Utility-based proof for the existence of strictly consistent price processes under proportional transaction costs

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Contents

Introduction

The classical *Fundamental Theorem of Asset Pricing* (FTAP), which goes back to Harrison, Kreps [6] and Harrison, Pliska [7], roughly states that in a financial market model there exist no arbitage opportunities if and only if there exists a fair price system for financial derivatives. Without transaction costs such a price system is given by taking expectation with respect to an equivalent martingale measure.

There are several proofs of the FTAP in finite discrete time for financial markets without transaction costs. Besides the original proof for a general probability space of Dalang, Morton and Willinger [2] based on a measurable selection theorem, there are many other proofs, e.g. Schachermayer's Hilbert Space proof [24] or Rogers' [20] utility-based proof.

Now, for markets with proportional transaction costs the reasoning in Kabanov, Ràsonyi and Stricker [10], [11] and in Schachermayer [25] can be compared to [24]. Once it is shown with help of a no-arbitrage condition that the set of all final portfolios achieved by self-financed trading is closed in $\mathbf{L}^0$ and intersects trivially with all 'positive' portfolios, one can apply the Hahn-Banach separation theorem and an exhaustion argument, as in the proof of the Kreps-Yan theorem, to seperate all final portfolios from all 'positive' portfolios. The terminal value of the consistent price process is then given by the vector which generates the separating hyperplane. Aside from these 'functional analytic' proofs there are results which are based on methods from the theory of random sets such as Ràsonyi [17] and Rokhlin [23]. The reader should consult the book 'Markets with Transaction Costs - Mathematical Theory' by Kabanov and Safarian [12] for a thorough exposition of the arbitrage theory under transaction costs.

Our goal is to give an elementary utility-based proof in the context of proportional transaction costs as was done by Rogers [20] for frictionless markets. The problems are twofold: Firstly for frictionless markets, in contrast to markets with transaction costs, there is equivalence between the no-arbitrage condition for the multi-period market and the no-arbitrage condition for each single-period market. Hence, it is basically enough to consider a single-period market, find a utility-maximizing portfolio and deduce from the maximizer using the first-order condition an equivalent martingale measure for the single-period market. In a multi-period market a martingale measure can then be constructed as a product of the single-period martingale measures when these are normalized appropriately. Under transaction costs several questions arise: Will it be enough to perform utility-maximization in a single-period market in the presence of transaction costs? What do the consistent price-processes from the single-period markets have to do with the consistent price-processes for the multi-period market? Can we construct a consistent price process for the multi-period market using consistent price processes from single-period markets?

Secondly, in frictionless markets we are naturally interested in the utility from the gain of our portfolio. Is it enough to naively liquidate our portfolio and consider the utility from this scalar value? If so, we will have to deal with a functional that is not differentiable.

We want to state the FTAP under proportional transaction costs. On a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t=0}^T, \mathbf{P})$ we consider for simplicity one risk-free and one risky asset. The risk-free asset serves as a numeraire. $\underline{S}_t$ and $\overline{S}_t$ denote the bid and ask prices respectively of the risky asset in terms of the numeraire at time t. The processes $(\underline{S}_t)_{t=0}^T$ and $(\overline{S}_t)_{t=0}^T$ are adapted, strictly positive and satisfy $\underline{S}_t \leq \overline{S}_t$ for every $t = 0, \ldots, T$. $\left(\underline{S}_t, \overline{S}_t\right)_{t=0}^T$ is called bid-ask process.

The following theorem is the one-dimensional version of Theorem 1.7 in Schachermayer [25].

Theorem. *The market satisfies the robust no-arbitrage condition if and only if there exists an equivalent probability measure* $\mathbf{Q}$ *and a process* $(S_t)_{t=0}^T$, $S_t \in \operatorname{ri}\left[\underline{S}_t, \overline{S}_t\right]$, *such that* $(S_t)_{t=0}^T$ *is a* $\mathbf{Q}$*-martingale.*

So, our goal as formulated above is to provide a new utility-based proof of the difficult sufficiency part of this theorem, i.e. of the existence of such pairs $(\mathbf{Q}, (S_t))$ in markets which satisfy the robust no-arbitrage condition. These pairs allow to define arbitrage-free prices for contingent claims in the market with transaction costs and thus extend the concept of an equivalent martingale measure from frictionless markets to markets with transaction costs.

The thesis is organized as follows: In Chapter 1 we first consider a model without transaction costs and describe the idea of Rogers' utility-based proof of the FTAP. Then we introduce a fairly general model which describes a market with proportional transaction costs. After we have clarified the appropriate concept of no-arbitrage and introduced consistent price processes, we sketch a first approach to a utility-based proof. The first chapter is more of introductory nature. The aim is to describe ideas and concepts, some calculations in between are just formal.

In Chapter 2 we consider a market with one risky asset. We first show in a generic single-period market that under a 'vector space assumption', which is in fact equivalent to the robust no-arbitrage condition, we can find a utility maximizing portfolio. With help of the first order condition we can construct a consistent price-process for the single-period market. Then, we apply the result from the single-period market to a general multi-period market. The basic idea, based on an inductive argument similar to [17] and [23], is to replace every single-period market by an auxiliary market with a smaller bid-ask spread hence more investment opportunities. This is outlined in detail at the beginning of Section 2.3, before we prove our main result in Theorem 2.16. These auxiliary bid-ask prices are not only sufficient but also necessary for the existsence of a strictly consistent price process for the multi-period market, see Corollary 2.20.

In Chapter 3 we apply the idea to a general multidimensional model. Again we first consider a generic single-period market. But, it turns out that we have to choose a different approach compared to the one-dimensional case. The main reason is that an analog result to Corollary 2.11 a) is missing in the multidimensional case. We will not maximize expected utility in the given market. Starting again from a 'vector space assumption' we show directly that the market satisfies the robust no-arbitrage condition. Therefore, we can reduce the transaction costs a little and maximize expected utility in this extended market. This will help us to circumvent Corollary 2.11 a) and get the result from Corollary 2.11 b) in the multidimensional case directly. It also takes much more work to show from the first-order condition that there actually is a consistent price process in the multidimensional case. The reason for this is that the first-order condition is only an inequality and the 'sandwich argument' from the one-dimensional case is no longer applicable here. Since we have more than one asset, we have to find a consistent price process for all assets simultaneously. After we have understood the single-period case we apply the idea to a multiperiod market. This is again similar to the one-dimensional case but more demanding technically.

In the Appendix we collect some advanced results from measure theory and probability theory which are used especially in Chapter 3.

Notation

A few words are in order concerning notation. Whenever a relation between random objects is stated, it is always supposed to hold almost surely. For example, when X, Y are random variables on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ and we write $X = Y$ on a set $B \in \mathcal{F}$, we mean that $X(\omega) = Y(\omega)$ for $\mathbf{P}$-almost every $\omega \in B$. Or when C is a random set, $X \in C$ means that $X(\omega) \in C(\omega)$ for $\mathbf{P}$-almost every $\omega \in \Omega$.

As usual we write $\mathbf{L}^0(C, \mathcal{F}; \mathbf{P})$ for the set of $\mathcal{F}$-measurable random variables which take their values $\mathbf{P}$-almost surely in C. Most of the time the probability measure $\mathbf{P}$ will be clear from the context, then we simply write $\mathbf{L}^0(C, \mathcal{F})$.

In a model with bank account we denote by d the number of risky assets, this corresponds to a model with $D \times D$ bid-ask matrices, where $D = d + 1$.

Chapter 1

Overview and first approach

1.1 Markets without transaction costs

Among the many proofs of the *Fundamental Theorem of Asset Pricing* (FTAP) there is Rogers' [20] utility-based proof. In this section we want to sketch his idea and point out some important properties of markets without transaction costs.

In our model we have one risk-free asset which serves as a numeraire and d risky assets whose prices are quoted in units of the numeraire. We fix a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t=0}^T, \mathbf{P})$ and consider an $(\mathcal{F}_t)_{t=0}^T$-adapted, $\mathbb{R}^d$-valued stochastic process $(S_t)_{t=0}^T = (S_t^1, \ldots, S_t^d)_{t=0}^T$, where each component S^i denotes the price of asset i in units of the numeraire.

When the investor decides to hold at time t an amount of h_t^i units of asset i, she has to pay $h_t^i S_t^i$ units from the numeraire for it. Her holdings are described by the vector $h_t = (h_t^1, \ldots, h_t^d)$. In the next step, at time $t+1$, the asset prices have changed so that her total wealth, measured in units of the numeraire, has changed by:

$$
\begin{aligned}
&-(h_t^1 S_t^1 + \cdots + h_t^d S_t^d) + (h_t^1 S_{t+1}^1 + \cdots + h_t^d S_{t+1}^d) \\
&= \begin{pmatrix} -(h_t^1 S_t^1 + \cdots + h_t^d S_t^d) \\ h_t^1 \\ \ldots \\ h_t^d \end{pmatrix} \cdot \begin{pmatrix} 1 \\ S_{t+1}^1 \\ \ldots \\ S_{t+1}^d \end{pmatrix} \\
&= h_t \cdot (S_{t+1} - S_t)
\end{aligned}
$$

With help of the second line we see that the investors gain corresponds to the liquidation value of her portfolio which she has acquired in a self-financing way. The investor starts with some wealth x_0 at the beginning. Then she trades according to h_0 at time 0. Her new wealth at time 1 is then $x_0 + h_0 \cdot (S_1 - S_0)$. This procedure continues until her wealth at terminal time T becomes

$$x_0 + h_0 \cdot (S_1 - S_0) + \cdots + h_{T-1} \cdot (S_T - S_{T-1}).$$

We call a d-dimensional process $(h_t)_{t=0}^{T-1}$ a *trading strategy* if, for every $t = 0, \ldots, T-1$, h_t is $\mathcal{F}_t$-measurable.

Definition 1.1. A trading strategy $(h_t)_{t=0}^{T-1}$ is called an *arbitrage opportunity* if

$$\sum_{t=0}^{T-1} h_t \cdot (S_{t+1} - S_t) \geq 0$$

and

$$\mathbf{P}\left(\sum_{t=0}^{T-1} h_t \cdot (S_{t+1} - S_t) > 0\right) > 0.$$

Of course, it is the investor's hope to find a trading strategy such that her gain is non-negative for sure at the end and strictly positive with some positive probability.

We say that $(S_t)_{t=0}^{T}$ satisfies the *no-arbitrage condition* if there is no arbitrage opportunity.

The FTAP relates the no-arbitrage condition to the existence of equivalent martingale measures.

Theorem 1.2. *For an $\mathbb{R}^d$-valued process $(S_t)_{t=0}^{T}$ the following are equivalent:*

(i) $(S_t)_{t=0}^{T}$ satisfies the no-arbitrage condition,

(ii) there exists a probabilty measure $\mathbf{Q} \sim \mathbf{P}$ such that $(S_t)_{t=0}^{T}$ is a $\mathbf{Q}$-martingale.

The probability measure $\mathbf{Q}$ in (ii) is called equivalent martingale measure for $(S_t)_{t=0}^{T}$.

Clearly, the difficult part in the proof is how to find an equivalent martingale measure assuming the no-arbitrage condition. In the case of finite Ω the theorem goes back to Harrison and Pliska [7]. For an arbitrary Ω it is due to Dalang, Morton and Willinger [2]. Many other proofs were given e.g. by Schachermayer [24], Kabanov and Kramkov [9] or Rogers [20].

The concept of equivalent martingale measures has many fruitful applications in the modern literature of mathematical finance such as the arbitrage-free pricing and hedging of contingent claims. Another very important application of equivalent martingale measures is to be found in the duality methods of portfolio optimization. The interested reader shall consult the book 'The Mathematics of Arbitrage' by Delbaen and Schachermayer [3] for a thorough exposition of arbitrage theory and its applications.

In contrast to models with transaction costs it is a distinctive feature for frictionless models that the no-arbitrage condition for every single-period model (S_{t-1}, S_t) is equivalent to the no-arbitrage condition for the multi-period model $(S_t)_{t=0}^T$. This is the main reason, why it is enough to consider the single-period models separately in search of an equivalent martingale measure.

Proposition 1.3. *For an $\mathbb{R}^d$-valued process $(S_t)_{t=0}^T$ the following are equivalent:*

(i) $(S_t)_{t=0}^T$ satisfies the no-arbitrage condition,

(ii) (S_{t-1}, S_t) satisfies the no-arbitrage condition for every $t = 1, \dots, T$.

For a proof see e.g. Proposition 5.11 in [4].

It is in fact enough to find an equivalent martingale measure for every single-period model. This is based on the following observation:

Assume that for some $t \in \{1, \dots, T\}$ we already have an equivalent martingale measure for $S_t, \dots, S_T$ which we call $\mathbf{Q}_t$. We can w.l.o.g. assume that $\mathbf{Q}_t | \mathcal{F}_t = \mathbf{P} | \mathcal{F}_t$, as we can switch to an equivalent probabilty measure whose density is given by

$$\frac{\frac{\mathrm{d}\mathbf{Q}_t}{\mathrm{d}\mathbf{P}}}{\mathbf{E}_{\mathbf{P}}\left[\frac{\mathrm{d}\mathbf{Q}_t}{\mathrm{d}\mathbf{P}} | \mathcal{F}_t\right]}$$

and for which the martingale property of $S_t, \dots, S_T$ is preserved. Now, if $\mathbf{P}_{t-1}$ is equivalent to $\mathbf{P}$ such that

$$\mathbf{E}_{\mathbf{P}_{t-1}}[S_t | \mathcal{F}_{t-1}] = S_{t-1},$$

then the probabilty measure $\mathbf{Q}_{t-1}$ defined by

$$\frac{\mathrm{d}\mathbf{Q}_{t-1}}{\mathrm{d}\mathbf{P}} = \frac{\mathrm{d}\mathbf{P}_{t-1}}{\mathrm{d}\mathbf{P}} \frac{\mathrm{d}\mathbf{Q}_t}{\mathrm{d}\mathbf{P}}$$

is an equivalent martingale measure for $S_{t-1}, \dots, S_T$. This follows from a straightforward application of Bayes' formula.

Thus, going backwards in time one has to use the no-arbitrage condition for every single-period market (S_{t-1}, S_t) separately and find an equivalent martingale measure $\mathbf{P}_{t-1}$ for this single-period market. Multiplying the denisities successively results in an equivalent martingale measure for $S_0, \dots, S_T$.

This concatenation property and the equivalence in the previous proposition break down under transaction costs as we shall see in the next chapter.

For the remainder of this section we want to describe Rogers' idea of his uility-based proof.

From an economic point of view it is very plausible that in an arbitrage-free market the investor can maximize her expected utility from terminal wealth. Thus, we fix a utility function u on $\mathbb{R}$ and aim to find a portfolio $\hat{h}_{t-1}$ which maximizes the expected utility from terminal wealth in the single-period market (S_{t-1}, S_t), i.e.

$$\mathbf{E}\left[u\left(\hat{h}_{t-1} \cdot (S_t - S_{t-1})\right)\right] = \max_{h_{t-1}} \mathbf{E}\left[u\left(h_{t-1} \cdot (S_t - S_{t-1})\right)\right],$$

where h_{t-1} ranges over $\mathbf{L}^0(\mathbb{R}^d, \mathcal{F}_{t-1})$. Besides Rogers' original work [20] the interested reader shall consult Section 6.6 in [3] for a different approach to find a utility-maximizing portfolio.

Once a maximizer $\hat{h}_{t-1}$ is found, the function

$$\mathbb{R} \ni \alpha \mapsto \mathbf{E}\left[u\left((\hat{h}_{t-1} + \alpha \mathbf{1}_A e^i) \cdot (S_t - S_{t-1})\right)\right]$$

has a maximum at $\alpha = 0$ for every $A \in \mathcal{F}_{t-1}$ and $i = 1 \dots, d$. Thus, taking the derivative at $\alpha = 0$ yields

$$0 = \mathbf{E}\left[u'(\hat{h}_{t-1} \cdot (S_t - S_{t-1}))(S_t^i - S_{t-1}^i)\mathbf{1}_A\right]$$

for every $A \in \mathcal{F}_{t-1}$ and $i = 1, \dots, d$.

When we define (up to a normalizing constant) $u'(\hat{h}_{t-1} \cdot (S_t - S_{t-1}))$ as the density of a measure $\mathbf{P}_{t-1}$, then the last equation is precisely the martingale property for (S_{t-1}, S_t) under $\mathbf{P}_{t-1}$. For the multiperiod model $(S_0, \dots, S_T)$ an equivalent martingale measure $\mathbf{Q}$ is then built by

$$\frac{\mathrm{d}\mathbf{Q}}{\mathrm{d}\mathbf{P}} = cu'(\hat{h}_0 \cdot (S_1 - S_0)) \dots u'(\hat{h}_{T-1} \cdot (S_T - S_{T-1}))$$

where c is a normalizing constant.

Note that, when (S_{t-1}, S_t) allows for an arbitrage opportunity, say h_{t-1}^a, there cannot be a utility maximizing portfolio: For any potfolio h_{t-1} we have

$$(h_{t-1} + h_{t-1}^a) \cdot (S_t - S_{t-1}) \geq h_{t-1} \cdot (S_t - S_{t-1})$$

and this inequality is strict with positive probabililty. Since u is strictly increasing, it follows that

$$\mathbf{E}\left[u\left(h_{t-1}+h^a_{t-1}\cdot(S_t-S_{t-1})\right)\right] > \mathbf{E}\left[u\left(h_{t-1}\cdot(S_t-S_{t-1})\right)\right].$$

1.2 Markets with transaction costs

In this section we want to describe a fairly general model for a market with transaction costs in finite discrete time. We will follow thereby mainly Schachermayer's work [25]; compare also Section 3.1.1/3.1.2 of Kabanov and Safarian [12]. We will introduce various notions of the no-arbitrage condition and the concept of consistent price processes which extends the concept of equivalent martingale measures to models with transaction costs. Our goal is to formulate the FTAP under transaction costs and point out the main differences to models without transaction costs.

The market consists of D assets and we can exchange every asset i for any asset j. π^{ij} denotes the number of units of asset i needed to buy one unit of asset j. We need the following properties for the matrix (π^{ij}).

Definition 1.4. A matrix $(\pi^{ij}) \in \mathbb{R}^{D\times D}$ is called *bid-ask matrix* if

(i) $\pi^{ij} > 0$, for $i, j = 1, \dots, D$,

(ii) $\pi^{ii} = 1$, for $i = 1, \dots, D$,

(iii) $\pi^{ij} \leq \pi^{ik}\pi^{kj}$, for $i, j, k = 1, \dots, D$.

The first two properties are obvious. The last property is due to transaction costs, i.e. a direct exchange between any two assets should be cheaper than an indirect exchange via a third asset.

Remark 1.5. a) Assume that we have d risky assets and one risk-free asset which serves as a numeraire. The bid and ask prices of the risky assets are given by $\underline{S}^1, \overline{S}^1, \dots, \underline{S}^d, \overline{S}^d$, quoted in units of the numeraire. For every asset i we require $(0 <)\underline{S}^i \leq \overline{S}^i$ due to transaction costs and we set $\underline{S}^0 = \overline{S}^0 = 1$. Then (π^{ij}), defined by

$$\pi^{ij} = \frac{\overline{S}^j}{\underline{S}^i} \quad i \neq j, \quad \text{and} \quad \pi^{ii} = 1,$$

is a bid-ask matrix with $D = d+1$. Since $\pi^{ij} = \pi^{i0}\pi^{0j}$, the entries (π^{i0}) and (π^{0j}) are enough to describe all possible transactions. Exchanging asset i for asset j directly costs as much as first selling asset i and then taking an appropriate amount of money from the bank account and buying asset j.

Note that, when $D = 2$, every bid-ask matrix is given by bid and ask prices, with $\frac{1}{\pi^{21}} = \underline{S}$ and $\pi^{12} = \overline{S}$.

b) A model based on bid-ask matrices generalizes a market where transactions occur only via the bank account. We can think of an exchange market for different currencies where direct transactions are in general cheaper than indirect ones.

We fix a filtered probabilty space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t=0}^T, \mathbf{P})$ and consider an $(\mathcal{F}_t)_{t=0}^T$-adapted, $(D \times D)$-dimensional process $(\pi_t)_{t=0}^T$. For every t and ω, $\pi_t(\omega)$ is supposed to be a bid-ask matrix. We call $(\pi_t)_{t=0}^T$ a *bid-ask process*.

We need to model self-financing trading.

Definition 1.6. a) A *portfolio process* is an adapted $\mathbb{R}^D$-valued process $v = (v_t)_{t=0}^T = \left(v_t^1, \dots, v_t^D\right)_{t=0}^T$.

a) The set of all *solvent portfolios* in state ω and at time t, the *solvency cone* $K_t(\omega)$, is defined as

$$K_t(\omega) = \text{cone}\left\{\pi_t^{ij}(\omega)e^i - e^j, e^k \,:\, i, j, k = 1, \dots, D\right\}.$$

b) $-K_t(\omega)$ is called the *cone of portfolios available at price* 0 in state ω at time t, i.e.

$$-K_t(\omega) = \text{cone}\left\{e^j - \pi_t^{ij}(\omega)e^i, -e^k \,:\, i, j, k = 1, \dots, D\right\}.$$

c) A portfolio process $v = (v_t)_{t=0}^T$ is called *self-financing* if

$$v_0 \in -K_0 \quad \text{and} \quad v_t - v_{t-1} \in -K_t$$

for every $t = 1, \dots, T$.

d) The set A_T of all *hedgable claims* is defined as

$$A_T = \mathbf{L}^0\left(-K_0, \mathcal{F}_0\right) + \dots + \mathbf{L}^0\left(-K_T, \mathcal{F}_T\right),$$

i.e. $A_T = \left\{v_T \,:\, (v_t)_{t=0}^T \text{ \textit{self-financing portfolio process}}\right\}$.
Similarly A_t, the set of hedgable claims up to time t, is defined as $A_t = \mathbf{L}^0\left(-K_0, \mathcal{F}_0\right) + \dots + \mathbf{L}^0\left(-K_t, \mathcal{F}_t\right)$.

Every portfolio of the form $\pi_t^{ij} e^i - e^j$ is solvent, because there are enough units of asset i to clear the short position in asset j. Therefore, each conic combination, i.e. linear combination with non-negativ coefficients, is solvent. Additionally, every portfolio with non-negative entries is solvent. We point out that, if $\pi_t^{i_0 j_0} \pi_t^{j_0 i_0} > 1$ for some i_0, j_0, then the vectors e^k can already be obtained from conic combinations of the vectors $\pi_t^{ij} e^i - e^j$. Since

$$\pi_t^{j_0 i_0} (\pi_t^{i_0 j_0} e^{i_0} - e^{j_0}) + \pi_t^{j_0 i_0} e^{j_0} - e^{i_0} = (\pi_t^{i_0 j_0} \pi_t^{j_0 i_0} - 1) e^{i_0},$$

we can obtain e^{i_0} and then every e^k from conic combinations of $\pi_t^{k i_0} e^k - e^{i_0}$ and e^{i_0}.

We need π_t^{ij} units of asset i if we want to buy one unit of asset j. Hence, $e^j - \pi_t^{ij} e^i$ is available at price 0, i.e. this portfolio can be achieved by trading according to the bid-ask matrix and starting from the zero portfolio $0 \in \mathbb{R}^D$. Eventually we consume assets, hence we include portfolios with non-positive entries.

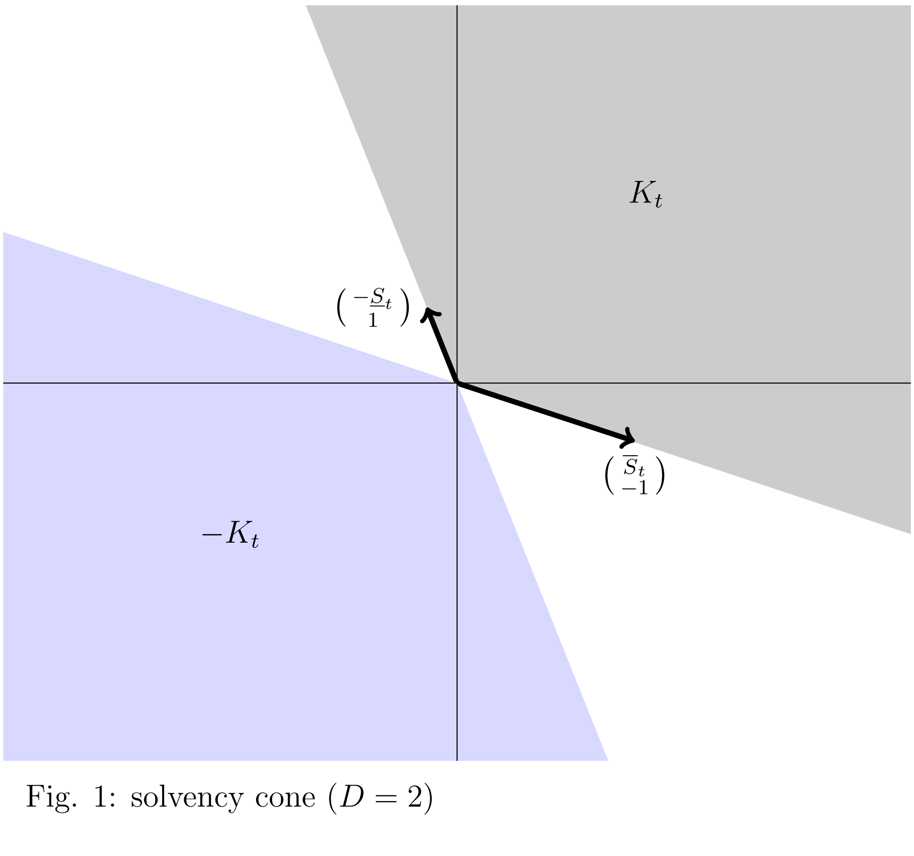

Fig. 1: solvency cone ($D = 2$)

Definition 1.7. The *dual cone* K_t^* is defined by

$$\begin{aligned} K_t^*(\omega) &= \left\{ z \in \mathbb{R}^D \,:\, z \cdot v \geq 0, \quad \text{for } v \in K_t(\omega) \right\} \\ &= \left\{ z \in \mathbb{R}_+^D \,:\, \pi_t^{ij}(\omega) z^i \geq z^j, \quad \text{for } i, j = 1, \dots, D \right\}. \end{aligned}$$

An $\mathcal{F}_t$-measurable random variable $Z \in K_t^* \setminus \{0\}$ is called *consistent price system* (at time t). If $Z \in \operatorname{ri} K_t^*$, then Z is called *strictly consistent price system* (at time t).

For a consistent price system Z we have

$$\frac{1}{\pi_t^{ji}} \leq \frac{Z^j}{Z^i} \leq \pi_t^{ij}.$$

In this sense the (frictionless) exchange rates $\frac{Z^j}{Z^i}$ are consistent with the rates given by the bid-ask matrix π_t^{ij}. Z being strictly consistent means that

$$\frac{Z^j}{Z^i} \in \operatorname{ri} \left[\frac{1}{\pi_t^{ji}}, \pi_t^{ij} \right].$$

In a model with bank account, where we have one risk-free asset which serves as a numeraire, and d risky assets with bid-ask prices $\underline{S}^1, \overline{S}^1, \dots, \underline{S}^d, \overline{S}^d$, $Z = (Z^0, Z^1, \dots, Z^d)$ is a (strictly) consistent price system if and only if

$$\frac{Z^i}{Z^0} \in \left[\underline{S}_t^i, \overline{S}_t^i \right] \quad \left(\text{resp. } \frac{Z^i}{Z^0} \in \operatorname{ri} \left[\underline{S}_t^i, \overline{S}_t^i \right] \right).$$

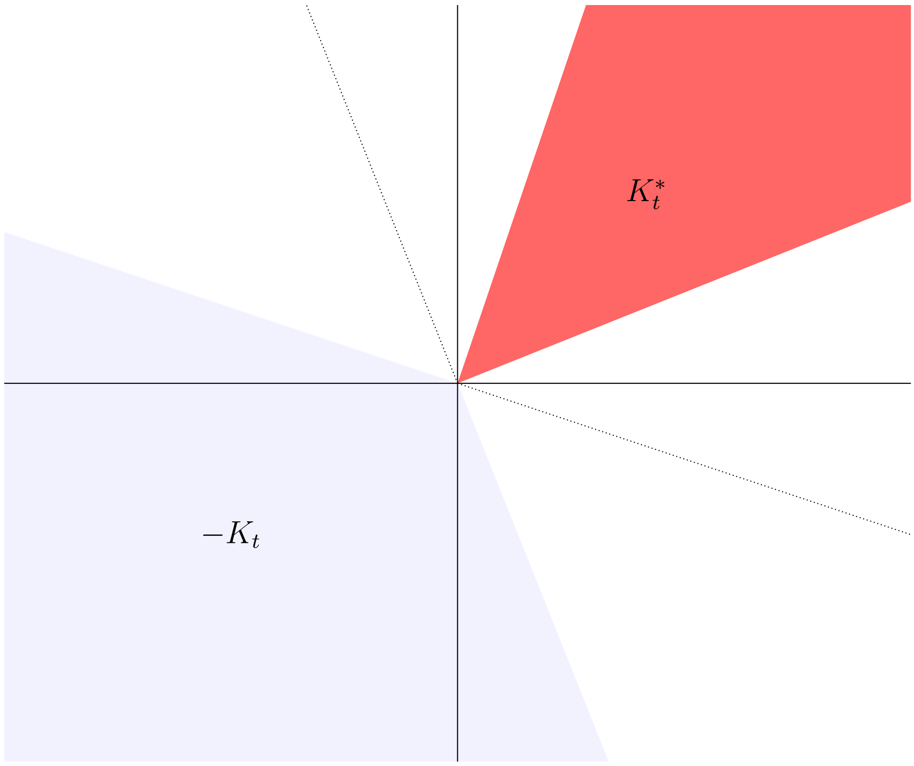

Fig. 2: the dual of the solvency cone ($D = 2$)

We now extend the key concept of equivalent martingale measures to models with transaction costs.

Definition 1.8. A D-dimensional martingale $(Z_t)_{t=0}^T$ is called

a) *consistent price process* if $Z_t \in K_t^* \setminus \{0\}$, for $t = 0, \dots, T$,

b) *strictly consistent price process* if $Z_t \in \operatorname{ri} K_t^*$, for $t = 0, \dots, T$.

Remark 1.9. In a model with bank account, where the bid and ask prices of the d risky assets are given by $\underline{S}^1, \overline{S}^1, \dots, \underline{S}^d, \overline{S}^d$, it follows by a straightforward application of Bayes' formula, that there exists a (strictly) consistent price process $(Z_t)_{t=0}^T$, if and only if there exists an equivalent probability measure $\mathbf{Q} \sim \mathbf{P}$ and a $\mathbf{Q}$-martingale $(S_t)_{t=0}^T = (S_t^1, \dots, S_t^d)_{t=0}^T$ such that $S_t^i \in [\underline{S}_t^i, \overline{S}_t^i]$, $(S_t^i \in \operatorname{ri}[\underline{S}_t^i, \overline{S}_t^i])$. In this case, we have $\frac{\mathrm{d}\mathbf{Q}}{\mathrm{d}\mathbf{P}} = \frac{Z_T^0}{Z_0^0}$ and $S^i = \frac{Z^i}{Z^0}$.

The *no-arbitrage condition* prevents that we can create money out of nothing.

Definition 1.10. The bid-ask process $(\pi_t)_{t=0}^T$ satisfies the *no-arbitrage condition* if for every self-financing portfolio process $v = (v_t)_{t=0}^T$ with $v_T \in \mathbb{R}_+^D$ it follows that $v_T = 0$.

Let us assume for a moment that there is a consistent price process $(Z_t)_{t=0}^T$. Ignoring integrability questions we conclude for every $v_T = \xi_0 + \cdots + \xi_T \in A_T$, where $\xi_t \in \mathbf{L}^0(-K_t, \mathcal{F}_t)$, that

$$\begin{aligned}\mathbf{E}[v_T \cdot Z_T] &= \mathbf{E}[(\xi_0 + \cdots + \xi_{T-1}) \cdot Z_{T-1} + \mathbf{E}[\underbrace{\xi_T \cdot Z_T}_{\leq 0} | \mathcal{F}_{T-1}]] \\ &\leq \mathbf{E}[(\xi_0 + \cdots + \xi_{T-1}) \cdot Z_{T-1}] \\ &\leq \mathbf{E}[(\xi_0 + \cdots + \xi_{T-2}) \cdot Z_{T-2} + \mathbf{E}[\xi_{T-1} \cdot Z_{T-1} | \mathcal{F}_{T-2}]] \\ &\leq \cdots \leq \mathbf{E}[\xi_0 \cdot Z_0] \leq 0.\end{aligned}$$

Consequently, when $v_T \in \mathbb{R}_+^D$, then $v_T = 0$ and the no-arbitrage condition is satisfied.

For a finite state space Ω or if $D = 2$ the following version of the Fundamental Theorem of Asset Pricing holds true. For finite Ω it is proved in Kabanov and Stricker [13], when $D = 2$ it is due to Grigoriev [5].

Theorem 1.11. *Let $(\pi_t)_{t=0}^T$ be a bid-ask process. Then, if Ω is finite or $D = 2$, the following are equivalent:*

(i) $(\pi_t)_{t=0}^T$ satisfies the no-arbitrage condition,

(ii) there exists a consistent price process.

The proof of how the no-arbitrage condition implies existence of a consistent price process relies on a separation argument in finite-dimenstional spaces. It is the same idea as for models without transaction costs. The set of hedgable claims $A_T = \mathbf{L}^0(-K_0, \mathcal{F}_0) + \cdots + \mathbf{L}^0(-K_T, \mathcal{F}_T)$ is a sum of finitely many polyhedral cones and is thus closed as a set in $\mathbb{R}^N$, $(N = D \cdot |\Omega|)$. The no-arbitrage condition ensures that $A_T \cap \mathbb{R}_+^N = \{0\}$ and we can separate with a hyperplane the cone A_T from the convex set $conv\big\{e^i \mathbf{1}_{\{\omega\}} : \omega \in \Omega, i = 1, \ldots, D\big\}$. The vector, which generates the separating hyperplane, can be taken as the terminal value of the consistent price process.

For infinite Ω, even when $D = 2$, the no-arbitrage condition is not enough to conclude that A_T is 'closed', see e.g. Example 1.3 in [5]. Thus the standard separation arguments are not applicable here. The theorem is even wrong when Ω is infinite and $D \geq 3$, see e.g. Example 3.1 in [25] or Example 2 in Section 3.2.4 of [12]. However, for $D = 2$, Grigoriev [5] managed to prove the theorem with new methods.

The concept of consistent price processes allows to describe the set of hedgable claims in a dual way. This is intimately connected with arbitrage-free pricing of contingent claims and essential for duality methods in portfolio optimization. Without being rigorous with integrability issues, by dual description we mean the equivalence

$$c \in A_T$$
$$\Leftrightarrow$$
$$\mathbf{E}\left[c \cdot Z_T\right] \leq 0 \text{ for every consistent price process } (Z_t)_{t=0}^{T}.$$

Of course, such a relation can only hold when A_T is 'closed', which is not always the case under the mere no-arbitrage condition.

Let us look at some example which illustrates that the no-arbitrage condition is insufficient in the presence of transaction costs. Even though a market with transaction costs satisfies the no-arbitrage condition, further market insufficiencies, so-called weak arbitrage opportunities, can appear.

