Analogies between Proofs – A Case Study

Erica Melis*
Universität Saarbrücken
Fachbereich Informatik
6600 Saarbrücken
email: melis@cs.uni-sb.de

This case study examines in detail the theorems and proofs that are shown by analogy in a mathematical textbook on semigroups and automata, that is widely used as an undergraduate textbook in theoretical computer science at German universities (P. Deussen, Halbgruppen und Automaten, Springer 1971). The study shows the important rôle of *restructuring* a proof for finding analogous subproofs, and of *reformulating* a proof for the analogical transformation. It also emphasizes the importance of the *relevant* assumptions of a known proof, i.e., of those assumptions actually used in the proof. In this document we show the theorems, the proof structure, the subproblems and the proofs of subproblems and their analogues with the purpose to provide an empirical test set of cases for automated analogy-driven theorem proving. Theorems and their proofs are given in natural language augmented by the usual set of mathematical symbols in the studied textbook. As a first step we encode the theorems in logic and show the actual restructuring. Secondly, we code the proofs in a Natural Deduction calculus such that a formal analysis

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becomes possible and mention reformulations that are necessary in order to reveal the analogy.

Introduction

Justified analogical reasoning proceeds by transferring an aspect from the base case s to a target case t based on the similarity of these cases with respect to a second aspect. The second and the first aspect have to be inherently connected. For example, analogical reasoning takes as input the similarity of s and t with respect to their function and the connection between function and structure.

Then it yields the commonality of s and t with respect to their structure.

Within the context of (automated) theorem proving, problems and proofs are usually inherently connected, in the sense that a basic heuristic assumption stipulates that analogous theorems can be proved analogously also. This assumption is true in many cases. For analogical theorem proving the first aspect of a connection is the pair (ass, thm) which we call *problem*, that consists of the set of relevant assumptions ass and of the theorem thm, and the second aspect is the proof of the theorem thm from the assumptions ass.

To obtain an empirical test set and in order to gain practical experience with analogical reasoning in mathematical theorem proving we have studied the textbook "Halbgruppen und Automaten" (abbreviated as HUA in the following) [1], since it is particularly rich in proofs that are explicitly stated as analogous to previous proofs by the author. Furthermore the book already served as a test case for automated theorem proving for the Margraf Karl Refutation Procedure [2].

This empirical study is the basis for our own approach to analogy-driven theorem proving that is presented in detail in [5]. This approach is inherently based on

- the *reformulation* of the base problem together with the base proof, and on the reformulation of the target problem. The aim of the reformulation is to make the representation of the base and the target problem compatible such that the essential analogy is revealed,
- carrying over certain reformulated parts of the base proof as parts of a hypothetical target proof.

The following study presents all theorems in HUA that are explicitly marked as analogous by the author. Theorems are first given in English (our translation) and coded in predicate logic. The proofs are then coded in a Natural Deduction format, such that a formal analysis becomes possible.

The main finding is that a problem $P2$ is called *analogous* to a problem $P1$ in the textbook, if $P1$ can be reformulated to a problem equal to $P2$, or $P1$ *and* $P2$ can be reformulated to a common abstraction. Actually, $P2$ is often called analogous to $P1$ even if only an important subproblem of $P2$ is analogous to a subproblem of $P1$. Hence the standard approach to automated theorem proving by analogy (e.g., [6]), which is mainly based on symbol mapping of the base case to the target case for a given representation *fails* in many cases: There is no such symbol map unless the actually given representation is reformulated such that the analogy becomes visible.

Why "Halbgruppen und Automaten"?

The textbook "Halbgruppen und Automaten" [1] was chosen for this case study since it consists of the three chapters each of which is built upon the previous one, partially by analogies. This is the reason, why this particular textbook is so rich in explicit proofs by analogy and actually very much liked by students because of its uniform structure. The actual chapters are:

- Semigroups and relations
- Semigroups and semimoduls
- Automata.

Notation

The study is based on Natural Deduction (ND) proofs, since it turned out to be most natural to code the proofs given within the textbook in a proof calculus, whose (primitive) rules are presented in [4] which in turn is based on [3]. The displayed ND-proof lines do not always correspond to primitive rules but can easily splitted into several lines that correspond to primitive ND-rules. The reason is to keep the proofs more readable.

In the following we quote the theorems by their original decimal numbering from HUA, for example, Theorem 17.6. refers to Satz 17.6. on page 182 in HUA. Sometimes a theorem is not explicitly stated in the textbook but just mentioned as analogous to some previous theorem, for example the existence of certain homomorphisms in semimoduls is directly carried over (i.e.

is analogous) from the existence of homomophisms in semigroups. Then, for instance, the theorem mentioned as analogous to theorem 5.7 in section 7 of HUA is denoted as 7.5.7. As another notational convention, we denote part n of theorem m by $m.n$.

The ND-proofs contain parts that are called *relevant assumptions*, and these may correspond to applications of the ND-rule called HYP (hypothesis introduction). The relevant assumptions are those hypotheses which cannot be omitted in the proof. As a further refinement, the origin of the HYP-rule is replaced by the name of the assumption that was introduced by the HYP-rule. For example, ASS means that the formula is an assumption of the problem, DEF points to a definition, and AX means that the hypothesis is an axiom.

Analogous Theorems and Proofs in HUA

The following theorems are marked by the author of HUA to be shown by analogy:

Theorem 6.3. (Chapter II) is analogous to theorem 3.3. (Ch.I).
Theorem 6.6. (Ch.II) is analogous to theorem 3.6. (Ch.I).
Theorem 7.5.2. (Ch.II) is analogous to theorem 5.2. (Ch.I).
Theorems 4.10, 4.11, and 4.12 (Ch.I) for sets can be taken over for the corresponding (sub-)theorems of 5.6, 5.7, and 5.8 for semigroups.
Theorems 5.6, 5.7, and 5.8 (Ch.I) for semigroups are supposed to be carried over analogously to the corresponding theorems 7.5.6, 7.5.7, and 7.5.8 for semimoduls (in Ch.II).
Theorem 5.3 (Ch.I) is analogous to theorem 4.8 (Ch.I).
Theorem 10.9.8 is analogous to 9.8 (Ch.II).
Theorem 13.7 (Ch.III) is analogous to theorem 6.9 (Ch.II).
Theorem 17.17.6 (Ch.III) is analogous to theorem 17.6 in the same section (Ch.III).
Theorem 17.9 part 2 (Ch.III) is analogous to 17.9 part 1 (Ch.III).
Two subproofs of theorem 17.6 are analogous.

The more interesting analogies are examined in the following.

CASE 1: THEOREMS 3.3 and 6.3

The proof of theorem 6.3 is analogous to the proof of theorem 3.3 in HUA, where the respective theorems are given as:

Theorem 3.3

Let $\{T_i : i \in I\}$ be a family of leftideals in the semigroup F.
1. Then $\bigcup_i T_i$ is a leftideal in F.
2. If $\bigcap_i T_i$ is not empty then $\bigcap_i T_i$ is a leftideal in F.

Theorem 6.3

Let $\{T_i : i \in I\}$ be a family of F-subsemimoduls in the F-semimodul S.
1. Then $\bigcup_i T_i$ is an F-subsemimodul in S.
2. If $\bigcap_i T_i$ is not empty then $\bigcap_i T_i$ is an F-subsemimodul in S.

The analogy of theorem 3.3 and theorem 6.3 is based on the correspondence between the definitions of a leftideal and of a subsemimodul (definition 3.1 and definition 6.2 in HUA) which are given as:

Definition 3.1 A nonempty subset T of a semigroup F is called *leftideal* if $FT \subset T$, where $FT = \{ft : f \in F, t \in T\}$.

Definition 6.2 A nonempty subset T of an F-semimodul S is called *F-subsemimodul* if $FT \subset T$, where $FT = \{ft : f \in F, t \in T\}$.

Consider the analogy between the subproof of 3.3.1 and the subproof of 6.3.1: Corresponding to the definition of a leftideal, it is shown for 3.3.1 that $\bigcup_i T_i$ is nonempty, $\bigcup_i T_i$ is a subset of F, and $F \cdot \bigcup_i T_i \subset \bigcup_i T_i$. Thus splitting the theorem 3.3.1 into its conjunctive subparts, a straightforward **proof structure** of 3.3.1 is the following:

- Part 1:
  Theorem: $\bigcup_i T_i$ is nonempty, i.e. (expanding the definition of 'nonempty'): $\exists x(x \in \bigcup_i T_i)$.
  Relevant assumptions: $\forall i(i \in I \rightarrow \exists x(x \in T_i))$; the definition of $\bigcup_i T_i$.

- Part 2:
  Theorem: $\bigcup_i T_i \subset$ F.
  Relevant assumptions: the definition of $\bigcup$; the definition of $\subset$; the assumption

$\forall i(i \in I \rightarrow (T_i \subset F))$.
This part is demonstrated in more detail below.

- Part3:
Theorem: $F \cdot \bigcup_i T_i \subset \bigcup_i T_i$, e.g.(after expanding the definition of $F \cdot T$ and of $\subset$): $\forall i, x, f(i \in I \wedge x \in T_i \wedge f \in F \rightarrow f \cdot x \in T_i)$
Relevant assumptions: the definition of $F \cdot T$; the definition of $\subset$; the definition of $\bigcup$; the assumption $\forall i(i \in I \rightarrow F \cdot T_i \subset T_i)$.

Given the definition of an F-subsemimodul, it has to be shown in a proof of 6.3.1 that $\bigcup_i T_i$ is nonempty, $\bigcup_i T_i$ is a subset of S, and $F \cdot \bigcup_i T_i \subset \bigcup_i T_i$. Splitting the theorem of 6.3.1 into its conjunctive subparts, a straightforward **proof structure** of 6.3.1 becomes:

- Part 1:
Theorem: $\bigcup_i T_i$ is nonempty, i.e.(expanding the definition of 'nonemptyness'):
$\exists x(x \in \bigcup_i T_i)$.
Relevant assumptions: $\forall i(i \in I \rightarrow \exists x(x \in T_i))$; the definition of $\bigcup_i T_i$.

- Part 2:
Theorem: $\bigcup_i T_i \subset$ S.
Relevant assumptions of the completed proof: the definition of $\bigcup$; the definition of $\subset$; the assumption $\forall i(i \in I \rightarrow (T_i \subset S))$.

- Part3:
Theorem: $F \cdot \bigcup_i T_i \subset \bigcup_i T_i$, i.e.(after expanding the definition of $F \cdot T$ and of $\subset$): $\forall i, x, f(i \in I \wedge x \in T_i \wedge f \in F \rightarrow f \cdot x \in T_i)$
Relevant assumptions: the definition of $F \cdot T$; the definition of $\subset$; the definition of $\bigcup$; the assumption $\forall i(i \in I \rightarrow F \cdot T_i \subset T_i)$.

We shall now give the explicit ND proofs of theorem 3.3.1 part 2 and of theorem 6.3.1 part 2. The first proof is a translation of the given natural language proof in HUA into an ND-calculus, while the second proof is an analogical reconstruction (it is not given in the textbook but just mentioned as "to be shown analogously").

ND proof for theorem 3.3.1 part 2.

NNo	S;D		Formula	Reason
			relevant assumptions	
1.	; 1	$\vdash$	$\forall I, x(x \in \bigcup_{i \in I} T_i \leftrightarrow \exists i(i \in I \wedge x \in T_i))$	(DEF $\bigcup$)
2.	; 2	$\vdash$	$\forall M, N(M \subset N \leftrightarrow \forall x(x \in M \rightarrow x \in N))$	(DEF $\subset$)
3.	; 3	$\vdash$	$\forall i(i \in I \rightarrow T_i \subset F)$	(ASS)
			the proof	
4.	4;	$\vdash$	$t \in \bigcup_i T_i$	(HYP)
5.	4; 1	$\vdash$	$t \in \bigcup_i T_i \leftrightarrow \exists i(i \in I \wedge t \in T_i)$	($\forall D$ 1)
6.	4; 1	$\vdash$	$t \in \bigcup_i T_i \rightarrow \exists i(i \in I \wedge t \in T_i)$	($\leftrightarrow D$ 5)
7.	4; 1	$\vdash$	$\exists i(i \in I \wedge t \in T_i)$	($\rightarrow D$ 6)
8.	8;	$\vdash$	$i_0 \in I \wedge t \in T_{i_0}$	(HYP)
9.	8;	$\vdash$	$t \in T_{i_0}$	($\wedge D$)
10.	8;	$\vdash$	$i_0 \in I$	($\wedge D$)
11.	; 3	$\vdash$	$i_0 \in I \rightarrow T_{i_0} \subset F$	($\forall D$)
12.	8; 3	$\vdash$	$T_{i_0} \subset F$	($\rightarrow D$ 10 11)
13.	8; 1, 2, 3	$\vdash$	$t \in F$	($\forall D, \leftrightarrow D, \rightarrow D$ 9 12 3)
14.	4; 1, 2, 3	$\vdash$	$t \in F$	(Choice 7 8)
15.	; 1, 2, 3	$\vdash$	$t \in \bigcup_i T_i \rightarrow t \in F$	(DED 14)
16.	; 1, 2, 3	$\vdash$	$\forall x(x \in \bigcup_i T_i \rightarrow x \in F)$	($\forall I$)
17.	; 2, 3	$\vdash$	$\bigcup_i T_i \subset F$	($\leftrightarrow D, \rightarrow D$ 16 2)
Thm.	;	$\vdash$	$\bigcup_i T_i \subset F$	()

ND proof for theorem 6.3.1 part 2.

NNo	S;D		Formula	Reason
			relevant assumptions	
1.	; 1	$\vdash$	$\forall I, x(x \in \bigcup_{i \in I} T_i \leftrightarrow \exists i(i \in I \wedge x \in T_i))$	(DEF $\bigcup$)
2.	; 2	$\vdash$	$\forall M, N(M \subset N \leftrightarrow \forall x(x \in M \rightarrow x \in N))$	(DEF $\subset$)
3.	; 3	$\vdash$	$\forall i(i \in I \rightarrow T_i \subset S)$	(ASS)
			the proof	
4.	4;	$\vdash$	$t \in \bigcup_i T_i$	(HYP)
5.	4; 1	$\vdash$	$t \in \bigcup_i T_i \leftrightarrow \exists i(i \in I \wedge t \in T_i)$	($\forall D$ 1)
6.	4; 1	$\vdash$	$t \in \bigcup_i T_i \rightarrow \exists i(i \in I \wedge t \in T_i)$	($\leftrightarrow D$ 5)
7.	4; 1	$\vdash$	$\exists i(i \in I \wedge t \in T_i)$	($\rightarrow D$ 6)
8.	8;	$\vdash$	$i_0 \in I \wedge t \in T_{i_0}$	(HYP)
9.	8;	$\vdash$	$t \in T_{i_0}$	($\wedge D$)
10.	8;	$\vdash$	$i_0 \in I$	($\wedge D$)

11.	; 3	$\vdash$	$i_0 \in I \rightarrow T_{i_0} \subset S$	$(\forall D)$
12.	8; 3	$\vdash$	$T_{i_0} \subset S$	$(\rightarrow D\ 10\ \ 11)$
13.	8; 2, 3	$\vdash$	$t \in S$	$(\forall D, \leftrightarrow D, \rightarrow D\ 9\ \ 12\ \ 3)$
14.	4; 1, 2, 3	$\vdash$	$t \in S$	(Choice 7 8)
15.	; 1, 2, 3	$\vdash$	$t \in \bigcup_i T_i \rightarrow t \in S$	(DED 14)
16.	; 1, 2, 3	$\vdash$	$\forall x(x \in \bigcup_i T_i \rightarrow x \in S)$	$(\forall I)$
17.	; 1, 2, 3	$\vdash$	$\bigcup_i T_i \subset S$	$(\leftrightarrow D, \rightarrow D\ 16\ \ 2)$
Thm.	;	$\vdash$	$\bigcup_i T_i \subset S$	()

Discussion

Conjunctive goal splitting yields the same proof structure for theorem 3.3.1 and theorem 6.3.1. Expanding the respective definitions then yields the same subtheorems and relevant assumptions for parts 1 and 3, respectively. For these parts the proofs are equal as well, due to the commonality of the subtheorems and of the *relevant* assumptions.

