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Hamiltonian Path Integrals in White Noise Analysis

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For my family and for my friends.

"*It doesn't matter how beautiful your theory is, it doesn't matter how smart you are. If it doesn't agree with experiment, it's wrong*" -Richard P. Feynman-

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Introduction

As an important special case of Gaussian Analysis, the framework of White Noise Analysis was developed to a tool used in various fields in Mathematical Physics as Statistical Mechanics, Quantum Field Theory, Quantum Mechanics and Polymer Physics as well as in Applied Mathematics and Stochastic Analysis, Dirichlet Forms, Stochastic (Partial) Differential Equations and Financial Mathematics. With the help of Gaussian Analysis problems from these fields can be represented and solved in a mathematical rigorous way. The mathematical framework offers many generalizations of methods and concepts known from finite dimensional analysis as differential operators and Fourier transform. Detailed information concerning White Noise Analysis and Gaussian Analysis can be found in the monographs [40, 6, 57, 64] and the articles [50, 81, 78].
This thesis is separated into three main parts:

- Development of Gaussian and White Noise Analysis.
- Hamiltonian Path Integrals as White Noise Distributions.
- Numerical methods for polymers driven by fractional Brownian motion.

Gaussian Analysis and generalized functions

Based on its rapid development in structure and applications, Gaussian Analysis and White Noise Analysis became more and more interesting in the last 30 years. The progress in the development of the mathematical concepts is based on various characterization theorems. These theorems and their succeeding variants and corollaries were the starting point for a deep analysis of the structure of smooth and generalized random variables in the white noise spaces, or more generally Gaussian spaces. For detailed information we refer to [40, 6, 64] and [57]. The main idea in the development of the theory is the use of test and distribution spaces. There are various triples of test and generalized functions which can be

used dependent on the application. We give a brief introduction into the characterization and construction of these spaces in Chapter 2. Preliminaries are given in Chapter 1. Throughout this thesis we will mainly use the following spaces:
The Hida spaces: We will sketch the construction of the Gel'fand triple

$$(\mathcal{N}) \subset L^2(\mu) \subset (\mathcal{N})'$$

and give the characterization and topological properties of these spaces in Section 2.2.2. and 2.2.3, respectively.
The spaces $\mathcal{G}$ and $\mathcal{M}$: In Section 2.2.4. the spaces of regular test and generalized functions are discussed. As introduced in [87] we also introduce the space $\mathcal{M}$ and its dual space $\mathcal{M}'$. The spaces $\mathcal{M}'$ and $\mathcal{G}'$ are of interest, since their kernels in the chaos decomposition are from the underlying Hilbert space. We have the chain

$$\mathcal{G} \subset \mathcal{M} \subset L^2(\mu) \subset \mathcal{M}' \subset \mathcal{G}'.$$

The triple $\mathcal{G} \subset L^2(\mu) \subset \mathcal{G}'$ was first introduced by [68] and characterized via the Bargmann-Segal space in [30].
Throughout this thesis the Donsker's delta function plays a key role. We investigate this generalized function also in Chapter 2. Moreover we show by giving a counterexample, that the general definition for complex kernels is not true as stated in [87]. Furthermore we fix this mistake by strengthen the requirements on the kernel of the monomial.

In Chapter 3 we take a closer look to generalized Gauss kernels. The results from this chapter are based on [41] and [34]. Here we generalized these results to the case of vector-valued White Noise. These results are the basis for Hamiltonian path integrals of quadratic type. The core result of this chapter is Lemma 3.2.6, which gives conditions under which pointwise products of generalized Gauss kernels and certain Hida distributions have a mathematical rigorous meaning as distributions in the Hida space.

In Chapter 4 we discuss operators which are related to applications for Feynman Integrals as differential operators, scaling, translation and projection. We show the relation of these operators to differential operators as in [86], which leads to the notion of so called convolution operators, introduced by [65]. We generalize the central homomorphy theorem in [65] to regular generalized functions from $\mathcal{G}'$. This leads to crucial simplifications in the following calculations.

We generalize the concept of complex scaling to scaling with bounded operators. Furthermore for this generalized scaling we discuss the relation to generalized Radon-Nikodym derivatives. This is done to sum up a toolbox to investigate products of generalized functions in chapter 5. At the end of the chapter, as a special case, we show that the projection operator from [86, 36] and [87] is not closable on $L^2(\mu)$. This leads to an ambiguity for approximations with the Wick formula beyond this space.

In Chapter 5 we discuss products of generalized functions. Moreover the Wick formula from [86, 36] is revisited. We investigate under which conditions and on which spaces the Wick formula can be generalized to. At the end of the chapter we consider the products of Donsker's delta function with a generalized function with help of a measure transformation. Here also problems as measurability are concerned.

Hamiltonian Path Integrals

As an alternative approach to quantum mechanics Feynman introduced the concept of path integrals ([23, 24, 25]), which was developed into an extremely useful tool in many branches of theoretical physics. In this thesis we develop the concepts for realizing Feynman integrals in phase space in the framework of White Noise Analysis. This also called Hamiltonian path integral for a particle moving from y_0 at time 0 to y at time t under the potential V in coordinate space representation is given by

$$\mathrm{N}\int_{x(0)=y_0,x(t)=y}\int \exp\left(\frac{i}{\hbar}\int_0^t p\dot{x}-\frac{p^2}{2}-V(x,p)\,d\tau\right)\prod_{0<\tau<t}dp(\tau)dx(\tau),\quad \hbar=\frac{h}{2\pi}. \qquad (1)$$

Here h is Planck's constant, and the integral is thought of being over all position paths with $x(0) = y_0$ and $x(t) = y$ and all momentum paths. The missing restriction on the momentum variable at time 0 and time t is an immediate consequence of the Heisenberg uncertainty relation, i.e. the fact that one can not measure momentum and space variable at the same time.
The measure is thought to be a flat measure on the infinite dimensional space of paths in phase space. Since such a measure does not exist the Hamiltonian path intgral as it stands is an object which is not mathematical justified. On the other hands its relevance in physics is outstanding.
The path integral to the phase space has several advantages. Firstly the semi-classical approximation can be validated easier in a phase space formulation and secondly that

quantum mechanics are founded on the phase space, i.e. every quantum mechanical observable can be expressed as a function of the space and momentum. A discussion about phase space path integrals can be found in the monograph [2] and in the references therein. In the last fifty years there have been many approaches for giving a mathematically rigorous meaning to the phase space path integral by using e.g. analytic continuation, see [45, 46] or Fresnel integrals [2, 1]. The idea of realizing Feynman integrals within the white noise framework goes back to [41]. There the authors used exponentials of quadratic (generalized) functions in order to give meaning to the Feynman integral in configuration space representation

$$\mathrm{N}\int_{x(0)=y_0,x(t)=y} \exp\left(\frac{i}{\hbar}S(x)\right) \prod_{0<\tau<t} dx(\tau), \quad \hbar = \frac{h}{2\pi},$$

with the classical action $S(x) = \int_0^t \frac{1}{2}m\dot{x}^2 - V(x)\, d\tau$. We use these concepts of quadratic actions in White Noise Analysis, which were further developed in [34] to give a rigorous meaning to the Feynman integrand in phase space.

Hamiltonian Path Integral in coordinate space First we introduce the space trajectories involving a Brownian motion $B(\tau)$ starting in 0.

$$x(\tau) = x_0 + \sqrt{\frac{\hbar}{m}}B(\tau), \quad 0 \leq \tau \leq t.$$

Furthermore the momentum variable is modeled with help of white noise, i.e.

$$p(\tau) = \sqrt{\hbar m}\omega_p(\tau), \quad 0 \leq \tau \leq t.$$

This is a meaningful definition, since a path has always start and end points which a noise does not have. Moreover since we have that if the initial and end conditions are fully known, the momentum is completely uncertain, which means has variance infinity. The white noise process is intrinsically fulfilling the no boundary condition property and has as well infinite variance. Furthermore one can think of for a potential just depending on the space variable the momentum to be $p = m\dot{x}$, which in our approach would correspond to a noise in terms of derivative of the Brownian path.
The model for the space path can be found in [40] to model the momentum path we take a closer look to the physical dimensions of $x(\tau)$.
$x(\tau)$ has as a space variable the dimension of a length, i.e. also $\sqrt{\frac{\hbar}{m}}B(\tau)$ has to have the

dimension of a length. We have

$$\left[\sqrt{\frac{\hbar}{m}}\right] = \sqrt{\frac{Js}{kg}} = \sqrt{\frac{kgm^2}{skg}} = \frac{m}{\sqrt{s}}.$$

Thus since the norm of the Brownian motion gives again a $\sqrt{t}$ which has the dimension $\sqrt{s}$ we have that $x(\tau)$ has the dimension of a length.
Considering the momentum variable we have to obtain that the dimension is the dimension of a momentum. We have

$$[\sqrt{\hbar m}] = \sqrt{Nmskg} = \sqrt{\frac{kg^2m^2s}{s^2}} = \frac{kgm}{\sqrt{s}},$$

hence ω_p has the dimension $\frac{1}{\sqrt{s}}$, such that $p(\tau)$ has the dimension of a momentum.
A definition which goes in the same direction using the momentum as a kind of derivative of the path can also be found in [2] and [1]. Here the authors modeled the path space as the space of absolutely continuous functions and the momentum to be in $L^2(\mathbb{R})$. Then we propose the following formal ansatz for the Feynman integrand in Phase space with respect to the Gaussian measure μ,

$$\begin{aligned} I_V = \mathrm{N}\exp\left(\frac{i}{\hbar}\int_{t_0}^{t} p(\tau)\dot{x}(\tau) - \frac{p(\tau)^2}{2m}d\tau + \frac{1}{2}\int_{t_0}^{t} \dot{x}(\tau)^2 + p(\tau)^2 d\tau\right) \\ \times \exp\left(-\frac{i}{\hbar}\int_{t_0}^{t} V(x(\tau),p(\tau),\tau)\,d\tau\right)\cdot\delta(x(t)-y) \end{aligned} \tag{2}$$

In this expression the sum of the first and the third integral is the action $S(x,p)$, and the Donsker's delta function serves to pin trajectories to y at time t. The second integral is introduced to simulate the Lebesgue measure by compensation of the fall-off of the Gaussian measure in the time interval (t_0,t). Furthermore, as in Feynman's formula we need a normalization which turns out to be infinity and is implemented by the use of a normalized exponential as in chapter 3.

Hamiltonian path integral in momentum space If we know the initial and the end momentums it is clear by Heisenbergs uncertainty principle that we have no certain information about the corresponding space variables. This means we model the momentum trajectories as a Brownian fluctuation starting in the initial momentum p_0.

$$p(\tau) = p_0 + \frac{\sqrt{\hbar m}}{t - t_0} B(\tau), \quad 0 \leq \tau \leq t. \tag{3}$$

Furthermore the space variable is modeled by white noise, i.e.

$$x(\tau) = \sqrt{\frac{\hbar}{m}} \cdot (t - t_0)\omega_x(\tau), \quad 0 \leq \tau \leq t. \tag{4}$$

The Hamiltonian path integral for the momentum space propagator is formally given by, see e.g. [48]

$$K(p', t', p_0, t_0) = \mathcal{N} \int_{p(t_0)=p_0, p(t)=p'} \exp(\frac{i}{\hbar} \int_{t_0}^{t} -q(s)\dot{p}(s) - H(p, q)\ ds)\, DpDq. \tag{5}$$

This path integral can be obtained by a Fourier transform of the coordinate space path integral in both variables, see e.g. [47]. Then we propose the following formal ansatz for the Feynman integrand in Phase space with respect to the Gaussian measure μ,

$$\begin{aligned} I_V = \mathrm{N} \exp\left(\frac{i}{\hbar} \int_{t_0}^{t} -x(\tau)\dot{p}(\tau) - \frac{p(\tau)^2}{2m} d\tau + \frac{1}{2} \int_{t_0}^{t} \omega_x(\tau)^2 + \omega_p(\tau)^2 d\tau\right) \\ \times \exp\left(-\frac{i}{\hbar} \int_{t_0}^{t} V(x(\tau), p(\tau), \tau)\, d\tau\right) \cdot \delta(p(t) - p') \end{aligned} \tag{6}$$

In Chapter 6 we characterize Hamiltonian path integrands for the free particle, the harmonic oscillator and the charged particle in a constant magnetic field as Hida distributions. Partially the results can also be found in [10]. This is done in terms of the T-transform and with the help of Lemma 3.2.6. For the free particle and the harmonic oscillator we also investigate the momentum space propagators. At the same time, the T-transform of the constructed Feynman integrands provides us with their generating functional. Finally using the generating functional, in Chapter 7, we can show that the generalized expectation (generating functional at zero) gives the Greens function to the corresponding Schrödinger equation.

Here for the charged particle we used an ansatz for the integrand using an upper triangular block operator matrix. Since the corresponding quadratic form coincides with the quadratic form from a symmetric approach, this ansatz is also justified physically. By the easy form, calculations can be done explicitly, such that not only the propagator, but also

the generating functional can be obtained. The charged particle plays a central role in this thesis, since for a velocity dependent potential it is crucial to prove the physical meaning of our ansatz. In this case the momentum is not equivalent to the velocity but differs from it by a non-trivial additional term.
Moreover, with help of the generating functional we can show that the canonical commutation relations for the free particle and the harmonic oscillator in phase space are fulfilled. This confirms on a mathematical rigorous level the heuristics developed in [25].
In Chapter 8 we give an outlook, how the scaling approach which is successfully applied in the Feynman integral setting in [35, 37, 36, 86] can be transferred to the phase space setting. Here we just list the ideas and go the way of [86, 36]. We give a mathematical rigorous meaning to an analogue construction to the scaled Feynman-Kac kernel. It is open if the expression solves the Schrödinger equation. At least for quadratic potentials we can get the right physics.

Off-lattice Simulation of the Discrete Edwards model for polymer chains driven by fractional Brownian motion

In the last part and the last chapter of this thesis, we focus on the numerical analysis of polymer chains driven by fractional Brownian motion.
The Edwards model for polymer chains [22] describes a nearest neighbour interaction including the effect, that two monomers, e.g. molecules cannot be at the same place in a polymer. This excluded-volume effect can be modeled by a density involving the so called interaction local time and is still in the focus of mathematical research, see e.g.[85, 33].
Here we focus on a numerical simulation of such polymers with an off-lattice discretization. Instead of complicated lattice algorithms, see e.g.[53] our discretization is based on the correlation matrix or the Hamiltonian. Using fBm one can achieve a long-range dependence of the interaction of the monomers inside a polymer chain. We use a Metropolis algorithm to create the paths of a polymer driven by fractional Brownian motion taking the excluded volume effect in account.
This is the first step to a deeper analysis of this model.
Recently Bornales, Oliveira, and Streit [14] have proposed a generalization of Flory's con-

jecture to the fractional case with general Hurst index H:

$$\upsilon_H = \begin{cases} 1 & \text{if} \quad d = 1 \text{ and } H > 1/2 \\ H & \text{if} \quad dH > 2 \\ \frac{2H+2}{d+2} & \text{if} \quad \text{otherwise.} \end{cases}$$

First numerical results basing on the joint research with L. Streit(Universidade Madeira), S. Eleuterio and M. J. Oliveira (both Lissabon), as well as J.Bornales et. al. from Iligan, Philippines are presented.

Part I

White Noise Analysis

Chapter 1

Preliminaries

In this chapter we list facts on nuclear triples which we need throughout this thesis. In fact the notion of a nuclear triple can be described more general than in this chapter. For this we refer to [26, 75, 66]. Here we consider such nuclear spaces which are generated by chains of Hilbert spaces and end up with so called countably Hilbert spaces or CH-spaces (see e.g. [57]). These spaces have the advantage that we can use the underlying Hilbert space structure. Furthermore the abstract kernel theorem is listed and the complexification of nuclear spaces and the Boson Fock space are defined.

1.1 Facts about nuclear triples

Initial point of a nuclear triple is a real separable Hilbert space $\mathcal{H}$ with inner product $(\cdot,\cdot)$ and corresponding norm $|\cdot|_{\mathcal{H}}$. For a given separable nuclear Fréchet space $\mathcal{N}$ (in the sense of Grothendieck, see e.g. [66, 26, 27]), which is densely topologically embedded in $\mathcal{H}$ we can construct the nuclear triple

$$\mathcal{N} \subset \mathcal{H} \subset \mathcal{N}'.$$

Here we identified the Hilbert space $\mathcal{H}$ with its dual space $\mathcal{H}'$ via Riesz isomorphism. $\mathcal{N}'$ denotes the topological dual of $\mathcal{N}$, i.e. the space of linear and continuous mappings from $\mathcal{N}$ to $\mathbb{R}$. As an extension of the scalar product on $\mathcal{H}$ one realizes the dual pairing $\langle\cdot,\cdot\rangle$ of $\mathcal{N}$ and $\mathcal{N}'$, i.e.

$$\langle \xi, f\rangle = (\xi, f), \quad f \in \mathcal{H},\, \xi \in \mathcal{N}.$$

Nuclear spaces can be characterized conveniently in terms of projective limits of Hilbert spaces, see e.g. [75]. Since we use this fact throughout this thesis we will not give an abstract definition of nuclear spaces.

Theorem 1.1.1. *The nuclear Fréchet space $\mathcal{N}$ can be represented as*

$$\mathcal{N} = \bigcap_{p\in\mathbb{N}} \mathcal{H}_p := \operatorname*{proj\,lim}_{p\in\mathbb{N}} \mathcal{H}_p,$$

where $\{\mathcal{H}_p,\, p \in \mathbb{N}\}$ is a family of Hilbert spaces such that for all $l,k \in \mathbb{N}$ there exists a $p \in \mathbb{N}$ such that the embeddings $\mathcal{H}_p \hookrightarrow \mathcal{H}_l$ and $\mathcal{H}_p \hookrightarrow \mathcal{H}_k$ are Hilbert-Schmidt embeddings. The topology in $\mathcal{N}$ is given by the projective limit topology, i.e. the coarsest topology on $\mathcal{N}$ such that the canonical embeddings $\mathcal{N} \hookrightarrow \mathcal{H}_p$ are continuous for all $p \in \mathbb{N}$.
Note that one can choose the family $\{\mathcal{H}_p \big| p \in \mathbb{N}\}$ such that for $p < q$ the spaces $\mathcal{H}_q$ is densely embedded in $\mathcal{H}_p$.

We denote the inner product norms on $\mathcal{H}_p$ by $|\cdot|_p$. We can suppose that the system of norms is ordered, i.e. $|\cdot|_p \le |\cdot|_q$, if $p < q$ and thus $|\xi|_{\mathcal{H}} \le |\xi|_p$, for all $p \in \mathbb{N}$.

Also the dual space of $\mathcal{N}$, the co-nuclear space $\mathcal{N}'$ can be written in a convenient way by using a chain of Hilbert spaces. Indeed by general duality theory one has

$$\mathcal{N}' = \bigcup_{p\in\mathbb{N}} \mathcal{H}_{-p} := \operatorname*{ind\,lim}_{p\in\mathbb{N}} \mathcal{H}_{-p},$$

where we use the dual family of spaces $\{\mathcal{H}_{-p} := \mathcal{H}_p',\, p \in \mathbb{N}\}$. The inductive limit topology

τ_{ind} corresponding to that family of Hilbert spaces is the finest topology on $\mathcal{N}'$ such that the embeddings $\mathcal{H}_{-p} \hookrightarrow \mathcal{N}'$ are continuous for all $p \in \mathbb{N}$. We denote the inner product norms on $\mathcal{H}_{-p}$ by $|\cdot|_{-p}$.

Next we list some facts about the inductive limit topology, see e.g. [75, 40]

Proposition 1.1.2. *In the above mentioned setting one has*

a) τ_{ind} *coincides with the Mackey topology* $\tau(\mathcal{N}', \mathcal{N})$*, i.e. the finest topology which is dual continuity preserving.*

b) τ_{ind} *coincides with the strong topology* $\beta(\mathcal{N}', \mathcal{N})$*, i.e. the topology with the most open sets on a dual pair.*

c) The dual pair $\langle \mathcal{N}, \mathcal{N}' \rangle$ *is reflexive if* $\mathcal{N}'$ *is equipped with the strong topology.*

d) A sequence converges in the in the strong topology $\beta(\mathcal{N}', \mathcal{N})$ *if and only if it converges in the weak topology* $\sigma(\mathcal{N}', \mathcal{N})$*.*

We introduce tensor powers of nuclear spaces via tensor powers of the Hilbert spaces $\mathcal{H}_p^{\otimes n}$, $n \in \mathbb{N}$. If there is no danger of confusion we denote the norms for the tensor powers of $\mathcal{H}_p$ and $\mathcal{H}_{-p}$ also by $|\cdot|_p$ and $|\cdot|_{-p}$, respectively. Then we define

$$\mathcal{N}^{\otimes n} := \operatorname*{proj\,lim}_{p \in \mathbb{N}} \mathcal{H}_p^{\otimes n}.$$

It is possible to show, that indeed $\mathcal{N}^{\otimes n}$ is a nuclear space, see e.g. [75], which is denoted by the n^{th} tensor power of $\mathcal{N}$. Again by general duality theory we obtain the dual space of $\mathcal{N}^{\otimes n}$:

$$\left(\mathcal{N}^{\otimes n}\right)' = \operatorname*{ind\,lim}_{p \in \mathbb{N}} \mathcal{H}_{-p}^{\otimes n}.$$

The well-known Schwartz kernel theorem states that every continuous bilinear form $b(\cdot, \cdot) : \mathcal{N} \times \mathcal{N} \to \mathbb{R}$ can be written in the form

$$b(\xi, \psi) = \langle \xi \otimes \psi, \Phi \rangle,$$

where $\Phi \in \mathcal{N}'^{\otimes 2}$.

A bilinear form $b(\cdot, \cdot) : \mathcal{H}_p \times \mathcal{H}_p \to \mathbb{R}$ can be identified with $\Phi \in (\mathcal{H}_p \otimes_\pi \mathcal{H}_p)'$ where $\otimes_\pi$ denotes the tensor product w.r.t the π-topology, see e.g. [26]. Since we consider chains of Hilbert spaces, it is important to find a $q \in \mathbb{Z}$, such that we can identify Φ as an element of $\mathcal{H}_q \otimes \mathcal{H}_q$. This is stated in the following "kernel theorem" for multlinear forms, see e.g. [6]

Theorem 1.1.3. *Let $\xi_1, \ldots, \xi_n \mapsto F_n(\xi_1, \ldots, \xi_n)$ be a $\mathcal{H}_p$-continuous n-linear form on $\mathcal{N}^n$, i.e.,*

$$|F_n(\xi_1, \ldots \xi_n)| \leq C \prod_{k=1}^{n} |\xi_k|_p \,,$$

for some $p \in \mathbb{N}$ and $C > 0$.
Then for all $q > p$, where the embedding $i_{q,p} : \mathcal{H}_q \hookrightarrow \mathcal{H}_p$ is Hilbert-Schmidt, there exists a unique $\Phi^{(n)} \in \mathcal{H}_{-q}^{\otimes n}$ such that

$$F_n(\xi_1, \ldots, \xi_n) = \langle \xi_1 \otimes \cdots \otimes \xi_n, \Phi^{(n)} \rangle, \quad \xi_1, \ldots \xi_n \in \mathcal{N}$$

with

$$\left|\Phi^{(n)}\right|_{-q} \leq \|i_{q,p}\|_{HS}^n,$$

where $\|i_{q,p}\|_{HS}$ denotes the Hilbert-Schmidt norm of the embedding $i_{q,p}$.

The next corollary can be found in e.g. [87, 6, 86].

Corollary 1.1.4. *Let $\xi_1, \ldots, \xi_n \mapsto F_n(\xi_1, \ldots, \xi_n)$ be a $\mathcal{H}_{-p}$-continuous n-linear form on $\mathcal{N}^n$, i.e.,*

$$|F_n(\xi_1, \ldots \xi_n)| \leq C \prod_{k=1}^{n} |\xi_k|_{-p} \,,$$

for some $p \in \mathbb{N}$ and $C > 0$.
Then for all $q < p$, where the embedding $i_{p,q} : \mathcal{H}_p \hookrightarrow \mathcal{H}_q$ is Hilbert-Schmidt, there exists a unique $\Phi^{(n)} \in \mathcal{H}_{q}^{\otimes n}$ such that

$$F_n(\xi_1, \ldots, \xi_n) = \langle \xi_1 \otimes \cdots \otimes \xi_n, \Phi^{(n)} \rangle, \quad \xi_1, \ldots \xi_n \in \mathcal{N}$$

with

$$\left|\Phi^{(n)}\right|_{q} \leq \|i_{p,q}\|_{HS}^n.$$

If in Theorem 1.1.3 and Corollary 1.1.4 we consider symmetric n-linear forms F_n on $\mathcal{N}^{\otimes n}$, i.e. for every permutation π of the set $\{1, \ldots, n\}$ we have $F_n(\xi_{\pi_1}, \ldots \xi_{\pi_n}) = F_n(\xi_1, \ldots \xi_n)$, the corresponding kernel $\Phi^{(n)}$ is in the n^{th} symmetric tensor power of $\mathcal{H}_q$, denoted by $\mathcal{H}_q^{\hat{\otimes} n} \subset \mathcal{H}_q^{\otimes n}$. By $\hat{\otimes}$ we denote the symmetrization of the tensor product

$$f_1 \hat{\otimes} \cdots \hat{\otimes} f_n := \frac{1}{n!} \sum_{\pi} f_{\pi_1} \otimes \cdots \otimes f_{\pi_n},$$

for $f_1, \dots f_n \in \mathcal{H}$ and where the sum extends over all permutations of n letters.
The theorems mentioned above also hold for the complexified spaces, $\mathcal{N}_{\mathbb{C}}$ and $\mathcal{N}'_{\mathbb{C}}$, respectively. By definition every element $\chi \in \mathcal{N}_{\mathbb{C}}$ can be decomposed into $\chi = \xi + i\psi,\ \ \xi, \psi \in \mathcal{N}$. The corresponding complexified Hilbert spaces $\mathcal{H}_{p,\mathbb{C}}$ are equipped with the inner product

$$(\chi_1, \chi_2)_{\mathcal{H}_{p,\mathbb{C}}} := (\xi_1, \xi_2)_{\mathcal{H}_p} + (\psi_1, \psi_2)_{\mathcal{H}_p} + i(\psi_1, \xi_2)_{\mathcal{H}_p} - i(\xi_1, \psi_2)_{\mathcal{H}_p},$$

where $\chi_1, \chi_2 \in \mathcal{H}_{p,\mathbb{C}}$, with $\chi_1 = \xi_1 + i\psi_1$ and $\chi_2 = \xi_2 + i\psi_2,\ \xi_1, \xi_2, \psi_1, \psi_2 \in \mathcal{H}_p$. Finally we obtain the nuclear triple

$$\mathcal{N}_{\mathbb{C}}^{\otimes n} \subset \mathcal{H}_{\mathbb{C}}^{\otimes n} \subset \left(\mathcal{N}_{\mathbb{C}}^{\otimes n}\right)'.$$

The dual pairing on the complexified spaces is chosen to be bilinear, which means that an additional complex conjugation is required to relate it to the inner product, i.e.

$$\langle \xi, \eta \rangle = (\xi, \overline{\eta})_{\mathcal{H}_{\mathbb{C}}}, \quad \xi, \eta \in \mathcal{H}_{\mathbb{C}}.$$

Remark 1.1.5. *In many monographs concerning White Noise Analysis, the dual pairing is defined by*

$$\langle \cdot, \cdot \rangle : D' \times D \to \mathbb{C}, \tag{1.1}$$

i.e. the test function is on the right hand side of the dual pairing. This is based on physics notation, see e.g. [40, 64, 57]. Throughout this thesis, we use the representation

$$\langle \cdot, \cdot \rangle : D \times D' \to \mathbb{C}, \tag{1.2}$$

i.e. the test function is on the left hand side, which is the common notation in functional analysis, see e.g. [3].
All the spaces we use in the following are reflexive (i.e. $D'' = D$), hence (1.1) *can be considered as the bidual pairing, i.e.*

$$\langle \xi, \omega \rangle = \langle \omega, \xi \rangle, \quad \xi \in D, \omega \in D',$$

where the left hand side is the dual and the right hand side the bidual pairing.

We also introduce the Fock space $\Gamma(\mathcal{H})$. The next definition can be found in [64].

Definition 1.1.6. *Let $\mathcal{H}$ be a Hilbert space with norm $|\cdot|$. Let $\Gamma(\mathcal{H})$ be the space of all sequences $\mathbb{f} = (f_n)_{n=0}^{\infty}$, $f_n \in \mathcal{H}_{\mathbb{C}}^{\hat{\otimes} n}$, (with the convention that $\mathcal{H}_{\mathbb{C}}^{\hat{\otimes} 0} = \mathbb{C}$) such that*

$\sum\limits_{n=0}^{\infty} n! \, |f_n|^2 < \infty$. *Equipped with the norm*

$$\|\mathfrak{f}\|^2_{\Gamma(\mathcal{H})} = \sum_{n=0}^{\infty} n! \, |f_n|^2 \, ,$$

the Hilbert space $\Gamma(\mathcal{H})$ *is called the (Boson or symmetric) Fock space over* $\mathcal{H}$.

1.2 Facts on CH-spaces

1.2.1 The Schwartz Spaces

Standard White Noise Analysis is based on the nuclear real countably Hilbert space (CH-space) of rapidly decreasing functions (Schwartz test functions) $S := S(\mathbb{R}, \mathbb{R})$. The topology of S is induced by the family of seminorms

$$\|f\|_{\alpha,\beta} = \left\| x^\alpha \frac{d^\beta}{dx^\beta} f \right\|_\infty , \quad \alpha, \ \beta \in \mathbb{N}.$$

More adapted to White Noise analysis is the reconstruction of the Schwartz test function space S by an equivalent system of seminorms, which is based on the so called N-representation of $S(\mathbb{R}, \mathbb{R})$, see [70, Appendix to V.3, p. 141ff.]. For technical reasons we need an operator H such that $\inf \sigma(H) > 1$. The topology of S is induced by the positive (unbounded) self-adjoint Operator

$$H = -\frac{d^2}{dx^2} + x^2 + 1$$

on the space $\mathcal{H} = (L^2(\mathbb{R}, dt, \mathbb{R}), dx)$ of (equivalence classes of) real-valued square integrable functions w.r.t. Lebesgue measure. The (point) spectrum of H is given by

$$\sigma(H) = \{2n + 2 \mid n \in \mathbb{N}\}$$

, hence $inf \ \sigma(H) > 1$. The eigenfunctions of H are the so-called Hermite functions $h_n, \ n \in \mathbb{N}$ with

$$Hh_n = \lambda_n h_n := (2n + 2)h_n, \quad n \in \mathbb{N}.$$

Note that the operator H has a Hilbert-Schmidt inverse H^{-1} . We have $\|H^{-1}\|_{OP} = \frac{1}{2}$ and denote $\delta := \|H^{-1}\|_{HS}$. The complexification $S_{\mathbb{C}}$ is equipped with the Hilbertian norms

$$|\xi|_p := |H^p \xi|_0 ,$$

for $p \in \mathbb{R}$. We denote $\mathcal{H}_{\mathbb{C}} := L^2(\mathbb{R}, dt, \mathbb{C})$ and $\mathcal{H}_{\mathbb{C},p} := \left\{ \xi \in S'_{\mathbb{C}} | \ |\xi|_p < \infty \right\}$ for $p \in \mathbb{R}$, respectively, where $\mathcal{H}_p := \left\{ \xi \in S' | \ |\xi|_p < \infty \right\}$, respectively. In the case of the standard CH-space $S(\mathbb{R})$ one often uses the notation S_p instead of $\mathcal{H}_p$ and $S_{p,\mathbb{C}}$ instead of $\mathcal{H}_{p,\mathbb{C}}$. Next we list some properties of the $S_{\mathbb{C}}$, which are used in calculations throughout the following chapters.

Proposition 1.2.1.

(i) For $p \le q$ we have

$$|\cdot|_p \le \frac{1}{2^{(q-p)}} |\cdot|_q .$$

(ii) Let $\xi \in S'_{\mathbb{C}}$. Then

$$\lim_{p \to \infty} |\xi|_{-p} = 0.$$

(iii) For any $p \ge 0$, the embedding $S_{p+1} \hookrightarrow S_p$ is of Hilbert-Schmidt type.

(iv) For $\xi \in S'_{\mathbb{C}}$, we have

$$\xi = \sum_{n=0}^{\infty} \langle h_i, \xi \rangle h_i,$$

where the sum converges in $S'_{\mathbb{C}}$.

(v) The Hermite functions $(h_i)_{i \in \mathbb{N}}$ is a family of functions which are orthogonal in every Hilbert space $S_{p,\mathbb{C}}$.

1.2.2 Facts about Trace operators

The general construction of CH-spaces, has the advantage of being manifestly independent of the choice of any concrete system of Hilbertian norms topologizing $\mathcal{N}$, see [87, Remark, p. 48]. But by the lack of a property like 1.2.1(i) we cannot conclude that

$$\lim_{q \to \infty} \|i_{q,p}\|_{HS} = 0 \text{ or } \lim_{n \to \infty} |\xi|_{-p} = 0, \quad \xi \in S'_{\mathbb{C}}$$

.
Since we have no knowledge about the inner structure of the Hilbert spaces $\mathcal{H}_p$, it is not possible to assume the existence of a sequence $(e_i)_{i\in\mathbb{N}}$, which fulfills the properties (iv) and (v) of Proposition 1.2.1.
The standard CH-space S has the advantage that many questions can be answered more precisely by an explicit calculation while in the general case the use of abstract theorems as e.g. the abstract kernel theorem as in the previous section is needed.

We state some general facts on CH-spaces.

Lemma 1.2.2. *Let $B \in L(\mathcal{N}_{\mathbb{C}}, \mathcal{N}_{\mathbb{C}})$. Then for all $p \in \mathbb{N}$ there exists a $q \in \mathbb{N}$ such that*

$$B^* \in L(\mathcal{H}_{\mathbb{C},-p}, \mathcal{H}_{\mathbb{C},-q})$$

Proof. Let $p \in \mathbb{N}$. Then there exists $C > 0,\ q \in \mathbb{N}$, such that for all $\xi \in \mathcal{N}_{\mathbb{C}}$ we have

$$|B(\xi)|_p \leq C \cdot |\xi|_q$$

Hence $B \in L(\mathcal{H}_{\mathbb{C},q}, \mathcal{H}_{\mathbb{C},p})$ and consequently

$$B^* \in L(\mathcal{H}_{\mathbb{C},-p}, \mathcal{H}_{\mathbb{C},-q}).$$

□

Proposition 1.2.3. *Let $B \in L(\mathcal{N}_{\mathbb{C}}, \mathcal{N}_{\mathbb{C}})$ and $n \in \mathbb{N},\ n > 0$. Then for all $p \in \mathbb{N}$ there exists a $K > 0$ and $q_1, q_2 \in \mathbb{N}$, with $p < q_1 < q_2$ such that for all $\theta \in \mathcal{N}_{\mathbb{C}}^{\otimes n}$ we have*

$$\left|B^{\otimes n}\theta\right|_p \leq \left(K \left\|B\right\|_{q_2,q_1} \left\|i_{q_1,p}\right\|_{HS}\right)^n \cdot |\theta|_{q_2}.$$

Proof. Choose $q_1,\ q_2 \in \mathbb{N}$ with $q_2 > q_1 > p$ and

$$B \in L(\mathcal{H}_{\mathbb{C},q_1}, \mathcal{H}_{\mathbb{C},p}) \text{ and } B \in L(\mathcal{H}_{\mathbb{C},q_2}, \mathcal{H}_{\mathbb{C},q_1}).$$

For a shorter notation we write e_J for $e_{j_1} \otimes \cdots \otimes e_{j_n}$. Now let $(e_J)_J$ be an orthonormal basis of $\mathcal{H}_p^{\otimes n}$. Further let $\mathcal{I}_p :\ (\mathcal{H}_{\mathbb{C},p}^{\otimes n})' \to \mathcal{H}_{\mathbb{C},p}^{\otimes n}$ be the Riesz isomorphism. Then for $\theta \in \mathcal{N}_{\mathbb{C}}^{\otimes n}$

$$\left|B^{\otimes n}\theta\right|_p^2 = \sum_J \left|(B^{\otimes n}\theta, e_J)_p\right|^2 = \sum_J \left|\left\langle B^{\otimes n}\theta, \overline{\mathcal{I}_p^{-1}(e_J)}\right\rangle\right|^2$$

$$= \sum_J \left|\left\langle B^{\otimes n}\theta, \mathcal{I}_p^{-1}(e_J)\right\rangle\right|^2 = \sum_J \left|\left\langle \theta, (B^*)^{\otimes n}\mathcal{I}_p^{-1}(e_J)\right\rangle\right|^2$$
$$\leq |\theta|_{q_2}^2 \cdot \sum_J \left|(B^*)^{\otimes n}\mathcal{I}_p^{-1}(e_J)\right|_{-q_2}^2 ,$$

We can estimate this with the norm of $B^* : \mathcal{H}_{\mathbb{C},-q_1} \to \mathcal{H}_{\mathbb{C},-q_2}$, denoted by $\|B^*\|_{-q_1,-q_2}$. Then for a $K > 0$ we have

$$\begin{aligned} |\theta|_{q_2}^2 \cdot \sum_J \left|(B^*)^{\otimes n}\mathcal{I}_p^{-1}(e_J)\right|_{-q_2}^2 &\leq |\theta|_{q_2}^2 \cdot K^{2n} \|B^*\|_{-q_1,-q_2}^{2n} \sum_J \left|\mathcal{I}_p^{-1}(e_J)\right|_{-q_1}^2 , \\ &= |\theta|_{q_2}^2 \cdot K^{2n} \|B^*\|_{-q_1,-q_2}^{2n} \|i_{-p,-q_1}\|_{HS}^{2n} \\ &= |\theta|_{q_2}^2 \cdot K^{2n} \|B\|_{q_1,q_2}^{2n} \|i_{q_1,p}\|_{HS}^{2n} , \end{aligned}$$

where the last equation is due to [72, Theorem 4.10(2), p. 93] □

Definition 1.2.4. *By the canonical correspondence between the bilinear forms* $\mathfrak{B}(\mathcal{N}_{\mathbb{C}}, \mathcal{N}_{\mathbb{C}})$ *and* $(\mathcal{N}_{\mathbb{C}} \otimes \mathcal{N}_{\mathbb{C}})'$*, the trace operator* tr *is uniquely defined via the formula:*

$$\langle \xi \otimes \eta, tr \rangle = \langle \xi, \eta \rangle = \langle \xi, Id(\eta) \rangle , \ \xi, \eta \in \mathcal{N}_{\mathbb{C}}.$$

Lemma 1.2.5. *The trace operator* tr *is an element of* $\mathcal{N}_{\mathbb{C}} \otimes \mathcal{N}_{\mathbb{C}}'$ *and* $\mathcal{N}_{\mathbb{C}}' \otimes \mathcal{N}_{\mathbb{C}}$*.*

Proof. By definition, the corresponding operator under the isomorphism $L(\mathcal{N}_{\mathbb{C}}, \mathcal{N}_{\mathbb{C}}) \approx \mathcal{N}_{\mathbb{C}}' \otimes \mathcal{N}_{\mathbb{C}}$ is the identity, compare [64, Theorem 1.3.10(Kernel Theorem), p. 11]. Hence $tr \in \mathcal{N}_{\mathbb{C}}' \otimes \mathcal{N}_{\mathbb{C}}$. Using the isomorphism $\mathcal{N}_{\mathbb{C}}' \otimes \mathcal{N}_{\mathbb{C}} \longrightarrow \mathcal{N}_{\mathbb{C}} \otimes \mathcal{N}_{\mathbb{C}}',\ \omega \otimes \eta \mapsto \eta \otimes \omega$, we obtain the desired result. Compare to the proof of [64, Proposition 5.3.2, p. 121]). □

Proposition 1.2.6. *The trace operator can be represented by*

$$tr = \sum_{n=0}^{\infty} e_n \otimes e_n,$$

where $(e_n)_{n\in\mathbb{N}}$ *is an arbitrary orthonormal basis of* $\mathcal{H}$*.*

Proof. For an elementary proof, see [64, Proposition 2.2.1, p. 24] Note that this proof is independent of the internal structure of the Hilbert spaces which define the CH-space. □

We give another example, where a representation by a basis is possible even in the case of a general CH-space.

Definition 1.2.7. *Let $B \in L(\mathcal{N}_{\mathbb{C}}, \mathcal{N}'_{\mathbb{C}})$. By tr_B we denote the element in $\mathcal{N}'_{\mathbb{C}} \otimes \mathcal{N}'_{\mathbb{C}}$, which is defined by*

$$\forall \xi,\ \eta \in \mathcal{N}_{\mathbb{C}}:\ tr_B(\xi \otimes \eta) := \langle \xi, B\eta \rangle\,.$$

Note that tr_B is not symmetric. Further there exists a $q \in \mathbb{Z}$ such that $tr_B \in \mathcal{H}_{\mathbb{C},q} \otimes \mathcal{H}_{\mathbb{C},q}$.

Proposition 1.2.8. *Let $B \in L(\mathcal{H}_{\mathbb{C}}, \mathcal{H}_{\mathbb{C}})$ be a Hilbert-Schmidt operator. Then $tr_B \in \mathcal{H}_{\mathbb{C}} \otimes \mathcal{H}_{\mathbb{C}}$. Further for each orthonormal basis $(e_j)_{j\in\mathbb{N}}$ of $\mathcal{H}_{\mathbb{C}}$ it follows:*

$$tr_B = \sum_{i=0}^{\infty} Be_i \otimes e_i.$$

Proof. $\left| \sum_{i,j=0}^{\infty} \langle e_i, Be_j \rangle \, e_i \otimes e_j \right|_0^2 = \sum_{i,j=0}^{\infty} |\langle e_i, Be_j \rangle|^2 = \sum_{j=0}^{\infty} |Be_j|_0^2 = \|B\|_{HS}^2 < \infty$. Hence the sum is a well-defined element in $\mathcal{H}_{\mathbb{C}} \otimes \mathcal{H}_{\mathbb{C}}$. The identity follows by verifying the formula for $\{e_k \otimes e_l\}_{k,l\in\mathbb{N}}$:

$$\left\langle e_k \otimes e_l, \sum_{i=0}^{\infty} Be_i \otimes e_i \right\rangle = (e_k, \overline{Be_l})_{\mathcal{H}} (e_l, \overline{e_l})_{\mathcal{H}} = (e_k, \overline{Be_l})_{\mathcal{H}} = \langle e_k, Be_l \rangle.$$

□

Proposition 1.2.9. *In the case $\mathcal{N} = S(\mathbb{R})$ and $B \in L(S_{\mathbb{C}}(\mathbb{R}), S'_{\mathbb{C}}(\mathbb{R}))$ we have*

$$tr_B = \sum_{i=0}^{\infty} Bh_i \otimes h_i$$

Proof. By the continuity of the bilinear form $\langle \cdot, B\cdot \rangle$ on $S(\mathbb{R})_{\mathbb{C}} \times S(\mathbb{R})_{\mathbb{C}}$ there exists $p \geq 0$ such that $B \in L(\mathcal{H}_{p,\mathbb{C}}, \mathcal{H}_{-p,\mathbb{C}})$. Let $q > p+1$. Then

$$|\sum_{n=0}^{\infty} Bh_n \otimes h_n|_{-q}^2 = \sum_{n=0}^{\infty} |Bh_n|_{-q}^2 \cdot |h_n|_{-q}^2 \leq K \sum_{n=0}^{\infty} |h_n|_p^2 \cdot |h_n|_{-q}^2,$$

for some $K > 0$. For the last expression we have

$$K \sum_{n=0}^{\infty} |h_n|_p^2 \cdot |h_n|_{-q}^2 \leq K \sum_{n=0}^{\infty} \left(\frac{1}{2n+2} \right)^2 < \infty.$$

Then as in the proof of Proposition 1.2.8 we obtain

$$tr_B = \sum_{n=0}^{\infty} Bh_n \otimes h_n,$$

since

$$\overline{\text{span}\{h_n \otimes h_l\}_{n,l}}^{S(\mathbb{R})\otimes S(\mathbb{R})} = S(\mathbb{R}) \otimes S(\mathbb{R}).$$

$\square$

Chapter 2

Gaussian Analysis and White Noise Analysis

In this chapter we give a brief introduction into the theory of White Noise Analysis. This framework generalizes structures known from finite dimensional analysis to the infinite dimensional setting. Basis of the calculus are spaces of generalized functions on which, by the theorem of Bochner and Minlos, a Gaussian measure is defined. Beneath powerful tools as Fourier transform and differential operators also a distribution theory can be found in the framework. Various triples of test and generalized functions and corresponding test functions could be characterized with the help of spaces of analytic functions by transforming them via the Gauss-Fourier or Gauss-Laplace transform. These characterization theorems also give the possibility to consider sequences and integrals of generalized functions, such that also these can be characterized as generalized functions of White Noise Analysis.

2.1 White Noise measure

Consider the Gel'fand triple or nuclear triple (compare to section 1.1)

$$\mathcal{N} \subset \mathcal{H} \subset \mathcal{N}'.$$

A standard example for such a triple is the so-called White Noise triple

$$S(\mathbb{R}) \subset L^2(\mathbb{R}, dx) \subset S'(\mathbb{R}),$$

where $S(\mathbb{R}) := \{f \in C^\infty(\mathbb{R}) \,|\, \forall k, n \in \mathbb{N}_0 \exists C \in \mathbb{R}_+ : \sup_{x\in\mathbb{R}} |x^k D^n f| \leq C\}$ is the space of Schwartz test functions and $S'(R) := \{\omega : S(\mathbb{R}) \to \mathbb{R} \,|\, \text{linear and continuous}\}$ its topological dual, the space of tempered distributions. Central space in this triple is the Hilbert space of (equivalence classes of) $\mathbb{R}$-valued square integrable functions w.r.t. the Lebesgue measure (equipped with its canonical inner product $(\cdot,\cdot)$ and norm $|\cdot|$) , more detailed see e.g. [87, 70].

It is not possible to construct a flat, i.e. translation invariant measure on an infinite dimensional space see e.g. [42]. Nor is it possible to construct a Gaussian measure on a separable Hilbert space.

Remark 2.1.1. *Assume that μ is a Gaussian measure on a separable Hilbert space $\mathcal{H}$. Let $(e_n)_{n\in\mathbb{N}}$ be an orthonormal basis for $\mathcal{H}$. One then has*

$$\int_{\mathcal{H}} \exp(i(h, e_n))\, d\mu(h) = \exp(-\frac{1}{2}(e_n, e_n)) = \exp(-\frac{1}{2}).$$

for all $n \in \mathbb{N}$.

Since for every $h \in \mathcal{H}$ one has that

$$h = \sum_{n=0}^{\infty} (h, e_n) e_n$$

it is known, that $((h, e_n))_{n\in\mathbb{N}}$ is a zero sequence for all $h \in \mathcal{H}$. Then, by the theorem of Lebesgues dominated convergence the integral tends to 1 as n tends to infinity, since μ is a probability measure. In the limit the equation yields

$$1 = \exp(-\frac{1}{2}),$$

which is a contradiction.

To overcome this, we construct a Gaussian measure on the space $\mathcal{N}'$ (or $S'(\mathbb{R})$ in the White Noise framework). Therefore we consider the σ-algebra generated by cylinder sets:

$$\mathcal{C}^{\xi_1,\dots,\xi_n}_{F_1,\dots,F_n} = \Big\{\omega \in \mathcal{N}' \,\big|\, \langle \xi_1, \omega\rangle \in F_1, \dots, \langle \xi_n, \omega\rangle \in F_n, \\ \xi_i \in \mathcal{N}, F_j \in \mathcal{B}(\mathbb{R}), j = 1, \dots, n, \; n \in \mathbb{N}, \Big\} \quad (2.1)$$

where $\mathcal{B}(\mathbb{R})$ denotes the σ-algebra of Borel sets in $\mathbb{R}$. We denote

$$\mathcal{C}_\sigma(\mathcal{N}') := \sigma(\mathcal{C}^{\xi_1,\dots,\xi_n}_{F_1,\dots,F_n}).$$

The next proposition can be found in [40] or for the proof [4].

Proposition 2.1.2. *Let $\mathfrak{N}$ be a nuclear CH-space, then we have*

$$\mathcal{C}_\sigma(\mathfrak{N}') = \mathcal{B}_w(\mathfrak{N}') = \mathcal{B}_s(\mathfrak{N}'),$$

where $\mathcal{B}_w(\mathfrak{N}')$ (resp. $\mathcal{B}_s(\mathfrak{N}')$) is the Borel σ-algebra generated by the weak (resp. strong) topology.

The existence of a probability measure on $\mathcal{N}'$ is given via the Theorem of Bochner and Minlos, see e.g. [61, 27, 6, 57].

Theorem 2.1.3 (Theorem of Bochner and Minlos)**.** *Let $\mathfrak{N}$ be a real nuclear space. A complex-valued function C on $\mathfrak{N}$ is the characteristic function of a unique probability measure ν on $\mathfrak{N}'$,i.e.*

$$C(\xi) = \int_{\mathfrak{N}'} \exp(i\langle \xi, \omega\rangle)\, d\nu(\omega), \quad \xi \in \mathfrak{N},$$

if and only if the following is fulfilled:

a) $C(0) = 1$,

b) C *is continuous,*

c) C *is positive definite, i.e.*

$$\sum_{j=1}^{n}\sum_{k=1}^{n} z_j \overline{z_k} C(\xi_j - \xi_k) \geq 0, \quad \textit{for all } z_1, \dots, z_n \in \mathbb{C} \textit{ and } \xi_1, \dots, \xi_n \in \mathfrak{N}.$$

Via the theorem of Bochner and Minlos one introduces the canonical Gaussian measure μ on $\mathcal{N}'$ by giving its characteristic function

$$C(\xi) = \int_{\mathcal{N}'} \exp(i\langle \xi, \omega\rangle)\,\mathrm{d}\mu(\omega) = \exp\left(-\frac{1}{2}\langle \xi, \xi\rangle\right), \quad \xi \in \mathcal{N}.$$

For a measurable function f defined on $\mathcal{N}'$ and which is integrable w.r.t. μ, i.e.

$$\int_{\mathcal{N}'} |f(\omega)|\, d\mu(\omega) < \infty,$$

we define the expectation of f (denoted by $\mathbb{E}_\mu(f)$) by

$$\mathbb{E}_\mu(f) = \int_{\mathcal{N}'} f(\omega)\, d\mu(\omega).$$

The space of integrable functions w.r.t μ we denote by $L^1(\mu) := L^1(\mathcal{N}', \mathcal{C}_{\sigma,\mu}(\mathcal{N}'), \mu)$.
For the use of test and generalized functions it is necessary to consider the square-integrable functions w.r.t μ, since in this case we end with a Hilbert space and obtain - as we will see later - a Gel'fand triple of spaces. The space of square-integrable functions we denote by

$$L^2(\mu) := L^2(\mathcal{N}', \mathcal{C}_{\sigma,\mu}(\mathcal{N}'), \mu).$$

This space becomes a Hilbert space with the inner product

$$(\!(f, g)\!) := \int_{\mathcal{N}'} f(\omega)\overline{g(\omega)}\, d\mu(\omega), \quad f, g \in L^2(\mu).$$

Example 2.1.4.

a) For $\phi^{(n)} \in \mathcal{N}_{\mathbb{C}}^{\hat{\otimes} n}$, $n \in \mathbb{N}$, and $\phi_0 \in \mathbb{C}$ we define the smooth Wick monomials of order n corresponding to the kernels $\phi^{(n)}$ by

$$I\left(\phi^{(n)}\right)(x) := \left\langle \phi^{(n)}, :\omega^{\otimes n}:\right\rangle, \quad \omega \in \mathcal{N}', n \in \mathbb{N}.$$

Here $:\omega^{\otimes n}: \in \left(\mathcal{N}^{\hat{\otimes} n}\right)'$ are the so-called Wick powers of ω of order n, $n \in \mathbb{N}$. For this we have the following recursion formula, compare to [40]

$$\begin{aligned} :\omega^{\otimes 0}: &= 1 \\ :\omega^{\otimes 1}: &= \omega \end{aligned}$$

$$:\omega^{\otimes n}: \;=\; :\omega^{\otimes n-1}:\hat{\otimes}\,\omega-(n-1):\omega^{\otimes n-2}:\hat{\otimes}\tau,$$

where $tr \in \mathcal{N}'^{\hat{\otimes}2}$ is the trace operator defined by

$$\langle f\otimes g, tr\rangle = \langle f, g\rangle, \quad f,g\in\mathcal{N}.$$

Note that smooth Wick monomials of different order are orthogonal in $L^2(\mu)$, see e.g.[40, 57, 64].

b) *Let $(f_n)_{n\in\mathbb{N}}$, $f_n\neq 0$ be a sequence in $\mathcal{N}$ converging to $f\in\mathcal{H}$. Then we have*

$$\int_{\mathcal{N}'}\langle f_n,\omega\rangle^2\,d\mu(\omega)=\frac{1}{\sqrt{2\pi\langle f_n,f_n\rangle}}\int_{\mathbb{R}}x^2\exp(-\frac{1}{2\langle f_n,f_n\rangle}x^2)\,dx$$
$$=\frac{1}{\sqrt{2\pi\langle f_n,f_n\rangle}}\langle f_n,f_n\rangle\cdot\sqrt{2\pi\langle f_n,f_n\rangle}=\langle f_n,f_n\rangle.$$

This gives the so-called Wiener-Itô-Segal-isometry which enables us to extend the dual pairing from $\mathcal{N}$ to $\mathcal{H}$, i.e. we define $\langle f,\cdot\rangle\in L^2(\mu)$ by

$$\langle f,\cdot\rangle := \lim_{n\to\infty}\langle f_n,\cdot\rangle.$$

Since f_n converges in $\mathcal{H}$ we have $\langle f_n,\cdot\rangle$ converges in $L^2(\mu)$ by the Wiener-Itô-Segal-isometry and is thus a well-defined object as an $L^2(\mu)$-limit.

c) *By approximation we also can construct Wick monomials $I(f^{(n)})$ with kernels $f^{(n)}\in\mathcal{H}_{\mathbb{C}}^{\hat{\otimes}n}$. Since we have for sequences $(f_k^{(n)})_{k\in\mathbb{N}}\subset\mathcal{N}_{\mathbb{C}}^{\hat{\otimes}n}, (g_k^{(m)})_{k\in\mathbb{N}}\subset\mathcal{N}_{\mathbb{C}}^{\hat{\otimes}m}$ converging to $f^{(n)}$ in $\mathcal{H}_{\mathbb{C}}^{\hat{\otimes}n}$ and $g^{(m)}$ in $\mathcal{H}_{\mathbb{C}}^{\hat{\otimes}m}$, respectively, we obtain*

$$\left(\left(I(f_k^{(n)}),I(g_k^{(m)})\right)\right)=\int_{\mathcal{N}'}\left\langle f_k^{(m)},:\omega^{\otimes m}:\right\rangle\overline{\left\langle g_k^{(n)},:\omega^{\otimes n}:\right\rangle}\,d\mu(\omega)=\delta_{n,m}\left(f_k^{(n)},g_k^{(n)}\right),$$

where $\delta_{n,m}$ is the Kronecker symbol. Thus again, convergence in $\mathcal{H}_{\mathbb{C}}^{\hat{\otimes}n}$ ensures convergence in $L^2(\mu)$ and $I(f^{(n)})$ and $I(g^{(m)})$ are rigorously defined as $L^2(\mu)$-limits.

d) *Important examples of $L^2(\mu)$-functions are the so-called coherent states or Wick exponentials*

$$:\!\exp(\langle\xi,\cdot\rangle)\!:=\exp(-\frac{1}{2}\langle\xi,\xi\rangle)\cdot\exp(\langle\xi,\cdot\rangle),\quad \xi\in\mathcal{N}$$

and the exponential vectors

$$\exp(i\langle \xi, \cdot \rangle), \quad \xi \in \mathcal{N}.$$

They also play a key role in the characterization of White Noise spaces. We will see later, that they are even more regular than mentioned here.

If we choose special nuclear spaces $\mathcal{N}$ and Hilbert spaces $\mathcal{H}$ we can give more examples and also related to stochastic processes as Brownian motion. This is the case for the White Noise triple.

Example 2.1.5.

a) *The Gel'fand triple for vector valued white noise is given by $S_d(\mathbb{R}) \subset L^2_d(\mathbb{R}, dx) \subset S'_d(\mathbb{R})$ of the $\mathbb{R}^d$-valued, Schwartz test functions and tempered distributions with the Hilbert space of (equivalence classes of) $\mathbb{R}^d$-valued square integrable functions w.r.t. the Lebesgue measure as central space (equipped with its canonical inner product $(\cdot, \cdot)$ and norm $|\cdot|$), more detailed see e.g. [87, Exam. 11]. $S_d(\mathbb{R})$ is a nuclear space and can be represented as projective limit of a decreasing chain of Hilbert spaces $(H_p)_{p\in\mathbb{N}}$.*

$$S_d(\mathbb{R}) = \bigcap_{p\in\mathbb{N}} H_p,$$

we have that $S_d(\mathbb{R})$ is a countably Hilbert space in the sense of Gel'fand and Vilenkin [27]. Note that $S'_d(\mathbb{R}) = \bigcup_{p\in\mathbb{N}} H_{-p}$, i.e. $S'_d(\mathbb{R})$ is the inductive limit of the increasing chain of Hilbert spaces $(H_{-p})_{p\in\mathbb{N}}$, see e.g. [27]. We denote the dual pairing of $S_d(\mathbb{R})$ and $S'_d(\mathbb{R})$ also by $\langle \cdot, \cdot \rangle$. Note that its restriction on $S_d(\mathbb{R}) \times L^2_d(\mathbb{R}, dx)$ is given by $(\cdot, \cdot)$. The space $(S'_d(\mathbb{R}), \mathcal{C}_\sigma(S'_d(\mathbb{R})), \mu)$ is the basic probability space in our setup. The central Gaussian spaces in this framework are the Hilbert spaces $(L^2) := L^2(S'_d(\mathbb{R}), \mathcal{C}_\sigma(S'_d(\mathbb{R})), \mu)$, $d \in \mathbb{N}$, of complex-valued square integrable functions w.r.t. the Gaussian measure μ. Here $\mathcal{C}_\sigma(S'_d(\mathbb{R}))$ is the σ-algebra generated by cylinder sets. Within this formalism a version of a d-dimensional Brownian motion is given by

$$\mathbf{B}(t, \boldsymbol{\omega}) := (\langle \mathbb{1}_{[0,t)}, \omega_1 \rangle, \dots \langle \mathbb{1}_{[0,t)}, \omega_d \rangle), \quad \boldsymbol{\omega} = (\omega_1, \dots \omega_d) \in S'_d(\mathbb{R}), \quad t \geq 0, \qquad (2.2)$$

in the sense of an (L^2)-limit. Here $\mathbb{1}_A$ denotes the indicator function of a set A.

b) *For d = 1 in a), we have the classical one-dimensional White Noise triple. Here a* *version of a Brownian motion is given by* $B(t,\omega) := \langle \mathbb{1}_{[0,t)}, \omega\rangle$*. Note that if we just consider* $\omega \in L^2(\mathbb{R}, dx)$ *the dual pairing can be written as an integral and we obtain*

$$\langle \mathbb{1}_{[0,t)}, \omega\rangle = \int_{\mathbb{R}} \mathbb{1}_{[0,t)}(s)\omega(s)\, ds = \int_0^t \omega(s)\, ds.$$

Thus one can formally consider ω *as the derivative of Brownian motion. Since the corresponding process is called White Noise process, we call this triple White Noise triple. White Noise Analysis uses this White Noise process as infinite-dimensional coordinate system.*

2.2 Spaces of test and generalized functions

2.2.1 Polynomials and Wiener-Itô-Segal decomposition

To characterize subspaces of $L^2(\mu)$ we start with an easy but very important subspace of $L^2(\mu)$, the space of so-called smooth polynomials on $\mathcal{N}'$.

