TU Kaiserslautern

Fachbereich Mathematik

Curve interactions in $\mathbb{R}^2$: An analytical and stochastical approach

Benedikt Heinrich

Datum der Disputation: 31.10.2013

Vom Fachbereich Mathematik der Technischen Universität Kaiserslautern zur Verleihung des akademischen Grades Doktor der Naturwissenschaften (Doctor rerum naturalium, Dr. rer. nat.) genehmigte Dissertation

1. Gutachter: Prof. Dr. Heinrich von Weizsäcker
2. Gutachter: Prof. Dr. Max von Renesse

D 386

Contents

Danke

Man kann einfach allen danken,
Familie, Professoren und Bekannten.
Dann vergisst man wirklich keinen,
man würde es auch ehrlich meinen,
doch verloren geht etwas Essentielles,
das Gefühl für etwas sehr Spezielles:
Dankbarkeit ist nicht zum Gießen,
zielgerichtet muss sie fließen.
D'rum lassen Sie mich Danke sagen,
bei Herrn von Weizsäcker für seine Fragen,
für seine Geduld und sein Vertrauen,
all dies ließ mich diese Arbeit bauen.
Zudem möcht' ich meinen Dank bekunden
bei den Drei von den Mittagessenrunden.
Beim Tudy, dem Herrn Beck,
für's Pasche würfeln und manch and'ren Gag,
Für das Lösen so mancher Logelei;
wo ist es sicherer als in Paraguay?
Und war ein Problem mal zu komplex,
dann erklärte es mir der Fex.
Nur eines bleibt für immer unerklärlich,
was macht das Planen des Zeltens so beschwerlich?
Und lag die Stimmung mal darnieder,
so pustete der Simon ein paar Lieder.
Ob Oper oder Schlager
alles hat der Jung' auf Lager.
Doch der Dank gilt nicht nur den drei Herr'n
bedanken möchte ich mich auch sehr gern
beim Herr'n Adam für all das Plaudern und den Beleg
Fußball schenkt einem ein Freundschaftsprivileg.
Zum Schluss möchte ich noch Danke sagen,
den beiden, die mich schon immer tragen,
die zwei sind einfach immer da,
Danke, Mama und Antonia.

Chapter 1

Introduction

In the last few years a lot of work has been done in the investigation of Brownian motion with point interaction(s) in one and higher dimensions. Roughly speaking a Brownian motion with point interaction is nothing else than a Brownian motion whose generator is disturbed by a measure supported in just one point. So in terms of measures we talk about point measures or in terms of distributions about dirac-delta distributions. By careful reading one can find the main stochastic results on this interactions in one dimension already in a paper by Ito and McKean Jr. published in the year 1963, see [20]. There a Feynman-type formula is proven which connects the question of point interaction in one dimension to a local time of Brownian motion in this point, i.e.

$$\mathbb{E}^x \left[f \left(X_t \right) \right] = \mathbb{E}^x \left[f \left(B_t \right) e^{-\kappa L_t} \right] \tag{1.1}$$

for suitable functions f. Here B_t is the usual one-dimensional Brownian motion and L_t the associated local time process in the interaction point. The generator of such a process is calculated explicitly given as restriction of the one dimensional Laplace operator to a set of functions which have a jump in the derivative in the interesting point given by the value of the function in this point. Also an stochastic interpretation of X_t is given: It is a process which behaves like a Brownian motion up to a stopping time τ with conditional law $e^{-\kappa L_t}$. Later Albeverio et al. could prove that these generators are all self-adjoint extensions of the one dimensional Laplace operator restricted to the set of infinitely often continuously differentiable functions with compact support and function value zero in this point, see [2]. Now, using this result Albeverio et al. constructed the heat kernels of these extensions and some similar extensions in two and three dimensions, see [1]. Based on this work Fleischmann and Mueller constructed superprocesses connected to the extensions in the two and three dimensional case, see [14], whereas Engländer and Pinsky constructed the superprocesses in one dimension by a very different approach, see [10].

The purpose of the present work is the generalization of these results to a curve interaction of the two dimensional Brownian motion for a closed curve $\mathcal{C}$. In view of the results of Albeverio et al. in [2] for the case of point interactions we will understand a curve interaction as a self-adjoint extension of the restriction of the Laplacian to the set of infinitely often continuously differentiable functions with compact support in $\mathbb{R}^2$ which are constantly 0 at the closed curve. This technical approach to curve interactions was chosen since there seems to be no natural way to extend the interpretation of the point interactions as $\Delta + \alpha\delta$ for some real α and a suitable Dirac-delta distribution δ to a curve interaction. There is no equivalent to the Dirac-delta distribution of one point in the case of a closed curve.

For a better understanding of curve interactions in $\mathbb{R}^2$ the first chapter gives a full description of all required self-adjoint extensions. Based especially on a work by Posiliciano, see [27], it is shown that these extensions are completely determined by a jump relation of the normal derivatives and the function value on the curve, which are given as generalized sobolev traces along the curve for the functions in the domain of the extensions. Moreover we will develop conditions under which a given jump relation really describes a self-adjoint extension we are interested in.

But if one has functions with a jump in the normal derivative along the closed curve, then these functions are not C^1-functions and therefore we cannot apply the classical Ito formula to them. To calculate the stochastic differential of such a function we will prove in the second chapter a generalization of Tanaka's formula to $\mathbb{R}^2$. Tanaka's formula gives us the stochastic differential of the absolute value function in $\mathbb{R}$ which is a continuous function harmonic outside 0 and having a jump in the first derivative at the point 0. Hence we define g to be a so-called harmonic single layer with continuous layer function η in $\mathbb{R}^2$. This function is continuous in $\mathbb{R}^2$, harmonic in the complement of our closed curve and has a jump in the normal derivative which is given by the continuous function η along the curve. For such a function g we prove

$$g(B_t) = g(B_0) + \int_0^t \nabla g(B_s)\,\mathrm{d}B_s + \int_0^t \eta(B_s)\,\mathrm{d}L(s,\mathcal{C}) \tag{1.2}$$

where B_t is just the usual Brownian motion in $\mathbb{R}^2$ and $L(t,\mathcal{C})$ is the connected unique local time process of B_t on the closed curve $\mathcal{C}$, which is constructed for example in the work of Blumenthal-Getoor, [4], pp.216.

