Technische Universität Kaiserslautern
Fachbereich Mathematik

Trading to stops

The investigation of state-based stopping rules

Nora Imkeller

1. Gutachter: Prof. Dr. L.C.G. Rogers
2. Gutachter: Prof. Dr. Ralf Korn

Datum der Disputation: 19. Juli 2013

Vom Fachbereich Mathematik der Technischen Universität Kaiserslautern
zur Verleihung des akademischen Grades
Doktor der Naturwissenschaften (Doctor rerum naturalium, Dr. rer. nat.)
genehmigte Dissertation

Technische Universität
Kaiserslautern

D 386

I dedicate this thesis to my beloved dog Jordy!

Acknowledgments

I would like to thank

- Chris Rogers for his support during the year I spent in Cambridge,
- Ralf Korn for giving me the opportunity to write this thesis,
- Fraunhofer ITWM for the scholarship,
- my family, especially Bobby,
- the Financial Mathematics Group at Fraunhofer ITWM for the nice atmosphere,
- the tower property for helping me to do the proofs,
- the semicolon, which makes writing a little easier.

Danksagungen

Ich möchte mich bedanken bei

- Chris Rogers für seine Unterstützung während des Jahres in Cambridge,
- Ralf Korn der mir die Möglichkeit zur Promotion gegeben hat,
- Fraunhofer ITWM für das Stipendium,
- meiner Familie, insbesondere Bobby,
- der Finanzmathe-Abteilung des ITWM für die schöne Atmosphäre,
- der tower property für die Hilfe bei den Beweisen,
- dem Semikolon, welches das Schreiben etwas leichter macht.

Abstract

The use of trading stops is a common practice in financial markets for a variety of reasons: it provides a simple way to control losses on a given trade, while also ensuring that profit-taking is not deferred indefinitely; and it allows opportunities to consider reallocating resources to other investments. In this thesis, it is explained why the use of stops may be desirable in certain cases. This is done by proposing a simple objective to be optimized. Some simple and commonly-used rules for the placing and use of stops are investigated; consisting of fixed or moving barriers, with fixed transaction costs. It is shown how to identify optimal levels at which to set stops, and the performances of different rules and strategies are compared. Thereby, uncertainty and altering of the drift parameter of the investment are incorporated.

Abstrakt

Die Nutzung von Stoppregeln ist aus folgenden Gründen eine gängige Praxis an den Finanzmärkten: sie bietet einen einfachen Weg, um Verluste einer Investition zu steuern, gleichzeitig wird gewährleistet dass Gewinnmitnahmen nicht auf unbestimmte Zeit aufgeschoben werden, und sie ermöglichen die Umverteilung der Ressourcen auf andere Investitionen. In dieser Dissertation wird erläutert, in welchen Fällen die Verwendung von Stoppregeln wünschenswert sein kann. Dies geschieht indem eine einfache Zielfunktion angenommen wird, die optimiert werden kann. Unter der Annahme von konstanten Transaktionskosten werden ein paar einfache und häufig verwendete Stoppregeln für die Platzierung und Nutzung der Stopps untersucht, die aus konstanten oder sich bewegenden Schranken bestehen. Es wird gezeigt wie optimale Parameter bestimmt werden können die angeben wie die Stopps platziert werden können und die Performances der verschiedenen Regeln und Strategien werden verglichen. Dabei wird die Ungewissheit und die Änderung der Drift der Investition in Betracht gezogen.

Contents

List of Figures

List of Tables

1 Introduction

When an investor has entered a position, the crucial question is when to get out of the position. There are two main reasons why to exit the investment. The first of them is to control losses; if an investment has gone wrong, setting a stop can prevent the investor from making severe losses. The second reason is to take profits; if the investment goes in the intended direction, the gains are just paper-gains unless the investment is closed out. Furthermore, setting stops gives the opportunity of reallocating the investment capital to a different investment. This first kind of stop is called a stop-loss and it is usually handled by using a lower barrier. This barrier could be a fixed stop such that the position is sold when it has fallen by a fixed amount; or it could be a trailing stop, which rises as the value of the position rises, thereby locking in any gains, while allowing the position to continue to rise in value. The take-profit stop is realized by setting an upper barrier. There are various stopping rules which can be imagined, but here the focus lies on quite simple state-based stopping rules. To justify whether those simple rules are sufficient, the solutions will be compared to an optimal stopping problem.

The question is how to place the stops. If the stop-loss is reached too early, the investment ends with a quick loss, although the investment might have turned into the profits afterwards. If the profit is taken too early, the gain remains small and is eaten away by transaction cost while the position moves further. Placing stops is a classical field of chart analysts who believe in the existence of support- and resistance levels that can be computed from historical data. The lower stop could then be placed below the support level to sell the position when it breaks though this level, which is seen as a sign for a turning point. The upper stop could be set below the resistance level. However, chart analysis contradicts the assumption that all information is contained in the price. A critical view on chart analysis can be found in [15]. In this thesis, the placing is done by setting up a simple objective which must be maximized over the parameters defining the stopping rule. The optimal choices highly depend on the chosen modeling assumptions; relaxing several assumptions, by incorporating parameter-uncertainty and allowing the parameter to change at random times, leads to a very different picture.

If the investor uses a certain stopping rule and the position is stopped out, the question of reallocation emerges; the investor could either re-invest into the same position again or choose a new investment. From that, some natural opportunities arise where the outcome of the trade might give a good advice to this question.

In the literature dealing with trading rules and trading strategies such as [13], the attention is manly focused on the empirical analysis. Theoretical results are rare; for fixed-stops based rules, the determination of the quantities needed for the objective is easy and can be found in many kind of sources as [8] or [5]; for the trailing stop, results have been found in [6] and [21], which were generalized in [14]. Although for obvious reasons it is good to combine a stop-loss with a take-profit stop, none of the existing literature seems to cover the combination of the trailing stop with an upper stop. Moreover, the reallocation part is not considered and the effects of incorporating uncertainty and altering of the parameters seem to be novel as well.

2 The basic model

The main part of this thesis will deal with a specific situation which is based on the following assumptions. Some of these assumptions will be relaxed later on.

2.1 Modeling assumptions

At time 0, a wealthy individual has a sum Y_0 of money to invest in a fund. The value of this investment at time t will be supposed to be an arithmetic Brownian motion

$$Y_t \equiv Y_0 + \mu t + \sigma W_t,$$

where W is a standard Brownian motion, and μ and $\sigma > 0$ are known constants until further notice[1].

It might be more conventional to use a geometric Brownian motion to model the fund process. In Section 6.2 it is shown that this can be done by a small variation. However, it turns out that this model leads to trivial solutions; the optimal strategy is either never to exit the position, or to exit immediately. This is a sign that the geometric Brownian motion is an inappropriate model for the particular study. In practice, a fund might be a basket of stocks, with each stock symbolizing a company's value. Although it is common practice to model stock dynamics with a geometric Brownian motion, the weighted sum can not be described by a geometric Brownian motion, even though it might be a reasonable approximation. A reason for the inappropriateness of the geometric Brownian motion for a company's value over an infinite time-horizon is given in [17]. Using a geometric Brownian motion to model the company's value embodies an assumption of constant returns to scale which can only be realized if the retained earnings will be re-invested. But, in an infinite time-horizon, the returns to scale have to decrease until the company reaches the optimal scale of operation. In contrast to that, the arithmetic Brownian motion describes a company which already has reached its optimal scale of operation. For sure, neither model perfectly describes reality, but either may be used as an adequate approximation. As the time-horizon will be infinite, the arithmetic Brownian motion is the right model to choose. The process can become

[1] The assumption that μ is known will be relaxed in Chapter 3 and the assumption that μ is constant will be relaxed in Chapter 4.

negative but this is unlikely for short holding periods; and especially when finite lower stops are involved, the 0-barrier becomes less important.

The gain process is defined to be

$$\begin{aligned} X_t &\equiv Y_t - Y_0 \\ &= \mu t + \sigma W_t. \end{aligned}$$

The investor's money is locked in the investment and can not be spent until the position is closed out. Thus, the investor has some incentive to take profits from the investment, withdrawing the gain $X_T = Y_T - Y_0$ (which might be negative), at some stopping time $T \equiv T_1$, for immediate consumption. Having closed out the position, the investor repeats the process, once again investing the remaining value Y_0 in the position, and using the same stopping rule applied to the rebased process $(Y(T_1 + t) - Y(T_1))_{t \geq 0}$. Thus the stopping times T_n (which are the times at which the position gets closed and immediately re-opened) form a renewal process. Withdrawal and reinvestment are supposed to incur a fixed cost c. The time-0 value of this repeated trading activity will be

$$\varphi \equiv \mathbf{E}\left[\sum_{n \geq 0} e^{-\rho T_{n+1}}\, U(Y(T_{n+1}) - Y(T_n) - c)\right] \tag{2.1}$$

where ρ is the (constant) rate of discounting, and the utility U is some concave strictly increasing function. A finite time-horizon is not taken into account because it leads to time-dependent solutions which can only be solved numerically; an infinite-horizon objective yields explicit solutions.

The following result reduces the calculation of φ to two simpler calculations.

Proposition 1 *The value φ of the trading activity is*

$$\varphi = \frac{\mathbf{E}[e^{-\rho T}\, U(X_T - c)]}{1 - \mathbf{E}[e^{-\rho T}]}. \tag{2.2}$$

Proof: By the strong Markov property and the stationarity of the increments of Y, by decomposing the objective (2.1) at the first time $T = T_1$ that the position gets closed out, it holds

$$\begin{aligned}\varphi &= \mathbf{E}[e^{-\rho T}U(X_T - c)] + \mathbf{E}\big[\sum_{n\geq 1} e^{-\rho T_{n+1}}U(Y(T_{n+1}) - Y(T_n) - c)\big] \\ &= \mathbf{E}[e^{-\rho T}U(X_T - c)] + \mathbf{E}\big[e^{-\rho T}\mathbf{E}[\sum_{n\geq 1} e^{-\rho(T_{n+1}-T)}U(Y(T_{n+1}) - Y(T_n) - c)|\mathcal{F}_T]\big] \\ &= \mathbf{E}[e^{-\rho T}U(X_T - c)] + \mathbf{E}[e^{-\rho T}]\varphi.\end{aligned}$$

Rearrangement gives the result (2.2). □

The investor's preferences are assumed to be given by a constant absolute risk aversion which leads to the exponential utility function

$$U(x) = 1 - \exp(-\gamma x) \tag{2.3}$$

for some $\gamma > 0$, the coefficient of absolute risk aversion[2]. In case of this utility function, the value of the problem (2.2) can be expressed as

$$\begin{aligned}\varphi &= \frac{\mathbf{E}[e^{-\rho T}] - e^{\gamma c}\mathbf{E}[e^{-\rho T - \gamma X_T}]}{1 - \mathbf{E}[e^{-\rho T}]} \\ &= \frac{L(\rho, 0) - e^{\gamma c}L(\rho, \gamma)}{1 - L(\rho, 0)},\end{aligned} \tag{2.4}$$

where for arbitrary $\rho,\ \gamma \geq 0$,

$$L(\rho, \gamma) \equiv \mathbf{E}[e^{-\rho T - \gamma X_T}] \tag{2.5}$$

is the joint Laplace transform of the time and place of stopping.
The stopping time T will now be defined for a small class of simple stopping rules.

2.2 Stopping examples

To set the stage, some natural examples of state-based stopping rules are introduced.

Example 1: fixed stops

This is the easiest example of all. The investor trades if a gain of b or a loss of a has emerged. The upper stop is the take-profit stop and the lower stop is used to limit the losses. So, for $a > 0$ and $b > 0$, the stopping time is defined by

[2]Other utility functions can be captured by regarding the generalization in Section 6.1

$$T \equiv \inf\{t : X_t = -a \text{ or } X_t = b\}. \tag{2.6}$$

Example 2: trailing stop and fixed stop

The trailing stop is a lower stop that is not fixed but moves with the investment price. More precisely, the trailing stop occurs if the investment price drops by a fixed value from its local maximum. This kind of stop is widely used by practitioners because once the investor has made a certain gain, the trailing stop preserves a part of this gain. Fix $a > 0$ and $b > 0$, and let $\bar{X}_t \equiv \sup_{0 \leq s \leq t} X_s$. Then the stopping time is defined by a trailing stop at $-a + \bar{X}_t$ and a fixed take-profit stop at $b > 0$.

$$T \equiv \inf\{t : \bar{X}_t - X_t = a \text{ or } X_t = b\} \tag{2.7}$$

Example 3: trailing stop

As a special case of Example 2, with $a > 0$

$$T \equiv \inf\{t : \bar{X}_t - X_t = a\}, \tag{2.8}$$

gives the stopping time of the trailing stop only.

In Section 2.5, the objective is to identify the joint Laplace transform L in (2.5) in each of the examples under investigation. Then, referring to (2.4), the value of the trading activity can be computed. Numerical results are given in Section 2.6, identifying the optimal parameters a and b. There, the stopping examples will be compared to a fixed-revision rule and an optimal stopping problem which are introduced in the following two sections.

2.3 Fixed-revision rule

The investor chooses $T > 0$ fixed, and then revises the position at multiples of T, regardless of the performance of the fund.

The objective is once again given by (2.4), though now of course T is constant. Then, X_T is a normally distributed random variable with mean μT and variance $\sigma^2 T$. The joint Laplace transform is then given by

$$\begin{aligned}\mathbf{E}[e^{-\rho T-\gamma X_T}] &= e^{-\rho T}\,\mathbf{E}[e^{-\gamma X_T}]\\ &= e^{-\rho T}\,e^{-\gamma\mu T+\frac{1}{2}\gamma^2\sigma^2 T}\\ &= e^{-(\rho+\gamma\mu-\frac{1}{2}\gamma^2\sigma^2)T}.\end{aligned}$$

2.4 Optimal stopping problem

The optimal stopping problem for multiple stopping times $0 \leq \tau \equiv \tau_1 \leq \tau_2 \leq \ldots$ is defined by

$$\begin{aligned}\bar{\varphi} &\equiv \sup_{0\leq\tau_1\leq\tau_2\leq\ldots} \mathbf{E}\Big[\sum_{n\geq 0} e^{-\rho\tau_{n+1}}\,U(Y(\tau_{n+1}) - Y(\tau_n) - c\,)\Big]\\ &= \sup_{0\leq\tau_1\leq\tau_2\leq\ldots} \mathbf{E}[e^{-\rho\tau}\{U(X_\tau - c) + \mathbf{E}[\sum_{n\geq 1} e^{-\rho(\tau_{n+1}-\tau)}U(Y(\tau_{n+1}) - Y(\tau_n) - c)\}|\mathcal{F}_\tau]]\\ &= \sup_{\tau\geq 0}\mathbf{E}[e^{-\rho\tau}\{U(X_\tau - c) + \sup_{\tau_2\leq\tau_3\leq\ldots}\mathbf{E}[\sum_{n\geq 1} e^{-\rho(\tau_{n+1}-\tau)}U(Y(\tau_{n+1}) - Y(\tau_n) - c)\}|\mathcal{F}_\tau]]\\ &= \sup_{\tau\geq 0}\mathbf{E}[e^{-\rho\tau}\{U(X_\tau - c) + \bar{\varphi}\}]. \end{aligned} \tag{2.9}$$

This can be solved by recursively solving

$$\bar{\varphi}_{n+1} = \sup_{\tau\geq 0}\mathbf{E}\big[e^{-\rho\tau}\big\{U(X_\tau - c) + \bar{\varphi}_n\big\}\ \big], \tag{2.10}$$

starting from $\bar{\varphi}_0 = 0$. The answer can be obtained by solving a Crank-Nicolson finite-difference scheme. The calculations are given in Appendix A.

2.5 Analysis of the examples

In this section, the examples of the stopping rules presented in Section 2.2 will be analyzed by deriving explicit solutions for the joint Laplace transform L in each case. The first example is solved using differential equations techniques, which can be thought of as an application of Itô calculus. Similar techniques may also be used to solve the other examples, but as the state variable is no longer one-dimensional, the construction of the correct functions is not as simple or transparent. For this reason, the answers are derived by using Itô excursion theory, introduced by Itô in [10]; see [19] or [20] for accessible accounts.

2.5.1 Example 1: fixed stops

Define

$$\mathcal{L} \equiv \tfrac{1}{2}\sigma^2 \frac{\mathrm{d}^2}{\mathrm{d}x^2} + \mu \frac{\mathrm{d}}{\mathrm{d}x} - \rho \tag{2.11}$$

for the generator of the diffusion X with killing rate ρ and

$$M(t, X_t) \equiv e^{-\rho t} f(X_t).$$

If $f : \mathbb{R} \mapsto \mathbb{R}$ is C^2, then by an application of Itô's formula [12],

$$\begin{aligned}
\mathrm{d}M(t, X_t) &= \frac{\mathrm{d}}{\mathrm{d}t} M(t, X_t)\mathrm{d}t + \frac{\mathrm{d}}{\mathrm{d}x} M(t, X_t)\mathrm{d}X_t + \tfrac{1}{2} \frac{\mathrm{d}^2}{\mathrm{d}x^2} M(t, X_t)\mathrm{d} < X >_t \\
&= -\rho e^{-\rho t} f(X_t)\mathrm{d}t + e^{-\rho t} \frac{\mathrm{d}}{\mathrm{d}x} f(X_t)(\mu \mathrm{d}t + \sigma \mathrm{d}W_t) + \tfrac{1}{2} e^{-\rho t} \frac{\mathrm{d}^2}{\mathrm{d}x^2} f(X_t)\sigma^2 \mathrm{d}t \\
&= e^{-\rho t} \left\{ \big(-\rho f(X_t) + \mu \frac{\mathrm{d}}{\mathrm{d}x} f(X_t) + \tfrac{1}{2}\sigma^2 \frac{\mathrm{d}^2}{\mathrm{d}x^2} f(X_t) \big)\mathrm{d}t + \sigma \frac{\mathrm{d}}{\mathrm{d}x} f(X_t)\mathrm{d}W_t \right\} \\
&= e^{-\rho t} \left\{ \mathcal{L} f(X_t)\mathrm{d}t + \sigma \frac{\mathrm{d}}{\mathrm{d}x} f(X_t)\mathrm{d}W_t \right\}. \qquad (2.12)
\end{aligned}$$

Then, by the martingale representation theorem [16], $\mathcal{L}f = 0$ is equivalent to M being a local martingale. As it is bounded on the interval $[0, T]$, with T from (2.6), $(M(t \wedge T))_{t \geq 0}$ is a martingale. By the Optional Sampling Theorem [11], it follows that

$$f(0) = \mathbf{E}^0[e^{-\rho T} f(X_T)], \tag{2.13}$$

where the notation $\mathbf{E}^x$ denotes the expectation under the initial condition $X_0 = x$. So, solving the ordinary differential equation $\mathcal{L}f = 0$ in $[-a, b]$ with boundary conditions, $f(-a) = A$ and $f(b) = B$, for appropriate A and B, is enough in order to determine the numerator and denominator in (2.2). Furthermore, for $f(x) = \exp(-\gamma x)$, the expectation in (2.13) is the joint Laplace transform of (2.5) which is needed to determine the numerator and denominator in (2.4); accordingly, the joint Laplace transform can be obtained by solving the ordinary differential equation with respect to the boundary conditions $f(-a) = \exp(\gamma a)$ and $f(b) = \exp(-\gamma b)$.

The solution to the ordinary differential equation

$$\mathcal{L}f = 0, \quad f(-a) = A, \quad f(b) = B \tag{2.14}$$

can be obtained by substituting a trial solution $f(x) = \exp(zx)$ with dummy variable z into $\mathcal{L}f = 0$, which leads to the polynomial equation of second order

$$-\rho + \mu z + \tfrac{1}{2}\sigma^2 z^2 = 0$$

that has two solutions $-\alpha < 0 < \beta$ given by

$$\alpha = \tfrac{1}{\sigma^2}(\mu + \sqrt{\mu^2 + 2\sigma^2\rho}) \tag{2.15}$$

$$\beta = \tfrac{1}{\sigma^2}(-\mu + \sqrt{\mu^2 + 2\sigma^2\rho}). \tag{2.16}$$

This yields two solutions to $\mathcal{L}f = 0$, namely $\exp(-\alpha x)$ and $\exp(\beta x)$. As any linear combination is a solution as well, the general solution is

$$f(x) = c_1\, e^{-\alpha x} + c_2\, e^{\beta x},$$

with c_1 and c_2 to be determined by the boundary conditions of (2.14), resulting in

$$c_1 = \frac{Ae^{\beta b} - Be^{-\beta a}}{e^{\alpha a + \beta b} - e^{-\alpha b - \beta a}}$$

$$c_2 = \frac{Be^{\alpha a} - Ae^{-\alpha b}}{e^{\alpha a + \beta b} - e^{-\alpha b - \beta a}}.$$

Then, the solution to (2.14) is

$$f(x) = \frac{(Ae^{\beta b} - Be^{-\beta a})e^{-\alpha x} + (Be^{\alpha a} - Ae^{-\alpha b})e^{\beta x}}{e^{\alpha a + \beta b} - e^{-\alpha b - \beta a}}. \tag{2.17}$$

Evaluating at $x = 0$ simplifies to

$$f(0) = \frac{A(e^{\beta b} - e^{-\alpha b}) + B(e^{\alpha a} - e^{-\beta a})}{e^{\alpha a + \beta b} - e^{-\alpha b - \beta a}}. \tag{2.18}$$

For $f(x) = \exp(-\gamma x)$, the boundary conditions give $A = \exp(\gamma a)$ and $B = \exp(-\gamma b)$ such that the joint Laplace transform L_1 for this first example is

$$L_1(\rho, \gamma) = \frac{e^{\gamma a}(e^{\beta b} - e^{-\alpha b}) + e^{-\gamma b}(e^{\alpha a} - e^{-\beta a})}{e^{\alpha a + \beta b} - e^{-\alpha b - \beta a}}. \tag{2.19}$$

Formula (2.19) can also be obtained by a variation of the calculations of §2 in [8]. Substituting the form of L_1 into the expression (2.4) gives the value φ for this stopping rule. The dependence of the right-hand side on ρ is of course through the dependence of α, β on ρ.

The mean of the hitting time can be derived from the Laplace transform as

$$
\begin{aligned}
\mathbf{E}[T] &= -\frac{\partial}{\partial\rho} L_1(\rho,0)\Big|_{\rho=0} \\
&= \frac{\partial}{\partial\rho} \frac{-e^{\beta b} + e^{-\alpha b} - e^{\alpha a} + e^{-\beta a}}{e^{\alpha a+\beta b} - e^{-\alpha b-\beta a}}\Big|_{\rho=0}
\end{aligned}
\tag{2.20}
$$

If $\mu > 0$, then $\alpha|_{\rho=0} = \frac{2\mu}{\sigma^2} \equiv k$, $\beta|_{\rho=0} = 0$, $\frac{\partial}{\partial\rho}\alpha|_{\rho=0} = \frac{1}{\mu}$ and $\frac{\partial}{\partial\rho}\beta|_{\rho=0} = \frac{1}{\mu}$. If $\mu < 0$, then $-\alpha$ and β switch roles but the expression in (2.20) remains the same. Hence, applying the quotient rule to (2.20) transforms the result into

$$
\begin{aligned}
\mathbf{E}[T] &= \frac{(-\frac{b}{\mu} - \frac{b}{\mu}e^{-kb} - \frac{a}{\mu}e^{ka} - \frac{a}{\mu})(e^{ka} - e^{-kb})}{(e^{ka} - e^{-kb})^2} \\
&\quad - \frac{(-1 + e^{-kb} - e^{ka} + 1)((\frac{a}{\mu} + \frac{b}{\mu})e^{ka} - (-\frac{b}{\mu} - \frac{a}{\mu})e^{-kb})}{(e^{ka} - e^{-kb})^2} \\
&= \frac{b(e^{ka} - 1) - a(1 - e^{-kb})}{\mu(e^{ka} - e^{-kb})}.
\end{aligned}
\tag{2.21}
$$

The special case $\mu = 0$ can be obtained from (2.21) by letting $k \to 0$ and applying l'Hôpital's rule two times giving a mean time of ab/σ^2.

2.5.2 Example 2: trailing stop and fixed stop

Recall that the stopping time is given by (2.7). The process $Y \equiv X - \bar{X}$ is a continuous strong Markov process with values in $\mathcal{X} \equiv (-\infty, 0]$, and 0 is a recurrent point for this process. Thus, Itô theory of excursions [10] can be applied to this process. The idea of excursion theory is to split the process into parts at the recurrent states. The resulting parts are called excursions. Due to the Markov property, the excursions are independent and identically distributed. In this case, the open set $Y^{-1}((-\infty,0)) = \{t : Y_t \neq 0\} = \bigcup_j (u_j, v_j)$ is the disjoint union of countably many excursion intervals $I_j = (u_j, v_j)$. Then, for each such interval $I_j = (u_j, v_j)$, the process ξ^j defined by

$$\xi^j \equiv Y|_{I_j}$$

is the excursion of Y from 0 on the interval I_j with lifetime $\zeta_j = v_j - u_j$. Let U denote the space of all excursions of Y away from 0, that is, continuous functions $\xi : \mathbb{R}^+ \to \mathcal{X}$ with the property that the set $\xi^{-1}((-\infty, 0))$ is of the form $(0, \zeta)$. Regarding U as a subset of $C(\mathbb{R}^+, \mathbb{R})$ induces the subset topology on U, and in fact U is a Polish space; see, for example, [20] for definitions and basic properties.

The process $\bar{X}$ is a continuous homogeneous additive functional of Y, growing only when $Y = 0$, and acts as the local time at zero for Y. This can be seen by a variation of Tanaka's formula [20] which says

$$(-Y_t)^+ = -\int_0^t \mathbf{1}_{\{-Y_s>0\}} \mathrm{d}Y_s + \tfrac{1}{2} l_t, \tag{2.22}$$

where $(\cdot)^+ \equiv \max\{\cdot, 0\}$, which can be omitted as $Y \leq 0$, and l_t is the so called semi-martingale local time of Y at level 0. The local time is only defined up to a multiplicative constant. Instead of the semimartingale local time l_t, the standard local time of Itô and McKean will be used, which is $L_t = \frac{1}{2} l_t$. Thus, with the definition of Y, (2.22) transforms to

$$-Y_t = -\int_0^t \mathbf{1}_{\{Y_s<0\}} \mathrm{d}X_s + \int_0^t \mathbf{1}_{\{Y_s<0\}} \mathrm{d}\bar{X}_s + L_t.$$

As $\bar{X}$ increases if and only if $Y = 0$, the second integral disappears. Furthermore, since $\{Y = 0\}$ is a null set and $\mathrm{d}X$ is bounded, $\int \mathbf{1}_{\{Y=0\}} \mathrm{d}X_s = 0$ can be subtracted. Hence,

$$\begin{aligned} -Y_t &= -\int_0^t \mathrm{d}X_s + L_t \\ &= -X_t + L_t. \end{aligned}$$

Comparing this result to the definition of Y yields $\bar{X} = L$. The local time can be used to label the excursions; the local time does not increase within an excursion interval I_j but it increases if Y hits 0 such that every excursion ξ^j is related to a unique local time $L^j \equiv L_t$, $t \in I_j$ with $L^j < L^k$ if excursion ξ^j occurs before excursion ξ^k. The point process $\Pi \equiv \{(L^j, \xi^j) : j \in \mathbb{Z}\}$ is a point process in $(0, \infty) \times U$. Then, (l, ξ) is a point in Π if and only if the process Y makes an excursion ξ at local time l. The corresponding point function Ξ which maps the local times to their corresponding excursions can be generalized to $\mathbb{R}^+$ by mapping all other local times to a graveyard state η

$$\Xi_l \equiv \begin{cases} \xi & \text{if } (l, \xi) \in \Pi \\ \eta & \text{else} \end{cases}$$

which allows to identify the first excursion in a certain subset $A \subset U$ by Ξ_{l_A} with $l_A \equiv \inf\{l \geq 0 : \Xi_l \in A\}$. Let $N_l(A)$ be the number of excursions of type A up to local time l.

The key of excursion theory is that Π is a Poisson point process with measure Leb $\times\, n$, where n is the σ-finite *excursion measure*: see Itô [10]. So, $(N_l(A))_{l\geq 0}$ is a Poisson process

with intensity $n(A)$, for fixed l, $N_l(A)$ is a Poisson random variable with intensity $l\,n(A)$, and l_A is exponentially distributed with intensity $n(A)$.
The missing link to make use of Itô excursion theory is an explicit characterization of the excursion measure n. This is given by the fact that once the excursion has escaped from 0, it evolves like the diffusion Y until it first hits zero, and it leaves 0 according to an entrance law.
Here, excursion theory is used to calculate the expectation $\mathbf{E}[e^{-\rho T} f(X_T)]$, in order to obtain the numerator and denominator in (2.2) and the joint Laplace transform in (2.5), by specifying $f(x) = \exp(-\gamma x)$. As explained in [19], expectations such as above can be dealt with by introducing an independent $\exp(\rho)$ time τ, and writing

$$\mathbf{E}[e^{-\rho T} f(X_T)] = \mathbf{E}[f(X_T) : T < \tau]. \tag{2.23}$$

The natural way of thinking of the event $\{T < \tau\}$ is to pass through the real time axis to verify whether the mark τ or the hitting time T occur first. The way this is handled by excursion theory is to think of τ as being the first event time τ_1 in a Poisson process on $\mathbb{R}^+$ of intensity ρ, with event times $\tau_1 < \tau_2 < \ldots$. This Poisson process of times can be dealt with by *marking* the excursions of Y, each independently of all others, according to a Poisson process of intensity ρ. The excursion point process Π gets modified to the marked excursion point process $\tilde{\Pi}$, where each excursion ξ^j gets augmented to $\tilde{\xi^j} \equiv (\xi^j, N^j)$, where N^j is an increasing $\mathbb{Z}^+$-valued path, representing the path of the marking process restricted to the excursion ξ^j. If $\tilde{\Xi}$ is the corresponding generalized point function which maps the local times to the marked excursions, the event $\{T < \tau\}$ can be verified by passing through the local time axis until one of the following three cases occurs

1. the local time $\bar{X}$ reaches b
2. there is an excursion which gets to $-a$ before any mark
3. an excursion gets marked before it reaches $\{0, -a\}$.