For part 2 the subtheorems in 3.3.1 and 6.3.1 are not equal right away, however the difference of the subtheorems 3.3.1 and 6.3.1 can be removed by *replacing* the constant F by the constant S.

The symbol mapping $F \Rightarrow S$ applied to theorem 3.3 in order to obtain theorem 6.3 can be extended by matching the assumptions as well. After unfolding the definitions, the proof of theorem 3.3 contains only parts of the definitions 3.1 and 6.2 that correspond directly. These are called assumptions *relevant in the base proof*. The proof does not use those parts of the definition that actually differ, such as an ideal being contained in a semigroup and a subsemimodul being contained in a semimodul.

Since the symbol mapping $\{F \Rightarrow S\}$ is applied to the second subproblem of 3.3.1 only, the mapping is consistent (one symbol is mapped to one symbol).

The mapped versions of all assumptions *relevant in the base proof* all occur in the knowledge base or in the assumptions of the target problem 6.3, and this serves as a strong justification for this analogy formation.

This example can be dealt with by standard techniques known from the literature on theorem proving by analogy, provided that means for structuring proofs and isolating relevant assumptions are present.

CASE 2: THEOREM 17.6 and ITS ANALOGUE

Theorem 17.6 Let $E \subset F$, then
1. π_E is a leftcongruence in the semigroup F,
2. π_E is compatible with E,
3. For all leftcongruences ρ in F, which are compatible with ρ, we have $\rho \subset \pi_E$, where π_E is defined in definition 17.5 (see below).

Theorem analogue Let $E \subset F$, then
1. $\breve{\pi}_E$ is a rightcongruence in the semigroup F,
2. $\breve{\pi}_E$ is compatible with E,
3. For all rightcongruences ρ in F, which are compatible with ρ, we have $\rho \subset \breve{\pi}_E$, where $\breve{\pi}_E$ is defined in a definition analogous to 17.5 (see below).

The following definitions are relevant assumptions:

Definiton 17.4
Let ρ be an equivalence relation on F and $E \subset F$. ρ is called *compatible with* E iff for all $f \in F$ with $\Omega(f) \cap E \neq \emptyset$ holds $\Omega(f) \subset E$, where $\Omega(x) = \{y : (y, x) \in \rho\}$.

Definition 5.1
Let ρ be an equivalence relation on a semigroup F. Then
ρ is called a *leftcongruence* iff for all $g, f_1, f_2 \in F$ holds
if $\rho(f_1, f_2)$ then $\rho(gf_1, gf_2)$.
ρ is called a *rightcongruence* iff for all $g, f_1, f_2 \in F$ holds
if $\rho(f_1, f_2)$ then $\rho(f_1 g, f_2 g)$.

The particular leftcongruence π_E in F is defined for $E \subset F$ as.

Definition 17.5.
$(f, g) \in \pi_E \leftrightarrow ((f \in E \leftrightarrow g \in E) \rightarrow \forall h(h \in F \rightarrow (hf \in E \leftrightarrow hg \in E)))$.

The particular rightcongruence $\breve{\pi}_E$ in F is defined for $E \subset F$ by the following analogous definition.

Analogue to definition 17.5.
$(f, g) \in \breve{\pi}_E \leftrightarrow ((f \in E \leftrightarrow g \in E) \rightarrow \forall h(h \in F \rightarrow (fh \in E \leftrightarrow gh \in E)))$.

The **proof structure of 17.6** is:

- Part 1:
  Theorem: π_E is a leftcongruence in F.
  Relevant assumptions: the definition of a leftcongruence, the definition of a semigroup, the definition 17.5.

- Part 2:
  Theorem: π_E is compatible with E.
  Relevant assumptions: the definition of $\cup$; the definition of $\subset$; definition 17.4; and definition 17.5.

- Part 3:
  Theorem: $leftcongruence(\rho) \wedge compatible(\rho, E) \rightarrow \rho \subset \pi_E$
  Relevant assumptions: the definition 17.4; the definition of leftcongruence; and the definition of $\subset$.

The **proof structure of the analogue of 17.6** is:

- Part 1:
  Theorem: $\breve{\pi}_E$ is a rightcongruence in F
  Relevant assumptions: the definition of a rightcongruence; the definition of a semigroup; the analogue of definition 17.5.

- Part 2:
  Theorem: $\breve{\pi}_E$ is compatible with E.
  Relevant assumptions: the definition of $\cup$; the definition of $\subset$; the analogue of definition 17.4; and the analogue of definition 17.5.

- Part 3:
  Theorem: $rightcongruence(\rho) \wedge compatible(\rho, E) \rightarrow \rho \subset \breve{\pi}_E$.
  Relevant assumptions: the analogue of definition 17.4; the definition of rightcongruence; and the definition of $\subset$.

Discussion

The symbol mapping $\{leftcongruence \Rightarrow rightcongruence, \pi_E \Rightarrow \breve{\pi}_E\}$ makes the subproblems of 17.6 and those of its analogous theorem equal but in this case the corresponding proofs still differ. This is due to the use of the different definitions of *leftcongruence* and *rightcongruence* within the proofs which belong to the relevant assumptions.

The definition 17.5 can be transformed to the analogous one by *term mapping* (i.e. not just symbol mapping, as in the previous example). There are two possibilities for the term mapping that transform the assumptions of 17.6 into the assumptions of its analogue:

- the concrete term mapping: $hf \Rightarrow fh; hg \Rightarrow gh; kf \Rightarrow fk; kg \Rightarrow gk; hkf \Rightarrow fhk; hkg \Rightarrow ghk$ for constants and variables h, f, g, k or,
- the term mapping based on the schema $term_1 \cdot term_2 \Rightarrow term_2 \cdot term_1$ which could be used as well.

The occurrence of the mapped versions of all relevant assumptions of the base proof in the knowledge base or in the assumptions of the analogue of problem 17.6. serves as a justification for this analogy formation.

This example could also be treated by techniques known from the literature, provided that means for isolating relevant assumptions are used in addition.

CASE 3: TWO ANALOGOUS PARTS OF 17.6.3

This example demonstrates a kind of analogy which is used very often in mathematics. It is shown on the third part of theorem 17.6 of HUA.

Theorem 17.6.3 Let $E \subset F$ and let ρ be a leftcongruence in F which is compatible with E, then $\rho \subset \pi_E$.

The definitions of $compatible(\rho, E), leftcongruence(\rho), \pi_E$, and $\Omega(x)$ are relevant assumptions. They have been given in the previous paragraph.

The problem is $\Delta \vdash \rho \subset \pi_E$ with
$\Delta = \{leftcongruence(\rho), compatible(\rho, E), (E \subset F), semigroup(F)\}$.

Some preparatory steps, usually not expressed explicitly by mathematicians, are necessary for the full ND-proof:

1. **Expanding the definition of** $\subset$ yields the problem
$\Delta \vdash \forall x, y((x, y) \in \rho \rightarrow (x, y) \in \pi_E)$.

2. **Expanding the definition of** π_E yields the problem
$\Delta \vdash \forall x, y((x, y) \in \rho \rightarrow (x \in E \leftrightarrow y \in E) \wedge \forall f(f \in F \rightarrow fx \in E \leftrightarrow fy \in E))$.

3. **Two applications of the Deduction Theorem** yield the problem
$\Delta \cup \{(x_0, y_0) \in \rho, (x_0 \in E \leftrightarrow y_0 \in E)\} \vdash \forall f(f \in F \rightarrow fx_0 \in E \leftrightarrow fy_0 \in E)$.

4. **Restructuring (splitting)** yields the subproblems

   - $\Delta \cup \{(x_0, y_0) \in \rho, (x_0 \in E \leftrightarrow y_0 \in E)\} \vdash \forall f(f \in F \rightarrow fx_0 \in E \rightarrow fy_0 \in E)$
   - $\Delta \cup \{(x_0, y_0) \in \rho, (x_0 \in E \leftrightarrow y_0 \in E)\} \vdash \forall f(f \in F \rightarrow fy_0 \in E \rightarrow fx_0 \in E)$.

5. **Application of the Deduction Theorem** yields the subproblems

   (a) $\Delta \cup \{(x_0, y_0) \in \rho, (x_0 \in E \leftrightarrow y_0 \in E)\} \cup \{f_0 \in F, f_0 x_0 \in E\} \vdash f_0 y_0 \in E$

   (b) $\Delta \cup \{(x_0, y_0) \in \rho, (x_0 \in E \leftrightarrow y_0 \in E)\} \cup \{f_0 \in F, f_0 y_0 \in E\} \vdash f_0 x_0 \in E$.

Thus we have obtained the subproblems (a) and (b) which are supposed to be proved *analogously* in HUA. We present the two ND-proofs in the following and discuss the respective transformation afterwards.

ND Proof of theorem 17.6.3 part (a)

NNo	S;D		Formula	Reason
			assumptions	
1.	; 1	$\vdash$	$\forall R, x, y \forall f(leftcongr(R) \leftrightarrow ((x, y) \in R \wedge f \in F \rightarrow (fx, fy) \in R))$	(DEF)
2.	; 2	$\vdash$	$leftcongr(\rho)$	(ASS)
3.	; 3	$\vdash$	$\forall x, y(x \in \Omega(y) \leftrightarrow (x, y) \in \rho)$	(DEF)
4.	; 4	$\vdash$	$\forall M, N : set, \forall x(x \in (M \cap N) \leftrightarrow x \in M \wedge x \in N)$	(DEF)
5.	; 5	$\vdash$	$\forall M : set(nonempty(M) \leftrightarrow \exists x(x \in M))$	(DEF)
6.	; 6	$\vdash$	$\forall \rho E(compatible(\rho, E) \leftrightarrow \forall x(nonempty(\Omega(x) \cap E) \rightarrow \Omega(x) \subset E))$	(DEF)
7.	; 7	$\vdash$	$compatible(\rho, E)$	(ASS)
8.	; 8	$\vdash$	$\forall M, N : set \forall x(M \subset N \rightarrow (x \in M \rightarrow x \in N))$	(DEF)
9.	; 9	$\vdash$	$\forall R \forall x, y, z(equivrel R \leftrightarrow (x, x) \in R \wedge ((x, y) \in R \rightarrow (y, x) \in R) \ldots)$	(DEF equiv-rel)
10.	; 10	$\vdash$	$equivrel(\rho)$	(ASS)
11.	; 11	$\vdash$	$(x_0, y_0) \in \rho$	(ASS)
12.	; 12	$\vdash$	$f_0 \in F$	(ASS)
13.	; 13	$\vdash$	$f_0 x_0 \in E$	(ASS)
			proof	
14.	; 9, 10	$\vdash$	$\forall x, y, z((x, x) \in \rho \wedge ((x, y) \in \rho \rightarrow (y, x) \in \rho) \ldots)$	(∀D,↔D 9)
15.	; 9, 10	$\vdash$	$\forall x((x, x) \in \rho)$	(∧D 14)
16.	; 1, 2, 12	$\vdash$	$(x_0, y_0) \in \rho \rightarrow (f_0 x_0, f_0 x_0) \in \rho$	(∀D,↔D,→D 1 2 12)
17.	; 1, 2, 11, 12	$\vdash$	$(f_0 x_0, f_0 y_0) \in \rho$	($\rightarrow D$ 11 16)
18.	; 3	$\vdash$	$(f_0 x_0, f_0 y_0) \in \rho \rightarrow f_0 x_0 \in \Omega(f_0 y_0)$	(∀D,↔D 3)
19.	; 1, 2, 3, 11, 12	$\vdash$	$f_0 x_0 \in \Omega(f_0 y_0)$	($\rightarrow D$ 17 18)
20.	; 1, 2, 3, 11, 12, 13	$\vdash$	$f_0 x_0 \in \Omega(f_0 y_0) \wedge f_0 x_0 \in E$	(∧I 19 13)
21.	; 13, 1, 2, 3, 4, 11, 12	$\vdash$	$f_0 x_0 \in (\Omega(f_0 y_0) \cap E)$	(∀D,↔D,→D 4 20)
22.	; 13, 1, 2, 3, 4, 11, 12	$\vdash$	$\exists x(x \in (\Omega(f_0 y_0) \cap E))$	($\exists I$ 21)
23.	; 5, 13, 1, 2, 3, 4, 11, 12	$\vdash$	$nonempty(\Omega(f_0 y_0) \cap E)$	(∀D, ↔D,→D 5 22)
24.	; 6, 7	$\vdash$	$\forall x(nonempty(\Omega(x) \cap E) \rightarrow \Omega(x) \subset E)$	(↔D,→D 6 7)
25.	; 5, 6, 7, 13, 1, 2, 3, 4, 11, 12	$\vdash$	$\Omega(f_0 y_0) \subset E$	(∀D→D 23 24)

26.	; 8, 5, 6, 7, 13, 1, 2, 3, 4, 11, 12	$\vdash$	$f_0 y_0 \in \Omega(f_0 y_0) \rightarrow f_0 y_0 \in E$	(∀D,→D 8 25)
27.	; 9, 10	$\vdash$	$(f_0 y_0, f_0 y_0) \in \rho$	(∀D 15)
28.	; 3, 9, 10	$\vdash$	$f_0 y_0 \in \Omega(f_0 y_0)$	(∀D,↔ D,→D 3 27)
29.	; 3, 9, 10, 8, 5, 6, 7, 13, 1, 2, 3, 4, 11, 12	$\vdash$	$f_0 y_0 \in E$	($\rightarrow D$ 28 26)
Thm.	;	$\vdash$	$f_0 y_0 \in E$	()

ND Proof of theorem 17.6.3 part (b)