Definition 2.2.1. *We call the space*

$$\begin{aligned}\mathcal{P}(\mathcal{N}') &:= \left\{ \phi \middle| \phi(\omega) = \sum_{n=0}^{N} \left\langle \tilde{\phi}^{(n)}, \omega^{\otimes n} \right\rangle,\ \textit{with}\ \tilde{\phi}^{(n)} \in \mathcal{N}_{\mathbb{C}}^{\hat{\otimes} n}, \omega \in \mathcal{N}', N \in \mathbb{N} \right\} \\ &= \left\{ \phi \middle| \phi(\omega) = \sum_{n=0}^{N} \left\langle \phi^{(n)}, : \omega^{\otimes n} : \right\rangle,\ \textit{with}\ \phi^{(n)} \in \mathcal{N}_{\mathbb{C}}^{\hat{\otimes} n}, \omega \in \mathcal{N}', N \in \mathbb{N} \right\}\end{aligned}$$

the space of smooth polynomials on $\mathcal{N}'$*. Note that the equality yields, since every smooth polynomial can be expressed in terms of a sum of smooth Wick monomials of the same order. Furthermore the smooth polynomials are dense in* $L^2(\mu)$*, i.e.* $\overline{\mathcal{P}(\mathcal{N}')}^{L^2(\mu)} = L^2(\mu)$*.*

Consequently the space $L^2(\mu)$ of equivalence classes of μ-square integrable complex valued functions on $\mathcal{N}'$ has a well-known Wiener-Itô-Segal chaos decomposition [63, 79, 77]. Within this decomposition there exists the so-called Segal isomorphism I mapping between the symmetric complex Fock space $\Gamma(\mathcal{H}_{\mathbb{C}})$ over the complexification $\mathcal{H}_{\mathbb{C}}$ of $\mathcal{H}$ and $L^2(\mu)$. To construct spaces of more regularity on $\mathcal{N}'$ one can use special mappings of proper subspaces of $\Gamma(\mathcal{H})$ into $L^2(\mu)$ via the unitary map $I : \Gamma(\mathcal{H}) \to L^2(\mu)$, see e.g. [6, 40].

Theorem 2.2.2 (Wiener-Itô-Segal decomposition)**.** *For each $\phi \in L^2(\mu)$ there exists a unique element $(f_n)_{n=0}^{\infty} \in \Gamma(\mathcal{H}_{\mathbb{C}})$ such that*

$$\phi(\omega) = \sum_{n=0}^{\infty} \left\langle f_n, : \omega^{\otimes n} : \right\rangle, \tag{2.3}$$

in $L^2(\mu)$-sense. On the other side every $(f_n)_{n=0}^{\infty} \in \Gamma(\mathcal{H}_{\mathbb{C}})$ defines a function in $L^2(\mu)$ via (2.3)*. Then we have*

$$\|\phi\|^2_{L^2(\mu)} = \sum_{n=0}^{\infty} n!\, |f_n|^2 = \|(f_n)_{n=0}^{\infty}\|_{\Gamma(\mathcal{H}_{\mathbb{C}})}.$$

We introduce the Gauss-Laplace transform also called S-transform on the space $L^2(\mu)$.

Definition 2.2.3. *Let $f \in L^2(\mu)$ and $\xi \in \mathcal{H}_{\mathbb{C}}$. We define the S-transform of f in ξ by*

$$S(f)(\xi) := \big(: \exp(\langle \xi, \cdot \rangle) :, \overline{f}\big)_{L^2(\mu)} = \exp(-\frac{1}{2}\,|\xi|_{L^2(\mathbb{R},dx)}) \int_{\mathcal{N}'} f(\omega) \exp(\langle \xi, \omega \rangle)\, d\mu(\omega).$$

the S-transform is well defined since $: \exp(\langle \xi, \cdot \rangle) :\in L^2(\mu)$ for $\xi \in \mathcal{H}_{\mathbb{C}}$.

Remark 2.2.4. *For $\xi \in \mathcal{H}$ the Wick-ordered exponential $: \exp(\langle \xi, \cdot \rangle) :$ is the Radon-Nikodym derivative of the measure $\mu_\xi := \mu(\cdot - \xi)$ w.r.t. the standard White Noise measure μ, i.e.*

$$: \exp(\langle \xi, \cdot \rangle) := \frac{d\mu_\xi}{d\mu}.$$

Then for $f \in L^2(\mu)$ and $\xi \in \mathcal{H}$

$$\int_{\mathcal{N}'} f(\omega) \cdot : \exp(\langle \xi, \omega \rangle) :\, d\mu(\omega) = \int_{\mathcal{N}'} f(\omega)\, d\mu_\xi(\omega) = \int_{\mathcal{N}'} f(\omega + \xi)\, d\mu(\omega),$$

compare [40, 64, 57]. This shows the quasi-translation invariance of the standard White Noise measure.

The S-transform is important since it enables us to determine the chaos decomposition of an $L^2(\mu)$ function, see e.g. [40, 64, 57]. This can be seen in the following, by applying the definition of Wick monomials.

$$S(f)(\xi) = \sum_{n=0}^{\infty} \langle \xi^{\otimes n}, f^{(n)} \rangle, \quad \xi \in \mathcal{N}, \tag{2.4}$$

where $f^{(n)}$ is the n^{th} kernel in the Wiener-Itô-Segal chaos decomposition of $f \in L^2(\mu)$. Furthermore the S-transform has an entire extension in $\xi \in \mathcal{N}$, i.e.

$$S(f)(\xi) = \sum_{n=0}^{\infty} \langle \xi^{\otimes n}, f^{(n)} \rangle), \quad \xi \in \mathcal{N}_{\mathbb{C}}.$$

2.2.2 Hida test functions and Hida distributions

We introduce spaces of test and generalized functions in infinite dimensional Gaussian analysis. Before we introduce the famous and important spaces of Hida and Kondratiev distributions, we come once more back to the smooth polynomials and their dual space, as a starting point for the construction of more applicable triples. The idea of using these spaces goes back to Kristensen et al., see e.g. [54]. In that work, the authors also discuss in which sense the space could be seen as minimal. The space of smooth polynomials $\mathcal{P}(\mathcal{N}')$ together with a natural topology is a nuclear space, see [6]. This topology is chosen such that we obtain an isomorphy between $\mathcal{P}(\mathcal{N}')$ and the topological direct sum of tensor powers $\mathcal{N}_{\mathbb{C}}^{\hat{\otimes} n}$, see e.g. [75]. Then we have

$$\mathcal{P}(\mathcal{N}') \simeq \bigoplus_{n=0}^{\infty} \mathcal{N}_{\mathbb{C}}^{\hat{\otimes} n}.$$

The isomorphism is the Wiener-Itô-Segal isomorphism mentioned in the previous section. I.e.

$$\mathcal{P}(\mathcal{N}') \ni \phi,\, \phi(\omega) = \sum_{n=0}^{\infty} \langle \phi^{(n)}, : \omega^{\otimes n} : \rangle, \omega \in \mathcal{N}' \leftrightarrow \left\{ \phi^{(n)} | n \in \mathbb{N} \right\} \in \bigoplus_{n=0}^{\infty} \mathcal{N}_{\mathbb{C}}^{\hat{\otimes} n},$$

where just a finite number of $\phi^{(n)}$ is non-zero, since ϕ is a polynomial.

Definition 2.2.5. *A sequence $\mathcal{P}(\mathcal{N}') \supset (\phi_j)_{j \in \mathbb{N}}$ with*

$$\phi_j := \sum_{n=0}^{N(\phi_j)} \langle \phi_j^{(n)}, : \cdot^{\otimes n} : \rangle$$

converges to an element

$$\mathcal{P}(\mathcal{N}') \ni \phi = \sum_{n=0}^{N(\phi)} \langle \phi^{(n)}, : \cdot^{\otimes n} : \rangle$$

if the set $\{N(\phi_j\}|j \in \mathbb{N}\}$ is bounded and for all $n \in \mathbb{N}$ one has that $\phi_j^{(n)} \to \phi^{(n)}$ as j tends to infinity.

Consider the Gel'fand triple

$$\mathcal{P}(\mathcal{N}') \subset L^2(\mu) \subset \mathcal{P}'(\mathcal{N}'),$$

where $\mathcal{P}'(\mathcal{N}')$ is the topological dual space of $\mathcal{P}(\mathcal{N}')$ w.r.t $L^2(\mu)$. Then the bilinear dual pairing $\langle\!\langle \cdot, \cdot \rangle\!\rangle$ between $\mathcal{P}(\mathcal{N}')$ and $\mathcal{P}'(\mathcal{N}')$ is an extension of the sesquilinear inner product on $L^2(\mu)$, i.e. $\langle\!\langle f, g \rangle\!\rangle = (\!(f, \overline{g})\!)$, for $g \in L^2(\mu)$ and $f \in \mathcal{P}(\mathcal{N}')$. Since the constant $\mathbb{1}$-function is a smooth polynomial, we can then generalize the notion of expectation to distributions from $\mathcal{P}'(\mathcal{N}')$ via the dual pairing. Since the dual pairing is an extension of the $L^2(\mu)$ inner product, i.e. the integral w.r.t. μ we define

$$\mathbb{E}_\mu(\Phi) := \langle\!\langle \mathbb{1}, \Phi \rangle\!\rangle, \quad \Phi \in \mathcal{P}'(\mathcal{N}').$$

As mentioned above the space $\mathcal{P}'(\mathcal{N}')$ is for applications often too large. To overcome this one considers projective and inductive limits of Hilbert spaces to obtain distribution spaces with higher regularity. To construct these spaces we consider the systems of Hilbertian norms on $\mathcal{P}(\mathcal{N}')$.

Definition 2.2.6. *Let $\phi = \sum_{n=0}^\infty \langle \phi^{(n)}, : \cdot^{\otimes n} : \rangle \in \mathcal{P}(\mathcal{N}')$. Then we define the Hilbertian norm $\| \cdot \|_{p,q,\beta}$ on $\mathcal{P}(\mathcal{N}')$ by*

$$\|\phi\|_{p,q,\beta}^2 := \sum_{n=0}^\infty (n!)^{1+\beta} 2^{nq} \left|\phi^{(n)}\right|_p^2,$$

for $p, q \in \mathbb{Z}$ and $-1 \leq \beta \leq 1$. We denote the completion of $\mathcal{P}(\mathcal{N}')$ w.r.t $\| \cdot \|_{p,q,\beta}$ by $(\mathcal{H})_{p,q}^\beta$. The spaces $(\mathcal{H})_{p,q}^\beta$ are Hilbert spaces with the inner product

$$(\!(\Psi, \Phi)\!) := \sum_{n=0}^\infty (n!)^{1+\beta} 2^{nq} (\psi^{(n)}, \phi^{(n)})_p,$$

with $\Psi = \sum_{n=0}^\infty \langle \psi^{(n)}, : \cdot^{\otimes n} : \rangle$ and $\Phi = \sum_{n=0}^\infty \langle \phi^{(n)}, : \cdot^{\otimes n} : \rangle \in \mathcal{P}(\mathcal{N}')$. In the following we also use the shorter notation

$$\| \cdot \|_p := \| \cdot \|_{p,p,0}.$$

Definition 2.2.7. *We define spaces of test functions $(\mathcal{N})^\beta$ as*

$$(\mathcal{N})^\beta := \operatorname*{proj\,lim}_{p,q \in \mathbb{N}} (\mathcal{H})_{p,q}^\beta,$$

for $0 \leq \beta \leq 1$*. By duality theory we define the corresponding generalized functions via the inductive limit, i.e.*

$$((\mathcal{N})^\beta)' := (\mathcal{N})^{-\beta} := \underset{p,q\in\mathbb{N}}{\mathrm{indlim}}(\mathcal{H})^\beta_{-p,-q}.$$

The following properties of the constructed test and generalized function spaces can be found e.g. in [87, 57, 40, 50]

Proposition 2.2.8. *(i)* $(\mathcal{N})^\beta$ *is nuclear for* $0 \leq \beta \leq 1$*.*

(ii) The topology on $(\mathcal{N}) := (\mathcal{N})^0$ *is uniquely determined by the topology on* $\mathcal{N}$*.*

Definition 2.2.9. *(i) In the case* $\beta = 0$ *the above mentioned construction gives the well-known triple of Hida test functions and distributions, i.e.*

$$(\mathcal{N}) \subset L^2(\mu) \subset (\mathcal{N})'.$$

For a different construction, see also [57] and the references therein.

(ii) The triple

$$(\mathcal{N})^1 \subset L^2(\mu) \subset (\mathcal{N})^{-1}$$

is called the Kondratiev triple of Kondratiev test and generalized functions. We will see in the characterization, that this triple gives the largest distribution space via this construction, see e.g.[87, 51].

Example 2.2.10. *(i) Consider again the case of vector-valued Schwartz test functions and tempered distributions. In this case the generalized function*

$$W_t := \langle \delta_t, \cdot \rangle \in (\mathcal{N})' := (S)',$$

where $\delta_t \in S'_d(\mathbb{R})$ *is the Dirac delta distribution.*
If we consider the parameter $t \in \mathbb{R}$ *as time,* W_t *gives in a weak sense the time derivative of a Brownian motion. The process* W_t *exists in a limit sense as a generalized function in* $(S)'$*. This can be seen by the characterization theorem for test and generalized functions in the following section.*

The next theorem is very important for the following chapters. It says that every test function has a continuous version. The theorem can be found in [64, Thm. 3.2.1, p.38].

Theorem 2.2.11 (Continuous version theorem)**.** *For each $\varphi \in (\mathcal{N})$ there exists a unique continuous function $\tilde{\varphi}$ on $\mathcal{N}'$ such that $\varphi(\omega) = \tilde{\varphi}(\omega)$ for $\mu-$a.e. $\omega \in \mathcal{N}'$. Moreover $\tilde{\varphi}$ is given by the absolutely continuous series*

$$\tilde{\varphi} = \sum_{n=0}^{\infty} \langle f_n, : \cdot^{\otimes n} : \rangle,$$

where $(f_n)_{n\in\mathbb{N}}$ are the corresponding kernels of φ.

2.2.3 Characterization of Test and Generalized Functions

The S-transform mentioned in the beginning of the chapter in Definition 2.2.3 can be used to characterize the generalized functions constructed in the subsection before. Therefore we have to make sure, that the function $: \exp(\langle \xi, \cdot \rangle) :$ for $\xi \in \mathcal{N}$ is an appropriate test function. Indeed we have

$$\| : \exp(\langle \xi, \cdot \rangle) : \|_{p,q,\beta}^2 = \sum_{n=0}^{\infty} (n!)^{1+\beta} 2^{nq} \left| \frac{1}{n!} \xi^{\otimes n} \right|_p^2 = \sum_{n=0}^{\infty} (n!)^{\beta - 1} 2^{nq} |\xi|_p^{2n} . \qquad (2.5)$$

The last expression in (2.5) is finite for all $\beta < 1$, thus the Wick exponential $: \exp(\langle \xi, \cdot \rangle) :\in (\mathcal{N})^{\beta}$. Then it is clear that a dual pairing between a generalized function from $(\mathcal{N})^{-\beta}$ and the Wick exponential with $\xi \in \mathcal{N}$ is well-defined. Therefore the S-transform extends naturally to elements from $(\mathcal{N})^{-\beta}$. I.e. , for $\Psi \in (\mathcal{N})^{-\beta}$ with kernels $\Psi^{(n)}$, $n \in \mathbb{N}$ we have

$$S(\Psi)(\xi) := \langle\langle : \exp(\langle \xi, \cdot \rangle) :, \Psi \rangle\rangle = \sum_{n=0}^{\infty} \langle \xi^{\otimes n}, \Psi^{(n)} \rangle,$$

for all $\xi \in \mathcal{N}$.

Remark 2.2.12. *In the special case of Kondratiev distributions, i.e. $\beta = 1$ the norm in (2.5) is finite if and only if $2^q |\xi|_p^2 < 1$. In this case the series is convergent and thus the dual pairing is finite. In this case therefore, we have that Kondratiev test functions fulfill $|\xi|_p^2 < 2^{-q}$, i.e. $: \exp(\langle \xi, \cdot) :\in (\mathcal{H})_{p,q}^1$ with $|\xi|_p^2 < 2^{-q}$.*
Since every distribution is of finite order, for all $\Psi \in (\mathcal{N})^{-1}$ there exist some $p, q \in \mathbb{N}$ such that $\Phi \in (\mathcal{H})_{-p,-q}^{-1}$. Then we can define the S-transform of $\Psi \in (\mathcal{N})^{-1}$ for all $\xi \in \mathcal{N}$ with

$2^q \left|\xi\right|_p < 1$. *Then we have*

$$S(\Psi)(\xi) := \langle\!\langle :\exp(\langle \xi, \cdot \rangle) :, \Psi \rangle\!\rangle = \sum_{n=0}^{\infty} \langle \xi^{\otimes n}, \Psi^{(n)} \rangle.$$

In this case the S-transform is defined for the set

$$\mathcal{U}_{p,q} := \{\theta \,|\, |\theta|_p^2 < 2^{-q}\}\, p, q \in \mathbb{N},$$

which is a neighborhood of zero.

By this remark, we see why the case $\beta = 1$ is extremal. For even larger β the above set is equal to zero, thus the S-transform can just be defined for $\xi = 0$. The Wick exponentials are in this case no longer in $(\mathcal{H})^\beta_{p,q}$.

Another transformation which is useful for the characterization of generalized functions is the so called T-transform, which is an analogue to the Fourier transform, see e.g. [40, 64, 57] .

Definition 2.2.13. *We define the T-transform of $\Psi \in (\mathcal{N})^{-\beta}$ for $0 \leq \beta < 1$ by*

$$T\Psi(\xi) := \langle\!\langle \exp(i\langle \xi, \cdot \rangle), \Psi \rangle\!\rangle, \quad \xi \in \mathcal{N}.$$

Remark 2.2.14. *(i) Since $\exp(i\langle \xi, \cdot \rangle) \in (\mathcal{N})^\beta$ for all $0 \leq \beta < 1$, $\xi \in \mathcal{N}$, the T-transform is well-defined, see e.g. [40, 57].*

(ii) For $\xi = 0$ the above expression yields $\langle\!\langle \Psi, 1\!\!1 \rangle\!\rangle$, thus $T\Psi(0)$ is the generalized expectation of $\Psi \in (\mathcal{N})^{-\beta}$.

(iii) For $\xi \in \mathcal{N}$ and $\Psi \in (\mathcal{N})^{-\beta}$ we have

$$T(\Psi)(\xi) = \exp(-\frac{1}{2}\langle \xi, \xi \rangle)(S(\Psi)(i\xi)).$$

The T- and S-transform enable us to characterize Hida distributions by entire functions. These functionals are called U-functionals, see also [40, 67, 57, 64, 87]

Definition 2.2.15. *A mapping $F : \mathcal{N} \to \mathbb{C}$ is called a* U*-functional if it satisfies the following conditions:*

*U1. For all $\xi, \eta \in \mathcal{N}$ the mapping $\mathbb{R} \ni \lambda \mapsto F(\lambda\xi + \eta) \in \mathbb{C}$ has an analytic continuation to $\lambda \in \mathbb{C}$ (***ray analyticity***).*

U2. There exist constants $0 < K, C < \infty$ and a $p \in \mathbb{N}$ such that

$$|F(z\xi)| \leq K \exp(C|z|^2\|\xi\|_p^2),$$

*for all $z \in \mathbb{C}$ and $\xi \in \mathcal{N}$ (***growth condition***).*

This is the basis of the following characterization theorem. For the proof we refer to [67, 49, 40, 50].

Theorem 2.2.16. *A mapping $F : \mathcal{N} \to \mathbb{C}$ is the T-transform (or S-transform) of an element in $(\mathcal{N})'$ if and only if it is a U-functional.*

Since by Theorem 2.2.16 the product of two U-functionals is still a U-functional, the following definitions make sense as a well-defined objects in $(\mathcal{N})'$. Compare to [57, Chap.7].

Definition 2.2.17. *Let $\Psi, \Phi \in (\mathcal{N})'$.*

(i) We define the Wick-product $\diamond : (\mathcal{N})' \times (\mathcal{N})' \to (\mathcal{N})'$ by

$$\Phi \diamond \Psi = S^{-1}(S(\Phi) \cdot S(\Psi)).$$

(ii) We define the convolution $* : (\mathcal{N})' \times (\mathcal{N})' \to (\mathcal{N})'$ by

$$\Phi * \Psi = T^{-1}(T(\Phi) \cdot T(\Psi)).$$

Theorem 2.2.16 enables us to discuss convergence of sequences of Hida distributions by considering the corresponding T-transforms (or S-transforms), i.e. by considering convergence on the level of U-functionals. The following corollary is proved in [67, 40, 50].

Corollary 2.2.18. *Let $(\Phi_n)_{n\in\mathbb{N}}$ denote a sequence in $(\mathcal{N})'$ such that*

(i) For all $\xi \in \mathcal{N}$, $((T\Phi_n)(\xi))_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{C}$.

(ii) There exist constants $0 < C, D < \infty$ such that for some $p \in \mathbb{N}$ one has

$$|(T\Phi_n)(z\xi)| \leq D \exp(C|z|^2\|\xi\|_p^2)$$

for all $\xi \in \mathcal{N}$, $z \in \mathbb{C}$, $n \in \mathbb{N}$.

Then $(\Phi_n)_{n\in\mathbb{N}}$ converges strongly in $(\mathcal{N})'$ to a unique Hida distribution.
This theorem also holds for the S-transform.

Based on the above theorem, we introduce the following Hida distribution.

Example 2.2.19 (Normalized exponential)**.** *Consider the formal expression*

$$J_c = \exp\left(c \cdot \langle \omega, \omega\rangle\right), \quad \omega \in \mathcal{N}', c \in \mathbb{C} \setminus \{\frac{1}{2}\}.$$

As it stands, the expression makes no mathematical sense, since the dual paining of two generalized functions is not defined. We will try to give a meaning to J_c as a Hida distribution with the help of Corollary 2.2.18. For this, we use Parseval equation to obtain the formal equality

$$c \cdot \langle \omega, \omega\rangle = c\sum_{n=0}^{\infty}(\langle e_n, \omega\rangle)^2,$$

where $(e_n)_n \in \mathcal{N}$ is a complete orthonormal system of the separable Hilbert space $\mathcal{H}$. Now we define

$$\Phi_N := \exp\left(c\sum_{n=0}^{N}(\langle e_n, \omega\rangle)^2\right).$$

Then for $\xi \in \mathcal{N}$ we obtain for the S-transform of Φ_N:

$$\begin{aligned}
S(\Phi_N)(\xi) &= \mathbb{E}_\mu\left(\exp\left(c\sum_{n=0}^{N}(\langle e_n, \omega\rangle)^2\right)\cdot \exp(\langle \xi, \omega\rangle)\cdot\exp(-\frac{1}{2}\langle \xi,\xi\rangle)\right)\\
&= \mathbb{E}_\mu\left(\exp\left(c\sum_{n=0}^{N}(\langle e_n, \omega\rangle)^2\right)\cdot \exp(\sum_{n=0}^{N}\langle \xi, e_n\rangle\cdot\langle e_n,\omega\rangle)\cdot\exp(-\frac{1}{2}\sum_{n=0}^{N}(\langle e_n,\xi\rangle)^2)\right)\\
&\qquad \times \mathbb{E}_\mu\left(\exp(\langle \xi_\perp, \omega\rangle)\cdot \exp(-\frac{1}{2}(\langle \xi_\perp, \xi_\perp\rangle)\right)\\
&= \mathbb{E}_\mu\left(\exp\big(c\sum_{n=0}^{N}(\langle e_n, \omega\rangle)^2\big)\cdot \exp\big(\sum_{n=0}^{N}\langle \xi, e_n\rangle\cdot\langle e_n,\omega\rangle\big)\cdot\exp\big(-\frac{1}{2}\sum_{n=0}^{N}(\langle e_n,\xi\rangle)^2\big)\right)\\
&= \prod_{n=0}^{N}\mathbb{E}_\mu\left(\exp(c\langle e_n,\omega\rangle^2 + \langle \xi, e_n\rangle\cdot\langle e_n,\omega\rangle - \frac{1}{2}(\langle e_n,\xi\rangle)^2)\right)\\
&= \prod_{n=0}^{N}\left(\frac{1}{\sqrt{2c-1}}\right)\exp\left(\frac{c}{1-2c}\sum_{n=0}^{N}\langle e_n,\xi\rangle^2\right)
\end{aligned}$$

Since the prefactor in front either converges to zero or diverges, we renormalize the expres-

sion and define

$$S\Phi_{ren}(\xi) = \lim_{N\to\infty} \frac{S\Phi^N(f)}{\mathbb{E}(\Phi^N)} = \lim_{N\to\infty} \exp\big(\frac{c}{1-2c}\sum_{n=0}^{N}\langle e_n, \xi\rangle^2\big)$$

$$= \exp(\frac{c}{1-2c}\langle \xi, \xi\rangle) = S\Big(\mathrm{Nexp}\,(c\cdot\langle\omega,\omega\rangle)\Big)(\xi).$$

We call this expression the normalized exponential and denote it by J_c or Nexp. *Is it obvious that the S-transform is a U-functional and hence* $\mathrm{Nexp} \in (\mathcal{N})'$. *Here we just used complex numbers c inside the exponential. We will generalize this to bounded operators in the last section of this chapter.*

Remark 2.2.20. *Normalized exponentials of the kind sketched in this example belong to the special class of Generalized Gauss kernels. Indeed we have*

$$T(\mathrm{Nexp}\,(c\cdot\langle\omega,\omega\rangle)\,(\xi) = T(J_c)(\xi)\exp(-\frac{1}{(1-2c)}|\xi|^2), \quad c \in (-\infty, \frac{1}{2})$$

which corresponds to the characteristic function of a Gaussian measure μ_c with covariance kernel $\frac{1}{1-2c}$. The normalized exponential plays therefore the role of a generalized Radon-Nikodym derivative of the measure μ_c w.r.t. the standard Gaussian measure μ, see e.g. [87].

Another useful corollary of Theorem 2.2.16 concerns integration of a family of generalized functions, see [67, 40, 50].

Corollary 2.2.21. *Let $(\Lambda, \mathcal{A}, \nu)$ be a measure space and $\Lambda \ni \lambda \mapsto \Phi(\lambda) \in (\mathcal{N})^{-\beta}$ a mapping. We assume that its T–transform $T\Phi$ satisfies the following conditions:*

(i) The mapping $\lambda \mapsto T(\Phi(\lambda))(\xi)$ is measurable for all $\xi \in \mathcal{N}$.

(ii) There exists a $p \in \mathbb{N}$ and functions $C \in L^\infty(\Lambda, \nu)$ and $D \in L^1(\Lambda, \nu)$ such that

$$|T(\Phi(\lambda))(z\xi)| \le D(\lambda)\exp(C(\lambda)\,|z|^2\,\|\xi\|_p^2),$$

for a.e. $\lambda \in \Lambda$ and for all $\xi \in \mathcal{N}$, $z \in \mathbb{C}$.

Then, in the sense of Bochner integration in $\mathcal{H}_{-p,-q,-\beta} \subset (\mathcal{N})^{-\beta}$ for a suitable $q \in \mathbb{N}$, the integral of the family of Hida distributions is itself a Hida distribution, i.e. $\int_\Lambda \Phi(\lambda)\,d\nu(\lambda) \in$

$(\mathcal{N})^{-\beta}$ *and the* T*–transform interchanges with integration, i.e.*

$$T\left(\int_\Lambda \Phi(\lambda)\, d\nu(\lambda)\right) = \int_\Lambda T(\Phi(\lambda))\, d\nu(\lambda).$$

This theorem also holds for the S*-transform.*

Based on the above theorem, we introduce the following Hida distribution.

Definition 2.2.22. *We define Donsker's delta at* $x \in \mathbb{R}$ *corresponding to* $0 \neq \eta \in \mathcal{H}$ *by*

$$\delta_x(\langle \eta, \cdot \rangle) := \frac{1}{2\pi} \int_{\mathbb{R}} \exp(i\lambda(\langle \eta, \cdot \rangle - x))\, d\lambda \tag{2.6}$$

in the sense of Bochner integration, see e.g. [40, 59, 87]. Its T*–transform in* $\xi \in \mathcal{N}$ *is given by*

$$T(\delta_x(\langle \eta, \cdot \rangle)(\xi) = \frac{1}{\sqrt{2\pi\langle \eta, \eta\rangle}} \exp\left(-\frac{1}{2\langle \eta, \eta\rangle}(i\,\langle \eta, \xi\rangle - x)^2 - \frac{1}{2}\langle \xi, \xi\rangle\right) . \tag{2.7}$$

Remark 2.2.23. *The Donsker's delta function serves to pin processes in a given point. Consider the vector-valued White Noise triple*

$$S_d(\mathbb{R}) \subset L^2_d(\mathbb{R}, dx) \subset S'_d(\mathbb{R}),$$

then the Donsker's delta function at 1 *corresponding to* $(\mathbb{1}_{[0,t)}, 0, \ldots 0)$*, serves to 'pin' Brownian motion in* 1 *at time* $0 < t < \infty$*.*
If one considers equation (2.6) *one can formally see Donsker's delta function as the composition of the Dirac delta distribution and the process* $\langle \eta, \cdot \rangle$ *or as above a Brownian motion. Clearly, as can be seen in* (2.7)*, Donskers delta function is a Hida distribution.*
The S*-transform of Donsker's delta function at* $x \in \mathbb{R}$ *corresponding to* $0 \neq \eta \in \mathcal{H}$ *is given by*

$$S(\delta_x(\langle \eta, \cdot \rangle))(\xi) = \frac{1}{\sqrt{2\pi(\eta, \eta)}} \exp\left(-\frac{1}{2\langle \eta, \eta\rangle}\big(\langle \eta, \xi\rangle - x\big)^2\right),$$

for all $\xi \in \mathcal{N}$*.*

If we expand the S*-transform in terms of Hermite polynomials we obtain as in [87],*

$$S(\delta_x(\langle \eta, \cdot \rangle))(\xi) = \frac{1}{\sqrt{2\pi\langle \eta, \eta\rangle}} \exp\left(-\frac{1}{2\langle \eta, \eta\rangle}\big(\langle \eta, \xi\rangle - x\big)^2\right)$$

$$= \frac{1}{\sqrt{2\pi\langle\eta,\eta\rangle}} \exp\left(-\frac{x^2}{2\langle\eta,\eta\rangle}\right) \sum_{n=0}^{\infty} \frac{1}{n!} H_n\left(\frac{x}{\sqrt{2\langle\eta,\eta\rangle}}\right) (2\langle\eta,\eta\rangle)^{-\frac{n}{2}} \langle\eta^{\otimes n}, \xi^{\otimes n}\rangle.$$

Hence, its chaos decomposition is given by

$$\delta_x(\langle\eta,\cdot\rangle) = \sum_{n=0}^{\infty} \langle f^{(n)}, :\cdot^{\otimes n}:\rangle,$$

where the kernels $f^{(n)}$ are given by

$$f^{(n)} = \frac{1}{n!\sqrt{2\pi\langle\eta,\eta\rangle}} \exp(-\frac{x^2}{2\langle\eta,\eta\rangle} H_n\left(\frac{x}{\sqrt{2\langle\eta,\eta\rangle}}\right) (2\langle\eta,\eta\rangle)^{-\frac{n}{2}} \eta^{\otimes n}.$$

2.2.4 The spaces $\mathcal{M}$ and $\mathcal{G}$

Next we introduce spaces which have the property that the kernels in the chaos decomposition are from the Hilbert space. Thus the test functions are more singular and the generalized functions are more regular than in the Hida triple. The spaces $\mathcal{G}$ and $\mathcal{G}'$ first were introduced by Potthoff and Timpel [69]. There are various applications from the probabilistic point of view as SPDE's and martingale properties, one can associate with these spaces, see [31, 30, 30]. Additionally Westerkamp introduced the spaces $\mathcal{M}$ and $\mathcal{M}'$, see [87] which leads to the chain

$$\mathcal{G} \subset \mathcal{M} \subset L^2(\mu) \subset \mathcal{M}' \subset \mathcal{G}'.$$

Consider the following norms for $q \in \mathbb{Z}$, $\beta \in [0,1]$

$$\|\varphi\|^2_{0,q,\beta} = \sum_{n=0}^{\infty} (n!)^{1+\beta} 2^{nq} |\varphi^{(n)}|^2_0,$$

for $\varphi = \sum_{n=0}^{N} \langle\varphi^{(n)}, :\cdot^{\otimes n}:\rangle$. We define

$$\mathcal{G}^\beta_q := \left\{ f \in L^2(\mu) \middle| f = \sum_{n=0}^{\infty} \langle f^{(n)}, :\cdot^{\otimes n}:\rangle,\ \|f\|_{0,q,\beta} < \infty \right\}.$$

Definition 2.2.24. *We define*

$$\mathcal{G}^\beta := \operatorname{proj}\lim_{q>0} (\mathcal{G})^\beta_q,$$

and call $\mathcal{G}^\beta$ the space of regular test functions. Moreover we define

$$\mathcal{M}^\beta := \operatorname*{indlim}_{q>0} (\mathcal{G})_q^\beta.$$

Note that $\mathcal{G}^\beta$ is a Fréchet space and both spaces are continuously embedded in $L^2(\mu)$, see also [87]. Moreover we have, that both spaces are **not** *nuclear.*
Furthermore we define the spaces of regular generalized functions

$$\mathcal{G}^{-\beta} := (\mathcal{G}^\beta)' := \operatorname*{indlim}_{q<0} (\mathcal{G})_q^\beta,$$

and

$$\mathcal{M}^{-\beta} := \operatorname*{proj\,lim}_{q<0} (\mathcal{G})_q^\beta.$$

Note that $\mathcal{M}^{-\beta}$ is a Fréchet space.

We end, for a fixed β with the chain

$$\mathcal{G}^\beta \subset \mathcal{M}^\beta \subset L^2(\mu) \subset \mathcal{M}^{-\beta} \subset \mathcal{G}^{-\beta}.$$

Moreover

$$\mathcal{G}^1 \subset \mathcal{G}^\beta \subset \mathcal{G} \subset L^2(\mu) \subset \mathcal{G}' \subset \mathcal{G}^{-\beta} \subset \mathcal{G}^{-1},$$

with $\mathcal{G} = \mathcal{G}^0$.

The next proposition is proved in [87], see also [69].

Proposition 2.2.25. *(i) Let $\varphi \in L^p(\mu)$ for $p > 1$ then $\varphi \in \mathcal{G}'$,i.e.*

$$\bigcup_{p>1} L^p(\mu) \subset \mathcal{G}'.$$

(ii) Let $\varphi \in L^p(\mu)$ for all $1 < p < 2$ then $\varphi \in \mathcal{M}'$, i.e.

$$\bigcap_{1<p<2} L^p(\mu) \subset \mathcal{M}'.$$

Compare to [86, 30], that one has for $\beta \in [-1, 1]$, $q \in \mathbb{Z}$

$$\mathcal{G}_q^\beta = \Big\{ \Phi = \sum_{n=0}^{\infty} \langle \Phi^{(n)}, :\cdot^{\otimes n}: \rangle \Big| \Phi^{(n)} \in \mathcal{H}^{\hat{\otimes} n}, \|\Phi\|_{q,\beta} < \infty \Big\}.$$

This shows that elements from $\mathcal{G}_q^\beta$ (hence also from $\mathcal{M}^\beta$ and $\mathcal{G}^\beta$) have the property, that the kernels in the Wiener-Itô-Segal chaos decomposition are elements from the symmetric tensor powers of the underlying Hilbert-space $\mathcal{H}_{\mathbb{C}}^{\hat{\otimes} n}$. The generalized functions are therefore called regular generalized functions. For the characterization of the spaces $\mathcal{G}$ and $\mathcal{G}'$ via the Bargmann-Segal isomorphism we refer to [30] and [32].

2.3 Donsker's Delta Function

This section is dedicated to Donsker's Delta function. We introduced this Hida distribution already in Definition 2.2.22. It serves to pin paths of processes at certain endpoints and gives therefore rise of various applications as Feynman integrals and polymer physics, see e.g.[59]. One can also describe the local time of a stochastic process with help of Donsker's Delta function by the so-called Takenaka formula see e.g. [57]. Many authors studied properties and applications of Donsker's Delta function, see e.g. [59, 87, 40] and the references therein. The generalized expectation of Donsker's Delta function is the heat kernel. Therefore also a complex-scaled Donsker's Delta function is of interest, since it formally gives the Schrödinger kernel. We will focus this in chapter 4.

2.3.1 Properties of Donsker's Delta Function

Consider again the S-transform of Donsker's Delta function corresponding to $\eta \in \mathcal{H}$ at $x \in \mathbb{R}$

$$S(\delta_x(\langle \eta, \cdot \rangle))(\xi) = \frac{1}{\sqrt{2\pi \langle \eta, \eta \rangle}} \exp(-\frac{1}{2\langle \eta, \eta \rangle} (x - \langle \xi, \eta \rangle)^2), \quad \xi \in \mathcal{N}.$$

Donsker's delta function is an element in $(\mathcal{N})'$, as we see by the above S-transform. Here we used the representation by Pettis integrals.

Example 2.3.1. *Let $\eta \in \mathcal{H}_{\mathbb{C}}, \Re(\langle \eta, \eta \rangle) > 0,\ x \in \mathbb{C}$, then $\int_{\mathbb{R}} \exp(i(\langle \eta, \cdot \rangle - x)s)\, ds$ is a well defined Pettis integral and*

$$S\left(\frac{1}{2\pi} \int_{\mathbb{R}} \exp(i(\langle \eta, \cdot \rangle - x)s)\, ds\right)(\xi) = \frac{1}{\sqrt{2\pi \langle \eta, \eta \rangle}} \exp(-\frac{1}{2} \frac{1}{\langle \eta, \eta \rangle} (\langle \xi, \eta \rangle - x)^2), \quad \xi \in \mathcal{N}_{\mathbb{C}}.$$

The Pettis integral $\frac{1}{2\pi}\int\limits_{\mathbb{R}} \exp(i(\langle \eta, \cdot\rangle - x)s)ds$ *is a Donsker's delta function.*

Proof. By [64, Corollary 3.3.8, p.51] it follows, that $\exp(i(\langle \eta, \cdot\rangle - x)t) \in (\mathcal{N})'$ for all $t \in \mathbb{R}$. We have

$$\begin{aligned}\exp(i(\langle \eta, \cdot\rangle - x)t) &= e^{-ixt} \cdot \exp(i\langle t\eta, \cdot\rangle) \\ &= \exp(-ixt - t^2 \frac{\langle \eta, \eta\rangle}{2}) \cdot :\exp(\langle it\eta, \cdot\rangle):\end{aligned}$$

Now let $\xi \in \mathcal{N}_{\mathbb{C}}$, then

$$\langle\langle :\exp(\langle \xi, \cdot\rangle):, \exp(i(\langle \eta, \cdot\rangle - x)t)\rangle\rangle = e^{-ixt - t^2\frac{\langle \eta,\eta\rangle}{2}} \cdot e^{it\langle \eta, \xi\rangle}. \tag{2.8}$$

Thus $t \mapsto \langle\langle \Psi_t, \Phi_\xi\rangle\rangle$ is Lebesgue measurable.
Furthermore, since for $K > 0$: $|t\langle \xi, \eta\rangle| = \left|\langle \sqrt{K}\xi, \frac{1}{\sqrt{K}} t\eta\rangle\right| \leq \frac{1}{2}\left(\frac{1}{K}t^2 |\eta|_0^2 + K|\xi|_0^2\right)$ we have

$$\begin{aligned}|S(\exp(i(\langle \eta, \cdot\rangle - x)t))(\xi)| \leq\ & e^{|xt|} \cdot e^{-t^2 \frac{Re(\langle \eta,\eta\rangle)}{2}} \cdot e^{t^2 \frac{|\eta|_0^2}{2K}} \cdot e^{K|\xi|_0^2} \\ \leq\ & e^{|xt|} \cdot e^{-t^2 \cdot c(K)} \cdot e^{K|\xi|_0^2},\end{aligned}$$

since $\Re(\langle \eta, \eta\rangle) > 0$ and for some suitable $c(K) > 0$, (for $K \to \infty$).
Since $t \mapsto e^{|xt|} \cdot e^{-c(K)t^2}$ in $L^1(\mathbb{R}, dt, \mathbb{R})$ fulfill the conditions of Corollary 2.2.21 the first assertion is proved.
To prove the second statement we substitute $u = s \cdot \sqrt{\langle \eta, \eta\rangle}$:

$$\begin{aligned}\langle\langle :\exp(\langle \xi, \cdot\rangle):, \frac{1}{2\pi}\int\limits_{\mathbb{R}} \exp(i(\langle \eta, \cdot\rangle - x)s\ ds\rangle\rangle &= \frac{1}{2\pi}\int\limits_{\mathbb{R}} e^{-i(x - \langle \eta, \xi\rangle)s} \cdot e^{\frac{-s^2\langle \eta,\eta\rangle}{2}}\ ds \\ &= \frac{1}{2\pi\sqrt{\langle \eta, \eta\rangle}}\int\limits_{\mathbb{R}} e^{-i\frac{(x-\langle \eta,\xi\rangle)}{\sqrt{\langle \eta,\eta\rangle}}\cdot u} \cdot e^{\frac{-u^2}{2}}\ du \\ &= \frac{1}{\sqrt{2\pi\langle \eta, \eta\rangle}}\int\limits_{\mathbb{R}} e^{-i\left(\frac{x-\langle \eta,\xi\rangle}{\sqrt{\langle \eta,\eta\rangle}}\right)u} \cdot \frac{e^{\frac{-u^2}{2}}}{\sqrt{2\pi}}\ du \\ &= \frac{1}{\sqrt{2\pi\langle \eta, \eta\rangle}} e^{-\frac{1}{2}\left(\frac{x-\langle \eta,\xi\rangle}{\sqrt{\langle \eta,\eta\rangle}}\right)^2},\end{aligned}$$

$\square$

Remark 2.3.2. *Note that the condition*

$$\eta \in \mathcal{H}_{\mathbb{C}} \text{ and } \Re(\langle \eta, \eta \rangle) > 0$$

in 2.3.1 is equivalent to the condition

$$\langle \eta, \eta \rangle \neq 0 \text{ and } \left(|\arg(\langle \eta, \eta \rangle)| < \frac{\pi}{4} \text{ or } \arg(\langle \eta, \eta \rangle) \in (\frac{3}{4}\pi, \frac{5}{4}\pi) \right).$$

Now we give a generalization of Donsker's delta function, which is quite standard, but it is necessary to give some comments to the domain of definition.

Definition 2.3.3. *Let $\eta \in \mathcal{H}_{\mathbb{C}}$, $\langle \eta, \eta \rangle \neq 0$ and $x \in \mathbb{C}$. The generalized function $\delta_x(\langle \eta, \cdot \rangle)$, which is defined via*

$$S(\delta_x(\langle \eta, \cdot \rangle))(\xi) := \frac{1}{\sqrt{2\pi \langle \eta, \eta \rangle}} \exp(-\frac{1}{2} \frac{1}{\langle \eta, \eta \rangle} (\langle \eta, \xi \rangle - x)^2), \quad \xi \in \mathcal{N}_{\mathbb{C}} :$$

is called a ***Donsker's delta function***.

In [87, Theorem 90, p. 77] it is stated, that $\delta_x(\langle \eta, \cdot \rangle) \in \mathcal{M}' := \bigcap_{p>0} \mathcal{G}_{-p}$. **This assertion is wrong**. As a counterexample let $\eta = e_1 + i\frac{1}{2} e_2$, where $\{e_1, e_2\}$ is an orthonormal System in $L^2(\mathbb{R}, dt, \mathbb{R})$. This we show in Example 2.3.4. Following the proof in [87, Theorem 90, p. 77], we correct the above statement as follows:

$$\delta_x(\langle \eta, \cdot \rangle) \in \bigcap_{p > log_2\left(\frac{|\eta|_0^2}{|\langle \eta, \eta \rangle|}\right)} \mathcal{G}_{-p} \subset \mathcal{G}'.$$

As a consequence, the assertion of Westerkamp is true for real η, but if η is a complex function, we can only say that $\delta_x(\langle \eta, \cdot \rangle) \in \mathcal{G}'$.

Recall that

$$\mathcal{M}' = \operatorname{proj} \lim_{p>0} \mathcal{G}_{-p}$$

is reflexive as countably Hilbert space with dual space

$$\mathcal{M} = \operatorname{indlim}_{p>0} \mathcal{G}_p.$$

Example 2.3.4. *Let $\eta = e_1 + i\frac{1}{2}e_2$, where $\{e_1, e_2\}$ are orthonormal vectors in $L^2(\mathbb{R}, dt, \mathbb{R})$. Then*

$$\textit{for all } \alpha \in \mathbb{R}: \ \delta_0(\langle \eta, \cdot \rangle) \in \mathcal{G}_\alpha \Leftrightarrow \alpha < log_2(\frac{3}{5})$$

Proof. First note that $\langle \eta, \eta \rangle = \frac{3}{4}$, such that $\langle \eta, \eta \rangle \neq 0$.
Then for arbitrary $\alpha \in \mathbb{R}$:

$$\begin{aligned} |\delta(\langle \eta, \cdot \rangle)|_\alpha^2 &= \sqrt{\frac{2}{3\pi}} \cdot \sum_{n=0}^{\infty} (2n)! \cdot 2^{2n\alpha} \left(\frac{1}{n!}\right)^2 \cdot \frac{1}{2^{2n}} \left|\langle \eta, \eta \rangle\right|^{-2n} \cdot |\eta|_0^{4n} \\ &= \sqrt{\frac{2}{3\pi}} \cdot \sum_{n=0}^{\infty} \frac{(2n)!}{2^{2n}\,(n!)^2} \cdot \left(\frac{2^{2\alpha}\,|\eta|_0^4}{|\langle \eta, \eta \rangle|^2}\right)^n, \end{aligned}$$

which converges if and only if $\left(\frac{2^\alpha |\eta|_0^2}{|\langle \eta, \eta \rangle|}\right) < 1$. □

Now we want to give as an example the calculation of a product of Donsker's Delta functions compare [59].

Example 2.3.5 (Products of Donsker's Delta functions)**.** *In [59] the authors used the following ansatz for a product of Donsker's Delta functions*

$$\prod_{n=1}^{N} \delta_{x_n}(\langle \eta_n, \cdot \rangle) = \prod_{n=1}^{N} \frac{1}{2\pi} \int_{\mathbb{R}} \exp(i\lambda_n \langle \eta_n, \cdot \rangle - \lambda_n x_n)\, d\lambda_n,$$

where $(\eta_n)_n$ is a linear independent family in $\mathcal{H}$ and $x_n \in \mathbb{C}, n = 1, \cdots N$.
Then the S-transform of $\prod_{n=1}^{N} \delta_{x_n}(\langle \eta_n, \cdot \rangle)$ is given by

$$\begin{aligned} S(\prod_{n=1}^{N} \delta_{x_n}(\langle \eta_n, \cdot \rangle))(\xi) &= \frac{1}{\sqrt{(2\pi)^n \det((\langle \eta_i, \eta_j \rangle)_{i,j})}} \\ &\times \exp\left(-\frac{1}{2}\Big(\langle \xi, \eta_1 \rangle - x_1, \ldots, \langle \xi, \eta_n \rangle - x_n\Big), \Big((\langle \eta_i, \eta_j \rangle)_{i,j}\Big)^{-1} \begin{pmatrix} \langle \xi, \eta_1 \rangle - x_1 \\ \cdots \\ \langle \xi, \eta_n \rangle - x_n \end{pmatrix}\right). \end{aligned}$$

In the case of vector-valued White Noise, i.e. $\mathcal{N} = S_d(\mathbb{R})$ we use the above product form to define Donsker's Delta function for a vector valued process by a product of the Donsker's Delta functions in the certain coordinates.

Let $\mathcal{N} = S_3(\mathbb{R})$. Recall that a three dimensional Brownian motion can be described as a vector of three independent Brownian motions. Then we have

$$\delta\left(\left(\langle \mathbb{1}_{[0,t)}, \cdot_1\rangle, \langle \mathbb{1}_{[0,t)}, \cdot_2, \rangle, \langle \mathbb{1}_{[0,t)}), \cdot_3\rangle\right) - \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}\right) := \prod_{n=1}^{3} \delta\left(\langle \mathbb{1}_{[0,t)}, \cdot_n\rangle - a_n\right),$$

for $a_1, a_2, a_3 \in \mathbb{C}$. The assumptions of the Theorem in [59] are clearly fulfilled, since the orthogonality of the coordinate processes guarantee the linear independence of the family in $\mathcal{H}^d$.

2.3.2 Donsker's Delta Function as Positive Generalized Function

For the following definition see [90, Definition 5.1, p. 150].

Definition 2.3.6. *A generalized white noise function $\Psi \in (S)'$ is said to be positive, denoted by $\Psi \geq 0$, if $\langle\langle \phi, \Psi\rangle\rangle \geq 0$ for all $\phi \in (S)$ such that $\phi(x) \geq 0$ at every point $x \in S'$. Hereby ϕ is identified with it's* ***continuous*** *version.*

The following theorem is due to Yokoi, see [90, Theorem 5.1, p. 151] and [90, Theorem 5.2, p. 152]. It is based on the continuity of the pointwise multiplication on $(\mathcal{N})$, see [64, Theorem 3.5.6, p. 61], and the theorem of Bochner and Minlos.

Theorem 2.3.7. *Suppose $\Psi \in (\mathcal{N})'$. Then the following statements are equivalent:*

(i) Ψ is a positive generalized White noise function.

(ii) The T-transform of Ψ is a positive definite function of $\xi \in \mathcal{N}$.

(iii) There exists a unique finite positive Borel measure ν on $\mathcal{N}'$, called a ***positive Hida measure****, such that*

(a) $(\mathcal{N}) \subset L^2(\mathcal{N}', d\nu, \mathbb{C})$

(b) $\forall \phi \in (\mathcal{N}) :\ \langle\langle \varphi, \Psi\rangle\rangle = \int_{\mathcal{N}'} \phi \, d\nu$

Hereby ϕ is identified with it's ***continuous*** *version.*

Proposition 2.3.8. *Let $\Psi \in (\mathcal{N})'$ be a positive generalized White noise function. Then the mapping*

$$[\cdot]_{\nu_\Psi} :\ (\mathcal{N}) \to L^2(\mathcal{N}', d\nu_\Psi, \mathbb{C})$$

is continuous. Hereby for $\phi \in (\mathcal{N})$ the expression $[\phi]_{\nu_\Psi}$ denotes the function class of the continuous version of ϕ in $L^2(\mathcal{N}', d\nu_\Psi, \mathbb{C})$.

Proof. Let $f \in (\mathcal{N})$. Note that by 2.3.7 we have $f \in L^2(\mathcal{N}', d\nu_\Psi, \mathbb{C})$. By the continuity of the pointwise multiplication on $(\mathcal{N})$ it follows $|f|^2 = f \cdot \overline{f} \in (\mathcal{N})$. Then

$$\int_{\mathcal{N}'} |f(x)|^2 \, d\nu_\Psi(x) = \langle\!\langle |f(\cdot)|^2, \Psi \rangle\!\rangle \leq \left|(f \cdot \overline{f})\right|_{p,p},$$

for some $p > 0$, by the continuity of Ψ on $(\mathcal{N})$. Moreover we have

$$\left|(f \cdot \overline{f})\right|_{p,p} \leq |f|_{p+q,p+q} \cdot |f|_{p+q,p+q},$$

for some $q > 0$, since the pointwise multiplication on $(\mathcal{N})$ is continuous. □

In order to investigate the convolution of two positive generalized white noise functions the definition of the convolution of two measures is useful.

Definition 2.3.9. *Let μ, ν be two measures on $\mathcal{N}'$. Then the image measure of $\mu \times \nu$ under the mapping $+: \mathcal{N}' \times \mathcal{N}' \to \mathcal{N}'$ is called the convolution of μ and ν and denoted by $\mu * \nu$.*

Note that the mapping $+: \mathcal{N}' \times \mathcal{N}' \to \mathcal{N}'$ is continuous and hence Borel-measurable. Further, if μ, ν are two finite positive Borel measures on $\mathcal{N}'$, so does $\mu * \nu$.

Theorem 2.3.10. *Let $\Phi, \Psi \in (\mathcal{N})'$ be positive generalized white noise functions and ν_Φ, ν_Ψ be the corresponding positive Hida measures. Then*

$$\forall \varphi \in (\mathcal{N}) : \ \Phi * \Psi(\varphi) = \int_{\mathcal{N}'} \varphi \, d\nu_\Phi * \nu_\Psi \tag{2.9}$$

Proof. By definition we have $T(\Phi * \Psi) = T(\Psi) \cdot T(\Phi)$. Since the product of two positive definite functions is positive definite, we have $\Phi * \Psi$ is a positive generalized white noise function. Then for all $\xi \in \mathcal{N}$:

$$\begin{aligned} T(\Phi * \Psi)(\xi) = & \ T(\Phi)(\xi) \cdot T(\Psi)(\xi) \\ & = \Big(\int_{\mathcal{N}'} e^{i\langle \xi, \rangle} \, d\nu_\Phi(x)\Big) \cdot \Big(\int_{\mathcal{N}'} e^{i\langle \xi, y\rangle} \, d\nu_\Psi(y)\Big) \end{aligned}$$

$$= \int_{\mathcal{N}'\times\mathcal{N}'} e^{i\langle\xi,x+y\rangle}\, d\nu_\Phi(x)\times\nu_\Psi(y) = \int_{\mathcal{N}'} e^{i\langle\xi,z\rangle}\, d\nu_\Phi * \nu_\Psi(z)$$

By the uniqueness of the characteristic function of a measure, (Bochner Minlos theorem), it follows for the corresponding measures:

$$\nu_{\Phi*\Psi} = \nu_\Phi * \nu_\Psi \tag{2.10}$$

□

Proposition 2.3.11. *Let $a \in \mathcal{N}'$ and δ_a be the point evaluation at a, i.e. for $\varphi \in (\mathcal{N})$:*

$$\delta_a(\varphi) = \varphi(a)$$

and ν be a positive Hida measure on $\mathcal{N}'$. Then, for $f \in (\mathcal{N})$, it follows:

$$\int_{\mathcal{N}'} f(z) d\nu_{\delta_a} * \nu = \int_{\mathcal{N}'} f(a+y) d\nu(y).$$

Proof. First notice that $T(\delta_a)(\xi) = \exp(i\,\langle\xi,a\rangle)$, $\xi \in \mathcal{N}$. Therefore δ_a is a positive generalized function. It holds by 2.3.7(iii)

$$\int_{\mathcal{N}'} f(z) d\nu_{\delta_a} * \nu = \int_{\mathcal{N}'\times\mathcal{N}'} f(x+y) d\nu_{\delta_a}\times\nu = \int_{\mathcal{N}'} \langle\langle f(.+y), \delta_a\rangle\rangle d\nu(y) = \int_{\mathcal{N}'} f(a+y) d\nu(y)$$

□

Proposition 2.3.12. *Let $\eta \in L^2(\mathbb{R}, dt)$ with $\eta \neq 0$ and $a \in \mathbb{R}$. Then $\delta(\langle\eta,\cdot\rangle - a)$ is a positive generalized White Noise function.*

Proof. By Definition 2.2.22 we have for $\xi \in \mathcal{N}$

$$T(\delta(\langle\eta,\cdot\rangle))(\xi) = \frac{1}{\sqrt{2\pi\langle\eta,\eta\rangle}} \exp(-\frac{1}{2}\langle\xi,\xi\rangle) \exp(\frac{1}{2\langle\eta,\eta\rangle}(\langle\eta,\xi\rangle^2 + 2ia\langle\eta,\xi\rangle - a^2)),$$

which is positive definite as a product of positive definite functions. Note that

$$|\langle\frac{\eta}{|\eta|_0},\xi\rangle| \le |\xi|_0.$$

□

Chapter 3

Generalized Gauss kernels

In this chapter we give properties of the so called generalized Gauss kernels. This class of Hida distributions plays an important role in the application of the theory of Feynman integrals. Moreover, since these generalized Gauss kernels are related to quadratic forms, one can also consider the expectation of a pointwise multiplication of such a GGK and a generalized function informally as the generalized expectation of the generalized function with respect to a Gaussian measure with a different covariance matrix, if the Gauss kernel fulfills certain positivity properties. We will give a rigorous meaning of this in the following chapter.

3.1 Exponentials of Quadratic Forms

Let $(e_n)_{n\in\mathbb{N}} \subset \mathcal{N}$ be an orthonormal basis of $\mathcal{H}$. Let P_{e_n} be the bounded linear operator on $\mathcal{H}$ defined by

$$P_{e_n} f = \langle f, e_n\rangle e_n, \qquad \text{with } f \in \mathcal{H}.$$

In [34] it is mentioned that this projection to the one-dimensional subspace spanned by e_n is a continuous mapping on $\mathcal{N}$ and can be extended to a continuous mapping on $\mathcal{N}'$.

Definition 3.1.1. *Let K be a normal compact operator on $\mathcal{H}$ with eigenvalues λ_n and corresponding eigenvectors $(e_n)_{n\in\mathbb{N}} \subset \mathcal{H}$, then we define the quadratic form*

$$\langle \omega, K\omega\rangle := \sum_{n=1}^{\infty} \lambda_n \langle e_n, \omega\rangle^2.$$

Remark 3.1.2. *With the help of Definition 3.1, we are also able to define* $\exp(\langle \cdot, K\cdot\rangle)$ *as a measurable function by the above limit procedure. Under special properties of K, the so constructed object is well-defined as a Hida distribution.*

Here we review a special class of Hida distributions which are defined by their T-transform, see e.g. [34], [40]. Let $\mathcal{B}$ be the set of all continuous bilinear mappings $\mathfrak{B} : \mathcal{N} \times \mathcal{N} \to \mathbb{C}$. Then the functions

$$\mathcal{N} \ni \xi \mapsto \exp\left(-\frac{1}{2}\mathfrak{B}(\xi,\xi)\right) \in \mathbb{C}$$

for all $\mathfrak{B} \in \mathcal{B}$ are U-functionals. Therefore, by using the characterization of Hida distributions in Theorem 2.2.16, the inverse T-transform of these functions

$$\Phi_{\mathfrak{B}} := T^{-1} \exp\left(-\frac{1}{2}\mathfrak{B}\right)$$

are elements of $(\mathcal{N})'$.

Definition 3.1.3. *The set of* **generalized Gauss kernels** *is defined by*

$$GGK := \{\Phi_{\mathfrak{B}},\ \mathfrak{B} \in \mathcal{B}\}.$$

In case the continuous bilinear form is given via the dual pairing and an operator $B : \mathcal{N}_{\mathbb{C}} \to \mathcal{N}'_{\mathbb{C}}$ we write $\Phi_B = T^{-1} \exp\left(-\frac{1}{2}\langle \cdot, B\cdot\rangle\right)$

Lemma 3.1.4. *Let $(\lambda_n)_{n\in\mathbb{N}}$ be a monotone decreasing sequence of nonnegative real numbers such that*

(i) $1 > \lambda_1 \geq \lambda_2 \geq \cdots \geq 0$

(ii) $\sum\limits_{n=1}^{\infty} \lambda_n < \infty$

then

$$\prod_{n=1}^{\infty}(1-\lambda_n) > 0$$

Proof. There exists an interval $[0, x_0]$ such that

$$\forall x \in [0, x_0] : 1 - x \geq e^{-2x}.$$

Because $\lim\limits_{n\to\infty} \lambda_n = 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ we have $\lambda_n \in [0, x_0]$. It follows

$$\prod_{n=n_0}^{\infty}(1-\lambda_n) \geq \prod_{n=n_0}^{\infty} e^{-2\lambda_n} = exp(-2\sum_{n=n_0}^{\infty} \lambda_n) > 0$$

□

Example 3.1.5. *[34], [10] We consider a symmetric trace class operator K on $L^2_d(\mathbb{R})$ such that $-\frac{1}{2} < K \leq 0$, then*

$$\int_{S'_d(\mathbb{R})} \exp\left(-\langle \omega, K\omega \rangle\right) \, d\mu(\omega) = \left(\det(Id + 2K)\right)^{-\frac{1}{2}} < \infty.$$

For the definition of $\langle \cdot, K \cdot \rangle$ see the remark below. Here Id denotes the identity operator on the Hilbert space $L^2_d(\mathbb{R})$, and $\det(A)$ of a symmetric trace class operator A on $L^2_d(\mathbb{R})$ denotes the infinite product of its eigenvalues. The trace class condition gives us condition (ii) in Lemma 3.1.4 and the boundedness of the spectrum is equivalent to condition (i) in Lemma 3.1.4. Hence the determinant is non-zero and the expectation makes sense.
In the present situation we have $\det(Id+2K) \neq 0$. Therefore we obtain that the exponential $g = \exp(-\frac{1}{2}\langle \cdot, K \cdot \rangle)$ is square-integrable and its T-transform is given by

$$Tg(\mathbf{f}) = \left(\det(Id + K)\right)^{-\frac{1}{2}} \exp\left(-\frac{1}{2}(\mathbf{f}, (Id + K)^{-1}\mathbf{f})\right), \quad \mathbf{f} \in S_d(\mathbb{R}).$$

Therefore $\left(\det(Id + K)\right)^{\frac{1}{2}} g$ is a generalized Gauss kernel.

