Some remarks on finitely presented monoids with automatic structure*

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Abstract

Groups with automatic structure (*automatic groups* for short) have recently received a lot of attention in the literature. They have been shown to have many nice properties, and consequently it has been asked whether these results can be carried over to *monoids with automatic structure.* Here we show that there exist finitely presented monoids with automatic structure that cannot be presented through finite and convergent string-rewriting systems, thus answering a question in the negative that is still open for the class of automatic groups. Secondly, we present an automatic monoid that has an exponential Dehn function, which establishes another difference to the class of automatic groups. In fact, both our example monoids are bi-automatic.

1 Introduction

In recent years the computational aspect has become more and more prominent in combinatorial group and semigroup theory. Given a semigroup, a monoid, or a group through some finite description, one wishes to effectively determine information about its algebraic structure. Although in general not much information on the algebraic structure presented

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can be extracted from a finite presentation, various methods have been used successfully in certain instances.

The classical approach is based on the *Todd-Coxeter* method of enumerating the cosets of a group G with respect to a subgroup H, which, however, is essentially limited to the case of subgroups of finite index [TC36, Lee63, Sim94]. For the case of monoids a related method for solving the task of determining the cardinality of a finite monoid can be found in [McN97].

Another approach is based on the notion of a *convergent presentation*. A (monoid-) presentation $(\Sigma; R)$ is called *convergent* if the string-rewriting system R is convergent, that is, the reduction relation induced by R is both noetherian and confluent. The presentations of this form are of particular interest, because the reduction relation induced by a convergent string-rewriting system R yields a unique irreducible string for each congruence class of the Thue congruence generated by R. Hence, if a monoid M has a finite presentation of this form, then it has a decidable *word problem*. In fact, for the class of finite convergent presentations, the *uniform word problem* is decidable, which implies that for this class various decision problems can be solved that are undecidable in general. For example, the problem of deciding whether the monoid presented is actually a group is decidable for this class [Ott86, BO93].

When dealing with finitely generated subgroups of groups that are given through finite convergent presentations, then in some instances information about the subgroups can be obtained by using the theory of *prefix-rewriting systems*, which can be seen as a variant of coset enumeration [Kuh91, KMO94]. Gilman [Gil84] considers groups that are given through finite, monadic, and convergent string-rewriting systems. His approach is based on performing computations with regular sets, and it can also be interpreted as a generalization of coset enumeration. Corresponding considerations had already led Book to the result that for monoid-presentations involving finite, monadic, and confluent string-rewriting systems, all those properties of Thue congruences are decidable that can be expressed through *linear sentences* [Boo82b, Boo83].

During the 1980's Epstein et al developed the notion of *groups with automatic structure* [Eps92]. A finitely generated group G has an *automatic structure* if it has a finite set of generators Σ and a regular set $S \subseteq \Sigma^*$ of representatives, which, however, need not be unique, such that the following tasks can be performed by finite state acceptors:

(1.) Given two elements $u, v \in S$, decide whether or not they both represent the same element of the group G.

(2.) For $a \in \Sigma$, if two elements $u, v \in S$ are given, decide whether or not ua and v represent the same element of the group G.

Thus, the set of representatives is accepted by some finite state acceptor, the so-called *word acceptor*, the finite state acceptor of (1.) is called the *equality recognizer*, and the finite state acceptors of (2.) are called the *multiplier automata*.

Although the definition mentions explicitly a set of generators Σ for the group considered, it turns out that the existence of an automatic structure is independent of the finite set of generators chosen. Further, the word problem for each automatic group can be solved in quadratic time, and the *Dehn function* of such a group has a quadratic upper bound, that is, automatic groups are *hyperbolic* [Eps92]. Finally, each automatic group is finitely presented and it satisfies the homological finiteness condition FP_∞ [Alo92], and hence, each automatic group has *finite derivation type* [CO96].

If the group G has a finite convergent presentation $(\Sigma; R)$, then the set of irreducible strings $\mathrm{IRR}(R)$ is a regular set of unique representatives for the group G, and so also condition (1.) above is satisfied. But in general there need not exist multiplier automata in this situation. And indeed, there exist finitely presented groups that do admit finite convergent presentations, but that do not have an automatic structure [Ger92]. However, it is still not known whether each automatic group admits a finite convergent presentation.

Recently, the notion of automatic structure has been generalized to monoids [Hud96]. *Automatic monoids* do also have word problems that are decidable in quadratic time, but it turns out that for monoids the existence of an automatic structure does indeed depend on the actually chosen set of generators. Also an automatic monoid need not be finitely presented [Tho97].

In this paper we also consider the class of automatic monoids. We will present some examples showing that further properties do not carry over from automatic groups to automatic monoids. First we present a monoid N that has a finite, noetherian, and weakly confluent presentation as well as an infinite left-regular, weight-reducing, and confluent presentation. This monoid has an automatic structure, but it does not have finite derivation type. Hence, it does not admit any finite convergent presentation. This shows that the class of monoids that have a finite convergent presentation and the class of automatic monoids are incomparable under inclusion. For the class of monoids this provides a negative anwer to the open problem mentioned above. Further, it proves that the property of having a finite, noetherian, and weakly confluent presentation is not sufficient to imply that the monoid considered has finite derivation type.

Our second example consists of a finitely presented monoid that is automatic, but that has an exponential Dehn function. In fact, both our example monoids are *bi-automatic*.

This paper is structured as follows. In the next section we restate some definitions and notation regarding monoid-presentations and string-rewriting systems, and we present the definition of an automatic structure for a finitely generated monoid in full detail. In Sections 3 and 4 we present the two example monoids mentioned above, providing detailed proofs for the properties we are interested in. The paper closes with a short discussion of the consequences of our results and some open problems.

2 Automatic monoids

First we restate some definitions regarding string-rewriting systems and monoid-presentations in order to establish notation. For additional information the reader is asked to consult the literature, where the monograph by Book and Otto [BO93] is our main reference on string-rewriting systems, and the monograph by Hopcroft and Ullmann [HU79] is our main reference on automata theory. Then we give the definition of an automatic structure for a monoid, and we restate some recent results on automatic monoids.

Let Σ be a finite alphabet. Then Σ^+ denotes the set of all non-empty strings over Σ, and $\Sigma^* := \Sigma^+ \cup \{\lambda\}$ denotes the set of all strings over Σ including the empty string λ. For $u, v \in \Sigma^*$, the concatenation of u and v is simply written as uv, and exponents are used to abbreviate strings, that is, $u^0 := \lambda$, $u^1 := u$, and $u^{n+1} := u^n u$ for all $u \in \Sigma^*$ and integers $n \geq 1$. For $u \in \Sigma^*$, the *length* of u is denoted by $|u|$.

For $\Gamma \subset \Sigma$, $\pi_\Gamma : \Sigma^* \to \Gamma^*$ denotes the projection, that is, π_Γ is the morphism that is induced by the mapping $a \mapsto a$ $(a \in \Gamma)$ and $b \to \lambda$ $(b \in \Sigma \setminus \Gamma)$. Further, the Γ-*length* of a string $u \in \Sigma^*$ is defined as $|u|_\Gamma := |\pi_\Gamma(u)|$.

A *string-rewriting system* R on Σ is a subset of $\Sigma^* \times \Sigma^*$, the elements of which are called (*rewrite*) *rules*. Usually these rules will be written in the form $\ell \to r$ to improve readability. By $\mathrm{dom}(R)$ we denote the *domain* of R, which is the set $\mathrm{dom}(R) := \{\ell \in \Sigma^* \mid \exists r \in \Sigma^* : (\ell \to r) \in R\}$. The system R is called *length-reducing* if $|\ell| > |r|$ holds for each rule $(\ell \to r)$ of R.

A string-rewriting system R on Σ induces several binary relations on Σ^*, the simplest of which is the *single-step reduction relation* $\rightarrow_R$:

$$u \rightarrow_R v \text{ iff } \exists x, y \in \Sigma^* \exists (\ell \rightarrow r) \in R : u = x\ell y \text{ and } v = xry.$$

Its reflexive and transitive closure $\rightarrow_R^*$ is the *reduction relation* induced by R. If $u \rightarrow_R^* v$, then u is an *ancestor* of v and v is a *descendant* of u (mod R). A string u is called *reducible* (mod R), if there exists a string v such that $u \rightarrow_R v$ holds; otherwise, u is *irreducible.* By $\mathrm{IRR}(R)$ we denote the set of all irreducible strings, and by $\mathrm{RED}(R)$ we denote the set of all reducible strings (mod R). Obviously, $\mathrm{RED}(R) = \Sigma^* \cdot \mathrm{dom}(R) \cdot \Sigma^*$ and $\mathrm{IRR}(R) = \Sigma^* \smallsetminus \mathrm{RED}(R)$. Thus, if R is finite or left-regular, then $\mathrm{RED}(R)$ and $\mathrm{IRR}(R)$ are both regular sets. Here the system R is called *left-regular* if $\mathrm{dom}(R)$ is a regular set.

The reflexive, symmetric, and transitive closure $\leftrightarrow_R^*$ of $\rightarrow_R$ is a congruence on Σ^*, the *Thue congruence* generated by R. For $u \in \Sigma^*$, $[u]_R := \{v \in \Sigma^* \mid u \leftrightarrow_R^* v\}$ is the *congruence class* of u (mod R). The set $M_R := \{[u]_R \mid u \in \Sigma^*\}$ of congruence classes is a monoid under the operation $[u]_R \circ [v]_R = [uv]_R$ with identity $[\lambda]_R$. It is the factor monoid $\Sigma^*/\leftrightarrow_R^*$, and it is uniquely determined (up to isomorphism) by Σ and R. Accordingly, the ordered pair $(\Sigma; R)$ is called a *(monoid-) presentation* of M_R with *generators* Σ and *defining relations* R. The monoid M_R is called *finitely generated* if it has a presentation $(\Sigma; R)$ with a finite set of generators Σ, and it is called *finitely presented* if it has a finite presentation.