NNo	S;D		Formula	Reason
			assumptions	
1.	; 1	$\vdash$	$\forall R, x, y \forall f(leftcongr(R) \leftrightarrow ((x, y) \in R \wedge f \in F \rightarrow (fx, fy) \in R))$	(DEF)
2.	; 2	$\vdash$	$leftcongr(\rho)$	(ASS)
3.	; 3	$\vdash$	$\forall x, y(x \in \Omega(y) \leftrightarrow (x, y) \in \rho)$	(DEF)
4.	; 4	$\vdash$	$\forall M, N : set, \forall x(x \in (M \cap N) \leftrightarrow x \in M \wedge x \in N)$	(DEF)
5.	; 5	$\vdash$	$\forall M : set(nonempty(M) \leftrightarrow \exists x(x \in M))$	(DEF)
6.	; 6	$\vdash$	$\forall \rho E(compatible(\rho, E) \leftrightarrow \forall x(nonempty(\Omega(x) \cap E) \rightarrow \Omega(x) \subset E))$	(DEF)
7.	; 7	$\vdash$	$compatible(\rho, E)$	(ASS)
8.	; 8	$\vdash$	$\forall M, N : set \forall x(M \subset N \rightarrow (x \in M \rightarrow x \in N))$	(DEF)
9.	; 9	$\vdash$	$\forall R \forall x, y, z(congruence R \leftrightarrow (x, x) \in R \wedge ((x, y) \in R \rightarrow (y, x) \in R) \ldots)$	(DEF equiv-rel)
10.	; 10	$\vdash$	$congruence(\rho)$	(ASS)
11.	; 11	$\vdash$	$(x_0, y_0) \in \rho$	(ASS)
12.	; 12	$\vdash$	$f_0 \in F$	(ASS)
13.	; 13	$\vdash$	$f_0 y_0 \in E$	(ASS)
			proof	
14.	; 9, 10	$\vdash$	$\forall x, y, z((x, x) \in \rho \wedge ((x, y) \in \rho \rightarrow (y, x) \in \rho) \ldots)$	($\forall$D,$\leftrightarrow$D 9)
15.	; 9, 10	$\vdash$	$\forall x((x, x) \in \rho)$	($\wedge$D 14)
16.	; 9, 10	$\vdash$	$(y_0, x_0) \in \rho \rightarrow (x_0, y_0) \in \rho$	($\forall$D,$\leftrightarrow$D,$\wedge$D 9 10)
17.	; 9, 11	$\vdash$	$(y_0, x_0) \in \rho$	($\rightarrow D$ 11 16)
18.	; 1, 2, 12	$\vdash$	$(y_0, x_0) \in \rho \rightarrow (f_0 y_0, f_0 x_0) \in \rho$	($\forall$D,$\leftrightarrow$D,$\rightarrow$D 1 2 12)
19.	; 1, 2, 9, 11, 12	$\vdash$	$(f_0 y_0, f_0 x_0) \in \rho$	($\rightarrow D$ 17 18)
20.	; 3	$\vdash$	$(f_0 y_0, f_0 x_0) \in \rho \rightarrow f_0 y_0 \in \Omega(f_0 x_0)$	($\forall$D,$\leftrightarrow$D 3)
21.	; 1, 2, 3, 9, 11, 12	$\vdash$	$f_0 y_0 \in \Omega(f_0 x_0)$	($\rightarrow D$ 19 20)
22.	; 1, 2, 3, 9, 11, 12, 13	$\vdash$	$f_0 y_0 \in \Omega(f_0 x_0) \wedge f_0 y_0 \in E$	($\wedge$I 21 13)
23.	; 13, 1, 2, 3, 4, 9, 11, 12	$\vdash$	$f_0 y_0 \in (\Omega(f_0 x_0) \cap E)$	($\forall$D,$\leftrightarrow$D,$\rightarrow$D 4 22)
24.	; 13, 1, 2, 3, 4, 9, 11, 12	$\vdash$	$\exists x(x \in (\Omega(f_0 x_0) \cap E))$	($\exists I$ 23)
25.	; 5, 13, 1, 2, 3, 4, 9, 11, 12	$\vdash$	$nonempty(\Omega(f_0 x_0) \cap E)$	($\forall$D,$\leftrightarrow$D,$\rightarrow$D 5 24)
26.	; 6, 7	$\vdash$	$\forall x(nonempty(\Omega(x) \cap E) \rightarrow \Omega(x) \subset E)$	($\leftrightarrow$D,$\rightarrow$D 6 7)

27.	; 5, 6, 7, 13, 1, 2, 3, 4, 9, 11, 12	$\vdash$	$\Omega(f_0x_0) \subset E$	($\forall$D$\rightarrow$D 25 26)
28.	; 8, 5, 6, 7, 13, 1, 2, 3, 4, 9, 11, 12	$\vdash$	$f_0x_0 \in \Omega(f_0x_0) \rightarrow f_0x_0 \in E$	($\forall$D$\rightarrow$D 8 27)
29.	; 9, 10	$\vdash$	$(f_0x_0, f_0x_0) \in \rho$	($\forall D$ 15)
30.	; 3, 9, 10	$\vdash$	$f_0x_0 \in \Omega(f_0x_0)$	($\forall$D,$\leftrightarrow$ D,$\rightarrow$D 3 29)
31.	; 9, 10, 8, 5, 6, 7, 13, 1, 2, 3, 4, 11, 12	$\vdash$	$f_0x_0 \in E$	($\rightarrow D$ 30 28)
Thm.	;	$\vdash$	$f_0x_0 \in E$	()

Discussion

An attempt to translate the first subproof to the second subproof by the symbol mapping $\{x_0 \Rightarrow y_0, y_0 \Rightarrow x_0\}$ fails, since the relevant assumptions differ in $((x_0, y_0) \in \rho)$ and $((y_0, x_0) \in \rho)$ after this mapping, respectively.

In order to obtain equal assumptions, $((x_0, y_0) \in \rho)$ is to be replaced by $((y_0, x_0) \in \rho)$ within the assumptions. $((y_0, x_0) \in \rho)$ becomes a new subtheorem, which is proven by the subproof of (b) that consists of the lines 15 and 16.

CASE 4: THEOREMS 5.7 and 7.5.7

Let us look at the analogy that provides a proof of theorem 5.7 that is based on the proof of theorem 7.5.7[1] by examining the proofs of theorem 7.5.7 and theorem 5.7: A stronger reformulation technique works for these examples, namely, *abstraction* based on the definition of a homomorphism.

Theorem 7.5.7 Let S, T_1, T_2 be F-semimoduls. Let $\phi_1 : S \mapsto T_1, \phi_2 : S \mapsto T_2$ be
two homomorphisms into the F-semimoduls T_1 and T_2, respectively, and let ρ_1, ρ_2 be the respectively induced leftcongruences.

1. If there exists a homomorphism $\Phi : T_1 \mapsto T_2$ with $\Phi\phi_1 = \phi_2$, then $\rho_1 \subset \rho_2$.
2. Let $\rho_1 \subset \rho_2$ and if ϕ_1 is surjective, then there is a unique homomorphism $\Phi : H_1 \mapsto H_2$ with $\Phi\phi_1 = \phi_2$. If in addition, ϕ_2 is surjective, then Φ is surjective as well.

Theorem 5.7 Let S', H_1, H_2 be semigroups. Let $\phi_1 : S' \mapsto H_1, \phi_2 : S' \mapsto H_2$ be two homomorphisms into the semigroups H_1 and H_2, respectively, and let ρ_1, ρ_2 be the respectively induced congruences.

1. If there exists a homomorphism $\Phi : H_1 \mapsto H_2$ with $\Phi\phi_1 = \phi_2$, then $\rho_1 \subset \rho_2$.
2. Let $\rho_1 \subset \rho_2$ and if ϕ_1 is surjective, then there is a unique homomorphism $\Phi : H_1 \mapsto H_2$ with $\Phi\phi_1 = \phi_2$. If in addition, ϕ_2 is surjective, then Φ is surjective as well.

The proofs are based on the definitions of a homomorphism in semigroups and a homomorphism in F-semimoduls, respectively:

Definition 2.1 Let F and H be semigroups. A mapping $\phi : F \mapsto H$ is called a homomorphism (from F to H) iff
$\forall f, g(f, g \in F \rightarrow \phi(f \cdot g) = \phi(f) \cdot \phi(g)$.

Definition 7.1 Let S and T be F-semimoduls. A mapping $\phi : F \mapsto H$

[1] This analogy is harder to find than the transformation of the proof of 5.7 to a proof of 7.5.7

is called a homomorphism (from S to T) iff
$\forall f, s(f \in F \wedge s \in S \rightarrow \phi(f \cdot s) = \phi(f) \cdot \phi(s)$.

The proofs of theorem 7.5.7 and of theorem 5.7 can now be structured as follows.

The proof structure of 7.5.7 becomes:

- Part 1:
  Theorem: $\rho_1 \subset \rho_2$
  Relevant assumptions: the definition of ρ_1 and of ρ_2; existence of a mapping Φ with $\Phi\phi_1 = \phi_2$.

- Part 2a:
  Theorem: There exists a function
  Φ with $(\forall z(z \in S \rightarrow \Phi\phi_1(z) = \phi_2(z)) \wedge \forall x \exists y((x \in T_1 \rightarrow y \in T_2) \wedge \Phi(x) = y)$.
  Relevant assumptions: ϕ_2 is a mapping $S \mapsto T_2$; ϕ_1 is a mapping $S \mapsto T_1$; ϕ_1 is surjective; the comprehension axiom; $=$ is an equivalence relation; the definition of ρ_1 and $\rho_2; \rho_1 \subset \rho_2$; the representation of functions as relations.

- Part 2b:
  Theorem: Φ is the only mapping for which the theorem of 2a holds, i.e.,
  $\forall \Phi' \forall x, y((x \in S \rightarrow \Phi'(\phi_1(x)) = \phi_2(x)) \rightarrow (y \in T_1 \rightarrow \Phi(y) = \Phi'(y)))$
  Relevant assumptions: the definition of $\Phi : \Phi(\phi_1) = \phi_2$; surjectivity of ϕ_1; transitivity of $=$.

- Part 2c:
  Theorem: Φ is a F-semimodul-homomorphism, i.e.,
  $\forall f, x(x \in T_1 \wedge f \in F \rightarrow \Phi(f \cdot x) = f \cdot \Phi(x))$.
  Relevant assumptions: surjective ϕ_1; ϕ_1 is a homomorphism in an F-semimodul; ϕ_2 is a homomorphism in an F-semimodul; the definition of Φ; theorem of 2a.

- Part 2d:
  Theorem: If ϕ_2 is surjective then Φ is surjective.

The proof structure of 5.7 becomes:

- Part 1:
  Theorem: $\rho_1 \subset \rho_2$
  Relevant assumptions: the definition of ρ_1 and of ρ_2; the existence of a mapping Φ with $\Phi\phi_1 = \phi_2$.

- Part 2a:
  Theorem: There exists a function Φ with
  $\forall z(z \in S' \to \Phi\phi_1(z) = \phi_2(z)) \wedge \forall x \exists y((x \in H_1 \to y \in H_2) \wedge \Phi(x) = y)$.
  Relevant assumptions: $\phi_1 : F \mapsto H_1$ is a mapping from a semigroup into a semigroup; $\phi_2 : F \mapsto H_2$ is a mapping from a semigroup into a semigroup $F \Rightarrow H_2$; ϕ_1 is surjective; the comprehension axiom; $=$ is an equivalence relation; the definitions of ρ_1 and ρ_2; $\rho_1 \subset \rho_2$; the representation of functions as relations.

- Part 2b:
  Theorem: Φ is the only mapping for which the theorem of 2a holds, i.e.,
  $\forall\Phi'\forall x, y((x \in S' \to \Phi'(\phi_1(x)) = \phi_2(x)) \to (y \in H_1 \to \Phi(y) = \Phi'(y)))$
  Relevant assumptions: the definition of $\Phi : \Phi(\phi_1) = \phi_2$; ϕ_1 is surjective; the transitivity of $=$.

- Part 2c:
  Theorem: Φ is a semigroup-homomorphism, i.e.,
  $\forall x, y(x \in H_1 \wedge y \in H_1 \to \Phi(x \cdot y) = \Phi(x) \cdot \Phi(y))$.
  Relevant assumptions: ϕ_1 is surjective; ϕ_1 is a semigroup-homomorphism; ϕ_2 is a semigroup-homomorphism; the definition of Φ; the theorem of 2a.

- Part 2d:
  If ϕ_2 is surjective then Φ is surjective.

Note that, the parts 1, 2a, 2b, 2d of theorem 7.5.7 are equal to the corresponding parts of theorem 4.11 of HUA as well, and in general play an important rôle.

The crucial point for the transformation of the proof of theorem 7.5.7.1 to the proof of theorem 5.7.1 are the relevant assumptions of the respective part 1 of theorem 5.7 and of theorem 7.5.7, which differ in symbols only. Hence, they become equal by the symbol mapping {F $\Rightarrow$ S , $H_1 \Rightarrow T_1$, and $H_2 \Rightarrow T_2$}. This symbol mapping is to be applied to the whole proof of 5.7.1.

The proofs of parts 2c of theorem 7.5.7 and theorem 5.7 are given next. For simplicity, let $\cdot$ be a polymorphic function.

ND Proof of theorem 7.5.7 part 2c

NNo	S;D		Formula	Reason
			relevant assumptions	
1.	; 1	$\vdash$	$\forall x, y, f(x \in T_2 \wedge y \in T_2 \wedge f \in F \wedge x = y \rightarrow f \cdot x = f \cdot y)$	(T_2 is semi-modul)
2.	; 2	$\vdash$	$\forall x, f(f \in F \wedge x \in S \rightarrow (f \cdot x) \in S)$	(S is semi-modul)
3.	; 3	$\vdash$	$\forall x, y, f(x \in T_1 \wedge y \in T_1 \wedge f \in F \wedge x = y \rightarrow f \cdot x = f \cdot y)$	(Ax =T_1-is-semimodul)
4.	; 4	$\vdash$	$hom\_from\_S(\phi) \leftrightarrow \forall f \forall x(f \in F \wedge x \in S \rightarrow \phi(f \cdot x) = f \cdot \phi(x))$	(DEF hom_from_S ϕ)
5.	; 5	$\vdash$	$hom\_from\_T_1(\Phi) \leftrightarrow \forall f \forall x(f \in F \wedge x \in T_1 \rightarrow \Phi(f \cdot x) = f \cdot \Phi(x))$	(hom_from_T_1)
6.	; 6	$\vdash$	$\forall x, f(f \in F \wedge x \in T_1 \rightarrow f \cdot x \in T_1)$	(T_1-is-semimodul)
7.	; 7	$\vdash$	$\forall x, y, z(x = y \wedge y = z \rightarrow x = z)$	(Ax =transitive)
8.	; 8	$\vdash$	$\forall x(x \in S \rightarrow \phi_2(x) \in T_2)$	(Def ϕ_2)
9.	; 9	$\vdash$	$\forall x(x \in T_1 \rightarrow \Phi(x) \in T_2)$	(lemma 2a)
10.	; 10	$\vdash$	$\forall x(x \in S \rightarrow \phi_1(x) \in T_1)$	(Def ϕ_1)
11.	; 11	$\vdash$	$\forall x, y(x \in T_1 \wedge y \in T_1 \wedge x = y \rightarrow \Phi(x) = \Phi(y))$	(lemma 2a)
12.	; 12	$\vdash$	$\forall x(x \in S \rightarrow \Phi(\phi_1(x)) = \phi_2(x))$	(lemma 2a)
13.	; 13	$\vdash$	$\forall y(y \in T_1 \rightarrow \exists x(x \in S \wedge \phi_1(x) = y))$	(ASS surjective ϕ_1)
14.	; 14	$\vdash$	$hom\_from\_S(\phi_1)$	(ASS)
15.	; 15	$\vdash$	$hom\_from\_S(\phi_2)$	(ASS)
			The Proof	
16.	; 4, 14	$\vdash$	$\forall f \forall x(f \in F \wedge x \in S \rightarrow \phi_1(f \cdot x) = f \cdot \phi_1(x))$	($\leftrightarrow D, \rightarrow D$ 4 14)
17.	; 4, 15	$\vdash$	$\forall f \forall x(f \in F \wedge x \in S \rightarrow \phi_2(f \cdot x) = f \cdot \phi_2(x))$	($\leftrightarrow D, \rightarrow D$ 4 15)
18.	18;	$\vdash$	$f \in F$	(HYP)
19.	19;	$\vdash$	$x_0 \in T_1$	(HYP)
			(*)	
20.	19; 13	$\vdash$	$\exists y(y \in S \wedge \phi_1(y) = x_0)$	($\forall D, \rightarrow D$ 19 13)
21.	19; 13	$\vdash$	$a \in S \wedge \phi_1(a) = x_0$	($\exists D$ 20)
22.	19; 10, 13	$\vdash$	$a \in S \wedge \phi_1(a) = x_0 \wedge \phi_1(a) \in T_1$	($\forall D, \wedge D, \leftrightarrow D, \wedge I$ 10 21)
23.	19; 10, 13	$\vdash$	$\phi_1(a) \in T_1 \wedge \phi_1(a) = x_0$	($\wedge$ D 22)
24.	19, 18; 10, 13	$\vdash$	$x_0 \in T_1 \wedge \phi_1(a) \in T_1 \wedge f \in F \wedge \phi_1(a) = x_0$	($\wedge$ I 18 19 23)

