Maximizing the Asymptotic Growth Rate under Fixed and Proportional Transaction Costs in a Financial Market with Jumps

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Contents

Chapter 1

Introduction

Consider an investor who is managing a pension fund or some other portfolio with a long time horizon. In a pension fund for example, there are many individuals with different utility preferences and retirement dates investing money. This makes it impossible for a manager to agree on a common time horizon or utility function. In this case it is reasonable to use the asymptotic growth rate

$$\liminf_{t\to\infty} \frac{1}{t}\mathbb{E}[\log(V_t)|V_0 = x]$$

as a measure of performance of the manager, see e.g. [Konno et al. (1993)]. Here V_t denotes the portfolio value at time t. This optimization problem was considered by various researchers in different settings. In the following we outline some of the main results in the literature on maximizing the asymptotic growth rate and the infinite-horizon discounted consumption problem under logarithmic or power utility.

In the Black-Scholes setting the problem of maximizing the growth rate was solved by [Merton (1969)]. He found that it is optimal to keep the fraction of wealth invested in the stock (risky fraction) at a constant level, the so-called Merton fraction. The drawback of this solution is that it is impossible for an investor to follow this strategy in markets with transaction costs, hence the strategy is not applicable in practice. Nevertheless it was the starting signal for the development of theory of optimal investment for various cost structures.

We consider first transaction costs which are paid proportionally to the transaction size. This kind of costs was first treated heuristically in continuous time by [Magill, Constantinides (1976)]. For the optimal strategy the authors define a region in which it is not optimal to trade. The investor needs to transact only if the vector of his portfolio proportions lies outside the region and the transaction will result in a new vector belonging to the same region. In case with one bond and one stock this strategy is illustrated in Figure 1.1. Here (a, b) denotes the region where no trading occurs. A rigorous proof for the optimality of such a strategy can be given using methods of singular stochastic control theory. These techniques were first introduced by [Taksar et al (1988)] in context of transaction cost problems. Under some assumptions on model parameters [Davis, Norman (1990)] give a rigorous proof using singular stochastic control methods. The problem was reconsidered and solved under weaker assumptions in [Shreve, Soner (1994)], [Akian et al. (1996)] and [Kabanov, Klüppelberg (2004)], where the authors showed the existence and uniqueness of a viscosity solution for the corresponding Hamilton-Jacobi-Bellman equation.

Because of very small transactions when the boundaries are reached the strategy described above is still not applicable in practice if the investor faces constant costs additionally to the proportional costs. In case of constant and proportional costs the optimal strategy belongs to the class of impulse control strategies and is characterized by a solution to a system of quasi variational inequalities. First the methods of the stochastic impulse control were applied by [Eastham, Hastings (1988)] for a finite horizon version of the problem. In the infinite horizon setting [Korn (1998)] presented the solution of the resulting impulse control problem via a formal optimal stopping approach and an approach using quasi variational inequalities. In [Øksendal, Sulem (2001)] the authors derive quasi-variational inequalities for the problem of maximizing discounted consumption under power utility for the infinite horizon and prove that the value function is the unique viscosity solution. The numerical results show that the optimal strategy consists of a no-trading region and two curves inside this region, such that by reaching the boundaries the wealth process restarts at the lower curve after buying and at the upper curve after

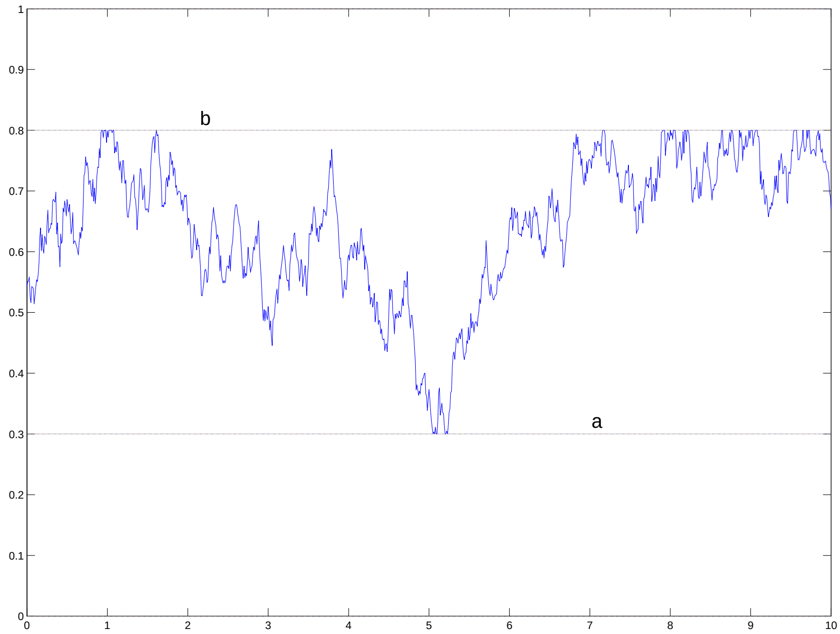

Figure 1.1: Risky fraction process controlled by the optimal strategy in case of proportional costs.

selling. Because of the constant part in the transaction costs it is not possible to express the optimal strategy in terms of the risky fraction.

Another type of transaction costs are fixed costs. These are proportional to the current wealth at the transaction time and can be interpreted as management costs. This kind of cost structure was considered e.g. in [Morton, Pliska (1995)]. Due to the multiplicative structure of fixed costs, the authors were able to factorize the wealth process and reduce the problem to solving an optimal stopping problem. The resulting optimal strategy is an impulse control strategy $(\tau_i, \eta_i)_{i\in\mathbb{N}}$, where $(\tau_i)_{i\in\mathbb{N}}$ are the intervention times and $(\eta_i)_{i\in\mathbb{N}}$ the optimal impulses. In case of one bond and one stock it is optimal to intervene if the risky fraction process reaches the boundaries of the no-trading interval. But in contrast to proportional costs, the investor trades to some optimal fraction near the Merton fraction, see Figure 1.2. This strategy seems reasonable for purely fixed costs since the size of transactions is not punished with costs. This approach was generalized in [Bielecki, Pliska (2000)] with a quite general cost structure. The authors characterize the optimal strategy in terms of a solution of the corresponding quasi variational inequalities.

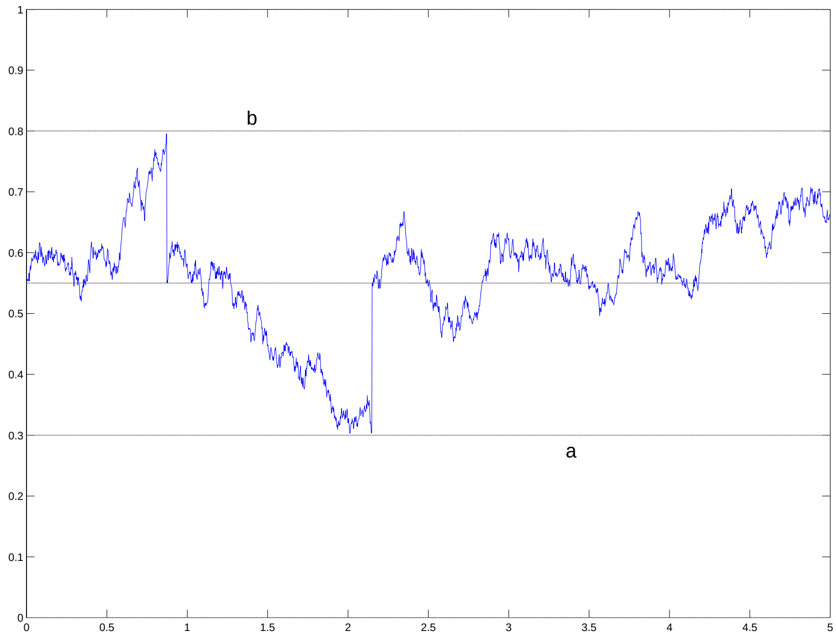

Figure 1.2: Risky fraction process controlled by the optimal strategy in case of fixed costs.

The combination of proportional and fixed costs seems to be a reasonable modeling of the real cost structure in financial markets in the following sense: the proportional part punishes the size of transactions and the fixed part the frequency at which transactions occur. Thus, we would also expect that the optimal strategy is an impulse control strategy, which is some combination of the optimal strategies described above. To be more precise, the strategy we expect to be optimal consists of a no-trading interval (a, b) and constants α, β, such that $a < \alpha \leq \beta < b$. The investor sells stocks such that the new risky fraction is β whenever the boundary b is crossed and buys stocks to reach the new risky fraction α when crossing the boundary a. We refer to such a strategy as a constant boundary strategy (a, α, β, b) or CB-strategy. The risky fraction process controlled by a CB-strategy is illustrated in Figure 1.3. Proportional and fixed costs in the Black-Scholes setting were considered in [Irle, Sass (2006)a], where the authors look at strategies with constant boundaries as candidates for optimal strategies. They use a renewal theory approach and reduce the problem of maximizing the growth rate to one period. [Irle, Prelle (2009)] use the same approach for solving the problem in multidimensional case. In [Irle, Sass (2006)b] the authors construct a solution to the quasi variational inequalities, such that the optimal strategy given by the solution of the quasi variational inequalities is a CB-strategy.

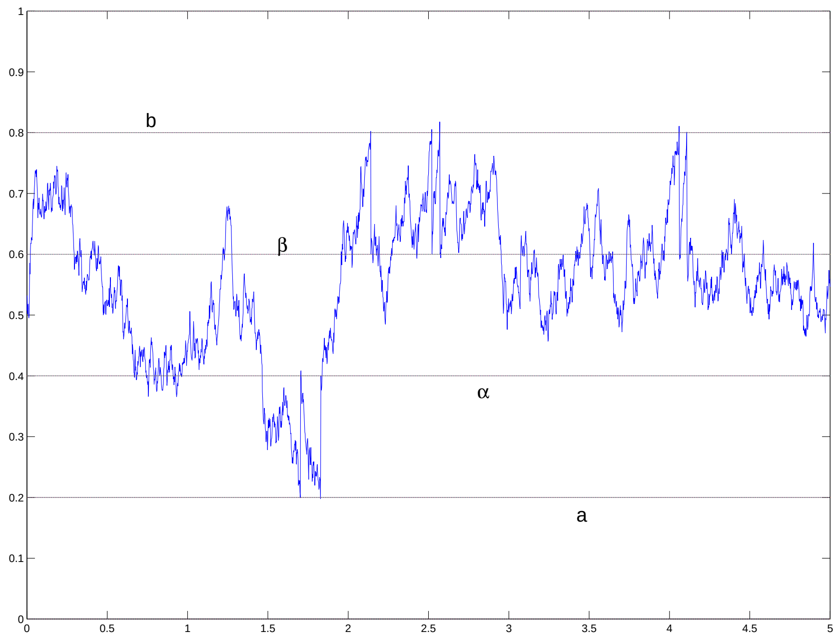

Figure 1.3: Risky fraction process controlled by a CB-strategy.

[Tamura (2006)] showed in a Black-Scholes setting with one bond and one stock, that the optimal impulse control strategy exists and is given by a solution of quasi variational inequalities. These are solved using a perturbation method. In [Tamura (2008)] the problem is solved in the multidimensional case. The difference in the cost structure in comparison to [Irle, Sass (2006)b] is that the fixed costs are paid from the stocks and the bond and not only from the bond. [Ludwig (2012)] points out the differences in these approaches and gives a connecting link between the different transaction cost models by letting the fixed costs go to zero.

The existence of the solution to the quasi variational inequalities in a framework with jumps is obtained in [Duncan et al. (2009)] under the assumption of obligatory diversification. The asset prices are modeled as exponents of a diffusion with jumps whose parameters depend on a finite state Markov process of economic factors. The obligatory diversification means that the investor is required to invest at least a fixed small fraction of his wealth in each asset. This forces the investor to rebalance the portfolio if the risky fractions become too large or too small. Obligatory diversification yields ergodic properties for the risky fraction process, which allow to solve the corresponding quasi variational inequalities.

In this thesis we consider the problem of maximizing the growth rate with proportional and fixed costs in a framework with one bond and one stock, which is modeled as a jump diffusion with compound Poisson jumps. Following the approach from [Irle, Sass (2006)b], we prove that in this framework it is optimal for an investor to follow a CB-strategy. The boundaries depend only on the parameters of the underlying stock and bond. Now it is natural to ask for the investor who follows a CB-strategy which is given by the stopping times $(\tau_i)_{i\in\mathbb{N}}$ and impulses $(\eta_i)_{i\in\mathbb{N}}$ how often he has to rebalance. In other words we want to obtain the limit of the inter trading times

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}(\tau_{i+1}-\tau_i).$$

For this purpose it is very useful to transform the risky fraction process into a Lévy process and since our transformation is bijective we obtain the same results on stopping times for the transformed and the original risky fraction process. We are able to obtain this limit which is given by the expected first exit time of the risky fraction process from some interval under the invariant measure of the Markov chain $(\eta_i)_{i\in\mathbb{N}}$ using the Ergodic Theorem from von Neumann and Birkhoff. In general, it is difficult to obtain the expectation of the first exit time for the process with jumps. Because of the jump part, when the process crosses the boundaries of the interval an overshoot may occur which makes it difficult to obtain the distribution. Nevertheless we can obtain the first exit time if the process has only negative jumps.

For spectrally negative Lévy processes there is a number of fluctuation identities, which yield explicit expressions for the expected first exit time in terms of scale functions, see e.g. [Kyprianou (2006)], [Bertoin (1996)]. The main difficulty of this approach is that the scale functions are known only up to their Laplace transforms. In [Egami, Yamazaki (2011)a] and [Egami, Yamazaki (2011)b] the closed-form expression for the scale function of the Lévy process with phase-type distributed jumps is obtained by using the structure of its Wiener-Hopf factors as obtained in [Asmussen (2004)]. Phase-type distributions build a rich class of positive-valued distributions, which are characterized by a continuous-time Markov chain with a given initial distribution and a state space consisting of a single absorbing state and a finite number of transient states. Then the phase-type distribution is the distribution of the time to absorption. Examples of phase-type distributions are the exponential, hyperexponential, Erlang, hyper-Erlang and Coxian distributions. Since the scale function is given as a function in a closed form we can differentiate to obtain the expected first exit time using the fluctuation identities explicitly.

This work is organized as follows. First we introduce some preliminary results on Lévy processes in Chapter 2. In Chapter 3 we follow the approach of [Irle, Sass (2006)b]. However, we need some additional assumption on the costs and the parameters of the stock to close a gap in the argument in [Irle, Sass (2006)b] Lemma 7.4(i). This assumption is not a big constraint since even for extreme costs it is fulfilled. Using some new arguments we carry over the results to a model, in which the stock is a jump diffusion with a compound Poisson process. We first solve the problem in framework without transaction costs. Then the trading in markets with proportional and fixed costs is introduced. Further we impose the QVIs corresponding to our optimization problem. We are able to solve the quasi variational inequality inside of the no-trading region explicitly. This allows us to construct a candidate for the solution of the QVIs via smooth pasting, such that the resulting impulse control strategy is a CB-strategy. The constructed function is flexible enough to yield the CB-strategy and fulfill necessary conditions for being a solution to the QVIs which is shown in Proposition 3.13 . In Theorem 3.14 we show that the constructed function is indeed a solution to the QVIs, hence the CB-strategy is optimal for the problem of maximizing the growth rate in presence of compound Poisson jumps. In the last section of this chapter numerical results are presented. In particular we investigate the impact of the jump intensity and jump distribution on the optimal boundaries and optimal growth rate. Furthermore we show that the technical assumptions we made in Proposition 3.13 and Theorem 3.14 are verified even for extreme transaction costs.

In Chapter 4 the problem of the frequency of trading is considered. Here we obtain the limit of the inter trading times

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} (\tau_{i+1} - \tau_i).$$

The random variables $(\tau_{i+1} - \tau_i)_{i\in\mathbb{N}}$ have the same distribution due to the Markov property of the risky fraction process. However, the $(\tau_{i+1} - \tau_i)_{i\in\mathbb{N}}$ are not independent in case $\alpha < \beta$ since after trading in τ_i we start the risky fraction in η_i, which implies some dependency. We use the Ergodic Theorem from von Neumann and Birkhoff and show in Theorem 4.5 and in Proposition 4.8 the convergence of the average inter trading time to the expected first exit time of the uncontrolled risky fraction under the invariant measure ν, which is given by

$$\mathbb{E}_\nu[\tau_{(a,b)}] = p \cdot \mathbb{E}^\alpha[\tau_{(a,b)}] + (1-p) \cdot \mathbb{E}^\beta[\tau_{(a,b)}],$$

where $(p, 1-p)$ is the invariant measure of the Markov chain $(\eta_n)_{n\in\mathbb{N}}$. In order to compute $\mathbb{E}_\nu[\tau_{(a,b)}]$ we need the expected first exit time from (a, b) and the transition probabilities of $(\eta_i)_{i\in\mathbb{N}}$. We can obtain these identities in the continuous setting using the Dynkin formula and the Optional Sampling Theorem. In the presence of jumps these methods are no longer sufficient because of the overshoot. Instead we apply the theory of scale functions. Thus we consider Lévy processes with only negative phase-type distributed jumps. The explicit representation of the scale function allows us to derive the desired identities. We calculate the scale function explicitly for the case of negative exponentially distributed jump and derive the limit of inter trading times $\mathbb{E}_\nu[\tau_{(a,b)}]$. We show, that these calculations coincide with Monte Carlo simulations. For increasing intensity of jumps the frequency of trading increases, although the trading is still not very frequent.

Chapter 2

Preliminaries

This chapter contains a brief introduction to Lévy processes which is mainly based on [Cont, Tankov (2004)]. Besides this reference we recommend [Bertoin (1996)], [Applebaum (2004)], [Sato (1999)], [Protter (2005)] for a detailed study of Lévy processes. We pay special attention to Lévy processes with jumps of bounded variation, since these will be used later in our model. We also introduce some basic tools in stochastic calculus such as Itô's formula and discuss some key properties of Lévy processes.

We consider a complete probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$, where $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ is a filtration which satisfies the usual conditions, i.e. $\mathbb{F}$ is right-continuous and $\mathcal{F}_0$ contains all P-null sets of $\mathcal{F}$.

Definition 2.1 ($\mathbb{F}$-Lévy Process)
A càdlàg $\mathbb{F}$-adapted stochastic process $(X_t)_{t \geq 0}$ defined on a probability space $(\Omega, \mathcal{F}, P)$ is called an $\mathbb{F}$-Lévy process, if it fulfills the following conditions:

- *$X_0 = 0$ a.s.;*

- *X has independent increments: $X_t - X_s$ is independent of $\mathcal{F}_s$, $0 \leq s < t < \infty$;*

- *X has stationary increments: given any two distinct times $0 \leq s < t < \infty$, the probability distribution of $X_t - X_s$ coincides with that of X_{t-s};*

- *X is stochastically continuous, i.e. for all $a > 0$ and for all $s \geq 0$ we have*

$$\lim_{t \to s} P(|X_t - X_s| > a) = 0.$$

We say that the process $(X_t)_{t\geq 0}$ is a Lévy process if $(X_t)_{t\geq 0}$ is a $\mathbb{F}^X$-Lévy process, where $\mathbb{F}^X$ is the natural filtration generated by $(X_t)_{t\geq 0}$.

A simple, but nevertheless very important example of a Lévy process is the Poisson process.

Definition 2.2 (Poisson Process)
Let $(\tau_i)_{i\geq 1}$be a sequence of independent exponentially distributed random variables with parameter λ and let $T_n = \sum_{i=1}^{n} \tau_i$. The Poisson process $(N_t)_{t\geq 0}$ with intensity λ is then defined by

$$N_t = \sum_{n\geq 1} \mathbb{1}_{\{t \geq T_n\}}.$$

From this definition it is not obvious that the Poisson process is in fact a Lévy process. $(N_t)_{t\geq 0}$ has independent and stationary increments due to the fact, that the jump-times are independent and have the exponential distribution, which is memoryless. Furthermore N_t is almost surely finite for any $t > 0$ and has the Poisson distribution with parameter λt, i.e.

$$P(N_t = n) = e^{-\lambda t} \frac{(\lambda t)^n}{n!},$$

for all $n \in \mathbb{N}$. In Proposition 2.12. from [Cont, Tankov (2004)] one can find the detailed proof of the above statements. However, the Poisson processes is a Lévy process with piecewise constant increasing paths, which moves only by jumps of size one. If we allow the jump sizes to be i.i.d. random variables, we obtain the compound Poisson process.

Definition 2.3 (Compound Poisson Process)
A compound Poisson process with intensity $\lambda > 0$ and jump size distribution f is a stochastic process $(X_t)_{t\geq 0}$ defined as

$$X_t = \sum_{i=1}^{N_t} Y_i, \quad t \geq 0,$$

where jumps sizes $(Y_i)_{i\geq 1}$ are i.i.d. random variables with distribution f and $(N_t)_{t\geq 0}$ is a Poisson process with intensity λ, independent of $(Y_i)_{i\geq 1}$.

Proposition 3.3 from [Cont, Tankov (2004)] shows that the compound Poisson process is again an example of a Lévy process. Even more, there is no other Lévy processes with piecewise constant paths.

A Poisson process is not a martingale. However, defining the compensated Poisson process by

$$\widetilde{N}_t = N_t - \lambda t,$$

it is easy to check that $(\widetilde{N}_t)_{t\geq 0}$ is a martingale. The only continuous Lévy process with infinite variation is the Brownian motion with drift. Thus it is reasonable to expect that every Lévy process can be decomposed into the sum of a Brownian motion with drift and a (possibly infinite) sum of independent compound Poisson processes. This statement is made precise in Proposition 2.4, the so-called Lévy-Itô decomposition.

Proposition 2.4 (Proposition 3.7, [Cont, Tankov (2004)])
Let $(X_t)_{t\geq 0}$ be a Lévy process on $\mathbb{R}$. Define a measure ν on $\mathbb{R}$ by

$$\nu(A) = \mathbb{E}[\#\{t \in [0,1] : \Delta X_t \neq 0, \Delta X_t \in A\}], \quad A \in \mathcal{B}(\mathbb{R}),$$

where $\Delta X_t = X_t - X_{t-}$. This is the so-called Lévy measure, it counts the jumps of $(X_t)_{t\geq 0}$ over the interval $[0,1]$ with jump sizes in A. Furthermore, define the jump measure of $(X_t)_{t\geq 0}$ on $[0,\infty)\times\mathbb{R}$ by

$$J_X([t_1,t_2]\times B) = \#\left\{s \in [t_1,t_2] : \Delta X_s \neq 0, \Delta X_s \in B\right\}, \quad B \in \mathcal{B}(\mathbb{R}).$$

Then the following statements hold:

- *ν is a Radon measure on $\mathbb{R}$ and satisfies*

$$\int_{|x|\leq 1} |x|^2\,\nu(dx) < \infty \qquad \int_{|x|\geq 1} \nu(dx) < \infty;$$

- *There exist constants* μ, σ^2, *such that*

$$
\begin{aligned}
X_t &= \mu t + \sigma W_t + X_t^l + \lim_{\epsilon \downarrow 0} X_t^\epsilon, \ \textit{where} \qquad (2.1) \\
X_t^l &= \int_{|x| \geq 1, s \in [0,t]} x J_X(ds, dx), \\
X_t^\epsilon &= \int_{\epsilon \leq |x| < 1, s \in [0,t]} x(J_X(ds, dx) - \nu(dx)ds).
\end{aligned}
$$

All terms in (2.1) are independent of each other and the convergence in the last term is almost sure and uniform in t *on* $[0, \infty)$. (μ, σ^2, ν) *is called the* characteristic triplet *of* $(X_t)_{t \geq 0}$.

Due to the càdlàg property of the trajectories of a Lévy process, the Lévy measure $\nu(A)$ is finite for any compact set A such that $0 \notin A$. However, ν may not necessarily be finite since, since $(X_t)_{t\geq 0}$ can have infinitely many small jumps. Thus, in oder to obtain convergence, we have to compensate the jump measure by its expectation $\nu(dx)ds$. We denote this centered version of the jump measure by $\widetilde{J}_X(dx, ds) = J_X(dx, ds) - \nu(dx)ds$. Note that we do not have to compensate if the Lévy process has only finitely many jumps on each finite time interval. We call this kind of process a *finite activity Lévy process.* Then, using the Lévy-Itô decomposition, we can write

$$
\begin{aligned}
X_t &= \mu t + \sigma W_t + \int_{\mathbb{R}} \int_0^t x J_X(ds, dx) \\
&= \mu t + \sigma W_t + \sum_{s \leq t, \Delta X_s \neq 0} \Delta X_s.
\end{aligned}
$$

The immediate conclusion from the Lévy-Itô decomposition, since all components are independent, is the Lévy-Itô representation for the characteristic function.

Theorem 2.5 (Lévy-Itô Representation, Theorem 3.1 [Cont, Tankov (2004)]) *Let* $(X_t)_{t\geq 0}$ *be a Lévy process on* $\mathbb{R}$ *with characteristic triplet* (μ, σ^2, ν). *Then*

$$
\mathbb{E}\left[e^{izX_t}\right] = e^{t\Psi(z)}, \quad z \in \mathbb{R},
$$

where Ψ *is the so-called characteristic exponent given by*

$$
\Psi(z) = \mu i z - \frac{1}{2}\sigma^2 z^2 + \int_{\mathbb{R}} (e^{izx} - 1 - izx \mathbb{1}_{\{|x| \leq 1\}}) \nu(dx).
$$

If $(X_t)_{t\geq 0}$ is a finite activity Lévy process, the characteristic exponent becomes

$$\Psi(z) = \mu i z - \frac{1}{2}\sigma^2 z^2 + \int_{\mathbb{R}} (e^{izx} - 1)\nu(dx).$$

Assume from now on that $(X_t)_{t\geq 0}$ is a finite activity Lévy process. Now we want to take a closer look at the jump measure J_X of $(X_t)_{t\geq 0}$. One can show that J_X is a Poisson random measure on $[0,\infty) \times \mathbb{R}$.

Definition 2.6 (Poisson Random Measure)
Let $E \subset \mathbb{R}^d$ and μ be a Radon measure on the measurable space $(E, \mathcal{E})$. A Poisson random measure on E with intensity measure μ is a random counting measure

$$\begin{aligned} M : \Omega \times \mathcal{E} &\longrightarrow \mathbb{N} \\ (\omega, A) &\mapsto M(\omega, A), \end{aligned}$$

such that

1. *For almost all $\omega \in \Omega$, $M(\omega, \cdot)$ is a Radon measure.*

2. *For any measurable set $A \subset E$, $M(A)$ is a Poisson random variable with parameter $\mu(A)$.*

3. *For disjoint measurable sets $A_1, \cdots, A_n \in \mathcal{E}$, the random variables $M(A_1), \cdots, M(A_n)$ are independent.*

This means that for a fixed $A = [s,t] \times B$, $A \in \mathcal{B}([0,\infty) \times \mathbb{R})$, the mapping $J_X(A)$ is a Poisson random variable with parameter $\nu(B)(t-s)$. J_X is a finite measure, since $(X_t)_{t\geq 0}$ has only a finite number of jumps on each finite time interval.

For a simple function

$$\phi(s,x) = \mathbb{1}_A(s,x), \quad A \in \mathcal{B}([0,\infty) \times \mathbb{R}),$$

the integral of ϕ with respect to J_X is given by

$$\begin{aligned}\int_{[0,t]\times\mathbb{R}} \phi(s,x)J_X(ds,dx) &= \int_{[0,t]\times\mathbb{R}} \mathbb{1}_A(s,x)J_X(ds,dx)\\ &= \#\left\{s\in[0,t] \mid \Delta X_s \neq 0, (s,\Delta X_s)\in A\right\}\\ &= \sum_{\substack{0<s\leq t\\ \Delta X_s\neq 0}} \mathbb{1}_A(s,\Delta X_s) = \sum_{\substack{0<s\leq t\\ \Delta X_s\neq 0}} \phi(s,\Delta X_s).\end{aligned}$$

With the usual extension arguments, we can define the integral with respect to the jump measure J_X for any measurable function ϕ which is bounded from below, by

$$\sum_{\substack{0<s\leq t\\ \Delta X_s\neq 0}} \phi(s,\Delta X_s) = \int_{[0,t]\times\mathbb{R}} \phi(s,x)J_X(ds,dx).$$

Proposition 2.7 (Martingale Property, Proposition 8.8 [Cont, Tankov (2004)])
For any predictable function $\phi:\Omega\times[0,\infty)\times\mathbb{R}\to\mathbb{R}$ *satisfying*

$$\mathbb{E}\left[\int_0^\infty \int_{\mathbb{R}} |\phi(t,y)|^2\,\nu(dy)dt\right] < \infty,$$

the process $(\widetilde{Y}_t)_{t\geq 0}$ *defined as*

$$\widetilde{Y}_t = \int_0^t \int_{\mathbb{R}} \phi(s,y)\widetilde{J}_X(ds,dy)$$

is a square-integrable martingale.

If $(X_t)_{t\geq 0}$ is a Lévy process, then $Y_t = f(X_t)$ may not necessarily be a Lévy process anymore. However, using Itô's formula we can express $(Y_t)_{t\geq 0}$ in terms of stochastic integrals. Hence $(Y_t)_{t\geq 0}$ is a discontinuous semimartingale, the largest class of processes for which the stochastic integral can be defined. We use the following version of Itô's formula for our calculations.

Proposition 2.8 (Itô's Formula, Theorem 71 [Protter (2005)])
Let $(X_t)_{t\geq 0}$ *be a semimartingale with finitely many jumps in any finite interval and let* $f\in C^1(\mathbb{R})$ *and its derivative* $f'\in C^1(\mathbb{R}\setminus\{x_1,\ldots,x_m\})$ *with finite*

one-sided derivatives f'' at $x_1, \dots, x_m$ for some $\{x_1, \dots, x_m\} \subseteq \mathbb{R}$. Then

$$f(X_t) = f(X_0) + \int_0^t f'(X_{s-})dX_s^c + \frac{1}{2}\int_0^t f''(X_{s-})d[X^c, X^c]_s + \sum_{\substack{0<s\le t\\ \Delta X_s\neq 0}} (f(X_s) - f(X_{s-})),$$

where X^c denotes the continuous part of the process $(X_t)_{t\geq 0}$.

An immediate consequence of the multidimensional version of Itô's formula is the following proposition.