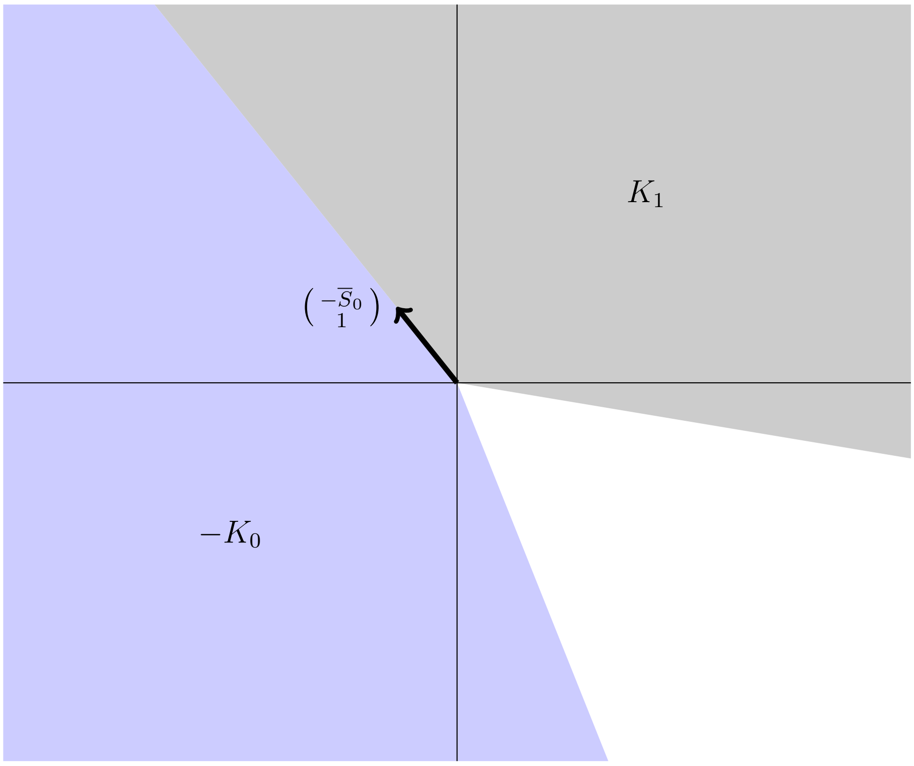

Fig. 3: weak arbitrage opportunity

Clearly, the market in Fig. 3 is free of arbitrage opportunities. But without incurring any loss we can buy one stock at time 0, $\binom{-\overline{S}_0}{1}$, and liquidate the portfolio into the bond at time 1, $\binom{-\overline{S}_0}{1} \cdot \binom{1}{\underline{S}_1} = 0$. We have to pay transaction costs for trading but the stock performance helps to cover it.

We can exclude weak arbitrage opportunitites if we assume that the broker can offer some discount on the transaction costs without creating arbitrage opportunities at the market.

Definition 1.12. The bid-ask process $(\pi_t)_{t=0}^T$ satisfies the *robust no-arbitrage condition* if there is a bid-ask process $(\sigma_t)_{t=0}^T$ such that

$$\left[\frac{1}{\sigma^{ji}}, \sigma_{ij}\right] \subset \mathrm{ri}\left[\frac{1}{\pi^{ji}}, \pi_{ij}\right]$$

and $(\sigma_t)_{t=0}^T$ satisfies the no-arbitrage condition.

Remark 1.13. In a model with one risk-free cash account and d risky assets with bid and ask prices $\underline{S}^1, \overline{S}^1, \dots, \underline{S}^d, \overline{S}^d$ the robust no-arbitrage condition is satisfied if and only if there exist bid-ask prices $\underline{U}^1, \overline{U}^1, \dots, \underline{U}^d, \overline{U}^d$ such that $\left[\underline{U}^i, \overline{U}^i\right] \subset \mathrm{ri}\left[\underline{S}^i, \overline{S}^i\right]$ and $\underline{U}^1, \overline{U}^1, \dots, \underline{U}^d, \overline{U}^d$ statsify the no-arbitrage condition.

Having established the definition of the robust no-arbitrage condition we formulate the FTAP under proportional transaction costs in finite discrete time.

Theorem 1.14. *Let $(\pi_t)_{t=0}^T$ be a bid-ask process. Then $(\pi_t)_{t=0}^T$ satisfies the robust no-arbitrage condition if and only if there exists a strictly consistent price process $(Z_t)_{t=0}^T$.*

The proof relies on the fact that under the robust no-arbitrage condition A_T is closed with respect to convergence in probability. Then separation arguments for infinite-dimensional spaces as in the Kreps-Yan theorem are applied to yield a consistent price process.

1.3 First approach and generalization

Our goal is to give a utility-based proof for consistent price processes in the spirit of Rogers [20] for frictionless models. We will describe a first approach now.

Let $U : \mathbb{R}^D \to \mathbb{R}$ be a multivariate utiltiy function, i.e. U is strictly concave, strictly increasing in every variable and differentiable. We assume that there is an optimal portfolio $\hat{v}_T \in A_T$, i.e.

$$\mathbf{E}\left[U(\hat{v}_T)\right] = \max_{v_T \in A_T} \mathbf{E}\left[U(v_T)\right].$$

Then, for every $\xi_t \in \mathbf{L}^0(-K_t, \mathcal{F}_t)$ and every $\alpha > 0$, we have

$$\mathbf{E}[U(\hat{v}_T)] \geq \mathbf{E}[U(\hat{v}_T + \alpha\xi_t)].$$

We take the one-sided derivative and get

$$0 \geq \lim_{\alpha \downarrow 0} \frac{\mathbf{E}[U(\hat{v}_T + \alpha\xi_t)] - \mathbf{E}[U(\hat{v}_T)]}{\alpha} = \mathbf{E}[\nabla U(\hat{v}_T) \cdot \xi_t].$$

$(Z_t)_{t=0}^T$ defined by $Z_T := \nabla U(\hat{v}_T)$ and $Z_t := \mathbf{E}[Z_T|\mathcal{F}_t]$ is a consistent price process.

Now, several questions arise in this approach:

- How can we find an optimal portfolio $\hat{v}_T$ under the robust no-arbitrage condition? In the multi-period model we have $\hat{v}_T = \hat{\xi}_0 + \ldots \hat{\xi}_T$, so $T+1$ decisions have to be made.
- Can we simplify the problem by looking on single-period models?
- Having solved the problem for single-period models, how can we construct a consistent price process for the multi-period model knowing that there are consistent price processes for each single-period model?
- What kind of utility functions can we use?

We want to address the first question in the case of finite Ω. We fix a utility function u on $\mathbb{R}$, which is bounded from above, and define a multivariate utility function

$$U(v^1, \ldots, v^D) := u(v^1) + \cdots + u(v^D).$$

We consider the concave function

$$\varphi : A_T \ni v_T \mapsto \mathbf{E}[U(v_T)].$$

It is well known that for a concave function φ, defined on a closed convex cone in a finite dimensional space, the condition

$$\lim_{s \to \infty} \varphi(sx) = -\infty, \quad \text{for every } x \neq 0,$$

is sufficent to show existence of a maximum for φ (see e.g. Lemma 3.5 in [4]). Well, whenever $v_T \in A_T$, $v_T \neq 0$, by the no-arbitrage condition we must have $\mathbf{P}(v_T^i < 0) > 0$ for some $i = 1, \ldots, D$. Since

$$\lim_{s \to \infty} u(sv_T^i) = -\infty \quad \text{on } \left\{v_T^i < 0\right\}.$$

and u is bounded from above, we conclude

$$\lim_{s\to\infty} \mathbf{E}\left[U(sv_T)\right] = -\infty, \quad \text{for every } v_T \in A_T, v_T \neq 0.$$

Thus, an optimal portfolio $\hat{v}_T$ exists when Ω is finite. For infinite Ω this argument fails since for example the condition

$$\lim_{s\to\infty} \varphi(sx) = -\infty, \quad \text{for every } x \neq 0,$$

is no longer sufficient for a maximum.

Generalization and vector space property

We want to generalize the framework a little bit. This will be similar to the model in Ràsonyi [17]. By the *Bipolar Theorem* we have for the solvency cone $K_t = K_t^{**}$, i.e.

$$K_t(\omega) = \left\{\xi \in \mathbb{R}^D \,:\, \xi \cdot z \geq 0, \quad \text{for } z \in K_t^*(\omega)\right\}.$$

Since

$$K_t^*(\omega) \setminus \{0\} = \left\{z \in (0,\infty)^D \,:\, \frac{z^j}{z^i} \in \left[\frac{1}{\pi_t^{ji}(\omega)}, \pi_t^{ij}(\omega)\right], \quad \text{for } i,j = 1,\dots,D\right\}$$

we can take a convex and compact set $\mathcal{S}_t(\omega) \subset (0,\infty)^{D-1}$ such that

$$K_t^*(\omega) = \left\{\lambda \left(\begin{smallmatrix} 1 \\ y \end{smallmatrix}\right) \,:\, \lambda \geq 0, y \in \mathcal{S}_t(\omega)\right\}$$

and thus

$$K_t(\omega) = \left\{\xi \in \mathbb{R}^D \,:\, \xi \cdot \left(\begin{smallmatrix} 1 \\ y \end{smallmatrix}\right) \geq 0, \quad \text{for } y \in \mathcal{S}_t(\omega)\right\}.$$

Writing $K_t^*(\omega)$ in this way means that we have chosen the first asset as a numeraire. When we have a model with bank account and d bid and ask prices $\underline{S}^1, \overline{S}^1, \dots, \underline{S}^d, \overline{S}^d$, then

$$\mathcal{S}_t(\omega) = [\underline{S}_t^{\mathbf{1}}(\omega), \overline{S}_t^{\mathbf{1}}(\omega)] \times \cdots \times [\underline{S}_t^d(\omega), \overline{S}_t^d(\omega)].$$

Thus, we will consider a model with one risk-free asset which is taken as a numeraire and for the remaining d risky assets we assume that prices are given by convex and compact subsets of $(0,\infty)^d$.

Given a filtered probabilty space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t=0}^T, \mathbf{P})$ we start with a process $(\mathcal{S}_t)_{t=0}^T$, where each $\mathcal{S}_t \subset (0,\infty)^d$ is a non-empty, convex, compact and $\mathcal{F}_t$-measurable random set. $\mathcal{S}_t$ is called the set of *consistent prices* at time t. The *solvency cone* at time t is defined by

$$K_t(\omega) := \left\{\xi \in \mathbb{R}^{d+1} \,:\, \xi \cdot \left(\begin{smallmatrix}1\\y\end{smallmatrix}\right) \geq 0, \quad \text{for } y \in \mathcal{S}_t(\omega)\right\}.$$

Of course, a portfolio process $(v_t)_{t=0}^T$ is called *self-financing* (for $(\mathcal{S}_t)_{t=0}^T$) if $v_0 \in -K_0$ and $v_t - v_{t-1} \in -K_t$ for every $t = 1, \ldots, T$. For the the dual cone of K_t we have

$$K_t^*(\omega) = \left\{\lambda \left(\begin{smallmatrix}1\\y\end{smallmatrix}\right) \,:\, \lambda \geq 0, y \in \mathcal{S}_t(\omega)\right\}$$

and

$$\operatorname{ri} K_t^*(\omega) = \left\{\lambda \left(\begin{smallmatrix}1\\y\end{smallmatrix}\right) \,:\, \lambda > 0, y \in \operatorname{ri} \mathcal{S}_t(\omega)\right\}.$$

$(\mathcal{S}_t)_{t=0}^T$ satisfies the *robust no-arbitrage condition* if there is a process $(\mathcal{W}_t)_{t=0}^T$ of consistent prices such that $\mathcal{W}_t \subset \operatorname{ri} \mathcal{S}_t$ and $(\mathcal{W}_t)_{t=0}^T$ satisfies the *no-arbitrage condition*, i.e. if $(v_t)_{t=0}^T$ is self-financing for $(\mathcal{W}_t)_{t=0}^T$ and $v_T \in \mathbb{R}_+^{d+1}$ then $v_T = 0$.

We cite the following important lemma from Schachermayer [25]. Its proof is a nice illustration for the robust no-arbitrage condition.

Lemma 1.15. *Assume that $(\mathcal{S}_t)_{t=0}^T$ satisfies the robust no-arbitrage condition. If $\xi_0 \in \mathbf{L}^0(-K_0, \mathcal{F}_0), \ldots, \xi_T \in \mathbf{L}^0(-K_T, \mathcal{F}_T)$ are such that*

$$\xi_0 + \cdots + \xi_T = 0,$$

then $\xi_0 \in \mathbf{L}^0(-K_0 \cap K_0, \mathcal{F}_0), \ldots, \xi_T \in \mathbf{L}^0(-K_T \cap K_T, \mathcal{F}_T)$.

Proof. Denote by $(\mathcal{W}_t)_{t=0}^T$ a process of consistent prices such that $\mathcal{W}_t \subset \operatorname{ri} \mathcal{S}_t$ and $(\mathcal{W}_t)_{t=0}^T$ satisfies the no-arbitrage condition. Assume that for some t_0 the event

$$A := \{\xi_{t_0} \in -K_{t_0} \setminus (-K_{t_0} \cap K_{t_0})\}$$

has positive probability. On A we have $\xi_{t_0} \cdot \left(\begin{smallmatrix}1\\y\end{smallmatrix}\right) < 0$ for every $y \in \operatorname{ri} \mathcal{S}_{t_0}$, especially $\xi_{t_0} \cdot \left(\begin{smallmatrix}1\\y\end{smallmatrix}\right) < 0$ for every $y \in \mathcal{W}_{t_0}$. Since $\mathcal{W}_{t_0}$ is compact, this implies $\max\left\{\xi_{t_0} \cdot \left(\begin{smallmatrix}1\\y\end{smallmatrix}\right) \,:\, y \in \mathcal{W}_{t_0}\right\} < 0$ on A.

Hence, for every $\omega \in A$, there exists $n \in \mathbb{N}$ such that

$$\left(\xi_{t_0}(\omega) + \frac{1}{n} e^0\right) \cdot \left(\begin{smallmatrix}1\\y\end{smallmatrix}\right) \leq 0$$

for all $y \in \mathcal{W}_{t_0}(\omega)$. Denote by $n^* = n^*(\omega)$ the minimal natural number with this property and define

$$\zeta_{t_0}(\omega) = \begin{cases} \xi_{t_0}(\omega) + \frac{1}{n^*(\omega)} e^0 & , \omega \in A \\ \xi_{t_0}(\omega) & , \text{otherwise} \end{cases}$$

and, for $t \neq t_0$, $\zeta_t = \xi_t$.

Then $(\zeta_t)_{t=0}^T$ are the increments of a self-financing portfolio process for $(\mathcal{W}_t)_{t=0}^T$ with terminal payoff

$$\zeta_0 + \cdots + \zeta_T = \frac{1}{n^*} \mathbf{1}_A e^0.$$

Since $P(A) > 0$, we have an arbitrage opportunity for $(\mathcal{W}_t)_{t=0}^T$. We conclude that the robust no-arbitrage condition implies

$$\xi_0 \in \mathbf{L}^0(-K_0 \cap K_0, \mathcal{F}_0), \ldots, \xi_T \in \mathbf{L}^0(-K_T \cap K_T, \mathcal{F}_T).$$

□

From this lemma it follows that the increments of self-financing portfolio processes, which sum up to 0, constitute a vector space under the robust no-arbitrage condition. This vector space property is indispensable for us, since it will help us reduce from the multi-period model to a single-period model. In the multidimensional case, Chapter 3, we will show the converse statement directly, i.e. that the vector space property implies the robust no-arbitrage condition.

Utility from liquidation value

We fix a utility function $u : \mathbb{R} \to \mathbb{R}$ and we aim to find a $\hat{v}_T = (\hat{v}_T^0, \ldots, \hat{v}_T^d) \in A_T$ subjet to $\hat{v}_T^{\mathbf{1}} = \cdots = \hat{v}_T^d = 0$ such that

$$\mathbf{E}\left[u(\hat{v}_T^0)\right] = \max\left\{\mathbf{E}\left[u(v_T^0)\right] : (v_T^0, \ldots, v_T^d) \in A_T,\ v_T^{\mathbf{1}} = \cdots = v_T^d = 0\right\}.$$

For a portfolio v_{T-1} we define the *liquidation value* by

$$l(v_{T-1}) = \min\left\{v_{T-1} \cdot \binom{1}{y} : y \in \mathcal{S}_T\right\}.$$

Given a portfolio in the form $(v_T^0, 0) = v_{T-1} + \xi_T$ and $\xi_T \in -K_T$, we have

$$v_T^0 = \min\left\{(v_{T-1} + \xi_T) \cdot \binom{1}{y} : y \in \mathcal{S}_T\right\} \leq l(v_{T-1}).$$

Conversely, for any v_{T-1} the increment $\xi_T := (l(v_{T-1}), 0) - v_{T-1}$ is in $-K_T$, since for any $y \in \mathcal{S}_T$

$$\xi_T \cdot \binom{1}{y} = l(v_{T-1}) - v_{T-1} \cdot \binom{1}{y} \leq 0.$$

As u is increasing, we conclude

$$\begin{aligned}&\sup\left\{\mathbf{E}\left[u(v_T^0)\right] \,:\, (v_T^0, \ldots, v_T^d) \in A_T,\ v_T^{\mathbf{1}} = \cdots = v_T^d = 0\right\}\\&= \sup\left\{\mathbf{E}\left[u(l(v_{T-1}))\right] \,:\, v_{T-1} \in A_{T-1}\right\}.\end{aligned}$$

It simplifies a lot that we are interested only in the utilty from liquidation value. The number of decisions to be made by the investor is reduced by one. In a single-period model this means that we have to make only one decision at the beginning. The downside is that $v \mapsto u(l(v))$ is no longer differentiable.

Chapter 2

One-dimensional case $(d = 1)$

In this chapter we give a utility-based proof for the existence of a consistent price process in a model with one risky asset. In Section 2.1 we introduce some technical results which we will need in the main parts, Section 2.2 and 2.3. In Section 2.2 we maximize expected utility from liquidation value in a single-period market. Using a variational argument, the existence of the maximizer will imply existence of a consistent price process for the single-period market. In Section 2.3 we show how the results from the single-period case can be applied in a multi-period market. With the results from this section we will fully describe the set of all strictly consistent price processes.

2.1 Extension property

It would be nice to show that there is a utility-maximizing portfolio in a multi-period model assuming just the mere robust no-arbitrage condition. The problem becomes more feasible in a single-period model where we are interested in utility from liquidation value. Then, only one decision is to be made at the beginning.

Assume that we can solve the portfolio problem for one period models and we can deduce from the existence of an optimal portfolio that there is a consistent price process for every one period model, how can we construct a consistent price process for the multi-period model? In markets without transaction costs it is enough to find an equivalent martingale measure for each single period. When the corresponding Radon-Nikodym deriviatives are normalized appropriately, we can multiply them to find an equivalent martingale measure for the whole time line. For models with transaction costs we replace this concatenation property by an extension property.

We consider a model with one risk-free asset, which serves as a numeraire, and one risky asset whose bid and ask prices are quoted in units of the numeraire.

On a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t=0}^T, \mathbf{P})$ we have a two-dimensional, $(\mathcal{F}_t)_{t=0}^T$-adapted process $(\underline{S}_t, \overline{S}_t)_{t=0}^T$, where $0 < \underline{S}_t \leq \overline{S}_t$ for every $t = 0, \ldots, T$. $\underline{S}_t$ denotes the bid price and $\overline{S}_t$ the ask price of the risky asset in units of the numeraire.

The solvency cone is given explicitly by

$$K_t(\omega) = \left\{ a \begin{pmatrix} \overline{S}_t(\omega) \\ -1 \end{pmatrix} + b \begin{pmatrix} -\underline{S}_t(\omega) \\ 1 \end{pmatrix} + c \begin{pmatrix} 1 \\ 0 \end{pmatrix} + d \begin{pmatrix} 0 \\ 1 \end{pmatrix} \;:\; a, b, c, d \geq 0 \right\}$$

and its dual is

$$\begin{aligned} K_t^*(\omega) &= \left\{ z \in \mathbb{R}^2 \;:\; z \cdot v \geq 0, \quad \text{for } v \in K_t(\omega) \right\} \\ &= \left\{ a \begin{pmatrix} 1 \\ \overline{S}_t(\omega) \end{pmatrix} + b \begin{pmatrix} 1 \\ \underline{S}_t(\omega) \end{pmatrix} \;:\; a, b \geq 0 \right\}. \end{aligned}$$

The main difficulty in a multi-period model is as follows. If we have, say in a two period model $(\underline{S}_0, \overline{S}_0, \underline{S}_1, \overline{S}_1, \underline{S}_2, \overline{S}_2)$, a consistent price process for the first period (Z_0, Z_1), then we may ask ourselves under what condition on Z_1 can we extend it to a consistent price process for the whole time line (Z_0, Z_1, Z_2)? Z_1 has to be in the range $\left\{ \mathbf{E}\left[Z_2 | \mathcal{F}_1\right] \;:\; Z_2 \in \mathbf{L}^0(G_2^*, \mathcal{F}_2) \right\}$.

We need the following generalization of the $\mathbf{L}^\infty$-norm.

Definition 2.1. Given a sub-σ-field $\mathcal{H} \subset \mathcal{F}$ and a random variable X in $\mathbb{R}$ we define

$$\begin{aligned} \sup\nolimits_{\mathcal{H}} X &:= \operatorname{ess\,inf} \left\{ Z : \Omega \to [-\infty, \infty] \;:\; Z \ \mathcal{H}\text{-measurable}, Z \geq X \right\} \\ \inf\nolimits_{\mathcal{H}} X &:= \operatorname{ess\,sup} \left\{ Z : \Omega \to [-\infty, \infty] \;:\; Z \ \mathcal{H}\text{-measurable}, Z \leq X \right\}. \end{aligned}$$

If $\mathcal{H} = \mathcal{F}_t$, we also write $\sup_t$ instead of $\sup_{\mathcal{F}_t}$ and $\inf_t$ instead of $\inf_{\mathcal{F}_t}$.

We now formulate what we call extension property. This is part b) of the following lemma.

Lemma 2.2. *Assume X, Y are strictly positive random variables with $X \leq Y$ and $\mathcal{H} \subset \mathcal{F}$ is a sub-σ-field.*

a) For any $Z^0 > 0, Z^1 > 0$ with $\frac{Z^1}{Z^0} \in \mathrm{ri}\,[X,Y]$, $\mathbf{E}\,[Z^0|\mathcal{H}] < \infty$ and $\mathbf{E}\,[Z^1|\mathcal{H}] < \infty$ we have

$$\frac{\mathbf{E}\,[Z^1|\mathcal{H}]}{\mathbf{E}\,[Z^0|\mathcal{H}]} \in \mathrm{ri}\,[\inf_{\mathcal{H}} X, \sup_{\mathcal{H}} Y]\,.$$

b) Conversely, for any $\mathcal{H}$-measurable $C^0 > 0, C^1 > 0$, with $\frac{C^1}{C^0} \in \mathrm{ri}\,[\inf_{\mathcal{H}} X, \sup_{\mathcal{H}} Y]$, there are $Z^0 > 0$, $Z^1 > 0$ such that

$$\frac{Z^1}{Z^0} \in \mathrm{ri}\,[X,Y] \;\; \textit{and} \;\; \begin{pmatrix} C^0 \\ C^1 \end{pmatrix} = \mathbf{E}\left[\begin{pmatrix} Z^0 \\ Z^1 \end{pmatrix}\middle|\mathcal{H}\right].$$

Proof. Let X, Y be strictly positive random variables with $X \le Y$.

a) Under the equivalent probability measure $\mathbf{Q}$, whose density is $\frac{\mathrm{d}\mathbf{Q}}{\mathrm{d}\mathbf{P}} = \frac{Z^0}{\mathbf{E}[Z^0|\mathcal{H}]}$, we have

$$\mathbf{E}_{\mathbf{Q}}\left[\frac{Z^1}{Z^0}\middle|\mathcal{H}\right] = \frac{\mathbf{E}\,[Z^1|\mathcal{H}]}{\mathbf{E}\,[Z^0|\mathcal{H}]}$$

and

$$\mathbf{E}_{\mathbf{Q}}\left[\frac{Z^1}{Z^0}\middle|\mathcal{H}\right] \in [\inf_{\mathcal{H}} X, \sup_{\mathcal{H}} Y]\,,$$

since $X \le \frac{Z^1}{Z^0} \le Y$, $\inf_{\mathcal{H}} X \le X$ as well as $Y \le \sup_{\mathcal{H}} Y$.

We have to show that on the event

$$A = \left\{\mathbf{E}_{\mathbf{Q}}\left[\frac{Z^1}{Z^0}\middle|\mathcal{H}\right] = \inf_{\mathcal{H}} X\right\}$$

$\inf_{\mathcal{H}} X$ and $\sup_{\mathcal{H}} Y$ are equal.

Indeed, on A we get $\inf_{\mathcal{H}} X = \mathbf{E}_{\mathbf{Q}}\,[X|\mathcal{H}]$, so that $\inf_{\mathcal{H}} X = X$ and $X = \frac{Z^1}{Z^0}$. It follows that $X = Y$ on A, since $\frac{Z^1}{Z^0} \in \mathrm{ri}\,[X,Y]$. But then $\inf_{\mathcal{H}} X = Y$ and thus $\inf_{\mathcal{H}} X = \sup_{\mathcal{H}} Y$ on A.

Similarly, it follows that $\inf_{\mathcal{H}} X$ and $\sup_{\mathcal{H}} Y$ coincide on $\left\{\mathbf{E}_{\mathbf{Q}}[\frac{Z^1}{Z^0}|\mathcal{H}] = \sup_{\mathcal{H}} Y\right\}$.

b) Put $c := \frac{C_2}{C_1}$. Then c is $\mathcal{H}$-measurable and $c \in \mathrm{ri}\,[\inf_{\mathcal{H}}(X), \sup_{\mathcal{H}}(Y)]$. We will find strictly positive a, b such that $\mathbf{E}\,[a+b|\mathcal{H}] = 1$ and $\mathbf{E}\,[aX + bY|\mathcal{H}] = c$.

$Z_1 := (a+b)\,C_1$ and $Z_2 := (aX + bY)\,C_1$ will then satisfy

$$\frac{Z^1}{Z^0} \in \mathrm{ri}\,[X,Y] \;\; \text{and} \;\; \begin{pmatrix} C^0 \\ C^1 \end{pmatrix} = \mathbf{E}\left[\begin{pmatrix} Z^0 \\ Z^1 \end{pmatrix}\middle|\mathcal{H}\right].$$

For $n \in \mathbb{N}$ put

$$\tilde{a}_n := \frac{\mathbf{1}_{\left\{X < \inf_{\mathcal{H}}(X) + \frac{1}{n}\right\}}}{\mathbf{P}\left(X < \inf_{\mathcal{H}}(X) + \frac{1}{n} | \mathcal{H}\right)} + \frac{1}{n} \frac{\mathbf{1}_{\left\{X \geq \inf_{\mathcal{H}}(X) + \frac{1}{n}\right\}}}{1 + X}$$

and

$$\tilde{b}_n := \frac{\mathbf{1}_{\{Y > f_n\}}}{\mathbf{P}(Y > f_n | \mathcal{H})} + \frac{1}{n} \frac{\mathbf{1}_{\{Y \leq f_n\}}}{1 + Y}, \text{ where}$$
$$f_n := \mathbf{1}_{\{\sup_{\mathcal{H}} Y < \infty\}} \left(\sup_{\mathcal{H}}(Y) - \frac{1}{n}\right) + \mathbf{1}_{\{\sup_{\mathcal{H}}(Y) = \infty\}} n.$$

Note that $\mathbf{P}\left(X < \inf_{\mathcal{H}}(X) + \frac{1}{n} | \mathcal{H}\right) > 0$ and $\mathbf{P}(Y > f_n | \mathcal{H}) > 0$. Then $\tilde{a}_n, \tilde{b}_n > 0$ and

$$\mathbf{E}[\tilde{a}_n X | \mathcal{H}] \xrightarrow[n \to \infty]{a.s.} \inf_{\mathcal{H}}(X), \mathbf{E}[\tilde{b}_n Y | \mathcal{H}] \xrightarrow[n \to \infty]{a.s.} \sup_{\mathcal{H}}(Y).$$

As

$$1 \leq \mathbf{E}[\tilde{a}_n | \mathcal{H}] \leq 1 + \frac{1}{n}, \quad 1 \leq \mathbf{E}[\tilde{b}_n | \mathcal{H}] \leq 1 + \frac{1}{n},$$

we can normalize $\tilde{a}_n$, $\tilde{b}_n$, i.e. replace $\tilde{a}_n$ by $\frac{\tilde{a}_n}{E(\tilde{a}_n|\mathcal{H})}$ and $\tilde{b}_n$ by $\frac{\tilde{b}_n}{E\left(\tilde{b}_n|\mathcal{H}\right)}$ and still have

$$\mathbf{E}[\tilde{a}_n X | \mathcal{H}] \xrightarrow[n \to \infty]{a.s.} \inf_{\mathcal{H}}(X), \mathbf{E}[\tilde{b}_n Y | \mathcal{H}] \xrightarrow[n \to \infty]{a.s.} \sup_{\mathcal{H}}(Y).$$

Now write $\Omega = \Omega_1 \cup \Omega_2$ with $\Omega_1 := \{\inf_{\mathcal{H}}(X) < \sup_{\mathcal{H}} Y\}$ and $\Omega_2 := \{\inf_{\mathcal{H}}(X) = \sup_{\mathcal{H}} Y\}$.
For $\omega \in \Omega_1$ (up to a null set) choose $n = n(\omega) \in \mathbb{N}$ minimal such that

$$\inf_{\mathcal{H}}(X)(\omega) \leq \mathbf{E}[\tilde{a}_n X | \mathcal{H}](\omega) < c(\omega).$$

Then $\omega \mapsto n(\omega)$ is $\mathcal{H}$-measurable on Ω_1, since

$$\begin{aligned}\{\omega \in \Omega_1 | n(\omega) = l\} = &\{\omega \in \Omega_1 | \mathbf{E}[\tilde{a}_l X | \mathcal{H}](\omega) < c(\omega)\} \cap \\ &\bigcap_{k<l} \{\omega \in \Omega_1 | \mathbf{E}[\tilde{a}_k X | \mathcal{H}](\omega) \geq c(\omega)\}.\end{aligned}$$

$\tilde{a}$, defined as $\tilde{a}(\omega) := \tilde{a}_{n(\omega)}(\omega)$ on Ω_1 and $\tilde{a}(\omega) = 1$ on Ω_2, is strictly positive and since $\tilde{a} = \left(\sum_{m \geq 1} \tilde{a}_m \mathbf{1}_{\{n=m\}} \mathbf{1}_{\Omega_1}\right) + \mathbf{1}_{\Omega_2}$ it satisfies

$$\mathbf{E}[\tilde{a} | \mathcal{H}] = \left(\sum_{m \geq 1} \mathbf{1}_{\{n=m\}} \mathbf{1}_{\Omega_1} \mathbf{E}[\tilde{a}_m | \mathcal{H}]\right) + \mathbf{1}_{\Omega_2} = 1,$$
$$\mathbf{E}[\tilde{a} X | \mathcal{H}] = \sum_{m \geq 1} \mathbf{1}_{\{n=m\}} \mathbf{E}[\tilde{a}_m X | \mathcal{H}] < c \quad \text{on } \Omega_1.$$

Similarly $\tilde{b}$, defined by $\tilde{b}(\omega) := \tilde{b}_{m(\omega)}(\omega)$ on Ω_1, where $m(\omega) \in \mathbb{N}$ is minimal such that $\mathbf{E}\left(\tilde{b}_m Y|\mathcal{H}\right)(\omega) > c(\omega)$, and $\tilde{b}(\omega) = 1$ on Ω_2, is strictly positive and satisfies

$$\mathbf{E}[\tilde{b}|\mathcal{H}] = 1, \quad \mathbf{E}[\tilde{b}Y|\mathcal{H}] > c \text{ on } \Omega_1.$$

Hence, $\mathbf{E}[\tilde{a}X|\mathcal{H}] < c < \mathbf{E}[\tilde{b}Y|\mathcal{H}]$ on Ω_1, $\mathbf{E}[\tilde{a}X|\mathcal{H}] = c = \mathbf{E}[\tilde{b}Y|\mathcal{H}]$ on Ω_2.

Define

$$a := \frac{c - \mathbf{E}[\tilde{a}X|\mathcal{H}]}{\mathbf{E}[\tilde{b}Y|\mathcal{H}] - \mathbf{E}[\tilde{a}X|\mathcal{H}]}\tilde{a}\mathbf{1}_{\Omega_1} + \frac{1}{2}\mathbf{1}_{\Omega_2}$$

$$b := \frac{\mathbf{E}[\tilde{b}Y|\mathcal{H}] - c}{\mathbf{E}[\tilde{b}Y|\mathcal{H}] - \mathbf{E}[\tilde{a}X|\mathcal{H}]}\tilde{b}\mathbf{1}_{\Omega_1} + \frac{1}{2}\mathbf{1}_{\Omega_2}.$$

then $c = \mathbf{E}[aX + bY|\mathcal{H}]$ and $\mathbf{E}[a + b|\mathcal{H}] = 1$.

□

The lemma will play a key role when we construct consistent price processes. In a model with strict transaction costs, i.e. $X < Y$, a similiar statement can be found in [18]. Here the costs are not necessarily strict and we gave a new and very elementary proof of it. Compare also the more abstract results for the multidimensional case in Chapter 3.

Remark 2.3. It follows from part b) of Lemma 2.2 that

$$\left\{\mathbf{E}_{\mathbf{Q}}[Z|\mathcal{H}] \,:\, Z \in [X,Y],\ \mathbf{Q} \sim \mathbf{P}\right\}$$

is dense in $[\inf_{\mathcal{H}} X, \sup_{\mathcal{H}} Y]$. In general it is only a strict subset. Take for example $X(\omega_1) < X(\omega_2)$, $X = Y$ and $\mathbf{P}(\omega_1) > 0, \mathbf{P}(\omega_2) > 0$ on $\Omega = \{\omega_1, \omega_2\}$. Then, for $\mathcal{H} = \{\emptyset, \Omega\}$,

$$\left\{\mathbf{E}_{\mathbf{Q}}[Z|\mathcal{H}] \,:\, Z \in [X,Y],\ \mathbf{Q} \sim \mathbf{P}\right\} = (X(\omega_1), X(\omega_2))$$

and $[\inf_{\mathcal{H}} X, \sup_{\mathcal{H}} Y] = [X(\omega_1), X(\omega_2)]$.

2.2 Utility maximization in a single-period model

We will show in this section that the investor can maximize her utility from liquidation value in a single-period market with proportional transaction costs. Once the existence of a maximum is established a variational argument will show how a consistent price process can be constructed. To be able to construct a strictly consistent price-processe and to use this results in the next section for the multi-period case we have to look in detail at the various scenarios which can occur for the optimal portfolio. A further analysis of it will help us to construct a (strictly) consistent price process in the multi-period case.

We fix a general probability space $(\Omega, \mathcal{F}, \mathbf{P})$. The bid and ask prices of the risky asset are described by $\left(\underline{S}_0, \overline{S}_0, \underline{S}_1, \overline{S}_1\right)$ which is adapted to a filtration $(\mathcal{F}_0, \mathcal{F}_1)$. We point out that $\mathcal{F}_0$ does not need to be trivial. We assume that $\underline{S}_t, \overline{S}_t$ are strictly positive and satisfy $\underline{S}_t \leq \overline{S}_t$ $(t = 0, 1)$.

2.2.1 Preliminaries

For our single-period market we make the following assumption about the set of all null portfolios. Later on it will turn out to be equivalent to the robust no-arbitrage condition.

Assumption 2.4. *The set of all self-financing portfolios with non-negative liquidation value constitutes a vector space, i.e.*

$$\mathcal{N} := \mathbf{L}^0\left(-K_0, \mathcal{F}_0\right) \cap \mathbf{L}^0\left(K_1, \mathcal{F}_1\right)$$

is a vector space in $\mathbf{L}^0\left(\mathbb{R}^2, \mathcal{F}_0\right)$.

It is easy to see how the robust no-arbitrage condition implies this assumption.