For $u, v \in \Sigma^*$, if $u \leftrightarrow_R^* v$, then there exists a *derivation*

$$u = w_0 \leftrightarrow_R w_1 \leftrightarrow_R \ldots \leftrightarrow_R w_m = v,$$

where $\leftrightarrow_R$ denotes the symmetric closure of the single-step reduction relation $\rightarrow_R$. The derivation above is said to be of *length m*. Let $d_R : \Sigma^* \times \Sigma^* \rightarrow \mathbb{N}$ be the function defined by

$$d_R(u,v) := \begin{cases} \min\{m \mid \exists w_0, w_1, \ldots, w_m \in \Sigma^* : \\ \qquad u = w_0 \leftrightarrow_R w_1 \leftrightarrow_R \ldots \leftrightarrow_R w_m = v\} & \text{, if } u \leftrightarrow_R^* v, \\ 0 & \text{, otherwise,} \end{cases}$$

that is, $d_R(u, v)$ is the length of the shortest derivation from u to v, if $u \leftrightarrow_R^* v$ holds. This function is called the *derivational complexity* of the string-rewriting system R.

Further, for $n \in \mathbb{N}$, we define

$$\delta_R(n) := \max\{d_R(u,v) \mid u, v \in \Sigma^*, |u| + |v| \leq n\}.$$

Then $\delta_R : \mathbb{N} \rightarrow \mathbb{N}$ is the *isoperimetric function* (or *Dehn function*) of R.

Although the Dehn function δ_R is defined explicitly for the string-rewriting system R (or the presentation $(\Sigma; R)$), it is essentially an invariant of the monoid M_R. For two functions $f, g : \mathbb{N} \rightarrow \mathbb{N}$, we write $f \leq g$ if there are positive integers α, β, and γ such that $f(n) \leq \alpha \cdot g(\beta \cdot n) + \gamma \cdot n$ holds for all $n \in \mathbb{N}$, and we call two functions $f, g : \mathbb{N} \rightarrow \mathbb{N}$ *equivalent* ($f \sim g$), if $f \leq g$ and $g \leq f$ both hold.

Proposition 2.1. [MO85, Pri95] *Let* $(\Sigma_1; R_1)$ *and* $(\Sigma_2; R_2)$ *be two finite presentations of the same monoid, that is,* $M_{R_1} \cong M_{R_2}$. *Then the Dehn functions* δ_{R_1} *and* δ_{R_2} *are equivalent.*

We are in particular interested in presentations involving string-rewriting systems of certain restricted forms. A string-rewriting system R on Σ is called

- *noetherian* if there is no infinite sequence of reduction steps of the form $u_0 \rightarrow_R u_1 \rightarrow_R u_2 \rightarrow_R \ldots \rightarrow_R u_i \rightarrow_R \ldots$;

– *weight-reducing* if there exists a *weight-function* $\varphi : \Sigma \to \mathbb{N}_+$ such that $\varphi(\ell) > \varphi(r)$ holds for each rule $(\ell \to r)$ of R, where φ is extended to a morphism from Σ^* to $\mathbb{N}$;

– *confluent*, if, for all $u, v, w \in \Sigma^*$, $u \to_R^* v$ and $u \to_R^* w$ imply that v and w have a common descendant (mod R);

– *convergent*, if it is noetherian and confluent.

If R is convergent, then each congruence class $[u]_R$ contains a unique irreducible string u_0, which thus can serve as the normal form for each string in $[u]_R$. Since u_0 can be obtained from u by a finite sequence of reduction steps, the *word problem* is decidable for each finite convergent string-rewriting system.

A presentation $(\Sigma; R)$ involving a convergent string-rewriting system R is called a *convergent presentation.* Since the decidability of the word problem is an invariant of finitely generated monoids, we have the following result.

Proposition 2.2. *If a monoid has a finite convergent presentation, then it has a decidable word problem.*

It is easily seen that a weight-reducing string-rewriting system is noetherian. In fact, if R is a finite, weight-reducing, and confluent string-rewriting system on Σ, then the word problem for R can be solved in linear time [BO93], and $\delta_R(n) \sim n$, that is, the Dehn function of R is linearly bounded.

Finally, we come to the main definition of this section, that is, the definition of an automatic structure for a monoid-presentation.

Let Σ be a finite alphabet. We will be interested in certain subsets of $\Sigma^* \times \Sigma^*$ that are accepted by *finite state acceptors* (fsa). To make this precise we proceed as follows.

Let # be an additional symbol. We define a finite alphabet $\Sigma_\#$ as

$$\Sigma_\# := ((\Sigma \cup \{\#\}) \times (\Sigma \cup \{\#\})) \smallsetminus \{(\#, \#)\}.$$

This alphabet is called the *padded extension* of Σ. A mapping $\nu : \Sigma^* \times \Sigma^* \to \Sigma_\#^*$ is then defined as follows:
if $u := a_1 a_2 \cdots a_n$ and $v := b_1 b_2 \cdots b_m$, where $a_1, \ldots, a_n, b_1, \ldots, b_m \in \Sigma$, then

$$\nu(u,v) := \begin{cases} (a_1,b_1)(a_2,b_2)\cdots(a_m,b_m)(a_{m+1},\#)\cdots(a_n,\#), & \text{if } m < n, \\ (a_1,b_1)(a_2,b_2)\cdots(a_m,b_m), & \text{if } m = n, \\ (a_1,b_1)(a_2,b_2)\cdots(a_n,b_n)(\#,b_{n+1})\cdots(\#,b_m), & \text{if } m > n. \end{cases}$$

Now a subset $L \subseteq \Sigma^* \times \Sigma^*$ will be called *synchronously regular*, if $\nu(L) \subseteq \Sigma_\#^*$ is accepted by some finite state acceptor.

An *automatic structure* for a finitely generated monoid-presentation $(\Sigma; R)$ consists of a fsa W over Σ, a fsa $M_=$ over $\Sigma_\#$, and fsa's M_a $(a \in \Sigma)$ over $\Sigma_\#$ satisfying the following conditions:

(0.) $L(W) \subseteq \Sigma^*$ is a complete set of (not necessarily unique) representatives for the monoid M_R, that is, $L(W) \cap [u]_R \neq \emptyset$ holds for each $u \in \Sigma^*$,

(1.) $L(M_=) = \{\nu(u,v) \mid u, v \in L(W) \text{ and } u \leftrightarrow_R^* v\}$, and

(2.) for all $a \in \Sigma$, $L(M_a) = \{\nu(u,v) \mid u, v \in L(W) \text{ and } ua \leftrightarrow_R^* v\}$.

The fsa W is called the *word acceptor*, $M_=$ is the *equality recognizer*, and the M_a ($a \in \Sigma$) are the *multiplier automata* for the automatic structure. The language $C := L(W)$ is a set of representatives for the monoid M_R, the language $L_= := L(M_=)$ is the *equality language*, and $L_a := L(M_a)$ is the *multiplication language* for a. A monoid-presentation is called *automatic* if it admits an automatic structure, and a monoid is called *automatic* if it has an automatic presentation.

Groups with automatic structure have been investigated thoroughly. It is known that the word problem of an automatic group can be solved in quadratic time, that its Dehn function is bounded from above by a polynomial of degree 2, that the existence of an automatic structure is independent of the chosen set of generators, and that each automatic group is finitely presented. Large classes of groups have been identified that are automatic, and various closure properties have been established for the class of automatic groups. For a detailed treatment of this class of groups see the monograph by Epstein [Eps92].

Until recently the automatic monoids did not receive much attention. This may partly be due to the fact that they do enjoy only a few of the nice properties that the automatic groups have.

Proposition 2.3. [Tho97]

(a) The word problem for an automatic monoid can be solved in quadratic time.

(b) For monoids the existence of an automatic structure depends on the chosen set of generators.

(c) An automatic monoid is not necessarily finitely presented.

Finally, an automatic structure for a monoid-presentation $(\Sigma; R)$ is a *bi-automatic structure* if there exist fsa's $\overline{M}_a$ ($a \in \Sigma$) over $\Sigma_\#$ such that,

(3.) for all $a \in \Sigma$, $L(\overline{M}_a) = \{\nu(u, v) \mid u, v \in L(W) \text{ and } au \leftrightarrow_R^* v\}$.

For $a \in \Sigma$, ${}_aL := L(\overline{M}_a)$ is the *left-multiplication language* for a.

For bi-automatic groups the conjugacy problem is decidable [Eps92], but it actually is an open problem whether or not each automatic group is in fact bi-automatic. Bi-automatic groups have been considered in detail by Gersten and Short [GS91].

3 Automatic monoids and convergent presentations

There exist finitely presented monoids, in fact groups, that admit finite convergent presentations, but that do not have an automatic structure [Ger92]. On the other hand, it is currently not known whether each automatic group admits a finite convergent presentation. Here we answer this question in the negative for the more general class of automatic monoids.

Let $\Sigma := \{a, b, c, d, A, W\}$, and let R be the following finite string-rewriting system on Σ:

$$R := \{abbc \to abc, da \to Aabb, WA \to \lambda\}.$$

By N we denote the monoid that is given through the presentation $(\Sigma; R)$.

In this section we will establish various properties of the monoid N. We begin with some simple observations.

Lemma 3.1.

(a) The string-rewriting system R is weight-reducing.

(b) The string-rewriting system R is not confluent.

Proof.

(a) Let $\varphi : \Sigma \to \mathbb{N}_+$ be the weight-function that is induced by the following assignment:

$$d \mapsto 4, \ \ s \mapsto 1 \text{ for all } s \in \{a, b, c, A, W\}.$$

Then $\varphi(\ell) > \varphi(r)$ holds for each rule $(\ell \to r) \in R$, and so R is indeed weight-reducing.

(b) Since $dabbc \to_R dabc \to_R Aab^3c$ and $dabbc \to Aab^4c$, where $Aab^3c, Aab^4c \in \mathrm{IRR}(R)$, we see that R is not confluent. □

Let R^∞ denote the string-rewriting system

$$R^\infty := R \cup \{ab^{2n}c \to ab^{2n-1}c \mid n \geq 1\}.$$

Lemma 3.2. *The string-rewriting system R^∞ is weight-reducing, confluent, left-regular, and equivalent to R.*

Proof. Obviously, R^∞ is weight-reducing with respect to the weight-function φ. The only critical pairs of R^∞ are of the form $(Aab^{2n+2}c, dab^{2n-1}c)$, since $Aab^{2n+2}c \leftarrow_{R^\infty} dab^{2n}c \to_{R^\infty} dab^{2n-1}c$. But $Aab^{2n+2}c \to_{R^\infty} Aab^{2n+1}c$ and $dab^{2n-1}c \to_{R^\infty} Aab^{2n+1}c$, and so we see that they resolve. Hence, R^∞ is indeed confluent.