Proposition 2.9 (Product Rule)
Let $(X_t)_{t\geq 0}$, $(Y_t)_{t\geq 0}$ be semimartingales with finite activity, then

$$X_tY_t = X_0Y_0 + \int_0^t X_{s-}dY_s^c + \int_0^t Y_{s-}dX_s^c + [X^c, Y^c]_t + \sum_{\substack{0<s\le t\\ \Delta X_s\neq 0, \Delta Y_s\neq 0}} (X_sY_s - X_{s-}Y_{s-}).$$

Proof. We can prove this proposition simply by applying the multidimensional Itô formula (Proposition 8.18, [Cont, Tankov (2004)]) to the function $f(x, y) = xy$. □

Now we want to introduce the concept of a *jump diffusion*, since we use jump diffusions later for modeling the stock price. First we present an important result on Lévy processes.

Proposition 2.10 (Theorem 32 [Protter (2005)])
Let $(X_t)_{t\geq 0}$ be a Lévy process and τ a stopping time. On the set $\{\tau < \infty\}$ the process $(Y_t)_{t\geq 0}$ defined by $Y_t = X_{\tau+t} - X_\tau$ is a Lévy process adapted to $(\mathcal{F}_{\tau+t})_{t\geq 0}$. $(Y_t)_{t\geq 0}$ is independent of $\mathcal{F}_\tau$ and has the same distribution as $(X_t)_{t\geq 0}$.

This means that the process $(X_t)_{t\geq 0}$ shifted by stopping time τ is again a Lévy process with the same distribution. This property implies that Lévy processes are Markov processes.

Theorem 2.11 (Existence and uniqueness of solutions of Lévy SDE's)
Consider the following SDE in $\mathbb{R}$:

$$\begin{aligned} X_0 &= x, \\ dX_t &= \mu(X_{t-})dt + \sigma(X_{t-})dW_t + \int_{\mathbb{R}} \gamma(X_{t-}, x)\widetilde{J}_X(dt, dx), \end{aligned} \tag{2.2}$$

where $\mu, \sigma : \mathbb{R} \to \mathbb{R}$ and $\gamma : \mathbb{R}^2 \to \mathbb{R}$ satisfy the following conditions:

Lipschitz condition: *There exists a constant $C_1 > 0$, such that for all $y_1, y_2 \in \mathbb{R}$*

$$\begin{aligned} &|\mu(y_1) - \mu(y_2)|^2 + |\sigma(y_1) - \sigma(y_2)|^2 \\ &\qquad + \int_{\mathbb{R}} |\gamma(y_1, x) - \gamma(y_2, x)|^2 \nu(dx) \le C_1 |y_1 - y_2|^2 ; \end{aligned}$$

Growth condition: *There exists a constant $C_2 > 0$, such that for all $y \in \mathbb{R}$*

$$|\mu(y)|^2 + |\sigma(y)|^2 + \int_{\mathbb{R}} |\gamma(y, x)|^2 \nu(dx) \le C_2(1 + |y|^2).$$

Then there exists a strong solution $(X_t)_{t\ge0}$ which solves (2.2) uniquely.

The proof of this theorem can be found in [Applebaum (2004)], Theorems 6.2.3, 6.4.5 and 6.4.6. Furthermore, it can be shown that $(X_t)_{t\ge0}$ is a homogeneous Markov process. Every such solution to a SDE of the form (1.2) will be called a *jump diffusion*.

Definition 2.12
Let $(X_t)_{t\ge0}$ be a jump diffusion on $\mathbb{R}$. Then the generator L_X of $(X_t)_{t\ge0}$ is defined on functions $f : \mathbb{R} \to \mathbb{R}$ by

$$L_X f(x) = \lim_{t\downarrow 0+} \frac{1}{t} \left(\mathbb{E}^x[f(X_t)] - f(x)\right),$$

if the limit exists.

The solutions of (2.2) form an important class of Markov processes where the infinitesimal generator can be constructed explicitly. Using the version of Itô's formula as in Proposition 2.8, we can derive the following result for continuously differentiable functions with piecewise continuous second derivatives with compact support.

Theorem 2.13 (Theorem 1.22 [Øksendal, Sulem (2005)])
Let $(X_t)_{t\geq 0}$ be a jump diffusion, let f be as in Proposition 2.8 and have a compact support, then L_X exists and is given by

$$\begin{aligned} L_X f(x) = \mu(x)\frac{\partial}{\partial x}f(x) + \frac{1}{2}\sigma^2(x)\frac{\partial^2}{\partial x^2}f(x) \\ + \int_{\mathbb{R}} \left(f(x+\gamma(x,z)) - f(x)\right)\nu(dz). \end{aligned}$$

Chapter 3

Maximizing the Growth Rate with Transaction Costs in a Framework with Jumps

3.1 Introduction

Our objective in this chapter is maximizing the asymptotic growth rate of the terminal wealth under proportional and fixed transaction costs. In this work we consider a financial market model consisting of one bond and one stock. Without costs, in the Black-Scholes setting, the optimal portfolio strategy consists of holding a constant fraction of wealth invested in the stock. This so-called Merton fraction lies between 0 and 1 if borrowing and short selling are not allowed. In the presence of costs it is impossible to follow this strategy, since the investor has to trade in each point of time to rebalance the portfolio. The transaction costs would lead to immediate bankruptcy. The natural class of trading strategies in a framework with proportional and fixed costs are impulse control strategies. In the Black-Scholes model (i.e. without jumps) it is known, see Introduction, that the optimal impulse control strategy exists and is given by a solution of quasi variational inequalities. Furthermore, [Irle, Sass (2006)b] showed that the so-called constant boundary impulse control strategy is optimal. The constant boundary strategy can be described by only four parameters (a, α, β, b). a and b are the stopping boundaries for rebalancing the portfolio and α, β the new fractions of wealth invested in stock after trading upon reaching a and b, respectively.

In this chapter, we carry over the results of [Irle, Sass (2006)b] to a model where the stock price is driven by a jump diffusion with compound Poisson jumps. We first describe the market model and trading in Section 3.2. Then we solve the optimization problem without costs via point-wise maximization in Section 3.3. The transaction costs and impulse control strategies are introduced in Section 3.4. Due to the logarithmic utility and the structure of the transaction costs we are able to write the wealth as a sum of gain and costs. We show in Section 3.5 that the strategy given by the solution of the QVIs, the so-called QVI-control, is optimal. Furthermore we assume that the QVI-control is a constant boundary strategy. This yields some necessary conditions on the solution of the QVIs, which we derive in Section 3.6. In Section 3.7 we construct a function, which fulfills the necessary conditions and show in Section 3.8 that this function is indeed a solution to the QVIs. Thus we show the existence and optimality of the constant boundary strategy for our optimization problem. In Section 3.9 we discuss some examples, in particular the influence of the intensity of jumps and of the jump distribution as well as the impact of costs on the optimal strategy.

3.2 Model Setting

We consider a probability space $(\Omega, \mathcal{F}, P)$ with a standard Brownian motion $(W_t)_{t\geq 0}$ and an independent compound Poisson process $(X_t)_{t\geq 0}$ defined by $X_t = \sum_{i=1}^{N_t} Z_i$, where $\lambda > 0$ is the jump intensity and f is the density of Z, which has the same distribution as Z_1. We assume that $f : E \to \mathbb{R}$, where $E \subseteq (-1, \infty)$ is some non-empty Borel set. We denote by $(\mathcal{F}_t)_{t\geq 0}$ the filtration generated by $(W_t)_{t\geq 0}$ and $(X_t)_{t\geq 0}$ and augmented by null sets, which satisfies the usual conditions. The jump measure J on $\mathbb{R} \times [0, \infty)$ associated with $(X_t)_{t\geq 0}$ is a Poisson random measure with intensity measure

$\lambda f(x)dxdt$. We denote by

$$\widetilde{J}(dt,dx) = J(dt,dx) - \lambda f(x)dxdt$$

the compensated Poisson random measure and by

$$\mathbb{E}[Z] = \int_E z f(z)dz$$

the expected jump size.

We consider a financial market which consists of two assets: one bond and one stock. Let us assume that the price process of the bond $(B_t)_{t\geq 0}$ follows the deterministic dynamics $dB_t = rB_t dt$ with interest rate $r \geq 0$ and initial value B_0. Let the price process of the stock $(S_t)_{t\geq 0}$ be the strong solution of the following SDE:

$$\begin{aligned}
S_0 &= y \text{ a.s.} \\
dS_t &= \mu S_{t-}dt + \sigma S_{t-}dW_t + \int_E S_{t-}x\widetilde{J}(dt,dx) \\
&= \mu S_{t-}dt + \sigma S_{t-}dW_t + \int_E S_{t-}x(J(dt,dx) - \lambda f(x)dxdt) \\
&= (\mu - \lambda\mathbb{E}[Z])S_{t-}dt + \sigma S_{t-}dW_t + \int_E S_{t-}xJ(dt,dx), \quad t > 0,
\end{aligned}$$

where $\mu \in \mathbb{R}$, $\sigma > 0$. The solution of this SDE is given by a jump diffusion

$$S_t = S_0 \exp\left\{\left(\mu - \lambda\mathbb{E}[Z] - \frac{1}{2}\sigma^2\right)t + \sigma W_t + \int_0^t \int_E \log(1+x)J(ds,dx)\right\}, \tag{3.1}$$

for $t > 0$, which can be easily proven by applying Itô's formula as given in Propostition 2.8.

We can describe the trading strategy by a two-dimensional predictable process $(N_t^B, N_t^S)_{t\geq 0}$, where N_t^B and N_t^S are the number of bonds and stocks, respectively, held by the investor at time t. The wealth of an investor with initial capital $x > 0$ is then given by

$$V_t = N_t^B B_t + N_t^S S_t, \quad V_0 = x, \quad t > 0.$$

The strategy $(N_t^B, N_t^S)_{t\geq 0}$ is self-financing if

$$dV_t = N_t^B dB_t + N_t^S dS_t, \quad V_0 = x, \quad t > 0, \tag{3.2}$$

holds. This equation means, that the changes in wealth are caused only by changes in stock and bond. Assuming, that the wealth process is positive, we can simplify the representation of a trading strategy significantly by introducing the *risky fraction process* $(\pi_t)_{t\geq 0}$:

$$\pi_t = \frac{S_{t-}N_t^S}{V_{t-}}, \quad t \geq 0.$$

The process $(\pi_t)_{t\geq 0}$ is predictable, it describes the fraction of the total wealth that the investor holds in the stock at time t. It is more convenient for us to use the risky fraction process $(\pi_t)_{t\geq 0}$ instead of the two-dimensional trading strategy $(N_t^B, N_t^S)_{t\geq 0}$. We have

$$N_t^B = \frac{(1-\pi_t)V_{t-}}{B_{t-}}, \qquad N_t^S = \frac{\pi_t V_{t-}}{S_{t-}}.$$

Using this representation in (3.2) yields

$$\begin{aligned} dV_t &= \frac{(1-\pi_t)V_{t-}}{B_{t-}} r B_t dt + \frac{\pi_t V_{t-}}{S_{t-}} S_{t-} \left(\mu dt + \sigma dW_t + \int_E x \widetilde{J}(dt, dx) \right) \\ &= r(1-\pi_t)V_{t-} dt + \pi_t V_{t-} \left(\mu dt + \sigma dW_t + \int_E x \widetilde{J}(dt, dx) \right). \end{aligned} \tag{3.3}$$

Applying Itô's formula to $\log(V_t)$ yields the log-wealth process

$$\begin{aligned} \log(V_t) = \log(V_0) &+ \int_0^t r + \pi_s \left(\mu - r - \frac{1}{2}\sigma^2 \pi_s - \lambda \mathbb{E}[Z] \right) ds \\ &+ \int_0^t \pi_s \sigma dW_s + \int_0^t \int_E \log(1 + \pi_s x) J(dx, ds), \quad t \geq 0. \end{aligned} \tag{3.4}$$

In the following, given initial capital $x > 0$, we describe a trading strategy by the predictable risky fraction process $(\pi_t)_{t\geq 0}$ with values in $(0, 1)$, such that the corresponding wealth process is given by (3.3). The assumption $\pi_t \in (0, 1)$ for all $t \geq 0$ means that we exclude borrowing and short selling.

3.3 Optimal Growth Rate without Costs

We call the trading strategy $(\pi_t)_{t\geq 0}$ with initial capital x *admissible* if the wealth process is a.s. positive, i.e. $P(V_t > 0 | V_0 = x) = 1$. Our objective is to maximize the asymptotic growth rate

$$\liminf_{t\to\infty} \frac{1}{t}\mathbb{E}[\log V_t \mid V_0 = x, \pi_0 = \pi]$$

over all admissible trading strategies. In the following we use the notation

$$\mathbb{E}[\,\cdot\,|V_0 = x, \pi_0 = \pi] = \mathbb{E}^{\pi,x}[\,\cdot\,].$$

Using representation (3.4) we have

$$\begin{aligned}
&\mathbb{E}^{\pi,x}[\log V_t] = \log(x) \\
&+ \mathbb{E}^{\pi}\left[\int_0^t r + \pi_s\left(\mu - r - \frac{1}{2}\sigma^2\pi_s - \lambda\mathbb{E}[Z]\right) ds + \int_0^t \pi_s \sigma dW_s \right.\\
&\left. + \int_0^t \int_E \log(1+\pi_s z) J(dz, ds)\right] \\
&= \log(x) + \mathbb{E}^{\pi}\left[\int_0^t r + \pi_s\left(\mu - r - \frac{1}{2}\sigma^2\pi_s - \lambda\mathbb{E}[Z]\right) ds + \int_0^t \pi_s \sigma dW_s \right.\\
&\left. + \int_0^t \int_E \log(1+\pi_s z)\widetilde{J}(dz, ds) + \lambda \int_0^t \int_E \log(1+\pi_s z) f(z) dz\, ds\right] \\
&= \log(x) + \mathbb{E}^{\pi}\left[\int_0^t r + \pi_s\left(\mu - r - \frac{1}{2}\sigma^2\pi_s - \lambda\mathbb{E}[Z]\right) ds \right.\\
&\left. + \lambda \int_0^t \int_E \log(1+\pi_s z) f(z) dz\, ds\right].
\end{aligned}$$

Thus, in order to maximize the asymptotic growth rate

$$\begin{aligned}
&\liminf_{t\to\infty} \frac{1}{t}\mathbb{E}^{\pi,x}[\log V_t] \\
&= \liminf_{t\to\infty} \frac{1}{t}\mathbb{E}\left[\int_0^t r + \pi_s\left(\mu - r - \frac{1}{2}\sigma^2\pi_s - \lambda\mathbb{E}[Z]\right) + \lambda\mathbb{E}[\log(1+\pi_s Z)] ds\right]
\end{aligned}$$

we can pointwisely maximize the integrand

$$g(x) := r - \frac{1}{2}\sigma^2 x^2 + (\mu - r - \lambda\mathbb{E}[Z])\, x + \lambda \int_E \log(1 + xz) f(z) dz.$$

We assume that $\mathbb{E}[Z] < \infty$, then using Jensen's inequality and dominated convergence it follows, that the function g is continuous and differentiable on $[0, 1]$. Thus, there exists a maximum. Assume further that

$$\mu - r > 0, \quad \mu - r - \sigma^2 < \lambda\mathbb{E}\left[\frac{Z^2}{1+Z}\right]. \tag{3.5}$$

The optimal risky fraction π^* is a root of

$$g'(x) = -\sigma^2 x + (\mu - r - \lambda\mathbb{E}[Z]) + \lambda\mathbb{E}\left[\frac{Z}{1+Zx}\right].$$

There exists a unique maximizer $\pi^* \in (0,1)$ because $g'(1) < 0$ and $g'(0) = \mu - r > 0$ by (3.5) and g is strictly concave since the second derivative is negative

$$g''(x) = -\sigma^2 - \lambda\mathbb{E}\left[\frac{Z^2}{(1+Zx)^2}\right] < 0.$$

In the following we simplify the notation by assuming $r = 0$ since the results for general r can be obtained by adjusting the drift μ by $-r$ and adding r to the optimal growth rate $R^* := g(\pi^*)$. We denote by $R_1 := g(1)$ the growth rate, which corresponds to the pure-stock buy-and-hold portfolio and assume that it is positive: $R_1 > 0$. We have

$$R_1 = -\frac{1}{2}\sigma^2 + (\mu - \lambda\mathbb{E}[Z]) + \lambda\mathbb{E}[\log(1+Z)] < R^*.$$

3.4 Fixed and Proportional Transaction Costs

In our framework we consider the following transaction costs. The investor with current wealth $V_t > 0$ has to pay costs in amount of

$$\delta V_t + \gamma\,|\Delta_t|$$

for a transaction of size $\Delta_t \in \mathbb{R}$, where $\delta \in (0,1)$ and $\gamma \in [0, 1-\delta)$. We call δV_t the fixed cost and $\gamma\,|\Delta_t|$ the proportional cost.

In Section 3.3 we have seen that without costs it is optimal to have a constant fraction of money π^* invested in the stock to achieve the optimal growth rate, i.e. we have to trade in each point of time to keep the risky fraction constant. The investor who faces fixed and proportional transaction costs following this strategy would go bankrupt immediately. Thus, it is reasonable to consider trading strategies with a finite number of trades in finite time. Hence we allow the risky fraction process to deviate from π^*. Since we have to pay a fraction of wealth each time we trade we do not want the deviation to become too large. Thus, we have to stop at some point and make a transaction which brings the process near the optimal fraction π^* and the transaction is not allowed to be too large because of the proportional transaction costs, which punish big transactions. Therefore, it is natural to consider impulse control strategies.

Definition 3.1 (Impulse Control Strategy)
A sequence $K = (\tau_n, \Delta_n)_{n\in\mathbb{N}}$ is called an impulse control strategy, if

(i) $(\tau_n)_{n\in\mathbb{N}}$ is a sequence of $(\mathcal{F}_t)_{t\geq 0}$-stopping times, such that

$$0 = \tau_0 \leq \tau_1 \leq \ldots \leq \infty, \quad \tau_n \to \infty \text{ a.s. and } \tau_n < \tau_{n+1} \text{ on } \{\tau_n < \infty\}\,.$$

(ii) $(\Delta_n)_{n\in\mathbb{N}}$ is a sequence of $(\mathcal{F}_{\tau_n})$-measurable random variables in $\mathbb{R}$.

We want to introduce the wealth process and the risky fraction process controlled by an impulse control strategy $K = (\tau_n, \Delta_n)_{n\in\mathbb{N}}$. After the first transaction of Δ_0 in $\tau_0 = 0$, the new wealth $\overline{V}_0$ is given by

$$\overline{V}_0 = V_0 - \delta V_0 - \gamma\,|\Delta_0|$$

and the new risky fraction $\overline{\pi}_0$ is given by

$$\overline{\pi}_0 = \frac{V_0\pi_0 + \Delta_0}{\overline{V}_0}$$

For $n \in \mathbb{N}$ on $\{\tau_n < \infty\}$ we have

$$\begin{aligned}
V_t &= \left(1 - \overline{\pi}_{n-1} + \overline{\pi}_{n-1} S_t/S_{\tau_{n-1}}\right)\overline{V}_{n-1}, && t \in (\tau_{n-1}, \tau_n],\\
\pi_t &= \overline{\pi}_{n-1}\overline{V}_{n-1}\frac{S_t}{S_{\tau_{n-1}}V_t}, && t \in (\tau_{n-1}, \tau_n],\\
\overline{V}_n &= V_{\tau_n} - \delta V_{\tau_n} - \gamma\,|\Delta_n|\,, &&\\
\overline{\pi}_n &= \frac{V_{\tau_n}\pi_{\tau_n} + \Delta_n}{\overline{V}_n}. &&
\end{aligned}$$

Note that the controlled process $(\pi_t)_{t\geq 0}$ is càdlàg, thus we use the predictable processes $(\pi_{t-})_{t\geq 0}$ for describing the trading strategy.

We consider only impulse control strategies $(\tau_n, \Delta_n)_{n\in\mathbb{N}}$, such that $V_t > 0$ and $\pi_t \in (0,1)$ a.s. for all $t \geq 0$. There is a one-to-one correspondence between controlling by transaction sizes Δ_n and controlling by the new risky fraction process after trading at τ_n, defined by

$$\eta_n := \frac{\pi_{\tau_n} V_{\tau_n} + \Delta_n}{(1-\delta)V_{\tau_n} - \gamma\,|\Delta_n|}, \tag{3.6}$$

see [Irle, Sass (2006)a], Lemma 3.5. We call $(\tau_n, \eta_n)_{n\in\mathbb{N}_0}$ the *new risky fraction strategies.* The wealth and risky fraction processes controlled by $(\tau_n, \eta_n)_{n\in\mathbb{N}_0}$ evolve in the same manner as above with $\eta_n = \overline{\pi}_n$ and

$$\Delta_n = \frac{(1-\delta)\eta_n - \pi_{\tau_n}}{1 + \gamma\eta_n \mathrm{sgn}((1-\delta)\eta_n - \pi_{\tau_n})} V_{\tau_n}. \tag{3.7}$$

We call the new risky fraction strategy *admissible* if $V_t > 0$ and $\pi_t \in (0,1)$ a.s. for all $t \geq 0$ and denote the family of these strategies by $\mathcal{A}(x,\pi)$, where $V_0 = x$ and $\pi_0 = \pi$ are the initial capital and initial risky fraction, respectively. In the following if we want to emphasize the control we use the notations $(V_t^K)_{t\geq 0}$ and $(\pi_t^K)_{t\geq 0}$ for controlled processes.

The next proposition points out the advantage of reformulating trading strategies in terms of new risky fractions.

Proposition 3.2

For any admissible strategy $(\tau_n, \eta_n)_{n\in\mathbb{N}_0}$ we have

$$\begin{aligned}\log(V_t) = \log(V_0) &+ \int_0^t \pi_{s-}\left(\mu - \frac{1}{2}\sigma^2\pi_{s-} - \lambda\mathbb{E}[Z]\right) ds \\ &+ \int_0^t \pi_{s-}\sigma dW_s + \int_0^t\int_E \log(1+\pi_{s-}x) J(dx,ds) + \sum_{n=0}^{M_t} \overline{\Gamma}(\pi_{\tau_n}, \eta_n),\end{aligned} \tag{3.8}$$

$$\begin{aligned}\pi_t = \pi_0 &+ \int_0^t \pi_{s-}(1-\pi_{s-})(\mu - \lambda E[Z] - \sigma^2\pi_{s-})ds + \int_0^t \pi_{s-}(1-\pi_{s-})\sigma dW_s \\ &+ \int_0^t\int_E \pi_{s-}(1-\pi_{s-})\frac{x}{1+\pi_{s-}x} J(ds,dx) + \sum_{n=0}^{M_t}(\eta_n - \pi_{\tau_n}),\end{aligned} \tag{3.9}$$

where $M_t = \sup\{n \in \mathbb{N} : \tau_n < t\}$ for $t > 0$ and

$$\overline{\Gamma}(x,y) = \begin{cases} \log\frac{1-\delta-\gamma x}{1-\gamma y}, & y < \frac{x}{1-\delta} \\ \log\frac{1-\delta+\gamma x}{1+\gamma y}, & y \geq \frac{x}{1-\delta} \end{cases}$$

denotes the costs, which the investor has to pay for the transaction of changing the risky fraction from x to y.

Proof. The wealth in the bank account V_t^B and in the stock V_t^S at $t \geq 0$ are given by

$$
\begin{aligned}
V_t^B &= (1-\pi_0)V_0 + \sum_{k\geq 0} \mathbb{1}_{\{\tau_k<t\}}(-\Delta_k - \gamma\,|\Delta_k| - \delta V_{\tau_k}) \\
V_t^S &= \pi_0 V_0 + \int_0^t \mu V_{s-}^S ds + \int_0^t \sigma V_{s-}^S dW_s + \int_0^t \int_E V_{s-}^S x \widetilde{J}(dt,dx) \\
&\quad + \sum_{k\geq 0} \mathbb{1}_{\{\tau_k<t\}}\Delta_k.
\end{aligned}
$$

The last term in V_t^B arise due to the transaction costs the investor has to pay for transactions in $\tau_0, \dots, \tau_{M_t}$. The wealth is given by the sum of V_t^B and V_t^S

$$
\begin{aligned}
V_t &= V_0 + \int_0^t \mu V_{s-}^S ds + \int_0^t \sigma V_{s-}^S dW_s + \int_0^t \int_E V_{s-}^S x \widetilde{J}(dt,dx) \\
&\quad + \sum_{k\geq 0} \mathbb{1}_{\{\tau_k<t\}}(-\gamma\,|\Delta_k| - \delta V_{\tau_k}).
\end{aligned}
$$

We can rewrite this dynamics in terms of the risky fraction

$$
\begin{aligned}
V_t &= V_0 + \int_0^t \mu \pi_{s-} V_{s-} ds + \int_0^t \sigma \pi_{s-} V_{s-} dW_s + \int_0^t \int_E \pi_{s-} V_{s-} x \widetilde{J}(dt,dx) \\
&\quad + \sum_{k\geq 0} \mathbb{1}_{\{\tau_k<t\}}(-\gamma\,|\Delta_k| - \delta V_{\tau_k}).
\end{aligned}
$$

Since the wealth is almost surely positive for all $t \geq 0$ we can apply Itô's formula to $f(V_t) = \log(V_t)$:

$$
\begin{aligned}
\log(V_t) &= \log(V_0) + \int_0^t \frac{1}{V_{s-}} dV_s^c - \frac{1}{2}\int_0^t \frac{1}{V_{s-}^2} d\,[V_s^c, V_s^c] + \sum_{\substack{0<s\leq t \\ \Delta V_s \neq 0}} (f(V_s) - f(V_{s-})) \\
&= \log(V_0) + \int_0^t \frac{\pi_{s-}V_{s-}}{V_{s-}}(\mu - \lambda \mathbb{E}[Z])ds + \int_0^t \frac{\pi_{s-}V_{s-}}{V_{s-}} \sigma dW_s \\
&\quad - \frac{1}{2}\int_0^t \frac{\pi_{s-}^2 V_{s-}^2}{V_{s-}^2}\sigma^2 ds + \sum_{\substack{0<s\leq t \\ \Delta V_s \neq 0}} (f(V_s) - f(V_{s-})) \\
&= \log(V_0) + \int_0^t \pi_{s-}(\mu - \frac{1}{2}\sigma^2 \pi_{s-} - \lambda \mathbb{E}[Z])ds + \int_0^t \pi_{s-}\sigma dW_s \\
&\quad + \sum_{\substack{0<s\leq t \\ \Delta V_s \neq 0}} (f(V_s) - f(V_{s-})).
\end{aligned}
$$

The last term in the equation is the sum of jumps caused by the compound Poisson process and jumps according to trading.

$$\sum_{\substack{0<s\le t\\ \Delta V_s\neq 0}} (f(V_s)-f(V_{s-})) = \sum_{\substack{0<s\le t\\ \Delta X_s\neq 0}} (f(V_s)-f(V_{s-})) + \sum_{n=0}^{M_t}(f(V_{\tau_n})-f(V_{\tau_n-})).$$

$$\sum_{\substack{0<s\le t\\ \Delta X_s\neq 0}} (f(V_s)-f(V_{s-})) = \sum_{\substack{0<s\le t\\ \Delta X_s\neq 0}} \log\left(\frac{V_s}{V_{s-}}\right) = \sum_{\substack{0<s\le t\\ \Delta X_s\neq 0}} \log\left(\frac{V_{s-}+\pi_{s-}V_{s-}\Delta X_s}{V_{s-}}\right)$$

$$= \int_0^t \int_E \log(1+\pi_{s-}x)J(dt,dx),$$

$$\sum_{n=0}^{M_t}(f(V_{\tau_n})-f(V_{\tau_n-})) = \sum_{n=0}^{M_t}\log\left(\frac{V_{\tau_n}}{V_{\tau_n-}}\right) = \sum_{n=0}^{M_t}\log\left(\frac{V_{\tau_n-}-\gamma\,|\Delta_n|-\delta V_{\tau_n-}}{V_{\tau_n-}}\right).$$

In case of buying stocks $\Delta_n > 0$, the representation (3.7) yields

$$V_{\tau_n-} - \gamma\Delta_n - \delta V_{\tau_n-} = V_{\tau_n-}\left(\frac{1+\gamma\pi_{\tau_n}-\delta}{1+\gamma\eta_n}\right).$$

In case of selling stock we have

$$V_{\tau_n-} + \gamma\Delta_n - \delta V_{\tau_n-} = V_{\tau_n-}\left(\frac{1-\gamma\pi_{\tau_n}-\delta}{1-\gamma\eta_n}\right).$$

Thus, summing up all terms we have

$$\sum_{\substack{0<s\le t\\ \Delta V_s\neq 0}} (f(V_s)-f(V_{s-})) = \int_0^t \int_E \log(1+\pi_{s-}x)J(dt,dx) + \sum_{n=0}^{M_t}\overline{\Gamma}(\pi_{\tau_n},\eta_n).$$

Altogether the controlled wealth process is given by (3.8).