Remark 2.5. a) Assume that $\left(\underline{S}_0, \overline{S}_0, \underline{S}_1, \overline{S}_1\right)$ satisfies the robust no-arbitrage condition and

$$v \in \mathbf{L}^0\left(-K_0, \mathcal{F}_0\right) \cap \mathbf{L}^0\left(K_1, \mathcal{F}_1\right).$$

We put $v_0 = v$ and $v_1 = -v$ and have $v_0 + v_1 = 0$. By Lemma 1.15 we get

$$v_0 \in \mathbf{L}^0\left(-K_0 \cap K_0, \mathcal{F}_0\right) \quad \text{and} \quad v_1 \in \mathbf{L}^0\left(-K_1 \cap K_1, \mathcal{F}_1\right).$$

$-K_0 \cap K_0$ (resp. $-K_1 \cap K_1$) being a vector space implies

$$-v \in \mathbf{L}^0\left(-K_0, \mathcal{F}_0\right) \cap \mathbf{L}^0\left(K_1, \mathcal{F}_1\right).$$

Since $\mathbf{L}^0\left(-K_0, \mathcal{F}_0;\right) \cap \mathbf{L}^0\left(K_1, \mathcal{F}_1\right)$ is à priori a convex cone we conclude that $\left(\underline{S}_0, \overline{S}_0, \underline{S}_1, \overline{S}_1\right)$ satisfies Assumption 2.4.

b) The assumption that the set

$$\begin{aligned}\mathcal{N} &= \mathbf{L}^0\left(-K_0, \mathcal{F}_0\right) \cap \mathbf{L}^0\left(K_1, \mathcal{F}_1\right)\\ &= \left\{v \in \mathbf{L}^0\left(-K_0, \mathcal{F}_0\right) \,:\, v \cdot \left(\begin{smallmatrix}1\\ \underline{S}_1\end{smallmatrix}\right) \geq 0, v \cdot \left(\begin{smallmatrix}1\\ \overline{S}_1\end{smallmatrix}\right) \geq 0\right\}\end{aligned}$$

is a vector space implies

$$\mathcal{N} = \left\{v \in \mathbf{L}^0\left(-K_0 \cap K_0, \mathcal{F}_0\right) \,:\, v \cdot \left(\begin{smallmatrix}1\\ \underline{S}_1\end{smallmatrix}\right) = 0, v \cdot \left(\begin{smallmatrix}1\\ \overline{S}_1\end{smallmatrix}\right) = 0\right\}.$$

Furthermore for any real-valued $\mathcal{F}_0$-measurable h and $v \in \mathcal{N}$ we have $hv \in \mathcal{N}$ and since K_0 (resp. K_1) is closed, $\mathcal{N}$ is closed with respect to convergence in probability.

In the proof of the main theorem in this section, Theorem 2.9, we will not need the portfolios from $\mathcal{N}$, since they have the same liquidation value as the zero-portfolio $\left(\begin{smallmatrix}0\\ 0\end{smallmatrix}\right)$. To get rid of the potfolios in $\mathcal{N}$, we will need to parameterize $\mathcal{N}$ in an $\mathcal{F}_0$-measurable way. We deal with this parameterization in the following lemma.

Lemma 2.6. *Under Assumption 2.4 we have*

a)

$$\left\{\mathbf{E}\left[\left|\underline{S}_1 - \overline{S}_0\right| |\mathcal{F}_0\right] = 0\right\} = \left\{\mathbf{E}\left[\left|\underline{S}_0 - \overline{S}_1\right| |\mathcal{F}_0\right] = 0\right\}$$

and $A := \left\{\mathbf{E}\left[\left|\underline{S}_1 - \overline{S}_0\right| |\mathcal{F}_0\right] = 0\right\}$ *is the biggest* $\mathcal{F}_0$*-measurable set such that*

$$\underline{S}_0 = \overline{S}_0 = \underline{S}_1 = \overline{S}_1 \quad \text{on } A.$$

b)

$$v \in \mathcal{N} \quad \Leftrightarrow \quad v = \mathbf{1}_A h \left(\begin{smallmatrix}-\overline{S}_0\\ 1\end{smallmatrix}\right)$$

for some $\mathcal{F}_0$*-measurable, real-valued* h.

c) For $\omega \in A$ *define* $P_0(\omega)$ *to be the orthogonal projection on the linear space generated by* $\left(\begin{smallmatrix}-\overline{S}_0(\omega)\\ 1\end{smallmatrix}\right)$ *and, for* $\omega \notin A$, $P_0(\omega) := 0$*. Then,* P_0 *is* $\mathcal{F}_0$*-measurable and for* $v \in \mathbf{L}^0\left(\mathbb{R}^2, \mathcal{F}_0\right)$ *we have*

$$v \in \mathcal{N} \quad \Leftrightarrow \quad P_0 v = v.$$

Proof. Let $\left(\underline{S}_0, \overline{S}_0, \underline{S}_1, \overline{S}_1\right)$ satisfy Assumption 2.4.

a) Put $A := \left\{\mathbf{E}\left[\left|\underline{S}_1 - \overline{S}_0\right| |\mathcal{F}_0\right] = 0\right\}$ and we have $\mathbf{E}\left[\left|\underline{S}_1 - \overline{S}_0\right| \mathbf{1}_A\right] = 0$, so that $\underline{S}_1 = \overline{S}_0$ on A. If B is $\mathcal{F}_0$-measuable and $\underline{S}_1 = \overline{S}_0$ on B we get

$$0 = \mathbf{E}\left[\left|\underline{S}_1 - \overline{S}_0\right| \mathbf{1}_B\right] = \mathbf{E}\left[\mathbf{E}\left[\left|\underline{S}_1 - \overline{S}_0\right| |\mathcal{F}_0\right] \mathbf{1}_B\right],$$

so that $\mathbf{E}\left[\left|\underline{S}_1 - \overline{S}_0\right| |\mathcal{F}_0\right] = 0$ on B, i.e. $B \subset A$. It follows that A is the greatest $\mathcal{F}_0$-measurable set where $\underline{S}_1 = \overline{S}_0$.

For the portfolio $v := \mathbf{1}_A \left(\begin{smallmatrix} -\overline{S}_0 \\ 1 \end{smallmatrix}\right)$ we calculate

$$v \cdot \left(\begin{smallmatrix} 1 \\ \underline{S}_1 \end{smallmatrix}\right) = 0 \quad \text{and} \quad v \cdot \left(\begin{smallmatrix} 1 \\ \overline{S}_1 \end{smallmatrix}\right) \geq 0,$$

so that $v \in \mathcal{N}$.

$\mathcal{N}$ is a vector space, so $-v = \mathbf{1}_A \left(\begin{smallmatrix} \overline{S}_0 \\ -1 \end{smallmatrix}\right) \in \mathcal{N}$ implies

$$-K_0 \cap K_0 \neq \{0\} \quad \text{on } A,$$

which is equivalent to $\underline{S}_0 = \overline{S}_0$ on A.

Since

$$v \cdot \left(\begin{smallmatrix} 1 \\ \underline{S}_1 \end{smallmatrix}\right) = 0, \quad v \cdot \left(\begin{smallmatrix} 1 \\ \overline{S}_1 \end{smallmatrix}\right) = 0$$

we need $\left(\begin{smallmatrix} 1 \\ \underline{S}_1 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 1 \\ \overline{S}_1 \end{smallmatrix}\right)$ to be linear dependent on A, i.e. $\underline{S}_1 = \overline{S}_1$ on A, altogether $\underline{S}_0 = \overline{S}_0 = \underline{S}_1 = \overline{S}_1$ on A.

As above, $C := \left\{\mathbf{E}\left[\left|\underline{S}_0 - \overline{S}_1\right| |\mathcal{F}_0\right] = 0\right\}$ is the greatest $\mathcal{F}_0$-measurable set where we have $\overline{S}_1 = \underline{S}_0$. It follows that $A \subset C$.

We repeat the reasoning from above applied to the portfolio

$$\mathbf{1}_C \left(\begin{smallmatrix} \underline{S}_0 \\ -1 \end{smallmatrix}\right)$$

and we see that $\underline{S}_0 = \overline{S}_0 = \underline{S}_1 = \overline{S}_1$ on C. It follows that $C \subset A$.

b) As the portfolio $\mathbf{1}_A \left(\begin{smallmatrix} -\overline{S}_0 \\ 1 \end{smallmatrix}\right)$ is in $\mathcal{N}$, which is a vector space, we also have $\mathbf{1}_A h \left(\begin{smallmatrix} -\overline{S}_0 \\ 1 \end{smallmatrix}\right)$ for every $\mathcal{F}_0$-measurable real-valued h.

Now, we take an arbitrary $v \in \mathcal{N}$. We have

$$v \cdot \left(\begin{smallmatrix} 1 \\ \underline{S}_0 \end{smallmatrix}\right) = 0, v \cdot \left(\begin{smallmatrix} 1 \\ \overline{S}_0 \end{smallmatrix}\right) = 0, v \cdot \left(\begin{smallmatrix} 1 \\ \underline{S}_1 \end{smallmatrix}\right) = 0, v \cdot \left(\begin{smallmatrix} 1 \\ \overline{S}_1 \end{smallmatrix}\right) = 0.$$

Thus on $\{v \neq 0\}$ all prices coincide $\underline{S}_0 = \overline{S}_0 = \underline{S}_1 = \overline{S}_1$, i.e. $\{v \neq 0\} \subset A$. It follows that

$$v = \mathbf{1}_A h \left(\begin{smallmatrix} -\overline{S}_0 \\ 1 \end{smallmatrix}\right)$$

for some $\mathcal{F}_0$-measurable, real-valued h.

c) Any $v \in \mathcal{N}$ is of the form $v = \mathbf{1}_A h \binom{-\overline{S}_0}{1}$ where h is $\mathcal{F}_0$-measurable. Hence, $v(\omega)$ is contained in the subspace generated by $\binom{-\overline{S}_0(\omega)}{1}$, i.e. $P_0 v = v$.

Conversely, any $\mathcal{F}_0$-measurable v with $P_0 v = v$ can be written as $v = \mathbf{1}_A h \binom{-\overline{S}_0}{1}$ where $h = \frac{-v^0 \overline{S}_0 + v^1}{\sqrt{\overline{S}_0^2 + 1}}$, so $v \in \mathcal{N}$.

$\square$

For the rest of this section we fix a utility function $u : \mathbb{R} \to \mathbb{R}$. We suppose that u is continuously differentiable, strictly concave, strictly increasing and bounded from above.

Further, given a portfolio $v = (v^0, v^1) \in \mathbb{R}^2$, we define by $l(v)$ the *liquidation value* in terms of the numeraire, i.e.

$$\begin{aligned} l(v) &:= \begin{cases} v^0 + v^1 \overline{S}_1, & v^1 \leq 0 \\ v^0 + v^1 \underline{S}_1, & v^1 \geq 0 \end{cases} \\ &= \min\left\{ v \cdot \binom{1}{\underline{S}_1}, v \cdot \binom{1}{\overline{S}_1} \right\}. \end{aligned}$$

Our goal is to find a portfolio $\hat{v} \in \mathbf{L}^0(-K_0, \mathcal{F}_0)$ such that

$$\mathbf{E}\left[u\left(l\left(\hat{v}\right)\right) | \mathcal{F}_0\right] = \operatorname*{ess\,sup}_{v \in \mathbf{L}^0(-K_0, \mathcal{F}_0)} \mathbf{E}\left[u\left(l\left(v\right)\right) | \mathcal{F}_0\right].$$

First, we have to make sure why we can w.l.o.g. assume that

$$\text{both} \quad \mathbf{E}\left[\left|u\left(l\left(v\right)\right)\right| | \mathcal{F}_0\right] < \infty \quad \text{and} \quad \mathbf{E}\left[u'\left(l\left(v\right)\right) | \mathcal{F}_0\right] < \infty$$

for all $v \in \mathbf{L}^0(-K_0, \mathcal{F}_0)$. Further the mapping

$$\mathbb{R}^2 \ni a \mapsto \mathbf{E}\left[u\left(l\left(a\right)\right) | \mathcal{F}_0\right]$$

shall satisfy some regularity conditions.

The following proposition is due to [20].

Proposition 2.7. *There exists a decreasing and continuous function $g : \mathbb{R}_+ \to (0, \infty)$ such that*

$$|u(a \cdot x)| g(|x|) \leq \frac{1}{g(|a|)} + const$$

for every $a, x \in \mathbb{R}^D$.

Proof. We first assume that u is bounded from above by 0. By the Cauchy-Schwarz inequality it follows $-|a|\,|x| \le a\cdot x$. Since u is increasing and takes values in $(-\infty, 0)$ we have

$$|u(a\cdot x)| = -u(a\cdot x) \le -u(-|a|\,|x|) = |u(-|a|\,|x|)| \le 1 \vee |u(-|a|\,|x|)|\,.$$

Note that $1 \vee |u(-t)|$ is an increasing function for t, so that

$$1 \vee |u(-|a|\,|x|)| \le \left(1 \vee \left|u\left(-|a|^2\right)\right|\right)\left(1 \vee \left|u\left(-|x|^2\right)\right|\right)$$

since $|a|\,|x| \le |a|^2$ or $|a|\,|x| \le |x|^2$.

We put $g(t) := \left(1 \vee |u(-t^2)|\right)^{-1}$ and get

$$|u(a\cdot x)|\, g(|x|) \le \frac{1}{g(|a|)}.$$

g is continuous, positive and decreasing.

If u is bounded from above by a constant c we apply the reasoning from above to the utility function $u(\cdot) - c$ and get

$$|u(a\cdot x) - c|\, g(|x|) \le \frac{1}{g(|a|)}$$

which implies

$$|u(a\cdot x)|\, g(|x|) \le \frac{1}{g(|a|)} + |c|\, g(|x|) \le \frac{1}{g(|a|)} + |c|\, g(0)\,.$$

□

Now, we want to clarify the integrability and regularity issues mentioned before this proposition.

Remark 2.8. a) We choose a decreasing and continuous function $g : \mathbb{R}_+ \to (0,\infty)$ such that

$$|u(a\cdot x)|g(|x|) \le \frac{1}{g(|a|)} + c, \quad \text{for every } a, x \in \mathbb{R}^2,$$

where c is a fixed constant.

Obviously, we then have

$$|u(\min\{a\cdot x, a\cdot y\})|\, g(\max\{|x|, |y|\}) \le \frac{1}{g(|a|)} + c, \quad \text{for } a, x, y \in \mathbb{R}^2.$$

Denote by $\mathbf{R}$ the equivalent probability measure with density

$$\frac{d\mathbf{R}}{d\mathbf{P}} = \frac{g\left(\max\left\{\left|\left(\begin{smallmatrix}1\\ \underline{S}_1\end{smallmatrix}\right)\right|, \left|\left(\begin{smallmatrix}1\\ \overline{S}_1\end{smallmatrix}\right)\right|\right\}\right)}{\mathbf{E_P}\left[g\left(\max\left\{\left|\left(\begin{smallmatrix}1\\ \underline{S}_1\end{smallmatrix}\right)\right|, \left|\left(\begin{smallmatrix}1\\ \overline{S}_1\end{smallmatrix}\right)\right|\right\}\right)\right]},$$

and denote by $\mathbf{k}$ the regular conditional $\mathbf{P}$-distribution for $\left(\underline{S}_1, \overline{S}_1\right)$ given $\mathcal{F}_0$. Then, for any $v \in \mathbf{L}^0\left(\Omega, \mathcal{F}_0; \mathbb{R}^2\right)$ we have by disintegration

$$\begin{aligned}\mathbf{E_R}\left[u\left(l\left(v\right)\right)|\mathcal{F}_0\right](\cdot) =& \frac{\mathbf{E_P}\left[u\left(l\left(v\right)\right)\frac{d\mathbf{R}}{d\mathbf{P}}|\mathcal{F}_0\right]}{\mathbf{E_P}\left[\frac{d\mathbf{R}}{d\mathbf{P}}|\mathcal{F}_0\right]}(\cdot) \\ =& h(\cdot)\int u\left(\min\left\{v(\cdot)\cdot\left(\begin{smallmatrix}1\\ x\end{smallmatrix}\right), v(\cdot)\cdot\left(\begin{smallmatrix}1\\ y\end{smallmatrix}\right)\right\}\right)\times \\ & g\left(\max\left\{\left|\left(\begin{smallmatrix}1\\ x\end{smallmatrix}\right)\right|, \left|\left(\begin{smallmatrix}1\\ y\end{smallmatrix}\right)\right|\right\}\right)\mathbf{k}\left(\cdot, dx, dy\right),\end{aligned}$$

with $h = \mathbf{E_P}\left[g\left(\max\left\{\left|\left(\begin{smallmatrix}1\\ \underline{S}_1\end{smallmatrix}\right)\right|, \left|\left(\begin{smallmatrix}1\\ \overline{S}_1\end{smallmatrix}\right)\right|\right\}\right)|\mathcal{F}_0\right]^{-1}$ and

$$\mathbf{E_R}\left[\left|u\left(l\left(v\right)\right)\right||\mathcal{F}_0\right](\cdot) \leq h(\cdot)\left(\frac{1}{g(|v(\cdot)|)} + c\right).$$

By dominated convergence, using the estimate from above, it follows that we can find a version of $\mathbf{E_R}\left[u\left(l\left(a\right)\right)|\mathcal{F}_0\right]$ such that for every $\omega \in \Omega$

$$\mathbb{R}^2 \ni a \mapsto \mathbf{E_R}\left[u\left(l\left(a\right)\right)|\mathcal{F}_0\right](\omega)$$

is finite and continuous. From

$$\mathbf{E_R}\left[\left|u\left(l\left(v\right)\right)\right||\mathcal{F}_0\right] < \infty, \quad \text{for all } v \in \mathbf{L}^0\left(-K_0, \mathcal{F}_0\right),$$

we obtain

$$\begin{aligned}\mathbf{E_R}\left[u'\left(l\left(v\right)\right)|\mathcal{F}_0\right] \leq & \mathbf{E_R}\left[\left.\frac{u\left(l\left(v\right)-1\right)-u\left(l\left(v\right)\right)}{-1}\right|\mathcal{F}_0\right] \\ =& \mathbf{E_R}\left[u\left(l\left(v\right)\right)|\mathcal{F}_0\right] - \mathbf{E_R}\left[\left.u\left(l\left(v+\left(\begin{smallmatrix}-1\\ 0\end{smallmatrix}\right)\right)\right)\right|\mathcal{F}_0\right] < \infty\end{aligned}$$

for all $v \in \mathbf{L}^0\left(-K_0, \mathcal{F}_0\right)$.

Now, if we switch to an equivalent probability measure Assumption 2.4 is still statisfied. So we can without loss of generality assume that

$$\mathbb{R}^2 \ni a \mapsto \mathbf{E_P}\left[u\left(l\left(a\right)\right)|\mathcal{F}_0\right]$$

is pointwise finite and continuous as well as

$$\mathbf{E_P}\left[\left|u\left(l\left(v\right)\right)\right||\mathcal{F}_0\right] < \infty, \quad \mathbf{E_P}\left[u'\left(l\left(v\right)\right)|\mathcal{F}_0\right] < \infty$$

for all $v \in \mathbf{L}^0\left(-K_0, \mathcal{F}_0\right)$.

b) If the ask price at time 0 is integrable and (S_0, S_1) is a process lying in the bid ask spread, i.e. $S_0 \in \left[\underline{S}_0, \overline{S}_0\right]$, $S_1 \in \left[\underline{S}_1, \overline{S}_1\right]$, such that $\mathbf{E}_{\mathbf{Q}}\left[S_1|\mathcal{F}_0\right] = S_0$ where the density $\frac{\mathrm{d}\mathbf{Q}}{\mathrm{d}\mathbf{P}}$ satisfies $\mathbf{E}_{\mathbf{P}}\left[\frac{\mathrm{d}\mathbf{Q}}{\mathrm{d}\mathbf{P}}|\mathcal{F}_0\right] \in \mathbf{L}^{\infty}(\mathbf{P})$, then clearly $S_0, S_1 \in \mathbf{L}^1(\mathbf{Q})$.

   Again the assumption $\overline{S}_0 \in \mathbf{L}^1(\mathbf{P})$ is without loss of generality as we can switch to an equivalent probabilty measure with density

$$\frac{\left(1+\overline{S}_0\right)^{-1}}{\mathbf{E}_{\mathbf{P}}\left[\left(1+\overline{S}_0\right)^{-1}\right]}.$$

We come to the main results for a single-period market.

2.2.2 Main results

Theorem 2.9. *Suppose that the single-period bid-ask process $\left(\underline{S}_0, \overline{S}_0, \underline{S}_1, \overline{S}_1\right)$ satisfies Assumption 2.4. Then:*

a) *There exists a portfolio $\hat{v} = (\hat{v}^0, \hat{v}^1) \in \mathbf{L}^0(-K_0, \mathcal{F}_0)$ such that*

$$\mathbf{E}\left[u\left(l\left(\hat{v}\right)\right)|\mathcal{F}_0\right] = \operatorname*{ess\,sup}_{v \in \mathbf{L}^0(-K_0,\mathcal{F}_0)} \mathbf{E}\left[u\left(l\left(v\right)\right)|\mathcal{F}_0\right].$$

   Further, $\hat{v}$ can be chosen such that it satisfies $P_0\hat{v} = 0$ where P_0 denotes the $\mathcal{F}_0$-measurable, projection-valued mapping from Lemma 2.6 corresponding to $\mathcal{N}$.

b) *Under the equivalent probabilty measure $\hat{\mathbf{Q}}$, where*

$$\frac{\mathrm{d}\hat{\mathbf{Q}}}{\mathrm{d}\mathbf{P}} := \frac{u'\left(l\left(\hat{v}\right)\right)}{\mathbf{E}\left[u'\left(l\left(\hat{v}\right)\right)|\mathcal{F}_0\right]},$$

   the bid-ask process $\left(\underline{S}_0, \overline{S}_0, \underline{S}_1, \overline{S}_1\right)$ satisfies

$$\begin{aligned}
&\mathbf{E}_{\hat{\mathbf{Q}}}\left[\underline{S}_1|\mathcal{F}_0\right] = \overline{S}_0 && \text{on } \left\{\hat{v}^1 > 0\right\},\\
&\mathbf{E}_{\hat{\mathbf{Q}}}\left[\overline{S}_1|\mathcal{F}_0\right] = \underline{S}_0 && \text{on } \left\{\hat{v}^1 < 0\right\},\\
&\mathbf{E}_{\hat{\mathbf{Q}}}\left[\underline{S}_1|\mathcal{F}_0\right] \leq \overline{S}_0, \quad \mathbf{E}_{\hat{\mathbf{Q}}}\left[\overline{S}_1|\mathcal{F}_0\right] \geq \underline{S}_0 && \text{on } \left\{\hat{v}^1 = 0\right\}.
\end{aligned}$$

   Hence, there exists a consistent price-process $\begin{pmatrix} 1 \\ \hat{Z}_0^1 \end{pmatrix}, \begin{pmatrix} \hat{Z}_1^0 \\ \hat{Z}_1^1 \end{pmatrix}$ such that

$\hat{Z}_1^0 = \frac{d\hat{\mathbf{Q}}}{d\mathbf{P}}$ *and*

$$\hat{Z}_0^1 = \overline{S}_0, \quad \frac{\hat{Z}_1^1}{\hat{Z}_1^0} = \underline{S}_1 \quad on \ \left\{\hat{v}^1 > 0\right\},$$
$$\hat{Z}_0^1 = \underline{S}_0, \quad \frac{\hat{Z}_1^1}{\hat{Z}_1^0} = \overline{S}_1 \quad on \ \left\{\hat{v}^1 < 0\right\}.$$

Proof. a) Denote by P_0 the $\mathcal{F}_0$-measurable, projection-valued mapping corresponding to $\mathcal{N}$ from Lemma 2.6. Then, for all $v \in \mathbf{L}^0(\mathbb{R}^2, \mathcal{F}_0)$, we have $P_0 v \in \mathcal{N}$, i.e.

$$P_0 v \in -K_0 \cap K_0 \text{ and } P_0 v \cdot \left(\begin{smallmatrix} 1 \\ \underline{S}_1 \end{smallmatrix}\right) = 0, P_0 v \cdot \left(\begin{smallmatrix} 1 \\ \overline{S}_1 \end{smallmatrix}\right) = 0.$$

So, if $v \in \mathbf{L}^0(-K_0, \mathcal{F}_0)$, then

$$v - P_0 v \in -K_0 \quad \text{and} \quad l(v - P_0 v) = l(v).$$

For every $v \in \mathbf{L}^0(-K_0, \mathcal{F}_0)$ we have

$$\mathbf{E}\left[u\left(l\left(v - P_0 v\right)\right)|\mathcal{F}_0\right] = \mathbf{E}\left[u\left(l\left(v\right)\right)|\mathcal{F}_0\right].$$

Therefore, it is enough to consider only portfolios $v \in \mathbf{L}^0(-K_0, \mathcal{F}_0)$ which satisfy $P_0 v = 0$.
Put $R := \mathrm{id} - P_0$ and $C := R(-K_0)$. Since R is linear and $-K_0$ is a polyhedral cone which is generated by

$$\left(\begin{smallmatrix} -\overline{S}_0 \\ 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} \underline{S}_0 \\ -1 \end{smallmatrix}\right), \left(\begin{smallmatrix} -1 \\ 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 \\ -1 \end{smallmatrix}\right)$$

C is generated by

$$R\left(\begin{smallmatrix} -\overline{S}_0 \\ 1 \end{smallmatrix}\right), R\left(\begin{smallmatrix} \underline{S}_0 \\ -1 \end{smallmatrix}\right), R\left(\begin{smallmatrix} -1 \\ 0 \end{smallmatrix}\right), R\left(\begin{smallmatrix} 0 \\ -1 \end{smallmatrix}\right).$$

Hence

$$\mathbf{L}^0(C, \mathcal{F}_0) = \left\{Rv \,:\, v \in \mathbf{L}^0(-K_0, \mathcal{F}_0)\right\}.$$

By Remark 2.8 we can assume that we can find a version of $\mathbf{E}\left[u\left(l\left(a\right)\right)|\mathcal{F}_0\right], a \in \mathbb{R}^2$, such that for all $\omega \in \Omega$

$$a \mapsto \mathbf{E}\left[u\left(l\left(a\right)\right)|\mathcal{F}_0\right](\omega)$$

is finite and continuous. Define for $\omega \in \Omega$, $a \in \mathbb{R}^2$

$$\varphi(\omega, a) := \begin{cases} \mathbf{E}\left[u\left(l\left(a\right)\right)|\mathcal{F}_0\right](\omega) & a \in C(\omega) \\ -\infty & a \notin C(\omega) \end{cases}.$$

Note that, if $v \in \mathbf{L}^0(C, \mathcal{F}_0)$, by disintegration

$$\varphi(\cdot, v) = E\left[u\left(l\left(v\right)\right)|\mathcal{F}_0\right].$$

For almost every ω an $\mathcal{F}_0$-measurable maximizer $v^*(\omega)$ can be found:

φ satisfies the properties of Lemma **??**, i.e. $\varphi(\omega, \cdot)$ is concave, continuous on $C(\omega)$ and $\varphi(\cdot, a)$ is $\mathcal{F}_0$-measurable.

We claim that $F(\omega) = \emptyset$ for almost every $\omega \in \Omega$ where

$$F(\omega) = \left\{a \in \mathbb{R}^d \,:\, |a| = 1, \lim_{s\to\infty} \varphi(\omega, sa) > -\infty\right\}.$$

Indeed, if $\{F \neq \emptyset\}$ has postive measure, then the $\mathcal{F}_0$-measurable selector of F, α, satisfies $P(\alpha \neq 0) > 0$ and $\alpha \in C$.
From

$$\lim_{s\to\infty} \mathbf{E}\left[u\left(l\left(s\alpha\right)\right)|\mathcal{F}_0\right](\omega) > -\infty, \quad \text{for every } \omega \in \Omega,$$

it follows with Fatou's Lemma

$$-\infty < \lim_{s\to\infty} \mathbf{E}\left[u\left(l\left(s\alpha\right)\right)|\mathcal{F}_0\right] \leq \mathbf{E}\left[\overline{\lim_{s\to\infty}} u\left(sl\left(\alpha\right)\right)|\mathcal{F}_0\right].$$

Since $\lim_{s\to-\infty} u(s) = -\infty$ we must have

$$\mathbf{P}\left(l\left(\alpha\right) < 0\right) = 0,$$

i.e. $\alpha \in \mathcal{N}$. It follows that $P_0\alpha = \alpha$, but also we have $R\alpha = \alpha$. Both relations hold simultaneously only if $\alpha = 0$. Hence $F(\omega) = \emptyset$ for almost every $\omega \in \Omega$ and for those ω a maximiser $\hat{v}(\omega) \in C(\omega)$ can be found by Lemma **??**.
We get $\hat{v} = (\hat{v}^0, \hat{v}^1) \in \mathbf{L}^0(-K_0, \mathcal{F}_0), P_0\hat{v} = 0$ and

$$\mathbf{E}\left[u\left(l\left(\hat{v}\right)\right)|\mathcal{F}_0\right] = \operatorname*{ess\,sup}_{v\in\mathbf{L}^0(-K_0,\mathcal{F}_0)} \mathbf{E}\left[u\left(l\left(v\right)\right)|\mathcal{F}_0\right].$$

b) Now, fix any $y \in \mathbf{L}^0(-K_0, \mathcal{F}_0)$ and note that the function

$$\mathbb{R} \ni h \mapsto u\left(l\left(\hat{v} + hy\right)\right)$$

is concave which implies that both

$$(0, \infty) \ni h \mapsto \frac{u\left(l\left(\hat{v} + hy\right)\right) - u\left(l\left(\hat{v}\right)\right)}{h}$$

and

$$(-\infty, 0) \ni h \mapsto \frac{u\left(l\left(\hat{v}+hy\right)\right)-u\left(l\left(\hat{v}\right)\right)}{h}$$

are decreasing functions.

By the monotone convergence theorem and optimality of $\hat{v}$ we get the following first order condition:

$$\begin{aligned} 0 &\geq \lim_{h\downarrow 0} \frac{\mathbf{E}\left[u\left(l\left(\hat{v}+hy\right)\right)-u\left(l\left(\hat{v}\right)\right)|\mathcal{F}_0\right]}{h} \\ &= \mathbf{E}\left[\lim_{h\downarrow 0} \frac{u\left(l\left(\hat{v}+hy\right)\right)-u\left(l\left(\hat{v}\right)\right)}{h} \middle| \mathcal{F}_0\right]. \end{aligned}$$

On $\{\hat{v}^1 > 0\}$ we have, when $h > 0$ is small enough,

$$l\left(\hat{v}+hy\right) = \left(\hat{v}+hy\right) \cdot \left(\begin{smallmatrix} 1 \\ \underline{S}_1 \end{smallmatrix}\right),$$

consequently

$$\begin{aligned} 0 &\geq \mathbf{E}\left[\lim_{h\downarrow 0} \frac{u\left(l\left(\hat{v}+hw\right)\right)-u\left(l\left(\hat{v}\right)\right)}{h} \middle| \mathcal{F}_0\right] \\ &= \mathbf{E}\left[\lim_{h\downarrow 0} \frac{u\left(\left(\hat{v}+hw\right)\cdot\left(\begin{smallmatrix} 1 \\ \underline{S}_1 \end{smallmatrix}\right)\right)-u\left(\hat{v}\cdot\left(\begin{smallmatrix} 1 \\ \underline{S}_1 \end{smallmatrix}\right)\right)}{h} \middle| \mathcal{F}_0\right] \\ &= \mathbf{E}\left[u'\left(l\left(\hat{v}\right)\right) w \cdot \left(\begin{smallmatrix} 1 \\ \underline{S}_1 \end{smallmatrix}\right) |\mathcal{F}_0\right]. \end{aligned}$$

For $w = \left(\begin{smallmatrix} -\overline{S}_0 \\ 1 \end{smallmatrix}\right)$ this yields the following inequality

$$0 \geq \mathbf{E}\left[u'\left(l\left(\hat{v}\right)\right)\left(\underline{S}_1 - \overline{S}_0\right)|\mathcal{F}_0\right]$$

which is in fact an equality: On $\{\hat{v}^1 > 0\}$ we have by optimality $\hat{v} = \hat{v}^1 \left(\begin{smallmatrix} -\overline{S}_0 \\ 1 \end{smallmatrix}\right)$ such that for $-\hat{v}^1 < h < 0$

$$\hat{v} + h\left(\begin{smallmatrix} -\overline{S}_0 \\ 1 \end{smallmatrix}\right) \in -K_0$$

and

$$l\left(\hat{v}+h\left(\begin{smallmatrix} -\overline{S}_0 \\ 1 \end{smallmatrix}\right)\right) = \left(\hat{v}+h\left(\begin{smallmatrix} -\overline{S}_0 \\ 1 \end{smallmatrix}\right)\right)\cdot\left(\begin{smallmatrix} 1 \\ \underline{S}_1 \end{smallmatrix}\right).$$

Again by the monotone convergence theorem and optimality of $\hat{v}$ we get on $\{\hat{v}^1 > 0\}$

$$\begin{aligned} 0 &\leq \lim_{h\uparrow 0} \frac{\mathbf{E}\left[u\left(l\left(\hat{v}+h\left(\begin{smallmatrix} -\overline{S}_0 \\ 1 \end{smallmatrix}\right)\right)\right)-u\left(l\left(\hat{v}\right)\right)|\mathcal{F}_0\right]}{h} \\ &= \mathbf{E}\left[u'\left(l\left(\hat{v}\right)\right)\left(\underline{S}_1 - \overline{S}_0\right)|\mathcal{F}_0\right]. \end{aligned}$$

Altogether on $\{\hat{v}^1 > 0\}$ we have

$$0 = \mathbf{E}\left[u'\left(l\left(\hat{v}\right)\right)\left(\underline{S}_1 - \overline{S}_0\right)|\mathcal{F}_0\right]$$

which is, by Bayes' formula, the same as

$$\mathbf{E}_{\hat{\mathbf{Q}}}\left[\underline{S}_1|\mathcal{F}_0\right] = \overline{S}_0, \quad \text{where } \frac{\mathrm{d}\hat{\mathbf{Q}}}{\mathrm{d}\mathbf{P}} = \frac{u'\left(l\left(\hat{v}\right)\right)}{\mathbf{E}\left[u'\left(l\left(\hat{v}\right)\right)|\mathcal{F}_0\right]}.$$

On $\{\hat{v}^1 < 0\}$ we proceed similarly. By optimaliy $\hat{v} = |\hat{v}^1|\left(\begin{smallmatrix} \underline{S}_0 \\ -1 \end{smallmatrix}\right)$ such that for h small enough

$$\hat{v} + h\left(\begin{smallmatrix} \underline{S}_0 \\ -1 \end{smallmatrix}\right) \in -K_0$$

and

$$l\left(\hat{v} + h\left(\begin{smallmatrix} \underline{S}_0 \\ -1 \end{smallmatrix}\right)\right) = \left(\hat{v} + h\left(\begin{smallmatrix} \underline{S}_0 \\ -1 \end{smallmatrix}\right)\right) \cdot \left(\begin{smallmatrix} 1 \\ \overline{S}_1 \end{smallmatrix}\right).$$

Now, taking the limit for $h \downarrow 0$ and $h \uparrow 0$, using the monotone convergence theorem and optimality of $\hat{v}$ we get on $\{\hat{v}^1 < 0\}$

$$\begin{aligned} 0 &= \lim_{h \to 0} \frac{\mathbf{E}\left[u\left(l\left(\hat{v} + h\left(\begin{smallmatrix} \underline{S}_0 \\ -1 \end{smallmatrix}\right)\right)\right) - u\left(l\left(\hat{v}\right)\right)|\mathcal{F}_0\right]}{h} \\ &= \mathbf{E}\left[u'\left(l\left(\hat{v}\right)\right)\left(\underline{S}_0 - \overline{S}_1\right)|\mathcal{F}_0\right] \end{aligned}$$

or equivalently

$$\mathbf{E}_{\hat{\mathbf{Q}}}\left[\overline{S}_1|\mathcal{F}_0\right] = \underline{S}_0.$$

On $\{\hat{v}^1 = 0\}$ it follows that $l\left(\hat{v} + hy\right) = l\left(\hat{v}\right) + hl\left(y\right)$, hence

$$\begin{aligned} 0 &\geq \lim_{h \downarrow 0} \frac{\mathbf{E}\left[u\left(l\left(\hat{v} + hy\right)\right) - u\left(l\left(\hat{v}\right)\right)|\mathcal{F}_0\right]}{h} \\ &= \mathbf{E}\left[\lim_{h \downarrow 0} \frac{u\left(l\left(\hat{v}\right) + hl\left(y\right)\right) - u\left(l\left(\hat{v}\right)\right)}{h}\middle|\mathcal{F}_0\right] \\ &= \mathbf{E}\left[u'\left(\hat{v}\right) l\left(y\right)|\mathcal{F}_0\right] \end{aligned}$$

for every $y \in \mathbf{L}^0\left(-K_0, \mathcal{F}_0\right)$. This is equivalent to

$$\overline{S}_0 \geq \mathbf{E}_{\hat{\mathbf{Q}}}\left[\underline{S}_1|\mathcal{F}_0\right] \quad \text{and} \quad \mathbf{E}_{\hat{\mathbf{Q}}}\left[\overline{S}_1|\mathcal{F}_0\right] \geq \underline{S}_0.$$

On $\{\hat{v}^1 = 0\}$ $\left(\overline{S}_0, \underline{S}_1\right)$ is a super- and $\left(\underline{S}_0, \overline{S}_1\right)$ a submartingale. The submartingale starts below the supermartingale and ends above the supermartingale, therefore there must be a martingale in between. For this we define

$$S_0 := \max\left\{\underline{S}_0, \mathbf{E}_{\hat{\mathbf{Q}}}\left[\underline{S}_1|\mathcal{F}_0\right]\right\}$$

$\big($or any convex combination between $\max\left\{\underline{S}_0, \mathbf{E}_{\hat{\mathbf{Q}}}\left[\underline{S}_1|\mathcal{F}_0\right]\right\}$ and $\min\left\{\overline{S}_0, \mathbf{E}_{\hat{\mathbf{Q}}}\left[\overline{S}_1|\mathcal{F}_0\right]\right\}\big)$. Then

$$\underline{S}_0 \le S_0 \le \overline{S}_0 \quad \text{and} \quad \mathbf{E}_{\hat{\mathbf{Q}}}\left[\underline{S}_1|\mathcal{F}_0\right] \le S_0 \le \mathbf{E}_{\hat{\mathbf{Q}}}\left[\overline{S}_1|\mathcal{F}_0\right].$$

We put

$$\begin{aligned} \lambda := & \frac{1}{2}\mathbf{1}_{\left\{\mathbf{E}_{\hat{\mathbf{Q}}}\left[\underline{S}_1|\mathcal{F}_0\right]=\mathbf{E}_{\hat{\mathbf{Q}}}\left[\overline{S}_1|\mathcal{F}_0\right]\right\}} \\ & + \frac{S_0 - \mathbf{E}_{\hat{\mathbf{Q}}}\left[\overline{S}_1|\mathcal{F}_0\right]}{\mathbf{E}_{\hat{\mathbf{Q}}}\left[\underline{S}_1|\mathcal{F}_0\right] - \mathbf{E}_{\hat{\mathbf{Q}}}\left[\overline{S}_1|\mathcal{F}_0\right]}\mathbf{1}_{\left\{\mathbf{E}_{\hat{\mathbf{Q}}}\left[\underline{S}_1|\mathcal{F}_0\right]<\mathbf{E}_{\hat{\mathbf{Q}}}\left[\overline{S}_1|\mathcal{F}_0\right]\right\}}, \end{aligned}$$

$$S_1 := \lambda \underline{S}_1 + (1-\lambda)\,\overline{S}_1$$

and conclude that

$$\mathbf{E}_{\hat{\mathbf{Q}}}\left[S_1|\mathcal{F}_0\right] = S_0.$$

Altogether there exists a consistent price-process $\begin{pmatrix} 1 \\ \hat{Z}_0^1 \end{pmatrix}, \begin{pmatrix} \hat{Z}_1^0 \\ \hat{Z}_1^1 \end{pmatrix}$ such that $\hat{Z}_1^0 = \frac{\mathrm{d}\hat{\mathbf{Q}}}{\mathrm{d}\mathbf{P}}$ and

$$\begin{aligned} \hat{Z}_0^1 &= \overline{S}_0, \quad \frac{\hat{Z}_1^1}{\hat{Z}_1^0} = \underline{S}_1 \quad \text{on } \left\{\hat{v}^1 > 0\right\}, \\ \hat{Z}_0^1 &= \underline{S}_0, \quad \frac{\hat{Z}_1^1}{\hat{Z}_1^0} = \overline{S}_1 \quad \text{on } \left\{\hat{v}^1 < 0\right\}. \end{aligned}$$

□

We want to show that the optimal portfolio $\hat{v}$ is unique when $P_0\hat{v} = 0$.

