New Abstract Criteria for Termination and Confluence of Conditional Rewrite Systems

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Abstract

We consider the problem of verifying confluence and termination of conditional term rewriting systems (TRSs). For unconditional TRSs the critical pair lemma holds which enables a finite test for confluence of (finite) terminating systems. And for ensuring termination of unconditional TRSs a couple of methods for constructing appropiate well-founded term orderings are known. If however termination is not guaranteed then proving confluence is much more difficult. Recently we have obtained some interesting results for unconditional TRSs which provide sufficient criteria for termination plus confluence in terms of restricted termination and confluence properties. In particular, we have shown that any innermost terminating and locally confluent overlay system is complete, i.e. terminating and confluent. Here we generalize our approach to the conditional case and show how to solve the additional complications due to the presence of conditions in the rules. Our main result can be stated as follows: Any conditional TRS which is an innermost terminating semantical overlay system such that all (conditional) critical pairs are joinable is complete.

Key Words: Conditional term rewriting systems, overlay systems, confluence, termination, weak termination, innermost termination.

1 Introduction

Due to the fact that termination is a fundamental property of term rewriting systems (TRSs for short) but undecidable in general (see [HL78]), many sufficient criteria, techniques and methods for proving termination have been developed (see [Der87] for

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a survey). Most practically applicable approaches are based on reduction orderings, i.e. well-founded term orderings which are stable w.r.t. substitutions and monotonic w.r.t. the term structure.

On the other hand, in many rewriting based computation models, e.g. in functional programming languages, the indeterminism of general rewriting is often restricted by imposing some fixed rewriting strategy. For instance, a frequent restriction is innermost reduction, i.e. to require that every reduction step takes place at an innermost position of the term to be reduced. Innermost reduction corresponds closely to the functional evaluation mechanism employed in functional programming languages like LISP or ML. Of course, it may be the case that correspondingly restricted computations, i.e. innermost reduction sequences always terminate but arbitrary computations (reduction sequences) do not necessarily terminate. A very simple example illustrating this gap is the following:

Example 1.1 *Let $\mathcal{R} = \{f(a) \to f(a), a \to b\}$. Then we have e.g. the infinite reduction sequence $f(a) \to f(a) \to f(a) \to \dots$, which uses only non-innermost reduction steps. But of course, every innermost derivation in R (e.g. $f(a) \to f(b)$) is terminating.*

Other kinds of restrictions imposed on rewriting steps might also be conceivable according to the intended purpose, e.g. leftmost outermost, top-down, bottom-up or other context-dependent strategies. Unfortunately, very little is known about termination of rewriting under such restrictions and its relation to (uniform) termination. In fact, there is one major exception, namely concerning the important and thoroughly investigated class of so-called *orthogonal* TRSs, i.e. TRSs which are left-linear and non-overlapping (see [Klo92] for a survey of basic ideas, concepts and results about the theory of orthogonal TRSs). It is well-known that any orthogonal (unconditional) TRSs is confluent notwithstanding the fact that it may be non-terminating.

Recently we have shown (in [Gra92], [Gra93]) that some basic termination (and confluence) properties of orthogonal TRSs can be generalized to the case of non-overlapping, but not necessarily left-linear TRSs, as well as to the more general case of locally confluent overlay systems. In particular these results include interesting sufficient conditions for inferring (uniform or strong) termination (plus conflence) from innermost termination and local confluence, for the case of unconditional TRSs.

Here we shall study for which cases and under what conditions these results can be generalized to conditional TRSs (CTRSs for short). In this paper we mainly concentrate on so-called join CTRSs where equality in the conditions is recursively interpreted as joinability. It seems plausible that most of the ideas, results and proofs can be carried over to other types of CTRSs, e.g. semi-equational and normal ones.

Since it is well-known that conditional rewriting is much more difficult to handle in theory and practice, it was not clear a priori that our results carry over to the conditional case. But this is indeed possible (for the main results) as we shall show by a careful analysis and inspection of the proofs for the unconditional case and by taking into account the additional complications arising with CTRSs. From a technical point of view one of the main problems in generalizing our results is the well-known fact that 'variable overlaps' may be critical for CTRSs and need to be explicitly handled.

Before going into details let us give a summary of our main results:[1]

- If a (join) CTRS $\mathcal{R}$ is semantically non-overlapping then weak innermost termination is equivalent to (strong) innermost termination of $\mathcal{R}$ (see Lemma 3.5).
- If a (join) CTRS $\mathcal{R}$ is semantically non-overlapping and (strongly) innermost terminating then it is (strongly) terminating (see Theorem 3.13).
- If a (join) CTRS $\mathcal{R}$ is semantically non-overlapping then there is no innermost reduction step $s \underset{i}{\rightarrow} t$ in $\mathcal{R}$ with $\infty(s)$ but $\neg\infty(t)$ (see Lemma 3.15).
- If a CTRS $\mathcal{R}$ is semantically non-overlapping, weakly terminating and non-erasing then it is (strongly) terminating (see Theorem 3.16).
- If a (join) CTRS $\mathcal{R}$ is a terminating semantical overlay system with joinable critical pairs then it is confluent, hence complete (see Theorem 3.10).
- If a (join) CTRS $\mathcal{R}$ is an innermost terminating, semantical overlay system with joinable critical pairs then it is (strongly) terminating and confluent, hence complete (see Theorem 3.19).

The rest of the paper is structured as follows. Firstly, we introduce the basic definitions and notions needed later on. In section 3 we study restricted termination (and confluence) properties of CTRSs and their interrelations, in particular innermost, weak and strong termination of non-overlapping CTRS. More generally we also investigate the termination (and confluence) behaviour of certain restricted classes of possibly overlapping CTRSs with joinable critical pairs. And finally, related work as well as some open problems are discussed.

2 Preliminaries

2.1 Basic Notions and Notations

We briefly recall the basic terminology needed for dealing with (C)TRSs (e.g. [Klo92], [DJ90]). Let $\mathcal{V}$ be a countably infinite set of *variables* and $\mathcal{F}$ be a set of *function symbols* with $\mathcal{V} \cap \mathcal{F} = \emptyset$. Associated to every $f \in \mathcal{F}$ is a natural number denoting its arity. Function symbols of arity 0 are called *constants*. The set $\mathcal{T}(\mathcal{F}, \mathcal{V})$ of *terms* over $\mathcal{F}$ and $\mathcal{V}$ is the smallest set with (1) $\mathcal{V} \subseteq \mathcal{T}(\mathcal{F}, \mathcal{V})$ and (2) if $f \in \mathcal{F}$ has arity n and $t_1, \ldots, t_n \in \mathcal{T}(\mathcal{F}, \mathcal{V})$ then $f(t_1, \ldots, t_n) \in \mathcal{T}(\mathcal{F}, \mathcal{V})$. If some function symbols are allowed to be varyadic then the definition of $\mathcal{T}(\mathcal{F}, \mathcal{V})$ is generalized in an obvious way. The set of all *ground terms* (over $\mathcal{F}$), i.e. terms with no variables, is denoted by $\mathcal{T}(\mathcal{F})$.

[1]The definitions involved here are presented below.

In the following we shall always assume that $\mathcal{T}(\mathcal{F})$ is non-empty, i.e. there is at least one constant in $\mathcal{F}$. Identity of terms is denoted by $\equiv$. The set of variables occurring in a term t is denoted by $V(t)$. Positions or occurrences of a term consist of sequences of natural numbers and are compared by the usual lexicographic ordering (which we shall ambiguously denote by $\leq$). The topmost position of a term, i.e. of its root symbol, is denoted by λ, the 'empty' string. Two uncomparable positions p and q are said to be *parallel* or *disjoint* which is denoted by $p|q$. If $p \leq q$ we say that p is above q or q is below p. The top symbol of a term t is denoted by $root(t)$. The set of all positions of a term t is denoted by $O(t)$. Concatenation of positions is denoted by juxtaposition. If p is a position and Π a set of positions then $p\Pi$ denotes $\{pq \mid q \in \Pi\}$ and, similarly, Πp stands for $\{qp \mid q \in \Pi\}$.

A *context* $C[,\ldots,]$ is a term with 'holes', i.e. a term in $\mathcal{T}(\mathcal{F} \uplus \{\Box\}, \mathcal{V})$ where $\Box$ is a new special constant symbol. If $C[,\ldots,]$ is a context with n occurrences of $\Box$ and $t_1,\ldots,t_n$ are terms then $C[t_1,\ldots,t_n]$ is the term obtained from $C[,\ldots,]$ by replacing from left to right the occurrences of $\Box$ by $t_1,\ldots,t_n$. A context containing precisely one occurrence of $\Box$ is denoted by $C[]$. If $C[s]$ is the context $C[]$ with $\Box$ replaced by s at position p in $C[]$, then we also write $C[s]_p$. If $C[s,\ldots,s]$ is $C[,\ldots,]$ with $\Box$ replaced by s at all positions from some set Π of mutually disjoint positions of $C[,\ldots,]$, then – slightly abusing notation – we also write $C[s]_\Pi$. A *non-empty* context is a term from $\mathcal{T}(\mathcal{F} \uplus \{\Box\}, \mathcal{V}) \setminus \mathcal{T}(\mathcal{F}, \mathcal{V})$ which is different from $\Box$. A term s is a *subterm* of a term t if there exists a context $C[]$ (and some position $p \in O(t)$) with $t \equiv C[s]$ $(= C[s]_p)$. If in addition $C[] \not\equiv \Box$ (or equivalently $p \neq \lambda$) then s is a *proper* subterm of t. The subterm of t at position $p \in O(t)$ is also denoted by t/p. The result of replacing in t the subterm at position $p \in O(t)$ by s is denoted by $t[p \leftarrow s]$.[2] A substitution σ is a mapping from $\mathcal{V}$ to $\mathcal{T}(\mathcal{F}, \mathcal{V})$ such that its domain $dom(\sigma)$ $\{x \in \mathcal{V} | \sigma x \not\equiv x\}$ is finite. Its homomorphic extension to a mapping from $\mathcal{T}(\mathcal{F}, \mathcal{V})$ to $\mathcal{T}(\mathcal{F}, \mathcal{V})$ is also denoted by σ.

A *term rewriting system (TRS)* is a pair $(\mathcal{R}, \mathcal{F})$ consisting of a signature $\mathcal{F}$ and a set $\mathcal{R} \subseteq \mathcal{T}(\mathcal{F}, \mathcal{V}) \times \mathcal{T}(\mathcal{F}, \mathcal{V})$ of (rewrite) rules (l, r) denoted by $l \to r$ with $l \notin \mathcal{V}$ and $V(r) \subseteq V(l)$. [3] Instead of $(\mathcal{R}, \mathcal{F})$ we also write $\mathcal{R}^{\mathcal{F}}$ or simply $\mathcal{R}$ when $\mathcal{F}$ is clear from the context or irrelevant.

Given a TRS $\mathcal{R}^{\mathcal{F}}$ the rewrite relation $\to_{\mathcal{R}^{\mathcal{F}}}$ for terms $s, t \in \mathcal{T}(\mathcal{F}, \mathcal{V})$ is defined as follows: $s \to_{\mathcal{R}^{\mathcal{F}}} t$ if there exists a rule $l \to r \in \mathcal{R}$, a substitution σ and a context $C[]$ such that $s \equiv C[\sigma l]$ and $t \equiv C[\sigma r]$. We also write $\to_{\mathcal{R}}$ or simply $\to$ when $\mathcal{F}$ or $\mathcal{R}^{\mathcal{F}}$ is clear from the context, respectively. The symmetric, transitive and transitive-reflexive closures of $\to$ are denoted by $\leftrightarrow$, $\to^+$ and $\to^*$, respectively. By $s \to^m t$ we mean that s is reduced to t in m steps. Accordingly $s \to^{\leq n} t$ means $s \to^m t$ for some $m \leq n$. Two terms s, t are *joinable in* $\mathcal{R}$, denoted by $s \downarrow_{\mathcal{R}} t$, if there exists a term u with $s \to_{\mathcal{R}}^* u \; {}_{\mathcal{R}}^{*}\!\leftarrow t$. A term s is *irreducible* or a *normal form* if there is no term t with $s \to t$. If $s \to^* t$ then t is said to be a *reduct* of s. The set of all terms which are irreducible w.r.t. some TRS $\mathcal{R}$ is denoted by $NF(\mathcal{R})$ (more precisely for

[2]This notation for replacing subterms should not be confused with the notations for contexts introduced above.

[3]This restriction of excluding variable left-hand sides and right-hand side extra-variables is not a severe one. In particular, concerning termination of rewriting it only excludes trivial cases.

$\mathcal{R} = \mathcal{R}^{\mathcal{F}}$: $NF(\mathcal{R}^{\mathcal{F}}) := \{s \in \mathcal{T}(\mathcal{F}, \mathcal{V}) \mid \int \mathcal{R}^{\mathcal{F}}\text{-irreducible}\}$. Moreover we shall also use the notation $NF_{\mathcal{F}}(\mathcal{R}) := \{s \in \mathcal{T}(\mathcal{F}, \mathcal{V}) \mid \int \mathcal{R}\text{-irreducible}\}$. A TRS $\mathcal{R}$ is *terminating* or *strongly normalizing (SN)* if $\rightarrow$ is noetherian, i.e. if there is no infinite reduction sequence $s_1 \rightarrow s_2 \rightarrow s_3 \rightarrow \cdots$. It is said to be *weakly terminating* or *weakly normalizing (WN)* if for every term there exists a normal form. If $s \rightarrow t$ then – in order to make explicit the position p of the reduced subterm and the applied rule $l \rightarrow r$ – we shall sometimes use the notation $s \rightarrow_{p,l\rightarrow r} t$ or $s \rightarrow_p t$. A step of the form $s \rightarrow_\lambda t$ is said to be a *root reduction (step)*. If for a term s a root reduction at position $p \in O(t)$ is possible then s/p is said to be a redex.[4] A reduction step $s \rightarrow t$ by applying some rule of $\mathcal{R}$ at position p in s is *innermost* if every proper subterm of s/p is irreducible. In that case we also write $s \underset{i}{\rightarrow} t$. $\mathcal{R}$ is *(strongly) innermost terminating* or *(strongly) innermost normalizing (SIN)* if every sequence of innermost reduction steps terminates. It is *weakly innermost terminating* or *weakly innermost normalizing (WIN)* if for every term s there exists a terminating sequence of innermost reduction steps starting with s. By $\infty(s)$ we denote the property that there exists an infinite ($\mathcal{R}$-) derivation starting with s. Accordingly, $\neg\infty(s)$ means that every derivation starting with s is finite. By $\infty_i(s)$ we denote the property that there exists an infinite innermost derivation starting with s.