To obtain the dynamics of the risky fraction we apply Itô's formula to the function $\pi_t = f(V_t^B, V_t^S) = \frac{V_t^S}{V_t^B + V_t^S}$. Then we get

$$\begin{aligned}
\pi_t &= \pi_0 + \int_0^t \frac{V_{s-}^B}{V_{s-}^2} d(V_s^S)^c - \int_0^t \frac{V_{s-}^S}{V_{s-}^2} d(V_s^B)^c + \frac{1}{2}\int_0^t -\frac{2V_{s-}^B}{V_{s-}^3} d[(V_s^S)^c, (V_s^S)^c] \\
&+ \frac{1}{2}\int_0^t \frac{2V_{s-}^S}{V_{s-}^3} d[(V_s^B)^c, (V_s^B)^c] + \frac{1}{2}\int_0^t \frac{1}{V_{s-}^2} d[(V_s^S)^c, (V_s^B)^c] \\
&+ \frac{1}{2}\int_0^t -\frac{1}{V_{s-}^2} d[(V_s^B)^c, (V_s^B)^c] + \sum_{\substack{0<s\le t \\ \Delta V_s^S \neq 0}} (f(V_s^B, V_s^S) - f(V_{s-}^B, V_{s-}^S)) \\
&= \pi_0 + \int_0^t \pi_{s-}(1-\pi_{s-})(\mu - \lambda\mathbb{E}[Z] - \sigma^2\pi_{s-})ds + \int_0^t \pi_{s-}(1-\pi_{s-})\sigma dW_s \\
&+ \sum_{\substack{0<s\le t \\ \Delta X_s \neq 0}} (f(V_s^B, V_s^S) - f(V_{s-}^B, V_{s-}^S)) + \sum_{n=0}^{M_t}(\eta_n - \pi_{\tau_n}),
\end{aligned}$$

where the first sum is the sum of jumps caused by the compound Poisson process,

$$\begin{aligned}
\sum_{\substack{0<s\le t \\ \Delta X_s \neq 0}} (f(V_s^B, V_s^S) - f(V_{s-}^B, V_{s-}^S)) &= \sum_{\substack{0<s\le t \\ \Delta X_s \neq 0}} \left(\frac{V_s^S}{V_s} - \frac{V_{s-}^S}{V_{s-}} \right) \\
&= \sum_{\substack{0<s\le t \\ \Delta X_s \neq 0}} \left(\frac{V_{s-}^S + V_{s-}^S \Delta X_s}{V_{s-} + V_{s-}\pi_{s-}\Delta X_s} - \pi_{s-} \right) \\
&= \sum_{\substack{0<s\le t \\ \Delta X_s \neq 0}} \pi_{s-} \frac{(1-\pi_{s-})\Delta X_s}{1 + \pi_{s-}\Delta X_s} \\
&= \int_0^t \int_E \pi_{s-}(1-\pi_{s-}) \frac{x}{1+\pi_{s-}x} J(dt, dx).
\end{aligned}$$

Thus the dynamics of the risky fraction is given by

$$\begin{aligned}
\pi_t &= \pi_0 + \int_0^t \pi_{s-}(1-\pi_{s-})(\mu - \lambda\mathbb{E}[Z] - \sigma^2\pi_{s-})ds + \int_0^t \pi_{s-}(1-\pi_{s-})\sigma dW_s \\
&+ \int_0^t \int_E \pi_{s-}(1-\pi_{s-}) \frac{x}{1+\pi_{s-}x} J(dt, dx) + \sum_{n=0}^{M_t}(\eta_n - \pi_{\tau_n}).
\end{aligned}$$

□

We denote by R^K the growth rate corresponding to an admissible new risky fraction strategy K, i.e.

$$R^K = \liminf_{t\to\infty} \frac{1}{t}\mathbb{E}\left[\log V_t^K | V_0 = x, \pi_0 = \pi\right].$$

Our purpose is to find an optimal strategy $K^* = (\tau_n^*, \eta_n^*)_{n\in\mathbb{N}_0}$ within the class of admissible new risky fraction impulse control strategies $\mathcal{A}(x,\pi)$, which maximizes the growth rate of the controlled system:

$$R^{K^*} = \sup_{K\in\mathcal{A}(x,\pi)} \liminf_{t\to\infty} \frac{1}{t}\mathbb{E}\left[\log V_t^K | V_0 = x, \pi_0 = \pi\right]. \tag{3.10}$$

Plugging the representation (3.8) into (3.10) yields

$$\begin{aligned}
&\liminf_{t\to\infty} \frac{1}{t}\mathbb{E}^{\pi,x}[\log V_t^K] = \liminf_{t\to\infty} \frac{1}{t}\mathbb{E}\left[\int_0^t \pi_{s-}\left(\mu - \frac{1}{2}\sigma^2\pi_{s-} - \lambda\mathbb{E}[Z]\right) ds\right.\\
&\left. + \int_0^t \pi_{s-}\sigma dW_s + \int_0^t\int_E \log(1+\pi_{s-}x)J(dx,ds) + \sum_{n=0}^{M_t} \overline{\Gamma}(\pi_{\tau_n},\eta_n)\right]\\
&= \liminf_{t\to\infty} \frac{1}{t}\mathbb{E}\left[\int_0^t \pi_{s-}\left(\mu - \frac{1}{2}\sigma^2\pi_{s-} - \lambda\mathbb{E}[Z]\right) + \lambda\mathbb{E}[\log(1+\pi_{s-}Z)]ds\right.\\
&\left. + \sum_{n=0}^{M_t} \overline{\Gamma}(\pi_{\tau_n},\eta_n)\right] = \liminf_{t\to\infty} \frac{1}{t}\mathbb{E}\left[\int_0^t g(\pi_{s-})ds + \sum_{n=0}^{M_t} \overline{\Gamma}(\pi_{\tau_n},\eta_n)\right].
\end{aligned}$$

Thus, using representation (3.8), we can express our objective in terms of the risky fraction $(\pi_t)_{t\geq 0}$ and the new risky fractions $(\eta_n)_n$ only. Also, we see that the asymptotic growth rate does not depend on the initial capital V_0 and the initial risky fraction π_0. Hence, (3.10) can be equivalently written as

$$R^{K^*} = \sup_{K\in\mathcal{A}(x,\pi)} \liminf_{t\to\infty} \frac{1}{t}E\left[\int_0^t g(\pi_{s-})ds + \sum_{n=0}^{M_t} \overline{\Gamma}(\pi_{\tau_n},\eta_n)\right].$$

3.5 Quasi Variational Inequalities

As we will prove later the optimal strategy for maximizing the asymptotic growth rate in a framework with proportional and fixed costs is an impulse control strategy which is given by a solution of the quasi variational inequalities (QVIs).

Definition 3.3 (Solution of the QVIs)
A solution of the QVIs is a function $v : (0,1) \to \mathbb{R}$, which is continuously differentiable and piecewise twice continuously differentiable on $(0,1)$, and a constant $l \in \mathbb{R}$ such that

1. $L_\pi v + g - l \leq 0$ *on* $(0,1)$ *with*
$$L_\pi v(x) = x(1-x)\left(\mu - \lambda \int_E z f(z) dz - x\sigma^2\right) v'(x) + \frac{1}{2}\sigma^2 x^2 (1-x)^2 v''(x) + \lambda \int_E \left(v\left(\frac{x(1+z)}{1+zx}\right) - v(x)\right) f(z) dz.$$
2. $L_\pi v + g - l = 0$ *on* $D = \{x \in (0,1) : v(x) > \mathcal{M}v(x)\}$, *where* $\mathcal{M}v(x) = \sup\left\{v(y) + \overline{\Gamma}(x,y), y \in (0,1)\right\}$.
3. $v(x) \geq \mathcal{M}v(x)$ *for all* $x \in (0,1)$.

Proposition 3.4
Assume, that v and l are a solution of the QVIs and v, v' are bounded. Then $l \geq R^K$ for any admissible strategy K. Assume, that an impulse control strategy $K^* = (\tau_n^*, \eta_n^*)_{n \in \mathbb{N}_0}$ exists, such that the stopping times are given by
$$\tau_i^* = \inf\left\{t > \tau_{i-1}^* : v(\pi_t) = \mathcal{M}v(\pi_t)\right\}$$
or equivalently
$$\tau_i^* = \inf\left\{t > \tau_{i-1}^* : \pi_t \notin D\right\},$$
and impulses η_i^* have values in $(0,1)$, such that
$$\mathcal{M}v(\pi_{\tau_i^*}) = v(\eta_i^*) + \overline{\Gamma}(\pi_{\tau_i^*}, \eta_i^*).$$
Then, we have $l = R^{K^*}$. We call the strategy $K^* = (\tau_n^*, \eta_n^*)_{n \in \mathbb{N}_0}$ the QVI-control and D the no-trading region.

Proof: Consider the controlled risky fraction process
$$\pi_t = \pi_0 + \int_0^t \pi_{s-}(1-\pi_{s-})(\mu - \lambda \mathbb{E}[Z] - \sigma^2 \pi_{s-}) ds + \int_0^t \pi_{s-}(1-\pi_{s-})\sigma dW_s + \int_0^t \int_E \pi_{s-}(1-\pi_{s-}) \frac{x}{1+\pi_{s-}x} J(ds,dx) + \sum_{n=0}^{M_t} (\eta_n - \pi_{\tau_n})$$

and apply Itô's formula (2.8) to $v(\pi_t)$:

$$\begin{aligned} v(\pi_t) &= v(\pi_0) + \int_0^t L_\pi v(\pi_{s-})ds \\ &\quad + \int_0^t \pi_{s-}(1-\pi_{s-})v'(\pi_{s-})\sigma dW_s + \sum_{n=0}^{M_t}(v(\eta_n) - v(\pi_{\tau_n})). \end{aligned}$$

Adding $-v(\pi_t) + \int_0^t g(\pi_{s-})ds + \sum_{n=0}^{M_t} \overline{\Gamma}(\pi_{\tau_n}, \eta_n)$ to both sides yields

$$\begin{aligned} \int_0^t g(\pi_{s-})ds + \sum_{n=0}^{M_t} \overline{\Gamma}(\pi_{\tau_n}, \eta_n) &= lt + v(\pi_0) - v(\pi_t) \\ &\quad + \int_0^t \pi_{s-}(1-\pi_{s-})v'(\pi_{s-})\sigma dW_s \\ &\quad + \int_0^t \underbrace{L_\pi v(\pi_{s-}) + g(\pi_{s-}) - l}_{\leq 0} ds \\ &\quad + \underbrace{\sum_{n=0}^{M_t}(v(\eta_n) - v(\pi_{\tau_n}) + \overline{\Gamma}(\pi_{\tau_n}, \eta_n))}_{\leq 0} \\ &\leq lt + v(\pi_0) - v(\pi_t) + \int_0^t \pi_{s-}(1-\pi_{s-})v'(\pi_{s-})\sigma dW_s. \end{aligned}$$

Because v and v' are bounded on $(0,1)$ we have

$$\mathbb{E}\left[\int_0^t \pi_{s-}(1-\pi_{s-})v'(\pi_{s-})\sigma dW_s\right] = 0$$

and it holds

$$\begin{aligned} &\liminf_{t\to\infty} \frac{1}{t}\mathbb{E}\left[\int_0^t g(\pi_{s-})ds + \sum_{n=0}^{M_t} \overline{\Gamma}(\pi_{\tau_n}, \eta_n)\right] \\ &\qquad \leq \liminf_{t\to\infty} \frac{1}{t}\mathbb{E}\left[lt + v(\pi_0) - v(\pi_t) + \int_0^t \pi_{s-}(1-\pi_{s-})v'(\pi_{s-})\sigma dW_s\right] \\ &\qquad = \liminf_{t\to\infty} \frac{1}{t}\mathbb{E}[lt + v(\pi_0) - v(\pi_t)] = l. \end{aligned}$$

Thus $R^K \leq l$ for any admissible impulse control strategy. Applying the QVI-control $K^* = (\tau_n^*, \eta_n^*)_{n\in\mathbb{N}_0}$ yields

$$\sum_{n=0}^{M_t}(v(\eta_n^*) - v(\pi_{\tau_n^*}) + \overline{\Gamma}(\pi_{\tau_n^*}, \eta_n^*)) = 0$$

and

$$L_\pi v(x) + g(x) - l = 0$$

for all $x \in D$. Therefore the optimal growth rate is given by l, i.e. $l = R^{K^*}$. □

Now we want to interpret the function v, which together with l is a solution of the quasi variational inequalities. Assume that we have an optimal impulse control strategy $(\tau_n, \eta_n)_{n\in\mathbb{N}}$, which is given by a solution v and l of the QVIs. Applying Itô's formula to $v(\pi_t)$ for $\tau_i < t < \tau_{i+1}$ yields

$$\begin{aligned} v(\pi_t) =& v(\pi_0) + \int_0^t L_\pi v(\pi_{s-})ds + \int_0^t \pi_{s-}(1-\pi_{s-})v'(\pi_{s-})\sigma dW_s \\ &+ \sum_{n=0}^{M_t}(v(\pi_{\tau_n}) - v(\eta_n)). \end{aligned}$$

Using this identity we derive

$$\begin{aligned} \mathbb{E}[v(\pi_t) - v(\pi_0)] =& \mathbb{E}\Bigg[\int_0^t \underbrace{L_\pi v(\pi_{s-}) + g(\pi_{s-}) - l}_{=0 \text{ by the first QVI}}\, ds \\ &+ \int_0^t l - g(\pi_{s-})ds - \sum_{n=0}^{M_t} \overline{\Gamma}(\pi_{\tau_n}, \eta_n) \\ &+ \underbrace{\sum_{n=0}^{M_t} v(\eta_n) - v(\pi_{\tau_n}) + \overline{\Gamma}(\pi_{\tau_n}, \eta_n)}_{=0 \text{ by the second QVI}}\Bigg]. \end{aligned}$$

Thus we have

$$\mathbb{E}[v(\pi_t) - v(\pi_0)] = \mathbb{E}\left[\int_0^t l - g(\pi_{s-})ds - \sum_{n=0}^{M_t} \overline{\Gamma}(\pi_{\tau_n}, \eta_n)\right].$$

We can interpret $\mathbb{E}[v(\pi_t) - v(\pi_0)]$ as the deviation of the gain

$$\mathbb{E}\left[\int_0^t g(\pi_{s-})ds + \sum_{n=0}^{M_t} \overline{\Gamma}(\pi_{\tau_n}, \eta_n)\right]$$

from the accumulated optimal growth lt up to time t. Because of the transaction costs it is impossible to hold the risky fraction constant as is optimal in the case without costs. Letting $(\pi_t)_{t\geq 0}$ deviate from π^*, we sacrifice some of the return, thus we can also interpret $\mathbb{E}[v(\pi_t) - v(\pi_0)]$ as the displacement cost. One can find this kind of interpretation of v for example in [Taksar et al (1988)].

3.6 Necessary Conditions for the Optimality of Constant Boundary Strategy

In our framework the cost of a transaction which brings the risky fraction from x to y is given by the function

$$\overline{\Gamma}(x,y) = \begin{cases} \log \frac{1-\delta-\gamma x}{1-\gamma y}, & y < \frac{x}{1-\delta} \\ \log \frac{1-\delta+\gamma x}{1+\gamma y}, & y \geq \frac{x}{1-\delta}. \end{cases}$$

The derivative with respect to y,

$$\frac{\partial}{\partial y}\overline{\Gamma}(x,y) = \begin{cases} \frac{\gamma}{1-\gamma y}, & y < \frac{x}{1-\delta} \\ -\frac{\gamma}{1+\gamma y}, & y \geq \frac{x}{1-\delta} \end{cases}$$

has a discontinuity at $\frac{x}{1-\delta}$, which leads to problems in the later argumentation. Therefore in the following we use a modification of the cost function, namely

$$\Gamma(x,y) = \begin{cases} \log \frac{1-\delta-\gamma x}{1-\gamma y}, & y \leq x \\ \log \frac{1-\delta+\gamma x}{1+\gamma y}, & y > x. \end{cases} \tag{3.11}$$

The use of the modified cost function is justified in [Irle, Sass (2006)b], Section 5. The main idea is that

$$\overline{\Gamma}(x,y) \leq \Gamma(x,y)$$

for all $x, y \in (0,1)$. The growth rate corresponding to $\overline{\Gamma}$, which we denote by $\overline{R}^K$ is for each admissible strategy smaller than or equal to R^K. If for the optimal constant boundary strategy K^* it holds that $\alpha \geq a(1-\delta)$, then we have $\overline{R}^{K^*} = R^{K^*}$. Thus, K^* is also optimal for the original cost function.

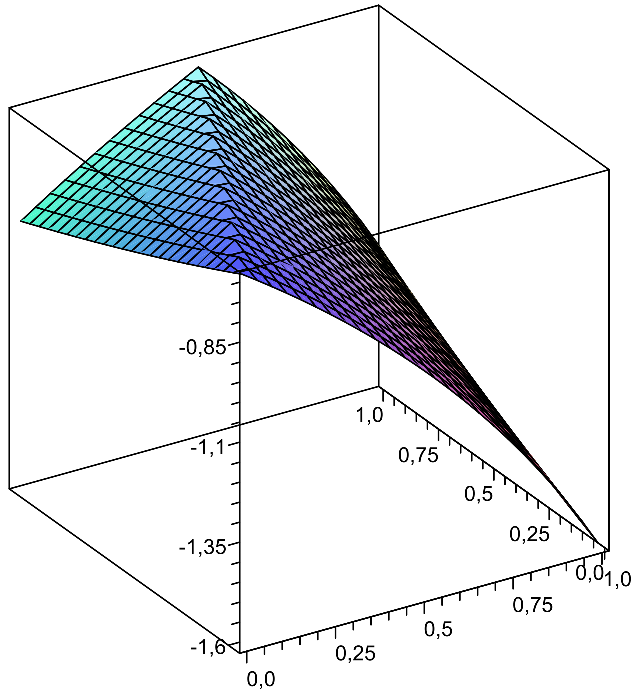

Figure 3.1: Original cost function $\overline{\Gamma}$.

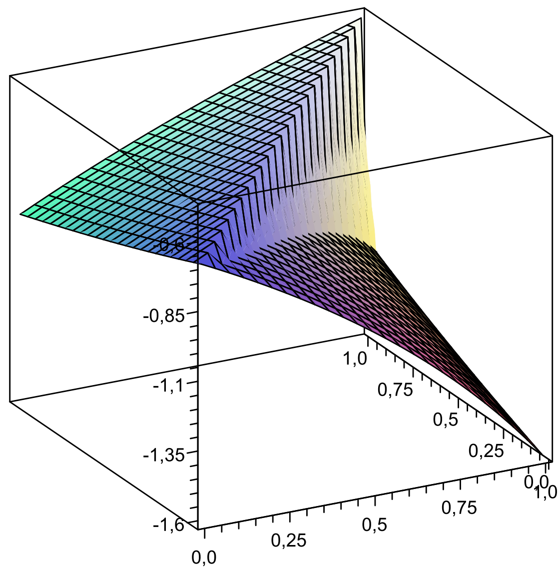

Figure 3.2: Modified cost function Γ.

The numerical examples will show later, that this assumption is fulfilled even for extreme costs.

The crucial point of the further argumentation is that we can obtain a solution to the first QVI explicitly.

Proposition 3.5
The function $v_0 : (0,1) \to \mathbb{R}$ which is given by

$$v_0(x) = \frac{1}{p}\left(c - \frac{l}{R_1}\right)\left(\frac{1-c}{c}\right)^p\left(\frac{x}{1-x}\right)^p + \frac{l}{R_1}\log\left(\frac{x}{1-x}\right) - \log\left(\frac{1}{1-x}\right)$$

solves $L_\pi v_0 + g - l = 0$ on $(0,1)$. Here $c \in (0,1)$ is some constant and $p \in (-1,0)$ is a root of

$$h(x) = \frac{1}{2}\sigma^2 x - \frac{1}{2}\sigma^2 + (\mu - \lambda\mathbb{E}[Z]) + \frac{1}{x}\lambda\mathbb{E}\left[(1+Z)^x - 1\right].$$

Proof. The derivatives of v_0 are given by

$$v_0'(x) = \frac{1}{x(1-x)}\left[\left(c - \frac{l}{R_1}\right)\left(\frac{1-c}{c}\right)^p\left(\frac{x}{1-x}\right)^p + \frac{l}{R_1} - x\right], \tag{3.12}$$

$$v_0''(x) = \frac{1}{x^2(1-x)^2}\left[\left(c - \frac{l}{R_1}\right)\left(\frac{1-c}{c}\right)^p\left(\frac{x}{1-x}\right)^p(p + 2x - 1) \right. \tag{3.13}$$
$$\left. -x(1-x) + (\frac{l}{R_1} - x)(2x-1)\right].$$

Plugging this derivatives and v_0 into $L_\pi v_0 + g - l = 0$ on $(0,1)$ yields the result. Further we show that there exists a root $p \in (-1,0)$ of

$$h(x) = \frac{1}{2}\sigma^2 x - \frac{1}{2}\sigma^2 + (\mu - \lambda\mathbb{E}[Z]) + \frac{1}{x}\lambda\mathbb{E}\left[(1+Z)^x - 1\right].$$

Since

$$\lim_{p \uparrow 0} \frac{1}{p}((1+Z)^p - 1) = \log(1+Z),$$

we have

$$\lim_{p\uparrow 0} h(p) = -\frac{1}{2}\sigma^2 + \mu - \lambda\mathbb{E}[Z] + \lambda\mathbb{E}[\log(1+Z)] = R_1 > 0.$$

Further $h(-1) < 0$ because of (3.5), hence there exists $p \in (-1,0)$, such that $h(p) = 0$. $\square$

Thus we have already a solution of the first QVI. Now we consider v_0 on $(a,b) \subset (0,1)$ and construct a function $v : (0,1) \to \mathbb{R}$, which extends v_0 outside of (a,b) such that v is bounded on $(0,1)$. Let us define v by

$$v(x) = \begin{cases} v_0(\alpha) + \Gamma(x,\alpha), & x \leq a \\ v_0(x), & a < x \leq b \\ v_0(\beta) + \Gamma(x,\beta), & x > b \end{cases}$$

for some constants $0 < a < \alpha < \beta < b < 1$. To find necessary conditions for a, α, β, b, l we suppose that this function is a solution of the QVIs and assume that the QVI-control given by v and l as in Proposition 3.4 is a constant boundary strategy (a, α, β, b). We will show in Theorem 3.14 that this assumption is fulfilled for the parameters defined in the following section (Proposition 3.13). A QVI-control given by (a, α, β, b) means, that the no-trading region is given by the interval $D = (a,b)$ and the optimal impulses are given by the new risky fraction α if we stop in a and β if we stop in b.

The strategy (a, α, β, b) has to fulfill the following conditions in order to be a QVI-control strategy given by v and l as in Proposition 3.4. We want v to be in $C^1(0,1)$, thus we have to choose a and b such that

$$\begin{aligned} &(I) && \frac{d}{dx} v_0(x)\Big|_a = \frac{\partial}{\partial x}\Gamma(x,\alpha)\Big|_a \\ &(II) && \frac{d}{dx} v_0(x)\Big|_b = \frac{\partial}{\partial x}\Gamma(x,\beta)\Big|_b . \end{aligned}$$

The optimal actions α, β are the solutions of

$$\begin{aligned} \mathcal{M}v(x) &= v(\alpha) + \Gamma(x,\alpha), \quad x \leq a \\ \mathcal{M}v(x) &= v(\beta) + \Gamma(x,\beta), \quad x \geq b, \end{aligned}$$

i.e. the function $v(y) + \Gamma(x,y)$ attains its maximum at α for $x \leq a$ and β for $x \geq b$. This means

$$\begin{aligned} &(III) && \frac{d}{dy} v_0(y)\Big|_\alpha = -\frac{\partial}{\partial y}\Gamma(x,y)\Big|_\alpha \\ &(IV) && \frac{d}{dy} v_0(y)\Big|_\beta = -\frac{\partial}{\partial y}\Gamma(x,y)\Big|_\beta . \end{aligned}$$

Furthermore we have $v(x) \leq \mathcal{M}v(x)$ for all $x \notin D$ and for all $x \in (0,1)$ we have $v(x) \geq \mathcal{M}v(x)$, thus

$$\mathcal{M}v(x) = v(x), \text{ for } x \leq a, x \geq b.$$

This implies

$$\begin{aligned}
&(V) && v_0(\alpha) - v_0(a) + \Gamma(a,\alpha) = 0 \\
&(VI) && v_0(\beta) - v_0(b) + \Gamma(b,\beta) = 0.
\end{aligned}$$

Summarizing the conditions for β, b yields

$$(II) \qquad v_0'(b) = -\frac{\gamma}{1-\delta-\gamma b} \tag{3.14}$$

$$(IV) \qquad v_0'(\beta) = -\frac{\gamma}{1-\gamma\beta} \tag{3.15}$$

$$(VI) \qquad v_0(b) - v_0(\beta) = \log\left(\frac{1-\delta-\gamma b}{1-\gamma\beta}\right). \tag{3.16}$$

Necessary conditions for a, α are

$$(I) \qquad v_0'(a) = \frac{\gamma}{1-\delta+\gamma a} \tag{3.17}$$

$$(III) \qquad v_0'(\alpha) = \frac{\gamma}{1+\gamma\alpha} \tag{3.18}$$

$$(V) \qquad v_0(a) - v_0(\alpha) = \log\left(\frac{1-\delta+\gamma a}{1+\gamma\alpha}\right). \tag{3.19}$$

Note, that by derivation of (3.18) the cost functions Γ and $\overline{\Gamma}$ coincide if $a \leq \alpha(1-\delta)$, in particular

$$\frac{\partial}{\partial y}\Gamma(a,y)\Big|_{\alpha} = \frac{\partial}{\partial y}\overline{\Gamma}(a,y)\Big|_{\alpha}.$$

3.7 Existence of a Constant Boundary Strategy as QVI-Control

In the previous section we assumed, that the function v and a constant l solve the QVIs and derived conditions under which the resulting QVI-control is a CB-strategy. The subject of this section is to show, that under some assumptions on the parameters of the stock, there exists a unique constant boundary strategy, such that the conditions $(I) - (VI)$ are fulfilled. We will mostly follow the approach from [Irle, Sass (2006)b]. But there are some differences we want to point out here. We need the Assumption (3.20)

$$\delta \leq 1 - \gamma + \frac{1 + \gamma p}{p} \exp\left(\frac{1 + p}{p(1 + \gamma p)}\right)$$

on costs and parameters of the stock in order to prove the existence of a CB-strategy. In fact, a corresponding assumption would be also needed in the continuous-time setting to close a gap in the argument of Lemma 7.4(i) in [Irle, Sass (2006)b]. Therefore in Lemma 3.9, Lemma 3.10 and in Theorem 3.14 we choose another approach as in [Irle, Sass (2006)b].

In the following lemma we discuss how the behavior of the function $v_0(x; c, l)$ depends on parameters c and l.

Lemma 3.6
Let $c \in (0, 1)$ and $l > R_1$.

(i) $\lim_{x \downarrow 0} v_0'(x) = -\infty, \quad \lim_{x \uparrow 1} v_0'(x) = \infty.$

(ii) If $l < \frac{R_1}{p} c(c - 1 + p)$, then v_0' has exactly three roots $x_1,\ x_2,\ c$ in $(0, 1)$, satisfying $x_1 < c < x_2$ and $v_0''(x_1) > 0,\ v_0''(c) < 0,\ v_0''(x_2) > 0$.

(iii) Let $l = \frac{R_1}{p} c(c - 1 + p)$, then there exist at most two roots:

- *If $c > \frac{1}{2}(1 - p)$, then $x_1 < c = x_2$ and $v_0''(x_1) > 0$.*
- *If $c < \frac{1}{2}(1 - p)$, then $x_1 = c < x_2$ and $v_0''(x_2) > 0$.*
- *If $c = \frac{1}{2}(1 - p)$, then $x_1 = c = x_2$ and $v_0''(c) = 0$.*

Proof. Since $c < 1$ and $\frac{l}{R_1} > 1$ it follows that $\left(c - \frac{l}{R_1}\right) \left(\frac{1-c}{c}\right)^p < 0$. The claim (i) follows using representation (3.12) and

$$\lim_{x \downarrow 0} \left(\frac{x}{1 - x}\right)^p = \infty \quad \lim_{x \uparrow 1} \left(\frac{x}{1 - x}\right)^p = 0$$
$$\lim_{x \downarrow 0} \frac{1}{x(1 - x)} = \infty \quad \lim_{x \uparrow 1} \frac{1}{x(1 - x)} = \infty.$$

For proving (ii) we show first, that there are at most three roots. We can decompose v_0' in the following way

$$v_0'(x) = \underbrace{\frac{\left(\frac{x}{1-x}\right)^p}{R_1 x(1-x)}}_{f_1(x)} \underbrace{\left((l - R_1 x)\left(\frac{1-x}{x}\right)^p - (l - R_1 c)\left(\frac{1-c}{c}\right)^p\right)}_{f_2(x)}.$$

f_1 has no roots in $(0,1)$. f_2 has at most three roots, because

$$f_2'(x) = -\left(\frac{1-x}{x}\right)^p \frac{p}{x(1-x)} \left(l - \underbrace{\frac{R_1}{p} x(x-1+p)}_{\text{Polynomial of degree 2}}\right)$$

has at most two roots. Hence v_0' has at most three roots in $(0,1)$. It is easy to see that c is a root of v_0'. There are at least three roots iff $v_0''(c) < 0$.

$$v_0''(c) = \frac{1}{c^2(1-c)^2}\left(c - \frac{l}{R_1}\right)\left(p - \frac{c(1-c)}{c - \frac{l}{R_1}}\right) < 0 \quad \Leftrightarrow \quad \left(p - \frac{c(1-c)}{c - \frac{l}{R_1}}\right) > 0$$

$$\Leftrightarrow \quad R_1 < l < \frac{R_1}{p} c(c-1+p)$$

To show (iii) we again look at the representation $v_0'(x) = f_1(x) \cdot f_2(x)$. It is sufficient to consider f_2 because the signs of f_2 and v_0' are the same. c is a local minimum of v_0' if $c > \frac{1}{2}(1-p)$ and c is a local maximum if $c < \frac{1}{2}(1-p)$, because $f_2'(c) = 0$ and

$$f_2''(c) = -\underbrace{\left(\frac{1-c}{c}\right)^p \frac{p}{c(1-c)}}_{>0} \left(-\frac{2R_1}{p} c + \frac{R_1}{p} - R_1\right) \gtrless 0 \Leftrightarrow c \lessgtr \frac{1}{2}(1-p).$$

$c = \frac{1}{2}(1-p)$ is a saddle point, because $f_2'''(\frac{1}{2}(1-p)) \neq 0$, where

$$f_2'''(x) = -R_1 \left(\frac{1-x}{x}\right)^p \frac{1}{x^2(1-x)^2}(p^2 - 2x^2 + 2x - 1).$$

$\square$

Figure 3.3 shows v_0' for $c \in (0,1)$, $l < \frac{R_1}{p} c(c-1+p)$ and the cost functions we used in conditions $(I) - (IV)$. Since we want to find some a, α

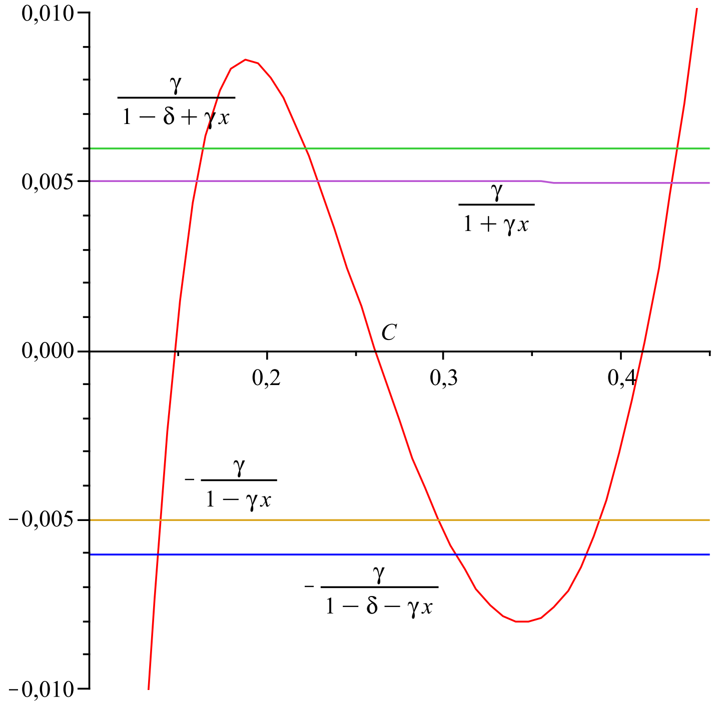

Figure 3.3: v_0' and cost functions for $c \in (0,1)$, $l < \frac{R_1}{p} c(c-1+p)$.

and β, b, such that equations $(I)-(IV)$ hold, it is reasonable to assume $c \in (0,1)$, $l < \frac{R_1}{p} c(c-1+p)$ for the right behavior of v_0'. Furthermore we expect, that the optimal growth rate l is greater than the growth rate corresponding to the pure-stock buy-and-hold portfolio R_1. This means that

$$l \in \left(R_1, \frac{R_1}{p} c(c-1+p) \right),$$

which holds iff $c \in (-p, 1)$. In the following we fix the parameter $c \in (-p,1)$ and consider v_0' on $(c,1)$. In the next lemma we show, that v_0' is strictly increasing in l on $(c,1)$ and there is an upper barrier for l, $\bar{l}_1 > R_1$, such that v_0' has only a contact point with the cost function $-\frac{\gamma}{1-\delta-\gamma x}$.

