z \begin{pmatrix} 0 & \delta & 0 & 0 \\ -\delta & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} Z^T\ .$$

With $\delta = 0.0001$, the system $\ddot{x} + (D + \alpha G_\delta)\dot{x} + Q^{-2}x = 0$ is stable for $\alpha \geq 25595$ as can also be seen in figure 4.6.

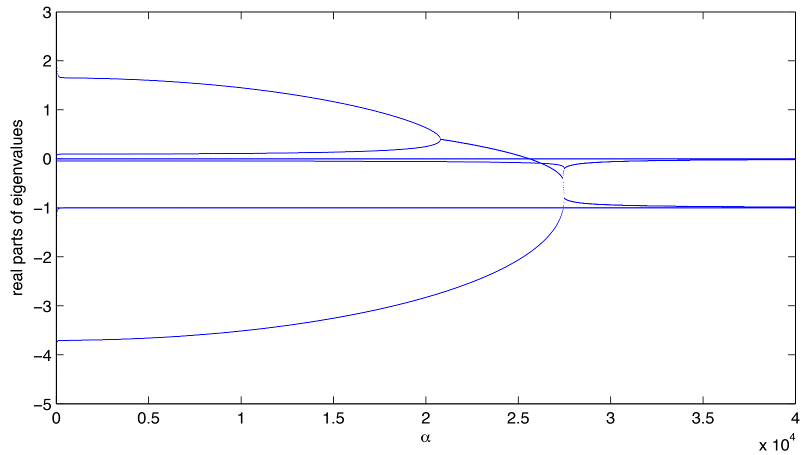

FIGURE 4.6. Plot of the real parts of the eigenvalues of $\ddot{x} + (D + \alpha G_{0.0001})\dot{x} + Kx = 0$ with α on the horizontal axis

CHAPTER 5

Space Dimension Five and Higher

The approach we take here is similar to our approach in the four dimensional case. Again the fact that the eigenspace for the eigenvalue zero of G results in an easy to identify eigenspace for the eigenvalue zero of QGQ is used.

Proposition 5.1. *For any $\delta \in \mathbb{R}$ and an orthogonal matrix $Z = [z_1, z_2, z_3, z_4, z_5] \in \mathbb{R}^{5\times 5}$ set*

$$G_\delta = Z \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \delta & 0 & 0 \\ 0 & -\delta & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 \end{pmatrix} Z^T \,. \tag{5.1}$$

If for $\tau > 0$ with $\operatorname{tr} D_\tau = 0,\ \operatorname{tr} QD_\tau Q \geq 0$ we have

$$\begin{aligned} &(i) && \operatorname{tr}_{\mathrm{Span}\{z_1\}} D_\tau = \operatorname{tr}_{\mathrm{Span}\{z_2,z_3\}} D_\tau = \operatorname{tr}_{\mathrm{Span}\{z_4,z_5\}} D_\tau = 0 \,, \\ &(ii) && \operatorname{tr}_{Q^{-1}\,\mathrm{Span}\{z_1,z_2,z_3\}} QD_\tau Q \geq \operatorname{tr}_{Q^{-1}\,\mathrm{Span}\{z_1\}} QD_\tau Q = 0 \,, \\ & && \operatorname{tr}_{Q\,\mathrm{Span}\{z_4,z_5\}} QD_\tau Q \geq 0 \,, \end{aligned}$$

then there exists $\delta \neq 0$ such that (a), (b), and (c) in Proposition 2.1 hold for G_δ and $\tau/2$.

The idea behind the above proposition is as follows. The matrix G_0 is constructed such that after a small perturbation δ, the triple eigenvalue zero of G_0 is split into a pair of imaginary eigenvalues $\pm i\delta$ and a single eigenvalue zero, such that the corresponding eigenvectors of G_δ and $QG_\delta Q$ satisfy the conditions (a), (b) and (c) in Proposition 2.1 for sufficiently small δ.

Proof. By continuity of eigenvalues, it is clear that (a) in Proposition 2.1 holds for small $|\delta| \neq 0$.
Using (2.6) and the structure of G_δ we conclude that assumption (b) is equivalent to $\operatorname{tr}_{\mathrm{Span}\{z_1\}} D_\tau \geq 0$, $\operatorname{tr}_{\mathrm{Span}\{z_2,z_3\}} D_\tau \geq 0$ and $\operatorname{tr}_{\mathrm{Span}\{z_4,z_5\}} D_\tau \geq 0$ for all $\delta \in \mathbb{R}\setminus\{0\}$.
To verify (c), first observe that for $\delta \neq 0$ the matrix $QG_\delta Q$ has a zero eigenvalue and two conjugate pairs of imaginary eigenvalues, which

we denote by $\pm\lambda_\delta$ and $\pm\mu_\delta$. These depend continuously on δ (where $\lambda_0 = 0$). The same is true (e.g. [19]) for the invariant subspaces

$$\begin{aligned}\mathcal{V}_{0,\lambda}(\delta) &:= \operatorname{Ker}\left(QG_\delta Q\left((QG_\delta Q)^2 + |\lambda_\delta|^2 I\right)\right),\\ \mathcal{V}_{\mu}(\delta) &:= \operatorname{Ker}\left((QG_\delta Q)^2 + |\mu_\delta|^2 I\right).\end{aligned}$$

Also, independently of δ, $Q^{-1}z_1$ is always an eigenvector for the eigenvalue zero of $QG_\delta Q$. By construction we have

$$\operatorname{tr}_{Q^{-1}\operatorname{Span}\{z_1\}} QD_\tau Q = 0 .$$

We define for any δ the orthogonal complement $\mathcal{V}_\lambda(\delta)$ of $Q^{-1}\operatorname{Span}\{z_1\}$ in $\mathcal{V}_{0,\lambda}(\delta)$ and get then $\mathcal{V}_\lambda(\delta) \oplus Q^{-1}\operatorname{Span}\{z_1\} = \mathcal{V}_{0,\lambda}(\delta)$.
By assumption for $\delta = 0$ and $\eta = \tau$ we have with $D = D_\tau + \tau P$:

$$\begin{aligned}\operatorname{tr}_{Q^{-1}\operatorname{Span}\{z_1\}} QDQ &= \eta\operatorname{tr}_{Q^{-1}\operatorname{Span}\{z_1\}} QPQ > 0 ,\\ \operatorname{tr}_{\mathcal{V}_{0,\lambda}(\delta)} QDQ &\geq \eta\operatorname{tr}_{\mathcal{V}_{0,\lambda}(\delta)} QPQ > 0 ,\\ \operatorname{tr}_{\mathcal{V}_{\mu}(\delta)} QDQ &\geq \eta\operatorname{tr}_{\mathcal{V}_{\mu}(\delta)} QPQ > 0 .\end{aligned}$$

Now, since $Q^{-1}\operatorname{Span}\{z_1\} \subset \mathcal{V}_{0,\lambda}(\delta)$ and QPQ is positive definite, it follows that

$$\operatorname{tr}_{\mathcal{V}_{0,\lambda}(\delta)} QDQ \geq \eta\operatorname{tr}_{\mathcal{V}_{0,\lambda}(\delta)} QPQ > \eta\operatorname{tr}_{\mathcal{V}_{0}(\delta)} QPQ > 0 ,$$

implying that

$$\operatorname{tr}_{\mathcal{V}_{\lambda}(\delta)} QDQ \geq \eta\operatorname{tr}_{\mathcal{V}_{\lambda}(\delta)} QPQ = \eta\big(\operatorname{tr}_{\mathcal{V}_{0,\lambda}(\delta)} QPQ - \operatorname{tr}_{\mathcal{V}_{0}(\delta)} QPQ\big) > 0 .$$

We altogether have

$$\begin{aligned}\operatorname{tr}_{Q^{-1}\operatorname{Span}\{z_1\}} QDQ &= \eta\operatorname{tr}_{Q^{-1}\operatorname{Span}\{z_1\}} QPQ > 0 ,\\ \operatorname{tr}_{\mathcal{V}_{\lambda}(\delta)} QDQ &\geq \eta\operatorname{tr}_{\mathcal{V}_{\lambda}(\delta)} QPQ > 0 ,\\ \operatorname{tr}_{\mathcal{V}_{\mu}(\delta)} QDQ &\geq \eta\operatorname{tr}_{\mathcal{V}_{\mu}(\delta)} QPQ > 0 .\end{aligned}$$

By continuity, the same holds for $\eta = \tau/2$ and sufficiently small δ. Together with (2.6) this completes the proof. □

Thus, as in the four dimensional case, we can relax the conditions in Problem 2.2 and reformulate it, such that it relates to two-dimensional spaces containing pairs of complex conjugate eigenvectors instead of relating directly to eigenvectors.

Problem 5.2. *For symmetric matrices $D, Q \in \mathbb{R}^{5\times 5}$ with $Q > 0$, $\operatorname{tr} D = 0$ and $\operatorname{tr} QDQ \geq 0$, find G_0 as in* (5.1) *so that*

(i) $\operatorname{tr}_{\operatorname{Span}\{z_1\}} D = \operatorname{tr}_{\operatorname{Span}\{z_2,z_3\}} D = \operatorname{tr}_{\operatorname{Span}\{z_4,z_5\}} D = 0$,
(ii) $\operatorname{tr}_{Q^{-1}\operatorname{Span}\{z_1,z_2,z_3\}} QDQ \geq 0$,
(iii) $\operatorname{tr}_{Q\operatorname{Span}\{z_4,z_5\}} QDQ \geq 0$.

The conditions in Problem 5.2 can be weakened even further:

Lemma 5.3. *Let* $\operatorname{tr} D = 0$ *and* $\operatorname{tr} QDQ \geq 0$ *and suppose that one of the following conditions holds.*

(i) *There exists a two-dimensional space* $\operatorname{Span}\{z_4, z_5\}$ *with*

$$\operatorname{tr}_{\operatorname{Span}\{z_4,z_5\}} D = 0$$

and

$$0 \leq \operatorname{tr}_{Q\operatorname{Span}\{z_4,z_5\}} QDQ \leq \operatorname{tr} QDQ\ .$$

(ii) *There exists a three-dimensional space* $\operatorname{Span}\{z_1, z_2, z_3\}$ *such that we have* $\operatorname{tr}_{\operatorname{Span}\{z_1,z_2,z_3\}} D = 0$ *and*

$$0 \leq \operatorname{tr}_{Q^{-1}\operatorname{Span}\{z_1,z_2,z_3\}} QDQ \leq \operatorname{tr} QDQ\ .$$

(iii) *There exists a two-dimensional space* $\operatorname{Span}\{z_4, z_5\}$ *with*

$$\operatorname{tr}_{\operatorname{Span}\{z_4,z_5\}} D = 0$$

and

$$0 \leq \operatorname{tr}_{Q\operatorname{Span}\{z_4,z_5\}} QDQ$$

and a three-dimensional space $\operatorname{Span}\{z_1, z_2, z_3\}$ *orthogonal on* $\operatorname{Span}\{z_4, z_5\}$ *with* $\operatorname{tr}_{\operatorname{Span}\{z_1,z_2,z_3\}} D$ *and*

$$0 \leq \operatorname{tr}_{Q^{-1}\operatorname{Span}\{z_1,z_2,z_3\}} QDQ\ .$$

Then there exists a solution to Problem 5.2.

Proof. We prove (i) exemplarily. The other cases can be handled analogously.
Without loss of generality, we assume $z_1, \ldots, z_5$ to be an orthonormal basis of $\mathbb{R}^5$. The condition $\operatorname{tr}_{\operatorname{Span}\{z_4,z_5\}} D = 0$ implies $\operatorname{tr}_{\operatorname{Span}\{z_1,z_2,z_3\}} D = 0$, which implies the existence of a D-isotropic vector in $\operatorname{Span}\{z_1, z_2, z_3\}$. By a change of the basis, we can assume this D-isotropic vector to be z_1. Thus, (i) in Problem 5.2 holds. Also,

$$0 \leq \operatorname{tr}_{Q\operatorname{Span}\{z_4,z_5\}} QDQ \leq \operatorname{tr} QDQ$$

implies

$$0 \leq \operatorname{tr}_{Q^{-1}\operatorname{Span}\{z_1,z_2,z_3\}} QDQ \leq \operatorname{tr} QDQ\ ,$$

so (ii) and (iii) in Problem 5.2 hold. □

In the subsequent sections, we will construct spaces according to the conditions given by Lemma 5.3.

Proposition 5.4. *Let* $\operatorname{tr} D = 0$ *and* $\operatorname{tr} QDQ \geq 0$ *and let* $u_1, \ldots, u_5$ *and* $\lambda_1, \ldots, \lambda_5$ *be a set of normalized eigenvectors and corresponding eigenvalues of* Q *in no particular order.*
Assume that $u_5^* D u_5 = 0$ *Then there exists a gyroscopic stabilizer.*

PROOF. We define

$$\begin{aligned} \hat{D} &= [u_1\, u_2\, u_3\, u_4]^* D [u_1\, u_2\, u_3\, u_4]\ , \\ \hat{Q} &= [u_1\, u_2\, u_3\, u_4]^* Q [u_1\, u_2\, u_3\, u_4]\ . \end{aligned}$$

It directly follows $\operatorname{tr}\hat{D} = 0, \operatorname{tr}\hat{Q}\hat{D}\hat{Q} = \operatorname{tr}QDQ \geq 0$. From section 4 it follows that there exists a twodimensional subspace $\mathcal{Z} \subset \mathbb{R}^4$ such that $\operatorname{tr}_{\mathcal{Z}}\hat{D} = 0$ and $0 \leq \operatorname{tr}_{\hat{Q}^{-1}\mathcal{Z}}\hat{Q}\hat{D}\hat{Q} \leq \operatorname{tr}\hat{Q}\hat{D}\hat{Q}$. Let $z_1, z_2 \in \mathbb{R}^4$ be an orthonormal basis of $\mathcal{Z}$ and define $\mathcal{Y}$ as the span of the columns of $[u_1\ u_2\ u_3\ u_4][z_1\ z_2] = [y_1\ y_2]$. Then by construction y_1, y_2, u_5 are orthonormal and $\operatorname{tr}_{\operatorname{Span}\{y_1,y_2,u_5\}}\hat{D} = 0$.
Since $Q^{-1}u_5 \perp Q^{-1}y_1, y_2$, it also follows

$$\begin{aligned} & \operatorname{tr}_{\hat{Q}^{-1}\operatorname{Span}\{y_1,y_2,u_5\}}\hat{Q}\hat{D}\hat{Q} \\ = \ & \operatorname{tr}_{\hat{Q}^{-1}\operatorname{Span}\{y_1,y_2\}}\hat{Q}\hat{D}\hat{Q} + \operatorname{tr}_{\hat{Q}^{-1}\operatorname{Span}\{u_5\}}\hat{Q}\hat{D}\hat{Q} \\ = \ & \operatorname{tr}_{\hat{Q}^{-1}\operatorname{Span}\{y_1,y_2\}}\hat{Q}\hat{D}\hat{Q}\ , \end{aligned}$$

which now implies $0 \leq \operatorname{tr}_{\hat{Q}^{-1}\operatorname{Span}\{y_1,y_2,u_5\}}QDQ \leq \operatorname{tr}QDQ$. Thus, according to lemma 5.3 (iii), there exists a gyroscopic stabilizer. □

The above proposition shows that in the case $u_i^* D u_i = 0$ for some eigenevector u_i of Q the construction of a gyroscopic stabilizer can be reduced to the four-dimensional case. Thus from here on, if not mentioned otherwise, we consider $u_i^* D u_i \neq 0$ for all eigenvectors.