Remark 3.1.6. *Since a symmetric trace class operator is compact, see e.g. [70], we have that K in the above example is diagonalizable, i.e.*

$$Kf = \sum_{k=0}^{\infty} k_n (f, e_n) e_n, \quad f \in \mathcal{H},$$

where $(e_n)_{n\in\mathbb{N}}$ denotes an eigenbasis of the corresponding eigenvalues $(k_n)_{n\in\mathbb{N}}$ with $k_n \in (-\frac{1}{2}, 0]$, for all $n \in \mathbb{N}$. Since K is compact, we have that $\lim_{n\to\infty} k_n = 0$ and since K is trace class we also have $\sum_{n=0}^{\infty}(e_n, -Ke_n) < \infty$. We define for $\omega \in \mathcal{N}'$

$$-\langle \omega, K\omega\rangle := \lim_{N\to\infty} \sum_{n=1}^{N} \langle e_n, \omega\rangle(-k_n)\langle e_n, \omega\rangle.$$

Then as a limit of measurable functions $\omega \mapsto -\langle \omega, K\omega\rangle$ is measurable and hence

$$\int_{\mathcal{N}'} \exp(-\langle \omega, K\omega\rangle)\, d\mu(\omega) \in [0, \infty].$$

The explicit formula for the T-transform and expectation then follow by a straightforward calculation with help of the above limit procedure.

Proposition 3.1.7. *Let $B \in L(\mathcal{N}_{\mathbb{C}}, \mathcal{N}'_{\mathbb{C}})$ and $\Phi_B \in GGK$ the associated generalized Gauss kernel, then Φ_B has the following properties:*

i) $S(\Phi_B)(\xi) = \exp(-\frac{1}{2}\langle \xi, (Id - B)\xi\rangle)$, *for $\xi \in \mathcal{N}$*

ii) $\Phi_B = \sum_{n=0}^{\infty} \frac{1}{n!}(-\frac{1}{2})^n \langle tr_{I-B}^{\hat{\otimes} n}, :\cdot^{\otimes 2n}:\rangle$. *I.e. for $\eta, \psi \in \mathcal{N}$, $\langle \eta\otimes\psi, tr_{I-B}\rangle = \langle \eta, (I-B)\psi\rangle$. For higher order of tensor powers it is the symmetric multilinearform given by $Id-B$ compare to [40]*

iii) $\Phi_B \in (\mathcal{H})_k$, *if $\|H^{-k}\Phi_B H^{-k}\|_{H.S.} < 1$.*

Proof. (i) We have

$$T(\Phi_B)(-i\xi) = \exp(\frac{1}{2}\langle \xi, \xi\rangle) S(\Phi_B)(\xi),$$

then

$$S(\Phi_B)(\xi) = \exp(\frac{1}{2}\langle \xi, B\xi\rangle)\exp(-\frac{1}{2}\langle \xi, \xi\rangle),$$

thus

$$S(\Phi_B)(\xi) = \exp(-\frac{1}{2}\langle \xi, (Id - B)\xi\rangle).$$

(ii) A Taylor expansion of the S-transform gives

$$S(\Phi_B)(\xi) = \sum_{n=0}^{\infty} \frac{1}{n!}(-\frac{1}{2})^n \langle \xi, (Id - B)\xi \rangle^n.$$

Then the chaos decomposition is obtained by the inverse S-transform.

(iii) Is proved in [40].

□

Remark 3.1.8. *For positive definite operator B, one can also consider Φ_B as generalized radon Nikodym-derivative as in [40]. The corresponding measure is then $\mu(\sqrt{B}^{-1}.)$, if B is self-adjoint w.r.t $\langle \cdot, \cdot \rangle$. The representation via trace operators or multilinearforms respectively is also valid in a more general setting, when the root of an operator does not exists, as in the case of non-positive definite operators B. One then still can consider Φ_B as a kind of change of measure, although, the "covariance matrix" might not be positive definite.*

Definition 3.1.9. *Let $K : \mathcal{H}_{\mathbb{C}} \to \mathcal{H}_{\mathbb{C}}$ be linear and continuous such that*

(i) $Id + K$ is injective,

(ii) there exists $p \in \mathbb{N}$ such that $(Id + K)(\mathcal{H}_{\mathbb{C}}) \subset \mathcal{H}_{p,\mathbb{C}}$ is dense,

(iii) there exist $q \in \mathbb{N}_0$ such that $(Id + K)^{-1} : \mathcal{H}_{p,\mathbb{C}} \to \mathcal{H}_{-q,\mathbb{C}}$ is continuous with p as in (ii).

Then we define the normalized exponential

$$\text{Nexp}(-\frac{1}{2}\langle \cdot, K \cdot \rangle) \tag{3.1}$$

by

$$T(\text{Nexp}(-\frac{1}{2}\langle \cdot, K \cdot \rangle))(\xi) := \exp(-\frac{1}{2}\langle \xi, (Id + K)^{-1}\xi \rangle), \quad \xi \in \mathcal{N}.$$

Example 3.1.10. *i) Let Φ_B be defined by $T(\Phi_B)(\xi) = \exp(-\frac{1}{2}\langle \xi, B\xi \rangle)$, $\xi \in S_2(\mathbb{R})$ where*

$$B = \begin{pmatrix} \mathbb{1}_{[0,t)^c} & 0 \\ 0 & \mathbb{1}_{[0,t)^c} \end{pmatrix} + \begin{pmatrix} \mathbb{1}_{[0,t)} & -\mathbb{1}_{[0,t)} \\ -\mathbb{1}_{[0,t)} & 0 \end{pmatrix} \in L(L_2^2(\mathbb{R}), L_2^2(\mathbb{R})).$$

Then B fulfills the requirements of definition 3.1.3 and we can find a K such that $B = (Id + K)^{-1}$. In this case

$$Id + K = \begin{pmatrix} 1\!\!1_{[0,t)^c} & 0 \\ 0 & 1\!\!1_{[0,t)^c} \end{pmatrix} + \begin{pmatrix} 0 & -1\!\!1_{[0,t)} \\ -1\!\!1_{[0,t)} & -1\!\!1_{[0,t)} \end{pmatrix}.$$

Thus K is given by

$$K = \begin{pmatrix} -1\!\!1_{[0,t)} & -1\!\!1_{[0,t)} \\ -1\!\!1_{[0,t)} & -2 1\!\!1_{[0,t)} \end{pmatrix}.$$

Hence $\Phi_B = \exp\big(-\frac{1}{2}\langle\cdot, K\cdot\rangle\big)$.

ii) Let B be the orthogonal projection on the complement of the subspace spanned by $\eta \in L^2(\mathbb{R})$, with $(\eta, \eta) = 1$. Then

$$\begin{aligned} T(\Phi_B)(\xi) &= \exp(-\frac{1}{2}\langle\xi, B\xi\rangle) \\ &= \exp(-\frac{1}{2}\langle\xi, \xi - \langle\xi,\eta\rangle\eta\rangle) \\ &= \exp(-\frac{1}{2}\langle\xi - \langle\xi,\eta\rangle\eta, \xi - \langle\xi,\eta\rangle\eta\rangle \\ &= \exp(-\frac{1}{2}\langle\xi,\xi\rangle)\exp(\frac{1}{2}(\langle\xi,\eta\rangle)^2) \\ &= \exp(-\frac{1}{2}\langle\xi,\xi\rangle)\exp(-\frac{1}{2}(i\langle\xi,\eta\rangle)^2) \\ &= T(\delta_0(\langle .,\eta\rangle))(\xi) \end{aligned}$$

Since the T-transform of Φ_B coincides with $T(\delta_0(\langle .,\eta\rangle))$ we have, that the Hida distributions are the same. Compare also [40]

Remark 3.1.11. *The "normalization" of the exponential in the above definition can be considered as a division of a divergent factor. In an informal way one can write*

$$T(\mathrm{N}\exp(-\frac{1}{2}\langle\cdot, K\cdot\rangle))(\xi) = \frac{T(\exp(-\frac{1}{2}\langle\cdot, K\cdot\rangle))(\xi)}{T(\exp(-\frac{1}{2}\langle\cdot, K\cdot\rangle))(0)} = \frac{T(\exp(-\frac{1}{2}\langle\cdot, K\cdot\rangle))(f)}{\sqrt{\det(Id + K)}}, \quad \xi \in \mathcal{N},$$

i.e. if the determinant in the Example 3.1.5 above is not defined, we can still define the normalized exponential by the T-transform without the diverging prefactor. The assumptions in the above definition then guarantee the existence of the generalized Gauss kernel in (3.1).

3.2 Pointwise Products with Generalized Gauss kernels

Example 3.2.1. *For sufficiently "nice" operators K and L on $\mathcal{H}$ we can define the product*

$$\mathrm{Nexp}\big(-\frac{1}{2}\langle\cdot, K\cdot\rangle\big)\cdot\exp\big(-\frac{1}{2}\langle\cdot, L\cdot\rangle\big)$$

of two square-integrable functions. Its T-transform is then given by

$$\begin{aligned} &T\Big(\mathrm{Nexp}(-\frac{1}{2}\langle\cdot, K\cdot\rangle)\cdot\exp(-\frac{1}{2}\langle\cdot, L\cdot\rangle)\Big)(\xi) \\ &\qquad = \sqrt{\frac{1}{\det(Id + L(Id+K)^{-1})}}\exp(-\frac{1}{2}\langle\xi, (Id+K+L)^{-1}\xi\rangle), \quad \xi\in\mathcal{N}, \end{aligned}$$

in the case the right hand side indeed is a U-functional.

Remark 3.2.2. *In Example 3.2.1*

In the case $g\in\mathcal{N}$, $c\in\mathbb{C}$ the product between the Hida distribution Φ and the Hida test function $\exp(i\langle g, .\rangle + c)$ is well-defined because $(\mathcal{N})$ is a continuous algebra under pointwise multiplication. The next definition is an extension of this product.

Definition 3.2.3. *The pointwise product of a Hida distribution $\Phi\in(\mathcal{N})'$ with an exponential of a linear term, i.e.*

$$\Phi\cdot\exp(i\langle\eta,\cdot\rangle + c), \quad \eta\in\mathcal{H}_{\mathbb{C}},\ c\in\mathbb{C},$$

is defined by

$$T(\Phi\cdot\exp(i\langle\eta,\cdot\rangle + c))(\xi) := T\Phi(\eta+\xi)\exp(c), \quad \xi\in\mathcal{N},$$

if $T\Phi$ has a continuous extension to $\mathcal{H}_{\mathbb{C}}$ and the term on the right-hand side is a U-functional in $\xi\in\mathcal{N}$.

Definition 3.2.4. *Let $D\subset\mathbb{R}$ such, that $0\in\overline{D}$. Under the assumption that $T\Phi$ has a continuous extension to $\mathcal{H}_{\mathbb{C}}$, $\eta\in\mathcal{H}_{\mathbb{C}}$, $y\in\mathbb{R}$, $\lambda\in\gamma_\alpha := \{\exp(-i\alpha)s \mid s\in\mathbb{R}\}$ and that the integrand*

$$\gamma_\alpha \ni \lambda \mapsto \exp(-i\lambda y)T\Phi(\xi+\lambda\eta)\in\mathbb{C}$$

fulfills the conditions of Corollary 2.2.21 for all $\alpha \in D$*, one can define the product*

$$\Phi \cdot \delta(\langle \eta, \cdot \rangle - y),$$

by

$$T(\Phi \cdot \delta(\langle \eta, \cdot \rangle - y))(\xi) := \lim_{\alpha \to 0} \frac{1}{2\pi} \int_{\gamma_\alpha} \exp(-i\lambda y) T\Phi(\xi + \lambda \eta)\, d\lambda.$$

Of course under the assumption that the right-hand side converges in the sense of Corollary 2.2.18, see e.g. [34].

Proposition 3.2.5. *Let* $K : \mathcal{H}_{\mathbb{C}} \to \mathcal{H}_{\mathbb{C}}$ *as in Definition 3.1.9 and let* $\eta_1, \ldots, \eta_n \in \mathcal{H}_{\mathbb{C}}$ *be a family of functions, such that the matrix*

$$M_{(Id+K)^{-1}} := (\langle \eta_k, (Id + K)^{-1} \eta_j \rangle)_{k,l}$$

is non–singular and

$$\Re(M_{(Id+K)^{-1}}) \text{ positive definite}$$

or

$$\Re(M_{(Id+K)^{-1}}) = 0 \text{ and } \Im(M_{(Id+K)^{-1}}) \text{ positive or negative definite.}$$

Then

$$\mathrm{Nexp}(-\frac{1}{2}\langle \cdot, K \cdot \rangle) \cdot \prod_{k=1}^{n} \delta_{y_k}(\langle \eta_k, \cdot \rangle), \quad y_k \in \mathbb{C}, k = 1, \ldots, n,$$

exists as an element in $(\mathcal{N})'$*. Moreover its* T*-transform in* $\xi \in \mathcal{N}$ *is given by*

$$T(\mathrm{Nexp}(-\frac{1}{2}\langle \cdot, K \cdot \rangle) \cdot \prod_{k=1}^{n} \delta_{x_k}(\langle \eta_k, \cdot \rangle)(\xi) = \frac{1}{\sqrt{(2\pi)^n \det(M_{(Id+K)^{-1}})}} \exp(-\frac{1}{2}\langle \xi, (Id+K)^{-1} \xi \rangle) \times \exp\left(\frac{1}{2} u^T \left(M_{(Id+K)^{-1}} \right)^{-1} u \right),$$

where

$$u = \begin{pmatrix} \frac{1}{2}\big\langle (Id + K)^{-1}\xi, \eta_1 \big\rangle + \frac{1}{2}\big\langle \xi, (Id + K)^{-1}\eta_1 \big\rangle + iy_1 \\ \cdot \\ \cdot \\ \frac{1}{2}\big\langle (Id + K)^{-1}\xi, \eta_n \big\rangle + \frac{1}{2}\big\langle \xi, (Id + K)^{-1}\eta_n \big\rangle + iy_n \end{pmatrix}.$$

Proof. We want to give meaning to the expression

$$\mathrm{Nexp}\big(-\frac{1}{2}\langle\cdot,K\cdot\rangle\big)\cdot\prod_{k=1}^{n}\delta(\langle\eta_k,\cdot\rangle-y_k),$$

using Definition 3.2.4 inductively.
Hence we obtain for the T-transform of the integrand

$$\gamma_\alpha^n \ \ni\ \lambda\ \mapsto\ \Phi_\lambda\ =\ \exp(-i\sum_{j=1}^{n}\lambda_j e^{-i\alpha}y_j)\ \cdot\ \exp(i\sum_{j=1}^{n}\lambda_j e^{-i\alpha}\langle\eta_j,\cdot\rangle)\ \cdot\ \mathrm{N}\exp(-\frac{1}{2}\langle\cdot,K\cdot\rangle)$$

in $\xi\in\mathcal{N}$,

$$\begin{aligned}
&T\big(\exp(-i\sum_{j=1}^{n}\lambda_j e^{-i\alpha}y_j)\exp(i\sum_{j=1}^{n}\lambda_j e^{-i\alpha}\langle\eta_j,\cdot\rangle)\cdot\mathrm{N}\exp(-\frac{1}{2}\langle\cdot,K\cdot\rangle)\big)(\xi)\\
&\quad=\exp(-i\sum_{j=1}^{n}\lambda_j e^{-i\alpha}y_j)T\big(\mathrm{N}\exp(-\frac{1}{2}\langle\cdot,K\cdot\rangle)\cdot\exp(i\sum_{j=1}^{n}\langle\lambda_j e^{-i\alpha}\eta_j,\cdot\rangle)\big)(\xi)\\
&\quad=\exp(-i\sum_{j=1}^{n}\lambda_j e^{-i\alpha}y_j)\exp(-\frac{1}{2}\big\langle(\xi+\sum_{j=1}^{n}\lambda_j e^{-i\alpha}\eta_j),(Id+K)^{-1}(\xi+\sum_{k=1}^{n}\lambda_k e^{-i\alpha}\eta_k)\big\rangle)
\end{aligned}$$

Then we can rewrite the above formula with the help of the matrix $M_{(Id+K)^{-1}}$ as

$$\begin{aligned}
\gamma_\alpha^n\ni\lambda\mapsto\ &T\big(\exp(-i\sum_{j=1}^{n}\lambda_j e^{-i\alpha}y_j)\exp(i\sum_{j=1}^{n}\lambda_j e^{-i\alpha}\langle\eta_j,\cdot\rangle)\cdot\mathrm{N}\exp(-\frac{1}{2}\langle\cdot,K\cdot\rangle)\big)(\xi)\\
&=\exp\big(-\frac{1}{2}\langle\xi,(Id+K)^{-1}\xi\rangle\big)\exp\Bigg(-\frac{1}{2}e^{-2i\alpha}(\lambda^T M_{(Id+K)^{-1}}\lambda)\\
&\quad-e^{-i\alpha}\lambda\bigg(\frac{1}{2}\Big(\big\langle\xi,(Id+K)^{-1}\eta_1\big\rangle,\dots,\big\langle\xi,(Id+K)^{-1}\eta_n\big\rangle\Big)\\
&\quad+\frac{1}{2}\Big(\big\langle(Id+K)^{-1}\xi,\eta_1\big\rangle,\dots,\big\langle(Id+K)^{-1}\xi,\eta_n\big\rangle\Big)+iy\bigg)\Bigg),
\end{aligned}\tag{3.2}$$

where $y=(y_1,\dots y_n)$ and $\lambda=(\lambda_1,\dots,\lambda_n)$ respectively. The function in (3.2) is integrable w.r.t. the Lebesgue measure, if the real part of $e^{-2i\alpha}M_{(Id+K)^{-1}}$, i.e. $\Re(e^{-2i\alpha}M_{(Id+K)^{-1}})=\cos(2\alpha)\Re(M_{(Id+K)^{-1}})+\sin(2\alpha)\Im(M_{(Id+K)^{-1}})$, is positive definite. Our assumptions on $M_{(Id+K)^{-1}}$ in Proposition 3.2.5 imply that this holds for α in a set D, as required in

Definition 3.2.4. Then we have

$$
\begin{aligned}
&\frac{1}{(2\pi)^n}\exp(-\frac{1}{2}\langle\xi,(Id+K)^{-1}\xi\rangle)\int_{\gamma_\alpha}\cdots\int_{\gamma_\alpha}\exp\Bigg(-\frac{1}{2}(\lambda, M_{(Id+K)^{-1}}\lambda)\\
&\qquad -\lambda\bigg(\frac{1}{2}\Big(\big\langle\xi,(Id+K)^{-1}\eta_1\big\rangle,\dots,\big\langle\xi,(Id+K)^{-1}\eta_n\big\rangle\Big)\\
&\quad+\frac{1}{2}\Big(\big\langle(Id+K)^{-1}\xi,\eta_1\big\rangle,\dots,\big\langle(Id+K)^{-1}\xi,\eta_n\big\rangle\Big)+i(y_1,\dots y_n)\bigg)\Bigg)d\lambda_1\cdots d\lambda_n\\
&\quad=\frac{e^{-\alpha n}}{(2\pi)^n}\exp(-\frac{1}{2}\langle\xi,(Id+K)^{-1}\xi\rangle)\int_{\mathbb{R}^n}\cdots\int_{\gamma_\alpha}\exp\Bigg(-\frac{1}{2}e^{-2i\alpha}(\lambda^T M_{(Id+K)^{-1}}\lambda)\\
&\qquad -e^{-i\alpha}\lambda\bigg(\frac{1}{2}\Big(\big\langle\xi,(Id+K)^{-1}\eta_1\big\rangle,\dots,\big\langle\xi,(Id+K)^{-1}\eta_n\big\rangle\Big)\\
&\quad+\frac{1}{2}\Big(\big\langle(Id+K)^{-1}\xi,\eta_1\big\rangle,\dots,\big\langle(Id+K)^{-1}\xi,\eta_n\big\rangle\Big)+i(y_1,\dots y_n)\bigg)\Bigg)d\lambda_1\cdots d\lambda_n\\
&\qquad=\frac{e^{-\alpha n}}{(2\pi)^n}\exp(-\frac{1}{2}\langle\xi,(Id+K)^{-1}\xi\rangle)\sqrt{\frac{(2\pi)^n}{e^{-i2\alpha n}\det(M_{(Id+K)^{-1}})}}\\
&\qquad\qquad\times\exp\left(\frac{e^{-2i\alpha}}{2}u^T\left(e^{-2i\alpha}M_{(Id+K)^{-1}}\right)^{-1}u\right)
\end{aligned}
$$

$$
=\frac{1}{\sqrt{(2\pi)^n\det(M_{(Id+K)^{-1}})}}\exp(-\frac{1}{2}\langle\xi,(Id+K)^{-1}\xi\rangle)\exp\left(\frac{1}{2}u^T\left(M_{(Id+K)^{-1}}\right)^{-1}u\right)
$$

□

Lemma 3.2.6. *Let L be a $d\times d$ block operator matrix on $\mathcal{H}_{\mathbb{C}}^d$ acting componentwise such that all entries are bounded operators on $\mathcal{H}_{\mathbb{C}}$. Let K be a $d\times d$ block operator matrix on $\mathcal{H}_{\mathbb{C}}$, such that $Id+K$ and $N=Id+K+L$ are bounded with bounded inverse. Furthermore assume that $\det(Id+L(Id+K)^{-1})$ exists and is different from zero (this is e.g. the case if L is trace class and -1 in the resolvent set of $L(Id+K)^{-1}$). Let $(\boldsymbol{\eta}_k)_{k=1,\dots J}$ be a family of non–zero functions from $\mathcal{H}^d$, $J\in\mathbb{N}$, such that the matrix*

$$M_{N^{-1}} := (\langle\boldsymbol{\eta}_k, N^{-1}\boldsymbol{\eta}_j\rangle)_{k,l}$$

is non–singular and

$$\Re(M_{N^{-1}}) \text{ positive definite or } \Re(M_{N^{-1}})=0 \text{ and } \Im(M_{N^{-1}}) \text{ positive or negative definite,}$$

where $M_{N^{-1}} = \Re(M_{N^{-1}}) + i\Im(M_{N^{-1}})$ *with real matrices* $\Re(M_{N^{-1}})$ *and* $\Im(M_{N^{-1}})$*,*
then

$$\Phi_{K,L} := \mathrm{Nexp}\big(-\frac{1}{2}\langle\cdot,K\cdot\rangle\big)\cdot\exp\big(-\frac{1}{2}\langle\cdot,L\cdot\rangle\big)\cdot\exp(i\langle\cdot,\mathbf{g}\rangle)\cdot\prod_{i=1}^{J}\delta(\langle\boldsymbol{\eta}_k,\cdot\rangle-y_k),$$

for $\mathbf{g}\in\mathcal{H}^d_{\mathbb{C}}$*,* $t>0$*,* $y_k\in\mathbb{R}$*,* $k=1\dots,J$*, exists as a Hida distribution.*
Moreover for $\mathbf{f}\in\mathcal{N}^d$

$$T\Phi_{K,L}(\mathbf{f}) = \frac{1}{\sqrt{(2\pi)^J\det((M_{N^{-1}}))}}\sqrt{\frac{1}{\det(Id+L(Id+K)^{-1})}}$$
$$\times\exp\Bigg(-\frac{1}{2}\Big\langle(\mathbf{f}+\mathbf{g}),N^{-1}(\mathbf{f}+\mathbf{g})\Big\rangle\Bigg)\exp\Bigg(\frac{1}{2}(u^T(M_{N^{-1}})^{-1}u)\Bigg),$$

where

$$u=\begin{pmatrix} iy_1+\frac{1}{2}\langle\boldsymbol{\eta}_1,N^{-1}(\mathbf{f}+\mathbf{g})\rangle\big)+\frac{1}{2}\langle N^{-1}\boldsymbol{\eta}_1,(\mathbf{f}+\mathbf{g})\rangle\\ \dots\\ iy_J+\frac{1}{2}\langle\boldsymbol{\eta}_J,N^{-1}(\mathbf{f}+\mathbf{g})\rangle+\frac{1}{2}\langle N^{-1}\boldsymbol{\eta}_J,(\mathbf{f}+\mathbf{g})\rangle\end{pmatrix}$$

Proof. We want to give meaning to the expression

$$\mathrm{Nexp}\big(-\frac{1}{2}\langle\cdot,K\cdot\rangle\big)\cdot\exp\big(-\frac{1}{2}\langle\cdot,L\cdot\rangle\big)\cdot\exp(i\langle\cdot,\mathbf{g}\rangle)\cdot\prod_{k=1}^{J}\delta(\langle\boldsymbol{\eta}_k,\cdot\rangle-y_k),$$

using Definition 3.2.4 inductively as in Proposition 3.2.5.
Note that $\mathrm{Nexp}\big(-\frac{1}{2}\langle\cdot,K\cdot\rangle\big)\cdot\exp\big(-\frac{1}{2}\langle\cdot,L\cdot\rangle\big)$ can be defined as in Example 3.2.1. Hence we obtain for the T-transform of the integrand

$$\gamma^J_\alpha\ni\lambda\mapsto\Phi_\lambda=\exp(-i\sum_{j=1}^{J}\lambda_je^{-i\alpha}y_j)\cdot\exp(i\sum_{j=1}^{J}\lambda_je^{-i\alpha}\langle\boldsymbol{\eta}_j,\cdot\rangle)$$
$$\cdot\,\mathrm{N}\exp(-\frac{1}{2}\langle\cdot,K\cdot\rangle)\exp(-\frac{1}{2}\langle\cdot,L\cdot\rangle)\exp(i\langle\cdot,\mathbf{g}\rangle)$$

in $\mathbf{f}\in\mathcal{N}^d$, Then with the notations $y=(y_1,\dots y_J)$ and $\lambda=(\lambda_1,\dots\lambda_J)$, respectively, we obtain the integrand

$$
\begin{aligned}
\gamma_\alpha^J \ni \lambda \mapsto T\big(&\exp(-i\sum_{j=1}^J \lambda_j e^{-i\alpha} y_j)\exp(i\sum_{j=1}^J \lambda_j e^{-i\alpha}\langle \boldsymbol{\eta}_j, \cdot\rangle) \\
&\cdot \mathrm{N}\exp(-\frac{1}{2}\langle \cdot, K\cdot\rangle)\cdot\exp(-\frac{1}{2}\langle\cdot, L\cdot\rangle)\cdot\exp(i\langle\cdot,\mathbf{g}\rangle)\big)(\mathbf{f}) \\
&= \frac{1}{\sqrt{\det(Id + L(Id+K)^{-1})}}\exp\Bigg(-\frac{1}{2}e^{-2i\alpha}(\lambda^T M_{N^{-1}}\lambda) \\
&- e^{-i\alpha}\lambda\Bigg((\frac{1}{2}\big\langle \mathbf{f}+\mathbf{g}), N^{-1}\boldsymbol{\eta}_1\big\rangle + \frac{1}{2}\langle \boldsymbol{\eta}_1, N^{-1}(\mathbf{f}+\mathbf{g})\rangle + iy_1), \\
&\dots, (\frac{1}{2}\big\langle(\mathbf{f}+\mathbf{g}), N^{-1}\boldsymbol{\eta}_d\big\rangle + \frac{1}{2}\langle\boldsymbol{\eta}_d, N^{-1}(\mathbf{f}+\mathbf{g})\rangle + iy_d)\Bigg)\Bigg).
\end{aligned} \tag{3.3}
$$

Again the function in (3.3) is integrable w.r.t. the Lebesgue measure, if the real part of $e^{-2i\alpha}M_{N^{-1}}$, i.e. $\Re(e^{-2i\alpha}M_{N^{-1}}) = \cos(2\alpha)\Re(M_{N^{-1}}) + \sin(2\alpha)\Im(M_{N^{-1}})$, is positive definite. Our assumptions on $M_{N^{-1}}$ in Lemma 3.2.6 imply that this holds for α in a set D, as required in Definition 3.2.4. The calculation of the T-transform then follows in analogous way to the calculation of the T-transform of a product of Donskers delta functions, see e.g. [59, 87] and as in the proof of Proposition 3.2.5. □

Chapter 4

Operators in White Noise Analysis

In this section we discuss operators which are related to applications for Feynman Integrals as differential operators, scaling, translation and projection. We show the relation of these operators to differential operators, which leads to the notion of so called convolution operators. We generalize the concept of complex scaling to scaling with bounded operators. Furthermore for this generalized scaling we discuss the relation to generalized Radon-Nikodym derivatives. This is done to sum up a toolbox to investigate products of generalized functions in chapter 5.

4.1 Differential operators

Many authors in White Noise Analysis studied several classes of differential operators on the Hida test functions or $(\mathcal{N})^{\beta}, 0 \leq \beta \leq 1$. Instead of listing a complete reference list, we refer to the monographs [64, 57, 40] and the references therein. First we clarify the notion of Gâteaux derivatives.

Definition 4.1.1. *Let $\mathfrak{X}$ be a locally convex Hausdorff space over the field $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. For a $\mathbb{C}$-valued function $F : \mathfrak{X} \to \mathbb{C}$ the Gâteaux-derivative at $\eta \in \mathfrak{X}$ in direction $\xi \in \mathfrak{X}$ is defined by*

$$D_{\xi}F(\eta) = \frac{\partial}{\partial \lambda} F(\lambda\xi + \eta)\Big|_{\lambda=0}, \quad \lambda \in \mathbb{K}$$

if this derivative exists, we call F Gâteaux-differentiable at $\eta \in \mathfrak{X}$ in direction $\xi \in \mathfrak{X}$. If F is Gâteaux differentiable in direction $\xi \in \mathfrak{X}$ for all $\eta \in \mathfrak{X}$ we call F Gâteaux-differentiable in direction $\xi \in \mathfrak{X}$.

Definition 4.1.2. *Let $\mathfrak{X}$ be a locally convex Hausdorff space over $\mathbb{C}$. A function $F : \mathfrak{X} \to \mathbb{C}$ is called entire holomorphic, if for all $\xi,\ \eta \in \mathfrak{X}$ the function $z \mapsto F(z\xi + \eta),\ z \in \mathbb{C}$ is entire holomorphic. The set of entire holomorphic functions of $\mathfrak{X}$ is denoted by $\mathfrak{A}(\mathfrak{X})$.*

Lemma 4.1.3. *Let $\mathfrak{X}$ be a locally convex Hausdorff space. Let $F \in \mathfrak{A}(\mathfrak{X})$. For $z \in \mathbb{C},\ \xi, \eta \in \mathfrak{X}$ let $g(z) := F(z\xi + \eta)$. Then*

(i)

$$D_{\xi}F \in \mathfrak{A}(\mathfrak{X}) \tag{4.1}$$

(ii)

$$\forall n \in \mathbb{N}_0 :\ g^{(n)}(z) = (D_{\xi}^{n}F)(z\xi + \eta) \tag{4.2}$$

(iii)

$$F(z\xi + \eta) = \sum_{n=0}^{\infty} \frac{(D_{\xi}^{n}F)(\eta)}{n!}\ z^{n} \tag{4.3}$$

Proof. Proof of (i):
Let $\xi_1, \xi_2, \eta \in \mathfrak{X}$ and $h(z_1, z_2) := F(z_1\xi_1 + z_2\xi_2 + \eta)$ for $z_1,\ z_2 \in \mathbb{C}$. Then h is separately holomorphic and by the Lemma of Hartog [18], h is holomorphic. Thus $\frac{\partial h}{\partial z_1}(0, z_2) = (D_{\xi_1}F)(z_2\xi_2 + \eta)$ is entire holomorphic.
Proof of (ii):

By definition we have for $z_0 \in \mathbb{C}$

$$\begin{aligned} g'(z_0) &= \lim_{z\to 0} \frac{g(z+z_0)-g(z_0)}{z} \\ &= \lim_{z\to 0} \frac{F((z+z_0)\xi+\eta)-F(z_0\xi+\eta)}{z} \\ &= (D_\xi F)(z_0\xi+\eta) \end{aligned}$$

Now the claim follows by iteration using (i).
Proof of (iii)
By (ii) the Taylor expansion of $g(z) = F(z\xi+\eta) = \sum_{n=0}^{\infty} \frac{(D_\xi^n F)(\eta)}{n!}\, z^n$ □

The next proposition can be found in [40, 64, 57].

Proposition 4.1.4. *For $\psi, \phi \in (\mathcal{N})^\beta$, $0 \le \beta \le 1$, and $\xi \in \mathcal{N}'$ the Gâteaux derivative has the following properties:*

(i) The Wiener-Itô-Segal chaos decomposition of $D_\xi\phi$ is given by

$$D_\xi\phi(\omega) = \sum_{n=0}^{\infty} n\langle \phi^{(n)}, :\omega^{\otimes(n-1)}: \hat{\otimes}\xi\rangle, \quad \omega \in \mathcal{N}', \tag{4.4}$$

where $\phi^{(n)}$ denotes the n-th kernel in the Wiener-Itô-Segal chaos decomposition of $\phi \in (\mathcal{N})^\beta$.

(ii)

$$D_\xi\phi\psi = (D_\xi\phi)\psi + \phi D_\xi\psi.$$

(iii) For $\omega \in \mathcal{N}'$ we have

$$(D_\xi\varphi)(\omega) = \lim_{h\to 0} \frac{\phi(\omega+h\xi)-\phi(\omega)}{h}, \tag{4.5}$$

see e.g. [64, Cor.4.3.6, p.79].

Definition 4.1.5. *For $\eta \in \mathcal{N}'_{\mathbb{C}}$ with $\eta = \eta_1 + i\eta_2$, $\eta_1, \eta_2 \in \mathcal{N}'$ we define*

$$D_\eta = D_{\eta_1} + iD_{\eta_2}.$$

Proposition 4.1.6. *Let $\xi, \eta \in \mathcal{N}_{\mathbb{C}}$ and $\Phi \in (\mathcal{N})'$. Let S denote the S-transform of Φ. Then*

$$(D_\xi^k S(\Phi))(\eta) = S(D_\xi^k \Phi)(\eta), \quad k \in \mathbb{N}$$

Proof. Note that $(\mathcal{N})'$ is a locally convex Hausdorff space. Suppose that

$$\Phi = \sum_{n=0}^{\infty} \langle F^{(n)}, :\cdot^{\otimes}:\rangle, \quad F^{(n)} \in \mathcal{N}_{\mathbb{C}}'^{\hat{\otimes} n}.$$

Then by [64, proof of Theorem 3.3.7, p.50] we have

$$\begin{aligned} S(\Phi)(z\xi + \eta) &= \sum_{k=0}^{\infty} \left(\sum_{n=0}^{\infty} \binom{n+k}{k} \langle F_{n+k}, \xi^{\otimes k} \otimes \eta^{\otimes n} \rangle) \right) z^k \\ &= \sum_{k=0}^{\infty} \left(\sum_{n=0}^{\infty} \binom{n+k}{k} \langle \xi^{\otimes k} \widehat{\otimes}_k F_{n+k}, \eta^{\otimes n} \rangle) \right) z^k \\ &= \sum_{k=0}^{\infty} \frac{S(D_\xi^k \Phi)(\eta)}{k!} z^k \end{aligned}$$

The claim follows by 4.1.3, (iii). □

Corollary 4.1.7. *Let $\xi_1, \cdots \xi_n,\ \eta \in \mathcal{N}_{\mathbb{C}}$ and $\Phi \in (\mathcal{N})'$. Let S denote the S-transform of Φ. Then*

$$(D_{\xi_1} \cdots D_{\xi_n} S(\Phi))(\eta) = S(D_{\xi_1} \cdots D_{\xi_n} \Phi)(\eta)$$

Proof. By [64, Theorem 3.3.7, p. 50] we have $S(\Phi) \in \mathfrak{A}(\mathcal{N}_{\mathbb{C}})$. By [64, Theorem 4.3.12, p. 88] we have $D_\xi \in L((\mathcal{N})', (\mathcal{N})')$ for all $\xi \in \mathcal{N}_{\mathbb{C}}$. Then the claim follows by iterated application of 4.1.6.
Another proof is given by the following: By [64, Lemma 3.6.5, p. 65].
The function

$$(\xi_1, \cdots, \xi_n) \mapsto D_{\xi_1} \cdots D_{\xi_n} \Phi(\eta)$$

is a symmetric n-linear form. The claim follows now by 4.1.6 and the polarization identity. □

With the help of (4.4) we can extend D_η also to regular distributions $\phi \in \mathcal{G}'$. This leads to the next proposition.

Proposition 4.1.8. *Let $\eta \in \mathcal{H}_{\mathbb{C}}$. Further let $\alpha, \gamma \in \mathbb{R}$, $\gamma < \alpha$. Then for all $\varphi \in \mathcal{G}'$ with $|\varphi|_{0,\alpha} < \infty$*

$$|D_\eta \varphi|_{0,\gamma} \leq |\eta|_0 \sqrt{\frac{2^{-\gamma}}{(\alpha-\gamma)e\ln(2)}} \quad |\varphi|_{0,\alpha} \tag{4.6}$$

As an immediate consequence we have

$$\begin{aligned} &D_\eta \in L(\mathcal{G}, \mathcal{G}) \\ &D_\eta \in L(\mathcal{M}, \mathcal{M}) \\ &D_\eta \in L(\mathcal{M}', \mathcal{M}') \\ &D_\eta \in L(\mathcal{G}', \mathcal{G}') \end{aligned}$$

Proof.

$$D_\eta(\varphi) = \sum_{n=0}^{\infty} (n+1)\langle\langle \eta, \varphi^{(n+1)}\rangle, :\cdot^{\otimes n}:\rangle$$

$$\begin{aligned} |D_\eta(\varphi)|_{0,\gamma}^2 &\leq \sum_{n=0}^{\infty} n!\ 2^{\gamma n} \cdot (n+1)^2\, |\eta|_0^2 \cdot \left|\varphi^{(n+1)}\right|_0^2 \\ &\leq |\eta|_0^2\ 2^{-\alpha} \left\{ \sup_{n\in\mathbb{N}} (2^{(\gamma-\alpha)n} \cdot (n+1)) \right\} |\varphi|_{0,\alpha}^2 , \end{aligned}$$

note that $\sup_{n\in\mathbb{N}}(2^{(\gamma-\alpha)n} \cdot (n+1)) < \infty$.

Furthermore

$$\begin{aligned} |\eta|_0^2\ 2^{-\alpha} \left\{ \sup_{n\in\mathbb{N}} (2^{(\gamma-\alpha)n} \cdot (n+1)) \right\} |\varphi|_{0,\alpha}^2 &\leq |\eta|_0^2\ 2^{-\alpha} \frac{2^{\alpha-\gamma}}{(\alpha-\gamma)e\ \ln(2)} |\varphi|_{0,\alpha}^2 \\ &= |\eta|_0^2\ \frac{2^{-\gamma}}{(\alpha-\gamma)e\ln(2)} |\varphi|_{0,\alpha}^2 \end{aligned}$$

(for the not straight-forward estimation in the proof, see [64, Eq. (4.4)ff, p. 72]) Concerning the further claims, we calculate explicitly the case of $\mathcal{M}$, which has an inductive topology. At this purpose we use the universal mapping property of $\mathcal{M}$. We show that for all $\alpha > 0$ the mapping $D_\eta :\ G_\alpha \longrightarrow \mathcal{M}$ is continuous. Let $\alpha > 0$. As just proven, we can choose $\gamma \in \mathbb{R}$ with $0 < \gamma < \alpha$ such that $D_\eta :\ G_\alpha \longrightarrow G_\gamma \hookrightarrow \mathcal{M}$ is continuous. Hence $D_\eta :\ \mathcal{M} \hookrightarrow \mathcal{M}$ is continuous by the universal mapping property.

We verify the case of $\mathcal{G}$ with a projective topology.
Let $\gamma > 0$. Then there exists $\alpha > 0$, such that (4.6) holds. This means that $D_\eta : \mathcal{G} \hookrightarrow \mathcal{G}$ is continuous.
Note the different assumptions in these cases. The other cases are similarly treated. □

Next we give the definitions for the so called Gross Laplacian. Compare e.g. [40, 57]

Definition 4.1.9. *Let $\varphi \in (\mathcal{N})^\beta$, $\varphi = \sum_{n=0}^\infty \langle \varphi^{(n)}, : \cdot^{\otimes n} :\rangle$ then we define the Gross Laplacian of φ by*

$$\Delta_G \varphi(\omega) = \sum_{n=0}^{\infty} (n+2)(n+1) \left\langle \langle \varphi^{(n+2)}, tr \rangle, : \omega^{\otimes n} : \right\rangle .$$

The next proposition can be found in [86].

Proposition 4.1.10. *The Gross Laplacian is a continuous operator from $(\mathcal{N})^\beta$ to $(\mathcal{N})^\beta$ for $0 \leq \beta \leq 1$.*

4.2 Scaling Operator

First note, that every test function $\varphi \in (\mathcal{N})$ can be extended to $\mathcal{N}'_{\mathbb{C}}$, see e.g. [57]. Thus the following definition makes sense.

Definition 4.2.1. *Let φ be the continuous version of an element of $(\mathcal{N})^\beta$ for $0 \leq \beta \leq 1$. Then for $0 \neq z \in \mathbb{C}$ we define the scaling operator σ_z by*

$$(\sigma_z \varphi)(\omega) = \varphi(z\omega), \quad \omega \in \mathcal{N}'.$$

Proposition 4.2.2. *Let $0 \leq \beta \leq 1$ then*

(i) for all $0 \neq z \in \mathbb{C}$ we have $\sigma_z \in L((\mathcal{N})^\beta, (\mathcal{N})^\beta)$,

(ii) for $\varphi, \psi \in (\mathcal{N})^\beta$ we have

$$\sigma_z(\varphi \cdot \psi) = (\sigma_z \varphi)(\sigma_z \psi).$$

Proof. (i) is proved in [87]
For (ii), first note that $(\mathcal{N})^\beta$ is an algebra under pointwise multiplication. Since the scaling operator is continuous from $(\mathcal{N})^\beta$ to itself by (i), it suffices to show the assumption for the set of Wick ordered exponentials. Since this set is total the rest follows by a density argument. We have for $\xi, \eta \in \mathcal{N}$,

$$\sigma_z(:\exp(\langle\xi,\cdot\rangle):\cdot:\exp(\langle\eta,\cdot\rangle):) = \sigma_z(\exp\left(-\frac{1}{2}(\langle\xi,\xi\rangle+\langle\eta,\eta\rangle)\right)\exp(\langle\xi+\eta,\omega\rangle))$$

$$= \exp\left(-\frac{1}{2}(\langle\xi,\xi\rangle+\langle\eta,\eta\rangle)\right)\exp(\langle\xi+\eta,z\omega\rangle)$$

$$= \exp\left(-\frac{1}{2}(\langle\xi,\xi\rangle+\langle\eta,\eta\rangle)\right)\exp(\langle z\xi,\omega\rangle)\exp(\langle z\eta,\omega\rangle)$$

on the other hand

$$\sigma_z(:\exp(\langle\xi,\cdot\rangle):)\cdot\sigma_z(:\exp(\langle\eta,\cdot\rangle:)) = \exp(-\frac{1}{2}\langle\xi,\xi\rangle)\exp(\langle\xi,z\omega\rangle)\exp(-\frac{1}{2}\langle\eta,\eta\rangle)\exp(\langle\eta,z\omega\rangle),$$

which proves the assumption. □

More precisely we have, compare to [86] and [87] the following proposition.

Proposition 4.2.3. *Let $\varphi \in (\mathcal{N})$, $z \in \mathbb{C}$, then*

$$\sigma_z\varphi = \sum_{n=0}^{\infty}\langle\varphi_z^{(n)}, :\cdot^n:\rangle,$$

with kernels

$$\varphi_z^{(n)} = z^n\sum_{k=0}^{\infty}\frac{(n+2k)!}{k!n!}\left(\frac{z^2-1}{2}\right)^k\cdot tr^k\varphi^{(n+2k)}.$$

Definition 4.2.4. *Since σ_z is a continuous mapping from $(\mathcal{N})^{\beta}$, $0\le\beta\le 1$ to itself we can define its dual operator $\sigma_z^{\dagger}:(\mathcal{N})^{-\beta}\to(\mathcal{N})^{-\beta}$ by*

$$\langle\!\langle\varphi,\sigma_z^{\dagger}\Phi\rangle\!\rangle = \langle\!\langle\sigma_z\varphi,\Phi\rangle\!\rangle,$$

for $\Phi\in(\mathcal{N})^{-\beta}$ and $\varphi\in(\mathcal{N})^{\beta}$.

The following proposition can be found in [87] and [86].

Proposition 4.2.5. *Let $\Phi\in(\mathcal{N})^{-\beta}$, $\varphi,\psi\in(\mathcal{N})^{\beta}$ and $z\in\mathbb{C}$ then we have*

(i)

$$\sigma_z^{\dagger}\Phi = J_z\diamond\Gamma_z\Phi,$$

where Γ_z *is defined by*

$$S(\Gamma_z\Phi)(\xi) = S(\Phi)(z\xi), \quad \xi \in \mathcal{N},$$

and $J_z = \mathrm{Nexp}(-\frac{1}{2}z^2\langle\cdot,\cdot\rangle)$. *In particular we have*

$$\sigma_z^\dagger \mathbb{1} = J_z.$$

(ii) $J_z\varphi = \sigma_z^\dagger(\sigma_z\varphi)$.

The next theorem gives the representation of the scaling operator by differential operators, see e.g. [86] and [87].

Theorem 4.2.6. *For any* $z \in \mathbb{C}$, $z \neq 0$, *we can represent the scaling operator on* $(\mathcal{N})^\beta$, $0 \leq \beta < 1$, *by*

$$\sigma_z = \Gamma_z \exp(\frac{z^2-1}{2}\Delta_G) := \Gamma_z \sum_{k=0}^{\infty} \frac{1}{k!}\left(\frac{z^2-1}{2}\right)^k \Delta_G^k,$$

where Δ_G *denotes Gross Laplacian. Moreover, for* $\varphi \in (\mathcal{N})^\beta$ *the* S*-transform is given by*

$$S(\sigma_z\varphi)(\xi) = S\left(\exp(\frac{z^2-1}{2}\Delta_G)\varphi\right)(z\xi), \quad \xi \in \mathcal{N}.$$

4.3 Translation Operator

In this section we investigate the so-called translation operator. Since the Gaussian measure on $\mathcal{N}'$ is not translation invariant, the translation operator gives - in comparison to the translation w.r.t. Lebesgue measure - a non-trivial contribution to the functional.
First not that, from the measure theoretical point of view, a shift of the variable by $y \in \mathcal{H}$ yields

$$\int_{\mathcal{N}'} \exp(i\langle\omega,\xi\rangle)d\mu(\omega - y) = \int_{\mathcal{N}'} \exp(i\langle\omega + y,\xi\rangle)d\mu(\omega)$$
$$= \exp(i\langle y,\xi\rangle)\int_{\mathcal{N}'} \exp(i\langle\omega,\xi\rangle)d\mu(\omega) = \exp(-\frac{1}{2}\langle\xi,\xi\rangle \exp(i\langle y,\xi\rangle).$$

On the other hand

$$\int_{\mathcal{N}'} \exp(i\langle\omega,\xi\rangle) : \exp(\langle\omega,y\rangle) : d\mu(\omega) = \exp(-\frac{1}{2}\langle y,y\rangle) \int_{\mathcal{N}'} \exp(i\langle\omega,\xi - iy\rangle) d\mu(\omega)$$
$$= \exp(-\frac{1}{2}\langle y,y\rangle) \exp(-\frac{1}{2}\langle \xi - iy, \xi - iy\rangle)$$
$$= \exp(-\frac{1}{2}\langle y,y\rangle) \exp(-\frac{1}{2}\langle \xi,\xi\rangle) \exp(i\langle y,\xi\rangle) \exp(\frac{1}{2}\langle y,y\rangle)$$
$$= \exp(-\frac{1}{2}\langle \xi,\xi\rangle \exp(i\langle y,\xi\rangle).$$

Thus $: \exp(\langle\omega,y\rangle) :$ can be considered as the generalized Radon-Nikodym density of $\mu_y := \mu(. - y)$ w.r.t. μ.

We define the translation operator (or shift operator) as follows:

Definition 4.3.1. *Let $y \in \mathcal{N}'$ and $\varphi \in (\mathcal{N})$ we define the translation operator τ_y by*

$$\tau_y \varphi(\omega) = \varphi(\omega + y).$$

See e.g. [87, 86]

Theorem 4.3.2. *The translation operator τ_y for $y \in \mathcal{N}'$ is continuous from $(\mathcal{N})$ into itself.*

Now we want to take a closer look on the action of the translation operator on a Hida test function. For this we first state the following lemma which can be found in [57].

Lemma 4.3.3 (Kuo 96, S.74 Lemma 7.16)**.**
For any $x, y \in \mathcal{N}'$ and any $n \geq 1$,

$$: (x+y) :^{\otimes n} := \sum_{k=0}^{n} \binom{n}{k} : x^{\otimes(n-k)} : \hat{\otimes} y^{\otimes k}. \tag{4.7}$$

There exists also a representation of the translation operator corresponding to differential operators. The next theorem is stated and proved in [86] compare also [87]

Theorem 4.3.4. *For $\eta \in \mathcal{N}'$ the translation operator on $(\mathcal{N})$ can be represented as*

$$\tau_\eta = \exp(D_\eta) := \sum_{k=0}^{\infty} \frac{1}{k!} D_\eta^k.$$

Thus we define for $\theta \in \mathcal{N}'_{\mathbb{C}}$

$$\tau_\theta = \exp(D_\theta) := \sum_{k=0}^{\infty} \frac{1}{k!} D_\theta^k, \tag{4.8}$$

which is a continuous operator on $(\mathcal{N})$. *Note that the series in* (4.8) *is convergent in* $L((\mathcal{N}),(\mathcal{N}))$. *Furthermore, for* $\varphi \in (\mathcal{N})$ *its* S*-transform in* $\xi \in \mathcal{N}$ *is given by*

$$S(\tau_\eta)(\xi) = S(\exp(D_\eta)\varphi)(\xi) = \exp(D_\eta)S(\varphi)(\xi) = \sum_{k=0}^{\infty} \frac{1}{k!} D_\eta^k S(\varphi)(\xi).$$

Proof. We give a sketch of the proof.
Let $\varphi \in (\mathcal{N})$ with

$$\varphi = \sum_{n=0}^{\infty} \langle \varphi^{(n)}, : \cdot^{\otimes n} : \rangle,$$

then for $y \in \mathcal{N}'$, one obtains

$$\tau_y \varphi(\omega) = \varphi(\omega + y) = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \binom{n+k}{n} \langle \langle \varphi^{(n+k)}, y^{\otimes k}\rangle, : \omega^{\otimes n} : \rangle = \sum_{k=0}^{\infty} \frac{1}{k!} D_\eta^k \varphi$$

□

Proposition 4.3.5. *Let* $\eta \in \mathcal{H}$. *Further let* $\alpha, \gamma \in \mathbb{R}$, $\gamma < \alpha$. *Then there exists* $K(\gamma, \alpha, \eta) > 0$, *such that for all* $\varphi \in \mathcal{G}'$ *with* $|\varphi|_{0,\alpha} < \infty$ *we have*

$$|\tau_\eta \varphi|_{0,\gamma} \le K(\gamma, \alpha, \eta)\, |\varphi|_{0,\alpha} \tag{4.9}$$

Proof. Consider τ_η as in Theorem 4.3.4 then we have

$$\begin{aligned}
|\tau_\eta \varphi|_{0,\gamma} = |\exp(D_\eta)\varphi|_{0,\gamma} &= \left|\sum_{n\in\mathbb{N}} \frac{1}{n!} D_\eta^n \varphi\right|_{0,\gamma} = \left|\sum_{n\in\mathbb{N}} \frac{1}{n!} \sum_{k\in\mathbb{N}} \frac{(k+n)!}{k!} \langle\langle \eta^{\otimes n}, \varphi_{k+n}\rangle, : \cdot^{\otimes k} : \rangle\right|_{0,\gamma} \\
&\le \sum_{n\in\mathbb{N}} \frac{1}{n!} \sum_{k\in\mathbb{N}} \frac{(k+n)!}{k!} \left|\langle\langle \eta^{\otimes n}, \varphi_{k+n}\rangle, : \cdot^{\otimes k} : \rangle\right|_{0,\gamma} \\
&\le \sum_{n\in\mathbb{N}}\sum_{k\in\mathbb{N}} \sqrt{k! 2^{k\gamma}} \frac{\sqrt{(k+n)!}}{k! n!} |\eta|_0^n\, 2^{-\frac{\alpha(k+n)}{2}} 2^{\frac{\alpha(k+n)}{2}} \sqrt{(k+n)!}\, |\varphi_{k+n}|_0 \\
&\le \sum_{n\in\mathbb{N}} \left(\sum_{k\in\mathbb{N}} 2^{k(\gamma-\alpha)} \frac{(k+n)!}{k! n!^2} 2^{-\alpha n} |\eta|_0^{2n} \right)^{\frac{1}{2}} \cdot |\varphi|_{0,\alpha},
\end{aligned}$$

by Cauchy-Schwartz inequality. Thus

$$\sum_{n\in\mathbb{N}}\left(\sum_{k\in\mathbb{N}}2^{k(\gamma-\alpha)}\frac{(k+n)!}{k!n!^2}2^{-\alpha n}\left|\eta\right|_0^{2n}\right)^{\frac{1}{2}}\cdot\left|\varphi\right|_{0,\alpha}$$

$$\leq\sum_{n\in\mathbb{N}}\frac{1}{\sqrt{n!}}2^{-\alpha n/2}\left|\eta\right|_0^n\left(\sum_{k\in\mathbb{N}}\binom{k+n}{k}2^{k(\gamma-\alpha)}\right)^{\frac{1}{2}}\cdot\left|\varphi\right|_{0,\alpha}$$

$$=\left\{\sum_{n\in\mathbb{N}}\frac{1}{\sqrt{n!}}2^{-\alpha n/2}\left|\eta\right|_0^n\left(1-2^{\gamma-\alpha}\right)^{-\frac{1}{2}(n+1)}\right\}\cdot\left|\varphi\right|_{0,\alpha},$$

where the infinite sum is finite by the root criterion and since $\gamma<\alpha$. Recall that we have the Stirling lower bound $\sqrt{2n\pi}\left(\frac{n}{e}\right)^n\leq n!$. □

Corollary 4.3.6. *As an immediate consequence we have for $\eta\in\mathcal{H}_{\mathbb{C}}$:*

$$\begin{aligned}&\tau_\eta\in L(\mathcal{G},\mathcal{G})\\&\tau_\eta\in L(\mathcal{M},\mathcal{M})\\&\tau_\eta\in L(\mathcal{M}',\mathcal{M}')\\&\tau_\eta\in L(\mathcal{G}',\mathcal{G}')\end{aligned}$$

Because of the continuity of the translation operator, we can define its adjoint operator $\tau_\eta^\dagger$.

Definition 4.3.7. *(i) Let $\eta\in\mathcal{H}_{\mathbb{C}}$ then we define $\tau_\eta^\dagger:(\mathcal{G})'\to(\mathcal{G})'$ by*

$$\langle\!\langle\psi,\tau_\eta^\dagger\varphi\rangle\!\rangle=\langle\!\langle\tau_\eta\psi,\varphi\rangle\!\rangle,$$

for $\varphi\in\mathcal{G}'$, $\psi\in\mathcal{G}$.

(ii) Let $\eta\in\mathcal{N}'_{\mathbb{C}}$ then we define $\tau_\eta^\dagger:(\mathcal{N})'\to(\mathcal{N})'$ by

$$\langle\!\langle\psi,\tau_\eta^\dagger\varphi\rangle\!\rangle=\langle\!\langle\tau_\eta\psi,\varphi\rangle\!\rangle,$$

for $\varphi\in(\mathcal{N})'$, $\psi\in(\mathcal{N})$.

For the adjoint operator we have the following properties, see e.g. [57, Thm. 10.25 p.142, Thm.10.26 p.143].

Proposition 4.3.8. *Let $y\in\mathcal{N}'$, then*

(i)

$$:\exp(\langle y,\cdot\rangle):\diamond\,\Phi=\tau_y^\dagger\Phi,\quad\Phi\in(\mathcal{N})^{-\beta}.$$

(ii)

$$: \exp(\langle y, \cdot \rangle) : \varphi = \tau_y^\dagger \tau_y \varphi, \quad \varphi \in (\mathcal{N})^\beta.$$

4.4 Generalized Scaling - Linear Transformation of the Measure

As mentioned in chapter 3 pointwise multiplication with a generalized Gauss kernel can be considered as a measure transformation. In a view to the previous sections we want to generalize the notion of scaling to bounded operators. More precisely we investigate for which kind of linear mappings $B \in L(\mathcal{N}', \mathcal{N}')$ there exists some operator $\sigma_B : (\mathcal{N}) \to (\mathcal{N})$ such that

$$\Phi_{(BB^*)} \cdot \varphi := \sigma_B^\dagger \sigma_B \varphi.$$

Further we state a generalization of the Wick formula to Gauss kernels. We start with the definition of σ_B.

Definition 4.4.1. *For $B \in L(\mathcal{N}'_{\mathbb{C}}, \mathcal{N}'_{\mathbb{C}})$ we define $\sigma_B \varphi$, $\varphi \in (\mathcal{N})$, via its chaos decomposition, which is given by*

$$\sigma_B \varphi = \sum_{n=0}^{\infty} \left\langle \varphi_B^{(n)}, : \cdot^{\otimes n} : \right\rangle, \tag{4.10}$$

with kernels

$$\varphi_B^{(n)} = \sum_{k=0}^{\infty} \frac{(n+2k)!}{k!n!} \left(-\frac{1}{2}\right)^k (B^*)^{\otimes n} (\mathrm{tr}^k_{(Id-BB^*)} \varphi^{(n+2k)}).$$

Here, B^* means the dual operator of B with respect to $\langle \cdot, \cdot \rangle$. Further for $A \in L(\mathcal{N}_{\mathbb{C}}, \mathcal{N}'_{\mathbb{C}})$, the expression $\mathrm{tr}^k_A \varphi^{(n+2k)}$ is defined by

$$\mathrm{tr}^k_A \varphi^{(n+2k)} := \left\langle \mathrm{tr}_A^{\otimes k}, \varphi^{(n+2k)} \right\rangle \in \mathcal{N}^{\hat{\otimes} n},$$

where the generalized trace kernel tr_A is defined in 1.2.7.

Next we show the continuity of the generalized scaling operator.