Lemma 3.7
There exists $\overline{l_1} > R_1$, such that

$$\overline{l_1} = \sup \left\{ l > R_1 \,\middle|\, v_0'(x;l) \leq -\frac{\gamma}{1-\delta-\gamma x} \text{ for some } x \in [c,1) \right\}.$$

Proof. $v_0'(x;l)$ is strictly increasing in l, since

$$\frac{dv_0'}{dl}(x;l) = \frac{\left(\frac{x}{1-x}\right)^p}{R_1 x(1-x)}\left(\left(\frac{1-x}{x}\right)^p - \left(\frac{1-c}{c}\right)^p\right) > 0$$

for all $x \in (c,1)$. Furthermore $v'(x;l)$ is continuous in l and we have

$$\lim_{x\uparrow 1} v_0'(x;R_1) = \lim_{x\uparrow 1}\left(\frac{1}{x} - \frac{1-c}{x(1-x)}\left(\frac{x}{1-x}\right)^p\left(\frac{1-c}{c}\right)^p\right) = -\infty,$$

thus

$$\lim_{l\downarrow R_1}\inf_{x\in(c,1)} v_0'(x;l) = -\infty \quad \text{and} \quad \overline{l_1} > R_1.$$

□

We can compute $\overline{l_1}$ explicitly, since we know that there must be a contact point $\bar{b}$ between $v_0'(x;\overline{l_1})$ and the cost function $-\frac{\gamma}{1-\delta-\gamma x}$, i.e. the functions have the same derivatives and the same values in $\bar{b}$:

$$v_0''(\bar{b}) = -\frac{\gamma^2}{(1-\delta-\gamma\bar{b})^2} = -(v_0'(\bar{b}))^2.$$

Thus we can calculate $\overline{l_1}$:

$$\overline{l_1} = \frac{R_1}{p}\frac{\gamma^2\bar{b}^2(1-\bar{b})^2}{(1-\delta-\gamma\bar{b})^2} + \frac{R_1}{p}\bar{b}(1-\bar{b})\left(-\frac{\gamma}{1-\delta-\gamma\bar{b}}(p+2\bar{b}-1)-1\right) + \bar{b}.$$

Lemma 3.8
For each parameter $l \in (R_1, \overline{l_1})$ there are unique $\beta, b \in (c,1)$, such that

$$v_0'(\beta;l) = -\frac{\gamma}{1-\gamma\beta} \quad v_0'(b;l) = -\frac{\gamma}{1-\delta-\gamma b},$$

$$v_0''(\beta;l) + \frac{\gamma^2}{(1-\gamma\beta)^2} < 0 \quad v_0''(b;l) + \frac{\gamma^2}{(1-\delta-\gamma b)^2} > 0$$

Proof. After the observations above, for each $l \in (R_1, \overline{l_1})$ there are at least three roots $\beta_0, \beta_1, \beta_2$ of

$$v_0'(x;l) + \frac{\gamma}{1-\gamma x} = 0$$

with $0 < \beta_1 < c < \beta_2 < \beta_3 < 1$. We want to show, that there are at most three roots. Consider the function

$$f(x) = \frac{1}{x^2(1-x)^2}\left[x(1-x)\left(-\frac{\gamma}{1-\gamma x}(p+2x-1)-1\right)-p\left(\frac{l}{R_1}-x\right)\right] + \frac{\gamma^2}{(1-\gamma x)^2},$$

which coincides with the derivative of $v_0'(x;c,l)+\frac{\gamma}{1-\gamma x}$ at β_0,β_1,β_2 and any other solution β_i, i.e.

$$v_0'(\beta_i)+\frac{\gamma}{1-\gamma\beta_i}=0, \quad v_0''(\beta_i)+\frac{\gamma^2}{(1-\gamma\beta_i)^2}=f(\beta_i).$$

Simplifying f shows, that there are at most two roots:

$$f(x) = \frac{\overbrace{-\frac{l}{R_1}p(1-\gamma x)^2 + x(\gamma-1)(p\gamma x - x - p + 1)}^{\text{Polynomial of degree 2}}}{x^2(1-x)^2(1-\gamma x)^2}.$$

Hence $v_0''(x)+\frac{\gamma^2}{(1-\gamma x)^2}$ has at most two roots and thus $v_0'(x;c,l)+\frac{\gamma}{1-\gamma x}$ has at most three roots. Furthermore

$$v_0''(\beta_1)+\frac{\gamma^2}{(1-\gamma\beta_1)^2}<0 \quad v_0''(\beta_2)+\frac{\gamma^2}{(1-\gamma\beta_2)^2}>0,$$

thus $\beta := \beta_1$ fulfills the claimed properties. We apply the same argumentation to

$$v_0'(x;l)+\frac{\gamma}{1-\delta-\gamma x}=0,$$

but take the biggest of three roots $b_0 < b_1 < b_2$, i.e. $b := b_2$. □

For a fixed $c \in (-p,1)$ and for each $l \in (R_1,\overline{l_1})$ there exist unique $\beta(l), b(l) \in (c,1)$, such that the conditions (3.14) and (3.15) are fulfilled. In the following we will show, that a unique $l_1 \in (R_1,\overline{l_1})$ exists, such that the condition (3.16) also holds. For this purpose we need the following lemma.

Lemma 3.9

$b(l)$ is strictly decreasing, $\beta(l)$ is strictly increasing in l with

$$\lim_{l\downarrow R_1} b(l) = 1 \quad \textit{and} \quad \lim_{l\downarrow R_1}\beta(l)=\beta^*, \qquad \textit{for some } \beta^*\in[c,1).$$

Proof. We define a function

$$F(x(l), v_0'(x(l), l)) := v_0'(x(l), l) + \frac{\gamma}{1 - \gamma x(l)}.$$

For each $l \in (R_1, \overline{l_1})$ there is a unique $\beta(l)$, such that $F(\beta(l), v_0'(\beta(l), l)) = 0$. Since

$$\frac{\partial}{\partial x} F(x(l), v_0'(x(l), l))\Big|_{\beta(l)} = v_0''(\beta(l), l) + \frac{\gamma^2}{(1 - \gamma\beta(l))^2} < 0,$$

$\beta(l)$ is in C^1 by the Implicit Function Theorem with derivative

$$\frac{\partial \beta}{\partial l}(l) = \frac{-\frac{\partial v_0'}{\partial l}(\beta(l); l)}{v_0''(\beta(l); l) + \frac{\gamma^2}{(1-\gamma\beta(l))^2}} > 0.$$

$\beta(l)$ is strictly increasing and continuous, thus there is $\beta^* \in [c, 1)$, such that $\lim_{l \downarrow R_1} \beta(l) = \beta^*$.

Analogously, $b(l)$ is also in C^1 and is strictly decreasing

$$\frac{\partial b}{\partial l}(l) = \frac{-\frac{\partial v_0'}{\partial l}(b(l); l)}{v_0''(b(l); l) + \frac{\gamma^2}{(1-\delta-\gamma b(l))^2}} < 0.$$

Thus, there exists $b^* \in (c, 1)$ such that $\lim_{l \downarrow R_1} b(l) = b^*$. Assume $b^* < 1$ and define

$$f(x, l) := v_0'(x, l) + \frac{\gamma}{1 - \delta - \gamma x}.$$

We have already seen that

$$\lim_{x \uparrow 1} f(x, R_1) = -\infty, \quad \lim_{x \uparrow 1} f(x, l) = +\infty \text{ for } l \in (R_1, \overline{l_1}).$$

$f(x, R_1)$ is strictly decreasing in x, thus the following holds for each $l \in (R_1, \overline{l_1})$

$$f(x_l^*, l) := \min_{x \in (c,1)} f(x, l) > f(x_l^*, R_1) > f(b^*, R_1),$$

which is a contradiction to

$$\lim_{l \downarrow R_1} \min_{x \in (c,1]} \left(v_0'(x, l) + \frac{\gamma}{1 - \delta - \gamma x} \right) = -\infty.$$

Thus, we have $\lim_{l \downarrow R_1} b(l) = 1$. □

Now we are able to prove that there is a unique $l_1 \in (R_1, \overline{l_1})$, such that the conditions (3.14), (3.15), (3.16) hold. In the further argumentation we will need the following assumption

$$\delta \leq 1 - \gamma + \frac{1+\gamma p}{p} \exp\left(\frac{1+p}{p(1+\gamma p)}\right). \tag{3.20}$$

Lemma 3.10

Under the assumption (3.20) for each $c \in (-p, 1)$ there exists a unique $l_1 \in (R_1, \overline{l_1})$, such that

$$v_0(b(l_1), l_1) - v_0(\beta(l_1), l_1) = \log\left(\frac{1-\delta-\gamma b(l_1)}{1-\gamma\beta(l_1)}\right).$$

l_1 is C^1 as a function of c.

Proof. Define $\psi(l) := v_0(b(l), l) - v_0(\beta(l), l)$. The function $\psi(l) - \log\left(\frac{1-\delta-\gamma b(l)}{1-\gamma\beta(l)}\right)$ is strictly increasing in l:

$$\begin{aligned}
\frac{\partial\psi}{\partial l}(l) - \frac{\partial}{\partial l}\log\left(\frac{1-\delta-\gamma b(l)}{1-\gamma\beta(l)}\right) &= \underbrace{\left(v_0'(b(l), l) - \frac{\gamma}{1-\delta-\gamma b(l)}\right)}_{=0} b_l(l) \\
&- \underbrace{\left(v_0'(\beta(l), l) - \frac{\gamma}{1-\gamma\beta(l)}\right)}_{=0} \beta_l(l) + \frac{\partial v_0}{\partial l}(b(l), l) - \frac{\partial v_0}{\partial l}(\beta(l), l) \\
&= \int_{\beta(l)}^{b(l)} \frac{\partial v_0'}{\partial l}(x, l)dx > 0,
\end{aligned}$$

since v_0' is strictly increasing in l on $(c, 1)$. Thus, we can prove the statement if we show that

$$\psi(\overline{l_1}) - \log\left(\frac{1-\delta-\gamma b(\overline{l_1})}{1-\gamma\beta(\overline{l_1})}\right) \geq 0 \quad \psi(R_1) - \log\left(\frac{1-\delta-\gamma b(R_1)}{1-\gamma\beta(R_1)}\right) \leq 0.$$

Following holds for all $x \in (\beta(\overline{l_1}), b(\overline{l_1}))$:

$$-\frac{\gamma}{1-\delta-\gamma x} \leq v_0'(x, \overline{l_1}) \leq -\frac{\gamma}{1-\gamma x}.$$

Thus, we have

$$\int_{\beta(\overline{l_1})}^{b(\overline{l_1})} -\frac{\gamma}{1-\delta-\gamma x}dx \leq \psi(\overline{l_1}).$$

Because

$$\int_{\beta(\overline{l_1})}^{b(\overline{l_1})} -\frac{\gamma}{1-\delta-\gamma x}dx = \log\left(\frac{1-\delta-\gamma b(\overline{l_1})}{1-\delta-\gamma\beta(\overline{l_1})}\right) > \log\left(\frac{1-\delta-\gamma b(\overline{l_1})}{1-\gamma\beta(\overline{l_1})}\right)$$

the first inequality holds:

$$\psi(\overline{l_1}) - \log\left(\frac{1-\delta-\gamma b(\overline{l_1})}{1-\gamma\beta(\overline{l_1})}\right) \geq 0.$$

For the second inequality we rewrite the function v_0 as follows

$$v_0(x,l) = \frac{1}{p}x(1-x)v_0'(x,l) - \frac{1}{p}\left(\frac{l}{R_1} - x\right) + \frac{l}{R_1}\log\left(\frac{x}{1-x}\right) - \log\left(\frac{1}{1-x}\right).$$

Using the assumption

$$\delta \leq 1-\gamma+\frac{1+\gamma p}{p}\exp\left(\frac{1+p}{p(1+\gamma p)}\right)$$

and the fact that

$$v_0'(\beta(l),l) = -\frac{\gamma}{1-\gamma\beta(l)} \quad \text{and} \quad v_0'(b(l),l) = -\frac{\gamma}{1-\delta-\gamma b(l)}$$

together with Lemma 3.9, we derive

$$\begin{aligned}\lim_{l\downarrow R_1}&\left(\psi(l) - \log\left(\frac{1-\delta-\gamma b(l)}{1-\gamma\beta(l)}\right)\right)\\ &= \frac{1}{p}(1-\beta^*)\frac{1}{1-\gamma\beta^*} - \log\left(\frac{\beta^*(1-\delta-\gamma)}{1-\gamma\beta^*}\right) < 0.\end{aligned}$$

The last statement follows by the Implicit Function Theorem. □

Summarizing the results of the previous lemmas, for each $c \in (-p,1)$ we can find a $l_1(c) \in (R_1, \overline{l_1})$, such that unique $\beta, b \in (c,1)$ exist which fulfill the conditions $(3.14), (3.15), (3.16)$. In the following lemma we will discuss the monotonicity of l_1 as function of c. Define $\varphi(x) := \frac{R_1}{p}x(x-1+p)$.

Lemma 3.11

There is a unique $c_1 \in (-p, \frac{1}{2}(1-p))$, such that $l_1(c) < \varphi(c)$ for all $c \in (c_1, 1)$ and $l_1(c) = \varphi(c)$ for $c = c_1$. The function $l_1(c)$ is continuous and strictly decreasing on $(c_1, 1)$.

Proof. The derivative of v_0' with respect to c is given by

$$\frac{\partial}{\partial c} v_0'(x) = \underbrace{\left(\frac{x}{1-x}\right)^p \frac{1}{x(1-x)} \left(\frac{1-c}{c}\right)^p}_{>0} \left(1 - \left(c - \frac{l}{R_1}\right) \frac{p}{c(1-c)}\right) > 0.$$

Thus, v_0' is strictly increasing in c iff $l < \varphi(c)$.

By the Implicit Function Theorem the functions $\beta(c,l)$ and $b(c,l)$ are continuously differentiable with derivatives

$$\frac{\partial}{\partial c}\beta(c,l) = \frac{-\frac{\partial v_0'}{\partial c}(\beta(c,l);l)}{v_0''(\beta(c,l);l) + \frac{\gamma^2}{(1-\gamma\beta(c,l))^2}} > 0$$
$$b(c,l) = \frac{-\frac{\partial v_0'}{\partial c}(b(c,l);l)}{v_0''(b(c,l);l) + \frac{\gamma^2}{(1-\delta-\gamma b(c,l))^2}} < 0.$$

The function $\psi(c,l) - \log\left(\frac{1-\delta-\gamma b(c,l)}{1-\gamma\beta(l)}\right)$ is strictly increasing in c because

$$\begin{aligned}
\frac{\partial\psi}{\partial c}(c,l) - \frac{\partial}{\partial c}\log\left(\frac{1-\delta-\gamma b(c,l)}{1-\gamma\beta(c,l)}\right) \\
&= \underbrace{\left(v'(b(c,l);c,l) - \frac{\gamma}{1-\delta-\gamma b(c,l)}\right)}_{=0} b_c(c,l) \\
&\quad - \underbrace{\left(v'(\beta(c,l);c,l) - \frac{\gamma}{1-\gamma\beta(c,l)}\right)}_{=0} \beta_c(c,l) \\
&\quad + \frac{\partial v}{\partial c}(b(c,l);c,l) - \frac{\partial v}{\partial c}(\beta(c,l);c,l) \\
&= \int_{\beta(c,l)}^{b(c,l)} \frac{\partial v'}{\partial c}(x;c,l)dx > 0.
\end{aligned}$$

Furthermore,

$$\psi(c,l_1(c)) - \log\frac{1-\delta-\gamma b(c,l_1(c))}{1-\gamma\beta(c,l_1(c))} = 0$$

for all $c \in (-p,1)$. Thus, we have

$$\begin{aligned}
&\frac{\partial}{\partial c}\left(\psi(c,l_1(c)) - \log\frac{1-\delta-\gamma b(c,l_1(c))}{1-\gamma\beta(c,l_1(c))}\right) \\
&\quad + \frac{\partial}{\partial l}\left(\psi(c,l_1(c)) - \log\frac{1-\delta-\gamma b(c,l_1(c))}{1-\gamma\beta(c,l_1(c))}\right)\frac{\partial}{\partial c}l_1(c) = 0,
\end{aligned}$$

and finally

$$\frac{\partial}{\partial c} l_1(c) = -\frac{\frac{\partial}{\partial c}\left(\psi(c, l_1(c)) - \log \frac{1-\delta-\gamma b(c,l_1(c))}{1-\gamma\beta(c,l_1(c))}\right)}{\frac{\partial}{\partial l}\left(\psi(c, l_1(c)) - \log \frac{1-\delta-\gamma b(c,l_1(c))}{1-\gamma\beta(c,l_1(c))}\right)} < 0,$$

if $l_1(c) < \varphi(c)$.

$l_1(c) < \overline{l_1}(c) < \varphi(c)$ for $c \geq \frac{1}{2}(1-p)$ by Lemma 3.6 (iii). Thus, $l_1(c)$ is strictly decreasing on $\left[\frac{1}{2}(1-p), 1\right)$, hence there is a unique $c_1 \in \left(-p, \frac{1}{2}(1-p)\right)$ such that $\varphi(c_1) = l_1(c)$, because φ is strictly increasing on $\left(-p, \frac{1}{2}(1-p)\right]$. For all $c > c_1$ we have $l_1(c) < \varphi(c)$. □

We will show the existence of a, α, which satisfy conditions (3.17),(3.18), (3.19) analogously.

Lemma 3.12

We consider the interval $(0, c)$ *and define*

$$\overline{l_2} := \inf_{l>0} \left\{ \sup_{x\in(0,c]} v_0'(x,l) \geq \frac{\gamma}{1-\delta+\gamma x} \right\}.$$

Then, the following statements hold:

(i) For each $c \in (-p, 1)$ *and* $l \in (0, \overline{l_2}(c))$ *there exist unique* $a = a(l,c), \alpha = \alpha(l,c)$ *with* $-p < a < \alpha < c$, *such that*

$$v_0'(a) = \frac{\gamma}{1-\delta+\gamma a} \quad v_0'(\alpha) = \frac{\gamma}{1+\gamma\alpha}.$$

(ii) There exists a unique $c_2 \in (\frac{1}{2}(1-p), 1)$ *such that for each* $c \in (-p, c_2]$ *there is a unique* $l_2(c) \in (0, \overline{l_2}(c))$ *which satisfies*

$$v_0(a(l_2(c), c)) - v_0(\alpha(l_2(c), c)) = \log\left(\frac{1-\delta+\gamma a(l_2(c), c)}{1+\gamma\alpha(l_2(c), c)}\right).$$

(iii) l_2 *is continuous and strictly increasing on* $(-p, c_2)$ *with* $l_2(c) < \varphi(c)$ *for* $c < c_2$ *and* $l_2(c_2) = \varphi(c_2) > R_1$.

Proof. To prove this Lemma we can proceed as in Lemmas 3.7-3.11. □

Combining the previous results we are now able to show the existence of a CB-strategy, which is a QVI-control as defined in Proposition 3.4.

Proposition 3.13
There exist unique a, α, β, b*, such that* $0 < a < \alpha < \beta < b < 1$ *which satisfy the necessary conditions*

$$
\begin{aligned}
v_0'(b) &= -\frac{\gamma}{1-\delta-\gamma b} & v_0'(a) &= \frac{\gamma}{1-\delta+\gamma a}\\
v_0'(\beta) &= -\frac{\gamma}{1-\gamma\beta} & v_0'(\alpha) &= \frac{\gamma}{1+\gamma\alpha}\\
v_0(b)-v_0(\beta) &= \log\left(\frac{1-\delta-\gamma b}{1-\gamma\beta}\right) & v_0(a)-v_0(\alpha) &= \log\left(\frac{1-\delta+\gamma a}{1+\gamma\alpha}\right).
\end{aligned}
$$

Proof. It remains to show that there exists a unique c^*, such that $l_1(c^*) = l_2(c^*)$. $l_1(c)$ is strictly decreasing on $(c_1, 1)$, $l_2(c)$ is strictly increasing on on $(-p, c_2)$. $l_1(c_1) = \varphi(c_1) > l_2(c_1)$ and $l_2(c_2) = \varphi(c_2) > l_1(c_2)$. Hence, there is an intersection point $c^* \in (c_1, c_2)$. □

3.8 Verification Theorem

In Section 3.6 we assumed, that the function

$$
v(x) = \begin{cases} v_0(\alpha) + \Gamma(x,\alpha), & x \le a\\ v_0(x), & a < x \le b\\ v_0(\beta) + \Gamma(x,\beta), & x > b \end{cases}
$$

is a solution to the quasi variational inequalities corresponding to

$$
R^{K^*} = \sup_{K \in \mathcal{A}(x,\pi)} \liminf_{t\to\infty} \frac{1}{t}\mathbb{E}\left[\int_0^t g(\pi_{s-})ds + \sum_{n=0}^{M_t} \overline{\Gamma}(\pi_{\tau_n}, \eta_n)\right],
$$

where the constants a, α, β, b are given, such that the resulting QVI-control is a CB-strategy (a, α, β, b). The existence of such constants was shown in Section 3.7. Thus, it remains to show that v is indeed a solution to the QVIs.

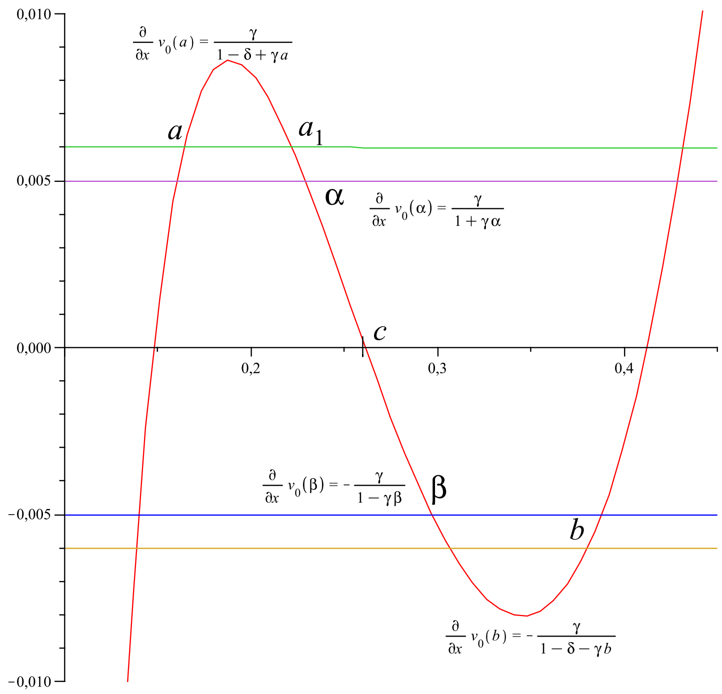

Figure 3.4: v_0' with cost functions and roots a, α, β, b which fulfill the necessary conditions. We will use a_1 in the proof of Verification Theorem.

Theorem 3.14

For the constants a, α, β, b as constructed in Proposition 3.13 under the assumption (3.20) and if additionally holds that

$$a < \frac{(1-\delta)\pi^*}{\gamma\pi^* + (1-\delta-\gamma)} < b,$$

the function

$$v(x) = \begin{cases} v_0(\alpha) + \Gamma(x,\alpha), & x \leq a \\ v_0(x), & a < x \leq b \\ v_0(\beta) + \Gamma(x,\beta), & x > b \end{cases}$$

satisfies the QVIs:

1. $L_\pi v + g - l \leq 0$ *on* $(0,1)$
2. $L_\pi v + g - l = 0$ *on* $D = (a,b)$

3. $\mathcal{M}v(x) - v(x) \leq 0$ *for all* $x \in (0,1)$.

Furthermore, the functions v *and* v' *are bounded.*

Proof: v_0 solves $L_\pi v_0(x)+g(x)-l = 0$ on (a,b) by Proposition 3.5. According to our construction of b and v_0 we have

$$L_\pi v(b) + g(b) - l = L_\pi v_0(b) + g(b) - l = 0.$$

For all $x \in (b,1)$ we have

$$\begin{aligned} L_\pi v(x) + g(x) &= -x(1-x)\left(\mu - \lambda \mathbb{E}[Z] - x\sigma^2\right)\frac{\gamma}{1-\delta-\gamma x} \\ &\quad - \frac{1}{2}\sigma^2 x^2 (1-x)^2 \left(\frac{\gamma}{1-\delta-\gamma x}\right)^2 \\ &\quad + \lambda \mathbb{E}\left[\log\left(\frac{1-\delta-\gamma\left(\frac{x(1+Z)}{1+Zx}\right)}{1-\delta-\gamma x}\right)\right] + g(x) \\ &= g\left(\frac{(1-\delta-\gamma)x}{1-\delta-\gamma x}\right), \end{aligned}$$

where

$$g(x) = -\frac{1}{2}\sigma^2 x^2 + (\mu - \lambda E[Z])x + \lambda E[\log(1+Zx)].$$

g is decreasing on $(\pi^*, 1)$ since $g'(1) < 0$ and there are no roots on $(\pi^*, 1)$ because

$$g'(x) = -\sigma^2 - \lambda E\left[\frac{Z^2}{(1+Zx)^2}\right] < 0.$$

Thus, under the assumption

$$\frac{(1-\delta)\pi^*}{\gamma\pi^* + (1-\delta-\gamma)} < b$$

the inequality

$$L_\pi v(x) + g(x) - l \leq 0$$

holds for all $x \geq b$. Using the same reasoning for a we show that

$$L_\pi v(x) + g(x) - l \leq 0$$

holds for all $x \leq a$ if

$$a < \frac{(1-\delta)\pi^*}{\gamma\pi^* + (1-\delta-\gamma)}.$$

So far, we showed that

$$\begin{aligned} L_\pi v(x) + g(x) - l &= 0 \quad \text{on } (a,b) \\ L_\pi v(x) + g(x) - l &\leq 0 \quad \text{on } (0,1). \end{aligned}$$

It remains to show that

$$\sup_{y\in(0,1)} \{v(y) + \Gamma(x,y)\} - v(x) \leq 0$$

for all $x \in (0,1)$. We assume first that $y > x$ and show

$$v(y) - v(x) + \Gamma(x,y) \leq 0$$

for all $x \in (0,1)$. We have

$$\begin{aligned} v(y) - v(x) + \Gamma(x,y) &= v(y) - v(x) + \log\left(\frac{1-\delta+\gamma x}{1+\gamma y}\right) \\ &\quad + \log\left(\frac{1+\gamma y}{1-\delta+\gamma y}\right) - \log\left(\frac{1+\gamma y}{1-\delta+\gamma y}\right) \\ &= \int_x^y v'(z) - \frac{\gamma}{1-\delta+\gamma z} dz - \log\left(\frac{1+\gamma y}{1-\delta+\gamma y}\right). \end{aligned}$$

$\log\left(\frac{1+\gamma y}{1-\delta+\gamma y}\right) > 0$ for all $y \in (0,1)$. Consider

$$v'(x) - \frac{\gamma}{1-\delta+\gamma x} = \begin{cases} \frac{\partial}{\partial x}\Gamma(x,\alpha) - \frac{\gamma}{1-\delta+\gamma x}, & x \leq a \\ v_0'(x) - \frac{\gamma}{1-\delta+\gamma x}, & a < y < a_1 \\ v_0'(x) - \frac{\gamma}{1-\delta+\gamma x}, & a_1 < y < b \\ \frac{\partial}{\partial x}\Gamma(x,\beta) - \frac{\gamma}{1-\delta+\gamma x}, & y \geq b, \end{cases}$$

where $a_1 \in (a,c)$ is a root of $v_0'(x) = \frac{\gamma}{1-\delta+\gamma a}$. Due to the construction of v we have (see Figure 3.4)

$$v'(x) - \frac{\gamma}{1-\delta+\gamma x} = \begin{cases} \frac{\gamma}{1-\delta+\gamma x} - \frac{\gamma}{1-\delta+\gamma x} = 0, & x \leq a \\ v_0'(x) - \frac{\gamma}{1-\delta+\gamma x} \geq 0, & a < y < a_1 \\ v_0'(x) - \frac{\gamma}{1-\delta+\gamma x} < 0, & a_1 < y < b \\ -\frac{\gamma}{1-\delta-\gamma x} - \frac{\gamma}{1-\delta+\gamma x} < 0, & y \geq b. \end{cases}$$

Thus, it remains to show that

$$f(y) := \int_a^y \left(v_0'(z) - \frac{\gamma}{1-\delta+\gamma z} \right) dz - \log\left(\frac{1+\gamma y}{1-\delta+\gamma y} \right) \leq 0$$

for all $a < y < b$. We know, that $f(\alpha) = 0$ and $f'(y) \geq 0$ for $y \leq \alpha$, where

$$\begin{aligned} f'(y) &= v_0'(y) - \frac{\gamma}{1-\delta+\gamma x} - \frac{\gamma}{1+\gamma y} + \frac{\gamma}{1-\delta+\gamma x} \\ &= v_0'(y) - \frac{\gamma}{1+\gamma y}. \end{aligned}$$

Thus, $f(y) \leq 0$ for all $a < y < b$.