On the one hand by use of (1.2) we can extend Ito's formula to functions with a jump in the normal derivative along $\mathcal{C}$. This extension will be used in the later chapters to determine the stochastic processes related to curve interactions. On the other hand we can show that (1.2) holds for every parallel curve $\mathcal{C}^r$ of $\mathcal{C}$ and hence we can prove for suitable $\varepsilon > 0$ the following Radon-Nikodyn property of $L(t, \mathcal{C}^r)$:

$$\int_0^t \mathbb{1}_{\mathcal{C}_\varepsilon}(B_s)\,\mathrm{d}s = \int_{-\varepsilon}^{\varepsilon} \frac{1}{2} L(t, \mathcal{C}^r)\,\mathrm{d}r \tag{1.3}$$

where $\mathcal{C}_\varepsilon$ is just the set of points in $\mathbb{R}^2$ with distance less than ε to $\mathcal{C}$. The proof of (1.3) is then an analogue to the proof in the one-dimensional case given in the book of Chung and Williams, see [5]. To the best of the author's knowledge a proof of (1.3) has not yet been published.

As already mentioned we will use (1.2) in the following chapter to construct classes of processes related to curve interactions. The first class is just the generalization of (1.1), i.e. we construct processes with

$$\mathbb{E}^x[f(X_t)] = \mathbb{E}^x\left[f(B_t)\exp\left(-\int_0^t \eta(B_s)\,\mathrm{d}L(s, \mathcal{C})\right)\right]$$

for non-negative and regular η on $\mathcal{C}$. We show that the generator of X_t is given by a curve interaction and that we have the same stochastic interpretation as in the one-dimensional case, i.e. a Brownian motion up to a stopping time with conditional law $\exp\left(-\int_0^t \eta(B_s)\,\mathrm{d}L(s, \mathcal{C})\right)$.

The other class of processes we are looking for is a class of processes which does not exist in the one point interaction case. Since $L(t, \mathcal{C})$ gives us a new time scale on the closed curve $\mathcal{C}$ we can take a process on $\mathcal{C}$ and add this movement in the time scale of $L(t, \mathcal{C})$ to a usual Brownian motion in $\mathbb{R}^2$. We end up with a process which behaves like a Brownian motion in the complement of $\mathcal{C}$ and has an additional movement along the curve if the Brownian motion hit $\mathcal{C}$. Such processes do not exist in the one point case since there we cannot move when the Brownian motion is in the point. By constructing the processes with additional movement on $\mathcal{C}$ and show that the generator is given by a curve interaction we really see a difference between point and curve interactions.

The next two chapters will deal with superprocesses related to some of the curve interactions. In the first one we will show existence of superprocesses, in the second one for a better understanding of the dynamics of these superprocesses an approximation by a system of branching particles is shown.

The existence of superprocesses related to some curve interactions are established using the results of the work of Engländer and Pinsky, see [10]. There it is shown that for "nice" potentials V we get superprocesses with generator $\Delta + V$. By establishing an approximation of a curve interaction by operators of the form $\Delta + V_n$ with "nice" potentials V_n we are able to deduce that the related sequence of superprocesses has a limit point which is again a superprocess and has the curve interaction as generator. To be more detailed we prove for a Hölder-continuous function η on the curve $\mathcal{C}$ the existence of a measure valued process X_t such that for any bounded and continuous function g

$$\mathbb{E}_\mu \left(\exp \langle X_t, -g\rangle\right) = \exp\left(-\int\limits_{\mathbb{R}^2} u\left(t,x\right)\mu(dx)\right)$$

holds, where $u\left(t,x\right)$ is the unique non-negative solution of

$$\begin{aligned} u\left(t,\cdot\right) = &\int\limits_{\mathbb{R}^2} g\left(y\right) p\left(t,\cdot,y\right) \mathrm{d}y \\ &+ \int\limits_0^t \int\limits_{\mathcal{C}} \eta\left(y\right) u\left(s,y\right) p\left(t-s,\cdot,y\right) \mathrm{d}S(y)\, \mathrm{d}s \\ &- \int\limits_0^t \int\limits_{\mathbb{R}^2} \alpha\left(y\right) u^2\left(s,y\right) p\left(t-s,\cdot,y\right) \mathrm{d}y\, \mathrm{d}s. \end{aligned}$$

In the following chapter we will give an approximation of the superprocesses of chapter 5 and the superprocesses for the one dimensional point interactions constructed in [10]. This approximation gives a better understanding of the related mass creation. Based on the fundamental work of A. Etheridge, see [12], we can prove that the approximation is given via a different mass creation of the branching particles w.rt. to the interaction point or curve, respectively. If the branching particles die outside of an ε-neighborhood of the point or the curve, respectively, no additional mass creation will happen. But if the particles die inside of the ε-neighborhood we will get additional mass creation given by the "strength" of the interaction, i.e. for the one dimensional case the prefactor of the dirac-distribution and in the case of the curve interactions the continuous function η. If we fix now the additional mass creation and choose the right proportion of $\varepsilon \downarrow 0$ and the lifetime of the branching particles tending to zero we can show that in the limiting case we just get the interactions we are interested in. Hence a point or curve interaction is nothing else but a super Brownian motion with additional mass creation near the point or the curve, respectively. The interesting point here is that we do not need the existence of a local time for performing the approximation. So maybe one can use this result to find a branching particle approximation of the superprocesses constructed by Fleischmann and Mueller in [14]. There we do not have a local time process since singular points are polar sets of Brownian motion in higher dimensions.

In the last section a new and short proof for the explicit formulas of the moments of a one-dimensional Brwonian local time which are given e.g. in a paper of Takacs, [31] is presented. The proof based on the interpretation of the point interaction via the semigroup

$$\mathbb{E}^x\left[f\left(X_t\right)\right] = \mathbb{E}^x\left[f\left(B_t\right) e^{-\kappa L_t}\right]$$

given in the work of Ito and McKean, [20], and the explicit formula for the heat kernel of the point interaction given by Albeverio et al. in [1]. Hence we are able to calculate $\mathbb{E}^x\left[e^{-\kappa L_t}\right]$ by choosing $f \equiv 1$ in $\mathbb{E}^x\left[f\left(B_t\right)e^{-\kappa L_t}\right]$.

Chapter 2

Self-adjoint extensions

As mentioned in the introduction we will understand a curve interaction as a self-adjoint extension of the Laplace operator restricted to functions which are zero at the curve. These functions do not “feel” the curve and hence any singular perturbation of the Laplace operator via the curve does not have any influence to such functions. Thus it seems to be natural to understand curve interactions as extensions of such restricted Laplace operators.