5.1. One D-negative Eigenvector of Q

In this section, we will prove the analogue of Proposition 4.4. Thus we consider the values $u_i^* D u_i$ for the eigenvectors u_i of Q. We assume that exactly one of the numbers $u_i^* D u_i$, $i = 1, \dots, 5$ is negative and all others are positive. Additionally, we assume the two-dimensional spaces $\operatorname{Span}\{u_j, u_k\}$ with $j, k \neq i$ to be positive definite, analogously to Proposition 4.4.
Note that, if $u_i = u_5$ is an eigenvector for the largest eigenvalue of Q, it follows that $Q = cI$ is a multiple of the unit matrix, see the proof of Proposition 4.4. This case is already solved in Proposition 2.9. Thus in this section, we only need to consider the cases $i = 1, \dots, 4$.
Let the eigenvectors $u_1, \dots u_5$ of Q be ordered such that $\lambda_1 \leq \dots \leq \lambda_5$. In this section, we then consider the cases

$$\begin{array}{llll} (i) & u_1^* D u_1 < 0 & u_j^* D u_j > 0 & \text{for all } j \neq 1 \\ (ii) & u_2^* D u_2 < 0 & [u_j\ u_k]^* D [u_j\ u_k] > 0 & \text{for all distinct } j, k \neq 2 \\ (iii) & u_3^* D u_3 < 0 & [u_j\ u_k]^* D [u_j\ u_k] > 0 & \text{for all distinct } j, k \neq 3 \\ (iv) & u_4^* D u_4 < 0 & [u_j\ u_k]^* D [u_j\ u_k] > 0 & \text{for all distinct } j, k \neq 4\ . \end{array}$$

It should be noted, that case (i) is more general than the others since $[u_j\ u_k]^* D[u_j\ u_k] > 0$ implies $u_j^* D u_j, u_k^* D u_k > 0$.

PROPOSITION 5.5. *Let* $\operatorname{tr} D = 0$ *and* $\operatorname{tr} QDQ \geq 0$ *and let* $u_1, \ldots, u_5$ *and* $\lambda_1 \leq \cdots \leq \lambda_5$ *be a set of normalized eigenvectors and corresponding eigenvalues of* Q.
Assume that $u_1^* D u_1 < 0$ *and* $u_j^* D u_j > 0$ *for all* $j \neq 1$. *Then there exists a gyroscopic stabilizer.*

PROOF. In the proof, we first show the existence of a 3-dimensional space $\mathcal{Y}_a$ with
(a) $\operatorname{tr}_{\mathcal{Y}_a} D = 0$ and $0 \leq \operatorname{tr}_{Q^{-1} \operatorname{Span} \mathcal{Y}_a} QDQ$,
(b) $0 \leq \operatorname{tr}_{Q\mathcal{Y}_b} QDQ$, where $\mathcal{Y}_b = \mathcal{Y}_a^{\perp}$.
Then according to Lemma 5.3 (iii), there exists a gyroscopic stabilizer.
(a) From $\operatorname{tr} D = \sum_{k=1}^{5} u_k D u_k = 0$ it follows that

$$u_1^* D u_1 < -(u_3^* D u_3 + u_4^* D u_4) < 0 < u_2^* D u_2 \ .$$

Then the number $-(u_3^* D u_3 + u_4^* D u_4)$ is in the numerical range of the gramian given by $[u_1 u_2]^* D [u_1 u_2]$, so there exists a normalized $y_1 = a u_1 + b u_2$ with $y_1^* D y_1 + u_3^* D u_3 + u_4^* D u_4 = 0$.
We define $\mathcal{Y}_a = \operatorname{Span}\{y_1, u_3, u_4\}$. By construction, we have $\operatorname{tr}_{\mathcal{Y}_a} D = 0$. Since y_1, u_3, u_4 remain orthogonal under multiplication with Q, we can compute

$$\begin{aligned}
\operatorname{tr}_{Q^{-1}\mathcal{Y}_a} QDQ &= \frac{(Q^{-1}y_1)^* QDQ (Q^{-1}y_1)}{\|Q^{-1}y_1\|^2} + \lambda_3^2 u_3^* D u_3 + \lambda_4^2 u_4^* D u_4 \\
&= \frac{y_1^* D y_1}{\|Q^{-1}y_1\|^2} + \lambda_3^2 u_3^* D u_3 + \lambda_4^2 u_4^* D u_4 \ . \qquad (5.2)
\end{aligned}$$

From the definition of $y_1 = a u_1 + b u_2$ with $a, b \in \mathbb{R}$ and $a^2 + b^2 = 1$ we get

$$\frac{1}{\|Q^{-1}y_1\|^2} \in [\lambda_1^2,\ \lambda_2^2] \ .$$

It follows with $y_1^* D y_1 < 0$ and an estimation for the denominator in equation (5.2) that

$$\begin{aligned}
\operatorname{tr}_{Q^{-1}\mathcal{Y}_a} QDQ &\geq \lambda_2^2 y_1^* D y_1 + \lambda_3^2 u_3^* D u_3 + \lambda_4^2 u_4^* D u_4 \\
&= \lambda_2^2(-u_3^* D u_3 - u_4^* D u_4) + \lambda_3^2 u_3^* D u_3 + \lambda_4^2 u_4^* D u_4 \\
&\geq -\lambda_3^2 u_3^* D u_3 - \lambda_4^2 u_4^* D u_4 + \lambda_3^2 u_3^* D u_3 + \lambda_4^2 u_4^* D u_4 = 0 \ .
\end{aligned}$$

(b) With $\mathcal{Y}_b = \mathcal{Y}_a^\perp$ and $\operatorname{tr}_{\mathcal{Y}_a} D = 0$ it follows that $\operatorname{tr}_{\mathcal{Y}_b} D = 0$. With $Q\mathcal{Y}_b \perp Q^{-1}\mathcal{Y}_a$ we get

$$\begin{aligned}
\operatorname{tr}_{Q\mathcal{Y}_b} QDQ &= \operatorname{tr} QDQ - \operatorname{tr}_{Q^{-1}\mathcal{Y}_a} QDQ \\
&= \sum_{k=1}^{5} \lambda_k^2 u_k^* D u_k - \frac{y_1^* D y_1}{\|Q^{-1} y_1\|^2} - \lambda_3^2 u_3^* D u_3 - \lambda_4^2 u_4^* D u_4 \\
&\geq \lambda_1^2 u_1^* D u_1 + \lambda_2^2 u_2^* D u_2 + \lambda_5^2 u_5^* D u_5 \\
&\quad + \lambda_1^2 (u_3^* D u_3 + u_4^* D u_4) \\
&\geq \lambda_1^2 \sum_{k=1}^{5} u_k^* D u_k = 0 \ ,
\end{aligned}$$

which shows that we fulfill the requirements of lemma 5.3 (iii). □

Proposition 5.6. *Let* $\operatorname{tr} D = 0$ *and* $\operatorname{tr} QDQ \geq 0$ *and let* $u_1, \ldots, u_5$ *and* $\lambda_1 \leq \cdots \leq \lambda_5$ *be a set of normalized eigenvectors and corresponding eigenvalues of* Q.
Assume that $u_2^* D u_2 < 0$ *and let*

$$K_{jk} = [u_j \; u_k]^* D [u_j \; u_k]$$

be positive definite for each pair $j, k \in \{1, 3, 4, 5\}$.
Then there exists a gyroscopic stabilizer.

Proof. The proof is divided into the following steps.
(a) We show the existence of a two-dimensional space $\mathcal{A}$ with

$$\operatorname{tr}_{\mathcal{A}} D = 0 \ , \quad \operatorname{tr}_{Q\mathcal{A}} QDQ \leq \operatorname{tr} QDQ \ .$$

(b) A second two-dimensional space $\mathcal{B}$ is constructed with

$$\operatorname{tr}_{\mathcal{B}} D = 0 \ , \quad \operatorname{tr}_{Q\mathcal{B}} QDQ \geq 0 \ .$$

(c) The existence of a continuous mapping $\mathcal{Y}$, whose image is in the set $\mathcal{P}$ of two-dimensional subspaces of $\mathbb{R}^5$, with the properties

$$\begin{aligned}
&\mathcal{Y} : [0, \tfrac{\pi}{2}] \mapsto \mathcal{P} \ , \quad t \mapsto \mathcal{Y}(t) \ , \quad \dim \mathcal{Y}(t) = 2 \text{ for all } t \\
&\mathcal{Y}(0) = \mathcal{A} \ , \quad \mathcal{Y}(\tfrac{\pi}{2}) = \mathcal{B} \ , \quad \operatorname{tr}_{\mathcal{Y}(t)} D = 0 \text{ for all } t
\end{aligned} \tag{5.3}$$

and the existence of a $t_0 \in [0, \frac{\pi}{2}]$ with $\operatorname{tr}_{Q\mathcal{Y}(t_0)} QDQ = 0$ is shown. Then from lemma 5.3 (i), the existence of a gyroscopic stabilizer follows.
Note that the specific parameterization of $\mathcal{Y}$ with $t \in [0, \frac{\pi}{2}]$ is completely arbitrary. It should also be mentioned that in general it holds that $(Q^{-1}\mathcal{Y})^\perp \neq Q^{-1}(\mathcal{Y}^\perp)$, which is why we introduced the notation with $\mathcal{Y}_a, \mathcal{Y}_b$ and avoided $\mathcal{Y}, \mathcal{Y}^\perp$.
(a) From $u_2^* D u_2 < 0$ and $u_j^* D u_j > 0$ for all $j \neq 2$ and $0 = \operatorname{tr} D = \sum_{i=1}^5 u_i^* D u_i$ it follows that there exists a normalized $x \in \operatorname{Span}\{u_1, u_2\}$

with $x^*Dx = -u_5^*Du_5$. We then have with $\mathcal{A} = \mathrm{Span}\{x, u_5\}$ and $x^\perp$ such that $\{x, x^\perp\}$ is an orthonormal basis of $\mathrm{Span}\{u_1, u_2\}$:

$$\begin{aligned}
\mathrm{tr}_{\mathcal{A}}\, D &= 0\,,\\
\mathrm{tr}_{Q\mathcal{A}}\, QDQ &= \lambda_5^2 u_5^* D u_5 + \frac{(Qx)^* QDQ(Qx)}{\|Qx\|^2}\\
&= \lambda_5^2 u_5^* D u_5 + \mathrm{tr}_{\mathrm{Span}\{u_1,u_2\}}\, QDQ\\
&\quad -\frac{(Q^{-1}x^\perp)^* QDQ(Q^{-1}x^\perp)}{\|Q^{-1}x^\perp\|^2}\\
&= \lambda_5^2 u_5^* D u_5 + \lambda_1^2 u_1^* D u_1 + \lambda_2^2 u_2^* D u_2 - \frac{(x^\perp)^* D x^\perp}{\|Q^{-1}x^\perp\|^2}\,.
\end{aligned}$$

Again from $\mathrm{tr}\, D = 0$ and $x^*Dx = -u_5^*Du_5$ it follows that

$$(x^\perp)^* D x^\perp = -(u_3^* D u_3 + u_4^* D u_4)$$

and thus

$$\mathrm{tr}_{Q\mathcal{A}}\, QDQ \;=\; \lambda_5^2 u_5^* D u_5 + \lambda_1^2 u_1^* D u_1 + \lambda_2^2 u_2^* D u_2 + \frac{u_3^* D u_3 + u_4^* D u_4}{\|Q^{-1}x^\perp\|^2}\,.$$

Now we have $\frac{1}{\|Q^{-1}x^\perp\|^2} \in [\lambda_1^2,\ \lambda_2^2]$ and therefore

$$\mathrm{tr}_{Q\mathcal{A}}\, QDQ \le \mathrm{tr}\, QDQ\,.$$

(b) On the other hand, there exists a normalized $y \in \mathrm{Span}\{u_2, u_4\}$ with $y^*Dy = -u_5^*Du_5$. With $\mathcal{B} = \mathrm{Span}\{y, u_5\}$ and $(y^\perp)^*Dy^\perp = -(u_1^*Du_1 + u_3^*Du_3)$ we get

$$\begin{aligned}
\mathrm{tr}_{Q\mathcal{B}}\, QDQ &= \lambda_5^2 u_5^* D u_5 + \frac{(Qy)^* QDQ(Qy)}{\|Qy\|^2}\\
&= \lambda_5^2 u_5^* D u_5 + \lambda_2^2 u_2^* D u_4 + \lambda_4^2 u_4^* D u_4 - \frac{(y^\perp)^* D y^\perp}{\|Q^{-1}y^\perp\|^2}\\
&= \lambda_5^2 u_5^* D u_5 + \lambda_2^2 u_2^* D u_4 + \lambda_4^2 u_4^* D u_4 + \frac{u_1^* D u_1 + u_3^* D u_3}{\|Q^{-1}y^\perp\|^2}\\
&\ge \lambda_5^2 u_5^* D u_5 + \lambda_2^2 u_2^* D u_4 + \lambda_4^2 u_4^* D u_4 + \lambda_2^2(u_1^* D u_1 + u_3^* D u_3)\\
&\ge \lambda_2^2 \sum_{i=1}^{5} u_i^* D u_i = 0\,.
\end{aligned}$$

(c) We consider the space $\mathrm{Span}\{u_1, u_2, u_4\}$. We already showed in (a) and (b) that $-u_5^*Du_5$ is in the numerical range of

$$K_{124} = [u_1, u_2, u_4]^* D [u_1, u_2, u_4]\,.$$

This implies that 0 is in the numerical range of $(K_{124} + u_5^*Du_5 I_3)$.
By construction of x and y, it follows that $x = [u_1, u_2, u_4]z_x$ for some

normalized $z_x \in \mathbb{R}^3$ with $z_x^*(K_{124} + u_5^* D u_5 I_3) z_x = 0$, and we have $y = [u_1, u_2, u_4] z_y$. Now by Proposition 2.7 there exists a continuous mapping $z : [0, \frac{\pi}{2}] \mapsto \mathbb{R}^5$ with $\|z(t)\| = 1$, $z(0) = z_x$ and either $z(\frac{\pi}{2}) = z_y$ or $z(\frac{\pi}{2}) = -z_y$. We define $\mathcal{Y}(t) = \operatorname{Span}\{[u_1, u_2, u_4] z(t), u_5\}$ for all t. Note that $\mathcal{Y}(\frac{\pi}{2}) = \operatorname{Span}\{y, u_5\}$ and that $\mathcal{Y}$ is continuous by definition. As seen above in (a) and (b), we have

$$\operatorname{tr}_{Q\mathcal{Y}(0)} QDQ = \operatorname{tr}_{Q\mathcal{A}} QDQ \leq \operatorname{tr} QDQ$$

and

$$\operatorname{tr}_{Q\mathcal{Y}(\frac{\pi}{2})} QDQ = \operatorname{tr}_{Q\mathcal{B}} QDQ \geq 0 \ .$$

If already either

$$0 \leq \operatorname{tr}_{Q\mathcal{Y}(0)} QDQ \leq \operatorname{tr} QDQ$$

or

$$0 \leq \operatorname{tr}_{Q\mathcal{Y}(\frac{\pi}{2}))} QDQ \leq \operatorname{tr} QDQ \ ,$$

we have found a space as in lemma 5.3 (ii), and the proposition is shown.