Proposition 4.4.2. *Let $B : \mathcal{N}'_{\mathbb{C}} \to \mathcal{N}'_{\mathbb{C}}$ a bounded operator. Then $\varphi \mapsto \sigma_B\varphi$ is continuous from $(\mathcal{N})$ into itself.*

Proof. Let $\varphi_B^{(n)}$ as in Definition 4.10. First choose $q_1 > 0$, such that $|tr_{Id-BB^*}|_{-q_1}^k < \infty$. Then, by 1.2.3, there exist $C(B) > 0,\ q_2 > q1$ such that

$$
\begin{aligned}
|\varphi_B^{(n)}|_p &= |\sum_{k=0}^{\infty} \frac{(n+2k)!}{k!n!}\left(-\frac{1}{2}\right)^k (B^*)^{\otimes n}(\mathrm{tr}_{Id-BB^*}^k \varphi^{(n+2k)})|_p \\
&\leq \frac{1}{\sqrt{n!}} C(B)^n \sum_{k=0}^{\infty} \sqrt{\binom{n+2k}{2k}} \frac{\sqrt{(2k)!}}{k!2^k} \sqrt{(n+2k)!} |tr_{Id-BB^*}|_{-q_2}^k |\varphi^{n+2k}|_{q_2}
\end{aligned}
$$

Since as in [87] it yields $\frac{\sqrt{(2k)!}}{k!2^k} < 1$ we have

$$
\begin{aligned}
&\frac{1}{\sqrt{n!}} C(B)^n \sum_{k=0}^{\infty} \sqrt{\binom{n+2k}{2k}} \frac{\sqrt{(2k)!}}{k!2^k} \sqrt{(n+2k)!} |tr_{Id-BB^*}|_{-q_2}^k |\varphi^{n+2k}|_{q_2} \\
&\qquad \leq \frac{1}{\sqrt{n!}} C(B)^n \sum_{k=0}^{\infty} \sqrt{\binom{n+k}{k}} \sqrt{(n+k)!} |tr_{Id-BB^*}|_{-q_2}^{\frac{k}{2}} |\varphi^{n+k}|_{q_2} \\
&\qquad \leq \frac{1}{\sqrt{n!}} C(B)^n 2^{-n\frac{q'}{2}} \left(\sum_{k=0}^{\infty} \binom{n+k}{k} 2^{-q'k} |tr_{Id-BB^*}|_{-q_2}^k \right)^{\frac{1}{2}} \\
&\qquad\qquad \times \left(\sum_{k=0}^{\infty} (n+k)! 2^{q'(n+k)} |\varphi^{(n+k)}|_{q_2}^2 \right)^{\frac{1}{2}} \\
&\qquad\qquad\qquad \leq \|\varphi\|_{q_2,q'} \frac{1}{\sqrt{n!}} 2^{-n\frac{q'}{2}} C(B)^n \left(1 - 2^{-q'} |tr_{Id-BB^*}|_{-q_2}\right)^{-\frac{n+1}{2}}.
\end{aligned}
$$

If q' fulfills

$$2^{-q'} |tr_{Id-BB^*}|_{-q_2} < 1,$$

we obtain

$$\|\sigma_B\varphi\|_{q_2,q}^2 \leq \|\varphi\|_{q_2,q'}^2 \cdot \sum_{n=0}^{\infty} 2^{n(q-q')} C(B)^{2n} \left(1 - 2^{-q'} |tr_{Id-BB^*}| - q_2\right)^{-(n+1)},$$

where the right hand side converges if $q' - q$ is large enough. □

Proposition 4.4.3. *Let $\varphi \in (\mathcal{N})$ given by its continuous version. Then it holds*

$$\sigma_B \varphi(\omega) = \varphi(B\omega),$$

if $B \in L(\mathcal{N}', \mathcal{N}')$, $\omega \in \mathcal{N}'$.

This can be proved directly by an explicit calculation on the the set of Wick exponentials, a density argument and a verifying of pointwise convergence, compare [64, Proposition 4.6.7, p. 104], last paragraph.
In the same manner the following statement is proved.

Proposition 4.4.4. *Let $B : \mathcal{N}' \to \mathcal{N}'$ be a bounded operator. For $\varphi, \psi \in (\mathcal{N})$ the following equation holds*

$$\sigma_B(\varphi\psi) = (\sigma_B\varphi)(\sigma_B\psi).$$

Since we consider a continuous mapping from $(\mathcal{N})$ into itself one can define the dual scaling operator with respect to $\langle \cdot, \cdot \rangle$, $\sigma_B^\dagger : (\mathcal{N})' \to (\mathcal{N})'$ by

$$\left\langle\!\left\langle \sigma_B^\dagger \Phi, \psi \right\rangle\!\right\rangle = \left\langle\!\left\langle \Phi, \sigma_B \psi \right\rangle\!\right\rangle,$$

The Wick formula as stated in [86, 36] for Donsker's delta function can be extended to Generalized Gauss kernels.

Proposition 4.4.5. *[Generalized Wick formula] Let $\Phi \in (\mathcal{N})^{-\beta}$, $0 \leq \beta < 1$, $\varphi, \psi \in (\mathcal{N})^{\beta}$ and $B \in L(\mathcal{N}', \mathcal{N}')$. then we have*

(i)

$$\sigma_B^\dagger = \Phi_{BB^*} \diamond {\Gamma_B}^* \Phi,$$

where ${\Gamma_B}^*$ is defined by

$$S({\Gamma_B}^*\Phi)(\xi) = S(\Phi)(B^*\xi), \quad \xi \in \mathcal{N}.$$

In particular we have

$$\sigma_B^\dagger \mathbb{1} = \Phi_{BB^*}.$$

(ii) $\Phi_{BB^*} \cdot \varphi = \sigma_B^\dagger(\sigma_B\varphi)$.

(iii) $\Phi_{BB^*} \cdot \varphi = \Phi_{BB^*} \diamond (\Gamma_{B^*} \circ \sigma_B(\varphi))$.

Proof. Proof of (i): Let $\Phi \in (\mathcal{N})^{-\beta}$ and $\xi \in \mathcal{N}$ then we have

$$\begin{aligned} S(\sigma_B^\dagger \Phi)(\xi) &= \langle\!\langle : \exp(\langle \xi, \cdot \rangle) :, \sigma_B^\dagger \Phi \rangle\!\rangle \qquad = \langle\!\langle \sigma_B : \exp(\langle \xi, \cdot \rangle) :, \Phi \rangle\!\rangle \\ &= \exp(-\frac{1}{2}\langle \xi, \xi \rangle) \langle\!\langle \exp(\langle B^* \xi, \cdot \rangle), \Phi \rangle\!\rangle \\ &= \exp(-\frac{1}{2}\langle \xi, Id - BB^* \xi \rangle) S(\Phi)(B^* \xi) \\ &= S(\Phi_{BB^*})(\xi) \cdot S(\Gamma_{B^*} \Phi)(\xi) \end{aligned}$$

Proof of (ii): First we have $\sigma^\dagger \mathbb{1} = \Phi_{BB^*} \diamond \Gamma_{B^*} \mathbb{1} = \Phi_{BB^*}$.
Thus for all $\varphi, \psi \in (\mathcal{N})^\beta$

$$\langle\!\langle \Phi_{BB^*} \varphi, \psi \rangle\!\rangle = \langle\!\langle \sigma^\dagger \mathbb{1}, \varphi \cdot \psi \rangle\!\rangle = \\ \langle\!\langle \mathbb{1}, (\sigma_B \varphi)(\sigma_B \psi) \rangle\!\rangle = \langle\!\langle (\sigma_B \varphi), (\sigma_B \psi) \rangle\!\rangle = \langle\!\langle \sigma_B^\dagger (\sigma_B \varphi), \psi \rangle\!\rangle$$

Proof of (iii): Immediate from (i) and (ii).

□

Remark 4.4.6. *The scaling operator can be considered as a linear measure transform. Let $\varphi \in \mathcal{N}$ and B a real bounded operator on $\mathcal{N}'$. Then we have*

$$\int_{\mathcal{N}'} \sigma_B \varphi(\omega) \, d\mu(\omega) = \int_{\mathcal{N}'} \varphi(B\omega) \, d\mu(\omega) = \int_{\mathcal{N}'} \varphi(\omega) \, d\mu(B^{-1}\omega)$$

Moreover we have

$$\int_{\mathcal{N}'} \exp(i \langle \xi, \omega) \, d\mu(B^{-1}\omega) = \exp(-\frac{1}{2}\langle B^* \xi, B^* \xi \rangle,$$

which is a characteristic function of a probability measure by the Theorem of Bochner and Minlos. Furthermore

$$\int_{\mathcal{N}'} \exp(i \langle \xi, \omega) \, d\mu(B^{-1}\omega) = T(\sigma^\dagger \mathbb{1})(\xi),$$

such that Φ_{BB^*} is represented by the positive Hida measure $\mu \circ B^{-1}$.

At the end of this section for the sake of completeness we present another formula for σ_B, which uses an integral kernel operator, for the definition see [64, Propsition 4.3.3, p.82].

Proposition 4.4.7. *Let $B \in L(\mathcal{N}', \mathcal{N}')$. Then*

$$\sigma_B = \Gamma_{B^*} \circ \exp(-\frac{1}{2}(\Xi_{0,2}((Id^{\otimes 2} - B^{\otimes 2})(\tau))))$$

Proof. Let $B \in L(\mathcal{N}', \mathcal{N}')$ and $\xi, \eta \in \mathcal{N}$, then

$$\begin{aligned}
S\left(\sigma_B : \exp(\langle \eta, \cdot \rangle) :\right)(\xi) &= \langle\!\langle : \exp(\langle \xi, \cdot \rangle) :, (\sigma_B : \exp(\langle \eta, \cdot \rangle) :\rangle\!\rangle \\
&= \langle\!\langle : \exp(\langle \xi, \cdot \rangle) :, \exp(\langle B^*\eta, \cdot \rangle\rangle\!\rangle \exp(-\frac{1}{2}\langle \eta, \eta \rangle) \\
&= \langle\!\langle : \exp(\langle \xi, \cdot \rangle) :, : \exp(\langle B^*\eta, \cdot \rangle :\rangle\!\rangle \exp(-\frac{1}{2}\langle \eta, \eta \rangle) \exp(\frac{1}{2}\langle B^*\eta, B^*\eta \rangle) \\
&= \langle\!\langle : \exp(\langle \xi, \cdot \rangle) :, : \exp(\langle B^*\eta, \cdot \rangle :\rangle\!\rangle \exp(-\frac{1}{2}\langle \eta, (Id - BB^*)\eta \rangle) \\
&= \exp(\langle \xi, B^*\eta \rangle) \exp(-\frac{1}{2}\langle \eta, (Id - BB^*)\eta \rangle) \\
&= S(\Gamma_{B^*} : \exp(\langle \eta, \cdot \rangle) :)(\xi) \cdot S(\Phi_{BB^*})(\eta) \\
&= S(\Gamma_{B^*} \{S(\Phi_{BB^*})(\eta) : \exp(\langle \eta, \cdot \rangle) :\})(\xi) \\
&= S(\Gamma_{B^*} \circ \exp(-\frac{1}{2}(\Xi_{0,2}((Id^{\otimes 2} - B^{\otimes 2})(\tau)))) : \exp(\langle \eta, \cdot \rangle) :)(\xi)
\end{aligned}$$

□

Notice that $\exp\Big(-\frac{1}{2}\big(\Xi_{0,2}(Id^{\otimes 2} - B^{\otimes 2})(\tau)\big)\Big)$ is just the convolution operator corresponding to Φ_{BB^*}. The convolution operator will be treated in detail in a later section.

4.5 Convolution Operators

In this section we introduce the so called convolution operators. We will see that these operators give an easy characterization of operators to be in $L(\mathcal{G}, \mathcal{G})$ with the help of e.g. Proposition 4.5.9. Furthermore a convolution operators allows a representation of a dual pairing as an integral w.r.t the Gaussian measure μ, i.e. we consider the expression $\langle\!\langle \varphi, \Psi \rangle\!\rangle$ with $\varphi \in (\mathcal{N})$ and $\Psi \in (\mathcal{N})'$ and ask ourselves, in which way can we represent this as an integral. We have already seen representations like that:

- $\Psi \in (L^2)$. In this case can consider Ψ as a Radon-Nikodym-density and have

$$\langle\!\langle \varphi, \Psi \rangle\!\rangle = \int_{\mathcal{N}'} \varphi(\omega) \cdot \Psi(\omega)\, d\mu(\omega).$$

- We can consider generalized Radon-Nikodym-derivatives given by a generalized scaling with an operator B. In this case $\Phi_{BB^*} = \Psi \in GGK$, i.e.

$$\langle\!\langle \varphi, \Psi \rangle\!\rangle = \int_{\mathcal{N}'} \varphi(\omega)\, d\mu \circ B^{-1}(\omega)$$

- The last case is the case when we use convolution operators. $\langle\!\langle \varphi, \Psi \rangle\!\rangle$ can be then written as an integral of a convolution operator w.r.t. the Gaussian measure μ, i.e.

$$\langle\!\langle \varphi, \Psi \rangle\!\rangle = \int_{\mathcal{N}'} C_\Psi(\varphi)\, d\mu,$$

  as in Corollary 4.5.5.

Recently in [65, Theorem 5.6, p. 671] it was stated, that there exists an injective vector algebra homomorphism between $(\mathcal{N})'$ with the Wick product and a subalgebra of $L((\mathcal{N}),(\mathcal{N}))$.
We show, that this homomorphism is even a topological isomorphism to the image space. Our aim is the application of the techniques from [65] to regular distributions as $\mathcal{G}'$ and $L(\mathcal{G},\mathcal{G})$. We give a proof in which the continuity of the Wick product plays a key role.
Let X be a reflexive (F)-space and X' its dual space with the strong dual topology. Recall that $L(X,X)$ is equipped with the *topology of bounded convergence*, namely the locally convex topology is defined by the seminorms:

$$\|T\|_{B,p} := \sup_{\xi \in B} |T\xi|_p\,,\ T \in L(X,X) \tag{4.11}$$

where B runs over the bounded subsets of X and $|\cdot|_p$ is an arbitrary seminorm on X. The topology on $L(X',X')$ is defined in the same way. We state the following proposition:

Proposition 4.5.1. *Let X be a reflexive (F)-space and X' it's dual space with the strong dual topology. Then the mapping $^*:\ L(X,X) \to L(X',X')$, $T \longmapsto T^*$ is continuous.*

Proof. Let B be a bounded subset of X, B' be a bounded subset of X' and $T \in L(X,X)$. Then, because X is reflexive, the claim follows by considering the strong dual topologies:

$$\sup_{b' \in B'} \left[\sup_{b \in B} |\langle T^*(b'), b \rangle| \right] = \sup_{b \in B} \left[\sup_{b' \in B'} |\langle b', T(b) \rangle| \right]. \tag{4.12}$$

□

Definition 4.5.2. ,

(i) *For all* $\phi \in (\mathcal{N})'$ *we denote by* $M_\phi^\diamond$ *the Wick multiplication operator* $\varphi \mapsto \phi \diamond \varphi$ *in* $L((\mathcal{N})', (\mathcal{N})')$

(ii) *For all* $\phi \in (\mathcal{N})'$ *we denote by* C_ϕ *the dual of the Wick multiplication operator, i.e.*

$$C_\phi := (M_\phi^\diamond)^* \in L((\mathcal{N}), (\mathcal{N}))$$

We summarize some properties of the convolution operators, see [65] for the proof of (i) and (ii).

Proposition 4.5.3. *Let* $\phi \in (\mathcal{N})'$. *Then*

(i) $C_\phi^*(\mathbb{1}) = \phi$,

(ii) $C_\phi(: \exp(\langle \xi, \cdot \rangle) :) = S(\phi)(\xi) \cdot : \exp(\langle \xi, \cdot \rangle) :, \quad \xi \in \mathcal{N}_\mathbb{C}$,

(iii) $C_{\langle \eta, \cdot \rangle} = D_\eta, \quad \eta \in \mathcal{N}'_\mathbb{C}$.

Proof. Proof of (iii): Note that we have for $\psi \in (\mathcal{N})'$ with

$$\psi = \sum_{n=0}^{\infty} \langle \psi^{(n)}, : \cdot^{\otimes n} : \rangle$$

with [64, Prop.4.1.4.,p.73] that

$$D_\eta^* \psi = \sum_{n=0}^{\infty} \langle \psi^{(n)} \hat{\otimes} \eta, : \cdot^{\otimes (n+1)} : \rangle = \langle \eta, \cdot \rangle \diamond \psi.$$

Then for any $\varphi \in (\mathcal{N})$, we obtain

$$\langle\!\langle D_\eta \varphi, \psi \rangle\!\rangle = \langle\!\langle \varphi, D_\eta^* \psi \rangle\!\rangle = \langle\!\langle \varphi, \langle \eta, \cdot \rangle \diamond \psi \rangle\!\rangle = \langle\!\langle C_{\langle \eta, \cdot \rangle}(\varphi), \psi \rangle\!\rangle.$$

□

Theorem 4.5.4. *The map*

$$C : ((\mathcal{N})', +, ., \diamond) \longrightarrow (L((\mathcal{N}), (\mathcal{N})), +, ., \circ)$$

$$\phi \mapsto C_\phi$$

is a continuous, injective homomorphism.
Further C is an isomorphism onto Im(C).

Proof. By Obata and Ouerdiane, see [65, Theorem 5.6, p. 671], the proof of the first claim is routine. To prove the last claim, we use 4.5.1 and and write C^{-1} as composition of two continuous mappings:

$$C^{-1} : \mathcal{C}_\phi \mapsto \mathcal{C}_\phi{}^* \mapsto \mathcal{C}_\phi{}^*(\mathbb{1}) = \phi.$$

□

The above theorem enables us to represent the action of $\psi \in (\mathcal{N})'$ by the measure μ.

Corollary 4.5.5. *Let $\psi \in (\mathcal{N})'$. Then it holds*

$$\forall \varphi \in (\mathcal{N}) : \langle\langle \psi, \varphi \rangle\rangle = \int_{\mathcal{N}'} C_\psi(\varphi)\ d\mu$$

Proof. $\langle\langle \psi, \varphi \rangle\rangle = \langle\langle C^*_\psi(\mathbb{1}), \varphi \rangle\rangle = \langle\langle \mathbb{1}, C_\psi(\varphi) \rangle\rangle = \int_{\mathcal{N}'} C_\psi(\varphi)\ d\mu.$ □

Example 4.5.6. *Let $\eta \in \mathcal{N}$. Then the convolution operator acts by*

(i) $\langle \eta, \cdot \rangle \mapsto D_\eta$

(ii) $\langle \eta^{\otimes k}, : \cdot^{\otimes k} : \rangle \mapsto D^k_\eta$

(iii) $: \exp(\langle \eta, \cdot \rangle) :\mapsto exp(D_\eta) = \tau_\eta$

Example 4.5.7. *For any orthonormal basis $(e_n)_{n \in \mathbb{N}}$ of $L^2(\mathbb{R}, dt, \mathbb{C})$ the trace operator tr has the representation $tr = \sum_{n=0}^{\infty} e_n \otimes e_n$, see [64, Proposition 2.2.1, p. 24]. By [64, Eq. (5.33), p. 121] and 4.5.4, we have the following representation of the Gross Laplace operator Δ_G*

$$\begin{aligned} \Delta_G = \ & \mathcal{C}_{\langle tr, :\cdot^{\otimes 2}: \rangle} = \sum_{n=0}^{\infty} \mathcal{C}_{(\langle e_n, \cdot \rangle^{\diamond 2})} \\ = & \sum_{n=0}^{\infty} \mathcal{C}_{\langle e_n, \cdot \rangle} \circ \mathcal{C}_{\langle e_n, \cdot \rangle} = \sum_{n=0}^{\infty} D_{e_n} \circ D_{e_n} \end{aligned}$$

A further application of 4.5.4 concerns Donsker's delta function. At this purpose we formulate the analogue of 4.5.4 for $\mathcal{G}'$, which furnishes the characterization of $\mathcal{G}'$ as subspace of $L(\mathcal{G}, \mathcal{G})$.

Lemma 4.5.8. *Let $\alpha \in \mathbb{R}$ and $f,\ g \in \mathcal{G}_{\alpha+2}$. Then it holds*

$$|f \diamond g|_{0,\alpha} \leq |f|_{0,\alpha+2} \cdot |g|_{0,\alpha+2}$$

The Wick product is separately continuous from $\mathcal{G}' \times \mathcal{G}' \longrightarrow \mathcal{G}'$ and continuous from $\mathcal{G} \times \mathcal{G} \to \mathcal{G}$

Proof. Let $f \sim (f_n)_{n\in\mathbb{N}_0}$ and $g \sim (g_n)_{n\in\mathbb{N}_0}$. For $n \in \mathbb{N}_0$ let $h_n := \sum_{m=0}^{n} f_{n-m}\widehat{\otimes} g_m$.
Then $f \diamond g \sim (h_n)_{n\in\mathbb{N}_0}$. Following the ideas in [40, proof of Theorem 4.21., p. 101 ff.] we get:

$$\begin{aligned}
|f \diamond g|_{0,\alpha}^2 &= \sum_{n=0}^{\infty} n! 2^{\alpha n} |h_n|_0^2 \leq \sum_{n=0}^{\infty} n! 2^{\alpha n}(n+1) \sum_{m=0}^{n} |f_{n-m}|_0^2 \cdot |g_m|_0^2 \\
&= \sum_{n=0}^{\infty} 2^{\alpha n}(n+1) \sum_{m=0}^{n} \binom{n}{m} (n-m)! \, |f_{n-m}|_0^2 \cdot m! \, |g_m|_0^2 \\
&\leq \sum_{n=0}^{\infty} 2^{\alpha n}(n+1)2^n \sum_{m=0}^{n} (n-m)! \, |f_{n-m}|_0^2 \cdot m! \, |g_m|_0^2 \\
&= \sum_{n=0}^{\infty} 2^{(\alpha+1)n}(n+1) \sum_{m=0}^{n} (n-m)! \, |f_{n-m}|_0^2 \cdot m! \, |g_m|_0^2 \\
&\leq \sum_{n=0}^{\infty} 2^{(\alpha+2)n} \sum_{m=0}^{n} (n-m)! \, |f_{n-m}|_0^2 \cdot m! \, |g_m|_0^2 \\
&= \sum_{n=0}^{\infty} \sum_{m=0}^{n} (n-m)! 2^{(\alpha+2)(n-m)} |f_{n-m}|_0^2 \cdot m! 2^{(\alpha+2)m} |g_m|_0^2 \\
&= |f|_{0,\alpha+2}^2 \cdot |g|_{0,\alpha+2}^2
\end{aligned}$$

□

Theorem 4.5.9. *The map*

$$C : (\mathcal{G}', +, ., \diamond) \longrightarrow (L(\mathcal{G}, \mathcal{G}), +, ., \circ)$$

$$\phi \mapsto C_\phi$$

is a continuous, injective homomorphism.
Further C is an isomorphism onto $Im(C)$.

Proof. The proof of the homomorphy follows immediately by the properties of the Wick product. For the proof of injectivity, note that $C_\phi = 0$ implies $C_\phi^* = 0$, which implies $C_\phi^*(1) = \phi = 0$.
Note that $\mathcal{G}$ is reflexive and the topology of $\mathcal{G}$ is defined by the collection of the following seminorms, where B' runs over all bounded subsets of $\mathcal{G}'$.

$$\forall g \in \mathcal{G}: \ |g|_{B'} = \sup_{b' \in B'} |\langle g, b' \rangle|$$

Now let $\beta > 0$, $\phi \in \mathcal{G}_{-\beta}$, $B \subset \mathcal{G}$ be bounded and $B' \subset \mathcal{G}'$ be bounded, i.e. there exists $\gamma > 0$, such that $\sup_{b' \in B'} |b'|_{0,-\gamma} = K(B') < \infty$. Then, using Lemma 4.5.8, we get

$$\begin{aligned}
\sup_{b \in B}(\sup_{b' \in B'} |\langle C_\phi(b), b' \rangle|) &= \sup_{b \in B}(\sup_{b' \in B'} \left|\langle b, M_\phi^\diamond(b') \rangle\right|) \\
&\leq \sup_{b \in B}(\sup_{b' \in B'} |b|_{0,\beta+\gamma+2} \cdot |\phi \diamond b'|_{0,-(\beta+\gamma+2)}) \\
&\leq \sup_{b \in B}(\sup_{b' \in B'} |b|_{0,\beta+\gamma+2} \cdot |\phi|_{0,-\beta} \ |b'|_{0,-\gamma}) \\
&\leq K(B) \ K(B') \, |\phi|_{0,-\beta}),
\end{aligned}$$

for some $K(B) > 0$, because B is bounded.
The last claim is proved in the same way as in the proof of 4.5.4. □

Corollary 4.5.10. *Let $\psi \in \mathcal{G}'$. Then it holds*

$$\forall \varphi \in \mathcal{G}: \ \langle\langle \psi, \varphi \rangle\rangle = \int_{\mathcal{N}'} C_\psi(\varphi) \ d\mu$$

Proof. Compare with the proof of 4.5.5. Note, that for $\varphi \in \mathcal{G}$, we have

$$C_\psi(\varphi) \in \mathcal{G} \subset (L^2).$$

□

Remark 4.5.11. *For the proof of 4.5.9 the well-definedness of a Wick product is crucial.*

An analogue can thus also be proven for the spaces $(\mathcal{G})^{-\beta}$, $0 \leq \beta \leq 1$. The continuity of the Wick product in $(\mathcal{G})^{-1}$ was shown in [32] and [30].

Next we present some properties of Donsker's delta function, which are based on Definition 4.5.2 and an application of 4.5.9 and 4.5.10. For (ii) compare with [87, Proposition 72, p. 67].

Proposition 4.5.12. *Let $\eta \in \mathcal{H}_{\mathbb{C}}$, $\langle \eta, \eta \rangle \neq 0$ and $a \in \mathbb{C}$. Then*

(i) It holds $C_{\delta(\langle \eta, \cdot \rangle - a)} \in L((\mathcal{N}), (\mathcal{N}))$ and $C_{\delta(\langle \eta, \cdot \rangle - a)} \in L((\mathcal{G}), (\mathcal{G}))$ and

$$C_{\delta(\langle \eta, \cdot \rangle - a)} = \frac{1}{\sqrt{2\pi \langle \eta, \eta \rangle}} \left(\exp(-\frac{1}{2\langle \eta, \eta \rangle}(D_\eta - a\ Id)^2) \right)$$

(ii) For all $\varphi \in \mathcal{G}$ (resp. $(\mathcal{N})$) it holds

$$\langle\!\langle \varphi, \delta_a(\langle \eta, \cdot \rangle) \rangle\!\rangle = \frac{1}{\sqrt{2\pi \langle \eta, \eta \rangle}} \int_{\mathcal{N}'} \left(\exp(-\frac{1}{2\langle \eta, \eta \rangle}(D_\eta - a\ Id)^2) \right) \varphi \, d\mu$$

Proof. The proof of (i) is a direct application of 4.5.4 resp. 4.5.9.
The proof of (ii) is a direct application of 4.5.5 resp. 4.5.10.

□

At the end of this section we present a situation where convolution operators appear in a natural way.

Remark 4.5.13. *Let A be a continuous linear operator on $\mathcal{N}'$. For the second quantized operator $\Gamma(A)$ we use the shorter notation Γ_A. Note that $\Gamma_A \in L((\mathcal{N}), (\mathcal{N})')$ and that Γ_A has a continuous extension to an operator in $L((\mathcal{N})', (\mathcal{N})')$, which formally operates in the same way. We define $J_A \in (\mathcal{N})'$ by it's S-transform:*

$$S(J_A)(\xi) := \exp(-\frac{1}{2}\langle \xi, (Id - AA^*)\xi \rangle), \ for\ \xi \in \mathcal{N}_{\mathbb{C}}$$

and the generalized scaling operator σ_A by

$$\sigma_A \phi(\omega) := \phi(A\omega), \phi \in (\mathcal{N}), \omega \in \mathcal{N}'.$$

We calculate for $\xi \in \mathcal{N}_{\mathbb{C}}$

$$\begin{aligned}\sigma_A : \exp(\langle \xi, \omega \rangle) : &=: \exp(\langle \xi, A\omega \rangle) : \\ &= \exp(\langle \xi, A\omega \rangle - \frac{1}{2}\langle \xi, \xi \rangle) \\ &= \exp(\langle A^*\xi, \omega \rangle - \frac{1}{2}\langle A^*\xi, A^*\xi \rangle) \cdot \exp(-\frac{1}{2}(\langle \xi, \xi \rangle - \langle \xi, AA^*\xi \rangle) \\ &=: \exp(\langle A^*\xi, \omega \rangle) : \cdot \langle\langle : \exp(\langle \xi, \omega \rangle) :, J_A \rangle\rangle.\end{aligned}$$

This implies the formula

$$\sigma_A = \Gamma_{A^*} \circ C_{J_A}, \tag{4.13}$$

by 4.5.3(ii).
Further it follows by (4.13) that $\sigma_A \in L((\mathcal{N}), (\mathcal{N}))$. *(Note that we have indeed* $\sigma_A \phi(\omega) := \phi(A\omega)$ *for all* $\omega \in \mathcal{N}'$. *It is shown above for* $\phi =: \exp(\langle \xi, \cdot \rangle) :$, *where* $\xi \in \mathcal{N}_{\mathbb{C}}$. *For a general* $\phi \in (\mathcal{N})$ *take an approximating sequence* ψ_n *each of which is a linear combination of exponential vectors. Then* $\sigma_A \psi_n \to \sigma_A \phi$ *in* $(\mathcal{N})$, *and therefore pointwisely by [64, Theorem 3.2.13, p. 47], i.e.* $\sigma_A \psi_n(\omega) \to \sigma_A \phi(\omega)$ *for any* $\omega \in \mathcal{N}'$. *On the other hand, since* $\sigma_A \psi_n(\omega) = \psi_n(A\omega)$ *and* $\psi_n(y) \to \phi(y)$ *especially for* $y = A\omega$, *we conclude that* $(\sigma_A \phi)(\omega) = \phi(A\omega)$. *Compare the proof of [64, Proposition 4.6.7, p. 104].)*

Example 4.5.14. *Now let* $\eta \in \mathcal{N}, |\eta|_0 = 1$ *and* $A\omega = \omega - \langle \eta, \omega \rangle \eta$ *for all* $\omega \in \mathcal{N}'$.
Then A *is an extension of* A^*. *It follows for* $\xi \in \mathcal{N}_{\mathbb{C}}$:

$$\langle \xi, (Id - AA^*)\xi \rangle = \langle \eta, \xi \rangle^2$$

It follows $J_A = \exp^{\diamond}(-\frac{1}{2}\langle \eta, \cdot \rangle^{\diamond 2})$ *and* $C_{J_A} = \exp(-\frac{1}{2}D_\eta^2)$.
Defining $P_{\eta,\perp}$ *by*

$$P_{\eta,\perp}\omega := \omega - \langle \eta, \omega \rangle \eta, \quad \omega \in \mathcal{N}',$$

we have, by (4.13), for all $\phi \in (\mathcal{N})$:

$$\left(\Gamma(P_{\eta,\perp}) \circ \exp(-\frac{1}{2}D_\eta^2)\phi \right)(\omega) = \phi(P_{\eta,\perp}\omega), \quad \omega \in \mathcal{N}'.$$

With $y \in \mathcal{N}'$ and the corresponding translation operator τ_y, we get

$$\forall \omega \in \mathcal{N}': \left(\Gamma(P_{\eta,\perp}) \circ \exp(-\frac{1}{2}D_\eta^2) \circ \tau_y \phi\right)(\omega) = (\tau_y \phi)(P_{\eta,\perp}\omega) = \phi(y + P_{\eta,\perp}\omega)$$

In Corollary 5.5.15 it will be shown that this statement also holds if η is chosen in $L^2(\mathbb{R}, dx)$.

4.6 An Example: The Projection Operator

In this section we consider the generalized scaling operator in the case when the linear operator is an orthogonal projection. This will lead to the projection operator which was considered in [87] and [86, Chap.3.2.2,] respectively. This operator plays a key role in the Wick formula from [86, 36]. Unfortunately we will see that this operator is not closable on (L^2), i.e. an approximation will always depend on the sequence one uses.

Definition 4.6.1. *Let $y \in L^2(\mathbb{R}, dt, \mathbb{C})$. We define the operator*

$$P_{y,\perp} \in L(L^2(\mathbb{R}, dt)_\mathbb{C}, L^2(\mathbb{R}, dt)_\mathbb{C})$$

by

$$P_{y,\perp}: \; L^2(\mathbb{R}, dt, \mathbb{C}) \longrightarrow L^2(\mathbb{R}, dt, \mathbb{C})$$
$$\xi \mapsto \xi - \frac{\langle y, \xi \rangle}{\langle y, y \rangle} y$$

$P_{y,\perp}$ is a linear projection, which is an orthogonal projection if y is a real valued function, (in this case we have $|< y, y >| = |y|_0^2$).
In order to estimate $\Gamma(P_{y,\perp})$ we state the following lemma.

Lemma 4.6.2. *Let $T \in L(\mathcal{H}_\mathbb{C}, \mathcal{H}_\mathbb{C})$. Then*

(i) $\Gamma(T) \in L(\mathcal{G}, \mathcal{G})$

(ii) $\Gamma(T) \in L(\mathcal{G}', \mathcal{G}')$

(iii) If $\|T\| \leq 1$ then, for all $\gamma \in \mathbb{R}$, $\Gamma(T)$ is a contraction on $\mathcal{G}_\gamma$.

Proof. Let $\gamma \in \mathbb{R}$. Choose $\alpha \geq 0$, such that $\|T\| \leq 2^{\frac{\alpha}{2}}$. Then, using [70, VIII.10 Tensor products, Proposition p. 299], for $\varphi \in \mathcal{G}$ with

$$\varphi = \sum_{n=0}^{\infty} \langle \varphi_n, :\cdot^{\otimes n}: \rangle,$$

we have

$$\begin{aligned} |\Gamma(T)(\varphi)|_{0,\gamma}^2 &= \sum_{n \in \mathbb{N}} n!\; 2^{\gamma n} \left|T^{\otimes n}\varphi_n\right|_0^2 \\ &\leq \sum_{n \in \mathbb{N}} n!\; (2^{-\alpha n}\|T\|^{2n}) \cdot (2^{(\alpha+\gamma)\cdot n} |\varphi_n|_0^2) \\ &\leq |\varphi|_{0,\alpha+\gamma}^2 \end{aligned}$$

In order to show that $\Gamma(T) \in L(\mathcal{G}', \mathcal{G}')$, choose $\beta \in \mathbb{R}$. Then

$$\mathcal{G}_\beta \hookrightarrow \mathcal{G}_{\beta-\alpha} \hookrightarrow \mathcal{G}'$$

is continuous. Therefore by the universal mapping property of the inductive limit it follows that $\Gamma(T) \in L(\mathcal{G}', \mathcal{G}')$. □

Example 4.6.3. *Now let $\eta \in \mathcal{N}, |\eta|_0 = 1$ and $P_{\eta,\perp}\omega := \omega - \langle \omega, \eta\rangle\eta$, $\omega \in \mathcal{N}'$. Then $\Phi_{P_{\eta,\perp}\circ P^*_{\eta,\perp}} = \sqrt{2\pi}\delta(\langle \eta, \cdot\rangle)$, by Proposition 3.1.7, Proposition 4.4.5(iii) and Proposition 4.5.12 we obtain*

$$\delta(\langle \eta, \cdot\rangle) \cdot \varphi = \delta(\langle \eta, \cdot\rangle) \diamond \left(\Gamma(P_{\eta,\perp}) \circ \exp(-\frac{1}{2}D_\eta^2)\varphi\right).$$

Definition 4.6.4. *Let $\eta \in \mathcal{H}, |\eta|_0 = 1$. Then we denote by P_η the operator $P_\eta := \Gamma(P_{\eta,\perp}) \circ \exp(-\frac{1}{2}D_\eta^2)$ and call it* ***the projection operator with respect to*** *η. Note that $P_\eta \in L(\mathcal{G}, \mathcal{G})$ and $P_\eta \in L((\mathcal{N}), (\mathcal{N})')$.*

Lemma 4.6.5. *Let $\eta \in \mathcal{H}, |\eta|_0 = 1$ and $Y \in \mathcal{H}_{\mathbb{C}}^{\widehat{\otimes} j}$, $j \in \mathbb{N}$. Then it holds for all $k \in \mathbb{N}$ and $\alpha \in \mathbb{C}$:*

$$\left|(\alpha\eta)^{\otimes k} \widehat{\otimes} P_{\eta,\perp}^{\otimes j} Y\right|_0^2 = \frac{j!k!}{(j+k)!} \cdot \left|(\alpha\eta)^{\otimes k}\right|_0^2 \left|P_{\eta,\perp}^{\otimes j} Y\right|_0^2$$

Proof. For $k = 0$ or $j = 0$ there is nothing to prove. Thus let $j,\ k > 0$. Using the Fourier expansion of Y we obtain

$$Y = \sum_{n=0}^{\infty} y_{n,1} \widehat{\otimes} \cdots \hat{\otimes} y_{n,j}.$$

Let M_k denote the set of the subsets of $\{1, \cdots, j+k\}$ consisting of k elements. For $I \in M_k$ and $\pi \in \mathfrak{S}_j$ we define

$$f_n(\pi, I) := x_1 \otimes \cdots \otimes x_{j+k},$$

where

$$x_i = \eta \text{ for } i \in I$$

and for

$$1 \leq i_1 < \cdots < i_j \leq j+k \text{ with } \{i_1, \cdots, i_j\} \cap I = \emptyset \text{ we set } x_{i_l} := P_{\eta,\perp} y_{n,\pi(l)},\ 1 \leq l \leq j.$$

Then by a rearrangement of the series we get

$$\eta^{\otimes k} \hat{\otimes} P_{\eta,\perp}^{\otimes j} Y = \frac{1}{(j+k)!} \sum_{I \in M_k} \sum_{n=0}^{\infty} k! \sum_{\pi \in \mathfrak{S}_j} f_n(\pi, I).$$

Note that each expression $f_n(\pi, I)$ appears $k!$ times since we use $(j+k)$-permutations by the definition of the symmetric tensor product (each expression $(\eta, \cdots, \eta)$ permutates $k!$ times with itself!). The following two arguments are important for the discussion. For $I_1,\ I_2 \in M_k$ with $I_1 \neq I_2$ we have:

$$\sum_{n=0}^{\infty} \sum_{\pi \in \mathfrak{S}_j} f_n(\pi, I_1) \ \perp\ \sum_{n=0}^{\infty} \sum_{\pi \in \mathfrak{S}_j} f_n(\pi, I_2)$$

and

$$\left| \sum_{n=0}^{\infty} \sum_{\pi \in \mathfrak{S}_j} f_n(\pi, I_1) \right|_0^2 = \left| \sum_{n=0}^{\infty} \sum_{\pi \in \mathfrak{S}_j} f_n(\pi, I_2) \right|_0^2$$

The last equation holds since only the sequel of the tensors is different. The first equation follows by $\langle \eta, P_{\eta,\perp}(h) \rangle = 0$ for all $h \in \mathcal{H}_{\mathbb{C}}$. Hence

$$\begin{aligned} \left| \eta^{\otimes k} \widehat{\otimes} P_{\eta,\perp}^{\otimes j} Y \right|_0^2 &= \frac{1}{(j+k)!^2} \cdot \binom{j+k}{k} \left| \sum_{n=0}^{\infty} k! \sum_{\pi \in \mathfrak{S}_j} \eta^{\otimes k} \otimes P_{\eta,\perp} y_{n,\pi(1)} \otimes \cdots \otimes P_{\eta,\perp} y_{n,\pi(j)} \right|_0^2 \\ &= \frac{1}{(j+k)!^2} \cdot \binom{j+k}{k} (k!)^2 (j!)^2 \left| \sum_{n=0}^{\infty} \sum_{\pi \in \mathfrak{S}_j} \eta^{\otimes k} \otimes \frac{1}{j!} P_{\eta,\perp} y_{n,\pi(1)} \otimes \cdots \otimes P_{\eta,\perp} y_{n,\pi(j)} \right|_0^2 \end{aligned}$$

$$= \frac{1}{(j+k)!^2} \cdot \binom{j+k}{k} (k!)^2 (j!)^2 \left| \eta^{\otimes k} \otimes \sum_{n=0}^{\infty} P_{\eta,\perp}^{\otimes j} \frac{1}{j!} \sum_{\pi \in \mathfrak{S}_j} y_{n,\pi(1)} \otimes \cdots \otimes y_{n,\pi(j)} \right|_0^2$$

$$= \frac{j!k!}{(j+k)!} \cdot \left|\eta^{\otimes k}\right|_0^2 \left|P_{\eta,\perp}^{\otimes j} Y\right|_0^2,$$

since $|\cdot|_0$ is a cross norm. Now multiply both sides with $|\alpha|^{2k}$. □

Example 4.6.6. *Consider $\eta^{\otimes 2} \widehat{\otimes} x$, where $x = P_{\eta,\perp} y$. Then*

$$\begin{aligned} \eta^{\otimes 2} \widehat{\otimes} x &= \frac{1}{3!}(\eta \otimes \eta \otimes x + \eta \otimes \eta \otimes x + \eta \otimes x \otimes \eta + \eta \otimes x \otimes \eta + x \otimes \eta \otimes \eta + x \otimes \eta \otimes \eta) \\ &= \frac{1}{3}(\eta \otimes \eta \otimes x + \eta \otimes x \otimes \eta + x \otimes \eta \otimes \eta), \end{aligned}$$

where the expressions in the last term are orthogonal to each other with the same norm. Then

$$\left|\eta^{\otimes 2} \widehat{\otimes} x\right|_0^2 = \frac{1}{9} \cdot 3 \left|\eta\right|_0^4 \cdot \left|P_{\eta,\perp} y\right|_0^2.$$

Proposition 4.6.7. *Let $\eta \in \mathcal{H}, |\eta|_0 = 1$ and $\varphi \in \mathcal{G}'$ with $\Gamma(P_{\eta,\perp})\varphi \neq 0$. Then for any $\alpha \in \mathbb{R}$:*

$$|\delta(\langle \eta, \cdot \rangle) \diamond \Gamma(P_{\eta,\perp})\varphi|_{0,\alpha}^2 = |\delta(\langle \eta, \cdot \rangle|_{0,\alpha}^2 \cdot |\Gamma(P_{\eta,\perp})\varphi|_{0,\alpha}^2,$$

where the result may be infinite.

Proof. Note that by Remark 2.2.23 we have

$$\delta_x(\langle \eta, \cdot \rangle) = \sum_{n=0}^{\infty} \langle f^{(n)}, :\cdot^{\otimes n}: \rangle,$$

where the kernels $f^{(n)}$ are given by

$$f^{(n)} = \frac{1}{n!\sqrt{2\pi \langle \eta, \eta \rangle}} \exp(-\frac{x^2}{2\langle \eta, \eta \rangle} H_n \left(\frac{x}{\sqrt{2\langle \eta, \eta \rangle}} \right) (2\langle \eta, \eta \rangle)^{-\frac{n}{2}} \eta^{\otimes n}.$$

Hence we can abbreviate the kernels by

$$f^{(k)} = \alpha_k \eta^{\otimes k}, \ \alpha_k \in \mathbb{C}.$$

$$
\begin{aligned}
|\delta(\langle\eta,\cdot\rangle)\diamond\Gamma(P_{\eta,\perp})\varphi|_{0,\alpha}^2 &= \left|\sum_{n=0}^{\infty}\left\langle\sum_{k=0}^{n} f^{(k)}\widehat{\otimes}P_{\eta,\perp}^{\otimes(n-k)\varphi^{(n-k)}}, :\cdot^{\otimes n}:\right\rangle\right|_{0,\alpha}^2 \\
&= \sum_{n=0}^{\infty} n!\cdot 2^{n\alpha}\left|\sum_{k=0}^{n} f^{(k)}\widehat{\otimes}P_{\eta,\perp}^{\otimes(n-k)}\varphi^{(n-k)}\right|_0^2 \\
&= \sum_{n=0}^{\infty} n!2^{n\alpha}\sum_{k=0}^{n}\left|f^{(k)}\widehat{\otimes}P_{\eta,\perp}^{\otimes(n-k)}\varphi^{(n-k)}\right|_0^2,
\end{aligned}
$$

since we have orthogonality. Then by 4.6.5

$$
\begin{aligned}
\sum_{k=0}^{n}\left|f^{(k)}\widehat{\otimes}P_{\eta,\perp}^{\otimes(n-k)}\varphi^{(n-k)}\right|_0^2 &= \sum_{n=0}^{\infty} n!2^{n\alpha}\sum_{k=0}^{n}\frac{k!(n-k)!}{n!}\left|f^{(k)}\right|_0^2\left|P_{\eta,\perp}^{\otimes(n-k)\varphi^{(n-k)}}\right|_0^2 \\
&= \sum_{n=0}^{\infty}\sum_{k=0}^{n} k!\; 2^{k\alpha}\left|f^{(k)}\right|_0^2 (n-k)!\; 2^{(n-k)\alpha}\left|P_{\eta,\perp}^{\otimes(n-k)\varphi^{(n-k)}}\right|_0^2 \\
&= |\delta(\langle\eta,\cdot\rangle|_{0,\alpha}^2\cdot|\Gamma(P_{\eta,\perp})\varphi|_{0,\alpha}^2 .
\end{aligned}
$$

□

Moreover we use [70, Theorem VIII.1, p. 252]

Lemma 4.6.8. *Let $\mathcal{H}$ be a Hilbert space and T be a densely defined operator on $\mathcal{H}$. Then T is closeable if and only if the domain $D(T^*)$ of T^* is dense in $\mathcal{H}$. Here T^* means the adjoint of T with respect to the sesquilinearform $(\cdot,\cdot)_{\mathcal{H}}$ on $\mathcal{H}\times\mathcal{H}$.*

Theorem 4.6.9. *Let $\eta\in\mathcal{G}, |\eta|_0 = 1$. Then $P_\eta:\mathcal{G}\longrightarrow\mathcal{G}$ is closeable in G_α if and only if $\alpha > 0$.*

Proof. Let $\alpha\in\mathbb{R}$. As in 4.6.8 let P_η^* denote the adjoint of P_η with respect to the sesquilinearform $((\cdot,\cdot))_\alpha$ on $G_\alpha\times G_\alpha$. Let $\varphi\in G_\alpha$. Then, by [70, Eq. (VIII.1), p. 252], φ is in $D(P_\eta^*)$ if and only if

$$
\mathcal{G}\longrightarrow\mathbb{C}
$$

$$
\psi\longmapsto((P_\eta\psi,\varphi))_\alpha
$$

has a continuous extension to $\mathcal{G}_\alpha$.

$$
((P_\eta\psi,\varphi))_\alpha = ((\Gamma(2^{\frac{\alpha}{2}}Id)P_\eta\psi,\Gamma(2^{\frac{\alpha}{2}}Id)\varphi))_0
$$

$$
\begin{aligned}
&= (P_\eta \psi, \Gamma(2^\alpha Id)\varphi))_0 \\
&= \sqrt{2\pi} \langle\langle \psi, \delta(\langle \eta, \cdot \rangle) \diamond \Gamma(P_{\eta,\perp})\Gamma(2^\alpha Id)\overline{\varphi} \rangle\rangle
\end{aligned}
$$

By the definition of the bilinearform $\langle\langle \cdot, \cdot \rangle\rangle$ on $\mathcal{G} \times \mathcal{G}'$ and the density of $\mathcal{G}$ in $\mathcal{G}_\alpha$ it follows that $\varphi \in D(P_\eta^*)$ if and only if $\delta(\langle \eta, \cdot \rangle) \diamond \Gamma(P_{\eta,\perp})\Gamma(2^\alpha Id)\overline{\varphi} \in \mathcal{G}_{-\alpha}$.
By 4.6.7 it holds

$$
\begin{aligned}
|\delta(\langle \eta, \cdot \rangle) \diamond \Gamma(P_{\eta,\perp})\Gamma(2^\alpha Id)\overline{\varphi}|_{0,-\alpha}^2 &= |\delta(\langle \eta, \cdot \rangle)|_{0,-\alpha}^2 \cdot |\Gamma(P_{\eta,\perp})\Gamma(2^\alpha Id)\overline{\varphi}|_{0,-\alpha}^2 \\
&= |\delta(\langle \eta, \cdot \rangle)|_{0,-\alpha}^2 \cdot |\Gamma(2^\alpha Id)\Gamma(P_{\eta,\perp})\overline{\varphi}|_0^2 \\
&= |\delta(\langle \eta, \cdot \rangle)|_{0,-\alpha}^2 \cdot |\Gamma(P_{\eta,\perp})\overline{\varphi}|_{0,\alpha}^2
\end{aligned}
$$

Since $\Gamma(P_{\eta,\perp}) \in L(\mathcal{G}_\alpha, \mathcal{G}_\alpha)$ by Lemma 4.6.2 it follows $|\Gamma(P_{\eta,\perp})\overline{\varphi}|_{0,\alpha}^2 < \infty$.
First let $\alpha > 0$. Then $|\delta(\langle \eta, \cdot \rangle|_{0,-\alpha}^2 < \infty$, because $\delta(\langle \eta, \cdot \rangle) \in \bigcap_{\beta > 0} \mathcal{G}_{-\beta}$, by [87, Sect.4.6]. Hence $\varphi \in D(P_\eta^*)$.
If $\alpha \leq 0$ we have $|\delta(\langle \eta, \cdot \rangle|_{0,-\alpha}^2 = \infty$. So, if $\Gamma(P_{\eta,\perp})\overline{\varphi} \neq 0$, we have $\varphi \notin D(P_\eta^*)$.
Consequently $D(P_\eta^*) \subset \ker(\Gamma(P_{\eta,\perp}))$. But $\ker(\Gamma(P_{\eta,\perp}))$ is a proper closed subspace of $\mathcal{G}_\alpha$, hence not dense in $\mathcal{G}_\alpha$. The claim follows now with 4.6.8. □

Chapter 5

A Wick representation of Wiener products

In this chapter we restrict ourselves to the case $\mathcal{N} = S(\mathbb{R})$, i.e. the case of the White Noise triple. We denote the spaces $(\mathcal{N})$ by (S) and $(\mathcal{N})'$ by $(S)'$. All the theorems also work for the multi-dimensional case $\mathcal{N} = S_d(\mathbb{R})$ and can be shown by the same techniques.
Throughout the next pages we want to give meaning to pointwise products of generalized with Donsker's Delta functions as well-defined objects of White Noise Analysis and investigate in which sense the Wick-formula in [86, 36] can be extended beyond (L^2). This is interesting for applications as e.g. Feynman integrals. Here the potential part is playing the role of the generalized function and Donsker's Delta function is used to pin the paths at the endpoints, see e.g. [16].

5.1 The Pointwise product

The test function space (S) is a Banach algebra with the pointwise product of two Hida test functions $\phi, \psi \in (S)$, which is defined by the algebraic product of their chaos decompositions. Thus the product of two kernels $\phi^{(n)} \in S(\mathbb{R})_{\mathbb{C}}^{\hat{\otimes} n} := S_{\mathbb{C}}$ and $\psi^{(m)} \in S_{\mathbb{C}}^{\hat{\otimes} m}$ is given by

$$\langle \phi^{(n)}, :\omega^{\otimes n}: \rangle \langle \psi^{(m)}, :\omega^{\otimes m}: \rangle = \sum_{k=0}^{n \wedge m} k! \binom{n}{k} \binom{m}{k} \langle \phi^{(n)} \hat{\otimes}_k \psi^m, :\omega^{\otimes (n+m-2k)}: \rangle, \quad \omega \in S', \tag{5.1}$$

where $\hat{\otimes}_k$ denotes the k-times symmetric tensor power, see [87] and [64, Lemma 3.5.1, p.58]. Hence the chaos decomposition of $\phi \cdot \psi$ is given by

$$\phi \cdot \psi := \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \sum_{k=0}^{n \wedge m} k! \binom{n}{k} \binom{m}{k} \langle :.^{\otimes n+m-2k}:, \phi^{(n)} \hat{\otimes}_k \psi^m \rangle.$$

Proposition 5.1.1. *The space $\mathcal{G}^\beta$,$0 \le \beta \le 1$, is closed under pointwise multiplication. More precisely, pointwise multiplication is a separately continuous bilinear map from $\mathcal{G}^\beta \times \mathcal{G}^\beta$ into $\mathcal{G}^\beta$.*

For the proof see [69]. For $\theta \in (S)$ and $\psi \in (S)'$ the pointwise product is defined in distribution sense, by

$$\langle\langle \theta \cdot \psi, \varphi \rangle\rangle = \langle\langle \psi, \theta \cdot \varphi \rangle\rangle, \quad \varphi \in (S).$$

Since also elements of $(S)'$ have a generalized chaos decomposition, by an extension of (5.1) one can also define the formal product of two Hida distributions $\theta \in (S)$ and $\Psi \in (S)'$ with corresponding kernels $\theta^{(n)}, n \in \mathbb{N}$ and $\Psi^{(m)}, m \in \mathbb{N}$ by

$$\Theta \cdot \Psi := \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \sum_{k=0}^{n \wedge m} k! \binom{n}{k} \binom{m}{k} \langle \Theta^{(n)} \hat{\otimes}_k \Psi^m, :.^{\otimes n+m-2k}: \rangle,$$

whenever the right hand side is in $(S)'$. In [64, Th.3.5.10., p.63] it is shown that in this case both distributions coincide.

5.2 The Wick formula revised

Proposition 5.2.1. *Let $\eta \in S_{\mathbb{C}}$. Further let $y \in L^2(\mathbb{R}, dt, \mathbb{C})$, $\langle y, y\rangle \neq 0$ and $a \in \mathbb{C}$. Then*

$$\delta(\langle y,\cdot\rangle - a)\cdot : \exp(\langle \eta,\cdot\rangle := \delta(\langle y,\cdot\rangle - a)\diamond\left[\left(\Gamma(P_{y,\perp})\circ \exp(-\frac{1}{2\langle y,y\rangle}D_y^2)\circ \tau_{a\frac{y}{\langle y,y\rangle}}\right)\exp(\langle \eta,\cdot\rangle :\right],$$

where $\tau_{a\frac{y}{\langle y,y\rangle}}$ is the translation operator corresponding to $a\frac{y}{\langle y,y\rangle}$.

Proof. Note that for all $\xi \in S_{\mathbb{C}}$ we obtain from definition that

$$\exp(\langle \xi,\cdot\rangle : \cdot \exp(\langle \eta,\cdot\rangle := e^{\langle \xi,\ \eta\rangle}\cdot \Phi_{\xi+\eta},$$

compare [64, proof of Proposition 4.6.4, p. 102]. By the definition of the Wiener product, we obtain

$$\begin{aligned}\langle\!\langle \delta_a(\langle y,\cdot\rangle)\cdot \exp(\langle \eta,\cdot\rangle :, \exp(\langle \xi,\cdot\rangle :\rangle\!\rangle &:= \langle\!\langle \delta_a(\langle y,\cdot\rangle), \Phi_\eta\Phi_\xi\rangle\!\rangle\\ &= e^{\langle \xi,\ \eta\rangle}\langle\!\langle \delta_a(\langle \cdot, y\rangle), \exp(\langle \xi+\eta,\cdot\rangle :\rangle\!\rangle\\ &= e^{\langle \xi,\ \eta\rangle}\frac{1}{\sqrt{2\pi}\langle y,y\rangle}\exp(-\frac{1}{2\langle y,y\rangle}(\langle \xi+\eta, y\rangle - a)^2),\end{aligned}$$

by definition of the S-transform we obtain for the above expression

$$\begin{aligned}&e^{\langle \xi,\ \eta\rangle}\frac{1}{\sqrt{2\pi}\langle y,y\rangle}\exp(-\frac{1}{2\langle y,y\rangle}(((\langle \xi, y\rangle - a) + \langle \eta, y\rangle)^2)\\ &\quad = S(\delta_a(\langle y,\cdot\rangle))(\xi)\cdot exp(-\frac{1}{2\langle y,y\rangle}(\langle \eta,y\rangle^2 - 2a\langle \eta, y\rangle))\cdot\\ &\quad\quad exp(-\frac{1}{2\langle y,y\rangle}(2\langle \eta,y\rangle y - 2\langle y,y\rangle\xi,\eta\rangle))\\ &\quad = S(\delta_a(\langle y,\cdot\rangle))(\xi)\cdot exp(-\frac{1}{2\langle y,y\rangle}(\langle \eta,y\rangle^2 - 2a\langle \eta, y\rangle))\cdot exp(\langle \eta - \frac{\langle \xi,\eta,y\rangle}{\langle y,y\rangle}y\rangle))\\ &\quad = S(\delta_a(\langle y,\cdot\rangle))(\xi)\cdot S\left(\Gamma(P_{y,\perp})\circ \exp(-\frac{1}{2\langle y,y\rangle}D_y^2)\circ \tau_{a\frac{y}{\langle y,y\rangle}}\exp(\langle \eta,\cdot\rangle :\right)(\xi)\end{aligned}$$

By the injectivity of the S-transform and the well-definiteness (on (S)) of the operators, which appear in the last equation the assertion is proved. □

Corollary 5.2.2 (Wick formula)**.** *Let $y \in L^2(\mathbb{R}, dt, \mathbb{C})$, $\langle y, y\rangle \neq 0$ and $a \in \mathbb{C}$.*

(i)

$$\forall \varphi \in (S):\ \delta(\langle y,\cdot\rangle - a)\cdot\varphi = \delta(\langle y,\cdot\rangle - a)\diamond\left[\left(\Gamma(P_{y,\perp})\circ\exp(-\frac{1}{2\langle y,y\rangle}D_y^2)\circ\tau_{a\frac{y}{\langle y,y\rangle}}\right)\varphi\right]$$

(ii)

$$\forall \varphi \in \mathcal{G}:\ \delta(\langle y,\cdot\rangle - a)\cdot\varphi = \delta(\langle y,\cdot\rangle - a)\diamond\left[\left(\Gamma(P_{y,\perp})\circ\exp(-\frac{1}{2\langle y,y\rangle}D_y^2)\circ\tau_{a\frac{y}{\langle y,y\rangle}}\right)\varphi\right]$$

Proof. (i) Pointwise multiplication with Donsker's delta is continuous from (S) to $(S)'$, see [64, Corollary 3.5.9, p. 63]. Note that, by 4.5.4, we have $\exp(-\frac{1}{2\langle y,y\rangle}D_y^2)$, $\tau_{a\frac{y}{\langle y,y\rangle}} \in L((S),(S))$, and therefore the composition of operators on the other side of the equation is well defined. The claim follows now because span $\left\{: \exp(\langle\eta,\cdot\rangle)\middle|\ \eta \in S(\mathbb{R})_{\mathbb{C}}\right\}$ is dense in (S).

(ii) Pointwise multiplication with Donsker's delta is continuous from $\mathcal{G}$ to $\mathcal{G}'$, by [87, Corollary 65, p. 62]. Because of 4.5.9, we have $\exp(-\frac{1}{2\langle y,y\rangle}D_y^2)$, $\tau_{a\frac{y}{\langle y,y\rangle}} \in L(\mathcal{G},\mathcal{G})$ and by 4.6.2 it holds $\Gamma(P_{y,\perp}) \in L(\mathcal{G},\mathcal{G})$. Then the claim follows by a density argument. □

Our aim is to investigate the validity of the Wick formula for larger spaces. Because [87, Theorem 90, p. 77] is only valid for real η, we can only expect extensions of the Wick formula to the space $\mathcal{M}$ for real y. We study the operators, mentioned above in detail.

We want to show that $exp(-\frac{1}{2}D_\eta^2) \in L(\mathcal{M},\mathcal{G}')$. At this purpose we use the universal mapping property of $\mathcal{M}$. We show that for all $p > 0$ the mapping $exp(-\frac{1}{2}D_\eta^2):\ G_p \longrightarrow \mathcal{G}'$ is continuous.

Theorem 5.2.3. *Let $\eta \in L^2(\mathbb{R},dt,\mathbb{C})$, $|\eta|_0 = 1$. Further let $\alpha,\gamma \in \mathbb{R},\alpha > 0$ and $2^\gamma < 2^\alpha - 1$. Then*

$$exp(-\frac{1}{2}D_\eta^2):\ \mathcal{G}_\alpha \longrightarrow \mathcal{G}_\gamma$$

is well defined and it holds for all $\varphi \in \mathcal{G}_\alpha$:

$$\left|exp(-\frac{1}{2}D_\eta^2)\varphi\right|_{0,\gamma} \leq \frac{1}{\sqrt{(1-2^{-\alpha})}-\sqrt{(2^{\gamma-\alpha})}}\cdot|\varphi|_{0,\alpha}$$

Further $\exp(-\frac{1}{2}D_\eta^2) \in L(\mathcal{G},\mathcal{G})$ *and* $\exp(-\frac{1}{2}D_\eta^2) \in L(\mathcal{M},\mathcal{G}')$.