A *partial ordering* $>$ on a set D is a transitive and irreflexive binary relation on D. A partial ordering $>$ on $\mathcal{T}(\mathcal{F}, \mathcal{V})$ is said to be *monotonic (w.r.t. the term structure)* if it possesses the *replacement property*

$$s > t \quad \Longrightarrow \quad C[s] > C[t]$$

for all $s, t, C[]$. It is *stable (w.r.t. substitutions)* if

$$s > t \quad \Longrightarrow \quad \sigma s > \sigma t$$

for all s, t, σ. A *term ordering on $\mathcal{T}(\mathcal{F}, \mathcal{V})$* is a monotonic and stable partial ordering on $\mathcal{T}(\mathcal{F}, \mathcal{V})$. A *reduction ordering* is a well-founded term ordering.

A TRS is *confluent* or has the *Church-Rosser property (CR)* if ${}^*\!\leftarrow \circ \rightarrow^* \subseteq \rightarrow^* \circ {}^*\!\leftarrow$ and *weakly Church-Rosser (WCR)* or *locally confluent* if $\leftarrow \circ \rightarrow \subseteq \rightarrow^* \circ {}^*\!\leftarrow$.[5] A confluent and terminating TRS is said to be *convergent* or *complete (COMP)*. If $l_1 \rightarrow r_1$, $l_2 \rightarrow r_2$ are two rules[6] of $\mathcal{R}$ with p some non-variable position of $l_2 \rightarrow r_2$ such that l_1 and l_2/p are unifiable with most general unifier σ then $\langle \sigma(l_2[p \leftarrow r_1]), \sigma(r_2) \rangle$ is said to be a *critical pair (CP)* of $\mathcal{R}$ (obtained by overlapping $l_1 \rightarrow r_1$ with $l_2 \rightarrow r_2$ at position p). It is well-known that for terminating TRSs local confluence is equivalent to joinability of all critical pairs *(JCP)*. A TRS $\mathcal{R}$ is said to be *non-overlapping (NO)* if there is no critical pair between rules of $\mathcal{R}$. It is *left-linear (LL)* if every variable occurs at most once in every left hand side of an $\mathcal{R}$-rule. $\mathcal{R}$ is *orthogonal (ORTH)*[7] if

[4]This is a slight abuse of the usual notion of a redex which also comprises the information which rule is applicable. For orthogonal TRSs the corresponding applicable rule is uniquely determined but not in general.

[5]Here, '$\circ$' denotes relation composition.

[6]W.l.o.g. we assume that they do not share common variables.

[7]In the literature this orthogonality property is sometimes called 'regularity'.

it is left-linear and non-overlapping. $\mathcal{R}$ is said to be *weakly orthogonal* (cf. [Klo92]) if it is left-linear and has only trivial critical pairs, i.e. if $\langle s, t\rangle$ is a critical pair then $s = t$. It is *non-erasing (NE)* if $V(r) = V(l)$ for every rule $l \to r \in \mathcal{R}$. If every critical pair of a TRS $\mathcal{R}$ is obtained by an *overlay*, i.e. by overlapping left hand sides of rules at top position then $\mathcal{R}$ is said to be an *overlay system (OS)*.

For the sake of readability let us summarize the abbreviating notions defined above which shall be freely used in the sequel.[8]

Abbreviations:

SN	=	strongly normalizing (terminating) [9]
WN	=	weakly normalizing (weakly terminating)
SIN	=	(strongly) innermost normalizing (innermost terminating)
WIN	=	weakly innermost normalizing (weakly innermost terminating)
NO	=	non-overlapping
LL	=	left-linear
$ORTH$	=	orthogonal (non-overlapping and left-linear)
NE	=	non-erasing
CP	=	critical pair(s)
JCP	=	joinable critical pairs
CR	=	confluence (Church-Rosser property)
WCR	=	local confluence (weak Church-Rosser property)
$COMP$	=	completeness (convergence)
OS	=	overlay system
$\infty(s)$	=	there exists an infinite derivation starting with s
$\infty_i(s)$	=	there exists an infinite innermost derivation starting with s
$\neg\infty(s)$	=	there exists no infinite derivation starting with s
$\neg\infty_i(s)$	=	there exists no infinite innermost derivation starting with s

By $P(\mathcal{R})$ we mean that the TRSs $\mathcal{R}$ has property P. Moreover we also ambiguously use the notation $P(t)$ for terms t provided there is a sensible local interpretation for $P(t)$. For instance, $CR(t)$ is to denote the property that whenever we have $t \to^* v$ and $t \to^* w$ then there exists a term s with $v \to^* s$ and $w \to^* s$.

2.2 Conditional Term Rewriting Systems

Moreover, we need some basic terminology about conditional term rewriting systems (CTRSs) (cf. e.g. [DOS88a], [DOS88b], [Klo92], [Mid93]).

[8] These abbreviations are mainly borrowed from [Klo87], [Klo92].

[9] In the sequel we shall prefer 'terminating' instead of 'normalizing' in verbal phrases since it seems to be the more usual notion in literature.

Definition 2.1 *A* CTRS *is a pair* $(\mathcal{R}, \mathcal{F})$ *consisting of a signature* $\mathcal{F}$ *and a set of* conditional rewrite rules *of the form*

$$s_1 = t_1 \ \wedge \ldots \wedge \ s_n = t_n \implies l \to r$$

with $s_1, \ldots, s_n, t_1, \ldots, t_n, l, r \in \mathcal{T}(\mathcal{F}, \mathcal{V})$. *Moreover, we require* $l \notin \mathcal{V}$ *and* $V(r) \subseteq V(l)$ *as for unconditional TRSs, i.e. no variable left hand sides and no extra variables on the right hand side. Extra variables in conditions are allowed if not stated otherwise. If the condition is empty, i.e.* $n = 0$, *we simply write* $l \to r$. *Instead of* $(\mathcal{R}, \mathcal{F})$ *we also write* $\mathcal{R}^{\mathcal{F}}$ *or simply* $\mathcal{R}$ *when* $\mathcal{F}$ *is clear from the context or irrelevant.*

Depending on the interpretation of the equality sign in the conditions of rewrite rules, different reduction relations may be associated with a given CTRS.

Definition 2.2

(1) In a join CTRS $\mathcal{R}$ *the equality sign in the conditions of rewrite rules is interpreted as joinability. Formally this means:* $s \to_{\mathcal{R}} t$ *if there exists a rewrite rule* $s_1 = t_1 \ \wedge \ldots \wedge \ s_n = t_n \implies l \to r \ \in \mathcal{R}$, *a substitution* σ *and a context* $C[]$ *such that* $s \equiv C[\sigma l]$, $t \equiv C[\sigma r]$ *and* $\sigma s_i \downarrow_{\mathcal{R}} \sigma t_i$ *for all* $i \in \{1, \ldots, n\}$. *For rewrite rules of a join CTRS we shall also use the notation* $s_1 \downarrow t_1 \ \wedge \ldots \wedge \ s_n \downarrow t_n \implies l \to r$.

(2) Semi-equational *CTRSs are obtained by interpreting the equality sign in the conditions as convertibility, i.e. as* $\overset{*}{\leftrightarrow}$.

(3) Normal *CTRSs have rules of the form* $s_1 \to_!^* t_1 \ \wedge \ \ldots \ \wedge \ s_n \to_!^* t_n \implies l \to r$ *where* $s \to_!^* t$ *means that* $s \to^* t$ *and* t *is a ground normal form w.r.t. the unconditional part of the CTRS* $\mathcal{R}$ *considered (which is obtained from* $\mathcal{R}$ *by removing all conditions).*

(4) A generalized *CTRS has rules of the form* $P_1 \wedge \ldots, \wedge P_n \implies l \to r$ *where the conditions* P_i, $1 \le i \le n$, *are formulated in a general mathematical framework, e.g. in some first order language.*

Definition 2.3 *The reduction relation corresponding to a given (join, semi-equational or normal) CTRS* $\mathcal{R}$ *is inductively defined as follows* ($\square$ *denotes* $\downarrow$, $\overset{*}{\leftrightarrow}$ *or* $\to_!^*$, *respectively):*

$$\begin{array}{rcl}
\mathcal{R}_0 & = & \emptyset \, , \\
\mathcal{R}_{i+1} & = & \{\sigma l \to \sigma r | \ s_1 \square t_1 \ \wedge \ldots \wedge \ s_n \square t_n \Longrightarrow l \to r \ \in \mathcal{R}, \\
 & & \sigma s_j \square_{\mathcal{R}_i} \sigma t_j \text{ for } j = 1, \ldots, n\} \, ,^{10} \\
s \to_{\mathcal{R}} t & :\Longleftrightarrow & s \to_{\mathcal{R}_i} t \text{ for some } i \ge 0, \text{ i.e. } \to_{\mathcal{R}} = \bigcup\limits_{i \ge 0} \to_{\mathcal{R}_i} \, .
\end{array}$$

[10] Note in particular that all unconditional rules of $\mathcal{R}$ are contained in $\mathcal{R}_1$ (because the empty conditions are vacuously satisfied) as well as all conditional rules with trivial conditions only, i.e. conditions of the form $s \square s$. In fact, rules of the latter class can be considered to be essentially unconditional.

Definition 2.4 *If $s \to_{\mathcal{R}} t$ then the* depth *of $s \to_{\mathcal{R}} t$ is defined to be the minimal n with $s \to_{\mathcal{R}_n} t$. For $s \to^*_{\mathcal{R}} t$ and $s \downarrow_{\mathcal{R}} t$ depths are defined analogously. More precisely, if $s \to^*_{\mathcal{R}} t$ then the* depth *of $s \to^*_{\mathcal{R}} t$ is defined to be the minimal n with $s \to^*_{\mathcal{R}_n} t$. The* depth *of $s \downarrow_{\mathcal{R}} t$ is the minimal n with $s \downarrow_{\mathcal{R}_n} t$.*

If the depth of $s \to^*_{\mathcal{R}} t$ is at most n we denote this by $s \xrightarrow{n}_{\mathcal{R}} t$.

For the sake of readability we shall use in the following some compact notations for conditional rules and conjunctions of conditions. When writing $P \Longrightarrow l \to r$ for some conditional rewrite rule then P stands for the conjunction of all conditions. Similarly, if P is $s_1 = t_1 \wedge \ldots \wedge s_n = t_n$, then $P\downarrow$ means $s_1 \downarrow t_1 \wedge \ldots \ldots \wedge s_n \downarrow t_n$, and $\sigma(P)$ is to denote $\sigma(s_1) = \sigma(t_1) \wedge \ldots \wedge \sigma(s_n) = \sigma(t_n)$.

Definition 2.5 *Let $\mathcal{R}$ be a join CTRS, and let $P_1 \Longrightarrow l_1 \to r_1$ and $P_2 \Longrightarrow l_2 \to r_2$ be two rewrite rules of $\mathcal{R}$ which have no variables in common.*[11] *Suppose $l_1 \equiv C[t]_p$ with $t \notin \mathcal{V}$ for some (possibly empty) context $C[,\ldots,]$ such that t and l_2 are unifiable with most general unifier σ, i.e. $\sigma(t) = \sigma(l_1/p) = \sigma(l_2)$. Then $\sigma(P_1) \wedge \sigma(P_2) \Longrightarrow \sigma(C[r_2]) = \sigma(r_1)$ is said to be a* (conditional) critical pair *of $\mathcal{R}$. If the two rules are renamed versions of the same rule of $\mathcal{R}$, we do not consider the case $C[] \equiv \Box$, i.e. we do not overlap a rule with itself at root position. A (conditional) critical pair $P \Longrightarrow s = t$ is said to be* joinable *if $\sigma(s) \downarrow_{\mathcal{R}} \sigma(t)$ for every substitution σ with $\sigma(P)\downarrow$. A substitution σ which satisfies the conditions, i.e. for which $\sigma(P)\downarrow$ holds, is said to be* feasible. *Otherwise σ is* unfeasible. *Analogously, a (conditional) critical pair is said to be* feasible (unfeasible) *if there exists some (no) feasible substitution for it.*

Note that testing joinability of conditional critical pairs is in general much more difficult than in the unconditional case since one has to consider all substitutions which satisfy the correspondingly instantiated conditions. Moreover, the critical pair lemma does not hold for CTRSs in general as shown e.g. by the following example.

Example 2.6 *([BK86]) Consider the join CTRS*

$$\mathcal{R} = \left\{ \begin{array}{rcl} x \;\downarrow\; f(x) \Longrightarrow f(x) & \to & a \\ b & \to & f(b)\,. \end{array} \right.$$

Here we get $f(b) \to a$ due to $b \downarrow f(b)$ and hence $f(f(b)) \to f(a)$. We also have $f(f(b)) \to a$ because of $f(b) \downarrow f(f(b))$. But a and $f(a)$ do not have a common reduct which is easily shown. Thus $\mathcal{R}$ is not locally confluent despite the lack of critical pairs. Note moreover that $\mathcal{R}$ is even orthogonal when considered as unconditional TRS, i.e. when omitting the condition in the first rule.