The idea of treating singular interactions with the help of self-adjoint extensions is presented in all its details in a joint work of S. Albeverio, F. Gesztesy, R. Hoegh-Krohn, and H. Holden, see [2]. There all possible point interactions of the Laplace operator in dimensions one, two and three are presented with the help of extension theory. The main advantage of this technical approach is the big box of mathematical tools to handle self-adjoint operators. For example, it is shown in the appendix of [2] that one can calculate the difference of the resolvent operators of two different self-adjoint extensions in an explicit form. Thus if one knows one self-adjoint extension and its resolvent operator one can easily answer questions on spectral properties or semiboundedness of the other self-adjoint extensions with the help of classical statements on self-adjoint operators, see e.g. [28].

Our aim in the present work is the construction of the self-adjoint extensions and a understanding with the help of boundary values. We will see that we can understand all self-adjoint extensions as restriction of the distributional Laplace operator to sets of functions with suitable boundary values. The biggest difficulty in treating these boundary conditions arises in the fact that the boundary values only exist in a fractional Sobolev space of negative order on the curve $\mathcal{C}$, i.e. the boundary values are only distributions.

The structure of the chapter is as follows: First of all we have to give mathematical preliminaries which we need for the construction and a better understanding of the self-adjoint extension. For example we will give the definition of fractional Sobolev spaces along $\mathcal{C}$. The next step is to construct the self-adjoint extensions in a general form using the results of A. Posiliciano, see [27]. Afterwards we will give an equivalent description with the help of boundary values. The last part is dedicated to the question how we can check if some boundary condition we have in mind really gives us a self-adjoint extension.

Preliminaries For analyzing the self-adjoint extensions we need the following technical framework.

For the whole of the present work let $\mathcal{C} \subseteq \mathbb{R}^2$ denote a closed, $C^{2,\alpha}$-curve with Lebesgue measure zero in $\mathbb{R}^2$, where $\alpha > 0$ is a Hölder parameter, i.e. the second derivative of the parametrization of $\mathcal{C}$ is Hölder-continuous with parameter α.

Denote by $H^{2,2}\left(\mathbb{R}^2\right)$ the Sobolev space of second order in $L^2\left(\mathbb{R}^2\right)$,i.e.

$$H^{2,2}\left(\mathbb{R}^2\right) = \left\{f \in L^2\left(\mathbb{R}^2\right) : D^\beta f \in L^2\left(\mathbb{R}^2\right) \text{ for every multiindex } \beta, \text{ s.t. } |\beta| \leq 2\right\},$$

where D^β is given by

$$D^\beta = \frac{\partial^{\beta_1}}{\partial_{x_1}^{\beta_1}} \frac{\partial^{\beta_2}}{\partial_{x_1}^{\beta_2}}.$$

For the rest of the work fix the notation Δ for the Laplace operator.

The spaces $H^s(\mathcal{C})$ The space $H^s(\mathcal{C})$ for any real s is constructed in the following form, see [23], Ch.1, Section 7.3:

Let Δ_{LB} denote the negative definite Laplace-Beltrami operator on $L^2(\mathcal{C})$ and for the self-adjoint operator $-\Delta_{LB} + 1$ one can define

$$\langle f, g\rangle_{H^s(\mathcal{C})} = \left(f, (-\Delta_{LB} + 1)^s g\right)_{L^2(\mathcal{C})}$$

for elements of $C^\infty(\mathcal{C})$. $H^s(\mathcal{C})$ is then the completion of $C^\infty(\mathcal{C})$ with respect to this scalar product.

Using this construction one regards for the restriction of $(-\Delta_{\mathrm{LB}} + 1)^{3/2}$ to $C^\infty(\mathcal{C})$ an unitary extension as mapping from $H^s(\mathcal{C})$ to $H^{s-3}(\mathcal{C})$ which we will denote by Λ. For further details of this construction see [23], Ch.1, Example 2.4.

The trace operators and normal derivatives For $f \in C^\infty\left(\mathbb{R}^2\right)$ and $x \in \mathcal{C}$ denote by

$$\left(\frac{\partial}{\partial n^+} f\right)(x) = n(x)_{\mathrm{ext}} \cdot \nabla f(x)$$

the outer normal derivative and by

$$\left(\frac{\partial}{\partial n^-} f\right)(x) = n(x)_{\mathrm{int}} \cdot \nabla f(x)$$

the inner normal derivative of f along $\mathcal{C}$ and by

$$(\mu f)(x) = f(x)$$

the restriction of f to $\mathcal{C}$,respectively. Here n_{int} is the inner, n_{ext} the outer normal vector on $\mathcal{C}$.

The operators

$$\begin{aligned} \tau : H^{2,2}\left(\mathbb{R}^2\right) &\rightarrow H^{3/2}\left(\mathcal{C}\right) \\ \gamma_+ : H^{2,2}\left(\mathbb{R}^2\right) &\rightarrow H^{1/2}\left(\mathcal{C}\right) \\ \gamma_- : H^{2,2}\left(\mathbb{R}^2\right) &\rightarrow H^{1/2}\left(\mathcal{C}\right) \end{aligned}$$

are defined as unique, bounded and surjective extensions of the operators μ, $\frac{\partial}{\partial n^+}$ and $\frac{\partial}{\partial n^-}$, respectively, see [23], Ch.1, Section 8.2. These operators are called trace operators.

Later on we will see that it is not enough to know trace operators for these Sobolev spaces. We will have less regularity. Hence denote by Ω the inner domain of $\mathcal{C}$ and take

$$W = \left\{ f \in L^2\left(\mathbb{R}^2\right) \cap H^{2,2}_{\text{loc}}\left(\mathbb{R}^2 \backslash \{\mathcal{C}\}\right) : \Delta f \in L^2\left(\Omega\right),\ \Delta f \in L^2\left(\mathbb{R}^2 \backslash \overline{\Omega}\right) \right\}.$$

The Laplacian here is understood in the sense of distributions and $H^{2,2}_{\text{loc}}\left(\mathbb{R}^2 \backslash \{\mathcal{C}\}\right)$ is defined in the following form

$$\begin{aligned} H^{2,2}_{\text{loc}}\left(\mathbb{R}^2 \backslash \{\mathcal{C}\}\right) =& \{ f \in L^2\left(\mathbb{R}^2\right) : \forall \text{ multiindex } \beta, \text{ s.t. } |\beta| \leq 2\ \exists\ v_{\beta,f} \in L^2\left(\mathbb{R}^2\right), \text{ s.t.} \\ & \int\limits_{\mathbb{R}^2} f D^\beta g \mathrm{d}x = (-1)^{|\beta|} \int\limits_{\mathbb{R}^2} v_{\beta,f} g \mathrm{d}x \ \ \forall g \in C_0^\infty\left(\mathbb{R}^2 \backslash \{\mathcal{C}\}\right) \}. \end{aligned}$$

It is proven in [23], Ch.2, Section 6.5, for the inner domain of $\mathcal{C}$, and in [18], Theorem 9.20, for the outer domain of $\mathcal{C}$, respectively, that one can extend the operator τ to an operator

$$\tilde{\tau}_{+,-} : W \rightarrow H^{-1/2}\left(\mathcal{C}\right)$$

where $+$ represents the extension from the outer domain and $-$ the extension from the inner domain, respectively. These extended trace operators are sometimes called generalized Sobolev trace operators.