Proof. We have

$$\left|\exp(-\frac{1}{2}D_\eta^2)\varphi\right|_{0,\gamma}$$

$$= \left|\sum_{n\in\mathbb{N}} \frac{1}{n!}(-\frac{1}{2})^n D_\eta^{2n}\varphi\right|_{0,\gamma}$$

$$= \left|\sum_{n\in\mathbb{N}} \frac{1}{n!}(-\frac{1}{2})^n \sum_{k\in\mathbb{N}} \frac{(k+2n)!}{k!}\langle\langle\eta^{\otimes 2n},\varphi_{k+2n}\rangle, :\cdot^{\otimes k}:\rangle\right|_{0,\gamma}$$

$$\leq \sum_{n\in\mathbb{N}} \frac{1}{n!}(\frac{1}{2})^n \sum_{k\in\mathbb{N}} \frac{(k+2n)!}{k!}\left|\langle\langle\eta^{\otimes 2n},\varphi_{k+2n}\rangle, :\cdot^{\otimes k}:\right|_{0,\gamma}\rangle$$

The following rearrangement of the series is justified by the proven absolute convergence of the rearranged series. Note that

$$\left|\langle\eta^{\otimes 2n},\varphi_{k+2n}\rangle\right|_0 \leq |\eta|_0^{2n}\cdot|\varphi_{k+2n}|_0$$

. Then

$$\left|\exp(-\frac{1}{2}D_\eta^2)\varphi\right|_{0,\gamma}$$

$$\leq \sum_{k\in\mathbb{N}}\sqrt{k!2^{k\gamma}}\sum_{n\in\mathbb{N}}(\frac{1}{2})^n\frac{\sqrt{(k+2n)!}}{k!n!}2^{-\alpha(k+2n)/2}2^{\alpha(k+2n)/2}\sqrt{(k+2n)!}\,|\varphi_{k+2n}|_0$$

$$= \sum_{k\in\mathbb{N}}\sqrt{k!2^{k(\gamma-\alpha)}}\sum_{n\in\mathbb{N}}(\frac{1}{2})^n\frac{\sqrt{(k+2n)!}}{k!n!}2^{-\alpha(2n)/2}2^{\alpha(k+2n)/2}\sqrt{(k+2n)!}\,|\varphi_{k+2n}|_0$$

$$\leq \sum_{k\in\mathbb{N}}\sqrt{k!2^{k(\gamma-\alpha)}}\left(\sum_{n\in\mathbb{N}}(\frac{1}{2})^{2n}\frac{(k+2n)!}{k!^2n!^2}2^{-\alpha(2n)}\right)^{\frac{1}{2}}\cdot|\varphi|_{0,\alpha}\,,$$

$$= \sum_{k\in\mathbb{N}}\sqrt{2^{k(\gamma-\alpha)}}\left(\sum_{n\in\mathbb{N}}\binom{k+2n}{2n}(2^{-\alpha})^{2n}\cdot\underbrace{\frac{(2n)!}{n!^2\cdot 2^{2n}}}_{\leq 1}\right)^{\frac{1}{2}}\cdot|\varphi|_{0,\alpha}$$

$$\leq \sum_{k\in\mathbb{N}}\sqrt{2^{k(\gamma-\alpha)}}(1-2^{-\alpha})^{-(k+1)/2}\cdot|\varphi|_{0,\alpha}$$

$$= \frac{1}{\sqrt{1-2^{-\alpha}}}\sum_{k\in\mathbb{N}}\left(\sqrt{\frac{2^{(\gamma-\alpha)}}{(1-2^{-\alpha})}}\right)^k\cdot|\varphi|_{0,\alpha}\,,$$

where

$$\sum_{k\in\mathbb{N}}\left(\sqrt{\frac{2^{(\gamma-\alpha)}}{(1-2^{-\alpha})}}\right)^k<\infty,$$

since $\frac{2^{(\gamma-\alpha)}}{(1-2^{-\alpha})}<1$ by the assumption $2^\gamma<2^\alpha-1$. Then the last term expression equals

$$\frac{1}{\sqrt{1-2^{-\alpha}}}\frac{\sqrt{(1-2^{-\alpha})}}{\sqrt{(1-2^{-\alpha})}-\sqrt{(2^{\gamma-\alpha})}}\cdot|\varphi|_{0,\alpha}=\frac{1}{\sqrt{(1-2^{-\alpha})}-\sqrt{(2^{\gamma-\alpha})}}\cdot|\varphi|_{0,\alpha}\,,$$

Since for all $\alpha>0$ it holds $2^\alpha-1>0$, there exists $\gamma(\alpha)\in\mathbb{R}$, such that $2^{\gamma(\alpha)}<2^\alpha-1$. Since $\gamma(\alpha)<\alpha$, we have

$$\mathcal{G}_\alpha\to\mathcal{G}_{\gamma(\alpha)}\hookrightarrow\mathcal{G}'$$

is continuous. □

If we omit the condition $|\eta|_0=1$ in 5.2.3, using the same argumentation as above, we get the following statement.

Corollary 5.2.4. *Let $\eta\in L^2(\mathbb{R},dt,\mathbb{C})$, $\langle\eta,\eta\rangle\neq0$ and $a\in\mathbb{C}$. Further let $\alpha,\gamma\in\mathbb{R},\alpha>\log_2(\frac{|\eta|_0^2}{|\langle\eta,\eta\rangle|})$ and $2^\gamma<2^\alpha-\frac{|\eta|_0^2}{|\langle\eta,\eta\rangle|}$. Then*

$$\exp(-\frac{1}{2\langle\eta,\eta\rangle}D_\eta^2):\ \mathcal{G}_\alpha\longrightarrow\mathcal{G}_\gamma$$

is well defined and it holds for all $\varphi\in\mathcal{G}_\alpha$:

$$\left|\exp(-\frac{1}{2\langle\eta,\eta\rangle}D_\eta^2)\varphi\right|_{0,\gamma}\leq\frac{1}{\sqrt{(1-2^{-\alpha}\frac{|\eta|_0^2}{|\langle\eta,\eta\rangle|})}-\sqrt{(2^{\gamma-\alpha})}}\cdot|\varphi|_{0,\alpha}$$

Note that, if η is not real valued the above corollary does not permit the extension of the linear operator $\exp(-\frac{1}{2\langle\eta,\eta\rangle}D_\eta^2)$ to an operator on $\mathcal{M}$.
We state the Wick formula for the real case, see [86, Theorem 4.24, p. 70] and [36].

Corollary 5.2.5 (Wick formula)**.** *Let $\eta\in L^2(\mathbb{R},dt,\mathbb{R})$, $|\eta|_0=1$, $a\in\mathbb{C}$, $\varphi\in\mathcal{M}$. With $P_\eta:=\Gamma(P_{\eta,\perp})\circ\exp(-\frac{1}{2}D_\eta^2)$ it holds*

$$\delta_a(\langle\eta,\cdot\rangle)\cdot\varphi=\delta_a(\langle\eta,\cdot\rangle)\diamond(P_\eta\circ\tau_{a\eta}\varphi)$$

Proof. By [87, Corollary 66, p. 62], Wick multiplication with real Donsker's delta is continuous from $\mathcal{M}$ to $\mathcal{G}'$. Note that $|\eta|_0=\langle\eta,\eta\rangle$, because η is real valued. The claim follows now by 5.2.1, because we have proven the well-definiteness of the concerned operators. (Recall 4.6.2(ii)). □

For the action of P_η on the test functions, see 4.5.14.

5.3 Extensibility of the Wick formula

We have established the use of the Wick formula 5.2.5 for general regular white noise distributions in $\mathcal{M}$. The question arises whether it is possible to use the Wick formula for an extension of the definition of the product with Donsker's delta function. The extensibility of the Wick formula to larger vector spaces by continuity depends highly on the continuity of the operator $\exp(-\frac{1}{2}D_\eta^2)$. The next result, which is stated exemplarily for real η, demonstrates the pathological situation. A similar investigation was made in [86, Corollary 3.19, p. 41].

Proposition 5.3.1. *Let* $\eta \in L^2(\mathbb{R}, dt, \mathbb{R})$, $|\eta|_0 = 1$. *Then the operator* $\exp(-\frac{1}{2}D_\eta^2)$ *is not extendable to an operator in* $L((L^2), \mathcal{G}')$

Proof. Suppose $\exp(-\frac{1}{2}D_\eta^2) \in L((L^2), \mathcal{G}')$. Then we have $\exp(-\frac{1}{2}D_\eta^2)^* \in L(\mathcal{G}, (L^2))$ and consequently, by 4.5.12, it follows that $\left(\frac{1}{\sqrt{2\pi}} \exp(-\frac{1}{2}D_\eta^2)^*\right)(\mathbb{1}) = \delta(\langle \eta, \cdot \rangle) \in (L^2)$, in contradiction to $\delta(\langle \eta, \cdot \rangle) \notin (L^2)$. □

Now we want to investigate if there exists a meaningful algebraic extension of the Wick formula to (L^2). For a meaningful extension of the notion of $\delta(\langle ., \eta \rangle) \cdot \varphi$ to (L^2), we demand at least that the expectation value, which is formally calculated by

$$\langle\!\langle \delta(\langle ., \eta \rangle), \varphi \rangle\!\rangle = \frac{1}{\sqrt{2\pi \langle \eta, \eta \rangle}} \sum_{n=0}^{\infty} (2n)! \frac{1}{n!} (\frac{-1}{2\langle \eta, \eta \rangle})^n \cdot \langle \varphi_{2n}, \eta^{\otimes 2n} \rangle \quad \varphi \in (L^2)$$

is well defined on (L^2), (perhaps not continuous). We present a counterexample.

Lemma 5.3.2. *Let* φ *defined by* $\varphi_n := \frac{i^n}{\sqrt{n!} \cdot n^{\frac{3}{4}}} \cdot 1_{[0,1)}^{\otimes n}$ *for* $n \in \mathbb{N}$ *and* $\varphi_0 := 0$. *Then* $\varphi \in (L^2) \setminus \mathcal{M}$.

Proof. By [74, Theorem 3.28, p.62] we have For $p < 1 : \sum_{n=1}^{\infty} \frac{1}{n^p}$ diverges and $\forall p \geq 1 : \sum_{n=1}^{\infty} \frac{1}{n^p}$ converges. Hence

$$|\varphi|_0^2 = \sum_{n \in \mathbb{N}} n! \, |\varphi_n|_0^2 = \sum_{n \in \mathbb{N}} n! \frac{1}{n!} \frac{1}{n^{\frac{3}{2}}} \cdot 1^{2n} < \infty$$

Now let $\alpha > 0$. Then

$$|\varphi|_{0,\alpha}^2 = \sum_{n\in\mathbb{N}} n!\,|\varphi_n|_0^2 = \sum_{n\in\mathbb{N}} n! 2^{\alpha n} \frac{1}{n!} \frac{1}{n^{\frac{3}{2}}} \cdot 1^{2n} = \infty,$$

because $\limsup_{n\in\mathbb{N}} \sqrt[n]{\frac{2^{\alpha n}}{n^{\frac{3}{2}}}} = 2^\alpha > 1$. □

We need further a more technical result.

Lemma 5.3.3.

$$\frac{2\sqrt{\pi}}{e^2 \cdot \sqrt{k}} \le \frac{(2k)!}{(2^k\ k!)^2} \le \frac{e}{\pi} \cdot \frac{1}{\sqrt{2k}}, \quad \text{for all } k \ge 0$$

Proof. The claim follows immediately by Stirlings formula

$$\sqrt{2n\pi}\left(\frac{n}{e}\right)^n \le n! \le e\sqrt{n}\left(\frac{n}{e}\right)^n$$

□

Proposition 5.3.4. *Let φ defined by $\varphi_n := \frac{i^n}{\sqrt{n!}\cdot n^{\frac{3}{4}}} \cdot \mathbb{1}_{[0,1)}^{\otimes n}$ for $n \in \mathbb{N}$ and $\varphi_0 := 0$. Then the generalized expectation value $\langle\!\langle \mathbb{1}, \delta(\langle \mathbb{1}_{[0,1),\cdot\rangle}) \cdot \varphi\rangle\!\rangle = \infty$*

Proof. Note that

$$\delta(\langle \mathbb{1}_{[0,1),\cdot}\rangle) = \sum_{n=0}^{\infty} \langle \frac{1}{\sqrt{2\pi}} \left(\frac{(-1)^n}{2^n \cdot n!} 1_{[0,1)}^{\otimes 2n}\right), :\cdot^{\otimes n}:\rangle$$

. Then

$$\begin{aligned}
\langle\!\langle \mathbb{1}, \delta(\langle \mathbb{1}_{[0,1)}, \cdot\rangle) \cdot \varphi\rangle\!\rangle &:= \langle\!\langle \delta(\langle ., 1_{[0,1)\rangle}), \varphi\rangle\!\rangle \\
&= \frac{1}{\sqrt{2\pi}} \sum_{n\in\mathbb{N}} (2n)! \frac{(-1)^n}{2^n \cdot n!} \cdot \frac{i^{2n}}{\sqrt{(2n)! \cdot (2n)^{\frac{3}{4}}}} \cdot 1^{2n} \\
&= \frac{1}{\sqrt{2\pi}} \sum_{n\in\mathbb{N}} \sqrt{\frac{(2n)!}{2^{2n} \cdot n!^2}} \cdot \frac{1}{(2n)^{\frac{3}{4}}} \\
&\ge \frac{1}{2^{\frac{3}{4}}\sqrt{2\pi}} \sum_{n\in\mathbb{N}} \left(\frac{\sqrt{4\pi}}{e^2}\right)^{\frac{1}{2}} \cdot \frac{1}{(n)^{\frac{1}{4}}} \cdot \frac{1}{(n)^{\frac{3}{4}}} = \infty,
\end{aligned}$$

by 5.3.3. □

We conclude that even an **algebraic extension** of the product with Donsker's delta to (L^2) is not meaningful by our minimal demands.

Consequently we can only expect that, outside $\mathcal{M}$, only in a special suitable situation, the product with Donsker's delta function is well defined.
Another approach to generalize the product with Donsker's delta function is to use the representation of Donsker's delta function as a Pettis integral, see 2.3.1. The problem of multiplication with Donsker's delta function will be simplified to a multiplication by $e^{i(\langle \cdot,\eta\rangle - a)t}$. But naturally we cannot find a better result as found above.

5.4 Extension of the Wick formula

Analyzing the proof of 5.2.5, the extension of the Wick formula depends highly on the domain of continuity of the operator $exp(-\frac{1}{2}D_\eta^2)$. We have seen in 5.3.1, that the extensibility of $exp(-\frac{1}{2}D_\eta^2)$ to a continuous operator from (L^2) to $\mathcal{G}'$ is not possible, which is expected by the fact, that Donsker's delta function is not in (L^2). The principle problem in the extension of the Wiener multiplication to some bigger subspace of $\mathcal{G}'$ is given by the fact, that we have tried to make continuous extensions which include whole White noise spaces, like $\mathcal{G}$, $\mathcal{M}$ or (L^2).
In this section we start with a more general setting. We discuss the question whether there exists a solution $X \in L(\mathcal{G}', \mathcal{G})$ of the equation

$$\delta(\langle \eta, \cdot \rangle - a) \cdot \varphi = \delta(\langle \eta, \cdot \rangle - a) \diamond X(\varphi),$$

where $\varphi \in \mathcal{G}'$.
Considering the whole space $\mathcal{G}$, we have for real η, $|\eta|_0 = 1$ the solution

$$X(\varphi) = P_\eta \circ \tau_{a\eta}(\varphi)$$

We restrict ourselves to some suitable subspaces of $\mathcal{G}$, where $X(\varphi)$ has a simple form which allows a continuous extension to some subspace of $\mathcal{G}'$, which of course does not contain the full space (L^2). For example, we consider the subspace

$$G_{Id} = \{g \in \mathcal{G} \mid \delta(\langle \eta, \cdot \rangle - a) \cdot \varphi = \delta(\langle \eta, \cdot \rangle - a) \diamond Id(\varphi)\}$$

Since $Id : \mathcal{G}' \longrightarrow \mathcal{G}'$ is continuous, we find a continuous extension of the Wiener product with Donsker's delta function from G_{Id} to the completion of G_{Id} in the topology of $\mathcal{G}'$, which indeed contains functions which are not in (L^2). Obviously we have a compatibility problem, which arises if we can define $\delta(\langle\eta,\cdot\rangle - a)\cdot\varphi = \delta(\langle\eta,\cdot\rangle - a)\diamond X(\varphi)$ and $\delta(\langle\eta,\cdot\rangle - a)\cdot\varphi = \delta(\langle\eta,\cdot\rangle - a)\diamond Y(\varphi)$, for different functions X, Y.

Lemma 5.4.1. *Let $\eta \in L^2(\mathbb{R}, dt, \mathbb{C})$, $\langle\eta,\eta\rangle \neq 0$ and $a \in \mathbb{C}$.*
Then for all $\varphi \in \mathcal{G}$ the equation

$$\delta(\langle\eta,\cdot\rangle - a)\cdot\varphi = \delta(\langle\eta,\cdot\rangle - a)\diamond X(\varphi)$$

has a unique solution. Further X defines a continuous linear mapping from $\mathcal{G}$ to $\mathcal{G}'$.

Proof. Verifying the S-transform of $\delta(\langle\eta,\cdot\rangle - a)$, it easy to see that $\delta(\langle\eta,\cdot\rangle - a)$ has a Wick inverse $\delta_a^{\diamond(-1)}(\langle\eta,\cdot\rangle)$ in $\mathcal{G}'$, i.e.

$$\delta(\langle\eta,\cdot\rangle - a)\diamond\delta_a^{\diamond(-1)}(\langle\eta,\cdot\rangle) = \mathbb{1}.$$

Consequently

$$X(\varphi) = \delta_a^{\diamond(-1)}(\langle\eta,\cdot\rangle)\diamond(\delta(\langle\eta,\cdot\rangle - a)\cdot\varphi),$$

i.e. $X(\varphi)$ is **uniquely defined** by this equation. The continuity of X follows, since the pointwise product with Donsker's delta function is a continuous linear mapping from $\mathcal{G}$ to $\mathcal{G}'$ by [87, Corollary 65, p. 62], and the Wick multiplication is separately continuous on $\mathcal{G}'$. □

In the following we need some notations

Definition 5.4.2. , *Let $\eta \in L^2(\mathbb{R}, dt, \mathbb{C})$, $\langle\eta,\eta\rangle \neq 0$ and $a \in \mathbb{C}$.*

Let D denote a complete locally convex Hausdorff space which is continuously embedded in $\mathcal{G}'$ and let X be an arbitrarily chosen operator in $L(D,\mathcal{G}')$.
We denote by G_X the following subspace of D:

$$G_X := \{g \in \mathcal{G} \cap D |\ \delta(\langle\eta,\cdot\rangle - a)\cdot g = \delta(\langle\eta,\cdot\rangle - a)\diamond X(g)\}.$$

Note that $0 \in G_X$
We denote by D_X the following subspace of D:

$$D_X := \overline{G_X}^D.$$

Further we use the notation

$$\begin{aligned} \Psi_X : \ & D \longrightarrow \mathcal{G}' \\ & d \mapsto \delta(\langle ., \eta \rangle - a) \diamond X(g) \end{aligned}$$

Proposition 5.4.3. *Ψ_X is a continuous extension of the pointwise multiplication with $\delta(\langle \eta, \cdot \rangle - a)$ from G_X to D_X.*

Proof. Note that Ψ_X is continuous as operator from D to $\mathcal{G}'$. The claim follows by the density of G_X in D_X. $\square$

Example 5.4.4. *We present an example where $D_X \setminus (L^2) \neq \emptyset$. We choose $X = Id$ and $D = \mathcal{G}'$. Note, that $\mathcal{G}'$ is complete as the strong dual space of a Fréchet space, see [66, Proposition 0.7.6, p. 11] and [64, Proposition 1.1.2, p. 3]. Further let $a = 0$, $\eta = \mathbb{1}_{[0,1)}$ and let φ defined by $\varphi_n := \frac{i^n}{\sqrt{n!} \cdot n} \cdot \mathbb{1}_{[1,2)}^{\otimes n}$ for $n \in \mathbb{N}$ and $\varphi_0 := 0$. Then $\varphi \in (L^2) \setminus \mathcal{M}$. Let $\Psi_n := \sum_{k=0}^{n} \langle \varphi_k, : \cdot^{\otimes k} : \rangle$. By the Wiener-Itô-Segal isomorphism, see [64, Theorem 2.3.5, p. 30], we have $|\varphi - \Psi_n|_0 \to 0$, hence $\Psi_n \to \varphi$ in $\mathcal{G}' = D$. Further, it holds for all $n \in \mathbb{N}$:*

$$\delta(\langle \eta, \cdot \rangle - a) \cdot \Psi_n = \delta(\langle \eta, \cdot \rangle - a) \diamond \Psi_n \in \mathcal{G}'.$$

Therefore, for all $n \in \mathbb{N}$, it holds $\Psi_n \in G_X$ and since $\Psi_n \to \varphi$ in D, we have $\varphi \in D_X$. Also it is easy to see, that for all $\alpha \in \mathbb{R}$, we have $\Gamma(2^\alpha \, Id)\varphi \in D_X$, i.e.

$$\forall \alpha \geq 0 : \ D_X \setminus \mathcal{G}_{-\alpha} \neq \emptyset.$$

Further, it's easy to see, that $\{\varphi \in \mathcal{G}' \mid D_\eta \varphi = 0\} \subset D_X$.

Example 5.4.5. *Let $D = \mathcal{G}'$. Further let $a \in \mathbb{C}$, $\eta = 1_{[0,1)}$ and let $X = \Gamma(P_{\eta,\perp})$. Then*

$$\text{span}\left\{ : \exp(\langle \xi, \cdot \rangle) : \ \mid \xi \in L^2(\mathbb{R}, dt, \mathbb{C}) \ \ \textit{and} \ \ (\langle \xi, \eta \rangle - 2a = 0 \ \vee \langle \xi, \eta \rangle = 0) \right\} \subset G_X$$

.

Because $\Gamma(P_{\eta,\perp})^2 = \Gamma(P_{\eta,\perp})$ on $\mathcal{G}'$ we have

$$\Gamma(P_{\eta,\perp})(\mathcal{G}') \subset D_X$$

5.5 Products with Donskers Delta function via measure transformation

In this section we consider the case, when X is a generalized scaling operator, i.e. for $g \in \mathcal{G}$, $\eta \in L^2(\mathbb{R}, dt, \mathbb{R})$, $|\eta|_0 = 1$, $a \in \mathbb{R}$, we discuss conditions for the validity of the equation

$$(\delta(\langle \eta, \cdot \rangle - a) \cdot g)(x) = \delta(\langle \eta, \cdot \rangle - a) \diamond \tilde{g}^{\eta,a}(a\eta + P_{\eta,\perp}(x)),\ x \in S'$$

Note that $\mathcal{G} \subset (L^2)$ and therefore each $g \in \mathcal{G}$ is a class of measurable functions relative to the measure μ. The above equation is in fact valid for the space $\mathcal{G}$, but note that we use a $\tilde{g}^{\eta,a}$ instead of g. We will see that $\tilde{g}^{\eta,a}$ is nothing but a function class w.r.t. the positive Hida measure $\nu_{\delta(\langle \eta, \cdot \rangle - a)}$. The following measure theoretical preliminaries are needed to show that we cannot replace $\tilde{g}^{\eta,a}$ by g in the above formula.

Definition 5.5.1.

(i) We denote by $M(S', \mathcal{B})$ the set of Borel measurable mappings from S' to S' and by $MF(S', \mathcal{B})$ the set of Borel measurable functions from S' to $\mathbb{C}$.

(ii) For $f \in MF(S', \mathcal{B})$ we denote by $f^\perp$ the measurable set:

$$f^\perp := f^{-1}(\{0\})$$

(iii) Let ν be a positive Borel measure on S' and $f \in M(S', \mathcal{B})$. Then we use the notation

$$[f]_\nu := \{g \in\ M(S', \mathcal{B}) \mid f =_{\nu-a.e.}\ g\}$$

For $f \in MF(S', \mathcal{B})$ we use $[f]_\nu$ analogously. Each element $\varphi \in (S)$ has a ***unique*** *continuous version f, see [64, Theorem 3.2.1, p. 38], with $\varphi = [f]_\mu$. Since of this uniqueness, we use the symbol f instead of $[f]_\mu$ or $[f]_\nu$, if the context allows no confusion.*

Remark 5.5.2. *Note that we must be careful in the use of compositions of measurable functions: if $f,\ \varphi \in M(S', \mathcal{B})$ and $f_1,\ f_2 \in [f]_\mu$ then* ***generally*** *we have*

$$[f_1 \circ \varphi]_\mu \neq [f_2 \circ \varphi]_\mu\,.$$

E.g. consider the case where μ and $\mu \circ \varphi^{-1}$ are mutually singular. (We can choose $f_1,\ f_2$

such that $f_1 \circ \varphi = 1$ and $f_2 \circ \varphi = 0$.)
But if $f_1,\ f_2 \in [f]_{\mu\circ\varphi^{-1}}$ then

$$[f_1 \circ \varphi]_\mu = [f_2 \circ \varphi]_\mu .$$

Lemma 5.5.3. *Let $\varphi :\ S' \longrightarrow S'$ be a Borel measurable mapping. Then for all $\phi \in [\varphi]_\mu$ it holds:*

$$\mu \circ \varphi^{-1} = \mu \circ \phi^{-1},$$

i.e. the construction of the image measure depends only on the class $[\varphi]_\mu$. So the expression

$$\mu \circ [\varphi]_\mu^{-1}$$

is well defined.

Proof. Let $\varphi_1,\ \varphi_1 \in M(S', \mathcal{B})$ and A, N Borel measurable sets, where

$$\mu(N) = 0 \ \text{ and } \ \varphi_1\big|_{S'\setminus N} = \varphi_2\big|_{S'\setminus N}$$

Then $\varphi_1^{-1}(A) \setminus N = \varphi_2^{-1}(A) \setminus N$ and

$$\mu(\varphi_1^{-1}(A)) = \mu(\varphi_1^{-1}(A) \cap N) + \mu(\varphi_1^{-1}(A) \setminus N) = \mu(\varphi_2^{-1}(A)).$$

□

Lemma 5.5.4. *For all $p \in \mathbb{R}$ it holds that S_p is a Borel measurable subset of S'. Also S is a Borel measurable subset of S'.*

Proof. Let $x \in\ S'$ In view of [64, Lemma 1.2.8, p.7] we obtain

$$|x|_p^2 = \sum_{n=0}^{\infty} |\langle x, H^p e_n \rangle|^2$$

Then $|x|_p^2$ is a countable infinite sum of continuous positive functions, hence measurable.The claim follows now with $S_p = \Big\{ x \in S' \mid |x|_p^2 < \infty \Big\}$.
Furthermore

$$S = \bigcap_{p\in\mathbb{N}} S_p$$

is Borel measurable as countable intersection of Borel measurable sets. □

Because of the independency of the image measure from the special defining measurable mapping see Lemma 5.5.3, we define suitable measurable functions, which allow an easy verification of properties.

Lemma 5.5.5.

(i) Let $\varphi \in L^2(\mathbb{R}, dt, \mathbb{R})$, $\varphi \neq 0$ and Y_φ be a Borel measurable version of $\langle \varphi, \cdot \rangle$. Then $\mu(Y_\varphi^\perp) = 0$.

(ii) $\mu(\text{span}\{\varphi\}) = 0, \quad \varphi \in S' \setminus \{0\}$

(iii) Let $\psi \in L^2(\mathbb{R}, dt, \mathbb{R})$. Then there exists a Borel measurable version X_ψ of $\langle \psi, \cdot \rangle \in (L^2)$ and a measurable subspace $V_\psi \subset S'$ with

(a) $\mu(V_\psi) = 1$ and $L^2(\mathbb{R}, dt, \mathbb{R}) \subset V_\psi$

(b) $X_\psi(\xi) = \langle \psi, \xi \rangle, \quad$ for all $\xi \in L^2(\mathbb{R}, dt, \mathbb{R})$

(c) X_ψ is a linear functional on V_ψ

(d) For all $\eta \in S' \setminus V_\psi$: $X_\psi(\eta) = 0$.

Proof. Proof of (i):
Note that $\{Y_\varphi\}^\perp$ is measurable by definition. Furthermore we have

$$\mathbb{1}_{\{Y_\varphi\}^\perp} = \mathbb{1}_{\{0\}} \circ (|\varphi|_0 \cdot \frac{Y_\varphi}{|\varphi|_0}).$$

Thus

$$\int_{S'} \mathbb{1}_{\{Y_\varphi\}^\perp} \, d\mu = \int_{\mathbb{R}} \mathbb{1}_{\{0\}}(|\varphi|_0 \, t) \, d\nu_{0,1} = \int_{\mathbb{R}} \mathbb{1}_{\{0\}}(t) \, d\nu_{0,1} = 0,$$

since $\{0\}$ is a set of Lebesgue measure zero.
Proof of (ii):
span $\{\varphi\}$ is a one dimensional, closed subset of S', hence measurable.
Then there exists $\xi \in S \setminus \{0\}$ with $\langle \xi, \varphi \rangle = 0$, (e.g. if $\varphi = \sum_{n \in \mathbb{N}} \langle e_i, \varphi \rangle e_i$ and $\langle e_1, \varphi \rangle$, $\langle e_2, \varphi \rangle \neq 0$.
We choose $\xi = \langle e_1, \varphi \rangle e_2 - \langle e_2, \varphi \rangle e_1)$. Then

$$\text{span}\{\varphi\} \subset X_\xi^\perp.$$

The assertion then follows by (i).
Proof of (iii):

Now let Y_ψ be a Borel measurable version of $\langle \psi, \cdot \rangle$. Moreover let $(f_n)_{n\in\mathbb{N}}$ be a sequence in S with $\langle f_n, \cdot \rangle \to Y_\psi$ in (L^2) and $\langle f_n, \cdot \rangle \to Y_\psi$ μ-a.e.. Then there exists a measurable set M of measure zero, such that for all $x \in S' \setminus M$: we have

$$\langle f_n, x \rangle \to Y_\psi(x)$$

Now we define

$$V_\psi := \{x \in S' |\ (\langle f_n, x\rangle)_{n\in\mathbb{N}} \text{ is convergent}\}$$

Then V_ψ is a vector space. Furthermore V_ψ is a measurable set, since $\limsup\langle f_n, \cdot\rangle$ and $\liminf\langle f_n, \cdot \rangle$ are measurable functions by [72, 1.14 Theorem, p. 15]. Thus

$$\begin{aligned} V_\psi &= \{x \in S' \mid |\limsup\langle f_n, x\rangle| < \infty\} \\ &\cap \{x \in S' \mid |\liminf\langle f_n, x\rangle| < \infty\} \\ &\cap \{x \in S' \mid \limsup\langle f_n, x\rangle = \liminf\langle f_n, x\rangle\} \end{aligned}$$

Finally we have $S' \setminus V_\psi \subset M$ and consequently $\mu(V_\psi) = 1$ and $\mu(S' \setminus V_\psi) = 0$. Now set

$$X_\psi := \begin{cases} 0, \ x \notin V_\psi \\ \lim\limits_{n\to\infty} \langle f_n, x\rangle, \quad x \in V_\psi \end{cases}$$

(Note that $x \mapsto \lim\limits_{n\to\infty} \langle f_n, x\rangle$ is measurable on V_ψ). □

The above situation is natural. It is well known from the theory of Gaussian measures on Banach spaces [58], that for all measurables vector spaces $V \subset S'$ with $\mu(V) = 1$ one has $L^2(\mathbb{R}, dt, \mathbb{R}) \subset V$. It holds even

$$L^2(\mathbb{R}, dt, \mathbb{R}) = \bigcap \{V \mid V \text{ is a measurable subspace of } S' \text{ and } \mu(V) = 1\}$$

.

Remark 5.5.6. *By the axiom of choice, we choose to each* $\psi \in L^2(\mathbb{R}, dt, \mathbb{R}) \setminus S$ *a* X_ψ *and a corresponding vectors pace* V_ψ *as in 5.5.5. In the case* $\psi \in S$*, we choose* X_ψ *as the uniquely defined continuous version of* ψ *and* $V_\psi := S'$*.*

For $\psi \in L^2(\mathbb{R}, dt, \mathbb{C})$, *with* $\psi = \psi_1 + \psi_2$, *where* $\psi_1,\ \psi_2 \in L^2(\mathbb{R}, dt, \mathbb{R})$, *we define*

$$X_\psi := X_{\psi_1} + iX_{\psi_2}$$

Definition 5.5.7. *Let* $y \in L^2(\mathbb{R}, dt, \mathbb{R}),\ |y|_0 = 1,\ a \in \mathbb{R}$.

(i) We define the measurable mapping $P_{y,\perp}$ *by*

$$\forall x \in S' :\ P_{y,\perp}(x) := x - X_y(x) \cdot y$$

Note that, by 5.5.5, this definition is compatible with 4.6.1

(ii) We define the measurable mapping $P_{y,a}$ *by*

$$\forall x \in S' :\ P_{y,a}(x) := ay + P_{y,\perp}(x)$$

(iii) The mapping

$$\begin{aligned} \sigma_{y,a} :\ M(S', \mathcal{B}) &\longrightarrow M(S', \mathcal{B}) \\ f &\longmapsto \sigma_{y,a}(f) = f \circ P_{y,a} \end{aligned}$$

is well defined.

Note that the above mappings are compositions of measurable mappings.

We want to present a statement about positive generalized functions with a normal distribution. Similar calculations are made in [57, Example 15.9, p.324-326]. Compare also [40, p.113]

Proposition 5.5.8. *Let* $u, \sigma \in \mathbb{R}, \sigma \neq 0$ *and* $\varphi \in L^2(\mathbb{R}, dt, \mathbb{R})$ *with* $|\varphi|_0 \leq |\sigma|,\ \ \tilde{\varphi} := \langle \varphi, :\cdot: \rangle$ *the corresponding element in* (L^2). *Let* $\Psi := \exp^\diamond(-\frac{1}{2\sigma^2}\tilde{\varphi}^{\diamond 2} + u \cdot \tilde{\varphi})$ *be the corresponding positive generalized white noise functional and let* ν_Ψ *be the corresponding unique finite positive Borel measure, such that for all* $\phi \in (S)$:

$$\langle\langle \phi, \Psi \rangle\rangle = \int_{S'} \phi \ d\nu_\Psi.$$

Then

(i) ν_Ψ *is a probability measure.*

(ii) *For all* $\xi \in S$ *the distribution of the test white noise functional* $X_\xi : S' \to \mathbb{C},\ x \mapsto \langle \xi, x\rangle$, *with respect to* ν_Ψ, *is* $N(m, \tilde{\sigma}^2)$ *distributed with* $m = u\langle\varphi,\xi\rangle$ *and* $\tilde{\sigma}^2 = B_\Psi(\xi,\xi)$, *where* B_Ψ *is the positive semidefinite bilinearform, defined by*

$$B_\Psi(\xi,\eta) := \langle \xi,\eta\rangle - \frac{1}{\sigma^2}\langle\varphi,\xi\rangle\langle\varphi,\eta\rangle, \quad \xi,\eta \in S_{\mathbb{C}}.$$

(iii) *Let* $|\varphi|_0 = |\sigma|$. *With the abbreviation*

$$K := P_{\frac{\varphi}{|\varphi|_0},\perp}$$

we have

$$\nu_\Psi = \ \nu_{\delta_{u\varphi}} \ * \ \mu \circ K^{-1},\ i.e. \forall \phi \in (S):\ \langle\langle \phi, \Psi\rangle\rangle \quad = \int_{S'} \phi(u\cdot\varphi + Kx)\ d\mu(x)$$

Here $\mu \circ K^{-1}$ *denotes the image measure of* μ *under the mapping* K.
Furthermore ν_Ψ *is the image measure of* μ *under the mapping* $P_{\frac{\varphi}{|\varphi|_0},|\varphi|_0 u}$ *i.e.*

$$\nu_\Psi = \mu \circ (P_{\frac{\varphi}{|\varphi|_0},|\varphi|_0 u})^{-1}.$$

Proof. Let $\xi \in \mathcal{N}$. Since $S(\Psi)(\xi) = \exp(-\frac{1}{2\sigma^2}\langle\xi,\varphi\rangle^2 + u\langle\xi,\varphi\rangle)$ we have by Theorem2.2.16 that $\Psi \in (S)'$.
The T-transform $T(\Psi)$ is positive definite, since we have

$$T(\Psi)(\xi) = \exp\big(-\frac{1}{2}(|\xi|_0^2 - \frac{1}{\sigma^2}\langle\xi,\varphi\rangle^2)\big)\cdot \exp(iu\langle\xi,\varphi\rangle),$$

where both terms are positive definite.
Proof of (i):
We use the analogy of [40, (A1.15), p. 453] for the White noise functions.

$$\int_{S'} \mathbb{1}\ d\nu_\Psi = \langle\langle \mathbb{1}, \Psi\rangle\rangle = \sum_{n=0}^{\infty} :u^n:_{\frac{1}{\sigma^2}}\ \langle\langle \mathbb{1}, \frac{\tilde{\varphi}^{\diamond n}}{n!}\rangle\rangle\ = :u^0:_{\frac{1}{\sigma^2}} \cdot 1 = 1.$$

Proof of (ii):
Let $\xi \in S$. It must be shown, that the image measure ν of ν_Ψ under the mapping $X_\xi : S' \to \mathbb{R}$ is the Gaussian measure $\nu_{m,\widetilde{\sigma}^2}$. For this we calculate the Fourier transform $\widehat{\nu}$ at $s \in \mathbb{R}$.

$$\begin{aligned}
\widehat{\nu} &= \int_{\mathbb{R}} e^{i \cdot st}\ dt \\
&= \int_{S'} e^{is\langle \xi, x\rangle} d\nu_\Psi(x) \\
&= \langle\!\langle \Psi, e^{i\langle s, x\xi\rangle} \rangle\!\rangle \\
&= e^{-\frac{1}{2}|s\xi|_0^2} \langle\!\langle : \exp(\langle is\xi, \cdot\rangle) :, \Psi \rangle\!\rangle \\
&= e^{-\frac{1}{2}|\xi|_0^2 s^2} \cdot \exp(-\frac{1}{2}\frac{1}{\sigma^2}\langle is\xi, \varphi\rangle^2 + u\langle is\xi, \varphi\rangle) \\
&= \exp(i(u\langle \xi, \varphi\rangle)s) \cdot \exp(-\frac{1}{2}(B_\Psi(\xi,\xi))s^2)
\end{aligned}$$

For the rest compare to [5, Bsp.3,p.192]
Proof of (iii):
First note that K is a Borel-measurable function from S' to S', which is defined μ-a.e.. By 5.5.5 we have

$$K\xi = \xi - \frac{\langle \xi, \varphi\rangle}{|\varphi|_0^2} \cdot \varphi, \quad \xi \in S.$$

Let B_Ψ as in (ii). Then $B_\Psi(\xi, \eta) = \langle \eta, K\xi\rangle$ for all $\xi, \eta \in S$. Further, by an explicit calculation, we have even
$B_\Psi(\xi,\xi) = \langle K\xi, \xi\rangle = \langle K\xi, K\xi\rangle$ for all $\xi \in S$, (or simply note that K just defines the orthogonal projection of S into $[\text{span}\,\{\varphi\}]^\perp$). As in the proof of (ii) we have:

$$\begin{aligned}
\widehat{\nu_\Psi} &= \exp(i(u\langle \xi, \varphi\rangle)) \cdot \exp(-\frac{1}{2}B_\Psi(\xi,\xi)) \\
&= \exp(i(u\langle \xi, \varphi\rangle)) \cdot \exp(-\frac{1}{2}\langle \xi, K\xi\rangle) \\
&= \exp(i(u\langle \xi, \varphi\rangle)) \cdot \exp(-\frac{1}{2}\langle K\xi, K\xi\rangle) \\
&= \exp(i(u\langle \xi, \varphi\rangle)) \cdot \int_{S'} e^{i\langle K\xi, x\rangle} d\mu
\end{aligned}$$

$$= \exp(i(u\langle\xi,\varphi\rangle)) \cdot \int_{S'} e^{i\langle\xi,Kx\rangle} d\mu,$$

since $X_{K\xi} = \langle\xi, K\cdot\rangle\mu-$a.e.. Finally we obtain

$$\hat{\nu}_\Psi = \int_{S'} e^{i\langle u\varphi+Kx,\xi\rangle} d\mu(x)$$

The claim follows now by the definitions of the image measure and the convolution of measures, compare Definition 2.2.17. □

Corollary 5.5.9. *Let* $y \in L^2(\mathbb{R}, dt, \mathbb{R}),\ |y|_0 = 1,\ a \in \mathbb{R}$. *Further let* ν_δ *denote the positive Hida measure corresponding to* $\delta(\langle y, \cdot\rangle - a)$. *Then we have for all* $f \in L^1(S', d\nu_\delta, \mathbb{C})$ *or Borel-measurable* $f \geq 0$*:*

$$\int_{S'} f(x)\ d\nu_\delta = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}a^2} \int_{S'} f(ay + (x - X_y(x)y))\ d\mu(x)$$

Compare with 5.5.1.

Corollary 5.5.10. , *Let* $y \in L^2(\mathbb{R}, dt, \mathbb{R}),\ |y|_0 = 1,\ a \in \mathbb{R}$ *and* $1 \leq p < \infty$. *Further let* ν_δ *denote the positive Hida measure corresponding to* $\delta(\langle y, \cdot\rangle - a)$ *and* $\mu_{y,a}$ *be the corresponding probability measure, i.e.* $\mu_{y,a} := (\sqrt{2\pi} e^{\frac{1}{2}a^2})\nu_\delta$. *Then we obtain*

(i)

$$\mu_{y,a} = \mu \circ P_{y,a}^{-1}$$

(ii) The mapping

$$\begin{aligned} \sigma_{y,a}:\ L^p((S', d\mu_{y,a}, \mathbb{C}) &\longrightarrow L^p((S', d\mu, \mathbb{C}) \\ f(x) &\longmapsto f(ay + (x - X_y(x)y)) \end{aligned}$$

defines an isometrical embedding from $L^p((S', \mu_{y,a}, \mathbb{C})$ *into* $L^p((S', d\mu, \mathbb{C})$.

Proof. The assertions follow immediately by 5.5.8. □

For the statements in the following definition compare [90] and Proposition 2.3.7.

Definition 5.5.11. *Let $y \in L^2(\mathbb{R}, dt, \mathbb{R})$, $|y|_0 = 1$, $a \in \mathbb{R}$. We denote*

$$\Psi_{y,a} := \sqrt{2\pi} exp(\frac{1}{2}a^2)\delta(\langle y, \cdot\rangle - a).$$

Note that $\Psi_{y,a} \in (S)'$ is a positive generalized White Noise functional and the corresponding Hida measure is a probability measure.

The next proposition is crucial for the discussion.

Proposition 5.5.12. *Let $\eta \in L^2(\mathbb{R}, dt, \mathbb{R})$, $|\eta|_0 = 1$, $a \in \mathbb{R}$. Then*

$$\forall \varphi \in S_{\mathbb{C}} : \ [X_\varphi(a\eta + P_{\eta,\perp}(\cdot))]_\mu \ = \ \Gamma(P_{\eta,\perp}) \exp(-\frac{1}{2}D_\eta^2) \circ \tau_{a\eta}(\langle \varphi, \cdot\rangle).$$

Proof. First, by 5.2.2, we have $\Gamma(P_{\eta,\perp}) \exp(-\frac{1}{2}D_\eta^2) \circ \tau_{a\eta}(\langle \varphi, \cdot\rangle) \in \mathcal{G}$. Note that $X_\varphi(a\eta + P_{\eta,\perp}(\cdot))$ is a Borel measurable function as composition of Borel measurable functions. Further note, that X_φ is the continuous version of φ and $V_\varphi = S'(\mathbb{R})$. Therefore

$$\forall x \in S' : \ X_\varphi(a\eta + P_{\eta,\perp}(x)) = \langle \varphi, a\eta\rangle + \langle \varphi, x\rangle - \langle \varphi, \eta\rangle X_\eta(x)$$

Hence

$$[X_\varphi(a\eta + P_{\eta,\perp}(\cdot))]_\mu = a\langle \varphi, \eta\rangle + \langle \varphi, \cdot\rangle - \langle \varphi, \eta\rangle\langle \eta, \cdot\rangle \in \mathcal{G}.$$

Further, by a simple calculation, it holds

$$\Gamma(P_{\eta,\perp}) \exp(-\frac{1}{2}D_\eta^2) \circ \tau_{a\eta}(\langle \varphi, \cdot\rangle) \ = a\langle \varphi, \eta\rangle + \langle P_{\eta,\perp}\varphi, \cdot\rangle$$

Now the claim follows by applying the S-transform. Indeed, for $\xi \in S_{\mathbb{C}}$, we have:

$$\langle\langle : \exp(\langle \xi, \cdot\rangle) :, [X_\varphi(a\eta + P_{\eta,\perp}(\cdot)]_\mu\rangle\rangle = \langle \varphi, a\eta\rangle + \langle \varphi, \xi\rangle - \langle \varphi, \eta\rangle\langle \xi, \eta\rangle = a\langle \varphi, \eta\rangle + \langle P_{\eta,\perp}\varphi, \xi\rangle$$

□

Lemma 5.5.13. *Let $y \in L^2(\mathbb{R}, dt, \mathbb{R})$, $|y|_0 = 1$, $a \in \mathbb{R}$. Let*

$$K := \Gamma(P_{y,\perp}) \exp(-\frac{1}{2}D_y^2)\tau_{ay}(\varphi)$$

Then

$$\forall \varphi_1, \ \varphi_2 \in \mathcal{G} : \ K(\varphi_1 \cdot \varphi_2) = K(\varphi_1) \cdot K(\varphi_2)$$

Proof. By Proposition 4.5.9, we have $\exp(-\frac{1}{2}D_y^2),\ \tau_{ay} \in L(\mathcal{G},\mathcal{G})$ and by Proposition 4.6.2 $\Gamma(P_{y,\perp}) \in L(\mathcal{G},\mathcal{G})$. Thus $K \in L(\mathcal{G},\mathcal{G})$. By the (joint) continuity of the pointwise multiplication on $\mathcal{G} \times \mathcal{G}$ it suffices to show the relation for Wick exponentials $:\exp(\langle \xi, \cdot \rangle):$ and $:\exp(\langle \eta, \cdot \rangle):$, where $\eta, \xi \in S_{\mathbb{C}}$. On the one hand we have

$$
\begin{aligned}
&K(:\exp(\langle \xi, \cdot \rangle) : \cdot : \exp(\langle \eta, \cdot \rangle) :) = \exp(\langle \xi, \eta \rangle) K(:\exp(\langle \xi + \eta, \cdot \rangle) :) \\
&\qquad\qquad = \exp(\langle \xi, \eta \rangle) \exp(-\frac{1}{2}\langle \xi + \eta, y \rangle^2) \exp(\langle \xi + \eta, ay \rangle) : \exp(\langle P_{y,\perp}(\xi + \eta), \cdot \rangle) :
\end{aligned}
$$

On the other hand we have

$$
\begin{aligned}
&K(:\exp(\langle \xi, \cdot \rangle) :) \cdot K(:\exp(\langle \eta, \cdot \rangle) :) \\
&\qquad = \exp(-\frac{1}{2}\langle \xi, y \rangle^2) \exp(\langle \xi, ay \rangle) : \exp(\langle P_{y,\perp}\xi, \cdot \rangle) : \\
&\qquad \times \exp(-\frac{1}{2}\langle \eta, y \rangle^2) \exp(\langle \eta, ay \rangle) : \exp(\langle P_{y,\perp}\eta, \cdot \rangle) : \\
&= \exp(-\frac{1}{2}\langle \xi, y \rangle^2) \exp(-\frac{1}{2}\langle \eta, y \rangle^2) \exp(\langle \xi + \eta, ay \rangle : \exp(\langle P_{y,\perp}\xi, \cdot \rangle) :: \exp(\langle P_{y,\perp}\eta, \cdot \rangle) : \\
&\qquad = \exp(-\frac{1}{2}\langle \xi, y \rangle^2) \exp(-\frac{1}{2}\langle \eta, y \rangle^2) \exp(\langle \xi + \eta, ay \rangle) \\
&\qquad \times \exp(\langle P_{y,\perp}\xi, P_{y,\perp}\eta \rangle) : \exp(\langle P_{y,\perp}(\xi + \eta), \cdot \rangle) : \\
&= \exp(\langle \xi, \eta \rangle) \exp(-\frac{1}{2}\langle \xi + \eta, y \rangle^2) \exp(\langle \xi + \eta, ay \rangle) : \exp(\langle P_{y,\perp}(\xi + \eta), \cdot \rangle) :
\end{aligned}
$$

□

Based on the previous results we are able to state the following theorem.

Theorem 5.5.14. *Let* $y \in L^2(\mathbb{R}, dt, \mathbb{R}),\ |y|_0 = 1,\ a \in \mathbb{R}$. *Then for all* $\varphi \in (S)$

$$
\Gamma(P_{y,\perp}) \exp(-\frac{1}{2}D_y^2)\tau_{ay}(\varphi) = \sigma_{y,a}(\varphi) = \varphi \circ P_{y,a}
$$

Proof. By 5.5.10 and 2.3.8 the mapping

$$
(S) \stackrel{[\cdot]_{\mu_{y,a}}}{\to} L^2(S', d\mu_{y,a}, \mathbb{C}) \stackrel{\sigma_{y,a}}{\to} (L^2)
$$

$$
\varphi \longmapsto \sigma_{y,a}(\varphi)
$$

is continuous. (Note that we use for $\varphi \in (S)$ the **unique** continuous version of φ, by [64, Theorem 3.2.1, p. 38] so the mapping is well defined.) The claim follows now by 5.2.2, 5.5.12, 5.5.13 and the density of the polynomials $\left\{ X_\xi^n \mid n \in \mathbb{N} \text{ and } \xi \in S \right\}$ in (S), compare [64, Proposition 2.2.3, p. 25] and [64, Theorem 3.2.1, p. 38]. □

As a corollary we state another Wick formula for (E) in the real case.

Corollary 5.5.15. *Let $y \in L^2(\mathbb{R}, dt, \mathbb{R})$, $|y|_0 = 1$, $a \in \mathbb{R}$. Then for all $\varphi \in (S)$ we have*

$$\delta(\langle y, \cdot \rangle - a) \cdot \varphi = \delta(\langle y, \cdot \rangle - a) \diamond \sigma_{y,a}(\varphi).$$

Now we want to answer the question, if it is possible, to present a similar statement in the case of $\mathcal{G}$.

Proposition 5.5.16. *Let $y \in L^2(\mathbb{R}, dt, \mathbb{R})$, $|y|_0 = 1$, $a \in \mathbb{R}$. Then to each $g \in \mathcal{G}$, there exists a unique $\widetilde{g}^{y,a} \in L^2(S', d\mu_{y,a}, \mathbb{C})$ with the following property:*
For each sequence $(f_n)_{n \in \mathbb{N}}$, $f_n \in (S)$ with $f_n \to g$ in the topology of $\mathcal{G}$ it holds $f_n \to \widetilde{g}^{y,a}$ in $L^2(S', d\mu_{y,a}, \mathbb{C})$. The mapping

$$\begin{aligned} \mathcal{G} &\longrightarrow L^2(S', d\mu_{y,a}, \mathbb{C}) \\ g &\longmapsto \widetilde{g}^{y,a} \end{aligned}$$

is linear and continuous.

Proof. Because $f_n \to g$ in the topology of $\mathcal{G}$ it holds $f_n \to g$ in the topology of (L^2), hence without loss of generality $f_n(x) \to g(x)$ μ-a.e.. By the continuity of the pointwise product on $\mathcal{G}$ it follows

$$\lim_{n,m \to \infty} (f_n - f_m)\overline{(f_n - f_m)} = 0.$$

Because $\Psi_{y,a} \in \mathcal{G}'$ we have

$$\langle\!\langle \Psi_{y,a}, (f_n - f_m)\overline{(f_n - f_m)} \rangle\!\rangle = \int_{S'} |f_n - f_m|^2 \, d\mu_{y,a} \to 0, \tag{5.2}$$

where equality follows by 5.5.11. Hence $(f_n)_{n \in \mathbb{N}}$ is a Cauchy-Sequence in $L^2(S', d\mu_{y,a}, \mathbb{C})$ and $f_n \to \widetilde{g}^{y,a} \in L^2(S', d\mu_{y,a}, \mathbb{C})$. It is easy to see, that $\widetilde{g}^{y,a}$ is independent of the choice of $(f_n)_{n \in \mathbb{N}}$.
The claimed continuity of $g \mapsto \widetilde{g}^{y,a}$ follows by (5.2), the continuity of $\Psi_{y,a}$ on $\mathcal{G}$ and the

joint continuity of the pointwise multiplication on $\mathcal{G} \times \mathcal{G}$:

$$\begin{aligned} \int\limits_{S'} |\widetilde{g}^{y,a}|^2 \; d\mu_{y,a} &= \lim_{n\to\infty} \int\limits_{S'} |f_n|^2 \; d\mu_{y,a} \\ &= \lim_{n\to\infty} \langle\!\langle |f_n|^2 , \Psi_{y,a} \rangle\!\rangle \\ &= \langle\!\langle g\overline{g}, \Psi_{y,a} \rangle\!\rangle \\ &\leq |g|_{0,\gamma}^2 , \end{aligned}$$

for some $\gamma > 0$. □

Definition 5.5.17. *Let $y \in L^2(\mathbb{R}, dt, \mathbb{R})$, $|y|_0 = 1$, $a \in \mathbb{R}$. Corresponding to 5.5.16 we use the notation.*

$$\widetilde{\mathcal{G}}^{y,a} := \{\widetilde{g}^{y,a} \mid g \in \mathcal{G}\} .$$

But we omit the identifiers $\cdot^{y,a}$, whenever there is no danger of confusion.

Corollary 5.5.18. *Let $y \in L^2(\mathbb{R}, dt, \mathbb{R})$, $|y|_0 = 1$, $a \in \mathbb{R}$. Then it holds*

$$\forall \varphi \in \mathcal{G} : \; \delta(\langle y, \cdot \rangle - a) \cdot \varphi = \delta(\langle y, \cdot \rangle - a) \diamond \sigma_{y,a}(\widetilde{\varphi}^{y,a}).$$

Proof. Choose a sequence $f_n \to \varphi$ as in 5.5.16. Then apply 5.5.10 and 5.2.2. □

There exists a similar convergence process for suitable spaces $\mathcal{G}_\alpha$.

Proposition 5.5.19. *Let $\alpha > log_2(3)$. Further let $y \in L^2(\mathbb{R}, dt, \mathbb{R})$, $|y|_0 = 1$, $a \in \mathbb{R}$. Then it holds for all $\varphi \in \mathcal{G}_\alpha$:*

$$\delta(\langle y, \cdot \rangle - a) \cdot \varphi = \delta(\langle y, \cdot \rangle - a) \diamond \sigma_{y,a}(\widetilde{\varphi}^{y,a}).$$

Proof. By [87, Theorem 64, p. 26] the pointwise multiplication $G_\alpha \times G_\alpha \to \mathcal{M}$ is continuous for $\alpha > \log_2(3)$. So we may repeat the same approximation procedure as above for this case, defining $\widetilde{\varphi}$ in the same way. Note that $\delta(\langle y, \cdot \rangle - a) \in \mathcal{M}'$ by [87, Theorem 90, p. 77], which is valid in the real case. □

By [87, 4.3.2 The pointwise product, p.61ff], the pointwise multiplication goes from $\mathcal{M} \times \mathcal{M} \longrightarrow \mathcal{G}'$, such that the above used method fails. So we can make at less the following statement:

Proposition 5.5.20. *Let* $y \in L^2(\mathbb{R}, dt, \mathbb{R})$, $|y|_0 = 1$, $a \in \mathbb{R}$, $m \in \mathcal{M}$ *and* $(g_n)_{n\in\mathbb{N}}$ *be a sequence of elements of* $\mathcal{G}$ *with* $g_n \to m$ *in the topology of* $\mathcal{M}$. *Then*

$$\delta(\langle y, \cdot\rangle - a) \cdot m = \lim_{n\to\infty} \delta(\langle y, \cdot\rangle - a) \diamond \sigma_{y,a}(\widetilde{g_n}^{y,a}),$$

Proof. $\delta(\langle y, \cdot\rangle - a) \cdot g_n \to \delta(\langle y, \cdot\rangle - a) \cdot m$ by 5.2.5. On the other hand

$$\delta(\langle y, \cdot\rangle - a) \cdot g_n = \delta(\langle y, \cdot\rangle - a) \diamond \sigma_{y,a}(\widetilde{g_n}^{y,a}),$$

by 5.5.18 □

Finally, as an application, we precise the meaning of the statement "The Brownian motion is pinned at time t at the point a."

Proposition 5.5.21. *The Brownian motion* $(B(s))_{s\geq 0}$ *defined by*

$$B(s) := \langle \mathbb{1}_{[0,s)}, \cdot\rangle$$

is pinned at time t at the point a, i.e. from

$$\int_{S'} \widetilde{B(s)} d\mu_{\frac{\mathbb{1}_{[0,t)}}{\sqrt{t}}, \frac{a}{\sqrt{t}}} = \frac{a}{t} \cdot (s \wedge t)$$

it follows

$$\int_{S'} \widetilde{B(t)} d\mu_{\frac{\mathbb{1}_{[0,t)}}{\sqrt{t}}, \frac{a}{\sqrt{t}}} = a$$

Proof. With the notations in 5.5.8 we have for $\xi \in S$:

$$\exp(i(\frac{a}{t}\langle \mathbb{1}_{[0,t)}, \xi\rangle)) \cdot \exp(-\frac{1}{2}(P_{\frac{\mathbb{1}_{[0,t)}}{\sqrt{t}},\perp}\xi, \xi)) = \int_{S'} e^{i\langle \xi,\omega\rangle} d\mu_{\frac{\mathbb{1}_{[0,t)}}{\sqrt{t}}, \frac{a}{\sqrt{t}}}(\omega)$$

Note that $\widehat{\mu_{\frac{\mathbb{1}_{[0,t)}}{\sqrt{t}}, \frac{a}{\sqrt{t}}}}$ is continuously extendable from S to $L^2(\mathbb{R}, dt, \mathbb{R})$ by the left side of the above equation. Using 5.5.16, an approximation of $\widetilde{B(s)}$ by a suitable sequence in (S) and the dominated convergence theorem, applied to the right side of the above equation, we get the result. Compare [40, Example 4.32, p.113-114]. □

On infinite-dimensional spaces measures have a strong tendency to be mutually singular. As an example we show, that the measures μ and $\mu_{y,a}$ are mutually singular, i.e. they have support on disjoint measurable sets. The following proposition shows that it is reasonable to introduce $\tilde{g}^{\eta,a}$.

Proposition 5.5.22. *Let* $y \in L^2(\mathbb{R}, dt, \mathbb{R})$ *and* $|y|_0 = 1$, $a \in \mathbb{R}$. *Then* μ *and* $\mu_{y,a}$ *are mutually singular. If* $a \neq b$ *then* $\mu_{y,a}$ *and* $\mu_{y,b}$ *are mutually singular.*

Proof. Since the Gaussian measure μ is quasi-invariant under translations by $\varphi \in L^2(\mathbb{R}, dt)$, see [64, Theorem 2.1.6, p.23], it follows by 5.5.5

$$\mu(ay + \{X_y\}^{\perp}) = 0.$$

Now we have to distinguish two cases:
First let $x \in S' \backslash V_y$. Then $X_y(x) = 0$ and we have $P_{y,\perp}(x) = x \notin V_y$, hence $X_y(P_{y,\perp}(x)) = 0$. In the case $x \in V_y$ this can be obtained immediately by the linearity of X_y on V_y. Hence $P_{y,\perp}(S') \subset X_y^{\perp}$. It follows that $P_{y,a}(S') \ = ay + P_{y,\perp}(S') \subset ay + \{X_y\}^{\perp}$.
Thus

$$\mu_{y,a}(S' \setminus \{ay + \{X_y\}^{\perp}\}) = \mu(P_{y,a}^{-1}(S' \setminus \{ay + \{X_y\}^{\perp}\})) = \mu(\emptyset) = 0.$$

The last claim yields since for $a \neq b$ we have

$$ay + \{X_y\}^{\perp} \cap by + \{X_y\}^{\perp} = \emptyset.$$

□

Chapter 6

Appendix

We transfer some results about Gaussian measures on Banach spaces to the White noise space. The results are typical properties of the $Cameron-Martin$ space. The following statement is quite standard and a well known characterization of the $Cameron-Martin$ space in the theory of Gaussian measures on Banach spaces. The idea of the proof is taken from lecture notes of [89] and [39], respectively.