Definition 2.7 *(cf. [BK86], [Klo92]) Let $\mathcal{R}$ be a CTRS and let $\mathcal{R}_u$ be its unconditional version, i.e. $\mathcal{R}_u := \{l \to r \mid P \Longrightarrow l \to r \in \mathcal{R}\}$. Then $\mathcal{R}$ is said to be* (syntactically) left-linear / non-overlapping / orthogonal / weakly orthogonal) *if $\mathcal{R}_u$ is left-linear / non-overlapping / orthogonal / weakly orthogonal.*

[11] Of course, this variable disjointness can always be achieved by appropiately renaming rules.

According to this definition example 2.6 above shows that orthogonal CTRSs need not be confluent. But note that the CTRS $\mathcal{R}$ defined in example 2.6 is not innermost terminating. This indicates that their might be some hope for generalizing (some of) our results for the unconditional case to the conditional one, in particular those involving the innermost termination property.

The careful reader may have observed that the definition of being (syntactically) non-overlapping (in 2.7) above is somehow rather restrictive. Namely, the case that there exist conditional critical pairs all of which are infeasible (and hence should not be 'properly critical') is not covered. This motivates the following.

Definition 2.8 *A CTRS $\mathcal{R}$ is said to be* semantically non-overlapping (*SEM-NO*) *if all its critical pairs are infeasible.*[12]

Clearly, a (syntactically) non-overlapping CTRS is semantically non-overlapping, too, but not vice-versa in general. Analogously the property of being an overlay system can be defined alternatively.

Definition 2.9 *A CTRS $\mathcal{R}$ is said to be a* (conditional syntactical) overlay system (OS) *if $\mathcal{R}_u$ is an unconditional overlay system (cf. [Klo92]). It is said to be a* (conditional) semantical overlay system (*SEM-OS*) *if all its feasible critical pairs are critical overlays.*

Note that the syntactical versions of the properties of being non-overlapping and of being an overlay system can be easily tested (for finite systems) whereas establishing their semantical versions may be very difficult. The reason is that e.g. for proving *SEM-NO* for some CTRS $\mathcal{R}$ one has to show that all (conditional) critical pairs of $\mathcal{R}$ are infeasible. But this is undecidable in general.

The other basic notions for unconditional TRSs introduced above generalize in a straightforward manner to CTRSs.

In general, conditional rewriting is much more complicated than unconditional rewriting. For instance, the rewrite relation may be undecidable even for complete CTRSs without extra variables in the conditions (cf. [Kap84]).

Definition 2.10 *([DOS88a]) A CTRS $\mathcal{R}$ is* decreasing *if there exists an extension $>$ of the reduction relation induced by $\mathcal{R}$ which satisfies the following properties:*

(1) $>$ is noetherian.

(2) $>$ has the subterm property, i.e. $C[s] > s$ for every term s and every non-empty context $C[]$.

[12]In [DOS88b] a CTRS $\mathcal{R}$ is said to be non-overlapping if it has no feasible, non-trivial critical pairs (where a critical pair $P \Longrightarrow s = t$ is trivial if s is identical to t). Hence, the definition of being non-overlapping in the sense of [DOS88b] is slightly more general than our notion of being semantically non-overlapping since trivial critical pairs are allowed in the former. Allowing trivial critical pairs in our definition one might speak of weakly semantically non-overlapping CTRSs.

(3) *If* $s_1 = t_1 \ \wedge \ldots \wedge \ s_n = t_n \implies l \to r$ *is a rule in* $\mathcal{R}$ *and* σ *is a substitution then* $\sigma l > \sigma s_i$ *and* $\sigma l > \sigma t_i$ *for* $i = 1, \ldots, n$.

A CTRS $\mathcal{R}$ *is* reductive *(cf. [JW86]) if there exists a well-founded monotonic extension* $>$ *of the reduction relation induced by* $\mathcal{R}$ *satisfying (3).*

Clearly, every reductive system is decreasing and any decreasing system is terminating. Decreasingness exactly captures the finiteness of recursive evaluation of terms (cf. [DO90]). For decreasing (join) CTRSs all the basic notions are decidable, e.g. reducibility and joinability. Moreover, fundamental results like the critical pair lemma hold for decreasing (join) CTRSs which is not the case in general for arbitrary (terminating join) CTRSs.

In the following we shall tacitly assume that all CTRSs considered are join CTRSs (which is the most important case in practice), except for cases where another kind of CTRSs is explicitly mentioned.

3 Restricted Termination and Confluence Properties of Conditional Term Rewriting Systems

We shall study now under which conditions various restricted kinds of termination imply (strong) termination (and also confluence under some additional assumptions) of (join) CTRSs. Firstly we summarize the most important known results on confluence and termination of unconditional orthogonal TRSs.

Theorem 3.1 *(c.f. e.g. [Ros73], [O'D77], [Klo92]) Let* $\mathcal{R}$ *be a TRS with* $ORTH(\mathcal{R})$. *Then we have:*

(1) $CR(\mathcal{R})$.

(2a) $\forall t : [WIN(t) \implies SIN(t)]$.

(2b) $WIN(\mathcal{R}) \implies SIN(\mathcal{R})$.

(3a) $\forall t : [SIN(t) \implies SN(t)]$.

(3b) $SIN(\mathcal{R}) \implies SN(\mathcal{R})$.

(4) *There is no (innermost) reduction step* $t \underset{i}{\to} t'$ *in* $\mathcal{R}$ *with* $\infty(t)$, $\neg\infty(t')$.

(5a) $NE(\mathcal{R}) \implies [\forall t : [WN(t) \implies SN(t)]]$.

(5b) $NE(\mathcal{R}) \implies [WN(\mathcal{R}) \implies SN(\mathcal{R})]$.

Our recently obtained results on confluence and termination of TRSs show that all statements of Theorem 3.1 above except (1) still hold for non-overlapping but not necessarily left-linear TRSs (cf. [Gra92], [Gra93]). Moreover we have shown there that any innermost terminating (unconditional) overlay system with joinable critical pairs is terminating and hence confluent and complete.

For CTRSs much less is known concerning similar criteria for termination and confluence (see example 2.6 above which shows that orthogonal CTRSs need not be confluent). Next we summarize the most important known criteria for confluence of (possibly non-terminating) CTRSs.

Definition 3.2 *(cf. [Klo92]) Let $\mathcal{R}$ be a CTRS with rewrite relation $\rightarrow$ and let P be an n-ary predicate on the set of terms of $\mathcal{R}$. Then P is said to be* closed with respect to $\rightarrow$ *if for all terms t_i, t_i' such that $t_i \rightarrow^* t_i'$ $(i = 1, \ldots, n)$:*

$$P(t_1, \ldots, t_n) \quad \Longrightarrow \quad P(t_1', \ldots, t_n') .$$

$\mathcal{R}$ is said to be closed *if all conditions (appearing in some conditional rewrite rule of $\mathcal{R}$), viewed as predicates with the variables ranging over $\mathcal{R}$-terms, are closed with respect to $\rightarrow$.*

Theorem 3.3

(1) *Any generalized, weakly orthogonal, closed CTRS is confluent (cf. [O'D77], [Klo92]).*

(2) *Any weakly orthogonal, semi-equational CTRS is confluent.*[13]

(3) *Any weakly orthogonal, normal CTRS is confluent (cf. [BK86], [Klo92]).*

In the following we shall show that most of our results on restricted termination and confluence properties of non-overlapping and even of overlay systems can be generalized to the conditional case. This generalization has to take into account the additional complications arising with CTRSs. In particular, we need a kind of 'local completeness' property implying in particular that variable overlaps are not critical for certain conditional overlay systems.

Let us start with an easy result about innermost reductions in semantically non-overlapping CTRSs.

Lemma 3.4 *Let $\mathcal{R}$ be a CTRS with SEM-$NO(\mathcal{R})$. Then we have:*

(a) *If $s \underset{i}{\rightarrow} t$, $s \underset{i}{\rightarrow} u$ then either $t = u$ or there exists a term v with $t \underset{i}{\rightarrow} v$ and $u \underset{i}{\rightarrow} v$.*

(b) *If $s \underset{i}{\rightarrow}^m t$, $s \underset{i}{\rightarrow}^n u$ then there exists a term v and $m' \leq m$, $n' \leq n$ with $t \underset{i}{\rightarrow}^{n'} v$ and $u \underset{i}{\rightarrow}^{m'} v$.*

[13] This is a corollary of (1).

(c) $\underset{i}{\rightarrow}$ *is confluent.*

Proof: It suffices to prove (a) since (b) is obtained from (a) by an easy induction, e.g. on (m, n), and (c) is a consequence from (b). Hence let $s \underset{i}{\rightarrow}_p t$ and $s \underset{i}{\rightarrow}_q u$. If the innermost redex positions p, q of s are the same then the applied rule is unique due to $SEM\text{-}NO(\mathcal{R})$ which implies $t = u$. Otherwise p and q are disjoint and v is uniquely defined by $s \underset{i}{\rightarrow}_p t \underset{i}{\rightarrow}_q v$ and $s \underset{i}{\rightarrow}_q u \underset{i}{\rightarrow}_p v$. ∎

Our next result shows that for semantically non-overlapping systems the existence of an innermost normal form for some term t implies that any innermost derivation initiated by t is finite.

Lemma 3.5 *Let $\mathcal{R}$ be a CTRS with $SEM\text{-}NO(\mathcal{R})$. Then we have:*

(a) $\forall t : [WIN(t) \implies SIN(t)]$.

(b) $WIN(\mathcal{R}) \implies SIN(\mathcal{R})$.

Proof: It suffices to prove the local version (a) which implies (b). For a proof by contradiction let t be a term with $WIN(t)$ but not $IN(t)$. Then we know that there exists some innermost derivation

$$t = t_0 \underset{i}{\rightarrow} t_1 \underset{i}{\rightarrow} t_2 \underset{i}{\rightarrow} \cdots \underset{i}{\rightarrow} t_{n-1} \underset{i}{\rightarrow} t_n$$

with t_n irreducible. Obviously we have $\infty_i(t_0)$ and $\neg\infty_i(t_n)$. Thus there exists some (unique) index k, $0 \leq k \leq n-1$ with $t_k \underset{i}{\rightarrow} t_{k+1}$ and $\infty_i(t_k)$, $\neg\infty_i(t_{k+1})$. Due to $\infty_i(t_k)$ there are terms $t'_k, t''_k, \ldots$ such that

$$t = t_k \underset{i}{\rightarrow} t'_k \underset{i}{\rightarrow} t''_k \underset{i}{\rightarrow} \cdots$$

is an infinite innermost derivation. By applying Lemma 3.4(a) and observing that $t_{k+1} \neq t'_k$ due to $\infty_i(t'_k)$, $\neg\infty_i(t_{k+1})$ we know that there exists a term t'_{k+1} with $t'_k \underset{i}{\rightarrow} t'_{k+1}$, $t_{k+1} \underset{i}{\rightarrow} t'_{k+1}$ and $\infty_i(t'_k)$, $\neg\infty_i(t'_{k+1})$. By induction we can conclude that there is an infinite sequence of terms $t'_{k+1}, t''_{k+1}, \ldots$ such that

$$t_{k+1} \underset{i}{\rightarrow} t'_{k+1} \underset{i}{\rightarrow} t''_{k+1} \underset{i}{\rightarrow} \cdots$$

is an infinite innermost derivation. But this is a contradiction to $\neg\infty_i(t_{k+1})$. Hence we are done. ∎

For the sake of readability we shall use subsequently some more compact notations for special (sequences of) reductions which are introduced now. For parallel (innermost) reduction and normalization (w.r.t. some given CTRS $\mathcal{R}$) we use the following notations. We write $s \mathrel{-\!\!\!\Vert\!\!\!\rightarrow}_P t$ if P is a non-empty set of mutually disjoint positions of s and $s \rightarrow^+ t$ by (parallel) one-step root reductions of all the redexes s/p, $p \in P$.[14] In

[14]Note that — for proof-technical reasons which will become clearer later on — we do not require that s/p is a redex for all $p \in P$ but only that at least one subterm s/p of s with $p \in P$ is a redex. This is reflected by the requirement $s \rightarrow^+ t$.

particular we write $s \mathrel{\underset{i}{-\!\!\|\!\!\rightarrow}}_P t$ if s/p is an innermost redex of s for all redex positions $p \in P$. We write $s \mathrel{-\!\!\downdownarrows\!\!\rightarrow}_P t$ if P is a non-empty set of mutually disjoint positions of s and $s \rightarrow^+ t$ by normalizing all the subterms s/p with $p \in P$. In particular we write $s \mathrel{\underset{i}{-\!\!\downdownarrows\!\!\rightarrow}}_P t$ if for all $p \in P$ the subterm s/p is normalized using only innermost reduction steps. By normalizing a term t we mean reducing it some normal form. If $WN(t)$ holds then normalization of t is possible but need not yield a unique result. We write $s \mathrel{-\!\!\|\!\!\rightarrow} t$, $s \mathrel{\underset{i}{-\!\!\|\!\!\rightarrow}} t$, $s \mathrel{-\!\!\downdownarrows\!\!\rightarrow} t$ or $s \mathrel{\underset{i}{-\!\!\downdownarrows\!\!\rightarrow}} t$ if there exists a non-empty set P of mutually disjoint positions of s with $s \mathrel{-\!\!\|\!\!\rightarrow}_P t$, $s \mathrel{\underset{i}{-\!\!\|\!\!\rightarrow}}_P t$, $s \mathrel{-\!\!\downdownarrows\!\!\rightarrow}_P t$ or $s \mathrel{\underset{i}{-\!\!\downdownarrows\!\!\rightarrow}}_P t$, respectively. Moreover, for the sake of readability we also write $s \mathrel{-\!\!\downdownarrows\!\!\rightarrow}_P^{\leq 1} t$ if $s \equiv t$ or $s \mathrel{-\!\!\downdownarrows\!\!\rightarrow}_P t$. In the latter case P must clearly be non-empty and s/p reducible for some $p \in P$. Analogously, $s \mathrel{\underset{i}{-\!\!\downdownarrows\!\!\rightarrow}}_P^{\leq 1} t$ means $s \equiv t$ or $s \mathrel{\underset{i}{-\!\!\downdownarrows\!\!\rightarrow}}_P t$.