Since $\tilde{\tau}_+$ and $\tilde{\tau}_-$ are the results of two different extension procedures, namely the extension from the outer domain and the extension from the inner domain, it is not necessarily true that

$$\tilde{\tau}_+ f = \tilde{\tau}_- f.$$

But if this is the case we just write $\tilde{\tau} f$.

In the same way see [18], Theorem 9.20, the operators γ_+ and γ_- can be extended to operators

$$\tilde{\gamma}_+ : W \rightarrow H^{-3/2}\left(\mathcal{C}\right)$$

and

$$\tilde{\gamma}_- : W \rightarrow H^{-3/2}\left(\mathcal{C}\right).$$

In the following work the jump operator κ defined as

$$\kappa = \tilde{\gamma}_+ + \tilde{\gamma}_- \tag{2.1}$$

plays an important role in the description of our self-adjoint extensions. The titling jump operator for κ is quite natural since κ represents the difference between the outer and the inner normal derivative, respectively. We have to take here the sum of the derivatives to get the difference since the derivatives are directed derivatives in diametrical directions which implies for differentiable functions that $\left(\frac{\partial}{\partial n^+}\right) = -\left(\frac{\partial}{\partial n^-}\right)$ and thus the sum represents the jump in the normal derivatives. For the reader which are familiar with the classical jump operators in potential theory we remark that the classical jump operator defined as the difference of the outer and the inner limit of the inner normal derivative is nothing else but $-\kappa$. For further information about the classical jump operator and the normal derivatives involved see e.g. [30].
Remember, for a bounded domain $U \subseteq \mathbb{R}^2$ with boundary $\Gamma(U)$ and $f, g \in H^{2,2}(U)$ Green's formula states

$$\langle \Delta f, g\rangle_{L^2(U)} - \langle f, \Delta g\rangle_{L^2(U)} = \left\langle \left(\frac{\partial}{\partial n^-} f\right), \tau(g)\right\rangle_{L^2(\Gamma(U))} - \left\langle \tau(f), \left(\frac{\partial}{\partial n^-} g\right)\right\rangle_{L^2(\Gamma(U))}.$$

It is shown in [23], Ch.2, Section 6.5, that this formula holds for elements of W restricted to Ω by replacing τ and $\frac{\partial}{\partial n^-}$ by its extensions $\tilde{\tau}_- f$ and $\tilde{\gamma}_- : W \to H^{-3/2}(\mathcal{C})$, respectively. For the outer domain Green's formula holds for functions with compact support and there we can also exchange the classical trace operators by its extensions, see e.g. [18], Theorem 9.21.

2.1 The operator T

In the following we want to give a precise definition of the operator whose self-adjoint extensions we want to determine. Afterwards we will show with the help of the fundamental results in [27] how we can represent the self-adjoint extensions in a general form. In a last step we use the operator κ to give a description of the self-adjoint extensions with the help of boundary conditions.

The operator T We are looking for the self-adjoint extensions of the operator

$$T : D(T) \subseteq L^2\left(\mathbb{R}^2\right) \to L^2\left(\mathbb{R}^2\right)$$
$$Tf = \frac{1}{2}\Delta f$$

with

$$D(T) = \left\{f \in C_0^\infty\left(\mathbb{R}^2\right) : \tau(f) = 0\right\}.$$

It is obvious that

$$L : H^{2,2}\left(\mathbb{R}^2\right) \subseteq L^2\left(\mathbb{R}^2\right) \to L^2\left(\mathbb{R}^2\right)$$
$$Lf = \frac{1}{2}\Delta f$$

is a self-adjoint extension of T and therefore the existence of self-adjoint extensions of T is clear.

The theory of self-adjoint extensions Following the work of Posiliciano, see [27], Theorem 2.1, we get the self-adjoint extensions in the following way:
For $z \in \rho\left(\frac{1}{2}\Delta\right)$, the resolvent set of Δ, define

$$\begin{aligned} G(z) &: H^{3/2}(\mathcal{C}) \to L^2\left(\mathbb{R}^2\right) \\ G(z) &= \left(\tau\left(-\frac{1}{2}\Delta + \overline{z}\right)^{-1}\right)^* \end{aligned}$$

where $*$ denotes the Hilbert space adjoint.
It was shown, see [27], that there is a family of bounded, linear operators

$$\Gamma(z) : H^{3/2}(\mathcal{C}) \to H^{3/2}(\mathcal{C})$$

with the properties

$$\begin{aligned} \Gamma(z)^* &= \Gamma(\overline{z}) \\ \Gamma(z) - \Gamma(\omega) &= (z - \omega)\, G(\overline{\omega})^* G(z) \end{aligned}$$

for $\omega \in \rho(\Delta)$.

Remark 1 It is also shown that one possible choice of Γ is

$$\Gamma(z) = \tau\left(G(\alpha) - G(z)\right)$$

for $\alpha \in \mathbb{R} \cap \rho(\Delta)$. For the rest of this chapter we will take $\Gamma(z)$ in this form for one arbitrary but fixed $\alpha \in \mathbb{R} \cap \rho(\Delta)$.

For the description of all self-adjoint extensions we need even more technical framework. Take an orthogonal projection

$$\Pi : H^{3/2}(\mathcal{C}) \to V \subseteq H^{3/2}(\mathcal{C})$$

and a self-adjoint operator

$$\theta : D(\theta) \subseteq V \to V$$

where V is the range of Π. Define

$$\Gamma_{\Pi,\theta}(z) : D(\theta) \subseteq V \to V$$

by

$$\Gamma_{\Pi,\theta}(z) = \theta + \Pi(\Gamma(z))\Pi$$

and take $z \in \rho\left(\frac{1}{2}\Delta\right)$ such that $0 \in \rho(\Gamma_{\Pi,\theta}(z))$.

Remark 2 That we can find a $z \in \rho\left(\frac{1}{2}\Delta\right)$ such that $0 \in \rho(\Gamma_{\Pi,\theta}(z))$ is also shown in [27], Theorem 2.1.