Obviously, R^∞ is left-regular, since $\mathrm{dom}(R) = \{ab^{2n}c \mid n \geq 1\} \cup \{da, WA\}$ is a regular set. Finally, $\to_R \subseteq \to_{R^\infty}$, since $R \subseteq R^\infty$, and $\to_{R^\infty} \subseteq \leftrightarrow_R^*$, since, for all $n \geq 1$, $ab^{2n+2}c \leftrightarrow_R WAab^{2n+2}c \leftrightarrow_R Wdab^{2n}c \leftrightarrow_R^* Wdab^{2n-1}c \leftrightarrow_R WAab^{2n+1}c \leftrightarrow_R ab^{2n+1}c$ holds. Thus, $\leftrightarrow_R^* = \leftrightarrow_{R^\infty}^*$, that is, R and R^∞ are equivalent. □

Using the left-regular convergent string-rewriting system R^∞ each string can be reduced to its irreducible descendant. Hence, the word problem is decidable for the monoid N. Observe, however, that the results of Ó'Dúnlaing [Ó'D83] do not apply to the system R^∞, since the left-regular systems considered by Ó'Dúnlaing are of a more restricted form than R^∞. Nevertheless, the standard method of reducing a string to its irreducible descendant by applying only left-most reduction steps, which Book originally developed for finite, length-reducing, and confluent string-rewriting systems [Boo82a], can be adjusted to apply to R^∞. Here it is important to notice that the string $uab^{2n-1}c$ is irreducible mod R^∞, whenever the string uab^{2n} is. Hence, after reducing the prefix $uab^{2n}c$ of the string w considered to $uab^{2n-1}c$ the reduction process can proceed without scanning a suffix of $uab^{2n-1}c$. This yields the following result.

Corollary 3.3. *There is a linear-time algorithm that, given a string $w \in \Sigma^*$ as input, determines the irreducible descendant w_0 of w mod R^∞.*

In particular, this gives the following.

Corollary 3.4. *The word problem for the monoid N can be solved in linear time.*

The subsystem $R_0 := \{da \to Aabb, WA \to \lambda\}$ of R is convergent. Further, for each $w_1 \in \mathrm{IRR}(R_0)$, if $w_1 \to^*_{R^\infty \smallsetminus R_0} w_2$, then $w_2 \in \mathrm{IRR}(R_0)$. Hence, for each string $w \in \Sigma^*$, there is a reduction sequence of the form

$$w \to^*_{R_0} w_1 \to^*_{R^\infty \smallsetminus R_0} w_0 \in \mathrm{IRR}(R^\infty).$$

Thus, w_1 is of the form $w_1 = u_0ab^{2n_1}cu_1 \cdots u_{m-1}ab^{2n_m}cu_m$, where $u_0, u_1, \ldots, u_m \notin \Sigma^* \cdot \{ab^{2n}c \mid n \geq 1\} \cdot \Sigma^*$, and $w_0 = u_0ab^{2n_1-1}cu_1 \cdots u_{m-1}ab^{2n_m-1}cu_m$.

Since R^∞ is weight-reducing, the reduction $w \to^*_{R_0} w_1$ is of length at most $c \cdot |w|$ for some constant $c \geq 1$, and $|w_1| \leq |w| + 2 \cdot |w|_d \leq 3 \cdot |w|$. Further, we have the following technical result.

Lemma 3.5. *For all* $n \geq 1$, $ab^{2n}c \leftrightarrow^{4n-3}_R ab^{2n-1}c$.

Proof by induction on n:

$\boldsymbol{n = 1}$**:** $ab^2c \leftrightarrow_R abc$.

$\boldsymbol{n \to n+1}$**:** $ab^{2n+2}c \leftrightarrow_R WAab^{2n+2}c \leftrightarrow_R Wdab^{2n}c \leftrightarrow^{4n-3}_R Wdab^{2n-1}c$ (by the induction hypothesis) $\leftrightarrow_R WAab^{2n+1}c \leftrightarrow_R ab^{2n+1}c$, and so, $ab^{2n+2}c \leftrightarrow^{4(n+1)-3}_R ab^{2n+1}c$. □

Thus, in R the above reduction sequence from w to w_0 can be simulated as follows:

$$\begin{array}{lll} w & \to^*_{R_0} & w_1 = u_0ab^{2n_1}cu_1 \cdots u_{m-1}ab^{2n_m}cu_m \\ & \leftrightarrow^{4n_1-3}_R & u_0ab^{2n_1-1}cu_1 \cdots u_{m-1}ab^{2n_m}cu_m \\ & \cdots & \\ & \leftrightarrow^{4n_m-3}_R & u_0ab^{2n_1-1}cu_1 \cdots u_{m-1}ab^{2n_m-1}cu_m = w_0, \end{array}$$

that is, this derivation has length at most

$$c \cdot |w| + \sum_{i=1}^{m}(4n_i - 3) \leq c \cdot |w| + 4 \cdot |w_1| \leq (c + 12) \cdot |w|.$$

Since $w_0 \in \mathrm{IRR}(R^\infty)$ is the unique representative of $[w_0]_R$, this gives the following result on the Dehn function δ_R for N.

Corollary 3.6. δ_R *is linearly bounded, that is,* $\delta_R(n) \leq c_0 \cdot n$ *for some integer constant* $c_0 \geq 1$.

A string-rewriting system S is called *confluent on* $[w]_S$ for some string w if, for all $x, y, z \in [w]_S$ satisfying $x \to^*_S y$ and $x \to^*_S z$, y and z have a common descendant (mod S). The system S is called *weakly confluent* if it is confluent on $[r]_S$ for each rule $(\ell \to r) \in S$ [MNOZ93].

Although the system R is not confluent, it has at least the following weaker properties.

Lemma 3.7.

(a) The string-rewriting system R *is confluent on* $[s]_R$ *for each* $s \in \Sigma \cup \{\lambda\}$.

(b) The string-rewriting system R *is weakly confluent.*

Proof.

(a) For all rules $(\ell \to r) \in R^\infty \smallsetminus \{WA \to \lambda\}$, we have $|\ell|_a = |r|_a = 1$, while $|WA|_a = 0$. Hence, for all $u, v \in \Sigma^*$, if $u \leftrightarrow^*_{R^\infty} v$, then $|u|_a = |v|_a$. In particular, if $u \to^*_{R^\infty} v$ and $|v|_a = 0$, then $|u|_a = 0$ and $u \to^*_{\{WA\to\lambda\}} v$.

Let $s \in \{\lambda, b, c, d, A, W\}$, and let $u \in [s]_R$. Then $u \to^*_{R^\infty} s$, and hence, $|u|_a = 0$ and $u \to^*_{\{WA\to\lambda\}} s$, that is, $u \to^*_R s$. Hence, R is confluent on $[s]_R$.

Finally, observe that $|r|_b \geq 1$ holds for each rule $(\ell \to r) \in R^\infty \smallsetminus \{WA \to \lambda\}$. Hence, for all $u, v \in \Sigma^*$, if $u \to^*_{R^\infty} v$ and $|v|_b = 0$, then $u \to^*_{\{WA\to\lambda\}} v$. Thus, for $u \in [a]_R$, we see that $u \to^*_{R^\infty} a$ implies that $u \to^*_{\{WA\to\lambda\}} a$, and hence, $u \to^*_R a$. So R is confluent on $[a]_R$, too.

(b) For each rule $(\ell \to r) \in R^\infty \smallsetminus \{da \to Aabb, WA \to \lambda\}$, we have $|\ell|_c = |r|_c = 1$, while $|da|_c = |Aabb|_c = |WA|_c = 0$. Hence, for all $u, v \in \Sigma^*$, if $u \leftrightarrow^*_{R^\infty} v$, then $|u|_c = |v|_c$. In particular, if $u \to^*_{R^\infty} v$ and $|v|_c = 0$, then $|u|_c = 0$ and $u \to^*_{\{da\to Aabb, WA\to\lambda\}} v$. Thus, for $u \in [Aabb]_R$, we have $u \to^*_{R^\infty} Aabb$, since $Aabb \in \mathrm{IRR}(R^\infty)$, and so $u \to^*_R Aabb$. Hence, R is confluent on $[Aabb]_R$.

It remains to show that R is confluent on $[abc]_R$. From the proof of Lemma 3.1(b) we see that R has the critical pair $(Aab^4c, dabc)$, which is in fact the only critical pair of R. For $u, v \in \mathrm{IRR}(R)$, if $uAab^4cv \to^*_R w$, then $w = w_1 b^4 cv$ for some $w_1 \in \Sigma^*$. Hence, the set

$$\{u\#v \mid u, v \in \mathrm{IRR}(R), uAab^4cv \to^*_R abc\}$$

is empty. Analogously, for $u, v \in \mathrm{IRR}(R)$, $udabcv$ only has the direct descendant $uAab^3cv$, and hence, if $udabcv \to^*_R w \in \mathrm{IRR}(R)$, then $w = w_1 b^3 cv$ for some $w_1 \in \Sigma^*$. Hence, the set

$$\{u\#v \mid u, v \in \mathrm{IRR}(R), udabcv \to^*_R abc\}$$

is empty, too. Thus, R is confluent on $[abc]_R$ by Theorem 2.1 of Otto [Ott87]. Hence, the system R is weakly confluent. □

Based on the infinite convergent string-rewriting system R^∞ we now establish the existence of an automatic structure for the presentation $(\Sigma; R)$.