Remark 2.10. The optimal portfolio $\hat{v}$ is unique when $P_0\hat{v} = 0$. Indeed, let be $\hat{w} \in \mathbf{L}^0\left(-K_0, \mathcal{F}_0\right)$ such that $P_0\widehat{w} = 0$ and

$$\mathbf{E}\left[u\left(l\left(\hat{v}\right)\right)|\mathcal{F}_0\right] = \mathbf{E}\left[u\left(l\left(\hat{w}\right)\right)|\mathcal{F}_0\right].$$

We consider the portfolio $\frac{1}{2}\left(\hat{v} + \hat{w}\right)$ and deduce

$$\begin{aligned} \mathbf{E}\left[u\left(l\left(\frac{1}{2}\left(\hat{v}+\hat{w}\right)\right)\right)\middle|\mathcal{F}_0\right] &\ge \mathbf{E}\left[u\left(\frac{1}{2}\left(l\left(\hat{v}\right)+l\left(\hat{w}\right)\right)\right)\middle|\mathcal{F}_0\right] \\ &\ge \frac{1}{2}\left(\mathbf{E}\left[u\left(l\left(\hat{v}\right)\right)|\mathcal{F}_0\right] + \mathbf{E}\left[u\left(l\left(\hat{w}\right)\right)|\mathcal{F}_0\right]\right) \\ &= \mathbf{E}\left[u\left(l\left(\hat{v}\right)\right)|\mathcal{F}_0\right]. \end{aligned}$$

If $\mathbf{P}(l(\hat{v}) \neq l(\hat{w})) > 0$, the last inequality cannot be a an equality because u is strictly concave. For the two maxima it follows $l(\hat{v}) = l(\hat{w})$.

We partition Ω into several events and show that $\hat{v} = \hat{w}$ on all those events. First, we look at the event $B := \{\hat{v}^1 \geq \hat{w}^1 \geq 0\}$ and get

$$l(\mathbf{1}_B(\hat{v} - \hat{w})) = \mathbf{1}_B(\hat{v} - \hat{w}) \cdot \left(\begin{smallmatrix} 1 \\ \underline{S}_1 \end{smallmatrix}\right) = \mathbf{1}_B(l(\hat{v}) - l(\hat{w})) = 0.$$

Since $\mathbf{1}_B(\hat{v} - \hat{w}) \in -K_0$, it follows that $\mathbf{1}_B(\hat{v} - \hat{w}) \in \mathcal{N}$, i.e.

$$P_0(\mathbf{1}_B(\hat{v} - \hat{w})) = \mathbf{1}_B(\hat{v} - \hat{w}).$$

This is the same as $\mathbf{1}_B(\hat{v} - P_0\hat{v}) = \mathbf{1}_B(\hat{w} - P_0\hat{w})$, hence

$$\mathbf{1}_B\hat{v} = \mathbf{1}_B\hat{w}.$$

Of course, the events $\{\hat{w}^1 \geq \hat{v}^1 \geq 0\}$, $\{\hat{w}^1 \leq \hat{v}^1 \leq 0\}$ or $\{\hat{v}^1 \leq \hat{w}^1 \leq 0\}$ can be treated in the same way.
Next, we look at $B := \{\hat{v}^1 > 0, \hat{w}^1 < 0\}$ (resp. $\{\hat{v}^1 < 0, \hat{w}^1 > 0\}$) and show that this is a null set. We have $\mathbf{1}_B\underline{S}_1 = \mathbf{1}_B\overline{S}_1$, otherwise

$$\mathbf{P}\left(l\left(\frac{1}{2}(\mathbf{1}_B\hat{v} + \mathbf{1}_B\hat{w})\right) > \frac{1}{2}(l(\mathbf{1}_B\hat{v}) + l(\mathbf{1}_B\hat{w}))\right) > 0$$

and, since u is strictly increasing,

$$\mathbf{P}\left(\mathbf{1}_B\mathbf{E}\left[u\left(l\left(\frac{1}{2}(\hat{v} + \hat{w})\right)\right)\middle|\mathcal{F}_0\right] > \mathbf{1}_B\mathbf{E}[u(l(\hat{v}))|\mathcal{F}_0]\right) > 0$$

contradicting the opimality of $\hat{v}$. Similarly to the first event $\{\hat{v}^1 \geq \hat{w}^1 \geq 0\}$ we proceed on $B \cap \{\underline{S}_0 = \overline{S}_0\}$ where we also have $\hat{v} - \hat{w} \in -K_0$. From $l(\hat{v} - \hat{w}) = l(\hat{v}) - l(w) = 0$ it develops, using the property $P_0\hat{v} = 0$ and $P_0\hat{w} = 0$, that

$$\hat{v} = \hat{w} \quad \text{on } B \cap \{\underline{S}_0 = \overline{S}_0\}.$$

$B \cap \{\underline{S}_0 = \overline{S}_0\}$ must be a null set. It remains to consider the event $B \cap \{\underline{S}_0 < \overline{S}_0\}$. Since the costs are strict here at time 0 we can write

$$\frac{1}{2}(\hat{v} + \hat{w}) = y + a\left(\begin{smallmatrix} -1 \\ 0 \end{smallmatrix}\right)$$

for some $y \in \mathbf{L}^0(-K_0, \mathcal{F}_0)$ and some $\mathcal{F}_0$-measurable $a > 0$. Clearly $l\left(y + a\left(\begin{smallmatrix} -1 \\ 0 \end{smallmatrix}\right)\right) = l(y) - a$ so that

$$\mathbf{E}[u(y)|\mathcal{F}_0] \geq \mathbf{E}\left[u\left(l\left(\frac{1}{2}(\hat{v} + \hat{w})\right)\right)\middle|\mathcal{F}_0\right] \geq \mathbf{E}[u(l(\hat{v}))|\mathcal{F}_0]$$

and the first inequality is strict on $B \cap \{\underline{S}_0 < \overline{S}_0\}$. It follows that $B \cap \{\underline{S}_0 < \overline{S}_0\}$ is a null set. We conclude that $\hat{v} = \hat{w}$.

The results in Theorem 2.9 are not enough for our later purpose when we want construct a consistent price process in a multi-period model. We have to refine them in the next corollary.

Corollary 2.11. *Let $\hat{v} = (\hat{v}^0, \hat{v}^1) \in \mathbf{L}^0(-K_0, \mathcal{F}_0)$ be an optimal portfolio, i.e.*

$$\mathbf{E}\left[u\left(l\left(\hat{v}\right)\right)|\mathcal{F}_0\right] = \operatorname*{ess\,sup}_{v \in \mathbf{L}^0(-K_0,\mathcal{F}_0)} \mathbf{E}\left[u\left(l\left(v\right)\right)|\mathcal{F}_0\right],$$

such that $P_0\hat{v} = 0$, where P_0 denotes the $\mathcal{F}_0$-measurable, projection-valued mapping from Lemma 2.6 corresponding to $\mathcal{N}$. Then

a)

$$\begin{aligned}
&\overline{S}_0 \in (\inf\nolimits_0 \underline{S}_1, \sup\nolimits_0 \underline{S}_1) && \text{on } \left\{\hat{v}^1 > 0\right\}, \\
&\underline{S}_0 \in \left(\inf\nolimits_0 \overline{S}_1, \sup\nolimits_0 \overline{S}_1\right) && \text{on } \left\{\hat{v}^1 < 0\right\}, \\
&\overline{S}_0 > \inf\nolimits_0 \underline{S}_1, \quad \underline{S}_0 < \sup\nolimits_0 \overline{S}_1 && \text{on } A^c \cap \left\{\hat{v}^1 = 0\right\},
\end{aligned}$$

where A is the biggest $\mathcal{F}_0$-measurable set such that

$$\underline{S}_0 = \overline{S}_0 = \underline{S}_1 = \overline{S}_1 \quad \text{on } A,$$

i.e. $A = \left\{\mathbf{E}\left[\left|\underline{S}_1 - \overline{S}_0\right| |\mathcal{F}_0\right] = 0\right\} = \left\{\mathbf{E}\left[\left|\underline{S}_0 - \overline{S}_1\right| |\mathcal{F}_0\right] = 0\right\}$.

b) Further, we have

$$\operatorname{ri}\left[\underline{S}_0, \overline{S}_0\right] \cap \operatorname{ri}\left[\inf\nolimits_0 \underline{S}_1, \sup\nolimits_0 \overline{S}_1\right] \neq \emptyset,$$

and there exists a strictly consistent price process.

Proof. Let $\hat{v}$ be an optimal portfolio with $P_0\hat{v} = 0$. As in the proof of Theorem 2.9 we set $R := \mathrm{id} - P_0$.

a) On $\{\hat{v}^1 > 0\}$ we have by Theorem 2.9 b)

$$\mathbf{E}_{\hat{\mathbf{Q}}}\left[\underline{S}_1|\mathcal{F}_0\right] = \overline{S}_0, \quad \frac{\mathrm{d}\hat{\mathbf{Q}}}{\mathrm{d}\mathbf{P}} = \frac{u'\left(l\left(\hat{v}\right)\right)}{\mathbf{E}\left(u'\left(l\left(\hat{v}\right)\right)|\mathcal{F}_0\right)}.$$

Now

$$\inf\nolimits_0 \underline{S}_1 = \mathbf{E}_{\hat{\mathbf{Q}}}\left[\inf\nolimits_0 \underline{S}_1|\mathcal{F}_0\right] \leq \mathbf{E}_{\hat{\mathbf{Q}}}\left[\underline{S}_1|\mathcal{F}_0\right] \leq \mathbf{E}_{\hat{\mathbf{Q}}}\left[\sup\nolimits_0 \underline{S}_1|\mathcal{F}_0\right] = \sup\nolimits_0 \underline{S}_1$$

implies

$$\overline{S}_0 \in \left[\inf\nolimits_0 \underline{S}_1, \sup\nolimits_0 \underline{S}_1\right].$$

We look at $D := \{\hat{v}^1 > 0\} \cap \left\{\overline{S}_0 = \inf_0 \underline{S}_1\right\}$ and consider the portfolio $w := \mathbf{1}_D \hat{v}$. Since $w = \mathbf{1}_D \hat{v}^1 \left(\begin{smallmatrix} -\overline{S}_0 \\ 1 \end{smallmatrix}\right)$ and $\overline{S}_0 \leq \underline{S}_1 \leq \overline{S}_1$ on D, we get

$$w \cdot \left(\begin{smallmatrix} 1 \\ \underline{S}_1 \end{smallmatrix}\right) \geq 0, \quad w \cdot \left(\begin{smallmatrix} 1 \\ \overline{S}_1 \end{smallmatrix}\right) \geq 0.$$

This implies $w \in \mathcal{N}$, i.e. $P_0 w = w$. Then

$$0 = Rw = R\left(\mathbf{1}_D \hat{v}\right) = \mathbf{1}_D \left(R\hat{v}\right) = \mathbf{1}_D \hat{v}.$$

Hence $\mathbf{P}\left(D\right) = 0$ and on $\{\hat{v}^1 > 0\}$ it is necessary that

$$\overline{S}_0 > \inf_0 \underline{S}_1.$$

Next we look at $D := \{\hat{v}^1 > 0\} \cap \left\{\overline{S}_0 = \sup_0 \underline{S}_1\right\}$ and consider $w := \mathbf{1}_D \hat{v}$. On D we have $\underline{S}_1 \leq \overline{S}_0$ which implies $l\left(w\right) \leq 0$. But $\hat{v}$ is a maximum for

$$\mathbf{L}^0\left(-K_0, \mathcal{F}_0\right) \ni v \mapsto \mathbf{E}\left[u\left(l\left(v\right)\right)|\mathcal{F}_0\right],$$

thus optimality of $\hat{v}$ enforces

$$l\left(w\right) = 0, \quad \text{i.e. } w \in \mathcal{N}.$$

Again $0 = Rw = R\left(\mathbf{1}_D \hat{v}\right) = \mathbf{1}_D \left(R\hat{v}\right) = \mathbf{1}_D \hat{v}$. Thus on $\{\hat{v}^1 > 0\}$ it is necessary that

$$\overline{S}_0 < \sup_0 \underline{S}_1.$$

Altogether on $\{\hat{v}^1 > 0\}$ we have

$$\overline{S}_0 \in \left(\inf_0 \underline{S}_1, \sup_0 \underline{S}_1\right).$$

On $\{\hat{v}^1 < 0\}$ we proceed in the same manner. The martingale property

$$\mathbf{E}_{\hat{\mathbf{Q}}}\left[\overline{S}_1 | \mathcal{F}_0\right] = \underline{S}_0$$

implies

$$\underline{S}_0 \in \left[\inf_0 \overline{S}_1, \sup_0 \overline{S}_1\right].$$

We conside the portfolio $w := \mathbf{1}_D \hat{v}$, where $D := \{\hat{v}^1 < 0\} \cap \left\{\underline{S}_0 = \sup_0 \overline{S}_1\right\}$ and calculate

$$w \cdot \left(\begin{smallmatrix} 1 \\ \overline{S}_1 \end{smallmatrix}\right) = \mathbf{1}_D \left|\hat{v}^1\right| \left(\begin{smallmatrix} \underline{S}_0 \\ -1 \end{smallmatrix}\right) \cdot \left(\begin{smallmatrix} 1 \\ \overline{S}_1 \end{smallmatrix}\right) \geq 0, \quad w \cdot \left(\begin{smallmatrix} 1 \\ \underline{S}_1 \end{smallmatrix}\right) \geq 0.$$

This implies $w \in \mathcal{N}$, which leads to $0 = Rw = \mathbf{1}_D \hat{v}$, i.e. on $\{\hat{v}^1 < 0\}$ we have $\underline{S}_0 < \sup_0 \overline{S}_1$.

Next, we consider $D := \{\hat{v}^1 < 0\} \cap \left\{\underline{S}_0 = \inf_0 \overline{S}_1\right\}$ and calculate for $w := \mathbf{1}_D \hat{v}$ that $l(w) \leq 0$. Optimality of $\hat{v}$ enforces $l(w) = 0$, so that $w \in \mathcal{N}$ and $0 = Rw = \mathbf{1}_D \hat{v}$.

Altogether on $\{\hat{v}^1 < 0\}$ we have

$$\underline{S}_0 \in \left(\inf_0 \overline{S}_1, \sup_0 \overline{S}_1\right).$$

In the last case we consider $A^c \cap \{\hat{v}^1 = 0\}$, where
$A = \left\{\mathbf{E}\left[\left|\underline{S}_1 - \overline{S}_0\right| |\mathcal{F}_0\right] = 0\right\} = \left\{\mathbf{E}\left[\left|\underline{S}_0 - \overline{S}_1\right| |\mathcal{F}_0\right] = 0\right\}$. Clearly, for any $C \in \mathcal{F}_0, C \subset A^c \cap \{\hat{v}^1 = 0\}$, with $\overline{S}_0 = \underline{S}_1$ on C or $\underline{S}_0 = \overline{S}_1$ on C, it follows that $\mathbf{P}(C) = 0$.

From

$$\mathbf{E}_{\hat{\mathbf{Q}}}\left[\underline{S}_1 | \mathcal{F}_0\right] \geq \inf_0 \underline{S}_1$$

and

$$\overline{S}_0 \geq \mathbf{E}_{\hat{\mathbf{Q}}}\left[\underline{S}_1 | \mathcal{F}_0\right] \quad \text{on } \left\{\hat{v}^1 = 0\right\}$$

we get

$$\overline{S}_0 = \underline{S}_1$$

on $C := A^c \cap \{\hat{v}^1 = 0\} \cap \left\{\overline{S}_0 = \inf_0 \underline{S}_1\right\}$.
Thus, $\mathbf{P}(C) = 0$, i.e. on $A^c \cap \{\hat{v}^1 = 0\}$ we get

$$\overline{S}_0 > \inf_0 \underline{S}_1.$$

Next, we look at $C := A^c \cap \{\hat{v}^1 = 0\} \cap \left\{\underline{S}_0 = \sup_0 \overline{S}_1\right\}$ and deduce from

$$\underline{S}_0 \leq \mathbf{E}_{\hat{\mathbf{Q}}}\left[\overline{S}_1 | \mathcal{F}_0\right] \leq \sup_0 \overline{S}_1 \quad \text{on } \left\{\hat{v}^1 = 0\right\}$$

that

$$\underline{S}_0 = \overline{S}_1 \quad \text{on } C.$$

Thus, $\mathbf{P}(C) = 0$, i.e.

$$\underline{S}_0 < \sup_0 \overline{S}_1$$

on $A^c \cap \{\hat{v}^1 = 0\}$.

Altogether, when it is optimal not to trade, we get necessarily that

$$\overline{S}_0 > \inf_0 \underline{S}_1 \quad \text{and} \quad \underline{S}_0 < \sup_0 \overline{S}_1 \quad \text{on } A^c.$$

b) We have the following conditions

$$\begin{aligned}
&\overline{S}_0 \in \left(\inf_0 \underline{S}_1, \sup_0 \underline{S}_1\right) && \text{on } \left\{\hat{v}^1 > 0\right\}, \\
&\underline{S}_0 \in \left(\inf_0 \overline{S}_1, \sup_0 \overline{S}_1\right) && \text{on } \left\{\hat{v}^1 < 0\right\}, \\
&\overline{S}_0 > \inf_0 \underline{S}_1, \quad \underline{S}_0 < \sup_0 \overline{S}_1 && \text{on } A^c \cap \left\{\hat{v}^1 = 0\right\},
\end{aligned}$$

where $A = \left\{\mathbf{E}\left[\left|\underline{S}_1 - \overline{S}_0\right| |\mathcal{F}_0\right] = 0\right\} = \left\{\mathbf{E}\left[\left|\underline{S}_0 - \overline{S}_1\right| |\mathcal{F}_0\right] = 0\right\}$.
By Lemma 2.6 we have

$$\underline{S}_0 = \overline{S}_0 = \underline{S}_1 = \overline{S}_1 \quad \text{on } A,$$

hence

$$\underline{S}_0 = \overline{S}_0 = \inf\nolimits_0 \underline{S}_1 = \sup\nolimits_0 \overline{S}_1 \quad \text{on } A.$$

All that yields

$$\operatorname{ri}\left[\underline{S}_0, \overline{S}_0\right] \cap \operatorname{ri}\left[\inf\nolimits_0 \underline{S}_1, \sup\nolimits_0 \overline{S}_1\right] \neq \emptyset.$$

By Lemma 2.2 for any $\mathcal{F}_0$-measurable $Z_0 = (Z_0^0, Z_0^1)$ with

$$Z_0^0 > 0, Z_0^1 > 0 \quad \text{and} \quad \frac{Z_0^1}{Z_0^0} \in \operatorname{ri}\left[\underline{S}_0, \overline{S}_0\right] \cap \operatorname{ri}\left[\inf\nolimits_0 \underline{S}_1, \sup\nolimits_0 \overline{S}_1\right]$$

we can find an $\mathcal{F}_1$-measurble $Z_1 = (Z_1^0, Z_1^1)$ such that

$$Z_1^0 > 0, Z_1^1 > 0, \quad \frac{Z_1^1}{Z_1^0} \in \operatorname{ri}\left[\underline{S}_1, \overline{S}_1\right]$$

and

$$Z_0 = \mathbf{E}\left[Z_1 | \mathcal{F}_0\right].$$

Integrabililty of Z_0 and Z_1 is assured when we replace Z_0 by $\frac{1}{Z_0^0 + Z_0^1}\begin{pmatrix} Z_0^0 \\ Z_0^1 \end{pmatrix}$ and Z_1 by $\frac{1}{Z_0^0 + Z_0^1}\begin{pmatrix} Z_1^0 \\ Z_1^1 \end{pmatrix}$.

□

We want to comment the results of Corollary 2.11.

Remark 2.12. a) The conditions

$$\begin{aligned}
&\overline{S}_0 \in (\inf\nolimits_0 \underline{S}_1, \sup\nolimits_0 \underline{S}_1) && \text{on } \left\{\hat{v}^1 > 0\right\}, \\
&\underline{S}_0 \in \left(\inf\nolimits_0 \overline{S}_1, \sup\nolimits_0 \overline{S}_1\right) && \text{on } \left\{\hat{v}^1 < 0\right\}, \\
&\overline{S}_0 > \inf\nolimits_0 \underline{S}_1, \quad \underline{S}_0 < \sup\nolimits_0 \overline{S}_1 && \text{on } A^c \cap \left\{\hat{v}^1 = 0\right\},
\end{aligned}$$

are very plausible and easily seen to be true with heuristic arguments. For example when it is optimal to buy the second asset $\{\hat{v}^1 > 0\}$, the probability that $\underline{S}_1 > \overline{S}_0$ given $\mathcal{F}_0$ should be positive otherwise the investor would be better off when she decides not to trade. And since there are no arbitrage oportunities there should always be a chance that the investor incurs a loss, i.e. the probability that $\overline{S}_0 > \underline{S}_1$ given

$\mathcal{F}_0$ should also be positive. Similarly one argues when it is optimal to sell the second asset $\{\hat{v}^1 < 0\}$. Now, in the last case $A^c \cap \{\hat{v}^1 = 0\}$ it is optimal not to trade and the bid-ask process always shows a random behavior in the sense that there are no non-trivial events $B \in \mathcal{F}_0$ on which $\overline{S}_0 = \underline{S}_1$ or $\underline{S}_0 = \overline{S}_0$. Here both buying and selling the second asset should incur a loss simultaneously in every state of the world. Hence the probabilty that $\overline{S}_0 > \underline{S}_1$ and $\underline{S}_0 < \overline{S}_1$ given $\mathcal{F}_0$ should be positive.

b) For a single-period model we have seen that Assumption 2.4 implies existence of a strictly consistent price process. By the 'easy' direction in the FTAP the bid-ask process $\left(\underline{S}_0, \overline{S}_0, \underline{S}_1, \overline{S}_1\right)$ satisfies the robust no-arbitrage condition, hence Assumption 2.4 and the no-arbitrage condition are equivalent.

   In the multidimensional case the analog result of Corollary 2.11 is not available, so we will have to proceed differently there. We will show directly that Assumption 2.4 implies the robust no-arbitrage condition. This will allow us to maximize expected utility in a market with a reduced bid-ask spread.

2.3 Multi-period model

In markets without transaction costs the problem to find an equivalent martingale measure for the multi-period model is simplified by looking at each single-period model separately. This is due to the fact that each single-period model is free of arbitrage opportunities if and only if the multi-period model is free of arbitrage opportunities.

This equivalence breaks down in the presence of proportional transaction costs.

Example 2.13. The following two-period model $\left(\underline{S}_0, \overline{S}_0, \underline{S}_1, \overline{S}_1, \underline{S}_2, \overline{S}_2\right)$

$$\underline{S}_0 = 1, \quad \overline{S}_0 = 3, \quad \underline{S}_1 = 2, \quad \overline{S}_1 = 5, \quad \underline{S}_2 = 4, \quad \overline{S}_2 = 6$$

contains an arbitrage opportunity $\left(\underline{S}_2 - \overline{S}_0 > 0\right)$ but the single-period models $\left(\underline{S}_0, \overline{S}_0, \underline{S}_1, \overline{S}_1\right)$ and $\left(\underline{S}_1, \overline{S}_1, \underline{S}_2, \overline{S}_2\right)$ satisfy the robust no-arbitrage condition. Since the costs are strict one can easily extend this deterministic example to a random setting.

In an example as above we can find an optimal portfolio for the market $\left(\underline{S}_0, \overline{S}_0, \underline{S}_1, \overline{S}_1\right)$ and an optimal portfolio for the market $\left(\underline{S}_1, \overline{S}_1, \underline{S}_2, \overline{S}_2\right)$. In both cases we can deduce from the first order condition a consistent price process for both single-period models. But in contrast to frictionless markets the two consistent price processes cannot be concatenated to yield a consistent price process for the two-period model.

A first attempt to construct a consistent price process in a two-period model $(\underline{S}_0, \overline{S}_0, \underline{S}_1, \overline{S}_1, \underline{S}_2, \overline{S}_2)$ given that we have already found a consistent price process for the last period (Z_1, Z_2), could be to maximize

$$\mathbf{E}\left[u\left(v_0 \cdot \left(\begin{smallmatrix} 1 \\ S_1 \end{smallmatrix}\right)\right)\right] \to \mathrm{MAX}, \quad v_0 \in L^0(-K_0, \mathcal{F}_0),$$

where $S_1 = Z_1^1/Z_1^0$. If we could succeed in finding a maximizer, the consistent price process coming from the first order condition, $(\hat{Z}_0, \hat{Z}_1)$, would already be concatenated with (Z_1, Z_2), i.e. $\hat{Z}_1^1/\hat{Z}_1^0 = Z_1^1/Z_1^0$ and $\hat{Z}_1 = cZ_1$ where $c > 0$ is a normalizing constant. Of course, we would have to pick S_1 in such a way that $(\underline{S}_0, \overline{S}_0, S_1)$ is free of arbitrage opportunities otherwise a maximizer cannot exist. Finding such an S_1 seems a little too much to ask. But, if S_1 is chosen in a 'least favorable' way for the investor, then arbitrage opportunities are ruled out and a maximum must exist. Thus, we try to maximize

$$\inf_{S_1 \in \mathcal{M}_1} \mathbf{E}\left[u\left(v_0 \cdot \left(\begin{smallmatrix} 1 \\ S_1 \end{smallmatrix}\right)\right)\right] \to \mathrm{MAX}, \quad v_0 \in L^0(-K_0, \mathcal{F}_0),$$

where $\mathcal{M}_1 = \left\{Z_1^1/Z_1^0 \,:\, (Z_1, Z_2) \text{ consistent price process}\right\}$. We show that

$$\sup_{v_0 \in L^0(-K_0, \mathcal{F}_0)} \inf_{S_1 \in \mathcal{M}_1} \mathbf{E}\left[u\left(v_0 \cdot \left(\begin{smallmatrix} 1 \\ S_1 \end{smallmatrix}\right)\right)\right] = \sup_{v_0 \in L^0(-K_0, \mathcal{F}_0)} \mathbf{E}\left[u\left(l_1(v_0)\right)\right],$$

where

$$l_1(v_0^0, v_0^1) = \begin{cases} v_0^0 + v_0^1 \max\left\{\underline{S}_1, \inf_1 \underline{S}_2\right\}, & v_0^1 \geq 0 \\ v_0^0 + v_0^1 \min\left\{\overline{S}_1, \sup_1 \overline{S}_2\right\}, & v_0^1 \leq 0. \end{cases}$$

Hence, we replace $\underline{S}_1$ by $\max\left\{\underline{S}_1, \inf_1 \underline{S}_2\right\}$ and $\overline{S}_1$ by $\min\left\{\overline{S}_1, \sup_1 \overline{S}_2\right\}$. Conditional on the information at time 1 the investor expects the worst case to happen for the prices $\underline{S}_2, \overline{S}_2$. E.g. when the investor buys the second asset at time 0 she wants to sell it at the best bid price at time 1 expecting the worst case to happen at time 2. In this way we will reduce to a single-period market with more investment opportunities than the original single-period market. This will 'replace' the missing investment opportunities from the future. As in Rogers' original work [20] our proof is based on a backwards induction. Motivated by this discussion and Lemma 2.2 we are led to the following recursion.

On a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t=0}^T, \mathbf{P})$ we have a two-dimensional, $(\mathcal{F}_t)_{t=0}^T$-adapted process $(\underline{S}_t, \overline{S}_t)_{t=0}^T$, where $0 < \underline{S}_t \leq \overline{S}_t$ for every $t = 0, \ldots, T$.

We define $X_T := \underline{S}_T, Y_T := \overline{S}_T$ and, for every $t = T-1, \ldots, 0$,

$$\begin{aligned}
&X_t := \max\left\{\underline{S}_t, \inf\nolimits_t X_{t+1}\right\}, \\
&Y_t := \min\left\{\overline{S}_t, \sup\nolimits_t Y_{t+1}\right\}, \\
&l_t(v) := \min\left\{v \cdot \left(\begin{smallmatrix}1\\X_t\end{smallmatrix}\right), v \cdot \left(\begin{smallmatrix}1\\Y_t\end{smallmatrix}\right)\right\}, \quad v \in \mathbb{R}^2.
\end{aligned}$$

As before we fix a utility function $u : \mathbb{R} \to \mathbb{R}$ such that u is continuously differentiable, strictly concave, strictly increasing and bounded from above.

Proposition 2.14. *Fix $t \in \{1, \ldots, T\}$, suppose that there exists a strictly consistent price process for $(\underline{S}_r, \overline{S}_r)_{r=t}^T$ and put*

$$\mathcal{M}_t = \left\{Z_t^1/Z_t^0 \,:\, (Z_r)_{r=t}^T \text{ a consistent price process for } (\underline{S}_r, \overline{S}_r)_{r=t}^T\right\}.$$

Then

$$\mathbf{E}\left[u\left(l_t\left(v\right)\right)|\mathcal{F}_{t-1}\right] = \operatorname*{ess\,inf}_{S_t \in \mathcal{M}_t} \mathbf{E}\left[u\left(v \cdot \left(\begin{smallmatrix}1\\S_t\end{smallmatrix}\right)\right)|\mathcal{F}_{t-1}\right].$$

Proof. Let $(Z_r)_{r=t}^T = (Z_r^0, Z_r^1)_{r=t}^T$ be a strictly consistent price process for the bid-ask process $\left(\underline{S}_r, \overline{S}_r\right)_{r=t}^T$, i.e. $(Z_r)_{r=t}^T$ is a martingale and $\frac{Z_r^1}{Z_r^0} \in \operatorname{ri}\left[\underline{S}_r, \overline{S}_r\right]$ for every $r = t, \ldots, T$. By Lemma 2.2 a) we have

$$\frac{Z_T^1}{Z_T^0} \in \operatorname{ri}\left[X_T, Y_T\right] \Rightarrow \frac{Z_{T-1}^1}{Z_{T-1}^0} \in \operatorname{ri}\left[\inf\nolimits_{T-1} X_T, \sup\nolimits_{T-1} Y_T\right],$$

hence

$$\frac{Z_{T-1}^1}{Z_{T-1}^0} \in \operatorname{ri}\left[\underline{S}_{T-1}, \overline{S}_{T-1}\right] \cap \operatorname{ri}\left[\inf\nolimits_{T-1} X_T, \sup\nolimits_{T-1} Y_T\right] = \operatorname{ri}\left[X_{T-1}, Y_{T-1}\right].$$

When we repeat this procedure until t, we end up with

$$\frac{Z_r^1}{Z_r^0} \in \operatorname{ri}\left[X_r, Y_r\right] \text{ for every } r = t, \ldots, T.$$

When $(Z_r)_{r=t}^T$ is only a consistent price process the same statement holds without taking the relative interior. Thus, $S_t \in [X_t, Y_t]$ when $S_t \in \mathcal{M}_t$. It follows $l_t\left(v\right) \leq v \cdot \left(\begin{smallmatrix}1\\S_t\end{smallmatrix}\right)$ and hence

$$\mathbf{E}\left[u\left(l_t\left(v\right)\right)|\mathcal{F}_{t-1}\right] \leq \operatorname*{ess\,inf}_{S_t \in \mathcal{M}_t} \mathbf{E}\left[u\left(v \cdot \left(\begin{smallmatrix}1\\S_t\end{smallmatrix}\right)\right)|\mathcal{F}_{t-1}\right].$$

For the reverse inequality assume that $v^1 \geq 0$, so that $l_t(v) = v^0 + v^1 X_t$. By Lemma 2.2 b) we can find a sequence of strictly consistent price processes $(Z_r^n)_{r=t}^T$ such that $S_t^n = \frac{Z_t^{n,1}}{Z_t^{n,0}}$ decreases towards X_t. By monotone convergence it follows

$$\mathbf{E}\left[u\left(l_t\left(v\right)\right)|\mathcal{F}_{t-1}\right] = \lim_{n\to\infty} \mathbf{E}\left[u\left(v \cdot \left(\begin{smallmatrix}1\\ S_t^n\end{smallmatrix}\right)\right)|\mathcal{F}_{t-1}\right] \geq \operatorname*{ess\,inf}_{S_t \in \mathcal{M}_t} \mathbf{E}\left[u\left(v \cdot \left(\begin{smallmatrix}1\\ S_t\end{smallmatrix}\right)\right)|\mathcal{F}_{t-1}\right].$$

When $v^1 \leq 0$, we take S_t^n to be increasing towards Y_t. $\square$

The following lemma is crucial for the main theorem. It assures that the single-period market consisting of $\left(\underline{S}_{t-1}, \overline{S}_{t-1}\right)$ and (X_t, Y_t) satisfies Assumption 2.4.