Otherwise, we have

$$\begin{aligned} \operatorname{tr}_{Q\mathcal{Y}(0)} QDQ &\leq 0 \ , \\ \operatorname{tr}_{Q\mathcal{Y}(\frac{\pi}{2})} QDQ &\geq \operatorname{tr} QDQ \ . \end{aligned}$$

But then, by continuity of the trace and the intermediate value theorem, foy any value $c \in [\operatorname{tr}_{Q\mathcal{Y}(0)} QDQ, \operatorname{tr}_{Q\mathcal{Y}(\frac{\pi}{2})} QDQ]$ there exists a $t_0 \in [0, \frac{\pi}{2}]$ such that $c = \operatorname{tr}_{Q\mathcal{Y}(t_0)} QDQ$. Choose $c = \frac{\operatorname{tr} QDQ}{2}$ and t_0 accordingly. Then

$$0 \leq \operatorname{tr}_{Q\mathcal{Y}(t_0)} QDQ \leq \operatorname{tr} QDQ \ ,$$

as in Lemma 5.3 (i). □

Proposition 5.7. *Let* $\operatorname{tr} D = 0$ *and* $\operatorname{tr} QDQ \geq 0$ *and let* $u_1, \ldots, u_5$ *and* $\lambda_1 \leq \cdots \leq \lambda_5$ *be a set of normalized eigenvectors and corresponding eigenvalues of* Q.
Assume that $u_3^* D u_3 < 0$ *and let*

$$K_{jk} = [u_j u_k]^* D [u_j u_k]$$

be positive definite for each pair $j, k \in \{1, 2, 4, 5\}$.
Then there exists a gyroscopic stabilizer.

Proof. The strategy for the proof is as follows.
(a) We show the existence of a space $\mathcal{Y}_a$ with

$$\operatorname{tr}_{\mathcal{Y}_a} D = 0 \ , \quad \operatorname{tr}_{Q\mathcal{Y}_a} QDQ \geq 0 \ ,$$

(b) and the existence of a second space $\mathcal{Y}_b$ with

$$\operatorname{tr}_{\mathcal{Y}_b} D = 0\ , \quad \operatorname{tr}_{Q\mathcal{Y}_b} QDQ \le 0\ .$$

(c) We construct a continuous mapping $\mathcal{Y}$, whose image is in the set $\mathcal{P}$ of two-dimensional subspaces of $\mathbb{R}^5$, with the properties

$$\begin{aligned} &\mathcal{Y} : [0,\pi] \mapsto \mathcal{P}\ , \quad t \mapsto \mathcal{Y}(t)\ , \quad \dim \mathcal{Y}(t) = 2 \text{ for all } t \\ &\mathcal{Y}(0) = \mathcal{Y}_a\ , \quad \mathcal{Y}(\pi) = \mathcal{Y}_b\ , \quad \operatorname{tr}_{\mathcal{Y}(t)} D = 0 \text{ for all } t\ , \end{aligned} \tag{5.4}$$

and show the existence of a $t_0 \in [0,\pi]$ with $\operatorname{tr}_{Q\mathcal{Y}(t_0)} QDQ = 0$.
(a) From our assumptions it follows that $u_3^* D u_3 < -u_5^* D u_5 < u_4^* D u_4$, thus there exists a normalized $x \in \operatorname{Span}\{u_3, u_4\}$ with $x^* D x = -u_5^* D u_5$. We then have with $\mathcal{Y}_a = \operatorname{Span}\{x, u_5\}$ and an $x^\perp$ such that $x, x^\perp$ is an orthonormal basis of $\operatorname{Span}\{u_3, u_4\}$:

$$\begin{aligned} \operatorname{tr}_{Q\mathcal{Y}_a} QDQ &= \lambda_5^2 u_5^* D u_5 + \frac{(Qx)^* QDQ(Qx)}{\|Qx\|^2} \\ &= \lambda_5^2 u_5^* D u_5 + \operatorname{tr}_{\operatorname{Span}\{u_3,u_4\}} QDQ \\ &\quad - \frac{(Q^{-1}x^\perp)^* QDQ(Q^{-1}x^\perp)}{\|Q^{-1}x^\perp\|^2} \\ &= \lambda_5^2 u_5^* D u_5 + \lambda_3^2 u_3^* D u_3 + \lambda_4^2 u_4^* D u_4 - \frac{(x^\perp)^* D x^\perp}{\|Q^{-1}x^\perp\|^2}\ . \end{aligned}$$

From $\operatorname{tr} D = 0$ and

$$u_3^* D u_3 + u_4^* D u_4 = x^* D x + (x^\perp)^* D x^\perp = -u_5^* D u_5 + (x^\perp)^* D x^\perp$$

it follows that

$$(x^\perp)^* D x^\perp = u_3^* D u_3 + u_4^* D u_4 + u_5 D u_5$$

and thus

$$\begin{aligned} \operatorname{tr}_{Q\mathcal{Y}_a} QDQ &= \lambda_3^2 u_3^* D u_3 + \lambda_4^2 u_4^* D u_4 + \lambda_5^2 u_5^* D u_5 \\ &\quad - \frac{u_3^* D u_3 + u_4^* D u_4 + u_5 D u_5}{\|Q^{-1}x^\perp\|^2}\ . \end{aligned}$$

We have $\frac{1}{\|Q^{-1}x^\perp\|^2} \in [\lambda_3^2,\ \lambda_4^2]$ since $x^\perp$ is a normalized element of $\operatorname{Span}\{u_3, u_4\}$, therefore we get with $-\big(u_3^* D u_3 + u_4^* D u_4 + u_5 D u_5\big) > 0$:

$$\begin{aligned} \operatorname{tr}_{Q\mathcal{Y}_a} QDQ &\ge \lambda_3^2 u_3^* D u_3 + \lambda_4^2 u_4^* D u_4 + \lambda_5^2 u_5^* D u_5 \\ &\quad -\lambda_3^2\big(u_3^* D u_3 + u_4^* D u_4 + u_5 D u_5\big) \\ &= \big(\lambda_5^2 - \lambda_3^2\big) u_5^* D u_5 + \big(\lambda_4^2 - \lambda_3^2\big) u_4^* D u_4 \ge 0\ . \end{aligned}$$

(b) On the other hand, there exists a normalized $y \in \operatorname{Span}\{u_2, u_3\}$ with $y^* D y = -u_1^* D u_1$. With $\mathcal{Y}_b = \operatorname{Span}\{y, u_1\}$ and $y^\perp$ such that $\{y, y^\perp\}$ is an orthonormal basis of $\mathrm{Span}\{u_2, u_3\}$, we get

$$\begin{aligned}
\mathrm{tr}_{Q\mathcal{Y}_b} QDQ &= \lambda_1^2 u_1^* D u_1 + \frac{(Qy)^* QDQ(Qy)}{\|Qy\|^2} \\
&= \lambda_1^2 u_1^* D u_1 + \mathrm{tr}_{\mathrm{Span}\{u_2,u_3\}} QDQ \\
&\quad - \frac{(Q^{-1}y^\perp)^* QDQ(Q^{-1}y^\perp)}{\|Q^{-1}y^\perp\|^2} \\
&= \lambda_1^2 u_1^* D u_1 + \lambda_2^2 u_2^* D u_2 + \lambda_3^2 u_3^* D u_3 - \frac{(y^\perp)^* D y^\perp}{\|Q^{-1}y^\perp\|^2} \,.
\end{aligned}$$

From $\mathrm{tr}\, D = 0$ and

$$u_2^* D u_2 + u_3^* D u_3 = y^* D y + (y^\perp)^* D y^\perp = -u_1^* D u_1 + (y^\perp)^* D y^\perp$$

it follows that

$$(y^\perp)^* D y^\perp = u_1^* D u_1 + u_2^* D u_2 + u_3^* D u_3$$

and

$$\begin{aligned}
\mathrm{tr}_{Q\mathcal{Y}_b} QDQ &= \lambda_1^2 u_1^* D u_1 + \lambda_2^2 u_2^* D u_2 + \lambda_3^2 u_3^* D u_3 \\
&\quad - \frac{u_1^* D u_1 + u_2^* D u_2 + u_3^* D u_3}{\|Q^{-1}y^\perp\|^2} \,.
\end{aligned}$$

We have $\frac{1}{\|Q^{-1}y^\perp\|^2} \in [\lambda_2^2,\ \lambda_3^2]$ since $y^\perp$ is a normalized element of $\mathrm{Span}\{u_2, u_3\}$, therefore, since $-\big(u_1^* D u_1 + u_2^* D u_2 + u_3^* D u_3\big) > 0$,

$$\begin{aligned}
\mathrm{tr}_{Q\mathcal{Y}_b} QDQ &\le \lambda_1^2 u_1^* D u_1 + \lambda_2^2 u_2^* D u_2 + \lambda_3^2 u_3^* D u_3 \\
&\quad -\lambda_3^2 \big(u_1^* D u_1 + u_2^* D u_2 + u_3^* D u_3\big) \\
&= \big(\lambda_1^2 - \lambda_3^2\big) u_1^* D u_1 + \big(\lambda_2^2 - \lambda_3^2\big) u_2^* D u_2 \le 0 \,.
\end{aligned}$$

(c) From $u_3^* D u_3 < -u_5^* D u_5 < u_4^* D u_4, u_2^* D u_2$ it follows that $-u_5^* D u_5$ is in the numerical range of

$$K_{234} = [u_2, u_3, u_4]^* D [u_2, u_3, u_4] \,,$$

in particular there exists $x \in \mathrm{Span}\{u_3, u_4\}$ with $x^* D x = -u_5^* D u_5$ as in (a), and there exists $w \in \mathrm{Span}\{u_2, u_3\}$ with $w^* D w = -u_5^* D u_5$. If we put $x = [u_2, u_3, u_4] z_x$ and $w = [u_2, u_3, u_4] z_w$, then both z_x, z_w are zeros of the quadratic form given by $(K_{234} + u_5^* D u_5 I_3)$.
By Proposition 2.7 there is a continuous $z = z(t)$ with $z(0) = z_x$ and either $z(\frac{\pi}{2}) = z_w$ or $z(\frac{\pi}{2}) = -z_w$. We put $\mathcal{Y}(t) = \mathrm{Span}\{[u_2, u_3, u_4] z(t), u_5\}$ for all $t \in [0, \frac{\pi}{2}]$. Note that $\mathcal{Y}(\frac{\pi}{2}) = \mathrm{Span}\{w, u_5\}$, and $\mathcal{Y}$ is continuous by definition.
We will now extend the map $\mathcal{Y}$ and use an argument analogous to the

first part of the proof of Proposition 4.4. By assumption, $\operatorname{Span}\{u_1, u_5\}$ is D-positive definite, thus the numerical range of

$$K_{15} = [u_1, u_5]^* D [u_1, u_5]$$

is completely contained in the interval $[0, u_1^* D u_1 + u_5^* D u_5]$, we write $\operatorname{nr}(K_{15}) \subseteq [0, u_1^* D u_1 + u_5^* D u_5]$. Since we have

$$u_3^* D u_3 \leq -(u_5^* D u_5 + u_1^* D u_1) \leq u_2^* D u_2 \ ,$$

it follows that $\operatorname{nr}(-K_{15}) \subseteq \operatorname{nr}(K_{23})$. Thus for each normalized element $v_2 \in \operatorname{Span}\{u_1, u_5\}$ there exists a normalized element $v_1 = v_1(v_2) \in \operatorname{Span}\{u_2, u_3\}$ so that $v_1^* D v_1 = -v_2^* D v_2$. We can choose $v_1 = f(\alpha) = \cos(\alpha)\tilde{u}_2 + \sin(\alpha)\tilde{u}_3$ with $\alpha \in [0, \frac{\pi}{2}]$ and $\tilde{u}_2, \tilde{u}_3$ are orthonormal such that $[\tilde{u}_2\tilde{u}_3]^* D [\tilde{u}_2\tilde{u}_3] = \operatorname{diag}(\mu_2, \mu_3)$ where $\mu_3 < 0 < \mu_2$. Then the mapping $g : \alpha \mapsto f(\alpha)^* D f(\alpha)$ is continuous and strictly monotonically decreasing and therefore continuously invertible. Since the mapping $v_2 \mapsto v_2^* D v_2$ is also continuous, the mapping

$$v_2 \mapsto v_1(v_2) = v_1(g^{-1}(v_2^* D v_2))$$

is continuous as well.
We set $v_2(t) := \cos(t) u_5 + \sin(t) u_1$ and $v_1(t) := v_1(v_2(t))$. Then the space $\mathcal{Y}(t) = \operatorname{Span}\{v_1(t), v_2(t)\}$ is continuous on $[\frac{\pi}{2}, \pi]$ with $\mathcal{Y}(\frac{\pi}{2}) = \operatorname{Span}\{w, u_5\}$ and $\mathcal{Y}(\pi) = \operatorname{Span}\{y, u_1\}$, where y is as in (b).
Altogether $\mathcal{Y}$ is continuous on the complete interval $[0, \pi]$ and

$$\begin{aligned} \operatorname{tr}_{Q\mathcal{Y}(0)} QDQ &\geq 0 \ , \\ \operatorname{tr}_{Q\mathcal{Y}(\pi)} QDQ &\leq 0 \ . \end{aligned}$$

The intermediate value theorem implies the existence of $t_0 \in [0, \pi]$ with

$$\operatorname{tr}_{Q\mathcal{Y}(t_0)} QDQ = 0 \ .$$

Thus by Lemma 5.3 (i), there exists a gyroscopic stabilizer. □

Proposition 5.8. *Let* $\operatorname{tr} D = 0$ *and* $\operatorname{tr} QDQ \geq 0$ *and let* $u_1, \ldots, u_5$ *and* $\lambda_1 \leq \cdots \leq \lambda_5$ *be a set of normalized eigenvectors and corresponding eigenvalues of* Q.
Assume that $u_4^* D u_4 < 0$ *and let*

$$K_{jk} = [u_j u_k]^* D [u_j u_k]$$

be positive definite for each pair $j, k \in \{1, 2, 3, 5\}$.
Then there exists a gyroscopic stabilizer.