First we state the following Lemma:

Lemma 6.0.23. *For $n \in \mathbb{N}$, let h_n be the n-th Hermite function. Then*

$$\sup_{n\in\mathbb{N}} |\langle h_n, \omega\rangle| = \infty,$$

for μ-almost all $\omega \in S'(\mathbb{R})$.

Proof. Let F_1 denote the cumulative distribution function corresponding to the normal distribution $\mathcal{N}(0,1)$. Further let $0 < m < \infty$ and $F(m) := F_1(m) - F_1(-m)$.
By [64, Eq. (2.7), (2.8), p. 21], each h_n is normally distributed by $\mathcal{N}(0,1)$, hence $\{\langle h_n, \cdot\rangle\}_{n\in\mathbb{N}}$ is a sequence of identically distributed and independent random variables, see [64, Lemma 2.1.3, p. 20], with

$$\forall n \in \mathbb{N}: \ \mu(\{x \in S'(\mathbb{R}) \mid |h_n(x)| \leq m\}) = \int_{\mathbb{R}} \mathbb{1}_{[0,m]}(|t|)\, d\nu_{0,1} = \int_{-m}^{m} d\nu_{0,1} = F(m) < 1.$$

It follows:

$$\mu\Big(\Big\{x \in S'(\mathbb{R}) \mid \max_{0\leq i\leq n} |h_i(x)| \leq m\Big\}\Big) = \mu\Big(\bigcap_{i=0}^{n} \{x \in S'(\mathbb{R}) \mid |h_i(x)| \leq m\}\Big)$$

$$
\begin{aligned}
&= (\prod_{i=0}^{n} \mu \{x \in S'(\mathbb{R}) \mid |h_i(x)| \leq m\}), \textit{ independency} \\
&= (F(m))^{n+1}, \textit{ normally distributed} \\
&\to 0, \textit{ for } n \to \infty
\end{aligned}
$$

For $n \in \mathbb{N}$ let $A_n := \left\{ x \in S'(\mathbb{R}) \mid \max_{0 \leq i \leq n} |h_i(x)| \leq m \right\}$. By [73, 1.19 Theorem(e), p. 17] it follows that

$$
\mu \left(\bigcap_{n=0}^{\infty} A_n \right) = \lim_{n \to \infty} \mu(A_n) = 0,
$$

hence

$$
\mu(x \in S'(\mathbb{R}) \mid \sup_{n \in \mathbb{N}} |h_n(x)| \leq m) = 0
$$

and

$$
\mu(\bigcup_{m \in \mathbb{N}} \left\{ x \in S'(\mathbb{R}) \mid \sup_{n \in \mathbb{N}} |h_n(x)| \leq m \right\}) = 0
$$

and finally

$$
\mu(\left\{ x \in S'(\mathbb{R}) \mid \sup_{n \in \mathbb{N}} |h_n(x)| = \infty \right\}) = 1
$$

□

Another comparable proof, using the first lemma of Borel-Cantelli:

Proof. Let F_1 denote the cumulative distribution function corresponding to the normal distribution $\mathcal{N}(0,1)$. Further let $0 < m < \infty$ and $F(m) := F_1(m) - F_1(-m)$.
By [64, Eq. (2.7), (2.8), p. 21], each h_n is normally distributed by $\mathcal{N}(0,1)$, hence $\{\langle h_n, \cdot \rangle\}_{n \in \mathbb{N}}$ is a sequence of identically distributed and independent random variables, see [64, Lemma 2.1.3, p. 20], with

$$
\forall n \in \mathbb{N}: \ \mu(\{x \in S'(\mathbb{R}) \mid |h_n(x)| \leq m\}) = \int_{\mathbb{R}} \mathbb{1}_{[0,m]}(|t|) \, d\nu_{0,1} = \int_{-m}^{m} d\nu_{0,1} = F(m) < 1.
$$

It follows:

$$\begin{aligned}\mu(\left\{x \in S'(\mathbb{R}) \mid \max_{0\le i\le n} |h_i(x)| \le m\right\}) &= \mu(\bigcap_{i=0}^{n} \{x \in S'(\mathbb{R}) \mid |h_i(x)| \le m\}) \\ &= (\prod_{i=0}^{n} \mu\{x \in S'(\mathbb{R}) \mid |h_i(x)| \le m\}),\ \textit{independency} \\ &= (F(m))^{n+1},\ \textit{normally distributed}\end{aligned}$$

It follows
$$\sum_{n\in\mathbb{N}} \mu(\left\{x \in S'(\mathbb{R}) \mid \max_{0\le i\le n} |h_i(x)| \le m\right\}) = \frac{F(m)}{1-F(m)} < \infty.$$

Hence by the first lemma of Borel-Cantelli, see [5, 11.8 Satz, p. 74], it follows

$$\mu(\left\{x \in S'(\mathbb{R}) \mid \max_{0\le i\le n} |h_i(x)| \le m \ for\ infinitely\ many\ n\right\}) = 0,$$

(this means here for all $n \in \mathbb{N}$), hence

$$\mu(\left\{x \in S'(\mathbb{R}) \mid \sup_{n\in\mathbb{N}} |h_n(x)| \le m\right\}) = 0$$

and
$$\mu(\bigcup_{m\in\mathbb{N}} \left\{x \in S'(\mathbb{R}) \mid \sup_{n\in\mathbb{N}} |h_n(x)| \le m\right\}) = 0$$

and finally
$$\mu(\left\{x \in S'(\mathbb{R}) \mid \sup_{n\in\mathbb{N}} |h_n(x)| = \infty\right\}) = 1$$

□

Proposition 6.0.24. *$L^2(\mathbb{R}, dt, \mathbb{R})$ is a Borel measurable subset of $S'(\mathbb{R})$ and it holds*

$$\mu(L^2(\mathbb{R}, dt, \mathbb{R})) = 0$$

Proof. The first claim follows by 5.5.4. For $n \in \mathbb{N}$, let h_n be the n-th Hermite function.

Then, by 6.0.23 it follows

$$\mu(L^2(\mathbb{R}, dt, \mathbb{R})) = \mu(\left\{x \in S'(\mathbb{R}) \mid \sum_{n\in\mathbb{N}} |\langle h_n, x\rangle|^2 \ < \ \infty\right\}) \leq \mu(\left\{\sup_{n\in\mathbb{N}} |h_n(x)| < \infty\right\}) = 0$$

□

Proposition 6.0.25.

(i) For all measurables vector spaces $V \subset S'(\mathbb{R})$ with $\mu(V) = 1$ it holds

$$L^2(\mathbb{R}, dt, \mathbb{R}) \subset V$$

(ii)

$$L^2(\mathbb{R}, dt, \mathbb{R}) = \bigcap \{V \mid V \text{ measurable subspace of } S'(\mathbb{R}) \text{ and } \mu(V) = 1\}$$

Proof. (i) Let $y \in L^2(\mathbb{R}, dt, \mathbb{R})$. Suppose $y \notin V$. Then $(y + V) \cap V = \emptyset$. Since the Gaussian measure μ is quasi-invariant under translations by $\varphi \in L^2(\mathbb{R}, dt, \mathbb{R})$, see [64, Theorem 2.1.6, p.23], it follows $\mu(y + V) = 1$, hence $\mu((y + V) \cup V) = 2$. Contradiction!

(ii) On the other hand, let $x \notin L^2(\mathbb{R}, dt, \mathbb{R})$. Denote $\mathcal{H} := L^2(\mathbb{R}, dt, \mathbb{R})$. Then

$$\forall \Psi \in S'(\mathbb{R}) : \quad (|\Psi|_0 = \sup_{h\in\mathcal{H},\ |h|_0=1} |< h, \Psi >| < \infty) \Leftrightarrow (\Psi \in \mathcal{H})$$

So there exists a sequence $(\varphi_n)_{n\in\mathbb{N}}$, $\varphi_n \in \mathcal{H}$ with $|\varphi_n|_0 = 1$ and $|\langle\varphi_n, x\rangle| \geq n$.
Now define

$$\forall \eta \in S'(\mathbb{R}) : \ \|y\| := \left(\sum_{n\in\mathbb{N}} \frac{1}{n^2} |< \varphi_n, y >|^2\right)^{\frac{1}{2}}.$$

It follows

$$\int_{S'(\mathbb{R})} \|y\|^2 \ d\mu(y) = \sum_{n\in\mathbb{N}} \frac{1}{n^2} < \infty$$

Consequently $\mu(\left\{y \in S'(\mathbb{R}) \mid \|y\|^2 = \infty\right\}) = 0$ This implies that the linear space $V := \left\{y \in S'(\mathbb{R}) \mid \|y\|^2 < \infty\right\}$, which is by definition measurable, has full measure one. But $x \notin V$, hence

$$x \notin \bigcap \{V \mid V \text{ measurable subspace of } S'(\mathbb{R}) \text{ and } \mu(V) = 1\}$$

□

As a further example we investigate the measurability of $\{\delta_t \mid t \in \mathbb{R}\}$.

Lemma 6.0.26. *The mapping*

$$\begin{aligned} \mathbb{R} &\to S'(\mathbb{R}) \\ t &\mapsto \delta_t \end{aligned}$$

is continuous.

Proof. Note that $S'(\mathbb{R})$ is endowed with the strong dual topology. Let B be a bounded subset of $S(\mathbb{R})$. Then there exists a $K > 0$ such that $\sup_{f \in B} \|f'\|_\infty \leq K$. For $s, t \in \mathbb{R}$ it follows:

$$\begin{aligned} \sup_{f \in B} |\langle f, \delta_t - \delta_s \rangle| = & \sup_{f \in B} |f(t) - f(s)| \\ & \leq \sup_{f \in B} \|f'\|_\infty \cdot |s - t| \\ & \leq K \cdot |s - t| \end{aligned}$$

□

Example 6.0.27.

$$\mu(\{\delta_t \mid t \in \mathbb{R}\}) = 0.$$

Proof. First note that by 6.0.26

$$\{\delta_t \mid t \in \mathbb{R}\} = \bigcup_{n \in \mathbb{N}} \{\delta_t \mid t \in [-n, n]\}$$

is Borel measurable as union of compact subsets of $S'(\mathbb{R})$. By [40, (A.1.11), p. 453], we have

$$|h_n(t)| = O(n^{-\frac{1}{4}}).$$

It follows

$$\{\delta_t \mid t \in \mathbb{R}\} \subset \left\{ x \in S'(\mathbb{R}) \mid \sup_{n \in \mathbb{N}} |h_n(x)| < \infty \right\}$$

□

The following theorem is remarkable since it implies that even though $L^2(\mathbb{R}, dt, \mathbb{R})$ itself has μ-measure 0, whenever A is a set of positive measure, no matter how small, the set $A + L^2(\mathbb{R}, dt, \mathbb{R})$ has full μ-measure one!

Theorem 6.0.28 (Borell-Sudakov-Cirelson)**.** *Let F_1 denote the cumulative distribution function corresponding to the normal distribution $\mathcal{N}(0,1)$ and for $\epsilon > 0$ let*

$$B_\epsilon := \left\{ f \in L^2(\mathbb{R}, dt, \mathbb{R}) \mid |f|_0 \leq \epsilon \right\}.$$

Let A be a μ-measurable set with $\mu(A) \geq F_1(\alpha)$ for some $\alpha \in \mathbb{R}$. Then

$$\mu_*(A + B_\epsilon) \geq F_1(\alpha + \epsilon).$$

Note that μ_* denotes the inner measure corresponding to μ.

Proof. We are not going to give a proof of 6.0.28 because these arguments are completely out of the scope of this work, see [60]. □

In the proof of the following statement, we use 6.0.28. Note that there no measurability questions occur and we can use the measure μ instead of μ_*.

Corollary 6.0.29. *Let V be a measurable vector subspace of $S'(\mathbb{R})$. Then*

$$\mu(V) = 0 \ \vee \ \mu(V) = 1.$$

Further

$$\mu(V) = 1 \Leftrightarrow \mu(V) > 0 \textit{ and } L^2(\mathbb{R}, dt, \mathbb{R}) \subset V.$$

Proof. Let $L^2(\mathbb{R}, dt, \mathbb{R}) \not\subset V$. Choose $h \in V \setminus L^2(\mathbb{R}, dt, \mathbb{R})$. For $r \geq 0$ set $V_r := rh + V$. Then

$$\forall r_1, r_2 \geq 0, r_1 \neq r_2 : \ r_1 h + V \cap r_2 h + V = \emptyset.$$

Consequently for all $n \in \mathbb{N}$ it follows that the set $\left\{ r \geq 0 \mid \mu(V_r) > \frac{1}{n} \right\}$ is finite. Hence the set $\{r \geq 0 \mid \mu(V_r) > 0\}$ is at most countable. But $\mathbb{R}_+$ is uncountable, such that there exists a $r > 0$ with $\mu(V_r) = 0$. By the quasi-invariance of the Gaussian measure μ under translations by any element of $L^2(\mathbb{R}, dt, \mathbb{R})$ it follows $\mu(V) = 0$.
Let $L^2(\mathbb{R}, dt, \mathbb{R}) \subset V$. For $\epsilon > 0$ let $B_\epsilon := \{ f \in L^2(\mathbb{R}, dt, \mathbb{R}) \mid |f|_0 \leq \epsilon \}$. Then $V = V + B_\epsilon$.

Suppose $\mu(V) > 0$. Then, by 6.0.28, with $\alpha \in \mathbb{R}$ and $\mu(V) \geq F_1(\alpha)$:

$$\mu(V) \geq \lim_{\epsilon \to \infty} F_1(\alpha + \epsilon) = 1.$$

Hence $\mu(V) = 1$. □

We investigate some often used measures in White noise theory. The proofs are translated from theorems about the Cameron-Martin space in the lecture notes of LIHU XU and Martin Hairer to the White noise space.
Measures in infinite dimensional spaces have a strong tendency to being mutually singular, i.e. they reside on disjoint measurable sets. We give a striking illustration of this phenomenon.
Given $c > 0$ we consider the dilatation operator $D_c : \ x \mapsto cx$ for all $x \in S'(\mathbb{R})$. The following theorem is a surprising result for the infinite dimensional Gaussian measure.

Proposition 6.0.30. *Let $c > 0$. If $c \neq 1$ then the measures μ and $\mu \circ D_c^{-1}$ are mutually singular:*

$$\mu \circ D_c^{-1} \perp \mu$$

Proof. By [64, Eq. (2.7), (2.8), p. 21] and [64, Lemma 2.1.3, p. 20], the Hermite functions $\{\langle h_n, \cdot \rangle\}_{n \in \mathbb{N}}$ are a set of identically distributed and independent (i.i.d.) random variables, with distribution $\mathcal{N}(0,1)$ relative to μ and distribution $\mathcal{N}(0,c^2)$ relative to $\mu \circ D_c^{-1}$. Now we apply the strong Law of Large Numbers (L.L.N.) of Kolmogoroff (see [5, 12.2 Korollar, p. 86] to the i.i.d. random variables

$$\left\{ |\langle h_n, \cdot \rangle|^2 \right\}_{n \in \mathbb{N}},$$

to get

$$\lim_{n \to \infty} \frac{1}{n+1} \sum_{k=0}^{n} |h_k(x)|^2 \longrightarrow 1, \ \mu - a.e. \tag{6.1}$$

and

$$\lim_{n \to \infty} \frac{1}{n+1} \sum_{k=0}^{n} |h_k(x)|^2 \longrightarrow c^2, \ \mu \circ D_c^{-1} - a.e. \tag{6.2}$$

It follows $\mu \left(\left\{ x \in S'(\mathbb{R}) \mid \lim_{n \to \infty} \frac{1}{n+1} \sum_{k=0}^{n} |h_k(x)|^2 \longrightarrow c^2 \right\} \right) = 0$ and
$\mu \circ D_c^{-1} \left\{ x \in S'(\mathbb{R}) \mid \lim_{n \to \infty} \frac{1}{n+1} \sum_{k=0}^{n} |h_k(x)|^2 \longrightarrow 1 \right\} = 0.$ □

Theorem 6.0.31 (Cameron-Martin)**.** *For $\xi \in S'(\mathbb{R})$, define the map $T_\xi : S'(\mathbb{R}) \to S'(\mathbb{R})$ by $T_\xi(x) = x + \xi$. Then, the measure $\mu \circ T_\xi^{-1}$ is absolutely continuous with respect to μ if and only if $\xi \in L^2(\mathbb{R}, dt, \mathbb{R})$.*
If $\xi \notin L^2(\mathbb{R}, dt, \mathbb{R})$ then the measures $\mu \circ T_\xi^{-1}$ and μ are mutually singular.

Proof. For the one part of the first claim, see [64, Proposition 2.1.6., p. 23]. Now let $\xi \notin L^2(\mathbb{R}, dt, \mathbb{R})$. Then, as in the proof of 6.0.25 (ii), we find a measurable vector space $V \subset S'(\mathbb{R})$ with $\mu(V) = 1$ and $\xi \notin V$, hence $-\xi \notin V$. Let $N := S'(\mathbb{R}) \setminus V$. Because $V \cap (-\xi + V) = \emptyset$ it follows $\mu(-\xi + V) = 0$. By $S'(\mathbb{R}) = V \cup N = (-\xi + V) \cup (-\xi + N)$ it follows

$$\mu(-\xi + V) = 0 \text{ and } \mu(-\xi + N) = 1$$

Consequently

$$\int_V d\mu \circ T_\xi^{-1} = 0 \quad \text{and} \quad \int_N d\mu \circ T_\xi^{-1} = 1$$

i.e.

$$\mu \perp \mu \circ T_\xi^{-1}$$

□

Part II

Hamiltonian Path Integrals - Feynman Integrals in Phase Space

Chapter 7

Fundamental Hamiltonian Path Integrals

7.1 Hamiltonian Path Integrals

As proposed by Feynman [23, 24], quantum mechanical transition amplitudes may be thought of as a kind of averaging over fluctuating paths, with oscillatory weight functions given in terms of the classical action

$$S(x) = \int_0^t L(x(\tau), \dot{x}(\tau), \tau)\, d\tau.$$

The ideas based on the previous work of Norbert Wiener and the succeeding work of Paul Dirac in 1933 [19]. The Lagrangian (hence the action as the time integral of the Lagrangian) is given by the difference of the kinetic energy and the potential, e.g.

$$L(x(t), \dot{x}(t), t) = -\frac{1}{2} m(\dot{x}(t))^2 - V(x(t), \dot{x}(t), t)$$

Formally, the Feynman path integral is then expressed as

$$K(t, y|0, y_0) = \mathrm{N} \int \exp\left(\frac{i}{\hbar} S(x)\right) \prod_{0<\tau<t} dx(\tau)$$

The integral is thought of as being over all paths with $x(0) = y_0 \in \mathbb{R}^d$ and $x(t) = y \in \mathbb{R}^d$. The quantum mechanical propagator $K(t, y|0, y_0)$ represents the transition amplitude for a particle to be found at position y at time t given that the particle was at position y_0 at an earlier time 0. The propagator $K(t,y|0,y_0)$ is the integral kernel of the unitary operator $U_t = \exp(it\hat{H})$, where $\hat{H}$ is the Hamilton operator. The operator U_t gives the semi-group of the Schrödinger equation, i.e.

$$i\hbar\partial_t\Big(U(t,t_0)\Psi\Big) = H(t)\Big(U(t,t_0)\Psi\Big), \quad U(t_0,t_0)\Psi = \Psi.$$

Although the first aim of Feynman was to develop path integrals based on a Lagrangian, they also can be used for various systems which have a law of least action, see e.g.[23]. Since classical quantum mechanics is based on a Hamiltonian formulation rather than a Lagrangian one, it is worthwhile to take a closer look to the so-called Hamiltonian path integral, which means the Feynman integral in phase space. This has therefore many advantages:

- At first the semi-classical limit of quantum mechanics is more natural in an Hamiltonian setting, i.e. the phase space is more natural in classical mechanics than the configuration space, see also [2, 45] and the references therein.
- In [21] the authors state, that potentials which are time-dependent or velocity dependent should be treated with the Hamiltonian path integral.
- The idea of canonical transformations due to a Hamiltonian system can be done easier in a Hamiltonian setting.
- Also momentum space propagators can be investigated.

Feynman gave a heuristic formulation of the phase space Feynman Integral in [24]

$$K(t,y|0,y_0) = \mathrm{N}\int_{x(0)=y_0,x(t)=y}\int \exp\left(\frac{i}{\hbar}S(x)\right)\prod_{0<\tau<t}\frac{dp(\tau)}{(2\pi)^d}dx(\tau) \tag{7.1}$$

Here the action (and hence the dynamic) is expressed by a canonical (Hamiltonian) system of generalized space variables and their corresponding conjugate momentums. The canonical variables can be found by a Legendre-transformation, see e.g. [76]. The Hamiltonian action:

$$S(x,p,t) = \int_0^t p(\tau)\cdot\dot{x}(\tau) - H(x(\tau),p(\tau),\tau)d\tau,$$

where H is the Hamilton function and given by the sum of the kinetic energy and the potential.

$$H(x,p,t) = \frac{1}{2m}p^2 + V(x,p,t).$$

The phase space is therefore $2n$-dimensional if we have n so called degrees of freedom. Furthermore the variables are independent.
By the result of Heisenberg uncertainty principle the momentum variables are fulfilling no boundary conditions, because the space variables are fixed.
Note that both integrals, the Feynman integral as well as the Hamiltonian path integral are thought to be integrals w.r.t a flat, i.e. translation invariant measure on the infinite dimensional path space. Such a measure does not exist, hence the integral at first - as it stands - is not a mathematical rigorous object. The normalization constant in both integrals turns out to be infinity. Nevertheless there is no doubt that it has a physical meaning.

There are many attempts to give a meaning to the Hamiltonian path integral as a mathematical rigorous object. Among these are analytic continuation of probabilistic integrals via coherent states [45, 46] and infinite dimensional distributions e.g. [17]. Most recently also an approach using time-slicing was developed by Naoto Kumano-Go [56] and also by Albeverio et al. using Fresnel integrals [2, 1]. As a guide to the literature on many attempts to formulate these ideas we point out the list in [2].

Ansatz for the Phase Space In the following we use throughout the thesis the vector-valued White Noise space as underlying space, i.e. we choose $\mathcal{N} = S_{2d}(\mathbb{R})$ and $\mathcal{N}' = S'_{2d}(\mathbb{R})$ such that we have with the central space $L^2_{2d}(\mathbb{R})$ the chain of spaces

$$S_{2d}(\mathbb{R}) \subset L^2_{2d}(\mathbb{R}) \subset S'_{2d}(\mathbb{R}).$$

In our approach we choose a Gaussian measure on the vector-valued White noise space $S'_{2d}(\mathbb{R})$ as a reference measure as in Chapter 2.
In the following for the two settings and objects we give a meaning to as distributions of White Noise Analysis. First we consider the ansatz for the configuration space Hamiltonian path integrand then for one in momentum space. The propagators are related to each other by Fourier transform. Hence we can check if the propagators are fulfilling the right physics.

Hamiltonian Path Integral in coordinate space First we introduce the space trajectories as Brownian motion starting in x_0.

$$x(\tau) = x_0 + \sqrt{\frac{\hbar}{m}} B(\tau), \quad 0 \leq \tau \leq t.$$

Furthermore the momentum variable is modeled by white noise, i.e.

$$p(\tau) = \sqrt{\hbar m}\omega_p(\tau), \quad 0 \leq \tau \leq t.$$

This is a meaningful definition, since a path has always start and end points which a noise does not have. Moreover since we have that if the initial and end conditions are fully known, the momentum is completely uncertain, which means has variance infinity. The white noise process is intrinsically fulfilling the no boundary condition property and has as well infinite variance. Furthermore one can think of for a potential just depending on the space variable the momentum to be $p = m\dot{x}$, which in our approach would correspond to a noise in terms of derivative of the Brownian path.
The model for the space path can be found in [40] to model the momentum path we take a closer look to the physical dimensions of $x(\tau)$.
$y(s)$ has as a space variable the dimension of a length, i.e. also $\sqrt{\frac{\hbar}{m}}B(\tau)$ has to have the dimension of a length. We have

$$[\sqrt{\frac{\hbar}{m}}] = \sqrt{\frac{Js}{kg}} = \sqrt{\frac{kgm^2}{skg}} = \frac{m}{\sqrt{s}}.$$

Thus since the norm of the Brownian motion gives again a $\sqrt{t}$ which has the dimension $\sqrt{s}$ we have that $x(\tau)$ has the dimension of a length.
Considering the momentum variable we have to obtain that the dimension is the dimension of a momentum. We have

$$[\sqrt{\hbar m}] = \sqrt{Nmskg} = \sqrt{\frac{kg^2m^2s}{s^2}} = \frac{kgm}{\sqrt{s}},$$

hence ω_p has the dimension $\frac{1}{\sqrt{s}}$, such that $p(\tau)$ has the dimension of a momentum.
A definition which goes in the same direction using the momentum as a kind of derivative of the path can also be found in [2] and [1]. Here the authors modeled the path space as the space of absolutely continuous functions and the momentum to be in $L^2(\mathbb{R})$. Then we propose the following formal ansatz for the Feynman integrand in Phase space with respect to the Gaussian measure μ,

$$I_V = \mathrm{N} \exp\left(\frac{i}{\hbar}\int_{t_0}^{t} p(\tau)\dot{x}(\tau) - \frac{p(\tau)^2}{2m} d\tau + \frac{1}{2}\int_{t_0}^{t} \dot{x}(\tau)^2 + p(\tau)^2 d\tau\right) \tag{7.2}$$

$$\times \exp\left(-\frac{i}{h}\int_{t_0}^{t} V(x(\tau),p(\tau),\tau)\,d\tau\right)\cdot \delta(x(t)-y)$$

In this expression the sum of the first and the third integral is the action $S(x,p)$, and the Donsker's delta function serves to pin trajectories to y at time t. The second integral is introduced to simulate the Lebesgue integral by compensation of the fall-off of the Gaussian measure in the time interval (t_0,t). Furthermore, as in Feynman's formula we need a normalization which turns out to be infinity and will be implemented by the use of a normalized exponential as in Chapter 3.

Hamiltonian path integral in momentum space If we know the initial and the end momentums it is clear by Heisenbergs uncertainty principle that we have no certain information about the corresponding space variables. This means we model the momentum trajectories as a Brownian fluctuation starting in the initial momentum p_0.

$$p(\tau) = p_0 + \frac{\sqrt{\hbar m}}{t-t_0}B(\tau), \quad 0 \leq \tau \leq t. \tag{7.3}$$

Furthermore the space variable is modeled by white noise, i.e.

$$x(\tau) = \sqrt{\frac{\hbar}{m}}\cdot (t-t_0)\omega_x(\tau), \quad 0 \leq \tau \leq t. \tag{7.4}$$

The Hamiltonian path integral for the momentum space propagator is formally given by, see e.g. [48]

$$K(p',t',p_0,t_0) = \mathcal{N}\int_{p(t_0)=p_0,p(t)=p'} \exp(\frac{i}{\hbar}\int_{t_0}^{t} -q(s)\dot{p}(s) - H(p,q)\ ds)\, DpDq. \tag{7.5}$$

This path integral can be obtained by a Fourier transform of the coordinate space path integral in both variables, see e.g. [47]. Then we propose the following formal ansatz for the Feynman integrand in Phase space with respect to the Gaussian measure μ,

$$I_V = \mathrm{N}\exp\left(\frac{i}{\hbar}\int_{t_0}^{t} -x(\tau)\dot{p}(\tau) - \frac{p(\tau)^2}{2m}d\tau + \frac{1}{2}\int_{t_0}^{t}\omega_x(\tau)^2+\omega_p(\tau)^2 d\tau\right) \tag{7.6}$$
$$\times \exp\left(-\frac{i}{h}\int_{t_0}^{t} V(x(\tau),p(\tau),\tau)\,d\tau\right)\cdot \delta(p(t)-p')$$

7.2 The free Feynman integrand in phase space

First we consider $V = 0$ (free particle). For simplicity let $\hbar = m = 1$ and $t_0 = 0$. Furthermore we choose to have one space dimension and one dimension for the corresponding momentum variable, i.e. the underlying space is $S_2(\mathbb{R})$. Note that higher dimensions can be obtained by multiplication of the generating functionals, since the used variables are independent.

7.2.1 Coordinate space

Note that the first term in (7.2) can be considered as a exponential of a quadratic type:

$$\mathrm{Nexp}\left(i\int_0^t (p(\tau)\dot{x}(\tau) - \frac{p(\tau)^2}{2})d\tau + \frac{1}{2}\int_0^t \omega_x(\tau)^2 + \omega_p(\tau)^2 d\tau\right) = \mathrm{Nexp}\Big(-\frac{1}{2}\big\langle(\omega_x,\omega_p), K(\omega_x,\omega_p)\big\rangle\Big),$$

where the operator matrix K on $L_2^2(\mathbb{R})_{\mathbb{C}}$ can be written as

$$K = \begin{pmatrix} -\mathbb{1}_{[0,t)} & -i\mathbb{1}_{[0,t)} \\ -i\mathbb{1}_{[0,t)} & -(1-i)\mathbb{1}_{[0,t)} \end{pmatrix}. \tag{7.7}$$

Here the operator $\mathbb{1}_{[0,t)}$ denotes the multiplication with $\mathbb{1}_{[0,t)}$. Hence, the integrand in (7.2) can then be written as

$$I_0 = \mathrm{Nexp}\left(-\frac{1}{2}\langle(\omega_x,\omega_p), K(\omega_x,\omega_p)\rangle\right)\cdot\delta\Big(\langle(\omega_x,\omega_p),(\mathbb{1}_{[0,t)},0)\rangle - (y-x_0)\Big),$$

where the last term pins the position variable to y at t. Note that the momentum variable is not pinned. Our aim is to apply Lemma 3.2.6 with K as above and $\mathbf{g} = 0$, $L = 0$ and as $\boldsymbol{\eta} = (\mathbb{1}_{[0,t)}, 0)$. The inverse of $(Id + K)$ is given by

$$N^{-1} = (Id+K)^{-1} = \begin{pmatrix} \mathbb{1}_{[0,t)^c} & 0 \\ 0 & \mathbb{1}_{[0,t)^c} \end{pmatrix} + i\begin{pmatrix} \mathbb{1}_{[0,t)} & \mathbb{1}_{[0,t)} \\ \mathbb{1}_{[0,t)} & 0 \end{pmatrix}, \tag{7.8}$$

hence $(\boldsymbol{\eta}, N^{-1}\boldsymbol{\eta}) = i\cdot t$. Therefore the assumptions of Lemma 3.2.6 are fulfilled. Thus I_0 exists as a Hida distribution. By applying Lemma 3.2.6 and using $\langle N^{-1}\eta, f\rangle = \langle \eta, N^{-1}f\rangle$, its T-transform in $(f_x, f_p) \in S_2(\mathbb{R})$ is given by

$$\begin{aligned}
&T\left(\mathrm{Nexp}\left(-\frac{1}{2}\langle(\omega_x,\omega_p),K(\omega_x,\omega_p)\rangle\right)\cdot\delta\left(\langle(\omega_x,\omega_p),(\mathbb{1}_{[0,t)},0)\rangle - y\right)\right)(f_x,f_p)\\
&=\frac{1}{\sqrt{2\pi i t}}\exp\left(\frac{1}{2it}\left(i(y-x_0)+\langle\boldsymbol{\eta},N^{-1}(f_x,f_p)\rangle\right)^2-\frac{1}{2}\left((f_x,f_p),N^{-1}(f_x,f_p)\right)\right)\\
&=\frac{1}{\sqrt{2\pi i t}}\exp\left(\frac{1}{2it}\left(i(y-x_0)+i\int_0^t f_x+f_p\,ds\right)^2\right)\\
&\quad\times\exp\left(-\frac{1}{2}\left((f_x,f_p),\left(\begin{pmatrix}\mathbb{1}_{[0,t)^c}+i\mathbb{1}_{[0,t)} & i\mathbb{1}_{[0,t)}\\ i\mathbb{1}_{[0,t)} & \mathbb{1}_{[0,t)^c}\end{pmatrix}\right)(f_x,f_p)\right)\right)\\
&=\frac{1}{\sqrt{2\pi i t}}\exp\left(\frac{1}{2it}\left(i(y-x_0)+i\int_0^t f_x+f_p\,ds\right)^2\right)\\
&\quad\times\exp\left(-\frac{1}{2}\left(\int_{[0,t)^c} f_x^2+f_p^2\,ds+i\int_{[0,t)} f_x^2\,ds+2i\int_{[0,t)} f_x(s)f_p(s)\,ds\right)\right).
\end{aligned}\tag{7.9}$$

Hence its generalized expectation

$$\mathbb{E}(I_0) = TI_0(0) = \frac{1}{\sqrt{2\pi i t}}\exp(-\frac{1}{2it}(y-x_0)^2) = K(y,t,x_0,0)$$

gives indeed the Greens function to the Schrödinger equation for a free particle, see e.g. [40]. Summarizing we have the following Theorem:

Theorem 7.2.1. *Let $y \in \mathbb{R}$, $0 < t < \infty$, then the free Feynman integrand in phase space I_0 exists as a Hida distribution. Its generating functional TI_0 is given by (7.9) and its generalized expectation $\mathbb{E}(I_0) = TI_0(0)$ is the Greens function to the Schrödinger equation for the free particle.*

7.2.2 Momentum space

It is well known, see e.g. [48] that the momentum space propagator for a free particle is given in form of a Dirac Delta function. We want to show therefore at least that we can find an expression which converges to this propagator. As above we consider first the action

$$S = \int_0^t -x(\tau)\dot{p}(\tau) - \frac{1}{2m}p^2(\tau)d\tau$$

then we have with (7.3) and (7.4)

$$S = \int_0^t -\sqrt{\frac{\hbar}{m}}\sqrt{\hbar m}\omega_p(\tau)\omega_x(\tau) - \frac{1}{2m}(p_0 + \frac{\sqrt{\hbar m}}{t}\langle \mathbb{1}_{[0,\tau)}, \omega_p\rangle)^2 \, d\tau$$

$$= -\hbar \int_0^t \omega_p(\tau)\omega_x(\tau) d\tau - \frac{p_0^2}{2m} t + \langle p_0 \sqrt{\frac{\hbar}{mt^2}}(s-t)\mathbb{1}_{[0,t)}(s), \omega_p(s)\rangle - \frac{1}{2}\int_0^t \frac{\hbar}{t^2}\langle \mathbb{1}_{[0,\tau)}, \omega_p\rangle^2 \, d\tau$$

Then we can write (7.6) in the following form using $m = \hbar = 1$:

$$\mathrm{Nexp}\Big(-\frac{1}{2}\big\langle(\omega_x,\omega_p), K_{mom}(\omega_x,\omega_p)\big\rangle\Big)$$
$$\times \exp(\langle p_0 \frac{1}{t}(\cdot - t)\mathbb{1}_{[0,t)}(\cdot), \omega_p\rangle)\delta(p' - p_0 - \langle \frac{1}{t}\mathbb{1}_{[0,t)}, \omega_p\rangle)$$

where the operator matrix K on $L^2_2(\mathbb{R})_{\mathbb{C}}$ can be written as

$$K_{mom} = \begin{pmatrix} -\mathbb{1}_{[0,t)} & i\mathbb{1}_{[0,t)} \\ i\mathbb{1}_{[0,t)} & -\mathbb{1}_{[0,t)} + \frac{i}{t^2}A \end{pmatrix}. \tag{7.10}$$

Here

$$A\,f(s) = \mathbb{1}_{[0,t)}(s)\int_s^t \int_0^\tau f(r)\,dr\,d\tau, f \in L^2(\mathbb{R},\mathbb{C}), s \in \mathbb{R},$$

we refer to [34] for properties of the operator as the trace class property, invertibility and spectrum.

We have for $f, g \in L^2(\mathbb{R})_{\mathbb{C}}$

$$\langle f, Ag\rangle = \int_0^t \int_0^\tau f(s)\,ds \cdot \int_0^\tau g(s)\,ds\; d\tau.$$

Hence the operator is used to implement the integral over the squared Brownian motion. The last term pins the momentum variable to p' at t. Note that the space variable is not pinned.

Our aim is to apply Lemma 3.2.6 with K as above and $\mathbf{g} = (0, \frac{p_0}{t}(s-t)\mathbb{1}_{[0,t)}(s))$, $L = 0$ and as $\boldsymbol{\eta} = (0, \frac{1}{t}\mathbb{1}_{[0,t)})$. The inverse of $(Id + K)$ is given by

$$N^{-1} = (Id + K_{mom})^{-1} = \begin{pmatrix} \mathbb{1}_{[0,t)^c} & 0 \\ 0 & \mathbb{1}_{[0,t)^c} \end{pmatrix} + i\begin{pmatrix} \frac{1}{t^2}A & -\mathbb{1}_{[0,t)} \\ -\mathbb{1}_{[0,t)} & 0 \end{pmatrix}, \tag{7.11}$$

hence $(\boldsymbol{\eta}, N^{-1}\boldsymbol{\eta}) = 0$.
To apply Lemma 3.2.6 we use a small perturbation of the matrix N^{-1}.
Let $\epsilon > 0$ then we define

$$N_\epsilon^{-1} = \begin{pmatrix} 1\!\!1_{[0,t)^c} & 0 \\ 0 & 1\!\!1_{[0,t)^c} \end{pmatrix} + \begin{pmatrix} \frac{i}{t^2}A & -i1\!\!1_{[0,t)} \\ -i1\!\!1_{[0,t)} & +\epsilon \end{pmatrix},$$

then we have $(\boldsymbol{\eta}, N_\epsilon^{-1}\boldsymbol{\eta}) = \frac{\epsilon}{t}$ and Lemma 3.2.6 can be applied. Therefore the assumptions of Lemma 3.2.6 are fulfilled. Thus we define the regularized free momentum integrand by its T-transform in $(f_x, f_p) \in S_2(\mathbb{R})$

$$\begin{aligned} T(I_{0mom,\epsilon})(f_x, f_p) &= \frac{1}{\sqrt{2\pi\frac{\epsilon}{t}}} \exp(-\frac{ip_0^2}{2}t) \\ &\times \exp\Big(-\frac{1}{2}\left\langle \begin{pmatrix} f_x \\ f_p + \frac{p_0}{t}(\cdot - t)1\!\!1_{[0,t)} \end{pmatrix}, N_\epsilon^{-1} \begin{pmatrix} f_x \\ f_p + \frac{p_0}{t}(\cdot - t)1\!\!1_{[0,t)} \end{pmatrix}\right\rangle\Big) \\ &\times \exp\left(\frac{1}{2\frac{\epsilon}{t}}\left(i(p' - p_0) + \left\langle \begin{pmatrix} f_x \\ f_p + (\cdot - t)1\!\!1_{[0,t)} \end{pmatrix}, N_\epsilon^{-1} \begin{pmatrix} 0 \\ 1\!\!1_{[0,t)} \end{pmatrix}\right\rangle\right)^2\right) \end{aligned}$$

Hence its generalized expectation

$$\begin{aligned} \mathbb{E}(I_{0,mom,\epsilon}) &= TI_{0,mom,\epsilon}(0) \\ &= \frac{\sqrt{t}}{\sqrt{2\pi\epsilon}} \exp(-\frac{1}{2}\left\langle \begin{pmatrix} 0 \\ 0 + \frac{p_0}{t}(\cdot - t)1\!\!1_{[0,t)} \end{pmatrix}, N_\epsilon^{-1} \begin{pmatrix} 0 \\ \frac{p_0}{t}(\cdot - t)1\!\!1_{[0,t)} \end{pmatrix}\right\rangle) \\ &\times \exp\left(\frac{t}{2\epsilon}\left(i(p' - p_0) + \left\langle \begin{pmatrix} 0 \\ 0 + \frac{p_0}{t}(\cdot - t)1\!\!1_{[0,t)} \end{pmatrix}, N_\epsilon^{-1} \begin{pmatrix} 0 \\ \frac{1}{t}1\!\!1_{[0,t)} \end{pmatrix}\right\rangle\right)^2\right) \cdot \exp(-\frac{ip_0^2}{2}t) \\ &= \frac{\sqrt{t}}{\sqrt{2\pi\epsilon}} \exp\left(-\frac{\epsilon}{2t^2}p_0^2 \int_0^t (s-t)^2 \, ds\right) \\ &\times \exp\left(\frac{t}{2\epsilon}\big(i(p' - p_0) + \frac{p_0\epsilon}{t^2}\int_0^t (s-t)\, ds\big)^2\right) \exp(-\frac{ip_0^2}{2}t) \\ &= \frac{\sqrt{t}}{\sqrt{2\pi\epsilon}} \exp\left(-\frac{\epsilon}{2t^2}p_0^2 \int_0^t (s-t)^2 \, ds\right) \exp(-\frac{ip_0^2}{2}t) \\ &\times \exp\left(\frac{t}{2\epsilon}\Big(-(p'-p_0)^2 + \frac{2ip_0\epsilon}{t^2}(p'-p_0)(\int_0^t (s-t)\, ds) + \frac{p_0^2\epsilon^2}{t^4}\big(\int_0^t (s-t)\, ds\big)^2\right) \\ &= \frac{\sqrt{t}}{\sqrt{2\pi\epsilon}} \exp(-\frac{t}{2\epsilon}(p'-p_0)^2) \exp(-\frac{ip_0^2}{2}t) \end{aligned}$$

$$\times \exp\left(-\frac{\epsilon}{2t^2}p_0^2\int_0^t (s-t)^2\ ds\right)\exp\left(\frac{p_0^2\epsilon}{2t^3}\big(\int_0^t (s-t)\,ds\big)^2\right)\cdot\exp(i\frac{p_0}{t}(p'-p_0)\int_0^t (s-t)\,ds)$$

In the limit $\epsilon \to 0$ we obtain:

$$\begin{aligned}\lim_{\epsilon\to 0}\mathbb{E}(I_{0mom,\epsilon}) &= \delta(p'-p_0)\cdot\exp(-\frac{ip_0^2}{2}t)\cdot\exp\big(\frac{ip_0}{t}(p'-p_0)\int_0^t (s-t)\,ds\big)\\ &= \delta(p'-p_0)\cdot\exp(-\frac{ip_0^2}{2}t),\end{aligned}$$

Since the last term vanishes note that the Delta function just gives values if $p' = p_0$. The generalized expectation is up to a factor 2π exactly the propagator of the free particle in momentum space, see [48]. Note that the Delta function serves to conserve the momentum of the free particle. If there is no potential the momentum must be the same as the initial momentum since the space is free of any force.

7.3 Harmonic oscillator

In this section we construct the Feynman integrand for the harmonic oscillator in phase space. I.e. the potential is given by $x \mapsto V(x) = \frac{1}{2}kx^2$, $0 \le k < \infty$.

7.3.1 Coordinate Space

The corresponding Lagrangian in phase space representation in coordinate space is given by

$$(x(\tau), p(\tau)) \mapsto L((x(\tau), p(\tau))) = p(\tau)\dot{x}(\tau) - \frac{p(\tau)^2}{2} - \frac{1}{2}kx(\tau)^2.$$

In addition to the matrix K from the free case, see (7.7), we have a matrix L which includes the information about the potential, see also [34]. In order to realize (7.2) for the harmonic oscillator we consider

$$\begin{aligned}I_{HO} = \mathrm{N}\exp\big(-\frac{1}{2}\langle(\omega_x,\omega_p), K(\omega_x,\omega_p)\rangle\big)\cdot\exp\big(-\frac{1}{2}\langle(\omega_x,\omega_p), L(\omega_x,\omega_p)\rangle\big)\\ \cdot\,\delta\big(\langle(\omega_x,\omega_p),(\mathbb{1}_{[0,t)},0)\rangle - y\big),\end{aligned}$$

with

$$L = \begin{pmatrix} ikA & 0\\ 0 & 0\end{pmatrix},\ y\in\mathbb{R},\ t>0.$$

Here again $A\,f(s) = \mathbb{1}_{[0,t)}(s)\int_s^t \int_0^\tau f(r)\,dr\,d\tau$, $f \in L^2(\mathbb{R},\mathbb{C})$, $s \in \mathbb{R}$. Hence we apply Lemma 3.2.6 to the case

$$N = \begin{pmatrix} \mathbb{1}_{[0,t)^c} + ikA & -i\mathbb{1}_{[0,t)} \\ -i\mathbb{1}_{[0,t)} & \mathbb{1}_{[0,t)^c} + i\mathbb{1}_{[0,t)} \end{pmatrix} = \begin{pmatrix} \mathbb{1}_{[0,t)^c} & 0 \\ 0 & \mathbb{1}_{[0,t)^c} \end{pmatrix} + i \begin{pmatrix} kA & -\mathbb{1}_{[0,t)} \\ -\mathbb{1}_{[0,t)} & \mathbb{1}_{[0,t)} \end{pmatrix}.$$

For determining the inverse of N we use the decomposition of $L^2_2(\mathbb{R})_\mathbb{C}$ into the orthogonal subspaces $L^2_2([0,t))_\mathbb{C}$ and $L^2_2([0,t)^c)_\mathbb{C}$. The operator N leaves both spaces invariant and on $L^2_2([0,t)^c)$ it is already the identity. Therefore we need just an inversion of N on $L^2_2([0,t))$. By calculation we obtain

$$N^{-1} = \begin{pmatrix} \mathbb{1}_{[0,t)^c} & 0 \\ 0 & \mathbb{1}_{[0,t)^c} \end{pmatrix} + \frac{1}{i}\mathbb{1}_{[0,t)} \begin{pmatrix} (kA - \mathbb{1}_{[0,t)})^{-1} & (kA - \mathbb{1}_{[0,t)})^{-1} \\ (kA - \mathbb{1}_{[0,t)})^{-1} & kA(kA - \mathbb{1}_{[0,t)})^{-1} \end{pmatrix}, \tag{7.12}$$

if $(kA - \mathbb{1}_{[0,t)})^{-1}$ exists, i.e. $kA - \mathbb{1}_{[0,t)}$ is bijective on $L^2_2([0,t))$. The operator $kAf(s) = \mathbb{1}_{[0,t)}(s)k\int_s^t \int_0^\tau f(r)\,dr\,d\tau$, $f \in L^2_2([0,t))_\mathbb{C}$, $s \in [0,t)$, diagonalizes and the eigenvalues l_n different from zero have the form:

$$l_n = k\bigg(\frac{t}{(n-\frac{1}{2})\pi}\bigg)^2, \quad n \in \mathbb{N}.$$

Thus $(kA - \mathbb{1}_{[0,t)})^{-1}$ exists if $l_n \neq 1$ for all $n \in \mathbb{N}$. For $0 < t < \pi/(2\sqrt{k})$ this is true. The corresponding normalized eigenvectors to l_n are

$$[0,t) \ni s \mapsto e_n(s) = \sqrt{\frac{2}{t}}\cos\left(\frac{s}{t}\left(n - \frac{1}{2}\right)\pi\right), \quad s \in [0,t) \quad n \in \mathbb{N}.$$

Hence we obtain using [28, p. 431, form. 1]:

$$\begin{aligned} \frac{1}{\det(Id + L(Id+K)^{-1})} &= \left(\det\left(Id + \begin{pmatrix} -kA & -kA \\ 0 & 0 \end{pmatrix}\right)\right)^{-1} \\ &= (\prod_{n=1}^\infty (1 - k(\frac{t}{(n-\frac{1}{2})\pi})^2))^{-1} = \frac{1}{\cos(\sqrt{k}t)}. \end{aligned}$$

Furthermore, again with $\boldsymbol{\eta} = (\mathbb{1}_{[0,t)}, 0)$ we obtain

$$(\boldsymbol{\eta}, N^{-1}\boldsymbol{\eta}) = (\mathbb{1}_{[0,t)}, (\mathbb{1}_{[0,t)^c} + i(\mathbb{1}_{[0,t)} - kA)^{-1})\,\mathbb{1}_{[0,t)}) = i\sum_{n=1}^\infty \big(1 - l_n\big)^{-1}(\mathbb{1}_{[0,t)}, e_n)^2$$

$$= i\sum_{n=1}^{\infty}\frac{1}{1-k\left(\frac{t}{((n-\frac{1}{2})\pi)}\right)^2}\frac{2t}{(n-\frac{1}{2})\pi)^2} = 2it\sum_{n=1}^{\infty}\frac{1}{((n-\frac{1}{2})\pi)^2-kt^2}$$

$$= \frac{i}{\sqrt{k}}8\sqrt{k}t\sum_{n=1}^{\infty}\frac{1}{((2n-1)\pi)^2-4kt^2} = \frac{i}{\sqrt{k}}\tan(\sqrt{k}t) = i\frac{\tan(\sqrt{k}t)}{\sqrt{k}},$$

by using [28, p. 421,form. 1]. Hence we have for the T-transform in $\mathbf{f} \in S_2(\mathbb{R})$ by applying Lemma 3.2.6

$$TI_{HO}(\mathbf{f}) = \sqrt{\left(\frac{\sqrt{k}}{2\pi i\sin(\sqrt{k}t)}\right)}\exp\left(\frac{1}{2}\frac{\sqrt{k}}{i\tan(\sqrt{k}t)}\Big(iy+(\boldsymbol{\eta},\mathbf{f}+\mathbf{g})\Big)^2\right)$$
$$\times\exp\left(-\frac{1}{2}\left(\big(\mathbf{f}+\mathbf{g}\big),\begin{pmatrix}\mathbb{1}_{[0,t)^c} & 0\\ 0 & \mathbb{1}_{[0,t)^c}\end{pmatrix}\big(\mathbf{f}+\mathbf{g}\big)\right)\right)$$
$$\times\exp\left(-\frac{1}{2}\left(\big(\mathbf{f}+\mathbf{g}\big),\frac{t}{i}\mathbb{1}_{[0,t)}\begin{pmatrix}\frac{1}{t}(kA-\mathbb{1}_{[0,t)})^{-1} & (kA-\mathbb{1}_{[0,t)})^{-1}\\ (kA-\mathbb{1}_{[0,t)})^{-1} & ktA(kA-\mathbb{1}_{[0,t)})^{-1}\end{pmatrix}\big(\mathbf{f}+\mathbf{g}\big)\right)\right) \quad (7.13)$$

Summarizing we have the following theorem:

Theorem 7.3.1. *Let $y \in \mathbb{R}$, $0 < t < \frac{\pi}{2\sqrt{k}}$, then the Feynman integrand for the harmonic oscillator in phase space I_{H0} exists as a Hida distribution and its generating functional is given by* (7.13)*. Moreover its generalized expectation*

$$\mathbb{E}(I_{HO}) = T(I_{HO})(0) = \sqrt{\left(\frac{\sqrt{k}}{2\pi i\sin(\sqrt{k}t)}\right)}\exp\left(i\frac{\sqrt{k}}{2\tan(\sqrt{k}t)}y^2\right)$$

is the Greens function to the Schrödinger equation for the harmonic oscillator, compare e.g. with [44].

7.3.2 Momentum Space

The corresponding Lagrangian in phase space representation in momentum space is given by

$$(x(\tau),p(\tau)) \mapsto L((x(\tau),p(\tau))) = -\dot{p}(\tau)x(\tau) - \frac{p(\tau)^2}{2} - \frac{1}{2}kx(\tau)^2.$$

In addition to the matrix K from the free case, see (7.7), we have a matrix L which includes the information about the potential, see also [34]. For the sake of simplicity we set $p_0 = 0$.

In order to realize (7.6) for the harmonic oscillator we consider

$$I_{HO,mom} = \mathrm{Nexp}\big(- \frac{1}{2}\langle(\omega_x,\omega_p), K_{mom}(\omega_x,\omega_p)\rangle\big) \cdot \exp\big(- \frac{1}{2}\langle(\omega_x,\omega_p), L(\omega_x,\omega_p)\rangle\big) \cdot \delta\big(\langle(\omega_x,\omega_p),(0,\mathbb{1}_{[0,t)})\rangle - (p')\big).$$

with

$$L = \begin{pmatrix} ikt^2\mathbb{1}_{[0,t)} & 0 \\ 0 & 0 \end{pmatrix}, \; p' \in \mathbb{R}, \; t > 0.$$

Then we have

$$N = (Id + K + L) = \begin{pmatrix} \mathbb{1}_{[0,t]^c} & 0 \\ 0 & \mathbb{1}_{[0,t)^c} \end{pmatrix} + i \begin{pmatrix} kt^2\mathbb{1}_{[0,t]} & \mathbb{1}_{[0,t)} \\ \mathbb{1}_{0,t)} & \frac{1}{t^2}A \end{pmatrix}.$$

Its inverse is then given by

$$N^{-1} = (Id+K+L)^{-1} = \begin{pmatrix} \mathbb{1}_{[0,t]^c} & 0 \\ 0 & \mathbb{1}_{[0,t)^c} \end{pmatrix} + \frac{1}{i}\begin{pmatrix} \frac{A}{t^2}(kA - \mathbb{1}_{[0,t)})^{-1} & -(kA - \mathbb{1}_{[0,t)})^{-1} \\ -(kA - \mathbb{1}_{[0,t)})^{-1} & kt^2(kA - \mathbb{1}_{[0,t)})^{-1} \end{pmatrix},$$

if $(kA - \mathbb{1}_{[0,t)})^{-1}$ exists, i.e. $kA - \mathbb{1}_{[0,t)}$ is bijective on $L_2^2([0,t))$. Using again the eigenvalues l_n different from zero as in coordinate space:

$$l_n = k\bigg(\frac{t}{(n-\frac{1}{2})\pi}\bigg)^2, \quad n \in \mathbb{N},$$

we have $(kA - \mathbb{1}_{[0,t)})^{-1}$ exists if $l_n \neq 1$ for all $n \in \mathbb{N}$ as before for $0 < t < \pi/(2\sqrt{k})$ this is true. Hence we obtain again using [28, p. 431, form. 1]:

$$\frac{1}{\det(Id + L(Id+K)^{-1})} = \det\left(Id + \begin{pmatrix} -kA & -kt^2\mathbb{1}_{[0,t)} \\ 0 & 0 \end{pmatrix}\right)^{-1} = \big(\prod_{n=1}^{\infty}(1 - k(\frac{t}{(n-\frac{1}{2})\pi})^2)\big)^{-1} = \frac{1}{\cos(\sqrt{k}t)}.$$

As in the coordinate space we use the eigenstructure of A to determine the matrix $M_{N^{-1}}$. We have

$$M_{N^{-1}} = \langle(0,\frac{1}{t}\mathbb{1}_{[0,t)}), N^{-1}(0,\frac{1}{t}\mathbb{1}_{[0,t)})\rangle = \frac{ikt^2}{t^2}\sum_{n=0}^{\infty}\frac{1}{1-l_n}\langle e_n, \mathbb{1}_{[0,t)}\rangle^2$$

$$= ik\frac{\tan(\sqrt{k}t)}{\sqrt{k}} = i\sqrt{k}\tan(\sqrt{k}t),$$

where we used the results of the previous subsection to calculate the explicit value of the series. We see that at least for small times the assumptions of Lemma 3.2.6 are fulfilled, since $M_{N^{-1}}$ is completely imaginary with positive imaginary part.

The T-transform of $I_{HO,mom}$ in $\mathbf{f} \in S_2(\mathbb{R})$ with $\boldsymbol{\eta} = (0, \frac{1}{t}\mathbb{1}_{[0,t)})$ is then given by

$$\begin{aligned} TI_{HO}(\mathbf{f}) &= \sqrt{\left(\frac{1}{2\pi i\sqrt{k}\sin(\sqrt{k}t)}\right)} \\ \times \exp&\left(\frac{1}{2}\frac{1}{i\sqrt{k}\tan(\sqrt{k}t)}\Big(ip' + \langle \boldsymbol{\eta}, \frac{1}{i}\begin{pmatrix} \frac{A}{t^2}(kA - \mathbb{1}_{[0,t)})^{-1} & -(kA - \mathbb{1}_{[0,t)})^{-1} \\ -(kA - \mathbb{1}_{[0,t)})^{-1} & kt^2(kA - \mathbb{1}_{[0,t)})^{-1} \end{pmatrix}(\mathbf{f}+\mathbf{g})\rangle\Big)^2\right) \\ &\times \exp\left(-\frac{1}{2}\Big((\mathbf{f}+\mathbf{g}), \begin{pmatrix} \mathbb{1}_{[0,t)^c} & 0 \\ 0 & \mathbb{1}_{[0,t)^c} \end{pmatrix}(\mathbf{f}+\mathbf{g})\Big)\right) \\ &\times \exp\left(-\frac{1}{2}\Big((\mathbf{f}+\mathbf{g}), \frac{1}{i}\begin{pmatrix} \frac{A}{t^2}(kA - \mathbb{1}_{[0,t)})^{-1} & -(kA - \mathbb{1}_{[0,t)})^{-1} \\ -(kA - \mathbb{1}_{[0,t)})^{-1} & kt^2(kA - \mathbb{1}_{[0,t)})^{-1} \end{pmatrix}(\mathbf{f}+\mathbf{g})\Big)\right) \end{aligned} \tag{7.14}$$

Finally we obtain the following theorem.

Theorem 7.3.2. *Let $y \in \mathbb{R}$, $0 < t < \frac{\pi}{2\sqrt{k}}$, then the Feynman integrand for the harmonic oscillator in phase space in momentum space $I_{H0,mom}$ exists as a Hida distribution and its generating functional is given by (7.13). Moreover its generalized expectation*

$$\mathbb{E}(I_{HO,mom}) = T(I_{HO,mom})(0) = \sqrt{\left(\frac{1}{2\pi i\sqrt{k}\sin(\sqrt{k}t)}\right)}\exp\left(i\frac{1}{2\sqrt{k}\tan(\sqrt{k}t)}p'^2\right)$$

is the Greens function to the Schrödinger equation for the harmonic oscillator in momentum space, compare e.g. with [48, p.118, form.2.187].