Moreover we shall tacitly make use of the following basic uniqueness properties of parallel reduction and normalization:

$$SEM\text{-}NO(\mathcal{R}) \quad \Longrightarrow \quad [\, s \mathrel{-\!\!\|\!\!\rightarrow}_P t_1 \quad \wedge \quad s \mathrel{-\!\!\|\!\!\rightarrow}_P t_2 \quad \Longrightarrow \quad t_1 = t_2 \,],$$

and

$$s \mathrel{-\!\!\downdownarrows\!\!\rightarrow}_P t_1 \quad \wedge \quad s \mathrel{-\!\!\downdownarrows\!\!\rightarrow}_P t_2 \quad \wedge \quad \forall\, p \in P : COMP(s/p) \quad \Longrightarrow \quad t_1 = t_2 \,.$$

The next result is a significantly generalized local version of Lemma 2 in [DOS88b][15] which in turn is the main technical result for inferring confluence of terminating CTRSs, provided that all conditional critical pairs are joinable overlays (cf. Theorem 4 in [DOS88b], cf. also Theorem 6.2 in [WG93] which handles the more general case of positive / negative conditional rewrite systems). Note that extra variables (in conditions) are allowed here.

Theorem 3.6 *Let $\mathcal{R}$ be a CTRS with $SEM\text{-}OS(\mathcal{R})$ and $JCP(\mathcal{R})$ and let s be a term with $SN(s)$. Furthermore let $C[,\ldots,]$ be a context, $\Pi \subseteq O(C[,\ldots,])$ a set of mutually disjoint positions of $C[,\ldots,]$ and t, u, v be terms. Then we have the following implication:*

$$u = C[s]_\Pi \rightarrow^* v \,\wedge\, s \rightarrow^* t \quad \Longrightarrow \quad C[t]_\Pi \downarrow v \,.$$

Proof: Let $\mathcal{R}$ be given as above. For a context $C[,\ldots,]$, a set $\Pi \subseteq O(C[,\ldots,])$ of mutually disjoint positions of $C[,\ldots,]$ and terms s, t, u, v we define the predicate $P(s,t,u,v,\Pi)$ by the following implication:

$$SN(s) \,\wedge\, u = C[s]_\Pi \rightarrow^* v \,\wedge\, s \rightarrow^* t \quad \Longrightarrow \quad C[t]_\Pi \downarrow v \,.$$

We have to show $P(s,t,u,v,\Pi)$ for all s, t, u, v, Π. To this end it is sufficient that $Q(s,n,k)$ defined by

$$\forall\, C[,\ldots,],\, \Pi,\, t,\, u,\, v, : SN(s) \,\wedge\, u = C[s]_\Pi \xrightarrow{n}{}^{k} v \,\wedge\, s \rightarrow^* t \quad \Longrightarrow \quad C[t]_\Pi \downarrow v$$

[15] In Lemma 2 of [DOS88b] only syntactical conditional overlay systems are considered and the proof (i.e. the induction ordering) makes use of the general termination assumption $SN(\mathcal{R})$ for the considered CTRS $\mathcal{R}$.

holds for all s with $SN(s)$ and for all n, k. We will show this by contradiction as follows. Assume that there exists a counterexample, i.e. $\langle s, n, k\rangle$ with $Q(s, n, k)$ not holding for. That means we have

$$(*) \quad SN(s) \quad \wedge \quad u = C[s]_\Pi \xrightarrow{n}{}^{k} v \quad \wedge \quad s \to^* t$$

for some $C[, \ldots,]$, Π, t, u, v, but

$$(**) \quad \neg\,(\,C[t]_\Pi \downarrow v\,)\,.$$

Now we define a complexity measure for $Q(\bar{s}, \bar{n}, \bar{k})$ by the triple $\langle \bar{s}, \bar{n}, \bar{k}\rangle$ using the lexicographic combination $\succ := lex(>_1, >_2, >_3)$ with $>_1 := (\to|_{Red(s)} \cup >_{st})^+$, $>_2 := >_3 := > := >_{\mathbb{N}}$, where $\to|_{Red(s)}$ is the reduction relation restricted to all reducts of s, for comparing these triples. Now, $>_1$ is well-founded — due to $SN(s)$ — and $>_2 = >_3 = >_{\mathbb{N}}$ is obviously well-founded, too. Hence, their lexicographic combination $\succ$ is also well-founded. Thus, we may assume w.l.o.g. that

$$(***) \quad \langle s, n, k\rangle \text{ is a minimal counterexample w.r.t. } \succ\,.$$

In order to obtain a contradiction we proceed by case analysis and induction[16] showing that $\langle s, n, k\rangle$ cannot be a (minimal) counterexample.

If $u = v$ (i.e. $n = k = 0$) or $s = t$ we are done since $(**)$ is violated. Otherwise, let $s \to s' \to^* t$. If we can show that $C[s']_\Pi \downarrow v$ holds then by induction (on the first component) we get $C[t]_\Pi \downarrow v$ because we have $s \to s'$, hence $s >_1 s'$. But this is a contradiction to $(***)$. We shall distinguish the following cases:

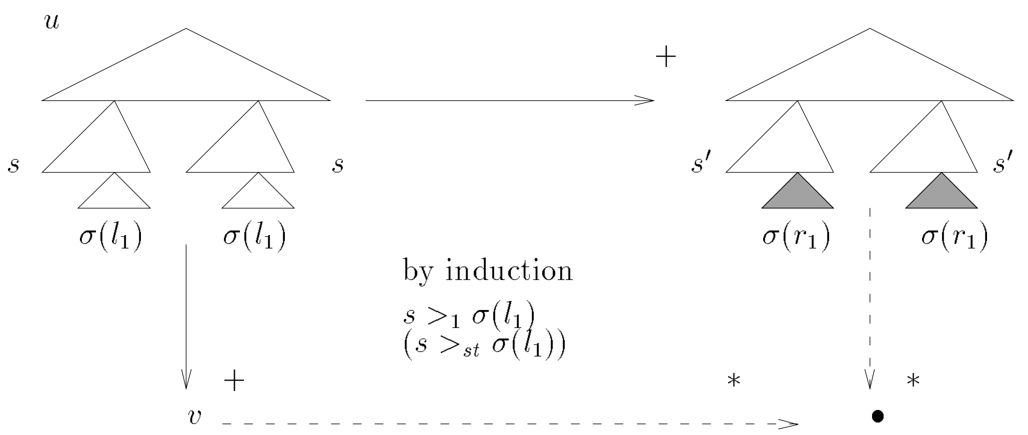
[16]This means that we shall exploit the minimality assumption of the counterexample $\langle s, n, k\rangle$.

(1) Proper subterm case (see figure 1): If the first step $s \to s'$ reduces a proper subterm of s, i.e. $s \to_p s'$ for some $p > \lambda$, then we have

$$C[s]_\Pi = C[C'[s/p]_p]_\Pi = (C[C'[]_p]_\Pi)[s/p]_{\Pi p} \to^+ C[s']_\Pi = (C[C'[]_p]_\Pi)[s'/p]_{\Pi p}$$

with $s = C'[s/p]_p$ for some context $C'[]_p$, hence $C[s']_\Pi \downarrow v$ as desired by induction on the first component because $s >_{st} s/p$ implies $s >_1 s/p$.

Figure 1 : Proper subterm case



(2) Otherwise, we may suppose

$$s \to_{\lambda,\sigma,P_1 \Longrightarrow l_1 \to r_1} s', \text{ i.e. } s = \sigma(l_1)\ ,\ s' = \sigma(r_1) \text{ and } \sigma(P_1)\downarrow$$

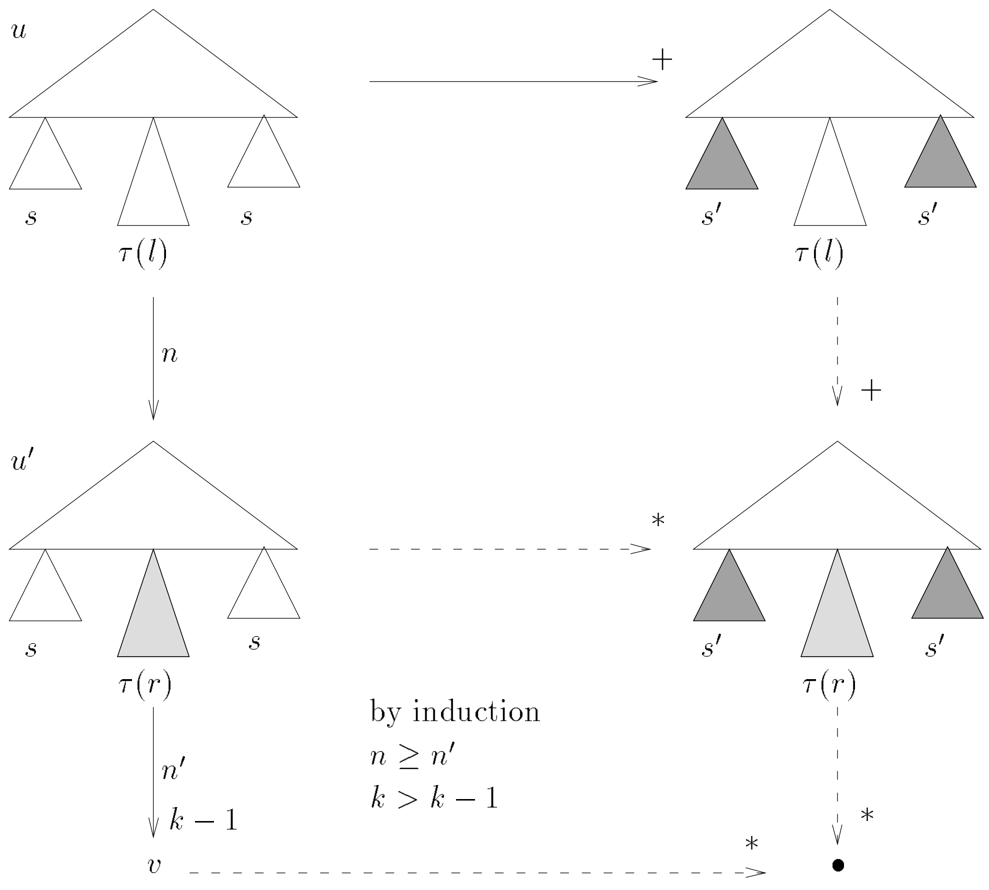
for some rule $P_1 \Longrightarrow l_1 \to r_1 \in \mathcal{R}$ and some substitution σ. Moreover assume

$$u = C[s]_\Pi \xrightarrow{n}_{q,\tau,P \Longrightarrow l \to r} u' \xrightarrow{n}^{k-1} v\,,$$

i.e. $C[s]_\Pi/q = \tau(l)$, $u'/q = \tau(r)$ and $\tau(P)\downarrow$, for n minimal with $u \xrightarrow{n} v$ and $k \geq 1$ minimal with $u \xrightarrow{n}^k v$. Then we have to distinguish the following four subcases according to the relative positions of q and Π:

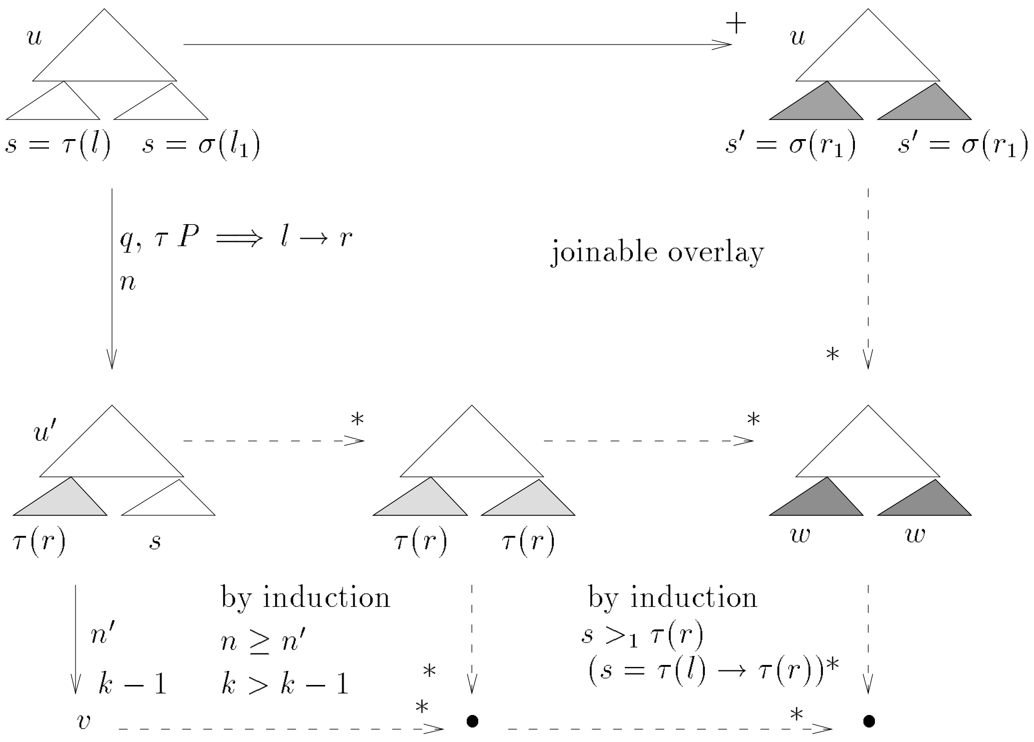
(2.1) $q \mid \Pi$ (disjoint peak, see figure 2.1): Then we have $u = C[s]_\Pi \xrightarrow{n}_q C'[s]_\Pi = u' \xrightarrow{n}{}^{k-1} v$, $C'[s]_\Pi \rightarrow^* C'[s']_\Pi$ and $C[s]_\Pi \rightarrow^* C[s']_\Pi \rightarrow_q C'[s']_\Pi$ for some context $C'[,\ldots,]$, hence by induction on the second or third component ($n \geq n'$, $k > k-1$) $C'[s']_\Pi \downarrow v$ and thus $C[s']_\Pi \rightarrow C'[s']_\Pi \downarrow v$ as desired.