Proof. The strategy is basically the same as in the previous proof.
(a) We show the existence of a space $\mathcal{Y}_a$ with

$$\operatorname{tr}_{\mathcal{Y}_a} D = 0 \ , \quad \operatorname{tr}_{Q\mathcal{Y}_a} QDQ \geq \operatorname{tr} QDQ \ ,$$

(b) then the existence of a second space $\mathcal{Y}_b$ with

$$\operatorname{tr}_{\mathcal{Y}_b} D = 0 \ , \quad \operatorname{tr}_{Q\mathcal{Y}_b} QDQ \le 0 \ .$$

(c) We construct a continuous mapping $\mathcal{Y}$, whose image is in the set $\mathcal{P}$ of two-dimensional subspaces of $\mathbb{R}^5$, with the properties

$$\begin{aligned} &\mathcal{Y} : [0, \tfrac{\pi}{2}] \mapsto \mathcal{P} \ , \quad t \mapsto \mathcal{Y}(t) \ , \quad \dim \mathcal{Y}(t) = 2 \text{ for all } t \\ &\mathcal{Y}(0) = \mathcal{Y}_a \ , \quad \mathcal{Y}(\tfrac{\pi}{2}) = \mathcal{Y}_b \ , \quad \operatorname{tr}_{\mathcal{Y}(t)} D = 0 \text{ for all } t \ . \end{aligned} \tag{5.5}$$

Then the existence of a $t_0 \in [0, \frac{\pi}{2}]$ with $\operatorname{tr}_{Q\mathcal{Y}(t_0)} QDQ = 0$ follows.
(a) From our assumptions it follows that there exists a normalized $x \in \operatorname{Span}\{u_3, u_4\}$ with $x^*Dx = -u_5^*Du_5$. We then have with $\mathcal{Y}_a = \operatorname{Span}\{x, u_5\}$ and $x^\perp$ such that $\{x, x^\perp\}$ is an orthonormal basis of the space $\operatorname{Span}\{u_3, u_4\}$:

$$\begin{aligned} \operatorname{tr}_{Q\mathcal{Y}_a} QDQ &= \lambda_5^2 u_5^* D u_5 + \frac{(Qx)^* QDQ(Qx)}{\|Qx\|^2} \\ &= \lambda_5^2 u_5^* D u_5 + \operatorname{tr}_{\operatorname{Span}\{u_3,u_4\}} QDQ \\ &\quad - \frac{(Q^{-1}x^\perp)^* QDQ (Q^{-1}x^\perp)}{\|Q^{-1}x^\perp\|^2} \\ &= \lambda_5^2 u_5^* D u_5 + \lambda_3^2 u_3^* D u_3 + \lambda_4^2 u_4^* D u_4 - \frac{(x^\perp)^* D x^\perp}{\|Q^{-1}x^\perp\|^2} \ . \end{aligned}$$

From $\operatorname{tr} D = 0$ and

$$u_3^* D u_3 + u_4^* D u_4 = x^* D x + (x^\perp)^* D x^\perp = -u_5^* D u_5 + (x^\perp)^* D x^\perp$$

it follows that

$$(x^\perp)^* D x^\perp = u_3^* D u_3 + u_4^* D u_4 + u_5 D u_5 = -\big(u_1^* D u_1 + u_2^* D u_2\big)$$

and thus

$$\operatorname{tr}_{Q\mathcal{Y}_a} QDQ \ = \ \lambda_3^2 u_3^* D u_3 + \lambda_4^2 u_4^* D u_4 + \lambda_5^2 u_5^* D u_5 + \frac{u_1^* D u_1 + u_2^* D u_2}{\|Q^{-1}x^\perp\|^2} \ .$$

We have $\frac{1}{\|Q^{-1}x^\perp\|^2} \in [\lambda_3^2, \ \lambda_4^2]$, since $x^\perp$ is a normalized element of $\operatorname{Span}\{u_3, u_4\}$, therefore

$$\begin{aligned} &\operatorname{tr}_{Q\mathcal{Y}_a} QDQ \\ \ge\ & \lambda_3^2 u_3^* D u_3 + \lambda_4^2 u_4^* D u_4 + \lambda_5^2 u_5^* D u_5 + \lambda_3^2 \big(u_1^* D u_1 + u_2^* D u_2\big) \\ \ge\ & \sum_{i=1}^{5} \lambda_i^2 u_i^* D u_i = \operatorname{tr} QDQ \ge 0 \ . \end{aligned}$$

(b) On the other hand, there exists a normalized $y \in \operatorname{Span}\{u_3, u_4\}$ with $y^* D y = -u_2^* D u_2$. With $\mathcal{Y}_b = \operatorname{Span}\{y, u_2\}$ and $y^\perp$ such that $y, y^\perp$ is an orthonormal basis of $\operatorname{Span}\{u_3, u_4\}$ we get

$$\begin{aligned}
\operatorname{tr}_{Q\mathcal{Y}_b} QDQ &= \lambda_2^2 u_2^* D u_2 + \frac{(Qy)^* QDQ(Qy)}{\|Qy\|^2} \\
&= \lambda_2^2 u_2^* D u_2 + \operatorname{tr}_{\operatorname{Span}\{u_3,u_4\}} QDQ \\
&\quad - \frac{(Q^{-1}y^\perp)^* QDQ(Q^{-1}y^\perp)}{\|Q^{-1}y^\perp\|^2} \\
&= \lambda_2^2 u_2^* D u_2 + \lambda_3^2 u_3^* D u_3 + \lambda_4^2 u_4^* D u_4 - \frac{(y^\perp)^* D y^\perp}{\|Q^{-1}y^\perp\|^2} \,.
\end{aligned}$$

From $\operatorname{tr} D = 0$ and

$$u_3^* D u_3 + u_4^* D u_4 = y^* D y + (y^\perp)^* D y^\perp = -u_2^* D u_2 + (y^\perp)^* D y^\perp$$

it follows that

$$(y^\perp)^* D y^\perp = u_2^* D u_2 + u_3^* D u_3 + u_4^* D u_4$$

and

$$\begin{aligned}
\operatorname{tr}_{Q\mathcal{Y}_b} QDQ &= \lambda_2^2 u_2^* D u_2 + \lambda_3^2 u_3^* D u_3 + \lambda_4^2 u_4^* D u_4 \\
&\quad - \frac{u_2^* D u_2 + u_3^* D u_3 + u_4^* D u_4}{\|Q^{-1}y^\perp\|^2} \,.
\end{aligned}$$

We have $\frac{1}{\|Q^{-1}y^\perp\|^2} \in [\lambda_3^2,\ \lambda_4^2]$ since $y^\perp$ is a normalized element of $\operatorname{Span}\{u_3, u_4\}$, therefore, since $-\big(u_2^* D u_2 + u_3^* D u_3 + u_4^* D u_4\big) > 0$,

$$\begin{aligned}
\operatorname{tr}_{Q\mathcal{Y}_b} QDQ &\leq \lambda_2^2 u_2^* D u_2 + \lambda_3^2 u_3^* D u_3 + \lambda_4^2 u_4^* D u_4 \\
&\quad -\lambda_4^2\big(u_2^* D u_2 + u_3^* D u_3 + u_4^* D u_4\big) \\
&= \big(\lambda_2^2 - \lambda_4^2\big) u_2^* D u_2 + \big(\lambda_3^2 - \lambda_4^2\big) u_3^* D u_3 \leq 0 \,.
\end{aligned}$$

(c) We will now construct the map $\mathcal{Y}$ exactly as in the proof of Proposition 5.7. By assumption, $\operatorname{Span}\{u_2, u_5\}$ is D-positive definite, thus we have $\operatorname{nr}(-K_{25}) \subseteq \operatorname{nr}(K_{34})$. It follows that for each normalized element $v_2 \in \operatorname{Span}\{u_2, u_5\}$ there exists a normalized element $v_1 = v_1(v_2) \in \operatorname{Span}\{u_3, u_4\}$ so that $v_1^* D v_1 = -v_2^* D v_2$. We can choose $v_1 = f(\alpha) = \cos(\alpha)\tilde{u}_3 + \sin(\alpha)\tilde{u}_4$ with $\alpha \in [0, \frac{\pi}{2}]$ and $\tilde{u}_3, \tilde{u}_4$ are orthonormal such that $[\tilde{u}_3\tilde{u}_4]^* D[\tilde{u}_3\tilde{u}_4] = \operatorname{diag}(\mu_3, \mu_4)$ where $\mu_4 < 0 < \mu_3$. Then the mapping $g : \alpha \mapsto f(\alpha)^* D f(\alpha)$ is continuous and strictly monotonically decreasing and therefore continuously invertible. Since the mapping $v_2 \mapsto v_2^* D v_2$ is also continuous, we can assume the mapping

$$v_2 \mapsto v_1(v_2) = v_1(g^{-1}(v_2^* D v_2))$$

to be continuous.
In particular we can assume, after choosing the parameter t accordingly, that $\mathcal{Y}(t) = \mathrm{Span}\{v_1(t), v_2(t)\}$ is continuous on $[0, \frac{\pi}{2}]$ with $\mathcal{Y}(0) = \mathrm{Span}\{x, u_5\}$ and $\mathcal{Y}(\frac{\pi}{2}) = \mathrm{Span}\{y, u_2\}$ where x, y are as constructed in (a) and(b).
Now, $\mathrm{tr}_{Q\mathcal{Y}(0)} QDQ \geq \mathrm{tr}\, QDQ \geq 0$ and $\mathrm{tr}_{Q\mathcal{Y}(\frac{\pi}{2})} QDQ \leq 0$. By continuity of $\mathcal{Y}$, the intermediate value theorem implies the existence of $t_0 \in [0, \frac{\pi}{2}]$ with

$$\mathrm{tr}_{Q\mathcal{Y}(t_0)} QDQ = 0 \ .$$

Then our proposition follows from Lemma 5.3 (i). □

5.2. One D-positive Eigenvector of Q

Here we prove the analogue of Proposition 4.6 with an approach as in section 5.1. Thus, we consider the following cases:

(i)	$u_5^* D u_5 > 0$	$u_j^* D u_j < 0$	for all $j \neq 5$
(ii)	$u_4^* D u_4 > 0$	$[u_j\ u_k]^* D [u_j\ u_k] < 0$	for all distinct $j, k \neq 4$
(iii)	$u_3^* D u_3 > 0$	$[u_j\ u_k]^* D [u_j\ u_k] < 0$	for all distinct $j, k \neq 3$
(iv)	$u_2^* D u_2 > 0$	$[u_j\ u_k]^* D [u_j\ u_k] < 0$	for all distinct $j, k \neq 2$.

As was the case in section 5.1, case (i) is more general than the others since $[u_j\ u_k]^* D [u_j\ u_k] < 0$ implies $u_j^* D u_j, u_k^* D u_k < 0$.

Proposition 5.9. *Let* $\mathrm{tr}\, D = 0$ *and* $\mathrm{tr}\, QDQ \geq 0$ *and let* $u_1, \ldots, u_5$ *and* $\lambda_1 \leq \cdots \leq \lambda_5$ *be a set of normalized eigenvectors and corresponding eigenvalues of* Q.
Assume that $u_5^* D u_5 > 0$ *and* $u_j^* D u_j < 0$ *for all* $j \neq 5$. *Then there exists a gyroscopic stabilizer.*

Proof. We construct a 3-dimensional space $\mathcal{Y}_a$ with
(a) $\mathrm{tr}_{\mathcal{Y}_a} D = 0$ and $0 \leq \mathrm{tr}_{Q^{-1}\mathcal{Y}_a} QDQ$,
(b) $0 \leq \mathrm{tr}_{Q\mathcal{Y}_b} QDQ$, where $\mathcal{Y}_b = \mathcal{Y}_a^{\perp}$.
Then according to lemma 5.3 (iii), there exists a gyroscopic stabilizer.
(a) From $\mathrm{tr}\, D = \sum_{k=1}^{5} u_k D u_k = 0$ it follows that

$$u_5^* D u_5 \geq -(u_2^* D u_2 + u_3^* D u_3) \geq 0 \geq u_4^* D u_4 \ .$$

This implies that there exists a normalized $y_1 \in \mathrm{Span}\{u_4, u_5\}$ with

$$y_1^* D y_1 + u_2^* D u_2 + u_3^* D u_3 = 0 \ .$$

We define $\mathcal{Y}_a = \mathrm{Span}\{y_1, u_2, u_3\}$. By construction, we have

$$\mathrm{tr}_{\mathcal{Y}_a} D = 0 \ .$$

Also by construction, y_1, u_3, u_4 remain orthogonal under multiplication with Q or Q^{-1}, thus we can compute

$$\begin{aligned} \operatorname{tr}_{Q^{-1}\mathcal{Y}_a} QDQ &= \lambda_2^2 u_2^* D u_2 + \lambda_3^2 u_3^* D u_3 + \frac{(Q^{-1}y_1)^* QDQ (Q^{-1}y_1)}{\|Q^{-1}y_1\|^2} \\ &= \lambda_2^2 u_2^* D u_2 + \lambda_3^2 u_3^* D u_3 + \frac{y_1^* D y_1}{\|Q^{-1}y_1\|^2}\ . \end{aligned} \tag{5.6}$$

Now, since $y_1 \in \operatorname{Span}\{u_4, u_5\}$, we have

$$\frac{1}{\|Q^{-1}y_1\|^2} \in [\lambda_4^2,\ \lambda_5^2]\ .$$

It follows with $y_1^* D y_1 > 0$ and an estimation for the denominator of (5.6) that

$$\begin{aligned} \operatorname{tr}_{Q^{-1}\mathcal{Y}_a} QDQ &\geq \lambda_2^2 u_2^* D u_2 + \lambda_3^2 u_3^* D u_3 + \lambda_4^2 y_1^* D y_1 \\ &= \lambda_2^2 u_2^* D u_2 + \lambda_3^2 u_3^* D u_3 + \lambda_4^2 \big(- u_2^* D u_2 - u_3^* D u_3 \big) \\ &= \big(\lambda_2^2 - \lambda_4^2\big) u_2^* D u_2 + \big(\lambda_3^2 - \lambda_4^2\big) u_3^* D u_3 \geq 0\ . \end{aligned}$$

(b) We define $\mathcal{Y}_b = \mathcal{Y}_a^{\perp}$. From $\operatorname{tr}_{\mathcal{Y}_a} D = 0$ and $\operatorname{tr} D = 0$ it follows that $\operatorname{tr}_{\mathcal{Y}_b} D = 0$. We get

$$\begin{aligned} \operatorname{tr}_{Q\mathcal{Y}_b} QDQ &= \operatorname{tr} QDQ - \operatorname{tr}_{Q^{-1}\mathcal{Y}_a} QDQ \\ &= \sum_{k=1}^{5} \lambda_k^2 u_k^* D u_k - \frac{y_1^* D y_1}{\|Q^{-1}y_1\|^2} - \lambda_2^2 u_2^* D u_2 - \lambda_3^2 u_3^* D u_3 \\ &\geq \lambda_1^2 u_1^* D u_1 + \lambda_4^2 u_4^* D u_4 + \lambda_5^2 u_5^* D u_5 \\ &\quad + \lambda_5^2 (u_2^* D u_2 + u_3^* D u_3) \\ &\geq \lambda_5^2 \sum_{k=1}^{5} u_k^* D u_k = 0\ , \end{aligned}$$

which shows that we fulfill the requirements of Lemma 5.3 (iii). Thus there exists a gyroscopic stabilizer. □

PROPOSITION 5.10. *Let $\operatorname{tr} D = 0$ and $\operatorname{tr} QDQ \geq 0$ and let $u_1, \dots, u_5$ and $\lambda_1 \leq \dots \leq \lambda_5$ be a set of normalized eigenvectors and corresponding eigenvalues of Q.*
Assume that $u_4^* D u_4 > 0$ and let let

$$K_{jk} = [u_j\ u_k]^* D [u_j\ u_k]$$

be negative definite for each pair $j, k \in \{1, 2, 3, 5\}$.
Then there exists a gyroscopic stabilizer.