Lemma 2.15. *Assume $(\underline{S}_0, \overline{S}_0, \dots, \underline{S}_T, \overline{S}_T)$ satisfies the robust no-arbitrage condition. Fix $t \in \{1, \dots, T\}$ and suppose that there is a strictly consistent price process for $(\underline{S}_r, \overline{S}_r)_{r=t}^T$.*

Then

$$\mathcal{N} := \left\{v \in \mathbf{L}^0\left(-K_{t-1}, \mathcal{F}_{t-1}\right) \,:\, v \cdot \left(\begin{smallmatrix}1\\ X_t\end{smallmatrix}\right) \geq 0, v \cdot \left(\begin{smallmatrix}1\\ Y_t\end{smallmatrix}\right) \geq 0\right\}$$

is a subspace closed with respect to convergence in probability. Especially

$$\mathcal{N} = \left\{v \in \mathbf{L}^0\left(-K_{t-1} \cap K_{t-1}, \mathcal{F}_{t-1}\right) \,:\, v \cdot \left(\begin{smallmatrix}1\\ X_t\end{smallmatrix}\right) = 0, v \cdot \left(\begin{smallmatrix}1\\ Y_t\end{smallmatrix}\right) = 0\right\}$$

and for real-valued $\mathcal{F}_{t-1}$-measurable h and $v \in \mathcal{N}$ we have $hv \in \mathcal{N}$.

The lemma will be proved in the last section of this chapter. We will take it for granted and use it now.

2.3.1 Main theorem

Theorem 2.16. *Assume that $(\underline{S}_0, \overline{S}_0, \dots, \underline{S}_T, \overline{S}_T)$ satisfies the robust no-arbitrage condition. Fix $t \in \{1, \dots, T\}$ and suppose that there is a strictly consistent price process for $(\underline{S}_r, \overline{S}_r)_{r=t}^T$.*

Then, there exists a portfolio $\hat{v}_{t-1} \in \mathbf{L}^0\left(-K_{t-1}, \mathcal{F}_{t-1}\right)$ such that

$$\mathbf{E}\left[u\left(l_t\left(\hat{v}_{t-1}\right)\right)|\mathcal{F}_{t-1}\right] = \operatorname*{ess\,sup}_{v \in \mathbf{L}^0(-K_{t-1}, \mathcal{F}_{t-1})} \mathbf{E}\left[u\left(l_t\left(v\right)\right)|\mathcal{F}_{t-1}\right]$$

and there exists a strictly consistent price process for $\left(\underline{S}_r, \overline{S}_r\right)_{r=t-1}^T$.

Especially, there exists a strictly consistent price process for $(\underline{S}_r, \overline{S}_r)_{r=0}^T$.

Proof. By Lemma 2.15 we can apply Theorem 2.9 to the single-period model $(\underline{S}_{t-1}, \overline{S}_{t-1}, X_t, Y_t)$ to find an optimal portfolio $\hat{v}_{t-1}$. Then, Corollary 2.11 implies that $X_{t-1} \leq Y_{t-1}$ and

$$\mathrm{ri}\left[\underline{S}_{t-1}, \overline{S}_{t-1}\right] \cap \mathrm{ri}\left[\inf_{t-1} X_t, \sup_{t-1} Y_t\right] = \mathrm{ri}\left[X_{t-1}, Y_{t-1}\right].$$

Since there exists a strictly consistent price process for $(\underline{S}_r, \overline{S}_r)_{r=t}^T$ by assumption, we have $X_r \leq Y_r$ and

$$\mathrm{ri}\left[\underline{S}_r, \overline{S}_r\right] \cap \mathrm{ri}\left[\inf_r X_{r+1}, \sup_r Y_{r+1}\right] = \mathrm{ri}\left[X_r, Y_r\right]$$

for all $r = t-1, \ldots, T$.

We can take any $\mathcal{F}_{t-1}$-measurable $Z_{t-1}^0 > 0, Z_{t-1}^1 > 0$ such that

$$\frac{Z_{t-1}^1}{Z_{t-1}^0} \in \mathrm{ri}\left[X_{t-1}, Y_{t-1}\right]$$

and we find by Lemma 2.2 $\mathcal{F}_r$-measurable $Z_r^0 > 0, Z_r^1 > 0$, $r = t, \ldots, T$, such that

$$\frac{Z_r^1}{Z_r^0} \in \mathrm{ri}\left[X_r, Y_r\right]$$

and

$$\mathbf{E}\left[Z_r | \mathcal{F}_{t-1}\right] = Z_{r-1}.$$

To ensure integrability, we replace every Z_r by $\frac{1}{Z_{t-1}^0 + Z_{t-1}^1} \begin{pmatrix} Z_r^0 \\ Z_r^1 \end{pmatrix}$. □

Remark 2.17. In [17] Ràsonyi implements a similar inductive argument in the presence of strict transaction costs under the strict no-arbitrage condition. There, he replaces, going backwards in time, G_t^* by

$$G_t^* \cap \mathbf{E}\left[G_{t+1}^* \cap \left[\ldots \mathbf{E}\left[K_{T-1} \cap \mathbf{E}\left[G_T^* \cap \overline{B_1(0)} | \mathcal{F}_{T-1}\right] | \mathcal{F}_{T-2}\right] \ldots\right] | \mathcal{F}_t\right]$$

and shows indirectly, that the interior of this intersection is non-empty. He applies methods from the theory of random sets such as conditional expectation of random sets and measurable selection in combination with separation arguments in finite dimensional spaces. In our approach it follows directly from the existence of the optimal portfolio that this intersection is non-empty.

In the same spirit as Ràsonyi's is Rokhlin's work [23], compare also [21] and [22]. Instead of conditional expectation for random sets he introduces the notion of the regular conditional upper distribution of a random set and its support. We will come back to that in Chapter 3 when we deal with the multidimensional case.

We want to comment on the relationship between the single-period consistent price process coming from the first-order condtion and the consistent price processes for the multi-period market. In the next remark and the following example we will make clear why the first-order condition in Theorem 2.9 b) is not enough and why we need the finer results from Corollary 2.11.

Remark 2.18. For every single-period market $\left(\underline{S}_{t-1}, \overline{S}_{t-1}, X_t, Y_t\right)$ we have solved

$$\mathbf{E}\left[u\left(l_t\left(\hat{v}_{t-1}\right)\right)|\mathcal{F}_{t-1}\right] = \underset{v \in \mathbf{L}^0(-K_{t-1}, \mathcal{F}_{t-1})}{\operatorname{ess\,sup}} \mathbf{E}\left[u\left(l_t\left(v\right)\right)|\mathcal{F}_{t-1}\right].$$

and found a consistent price process $\hat{Z}_{t-1} = (1, \hat{Z}^1_{t-1})$, $\hat{Z}_t = (\hat{Z}^0_t, \hat{Z}^1_t)$ for the single-period market $(\underline{S}_{t-1}, \overline{S}_{t-1}, X_t, Y_t)$ where

$$\hat{Z}^0_t := \frac{u'\left(l_t\left(\hat{v}_{t-1}\right)\right)}{\mathbf{E}\left[u'\left(l_t\left(\hat{v}_{t-1}\right)\right)|\mathcal{F}_{t-1}\right]},$$

$$\hat{Z}^1_{t-1} := \begin{cases} \overline{S}_{t-1}, & \hat{v}^2_{t-1} > 0 \\ \underline{S}_{t-1}, & \hat{v}^2_{t-1} < 0 \\ S_{t-1}, & \hat{v}^2_{t-1} = 0 \end{cases}, \quad \frac{\hat{Z}^1_t}{\hat{Z}^0_t} := \begin{cases} X_t, & \hat{v}^2_{t-1} > 0 \\ Y_t, & \hat{v}^2_{t-1} < 0 \\ S_t, & \hat{v}^2_{t-1} = 0 \end{cases}$$

for some appropriate $S_{t-1} \in [\underline{S}_{t-1}, \overline{S}_{t-1}]$, $S_t \in [X_t, Y_t]$.

Since

$$\operatorname{ri}\left[\underline{S}_{t-1}, \overline{S}_{t-1}\right] \cap \operatorname{ri}\left[\inf_{t-1} X_t, \sup_{t-1} Y_t\right] \neq \emptyset$$

and

$$[\underline{S}_{t-1}, \overline{S}_{t-1}] \cap [\mathbf{E}[\hat{Z}^0_t X_t|\mathcal{F}_{t-1}], \mathbf{E}[\hat{Z}^0_t Y_t|\mathcal{F}_{t-1}]] \neq \emptyset$$

we can find S_{t-1} such that

$$S_{t-1} \in [\underline{S}_{t-1}, \overline{S}_{t-1}] \cap [\mathbf{E}[\hat{Z}^0_t X_t|\mathcal{F}_{t-1}], \mathbf{E}[\hat{Z}^0_t Y_t|\mathcal{F}_{t-1}]] \cap \operatorname{ri}[\inf_{t-1} X_t, \sup_{t-1} Y_t].$$

Furthermore, we have

$$\begin{aligned} \overline{S}_{t-1} &\in \left(\inf_{t-1} X_t, \sup_{t-1} X_t\right) && \text{on } \left\{\hat{v}^2_{t-1} > 0\right\}, \\ \underline{S}_{t-1} &\in \left(\inf_{t-1} Y_t, \sup_{t-1} Y_t\right) && \text{on } \left\{\hat{v}^2_{t-1} < 0\right\}, \end{aligned}$$

i.e. $Z^2_{t-1} \in \operatorname{ri}\left[\inf_{t-1} X_t, \sup_{t-1} Y_t\right]$.

By Lemma 2.2 there exists a $Z_t = (Z^1_t, Z^2_t)$ such that

$$\frac{Z^2_t}{Z^t_1} \in \operatorname{ri}\left[X_t, Y_t\right] \quad \text{and} \quad \mathbf{E}\left[Z_t|\mathcal{F}_{t-1}\right] = Z_{t-1}.$$

As
$$\mathrm{ri}\,[X_s, Y_s] = \mathrm{ri}\,\left[\underline{S}_s, \overline{S}_s\right] \cap \mathrm{ri}\,[\inf_s X_{s+1}, \sup_s Y_{s+1}]$$
for every $s = t, \ldots, T-1$, the pair $(\hat{Z}_{t-1}, Z_t)$ can be extended to a consistent price process $(\hat{Z}_{t-1}, (Z_s)_{s=t}^T)$ for the market $(\underline{S}_{t-1}, \overline{S}_{t-1}, \ldots, \underline{S}_T, \overline{S}_T)$.

Of course $\hat{Z}_t \neq Z_t$ in general, and we will see in the next example that the pair $(\hat{Z}_{t-1}, \hat{Z}_t)$ cannot be extended and the 'ess inf' in Proposition 2.14 cannot be attained.

Example 2.19. We consider the following two period market.

On a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ let be R_1 and R_2 two strictly positive, independent random variables and $\lambda, \mu \in (0,1)$. We set $\mathcal{F}_0 = \{\emptyset, \Omega\}$, $\mathcal{F}_1 = \sigma(R_1)$, $\mathcal{F}_2 = \sigma(R_1, R_2)$ and

$$\begin{aligned} \underline{S}_0 &= 1 - \lambda & \overline{S}_0 &= 1 + \lambda \\ \underline{S}_1 &= (1-\lambda)\, R_1 & \overline{S}_1 &= (1+\lambda)\, R_1 \\ \underline{S}_2 &= (1-\mu)\, R_1 R_2 & \overline{S}_2 &= (1+\mu)\, R_1 R_2. \end{aligned}$$

For R_2 we assume that

$$\frac{1-\lambda}{1-\mu} < \inf\nolimits_0 R_2 < \frac{1+\lambda}{1-\mu} \quad \text{and} \quad \mathbf{P}\,(R_2 > M) > 0$$

for all $M > 0$, i.e. $\sup_0 R_2 = \infty$. Then, since R_1 and R_2 are independent,

$$\inf\nolimits_1 \underline{S}_2 = (1-\mu)\,(\inf\nolimits_0 R_2)\, R_1, \quad \sup\nolimits_1 \overline{S}_2 = \infty,$$

i.e.
$$X_1 = (1-\mu)\,(\inf\nolimits_0 R_2)\, R_1, \quad Y_1 = \overline{S}_1.$$

If R_1 is chosen such that

$$\frac{1-\lambda}{1+\lambda} < \inf\nolimits_0 R_1 < \frac{1+\lambda}{(1-\mu)\inf_0 R_2} \quad \text{and} \quad \mathbf{P}\,(R_1 > M) > 0$$

for all $M > 0$, then

$$\overline{S}_0 \in (\inf\nolimits_0 X_1, \infty)\,, \quad \underline{S}_0 \notin (\inf\nolimits_0 Y_1, \infty)\,.$$

Hence, if it is optimal to trade at time 0 then it is optimal to buy the second asset. We can assure trading if we assume that

$$\mathbf{E}\,[X_1] > \overline{S}_0, \text{ i.e. } \mathbf{E}\,[R_1] > \frac{1+\lambda}{(1-\mu)\inf_0 R_2}.$$

With the notation from the previous remark we have

$$\frac{\hat{Z}_1^1}{\hat{Z}_1^0} = X_1.$$

But for every equivalent probability measure $\mathbf{Q}_2$ and every $S_2 \in \left[\underline{S}_2, \overline{S}_2\right]$ we have

$$\mathbf{E}_{\mathbf{Q}_2}\left[S_2|\mathcal{F}_1\right] \geq \mathbf{E}_{\mathbf{Q}_2}\left[\underline{S}_2|\mathcal{F}_1\right] = (1-\mu) R_1 \mathbf{E}_{\mathbf{Q}_2}\left[R_2\right] > X_1,$$

hence there is no $Z_2 \in \mathbf{L}^0\left(G_2^* \setminus \{0\}, \mathcal{F}_2\right)$ such that

$$\mathbf{E}\left[Z_2|\mathcal{F}_1\right] = \hat{Z}_1.$$

In the one-dimensional setting we can describe nicely the set of all strictly consistent price processes.

Corollary 2.20. *a) If $(Z_t)_{t=0}^T = (Z_t^0, Z_t^1)_{t=0}^T$ is a strictly consisent price process for $\left(\underline{S}_t, \overline{S}_t\right)_{t=0}^T$, then*

$$\frac{Z_t^1}{Z_t^0} \in \operatorname{ri}\left[X_t, Y_t\right] = \operatorname{ri}\left[\underline{S}_t, \overline{S}_t\right] \cap \operatorname{ri}\left[\inf_t X_{t+1}, \sup_t Y_{t+1}\right]$$

for every $t = 0, \ldots, T$.

b) If $\left(\underline{S}_t, \overline{S}_t\right)_{t=0}^T$ satisfies the robust no-arbitrage condition, then any $\mathcal{F}_0$-measurable $Z_0^0, Z_0^1 > 0$ with $\frac{Z_0^1}{Z_0^0} \in \operatorname{ri}\left[X_0, Y_0\right]$ can be extended to a strictly consistent price process $(Z_t)_{t=0}^T$ for $\left(\underline{S}_t, \overline{S}_t\right)_{t=0}^T$.

Proof. a) See the proof of Proposition 2.14.

b) See the proof of Theorem 2.16.

□

Given a bid-ask price process $\left(\underline{S}_t, \overline{S}_t\right)_{t=0}^T$, which statisfies the robust no-arbitrage condition, it is both necessary and sufficient to reduce the bid-ask spread to an eventually smaller bid-ask spread related to $(X_t, Y_t)_{t=0}^T$ to find all strictly consistent price processes. $(X_t, Y_t)_{t=0}^T$ has the crucial property

$$\operatorname{ri}\left[X_t, Y_t\right] \subset \operatorname{ri}\left[\inf_t X_{t+1}, \sup_t Y_{t+1}\right]$$

for every $t = 0, \ldots, T-1$.

What does it mean if the given bid-ask price process fails this property, i.e. for some t_0 we have

$$\mathbf{P}\left(\operatorname{ri}\left[\underline{S}_{t_0}, \overline{S}_{t_0}\right] \not\subset \operatorname{ri}\left[\inf_{t_0} \underline{S}_{t_0+1}, \sup_{t_0} \overline{S}_{t_0+1}\right]\right) > 0?$$

Put

$$B := \left\{ \mathrm{ri}\left[\underline{S}_{t_0}, \overline{S}_{t_0}\right] \not\subset \mathrm{ri}\left[\inf_{t_0} \underline{S}_{t_0+1}, \sup_{t_0} \overline{S}_{t_0+1}\right] \right\}$$

and

$$B_1 := B \cap \left\{ \underline{S}_{t_0} = \overline{S}_{t_0} \right\}, \quad B_2 := B \cap \left\{ \underline{S}_{t_0} < \overline{S}_{t_0} \right\}.$$

When we replace Ω by B_1 and $\mathcal{F}_{t_0}$ by $\mathcal{F}_{t_0} \cap B_1$ as well as $\mathcal{F}_{t_0+1}$ by $\mathcal{F}_{t_0+1} \cap B_1$ then Corollary 2.11 b) implies that

$$\underline{S}_{t_0} = \overline{S}_{t_0} \in \mathrm{ri}\left[\inf_{t_0} \underline{S}_{t_0+1}, \sup_{t_0} \overline{S}_{t_0+1}\right] \quad \text{on } B_1.$$

So, B_1 is a null set.

On B_2 we have

$$\underline{S}_{t_0} < \inf_{t_0} \underline{S}_{t_0+1} \quad \text{or} \quad \overline{S}_{t_0} > \sup_{t_0} \overline{S}_{t_0+1}.$$

One of the following portfolios,

$$\begin{aligned} v &:= \mathbf{1}_{B_2 \cap \left\{ \underline{S}_{t_0} < \inf_{t_0} \underline{S}_{t_0+1} \right\}} \begin{pmatrix} -\left(\frac{1}{2}\underline{S}_{t_0} + \frac{1}{2}\inf_{t_0} \underline{S}_{t_0+1}\right) \\ 1 \end{pmatrix}, \\ w &:= \mathbf{1}_{B_2 \cap \left\{ \overline{S}_{t_0} > \sup_{t_0} \overline{S}_{t_0+1} \right\}} \begin{pmatrix} \frac{1}{2}\overline{S}_{t_0} + \frac{1}{2}\sup_{t_0} \overline{S}_{t_0+1} \\ -1 \end{pmatrix} \end{aligned}$$

is not identically zero, say $v \neq 0$.

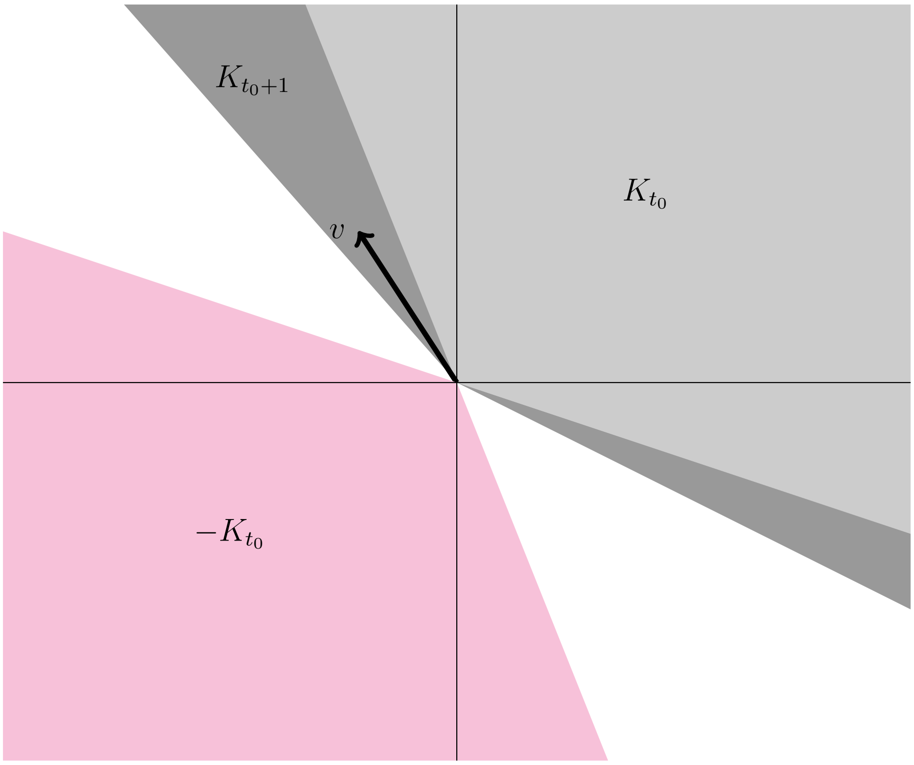

Fig. 4: v an arbitrage opportunity of the 2nd kind

The liquidation values today, $l_{t_0}(v)$ (here in terms of $\underline{S}_{t_0}, \overline{S}_{t_0}$), satisfies

$$l_{t_0}(v) \leq 0, \quad \mathbf{P}\left(l_{t_0}(v) < 0\right) > 0$$

and tomorrow, $l_{t_0+1}(v)$ (here in terms of $\underline{S}_{t_0+1}, \overline{S}_{t_0+1}$),

$$l_{t_0+1}(v) \geq 0, \quad \mathbf{P}\left(l_{t_0+1}(v) > 0\right) > 0.$$

v is a so-called arbitrage opportunity of the 2nd kind. The notion of an arbitrage opportunity of the 2nd kind was introduced in [18] for a model with strict costs in every state and at every time. Clearly, in an arbitrage-free model without costs arbitrage opportunities of the 2nd kind cannot exist. We see, that in a model with not necessarily strict costs arbitrage opportunities of the 2nd kind can still occur. But, when we replace the bid-ask price process by $(X_t, Y_t)_{t=0}^T$ the arbitrage opportunities of the 2nd kind vanish.

2.4 Conclusion

We have given a utility-based proof for the existence of consistence price processes in a discrete-time market with transaction costs, Theorem 2.16. In contrast to markets without transaction costs, it is not enough to determine a consistent price process for each given single-period market. Even if we reduce to a single-period market with more investment opportunities, i.e. $\left(\underline{S}_{t-1}, \overline{S}_{t-1}, X_t, Y_t\right)$, the single-period consistent price process related to the first order condition cannot be extended in general. This is contrary to markets without transaction costs since there the consistent price systems are valued in one-dimensional rays, i.e. there we always have $X_t = Y_t = \underline{S}_t = \overline{S}_t$. With costs going backwards in time we had to replace the original bid-ask prices $\underline{S}_t, \overline{S}_t$ by X_t, Y_t with a possibly smaller bid-ask spread. Yet, the strictly consistent price process for the market $\left(\underline{S}_0, \overline{S}_0, \dots, \underline{S}_T, \overline{S}_T\right)$ are exactly those from the market $(X_0, Y_0, \dots, X_T, Y_T)$. The latter market satisfies an extension property which has been mentioned by Ràsonyi [18] in the presence of strict costs. The market $(X_0, Y_0, \dots, X_T, Y_T)$ excludes arbitrage opportunities of the 2nd kind.

2.5 Proof of Lemma 2.15

We need to prove Lemma 2.15. Given an $(\mathcal{F}_t)_{t=0}^T$-adapted process $(\underline{S}_t, \overline{S}_t)_{t=0}^T$ with $0 < \underline{S}_t \leq \overline{S}_t$ for every $t = 0, \dots, T$, we define

$$\begin{aligned} X_T &:= \underline{S}_T, \quad Y_T := \overline{S}_T, \\ X_t &:= \max\left\{\underline{S}_t, \inf\nolimits_t X_{t+1}\right\}, \quad Y_t := \min\left\{\overline{S}_t, \sup\nolimits_t Y_{t+1}\right\}. \end{aligned}$$

Lemma 2.15 *Assume that $(\underline{S}_0, \overline{S}_0, \dots, \underline{S}_T, \overline{S}_T)$ satisfies the robust no-arbitrage condition. Fix $t \in \{1, \dots, T\}$ and suppose that there is a strictly consistent price process for $(\underline{S}_r, \overline{S}_r)_{r=t}^T$. Then,*

$$\mathcal{N} := \left\{ v \in \mathbf{L}^0\left(-K_{t-1}, \mathcal{F}_{t-1}\right) \,:\, v \cdot \left(\begin{smallmatrix} 1 \\ X_t \end{smallmatrix}\right) \geq 0, v \cdot \left(\begin{smallmatrix} 1 \\ Y_t \end{smallmatrix}\right) \geq 0 \right\}$$

is a subspace closed with respect to convergence in probability. Especially

$$\mathcal{N} = \left\{ v \in \mathbf{L}^0\left(-K_{t-1} \cap K_{t-1}, \mathcal{F}_{t-1}\right) \,:\, v \cdot \left(\begin{smallmatrix} 1 \\ X_t \end{smallmatrix}\right) = 0, v \cdot \left(\begin{smallmatrix} 1 \\ Y_t \end{smallmatrix}\right) = 0 \right\}$$

and for real-valued $\mathcal{F}_{t-1}$-measurable h and $v \in \mathcal{N}$ we have $hv \in \mathcal{N}$.

We begin with one auxiliary result needed for the proof.

Proposition. *Fix $t \in \{1, \dots, T\}$ and assume $X_t \leq Y_t$ for all $s = t, \dots, T$. Further let be $v_t \in \mathbf{L}^0\left(K_t, \mathcal{F}_t\right), \dots, v_T \in \mathbf{L}^0\left(K_T, \mathcal{F}_T\right)$ such that*

$$w_{s-1} := \sum_{r=s}^{T} v_r \text{ is } \mathcal{F}_{s-1}\text{-measurable, } s = t, \dots, T.$$

Then,

$$w_{s-1} \cdot \begin{pmatrix} 1 \\ X_s \end{pmatrix} \geq 0 \quad \textit{and} \quad w_{s-1} \cdot \begin{pmatrix} 1 \\ Y_s \end{pmatrix} \geq 0$$

for all $s = t, \dots, T$.

Proof. For $s = T$ it is clear that $w_{T-1} \cdot \begin{pmatrix} 1 \\ X_T \end{pmatrix} \geq 0$ and $w_{T-1} \cdot \begin{pmatrix} 1 \\ Y_T \end{pmatrix} \geq 0$, because $w_{T-1} = v_T \in K_T$ and $X_T = \underline{S}_T,\ Y_T = \overline{S}_T$.

Now, assume $t \leq s < T$ and write $w_{s-1} = v_s + w_s$. By induction hypothesis we have

$$w_s \cdot \begin{pmatrix} 1 \\ X_{s+1} \end{pmatrix} \geq 0, \quad w_s \cdot \begin{pmatrix} 1 \\ Y_{s+1} \end{pmatrix} \geq 0.$$

Equivalently,

$$w_s \cdot \left\{ a \begin{pmatrix} 1 \\ X_{s+1} \end{pmatrix} + b \begin{pmatrix} 1 \\ Y_{s+1} \end{pmatrix} \right\} \geq 0$$

for all $\mathcal{F}_{s+1}$-measurable, strictly positive a, b.

From $X_s \leq Y_s$, $X_s = \max\left\{\underline{S}_s, \inf_s X_{s+1}\right\}$ and $Y_s = \min\left\{\overline{S}_s, \sup_s Y_{s+1}\right\}$ we get

$$\inf_s X_{s+1} \leq X_s \leq Y_s \leq \sup_s Y_{s+1},$$

so we can write

$$\begin{aligned} X_s = {} & \mathbf{1}_{\{\sup_s Y_{s+1} < \infty\}} \left\{\lambda \inf_s X_{s+1} + (1-\lambda) \sup_s Y_{s+1}\right\} \\ & + \mathbf{1}_{\{\inf_s X_{s+1} > 0, \sup_s Y_{s+1} = \infty\}} \left\{\lambda \inf_s X_{s+1}\right\} \\ & + \mathbf{1}_{\{\inf_s X_{s+1} = 0, \sup_s Y_{s+1} = \infty\}} X_s \end{aligned}$$

with $\mathcal{F}_s$-measurable λ valued in $[0, 1]$ on the event $\{\sup_s Y_{s+1} < \infty\}$ and $\lambda \geq 1$ on the event $\{\inf_s X_{s+1} > 0, \sup_s Y_{s+1} = \infty\}$.

Now, define for $n \geq 2$

$$\begin{aligned} \lambda^n := {} & \mathbf{1}_{\{\sup_s Y_{s+1} < \infty\}} \left\{ \mathbf{1}_{\{\lambda = 0\}} \frac{1}{n} + \mathbf{1}_{\{\lambda = 1\}} \left(1 - \frac{1}{n}\right) + \mathbf{1}_{\{0 < \lambda < 1\}} \lambda \right\} \\ & + \mathbf{1}_{\{\inf_s X_{s+1} > 0, \sup_s Y_{s+1} = \infty\}} \left(\lambda + \frac{1}{n}\right) \end{aligned}$$

and put

$$\begin{aligned} X_s^n = {} & \mathbf{1}_{\{\sup_s Y_{s+1} < \infty\}} \left\{\lambda^n \inf_s X_{s+1} + (1-\lambda^n) \sup_s Y_{s+1}\right\} \\ & + \mathbf{1}_{\{\inf_s X_{s+1} > 0, \sup_s Y_{s+1} = \infty\}} \left\{\lambda^n \inf_s X_{s+1}\right\} \\ & + \mathbf{1}_{\{\inf_s X_{s+1} = 0, \sup_s Y_{s+1} = \infty\}} X_s. \end{aligned}$$

Clearly, $X_s^n \in \mathrm{ri}\,[\inf_s X_{s+1} \sup_s Y_{s+1}]$ and $X_s^n \to X_s$. By Lemma 2.2 b) we can find strictly positive $\mathcal{F}_{s+1}$-measurable a^n, b^n such that

$$\begin{pmatrix} 1 \\ X_s^n \end{pmatrix} = \mathbf{E}\left[a^n \begin{pmatrix} 1 \\ X_{s+1} \end{pmatrix} + b^n \begin{pmatrix} 1 \\ Y_{s+1} \end{pmatrix} \middle| \mathcal{F}_s \right].$$

Since w_s is $\mathcal{F}_s$-measurable, we get

$$0 \leq \mathbf{E}\left[w_s \cdot \left\{ a^n \begin{pmatrix} 1 \\ X_{s+1} \end{pmatrix} + b^n \begin{pmatrix} 1 \\ Y_{s+1} \end{pmatrix} \right\} \middle| \mathcal{F}_s \right] = w_s \cdot \begin{pmatrix} 1 \\ X_s^n \end{pmatrix}$$

and when $n \to \infty$ then $0 \leq w_s \cdot \begin{pmatrix} 1 \\ X_s \end{pmatrix}$. In the same manner we get $0 \leq w_s \cdot \begin{pmatrix} 1 \\ Y_s \end{pmatrix}$.

Finally, $v_s \in \mathbf{L}^0\left(K_s, \mathcal{F}_s\right)$ so that

$$v_s \cdot \begin{pmatrix} 1 \\ \underline{S}_s \end{pmatrix} \geq 0, \quad v_s \cdot \begin{pmatrix} 1 \\ \overline{S}_s \end{pmatrix} \geq 0.$$

Especially

$$v_s \cdot \begin{pmatrix} 1 \\ X_s \end{pmatrix} \geq 0, \quad v_s \cdot \begin{pmatrix} 1 \\ Y_s \end{pmatrix} \geq 0,$$

since

$$\underline{S}_s \leq X_s \leq Y_s \leq \overline{S}_s.$$

Altogether

$$(v_s + w_s) \cdot \begin{pmatrix} 1 \\ X_s \end{pmatrix} \geq 0, \quad (v_s + w_s) \cdot \begin{pmatrix} 1 \\ Y_s \end{pmatrix} \geq 0,$$

□

Proof of Lemma 2.15: Since there exists a consistent price process for $(\underline{S}_r, \overline{S}_r)_{r=t}^T$ by assumption, we have $X_r \leq Y_r$ for all $r = t, \ldots, T$.

Let v be $\mathcal{F}_t$-measurable such that

$$v \cdot \begin{pmatrix} 1 \\ X_t \end{pmatrix} \geq 0, \quad v \cdot \begin{pmatrix} 1 \\ Y_t \end{pmatrix} \geq 0.$$

With approriate $\mathcal{F}_t$-measurable $a_t, b_t, c_t, d_t \geq 0$ we can write

$$\begin{aligned} v &= a_t \begin{pmatrix} Y_t \\ -1 \end{pmatrix} + b_t \begin{pmatrix} -X_t \\ 1 \end{pmatrix} + c_t \begin{pmatrix} 1 \\ 0 \end{pmatrix} + d_t \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ &= a_t \mathbf{1}_{\left\{\overline{S}_t \leq \sup_t Y_{t+1}\right\}} \begin{pmatrix} \overline{S}_t \\ -1 \end{pmatrix} + b_t \mathbf{1}_{\{\underline{S}_t \geq \inf_t X_{t+1}\}} \begin{pmatrix} -\underline{S}_t \\ 1 \end{pmatrix} + c_t \begin{pmatrix} 1 \\ 0 \end{pmatrix} + d_t \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ &\quad + a_t \mathbf{1}_{\left\{\overline{S}_t > \sup_t Y_{t+1}\right\}} \begin{pmatrix} \sup_t Y_{t+1} \\ -1 \end{pmatrix} + b_t \mathbf{1}_{\{\underline{S}_t < \inf_t X_{t+1}\}} \begin{pmatrix} -\inf_t X_{t+1} \\ 1 \end{pmatrix}. \end{aligned}$$

We put

$$v_t := a_t \mathbf{1}_{\left\{\overline{S}_t \leq \sup_t Y_{t+1}\right\}} \begin{pmatrix} -\overline{S}_t \\ 1 \end{pmatrix} + b_t \mathbf{1}_{\{\underline{S}_t \geq \inf_t X_{t+1}\}} \begin{pmatrix} \underline{S}_t \\ -1 \end{pmatrix} + c_t \begin{pmatrix} -1 \\ 0 \end{pmatrix} + d_t \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

then $v_t \in \mathbf{L}^0(-K_t, \mathcal{F}_t)$ and we have

$$v + v_t = a_t \mathbf{1}_{\left\{\overline{S}_t > \sup_t Y_{t+1}\right\}} \begin{pmatrix} \sup_t Y_{t+1} \\ -1 \end{pmatrix} + b_t \mathbf{1}_{\{\underline{S}_t < \inf_t X_{t+1}\}} \begin{pmatrix} -\inf_t X_{t+1} \\ 1 \end{pmatrix}.$$

Using $Y_{t+1} \leq \sup_t Y_{t+1}$ and $X_{t+1} \geq \inf_t X_{t+1}$ allows us to write

$$\begin{aligned} v + v_t = {} & a_t \mathbf{1}_{\left\{\overline{S}_t > \sup_t Y_{t+1}\right\}} \begin{pmatrix} Y_{t+1} \\ -1 \end{pmatrix} + a_t \mathbf{1}_{\left\{\overline{S}_t > \sup_t Y_{t+1}\right\}} \begin{pmatrix} \sup_t Y_{t+1} - Y_{t+1} \\ 0 \end{pmatrix} \\ & + b_t \mathbf{1}_{\{\underline{S}_t < \inf_t X_{t+1}\}} \begin{pmatrix} -X_{t+1} \\ 1 \end{pmatrix} + b_t \mathbf{1}_{\{\underline{S}_t < \inf_t X_{t+1}\}} \begin{pmatrix} X_{t+1} - \inf_t X_{t+1} \\ 0 \end{pmatrix}. \end{aligned}$$

and recognize that

$$(v + v_t) \cdot \begin{pmatrix} 1 \\ X_{t+1} \end{pmatrix} \geq 0, \quad (v + v_t) \cdot \begin{pmatrix} 1 \\ Y_{t+1} \end{pmatrix} \geq 0.$$

Thus, we construct inductively

$$v_t \in \mathbf{L}^0(-K_t, \mathcal{F}_t), \ldots, v_{T-1} \in \mathbf{L}^0(-K_{T-1}, \mathcal{F}_{T-1})$$

such that

$$(v + v_t + \cdots + v_{T-1}) \cdot \begin{pmatrix} 1 \\ \underline{S}_T \end{pmatrix} \geq 0,\ (v + v_t + \cdots + v_{T-1}) \cdot \begin{pmatrix} 1 \\ \overline{S}_T \end{pmatrix} \geq 0,$$

i.e. $-v_T := v + v_t + \cdots + v_{T-1} \in \mathbf{L}^0(K_T, \mathcal{F}_T)$.