7.4 Charged particle in a constant magnetic field

In this subsection we want to calculate the transition amplitude for the movement of a charged particle in a constant magnetic field. Investigations of this system for the Feynman integrand had been done in White Noise in [29], [43] and [11]. Here we just consider the motion in the plane orthogonal to the direction of the magnetic field. Note that the propagator in three dimension can just be obtained by multiplying the expression with the free motion propagator along the axis of the magnetic field vector. For the corresponding Hamiltonian action one finds see e.g. [76, form.(2.49), p.103]:

$$S(q,p,T) = \int_0^T \mathbf{p}\dot{\vec{\mathbf{x}}} - \frac{1}{2m}\left(\mathbf{p} - \frac{q}{c}\mathbf{A}(\mathbf{x})\right)^2 d\tau,$$

where

$$\mathbf{p} = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} \text{ and } \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix},$$

respectively. Moreover q is the charge of the particle, c is the speed of light and $\mathbf{A}(\mathbf{x})$ a is the two dimensional vector potential. Note that a multiplication of the vectors above is thought of a the euclidean scalar product, e.g.

$$\mathbf{px} = p_1 x_1 + p_2 x_2.$$

Here we consider the case of a constant magnetic field along the x_3-axis, i.e. the axis orthogonal to the plane spanned by x_1 and x_2. We have $\mathbf{B} = (0, 0, B_z)$. With the relation

$$\mathbf{B} = \mathrm{rot}(\mathbf{A}),$$

we have

$$A = B_z \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix}.$$

Thus we have

$$\begin{aligned} S(q,p,t) &= \int_0^t \mathbf{p}\dot{\vec{\mathbf{x}}} - \frac{1}{2m}\left(\mathbf{p} - \frac{q}{c}B_z \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix}\right)^2 d\tau \\ &= \int_0^t \mathbf{p}\dot{\vec{\mathbf{x}}} - \frac{1}{2m}(p_1^2 + p_2^2) + \frac{q}{mc}B_z(x_1 p_2 - x_2 p_1) - \frac{q^2 B_z^2}{2mc^2}(x_1^2 + x_2^2) d\tau. \end{aligned}$$

At the beginning for simplicity we set $x_{1,0} = x_{2,0} = 0, t_0 = 0$ and $m = \hbar = 1$. Then with $k = \frac{qB_z}{mc}$ we consider the following ansatz for the integrand:

$$I_{CP} = \mathrm{N} \exp\big(-\frac{1}{2}\langle\cdot, K\cdot\rangle\big) \cdot \exp\big(-\frac{1}{2}\langle\cdot, L\cdot\rangle\big) \cdot \delta\big(\langle\cdot_{x1}, \mathbf{1}_{[0,t)}\rangle - y_1\big)\delta\big(\langle\cdot_{x2}, \mathbf{1}_{[0,t)}\rangle - y_2\big). \tag{7.15}$$

For the kinetic energy part and the locally simulation of the flat measure we have for the matrix K as in the previous section:

$$K = \begin{pmatrix} -\mathbf{1}_{[0,t)} & -i\mathbf{1}_{[0,t)} & 0 & 0 \\ -i\mathbf{1}_{[0,t)} & -\mathbf{1}_{[0,t)} + i\mathbf{1}_{[0,t)} & 0 & 0 \\ 0 & 0 & -\mathbf{1}_{[0,t)} & -i\mathbf{1}_{[0,t)} \\ 0 & 0 & -i\mathbf{1}_{[0,t)} & -\mathbf{1}_{[0,t)} + i\mathbf{1}_{[0,t)} \end{pmatrix}$$

and in addition we have to model the potential. We use an ansatz where we have an upper triangular block matrix, i.e.

$$L = \begin{pmatrix} ik^2A & 0 & 0 & -2ikB^* \\ 0 & 0 & 2ikB & 0 \\ 0 & 0 & ik^2A & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

with $\mathbb{R} \ni s \mapsto Af(s) = \mathbb{1}_{[0,t)}(s) \int_s^t \int_0^r f(\tau)\, d\tau\, dr$ and $\mathbb{R} \ni s \mapsto Bf(s) = \mathbb{1}_{[0.t)}(s) \int_0^s f(r)\, dr$ for $f \in L^2(\mathbb{R})$. With B^* we denote the dual operator of B w.r.t. the dual pairing. Note that we have $(Ag, f) = (Bg, Bf)$ for all $f, g \in L^2(\mathbb{R})$. Thus

$$\begin{aligned} Id + K + L = N \\ = \begin{pmatrix} \mathbf{1}_{[0,t)^c} & 0 & 0 & 0 \\ 0 & \mathbf{1}_{[0,t)^c} & 0 & 0 \\ 0 & 0 & \mathbf{1}_{[0,t)^c} & 0 \\ 0 & 0 & 0 & \mathbf{1}_{[0,t)^c} \end{pmatrix} + \begin{pmatrix} ik^2A & -i\mathbf{1}_{[0,t)} & 0 & -2ikB^* \\ -i\mathbf{1}_{[0,t)} & i\mathbf{1}_{[0,t)} & 2ikB & 0 \\ 0 & 0 & ik^2A & -i\mathbf{1}_{[0,t)} \\ 0 & 0 & -i\mathbf{1}_{[0,t)} & i\mathbf{1}_{[0,t)} \end{pmatrix} \end{aligned}$$

We have the following proposition:

Proposition 7.4.1. *The operator*

$$\begin{aligned} Id + K + L = N \\ = \begin{pmatrix} \mathbf{1}_{[0,t)^c} & 0 & 0 & 0 \\ 0 & \mathbf{1}_{[0,t)^c} & 0 & 0 \\ 0 & 0 & \mathbf{1}_{[0,t)^c} & 0 \\ 0 & 0 & 0 & \mathbf{1}_{[0,t)^c} \end{pmatrix} + \begin{pmatrix} ik^2A & -i\mathbf{1}_{[0,t)} & 0 & -2ikB^* \\ -i\mathbf{1}_{[0,t)} & i\mathbf{1}_{[0,t)} & 2ikB & 0 \\ 0 & 0 & ik^2A & -i\mathbf{1}_{[0,t)} \\ 0 & 0 & -i\mathbf{1}_{[0,t)} & i\mathbf{1}_{[0,t)} \end{pmatrix} \end{aligned}$$

is linear and bounded on $L^2_4(\mathbb{R})_\mathbb{C}$ and has a bounded inverse.

Proof. The operator A is in the trace class, moreover B and B^* are compact operators. Moreover by 7.13 we know that A is invertible on the orthogonal subspace $L^2([0,t),dx)$. Since N is already the identity restricted on $L^2_4([0,t)^c,dx)$ we can restrict ourselves to $L^2_4([0,t),dx)$. We then have

$$N_{|[0,t)} = \begin{pmatrix} ik^2A & -i\mathbf{1}_{[0,t)} & 0 & -2ikB^* \\ -i\mathbf{1}_{[0,t)} & i\mathbf{1}_{[0,t)} & 2ikB & 0 \\ 0 & 0 & ik^2A & -i\mathbf{1}_{[0,t)} \\ 0 & 0 & -i\mathbf{1}_{[0,t)} & i\mathbf{1}_{[0,t)} \end{pmatrix} = i \begin{pmatrix} k^2A & -\mathbf{1}_{[0,t)} & 0 & -2kB^* \\ -\mathbf{1}_{[0,t)} & \mathbf{1}_{[0,t)} & 2kB & 0 \\ 0 & 0 & k^2A & -\mathbf{1}_{[0,t)} \\ 0 & 0 & -\mathbf{1}_{[0,t)} & \mathbf{1}_{[0,t)} \end{pmatrix},$$

which is of the form

$$R = \begin{pmatrix} M & P \\ 0 & M \end{pmatrix},$$

with a bounded invertible matrix M. The inverse of such a matrix is given by

$$R^{-1} = \begin{pmatrix} M^{-1} & -M^{-1}PM^{-1} \\ 0 & M^{-1} \end{pmatrix}.$$

Indeed

$$\begin{pmatrix} M & P \\ 0 & M \end{pmatrix}\begin{pmatrix} M^{-1} & -M^{-1}PM^{-1} \\ 0 & M^{-1} \end{pmatrix} = \begin{pmatrix} MM^{-1} & -MM^{-1}PM^{-1} + PM^{-1} \\ 0 & MM^{-1} \end{pmatrix} = \begin{pmatrix} Id & 0 \\ 0 & Id \end{pmatrix}.$$

Now in our case

$$M^{-1} = \begin{pmatrix} (kA - \mathbb{1}_{[0,t)})^{-1} & (kA - \mathbb{1}_{[0,t)})^{-1} \\ (kA - \mathbb{1}_{[0,t)})^{-1} & kA(kA - \mathbb{1}_{[0,t)})^{-1} \end{pmatrix}.$$

Thus

$$
\begin{aligned}
M^{-1}PM^{-1} &= \begin{pmatrix} (k^2A - \mathbb{1}_{[0,t)})^{-1} & (k^2A - \mathbb{1}_{[0,t)})^{-1} \\ (k^2A - \mathbb{1}_{[0,t)})^{-1} & k^2A(k^2A - \mathbb{1}_{[0,t)})^{-1} \end{pmatrix} \\
&\times \begin{pmatrix} 0 & -2kB^* \\ 2kB & 0 \end{pmatrix} \begin{pmatrix} (k^2A - \mathbb{1}_{[0,t)})^{-1} & (k^2A - \mathbb{1}_{[0,t)})^{-1} \\ (k^2A - \mathbb{1}_{[0,t)})^{-1} & k^2A(k^2A - \mathbb{1}_{[0,t)})^{-1} \end{pmatrix} \\
&= \begin{pmatrix} (k^2A - \mathbb{1}_{[0,t)})^{-1} & (k^2A - \mathbb{1}_{[0,t)})^{-1} \\ (k^2A - \mathbb{1}_{[0,t)})^{-1} & k^2A(k^2A - \mathbb{1}_{[0,t)})^{-1} \end{pmatrix} \\
&\times \begin{pmatrix} -2kB^*(k^2A - \mathbb{1}_{[0,t)})^{-1} & -2k^3B^*A(kA - \mathbb{1}_{[0,t)})^{-1} \\ 2kB(k^2A - \mathbb{1}_{[0,t)})^{-1} & 2kB(k^2A - \mathbb{1}_{[0,t)})^{-1} \end{pmatrix}
\end{aligned}
$$

$$
= 2 \begin{pmatrix} k(k^2A - \mathbb{1}_{[0,t)})^{-1}(B - B^*)(k^2A - \mathbb{1}_{[0,t)})^{-1} & (k^2A - \mathbb{1}_{[0,t)})^{-1}(kB - k^3B^*A)(k^2A - \mathbb{1}_{[0,t)})^{-1} \\ -(k^2A - \mathbb{1}_{[0,t)})^{-1}(kB^* - k^3AB)(k^2A - \mathbb{1}_{[0,t)})^{-1} & k^3(k^2A - \mathbb{1}_{[0,t)})^{-1}(AB - B^*A)(k^2A - \mathbb{1}_{[0,t)})^{-1} \end{pmatrix}.
$$

Moreover we have that N is bounded invertible with

$$
\begin{aligned}
N^{-1} &= \begin{pmatrix} \mathbf{1}_{[0,t)^c} & 0 & 0 & 0 \\ 0 & \mathbf{1}_{[0,t)^c} & 0 & 0 \\ 0 & 0 & \mathbf{1}_{[0,t)^c} & 0 \\ 0 & 0 & 0 & \mathbf{1}_{[0,t)^c} \end{pmatrix} \\
&+ \frac{1}{i} \begin{pmatrix} \mathbb{1}_{[0,t)} & \mathbb{1}_{[0,t)} & 2k(k^2A - \mathbb{1}_{[0,t)})^{-1}(B - B^*) & 2(k^2A - \mathbb{1}_{[0,t)})^{-1}(kB - k^3B^*A) \\ \mathbb{1}_{[0,t)} & k^2A & -2(k^2A - \mathbb{1}_{[0,t)})^{-1}(kB^* - k^3AB) & 2k^3(k^2A - \mathbb{1}_{[0,t)})^{-1}(AB - B^*A) \\ 0 & 0 & \mathbb{1}_{[0,t)} & \mathbb{1}_{[0,t)} \\ 0 & 0 & \mathbb{1}_{[0,t)} & k^2A \end{pmatrix} \\
&\times \begin{pmatrix} (k^2A - \mathbb{1}_{[0,t)})^{-1} & 0 & 0 & 0 \\ 0 & (k^2A - \mathbb{1}_{[0,t)})^{-1} & 0 & 0 \\ 0 & 0 & (k^2A - \mathbb{1}_{[0,t)})^{-1} & 0 \\ 0 & 0 & 0 & (k^2A - \mathbb{1}_{[0,t)})^{-1} \end{pmatrix} \qquad (7.16)
\end{aligned}
$$

□

Next we calculate the matrix $M_{N^{-1}}$ from Lemma 3.2.6 to take care if it is applicable.

Proposition 7.4.2. *For N as is Proposition 7.4.1 we have for $\boldsymbol{\eta}_1 = (\mathbb{1}_{[0,t)}, 0, 0, 0)$ and*

$\boldsymbol{\eta}_3 = (0, 0, \mathbb{1}_{[0,t)}, 0)$, *that*

$$M_{N^{-1}} = \begin{pmatrix} \langle \boldsymbol{\eta}_1, N^{-1}\boldsymbol{\eta}_1 \rangle & \langle \boldsymbol{\eta}_1, N^{-1}\boldsymbol{\eta}_3 \rangle \\ \langle \boldsymbol{\eta}_3, N^{-1}\boldsymbol{\eta}_1 \rangle & \langle \boldsymbol{\eta}_3, N^{-1}\boldsymbol{\eta}_3 \rangle \end{pmatrix} = i \begin{pmatrix} \frac{\tan(k\,t)}{k} & 0 \\ 0 & \frac{\tan(k\,t)}{k} \end{pmatrix},$$

moreover the assumptions on Lemma 3.2.6 are fulfilled.

Proof. We have $\boldsymbol{\eta}_1 = (\mathbb{1}_{[0,t)}, 0, 0, 0)$ and $\boldsymbol{\eta}_3 = (0, 0, \mathbb{1}_{[0,t)}, 0)$. Hence by Proposition 7.4.1, we just have to consider the part of the inverse which has support on $[0, t)$.
Instead of calculating the inverse directly we find a preimage of $\boldsymbol{\eta}_1$ and $\boldsymbol{\eta}_3$, respectively under the operator N. We have

$$\begin{pmatrix} k^2 A & -\mathbf{1}_{[0,t)} & 0 & -2kB^* \\ -\mathbf{1}_{[0,t)} & \mathbf{1}_{[0,t)} & 2kB & 0 \\ 0 & 0 & k^2 A & -\mathbf{1}_{[0,t)} \\ 0 & 0 & -\mathbf{1}_{[0,t)} & \mathbf{1}_{[0,t)} \end{pmatrix} \begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{pmatrix} = -i\boldsymbol{\eta}_k \quad k = 1, 3.$$

We can transfer this to a system of differential equations, note that the function on the right-hand-side is almost surely constant. We obtain

$$(I) - k^2 f_1 + 2k f_4' = f_2'' \tag{7.17}$$

$$(II) f_1' - 2k f_3 = f_2' \tag{7.18}$$

$$(III) - k^2 f_3 = f_4'' \tag{7.19}$$

$$(IV) f_3 = f_4. \tag{7.20}$$

Taking into account that $f_3 = f_4$ and deriving equation (II) and setting it equal to (I) we obtain.

$$(I) - k^2 f_1 + 2k f_4' = f_1'' - 2k f_3' \tag{7.21}$$

$$(II) f_1' - 2k f_3 = f_2' \tag{7.22}$$

$$(III) - k^2 f_3 = f_3'' \tag{7.23}$$

$$(IV) f_3 = f_4. \tag{7.24}$$

Then (I) can be written as

$$f_1'' = -k^2 f_1 + 4k f_3'.$$

To obtain now the preimages we have to take the boundary conditions into account. We have by the definition of B and B^* and taking into account that $\boldsymbol{\eta}_{k2} = \boldsymbol{\eta}_{k4} = 0$ the following boundary conditions:

$$f_1(0) = f_2(0)$$

$$f_2'(0) = f_3(0)$$

$$f_4'(0) = f_3'(0) = 0.$$

The additionally two boundary conditions are obtained by inserting $\boldsymbol{\eta}_k$. For $\boldsymbol{\eta}_1$ we have

$$f_2(t) = i,$$

$$f_4(t) = f_3(t) = 0.$$

For $\boldsymbol{\eta}_3$ we have

$$f_2(t) = 0,$$

$$f_4(t) = f_3(t) = i.$$

We solved this system of differential equations with the dsolve-routine in MATLAB and obtained

$$N\mathbf{f} = (\mathbb{1}_{[0,t)}, 0, 0, 0)$$

with

$$\mathbf{f} = \begin{pmatrix} \frac{\cos(k\,s)\,\mathrm{i}}{\cos(k\,t)} \\ \frac{\cos(k\,s)\,\mathrm{i}}{\cos(k\,t)} \\ 0 \\ 0 \end{pmatrix}$$

and

$$N\mathbf{h} = (0, 0, \mathbb{1}_{[0,t)}, 0)$$

with

$$\mathbf{h} = \begin{pmatrix} \frac{2\,\sin(k\,s)\,\mathrm{i} + 2\,k\,s\,\cos(k\,s)\,\mathrm{i} - 2\,k\,t\,\cos(k\,s)\,\mathrm{i}}{\cos(k\,t)} \\ \frac{2\,k\,\cos(k\,s)\,(s-t)\,\mathrm{i}}{\cos(k\,t)} \\ \frac{\cos(k\,s)\,\mathrm{i}}{\cos(k\,t)} \\ \frac{\cos(k\,s)\,\mathrm{i}}{\cos(k\,t)} \end{pmatrix}$$

Then we have

$$M_{N^{-1}} = \begin{pmatrix} \langle \boldsymbol{\eta}_1, \mathbf{f} \rangle & \langle \boldsymbol{\eta}_1, \mathbf{h} \rangle \\ \langle \boldsymbol{\eta}_3, \mathbf{f} \rangle & \langle \boldsymbol{\eta}_3, \mathbf{h} \rangle \end{pmatrix} = i \begin{pmatrix} \frac{\tan(k\,t)}{k} & 0 \\ 0 & \frac{\tan(k\,t)}{k} \end{pmatrix}.$$

□

Now, to have all ingredients for the integrand, we calculate the determinant of $(Id + L(I+K)^{-1})$ We have

$$(Id + L(I+K)^{-1}) = \begin{pmatrix} \mathbb{1}_{[0,t)^c} & 0 & 0 & 0 \\ 0 & \mathbb{1}_{[0,t)^c} & 0 & 0 \\ 0 & 0 & \mathbb{1}_{[0,t)^c} & 0 \\ 0 & 0 & 0 & \mathbb{1}_{[0,t)^c} \end{pmatrix}$$
$$+ \begin{pmatrix} \mathbb{1}_{[0,t)} - k^2 A & -k^2 A & 2kB^* & 0 \\ 0 & \mathbb{1}_{[0,t)} & -2kB & -2kB \\ 0 & 0 & \mathbb{1}_{[0,t)} - k^2 A & -k^2 A \\ 0 & 0 & 0 & \mathbb{1}_{[0,t)} \end{pmatrix}$$

We have the following structure of the spectrum of this operator.

Proposition 7.4.3. *Let $0 < t < \frac{\pi}{2k}$. Then the eigenvalues of $Id + L(Id+K)^{-1} \mid_{L^2([0,t))}$ are $v_0 = 1$ and*

$$v_n = 1 - k^2 \left(\frac{t}{(n - \frac{1}{2})\pi} \right)^2, \quad n = 1, 2, 3 \cdots .$$

The algebraic multiplicity of each v_n, $n = 1, 2, 3, \cdots$ is 2.

The eigenvectors to v_0 have the form

$$\begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ -f_3 \end{pmatrix},$$

where f_2, f_3 are arbitrarily choosen in $L^2([0,t))$ and f_1 solves the equation

$$k^2 A f_1 = -k^2 A f_2 + 2kB^* f_3$$

.

The eigenvectors to $v_n,\ n = 1, 2, 3, \cdots$ is the set

$$\left\{ \alpha \begin{pmatrix} cos(\frac{k}{\sqrt{1-v_n}}s) \\ 0 \\ 0 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} \frac{2k}{1-v_n} \cdot s\ cos(\frac{k}{\sqrt{1-v_n}}s) \\ \frac{2}{\sqrt{1-v_n}} sin(\frac{k}{\sqrt{1-v_n}}s) \\ cos(\frac{k}{\sqrt{1-v_n}}s) \\ 0 \end{pmatrix} \mid \alpha, \beta \in \mathbb{C} \right\}.$$

Finally $det(Id + L(Id + K)^{-1} \mid_{L^2([0,t))}) = cos^2(kt)$.

Proof. First note that we need the restriction $0 < t < \frac{\pi}{2k}$ for the well-definiteness of $(Id + K)^{-1}$. We consider the equation

$$(Id + L(Id + K)^{-1}) \begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{pmatrix} = \lambda \begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{pmatrix},\ \lambda \in \mathbb{C},\ f_1,\ f_2,\ f_3,\ f_4\ \in L^2([0,t))$$

Note that $Id + L(Id + K)^{-1} = N(Id + K)^{-1}$ is invertible because N is invertible. Hence 0 is not an eigenvalue of $Id + L(Id + K)^{-1}$. Further $L(Id + K)^{-1}$ is a Hilbert-Schmidt operator, where each eigenvalue different from zero has a finite algebraic multiplicity. Consequently each eigenvalue of $Id + L(Id + K)^{-1}$ which is different from 1 has a finite algebraic multiplicity. Because we do not know whether $L(Id + K)^{-1}$ is a trace class operator, it is not sure that the determinant of $Id + L(Id + K)^{-1}$ has a finite value. It must be calculated.
1.case: $\lambda = 1$:
By the last equation we choose f_4 arbitrarily. The third equation furnishes $(1 - k^2 A)f_3 - k^2 A f_4 = f_3$, hence $f_3 = -f4$. Using this result in the second equation we get $f_2 = f_2$, hence f_2 can be choosen arbitrarily. The first equation leads to

$$k^2 A f_1 = -k^2 A f_2 + 2kB^* f_3$$

as condition for f_1. Examples of eigenvectors of $Id + L(Id + K)^{-1} \mid_{L^2([0,t))}$ to the eigenvalue

1 are given by $\begin{pmatrix} f_1 \\ f_2 \\ \frac{k}{2}B(f_1+f_2) \\ -\frac{k}{2}B(f_1+f_2) \end{pmatrix}$, with $f_1,\ f_2 \in L^2([0,t))$.

2.case: $\lambda \neq 1$:

The last equation implies $f_4 = 0$. Assumimg $f_3 = 0$ implies $f_2 = 0$ and f_1 is an eigenvector of $1 - k^2A$ corresponding to λ, hence $\lambda = v_n$ for some $n > 0$. Now we consider the last case:

Assume $f_3 \neq 0$. Then f_3 is an eigenvector of $1 - k^2A$ corresponding to λ, hence $\lambda = v_n$ for some $n > 0$. Further $f_2 = \frac{2k}{1-\lambda}Bf_3$ by the second equation and $f_1 \in (1 - \lambda - k^2A)^{-1}\left(\left\{(\frac{2k^3}{1-\lambda}AB - 2kB^*)f_3\right\}\right)$ by the first equation. Note that the set $(1 - \lambda - k^2A)^{-1}\left(\left\{(\frac{2k^3}{1-\lambda}AB - 2kB^*)f_3\right\}\right)$ is not empty, because using $f_3 \in LH\left\{cos(\frac{k}{\sqrt{1-v_n}}s)\right\}$ it holds

$$(1-\lambda-k^2A)^{-1}\left(\left\{(\frac{2k^3}{1-\lambda}AB - 2kB^*)f_3\right\}\right) = \frac{2k}{1-\lambda}\cdot(s\ f_3) + ker((1-\lambda) - k^2A).$$

So the eigenspace of $Id + L(Id+K)^{-1}\mid_{L^2([0,t))}$ corresponding to v_n is the set

$$\left\{\alpha\begin{pmatrix} f_1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \beta\begin{pmatrix} \frac{2k}{1-v_n}\cdot s\ f_3 \\ \frac{2k}{1-v_n}Bf_3 \\ f_3 \\ 0 \end{pmatrix} \mid f_1,\ f_3\ eigenvectors\ of\ 1-k^2A\ to\ v_n,\ \alpha,\beta \in \mathbb{C}\right\}.$$

Finally

$$\det(Id + L(Id+K)^{-1}\mid_{L^2([0,t))}) = \prod_{n=1}^{\infty}\left(1 - k^2\left(\frac{t}{(n-\frac{1}{2})\pi}\right)^2\right)\left(1 - k^2\left(\frac{t}{(n-\frac{1}{2})\pi}\right)^2\right).$$

□

Thus

$$\det(Id + L(I+K)^{-1}) = det(Id - k^2A)^2 = \left(\prod_{n=1}^{\infty} 1 - k^2(\frac{t^2}{(n-\frac{1}{2})^2\pi^2}\right)^2 = \cos(kt)^2,$$

as in Theorem 7.3.1.

Altogether we have with N^{-1} as in (7.16):

$$T(I_{cp})(\boldsymbol{\xi}) = \left(\frac{kt}{2\pi i \sin(kt)}\right) \exp\left(-\frac{1}{2}\langle \boldsymbol{\xi}, N^{-1}\boldsymbol{\xi}\rangle\right) \exp\left(\frac{1}{2i}(u^T \begin{pmatrix} \frac{kt}{\tan(kt)} & 0 \\ 0 & \frac{kt}{\tan(kt)} \end{pmatrix} u)\right),$$

$$\text{with } u = \begin{pmatrix} iy_1 + \frac{1}{2}\langle \boldsymbol{\eta}_1, N^{-1}\boldsymbol{\xi}\rangle) + \frac{1}{2}\langle N^{-1}\boldsymbol{\eta}_1, \boldsymbol{\xi}\rangle \\ \dots \\ iy_2 + \frac{1}{2}\langle \boldsymbol{\eta}_3, N^{-1}\boldsymbol{\xi}\rangle + \frac{1}{2}\langle N^{-1}\boldsymbol{\eta}_3, \boldsymbol{\xi}\rangle \end{pmatrix}. \quad (7.25)$$

Thus we can state the following theorem.

Theorem 7.4.4. *Let $y_1, y_2 \in \mathbb{R}$, $0 < t < \frac{\pi}{k}$, then the Feynman integrand for the charged particle in a constant magnetic field in phase space I_{cp} exists as a Hida distribution and its generating functional is given by (7.25). Moreover its generalized expectation*

$$\mathbb{E}(I_{cp}) = T(I_{cp})(0) = \left(\frac{k}{2\pi i \sin(kt)}\right) \exp\left(i\frac{k}{2\tan(kt)}(y_1^2 + y_2^2)\right)$$

is the Greens function to the Schrödinger equation for the charged particle in a constant magnetic field, compare e.g. with [44].

Chapter 8

Canonical commutation relations

In this section we give a functional form of the quantum mechanical commutator relations. The definition can be found in [25], for their realization in the white noise framework, we refer to [87, Chap. 9]. With the help of these relations we can confirm that the choice of the phase space variables gives the right physics. I.e. the variables fulfill the non-commutativity of momentum and position variables at equal times. This seemed to have no direct translation in a path integral formulation of quantum mechanics. But on a heuristic level Feynman and Hibbs [25] found an argument to show that $\mathbb{E}(p(t+\varepsilon)x(t)I_V) \neq \mathbb{E}(p(t-\varepsilon)x(t)I_V)$ for infinitesimal small ε and that the difference is given by the commutator. First we collect some helpful formulas.

Lemma 8.0.5. *Let* $\Phi \in (\mathcal{N})'$, $\mathbf{k} \in S_d(\mathbb{R})$ *and* $n \in \mathbb{N}$, *then*

$$(-i)^n \frac{d^n}{d\lambda^n} T\Phi(\lambda \mathbf{k} + \mathbf{f})_{|\lambda=0} = T(\langle \mathbf{k}, \cdot \rangle^n \cdot \Phi)(\mathbf{f}), \quad \mathbf{f} \in S_d(\mathbb{R}).$$

Proof. By [40, Thm.5.36.(iii),p.178] we have

$$T(\langle \mathbf{k}, \cdot \rangle \Phi)(\mathbf{f}) = (-i) \frac{d}{d\lambda} T\Phi(\lambda \mathbf{k} + \mathbf{f})_{|\lambda=0}. \tag{8.1}$$

The claim then follows by iterated application of (8.1). In the following for $\boldsymbol{\eta}_i, \mathbf{k} \in L^2_d(\mathbb{R})$ and $y_i \in \mathbb{R}$, $i \in (1, \dots, J)$, we use the abbreviations:

$$\langle \boldsymbol{\eta}, N^{-1}\mathbf{k} \rangle = \big((\boldsymbol{\eta}_1, N^{-1}\mathbf{k}), \dots, (\boldsymbol{\eta}_J, N^{-1}\mathbf{k}) \big) \in \mathbb{R}^J$$

and

$$y = (y_1, \dots, y_J) \in \mathbb{R}^J$$

. $\square$

In the following we use the notation $(v, Aw) := v^T \cdot A \cdot w$ for vector matrix vector multiplication, where $v, w \in \mathbb{C}^n$ and $A \in \mathrm{Mat}(\mathbb{C}^n, \mathbb{C}^n)$. Note that the above pairing is bilinear.

8.1 Canonical Commutation Relations for the free integrand in Phase Space

Proposition 8.1.1. *Let* $\Phi_{K,L} := \mathrm{Nexp}(-\frac{1}{2}\langle\cdot, K\rangle) \cdot \exp(-\frac{1}{2}\langle\cdot, L\cdot\rangle \cdot \prod_{k=1}^{J} \delta(\langle \boldsymbol{\eta}_k, \cdot\rangle - y_k)$ *be as in Lemma 3.2.6. Then for* $\mathbf{k}, \mathbf{h} \in L^2_2(\mathbb{R})$ $\langle \mathbf{k}, \cdot\rangle \cdot \Phi_{K,L}$ *and* $\langle \mathbf{h}, \cdot\rangle \cdot \langle \mathbf{k}, \cdot\rangle \cdot \Phi_{K,L}$ *exist as Hida distributions. Furthermore for* $\mathbf{f} \in S_d(\mathbb{R})$ *and with* $y = (y_1, \dots y_J) \in \mathbb{R}^k$*, we have*

$$T(\langle \mathbf{k}, \cdot\rangle \cdot \Phi_{K,L})(\mathbf{f}) = iT\Phi_{K,L}(\mathbf{f})\Bigg(\langle \mathbf{f}, N^{-1}\mathbf{k}\rangle - \Big(\langle \boldsymbol{\eta}, N^{-1}\mathbf{k}\rangle, M^{-1}_{N^{-1}}\Big(iy + \langle \boldsymbol{\eta}, N^{-1}\mathbf{f}\rangle\Big)\Big)\Bigg)$$

and

$$\begin{aligned} T(\langle \mathbf{k}, \cdot\rangle \cdot \langle \mathbf{h}, \cdot\rangle \cdot \Phi_{K,L})(\mathbf{f}) = T(\Phi_{K,L})(\mathbf{f})\Bigg(\Bigg(\langle \mathbf{k}, N^{-1}\mathbf{h}\rangle - \Big(\langle \boldsymbol{\eta}, N^{-1}\mathbf{h}\rangle, M^{-1}_{N^{-1}}\langle \boldsymbol{\eta}, N^{-1}\mathbf{k}\rangle\Big)\Bigg) \\ - \Bigg(\langle \mathbf{f}, N^{-1}\mathbf{h}\rangle - \Big(\Big(iy + \langle \boldsymbol{\eta}, N^{-1}\mathbf{f}\rangle\Big), M^{-1}_{N^{-1}}\langle \boldsymbol{\eta}, N^{-1}\mathbf{h}\rangle\Big)\Bigg) \\ \times \Bigg(\langle \mathbf{f}, N^{-1}\mathbf{k}\rangle - \Big(\Big(iy + \langle \boldsymbol{\eta}, N^{-1}\mathbf{f}\rangle\Big), M^{-1}_{N^{-1}}\langle \boldsymbol{\eta}, N^{-1}\mathbf{k}\rangle\Big)\Bigg)\Bigg). \end{aligned}$$

Proof. We have from Lemma 8.0.5 that $T(\langle \mathbf{k}, \cdot\rangle \cdot \Phi_{K,L})(\mathbf{f}) = \frac{1}{i}\frac{d}{d\lambda} T(\Phi_{K,L})(\mathbf{f} + \lambda\mathbf{k})_{|\lambda=0}$, $\mathbf{k} \in S_d(\mathbb{R})$. Then by Lemma 3.2.6,

$$\begin{aligned} T(\Phi_{K,L})(\mathbf{f} + \lambda\mathbf{k}) = T(\Phi_{K,L})(\mathbf{f}) \exp\Big(-\frac{1}{2}\lambda^2\langle \mathbf{k}, N^{-1}\mathbf{k}\rangle - \lambda\big(\mathbf{f}, N^{-1}\mathbf{k}\big)\Big) \\ \exp\Bigg(\frac{1}{2}\lambda^2\Big(\langle \boldsymbol{\eta}, N^{-1}\mathbf{k}\rangle, M^{-1}_{N^{-1}}\langle \boldsymbol{\eta}, N^{-1}\mathbf{k}\rangle\Big) \\ + \lambda\Big(\langle \boldsymbol{\eta}, N^{-1}\mathbf{k}\rangle, M^{-1}_{N^{-1}}\Big(iy + \langle \boldsymbol{\eta}, N^{-1}(\mathbf{f} + \mathbf{g})\rangle\Big)\Big)\Bigg). \end{aligned}$$

Thus, by the above formula we get

$$\frac{1}{i}\frac{d}{d\lambda}T(\Phi_{K,L})(\mathbf{f}+\lambda\mathbf{k}) = -iT\Phi_{K,L}(\mathbf{f}) \times \Bigg(-\Big(\langle \mathbf{f}, N^{-1}\mathbf{k}\rangle - \Big(\langle \boldsymbol{\eta}, N^{-1}\mathbf{k}\rangle, M_{N^{-1}}^{-1}\Big(iy + \langle \boldsymbol{\eta}, N^{-1}(\mathbf{f})\rangle\Big)\Big)\Big) - \lambda\Big(\langle \mathbf{k}, N^{-1}\mathbf{k}\rangle - \big(\langle \boldsymbol{\eta}, N^{-1}\mathbf{k}\rangle, M_{N^{-1}}^{-1}\langle \boldsymbol{\eta}, N^{-1}\mathbf{k}\rangle\big)\Big)\Bigg).$$

Then by an approximation in the sense of Corollary 2.2.18 we get $\langle \mathbf{k}, \cdot\rangle \cdot \Phi_{K,L} \in (\mathcal{N})'$ for $\mathbf{k} \in L^2_d(\mathbb{R})$. Setting $\lambda = 0$ we obtain the desired expression. In an analogue way one can show the second formula by using the second derivative, see Lemma 8.0.5 and polarization identity. □

Next we extend this to the case, where just one of the functions is in $L^2_2(\mathbb{R})$, but the other one is a tempered distribution.

Remark 8.1.2. *Since the dual pairing between a Dirac delta function and an indicator function is defined as a limit object, we can not expect that the approximation is independent of the choice of the approximating sequence. We choose the following approximation.*

Definition 8.1.3. *Let $\mathbf{h} \in L^2_d(\mathbb{R})$ and $\mathbf{k} \in S'_d(\mathbb{R})$ with compact support and let $(\psi_n)_{n\in\mathbb{N}}$ be a standard approximate identity. Since the convolution of a compactly supported smooth function with a compactly supported tempered distribution gives a Schwartz test function, i.e. $\psi_n * \mathbf{k} \in S_d(\mathbb{R})$, $n \in \mathbb{N}$, see e.g. [71, Chap.9] we may define*

$$\langle \mathbf{k}, \cdot\rangle \cdot \langle \mathbf{h}, \cdot\rangle \cdot \Phi_{K,L} := \lim_{n\to\infty} \langle \psi_n * \mathbf{k}, \cdot\rangle \cdot \langle \mathbf{h}, \cdot\rangle \cdot \Phi_{K,L},$$

in the case the limit exists in the sense of Corollary 2.2.18, i.e. the limit on the right-hand side is a U-functional.

In the following for convenience we restrict ourselves to the case $d = 2$.

For the free Feynman Integrand we have then as an analogue to [87]:

Theorem 8.1.4. *Let $0 < s - \varepsilon < s < s + \varepsilon < t < \infty$, then*

$$\langle \big(0, \delta_{s\pm\varepsilon}\big), \cdot\rangle \cdot \langle \big(\mathbb{1}_{0,s)}, 0\big), \cdot\rangle \cdot I_0 \in (\mathcal{N})'$$

and

$$\lim_{\varepsilon\to 0}\Big(T(\langle\delta_{s+\varepsilon},\cdot\rangle\cdot\langle\mathbb{1}_{[0,s)},\cdot\rangle\cdot I_0))(\mathbf{0})-T(\langle\delta_{s-\varepsilon},\cdot\rangle\cdot\langle\mathbb{1}_{[0,s)},\cdot\rangle\cdot I_0))(\mathbf{0})\Big)=-iT(I_0)(\mathbf{0}).$$

Proof. Set $\boldsymbol{\psi}_n^{\pm}:=\Big(0,\psi_n*\delta_{s\pm\varepsilon}\Big)$, $n\in\mathbb{N}$, where $(\psi_n)_{n\in\mathbb{N}}$ is a standard approximate identity. Note that $\lim\limits_{n\to\infty}\langle\boldsymbol{\psi}_n^{\pm},(0,\mathbb{1}_{[0,s)})\rangle=\frac{1}{2}\pm\frac{1}{2}$. Using (7.9) in the case $\boldsymbol{\eta}=\big(\mathbb{1}_{[0,t)},0\big)$ with $N^{-1}=\begin{pmatrix}\mathbb{1}_{[0,t)^c}+i\mathbb{1}_{[0,t)} & i\mathbb{1}_{[0,t)}\\ i\mathbb{1}_{[0,t)} & \mathbb{1}_{[0,t)^c}\end{pmatrix}$ as in (7.8), we have $(M_{N^{-1}})^{-1}=\frac{1}{it}$. Thus, together with Proposition 8.1.1 we obtain

$$\begin{aligned}
&T(\langle\boldsymbol{\psi}_n^{\pm},\cdot\rangle\cdot\langle(\mathbb{1}_{[0,s)},0),\cdot\rangle\cdot I_0)(\mathbf{f})=T(I_0)(\mathbf{f})\\
&\quad\times\Bigg(\bigg(\langle\Psi_n^{\pm},(i\mathbb{1}_{[0,s)},i\mathbb{1}_{[0,s)})\rangle-\Big(\langle(i\mathbb{1}_{[0,t)},i\mathbb{1}_{[0,t)}),(\mathbb{1}_{[0,s)},0)\rangle,\frac{1}{it}\langle(i\mathbb{1}_{[0,t)},i\mathbb{1}_{[0,t)}),\Psi_n^{\pm})\rangle\Big)\bigg)\\
&\quad-\Bigg(\bigg(\langle\mathbf{f},(i\mathbb{1}_{[0,s)},i\mathbb{1}_{[0,s)})\rangle\langle-\Big(iy+\langle(i\mathbb{1}_{[0,t)},i\mathbb{1}_{[0,t)}),\mathbf{f}\rangle,\frac{1}{it}\langle(i\mathbb{1}_{[0,t)},i\mathbb{1}_{[0,t)}),(\mathbb{1}_{[0,s)},0)\rangle\Big)\bigg)\\
&\qquad\times\bigg(\Big(N^{-1}\mathbf{f},\Psi_n^{\pm}\Big)-\Big(iy+\langle((i\mathbb{1}_{[0,t)},i\mathbb{1}_{[0,t)}),\mathbf{f})\rangle,\frac{1}{it}\langle(i\mathbb{1}_{[0,t)},i\mathbb{1}_{[0,t)}),\Psi_n^{\pm}\rangle\Big)\bigg)\Bigg)
\end{aligned}\tag{8.2}$$

Now let us take a look at the terms which include the sequence $\Psi_n^{\pm}$.
Since N^{-1} consists of projections on $[0,t)$ or $[0,t)^c$ respectively and $\int_{\mathbb{R}}\Psi_n^{\pm}(s)\,ds=1$, we have $|(N^{-1}\mathbf{f},\Psi_n^{\pm})|\leq\|\mathbf{f}\|_{\sup}$. Furthermore $|((i\mathbb{1}_{[0,u)},i\mathbb{1}_{[0,u)}),\Psi_n^{\pm})|\leq 1$, for all $n\in\mathbb{N}$ and $0<u\leq t$. Therefore the expression can be bounded uniformly in $n\in\mathbb{N}$ in the sense of Corollary 2.2.18 (note that $\|\cdot\|_{\sup}\leq\|\cdot\|_p$ for some $p\in\mathbb{N}$). Obviously the T-transform in (8.2) is convergent as $n\to\infty$, thus the limit exists as a Hida distribution by Corollary 2.2.18. Taking the limit leads us to

$$\begin{aligned}
&T(I_0)(\mathbf{f})\bigg(i\mathbb{1}_{[0,s)}(s\pm\varepsilon)-\frac{is}{t}-\Big(i\int_0^s f_x(r)+f_p(r)\;dr-\frac{s}{t}\Big(iy+i\int_0^t f_x(r)+f_p(r)\;dr\Big)\Big)\times\\
&\qquad\Big(if_x(s\pm\varepsilon)-\frac{1}{t}\Big(iy+i\int_0^t f_x(r)+f_p(r)\;dr\Big)\Big)\bigg),\quad \mathbf{f}=(f_x,f_p)\in S_2(\mathbb{R}).
\end{aligned}$$

For the difference $\mathbb{E}(\langle\delta_{s+\varepsilon},\cdot\rangle\langle\mathbb{1}_{[0,s)},\cdot\rangle I_0)-\mathbb{E}(\langle\delta_{s-\varepsilon},\cdot\rangle\langle\mathbb{1}_{[0,s)},\cdot\rangle I_0)$ we have

$$\begin{aligned}
&\lim_{\varepsilon\to 0}T(\langle\delta_{s+\varepsilon},\cdot\rangle\langle\mathbb{1}_{[0,s)},\cdot\rangle I_0))(\mathbf{0})-T(\langle\delta_{s-\varepsilon},\cdot\rangle\langle\mathbb{1}_{[0,s)},\cdot\rangle I_0))(\mathbf{0})\\
&\qquad=\lim_{\varepsilon\to 0}T(I_0)(\mathbf{0})\big(i\mathbb{1}_{[0,s)}(s+\varepsilon)-i\mathbb{1}_{[0,s)}(s-\varepsilon)\big)=T(I_0)(\mathbf{0})\cdot(0-i)=-iT(I_0)(\mathbf{0}),
\end{aligned}$$

which completes the proof. ∎

Thus, the commutation law for the free Feynman integrand in phase space is fulfilled in the sense of Feynman and Hibbs [25].

8.2 Canonical Commutation Relations for the Harmonic Oscillator in Phase Space

Commutation relations for the harmonic oscillator in coordinate space

For the harmonic oscillator we have

Theorem 8.2.1. *Let* $0 < s - \varepsilon < s < s + \varepsilon < t < \infty$, *then*

$$\langle \big(0, \delta_{s\pm\varepsilon}\big), \cdot \rangle \cdot \langle \big(\mathbb{1}_{0,s)}, 0\big), \cdot \rangle \cdot I_{HO} \in (\mathcal{N})'$$

and

$$\lim_{\varepsilon \to 0} \Big(T(\langle \delta_{s+\varepsilon}, \cdot \rangle \cdot \langle \mathbb{1}_{[0,s)}, \cdot \rangle \cdot I_{HO}))(\mathbf{0}) - T(\langle \delta_{s-\varepsilon}, \cdot \rangle \cdot \langle \mathbb{1}_{[0,s)}, \cdot \rangle \cdot I_{HO}))(\mathbf{0}) \Big) \\ = -iT(I_{HO})(\mathbf{0}).$$

Proof. Set $\boldsymbol{\psi}_n^{\pm} := \Big(0, \psi_n * \delta_{s\pm\varepsilon}\Big)$, $n \in \mathbb{N}$, where $(\psi_n)_{n\in\mathbb{N}}$ is a standard approximate identity. Note that $\lim_{n\to\infty} \langle \boldsymbol{\psi}_n^{\pm}, (0, \mathbb{1}_{[0,s)}) \rangle = \frac{1}{2} \pm \frac{1}{2}$. Using (7.9) in the case $\boldsymbol{\eta} = \big(\mathbb{1}_{[0,t)}, 0\big)$ with

$$N^{-1} = \begin{pmatrix} \mathbb{1}_{[0,t)^c} & 0 \\ 0 & \mathbb{1}_{[0,t)^c} \end{pmatrix} + \frac{1}{i} \mathbb{1}_{[0,t)} \begin{pmatrix} (kA - \mathbb{1}_{[0,t)})^{-1} & (kA - \mathbb{1}_{[0,t)})^{-1} \\ (kA - \mathbb{1}_{[0,t)})^{-1} & kA(kA - \mathbb{1}_{[0,t)})^{-1} \end{pmatrix}$$

as in (7.12), we have $(M_{N^{-1}})^{-1} = \frac{1}{\frac{\tan(\sqrt{k}t)}{\sqrt{k}}}$. Thus, together with Proposition 8.1.1 we obtain

$$T(\langle \boldsymbol{\psi}_n^{\pm}, \cdot \rangle \cdot \langle (\mathbb{1}_{[0,s)}, 0), \cdot \rangle \cdot I_{HO})(\mathbf{f}) = T(I_{HO})(\mathbf{f}) \\ \times \Bigg(\Bigg(\langle \Psi_n^{\pm}, N^{-1}(\mathbb{1}_{[0,s)}, 0) \rangle - \Bigg(\langle (N^{-1}(\mathbb{1}_{[0,t)}, 0), (\mathbb{1}_{[0,s)}, 0) \rangle, \frac{1}{i\frac{\tan(\sqrt{k}t)}{\sqrt{k}}} \langle N^{-1}(\mathbb{1}_{[0,t)}, 0), \Psi_n^{\pm} \rangle \Bigg) \Bigg) \\ - \Bigg(\Bigg(\langle \mathbf{f}, N^{-1}(\mathbb{1}_{[0,s)}, 0) \rangle - \Bigg(\big(iy + \langle N^{-1}(\mathbb{1}_{[0,t)}, 0), \mathbf{f} \rangle\big), \frac{1}{i\frac{\tan(\sqrt{k}t)}{\sqrt{k}}} \Big(\langle N^{-1}(\mathbb{1}_{[0,t)}, 0), (\mathbb{1}_{[0,s)}, 0) \rangle \Big) \Bigg) \Bigg)$$

$$\times\left(\left\langle (N^{-1}\mathbf{f},\Psi_n^\pm\right\rangle - \left(\left(iy+\langle N^{-1}(\mathbb{1}_{[0,t)},0),\mathbf{f}\rangle\right),\frac{1}{i\frac{\tan(\sqrt{k}t)}{\sqrt{k}}}\langle N^{-1}(\mathbb{1}_{[0,t)},0),\Psi_n^\pm\rangle\right)\right)\right) \quad (8.3)$$

Now let us take a look at the terms which include the sequence $\Psi_n^\pm$.
Since N^{-1} consists of operators on $[0,t)$ or $[0,t)^c$ respectively and $\int_{\mathbb{R}}\Psi_n^\pm(s)\,ds = 1$, we have $|(N^{-1}\mathbf{f},\Psi_n^\pm)| \leq \|N^{-1}\mathbf{f}\|_{\sup}$. Note that for $\mathbf{f}\in S_2(\mathbb{R})$ we have $N^{-1}\mathbf{f}_{|_{[0,t)}} \in C([0,t))$. Furthermore by taking n so large that $\text{supp}(\Psi_n^\pm)\subset[0,t)$ we have

$$\begin{aligned}
&\langle N^{-1}(\mathbb{1}_{[0,u)},\Psi_n^\pm)\rangle \\
&\qquad = 2\sum_{l=0}^{\infty}\frac{1}{1-\frac{k^2t^2}{(l-\frac{1}{2})^2\pi^2}}\frac{1}{(l-\frac{1}{2})\pi}\sin((l-\frac{1}{2})\pi\frac{u}{t})\langle\cos((l-\frac{1}{2})\pi\frac{\cdot}{t}),\psi_n^\pm\rangle| \\
&\qquad\qquad \leq 2\sum_{l=0}^{\infty}2|\frac{1}{1-\frac{k^2t^2}{(l-\frac{1}{2})^2\pi^2}}\frac{1}{(l-\frac{1}{2})\pi}\sin((l-\frac{1}{2})\pi\frac{u}{t})|,
\end{aligned}$$

for n large enough, since then $|\langle\cos((l-\frac{1}{2})\pi\frac{\cdot}{t}),\psi_n^\pm\rangle| < 2$.
Then we have

$$\begin{aligned}
&\frac{2}{\sum_{l=0}}^{\infty}2|\frac{1}{1-\frac{k^2t^2}{(l-\frac{1}{2})^2\pi^2}}\frac{1}{(l-\frac{1}{2})\pi}\sin((l-\frac{1}{2})\pi\frac{u}{t})| \\
&\quad = 4\sum_{l=0}^{\infty}\frac{(l-\frac{1}{2})^2\pi^2}{(l-\frac{1}{2})^2\pi^2-k^2t^2}\frac{1}{(l-\frac{1}{2})\pi}\sin((l-\frac{1}{2})\pi\frac{u}{t})| \leq 4\sum_{l=0}^{\infty}|\frac{1}{(l-\frac{1}{2})\pi}\sin((l-\frac{1}{2})\pi\frac{u}{t})|,
\end{aligned}$$

which is convergent for all $0<u\leq t$ by the Dirichlet test.
Therefore the expression can be bounded uniformly in $n\in\mathbb{N}$ in the sense of Corollary 2.2.18 (note that $\|\cdot\|_{\sup}\leq\|\cdot\|_p$ for some $p\in\mathbb{N}$). Obviously the T-transform in (8.3) is convergent as $n\to\infty$, thus the limit exists as a Hida distribution by Corollary 2.2.18. Taking the limit leads us to

$$\begin{aligned}
&T(I_{HO})(\mathbf{f})\times \\
&\left(\left(\langle(0,\delta_{s\pm\varepsilon}),N^{-1}(\mathbb{1}_{[0,s)},0)\rangle - \left(\langle N^{-1}(\mathbb{1}_{[0,t)},0),(\mathbb{1}_{[0,s)},0)\rangle\frac{1}{i\frac{\tan(\sqrt{k}t)}{\sqrt{k}}}\langle N^{-1}(\mathbb{1}_{[0,t)},0),(0,\delta_{s\pm\varepsilon})\rangle\right)\right)\right. \\
&-\left(\left(\langle\mathbf{f},N^{-1}(\mathbb{1}_{[0,s)},0)\rangle - \left(\left(iy+\langle N^{-1}(\mathbb{1}_{[0,t)},0),\mathbf{f}\rangle\right)\frac{1}{i\frac{\tan(\sqrt{k}t)}{\sqrt{k}}}\left(\langle N^{-1}(\mathbb{1}_{[0,t)},0),(\mathbb{1}_{[0,s)},0)\rangle\right)\right)\right)
\end{aligned}$$

$$\times\Bigg(\langle N^{-1}\mathbf{f},(0,\delta_{s\pm\varepsilon})\rangle-\Bigg(\Big(iy+\langle N^{-1}(\mathbb{1}_{[0,t)},0),\mathbf{f}\rangle\Big)\frac{1}{i\frac{\tan(\sqrt{k}t)}{\sqrt{k}}}\langle N^{-1}(\mathbb{1}_{[0,t)},0),(0,\delta_{s\pm\varepsilon})\rangle\Bigg)\Bigg)\Bigg) \tag{8.4}$$

For the difference $\mathbb{E}(\langle\delta_{s+\varepsilon},\cdot\rangle\langle\mathbb{1}_{[0,s)},\cdot\rangle I_0)-\mathbb{E}(\langle\delta_{s-\varepsilon},\cdot\rangle\langle\mathbb{1}_{[0,s)},\cdot\rangle I_0)$ we have

$$\begin{aligned}
&\lim_{\varepsilon\to 0}T(I_{HO})(0)\Bigg(\Bigg(\langle(0,\delta_{s+\varepsilon}),N^{-1}(\mathbb{1}_{[0,s)},0)\rangle-\langle(0,\delta_{s-\varepsilon}),N^{-1}(\mathbb{1}_{[0,s)},0)\rangle\\
&\qquad-\Bigg(\langle N^{-1}(\mathbb{1}_{[0,t)},0),(\mathbb{1}_{[0,s)},0)\rangle,\frac{1}{i\frac{\tan(\sqrt{k}t)}{\sqrt{k}}}\langle N^{-1}(\mathbb{1}_{[0,t)},0),(0,\delta_{s+\varepsilon})\rangle\Bigg)\\
&\qquad\qquad-\langle N^{-1}(\mathbb{1}_{[0,t)},0),(0,\delta_{s-\varepsilon})\rangle\Bigg)\Bigg)\Bigg)\\
&\qquad-\Bigg(\Bigg(\Big(iy\frac{1}{i\frac{\tan(\sqrt{k}t)}{\sqrt{k}}}\Big(\langle N^{-1}(\mathbb{1}_{[0,t)},0),(\mathbb{1}_{[0,s)},0)\rangle\Big)\Bigg)\\
&\quad\times\Bigg(\Big(iy\frac{1}{i\frac{\tan(\sqrt{k}t)}{\sqrt{k}}}(\langle N^{-1}(\mathbb{1}_{[0,t)},0),(0,\delta_{s+\varepsilon})\rangle-\langle N^{-1}(\mathbb{1}_{[0,t)},0),(0,\delta_{s-\varepsilon})\rangle)\Big)\Bigg)\\
&\qquad=\lim_{\varepsilon\to 0}T(I_{HO})(\mathbf{f})\Bigg(\Big((0,\delta_{s+\varepsilon}),N^{-1}(\mathbb{1}_{[0,s)},0)\Big)-\Big((0,\delta_{s-\varepsilon}),N^{-1}(\mathbb{1}_{[0,s)},0)\Big),
\end{aligned}$$

because

$$\lim_{\varepsilon\to 0}(\langle N^{-1}(\mathbb{1}_{[0,t)},0),(0,\delta_{s+\varepsilon})\rangle-\langle N^{-1}(\mathbb{1}_{[0,t)},0),(0,\delta_{s-\varepsilon})\rangle)=0,$$

since $0<s<t$. Hence we take a look at

$$\Bigg(\Big((0,\delta_{s+\varepsilon}),N^{-1}(\mathbb{1}_{[0,s)},0)\Big)-\Big((0,\delta_{s-\varepsilon}),N^{-1}(\mathbb{1}_{[0,s)},0)\Big).$$

We have

$$\begin{aligned}
&\Big((0,\delta_{s\pm\varepsilon}),N^{-1}(\mathbb{1}_{[0,s)},0)\Big)\\
&\qquad=i\sum_{n=1}^{\infty}\frac{1}{1-\frac{(kt)^2}{(n-\frac{1}{2})^2\pi^2}}\frac{2}{(n-\frac{1}{2})\pi}\sin(\frac{s}{t}(n-\frac{1}{2})\pi)\cos(\frac{s\pm\varepsilon}{t}(n-\frac{1}{2})\pi)\\
&\quad=i\sum_{n=1}^{\infty}\frac{1}{1-\frac{(kt)^2}{(n-\frac{1}{2})^2\pi^2}}\frac{2}{(n-\frac{1}{2})\pi}\big(\sin(\frac{s}{t}(n-\frac{1}{2})\pi)\cos(\frac{s}{t}(n-\frac{1}{2})\pi)\cos(\frac{\varepsilon}{t}(n-\frac{1}{2})\pi)\\
&\qquad\qquad\qquad\qquad-\sin^2(\frac{s}{t}(n-\frac{1}{2})\pi)\sin(\frac{\varepsilon}{t}(n-\frac{1}{2})\pi)\big),
\end{aligned}$$

by the addition theorems of sine and cosine. Now since $\sin(x)\cos(x) = \frac{1}{2}\sin(2x)$ and with $cos^2(x) - sin^2(x) = cos(2x)$ we obtain

$$\sum_{n=1}^{\infty} \frac{1}{1 - \frac{(kt)^2}{(n-\frac{1}{2})^2\pi^2}} \frac{2}{(n-\frac{1}{2})\pi} \big(\sin(\frac{s}{t}(n-\frac{1}{2})\pi)\cos(\frac{s}{t}(n-\frac{1}{2})\pi)\cos(\frac{\varepsilon}{t}(n-\frac{1}{2})\pi)$$

$$- \sin^2(\frac{s}{t}(n-\frac{1}{2})\pi)\sin(\frac{\varepsilon}{t}(n-\frac{1}{2})\pi)\big)$$

$$= \sum_{n=1}^{\infty} \frac{1}{1 - \frac{(kt)^2}{(n-\frac{1}{2})^2\pi^2}} \frac{2}{(n-\frac{1}{2})\pi} \frac{1}{2} \sin(\frac{s}{t}(2n-1)\pi)\cos(\frac{\varepsilon}{t}(n-\frac{1}{2})\pi)$$

$$+ \sum_{n=1}^{\infty} \frac{1}{1 - \frac{(kt)^2}{(n-\frac{1}{2})^2\pi^2}} \frac{2}{(n-\frac{1}{2})\pi t} \frac{1}{2} \sin(\frac{\varepsilon}{t}(n-\frac{1}{2})\pi)\cos(\frac{s}{t}(2n-1)\pi)$$

$$- \sum_{n=1}^{\infty} \frac{1}{1 - \frac{(kt)^2}{(n-\frac{1}{2})^2\pi^2}} \frac{2}{(n-\frac{1}{2})\pi} \frac{1}{2} \sin(\frac{\varepsilon}{t}(n-\frac{1}{2})\pi).$$

Note that the first two series exist as Fourier series with square summable coefficients. Moreover both series are absolutely convergent. Hence we have continuity in ε. Thus for the difference we consider we just have to focus on the last series. We have

$$\sum_{n=1}^{\infty} \frac{1}{1 - \frac{(kt)^2}{(n-\frac{1}{2})^2\pi^2}} \frac{2}{(n-\frac{1}{2})\pi t} \frac{1}{2} \sin(\frac{\varepsilon}{t}(n-\frac{1}{2})\pi)$$

$$= -\sum_{n=1}^{\infty} 1 - \frac{1}{1 - \frac{(kt)^2}{(n-\frac{1}{2})^2\pi^2}} \frac{2}{(n-\frac{1}{2})\pi t} \frac{1}{2} \sin(\frac{\varepsilon}{t}(n-\frac{1}{2})\pi) + \sum_{n=1}^{\infty} \frac{2}{(n-\frac{1}{2})\pi t} \frac{1}{2} \sin(\frac{\varepsilon}{t}(n-\frac{1}{2})\pi).$$

Since

$$1 - \frac{1}{1 - \frac{(kt)^2}{(n-\frac{1}{2})^2\pi^2}} = \frac{-\frac{(kt)^2}{(n-\frac{1}{2})^2\pi^2}}{1 - \frac{(kt)^2}{(n-\frac{1}{2})^2\pi^2}},$$

the first series is uniformly convergent thus continuous in ε. For the last series we have

$$\sum_{n=1}^{\infty} \frac{2}{(n-\frac{1}{2})\pi t} \frac{1}{2} \sin(\frac{\varepsilon}{(n-\frac{1}{2})\pi}) = \frac{2}{t} \sum_{n=1}^{\infty} \frac{1}{(n-\frac{1}{2})\pi t} \frac{1}{2} \sin(\frac{\varepsilon}{(n-\frac{1}{2})\pi}),$$

which is the Fourier series of a rectangular pulse between 0 and ε. Altogether we have

$$\mathbb{E}(\langle \delta_{s+\varepsilon}, \cdot \rangle \langle \mathbb{1}_{[0,s)}, \cdot \rangle I_{HO}) - \mathbb{E}(\langle \delta_{s-\varepsilon}, \cdot \rangle \langle \mathbb{1}_{[0,s)}, \cdot \rangle I_{HO}) = iT(I_{HO})(0)$$

which completes the proof. □

Chapter 9

A scaling approach to the Hamiltonian Path Integral in White Noise Analysis

In this chapter we want to give an outlook, how one can realize the ideas of complex scaling from [36], [35], [37] to phase space path integrals. The idea of this scaling method goes back to [20]. We use again the Wick representation of a product with Donsker's Delta function. Note that, since the projection operator is not not closeable we can not expect, that a limit we obtain by approximating the potential is unique. At least we can say that if the sequence converges, it will be a Hida distribution. The basis of this chapter is a representation of the Feynman-Kac formula with Brownian bridges, see e.g. [38].