Figure 2.1 : Disjoint peak case



(2.2) $q \in \Pi$ (critical peak, see figure 2.2): In this case we have a critical peak which is an instance of a critical overlay of $\mathcal{R}$, i.e. $s = \sigma(l_1) = \tau(l)$. Since all conditional critical pairs are joinable (overlays) we know that there exists some term w with $s = \sigma(l_1) \to \sigma(r_1) = s' \to^* w$ and $s = \tau(l) \to \tau(r) \to^* w$. Obviously, we have $u = C[s]_\Pi \to_{P \Longrightarrow l \to r} (C[\tau(l)]_\Pi)[q \leftarrow \tau(r)] = u' \to^*_{P \Longrightarrow l \to r} C[\tau(r)]_\Pi$. For $|\Pi| = 1$ we obtain $\Pi = \{q\}$ and $(C[\tau(l)]_\Pi)[q \leftarrow \tau(r)] = u' = C[\tau(r)]_\Pi \to^* v$. Otherwise, we have $C[\tau(l)]_\Pi[q \leftarrow \tau(r)] \xrightarrow{n'}{}^{k-1} v$ with $n' \leq n$. Hence, by induction on the second or third component we obtain $C[\tau(r)]_\Pi \downarrow v$ (due to $n' \leq n$, $k-1 < k$). Moreover, $\tau(r) \to^* w$ yields $C[\tau(r)]_\Pi \to^* C[w]_\Pi$ which by induction on the first component implies $C[w]_\Pi \downarrow v$ (due to $s = \tau(l) \to \tau(r)$, hence $s >_1 \tau(r)$). Thus, $C[s']_\Pi = C[\sigma(r_1)]_\Pi \to^* C[w]_\Pi \downarrow v$ because of $\sigma(r_1) \to^* w$. Hence we get $C[s']_\Pi \downarrow v$ as desired.

Figure 2.2 : Critical peak (overlay)



The remaining case is that of a *variable overlap*, either above or in some subterm $C[s]_\Pi/\pi = s$ ($\pi \in \Pi$) of $C[s]_\Pi$. Note that a critical peak which is not an overlay cannot occur due to $SEM\text{-}OS(\mathcal{R})$.

(2.3) $q < \pi$ for some $\pi \in \Pi$ (variable overlap above, see figure 2.3): Let Π' be the set of positions of those subterms $s = \sigma(l_1)$ of $u/q = \tau(l)$ which correspond to some $u/\pi = s$, $\pi \in \Pi$. Formally, $\Pi' := \{\pi' \mid q\pi' \in \Pi\}$. Moreover, for every $x \in dom(\tau)$, let $\Delta(x)$ be the set of positions of those subterms s in $\tau(x)$ which are rewritten into s' in the derivation $u = C[s]_\Pi \to^+ C[s']_\Pi$, i.e. $\Delta(x) := \{\rho' \mid \exists\, \rho : l/\rho = x \wedge \rho\rho' \in \Pi'\}$. Then τ' is defined by $\tau'(x) := \tau(x)[\rho' \leftarrow s' \mid \rho' \in \Delta(x)]$ for all $x \in dom(\tau)$. Obviously, we have $\tau(x) \to^*_{P_1 \Longrightarrow l_1 \to r_1} \tau'(x)$ for all $x \in dom(\tau)$. Thus we get

$$u = C[s]_\Pi = C'[\tau(l)]_q \xrightarrow{n}_{q,\tau,P \Longrightarrow l \to r} C'[\tau(r)]_q = u' \stackrel{n'}{\longrightarrow}{}^{k-1} v$$

for some context $C'[]_q$ and some $n' \leq n$,

$$u = C[s]_\Pi \to^*_{P_1 \Longrightarrow l_1 \to r_1} C[s']_\Pi \to^*_{P_1 \Longrightarrow l_1 \to r_1} C''[\tau'(l)]_q$$

for some context $C''[]_q$, and

$$u' = C'[\tau(r)]_q \to^*_{P_1 \Longrightarrow l_1 \to r_1} C''[\tau'(r)]_q\,.$$

Moreover we have

$$C''[\tau'(l)]_q \to_{q,\tau',P \Longrightarrow l \to r} C''[\tau'(r)]_q$$

by induction (due to $\tau'(P)\downarrow$ as shown below) and finally

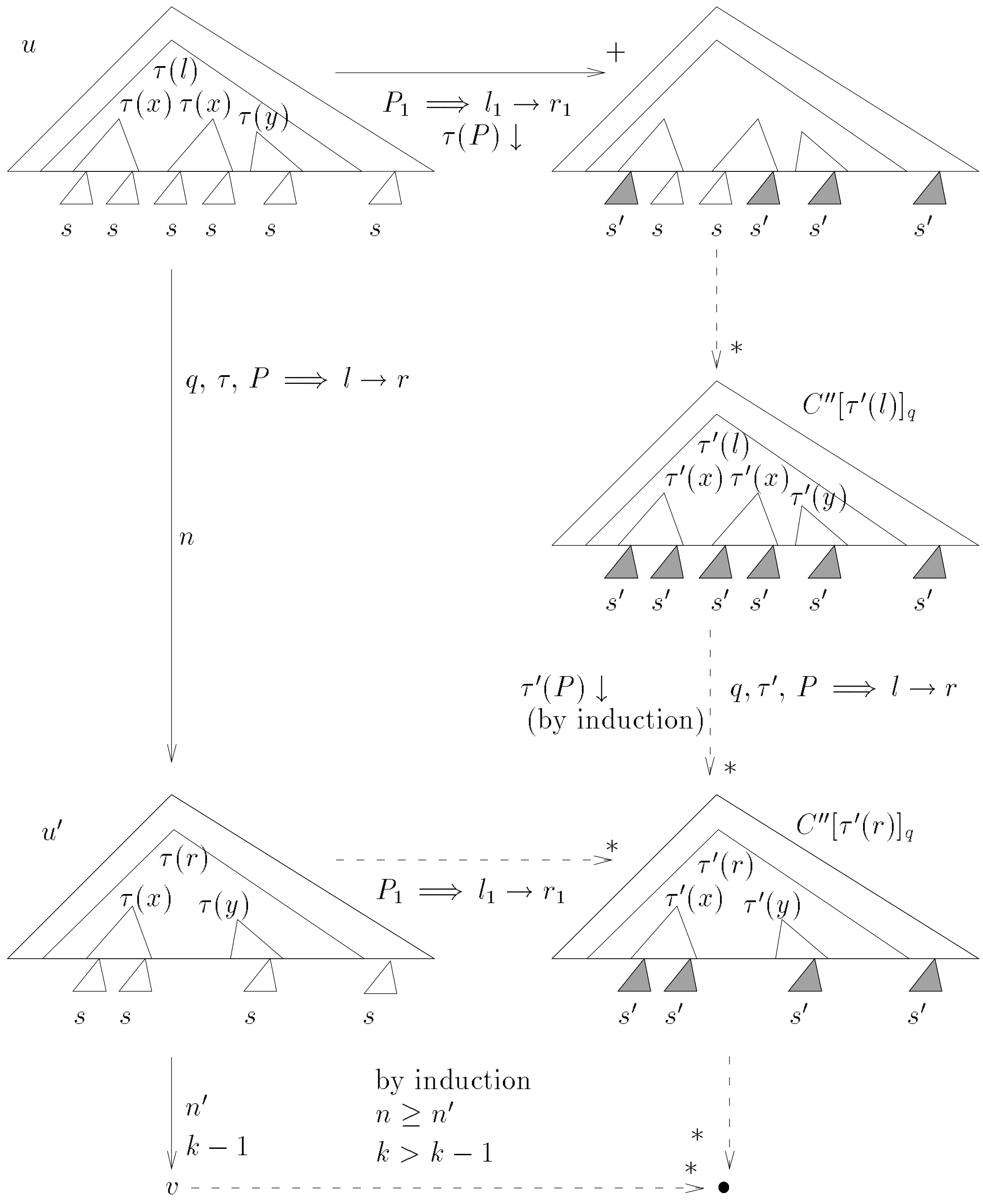
$$C[s']_\Pi \to^* C''[\tau'(r)]_q \downarrow v$$

as desired by induction (on the second or third component) due to $n \geq n'$, $k > k-1$.

It remains to prove the claim $\tau'(P)\downarrow$. This means that we have to show $\tau'(z_1) \downarrow \tau'(z_2)$ for all $z_1 \downarrow z_2 \in P$. If P is empty or trivially satisfied (i.e. $n \leq 1$) we are done. Otherwise, we know by assumption that $\tau(z_1) \downarrow \tau(z_2)$ for all $z_1 \downarrow z_2 \in P$ in depth $n-1$. This means that there exists some term w with $\tau(z_1) \stackrel{n-1}{\longrightarrow}{}^* w$, $\tau(z_2) \stackrel{n-1}{\longrightarrow}{}^* w$. Hence, $eq(\tau(z_1), \tau(z_2)) \stackrel{n-1}{\longrightarrow}{}^* eq(w,w)$ where eq is to denote some fresh binary function symbol not occurring in $\mathcal{R}$. By construction of τ' we know $\tau(z_1) \to^*_{P_1 \Longrightarrow l_1 \to r_1} \tau'(z_1)$, $\tau(z_2) \to^*_{P_1 \Longrightarrow l_1 \to r_1} \tau'(z_2)$. Moreover, $eq(\tau(z_1), \tau(z_2))$ is of the form $E[s]_Q$, for some context $E[,\ldots,]$ and some set Q of mutually disjoint positions of $E[,\ldots,]$, such that $E[s]_Q \stackrel{n-1}{\longrightarrow}{}^* eq(w,w)$ and $E[s]_Q \to^* E[s']_Q = eq(\tau'(z_1), \tau'(z_2))$. By induction on the second component (due to $n > n-1$) we obtain $E[s']_Q = eq(\tau'(z_1), \tau'(z_2)) \downarrow eq(w,w)$. Since there are no rules for 'eq' we conclude $\tau'(z_1) \downarrow \tau'(z_2)$. [17] This finishes the proof of the claim $\tau'(P)\downarrow$. Summarizing we have shown $C[s']_\Pi \downarrow v$ as desired.

[17]This reasonig can be easily made formally precise by adding a rule $eq(x,x) \to true$ with eq and $true$ new symbols (of a new sort). Then, with $\mathcal{R}' := \mathcal{R} \cup \{eq(x,x) \to true\}$, one easily shows: $eq(s,t) \to_{\mathcal{R}'} true \iff s \downarrow_{\mathcal{R}} t$.