25.	18, 19; 10, 13, 3	⊢	$f \cdot \phi_1(a) = f \cdot x_0$	$(\forall D, \to D$ 3 24)
26.	19, 18; 10, 13, 3	⊢	$a \in S \wedge \phi_1(a) = x_0 \wedge \phi_1(a) \in T_1 \wedge f \cdot \phi_1(a) = f \cdot x_0$	$(\wedge$ I 25 22)
			(**)	
27.	19, 18; 10, 3, 4, 11, 6	⊢	$\Phi(f \cdot x_0) = \Phi(f \cdot \phi_1(a))$	$(\wedge D, \to D$ 11 6 26 19)
28.	18, 19; 10, 3, 4, 14, 11	⊢	$\Phi(f \cdot \phi_1(a)) = \Phi(\phi_1(f \cdot a))$	$(\forall D, \to D$ 11 26 16)
29.	19, 18; 10, 3, 4, 2, 12	⊢	$\Phi(\phi_1(f \cdot a)) = \phi_2(f \cdot a)$	$(\forall D, \to D$ 12 26 2)
30.	18, 19; 10, 3, 4, 15	⊢	$\phi_2(f \cdot a) = f \cdot \phi_2(a)$	$(\forall D, \to D$ 17 18 26)
31.	19, 18; 10, 3, 4, 12, 1, 8, 9	⊢	$f \cdot \phi_2(a) = f \cdot \Phi(\phi_1(a))$	$(\forall D$ 26 12 1 8 9)
32.	19, 18; 10, 3, 4, 11, 9	⊢	$f \cdot \Phi(\phi_1(a)) = f \cdot \Phi(x_0)$	$(\to D$ 1 26 11 9)
33.	19, 18; 10, 3, 4, 14, 15, 11, 7, 2, 8, 9, 6, 1	⊢	$\Phi(f \cdot x_0) = f \cdot \Phi(x_0)$	$(\to D$ 7 27 28 29 30 31 32)
34.	; 3, 4, 14, 4, 15, 11, 7, 2, 10, 8, 9, 6, 13, 12, 1	⊢	$f \in F \wedge x_0 \in T_1 \to \Phi(f \cdot x_0) = f \cdot \Phi(x_0))$	(DED 33)
35.	; 3, 4, 14, 15, 11, 7, 2, 10, 8, 9, 6, 13, 12, 1	⊢	$\forall f \forall x(f \in F \wedge x \in T_1 \to \Phi(f \cdot x) = f \cdot \Phi(x))$	$(\forall I$ 34)
36.	; 3, 4, 14, 15, 11, 7, 2, 10, 8, 9, 6, 13, 12, 1, 5	⊢	$hom\_from\_T_1(\Phi)$	$(\leftrightarrow D, \to D, \forall D$ 5 35)

ND Proof of theorem 5.7 part 2c

NNo m	S;D		Formula	Reason
			relevant assumptions	
1.	; 1	⊢	$\forall x_1, x_2, y_1, y_2(x_1, x_2, y_1, y_2 \in H_2 \wedge x_1 = x_2 \wedge y_1 = y_2 \to x_1 \cdot y_1 = x_2 \cdot y_2)$	(Ax semigroup H_2)
2.	; 2	⊢	$\forall x_1, x_2(x_1 \in S' \wedge x_2 \in S' \to x_1 \cdot x_2 \in S')$	(Ax S' semigroup)

3.	; 3	$\vdash$	$\forall x_1, x_2, y_1, y_2(x_1, x_2, y_1, y_2 \in H_1 \wedge x_1 = x_2 \wedge y_1 = y_2 \rightarrow x_1 \cdot y_1 = x_2 \cdot y_2)$	(Ax semigroupH_1)
4.	; 4	$\vdash$	$hom\_from\_S'(\phi) \leftrightarrow \forall x \forall y(x \in S' \wedge y \in S' \rightarrow \phi(x \cdot y) = \phi(x) \cdot \phi(y))$	(DEF hom_from_S' ϕ)
5.	; 5	$\vdash$	$hom\_from\_H_1(\Phi) \leftrightarrow \forall x \forall y(x \in H_1 \wedge y \in H_1 \rightarrow \Phi(x \cdot y) = \Phi(x) \cdot \Phi(y))$	(DEF hom_from_H_1)
6.	; 6	$\vdash$	$\forall x, y(x \in H_1 \wedge y \in H_1 \rightarrow x \cdot y \in H_1)$	(Ax semi-group H_1)
7.	; 7	$\vdash$	$\forall x, y, z(x = y \wedge y = z \rightarrow x = z)$	(Ax= transitive)
8.	; 8	$\vdash$	$\forall x(x \in S' \rightarrow \phi_2(x) \in H_2)$	(ASS Def ϕ_2)
9.	; 9	$\vdash$	$\forall x(x \in H_1 \rightarrow \Phi(x) \in H_2)$	(lemma 2a)
10.	; 10	$\vdash$	$\forall x(x \in S' \rightarrow \phi_1(x) \in H_1)$	(Def ϕ_1)
11.	; 11	$\vdash$	$\forall x, y(x \in H_1 \wedge y \in H_1 \wedge x = y \rightarrow \Phi(x) = \Phi(y))$	(lemma 2a)
12.	; 12	$\vdash$	$\forall x(x \in S' \rightarrow \Phi(\phi_1(x)) = \phi_2(x))$	(lemma 2a)
13.	; 13	$\vdash$	$\forall y(y \in H_1 \rightarrow \exists x(x \in S' \wedge \phi_1(x) = y))$	(ASS surj ϕ_1)
14.	; 14	$\vdash$	$hom\_in\_S'(\phi_1)$	(ASS)
15.	; 15	$\vdash$	$hom\_in\_S'(\phi_2)$	(ASS)
			The Proof	
16.	; 4, 14	$\vdash$	$\forall y \forall x(y \in S' \wedge x \in S' \rightarrow \phi_1(x \cdot y) = \phi_1(x) \cdot \phi_1(y)$	($\leftrightarrow D, \rightarrow D$ 4 14)
17.	; 4, 15	$\vdash$	$\forall y \forall x(y \in S' \wedge x \in S' \rightarrow \phi_2(x \cdot y) = \phi_2(x) \cdot \phi_2(y)$	($\leftrightarrow D, \rightarrow D$ 4 15)
18.	18;	$\vdash$	$x_{10} \in H_1$	(HYP)
19.	19;	$\vdash$	$x_{20} \in H_1$	(HYP)
			(*)	
20.	18; 13	$\vdash$	$\exists y(y \in S' \wedge \phi_1(y) = x_{10})$	($\rightarrow D, \forall D$ 18 13)
21.	19; 13	$\vdash$	$\exists z(z \in S' \wedge \phi_1(z) = x_{20})$	($\rightarrow D, \forall D$ 19 13)
22.	18; 13	$\vdash$	$a_1 \in S' \wedge \phi_1(a_1) = x_{10}$	($\exists D$ 20)
23.	19; 13	$\vdash$	$a_2 \in S' \wedge \phi_1(a_2) = x_{20}$	($\exists D$ 21)
24.	18; 10, 13	$\vdash$	$a_1 \in S' \wedge \phi_1(a_1) = x_{10} \wedge \phi_1(a_1) \in H_1$	($\wedge D, \forall D, \wedge I, \rightarrow D$ 22 10)
25.	19; 10, 13	$\vdash$	$a_2 \in S' \wedge \phi_1(a_2) = x_{20} \wedge \phi_1(a_2) \in H_1$	($\wedge D, \forall D, \wedge I, \rightarrow D$ 23 10)
26.	18; 10, 13	$\vdash$	$\phi_1(a_1) \in H_1 \wedge \phi_1(a_1) = x_{10}$	($\wedge$ D 24)
27.	19; 10, 13	$\vdash$	$\phi_1(a_2) \in H_1 \wedge \phi_1(a_2) = x_{20}$	($\wedge$ D 25)
28.	18; 10, 13	$\vdash$	$x_{10} \in H_1 \wedge \phi_1(a_1) \in H_1 \wedge \phi_1(a_1) = x_{10}$	($\wedge$ I 18 26)
29.	19; 10, 13	$\vdash$	$x_{20} \in H_1 \wedge \phi_1(a_2) \in H_1 \wedge \phi_1(a_2) = x_{20}$	($\wedge$ I 19 26)
30.	18, 19; 10, 13	$\vdash$	$x_{10} \in H_1 \wedge \phi_1(a_1) \in H_1 \wedge \phi_1(a_1) = x_{10} \wedge x_{20} \in H_1 \wedge \phi_1(a_2) \in H_1 \wedge \phi_1(a_2) = x_{20}$	($\wedge$ I 28 29)

31.	18, 19; 10, 13, 3	$\vdash$	$\phi_1(a_1) \cdot \phi_1(a_2) = x_{10} \cdot x_{20}$	$(\forall D, \rightarrow D$ 3 30)
32.	18, 19; 10, 13, 3	$\vdash$	$a_1 \in S' \wedge \phi_1(a_1) = x_{10} \wedge \phi_1(a_1) \in H_1 \wedge a_2 \in S' \wedge \phi_1(a_2) = x_{20} \wedge \phi_1(a_2) \in H_1 \wedge \phi_1(a_1) \cdot \phi_1(a_2) = x_{10} \cdot x_{20}$	$(\wedge$ I 31 24 25)

(**)

33.	18, 19; 10, 3, 13, 11, 6	$\vdash$	$\Phi(x_{10} \cdot x_{20}) = \Phi(\phi_1(a_1) \cdot \phi_1(a_2))$	$(\wedge D, \rightarrow D$ 11 32 18 19 6)
34.	18, 19; 10, 3, 13, 4, 14, 11, 6, 2	$\vdash$	$\Phi(\phi_1(a_1) \cdot \phi_1(a_2)) = \Phi(\phi_1(a_1 \cdot a_2))$	$(\wedge D, \forall D, \rightarrow D$ 11 32 16 6)
35.	18, 19; 10, 3, 13, 2, 12	$\vdash$	$\Phi(\phi_1(a_1 \cdot a_2)) = \phi_2(a_1 \cdot a_2)$	$(\wedge D, \forall D, \rightarrow D$ 32 12 2)
36.	18, 19; 10, 3, 13, 4, 15	$\vdash$	$\phi_2(a_1 \cdot a_2) = \phi_2(a_1) \cdot \phi_2(a_2)$	$(\wedge D, \forall D, \rightarrow D$ 17 32)
37.	18, 19; 10, 3, 13, 12, ??, 8, 9	$\vdash$	$\phi_2(a_1) \cdot \phi_2(a_2) = \Phi(\phi_1(a_1)) \cdot \Phi(\phi_1(a_2))$	$(\wedge D, \forall D$ 32 12 1 8 9)
38.	18, 19; 10, 3, 13, 11, 1, 9	$\vdash$	$\Phi(\phi_1(a_1)) \cdot \Phi(\phi_1(a_2)) = \Phi(x_{10}) \cdot \Phi(x_{20})$	$(\wedge D, \rightarrow D$ 1 32 11 9 10)
39.	18, 19; 10, 3, 13, 1, 4, 14, 15, 11, 7, 2, 8, 9, 12, 6	$\vdash$	$\Phi(x_{10} \cdot x_{20}) = \Phi(x_{10}) \cdot \Phi(x_{20})$	$(\wedge D, \rightarrow D$ 7 33 34 35 36 37 38)
40.	; 3, 13, 1, 4, 14, 15, 11, 7, 2, 8, 9, 12, 6, 10	$\vdash$	$x_{10} \in H_1 \wedge x_{20} \in H_1 \rightarrow \Phi(x_{10} \cdot x_{20}) = \Phi(x_{10}) \cdot \Phi(x_{20})$	(DED 39)

41.	; 3, 13, 1, 4, 14, 15, 11, 7, 2, 8, 9, 12, 6, 10	$\vdash$	$\forall x_1 \forall x_2 (x_1 \in H_1 \wedge x_2 \in H_1 \rightarrow \Phi(x_1 \cdot x_2) = \Phi(x_1) \cdot \Phi(x_2))$	$(\forall I$ 40)
42.	; 3, 13, 1, 4, 14, 15, 11, 7, 2, 8, 9, 12, 6, 5, 10	$\vdash$	$hom\_from\_H_1(\Phi)$	$(\leftrightarrow D, \rightarrow D, \forall D$ 5 41)

Discussion

The subtheorems 2c of 7.5.7 and 5.7 differ in more than one corresponding symbol. Thus symbol mapping is not sufficient to obtain equal theorems and assumptions of 5.7.2c and 7.5.7.2c.

Term mapping, e.g., $\{f \cdot term(x) \Rightarrow term(x) \cdot term(y)\}$ is not sufficient either, since part (*) would differ after the term mapping and a proof checker would not accept the transformation as a proof of 5.7.2c. Probably more importantly however, the theorem and the assumptions of 5.7.2c and 7.5.7.2c contain different subformulae of the form $x \in M$ and quantifiers which have to be modified by the mapping as well. This problem is due to fact that the mapping of terms is essentially an abstraction by which some irrelevant symbols disappear. The actual justification for this abstraction is the occurrence of the definition of a homomorphism within the relevant assumptions. An analogy based on pure term mapping is not justified at all and hence the abstracting reformulation has to be preferred. Another reason for this preference is that less modification is to be done after this analogy formation.

An reformulation of theorem 7.5.7 and its proof to theorem 5.7 and its proof consists of three steps.

1. Abstraction of both problems (i.e., theorem and assumptions) 7.5.7.2.c and 5.7.2.c based on the meaning of the two respective definitions of homomorphism.
   The key is a reformulation of terms of the form $f \cdot \texttt{term}(x)$ to terms $Op(\texttt{term}(x))$ for 7.5.7.2c and $\texttt{term1}\cdot\texttt{term2}$ to $Op(\texttt{term1},\texttt{term2})$ with a function variable Op. This reformulation affects the definitions of homomorphism within the relevant assumptions:
   $\forall f \forall x (f \in F \wedge x \in S \rightarrow \phi(f \cdot x) = f \cdot \phi(x))$ becomes
   $\forall x (x \in S \rightarrow \phi(Op(x)) = Op(\phi(x)))$
   by the mapping $\texttt{f} \cdot \texttt{term} \Rightarrow \text{Op}(\texttt{term})$

   $\forall x, y (x \in S' \wedge y \in S' \rightarrow \phi(x \cdot y) = \phi(x) \cdot \phi(y))$ becomes
   $\forall x, y (x \in S' \wedge y \in S' \rightarrow \phi(Op'(x, y)) = Op'(\phi(x), \phi(y)))$
   by the mapping $\texttt{term1}\cdot\texttt{term2} \Rightarrow \text{Op}(\texttt{term1},\texttt{term2})$.
   The reformulation affects also the corresponding terms within the whole proof. Certain subformulae and quantifiers become superfluous and, hence, can be omitted. As a result we obtain the theorems and reformulated proofs 7.5.7.2c′ and 5.7.2.c′.