To set some notation, let

$$A \equiv \{\text{excursions which are marked before reaching } 0 \text{ or } -a\}; \tag{2.24}$$
$$B \equiv \{\text{excursions which get to } -a \text{ with no mark before reaching } -a\ \}. \tag{2.25}$$

Then the event $\{T < \tau\}$ could *either* happen because $\bar{X}$ reaches b before the first excursion in $A \cup B$; *or* because the first excursion in $A \cup B$ happens before $\bar{X}$ reaches b, and is in fact an excursion in B. In the first case, the process stops at the high side, denoted by H, where it is $X_T = b$ and there is no excursion of type $A \cup B$ before local time b. Thus, using the Poisson property stated above and denoting $\nu \equiv n(A \cup B)$, the expectation (2.23) in this case is

$$\begin{aligned} \mathbf{E}[f(X_T) : T < \tau, H] &= f(b)\,\mathbf{P}(N_b(A \cup B = 0)) \\ &= f(b)\,\frac{(n(A \cup B)b)^0}{0!} e^{-n(A\cup B)b} \\ &= f(b)\,e^{-\nu b}. \end{aligned} \tag{2.26}$$

In the second case, the down side case, denoted by H^c, it is $X_T = -a + \bar{X}_T$ and $\bar{X}_T$ equals $l_{A\cup B} \equiv \inf\{l \geq 0 : \Xi_l \in A \cup B\}$, the local time of the first excursion in $A \cup B$, which is exponentially distributed as stated above. Thus, with the help of the tower property [11] and an immediate consequence of the Poisson property, the expectation (2.23) in this case is

$$\begin{aligned} \mathbf{E}[f(X_T) : T < \tau, H^c] &= \mathbf{E}[\mathbf{E}[f(-a + \bar{X}_t) : T < \tau, H^c | l_{A\cup B}]] \\ &= \mathbf{E}[f(-a + l_{A\cup B})\,\mathbf{P}(\Xi_{l_{A\cup B}} \in B | l_{A\cup B}) : l_{A\cup B} < b] \\ &= \frac{n(B)}{\nu}\,\mathbf{E}[f(-a + l_{A\cup B}) : l_{A\cup B} < b] \\ &= \frac{n(B)}{\nu} \int_0^b f(-a+y)\,\nu\,e^{-\nu y}\mathrm{d}y. \end{aligned}$$

So, the overall expectation (2.23) is the sum of these partial expectations

$$\mathbf{E}[e^{-\rho T} f(X_T)] = f(b)\,e^{-\nu b} + n(B) \int_0^b f(-a+y)\,e^{-\nu y}\mathrm{d}y, \tag{2.27}$$

and it remains to determine $n(A)$ and $n(B)$ because $\nu = n(A) + n(B)$ as the two sets are disjoint. As written above, away from 0, the excursion evolves like the diffusion $Y = X - \bar{X} = X - L$. Within an excursion, the local time does not change, so the excursion has the same dynamics as the process X once it has escaped from 0. So, it is worth to restrict the desired excursion sets to those which get to $-\epsilon$ for some $\epsilon > 0$. Therefore, let

$$E \equiv \{\text{excursions which get to } -\epsilon\} \tag{2.28}$$

be the corresponding set and $\tau_{-\epsilon}(\xi) \equiv \inf\{t : \xi_t = -\epsilon\}$ be the hitting time of $-\epsilon$ of such an excursion $\xi \in E$. The corresponding shift operator θ_ϵ, defined by $(\theta_\epsilon\,\xi)_t \equiv \xi_{\tau_{-\epsilon}(\xi)+t}$ shifts the excursion such that it starts at $-\epsilon$. The inverse operator will be denoted by

θ_ϵ^{-1}. Then, referring to [20], for any set[3] D, the probability law of a process starting at $-\epsilon$ with dynamics $\mu \mathrm{d}t + \sigma \mathrm{d}W_t$ and stopped at 0 is fulfills

$$\mathbf{P}^{-\epsilon}(D) = \frac{n((\theta_\epsilon^{-1} D) \cap E)}{n(E)}.$$

Thus, the excursion measure of an excursion set D is

$$\begin{aligned} n(D) &= \lim_{\epsilon \to 0} n((\theta_\epsilon^{-1} D) \cap E) \\ &= \lim_{\epsilon \to 0} n(E)\, \mathbf{P}^{-\epsilon}(D). \end{aligned}$$

Since the measure of excursions which reach $-\epsilon$ is asymptotic to ϵ^{-1} as $\epsilon \to 0$ (see Williams' decomposition of the Brownian excursion law [22], II.67), $\lim_{\epsilon \to 0} n(E)$ can be replaced by $\lim_{\epsilon \to 0} \epsilon^{-1}$ such that

$$n(D) = \lim_{\epsilon \to 0} \epsilon^{-1}\, \mathbf{P}^{-\epsilon}(D). \tag{2.29}$$

In order to compute $n(A)$ and $n(B)$, the corresponding probabilities $\mathbf{P}^{-\epsilon}(A)$ and $\mathbf{P}^{-\epsilon}(B)$ are needed. So, let Z be a process starting at $-\epsilon$ with dynamics $\mu \mathrm{d}t + \sigma \mathrm{d}W_t$ which is stopped when hitting 0. Then, $\mathbf{P}^{-\epsilon}(A)$ is the probability that an independent $\exp(\rho)$ time τ occurs before the process hits 0 or $-a$. Defining the corresponding stopping time by $T_{0,-a} \equiv \inf\{t : Z_t = 0 \text{ or } Z_t = -a\}$, then

$$\begin{aligned} \mathbf{P}^{-\epsilon}(A) &= \mathbf{P}^{-\epsilon}(\tau < T_{0,-a}) \\ &= \mathbf{E}^{-\epsilon}[\mathbf{P}(\tau < T_{0,-a} | T_{0,-a})] \\ &= \mathbf{E}^{-\epsilon}[1 - e^{-\rho T_{0,-a}}], \end{aligned}$$

where the Laplace transform can be obtained by the results of 2.5.1, because the stopping time is as in Example 1, with $b = 0$. There, the general result for an arbitrary starting value $x \in (-a, b)$ was given in (2.17). Using $x = -\epsilon$ and the boundary conditions $f(0) = 1$ and $f(-a) = 1$ gives

$$\mathbf{E}^{-\epsilon}[e^{-\rho T_{0,-a}}] = \frac{(1 - e^{-\beta a}) e^{\alpha \epsilon} + (e^{\alpha a} - 1) e^{-\beta \epsilon}}{e^{\alpha a} - e^{-\beta a}}.$$

Thus,

[3] To simplify notation, the set may be a set of excursions or processes which are stopped at 0.

$$\begin{aligned}\mathbf{P}^{-\epsilon}(A) &= 1 - \frac{(1-e^{-\beta a})e^{\alpha\epsilon} + (e^{\alpha a}-1)e^{-\beta\epsilon}}{e^{\alpha a} - e^{-\beta a}} \\ &= \frac{(1-e^{-\beta a})(1-e^{\alpha\epsilon}) + (e^{\alpha a}-1)(1-e^{-\beta\epsilon})}{e^{\alpha a} - e^{-\beta a}}. \end{aligned} \tag{2.30}$$

$\mathbf{P}^{-\epsilon}(B)$ is the probability that the process hits $-a$ before it hits 0 and before τ occurs. Using the stopping time $T_{0,-a}$, this means the stopping time occurs before τ and the process stops at the down side. Denoting the event that the process stops at the high side by H and the event that the process stops at the down side by H^c, then it holds

$$\begin{aligned}\mathbf{P}^{-\epsilon}(B) &= \mathbf{P}^{-\epsilon}(\tau > T_{0,-a} : H^c) \\ &= \mathbf{E}^{-\epsilon}[\mathbf{P}(\tau > T_{0,-a} | T_{0,-a}) : H^c] \\ &= \mathbf{E}^{-\epsilon}[e^{-\rho T_{0,-a}} : H^c],\end{aligned}$$

where the partial Laplace transform can be obtained from (2.17) by using $x = -\epsilon$ and the boundary conditions $f(0) = 0$ and $f(-a) = 1$ which results in

$$\mathbf{P}^{-\epsilon}(B) = \frac{e^{\alpha\epsilon} - e^{-\beta\epsilon}}{e^{\alpha a} - e^{-\beta a}}. \tag{2.31}$$

Substituting (2.30) and (2.31), respectively, into (2.29) and applying l'Hôpital's rule yields

$$\begin{aligned}n(A) &= \frac{1-e^{-\beta a}}{e^{\alpha a} - e^{-\beta a}} \lim_{\epsilon\to 0} \frac{1-e^{\alpha\epsilon}}{\epsilon} + \frac{e^{\alpha a}-1}{e^{\alpha a} - e^{-\beta a}} \lim_{\epsilon\to 0} \frac{1-e^{-\beta\epsilon}}{\epsilon} \\ &= \frac{-\alpha(1-e^{-\beta a}) + \beta(e^{\alpha a}-1)}{e^{\alpha a} - e^{-\beta a}} \\ &= \frac{\beta e^{\alpha a} + \alpha e^{-\beta a} - (\alpha+\beta)}{e^{\alpha a} - e^{-\beta a}}\end{aligned}$$

and

$$\begin{aligned}n(B) &= \frac{1}{e^{\alpha a} - e^{-\beta a}} \lim_{\epsilon\to 0} \frac{e^{\alpha\epsilon} - e^{-\beta\epsilon}}{\epsilon} \\ &= \frac{\alpha+\beta}{e^{\alpha a} - e^{-\beta a}}.\end{aligned} \tag{2.32}$$

As ν is the sum of these measures, it holds,

$$\nu = \frac{\beta e^{\alpha a} + \alpha e^{-\beta a}}{e^{\alpha a} - e^{-\beta a}}. \tag{2.33}$$

The last two expressions determine the unknowns in equation (2.27). In the special case of $f(x) = \exp(-\gamma x)$, this leads to the joint Laplace transform for Example 2,

$$\begin{aligned} L_2(\rho,\gamma) &= \mathbf{E}[e^{-\rho T - \gamma X_T}] \\ &= e^{-(\nu+\gamma)b} + n(B)e^{\gamma a}\int_0^b e^{-(\nu+\gamma)y}\mathrm{d}y \\ &= e^{-(\nu+\gamma)b} + \frac{n(B)e^{\gamma a}}{\nu+\gamma}(1-e^{-(\nu+\gamma)b}). \end{aligned} \tag{2.34}$$

As before, the mean of T can be computed by differentiating the Laplace transform with respect to ρ at zero.

$$\begin{aligned} \mathbf{E}[T] &= -\frac{\partial}{\partial\rho}L_1(\rho,0)\Big|_{\rho=0} \\ &= \frac{\partial}{\partial\rho}\, e^{-\nu b}\left(\frac{n(B)}{\nu} - 1\right) - \frac{n(B)}{\nu}\Big|_{\rho=0} \\ &= \left(\frac{\partial}{\partial\rho}\, e^{-\nu b}\Big|_{\rho=0}\right)\left(\frac{n(B)}{\nu}\Big|_{\rho=0} - 1\right) + \left(e^{-\nu b}\Big|_{\rho=0} - 1\right)\left(\frac{\partial}{\partial\rho}\frac{n(B)}{\nu}\Big|_{\rho=0}\right) \end{aligned} \tag{2.35}$$

Analogously to Subsection 2.5.1, it can be assumed that $\mu > 0$, giving $\alpha|_{\rho=0} = \frac{2\mu}{\sigma^2} \equiv k$, $\beta|_{\rho=0} = 0$, $\frac{\partial}{\partial\rho}\alpha|_{\rho=0} = \frac{1}{\mu}$ and $\frac{\partial}{\partial\rho}\beta|_{\rho=0} = \frac{1}{\mu}$, because for $\mu < 0$, $-\alpha$ and β switch roles but the expressions $n(B)$ and ν remain the same.
Then, it is

$$\begin{aligned} \frac{n(B)}{\nu}\Big|_{\rho=0} &= \frac{\alpha+\beta}{\beta e^{\alpha a} + \alpha e^{-\beta a}}\Big|_{\rho=0} \\ &= \frac{k}{k} \\ &= 1, \end{aligned}$$

$$\begin{aligned} \nu|_{\rho=0} &= \frac{\beta e^{\alpha a} + \alpha e^{-\beta a}}{e^{\alpha a} - e^{-\beta a}}\Big|_{\rho=0} \\ &= \frac{k}{e^{ka} - 1} \\ &\equiv m, \end{aligned}$$

$$
\begin{aligned}
\frac{\partial}{\partial\rho}\frac{n(B)}{\nu}\Big|_{\rho=0} &= \frac{\partial}{\partial\rho}\frac{\alpha+\beta}{\beta e^{\alpha a}+\alpha e^{-\beta a}}\Big|_{\rho=0} \\
&= \frac{(\frac{1}{\mu}+\frac{1}{\mu})k-k(\frac{1}{\mu}e^{ka}+\frac{1}{\mu}-ka\frac{1}{\mu})}{k^2} \\
&= \frac{1-e^{ka}+ka}{\mu k} \\
&= \frac{1}{\mu}\left(a-\frac{1}{m}\right),
\end{aligned}
$$

which transform (2.35) into

$$
\mathbf{E}[T] = \tfrac{1}{\mu}\left(e^{-mb}-1\right)\left(a-\tfrac{1}{m}\right). \tag{2.36}
$$

Again, the special case $\mu = 0$ can be obtained from (2.36) by letting $k \to 0$ and applying l'Hôpital's rule two times giving an expected hitting time of $\frac{a^2}{\sigma^2}(1-e^{-\frac{b}{a}})$.

2.5.3 Example 3: trailing stop

As this example is a special case of Example 2 with $b = \infty$, the results of Subsection 2.5.2 reduce to simpler expressions. The Laplace transform is

$$
\begin{aligned}
L_3(\rho,\gamma) &= \lim_{b\to\infty} e^{-(\nu+\gamma)b} + \frac{n(B)e^{\gamma a}}{\nu+\gamma}(1-e^{-(\nu+\gamma)b}) \\
&= \frac{n(B)e^{\gamma a}}{\nu+\gamma},
\end{aligned} \tag{2.37}
$$

which agrees with the result of Taylor [21], equation (1.1). It can easily be obtained from a result of Glynn & Iglehart [6], who have determined the joint Laplace transform of T and $\bar{X}_T$, by using the relation $\bar{X}_T = a + X_T$. Lehoczky determined this quantity in [14], equation 4, for the larger class of time homogeneous processes, given by the stochastic differential equation $\mathrm{d}X_t = \mu(X_t)\mathrm{d}t + \sigma(X_t)\mathrm{d}W_t$, with initial condition $X_0 = 0$.
As $m > 0$ for $\mu > 0$ and $\mu < 0$, the mean of the stopping time is

$$
\begin{aligned}
\mathbf{E}[T] &= \lim_{b\to\infty} \tfrac{1}{\mu}\left(e^{-mb}-1\right)\left(a-\tfrac{1}{m}\right) \\
&= \tfrac{1}{\mu}\left(\tfrac{1}{m}-a\right).
\end{aligned} \tag{2.38}
$$

Also the special case $\mu = 0$ follows from the previous subsection with $b = \infty$, leading to a mean time of a^2/σ^2.

2.6 Numerical results

The identification of the joint Laplace transform for each of the stopping examples of Section 2.2 now allows evaluating the objective φ (2.4). This value can be optimized by varying the parameters a and b, to find the best choice of these parameters, giving the objective

$$\varphi^* \equiv \sup_{a,b} \varphi.$$

The performance of these state-based rules can be compared to the non-state-based fixed-revision rule, introduced in Section 2.3, where the investor's best choice of fixed T has to be found to get the objective

$$\hat{\varphi}^* \equiv \sup_{T} \hat{\varphi}.$$

Furthermore, the results will be compared to the optimal stopping problem of Section 2.4. As this stopping problem is not restricted to any state- or time-based rules, its value cannot be worse than the restricted objectives.
For all rules under investigation, it will be assumed that the modeling parameters are $\mu = 0.15$, $\sigma = 0.3$, $\gamma = 2.5$, $c = 0.0005$, and $\rho = 0.1$. Various other parameters have been explored and the behavior appears to be quite the same.
Table 2.1 summarizes the results of the state-based stopping rules with corresponding objectives, best choices of a and b, and mean trading times; the fixed-revision rule with objective and best choice of T, which is given as a fixed mean time; and the optimal stopping problem with its objective.

	Best a	Best b	Objective	$\mathbf{E}[T]$
Fixed stops	∞	0.0184	4.1553	0.1224
Trailing stop and fixed stop	∞	0.0184	4.1553	0.1224
Trailing stop	0.0894		0.5314	0.0983
Fixed-revision			0.8724	0.3780
Optimal stopping			4.1553	

Table 2.1: Numerical results with certain drift parameter.

The stopping boundary of the optimal stopping problem is plotted in Figure 2.1 for a reasonable range. The blue area is the continuation region and the yellow area is the stopping region. The shape of the stopping boundary corresponds to a single upper stop and the value at the boundary agrees with the value b of the stopping examples. Accordingly, with fixed stops or with a fixed upper stop and a trailing stop, the best

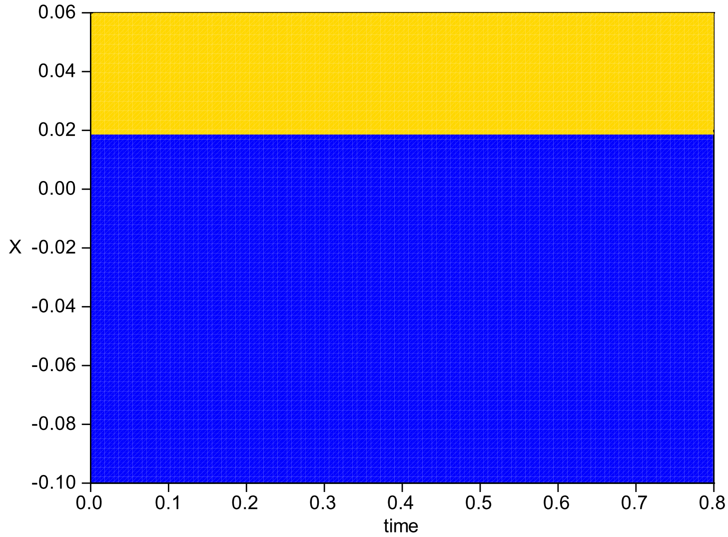

Figure 2.1: Stopping boundary of the optimal stopping problem with certain drift parameter.

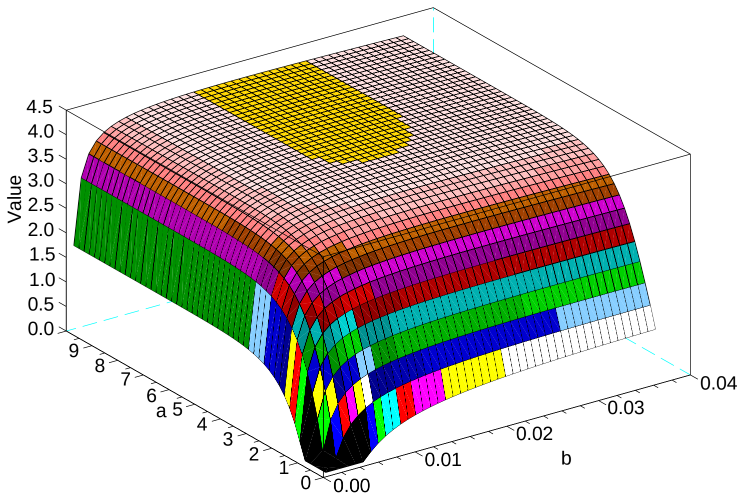

Figure 2.2: Value of the two-stops rules with certain drift parameter.

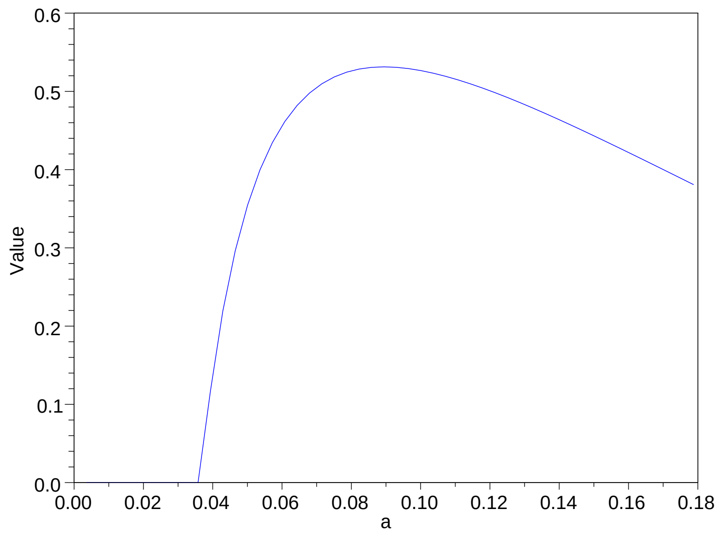

Figure 2.3: Value of the trailing stop rule with certain drift parameter.

choice of a is $a = \infty$; it always pays to push the lower stop all the way down. If this is done, then of course the two stopping rules amount to stopping at b, and so it is no surprise that the values, the optimal choices of b, and the mean time per trade all agree. The reason for this behavior is explained in more detail in Section 2.7. The value φ as a function of a and b is displayed in Figure 2.2; for a finite range of a, the pictures for Examples 1 and 2 are in principle different, but in this example they are not visibly different. The colors of the color level plot are used to illustrate the regions of values (a, b) where the value φ is nearly equal. The yellow area represents the set of values which are close to optimal. Notice that the value for a fixed upper and trailing stop is substantially higher than for a trailing stop only; this is of course to be expected, as the optimization is done over a larger set, but the magnitude of the improvement is noteworthy. The trailing stop example, Example 3, is quite different in character, with a much shorter mean time in trade. The corresponding value φ as a function of a is displayed in Figure 2.3. The fixed-revision rule performs very poorly relative to the two-sided stops rules, Examples 1 and 2.

2.7 Placing of the stops

The numerical investigation has shown that in many cases it is optimal to let $a \to \infty$. This is not a particular feature of the stopping examples, as the continuation region of the optimal stopping problem 2.4 is unbounded from below as well as seen in Figure 2.1. If this happens, then there would be no reason to place a lower stop, which is somewhat unexpected. This phenomenon can be analyzed quite completely for the case of fixed stops, which is done below. The other examples are more complicated, and the phenomenon has not been pursued analytically in those instances; numerical investigations show similar behavior.

Accordingly, the attention will be restricted to the fixed-stops example, Example 1. The value φ can be obtained explicitly by using (2.4) with the joint Laplace transform given in (2.19). Since the behavior of this value should be examined as $a \to \infty$ with all other parameters fixed, the (local) notation $\varphi(a)$ will be used, which is

$$\begin{aligned}\varphi(a) &= \frac{L_1(\rho,0) - e^{\gamma c} L_1(\rho,\gamma)}{1 - L_1(\rho,0)} \\ &= -1 + \frac{1 - e^{\gamma c} L_1(\rho,\gamma)}{1 - L_1(\rho,0)} \\ &= -1 + \frac{1 - B_1 e^{-(\alpha+\beta)a} - e^{\gamma c}(1-B_1) e^{(\gamma-\alpha)a} - B_2 e^{\gamma c}(1 - e^{-(\alpha+\beta)a})}{1 - B_1 e^{-(\alpha+\beta)a} - (1-B_1)\, e^{-\alpha a} - B_3(1 - e^{-(\alpha+\beta)a})}, \end{aligned} \tag{2.39}$$

where $B_1 = e^{-(\alpha+\beta)b}$, $B_2 = e^{-(\gamma+\beta)b}$, $B_3 = e^{-\beta b}$ are all positive constants less than 1. The large-a behavior of this expression is determined in the following result.

Proposition 2 *Consider the behavior of the objective* (2.4) *in the case of fixed stops* (2.6) *as $a \to \infty$, with b fixed.*

(i) If $\gamma > \alpha$ then

$$\lim_{a\to\infty} \varphi(a) = -\infty \tag{2.40}$$

(ii) If $\alpha > \gamma$ and $b > c$ then

$$\varphi(a) < \varphi(\infty) \tag{2.41}$$

for all $a > 0$.

Proof: For $a \to \infty$, the quantity in (2.39) becomes

$$\begin{aligned}\lim_{a\to\infty}\varphi(a) &= \lim_{a\to\infty} -1 + \frac{1 - B_1 e^{-(\alpha+\beta)a} - e^{\gamma c}(1-B_1)\, e^{(\gamma-\alpha)a} - B_2 e^{\gamma c}(1 - e^{-(\alpha+\beta)a})}{1 - B_1 e^{-(\alpha+\beta)a} - (1-B_1)e^{-\alpha a} - B_3(1 - e^{-(\alpha+\beta)a})} \\ &= -1 + \frac{1 - e^{\gamma c}(1-B_1)\, e^{\lim_{a\to\infty}(\gamma-\alpha)a} - B_2 e^{\gamma c}}{1 - B_3}.\end{aligned}$$

In case of $\gamma > \alpha$, the representation above makes it obvious that $\varphi(a) \to -\infty$ as $a \to \infty$.

The second case is more complicated. As $\gamma < \alpha$, from the above representation it can be seen that the limit of $\varphi(a)$ is

$$\varphi(\infty) = -1 + \frac{1 - B_2 e^{\gamma c}}{1 - B_3}.$$

Considering $\varphi(\infty) - \varphi(a)$, then a meticulous calculation leads to a rational expression whose denominator is positive, and whose numerator is (a positive multiple of)

$$H \equiv (1 - B_3)z - (B_2 e^{\gamma c} - B_3)y - (1 - B_2 e^{\gamma c}),$$

where $z \equiv e^{\gamma(a+c)}$ and $y \equiv e^{-\beta a}$ are set for brevity. Thus it will be sufficient to prove that the expression H is non-negative.

Since $b > c$, it holds $\varepsilon \equiv b - c > 0$, and then H becomes

$$\begin{aligned} H &= (1 - B_3)z + B_3(1 - e^{-\gamma-\varepsilon})y - (1 - B_3 e^{-\gamma\varepsilon}) \\ &= (1 - B_3)(z - 1) + B_3(1 - e^{-\gamma\varepsilon})y - B_3(1 - e^{-\gamma\varepsilon}) \\ &= (1 - B_3)(z - 1) - B_3(1 - e^{-\gamma\varepsilon})(1 - y).\end{aligned}$$

It is clear from the final equation that for $a > 0$ fixed, and if H is considered as a function of γ, then H is convex, and vanishes as $\gamma \downarrow 0$. To prove non-negativity of H, the gradient of H with respect to γ will be investigated, which is

$$\begin{aligned}\frac{\partial H}{\partial \gamma} &= (1 - B_3)(a + c)e^{\gamma(a+c)} - \varepsilon B_3(1 - y)e^{-\gamma\varepsilon} \\ &= e^{-\gamma\varepsilon}\big[(1 - B_3)(a + c)e^{\gamma(a+b)} - (1 - y)B_3(b - c)\big].\end{aligned}$$

As $\gamma \downarrow 0$, the limit of this gradient is

$$
\begin{aligned}
\lim_{\gamma \downarrow 0} \frac{\partial H}{\partial \gamma} &= (1 - B_3)(a + c) - (1 - y)B_3(b - c) \\
&= (1 - e^{-\beta b})(a + c) - e^{-\beta b}(b - c)(1 - e^{-\beta a}) \\
&= e^{-\beta b}\big[\, (a + c)e^{\beta b} + (b - c)e^{-\beta a} - (a + b) \,\big] \\
&= (a + b)e^{-\beta b}\left[\, \frac{a + c}{a + b}\, e^{\beta b} + \frac{b - c}{a + b}\, e^{-\beta a} - 1 \,\right] \\
&\geq (a + b)e^{-\beta b}\big[\, e^{\beta c} - 1 \,\big] \\
&> 0,
\end{aligned}
$$

where the convexity of the exponential function has been used for the first inequality. Since H is convex, and its derivative at zero is positive, it follows that H is increasing, and therefore is everywhere non-negative, since it is zero at $\gamma = 0$.

□

Proposition 2 says that in the case where $\gamma > \alpha$, it is *not* advantageous to let $a \to \infty$. The reason for that is that although the expectation of the discount factor at the down side, which can be read off (2.18) with boundary conditions $f(-a) = 1$ and $f(b) = 0$,

$$
\begin{aligned}
\mathbf{E}[e^{-\rho T} : H^c] &= \frac{e^{\beta b} - e^{-\alpha b}}{e^{\alpha a + \beta b} - e^{-\alpha b - \beta a}} \\
&= e^{-\alpha a}\, \frac{e^{\beta b} - e^{-\alpha b}}{e^{\beta b} - e^{-\alpha b} e^{-(\alpha + \beta) a}} \\
&\sim e^{-\alpha a},
\end{aligned}
\tag{2.42}
$$

is getting exponentially small when $a \to \infty$, the utility when this event happens is getting large negative exponentially, and *at a greater rate*.

In contrast, if $\alpha > \gamma$, the exponential decay of the expectation (2.42) beats the growth of the penalty, and the investor can ignore the penalty for stopping at a low negative level. The condition $b > c$ is needed for the proof, but has a natural interpretation; if $b < c$, the investor will certainly loose money every time the portfolio is reviewed as the gains can not even outperform the transaction costs.

So it seems that a finite lower stop is only useful in the case $\gamma > \alpha$. However, in typical examples, this can lead to coefficients γ of absolute risk aversion so high that the value φ is always negative, so an investor would never engage in this trade! The point is that α is given by (2.15), and if $\mu > 0$, it will always be $\alpha > 2\mu/\sigma^2$, a lower bound which need not be small. For realistic values of γ, $\gamma > \alpha$ only holds in situations where μ is negative. But if the growth rate of the trade was negative, and transaction costs have to be paid, an investor would certainly never want to enter into it!

Hence, with realistic values, an investor who enters a position has no advantage of using a lower stop.
The reason for this counterintuitive observation is that the modeling assumptions in Section 2.1 were too restrictive and have to be re-assessed. In a real world situation, an investor is *not certain* about the true value of μ. Thus, this assumption will be relaxed in the next chapter.

3 Uncertainty of the drift parameter

The modeling assumptions of Section 2.1 are generalized by incorporating an uncertainty risk to the model, which meets the demand that the investor who picks a fund is not certain about the drift parameter[4]. Merely, it will be assumed that the investor has some prior distribution over possible μ values with a positive probability that μ is negative. Then, even for small values of γ the punishment for stopping at very low levels really hurts, and the investor has an advantage to use a finite lower stop. On the other hand, if the probability of decently positive values of μ is quite high, the investor will be emboldened to take part in the trade.

3.1 Reallocating strategies

When μ was assumed to be known, it made no difference whether the investor picked the same fund every time or invested in an other one with the same drift. Allowing the drift to be random leads to the possibility of reallocating the investment capital to a new fund. There are several reallocating strategies which will be investigated.
For the first three of the following strategies, the position is closed out according to the stopping examples of Sections 2.2 and 2.3, respectively.
The value function to be optimized is given by

$$\bar{\varphi} = \int \varphi(\mu)\, m(\mathrm{d}\mu), \tag{3.1}$$

where $\varphi(\mu)$ is the value when the investor has picked a fund with drift μ, and m is the distribution of μ.