Figure 2.3 : Variable overlap above



(2.4) $\pi < q$ for some $\pi \in \Pi$ (variable overlap below, see figure 2.4): Remember that we have $u/\pi = \sigma(l_1) = s$ and $u/q = \tau(l)$. Now let q', q'', q''', Π', Π'' and contexts $C'[]_q$, $D[]_{q'''}$, $D'[]_{\Pi'}$, $D''[]_{\Pi''}$ be (uniquely) defined by $u = C'[\tau(l)]_q$, $q = \pi q'$, $q' = q''q'''$, $l_1/q'' = x \in \mathcal{V}$, $\sigma(x) = D[\tau(l)]_{q'''}$, $\Pi' = \{\pi' \mid l_1/\pi' = x\}$, $\Pi'' = \{\pi'' \mid r_1/\pi'' = x\}$, $\sigma(l_1) = D'[D[\tau(l)]_{q'''}]_{\Pi'}$, $\sigma(r_1) = D''[D[\tau(l)]_{q'''}]_{\Pi''}$. Moreover let σ' be the substitution on $V(l_1)$ defined by

$$\sigma'(y) = \begin{cases} \sigma(y)\,,\, y \neq x \\ D[\tau(r)]_{q'''}\,,\, y = x, \sigma(x) = D[\tau(l)]_{q'''}\,. \end{cases}$$

Then we get

$$\begin{aligned} C[s]_\Pi &= C[\sigma(l_1)]_\Pi \\ &= C[D'[D[\tau(l)]_{q'''}]_{\Pi'}]_\Pi \to^+_{\sigma,P_1\Longrightarrow l_1\to r_1} C[s']_\Pi = C[\sigma(r_1)]_\Pi \\ &= C[D''[D[\tau(l)]_{q'''}]_{\Pi''}]_\Pi \to^*_{\tau,R\Longrightarrow l\to r} C[D''[D[\tau(r)]_{q'''}]_{\Pi''}]_\Pi = C[\sigma'(r_1)]_\Pi\,, \end{aligned}$$

and

$$\begin{aligned} C[s]_\Pi &= C[\sigma(l_1)]_\Pi = C[D'[D[\tau(l)]_{q'''}]_{\Pi'}]_\Pi \\ &= C'[\tau(l)]_q \to_{q,\tau,P\Longrightarrow l\to r} u' \\ &= C'[\tau(r)]_q \to^*_{\tau,P\Longrightarrow l\to r} C[D'[D[\tau(r)]_{q'''}]_{\Pi'}]_\Pi = C[\sigma'(l_1)]_\Pi\,. \end{aligned}$$

By induction on the first component we obtain

$$C[s]_\Pi \;=\; C[\sigma(l_1)]_\Pi \to^+ C[\sigma'(l_1)]_\Pi \downarrow v$$

(due to $s >_{st} \tau(l)$, hence $s >_1 \tau(l)$). Moreover, we get

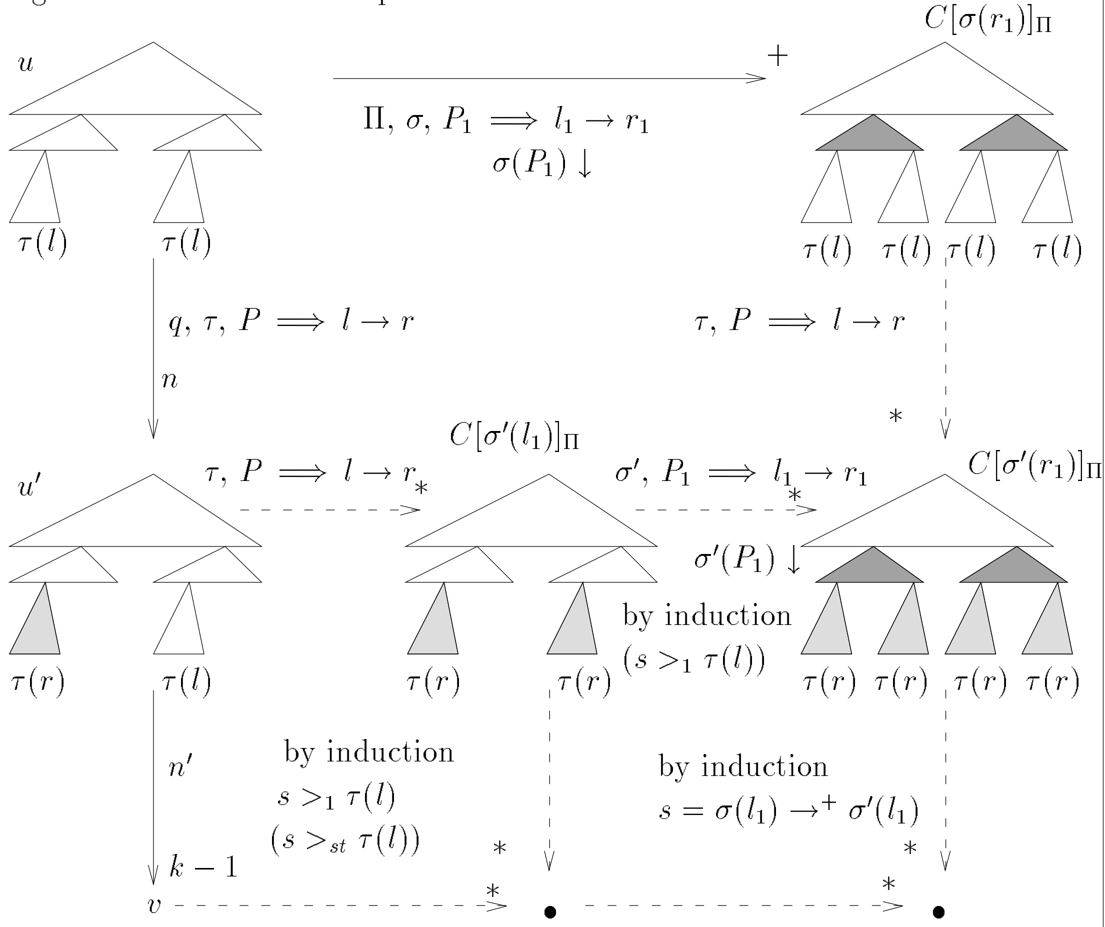
$$C[\sigma'(l_1)]_\Pi \to^*_{\sigma',P_1\Longrightarrow l_1\to r_1} C[\sigma'(r_1)]_\Pi$$

since $\sigma'(P_1)\downarrow$ is satisfied by induction on the first component ($s >_{st} \tau(l)$, hence $s >_1 \tau(l)$). Furthermore we have $C[\sigma'(r_1)]_\Pi \downarrow v$ by induction on the first component (due to $s = \sigma(l) \to^+ \sigma'(l_1)$, hence $s >_1 \sigma'(l_1)$). Summarizing we have shown

$$C[s']_\Pi \to^* C[\sigma'(r_1)]_\Pi \downarrow v$$

as desired.

Figure 2.4 : Variable overlap below



Thus, for all cases we have shown $C[s']_\Pi \downarrow v$ yielding a contradiction to $(***)$, hence we are done. ■

As an easy consequence of theorem 3.6 we obtain the following sufficient criterion for a variable overlap to be non-critical.

Lemma 3.7 *Let $\mathcal{R}$ be a CTRS with $SEM\text{-}OS(\mathcal{R})$ and $JCP(\mathcal{R})$, and let s, t be terms with $s \to_{p,\sigma,P\Longrightarrow l\to r} t$. Furthermore let σ' be given with $\sigma \to^* \sigma'$, i.e. $\sigma(x) \to^* \sigma'(x)$ for all $x \in dom(\sigma)$, such that $SN(\sigma(x))$ holds for all $x \in dom(\sigma)$. Then we have: $s = C[\sigma(l)]_p \to^* C[\sigma'(l)]_p \to_{p,\sigma',P\Longrightarrow l\to r} C[\sigma'(r)]_p$ (due to $\sigma'(P)\downarrow$) and $t = C[\sigma(r)]_p \to^* C[\sigma'(r)]_p$ for some context $C[]_p$.*

Proof: Straightforward by repeated application of theorem 3.6 choosing $s := \sigma(x)$ for all $x \in dom(\sigma)$. ■

Choosing $C[]_\Pi$ to be the empty context (and accordingly $\Pi = \{\lambda\}$) in theorem 3.6 we obtain as corollary the following local version of a confluence criterion.

Corollary 3.8 *Let $\mathcal{R}$ be a CTRS with $SEM\text{-}OS(\mathcal{R})$ and $JCP(\mathcal{R})$, and let s be a term with $SN(s)$. Then we have $CR(s)$ and hence $COMP(s)$, too.*

The termination assumption concerning s in this result is crucial as demonstrated by the following example.

Example 3.9 *(example 2.6 continued) Here $\mathcal{R} = \{x \downarrow f(x) \implies f(x) \to a,\ b \to f(b)\}$ is clearly a semantical overlay system with joinable critical pairs (it is even syntactically non-overlapping). Moreover we have $f(f(b)) \to a$ and $f(f(b)) \to f(a)$ but not $a \downarrow f(a)$. Obviouly, $SN(f(f(b)))$ does not hold due to the presence of the rule $b \to f(b)$ in $\mathcal{R}$ (note that we even do not have $SIN(\,f(f(b)))\,)$.*

Under the stronger assumption of global termination we get from corollary 3.8 the following critical pair criterion for confluence of conditional semantical overlay systems which is a slightly generalized version of Theorem 4 in [DOS88b] in the sense that it holds not only for syntactical but also for semantical overlay systems.

Theorem 3.10 *A terminating CTRS which is a semantical overlay system such that all its conditional critical pairs are joinable is confluent, hence complete.*

Next we shall show that any semantically non-overlapping and innermost terminating CTRS is terminating. The following two technical lemmas are useful for giving a shorter proof of this result.

Lemma 3.11 *Let $\mathcal{R}$ be given with $SEM\text{-}NO(\mathcal{R})$ and let $s \to_{p,P \Longrightarrow l \to r} t$ be a non-innermost reduction step with $SN(u)$ for every proper subterm u of s/p. Then there exist a set P of mutually distinct positions of s strictly below p and terms s', t' with $s \underset{i}{\Vert\!\!\!\longrightarrow}_P s' \to_{p,P \Longrightarrow l \to r} t'$ and $t \underset{i}{\Vert\!\!\!\longrightarrow}^{\leq 1} t'$.*

Proof: Let $\mathcal{R}$ be a CTRS with $SEM\text{-}NO(\mathcal{R})$ and $s \to_{p,\sigma,P \Longrightarrow l \to r} t$ be a non-innermost reduction step in $\mathcal{R}$. Define $Q_l := \{u \in O(l) \mid l/u \in \mathcal{V}\}$ and $Q_r := \{u \in O(r) \mid r/u \in \mathcal{V}\}$. Since $s \to_{p,\sigma,P \Longrightarrow l \to r} t$ is non-innermost at least one proper subterm of s/p is reducible. From $SEM\text{-}NO(\mathcal{R})$ we know that all innermost redexes of s strictly below p are below positions pq with $q \in Q_l$. This means that we can define s' by $s \underset{i}{\Vert\!\!\!\longrightarrow}_P s'$ where P is the set of positions of all innermost redexes of s strictly below p and $s'/p = \sigma'(l)$ for some substitution σ'. Moreover, t' is defined by $s' \to_{p,\sigma',P \Longrightarrow l \to r} t'$ (note that this step is possible by application of Lemma 3.7) such that we get $s \underset{i}{\Vert\!\!\!\longrightarrow}_P s' \to_{p,l \to r} t'$ and $s \to_{p,P \Longrightarrow l \to r} t \underset{i}{\Vert\!\!\!\longrightarrow}^{\leq 1}_{P'} t'$, where P' is the set of positions of all innermost redexes of t' strictly below pq, $q \in Q_R$, as desired. ■

Lemma 3.12 *Let $\mathcal{R}$ be given with $SEM\text{-}NO(\mathcal{R})$ and let s, t, t' be terms with $s \to_p t$, $s \underset{i}{+\!\!\!\!\mid\!\mid\!\!\!\rightarrow}_Q s'$ such that $p < p'$ and s/p' is reducible for at least one $p' \in Q$. Moreover, assume $SN(u)$ for every proper subterm u of s/p. Then there exists a term t' with $s' \to^+ t'$ and $t \underset{i}{+\!\!\!\!\mid\!\mid\!\!\!\rightarrow}^{\leq 1} t'$.*

Proof: Let $\mathcal{R}$ be given with $SEM\text{-}NO(\mathcal{R})$ and let s, t, t' be terms with $s \to_p t$, $s \underset{i}{+\!\!\!\!\mid\!\mid\!\!\!\rightarrow}_Q s'$ such that $p < p'$ and s/p' is reducible for at least one $p' \in Q$. Hence we know that $s \to_p t$ is not an innermost step. W.l.o.g. we may further assume that $p = \lambda$ (and hence $p < p'$ for all $p' \in Q$). From $s \to_p t$ we deduce that $s/p = \sigma(l)$ for some rule $P \Longrightarrow l \to r$ and some substitution σ. Now we would like to apply the same rule $P \Longrightarrow l \to r$ to s'/p. Remember that s' is obtained from s by parallel innermost reduction at the redex positions from Q. Due to $SEM\text{-}NO(\mathcal{R})$ all these redex positions are below variable positions of l. Thus the only potential reason for non-applicability of $P \Longrightarrow l \to r$ to s' is that s' is no longer an instance of l due to the fact that $P \Longrightarrow l \to r$ might be non-left-linear. But this problem is easily solved by an additional parallel reduction of s at all innermost redex positions (strictly below $p = \lambda$) which were not contained in Q. Let us denote this set of all innermost redex positions of s not contained in Q by Q'. Then – using Lemma 3.7 – we can (uniquely) define terms s'' and t' by $s \underset{i}{+\!\!\!\!\mid\!\mid\!\!\!\rightarrow}_Q s' \underset{i}{+\!\!\!\!\mid\!\mid\!\!\!\rightarrow}^{\leq 1}_{Q'} s'' \to_{p,P\Longrightarrow l\to r} t'$ such that $s \to_{p,P\Longrightarrow l\to r} t \underset{i}{+\!\!\!\!\mid\!\mid\!\!\!\rightarrow}^{\leq 1} t'$ (with $s''/p = \sigma'(l)$, $t'/p = \sigma'(r)$ for some substitution σ') as desired. ■

Theorem 3.13 *For any CTRS R we have:*

(a) $SEM\text{-}NO(\mathcal{R}) \wedge SIN(\mathcal{R}) \quad\Longrightarrow\quad SN(\mathcal{R})$.

(b) $SEM\text{-}NO(\mathcal{R}) \quad\Longrightarrow\quad [\forall t : IN(t) \quad\Longrightarrow\quad SN(t)]$.

Proof: Although we shall prove a more general result later on (cf. Theorem 3.19) we will give a relatively simple proof here since it cannot be used for the more general case.

For a proof by contradiction let t be a term with $SIN(t)$ but not $SN(t)$. Hence there exists an infinite derivation initiated by t. Due to $SIN(t)$ every such counterexample contains at least one reduction step which is non-innermost. We consider now a counterexample

$$D : t = t_0 \to t_1 \to t_2 \to \cdots$$

which is minimal in the sense that reduction steps are performed at deepest possible positions. Formally this means that D satisfies

(1) $\forall j \geq 0 : \infty(t_j)$, and

(2) $\forall j \geq 0\ \forall s_j : [t_j \to_p t_{j+1} \wedge t_j \to_q s_j \text{ with } q > p] \Longrightarrow \neg\infty(s_j)$.

