Effective properties of textiles by asymptotic homogenization

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List of notations

ε - small parameter

$\xi = \frac{x}{\varepsilon}$ - fast variable

Y - unit cell

S - contact interface in the unit cell

n - dimension of the Euclidean space $\mathbb{R}^n$

Ω - open and bounded domain in $\mathbb{R}^n$

S^ε - oscillating contact interface

Ω^ε - periodic open and bounded domain in $\mathbb{R}^n$ with period ε

Ω^ε_N - subset of the outer boundary where Neumann boundary conditions are applied

Ω^ε_D - subset of the outer boundary where Dirichlet boundary conditions are applied

$u^\varepsilon(x)$ - displacement vector

$a^\varepsilon_{ijkl}(x)$ - components of the elasticity tensor

$\sigma^\varepsilon_{ij}(x)$ - components of the Cauchy stress tensor

$g^\varepsilon(x)$ - gap function

$t(x)$ - boundary tractions vector

$f^\varepsilon(x)$ - body force vector

$\mathcal{F}^\varepsilon(x)$ - function of the friction law

$G^\varepsilon(x)$ - Tresca friction function

$\mu^\varepsilon(x)$ - Coulomb friction coefficient

δ^ε - penalty parameter

γ^ε - regularization parameter

N_q - auxiliary functions

$w(x)$ - vector representing the column of some auxiliary function N_q

$A^{hom} = \{\bar{a}_{ijkl}\}$ - homogenized elasticity tensor

$\mathbf{R}$ - finite element interpolation matrix

$\mathbf{B}$ - finite element strain-displacement matrix

$\mathbf{K}$ - global stiffness matrix

$\mathbf{K_N}$ - contribution of the normal contact

$\mathbf{K_T}$ - contribution of the tangential contact

$\mathbf{Q}$ - global displacement vector

Chapter 1

Introduction

1.1 Motivation

Woven, non-woven and knitted textiles are common materials for many technical and medical applications. In particular, technical textiles and fiber composites are subjected to contact with sliding and friction on the micro scale, which results in elasto-plastic constitutive behavior on the macro scale, see [Wriggers, Hain, Wellmann, Temizer, 2007].

Special constraints on stiffness or strength of technical textiles or composites are required, depending on the type of application, e.g. knitted medical stents should provide certain resistance against the blood pressure; geo-textiles, ropes and belts, protection wear should also provide certain stiffness and strength against external mechanical loading.

The aim of this thesis is to develop a simulation-based algorithm, allowing the prediction of the effective mechanical properties of textiles on the basis of their microstructure and corresponding properties of fibers. This method can be used for optimization of the microstructure, in order to obtain a better stiffness or strength of the corresponding fiber material later on.

Often, technical textiles or composites, such as in Fig. 1.1, have nearly periodic structure, (examples of a periodicity cell see in Fig. 1.2 (a)–(c)), and the period of their structure is much smaller than the characteristic size of a textile or composite. If the period is of the order of 0.01 or even less, direct numerical computations of the solution of the elasticity boundary value problem by means of commonly used methods like finite differences or finite elements become very expensive. This is caused by the need to use a very fine mesh in order to capture the periodic microstructure. On the other hand, by using homogenization

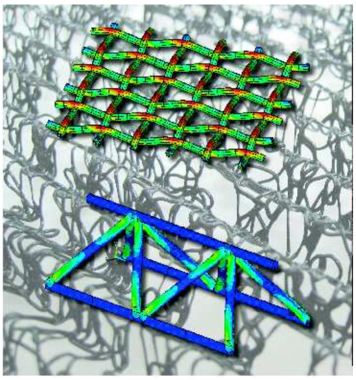


Figure 1.1: Technical textiles.

technique, one can obtain a solution with an error of the same order as the small parameter, namely, less than one percent. Such an idea was introduced in eighties, for instance, in the [Panasenko, 1982]. The approach was applied for so-called lattice structures, namely the fiber structures where only fixed junctions are considered between fibers. In the case of a textile-like material, the fibers may slide in the contact, and such a case was not considered in these works.

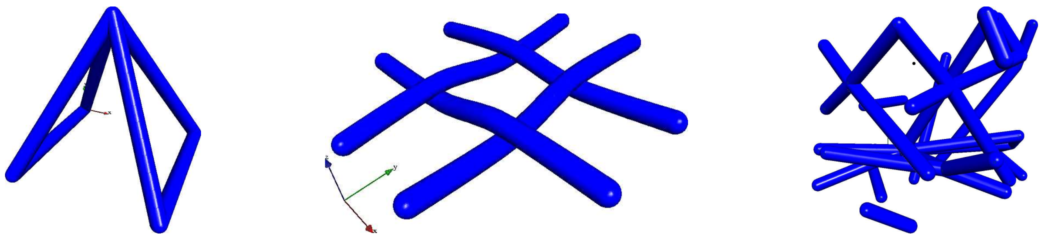
Homogenization technique allows to consider the whole problem asymptotically with respect to a small parameter, the relation between the micro and macro sizes of the textile or composite. This should lead in the limit to an equivalent homogenized problem with some averaged properties. The obtained solution of the homogenized problem then, is an approximation of the exact solution with a certain accuracy in some sense of convergence (see e.g., [Allaire, 1992], [Bakhvalov, Panasenko, 1984], [Hornung, 1997]).

This thesis is devoted to the homogenization of the contact conditions on a highly oscillating inner interface, which is the contact surface of fibers in the considered textile or components of the considered composite. That is, the homogenization technique from [Allaire, 1992] should be extended to the microscale contact.

The next aspect of the thesis is the accounting for the thickness of thin fibers in the textile. An introduction of an additional asymptotics with respect to the small parameter, a relation between the thickness and the representative length of the fibers, will allow a reduction of local contact problems between fibers to 1-dimensional problems, which will reduce numerical computations significantly.

1.2 State of the art

The theory and analysis of Signorini contact problem in elasticity with friction and sliding can be found in [Kikuchi, Oden, 1988], [Eck, PhD thesis, 1996], [Han, Sofonea]. The main reason that makes the contact problem challenging lies in the nonlinear nature of contact conditions and the non-smoothness of the functional. The key point in different methods for solving frictional contact problems is how to handle the contact conditions. The non-penetration condition can be represented as a constraint on the set of the admissible functions, while the contribution of friction is given by an additional functional. In this case, the weak formulation of the contact problem is in the form of a variational inequality, where the contact conditions take form of a friction functional over the contact interface and non-penetration constraint on the set of admissible functions. The proof of existence of a solution can be found in [Eck, PhD thesis, 1996], [Lions, Stampacchia, 1967]. Then the contact minimization problem can be solved using the Lagrange multiplier method, where the constraint is resolved by introducing additional unknowns, and the problem can be reduced to a saddle-point problem. Another approach is based on the reformulation of the problem by adding a penalty functional, which penalizes the violation of the geometrical non-penetration constraint. A disadvantage of the Lagrange multiplier method is an introduction of additional unknowns, while the penalty method often leads to bad-conditioned systems of discretized equations. The penalty method gives an approximation of the problem by penalization, and this requires a sensitivity analysis with respect to the penalty parameter. The combination of the both approaches is the augmented Lagrange method, well-known in optimization theory, see [Nocedal, Wright]. In this case, the penalty term is added to the Lagrangian, rather than to the minimization functional. The augmented Lagrangian at the minimizer coincides with the Lagrangian, and the penalty parameter no longer needs to be small. This allows to avoid ill-conditioning of the penalty method. For details, see [Kikuchi, Oden, 1988], [Christensen, 1998], [Laursen, 2002]. However, the system of equations is still non-differentiable. Therefore, the classical Newton method is not applicable. Then the so-called non-smooth Newton method or an active set strategy can be used, see [Kunish, Stadler, 2005], [Pang, 1990]. This method observes that, in fact, the system is B-differentiable, see [Pang, 1990]. Then the system can be solved by the



(a) (b) (c)

Figure 1.2: Examples of periodicity cells: (a) a fixed structure, (b) a woven structure, (c) a non-woven structure.

extended Newton method using an iterative procedure.

The finite element (FE) approximation for the solution of contact problems using different methods is considered in [Kikuchi, Oden, 1988] and [Wriggers, 2002]. The last reference gives more detailed results on the numerical treatment of the Signorini contact problem, including some special cases, e.g. the beam contact, see Chapter 11 in [Wriggers, 2002].

Some special cases of finite element analysis of beam-to-beam contact are considered in the work of [Litewka, 2010]: the case of frictionless contact of beams with rectangular cross-section, the frictional contact model for beams, the smoothed beam contact, the electric contact and thermo-mechanical coupling. The interesting part is the smoothing of beam contact: the C^1 smoothness of the contact interface, achieved by applying Hermite's polynomials or the Bezier's curves, is the smoothness of the geometry, not the solution. This allows to formulate a smooth inscribed Hermite beam finite element by using Hermite's inscribed curve approximation. The numerical examples of frictionless and frictional contact, which are presented in this work, are implemented using penalty and Lagrange multiplier methods.

The domain decomposition methods or extensions of mortar methods exploit non-matching properties of the spacial discretization on the contact interface, what yields an alternative spacial formulation of the contact conditions, see the works of [Hueber, Wohlmuth, 2006], [McDevitt, Laursen], and [Belgacem, Hild, Laborde, 1999]. Together with the active set strategies or the non-smooth Newton method, these methods produce results of a better accuracy than the standard approaches.

Contact problems for the inner oscillating interface were considered in papers, including

[Yosifian, 1997], [Yosifian, 1999], [Mikelic, Shillor, 1998], and, for instance, in the book [Sanchez-Palencia, 1980]. Also papers [Yosifian, 1997], [Yosifian, 1999] provide tools for homogenization of some non-linear Robin-type conditions on the inner oscillating interface under some assumptions on the nonlinearity. Although these assumptions are satisfied for the penalized non-penetration functional (we should make a remark that not all of them are satisfied for the Tresca friction and, hence, can be applied for it), the interesting for our consideration penalty functional, $\lim_{\varepsilon\to 0} \frac{1}{\varepsilon} \int_{S^\varepsilon} g(u^\varepsilon) v ds$, for the inner oscillating interface is not considered there. Furthermore, these results refer to the weak convergence of the solution, and the limit above will provide a contact condition for an auxiliary problem for some corrector of the solution. Hence, we need some generalization of the two-scale convergence for the nonlinear Robin-type conditions.

[Mikelic, Shillor, 1998] and Section 5 in Chapter 6 of [Sanchez-Palencia, 1980] deal with frictionless Signorini problems on the oscillating interface. In [Sanchez-Palencia, 1980] small cracks are imposed, and the homogenization is performed in a more formal way, without penalization. The results of this chapter coincide with those formulated in the paper [Sanchez-Palencia, Suquet, 1983] up to the frictional term, and we can use the proofs given in [Sanchez-Palencia, 1980] for solvability and uniqueness of the macroscopic problem and the auxiliary periodic contact problems. In [Mikelic, Shillor, 1998], particles, diluted in a matrix material, are considered, and their possible rotation is assumed. It is also assumed that the stiffnesses of these particles and of the matrix differ by several orders of magnitude. The problem is handled by the two-scale homogenization method in [Allaire, 1992]. Using the assumptions on the geometry, the coefficients are separated into the macroscopic one and one for the inclusion.

In this thesis we are thinking on application of the homogenization to some periodic fiber structures or textiles. Since our assumptions on the geometry are different from those given in [Mikelic, Shillor, 1998], and stiffnesses of the structural components are assumed to be of the same oder, the results of [Mikelic, Shillor, 1998] can not be used here. In [Hummel, 1999] the Robin-type interface conditions for linear heat equations on the oscillating interface are considered. Nevertheless, in this thesis we will use some results from these works as auxiliary tools.

The main expected contribution of this thesis is the homogenization algorithm. Another aspect of the work is the homogenization of the boundary conditions on the out-of-plane vanishing interface and the reduction of the dimension of contact problems between thin fibers using asymptotics with respect to their cross-sectional characteristic size.

In this connection we refer to the papers and books [Panasenko, 2005], [Pastukhova, 2005], [Pastukhova, 2006], [Pantz, 2003]. Homogenization based on the two-scale analysis and convergence for periodic box and rod frames structures were considered in the works [Pastukhova, 2005], [Pastukhova, 2006]. In these works, the thickness of walls or rods was introduced as an additional small parameter, and the homogenization results via two-scale convergence were obtained for certain relationships between the thickness and the period of the microstructure. The proof of the two-scale convergence is based on a special type of the Korn inequality derived in [Zhikov, Pastukhova, 2003] for thin periodic structures. Moreover, the two-scale limit can be represented as a sum of the homogenized solution for longitudinal beam or plate displacement and the transversal (bending) displacement, which is a solution of a separate boundary value problem of the fourth order.

Another approach for the homogenization of periodic finite rod structures with fixed junctions was proposed by Panasenko in [Panasenko, 2005]. The approach is based on the formal asymptotic expansion with respect to the period of the structure, and the inner expansion with respect to the thickness of the rods. The last expansion is considered on each segment of the finite rod structure, and additional boundary layer correctors are introduced in the neighborhood of the nodes. As a result, the elasticity problem can be asymptotically reduced to the one-dimensional problems in the form of ordinary differential equations for longitudinal, transversal (bending) and torsion components with matching conditions at the nodes. The fourth order equations for the bending components are obtained using the assumption that the transversal components of the body force should be of the second order of the period.

The same assumption is used in the work [Pantz, 2003], where an asymptotic analysis of the total strain energy with respect to the thickness is considered for a plate or a three-dimensional cylindrical body made of a hyperelastic Saint Venant-Kirchhoff material. The minimizer of the total energy converges to the minimizer of the inextensional nonlinear bending energy. The result is obtained by the Γ-convergence.

The two-scale analysis with respect to the thickness for thin curved rods was implemented in [Sanchez-Hubert, Sanchez-Palencia, 1999]. The three-dimensional elasticity problem for curved rods, which takes into account torsion and flexion, was considered, and the reduction of a dimension by a two-scale procedure was made. The results show that the two leading order terms have the Bernoulli's structure, and the flexion and torsion effects are of the same order. The asymptotic analysis also shows, as in [Panasenko, 2005], [Pantz, 2003], that one of the Bernoulli's hypothesis, the inextensibility of the middle line of the rod, cannot be obtained at the loading order of the expansion unless the body force is of the second order of the thickness of the rod.

1.3 Outline of the work

A fiber composite material with periodic microstructure and multiple micro contacts between fibers or inclusions and matrix are considered. In Chapter 2, the Signorini and friction contact conditions are prescribed on the microcontact surface. Two types of friction conditions are imposed in the form of the Tresca and Coulomb frictions. A two-scale asymptotic approach with respect to a small parameter, which describes the period of the microstructure, was suggested for the solution of the weak penalized contact problem.

The dependence of all contact parameters in the penalty and frictional functionals on the small geometric parameter, denoting the period of the structure, was investigated from the physical point of view in Chapter 2. Then the penalized Signorini conditions were interpreted in the form of Caratheodory monotone functionals of the traces of the solution on an oscillating interface, i.e. as the nonlinear Robin-type boundary conditions, while the Tresca frictional term was interpreted as the Neumann boundary condition on the oscillating interface. The known mathematical results on the convergence for Neumann and third-kind conditions on the oscillating interface were recalled in the thesis as an auxiliary machinery. The results of [Allaire, 1992], [Braides, Defranceschi, 1998] on the two-scale convergence of convex differentiable Caratheodory functions of a two-scale convergent argument were applied to penalty functionals, while for the friction it was shown that $\mathbf{v}_\tau$ is an admissible test function for the two-scale convergence.

Passing to the limit in the penalized variational inequality yields the homogenized problem and the auxiliary cell contact problems in the periodicity cell. The obtained coupled micro-macro contact problems can be used for future numerical computations by the finite element method (FEM).

The domain of a technical textile is usually represented as a thin layer. In Chapter 3 we consider a domain in the form of a layer with the thickness of the order of the in-plane periodicity. Thus, one needs to take into account the fact that the layer is vanishing. Therefore, in-plane homogenization decreases the dimension of the homogenized problem. The homogenization of the elasticity problem for such a domain was considered in [Panasenko, 2005]. In this homogenization procedure the zero Neumann boundary conditions on the out-of-plane boundaries were considered.

In Chapter 3 we deduce how to incorporate the non-homogeneous Neumann boundary conditions on the vanishing out-of-plane boundaries into the homogenized problem. We introduce additional terms in the formal asymptotic expansion and deduce the auxiliary problems and the homogenized problem. It turns out that the non-homogeneous Neumann boundary condition is incorporated into the homogenized problem as an additional term to the right hand side of the homogenized equation by the out-of-plane moduli.

Another aspect of the homogenization of a technical textile is that the thickness of the fibers is a small value compared to the size of the periodicity cell. This allows to consider the asymptotics with respect to the thickness of the fibers. In Chapter 3 we present known results of [Panasenko, 2005] and [Sanchez-Hubert, Sanchez-Palencia, 1999] and, using them, formulate reduced one-dimensional problems for thin fibers. We also present assumed one-dimensional contact conditions. This allows to reduce the dimension of the auxiliary problems.

In order to solve reduced contact auxiliary problems we use beam finite elements with hermitian shape functions. The construction of the finite element space is made naturally by taking into account the fiber structure of the geometry. In Chapter 4 we give the finite element formulation of the contact elasticity problem, where the one-dimensional contact conditions from Chapter 3 are introduced using the penalty method. We describe the contact algorithm given in [Wriggers, 2002] and deduce the matrix contribution of the normal and tangential contact to the global stiffness matrix.

In Chapter 5 we formulate an algorithm for computation of effective material properties of technical textiles using results of Chapters 2-4. We present numerical results obtained by our software FiberFEM.

Chapter 6 summarizes the results and gives conclusions.

1.4 Statement of the problem

Our goal is to homogenize the media and to obtain the effective material law of the textile or composite with periodic microstructure, using assumptions on its geometry and material properties.

We start this section with the assumptions on the geometry similar to those given in [Orlik, 2000].

Assumption 1.1. ***(Assumptions on the geometry)***. *We consider a heterogeneous solid which consists of linear elastic materials and has a microstructure with a period εY, where Y is the unit cell and ε is a scaling parameter. Consider l mutually disjoint generally non-connected Y-periodic Lipschitz domains $\Omega_i^{per} \subset \mathbb{R}^n$, $i = 1, ..., l$, $Y_i := \Omega_i^{per} \cap Y$, $i = 1, ..., l$. We assume that Ω_i^{per} are in multiple contact and define by $S^{per} = \cup_{i=1}^{l} \partial\Omega_i^{per}$ the periodic contact interface. We define further $S = Y \cap S^{per}$.*

Let $\Omega \subset \mathbb{R}^n$ be an open and bounded Lipschitz domain, we define $\Omega_i^\varepsilon = \varepsilon\Omega_i^{per} \cap \Omega$ and $\Omega^\varepsilon = \left(\cup_{i=1}^{l} \varepsilon\Omega_i^{per}\right) \cap \Omega$. Denote by $S^\varepsilon = \Omega \cap \left(\cup_{i=1}^{l} \partial\Omega_i^\varepsilon\right)$ the oscillating contact interface, $S_i^\varepsilon = \Omega \cap \partial\Omega_i^\varepsilon$, and by $\partial\Omega_{outer}^\varepsilon = \partial\Omega^\varepsilon \backslash S^\varepsilon$ the Lipschitz external boundary. Let $\partial\Omega_D^\varepsilon \subset \partial\Omega_{outer}^\varepsilon$ and $\partial\Omega_N^\varepsilon = \partial\Omega_{outer}^\varepsilon \backslash \partial\Omega_D^\varepsilon$ be the subsets of outer boundary, where Dirichlet and Neumann boundary conditions are applied.

The following remark can be found in [Orlik, 2000].

Remark 1.2. *Let us denote by $meas(\Omega^\varepsilon)$ the Lebesgue measure of the domain Ω^ε. It is obvious that $meas(\cup_{i=1}^{l} \bar{\Omega}_i^\varepsilon) = meas(\bar{\Omega}^\varepsilon)$ $\forall\varepsilon$. The number of cells in $\cup_{i=1}^{l} \bar{\Omega}_i^\varepsilon$ is approximately equal to $N^\varepsilon = meas(\Omega^\varepsilon)/meas(\varepsilon Y) = meas(\Omega^\varepsilon)/(meas(Y)\varepsilon^n)$. Define by $meas(S^\varepsilon)$ the Lebesgue measure of surface S^ε. Obviously, $meas(S^\varepsilon) = N^\varepsilon meas(\varepsilon S) = N^\varepsilon meas(S)\varepsilon^{n-1} = \frac{meas(\Omega)meas(S)}{meas(Y)}\frac{1}{\varepsilon}$. Consequently, the measure of S^ε grows as $\frac{1}{\varepsilon}$.*

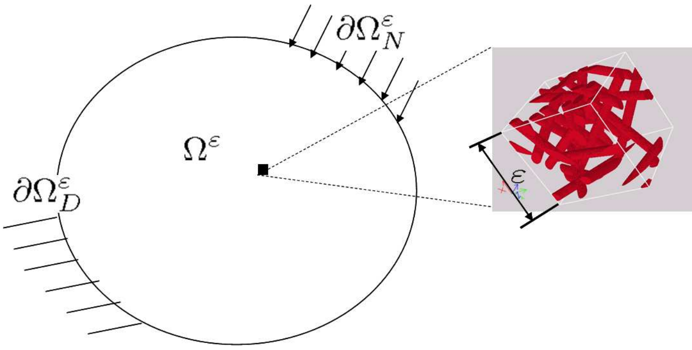


Figure 1.3: Domain Ω^ε.

1.4.1 Strong formulation of the multiscale contact problem

For fixed ε, we consider the Signorini contact problem with friction for a periodic microstructured solid in Ω^ε, satisfying Assumption 1.1, subjected to the body force $f(x)$ and the surface tractions $t(x)$ applied to the Neumann part of the boundary, $\partial\Omega_N^\varepsilon$, with prescribed displacements $g_0(x)$ applied on the Dirichlet boundary $\partial\Omega_D^\varepsilon$.

For the displacement vector $u^\varepsilon(x)$ and the symmetric 4th order elasticity tensor $A^\varepsilon(x) = (a_{ijkl}(\frac{x}{\varepsilon}))$, we consider the contact problem

$$\begin{aligned}
&\tfrac{\partial\sigma_{ij}^\varepsilon(x)}{\partial x_j} = f_i^\varepsilon(x), \quad \sigma_{ij}^\varepsilon(x) = a_{ijkl}^\varepsilon\tfrac{\partial u_k^\varepsilon(x)}{\partial x_l}, && x \in \Omega^\varepsilon,\\
&[u^\varepsilon(x)]_n \le g^\varepsilon(x), \quad \sigma_n^\varepsilon(x) \le 0, \quad \sigma_{\mathrm{n}}^\varepsilon(x)\left([u^\varepsilon(x)]_{\mathrm{n}} - g^\varepsilon(x)\right) = 0, && x \in S^\varepsilon,\\
&|\sigma_{\mathrm{t}}^\varepsilon(x)| < \mathcal{F}^\varepsilon(u^\varepsilon, x) \quad \Rightarrow \quad [u^\varepsilon(x)]_{\mathrm{t}} = 0, && x \in S^\varepsilon,\\
&|\sigma_{\mathrm{t}}^\varepsilon(x)| = \mathcal{F}^\varepsilon(u^\varepsilon, x) \;\Rightarrow\; \exists\, \lambda^\varepsilon \ge 0 \text{ s.t. } [u^\varepsilon(x)]_{\mathrm{t}} = -\lambda^\varepsilon\sigma_{\mathrm{t}}^\varepsilon(x), && x \in S^\varepsilon,\\
&\sigma_{ij}^\varepsilon n_j(x) = t_i(x), && x \in \partial\Omega_N^\varepsilon,\\
&u^\varepsilon(x) = g_0(x), && x \in \partial\Omega_D^\varepsilon,
\end{aligned} \tag{1.1}$$

where Einstein notation for repeating indices is used. σ_{ij}^ε are the components of the Cauchy stress tensor, $g^\varepsilon(x)$ is the initial gap function, which measures the distance between contacting bodies along the contact normal, and the friction law is given by the function

$$\mathcal{F}^\varepsilon(u^\varepsilon, x) := \begin{cases} G^\varepsilon(x), & \text{Tresca's friction,}\\ \mu^\varepsilon(x)|\sigma_n^\varepsilon(x)|, & \text{Coulomb's friction,}\end{cases} \tag{1.2}$$

where $\mu^\varepsilon(x)$ is the friction coefficient and $G^\varepsilon(x)$ is the friction traction, ε is a small parameter denoting the period of the microstructure. $n(x)$ is the normal unit outward vector.

$[u^\varepsilon(x)]_n$ can be represented as

$$[u^\varepsilon(x)]_n = [u^\varepsilon](x) \cdot n$$

and at the same time

$$[u^\varepsilon(x)]_t = [u^\varepsilon(x)] - [u^\varepsilon(x)]_n n.$$

$$\sigma_n^\varepsilon = (\sigma \cdot n) \cdot n, \qquad \sigma_t^\varepsilon = \sigma^\varepsilon \cdot n - \sigma_n^\varepsilon n.$$

Let $\xi = \frac{x}{\varepsilon} \in Y$ denote a fast variable.

Assumption 1.3. *The elasticity tensor $(a_{ijkl}^\varepsilon(x))$ is assumed to be symmetric at each point $x \in \Omega^\varepsilon$,*

$$a_{ijkl}^\varepsilon(x) = a_{jikl}^\varepsilon(x) = a_{ijlk}^\varepsilon(x) = a_{klij}^\varepsilon(x), \tag{1.3}$$

and positive-definite, with elements $a_{ijkl}^\varepsilon(x) = a_{ijkl}(\frac{x}{\varepsilon}) \in L_{per}^\infty(Y)$ bounded at each point $x \in \Omega^\varepsilon$,

$$c_0 \eta_l^k \eta_l^k \leq a_{ijkl}^\varepsilon(x) \eta_j^i \eta_l^k \leq C_0 \eta_l^k \eta_l^k, \tag{1.4}$$

for all $\eta_k^j = \eta_j^k \in \mathbb{R}$, where the constants $0 < c_0 \leq C_0 < \infty$ are independent of ξ.

For isotropic materials the elasticity tensor can be expressed using two material constants, for instance, two Lame parameters λ, μ:

$$a_{ijkl}^\varepsilon = \lambda \delta_{ij} \delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}). \tag{1.5}$$

The elements of the elasticity tensor can be piecewise constant functions, for instance as given in the following example.

Example 1.4.

$$a_{ijkm}^\varepsilon(x) = \begin{cases} a_{ijkm}^1, & x \in \Omega_1^\varepsilon, \\ \dots & \dots \\ a_{ijkm}^l, & x \in \Omega_l^\varepsilon. \end{cases}$$

Chapter 2

Homogenization of microcontact elasticity problems

The microstructure of a textile consists of fibers in frictional contact on the oscillating contact interface. Two-scale homogenization of periodic elasticity problems with frictional contact on the microstructure is considered here.

In this chapter we give the weak problem formulation and present two-scale convergence results for normal and tangential contact conditions. The rigorous derivation and proofs are given in [Orlik, in preparation]. The formal asymptotics for normal contact can be found in [Sanchez-Palencia, 1980]. The two-scale convergence approach for frictional term is used in the work of Julia Orlik [Orlik, in preparation]. The diagram in Fig. 2.1 shows the block scheme of this chapter.

2.1 Weak formulation of the problem

In order to introduce the weak formulation of the problem (1.1), we make the following assumption and definitions:

Assumption 2.1. ***(On the regularity)***. *Assume that the following properties hold: the elasticity coefficients $a^{\varepsilon}_{ijkl} \in L^{\infty}(\Omega^{\varepsilon})$, the boundary tractions $t_i \in H^{-1/2}(\partial\Omega^{\varepsilon}_N)$, the system of boundary values $g_{0_i} \in H^{1/2}(\partial\Omega^{\varepsilon}_D)$, and the body force components $f^{\varepsilon}_i \in L^2(\Omega^{\varepsilon})$. The non-penetration function $g^{\varepsilon} \in H^{1/2}(S^{\varepsilon})$ and $g^{\varepsilon} \geq 0$ a.e. in S^{ε}, the Coulomb friction coefficient $\mu^{\varepsilon} \in L^{\infty}(S^{\varepsilon})$ is globally bounded, and $\mu^{\varepsilon} > 0$ a.e. on S^{ε}. The Tresca friction function is $G^{\varepsilon} \in L^{\infty}(S^{\varepsilon})$.*

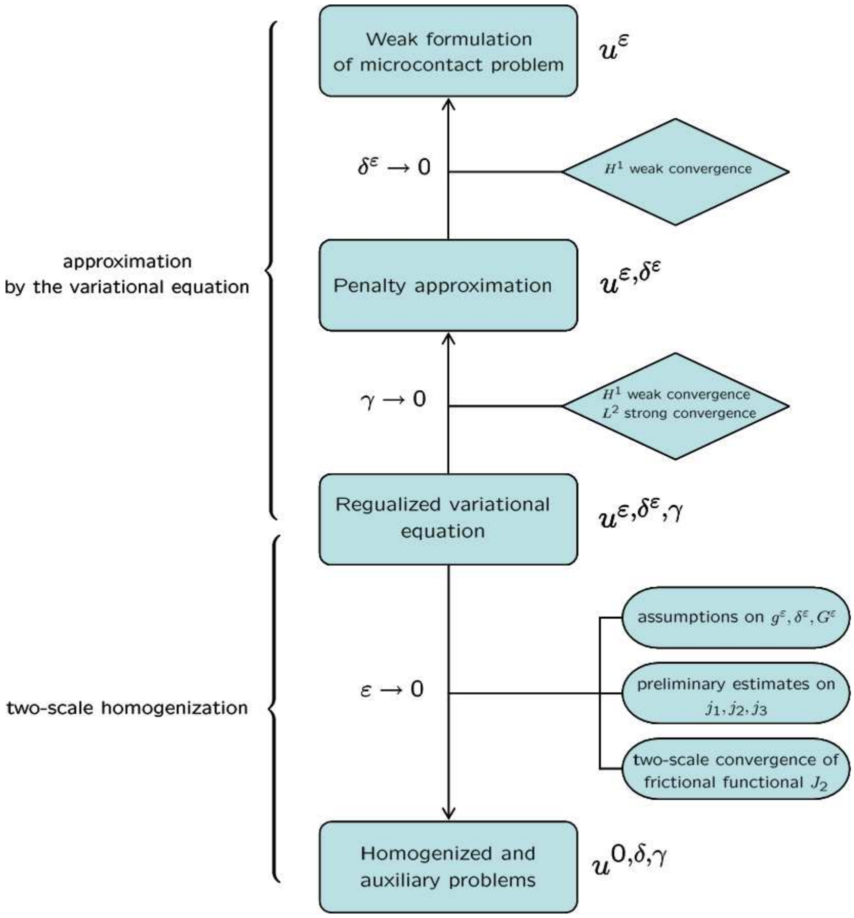


Figure 2.1: A block diagram overview of Chapter 2.