Theorem 3.8. *The monoid N is automatic.*

Proof. We consider the finite presentation $(\Sigma; R)$. Since R^∞ is left-regular, convergent, and equivalent to R, the set $C := \mathrm{IRR}(R^\infty)$ is a complete set of unique representatives of the monoid N, and in addition C is a regular set. Hence, there exist a finite state acceptor W over Σ and a finite state acceptor $M_=$ over $\Sigma_\#$ such that $L(W) = C$ and $L(M_=) = \{\nu(u, v) \mid u, v \in C \text{ and } u \leftrightarrow^*_R v\} = \{\nu(u, u) \mid u \in C\}$. It remains to verify the existence of the multiplier automata M_s for $s \in \Sigma$.

(1.) $\boldsymbol{s := a}$ **:** Let $u \in C$. Then u can uniquely be written as $u = u_1(Wd)^m$ for some $u_1 \notin \Sigma^* \cdot \{Wd\}$ and some $m \geq 0$. Hence, $ua = u_1(Wd)^m a \to^*_{R^\infty} u_1 ab^{2m}$. If u_1 does not end in d, then $u_1 ab^{2m} \in C$; otherwise, $u_1 = u_2 d$ for some $u_2 \notin \Sigma^* \cdot \{W\}$, and $u_1 ab^{2m} = u_2 dab^{2m} \to_{R^\infty} u_2 Aab^{2m+2} \in C$. Hence, the multiplier automaton M_a is to accept the language L_a, where

$$\begin{aligned} L_a \ := \ & \{\nu(u, ua) \mid ua \in C\} \cup \\ & \{\nu(u_1(Wd)^m, u_1 ab^{2m}) \mid u_1 \in C, u_1 \notin \Sigma^* \cdot \{d\}, m \geq 1\} \cup \\ & \{\nu(u_2 d(Wd)^m, u_2 Aab^{2m+2}) \mid u_2 \in C, u_2 \notin \Sigma^* \cdot \{W\}, m \geq 0\}. \end{aligned}$$

Since L_a is easily seen to be regular, a finite state acceptor M_a accepting this language clearly exists.

(2.) $\boldsymbol{s := b}$ **:** For each $u \in C$, also $ub \in C$. Hence, $L_b = \{\nu(u, ub) \mid u \in C\}$, which is obviously regular.

(3.) $\boldsymbol{s := c}$ **:** Let $u \in C$. If $u = u_1ab^{2n}$ for some $n \geq 1$, then $uc = u_1ab^{2n}c \to_{R^\infty} u_1ab^{2n-1}c \in C$. Thus,

$$\begin{aligned} L_c &= \{\nu(u_1ab^{2n}, u_1ab^{2n-1}c) \mid u_1 \in C, u_1 \notin \Sigma^* \cdot \{d\}, n \geq 1\} \cup \\ &\quad \{\nu(u, uc) \mid uc \in C\}, \end{aligned}$$

which is regular.

(4.) $\boldsymbol{s \in \{d, W\}}$ **:** For each $u \in C$, $ud \in C$ and $uW \in C$. Thus, L_d and L_W are regular as in (2.).

(5.) $\boldsymbol{s := A}$ **:** Let $u \in C$. If $u = u_1W$, then $uA = u_1WA \to_{R^\infty} u_1 \in C$; otherwise, $uA \in C$. Thus

$$L_A = \{\nu(u_1W, u_1) \mid u_1W \in C\} \cup \{\nu(u, uA) \mid uA \in C\},$$

which is certainly regular.

Thus, C and the automata M_s $(s \in \Sigma)$ for the languages L_s constitute an automatic structure for the presentation $(\Sigma; R)$ of N. □

Actually, we obtain the following stronger result.

Corollary 3.9. *The monoid N is bi-automatic.*

Proof. We show that the automatic structure above can be extended to a bi-automatic structure by providing the multiplier automata $\overline{M}_s$ $(s \in \Sigma)$ for left-multiplication.

(1.) $\boldsymbol{s := a}$ **:** Let $u \in C$. If $u = b^{2n}cu_1$ for some $n \geq 1$, then $au = ab^{2n}cu_1 \to_{R^\infty} ab^{2n-1}cu_1 \in C$. Hence,

$$\begin{aligned} {}_aL &= \{\nu(b^{2n}cu_1, ab^{2n-1}cu_1) \mid u_1 \in C, n \geq 1\} \cup \\ &\quad \{\nu(u, au) \mid au \in C\}, \end{aligned}$$

which is clearly regular.

(2.) $\boldsymbol{s \in \{b, c, A\}}$ **:** If $u \in C$, then $bu, cu, Au \in C$. Hence, ${}_bL = \{\nu(u, bu) \mid u \in C\}$ is regular, and analogously for ${}_cL$ and ${}_AL$.

(3.) $\boldsymbol{s := d}$ **:** Let $u \in C$. If $u = ab^{2n-1}u_1$ for some $n \geq 1$ such that $u_1 \notin \{b\} \cdot \Sigma^*$, then $du = dab^{2n-1}u_1 \to_{R^\infty} Aab^{2n+1}u_1 \in C$. Analogously, if $u = ab^{2n}u_1$ for some $n \geq 1$ such that $u_1 \notin \{c\} \cdot \Sigma^*$, then $du = dab^{2n}u_1 \to_{R^\infty} Aab^{2n+2}u_1 \in C$. Finally, if $u = acu_1$, then $du = dacu_1 \to_{R^\infty} Aab^2cu_1 \to_{R^\infty} Aabcu_1 \in C$. Hence,

$$\begin{aligned} {}_dL &= \{\nu(au_1, Aab^2u_1) \mid au_1 \in C, u_1 \notin \{c\} \cdot \Sigma^*\} \cup \\ &\quad \{\nu(acu_1, Aabcu_1) \mid u_1 \in C\} \cup \\ &\quad \{\nu(u, du) \mid du \in C\}, \end{aligned}$$

which is also regular.

(4.) $\boldsymbol{s := W}$ **:** For $u \in C$, if $u = Au_1$, then $Wu = WAu_1 \to_{R^\infty} u_1$. Hence,

$${}_WL = \{\nu(Au_1, u_1) \mid Au_1 \in C\} \cup \{\nu(u, Wu) \mid Wu \in C\},$$

which is also regular.

Thus, the multiplier automata $\overline{M}_s$ $(s \in \Sigma)$ do exist, and hence, $C, M_s (s \in \Sigma)$, and $\overline{M}_s (s \in \Sigma)$ give a bi-automatic structure for the presentation $(\Sigma; R)$ of N. □

Despite all the nice properties that the monoid N has, we will show in the following that it does not admit a finite convergent presentation. To this end we will prove that the monoid N does not satisfy one of the conditions that are necessary for admitting a finite convergent presentation. See the recent paper by Kobayashi and Otto [OK97] for an overview on these conditions. One of these conditions is the property of having finite derivation type, which was introduced by Squier in [SOK94]. Here we will verify that the monoid N does not have finite derivation type.

We restate the definitions in short that underly the notion of finite derivation type. In our presentation we follow the exposition in [CO94], which differs slightly from the original definitions given by Squier [SOK94].

Let $(\Sigma; S)$ be a monoid-presentation. With $(\Sigma; S)$ we associate an infinite graph $\Gamma(\Sigma; S) := (V, E, \sigma, \tau, ^{-1})$, which is defined as follows:

(a) $V := \Sigma^*$ is the set of vertices,

(b) $E := \{(u, (\ell, r), v, \varepsilon) \mid u, v \in \Sigma^*, (\ell, r) \in S, \varepsilon \in \{1, -1\}\}$ is the set of edges,

(c) $\sigma, \tau : E \to V$ are mappings, which assign an initial vertex $\sigma(e)$ and a terminal vertex $\tau(e)$ to each edge $e \in E$ as follows:

$$\sigma(u, (\ell, r), v, \varepsilon) := \left\{ \begin{array}{ll} u\ell v, & \text{if} \quad \varepsilon = 1 \\ urv, & \text{if} \quad \varepsilon = -1 \end{array} \right\}, \tau(u, (\ell, r), v, \varepsilon) := \left\{ \begin{array}{ll} urv, & \text{if} \quad \varepsilon = 1 \\ u\ell v, & \text{if} \quad \varepsilon = -1 \end{array} \right\},$$

(d) and $^{-1} : E \to E$ is a mapping, which assigns an inverse edge $e^{-1} \in E$ to each edge $e \in E$ as follows:

$$(u, (\ell, r), v, \varepsilon)^{-1} := (u, (\ell, r), v, -\varepsilon).$$

By $P(\Gamma(\Sigma; S))$ we denote the set of all paths in $\Gamma(\Sigma; S)$, where a path (w) of length 0 from w to w is added for each $w \in V$. The mappings σ, τ, and $^{-1}$ can easily be extended to paths. Further, $\circ$ denotes the concatenation of paths. The free monoid Σ^* induces a two-sided action on $\Gamma(\Sigma; S)$ through $xey := (xu, (\ell, r), vy, \varepsilon)$ for all $x, y \in \Sigma^*$ and $e = (u, (\ell, r), v, \varepsilon) \in E$. Obviously, this action can be extended to a two-sided action on $P(\Gamma(\Sigma; S))$. Finally, by $P^{(2)}(\Gamma(\Sigma; S))$ we denote the set of all pairs of parallel paths in $\Gamma(\Sigma; S)$, that is,

$$P^{(2)}(\Gamma(\Sigma; S)) := \{(p, q) \mid p, q \in P(\Gamma(\Sigma; S)), \sigma(p) = \sigma(q), \text{ and } \tau(p) = \tau(q)\}.$$