The case $y \leq x$ can be proven analogously.

v and v' are continuous on $[0,1]$, hence the functions are bounded. □

3.9 Numerical Results

In order to derive the existence and optimality of a CB-strategy for our optimization problem we made three assumptions on the costs and the optimal CB-strategy. The first assumption

$$\alpha \geq a(1-\delta)$$

was made by transition from the original cost function $\overline{\Gamma}$ to the modified cost function Γ, see (3.11). We used the second assumption

$$\delta \leq 1 - \gamma + \frac{1+\gamma p}{p} \exp\left(\frac{1+p}{p(1+\gamma p)} \right)$$

in Lemma 3.10. The third assumption

$$a < \frac{(1-\delta)\pi^*}{\gamma\pi^* + (1-\delta-\gamma)} < b$$

was used in the proof of the Verification Theorem 3.14. In the following we see from numerical examples that, even if the costs are extremely high or extremely low, these assumptions are fulfilled. With parameters $\mu = 0.22, \sigma = 0.4, \lambda = 1$ and lognormally distributed jumps $(1+Z) \sim \mathcal{LN}(-0.15, 0.5)$ we calculate the growth rate of the pure-stock buy-and-hold

Table 3.1: Optimal CB-strategies for extreme costs.

Sc.	γ	δ	a	α	β	b
1	0.0001	0.0001	0.4398792	0.5317788	0.5359239	0.6263818
2	0.003	0.001	0.3613570	0.5170684	0.5513334	0.7021386
3	0.25	0.25	0.0217726	0.4151983	0.6391181	0.9943895
4	0.2	0.7	0.0007713	0.4588643	0.5814076	0.9999985
5	0.99	0.001	0.0501556	0.0805773	0.9978468	0.9999138
6	0.001	0.93	0.0000309	0.5898451	0.5904173	0.9999999

portfolio $R_1 = 0.0147$, the optimal risky fraction without costs $\pi^* = 0.5572$ and the parameter $p = -0.0675$. Note that, $R_1 > 0$, $\pi^* \in (0,1)$ and $p \in (-1,0)$ as required for the existence of the optimal CB-strategy. Now, we consider different scenarios for the costs δ, γ and compute optimal CB-strategies. The results are gathered in Table 3.1.

In all scenarios, even in 5 and in 6, where $\gamma = 0.99$ and $\delta = 0.93$, the assumptions above are fulfilled. Because of high proportional costs in scenario 5 the boundaries β, b and α, a are very close. The strategy is similar to the strategy with only fixed costs, see Figure 1.2. High fixed costs in scenario 6 push apart the outer boundaries a, b. Thus, the risky fraction evolves freely on almost the whole interval $(0,1)$. This is similar to pure proportional costs, see Figure 1.3. The functions v corresponding to Scenarios 1-6 are shown in Figure 3.5 on page 63 and the corresponding growth rates l are shown in Table 3.2. Note that v is not a value function. We interpret v as displacement cost, therefore v does not have to be concave. We see that in all cases $l \geq R_1$ and, as one might have expected, higher costs cause a smaller growth rate.

Figure 3.6 clarifies the dependence between the optimal strategies and costs. Here, we use the same parameters as in the beginning of the section. The left picture in Figure 3.6 shows the boundaries for increasing fixed costs δ and fixed $\gamma = 0.001$. The no-trading region $D = (a, b)$ becomes larger and the new risky fractions α and β move closer together, since the frequency of trading is punished more than the transaction size. This effect is more

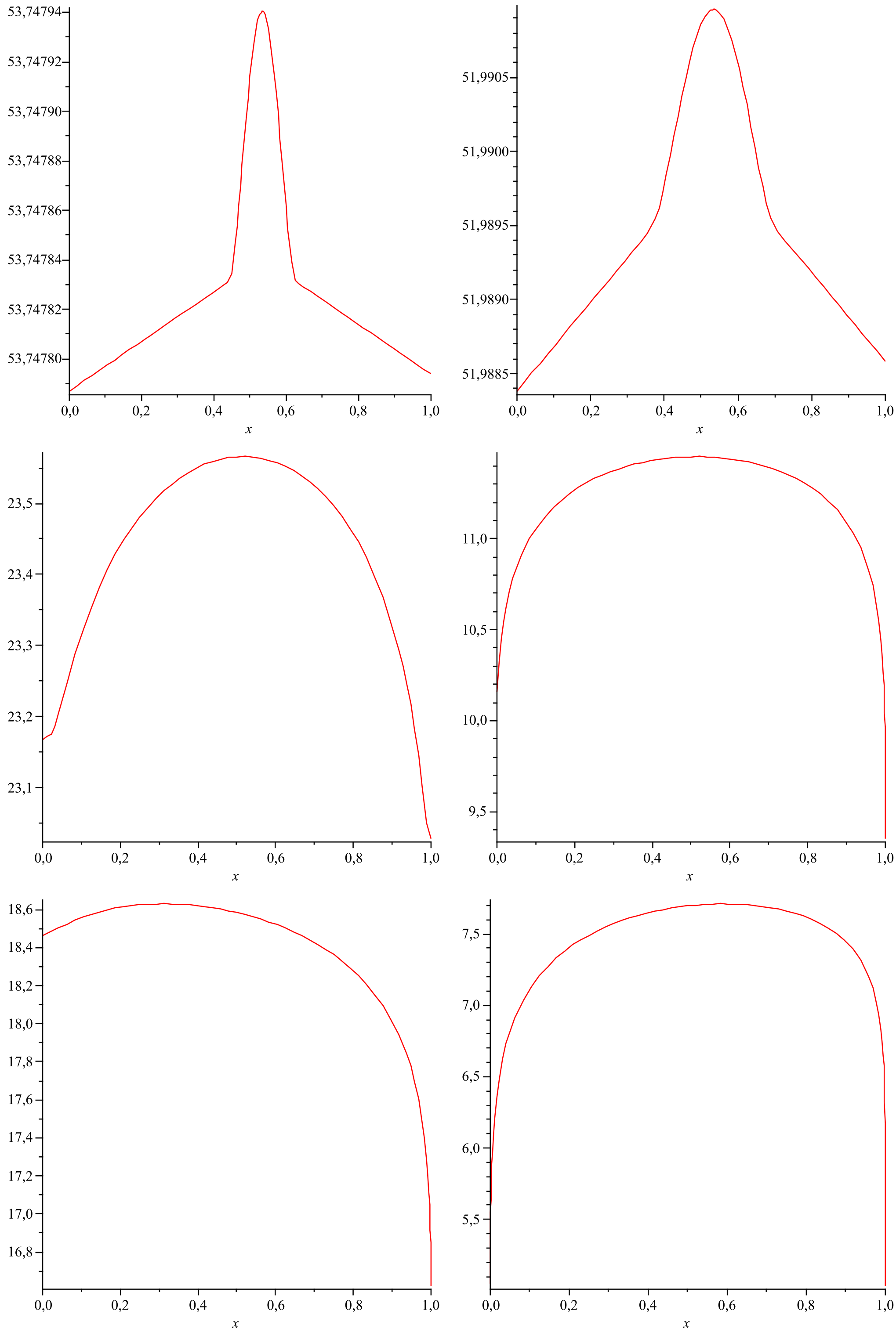

Figure 3.5: v for Scenarios 1-6.

Table 3.2: Growth rate for extreme costs.

Sc.	γ	δ	c	l
1	0.0001	0.0001	0.438853864	0.061341581
2	0.003	0.001	0.709147677	0.059615174
3	0.25	0.25	0.515225067	0.031528284
4	0.2	0.7	0.000767177	0.019557254
5	0.99	0.001	0.999929623	0.024714287
6	0.001	0.93	0.999999999	0.016801069

visible in the left picture in Figure 3.7, where we consider very small fixed costs. Due to increasing proportional costs, the new risky fractions α and β move closer to the outer boundaries, see the right picture in Figure 3.6. This becomes more evident for large γ in the right picture in Figure 3.7. This is explained through the fact that the transaction size is punished by higher proportional costs. The no-trading region (a, b) becomes larger since we do not want to trade very often due to the increasing costs. Figure 3.8 shows that the growth rate l decrease faster for increasing δ then for increasing γ, because the percentage of the wealth has higher impact on the growth rate, than the percentage of transaction size.

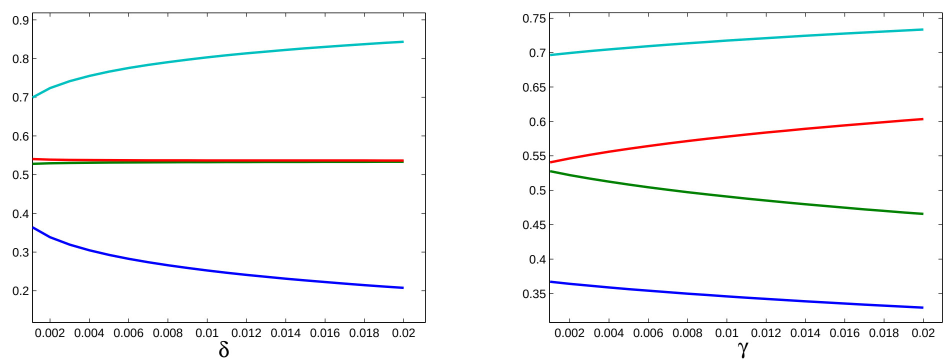

Figure 3.6: The constant boundary strategies against fixed costs (left) and proportional costs.

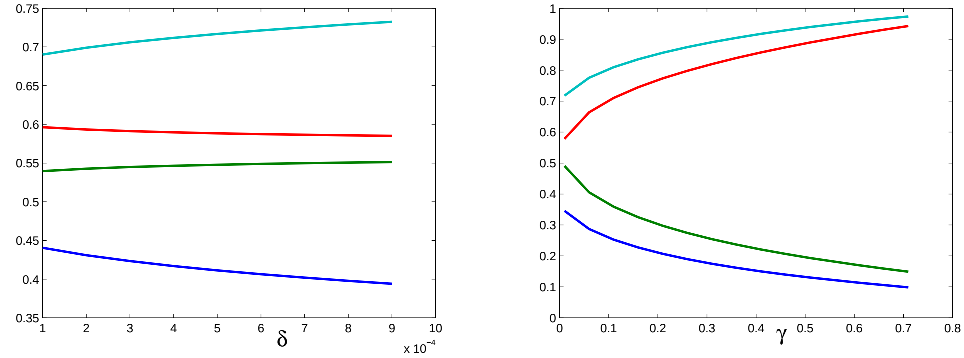

Figure 3.7: Boundaries for small δ (left) and large γ.

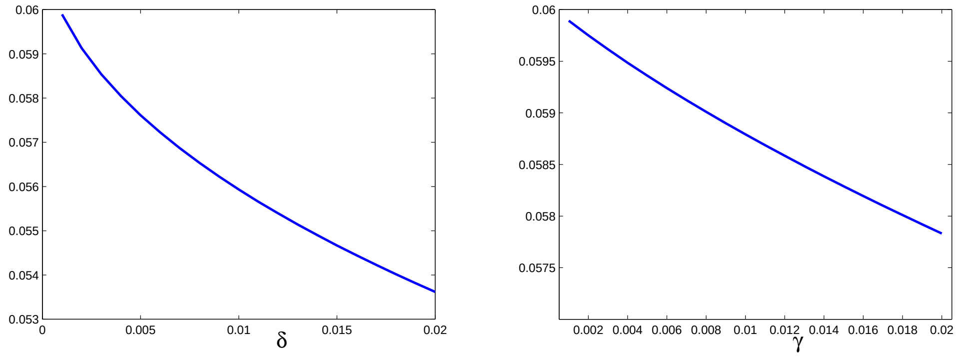

Figure 3.8: Optimal growth rate against δ (left) and γ.

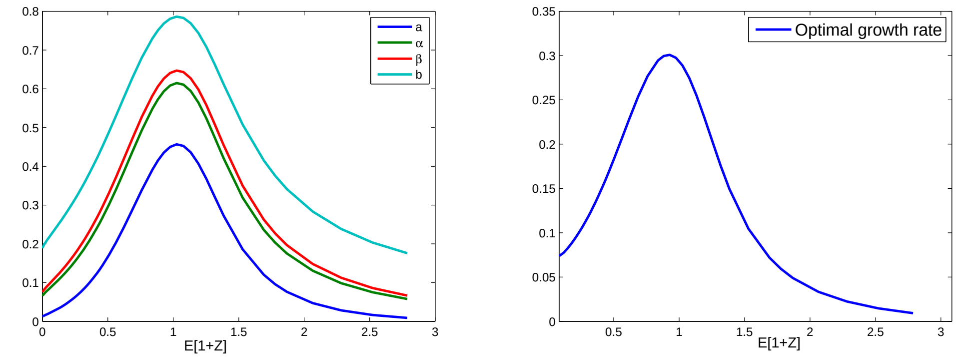

Figure 3.9: Boundaries (left) and optimal growth rate l against $\mathbb{E}[1+Z]$.

In the following we want to illustrate the influence of the jump parameters $\mathbb{E}[Z]$ and λ on the strategies and optimal growth rate. We recapitulate, that the stock evolves according to

$$S_t = S_0 \exp\left\{\left(\mu - \lambda\mathbb{E}[Z] - \frac{1}{2}\sigma^2\right)t + \sigma W_t\right\}\prod_{i=0}^{N_t}(1+Z_i), \tag{3.21}$$

where $(N_t)_{t\geq 0}$is a Poisson process with intensity $\lambda > 0$ and $(Z_i)_i$ are i.i.d random variables. Note that the stock jumps downwards if the jump $1+Z$ is between zero and one and upwards if $1+Z$ is bigger then one. Consider the following parameters $\sigma = 0.4,\ \mu = 0.9,\ \lambda = 5,\ \gamma = 0.003,\ \delta = 0.001$ and $(1+Z) \sim \mathcal{LN}(m, 0.5)$, where $m \in \mathbb{R}$, such that $\mathbb{E}[1+Z] \in (0.044, 2.7871)$. In the left picture of Figure 3.9 one can see the optimal boundaries for different values of $\mathbb{E}[1+Z]$. The boundaries increase on $(0.044, 1)$ and decrease on $(1, 2.7871)$, i.e. the investor puts less money in the stock. This effect is explained by higher riskiness of the stock with increasing jumps in both directions, which is also the reason for decreasing growth rate. The optimal growth rate l lies as expected between the optimal growth rate without costs R^* and pure-stock buy-and-hold growth rate R_1, see Figure 3.10 and Figure 3.13. Note that the transaction costs do not have much impact on the optimal growth rate.

Figure 3.12 shows the influence of increasing intensity of jumps λ on the optimal boundaries. For the calculation of the boundaries we used the following parameters: $\sigma = 0.4,\ \mu = 0.0999,\ \gamma = 0.006,\ \delta = 0.0001$ and $1+Z \sim \mathcal{LN}(0.5, 0.5)$. With increasing intensity of jumps the stock becomes more risky and the boundaries decrease rapidly.

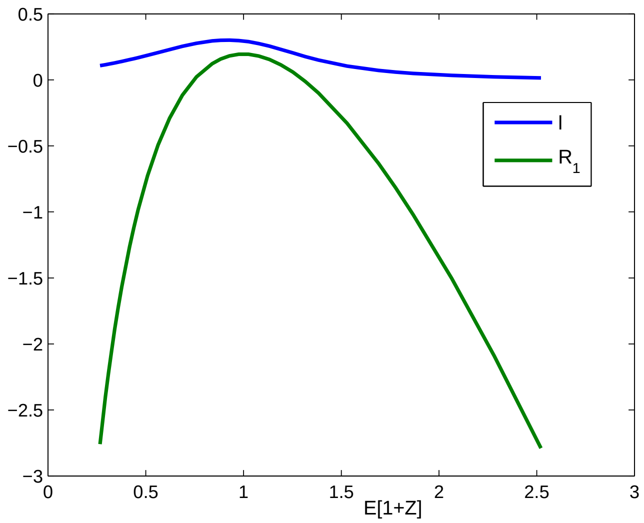

Figure 3.10: Optimal growth rate l and pure-stock buy-and-hold rate R_1 against $\mathbb{E}[1+Z]$.

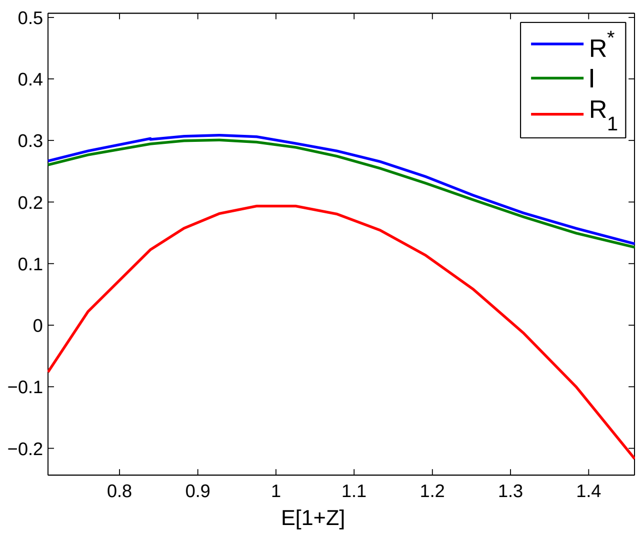

Figure 3.11: Optimal growth rate l, pure-stock buy-and-hold rate R_1 and optimal growth rate without costs R^* against $\mathbb{E}[1+Z]$.

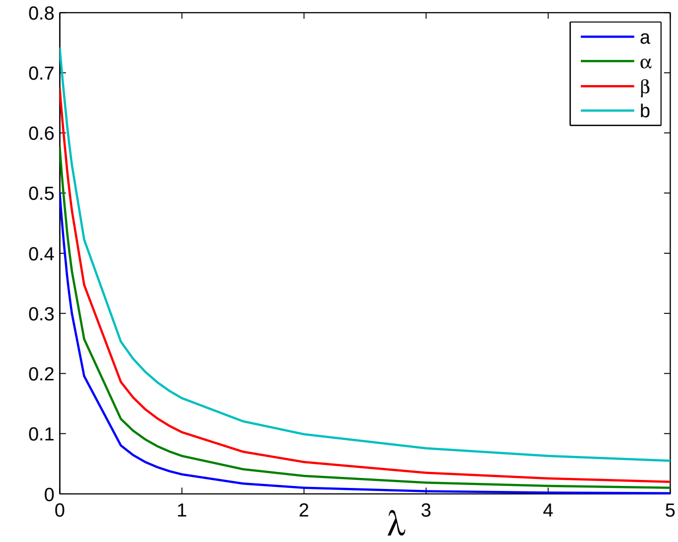

Figure 3.12: Boundaries against jump intensity λ.

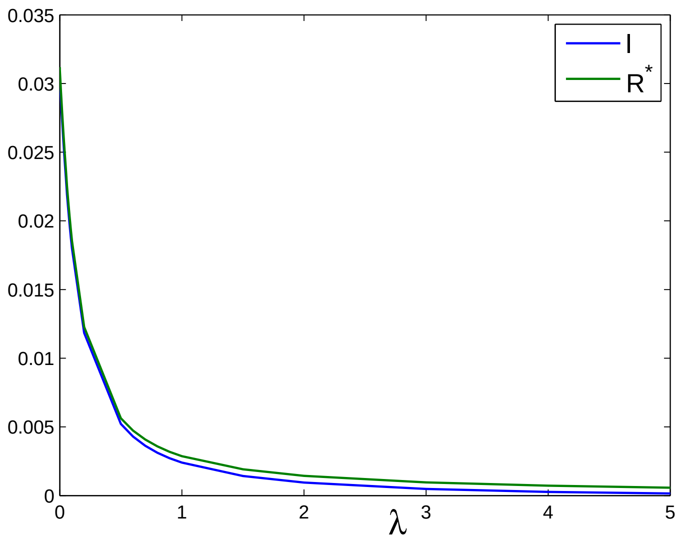

Figure 3.13: Optimal growth rate l and R^* against jump intensity λ.

Chapter 4

Expected Trading Frequency

4.1 Introduction

In the previous chapter we showed, that the optimal impulse control strategy for maximizing the expected growth rate can be found within the class of CB-strategies. These strategies are described by four parameters a, α, β, b, such that $a < \alpha \leq \beta < b$. In this chapter we consider the case $\alpha < \beta$ ($\gamma > 0$). The case $\alpha = \beta$ would simplify the arguments. The controls take place each time the boundaries of the no-trading region (a, b) are crossed and bring the risky fraction down to β or up to α. Each time a control occurs, the investor has to pay transaction costs, therefore it is important for her to know how often she has to rebalance her portfolio. In other words we are interested in the average inter trading time

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} (\tau_{i+1} - \tau_i).$$

We will apply a bijective transformation ϕ to the risky fraction process $(\pi_t)_{t\geq 0}$ and obtain the transformed risky fraction

$$\phi(\pi_t) = \phi(\pi) + (\mu - \lambda\mathbb{E}[Z] - \frac{1}{2}\sigma^2)t + \sigma W_t + \sum_{i=1}^{N_t} \log(1 + Z_i).$$

The advantage of this transformation is that the resulting process is a Lévy process with compound Poisson jumps. Because of the nice properties of this process we use the transformed risky fraction in the following. Since the transformation is bijective we obtain the same expected inter trading times for $(\pi_t)_{t\geq 0}$ and $(\phi(\pi_t))_{t\geq 0}$. Using the Ergodic Theorem from von Neumann and Birkhoff we show in Section 4.2 the convergence of the average inter trading time to the expected first exit time of the uncontrolled risky fraction under the invariant measure ν, which is given by

$$\mathbb{E}_\nu[\tau_{(a,b)}] = p \cdot \mathbb{E}^\alpha[\tau_{(a,b)}] + (1-p) \cdot \mathbb{E}^\beta[\tau_{(a,b)}],$$

where $(p, 1-p)$ is the invariant measure of the Markov chain $(\eta_n)_{n\in\mathbb{N}}$. In order to calculate $\mathbb{E}_\nu[\tau_{(a,b)}]$ we have to determine the expected first exit time starting in α and in β, i.e. $\mathbb{E}^\alpha[\tau_{(a,b)}]$, $\mathbb{E}^\beta[\tau_{(a,b)}]$ and the transition probabilities

$$p_{\alpha,\beta} = P^\alpha(\eta_1 = \beta),\ p_{\beta,\alpha} = P^\beta(\eta_1 = \alpha)$$

for the Markov chain $(\eta_n)_{n\in\mathbb{N}}$ to calculate the probability p.

In the continuous setting with Black-Scholes-driven stock prices these quantities are obtained in [Irle, Sass (2006)a] using the Optional Sampling Theorem and Dynkin's formula. We illustrate this method in Section 4.3. Unfortunately this approach fails if the stock is driven by a Lévy process. Nevertheless we can apply the theory of scale functions for spectrally negative Lévy processes and obtain the above quantities.

In Section 4.4 we introduce spectrally negative Lévy processes and show how $\mathbb{E}^\alpha[\tau_{(a,b)}]$, $\mathbb{E}^\beta[\tau_{(a,b)}]$, $p_{\alpha,\beta} = P^\alpha(\eta_1 = \beta)$, $p_{\beta,\alpha} = P^\beta(\eta_1 = \alpha)$ can be computed via scale functions. Since the scale functions are defined by their Laplace transforms, there are only few examples of processes where the scale function can be obtained explicitly. One of these examples are Lévy processes with phase-type distributed jumps. We introduce this kind of processes in Section 4.5 and present the results on scale functions from [Egami, Yamazaki (2011)a], [Egami, Yamazaki (2011)b]. Here, the authors obtain the scale functions explicitly using the roots of the Laplace transform of the process. We apply these results to the Lévy process with negative exponentially and hyperexponentially distributed jumps. With help of the scale functions, we are then able to compute $\mathbb{E}_\nu[\tau_{(a,b)}]$.

In Section 4.6 we compare the results for $\mathbb{E}_\nu[\tau_{(a,b)}]$ obtained by using the scale functions with a Monte Carlo simulation.

4.2 Convergence of the Average Inter Trading Time

On a probability space $(\Omega, \mathcal{F}, P)$ we consider a standard Brownian motion $(W_t)_{t\geq 0}$ and a compound Poisson process $(X_t)_{t\geq 0}$ defined by $X_t = \sum_{i=1}^{N_t} Z_i$ with $\mathbb{E}[Z] < \infty$, where $Z \sim Z_i$. Furthermore we consider a random variable π_0. $(W_t)_{t\geq 0}$, $(X_t)_{t\geq 0}$ and π_0 are supposed to be independent. Let $(\mathcal{G}_t)_{t\geq 0}$ be the filtration generated by $(W_t)_{t\geq 0}$, $(X_t)_{t\geq 0}$ and π_0 augmented with null-sets to satisfy the usual conditions. As usual we write $P^\pi(\cdot) = P(\cdot|\pi_0 = \pi)$.

We consider a risky fraction process $(\pi_t)_{t\geq 0}$ which evolves freely without interventions by the investor. The dynamics of the uncontrolled risky fraction process can be obtained similarly to the proof of Proposition 3.2, if we set $\Delta_n = 0$ for all n:

$$\begin{aligned} d\pi_t = \pi_{t-}(1-\pi_{t-})(\mu - \lambda\mathbb{E}[Z] - \sigma^2\pi_{t-})dt + \pi_{t-}(1-\pi_{t-})\sigma dW_t \\ + \int_E \pi_{t-}(1-\pi_{t-})\frac{x}{1+\pi_{t-}x}J(dx,dt). \end{aligned} \tag{4.1}$$

In the previous chapter we solved the problem of maximizing the asymptotic growth rate. We found an optimal impulse control strategy $(\tau_n, \eta_n)_{n\in\mathbb{N}}$ within the class of constant boundary strategies. Let (a, α, β, b) be an optimal CB-strategy. By Proposition 3.4, the intervention times $0 = \tau_0 \leq \tau_1 ... \leq \infty$ are given by the exit times from the no-trading region (a, b):

$$\tau_n := \inf\{t \geq \tau_{n-1} : \pi_t \notin (a,b)\}, n \in \mathbb{N},$$

and the new risky fractions are $\mathcal{F}_{\tau_n}$-measurable variables $(\eta_n)_{n\in\mathbb{N}}$ satisfying

$$\begin{aligned} &\mathcal{M}v(\pi_{\tau_n}) = v(\eta_n) + \Gamma(\pi_{\tau_n}, \eta_n), \ n \in \mathbb{N}, \text{ where} \\ &\mathcal{M}v(x) = \sup\{v(y) + \Gamma(x,y), y \in (0,1)\} \text{ and} \\ &\Gamma(x,y) = \begin{cases} \log\frac{1-\delta-\gamma x}{1-\gamma y}, & y \leq x \\ \log\frac{1-\delta+\gamma x}{1+\gamma y}, & y > x. \end{cases} \end{aligned}$$

In particular,

$$\eta_n = \begin{cases} \alpha, & \pi_{\tau_n} \leq a \\ \beta, & \pi_{\tau_n} \geq b. \end{cases}$$

We denote by $(\pi_t^{CB})_{t\geq 0}$ the risky fraction process, controlled by a constant boundary strategy. $(\pi_t^{CB})_{t\geq 0}$ can be understood as a composition of independent copies of the uncontrolled risky fraction $(\pi_t)_{t\geq 0}$ as given in (4.1), which starts after the n-th trading in τ_n with initial value η_n. Note that using the initial values η_n may induce some dependency.

For further investigation the risky fraction process (4.1) can be simplified significantly via the transformation

$$\begin{aligned} \phi : (0,1) &\to \mathbb{R} \\ x &\mapsto \log\left(\frac{x}{1-x}\right) \end{aligned}$$

into a Lévy process with constant volatility, drift and compound Poisson process. Since $\pi_t \in (0,1)$ for all $t \geq 0$, we can apply Itô's formula to obtain:

$$\begin{aligned} \phi(\pi_t) =& \phi(\pi_0) + \int_0^t \frac{1}{\pi_{s-}(1-\pi_{s-})} d\pi_s^c + \frac{1}{2}\int_0^t \frac{2\pi_{s-}}{\pi_{s-}^2(1-\pi_{s-})^2} d[\pi_s^c, \pi_s^c] \\ &+ \sum_{\substack{0<s\leq t \\ \Delta X_s \neq 0}} \log\left(\frac{\pi_s}{1-\pi_s}\right) - \log\left(\frac{\pi_{s-}}{1-\pi_{s-}}\right) \\ =& \phi(\pi_0) + \int_0^t \left(\mu - \lambda\mathbb{E}[Z] - \frac{1}{2}\sigma^2\right) ds + \int_0^t \sigma dW_s \\ &+ \sum_{\substack{0<s\leq t \\ \Delta X_s \neq 0}} \log\left(\frac{\pi_{s-} + \Delta X_s \frac{\pi_{s-}(1-\pi_{s-})}{1+\Delta X_s \pi_{s-}}}{1-\pi_{s-} - \Delta X_s \frac{\pi_{s-}(1-\pi_{s-})}{1+\Delta X_s \pi_{s-}}}\right) - \log\left(\frac{\pi_{s-}}{1-\pi_{s-}}\right) \\ =& \phi(\pi_0) + \left(\mu - \lambda\mathbb{E}[Z] - \frac{1}{2}\sigma^2\right) t + \sigma W_t + \sum_{i=1}^{N_t} \log(1+Z_i). \end{aligned}$$

Since ϕ is a bijective and increasing transformation, the following holds for the first exit time

$$\tau_{(a,b)} := \inf\left\{t \geq 0 : \pi_t \notin (a,b)\right\} = \inf\left\{t \geq 0 : \phi(\pi_t) \notin (\phi(a), \phi(b))\right\}.$$

Thus we will obtain the same results for quantities such as $\mathbb{E}[\tau_{(a,b)}|\pi_0 = \pi]$ if we work with the transformed process $(\phi(\pi_t))_{t\geq 0}$ and the transformed boundaries $(\phi(a), \phi(b))$. In the following we denote the transformed risky fraction process by $\overline{\pi}_t = \phi(\pi_t)$. For simplicity we continue using the notation a, α, β, b and $(\eta_n)_{n\in\mathbb{N}}$ for the transformed boundaries instead of $\overline{a}, \overline{\alpha}, \overline{\beta}, \overline{b}$ and $(\overline{\eta}_n)_{n\in\mathbb{N}}$.