Proof. The proof is divided into the following steps.
(a) We show the existence of a two-dimensional space $\mathcal{A}$ with

$$\operatorname{tr}_{\mathcal{A}} D = 0 \;, \quad \operatorname{tr}_{Q\mathcal{A}} QDQ \le \operatorname{tr} QDQ \;.$$

(b) A second two-dimensional space $\mathcal{B}$ is constructed with

$$\operatorname{tr}_{\mathcal{B}} D = 0 \;, \quad \operatorname{tr}_{Q\mathcal{B}} QDQ \ge 0 \;.$$

(c) The existence of a continuous mapping $\mathcal{Y}$, whose image is in the set $\mathcal{P}$ of two-dimensional subspaces of $\mathbb{R}^5$, with the properties

$$\begin{aligned} &\mathcal{Y} : [0, \tfrac{\pi}{2}] \mapsto \mathcal{P} \;, \quad t \mapsto \mathcal{Y}(t) \;, \quad \dim \mathcal{Y}(t) = 2 \text{ for all } t \;, \\ &\mathcal{Y}(0) = \mathcal{A} \;, \quad \mathcal{Y}(\tfrac{\pi}{2}) = \mathcal{B} \;, \quad \operatorname{tr}_{\mathcal{Y}(t)} D = 0 \text{ for all } t \;, \end{aligned} \tag{5.7}$$

and the existence of a $t_0 \in [0, \frac{\pi}{2}]$ with $\operatorname{tr}_{Q\mathcal{Y}(t_0)} QDQ = 0$ is shown. Then our proposition follows from Lemma 5.3 (i).
(a) From $u_4^* D u_4 > 0$ and $u_j^* D u_j < 0$ for all $j \neq 4$ and $0 = \operatorname{tr} D = \sum_{i=1}^5 u_i^* D u_i$ it follows that there exists a normalized $x \in \operatorname{Span}\{u_4, u_5\}$ with $x^* D x = -u_1^* D u_1$. We then have with $\mathcal{A} = \operatorname{Span}\{x, u_1\}$ and $x^\perp$ such that $\{x, x^\perp\}$ is an orthonormal basis of $\operatorname{Span}\{u_4, u_5\}$:

$$\begin{aligned} \operatorname{tr}_{Q\mathcal{A}} QDQ &= \lambda_1^2 u_1^* D u_1 + \frac{(Qx)^* QDQ(Qx)}{\|Qx\|^2} \\ &= \lambda_1^2 u_1^* D u_1 + \operatorname{tr}_{\operatorname{Span}\{u_4,u_5\}} QDQ \\ &\quad - \frac{(Q^{-1}x^\perp)^* QDQ(Q^{-1}x^\perp)}{\|Q^{-1}x^\perp\|^2} \\ &= \lambda_1^2 u_1^* D u_1 + \lambda_4^2 u_4^* D u_4 + \lambda_5^2 u_5^* D u_5 - \frac{(x^\perp)^* D x^\perp}{\|Q^{-1}x^\perp\|^2} \;. \end{aligned}$$

Again from $\operatorname{tr} D = 0$ and $x^* D x = -u_1^* D u_1$ it follows that

$$(x^\perp)^* D x^\perp = -(u_2^* D u_2 + u_3^* D u_3)$$

and thus

$$\operatorname{tr}_{Q\mathcal{A}} QDQ = \lambda_1^2 u_1^* D u_1 + \lambda_4^2 u_4^* D u_4 + \lambda_5^2 u_5^* D u_5 + \frac{u_2^* D u_2 + u_3^* D u_3}{\|Q^{-1}x^\perp\|^2} \;.$$

Now we have $\frac{1}{\|Q^{-1}x^\perp\|^2} \in [\lambda_4^2,\ \lambda_5^2]$ and $u_2^* D u_2 + u_3^* D u_3 < 0$, therefore

$$\begin{aligned} \operatorname{tr}_{Q\mathcal{A}} QDQ &\le \lambda_1^2 u_1^* D u_1 + \lambda_4^2 u_4^* D u_4 + \lambda_5^2 u_5^* D u_5 \\ &\quad + \lambda_4^2 \big(u_2^* D u_2 + u_3^* D u_3\big) \\ &\le \sum_{i=1}^5 \lambda_i^2 u_i^* D u_i = \operatorname{tr} QDQ \;. \end{aligned}$$

(b) On the other hand, there exists a normalized $y \in \operatorname{Span}\{u_2, u_4\}$ with $y^*Dy + u_1^*Du_1 = 0$. With $\mathcal{B} = \operatorname{Span}\{y, u_1\}$ and $(y^\perp)^*Dy^\perp = -(u_3^*Du_3 + u_5^*Du_5)$ we get

$$\begin{aligned}
\operatorname{tr}_{Q\mathcal{B}} QDQ &= \lambda_1^2 u_1^*Du_1 + \frac{(Qy)^*QDQ(Qy)}{\|Qy\|^2} \\
&= \lambda_1^2 u_1^*Du_1 + \lambda_2^2 u_2^*Du_2 + \lambda_4^2 u_4^*Du_4 - \frac{(y^\perp)^*Dy^\perp}{\|Q^{-1}y^\perp\|^2} \\
&= \lambda_1^2 u_1^*Du_1 + \lambda_2^2 u_2^*Du_2 + \lambda_4^2 u_4^*Du_4 + \frac{u_3^*Du_3 + u_5^*Du_5}{\|Q^{-1}y^\perp\|^2} \\
&\geq \lambda_1^2 u_1^*Du_1 + \lambda_2^2 u_2^*Du_2 + \lambda_4^2 u_4^*Du_4 + \lambda_4^2(u_3^*Du_3 + u_5^*Du_5) \\
&\geq \lambda_4^2 \sum_{i=1}^{5} u_i^*Du_i = 0 \, .
\end{aligned}$$

(c) We consider the space $\operatorname{Span}\{u_2, u_4, u_5\}$. We already showed in (a) and (b) that $-u_1^*Du_1$ is in the numerical range of

$$K_{245} = [u_2, u_4, u_5]^*D[u_2, u_4, u_5] \, .$$

This implies that 0 is in the numerical range of $(K_{245} + u_1^*Du_1I_3)$.
By construction of x and y, it follows that $x = [u_2, u_4, u_5]z_x$ for some normalized $z_x \in \mathbb{R}^3$ with $z_x^*(K_{245} + u_1^*Du_1I_3)z_x = 0$ and we have $y = [u_2, u_4, u_5]z_y$. Now by Proposition 2.7 there exists a continuous mapping $z : [0, \frac{\pi}{2}] \mapsto \mathbb{R}^5$ with $\|z(t)\| = 1$, $z(0) = z_x$ and either $z(\frac{\pi}{2}) = z_y$ or $z(\frac{\pi}{2}) = -z_y$. We put

$$\mathcal{Y}(t) = \operatorname{Span}\{[u_2, u_4, u_5]z(t), u_1\}$$

for all t. Note that $\mathcal{Y}(\frac{\pi}{2}) = \operatorname{Span}\{y, u_1\}$ and $\mathcal{Y}$ is continuous by definition.
As seen above in (a) and (b), we have

$$\operatorname{tr}_{Q\mathcal{Y}(0)} QDQ = \operatorname{tr}_{Q\mathcal{A}} QDQ \leq \operatorname{tr} QDQ$$

and

$$\operatorname{tr}_{Q\mathcal{Y}(\frac{\pi}{2})} QDQ = \operatorname{tr}_{Q\mathcal{B}} QDQ \geq 0 \, .$$

If we already have either

$$0 \leq \operatorname{tr}_{Q\mathcal{Y}(0)} QDQ \leq \operatorname{tr} QDQ$$

or

$$0 \leq \operatorname{tr}_{Q\mathcal{Y}(\frac{\pi}{2}))} QDQ \leq \operatorname{tr} QDQ \, ,$$

we have found a space as proposed.
Otherwise, we have

$$\begin{aligned} \operatorname{tr}_{Q\mathcal{Y}(0)} QDQ &\leq 0 , \\ \operatorname{tr}_{Q\mathcal{Y}(\frac{\pi}{2})} QDQ &\geq \operatorname{tr} QDQ . \end{aligned}$$

But then, by continuity of the trace and the intermediate value theorem, for any value $c \in [\operatorname{tr}_{Q\mathcal{Y}(0)} QDQ, \operatorname{tr}_{Q\mathcal{Y}(\frac{\pi}{2})} QDQ]$ there exists a $t_0 \in [0, \frac{\pi}{2}]$ such that $c = \operatorname{tr}_{Q\mathcal{Y}(t_0)} QDQ$. Choose $c = \frac{\operatorname{tr} QDQ}{2}$ and t_0 accordingly. Then

$$0 \leq \operatorname{tr}_{Q\mathcal{Y}(t_0)} QDQ \leq \operatorname{tr} QDQ .$$

Then our proposition follows from Lemma 5.3 (i). □

Proposition 5.11. *Let* $\operatorname{tr} D = 0$ *and* $\operatorname{tr} QDQ \geq 0$ *and let* $u_1, \dots, u_5$ *and* $\lambda_1 \leq \dots \leq \lambda_5$ *be a set of normalized eigenvectors and corresponding eigenvalues of* Q. *Assume that* $u_3^* D u_3 > 0$ *and let*

$$K_{jk} = [u_j u_k]^* D [u_j u_k]$$

be negative definite for each pair $j, k \in \{1, 2, 4, 5\}$.
Then there exists a gyroscopic stabilizer.

Proof. The strategy for the proof is as follows.
(a) We show the existence of a two-dimensional space $\mathcal{Y}_a$ with

$$\operatorname{tr}_{\mathcal{Y}_a} D = 0 , \quad \operatorname{tr}_{Q\mathcal{Y}_a} QDQ \geq 0 .$$

(b) Then we show the existence of a second two-dimensional space $\mathcal{Y}_b$ with

$$\operatorname{tr}_{\mathcal{Y}_b} D = 0 , \quad \operatorname{tr}_{Q\mathcal{Y}_b} QDQ \leq 0 .$$

(c) We construct a continuous mapping $\mathcal{Y}$, whose image is in the set $\mathcal{P}$ of two-dimensional subspaces of $\mathbb{R}^5$, with the properties

$$\begin{aligned} &\mathcal{Y} : [0, \pi] \mapsto \mathcal{P} , \quad t \mapsto \mathcal{Y}(t) , \quad \dim \mathcal{Y}(t) = 2 \text{ for all } t , \\ &\mathcal{Y}(0) = \mathcal{Y}_a , \quad \mathcal{Y}(\pi) = \mathcal{Y}_b , \quad \operatorname{tr}_{\mathcal{Y}(t)} D = 0 \text{ for all } t . \end{aligned} \tag{5.8}$$

We show the existence of a $t_0 \in [0, \pi]$ with $\operatorname{tr}_{Q\mathcal{Y}(t_0)} QDQ = 0$, then from Lemma 5.3 (i), the existence of a gyroscopic stabilizer follows.
(a) From our assumptions it follows that $u_3^* D u_3 > -u_1^* D u_1 > u_2^* D u_2$, thus there exists a normalized $x \in \operatorname{Span}\{u_2, u_3\}$ with $x^* D x = -u_1^* D u_1$. We then have with $\mathcal{Y}_a = \operatorname{Span}\{x, u_1\}$ and $x^\perp$ such that $x, x^\perp$ is an

orthonormal basis of $\operatorname{Span}\{u_2, u_3\}$:

$$\begin{aligned}
\operatorname{tr}_{Q\mathcal{Y}_a} QDQ &= \lambda_1^2 u_1^* D u_1 + \frac{(Qx)^* QDQ(Qx)}{\|Qx\|^2} \\
&= \lambda_1^2 u_1^* D u_1 + \operatorname{tr}_{\operatorname{Span}\{u_2,u_3\}} QDQ \\
&\quad - \frac{(Q^{-1}x^\perp)^* QDQ(Q^{-1}x^\perp)}{\|Q^{-1}x^\perp\|^2} \\
&= \lambda_1^2 u_1^* D u_1 + \lambda_2^2 u_2^* D u_2 + \lambda_3^2 u_3^* D u_3 - \frac{(x^\perp)^* D x^\perp}{\|Q^{-1}x^\perp\|^2} \, .
\end{aligned}$$

From $\operatorname{tr} D = 0$ and

$$u_2^* D u_2 + u_3^* D u_3 = x^* D x + (x^\perp)^* D x^\perp = -u_1^* D u_1 + (x^\perp)^* D x^\perp$$

it follows that

$$(x^\perp)^* D x^\perp = u_1^* D u_1 + u_2^* D u_2 + u_3^* D u_3$$

and thus

$$\begin{aligned}
\operatorname{tr}_{Q\mathcal{Y}_a} QDQ &= \lambda_1^2 u_1^* D u_1 + \lambda_2^2 u_2^* D u_2 + \lambda_3^2 u_3^* D u_3 \\
&\quad - \frac{u_1^* D u_1 + u_2^* D u_2 + u_3^* D u_3}{\|Q^{-1}x^\perp\|^2} \, .
\end{aligned}$$