The self-adjoint extension $T_{\Pi,\theta}$ of T is then given by

$$D(T_{\Pi,\theta}) = \left\{\phi \in L^2\left(\mathbb{R}^2\right) : \phi = \phi_z + G(z)\Pi\left(\Gamma_{\Pi,\theta}(z)^{-1}\right)\Pi\tau\phi_z,\ \phi_z \in H^{2,2}\left(\mathbb{R}^2\right)\right\} \quad (2.2)$$

with the calculation rule

$$(-T_{\Pi,\theta} + z)\phi = \left(-\frac{1}{2}\Delta + z\right)\phi_z.$$

This definition is z-independent and the decomposition of ϕ is unique, see [27], Theorem 2.1. It was also shown that one gets all self-adjoint extensions in this way.
To understand the self-adjoint extensions it seems to be necessary to investigate the operator $G(z)$.

The properties of $G(z)$ We will start with a description of $D(T^*)$ and then we will show how we can use the operator κ to describe $G(z)$.
In the following denote by $\mathcal{R}(\cdot)$ the range and by $\mathcal{N}(\cdot)$ the kernel of an operator, respectively.

Lemma 3

$$D(T^*) = \{f \in W : \tilde{\tau}_- f = \tilde{\tau}_+ f\}$$
$$T^* f = \frac{1}{2}\Delta f$$

Proof. Following [7], Ch.XIII.2, we get $T^* f = \frac{1}{2}\Delta f$ and $D(T^*) \subseteq W$, hence we only have to show the boundary condition $\tilde{\tau}_- f = \tilde{\tau}_+ f$.
"$\subseteq$" Denote by $d(x, \mathcal{C})$ the distance of a point $x \in \mathbb{R}^2$ to $\mathcal{C}$. Define the set $\mathcal{C}_\varepsilon$ by

$$\mathcal{C}_\varepsilon = \{x \in \mathbb{R}^2 :\ d(x, \mathcal{C}) < \varepsilon\}$$

Take $g \in D(T)$ and $f \in D(T^*)$. Then by Green's formula

$$
\begin{aligned}
0 &= (Tg, f)_{L^2(\mathbb{R}^2)} - (g, T^* f)_{L^2(\mathbb{R}^2)} \\
&= \lim_{\varepsilon \to 0} \left[(Tg, f)_{L^2(\mathbb{R}^2 \backslash \{\mathcal{C}_\varepsilon\})} - (g, T^* f)_{L^2(\mathbb{R}^2 \backslash \{\mathcal{C}_\varepsilon\})} \right] \\
&= \lim_{\varepsilon \to 0} \left[\left(\frac{1}{2} \Delta g, f \right)_{L^2(\mathbb{R}^2 \backslash \{\mathcal{C}_\varepsilon\})} - \left(g, \frac{1}{2} \Delta f \right)_{L^2(\mathbb{R}^2 \backslash \{\mathcal{C}_\varepsilon\})} \right] \\
&= \frac{1}{2} \left(\langle \gamma_+ (g), (\tilde{\tau}_+) f \rangle_{L^2(\mathcal{C})} - \langle \tau (g), (\tilde{\gamma}_+) f \rangle_{L^2(\mathcal{C})} \right) \\
&\quad + \frac{1}{2} \left(\langle \gamma_- (g), (\tilde{\tau}_-) f \rangle_{L^2(\mathcal{C})} - \langle \tau (g), (\tilde{\gamma}_-) f \rangle_{L^2(\mathcal{C})} \right).
\end{aligned}
$$

But $g \in D(T)$ and therefore $\tau(g) = 0$ and we have that g is continuously differentiable and hence there is no jump in the normal derivatives along $\mathcal{C}$ and hence the outer normal derivative is just the negative inner normal derivative. Therefore we got

$$
\begin{aligned}
&\langle \gamma_+ (g), (\tilde{\tau}_+) f \rangle_{L^2(\mathcal{C})} - \langle \tau (g), (\tilde{\gamma}_+) f \rangle_{L^2(\mathcal{C})} + \langle \gamma_- (g), (\tilde{\tau}_-) f \rangle_{L^2(\mathcal{C})} - \langle \tau (g), (\tilde{\gamma}_-) f \rangle_{L^2(\mathcal{C})} \\
&= \langle \gamma_+ (g), (\tilde{\tau}_+ - \tilde{\tau}_-) f \rangle_{L^2(\mathcal{C})}.
\end{aligned}
$$

The assertion follows now by the definition of the operator γ_+. This argument will be shown in detail in the proof of Lemma 2 in the case of the operator τ.

"$\supseteq$" Obvious. Take $g \in D(T)$ and make use of Green's formula. □

In [27], Theorem 2.1, it was shown, that $G(z)(\cdot) \in \mathcal{R}(-T+z)^\perp$. This is equivalent to $G(z)(\cdot) \in \mathcal{N}(-T^*+z)$ and therefore $G(z)(\cdot)$ fulfills the so-called Helmholtz-equation in $\mathbb{R}^2 \backslash \{\mathcal{C}\}$, i.e.

$$
\left(-\frac{1}{2} \Delta + z \right) G(z)(\cdot) = 0
$$

in $\mathbb{R}^2 \backslash \{\mathcal{C}\}$.

Lemma 4 For any possible choice of z we have $\kappa(G(z) f) = 2\Lambda f$ in $H^{-3/2}(\mathcal{C})$.

Proof. Take $g \in C_0^\infty(\mathbb{R}^2)$. Calculate

$$
\left\langle \left(-\frac{1}{2} \Delta + z \right) g, G(z) f \right\rangle_{L^2(\mathbb{R}^2)}
$$

in two different ways:

By the section above $G(z) = \left(\tau \left(-\frac{1}{2} \Delta + \overline{z} \right)^{-1} \right)^*$ and therefore

$$
\begin{aligned}
\left\langle \left(-\frac{1}{2} \Delta + \overline{z} \right) g, G(z) f \right\rangle_{L^2(\mathbb{R}^2)} &= \langle \tau(g), f \rangle_{H^{3/2}(\mathcal{C})} \\
&= \langle \tau(g), \Lambda f \rangle_{L^2(\mathcal{C})}.
\end{aligned}
$$

Here $\langle \cdot, \cdot \rangle_{L^2(\mathcal{C})}$ denotes the $L^2(\mathcal{C})$-dual pairing.