Proposition 4.1
$P^{\overline{\pi}}(\tau_{(a,b)} < \infty) = 1$ for $\overline{\pi} \in (a,b)$, where

$$\tau_{(a,b)} = \inf\left\{t \geq 0 : \overline{\pi}_t \notin (a,b)\right\}.$$

Proof. We have

$$\mathbb{E}^{\overline{\pi}}[\overline{\pi}_1] = \overline{\pi} + \mu - \lambda\mathbb{E}[Z] - \frac{1}{2}\sigma^2 + \lambda\mathbb{E}[\log(1+Z)].$$

Since $\mathbb{E}[Z] < \infty$ we have $\mathbb{E}[\overline{\pi}_1] \in (-\infty, \infty)$. By Theorem 7.2 from [Kyprianou (2006)] it holds

(1) $\mathbb{E}^{\overline{\pi}}[\overline{\pi}_1] > 0 \Rightarrow \lim_{t\to\infty} \overline{\pi}_t = \infty$,

(2) $\mathbb{E}^{\overline{\pi}}[\overline{\pi}_1] < 0 \Rightarrow \lim_{t\to\infty} \overline{\pi}_t = -\infty$,

(3) $\mathbb{E}^{\overline{\pi}}[\overline{\pi}_1] = 0 \Rightarrow \limsup_{t\to\infty} \overline{\pi}_t = -\liminf_{t\to\infty} \overline{\pi}_t = \infty$.

These three cases imply the assertion. □

Consider the uncontrolled risky fraction process

$$\overline{\pi}_t = \overline{\pi}_0 + \mu' t + \sigma W_t + \underbrace{\sum_{i=1}^{N_t} \log(1+Z_i) - \lambda\mathbb{E}[\log(1+Z)]t}_{:=Y_t}, \tag{4.2}$$

where $\mu' = \mu - \lambda\mathbb{E}[Z - \log(1+Z)] - \frac{1}{2}\sigma^2$ and $(Y_t)_{t\geq 0}$ is a compensated compound Poisson process. Further consider a CB-strategy $(\tau_n, \eta_n)_{n\in\mathbb{N}}$, which controls the process $(\overline{\pi}_t^{CB})_{t\geq 0}$. Between τ_n and τ_{n+1} the process evolves like

$$\overline{\pi}_t^{CB} = \eta_n + \mu'(t - \tau_n) + \sigma(W_t - W_{\tau_n}) + (Y_t - Y_{\tau_n}),$$

for $\tau_n \le t < \tau_{n+1}$. We denote by

$$W_t^{\tau_i} := W_{\tau_i+t} - W_{\tau_i}, \quad Y_t^{\tau_i} := Y_{\tau_i+t} - Y_{\tau_i}, \quad t \ge 0,$$

the Brownian motion and compensated compound Poisson process shifted by τ_i. By Proposition 2.10, W^{τ_i} is again a Brownian motion and Y^{τ_i} a compensated compound Poisson process with the same distribution as $(W_t)_{t\ge0}$ and $(Y_t)_{t\ge0}$, respectively. Now starting in τ_1 with η_1, the next stopping time is given by

$$\begin{aligned}\tau_2 &= \inf\left\{t \ge \tau_1 : \eta_1 + \mu'(t-\tau_1) + \sigma(W_t - W_{\tau_1}) + (Y_t - Y_{\tau_1}) \notin (a,b)\right\}\\ &= \inf\left\{t \ge 0 : \eta_1 + \mu' t + \sigma W_t^{\tau_1} + Y_t^{\tau_1} \notin (a,b)\right\} + \tau_1.\end{aligned}$$

Thus we can write

$$\tau_2 - \tau_1 = \inf\left\{t \ge 0 : \eta_1 + \mu' t + \sigma W_t^{\tau_1} + Y_t^{\tau_1} \notin (a,b)\right\}.$$

This means that we can use a measurable function

$$f : S \to \mathbb{R}_+$$

on $S := \{\alpha, \beta\} \times D([0,\infty),\mathbb{R})$, where $D([0,\infty),\mathbb{R})$ is the set of all càdlàg-functions from $[0,\infty)$ to $\mathbb{R}$, to write

$$\begin{aligned}f(\eta_i, \sigma W^{\tau_i} + Y^{\tau_i}) &= \inf\left\{t \ge 0 : \eta_i + \mu' t + \sigma W_t^{\tau_i} + Y_t^{\tau_i} \notin (a,b)\right\}\\ &= \tau_{i+1} - \tau_i \end{aligned} \tag{4.3}$$

for all $i \in \mathbb{N}$ and the initial risky fraction $\overline{\pi}_0 = \overline{\pi}$ a.s. We equip S with the σ-field

$$\mathcal{S} = \mathcal{P}(\{\alpha,\beta\}) \otimes \mathcal{B}(D([0,\infty),\mathbb{R})).$$

The new risky fraction η_2 is determined by

$$\begin{aligned}\eta_2 =& \alpha \cdot \mathbb{1}_{(-\infty,a]}(\eta_1 + \mu'(\tau_2-\tau_1) + \sigma W^{\tau_1}_{\tau_2-\tau_1} + Y^{\tau_1}_{\tau_2-\tau_1})\\ &+ \beta \cdot \mathbb{1}_{[b,\infty)}(\eta_1 + \mu'(\tau_2-\tau_1) + \sigma W^{\tau_1}_{\tau_2-\tau_1} + Y^{\tau_1}_{\tau_2-\tau_1}).\end{aligned}$$

Since $\tau_2 - \tau_1 = f(\eta_1, \sigma W^{\tau_1} + Y^{\tau_1})$, η_2 depends only on η_1 and $\sigma W^{\tau_1} + Y^{\tau_1}$. Thus, η_{i+1} is given by a measurable function F on S such that

$$F : S \to \{\alpha, \beta\},$$

$$
\begin{aligned}
&F(\eta_i, \sigma W^{\tau_i} + Y^{\tau_i}) \\
&= \alpha \mathbb{1}_{(-\infty,a]} \left(\eta_i + \mu' f(\eta_i, \sigma W^{\tau_i} + Y^{\tau_i}) + \sigma W^{\tau_i}_{f(\eta_i, \sigma W^{\tau_i} + Y^{\tau_i})} + Y^{\tau_i}_{f(\eta_i, \sigma W^{\tau_i} + Y^{\tau_i})} \right) \\
&+ \beta \mathbb{1}_{[b,\infty)} \left(\eta_i + \mu' f(\eta_i, \sigma W^{\tau_i} + Y^{\tau_i}) + \sigma W^{\tau_i}_{f(\eta_i, \sigma W^{\tau_i} + Y^{\tau_i})} + Y^{\tau_i}_{f(\eta_i, \sigma W^{\tau_i} + Y^{\tau_i})} \right) \\
&= \eta_{i+1} \qquad (4.4)
\end{aligned}
$$

for all $i \in \mathbb{N}$.

Lemma 4.2

Consider the risky fraction process $(\overline{\pi}_t^{CB})_{t \geq 0}$ controlled by a constant boundary strategy (a, α, β, b) and the first exit time of the uncontrolled process $\tau_{(a,b)} = \inf\{t \geq 0 | \overline{\pi}_t \notin (a,b)\}$, then the following holds:

(i) $P^{\overline{\pi}}(\tau_{i+1} - \tau_i \in \cdot | \mathcal{G}_{\tau_i}) = P^{\eta_i}(\tau_{(a,b)} \in \cdot)$;

(ii) The process $(\eta_i)_{i \in \mathbb{N}}$ is a homogeneous Markov chain with state space $\{\alpha, \beta\}$, initial distribution $P^{\overline{\pi}}(\eta_1 \in \cdot)$, and transition probabilities

$$
\begin{aligned}
p_{\alpha,\alpha} &:= P^{\overline{\pi}}(\eta_i = \alpha | \eta_{i-1} = \alpha) = P^{\alpha}(\eta_1 = \alpha), \\
p_{\alpha,\beta} &:= P^{\overline{\pi}}(\eta_i = \beta | \eta_{\tau_{i-1}} = \alpha) = P^{\alpha}(\eta_1 = \beta), \\
p_{\beta,\alpha} &:= P^{\overline{\pi}}(\eta_i = \alpha | \eta_{\tau_{i-1}} = \beta) = P^{\beta}(\eta_1 = \alpha), \\
p_{\beta,\beta} &:= P^{\overline{\pi}}(\eta_i = \beta | \eta_{\tau_{i-1}} = \beta) = P^{\beta}(\eta_1 = \beta).
\end{aligned}
$$

The invariant distribution $(p, 1-p)^\top$ of $(\eta_i)_{i \in \mathbb{N}}$ is given by

$$p = \frac{p_{\beta,\alpha}}{p_{\alpha,\beta} + p_{\beta,\alpha}}.$$

Proof. (i) As we have seen above, there is a measurable function f, such that

$$\tau_{i+1} - \tau_i = f(\eta_i, \sigma W^{\tau_i} + Y^{\tau_i}).$$

Note that η_i is independent of $\sigma W^{\tau_i} + Y^{\tau_i}$ and η_i is $\mathcal{G}_{\tau_i}$-measurable. Thus the claim (i) follows by

$$
\begin{aligned}
P^{\overline{\pi}}(\tau_{i+1} - \tau_i \in \cdot | \mathcal{G}_{\tau_i}) &= P^{\overline{\pi}}(f(\eta_i, \sigma W^{\tau_i} + Y^{\tau_i}) \in \cdot | \mathcal{G}_{\tau_i}) \\
&= P^{\overline{\pi}}(f(y, \sigma W + Y) \in \cdot)|_{y = \eta_i} \\
&= P^{\eta_i}(\tau_{(a,b)} \in \cdot).
\end{aligned}
$$

(ii) The new risky fraction η_{i+1} after trading in τ_{i+1} is given by a measurable function F such that

$$\eta_{i+1} = F(\eta_i, \sigma W^{\tau_i} + Y^{\tau_i}),$$

see (4.4). $(\eta_i)_{i\in\mathbb{N}}$ is a homogeneous Markov chain, since it holds

$$\begin{aligned} P^{\overline{\pi}}(\eta_{i+1} \in \cdot | \mathcal{G}_{\tau_i}) &= P^{\overline{\pi}}(F(\eta_i, \sigma W^{\tau_i} + Y^{\tau_i}) \in \cdot | \mathcal{G}_{\tau_i}) \\ &= P^{\overline{\pi}}(F(y, \sigma W + Y) \in \cdot)|_{y=\eta_i} \\ &= P^{\eta_i}(\eta_1 \in \cdot). \end{aligned}$$

The invariant distribution of $(\eta_i)_{i\in\mathbb{N}}$ is obtained from the equation

$$(p, 1-p) \cdot \begin{pmatrix} p_{\alpha,\alpha} & p_{\alpha,\beta} \\ p_{\beta,\alpha} & p_{\beta,\beta} \end{pmatrix} = \begin{pmatrix} p \\ 1-p \end{pmatrix},$$

where we find a solution since $p_{\alpha,\beta} > 0$ or $p_{\beta,\alpha} > 0$ by Proposition 4.1. □

Definition 4.3
For $\alpha < \beta$ we define a measure ν on $\{\alpha, \beta\}$ by the invariant distribution

$$\nu(\alpha) = 1 - \nu(\beta) = p.$$

The probability measure with ν as initial distribution is defined by

$$P_\nu = pP^\alpha + (1-p)P^\beta.$$

Recall that we consider $\alpha < \beta$. We assume that $(\eta_i)_{i\in\mathbb{N}}$ is stationary, i.e. $\eta_i \sim \nu$. This is the case if we start with $\pi_0 \sim \nu$.

The average inter trading time of the constant boundary strategy $(\tau_i, \eta_i)_{i\in\mathbb{N}}$ is given by

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} (\tau_{i+1} - \tau_i).$$

Since the variables $(\tau_{i+1} - \tau_i)_{i\in\mathbb{N}}$ are stationary by Lemma 4.2, but not necessarily independent we cannot obtain the limit by applying the law of large numbers. One could try the renewal theoretic arguments here. Instead we use an extension of the law of large numbers to stationary random sequences. This extension is the mean and a.s. ergodic theorem, the proof of which can be found in [Kallenberg (1997)] (Theorem 9.6):

Theorem 4.4 (Ergodic Theorem, von Neumann, Birkhoff)
Fix a measurable space $(S,\mathcal{S})$, *a measurable transformation* T *on* S *with associated invariant* σ*-field* $\mathcal{I}$, *i.e.* $\mathcal{I}=\{A\in\mathcal{S}:T^{-1}(A)=A\}$ *and a random element* ξ *in* S *with* $T\xi := T\circ\xi \stackrel{d}{=} \xi$. *Consider a measurable function* $f:S\to\mathbb{R}_+$. *Then*

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n} f(T^{i-1}\xi)=\mathbb{E}[f(\xi)|\xi^{-1}(\mathcal{I})] \ \text{a.s.}$$

Due to this theorem it is sufficient to find a measure preserving mapping T, a random variable ξ and a function f, such that $(\tau_{i+1}-\tau_i)=f(T^{i-1}\xi)$ to obtain the limit of the inter trading time.

Theorem 4.5
Consider a constant boundary impulse control strategy $(\tau_i,\eta_i)_{i\in\mathbb{N}}$, *with underlying risky fraction process* $(\overline{\pi}_t)_{t\geq 0}$. *Then for* $\xi := (\eta_1, \sigma W^{\tau_1}+Y^{\tau_1})$ *there exists a measurable function*

$$T:(S,\mathcal{S})\to(S,\mathcal{S})$$

such that

$$T(\eta_i,\sigma W^{\tau_i}+Y^{\tau_i})=(\eta_{i+1},\sigma W^{\tau_{i+1}}+Y^{\tau_{i+1}}),$$

in particular

$$(\eta_i,\sigma W^{\tau_i}+Y^{\tau_i})=T^{i-1}(\eta_1,\sigma W^{\tau_1}+Y^{\tau_1}),$$

for $i\in\mathbb{N},\ i>1$. *Furthermore* T *is measure preserving, i.e.* $T\xi\stackrel{d}{=}\xi$ *and the limit of the average inter trading time is given by*

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}(\tau_{i+1}-\tau_i)=\mathbb{E}[\tau_2-\tau_1|\xi^{-1}(\mathcal{I})] \ \text{a.s.},$$

where $\mathcal{I}=\{A\in\mathcal{S}:T^{-1}(A)=A\}$ *is the* T*-invariant* σ*-field.*

Proof. Clearly, we can compose the function F from (4.4) and f from (4.3) to obtain the function T on the measurable space $(S,\mathcal{S})$ such that

$$T(\eta_i,\sigma W^{\tau_i}+Y^{\tau_i})=\left(F(\eta_i,\sigma W^{\tau_i}+Y^{\tau_i}),(\sigma W^{\tau_i}+Y^{\tau_i})^{\tau_{i+1}-\tau_i}\right),$$

where

$$f(\eta_i, \sigma W^{\tau_i} + Y^{\tau_i}) = \tau_{i+1} - \tau_i.$$

Plugging in the function F defined in (4.4), we have

$$T(\eta_i, \sigma W^{\tau_i} + Y^{\tau_i}) = (\eta_{i+1}, \sigma W^{\tau_{i+1}} + Y^{\tau_{i+1}}).$$

It is clear that

$$(\eta_i, \sigma W^{\tau_i} + Y^{\tau_i}) = T^{i-1}(\eta_1, \sigma W^{\tau_1} + Y^{\tau_1}).$$

The function T is measure preserving since η_1 is independent of $\sigma W^{\tau_1} + Y^{\tau_1}$ and the shifted processes $\sigma W^{\tau_i} + Y^{\tau_i}$ have the same distribution for all $i \in \mathbb{N}$ due to the Theorem 2.10 as well as $\eta_i \sim \eta_1$ for all $i \in \mathbb{N}$. This means

$$P^{(\eta_1, \sigma W^{\tau_1} + Y^{\tau_1})} = P^{\eta_1} \otimes P^{\sigma W^{\tau_1} + Y^{\tau_1}} = P^{\eta_i} \otimes P^{\sigma W^{\tau_i} + Y^{\tau_i}}.$$

Thus the requirements of Theorem 4.4 are fulfilled and we obtain the limit

$$\begin{aligned} \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} (\tau_{i+1} - \tau_i) &= \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} f(T^{i-1}\xi) \\ &= \mathbb{E}[f(\eta_1, \sigma W^{\tau_1} + Y^{\tau_1}) | \xi^{-1}\mathcal{I}] = \mathbb{E}[\tau_2 - \tau_1 | \xi^{-1}\mathcal{I}]. \end{aligned}$$

□

Remark 4.6. For the σ-field $\mathcal{S} = \mathcal{P}(\{\alpha, \beta\}) \otimes \mathcal{B}(D([0,\infty), \mathbb{R}))$ the sets can be written as

$$A = (\{\alpha\} \times E) \cup (\{\beta\} \times F),$$

where $E, F \in \mathcal{B}(D([0,\infty), \mathbb{R}))$. Thus we can also write

$$\mathcal{S} = \{(\{\alpha\} \times E) \cup (\{\beta\} \times F) \,|\, E, F \in \mathcal{B}(D([0,\infty), \mathbb{R}))\}.$$

Proposition 4.7

Consider a set $A \in \mathcal{I}$, where

$$\mathcal{I} = \left\{A \in \mathcal{S} : T^{-1}A = A\right\},$$

and A has the form

$$A = (\{\alpha\} \times E) \cup (\{\beta\} \times F),$$

with some $E,\ F \in \mathcal{B}(D([0,\infty), \mathbb{R}))$. Then it holds that $\mu(E) = \mu(F)$, where $\mu := P^{\sigma W + Y}$.

Proof. Since A is T-invariant, it holds for $\mathbb{1}_A(\eta_1, \sigma W^{\tau_1} + Y^{\tau_1})$

$$\begin{aligned}\mathbb{1}_A(\eta_1, \sigma W^{\tau_1} + Y^{\tau_1}) &= \mathbb{1}_{T^{-1}A}(\eta_1, \sigma W^{\tau_1} + Y^{\tau_1})\\ &= \mathbb{1}_A(T(\eta_1, \sigma W^{\tau_1} + Y^{\tau_1}))\\ &= \mathbb{1}_A(\eta_2, \sigma W^{\tau_2} + Y^{\tau_2}).\end{aligned}$$

We apply $\mathbb{E}(\cdot|\eta_1)$ on both sides of the above equation. The following holds for the right-hand side using Lemma 4.2 and the fact that $\mathcal{G}_{\tau_2}$ and $\sigma W^{\tau_2} + Y^{\tau_2}$ are independent

$$\begin{aligned}&\mathbb{E}[\mathbb{1}_A(\eta_2, \sigma W^{\tau_2} + Y^{\tau_2})|\eta_1]\\ &= \mathbb{E}[\mathbb{E}[\mathbb{1}_A(\eta_2, \sigma W^{\tau_2} + Y^{\tau_2})|\mathcal{G}_{\tau_2}]|\eta_1]\\ &= \mathbb{E}[\mathbb{E}[\mathbb{1}_{\{\alpha\}\times E}(\eta_2, \sigma W^{\tau_2} + Y^{\tau_2})|\mathcal{G}_{\tau_2}]|\eta_1] + \mathbb{E}[\mathbb{E}[\mathbb{1}_{\{\beta\}\times F}(\eta_2, \sigma W^{\tau_2} + Y^{\tau_2})|\mathcal{G}_{\tau_2}]|\eta_1]\\ &= \left(\mathbb{1}_{\{\eta_1=\alpha\}}p_{\alpha,\alpha} + \mathbb{1}_{\{\eta_1=\beta\}}p_{\beta,\alpha}\right)\mu(E) + \left(\mathbb{1}_{\{\eta_1=\alpha\}}p_{\alpha,\beta} + \mathbb{1}_{\{\eta_1=\beta\}}p_{\beta,\beta}\right)\mu(F).\end{aligned}$$

On the left-hand side we use that η_1 and $\sigma W^{\tau_1} + Y^{\tau_1}$ are independent, thus it holds

$$\begin{aligned}&\mathbb{E}[\mathbb{1}_A(\eta_1, \sigma W^{\tau_1} + Y^{\tau_1})|\eta_1) = \mathbb{E}[\mathbb{1}_A(\eta_1, \sigma W^{\tau_1} + Y^{\tau_1})]\\ &= \mathbb{1}_{\{\eta_1=\alpha\}}\mu(E) + \mathbb{1}_{\{\eta_1=\beta\}}\mu(F).\end{aligned}$$

Combining the left and the right side using the assumption $p_{\alpha,\beta}$, $p_{\beta,\alpha} > 0$ yields

$$\begin{aligned}&\mathbb{1}_{\{\eta_1=\alpha\}}(p_{\alpha,\alpha}\mu(E) + p_{\alpha,\beta}\mu(F) - \mu(E)) + \mathbb{1}_{\{\eta_1=\beta\}}(p_{\beta,\alpha}\mu(E) + p_{\beta,\beta}\mu(F) - \mu(F))\\ &= \mathbb{1}_{\{\eta_1=\alpha\}}p_{\alpha,\beta}(\mu(F) - \mu(E)) + \mathbb{1}_{\{\eta_1=\beta\}}p_{\beta,\alpha}(\mu(E) - \mu(F)) = 0.\end{aligned}$$

This equation holds iff $\mu(E) = \mu(F)$. □

We use this result to show in the next proposition, that the limit of the average inter trading time is in fact the expected first exit time of the risky fraction from (a, b) under the invariant measure ν of the Markov chain $(\eta_i)_{i\in\mathbb{N}}$.

Proposition 4.8

$$\mathbb{E}[\tau_2 - \tau_1|\xi^{-1}(\mathcal{I})] = \mathbb{E}_\nu[\tau_{(a,b)}]$$

Proof. Recall that $\mathcal{I} = \{A \in \mathcal{S} | T^{-1}A = A\}$ is the T-invariant σ-field and $\xi^{-1}\mathcal{I} = \{\{\xi \in A\} | A \in \mathcal{I}\}$, where $\xi = (\eta_1, \sigma W^{\tau_1} + Y^{\tau_1})$. We show first that

$$\mathbb{E}[\tau_2 - \tau_1 | \xi^{-1}\mathcal{I}] = \mathbb{E}[\tau_2 - \tau_1].$$

Using the definition of the conditional expectation we have to show that for all $A \in \mathcal{I}$ it holds

$$\begin{aligned}\mathbb{E}[\mathbb{1}_{\{\xi \in A\}}(\tau_2 - \tau_1)] &= \mathbb{E}[\mathbb{1}_{\{\xi \in A\}}\mathbb{E}[\tau_2 - \tau_1]] \\ &= P(\xi \in A)\mathbb{E}[\tau_2 - \tau_1].\end{aligned}$$

We begin with the left side of the equation:

$$\begin{aligned}\mathbb{E}[\mathbb{1}_{\{\xi \in A\}}(\tau_2 - \tau_1)] &= \mathbb{E}[\mathbb{1}_{\{(\eta_1, \sigma W^{\tau_1} + Y^{\tau_1}) \in T^{-1}A\}}(\tau_2 - \tau_1)] \\ &= \mathbb{E}[\mathbb{E}[\mathbb{1}_{\{(\eta_2, \sigma W^{\tau_2} + Y^{\tau_2}) \in A\}}(\tau_2 - \tau_1) | \mathcal{G}_{\tau_2}]] \\ &= \mathbb{E}[(\tau_2 - \tau_1)\mathbb{E}[\mathbb{1}_{\{(\eta_2, \sigma W^{\tau_2} + Y^{\tau_2}) \in A\}} | \mathcal{G}_{\tau_2}]].\end{aligned}$$

Remark 4.6 says that the sets $A \in \mathcal{I}$ have the form $A = \{\alpha\} \times E \cup \{\beta\} \times F$ with some $E, F \in \mathcal{B}(D([0,\infty), \mathbb{R}))$. We insert this representation into the conditional expectation above, furthermore we use that η_2 is $\mathcal{G}_{\tau_2}$-measurable and $\sigma W^{\tau_2} + Y^{\tau_2}$ is independent of $\mathcal{G}_{\tau_2}$:

$$\begin{aligned}&\mathbb{E}[\mathbb{1}_{\{(\eta_2, \sigma W^{\tau_2} + Y^{\tau_2}) \in A\}} | \mathcal{G}_{\tau_2}] \\ &= \mathbb{1}_{\{\eta_2 = \alpha\}}\mathbb{E}[\mathbb{1}_{\{(\alpha, \sigma W^{\tau_2} + Y^{\tau_2}) \in A\}} | \mathcal{G}_{\tau_2}] + \mathbb{1}_{\{\eta_2 = \beta\}}\mathbb{E}[\mathbb{1}_{\{(\beta, \sigma W^{\tau_2} + Y^{\tau_2}) \in A\}} | \mathcal{G}_{\tau_2}] \\ &= \mathbb{1}_{\{\eta_2 = \alpha\}}\mathbb{E}[\mathbb{1}_{\{(\alpha, \sigma W^{\tau_2} + Y^{\tau_2}) \in A\}}] + \mathbb{1}_{\{\eta_2 = \beta\}}\mathbb{E}[\mathbb{1}_{\{(\beta, \sigma W^{\tau_2} + Y^{\tau_2}) \in A\}}] \\ &= \mathbb{1}_{\{\eta_2 = \alpha\}}\mu(E) + \mathbb{1}_{\{\eta_2 = \beta\}}\mu(F).\end{aligned}$$

Using Proposition 4.7 we have $\mu(E) = \mu(F)$. Altogether the assertion follows by

$$\begin{aligned}\mathbb{E}[\mathbb{1}_{\{\xi \in A\}}(\tau_2 - \tau_1)] &= \mathbb{E}[(\tau_2 - \tau_1)\mathbb{E}[\mathbb{1}_{\{(\eta_2, \sigma W^{\tau_2} + Y^{\tau_2}) \in A\}} | \mathcal{G}_{\tau_2}]] \\ &= \mathbb{E}[(\tau_2 - \tau_1)(\mathbb{1}_{\{\eta_2 = \alpha\}}\mu(E) + \mathbb{1}_{\{\eta_2 = \beta\}}\mu(F))] \\ &= \mathbb{E}[(\tau_2 - \tau_1)]\mu(E) \\ &= \mathbb{E}[(\tau_2 - \tau_1)](\mu(E) \cdot \nu(\alpha) + \mu(F) \cdot \nu(\beta)) \\ &= \mathbb{E}[(\tau_2 - \tau_1)] \cdot P(\xi \in A).\end{aligned}$$

Finally we use Lemma 4.2 to obtain

$$\begin{aligned}\mathbb{E}[(\tau_2 - \tau_1)] &= \mathbb{E}[\,\mathbb{E}\,[(\tau_2 - \tau_1)|\eta_1]\,] = \mathbb{E}\,[\,\mathbb{E}^{\eta_1}[\tau_1]\,] \\ &= \nu(\alpha) \cdot \mathbb{E}^{\alpha}[\tau_1] + \nu(\beta) \cdot \mathbb{E}^{\beta}[\tau_1] \\ &= \mathbb{E}_{\nu}[\tau_{(a,b)}].\end{aligned}$$

□

Corollary 4.9
For a CB-strategy $(\tau_i, \eta_i)_{i\in\mathbb{N}}$ which is given by (a, α, β, b) it holds

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} (\tau_{i+1} - \tau_i) = \mathbb{E}_{\nu}[\tau_{(a,b)}].$$

Proof. The assertion follows from Theorem 4.5 and Proposition 4.8. □

4.3 Expected Frequency of Trading in the Continuous Setting

Let us consider the risky fraction process $(\pi_t)_{t\geq 0}$ in the continuous setting, i.e. we assume that the process $(\pi_t)_{t\geq 0}$ has the following dynamics:

$$d\pi_t = \pi_t(1 - \pi_t)(\mu - \sigma^2\pi_t)dt + \pi_t(1 - \pi_t)\sigma dW_t. \tag{4.5}$$

Assume the initial value is inside of the no-trading region, i.e. $\pi \in (a, b)$. The diffusion $(\pi_t)_{t\geq 0}$ is a strong Markov process taking values in $(0, 1)$. Furthermore $(\pi_t)_{t\geq 0}$ is a regular diffusion, because the probability that $(\pi_t)_{t\geq 0}$ hits an arbitrary $y \in (0, 1)$ in finite time, starting at some $\pi \in (0, 1)$ is positive. The generator of $(\pi_t)_{t\geq 0}$ is a differential operator L_π, defined for all $f \in C_0^2(0, 1)$ by

$$L_\pi f = x(1 - x)(\mu - \sigma^2 x)\frac{\partial}{\partial x} f + \frac{1}{2}x^2(1 - x)^2\sigma^2 \frac{\partial^2}{\partial x^2} f.$$

Dynkin's formula states, that for stopping times τ with $\mathbb{E}[\tau] < \infty$ and $f \in C_0^2(0, 1)$ the following holds:

$$\mathbb{E}^{\pi}[f(\pi_\tau)] = f(\pi) + \mathbb{E}^{\pi}\left[\int_0^{\tau} L_\pi f(\pi_t)dt\right].$$