Strategy A

Having closed out the position, the investor repeats the process, investing in the *same* fund.
When a fund with drift μ is chosen, as in (2.2), the value will be

[4] Similarly, it can be assumed that there is uncertainty in the volatility parameter, but considering the drift is the more interesting story.

$$\varphi(\mu) = \mathbf{E}_\mu[e^{-\rho T} U(X_T - c)] + \mathbf{E}_\mu[e^{-\rho T}]\varphi(\mu),$$

where $\mathbf{E}_\mu$ denotes the expectation under the drift μ. This equation is equivalent to

$$\varphi(\mu) = \frac{\mathbf{E}_\mu[e^{-\rho T} U(X_T - c)]}{1 - \mathbf{E}_\mu[e^{-\rho T}]},$$

so the overall value (3.1) is given by

$$\bar{\varphi} = \int \frac{\mathbf{E}_\mu[e^{-\rho T} U(X_T - c)]}{1 - \mathbf{E}_\mu[e^{-\rho T}]} \, m(\mathrm{d}\mu). \tag{3.2}$$

Strategy B

Having closed out the position, the investor picks an *independent* fund with the same probabilistic structure.
When a fund with drift μ is chosen, the value will be

$$\varphi(\mu) = \mathbf{E}_\mu[e^{-\rho T} U(X_T - c)] + \mathbf{E}_\mu[e^{-\rho T}]\bar{\varphi},$$

as the fund will be discarded at T. Thus, the overall value (3.1) is given by

$$\begin{aligned} \bar{\varphi} &= \int \mathbf{E}_\mu[e^{-\rho T} U(X_T - c)] + \mathbf{E}_\mu[e^{-\rho T}] \, \bar{\varphi} \, m(\mathrm{d}\mu) \\ &= \int \mathbf{E}_\mu[e^{-\rho T} U(X_T - c)] \, m(\mathrm{d}\mu) + \int \mathbf{E}_\mu[e^{-\rho T}] \, m(\mathrm{d}\mu) \, \bar{\varphi}, \end{aligned}$$

which can be rearranged to give

$$\bar{\varphi} = \frac{\int \mathbf{E}_\mu[e^{-\rho T} U(X_T - c)] \, m(\mathrm{d}\mu)}{1 - \int \mathbf{E}_\mu[e^{-\rho T}] \, m(\mathrm{d}\mu)}. \tag{3.3}$$

Strategy C

If the position is closed out on the down side, the investor picks an *independent* fund with the same probabilistic structure. On the high side, the *same* investment will be chosen.
As before, H and H^c denote the events that the position is closed out on the high side and the down side, respectively; with $\mathbf{E}_\mu[\,\cdot : H]$ and $\mathbf{E}_\mu[\,\cdot : H^c]$ being the corresponding partial expectations. Then, the value given μ is

$$\varphi(\mu) = \mathbf{E}_\mu[e^{-\rho T} U(X_T - c)] + \mathbf{E}_\mu[e^{-\rho T} : H^c]\,\bar{\varphi} + \mathbf{E}_\mu[e^{-\rho T} : H]\,\varphi(\mu),$$

which is equivalent to

$$\varphi(\mu) = \frac{\mathbf{E}_\mu[e^{-\rho T} U(X_T - c)] + \mathbf{E}_\mu[e^{-\rho T} : H^c]\,\bar{\varphi}}{1 - \mathbf{E}_\mu[e^{-\rho T} : H]},$$

and the overall value (3.1) is

$$\begin{aligned}\bar{\varphi} &= \int \frac{\mathbf{E}_\mu[e^{-\rho T} U(X_T - c)] + \mathbf{E}_\mu[e^{-\rho T} : H^c]\,\bar{\varphi}}{1 - \mathbf{E}_\mu[e^{-\rho T} : H]}\, m(\mathrm{d}\mu) \\ &= \int \frac{\mathbf{E}_\mu[e^{-\rho T} U(X_T - c)]}{1 - \mathbf{E}_\mu[e^{-\rho T} : H]}\, m(\mathrm{d}\mu) + \int \frac{\mathbf{E}_\mu[e^{-\rho T} : H^c]}{1 - \mathbf{E}_\mu[e^{-\rho T} : H]}\, m(\mathrm{d}\mu)\,\bar{\varphi}.\end{aligned}$$

Rearranging leads to

$$\bar{\varphi} = \frac{\int \frac{\mathbf{E}_\mu[e^{-\rho T} U(X_T - c)]}{1 - \mathbf{E}_\mu[e^{-\rho T} : H]}\, m(\mathrm{d}\mu)}{1 - \int \frac{\mathbf{E}_\mu[e^{-\rho T} : H^c]}{1 - \mathbf{E}_\mu[e^{-\rho T} : H]}\, m(\mathrm{d}\mu)}. \tag{3.4}$$

For the above three stories, the main point is that the value $\bar{\varphi}$ can be deduced quite explicitly by doing at most two integrations. The expectations above are given by the Laplace transforms

$$\begin{aligned} L^\mu(\rho, \gamma) &\equiv \mathbf{E}_\mu[\exp(-\rho T - \gamma X_T)]; \\ L^\mu(\rho, \gamma : H) &\equiv \mathbf{E}_\mu[\exp(-\rho T - \gamma X_T) : H]; \\ L^\mu(\rho, \gamma : H^c) &\equiv \mathbf{E}_\mu[\exp(-\rho T - \gamma X_T) : H^c] \end{aligned}$$

for the different stopping rules. These Laplace transforms are restricted to a certain drift and have been determined explicitly in Section 2.5. There, also the partial joint Laplace transforms can be found. Using the boundary conditions $f(-a) = 0$ and $f(b) = \exp(-\gamma b)$ in (2.18) gives

$$L_1^\mu(\rho, \gamma : H) = \frac{(e^{\alpha a} - e^{-\beta a})e^{-\gamma b}}{e^{\alpha a + \beta b} - e^{-\alpha b - \beta a}},$$

for stopping Example 1, where μ is hidden in α and β. For Example 2, this quantity is given in equation (2.26) with $f(b) = \exp(-\gamma b)$ and coincides with the first summand of (2.34),

$$L_2^\mu(\rho,\gamma : H) = e^{-(\nu+\gamma)b}.$$

The remaining partial Laplace transform can be found similarly or by using the connection $L^\mu(\rho,\gamma : H^c) = L^\mu(\rho,\gamma) - L^\mu(\rho,\gamma : H)$.

In contrast to using the stopping examples, the next strategy uses an optimal stopping analysis for a Bayesian learning model.

Strategy D

The investor considers a learning process for the drift of a chosen fund. Each time the position reaches the stopping region of the optimal stopping problem, the investor picks an *independent* fund with the same probabilistic structure.

More precisely, having observed the data up to some time τ_0, resulting in a prior μ_0, the distribution of the drift is assumed to be normally distributed with mean μ_0 and variance σ^2/τ_0. Then, by Bayes' formula [9], the density of μ conditional on the observation filtration $\mathcal{X}_t$ yields

$$\begin{aligned} p_{\mu|X_t=x}(y) &= \frac{p_\mu(y)\, p_{X_t|\mu=y}(x)}{\int p_\mu(l)\, p_{X_t|\mu=l}(x)\mathrm{d}l} \\ &= \frac{\frac{1}{\sqrt{2\pi\sigma^2/\tau_0}} e^{-\frac{(y-\mu_0)^2}{2\sigma^2/\tau_0}} \frac{1}{\sqrt{2\pi\sigma^2 t}} e^{-\frac{(x-yt)^2}{2\sigma^2 t}}}{\int \frac{1}{\sqrt{2\pi\sigma^2/\tau_0}} e^{-\frac{(l-\mu_0)^2}{2\sigma^2/\tau_0}} \frac{1}{\sqrt{2\pi\sigma^2 t}} e^{-\frac{(x-lt)^2}{2\sigma^2 t}} \mathrm{d}l} \\ &= \frac{1}{\sqrt{2\pi v_t}} e^{-\frac{(y-\hat{\mu}(t,x))^2}{2v_t}}, \end{aligned} \tag{3.5}$$

with

$$\hat{\mu}(t,x) = \frac{\tau_0\mu_0 + x}{\tau_0 + t} \quad \text{and} \quad v_t = \frac{\sigma^2}{\tau_0 + t}. \tag{3.6}$$

The last equation of (3.5) is shown by rewriting the quantity which occurs in the numerator and denominator in terms of the final normal density multiplied by a constant which is independent from y or l; the constant cancels out and the integral of the density equals 1.

(3.5) demonstrates that μ conditional on $\mathcal{X}_t$ is normally distributed with mean $\hat{\mu}(t, X_t)$ and variance v_t. Furthermore, the gain process fulfills

$$
\begin{aligned}
\mathrm{d}\hat{X}_t &= \left(\int y\, p_{\mu|X_t}(y)\mathrm{d}y \right) \mathrm{d}t + \sigma \mathrm{d}\hat{W}_t \\
&= \mathbf{E}[\mu|X_t]\mathrm{d}t + \sigma \mathrm{d}\hat{W}_t \\
&= \hat{\mu}(t, X_t)\mathrm{d}t + \sigma \mathrm{d}\hat{W}_t,
\end{aligned} \tag{3.7}
$$

where $\hat{W}_t$ is a standard Brownian motion in the observation filtration $\mathcal{X}_t$. See [3] for more details of this derivation.
As in (2.9), the value of the optimal stopping problem is given by

$$
\bar{\varphi} = \sup_{\tau \geq 0} \mathbf{E}\big[e^{-\rho\tau} \big\{ U(\hat{X}_\tau - c) + \bar{\varphi} \big\} \big], \tag{3.8}
$$

but $\hat{X}$ (3.7) has a non-constant drift process (3.6). Again, this can be solved recursively as in (2.10) with the Crank-Nicolson finite-difference scheme given in Appendix A.

3.2 Numerical results

For the reallocating strategies A, B and C, the distribution of the drift is assumed to be a normal distribution with mean μ_0 and variance σ_μ^2. These values can be seen as the market drift and its variance. The drift will be supposed to be $\mu_0 = 0.15$ and two different values are taken for the standard deviation: firstly, $\sigma_\mu = 0.3$; and secondly the more uncertain case $\sigma_\mu = 0.7$. All other parameters are as in Section 2.6.

Strategy A

The results obtained for the strategy where the investor goes back into the same fund are reported in the following tables. Table 3.1 is for $\sigma_\mu = 0.3$ and Table 3.2 is for $\sigma_\mu = 0.7$. For $\sigma_\mu = 0.3$, the calculated values of the three stopping examples are displayed in Figures 3.1, 3.2 and 3.3, respectively. Due to the risk aversion, compared to the known-drift case in Table 2.1, the values of all the rules have dropped, particularly the state-based stops trading rules. As with the certain growth rate, the two-stops rules do substantially better than either the trailing stop alone or the fixed time to revision. Mean times in trades have fallen in all cases. As before, there is no appreciable difference between Examples 1 and 2; the trailing stop has very little effect. Increasing the deviation of the drift to $\sigma_\mu = 0.7$ leads to even smaller objectives. In all cases, the parameter a has fallen to protect against huge losses.

	Best a	Best b	Objective	$\mathbf{E}[T]$
Fixed stops	0.2159	0.0470	0.8416	0.0998
Trailing stop and fixed stop	0.2375	0.0464	0.8398	0.0984
Trailing stop	0.0603		0.2397	0.0439
Fixed-revision			0.5627	0.0670

Table 3.1: Numerical results for reallocating Strategy A with $\sigma_\mu = 0.3$.

	Best a	Best b	Objective	$\mathbf{E}[T]$
Fixed stops	0.0837	0.0559	0.4410	0.0475
Trailing stop and fixed stop	0.1069	0.0523	0.4223	0.0452
Trailing stop	0.0411		-0.4264	0.0203
Fixed-revision			0.0698	0.0290

Table 3.2: Numerical results for reallocating Strategy A with $\sigma_\mu = 0.7$.

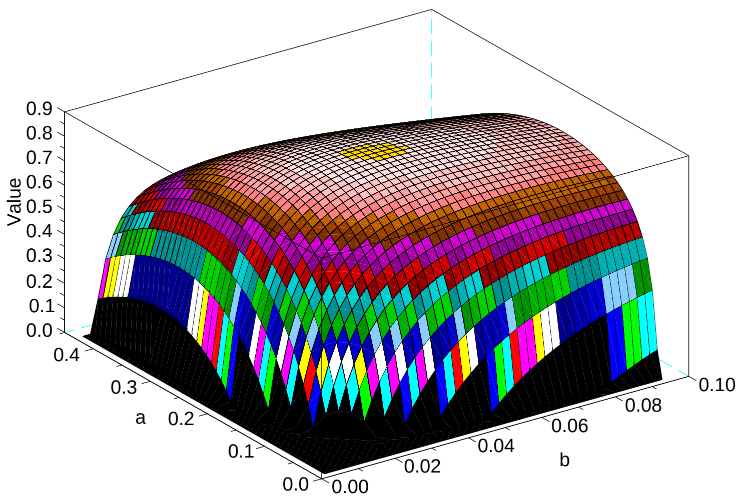

Figure 3.1: Value of the fixed stops rule for reallocating Strategy A.

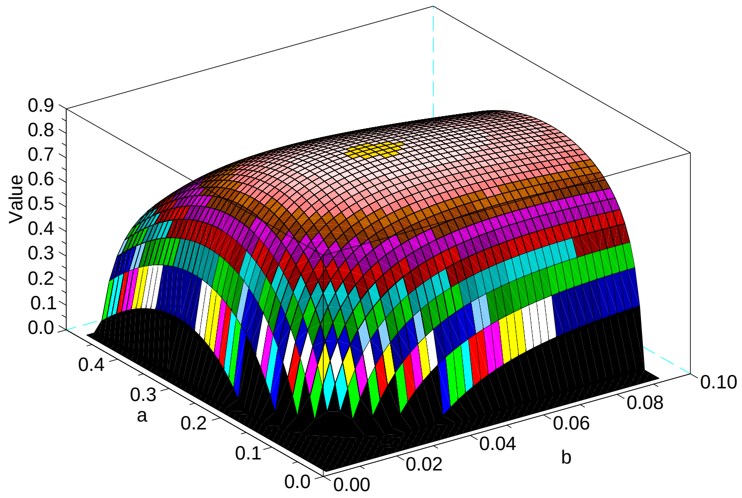

Figure 3.2: Value of the trailing stop and fixed stop rule for reallocating Strategy A.

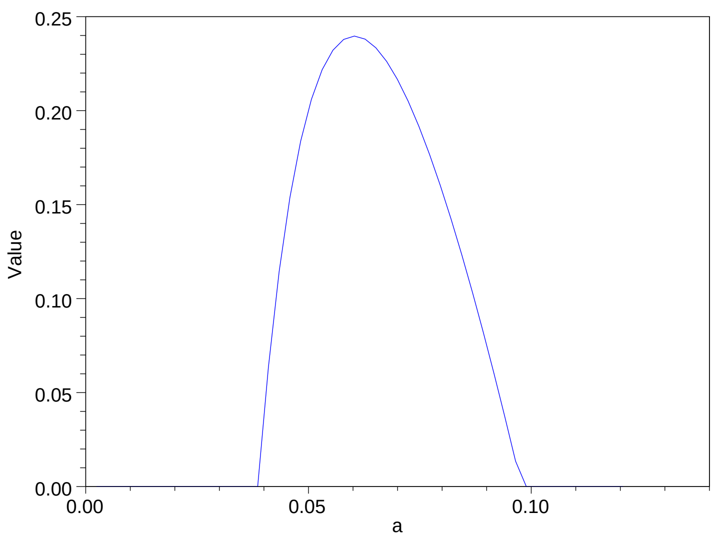

Figure 3.3: Value of the trailing stop rule for reallocating Strategy A.

Strategy B

The next tables, Table 3.3 and Table 3.4, show the results for $\sigma_\mu = 0.3$ and $\sigma_\mu = 0.7$, respectively, if the investor picks a new independent fund each time the position is closed out.

	Best a	Best b	Objective	$\mathbf{E}[T]$
Fixed stops	0.0191	0.3409	1.4071	0.0804
Trailing stop and fixed stop	0.0985	0.6558	1.3431	0.1236
Trailing stop	0.0952		1.3354	0.1159
Fixed-revision			0.5596	0.0662

Table 3.3: Numerical results for reallocating Strategy B with $\sigma_\mu = 0.3$.

	Best a	Best b	Objective	$\mathbf{E}[T]$
Fixed stops	0.0083	0.3426	4.8955	0.0338
Trailing stop and fixed stop	0.0767	0.7572	4.1695	0.0792
Trailing stop	0.0728		4.1224	0.0720
Fixed-revision			0.0698	0.0290

Table 3.4: Numerical results for reallocating Strategy B with $\sigma_\mu = 0.7$.

The values of the three stopping examples which were calculated with respect to $\sigma_\mu = 0.3$ are displayed in Figures 3.4, 3.5 and 3.6, respectively. Compared to Strategy A, the values of the stops trading rules have grown. It can be seen that the a-parameters of the optimal lower stops have fallen to much lower values, while the b-values for the upper stops are much larger. The reason for this is that a *good* investment having a large positive drift should not be stopped, but those investments with poor drifts should be discarded quickly. The difference in value of Examples 2 and 3 is comparatively small because the gain process in Example 2 will only occasionally get stopped at b; most will be caught by the trailing stop. It can be seen that the fixed stops rule, Example 1, performs better than Example 2 with a trailing stop; presumably because the trailing stop may prematurely close out a trade which might have turned out to be profitable.

Interestingly, if the values for Examples 1 and 2 for Strategy B are compared to the values for Examples 1 and 2 in the certain-drift case, it emerges that for the smaller value $\sigma_\mu = 0.3$ it is better if the drift is known, while for the larger value $\sigma_\mu = 0.7$ the investment does better if there is uncertainty in the drift. The reason is not hard to discern. For small σ_μ, risk aversion is the dominant effect, but for larger σ_μ the investment benefits from the wider spread of μ-values; the lower stop closes down the unprofitable trades, but the investor gets more of an upside from the profitable trades.

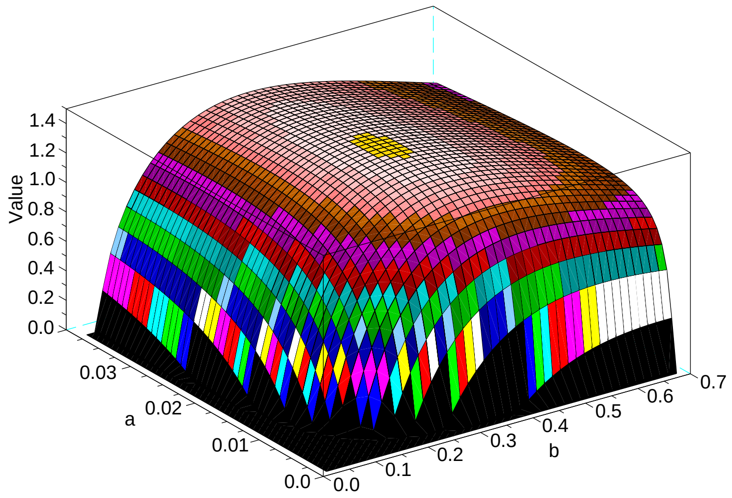

Figure 3.4: Value of the fixed stops rule for reallocating Strategy B.

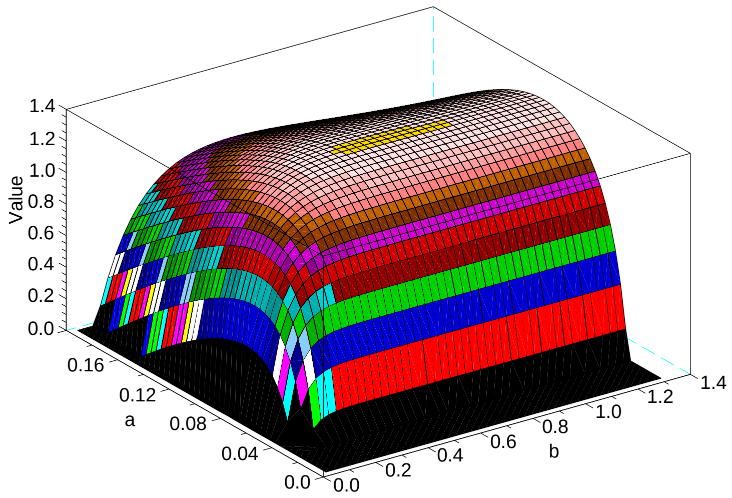

Figure 3.5: Value of the trailing stop and fixed stop rule for reallocating Strategy B.

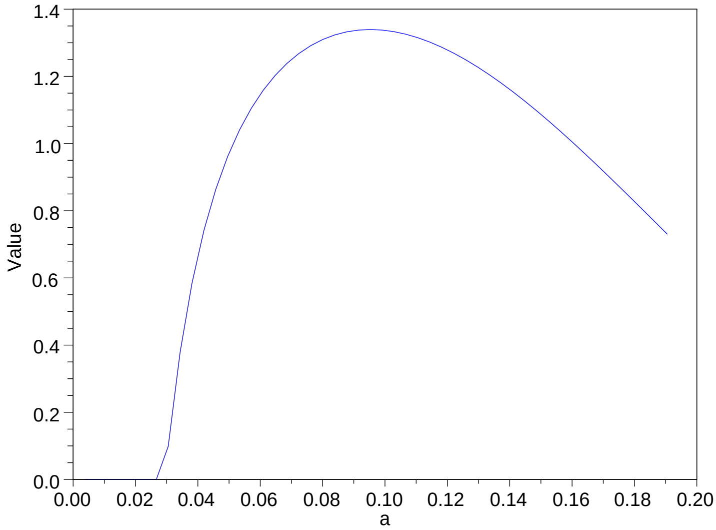

Figure 3.6: Value of the trailing stop rule for reallocating Strategy B.

Strategy C

For Strategy B it was shown that the two-stops examples have small a-values to shut down the unprofitable trades, and large b-values to let the gains accumulate when a profitable trade has been found. In contrast, with Strategy C which only changes funds if the process comes out at the lower stop, the results in the Tables 3.5 and 3.6 look quite different[5].

For $\sigma_\mu = 0.3$, the values are displayed in Figures 3.7 and 3.8, respectively. Notice firstly that the values of the objectives are substantially higher, because the strategy allows to shop around for good funds, and once one is found, the investor is allowed to hold that fund until it gets stopped out at a lower stop. Because of this, it has to be avoided to stop out a fund at the lower stop unless it is confident that it is a poor performer; the loss of profit from killing a good fund too early would be considerable. So this explains why there are larger a values than for Strategy B. Also the b values are much smaller, which can be understood as a desire to book profits quickly and avoid discounting them away; if the fund is seemingly good, the investor will gladly do this, because after that the same good fund can be played again, in contrast of Strategy B where a new independent fund has to be picked with unknown chances on its quality.

[5] Of course, it does not make any sense to consider the single-stop rules, trailing stop and fixed-revision, because there it cannot be distinguished between upper and lower outcomes.

	Best a	Best b	Objective	$\mathbf{E}[T]$
Fixed stops	0.3620	0.0239	6.8525	0.0370
Trailing stop and fixed stop	0.3735	0.0240	6.8539	0.0371

Table 3.5: Numerical results for reallocating Strategy C with $\sigma_\mu = 0.3$.

	Best a	Best b	Objective	$\mathbf{E}[T]$
Fixed stops	0.2020	0.0213	20.216	0.0125
Trailing stop and fixed stop	0.2119	0.0219	20.231	0.0129

Table 3.6: Numerical results for reallocating Strategy C with $\sigma_\mu = 0.7$.

For the first time, Example 2 outperforms Example 1 (but only very slightly). This seems to be because the trailing stop will allow a slightly quicker closing out of bad trades, and since the lower stop is initially quite far from 0, this difference matters.
Another way of capturing this advantage would be by adding a time-dependent slope to the barriers, and this is examined in Section 3.3.
For the same reasons as with Strategy B, a larger standard deviation $\sigma_\mu = 0.7$ yields a better objective.

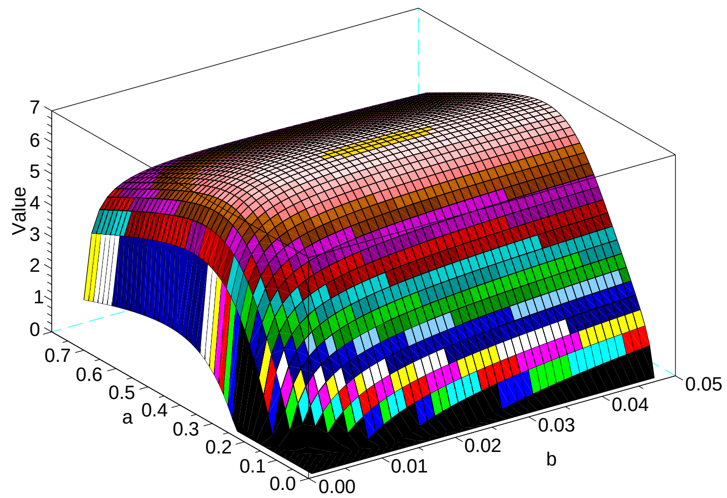

Figure 3.7: Value of the fixed stops rule for reallocating Strategy C.

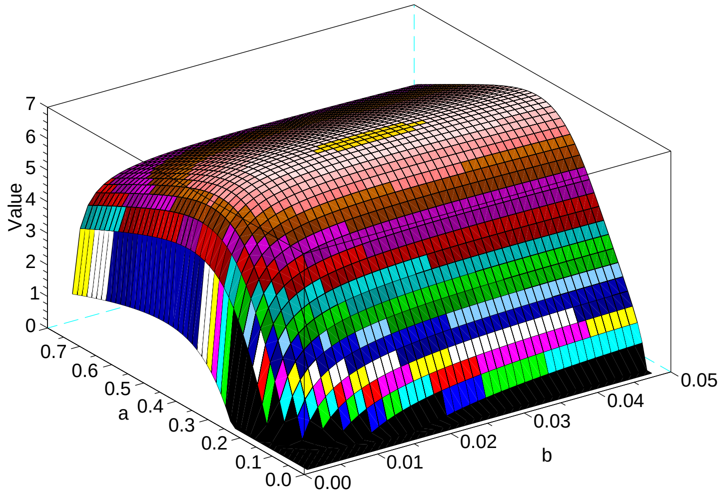

Figure 3.8: Value of the trailing stop and fixed stop rule for reallocating Strategy C.

Hence, when using the stops trading rules, by far the best strategy is to change the fund only on those occasions when the return process comes out at the lower stop.

Strategy D

The Bayesian strategy has similarities to Strategy B; the stochastic nature of the funds is identical, but this time *any* stopping rule is allowed. As was recorded at (3.7), the gain process in the observation filtration can be modeled as the solution of a stochastic differential equation, and the optimal stopping problem for this is found by solving the recursive scheme (2.10) by Crank-Nicolson. To compare with the results of Strategy B (only the case $\sigma_\mu = 0.3$), the prior distribution for μ to be chosen will have a mean $\mu_0 = 0.15$ as before, and precision $\tau_0 = {}^{\sigma^2}\!/_{\sigma_\mu^2} = 1$. Table 3.7 compares the results from Strategy B with the optimal solution obtained using Strategy D. Of course, no fixed values can be reported for the optimal stopping solution as the stopping boundary is a curve, which can be seen in Figure 3.9.
The shape of the stopping boundary can be interpreted as a time-dependent decreasing upper stop $\eta(t)$ and an increasing lower stop $\xi(t)$. The upper stop η begins at a high level to let *good* investments run and the lower stop ξ starts at a small negative value to immediately get rid of *bad* investments. As time goes by, the state of the gain process

	Best a	Best b	Objective	$\mathbf{E}[T]$
Fixed stops	0.0191	0.3409	1.4071	0.0804
Trailing stop and fixed stop	0.0985	0.6558	1.3431	0.1236
Trailing stop	0.0952		1.3354	0.1159
Fixed-revision			0.5596	0.0662
Optimal stopping			1.6770	

Table 3.7: Numerical results for reallocating strategies B and D.

updates the drift estimator (3.6). The threshold $\eta(t)$ will decrease with t; if it was advantageous to stop at time t when the gain value X_t was $y > 0$, then it is certainly advantageous to stop at level y at any later time, because the estimate of μ would then be even smaller by being more reliable. A corresponding argument applies to the lower stop $\xi(t)$. This kind of stopping boundary can not be captured by the stopping examples defined in Section 2.2; this reflects in their objectives which are way off the optimum. However, by a slight modification, the stopping examples can be improved a lot.

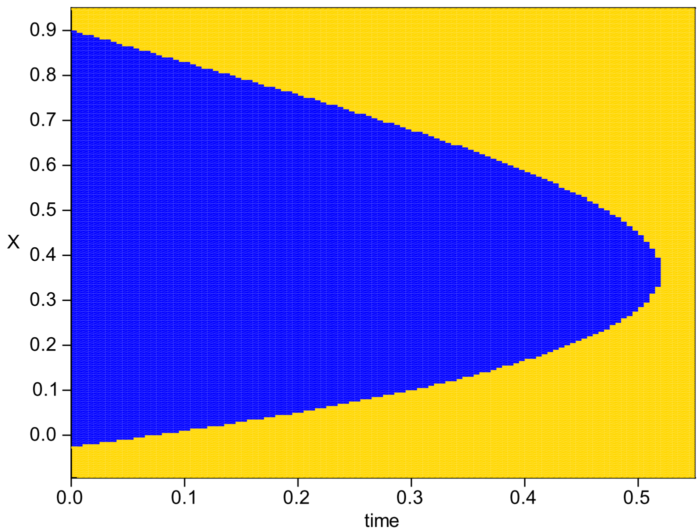

Figure 3.9: Stopping boundary of the Bayesian optimal stopping problem with uncertain drift parameter.

3.3 Time-dependent slope

As written in the previous section, the stopping examples of Section 2.2 have to be modified in order to enable the time-dependent structure seen from the optimal stopping problem. The best option would be to use a falling upper stop with a rising lower stop. For fixed stops, this is done in Section 3.5. Unfortunately, then the joint Laplace transform cannot be computed explicitly as this leads to a finite time horizon for each stop with solutions which can only be solved numerically. However, most of the benefit can be obtained by using just one slope; then the Laplace transform is still explicitly solvable and the objectives improve substantially.
Therefore, let q be the slope parameter, then the modified stopping examples are defined as below.

Example 1: fixed stops

For $a > 0$ and $b > 0$, the stopping time is defined by

$$T \equiv \inf\{t : X_t = -a + q\,t \text{ or } X_t = b + q\,t\}.$$

Example 2: trailing stop and fixed stop

Fix $a > 0$ and $b > 0$, and define $\hat{X}_t = \sup_{0\leq s\leq t}\{X_s - q\,s\}$. Then the stopping time is

$$T \equiv \inf\{t : X_t = \hat{X}_t - a + q\,t \text{ or } X_t = b + q\,t\}.$$

Example 3: trailing stop

As a special case of Example 2, with $a > 0$,

$$T \equiv \inf\{t : X_t = \hat{X}_t - a + q\,t\}.$$

Regarding a process $Y_t \equiv X_t - q\,t$, then for Y, the above stopping rules correspond to the time-independent ones defined in Section 2.2. Thus, for process X and the time-dependent stops, the joint Laplace transforms can be computed from the Laplace transforms of Section 2.5 with respect to process Y,

$$\begin{aligned} L_X(\rho,\gamma) &= \mathbf{E}[e^{-\rho T-\gamma X_T}] \\ &= \mathbf{E}[e^{-\rho T-\gamma(Y_T+qT)}] \\ &= \mathbf{E}[e^{-(\rho+\gamma q)T-\gamma Y_T}] \\ &= L_Y(\rho+\gamma q,\gamma). \end{aligned}$$

3.4 Numerical results

Having added an additional parameter q, the value φ can be optimized over a, b and q to obtain the objective. As $q = 0$ is a feasible choice, the objective cannot be smaller than those found in Section 2.6 and Section 3.2. For a better comparison, these values are given in the following tables in an extra column marked with $q = 0$.
As it was found that the fixed-revision rule underperforms the state-based stopping rules and the take-profit stop is beneficial, results are only given for Example 1 and Example 2. Furthermore, only $\sigma_\mu = 0.3$ will be considered. All other parameters are as in Section 3.2.
In case of certainty, it was shown that the optimal solution is a constant upper stop as seen from the stopping barrier of the optimal stopping problem in Figure 2.1. Hence, it is no surprise that the optimal q equals 0 and all results are as in Table 2.1.
The uncertain case is more interesting; the results are given for the corresponding reallocating strategies.