By the Helmholtz-equation, the regularity of g and Green's formula one gets:

$$\begin{aligned}
&\left\langle \left(-\frac{1}{2}\Delta + \overline{z}\right) g, G(z) f \right\rangle_{L^2(\mathbb{R}^2)} \\
= &\left\langle \left(-\frac{1}{2}\Delta + \overline{z}\right) g, G(z) f \right\rangle_{L^2(\mathbb{R}^2)} - \left\langle g, \left(-\frac{1}{2}\Delta + z\right) G(z) f \right\rangle_{L^2(\mathbb{R}^2)} \\
= &\left\langle \left(-\frac{1}{2}\Delta\right) g, G(z) f \right\rangle_{L^2(\mathbb{R}^2)} - \left\langle g, \left(-\frac{1}{2}\Delta\right) G(z) f \right\rangle_{L^2(\mathbb{R}^2)} \\
= &\frac{1}{2} \langle \tau(g), (\tilde{\gamma}_+ + \tilde{\gamma}_-) G(z) f \rangle_{L^2(\mathcal{C})} - \frac{1}{2} \langle (\tilde{\gamma}_+ + \tilde{\gamma}_-)(g), \tilde{\tau} G(z) f \rangle_{L^2(\mathcal{C})} \\
= &\frac{1}{2} \langle \tau(g), (\tilde{\gamma}_+ + \tilde{\gamma}_-) G(z) f \rangle_{L^2(\mathcal{C})}.
\end{aligned}$$

Now the assertion follows by the definition of the operator τ as unique, bounded and surjective extension of the restriction operator μ. Hence for $g \in H^{3/2}(\mathcal{C})$ it exists $g_n \in C_0^\infty(\mathbb{R}^2)$ such that $\tau(g_n) \to g$ in $H^{3/2}(\mathcal{C})$. Therefore, since Λf and $\frac{1}{2}(\tilde{\gamma}_+ + \tilde{\gamma}_-) G(z) f \in H^{-3/2}(\mathcal{C})$, we have

$$\left\langle g, \Lambda f - \frac{1}{2}(\tilde{\gamma}_+ + \tilde{\gamma}_-) G(z) f \right\rangle_{L^2(\mathcal{C})} = \lim_{n\to\infty} \left\langle \tau(g_n), \Lambda f - \frac{1}{2}(\tilde{\gamma}_+ + \tilde{\gamma}_-) G(z) f \right\rangle_{L^2(\mathcal{C})} = 0$$

and the lemma is shown. □

By [6], it can be seen that $G(z) f = S_z(\Lambda f)$, where $S_z(\cdot)$ is the so called acoustic single-layer potential. There it is mentioned that the operator $G(z)$ can be written as integral operator along $\mathcal{C}$ with the fundamental solution of $\frac{1}{2}\Delta + z$ as integral kernel. This result gives us a nice representation of $G(z) f$ for $f \in H^3(\mathcal{C})$:

$$(G(z) f)(x) = (S_z(\Lambda f))(x) = \int_{\mathcal{C}} (\Lambda f)(y) K_{2z}(x - y) \, \mathrm{d}S(y)$$

Here K_z is the so-called MacDonald function, see e.g [17], pp.951.

2.2 The self-adjoint extensions of T

The section above gives us a new description of the self-adjoint extensions of T by the use of the proven jump relations. For the description we need the following lemma

Lemma 5 We have

$$D(T^*) \cap \mathcal{N}(\kappa) = H^{2,2}(\mathbb{R}^2).$$

Proof. Take $\phi \in H^{2,2}(\mathbb{R}^2)$ then

$$\kappa(\phi) = (\tilde{\gamma}_+ + \tilde{\gamma}_-)\phi = (\gamma_+ + \gamma_-)\phi$$

by the definition of $\tilde{\gamma}_{\pm}$ as extensions of $\gamma_{\pm}$. Due to [23], Ch.1, Remark 7.3 we have that $C_0^{\infty}\left(\mathbb{R}^2\right)$ is dense in $H^{2,2}\left(\mathbb{R}^2\right)$. Take a sequence $\phi_n \in C_0^{\infty}\left(\mathbb{R}^2\right)$ which converges to ϕ in $H^{2,2}\left(\mathbb{R}^2\right)$. Then we have

$$\kappa(\phi) = \lim_{n\to\infty}\left(\gamma_+ + \gamma_-\right)\phi_n = \lim_{n\to\infty}\left(\frac{\partial}{\partial n^+} + \frac{\partial}{\partial n^-}\right)\phi_n$$

by the definition of $\gamma_{\pm}$ as bounded extensions of $\frac{\partial}{\partial n^{\pm}}$. But $\phi_n \in C_0^{\infty}\left(\mathbb{R}^2\right)$ hence it has no jump in the derivatives and therefore

$$\kappa(\phi) = \lim_{n\to\infty}\left(\frac{\partial}{\partial n^+} + \frac{\partial}{\partial n^-}\right)\phi_n = 0$$

Take now $\phi \in D\left(T^*\right) \cap \mathcal{N}(\kappa)$. It is mentioned in [27], Theorem 3.1 that we can write every element ϕ of $D\left(T^*\right)$ as

$$\phi = g + G(i)h_1 + G(-i)h_2$$

for $g \in H^{2,2}\left(\mathbb{R}^2\right)$ with $\tau(g) = 0$ and $h_1,\, h_2 \in H^{3/2}(\mathcal{C})$. Due to Lemma 2 and the first part of this proof we have

$$\kappa(\phi) = 2\Lambda\left(h_1 + h_2\right)$$

If we now assume that $\kappa(\phi) = 0$ we have $2\Lambda\left(h_1 + h_2\right) = 0$ and this implies by the unitarity of Λ that $h_2 = -h_1$ in $H^{3/2}(\mathcal{C})$. Hence

$$\phi = g + \left(G(i) - G(-i)\right)h_1.$$

But it is shown in [27], Theorem 2.1 that $G(i) - G(-i)$ maps into $H^{2,2}\left(\mathbb{R}^2\right)$. Hence ϕ is as sum of elements of $H^{2,2}\left(\mathbb{R}^2\right)$ an element of $H^{2,2}\left(\mathbb{R}^2\right)$. □

The lemmata 2 and 3 give us a new understanding of elements of $D\left(T_{\Pi,\theta}\right)$. It can be seen in (2.2) that one can decompose an element of $D\left(T_{\Pi,\theta}\right)$ into the sum of an element of $H^{2,2}\left(\mathbb{R}^2\right)$ and a function given by $G(z)(\cdot)$. Due to lemma 2 and lemma 3 we know that the boundary conditions are totally described by $G(z)(\cdot)$ and we see the independence of z since the boundary condition of lemma 2 and therefore the boundary condition of $T_{\Pi,\theta}$ is independent of z.