We have $\frac{1}{\|Q^{-1}x^\perp\|^2} \in [\lambda_2^2, \ \lambda_3^2]$ since $x^\perp$ is a normalized element of $\operatorname{Span}\{u_2, u_3\}$, therefore, since $-\big(u_1^* D u_1 + u_2^* D u_2 + u_3^* D u_3\big) < 0$,

$$\begin{aligned}
\operatorname{tr}_{Q\mathcal{Y}_a} QDQ &\geq \lambda_1^2 u_1^* D u_1 + \lambda_2^2 u_2^* D u_2 + \lambda_3^2 u_3^* D u_3 \\
&\quad - \lambda_3^2 \big(u_1^* D u_1 + u_2^* D u_2 + u_3^* D u_3\big) \\
&= \big(\lambda_1^2 - \lambda_3^2\big) u_1^* D u_1 + \big(\lambda_2^2 - \lambda_3^2\big) u_2^* D u_2 \geq 0 \, .
\end{aligned}$$

(b) On the other hand, there exists a normalized $y \in \operatorname{Span}\{u_3, u_4\}$ with $y^* D y = -u_5^* D u_5$. With $\mathcal{Y}_b = \operatorname{Span}\{y, u_5\}$ and $y^\perp$ such that $y, y^\perp$ is an orthonormal basis of $\operatorname{Span}\{u_3, u_4\}$ we obtain

$$\begin{aligned}
\operatorname{tr}_{Q\mathcal{Y}_b} QDQ &= \lambda_5^2 u_5^* D u_5 + \frac{(Qy)^* QDQ(Qy)}{\|Qy\|^2} \\
&= \lambda_5^2 u_5^* D u_5 + \operatorname{tr}_{\operatorname{Span}\{u_3,u_4\}} QDQ \\
&\quad - \frac{(Q^{-1}y^\perp)^* QDQ(Q^{-1}y^\perp)}{\|Q^{-1}y^\perp\|^2} \\
&= \lambda_3^2 u_3^* D u_3 + \lambda_4^2 u_4^* D u_4 + \lambda_5^2 u_5^* D u_5 - \frac{(y^\perp)^* D y^\perp}{\|Q^{-1}y^\perp\|^2} \, .
\end{aligned}$$

From $\operatorname{tr} D = 0$ and

$$u_3^* D u_3 + u_4^* D u_4 = y^* D y + (y^\perp)^* D y^\perp = -u_5^* D u_5 + (y^\perp)^* D y^\perp$$

it follows that

$$(y^\perp)^* D y^\perp = u_3^* D u_3 + u_4^* D u_4 + u_5^* D u_5$$

and

$$\begin{aligned} \operatorname{tr}_{Q\mathcal{Y}_b} QDQ &= \lambda_3^2 u_3^* D u_3 + \lambda_4^2 u_4^* D u_4 + \lambda_5^2 u_5^* D u_5 \\ &\quad - \frac{u_3^* D u_3 + u_4^* D u_4 + u_5^* D u_5}{\|Q^{-1} y^\perp\|^2} . \end{aligned}$$

We have $\frac{1}{\|Q^{-1}y^\perp\|^2} \in [\lambda_3^2,\ \lambda_4^2]$ since $y^\perp$ is a normalized element of $\operatorname{Span}\{u_3, u_4\}$, therefore, since $-\big(u_3^* D u_3 + u_4^* D u_4 + u_5^* D u_5\big) < 0$,

$$\begin{aligned} \operatorname{tr}_{Q\mathcal{Y}_b} QDQ &\leq \lambda_3^2 u_3^* D u_3 + \lambda_4^2 u_4^* D u_4 + \lambda_5^2 u_5^* D u_5 \\ &\quad -\lambda_3^2 \big(u_3^* D u_3 + u_4^* D u_4 + u_5^* D u_5\big) \\ &= \big(\lambda_4^2 - \lambda_3^2\big) u_4^* D u_4 + \big(\lambda_5^2 - \lambda_3^2\big) u_5^* D u_5 \leq 0 . \end{aligned}$$

(c) From $u_3^* D u_3 \geq -u_1^* D u_1 \geq u_2^* D u_2$ it follows that $-u_1^* D u_1$ is in the numerical range of

$$K_{234} = [u_2, u_3, u_4]^* D [u_2, u_3, u_4] ,$$

in particular there exists $x \in \operatorname{Span}\{u_2, u_3\}$ with $x^* D x = -u_1^* D u_1$ as in part (a) and there exists also $w \in \operatorname{Span}\{u_3, u_4\}$ with $w^* D w = -u_1^* D u_1$. If we put $x = [u_2, u_3, u_4] z_x$ and $w = [u_2, u_3, u_4] z_w$, then both z_x, z_w are zeros of the quadratic form given by $(K_{234} + u_1^* D u_1 I_3)$.
Now by Proposition 2.7 there exists a continuous mapping $z : [0, \frac{\pi}{2}] \mapsto \mathbb{R}^5$ with $\|z(t)\| = 1$, $z(0) = z_x$ and either $z(\frac{\pi}{2}) = z_w$ or $z(\frac{\pi}{2}) = -z_w$. We put $\mathcal{Y}(t) = \operatorname{Span}\{[u_2, u_3, u_4] z(t), u_1\}$ for all $t \in [0, \frac{\pi}{2}]$. Note that $\mathcal{Y}(\frac{\pi}{2}) = \operatorname{Span}\{w, u_1\}$, and $\mathcal{Y}$ is continuous by definition.
We will now extend the map $\mathcal{Y}$ and use an argument analogous to the first part of the proof of Proposition 4.3. By assumption, $\operatorname{Span}\{u_1, u_5\}$ is D-negative definite, thus the numerical range of

$$K_{15} = [u_1, u_5]^* D [u_1, u_5]$$

is completely contained in the interval $[u_1^* D u_1 + u_5^* D u_5, 0]$, we write $\operatorname{nr}(K_{15}) \subseteq [u_1^* D u_1 + u_5^* D u_5, 0]$. Since we have

$$u_3^* D u_3 \geq -(u_5^* D u_5 + u_1^* D u_1) \geq 0 \geq u_4^* D u_4 ,$$

it follows that $\operatorname{nr}(-K_{15}) \subseteq \operatorname{nr}(K_{34})$. Thus for each normalized element $v_2 \in \operatorname{Span}\{u_1, u_5\}$ there exists a normalized element $v_1 = v_1(v_2) \in \operatorname{Span}\{u_3, u_4\}$ such that $v_1^* D v_1 = -v_2^* D v_2$. We can choose $v_1 = f(\alpha) = \cos(\alpha)\tilde{u}_3 + \sin(\alpha)\tilde{u}_4$ with $\alpha \in [0, \frac{\pi}{2}]$ and $\tilde{u}_3, \tilde{u}_4$ are orthonormal such that $[\tilde{u}_3 \tilde{u}_4]^* D [\tilde{u}_3 \tilde{u}_4] = \operatorname{diag}(\mu_3, \mu_4)$ where $\mu_3 > 0 \geq \mu_2$. Then the mapping $g : \alpha \mapsto f(\alpha)^* D f(\alpha)$ is continuous and strictly monotonically decreasing and therefore continuously invertible. Since the mapping $v_2 \mapsto v_2^* D v_2$ is also continuous, we can assume the mapping

$$v_2 \mapsto v_1(v_2) = v_1(g^{-1}(v_2^* D v_2))$$

to be continuous.
In particular we can assume, after choosing the parameter t accordingly, that $\mathcal{Y}(t) = \operatorname{Span}\{v_1(t), v_2(t)\}$ is continuous on $[\frac{\pi}{2}, \pi]$ with $\mathcal{Y}(\frac{\pi}{2}) = \operatorname{Span}\{w, u_1\}$ and $\mathcal{Y}(\pi) = \operatorname{Span}\{y, u_5\}$ where y is as constructed in (b). Altogether $\mathcal{Y}$ is continuous on the complete interval $[0, \pi]$ and

$$\begin{aligned} \operatorname{tr}_{Q\mathcal{Y}(0)} QDQ &\geq 0 , \\ \operatorname{tr}_{Q\mathcal{Y}(\pi)} QDQ &\leq 0 . \end{aligned}$$

The intermediate value theorem implies the existence of $t_0 \in [0, \pi]$ with

$$\operatorname{tr}_{Q\mathcal{Y}(t_0)} QDQ = 0 .$$

Thus by Lemma 5.3 (i), there exists a gyroscopic stabilizer. $\square$

Proposition 5.12. *Let* $\operatorname{tr} D = 0$ *and* $\operatorname{tr} QDQ \geq 0$ *and let* $u_1, \ldots, u_5$ *and* $\lambda_1 \leq \cdots \leq \lambda_5$ *be a set of normalized eigenvectors and corresponding eigenvalues of* Q. *Assume that* $u_2^* D u_2 > 0$ *and let*

$$K_{jk} = [u_j \; u_k]^* D [u_j \; u_k]$$

be negative definite for each pair $j, k \in \{1, 3, 4, 5\}$.
Then there exists a gyroscopic stabilizer.

Proof. The strategy is basically the same as in the previous proof.
(a) We show the existence of a space $\mathcal{Y}_a$ with

$$\operatorname{tr}_{\mathcal{Y}_a} D = 0 , \quad \operatorname{tr}_{Q\mathcal{Y}_a} QDQ \geq \operatorname{tr} QDQ ,$$

(b) then the existence of a second space $\mathcal{Y}_b$ with

$$\operatorname{tr}_{\mathcal{Y}_b} D = 0 , \quad \operatorname{tr}_{Q\mathcal{Y}_b} QDQ \leq 0 .$$

(c) We construct a continuous mapping $\mathcal{Y}$, whose image is in the set $\mathcal{P}$ of two-dimensional subspaces of $\mathbb{R}^5$, with the properties

$$\begin{gathered} \mathcal{Y} : [0, \tfrac{\pi}{2}] \mapsto \mathcal{P} , \quad t \mapsto \mathcal{Y}(t) , \quad \dim \mathcal{Y}(t) = 2 \text{ for all } t \\ \mathcal{Y}(0) = \mathcal{Y}_a , \quad \mathcal{Y}(\tfrac{\pi}{2}) = \mathcal{Y}_b , \quad \operatorname{tr}_{\mathcal{Y}(t)} D = 0 \text{ for all } t . \end{gathered} \tag{5.9}$$

Then the existence of a $t_0 \in [0, \frac{\pi}{2}]$ with $\operatorname{tr}_{Q\mathcal{Y}(t_0)} QDQ = 0$ follows, and from Remark 5.3 (i), the existence of a gyroscopic stabilizer follows.
(a) By assumption there exists a normalized $x \in \operatorname{Span}\{u_2, u_3\}$ with

$x^*Dx = -u_1^*Du_1$. Then we have with $\mathcal{Y}_a = \mathrm{Span}\{x, u_1\}$ and $x^\perp$ such that $\{x, x^\perp\}$ is an orthonormal basis of the space $\mathrm{Span}\{u_2, u_3\}$:

$$\begin{aligned}
\mathrm{tr}_{Q\mathcal{Y}_a} QDQ &= \lambda_1^2 u_1^* Du_1 + \frac{(Qx)^*QDQ(Qx)}{\|Qx\|^2} \\
&= \lambda_1^2 u_1^* Du_1 + \mathrm{tr}_{\mathrm{Span}\{u_2,u_3\}} QDQ \\
&\quad - \frac{(Q^{-1}x^\perp)^*QDQ(Q^{-1}x^\perp)}{\|Q^{-1}x^\perp\|^2} \\
&= \lambda_1^2 u_1^* Du_1 + \lambda_2^2 u_2^* Du_2 + \lambda_3^2 u_3^* Du_3 - \frac{(x^\perp)^* Dx^\perp}{\|Q^{-1}x^\perp\|^2} \, .
\end{aligned}$$

From $\mathrm{tr}\, D = 0$ and

$$u_2^*Du_2 + u_3^*Du_3 = x^*Dx + (x^\perp)^*Dx^\perp = -u_1^*Du_1 + (x^\perp)^*Dx^\perp$$

it follows that

$$(x^\perp)^*Dx^\perp = u_1^*Du_1 + u_2^*Du_2 + u_3Du_3 = -\big(u_4^*Du_4 + u_5^*Du_5\big)$$

and thus

$$\mathrm{tr}_{Q\mathcal{Y}_a} QDQ = \lambda_1^2 u_1^* Du_1 + \lambda_2^2 u_2^* Du_2 + \lambda_3^2 u_3^* Du_3 + \frac{u_4^*Du_4 + u_5^*Du_5}{\|Q^{-1}x^\perp\|^2} \, .$$

We have $\frac{1}{\|Q^{-1}x^\perp\|^2} \in [\lambda_2^2,\ \lambda_3^2]$ since $x^\perp$ is a normalized element of $\mathrm{Span}\{u_2, u_3\}$, and with $u_4^*Du_4 + u_5^*Du_5 < 0$ we get

$$\begin{aligned}
\mathrm{tr}_{Q\mathcal{Y}_a} QDQ &\geq \lambda_1^2 u_1^* Du_1 + \lambda_2^2 u_2^* Du_2 + \lambda_3^2 u_5^* Du_3 \\
&\quad + \lambda_3^2\big(u_4^*Du_4 + u_5^*Du_5\big) \\
&\geq \sum_{i=1}^5 \lambda_i^2 u_i^* Du_i = \mathrm{tr}\, QDQ \geq 0 \, .
\end{aligned}$$

(b) On the other hand there exists a normalized $y \in \mathrm{Span}\{u_2, u_3\}$ with $y^*Dy = -u_4^*Du_4$. With $\mathcal{Y}_b = \mathrm{Span}\{y, u_4\}$ and $y^\perp$ such that $y, y^\perp$ is an orthonormal basis of $\mathrm{Span}\{u_2, u_3\}$ we get

$$\begin{aligned}
\mathrm{tr}_{Q\mathcal{Y}_b} QDQ &= \lambda_4^2 u_4^* Du_4 + \frac{(Qy)^*QDQ(Qy)}{\|Qy\|^2} \\
&= \lambda_4^2 u_4^* Du_4 + \mathrm{tr}_{\mathrm{Span}\{u_2,u_3\}} QDQ \\
&\quad - \frac{(Q^{-1}y^\perp)^*QDQ(Q^{-1}y^\perp)}{\|Q^{-1}y^\perp\|^2} \\
&= \lambda_2^2 u_2^* Du_2 + \lambda_3^2 u_3^* Du_3 + \lambda_4^2 u_4^* Du_4 - \frac{(y^\perp)^* Dy^\perp}{\|Q^{-1}y^\perp\|^2} \, .
\end{aligned}$$

From $\mathrm{tr}\, D = 0$ and

$$u_2^*Du_2 + u_3^*Du_3 = y^*Dy + (y^\perp)^*Dy^\perp = -u_4^*Du_4 + (y^\perp)^*Dy^\perp$$

it follows that

$$(y^{\perp})^* D y^{\perp} = u_2^* D u_2 + u_3^* D u_3 + u_4^* D u_4$$

and

$$\begin{aligned} \operatorname{tr}_{Q\mathcal{Y}_b} QDQ &= \lambda_2^2 u_2^* D u_2 + \lambda_3^2 u_3^* D u_3 + \lambda_4^2 u_4^* D u_4 \\ &\quad - \frac{u_2^* D u_2 + u_3^* D u_3 + u_4^* D u_4}{\|Q^{-1} y^{\perp}\|^2} \,. \end{aligned}$$