We introduce the functional space of displacements

$$\mathcal{V}^\varepsilon = \{v^\varepsilon, v_i^\varepsilon \in H^1(\Omega^\varepsilon) : v^\varepsilon = g_0 \text{ on } \partial\Omega_D^\varepsilon\}$$

and the closed, convex cone $\mathcal{K}^\varepsilon \subset \mathcal{V}^\varepsilon$,

$$\mathcal{K}^\varepsilon = \{v^\varepsilon \in \mathcal{V}^\varepsilon : [v^\varepsilon]_n \leq g^\varepsilon(\xi) \text{ on } S^\varepsilon\},$$

which describes the non-penetration constraint on the set of admissible functions. We define the bilinear form

$$a^\varepsilon(u^\varepsilon, v^\varepsilon) = \int_{\Omega^\varepsilon} a_{ijkl}^\varepsilon \frac{\partial u_i^\varepsilon}{dx_j} \frac{\partial v_k^\varepsilon}{dx_l} dx, \quad u^\varepsilon, v^\varepsilon \in \mathcal{V}^\varepsilon,$$

the functional

$$F^\varepsilon(v^\varepsilon) = \int_{\Omega^\varepsilon} f^\varepsilon \cdot v^\varepsilon dx + \int_{\partial\Omega_N^\varepsilon} t \cdot v^\varepsilon ds, \quad v^\varepsilon \in \mathcal{V}^\varepsilon$$

and the non-smooth frictional functional

$$J_2^\varepsilon(u^\varepsilon, v^\varepsilon) = \int_{S^\varepsilon} \mathcal{F}^\varepsilon(u^\varepsilon, x)\, |[v^\varepsilon]_t|\, ds, \quad u^\varepsilon, v^\varepsilon \in \mathcal{V}^\varepsilon. \tag{2.1}$$

Then, integrating by parts (1.1) and introducing the Neumann boundary conditions (see [Duvaut, Lions, 1976], [Kikuchi, Oden, 1988]), we get the weak formulation of the problem (1.1) in the form of a variational inequality.

Definition 2.2. *The weak formulation of the problem is: find $u^\varepsilon \in \mathcal{K}^\varepsilon$, s.t.*

$$a^\varepsilon(u^\varepsilon, v^\varepsilon - u^\varepsilon) + J_2^\varepsilon(u^\varepsilon, v^\varepsilon) - J_2^\varepsilon(u^\varepsilon, u^\varepsilon) \geq F^\varepsilon(v^\varepsilon - u^\varepsilon) \quad \forall v^\varepsilon \in \mathcal{K}^\varepsilon. \tag{2.2}$$

The contact conditions in (1.1) take the form of a non-penetration constraint on the set of admissible functions and a non-smooth frictional functional. There are different approaches to solve the contact minimization problem, see Section 1.2. In particular, the constraint on the set of admissible functions can be substituted by adding an additional functional, which penalizes violation of the constraint. We would like to use the penalty formulation, i.e. include the non-penetration condition in the form of a functional in the weak formulation in order to apply the two-scale convergence.

Remark 2.3. *The main difficulty in the study of solvability of the problem* (2.2) *for the general frictional law $\mathcal{F}^\varepsilon$ is that the frictional functional J_2^ε is non-smooth and non-convex, and the question of existence of solution, therefore, cannot be analyzed by the classical methods of the constrained optimization theory. In this work we give the theorems of existence of solutions for the Tresca and Coulomb frictional laws given by* (2.1) *with assumed regularity conditions 2.1.*

2.2 Approximation of the variational inequality by variational equation

Throughout this section we fix ε and consider the contact problem only. The penalty formulation for the contact problem comes from the penalty methods in minimization problems. They introduce an approach to a constrained optimization problem which avoids the necessity of introducing additional unknowns in the form of Langrange multipliers. In the content of the constrained minimization problem (2.2) we want to substitute the non-penetration constraint by adding a penalty functional.

2.2.1 Penalty formulation

Remark 2.4. *In Section 5.5 in [Kikuchi, Oden, 1988], the Coulomb law is associated with the nonlinear boundary conditions of the third or Robin type. The Tresca friction can be associated with the Neumann-type boundary condition.*

Consider the weak formulation of problem (2.2), which is equivalent to the variational inequality

$$\int_{\Omega^\varepsilon} a^\varepsilon_{ijkl}(x)\frac{\partial u^\varepsilon_k(x)}{\partial x_l}\frac{\partial(v^\varepsilon_i(x)-u^\varepsilon_i(x))}{\partial x_j}\,dx + J^\varepsilon_2(u^\varepsilon,v^\varepsilon) - J^\varepsilon_2(u^\varepsilon,u^\varepsilon)$$
$$\geq \int_{\Omega^\varepsilon} f_i(x)(v^\varepsilon_i(x)-u^\varepsilon_i(x))dx + \int_{\partial\Omega^\varepsilon_N} t_i(x)(v^\varepsilon_i(x)-u^\varepsilon_i(x))ds, \quad \forall v^\varepsilon \in \mathcal{K}^\varepsilon. \tag{2.3}$$

We replace the constraint $[v^\varepsilon]_n \leq g^\varepsilon$ by adding the penalty functional

$$J^{\varepsilon,\delta^\varepsilon}_1(u^{\varepsilon,\delta^\varepsilon},v^\varepsilon) = \frac{1}{\delta^\varepsilon}\int_{S^\varepsilon}[[u^{\varepsilon,\delta^\varepsilon}]\cdot n^\varepsilon - g^\varepsilon]_+ n^\varepsilon\cdot([v^\varepsilon]-[u^{\varepsilon,\delta^\varepsilon}])ds \tag{2.4}$$

with a small penalty parameter $\delta^\varepsilon > 0$ and $[\cdot]_+ := \max\{0,\cdot\}$. Let us make a remark, that the expression $\frac{1}{\delta^\varepsilon}(u^{\varepsilon,\delta^\varepsilon}(x)\cdot n^\varepsilon(x) - g^\varepsilon(x))$ represents the jump in the normal stress $\sigma^\varepsilon_n|_{S^{\varepsilon+}} - \sigma^\varepsilon_n|_{S^{\varepsilon-}}$ at the contact interface.

The following remark is taken from [Orlik, in preparation].

Remark 2.5. *Mechanical engineers interpret the functional $J^{\varepsilon,\delta^\varepsilon}_1$ as a contact layer of small penetration δ^ε. This contact layer can be mechanically represented by a spring of the stiffness $k^{\varepsilon,\delta}_n := 1/\delta^\varepsilon$, where $k^{\varepsilon,\delta}_n$ is called normal microcontact stiffness. Usually in the mechanical literature, for instance in [Goryacheva, 1988], it is taken as $k_n = E/\delta^\varepsilon$, where E is the Young's modulus of the material and δ^ε is the thickness of the artificial contact layer (it is also mentioned in [Kikuchi, Oden, 1988]).*

Definition 2.6. *The penalty formulation for δ^ε is: find $u^{\varepsilon,\delta^\varepsilon} \in \mathcal{V}^\varepsilon$, s.t.*

$$\begin{aligned}\int_{\Omega^\varepsilon} a^\varepsilon_{ijkl}(x)\frac{\partial u_k^{\varepsilon,\delta^\varepsilon}(x)}{\partial x_l}\frac{\partial(v_i^\varepsilon(x)-u_i^{\varepsilon,\delta^\varepsilon}(x))}{\partial x_j}\,dx \\ + \frac{1}{\delta^\varepsilon}\int_{S^\varepsilon}[[u^{\varepsilon,\delta^\varepsilon}(x)]\cdot n^\varepsilon(x) - g^\varepsilon(x)]_+ n^\varepsilon(x)\cdot([v^\varepsilon(x)]-[u^{\varepsilon,\delta^\varepsilon}(x)])ds \\ + \int_{S^\varepsilon}\mathcal{F}^\varepsilon(u^\varepsilon,x)(|[v^\varepsilon(x)]_t| - |[u^{\varepsilon,\delta^\varepsilon}(x)]_t|)ds \\ \geq \int_{\Omega^\varepsilon} f_i(x)(v_i^\varepsilon(x)-u_i^{\varepsilon,\delta^\varepsilon}(x))dx + \int_{\partial\Omega_N^\varepsilon} t_i(x)(v_i^\varepsilon(x)-u_i^{\varepsilon,\delta^\varepsilon}(x))ds \quad \forall v^\varepsilon\in\mathcal{V}^\varepsilon. \quad (2.5)\end{aligned}$$

The existence of a solution of the contact problem (2.5) for Coulomb and Tresca friction laws is a well-studied subject (see Theorem 2.2 in [Han, 1996] or Theorem 5.1 in [Duvaut, Lions, 1976]). We recall it as an auxiliary result for further analysis of our multiscale problem.

Theorem 2.7. *If measures of $\partial\Omega_D^\varepsilon, \partial\Omega_N^\varepsilon$ are strictly positive and regularity conditions 2.1 hold, then, for every $\delta^\varepsilon > 0$ and fixed $\varepsilon > 0$, the problem (2.5) has a solution $u^{\varepsilon,\delta^\varepsilon} \in \mathcal{V}^\varepsilon(\Omega^\varepsilon)$.*

Furthermore, for every fixed ε, there exists a subsequence of solutions $u^{\varepsilon,\delta^\varepsilon} \in \mathcal{V}^\varepsilon(\Omega^\varepsilon) \subset H^1(\Omega^\varepsilon)$ w.r.t. δ^ε, which converges weakly in $H^1(\Omega^\varepsilon)$, as $\delta^\varepsilon \to 0$, to at least one solution u^ε of the constrained problem (2.5) with the Coulomb's friction law, and if one replaces the Coulomb's friction by a given friction $G^\varepsilon(x)$ (Tresca condition), the limit solution will be unique.

The proof is based on the convexity and Gateaux-differentiability of the functional $I^{\varepsilon,\delta^\varepsilon}(v^\varepsilon) := \frac{1}{2}a^\varepsilon(v^\varepsilon,v^\varepsilon) + \frac{1}{2}J_1^{\varepsilon,\delta^\varepsilon}(v^\varepsilon,v^\varepsilon) + J_2^\varepsilon(v^\varepsilon,v^\varepsilon) - F^\varepsilon(v^\varepsilon)$, which implies its weak lower semicontinuity and the coercivity. The last one is based on the coercivity of the bilinear form $a^\varepsilon(v^\varepsilon,v^\varepsilon)$ and Korn's inequality (see Chapter 3 in [Kikuchi, Oden, 1988] and Theorem 3.1 in [Duvaut, Lions, 1976]). For the Coulomb's friction see Theorem 2.7 in [Eck, PhD thesis, 1996].

Furthermore, Theorem 2.3 in [Han, 1996] proves the estimate

$$||u^{\varepsilon,\delta^\varepsilon}||_{H^1(\Omega^\varepsilon)} \leq c\left(||f^\varepsilon||_{L^2(\Omega^\varepsilon)} + ||t||_{L^2(\partial\Omega_N^\varepsilon)} + ||g_0||_{H^{1/2}(\partial\Omega_D^\varepsilon)}\right) \quad (2.6)$$

for the solution of problem (2.5), where the constant c depends only on the measure of domain Ω^ε and the constants from the conditions of the positive definiteness and boundedness of the elasticity tensor (a^ε_{ijkl}). To prove, it is enough to observe that the under-integral expressions in the penalty and friction functions are non-negative (since $g^\varepsilon \geq 0$) and, hence, can be skipped in the inequality (2.5) considered for $v^\varepsilon = 0$.

2.2.2 Regularized variational equation

Although we have studied the solvability of problem (2.5), solving it numerically would be difficult not only because of its different scales, but also due to the presence of the non-differentiable Euclidean norms $|[v]_t^\varepsilon|$ and $|[u^{\varepsilon,\delta^\varepsilon}]_t|$. For regularization purposes, we follow [Kikuchi, Oden, 1988] and replace them by the smooth and convex approximation $\phi^{\gamma^\varepsilon}(v)$, where γ^ε denotes the regularization parameter:

$$\phi^{\gamma^\varepsilon}(v^\varepsilon) = \begin{cases} |v^\varepsilon| - \frac{1}{2}\gamma^\varepsilon & \text{if } |v^\varepsilon| \geq \gamma^\varepsilon, \\ \frac{1}{2\gamma^\varepsilon}|v^\varepsilon|^2 & \text{if } |v^\varepsilon| < \gamma^\varepsilon, \end{cases} \tag{2.7}$$

$$\sigma_t([u^{\varepsilon,\delta^\varepsilon,\gamma^\varepsilon}]_t) = -\mathcal{F}^\varepsilon \frac{\partial \phi}{\partial [u^{\varepsilon,\delta^\varepsilon,\gamma^\varepsilon}]_t}([u^{\varepsilon,\delta^\varepsilon,\gamma^\varepsilon}]_t) = \begin{cases} -\mathcal{F}^\varepsilon [u^{\varepsilon,\delta^\varepsilon,\gamma^\varepsilon}]_t / |[u^{\varepsilon,\delta^\varepsilon,\gamma^\varepsilon}]_t|, & \text{if } |[u^{\varepsilon,\delta^\varepsilon,\gamma^\varepsilon}]_t| \geq \gamma^\varepsilon, \\ -\mathcal{F}^\varepsilon [u^{\varepsilon,\delta^\varepsilon,\gamma^\varepsilon}]_t / \gamma^\varepsilon, & \text{if } |[u^{\varepsilon,\delta^\varepsilon,\gamma^\varepsilon}]_t| < \gamma^\varepsilon. \end{cases} \tag{2.8}$$

Then, after regularization we obtain the variational equation.

Definition 2.8. *Find $u_i^{\varepsilon,\delta^\varepsilon,\gamma^\varepsilon} \in \mathcal{V}^\varepsilon$, such that*

$$\begin{aligned} &\int_{\Omega^\varepsilon} a_{ijkl} \frac{\partial u_k^{\varepsilon,\delta^\varepsilon,\gamma^\varepsilon}(x)}{x_l} \frac{\partial v_i^\varepsilon(x)}{x_j}\, dx + \frac{1}{\delta^\varepsilon} \int_{S^\varepsilon} [[u^{\varepsilon,\delta^\varepsilon,\gamma^\varepsilon}(x)]_{\mathrm{n}} - g^\varepsilon(x)]_+ [v^\varepsilon(x)]_{\mathrm{n}}\, ds \\ &+ \int_{S^\varepsilon} \mathcal{F}^\varepsilon(u^{\varepsilon,\delta^\varepsilon,\gamma^\varepsilon}, x) \frac{\partial \phi}{\partial [u^{\varepsilon,\delta^\varepsilon,\gamma^\varepsilon}]_t}([u^{\varepsilon,\delta^\varepsilon,\gamma^\varepsilon}]_t)(x)[v^\varepsilon(x)]_t\, ds \\ &= \int_{\Omega^\varepsilon} f_i(x) v_i^\varepsilon(x) dx + \int_{\partial\Omega_N^\varepsilon} t_i(x) v_i^\varepsilon(x) ds \qquad \forall v^\varepsilon \in \mathcal{V}^\varepsilon. \end{aligned} \tag{2.9}$$

The question of existence of solution of problem (2.9) for Tresca friction and convergence with respect to the regularization parameter δ^ε are formulated in Theorems 10.3 and 10.4 in [Kikuchi, Oden, 1988]. For the Coulomb friction, the solvability and convergence results can be found in Proposition 2 in [Eck, Jarusek, 1998]. We summarize these results in the next theorem.

Theorem 2.9. *Let the measure of $\partial\Omega_D^\varepsilon$ be strictly positive and regularity conditions 2.1 hold. Then, for each $\delta^\varepsilon > 0$ and each $\gamma^\varepsilon > 0$, there exists a solution $u^{\varepsilon,\delta^\varepsilon,\gamma^\varepsilon} \in \mathcal{V}^\varepsilon(\Omega^\varepsilon)$*

of problem (2.9). An appropriate sequence $u^{\varepsilon,\delta^\varepsilon,\gamma_k} \in \mathcal{V}^\varepsilon(\Omega^\varepsilon) \subset H^1(\Omega^\varepsilon)$ of these solutions converges weakly in $H^1(\Omega^\varepsilon)$ and strongly in $L^2(\Omega^\varepsilon)$ to a solution $u^{\varepsilon,\delta^\varepsilon} \in \mathcal{V}^\varepsilon(\Omega^\varepsilon)$ of variational inequality (2.5) as $\gamma_k^\varepsilon \to 0$. Moreover, in the case of Tresca friction, the solution $u^{\varepsilon,\delta^\varepsilon,\gamma^\varepsilon} \in \mathcal{V}^\varepsilon(\Omega^\varepsilon)$ is unique and an appropriate sequence $u^{\varepsilon,\delta^\varepsilon,\gamma_k^\varepsilon} \in \mathcal{V}^\varepsilon(\Omega^\varepsilon) \subset H^1(\Omega^\varepsilon)$ of these solutions converges strongly in $H^1(\Omega^\varepsilon)$.

2.3 Two-scale convergence

The previous subsection was devoted to the questions of existence of solution for fixed ε and to equivalence of solutions of different variational formulations of the contact elasticity problem. This section will provide the results on two-scale convergence of solution of the resulted regularized variational equation. To apply them, we have to recall some known preliminary results.

For further analysis we use Tresca friction, i.e. $\mathcal{F}^\varepsilon(u^{\varepsilon,\delta^\varepsilon,\gamma^\varepsilon}, x) = G^\varepsilon(x)$, and introduce the following integrals

$$j_2^{\varepsilon,\delta^\varepsilon,\gamma^\varepsilon}(u^{\varepsilon,\delta^\varepsilon,\gamma^\varepsilon}, v^\varepsilon) = -\frac{1}{\gamma^\varepsilon}\int_{S^\varepsilon} G^\varepsilon(x)[u^{\varepsilon,\delta^\varepsilon,\gamma^\varepsilon}]_t[v^\varepsilon]_t ds,$$

$$j_3^{\varepsilon,\delta^\varepsilon,\gamma}(u^{\varepsilon,\delta^\varepsilon,\gamma^\varepsilon}, v^\varepsilon) = -\int_{S^\varepsilon} G^\varepsilon(x)\frac{[u^{\varepsilon,\delta^\varepsilon,\gamma}]_t}{|[u^{\varepsilon,\delta^\varepsilon,\gamma}]_t|}[v^\varepsilon]_t ds.$$

Then the frictional functional $J_2^{\varepsilon,\delta^\varepsilon,\gamma^\varepsilon}$ is expressed as

$$J_2^{\varepsilon,\delta^\varepsilon,\gamma^\varepsilon} = \begin{cases} j_2^{\varepsilon,\delta^\varepsilon,\gamma^\varepsilon}, & \text{if } |[u]_t^{\varepsilon,\delta^\varepsilon,\gamma^\varepsilon}| < \gamma^\varepsilon, \\ j_3^{\varepsilon,\delta^\varepsilon,\gamma^\varepsilon}, & \text{if } |[u]_t^{\varepsilon,\delta,\gamma}| \geq \gamma^\varepsilon. \end{cases}$$

The following assumption in [Orlik, in preparation] is based on Remark 2.5.

Assumption 2.10. ***(Physical assumption)***. *Assume that the initial gap function is of the order of the size of the periodicity cell ε, i.e. $g^\varepsilon(x) = \varepsilon\bar{g}(x/\varepsilon)$. The thicknesses of the penetration layers δ^ε, γ^ε on the interface S^ε are small parameters, which can be assumed to be at least of the order of the periodicity cell, i.e, we can assume that $\delta^\varepsilon = \delta\varepsilon$, $\gamma^\varepsilon = \gamma\varepsilon$ on S^ε. The friction coefficient is $\mu^\varepsilon(x) = \mu(\varepsilon^{-1}x)$ and $G^\varepsilon(x) = G(x, \varepsilon^{-1}x)$ on S^ε.*

Remark 2.11. *Since we are in the framework of the theory of the infinitesimal deformations, $[u]_t$ should not exceed ε, i.e. the tangential sliding represented by the $j_3^{\varepsilon,\delta^\varepsilon,\gamma^\varepsilon}$ can be neglected. Therefore, to estimate the frictional functional, we have to consider only $j_2^{\varepsilon,\delta^\varepsilon,\gamma^\varepsilon}$.*

Remark 2.12. *To obtain the bounded invertible operator on the left-hand side of* (2.9)*, we need to estimate the penalty functionals J_1, j_2 from below. Furthermore, these functionals are analogous to the Robin-type interface conditions considered in Subsection 2.3.1.*

2.3.1 Auxiliary results

The following Definitions and Theorems 2.13-2.15 can be found in [Allaire, 1992] and will be used for the proof of our main results.

Definition 2.13. *A sequence of functions u^ε in $L^2(\Omega)$ is said to two-scale converge to a limit $u_0(x,\xi) \in L^2(\Omega \times Y)$, iff for any function $\psi(x,\xi) \in D(\Omega, C^\infty_{per}(Y))$ we have*

$$\lim_{\varepsilon \to 0} \int_\Omega u^\varepsilon(x)\psi(x, \tfrac{x}{\varepsilon})dx = \frac{1}{|Y|}\int_\Omega \int_Y u_0(x,\xi)\psi(x,\xi)dxd\xi. \tag{2.10}$$

This makes sense because of the following compactness theorem.

Theorem 2.14. *From each bounded sequence $u^\varepsilon \in L^2(\Omega)$ we can extract a subsequence that two-scale converges to $u_0(x,\xi) \in L^2(\Omega \times Y)$.*

The following Theorem is presented in [Allaire, 1992], see Lemma 1.3. We only replace the space $L^2(\Omega, C_{per}(Y))$ by $L^2_{per}(Y, C(\bar{\Omega}))$, owing Remark 1.5 and Corollary 5.4 from the same source.

Theorem 2.15. *Let u^ε be a bounded sequence in $H^1(\Omega)$ that converges weakly to a limit $u \in H^1(\Omega)$. Then u^ε two-scale converges to $u_0(x,\xi) := u(x)$, and there exists a function $u_1(x,\xi)$ in $L^2(\Omega, H^1_{per}(Y))$, such that, up to a subsequence, ∇u^ε two-scale converges to $\nabla_x u(x) + \nabla_\xi u_1(x,\xi)$.*

Remark 2.16. *"Converges up to a subsequence" means that the sequence u^ε has a convergent subsequence, $u^{\varepsilon'}$, which we again redenote by u^ε.*

The following theorem gives an estimate for the solution with a corrector and can be found in Theorem 3.6. in [Allaire, 1992] or Theorem 9.7. in [Oleinik, Schamaev, Yosifian].

Theorem 2.17. *Let the elasticity coefficients in the elasticity operator be smooth periodic functions and the right-hand side function $f \in H^1(\Omega)$. Then the following estimate is valid*

$$||u^\varepsilon(x) - u^0(x) - \varepsilon u^1(x, \frac{x}{\varepsilon})||_{H^1(\Omega)} \le c\sqrt{\varepsilon}||f||_{H^1(\Omega)}. \tag{2.11}$$

The two-scale convergence was also considered for $(n-1)$-dimensional structures, see [Neuss-Radu, 1996] and [Allaire, Damlamian, Hornung, 1995].

Definition 2.18. *A sequence of functions $u^\varepsilon \in L^2(S^\varepsilon)$ equipped with the scaled norm $||u^\varepsilon||^2_{L^2(S^\varepsilon)} := \varepsilon \int_S u^\varepsilon(x)^2 dx$ is said to two-scale converge to a limit $u_0 \in L^2(\Omega \times S)$ iff for any $\psi \in C(\bar{\Omega}, C_{per}(Y))$*

$$\lim_{\varepsilon\to 0} \varepsilon \int_{S^\varepsilon} u^\varepsilon(x)\psi(x, \tfrac{x}{\varepsilon})ds = \frac{1}{|Y|}\int_\Omega \int_S u_0(x,\xi)\psi(x,\xi)dxds_\xi \tag{2.12}$$

holds, where ds, ds_ξ have to be understood as the Hausdorff measures on S^ε and S respectively.

The following compactness theorem is also given by the same authors.

Theorem 2.19. *From each sequence $u^\varepsilon \in L^2(S^\varepsilon)$, bounded w.r.t. the scaled norm, we can extract a subsequence that two-scale converges to $u_0 \in L^2(\Omega \times S)$.*

For estimation of functionals $J_1^{\varepsilon,\delta^\varepsilon,\gamma}$ and $j_2^{\varepsilon,\delta,\gamma^\varepsilon}$, as well as for the two-scale convergence, we recall the results from [Hummel, 1999], providing convergence for a problem with linear Robin-type interface conditions:

Boundedness, compactness and convergence for a problem with linear Robin-type interface conditions

Consider a two-scale elasticity problem with Robin-type conditions on the interface

$$
\begin{aligned}
\frac{\partial}{\partial x_h}\Big[a_{ihjk}(\tfrac{x}{\varepsilon})\frac{\partial u_j^\varepsilon(x)}{\partial x_k}\Big] &= f_i(x), \text{ in } \Omega^\varepsilon, \\
a_{ihjk}(\tfrac{x}{\varepsilon})\frac{\partial u_j^\varepsilon(x)}{\partial x_k}n_h(x)\mid_{S_+^\varepsilon} &= a_{ihjk}(\tfrac{x}{\varepsilon})\frac{\partial u_j^\varepsilon(x)}{\partial x_k}n_h(x)\mid_{S_-^\varepsilon}, \\
a_{ihjk}(\tfrac{x}{\varepsilon})\frac{\partial u_j^\varepsilon(x)}{\partial x_k}n_h(x)\mid_{S_+^\varepsilon} &= \varepsilon^{-1}h^\varepsilon(x)[u_i^\varepsilon(x)], \ x\in S^\varepsilon, \\
u_i^\varepsilon(x) &= 0, \qquad x\in\partial\Omega_D^\varepsilon,
\end{aligned}
\tag{2.13}
$$

with $h^\varepsilon(x) := h(\frac{x}{\varepsilon}) \geq c_a, \ \forall x\in\Omega^\varepsilon$, and $\int_S h^\varepsilon(\xi)ds_\xi = 0$.
All following results are recalled from [Hummel, 1999]. The following Korn's inequality for discontinuous on S^ε periodic functions is analogous to the Poincaré inequality from [Hummel, 1999].

Lemma 2.20. (Korn's inequality in $\Omega^\varepsilon \setminus S^\varepsilon$). *Let Ω^ε be a bounded domain with a periodic structure, S^ε be an oscillating interface and $u_i^\varepsilon \in H_0^1(\Omega^\varepsilon\setminus S^\varepsilon)$. Then there exists a constant $C_0 > 0$ independent of ε, such that*

$$
||u^\varepsilon||^2_{L^2(\Omega^\varepsilon)} \leq C_0\left(\frac{1}{4}\int_{\Omega^\varepsilon\setminus S^\varepsilon}\Big(\frac{\partial u_i^\varepsilon}{\partial x_h}+\frac{\partial u_h^\varepsilon}{\partial x_i}\Big)\Big(\frac{\partial u_i^\varepsilon}{\partial x_h}+\frac{\partial u_h^\varepsilon}{\partial x_i}\Big)dx + \varepsilon^{-1}\int_{S^\varepsilon\cap\Omega^\varepsilon}[u^\varepsilon]^2 d\mathcal{H}^{n-1}\right), \tag{2.14}
$$

where $\mathcal{H}^{n-1}$ is the $(n-1)$-dimensional Hausdorff measure.

The proof coincides with the proof of the Poincaré inequality in $\Omega^\varepsilon\setminus S^\varepsilon$ given in [Hummel, 1999]. Let us study the ε-problem. We define the bilinear form

$$
a^\varepsilon(\varphi,\psi) := \int_{\Omega^\varepsilon\setminus S^\varepsilon}\nabla\psi\cdot A^\varepsilon\nabla\varphi dx + \varepsilon^{-1}\int_{S^\varepsilon\cap\Omega^\varepsilon}h^\varepsilon[\varphi][\psi]ds, \qquad \varphi_i,\psi_i\in H^1(\Omega^\varepsilon\setminus S^\varepsilon). \tag{2.15}
$$

Then the weak problem formulation can be written as

$$
a^\varepsilon(u^\varepsilon,\varphi) = \int_{\Omega^\varepsilon} f\varphi dx \qquad \forall\varphi\in(H_0^1(\Omega^\varepsilon\setminus S^\varepsilon))^n. \tag{2.16}
$$

Theorem 2.21. (Compactness of the solution to the two-scale elasticity problem with the Robin-type interface conditions). *Let Ω^ε be a bounded domain with a periodic structure, S^ε be an oscillating interface, and $u^\varepsilon \in H_0^1(\Omega^\varepsilon\setminus S^\varepsilon)$ be the solution to*

the two-scale elasticity problem with the Robin-type interface conditions (2.13).
(1) Then there exists a constant $C > 0$ independent of ε, such that the solution u^ε satisfies the energy estimate:

$$\frac{1}{4}\int_{\Omega^\varepsilon\setminus S^\varepsilon}\Big(\frac{\partial u_i}{\partial x_h}+\frac{\partial u_h}{\partial x_i}\Big)\Big(\frac{\partial u_i}{\partial x_h}+\frac{\partial u_h}{\partial x_i}\Big)dx+\varepsilon^{-1}\int_{S^\varepsilon\cap\Omega^\varepsilon}[u^\varepsilon]^2 d\mathcal{H}^{n-1} < C||f||^2_{L^2(\Omega^\varepsilon)}. \quad (2.17)$$

(2) There exists a function $u \in L^2(\Omega^\varepsilon)$, such that, up to a subsequence, $u^\varepsilon \to u$ strongly in L^2 as $\varepsilon \to 0$.