Using this result one can prove that the process

$$\left(f(\pi_t) - \int_0^t L_\pi f(\pi_t) dt \right)_{t \geq 0}$$

is a martingale with initial value $f(\pi)$, see e.g. Proposition VII.1.6 in [Revuz, Yor (1991)]. This is a key result for computing the expected first exit time of the interval (a,b). Let us define the *first exit time* by

$$\tau_{(a,b)} := \inf \{ t \geq 0 : \pi_t \notin (a,b) \} .$$

Applying Itô's formula to $(h_i(\pi_t))_{t \geq 0}$ for $i = 0,1$, one can verify that the functions

$$h_0(x) = \begin{cases} \left(\frac{1-x}{x} \right)^{2\frac{\mu}{\sigma^2} - 1}, & \text{if } \frac{\mu}{\sigma^2} \neq \frac{1}{2} \\ -\log \left(\frac{1-x}{x} \right), & \text{if } \frac{\mu}{\sigma^2} = \frac{1}{2} \end{cases}$$

$$h_1(x) = \begin{cases} -\frac{2 \log \left(\frac{1-x}{x} \right)}{\sigma^2 (2 \frac{m}{\sigma^2} - 1)}, & \text{if } \frac{\mu}{\sigma^2} \neq \frac{1}{2} \\ \frac{1}{\sigma^2} \log \left(\frac{1-x}{x} \right)^2, & \text{if } \frac{\mu}{\sigma^2} = \frac{1}{2}, \end{cases}$$

solves $L_\pi h_0(\pi_t) = 0$ and $L_\pi h_1(\pi_t) = 1$ (see Proposition 5.3.6 [Sass (2001)]). Thus the processes

$$\left(h_i(\pi_t) \mathbb{1}_{[a,b]}(\pi_t) - \int_0^t L_\pi h_i(\pi_s) \mathbb{1}_{[a,b]}(\pi_s) ds \right)_{t \geq 0}$$

are martingales for $i = 0,1$. This implies for $\tau_{(a,b)}$:

$$\begin{aligned} h_0(\pi) &= \mathbb{E}^\pi \left[h_0(\pi_{\tau_{(a,b)}}) - \int_0^{\tau_{(a,b)}} L_\pi h_0(\pi_s) ds \right] \\ &= \mathbb{E}^\pi [h_0(\pi_{\tau_{(a,b)}})] \\ &= h_0(a) P^\pi(\pi_{\tau_{(a,b)}} = a) + h_0(b) P^\pi(\pi_{\tau_{(a,b)}} = b). \end{aligned}$$

Using the fact that $P^\pi(\pi_{\tau_{(a,b)}} = a) + P^\pi(\pi_{\tau_{(a,b)}} = b) = 1$ we derive

$$P^\pi(\pi_{\tau_{(a,b)}} = b) = \frac{h_0(\pi) - h_0(a)}{h_0(b) - h_0(a)}.$$

This probability and the function h_1 yield the expected first exit time:

$$\begin{aligned}\mathbb{E}^\pi[\tau_{(a,b)}] &= -h_1(\pi) + \mathbb{E}^\pi\left[h_1(\pi_{\tau_{(a,b)}})\right] \\ &= -h_1(\pi) + h_1(a)P^\pi(\pi_{\tau_{(a,b)}} = a) + h_1(b)P^\pi(\pi_{\tau_{(a,b)}} = b) \\ &= -h_1(\pi) + h_1(a) + \frac{h_0(\pi) - h_0(a)}{h_0(b) - h_0(a)}(h_1(b) - h_1(a)).\end{aligned}$$

The crucial point of derivation above is to find a function h_0, which transforms the diffusion process into a martingale. This can be done by using a suitable change of scale, i.e a function which removes the drift in (4.5). The existence of the so-called *scale function* is given in Theorem 20.7 of [Kallenberg (1997)]:

Theorem 4.10 (Scale Function)
Given any regular diffusion X on the interval $I \subset \mathbb{R}$, there exists a continuous and strictly increasing function $h : I \to \mathbb{R}$ such that $(h(X_{\tau_{(a,b)}\wedge t}))_{t\geq 0}$ is a P^x-martingale for any $a < x < b$ in I. Furthermore, an increasing function h has the stated property iff

$$P^x(X_{\tau_{(a,b)}} = b) = \frac{h(x) - h(a)}{h(b) - h(a)}. \tag{4.6}$$

Once we have the functions h_0 and h_1 we can derive all components of the expected inter trading time:

$$\mathbb{E}^\nu[\tau_{(a,b)}] = p \cdot \mathbb{E}^\alpha[\tau_{(a,b)}] + (1-p) \cdot \mathbb{E}^\beta[\tau_{(a,b)}],$$

where $p = \frac{p_{\beta,\alpha}}{p_{\alpha,\beta}+p_{\beta,\alpha}}$ with

$$\begin{aligned}p_{\alpha,\beta} &= P^\alpha(\pi_{\tau_{(a,b)}} = b) = \frac{h_0(\alpha) - h_0(a)}{h_0(b) - h_0(a)} \\ p_{\beta,\alpha} &= P^\beta(\pi_{\tau_{(a,b)}} = a) = \frac{h_0(\beta) - h_0(b)}{h_0(a) - h_0(b)} \\ \mathbb{E}^\alpha[\tau_{(a,b)}] &= -h_1(\alpha) + h_1(a) + \frac{h_0(\alpha) - h_0(a)}{h_0(b) - h_0(a)}(h_1(b) - h_1(a)) \\ \mathbb{E}^\beta[\tau_{(a,b)}] &= -h_1(\beta) + h_1(a) + \frac{h_0(\beta) - h_0(a)}{h_0(b) - h_0(a)}(h_1(b) - h_1(a)).\end{aligned}$$

Unfortunately the techniques we used in this section fail if we face a process with jumps. Nevertheless, in the next section we can apply the theory of scale functions for spectrally negative Lévy processes and obtain similar results for the expected inter trading time.

4.4 Spectrally Negative Lévy Processes

Consider a Lévy process $(\pi_t)_{t\geq 0}$ with characteristic triplet (μ, σ, ν). The characteristic function of $(\pi_t)_{t\geq 0}$ can be written as $\mathbb{E}[e^{iz\pi_t}] = e^{t\Psi(z)}$ for $z \in \mathbb{R}$ due to the infinite divisibility of increments. The characteristic exponent Ψ is given by the Lévy-Khinchin formula

$$\Psi(z) = i\mu z - \frac{1}{2}\sigma^2 z^2 + \int_{\mathbb{R}} (e^{izx} - 1 - izx\mathbb{1}_{\{|x|\leq 1\}})\nu(dx).$$

We assume here and in the following that $\sigma > 0$, which means that $(\pi_t)_{t\geq 0}$ has a Gaussian component. Furthermore we assume that $(\pi_t)_{t\geq 0}$ has only negative jumps, i.e. the Lévy measure of $(\pi_t)_{t\geq 0}$ is zero on $(0, \infty)$. Lévy processes with only negative jumps are called *spectrally negative*. We consider this class of processes, because due to the absence of positive jumps many important identities in fluctuation theory can be derived explicitly.

Because of absence of positive jumps the characteristic exponent Ψ is well defined on the complex lower-half plane. Thus the following definition makes sense for $s \geq 0$

$$\psi(s) := \Psi(-is) = \mu s + \frac{1}{2}\sigma^2 s^2 + \int_{-\infty}^{0} (e^{sx} - 1 - sx\mathbb{1}_{\{-1<x<0\}})\nu(dx).$$

This implies

$$\mathbb{E}[e^{s\pi_t}] = e^{t\psi(s)}.$$

The function $\psi : [0, \infty) \to \mathbb{R}$ is called the *Laplace exponent* of $(\pi_t)_{t\geq 0}$. ψ is strictly convex and infinitely often differentiable with

$$\psi(0) = 0, \quad \lim_{s\to\infty} \psi(s) = \infty.$$

We define the right inverse of ψ by

$$\Phi(q) := \sup \{\theta \geq 0 : \psi(\theta) = q\}.$$

In case $\psi(0+) = \mathbb{E}[\pi_1] \geq 0$ there exists only one solution θ such that $\psi(\theta) = 0$, otherwise we have two solutions $\theta_1 = 0$ and $\theta_2 > 0$ with $\psi(\theta_2) > 0$.

Equipped with a spectrally negative Lévy process and its Laplace transform, we want to investigate the two sided exit time of the process $(\pi_t)_{t\geq 0}$. Consider stopping times

$$\tau_0 = \inf \{t \geq 0 : \pi_t \leq 0\} \quad \tau_c = \inf \{t \geq 0 : \pi_t \geq c\}.$$

Two sided exit time from the interval $(0, c)$ is given by $\tau_0 \wedge \tau_c = \tau_{(0,c)}$. One of the most important results of the fluctuation theory for spectrally negative Lévy processes is the following result from [Kyprianou (2006)], Chapter 8:

Theorem 4.11 (Two sided exit time)
There exists a family of functions $W^{(q)} : \mathbb{R} \to [0, \infty)$ and

$$Z^{(q)}(x) = 1 + q \int_0^x W^{(q)}(y)dy, \quad x \in \mathbb{R},$$

defined for each $q \geq 0$ such that the following holds.

(i) For any $q \geq 0$, we have $W^{(q)}(x) = 0$ for $x < 0$ and $W^{(q)}$ is characterized on $[0, \infty)$ as a strictly increasing and continuous function whose Laplace transform satisfies

$$\int_0^\infty e^{-sx} W^{(q)}(x)dx = \frac{1}{\psi(s) - q} \quad \text{for } s > \Phi(q).$$

(ii) For any $x \leq a$ and $q \geq 0$,

$$\mathbb{E}^x \left[e^{-q\tau_c} \mathbb{1}_{\{\tau_0 > \tau_c\}}\right] = \frac{W^{(q)}(x)}{W^{(q)}(c)} \tag{4.7}$$

$$\mathbb{E}^x \left[e^{-q\tau_0} \mathbb{1}_{\{\tau_0 < \tau_c\}}\right] = Z^{(q)}(x) - Z^{(q)}(c)\frac{W^{(q)}(x)}{W^{(q)}(c)}. \tag{4.8}$$

The function $W^{(q)}$ defined in this theorem is called the *q-scale function*. This name is explained by the analogous role $W^{(q)}$ plays in (4.7) compared with the scale function h in (4.6) in the continuous setting. We can see this more clearly if we put $q = 0$, the function $W^{(0)}$ satifies the same identity as in Theorem 4.10:

$$P^x(\tau_0 > \tau_c) = \frac{W^{(0)}(x) - W^{(0)}(0)}{W^{(0)}(c) - W^{(0)}(0)}.$$

As in the continuous setting we are interested in finding an analytic formula for the expected exit time of the interval (a, b). The constants a, b need not to be in $(0, 1)$ or positive. First we adopt Theorem 4.11 to work with the stopping times

$$\tau_a = \inf\{t \geq 0 : \pi_t \leq a\} \quad \tau_b = \inf\{t \geq 0 : \pi_t \geq b\}.$$

This can be done using existing results relating to spectrally negative Lévy processes. For any $a < \pi < b$ and $q \geq 0$ we have

$$\mathbb{E}^\pi\left[e^{-q\tau_b}\mathbb{1}_{\{\tau_a > \tau_b\}}\right] = \mathbb{E}^{\pi - a}\left[e^{-q\tau_{b-a}}\mathbb{1}_{\{\tau_0 > \tau_{b-a}\}}\right] = \frac{W^{(q)}(\pi - a)}{W^{(q)}(b - a)}. \tag{4.9}$$

This equation holds due to the stationary and independent increments of the underlying process $(\pi_t)_{t\geq 0}$, see also Theorem VII.8 of [Bertoin (1996)]. With the same arguments we obtain

$$\begin{aligned}\mathbb{E}^\pi\left[e^{-q\tau_a}\mathbb{1}_{\{\tau_a < \tau_b\}}\right] &= \mathbb{E}^{\pi - a}\left[e^{-q\tau_0}\mathbb{1}_{\{\tau_0 < \tau_{b-a}\}}\right] \\ &= Z^{(q)}(\pi - a) - Z^{(q)}(b - a)\frac{W^{(q)}(\pi - a)}{W^{(q)}(b - a)}.\end{aligned} \tag{4.10}$$

Once we have these identities, we can compute the expected first exit time $\mathbb{E}^\pi\left[\tau_{(a,b)}\right]$.

Proposition 4.12
The expected first exit time from (a, b) of a Lévy process with initial value $\pi \in (a, b)$ is given by

$$\mathbb{E}^\pi\left[\tau_{(a,b)}\right] = -\frac{\partial}{\partial q}\left(\frac{W^{(q)}(\pi - a)}{W^{(q)}(b - a)} + Z^{(q)}(\pi - a) - Z^{(q)}(b - a)\frac{W^{(q)}(\pi - a)}{W^{(q)}(b - a)}\right)\Big|_{q=0}.$$

Proof. From Theorem 4.11 and the observations above we obtain

$$
\begin{aligned}
&E^{\pi}[e^{-q\tau_b}\mathbb{1}_{\{\tau_a>\tau_b\}} + e^{-q\tau_a}\mathbb{1}_{\{\tau_a<\tau_b\}}] \\
&\quad = \frac{W^{(q)}(\pi-a)}{W^{(q)}(b-a)} + Z^{(q)}(\pi-a) - Z^{(q)}(b-a)\frac{W^{(q)}(\pi-a)}{W^{(q)}(b-a)}.
\end{aligned}
$$

Furthermore, differentiation with respect to q yields:

$$
\begin{aligned}
&\frac{\partial}{\partial q}\mathbb{E}^{\pi}\left[e^{-q\tau_b}\mathbb{1}_{\{\tau_a>\tau_b\}} + e^{-q\tau_a}\mathbb{1}_{\{\tau_a<\tau_b\}}\right]\Big|_{q=0} \\
&= \mathbb{E}^{\pi}\left[-\tau_b e^{-q\tau_b}\mathbb{1}_{\{\tau_a>\tau_b\}} - \tau_a e^{-q\tau_a}\mathbb{1}_{\{\tau_a<\tau_b\}}\right]\Big|_{q=0} \\
&= -\mathbb{E}^{\pi}\left[\tau_b\mathbb{1}_{\{\tau_a>\tau_b\}} + \tau_a\mathbb{1}_{\{\tau_a<\tau_b\}}\right] \\
&= -\mathbb{E}^{\pi}\left[\tau_b\wedge\tau_a\right] = -\mathbb{E}^{\pi}\left[\tau_{(a,b)}\right].
\end{aligned}
$$

Thus we have

$$
\begin{aligned}
&\mathbb{E}^{\pi}[\tau_{(a,b)}] \\
&\quad = -\frac{\partial}{\partial q}\left(\frac{W^{(q)}(\pi-a)}{W^{(q)}(b-a)} + Z^{(q)}(\pi-a) - Z^{(q)}(b-a)\frac{W^{(q)}(\pi-a)}{W^{(q)}(b-a)}\right)\Big|_{q=0}.
\end{aligned}
$$

□

Proposition 4.13

Consider a Lévy process $(\overline{\pi}_t)_{t\geq 0}$ as given in (4.2) with only negative jumps and a CB-strategy $(\eta_i,\tau_i)_{i\in\mathbb{N}}$, then the transition probabilities of the homogeneous Markov chain $(\eta_i)_{i\in\mathbb{N}}$ are given by

$$
p_{\alpha,\beta} = P^{\alpha}(\overline{\pi}_{\tau_{(a,b)}} \geq b) = \mathbb{E}^{\alpha}[\mathbb{1}_{\{\tau_b<\tau_a\}}] = \frac{W^{(0)}(\alpha-a)}{W^{(0)}(b-a)} \tag{4.11}
$$

$$
p_{\beta,\alpha} = P^{\beta}(\overline{\pi}_{\tau_{(a,b)}} \leq a) = \mathbb{E}^{\beta}[\mathbb{1}_{\{\tau_b>\tau_a\}}] = 1 - \frac{W^{(0)}(\beta-a)}{W^{(0)}(b-a)} \tag{4.12}
$$

Proof. It holds

$$
\begin{aligned}
p_{\alpha,\beta} &= P^{\alpha}(\eta_1=\beta) = P^{\alpha}(\overline{\pi}_{\tau_{(a,b)}} \geq b) = \mathbb{E}^{\alpha}[\mathbb{1}_{\{\tau_b<\tau_a\}}] \\
p_{\beta,\alpha} &= P^{\beta}(\eta_1=\alpha) = P^{\beta}(\overline{\pi}_{\tau_{(a,b)}} \leq a) = \mathbb{E}^{\beta}[\mathbb{1}_{\{\tau_b>\tau_a\}}].
\end{aligned}
$$

Using (4.9) and (4.10) we have

$$E^{\alpha}[\mathbb{1}_{\{\tau_b<\tau_a\}}] = \mathbb{E}^{\alpha}\left[e^{-q\tau_b}\mathbb{1}_{\{\tau_a>\tau_b\}}\right]\Big|_{q=0} = \frac{W^{(0)}(\alpha-a)}{W^{(0)}(b-a)}$$

$$E^{\beta}[\mathbb{1}_{\{\tau_b>\tau_a\}}] = \mathbb{E}^{\beta}\left[e^{-q\tau_a}\mathbb{1}_{\{\tau_a<\tau_b\}}\right]\Big|_{q=0} = 1-\frac{W^{(0)}(\beta-a)}{W^{(0)}(b-a)}.$$

□

Thus, if we know the q-scale function, we have all ingredients for the computation of the inter trading time

$$\mathbb{E}^{\nu}[\tau_{(a,b)}] = p\cdot\mathbb{E}^{\alpha}[\tau_{(a,b)}] + (1-p)\cdot\mathbb{E}^{\beta}[\tau_{(a,b)}],$$

where

$$p = \frac{p_{\beta,\alpha}}{p_{\alpha,\beta}+p_{\beta,\alpha}},$$

if we consider only Lévy process with negative jumps.

4.5 Scale Functions for Spectrally Negative Lévy Processes with Phase-Type Distribution of Jumps.

In general it is difficult to obtain scale functions explicitly. Several examples for scale functions are given in [Hubalek, Kyprianou (2010)]. We will mention here just two examples: Brownian motion with drift and compound Poisson process with drift.

Consider a compound Poisson process with negative exponentially distributed jumps with mean μ, intensity λ and positive drift c, such that $c-\frac{\lambda}{\mu}>0$. Then the q-scale function for $q=0$ is given by

$$W^{(0)}(x) = \frac{1}{c}\left(1+\frac{\lambda}{c\mu-\lambda}\left(1-e^{(\mu-\lambda c^{-1})x}\right)\right).$$

The exponential distribution can be replaced by any other distribution which has a rational Laplace transform, see [Mordecki, Lewis (2005)].

The q-scale function of the Brownian motion with drift μ is given by

$$W^{(q)}(x) = \frac{2}{\sqrt{2q\sigma^2+\mu}} e^{-\mu x/\sigma^2} \sinh\left(\frac{x}{\sigma^2}\sqrt{2q\sigma^2+\mu}\right).$$

In this section we focus on spectrally negative Lévy processes with phase-type distributed jumps. The scale function for this class of processes is obtained in [Egami, Yamazaki (2011)a]. Phase-type distributions are dense in the class of all positive valued distributions in the sense of weak convergence (Chapter III, Theorem 4.2, [Asmussen (2004)]). In the following we give a brief introduction to the class of phase-type distributions, which includes for example exponential, hyperexponential, Erlang, hyper-Erlang and Coxian distributions.

4.5.1 Phase-Type Distributions

Consider a continuous-time Markov chain $(X_t)_{t\geq 0}$ with finite state space $\{1, 2...m, \Delta\}$, where $1, ..., m$ are transient and Δ is absorbing. Then $(X_t)_{t\geq 0}$ has an intensity matrix of the form

$$\mathbf{Q} = \begin{pmatrix} \mathbf{T} & \mathbf{t} \\ \mathbf{0} & 0 \end{pmatrix},$$

where $\mathbf{T}$ is an $m \times m$ matrix, $\mathbf{t}$ is $m \times 1$ vector and $\mathbf{0}$ a $1 \times m$ row vector of zeros. The sum of the rows in $\mathbf{Q}$ must be zero, thus $\mathbf{t} = -\mathbf{T} \cdot \mathbf{1}$, where $\mathbf{1} = (1, \cdots, 1)'$. The initial distribution of $(X_t)_{t\geq 0}$ is $\delta = (\delta_1, ..., \delta_m)$, which is given for transient states by $\delta_i = P(X_0 = i)$ for $i = 1, ..., m$ and for the absorbing state by $P(X_0 = \Delta) = 0$.

Definition 4.14 (Phase-type Distribution)
The time until absorption

$$\tau = \inf\{t \geq 0 : X_t = \Delta\}$$

is said to have a phase-type distribution with parameter $(m, \delta, \boldsymbol{T})$.

We present two examples of phase-type distributions from [Bladt (2005)].

Example 4.15. Let $(X_i)_{0<i\leq m}$ be a sequence of independent exponentially distributed random variables with parameters $\lambda_1, ..., \lambda_m$. The sum $S_m = X_1 + ... + X_m$ has a phase-type distribution with initial distribution $\delta = (1, 0, ..., 0)$ and intensity matrix

$$\mathbf{T} = \begin{pmatrix} -\lambda_1 & \lambda_1 & 0 & \cdots & 0 & 0 \\ 0 & -\lambda_2 & \lambda_2 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & 0 & -\lambda_m \end{pmatrix}.$$

The sum S_m can be interpreted as the waiting time till absorption of a Markov chain with m transient states. The chain starts in state 1 and waits for X_1 units of time until the first jump into state 2 occurs and so on. This representation is not unique, because we can interchange the summands in S_m. If the waiting times are all distributed with the same parameter λ, the resulting distribution of S_m is *Erlang* with parameter λ and m. The Erlang distribution consists of m identical phases in a sequence.

Example 4.16. We assume the same setting as in the previous example: $(X_i)_{0<i\leq m}$ independent exponentially distributed with parameter $\lambda_1, ..., \lambda_m$. We define a probability density f by a linear combination of exponential densities $(f_i)_{0<i\leq m}$ of $(X_i)_{0<i\leq m}$ with positive constants $(\delta_i)_{0<i\leq m}$, such that $\sum_{i=1}^m \delta_i = 1$.

$$f := \sum_{i=1}^{m} \delta_i f_i$$

Then the distribution defined by f is phase-type with intensity

$$\mathbf{T} = \begin{pmatrix} -\lambda_1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & -\lambda_2 & 0 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & 0 & -\lambda_m \end{pmatrix} \tag{4.13}$$

and initial distribution $\delta = (\delta_1, ..., \delta_m)$. This distribution is called *hyperexponential*, it consists of m non-identical parallel phases with probability of occurrence equal to δ.

The distribution function and density of the phase-type distributed random variable is given in Theorem 4.1, 4.2 of [Bladt (2005)] for $z \geq 0$:

$$f(z) = \delta e^{\mathbf{T}z}\mathbf{t}, \quad F(z) = 1 - \delta e^{\mathbf{T}z}\mathbf{t}. \tag{4.14}$$

Although the above formulas seem simple, the explicit calculation of matrix exponentials can be complex in higher dimensions.

4.5.2 Scale Function

Consider a Lévy process $(\pi_t)_{t\geq 0}$ given by

$$\pi_t - \pi = \mu t + \sigma W_t - \sum_{i=1}^{N_t} Z_i,$$

where $\sigma > 0$ and where the jump part is a compound Poisson process with intensity λ and phase-type distributed jumps $(Z_i)_{i\geq 0}$ with parameters $(\delta, \mathbf{T})$.

Due to the phase-type distribution the Laplace exponent takes the form

$$\begin{aligned}
\psi(s) &= \mu s + \frac{1}{2}\sigma^2 s^2 + \int_{-\infty}^{0} (e^{sx} - 1)\nu(dx) \\
&= \mu s + \frac{1}{2}\sigma^2 s^2 + \int_{-\infty}^{0} (e^{sx} - 1)\lambda f(x)dx \\
&= \mu s + \frac{1}{2}\sigma^2 s^2 + \int_{-\infty}^{0} (e^{sx} - 1)\lambda\delta e^{\mathbf{T}x}\mathbf{t}dx \\
&= \mu s + \frac{1}{2}\sigma^2 s^2 + \lambda(\delta(s\mathbf{I}\text{-}\mathbf{T})^{-1}\mathbf{t} - 1). \qquad (4.15)
\end{aligned}$$

ψ is analytic for $s \in \mathbb{C}$ except for poles, which are the eigenvalues of $\mathbf{T}$. Further we define the *running maximum* and the *running minimum* by

$$\pi_t^{max} := \sup_{s\leq t} \pi_s \quad \pi_t^{min} := \inf_{s\leq t} \pi_s.$$

Let e_q be an independent random variable with exponential distribution with parameter q. For all $s \in \mathbb{C}$ define functions

$$\varphi_q^-(s) := \mathbb{E}[\exp(s\pi_{e_q}^{min})] \quad \varphi_q^+(s) := \mathbb{E}[\exp(s\pi_{e_q}^{max})].$$

φ_q^- is analytic for $Re(s) > 0$ and φ_q^+ for $Re(s) < 0$. The *Wiener-Hopf Factorization* states that the following holds:

$$\frac{q}{q - \psi(s)} = \varphi_q^-(s)\varphi_q^+(s)$$

for all $s \in \mathbb{C}$, with $Re(s) = 0$. There is an explicit formula for φ_q^-, which is derived in [Asmussen (2004)] Lemma 1:

$$\varphi_q^-(s) = \frac{\prod_{j \in J_q}(s + \eta_j)}{\prod_{j \in J_q} \eta_j} \frac{\prod_{i \in I_q} \xi_{i,q}}{\prod_{i \in I_q}(s + \xi_{i,q})},$$

where

$$I_q = \{i : \psi(-\xi_{i,q}) = q, Re(\xi_{i,q}) > 0\}$$

is the set of solutions of the Cramer-Lundberg equation $\psi(s) = q$ with negative real part and

$$J_q = \left\{ j : \frac{q}{q - \psi(-\eta_j)} = 0, Re(\eta_j) > 0 \right\}$$

is the set of poles of the Laplace transform. Using the Laplace inversion on φ_q^- we can obtain the density of the running minimum (See [Asmussen (2004)] Lemma 1. (3)):

$$P(-\pi_{e_q}^{min} \in dx) = \sum_{i=1}^{n} \sum_{k=1}^{m_i} A_{i,q}^{(k)} \xi_{i,q} \frac{(\xi_{i,q} x)^{k-1}}{(k-1)!} e^{-\xi_{i,q} x} dx,$$

where

$$A_{i,q}^{(k)} = \frac{1}{(m_i - k)!} \frac{\partial^{m_i - k}}{\partial s^{m_i - k}} \frac{\varphi_q^-(s)(s + \xi_{i,q})^{m_i}}{\xi_{i,q}^k} \bigg|_{s = -\xi_{i,q}},$$

n denotes the number of different solutions in I_q, m_i denotes the multiplicity of each solution $-\xi_{i,q}$. The random variable e_q is independent, thus the following holds:

$$\begin{aligned}
\mathbb{E}^\pi[e^{-q\tau_a} \mathbb{1}_{\tau_a < \infty}] &= \mathbb{E}^\pi[P(e_q > \tau_a | \tau_a)] = \mathbb{E}^\pi[\mathbb{1}_{e_q > \tau_a}] = P^\pi(e_q > \tau_a) \\
&= P^\pi(\pi_{e_q}^{min} < a) = P(-\pi_{e_q}^{min} > \pi - a) \\
&= \sum_{i=1}^{n} \sum_{k=1}^{m_i} A_{i,q}^{(k)} \xi_{i,q} \int_{\pi - a}^{\infty} \frac{(\xi_{i,q} y)^{k-1}}{(k-1)!} e^{-\xi_{i,q} y} dy.
\end{aligned}$$

On the other hand, Theorem 4.11 states that

$$\mathbb{E}^\pi[e^{-q\tau_a}\mathbb{1}_{\tau_a<\infty}] = Z^{(q)}(\pi - a) - \frac{q}{\Phi(q)}W^{(q)}(\pi - a).$$

Using these identities one can derive the formula for $W^{(q)}$, see [Egami, Yamazaki (2011)a] Proposition 2.1:

Theorem 4.17
For a fixed $q > 0$ and $x \geq 0$ the scale function of the process $(\pi_t)_{t\geq 0}$ is given by

$$\begin{aligned} W^{(q)}(x) =& \frac{\Phi(q)}{q}\sum_{i=1}^{n}\sum_{k=1}^{m_i} A_{i,q}^{(k)}\left(\frac{\xi_{i,q}}{\Phi(q)+\xi_{i,q}}\right)^k \\ &\times\left[e^{\Phi(q)x} - e^{-\xi_{i,q}x}\sum_{j=0}^{k-1}\frac{((\Phi(q)+\xi_{i,q})x)^j}{j!}\right]. \end{aligned}$$

Remark 4.18. According to Corollary 2.1 (1) in [Egami, Yamazaki (2011)a], if all solutions in I_q are distinct, the scale function can be simplified to

$$W^{(q)} = \frac{\Phi(q)}{q}\sum_{i\in I_q} A_{i,q}^{(1)}\left(\frac{\xi_{i,q}}{\Phi(q)+\xi_{i,q}}\right)\left[e^{\Phi(q)x} - e^{-\xi_{i,q}x}\right], \text{ where} \tag{4.16}$$

$$A_{i,q}^{(1)} = \frac{\prod_{j\in J_q}(-\xi_{i,q}+\eta_j)}{\prod_{j\in J_q}\eta_j}\frac{\prod_{k\in I_q,k\neq i}\xi_{k,q}}{\prod_{k\in I_q,k\neq i}(-\xi_{i,q}+\xi_{k,q})}. \tag{4.17}$$

4.5.3 Example with Exponential Distribution

As in the previous section we consider a Lévy process

$$\pi_t - \pi = \mu t + \sigma W_t - \sum_{i=1}^{N_t} Z_i. \tag{4.18}$$

We assume, that $\sigma > 0$ and the distribution of the jumps $(Z_i)_{i\in\mathbb{N}}$ is exponential with parameter $d > 0$. The exponential distribution is a phase-type distribution with $\delta = 1$ and intensity matrix $\mathbf{T} = -d$. We can see this by calculating the distribution and density functions of a random variable with phase-type distribution $(1, \delta, \mathbf{T})$ using (4.14). We have

$$F(z) = 1 - e^{-dz} \text{ and } f(z) = de^{-dz},$$

which characterize the exponential distribution.

Within this setting we can use Theorem 4.17 to derive the scale function. The Laplace exponent of $(\pi_t)_{t\geq 0}$ is given by

$$\begin{aligned}\psi(s) =&\mu s + \frac{1}{2}\sigma^2 s^2 + \lambda(\delta(sI - \mathbf{T})^{-1}(-\mathbf{T}) \cdot 1 - 1) \\ =&\mu s + \frac{1}{2}\sigma^2 s^2 + \lambda\left(\frac{d}{d+s} - 1\right).\end{aligned}$$

Lemma 4.19

Assume, that the right derivative of ψ at zero is negative, i.e.

$$\psi'(0+) = \mathbb{E}^0[\pi_1] = \mu - \frac{\lambda}{d} < 0, \tag{4.19}$$

which implies $\Phi(q) > \Phi(0) > 0$. Then the Cramer-Lundberg equation $\psi(s) = q,\ q \geq 0$, where

$$\psi(s) = \mu s + \frac{1}{2}\sigma^2 s^2 + \lambda\left(\frac{d}{d+s} - 1\right),$$

has exactly three different roots $\Phi(q),\ -\xi_{1,q},\ -\xi_{2,q}$ satisfying

$$-\xi_{2,q} < -d < -\xi_{1,q} < 0 < \Phi(q).$$

In particular in case $q = 0$ we have $\xi_{1,0} = 0$.