Now let $t_n \rightarrow_p t_{n+1}$ be the first non-innermost step in D. By the minimality assumption (2) above we know $SN(t)$ for every proper subterm t of t_n/p. Hence, by applying lemma 3.11 we know that there exist terms s_n, s_{n+1} with

$$t_n \underset{i}{\Vert\!\!\!\longrightarrow} s_n \rightarrow s_{n+1} \quad \text{and} \quad t_n \rightarrow t_{n+1} \underset{i}{\Vert\!\!\!\longrightarrow}^{\leq 1} s_{n+1}$$

and $\neg\infty(s_n)$, $\neg\infty(s_{n+1})$. Moreover, $\infty(t_{n+1})$ implies $t_{n+1} \neq s_{n+1}$, hence $t_{n+1} \underset{i}{\Vert\!\!\!\longrightarrow} s_{n+1}$. In order to obtain a contradiction to $\neg\infty(s_n)$ it suffices to prove that $s_n \rightarrow s_{n+1}$ can be extended to an infinite reduction sequence. For that purpose it is sufficient to show that whenever we have $t_m \underset{i}{\Vert\!\!\!\longrightarrow} s_m$ with $\neg\infty(s_m)$ then there exists an index $m' > m$ and a term $s_{m'}$ with $s_m \rightarrow^+ s_{m'}$, $t_{m'} \underset{i}{\Vert\!\!\!\longrightarrow} s_{m'}$ and $\neg\infty(s_{m'})$. Hence, let $t_m \underset{i}{\Vert\!\!\!\longrightarrow}_Q s_m$ with $\neg\infty(s_m)$ and $t_m \rightarrow_p t_{m+1}$ such that w.l.o.g. t_m/q is an innermost redex of t_m for all $q \in Q$. Moreover – due to the minimality assumption (2) above – we may assume $SN(t)$ for every proper subterm t of t_m/p. Now we distinguish three cases:

(a) $\forall\, q \in Q \,:\, p|q$: In this case we can choose $m' = m+1$ and $s_{m'}$ is obtained from s_m by reduction at position p, i.e. we get $t_m \Vert\!\!\!\longrightarrow_Q s_m \rightarrow_p s_{m+1}$, $t_m \rightarrow_p t_{m+1} \Vert\!\!\!\longrightarrow_Q s_{m+1}$ with $\neg\infty(s_{m+1})$.

(b) Some of the positions $q \in Q$ are strictly below p: In this case we can again choose $m' = m+1$ by Lemma 3.12 (note that the termination condition needed for applying Lemma 3.12 is satisfied since $SN(t)$ holds for every proper subterm t of t_m/p by assumption) which yields the existence of a term s_{m+1} with $s_m \rightarrow^+ s_{m+1}$, $\neg\infty(s_{m+1})$ and $t_{m+1} \underset{i}{\Vert\!\!\!\longrightarrow}^{\leq 1} s_{m+1}$. Due to $\infty(t_{m+1})$, $\neg\infty(s_{m+1})$ we have $t_{m+1} \not\equiv s_{m+1}$, hence $t_{m+1} \underset{i}{\Vert\!\!\!\longrightarrow} s_{m+1}$.

(c) p is below one of the positions from Q, let's say $p \geq q \in Q$. Then we have $p = q$ because p is an innermost redex position of t_m. From $SEM\text{-}NO(\mathcal{R})$ we know that reducing t_m at position $p = q$ yields a unique result. Thus, choosing $m' = m+1$ and $s_{m+1} = s_m$ we get $t_m \underset{i}{\Vert\!\!\!\longrightarrow}_Q s_m \rightarrow^* s_{m+1}$, $t_m \rightarrow_p t_{m+1} \underset{i}{\Vert\!\!\!\longrightarrow}^{\leq 1}_{Q'} s_{m+1}$ with $Q' = Q \setminus \{q\}$. Using $\infty(t_{m+1})$, $\neg\infty(s_{m+1})$ we can conclude $|Q| \geq 2$ and $t_{m+1} \underset{i}{\Vert\!\!\!\longrightarrow}_{Q'} s_{m+1}$. Obviously, the reduction $s_m \rightarrow^* s_{m+1}$ is not a proper one (since we have $s_{m+1} = s_m$) but we know that after at most $|Q| - 1$ steps (in D) we must be back in case (a) or (b) in which a proper reduction of s_m is enabled as desired.

By induction we can finally conclude now that there exists an infinite derivation initiated by s_n. But this is a contradiction to $\neg\infty(s_n)$. Hence we are done. ■

Note that any semantically non-overlapping CTRS is in particluar a (conditional) semantical overlay system with joinable critical pairs. Hence, combining Theorem 3.13 with Theorem 3.10 yields the following result.

Corollary 3.14 *Let $\mathcal{R}$ be a CTRS. Then the following holds:*

(a) $SEM\text{-}NO(\mathcal{R}) \wedge SIN(\mathcal{R}) \Longrightarrow COMP(\mathcal{R})$.

(b) $SEM\text{-}NO(\mathcal{R}) \Longrightarrow [\forall t : SIN(t) \Longrightarrow COMP(t)]$.

The next result says that innermost reduction steps in semantically non-overlapping CTRSs cannot be critical in the sense that they may destroy the possibility of infinite derivations.

Lemma 3.15 *Let $\mathcal{R}$ be a CTRS with $SEM\text{-}NO(\mathcal{R})$. Then there is no innermost reduction step $s \underset{i}{\rightarrow} t$ in $\mathcal{R}$ with $\infty(s)$ but $\neg\infty(t)$.*

Proof: For a proof by contradiction assume $s \underset{i}{\rightarrow} t$ with $\infty(s)$ but $\neg\infty(t)$, hence $SN(t)$. Together with $s \underset{i}{\rightarrow} t$ this implies $WIN(s)$. Using lemma 3.5 we get $SIN(s)$ which by theorem 3.13 yields $SN(s)$. But this is a contradiction to $\infty(s)$. ∎

Obviously, Lemma 3.5, Theorem 3.13 and Lemma 3.15 express generalizations of theorem 3.1 (2)-(4). Indeed, it is also possible to prove the following generalization of theorem 3.1(5).

Theorem 3.16 *For any CTRS $\mathcal{R}$ the following holds:*

(a) $SEM\text{-}NO(\mathcal{R}) \wedge NE(\mathcal{R}) \wedge WN(\mathcal{R}) \Longrightarrow SN(\mathcal{R})$.

(b) $SEM\text{-}NO(\mathcal{R}) \wedge NE(\mathcal{R}) \Longrightarrow [\forall t : WN(t) \Longrightarrow SN(t)]$

Proof: It suffices to prove the stronger (b) from which (a) follows easily. For a proof of (b) by contradiction let us assume that $\mathcal{R}$ is a CTRS with $SEM\text{-}NO(\mathcal{R}) \wedge NE(\mathcal{R})$. Moreover, let t be a term with $WN(t)$ but not $SN(t)$. Hence there exists a normalizing derivation initiated by t, e.g. a derivation of the form

$$D \,:\, t =: t_0 \rightarrow t_1 \rightarrow t_2 \rightarrow \cdots \rightarrow t_{n-1} \rightarrow t_n \;, n > 0$$

with t_n irreducible, hence $\neg\infty(t_n)$ and $\infty(t_0)$. This implies that there is some (unique) index k, $0 \leq k < n$, with $t_k \rightarrow_{p,P \Longrightarrow l \rightarrow r} t_{k+1}$, $\infty(t_k)$ and $\neg\infty(t_{k+1})$. Let us denote the set of variable occurrences of the left and right hand side of the applied rule $P \Longrightarrow l \rightarrow r$ by $Q_l := \{q \in O(l) \,|\, l/q \in \mathcal{V}\}$ and $Q_r := \{q \in O(r) \,|\, r/q \in \mathcal{V}\}$, respectively. By lemma 3.15 we know that t/p must be a non-innermost redex of t. Since $\mathcal{R}$ is semantically non-overlapping t_k/pq must be reducible for at least one $q \in Q_l$. From $NE(\mathcal{R})$ we know moreover that $\{t_k/pq | q \in Q_l\} = \{t_{k+1}/pq | q \in Q_r\}$. Furthermore $\neg\infty(t_{k+1})$ implies $SN(t_{k+1}/pq)$ for all $q \in Q_r$ which – by Corollary 3.14(b) – yields $COMP(t)$ for all $t \in \{t_k/pq | q \in Q_l\} = \{t_{k+1}/pq | q \in Q_r\}$. Hence there exist (uniquely defined) terms s_k, s_{k+1} with $t_k \longrightarrow\!\!\!\!\!\!\!\!\downarrow\!\downarrow\;\;_Q\, s_k \underset{i}{\rightarrow}_{p,P \Longrightarrow l \rightarrow r} s_{k+1}$, $t_k \rightarrow_{p,P \Longrightarrow l \rightarrow r} t_{k+1} \longrightarrow\!\!\!\!\!\!\!\!\downarrow\!\downarrow\;\;_{P'}\, s_{k+1}$ for $Q := \{pq | q \in Q_l\}$, $P' := \{pq | q \in Q_r\}$ such that $\infty(t_k)$, $\neg\infty(t_{k+1})$, $\neg\infty(s_{k+1})$. Since parallel normalization of t_k at all positions from Q can be achieved by using only innermost steps we obtain $t_k \underset{i}{\rightarrow}^+ s_{k+1}$ with $\infty(t_k)$ and $\neg\infty(s_{k+1})$. This implies

$WIN(t_k)$ which yields $SIN(t_k)$ by lemma 3.5 and $SN(t_k)$ by theorem 3.13. But this is a contradiction to $\infty(t_k)$. ■

We shall prove now that the rather restrictive property *SEM-NO* in Theorem 3.13 may be further weakened by allowing possible overlaps but guaranteeing joinability of critical pairs. To be precise, *SEM-NO* is replaced by *SEM-OS* $\wedge$ *JCP*, i.e. corresponding CTRSs have to be semantical overlay systems (i.e. all feasible critical pairs are obtained by overlapping at top positions) such that all its (conditional) critical pairs are joinable.

In order to enable a simpler proof of this main result we need the following two auxiliary lemmas.

Lemma 3.17 *Let $\mathcal{R}$ be a CTRS with $SEM\text{-}OS(\mathcal{R})$ and $JCP(\mathcal{R})$. Moreover let $s \rightarrow_{p,C \Longrightarrow l \rightarrow r} t$ be a non-innermost reduction step with $SN(\bar{s})$ for all proper subterms $\bar{s}$ of s/p. Then there exist terms s', t' and $P \subseteq O(s)$, $Q \subseteq O(t)$ such that*

$$s \mathrel{-\!\!\!\Downarrow\!\!\!\!\rightarrow}_P s' \rightarrow_{p,C \Longrightarrow l \rightarrow r} t' \quad \textit{and} \quad s \rightarrow_{p,C \Longrightarrow l \rightarrow r} t \mathrel{-\!\!\!\Downarrow\!\!\!\!\rightarrow}^{\leq 1}_{Q} t'$$

with $p < u$ for all $u \in P$, $COMP(s/u)$ for all $u \in P$, and $COMP(t/v)$ for all $v \in Q$.

Proof: Under the assumptions of the lemma let $Q_l = \{q \in O(l) \,|\, l/q \in \mathcal{V}\}$ and $Q_r = \{q \in O(r) \,|\, r/q \in \mathcal{V}\}$ be the sets of variable positions of l and r, respectively, and define $P := \{pq | q \in Q_l\}$, $Q := \{pq | q \in Q_r\}$. By assumption we know $SN(\bar{s})$ for all proper subterms $\bar{s}$ of s/p. Due to $JCP(\mathcal{R})$ and $SEM\text{-}OS(\mathcal{R})$ this implies by Corollary 3.8 that $COMP(\bar{s})$ holds for all proper subterms $\bar{s}$ of s/p. From $SEM\text{-}OS(\mathcal{R})$ and the fact that the step $s \rightarrow_{p,C \Longrightarrow l \rightarrow r} t$ is non-innermost we conclude that at least one subterm s/u of s, $u \in P$, is reducible. Thus s' defined by $s \mathrel{-\!\!\!\Downarrow\!\!\!\!\rightarrow}_P s'$ exists (and is unique). Since s' is an instance of l, we can – by applying Lemma 3.7 – also uniquely define t' by $s' \rightarrow_{p,C \Longrightarrow l \rightarrow r} t'$). Moreover, t' can be obtained from t by (uniquely) normalizing all subterms t/u of t with $u \in Q$, e.g. $t \mathrel{-\!\!\!\Downarrow\!\!\!\!\rightarrow}^{\leq 1}_{Q}$. Finally, $\{t/v \,|\, v \in Q\} \subseteq \{s/u \,|\, u \in P\}$ implies $COMP(t/v)$ for all $v \in Q$ as desired. ■

The next result is a technical key lemma which will be used below in the proof of theorem 3.19 for properly extending some given finite derivation to an infinite one.