9.1 Complex-scaled heat kernel

From standard results of Stochastic Analysis e.g. [38] it is known that a solution of the heat equation

$$\begin{cases} \frac{\partial}{\partial t}\psi(t,x) = \frac{1}{2}\Delta\psi(t,x) + V(x)\psi(t,x) \\ \psi(0,x) = f(x), \quad 0 \leq t \leq T < \infty,\, x \in \mathbb{R}^d, \end{cases} \tag{9.1}$$

is given by the Feynman-Kac formula. In (9.1) Δ denotes the Laplace operator on $\mathbb{R}^d$. For a suitably nice potential $V : \mathcal{O} \to \mathbb{C}$, $\mathcal{O} \subset \mathbb{R}^d$ open, $d \geq 1$ and a source function

$f : \mathbb{R}^d \to \mathbb{C}$ the unique solution of (9.1) is given by

$$\psi(t,x) = \mathbb{E}(\exp(\int_0^t V(x+B_r)\, dr) f(x+B_t)), \quad t \in [0,T], x \in \mathcal{O}, \tag{9.2}$$

where $\mathbb{E}$ denotes the expectation w.r.t. a Brownian motion B starting at 0. For suitable potentials the heat kernel can be written as follows, see e.g. [38]

$$\begin{aligned} K_V(x,t;x_0,t_0) = & \frac{1}{\sqrt{2\pi(t-t_0)}} \exp(-\frac{1}{2(t-t_0)}(x_0-x)^2) \\ & \times \mathbb{E}(\exp(\int_{t_0}^t V(x_0 - \frac{s-t_0}{t-t_0}(x_0-x) + B_s - \frac{s-t_0}{t-t_0}B_t)\, ds)). \end{aligned} \tag{9.3}$$

In the following, since we are interested in solutions to the Schrödinger equation we focus on complex scaled heat equations, i.e. for $z \in \mathbb{C}$, we consider

$$\begin{cases} \frac{\partial}{\partial t}\psi(t,x) = -z^2 \frac{1}{2}\Delta\psi(t,x) + \frac{1}{z^2}V(x)\psi(t,x) \\ \psi(0,x) = f(x), \quad 0 \le t \le T < \infty,\, x \in \mathcal{O} \subset \mathbb{R}^d, \end{cases} \tag{9.4}$$

for suitable functions f and suitable time-independent potentials V.
In the configuration space, this has been done in [87] and [86, 36]. This scaling approach has several advantages:

- Treatable potentials are beyond perturbation theory such as

$$V(x) = (-1)^{n+1} a_{4n+2} x^{4n+2} + \sum_{j=1}^{4n+1} a_j x^j, \quad x \in \mathbb{R}, n \in \mathbb{N}, \text{ with } a_{4n+2} > 0, a_j \in \mathbb{C}.$$

- Due to a Wick formula we have a convenient structure (i.e. "Brownian motion is replaced by a Brownian bridge")

- The kinetic energy "'$\sigma_z\delta$"' and the potential can be treated separately.

- The Wick product of two Hida distributions exists always as Hida distribution, thus one does not need to justify the well-definedness of the pointwise product.

We give an idea how to implement this approach to phase space.

9.2 Construction of the generalized scaled heat kernel

Within this section we consider the case of one degree of freedom, i.e. the underlying space is the space $S_2'(\mathbb{R})$. In the euclidean configuration space a solution to the heat equation is given by the Feynman-Kac formula with its corresponding heat kernel. In White Noise Analysis one constructs the integral kernel by inserting Donsker's delta function to pin the final point $x \in \mathbb{R}$ and taking the expectation, i.e.,

$$K_V(x,t,x_0,t_0) = \mathbb{E}\left(\exp(\int_{t_0}^{t} V(x_0 + \langle \mathbb{1}_{[t_0,r)}, \cdot \rangle)\, dr)\delta(x_0 + \langle \mathbb{1}_{[t_0,t)}, \cdot \rangle - x)\right),$$

where the integrand is a suitable distribution in White Noise Analysis (e.g. a Hida distribution).

We will construct in this section by a suitable generalized scaling the Hamiltonian Path Integral as an expectation based on the formula above.

First we construct the scaling operator we need.

Proposition 9.2.1. *Let* $N^{-1} = \begin{pmatrix} \mathbb{1}_{[0,t)^c} & 0 \\ 0 & \mathbb{1}_{[0,t)^c} \end{pmatrix} + i \begin{pmatrix} \mathbb{1}_{[0,t)} & \mathbb{1}_{[0,t)} \\ \mathbb{1}_{[0,t)} & 0 \end{pmatrix}$ *as in the case of the free Hamiltonian integrand. Let* R *be a symmetric operator (w.r.t. the dual pairing) with* $R^2 = N^{-1}$*. Indeed we have:*

$$R = \begin{pmatrix} \mathbb{1}_{[0,t)^c} & 0 \\ 0 & \mathbb{1}_{[0,t)^c} \end{pmatrix} + \frac{\sqrt{i}}{1 + (\frac{\sqrt{5}+1}{2})^2} U^T \begin{pmatrix} \frac{1+\sqrt{5}}{2}\mathbb{1}_{[0,t)} & 0 \\ 0 & \frac{1-\sqrt{5}}{2}\mathbb{1}_{[0,t)} \end{pmatrix} U,$$

with

$$U = \begin{pmatrix} -\frac{\sqrt{5}+1}{2} & 1 \\ -1 & -\frac{\sqrt{5}+1}{2} \end{pmatrix}$$

Then under the assumption that $\sigma_R \delta(\langle (\mathbb{1}_{[0,t)}, 0), \cdot \rangle) = \delta(\langle R(\mathbb{1}_{[0,t)}, 0), \cdot \rangle) \in (S)'$*, we have*

$$I_0 = \sigma_R^{\dagger} \sigma_R \delta(\langle (\mathbb{1}_{[0,t)}, 0), \cdot \rangle).$$

Consequently the Hamiltonian path integrand for an arbitrary space dependent potential V, can be informally written as

$$I_V = \mathrm{Nexp}\left(-\frac{1}{2}\langle \cdot, K \cdot \rangle\right) \exp\left(-i \int_0^t V(x_0 + \langle (\mathbb{1}_{[0,r)}, 0), \cdot \rangle)\, dr\right) \delta(x_0 + \langle (\mathbb{1}_{[t_0,t)}, 0), \cdot \rangle - x)$$

$$= \sigma_R^\dagger \left(\sigma_R \Big(\exp \left(-i \int_0^t V(x_0 + \langle (\mathbb{1}_{[0,r)}, 0), \cdot \rangle) \, dr \right) \Big) \sigma_R \delta(x_0 + \langle (\mathbb{1}_{[t_0,t)}, 0), \cdot \rangle - x) \right), \quad (9.5)$$

for $x, x_0 \in \mathbb{R}$ and $0 < t_0 < t < \infty$.

In the following we give some ideas to give a mathematical meaning to the expression in (9.5). First we consider a quadratic potential, i.e. we consider

$$\exp(-\frac{1}{2}\langle \cdot L \cdot \rangle)\delta(\langle (\mathbb{1}_{[t_0,t)}, 0), \cdot \rangle - x).$$

Definition 9.2.2. *For L fulfilling the assumption of Lemma 3.2.6 and $\delta(\langle (\mathbb{1}_{[t_0,t)}, 0), \cdot \rangle - x)$ we define*

$$\sigma_R \left(\exp(-\frac{1}{2}\langle \cdot L \cdot \rangle)\delta(\langle (\mathbb{1}_{[t_0,t)}, 0), \cdot \rangle - x) \right) := \exp(-\frac{1}{2}\langle \cdot RLR \cdot \rangle)\delta(\langle R(\mathbb{1}_{[t_0,t)}, 0), \cdot \rangle - x).$$

We now take a look at the T-transform of this expression. We have

$$\begin{aligned} &T(\sigma_R^\dagger \sigma_R \left(\exp(-\frac{1}{2}\langle \cdot L \cdot \rangle)\delta(\langle (\mathbb{1}_{[t_0,t)}, 0), \cdot \rangle - y) \right))(\xi) \\ &\qquad = T(\sigma_R \left(\exp(-\frac{1}{2}\langle \cdot L \cdot \rangle)\delta(\langle (\mathbb{1}_{[t_0,t)}, 0), \cdot \rangle - y) \right))(R\xi) \\ &\qquad = \frac{1}{\sqrt{2\pi \det(Id + RLR)}} \exp(-\frac{1}{2}\langle R\xi, (Id + RLR)^{-1} R\xi \rangle) \\ &\exp\big(\frac{1}{2\langle R(\mathbb{1}_{[t_0,t)}, 0), (Id + RLR)^{-1} R(\mathbb{1}_{[t_0,t)}, 0)\rangle}(iy - \langle R\xi, (Id + RLR)^{-1} R(\mathbb{1}_{[t_0,t)}, 0)\rangle)^2\big). \end{aligned}$$

Now with $R^2 = N^{-1}$ and since R is invertible with $R^{-1}R^{-1} = N$, we have

$$Id + RLR = RR^{-1}R^{-1}R + RLR = R(Id + K + L)R$$

and

$$(Id + RLR)^{-1} = R^{-1}(Id + K + L)^{-1}R^{-1}.$$

Thus

$$\begin{aligned} &T(\sigma_R^\dagger \sigma_R \left(\exp(-\frac{1}{2}\langle \cdot L \cdot \rangle)\delta(\langle (\mathbb{1}_{[t_0,t)}, 0), \cdot \rangle - y) \right))(\xi) \\ &\qquad = \frac{1}{\sqrt{2\pi \det((N + L)N^{-1})}} \exp(-\frac{1}{2}\langle \xi, (N + L)^{-1} \xi \rangle) \end{aligned}$$

$$\exp\Big(\frac{1}{2\langle(\mathbb{1}_{[t_0,t)},0),(N+L)^{-1}(\mathbb{1}_{[t_0,t)},0)\rangle}(iy-\langle\xi,(N+L)^{-1}(\mathbb{1}_{[t_0,t)},0)\rangle)^2\Big),$$

which equals the expression from Lemma 3.2.6. Hence we have that for a suitable quadratic potential

$$\sigma_R^\dagger\sigma_R\left(\exp(-\frac{1}{2}\langle\cdot L\cdot\rangle)\delta(\langle(\mathbb{1}_{[t_0,t)},0),\cdot\rangle-y)\right),$$

exists as a Hida distribution. Moreover for all quadratic potentials from the previous chapter, the T-transform obtained via scaling gives the generating functional as in chapter 8. Since the T-transforms coincide, also the distributions are the same.
For the case of quadratic potentials we obtained the correct physics also by the scaling approach. Now we generalize this to more complicated potentials. Therefore we follow the way from [86] and [36] by the use of the Wick-formula for generalized function with Donsker's delta function and so-called finitely based Hida distributions, compare to [87].
First we have to list properties, which we demand from the potentials we investigate, compare also to [86, Ch. 7]

Assumption 9.2.3. *Let $0<t<T<\infty$ and $\mathcal{O}\subset\mathbb{R}$ open such that $\mathbb{R}\setminus\mathcal{O}$ is of Lebesgue measure zero. We assume that the potential $V:\mathcal{D}_R\to\mathbb{C}$ is analytic, with*

$$\mathcal{D}_R=\{x_0+\langle(y_1,y_2),R\begin{pmatrix}1\\0\end{pmatrix}\rangle|y_1,y_2\in\mathbb{R}\}$$

and there exist a constant $0<A<\infty$, a locally bounded function $B:\mathcal{O}\to\mathbb{R}$ and some $\varepsilon<\frac{1}{8T}$ such that for all $x_0\in\mathcal{O}$ and $y\in\mathbb{R}$ one has that

$$|\exp(-iV(x)|\leq A\exp(\varepsilon x^2),$$

and

$$|\exp(-iV(x_0+\langle(y_1,y_2),R\begin{pmatrix}1\\0\end{pmatrix}\rangle)|\leq B(x_0)\exp(\varepsilon(y_1^2+y_2^2)).$$

Furthermore we assume the following for the potential and its derivative:

Assumption 9.2.4. *Let $0<T<\infty$ and $V:\mathcal{D}_R\to\mathbb{C}$ such that Assumption 9.2.3 is fulfilled. We furthermore assume the existence of a locally bounded function $C:\mathcal{O}\times\mathcal{O}\to\mathbb{R}$ and some $0<\varepsilon<\frac{1}{8T}$ such that for all $x_0,x_1\in\mathcal{O}$ and $y_1,y_2\in\mathbb{R}$ we have*

$$|\exp(V(x_0+\langle(y_1,y_2),R\begin{pmatrix}1\\0\end{pmatrix}\rangle)\exp(-iV(x_1+\langle(y_1,y_2),R\begin{pmatrix}1\\0\end{pmatrix}\rangle)| \\ \leq C(x_0,x_1)\exp(\varepsilon(y_1^2+y_2^2))$$

and

$$|\frac{\partial}{\partial z}\exp(V(x_0+\langle(y_1,y_2),R\begin{pmatrix}1\\0\end{pmatrix}\rangle)\exp(-iV(x_1+\langle(y_1,y_2),R\begin{pmatrix}1\\0\end{pmatrix}\rangle)| \\ \leq C(x_0,x_1)\exp(\varepsilon(y_1^2+y_2^2)),$$

where $\frac{\partial}{\partial z}$ *denotes the derivative of* $z \to V(z)$ *w.r.t.* z*.*

9.3 Approximation by finitely based Hida distributions

In this section we give an outlook how to give a meaning to a scaling in phase space for potentials V fulfilling the Assumptions 9.2.3 and 9.2.4. For simplicity we consider the case $t_0 = x_0 = 0$.
First we consider the approximation by finitely based Hida distributions as in [86, 36], compare also [87]. The main use of finitely based Hida distributions in this work is the fact, that they allow us to extend the generalized scaling operator.
Let $\eta_j \in L^2_2(\mathbb{R})$, $j = 1, \dots n$ a system of linear independent vectors and $G : \mathbb{R}^n \to \mathbb{C}$ such that $G \in L^p(\nu_M)$ for some $p > 1$, where ν_M denotes the measure on $\mathbb{R}^n$ with density

$$\exp(-\frac{1}{2}\sum_{k,l=1}^{n} x_k M^{-1}_{k,j} x_j),$$

w.r.t. the Lebesgue measure on $\mathbb{R}^n$, where $M_{k,j} := \langle \eta_k, \eta_l \rangle$. Then one can define compare [86, p.66]

$$\phi(\cdot) := G(\langle \eta_1, \cdot \rangle, \dots, \langle \eta_n, \cdot \rangle) \in L^p(\mu).$$

Such elements are called finitely based Hida distributions, since they just depend on a finite number of basis elements $\langle \eta_k, \cdot \rangle$, $k \in \mathbb{N}$. The definition goes back to [55], see [86, 36] for non-smooth η_k.

Lemma 9.3.1 ([86]). *If $G \in L^p(\nu_M)$ the following relation holds*

$$G(\langle\eta_1,\cdot\rangle,\ldots,\langle\eta_n,\cdot\rangle) = \int_{\mathbb{R}^n} G(x_1,\ldots,x_n)\prod_{j=1}^n \delta(\langle\eta_j,\cdot\rangle - x_j)\,dx_1\ldots dx_n,$$

where the integral exists in $(S)'$ in the sense of Corollary 2.2.21.

The next lemma is a modification of [86, Thm.4.19,p.66]

Lemma 9.3.2. *Let $\eta_j \subset L^2_2(\mathbb{R})$, $j = 1,\ldots,n$ be a system of linear independent vectors. Let $\nu_{M_{N^{-1}},\varepsilon}$ the measure having the density*

$$\exp\left(-\frac{1}{2}\begin{pmatrix} x_1 \\ . \\ .x_n \end{pmatrix}^T \Re((M_{N^{-1}})^{-1} - \varepsilon Id)\begin{pmatrix} x_1 \\ . \\ .x_n \end{pmatrix}\right),$$

w.r.t. the Lebesgue measure on $\mathbb{R}^n$, where $(M_{N^{-1}})_{k,l} = \langle\eta_k,\eta_l\rangle$. Let $G \in L^p(\nu_{M_{N^{-1}},\varepsilon})$. Then

$$\sigma_R\phi := \int_{\mathbb{R}^n} G(x_1,\ldots,x_n)\prod_{j=1}^n \sigma_R\delta(\langle\eta_j,\cdot\rangle - x_j)dx_1\ldots dx_n,$$

is a well-defined Hida distribution as a Bochner integral in $(S)'$.

This can be proven analogously to the proof in [86]. Note that the density is analogue to the density in the complex scaling case.

The next assumption is based on [86].

Assumption 9.3.3. *We consider the decomposition of the interval $[0,t)$ given by $t_k := t\frac{k}{n}$, $k = 1,\ldots,n$. We assume that the Riemann approximation*

$$\phi_n := \exp\left(-i\frac{t}{n}\sum_{k=1}^{n-1} V(\langle R(\mathbb{1}_{[0,t_k)},0),\cdot\rangle)\right) \in L^2(\mu). \tag{9.6}$$

Then we have the following proposition.

Proposition 9.3.4. *The product of the generalized scaled Donsker's delta function with the Riemann approximation defined as in* (9.6) *can be defined as a Hida distribution, i.e.*

$$\Phi_n := \exp\left(-i\frac{t}{n}\sum_{k=1}^{n-1} V(\langle R(\mathbb{1}_{[0,t_k)},0),\cdot\rangle)\right)\sigma_R\delta(\langle(\mathbb{1}_{[0,t)},0) - y,\cdot\rangle) \in (S)', \tag{9.7}$$

for all $y \in \mathbb{R}$, $0 < t < \infty$ *and* $n \in \mathbb{N}$.

Proof. We define

$$\begin{aligned} G : \mathbb{R}^{n-1} &\to \mathbb{C} \\ y = (y_1, \dots, y_{n-1}) &\mapsto \exp\left(-i\frac{t}{n}\sum_{k=1}^{n-1} V(y_k)\right) \end{aligned} \tag{9.8}$$

Then we have

$$\phi_n = \int_{\mathbb{R}^{n-1}} G(y) \prod_{k=1}^{n-1} \delta(\langle R(\mathbb{1}_{[0,t_k)}, 0), \cdot\rangle - y_k) d^{n-1}y,$$

for all $n \in \mathbb{N}$ with $y = (y_1, \dots, y_{n-1})$.
Since R is invertible and $(\mathbb{1}_{[0,t_k)}, 0)$, $k = 1, \dots, n-1$ and $(\mathbb{1}_{[0,t]}, 0)$ form a linear independent system also their images under R form a linear independent system. Hence we have

$$\left(\prod_{k=1}^{n-1} \delta(\langle R(\mathbb{1}_{[0,t_k)}, 0), \cdot\rangle)\right) \delta(\langle R(\mathbb{1}_{[0,t)}, 0), \cdot\rangle),$$

is a Hida distribution for all $n \in \mathbb{N}$. Moreover we have

$$\begin{aligned} &T(\exp\left(-i\frac{t}{n}\sum_{k=1}^{n-1} V(\langle R(\mathbb{1}_{[0,t_k)}, 0), \cdot\rangle)\right) \sigma_R \delta(\langle(\mathbb{1}_{[0,t)}, 0) - y, \cdot\rangle))(\xi) \\ &\qquad = \int_{\mathbb{R}^{n-1}} G(y) T\left(\left(\prod_{k=1}^{n-1} \delta(\langle R(\mathbb{1}_{[0,t_k)}, 0), \cdot\rangle)\right) \delta(\langle R(\mathbb{1}_{[0,t)}, 0), \cdot\rangle)\right)(\xi) dy, \end{aligned}$$

for all $\xi \in S_2(\mathbb{R})$. □

Note that we can prove that the sequence converges in $(S)'$, since the T-transform can not be estimated independently on n, see also [86]. The following assumption can be compared to [86, Prop.5.11.,p.81].

Assumption 9.3.5. *Let* $h_k := (\mathbb{1}_{[0,t_k)} - \frac{k}{n}\mathbb{1}_{[0,t)}, 0)$, $1 \leq k \leq n-1$. *Then*

$$\Psi_n := \exp\left(-i\frac{t}{n}\sum_{k=1}^{n-1} V(\frac{k}{n}y + \langle Rh_k, \cdot\rangle)\right) \in L^2(\mu)$$

and

$$\Phi_n = \Psi_n \diamond \sigma_R \delta(\langle(\mathbb{1}_{[0,t)}, 0), \cdot\rangle - y) \in (S)',$$

with Φ_n, $n \in \mathbb{N}$ as in Proposition 9.3.4.

Proposition 9.3.6. *Let ϕ_n and ψ_n, $n \in \mathbb{N}$, be defined as in Assumption 9.3.5. Then $\phi_n, \Psi_n \in L^2(\mu)$ and*

$$\lim_{n\to\infty} \phi_n = \exp\left(-i\int_0^t V(\langle R(\mathbb{1}_{[0,r)}, 0), \cdot\rangle)\ dr\right),$$

$$\lim_{n\to\infty} \psi_n = \exp\left(-i\int_0^t V(\langle R(\mathbb{1}_{[0,r)} - \frac{r}{t}\mathbb{1}_{[0,t)}, 0), \cdot\rangle - \frac{r}{t}y)\ dr\right).$$

This can be proven by Lebesgues dominated convergence for suitable potentials V. Then we can as in [86] state the following crucial theorem.

Theorem 9.3.7. *Under the Assumption 9.3.5 and for suitable potentials V the sequence Φ_n converges in $(S)'$. Then it is natural to identify the limit object with*

$$\Phi := \exp\left(-i\int_0^t V(\langle R(\mathbb{1}_{[0,r)}, 0), \cdot\rangle)\ dr\right)\sigma_R\delta(\langle(\mathbb{1}_{[0,t)}, 0), \cdot\rangle - y) := \lim_{n\to\infty} \Phi_n.$$

Moreover we have

$$S(\Phi)(\xi) = S(\psi)(\xi)S(\sigma_R\delta(\langle(\mathbb{1}_{[0,t)}, 0), \cdot\rangle - y))(\xi), \quad \xi \in S_2(\mathbb{R}).$$

This means we have given a meaning to the generalized scaled heat kernel as a Hida distribution. Note that here we just gave the ideas and the heuristics to achieve this goal. Moreover the limit here is strongly dependent on the sequence, since the projection operator, where the relation between ψ and ϕ is based on, is not closable on $L^2(\mu)$. The same hold for the generalized scaling operator. Never the less, the particular choice of the approximation converges to a well-defined Hida distribution. The object then is mathematical rigorously defined. The last step now would be to check if the so achieved integrands solve the Schrödinger equation, as it is done in [86, 36] and [87]. This is not done in this work.

Part III

Numerical Investigation of Fractional Polymers

Chapter 10

Off-Lattice Discretization of Fractional Polymers - First Numerical Results

10.1 The Edwards model for Polymers

In 1964 Edwards proposed that polymer chains can be described in an idealized manner as elastic strings which are Gaussian distribibuted and two point interaction. Although the Gaussian statistics would lead at first to a classical Brownian path, the number of self-crossings would not fit to the behaviour of a polymer chain. To overcome this, there is also taken in account, that two monomers can not occupy the same place in a polymer. This effect is called excluded-volume effect, see e.g. [22], see also [48]. This self-repellence property causes the polymers to spread itself in the space.

In the Edwards model[22] this is implemented by using the so-called self-intersection local time

$$L = \int \int ds dt \delta \left(x(s) - x(t) \right)$$

of Brownian motion. That means that formally we multiply the Wiener measure $d\mu(x)$ with a density function $Z^{-1} \exp(-gL)$ where the normalization constant

$$Z = E\left(\exp\left(-g \int_0^N \int_0^N \delta\left(x(s) - x(t) \right) \right) \, dt \, ds \right) = \int \exp\left(-gL\right) d\mu$$

is used to obtain $Z^{-1}\exp(-gL)d\mu$ as a probability measure.
In [84] one gave a mathematical meaning to the objects above for dimension $d = 1$, e.g. as limit of suitably regularized versions with

$$Z = \lim_{\varepsilon} \int \exp(-gL_\varepsilon)\, d\mu.$$

For planar Brownian motion Varadhan [85] showed that the expectation value $\mathbb{E}(L_\varepsilon(T))$ has a logarithmic divergence but after its subtraction the centered $L_{\varepsilon,c}(T)$ converges in L^2, with a suitable rate of convergence. From this, Varadhan could conclude the integrability of $\exp(-gL_c(T))$, thus giving a proper meaning to the Edwards model. In the three-dimensional case the problem is being more complicated and one kind of renormalization has been constructed to make the model well defined [88] and [12].

10.2 Fractional Brownian Motion Models.

In recent years the fractional Brownian motion (fBm) has become an object of intense study due to its special properties, such as short/long range dependence and self-similarity, leading to proper and natural applications in different fields. In particular, the specific properties of fractional Brownian motion paths have been used in the modelling of polymers. We note the relevance of fractional Brownian motion for modelling of polymers as follows: "...by suitable choice of the parameter H, the average configurational behaviour of the chain can be made to correspond to its actual behaviour in solvents of different quality, thereby eliminating the need to account in detail, for the nature of the intermolecular potential appropriate to the given solvent. For instance, the choice $H = 3/5$ models polymers in good solvent, while the choice $H = 1/3$ models polymers in compact or collapsed phases.",see [9].
Moreover there is a large class of systems which are not behaving by a nearest neighbour interaction as in the classical Brownian model by Edwards. It can be found in Stanley et. al. [80] that long-range correlations can be found e.g. in DNA configurations. This lang-range interaction of monomers (i.e. changes of a single monomer influence other monomers 'far away' in the polymer chain) can be covered by the use of fBm.
Fractional Brownian motion with "Hurst parameter" $H \in (0, 1)$ can be characterized as a

centered Gaussian process with covariance

$$\mathbb{E}(B_i^H(s)B_j^H(t)) = \frac{\delta_{ij}}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right), \quad i,j = 1,\dots,d, \; s,t \geq 0, \tag{10.1}$$

[8, 62]. In the special case $H = 1/2$ is standard Brownian motion.
It is important to note that fBm is a process with stationary increments and that it is not Markovian. This of course makes the mathematical analysis of the fBm models more complicated.
Again a more realistic model would also involve te exculded volume effect. In [33] the authors showed that the Edwards model is well defined for $x = B^H$ whenever $H \leq 1/d$.

10.3 Monomer Interaction

In contrast to the classical methods of path integrals, where a finite dimensional approximation is used to give a meaning to the infinite dimensional path integral, the situation in the context of polymers is different. Here the path integral is an idealization of the finite dimensional integration over all monomer positions $x(k)$.
Considering N monomers we set

$$x_k = x(k) = B^H(k), \quad k = 0, ..., N-1$$

$$\int d\mu(x) \approx C \prod_{k=0}^{N-1} \int \exp\left(-H_0(x)\right) dx_k,$$

for a constant C.
In the following we want to the quadratic form $H_0(x) \equiv \frac{1}{2}\left(x, h_0 x\right)$. Consider the covariance of the fractional Brownian motion at the time points k and l. We have:

$$(A)_{k,l} = \mathbb{E}(B^H(k)B^H(l)) = \frac{1}{2}(k^{2H} + l^{2H} - |k-l|^{2H}) \tag{10.2}$$

Then since for a function $f : \mathbb{R}^N \to \mathbb{R}$ we have as expectation w.r.t. the fractional Brownian motion

$$\mathbb{E}(f(x)) = \frac{1}{\sqrt{2\pi}^N \cdot \sqrt{\det(h_0)}} \int_{\mathbb{R}^N} f(x) \exp(-\frac{1}{2}x, h_0 x)\, dx,$$

we can especially calculate the characteristic function by

$$\mathbb{E}(e^{i\lambda x}) = \frac{1}{\sqrt{2\pi}^N \cdot \sqrt{\det(h_0)}} \int_{\mathbb{R}^N} e^{i\lambda x} \exp(-\frac{1}{2}(x, h_0 x)\, dx = \exp(-\frac{1}{2}\lambda h_0^{-1}\lambda).$$

With this result we have for the second moment, i.e. for the covariance

$$\mathbb{E}(B^H(k)B^H(l)) = \frac{1}{\sqrt{2\pi}^N \cdot \sqrt{\det(h_0)}} \int_{\mathbb{R}^N} -\frac{\partial^2}{\partial\lambda_k \partial\lambda_l} e^{i\lambda x}_{|\lambda_k=\lambda_l=0} \exp(-\frac{1}{2}(x, h_0 x))\, dx$$

$$= -\frac{\partial^2 \exp(-\frac{1}{2}(\lambda h_0^{-1}\lambda))}{\partial\lambda_k \partial\lambda_l}_{|\lambda_k=\lambda_l=0}.$$

Hence

$$h_0^{-1} = A.$$

For the standard Brownian motion the well-known result is

$$H_0(x) = \frac{1}{2} \sum (x_k - x_{k-1})^2$$

i.e. an attractive quadratic interaction between nearest neighbors. For $N = 51$ we display the matrix elements

$$h_{25,k} = \begin{cases} 2 & \text{if} \quad k = 26 \\ -1 & \text{if} \quad k = 25, 27 \\ 0 & \quad \text{otherwise} \end{cases}.$$

in Figure 10.4.

A numerical inversion of the covariance matrix (10.2) can be done easily and generally. For small H we have as expected a long range attraction which corresponds to curlier polymers (Fig. 10.1-10.2), while for bigger H we observe a next-to-nearest neighbor repulsion (Fig. 10.3) which stretches the polymers.

10.4 An off-lattice discretization of fractional Brownian paths

In this section we present an off-lattice discretization of fractional Braownian motion. For the excluded volume effect for the discrete Edwards model off-lattice methods were also used by [7].

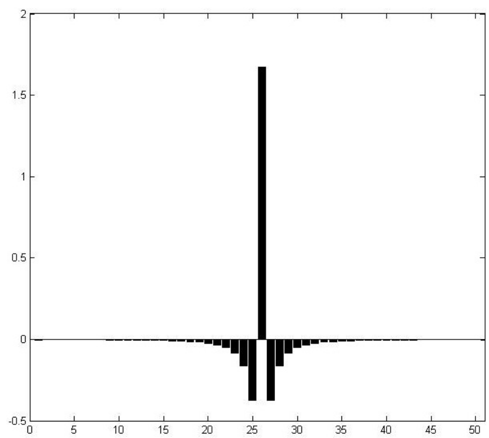

Figure 10.1: Correlation for H=0.1

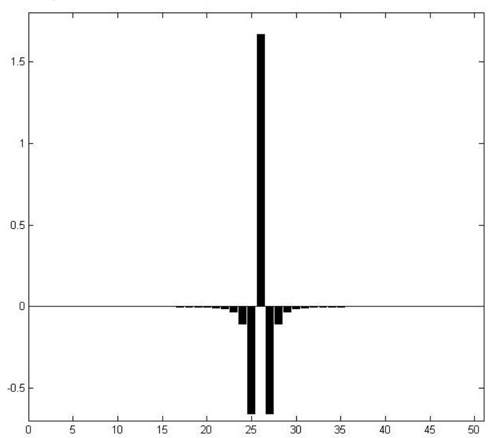

Figure 10.2: Correlation for H=0.3

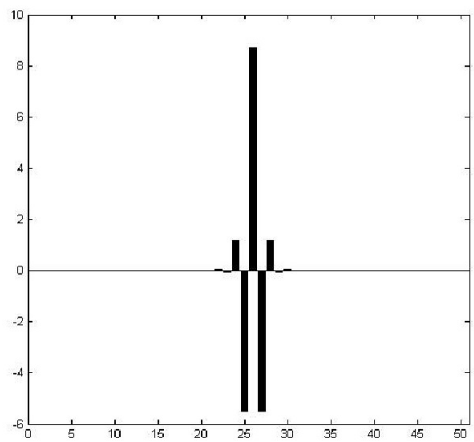

Figure 10.3: Correlation for H=0.9

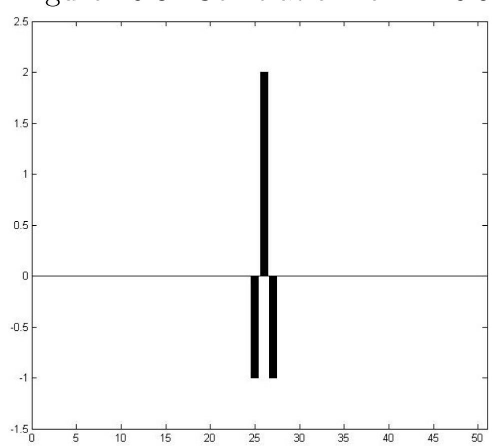

Figure 10.4: Correlation for H=0.5

The advantage of off-lattice discretizations compared to on-lattics discretizations of fBm are less complicated from their mathematical structure and their numerical efford. Due to the long-range correlation of the increments, a simulation, which uses a lattice has to take this into account in every refinement step. A list of approaches for simulating fBm can be found in [15].

The derivation in Section 10.3 is the basis of the for the discretization of fBm paths. We have that the correlation matrix of a fBm can be used to derive the free energy matrix of a polymer driven by fractional Brownian motion.

The free fBm polymer

Now we consider the case of a free polymer in one dimension. To obtain such a polymer path we use a Metropolis routine. The idea is to minimze the free energy of the system by varying the position of the monomers.
Here the basis is the free energy matrix calculated from the covariance matrix.

Starting with a random configuration of increments, during the algorithm monomer increment positions are chosen randomly and set to a random number. Then the change of energy, which is caused by this influence is tested.
If the energy is reduced by the above mentioned random change, the new configuration is taken as an update and the procedure starts again with this update as initial configuration. Otherwise, if the change of configuration increases the energy, it is tested with a random number r if $\exp(-\frac{1}{2}(E_{new} - E_{old})) > r$.
If this is the case, we take also the new configuration as an update. Otherwise we restart with the old one.

With this routine it is possible to create discretized fBm paths just with the help of the covariance matrix.
Of course, as usual in Metropolis routines, the algorithm needs a certain number of updates until covergence starts.
In the graphics below (Fig. 10.5-Fig. 10.8), we displayed 1d-fBm paths created with this algorithm. Here the start configuration was always the same in all examples. Moreover we used $N = 30$ and 100000 iterations.

In the figures the x-axis gives the monomer number. Note that we are in the $1d$-case. Here a self-crossing takes already part if the path is changing its direction, i.e. if the path

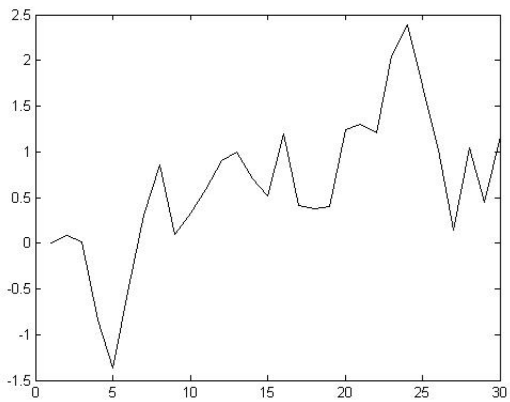

Figure 10.5: 1d-fBm path for H=0.1

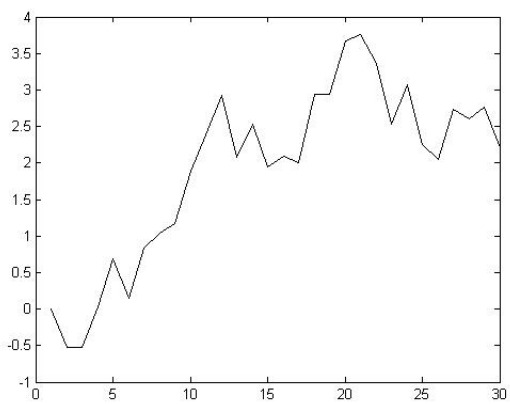

Figure 10.6: 1d-fBm path for H=0.3

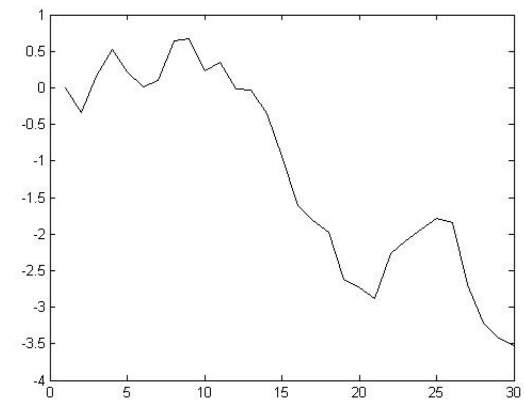

Figure 10.7: 1d-fBm path for H=0.9

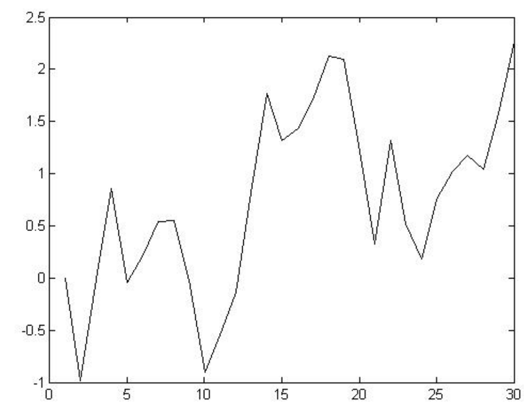

Figure 10.8: 1d-fBm path for H=0.5

runs up between monomer 4 and 5 and after that runs down to monomer 6.
From the sample paths we already see, that the paths with $H < \frac{1}{2}$ are rougher than the path with $H = 0.9$. In the case of $H = 0.5$ we have the shape of a classical Brownian random walk.

Simulation including the excluded volume

The routine sketched above can also be modified to incorporate the excluded volume effect. Here we have to model the self-intersection local time mentioned above. The idea is to put a grid around the monomer position and count the number of monomers are inside one box. For every self-crossing the energy is increased by a factor g, which is making in the routine paths with a lot of self-crossings unlikely. We obtain the following graphs for $d = 1$. Here again we used the same start configuration as in the examples before. Moreover we have $N = 30$, $g = 0.5$ and 100000 iterations.

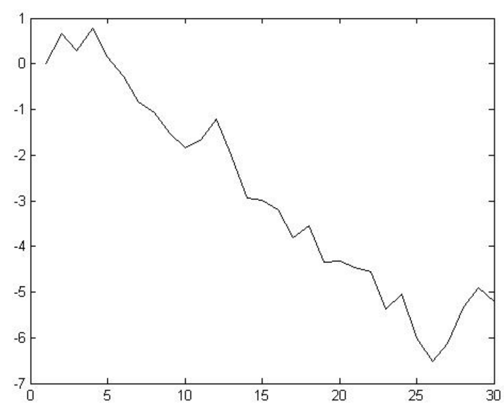

Figure 10.9: g=0.5, H=0.1

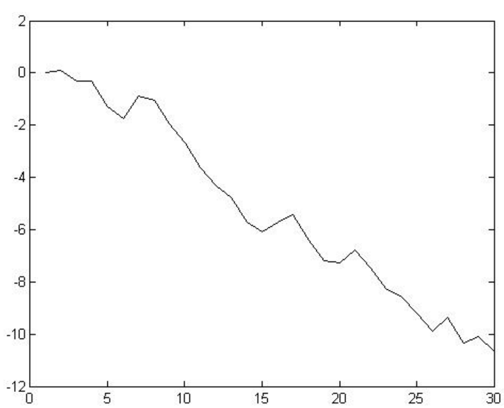

Figure 10.10: g=0.5, H=0.3

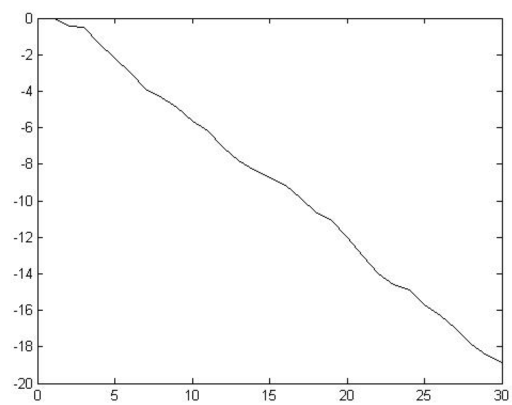

Figure 10.11: g=0.5, H=0.9

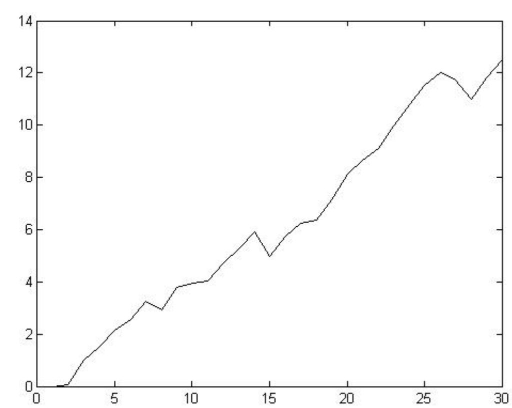

Figure 10.12: g=0.5, H=0.5

In the pictures we see, that the paths have a straight direction upwards or downwards. Still the paths with lower Hurst index are rougher. Nevertheless the number of self-crossings is reduced.

In the follwing to picures we one more displayed paths with the same initial conditions but with a lower self-repellence factor g (here g=0.3). Compared to Figures 10.9-10.12., we see that the number of self-crossings is much higher but the strong tendency to a certain direction is still recovered. The algorithms above can also be used to obtain $2d$-

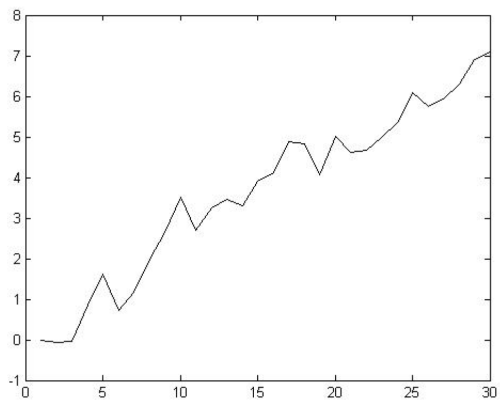

Figure 10.13: g=0.3, H=0.1

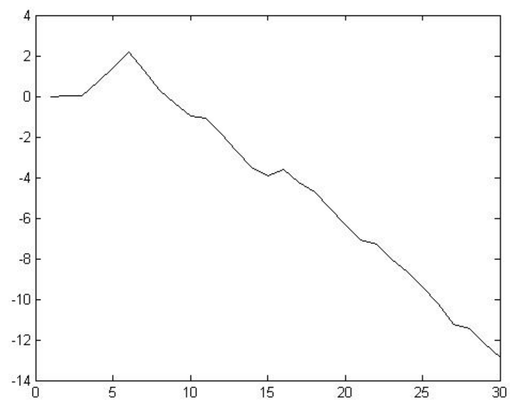

Figure 10.14: g=0.3, H=0.3

paths of fBm polymers. This is at the moment object of a research project with Ludwig Streit(CCM Madeira), Samuel Eleuterio, Maria Joao Oliveira (Lisbon), Jinky Bornales and Dennis Arogancia (both Iligan, Phillippines). Simulations based on this algorithms to investigate the end-to-end length of the polymer are actual research projects.

10.5 The End-to-End Length.

A crucial object of interest is the investigation of how the end-to-end distance R of a polymer, with

$$R^2 = E(|x_{N-1}|^2),$$

scales with the number of monomers N for N large. In polymer physics we have the ansatz

$$R \sim N^{\nu}, \text{see [83]}$$

where ν is the famous "Flory index". This exponent has the following conjectured dependence to the dimension n d:

$$\upsilon = \frac{3}{d+2} \ . \tag{10.3}$$

Theoretical studies support this conjecture for $d = 1, 2$ while various less rigorous methods, from computer simulations to renormalization group theory, point to a slight deviation for $d = 3$, in agreement with experimental results, see e.g.[13]:

	Flory	*Theory*	*Exp.*
$d = 1:$	1	1	
$d = 2:$	0.75	0.75	0.79
$d = 3:$	0.60	0.588	0.59
$d = 4:$	0.5	0.5	

10.5.1 Extension to the fractional case.

Generalization of Flory's conjecture to the fractional case with general Hurst index H was proposed in [14]:

$$\upsilon_H = \begin{cases} 1 & \text{if} \quad d = 1 \text{ and } H > 1/2 \\ H & \text{if} \quad dH > 2 \\ \frac{2H+2}{d+2} & \text{if} \quad \text{otherwise.} \end{cases} \ . \tag{10.4}$$

Here the first condition stems from the fact that the scaling exponent $\upsilon = 1$ is maximal, polymers will not grow faster than the number of monomers. The second condition reflects the fact that for $dH = 2$ fBm becomes self-intersection free[82], beyond that value the end-to-end-length will scale like for free fBm.

We mention in passing that there is also a proposition for a recursion formula, generalizing the one given by Kosmas and Freed [52] for the case of conventional Brownian motion.

10.5.2 Computer simulations - first results.

Preliminary results for the Flory index in the one-dimensional case could be obtained based on the algorithm sketched above. Here the proposed result is given by

$$\upsilon_H = \frac{2H+2}{3}$$

for $H \leq 1/2$ and $\upsilon_H = 1$ otherwise. Numerically in [13] the authors found

H	υ_H	Simulation
0.10	0.73	0.772 ± 0.037
0.20	0.80	0.803 ± 0.012
0.35	0.90	0.917 ± 0.013

in preliminary computations.
More systematic and detailed simulations of the Flory index in the one-and two-dimensional case are in work. The preliminary results are very good comparable to the proposed values. A publication together with Streit, Eleuterio, Bornales et. al. about this topic is in preparation.

Bibliography

[1] S. Albeverio, G. Guatteri, and S. Mazzucchi. Phase space Feynman path integrals. 43(6):2847–2857, June 2002.

[2] S. Albeverio, R. Høegh-Krohn, and S. Mazzucchi. *Mathematical Theory of Feynman Path Integrals: An Introduction*, volume 523 of *Lecture Notes in Mathematics*. Springer Verlag, Berlin, Heidelberg, New York, 2008.

[3] Hans Wilhelm Alt. *Lineare Funktionalanalysis*. Springer-Verlag Berlin Heidelberg, 2006.

[4] A. Badrikian. *Seminaire sur les Fonctions Aleatoires Lineaires et les Mesures Cylindriques*. Lecture Notes in Mathematics. Springer, 1970.

[5] H. Bauer. *Wahrscheinlichkeitstheorie*. De Gruyter Lehrbuch. De Gruyter, 2002.

[6] Y. M. Berezansky and Y. G. Kondratiev. *Spectral methods in infinite-dimensional analysis. Vol. 2*, volume 12/2 of *Mathematical Physics and Applied Mathematics*. Kluwer Academic Publishers, Dordrecht, 1995. Translated from the 1988 Russian original by P. V. Malyshev and D. V. Malyshev and revised by the authors.

[7] G. Besold, H. Guo, and M. J. Zuckermann. Off-lattice monte carlo simulation of the discrete edwards model. *Journal of Polymer Science Part B: Polymer Physics*, 38(8):1053–1068, 2000.

[8] F. Biagini, Y. Hu, B. Øksendal, and T. Zhang. *Stochastic Calculus for Fractional Brownian Motion and Applications*. Probability and Its Applications. Springer, 2007.

[9] P. Biswas and B.J. Cherayil. Dynamics of fractional brownian walks. *The Journal of Physical Chemistry*, 99(2):816–821, 1995.

[10] W. Bock and M. Grothaus. A White Noise Approach to Phase Space Feynman Path Integrals. *Teor. Imovir. ta Matem. Statyst.*, (85):7–21, 2011.

[11] W. Bock, M. Grothaus, and S. Jung. The feynman integrand for the charged particle in a constant magnetic field as white noise distribution, 2012.

[12] E. Bolthausen. On the construction of the three dimensional polymer measure. *Probability Theory and Related Fields*, 97:81–101, 1993. 10.1007/BF01199313.

[13] J. Bornales, S. Eleuterio, and Streit L Oliveira, M. J. Fractional brownian polymers - some first results. *to appear in Proceedings of the 6th Jagna International Workshop: Mathematical Analysis, Modelling, and Simulation in Interdisciplinary Sciences*, 2012.

[14] J. Bornales, M. O. Oliveira, and L. Streit. Self-repelling fractional brownian motion - a generalized edwards model for chain polymers, 2011.

[15] Jean-Francois Coeurjolly. Simulation and identification of the fractional brownian motion: a bibliographical and comparative study. *Journal of Statistical Software*, 5(7):1–53, 9 2000.

[16] M. de Faria, J. Potthoff, and L. Streit. The Feynman integrand as a Hida distribution. *J. Math. Phys.*, 32:2123–2127, 1991.

[17] C. DeWitt-Morette, A. Maheshwari, and B. Nelson. Path integration in phase space. *General Relativity and Gravitation*, 8(8):581–593, 1977.

[18] S. Dineen. *Complex Analysis in Locally Convex Spaces*, volume 57 of *Mathematical Studies.* Elsevier Science Publisher B.V., Amsterdam, 1981.

[19] Paul A.M. Dirac. The Lagrangian in quantum mechanics. *Phys.Z.Sowjetunion*, 3:64–72, 1933.

[20] H. Doss. Sur une Resolution Stochastique de l'Equation de Schrödinger à Coefficients Analytiques. *Communications in Mathematical Physics*, 73:247–264, October 1980.

[21] I. Duru and H. Kleinert. Quantum mechanics of h-atom from path integrals. *Fortsch.d.Physik*, 30:401–435, 1982.

[22] S. F. Edwards. The statistical mechanics of polymers with excluded volume. *Proc. Phys. Soc.*, 85:613–624, 1965.

[23] R. P. Feynman. Space-time approach to non-relativistic quantum mechanics. *Rev. Modern Physics*, 20:367–387, 1948.

[24] R.P. Feynman. An operator calculus having applications in quantum electrodynamics. *Physical Review*, 84(1):108–124, 1951.

[25] R.P. Feynman and A.R. Hibbs. *Quantum Mechanics and Path Integrals*. McGraw-Hill, London, New York, 1965.

[26] K. Floret and J. Wloka. *Einführung in die Theorie der lokalkonvexen Räume*. Lecture notes in mathematics. Springer-Verlag, 1968.

[27] I.M. Gel'fand and N.Ya. Vilenkin. *Generalized Functions*, volume 4. Academic Press, New York, London, 1968.

[28] I. S. Gradshteyn and I. M. Ryzhik. *Table of integrals, series, and products*. Fourth edition prepared by Ju. V. Geronimus and M. Ju. Ceĭtlin. Translated from the Russian by Scripta Technica, Inc. Translation edited by Alan Jeffrey. Academic Press, New York, 1965.

[29] M. Grothaus. White noise analysis and feynman integrals.

[30] M. Grothaus, Y. G. Kondratiev, and L. Streit. Complex gaussian analysis and the Bargmann-Segal space. *Methods Funct. Anal. Topology*, 3(2), 1997.

[31] M. Grothaus, Y. G. Kondratiev, and G.F. Us. Wick calculus for regular generalized stochastic functions. *Random Operators and Stochastic Equations*, 7(4):301–328, 1999.

[32] M. Grothaus, Yu.G. Kondratiev, and L. Streit. Regular generalized functions in Gaussian analysis. *Infinite Dim. Anal. Quantum Prob.*, 2(1):1–24, 1999.

[33] M. Grothaus, M. J. Oliveira, J. L. Silva, and L. Streit. Self-avoiding fractional brownian motion - the edwards model. *J. Stat. Phys.*, 145(6):1513–1523, 2011.

[34] M. Grothaus and L. Streit. Quadratic actions, semi-classical approximation, and delta sequences in Gaussian analysis. *Rep. Math. Phys.*, 44(3):381–405, 1999.

[35] M. Grothaus, L. Streit, and A. Vogel. The feynman integrand as a white noise distribution for non-perturbative potentials. 2008. to appear in the festschrift in honor of Jean-Michel Bismut's sixtieth birthday.

[36] M. Grothaus, L. Streit, and A. Vogel. The complex scaled feynman–kac formula for singular initial distributions. *Stochastics*, 84(2-3):347–366, April-June 2012.

[37] M. Grothaus and A. Vogel. The Feynman integrand as a white noise distribution beyond perturbation theory. To appear in the proceedings of the "5th Jagna International Workshop: Stochastic and Quantum Dynamics of Biomolecular Systems", 2008.

[38] W. Hackenbroch and A. Thalmaier. *Stochastische Analysis: eine Einführung in die Theorie der stetigen Semimartingale*. Mathematische Leitfäden. B.G. Teubner, 1994.

[39] M. Hairer. An Introduction to Stochastic PDEs. *http://www.hairer.org/notes/SPDEs.pdf*, 2009.

[40] T. Hida, H.-H. Kuo, J. Potthoff, and L. Streit. *White Noise. An infinite dimensional calculus*. Kluwer Academic Publisher, Dordrecht, Boston, London, 1993.

[41] T. Hida and L. Streit. Generalized brownian functionals and the feynman integral. *Stoch. Proc. Appl.*, 16:55–69, 1983.

[42] Brian R. Hunt, Tim Sauer, and James A. Yorke. Prevalence: a translation-invariant "almost every" on infinite-dimensional spaces. *Bull. Amer. Math. Soc.*, 27(2):217–238, 1992.

[43] S. Jung. Concepts of vector valued white noise and construction of the feynman integrand for the magnetic field in the space of hida distributions.

[44] D.C. Khandekar and S.V. Lawande. Feynman path integrals: Some exact results and applications. *Physics Reports*, 137(2), 1986.

[45] J. R. Klauder and I. Daubechies. Measures for path integrals. *Physical Review Letters*, 48(3):117–120, 1982.

[46] J. R. Klauder and I. Daubechies. Quantum mechanical path integrals with wiener measures for all polynomial hamiltonians. *Physical Review Letters*, 52(14):1161–1164, 1984.

[47] John R. Klauder. The feynman path integral: An historical slice, 2003.

[48] H. Kleinert. *Path Integrals in Quantum Mechanics, Statistics, Polymer Physics, and Financial Markets*. World Scientific, 2004.

[49] Yu.G. Kondratiev. Spaces of entire functions of an infinite number of variables, connected with the rigging of a Fock space. *Selecta Mathematica Sovietica*, 10(2):165–180, 1991. Originally published in Spectral Analysis of Differential Operators, Mathematical Institute of the Academy of Sciences of the Ukrainian SSR, Kiev, 1980, pp. 18–37.

[50] Yu.G. Kondratiev, P. Leukert, J. Potthoff, L. Streit, and W. Westerkamp. Generalized functionals in Gaussian spaces: The characterization theorem revisited. *J. Funct. Anal.*, 141(2):301–318, 1996.

[51] Yu.G. Kondratiev, P. Leukert, and L. Streit. Wick Calculus in Gaussian Analysis. *Acta Appl. Math.*, 44:269–294, 1996.

[52] Marios K. Kosmas and Karl F. Freed. On scaling theories of polymer solutions. *The Journal of Chemical Physics*, 69(8):3647–3659, 1978.

[53] K. Kremer and K. Binder. *Monte Carlo Simulations of Lattice Models for Macromolecules*. North-Holland, Amsterdam, 1988.

[54] P. Kristensen, L. Mejlbo, and E.T. Poulsen. Tempered distributions in infinitely many dimensions. I. Canonical field operators. *Comm. Math. Phys.*, 1:175–214, 1965.

[55] Izumi Kubo and Hui-Hsiung Kuo. Finite-dimensional Hida distributions. *J. Funct. Anal.*, 128(1):1–47, 1995.

[56] Naoto Kumano-Go. Phase space Feynman path integrals with smooth functional derivatives by time slicing approximation. *Bull. Sci. Math.*, 135(8):936–987, 2011.

[57] H.-H. Kuo. *White Noise Distribution Theory*. CRC Press, Boca Raton, New York, London, Tokyo, 1996.

[58] H.H. Kuo. *Gaussian Measures in Banach Spaces*. Lecture Notes in Mathematics. Springer, 1975.

[59] Angelika Lascheck, Peter Leukert, Ludwig Streit, and Werner Westerkamp. More about Donsker's delta function. *Soochow J. Math.*, 20(3):401–418, 1994.

[60] M. Ledoux and M. Talagrand. *Probability in Banach Spaces: Isoperimetry and Processes*. Ergebnisse der Mathematik Und Ihrer Grenzgebiete. Springer, 2011.

[61] R. A. Minlos. Generalized random processes and their extension to a measure. *Selected Transl. Math. Statist. and Prob.*, 3:291–313, 1963.

[62] Y. Mishura. *Stochastic Calculus for Fractional Brownian Motion and Related Processes*. Number Nr. 1929 in Lecture Notes in Mathematics. Springer, 2008.

[63] E. Nelson. Probability theory and euclidean quantum field theory. In *Constructive Quantum Field Theory*, pages 187–214. Springer, 1973.

[64] N. Obata. *White Noise Calculus and Fock Spaces*, volume 1577 of LNM. Springer Verlag, Berlin, Heidelberg, New York, 1994.

[65] Nobuaki Obata and Habib Ouerdiane. A note on convolution operators in white noise calculus. *Infin. Dimens. Anal. Quantum Probab. Relat. Top.*, 14(4):661–674, 2011.

[66] A. Pietsch. *Nuclear Locally Convex Spaces*. Springer Verlag, Berlin, Heidelberg, New York, 1972.

[67] J. Potthoff and L. Streit. A characterization of Hida distributions. *J. Funct. Anal.*, 101:212–229, 1991.

[68] J. Potthoff and M. Timpel. On a dual pair of spaces of smooth and generalized random variable s. 1995.

[69] J. Potthoff and M. Timpel. On a dual pair of spaces of smooth and generalized random variables. *Potential Anal.*, 4(6):637–654, 1995.

[70] M. Reed and B. Simon. *Methods of modern mathematical physics*, volume I. Academic Press, New York, London, 1975.

[71] M. Reed and B. Simon. *Methods of modern mathematical physics*, volume II. Academic Press, New York and London, 1975.

[72] Walter Rudin. *Functional analysis*. International Series in Pure and Applied Mathematics. McGraw-Hill Inc., New York, first edition, 1974.

[73] Walter Rudin. *Real and complex analysis*. McGraw-Hill Book Co., New York, 1974.

[74] Walter Rudin. *Principles of mathematical analysis.* McGraw-Hill Book Co., New York, third edition, 1976. International Series in Pure and Applied Mathematics.

[75] H.H. Schaefer. *Topological Vector Spaces.* Springer Verlag, Berlin, Heidelberg, New York, 1971.

[76] F. Scheck. *Theoretische Physik 1.* Springer-Verlag Berlin Heidelberg, Berlin, Heidelberg, achte auflage edition, 2007.

[77] I. E. Segal. Tensor algebras over hilbert spaces. ii. *The Annals of Mathematics*, 63(1):pp. 160–175, 1956.

[78] José L. Silva and Ludwig Streit. Feynman integrals and white noise analysis. In *Stochastic analysis and mathematical physics (SAMP/ANESTOC 2002)*, pages 285–303. World Sci. Publ., River Edge, NJ, 2004.

[79] B. Simon. *The $P(\Phi)_2$ Euclidean (quantum) Field Theory.* Pinceton University Press, Princeton, 1974.

[80] H.E. Stanley, S.V. Buldyrev, A.L. Goldberger, S. Havlin, R.N. Mantegna, C.-K. Peng, M. Simons, and M.H.R. Stanley. *Long-range correlations and generalized Lévy walks in DNA sequences*, volume 450 of *Lecture Notes in Physics*, pages 331–347. Springer Berlin Heidelberg, 1995.

[81] L. Streit. Feynman paths, sticky walls, white noise. In *A garden of quanta*, pages 105–113. World Sci. Publ., River Edge, NJ, 2003.

[82] Michel Talagrand. Multiple points of trajectories of multiparameter fractional brownian motion. *Probability Theory and Related Fields*, 112:545–563, 1998. 10.1007/s004400050200.

[83] I. Teraoka. *Polymer Solutions: An Introduction to Physical Properties.* John Wiley & Sons, 2002.

[84] Remco van der Hofstad and Wolfgang KÃ¶nig. A survey of one-dimensional random polymers. *Journal of Statistical Physics*, 103:915–944, 2001. 10.1023/A:1010309005541.

[85] S.R.S. Varadhan. *Appendix to "Euclidean quantum field theory" by K. Symanzik.* Academic Press, New York, 1969.

[86] A. Vogel. A new wick formula for products of white noise distributions and application to feynman path integrands, 2010.

[87] W. Westerkamp. *Recent Results in Infinite Dimensional Analysis and Applications to Feynman Integrals*. PhD thesis, University of Bielefeld, 1995.

[88] M.J. Westwater. On Edwards' model for long polymer chains. *Commun. Math. Phys.*, 72:131–174, 1980.

[89] L. Xu. An Introduction to Stochastic Partial Differential Equations. *http://page.math.tu-berlin.de/ xu/teaching.html*, 2012.

[90] Yoshitaka Yokoi. Positive generalized white noise functionals. *Hiroshima Math. J.*, 20(1):137–157, 1990.

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