Proof. Since A^ε and h^ε are bounded, the bilinear form a^ε is also bounded. Also, from $h^\varepsilon \geq c_a$ for all $x \in S^\varepsilon$, and Korn's inequality (2.14), it follows that a^ε is elliptic on $H_0^1(\Omega^\varepsilon \setminus S^\varepsilon)$. Hence, existence and uniqueness of the solutions follow from the Lax-Milgram lemma.

$$c_a\left(\frac{1}{4}\int_{\Omega^\varepsilon\setminus S^\varepsilon}\Big(\frac{\partial u_i}{\partial x_h}+\frac{\partial u_h}{\partial x_i}\Big)\Big(\frac{\partial u_i}{\partial x_h}+\frac{\partial u_h}{\partial x_i}\Big)dx+\varepsilon^{-1}\int_{S^\varepsilon\cap\Omega^\varepsilon}[u^\varepsilon]^2 ds\right)$$
$$\leq a^\varepsilon(u^\varepsilon,u^\varepsilon)=\int_{\Omega^\varepsilon} f u^\varepsilon dx \leq ||f||^2_{L^2(\Omega^\varepsilon)}||u^\varepsilon||^2_{L^2(\Omega^\varepsilon)}. \quad (2.18)$$

The Korn inequality (2.14) completes the proof of the part (1). The statement of the part (2) follows directly from the estimate (2.17) and Theorem 2.14. □

Theorem 2.22. (Convergence theorem for the two-scale elasticity problem with the Robin-type interface conditions). *Let Ω^ε be a bounded domain with a periodic structure, S^ε be an oscillating interface, and $u^\varepsilon \in H_0^1(\Omega^\varepsilon \setminus S^\varepsilon)$ be the solution to the two-scale elasticity problem with the Robin-type interface conditions (2.13).*
Then there exist functions $u \in H_0^1(\Omega^\varepsilon)$ and $u_1 \in L^2(\Omega^\varepsilon; H^1_{per}(\Omega^\varepsilon \setminus S))$, such that (up to a subsequence) $u^\varepsilon \to u$, $\mathbf{1}_{\Omega^\varepsilon\setminus S^\varepsilon}\nabla u^\varepsilon \to \nabla u + \nabla u_1$ and $\mathbf{1}_{S^\varepsilon\cap\Omega^\varepsilon}\varepsilon^{-1}[u^\varepsilon]n^\varepsilon \to \mathbf{1}_{\Omega^\varepsilon\times(S\cap Y)}[u_1]n$ for $\varepsilon \to 0$ in the two-scale sense. That is, for all suitable test functions $\psi \in C_0^\infty(\Omega^\varepsilon, C_Y^\infty(S))$ the following holds

$$\int_{\Omega^\varepsilon\setminus S^\varepsilon}\nabla u^\varepsilon(x)\psi(x,\varepsilon^{-1}x)dx \to \int_\Omega^\varepsilon\int_{Y\setminus S}(\nabla_x u(x)+\nabla_\xi u_1(x,\xi))\psi(x,\xi)d\xi dx, \quad (2.19)$$
$$\int_{S^\varepsilon\cap\Omega^\varepsilon}[u^\varepsilon](x)n^\varepsilon(x)\psi(x,\varepsilon^{-1}x)ds \to \int_\Omega^\varepsilon\int_{S\cap Y}[u_1](x,\xi)n(x,\xi)\psi(x,\xi)ds_\xi. \quad (2.20)$$

For a proof see Proposition 5.5 in [Hummel, 1999].

Homogenization of the two-scale Robin-type elasticity problem

The auxiliary problem for an auxiliary function N_q, which consists of the columns n^{qp}, on the periodicity cell will be

$$\begin{aligned}
\frac{\partial}{\partial \xi_h}\Big[a_{ihjk}(\xi)\frac{\partial(n_j^{qp}(\xi)+\delta_{pj}\xi_q)}{\partial \xi_k}\Big] &= 0, \text{ in } Y, &(2.21)\\
a_{ihjk}(\xi)\frac{\partial(n_j^{qp}(\xi)+\delta_{pj}\xi_q)}{\partial \xi_k}n_h(\xi)\mid_{S^+} &= a_{ihjk}(\xi)\frac{\partial(n_j^{qp}(\xi)+\delta_{pj}\xi_q)}{\partial \xi_k}n_h(\xi)\mid_{S^-},\ \xi\in S,\\
a_{ihjk}(\xi)\frac{\partial(n_j^{qp}(\xi)+\delta_{pj}\xi_q)}{\partial \xi_k}n_h(\xi)\mid_{S^+} &= h[n_i^{qp}(\xi)],\ \xi\in S,\\
& \quad n_j^{qp} \text{ is } Y-\text{periodic}. &(2.22)
\end{aligned}$$

The homogenized elasticity tensor will be given by

$$A^{hom}_{i,k} := \int_{Y\setminus S}(\nabla n^{qi}+e_i)\cdot A^\varepsilon(\nabla n^{qk}+e_k)d\xi + \int_{S\cap Y} h[n^{qi}][n^{qk}]ds_\xi .$$

It is positive definite, since A^ε and h^ε are positive definite. Hence, there exists a unique solution $u\in H_0^1(\Omega)$ of the homogenized problem

$$\int_\Omega \nabla\varphi\cdot A^{hom}\nabla u = \int_\Omega f\varphi dx,\ \forall\varphi\in (H_0^1(\Omega))^n. \tag{2.23}$$

Theorem 2.23. *Let Ω^ε be a bounded domain with a periodic structure, S^ε be an oscillating interface, and $u^\varepsilon\in H_0^1(\Omega^\varepsilon\setminus S^\varepsilon)$. Then the problem (2.15)-(2.16) converges to*

$$\int_\Omega^\varepsilon\int_{Y\setminus S}(\nabla u+\nabla_\xi u_1)\cdot A^\varepsilon(\nabla\varphi+\nabla_\xi\varphi_1)d\xi dx + \int_\Omega^\varepsilon\int_{S\cap Y}h[u_1][\varphi_1]ds_\xi dx = \int_\Omega^\varepsilon f\varphi dx, \tag{2.24}$$

or, in the strong formulation,

$$\begin{aligned}
&-\mathrm{div}_x\,(\int_Y A^\varepsilon(\xi)(\nabla_x u(x)+\nabla_\xi u_1(\cdot,\xi))d\xi) = f, \qquad x\in\Omega^\varepsilon, &(2.25)\\
&-\mathrm{div}_\xi\,(A^\varepsilon(\xi)(\nabla_x u(x)+\nabla_\xi u_1(x,\cdot))) = 0, \qquad x\in\Omega,\ \xi\in Y\setminus S,\\
&[A^\varepsilon(\xi)(\nabla_x u(x)+\nabla_\xi u_1(x,\cdot))] = 0, \qquad x\in\Omega^\varepsilon,\ \xi\in S\cap Y,\\
&A^\varepsilon(\xi)(\nabla_x u(x)+\nabla_\xi u_1(x,\cdot)) = h[u_1(x,\xi)], \qquad x\in\Omega^\varepsilon,\ \xi\in S\cap Y,\\
&(u,u_1)\in H_0^1(\Omega^\varepsilon)\times L^2(\Omega^\varepsilon, H^1_{per}(Y\setminus S)),
\end{aligned}$$

as ε tends to 0.

The proof is similar to the proof of Theorem 2.15. One should take the test function in (2.16) $\varphi(x) + \varepsilon\varphi_1(x, \varepsilon^{-1}\varphi)$ with $\varphi \in C_0^\infty(\Omega^\varepsilon)$, $\varphi_1 \in C_0^\infty(\Omega^\varepsilon, H^1_{per}(Y))$, and then in each term go to the limit known from the previous lemmas.

2.4 The main results of the chapter

At this point, the results presented in this section are outlined without the proofs. However, the proofs of the theorems will be given in [Orlik, in preparation]. Note that these results are formulated for any microcontact problem with highly oscillating inner contact interface, not restricted to textiles with fiber microstructure.

In order to construct the algorithm given in Chapter 5, the auxiliary problem and the limit problem are presented here.

Proposition 2.24. (Convergence for penalty terms). *Let Ω^ε be a bounded domain with a periodic structure, S^ε be an oscillating interface and $u^\varepsilon \in H_0^1(\Omega^\varepsilon \setminus S^\varepsilon)$ be the solution to the two-scale elasticity problem with the contact interface condition. Let further $u \in H_0^1(\Omega^\varepsilon)$ and $u_1 \in L^2(\Omega^\varepsilon; H^1_{per}(\Omega^\varepsilon \setminus S))$ be such that (up to a subsequence) $u^\varepsilon \to u$, $\mathbf{1}_{\Omega^\varepsilon\setminus S^\varepsilon}\nabla u^\varepsilon \to \nabla u + \nabla u_1$ and $\mathbf{1}_{S^\varepsilon\cap\Omega^\varepsilon}\varepsilon^{-1}[u^\varepsilon]n^\varepsilon \to \mathbf{1}_{\Omega^\varepsilon\times(S\cap Y)}[u_1]n$ for $\varepsilon \to 0$ in the two-scale sence. Let $v^\varepsilon(x) := v(x) + \varepsilon v_1(x, \frac{x}{\varepsilon})$ be a suitable test function with $v \in D(\Omega^\varepsilon \cup \partial\Omega^\varepsilon_N)$ and $v_1 \in D(\Omega^\varepsilon, C^\infty_{per}(S \cap Y))$. Then the macroscopic non-penetration condition is*

$$\lim_{\varepsilon\to 0} J_i^{\varepsilon,\delta}(u^\varepsilon, v + \varepsilon v_1, x)|_{\Omega^\varepsilon} = J_1^{1,\delta}(u_1, v_1, x)|_{\Omega^\varepsilon}, \quad i = 1, 2,$$

i.e. the non-penetration condition for the auxiliary periodicity cell problem on S, where $[v] = [u_0] = 0$ is given by

$$\frac{1}{\delta\varepsilon}\int_{S^\varepsilon}[[u^\varepsilon](x)\cdot n^\varepsilon(x) - g^\varepsilon(x)]_+ n^\varepsilon(x)\cdot([u_0 + \varepsilon v_1] - [u^{\varepsilon,\delta}])\,ds$$
$$\to \frac{1}{\delta|T|}\int_\Omega\int_S[[u_1](x,\xi)\,n(x,\xi) - \bar{g}(\xi)]_+ n(x,\xi)([v_1](x,\xi) - [u_1](x,\xi))\,ds_\xi\,dx,$$

and the friction condition is

$$\lim_{\varepsilon\to 0}\frac{1}{\gamma^\varepsilon}\int_{S^\varepsilon} G^\varepsilon(x)[u^\varepsilon]_t[v\varepsilon]_t ds = \frac{1}{\gamma|T|}\int_S G(x,\xi)[u_1]_t[v_1]_t ds_\xi dx, \quad [u_0]_t = 0.$$

The proofs of both statements will be given in [Orlik, in preparation], looking through the results of [Hummel, 1999] given in Subsection 2.3.1.

2.4.1 Homogenized contact problem

Theorem 2.25. (Homogenized system).

The sequence u^ε of solutions of (1.1) *has a subsequence strongly convergent to $u_0(x)$ and the sequence ∇u^ε has a subsequence, two-scale convergent to $\nabla u_0(x) + \nabla_\xi u_1(x,\xi)$, where $(u_0, u_1) \in H^1(\Omega^\varepsilon, \partial\Omega^\varepsilon_D) \times H^1(\Omega^\varepsilon, H^1_{per[0]}(Y \setminus S))$ is the unique solution of the two-scale homogenized system.*

Limit problem

$$\begin{aligned}&\int_{\Omega^\varepsilon}\left[\frac{1}{|Y|}\int_Y a_{ijkl}\left(\frac{\partial(u_0)_k(x)}{\partial x_l}+\frac{\partial(u_1)_k(x,\xi)}{\partial\xi_l}\right)\left(\frac{\partial((v_0)_i(x,\xi)-(u_0)_i(x,\xi))}{\partial x_j}\right.\right.\\&+\left.\left.\frac{\partial((v_1)_i(x,\xi)-(u_1)_i(x,\xi))}{\partial\xi_j}\right)d\xi+\frac{1}{|Y|}\int_S\mathcal{F}(u_1,\hat{x},\hat{\xi})(|[v_1]_\tau(x,\xi)|-|[{u_1}^\delta]_\tau(x,\xi)|)ds_\xi\right]dx\\&\qquad\geq-\int_{\Omega^\varepsilon}f_i(x)((v_0)_i(x)-(u_0)_i(x))dx+\int_{\partial\Omega^\varepsilon_N}t_i(x)((v_0)_i(x)-(u_0)_i(x))ds_x\end{aligned}$$

for any $v_0 \in H^1(\Omega^\varepsilon, \partial\Omega^\varepsilon_D)$, where $H^1(\Omega^\varepsilon, \partial\Omega^\varepsilon_D) = \{v \in H^1(\Omega^\varepsilon)^n \mid v_n = g_0(x)$ on $\partial\Omega^\varepsilon_D\}$, and any $v_1 \in K^1$, where the cone $K^1 := \{v_1 \in H^1(\Omega^\varepsilon, H^1_{per}(\hat{Y}))^n \mid [v_1]_n(x,\xi) \leq \bar{g}(\xi)$ for $\xi \in S$, a.e. $x \in \Omega^\varepsilon\}$, which yields the following

(i) Homogenized elasto-plastic problem

$$\begin{aligned}&\int_{\Omega^\varepsilon}\left[\int_Y a_{ijkl}\left(\frac{\partial(u_0)_k(x)}{\partial x_l}+\frac{\partial(u_1)_k(x,\xi)}{\partial\xi_l}\right)\frac{\partial((v_0)_i(x,\xi)-(u_0)_i(x,\xi))}{\partial x_j}d\xi\right.\\&\left.+\int_S\mathcal{F}(u_1,\hat{x},\hat{\xi})(|[v_1]_\tau(x,\xi)|-|[u_1^\delta]_\tau(x,\xi)|)ds_\xi\right]dx\\\geq\ &-\int_{\Omega^\varepsilon}f_i(x)((v_0)_i(x)-(u_0)_i(x))dx+\int_{\partial\Omega^\varepsilon_N}t_i(x)((v_0)_i(x)-(u_0)_i(x))ds_x,\end{aligned}$$

for any $v_0 \in H^1(\Omega^\varepsilon, \partial\Omega^\varepsilon_D)$, where $H^1(\Omega^\varepsilon, \partial\Omega^\varepsilon_D) := \{v \in H^1(\Omega^\varepsilon)^n \mid v_n = g_0(x)$ on $\partial\Omega^\varepsilon_D\}$, and any $v_1 \in K^1$, where the cone $K^1 := \{v_1 \in L^2(\Omega, H^1_{per}(\hat{Y}))^n \mid [v]_n(x,\xi) \leq \bar{g}(\xi)$ for $\xi \in S$, a.e. $x \in \Omega^\varepsilon\}$.

(ii) Auxiliary problem in the periodicity cell

$$\int_Y a_{ijkl}\left(\frac{\partial(u_0)_k(x)}{\partial x_l}+\frac{\partial(u_1)_k(x,\xi)}{\partial\xi_l}\right)\frac{\partial((v_1)_i(x,\xi)-(u_1)_i(x,\xi))}{\partial\xi_j}d\xi dx \tag{2.26}$$

$$+\int_S\mathcal{F}(u_1,\hat{x},\hat{\xi})(|[v_1]_\tau(x,\xi)|-|[u_1^\delta]_\tau(x,\xi)|)ds_\xi dx\geq 0, \tag{2.27}$$

for any $v_1 \in K^1$ *and*

$$\mathcal{F}(u_1, \hat{x}, \hat{\xi}) := \begin{cases} G(\hat{x}, \hat{\xi}), & \textit{Tresca friction,} \\ \frac{\mu(x,\xi)}{\delta}(u_1^\delta(x)\, n(x,\xi) - \bar{g}(\xi)), & \textit{Coulomb's friction.} \end{cases} \tag{2.28}$$

2.4.2 Strong homogenized and auxiliary contact problems

Let us make an ansatz like in the linear elliptic problems and look for $u_1(x,\xi)$ in the form

$$u_1(x,\xi) \equiv N(\xi)\nabla u_0(x). \tag{2.29}$$

Assume that boundaries, elastic coefficients and right-hand side functions are smooth.

The auxiliary problem on the periodicity cell will be

$$\begin{aligned}
&\frac{\partial}{\partial \xi_h}\Big[a_{ihjk}(\xi)\frac{\partial(n_j^{qp}(\xi) + \delta_{pj}\xi_q)}{\partial \xi_k}\Big] = 0, \quad \sigma^1_{ih} \equiv a_{ihjk}(\xi)\frac{\partial(n_j^{qp}(\xi) + \delta_{pj}\xi_q)}{\partial \xi_k} \text{ in } Y, \\
&\sigma^1_n(\xi) \le 0, \quad [n^{qp}(\xi)] \le \bar{g}(\xi), \quad \sigma^1_n(x,\xi)[n^{1p}(\xi) - \bar{g}(\xi)] = 0 \text{ on } S, \\
&|\sigma^1_\tau(\xi)| \le \frac{\mathcal{F}(N(\xi)\nabla u_0, x, \xi)}{|\nabla u_0|} \quad \Rightarrow \quad [n^{qp}_\tau] = 0, \quad \xi \in S, \\
&|\sigma^1_\tau(\xi)| = \frac{\mathcal{F}(N(\xi)\nabla u_0, x, \xi)}{|\nabla u_0(x)|} \quad \Rightarrow \quad \exists\lambda \ge 0 : \quad [n^{qp}_\tau] = -\lambda\sigma^1_\tau, \quad \xi \in S, \\
&n_j^{qp} \text{ is } Y-\text{periodic.}
\end{aligned} \tag{2.30}$$

The homogenized elasto-plasticity tensor will be computed from

$$A^{hom}_{i,k}(\nabla u_0) \equiv \int_{Y\setminus S}(\nabla n^{qi} + e_i)\cdot A(\nabla n^{qk} + e_k)\, d\xi + \int_{S\cap Y}\frac{\mathcal{F}(N(\xi)\nabla u_0(x), x, \xi)}{|\nabla u_0(x)|}[w_\tau] ds_\xi.$$

It is positive definite, since A and $\mathcal{F}$ are positive definite.

The proof is similar to those for analogous theorems from [Allaire, 1992] and [Hummel, 1999].

Chapter 3

Homogenization of textiles

The motivation for this chapter is the following: the homogenization of the microcontact elasticity problem described in Chapter 2 can be applied for homogenization of textiles with microcontact between fibers. However, a textile the on macro scale is geometrically represented as a thin plate with in-plane periodic micro structure. On the other hand, on the microscale the microstructure of a textile is characterized by thin fibers with the characteristic diameter much smaller than the size of the periodicity cell. This gives a motivation to come to the asymptotics with respect to the thickness of the textile plate (see Fig.3.1) and the diameter of the fibers.

The chapter is organized in the following way. Section 3.1 presents the results on homogenization of the heterogeneous plate given in [Panasenko, 2005]. In Section 3.2 we consider the problem with non-homogeneous Neumann boundary conditions on the vanishing out-of-plane interface. By introducing an additional expansion in the formal asymptotic expansion, we deduce the auxiliary problems and homogenized problem. In Section 3.3 we treat the microstructure of the fibers on the unit cell as a finite rod structure and use results of [Panasenko, 2005] in order to obtain the reduced problem.

3.1 Homogenization of a heterogeneous plate

In this section we recall some results of [Panasenko, 2005] on homogenization of linear elasticity equations considered in the thin plate domain. The homogenization is made for zero Neumann out-of-plane boundary conditions on the vanishing interface. The formal asymptotic solution is sought in the form of series. This allows to construct a recurrent

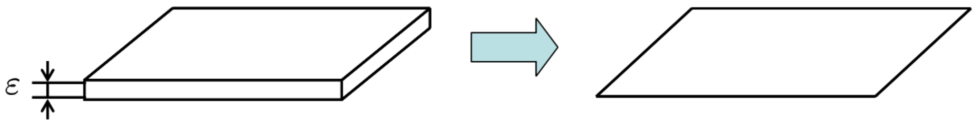


Figure 3.1: Asymptotics with respect to thickness of the plate.

chain of problems and to obtain the complete formal asymptotic solution. The in-plane boundary conditions can be introduced by assuming a boundary layer corrector. The zero order homogenized problem has the form of an in-plane elasticity problem with effective in-plane elasticity coefficients and a 4th order problem with effective bending coefficients.

3.1.1 Statement of ε problem

We consider that our domain Ω^ε described in Chapter 1 is given by the layer $\{x \in \mathbb{R}^n : x_n/\varepsilon \in (-1/2, 1/2)\}$ periodic in in-plane variables $x_1, x_2, \ldots, x_{n-1}$ with the period and thickness equal to ε.

Let u^ε be a n dimensional displacement vector, and $A_{ij} = \{a_{ijkl}\}$ be $n \times n$ matrices of the linear elasticity tensor. In Ω^ε we consider the linear elasticity problem:

$$\mathcal{A}^\varepsilon u^\varepsilon = f(x_1, \ldots, x_{n-1}), \qquad x \in \Omega^\varepsilon, \tag{3.1}$$

with the boundary conditions:

$$\frac{\partial u^\varepsilon}{\partial \bar{n}} = 0, \qquad x_n/\varepsilon = 1/2, \tag{3.2}$$

$$\frac{\partial u^\varepsilon}{\partial \bar{n}} = 0, \qquad x_n/\varepsilon = -1/2, \tag{3.3}$$

where $\bar{n}$ is the outer normal to the boundaries $x_n/\varepsilon = \pm 1/2$,

$$\mathcal{A}^\varepsilon = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} \left(A_{ij}(\tfrac{x}{\varepsilon}) \frac{\partial u}{\partial x_j} \right),$$

$A_{ij} = \{a_{ijkl}\}$ and

$$\frac{\partial u^\varepsilon}{\partial \bar{n}}|_{x_n/\varepsilon = \pm 1/2} \equiv \pm \sum_{j=1}^{n} A_{3j} \left(\frac{x_1}{\varepsilon}, \frac{x_2}{\varepsilon}, \ldots, \pm\frac{1}{2} \right) \frac{\partial u^\varepsilon}{\partial x_j}. \tag{3.4}$$

Assumption 3.1. *The elements of matrices* $A_{ij}(\xi) = \{a_{ijkl}(\xi)\}$ *are assumed to be periodic in* $\xi_1, \ldots, \xi_{n-1}$ *and satisfy conditions given in Chapter 1, Assumption 1.3. The right hand side* f *is a* C^∞ *vector valued function.*

Remark 3.2. *In [Panasenko, 2005] the statement of the problem is made for the cases* $n = 2, 3$. *However, the results can be extended to the general,* n*-dimensional case. Besides, one can consider the in-plane boundary conditions, for instance, the homogeneous Dirichlet boundary condition at* $x_1 = 0, b;\ b \in \mathbb{R}$. *In this case, a boundary layer corrector is used to impose the boundary condition, for details see Subsection 3.2 in Chapter 3 in [Panasenko, 2005].*

According to Section 2 of Chapter 3 in [Panasenko, 2005], the formal asymptotic solution can be sought in the form of series

$$u^\varepsilon\left(x, \frac{x}{\varepsilon}\right) = \sum_{l=0}^{\infty} \varepsilon^l \sum_{|q|=l} N_q\left(\frac{x}{\varepsilon}\right) D^q u_0(x_1, \ldots, x_{n-1}), \tag{3.5}$$

where $q = (q_1, \ldots, q_l)$ is a multi index, $q_j \in \{1, \ldots, n-1\}$, the auxiliary functions N_q are $n \times n$ matrix valued functions periodic in $x_1, \ldots, x_{n-1}$, D^q is a multiderivative.

3.1.2 Auxiliary in-plane problems

Plugging expansion (3.5) into (3.1), and grouping the terms of the same order one can obtain (see [Panasenko, 2005]) a recurrent chain of auxiliary problems

$$\begin{aligned}
&\sum_{i,j=1}^{n} \frac{\partial}{\partial \xi_i}\left(A_{ij}(\xi)\frac{\partial N_q}{\partial \xi_j}\right) = -T_q(\xi) + A_q^{hom}, \qquad \xi \in Y, && (3.6)\\
&\frac{\partial N_q}{\partial n} = -A_{3q_1} N_{q_2 \ldots q_l}, \qquad \xi_n = \pm 1/2,\\
&N_q \text{ is } \xi_1, \ldots, \xi_{n-1} \text{ periodic},
\end{aligned}$$

where

$$T_q(\xi) = \sum_{j=1}^{n} \frac{\partial}{\partial \xi_j}\left(A_{jq_1} N_{q_2 \ldots q_l}\right) + \sum_{j=1}^{n} A_{q_1 j}\frac{\partial N_{q_2 \ldots q_l}}{\partial \xi_j} + A_{q_1 q_2} N_{q_3 \ldots q_l}, \tag{3.7}$$

$N_\emptyset = I$, and from the solvability conditions

$$A_q^{hom} = \frac{1}{|Y|}\int_Y \left(A_{q_1 q_2} N_{q_3 \ldots q_l} + \sum_{j=1}^{n} A_{i_1 j}\frac{\partial N_{q_2 \ldots q_l}}{\partial \xi_j}\right) d\xi, \qquad |q| \geq 2, \tag{3.8}$$

$A_\emptyset^{hom} = 0$, $A_{q_1}^{hom} = 0$.

3.1.3 Homogenized problem

The homogenized equation of the infinite order (see [Panasenko, 2005]) is

$$\mathcal{A}^\varepsilon u^\varepsilon - f = \sum_{q_1,q_2=1}^{n-1} A^{hom}_{q_1,q_2} \frac{\partial^2 u_0}{\partial x_{q_1} \partial x_{q_2}} + \varepsilon \sum_{q_1,q_2,q_3=1}^{n-1} A^{hom}_{q_1,q_2,q_3} \frac{\partial^3 u_0}{\partial x_{q_1} \partial x_{q_2} \partial x_{q_3}} + \varepsilon^2 \sum_{q_1,q_2,q_3,q_4=1}^{n-1} A^{hom}_{q_1,q_2,q_3,q_4} \frac{\partial^4 u_0}{\partial x_{q_1} \partial x_{q_2} \partial x_{q_3} \partial x_{q_4}} + O(\varepsilon^3) - f = 0, \quad (3.9)$$

and the zero order homogenized problem is

$$\sum_{q_1,q_2=1}^{n-1} A^{hom}_{q_1 q_2} \frac{\partial^2 u_0}{\partial x_{q_1} \partial x_{q_2}} = f. \quad (3.10)$$

For bending properties, one has to consider the formal asymptotic expansion of u_0 with respect to the thickness of the heterogeneous plate equal to ε

$$u_0(x_1, \ldots, x_{n-1}) = \sum_{j=-2}^{\infty} \varepsilon^j u_0^j(x_1, \ldots, x_{n-1}), \quad (3.11)$$

where u_0^j for $j = -2, -1$ has the form $u_0^j = (0, \ldots, 0, (u_0^j)_n)^T$.

Let us redenote u_0^0 by u_0. One can write (see [Panasenko, 2005]) the zero order homogenized equation as

$$\sum_{q_1,q_2=1}^{n-1} A^{hom}_{q_1 q_2} \frac{\partial^2 u_0}{\partial x_{q_1} \partial x_{q_2}} + \sum_{q_1,q_2,q_3,q_4=1}^{n-1} A^{hom}_{q_1 q_2 q_3 q_4} \frac{\partial^4 u_0}{\partial x_{q_1} \partial x_{q_2} \partial x_{q_3} \partial x_{q_4}} = f. \quad (3.12)$$

Remark 3.3. *It is proven (see Theorem 3.2.1 in [Panasenko, 2005]) that matrices A^{hom}_q have a block form, which allows to write homogenized equations componentwise.*

3.1.4 Algorithm for computation of the effective stiffness of a plate

This subsection represents the implementation-ready form of the algorithm described in Subsection 3.2.5 of the book [Panasenko, 2005]. The algorithm is based on the solution derived in the previous subsection and allows to compute elements of $A^{hom}_{q_1 q_2}$ and $A^{hom}_{q_1 q_2 q_3 q_4}$, where $q_1, q_2, q_3, q_4 \in \{1, \ldots, n-1\}$, in (3.12).

The case $n = 3$ is considered. Let us solve the elasticity theory system of equations for $q_1, s = 1, 2$:

$$\sum_{m,j=1}^{3} \frac{\partial}{\partial \xi_m}(A_{mj}(\xi)\frac{\partial}{\partial \xi_j}(N_{q_1}^{(s)} + e_s\xi_{q_1})) = 0, \qquad \xi_3 \in \left(-\frac{1}{2}, \frac{1}{2}\right), \tag{3.13}$$
$$\sum_{j=1}^{3} A_{3j}(\xi)\frac{\partial}{\partial \xi_j}(N_{q_1}^{(s)} + e_s\xi_{q_1}) = 0 \qquad \xi_3 = \pm\frac{1}{2},$$
$$N_{q_1}^{(s)} \text{ is } \xi_1, \xi_2 \text{ periodic},$$

where $N_{q_1}^{(s)}$ is a 3-dimensional vector, the s-th column in the auxiliary matrix-valued functions N_q with the length of multi-index q equal to 1, i.e. $q = \{q_1\}$. The e_s is the unit vector defined as $(\delta_{s1}, \delta_{s2}, \delta_{s3})$. Then the elements of the effective tensor $A_{q_1q_2}^{hom} = \{\bar{a}_{q_1q_2}^{ks}\}$ can be computed from the following formula:

$$\bar{a}_{q_1q_2}^{ks} = \langle a_{q_1j}^{kr}\frac{\partial}{\partial \xi_j}(n_{q_2}^{rs} + \delta_{rs}\xi_{q_1})\rangle, \tag{3.14}$$

where $n_{q_2}^{rs}$ is the s-th element of the vector $N_{q_2}^{(s)}$, and $\langle\cdot\rangle = \frac{1}{|Y|}\int_Y \cdot d\xi$.

Now consider the elasticity theory system of equations for $q_1, q_2 = 1, 2$:

$$\sum_{m,j=1}^{3} \frac{\partial}{\partial \xi_m}(A_{mj}(\xi)\frac{\partial}{\partial \xi_j}(N_{q_1q_2}^{(3)} - A_{mq_1}e_{q_2}\xi_3)) = 0, \qquad \xi_3 \in \left(-\frac{1}{2}, \frac{1}{2}\right), \tag{3.15}$$
$$\sum_{j=1}^{3} A_{3j}\frac{\partial}{\partial \xi_j}N_{q_1q_2}^{(3)} - A_{3q_1}e_{q_2}\xi_3 = 0, \qquad \xi_3 = \pm\frac{1}{2},$$
$$N_{q_1q_2}^{(3)} \text{ is } \xi_1, \xi_2 \text{ periodic.}$$

Then the elements of the effective tensor $A_{q_1q_2q_3q_4}^{hom} = \{\bar{a}_{q_1q_2q_3q_4}\}$ are given by

$$\bar{a}_{q_1q_2q_3q_4} = -\left\langle \xi_3\left(\sum_{j,r=1}^{3} a_{q_2j}^{q_1r}\frac{\partial n_{q_3q_4}^{r3}}{\partial \xi_j} - a_{q_1q_4}^{q_2q_3}\xi_3\right)\right\rangle, \tag{3.16}$$

where $\langle\cdot\rangle = \frac{1}{|Y|}\int_Y \cdot d\xi$.

3.2 Non-homogeneous Neumann BCs

The aim of this section is to homogenize the non-zero Neumann conditions. The right hand side of the ε problem is considered to have the form which is connected to the out-of-plane pressure.

The introduction of an additional asymptotic expansion in the ansatz allows to deduce the recurrent chain of cell problems and to obtain the homogenized problem where the the out-of-plane pressure is incorporated into the right hand side.