We have $\frac{1}{\|Q^{-1}y^{\perp}\|^2} \in [\lambda_2^2,\ \lambda_3^2]$ since $y^{\perp}$ is a normalized element of $\operatorname{Span}\{u_2, u_3\}$, therefore, since $-\big(u_2^* D u_2 + u_3^* D u_3 + u_4^* D u_4\big) < 0$,

$$\begin{aligned} \operatorname{tr}_{Q\mathcal{Y}_b} QDQ &\leq \lambda_2^2 u_2^* D u_2 + \lambda_3^2 u_3^* D u_3 + \lambda_4^2 u_4^* D u_4 \\ &\quad -\lambda_2^2\big(u_2^* D u_2 + u_3^* D u_3 + u_4^* D u_4\big) \\ &= \big(\lambda_3^2 - \lambda_2^2\big) u_3^* D u_3 + \big(\lambda_4^2 - \lambda_2^2\big) u_4^* D u_4 \leq 0 \,. \end{aligned}$$

(c) We will now construct the map $\mathcal{Y}$ as in the proof of Proposition 5.7. By assumption, $\operatorname{Span}\{u_1, u_4\}$ is D-negative definite, thus we have $\operatorname{nr}(-K_{14}) \subseteq \operatorname{nr}(K_{23})$. It follows that for each normalized element $v_2 \in \operatorname{Span}\{u_1, u_4\}$ there exists a normalized element $v_1 = v_1(v_2) \in \operatorname{Span}\{u_2, u_3\}$ so that $v_1^* D v_1 = -v_2^* D v_2$. We can choose $v_1 = f(\alpha) = \cos(\alpha)\tilde{u}_2 + \sin(\alpha)\tilde{u}_3$ with $\alpha \in [0, \frac{\pi}{2}]$ and $\tilde{u}_2, \tilde{u}_3$ are orthonormal such that $[\tilde{u}_2 \tilde{u}_3]^* D [\tilde{u}_2 \tilde{u}_3] = \operatorname{diag}(\mu_2, \mu_3)$ where $\mu_2 > 0 \geq \mu_3$. Then the mapping $g : \alpha \mapsto f(\alpha)^* D f(\alpha)$ is continuous and strictly monotonically decreasing and therefore continuously invertible. Since the mapping $v_2 \mapsto v_2^* D v_2$ is also continuous, we can assume the mapping

$$v_2 \mapsto v_1(v_2) = v_1(g^{-1}(v_2^* D v_2))$$

to be continuous.

In particular we can assume, after choosing the parameter t accordingly, that $\mathcal{Y}(t) = \operatorname{Span}\{v_1(t), v_2(t)\}$ is continuous on $[0, \frac{\pi}{2}]$ with $\mathcal{Y}(0) = \operatorname{Span}\{x, u_1\}$ and $\mathcal{Y}(\frac{\pi}{2}) = \operatorname{Span}\{y, u_4\}$ where x, y are as constructed in (a) and(b).

Now, $\operatorname{tr}_{Q\mathcal{Y}(0)} QDQ \geq \operatorname{tr} QDQ \geq 0$ and $\operatorname{tr}_{Q\mathcal{Y}(\frac{\pi}{2})} QDQ \leq 0$. By continuity of $\mathcal{Y}$, the intermediate value theorem implies the existence of $t_0 \in [0, \frac{\pi}{2}]$ with

$$\operatorname{tr}_{\mathcal{Y}(t_0)} D = 0$$

and

$$\operatorname{tr}_{Q\mathcal{Y}(t_0)} QDQ = 0 \,.$$

Then our proposition follows from Lemma 5.3 (i). $\square$

5.3. Two D-indefinite Subspaces

Proposition 4.3
In section 5.1 we were considered all cases where one of the $u_i^* D u_i$ was negative and the two-dimensional spaces $\operatorname{Span}\{u_j, u_k\}$ with $j, k \neq i$ were each positive definite, in section 5.2 equivalently. Thus the case where we have two indefinite spaces each spanned by a pair of eigenvectors remains to be investigated.
So far we were not able to give a complete proof of the existence of a gyroscopic stabilizer in this case. Thus in this section we will present a construction that at least partly solves of our problem of constructing G.
Let $\operatorname{tr} D = 0$ and $\operatorname{tr} QDQ \geq 0$ and let $u_1, \ldots, u_5$ and $\lambda_1, \ldots, \lambda_5$ be a set of normalized eigenvectors and corresponding eigenvalues of Q, given in no particular order. Assume that with $\{i, j, k, m, n\} = \{1, 2, 3, 4, 5\}$ the matrices

$$K_{jk} = [u_j u_k]^* D [u_j u_k] \ , \quad K_{mn} = [u_m u_n]^* D [u_m u_n]$$

are each indefinite or singular.
The goal is to construct a three-dimensional space such that the conditions of Lemma 5.3 (ii) are satisfied. Since K_{jk}, K_{mn} are indefinite or singular, there exist normalized $z_1 \in \operatorname{Span}\{u_j, u_k\}$ and $z_2 \in \operatorname{Span}\{u_m, u_n\}$ with $z_1^* D z_1 = z_2^* D z_2 = 0$. By construction, $Q^{-1} z_1 \perp Q^{-1} z_2$, so we can also compute

$$\begin{aligned} & \operatorname{tr}_{Q^{-1}\operatorname{Span}\{z_1,z_2\}} QDQ \\ = \ & \frac{(Q^{-1}z_1)^* QDQ * Q^{-1} z_1}{\|Q^{-1} z_1\|^2} + \frac{(Q^{-1}z_2)^* QDQ * Q^{-1} z_2}{\|Q^{-1} z_2\|^2} = 0 \ . \end{aligned}$$

We will extend $Q^{-1} \operatorname{Span}\{z_1, z_2\}$ to a three-dimensional space, therefore we analize the value $\operatorname{tr}_{Q^{-1}\operatorname{Span}\{z_1,z_2,z_3\}} QDQ$ for a given z_3 orthogonal on z_1, z_2. Let v_3 be orthonormal on z_1 in $\operatorname{Span}\{u_j, u_k\}$ and v_4 orthonormal on z_2 in $\operatorname{Span}\{u_m, u_n\}$. Then by construction, v_3, v_4, u_i are pairwise orthogonal and remain so under multiplication with Q or Q^{-1}.
Now $\{\frac{Qv_3}{\|Qv_3\|}, \frac{Q^{-1}z_1}{\|Q^{-1}z_1\|}\}$ is an orthonormal basis of

$$\operatorname{Span}\{u_j, u_k\} = Q^{-1} \operatorname{Span}\{u_j, u_k\} = Q \operatorname{Span}\{u_j, u_k\} \ ,$$

and $\{\frac{Qv_4}{\|Qv_4\|}, \frac{Q^{-1}z_2}{\|Q^{-1}z_2\|}\}$ is an orthonormal basis of

$$\operatorname{Span}\{u_m, u_n\} = Q^{-1} \operatorname{Span}\{u_m, u_n\} = Q \operatorname{Span}\{u_m, u_n\} \ .$$

Let z_3 be a normalized vector in $\operatorname{Span}\{v_3, v_4, u_i\}$. Then

$$\{\tfrac{Q^{-1}z_1}{\|Q^{-1}z_1\|}, \tfrac{Q^{-1}z_2}{\|Q^{-1}z_2\|}, \tfrac{PQ^{-1}z_3}{\|PQ^{-1}z_3\|}\}$$

is an orthonormal basis of $Q^{-1}\,\mathrm{Span}\{z_1, z_2, z_3\}$, where P denotes the projection onto the space $Q\,\mathrm{Span}\{v_3, v_4, u_i\}$:

$$\begin{aligned} P &= \frac{Qv_3v_3^*Q}{\|Qv_3\|^2} + \frac{Qv_4v_4^*Q}{\|Qv_4\|^2} + \frac{Qu_iu_i^*Q}{\|Qu_i\|^2} \\ &= \frac{Qv_3v_3^*Q}{\|Qv_3\|^2} + \frac{Qv_4v_4^*Q}{\|Qv_4\|^2} + u_iu_i^*\,, \\ PQ^{-1} &= \frac{Qv_3v_3^*}{\|Qv_3\|^2} + \frac{Qv_4v_4^*}{\|Qv_4\|^2} + \frac{u_iu_i^*}{\lambda_i}\,. \end{aligned}$$

With $z_3 = av_3 + bv_4 + cu_i$ we get

$$\begin{aligned} PQ^{-1}(av_3 + bv_4 + cu_i) &= a\frac{Qv_3}{\|Qv_3\|^2} + b\frac{Qv_4}{\|Qv_4\|^2} + c\frac{u_i}{\lambda_i} \\ &= [\frac{Qv_3}{\|Qv_3\|^2}\ \frac{Qv_4}{\|Qv_4\|^2}\ \frac{u_i}{\lambda_i}]\begin{pmatrix} a \\ b \\ c \end{pmatrix}\,, \end{aligned}$$

and

$$\|PQ^{-1}(av_3 + bv_4 + cu_i)\|^2 = \frac{a^2}{\|Qv_3\|^2} + \frac{b^2}{\|Qv_4\|^2} + \frac{c^2}{\lambda_i^2}\,.$$

Now we have

$$\begin{aligned} &\mathrm{tr}_{Q^{-1}\,\mathrm{Span}\{z_1,z_2,z_3\}}\,QDQ \\ =\ &\mathrm{tr}_{Q^{-1}\,\mathrm{Span}\{z_1,z_2\}}\,QDQ + (\frac{PQ^{-1}z_3}{\|PQ^{-1}z_3\|})^*QDQ(\frac{PQ^{-1}z_3}{\|PQ^{-1}z_3\|}) \\ =\ &\frac{z_3^*Q^{-1}PQDQPQ^{-1}z_3}{\|PQ^{-1}z_3\|^2}\,. \end{aligned}$$

We put $F = Q^{-1}PQDQPQ^{-1}$ and get

$$F = \begin{pmatrix} \frac{v_3^*Q^2DQ^2v_3}{\|Qv_3\|^4} & \frac{v_3^*Q^2DQ^2v_4}{\|Qv_3\|^2\|Qv_4\|^2} & \frac{v_3^*Q^2Du_i}{\|Qv_3\|^2} \\ \frac{v_4^*Q^2DQ^2v_3}{\|Qv_4\|^2\|Qv_3\|^2} & \frac{v_4^*Q^2DQ^2v_4}{\|Qv_4\|^4} & \frac{v_4^*Q^2Du_i}{\|Qv_4\|^2} \\ \frac{u_i^*DQ^2v_3}{\|Qv_3\|^2} & \frac{u_i^*DQ^2v_4}{\|Qv_4\|^2} & u_i^*Du_i \end{pmatrix}$$

and

$$\begin{aligned} Q_F &= \big(PQ^{-1}[\frac{Qv_3}{\|Qv_3\|^2}\ \frac{Qv_4}{\|Qv_4\|^2}\ \frac{u_i}{\lambda_i}]\big)^*PQ^{-1}[\frac{Qv_3}{\|Qv_3\|^2}\ \frac{Qv_4}{\|Qv_4\|^2}\ \frac{u_i}{\lambda_i}] \\ &= \begin{pmatrix} \frac{1}{\|Qv_3\|^2} & & \\ & \frac{1}{\|Qv_4\|^2} & \\ & & \frac{1}{\lambda_i^2} \end{pmatrix}\,, \end{aligned}$$

this allows us to write with $\|v\| = 1$ and $v = (a\ b\ c)$:

$$\operatorname{tr}_{Q^{-1}\,\mathrm{Span}\{z_1,z_2,z_3\}} QDQ = \frac{v^* F v}{v^* Q_F v}$$

and $\operatorname{tr} F_0 = \operatorname{tr} QDQ$. We also define

$$\tilde{D} = [v_3\ v_4\ u_i]^* D [v_3\ v_4\ u_i]\ .$$

In this formulation, what we need is a normalized v such that

$$\begin{aligned} 0 &= v^* \tilde{D} v\ , \\ 0 &\leq \frac{v^* F v}{v^* Q_F v} \leq \operatorname{tr} QDQ\ . \end{aligned} \tag{5.10}$$

Now either the quadratic forms given by F and $\tilde{D}$ have a nontrivial common zero v_0, then we put $z_3 = [v_3\ v_4\ u_i] v_0$ and have

$$\begin{aligned} \operatorname{tr}_{\mathrm{Span}\{z_1,z_2,z_3\}} D &= 0\ , \\ \operatorname{tr}_{Q^{-1}\,\mathrm{Span}\{z_1,z_2,z_3\}} QDQ &= 0\ , \end{aligned}$$

and our statement is shown.
Or there exist according to [9] numbers $g, h \in \mathbb{R}$ such that the matrix given by $g\tilde{D} + hF$ is positive definite, which just means that the sign of $v^* F v$ is invariant on the isotropic vectors of $\tilde{D}$. In that case, one needs to establish that a normalized v as in (5.10) can be found, which so far we were not able to prove.
Our approach here might be too restrictive since with Lemma 5.3, we already demand a very specific form for a gyroscopic stabilizer with a pair of eigenvalues close to zero.
But still, for given pairs D, Q, we were always able to find a gyroscopic stabilizer G in $\mathbb{R}^5$ by doing a random search in Matlab. A counterexample in $\mathbb{R}^5$, where $\operatorname{tr} D, \operatorname{tr} QDQ > 0$ and no gyroscopic stabilizer exists, has not yet been found. We therefore formulate our *missing case* in $\mathbb{R}^5$ as an open problem.

Problem 5.13. *Let* $\operatorname{tr} D = 0$ *and* $\operatorname{tr} QDQ \geq 0$ *and let* $u_1, \ldots, u_5$ *be a set of eigenvectors of* Q*. For* $\{i, j, k, m, n\} = \{1, 2, 3, 4, 5\}$ *in no particular order, let the matrices*

$$K_{jk} = [u_j\ u_k]^* D [u_j\ u_k]\ , \quad K_{mn} = [u_m\ u_n]^* D [u_m\ u_n]$$

be indefinite. Show that there exists a gyroscopic stabilizer.

5.4. Example

In this section, we will demonstrate the construction of a gyroscopic stabilizer in $\mathbb{R}^5$ as in Proposition 5.7. The construction in each case in the first two sections of this chapter is very similar. The difficulty lies is constructing a function $\mathcal{Y}$.