2. The problems 7.5.7.2.c′ and 5.7.2.c′ are not equal yet. Their comparison suggests another reformulation of 7.5.7.2.c′ to 7.5.7.2.c″ in order to obtain equal abstracted assumptions and theorems, which increases

the number of arguments of Op in 7.5.7.2.c.$'$. This reformulation causes several additional changes within the reformulated proof.

3. Finally, to return to the original theorem and assumptions of 5.7.2.c, a *reversion* of the abstraction of 5.7.2.c has to be applied to 7.5.7.2.c$''$.

All these reformulations have to be applied to the whole proofs and not only to the assumptions and the theorem.

CASE 5: THEOREMS 4.8 and 5.3

Theorem 4.8 Let ρ and σ be two equivalence relations, then
1. $(\rho \cap \sigma)$ is an equivalence relation and
2. $(\rho \cup \sigma)^t$ is the smallest equivalence relation, containing ρ and σ.

Theorem 5.3 Let ρ and σ be two leftcongruences, then
1. $(\rho \cap \sigma)$ is a leftcongruence and
2. $(\rho \cup \sigma)^t$ is the smallest leftcongruence containing ρ and σ.

The proofs of theorem 4.8 and theorem 5.3 can be structured as follows.
The proof structure for theorem 4.8 becomes:

- Part 1:
  1. *Theorem*: $(\rho \cap \sigma)$ is an equivalence relation.
  1.1. Subtheorem: reflexivity of $(\rho \cap \sigma)$,
  1.2. Subtheorem: symmetry of $(\rho \cap \sigma)$,
  1.3. Subtheorem: transitivity of $(\rho \cap \sigma)$.
- Part 2:
  2. *Theorem*: $(\rho \cup \sigma)^t$ is an equivalence relation.
  2.1. Subtheorem: reflexivity of $(\rho \cup \sigma)^t$,
  2.2. Subtheorem: symmetry of $(\rho \cup \sigma)^t$,
  2.3. Subtheorem: transitivity of $(\rho \cup \sigma)^t$.
- Part 3:
  Theorem: $(\rho \cup \sigma)^t$ is the smallest equivalence relation.

The proof structure of 5.3 becomes:

- Part 1:
  1.a. *Subtheorem*: $(\rho \cap \sigma)$ is a leftcongruence.
  1.a.1. Subsubtheorem: reflexivity of $(\rho \cap \sigma)$,
  1.a.2. Subsubtheorem: symmetry of $(\rho \cap \sigma)$,
  1.a.3. Subsubtheorem: transitivity of $(\rho \cap \sigma)$,
  1.b. *Subtheorem*: $(f_1, f_2) \in (\rho \cap \sigma) \rightarrow (gf_1, gf_2) \in (\rho \cap \sigma)$.
- Part 2:
  2. *Theorem*: $(\rho \cup \sigma)^t$ is a leftcongruence

2.a. Subtheorem: $(\rho \cup \sigma)^t$ is an equivalence relation.
2.a.1. Subsubtheorem: reflexivity of $(\rho \cup \sigma)^t$,
2.a.2. Subsubtheorem: symmetry of $(\rho \cup \sigma)^t$,
2.a.3. Subsubtheorem: transitivity of $(\rho \cup \sigma)^t$,
2.b. Subtheorem: $(f_1, f_2) \in (\rho \cup \sigma)^t \rightarrow (gf_1, gf_2) \in (\rho \cup \sigma)^t$.

- Part 3:
  Theorem: $(\rho \cup \sigma)^t$ is the smallest equivalence relation.

Discussion

This example illustrates particularly well the importance of structuring theorems and proofs for analogy-driven theorem proving: Some parts of the proofs become identical. For example, the proofs of the parts 1a, 2a, and 3 of theorem 5.3 are identical to the corresponding subproofs of theorem 4.8 since the problems have identical theorems and assumptions.

Looking at the remaining parts it turns out that
5.3.1.b can be proved analogously to 4.8.1.2 and
5.3.2.b can be proved analogously to 4.8.2.2. These proofs are given in the following:

ND Proof of theorem 4.8 part 1.2

NNo	S;D		Formula	Reason
			relevant assumptions	
1.	; 1	$\vdash$	$\forall R(symm(R) \leftrightarrow \forall x_1, x_2((x_1, x_2) \in R \rightarrow (x_2, x_1) \in R))$	(DEF-symm)
2.	; 2	$\vdash$	$\forall R_1, R_2, x((x \in (R_1 \cap R_2) \leftrightarrow (x \in R_1 \wedge x \in R_2))$	(DEF-$\cap$)
3.	; 3	$\vdash$	$symm(\rho)$	(ASS)
4.	; 4	$\vdash$	$symm(\sigma)$	(ASS)
			The proof	
5.	5;	$\vdash$	$(f_1, f_2) \in (\sigma \cap \rho)$	(HYP)
6.	; 2	$\vdash$	$(f_1, f_2) \in (\sigma \cap \rho) \rightarrow (f_1, f_2) \in \sigma \wedge (f_1, f_2) \in \rho$	($\forall D, \leftrightarrow D$ 2)
7.	5; 2	$\vdash$	$(f_1, f_2) \in \rho$	($\rightarrow D, \wedge D$ 6)
8.	5; 2	$\vdash$	$(f_1, f_2) \in \sigma$	($\rightarrow D, \wedge D$ 6)
9.	; 1	$\vdash$	$symm(\rho) \rightarrow \forall x_1, x_2((x_1, x_2) \in \rho \rightarrow (x_2, x_1) \in \rho)$	($\forall D, \leftrightarrow D$ 1)
10.	; 1, 3	$\vdash$	$\forall x_1, x_2((x_1, x_2) \in \rho \rightarrow (x_2, x_1) \in \rho)$	($\rightarrow D$ 9 3)
11.	; 1, 3	$\vdash$	$(f_1, f_2) \in \rho \rightarrow (f_2, f_1) \in \rho$	($\forall D$ 10)
12.	; 1	$\vdash$	$symm(\sigma) \rightarrow \forall x_1, x_2((x_1, x_2) \in \sigma \rightarrow (x_2, x_1) \in \sigma)$	($\forall D, \leftrightarrow D$ 1)
13.	; 1, 4	$\vdash$	$\forall x_1, x_2((x_1, x_2) \in \sigma \rightarrow (x_2, x_1) \in \sigma)$	($\rightarrow D$ 12 4)

14.	; 1, 4	$\vdash$	$(f_1, f_2) \in \sigma \rightarrow (f_2, f_1) \in \sigma$	($\forall D$ 13)
15.	5; 2, 1, 3	$\vdash$	$(f_2, f_1) \in \rho$	($\rightarrow D$ 11 7)
16.	5; 2, 1, 4	$\vdash$	$(f_2, f_1) \in \sigma$	($\rightarrow D$ 14 8)
17.	5; 2, 1, 4, 3	$\vdash$	$(f_2, f_1) \in (\sigma \cap \rho)$	($\forall D, \leftrightarrow D$ 2 16)
18.	; 2, 1, 4, 3	$\vdash$	$(f_1, f_2) \in (\sigma \cap \rho) \rightarrow (f_2, f_1) \in (\sigma \cap \rho)$	(DED 17)
19.	; 2, 1, 4, 3	$\vdash$	$\forall f_1, f_2((f_1, f_2) \in (\sigma \cap \rho) \rightarrow (f_2, f_1) \in (\sigma \cap \rho))$	($\forall I$ 18)
20.	; 2, 1, 4, 3	$\vdash$	$symm(\sigma \cap \rho)$	($\forall D, \leftrightarrow D, \rightarrow D$ 1 19)
Thm.	;	$\vdash$	$symm(\sigma \cap \rho)$	()

ND Proof of theorem 5.3 part 1.b

NNo	S;D		Formula	Reason
			relevant assumptions	
1.	; 1	$\vdash$	$\forall R(leftcongruence(R) \leftrightarrow \forall g, x_1, x_2(g \in H \wedge (x_1, x_2) \in R \rightarrow (gx_1, gx_2) \in R))$	(DEF-leftcongruence)
2.	; 2	$\vdash$	$\forall R_1, R_2, x((x \in (R_1 \cap R_2) \leftrightarrow (x \in R_1 \wedge x \in R_2))$	(DEF-$\cap$)
3.	; 3	$\vdash$	$leftcongruence(\rho)$	(ASS)
4.	; 4	$\vdash$	$leftcongruence(\sigma)$	(ASS)
			The proof	
5.	5;	$\vdash$	$(f_1, f_2) \in (\sigma \cap \rho)$	(HYP)
6.	; 2	$\vdash$	$(f_1, f_2) \in (\sigma \cap \rho) \rightarrow (f_1, f_2) \in \sigma \wedge (f_1, f_2) \in \rho$	($\forall D, \leftrightarrow D$ 2)
7.	7;	$\vdash$	$g_0 \in H$	(HYP)
8.	5; 2	$\vdash$	$(f_1, f_2) \in \rho$	($\rightarrow D, \wedge D$, 6)
9.	5; 2	$\vdash$	$(f_1, f_2) \in \sigma$	($\rightarrow D, \wedge D$, 6)
10.	; 1	$\vdash$	$leftcongruence(\rho) \leftrightarrow \forall g, x_1, x_2(g \in H \wedge (x_1, x_2) \in \rho \rightarrow (gx_1, gx_2) \in \rho)$	($\forall D, \leftrightarrow D$ 1)
11.	; 1	$\vdash$	$\forall x_1, x_2((x_1, x_2) \in \rho \rightarrow (g_0x_1, g_0x_2) \in \rho)$	($\rightarrow D$ 10 3)
12.	; 1	$\vdash$	$leftcongruence(\sigma) \leftrightarrow \forall g, x_1, x_2(g \in H \wedge (x_1, x_2) \in \sigma \rightarrow (gx_1, gx_2) \in \sigma)$	($\forall D, \leftrightarrow D$ 1)
13.	7; 1	$\vdash$	$\forall x_1, x_2((x_1, x_2) \in \sigma \rightarrow (g_0x_1, g_0x_2) \in \sigma)$	($\rightarrow D$ 12 4)
14.	7; 1	$\vdash$	$(f_1, f_2) \in \rho \rightarrow (g_0f_1, g_0f_2) \in \rho$	($\forall D$ 11)
15.	5, 7; 2, 1, 3	$\vdash$	$(g_0f_1, g_0f_2) \in \rho$	($\rightarrow D$, 14 8)
16.	7; 1	$\vdash$	$(f_1, f_2) \in \sigma \rightarrow (g_0f_1, g_0f_2) \in \sigma$	($\forall D$ 13)
17.	5, 7; 2, 1, 4	$\vdash$	$(g_0f_1, g_0f_2) \in \sigma$	($\rightarrow D$, 16 9)
18.	5, 7; 2, 1, 4, 3	$\vdash$	$(g_0f_1, g_0f_2) \in (\sigma \cap \rho)$	($\forall D, \leftrightarrow D$ 2 17)
19.	7; 2, 1, 4, 3	$\vdash$	$g_0 \in H \wedge (f_1, f_2) \in (\sigma \cap \rho) \rightarrow (g_0f_1, g_0f_2) \in (\sigma \cap \rho)$	(DED 18)
20.	; 2, 1, 4, 3	$\vdash$	$\forall g, f_1, f_2(g \in H \wedge (f_1, f_2) \in (\sigma \cap \rho) \rightarrow (gf_1, gf_2) \in \sigma \cap \rho)$	($\forall I$ 19)
21.	; 2, 1, 4, 3	$\vdash$	$leftcongruence(\sigma \cap \rho)$	($\forall D, \leftrightarrow D, \rightarrow D$, 1 20)
Thm.	;	$\vdash$	$leftcongruence(\sigma \cap \rho)$	()

Discussion

Symbol- or term mappings are not sufficient for a transformation of 4.8.1.2 to 5.3.1.b. For example, the symbol mapping $\{symm \Rightarrow leftcongrunce\}$ is not sufficient, since the definitions of $symm$ and $leftcongruence$ (which are part of the relevant assumptions) are not equal after this mapping. An additional term mapping would have to be restricted to certain occurrences of terms, because the overall mapping $\{(f_2, f_1) \Rightarrow (g \cdot f_1, g \cdot f_2)\}$ or $\{(term1, term2) \Rightarrow (g \cdot term1, g \cdot term2)\}$ also yields $\{(f_1, f_2) \Rightarrow (gf_2, gf_1)\}$ which is not desired at all. Furthermore, the reformulated proof cannot be verified for 5.3.1.b because of missing sort declarations and quantifiers.

Furthermore, the theorem and the assumptions of 4.8.1.2 and 5.3.1.b contain subformulae of the form $x \in M$ and quantifiers which have to be modified by the mapping. This problem is due to fact that the necessary mapping of terms is essentially an abstraction by which some symbols irrelevant for the proof disappear, just as in the previous case. The actual justification for this abstraction is the occurrence within the relevant assumptions of the definitions of $symm$ and $leftcongruence$, which have the same characteristic structure. An analogy formation based on pure term mapping is not justified at all and hence the abstracting reformulation has to be preferred.

A successful transformation is composed of an abstraction followed by a symbol mapping, and a subsequent reverse abstraction.

1. The abstraction of problem 4.8.1.2 that yields problem 4.8.1.2' changes the pairs $(term_2, term_1)$, which are determined by the definition of $symm$, to terms $\mathrm{f}_{rev}(term_1, term_2)$. The abstraction of problem 5.3.1.b that yields problem 5.3.1.b' transforms the pairs $(g \cdot term_1, g \cdot term_2)$ to $\mathrm{f}_g(term_1, term_2)$. These reformulations affect the pairs contained in the definition of $symm(R)$ and $leftcongruence(R)$, respectively. It affects the derived terms within the whole proof and in addition, certain formulae and quantifiers have to be removed.

2. The symbol mapping $\{\mathrm{symm} \Rightarrow \mathrm{leftcongruence}, f_{rev} \Rightarrow f_g\}$ is applied to problem $4.8.1.2'$ and yields problem $4.8.1.2''$ which is equal to problem $5.3.1.\mathrm{b}'$.

3. Finally, to return to the original problem 5.3.1.b, a reversion of the abstraction of problem 5.3.1.b has to be applied to problem $4.8.1.2''$.

In the following the ND-proofs of theorem 4.8 part 2.2 and of theorem 5.3 part 2.b are given.