3.2.1 Statement of ε problem

We consider the linear elasticity problem in Ω^ε,

$$\mathcal{A}^\varepsilon u^\varepsilon = f(x_1, \dots, x_{n-1}), \qquad x \in \Omega^\varepsilon, \tag{3.17}$$

with the Neumann out-of-plane boundary conditions

$$\frac{\partial u^\varepsilon}{\partial n} = \varepsilon p(x_1, \dots, x_{n-1}), \qquad x_n/\varepsilon = 1/2, \tag{3.18}$$

$$\frac{\partial u^\varepsilon}{\partial n} = 0, \qquad x_n/\varepsilon = -1/2, \tag{3.19}$$

where n is the outer normal to the boundaries, $\frac{\partial u^\varepsilon}{\partial n}$ is given by (3.4), and the elements of matrices A_{ij} satisfy the conditions given in Assumption 3.1.

Assumption 3.4. *We assume that*

$$\int_{\Omega^\varepsilon} f dx = \varepsilon \int_{\{\xi_n = 1/2\} \cap \partial\Omega^\varepsilon} p ds.$$

Also, f and p are assumed to be orthogonal to all rigid rotations.

We look for a solution in the form

$$u^\varepsilon = u_f^\varepsilon + u_p^\varepsilon, \tag{3.20}$$

where

$$u_f^\varepsilon = \sum_{l=0}^{\infty} \varepsilon^l \sum_{|q|=l} N_q(\tfrac{x}{\varepsilon}) D^q u_0(x_1, \dots, x_{n-1})$$

is the solution of the problem (3.17)-(3.19) with $p = 0$, $f \neq 0$, and is constructed in the way described in Section 3.1.

$$u_p^\varepsilon = \varepsilon^2 \sum_{l=0}^{\infty} \varepsilon^l \sum_{|q|=l} \Theta_q(\tfrac{x}{\varepsilon}) D^q p(x_1, \dots, x_{n-1}) \tag{3.21}$$

is the solution of problem (3.17)-(3.19) with $p \neq 0$, $f = 0$, and is constructed below. The auxiliary functions Θ_q are $n \times n$ matrix-valued functions periodic in $\xi_1, \dots, \xi_n$.

3.2.2 Auxiliary problems

Substituting series (3.21) in (3.17) with $f = 0$ and grouping the terms of the same order we get

$$A^\varepsilon u_p^\varepsilon = \sum_{l=0}^{\infty} \varepsilon^l \sum_{|q|=l} H_q^\Theta(\xi) D^q p = 0, \tag{3.22}$$

where

$$H_q^\Theta = \sum_{i,j=1}^{n} \frac{\partial}{\partial \xi_i}\left(A_{ij}\frac{\partial \Theta_q}{\partial \xi_j}\right) + T_q^\Theta, \tag{3.23}$$

$$T_q^\Theta = \sum_{j=1}^{n} A_{q_1 j}\frac{\partial \Theta_{q_2\ldots q_l}}{\partial \xi_j} + \sum_{j=1}^{n} \frac{\partial}{\partial \xi_j}(A_{jq_1}\Theta_{q_2\ldots q_l}) + A_{q_1 q_2}\Theta_{q_3\ldots q_l}. \tag{3.24}$$

Suppose that

$$H_q^\Theta(\xi) = B_q^{hom},$$

where B_q^{hom} are constant $n\times n$ matrices, and $B_\emptyset^{hom} = I$. Then we get the following recurrent chain of equations:

$$\sum_{i,j=1}^{n} \frac{\partial}{\partial \xi_i}\left(A_{ij}\frac{\partial \Theta_q}{\partial \xi_j}\right) = -T_q^\Theta + B_q^{hom}. \tag{3.25}$$

Substituting series (3.21) in (3.18), (3.19) and grouping terms of the same order we get

$$\frac{\partial u_p^\varepsilon}{\partial n} = \sum_{l=0}^{\infty} \varepsilon^{l+1} \sum_{|q|=l}\left(\sum_{j=1}^{n} A_{nj}\frac{\partial \Theta_q}{\partial \xi_j} + A_{nq_1}\Theta_{q_2\ldots q_l}\right) D^q p = \varepsilon p, \quad x_n/\varepsilon = 1/2, \tag{3.26}$$

$$\frac{\partial u_p^\varepsilon}{\partial n} = -\sum_{l=0}^{\infty} \varepsilon^{l+1} \sum_{|q|=l}\left(\sum_{j=1}^{n} A_{nj}\frac{\partial \Theta_q}{\partial \xi_j} + A_{nq_1}\Theta_{q_2\ldots q_l}\right) D^q p = 0, \quad x_n/\varepsilon = -1/2. \tag{3.27}$$

Thus, we obtain the following recurrent chain of auxiliary problems for Θ_q:

$$\begin{aligned}
&\sum_{i,j=1}^{n} \frac{\partial}{\partial \xi_i}(A_{ij}(\xi)\frac{\partial \Theta_q}{\partial \xi_j}) = -T_q(\xi) + B_q^{hom}, \qquad \xi \in Y,\\
&\frac{\partial \Theta_q}{\partial n} = I\delta_{l0} - A_{nq_1}\Theta_{q_2\ldots q_l}, \qquad \xi_n = 1/2,\\
&\frac{\partial \Theta_q}{\partial n} = -A_{nq_1}\Theta_{q_2\ldots q_l}, \qquad \xi_n = -1/2,\\
&\Theta_q \text{ is periodic in } \xi_1, \ldots, \xi_{n-1},
\end{aligned} \tag{3.28}$$

where B_q^{hom} are chosen from the solvability conditions for problem (3.28).

$$B_q^{hom} = \frac{1}{|Y|}\int_Y \left(\sum_{j=1}^{n} A_{q_1 j} \frac{\partial \Theta_{q_2 \dots q_l}}{\partial \xi_j} + A_{q_1 q_2}\Theta_{q_3\dots q_l} \right) d\xi + I\delta_{l0}, \quad l > 1, \tag{3.29}$$

and $B_\emptyset^{hom} = I$.

One has to point out that the problem for $l = 0$, i.e. for the auxiliary function $\Theta_\emptyset$, has a non-trivial solution.

3.2.3 Homogenized problem

Substituting (3.20) in (3.17), we obtain

$$A^\varepsilon(u_f^\varepsilon + u_p^\varepsilon) = \sum_{l=0}^{\infty} \varepsilon^{l-2} \sum_{|q|=l} A_q^{hom} D^q u_0 + \sum_{l=0}^{\infty} \varepsilon^{l} \sum_{|q|=l} B_q^{hom} D^q p = f, \tag{3.30}$$

the homogenized equation of the infinite order with respect to u_0, where the out-of-plane pressure in problem (3.17)-(3.19) is incorporated into the right hand side of the homogenized equation via B_q^{hom}.

The homogenized equation of the zeroth order is

$$\sum_{i,j=1}^{n-1} A_{ij}^{hom} \frac{\partial^2 u_0}{\partial x_i \partial x_j} = f - p, \tag{3.31}$$

where

$$A_{ij}^{hom} := \frac{1}{|Y|}\int_Y \left(A_{ij}(\xi) + \sum_{q=1}^{n} A_{iq} \frac{\partial N_j(\xi)}{\partial \xi_q} \right) d\xi. \tag{3.32}$$

3.3 Auxiliary problems: asymptotics with respect to the thickness of the fibers

The nature of a textile composite or fiber structure Y is such that the geometry of fibers, rods or beams on the unit cell can be represented as connections of straight or curved slender bodies with fixed junctions or contacting points. This gives us a natural mathematical description of the fiber microstructure on the unit cell as the union $Y^h = \cup_{e=1}^{l} Y_e^h$ of curved or straight rods of the thickness h with cross-sections $\beta_h^e = \{(\hat{x}_2/h, \dots, \hat{x}_n/h) \in \beta^e\}$, $e =$

$1, \dots, l$, where $\beta^e \subset \mathbb{R}^{n-1}$ are bounded domains and $\hat{x}$ is the local coordinate system of the rod Y^h. The incorporation of the orientation is given below, in Definition 3.10.

For simplicity we assume a 3-dimensional case, $n = 3$, and that all fibers have the same cross-section β_h. The question of the approximation of the geometry Y by Y^h is not considered here, and we assume that the geometry on the unit cell is already given by Y^h.

Assumption 3.5. *Assume that domain $Y^h = \cup_{e=1}^l Y_e^h$ is the union of fiber domains $Y_e^h = \beta_h \times Y_e$, where Y_e is a straight segment in $\mathbb{R}^n$, representing the neutral line of the fiber domain Y_e^h.*

Y^h is a finite rod structure, described in [Panasenko, 2005]. Therefore, all results on L convergence for an elasticity problem on finite rod structures are applied here. Recall the definitions and notations for a finite rod structure.

Definition 3.6. *Let $\partial Y^h = \partial Y \cup Y^h$ be the outer boundary of the microstructure Y^h on the unit cell Y. Then we denote by $\partial Y_D^h \neq \emptyset$ and $\partial Y_N^h = \partial Y^h \setminus \partial Y_D^h$ the parts of the outer boundary where Dirichlet and Neumann boundary conditions apply.*

Definition 3.7. *Let $Y_E = \cup_{e=1}^l Y_e$ be the union of all neutral lines of all fiber domains. Let Y_E be such that intersection of any two neutral lines can only be the end point for both lines. The end points s of neutral lines are called nodes, and the set Y_E is called skeleton. The nodes $s \notin \partial Y^h$ are called internal nodes.*

Definition 3.8. *Let S_e be the maximal subset of Y_e^h, such that any cross-section by a plane perpendicular to the neutral line Y_e is free of points of any other fiber domain. Then S_e is called a section of the fiber domain Y_e^h. The union of all sections is denoted by S_0.*

Definition 3.9. *For each element Y_e, let E be the Young modulus, $A = |\beta_h|$ be the area of cross section. Let $I_2 = \int_{\beta_h} x_2^2 dx_2 dx_3, I_3 = \int_{\beta_h} x_3^2 dx_2 dx_3$ be the second area moments, G — the torsion stiffness. Let us fix the coordinate system in $\mathbb{R}^n$, then let γ_e be the n-dimensional orientation vector of the element Y_e.*

Definition 3.10. *Let $\hat{e}_1 = (c_{11}, c_{21}, c_{31})^T$, $\hat{e}_2 = (c_{12}, c_{22}, c_{32})^T$, and $\hat{e}_3 = (c_{13}, c_{23}, c_{33})^T$ be the unit coordinate vectors of an element's local coordinate system. Then C_e defined as $(\hat{e}_1, \hat{e}_2, \hat{e}_3)$ is the 3×3 orthogonal matrix of transformation of the local coordinate system of the element Y_e to the global coordinate system.*

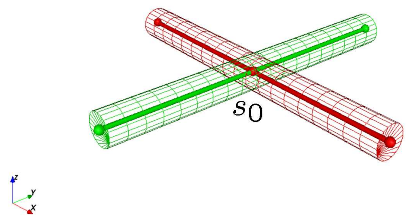


Figure 3.2: A finite rod structure.

Definition 3.11. *For each element Y_e define the matrix*

$$\bar{\Gamma}_e = \begin{pmatrix} C_e^{32}C_e^{23} - C_e^{22}C_e^{33} & C_e^{31}C_e^{22} - C_e^{21}C_e^{32} & C_e^{31}C_e^{23} - C_e^{21}C_e^{33} \\ C_e^{22}C_e^{13} - C_e^{12}C_e^{23} & C_e^{21}C_e^{12} - C_e^{11}C_e^{22} & C_e^{21}C_e^{13} - C_e^{11}C_e^{23} \\ C_e^{32}C_e^{13} - C_e^{12}C_e^{33} & C_e^{31}C_e^{12} - C_e^{11}C_e^{32} & C_e^{31}C_e^{13} - C_e^{11}C_e^{33} \end{pmatrix}^T,$$

where C_e^{ij} are elements of the matrix C_e.

We consider auxiliary problems for N_q, Θ_q on the set Y^h.

3.3.1 Microstructure of fibers as a finite rod structure

We formulate the statement of the linear elasticity problem for a displacement vector w^h, a column of the matrix-valued auxiliary functions $N_q, \Theta_\emptyset$. We treat the domain Y^h as a finite rod structure. Then, according to [Panasenko, 2005], the linear elasticity problem in Y^h is

$$\begin{aligned} &\sum_{i,j=1}^{n} \frac{\partial}{\partial \xi_i}(A_{ij}(\xi)\frac{\partial w^h}{\partial \xi_j}) = \psi, \qquad \xi \in Y^h, \\ &\frac{\partial w^h}{\partial n} = 0, \qquad \xi \in \partial Y_N^h, \\ &w^h = 0, \qquad \xi \in \partial Y_D^h, \end{aligned} \tag{3.33}$$

where Y_N^h and Y_D^h are parts of the boundary that coincides with the base of the fiber domain Y_e^h and contains the nodal end point.

Assumption 3.12. *The bending components of the body force ψ acting on each section S_e are of the order $O(h^2)$. Namely, in the local coordinate system (s_1, s_2, s_3) of the element Y_e, $\psi(s_1)$ can be written as $(\psi_1^e(s_1), h^2\psi_2^e(s_1), h^2\psi_3^e(s_1))$.*

Theorem 3.13. *The solution of the problem* (3.33) *exists and is unique.*

Proof. Problem (3.33) is the well-known mixed boundary value problem of elasticity. This problem has a unique solution (see, for example, [Fishera, 1972]). □

Using results of [Panasenko, 2005] on finite rod structures, one can construct the asymptotic expansion for w_h with respect to h

$$w^h = w_0(s_1) + \sum_{i=1}^{\infty} h^i M_i(\tfrac{s_2}{h}, \tfrac{s_3}{h}) \frac{d^i w_0(s_1)}{ds_1^i}, \quad s \in Y_e^h, \tag{3.34}$$

where $s_1 \in Y_e$ and s_2, s_3 are cross-sectional variables, $M_i(\frac{s_2}{h}, \frac{s_3}{h})$ are 3×4 matrix-valued functions, and $w_0(s_1)$ is a 4-dimensional vector-valued function.

Passing to the asymptotic limit with respect to h gives the limit problem for w_0 of the asymptotic expansion (3.34), where E, A, I_1, I_2, G are given by Definition 3.9.

Theorem 3.14. *The limit problem for $w_0 = (w_{0_1}, w_{0_2}, w_{0_3}, w_{0_4})^T$ is: for each element Y_e*

- *for $w_{0_1}^e$,*

$$EA\frac{d^2 w_{0_1}^e(s_1)}{ds_1^2} = \psi_1^e(s_1), \quad s_1 \in Y_e, \tag{3.35}$$

with the matching condition

$$EA\sum_{e(s_0)} \gamma_e \frac{dw_0^1(s_0)}{ds} = 0 \tag{3.36}$$

at all nodes common for at least two segments, except $s_0 \in \partial Y_D^h$, the condition

$$w_{0_1}^e = 0 \tag{3.37}$$

at all nodes $s_0 \in \partial Y_D^h$, and the condition

$$EA\gamma_e \frac{dw_{0_1}^e(s_0)}{ds} = 0 \tag{3.38}$$

at all nodes initial for one element only;

- *for* $(w^e_{0_2}, w^e_{0_3}, w^e_{0_4})$,

$$G\frac{d^2 w^e_{0_4}(s_1)}{ds_1^2} = 0, \quad s_1 \in Y_e, \tag{3.39}$$

$$EI_2\frac{d^4 w^e_{0_2}(s_1)}{ds_1^4} = \psi^e_2(s_1), \quad s_1 \in Y_e, \tag{3.40}$$

$$EI_3\frac{d^4 w^e_{0_3}(s_1)}{ds_1^4} = \psi^e_3(s_1), \quad s_1 \in Y_e, \tag{3.41}$$

with the matching conditions

$$\sum_{e(s_0)} \bar{\Gamma}_e \left(G\frac{dw^e_{0_4}(s_0)}{ds}, EI\frac{d^2 w^e_{0_2}(s_0)}{ds^2}, EI\frac{d^2 w^e_{0_3}(s_0)}{ds^2} \right)^T = 0, \tag{3.42}$$

$$w^e_{0_2} = 0, \quad w^e_{0_3} = 0 \tag{3.43}$$

at all internal nodes and

$$\bar{\Gamma}_{e_1} \left(w^e_{0_4}, \frac{dw^e_{0_2}(s_0)}{ds}, \frac{dw^e_{0_3}(s_0)}{ds} \right)^T = \bar{\Gamma}_{e_2} \left(w^e_{0_4}, \frac{dw^e_{0_2}(s_0)}{ds}, \frac{dw^e_{0_3}(s_0)}{ds} \right)^T \tag{3.44}$$

at nodes s_0 *common for at least two segments. The boundary conditions are*

$$w^e_{0_4} = 0, \quad w^e_{0_2} = 0, \quad \frac{dw^e_{0_2}(s_0)}{ds} = 0, \quad w^e_{0_3} = 0, \quad \frac{dw^e_{0_3}(s_0)}{ds} = 0 \tag{3.45}$$

at all nodes $s_0 \in \partial Y^h_D$ *and*

$$w^e_{0_2} = 0, \quad w^e_{0_3} = 0, \quad C_e(w^e_{0_4}, w^e_{0_2}, w^e_{0_3})^T = 0 \tag{3.46}$$

at all nodes initial only for one element.

Proof. For details, see Chapter 4 in [Panasenko, 2005]. □

For solvability of the reduced one-dimensional problem (3.35)-(3.46) we need the assumptions $PF1, PF2$ given in [Panasenko, 2005]:

Assumption 3.15. ***(PF1)***. *Let* $\phi(s)$ *be a 3-dimensional vector-valued function defined on* Y_E*, vanishing at the nodes in* ∂Y^h_D*, such that*

- $\phi(s)$ *has a generalized derivative along each element* Y_e*,*
- *all components of* $\phi(s)$*, except the first one, are linear on each element* Y_e*,*

- *$\phi(s)$ satisfies the matching condition $C_{e_1}\phi = C_{e_2}\phi$ for each two elements having the common node.*

Then it is assumed that

$$\sum_e \int_{Y_e} (\phi_1(s))^2 ds \leq c \sum_e \int_{Y_e} (\frac{d\phi_1(s)}{ds})^2 ds, \tag{3.47}$$

where the constant c depends only on Y_E.

Assumption 3.16. ***(PF2)**. Let $\phi(s)$ be 3-dimensional vector-valued function defined on Y_E, such that*

- *$\phi_1(s)$ has generalized derivative along each element Y_e,*
- *$\phi_2(s), \phi_3(s)$ have two generalized derivatives along each element Y_e,*
- *$\phi_2(s_0) = \phi_3(s_0) = 0$ at all nodes,*
- *the matching condition*

$$\bar{\Gamma}_{e_1}\left(\phi_1, \frac{d\phi_2(s_0)}{ds}, \frac{d\phi_3(s_0)}{ds}\right)^T = \bar{\Gamma}_{e_2}\left(\phi_1, \frac{d\phi_2(s_0)}{ds}, \frac{d\phi_3(s_0)}{ds}\right)^T \tag{3.48}$$

  holds for each two elements having the common node,

- *$\phi(s)$ satisfies the boundary conditions*

$$\phi = 0, \quad \frac{d\phi_2}{ds} = 0, \quad \frac{d\phi_3}{ds} = 0. \tag{3.49}$$

Then it is assumed that

$$\int_{Y_E} \left(\phi_1^2(s) + (\frac{d\phi_2(s)}{ds})^2 + (\frac{d\phi_3(s)}{ds})^2\right) ds \leq c \int_{Y_E} \left(\phi_1^2(s) + (\frac{d^2\phi_2(s)}{ds^2})^2 + (\frac{d^2\phi_3(s)}{ds^2})^2\right) ds, \tag{3.50}$$

where the constant c depends only on Y_E.

The solution w_0 of the problem (3.35)-(3.46) can be extended from each element Y_e to S_e by

$$w_0^h(x) := \begin{cases} C_e\left[(w_{0_1}^e(s(x)), w_{0_2}^e(s(x)), w_{0_3}^e(s(x))) + \frac{1}{h}(0, -s_3(x), s_2(x)) w_{0_4}^e(s(x))\right], & x \in S_0, \\ C_e w_{0_1}^e(s_0) = \text{const}, & x \in Y^h \setminus S_0. \end{cases} \tag{3.51}$$

Theorem 3.17. *Let the assumptions PF1, PF2 hold. Then there exists a solution of the problem* (3.35)-(3.46), *and for* w_0^h *defined by* (3.51) *the following is true:*

$$\frac{1}{|Y^h|}\|w^h - w_0^h\|_{L^2(Y^h)} = O(\sqrt{h}). \tag{3.52}$$

Proof. Existence of the solution of the problem (3.35)-(3.46) follows from Lemma 4.4.2 in [Panasenko, 2005], while the estimate (3.52) is the corollary from Theorem 4.4.5 in [Panasenko, 2005] □

Assumption 3.18. *Assume that PF1, PF2 for solution of the limiting problem are true.*

Chapter 4

FEM for cell problems

The reduced problem (3.35)-(3.46) with contact conditions in the fiber skeleton Y_E can be solved numerically using the finite element method. The representation of the skeleton $Y_E = \cup_{e=1}^{l} Y_e$ as the union of the fiber elements Y_e gives a natural partition into finite elements. Using the frame finite element formulation we construct the finite element space and give the finite element formulation of the problem (3.35)-(3.46) with contact conditions.

The structure of Chapter 4 is the following: Section 4.1 gives the weak statement of the reduced contact problem. In Section 4.2 the finite element formulation is given, and the finite element space is constructed. The contribution of the normal and tangential contact in the matrix formulation of the problem is derived in Section 4.3. This contribution depends on the type of the friction law.

The construction of the finite element approximation of the reduced problem (3.35)-(3.46) with contact conditions consists of the weak statement of the problem, construction of the finite element space, and the finite element formulation of the problem.

Remark 4.1. *The sliding in the contact conditions is allowed to happen only on one of the directions: along one fiber or along another fiber in the contact pair.*

Remark 4.2. *It is important to remember that the contact interface is not known a priori. The evolution of the system under contact constraints makes the task of the contact search and identification of the contact points difficult. In the case of a one-dimensional beam contact, the normal contact vector can be defined by solving the minimum distance problem between neutral lines of the contacting beams (see Section 11.1.1 in [Wriggers, 2002]). The tangential contact contribution can be computed by expressing the overall tangential contact* *force in terms of the projections on the tangential gap vectors along the neutral lines of the contacting beams, see equation (11.26) in [Wriggers, 2002].*

Remark 4.3. *The work of [Litewka, 2010] considers the finite element formulations of the beam-to-beam frictionless contact for beams with a rectangular cross-section and of the frictional contact of beams with a circular cross-section.*

4.1 Weak formulation of the contact auxiliary problem

Problem (3.35)-(3.46) consists of the sets of problems for axial tension, bending and torsion on each element Y_e with matching and boundary conditions.

In order to formulate the weak statement of the problem, we need the following definition.

Definition 4.4. *The fiber element Y_e is called internal, if $\partial Y_e \cap \partial Y_D^h = \emptyset$, otherwise the element is called a boundary element.*

Let Y_e be a boundary element. We denote $\partial Y_e^D = \partial Y_e \cap \partial Y_D^h$ and define the vector spaces of axial and torsion displacements $w_{0_q},\ q = 1, 4$,

$$\mathcal{V}^e = \{w_{0_q}^e \in H^1(Y_e) : w_{0_q}^e(s) = 0, s \in \partial Y_e\},$$

and the vector space of bending displacements $w_{0_q},\ q = 2, 3$,

$$\mathcal{W}^e = \{w_{0_q}^e \in H^2(Y_e) : w_{0_q}^e(s) = 0, \frac{dw_{0_q}^e}{ds}(s) = 0, s \in \partial Y_e\}.$$

We recall the weak statements of the problems for axial tension, bending, and torsion for the element Y_e.

Definition 4.5. *The weak formulation of the problem for the axial displacement is: find $u \in \mathcal{V}^e$, s.t.*

$$a_1(u, v) = f_1(v), \quad \forall v \in \mathcal{V}^e, \tag{4.1}$$

where

$$a_1(u, v) = \int_{Y_e} \frac{dv}{ds} EA \frac{du}{ds} ds, \quad f_1(v) = -\int_{Y_e} v\psi_1^e ds. \tag{4.2}$$

Indeed, recall the equation (3.35) for the axial displacement $w^e_{0_1}(s)$ on a fiber element Y_e:

$$EA\frac{d^2 w^e_{0_1}(s)}{ds^2} = \psi^e_1(s), \quad s \in Y_e, \quad \psi^e_1 \in C^0(Y_e). \tag{4.3}$$

Multiplying the both sides by a test function, $v \in \mathcal{V}^e$, and integrating by parts we get

$$\int_{Y_e} \frac{dv}{ds} EA \frac{dw^e_{0_1}}{ds} ds = -\int_{Y_e} v\psi^e_1 ds + (v\frac{dw^e_{0_1}}{ds})|_{\partial Y_e} - (v\frac{dw^e_{0_1}}{ds})|_{\partial Y_e} = -\int_{Y_e} v\psi^e_1 ds. \tag{4.4}$$

Definition 4.6. *The weak formulation of the problem for the bending displacements* w_{0_q}, $q = 2, 3$, *is: find* $u \in \mathcal{W}^e$ *s.t.*

$$a_q(u, v) = f_q(v), \quad \forall v \in \mathcal{W}^e, \tag{4.5}$$

where

$$a_q(u, v) = \int_{Y_e} \frac{d^2 v}{ds^2} EI_q \frac{d^2 u}{ds^2} ds, \quad f_q(v) = \int_{Y_e} v\psi^e_q ds. \tag{4.6}$$

Similarly, the equation for bending components $w_{0_q}, q = 2, 3$, is

$$EI_q \frac{d^4 w_{0_q}(s)}{ds^4} = \psi^e_q(s), \quad s \in Y_e, \quad \psi^e_q \in C^0(Y_e). \tag{4.7}$$

Multiplying the equation by a test function, $v \in \mathcal{W}^e$, and integrating by parts we get

$$\int_{Y_e} \frac{d^2 v}{ds^2} EI_q \frac{d^2 w_{0_q}}{ds^2} ds = \int_{Y_e} v\psi^e_q ds + \left(\frac{dv}{ds} EI_q \frac{d^2 w_{0_q}}{ds^2}\right)|_{\partial Y_e} + \left(vEI_q \frac{d^3 w_{0_q}}{ds^3}\right)|_{\partial Y_e} = \int_{Y_e} v\psi^e_q ds. \tag{4.8}$$

Definition 4.7. *The weak formulation of the problem for the torsion displacement is: find* $u \in \mathcal{V}^e$, *s.t.*

$$a_4(u, v) = 0, \quad \forall v \in \mathcal{V}^e, \tag{4.9}$$

where

$$a_4(u, v) = \int_{Y_e} \frac{dv}{ds} EA \frac{du}{ds} ds. \tag{4.10}$$

We denote

$$D = \begin{pmatrix} EA & 0 & 0 & 0 \\ 0 & EI_2 & 0 & 0 \\ 0 & 0 & EI_3 & 0 \\ 0 & 0 & 0 & G \end{pmatrix}, \quad B(w) = \begin{pmatrix} \frac{dw_1}{ds} \\ \frac{d^2 w_2}{ds^2} \\ \frac{d^2 w_3}{ds^2} \\ \frac{dw_4}{ds} \end{pmatrix}. \tag{4.11}$$

Furthermore, we define the bilinear form

$$a_e(u,v) := \int_{Y_e} B(u)^T DB(v)ds = a_1(u_1,v_1) + a_2(u_2,v_2) + a_3(u_3,v_3) + a_4(u_4,v_4), \quad (4.12)$$

and a functional of the right hand side

$$f_e(v) := \int_{Y_e} v^T \psi^e ds = f_1(v_1) + f_2(v_2) + f_3(v_3), \quad (4.13)$$

where $\psi^e = (\psi_1^e, \psi_2^e, \psi_3^e, 0)^T$. The frictional functional is given by

$$J_2^e(u,v) := \int_{J_c} \mathcal{F}(u,s)|[v]_\tau(s)|ds. \quad (4.14)$$

We also define the penalty functional

$$J_1^e(u,v) := \frac{1}{\delta}\int_{J_c} [[u]_n - g_N]_+ [v]_n(s)ds \quad (4.15)$$

and the functional

$$P_{\partial Y_e}(u,v) := \int_{\partial Y_e} v^T B(u)ds = \int_{\partial Y_e} v_1 \frac{du_1}{ds}ds + \int_{\partial Y_e} v_2 \frac{d^2u_2}{ds^2}ds + \int_{\partial Y_e} v_3 \frac{d^2u_3}{ds^2}ds + \int_{\partial Y_e} v_4 \frac{du_4}{ds}ds. \quad (4.16)$$

We introduce the vector spaces

$$H^e := H^1(Y_e) \times H^2(Y_e) \times H^2(Y_e) \times H^1(Y_e),$$

$$V^e := \mathcal{V}^e \times \mathcal{W}^e \times \mathcal{W}^e \times \mathcal{V}^e.$$

Definition 4.8. *Let $\delta > 0$, then the weak penalty formulation of the reduced contact auxiliary problem is*

- *for internal Y_e: find $w_0^e \in H^e$ s.t.*

  $$a_e(w_0^e, v) + J_1^e(w_0^e, v) + J_2^e(w_0^e, v) \geq f_e(v) + P_{\partial Y_e}(w_0^e|_{\partial Y_e}, v), \quad \forall v \in H^e, \quad (4.17)$$

  where $B(w_0^e|_{\partial Y_e})$ in $P_{\partial Y_e}(w_0^e, v)$ satisfy the matching conditions (3.36), (3.37), (3.42)-(3.44);

- *for boundary Y_e: find $w_0^e \in V^e$ s.t.*

  $$a_e(w_0^e, v) + J_1^e(w_0^e, v) + J_2^e(w_0^e, v) \geq f_e(v) + P_{\partial Y_e \backslash \partial Y_e^D}(w_0^e|_{\partial Y_e \backslash \partial Y_e^D}, v), \quad \forall v \in V^e, \quad (4.18)$$

  where $B(w_0^e|_{\partial Y_e \backslash \partial Y_e^D})$ in $P_{\partial Y_e \backslash \partial Y_e^D}(w_0^e, v)$ satisfy the matching conditions (3.36), (3.37).

Theorem 4.9. *Let $\delta > 0$, then the solution of the problem* (4.17)-(4.18) *exists.*

Proof. Define $F_1^e(v) = f_e(v) + P_{\partial Y_e}(w_0^e|_{\partial Y_e}, v)$ and $F_2^e(v) = f_e(v) + P_{\partial Y_e \backslash \partial Y_e^D}(w_0^e|_{\partial Y_e \backslash \partial Y_e^D}, v)$ for problems (4.17) and (4.18) respectively. Then the existence follows from Theorem 2.7 applied for each of the problems (4.17), (4.18) with the right hand side functionals F_1^e and F_2^e respectively. □

Let

$$\tilde{J}_2^e(u,v) := \int_{J_c} \mathcal{F}(u,s)\nabla\phi([u]_\tau)[v]_\tau(s)ds \tag{4.19}$$

be a regularized frictional functional.