EXAMPLE 5.14. Consider the unstable system $\ddot{x} + D\dot{x} + Q^{-2}x = 0$ with

$$D = \begin{pmatrix} 10 & 4 & -8 & -3 & -8 \\ 4 & 6 & -1 & -2 & -2 \\ -8 & -1 & -25 & 12 & 3 \\ -3 & -2 & 12 & 3 & 1 \\ -8 & -2 & 3 & 1 & 11 \end{pmatrix}, \quad Q = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 5 \end{pmatrix}.$$

With $\frac{\operatorname{tr} D}{n} = 1 < \frac{132}{55} = \frac{\operatorname{tr} QDQ}{\operatorname{tr} Q^2}$ we put $\tau = 1$ and $P = I$ in Proposition 2.1 and get

$$D_\tau = \begin{pmatrix} 9 & 4 & -8 & -3 & -8 \\ 4 & 5 & -1 & -2 & -2 \\ -8 & -1 & -26 & 12 & 3 \\ -3 & -2 & 12 & 2 & 1 \\ -8 & -2 & 3 & 1 & 10 \end{pmatrix}.$$

Any space $\operatorname{Span}\{e_3, e_j\}$ with $j \neq 3$ is D-indefinite, while the space $\operatorname{Span}\{e_1, e_2, e_4, e_5\}$ is D-positive definite, thus we are in the situation of 5.7.
The proposed spaces $\mathcal{Y}_a$ and $\mathcal{Y}_b$ can be chosen as

$$\mathcal{Y}_a = \operatorname{Span}\{\begin{pmatrix} 0 \\ 0 \\ -0.3679 \\ 0.9299 \\ 0 \end{pmatrix}, e_5\}, \quad \mathcal{Y}_b = \operatorname{Span}\{\begin{pmatrix} 0 \\ 0.7184 \\ -0.6956 \\ 0 \\ 0 \end{pmatrix}, e_1\}.$$

Note that even if we stick to the proposed construction, the spaces $\mathcal{Y}_a, \mathcal{Y}_b$ are not unique, since in $\operatorname{Span}\{e_3, e_4\}$ there are in general at least two linearly independent normalized vectors x, y with $x^*Dx = y^*Dy = -e_5^*De_5$. For $\mathcal{Y}_b$ holds the same.
For $t \in [0, \frac{\pi}{2}]$, we put $u(t) = \cos(t)u_4 + \sin(t)u_2$. The proposed normalized $x(t)$ with $x(t)^*Dx(t) + e_5^*De_5 = 0$ can then be constructed as follows. Define

$$K(t) = [e_3\ u(t)]^*D[e_3\ u(t)] + e_5^*De_5I.$$

Let $U(t)$ be a matrix whose columns are an orthonormal basis of eigenvectors of $K(t)$ and $V(t)$ a diagonal matrix with corrsponding eigenvalues of $K(t)$ on the diagonal. As the components of $K(t)$ are continuous, so are the eigenvectors and eigenvalues. Thus $U(t), V(t)$ can also be chosen to depend continuously on t. Now choose an isotropic vector $\hat{y}$ of $V(t)$, see example 4.8. Since isotropy now only depends on the eigenvalues on the diagonal of $V(t)$, we can choose $\hat{y}$ such that it depends continuously on t. Then $y(t) = U(t)(1\ 1)^T$ is isotropic for $K(t)$. In particular, it also depends continuously on t. Then for

$$x(t) = [e_3\ u(t)]y(t)$$

we have $x(t)^* Dx(t) = -e_5^* De_5$ by construction.
For $t \in [\frac{\pi}{2}, \pi]$, we define $\hat{u}(t) = \sin(t)e_5 - \cos(t)u_1$ and

$$\hat{K}(t) = [e_3\ e_2]^* D[e_3\ e_2] + \hat{u}(t)^* D\hat{u}(t)I \ .$$

As above, we construct $y(t)$ as an isotropic vector of $\hat{K}(t)$ and put $x(t) = [e_3\ e_2]y(t)$ for $t \in [\frac{\pi}{2}, \pi]$. With

$$\mathcal{Y}(t) = \begin{cases} \mathrm{Span}\{x(t), u_5\} \ , & t \in [0, \frac{\pi}{2}] \ , \\ \mathrm{Span}\{x(t), \hat{u}(t)\} \ , & t \in [\frac{\pi}{2}, \pi] \ . \end{cases}$$

We now have $\mathrm{tr}_{\mathcal{Y}(t)} D = 0$ for all $t \in [0, \pi]$. Via Matlab we get

$$\begin{aligned} \mathrm{tr}_{Q^{-1}\mathcal{Y}(0)} QDQ &= 181.9306 \\ \mathrm{tr}_{Q^{-1}\mathcal{Y}(\pi)} QDQ &= -137.7026 \ , \end{aligned}$$

so none of the above two spaces satisfies the desired trace condition

$$0 < \mathrm{tr}_{Q^{-1}\mathcal{Y}(0)} QDQ < \mathrm{tr}\, QDQ \ ,$$

but there exists an intermediate value such that our condition is satisfied.

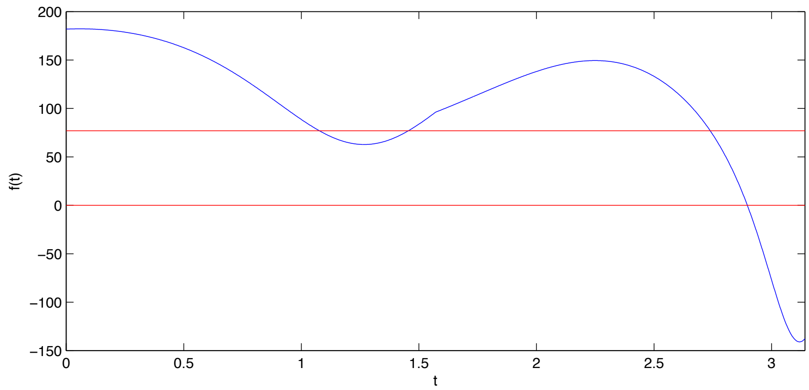

FIGURE 5.1. Plot of the function f on $[0\ \pi]$

In figure 5.1, we see a plot of $f(t) = \operatorname{tr}_{Q^{-1}\mathcal{Y}(t)} QDQ$, the horizontal red lines show the values 0 and $\operatorname{tr} QDQ$. By construction, there is now an area where $f(t)$ is in between 0 and $\operatorname{tr} QDQ$. We pick $t_0 = 1.2563$ and get $f(1.2563) = 62.8646$ and

$$\mathcal{Y}(t_0) = \operatorname{Span}\{\begin{pmatrix} 0 \\ 0.7559 \\ -0.6067 \\ 0.2459 \\ 0 \end{pmatrix} e_5\} .$$

We now choose a normalized D-isotropic vector z_1 in $\mathcal{Y}(t_0)^\perp$, complete z_1 to an orthonormal basis $\{z_1, z_2, z_3\}$ of $\mathcal{Y}(t_0)^\perp$ and put

$$\begin{aligned} Z &= [z_1\ z_2\ z_3\ x(t_0)\ u_5] \\ &= \begin{pmatrix} -0.5314 & -0.8170 & -0.2238 & 0 & 0 \\ 0.3367 & -0.0663 & -0.5575 & 0.7559 & 0 \\ 0.1075 & 0.1423 & -0.7746 & -0.6067 & 0 \\ -0.7698 & 0.5549 & -0.1975 & 0.2459 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} . \end{aligned}$$

By construction,

$$G = Z \operatorname{diag}\left(0, \begin{pmatrix} 0 & \epsilon \\ -\epsilon & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right) Z^*$$

now satisfies our eigenvector conditions. Via Matlab we find that the system is stabilized by αG with $\alpha \geq 296097$.

5.5. A Reduction Method

In some cases, the problem of constructing a gyroscopic stabilizer G as in Proposition 2.1 can be reduced to lower space dimensions. The basic idea is similar to case 3.1 in [15].

Proposition 5.15. *Let* $\operatorname{tr} D, \operatorname{tr} QDQ > 0$ *and let* $\mathcal{B} = \{u_1, \dots, u_n\}$ *be a basis of* $\mathbb{R}^n$ *consisting of normalized eigenvectors of* Q. *Suppose there is a partition*

$$\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2 = \{u_1, \dots, u_t\} + \{u_{t+1}, \dots, u_n\}$$

such that t *is even and*

$$\begin{array}{ll} \operatorname{tr}_{\operatorname{Span}\mathcal{B}_1} D > 0, & \operatorname{tr}_{\operatorname{Span}\mathcal{B}_1} QDQ > 0 \, , \\ \operatorname{tr}_{\operatorname{Span}\mathcal{B}_2} D > 0, & \operatorname{tr}_{\operatorname{Span}\mathcal{B}_2} QDQ > 0 \, . \end{array}$$

With $B_1 = [u_1, \dots, u_t], B_2 = [u_{t+1}, \dots, u_n]$ *define* $D_1 = B_1^* D B_1, Q_1 = B_1^* Q B_1$ *and* $D_2 = B_2^* D B_2, Q_2 = B_2^* Q B_2$.
Let G_1 *be a gyroscopic stabilizer such that*

$$v_1^* D_1 v_1 > 0 \, , \quad w_1^* Q_1 D_1 Q_1 w_1 > 0$$

for all eigenvectors v_1 *of* G_1 *and all eigenvectors* w_1 *of* $Q_1 G_1 Q_1$, *for* G_2 *equivalently. Then*

$$G = [B_1 \; B_2] \begin{pmatrix} G_1 & 0 \\ 0 & G_2 \end{pmatrix} [B_1 \; B_2]^*$$

is a gyroscopic stabilizer for the pair D, Q.

The assumption that at least one of G_1, G_2 is of even dimension is necessary to ensure that the eigenvalues of G are simple. If both G_1 and G_2 were of odd dimension, then both would have a zero eigenvalue, resulting in a double eigenvalue zero in G.

Proof. By scaling G_2, we can assume that $\sigma(G_1) \cap \sigma(G_2) = \emptyset$, thus no multiple eigenvalues appear in G.
Suppose v_1 is an eigenvector of G_1, then $v = B_1 v_1$ is an eigenvector of G. We get

$$v^* D v = v_1^* B_1^* D B_1 v_1 = v_1^* D_1 v_1 > 0 \, .$$

Now let w_1 be an eigenvector of $Q_1 G_1 Q_1$, then $w = B_1 w_1$ is an eigenvector of QGQ. We get, using that $\operatorname{im}(QB_1) = \operatorname{Span}\mathcal{B}_1$,

$$\begin{aligned} w^* QDQw &= w_1^* B_1^* QDQ B_1 w_1 \\ &= w_1^* B_1^* Q B_1 B_1^* D B_1 B_1^* Q B_1 w_1 \\ &= w_1^* Q_1 D_1 Q_1 w_1 > 0 \, . \end{aligned}$$

The same holds for the eigenvectors of G_2 and $Q_2 G_2 Q_2$. □

In case 3.1 in [15] the existence of a gyroscopic stabilizer G is shown under the condition that $v^*Dv > 0$ for every eigenvector of the eigenproblem $(\lambda^2 I + K)v = 0$. Clearly $(\lambda^2 I + K)v = 0$ if and only if v is an eigenvector of Q.
If $v^*Dv > 0$ for every eigenvector of Q, then the conditions of Proposition 5.15 are met for any partition of the set of eigenvectors $\mathcal{B}$. Thus G can be defined for odd $n = 2m + 1$ as

$$\begin{aligned} G &= [v_1 \dots v_n] \operatorname{diag}(0,\ \rho_1 G_0,\ \dots,\ \rho_m G_0)[v_1 \dots v_n]^* , \\ G_0 &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} , \end{aligned}$$

where the $\rho_i \in \mathbb{R}\backslash\{0\}$ are pairwise distinct. For even n, the construction is equivalent.
Finally, we present a condition that guarantees that every G with D-isotropic eigenvectors is a gyroscopic stabilizer.

Proposition 5.16. *Let $\operatorname{tr} D = 0$ and $\operatorname{tr} QDQ \geq 0$ and let $\mu_1 \leq \dots \leq \mu_n$ be the eigenvalues of QDQ. If $\mu_1 < 0 < \mu_2$ and $|\mu_1| \leq |\mu_2|$, then for any G with $v^*Dv = 0$ for all eigenvectors v of G and pairwise distict eigenvalues also satisfies $w^*QDQw \geq 0$ for all eigenvectors w of QGQ.*

Proof. By assumption, condition (a) and (b) of Problem 2.2 are satisfied, thus we just have to verify (c).
If the space dimension n is odd, then there is an an eigenvalue zero of G with eigenvector v_0. By assumption $v_0^*Dv_0 = 0$, which implies that for the eigenvector $w_0 = Q^{-1}v_0$ for the eigenvalue zero of QGQ we have $w_0^*QDQw_0 = 0$ as well.
Now let w_k, w_l be the eigenvectors for a complex conjugate pair of eigenvalues of QGQ. By assumption on the eigenvalues of QDQ, it follows that

$$\operatorname{tr}_{\mathcal{W}} QDQ \geq \mu_1 + \mu_2 \geq 0$$

for any two-dimensional subspace $\mathcal{W}$. In particular,

$$\operatorname{tr}_{\operatorname{Span}\{w_k, w_l\}} QDQ \geq 0$$

for every pair of eigenvectors w_k, w_l, implying $w^*QDQw \geq 0$ for all eigenvectors w of QDQ. Thus (c) in Problem 2.2 holds. □

5.6. Conclusion

In our work we were able to devise a new sufficient condition for gyroscopic stabilizabilty. This condition allowed us to show the necessity of $\operatorname{tr} D > 0$ and $\operatorname{tr} QDQ > 0$ for the existence of G in the space dimensions 3 and 4. We were able to develop a construction method for G

in these dimensions and by means of several examples showed how our construction can be applied. The case $n = 5$ was not solved completely in our work. For $n \geq 5$ the construction of a gyroscopic stabilizer so far remains an open problem.
Clearly it is desirable to develop an inductive algorithm for the construction of gyroscopic stabilizers that would allow us to reduce the construction problem to lower space dimensions. We were not yet able to devise a method to achieve that. The difficulty in generalizing our approaches is mainly the fact that in general orthogonality of vectors and/or spaces is destroyed once they are multiplied with Q or Q^{-1}. That was the reason we relied so much on eigenvectors of Q in our construction because for them orthogonality was preserved.
For further research we suggest to analyze for the shifted problem with $\operatorname{tr} D = 0$ the elements of the set

$$\mathcal{A} \quad = \quad \{T \in O(n) \mid \text{the diagonal of } T^T DT \text{ consists of zeros } \} \ ,$$

where $O(n) \subset \mathbb{R}^{n \times n}$ denotes the orthogonal group. For $n \geq 3$, this set appears to become decomposed into a finite number of path-connected components. Then by using continuity arguments it might to be possible to find a $T \in \mathcal{A}$ that qualifies for the construction of G via

$$\begin{aligned} G &= T \operatorname{diag}(0, \ \rho_1 G_0, \ \ldots, \ \rho_m G_0) T^* \ , \ \rho_i \neq \rho_j \text{ for } i \neq j \ , \\ G_0 &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} . \end{aligned}$$

This approach involves differential geometry and topology and for the moment is out of reach for us.

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