Strategy A

If the investor sticks with the same fund forever, the results given in Table 3.8 emerge.

	Best a	Best b	Best q	Objective	$\mathbf{E}[T]$	$q = 0$
Fixed stops	0.2244	0.0447	0.0347	0.8426	0.1005	0.8416
Trailing stop and fixed stop	0.2450	0.0441	0.0346	0.8407	0.0991	0.8398

Table 3.8: Numerical results for slope-stops and reallocating Strategy A.

The results are very close to those in Section 3.2 because the optimal parameter q is close to 0. In other words, allowing the stops to have a time-dependent slope does not yield a substantial improvement.

Strategy B

Choosing a different investment from the marketplace on each side yields Table 3.9.

	Best a	Best b	Best q	Objective	$\mathbf{E}[T]$	$q = 0$
Fixed stops	0.0345	0.2944	0.4949	1.5976	0.0782	1.4071
Trailing stop and fixed stop	0.1102	0.5445	0.2789	1.3519	0.1251	1.3431

Table 3.9: Numerical results for slope-stops and reallocating Strategy B.

This time, q is significantly larger than 0, leading to an improvement of the objective, which is much larger for Example 1 than for Example 2. As in Section 3.2, large b-values to let *good* investments run and small a-values to quickly stop *bad* investments are found.

Strategy C

If the investor picks an independent fund if the process ends up on the down side gives the results which are summarized in Table 3.10.

	Best a	Best b	Best q	Objective	$\mathbf{E}[T]$	$q = 0$
Fixed stops	0.4181	0.0192	0.1212	6.9719	0.0330	6.8525
Trailing stop and fixed stop	0.4274	0.0192	0.1209	6.9721	0.0330	6.8539

Table 3.10: Numerical results for slope-stops and reallocating Strategy C.

In this case there is an 1.7% improvement of the objective due to the slope q. As guessed above, the increasing lower stop tackles below-average investments. The slope parameter q is considerably larger than 0 but it is not as high as with Strategy B, which reflects the risk to accidentally stop a *good* investment.

Strategy D

In Section 3.2, it was found that the objective for the best fixed-stops rule is quite far from the optimum. However, adding a time-dependent drift yields the situation given in Table 3.11. This is a summary of the results of Table 3.9 and the optimal stopping rule of Table 3.7.

The result of the optimal stopping problem cannot be improved by the slope, so the same objective as in Section 3.2 is found. But the time-dependent slope pushes the fixed stop's objective up by 13.5%, bringing the value much closer to the optimum, remarkably so given the very simple-minded nature of the stopping rule.

	Best a	Best b	Best q	Objective	$\mathbf{E}[T]$	$q = 0$
Fixed stops	0.0345	0.2944	0.4949	1.5976	0.0782	1.4071
Trailing stop and fixed stop	0.1102	0.5445	0.2789	1.3519	0.1251	1.3431
Optimal stopping				1.6770		

Table 3.11: Numerical results for slope-stops and reallocating Strategies B and D.

As a conclusion of the results, it can be mentioned that the simple fixed-stops rule with a suitably-chosen slope is close to optimal. With Strategy C, the objectives of Example 1 and Example 2 nearly coincide while with Strategy B, the fixed-stops rule does substantially better. This observation might lead to the assumption that the fixed-stops rule should be preferred. However, the trailing stop guards against the possibility that the drift might deteriorate while the fund is hold. This effect is considered in Chapter 4.

3.5 Linear stops and vertical stop

As stated in Section 3.3, for Strategy B, the best option would be to use a falling upper stop with a rising lower stop. Hence, two different slope parameters q_L and q_H are needed. Additionally, a vertical stop $t_0 > 0$ can be considered. Then, for $a > 0$ and $b > 0$, the stopping time is defined by

$$T \equiv \inf\{t : X_t = -a + q_L t \text{ or } X_t = b + q_H t \text{ or } t = t_0\}. \tag{3.9}$$

If $q_H < q_L$, then the linear barriers will intersect at $t_{\max} = \frac{a+b}{q_L - q_H}$, thus, the vertical stop should be $t_0 \leq t_{\max}$. Although the joint Laplace transform cannot be computed explicitly, Hall [7] and Anderson [2] derived formulas for the densities of the stopped process along the boundaries. Hall used a drifted Brownian motion; by defining $Y_t \equiv \frac{X_t}{\sigma}$, the results can be translated for the arithmetic Brownian motion. If H, L and V denote the cases that the process stops at the upper, lower and vertical stop, respectively, the joint Laplace transform can then be computed as

$$\begin{aligned}
\mathbf{E}[e^{-\rho T-\gamma X_T}] &= \mathbf{E}[e^{-\rho T-\gamma X_T} : H] + \mathbf{E}[e^{-\rho T-\gamma X_T} : L] + \mathbf{E}[e^{-\rho T-\gamma X_T} : V] \\
&= \mathbf{E}[e^{-\rho T-\gamma(b+q_H T)} : H] + \mathbf{E}[e^{-\rho T-\gamma(-a+q_L t)} : L] + \mathbf{E}[e^{-\rho t_0-\gamma X_{t_0}} : V] \\
&= e^{-\gamma b}\,\mathbf{E}[e^{-(\rho+\gamma q_H)T} : H] + e^{\gamma a}\,\mathbf{E}[e^{-(\rho+\gamma q_L)T} : L] + e^{-\rho t_0}\,\mathbf{E}[e^{-\gamma X_{t_0}} : V] \\
&= e^{-\gamma b}\int_0^{t_0} e^{-(\rho+\gamma q_H)t} p_H(t)\mathrm{d}t + e^{\gamma a}\int_0^{t_0} e^{-(\rho+\gamma q_L)t} p_L(t)\mathrm{d}t \\
&\quad + e^{-\rho t_0}\int_{-a+q_L t_0}^{b+q_H t_0} e^{-\gamma x} p_V(x)\mathrm{d}x, \qquad (3.10)
\end{aligned}$$

where p_H and p_L are the corresponding densities of the stopping time on the linear stops and p_V is the density of the state on the vertical stop. With a slight modification of the formulas in [7], the densities are

$$p_H(t) = \frac{1}{\sqrt{2\pi\sigma^2}} t^{-\frac{3}{2}} e^{-b^2\frac{q_L-q_H}{2\sigma^2(a+b)}+b\frac{\mu-q_H}{\sigma^2}-\frac{(\mu-q_H)^2}{2\sigma^2}t} \sum_{j=0}^{\infty}(-1)^j\, r_j\, e^{\frac{1}{2\sigma^2}(\frac{q_L-q_H}{a+b}-\frac{1}{t})r_j^2},$$

for $t < t_0$ with

$$r_j = \begin{cases} j(a+b)+b & \text{if } j \text{ even} \\ j(a+b)+a & \text{if } j \text{ odd} \end{cases}$$

and

$$p_L(t) = \frac{1}{\sqrt{2\pi\sigma^2}} t^{-\frac{3}{2}} e^{-a^2\frac{q_L-q_H}{2\sigma^2(a+b)}+a\frac{q_L-\mu}{\sigma^2}-\frac{(q_L-\mu)^2}{2\sigma^2}t} \sum_{j=0}^{\infty}(-1)^j\, s_j\, e^{\frac{1}{2\sigma^2}(\frac{q_L-q_H}{a+b}-\frac{1}{t})s_j^2},$$

for $t < t_0$ with

$$s_j = \begin{cases} j(a+b)+a & \text{if } j \text{ even} \\ j(a+b)+b & \text{if } j \text{ odd} \end{cases}$$

and for $x \in (-a+q_L t_0, b+q_H t_0)$,

$$p_V(x) = \frac{1}{\sqrt{\sigma^2 t_0}} e^{\frac{1}{\sigma^2}(\mu-(q_H+q_L))x-\frac{1}{2\sigma^2}(\mu^2-(q_H+q_L)^2)t_0}\, \tilde{p}_V(x),$$

with

$$\begin{aligned}\tilde{p}_V(x) = \phi\left(\tfrac{h}{\sigma\sqrt{t_0}}\right) + \sum_{j=1}^{\infty} \Big\{ & e^{\frac{q_L-q_H}{\sigma^2} j(2j(a+b)-(b-a))}\, \phi\left(\tfrac{h-2j(a+b)}{\sigma\sqrt{t_0}}\right) \\ & - e^{\frac{q_L-q_H}{\sigma^2}(2j-1)(j(a+b)-b)}\, \phi\left(\tfrac{h+2(j(a+b)-b)}{\sigma\sqrt{t_0}}\right) \\ & + e^{\frac{q_L-q_H}{\sigma^2} j(2j(a+b)+(b-a))}\, \phi\left(\tfrac{h+2j(a+b)}{\sigma\sqrt{t_0}}\right) \\ & - e^{\frac{q_L-q_H}{\sigma^2}(2j-1)(j(a+b)-a)}\, \phi\left(\tfrac{h+2(j(a+b)-a)}{\sigma\sqrt{t_0}}\right) \Big\},\end{aligned}$$

where $h = x - \frac{q_H+q_L}{2} t_0$ and

$$\phi(z) = \tfrac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}$$

is the density of the standard-normal distribution.

3.6 Numerical results

The task is to determine if the generalization of using two different slope parameters is substantially better than using just one as in Section 3.4. Surely, the objective cannot be worse, because $q_H = q_L$ is a feasible choice. It needs to be figured out if it is worth to take on the additional numerical complexity which should only be justified if the objective gets close to the objective of the optimal stopping problem. Therefore, only Strategy B will be considered.

Strategy B

If $q_H \geq q_L$, then $t_{\max} = \infty$ and if t_0 is chosen to be $t_{\max}$, then the third integral of (3.10) vanishes and the first two integrals can be computed explicitly; but numerical results show that the best choice is to take $q_H = q_L$ which both coincide with the optimal slope parameter q found in Section 3.4. This observation does not come as a surprise as the stopping region of the optimal stopping problem has a rising lower barrier and a decreasing upper barrier. Hence, it is reasonable to concentrate on the case $q_H > q_L$; then it is $t_{\max} < \infty$ and the integrals in (3.10) have to be computed numerically. For a less costly computation, let $t_0 = t_{\max}$. Then, the third integral of (3.10) vanishes. The results for the linear stops are found in Table 3.12. This table also shows the results for the fixed slope-stops $q_H = q_L$ and the optimal stopping problem, which can be found in Table 3.11.
The results show that linear stops only get a little closer to the optimal stopping result. In Figure 3.10, the linear stops are overlapping the figure of the optimal stopping problem.

	Best a	Best b	Best q_L	Best q_H	Objective
Fixed (slope) stops	0.0345	0.2944	0.4949	0.4949	1.5976
Linear stops	0.0324	1.1400	0.4198	-1.4532	1.6305
Optimal stopping					1.6770

Table 3.12: Numerical results for fixed slope-stops, linear stops and reallocating Strategy B; and reallocating Strategy D.

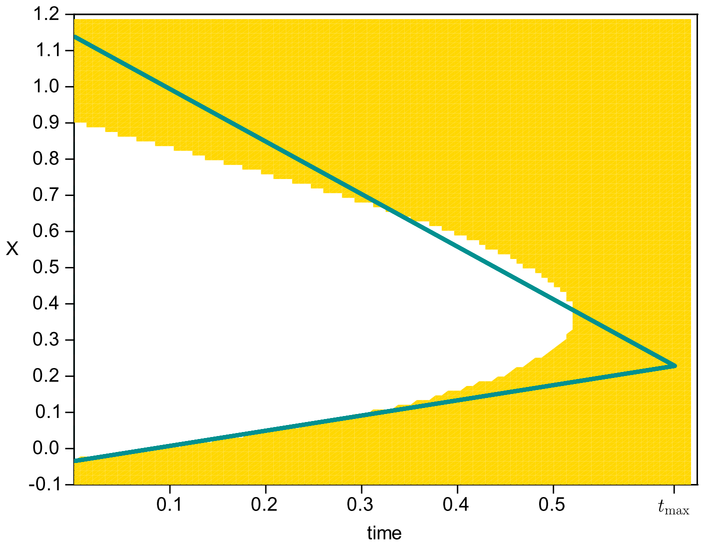

Figure 3.10: Linear stops and stopping boundary of the optimal stopping problem.

This time, the continuation region is white and the stopping region is yellow. The upper linear stop is a more or less a regression line of the curve $\xi(t)$ of the optimal stopping problem. The parameter q_L is not far from the parameter of the fixed slope-stops; and the lower stop is the more important one as the a values are small. So, the difference to the single q case is not that big; it is just an improvement of 2 %. However, the additional computational complexity is enormous such that for fast results the simple stopping examples of Section 3.3 should be preferred.

4 Switching drift

In the previous chapters, each fund had a constant drift which did not change during the holding period. As a consequence, the fixed-stops rule exceeded the fixed-and-trailing-stop rule in most cases. In a real world situation, it cannot be assumed that the drift is a constant. Therefore, the assumption of having a constant drift will be replaced by assuming to have change-points τ_l where the drift changes from μ_l to an independent value μ_{l+1}. The change-points are supposed to be the event times of a Poisson process with intensity λ. Then, $\tau_0 = 0$ and $\tau_l = \tau_{l-1} + s_l$ with s_l being independent and exponentially distributed random variables with intensity λ. A connection of the variables is displayed in Figure 4.1.

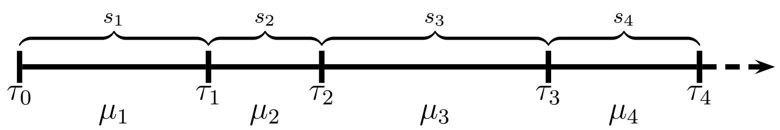

Figure 4.1: Time-line with some realizations of s_l and τ_l and μ_l.

As in the derivation of the analysis in the previous chapters, the drift parameters μ_l, $l \geq 1$ are assumed to be known for the time being; this unrealistic assumption will be relaxed afterwards. In order to keep the renewal property of Proposition 1, the story is as following: the investor will sell the fund at the stopping time and reinvests in a fund starting with drift μ_1 again. This story does not correspond to a realistic setting as the second fund's drift is assumed to have not changed. However, it helps to understand why the trailing stop might be advantageous in some cases. Furthermore, the analytical results found here are used to determine the values in a more realistic case; when the drift is uncertain and the investor uses reallocating Strategy B, where the drift of the new investment is an independent random variable. With (2.2) still valid, the value of the trading activity can be computed by the joint Laplace transforms as in (2.4).

4.1 Analysis of the examples

The Laplace transforms can be split according to the events $T \in (\tau_{l-1}, \tau_l)$, giving

$$L(\rho,\gamma) = \sum_{l=1}^{\infty} \mathbf{E}[e^{-\rho T - \gamma X_T} : T \in (\tau_{l-1}, \tau_l)]. \tag{4.1}$$

For some fixed l, the partial expectation in (4.1) only depends on the drift values $\mu_1, \ldots, \mu_l$. In order to find the joint Laplace transform, the task will be to determine this partial expectation for the three stopping examples of Section 2.2. For numerical results, the infinite sum has to be approximated in an appropriate way.

4.1.1 Example 1: fixed stops

For a compact appearance of the result, it is given first with the derivation following.

Proposition 3 *For fixed stops, the partial expectations of* (4.1) *are*

$$\mathbf{E}[e^{-\rho T - \gamma X_T} : T \in (\tau_{l-1}, \tau_l)] = \left(\tfrac{2\lambda}{\sigma^2}\right)^{l-1} \sum_{i_1=1}^{2} c_{i_1}^{l}(\gamma) \sum_{\substack{i_m=1\\ 2\le m\le l}}^{3} \prod_{j=2}^{l} \hat{c}_{i_j}^{l-(j-1)}\left(\delta_{i_{(j-1)}}^{l-(j-2)}\right),$$

where

$$\alpha_j = \tfrac{1}{\sigma^2}(\mu_j + \sqrt{\mu_j^2 + 2\sigma^2(\rho+\lambda)}) \tag{4.2}$$

$$\beta_j = \tfrac{1}{\sigma^2}(-\mu_j + \sqrt{\mu_j^2 + 2\sigma^2(\rho+\lambda)}) \tag{4.3}$$

and

$$\delta_1^j = -\alpha_j \tag{4.4}$$

$$\delta_2^j = \beta_j \tag{4.5}$$

$$\delta_3^j = \delta_{i_{(l-j)}}^{j+1} \tag{4.6}$$

and

$$\hat{c}_i^j(\delta) = \frac{c_i^j(-\delta)}{(\delta+\alpha_j)(\delta-\beta_j)} \tag{4.7}$$

with

$$c_1^j(x) = \frac{e^{xa+\beta_j b} - e^{-xb-\beta_j a}}{e^{\alpha_j a+\beta_j b} - e^{-\alpha_j b-\beta_j a}} \tag{4.8}$$

$$c_2^j(x) = \frac{e^{-xb+\alpha_j a} - e^{xa-\alpha_j b}}{e^{\alpha_j a+\beta_j b} - e^{-\alpha_j b-\beta_j a}} \tag{4.9}$$

$$c_3^j(x) = -1. \tag{4.10}$$

Proof: The idea of this derivation is to go backwards in time, assuming to know all the information up to the preceding change-point, which will lead to a recursive equation. Therefore, let $\mathcal{F}_{\tau_k} \equiv \sigma(\tau_1, \dots, \tau_k, (X_t)_{0 \le t \le \tau_k})$ be the filtration that contains all information about the change-points $\tau_1, \dots, \tau_k$ as well as the process X up to τ_k. The first step is to use the tower property with respect to the filtration $\mathcal{F}_{\tau_{l-1}}$. Let $T' \equiv T - \tau_{l-1}$, then the restriction $T \in (\tau_{l-1}, \tau_l)$ can be partitioned into an $\mathcal{F}_{\tau_{l-1}}$-measurable part $T > \tau_{l-1}$ and $T < \tau_l$ which is equivalent to $T' < s_l$. As displayed in Figure 4.2, T' can be considered to be the fixed-stops stopping time $T' = \inf\{t : Y_t = b \text{ or } Y_t = -a\}$ of the shifted process $Y_t \equiv X_{\tau_{l-1}+t}$, for $t \le s_l$, which starts at $X_{\tau_{l-1}}$ and stops at $Y_{T'} = X_T$. As Y represents the process X in the interval (τ_{l-1}, τ_l), it possesses the constant drift μ_l.

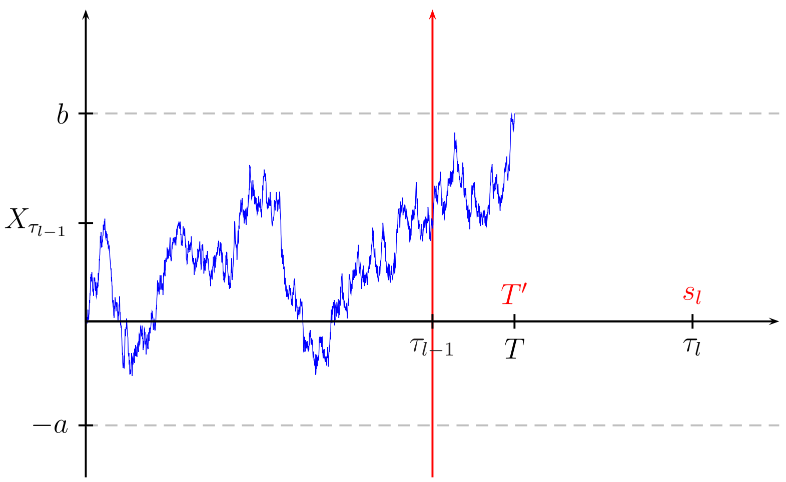

Figure 4.2: Realization of a gain process X with stopping time $T \in (\tau_{l-1}, \tau_l)$ and the shifted process Y with stopping time $T' < s_l$.

The expectation can therefore be written as

$$\mathbf{E}[e^{-\rho T - \gamma X_T} : T \in (\tau_{l-1}, \tau_l)] = \mathbf{E}[e^{-\rho \tau_{l-1}} \mathbf{E}_{\mu_l}^{X_{\tau_{l-1}}} [e^{-\rho T' - \gamma Y_{T'}} : T' < s_l | \mathcal{F}_{\tau_{l-1}}] : T > \tau_{l-1}], \tag{4.11}$$

where $\mathbf{E}^x_\mu$ is the notation indicating starting value and drift. Due to the Markov property, the inner expectation of (4.11) only depends on the starting value $X_{\tau_{l-1}}$. Let $\mathcal{G}_t \equiv \sigma((X_t)_{0\leq s\leq t})$ be the smaller filtration that contains all information about the process X, but no information about the change-points. Then, this expectation can be determined by using the tower property with respect to the filtration $\mathcal{G}_{T'}$; and that s_l is exponentially distributed with parameter λ and independent from the process X,

$$\begin{aligned}\mathbf{E}^{X_{\tau_{l-1}}}_{\mu_l}[e^{-\rho T'-\gamma Y_{T'}} : T' < s_l|\mathcal{F}_{\tau_{l-1}}] &= \mathbf{E}^{X_{\tau_{l-1}}}_{\mu_l}[e^{-\rho T'-\gamma Y_{T'}}\,\mathbf{E}[\mathbf{1}_{\{T'<s_l\}}|\mathcal{G}_{T'}]|X_{\tau_{l-1}}]\\ &= \mathbf{E}^{X_{\tau_{l-1}}}_{\mu_l}[e^{-\rho T'-\gamma Y_{T'}}\,\mathbf{P}(T' < s_l|T')|X_{\tau_{l-1}}]\\ &= \mathbf{E}^{X_{\tau_{l-1}}}_{\mu_l}[e^{-(\rho+\lambda)T'-\gamma Y_{T'}}|X_{\tau_{l-1}}].\end{aligned}$$

This is the joint Laplace transform for process Y, starting at $X_{\tau_{l-1}}$ with constant drift μ_l, and fixed-stops stopping time T'. Its solution can therefore be obtained from (2.17) with boundary conditions $f(-a) = \exp(\gamma a)$ and $f(b) = \exp(-\gamma b)$. The only differences are that the drift is μ_l and the rate of discounting is $\rho + \lambda$ such that the roots of the polynomial equation are $-\alpha_l$ and β_l with definitions in (4.2) and (4.3), respectively. Hence,

$$\begin{aligned}\mathbf{E}^{X_{\tau_{l-1}}}[e^{-(\rho+\lambda)T'-\gamma Y_{T'}}|X_{\tau_{l-1}}] &= \frac{(e^{\gamma a+\beta_l b} - e^{-\gamma b-\beta_l a})e^{-\alpha_l X_{\tau_{l-1}}} + (e^{-\gamma b+\alpha_l a} - e^{\gamma a-\alpha_l b})e^{\beta_l X_{\tau_{l-1}}}}{e^{\alpha_l a+\beta_l b} - e^{-\alpha_l b-\beta_l a}}\\ &= c^l_1(\gamma)e^{\delta^l_1 X_{\tau_{l-1}}} + c^l_2(\gamma)e^{\delta^l_2 X_{\tau_{l-1}}}\\ &= \sum_{i_1=1}^{2} c^l_{i_1}(\gamma)e^{\delta^l_{i_1} X_{\tau_{l-1}}},\end{aligned}$$

where the notations (4.4), (4.5), (4.8) and (4.9) have been used for brevity. Successive reinserting gives

$$\mathbf{E}[e^{-\rho T-\gamma X_T} : T \in (\tau_{l-1}, \tau_l)] = \sum_{i_1=1}^{2} c^l_{i_1}(\gamma)\,\mathbf{E}[e^{-\rho\tau_{l-1}+\delta^l_{i_1} X_{\tau_{l-1}}} : T > \tau_{l-1}]. \tag{4.12}$$

The last expectation only depends on the drifts $\mu_1, \ldots, \mu_{l-1}$. Now, the tower property with respect to the filtration $\mathcal{F}_{\tau_{l-2}}$ can be used. This time, let $T' \equiv T - \tau_{l-2}$, then the restriction $T > \tau_{l-1}$ can be split into an $\mathcal{F}_{\tau_{l-2}}$-measurable part $T > \tau_{l-2}$ and $T' > s_{l-1}$. Defining the shifted and rebased process $Y_t \equiv X_{\tau_{l-2}+t} - X_{\tau_{l-2}}$, for $t \leq s_{l-1}$, then T' can be seen to be the fixed-stops stopping time $T' = \inf\{t : Y_t = b' \equiv b - X_{\tau_{l-2}}$ or $Y_t = -a' \equiv -(a + X_{\tau_{l-2}})\}$. The process Y starts at 0 and as it represents the process X in the interval (τ_{l-2}, τ_{l-1}), the drift is constantly μ_{l-1}. The connection of the two processes is displayed in Figure 4.3.

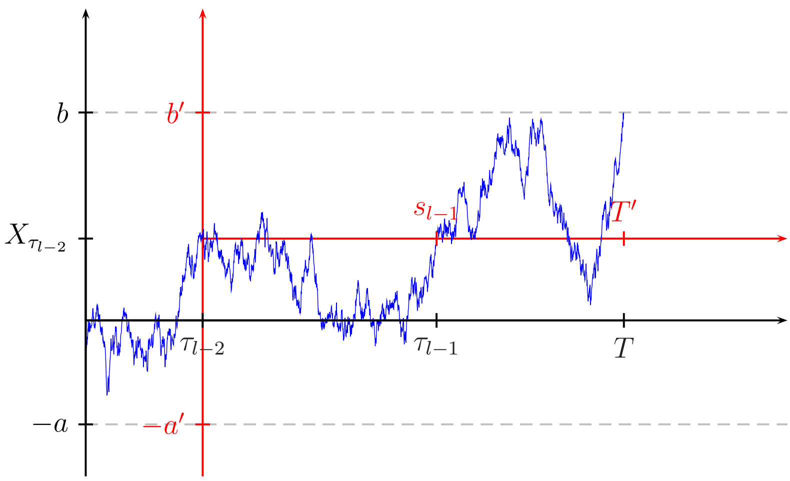

Figure 4.3: Realization of a gain process X with stopping time $T > \tau_{l-1}$ and the process Y with stopping time $T' > s_{l-1}$.

Since $\tau_{l-1} = \tau_{l-2} + s_{l-1}$ by definition, it holds $X_{\tau_{l-1}} = Y_{s_{l-1}} + X_{\tau_{l-2}}$. Due to the Markov property, Y is independent from the process X up to time τ_{l-2}. Only the value $X_{\tau_{l-2}}$ has to be remembered because the barriers of the stopping time T' depend on it. Thus, the expectation in (4.12) yields

$$\begin{aligned}&\mathbf{E}[e^{-\rho\tau_{l-1}+\delta^l_{i_1}X_{\tau_{l-1}}} : T > \tau_{l-1}] \\ &\quad = \mathbf{E}[e^{-\rho\tau_{l-2}+\delta^l_{i_1}X_{\tau_{l-2}}}\mathbf{E}_{\mu_{l-1}}[e^{-\rho s_{l-1}+\delta^l_{i_1}Y_{s_{l-1}}} : T' > s_{l-1}|X_{\tau_{l-2}}] : T > \tau_{l-2}]. \qquad (4.13)\end{aligned}$$

The inner expectation of (4.13) is given by the following Lemma.

Lemma 1 *Let τ be an exponentially distributed random variable with intensity λ and let T be the fixed-stops stopping time $T = \inf\{t : X_t = b \text{ or } X_t = -a\}$ of a process $X_t = \mu_k t + \sigma W_t$, then it holds*

$$\mathbf{E}[e^{-\rho\tau+\delta X_\tau} : T > \tau] = \tfrac{2\lambda}{\sigma^2}\tfrac{L_1^{\mu_k}(\rho+\lambda,-\delta)-1}{(\delta+\alpha_k)(\delta-\beta_k)},$$

where $L_1^{\mu_k}$ is the joint Laplace transform for fixed stops from (2.19) *with respect to drift μ_k.*

The proof is given in Appendix B.1.