Definition 4.10. *Let $\delta > 0, \gamma > 0$, then the regularized weak formulation of the reduced contact auxiliary problem is*

- *for internal Y_e: find $w_0^e \in H^e$ s.t.*

  $$a_e(w_0^e, v) + J_1^e(w_0^e, v) + \tilde{J}_2^e(w_0^e, v) = f_e(v) + P_{\partial Y_e}(w_0^e|_{\partial Y_e}, v), \quad \forall v \in H^e, \tag{4.20}$$

  where $w_0^e|_{\partial Y_e}$ in $P_{\partial Y_e}(w_0^e, v)$ satisfy the matching conditions (3.36), (3.37), (3.42)-(3.44)*;*

- *for boundary Y_e: find $w_0^e \in V^e$ s.t.*

  $$a_e(w_0^e, v) + J_1^e(w_0^e, v) + \tilde{J}_2^e(w_0^e, v) = f_e(v) + P_{\partial Y_e \backslash \partial Y_e^D}(w_0^e|_{\partial Y_e \backslash \partial Y_e^D}, v), \quad \forall v \in V^e, \tag{4.21}$$

  where $w_0^e|_{\partial Y_e \backslash \partial Y_e^D}$ in $P_{\partial Y_e \backslash \partial Y_e^D}(w_0^e, v)$ satisfy the matching conditions (3.36), (3.37).

Theorem 4.11. *Let $\delta > 0, \gamma > 0$, then the solution of the problem* (4.20)-(4.21) *exists.*

Proof. Define $F_1^e(v) = f_e(v) + P_{\partial Y_e}(w_0^e|_{\partial Y_e}, v)$ and $F_2^e(v) = f_e(v) + P_{\partial Y_e \backslash \partial Y_e^D}(w_0^e|_{\partial Y_e \backslash \partial Y_e^D}, v)$ for the problems (4.20) and (4.21) respectively. Then the existence follows from Theorem 2.9 applied for each of the problems (4.20), (4.21) with the right hand side functionals F_1^e and F_2^e respectively. □

4.2 Finite element formulation of a contact auxiliary problem

Let H_d^e, V_d^e be finite dimensional subspaces of H^e and V^e respectively, where d is the parameter of discretization of Y_e.

Definition 4.12. *Let $\delta > 0, \gamma > 0$, the finite element formulation of the contact auxiliary problem* (4.20), (4.21) *in Y_E is*

- *for internal Y_e: find $w_{0_d}^e \in H_d^e$ s.t.*

$$a_e(w_{0_d}^e, v_d) + J_1^e(w_{0_d}^e, v_d) + \tilde{J}_2^e(w_{0_d}^e, v_d) = f_e(v_d) + P_{\partial Y_e}(w_{0_d}^e|_{\partial Y_e}, v_d), \quad \forall v_d \in H_d^e, \tag{4.22}$$

  where $B(w_{0_d}^e|_{\partial Y_e})$ in $P_{\partial Y_e}(w_{0_d}^e, v_d)$ satisfy the matching conditions (3.36), (3.37), (3.42)-(3.44)*;*

- *for boundary Y_e: find $w_{0_d}^e \in V_d^e$ s.t.*

$$a_e(w_{0_d}^e, v_d) + J_1^e(w_{0_d}^e, v_d) + \tilde{J}_2^e(w_{0_d}^e, v_d) = f_e(v_d) + P_{\partial Y_e \setminus \partial Y_e^D}(w_{0_d}^e|_{\partial Y_e \setminus \partial Y_e^D}, v_d), \quad \forall v_d \in V_d^e, \tag{4.23}$$

  where $B(w_{0_d}^e|_{\partial Y_e \setminus \partial Y_e^D})$ in $P_{\partial Y_e \setminus \partial Y_e^D}(w_{0_d}^e, v_d)$ satisfy the matching conditions (3.36), (3.37).

Theorem 4.13. *Let $\delta > 0, \gamma > 0$, then the solution of the problem* (4.22)-(4.23) *exists.*

Proof. Follows from Theorem 4.11. □

Theorem 4.14. *Let $\delta > 0, \gamma > 0$, then the following estimate holds:*

$$\|w_0^e - w_{0_d}^e\|_{H^1(Y_e)} = O(\sqrt{\gamma} + d). \tag{4.24}$$

Proof. The estimate follows from Theorem 10.5 in [Kikuchi, Oden, 1988]. □

Remark 4.15. *The partition $Y_E = \cup_{e=1}^l Y_e$ is a natural mesh for construction of the finite element space of the problem* (4.22)-(4.23)*. On the other hand, one can consider mesh partition for each fiber element Y_e, for instance in the case of curved fiber elements. Without loss of generality we define a finite element in Y_e.*

The construction of the finite element spaces H_d^e and V_d^e consists of describing the local basis and a finite element. We construct them componentwise, by describing the finite elements for the axial tension, bending and torsion.

Consider an element Y_e with the length l_e, the start node 1 and the end node 2, and $H^1(Y_e)$, the space of the axial displacement. Let $H_d^1(Y_e)$ be a finite dimensional subspace of $H^1(Y_e)$ with the basis $\Big[N_1(s), N_2(s)\Big]$, where N_i are linear shape functions,

$$N_1(s) = 1 - \frac{s}{l_e}, \quad N_2(s) = \frac{s}{l_e}. \tag{4.25}$$

Let $q_1 = \Big[w_1^1, w_1^2\Big]^T$ be basis of nodal variables for the dual space $(H_d^1(Y_e))'$, where w_1^1 is the axial displacement at the node 1, and w_1^2 is the axial displacement at the node 2.

Definition 4.16. $(Y_e, H_d^1(Y_e), q_1)$ *is called a bar finite element with the element domain* Y_e*, the space of shape functions* $H_d^1(Y_e)$*, and the set of nodal variables* q_1*.*

With the basis $\Big[N_1(s), N_2(s)\Big]$, the axial displacement is interpolated as

$$w_{0_1}^e(x) = \Big[N_1(x), N_2(x)\Big] \begin{bmatrix} w_1^1 \\ w_1^2 \end{bmatrix} = \bar{N} q_1.$$

Consider $H^2(Y_e)$, the space of bending displacements. Let $H_d^2(Y_e)$ be a finite dimensional subspace of $H^2(Y_e)$ with the basis

$$\begin{array}{rcl} H_1(s) & = & 1 - \frac{3s^2}{l_e^2} + \frac{2s^3}{l_e^3}, \\ H_2(s) & = & x - \frac{2s^2}{l_e} + \frac{s^3}{l_e^2}, \\ H_3(s) & = & \frac{3s^2}{l_e^2} - \frac{2s^3}{l_e^3}, \\ H_4(s) & = & -\frac{s^2}{l_e} + \frac{s^3}{l_e^2}, \end{array} \tag{4.26}$$

where H_i are Hermitian shape functions. Let $q_2 = \Big[w^1, \theta^1, w^2, \theta^2\Big]$ be the basis of nodal variables for $(H_d^2(Y_e))'$, where w^1 and θ^1 are the bending displacement and the slope at the node 1, while w^2 and θ^2 are the bending displacement and the slope at the node 2.

Definition 4.17. $(Y_e, H_d^2(Y_e), q_2)$ *is called a beam finite element with the element domain* Y_e*, the space of shape functions* $H_d^2(Y_e)$*, and the set of nodal variables* q_2*.*

With the basis $\left[H_1(s), H_2(s), H_3(s), H_4(s)\right]$, the bending displacement is interpolated as

$$w_{0_2} = \left[H_1(s), H_2(s), H_3(s), H_4(s)\right] \begin{bmatrix} w^1 \\ \theta^1 \\ w^2 \\ \theta^2 \end{bmatrix} = \bar{H} q_2.$$

The definition of the torsion element repeats definition of the bar element. Let $q_4 = (w_4^1, w_4^2)$ be the nodal basis of $H_d^1(Y_e)$.

Definition 4.18. $(Y_e, H_d^1(Y_e), q_4)$ *is called a torsion finite element with the element domain* Y_e*, the space of shape functions* $H_d^1(Y_e)$*, and the set of nodal variables* q_2*.*

Lemma 4.19. *Let* $H_d^1(Y_e)$ *and* $H_d^2(Y_e)$ *be finite dimensional spaces with the bases* (4.25) *and* (4.26) *respectively. Then*

$$H_d^e = H_d^1(Y_e) \times H_d^2(Y_e) \times H_d^2(Y_e) \times H_d^1(Y_e)$$

is a finite dimensional subspace of H^e*.*

Proof. Indeed, by the properties of the direct product of finite dimensional vector spaces, H_d^e is a finite dimensional space and, obviously, a subspace of H^e. $\square$

Remark 4.20. *The construction of the finite dimensional space* V_d^e *is analogous. One needs to specify finite dimensional subspaces of* $\mathcal{V}^e$ *and* $\mathcal{W}^e$ *for the axial tension, bending, and torsion, then take their direct product.*

Consider an element Y_e with the start node 1 and the end node 2. Let

$$\mathbf{q} = (w_1^1, w_2^1, w_3^1, w_4^1, \theta_2^1, \theta_3^1, w_1^2, w_2^2, w_3^2, w_4^2, \theta_2^2, \theta_3^2)^T$$

be a basis of $(H_d^e)'$, where w_j^i and θ_j^i are the j-th displacement and the slope at the node

i. Let us further define the matrix

$$\mathbf{R} = \begin{pmatrix} N_1(s) & 0 & 0 & 0 \\ 0 & H_1(s) & 0 & 0 \\ 0 & 0 & H_1(s) & 0 \\ 0 & 0 & 0 & N_1(s) \\ 0 & H_2(s) & 0 & 0 \\ 0 & 0 & H_2(s) & 0 \\ N_2(s) & 0 & 0 & \\ 0 & H_3(s) & 0 & 0 \\ 0 & 0 & H_3(s) & 0 \\ 0 & 0 & 0 & N_2(s) \\ 0 & H_4(s) & 0 & 0 \\ 0 & 0 & H_4(s) & 0 \end{pmatrix}^T .$$

Then an interpolation over a finite element is given by

$$w^e_{0_d} = \mathbf{R} \cdot \mathbf{q}^e. \tag{4.27}$$

Definition 4.21. $(Y_e, H^e_d, \mathbf{q}^e)$ *is called a frame element with the element domain* Y_e*, the space of shape functions* H^e_d*, and the set of nodal variables* $\mathbf{q}$.

Thus, the finite element space of the problem (4.22), (4.23) has been constructed.

4.3 Equivalent matrix form of the problem

The Ritz-Galerkin method allows to obtain the equivalent matrix formulation of the problem (4.22), (4.23). Substituting (4.27) into equations (4.22), (4.23) and using the bilinear properties of $a_e(\cdot,\cdot)$, $J^e_1(\cdot,\cdot)$, $\tilde{J}^e_2(\cdot,\cdot)$, as well as the linear properties of $f_e(\cdot)$, $P_{\cdot}(\cdot)$, one can obtain (see [Axelsson, 2001])

$$(\mathbf{K} + \mathbf{K}_N + \mathbf{K}_T) \cdot \mathbf{Q} = \mathbf{F} + \mathbf{F}_N + \mathbf{F}_T, \tag{4.28}$$

where $\mathbf{K}$ is the global stiffness matrix, $\mathbf{Q}$ is the global vector of degrees of freedom, $\mathbf{F}$ is the global load vector, $\mathbf{K}_N$ and $\mathbf{K}_T$ are the contributions of the normal and tangential contact to the global stiffness matrix, $\mathbf{F}_N$ and $\mathbf{F}_T$ are the contributions of the normal and tangential contact to the global load vector.

The global system (4.28) can be constructed using the assembling procedure from the local quantities $\mathbf{k}^e$, $\mathbf{q}^e$, $\mathbf{f}^e$ of the finite elements. However, the contact contribution of $\mathbf{K}_N$ and $\mathbf{K}_T$ is not local, since it involves at least two elements.

Assumption 4.22. *During assembling procedure, the matching conditions* (3.36) *and* (3.42) *imply, that sums over internal nodes of* $\mathbf{p}^e$ *equal to zero. Therefore global* $\mathbf{P}$ *can be droped.*

Consider the variational equation on the element Y_e,

$$a_e(u,v) + J_1^e(u,v) + \tilde{J}_2^e(u,v) = f_e(v) + P_{\partial Y_e}(u|_{\partial Y_e}, v). \tag{4.29}$$

Recall that

$$a_e(u,v) = \int_{Y_e} B(v)^T \cdot D \cdot B(u) ds. \tag{4.30}$$

Consider the interpolations

$$u = \mathbf{R} \cdot \mathbf{q}_u^e, \tag{4.31}$$

$$v = \mathbf{R} \cdot \mathbf{q}_v^e. \tag{4.32}$$

Substitution of (4.31), (4.32) in (4.30) gives

$$\begin{aligned} a_e(u,v) = \int_{Y_e} B(\mathbf{R}\cdot\mathbf{q}_v^e)^T D B(\mathbf{R}\cdot\mathbf{q}_u^e) ds = \int_{Y_e} (\mathbf{B}\mathbf{q}_v^e)^T \cdot D \cdot (\mathbf{B}\mathbf{q}_u^e) ds \\ = (\mathbf{q}_v^e)^T \left(\int_{Y_e} \mathbf{B}^T \cdot D \cdot \mathbf{B}\ ds \right) \mathbf{q}_u^e = (\mathbf{q}_v^e)^T \mathbf{K}^e \mathbf{q}_u^e, \end{aligned} \tag{4.33}$$

where D and $B(\cdot)$ are defined in (4.11), and $\mathbf{B}$ is the strain-displacement matrix,

$$\mathbf{B} = \begin{pmatrix} \frac{dN_1(s)}{ds} & 0 & 0 & \\ 0 & \frac{d^2H_1(s)}{ds^2} & 0 & 0 \\ 0 & 0 & \frac{d^2H_1(s)}{ds^2} & 0 \\ 0 & 0 & 0 & \frac{dN_1(s)}{ds} \\ 0 & \frac{d^2H_2(s)}{ds^2} & 0 & 0 \\ 0 & 0 & \frac{d^2H_2(s)}{ds^2} & 0 \\ \frac{dN_2(s)}{ds} & 0 & 0 & 0 \\ 0 & \frac{d^2H_3(s)}{ds^2} & 0 & 0 \\ 0 & 0 & \frac{d^2H_3(s)}{ds^2} & 0 \\ 0 & 0 & 0 & \frac{dN_2(s)}{ds} \\ 0 & \frac{d^2H_4(s)}{ds^2} & 0 & 0 \\ 0 & 0 & \frac{d^2H_4(s)}{ds^2} & 0 \end{pmatrix}^T.$$

Definition 4.23. $\mathbf{k}^e$ *given by*

$$\mathbf{k}^e := \int_{Y_e} \mathbf{B}^T \cdot D \cdot \mathbf{B}\ ds \tag{4.34}$$

is called a local stiffness matrix.

The local stiffness matrix can be computed explicitly, see A.1.

Recall that

$$f_e(v) := \int_{Y_e} v^T \psi^e ds. \tag{4.35}$$

Substitution of (4.32) in (4.35) gives

$$f_e(v) := \int_{Y_e} (\mathbf{R}\cdot\mathbf{q}_v^e)^T \psi^e ds = \int_{Y_e} (\mathbf{q}_v^e)^T \mathbf{R}^T \psi^e ds = (\mathbf{q}_v^e)^T \left(\int_{Y_e} \mathbf{R}^T \psi^e ds \right) = (\mathbf{q}_v^e)^T \mathbf{F}^e. \tag{4.36}$$

Definition 4.24. $\mathbf{f}^e$ *defined by*

$$\mathbf{f}^e := \int_0^1 \mathbf{R}^T \cdot \psi^e ds \tag{4.37}$$

is called a local load vector.

Recall that

$$P_{\partial Y_e}(u|_{\partial Y_e}, v) := \int_{\partial Y_e} v^T B(u|_{\partial Y_e}) ds. \tag{4.38}$$

Substitution of (4.32) in (4.38) gives

$$\begin{aligned} P_{\partial Y_e}(u|_{\partial Y_e}, v) := \int_{\partial Y_e} (\mathbf{R} \cdot \mathbf{q}_v^e)^T B(u|_{\partial Y_e}) ds &= \int_{\partial Y_e} (\mathbf{q}_v^e)^T \mathbf{R}^T B(u|_{\partial Y_e}) ds \\ &= (\mathbf{q}_v^e)^T \int_{\partial Y_e} \mathbf{R}^T B(u|_{\partial Y_e}) ds = (\mathbf{q}_v^e)^T \mathbf{P}^e. \end{aligned} \tag{4.39}$$

Definition 4.25. $\mathbf{p}^e$ *given by*

$$\mathbf{p}^e := \int_{\partial Y_e} \mathbf{R}^T B(u|_{\partial Y_e}) ds \tag{4.40}$$

is called a nodal load vector, where $B(\cdot)$ *is defined in* (4.11).

Passing from local to global quantities

Recall C^e, the matrix of transformation of the local element coordinate system to the global coordinate system. We define

$$\mathbf{C}_e := \begin{pmatrix} C^e & O & O & O \\ O & C^e & O & O \\ O & O & C^e & O \\ O & O & O & C^e \end{pmatrix}_{12\times 12}, \quad \bar{\mathbf{C}}_e := \begin{pmatrix} C^e & 0 \\ 0 & 1 \end{pmatrix}_{4\times 4}, \tag{4.41}$$

where I is the 3×3 identity matrix and O is the 3×3 zero matrix. Then the transformation of the matrices is given by

$$\mathbf{K}^e = \mathbf{C}_e \mathbf{k}^e \mathbf{C}_e^T. \tag{4.42}$$

The vector of the axial tension, bending, and torsion displacements in the global coordinate system is given by

$$U^e = \bar{\mathbf{C}}_e u^e. \tag{4.43}$$

Matrix contribution of normal and tangential contact

It remains to obtain the contributions of J_1^e and $\tilde{J}_2^e$ into the matrix formulation of the problem. However, derivation of the matrix form of the contribution of the normal and

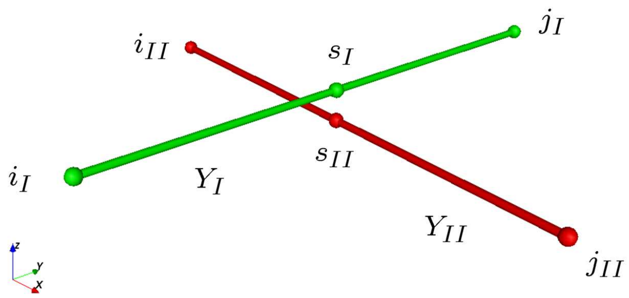


Figure 4.1: Contacting fiber elements.

tangential contact cannot be obtained locally, since the non-penetration and friction functionals J_1^e and $\tilde{J}_2^e$ involve at least two contacting elements with different orientations.

Consider two contacting fiber elements Y_I, Y_{II} (see Fig. 4.1). Let i_I and j_I be the start and the end node of the element Y_I, i_{II} and j_{II} be the start and the end node of the element Y_{II}, s_I, s_{II} be the contact points. Then we define

$$n_c := s_{II} - s_I, \quad \text{the contact normal,} \tag{4.44}$$

$$t_e := j_e - i_e, \quad \text{the contact tangential vector.} \tag{4.45}$$

Denote

$$\mathbf{n}_c = \begin{pmatrix} n_c \\ 0 \end{pmatrix}_{4\times 1}, \quad \mathbf{t}_e = \begin{pmatrix} t_e \\ 0 \end{pmatrix}_{4\times 1}, \tag{4.46}$$

and $\mathbf{R}_e = \mathbf{R}(s_e)$, $e = I, II$.

Lemma 4.26. *Let $s_I, s_{II} \in J_c$ be a contact pair, then*

$$[u]_n|_{s_I,s_{II}} = (\mathbf{Q}_v^I)^T\mathbf{a}_1 \cdot \mathbf{n}_c - (\mathbf{Q}_v^{II})^T\mathbf{a}_2 \cdot \mathbf{n}_c = \mathbf{n}_c{}^T \cdot \mathbf{a}_1^T\mathbf{Q}_u^I - \mathbf{n}_c{}^T \cdot \mathbf{a}_2^T\mathbf{Q}_u^{II}, \tag{4.47}$$

$$[u]_{\mathbf{t}_e}|_{s_I,s_{II}} = (\mathbf{Q}_v^I)^T\mathbf{a}_1 \cdot \mathbf{t}_e - (\mathbf{Q}_v^{II})^T\mathbf{a}_2 \cdot \mathbf{t}_e = \mathbf{t}_e^T \cdot \mathbf{a}_1^T\mathbf{Q}_u^I - \mathbf{t}_e^T \cdot \mathbf{a}_2^T\mathbf{Q}_u^{II}, \tag{4.48}$$

where

$$\mathbf{a}_1 = \mathbf{C}_I\mathbf{R}_I^T\bar{\mathbf{C}}_I, \tag{4.49}$$

$$\mathbf{a}_2 = \mathbf{C}_{II}\mathbf{R}_{II}^T\bar{\mathbf{C}}_{II}. \tag{4.50}$$

Proof. By definition

$$[u]_n = (u^I, \mathbf{n}_c) - (u^{II}, \mathbf{n}_c) = (u^I - u^{II})^T \cdot \mathbf{n}_c = {\mathbf{n}_c}^T \cdot (u^I - u^{II}),$$

we consider $u^I - u^{II}$ in the global coordinate system and substitute local interpolation (4.31).

$$\begin{aligned}(u^I - u^{II})|_{s_I,s_{II}} &= \bar{\mathbf{C}}_I^T \mathbf{R}_I \mathbf{q}_u^I - \bar{\mathbf{C}}_{II}^T \mathbf{R}_{II} \mathbf{q}_u^{II} \\ &= \bar{\mathbf{C}}_I^T \mathbf{R}_I \mathbf{C}_I^T \mathbf{C}_I \mathbf{q}_u^I - \bar{\mathbf{C}}_{II}^T \mathbf{R}_{II} \mathbf{C}_{II}^T \mathbf{C}_{II} \mathbf{q}_u^{II} \\ &= \bar{\mathbf{C}}_I^T \mathbf{R}_I \mathbf{C}_I^T \mathbf{Q}_u^I - \bar{\mathbf{C}}_{II}^T \mathbf{R}_{II} \mathbf{C}_{II}^T \mathbf{Q}_u^{II},\end{aligned}$$

and hence

$${\mathbf{n}_c}^T \cdot (u^I - u^{II})|_{s_I,s_{II}} = ({\mathbf{n}_c}^T \bar{\mathbf{C}}_I^T \mathbf{R}_I \mathbf{C}_I^T) \mathbf{Q}_u^I - ({\mathbf{n}_c}^T \bar{\mathbf{C}}_{II}^T \mathbf{R}_{II} \mathbf{C}_{II}^T) \mathbf{Q}_u^{II} = {\mathbf{n}_c}^T \mathbf{a}_1^T \mathbf{Q}_u^I - {\mathbf{n}_c}^T \mathbf{a}_2^T \mathbf{Q}_u^{II}.$$

On the other hand,

$$\begin{aligned}((v^I - v^{II})|_{s_I,s_{II}})^T \cdot \mathbf{n}_c &= \left(\bar{\mathbf{C}}_I^T \mathbf{R}_I \mathbf{C}_I^T \mathbf{Q}_v^I - \bar{\mathbf{C}}_{II}^T \mathbf{R}_{II} \mathbf{C}_{II}^T \mathbf{Q}_v^{II}\right)^T \cdot \mathbf{n}_c \\ &= \left((\mathbf{Q}_v^I)^T (\mathbf{C}_I \mathbf{R}_I^T \bar{\mathbf{C}}_I) - (\mathbf{Q}_v^{II})^T (\mathbf{C}_{II} \mathbf{R}_{II}^T \bar{\mathbf{C}}_{II})\right) \cdot \mathbf{n}_c \\ &= (\mathbf{Q}_v^I)^T (\mathbf{C}_I \mathbf{R}_I^T \bar{\mathbf{C}}_I \mathbf{n}_c) - (\mathbf{Q}_v^{II})^T (\mathbf{C}_{II} \mathbf{R}_{II}^T \bar{\mathbf{C}}_{II} \mathbf{n}_c) = (\mathbf{Q}_v^I)^T \mathbf{a}_1 \mathbf{n}_c - (\mathbf{Q}_v^{II})^T \mathbf{a}_2 \mathbf{n}_c.\end{aligned}$$

Substituting $\mathbf{n}_c$ by $\mathbf{t_e}$, we get (4.48). ☐

Proposition 4.27. *Let $s_I, s_{II} \in J_c$ be a contact pair, then non-penetration functional J_1^e can be represented as*

$$J_1^e = \frac{1}{\delta} \begin{pmatrix} (\mathbf{Q}_v^I)^T \\ (\mathbf{Q}_v^{II})^T \end{pmatrix}^T \begin{pmatrix} \mathbf{a}_1 \mathbf{n}_c {\mathbf{n}_c}^T \mathbf{a}_1^T & -\mathbf{a}_1 \mathbf{n}_c {\mathbf{n}_c}^T \mathbf{a}_2^T \\ -\mathbf{a}_2 \mathbf{n}_c {\mathbf{n}_c}^T \mathbf{a}_1^T & \mathbf{a}_2 \mathbf{n}_c {\mathbf{n}_c}^T \mathbf{a}_2^T \end{pmatrix} \begin{pmatrix} \mathbf{Q}_u^I \\ \mathbf{Q}_u^{II} \end{pmatrix} - \frac{\bar{g}}{\delta} \begin{pmatrix} (\mathbf{Q}_v^I)^T \\ (\mathbf{Q}_v^{II})^T \end{pmatrix}^T \begin{pmatrix} \mathbf{a}_1 \mathbf{n}_c \\ -\mathbf{a}_2 \mathbf{n}_c \end{pmatrix}. \tag{4.51}$$

Proof. Recall the non-penetration functional

$$J_1^e(u, v) := \frac{1}{\delta} \int_{J_c} ([u]_n - \bar{g})_+ [v]_n(s) ds. \tag{4.52}$$

J_c is a discrete set of contact points. Then, considering those points where the integral does not vanish, we obtain

$$\begin{aligned}J_1^e(u, v) &= \frac{1}{\delta} \int_{J_c} ([u]_n - \bar{g})_+ [v]_n(s) ds = \frac{1}{\delta} \int_{J_c} ([u]_n - \bar{g}) [v]_n(s) ds \\ &= \frac{1}{\delta} [([u]_n - \bar{g})[v]_n]|_{s_I,s_{II}} = \frac{1}{\delta} ([u]_n [v]_n)|_{s_I,s_{II}} - \frac{1}{\delta} (\bar{g}[v]_n)|_{s_I,s_{II}}.\end{aligned}$$

By Lemma 4.26, the first term becomes

$$\begin{aligned}\frac{1}{\delta}([u]_n[v]_n)|_{s_I,s_{II}} &= \frac{1}{\delta}([v]_n[u]_n)|_{s_I,s_{II}}\\ &= \frac{1}{\delta}\left[((v^I - v^{II})|_{s_I,s_{II}})^T \cdot \mathbf{n}_c \cdot {\mathbf{n}_c}^T \cdot (u^I - u^{II})|_{s_I,s_{II}}\right]\\ &= ((\mathbf{Q}_v^I)^T\mathbf{a}_1\mathbf{n}_c - (\mathbf{Q}_v^{II})^T\mathbf{a}_2\mathbf{n}_c)({\mathbf{n}_c}^T\mathbf{b}_1\mathbf{Q}_u^I - {\mathbf{n}_c}^T\mathbf{b}_1\mathbf{Q}_u^{II}).\end{aligned}$$

Then the opening of the brackets and rearrangement of the terms give

$$\frac{1}{\delta}([u]_n[v]_n)|_{s_I,s_{II}} = \frac{1}{\delta}\begin{pmatrix}(\mathbf{Q}_v^I)^T\\ (\mathbf{Q}_v^{II})^T\end{pmatrix}^T \begin{pmatrix}\mathbf{a}_1\mathbf{n}_c{\mathbf{n}_c}^T\mathbf{a}_1^T & -\mathbf{a}_1\mathbf{n}_c{\mathbf{n}_c}^T\mathbf{a}_2^T\\ -\mathbf{a}_2\mathbf{n}_c{\mathbf{n}_c}^T\mathbf{a}_1^T & \mathbf{a}_2\mathbf{n}_c{\mathbf{n}_c}^T\mathbf{a}_2^T\end{pmatrix}\begin{pmatrix}\mathbf{Q}_u^I\\ \mathbf{Q}_u^{II}\end{pmatrix}.$$

Lemma 4.26 implies that the second term is

$$\frac{1}{\delta}(\bar{g}[v]_n)|_{s_I,s_{II}} = \frac{1}{\delta}\bar{g}((\mathbf{Q}_v^I)^T\mathbf{a}_1\mathbf{n}_c - (\mathbf{Q}_v^{II})^T\mathbf{a}_2\mathbf{n}_c),$$

which can be rewritten as

$$\frac{1}{\delta}(\bar{g}[v]_n)|_{s_I,s_{II}} = \frac{\bar{g}}{\delta}\begin{pmatrix}(\mathbf{Q}_v^I)^T\\ (\mathbf{Q}_v^{II})^T\end{pmatrix}^T\begin{pmatrix}\mathbf{a}_1\mathbf{n}_c\\ -\mathbf{a}_2\mathbf{n}_c\end{pmatrix},$$

what finishes the proof. □

Definition 4.28. $\mathbf{K}_N(s_I, s_{II})$ *defined by*

$$\mathbf{K}_N(s_I, s_{II}) := \frac{1}{\delta}\begin{pmatrix}\mathbf{a}_1\mathbf{n}_c{\mathbf{n}_c}^T\mathbf{a}_1^T & -\mathbf{a}_1\mathbf{n}_c{\mathbf{n}_c}^T\mathbf{a}_2^T\\ -\mathbf{a}_2\mathbf{n}_c{\mathbf{n}_c}^T\mathbf{a}_1^T & \mathbf{a}_2\mathbf{n}_c{\mathbf{n}_c}^T\mathbf{a}_2^T\end{pmatrix} \tag{4.53}$$

is called the normal contact matrix.

Definition 4.29. $\mathbf{F}_N(s_I, s_{II})$ *defined by*

$$\mathbf{F}_N(s_I, s_{II}) := \frac{\bar{g}}{\delta}\begin{pmatrix}\mathbf{a}_1\mathbf{n}_c\\ -\mathbf{a}_2\mathbf{n}_c\end{pmatrix} \tag{4.54}$$

is called the normal contact load vector.