Next we consider certain equivalence relations on $P(\Gamma(\Sigma; S))$ that are called homotopy relations. Let

$$\begin{array}{rcl} D & := & \Big\{((\lambda, (\ell_1, r_1), xu_2, \varepsilon_1) \circ (v_1 x, (\ell_2, r_2), \lambda, \varepsilon_2), \\ & & \quad (u_1 x, (\ell_2, r_2), \lambda, \varepsilon_2) \circ (\lambda, (\ell_1, r_1), xv_2, \varepsilon_1)) \mid \\ & & x \in \Sigma^*, \varepsilon_1, \varepsilon_2 \in \{1, -1\}, (\ell_1, r_1), (\ell_2, r_2) \in S \text{ satisfying} \\ & & u_i = \left\{ \begin{array}{ll} \ell_i, & \text{if} \quad \varepsilon_i = 1 \\ r_i, & \text{if} \quad \varepsilon_i = -1 \end{array} \right\}, \text{ and } \; v_i = \left\{ \begin{array}{ll} r_i, & \text{if} \quad \varepsilon_i = 1 \\ \ell_i, & \text{if} \quad \varepsilon_i = -1 \end{array} \right\}, \; i = 1, 2\Big\}, \end{array}$$

which is called the set of *disjoint derivations*, and let

$$I := \{(e \circ e^{-1}, (w)) \mid e \text{ is an edge with } \sigma(e) = w, w \in \Sigma^*\},$$

which is the set of *inverse derivations*. Notice that $D \cup I \subseteq P^{(2)}(\Gamma(\Sigma; S))$ holds. Now an equivalence relation $\simeq \subseteq P^{(2)}(\Gamma(\Sigma; S))$ is called a *homotopy relation* if it satisfies the following conditions:

(a) $D \cup I \subseteq \simeq$,

(b) if $p \simeq q$, then $xpy \simeq xqy$ for all $x, y \in \Sigma^*$, and

(c) if $p, q_1, q_2, r \in P(\Gamma(\Sigma; S))$ satisfy $\tau(p) = \sigma(q_1) = \sigma(q_2)$, $\tau(q_1) = \tau(q_2) = \sigma(r)$, and $q_1 \simeq q_2$, then $p \circ q_1 \circ r \simeq p \circ q_2 \circ r$.

The collection of all homotopy relations on $P(\Gamma(\Sigma; S))$ is closed under arbitrary intersection. Since $P^{(2)}(\Gamma(\Sigma; S))$ itself is a homotopy relation, each subset $B \subseteq P^{(2)}(\Gamma(\Sigma; S))$ is contained in a smallest homotopy relation $\simeq_B$ on $P(\Gamma(\Sigma; S))$, which is accordingly called the homotopy relation *generated* by B.

The monoid-presentation $(\Sigma; S)$ is said to have *finite derivation type*, FDT for short, if there exists a finite subset $B \subseteq P^{(2)}(\Gamma(\Sigma; S))$ which generates $P^{(2)}(\Gamma(\Sigma; S))$ as a homotopy relation.

Squier has established the following results on the property FDT.

Proposition 3.10. [SOK94]

(a) If $(\Sigma_1; S_1)$ and $(\Sigma_2; S_2)$ are two finite presentations of the same monoid, then $(\Sigma_1; S_1)$ has FDT if and only if $(\Sigma_2; S_2)$ has FDT.

(b) If M has a finite convergent presentation, then M has FDT.

Because of part (a) of this proposition the property FDT is actually a property of finitely presented monoids.

We claim that the monoid N does not have FDT. In [SOK94] Squier presents a finitely presented monoid with a decidable word problem that does not have FDT. Our proof will follow the one given by Squier.

Theorem 3.11. *The monoid N does not have FDT.*

Proof. Let Γ denote the graph that is associated with the finite presentation $(\Sigma; R)$, and let Γ_∞ denote the graph that is associated with the presentation $(\Sigma; R^\infty)$. Obviously, Γ is a subgraph of Γ_∞. In order to simplify the notation we introduce names for the rules of R^∞:

$$\begin{array}{lrcl} (r_a) & da & \rightarrow & Aabb \\ (r_A) & WA & \rightarrow & \lambda \\ (P_n) & ab^{2n}c & \rightarrow & ab^{2n-1}c, \ n \geq 1. \end{array}$$

In R^∞ we have the following critical pairs:

$$Aab^{2n+2}c \leftarrow_{r_a} dab^{2n}c \rightarrow_{P_n} dab^{2n-1}c \ (n \geq 1).$$

These pairs can be resolved as follows:

$$Aab^{2n+2}c \rightarrow_{P_{n+1}} Aab^{2n+1}c \leftarrow_{r_a} dab^{2n-1}c \ (n \geq 1).$$

Let B_1 be the following set of pairs of parallel paths in Γ_∞:

$$B_1 := \{(r_a \cdot b^{2n}c \circ A \cdot P_{n+1}, d \cdot P_n \circ r_a \cdot b^{2n-1}c) \mid n \geq 1\},$$

where the names of the rules of R^∞ are also used to denote the corresponding edges of Γ_∞. From the results of Squier [SOK94] it follows that $\simeq_{B_1} = P^{(2)}(\Gamma_\infty)$.

For each $n \geq 1$, the rule P_{n+1} is a consequence of the rules r_a, r_A, and P_n:

$$abbb^{2n}c \leftarrow_{r_A} WAabbb^{2n}c \leftarrow_{r_a} Wdab^{2n}c \rightarrow_{P_n} Wdab^{2n-1}c \rightarrow_{r_a} WAab^{2n+1}c \rightarrow_{r_A} ab^{2n+1}c.$$

Accordingly, for all $n \geq 1$, we can define a path $\tilde{P}_n$ inductively as follows:

$$\begin{array}{lcl} \tilde{P}_1 & := & P_1, \\ \tilde{P}_{n+1} & := & r_A^{-1} \cdot ab^{2n+2}c \circ W \cdot r_a^{-1} \cdot b^{2n}c \circ Wd \cdot \tilde{P}_n \circ W \cdot r_a \cdot b^{2n-1}c \circ r_A \cdot ab^{2n+1}c, \ n \geq 1. \end{array}$$

Observe that $\tilde{P}_n \in P(\Gamma)$, and that $(P_n, \tilde{P}_n) \in P^{(2)}(\Gamma_\infty)$ hold for all $n \geq 1$. We define a mapping f from Γ_∞ to Γ as follows. On the subgraph $\Gamma \subseteq \Gamma_\infty$, f is the identity. On the additional edges of Γ_∞, which are of the form $u \cdot P_n \cdot v$ and $u \cdot P_n^{-1} \cdot v$, $u, v \in \Sigma^*$, $n \geq 2$, f is defined by

$$\left.\begin{array}{lcl} f(u \cdot P_n \cdot v) & := & u \cdot \tilde{P}_n \cdot v \\ f(u \cdot P_n^{-1} \cdot v) & := & u \cdot \tilde{P}_n^{-1} \cdot v \end{array}\right\} \text{for all } u, v \in \Sigma^*, \text{ and } n \geq 2.$$

Then $f : \Gamma_\infty \rightarrow \Gamma$ is a mapping of graphs, and it follows from [SOK94], Corollary 3.7, that $\simeq_B = P^{(2)}(\Gamma)$ holds, where $B := \{(f(p), f(q)) \mid (p,q) \in B_1\}$. Thus, B is an infinite homotopy basis for $P^{(2)}(\Gamma)$.

Now assume that the monoid N does have FDT. Then there exists a finite subset $B_0 \subseteq B$ such that B_0 is already a homotopy basis for $P^{(2)}(\Gamma)$, that is, $\simeq_{B_0} = P^{(2)}(\Gamma)$. Let C_1 denote the following set of pairs of parallel paths in Γ_∞:

$$C_1 := \{(P_n, \tilde{P}_n) \mid n \geq 2\},$$

and let $C_2 := C_1 \cup B_0$. For $(p,q) \in P^{(2)}(\Gamma_\infty)$,

$$p \simeq_{C_1} f(p) \simeq_{B_0} f(q) \simeq_{C_1} q,$$

and so $\simeq_{C_2} = P^{(2)}(\Gamma_\infty)$. Since $f(B_1) = B \supset B_0$, we can choose a subset B_0' of B_1 as follows:

$$B_0' := \{(r_a \cdot b^{2n}c \circ A \cdot P_{n+1}, d \cdot P_n \circ r_a \cdot b^{2n-1}c) \mid (r_a \cdot b^{2n}c \circ A \cdot \tilde{P}_{n+1}, d \cdot \tilde{P}_n \circ r_a \cdot b^{2n-1}c) \in B_0\}.$$

For $(p,q) \in B_0$, $p \simeq_{B_0' \cup C_1} q$, and hence, $C_2' := C_1 \cup B_0'$ is another homotopy basis for $P^{(2)}(\Gamma_\infty)$.

Finally, let

$$D_1 := \{(P_{n+1}, r_A^{-1} \cdot ab^{2n+2}c \circ W \cdot r_a^{-1} \cdot b^{2n}c \circ Wd \cdot P_n \circ W \cdot r_a \cdot b^{2n-1}c \circ r_A \cdot ab^{2n+1}c) \mid n \geq 1\},$$

and let $D_2 := D_1 \cup B_0'$. Then $P_n \simeq_{D_2} \tilde{P}_n$ for all $n \geq 1$, and so $\simeq_{D_2} = P^{(2)}(\Gamma_\infty)$.

Next we decompose the system R^∞ into two parts:

$$R' := R^\infty \setminus \{(r_a)\ da \rightarrow Aabb\} \text{ and } R'' := \{(r_a)\ da \rightarrow Aabb\}.$$

By N' we denote the monoid that is given through the presentation $(\Sigma; R')$, and by $g : \Sigma^* \rightarrow N'$ we denote the corresponding quotient morphism. On N' we consider the binary relation that is induced by $R_{N'} := \{(da, Aabb)\}$, and we introduce the graph $\Gamma_{N'} := \Gamma(N', R_{N'})$ that is associated with $(N', R_{N'})$. The vertices of $\Gamma_{N'}$ are the elements of the monoid N', and its edges are of the form $(g(u), (da, Aabb), g(v), \varepsilon)$ with $u, v \in \Sigma^*$ and $\varepsilon \in \{1, -1\}$. In order to simplify the notation this edge will be denoted as $g(u) \cdot r_a^\varepsilon \cdot g(v)$ or even as $u \cdot r_a^\varepsilon \cdot v$.