If we start with $v \in \mathbf{L}^0(-K_{t-1}, \mathcal{F}_{t-1})$, we get from Lemma 1.15 that

$$\begin{aligned} & v \in \mathbf{L}^0(-K_{t-1} \cap K_{t-1}, \mathcal{F}_{t-1}), v_t \in \mathbf{L}^0(-K_t \cap K_t, \mathcal{F}_t), \ldots, \\ & v_T \in \mathbf{L}^0(-K_T \cap K_T, \mathcal{F}_T). \end{aligned}$$

By construction we have

$$\sum_{r=s}^{T} v_r \text{ is } \mathcal{F}_{s-1}\text{-measurable for every } s = t, \ldots, T.$$

By the proposition it develops

$$-v \cdot \begin{pmatrix} 1 \\ X_t \end{pmatrix} \geq 0,\ -v \cdot \begin{pmatrix} 1 \\ Y_t \end{pmatrix} \geq 0,$$

hence $-v \in \mathcal{N}$. □

Chapter 3

Multidimensional case

In this chapter we consider a market with d risky assets and one risk-free asset which serves as a numeraire. We show that the idea from Chapter 2 carries over to multidimensional models. First, we consider a generic single-period model in Section 3.1. Afterwards, we show in Section 3.2 how the idea applies to a multi-period model. The single-period case requires a new approach in the multidimensional setting whereas the multi-period case is more or less a straightforward generalization of the one-dimensional setting, only more demanding with regard to technical questions.

3.1 Utility maximization in a single-period model

Let $(\Omega, \mathcal{F}, (\mathcal{F}_0, \mathcal{F}_1), \mathbf{P})$ be a filtered probability space. The consistent prices in units of the numeraire for the d risky assets are given by non-empty convex and compact random sets $\mathcal{S}_0, \mathcal{S}_1 \subset (0,\infty)^d$ where $\mathcal{S}_0$ is $\mathcal{F}_0$-measurable and $\mathcal{S}_1$ is $\mathcal{F}_1$-measurable. If bid and ask prices in units of the numeraire were given, e.g. $\underline{S}_t^i$ and $\overline{S}_t^i$, then we would have $\mathcal{S}_t = [\underline{S}_t^1, \overline{S}_t^1] \times \cdots \times [\underline{S}_t^d, \overline{S}_t^d]$. But to adapt for later purposes in the multi-period case we need $\mathcal{S}_1$ to be an arbitrary convex set. And since our arguments do not rely on concrete bid and ask prices at time 0, we can assume that $\mathcal{S}_0$ is an arbitrary convex set.

3.1.1 Preliminaries

Definition 3.1. a) The *solvency cone* at time t, denoted by $K(\mathcal{S}_t)$, is defined as

$$K(\mathcal{S}_t) := \left\{ v \in \mathbb{R}^{d+1} \, : \, v \cdot \left(\begin{smallmatrix} 1 \\ y \end{smallmatrix}\right) \geq 0, \quad \text{for } y \in \mathcal{S}_t \right\}.$$

$-K(\mathcal{S}_t)$ is called the *cone of portfolios available at price zero.*

b) The *liquidation value* (corresponding to $\mathcal{S}_1$) in units of the numeraire at time 1 of a portfolio v is defined as

$$l^{\mathcal{S}_1}(v) := \min\left\{v \cdot \left(\begin{smallmatrix}1\\y\end{smallmatrix}\right) \;:\; y \in \mathcal{S}_1\right\}$$

Similarly to the one-dimensional case we make the following assumption about the portfolios which are available at price zero today and solvent tomorrow.

Assumption 3.2. *The set*

$$\mathcal{N} := \mathbf{L}^0\left(-K(\mathcal{S}_0), \mathcal{F}_0\right) \cap \mathbf{L}^0\left(K(\mathcal{S}_1), \mathcal{F}_1\right)$$

is a subspace of $\mathbf{L}^0\left(\mathbb{R}^{d+1}, \mathcal{F}_0\right)$.

Obviously, we can write

$$\mathcal{N} = \left\{v \in \mathbf{L}^0\left(-K(\mathcal{S}_0) \cap K(\mathcal{S}_0), \mathcal{F}_0\right) \;:\; v \cdot \left(\begin{smallmatrix}1\\y\end{smallmatrix}\right) = 0, \quad \text{for } y \in \mathcal{S}_1\right\}$$

and we have $fv \in \mathcal{N}$ whenever $v \in \mathcal{N}$ and $f \in \mathbf{L}^0(\mathbb{R}, \mathcal{F}_0)$. Since $K(\mathcal{S}_0)$ and $K(\mathcal{S}_1)$ are closed, $\mathcal{N}$ is closed with respect to convergence in probability.

We will apply the following measure theoretic result from the book of Delbaen, Schachermayer [3] to parameterize the portfolios in $\mathcal{N}$.

Lemma 3.3. *Let $E \subset \mathbf{L}^0\left(\Omega, \mathcal{F}; \mathbb{R}^{d+1}\right)$ be a subspace which is closed with respect to convergence in probability. We suppose that E satisfies the following stability property: If $f \in E$ and h is real-valued and $\mathcal{F}$-measurable, then $hf \in E$.*

Under these assumptions there exists an $\mathcal{F}$-measurable mapping P_0 taking values in the orthogonal projections in $\mathbb{R}^{d+1}$, such that $f \in E$ if and only if $P_0 f = f$.

For the rest of this section P_0 shall always denote the $\mathcal{F}_0$-measurable projection-valued mapping corresponding to $\mathcal{N}$.

Compared to the one-dimensional case we will proceed differently. We first show directly that the market corresponding to $\mathcal{S}_0$ and $\mathcal{S}_1$ satisfies the robust no-arbitrage condition when Assumption 3.2 is valid. So we will have to show that the transaction costs can be slightly reduced, i.e. we can find convex and compact $\mathcal{W}_t \subset \operatorname{ri}\mathcal{S}_t$ such that the market corresponding to $\mathcal{W}_0$ and $\mathcal{W}_1$ is arbitrage free. This reduction is based on the next lemma which is adapted from the frictionless setting in [16] to our model with transaction costs. It entails a robustness property for the liquidation value of all normed portfolios which are 'orthogonal' to $\mathcal{N}$.

Lemma 3.4. *Under Assumption 3.2 there exists a strictly positive $\mathcal{F}_0$-measurable γ_0 such that*

$$\mathbf{P}\left(l^{\mathcal{S}_1}(v) < -\gamma_0 | \mathcal{F}_0\right) > 0$$

for every $v \in \mathbf{L}^0\left(-K\left(\mathcal{S}_0\right), \mathcal{F}_0\right)$ which satisfies $P_0 v = 0$ and $|v| = 1$.

Proof. We define $I := \left\{v \in \mathbf{L}^0\left(-K\left(\mathcal{S}_0\right), \mathcal{F}_0\right) \ : \ |v| = 1, P_0 v = 0\right\}$ and observe that for any $v_1, v_2 \in I$ there is $w \in I$ such that

$$\mathbf{P}\left(l^{\mathcal{S}_1}(w) < -1/n|\mathcal{F}_0\right) \leq \min\left\{\mathbf{P}\left(l^{\mathcal{S}_1}(v_1) < -1/n|\mathcal{F}_0\right), \mathbf{P}\left(l^{\mathcal{S}_1}(v_2) < -1/n|\mathcal{F}_0\right)\right\}.$$

So we can find a sequence $(v_k^n)_{k=1}^{\infty}$ in I such that

$$\lim_{k\to\infty} \mathbf{P}\left(l^{\mathcal{S}_1}\left(v_k^n\right) < -1/n|\mathcal{F}_0\right) = \operatorname{ess\,inf}\left\{\mathbf{P}\left(l^{\mathcal{S}_1}(v) < -1/n|\mathcal{F}_0\right) \ : \ v \in I\right\}.$$

As $|v_k^n| = 1$, by Proposition A.1 in the Appendix we can find an $\mathcal{F}_0$-measurable, strictly increasing sequence in $\mathbb{N}$, $(\tau_k)_{k=1}^{\infty}$, such that the random subsequence $(\tilde{v}_k^n)_{k=1}^{\infty} := \left(v_{\tau_k}^n\right)_{k=1}^{\infty}$ converges towards a v^n. Obviously $v^n \in \mathbf{L}^0\left(-K\left(\mathcal{S}_0\right), \mathcal{F}_0\right)$, $|v^n| = 1$ and $P_0 v^n = 0$. By Fatou's Lemma it follows that v^n attains the 'ess inf'

$$\begin{aligned}
\mathbf{P}\left(l^{\mathcal{S}_1}(v^n) < -1/n|\mathcal{F}_0\right) &= \mathbf{E}\left[\mathbf{1}_{(-\infty,-1/n)}(l^{\mathcal{S}_1}(v^n))|\mathcal{F}_0\right] \\
&\leq \mathbf{E}\left[\varliminf_{k\to\infty} \mathbf{1}_{(-\infty,-1/n)}(l^{\mathcal{S}_1}(\tilde{v}_k^n))\right] \\
&\leq \varliminf_{k\to\infty} \mathbf{P}\left(l^{\mathcal{S}_1}(\tilde{v}_k^n) < -1/n|\mathcal{F}_0\right) \\
&= \lim_{k\to\infty} \mathbf{P}\left(l^{\mathcal{S}_1}(v_k^n) < -1/n|\mathcal{F}_0\right) \\
&= \operatorname{ess\,inf}\left\{\mathbf{P}\left(l^{\mathcal{S}_1}(v) < -1/n|\mathcal{F}_0\right) \ : \ v \in I\right\}.
\end{aligned}$$

Again we extract a random subsequence $(\tilde{v}^n)_{n=1}^{\infty}$ from $(v^n)_{n=1}^{\infty}$ which converges towards a $v^* \in I$. We set

$$\begin{aligned}
A_n &:= \left\{\operatorname{ess\,inf}\left\{\mathbf{P}\left(l^{\mathcal{S}_1}(v) < -1/n|\mathcal{F}_0\right) \ : \ v \in I\right\} = 0\right\} \\
&= \left\{\mathbf{P}\left(l^{\mathcal{S}_1}(v^n) < -1/n|\mathcal{F}_0\right) = 0\right\}
\end{aligned}$$

and $A := \bigcap_{n=1}^{\infty} A_n$, so that $\mathbf{P}\left(l^{\mathcal{S}_1}(\tilde{v}^n) < -1/n|\mathcal{F}_0\right) = 0$ on A. By Fatou's Lemma we deduce

$$\mathbf{P}\left(l^{\mathcal{S}_1}(v^*) < 0|\mathcal{F}_0\right) \leq \varliminf_{n\to\infty} \mathbf{P}\left(l^{\mathcal{S}_1}(\tilde{v}^n) < -1/n|\mathcal{F}_0\right),$$

hence $\mathbf{P}\left(l^{\mathcal{S}_1}(\mathbf{1}_A v^*) \geq 0 | \mathcal{F}_0\right) = 1$, i.e. $\mathbf{1}_A v^* \in \mathcal{N}$ and $\mathbf{1}_A v^* = P_0\left(\mathbf{1}_A v^*\right) = 0$. It follows that A is a null set. Note that $A_{n+1} \subset A_n$, so that

$$\gamma_0 := \mathbf{1}_{A_1^c} + \sum_{n=2}^{\infty} \frac{1}{n} \mathbf{1}_{A_n^c \setminus A_{n-1}^c}$$

is strictly positive and satisfies by construction

$$\mathbf{P}\left(l^{\mathcal{S}_1}(v) < -\gamma_0 | \mathcal{F}_0\right) > 0$$

for every $v \in I$. □

Now, we slightly reduce the transaction costs.

Lemma 3.5. *There exists an $\mathcal{F}_0$-measurable, non-empty, convex and compact random set $\mathcal{W}_0 \subset \operatorname{ri} \mathcal{S}_0$ and an $\mathcal{F}_1$-measurable, non-empty, convex and compact random set $\mathcal{W}_1 \subset \operatorname{ri} \mathcal{S}_1$ such that*

$$\mathbf{P}\left(l^{\mathcal{W}_1}(v) < 0 | \mathcal{F}_0\right) > 0 \quad \text{on } \{v \neq 0\}$$

for every $v \in \mathbf{L}^0\left(-K(\mathcal{W}_0), \mathcal{F}_0\right)$, $P_0 v = 0$.

In particular, $\mathcal{W}_0, \mathcal{W}_1$ satisfy the no-arbitrage condition.

Proof. For $t = 0, 1$ we pick an $\mathcal{F}_t$-measurable $y_t \in \operatorname{ri} \mathcal{S}_t$; see Example A.7 c) in the Appendix for the existence of such selectors. Then, for $\alpha_t \in (0,1)$ $\mathcal{W}^{\alpha_t} := (1-\alpha_t) y_t + \alpha_t \mathcal{S}_t \subset \operatorname{ri} \mathcal{S}_t$. The idea is to find α_t near 1 such that the liquidatation value $l^{\mathcal{W}^{\alpha_1}}$ is near $l^{\mathcal{S}_1}$ in combination with the previous lemma. Clearly, $l^{\mathcal{W}_{\alpha_1}}(v) = (1-\alpha_1) v \cdot \binom{1}{y_1} + \alpha l^{\mathcal{S}_1}(v)$ i.e.

$$l^{\mathcal{W}_{\alpha_1}}(v) - l^{\mathcal{S}_1}(v) = (1-\alpha_1)\left(v \cdot \binom{1}{y_1} - l^{\mathcal{S}_1}(v)\right).$$

For $|v| = 1$ we estimate with Cauchy-Schwartz inequality

$$\left|v \cdot \binom{1}{y_1} - l^{\mathcal{S}_1}(v)\right| \leq 2 \max\left\{\left|\binom{1}{y}\right| \,:\, y \in \mathcal{S}_1\right\}.$$

Now, we fix $\alpha_1 \in (0,1)$ such that $2(1-\alpha_1) \max\left\{\left|\binom{1}{y}\right| \,:\, y \in \mathcal{S}_1\right\} < \gamma_0/2$. It follows with Lemma 3.4 that

$$\mathbf{P}\left(l^{\mathcal{W}_{\alpha_1}}(v) < -\gamma_0/2 | \mathcal{F}_0\right) \geq \mathbf{P}\left(l^{\mathcal{S}_1}(v) < -\gamma_0 | \mathcal{F}_0\right) > 0$$

for every $v \in \mathbf{L}^0\left(-K\left(\mathcal{S}_0\right), \mathcal{F}_0\right)$ which satisfies $P_0 v = 0$ and $|v| = 1$.

Now, we adapt α_0. Let us take an $\mathcal{F}_0$-measurable $v \in -K\left(\mathcal{W}_{\alpha_0}\right)$. We decompose v into $v = (\operatorname{id} - P_0)\, v + P_0 v$. Since P_0 maps into

$$-K(\mathcal{S}_0) \cap K(\mathcal{S}_0) \subset -K(\mathcal{W}_{\alpha_0}) \cap K(\mathcal{W}_{\alpha_0}),$$

$(P_0 v) \cdot \left(\begin{smallmatrix} 1 \\ z \end{smallmatrix}\right) = 0$ for all $z \in \mathcal{W}_{\alpha_1} (\subset \mathcal{S}_1)$ and $l^{\mathcal{W}_{\alpha_1}}(v) = l^{\mathcal{W}_{\alpha_1}}((\mathrm{id} - P_0)v)$ we can assume that $v = (\mathrm{id} - P_0)v$. We normalize, i.e. $|v| = 1$ on $\{v \neq 0\}$.
$v \cdot \left(\begin{smallmatrix} 1 \\ z \end{smallmatrix}\right) \leq 0$ for every $z \in \mathcal{W}_{\alpha_0}$ implies

$$v \cdot \left(\begin{smallmatrix} 1 \\ y \end{smallmatrix}\right) \leq -(1/\alpha_0 - 1) v \cdot \left(\begin{smallmatrix} 1 \\ y_0 \end{smallmatrix}\right) \leq (1/\alpha_0 - 1) \left|\left(\begin{smallmatrix} 1 \\ y_0 \end{smallmatrix}\right)\right|$$

for every $y \in \mathcal{S}_0$. We put $c(\alpha_0) := \left|\left(\begin{smallmatrix} 1 \\ y_0 \end{smallmatrix}\right)\right| (1/\alpha_0 - 1)$ then $v - c(\alpha_0)e^0 \in -K(\mathcal{S}_0)$. Note that $c(\alpha_0) \downarrow 0$ as $\alpha_0 \uparrow 1$.

We have $(\mathrm{id} - P_0)(v - c(\alpha_0)e^0) \neq 0$, otherwise $v - c(\alpha_0)e^0 = P_0(v - c(\alpha_0)e^0)$, which would imply $(v - c(\alpha_0)e^0) \cdot \left(\begin{smallmatrix} 1 \\ z \end{smallmatrix}\right) = 0$ for all $z \in \mathcal{S}_0$, especially $v \cdot \left(\begin{smallmatrix} 1 \\ z \end{smallmatrix}\right) = c(\alpha_0) > 0$ for all $z \in \mathcal{W}_{\alpha_0}$.

Now

$$l^{\mathcal{W}_{\alpha_1}} \left(\frac{v - c(\alpha_0)e^0}{|(\mathrm{id} - P_0)(v - c(\alpha_0)e^0)|} \right) = l^{\mathcal{W}_{\alpha_1}} \left(\frac{(\mathrm{id} - P_0)(v - c(\alpha_0)e^0)}{|(\mathrm{id} - P_0)(v - c(\alpha_0)e^0)|} \right) < -\gamma_0/2$$

implies on $\{v \neq 0\}$

$$\begin{aligned} l^{\mathcal{W}_{\alpha_1}}(v) &< -\gamma_0/2 \left|(\mathrm{id} - P_0)(v - c(\alpha_0)e^0)\right| + c(\alpha_0) \\ &\leq -\gamma_0/2 \left(|(\mathrm{id} - P_0)v| - \left|(\mathrm{id} - P_0)c(\alpha_0)e^0\right|\right) + c(\alpha_0) \\ &= -\gamma_0/2 \left(|v| - \left|(\mathrm{id} - P_0)c(\alpha_0)e^0\right|\right) + c(\alpha_0) \\ &= -\gamma_0/2 \left(1 - \left|(\mathrm{id} - P_0)c(\alpha_0)e^0\right|\right) + c(\alpha_0). \end{aligned}$$

We fix α_0 ($\mathcal{F}_0$-measurable) near 1 such that the last expression is strictly negative. It follows

$$\mathbf{P}\left(l^{\mathcal{W}_{\alpha_1}}(v) < 0 | \mathcal{F}_0\right) > 0 \quad \text{on } \{v \neq 0\}$$

for every $\mathcal{F}_0$-measurable $v \in -K(\mathcal{W}_{\alpha_0})$ with $P_0 v = 0$. Thus, we have shown that $\mathcal{W}_{\alpha_0}, \mathcal{W}_{\alpha_1}$ satisfy the no-arbitrage condition. We put

$$\mathcal{W}_0 := \mathcal{W}_{\alpha_0} \quad \text{and} \quad \mathcal{W}_1 := \mathcal{W}_{\alpha_1}$$

and the proof is complete. $\square$

For the rest of this section we focus on $\mathcal{W}_0$ and $\mathcal{W}_1$.

As in the one-dimensional case we fix a utility function $u : \mathbb{R} \to \mathbb{R}$, i.e. u is strictly concave, strictly increasing, bounded from above and continuously differentiable. We wish to find a portfolio $\hat{v} \in \mathbf{L}^0(-K(\mathcal{W}_0), \mathcal{F}_0)$ which maximizes all expected utilities from terminal wealth

$$\mathbf{E}\left[u\left(l^{\mathcal{W}_1}(v)\right) | \mathcal{F}_0\right], \quad v \in \mathbf{L}^0(-K(\mathcal{W}_0), \mathcal{F}_0).$$

Again, we first have to clarify some integrability and regularity questions.

Remark 3.6. As in the one-dimensional case we use, by Proposition 2.7, a strictly positive, continuous and decreasing function g which satisfies

$$|u(a\cdot x)|\, g(|x|) \leq \frac{1}{g(|a|)} + c$$

for every $a, x \in \mathbb{R}^{d+1}$ where c is a fixed constant. Then, since g is decreasing,

$$\left|u\left(\inf\left\{a\cdot x_n \,:\, n\geq 1\right\}\right)\right| g\left(\sup\left\{|x_n| \,:\, n\geq 1\right\}\right) \leq \frac{1}{g(|a|)} + c$$

for every bounded sequence $(x_n)_{n=1}^{\infty}$ in $\mathbb{R}^{d+1}$. This inequality still holds for every unbounded sequence $(x_n)_{n=1}^{\infty}$ with the understanding that $g(\infty) := 0$ and $\infty \cdot 0 := 0$.

We fix a Castaing representation for $\mathcal{W}_1$, i.e. a sequence $(w_n)_{n=1}^{\infty}$ of $\mathcal{F}_1$-measurable random variables such that $\mathcal{W}_1(\omega) = \overline{\left\{w_n(\omega) \,:\, n\geq 1\right\}}$ for every $\omega\in\Omega$. We define an equivalent probability measure $\mathbf{R}$ by

$$\frac{\mathrm{d}\mathbf{R}}{\mathrm{d}\mathbf{P}} := \frac{g\left(\sup\left\{\left|\left(\begin{smallmatrix}1\\ w_n\end{smallmatrix}\right)\right| \,:\, n\geq 1\right\}\right)}{\mathbf{E}_{\mathbf{P}}\left[g\left(\sup\left\{\left|\left(\begin{smallmatrix}1\\ w_n\end{smallmatrix}\right)\right| \,:\, n\geq 1\right\}\right) |\mathcal{F}_0\right]}$$

and use $\mathbf{k}$, a regular conditional $\mathbf{P}$-distribution for $(w_n)_{n=1}^{\infty}$ given $\mathcal{F}_0$, to see that for every $v \in \mathbf{L}^0\left(\mathbb{R}^{d+1}, \mathcal{F}_0\right)$

$$\begin{aligned}
\mathbf{E}_{\mathbf{R}}\left[\left|u\left(l^{\mathcal{W}_1}(v)\right)\right| |\mathcal{F}_0\right](\cdot) =& \mathbf{E}_{\mathbf{P}}\left[\left|u\left(l^{\mathcal{W}_1}(v)\right)\right| \frac{d\mathbf{R}}{d\mathbf{P}}\middle|\mathcal{F}_0\right](\cdot) \\
=& h(\cdot)\int \left|u\left(\inf\left\{v(\cdot)\cdot\left(\begin{smallmatrix}1\\ x_n\end{smallmatrix}\right) \,:\, n\geq 1\right\}\right)\right| \\
& \times g\left(\sup\left\{\left|\left(\begin{smallmatrix}1\\ x_n\end{smallmatrix}\right)\right| \,:\, n\geq 1\right\}\right) \mathbf{k}(\cdot, d(x_n)) \\
\leq& h(\cdot)\left(\frac{1}{g(|v(\cdot)|)} + c\right)
\end{aligned}$$

with $h = \mathbf{E}_{\mathbf{P}}\left[g\left(\sup\left\{\left|\left(\begin{smallmatrix}1\\ w_n\end{smallmatrix}\right)\right| \,:\, n\geq 1\right\}\right) |\mathcal{F}_0\right]^{-1}$. Again we can conclude that

$$\mathbf{E}\left[\left|u\left(l^{\mathcal{W}_1}(v)\right)\right| |\mathcal{F}_0\right] < \infty, \quad \mathbf{E}\left[u'\left(l^{\mathcal{W}_1}(v)\right) |\mathcal{F}_0\right] < \infty$$

and there are versions of $\mathbf{E}_{\mathbf{R}}\left[u\left(l^{\mathcal{W}_1}(a)\right) |\mathcal{F}_0\right]$, $a\in\mathbb{R}^{d+1}$, such that for every $\omega\in\Omega$

$$\mathbb{R}^{d+1} \ni a \mapsto \mathbf{E}_{\mathbf{R}}\left[u\left(l^{\mathcal{W}_1}(a)\right) |\mathcal{F}_0\right](\omega)$$

is finite and continuous. Switching to the equivalent probability measure $\mathbf{R}$ preserves Assumption 3.2, so we can assume that all these properties already hold under $\mathbf{P}$.

For the same reason we can assume that every $\mathcal{F}_0$-measurable selector of $\mathcal{W}_0$ is integrable with resprect to $\mathbf{P}$. If not we can switch to an equivalent measure with density

$$\frac{\left(1+\sup\{f_n \,:\, n\geq 1\}\right)^{-1}}{\mathbf{E}_{\mathbf{P}}\left[\left(1+\sup\{f_n \,:\, n\geq 1\}\right)^{-1}\right]}$$

where $(f_n)_{n=1}^{\infty}$ is a Castaing representation of $\mathcal{W}_0$.

3.1.2 Main results

Theorem 3.7. *There exists a portfolio $\hat{v}\in\mathbf{L}^0\left(-K\left(\mathcal{W}_0\right),\mathcal{F}_0\right)$ such that*

a)

$$\mathbf{E}\left[u\left(l^{\mathcal{W}_1}\left(\hat{v}\right)\right)|\mathcal{F}_0\right]=\operatorname*{ess\,sup}_{v\in\mathbf{L}^0\left(-K\left(\mathcal{W}_0\right),\mathcal{F}_0\right)}\mathbf{E}\left[u\left(l^{\mathcal{W}_1}\left(v\right)\right)|\mathcal{F}_0\right]$$

b)

$$\mathbf{E}_{\mathbf{Q}}\left[l^{\mathcal{W}_1}\left(v\right)|\mathcal{F}_0\right]\leq 0$$

for every $v\in\mathbf{L}^0\left(-K\left(\mathcal{W}_0\right),\mathcal{F}_0\right)$, where $\mathbf{Q}$ is defined by

$$\frac{\mathrm{d}\mathbf{Q}}{\mathrm{d}\mathbf{P}}:=\frac{u'\left(l^{\mathcal{W}_1}\left(\hat{v}\right)\right)}{\mathbf{E}\left[u'\left(l^{\mathcal{W}_1}\left(\hat{v}\right)\right)|\mathcal{F}_0\right]}.$$

Proof. a) For every $v\in\mathbf{L}^0\left(\mathbb{R}^{d+1},\mathcal{F}_0\right)$ we have $P_0v\in\mathcal{N}$, which means that $P_0v\in -K\left(\mathcal{S}_0\right)\cap K\left(\mathcal{S}_0\right)\subset K\left(\mathcal{W}_0\right)$ and $P_0v\cdot\binom{1}{y}=0$ for all $y\in\mathcal{S}_1$. As $W_1\subset\operatorname{ri}\mathcal{S}_1$ this implies especially $P_0v\cdot\binom{1}{w}=0$ for all $w\in\mathcal{W}_1$. Now we can copy almost everything from the proof in the one-dimensional case. If $v\in\mathbf{L}^0\left(-K\left(\mathcal{W}_0\right),\mathcal{F}_0\right)$ then $v-P_0v\in\mathbf{L}^0\left(-K\left(\mathcal{W}_0\right),\mathcal{F}_0\right)$ and $l^{\mathcal{W}_1}\left(v-P_0v\right)=l^{\mathcal{W}_1}\left(v\right)$. So it is enough to consider portfolios v which satisfy $P_0v=0$. This is why we define $C:=\left(\mathrm{id}-P_0\right)\left(-K\left(\mathcal{W}_0\right)\right)$ and

$$\varphi\left(\omega,a\right):=\begin{cases}\mathbf{E}\left[u\left(l^{\mathcal{W}_1}\left(a\right)\right)|\mathcal{F}_0\right]\left(\omega\right) & ,\quad a\in C(\omega)\\ -\infty & ,\quad a\notin C(\omega)\end{cases}.$$

For almost every $\omega\in\Omega$ the set

$$F(\omega):=\left\{a\in\mathbb{R}^{d+1}\,:\,|a|=1,\lim_{s\to\infty}\varphi(\omega,sa)>-\infty\right\}$$

must be empty, otherwise there is a selector α, $\mathbf{P}\left(\alpha\neq 0\right)>0$ and $\alpha=0$ on $\{F=\emptyset\}$. Clearly we have $\alpha\in C$, i.e. $P_0\alpha=0$ and thus by Lemma 3.5

$$\mathbf{P}\left(l^{\mathcal{W}_1}\left(\alpha\right)<0\right)>0\quad\text{on}\ \left\{\alpha\neq 0\right\}.$$

On the event $\left\{l^{\mathcal{W}_1}(\alpha) < 0\right\}$ we have $\lim_{s\to\infty} u\left(l^{\mathcal{W}_1}(s\alpha)\right) = -\infty$. Fatou's lemma implies

$$\lim_{s\to\infty} \mathbf{E}\left[u\left(l^{\mathcal{W}_1}(\alpha)\right)|\mathcal{F}_0\right] \leq \mathbf{E}\left[\overline{\lim_{s\to\infty}} u\left(l^{\mathcal{W}_1}(s\alpha)\right)|\mathcal{F}_0\right]$$

so that with positive probabilty we get

$$\lim_{s\to\infty} \varphi(\omega, s\alpha(\omega)) = -\infty.$$

This is a contradiction to $\alpha(\omega) \in F(\omega)$, when $\alpha(\omega) \neq 0$. Thus $F = \emptyset$ and by Lemma **??** there is an $\mathcal{F}_0$-measurable maximizer $\hat{v}$, i.e. $\hat{v} \in \mathbf{L}^0(-K(\mathcal{W}_0), \mathcal{F}_0)$, $P_0\hat{v} = 0$ and

$$\mathbf{E}\left[u\left(l^{\mathcal{W}_1}(\hat{v})\right)|\mathcal{F}_0\right] = \operatorname*{ess\,sup}_{v\in\mathbf{L}^0(-K(\mathcal{W}_0),\mathcal{F}_0)} \mathbf{E}\left[u\left(l^{\mathcal{W}_1}(v)\right)|\mathcal{F}_0\right].$$

b) We compare the maximiser $\hat{v}$ with any other portfolio of the form $\hat{v}+hv$, $v \in \mathbf{L}^0(-K(\mathcal{W}_0), \mathcal{F}_0)$, $h > 0$, and apply the monotone convergence theorem:

$$\begin{aligned}
0 &\geq \lim_{h\downarrow 0} \frac{\mathbf{E}\left[u\left(l^{\mathcal{W}_1}(\hat{v}+hv)\right)|\mathcal{F}_0\right] - \mathbf{E}\left[u\left(l^{\mathcal{W}_1}(\hat{v})\right)|\mathcal{F}_0\right]}{h} \\
&= \mathbf{E}\left[\lim_{h\downarrow 0} \frac{u\left(l^{\mathcal{W}_1}(\hat{v}+hv)\right) - u\left(l^{\mathcal{W}_1}(\hat{v})\right)}{h}\middle|\mathcal{F}_0\right] \\
&\geq \mathbf{E}\left[\lim_{h\downarrow 0} \frac{u\left(l^{\mathcal{W}_1}(\hat{v}) + hl^{\mathcal{W}_1}(v)\right) - u\left(l^{\mathcal{W}_1}(\hat{v})\right)}{h}\middle|\mathcal{F}_0\right] \\
&= \mathbf{E}\left[u'\left(l^{\mathcal{W}_1}(\hat{v})\right) l^{\mathcal{W}_1}(v)|\mathcal{F}_0\right].
\end{aligned}$$

Bayes' formula yields $0 \geq \mathbf{E}_{\mathbf{Q}}\left[l^{\mathcal{W}_1}(v)|\mathcal{F}_0\right]$, where $\frac{\mathrm{d}\mathbf{Q}}{\mathrm{d}\mathbf{P}} := \frac{u'\left(l^{\mathcal{W}_1}(\hat{v})\right)}{\mathbf{E}\left[u'\left(l^{\mathcal{W}_1}(\hat{v})\right)|\mathcal{F}_0\right]}$.

□

It should be mentioned that in the proof of part b) the expectation after the second inequality never assumes the value $-\infty$. This is due to the following observation:

We already know that, for every $v \in \mathbf{L}^0(-K(\mathcal{W}_0), \mathcal{F}_0)$,

$$\mathbf{E}\left[\left|u\left(l^{\mathcal{W}_1}(v)\right)\right| |\mathcal{F}_0\right] < \infty.$$

From

$$u\left(l^{\mathcal{W}_1}(v) + l^{\mathcal{W}_1}(w)\right) \geq \begin{cases} u\left(2\min\left\{l^{\mathcal{W}_1}(v), l^{\mathcal{W}_1}(w)\right\}\right), l^{\mathcal{W}_1}(v) < 0, l^{\mathcal{W}_1}(w) < 0 \\ u\left(\min\left\{l^{\mathcal{W}_1}(v), l^{\mathcal{W}_1}(w)\right\}\right), l^{\mathcal{W}_1}(v) \geq 0 \text{ or } l^{\mathcal{W}_1}(w) \geq 0 \end{cases}$$

and

$\mathbf{E}\left[\left|u\left(\min\left\{l^{\mathcal{W}_1}(v), l^{\mathcal{W}_1}(w)\right\}\right)\right| |\mathcal{F}_0\right] \leq \mathbf{E}\left[\left|u\left(l^{\mathcal{W}_1}(v)\right)\right| + \left|u\left(l^{\mathcal{W}_1}(w)\right)\right| |\mathcal{F}_0\right] < \infty$

and the fact, that u is bounded from above, we conclude

$$\mathbf{E}\left[\left|u\left(l^{\mathcal{W}_1}(v) + l^{\mathcal{W}_1}(w)\right)\right| |\mathcal{F}_0\right] < \infty$$

for every $v, w \in \mathbf{L}^0\left(-K\left(\mathcal{W}_0\right), \mathcal{F}_0\right)$. Especially,

$$0 \geq \mathbf{E}\left[\frac{u\left(l^{\mathcal{W}_1}(\hat{v}) + h l^{\mathcal{W}_1}(v)\right) - u\left(l^{\mathcal{W}_1}(\hat{v})\right)}{h}\middle|\mathcal{F}_0\right] > -\infty$$

and $0 \geq \mathbf{E}_{\mathbf{Q}}\left[l^{\mathcal{W}_1}(v)|\mathcal{F}_0\right] > -\infty$ for every $v \in \mathbf{L}^0\left(-K\left(\mathcal{W}_0\right), \mathcal{F}_0\right)$ by monotone convergence. For $v = -e^i$ this yields

$$0 \geq \mathbf{E}_{\mathbf{Q}}\left[\min\left\{-e^i \cdot w \,:\, w \in \mathcal{W}_1\right\}|\mathcal{F}_0\right] > -\infty$$

and hence

$$\mathbf{E}_{\mathbf{Q}}\left[\max\left\{e^i \cdot w \,:\, w \in \mathcal{W}_1\right\}|\mathcal{F}_0\right] < \infty.$$

Especially, for every $z \in \mathbf{L}^0\left(\mathcal{W}_1, \mathcal{F}_1\right)$ we have $\mathbf{E}_{\mathbf{Q}}\left[|z|\,|\mathcal{F}_0\right] < \infty$.

It is not clear how Theorem 3.7 part b) should imply existence of a consistent price process, not even in the polyhedral case $\mathcal{W}_0 = [\underline{S}_0^1, \overline{S}_0^1] \times \cdots \times [\underline{S}_0^d, \overline{S}_0^d]$. If we plug in for v the portfolios $e^i - \overline{S}_0^i e^0$ and $-e^i + \underline{S}_0^i e^0$ we get the following inequalities

$$\mathbf{E}_{\mathbf{Q}}\left[\min\left\{e^i \cdot w \,:\, w \in W_1\right\}\right] \leq \overline{S}_0^i,$$
$$\mathbf{E}_{\mathbf{Q}}\left[\max\left\{e^i \cdot w \,:\, w \in W_1\right\}\right] \geq \underline{S}_0^i.$$

Without looking too much on technical details now, we write $\min\left\{e^i \cdot w \,:\, w \in \mathcal{W}_1\right\} = e^i \cdot m_i$ and $\max\left\{e^i \cdot w \,:\, w \in \mathcal{W}_1\right\} = e^i \cdot M_i$ for some appropriate $m_i, M_i \in \mathbf{L}^0\left(\mathcal{W}_1, \mathcal{F}_1\right)$. It follows from a 'sandwich' argument (see the proof of Theorem 2.9 part b)) that we can find a $w_i = \left(w_i^1, \ldots, w_i^d\right)$ on the line segment between m_i and M_i such that $\underline{S}_0^i \leq \mathbf{E}_{\mathbf{Q}}\left[w_i^i|\mathcal{F}_0\right] \leq \overline{S}_0^i$. So we have shown that, when we restrict to the i-th asset, we get a consistent price process. But it is by no means clear, why there is a consistent price process for all assets simultaneously.