ND Proof for theorem 4.8 part 2.2

NNo	S;D		Formula	Reason
			relevant assumptions	
1.	; 1	$\vdash$	$\forall R(symm(R) \leftrightarrow \forall x_1, x_2((x_1, x_2) \in R \rightarrow (x_2, x_1) \in R))$	(DEF symm)
2.	; 2	$\vdash$	$\forall x(x \in (\rho \cup \sigma) \leftrightarrow x \in \rho \vee x \in \sigma)$	(DEF-$\cup$)
3.	; 3	$\vdash$	$\forall x, y, k(k \in N \rightarrow ((x, y) \in R^1 \rightarrow (y, x) \in R^1) \wedge (((x, y) \in R^k \rightarrow (y, x) \in R^k) \rightarrow ((x, y) \in R^{k+1} \rightarrow (y, x) \in R^{k+1})) \rightarrow \forall n(n \in N \rightarrow (x, y) \in R^n \rightarrow (y, x) \in R^n))$	(Induction-AX)
4.	; 4	$\vdash$	$\forall x, y((x, y) \in (\rho \cup \sigma)^1 \leftrightarrow (x, y) \in (\rho \cup \sigma))$	(DEF-$(\rho \cup \sigma)^1$)
5.	; 5	$\vdash$	$\forall n, x, y(n \in N \rightarrow ((x, y) \in (\rho \cup \sigma)^{n+1} \leftrightarrow \exists z((x, z) \in (\rho \cup \sigma)^n \wedge (z, y) \in (\rho \cup \sigma)^1) \vee ((z, y) \in (\rho \cup \sigma)^n \wedge (x, z) \in (\rho \cup \sigma)^1))$	(DEF $(\rho \cup \sigma)^{n+1}$)
6.	; 6	$\vdash$	$\forall x, y(x, y) \in (\rho \cup \sigma)^t \leftrightarrow \exists n(n \in N \wedge (x, y) \in (\rho \cup \sigma)^n)$	(DEF-$(\rho \cup \sigma)^t$)
7.	; 7	$\vdash$	$symm(\rho)$	(ASS)
8.	; 8	$\vdash$	$symm(\sigma)$	(ASS)
			induction base	
9.	9;	$\vdash$	$(f_1, f_2) \in (\rho \cup \sigma)^1$	(HYP)
10.	9;	$\vdash$	$(f_1, f_2) \in \rho \vee (f_1, f_2) \in \sigma$	($\forall D, \rightarrow D$ 2 9)
11.	11;	$\vdash$	$(f_1, f_2) \in \rho$	(HYP)
12.	11; 1, 7	$\vdash$	$(f_2, f_1) \in \rho$	($\forall D, \rightarrow D$ 7 11 1)
13.	11; 1, 7, 2	$\vdash$	$(f_2, f_1) \in (\rho \cup \sigma)$	($\forall D, \leftrightarrow D, \rightarrow D$ 2 12)
14.	11; 1, 7, 2, 4	$\vdash$	$(f_2, f_1) \in (\rho \cup \sigma)^1$	($\forall D, \leftrightarrow D, \rightarrow D$ 13 4)
15.	; 1, 7, 2, 4	$\vdash$	$(f_1, f_2) \in \rho \rightarrow (f_2, f_1) \in (\rho \cup \sigma)^1$	(DED 14)
16.	16;	$\vdash$	$(f_1, f_2) \in \sigma$	(HYP)
17.	16; 1, 8	$\vdash$	$(f_2, f_1) \in \sigma$	($\forall D, \rightarrow D$ 8 16 1)
18.	16; 1, 2, 8	$\vdash$	$(f_2, f_1) \in (\rho \cup \sigma)$	($\forall D, \leftrightarrow D, \rightarrow D$ 2 17)
19.	16; 1, 2, 8, 4	$\vdash$	$(f_2, f_1) \in (\rho \cup \sigma)^1$	($\forall D, \leftrightarrow D, \rightarrow D$ 18 4)
20.	; 1, 2, 8, 4	$\vdash$	$(f_1, f_2) \in \sigma \rightarrow (f_2, f_1) \in (\rho \cup \sigma)^1$	(DED 19)

21.	9; 1, 2, 8, 7, 4	$\vdash$	$(f_2, f_1) \in (\rho \cup \sigma)^1$	($\vee D$ 10 15 20)
22.	; 1, 2, 8, 7, 4	$\vdash$	$\forall f_1, f_2((f_1, f_2) \in (\rho \cup \sigma)^1 \rightarrow (f_2, f_1) \in (\rho \cup \sigma)^1$	(DED,$\forall I$ 21)
			The proof induction step	
23.	23;	$\vdash$	$k \in N$	(HYP)
24.	24;	$\vdash$	$\forall x_1, x_2((x_1, x_2) \in (\rho \cup \sigma)^k \rightarrow (x_2, x_1) \in (\rho \cup \sigma)^k)$	(Induction-HYP)
25.	25;	$\vdash$	$(f_1, f_2) \in (\rho \cup \sigma)^{k+1}$	(HYP)
26.	25, 23; 5	$\vdash$	$\exists z(((f_1, z) \in (\rho \cup \sigma)^k \wedge (z, f_2) \in (\rho \cup \sigma)^1) \vee ((z, f_2) \in (\rho \cup \sigma)^k \wedge (f_1, z) \in (\rho \cup \sigma)^1))$	($\leftrightarrow D, \rightarrow D$ 5 25)
27.	27;	$\vdash$	$((f_1, x_0) \in (\rho \cup \sigma)^k \wedge (x_0, f_2) \in (\rho \cup \sigma)^1) \vee ((x_0, f_2) \in (\rho \cup \sigma)^k \wedge (f_1, x_0) \in (\rho \cup \sigma)^1)$	(HYP)
			case 1	
28.	28;	$\vdash$	$(f_1, x_0) \in (\rho \cup \sigma)^k \wedge (x_0, f_2) \in (\rho \cup \sigma)^1$	(HYP)
29.	28;	$\vdash$	$(f_1, x_0) \in (\rho \cup \sigma)^k$	($\wedge D$ 28)
30.	28;	$\vdash$	$(x_0, f_2) \in (\rho \cup \sigma)^1$	($\wedge D$ 28)
31.	28, 23;	$\vdash$	$(x_0, f_1) \in (\rho \cup \sigma)^k$	($\forall D, \rightarrow D$ 24 29)
32.	28, 24;	$\vdash$	$(f_2, x_0) \in (\rho \cup \sigma)^1$	($\forall D, \rightarrow D$ 24 30)
33.	24, 28, 23;	$\vdash$	$(f_2, x_0) \in (\rho \cup \sigma)^1 \wedge (x_0, f_1) \in (\rho \cup \sigma)^k)$	($\wedge I$ 31 32)
34.	; 24, 28, 23	$\vdash$	$(f_2, x_0) \in (\rho \cup \sigma)^1 \wedge (x_0, f_1) \in (\rho \cup \sigma)^k \vee (f_2, x_0) \in (\rho \cup \sigma)^k \wedge (x_0, f_1) \in (\rho \cup \sigma)^1$	($\vee I$ 33)
35.	28, 24, 23;	$\vdash$	$\exists z((f_2, z) \in (\rho \cup \sigma)^1 \wedge (z, f_1) \in (\rho \cup \sigma)^k \vee (f_2, z) \in (\rho \cup \sigma)^k \wedge (z, f_1) \in (\rho \cup \sigma)^1)$	($\wedge I, \exists I$ 34)
36.	28, 24, 23; 5	$\vdash$	$(f_2, f_1) \in (\rho \cup \sigma)^{k+1}$	($\forall D, \leftrightarrow D, \rightarrow D$ 35 5)
			case 2	
37.	37;	$\vdash$	$(x_0, f_2) \in (\rho \cup \sigma)^k \wedge (f_1, x_0) \in (\rho \cup \sigma)^1$	(HYP)
38.	37;	$\vdash$	$(x_0, f_2) \in (\rho \cup \sigma)^k$	($\wedge D$ 37)
39.	37;	$\vdash$	$(f_1, x_0,) \in (\rho \cup \sigma)^1$	($\wedge D$ 37)
40.	37, 23;	$\vdash$	$(f_2, x_0) \in (\rho \cup \sigma)^k$	($\forall D, \rightarrow D$ 24 38)
41.	37, 24;	$\vdash$	$(x_0, f_1) \in (\rho \cup \sigma)^1$	($\forall D, \rightarrow D$ 24 39)
42.	24, 37, 23;	$\vdash$	$(f_2, x_0) \in (\rho \cup \sigma)^1 \wedge (x_0, f_1) \in (\rho \cup \sigma)^k)$	($\wedge I$ 40 41)
43.	; 24, 37, 23	$\vdash$	$(f_2, x_0) \in (\rho \cup \sigma)^1 \wedge (x_0, f_1) \in (\rho \cup \sigma)^k) \vee (f_2, x_0) \in (\rho \cup \sigma)^k \wedge (x_0, f_1) \in (\rho \cup \sigma)^1)$	($\vee I$ 42)
44.	37, 24, 23;	$\vdash$	$\exists z((z, f_1) \in (\rho \cup \sigma)^1 \wedge (f_2, z) \in (\rho \cup \sigma)^k \vee (f_2, z) \in (\rho \cup \sigma)^k \wedge (z, f_1) \in (\rho \cup \sigma)^1)$	($\exists I$ 43)
45.	37, 24, 23; 5	$\vdash$	$(f_2, f_1) \in (\rho \cup \sigma)^{k+1}$	($\forall D, \leftrightarrow D, \rightarrow D$ 44 5)

46.	37, 28, 24, 23, 27; 5	$\vdash$	$(f_2, f_1) \in (\rho \cup \sigma)^{k+1}$	($\vee D$ 45 36 27)
47.	23, 24, 37, 25, 28; 5	$\vdash$	$(f_2, f_1) \in (\rho \cup \sigma)^{k+1}$	(CHOICE 46 27 26)
48.	23, 24, 28, 37; 5	$\vdash$	$\forall f_1, f_2((f_1, f_2) \in (\rho \cup \sigma)^{k+1} \rightarrow (f_2, f_1) \in (\rho \cup \sigma)^{k+1})$	(DED,$\forall I$ 47)
49.	; 5	$\vdash$	$k \in N \wedge \forall x_1, x_2(((x_1, x_2) \in (\rho \cup \sigma)^k \rightarrow (x_2, x_1) \in (\rho \cup \sigma)^k) \rightarrow \forall f_1, f_2((f_1, f_2) \in (\rho \cup \sigma)^{k+1} \rightarrow (f_2, f_1) \in (\rho \cup \sigma)^{k+1}))$	(DED 48)
50.	; 1, 2, 3, 4, 5, 7, 8	$\vdash$	$\forall f_1, f_2 \forall n(n \in N \wedge (f_1, f_2) \in (\rho \cup \sigma)^n \rightarrow (f_2, f_1) \in (\rho \cup \sigma)^n)$	($\forall I, \wedge I, \rightarrow D$ 49 22 3)
			for $(\rho \cup \sigma)^t$	
51.	51;	$\vdash$	$(f_1, f_2) \in (\rho \cup \sigma)^t$	(HYP)
52.	51; 6	$\vdash$	$\exists n(n \in N \wedge (f_1, f_2) \in (\rho \cup \sigma)^n)$	($\forall D, \leftrightarrow D, \rightarrow D$ 6 51)
53.	53;	$\vdash$	$m_0 \in N \wedge (f_1, f_2) \in (\rho \cup \sigma)^{m_0}$	(HYP)
54.	53; 1, 2, 3, 4, 5, 7, 8	$\vdash$	$m_0 \in N \wedge (f_2, f_1) \in (\rho \cup \sigma)^{m_0}$	($\forall D, \rightarrow D$ 53 50)
55.	53; 1, 2, 3, 4, 5, 7, 8	$\vdash$	$\exists n(n \in N \wedge (f_2, f_1) \in (\rho \cup \sigma)^n)$	($\exists I$ 54)
56.	53; 1, 2, 3, 4, 5, 7, 8	$\vdash$	$(f_2, f_1) \in (\rho \cup \sigma)^t$	($\rightarrow D$ 6 55)
57.	51; 6, 1, 2, 3, 4, 5, 7, 8	$\vdash$	$f_2, f_1) \in (\rho \cup \sigma)^t$	(CHOICE 56 52 53)
58.	; 6, 1, 2, 3, 4, 5, 7, 8	$\vdash$	$(f_1, f_2) \in (\rho \cup \sigma)^t \rightarrow (f_2, f_1) \in (\rho \cup \sigma)^t$	(DED 57)
59.	; 6, 1, 2, 3, 4, 5, 7, 8	$\vdash$	$\forall f_1, f_2((f_1, f_2) \in (\rho \cup \sigma)^t \rightarrow (f_2, f_1) \in (\rho \cup \sigma)^t)$	($\forall I$ 58)
60.	; 6, 1, 2, 3, 4, 5, 7, 8	$\vdash$	$symm(\rho \cup \sigma)^t$	($\leftrightarrow D$ 59 1)
Thm.	;	$\vdash$	$symm(\rho \cup \sigma)^t$	()

ND Proof for theorem 5.3 part 2.b

NNo	S;D		Formula	Reason
			relevant assumptions	
1.	; 1	$\vdash$	$\forall R(leftcongruence(R) \leftrightarrow \forall x_1, x_2, g(g \in F(x_1, x_2) \in R \rightarrow (gx_1, gx_2) \in R))$	(DEF-leftcongruence)
2.	; 2	$\vdash$	$\forall x(x \in (\rho \cup \sigma) \leftrightarrow x \in \rho \vee x \in \sigma)$	(DEF-$\cup$)
3.	; 3	$\vdash$	$\forall k, x, y, x_1, y_1, x_2, y_2, x_3, y_3, g(k \in N \wedge g \in F \rightarrow ((x, y) \in R^1 \rightarrow (gx, gy) \in R^1) \wedge (((x_1, y_1) \in R^k \rightarrow (gx_1, gy_1) \in R^k) \rightarrow ((x_2, y_2) \in R^{k+1} \rightarrow (gx_2, gy_2) \in R^{k+1})) \rightarrow \forall n(n \in N \rightarrow (x_3, y_3) \in R^n \rightarrow (gx, gy) \in R^n))$	(Induction-AX)
4.	; 4	$\vdash$	$\forall x, y((x, y) \in (\rho \cup \sigma)^1 \leftrightarrow (x, y) \in (\rho \cup \sigma))$	(DEF-$(\rho \cup \sigma)^1$)
5.	; 5	$\vdash$	$\forall n, x, y(n \in N \rightarrow ((x, y) \in (\rho \cup \sigma)^{n+1} \leftrightarrow \exists z((x, z) \in (\rho \cup \sigma)^n \wedge (z, y) \in (\rho \cup \sigma)^1) \vee ((z, y) \in (\rho \cup \sigma)^n \wedge (x, z) \in (\rho \cup \sigma)^1)))$	(DEF $(\rho \cup \sigma)^{n+1}$)
6.	; 6	$\vdash$	$\forall x, y(x, y) \in (\rho \cup \sigma)^t \leftrightarrow \exists n(n \in N \wedge (x, y) \in (\rho \cup \sigma)^n)$	(DEF-$(\rho \cup \sigma)^t$)
7.	; 7	$\vdash$	$leftcongruence(\rho)$	(ASS)
8.	; 8	$\vdash$	$leftcongruence(\sigma)$	(ASS)
			induction base	
9.	9;	$\vdash$	$g_0 \in F$	(HYP)
10.	10;	$\vdash$	$(f_1, f_2) \in (\rho \cup \sigma)^1$	(HYP)
11.	10; 2	$\vdash$	$(f_1, f_2) \in \rho \vee (f_1, f_2) \in \sigma$	($\forall D, \rightarrow D$ 2 10)
12.	12;	$\vdash$	$(f_1, f_2) \in \rho$	(HYP)
13.	12, 9; 1, 7	$\vdash$	$(g_0 f_1, g_0 f_2) \in \rho$	($\forall D, \leftrightarrow D, \wedge I, \rightarrow D$ 7 12 9 1)
14.	12, 9; 1, 2, 7	$\vdash$	$(g_0 f_1, g_0 f_2) \in (\rho \cup \sigma)$	($\forall D, \leftrightarrow D, \rightarrow D$ 2 13)
15.	12, 9; 1, 2, 4, 7	$\vdash$	$(g_0 f_1, g_0 f_2) \in (\rho \cup \sigma)^1$	($\forall D, \leftrightarrow D, \rightarrow D$ 14 4)
16.	9; 1, 2, 4, 7	$\vdash$	$(f_1, f_2) \in \rho \rightarrow (g_0 f_1, g_0 f_2) \in (\rho \cup \sigma)^1$	(DED 15)
17.	17;	$\vdash$	$(f_1, f_2) \in \sigma$	(HYP)
18.	17, 9; 1, 8	$\vdash$	$(g_0 f_1, g_0 f_2) \in \sigma$	($\forall D, \leftrightarrow D, \wedge I \rightarrow D$ 8 17 9 1)
19.	17, 9; 1, 2, 8	$\vdash$	$(g_0 f_1, g_0 f_2) \in (\rho \cup \sigma)$	($\forall D, \leftrightarrow D, \rightarrow D$ 2 18)