In the case of friction, two states must be taken into account. The stick case — when the tangential contact force is the reaction from the contact constraint at the contact interface. The second, sliding case — when the tangential contact force is the friction force obtained from the law $\mathcal{F}(u, s)$.

Assumption 4.30. *The contact contribution in the stick case is given by the functional*

$$\frac{1}{\delta}\int_{J_c}[u]_{\mathbf{t}_e}[v]_{\mathbf{t}_e}ds. \tag{4.55}$$

The assumption is motivated by equation (5.32) in [Wriggers, 2002].

Proposition 4.31. *Let $s_I, s_{II} \in J_c$ be a contact pair, then*

$$\frac{1}{\delta}\int_{J_c}[u]_\tau[v]_\tau ds = \frac{1}{\delta}\begin{pmatrix}(\mathbf{Q}_v^I)^T\\(\mathbf{Q}_v^{II})^T\end{pmatrix}^T\begin{pmatrix}\mathbf{a}_1\mathbf{t}_e\mathbf{t}_e^T\mathbf{a}_1^T & -\mathbf{a}_1\mathbf{t}_e\mathbf{t}_e^T\mathbf{a}_2^T\\-\mathbf{a}_2\mathbf{t}_e\mathbf{t}_e^T\mathbf{a}_1^T & \mathbf{a}_2\mathbf{t}_e\mathbf{t}_e^T\mathbf{a}_2^T\end{pmatrix}\begin{pmatrix}\mathbf{Q}_u^I\\\mathbf{Q}_u^{II}\end{pmatrix}. \tag{4.56}$$

Proof. Lemma 4.26 implies that

$$\begin{aligned}\frac{1}{\delta}\int_{J_c}[u]_{\mathbf{t}_e}[v]_{\mathbf{t}_e}ds &= \frac{1}{\delta}([u]_{\mathbf{t}_e}[v]_{\mathbf{t}_e})|_{s_I,s_{II}} = \frac{1}{\delta}([v]_{\mathbf{t}_e}[u]_{\mathbf{t}_e})|_{s_I,s_{II}}\\&= \frac{1}{\delta}\left[((v^I-v^{II})|_{s_I,s_{II}})^T\cdot\mathbf{t}_e\cdot\mathbf{t}_e^T\cdot(u^I-u^{II})|_{s_I,s_{II}}\right]\\&= ((\mathbf{Q}_v^I)^T\mathbf{a}_1\mathbf{t}_e-(\mathbf{Q}_v^{II})^T\mathbf{a}_2\mathbf{t}_e)(\mathbf{t}_e^T\mathbf{b}_1\mathbf{Q}_u^I-\mathbf{t}_e^T\mathbf{b}_1\mathbf{Q}_u^{II}).\end{aligned}$$

Rearranging the terms we obtain

$$\frac{1}{\delta}([u]_{\mathbf{t}_e}[v]_{\mathbf{t}_e})|_{s_I,s_{II}} = \frac{1}{\delta}\begin{pmatrix}(\mathbf{Q}_v^I)^T\\(\mathbf{Q}_v^{II})^T\end{pmatrix}^T\begin{pmatrix}\mathbf{a}_1\mathbf{t}_e\mathbf{t}_e^T\mathbf{a}_1^T & -\mathbf{a}_1\mathbf{t}_e\mathbf{t}_e^T\mathbf{a}_2^T\\-\mathbf{a}_2\mathbf{t}_e\mathbf{t}_e^T\mathbf{a}_1^T & \mathbf{a}_2\mathbf{t}_e\mathbf{t}_e^T\mathbf{a}_2^T\end{pmatrix}\begin{pmatrix}\mathbf{Q}_u^I\\\mathbf{Q}_u^{II}\end{pmatrix}. \tag{4.57}$$

□

Definition 4.32. $\mathbf{K}_T^{st}(s_I, s_{II})$ *defined by*

$$\mathbf{K}_T^{st}(s_I,s_{II}) := \frac{1}{\delta}\begin{pmatrix}\mathbf{a}_1\mathbf{t}_e\mathbf{t}_e^T\mathbf{a}_1^T & -\mathbf{a}_1\mathbf{t}_e\mathbf{t}_e^T\mathbf{a}_2^T\\-\mathbf{a}_2\mathbf{t}_e\mathbf{t}_e^T\mathbf{a}_1^T & \mathbf{a}_2\mathbf{t}_e\mathbf{t}_e^T\mathbf{a}_2^T\end{pmatrix} \tag{4.58}$$

is called the stick tangential contact matrix.

Remark 4.33. *The definition 4.32 is related to the matrix form of the contact contribution in the stick phase.*

Consider the regularized friction functional

$$\tilde{J}_2^e(u,v) := \int_{J_c}\mathcal{F}(u,s)\nabla\phi([u]_{\mathbf{t}_e})[v]_{\mathbf{t}_e}(s)ds,$$

where

$$\nabla\phi([u]_{\mathbf{t}_e}) = \begin{cases} \frac{[u]_{\mathbf{t}_e}}{|[u]_{\mathbf{t}_e}|}, & \text{if } |[u]_{\mathbf{t}_e}| \geq \gamma, \\ \frac{1}{\gamma}[u]_{\mathbf{t}_e}, & \text{if } |[u]_{\mathbf{t}_e}| < \gamma. \end{cases} \tag{4.59}$$

Following equation (11.26) in [Wriggers, 2002], if two contacting elements are orthogonal the tangential force $\mathbf{F}_t$ can be decomposed as follows

$$\mathbf{F}_t = F_{\mathbf{t}_I}\mathbf{t}_I + F_{\mathbf{t}_{II}}\mathbf{t}_{II},$$

where $F_{\mathbf{t}_e}$ is a projection of $\mathbf{F}_t$ onto the e-element axis. Therefore, the sliding caused by the force F_t can be replaced by two sliding displacements of one contacting beam along another contacting beam. Each such sliding displacement along element e is caused by the force $F_{\mathbf{t}_e}$

$$[u]_t = [u_{\mathbf{t}_I}]\mathbf{t}_I + [u_{\mathbf{t}_{II}}]\mathbf{t}_{II}.$$

Let us decompose the term

$$\frac{[u]_{\mathbf{t}}}{|[u]_{\mathbf{t}}|} = z_u^I\mathbf{t}_I + z_u^{II}\mathbf{t}_{II},$$

where

$$z_u^e = \frac{[u]_{\mathbf{t}}}{|[u]_{\mathbf{t}}|}\cdot\mathbf{t}_e,$$

with discretization given by

$$z_Q^e = \frac{\mathbf{Q}_u}{|\mathbf{Q}_u|}\cdot\mathbf{t}_e$$

Proposition 4.34. *Let $s_I, s_{II} \in J_c$ be a contact pair and $\mathcal{F}(s,u) = \frac{1}{\delta}\mu(s)([u]_n - \bar{g})_+$ be Coulomb friction, then, if $|[u]_{\mathbf{t}_e}| > \gamma$,*

$$\tilde{J}_2^e(u,v) = \frac{1}{\delta}z_Q^e\mu(s_e)\begin{pmatrix}(\mathbf{Q}_v^I)^T \\ (\mathbf{Q}_v^{II})^T\end{pmatrix}^T\begin{pmatrix}\mathbf{a}_1\mathbf{t}_e\mathbf{n}_c{}^T\mathbf{a}_1^T & -\mathbf{a}_1\mathbf{t}_e\mathbf{n}_c{}^T\mathbf{a}_2^T \\ -\mathbf{a}_2\mathbf{t}_e\mathbf{n}_c{}^T\mathbf{a}_1^T & \mathbf{a}_2\mathbf{t}_e\mathbf{n}_c{}^T\mathbf{a}_2^T\end{pmatrix}\begin{pmatrix}\mathbf{Q}_u^I \\ \mathbf{Q}_u^{II}\end{pmatrix}$$
$$- \frac{1}{\delta}z_Q^e\mu(s_e)\bar{g}\begin{pmatrix}(\mathbf{Q}_v^I)^T \\ (\mathbf{Q}_v^{II})^T\end{pmatrix}^T\begin{pmatrix}\mathbf{a}_1\mathbf{t}_e \\ -\mathbf{a}_2\mathbf{t}_e\end{pmatrix}. \tag{4.60}$$

Proof. Substituting (4.59) for $\gamma \leq |[u]_{\mathbf{t}_e}|$ we get

$$\tilde{J}_2^e(u,v) = \int_{J_c} z_Q^e\frac{1}{\delta}\mu(s)([u]_n - \bar{g})_+[v]_{\mathbf{t}_e}(s)ds.$$

Consider only those contact points where integral does not vanish, then

$$\int_{J_c} z_Q^e \frac{1}{\delta}\mu(s)([u]_n - \bar{g})_+ [v]_{\mathbf{t}_e}(s)ds = \frac{1}{\delta}\int_{J_c} z_Q^e \mu(s)([u]_n - \bar{g})[v]_{\mathbf{t}_e}(s)ds$$
$$= \frac{1}{\delta} z_Q^e \mu(s_I)([u]_n[v]_{\mathbf{t}_e})|_{s_I,s_{II}} - \frac{1}{\delta} z_Q^e \mu(s_I)(\bar{g}[v]_{\mathbf{t}_e})|_{s_I,s_{II}}.$$

Lemma 4.26 implies that

$$\mu(s_e)([u]_n[v]_{\mathbf{t}_e})|_{s_I,s_{II}} = \mu(s_e)([v]_{\mathbf{t}_e}[u]_n)|_{s_I,s_{II}} = \mu(s_e)\left[(v^I - v^{II})^T \cdot \mathbf{t}_e \cdot \mathbf{n}_c{}^T \cdot (u^I - u^{II})\right]|_{s_I,s_{II}}$$
$$= ((\mathbf{Q}_v^I)^T \mathbf{a}_1 \mathbf{t}_e - (\mathbf{Q}_v^{II})^T \mathbf{a}_2 \mathbf{t}_e)(\mathbf{n}_c{}^T \mathbf{a}_1^T \mathbf{Q}_u^I - \mathbf{n}_c{}^T \mathbf{a}_2^T \mathbf{Q}_u^{II}).$$

Rearranging the terms we get

$$z_Q^e \mu(s_I)([u]_n[v]_{\mathbf{t}_e})|_{s_I,s_{II}} = z_Q^e \mu(s_e) \begin{pmatrix} (\mathbf{Q}_v^I)^T \\ (\mathbf{Q}_v^{II})^T \end{pmatrix}^T \begin{pmatrix} \mathbf{a}_1 \mathbf{t}_e \mathbf{n}_c{}^T \mathbf{a}_1^T & -\mathbf{a}_1 \mathbf{t}_e \mathbf{n}_c{}^T \mathbf{a}_2^T \\ -\mathbf{a}_2 \mathbf{t}_e \mathbf{n}_c{}^T \mathbf{a}_1^T & \mathbf{a}_2 \mathbf{t}_e \mathbf{n}_c{}^T \mathbf{a}_2^T \end{pmatrix} \begin{pmatrix} \mathbf{Q}_u^I \\ \mathbf{Q}_u^{II} \end{pmatrix}.$$

The second term is

$$z_Q^e \mu(s_e)(\bar{g}[v]_{\mathbf{t}_e})|_{s_I,s_{II}} = z_Q^e \mu(s_e)\bar{g}((\mathbf{Q}_v^I)^T \mathbf{a}_1 \mathbf{t}_e - (\mathbf{Q}_v^{II})^T \mathbf{a}_2 \mathbf{t}_e) = z_Q^e \mu(s_e)\bar{g} \begin{pmatrix} (\mathbf{Q}_v^I)^T \\ (\mathbf{Q}_v^{II})^T \end{pmatrix}^T \begin{pmatrix} \mathbf{a}_1 \mathbf{t}_e \\ -\mathbf{a}_2 \mathbf{t}_e \end{pmatrix}. \tag{4.61}$$

□

In the case of Tresca friction, the sliding can be described by

Proposition 4.35. *Let $s_I, s_{II} \in J_c$ be a contact pair and $\mathcal{F}(s,u) = G(s)$ be Tresca friction, then*

$$\tilde{J}_2^e(u,v) = z_Q^e G(s_e) \begin{pmatrix} (\mathbf{Q}_v^I)^T \\ (\mathbf{Q}_v^{II})^T \end{pmatrix}^T \begin{pmatrix} \mathbf{a}_1 \mathbf{t}_e \\ -\mathbf{a}_2 \mathbf{t}_e \end{pmatrix}. \tag{4.62}$$

Proof. In the case of Tresca friction, choosing positive contact orientation,

$$\tilde{J}_2^e(u,v) = \int_{J_c} z_Q^e G(s)[v]_{\mathbf{t}_e}(s)ds.$$

Lemma 4.26 implies that

$$\int_{J_c} z_Q^e G(s)[v]_{\mathbf{t}_e}(s)ds = z_Q^e G(s_e)((\mathbf{Q}_v^I)^T \mathbf{a}_1 \mathbf{t}_e - (\mathbf{Q}_v^{II})^T \mathbf{a}_2 \mathbf{t}_e) = z_Q^e G(s_e) \begin{pmatrix} (\mathbf{Q}_v^I)^T \\ (\mathbf{Q}_v^{II})^T \end{pmatrix}^T \begin{pmatrix} \mathbf{a}_1 \mathbf{t}_e \\ -\mathbf{a}_2 \mathbf{t}_e \end{pmatrix}.$$

□

Definition 4.36. $\mathbf{K}_T^{sl}(s_I, s_{II})$ *defined by*

$$\mathbf{K}_T^{sl}(s_I, s_{II}) := z_Q^e \mu(s_e) \begin{pmatrix} \mathbf{a}_1 \mathbf{t}_e \mathbf{n}_c{}^T \mathbf{a}_1^T & -\mathbf{a}_1 \mathbf{t}_e \mathbf{n}_c{}^T \mathbf{a}_2^T \\ -\mathbf{a}_2 \mathbf{t}_e \mathbf{n}_c{}^T \mathbf{a}_1^T & \mathbf{a}_2 \mathbf{t}_e \mathbf{n}_c{}^T \mathbf{a}_2^T \end{pmatrix} \tag{4.63}$$

is called sliding tangential contact matrix for Coulomb friction.

Definition 4.37. $\mathbf{F}_T^{sl}(s_I, s_{II})$ *defined by*

$$\mathbf{F}_T^{sl}(s_I, s_{II}) := z_Q^e \mu(s_e) \bar{g} \begin{pmatrix} \mathbf{a}_1 \mathbf{t}_e \\ -\mathbf{a}_2 \mathbf{t}_e \end{pmatrix} \tag{4.64}$$

in the case of Coulomb friction, and by

$$\mathbf{F}_T^{sl}(s_I, s_{II}) := z_Q^e G(s_e) \begin{pmatrix} \mathbf{a}_1 \mathbf{t}_e \\ -\mathbf{a}_2 \mathbf{t}_e \end{pmatrix} \tag{4.65}$$

in the case of Tresca friction, is called the sliding load vector.

Remark 4.38. *In order to linearize the problem, we take the direction vector* $\frac{\mathbf{Q}_u}{|\mathbf{Q}_u|}$ *from a preliminary simulation with no friction (*$F = 0$ *or* $G = 0$*).*

Assembling procedure

Detailed algorithms and the description for the assembling procedure of the global stiffness matrix can be found in FEM literature, for instance, in the works of [Axelsson, 2001] and [Gockenbach, 2006]. Here we describe only the general procedure. The idea which stands behind is that the assembling procedure is implemented for the whole block of degrees of freedom for each node in the finite element mesh. Namely, let the vector and matrix quantities be subdivided to the blocks for the corresponding start and end nodes i, j

$$\mathbf{K}^e = \begin{pmatrix} \mathbf{K}_{ii}^e & \mathbf{K}_{ij}^e \\ (\mathbf{K}_{ij}^e)^T & \mathbf{K}_{jj}^e \end{pmatrix}, \quad \mathbf{F}^e = \begin{pmatrix} \mathbf{F}_i^e \\ \mathbf{F}_j^e \end{pmatrix}, \tag{4.66}$$

then the contribution of the element e in the global system takes the form

$$\begin{array}{cc} & \begin{array}{ccccc} & i\text{ block} & & j\text{ block} & \end{array} \\ \begin{array}{c} \\ i\text{ block} \\ \\ j\text{ block} \\ \\ \end{array} & \begin{pmatrix} 0 & \overbrace{\cdot} & \cdot & \overbrace{\cdot} & 0 \\ \cdot & \mathbf{K}^e_{ii} & \cdot & \mathbf{K}^e_{ij} & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & (\mathbf{K}^e_{ij})^T & \cdot & \mathbf{K}^e_{jj} & \cdot \\ 0 & \cdot & \cdot & \cdot & 0 \end{pmatrix} \end{array}, \qquad \begin{pmatrix} \cdot \\ \mathbf{F}^e_i \\ \cdot \\ \mathbf{F}^e_j \\ \cdot \end{pmatrix}, \tag{4.67}$$

and the global vectors and matrices are obtained via

$$\mathbf{K} = \sum_e \mathbf{K}^e, \qquad \mathbf{F} = \sum_e \mathbf{F}^e. \tag{4.68}$$

Remark 4.39. *The functional P_e drops due to the matching conditions listed in Theorem 3.14.*

In the case of the contact contribution, one needs to notice that for each contact pair the contact affects two elements and, hence, four nodes.

$$\mathbf{K}_N(s_I, s_{II}) = \begin{pmatrix} (\mathbf{K}_N)_{i_I i_I} & (\mathbf{K}_N)_{i_I j_I} & (\mathbf{K}_N)_{i_I i_{II}} & (\mathbf{K}_N)_{i_I j_{II}} \\ (\mathbf{K}_N)_{j_I i_I} & (\mathbf{K}_N)_{j_I j_I} & (\mathbf{K}_N)_{j_I i_{II}} & (\mathbf{K}_N)_{j_I j_{II}} \\ (\mathbf{K}_N)_{i_{II} i_I} & (\mathbf{K}_N)_{i_I j_I} & (\mathbf{K}_N)_{i_{II} i_{II}} & (\mathbf{K}_N)_{i_{II} j_{II}} \\ (\mathbf{K}_N)_{j_{II} i_I} & (\mathbf{K}_N)_{j_{II} j_I} & (\mathbf{K}_N)_{j_{II} i_{II}} & (\mathbf{K}_N)_{j_{II} j_{II}} \end{pmatrix}, \quad \mathbf{F}_N(s_I, s_{II}) = \begin{pmatrix} (\mathbf{F}_N)_{i_I} \\ (\mathbf{F}_N)_{j_I} \\ (\mathbf{F}_N)_{i_{II}} \\ (\mathbf{F}_N)_{j_{II}} \end{pmatrix}, \tag{4.69}$$

what results in the contribution

$$\begin{array}{cc} & \begin{array}{ccccccccc} & i_I\text{block} & & j_I\text{block} & & i_{II}\text{block} & & j_{II}\text{block} & \end{array} \\ \begin{array}{c} \\ i_I\text{ block} \\ \\ j_I\text{ block} \\ \\ i_{II}\text{ block} \\ \\ j_{II}\text{ block} \\ \\ \end{array} & \begin{pmatrix} 0 & \overbrace{\cdot} & \cdot & \overbrace{\cdot} & \cdot & \overbrace{\cdot} & \cdot & \overbrace{\cdot} & 0 \\ \cdot & (\mathbf{K}_N)_{i_I i_I} & \cdot & (\mathbf{K}_N)_{i_I j_I} & \cdot & (\mathbf{K}_N)_{i_I i_{II}} & \cdot & (\mathbf{K}_N)_{i_I j_{II}} & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & (\mathbf{K}_N)_{j_I i_I} & \cdot & (\mathbf{K}_N)_{j_I j_I} & \cdot & (\mathbf{K}_N)_{j_I i_{II}} & \cdot & (\mathbf{K}_N)_{j_I j_{II}} & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & (\mathbf{K}_N)_{i_{II} i_I} & \cdot & (\mathbf{K}_N)_{i_I j_I} & \cdot & (\mathbf{K}_N)_{i_{II} i_{II}} & \cdot & (\mathbf{K}_N)_{i_{II} j_{II}} & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & (\mathbf{K}_N)_{j_{II} i_I} & \cdot & (\mathbf{K}_N)_{j_{II} j_I} & \cdot & (\mathbf{K}_N)_{j_{II} i_{II}} & \cdot & (\mathbf{K}_N)_{j_{II} j_{II}} & \cdot \\ 0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 \end{pmatrix} \end{array}, \qquad \begin{pmatrix} \cdot \\ (\mathbf{F}_N)_{i_I} \\ \cdot \\ (\mathbf{F}_N)_{j_I} \\ \cdot \\ (\mathbf{F}_N)_{i_{II}} \\ \cdot \\ (\mathbf{F}_N)_{j_{II}} \\ \cdot \end{pmatrix}. \tag{4.70}$$

The total contribution can be obtained by summing over all contact pairs,

$$\mathbf{K}_N = \sum_{s_I, s_{II} \in J_c} \mathbf{K}_N(s_I, s_{II}), \qquad \mathbf{F}_N = \sum_{s_I, s_{II} \in J_c} \mathbf{F}_N(s_I, s_{II}). \tag{4.71}$$

The contribution of the tangential contact is assembled similarly,

$$\mathbf{K}_T^{st} = \sum_{s_I, s_{II} \in J_c} \mathbf{K}_T^{st}(s_I, s_{II}), \quad \mathbf{K}_T^{sl} = \sum_{s_I, s_{II} \in J_c} \mathbf{K}_T^{sl}(s_I, s_{II}), \quad \mathbf{F}_T^{sl} = \sum_{s_I, s_{II} \in J_c} \mathbf{F}_T^{sl}(s_I, s_{II}). \tag{4.72}$$

Matrix formulation of the problem

Substitution of the matrix formulations for the bilinear forms $a_e(\cdot,\cdot)$, $J_1^e(\cdot,\cdot)$, $\tilde{J}_2^e(\cdot,\cdot)$ and for the functional $f_e(\cdot)$ yields

$$\mathbf{Q}^T \left(\mathbf{K} + \mathbf{K_N} + \mathbf{K}_T\right) \mathbf{Q} = \mathbf{Q}^T \left(\mathbf{F} + \mathbf{F}_N + \mathbf{F}_T\right). \tag{4.73}$$

Define

$$\mathbf{G}(\mathbf{Q}) := \frac{1}{2}\mathbf{Q}^T \left(\mathbf{K} + \mathbf{K_N} + \mathbf{K}_T\right) \mathbf{Q} - \mathbf{Q}^T \left(\mathbf{F} + \mathbf{F}_N + \mathbf{F}_T\right), \tag{4.74}$$

then the minimizer of $\mathbf{G}$ over all $\mathbf{Q} \in \mathbb{R}^M$, where M is the number of degrees of freedom, determines the solution of our finite element problem (4.22), (4.23).

Theorem 4.40. *Solution of the minimization problem*

$$\mathbf{G}(\mathbf{Q}) \to \min, \quad \mathbf{Q} \in \mathbb{R}^M, \tag{4.75}$$

exists and satisfies

$$\left(\mathbf{K} + \mathbf{K_N} + \mathbf{K}_T\right) \mathbf{Q} = \left(\mathbf{F} + \mathbf{F}_N + \mathbf{F}_T\right). \tag{4.76}$$

Proof. Existence of solution follows from Theorem 4.13. $\mathbf{G}$ is the quadratic functional, and its minimizer must satisfy the necessary condition of extrema

$$\nabla \mathbf{G} = \left(\mathbf{K} + \mathbf{K_N} + \mathbf{K}_T\right) \mathbf{Q} - \left(\mathbf{F} + \mathbf{F}_N + \mathbf{F}_T\right) = 0.$$

□

The equation (4.76) is the equivalent matrix formulation of the problem (4.22), (4.23).

Contact solution algorithm

All aspects of many approaches to treat contact problems can be found in [Wriggers, 2002]. Globally the contact solution algorithm consists of two parts: the contact search and an update of the contact contribution to the matrix formulation of the problem.

For a fiber structure the contact search can be implemented in the following way:

1. run over all elements,

2. for each master element j_k find a slave element i_k with the minimum distance between neutral lines defined by the master element j_k and the slave element i_k:

$$d = \min_{i,j} \|\mathbf{r}^i - \mathbf{r}^j\|,$$

   satisfying additional constraints:

$$\mathbf{r}^i \in element\ i_k, \quad \mathbf{r}^j \in element\ j_k,$$

3. if the penetration is negative,

$$g = d - 2R < 0,$$

   then elements i_k, j_k are contacting elements, and the contact points can be obtained from the minimum distance problem.

The update of the contact contribution is important especially for the tangential contact, where it is required to check whenever the tangential contact is in the stick or sliding phase. The situation becomes even more complicated since under the external loading, Neumann or Dirichlet boundary conditions, the state may change from stick to sliding. Then the tangential gap at the contact point splits into the stick and sliding parts $[u]_t = ([u]_t)_{st} + ([u]_t)_{sl}$. Therefore, an iterative procedure is required. In the case of finite deformations, the contact point may slide to a neighboring element, what also requires a modification of the algorithm. The iteration can be made by

$$\left(\mathbf{K} + \mathbf{K_N} + \mathbf{K}_T\right)\mathbf{Q} = \alpha\mathbf{F} + \left(\mathbf{F}_N + \mathbf{F}_T\right),$$

where α can be taken as $\alpha_n = \Delta t \cdot n$ $(\Delta t = 1/N,\ n = 1 \dots N)$. Let f_{fr} be a friction force and F_t be the tangential force acting along the element.

Assume that the beam come to contact orthoganally at the contact points.

The switching between different states of the tangential contact is made according to the following criteria:

- Stick phase: beam I - stick $(f^n_{I_s} < 0)$, beam II - stick $(f^n_{I_s} < 0)$
  (*full stick*)

- Slip phase:

  - beam I - stick $(f^n_I < 0)$, beam II - slip $(f^n_{II} > 0)$,
  - beam I - slip $(f^n_I > 0)$, beam II - stick $(f^n_{II} < 0)$,
  - beam I - slip $(f^n_I > 0)$, beam II - slip $(f^n_{II} > 0)$
    (*full slip*),

where

$$f^n := |(F_t)^n| - |(f_{fr})^n|.$$

Then a possible algorithm for determining the stick or sliding state is

1. initialize the algorithm: set all contact points into the stick phase,

2. iterating over $n = 1 \dots N$,

   - update the tangential force F_t,
   - update the friction force f^n_{fr},
   - check for a contact: $g_N \leq 0 \Rightarrow$ active element,
   - check whether $f^n_e < 0$ for each element,
   - update $\mathbf{K}$, $\mathbf{K}_N$ and $\mathbf{K}_T$, $\mathbf{F}_N$ and $\mathbf{F}_T$,
   - solve $(\mathbf{K}^n + \mathbf{K_N}^n + \mathbf{K}^n_T)\,\mathbf{Q} = \alpha^n \mathbf{F} + (\mathbf{F}^n_N + \mathbf{F}^n_T)$.

Chapter 5

The algorithm and numerical examples

Chapter 5 presents the homogenization algorithm for computation of the effective elasto-plastic material law of a technical textile. The textile is characterized by the microstructure of fibers on the periodicity cell. The algorithm is based on the results of homogenization of the contact conditions given in Chapter 2, what allows to interpret the averaged frictional microsliding as effective plasticity. The proposed algorithm for construction of the piece-wise linear elasto-plastic curve is implemented numerically, using the finite element with the finite-element contact formulation derived in Chapter 4. The implementation is made in the form of an own code, named FiberFEM. Numerical examples can be found in Section 5.3. The author would like to thank students Albina Davletkulova and Iliya Prozorov for preparing some numerical examples using FiberFEM.

5.1 The algorithm for computation of effective mechanical properties of textiles

The algorithm is based on homogenization results of Chapters 2 and 3, the numerical solution of the auxiliary problem is based on the results of Chapter 4. The proposed algorithm consists of the next steps:

- solve the reduced contact auxiliary problem for N_q,
- solve the reduced contact auxiliary problem for $\Theta_\emptyset$,

- obtain elasto-plastic moduli.

5.1.1 Auxiliary problems

Construction of the algorithm

The construction of the algorithm consists of two main steps. First of all, the textile is represented by the layer of thickness ε with the microstructure periodic in the in-plane variables $x_1, \ldots, x_{n-1}$. As it was shown in Chapter 3, the auxiliary problems in this case have out-of-plane boundary conditions, for instance, for the columns n^{qp} of the auxiliary functions N_q in (5.1) the conditions

$$\begin{aligned} \frac{\partial(n_i^{qp} + \delta_{pj}\xi_q)}{\partial n} &= 0, \qquad \xi_n = \frac{1}{2}, \\ \frac{\partial(n_i^{qp} + \delta_{pj}\xi_q)}{\partial n} &= 0, \qquad \xi_n = -\frac{1}{2}, \end{aligned}$$

which come from the macro boundary conditions, and the condition of periodicity in the in-plane directions.

Second, the consideration of the microcontact with friction adds to the formulation of the cell problem the contact conditions given in (2.30) of Chapter 2. For instance, for the columns n^{qp} of the auxiliary functions N_q in (5.1) they take the form:

$$\begin{aligned} &\sigma_n^1(\xi) \le 0, \quad [n^{qp}(\xi)]_n \le \bar{g}(\xi), \quad \sigma_n^1(x,\xi)[n^{qp}(\xi) - \bar{g}(\xi)]_n = 0 \ \text{ on } S, \\ &|\sigma_\tau^1(\xi)| \le \frac{\mathcal{F}(n^{qp}(\xi)\nabla u_0, x, \xi)}{|\nabla u_0|} \quad \Rightarrow \quad [n_\tau^{qp}] = 0, \ \ \xi \in S, \\ &|\sigma_\tau^1(\xi)| = \frac{\mathcal{F}(n^{qp}(\xi)\nabla u_0, x, \xi)}{|\nabla u_0|} \quad \Rightarrow \quad \exists \lambda \ge 0 : \ \ [n_\tau^{qp}] = -\lambda\sigma_\tau^1, \ \ \xi \in S. \end{aligned}$$

In-plane auxiliary problems

Taking into account Theorem 2.25 and equation (3.28), we state the following auxiliary problems for Θ, N_q $(q = 1, \ldots, n-1)$.