So, here in the multidimensional case, we have to proceed differently and work harder to deduce from the first-order condition

$$\mathbf{E}_{\mathbf{Q}}\left[l^{\mathcal{W}_1}\left(v\right)|\mathcal{F}_0\right] \leq 0, \quad \text{for } v \in \mathbf{L}^0\left(-K\left(\mathcal{W}_0\right), \mathcal{F}_0\right),$$

that there is a consistent price process. The basic idea is in the next lemma which we will adapt to a random setting.

Lemma 3.8. *Let C_0 and C_1 be non-empty convex subsets of $\mathbb{R}^d$ such that C_0 is compact and C_1 is closed. Suppose that*

$$\inf\left\{v \cdot \left(\begin{smallmatrix}1\\ y\end{smallmatrix}\right) \,:\, y \in C_1\right\} \le 0$$

for every $v \in \mathbb{R}^{d+1}$ which satisfies $v \cdot \left(\begin{smallmatrix}1\\ x\end{smallmatrix}\right) \le 0$ for every $x \in C_0$. Then

$$C_0 \cap C_1 \neq \emptyset.$$

Proof. If $C_0 \cap C_1 = \emptyset$, we can separate them with a hyperplane, i.e. there are $h \in \mathbb{R}^d, c \in \mathbb{R}$ such that

$$\sup\left\{h \cdot x \,:\, x \in C_0\right\} < c < \inf\left\{h \cdot y \,:\, y \in C_1\right\}.$$

Especially $\left(\begin{smallmatrix}-c\\ h\end{smallmatrix}\right) \cdot \left(\begin{smallmatrix}1\\ x\end{smallmatrix}\right) \le 0$, for $x \in C_0$, so that by assumption

$$\inf\left\{\left(\begin{smallmatrix}-c\\ h\end{smallmatrix}\right) \cdot \left(\begin{smallmatrix}1\\ y\end{smallmatrix}\right) \,:\, y \in C_1\right\} \le 0$$

or equivalently

$$\inf\left\{h \cdot y \,:\, y \in C_1\right\} \le c$$

which is a contradiction. $\square$

To adapt this lemma to a random setting we show that the 'inf' coming from the liquidation function $l^{\mathcal{W}_1}$ commutates with the expectation operator $\mathbf{E}_{\mathbf{Q}}$.

Lemma 3.9. *For every $v \in \mathbf{L}^0\left(-K\left(\mathcal{W}_0\right), \mathcal{F}_0\right)$ we have*

$$\mathbf{E}_{\mathbf{Q}}\left[l^{\mathcal{W}_1}(v)|\mathcal{F}_0\right] = \operatorname{ess\,inf}\left\{v \cdot \left(\begin{smallmatrix}1\\ \mathbf{E}_{\mathbf{Q}}[w|\mathcal{F}_0]\end{smallmatrix}\right) \,:\, w \in \mathbf{L}^0\left(\mathcal{W}_1, \mathcal{F}_1\right)\right\}.$$

Proof. Clearly for every $w \in \mathbf{L}^0\left(\mathcal{W}_1, \mathcal{F}_1\right)$ $v \cdot \left(\begin{smallmatrix}1\\ w\end{smallmatrix}\right) \ge l^{\mathcal{W}_1}(v)$ so that

$$v \cdot \left(\begin{smallmatrix}1\\ \mathbf{E}_{\mathbf{Q}}[w|\mathcal{F}_0]\end{smallmatrix}\right) = \mathbf{E}_{\mathbf{Q}}\left[v \cdot \left(\begin{smallmatrix}1\\ w\end{smallmatrix}\right)|\mathcal{F}_0\right] \ge \mathbf{E}_{\mathbf{Q}}\left[l^{\mathcal{W}_1}(v)|\mathcal{F}_0\right].$$

Hence,

$$E_{\mathbf{Q}}\left[l^{\mathcal{W}_1}(v)|\mathcal{F}_0\right] \le \operatorname{ess\,inf}\left\{v \cdot \left(\begin{smallmatrix}1\\ \mathbf{E}_{\mathbf{Q}}[w|\mathcal{F}_0]\end{smallmatrix}\right) \,:\, w \in \mathbf{L}^0\left(\mathcal{W}_1, \mathcal{F}_1\right)\right\}.$$

To show the other inequality we observe that

$$\begin{aligned} l^{\mathcal{W}_1}(v) &= \min\left\{v \cdot \left(\begin{smallmatrix}1\\ w\end{smallmatrix}\right) \,:\, w \in \mathcal{W}_1\right\} = \inf\left\{v \cdot \left(\begin{smallmatrix}1\\ w_n\end{smallmatrix}\right) \,:\, n \ge 1\right\} \\ &= \operatorname{ess\,inf}\left\{v \cdot \left(\begin{smallmatrix}1\\ w\end{smallmatrix}\right) \,:\, w \in \mathbf{L}^0\left(\mathcal{W}_1, \mathcal{F}_1\right)\right\} \end{aligned}$$

where $(w_n)_{n=1}^{\infty}$ is a Castaing representation of $\mathcal{W}_1$. For any $z, y \in \mathbf{L}^0(\mathcal{W}_1, \mathcal{F}_1)$ there is a $u \in \mathbf{L}^0(\mathcal{W}_1, \mathcal{F}_1)$ such that

$$v \cdot \left(\begin{smallmatrix} 1 \\ u \end{smallmatrix}\right) \leq \min\left\{v \cdot \left(\begin{smallmatrix} 1 \\ z \end{smallmatrix}\right), v \cdot \left(\begin{smallmatrix} 1 \\ y \end{smallmatrix}\right)\right\}.$$

So, we can take a sequence $(u_n)_{n=1}^{\infty}$ in $\mathbf{L}^0(\mathcal{W}_1, \mathcal{F}_1)$ such that

$$v \cdot \left(\begin{smallmatrix} 1 \\ u_n \end{smallmatrix}\right) \underset{n\to\infty}{\longrightarrow} l^{\mathcal{W}_1}(v)$$

in a monotone decreasing way. The monotone convergence theorem implies

$$\mathbf{E_Q}\left[l^{\mathcal{W}_1}(v)|\mathcal{F}_0\right] = \mathbf{E_Q}\left[\lim_{n\to\infty} v \cdot \left(\begin{smallmatrix} 1 \\ u_n \end{smallmatrix}\right)|\mathcal{F}_0\right] = \lim_{n\to\infty} v \cdot \left(\begin{smallmatrix} 1 \\ \mathbf{E_Q}[u_n|\mathcal{F}_0] \end{smallmatrix}\right).$$

But, for every n we have

$$v \cdot \left(\begin{smallmatrix} 1 \\ \mathbf{E_Q}[u_n|\mathcal{F}_0] \end{smallmatrix}\right) \geq \operatorname{ess\,inf}\left\{v \cdot \left(\begin{smallmatrix} 1 \\ \mathbf{E_Q}[w|\mathcal{F}_0] \end{smallmatrix}\right) : w \in \mathbf{L}^0(\mathcal{W}_1, \mathcal{F}_1)\right\}.$$

It follows

$$\mathbf{E_Q}\left[l^{\mathcal{W}_1}(v)|\mathcal{F}_0\right] \geq \operatorname{ess\,inf}\left\{v \cdot \left(\begin{smallmatrix} 1 \\ \mathbf{E_Q}[w|\mathcal{F}_0] \end{smallmatrix}\right) : w \in \mathbf{L}^0(\mathcal{W}_1, \mathcal{F}_1)\right\}.$$

□

Before we continue with the main text, we should talk about an important tool namely conditional expectation of a random set. We follow the book of Molchanov [15] and the Appendix from the book of Kabanov, Safarian [12]. There the conditional expectation has been introduced with the help of selectors which are integrable in the ordinary sense. We will work here with selectors whose conditional expectation exists in the generalized sense, i.e. we will consider selectors f of a random set F with $\mathbf{E_P}[|f|\,|\mathcal{G}] < \infty$. But of course everything works mutatis mutandis as in [15].

Theorem 3.10. *Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space, $\mathcal{G} \subset \mathcal{F}$ a sub-σ-field and F an $\mathcal{F}$-measurable closed random set. Then there exists a unique $\mathcal{G}$-measurable closed random set, denoted by $\mathbf{E_P}[F|\mathcal{G}]$, such that*

$$\mathbf{L}^0(\mathbf{E_P}[F|\mathcal{G}], \mathcal{G}) = \overline{\left\{\mathbf{E_P}[f|\mathcal{G}] : f \in \mathbf{L}^0(F, \mathcal{F}), \mathbf{E_P}[|f|\,|\mathcal{G}] < \infty\right\}}$$

where the closure is taken with respect to convergence in probability.

Proof. A set $S \subset \mathbf{L}^0(\mathbb{R}^d, \mathcal{G})$ is called decomposable, if for any $f_1, f_2 \in S$ and $A \in \mathcal{G}$ we have $\mathbf{1}_A f_1 + \mathbf{1}_{A^c} f_2 \in S$.
Clearly $\left\{\mathbf{E_P}[f|\mathcal{G}] : f \in \mathbf{L}^0(F, \mathcal{F}), \mathbf{E_P}[|f|\,|\mathcal{G}] < \infty\right\}$ is decomposable and so is its closure. By Proposition A.9 from the Appendix there exists a unique $\mathcal{G}$-measurable random closed set with the desired property. □

Before we can apply Lemma 3.8 we show that the set

$$\left\{\mathbf{E}_{\mathbf{Q}}\left[w|\mathcal{F}_0\right] \,:\, w \in \mathbf{L}^0\left(\mathcal{W}_1, \mathcal{F}_1\right)\right\}$$

is closed.

Lemma 3.11. *The set*

$$\left\{\mathbf{E}_{\mathbf{Q}}\left[w|\mathcal{F}_0\right] \,:\, w \in \mathbf{L}^0\left(\mathcal{W}_1, \mathcal{F}_1\right)\right\}$$

is closed with respect to convergence in probability. $\mathbf{E}_{\mathbf{Q}}\left[\mathcal{W}_1|\mathcal{F}_0\right]$ *is compact and convex valued.*

Proof. Let $(z_n)_{n=1}^{\infty}$ be a sequence in $\mathbf{L}^0\left(\mathcal{W}_1, \mathcal{F}_1\right)$ such that $\mathbf{E}_{\mathbf{Q}}\left[z_n|\mathcal{F}_0\right]$ converges in probability to a random variable g. By passing to a subsequence we can assume that $\mathbf{E}_{\mathbf{Q}}\left[z_n|\mathcal{F}_0\right]$ converges almost surely pointwise to g. Each component z_n^i is non-negative and we can apply Proposition A.2 from the Appendix to yield for every n a $u_n \in \operatorname{conv}\left\{z_m \,:\, m \geq n\right\}$ so that the sequence $(u_n)_{n=1}^{\infty}$ converges almost surely to an $\eta \in \mathbf{L}^0\left([0,\infty]^d, \mathcal{F}_1\right)$. But $z_m \in \mathcal{W}_1$ and $\mathcal{W}_1$ is convex. Hence, $u_n \in \mathcal{W}_1$ and, as $\mathcal{W}_1$ is closed, we have $\eta \in \mathcal{W}_1$. Note that with $\mathbf{E}_{\mathbf{Q}}\left[z_n|\mathcal{F}_0\right] \to g$ we also have $\mathbf{E}_{\mathbf{Q}}\left[u_n|\mathcal{F}_0\right] \to g$.

Now, the random variable

$$M := \max\left\{z \cdot e^1 \,:\, z \in \mathcal{W}_1\right\}e^1 + \cdots + \max\left\{z \cdot e^d \,:\, z \in \mathcal{W}_1\right\}e^d$$

serves as a majorant and we can apply the dominated convergence theorem

$$\mathbf{E}_{\mathbf{Q}}\left[\eta|\mathcal{F}_0\right] = \mathbf{E}_{\mathbf{Q}}\left[\lim_{n\to\infty} u_n|\mathcal{F}_0\right] = \lim_{n\to\infty} \mathbf{E}_{\mathbf{Q}}\left[u_n|\mathcal{F}_0\right] \ = g.$$

Thus, the set

$$\left\{\mathbf{E}_{\mathbf{Q}}\left[w|\mathcal{F}_0\right] \,:\, w \in \mathbf{L}^0\left(\mathcal{W}_1, \mathcal{F}_1\right)\right\}$$

is closed with respect to convergence in probability. $\mathbf{E}_{\mathbf{Q}}\left[\mathcal{W}_1|\mathcal{F}_0\right]$ is also bounded because we have

$$\left|\mathbf{E}_{\mathbf{Q}}\left[w|\mathcal{F}_0\right]\right| \leq \mathbf{E}_{\mathbf{Q}}\left[|M|\,|\mathcal{F}_0\right]$$

for every $w \in \mathbf{L}^0\left(\mathcal{W}_1, \mathcal{F}_1\right)$. So it is compact and by Proposition A.10 from the Appendix also convex-valued. ☐

In Theorem 3.7 we have found an optimal portfolio $\hat{v}$ and the first-order condition implied that

$$\mathbf{E}_{\mathbf{Q}}\left[l^{\mathcal{W}_1}\left(v\right)|\mathcal{F}_0\right] \leq 0$$

for every $v \in \mathbf{L}^0\left(-K\left(\mathcal{W}_0\right), \mathcal{F}_0\right)$, where $\mathbf{Q}$ is defined by $\frac{\mathrm{d}\mathbf{Q}}{\mathrm{d}\mathbf{P}} := \frac{u'\left(l^{\mathcal{W}_1}(\hat{v})\right)}{\mathbf{E}\left[u'\left(l^{\mathcal{W}_1}(\hat{v})\right)|\mathcal{F}_0\right]}$. We are now in a position to show how this condition implies existence of a consistent price process.

Corollary 3.12. *Under Assumption 3.2 there exists a* $y_1 \in \mathbf{L}^0(\operatorname{ri}\mathcal{S}_1, \mathcal{F}_1)$ *such that*

$$\mathbf{E}_{\mathbf{Q}}[y_1|\mathcal{F}_0] \in \operatorname{ri}\mathcal{S}_0.$$

Hence, there exists a strictly consistent price process for $\mathcal{S}_0, \mathcal{S}_1$*.*

Proof. For every $v \in \mathbf{L}^0(-K(\mathcal{W}_0), \mathcal{F}_0)$ we have

$$\begin{aligned} 0 &\geq \mathbf{E}_{\mathbf{Q}}\left[l^{\mathcal{W}_1}(v)|\mathcal{F}_0\right] \\ &= \operatorname{ess\,inf}\left\{v \cdot \left(\begin{smallmatrix} 1 \\ \mathbf{E}_{\mathbf{Q}}[w|\mathcal{F}_0] \end{smallmatrix}\right) \,:\, w \in \mathbf{L}^0(\mathcal{W}_1, \mathcal{F}_1)\right\} \\ &= \inf\left\{v \cdot \left(\begin{smallmatrix} 1 \\ u \end{smallmatrix}\right) \,:\, u \in \mathbf{E}_{\mathbf{Q}}[W_1|\mathcal{F}_0]\right\}. \end{aligned}$$

Note that for a fixed $\omega \in \Omega$, since $\mathbf{E}_{\mathbf{Q}}[\mathcal{W}_1|\mathcal{F}_0](\omega)$ is compact, the mapping

$$\mathbb{R}^{d+1} \ni a \mapsto \inf\left\{a \cdot \left(\begin{smallmatrix} 1 \\ u \end{smallmatrix}\right) \,:\, u \in \mathbf{E}_{\mathbf{Q}}[\mathcal{W}_1|\mathcal{F}_0](\omega)\right\}$$

is continuous. Using a Castaing representation $(v_n)_{n=1}^{\infty}$ of $-K(\mathcal{W}_0)$ it follows that

$$0 \geq \inf\left\{v \cdot \left(\begin{smallmatrix} 1 \\ u \end{smallmatrix}\right) \,:\, u \in \mathbf{E}_{\mathbf{Q}}[W_1|\mathcal{F}_0]\right\}$$

for every $v \in -K(\mathcal{W}_0)$. So we can apply Lemma 3.8 and we obtain

$$\mathcal{W}_0 \cap \mathbf{E}_{\mathbf{Q}}[\mathcal{W}_1|\mathcal{F}_0] \neq \emptyset.$$

We pick an $\mathcal{F}_0$-measurable selection of this intersection which can be written as $\mathbf{E}_{\mathbf{Q}}[y_1|\mathcal{F}_0]$ for some $y_1 \in \mathbf{L}^0(\mathcal{W}_1, \mathcal{F}_1)$. Clearly $Z_1 := \frac{\mathrm{d}\mathbf{Q}}{\mathrm{d}\mathbf{P}}\left(\begin{smallmatrix} 1 \\ y_1 \end{smallmatrix}\right)$ and $Z_0 := \mathbf{E}[Z_1|\mathcal{F}_0]$ define a strictly consistent price process for $\mathcal{S}_0, \mathcal{S}_1$. □

A few words are in order to compare the multidimensional to the one-dimensional case.

Remark 3.13. a) We have started with $\mathcal{S}_0$ and $\mathcal{S}_1$ for which Assumption 3.2 holds. Then we have constructed a smaller 'bid-ask spread' $\mathcal{W}_0 \subset \operatorname{ri}\mathcal{S}_0$ and $\mathcal{W}_1 \subset \operatorname{ri}\mathcal{S}_1$ such that $\mathcal{W}_0$ and $\mathcal{W}_1$ are still free of arbitrage opportunities by Lemma 3.5. This allowed us to find an optimal portfolio $\hat{v}$. From the first-order condition corresponding to the optimal portfolio $\hat{v}$

$$\mathbf{E}\left[u'\left(l^{\mathcal{W}_1}(\hat{v})\right) l^{\mathcal{W}_1}(v)|\mathcal{F}_0\right] \leq 0, \quad \text{for } v \in \mathbf{L}^0(-K(\mathcal{W}_0), \mathcal{F}_0),$$

we could show that there is a consistent price process for $\mathcal{W}_0, \mathcal{W}_1$ which is strictly consistent for $\mathcal{S}_0, \mathcal{S}_1$.

b) In the one-dimensional case part a) of Corollary 2.11 was essential to get in part b)

$$\operatorname{ri}\left[\overline{S}_0, \underline{S}_0\right] \cap \operatorname{ri}\left[\inf_0 \underline{S}_1, \sup_0 \overline{S}_1\right] \neq \emptyset.$$

This condition appeared in the multi-period market $\underline{S}_0, \overline{S}_0, \ldots, \underline{S}_T, \overline{S}_T$ as

$$\operatorname{ri}\left[\underline{S}_t, \overline{S}_t\right] \cap \operatorname{ri}\left[\operatorname{ess\,inf}_t X_{t+1}, \operatorname{ess\,sup}_t Y_{t+1}\right] = \operatorname{ri}\left[X_t, Y_t\right]$$

for every $t = 0, \ldots, T-1$, $X_T = \underline{S}_T, Y_T = \overline{S}_T$, which is crucial for contructing a strictly consistent price process. Corollary 2.11 depends very much on the form of the projection-valued mapping P_0 (Lemma 2.6). Since an analogue of Corollary 2.11 in the multidimensional case is out of reach, we have to proceed differently here. This is why we first make the 'bid-ask spread' a little bit smaller. Then the consistent price process coming from the first-order condition already implies

$$\operatorname{ri} G\left(\mathcal{S}_0\right)^* \cap \operatorname{ri} \mathbf{E}\left[G\left(\mathcal{S}_1\right)^* | \mathcal{F}_0\right] \neq \emptyset.$$

In fact, we will prove this in the next section.

3.2 Multi-period model

We now apply the single-period results from the previous section to a multi-period financial market. As in the one-dimensional case we will apply an inductive argument similar to [17] and [23].

Given a filtered probability space $\left(\Omega, \mathcal{F}, \left(\mathcal{F}_t\right)_{t=0}^T, \mathbf{P}\right)$ we start with $\left(\mathcal{S}_t\right)_{t=0}^T$ where each $\mathcal{S}_t \subset (0, \infty)^d$ is a compact, convex and $\mathcal{F}_t$-measurable random set. We assume that $\left(\mathcal{S}_t\right)_{t=0}^T$ satisfies the robust no-arbitrage condition.

Analogously to the one-dimensional case we go backwards in time and adapt for every period $(t-1 \to t)$ the consistent prices at time t. For this pupose we denote by $\left(K_t\right)_{t=0}^T = \left(K\left(\mathcal{S}_t\right)\right)_{t=0}^T$ the solvency cones corresponding to $\left(\mathcal{S}_t\right)_{t=0}^T$ and define

$$\begin{aligned} H_T &:= K_T^* \\ H_t &:= K_t^* \cap \mathbf{E}\left[H_{t+1} | \mathcal{F}_t\right], \quad \text{for } t = 1, \ldots, T-1. \end{aligned}$$

By Proposition A.10 from the Appendix every H_t is convex and cone-valued.

As in the one-dimensional case the crucial point for the induction step is to verify Assumption 3.2.

Lemma 3.14. *Assume that the robust no-arbitrage condition is satisfied by $(\mathcal{S}_t)_{t=0}^T$. Then, for every $t = 1, \ldots, T$,*

$$\mathcal{N} := \left\{ v \in \mathbf{L}^0 \left(-K_{t-1}, \mathcal{F}_{t-1} \right) \, : \, v \cdot z \geq 0, \quad \textit{for } z \in H_t \right\}$$

is a subspace of $\mathbf{L}^0 \left(\mathbb{R}^{d+1}, \mathcal{F}_{t-1} \right)$ which is closed with respect to convergence in probability and stable under multiplication with every $f \in \mathbf{L}^0 (\mathbb{R}, \mathcal{F}_{t-1})$. Further

$$K_{t-1} + \mathbf{E} \left[H_t | \mathcal{F}_{t-1} \right]^*$$

is closed.

Proof by induction on t. Let be $t = T$ and $v \in \mathbf{L}^0 (-K_{T-1}, \mathcal{F}_{T-1})$ such that $v \cdot z \geq 0$ for every $z \in K_T^*$, i.e. $v \in K_T^{**} = K_T$. We set $v_T := -v$ and have $v + v_T = 0$ as well as $v \in -K_{T-1}$ and $v_T \in -K_T$. By Lemma 1.15 $v \in -K_{T-1} \cap K_{T-1}$ and $v_T \in -K_T \cap K_T$, especially $-v \in -K_{T-1} \cap K_T$. Hence, $\mathcal{N}$ is a vector space. Clearly, it is closed and stable under multiplication with every $f \in \mathbf{L}^0 (\mathbb{R}, \mathcal{F}_{T-1})$.

Further, if $z \in \mathbf{L}^0 (K_T^*, \mathcal{F}_T)$ such that $\mathbf{E} \left[|z| \, | \mathcal{F}_{T-1} \right] < \infty$ we get $v \cdot \mathbf{E} \left[z | \mathcal{F}_{T-1} \right] = \mathbf{E} \left[v \cdot z | \mathcal{F}_{T-1} \right] \geq 0$ and deduce that $v \in \mathbf{E} \left[K_T^*, \mathcal{F}_{T-1} \right]^*$. Conversely for any $\mathcal{F}_{T-1}$-measurable $v \in \mathbf{E} \left[K_T^*, \mathcal{F}_{T-1} \right]^*$ we have $0 \leq v \cdot \mathbf{E} \left[z | \mathcal{F}_{T-1} \right] = \mathbf{E} \left[v \cdot z | \mathcal{F}_{T-1} \right]$ for every such z. This implies $v \cdot z \geq 0$ hence $v \in K_T$. This allows us to write for $t = T$

$$\mathcal{N} = \mathbf{L}^0 \left(-K_{T-1} \cap \mathbf{E} \left[H_T | \mathcal{F}_{T-1} \right]^*, \mathcal{F}_{T-1} \right).$$

From Lemma 3.3 it follows that $-K_{T-1} \cap \mathbf{E} \left[H_T | \mathcal{F}_{T-1} \right]^*$ is an $\mathcal{F}_{T-1}$-measurable subspace and hence Proposition A.11 from the Appendix implies that $K_{T-1} + \mathbf{E} \left[H_T | \mathcal{F}_{T-1} \right]^*$ is closed.

Now, let be $t < T$. We take a $v \in \mathcal{N}$ and have to show that $-v \in \mathcal{N}$. From $v \cdot z \geq 0$ for every $z \in H_t$, it follows that $v \in \left(K_t^* \cap \mathbf{E} \left[H_{t+1} | \mathcal{F}_t \right] \right)^*$. By induction hypothesis $K_t + \mathbf{E} \left[H_{t+1} | \mathcal{F}_t \right]^*$ is closed, so that $\left(K_t^* \cap \mathbf{E} \left[H_{t+1} | \mathcal{F}_t \right] \right)^* = \overline{K_t + \mathbf{E} \left[H_{t+1} | \mathcal{F}_t \right]^*} = K_t + \mathbf{E} \left[H_{t+1} | \mathcal{F}_t \right]^*$. By Proposition A.12 from the Appendix we can write $v = w_t + r_t$ where $w_t \in \mathbf{L}^0 (K_t, \mathcal{F}_t), r_t \in \mathbf{L}^0 \left(\mathbf{E} \left[H_{t+1} | \mathcal{F}_t \right]^*, \mathcal{F}_t \right)$. Again we deduce that $r_t \cdot y \geq 0$ for every $y \in H_{t+1}$ and hence $r_t \in H_{t+1}^* = \left(K_{t+1}^* \cap \mathbf{E} \left[H_{t+2} | \mathcal{F}_{t+1} \right] \right)^* = K_{t+1} + \mathbf{E} \left[H_{t+2} | \mathcal{F}_{t+1} \right]^*$. Continuing in this manner we get $v = w_t + \cdots + w_T$ where $w_t \in \mathbf{L}^0 (K_t, \mathcal{F}_t), \ldots, w_T \in \mathbf{L}^0 (K_T, \mathcal{F}_T)$. From $v - w_t - \cdots - w_T = 0$ and Lemma 1.15 it follows that $v \in -K_{t-1} \cap K_{t-1}, w_t \in -K_t \cap K_t, \ldots, w_T \in -K_T \cap K_T$. Note that for any

$r \geq t$ $-w_r - w_{r+1} - \cdots - w_T$ is $\mathcal{F}_{r-1}$-measurable. This allows us to conclude

$$\begin{aligned}
-w_T \cdot z \geq 0, &\quad \text{for } z \in K_T^* (= H_T) \\
\Rightarrow -w_T \cdot z \geq 0, &\quad \text{for } z \in \mathbf{E}\left[H_T | \mathcal{F}_{T-1}\right], \\
\Rightarrow (-w_{T-1} - w_T) \cdot z \geq 0, &\quad \text{for } z \in K_{T-1}^* \cap \mathbf{E}\left[H_T | \mathcal{F}_{T-1}\right] (= H_{T-1}), \\
\Rightarrow (-w_{T-1} - w_T) \cdot z \geq 0, &\quad \text{for } z \in \mathbf{E}\left[H_{T-1} | \mathcal{F}_{T-2}\right], \\
&\quad \cdots \\
\Rightarrow (-w_t - \cdots - w_T) \cdot z \geq 0, &\quad \text{for } z \in K_t^* \cap \mathbf{E}\left[H_{t+1} | \mathcal{F}_t\right],
\end{aligned}$$

i.e. $-v \cdot z \geq 0,$ for $z \in H_t$, so that $-v \in \mathcal{N}$.

As above we can write

$$\mathcal{N} = \mathbf{L}^0 \left(-K_{t-1} \cap \mathbf{E}\left[H_t | \mathcal{F}_{t-1}\right]^*, \mathcal{F}_{t-1}\right)$$

and conclude from Lemma 3.3 that $-K_{t-1} \cap \mathbf{E}\left[H_t | \mathcal{F}_{t-1}\right]^*$ is an $\mathcal{F}_{t-1}$-measurable subspace. Proposition A.11 from the Appendix implies that $K_{t-1} + \mathbf{E}\left[H_t | \mathcal{F}_{t-1}\right]^*$ is closed. □

We need to adapt Lemma 2.2 to the multidimensional case.

Lemma 3.15. *Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probabilty space, $\mathcal{H} \subset \mathcal{F}$ a sub-σ-field and $F \subset \mathbb{R}^{d+1}$ a non-empty, closed, convex and conic random set. Then:*

a) For every $Z \in \mathbf{L}^0 (\operatorname{ri} F, \mathcal{F})$ such that $\mathbf{E}\left[|Z| \,|\mathcal{H}\right] < \infty$ we have $\mathbf{E}\left[Z | \mathcal{H}\right] \in \operatorname{ri} \mathbf{E}\left[F | \mathcal{H}\right]$.

b) Conversely, for every $C \in \mathbf{L}^0 (\operatorname{ri} \mathbf{E}\left[F|\mathcal{H}\right], \mathcal{H})$ there is a $Z \in \mathbf{L}^0 (\operatorname{ri} F, \mathcal{F})$ such that $\mathbf{E}\left[Z|\mathcal{H}\right] = C$.

Note that, when Ω is finite, this Lemma follows from the fact that

$$A (\operatorname{ri} F) = \operatorname{ri} A (F)$$

whenever $A : \mathbb{R}^n \to \mathbb{R}^m$ is a linear mapping and $F \subset \mathbb{R}^n$ is convex (see Theorem 6.6 in [19]). For a general Ω a version of this Lemma has been proven by Ràsonyi in [17] under the assumption that F is closed, convex, bounded and $\operatorname{int} F \neq \emptyset$. We will use deep results from Rokhlin [22] to prove part b).

We postpone the proof of Lemma 3.15 to the end of this section.

3.2.1 Main theorem

Theorem 3.16. *Assume that* $(\mathcal{S}_0, \ldots, \mathcal{S}_T)$ *satisfies the robust no-arbitrage condition. Fix* $t \in \{1, \ldots, T\}$ *and suppose that there exists a strictly consistent price process for* $(\mathcal{S}_r)_{r=t}^T$.

Then:

a) There exists an $\mathcal{F}_t$*-measurable, non-empty, compact and convex random set* $\mathcal{W}_t \subset (0,\infty)^d$ *and an* $\mathcal{F}_{t-1}$*-measurable, non-empty, compact and convex random set* $\mathcal{W}_{t-1} \subset (0,\infty)^d$ *such that*

$$\left\{\lambda \begin{pmatrix} 1 \\ y \end{pmatrix} : \lambda > 0, y \in \mathcal{W}_t\right\} \subset \operatorname{ri} H_t,$$
$$\left\{\lambda \begin{pmatrix} 1 \\ y \end{pmatrix} : \lambda > 0, y \in \mathcal{W}_{t-1}\right\} \subset \operatorname{ri} K_{t-1}^*$$

and $\mathcal{W}_{t-1}, \mathcal{W}_t$ *satisfy the no-arbitrage condition.*

b) There exists a portfolio $\hat{v}_{t-1} \in \mathbf{L}^0(-K(\mathcal{W}_{t-1}), \mathcal{F}_{t-1})$ *such that*

$$\mathbf{E}\left[u\left(l^{\mathcal{W}_t}\left(\hat{v}_{t-1}\right)\right)|\mathcal{F}_{t-1}\right] = \operatorname*{ess\,sup}_{v_{t-1} \in \mathbf{L}^0(-K(\mathcal{W}_{t-1}), \mathcal{F}_{t-1})} \mathbf{E}\left[u\left(l^{\mathcal{W}_t}\left(v_{t-1}\right)\right)|\mathcal{F}_{t-1}\right].$$

c) There exists a strictly consistent price process for $(\mathcal{S}_r)_{r=t-1}^T$. *Especially, there exists a strictly consistent price process for* $(\mathcal{S}_t)_{t=0}^T$.

Proof. a) Let $(Z_r)_{r=t}^T$ be a strictly consistent price process for $(\mathcal{S}_r)_{r=t}^T$. The martingale property of $(Z_r)_{r=t}^T$ and Lemma 3.15 a) imply that

$$Z_{T-1} \in \operatorname{ri} K_{T-1}^* \cap \operatorname{ri} \mathbf{E}\left[H_T|\mathcal{F}_{T-1}\right].$$

Theorem 6.5 in [19] implies that

$$\operatorname{ri} H_{T-1} = \operatorname{ri} K_{T-1}^* \cap \operatorname{ri} \mathbf{E}\left[H_T|\mathcal{F}_{T-1}\right].$$

We continue in this manner and get $H_r \neq \{0\}$ as well as

$$\operatorname{ri} H_r = \operatorname{ri} K_r^* \cap \operatorname{ri} \mathbf{E}\left[H_{r+1}|\mathcal{F}_r\right]$$

for every $r = t, \ldots, T-1$.

Since $H_t \neq \{0\}$ we can write $H_t = \left\{\lambda \begin{pmatrix} 1 \\ y \end{pmatrix} : \lambda \geq 0, y \in \mathcal{Y}_t\right\}$ for some $\mathcal{F}_t$-measurable, non-empty, compact and convex $\mathcal{Y}_t \subset (0,\infty)^d$. By Lemma 3.14 $(\mathcal{S}_{t-1}, \mathcal{Y}_t)$ satisfy Assumption 3.2. By Lemma 3.5 there exist $\mathcal{W}_{t-1} \subset \operatorname{ri} \mathcal{S}_{t-1}$ and $\mathcal{W}_t \subset \operatorname{ri} \mathcal{Y}_t$ with the desired properties.

b), c) By Theorem 3.7 there exists an optimal portfolio for the market $\mathcal{W}_{t-1}, \mathcal{W}_t$ and by Corollary 3.12 this implies that there is a strictly consistent price process for $\mathcal{S}_{t-1}, \mathcal{Y}_t$, i.e. $H_{t-1} \neq \{0\}$ and

$$\operatorname{ri} H_{t-1} = \operatorname{ri} K_{t-1}^* \cap \operatorname{ri} \mathbf{E}\left[H_t | \mathcal{F}_{t-1}\right].$$

Let be $Z_{t-1} \in \mathbf{L}^0\left(\operatorname{ri} H_{t-1}, \mathcal{F}_{t-1}\right)$. Since $Z_{t-1} \in \operatorname{ri} \mathbf{E}\left[H_t | \mathcal{F}_{t-1}\right]$ we find by Lemma 3.15 b) an $\mathcal{F}_t$-measurable $Z_t \in \operatorname{ri} H_t$ such that $\mathbf{E}\left[Z_t | \mathcal{F}_{t-1}\right] = Z_{t-1}$. After finitely many steps we have found a strictly consistent price process $\left(Z_r\right)_{r=t-1}^T$ for $\left(\mathcal{S}_r\right)_{r=t-1}^T$.

□

Extension property

We want to prepare the proof of Lemma 3.15. For this we have to go into greater detail of random sets and study the relationship between *conditional expectatation* and the so-called *regular conditional upper distribution* of a set-valued map. We follow here [21] and [22] and use the notation from therein. We assume first that all appearing σ-fields are complete.

Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space and $\mathcal{H} \subset \mathcal{F}$ a sub-σ-field. Let F be a random set, assigning some non-empty closed set $F(\omega) \subset \mathbb{R}^D$ for every $\omega \in \Omega$. Measurability of F means that $\left\{\omega \in \Omega \,:\, F(\omega) \cap V \neq \emptyset\right\} \in \mathcal{F}$ for every open $V \subset \mathbb{R}^D$. We equip CL, the family of closed subset of $\mathbb{R}^D$, with the so-called *Effros σ-algebra* $\mathcal{E}$, generated by

$$A_V := \left\{D \in \mathrm{CL} \,:\, D \cap V \neq \emptyset\right\}$$

where $V \subset \mathbb{R}^D$ is open. Then F is measurable with respect to $\mathcal{E}$.

$(\mathrm{CL}, \mathcal{E})$ is a *Borel*-space, so there exists a regular conditional $\mathbf{P}$-distribution for F given $\mathcal{H}$ which we denote by $\mathbf{P}^*$. The function

$$\mu_F(\omega, V) := \mathbf{P}^*\left(\omega, A_V\right),$$

$V \subset \mathbb{R}^D$ open, is called the *regular conditional upper distribution* of F with respect to $\mathcal{H}$ and the set

$$\mathcal{K}(F, \mathcal{H}, \omega) := \left\{x \in \mathbb{R}^D \,:\, \mu_F\left(\omega, B_\varepsilon(x)\right) > 0, \quad \text{for } \varepsilon > 0\right\}$$

is called the support of $\mu_F(\omega, \cdot)$. The mapping $\omega \mapsto \mathcal{K}(F, \mathcal{H}, \omega)$ has non-empty closed values and is $(\mathcal{H}, \mathcal{E})$-measurable.