The morphism $g : \Sigma^* \rightarrow N'$ can be extended to a mapping from Γ_∞ to $\Gamma_{N'}$. If $e = (u, (\ell \rightarrow r), v, \varepsilon)$ is an edge of Γ_∞ such that $(\ell \rightarrow r) \in R'$, then $g(\sigma(e)) = g(\tau(e))$ in N', and so $g(e) := (g(u\ell v))$ is just the corresponding path of length 0. If $e = (u, (da \to Aabb), v, \varepsilon)$, then $g(e) := (g(u), (da, Aabb), g(v), \varepsilon) = g(u) \cdot r_a^\varepsilon \cdot g(v)$ is an edge of $\Gamma_{N'}$.

In $\Gamma_{N'}$ we choose a set of pairs of parallel paths as follows:

$$E_{N'} := \{(g(p), g(q)) \mid (p, q) \in D_2\},$$

that is,

$$\begin{array}{rcl} E_{N'} & = & \{((ab^{2n+1}c), W \cdot r_a^{-1} \cdot b^{2n}c \circ W \cdot r_a \cdot b^{2n-1}c) \mid n \geq 1\} \\ & \cup & \{(r_a \cdot b^{2n}c, r_a \cdot b^{2n-1}c) \mid (r_a \cdot b^{2n}c \circ A \cdot \tilde{P}_{n+1}, d \cdot \tilde{P}_n \circ r_a \cdot b^{2n-1}c) \in B_0\}. \end{array}$$

Claim 1: $\simeq_{E_{N'}} = P^{(2)}(\Gamma_{N'})$.

Proof. Let $(p, q) \in P^{(2)}(\Gamma_{N'})$. Choose $w_1, w_2 \in \Sigma^*$ such that $g(w_1) = \sigma(p)(= \sigma(q))$ and $g(w_2) = \tau(p)(= \tau(q))$. Then $w_1 \leftrightarrow_{R^\infty}^* w_2$, and there are paths $p_1, q_1 \in P(\Gamma_\infty)$, both leading from w_1 to w_2, such that $g(p_1) = p$ and $g(q_1) = q$. Hence, $(p_1, q_1) \in P^{(2)}(\Gamma_\infty)$, and thus, $p_1 \simeq_{D_2} q_1$. By [SOK94], Corollary 3.7, this yields $p = g(p_1) \simeq_{E_{N'}} g(q_1) = q$. Hence, $E_{N'}$ is a homotopy basis for $P^{(2)}(\Gamma_{N'})$. □

Let $I := \{n \geq 1 \mid (r_a \cdot b^{2n}c \circ A \cdot \tilde{P}_{n+1}, d \cdot \tilde{P}_n \circ r_a \cdot b^{2n-1}c) \in B_0\}$, that is, I is a finite subset of $\mathbb{N}$. Let $m \in \mathbb{N}$ be an index such that $m \notin I$. By E_m we denote the following subset of $P^{(2)}(\Gamma_{N'})$:

$$\begin{array}{rcl} E_m & := & \{((ab^{2m+1}c), W \cdot r_a^{-1} \cdot b^{2m}c \circ W \cdot r_a \cdot b^{2m-1}c)\} \\ & \cup & \{(r_a \cdot b^{2n}c, r_a \cdot b^{2n-1}c) \mid n \in \mathbb{N} \smallsetminus \{m\}\}. \end{array}$$

Since $r_a \cdot b^{2n}c \simeq_{E_m} r_a \cdot b^{2n-1}c$ holds for all $n \in \mathbb{N} \smallsetminus \{m\}$, it follows that

$$(ab^{2n+1}c) \simeq_{E_m} W \cdot r_a^{-1} \cdot b^{2n}c \circ W \cdot r_a \cdot b^{2n}c \simeq_{E_m} W \cdot r_a^{-1} \cdot b^{2n}c \circ W \cdot r_a \cdot b^{2n-1}c$$

holds for all $n \geq 1$. Hence, $E_{N'} \subseteq \simeq_{E_m}$, which in turn implies that $\simeq_{E_m} = P^{(2)}(\Gamma_{N'})$ by Claim 1.

This statement will now lead to a contradiction. Let $h : N' \to N$ denote the quotient morphism, and let T_1 and T_2 denote the following subsets of N:

$$\begin{array}{rcl} T_1 & := & N \cdot W, \text{ and} \\ T_2 & := & b^{2m}c \cdot N. \end{array}$$

Then, for $u \in \Sigma^*$, $[u]_R \in T_1$ if and only if there exists some $u_1 \in \Sigma^*$ such that $u \leftrightarrow_R^* u_1W$, and $[u]_R \in T_2$ if and only if $u \leftrightarrow_R^* b^{2m}cu_1$ for some $u_1 \in \Sigma^*$.

Since the letter W does not occur as a suffix of the left-hand side of any rule of R^∞, the irreducible descendant of a string of the form u_1W has necessarily the form $u_1'W$. Since W does not occur in the right-hand side of any rule of R^∞, we see that, for $u \in \Sigma^*$, $[u]_R \in T_1$ holds if and only if $u = u_1Wu_2$ for some $u_1, u_2 \in \Sigma^*$ satisfying $u_2 \to_{R^\infty}^* \lambda$. Analogously, since no non-empty suffix of $b^{2m}c$ is a prefix of the left-hand side of any rule of R^∞, we have, for all $u \in \Sigma^*$, $[u]_R \in T_2$ if and only if the normal form u_0 of u mod R^∞ has the prefix $b^{2m}c$. Thus, $[b^{2m}c]_R \in T_2$, but $[b^ncv]_R \notin T_2$ for all $n \in \mathbb{N} \smallsetminus \{2m\}$ and all $v \in \Sigma^*$.

Finally, we define a mapping $h_T : P(\Gamma_{N'}) \to \mathbb{N}$ as follows:

$h_T(p)$:= the number of edges $e = (g(u), (da, Aabb), g(v), \varepsilon)$ of p for which $h(g(u)) \notin T_1$ and $h(g(v)) \in T_2$ hold.

Claim 2: For all $(p, q) \in P^{(2)}(\Gamma_{N'})$, if $p \simeq_{E_m} q$, then $h_T(p) \equiv h_T(q)$ mod 2.

Proof. It suffices to show that $h_T(p) \equiv h_T(q)$ mod 2 holds for all pairs (p, q) of the form $(p, q) = (g(u)p_1g(v), g(u)q_1g(v)) \in P^{(2)}(\Gamma_{N'})$, where $(p_1, q_1) \in D \cup I \cup E_m$ and $u, v \in \Sigma^*$.

(1.) For $(p_1, q_1) = (e \circ e^{-1}, (\sigma(e)))$, where e is an edge of $\Gamma_{N'}$, the result is trivially true.

(2.) Let $(p_1, q_1) = ((g(\lambda), (da \to Aabb), g(xu_2), \varepsilon_1) \circ (g(v_1x), (da \to Aabb), g(\lambda), \varepsilon_2),$
$(g(u_1x), (da \to Aabb), g(\lambda), \varepsilon_2) \circ (g(\lambda), (da \to Aabb), g(xv_2), \varepsilon_2)),$

where $u_i = \left\{ \begin{array}{ll} da & \text{if } \varepsilon_i = 1 \\ Aabb & \text{if } \varepsilon_i = -1 \end{array} \right\}$ and $v_i = \left\{ \begin{array}{ll} Aabb & \text{if } \varepsilon_i = 1 \\ da & \text{if } \varepsilon_i = -1 \end{array} \right\}$, $i = 1, 2$,

and $x \in \Sigma^*$. Then

$$h_T((g(u), (da \to Aabb), g(xu_2v), \varepsilon_1)) = h_T((g(u), (da \to Aabb), g(xv_2v), \varepsilon_2))$$

and

$$h_T((g(uv_1x), (da \to Aabb), g(v), \varepsilon_2)) = h_T((g(uu_1x), (da \to Aabb), g(v), \varepsilon_2)),$$

since $xu_2v \leftrightarrow_R^* xv_2v$ and $uv_1x \leftrightarrow_R^* uu_1x$ hold. Thus, $h_T(p) = h_T(q)$.

(3.) If $(p_1, q_1) \in E_m$, we have to consider two subcases:

(i) If $(p_1, q_1) = ((ab^{2m+1}c), W \cdot r_a^{-1} \cdot b^{2m}c \circ W \cdot r_a \cdot b^{2m-1}c)$, then
$h_T(p) = h_T(g(u)p_1g(v)) = h_T((uab^{2m+1}cv)) = 0$, and
$h_T(q) = h_T(g(u)q_1g(v)) = h_T(g(uW) \cdot r_a^{-1} \cdot g(b^{2m}cv) \circ g(uW) \cdot r_a \cdot g(b^{2m-1}cv)) = 0,$

since $h(g(uW)) \in T_1$.

(ii) If $(p_1, q_1) = (r_a \cdot b^{2n}c, r_a \cdot b^{2n-1}c)$ for some $n \in \mathbb{N} \setminus \{m\}$, then

$$h_T(p) = h_T(g(u)p_1g(v)) = h_T(g(u) \cdot r_a \cdot g(b^{2n}cv)) = 0,$$

since $h(g(b^{2n}cv)) \notin T_2$, and

$$h_T(q) = h_T(g(u)q_1g(v)) = h_T(g(u) \cdot r_a \cdot g(b^{2n-1}cv)) = 0,$$

since $h(g(b^{2n-1}cv)) \notin T_2$. □

To conclude we consider the paths $p := r_a \cdot b^{2m}c$ and $q := r_a \cdot b^{2m-1}c$ in $\Gamma_{N'}$. We have $\sigma(p) = g(dab^{2m-1}c) = \sigma(q)$ and $\tau(p) = g(Aab^{2m+1}c) = \tau(q)$ in $\Gamma_{N'}$, that is, $(p, q) \in P^{(2)}(\Gamma_{N'})$. Since $h(g(\lambda)) \notin T_1$ and $h(g(b^{2m}c)) \in T_2$, we have $h_T(p) = 1$, while $h(g(b^{2m-1}c)) \notin T_2$ implies that $h_T(q) = 0$. Thus, $(p, q) \notin \simeq_{E_m}$, contradicting the above statement that $\simeq_{E_m} = P^{(2)}(\Gamma_{N'})$ holds. Hence, the monoid N does not have finite derivation type. □

Because of Proposition 3.10(b) this result has the following consequence.