Thus,

$$\mathbf{E}_{\mu_{l-1}}[e^{-\rho s_{l-1}+\delta^l_{i_1}Y_{s_{l-1}}} : T' > s_{l-1}|X_{\tau_{l-2}}] = \frac{2\lambda}{\sigma^2}\frac{L_1^{\mu_{l-1}}(\rho+\lambda,-\delta^l_{i_1}|X_{\tau_{l-2}})-1}{(\delta^l_{i_1}+\alpha_{l-1})(\delta^l_{i_1}-\beta_{l-1})},$$

where the joint Laplace transform is with respect to the stopping time T' with $X_{\tau_{l-2}}$ known. Its solution is obtained from (2.19),

$$\begin{aligned}
&L_1^{\mu_{l-1}}(\rho+\lambda,-\delta^l_{i_1}|X_{\tau_{l-2}})\\
&\quad=\frac{e^{-\delta^l_{i_1}a'}(e^{\beta_{l-1}b'}-e^{-\alpha_{l-1}b'})+e^{\delta^l_{i_1}b'}(e^{\alpha_{l-1}a'}-e^{-\beta_{l-1}a'})}{e^{\alpha_{l-1}a'+\beta_{l-1}b'}-e^{-\alpha_{l-1}b'-\beta_{l-1}a'}}\\
&\quad=\frac{e^{(-\alpha_{l-1}-\delta^l_{i_1})X_{\tau_{l-2}}}(e^{-\delta^l_{i_1}a+\beta_{l-1}b}-e^{\delta^l_{i_1}b-\beta_{l-1}a})+e^{(\beta_{l-1}-\delta^l_{i_1})X_{\tau_{l-2}}}(e^{\delta^l_{i_1}b+\alpha_{l-1}a}-e^{-\delta^l_{i_1}a-\alpha_{l-1}b})}{e^{\alpha_{l-1}a+\beta_{l-1}b}-e^{-\alpha_{l-1}b-\beta_{l-1}a}}\\
&\quad=c_1^{l-1}(-\delta^l_{i_1})\,e^{(\delta_1^{l-1}-\delta^l_{i_1})X_{\tau_{l-2}}}+c_2^{l-1}(-\delta^l_{i_1})\,e^{(\delta_2^{l-1}-\delta^l_{i_1})X_{\tau_{l-2}}},
\end{aligned}$$

where the notations (4.4), (4.5), (4.8) and (4.9) have been used for the last equality. Hence, continuing the calculation of the expectation, it is

$$\begin{aligned}
&\mathbf{E}_{\mu_{l-1}}[e^{-\rho s_{l-1}+\delta^l_{i_1}Y_{s_{l-1}}} : T' > s_{l-1}|X_{\tau_{l-2}}]\\
&\quad=\frac{2\lambda}{\sigma^2}\frac{c_1^{l-1}(-\delta^l_{i_1})\,e^{(\delta_1^{l-1}-\delta^l_{i_1})X_{\tau_{l-2}}}+c_2^{l-1}(-\delta^l_{i_1})\,e^{(\delta_2^{l-1}-\delta^l_{i_1})X_{\tau_{l-2}}}-1}{(\delta^l_{i_1}+\alpha_{l-1})(\delta^l_{i_1}-\beta_{l-1})}\\
&\quad=\frac{2\lambda}{\sigma^2}\Bigg(\frac{c_1^{l-1}(-\delta^l_{i_1})}{(\delta^l_{i_1}+\alpha_{l-1})(\delta^l_{i_1}-\beta_{l-1})}\,e^{(\delta_1^{l-1}-\delta^l_{i_1})X_{\tau_{l-2}}}+\frac{c_2^{l-1}(-\delta^l_{i_1})}{(\delta^l_{i_1}+\alpha_{l-1})(\delta^l_{i_1}-\beta_{l-1})}\,e^{(\delta_2^{l-1}-\delta^l_{i_1})X_{\tau_{l-2}}}\\
&\qquad+\frac{c_3^{l-1}(-\delta^l_{i_1})}{(\delta^l_{i_1}+\alpha_{l-1})(\delta^l_{i_1}-\beta_{l-1})}\,e^{(\delta_3^{l-1}-\delta^l_{i_1})X_{\tau_{l-2}}}\Bigg)\\
&\quad=\frac{2\lambda}{\sigma^2}\sum_{i_2=1}^{3}\hat{c}_{i_2}^{l-1}(\delta^l_{i_1})\,e^{(\delta_{i_2}^{l-1}-\delta^l_{i_1})X_{\tau_{l-2}}},
\end{aligned}$$

where δ_3^{l-1} is chosen as in (4.6) such that the third exponent cancels out. Additionally, the definitions (4.10) and (4.7) have been used. Plugging in this result into (4.13) then gives the recursive equation

$$\begin{aligned}
\mathbf{E}[e^{-\rho\tau_{l-1}+\delta^l_{i_1}X_{\tau_{l-1}}} : T > \tau_{l-1}] &= \frac{2\lambda}{\sigma^2}\sum_{i_2=1}^{3}\hat{c}_{i_2}^{l-1}(\delta^l_{i_1})\,\mathbf{E}[e^{-\rho\tau_{l-2}+\delta^l_{i_1}X_{\tau_{l-2}}}\,e^{(\delta_{i_2}^{l-1}-\delta^l_{i_1})X_{\tau_{l-2}}} : T > \tau_{l-2}]\\
&= \frac{2\lambda}{\sigma^2}\sum_{i_2=1}^{3}\hat{c}_{i_2}^{l-1}(\delta^l_{i_1})\,\mathbf{E}[e^{-\rho\tau_{l-2}+\delta_{i_2}^{l-1}X_{\tau_{l-2}}} : T > \tau_{l-2}]. \qquad (4.14)
\end{aligned}$$

This result can be substituted into (4.12). Furthermore, equation (4.14) can be applied recursively to obtain the desired result,

$$
\begin{aligned}
&\mathbf{E}[e^{-\rho T-\gamma X_T} : T \in (\tau_{l-1}, \tau_l)] \\
&= \tfrac{2\lambda}{\sigma^2} \sum_{i_1=1}^{2} c_{i_1}^l(\gamma) \sum_{i_2=1}^{3} \hat{c}_{i_2}^{l-1}(\delta_{i_1}^l)\, \mathbf{E}[e^{-\rho\tau_{l-2}+\delta_{i_2}^{l-1} X_{\tau_{l-2}}} : T > \tau_{l-2}] \\
&\cdots \\
&= \left(\tfrac{2\lambda}{\sigma^2}\right)^{l-1} \sum_{i_1=1}^{2} c_{i_1}^l(\gamma) \sum_{i_2=1}^{3} \hat{c}_{i_2}^{l-1}(\delta_{i_1}^l) \cdots \sum_{i_l=1}^{3} \hat{c}_{i_l}^1(\delta_{i_{l-1}}^2) \\
&= \left(\tfrac{2\lambda}{\sigma^2}\right)^{l-1} \sum_{i_1=1}^{2} c_{i_1}^l(\gamma) \sum_{i_2=1}^{3} \cdots \sum_{i_l=1}^{3} \hat{c}_{i_2}^{l-1}(\delta_{i_1}^l) \cdots \hat{c}_{i_l}^1(\delta_{i_{l-1}}^2) \\
&= \left(\tfrac{2\lambda}{\sigma^2}\right)^{l-1} \sum_{i_1=1}^{2} c_{i_1}^l(\gamma) \sum_{\substack{i_m=1 \\ 2\le m\le l}}^{3} \prod_{j=2}^{l} \hat{c}_{i_j}^{l-(j-1)} \left(\delta_{i_{(j-1)}}^{l-(j-2)}\right).
\end{aligned}
$$

□

4.1.2 Example 2: trailing stop and fixed stop

Proposition 4 *For the trailing stop and fixed stop, the partial expectations of* (4.1) *are*

$$
\mathbf{E}[e^{-\rho T-\gamma X_T} : T \in (\tau_{l-1}, \tau_l)] = \left(\tfrac{2\lambda}{\sigma^2}\right)^{l-1} \sum_{i_1=1}^{4} c_{i_1}^l(\gamma,\gamma) \sum_{\substack{i_m=1 \\ 2\le m\le l}}^{5} \prod_{j=2}^{l} \hat{c}_{i_j}^{l-(j-1)} \left(\delta_{i_{(j-1)}}^{l-(j-2)}, \gamma_{i_{(j-1)}}^{l-(j-2)}\right),
$$

with

$$\alpha_j = \tfrac{1}{\sigma^2}(\mu_j + \sqrt{\mu_j^2 + 2\sigma^2(\rho+\lambda)}) \tag{4.15}$$

$$\beta_j = \tfrac{1}{\sigma^2}(-\mu_j + \sqrt{\mu_j^2 + 2\sigma^2(\rho+\lambda)}) \tag{4.16}$$

$$\nu_j = \tfrac{\beta_j e^{\alpha_j a} + \alpha_j e^{-\beta_j a}}{e^{\alpha_j a} - e^{-\beta_j a}} \tag{4.17}$$

$$n_j(B) = \tfrac{\alpha_j+\beta_j}{e^{\alpha_j a} - e^{-\beta_j a}} \tag{4.18}$$

$$r_j(x) = -\nu_j - x + n_j(B)\, e^{xa} \tag{4.19}$$

$$\delta_1^j = \delta_3^j = -\alpha_j \tag{4.20}$$
$$\delta_2^j = \delta_4^j = \beta_j \tag{4.21}$$
$$\delta_5^j = \delta_{i_{(l-j)}}^{j+1} \tag{4.22}$$

$$\gamma_1^j = \gamma_2^j = -\nu_j \tag{4.23}$$
$$\gamma_3^j = \gamma_4^j = \begin{cases} \gamma_{i_{(l-j)}}^{j+1} & for\ j < l \\ \gamma & for\ j = l \end{cases} \tag{4.24}$$
$$\gamma_5^j = \gamma_{i_{(l-j)}}^{j+1} \tag{4.25}$$

and

$$\hat{c}_i^j(\delta, \gamma) = \frac{c_i^j(-\delta, \gamma)}{(\delta + \alpha_j)(\delta - \beta_j)} \tag{4.26}$$

with

$$c_1^j(x, y) = \tfrac{1}{e^{\alpha_j a} - e^{-\beta_j a}}\, e^{-(\nu_j + y)b}\, \tfrac{r_j(x)}{\nu_j + y}\, e^{-\beta_j a} \tag{4.27}$$
$$c_2^j(x, y) = -\tfrac{1}{e^{\alpha_j a} - e^{-\beta_j a}}\, e^{-(\nu_j + y)b}\, \tfrac{r_j(x)}{\nu_j + y}\, e^{\alpha_j a} \tag{4.28}$$
$$c_3^j(x, y) = -\tfrac{1}{e^{\alpha_j a} - e^{-\beta_j a}} \left(\left(\tfrac{r_j(x)}{\nu_j + y} + 1 \right) e^{-\beta_j a} - e^{xa} \right) \tag{4.29}$$
$$c_4^j(x, y) = \tfrac{1}{e^{\alpha_j a} - e^{-\beta_j a}} \left(\left(\tfrac{r_j(x)}{\nu_j + y} + 1 \right) e^{\alpha_j a} - e^{xa} \right) \tag{4.30}$$
$$c_5^j(x, y) = -1. \tag{4.31}$$

Proof: Similarly to the proof of Proposition 3, the tower property is used recursively at the change-points. Hence, the first step is to use the tower property with respect to the filtration $\mathcal{F}_{\tau_{l-1}}$. Let $S \equiv T - \tau_{l-1}$, then the restriction $T \in (\tau_{l-1}, \tau_l)$ can be split into $T > \tau_{l-1}$ and $S < s_l$. Define the shifted process $Y_t \equiv X_{\tau_{l-1}+t}$, for $t \leq s_l$, with constant drift μ_l. Then it holds $Y_0 = X_{\tau_{l-1}}$ and $Y_S = X_T$. Thus,

$$\mathbf{E}[e^{-\rho T - \gamma X_T} : T \in (\tau_{l-1}, \tau_l)] = \mathbf{E}[e^{-\rho \tau_{l-1}}\, \mathbf{E}_{\mu_l}^{X_{\tau_{l-1}}}[e^{-\rho S - \gamma Y_S} : S < s_l | \mathcal{F}_{\tau_{l-1}}] : T > \tau_{l-1}]. \tag{4.32}$$

Analogously to the proof of Proposition 3, $S < s_l$ can be dropped by changing the rate of discounting. Due to the Markov property, the process Y is independent from $\mathcal{F}_{\tau_{l-1}}$, except for the starting value $X_{\tau_{l-1}}$. As seen in Figure 4.4, the difference here is that stopping time S is not of Example 2 because the lower stop is a fixed stop until the process exceeds $\bar{X}_{\tau_{l-1}}$. Therefore, also the local maximum $\bar{X}_{\tau_{l-1}}$ has to be remembered. Consequently, the inner expectation of (4.32) yields

$$\mathbf{E}_{\mu_l}^{X_{\tau_{l-1}}}[e^{-\rho S-\gamma Y_S} : S < s_l|\mathcal{F}_{\tau_{l-1}}] = \mathbf{E}_{\mu_l}^{X_{\tau_{l-1}}}[e^{-(\rho+\lambda)S-\gamma Y_S}|X_{\tau_{l-1}}, \bar{X}_{\tau_{l-1}}]. \tag{4.33}$$

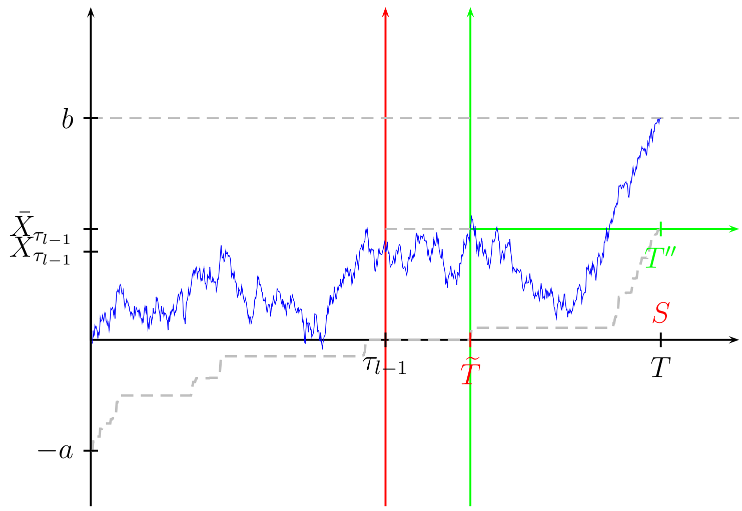

Figure 4.4: Realization of a gain process X with stopping time T, the shifted process Y with fixed-stops stopping time $\widetilde{T}$ and the rebased process Z with stopping time T''.

Since the lower stop of the stopping time S is constant, it is useful to consider the fixed-stops stopping time $\widetilde{T} \equiv \inf\{t : Y_t = \tilde{b} \equiv \bar{X}_{\tau_{l-1}}$ or $Y_t = -\tilde{a} \equiv -a + \bar{X}_{\tau_{l-1}}\}$, because if the process reaches the high side, the remaining time is a stopping time of Example 2. See Figure 4.4 for visualization. Thus, expectation (4.33) will be split according to the two outcomes,

$$\begin{aligned}\mathbf{E}_{\mu_l}^{X_{\tau_{l-1}}}[e^{-(\rho+\lambda)S-\gamma Y_S}|X_{\tau_{l-1}}, \bar{X}_{\tau_{l-1}}] &= \mathbf{E}_{\mu_l}^{X_{\tau_{l-1}}}[e^{-(\rho+\lambda)S-\gamma Y_S} : H|X_{\tau_{l-1}}, \bar{X}_{\tau_{l-1}}] \\ &+ \mathbf{E}_{\mu_l}^{X_{\tau_{l-1}}}[e^{-(\rho+\lambda)S-\gamma Y_S} : H^c|X_{\tau_{l-1}}, \bar{X}_{\tau_{l-1}}]. \end{aligned}\tag{4.34}$$

If the process reaches the down side, then $S = \widetilde{T}$ and $Y_S = Y_{\widetilde{T}}$. The resulting expectation is the partial joint Laplace transform of Y hitting the lower stop $-\tilde{a}$ before the upper stop $\tilde{b}$, which is given in (2.17) with boundary conditions $f(-\tilde{a}) = \exp(\gamma\tilde{a})$ and $f(\tilde{b}) = 0$, initial value $X_{\tau_{l-1}}$ and drift μ_l,

$$\begin{aligned}
&\mathbf{E}_{\mu_l}^{X_{\tau_{l-1}}}[e^{-(\rho+\lambda)S-\gamma Y_S} : H^c | X_{\tau_{l-1}}, \bar{X}_{\tau_{l-1}}] \\
&\quad = \mathbf{E}_{\mu_l}^{X_{\tau_{l-1}}}[e^{-(\rho+\lambda)\widetilde{T}-\gamma Y_{\widetilde{T}}} : H^c | X_{\tau_{l-1}}, \bar{X}_{\tau_{l-1}}] \\
&\quad = \frac{e^{\gamma\tilde{a}}(e^{\beta_l\tilde{b}-\alpha_l X_{\tau_{l-1}}} - e^{-\alpha_l\tilde{b}-\beta_l X_{\tau_{l-1}}})}{e^{\alpha_l\tilde{a}+\beta_l\tilde{b}} - e^{-\alpha_l\tilde{b}-\beta_l\tilde{a}}} \\
&\quad = \tfrac{e^{\gamma a}}{e^{\alpha_l a} - e^{-\beta_l a}}(e^{-\alpha_l X_{\tau_{l-1}} - (-\alpha_l+\gamma)\bar{X}_{\tau_{l-1}}} - e^{-\beta_l X_{\tau_{l-1}} - (\beta_l+\gamma)\bar{X}_{\tau_{l-1}}}),
\end{aligned}$$

with α_l and β_l as in (4.15) and (4.16).

If the process reaches the high side, the remaining time $T'' \equiv S - \widetilde{T}$ is the trailing-and-fixed-stops stopping time $T'' = \inf\{t : Z_t = b'' \equiv b - \bar{X}_{\tau_{l-1}} \text{ or } Z_t = -a + \bar{Z}_t\}$ of the rebased process $Z_t \equiv Y_{\widetilde{T}+t} - Y_{\widetilde{T}} = Y_{\widetilde{T}+t} - \bar{X}_{\tau_{l-1}}$, which starts at 0. This connection can also be seen in Figure 4.4. The tower property with respect to $\mathcal{F}_{\widetilde{T}}$ can be applied; Z is independent of the filtration and the T'' only depends on the value $\bar{X}_{\tau_{l-1}}$. Furthermore, by definition of Z it is $Y_S = Z_{T''} + \bar{X}_{\tau_{l-1}}$ thus,

$$\begin{aligned}
&\mathbf{E}_{\mu_l}^{X_{\tau_{l-1}}}[e^{-(\rho+\lambda)S-\gamma Y_S} : H | X_{\tau_{l-1}}, \bar{X}_{\tau_{l-1}}] \\
&\quad = e^{-\gamma\bar{X}_{\tau_{l-1}}}\,\mathbf{E}_{\mu_l}^{X_{\tau_{l-1}}}[e^{-(\rho+\lambda)\widetilde{T}}\,\mathbf{E}_{\mu_l}[e^{-(\rho+\lambda)T''-\gamma Z_{T''}} | \bar{X}_{\tau_{l-1}}] : H | X_{\tau_{l-1}}, \bar{X}_{\tau_{l-1}}]. \qquad (4.35)
\end{aligned}$$

The inner expectation of (4.35) is the joint Laplace transform of Z hitting the trailing-stop-and-fixed-stop stopping time T'', which can be read off (2.34),

$$\begin{aligned}
\mathbf{E}_{\mu_l}[e^{-(\rho+\lambda)T''-\gamma Z_{T''}} | \bar{X}_{\tau_{l-1}}] &= e^{-(\nu_l+\gamma)b''}\left(1 - \tfrac{n_l(B)e^{\gamma a}}{\nu_l+\gamma}\right) + \tfrac{n_l(B)e^{\gamma a}}{\nu_l+\gamma} \\
&= e^{-(\nu_l+\gamma)b}\left(1 - \tfrac{n_l(B)e^{\gamma a}}{\nu_l+\gamma}\right)e^{(\nu_l+\gamma)\bar{X}_{\tau_{l-1}}} + \tfrac{n_l(B)e^{\gamma a}}{\nu_l+\gamma}, \qquad (4.36)
\end{aligned}$$

with the notation corresponding to (4.15) - (4.18). Plugging in (4.36) into (4.35) yields

$$\begin{aligned}
&\mathbf{E}_{\mu_l}^{X_{\tau_{l-1}}}[e^{-(\rho+\lambda)S-\gamma Y_S} : H | X_{\tau_{l-1}}, \bar{X}_{\tau_{l-1}}] \\
&\quad = e^{-(\nu_l+\gamma)b}\left(1 - \tfrac{n_l(B)e^{\gamma a}}{\nu_l+\gamma}\right)\mathbf{E}_{\mu_l}^{X_{\tau_{l-1}}}[e^{-(\rho+\lambda)\widetilde{T}} : H | X_{\tau_{l-1}}, \bar{X}_{\tau_{l-1}}]\,e^{\nu_l\bar{X}_{\tau_{l-1}}} \\
&\qquad + \tfrac{n_l(B)e^{\gamma a}}{\nu_l+\gamma}\,\mathbf{E}_{\mu_l}^{X_{\tau_{l-1}}}[e^{-(\rho+\lambda)\widetilde{T}} : H | X_{\tau_{l-1}}, \bar{X}_{\tau_{l-1}}]\,e^{-\gamma\bar{X}_{\tau_{l-1}}}. \qquad (4.37)
\end{aligned}$$

The remaining expectation of (4.37) is the partial Laplace transform of a fixed-stops stopping time $\widetilde{T}$, which is given in (2.17) with boundary conditions $f(-\tilde{a}) = 0$ and $f(\tilde{b}) = 1$, with starting value $X_{\tau_{l-1}}$, such that

$$\begin{aligned}\mathbf{E}_{\mu_l}^{X_{\tau_{l-1}}}[e^{-(\rho+\lambda)\widetilde{T}} : H|X_{\tau_{l-1}},\bar{X}_{\tau_{l-1}}] &= \frac{-e^{-\beta_l\tilde{a}-\alpha_l X_{\tau_{l-1}}}+e^{\alpha_l\tilde{a}-\beta_l X_{\tau_{l-1}}}}{e^{\alpha_l\tilde{a}+\beta_l\tilde{b}}-e^{-\alpha_l\tilde{b}-\beta_l\tilde{a}}}\\ &= \tfrac{e^{\alpha_l a}}{e^{\alpha_l a}-e^{-\beta_l a}}\, e^{\beta_l X_{\tau_{l-1}}-\beta_l\bar{X}_{\tau_{l-1}}}\\ &\quad - \tfrac{e^{-\beta_l a}}{e^{\alpha_l a}-e^{-\beta_l a}}\, e^{-\alpha_l X_{\tau_{l-1}}+\alpha_l\bar{X}_{\tau_{l-1}}}.\end{aligned}$$

Reinserting everything into (4.34) leads to

$$\begin{aligned}&\mathbf{E}_{\mu_l}^{X_{\tau_{l-1}}}[e^{-(\rho+\lambda)S-\gamma X_S}|X_{\tau_{l-1}},\bar{X}_{\tau_{l-1}}]\\ &\quad= e^{-(\nu_l+\gamma)b}\left(1-\tfrac{n_l(B)e^{\gamma a}}{\nu_l+\gamma}\right)\tfrac{e^{\alpha_l a}}{e^{\alpha_l a}-e^{-\beta_l a}}\, e^{\beta_l X_{\tau_{l-1}}-(\beta_l-\nu_l)\bar{X}_{\tau_{l-1}}}\\ &\quad\quad - e^{-(\nu_l+\gamma)b}\left(1-\tfrac{n_l(B)e^{\gamma a}}{\nu_l+\gamma}\right)\tfrac{e^{-\beta_l a}}{e^{\alpha_l a}-e^{-\beta_l a}}\, e^{-\alpha_l X_{\tau_{l-1}}-(-\alpha_l-\nu_l)\bar{X}_{\tau_{l-1}}}\\ &\quad\quad + \tfrac{n_l(B)e^{\gamma a}}{\nu_l+\gamma}\tfrac{e^{\alpha_l a}}{e^{\alpha_l a}-e^{-\beta_l a}}\, e^{\beta_l X_{\tau_{l-1}}-(\beta_l+\gamma)\bar{X}_{\tau_{l-1}}}\\ &\quad\quad - \tfrac{n_l(B)e^{\gamma a}}{\nu_l+\gamma}\tfrac{e^{-\beta_l a}}{e^{\alpha_l a}-e^{-\beta_l a}}\, e^{-\alpha_l X_{\tau_{l-1}}-(-\alpha_l+\gamma)\bar{X}_{\tau_{l-1}}}\\ &\quad\quad + \tfrac{e^{\gamma a}}{e^{\alpha_l a}-e^{-\beta_l a}}\big(e^{-\alpha_l X_{\tau_{l-1}}-(-\alpha_l+\gamma)\bar{X}_{\tau_{l-1}}}-e^{-\beta_l X_{\tau_{l-1}}-(\beta_l+\gamma)\bar{X}_{\tau_{l-1}}}\big).\end{aligned}$$

Using the definitions in (4.19), (4.20), (4.21), (4.23), (4.24), (4.27), (4.28), (4.29) and (4.30) then yields

$$\begin{aligned}&\mathbf{E}_{\mu_l}^{X_{\tau_{l-1}}}[e^{-(\rho+\lambda)S-\gamma X_S}|X_{\tau_{l-1}},\bar{X}_{\tau_{l-1}}]\\ &\quad= \tfrac{1}{e^{\alpha_l a}-e^{-\beta_l a}}\, e^{-(\nu_l+\gamma)b}\,\tfrac{r_l(\gamma)}{\nu_l+\gamma}\, e^{-\beta_l a}\, e^{-\alpha_l X_{\tau_{l-1}}-(-\alpha_l-\nu_l)\bar{X}_{\tau_{l-1}}}\\ &\quad\quad - \tfrac{1}{e^{\alpha_l a}-e^{-\beta_l a}}\, e^{-(\nu_l+\gamma)b}\,\tfrac{r_l(\gamma)}{\nu_l+\gamma}\, e^{\alpha_l a}\, e^{\beta_l X_{\tau_{l-1}}-(\beta_l-\nu_l)\bar{X}_{\tau_{l-1}}}\\ &\quad\quad - \tfrac{1}{e^{\alpha_l a}-e^{-\beta_l a}}\left(\left(\tfrac{r_l(\gamma)}{\nu_l+\gamma}+1\right)e^{-\beta_l a}-e^{\gamma a}\right)e^{-\alpha_l X_{\tau_{l-1}}-(-\alpha_l+\gamma)\bar{X}_{\tau_{l-1}}}\\ &\quad\quad + \tfrac{1}{e^{\alpha_l a}-e^{-\beta_l a}}\left(\left(\tfrac{r_l(\gamma)}{\nu_l+\gamma}+1\right)e^{\alpha_l a}-e^{\gamma a}\right)e^{\beta_l X_{\tau_{l-1}}-(\beta_l+\gamma)\bar{X}_{\tau_{l-1}}}\\ &\quad= c_1^l(\gamma,\gamma)\, e^{\delta_1^l X_{\tau_{l-1}}-(\delta_1^l+\gamma_1^l)\bar{X}_{\tau_{l-1}}} + c_2^l(\gamma,\gamma)\, e^{\delta_2^l X_{\tau_{l-1}}-(\delta_2^l+\gamma_2^l)\bar{X}_{\tau_{l-1}}}\\ &\quad\quad + c_3^l(\gamma,\gamma)\, e^{\delta_3^l X_{\tau_{l-1}}-(\delta_3^l+\gamma_3^l)\bar{X}_{\tau_{l-1}}} + c_4^l(\gamma,\gamma)\, e^{\delta_4^l X_{\tau_{l-1}}-(\delta_4^l+\gamma_4^l)\bar{X}_{\tau_{l-1}}}\\ &\quad= \sum_{i_1=1}^{4} c_{i_1}^l(\gamma,\gamma)\, e^{\delta_{i_1}^l X_{\tau_{l-1}}-(\delta_{i_1}^l+\gamma_{i_1}^l)\bar{X}_{\tau_{l-1}}}.\end{aligned}$$

Having found the solution to (4.33), equation (4.32) transforms to

$$\mathbf{E}[e^{-\rho T-\gamma X_T} : T \in (\tau_{l-1}, \tau_l)] = \sum_{i_1=1}^{4} c^l_{i_1}(\gamma, \gamma)\, \mathbf{E}[e^{-\rho\tau_{l-1}+\delta^l_{i_1} X_{\tau_{l-1}}-(\delta^l_{i_1}+\gamma^l_{i_1})\bar{X}_{\tau_{l-1}}} : T > \tau_{l-1}]. \tag{4.38}$$

Now, the tower property with respect to the filtration $\mathcal{F}_{\tau_{l-2}}$ can be used. Let $S \equiv T-\tau_{l-2}$, then the restriction $T > \tau_{l-1}$ can be split into an $\mathcal{F}_{\tau_{l-2}}$-measurable part $T > \tau_{l-2}$ and $S > s_{l-1}$. Defining the shifted and rebased process $Y_t \equiv X_{\tau_{l-2}+t} - X_{\tau_{l-2}}$ for $t \leq s_{l-1}$, then $Y_0 = 0$ and the drift is constantly μ_{l-1}. Moreover, as $\tau_{l-1} = \tau_{l-2} + s_{l-1}$, it holds $X_{\tau_{l-1}} = Y_{s_{l-1}} + X_{\tau_{l-2}}$. The expectation in (4.38) is then

$$\begin{aligned} &\mathbf{E}[e^{-\rho\tau_{l-1}+\delta^l_{i_1} X_{\tau_{l-1}}-(\delta^l_{i_1}+\gamma^l_{i_1})\bar{X}_{\tau_{l-1}}} : T > \tau_{l-1}] \\ &= \mathbf{E}[e^{-\rho\tau_{l-2}}\, e^{\delta^l_{i_1} X_{\tau_{l-2}}}\, \mathbf{E}_{\mu_{l-1}}[e^{-\rho s_{l-1}+\delta^l_{i_1} Y_{s_{l-1}}-(\delta^l_{i_1}+\gamma^l_{i_1})\bar{X}_{\tau_{l-1}}} : S > s_{l-1}|\mathcal{F}_{\tau_{l-2}}] : T > \tau_{l-2}]. \end{aligned} \tag{4.39}$$

Replacing the quantity $\bar{X}_{\tau_{l-1}}$ is a bit more complicated. At time τ_{l-2}, the trailing stop does not rise until X hits $\bar{X}_{\tau_{l-2}}$. For $S > s_{l-1}$, it might happen that the process X stays inside the corridor $(-a+\bar{X}_{\tau_{l-2}}, \bar{X}_{\tau_{l-2}})$ for the entire time-interval (τ_{l-2}, τ_{l-1}). This case will be denoted by V and a realization is shown in Figure 4.5. Here, it holds $\bar{X}_{\tau_{l-1}} = \bar{X}_{\tau_{l-2}}$. By definition of the process Y, this case is equivalent to the case that Y stays inside the corridor $(-a + \bar{X}_{\tau_{l-2}} - X_{\tau_{l-2}}, \bar{X}_{\tau_{l-2}} - X_{\tau_{l-2}})$ for the time-interval $(0, s_{l-1})$. Defining the stopping time $\widetilde{T} \equiv \inf\{t : Y_t = \tilde{b} \equiv \bar{X}_{\tau_{l-2}} - X_{\tau_{l-2}}$ or $Y_t = -\tilde{a} \equiv -a+\bar{X}_{\tau_{l-2}}-X_{\tau_{l-2}}\}$, then the event V is equivalent to $\widetilde{T} > s_{l-1}$ and $S > s_{l-1}$ is redundant. Hence, the inner expectation of (4.39) multiplied by its prefactor can be determined for the case V. Then, the functionals inside the resulting expectation are independent of the filtration $\mathcal{F}_{\tau_{l-2}}$ and the stopping time $\widetilde{T}$ only depends on the values $X_{\tau_{l-2}}$ and $\bar{X}_{\tau_{l-2}}$.

$$\begin{aligned} &e^{\delta^l_{i_1} X_{\tau_{l-2}}}\, \mathbf{E}_{\mu_{l-1}}[e^{-\rho s_{l-1}+\delta^l_{i_1} Y_{s_{l-1}}-(\delta^l_{i_1}+\gamma^l_{i_1})\bar{X}_{\tau_{l-1}}} : S > s_{l-1}, V|\mathcal{F}_{\tau_{l-2}}] \\ &\quad = e^{\delta^l_{i_1} X_{\tau_{l-2}}-(\delta^l_{i_1}+\gamma^l_{i_1})\bar{X}_{\tau_{l-2}}}\, \mathbf{E}_{\mu_{l-1}}[e^{-\rho s_{l-1}+\delta^l_{i_1} Y_{s_{l-1}}} : \widetilde{T} > s_{l-1}|X_{\tau_{l-2}}, \bar{X}_{\tau_{l-2}}] \end{aligned} \tag{4.40}$$

As $\widetilde{T}$ is a fixed-stops stopping time, the expectation in (4.40) is given by Lemma 1,

$$\mathbf{E}_{\mu_{l-1}}[e^{-\rho s_{l-1}+\delta^l_{i_1} Y_{s_{l-1}}} : \widetilde{T} > s_{l-1}|X_{\tau_{l-2}}, \bar{X}_{\tau_{l-2}}] = \frac{2\lambda}{\sigma^2}\frac{L_1^{\mu_{l-1}}(\rho+\lambda, -\delta^l_{i_1}|X_{\tau_{l-2}}, \bar{X}_{\tau_{l-2}})-1}{(\delta^l_{i_1}+\alpha_{l-1})(\delta^l_{i_1}-\beta_{l-1})},$$

where L_1 is the joint Laplace transform given in (2.19), for the stopping time $\widetilde{T}$ with $X_{\tau_{l-2}}$ and $\bar{X}_{\tau_{l-2}}$ known. Here, it is