The auxiliary functions N_q consist of the columns n^{qp}, the solutions of the problems

$$
\begin{aligned}
&\frac{\partial}{\partial \xi_h}\Big[a_{ihjk}(\xi)\frac{\partial (n_j^{qp}(\xi)+\delta_{pj}\xi_q)}{\partial \xi_k}\Big]=0, \quad \sigma_{ih}^1 \equiv a_{ihjk}(\xi)\frac{\partial (n_j^{qp}(\xi)+\delta_{pj}\xi_q))}{\partial \xi_k} \text{ in } Y^h, \qquad (5.1)\\
&\sigma_n^1(\xi)\le 0, \quad [n^{qp}(\xi)]_n \le \bar{g}(\xi), \quad \sigma_n^1(x,\xi)[n^{qp}(\xi)-\bar{g}(\xi)]_n = 0 \text{ on } S,\\
&|\sigma_\tau^1(\xi)| \le \frac{\mathcal{F}(n^{qp}(\xi)\nabla u_0, x, \xi)}{|\nabla u_0|} \quad \Rightarrow \quad [n_\tau^{qp}]=0, \quad \xi\in S,\\
&|\sigma_\tau^1(\xi)| = \frac{\mathcal{F}(n^{qp}(\xi)\nabla u_0, x, \xi)}{|\nabla u_0|} \quad \Rightarrow \quad \exists\lambda\ge 0: \quad [n_\tau^{qp}]=-\lambda\sigma_\tau^1, \quad \xi\in S,\\
&\frac{\partial (n_i^{qp}+\delta_{pj}\xi_q)}{\partial n}=0, \qquad \xi_n=\frac{1}{2},\\
&\frac{\partial (n_i^{qp}+\delta_{pj}\xi_q)}{\partial n}=0, \qquad \xi_n=-\frac{1}{2},\\
&n^q \text{ is } \xi_1,\dots,\xi_{n-1} - \text{periodic.}
\end{aligned}
$$

Auxiliary function $\Theta_\emptyset$ consists of the columns θ^q, the solution of the problem

$$
\begin{aligned}
&\frac{\partial}{\partial \xi_h}\Big[a_{ihjk}(\xi)\frac{\partial \theta_j^q(\xi)}{\partial \xi_k}\Big]=\delta_{jq}, \quad \sigma_{ih}^1 \equiv a_{ihjk}(\xi)\frac{\partial \theta_j^q(\xi)}{\partial \xi_k} \text{ in } Y^h, \qquad (5.2)\\
&\sigma_n^1(\xi)\le 0, \quad [\theta^q(\xi)]_n \le \bar{g}(\xi), \quad \sigma_n^1(x,\xi)[\theta^q(\xi)-\bar{g}(\xi)]_n = 0 \text{ on } S,\\
&|\sigma_\tau^1(\xi)| \le \frac{\mathcal{F}(\theta^q(\xi)p_1, x, \xi)}{|p_1|} \quad \Rightarrow \quad [\theta_\tau^q]=0, \quad \xi\in S,\\
&|\sigma_\tau^1(\xi)| = \frac{\mathcal{F}(\theta^q(\xi)p_1, x, \xi)}{|p_1|} \quad \Rightarrow \quad \exists\lambda\ge 0: \quad [\theta_\tau^q]=-\lambda\sigma_\tau^1, \quad \xi\in S,\\
&\frac{\partial \theta_i^q}{\partial n}=\delta_{iq}, \qquad \xi_n=\frac{1}{2},\\
&\frac{\partial \theta_i^q}{\partial n}=0, \qquad \xi_n=-\frac{1}{2},\\
&\theta^q \text{ is } \xi_1,\dots,\xi_{n-1} - \text{periodic.}
\end{aligned}
$$

In the case of symmetry of the fiber structure, the periodicity condition for Θ and N_q can be substituted by the third type boundary conditions, see [Bakhvalov, Panasenko, 1984].

Furthermore, the asymptotics with respect to h can be considered, and the corresponding reduced contact auxiliary problem can be obtained and solved numerically using the finite element method described in Chapter 4.

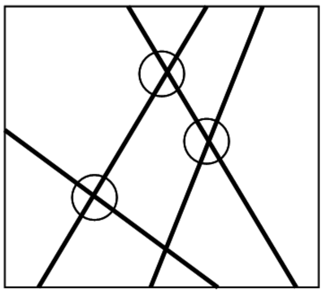


Figure 5.1: Contact points in the sliding state.

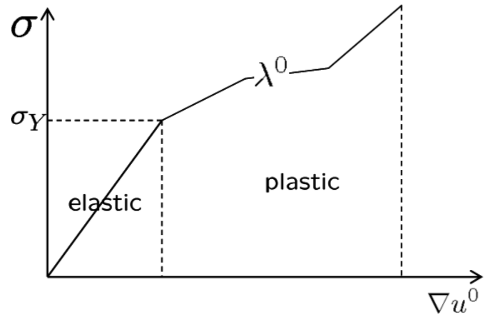


Figure 5.2: A piecewise linear elasto-plastic curve.

5.1.2 Effective properties, frictional microsliding as effective plasticity

The effective properties can be computed using

$$A_{ij}^{hom} := \frac{1}{|Y|}\int_Y \left(A_{ij}(\xi) + \sum_{q=1}^{n} A_{iq}\frac{\partial N_j(\xi)}{\partial \xi_q} \right) d\xi + \int_S \frac{\mathcal{F}(N_j(\xi)\nabla u^0(x), x, \xi)}{|\nabla u^0(x)|}[N_j]_t ds_\xi \quad (5.3)$$

and

$$B_{i}^{hom} := \frac{1}{|Y|}\int_Y \left(\sum_{j=1}^{n} A_{ij}(\xi)\frac{\partial \Theta_\emptyset}{\partial \xi_j} \right) d\xi + \int_S \frac{\mathcal{F}(\Theta_\emptyset(\xi)p(x), x, \xi)}{|p(x)|}[\Theta_\emptyset]_t ds_\xi. \quad (5.4)$$

The contribution of the contact can be divided into two parts: the contribution of the stick phase into the elastic part and the contribution of the sliding phase into the plastic part,

$$A_{ij}^{hom} := \bar{A}_{ij}^{elas} + \bar{A}_{ij}^{plas}, \quad (5.5)$$

$$B_{i}^{hom} := \bar{B}_{i}^{elas} + \bar{B}_{i}^{plas}. \quad (5.6)$$

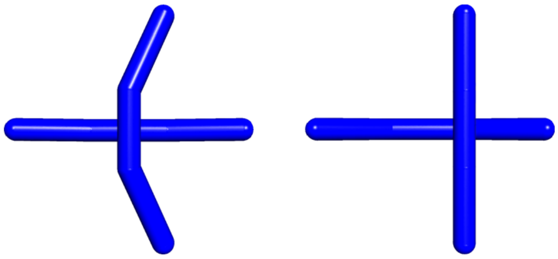
Consider

$$\int_S \frac{\mathcal{F}(N_j(\xi)\nabla u^0(x), x, \xi)}{|\nabla u^0(x)|}[N_j]_t ds_\xi.$$

Recall that under the sliding

$$[N_j]_t = -\lambda \sigma_t^1.$$

The evolution of $[N_j]_t$ depending on the strain on the fiber structure is characterized by the set of contacting points in the sliding state, see Fig. 5.1. The sequence in which these



$$|\sigma_\tau^{\mathbf{1}}(\xi)| \leq \frac{\mathcal{F}(\mathbf{w}(\xi)\nabla\mathbf{u^0},\xi)}{|\nabla\mathbf{u^0}|} \qquad |\sigma_\tau^{\mathbf{1}}(\xi)| = \frac{\mathcal{F}(\mathbf{w}(\xi)\nabla\mathbf{u^0},\xi)}{|\nabla\mathbf{u^0}|}$$

Figure 5.3: The gap between stick and sliding states.

contacting points turn to the sliding state defines the linear pieces in the stress-strain curve, see Fig. 5.2. The number of linear parts is equal to the number of switches to sliding. λ is the slope on each plastic part, which can be represented as the difference gap between sliding and stick states, see Fig. 5.3. An illustrative numerical example obtained using the FiberFEM code described in the next section, is given in Subsection 5.3.2.

Recall that in the case of stick state $[N_j]_t = 0$, the contact point does not slide, and in this case the contribution of the frictional contact goes into the elastic part

$$\bar{A}_{ij}^{elas} = \{\bar{a}_{ij}^{kl}\},$$

which in general will be fully anisotropic. However, in the case of plane symmetry of the fiber configuration Y^h, the set Y^h is invariant regarding to all mappings S_h ($h = 1, \ldots, n-1$), where

$$S_h(\xi) = ((-1)^{\delta_{h1}}\xi_1, \ldots, (-1)^{\delta_{hn}}\xi_n)$$

and

$$\delta_{hi} = \begin{cases} 1, h = i, \\ 0, h \neq i \end{cases}$$

is the Kronecker delta, therefore in-plane orthotropy can be obtained. Indeed, according to Chapter 6 of [Bakhvalov, Panasenko, 1984], the only nonzero effective elasticity moduli are $\bar{a}_{11}^{11}$, $\bar{a}_{22}^{22}$, $\bar{a}_{12}^{12}$, $\bar{a}_{12}^{21}$. Recall that for an orthotropic material

$$\begin{pmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ 2\varepsilon_{12} \end{pmatrix} = \begin{pmatrix} \frac{1}{E_1} & -\frac{\nu_{12}}{E_1} & 0 \\ -\frac{\nu_{21}}{E_2} & \frac{1}{E_2} & 0 \\ 0 & 0 & \frac{1}{G_{12}} \end{pmatrix} \begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{pmatrix}.$$

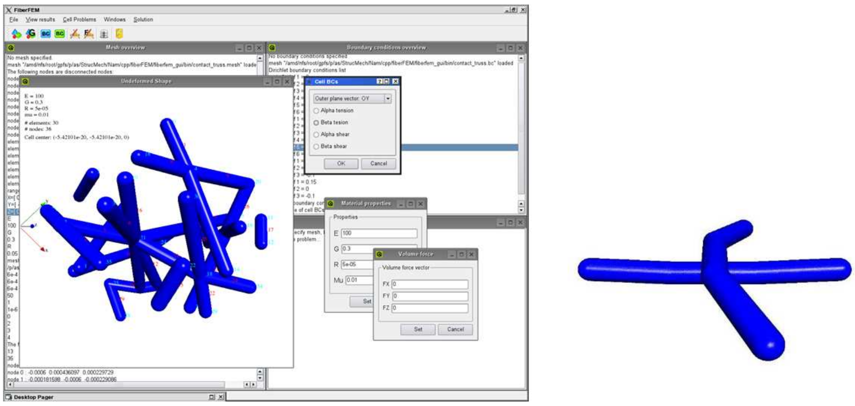


Figure 5.4: FiberFEM.

Inverting the compliance matrix given above we get

$$\begin{pmatrix} \bar{a}_{11}^{11} & \bar{a}_{12}^{12} & 0 \\ \bar{a}_{12}^{12} & \bar{a}_{22}^{22} & 0 \\ 0 & 0 & \bar{a}_{22}^{11} \end{pmatrix} = \begin{pmatrix} \frac{E_1}{1-\nu_{12}\nu_{21}} & \frac{E_1\nu_{12}}{1-\nu_{12}\nu_{21}} & 0 \\ \frac{E_2\nu_{21}}{1-\nu_{12}\nu_{21}} & \frac{E_2}{1-\nu_{12}\nu_{21}} & 0 \\ 0 & 0 & G_{12} \end{pmatrix}.$$

Then the orthotropic material constants can be obtained in the form:

$$\nu_{12} = \frac{\bar{a}_{12}^{12}}{\bar{a}_{11}^{11}}, \quad \nu_{21} = \frac{\bar{a}_{12}^{12}}{\bar{a}_{22}^{22}},$$

$$E_1 = \bar{a}_{11}^{11} - \frac{(\bar{a}_{12}^{12})^2}{\bar{a}_{22}^{22}}, \quad E_2 = \bar{a}_{22}^{22} - \frac{(\bar{a}_{12}^{12})^2}{\bar{a}_{11}^{11}}, \quad G_{12} = \bar{a}_{22}^{11}.$$

Remark 5.1. *The question of existence of an equivalent homogeneous elastic plate with elastic material properties given by* $\bar{A}_{ij}^{elas}$ *was studied in [Panasenko, 2005]. The necessary and sufficient conditions were derived.*

5.2 FiberFEM

FiberFEM is a finite element code for simulation of 3D woven, non-woven, and knitted fiber-yarn structures with frictional contacts. It was developed using the beam finite element formulation given in Chapter 4. FiberFEM is based on the standalone linear algebra library GMM++, a generic C++ template library for sparse, dense, and skyline matrices. The GUI interface (see Fig.5.4) is based on Qt4, a cross-platform application development framework. The visualization is implemented using OpenGL, a cross-platform API for writing applications that produce 2D and 3D computer graphics.

The motivation for developing the code FiberFEM is the following. First, even though some commercial or open source finite element (FE) software are claimed to be able to solve beam contact problems, due to the highly nonlinear nature of contact problems these software packages are not very effective. For instance, trying to solve a simple example with sliding involving 4 knitted fibers, ANSYS did not show good results. Second and the most important reason for developing an own code is that the stick/slip switching at the contact points in the algorithm of the construction of the effective stress-strain curve, described in Subsection 5.1.2, is not available in these FE packages. This requires an own implementation of the FE solution of the auxiliary contact problems.

Thus, the similarity of FiberFEM with other FE codes is that they share the well-known approaches (see [Wriggers, 2002]) in the contact mechanics. The main distinction of FiberFEM is that it allows to construct the effective stress-strain curve by taking into account the microsliding as a plasticity effect.

The contact in FiberFEM is resolved by the penalty method, and the algorithm for stick/sliding state detection and the sliding gap λ, described in Section 4.3, is implemented. The fiber geometries are created in fViz, a visualization code for FiberFEM.

Input:

- the geometry of the cross-section,
- the stiffness of each yarn or fiber,
- the microstructure or the pattern.

Output:

- effective properties of textile.

5.2.1 Validation with ANSYS

A 29-node mesh of a fiber structure with fixed junctions is taken to validate the FiberFEM finite element code. The same mesh is used with the commercial code ANSYS. The material of fibers is isotropic with the Young's modulus $E = 1000000\,\mathrm{Pa}$, the shear modulus $G = 0.3$, and the radius of fibers $r = 0.0005\,\mathrm{m}$. The size of the periodicity cell is equal to 0.01. The deformed and undeformed shape obtained by FiberFEM and ANSYS, the table of the components UX_2, UY_2, UZ_2 of the displacement vector at the node 2 and the components $UX_{18}, UY_{18}, UZ_{18}$ of the displacement vector at the node 18 are given in Fig. 5.5-5.7 and Tab. 5.1.

5.2.2 Validation of contact

First testing example

The first testing example is taken from [Wriggers, 2002], p.102. The system consists of two beams of the equal length l, with the equal tension stiffness. The right end is fixed, and at the left end the displacement $\bar{u}$ is applied in the axial direction. The initial gap g_o is assumed to be closed. The contact constraint is given by the non-penetration condition $|u_2^1 - u_1^2| \leq 0$. The penalized normal contact force is given explicitly. After imposing boundary conditions and introduction of the penalty form into the global stiffness matrix we get

$$\begin{pmatrix} \frac{EA}{l} + 1/\delta & -1/\delta \\ -1/\delta & \frac{EA}{l} + 1/\delta \end{pmatrix} \left\{ \begin{array}{c} u_2^1 \\ u_1^2 \end{array} \right\} = \left\{ \begin{array}{c} \frac{EA}{l}\bar{u} \\ 0 \end{array} \right\}.$$

The solution is

$$\left\{ \begin{array}{c} u_2^1 \\ u_1^2 \end{array} \right\} = \frac{\bar{u}}{2 + \frac{EA}{l(1/\delta)}} \left\{ \begin{array}{c} 1 + \frac{EA}{l(1/\delta)} \\ 1 \end{array} \right\}.$$

The contact force at the right end of the left beam is given by

$$F_N = \frac{EA}{l}(u_2^1 - \bar{u}) = \frac{EA}{l} \left(\frac{1 + \frac{EA}{l(1/\delta)}}{2 + \frac{EA}{l(1/\delta)}} - 1 \right) \bar{u}.$$

The exact contact force is $\bar{F}_N = \lim_{\delta \to 0} F_N$.

Tab. 5.2 gives the relative error of the contact force computed by FiberFEM, compared with the exact contact force $N = -5\,\mathrm{N}$, for a truss construction with the axial stiffness $1000\,\mathrm{N/m}$ and the length of each beam $l = 1\,\mathrm{m}$, for different values of the penalty parameter ε.

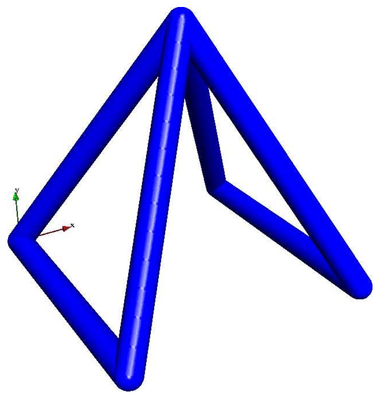


Figure 5.5: The undeformed shape, FiberFEM.

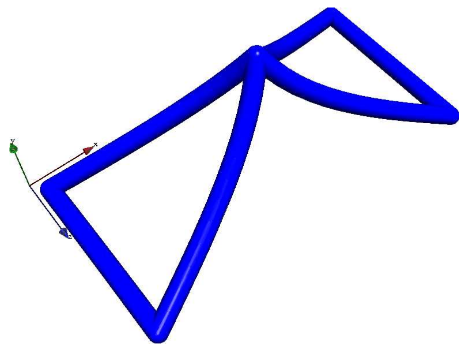


Figure 5.6: The deformed shape, FiberFEM.

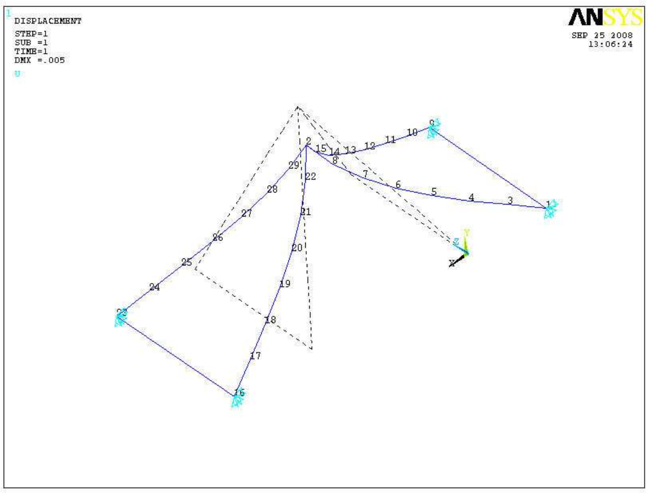


Figure 5.7: The deformed and undeformed shape, ANSYS.

	UX_2	UY_2	UZ_2
ANSYS	3.5378e-17	-0.0024968	0.18021e-17
fiberFEM	-5.4651e-18	-0.0024971	-7.78662e-19
relative error	0.544745	0.0000801	0.5679141
	UX_{18}	UY_{18}	UZ_{18}
ANSYS	0.0029442	-0.00102701	-0.52330e-07
fiberFEM	0.0029156	-0.00102129	-1.31143e-07
relative error	0.0097208	0.00555988	1.506076

Table 5.1: Displacements at the node 2 and at the node 18.

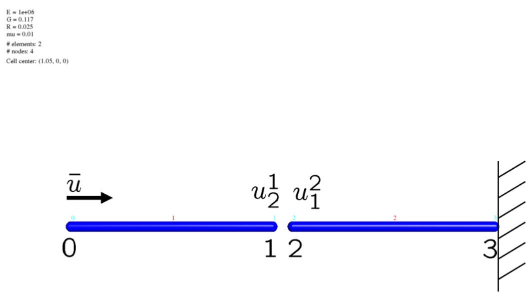


Figure 5.8: The first testing example.

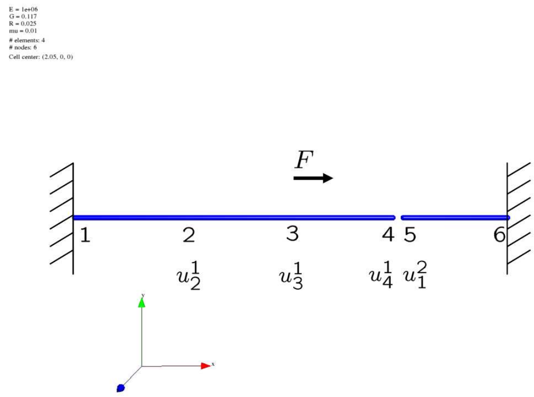


Figure 5.9: The second testing example.

Second testing example

The second testing example is taken from [Wriggers, 2002], p.113. It consists of a truss with 2 beams. The left beam is discretized by 3 elements, the left end of the beam is fixed, and a point load F is applied. The right beam is discretized by 1 element with the fixed right end. The initial gap between beams is g, and it closes at a certain point under the load F. The axial stiffness of the both, right and left beams, is the same.

The matrix form of the finite element formulation after imposing the boundary conditions

$1/\delta$	FiberFEM	relative error
1000	-3.3333299999999992	0.33333400000000013
10000	-4.7618999999999989	0.04762000000000021
100000	-4.9751243781094514	0.00497600000000009
1000000	-4.9974999999999987	0.00050000000000025

Table 5.2: The relative error $\frac{\|\bar{F}_N - F_N^f\|}{\|\bar{F}_N\|}$ for the exact contact force $\bar{F}_N = -5\,\mathrm{N}$. F_N^f is the contact force computed by FiberFEM.

$1/\delta$	FiberFEM	relative error
100	7.1429499999999999	1.8571800000000071
1000	2.0000200000000001	0.19999199999999981
10000	2.4390499999999999	0.024379999999999985
100000	2.4937900000000002	0.0024839999999999975
1000000	2.4994000000000001	0.00023999999999999979

Table 5.3: The relative error for different δ. The exact contact force $\bar{F}_N = 2.4999999999999996$ N.

and introducing the penalization parameter becomes

$$\begin{pmatrix} 2\frac{EA}{l} & -\frac{EA}{l} & 0 & 0 \\ -\frac{EA}{l} & 2\frac{EA}{l} & 0 & 0 \\ 0 & -\frac{EA}{l} & \frac{EA}{l} + 1/\delta & -1/\delta \\ 0 & 0 & -1/\delta & \frac{EA}{l} + 1/\delta \end{pmatrix} \begin{Bmatrix} u_2^1 \\ u_3^1 \\ u_4^1 \\ u_1^2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ F \\ g/\delta \\ -g/\delta \end{Bmatrix}.$$

The displacement u_1^2 is

$$u_1^2 = \frac{1}{4}\left(\frac{2Fl}{EA} - g - \delta\frac{EA}{l}\right).$$

The normal contact force at this node is then

$$F_N = \frac{EA}{l}u_1^2.$$

Let δ go to zero, and the exact contact force $N = 2.4999999999999996$ N. The calculated normal force for different values of the penalty parameter with the corresponding relative error is given in Tab. 5.3.

5.3 Numerical examples

The numerical example of Subsection 5.3.1 is computed using ANSYS. The numerical examples in Subsections 5.3.3, 5.3.4 and 5.3.2 are computed using the FiberFEM code. The cross-section is taken to be circular, with the radius equal to R. The material of the fibers is linear elastic with the Young's modulus E and the shear modulus G. The effective elasticity moduli $\bar{a}_{11}^{11}$, $\bar{a}_{22}^{22}$, $\bar{a}_{12}^{12}$, $\bar{a}_{11}^{22}$ are obtained from the auxiliary problems.

XY shear	$\bar{a}^{12}_{21} = 0.955574062947$ Pa	$\bar{a}^{22}_{22} = 0.634416066157$ Pa	$\bar{a}^{12}_{22} = 13.36879032292$ Pa
YX shear	$\bar{a}^{11}_{21} = 3.687156617357$ Pa	$\bar{a}^{21}_{22} = 1.352853122289$ Pa	$\bar{a}^{11}_{22} = 10.83857299372$ Pa
X extention	$\bar{a}^{11}_{11} = 1.290460597368$ Pa	$\bar{a}^{21}_{12} = 0.892342418811$ Pa	$\bar{a}^{11}_{12} = 0.004520425091$ Pa
Y extention	$\bar{a}^{12}_{11} = 14.34474768496$ Pa	$\bar{a}^{22}_{12} = 671.1632115605$ Pa	$\bar{a}^{12}_{12} = 0.692406897923$ Pa
bending	$\bar{c}_{1111} = 211.398488432$ Pa	$\bar{c}_{1122} = 83.960735206$ Pa	$\bar{c}_{2222} = 215.642053965$ Pa

Table 5.4: Effective elasticity and bending coefficients, transversal compliance moduli, where $h = 0.1$, $E = 0.01\,\mathrm{GPa}$, $\nu = 0.27$, the dimensions of the unit cell: $1 \times 0.85 \times 0.97$.

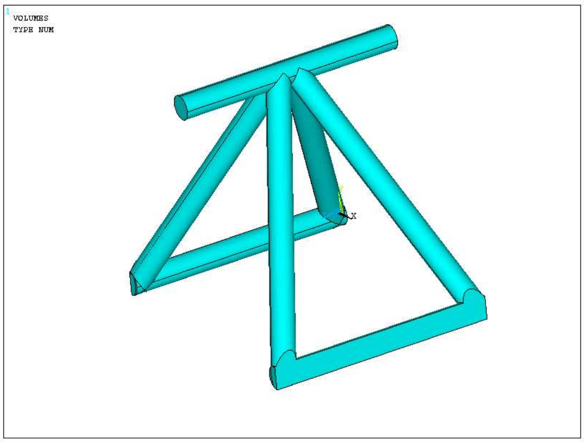


Figure 5.10: A rod structure in the unit cell.

5.3.1 Geotextile example using ANSYS

This numerical example can be found in [Orlik, Nam, ICIAM2007]. A geotextile with fixed junctions is considered as depicted in Fig. 5.10. The material constants are

$$E = 0.01 \text{ GPa}, \quad \nu = 0.27,$$

where ν is the Poisson ratio.

The computed effective elasticity and bending coefficients are listed in Tab. 5.4.

5.3.2 Stick/slip switching example using FiberFEM

To illustrate the effect of homogenization of the microcontact sliding we consider two fibers in frictional contact as depicted in Fig. 5.12. The material of fibers is linear with the constants:

$$E = 10^7 \text{ Pa}, \quad G = 0.117.$$

The lower fiber is fixed at the ends, the upper fiber is subjected to the prescribed displacement along the lower fiber. The coefficient of friction is artificially chosen in order to get stick and slip phases. The prescribed displacement is applied incrementally with the number of iterations equal to 100.

When the coefficient of friction $\mu = 300000$, the fibers stay in the stick phase. When the coefficient of friction is 76214, the stick/slip switching happens at the iteration 46. The evolution of the averaged stress is given in Fig. 5.11, where the blue line represents a full stick phase and the red line shows the switching.

Let us have a look at the red line in Fig. 5.11. From iteration 1 to iteration 46 fibers are in the stick phase, and the upper fiber bends as it is shown in Fig. 5.13. At the iteration 46 the switching from the stick to the slip phase happens, and the upper fiber bends only under the friction force (see Fig. 5.14), which is constant and depends only on the material of the fibers. That is why we can observe a constant value of the averaged stress.

Remark 5.2. *The nonlinear part of the stress-strain curve in Fig. 5.11 might be explained by the influence of the penalty parameter on the solution.*

The author would like to thank students Albina Davletkulova and Iliya Prozorov for helping to run FiberFEM in order to compute this numerical example.

5.3.3 Geotextile example using FiberFEM

A geotextile depicted in Fig. 5.15 is considered. The unit cell representing the geometry of the fibers with fixed junctions is shown in Fig. 5.16. The material constants are

$$E = 10^8 \text{ Pa}, \quad G = 0.3.$$

The finite element mesh of the fiber geometry consists of 72 elements, 71 nodes with the total number of degrees of freedom equal to 426.

The deformed shapes of the microstructure, representing the solutions of the auxiliary problems, are shown in Fig. 5.18-5.19. The obtained effective elasticity moduli are

$$\bar{a}_{11}^{11} = 0.247221 * 10^4 \text{ Pa}, \quad \bar{a}_{22}^{22} = 1.110013 * 10^8 \text{ Pa},$$

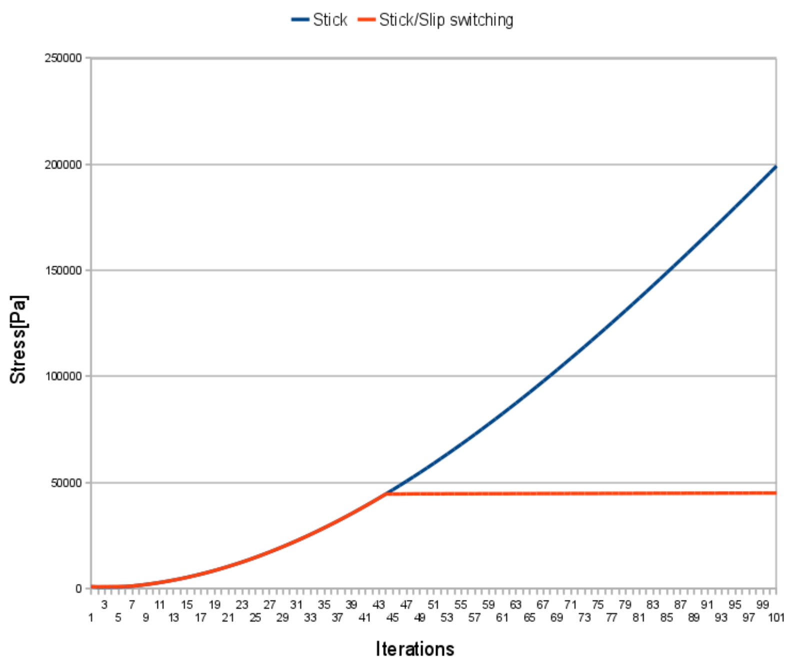


Figure 5.11: Stick/slip switching.

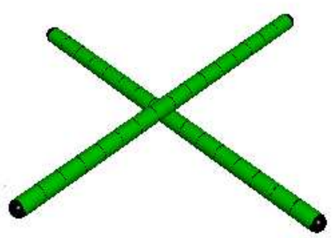


Figure 5.12: Two fibers, the undeformed shape.

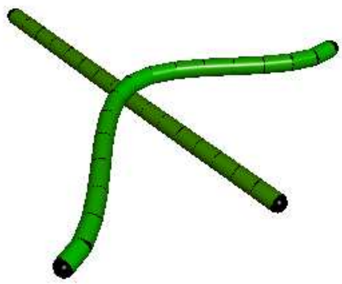


Figure 5.13: Two fibers, the stick phase.

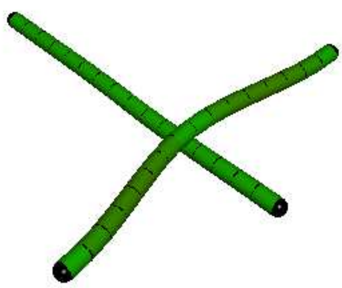


Figure 5.14: Two fibers, the slip phase.