Corollary 3.12. *The monoid N does not have a finite convergent presentation.*

Hence, we see that the class of monoids that have a finite convergent presentation and the class of finitely presented monoids that are (bi)-automatic are incomparable under inclusion. Further, this result shows that a monoid with an easily decidable word problem need not have finite derivation type even if it has a finite, noetherian, and weakly confluent presentation. Hence, Proposition 3.10(b) cannot be strengthened to this class of presentations.

4 Automatic monoids and Dehn functions

As mentioned before the Dehn function of an automatic group is bounded by a quadratic function. The Dehn function of the automatic monoid N considered in the previous section is even bounded by a linear function. Here, however, we will show that Dehn functions of automatic monoids are in general not bounded by polynomial functions at all. For this we present a finitely presented (bi-)automatic monoid the Dehn function of which grows exponentially.

Let $\Sigma = \{0, 1, ¢, \$\}$, and let R consist of the following 4 rules:

$$\begin{array}{rcl} 0\$ & \rightarrow & 1\$, \\ 1\$ & \rightarrow & ¢\$, \\ 0¢ & \rightarrow & 10, \\ 1¢ & \rightarrow & ¢0. \end{array}$$

Lemma 4.1. *The system R is convergent.*

Proof. If we order the alphabet Σ by taking $\$ > 0 > 1 > ¢$, then we see that with respect to the length-lexicographical ordering $>_{\text{lex}}$ induced by $>$, $\ell >_{\text{lex}} r$ holds for each rule $(\ell \rightarrow r) \in R$. Hence, R is noetherian.

The left-hand sides of the rules of R do not overlap. Hence, R does not admit any critical pairs, that is, R is also confluent. □

Thus, the set $\text{IRR}(R)$ of irreducible strings is a regular set of unique representatives of the monoid M presented by $(\Sigma; R)$. Obviously, $\text{IRR}(R) = \{\$, ¢\}^* \cdot \{0,1\}^*$. Thus, for all $n \geq 1$, $¢^n\$$ is the only irreducible element in the set $\{0, 1, ¢\}^n \cdot \{\$\}$. Since R is convergent and length-preserving, this immediately yields the following.

Lemma 4.2. *For all $u \in \{0, 1, ¢\}^*$, we have $u\$ \rightarrow_R^* ¢^{|u|}\$$.*

Based on this observation we now establish the following result.

Lemma 4.3. *The presentation $(\Sigma; R)$ is bi-automatic.*

Proof. We take C to be the set $\text{IRR}(R)$. Then

$$\begin{array}{rcl} L_= & = & \{\nu(w_1, w_2) \mid w_1, w_2 \in IRR(R),\ w_1 \leftrightarrow_R^* w_2\} \\ & = & \{\nu(w, w) \mid w \in \text{IRR}(R)\}, \ \text{which is obviously regular.} \end{array}$$

For $u \in \text{IRR}(R)$, we have $u0, u1 \in \text{IRR}(R)$. Hence,

$$\begin{array}{rcl} L_0 & = & \{\nu(w_1, w_2) \mid w_1, w_2 \in \text{IRR}(R),\ w_1 0 \leftrightarrow_R^* w_2\} \\ & = & \{\nu(u, u0) \mid u \in \text{IRR}(R)\}, \ \text{and} \\ L_1 & = & \{\nu(u, u1) \mid u \in \text{IRR}(R)\}, \ \text{which are regular, too.} \end{array}$$

Now let $u = u_1 u_2$, where $u_1 \in \{\$, ¢\}^*$ and $u_2 \in \{0, 1\}^*$. Then

$$u¢ \rightarrow_R^* \begin{cases} u_1 ¢ & \text{if } u_2 = \lambda, \\ u_1 ¢ 0^i & \text{if } u_2 = 1^i \quad (i \geq 1), \\ u_1 u_3 1 0^{i+1} & \text{if } u_2 = u_3 0 1^i \quad (i \geq 0), \end{cases}$$

and

$$u\$ \rightarrow_R^* \begin{cases} u_1 \$ & \text{if } u_2 = \lambda, \\ u_1 ¢^{|u_2|} \$ & \text{if } u_2 \neq \lambda \quad \text{(cf. Lemma 4.2).} \end{cases}$$

It is easily seen that the sets $L_{¢}$ and $L_{\$}$ are regular.

Further, $¢u, \$u \in \mathrm{IRR}(R)$,

$$0u \rightarrow_R^* \begin{cases} 0u_2 & \text{if } u_1 = \lambda, \\ 1^i 0u_2 & \text{if } u_1 = ¢^i \quad (i \geq 1), \\ ¢u_1u_2 & \text{if } |u_1|_\$ > 0 \quad \text{(cf. Lemma 4.2)} \end{cases}$$

and

$$1u \rightarrow_R^* \begin{cases} 1u_2 & \text{if } u_1 = \lambda, \\ ¢1^{i-1}0u_2 & \text{if } u_1 = ¢^i \quad (i \geq 1), \\ ¢u_1u_2 & \text{if } |u_1|_\$ > 0 \quad \text{(cf. Lemma 4.2).} \end{cases}$$

Hence, also the sets ${}_{¢}L$, ${}_{\$}L$, ${}_{0}L$, and ${}_{1}L$ are regular. Thus, $(\Sigma; R)$ is indeed bi-automatic. □

There are no overlaps between the left-hand sides of the rules of R. Thus, if $u = u_1\ell u_2$ for some $u_1, u_2 \in \Sigma^*$ and some rule $(\ell \rightarrow r) \in R$, and if $u = w_0 \rightarrow_R w_1 \rightarrow_R \ldots \rightarrow_R w_m \in \mathrm{IRR}(R)$ is a reduction sequence, then there exists an index $k \geq 0$ such that the following conditions are satisfied:

$$w_k = v_1\ell v_2, \; w_{k+1} = v_1 r v_2, \text{ and } u_i \rightarrow_R^* v_i, \;\; i = 1, 2.$$

Based on this observation we will now establish the following result.

Lemma 4.4. *Let $u \in \Sigma^*$ and $u_0 \in \mathrm{IRR}(R)$ such that $u \leftrightarrow_R^* u_0$. Then there exists an integer $n_u \in \mathbb{N}$ such that each reduction sequence $u = w_0 \rightarrow_R w_1 \rightarrow_R \ldots \rightarrow_R u_0$ has length n_u.*

Proof. If u is irreducible, then $u = u_0$ and $n_u = 0$. So let us assume that u is reducible. For n_u we choose the length of a shortest reduction sequence that reduces u to u_0.

Claim: Every reduction sequence $u = w_0 \rightarrow_R \ldots \rightarrow_R u_0$ has length n_u.

Proof. Assume that there is a reduction sequence $u = w_0 \rightarrow_R \ldots \rightarrow_R w_m \rightarrow_R u_0$ that is of length larger than n_u. Choose a reduction sequence $u = v_0 \rightarrow_R v_1 \rightarrow_R \ldots \rightarrow_R v_{n_u} = u_0$ of length n_u such that the length of the common initial part of these two sequences is as large as possible, that is, there is an index $j \geq 0$ such that, for $i = 0, 1, \ldots, j$, $w_i = v_i$, but $w_{j+1} \neq v_{j+1}$. Since $w_j = v_j$, this means that $w_j = v_j$ can be factored as $w_j = v_j = f\ell_1 g\ell_2 h$ for some $f, g, h \in \Sigma^*$ and some rules $(\ell_i \rightarrow r_i) \in R$, $i = 1, 2$, where $w_{j+1} = fr_1g\ell_2h$ and $v_{j+1} = f\ell_1gr_2h$, or vice versa, $w_{j+1} = f\ell_1gr_2h$ and $v_{j+1} = fr_1g\ell_2h$. Let us assume the former. By the observation above there exists an index $k \geq j+1$ such that $v_k = f'\ell_1g' \rightarrow_R f'r_1g' = v_{k+1}$, where $f \rightarrow_R^* f'$ and $gr_2h \rightarrow_R^* g'$ describe the effect of the subsequence $v_{j+1} \rightarrow_R^* v_k$. Now we can construct another reduction sequence as follows:

$$\begin{aligned} u \;\; = \;\; & v_0 \rightarrow_R \ldots \rightarrow_R v_j = f\ell_1g\ell_2h \rightarrow_R fr_1g\ell_2h \rightarrow_R fr_1gr_2h \rightarrow_R^* \\ & f'r_1g' = v_{k+1} \rightarrow_R v_{k+2} \rightarrow_R \ldots \rightarrow_R v_{n_u} = u_0. \end{aligned}$$

This sequence has length n_u, and its initial part of length $j+1$ coincides with the corresponding initial part of the given reduction sequence $u = w_0 \rightarrow_R \ldots \rightarrow_R w_m \rightarrow_R u_0$, contradicting the choice of the sequence $u = v_0 \rightarrow_R \ldots \rightarrow_R v_{n_u} = u_0$. Thus, all reduction sequences from u to u_0 have length n_u. □□

For $n \in \mathbb{N}$, let $f(n)$ denote the length of the reduction sequences that reduce the string $0^n\$$ to its irreducible representative $¢^n\$$.

Lemma 4.5. $f(n) = 2^{n+1} + \sum_{j=1}^{n-1} j \cdot 2^{n-j} - 2$ *for all* $n \geq 1$.

Proof. First we describe reduction sequences from $0^n\$$ to $¢^n\$$ inductively:

$$n = 1: \qquad 0\$ \to_R 1\$ \to_R ¢\$, \text{ that is, } f(1) = 2.$$

$$n \to n+1: \quad 0^{n+1}\$ = 00^n\$ \to_R^{f(n)} 0¢^n\$ \to_R^n 1^n0\$ \to_R^2 1^n¢\$ \to_R^n ¢0^n\$ \to_R^{f(n)} ¢¢^n\$ = ¢^{n+1}\$, \text{ that is, } f(n+1) = 2 \cdot f(n) + 2n + 2.$$

Based on this recursion we now prove the statement about $f(n)$ by induction:

$$n = 1: \qquad f(1) = 2 = 2^2 - 2$$

$$\begin{aligned} n \to n+1: \quad f(n+1) &= 2 \cdot f(n) + 2n + 2 = 2^{n+2} + 2 \cdot \sum_{j=1}^{n-1} j \cdot 2^{n-j} - 4 + 2n + 2 \\ &= 2^{n+2} + \sum_{j=1}^{n-1} j \cdot 2^{n+1-j} + 2n - 2 \\ &= 2^{n+2} + \sum_{j=1}^{n} j \cdot 2^{n+1-j} - 2. \end{aligned}$$

□

Finally, we complete our investigation of the presentation $(\Sigma; R)$ by proving the following result on the distance function d_R.