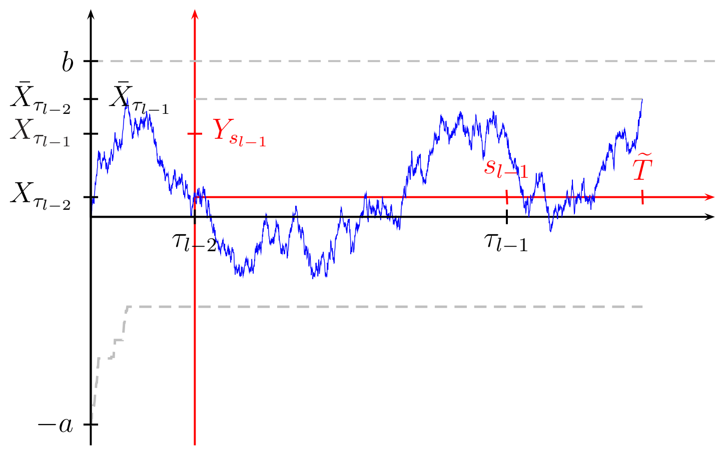

Figure 4.5: Realization of a gain process X not leaving the corridor and the shifted and rebased process Y with $\widetilde{T} > s_{l-1}$.

$$
\begin{aligned}
&L_1^{\mu_{l-1}}(\rho+\lambda, -\delta_{i_1}^l | X_{\tau_{l-2}}, \bar{X}_{\tau_{l-2}}) \\
&\quad = \frac{e^{-\delta_{i_1}^l \tilde{a}}\left(e^{\beta_{l-1}\tilde{b}} - e^{-\alpha_{l-1}\tilde{b}}\right) + e^{\delta_{i_1}^l \tilde{b}}\left(e^{\alpha_{l-1}\tilde{a}} - e^{-\beta_{l-1}\tilde{a}}\right)}{e^{\alpha_{l-1}\tilde{a}+\beta_{l-1}\tilde{b}} - e^{-\alpha_{l-1}\tilde{b}-\beta_{l-1}\tilde{a}}} \\
&\quad = \frac{1}{e^{\alpha_{l-1}a} - e^{-\beta_{l-1}a}}\left[-\left(e^{-\beta_{l-1}a} - e^{-\delta_{i_1}^l a}\right) e^{(\alpha_{l-1}+\delta_{i_1}^l)(\bar{X}_{\tau_{l-2}} - X_{\tau_{l-2}})}\right. \\
&\qquad\qquad \left. + \left(e^{\alpha_{l-1}a} - e^{-\delta_{i_1}^l a}\right) e^{(-\beta_{l-1}+\delta_{i_1}^l)(\bar{X}_{\tau_{l-2}} - X_{\tau_{l-2}})}\right].
\end{aligned}
$$

Reinserting into (4.40) gives

$$
\begin{aligned}
&e^{\delta_{i_1}^l X_{\tau_{l-2}}} \mathbf{E}_{\mu_{l-1}}[e^{-\rho s_{l-1} + \delta_{i_1}^l Y_{s_{l-1}} - (\delta_{i_1}^l + \gamma_{i_1}^l)\bar{X}_{\tau_{l-1}}} : S > s_{l-1}, V | \mathcal{F}_{\tau_{l-2}}] \\
&= \frac{2\lambda}{\sigma^2} \frac{1}{(\delta_{i_1}^l + \alpha_{l-1})(\delta_{i_1}^l - \beta_{l-1})} e^{\delta_{i_1}^l X_{\tau_{l-2}} - (\delta_{i_1}^l + \gamma_{i_1}^l)\bar{X}_{\tau_{l-2}}} \left\{L_1^{\mu_{l-1}}(\rho+\lambda, -\delta_{i_1}^l | X_{\tau_{l-2}}, \bar{X}_{\tau_{l-2}}) - 1\right\} \\
&= \frac{2\lambda}{\sigma^2} \frac{1}{(\delta_{i_1}^l + \alpha_{l-1})(\delta_{i_1}^l - \beta_{l-1})} \left\{\frac{1}{e^{\alpha_{l-1}a} - e^{-\beta_{l-1}a}}\left[-\left(e^{-\beta_{l-1}a} - e^{-\delta_{i_1}^l a}\right) e^{-\alpha_{l-1}X_{\tau_{l-2}} - (-\alpha_{l-1}+\gamma_{i_1}^l)\bar{X}_{\tau_{l-2}}}\right.\right. \\
&\quad \left.\left. + \left(e^{\alpha_{l-1}a} - e^{-\delta_{i_1}^l a}\right) e^{\beta_{l-1}X_{\tau_{l-2}} - (\beta_{l-1}+\gamma_{i_1}^l)\bar{X}_{\tau_{l-2}}}\right] - e^{\delta_{i_1}^l X_{\tau_{l-2}} - (\delta_{i_1}^l + \gamma_{i_1}^l)\bar{X}_{\tau_{l-2}}}\right\}. \qquad (4.41)
\end{aligned}
$$

The case that process X leaves the above mentioned corridor during the time-interval (τ_{l-2}, τ_{l-1}) will be denoted by V^c and a realization is shown in Figure 4.6.

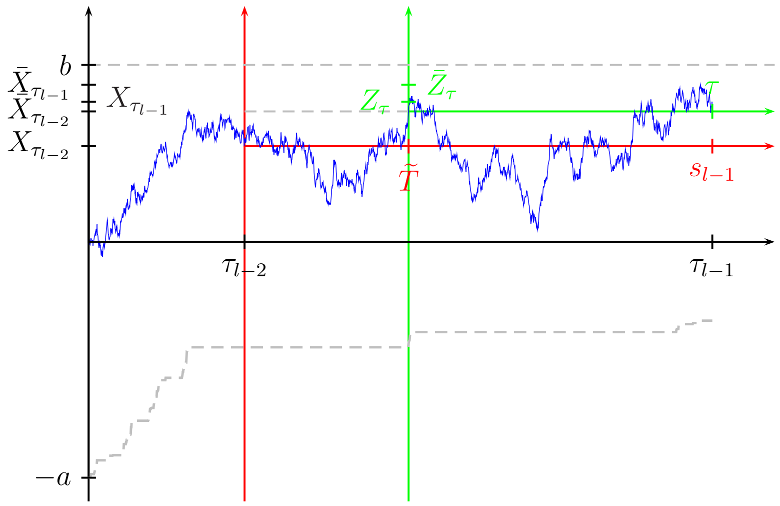

Figure 4.6: Realization of a gain process X leaving the corridor, the rebased process Y with $\widetilde{T} < s_{l-1}$ and the rebased process Z with $T'' > \tau$.

Now, the tower property with respect to the filtration $\mathcal{F}_{\widetilde{T}}$ can be applied to the inner expectation of (4.39) for the case V^c. For the process Y, the event V^c is equivalent to $\widetilde{T} < s_{l-1}$. The constraint $S > s_{l-1}$ can be separated into an $\mathcal{F}_{\widetilde{T}}$-measurable part $S > \widetilde{T}$ and an independent part $S - \widetilde{T} > \tau$, where $\tau \equiv s_{l-1} - \widetilde{T}$. The first of these restrictions means that the process Y ends up on its high side, $Y_{\widetilde{T}} = \bar{X}_{\tau_{l-2}} - X_{\tau_{l-2}}$. There, the shifted and rebased process $Z_t \equiv Y_{\widetilde{T}+t} - Y_{\widetilde{T}} = Y_{\widetilde{T}+t} - \bar{X}_{\tau_{l-2}} + X_{\tau_{l-2}}$, for $t \leq \tau$ can be defined. It starts at 0 and yields $Y_{s_{l-1}} = Z_\tau + \bar{X}_{\tau_{l-2}} - X_{\tau_{l-2}}$ as well as $\bar{X}_{\tau_{l-1}} = \bar{X}_{\tau_{l-2}} + \bar{Z}_\tau$. As the lower stop evolves like a trailing stop again, the remaining time $T'' \equiv S - \widetilde{T}$ is the trailing-stop-and-fixed-stop stopping time $T'' = \inf\{t : Z_t = b'' \equiv b - \bar{X}_{\tau_{l-2}} \text{ or } Z_t = -a + \bar{Z}_t\}$; and the other constraint is equivalent to $T'' > \tau$. Due to the Markov property, the functionals inside the resulting inner expectation are independent of the filtration $\mathcal{F}_{\widetilde{T}}$ and the stopping time T'' only depends on the value $\bar{X}_{\tau_{l-2}}$. Thus, the prefactor multiplied with the expectation is

$$
\begin{aligned}
&e^{\delta^l_{i_1} X_{\tau_{l-2}}} \mathbf{E}_{\mu_{l-1}}[e^{-\rho s_{l-1} + \delta^l_{i_1} Y_{s_{l-1}} - (\delta^l_{i_1} + \gamma^l_{i_1}) \bar{X}_{\tau_{l-1}}} : S > s_{l-1}, V^c | \mathcal{F}_{\tau_{l-2}}] \\
&= e^{-\gamma^l_{i_1} \bar{X}_{\tau_{l-2}}} \mathbf{E}_{\mu_{l-1}}[e^{-\rho \widetilde{T}} \mathbf{E}_{\mu_{l-1}}[e^{-\rho\tau + \delta^l_{i_1} Z_\tau - (\delta^l_{i_1} + \gamma^l_{i_1}) \bar{Z}_\tau} : T'' > \tau | \bar{X}_{\tau_{l-2}}] : \widetilde{T} < s_{l-1}, H | \mathcal{F}_{\tau_{l-2}}].
\end{aligned}
\tag{4.42}
$$

$\widetilde{T} < s_{l-1}$ means that the exponentially distributed random variable has not appeared at $\widetilde{T}$. Then, the memorylessness property of the exponential distribution states that τ

is exponentially distributed with parameter λ as well. Hence, the inner expectation of (4.42) can be obtained by the following lemma.

Lemma 2 *Let τ be an exponentially distributed random variable with parameter λ and T be the trailing-stop-and-fixed-stop stopping time $T = \inf\{t : X_t = b$ or $X_t = -a + \bar{X}_t\}$ of a process $X_t = \mu_k t + \sigma W_t$, then it holds*

$$\mathbf{E}_{\mu_k}[e^{-\rho\tau+\delta X_\tau-(\delta+\gamma)\bar{X}_\tau} : T > \tau] = \tfrac{2\lambda}{\sigma^2} \tfrac{1-e^{-(\nu_k+\gamma)b}}{(\delta+\alpha_k)(\delta-\beta_k)} \tfrac{r_k(-\delta)}{\nu_k+\gamma},$$

with ν_k and r_k according to the definitions (4.17) *and* (4.19), *respectively.*

The proof is given in Appendix B.2. For the special case $\gamma = -\delta$, Lemma 2 gives a similar result as Lemma 1 where the joint Laplace transform L_1 is replaced by L_2. Applying Lemma 2 to the inner expectation of (4.42) yields

$$\mathbf{E}_{\mu_{l-1}}[e^{-\rho\tau+\delta^l_{i_1} Z_\tau-(\delta^l_{i_1}+\gamma^l_{i_1})\bar{Z}_\tau} : T'' > \tau|\bar{X}_{\tau_{l-2}}] = \tfrac{2\lambda}{\sigma^2} \tfrac{1-e^{-(\nu_{l-1}+\gamma^l_{i_1})(b-\bar{X}_{\tau_{l-2}})}}{(\delta^l_{i_1}+\alpha_{l-1})(\delta^l_{i_1}-\beta_{l-1})} \tfrac{r_{l-1}(-\delta^l_{i_1})}{\nu_{l-1}+\gamma^l_{i_1}},$$

which transforms (4.42) into

$$\begin{aligned} &e^{\delta^l_{i_1} X_{\tau_{l-2}}} \mathbf{E}_{\mu_{l-1}}[e^{-\rho s_{l-1}+\delta^l_{i_1} Y_{s_{l-1}}-(\delta^l_{i_1}+\gamma^l_{i_1})\bar{X}_{\tau_{l-1}}} : S > s_{l-1}, V^c|\mathcal{F}_{\tau_{l-2}}] \\ &\quad = \tfrac{2\lambda}{\sigma^2} \tfrac{1-e^{-(\nu_{l-1}+\gamma^l_{i_1})(b-\bar{X}_{\tau_{l-2}})}}{(\delta^l_{i_1}+\alpha_{l-1})(\delta^l_{i_1}-\beta_{l-1})} \tfrac{r_{l-1}(-\delta^l_{i_1})}{\nu_{l-1}+\gamma^l_{i_1}} e^{-\gamma^l_{i_1}\bar{X}_{\tau_{l-2}}} \mathbf{E}_{\mu_{l-1}}[e^{-\rho\widetilde{T}} : \widetilde{T} < s_{l-1}, H|\mathcal{F}_{\tau_{l-2}}]. \end{aligned} \tag{4.43}$$

$\widetilde{T}$ is the fixed-stops stopping time, which depends on $X_{\tau_{l-2}}$ and$\bar{X}_{\tau_{l-2}}$; and Y is independent of $\mathcal{F}_{\tau_{l-2}}$. Thus, the remaining expectation in (4.43) can be obtained by changing the rate of discounting and reading off the solution of the partial joint Laplace transform for the high side from (2.18), with boundary conditions $f(-\tilde{a}) = 0$ and $f(\tilde{b}) = 1$,

$$\begin{aligned} &\mathbf{E}_{\mu_{l-1}}[e^{-\rho\widetilde{T}} : \widetilde{T} < s_{l-1}, H|\mathcal{F}_{\tau_{l-2}}] \\ &\quad = \mathbf{E}_{\mu_{l-1}}[e^{-(\rho+\lambda)\widetilde{T}} : H|X_{\tau_{l-2}}, \bar{X}_{\tau_{l-2}}] \\ &\quad = \tfrac{e^{\alpha_{l-1}\tilde{a}}-e^{\beta_{l-1}\tilde{a}}}{e^{\alpha_{l-1}\tilde{a}+\beta_{l-1}\tilde{b}}-e^{-\alpha_{l-1}\tilde{b}-\beta_{l-1}\tilde{a}}} \\ &\quad = \tfrac{e^{\alpha_{l-1}a}\, e^{\beta_{l-1}X_{\tau_{l-2}}-\beta_{l-1}\bar{X}_{\tau_{l-2}}}-e^{-\beta_{l-1}a}\, e^{-\alpha_{l-1}X_{\tau_{l-2}}+\alpha_{l-1}\bar{X}_{\tau_{l-2}}}}{e^{\alpha_{l-1}a}-e^{\beta_{l-1}a}}. \end{aligned} \tag{4.44}$$

Plugging in (4.44) into (4.43) gives the result for the case V^c,

$$
\begin{aligned}
&e^{\delta^l_{i_1} X_{\tau_{l-2}}} \mathbf{E}_{\mu_{l-1}}[e^{-\rho s_{l-1}+\delta^l_{i_1} Y_{s_{l-1}}-(\delta^l_{i_1}+\gamma^l_{i_1})\bar{X}_{\tau_{l-1}}} : S > s_{l-1}, V^c|\mathcal{F}_{\tau_{l-2}}] \\
&\quad = \tfrac{2\lambda}{\sigma^2} \tfrac{1}{(\delta^l_{i_1}+\alpha_{l-1})(\delta^l_{i_1}-\beta_{l-1})} \tfrac{1}{e^{\alpha_{l-1}a}-e^{\beta_{l-1}a}} \tfrac{r_{l-1}(-\delta^l_{i_1})}{\nu_{l-1}+\gamma^l_{i_1}} (1-e^{-(\nu_{l-1}+\gamma^l_{i_1})(b-\bar{X}_{\tau_{l-2}})}) \\
&\quad\cdot \left[e^{\alpha_{l-1}a} e^{\beta_{l-1}X_{\tau_{l-2}}-(\beta_{l-1}+\gamma^l_{i_1})\bar{X}_{\tau_{l-2}}} - e^{-\beta_{l-1}a} e^{-\alpha_{l-1}X_{\tau_{l-2}}-(-\alpha_{l-1}+\gamma^l_{i_1})\bar{X}_{\tau_{l-2}}} \right], \quad (4.45)
\end{aligned}
$$

which is the solution for the case V^c. As inner expectation in (4.39) is the sum of the two cases,

$$
\begin{aligned}
&\mathbf{E}_{\mu_{l-1}}[e^{-\rho s_{l-1}+\delta^l_{i_1} Y_{s_{l-1}}-(\delta^l_{i_1}+\gamma^l_{i_1})\bar{X}_{\tau_{l-1}}} : S > s_{l-1}|\mathcal{F}_{\tau_{l-2}}] \\
&\qquad = \mathbf{E}_{\mu_{l-1}}[e^{-\rho s_{l-1}+\delta^l_{i_1} Y_{s_{l-1}}-(\delta^l_{i_1}+\gamma^l_{i_1})\bar{X}_{\tau_{l-1}}} : S > s_{l-1}, V|\mathcal{F}_{\tau_{l-2}}] \\
&\qquad\quad + \mathbf{E}_{\mu_{l-1}}[e^{-\rho s_{l-1}+\delta^l_{i_1} Y_{s_{l-1}}-(\delta^l_{i_1}+\gamma^l_{i_1})\bar{X}_{\tau_{l-1}}} : S > s_{l-1}, V^c|\mathcal{F}_{\tau_{l-2}}],
\end{aligned}
$$

the inner expectation multiplied with its prefactor can be obtained by summing up (4.41) and (4.45).

$$
\begin{aligned}
&e^{\delta^l_{i_1} X_{\tau_{l-2}}} \mathbf{E}_{\mu_{l-1}}[e^{-\rho s_{l-1}+\delta^l_{i_1} Y_{s_{l-1}}-(\delta^l_{i_1}+\gamma^l_{i_1})\bar{X}_{\tau_{l-1}}} : S > s_{l-1}|\mathcal{F}_{\tau_{l-2}}] \\
&= \tfrac{2\lambda}{\sigma^2} \tfrac{1}{(\delta^l_{i_1}+\alpha_{l-1})(\delta^l_{i_1}-\beta_{l-1})} \left\{ \tfrac{1}{e^{\alpha_{l-1}a}-e^{-\beta_{l-1}a}} \left[-\left(e^{-\beta_{l-1}a}-e^{-\delta^l_{i_1}a}\right) e^{-\alpha_{l-1}X_{\tau_{l-2}}-(-\alpha_{l-1}+\gamma^l_{i_1})\bar{X}_{\tau_{l-2}}} \right.\right. \\
&\quad + \left(e^{\alpha_{l-1}a}-e^{-\delta^l_{i_1}a}\right) e^{\beta_{l-1}X_{\tau_{l-2}}-(\beta_{l-1}+\gamma^l_{i_1})\bar{X}_{\tau_{l-2}}} + \tfrac{r_{l-1}(-\delta^l_{i_1})}{\nu_{l-1}+\gamma^l_{i_1}} e^{\alpha_{l-1}a} e^{\beta_{l-1}X_{\tau_{l-2}}-(\beta_{l-1}+\gamma^l_{i_1})\bar{X}_{\tau_{l-2}}} \\
&\quad - \tfrac{r_{l-1}(-\delta^l_{i_1})}{\nu_{l-1}+\gamma^l_{i_1}} e^{-\beta_{l-1}a} e^{-\alpha_{l-1}X_{\tau_{l-2}}-(-\alpha_{l-1}+\gamma^l_{i_1})\bar{X}_{\tau_{l-2}}} \\
&\quad - e^{-(\nu_{l-1}+\gamma^l_{i_1})b} \tfrac{r_{l-1}(-\delta^l_{i_1})}{\nu_{l-1}+\gamma^l_{i_1}} e^{\alpha_{l-1}a} e^{\beta_{l-1}X_{\tau_{l-2}}-(\beta_{l-1}-\nu_{l-1})\bar{X}_{\tau_{l-2}}} \\
&\quad \left.\left. + e^{-(\nu_{l-1}+\gamma^l_{i_1})b} \tfrac{r_{l-1}(-\delta^l_{i_1})}{\nu_{l-1}+\gamma^l_{i_1}} e^{-\beta_{l-1}a} e^{-\alpha_{l-1}X_{\tau_{l-2}}-(-\alpha_{l-1}-\nu_{l-1})\bar{X}_{\tau_{l-2}}} \right] - e^{\delta^l_{i_1} X_{\tau_{l-2}}-(\delta^l_{i_1}+\gamma^l_{i_1})\bar{X}_{\tau_{l-2}}} \right\}
\end{aligned}
$$

$$
\begin{aligned}
&= \tfrac{2\lambda}{\sigma^2} \tfrac{1}{(\delta^l_{i_1}+\alpha_{l-1})(\delta^l_{i_1}-\beta_{l-1})} \Bigg\{ \tfrac{1}{e^{\alpha_{l-1}a}-e^{-\beta_{l-1}a}} \\
&\quad \cdot \Bigg[e^{-(\nu_{l-1}+\gamma^l_{i_1})b} \tfrac{r_{l-1}(-\delta^l_{i_1})}{\nu_{l-1}+\gamma^l_{i_1}} \, e^{-\beta_{l-1}a} \, e^{-\alpha_{l-1}X_{\tau_{l-2}}-(-\alpha_{l-1}-\nu_{l-1})\bar{X}_{\tau_{l-2}}} \\
&\qquad - e^{-(\nu_{l-1}+\gamma^l_{i_1})b} \tfrac{r_{l-1}(-\delta^l_{i_1})}{\nu_{l-1}+\gamma^l_{i_1}} \, e^{\alpha_{l-1}a} \, e^{\beta_{l-1}X_{\tau_{l-2}}-(\beta_{l-1}-\nu_{l-1})\bar{X}_{\tau_{l-2}}} \\
&\qquad - \left(\left(\tfrac{r_{l-1}(-\delta^l_{i_1})}{\nu_{l-1}+\gamma^l_{i_1}} + 1 \right) e^{-\beta_{l-1}a} - e^{-\delta^l_{i_1}a} \right) e^{-\alpha_{l-1}X_{\tau_{l-2}}-(-\alpha_{l-1}+\gamma^l_{i_1})\bar{X}_{\tau_{l-2}}} \\
&\qquad + \left(\left(\tfrac{r_{l-1}(-\delta^l_{i_1})}{\nu_{l-1}+\gamma^l_{i_1}} + 1 \right) e^{\alpha_{l-1}a} - e^{-\delta^l_{i_1}a} \right) e^{\beta_{l-1}X_{\tau_{l-2}}-(\beta_{l-1}+\gamma^l_{i_1})\bar{X}_{\tau_{l-2}}} \Bigg] \\
&\quad - e^{\delta^l_{i_1}X_{\tau_{l-2}}-(\delta^l_{i_1}+\gamma^l_{i_1})\bar{X}_{\tau_{l-2}}} \Bigg\} \\
&= \tfrac{2\lambda}{\sigma^2} \tfrac{1}{(\delta^l_{i_1}+\alpha_{l-1})(\delta^l_{i_1}-\beta_{l-1})} \Big\{ c^{l-1}_1(-\delta^l_{i_1},\gamma^l_{i_1}) \, e^{\delta^{l-1}_1 X_{\tau_{l-2}}-(\delta^{l-1}_1+\gamma^{l-1}_1)\bar{X}_{\tau_{l-2}}} \\
&\qquad + c^{l-1}_2(-\delta^l_{i_1},\gamma^l_{i_1}) \, e^{\delta^{l-1}_2 X_{\tau_{l-2}}-(\delta^{l-1}_2+\gamma^{l-1}_2)\bar{X}_{\tau_{l-2}}} \\
&\qquad + c^{l-1}_3(-\delta^l_{i_1},\gamma^l_{i_1}) \, e^{\delta^{l-1}_3 X_{\tau_{l-2}}-(\delta^{l-1}_3+\gamma^{l-1}_3)\bar{X}_{\tau_{l-2}}} \\
&\qquad + c^{l-1}_4(-\delta^l_{i_1},\gamma^l_{i_1}) \, e^{\delta^{l-1}_4 X_{\tau_{l-2}}-(\delta^{l-1}_4+\gamma^{l-1}_4)\bar{X}_{\tau_{l-2}}} \\
&\qquad + c^{l-1}_5(-\delta^l_{i_1},\gamma^l_{i_1}) \, e^{\delta^{l-1}_5 X_{\tau_{l-2}}-(\delta^{l-1}_5+\gamma^{l-1}_5)\bar{X}_{\tau_{l-2}}} \Big\} \\
&= \tfrac{2\lambda}{\sigma^2} \sum_{i_2=1}^{5} \hat{c}^{l-1}_{i_2}(\delta^l_{i_1},\gamma^l_{i_1}) \, e^{\delta^{l-1}_{i_2} X_{\tau_{l-2}}-(\delta^{l-1}_{i_2}+\gamma^{l-1}_{i_2})\bar{X}_{\tau_{l-2}}},
\end{aligned}
$$

where the definitions (4.20), (4.21), (4.22), (4.23), (4.24), (4.25), (4.26), (4.27), (4.28), (4.29), (4.30) and (4.31) have been used. Equation (4.39) then becomes

$$
\begin{aligned}
&\mathbf{E}[e^{-\rho\tau_{l-1}+\delta^l_{i_1}X_{\tau_{l-1}}-(\delta^l_{i_1}+\gamma^l_{i_1})\bar{X}_{\tau_{l-1}}} : T > \tau_{l-1}] \\
&\qquad = \tfrac{2\lambda}{\sigma^2} \sum_{i_2=1}^{5} \hat{c}^{l-1}_{i_2}(\delta^l_{i_1},\gamma^l_{i_1}) \, \mathbf{E}[e^{-\rho\tau_{l-2}+\delta^{l-1}_{i_2}X_{\tau_{l-2}}-(\delta^{l-1}_{i_2}+\gamma^{l-1}_{i_2})\bar{X}_{\tau_{l-2}}} : T > \tau_{l-2}]. \qquad (4.46)
\end{aligned}
$$

This result can be substituted in (4.38). Furthermore, equation (4.46) can be applied recursively to obtain the desired result.

$$
\begin{aligned}
&\mathbf{E}[e^{-\rho T-\gamma X_T} : T \in (\tau_{l-1}, \tau_l)] \\
&\quad = \tfrac{2\lambda}{\sigma^2} \sum_{i_1=1}^{4} c_{i_1}^l(\gamma,\gamma) \sum_{i_2=1}^{5} \hat{c}_{i_2}^{l-1}(\delta_{i_1}^l, \gamma_{i_1}^l)\, \mathbf{E}[e^{-\rho\tau_{l-2}+\delta_{i_2}^{l-1}X_{\tau_{l-2}}-(\delta_{i_2}^{l-1}+\gamma_{i_2}^{l-1})\bar{X}_{\tau_{l-2}}} : T > \tau_{l-2}] \\
&\quad \ldots \\
&\qquad = \left(\tfrac{2\lambda}{\sigma^2}\right)^{l-1} \sum_{i_1=1}^{4} c_{i_1}^l(\gamma,\gamma) \sum_{i_2=1}^{5} \hat{c}_{i_2}^{l-1}(\delta_{i_1}^l, \gamma_{i_1}^l) \cdots \sum_{i_l=1}^{5} \hat{c}_{i_l}^1(\delta_{i_{l-1}}^2, \gamma_{i_{l-1}}^2) \\
&\quad = \left(\tfrac{2\lambda}{\sigma^2}\right)^{l-1} \sum_{i_1=1}^{4} c_{i_1}^l(\gamma,\gamma) \sum_{i_2=1}^{5} \cdots \sum_{i_l=1}^{5} \hat{c}_{i_2}^{l-1}(\delta_{i_1}^l, \gamma_{i_1}^l) \cdots \hat{c}_{i_l}^1(\delta_{i_{l-1}}^2, \gamma_{i_{l-1}}^2) \\
&\quad = \left(\tfrac{2\lambda}{\sigma^2}\right)^{l-1} \sum_{i_1=1}^{4} c_{i_1}^l(\gamma,\gamma) \sum_{\substack{i_m=1 \\ 2\le m\le l}}^{5} \prod_{j=2}^{l} \hat{c}_{i_j}^{l-(j-1)} \left(\delta_{i_{(j-1)}}^{l-(j-2)}, \gamma_{i_{(j-1)}}^{l-(j-2)}\right)
\end{aligned}
$$

□

4.1.3 Example 3: trailing stop

Proposition 5 *For the trailing stop, the partial expectations of* (4.1) *are*

$$
\mathbf{E}[e^{-\rho T-\gamma X_T} : T \in (\tau_{l-1}, \tau_l)] = \left(\tfrac{2\lambda}{\sigma^2}\right)^{l-1} \sum_{i_1=3}^{4} c_{i_1}^l(\gamma,\gamma) \sum_{\substack{i_m=3 \\ 2\le m\le l}}^{5} \prod_{j=2}^{l} \hat{c}_{i_j}^{l-(j-1)} \left(\delta_{i_{(j-1)}}^{l-(j-2)}, \gamma_{i_{(j-1)}}^{l-(j-2)}\right),
$$

with the same definitions as in Proposition 4.

Proof: The trailing stop is a special case of Example 2 with $b = \infty$. Therefore, the limiting case $b \to \infty$ has to be regarded. The assertion of Proposition 5 looks like the result of Proposition 4 but the index of summation starts at 3. As for the indices $i_k \in \{3, 4, 5\}$, the crucial functionals (4.29), (4.30) and (4.31) are independent from b, it has to be shown that there is no index $i_k \in \{1, 2\}$. This will be done by induction for $n = 1, \ldots, l$.
Let $n = 1$ and $i_1 \in \{1, 2\}$. The corresponding functionals c_{i_1} in (4.27) and (4.28) can be expressed as

$$
c_{i_1}^l(\gamma,\gamma) = C_{i_1}^l\, e^{-(\nu_l+\gamma)b},
$$

where $C_{i_1}^l$ is independent from b. As $\nu_l > 0$ and $\gamma \ge 0$,

$$\lim_{b\to\infty} c^l_{i_1}(\gamma,\gamma) = C^l_{i_1} \lim_{b\to\infty} e^{-(\nu_l+\gamma)b}$$
$$= 0.$$

Hence, there is no index $i_1 \in \{1,2\}$. Now, it can be assumed that for $k < n$ there is no index $i_k \in \{1,2\}$ and it has to be shown that this also holds for n.
Let $i_n \in \{1,2\}$ with $n \geq 2$. The corresponding functionals c_{i_n} in (4.27) and (4.28) can be expressed as

$$c^{l-(n-1)}_{i_n}\left(\delta^{l-(n-2)}_{i_{(n-1)}}, \gamma^{l-(n-2)}_{i_{(n-1)}}\right) = C^{l-(n-1)}_{i_n} \exp\left(-\left(\nu_{l-(n-1)} + \gamma^{l-(n-2)}_{i_{(n-1)}}\right) b\right),$$

where $C^{l-(n-1)}_{i_n}$ is independent from b. $\nu_{l-(n-1)} > 0$ and it has to be checked that $\gamma^{l-(n-2)}_{i_{(n-1)}} \geq 0$ because then the functional c_{i_n} will vanish for $b \to \infty$. By the induction hypothesis $i_{(n-1)} \in \{3,4,5\}$. Therefore, the parameter is defined by (4.24) or (4.25), which is

$$\gamma^{l-(n-2)}_{i_{(n-1)}} = \begin{cases} \gamma & \text{if } i_{(n-1)} \in \{3,4\} \text{ and } n = 2 \\ \gamma^{l-(n-3)}_{i_{(n-2)}} & \text{else.} \end{cases}$$

By the induction hypothesis also $i_{(n-2)} \in \{3,4,5\}$ and this routine can be carried out recursively and will end in $\gamma > 0$ which proves the statement. $\square$

4.2 Numerical results

The sum in (4.1) could be approximated by using a large N as an upper bound of summation. However, when it comes to uncertainty of the drift parameters $\mu_1, \ldots, \mu_N$, for each μ_j a numerical integration has to be carried out which leads to a high computational complexity if N is big. For all three stopping examples, the probability that the stop occurs before the first change-point is $\mathbf{P}(T < \tau) = \mathbf{E}_{\mu_1}[e^{-\lambda T}] = L^{\mu_1}(\lambda, 0)$. For the parameters used in the numerical examples, this value is quite close to 1, which encourages using a smaller N. However, a small value N means that the Laplace transforms for the case $T > \tau_N$ will be ignored which makes the optimal stop to occur earlier. Hence, for a better approximation, the last change-point can be defined as $\tau_N \equiv \infty$, which means that the drift changes up to μ_N and then remains constant. Then, in the previous proofs, for the last summand, where $l = N$, the constraint $T > \tau_N$ is redundant; this leaves the rate of discounting unchanged to ρ instead of $\rho + \lambda$. This leads to different parameters,

$$\alpha_N = \frac{1}{\sigma^2}(\mu_N + \sqrt{\mu_N^2 + 2\sigma^2\rho})$$
$$\beta_N = \frac{1}{\sigma^2}(-\mu_N + \sqrt{\mu_N^2 + 2\sigma^2\rho}),$$

and everything else remains the same.