Corollary 6 Take α as in remark 1 and choose $f \in D\left(T^*\right)$. Then we have

$$f_\alpha = f - \left(\frac{1}{2}G(\alpha)\Lambda^{-1}\kappa f\right) \in H^{2,2}\left(\mathbb{R}^2\right) \tag{2.3}$$

Proof. In [27], Ch.2, it is shown that $G(\alpha)$ maps into $D\left(T^*\right)$. Hence f_α is an element of $D\left(T^*\right)$. The next step is to show that $f_\alpha \in \mathcal{N}(\kappa)$. But

$$\kappa\left(f_\alpha\right) = \kappa\left(f - \left(\frac{1}{2}G(\alpha)\Lambda^{-1}\kappa f\right)\right) = \kappa(f) - \kappa(f) = 0$$

where we used the result of Lemma 2 to see that $\kappa\left(\frac{1}{2}G(\alpha)\Lambda^{-1}\kappa f\right) = \kappa f$. Hence $f_\alpha \in D(T^*) \cap \mathcal{N}(\kappa)$ and therefore due to Lemma 3 it is in $H^{2,2}\left(\mathbb{R}^2\right)$. □

Remark 7 Later on we will see that we can use the function f_α defined in (2.3) to get a new description of elements of $D(T_{\Pi,\theta})$ with the help of a boundary condition given via the jump operator κ. Little warning: We only know that $\alpha \in \rho(\Delta)$ but we do not know whether $0 \in \rho(\Gamma_{\Pi,\theta}(\alpha))$ or not. Thus we cannot substitute the parameter z by our nice α in the definition of $D(T_{\Pi,\theta})$ given in (2.2).

Definition 8 Corollary 6 allows us to define an operator τ_α by

$$\begin{aligned} \tau_\alpha : D(T^*) &\to H^{3/2}(\mathcal{C}) \\ \tau_\alpha f &= \tau(f_\alpha). \end{aligned}$$

Taken in to account that due to lemma 3 we have $\kappa f = 0$ for $f \in H^{2,2}\left(\mathbb{R}^2\right)$ we get $f_\alpha = f$ and hence $\tau_\alpha = \tau$ on $H^{2,2}\left(\mathbb{R}^2\right)$. So the operator τ_α is an extended trace operator which we want to use to describe the self-adjoint extensions in a new way which immediately includes some boundary condition.
Now we are able to give the desired new description.

Theorem 9 If we denote by $T_{\Pi,\theta}$ the self-adjoint extension given by Π and θ we get

$$D(T_{\Pi,\theta}) = \left\{f \in D(T^*) \;:\; \Lambda^{-1}\kappa f \in D(\theta), 2\Pi\tau_\alpha f = \theta\Lambda^{-1}\kappa f\right\},$$

where τ_α is just the operator defined in definition 8.

Proof. Take $f \in D(T_{\Pi,\theta})$, then by the decomposition given in the definition of $D(T_{\Pi,\theta})$

$$f = f_z + G(z)\Pi\Gamma_{\Pi,\theta}(z)^{-1}\Pi\tau f_z \tag{2.4}$$

with $f_z \in H^{2,2}\left(\mathbb{R}^2\right)$ and therefore by Lemma 2 and Lemma 3:

$$\kappa f = 2\Lambda\Pi\left(\Gamma_{\Pi,\theta}(z)^{-1}\right)\Pi\tau(f_z).$$

Since $\Gamma_{\Pi,\theta}(z)$ maps from $D((\theta)) \subseteq V$ into the range of Π, which is given by V, we have

$$\frac{1}{2}\Lambda^{-1}\kappa(f) = \Gamma_{\Pi,\theta}(z)^{-1}\Pi\tau(f_z) \in D(\theta).$$

Therefore we have

$$\Gamma_{\Pi,\theta}(z)\frac{1}{2}\Lambda^{-1}\kappa(f) = \Pi\tau(f_z)$$

and hence by use of the definition of $\Gamma_{\Pi,\theta}(z)$ and $\Gamma(z)$ we get

$$\begin{aligned}\frac{1}{2}\theta\Lambda^{-1}\kappa(f) &= \Pi\tau(f_z)-\Pi\Gamma(z)\Pi\frac{1}{2}\Lambda^{-1}\kappa(f)\\ &= \Pi\left(\tau(f_z)-\Gamma(z)\Pi\frac{1}{2}\Lambda^{-1}\kappa(f)\right)\\ &= \Pi\left(\tau(f_z)-\tau(G(\alpha)-G(z))\frac{1}{2}\Lambda^{-1}\kappa(f)\right)\\ &= \Pi\left(\tilde{\tau}\left((f_z)+G(z)\left(\Pi\Gamma_{\Pi,\theta}(z)^{-1}\right)\Pi\tau(f_z)\right)-\frac{1}{2}\tilde{\tau}G(\alpha)\Lambda^{-1}\kappa(f)\right)\\ &= \Pi\left(\tilde{\tau}\left(f-\frac{1}{2}G(\alpha)\Lambda^{-1}\kappa(f)\right)\right)\end{aligned}$$

and this implies by use of (2.3)

$$\begin{aligned}\theta\Lambda^{-1}\kappa f &= 2\Pi\tilde{\tau}\left(f-\left(\frac{1}{2}G(\alpha)\Lambda^{-1}\kappa\right)f\right)\\ &= 2\Pi\tilde{\tau}(f_\alpha)\\ &= 2\Pi\tau_\alpha(f).\end{aligned}$$

Now take $f\in D(T^*)$ such that $\Lambda^{-1}\kappa(f)\in D(\theta)$ and $2\Pi\tau_\alpha(f)=\theta\Lambda^{-1}\kappa(f)$. We have

$$\begin{aligned}f &= \left(f-\frac{1}{2}G(z)\Lambda^{-1}\kappa(f)\right)+\frac{1}{2}G(z)\Lambda^{-1}\kappa(f)\\ &= f_z+\frac{1}{2}G(z)\Lambda^{-1}\kappa(f).\end{aligned}$$