Lemma 4.6. *$d_R(u\$, ¢^n\$) = n_{u\$}$ for all $u \in \{0, 1, ¢\}^n$, $n \geq 0$.*

Proof. By Lemma 4.2 $u\$ \leftrightarrow_R^* ¢^n\$$ for all $u \in \{0, 1, ¢\}^n$, $n \geq 0$, and by Lemma 4.4, $u\$ = v_0 \to_R \ldots \to_R v_m = ¢^n\$$ implies that $m = n_{u\$}$.

Now let $u\$ = w_0 \leftrightarrow_R \ldots \leftrightarrow_R w_\ell = ¢^n\$$ be a derivation of minimal length from $u\$$ to $¢^n\$$. We claim that this derivation is in fact a reduction sequence, which then yields $\ell = d_R(u\$, ¢^n\$) = n_{u\$}$.

Assume to the contrary that, for some index j, we have $w_j \leftarrow_R w_{j+1}$, and let this index j be chosen maximal. Then $u\$ = w_0 \leftrightarrow_R^j w_j \leftarrow_R w_{j+1} \to_R w_{j+2} \to_R \ldots \to_R w_\ell = ¢^n\$$. Observe that $j + 1 < \ell$, since $w_\ell = ¢^n\$$ is irreducible. Since R is convergent, there is a reduction sequence

$$w_j = g_0 \to_R g_1 \to_R \ldots \to_R g_k = ¢^n\$.$$

Hence,

$$w_{j+1} \to_R w_{j+2} \to_R \ldots \to_R w_\ell = ¢^n\$ \quad \text{and} \quad w_{j+1} \to_R w_j \to_R g_1 \to_R \ldots \to_R g_k = ¢^n\$$$

are two reduction sequences from w_{j+1} to $¢^n\$$. By Lemma 4.4 they both have the same length $\ell - j - 1$, which implies that $k = \ell - j - 2$. Hence, $u\$ = w_0 \leftrightarrow_R^j w_j \to_R g_1 \to_R \ldots \to_R g_k = ¢^n\$$ is a derivation from $u\$$ to $¢^n\$$ that has length $j + k = \ell - 2$. This obviously contradicts the choice of the original derivation. Thus, the derivations of minimal length from $u\$$ to $¢^n\$$ are in fact reduction sequences. □

Combining Lemma 4.5 and Lemma 4.6 we obtain the following result.

Corollary 4.7.

a) $d_R(0^n\$, ¢^n\$) = 2^{n+1} + \sum_{j=1}^{n-1} j \cdot 2^{n-j} - 2$ *for all* $n \geq 1$.

b) $\delta_R(2n+2) \geq 2^{n+1} + \sum_{j=1}^{n-1} j \cdot 2^{n-j} - 2$ *for all* $n \geq 1$.

Thus, we can summarize our results as follows.

Theorem 4.8. *There exists a finite convergent monoid-presentation that is bi-automatic such that the corresponding Dehn function grows exponentially.*

5 Concluding remarks

While the monoids with automatic structure share some of the nice properties of the groups with automatic structure [Tho97], we have seen that this is not true for the rate of growth of their Dehn functions. While the Dehn function of an automatic group only grows quadratically, that of an automatic monoid may grow exponentially. This is true even if the finitely presented monoid considered is bi-automatic, and if it is given through a finite convergent presentation.

On the other hand, we have seen that a bi-automatic monoid may not have a finite convergent presentation even if it is finitely presented, has an easily decidable word problem, and has a linearly bounded Dehn function. However, the question remains open whether each automatic group has a finite convergent presentation.

The monoid N considered in Section 3 is given through a finite, weight-reducing, and weakly confluent string-rewriting system, from which we obtained an infinite, weight-reducing, and confluent string-rewriting system that is left-regular. Since $abb \leftrightarrow_R^* Wda$ holds, but abb is not congruent to any string of shorter length, we see that no length-reducing and confluent string-rewriting system is equivalent to R. If, however, we introduce three additional letters f, g, and h as short forms for the strings bc, da, and dW, respectively, then we obtain a left-regular, length-reducing, and confluent string-rewriting system T^∞ on $\Gamma := \Sigma \cup \{f, g, h\}$ such that $(\Gamma; T^\infty)$ is another presentation of the monoid N.

References

[Alo92] J.M. Alonso. Combings of groups. In G. Baumslag and C.F. Miller III, editors, *Algorithms and Classification in Combinatorial Group Theory*, Math. Sciences Research Institute Publ. 23, pages 165–178. Springer-Verlag, New York, 1992.

[BO93] R.V. Book and F. Otto. *String-Rewriting Systems.* Springer-Verlag, New York, 1993.

[Boo82a] R.V. Book. Confluent and other types of Thue systems. *J. Association Computing Machinery*, 29:171–182, 1982.

[Boo82b] R.V. Book. The power of the Church-Rosser property in string-rewriting systems. In D.W. Loveland, editor, *6th Conference on Automated Deduction*, Lecture Notes in Computer Science 138, pages 360–368. Springer-Verlag, Berlin, 1982.

[Boo83] R.V. Book. Decidable sentences of Church-Rosser congruences. *Theoretical Computer Science*, 24:301–312, 1983.

[CO94] R. Cremanns and F. Otto. Finite derivation type implies the homological finiteness condition FP_3. *Journal of Symbolic Computation*, 18:91–112, 1994.

[CO96] R. Cremanns and F. Otto. For groups the property of having finite derivation type is equivalent to the homological finiteness condition FP_3. *Journal of Symbolic Computation*, 22:155–177, 1996.

[Eps92] D.B.A. Epstein. *Word Processing In Groups.* Jones and Bartlett Publishers, 1992.

[Ger92] S.M. Gersten. Dehn functions and l1-norms of finite presentations. In G. Baumslag and C.F. Miller III, editors, *Algorithms and Classification in Combinatorial Group Theory*, Math. Sciences Research Institute Publ. 23, pages 195–224. Springer-Verlag, New York, 1992.

[Gil84] R.H. Gilman. Computations with rational subsets of confluent groups. In J. Fitch, editor, *Proc. EUROSAM 84*, Lecture Notes in Computer Science 174, pages 207–212. Springer-Verlag, Berlin, 1984.

[GS91] S.M. Gersten and H.B. Short. Rational subgroups of biautomatic groups. *Annals of Mathematics*, 134:125–158, 1991.

[HU79] J.E. Hopcroft and J.D. Ullman. *Introduction to Automata Theory, Languages, and Computation.* Addison-Wesley, Reading, M.A., 1979.

[Hud96] J.F.P. Hudson. Regular rewrite systems and automatic structures. In J. Almeida, G.M.S. Gomes, and P.V. Silva, editors, *Semigroups, Automata and Languages*, pages 145–152. World Scientific, Singapure, 1996.

[KMO94] N. Kuhn, K. Madlener, and F. Otto. Computing presentations for subgroups of polycyclic groups and of context-free groups. *Applicable Algebra in Engineering, Communication and Computing*, 5:287–316, 1994.

[Kuh91] N. Kuhn. *Zur Entscheidbarkeit des Untergruppenproblems für Gruppen mit kanonischen Darstellungen.* Dissertation, Universität Kaiserslautern, 1991.

[Lee63] J. Leech. Coset enumeration on digital computers. *Proc. Cambridge Philos. Soc.*, 59:257–267, 1963.

[McN97] R. McNaughton. *The finiteness of finitely presented monoids.* Report No. 97-2, Department of Computer Science, Rensselaer Polytechnic Institute, Troy, N.Y., March 1997.

[MNOZ93] K. Madlener, P. Narendran, F. Otto, and L. Zhang. On weakly confluent monadic string-rewriting systems. *Theoretical Computer Science*, 113:119–165, 1993.

[MO85] K. Madlener and F. Otto. Pseudo-natural algorithms for the word problem for finitely presented monoids and groups. *Journal of Symbolic Computation*, 1:383–418, 1985.

[Ó'D83] C. Ó'Dúnlaing. Infinite regular Thue systems. *Theoretical Computer Science*, 25:171–192, 1983.

[OK97] F. Otto and Y. Kobayashi. Properties of monoids that are presented by finite convergent string-rewriting systems - a survey. In D.Z. Du and K. Ko, editors, *Advances in Algorithms, Languages and Complexity*, pages 226–266. Kluwer Academic Publ., Dordrecht, 1997.

[Ott86] F. Otto. On deciding whether a monoid is a free monoid or is a group. *Acta Informatica*, 23:99–110, 1986.

[Ott87] F. Otto. On deciding the confluence of a finite string-rewriting system on a given congruence class. *Journal Computer System Sciences*, 35:285–310, 1987.

[Pri95] S.J. Pride. Geometric methods in combinatorial semigroup theory. In J. Fountain, editor, *Proc. of Int. Conf. on Semigroups, Formal Languages, and Groups*, pages 215–232. Kluwer Academic Publ., Dordrecht, 1995.

[Sim94] C.C. Sims. *Computation With Finitely Presented Groups*, volume 48 of *Encyclopedia of Mathematics and its Applications.* Cambridge University Press, New York, 1994.

[SOK94] C.C. Squier, F. Otto, and Y. Kobayashi. A finiteness condition for rewriting systems. *Theoretical Computer Science*, 131:271–294, 1994.

[TC36] J.A. Todd and H.S.M. Coxeter. A practical method for enumerating cosets of a finite abstract group. *Proc. Edinburgh Math. Soc.*, 5:26–34, 1936.

[Tho97] R.M. Thomas. *Automatic semigroups.* Talk at the Conference on Semigroups and Applications, St. Andrews, Scotland, July 1997.