4.2.1 Certain drift parameters

In the first numerical results, there are two drift parameters μ_1 and μ_2 which are known with certainty. The results can be compared to the single drift case which corresponds to the special case $\mu_1 = \mu_2$ and has been examined in Table 2.1. As there the parameters were such that $\alpha > \gamma$, the optimal choice was a fixed upper stop without a lower stop. In the following tables, results are given for $\lambda = 2$, $\mu_1 = 0.15$ and three different values for μ_2. In Table 4.1, the second drift outperforms the first one: $\mu_2 = 0.2$. In Table 4.2, the drift decreases but remains positive: $\mu_2 = 0.05$. Finally, a crash scenario with $\mu_2 = -0.2$ is given in Table 4.3. All other parameters are as in the previous chapters.

	Best a	Best b	Objective
Fixed stops	∞	0.0203	4.6763
Trailing stop and fixed stop	∞	0.0203	4.6763
Trailing stop	0.1274		0.6948

Table 4.1: Numerical results for switching drift and certain drift parameters: $\mu_1 = 0.15$, $\mu_2 = 0.2$.

	Best a	Best b	Objective
Fixed stops	0.1264	0.0432	0.5541
Trailing stop and fixed stop	0.1413	0.0467	0.5633
Trailing stop	0.0680		0.3381

Table 4.2: Numerical results for switching drift and certain drift parameters: $\mu_1 = 0.15$, $\mu_2 = 0.05$.

As in the special case $\mu_1 = \mu_2$, the outperforming scenario recommends to use no lower stop and the results for Example 1 and 2 coincide. Due to the increased drift, the objective is larger compared to the single drift case $\mu = 0.15$. This result is not a surprise as both drifts are large enough to give $\alpha > \gamma$.
In case of $\mu_2 = 0.05$, the optimal lower stop is finite. This would have been the case in the single drift case with $\mu = 0.05$ since then it is $\alpha < \gamma$, but then, the objective

	Best a	Best b	Objective
Fixed stops	0.0601	0.0459	0.1098
Trailing stop and fixed stop	0.0724	0.0615	0.1623
Trailing stop	0.0539		0.0531

Table 4.3: Numerical results for switching drift and certain drift parameters: $\mu_1 = 0.15$, $\mu_2 = -0.2$.

would have been negative. Here, the objective has fallen but it is still positive because in around 90%, the process will hit a stop before the drift changes. The trailing-stop-and-fixed-stop example slightly outperforms the fixed-stops example.
For $\mu_2 = -0.2$, unsurprisingly, all objectives have fallen but the result of the trailing-stop-and-fixed-stop rule compared to the fixed-stops rule is much bigger.

4.2.2 Uncertain drift parameters

When it comes to uncertainty of the drift parameters, only Strategy B of Section 3.1 will be taken into account; Strategy A does not make sense in this setting and Strategy C would lead to a high computational complexity.

Strategy B

Analogously to (3.3), the overall value is

$$\bar{\varphi} = \frac{\int \cdots \int \mathbf{E}_{\mu_1,\ldots,\mu_N}[e^{-\rho T}\, U(X_T - c)]\, m_1(\mathrm{d}\mu_1) \cdots m_N(\mathrm{d}\mu_N)}{1 - \int \cdots \int \mathbf{E}_{\mu_1,\ldots,\mu_N}[e^{-\rho T}]\, m_1(\mathrm{d}\mu_1) \cdots m_N(\mathrm{d}\mu_N)},$$

where $\mathbf{E}_{\mu_1,\ldots,\mu_N}$ denotes the expectation under a certain drift-set $\{\mu_1, \ldots, \mu_N\}$ and m_k is the distribution of drift μ_k.
Here, the drift parameters are supposed to be independent and identically distributed random variables with prior $N(\mu_0, \sigma_\mu^2)$ distribution; with $\mu_0 = 0.15$ and $\sigma_\mu = 0.3$.
Table 4.4 shows the results for $\lambda = 2$.

	Best a	Best b	Objective
Fixed stops	0.0232	0.2366	1.1374
Trailing stop and fixed stop	0.0913	0.5857	1.2426
Trailing stop	0.0892		1.2379

Table 4.4: Numerical results for switching drift and Strategy B with $\lambda = 2$.

In the single drift case of Table 3.3, it was optimal to use small a values and large b values in order to stop *bad* investments let *good* investments run. Here, the b values have dropped because it is not certain that a *good* investment will stay a *good* investment. As the trailing stop is able to stop out investments which turn from *good* to *bad*, the stops trading rules with trailing stop outperform the fixed-stops trading rule by a remarkable margin.
Surely, it depends on the intensity λ whether the trailing-stop rules can beat the fixed-stop rule. For $\lambda = 0$, the drift does not change and the results are as in Table 3.3, where the trailing performed best. When λ increases, this relation is reversed; for $\lambda = 2$, the results were shown above, in Table 4.4. For these parameters and some more λ-values, $\lambda = 1/4$, $\lambda = 1/2$ and $\lambda = 1$, the objectives of the three stopping rules are shown in a bar chart in Figure 4.7. From this chart it can be seen that the trailing stop is more robust against drift-changes and for $\lambda = 1/2$ or bigger, the stopping rules with trailing stop outperform the fixed-stops rule.

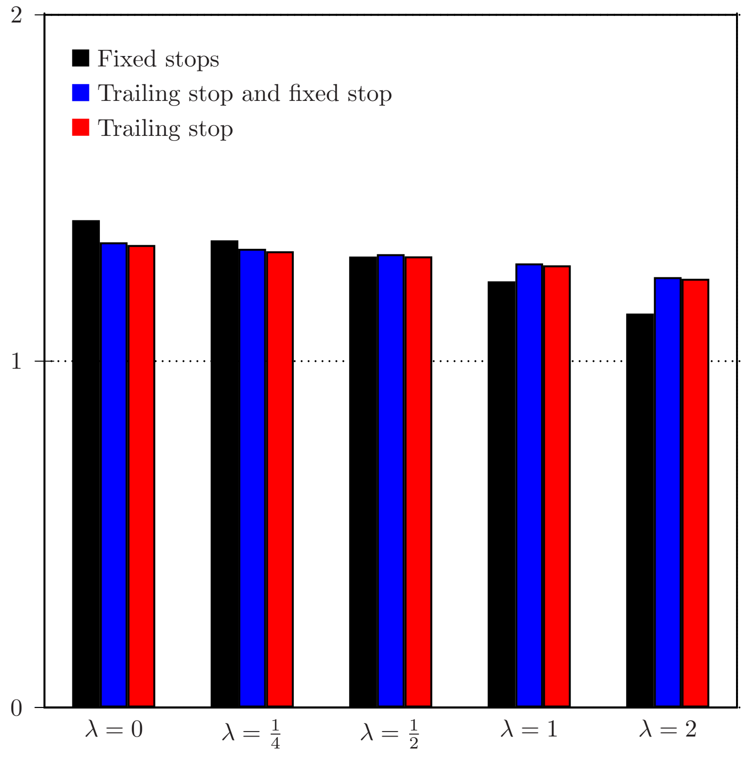

Figure 4.7: Objectives for different intensities λ.

5 Conclusion

In this study, several simple rules for placing stops were investigated. Their performances were compared by optimizing over an objective which was derived from a repeated trading activity. For the stopping rules considered, in case of an arithmetic Brownian motion and an exponential utility function, the value of the trading activity could be determined in closed form. For sake of comparison, a fixed revision rule and an optimal stopping problem were regarded.

It turned out that if the drift term is known and big enough, there is no reason to place a lower stop. The optimal choice is thus a fixed take-profit stop. When uncertainty over the growth rate was included to the model, the results were fundamentally different; the possibility that the drift might be small or even negative is what leads to put in a lower stop.

Hypothesizing the existence of many alternative investment opportunities, allows establishing a number of reallocation strategies which could be pursued when the position is stopped out. If the same investment is played forever, the two-stops rules do better than the alternatives as the take-profit stop is the crucial one. The uncertainty has caused a reduction in value.

If the investor starts with a new investment after being stopped out, this leads to stopping rules closest to the market wisdom of ‘run large gains and stop small losses’ with better values for that reallocation strategy. The investor can even profit from a high degree of uncertainty. The fixed-stops rule outperforms the trailing-stop rules as the latter ones might accidentally stop out a good investment.

However, the best reallocation strategy is to only shuffle investments when the position is closed out on the down side. Then, the fixed-stops rule and the mixed-stops rule are both well-suited.

The comparison to the optimal stopping problem shows that it is even better to use time-dependent stops. The stopping rules can be modified slightly by adding a time-dependent slope to the stops. If this is done, all results remain in closed form. Since the optimization is done over a larger class, the performance improves, but it improves by an appreciable margin. This means that the much simpler fixed-stops rule with a suitably-chosen slope is close to optimal. Trying to improve this rule by using a second slope and a vertical stop gets a little further but only at the expense of a higher computational complexity.

As the results of the fixed-stops rule were equal or better than the results of the mixed-stops rule, it seemed as if the fixed-stops rule should have been recommended. However, the trailing stop can guard against the possibility that the drift might deteriorate while the position is hold. To see this, the drift was assumed to change its value at random times. In situations where the drift gets worse, the trailing stop will stop out the position at a higher level. Thus, the trailing-stop-and-fixed-stop rule outperforms the fixed-stops rule if the frequency of changes is big enough.

6 Variations

6.1 General utility function

Instead of limiting the model to the exponential utility function (2.3), more general results can be obtained for an arbitrary utility. Then, the value of the trading activity is no longer given in terms of the joint Laplace transform as in (2.4); it is rather given by (2.2). Therefore, the quantity $\mathbf{E}[e^{-\rho T} f(X_T)]$, for a general function $f \in C^2$ has to be determined; The value (2.2) can then be computed by choosing $f(x) = U(x-c)$ and $f(x) = 1$, respectively. For the stopping examples of Section 2.2, the quantity can be obtained similarly to the joint Laplace transform.

Example 1: fixed stops

Going through the derivation in Subsection 2.5.1, (2.13) states that the desired quantity is given by $f(0)$ and this value is given in equation (2.18). The boundary conditions remain unspecified, $f(-a)$ and $f(b)$, which leads to

$$\mathbf{E}[e^{-\rho T} f(X_T)] = \frac{f(-a)(e^{\beta b} - e^{-\alpha b}) + f(b)(e^{\alpha a} - e^{-\beta a})}{e^{\alpha a + \beta b} - e^{-\alpha b - \beta a}}.$$

Example 2: trailing stop and fixed stop

In Subsection 2.5.2, the desired expectation is already given in equation (2.27).

$$\mathbf{E}[e^{-\rho T} f(X_T)] = f(b)\, e^{-\nu b} + n(B) \int_0^b f(-a+y)\, e^{-\nu y} \mathrm{d}y,$$

with $n(B)$ and ν as in (2.32) and (2.33), respectively. However, for a general utility function, an explicit solution is not guaranteed; the integral might only be solved numerically.

Example 3: trailing stop

The result for this case is given by setting $b = \infty$, which gives

$$\mathbf{E}[e^{-\rho T} f(X_T)] = n(B) \int_0^\infty f(-a+y)\, e^{-\nu y} \mathrm{d}y.$$

6.2 Geometric Brownian motion

In this setting, the value of the investment is supposed to be

$$S_t \equiv S_0 e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma W_t},$$

where W is a standard Brownian motion, and μ and $\sigma > 0$ are constants. Defining $\mu_X \equiv \mu - \frac{1}{2}\sigma^2$, then the process S can be written as

$$S_t = S_0 e^{X_t},$$

where $X_t = \mu_X t + \sigma W_t$ is an arithmetic Brownian motion. Then, by a slight modification of the stopping examples, the results obtained for the arithmetic Brownian motion can be used to determine the same values in case of a geometric Brownian motion.

Example 1: fixed stops

For a lower stop $a \in (0, S_0)$ and an upper stop $b > S_0$, the stopping time is defined by

$$T \equiv \inf\{t : S_t = a \text{ or } S_t = b\}.$$

This stopping time is equivalent to the fixed-stops stopping time of the arithmetic Brownian motion,

$$T = \inf\{t : X_t = -a_X \text{ or } X_t = b_X\},$$

with $a_X \equiv -\ln(\frac{a}{S_0}) > 0$ and $b_X \equiv \ln(\frac{b}{S_0}) > 0$.

Example 2: trailing stop and fixed stop

Let $p \in (0,1)$, $b > S_0$ and $\bar{S}_t \equiv \sup_{0 \le s \le t} S_s$ be the running maximum, then the stopping time is defined by a proportional trailing stop and a fixed take-profit stop,

$$T \equiv \inf\{t : S_t = p\bar{S}_t \text{ or } S_t = b\}.$$

By the monotonicity of the exponential function, this stopping time is equivalent to the trailing-stop-and-fixed-stop stopping time of the arithmetic Brownian motion,

$$T = \inf\{t : X_t = -a_X + \bar{X}_t \text{ or } X_t = b_X\},$$

with $a_X \equiv -\ln(p) > 0$ and $b_X \equiv \ln(\frac{b}{S_0}) > 0$.

Example 3: trailing stop

Accordingly, the special case of Example 2 is given by the stopping time

$$T \equiv \inf\{t : S_t = p\bar{S}_t\},$$

which is equivalent to

$$T = \inf\{t : X_t = -a_X + \bar{X}_t\},$$

with $a_X \equiv -\ln(p) > 0$.
A more extensive overview of the results for the trailing stop in the geometric Brownian motion model (and fractional Brownian motion) is given in [1].

The time-0 value of the repeated trading activity is given by

$$\varphi = \frac{\mathbf{E}[e^{-\rho T}\, U(S_T - S_0 - c)]}{1 - \mathbf{E}[e^{-\rho T}]}.$$

In the risk-neutral case of a linear[6] utility function, the value transforms to

$$\begin{aligned}
\varphi &= \frac{\mathbf{E}[e^{-\rho T}\,(S_T - S_0 - c)]}{1 - \mathbf{E}[e^{-\rho T}]} \\
&= \frac{S_0\,\mathbf{E}[e^{-\rho T + X_T}] - (S_0 + c)\,\mathbf{E}[e^{-\rho T}]}{1 - \mathbf{E}[e^{-\rho T}]} \\
&= \frac{S_0\,L(\rho, -1) - (S_0 + c)\,L(\rho, 0)}{1 - L(\rho, 0)}.
\end{aligned}$$

Hence, the value is given in terms of the joint Laplace transform for an arithmetic Brownian motion, which is known. Although this value can be computed explicitly, in most cases, this model leads to trivial solutions; the optimal strategy is either never to

[6] Other utility functions are given by the results of Section 6.1.

exit the position ($\mu > \rho$), or to exit immediately ($\mu < \rho$). This coincides with what is found in [17].

A Crank-Nicolson finite-difference scheme

In the sequel it will be shown how to calculate the value $\bar{\varphi}_{n+1}$ with $\bar{\varphi}_n$ given. The gain process satisfies the diffusion equation $\mathrm{d}X_t = \mu(t, X_t)\mathrm{d}t + \sigma \mathrm{d}W_t$ with drift process $\mu(t, X_t)$ which might be constant in the known drift case or as in (3.6) for the Bayesian strategy. The stopping reward process is of the form

$$\begin{aligned} Z(t, X_t) &= e^{-\rho t}(U(X_t - c) + \bar{\varphi}_n) \\ &\equiv e^{-\rho t} g(t, X_t), \end{aligned}$$

where $\rho \geq 0$. Fix a final time $\bar{T}$ which should be large enough to either be outside of the continuation region or irrelevant and define the value function

$$V(t, x) \equiv \sup_{t \leq \tau \leq \bar{T}} E[e^{-\rho(\tau - t)} g(\tau, X_\tau) | X_t = x]. \tag{A.1}$$

Then it holds $V \geq g$ everywhere, and that

$$\mathcal{L}V + V_t - \rho V \leq 0 \tag{A.2}$$

which holds with equality when $V > g$, where $\mathcal{L}$ is the generator of the diffusion,

$$\mathcal{L} \equiv \tfrac{1}{2}\sigma^2 \frac{\mathrm{d}^2}{\mathrm{d}x^2} + \mu(t, X_t)\frac{\mathrm{d}}{\mathrm{d}x}.$$

This problem can be solved numerically by using the Crank-Nicolson finite-difference scheme. See [4] for the original paper or [23] for a more general description. Set down a grid of x-values and a grid $0 = t_0 < t_1 < \ldots < t_N = \bar{T}$ of time values, and let $L^{(n)}$ be a discrete approximation of the diffusion generator at $t = t_n$. If $v^{(n)}$ denotes the approximation of the value function at time t_n, then the Crank-Nicolson method approximates (A.2) by

$$\tfrac{1}{2}\{L^{(n)}v^{(n)} + L^{(n+1)}v^{(n+1)}\} - \tfrac{1}{2}\rho(v^{(n)} + v^{(n+1)}) + \Delta t_n^{-1}(v^{(n+1)} - v^{(n)}) \leq 0, \tag{A.3}$$

with equality where it is optimal to continue. Here, it is $\Delta t_n = t_{n+1} - t_n$. The unknown in this equation is $v^{(n)}$ which can be obtained by working recursively back through the grid in the usual dynamic-programming fashion; starting with $v^{(N)}(x) = g(\bar{T}, x)$, since, by assumption, the final time point $\bar{T}$ is outside the continuation region.
Rewriting (A.3) to make the unknown the subject gives

$$\begin{aligned}(L^{(n)} - \rho - 2\Delta t_n^{-1})v^{(n)} &\leq -(L^{(n+1)} - \rho + 2\Delta t_n^{-1})v^{(n+1)} \\ &\equiv -\alpha^{(n)},\end{aligned} \tag{A.4}$$

say, with equality at all places where it is optimal to continue, and with $v^{(n)}(x) = g(t_n, x)$ in places where it is optimal to stop.
However, the problem (A.4) is an optimal stopping problem for the Markov chain with generator[7] $L^{(n)}$, with discount rate $\rho + 2\Delta t_n^{-1}$, and running reward $\alpha^{(n)}$. It is quite straightforward (and very fast) to solve this by policy improvement [18]. Probably the simplest thing to do at the boundaries is to insist that the process gets absorbed there, so in the original stopping problem, it has to be stopped when one end or the other end of the x-grid is reached.

[7]With a three-point finite difference scheme, the matrix $L^{(n)}$ will usually be a Q-matrix; the calculations need to check this.

B Proofs of the lemmas

B.1 Proof of Lemma 1

Proof of Lemma 1: The tower property and the restriction $Y_\tau \in (-a, b)$, which is obsolete for $T > \tau$ give

$$\begin{aligned}\mathbf{E}[e^{-\rho\tau+\delta X_\tau} : T > \tau] &= \mathbf{E}[e^{-\rho\tau+\delta X_\tau}\, \mathbf{E}[\mathbf{1}_{\{T>\tau\}}|\tau, X_\tau]] \\ &= \mathbf{E}[e^{-\rho\tau+\delta X_\tau}\, \mathbf{P}(T > \tau|\tau, X_\tau)] \\ &= \mathbf{E}[e^{-\rho\tau+\delta X_\tau}\, \mathbf{1}_{\{X_\tau\in(-a,b)\}}\,(1 - \mathbf{P}(T < \tau|\tau, X_\tau))]. \end{aligned} \tag{B.1}$$

For $X_\tau = x \in (-a, b)$, the probability in (B.1) will be transformed into a function $f(\tau, X_\tau)$ such that the above expectation only depends on the random variables τ and X_τ. This transformation can be carried out by using a result for a standard Brownian motion. The process X conditional on $X_\tau = x$ for some x is a generalized Brownian bridge whose distribution does not depend on the drift. Thus, for the calculation of this probability, the drift can be assumed to be 0 which yields a process $X_t = \sigma W_t$. Then, the probability can be expressed in terms of the standard Brownian motion W,

$$\mathbf{P}(T < \tau|X_\tau = x) = \mathbf{P}(T_W < \tau|W_\tau = \tfrac{x}{\sigma}),$$

where τ is treated as a given constant and $T_W \equiv \inf\{t : W_t = \frac{b}{\sigma} \text{ or } W_t = \frac{-a}{\sigma}\}$. Then, the partial probability for stopping at the high side is given by equation (15) of [7],

$$\mathbf{P}(T_W < \tau, H|W_\tau = \tfrac{x}{\sigma}) = \sum_{j=1}^{\infty} e^{\frac{2}{\tau\sigma^2}(j(a+b)-a)(x-(j(a+b)-a))} - e^{\frac{2}{\tau\sigma^2}j(a+b)(x-j(a+b))}.$$

As a Brownian motion without drift is symmetric, the probability of hitting the down side first, given $W_\tau = \frac{x}{\sigma}$ is equal to hitting an upper stop $\frac{a}{\sigma}$ before the lower one $\frac{-b}{\sigma}$, given $W_\tau = \frac{-x}{\sigma}$. Thus, the second partial probability can be obtained by equation (15) of [7] as well by switching the boundaries and the sign of x, giving

$$\mathbf{P}(T_W < \tau, H^c|W_\tau = \tfrac{x}{\sigma}) = \sum_{j=1}^{\infty} e^{\frac{2}{\tau\sigma^2}(j(a+b)-b)(-x-(j(a+b)-b))} - e^{\frac{2}{\tau\sigma^2}j(a+b)(-x-j(a+b))}.$$

Then, the desired probability is the sum of the partial ones,

$$\mathbf{P}(T<\tau|X_\tau) = \sum_{j=1}^{\infty} e^{\frac{2}{\tau\sigma^2}(j(a+b)-a)(X_\tau-(j(a+b)-a))} - e^{\frac{2}{\tau\sigma^2}j(a+b)(X_\tau-j(a+b))} + e^{\frac{2}{\tau\sigma^2}(j(a+b)-b)(-X_\tau-(j(a+b)-b))} - e^{\frac{2}{\tau\sigma^2}j(a+b)(-X_\tau-j(a+b))}.$$

To shorten notation, let $h_1 \equiv 1$, $h_2 \equiv -1$, $h_3 \equiv 1$, $h_4 \equiv -1$ and

$$\begin{aligned}
\psi_0(j,x) &\equiv 0\\
\psi_1(j,x) &\equiv (j(a+b)-a)(x-(j(a+b)-a))\\
\psi_2(j,x) &\equiv j(a+b)(x-j(a+b))\\
\psi_3(j,x) &\equiv (j(a+b)-b)(-x-(j(a+b)-b))\\
\psi_4(j,x) &\equiv j(a+b)(-x-j(a+b)),
\end{aligned}$$

then it holds

$$1-\mathbf{P}(T<\tau|X_\tau) = \left(\sum_{i=0}^{0} - \sum_{i=1}^{4} h_i \sum_{j=1}^{\infty}\right) e^{\frac{2}{\tau\sigma^2}\psi_i(j,X_\tau)}.$$

With the probability as requested, the expectation in (B.1) now only depends on the random variables τ and X_τ. By definition, τ is exponentially distributed with intensity λ. Given τ, then X_τ is normally distributed with mean $\mu_k\tau$ and variance $\sigma^2\tau$. Therefore, the corresponding densities can be used to calculate the expectation,

$$\begin{aligned}
&\mathbf{E}[e^{-\rho\tau+\delta X_\tau} : T>\tau]\\
&= \int_0^\infty \int_{-a}^{b} \lambda e^{-\lambda t} \frac{1}{\sqrt{2\pi\sigma^2 t}} e^{-\frac{(x-\mu_k t)^2}{2\sigma^2 t}} e^{-\rho t} e^{\delta x} \left[\left(\sum_{i=0}^{0} - \sum_{i=1}^{4} h_i \sum_{j=1}^{\infty}\right) e^{\frac{2}{\tau\sigma^2}\psi_i(j,X_\tau)}\right] \mathrm{d}x\,\mathrm{d}t\\
&= \left(\sum_{i=0}^{0} - \sum_{i=1}^{4} h_i \sum_{j=1}^{\infty}\right) \int_{-a}^{b}\int_0^\infty \lambda e^{-\lambda t} \frac{1}{\sqrt{2\pi\sigma^2 t}} e^{-\frac{\frac{x^2}{2\sigma^2}}{t}+\frac{\mu_k}{\sigma^2}x-\frac{\mu_k^2}{2\sigma^2}t} e^{-\rho t} e^{\delta x} e^{\frac{2}{\tau\sigma^2}\psi_i(j,X_\tau)} \mathrm{d}t\,\mathrm{d}x\\
&= \left(\sum_{i=0}^{0} - \sum_{i=1}^{4} h_i \sum_{j=1}^{\infty}\right) \frac{\lambda}{\sqrt{2\pi\sigma^2}} \int_{-a}^{b} e^{(\delta+\frac{\mu_k}{\sigma^2})x} \int_0^\infty e^{-(\rho+\lambda+\frac{\mu_k^2}{2\sigma^2})t} \frac{1}{\sqrt{t}} e^{-\frac{\frac{x^2}{2\sigma^2}-\frac{2}{\sigma^2}\psi_i(j,x)}{t}} \mathrm{d}t\,\mathrm{d}x\\
&= \left(\sum_{i=0}^{0} - \sum_{i=1}^{4} h_i \sum_{j=1}^{\infty}\right) \frac{\lambda}{\sqrt{2\pi\sigma^2}} \int_{-a}^{b} e^{(\delta+\frac{\mu_k}{\sigma^2})x} \int_0^\infty e^{-\kappa t} \frac{1}{\sqrt{t}} e^{-\frac{\xi_i(j,x)}{t}} \mathrm{d}t\,\mathrm{d}x, \qquad \text{(B.2)}
\end{aligned}$$

where the last equation uses the notations

$$\kappa \equiv \rho + \lambda + \tfrac{\mu_k^2}{2\sigma^2};$$
$$\xi_i(j,x) \equiv \tfrac{x^2}{2\sigma^2} - \tfrac{2}{\sigma^2}\psi_i(j,x).$$

Recognizing that $\kappa > 0$ as well as

$$\begin{aligned}
\xi_0(j,x) &= \tfrac{1}{2\sigma^2}\, x^2 \\
\xi_1(j,x) &= \tfrac{1}{2\sigma^2}(x - 2(j(a+b)-a))^2 \\
\xi_2(j,x) &= \tfrac{1}{2\sigma^2}(x - 2j(a+b))^2 \\
\xi_3(j,x) &= \tfrac{1}{2\sigma^2}(x + 2(j(a+b)-b))^2 \\
\xi_4(j,x) &= \tfrac{1}{2\sigma^2}(x + 2j(a+b))^2
\end{aligned}$$

are non-negative, then the integral in (B.2) has the solution[8]

$$\int_0^\infty e^{-\kappa t}\, \tfrac{1}{\sqrt{t}}\, e^{-\frac{\xi_i(j,x)}{t}} \mathrm{d}t = \tfrac{\sqrt{\pi}}{\sqrt{\kappa}}\, e^{-2\sqrt{\kappa}\sqrt{\xi_i(j,x)}}.$$

Substituting this into equation (B.2) gives

$$\mathbf{E}[e^{-\rho\tau+\delta X_\tau} : T > \tau] = \left(\sum_{i=0}^{0} - \sum_{i=1}^{4} h_i \sum_{j=1}^{\infty}\right) \frac{\lambda}{\sqrt{2\sigma^2\kappa}} \int_{-a}^{b} e^{(\delta+\frac{\mu_k}{\sigma^2})x - 2\sqrt{\kappa}\sqrt{\xi_i(j,x)}} \mathrm{d}x.$$

Using the quadratic representation of $\xi_i(j,x)$ and reinserting h_i for $i = 0,\ldots,4$ give

$$\begin{aligned}
\mathbf{E}[e^{-\rho\tau+\delta X_\tau} : T > \tau] = \tfrac{\lambda}{\sqrt{2\sigma^2\kappa}} \Big[& \int_{-a}^{b} e^{(\delta+\frac{\mu_k}{\sigma^2})x - \frac{\sqrt{2\kappa}}{\sigma}|x|} \mathrm{d}x \\
& - \sum_{j=1}^{\infty} \int_{-a}^{b} e^{(\delta+\frac{\mu_k}{\sigma^2})x - \frac{\sqrt{2\kappa}}{\sigma}|x-2(j(a+b)-a)|} \mathrm{d}x \\
& + \sum_{j=1}^{\infty} \int_{-a}^{b} e^{(\delta+\frac{\mu_k}{\sigma^2})x - \frac{\sqrt{2\kappa}}{\sigma}|x-2j(a+b)|} \mathrm{d}x \\
& - \sum_{j=1}^{\infty} \int_{-a}^{b} e^{(\delta+\frac{\mu_k}{\sigma^2})x - \frac{\sqrt{2\kappa}}{\sigma}|x+2(j(a+b)-b)|} \mathrm{d}x \\
& + \sum_{j=1}^{\infty} \int_{-a}^{b} e^{(\delta+\frac{\mu_k}{\sigma^2})x - \frac{\sqrt{2\kappa}}{\sigma}|x+2j(a+b)|} \mathrm{d}x \Big].
\end{aligned}$$

[8]The calculations were carried out by a symbolic mathematics package.