20.	17, 9; 1, 2, 4, 8	$\vdash$	$(g_0 f_1, g_0 f_2) \in (\rho \cup \sigma)^1$	($\forall D, \leftrightarrow D, \rightarrow D$ 19 4)
21.	9; 1, 2, 4, 8	$\vdash$	$(f_1, f_2) \in \sigma \rightarrow (g_0 f_1, g_0 f_2) \in (\rho \cup \sigma)^1$	(DED 20)
22.	10, 9; 1, 2, 4, 7, 8	$\vdash$	$(g_0 f_1, g_0 f_2) \in (\rho \cup \sigma)^1$	($\vee D$ 11 16 21)
23.	; 1, 2, 4, 7, 8	$\vdash$	$\forall f_1, f_2, g(g \in F \wedge (f_1, f_2) \in (\rho \cup \sigma)^1 \rightarrow (g f_1, g f_2) \in (\rho \cup \sigma)^1)$	(DED,$\forall I$ 22)

induction step

24.	24;	$\vdash$	$k \in N$	(HYP)
25.	25;	$\vdash$	$\forall x_1, x_2((x_1, x_2) \in (\rho \cup \sigma)^k \rightarrow (g x_1, g x_2) \in (\rho \cup \sigma)^k)$	(InductionHYP)
26.	26, 24;	$\vdash$	$(f_1, f_2) \in (\rho \cup \sigma)^{k+1}$	(HYP)
27.	26, 24; 5	$\vdash$	$\exists z(((f_1, z) \in (\rho \cup \sigma)^k \wedge (z, f_2) \in (\rho \cup \sigma)^1) \vee ((z, f_2) \in (\rho \cup \sigma)^k \wedge (f_1, z) \in (\rho \cup \sigma)^1))$	($\forall D, \leftrightarrow D, \rightarrow D$ 5 26)
28.	24, 26; 5	$\vdash$	$((f_1, x_0) \in (\rho \cup \sigma)^k \wedge (x_0, f_2) \in (\rho \cup \sigma)^1) \vee ((x_0, f_2) \in (\rho \cup \sigma)^k \wedge (f_1, x_0) \in (\rho \cup \sigma)^1)$	($\exists D$ 27)

case 1

29.	29;	$\vdash$	$(f_1, x_0) \in (\rho \cup \sigma)^k \wedge (x_0, f_2) \in (\rho \cup \sigma)^1$	(HYP)
30.	29, 24;	$\vdash$	$(f_1, x_0) \in (\rho \cup \sigma)^k$	($\wedge D$ 29)
31.	29, 24;	$\vdash$	$(x_0, f_2) \in (\rho \cup \sigma)^1$	($\wedge D$ 29)
32.	29, 24, 25;	$\vdash$	$(g f_1, g x_0) \in (\rho \cup \sigma)^k$	($\rightarrow D$ 25 30)
33.	29, 24, 25;	$\vdash$	$(g x_0, g f_2) \in (\rho \cup \sigma)^1$	($\rightarrow D$ 25 31)
34.	29, 24, 25;	$\vdash$	$\exists z((g f_1, z) \in (\rho \cup \sigma)^k \wedge (z, g f_2) \in (\rho \cup \sigma)^1)$	($\wedge I, \exists I$ 32 33)
35.	29, 24, 25; 5	$\vdash$	$(g f_1, g f_2) \in (\rho \cup \sigma)^{k+1}$	($\wedge I, \forall D, \leftrightarrow D, \rightarrow D$ 32 33 5)
36.	24, 25; 5	$\vdash$	$(f_1, x_0) \in (\rho \cup \sigma)^k \wedge (x_0, f_2) \in (\rho \cup \sigma)^1 \rightarrow (g f_1, g f_2) \in (\rho \cup \sigma)^{k+1}$	(DED 35)

case 2

37.	37;	$\vdash$	$(x_0, f_2) \in (\rho \cup \sigma)^k \wedge (f_1, x_0) \in (\rho \cup \sigma)^1$	(HYP)
38.	37, 24;	$\vdash$	$(f_1, x_0) \in (\rho \cup \sigma)^1$	($\wedge D$ 37)
39.	37, 24;	$\vdash$	$(x_0, f_2) \in (\rho \cup \sigma)^k$	($\wedge D$ 37)
40.	37, 24, 25;	$\vdash$	$(g f_1, g x_0) \in (\rho \cup \sigma)^1$	($\rightarrow D$ 25 38)
41.	37, 24, 25;	$\vdash$	$(g x_0, g f_2) \in (\rho \cup \sigma)^k$	($\rightarrow D$ 25 39)
42.	37, 24, 25;	$\vdash$	$\exists z((g f_1, z) \in (\rho \cup \sigma)^1 \wedge (z, g f_2) \in (\rho \cup \sigma)^k)$	($\wedge I, \exists I$ 40 41)
43.	37, 24, 25; 5	$\vdash$	$(g f_1, g f_2) \in (\rho \cup \sigma)^{k+1}$	($\wedge I, \forall D, \leftrightarrow D, \rightarrow D, \vee I$ 40 41 42 5)
44.	24, 25; 5	$\vdash$	$(f_1, x_0) \in (\rho \cup \sigma)^1 \wedge (x_0, f_2) \in (\rho \cup \sigma)^k \rightarrow (g f_1, g f_2) \in (\rho \cup \sigma)^{k+1}$	(DED 43)
45.	24, 26, 25; 5	$\vdash$	$(g f_1, g f_2) \in (\rho \cup \sigma)^{k+1}$	($\vee D$ 36 44 28)

46.	; 5	$\vdash$	$k \in N \wedge \forall x_1, x_2, f_1, f_2(((x_1, x_2) \in (\rho \cup \sigma)^k \rightarrow (gx_1, gx_2) \in (\rho \cup \sigma)^k) \wedge ((f_1, f_2) \in (\rho \cup \sigma)^{k+1} \rightarrow (gf_1, gf_2) \in (\rho \cup \sigma)^{k+1}))$	(DED,$\forall I$ 45)
			induction for $(\rho \cup \sigma)^n$	
47.	; 1, 2, 3, 4, 5, 7, 8	$\vdash$	$\forall f_1, f_2, g \forall n(n \in N \wedge g \in F \wedge (f_1, f_2) \in (\rho \cup \sigma)^n \rightarrow (gf_1, gf_2) \in (\rho \cup \sigma)^n)$	($\forall I, \wedge I, \rightarrow D$ 46 23 3)
			for $(\rho \cup \sigma)^t$	
48.	48;	$\vdash$	$(f_1, f_2) \in (\rho \cup \sigma)^t$	(HYP)
49.	48; 6	$\vdash$	$\exists n(n \in N \wedge (f_1, f_2) \in (\rho \cup \sigma)^n)$	($\forall D, \leftrightarrow D, \rightarrow D$ 6 48)
50.	48; 6	$\vdash$	$m_0 \in N \wedge (f_1, f_2) \in (\rho \cup \sigma)^{m_0}$	($\exists D$ 49)
51.	48, 9; 1, 2, 3, 4, 5, 6, 7, 8	$\vdash$	$m_0 \in N \wedge (gf_1, gf_2) \in (\rho \cup \sigma)^{m_0}$	($\forall D, \rightarrow D$ 50 47)
52.	48, 9; 1, 2, 3, 4, 5, 6, 7, 8	$\vdash$	$\exists n(n \in N \wedge (gf_1, gf_2) \in (\rho \cup \sigma)^n)$	($\exists I$ 51)
53.	48, 9; 1, 2, 3, 4, 5, 6, 7, 8	$\vdash$	$(gf_1, gf_2) \in (\rho \cup \sigma)^t$	($\rightarrow D$ 6 52)
54.	; 1, 2, 3, 4, 5, 6, 7, 8	$\vdash$	$(f_1, f_2) \in (\rho \cup \sigma)^t \rightarrow (gf_1, gf_2) \in (\rho \cup \sigma)^t$	(DED 53)
55.	; 1, 2, 3, 4, 5, 6, 7, 8	$\vdash$	$\forall f_1, f_2(g \in F \wedge (f_1, f_2) \in (\rho \cup \sigma)^t \rightarrow (gf_1, gf_2) \in (\rho \cup \sigma)^t)$	($\forall I$ 54)
56.	; 1, 2, 3, 4, 5, 6, 7, 8	$\vdash$	$leftcongruence(\rho \cup \sigma)^t$	($\leftrightarrow D$ 55 ??)
Thm.	;	$\vdash$	$leftcongruence(\rho \cup \sigma)^t$	()

Discussion

The same reformulation as presented for theorem 4.8.1.2 that leads to theorem 5.3.1.b works for theorem 4.8.2.2 and theorem 5.3.2.b as well.

CASE 6: THEOREM 5.2 and its ANALOGUE

Theorem 5.2
Let ρ be an equivalence relation on a semigroup F. The following assertions are equivalent
a) ρ is a leftcongruence in F
b) For all $f \in F$ and $h \in F$ holds $h \cdot \Omega(f) \subset \Omega(hf)$.

Theorem 7.5.2 (analogous to 5.2.)
Let ρ be an equivalence relation in an F-semimodul S. The following assertions are equivalent
a') ρ is a congruence in S
b') For all $f \in S$ and $h \in F$ holds $h \cdot \Omega(f) \subset \Omega(hf)$.

The proofs of these theorems are based on definition1 and definition1' of $x \in \Omega(z)$ (see Case 2) for semigroups F and S, which differ in some symbols and on definition2 and definition2' of $x \in h \cdot \Omega(z)$, which differ in some symbols as well.

Definition 2:
$\forall x, z, h(x \in F \wedge z \in F \wedge h \in F \rightarrow x \in h \cdot \Omega(z) \leftrightarrow \exists y(y \in F \wedge y \in \Omega(z) \wedge x = h \cdot y))$

Definition 2':
$\forall x, z, h(x \in S \wedge z \in S \wedge h \in F \rightarrow x \in h \cdot \Omega(z) \leftrightarrow \exists y(y \in S \wedge y \in \rho(z) \wedge x = h \cdot y))$

The proofs of theorem 5.2 and theorem 7.5.2 can be structured as follows:
The proof structure of 5.2. is:

- Part 1 (a$\rightarrow$b)
  Theorem: $\forall h, f(h \in F \wedge f \in F \rightarrow h \cdot \Omega(f) \subset \Omega(hf))$
  Relevant assumptions: definition1; assumption of ρ being a leftcongruence in a semigroup; the extensionality axiom of =; definition2.

- Part 2 (a$\leftarrow$b)
  Theorem: $\rho(f_1, f_2) \rightarrow \rho(hf_1, hf_2)$

Relevant assumptions: (a part of) the definition F is semigroup; definition1; the definition of $\subset$; definition2.

The proof structure of 7.5.2 is:

- Part 1 (a'$\rightarrow$b')
  Theorem: $\forall h, f(h \in F \wedge f \in S \rightarrow h \cdot \Omega(f) \subset \Omega(hf))$
  Relevant assumptions: definition1'; assumption of ρ being a congruence relation on a semimodul; the extensionality axiom of =; definition2'.

- Part 2 (a'$\leftarrow$b')
  Theorem: $\rho(f_1, f_2) \rightarrow \rho(hf_1, hf_2)$
  Relevant assumptions: (a part of) the definition that F is semimodul; definition1'; the definition of $\subset$; definition2'.

ND proof of the subtheorem 5.2.1

NNo	S;D		Formula	Reason
			relevant assumptions	
1.	; 1	$\vdash$	$\forall R \forall x, x', y, y'(x = x' \wedge y = y' \rightarrow (Rxy \leftrightarrow Rx'y'))$	(AX=)
2.	; 2	$\vdash$	$\forall h, f_1, f_2(h \in F \wedge f_1 \in F \wedge f_2 \in F \rightarrow (\rho(f_1, f_2) \rightarrow \rho(hf_1, hf_2)))$	(part of DEF leftcongruence(ρ))
3.	; 3	$\vdash$	$\forall h, z, x(h \in F \wedge z \in F \rightarrow (x \in h \cdot \Omega z \leftrightarrow x \in F \wedge \exists y(y \in F \wedge x = h \cdot y \wedge \rho(y, z))))$	(DEF1)
4.	; 4	$\vdash$	$\forall x, z(x \in F \wedge z \in F \rightarrow x \in \Omega z \leftrightarrow \rho(x, z))$	(DEF2)
5.	; 5	$\vdash$	$\forall x, y(x \in F \wedge y \in F \rightarrow x \cdot y \in F)$	(part of DEF semi-group(F))
6.	; 6	$\vdash$	$\forall M, N(M \subset N \leftrightarrow \forall x(x \in M \rightarrow x \in N))$	(DEF $\subset$)
			the proof	
7.	7;	$\vdash$	$h \in F$	(HYP)
8.	8;	$\vdash$	$f \in F$	(HYP)
9.	9;	$\vdash$	$x_0 \in h \cdot \Omega f$	(HYP)
10.	7, 8, 9; 3	$\vdash$	$x_0 \in F$	($\forall D, \leftrightarrow D, \wedge D$ 9 3 7 8)
11.	; 3	$\vdash$	$(x_0 \in F \wedge h \in F \wedge f \in F \rightarrow (x_0 \in h \cdot \Omega f \leftrightarrow \exists y(y \in F \wedge x_0 = h \cdot y \wedge \rho(y, f))))$	($\forall$ D 3)

12.	7, 8; 3	$\vdash$	$(x_0 \in F \to (x_0 \in h \cdot \Omega f \leftrightarrow \exists y(y \in F \wedge x_0 = h \cdot y \wedge \rho(y, f))))$	$(\to D$ 11 7 8)
13.	7, 8, 9; 3	$\vdash$	$\exists y(y \in F \wedge x_0 = h \cdot y \wedge \rho(y, f))$	$(\leftrightarrow D, \to D$ 12 9)
14.	14;	$\vdash$	$y_0 \in F \wedge x_0 = h \cdot y_0 \wedge \rho(y_0, f)$	(HYP)
15.	14;	$\vdash$	$y_0 \in F \wedge \rho(y_0, f)$	$(\wedge D$ 14)
16.	14, 8;	$\vdash$	$y_0 \in F \wedge f \in F \wedge \rho(y_0, f)$	$(\wedge I$ 15 8)
17.	14, 8; 2	$\vdash$	$\rho(hy_0, hf)$	$(\forall D, \to D$ 16 2)
18.	14, 8; 2, 1	$\vdash$	$\rho(x_0, hf)$	$(\forall D, \to D$ 17 14 1)
19.	7, 8, 9; 3, 2, 1	$\vdash$	$\rho(x_0, hf)$	(CHOICE 18 13)
20.	7, 8; 5	$\vdash$	$hf \in F$	$(\forall D, \to D$ 5 7 8)
21.	7, 8, 9; 4, 5, 1, 3, 2	$\vdash$	$x_0 \in \Omega(hf)$	$(\forall D, \wedge D, \to D$ 4 20 10)
22.	; 4, 5, 1, 3, 2	$\vdash$	$h \in F \wedge f \in F \to (x_0 \in h \cdot \Omega f \to x_0 \in \Omega(hf))$	(DED 21)
23.	; 4, 5, 1, 3, 2	$\vdash$	$\forall x, f, h(h \in F \wedge f \in F \to (x \in h \cdot \Omega f \to x \in \Omega(hf)))$	$(\forall I$ 22)
24.	; 4, 5, 1, 3, 2, 6	$\vdash$	$\forall h, f(h \in F \wedge f \in F \to h \cdot \Omega f \subset \Omega(hf))$	$(\leftrightarrow D$ 6 23)
Thm.	;	$\vdash$	$\forall h, f(h \in F \wedge f \in F \to h \cdot \Omega f \subset \Omega(hf))$	()

The operation $\cdot_F$ should have been used in the theorem, in the assumptions, and in the proof. Instead, the polymorphic symbol $\cdot$ is used for convenience and readability.