If f is a random variable in $\mathbb{R}^D$, then we also denote by $\mathcal{K}(f,\mathcal{H})$ the support of the regular condition $\mathbf{P}$-distribution of f with respect to $\mathcal{H}$. Let $(\xi_i)_{i=1}^{\infty}$ be a Castaing representation of F, then

$$\mathcal{K}(F,\mathcal{H}) = \overline{\bigcup_{i=1}^{\infty}\mathcal{K}(\xi_i,\mathcal{H})}.$$

Especiall for every $f \in \mathbf{L}^0(F,\mathcal{F})$ we have $\mathcal{K}(f,\mathcal{H}) \subset \mathcal{K}(F,\mathcal{H})$.

Proposition 3.17. *If F is a random set such that $F(\omega)$ is closed convex cone for every $\omega \in \Omega$ and $\mathcal{H} \subset \mathcal{F}$ a sub-σ-field then*

$$\mathbf{E}[F|\mathcal{H}] = \overline{\mathit{conv}\,\mathcal{K}(F,\mathcal{H})}.$$

Proof. Let $f \in \mathbf{L}^0(F,\mathcal{F})$ such that $\mathbf{E}[|f|\,|\mathcal{H}] < \infty$ and denote by μ_f a regular condition $\mathbf{P}$-distribution for f given $\mathcal{H}$. Then it follows by Theorem 3 in [8] (or Theorem 1.48 in [4] when $\mathcal{H} = \{\emptyset,\Omega\}$) that

$$\mathbf{E}[f|\mathcal{H}] = \int x\mu_f(\cdot,\mathrm{d}x) \in \operatorname{ri}\operatorname{conv}\mathcal{K}(f,\mathcal{H}).$$

Especially, $\mathbf{E}[f|\mathcal{H}] \in \operatorname{conv}\mathcal{K}(F,\mathcal{H})$. Since every element in $\mathbf{L}^0(\mathbf{E}[F|\mathcal{H}],\mathcal{H})$ is a limit of such conditional expectations we get

$$\mathbf{E}[F|\mathcal{H}] \subset \overline{\operatorname{conv}\mathcal{K}(F,\mathcal{H})}.$$

To show the reverse inclusion let $(\xi_i)_{i=1}^{\infty}$ be a Castaing representation of F such that

$$\mathcal{K}(F,\mathcal{H}) = \overline{\bigcup_{i=1}^{\infty}\mathcal{K}(\xi_i,\mathcal{H})}.$$

We fix i as well as a conditional $\mathbf{P}$-distribution μ_i for ξ_i given $\mathcal{H}$. For every $h \in \mathbf{L}^0(\mathcal{K}(\xi_i,\mathcal{H}),\mathcal{H})$ we find a nullset $N \in \mathcal{H}$ such that

$$\mu_i(\omega, B_\varepsilon(h(\omega))) > 0$$

for every $\omega \in N^c$ and every $\varepsilon > 0$. Since F is a convex cone it follows by A.10 from the Appendix that

$$\frac{1}{\mu_i(B_\varepsilon(h))}\mathbf{E}\left[\mathbf{1}_{B_\varepsilon(h)}(\xi_i)\xi_i|\mathcal{H}\right] \in \mathbf{E}[F|\mathcal{H}].$$

We get by disintegration

$$
\begin{aligned}
&\left|\frac{1}{\mu_i\left(B_\varepsilon\left(h\right)\right)}\mathbf{E}\left[\mathbf{1}_{B_\varepsilon(h)}(\xi_i)\xi_i|\mathcal{H}\right]-h\right| \\
=&\left|\frac{1}{\mu_i\left(B_\varepsilon\left(h\right)\right)}\mathbf{E}\left[\mathbf{1}_{B_\varepsilon(h)}(\xi_i)(\xi_i-h)|\mathcal{H}\right]\right| \\
=&\left|\frac{1}{\mu_i\left(B_\varepsilon\left(h\right)\right)}\int_{B_\varepsilon(h)} x-h\;\mu_i\left(\mathrm{d}x\right)\right|\leq\varepsilon,
\end{aligned}
$$

hence $h\in\mathbf{E}\left[F|\mathcal{H}\right]$ and $\mathcal{K}\left(\xi_i,\mathcal{H}\right)\subset\mathbf{E}\left[F|\mathcal{H}\right]$. Since $\mathbf{E}\left[F|\mathcal{H}\right]$ is closed and convex we conclude

$$\overline{\operatorname{conv}\mathcal{K}\left(F,\mathcal{H}\right)}\subset\mathbf{E}\left[F|\mathcal{H}\right].$$

□

We quote an important Lemma from [22] from which part b) of Lemma 3.15 will follow.

Lemma. *Suppose that F is an $\mathcal{F}$-measurable map with non-empty closed and convex values. For any $\mathcal{H}$-measurable selector ξ of the map* $\operatorname{ri}\left(\textit{conv}\,\mathcal{K}\left(F,\mathcal{H}\right)\right)$ *there exist $\mathcal{F}$-measurable $\eta\in\operatorname{ri}F$ and $\gamma>0$ such that*

$$\xi=\mathbf{E}\left[\gamma\eta|\mathcal{H}\right],\quad \mathbf{E}\left[\gamma|\mathcal{H}\right]=1.$$

Proof of Lemma 3.15. Let us assume that all σ-fields are complete.
Let $F\subset\mathbb{R}^{d+1}$ be a non-empty convex and conic random set. We first proof part b) which we will use then for the proof of part a).

b) Let be $u\in\mathbf{L}^0\left(\operatorname{ri}\mathbf{E}\left[F|\mathcal{H}\right],\mathcal{H}\right)$. Then, by Proposition 3.17, $u\in\operatorname{ri}\left(\overline{\operatorname{conv}\mathcal{K}\left(F,\mathcal{H}\right)}\right)$. From $\operatorname{ri}\left(\overline{\operatorname{conv}\mathcal{K}\left(F,\mathcal{H}\right)}\right)=\operatorname{ri}\left(\operatorname{conv}\mathcal{K}\left(F,\mathcal{H}\right)\right)$ it follows with the above Lemma that there exist $\mathcal{F}$-measurable $\eta\in\operatorname{ri}F$ and $\gamma>0$ such that $u=\mathbf{E}\left[\gamma\eta|\mathcal{H}\right]$. $z:=\gamma\eta\in\operatorname{ri}F$ because F is a cone and the claim follows.

a) Let be $z\in\mathbf{L}^0\left(\operatorname{ri}F,\mathcal{F}\right)$ such that $\mathbf{E}\left[|z|\,|\mathcal{H}\right]<\infty$ and suppose that $A:=\left\{\mathbf{E}\left[z|\mathcal{H}\right]\notin\operatorname{ri}\mathbf{E}\left[F|\mathcal{H}\right]\right\}$ has positive measure. In the Appendix (Lemma A.13) it is worked out that, on A, $\mathbf{E}\left[z|\mathcal{H}\right]$ can be separated from the convex set $\operatorname{ri}\mathbf{E}\left[F|\mathcal{H}\right]$ in an $\mathcal{H}$-measurable way, i.e. there exist $h\in\mathbf{L}^0\left(\mathbb{R}^{d+1},\mathcal{H}\right)$, $\bar{y}\in\mathbf{L}^0\left(\operatorname{ri}\mathbf{E}\left[F|\mathcal{H}\right],\mathcal{H}\right)$ such that

$$
\begin{aligned}
&h\cdot w\geq h\cdot\mathbf{E}\left[z|\mathcal{H}\right],\quad \text{for } w\in\operatorname{ri}\mathbf{E}\left[F|\mathcal{H}\right],\\
&h\cdot\bar{y}>h\cdot\mathbf{E}\left[z|\mathcal{H}\right]
\end{aligned}
$$

on A. Since $\overline{\operatorname{ri}\mathbf{E}[F|\mathcal{H}]} = \mathbf{E}[F|\mathcal{H}]$ we get on A: $\mathbf{E}[h \cdot v|\mathcal{H}] \geq \mathbf{E}[h \cdot z|\mathcal{H}]$ for every $v \in \mathbf{L}^0(F, \mathcal{F})$ with $\mathbf{E}[|v|\,|\mathcal{H}] < \infty$. It follows $h \cdot v \geq h \cdot z$ for all $v \in F$ on A.

Choose by part b) a $\bar{z} \in \mathbf{L}^0(\operatorname{ri} F, \mathcal{F})$ with $\mathbf{E}[\bar{z}|\mathcal{H}] = \bar{y}$. Then $A' := \{h \cdot \bar{z} > h \cdot z\} \cap A$ has positive measure. It follows that on A' z is separated from $\operatorname{ri} F$ with a hyperplane, from which we conclude that on A' $z \notin \operatorname{ri} F$. Thus, $\{z \notin \operatorname{ri} F\}$ has positive measure which is a contradiction.

Altogether it follows that $\mathbf{E}[z|\mathcal{H}] \in \operatorname{ri}\mathbf{E}[F|\mathcal{H}]$.

We can drop the assumption about completeness of the involved probability spaces if we argue as follows:

When $(\Omega, \mathcal{F}, \mathbf{P})$ is a probabilty space and $\mathcal{H} \subset \mathcal{F}$ a sub-σ-field field, we switch to the completions $\left(\Omega, \overline{\mathcal{F}}, \overline{\mathbf{P}}\right)$ and $\overline{\mathcal{H}}$ for which the satement of the lemma holds. Then for every $\overline{\mathcal{F}}$-measurable $\overline{z}$ there exists an $\mathcal{F}$-measurable z such that $\overline{z} = z$ a.s.. Also the conditional expectations $\overline{\mathbf{E}}\left[\overline{z}|\overline{\mathcal{H}}\right]$ and $\mathbf{E}[z|\mathcal{H}]$ coincide up to a null-set, from which $\overline{\mathbf{E}}\left[F|\overline{\mathcal{H}}\right] = \mathbf{E}[F|\mathcal{H}]$ a.s. follows. $\square$

3.3 Conclusion

The ideas from the one-dimensional case can be preserved in a multidimensional market with transaction costs. However, we had to find a different approach for the single-period case. The main reason is that an analog to part a) from Corollary 2.11 is out of reach here. So, in the first step we considered a generic single-period market with consistent prices given by $\mathcal{S}_0$ and $\mathcal{S}_1$ and showed directly in Lemma 3.5 that Assumption 3.2 implies the robust no-arbitrage condition. Thus, there are consistent prices $\mathcal{W}_0 \subset \operatorname{ri}\mathcal{S}_0$ and $\mathcal{W}_1 \subset \operatorname{ri}\mathcal{S}_1$ such that $\mathcal{W}_0, \mathcal{W}_1$ is free of arbitrage opportunities. This allowed us to find an optimal portfolio for $\mathcal{W}_0, \mathcal{W}_1$ in Theorem 3.7. It turned out that the first order condition for the optimal portfolio requires some extra work to show existence of a consistent price process. But still, the marginal utility evaluated at the liquidation value of the optimal portfolio gives an equivalent martingale measure for specific selectors of $\mathcal{W}_0$ and $\mathcal{W}_1$, see Corollary 3.12. Thus, we derived a strictly consistent price process for the original market $\mathcal{S}_0$ and $\mathcal{S}_1$.

The multi-period case can be seen as a straightforward generalization of the one-dimensional setting. It is only more demanding with regard to technical questions. Going backwards in time we replace in every given single-period market $\mathcal{S}_{t-1}, \mathcal{S}_t$ the consistent prices at time t by some smaller set

$\mathcal{Y}_t$, see the proof of Theorem 3.16. Then the strictly consistent price processes for $\mathcal{S}_{t-1}, \mathcal{Y}_t$ can be extended to a strictly consistent price processes for $\mathcal{S}_{t-1}, \mathcal{S}_t, \ldots, \mathcal{S}_T$.

The benefit of considering the one-dimensional case separately in Chapter 2 is that we can explicitly describe all strictly consistent price processes, see Corollary 2.20. Whereas in the multidimensional case we need to work with the rather abstract definition for conditional expectation of random sets.

Appendix A

In the Appendix we want to collect some advanced results from Measure Theory and Probability Theory which are used throughout the text.

We fix a probabilty space $(\Omega, \mathcal{A}, \mathbf{P})$.

The classical Bolzano-Weierstrass Theorem states that every bounded sequences $(x_n)_{n=1}^{\infty}$ in $\mathbb{R}^D$ has a convergent subsequence $(x_{n_k})_{k=1}^{\infty}$. This result is generalized to random sequences in the next proposition whose proof can be found in [3] (Proposition 6.3.3).

Proposition A.1. *Let $(X_n)_{n=1}^{\infty}$ be a sequence in $\mathbf{L}^0(K, \mathcal{A})$ where $K \subset \mathbb{R}^D$ is a compact set. Then there exists a sequence $(\tau_k)_{k=1}^{\infty}$, $\tau_k : \Omega \to \mathbb{N}$, such that every τ_k is $\mathcal{A}$-measurable, strictly increasing and $(X_{\tau_k}(\omega))_{k=1}^{\infty}$ converges for all $\omega \in \Omega$.*

When $(X_n)_{n=1}^{\infty}$ is only bounded from below in every component, then we can still extract a convergent sequence from $(X_n)_{n=1}^{\infty}$ by taking convex combinations. For a subset A of a vector space W we denote by

$$\operatorname{conv}(A) := \left\{ \sum_{i=1}^{m} \lambda_i x_i \,:\, x_i \in A, \lambda_i \in [0,1], \sum_{i=1}^{m} \lambda_i = 1 \right\}$$

the convex hull of A.

Proposition A.2. *Let $(X_n)_{n=1}^{\infty}$ be a sequence in $\mathbf{L}^0\left([0,\infty)^D, \mathcal{A}\right)$. Then there exists a sequence $(Y_n)_{n=1}^{\infty}$, $Y_n \in conv\{X_n, X_{n+1}, \dots\}$, and a random variable $Z \in \mathbf{L}^0\left([0,\infty]^D, \mathcal{A}\right)$ such that Y_n converges towards Z almost surely.*

Proof. This result is proved in Lemma 9.8.1 of [3] for $D = 1$. So we have to repeat the one-dimensional result D-times:

Take $Y_{n,1} = \left(Y_{n,1}^1, \dots, Y_{n,1}^D\right) \in \operatorname{conv}\{X_n, X_{n+1}, \dots\}$ such that $\left(Y_{n,1}^1\right)_{n=1}^{\infty}$ converges almost surely towards a random variable $Z^1 \in [0, \infty]$.

Take $Y_{n,2} = \left(Y^1_{n,2}, \dots, Y^D_{n,2}\right) \in \operatorname{conv}\{Y_{n,1}, Y_{n+1,1}, \dots\}$ such that $\left(Y^2_{n,2}\right)_{n=1}^{\infty}$ converges almost surely towards a random variable $Z^2 \in [0, \infty]$. Note that with $Y^1_{n,1} \to Z^1$ we also have $Y^1_{n,2} \to Z^1$. Since $\operatorname{conv}\{Y_{n,1}, Y_{n+1,1}, \dots\} \subset \operatorname{conv}\{X_n, X_{n+1}, \dots\}$, also $Y_{n,2} \in \operatorname{conv}\{X_n, X_{n+1}, \dots\}$.

We continue like this finitely many times until we have found $Y_{n,D} \in \operatorname{conv}\{Y_{n,D-1}, Y_{n+1,D-1}, \dots\} \subset \operatorname{conv}\{X_n, X_{n+1}, \dots\}$ and random variables $Z^1, \dots, Z^D \in [0, \infty]$ such that $Y_{n,D} \to \left(Z^1, \dots, Z^D\right)$ almost surely. $\square$

We continue with an overview about some basic results from the Theory of Random Sets. We follow the book by Castaing, Valadier [1] and the book by Molchanov [15].

Denote by CL the family of closed subsets of $\mathbb{R}^D$. We equip CL with the *Effros-σ-field* $\mathcal{E}$ which is generated by the sets

$$\left\{F \in \mathrm{CL} \,:\, F \cap V \neq \emptyset\right\}$$

where V ranges over the open subsets of $\mathbb{R}^D$.

Definition A.3. A random variable $F : \Omega \to \mathrm{CL}$, which is $(\mathcal{A}, \mathcal{E})$-measurable, is called ($\mathcal{A}$-measurable) closed random set.

Obviously we have the following equivalence for a mapping $F : \Omega \to \mathrm{CL}$:

$$\{F \cap V \neq \emptyset\} \in \mathcal{A} \quad \text{for every open } V \subset \mathbb{R}^D$$

$$\Leftrightarrow$$

$$\{F \cap K \neq \emptyset\} \in \mathcal{A} \quad \text{for every compact } K \subset \mathbb{R}^D$$

$$\Leftrightarrow$$

$$\{F \cap G \neq \emptyset\} \in \mathcal{A} \quad \text{for every closed } G \subset \mathbb{R}^D$$

Every random set with non-empty values admits a measurable selection by Theorem III.6 in [1].

Theorem A.4. *Let F be a closed random set such that $F(\omega) \neq \emptyset$ for every $\omega \in \Omega$. Then there exists an $\mathcal{A}$-measurable $f : \Omega \to \mathbb{R}^D$ with values in F, i.e. $f(\omega) \in F(\omega)$ for every $\omega \in \Omega$.*

Random sets can be described by a sequence of random variables. This is Theorem III.7 in [1].

Theorem A.5. *a) Let F be a non-empty closed random set. Then there exists a sequence of random variables $(f_n)_{n=1}^{\infty}$ such that*

$$F(\omega) = \overline{\{f_n(\omega) : n \geq 1\}}$$

for every $\omega \in \Omega$.
$(f_n)_{n=1}^{\infty}$ is called a Castaing-representation of F.

b) Conversely, for any sequence $(f_n)_{n=1}^{\infty}$ of random variables

$$\omega \mapsto \overline{\{f_n(\omega) : n \geq 1\}}$$

defines closed random set.

The methods to prove Theorem A.4 and Theorem A.5 are rather staight-forward in contrast to the following deep selection theorem. This is Theorem III.22 in [1].

Theorem A.6. *Assume that, for every $\omega \in \Omega$, $F(\omega)$ is a non-empty subset of $\mathbb{R}^D$ such that*

$$\mathit{Graph}(F) := \{(\omega, x) \in \Omega \times \mathbb{R}^D : x \in F(\omega)\} \in \mathcal{A} \otimes \mathcal{B}\left(\mathbb{R}^D\right).$$

Then, there exists a sequence $(f_n)_{n=1}^{\infty}$ of $\overline{\mathcal{A}}$-measurable selections of F such that, for every $\omega \in \Omega$, $(f_n(\omega))_{n=1}^{\infty}$ is dense in $F(\omega)$. Here, $\overline{\mathcal{A}}$ denotes the $\mathbf{P}$-completion of $\mathcal{A}$.

We want to illustrate the foregoing theorems.

Example A.7. Let F be an $\mathcal{A}$-measurable closed random set such that $F(\omega)$ is non-empty for every $\omega \in \Omega$. Denote by $(f_n)_{n=1}^{\infty}$ a Castaing representation of F.

a) Then

$$\begin{aligned}\mathrm{Graph}(F) &= \Big\{(\omega, x) : x \in F(\omega)\Big\} \\ &= \bigcap_{m=1}^{\infty}\bigcup_{n=1}^{\infty}\Big\{(\omega, x) : |x - f_n(\omega)| < 1/m\Big\} \in \mathcal{A} \otimes \mathcal{B}\left(\mathbb{R}^D\right)\end{aligned}$$

b) The affine hull, aff F, defines an $\mathcal{A}$-measurable closed random set, since

$$\begin{aligned}\text{aff } F(\omega) &= \Big\{\sum_{i=1}^{m}\lambda_i x_i : x_i \in F(\omega), \lambda_i \in \mathbb{R}, \sum_{i=1}^{m}\lambda_i = 1\Big\} \\ &= \overline{\Big\{\sum_{i=1}^{m}\lambda_i f_i(\omega) : \lambda_i \in \mathbb{Q}, \sum_{i=1}^{m}\lambda_i = 1\Big\}}.\end{aligned}$$

c) Assume that additionally F is convex valued and let $(K_m)_{m=1}^{\infty}$ be a Castaing-representation of aff F. Then $\operatorname{ri} F(\omega) \neq \emptyset$ for every $\omega \in \Omega$ and

$$\begin{aligned}
&\operatorname{Graph}(\operatorname{ri} F)\\
&\quad=\Big\{(\omega,x)\ :\ x \in \operatorname{aff} F(\omega), \exists \varepsilon > 0 : (x+\varepsilon B) \cap \operatorname{aff} F(\omega) \subset F(\omega)\Big\}\\
&\quad=\operatorname{Graph}(\operatorname{aff}\, F) \cap\\
&\quad\bigcup_{j=1}^{\infty}\bigcap_{m=1}^{\infty}\left(\{|K_m(\omega)-x| \geq 1/j\} \cup \bigcap_{i=1}^{\infty}\bigcup_{n=1}^{\infty}\{|K_m(\omega)-f_n(\omega)| < 1/i\}\right).
\end{aligned}$$

It follows that $\operatorname{Graph}(\operatorname{ri} F) \in \mathcal{A} \otimes \mathcal{B}\left(\mathbb{R}^D\right)$ and there exists a sequence $(h_n)_{n=1}^{\infty}$ of $\overline{\mathcal{A}}$-measurable selections such that $(h_n(\omega))_{n=1}^{\infty}$ is dense in $\operatorname{ri} F(\omega)$ for every $\omega \in \Omega$.

We can modify this sequence on a null-set to become $\mathcal{A}$-measurable. Then the modified sequence is dense in $\operatorname{ri} F(\omega)$ up to a null-set.

It is well known that for a concave function $\varphi : \mathbb{R}^D \to \mathbb{R}$ the condition

$$\lim_{t\to\infty} \varphi(tx) = -\infty, \quad \text{for } x \neq 0,$$

is sufficent to show existence of a maximum for φ (e.g. Lemma 3.5 in [4]). To find a utility maximizing portfolio we will need to maximize a random version of such a φ and select the maximizers in a measurable way. For this purpose we need the following lemma which generalizes a smiliar result in [20].

Lemma A.8. *Let C be an $\mathcal{A}$-measurable, closed convex cone in $\mathbb{R}^d$. Suppose that $\varphi : \Omega \times \mathbb{R}^d \to \mathbb{R} \cup \{-\infty\}$ is $\mathcal{A} \otimes \mathcal{B}(\mathbb{R}^d)$-measurable and has the following properties:*

(i) *$\varphi(\omega,\cdot)$ is concave for every $\omega \in \Omega$,*

(ii) *$\varphi(\omega,\cdot)$ is $\mathbb{R}$-valued and continuous on $C(\omega)$ for every $\omega \in \Omega$,*

(iii) *$\varphi(\omega,a) = -\infty$ for every $a \in \mathbb{R}^d \setminus C(\omega)$.*

Then for every $\omega \in \Omega$

$$F(\omega) := \left\{a \in \mathbb{R}^D\ :\ |a| = 1, \lim_{t\to\infty} \varphi(\omega, ta) > -\infty\right\}$$

is closed and we can define an $\mathcal{A}$-measurable mapping

$$\alpha : \omega \mapsto \begin{cases} \alpha(\omega) \in F(\omega) & , \textit{ if } F(\omega) \neq \emptyset \\ 0 & \textit{otherwise.} \end{cases}$$

For every $\omega \in \Omega$ the set of maximizers

$$A(\omega) := \left\{ a \in \mathbb{R}^D \,:\, \varphi(\omega, a) \geq \varphi(\omega, b), \quad \textit{for } b \in \mathbb{R}^D \right\}$$

is closed and we can define an $\mathcal{A}$-measurable mapping

$$\beta : \omega \mapsto \begin{cases} \beta(\omega) \in A(\omega) & , \textit{ if } F(\omega) = \emptyset \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. If $a \in F$ and $t > 0$, then $\varphi(0) \leq \varphi(ta)$. Otherwise $\lim_{t\to\infty} \varphi(ta) = -\infty$ would follow by concavity of φ. Then, since $F \subset C$, C is closed and φ is continuous on C, it follows that F is closed. We show that

$$\{\omega \in \Omega | F(\omega) \cap K \neq \emptyset\} \in \mathcal{A}, \quad K \subset \mathbb{R}^d \text{ closed},$$

and by Theorem A.4 we can then define an $\mathcal{A}$-measurable mapping α, where $\alpha(\omega) \in F(\omega)$, if $F(\omega) \neq \emptyset$ and $\alpha(\omega) = 0$ otherwise. Fix a closed $K \subset \mathbb{R}^d$, for which we can assume that $K \subset \{|x| = 1\}$, and choose a sequence $(x_l)_{l=1}^\infty$ of $\mathcal{A}$-measurable random variables with values in K such that $(x_l(\omega))_{l=1}^\infty$ is dense in $K \cap C(\omega)$ on the event $\{K \cap C \neq \emptyset\}$. Then

$$\left\{\omega \in \Omega \,:\, F(\omega) \cap K \neq \emptyset\right\} = \bigcup_{M=1}^\infty \bigcap_{k=1}^\infty \bigcup_{l=1}^\infty \left\{\omega \in \Omega \,:\, \varphi(\omega, kx_l(\omega)) \geq -M\right\} \in \mathcal{A}.$$

Indeed, if $\varphi(\omega, kx_{l_k}(\omega)) \geq -M$, let $x_0 \in K \cap C(\omega)$ be an accumulation point of $(x_{l_k}(\omega))_{k=1}^\infty$. Then for $t > 0$ and $k > t$

$$\varphi(\omega, tx_{l_k}(\omega)) \geq \frac{t}{k}\varphi(\omega, kx_{l_k}(\omega)) + \frac{k-t}{k}\varphi(\omega, 0) \geq \varphi(\omega, 0) \wedge -M.$$

Thus, by continuity $\varphi(\omega, tx_0) \geq \varphi(\omega, 0) \wedge -M$ for every $t > 0$ and we get $x_0 \in F(\omega) \cap K$.
Conversely, choose $x_0 \in F(\omega) \cap K$ and $M \in \mathbb{N}$ such that $\varphi(\omega, tx_0) \geq \varphi(\omega, 0) > -M$ for every $t > 0$. Then $x_0 \in K \cap C(\omega)$ and by continuity we can find for each $k \in \mathbb{N}$ an $l \in \mathbb{N}$ such that $\varphi(\omega, kx_l(\omega)) \geq -M$.

A is closed because φ is continuous on C and $A \subset C$. Now, for a fixed $K \subset \mathbb{R}^d$ which is supposed to be compact, we choose a sequence of $\mathcal{A}$-measurable random variables $(y_l)_{l=1}^\infty$ with values in K such that $(y_l(\omega))_{l=1}^\infty$

is dense in $K \cap C(\omega)$ on the event $\{K \cap C \neq \emptyset\}$. Further, let $(c_n)_{n=1}^{\infty}$ be a sequence of $\mathcal{A}$-measurable random variables such that $(c_n(\omega))_{n=1}^{\infty}$ is dense in $C(\omega)$. Then

$$\begin{aligned}&\left\{\omega \in \Omega \,:\, A(\omega) \cap K \neq \emptyset\right\}\\&= \bigcap_{m=1}^{\infty}\bigcup_{l=1}^{\infty}\bigcap_{n=1}^{\infty}\{\omega \in \Omega \,:\, \varphi(\omega, y_l(\omega)) \geq \varphi(\omega, c_n(\omega)) - \frac{1}{m}\} \in \mathcal{H}.\end{aligned}$$

Now, if $F(\omega) = \emptyset$, then $\lim_{t\to\infty} \varphi(\omega, ta) = -\infty$ for every $a \neq 0$. This condition implies that $A(\omega) \neq \emptyset$. Thus, by Theorem A.4 β can be defined in an $\mathcal{A}$-measurable way. $\square$

We want to describe when exactly a subset of $\mathbf{L}^0\left(\mathbb{R}^D, \mathcal{A}\right)$ can be written in the form $\mathbf{L}^0\left(F, \mathcal{A}\right)$ for some closed random set F and how the elements of $\mathbf{L}^0\left(F, \mathcal{A}\right)$ can be approximated using a Castaing-representation of F. In the next theorem we cite Lemma 1.6 and Theorem 1.6 from [15]. There, the statement is formulated for $L^p, p \geq 1$. But, as pointed out in the Appendix of Kabanov and Safarian [12], the proofs also work for $p = 0$.

$d(X, Y) := \mathbf{E}\left[|X - Y| \wedge 1\right]$ for $X, Y \in \mathbf{L}^0\left(\mathbb{R}^D, \mathcal{A}\right)$ denotes a metric which induces convergence in probability. A subset $\mathcal{V} \subset \mathbf{L}^0\left(\mathbb{R}^D, \mathcal{A}\right)$ is called *decomposable*, if for any $f_1, f_2 \in \mathcal{V}$ and $A \in \mathcal{A}$, we have $\mathbf{1}_A f_1 + \mathbf{1}_{A^c} f_2 \in \mathcal{V}$.

Theorem A.9. *Let $\mathcal{V}$ be a non-empty subset of $\mathbf{L}^0\left(\mathbb{R}^D, \mathcal{A}\right)$. Then:*

a) If $\mathcal{V} = \mathbf{L}^0\left(F, \mathcal{A}\right)$ for a closed random set F with Castaing-representation $(f_n)_{n=1}^{\infty}$, then for every $\varepsilon > 0$ and every $f \in \mathcal{V}$ there is a measurable finite partition $A_1, \ldots, A_m$ such that

$$d\left(f, \sum_{i=1}^{m} \mathbf{1}_{A_i} f_i\right) < \varepsilon.$$

b) $\mathcal{V} = \mathbf{L}^0\left(F, \mathcal{A}\right)$ for a closed random set F if and only if $\mathcal{V}$ is closed and decomposable.

The following proposition follows directly from Proposition 1.5 in [15]. There it is formulated only for the convex case. With the obvious changes in the proof it is also true for the convex and conic case.

Proposition A.10. *Let $\mathcal{V} \subset \mathbf{L}^0\left(\mathbb{R}^D, \mathcal{A}\right)$ be non-empty, closed and decomposable. Then:*

a) If $\mathcal{V}$ is convex then there is a closed and convex random F such that $\mathcal{V} = \mathbf{L}^0(F, \mathcal{A})$.

b) If $\mathcal{V}$ is a convex cone then there is a closed, convex and cone-valued F such that $\mathcal{V} = \mathbf{L}^0(F, \mathcal{A})$.

When C_1 and C_2 are closed and convex cones in $\mathbb{R}^D$ then $C_1 + C_2$ is not necessarily closed. The following proposition gives a sufficient criterion for the closedness of $C_1 + C_2$. Its proof is basically between page 32 and 33 of [25]

Proposition A.11. *Let C_1 and C_2 be two closed convex cones in $\mathbb{R}^D$. If $C_1 \cap (-C_2)$ is a subspace of $\mathbb{R}^D$, then the sum $C_1 + C_2$ is closed.*

Proof. Denote by p the orthogonal projection onto the subspace $C_1 \cap (-C_2)$. Let be $x_n \in C_1$ and $y_n \in C_2$ such that $(x_n + y_n)_{n=1}^{\infty}$ converges. We write

$$x_n + y_n = [x_n - p(x_n)] + [y_n + p(x_n)]$$

and note, since $C_1 \cap (-C_2)$ is a linear space, that $x_n - p(x_n) \in C_1$ and $y_n + p(y_n) \in C_2$.

We claim that $(x_n - p(x_n))_{n=1}^{\infty}$ is bounded. Otherwise, by passing to a subsequence, we can assume that $|x_n - p(x_n)| \to \infty$.

Since $\left(\frac{x_n - p(x_n)}{|x_n - p(x_n)|}\right)_{n=1}^{\infty}$ is situated on the unit sphere, we can again assume by passing to a subsequence, that $\left(\frac{x_n - p(x_n)}{|x_n - p(x_n)|}\right)_{n=1}^{\infty}$ converges towards some η on the unit sphere. From

$$\frac{x_n + y_n}{|x_n - p(x_n)|} = \frac{x_n - p(x_n)}{|x_n - p(x_n)|} + \frac{y_n + p(x_n)}{|x_n - p(x_n)|}$$

and the fact that $(x_n + y_n)_{n=1}^{\infty}$ converges, it follows that $\left(\frac{y_n + p(x_n)}{|x_n - p(x_n)|}\right)_{n=1}^{\infty}$ converges towards $-\eta$.

From the closedness of C_1 and C_2 we have $\eta \in C_1$ and $-\eta \in C_2$, i.e. $\eta \in C_1 \cap (-C_2)$. But by construction $\frac{x_n - p(x_n)}{|x_n - p(x_n)|} \in [C_1 \cap (-C_2)]^{\perp}$, which enforces $\eta \in [C_1 \cap (-C_2)]^{\perp}$. Thus $\eta = 0$, which is a contradiction.

From the boundedness of $(x_n - p(x_n))_{n=1}^{\infty}$ and by passing to a subsequence we can assume that $(x_n - p(x_n))_{n=1}^{\infty}$ converges. Since C_1 is closed, the limit must be in C_1. Now, $(y_n + p(x_n))_{n=1}^{\infty}$ must converge and its limit is in C_2. It follows that $\lim(x_n + y_n) \in C_1 + C_2$.

□

Proposition A.12. *Let F, G, H be closed random sets such that $F + G = H$. Then*

$$\mathbf{L}^0(F, \mathcal{A}) + \mathbf{L}^0(G, \mathcal{A}) = \mathbf{L}^0(H, \mathcal{A}).$$

Proof. The statement is proved in Lemma 2 of [23] under the assumption that the probability space is complete.

Let $\left(\Omega, \overline{\mathcal{A}}, \overline{\mathbf{P}}\right)$ denote the completion of $(\Omega, \mathcal{A}, \mathbf{P})$. Then for every random variable $\overline{z}$ measurable with respect to $\overline{\mathcal{A}}$ there exists an $\mathcal{A}$-measurable random variable z such that $\overline{z} = z$ a.s.. From this observation the statement follows immediately for an arbitrary probability space. $\square$

If $C \subset \mathbb{R}^D$ is a non-empty convex set and $x \notin C$ then we can separate x from C by a hyperplane, i.e. there exist $h \in \mathbb{R}^D$, $a \in C$ such that

$$h \cdot w \geq h \cdot x, \quad \text{for } w \in C, \quad \text{and} \quad h \cdot a > h \cdot x.$$

Lemma A.13. *Assume that $(\Omega, \mathcal{A}, \mathbf{P})$ is a complete probability space. Let F be a non-empty closed and convex random set and f a random variable. Then there exist $h \in \mathbf{L}^0\left(\mathbb{R}^D, \mathcal{A}\right)$ and $a \in \mathbf{L}^0(\operatorname{ri} F, \mathcal{A})$ such that on $\{f \notin \operatorname{ri} F\}$*

$$\begin{aligned} &h \cdot w \geq h \cdot f, \quad \textit{for } w \in \operatorname{ri} F, \\ &h \cdot a > h \cdot f. \end{aligned}$$

Proof. Put $B := \{f \notin \operatorname{ri} F\}$ and let $(f_n)_{n=1}^{\infty}$ be a sequence of $\mathcal{A}$-measurable random variables such that $(f_n(\omega))_{n=1}^{\infty}$ is dense in $\operatorname{ri} F(\omega)$ for every $\omega \in \Omega$ (see Example A.7). Then

$$\begin{aligned} W :=& \Big\{(\omega, x) \,:\, \forall w \in \operatorname{ri} F(\omega)\ x \cdot k \geq x \cdot f(\omega), \exists k \in \operatorname{ri} F(\omega)\ x \cdot k > x \cdot f(\omega)\Big\} \\ =& \left(\bigcap_{n=1}^{\infty} \Big\{(\omega, x) \,:\, x \cdot f_n(\omega) \geq x \cdot f(\omega)\Big\}\right) \\ & \cap \left(\bigcup_{n=1}^{\infty} \Big\{(\omega, x) \,:\, x \cdot f_n(\omega) > x \cdot f(\omega)\Big\}\right) \in \mathcal{A} \otimes \mathcal{B}(\mathbb{R}^D) \end{aligned}$$

and by Theorem A.6 it follows that we can find a random variable h such that $(\omega, h(\omega)) \in W$ for every $\omega \in B$. Further, we define, for $\omega \in B$, $a(\omega) := f_m(\omega)$ where m is minimal such that $h(\omega) \cdot f_m(\omega) > h(\omega) \cdot f(\omega)$. We extend a on B^c by some selector in $\operatorname{ri} F$. $\square$

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