It is possible to get rid of the absolute values by splitting the first integral into two integrals and recognizing that for $j \geq 1$ and $x \in (-a, b)$ the expressions $x-2(j(a+b)-a)$ and $x-2j(a+b)$ are negative whereas $x+2(j(a+b)-b)$ and $x+2j(a+b)$ are positive. These considerations lead to

$$\begin{aligned}
\mathbf{E}[e^{-\rho\tau+\delta X_\tau} : T > \tau] &= \frac{\lambda}{\sqrt{2\sigma^2\kappa}}\Big[\int_{-a}^{0} e^{(\delta+\frac{\mu_k}{\sigma^2})x+\frac{\sqrt{2\kappa}}{\sigma}x}\mathrm{d}x + \int_{0}^{b} e^{(\delta+\frac{\mu_k}{\sigma^2})x-\frac{\sqrt{2\kappa}}{\sigma}x}\mathrm{d}x \\
&\quad - \sum_{j=1}^{\infty}\int_{-a}^{b} e^{(\delta+\frac{\mu_k}{\sigma^2})x+\frac{\sqrt{2\kappa}}{\sigma}(x-2(j(a+b)-a))}\mathrm{d}x \\
&\quad + \sum_{j=1}^{\infty}\int_{-a}^{b} e^{(\delta+\frac{\mu_k}{\sigma^2})x+\frac{\sqrt{2\kappa}}{\sigma}(x-2j(a+b))}\mathrm{d}x \\
&\quad - \sum_{j=1}^{\infty}\int_{-a}^{b} e^{(\delta+\frac{\mu_k}{\sigma^2})x-\frac{\sqrt{2\kappa}}{\sigma}(x+2(j(a+b)-b))}\mathrm{d}x \\
&\quad + \sum_{j=1}^{\infty}\int_{-a}^{b} e^{(\delta+\frac{\mu_k}{\sigma^2})x-\frac{\sqrt{2\kappa}}{\sigma}(x+2j(a+b))}\mathrm{d}x\Big] \\
&= \frac{\lambda}{\sqrt{2\sigma^2\kappa}}\Big[\int_{-a}^{0} e^{(\delta+\frac{\mu_k}{\sigma^2}+\frac{\sqrt{2\kappa}}{\sigma})x}\mathrm{d}x + \int_{0}^{b} e^{(\delta+\frac{\mu_k}{\sigma^2}-\frac{\sqrt{2\kappa}}{\sigma})x}\mathrm{d}x \\
&\quad - \sum_{j=1}^{\infty} e^{-\frac{\sqrt{8\kappa}}{\sigma}(j(a+b)-a)}\int_{-a}^{b} e^{(\delta+\frac{\mu_k}{\sigma^2}+\frac{\sqrt{2\kappa}}{\sigma})x}\mathrm{d}x \\
&\quad + \sum_{j=1}^{\infty} e^{-\frac{\sqrt{8\kappa}}{\sigma}j(a+b)}\int_{-a}^{b} e^{(\delta+\frac{\mu_k}{\sigma^2}+\frac{\sqrt{2\kappa}}{\sigma})x}\mathrm{d}x \\
&\quad - \sum_{j=1}^{\infty} e^{\frac{\sqrt{8\kappa}}{\sigma}(j(a+b)-b)}\int_{-a}^{b} e^{(\delta+\frac{\mu_k}{\sigma^2}-\frac{\sqrt{2\kappa}}{\sigma})x}\mathrm{d}x \\
&\quad + \sum_{j=1}^{\infty} e^{\frac{\sqrt{8\kappa}}{\sigma}j(a+b)}\int_{-a}^{b} e^{(\delta+\frac{\mu_k}{\sigma^2}-\frac{\sqrt{2\kappa}}{\sigma})x}\mathrm{d}x\Big].
\end{aligned}$$

Using the definition of κ, α_k and β_k gives

$$\begin{aligned}
\frac{\mu_k}{\sigma^2} - \frac{\sqrt{2\kappa}}{\sigma} &= \frac{\mu_k}{\sigma^2} - \frac{\sqrt{\mu_k^2+2\sigma^2(\rho+\lambda)}}{\sigma^2} \\
&= -\beta_k \\
\frac{\mu_k}{\sigma^2} + \frac{\sqrt{2\kappa}}{\sigma} &= \frac{\mu_k}{\sigma^2} + \frac{\sqrt{\mu_k^2+2\sigma^2(\rho+\lambda)}}{\sigma^2} \\
&= \alpha_k
\end{aligned}$$

$$\tfrac{\sqrt{8\kappa}}{\sigma} = \alpha_k + \beta_k,$$

which simplifies the above expectation to

$$\begin{aligned}
\mathbf{E}[e^{-\rho\tau+\delta X_\tau} : T > \tau] = {} & \tfrac{2\lambda}{(\alpha_k+\beta_k)\sigma^2}\Big[\int_{-a}^{0} e^{(\delta+\alpha_k)x}\mathrm{d}x + \int_0^b e^{(\delta-\beta_k)x}\mathrm{d}x \\
& - \sum_{j=1}^{\infty} e^{-(\alpha_k+\beta_k)(j(a+b)-a)} \int_{-a}^{b} e^{(\delta+\alpha_k)x}\mathrm{d}x \\
& + \sum_{j=1}^{\infty} e^{-(\alpha_k+\beta_k)j(a+b)} \int_{-a}^{b} e^{(\delta+\alpha_k)x}\mathrm{d}x \\
& - \sum_{j=1}^{\infty} e^{(\alpha_k+\beta_k)(j(a+b)-b)} \int_{-a}^{b} e^{(\delta-\beta_k)x}\mathrm{d}x \\
& + \sum_{j=1}^{\infty} e^{(\alpha_k+\beta_k)j(a+b)} \int_{-a}^{b} e^{(\delta-\beta_k)x}\mathrm{d}x\Big] \\
= {} & \tfrac{2\lambda}{(\alpha_k+\beta_k)\sigma^2}\Big[\tfrac{1-e^{-(\delta+\alpha_k)a}}{\delta+\alpha_k} + \tfrac{e^{(\delta-\beta_k)b}-1}{\delta-\beta_k} \\
& - \sum_{j=1}^{\infty} e^{-(\alpha_k+\beta_k)(j(a+b)-a)}\ \tfrac{e^{(\delta+\alpha_k)b}-e^{-(\delta+\alpha_k)a}}{\delta+\alpha_k} \\
& + \sum_{j=1}^{\infty} e^{-(\alpha_k+\beta_k)j(a+b)}\ \tfrac{e^{(\delta+\alpha_k)b}-e^{-(\delta+\alpha_k)a}}{\delta+\alpha_k} \\
& - \sum_{j=1}^{\infty} e^{(\alpha_k+\beta_k)(j(a+b)-b)}\ \tfrac{e^{(\delta-\beta_k)b}-e^{-(\delta-\beta_k)a}}{\delta-\beta_k} \\
& + \sum_{j=1}^{\infty} e^{(\alpha_k+\beta_k)j(a+b)}\ \tfrac{e^{(\delta-\beta_k)b}-e^{-(\delta-\beta_k)a}}{\delta-\beta_k}\Big] \\
= {} & \tfrac{2\lambda}{(\alpha_k+\beta_k)\sigma^2}\Big[\tfrac{1-e^{-(\delta+\alpha_k)a}}{\delta+\alpha_k} + \tfrac{e^{(\delta-\beta_k)b}-1}{\delta-\beta_k} \\
& + \tfrac{e^{(\delta+\alpha_k)b}-e^{-(\delta+\alpha_k)a}}{\delta+\alpha_k}\left(1-e^{(\alpha_k+\beta_k)a}\right)\sum_{j=1}^{\infty} e^{-(\alpha_k+\beta_k)j(a+b)} \\
& + \tfrac{e^{(\delta-\beta_k)b}-e^{-(\delta-\beta_k)a}}{\delta-\beta_k}\left(1-e^{(\alpha_k+\beta_k)b}\right)\sum_{j=1}^{\infty} e^{-(\alpha_k+\beta_k)j(a+b)}\Big]. \qquad \text{(B.3)}
\end{aligned}$$

As $(\alpha_k+\beta_k)(a+b) > 0$ and therefore $e^{(\alpha_k+\beta_k)(a+b)} > 1$, the sum in the equation above is a geometric series, so it holds

$$\sum_{j=1}^{\infty} e^{-(\alpha_k+\beta_k)j(a+b)} = \sum_{j=1}^{\infty} (e^{(\alpha_k+\beta_k)(a+b)})^{-j} = \frac{1}{e^{(\alpha_k+\beta_k)(a+b)}-1},$$

which transforms (B.3) into the desired result.

$$\begin{aligned}
\mathbf{E}[e^{-\rho\tau+\delta X_\tau} : T > \tau] &= \tfrac{2\lambda}{(\alpha_k+\beta_k)\sigma^2}\Big[\tfrac{1-e^{-(\delta+\alpha_k)a}}{\delta+\alpha_k} + \tfrac{e^{(\delta-\beta_k)b}-1}{\delta-\beta_k} \\
&\quad + \tfrac{e^{(\delta+\alpha_k)b}-e^{-(\delta+\alpha_k)a}}{\delta+\alpha_k}\,\tfrac{1-e^{(\alpha_k+\beta_k)a}}{e^{(\alpha_k+\beta_k)(a+b)}-1} \\
&\quad + \tfrac{e^{(\delta-\beta_k)b}-e^{-(\delta-\beta_k)a}}{\delta-\beta_k}\,\tfrac{1-e^{(\alpha_k+\beta_k)b}}{e^{(\alpha_k+\beta_k)(a+b)}-1}\Big] \\
&= \tfrac{2\lambda}{(\alpha_k+\beta_k)\sigma^2}\Big[\tfrac{1}{\delta+\alpha_k}\Big(1-e^{-(\delta+\alpha_k)a} + \tfrac{(e^{(\delta+\alpha_k)b}-e^{-(\delta+\alpha_k)a})(1-e^{(\alpha_k+\beta_k)a})}{e^{(\alpha_k+\beta_k)(a+b)}-1}\Big) \\
&\quad + \tfrac{1}{\delta-\beta_k}\Big(-1+e^{(\delta-\beta_k)b} + \tfrac{(e^{(\delta-\beta_k)b}-e^{-(\delta-\beta_k)a})(1-e^{(\alpha_k+\beta_k)b})}{e^{(\alpha_k+\beta_k)(a+b)}-1}\Big)\Big] \\
&= \tfrac{2\lambda}{(\alpha_k+\beta_k)\sigma^2}\Big[\tfrac{1}{\delta+\alpha_k}\Big(1-\tfrac{e^{-\delta a}(e^{\beta_k b}-e^{-\alpha_k b})+e^{\delta b}(e^{\alpha_k a}-e^{-\beta_k a})}{e^{\alpha_k a+\beta_k b}-e^{-\alpha_k b-\beta_k a}}\Big) \\
&\quad + \tfrac{1}{\delta-\beta_k}\Big(-1+\tfrac{e^{-\delta a}(e^{\beta_k b}-e^{-\alpha_k b})+e^{\delta b}(e^{\alpha_k a}-e^{-\beta_k a})}{e^{\alpha_k a+\beta_k b}-e^{-\alpha_k b-\beta_k a}}\Big)\Big] \\
&= \tfrac{2\lambda}{(\alpha_k+\beta_k)\sigma^2}\Big[\tfrac{1}{\delta+\alpha_k}\left(1-L_1^{\mu_k}(\rho+\lambda,-\delta)\right) \\
&\quad + \tfrac{1}{\delta-\beta_k}\left(-1+L_1^{\mu_k}(\rho+\lambda,-\delta)\right)\Big] \\
&= \tfrac{2\lambda}{(\alpha_k+\beta_k)\sigma^2}\Big[\left(L_1^{\mu_k}(\rho+\lambda,-\delta)-1\right)\tfrac{\alpha_k+\beta_k}{(\delta+\alpha_k)(\delta-\beta_k)}\Big] \\
&= \tfrac{2\lambda}{\sigma^2}\,\tfrac{L_1^{\mu_k}(\rho+\lambda,-\delta)-1}{(\delta+\alpha_k)(\delta-\beta_k)}
\end{aligned}$$

□

B.2 Proof of Lemma 2

Proof of Lemma 2: The quantity $\mathbf{E}_{\mu_k}[e^{-\rho\tau+\delta X_\tau-(\delta+\gamma)\bar{X}_\tau} : T > \tau]$ for a trailing stop and fixed stop will be calculated by using a technique of Lehoczky [14]. Therefore, the interval $[0,b]$ will be split into M equidistant subintervals of length $\epsilon = {}^{b}/_{M}$ with boundaries $0 = 0\epsilon < 1\epsilon < 2\epsilon < \ldots < M\epsilon = b$. Then, the trailing stop can be approximated by a piecewise constant and increasing lower stop which will be defined as $-a + \hat{X}$, where

$$\hat{X}_t \equiv \begin{cases} 0, & 0 \leq \bar{X}_t < \epsilon \\ \epsilon, & \epsilon \leq \bar{X}_t < 2\epsilon \\ \dots & \\ (M-1)\epsilon, & (M-1)\epsilon \leq \bar{X}_t < M\epsilon. \end{cases}$$

Let

$$E(m) \equiv \mathbf{E}_{\mu_k}[e^{-\rho\tau+\delta X_\tau-(\delta+\gamma)\bar{X}_\tau} : \hat{T}_m > \tau], \tag{B.4}$$

where $\hat{T}_m \equiv \inf\{t : X_t = m\epsilon \text{ or } X_t = -a + \hat{X}_t\}$, then $E(M)$ approximates the desired expectation for the trailing stop and fixed stop and letting M converge to infinity gives the result.

The first stopping time $\hat{T}_1$ has fixed stops ϵ and $-a$ and the expectation can be partitioned according to the two events $\hat{T}_1 < \tau$ and $\hat{T}_1 > \tau$.

$$\begin{aligned} E(M) = &\ \mathbf{E}_{\mu_k}[e^{-\rho\tau+\delta X_\tau-(\delta+\gamma)\bar{X}_\tau} : \hat{T}_M > \tau, \hat{T}_1 < \tau] \\ &+ \mathbf{E}_{\mu_k}[e^{-\rho\tau+\delta X_\tau-(\delta+\gamma)\bar{X}_\tau} : \hat{T}_M > \tau, \hat{T}_1 > \tau] \end{aligned} \tag{B.5}$$

For $\hat{T}_1 < \tau$, $\hat{T}_M > \tau$ ensures that the process stops on the high side $H = \{X_{\hat{T}_1} = \epsilon\}$. Defining $s_1 \equiv \tau - \hat{T}_1$, then the memorylessness property states that s_1 is exponentially distributed with parameter λ as well. Furthermore, by defining $Y_t \equiv X_{\hat{T}_1+t} - X_{\hat{T}_1}$, it is $X_\tau = X_{\hat{T}_1} + Y_{s_1} = \epsilon + Y_{s_1}$ and $\bar{X}_\tau = \bar{X}_{\hat{T}_1} + \bar{Y}_{s_1} = \epsilon + \bar{Y}_{s_1}$. Hence, the tower property with respect to the filtration $\mathcal{F}_{\hat{T}_1}$ can be applied. The remaining stopping time $\hat{T}_M - \hat{T}_1$ corresponds to $\hat{T}_{M-1}$. Thus, $\hat{T}_M > \tau$ is equivalent to the $\mathcal{F}_{\hat{T}_1}$-independent constraint $\hat{T}_{M-1} > s_1$.

$$\begin{aligned} &\mathbf{E}_{\mu_k}[e^{-\rho\tau+\delta X_\tau-(\delta+\gamma)\bar{X}_\tau} : \hat{T}_M > \tau, \hat{T}_1 < \tau] \\ &\quad = \mathbf{E}_{\mu_k}[\mathbf{E}_{\mu_k}[e^{-\rho(\hat{T}_1+s_1)+\delta(\epsilon+Y_{s_1})-(\delta+\gamma)(\epsilon+\bar{Y}_{s_1})} : \hat{T}_{M-1} > s_1, \hat{T}_1 < \tau, H|\mathcal{F}_{\hat{T}_1}]] \\ &\quad = e^{-\gamma\epsilon}\,\mathbf{E}_{\mu_k}[e^{-\rho\hat{T}_1}\mathbf{E}_{\mu_k}[e^{-\rho s_1+\delta Y_{s_1}-(\delta+\gamma)\bar{Y}_{s_1}} : \hat{T}_{M-1} > s_1] : \hat{T}_1 < \tau, H] \end{aligned} \tag{B.6}$$

As s_1 is exponentially distributed, the inner expectation of (B.6) can be substituted by using the definition in (B.4) and the remaining expectation can be computed by changing the rate of discounting and reading off the partial Laplace transform from (2.18) with boundary conditions $f(-a) = 0$ and $f(\epsilon) = 1$.

$$
\begin{aligned}
\mathbf{E}_{\mu_k}[e^{-\rho\tau+\delta X_\tau-(\delta+\gamma)\bar{X}_\tau} : \hat{T}_M > \tau, \hat{T}_1 < \tau] &= e^{-\gamma\epsilon}\,\mathbf{E}_{\mu_k}[e^{-\rho\hat{T}_1} : \hat{T}_1 < \tau, H]\,E(M-1) \\
&= e^{-\gamma\epsilon}\,\mathbf{E}_{\mu_k}[e^{-(\rho+\lambda)\hat{T}_1} : H]\,E(M-1) \\
&= e^{-\gamma\epsilon}\,\tfrac{e^{\alpha_k a}-e^{-\beta_k a}}{e^{\alpha_k a+\beta_k\epsilon}-e^{-\alpha_k\epsilon-\beta_k a}}\,E(M-1) \qquad \text{(B.7)}
\end{aligned}
$$

Defining

$$
A_\epsilon \equiv e^{-\gamma\epsilon}\,\tfrac{e^{\alpha_k a}-e^{-\beta_k a}}{e^{\alpha_k a+\beta_k\epsilon}-e^{-\alpha_k\epsilon-\beta_k a}}, \tag{B.8}
$$

then (B.7) can be written in shorthand notation

$$
\mathbf{E}_{\mu_k}[e^{-\rho\tau+\delta X_\tau-(\delta+\gamma)\bar{X}_\tau} : \hat{T}_M > \tau, \hat{T}_1 < \tau] = A_\epsilon\,E(M-1). \tag{B.9}
$$

For $\hat{T}_1 > \tau$ in the other expectation of (B.5), the constraint $\hat{T}_M > \tau$ is redundant. Furthermore, $\bar{X}_\tau \in (0,\epsilon)$. Hence, the functional $e^{-(\delta+\gamma)\bar{X}_\tau}$ is bounded by 1 from above and by $e^{-(\delta+\gamma)\epsilon}$ from below. Thus, the expectation can be estimated from above by

$$
\mathbf{E}_{\mu_k}[e^{-\rho\tau+\delta X_\tau-(\delta+\gamma)\bar{X}_\tau} : \hat{T}_M > \tau, \hat{T}_1 > \tau] < \mathbf{E}_{\mu_k}[e^{-\rho\tau+\delta X_\tau} : \hat{T}_1 > \tau] \tag{B.10}
$$

and from below by

$$
\mathbf{E}_{\mu_k}[e^{-\rho\tau+\delta X_\tau-(\delta+\gamma)\bar{X}_\tau} : \hat{T}_M > \tau, \hat{T}_1 > \tau] > e^{-(\delta+\gamma)\epsilon}\,\mathbf{E}_{\mu_k}[e^{-\rho\tau+\delta X_\tau} : \hat{T}_1 > \tau]. \tag{B.11}
$$

The remaining expectation in (B.10) and (B.11) will be abbreviated by

$$
B_\epsilon \equiv \mathbf{E}_{\mu_k}[e^{-\rho\tau+\delta X_\tau} : \hat{T}_1 > \tau], \tag{B.12}
$$

and it is given by Lemma 1 because $\hat{T}_1$ is a fixed-stops stopping time and the resulting joint Laplace transform L_1 can be read off (2.19),

$$
\begin{aligned}
B_\epsilon &= \tfrac{2\lambda}{\sigma^2}\,\tfrac{L_1^{\mu_k}(\rho+\lambda,-\delta)-1}{(\delta+\alpha_k)(\delta-\beta_k)} \\
&= \tfrac{2\lambda}{\sigma^2}\,\frac{\tfrac{e^{-\delta a}(e^{\beta_k\epsilon}-e^{-\alpha_k\epsilon})+e^{\delta\epsilon}(e^{\alpha_k a}-e^{-\beta_k a})}{e^{\alpha_k a+\beta_k\epsilon}-e^{-\alpha_k\epsilon-\beta_k a}}-1}{(\delta+\alpha_k)(\delta-\beta_k)}.
\end{aligned}
$$

Reinserting into the inequality in (B.10) and using the equation in (B.9) gives an upper bound for the expression in (B.5),

$$
E(M) < B_\epsilon + A_\epsilon E(M-1). \tag{B.13}
$$

Similarly, by defining

$$C_\epsilon \equiv e^{-(\delta+\gamma)\epsilon}\, B_\epsilon, \tag{B.14}$$

which can be inserted into the inequality in (B.11), leads to a lower bound for the expression in (B.5),

$$E(M) > C_\epsilon + A_\epsilon E(M-1). \tag{B.15}$$

With $A_\epsilon > 0$, the inequality in (B.13) can be applied recursively up to $E(0) = 0$, giving

$$\begin{aligned} E(M) &< B_\epsilon + A_\epsilon E(M-1) \\ &< B_\epsilon + A_\epsilon(B_\epsilon + A_\epsilon E(M-2)) \\ &\dots \\ &< B_\epsilon \sum_{i=0}^{M-1} A_\epsilon^i \\ &= B_\epsilon \tfrac{1-A_\epsilon^M}{1-A_\epsilon} \\ &= (1-A_\epsilon^M)\, \tfrac{B_\epsilon}{1-A_\epsilon} \end{aligned} \tag{B.16}$$

and analogously the inequality (B.15) with definition (B.14) give

$$\begin{aligned} E(M) &> (1-A_\epsilon^M)\, \tfrac{C_\epsilon}{1-A_\epsilon} \\ &= (1-A_\epsilon^M)\, \tfrac{B_\epsilon}{1-A_\epsilon}\, e^{-(\delta+\gamma)\epsilon}. \end{aligned} \tag{B.17}$$

As stated above, $\epsilon = {}^{b}\!/\!{}_{M}$ and the desired expectation can be obtained by letting M converge to infinity. Therefore, the limits of the following converging parts have to be calculated. Using the definition in (B.8) and a symbolic mathematics package give

$$\begin{aligned} \lim_{M\to\infty} A_\epsilon^M &= \lim_{M\to\infty} \left(e^{-\gamma \frac{b}{M}}\, \frac{e^{\alpha_k a} - e^{-\beta_k a}}{e^{\alpha_k a + \beta_k \frac{b}{M}} - e^{-\alpha_k \frac{b}{M} - \beta_k a}} \right)^M \\ &= e^{-\gamma b} \lim_{M\to\infty} \left(\frac{e^{\alpha_k a} - e^{-\beta_k a}}{e^{\alpha_k a + \beta_k \frac{b}{M}} - e^{-\alpha_k \frac{b}{M} - \beta_k a}} \right)^M \\ &= e^{-\gamma b}\, e^{-\frac{b(\beta_k e^{\alpha_k a} + \alpha_k e^{-\beta_k a})}{e^{\alpha_k a} - e^{-\beta_k a}}} \\ &= e^{-(\nu_k+\gamma)b}, \end{aligned}$$

with ν_k as in (4.17). Using the definitions in (B.8) and (B.12) with a symbolic mathematics package yields

$$
\begin{aligned}
\lim_{M\to\infty} \frac{B_\epsilon}{1-A_\epsilon} &= \frac{2\lambda}{\sigma^2}\,\frac{1}{(\delta+\alpha_k)(\delta-\beta_k)}\,\lim_{M\to\infty} \frac{\frac{e^{-\delta a}\left(e^{\beta_k \frac{b}{M}}-e^{-\alpha_k \frac{b}{M}}\right)+e^{\delta \frac{b}{M}}\left(e^{\alpha_k a}-e^{-\beta_k a}\right)}{e^{\alpha_k a+\beta_k \frac{b}{M}}-e^{-\alpha_k \frac{b}{M}-\beta_k a}}-1}{1-e^{-\gamma \frac{b}{M}}\,\frac{e^{\alpha_k a}-e^{-\beta_k a}}{e^{\alpha_k a+\beta_k \frac{b}{M}}-e^{-\alpha_k \frac{b}{M}-\beta_k a}}} \\
&= \frac{2\lambda}{\sigma^2}\,\frac{1}{(\delta+\alpha_k)(\delta-\beta_k)}\,\frac{-(\beta_k e^{\alpha_k a}+\alpha_k e^{-\beta_k a})+\delta(e^{\alpha_k a}-e^{-\beta_k a})+(\alpha_k+\beta_k)e^{-\delta a}}{(\beta_k e^{\alpha_k a}+\alpha_k e^{-\beta_k a})+\gamma(e^{\alpha_k a}-e^{-\beta_k a})} \\
&= \frac{2\lambda}{\sigma^2}\,\frac{1}{(\delta+\alpha_k)(\delta-\beta_k)}\,\frac{-\nu_k+\delta+n_k(B)e^{-\delta a}}{\nu_k+\gamma} \\
&= \frac{2\lambda}{\sigma^2}\,\frac{1}{(\delta+\alpha_k)(\delta-\beta_k)}\,\frac{r_k(-\delta)}{\nu_k+\gamma},
\end{aligned}
$$

where ν_k, $n_k(B)$ and r_k are as in (4.17), (4.18) and (4.19), respectively. As

$$\lim_{M\to\infty} e^{-(\delta+\gamma)\frac{b}{M}} = 1,$$

the upper and lower limits are equal and it is

$$
\begin{aligned}
\mathbf{E}_{\mu_k}[e^{-\rho\tau+\delta X_\tau-(\delta+\gamma)\bar{X}_\tau} : T > \tau] &= \lim_{M\to\infty} E(M) \\
&= (1-e^{-(\nu_k+\gamma)b})\,\frac{2\lambda}{\sigma^2}\,\frac{1}{(\delta+\alpha_k)(\delta-\beta_k)}\,\frac{r_k(-\delta)}{\nu_k+\gamma},
\end{aligned}
$$

which is the claim of the lemma.

$\square$

Bibliography

[1] Abramov, V. *Stopping times related to trading strategies*. PhD thesis, Kent State University, 2008.

[2] Anderson, T. W. A modification of the sequential probability ratio test to reduce the sample size. *The Annals of Mathematical Statistics 31*, 1 (1960), 165–197.

[3] Bawa, V. S., Brown, S. J., and Klein, R. W. *Estimation risk and optimal portfolio choice*. North-Holland, Amsterdam and New York, 1979.

[4] Crank, J., and Nicolson, P. A practical method for numerical evaluation of solutions of partial differential equations of the heat-conduction type. In *Proc. Cambridge Phil. Soc.* (1947), vol. 43, pp. 50–67.

[5] Dixit, A. K., and Pindyck, R. S. *Investment under Uncertainty*. Princeton University Press, 1994.

[6] Glynn, P. W., and Iglehart, D. L. Trading securities using trailing stops. *Managm. Sci. 41*, 6 (1995), 1096–1106.

[7] Hall, W. J. The distribution of Brownian motion on linear stopping boundaries. *Sequential analysis 16*, 4 (1997), 345–352.

[8] Harrison, J. M. *Brownian motion and stochastic flow systems*. Kreiger Publishing Company, 1990.

[9] Hoff, P. D. *A first course in Bayesian statistical methods*. Springer, 2009.

[10] Itô, K. Poisson point processes attached to markov processes. In *Proc. Sixth Berkeley Symp. Math. Statist. Probab.* (1971), vol. 3, Univ. California Press, Berkeley, pp. 225–240.

[11] Jacod, J., and Protter, P. *Probability Essentials*, 2nd ed. Springer, 2004.

[12] Karatzas, I., and Shreve, S. *Brownian Motion and Stochastic Calculus*. Springer, 1991.

[13] Katz, J. O., and McCormick, D. L. *The encyclopedia of trading strategies.* McGraw-Hill New York, 2000.

[14] Lehoczky, J. P. Formulas for stopped diffusion processes with stopping times based on the maximum. *Ann. Probab. 5*, 4 (1977), 601–607.

[15] Malkiel, B. *A Random Walk Down Wall Street: The Time-Tested Strategy for Successful Investing*, 10th ed. W. W. Norton, 2011.

[16] Øksendal, B. K. *Stochastic differential equations: an introduction with applications*, 6th ed. Springer, 2010.

[17] Radner, R., and Shepp, L. Risk vs. profit potential: A model for corporate strategy. *Journal of Economic Dynamics and Control 20* (1996), 1373–1393.

[18] Reisinger, C., and Witte, J. H. On the use of policy iteration as an easy way of pricing american options. *SIAM Journal on Financial Mathematics 3*, 1 (2012), 459–478.

[19] Rogers, L. C. G. A guided tour through excursions. *Bull. London Math. Soc. 21*, 4 (1989), 305–341.

[20] Rogers, L. C. G., and Williams, D. *Diffusions, Markov processes and martingales. Vol. 2: Itô calculus.* Cambridge Univ. Press, 2000.

[21] Taylor, H. M. A stopped brownian motion formula. *Ann. Probab. 3*, 2 (1975), 234–246.

[22] Williams, D. *Diffusions, Markov processes and Martingales*, vol. 1. Wiley, Chichester, 1979.

[23] Wilmott, P. *Paul Wilmott on quantitative finance*, vol. 2. John Wiley & Sons, 2006.

Scientific and professional career

08/1995 - 06/2004	School education at Cecilien-Gymnasium in Düsseldorf Degree: general higher education entrance qualification
05/2002 - 08/2002	Temporary employee at AEGON LV AG in Düsseldorf
10/2004 - 05/2009	Study of Economathematics with a specialization in Financial Mathematics at the University of Kaiserslautern Degree: Diploma (Dipl.-Math. oec.)
11/2008 - 03/2009	Research assistant at the University of Kaiserslautern
07/2009 - 09/2009	Research assistant at the University of Kaiserslautern
10/2009 - 07/2013	Doctoral studies in Mathematics at the University of Kaiserslautern
11/2009 - 12/2010	Research visit at the University of Cambridge, UK
since 07/2011	Research assistant at the Fraunhofer Institute for Industrial Mathematics (ITWM) in Kaiserslautern

Wissenschaftlicher und beruflicher Werdegang

08/1995 - 06/2004	Schulausbildung am Cecilien-Gymnasium in Düsseldorf Abschluss: Allgemeine Hochschulreife
05/2002 - 08/2002	Aushilfskraft bei AEGON LV AG in Düsseldorf
10/2004 - 05/2009	Studium der Wirtschaftsmathematik mit dem Studienschwerpunkt Finanzmathematik an der Technischen Universität Kaiserslautern Abschluss: Diplom (Dipl.-Math. oec.)
11/2008 - 03/2009	Wissenschaftliche Hilfskraft an der Technischen Universität Kaiserslautern
07/2009 - 09/2009	Wissenschaftliche Hilfskraft an der Technischen Universität Kaiserslautern
10/2009 - 07/2013	Promotionsstudium im Fach Mathematik an der Technischen Universität Kaiserslautern
11/2009 - 12/2010	Forschungsaufenthalt an der Universität Cambridge, UK
seit 07/2011	Wissenschaftliche Hilfskraft am Fraunhofer Institut für Techno- und Wirtschaftsmathematik (ITWM) in Kaiserslautern