Proof. The equation $\psi(s) = q$ has one pole $-d$ and at most three roots. Consider ψ on $(-d, \infty)$, we know that

$$\psi(0) = 0, \quad \psi'(0+) < 0, \quad \lim_{s\to\infty} \psi(s) = \infty.$$

hence, there is a root $\Phi(q) > 0$. There exist a root $-\xi_{1,q}$ in $(-d, 0)$, because $\lim_{s\downarrow -d} \psi(s) = \infty$. Consider ψ on $(-\infty, -d)$,

$$\lim_{s\uparrow -d} \psi(s) = -\lim_{s\to -\infty} \psi(s) = -\infty,$$

thus there is a root $-\xi_{2,q}$ in $(-\infty, -d)$. Alltogether, there are exactly three roots of ψ. See for example Figure 4.1 on page 95. $\square$

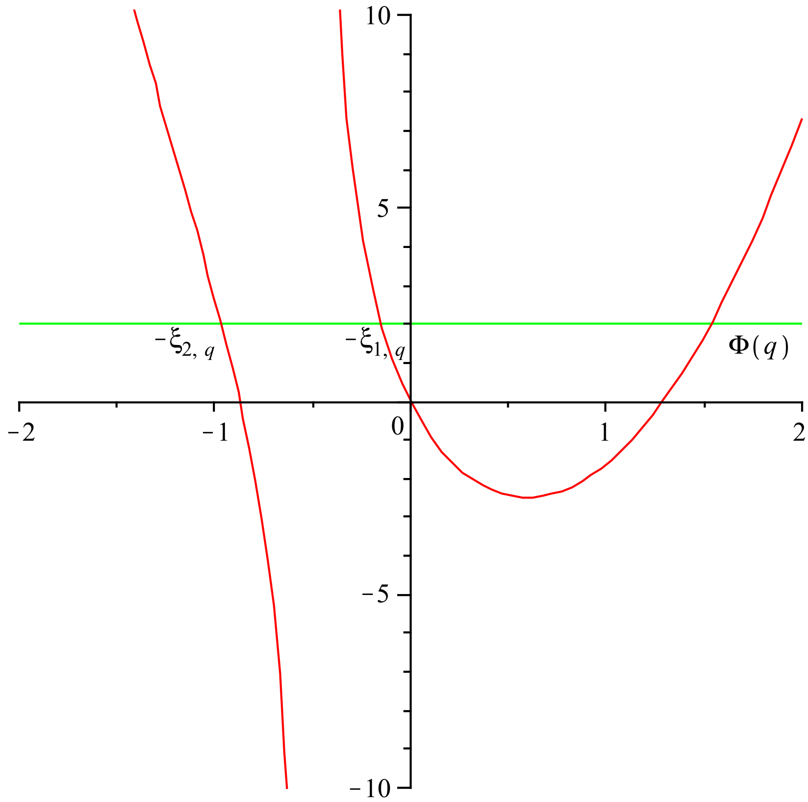

Figure 4.1: Laplace exponent with negative exponentially distributed jumps.

The previous lemma shows that the set I_q consists of two elements and the set J_q of one. The following Lemma establishes a relation between the roots of Cramer-Lundberg equation, which will be useful later in the derivation of the scale function.

Lemma 4.20 *(i) For $q > 0$ we have $\Phi(q) = \frac{2dq}{\sigma^2 \xi_{1,q}\, \xi_{2,q}}$.*

(ii) For $q = 0,\ \psi'(0+) < 0$ we have $\Phi(0) = \frac{2d|\psi'(0+)|}{\sigma^2 \xi_{2,0}}$.

Proof. Using the equation (4.10) from [Egami, Yamazaki (2011)b] with $h = 1$ we obtain

$$\frac{\partial}{\partial x}\mathbb{E}^x\left[\int_0^{\tau_a} e^{-qt}dt\right]\Big|_{x=0+} = W^{(q)\prime}(0+)\int_0^{\infty} e^{-\Phi(q)y}dy$$

for $0 < a < x$ and $\tau_a = \inf\{t \geq 0 : \pi_t \leq a\}$. By relation (4.7) in [Egami, Yamazaki (2011)b] we have

$$W^{(q)'}(0+) = \frac{2}{\sigma^2}.$$

This yields

$$\frac{\partial}{\partial x}\mathbb{E}^x\left[\int_0^{\tau_a} e^{-qt}dt\right]\Big|_{x=0+} = \frac{2}{\sigma^2\Phi(q)}.$$

On the other hand using Lemma 3.2 from [Egami, Yamazaki (2011)b] we obtain

$$\frac{\partial}{\partial x}\mathbb{E}^x\left[\int_0^{\tau_a} e^{-qt}dt\right]\Big|_{x=0+} = \frac{\xi_{1,q}\xi_{2,q}}{qd}.$$

Matching these equations yields the claim (i). The assertion (ii) follows by letting q go to zero. □

Plugging the formulas of Lemma 4.20 into (4.16) yields the following representation for the scale function

$$\begin{aligned} W^{(q)}(x) = &\frac{2}{\sigma^2}\left(e^{\Phi(q)x} - e^{-\xi_{1,q}x}\right)\frac{d-\xi_{1,q}}{(\xi_{2,q}-\xi_{1,q})(\xi_{1,q}+\Phi(q))} \\ &+ \frac{2}{\sigma^2}\left(e^{\Phi(q)x} - e^{-\xi_{2,q}x}\right)\frac{\xi_{2,q}-d}{(\xi_{2,q}-\xi_{1,q})(\xi_{2,q}+\Phi(q))}, \end{aligned} \tag{4.20}$$

$x, q \geq 0$. In particular, for $q = 0$ we have

$$W(x) = \frac{2}{\sigma^2}\left[\left(e^{\Phi(0)x} - 1\right)\frac{d}{\xi_{2,0}\Phi(0)} + \left(e^{\Phi(0)x} - e^{-\xi_{2,0}x}\right)\frac{\xi_{2,0}-d}{(\xi_{2,0}+\Phi(0))\xi_{2,0}}\right].$$

4.5.4 Example with Hyperexponential Distribution

Now we consider the case that the jumps $(Z_i)_{i\in\mathbb{N}}$ in

$$\pi_t - \pi = \mu t + \sigma W_t - \sum_{i=1}^{N_t} Z_i \tag{4.21}$$

have a hyperexponential distribution with density

$$f(x) = \sum_{i=1}^{m} \delta_i d_i e^{-d_i z}, \quad z \geq 0,$$

for some $0 < d_1 < d_2 < ... < d_m < \infty$ and $\delta_i > 0$, such that $\sum_i^m \delta_i = 1$. Using (4.15) we can compute the Laplace exponent, where we use $\mathbf{T}$ as given in (4.13), then

$$\psi(s) = \mu s + \frac{1}{2}\sigma^2 - \lambda \sum_{i=1}^{m} \delta_i \frac{s}{d_i + s}.$$

Lemma 4.21
There are m distinct poles of ψ, which are given by

$$0 > -d_1 > -d_2 > ... > -d_m > -\infty.$$

There are exactly $m+1$ distinct roots of $\psi(s) = q$, for $q \geq 0$ satisfying

$$0 < \xi_{1,q} < d_1 < \xi_{2,q} < d_2 \ldots < d_m < \xi_{m+1,q} < \infty.$$

Proof. We can prove this lemma analogously to Lemma 4.19, where we showed the case $m = 1$, see Figure 4.2. □

Since the roots are all distinct we can use (4.16) to compute the scale function.

4.6 Numerical Results

In this section we compare the results for the expected inter trading time computed using scale functions with results for a Monte Carlo simulation. We consider the risky fraction process $(\pi_t)_{t\geq 0}$ with dynamics

$$d\pi_t = \pi_{t-}(1-\pi_{t-})(\mu - \lambda\mathbb{E}[Z] - \sigma^2\pi_{t-})dt + \pi_{t-}(1-\pi_{t-})\sigma dW_t + \int_E \pi_{t-}(1-\pi_{t-})\frac{x}{1+\pi_{t-}x}J(dx,dt).$$

As we have seen in the previous sections, the transformation ϕ does not have an impact on the transition probabilities or the first exit time, thus we can work with the transformed process

$$\overline{\pi}_t = \overline{\pi} + (\mu - \lambda\mathbb{E}[Z] - \frac{1}{2}\sigma^2)t + \sigma W_t + \sum_{i=1}^{N_t} \log(1+Z_i).$$

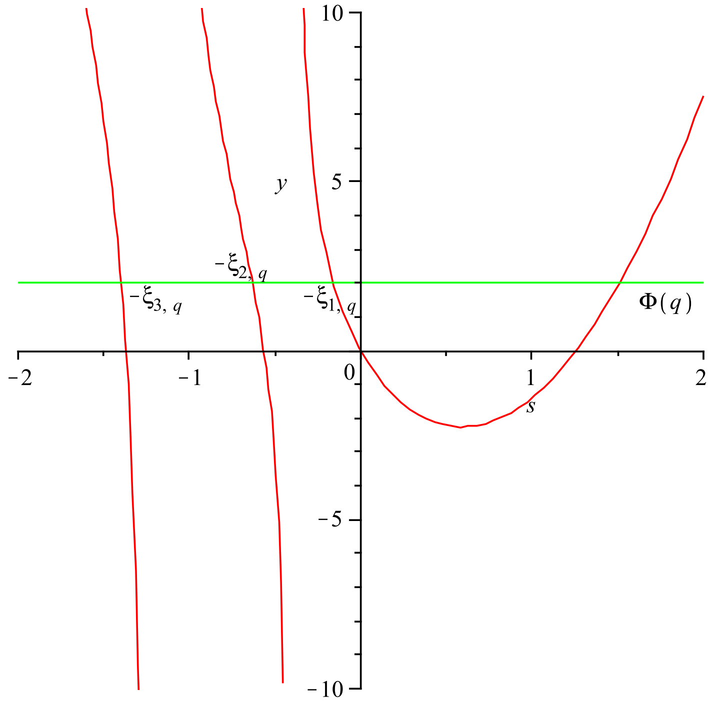

Figure 4.2: Laplace exponent with hyperexponential distribution of jumps, $m = 2$.

Since the scale functions are defined only for spectrally negative Lévy processes, we assume that the stock jumps only downwards, i.e. the jumps $(Z_i)_{i\in\mathbb{N}}$ are defined on $E = (-1, 0)$. Now we just have to rewrite the process

$$\overline{\pi}_t = \overline{\pi} + (\mu - \lambda\mathbb{E}[Z] - \frac{1}{2}\sigma^2)t + \sigma W_t - \sum_{i=1}^{N_t} \underbrace{-\log(1 + Z_i)}_{:=\widetilde{Z}_i},$$

and assume that the jumps $(\widetilde{Z}_i)_{i\in\mathbb{N}}$ are exponentially distributed with rate d, to obtain the same situation as in previous section: $(\overline{\pi}_t)_{t\geq 0}$ is a spectrally negative Lévy process with phase-type distributed jumps.

Here we want to analyze the impact of the intensity of jumps on the first exit time and the inter trading time. We fix the volatility $\sigma = 0.4$, the drift $\mu = 0.2$ and the costs $\gamma = 0.006$, $\delta = 0.0001$ and compute the optimal boundaries (a, α, β, b) for different λ and fixed $d = 0.2$. These strategies are gathered in Table 4.1 on page 99.

Table 4.1: Optimal CB-strategies for different λ.

λ	a	α	β	b
1	0.749444039	0.816005082	0.877047017	0.921057548
1.5	0.626378753	0.700147026	0.783474648	0.840899816
2	0.540047073	0.615835506	0.710034385	0.774677324
2.5	0.474966353	0.550856245	0.650500009	0.719507488
3	0.423730785	0.498866676	0.601015586	0.672800629
3.5	0.382165053	0.456138658	0.559082381	0.632669782
4	0.34768081	0.420298554	0.522999107	0.597751443
4.5	0.318566888	0.389747049	0.491560108	0.56704158
5	0.29363736	0.363358939	0.463881632	0.539785476
5.5	0.272039653	0.340315689	0.43929866	0.515403703

Note that even though the drift $(\mu-\lambda\mathbb{E}[Z]-\frac{1}{2}\sigma^2)$ increases with increasing λ, the boundaries decrease due to the increasing variance, see Figure 4.3 on page 100. Thus it is optimal for the investor to allocate less money in the stock.

The next step is to transform the boundaries (a, α, β, b) with ϕ and compute the expected exit time from the interval $(\overline{a}, \overline{b})$, where $\overline{a} = \phi(a)$ and $\overline{b} = \phi(b)$. We compute

$$\begin{aligned}
&\mathbb{E}^{\pi}\left[\tau_{(a,b)}\right] \\
&= -\frac{\partial}{\partial q}\left(\frac{W^{(q)}(\pi-a)}{W^{(q)}(b-a)} + Z^{(q)}(\pi-a) - Z^{(q)}(b-a)\frac{W^{(q)}(\pi-a)}{W^{(q)}(b-a)}\right)\Big|_{q=0},
\end{aligned}$$

where we use the formula (4.20) for the scale function $W^{(q)}(x)$. Now we define a function

$$\widetilde{W}(q,\pi,a,b) := \frac{W^{(q)}(\pi-a)}{W^{(q)}(b-a)} + Z^{(q)}(\pi-a) - Z^{(q)}(b-a)\frac{W^{(q)}(\pi-a)}{W^{(q)}(b-a)},$$

and we denote by $\mathbb{E}^{\overline{\pi}}[\tau_{(\overline{a},\overline{b})}]$ the first exit time computed with scale functions. With this definition we have

$$\mathbb{E}^{\overline{\pi}}[\tau_{(\overline{a},\overline{b})}]^{SF} = -\frac{\partial}{\partial q}\widetilde{W}(q,\overline{\pi},\overline{a},\overline{b})\Big|_{q=0}.$$

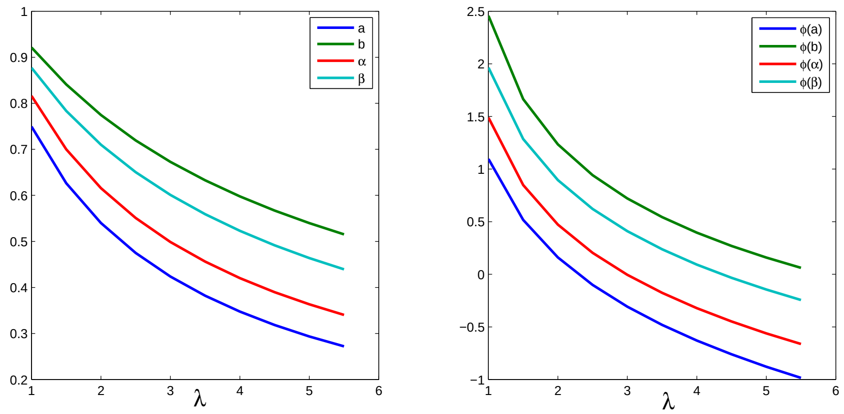

Figure 4.3: Optimal strategies (a, α, β, b) and transformed strategies $(\overline{a}, \overline{\alpha}, \overline{\beta}, \overline{b})$ for different λ.

We compare $\mathbb{E}^{\overline{\pi}}[\tau_{(\overline{a},\overline{b})}]^{SF}$ with the result of the Monte Carlo simulation, which we denote by $\mathbb{E}^{\overline{\pi}}[\tau_{(\overline{a},\overline{b})}]^{MC}$. We generate 100000 paths of the process $(\overline{\pi}_t)_{t\geq 0}$ with time horizon $T = 25$ years and 10000 discretization points per year. We choose first the initial value of $(\overline{\pi}_t)_{t\geq 0}$ in the middle of the no-trading region, i.e. $\overline{\pi} = \frac{\overline{a}+\overline{b}}{2}$.

The expectation of the first exit time computed by scale functions $\mathbb{E}^{\overline{\pi}}[\tau_{(\overline{a},\overline{b})}]^{SF}$ is slightly less than simulated results $\mathbb{E}^{\overline{\pi}}[\tau_{(\overline{a},\overline{b})}]^{MC}$, see Table 4.2 on page 101. These differences arise because $-\widetilde{W}(q)$ is concave at the origin. Figure 4.4 on page 102 shows $\widetilde{W}(q)$ for $\lambda = 1, 1.5, \ldots, 5.5$ as listed in Table 4.2. $\widetilde{W}(q)$ is decreasing in λ on $[0, \infty)$ and increasing on $(-\infty, 0)$.

We computed the scale function of risky fraction process for $\lambda = 1$ and $q = 0, 0.1, \ldots, 1$. As we see in Figure 4.5 $W^{(q)}(x)$ is increasing as expected and and $W^{(q)}(0) = 0$. Furthermore $W^{(q)}(x)$ is increasing in q. We also obtained $W^{(1)}(x)$ for different values of λ. Figure 4.6 shows that $W^{(1)}(x)$ is decreasing in λ.

Using the representations of the transition probabilities

$$p_{\overline{\alpha},\overline{\beta}} = \frac{W^{(0)}(\overline{\alpha} - \overline{a})}{W^{(0)}(\overline{b} - \overline{a})} \quad p_{\overline{\beta},\overline{\alpha}} = 1 - \frac{W^{(0)}(\overline{\beta} - \overline{a})}{W^{(0)}(\overline{b} - \overline{a})}$$

and $\mathbb{E}^{\overline{\alpha}}[\tau_{(\overline{a},\overline{b})}]$, $\mathbb{E}^{\overline{\beta}}[\tau_{(\overline{a},\overline{b})}]$, we can now compute the expected inter trading time,

Table 4.2: Expected first exit time.

λ	$\overline{b}-\overline{a}$	$\mathbb{E}^{\overline{\pi}}[\tau_{(\overline{a},\overline{b})}]^{SF}$	$\mathbb{E}^{\overline{\pi}}[\tau_{(\overline{a},\overline{b})}]^{MC}$
1	0.39386309	2.030486398	2.039054231
1.5	0.331285435	1.335973419	1.341960591
2	0.31137689	1.076607193	1.082514625
2.5	0.304349265	0.93487856	0.945259594
3	0.302943334	0.842998774	0.851727674
3.5	0.304472038	0.777333608	0.782179953
4	0.307701393	0.727465449	0.732162309
4.5	0.311990011	0.687909797	0.692020832
5	0.316968345	0.655537654	0.660630843
5.5	0.322411016	0.628417006	0.633832544

which we denote by $\mathbb{E}^{\nu}[\tau]^{SF}$. The results for $\mathbb{E}^{\nu}[\tau]^{SF}$ and expected inter trading time computed via Monte Carlo $\mathbb{E}^{\nu}[\tau]^{MC}$ are gathered in Table 4.3. For $\lambda = 1$ we have a small deviation because the number of crossings is not big enough to make the value of the inter trading time more precise. For other values of λ the expected inter trading time computed by scale functions correspond remarkably well to the Monte Carlo simulation.

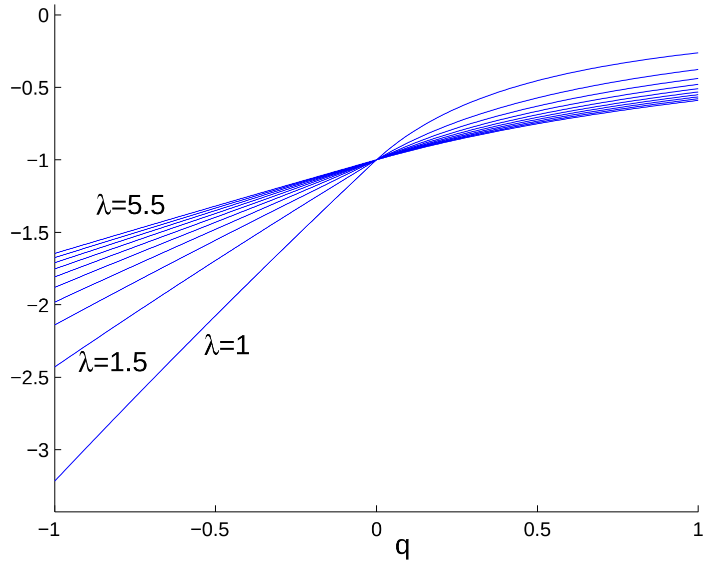

Figure 4.4: -$\widetilde{W}(q)$ for different intensities of jumps.

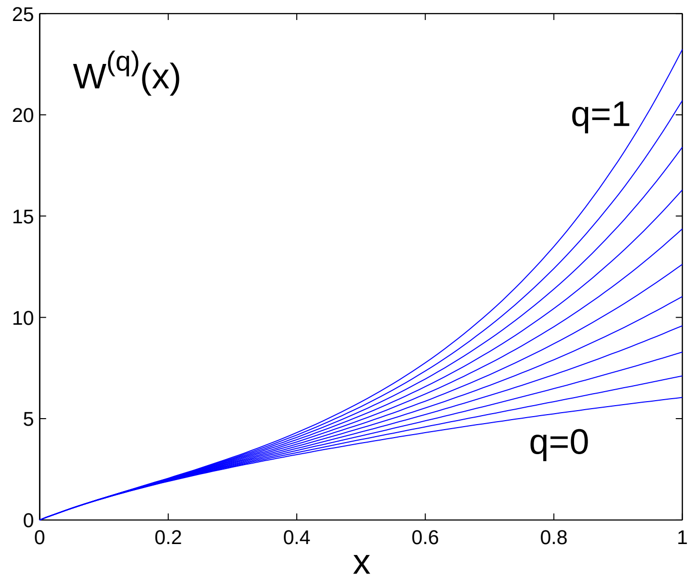

Figure 4.5: Scale Function of $(\overline{\pi}_t)_{t\geq 0}$ for different q.

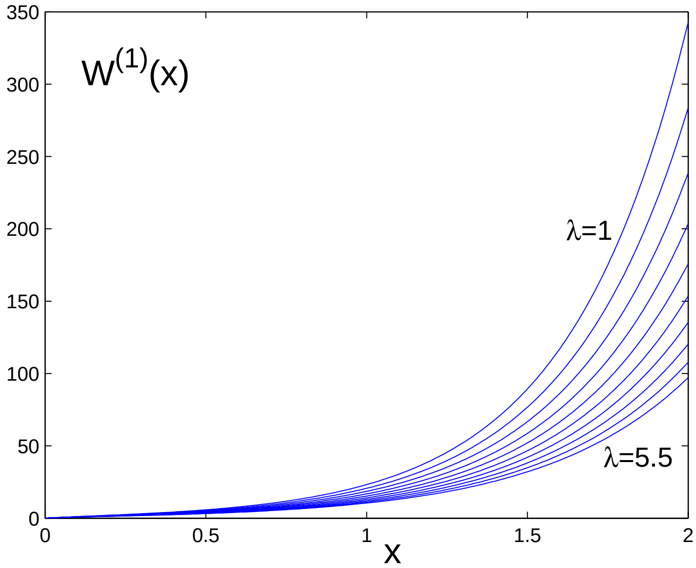

Figure 4.6: $W^{(1)}(x)$ for different λ.

Table 4.3: Expected inter trading time.

λ	$\bar{b} - \bar{a}$	$\mathbb{E}^{\nu}[\tau]^{SF}$	$\mathbb{E}^{\nu}[\tau]^{MC}$
1	0.39386309	1.810850627	1.743441095
1.5	0.331285435	1.159257341	1.144003546
2	0.31137689	0.925226782	0.920757064
2.5	0.304349265	0.800788908	0.803035315
3	0.302943334	0.721724343	0.724163212
3.5	0.304472038	0.666045277	0.670851931
4	0.307701393	0.624220091	0.629527937
4.5	0.311990011	0.591320137	0.599930725
5	0.316968345	0.56454934	0.573208515
5.5	0.322411016	0.542214039	0.548988395

Bibliography

[Akian et al. (1996)] AKIAN M., MENALDI, J.L., SULEM, A.: *On an investment-consumption model with transaction costs*, SIAM Journal on Control and Optimization 34, 329-364.

[Applebaum (2004)] APPLEBAUM D.: *Lévy-Processes and stochastic calculus*, Cambridge studies in advanced mathematics, Cambridge University Press.

[Asmussen (2004)] ASMUSSEN, S., AVRAM, F., PISTORIUS, M.R.: *Russian and American put option under exponential phase-type Lévy models*, Stochastic Processes and their Applications 109, 79-111.

[Asmussen (2003)] ASMUSSEN, S.: *Applied Probability and Queues*, Springer-Verlag, New York.

[Bertoin (1996)] BERTOIN, J.: *Lévy Processes*, Cambridge University Press.

[Bielecki, Pliska (2000)] BIELECKI, T., PLISKA, S.: *Risk sensitive asset management with transaction costs*, Finance and Stochastics 4, 1-33.

[Bladt (2005)] BLADT, M.: *A Review on phase-type distributions and their use in risk theory*, Austin Bulletin, 146-161.

[Cont, Tankov (2004)] CONT, R., TANKOV, P.: *Financial modeling with jump processes*, Chapman & Hall/CRC, Financial Mathematics Series.

[Davis, Norman (1990)] DAVIS, M.H.A., NORMAN, A.: *Portfolio selection with transaction costs.*, Mathematics of Operations Research 15, 676-713.

[Eastham, Hastings (1988)] EASTHAM, J.F., HASTINGS, K.J.: *Optimal impulse control of portfolios*, Mathematical Methods of Operations Research 13, 588-605.

[Egami, Yamazaki (2011)a] EGAMI, M., YAMAZAKI, K.: *On scale functions of spectrally negative Levy processes with phase-type jumps*, working paper, July 3.

[Egami, Yamazaki (2011)b] EGAMI, M., YAMAZAKI, K.: *Precautionary measures for credit risk management in jump models*, working paper, June 17.

[Hubalek, Kyprianou (2010)] HUBALEK, F., KYPRIANOU, A.: *Old and new examples of scale functions for spectrally negative Lévy Processes*, Progress in Probability, Vol. 63, 119-145.

[Irle, Prelle (2009)] IRLE, A., PRELLE, C.: *A renewal theoretic result in portfolio theory under transaction costs with multiple risky assets*, Statistics & Decisions: Vol. 27, No. 3, 211-233.

[Irle, Sass (2006)a] IRLE, A., SASS, J.: *Good portfolio strategies under transaction costs: a renewal theoretic approach*, In Stochastic Finance, eds M. do Rosario Grossinho, A.N.Shiryaev, M.L.Esquivel and P.E.Oliveira, Springer New York, 321-341.

[Irle, Sass (2006)b] IRLE, A. AND SASS,J.: *Optimal portfolio policies under fixed and proportional transaction costs*, Advances in Applied Probability 38, 916-942.

[Kabanov, Klüppelberg (2004)] KABANOV, Y., KLÜPPELBERG, C.: *A geometric approach to portfolio optimization in models with transaction costs*, Finance and Stochastics 8, 207-227.

[Kallenberg (1997)] KALLENBERG, O.: *Foundations of Modern Probability*, Springer.

[Konno et al. (1993)] KONNO, H., PLISKA, S., SUZUKI, K.-I.: *Optimal portfolios with asymptotic criteria*, Annals of Operations Research 45, 187-204.

[Korn (1998)] KORN, R.: *Portfolio optimisation with strictly positive transaction costs and impulse control*, Finance and Stochastics 2, 85-114.

[Kyprianou (2006)] KYPRIANOU, A.: *Introductory Lectures on Fluctuations of Lévy Processes with Applications*, Springer Berlin Heidelberg.

[Ludwig (2012)] LUDWIG, A.: *Comparisons and asymptotics in the theory of portfolio optimization under fixed and proportional transaction costs*, PhD Thesis, Christian-Albrechts-Universität zu Kiel.

[Magill, Constantinides (1976)] MAGILL, M.J., CONSTANTINIDES, G.M.: *Portfolio selection with transaction costs*, Journal of Economic Theory 13, 245-263.

[Merton (1969)] MERTON, R.C.: *Lifetime portfolio selection under uncertainty: the continuous time case.*, The Review of Economics and Statistics 51, 247-257.

[Mordecki, Lewis (2005)] MORDECKI, E., LEWIS, A.L.: *Wiener-Hopf factorization for Lévy processes having negative jumps with rational transforms*, working paper, august 4.

[Morton, Pliska (1995)] MORTON, A.J., PLISKA, S.R.: *Optimal portfolio management with fixed transaction costs*, Mathematical Finance 5, 337-356.

[Øksendal, Sulem (2005)] ØKSENDAL, B., SULEM, A.: *Applied Stochastic Control of Jump Diffusions*, Springer.

[Øksendal, Sulem (2001)] ØKSENDAL, B., SULEM, A.: *Optimal consumption and portfolio with both fixed and proportional transaction costs.*, SIAM Journal on Control and Optimization 40, 1765-1790.

[Protter (2005)] PROTTER, P.: *Stochastic Integration and Differential Equations*, Springer.

[Revuz, Yor (1991)] REVUZ, D., YOR, M.: *Continuous Martingales and Brownian Motion*, Springer Berlin.

[Sass (2001)] SASS, J.: *Portfoliooptimierung unter Transaktionskosten*, PhD Thesis, Christian-Albrechts-Universität zu Kiel.

[Sato (1999)] SATO, K.I.: *Lévy-processes and infinitely divisible distributions*, Cambridge studies in advanced mathematics, Cambridge University Press.

[Shreve, Soner (1994)] SHREVE, S.E, SONER, H.M.: *Optimal investment and consumption with transaction costs*, Annals of Applied Probability 4, 609-692.

[Duncan et al. (2009)] DUNCAN, T., PASIK DUNCAN, B., STETTNER, L.: *Growth optimal portfolio selection under proportional transaction costs with obligatory diversification*, Optimality and Risk - Modern Trends in Mathematical Finance, Springer Berlin Heidelberg.

[Taksar et al (1988)] TAKSAR, M., KLASS, M., ASSAF, D.: *A Diffusion model for optimal portfolio selection in presence of brokerage fees*, Mathematical Methods of Operations Research 13, 277-294.

[Tamura (2006)] TAMURA, T.: *Maximizing the growth rate of a portfolio with fixed and proportional transaction costs*, Applied Mathematics and Optimization 54, 95-116.

[Tamura (2008)] TAMURA, T.: *Maximization of the long-term growth rate for a portfolio with fixed and proportional transaction costs*, Advances in Applied Probability 40, 673-695.

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