ND proof of the subtheorem 7.5.2.1

NNo	S;D		Formula	Reason
			relevant assumption	
1.	; 1	$\vdash$	$\forall R \forall x, x', y, y'(x = x' \wedge y = y' \to (Rxy \leftrightarrow Rx'y'))$	(AX=)
2.	; 2	$\vdash$	$\forall h, f_1, f_2(h \in F \wedge f_1 \in S \wedge f_2 \in S \to (\rho(f_1, f_2) \to \rho(hf_1, hf_2)))$	(part of DEF congruence(ρ))
3.	; 3	$\vdash$	$\forall h, z, x(h \in F \wedge z \in S \to (x \in h \cdot \Omega z \leftrightarrow x \in S \wedge \exists y(y \in S \wedge x = h \cdot y \wedge \rho(y, z))))$	(DEF1')
4.	; 4	$\vdash$	$\forall x, z(x \in S \wedge z \in S \to x \in \Omega z \leftrightarrow \rho(x, z))$	(DEF2')
5.	; 5	$\vdash$	$\forall x, y(x \in F \wedge y \in S \to x \cdot y \in S)$	(part of DEF F-semimodul(S))
6.	; 6	$\vdash$	$\forall M, N(M \subset N \leftrightarrow \forall x(x \in M \to x \in N))$	(DEF $\subset$)
			the very proof	

7.	7;	$\vdash$	$h \in F$	(HYP)
8.	8;	$\vdash$	$f \in S$	(HYP)
9.	9;	$\vdash$	$x_0 \in h \cdot \Omega f$	(HYP)
10.	7, 8, 9; 3	$\vdash$	$x_0 \in S$	($\forall D, \leftrightarrow D, \wedge D$ 9 3 7 8)
11.	; 3	$\vdash$	$(x_0 \in S \wedge h \in F \wedge f \in S \rightarrow (x_0 \in h \cdot \Omega f \leftrightarrow \exists y(y \in S \wedge x_0 = h \cdot y \wedge \rho(y, f))))$	($\forall$ D 3)
12.	7, 8; 3	$\vdash$	$(x_0 \in S \rightarrow (x_0 \in h \cdot \Omega f \leftrightarrow \exists y(y \in S \wedge x_0 = h \cdot y \wedge \rho(y, f))))$	($\rightarrow D$ 11 7 8)
13.	7, 8, 9; 3	$\vdash$	$\exists y(y \in S \wedge x_0 = h \cdot y \wedge \rho(y, f))$	($\leftrightarrow D, \rightarrow D$ 12 9)
14.	14;	$\vdash$	$y_0 \in S \wedge x_0 = h \cdot y_0 \wedge \rho(y_0, f)$	(HYP)
15.	14;	$\vdash$	$y_0 \in S \wedge \rho(y_0, f)$	($\wedge D$ 14)
16.	14, 8;	$\vdash$	$y_0 \in S \wedge f \in S \wedge \rho(y_0, f)$	($\wedge I$ 15 8)
17.	14, 8; 2	$\vdash$	$\rho(hy_0, hf)$	($\forall D, \rightarrow D$ 16 2)
18.	14, 8; 2, 1	$\vdash$	$\rho(x_0, hf)$	($\forall D, \rightarrow D$ 17 14 1)
19.	7, 8, 9; 3, 2, 1	$\vdash$	$\rho(x_0, hf)$	(CHOICE 18 13)
20.	7, 8; 5	$\vdash$	$hf \in S$	($\forall D, \rightarrow D$ 5 7 8)
21.	7, 8, 9; 4, 5, 1, 3, 2	$\vdash$	$x_0 \in \Omega(hf)$	($\forall D, \wedge D, \rightarrow D$ 4 20 10)
22.	; 4, 5, 1, 3, 2	$\vdash$	$h \in F \wedge f \in S \rightarrow (x_0 \in h \cdot \Omega f \rightarrow x_0 \in \Omega(hf))$	(DED 21)
23.	; 4, 5, 1, 3, 2	$\vdash$	$\forall x, f, h(h \in F \wedge f \in S \rightarrow (x \in h \cdot \Omega f \rightarrow x \in \Omega(hf)))$	($\forall I$ 22)
24.	; 4, 5, 1, 3, 2, 6	$\vdash$	$\forall h, f(h \in F \wedge f \in S \rightarrow h \cdot \Omega f \subset \Omega(hf))$	($\leftrightarrow D$ 6 23)
Thm.	;	$\vdash$	$\forall h, f(h \in F \wedge f \in S \rightarrow h \cdot \Omega f \subset \Omega(hf))$	()

The operation $\cdot_S$ should have been used in the theorem, in the assumptions, and in the proof. The polymorphic symbol $\cdot$ is used as before for convenience and readability.

Discussion

At first sight, a symbol mapping with $\{\mathrm{F} \Rightarrow \mathrm{S}, \cdot_F \Rightarrow \cdot_S\}$ seems to be an appropriate technique to remove the superficial differences between the problems 5.2.1 and 7.5.2.1. However, applying this symbol mapping causes confusion, because the mapping $\{\mathrm{F} \Rightarrow \mathrm{S}\}$ has to be applied at some occurrences only.

Such a symbol mapping is called inconsistent.

Instead a reformulation by abstraction works: It transforms the theorem, the assumptions, and the proofs of 5.2.1 to the abstracted theorem 5.2.1′ etc. by the mapping $\{h \cdot_F \texttt{term} \Rightarrow a_F(\texttt{term})\}$, where a_F is a new variable, and by removing certain formulae and quantifiers which become superfluous, such as $(h \in F)$ and $\forall h$. This abstraction is based on line 2 of the proof, which is a part of the definition of a leftcongruence on a semigroup. The same abstraction as for 5.2.1 transforms the theorem, the assumptions, and the proof of 7.5.2.1 to the abstracted theorem 7.5.2.1′ using the mapping $\{h \cdot_S \texttt{term} \Rightarrow a_S(\texttt{term})\}$, and also removing $(h \in F)$ and $\forall h$. This abstraction is based on line 2 of the proof of 7.5.2.1, which is part of the definition of a congruence on a semimodul. The theorem and the relevant assumptions of 5.2.1′ and 7.5.2.1′ are equal except for the symbols $\mathbf{a}_F$ and F. Hence, we need a final symbol mapping $\{\ \mathrm{a}_F \Rightarrow a_S;\ \mathrm{F} \Rightarrow \mathrm{S}\}$, which yields identical problems 5.2.1″ and 7.5.2.1″. The proof of the original theorem 7.5.2.1 can be obtained by a reversion of the abstraction that was applied to theorem 7.5.2.1 and to its assumptions and proof.

CASE 7: THEOREMS 6.9 and 13.7

The proofs of theorems 6.9 and 13.7 are marked as analogous in HUA, which means in this case for each of the corresponding subproofs essentially the same method should be used for proving the theorem.

Theorem 6.9
Let $F = W(X)$ be a free semigroup, S a nonempty set, and let $\Phi_0 : X \times S \mapsto S$ be a mapping, then there exists one and only one mapping $\Phi : X \times S \mapsto S$ which satisfies the semimodul condition
(1) $\forall f, g, s(f \in F \wedge g \in F \wedge s \in S \rightarrow \Phi((fg), s) = \Phi(f, \Phi(g, s)))$ and secondly
(2) Φ restricted to $(X \times S)$ is Φ_0.
This mapping Φ is uniquely defined by:
(D) $\forall x, u, s(x \in S \wedge u \in F \wedge s \in S \rightarrow \Phi((xu), s) = \Phi_0(x, \Phi(u, s)))$
and $xs = \Phi(x, s) = \Phi_0(x, s)$.

Theorem 13.7
Let $F = W(X)$ be a free semigroup, S an F-semimodul, A a semigroup, and let $\lambda_0 : X \times S \mapsto A$ be a mapping.
Then there exists one and only one mapping $\lambda : X \times S \mapsto A$ satisfying the automata- condition
(1') $\forall f, g, s(f \in F \wedge g \in F \wedge s \in S \rightarrow \lambda((f, g), s) = \lambda(f, (gs)) \cdot \lambda(g, s)))$ and secondly
(2') λ restricted to $X \times S$ is λ_0,
This mapping λ is uniquely defined by:
(D') $\forall x, u, s(x \in S \wedge u \in F \wedge s \in S \rightarrow \lambda((xu), s) = \lambda_0(x, (us)) \cdot \lambda(u, s))$ and $xs = \lambda(x, s) = \lambda_0(x, s)$.

The proofs of theorem 6.9 and theorem 13.7 can be structured as follows.
The proof structure of 6.9. is:

- Part 1
  Theorem: Uniqueness of Φ, i.e.,
  If $(u \in F \wedge s \in S)$ and if there exists Φ' for which (1) and (2) are true, then $\Phi'(u, s) = \Phi(u, s))$
  The proof is by induction on the level of generation of u:

1. base step: the *relevant assumptions* are (2)
2. induction step: the *relevant assumptions* are the induction hypothesis; (1) is true for (Φ'); Φ' is a mapping; (2) is true for (Φ');(2) is true for (Φ); (1) is true for (Φ).

- Part 2
  Theorem: Φ is a mapping , i.e.,
  $\forall u_1, u_2, s_1, s_2 (u_1, u_2 \in F \wedge s_1, s_2 \in S \wedge u_1 = u_2 \wedge s_1 = s_2 \rightarrow \Phi(u_1, s_1) = \Phi(u_2, s_2)))$
  The proof is by induction on the level of generation of u_1, u_2:

  1. base step: the *relevant assumptions* are that Φ_0 is a mapping and that (2) is true for (Φ).
  2. induction step: the *relevant assumptions* are the induction hypothesis; that F is a free semigroup; that (D) is true for (Φ); transitivity of =; and the theorem of part 2.1.

- Part 3
  Theorem: The mapping Φ of part 2 fulfills condition (1), i.e.,
  $\forall u, w, s (u, w \in F \wedge s \in S \rightarrow \Phi((uw), s) = \Phi(u, \Phi(w, s)))$
  The proof is by induction on the level of generation of u:

  1. base step: the *relevant assumptions* are definition of Φ; (D) is true for Φ.
  2. induction step: the *relevant assumptions* are the induction hypothesis; the theorem of part 2; that F is a semigroup; that (D) is true for (Φ) with uw=u'; and that (D) is true for (Φ) with ws=s'.

The proof structure of 13.7 is:

- Part 1
  Theorem: Uniqueness of λ, i.e.,
  If $(u \in F \wedge s \in S)$ and if there exists λ' for which (1') and (2') are true, then $\lambda'(u, s) = \lambda(u, s)$
  The proof is by induction on the level of generation of u:

  1. base step: the *relevant assumptions* are (2')

2. induction step: the *relevant assumptions* are the induction hypothesis; (1) is true for (λ'); (2') is true for (λ'); (2') is true for (λ); (1') is true for (λ); and additionally, $\forall x, y, z, t(x = y \wedge z = t \rightarrow x \cdot_A z = y \cdot_A t)$.

- Part 2
  Theorem: Existence of a mapping λ, i.e.,
  $\forall u_1, u_2, s_1, s_2(u_1, u_2 \in F \wedge s_1, s_2 \in S \wedge u_1 = u_2 \wedge s_1 = s_2 \rightarrow \lambda(u_1, s_1) = \lambda(u_2, s_2)))$
  The proof is by induction on the level of generation of u_1, u_2:

  1. base step: the *relevant assumptions* are that λ_0 is a mapping and that (2') is true for (λ).
  2. induction step: the *relevant assumptions* are the induction hypothesis; that F is a free semigroup; that (D') is true for (λ); the transitivity of =; the theorem of part 2.1; and additionally, $\forall x, y, z, t(x = y \wedge z = t \rightarrow x \cdot_A z = y \cdot_A t)$.

- Part 3
  Theorem: The mapping λ of part 2 fulfills the condition (1'), i.e.,
  $\forall u, w, s(u, w \in F \wedge s \in S \rightarrow \lambda((uw), s) = \lambda(u, (ws)) \cdot \lambda(w, s))$
  The proof is by induction on the level of generation of u:

  1. base step: the *relevant assumptions* are the definition of λ; and that (D) is true for λ.
  2. induction step: the *relevant assumptions* are the induction hypothesis; the theorem of part 2.; that F is a semigroup; that (D') is true for λ with uw=u'; that (D') is true for λ with ws=s'; and additionally,
     (uw)s=u(ws) (since S is an F-semimodul);
     $\forall x, y, z, t(x = y \wedge z = t \rightarrow x \cdot_A z = y \cdot_A t)$.

Discussion

Based on the following correspondences between the assumptions
(1) – (1'),
(2) – (2'), and
(D) – (D'),

that are induced by the structure of the theorems a reformulation composed of the following steps succeeds in making the theorems and assumptions equal.

1. The term mapping
   $\{\Phi(f, \Phi(g, s)) \Rightarrow \lambda(f, (g \cdot s)) \cdot \lambda(g, s)) \ ; \ \Phi_0(x, \Phi(u, s)) \Rightarrow \lambda_0(x, u \cdot s) \cdot \lambda(u, s)\}$
   transforms the subtheorems and the assumptions of 6.9 to that of 13.7. However, the application of this term mapping, which is based on a comparison of the assumptions and of the theorems is not sufficient for producing a verifiable proof, for which we need additional assumptions.

2. These additional preconditions are necessary for the subproofs 1.2, 2.2, and 3.2 of theorem 13.7. These preconditions are
   $\forall x, y, z, t(x = y \wedge z = t \rightarrow x \cdot_A z = y \cdot_A t)$ for 1.2, 2.2, and 3.2, and for the subproof 3.2 $\forall x, y, z(x \in F \wedge y \in F \wedge z \in S \rightarrow (xy)z = x(yz))$. The first formula is assumed to be in the knowledge base, since $\cdot_A$ is a function by definition. The second formula is in the knowledge base since S is assumed to be an F-semimodul.

Conclusion

This report contains those theorems and their proofs in a Natural Deduction format that are explicitly marked as analogous in HUA. In addition, we gave some hints and a discussion for each case , to show how the actual analogy could be established. However, in order to keep this self-contained such that it may serve as a test set for other workers in the field as well, we did not show how our system actually established the analogy and how it finds the respective proofs.

An account of our approach to analogy-driven theorem proving, by which all of these cases could be solved, is given in [5].

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