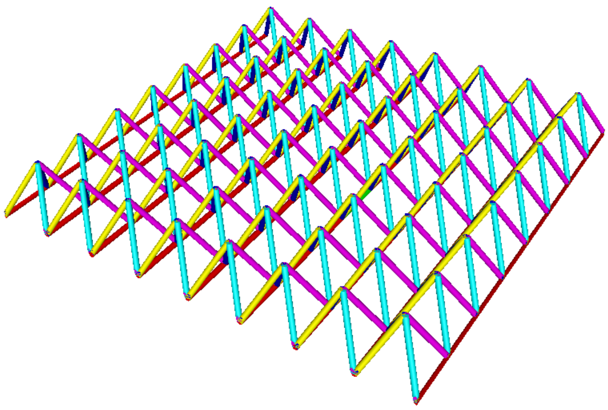


Figure 5.15: The first geotextile example.

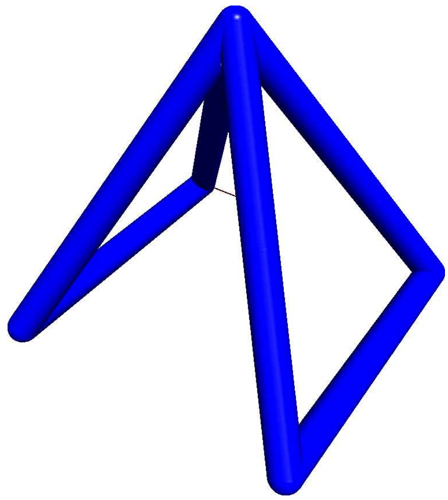


Figure 5.16: The geometry of the unit cell.

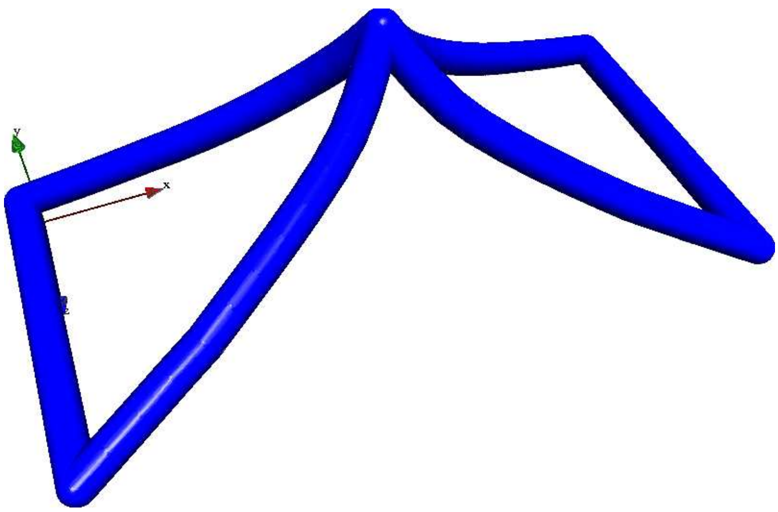


Figure 5.17: The ξ_1 extension experiment.

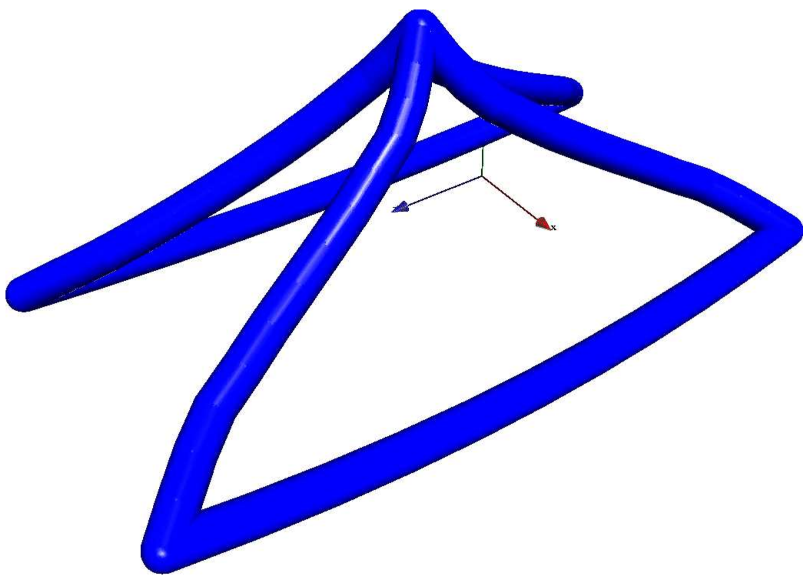


Figure 5.18: The ξ_2 extension experiment.

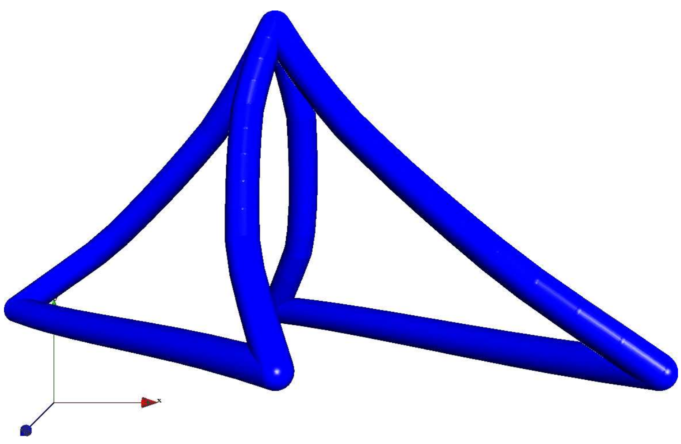


Figure 5.19: The shear experiment.

$$\bar{a}_{12}^{12} = 0.018418 * 10^4 \text{ Pa}, \quad \bar{a}_{22}^{11} = 0.498762 * 10^5 \text{ Pa},$$

and the effective elastic properties are

$$\bar{\nu}_{12} = 0.0745, \quad \bar{\nu}_{21} = 1.6592 * 10^{-6},$$

$$\bar{E}_1 = 2472 \text{ Pa}, \quad \bar{E}_2 = 1.1100 * 10^8 \text{ Pa}, \quad \bar{G} = 0.498762 * 10^5.$$

5.3.4 Geotextile versus woven

Two numerical examples are presented. The first geometry represents a geotextile with rigid fixed junctions, the second geometry is an example of a woven. Both geometries are chosen to have similar structure, but in the case of the woven geometry a contact between fibers is considered at the location of the fixed junctions of the geotextile.

Second geotextile

A geotextile depicted in Fig. 5.20 is considered. The unit cell representing the geometry of the fibers with fixed junctions is shown in Fig. 5.21.

The material of fibers is isotropic with the material constants

$$E = 10^6 \text{ Pa}, \quad G = 0.3.$$

The finite element mesh of the fiber geometry consists of 56 elements, 55 nodes with the total number of degrees of freedom equal to 330.

The deformed shapes of the microstructure, representing the solutions of the auxiliary problems are shown in Fig. 5.24, 5.26, 5.28. The obtained effective elasticity moduli are

$$\bar{a}_{11}^{11} = 0.415201 * 10^6 \text{ Pa}, \quad \bar{a}_{22}^{22} = 1.420008 * 10^6 \text{ Pa},$$

$$\bar{a}_{12}^{12} = 0.844399 * 10^4 \text{ Pa}, \quad \bar{a}_{22}^{11} = 1.359719 * 10^4 \text{ Pa},$$

and the effective elastic properties are

$$\bar{\nu}_{12} = 0.0203, \quad \bar{\nu}_{21} = 0.0059,$$

$$\bar{E}_1 = 0.4151 * 10^6 \text{ Pa}, \quad \bar{E}_2 = 1.4198 * 10^6 \text{ Pa}, \quad \bar{G} = 1.359719 * 10^4.$$

$1/\delta$	$\bar{a}_{11}^{11}$ [Pa]	$\bar{a}_{22}^{22}$ [Pa]	$\bar{a}_{12}^{12}$ [Pa]	$\bar{a}_{22}^{11}$ [Pa]
100	$0.956456 * 10^5$	$0.3883210 * 10^6$	$0.012082 * 10^5$	$0.9952 * 10^3$
1000	$0.142843 * 10^6$	$1.2157951 * 10^6$	$0.003058 * 10^6$	$1.5860 * 10^3$
10000	$0.294002 * 10^6$	$1.3040001 * 10^6$	$0.003529 * 10^6$	$1.9538 * 10^3$
100000	$0.311029 * 10^6$	$1.3211258 * 10^6$	$0.004211 * 10^6$	$2.1035 * 10^3$
1000000	$0.311992 * 10^6$	$1.3256281 * 10^6$	$0.004251 * 10^6$	$2.1042 * 10^3$

Table 5.5: Effective elasticity coefficients.

Woven

A woven textile is considered as depicted in Fig.5.22. The unit cell geometry is shown in Fig. 5.23. There are 4 contact points between yarns.

The material of fibers is isotropic with the material constants

$$E = 10^6 \text{ Pa}, \quad G = 0.3, \quad \mu = 0.3.$$

For a contact, the initial penetration is zero at all contact points. The friction law is modeled as a Coulomb one. All contact points are initially set to be in the stick state. The contact contribution and the sliding gap are implemented in the form derived in Chapter 4. The finite element mesh of the fiber geometry consists of 56 elements and 59 nodes, with the total number of degrees of freedom equal to 354. The deformed shapes of microstructure, representing the solutions of the auxiliary problems, are shown in Fig. 5.25, 5.27, 5.29. The obtained effective material constants are presented in Tab. 5.5.

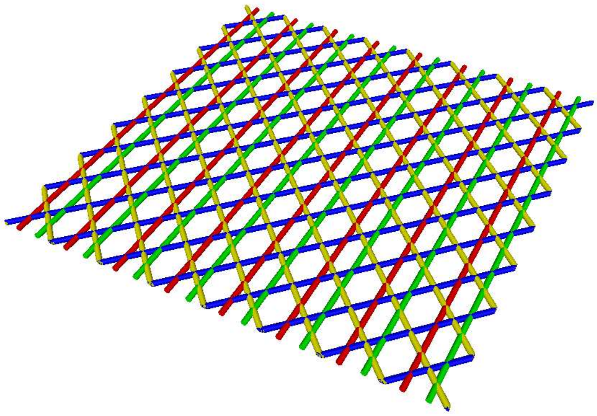


Figure 5.20: The second geotextile example.

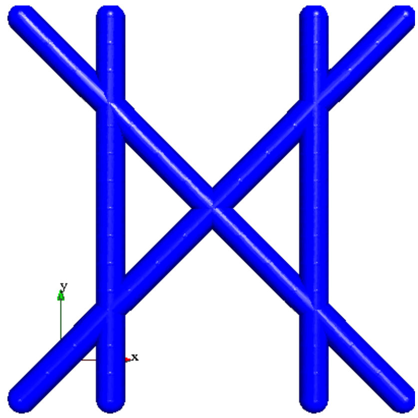


Figure 5.21: The geometry of the unit cell.

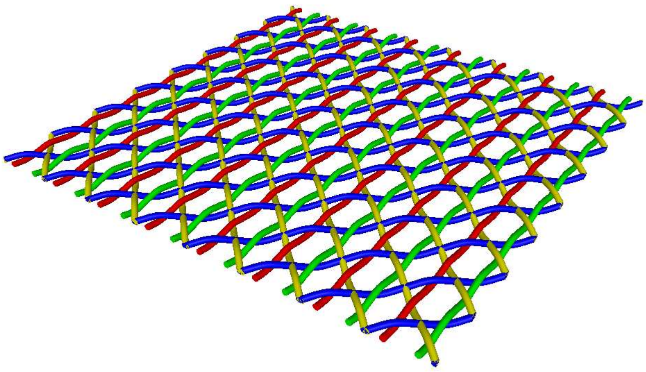


Figure 5.22: The woven.

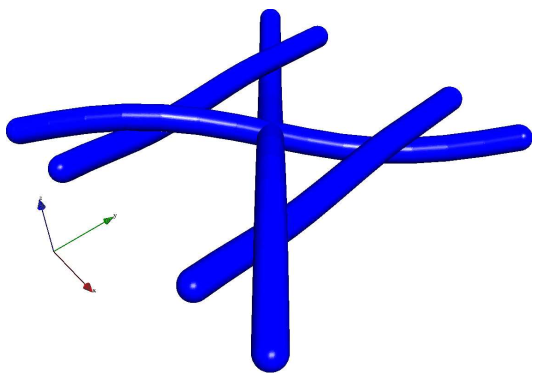


Figure 5.23: The geometry of the unit cell.

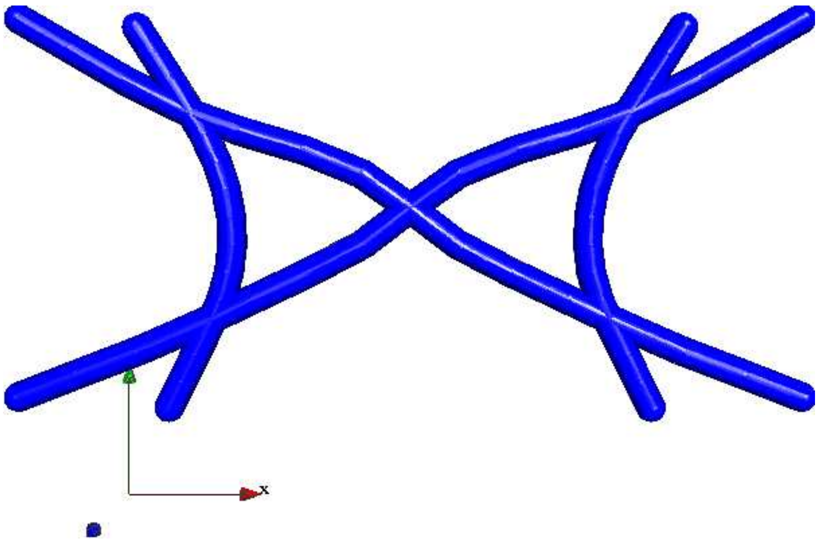


Figure 5.24: A ξ_1 extension, the second geotextile.

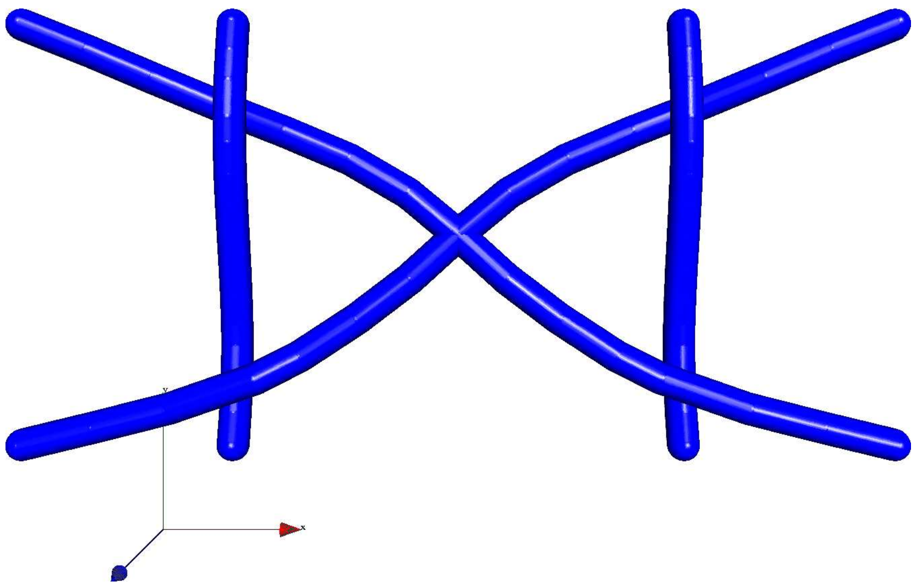


Figure 5.25: A ξ_1 extension, the woven.

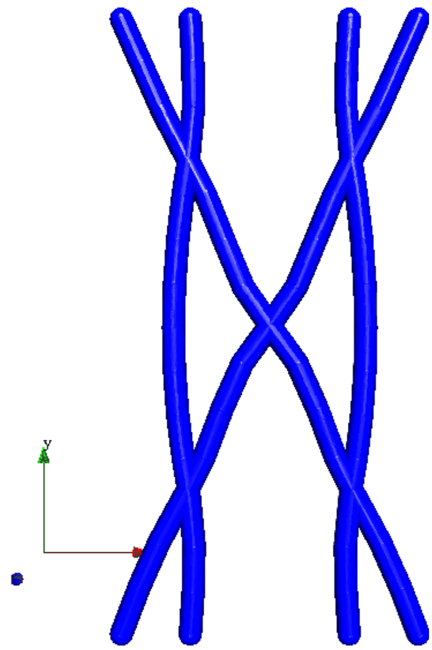


Figure 5.26: A ξ_2 extension, the second geotextile.

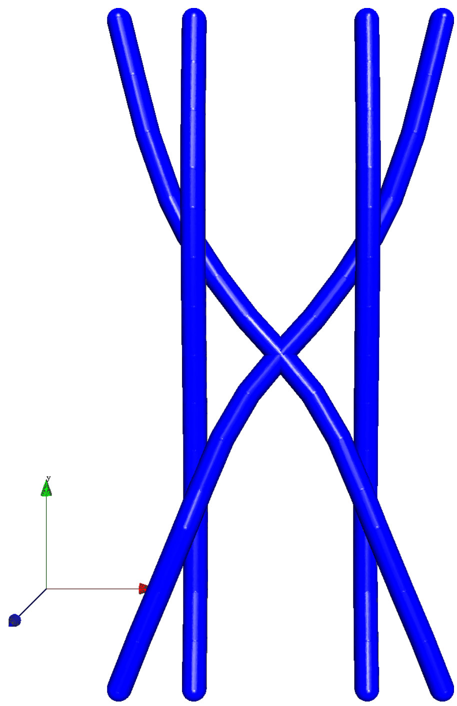


Figure 5.27: A ξ_2 extension, the woven.

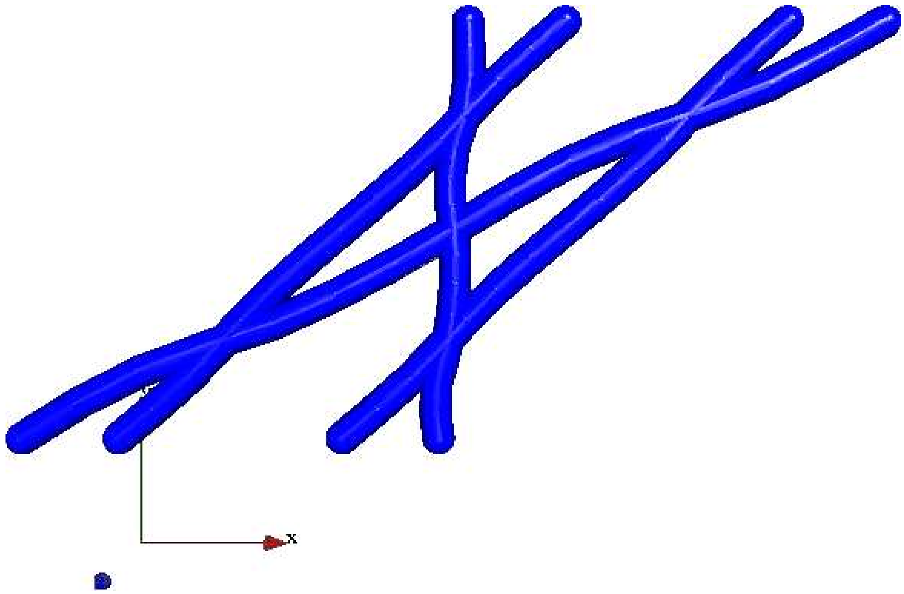


Figure 5.28: A shear, the second geotextile.

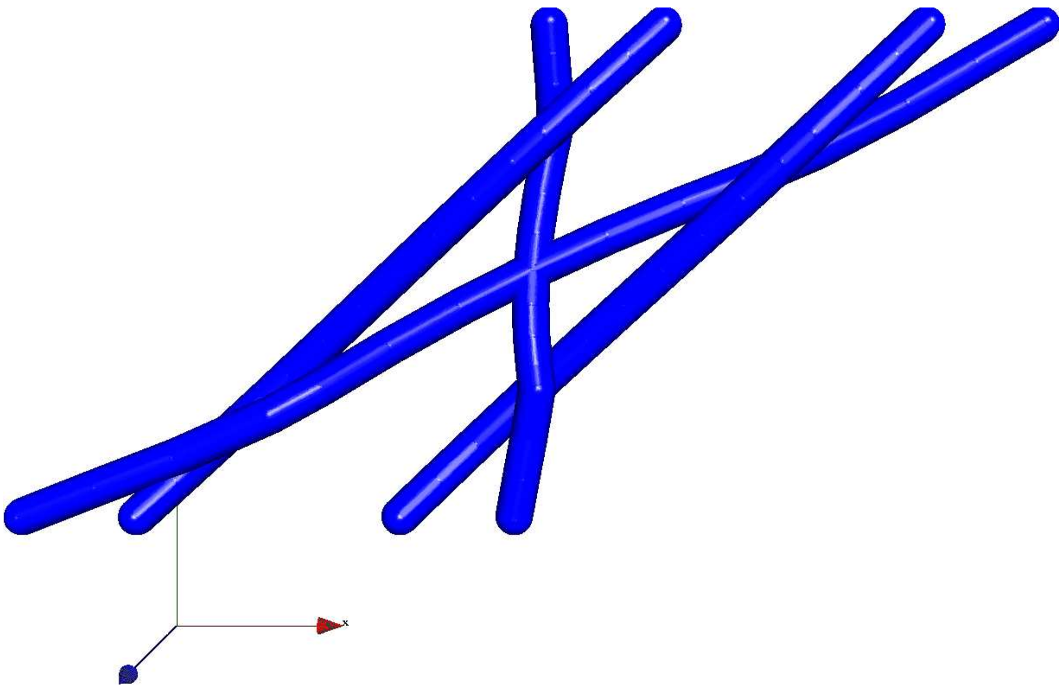


Figure 5.29: A shear, the woven.

Chapter 6

Conclusions

- A fiber composite material with periodic microstructure and multiple frictional micro-contacts between fibers is studied. The textile is modeled by introducing small geometrical parameters: the periodicity of the microstructure and the characteristic diameter of fibers. The contact linear elasticity problem is considered. A two-scale approach is used for obtaining the effective mechanical properties.

- The results of [Orlik, in preparation] on two-scale convergence of the microcontact elasticity problem are used in order to obtain auxiliary contact problems and the homogenized problem with the effective nonlinear elasto-plastic material law.

- The algorithm using asymptotic two-scale homogenization for computation of the effective mechanical properties of textiles with periodic rod or fiber microstructure is proposed. The algorithm is based on the consequent passing to the asymptotics with respect to the in-plane period and the characteristic diameter of fibers. This allows to come to the equivalent homogeneous problem and to reduce the dimension of the auxiliary problems. Further numerical simulations of the cell problems give the effective material properties of the textile.

- The homogenization of the boundary conditions on the vanishing out-of-plane interface of the textile or fiber structured layer has been studied. Introducing the additional auxiliary functions into the formal asymptotic expansion for a heterogeneous plate, the corresponding auxiliary and homogenized problems for a nonhomogeneous Neumann boundary condition were deduced. It is incorporated into the right hand side of the homogenized problem via effective out-of-plane moduli.

- By using the finite element formulation for the contact problem between fibers given in Chapter 4, the contact contribution of the non-penetration functional and frictional functional into the Ritz-Galerikin system has been explicitly obtained. The corresponding normal and tangential contact matrices and vectors are derived for Tresca and Coulomb friction laws.

- FiberFEM, a C++ finite element code for solving contact elasticity problems, is developed. The code is based on the implementation of the algorithm for the contact between fibers, for which the contact contribution into the Ritz-Galerkin system is deduced in Chapter 4. A code for visualization of fiber meshes, fViz, has been developed and used for generation of geotextile and woven unit cell meshes, listed in Appendix A.2.

- Numerical examples of homogenization of geotexiles and wovens are obtained in the work by implementation of the developed algorithm. The effective material moduli are computed numerically using the finite element solutions of the auxiliary contact problems obtained by FiberFEM.

Appendix A

A.1 Beam stiffness matrices

Definition A.1. *k_1^e given by*

$$k_1^e = \int_0^{l_e} B_1^T EAB_1 ds = \frac{EA}{l_e}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

is called the bar element stiffness matrix. Here B_1 is a strain-displacement matrix,

$$B_1 = \left[\frac{dN_1(x)}{dx}, \frac{dN_2(x)}{dx}\right]^T.$$

Definition A.2. *f_1^e defined by*

$$f_1^e = \int_0^{l_e} \bar{N}\psi_1^e ds$$

is the bar nodal force vector.

Definition A.3. *k_2^e given by*

$$k_2^e = \int_0^1 B_2^T EI_2 B_2 dx = \frac{EI_2}{l_e^3}\begin{pmatrix} 12 & 6l_e & -12 & 6l_e \\ 6l_e & 4l_e^2 & -6l_e & 2l_e^2 \\ -12 & -6l_e & 12 & -6l_e \\ 6l_e & 2l_e^2 & -6l_e & 4l_e^2 \end{pmatrix}$$

is called the beam element stiffness matrix. Here B_2 is the strain-displacement matrix,

$$B_2 = \left[\frac{d^2H_1(s)}{ds^2}, \frac{d^2H_2(s)}{ds^2}, \frac{d^2H_3(s)}{ds^2}, \frac{d^2H_4(s)}{ds^2}\right].$$

Definition A.4. *f_2^e defined by*

$$f_2^e = \int_0^{l_e} \bar{H}\psi_2^e ds$$

is the beam nodal force vector.

Definition A.5. *k_4^e given by*

$$k_4^e = \int_0^{l_e} B_1^T G B_1 ds = \frac{G}{l_e}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

is called the torsion element stiffness matrix.

The local stiffness matrix of the frame element is

$$\mathbf{k}^e = \begin{pmatrix} \mathbf{k}_{11}^e & \mathbf{k}_{12}^e \\ (\mathbf{k}_{12}^e)^T & \mathbf{k}_{22}^e \end{pmatrix},$$

where

$$\mathbf{k}_{11}^e = \begin{pmatrix} EA/l_e & 0 & 0 & 0 & 0 & 0 \\ 0 & 12EI_3/l_e^3 & 0 & 0 & 0 & 6EI_3/l_e^2 \\ 0 & 0 & 12EI_2/l_e^3 & 0 & -6EI_2/l_e^2 & \\ 0 & 0 & 0 & G/l_e & 0 & 0 \\ 0 & 0 & -6EI_2/l_e^2 & 0 & 4EI_2/l_e & 0 \\ 0 & 6EI_3/l_e^2 & 0 & 0 & 0 & 4EI_3/l_e \end{pmatrix},$$

$$\mathbf{k}_{22}^e = \begin{pmatrix} EA/l_e & 0 & 0 & 0 & 0 & 0 \\ 0 & 12EI_3/l_e^3 & 0 & 0 & 0 & -6EI_3/l_e^2 \\ 0 & 0 & 12EI_2/l_e^3 & 0 & 6EI_2/l_e^2 & \\ 0 & 0 & 0 & G/l_e & 0 & 0 \\ 0 & 0 & 6EI_2/l_e^2 & 0 & 4EI_2/l_e & 0 \\ 0 & -6EI_3/l_e^2 & 0 & 0 & 0 & 4EI_3/l_e \end{pmatrix},$$

$$\mathbf{k}_{12}^e = \begin{pmatrix} -EA/l_e & 0 & 0 & 0 & 0 & 0 \\ 0 & -12EI_2/l_e^3 & 0 & 0 & 0 & 6EI_2/l_e^2 \\ 0 & 0 & -12EI_3/l_e^3 & 0 & -6EI_3/l_e^2 & \\ 0 & 0 & 0 & -G/l_e & 0 & 0 \\ 0 & 0 & 6EI_2/l_e^2 & 0 & 2EI_2/l_e & 0 \\ 0 & -6EI_3/l_e^2 & 0 & 0 & 0 & 2EI_3/l_e \end{pmatrix}.$$

A.2 fViz

fViz is a C++ code for visualization of fiber geometries. The visualization is implemented by using OpenGL, and the GUI interface is based on Qt4, see Fig. A.1. Examples of fiber structures are shown in Fig. A.2, A.3.

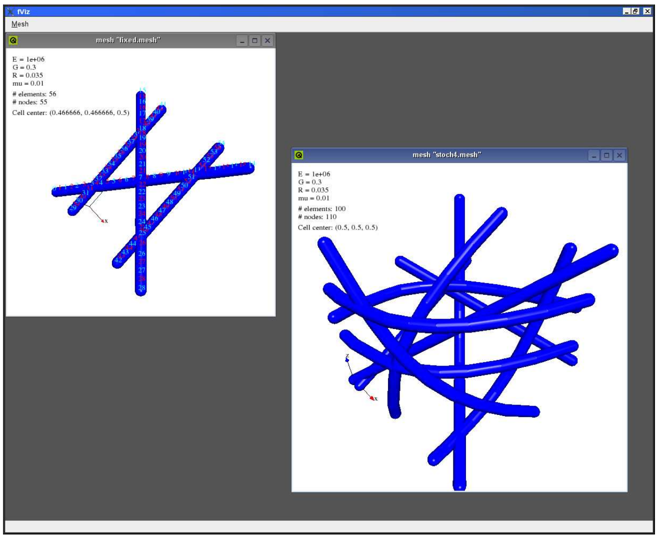


Figure A.1: A screen shot of fViz.

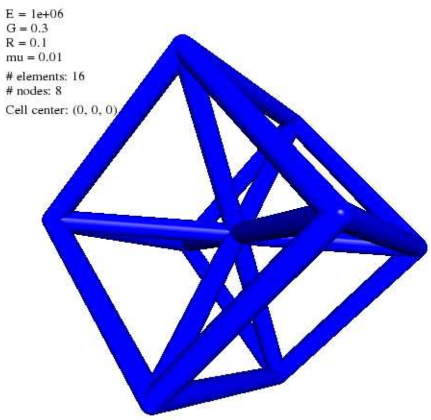


Figure A.2: A fiber mesh.

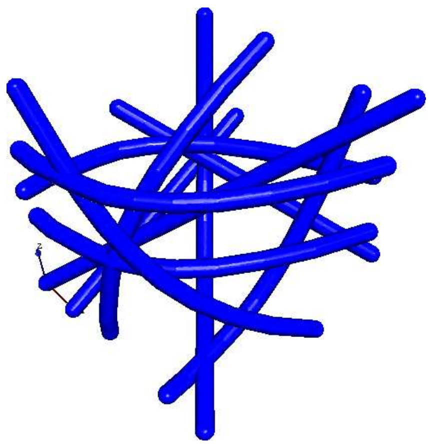


Figure A.3: A stochastic fiber mesh.

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