Lemma 3.18 *Let $\mathcal{R}$ be a CTRS with $SEM\text{-}OS(\mathcal{R})$ and $JCP(\mathcal{R})$. Moreover let $s \rightarrow_{p,C \Longrightarrow l \rightarrow r} t$ and $s \mathrel{-\!\!\!\Downarrow\!\!\!\!\rightarrow}_U s'$ with $COMP(s/u)$ for all $u \in U$, $COMP(s/v)$ for all $v \in O(s)$ with $v > p$, $U_{>p} := \{u \in U | u > p\} \neq \emptyset$ and s/u reducible for all $u \in U_{>p}$. Then there exists a term t' and $W \subseteq O(t)$ such that $s \mathrel{-\!\!\!\Downarrow\!\!\!\!\rightarrow}_U s' \rightarrow^{+} t'$ and $s \rightarrow_{p,C \Longrightarrow l \rightarrow r} t \mathrel{-\!\!\!\Downarrow\!\!\!\!\rightarrow}^{\leq 1}_{W} t'$ with $COMP(t/w)$ for all $w \in W$.*

Proof: Let $\mathcal{R}$, s, t, s', p, $C \Longrightarrow l \rightarrow r$ and U be given as above. W.l.o.g. we may assume $p = \lambda$, hence $s/p = s = \sigma(l) \rightarrow_{p,\sigma,P \Longrightarrow l \rightarrow r} \sigma(r) = t/p = t$, for some substitution σ with $\sigma(P)\downarrow$, and $COMP(s/u)$ for all $u \in U$. Define $P_l := \{q \in O(l) \,|\, l/q \in \mathcal{V}\} \subseteq O(s)$, $P_r := \{q \in O(r) \,|\, r/q \in \mathcal{V}\} \subseteq O(t)$. Due to $SEM\text{-}OS(\mathcal{R})$ and the fact that s/u is reducible for all $u \in U$ we know that for every redex s/q of s with $q > \lambda$ we have $q \geq q'$

for some $q' \in P_l$. Hence, every $u \in U$ is below or above some $q \in P_l$. Now we define s'' by $s \mathrel{-\!\!\!\!\downdownarrows\!\!\!\!\rightarrow}_{P_l} s''$, t' by $t \mathrel{-\!\!\!\!\downdownarrows\!\!\!\!\rightarrow}_{P_r}^{\le 1} t'$ and show that $s \mathrel{-\!\!\!\!\downdownarrows\!\!\!\!\rightarrow}_U s' \mathrel{-\!\!\!\!\downdownarrows\!\!\!\!\rightarrow}_{P_l}^{\le 1} s'' \rightarrow_{p,C \Longrightarrow l \rightarrow r} t'$ holds. To this end we consider all those positions from $U \cup P_l$ which are minimal among $U \cup P_l$ w.r.t. $\le$.

- If $p \in P_l$ is minimal among $U \cup P_l$ such that $u_1, \ldots, u_m$, $m \ge 1$, are all positions from U below p then due to $COMP(s/p)$ normalization of s at p can be achieved by normalizing s at $u_1, \ldots, u_m$ followed by normalizing the resulting term at p.

- If $u \in U$ is strictly minimal among $U \cup P_l$ such that $p_1, \ldots, p_n$ are all positions from P_l strictly below u then normalization of s at u can be achieved by first normalizing s at $p_1, \ldots, p_n$ yielding let's say $\hat{s}$ and then normalizing $\hat{s}$ at u. But the latter normalization must be empty, i.e. $\hat{s}/u$ must be irreducible. To see this, let us assume that $\hat{s}/u$ were reducible, let's say at position v with rule $C_1 \Longrightarrow l_1 \rightarrow r_1$ and matching substitution σ_1, i.e. $\sigma_1(l_1) = \hat{s}/uv$ with $\sigma_1(C_1) \downarrow$. By the construction of $\hat{s}$ we know that $\hat{s}/p_i$ is irreducible for $i = 1, \ldots, n$. Hence, $uv > \lambda$ is a non-variable position of both $\hat{s}$ and l. Moreover we know by the construction of $\hat{s}$ that $\hat{s}/u = \sigma'(l)/u$ for some substitution σ' with $\sigma \rightarrow^* \sigma'$ which implies $\sigma'(l)/uv = \sigma_1(l_1)$. Applying Lemma 3.7 (note that $SN(\sigma(x))$ holds for all $x \in V(l)$ due to the assumption $COMP(s/v)$ for all $v > p = \lambda$) we get $\sigma'(P) \downarrow$. But this means that there exists a feasible critical pair between the rules $C_1 \Longrightarrow l_1 \rightarrow r_1$ and $C \Longrightarrow l \rightarrow r$ which is not a critical overlay due to $uv > \lambda$. Hence we have a contradiction to $SEM\text{-}OS(\mathcal{R})$.

In summary this means that we have $s \mathrel{-\!\!\!\!\downdownarrows\!\!\!\!\rightarrow}_U s' \rightarrow^* s'' \rightarrow_{p,C \Longrightarrow l \rightarrow r} t'$ with $s''/p = \sigma'(l)$ for some substitution σ' with $\sigma \rightarrow^* \sigma'$, and $s \rightarrow_{p,C \Longrightarrow l \rightarrow r} t \mathrel{-\!\!\!\!\downdownarrows\!\!\!\!\rightarrow}_{P_r}^{\le 1} t'$ with $COMP(t/w)$ for all $w \in P_r$ due to $\{t/w | w \in P_r\} \subseteq \{s/v | v \in P_l\}$ and $COMP(s/v)$ for all $v > p = \lambda$. Hence, choosing $W := P_r$ we are done. ■

Now we are prepared to state and prove the following main result.

Theorem 3.19 *For any CTRS $\mathcal{R}$ we have:*

(a) $SEM\text{-}OS(\mathcal{R}) \wedge JCP(\mathcal{R}) \wedge SIN(\mathcal{R}) \Longrightarrow SN(\mathcal{R}) \wedge CR(\mathcal{R})$, *and*

(b) $SEM\text{-}OS(\mathcal{R}) \wedge JCP(\mathcal{R}) \quad \Longrightarrow \quad [\forall s : [SIN(s) \Longrightarrow SN(s) \wedge CR(s)]]$,

i.e. any locally confluent and innermost terminating overlay system is terminating and confluent, hence complete (part (a)) which also holds in the localized version (b).

Proof: It suffices to prove (b) since (a) follows from it. For a proof of (b) by contradiction let $\mathcal{R}$ be a CTRS with the assumed properties $SEM\text{-}OS(\mathcal{R})$ and $JCP(\mathcal{R})$ and let t_0 be a term with $SIN(t_0)$ but not $SN(t_0)$. Then we consider a minimal counterexample for t_0, i.e. an infinite derivation

$$D: \quad t_0 \rightarrow t_1 \rightarrow t_2 \rightarrow \cdots$$

satisfying

(1) $\forall j \geq 0 \,:\, \infty(t_j)$, and

(2) $\forall j \geq 0 \; \forall s_j \,:\, [\, t_j \rightarrow_p t_{j+1} \wedge t_j \rightarrow_q s_j \text{ with } q > p \,] \Longrightarrow \neg\infty(s_j)$.

The minimality assumption (2) says that reduction steps are performed at deepest possible positions. This means in particular that innermost reduction steps are preferred as long as possible. Due to $SIN(t_0)$ there must exist some first non-innermost step in D, let's say $t_n \rightarrow_{p,C \Longrightarrow l \rightarrow r} t_{n+1}$. The minimality assumption (2) implies $SN(t)$ and hence $COMP(t)$ (by Corollary 3.8) for every proper subterm t of t_n/p. By applying lemma 3.17 and using (2) we know that there exist (uniquely defined) terms s_n, s_{n+1} and $P \subseteq O(t_n)$, $p > P \neq \emptyset$, $Q \subseteq O(t_{n+1})$ with $t_n \mathrel{-\!\!\!\downdownarrows\!\!\!\rightarrow}_P s_n \rightarrow_{p,C \Longrightarrow l \rightarrow r} s_{n+1}$ and $t_n \rightarrow_{p,C \Longrightarrow l \rightarrow r} t_{n+1} \mathrel{-\!\!\!\downdownarrows\!\!\!\rightarrow}^{\leq 1}_Q s_{n+1}$ such that $SN(s_n)$ and $COMP(t_{n+1}/q)$ for all $q \in Q$. Moreover, $\infty(t_{n+1})$ and $\neg\infty(s_{n+1})$ imply $t_{n+1} \not\equiv s_{n+1}$, hence $t_{n+1} \mathrel{-\!\!\!\downdownarrows\!\!\!\rightarrow}_Q s_{n+1}$.

In order to obtain a contradiction to $SN(s_n)$, i.e. to $\neg\infty(s_n)$, it suffices to prove that $s_n \rightarrow s_{n+1}$ can be extended to an infinite reduction sequence. For that purpose it is sufficient to show that

> whenever we have $t_m \mathrel{-\!\!\!\downdownarrows\!\!\!\rightarrow}_U s_m$ with $\infty(t_m)$, $\neg\infty(s_m)$ and $COMP(t_m/u)$ for all $u \in U$
>
> then there exists an index $m' > m$, a term $s_{m'}$ and $U' \subseteq O(t_{m'})$ with $s_m \rightarrow^+ s_{m'}$, $t_{m'} \mathrel{-\!\!\!\downdownarrows\!\!\!\rightarrow}_{U'} s_{m'}$, $\neg\infty(s_{m'})$ and $COMP(t_{m'}/u)$ for all $u \in U'$.

Hence, let $t_m \rightarrow_{p,C \Longrightarrow l \rightarrow r} t_{m+1}$ be some step in D (for arbitrary $m > n$) and assume w.l.o.g. that t_m/u is reducible for all $u \in U$. Then we have to distinguish the following three cases:

(a) $\forall\, u \in U \,:\, u | p$: In this case we can choose $m' = m+1$ and $s_{m'} = s_{m+1}$ is obtained from s_m by applying $C \Longrightarrow l \rightarrow r$ at position p, i.e. we get $t_m \mathrel{-\!\!\!\downdownarrows\!\!\!\rightarrow}_U s_m \rightarrow_{p,C \Longrightarrow l \rightarrow r} s_{m+1}$, $t_m \rightarrow_{p,C \Longrightarrow l \rightarrow r} t_{m+1} \mathrel{-\!\!\!\downdownarrows\!\!\!\rightarrow}_U s_{m+1}$ with $\neg\infty(s_{m+1})$ and $COMP(t_{m+1}/u)$ – due to $t_{m+1}/u = t_m/u$ – for all $u \in U$ as desired.

(b) $\exists\, u \in U \,:\, p > u$: From the minimality assumption (2) we know $COMP(t_m/q)$ for all $q > p$, $q \in O(t_m)$. Hence, applying lemma 3.18 we get a term s_{m+1} and $W \subseteq O(t_{m+1})$ with $s_m \rightarrow^+ s_{m+1}$, $t_{m+1} \mathrel{-\!\!\!\downdownarrows\!\!\!\rightarrow}^{\leq 1}_W s_{m+1}$, $\neg\infty(s_{m+1})$ and $COMP(t_{m+1}/w)$ for all $w \in W$. Moreover, $\neg\infty(s_{m+1})$ and $\infty(t_{m+1})$ imply $t_{m+1} \not\equiv s_{m+1}$, hence $t_{m+1} \mathrel{-\!\!\!\downdownarrows\!\!\!\rightarrow}_W s_{m+1}$ as desired.

(c) $\exists\, u \in U \,:\, p \geq u$: In this case s_{m+1} is defined by $t_m \rightarrow_{p,C \Longrightarrow l \rightarrow r} t_{m+1} \mathrel{-\!\!\!\downdownarrows\!\!\!\rightarrow}^{\leq 1}_U s_{m+1}$. From $t_m \mathrel{-\!\!\!\downdownarrows\!\!\!\rightarrow}_U s_m$, $COMP(t_m/u)$ for all $u \in U$ and $p \geq u$ for some $u \in U$ we get $COMP(t_{m+1}/u)$ for all $u \in U$, and $s_m = s_{m+1}$. Moreover, $\neg\infty(s_m)$ implies $\neg\infty(s_{m+1})$ which together with $\infty(t_{m+1})$ yields $t_{m+1} \not\equiv s_{m+1}$, hence $t_{m+1} \mathrel{-\!\!\!\downdownarrows\!\!\!\rightarrow}_U s_{m+1}$. The only problem now is that the reduction sequence passing by s_m is not properly extended due to $s_m = s_{m+1}$. But from $COMP(t_m/u)$ for all $u \in U$ we know that only finitely many subsequent steps in D can take place below positions from U. Hence, eventually case (a) or case (b) applies again in which a proper extension of the reduction sequence passing by s_m is possible as desired.

By induction we can conclude that there exists an infinite derivation starting from s_n. But this is a contradiction to $\neg\infty(s_n)$. Thus we have proved the implication $SIN(t_0) \implies SN(t_0)$ under the assumptions $SEM\text{-}OS(\mathcal{R})$ and $JCP(\mathcal{R})$. Finally, applying Corollary 3.8 yields $CR(t_0)$. Hence we are done. ■

Note that for proving Theorem 3.19 we cannot apply the (simpler) construction used for proving Theorem 3.13 by means of parallel (unique) one-step reduction. The crucial point is that reduction of some term t at some position p need not be unique since critical overlays are allowed. But — as we have shown — it is possible to modify the construction by performing parallel normalization steps instead of parallel reduction steps.

Theorem 3.19 states that any (strongly) innermost terminating (conditional) semantical overlay system with joinable critical pairs is (strongly) terminating and confluent, hence complete, which even holds in a stronger local version. In other words, for (conditional) semantical overlay systems it suffices to verify innermost termination and joinability of all critical pairs in order to infer general termination and confluence, i.e. completeness. The non-triviality of this result is obvious taking into account the fact that for CTRSs the critical pair lemma does not hold in general and almost all known sufficient criteria for confluence presume even stronger properties than termination plus joinability of (conditional) critical pairs.

4 Conclusion

We have provided an abstract analysis of how various kinds of restricted termination (and confluence) properties of CTRSs are related to strong termination (and confluence). In particular, we have proved some new results about sufficient criteria for (strong) termination (and confluence) which can be considered as generalizations of known results about orthogonal unconditional TRSs.

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