Taken into account that f_z is nothing else but (2.3) with z instead of α we have $f_z\in H^{2,2}\left(\mathbb{R}^2\right)$ and therefore if we show

$$\frac{1}{2}\Lambda^{-1}\kappa(f)=\Gamma_{\Pi,\theta}(z)^{-1}\Pi\tau(f_z)$$

we would have

$$f=f_z+G(z)\Pi\Gamma_{\Pi,\theta}(z)^{-1}\Pi\tau(f_z)$$

for some $f_z\in H^{2,2}\left(\mathbb{R}^2\right)$ and hence $f\in D(T_{\Pi,\theta})$.
By assumption we have $\Lambda^{-1}\kappa(f)\in D(\theta)$ and therefore we can apply $\Gamma_{\Pi,\theta}(z)$ to it and reach at

$$\begin{aligned}\frac{1}{2}\Gamma_{\Pi,\theta}(z)\Lambda^{-1}\kappa(f) &= \frac{1}{2}(\theta+\Pi\Gamma(z)\Pi)\Lambda^{-1}\kappa(f)\\ &= \frac{1}{2}\theta\Lambda^{-1}\kappa(f)+\frac{1}{2}\Pi\Gamma(z)\Lambda^{-1}\kappa(f)\\ &= \Pi\tau_\alpha(f)+\frac{1}{2}\Pi\Gamma(z)\Lambda^{-1}\kappa(f)\end{aligned}$$

where we used the second assumption that $2\Pi\tau_\alpha(f)=\theta\Lambda^{-1}\kappa(f)$. Hence by (2.3) and the definition of $\Gamma(z)$ we get

$$\begin{aligned}\frac{1}{2}\Gamma_{\Pi,\theta}(z)\Lambda^{-1}\kappa(f) &= \Pi\left(\tilde{\tau}\left(f-\frac{1}{2}G(\alpha)\Lambda^{-1}\kappa(f)+\frac{1}{2}(G(\alpha)-G(z))\Lambda^{-1}\kappa(f)\right)\right)\\ &= \Pi\tau(f_z)\end{aligned}$$

and we are done. □

2.3 Boundary conditions and examples

This section is dedicated to the question how we can check if some boundary condition we have in mind belongs to a self-adjoint extension of T or more formally: Under which conditions on an operator A can we find Π and θ as in (2.2) such that

$$\{f \in D(T^*) \,:\, \kappa f = A\tilde{\tau} f\} = D(T_{\Pi,\theta})\,. \tag{2.5}$$

Before we start with the construction of Π and θ for some A we want to define an operator P_λ which we will use later on.

Definition 10 For any $\lambda \in \rho(\Delta)$ define $P_\lambda = \frac{1}{2}\tilde{\tau} S_\lambda$.

Let us collect the properties of P_λ which we will need for our calculations.

Proposition 11 For any $\lambda \in \rho(\Delta)$ we have that the operator P_λ is $L^2(\mathcal{C})$-symmetric and fulfills

$$P_\lambda : H^\sigma(\mathcal{C}) \to H^{\sigma+1}(\mathcal{C}) \text{ is bounded for all } \sigma \in [-2,2]\,, \tag{2.6}$$

$$\text{if } \phi \in H^s(\mathcal{C}) \text{ and } P_\lambda \phi \in H^{s+1+\sigma}(\mathcal{C}) \text{ then } \phi \in H^{s+\sigma}(\mathcal{C}) \text{ for all } s \in \left[-\frac{1}{2},\frac{3}{2}\right], \sigma \geq 0 \tag{2.7}$$

Proof. A proof of these properties can be found in [16], theorem 1 and theorem 2. □

For our α chosen due to remark 1 we have the following.

Corollary 12 Choose α as in remark 1 then we get

$$\tau_\alpha(f) = \tilde{\tau}(f) - P_\alpha \kappa(f)$$

for $f \in D(T^*)$.

Proof. The identity above follows directly by the formulas

$$G(\alpha)\, g = S_\alpha(\Lambda g)$$

and

$$\tau_\alpha(f) = \tau\left(f - \frac{1}{2} G(\alpha)\, \Lambda^{-1} \kappa f\right)$$

which were shown in the sections before. □

Boundary conditions We want to give conditions on an operator A such that

$$\{f \in D(T^*) \, : \, \kappa f = A\tilde{\tau} f\} = \left\{f \in D(T^*) \, : \, \Lambda^{-1}\kappa f \in D(\theta), 2\Pi\tau_\alpha f = \theta\Lambda^{-1}\kappa f\right\}$$

holds for an orthogonal projection Π and a self-adjoint operator θ.

Theorem 13 Take a symmetric operator $A : D(A) \subseteq L^2(\mathcal{C}) \to L^2(\mathcal{C})$ such that

$$\mathrm{Id} - PA : D(A) \subseteq L^2(\mathcal{C}) \to H^{3/2}(\mathcal{C}) \tag{2.8}$$

and

$$\mathcal{R}\left(\Lambda^{-1}A\right)^{\perp} = \mathcal{N}(A) \tag{2.9}$$

where the orthogonal complement is taken w.r.t. the $H^{3/2}(\mathcal{C})$-scalar product. Define Π to be the orthogonal projection on $H^{3/2}(\mathcal{C})$ to $\overline{\mathcal{R}(\Lambda^{-1}A)} = \mathcal{N}(A)^{\perp}$ and the operator

$$\theta : D(\theta) = \mathcal{R}\left(\Lambda^{-1}A\right) \to \mathcal{N}(A)^{\perp}$$

via

$$\theta\left(\Lambda^{-1}A\right) f = 2\Pi\left(\mathrm{Id} - PA\right) f. \tag{2.10}$$

(a) The operator θ is symmetric and

$$\{f \in D(T^*) \, : \, \kappa f = A\tilde{\tau} f\} = \left\{f \in D(T^*) \, : \, \Lambda^{-1}\kappa f \in D(\theta), 2\Pi\tau_\alpha f = \theta\Lambda^{-1}\kappa f\right\}.$$

(b) The restriction of T^* to $\{f \in D(T^*) \, : \, \kappa f = A\tilde{\tau} f\}$ is a self-adjoint extension of T if and only if θ is self-adjoint on $\mathcal{N}(A)^{\perp}$.

Proof. First of all we want to mention that condition (2.8) implies that $\mathcal{N}(A) \subseteq H^{3/2}(\mathcal{C})$ and hence condition (2.9) and the definition of Π is well defined.

Proof of part (a) Our first aim is to show that the operator θ given in (2.10) is really symmetric. For this purpose take $F, G \in D(\theta)$. Then there exist $f, g \in D(A)$ such that $\Lambda^{-1}Af = F$ and $\Lambda^{-1}Ag = G$, respectively. Hence by use of the definitions of the scalar product in $H^{3/2}(\mathcal{C})$ and of Π we reach at

$$\begin{aligned}
& \langle \theta F, G\rangle_{H^{3/2}(\mathcal{C})} - \langle F, \theta G\rangle_{H^{3/2}(\mathcal{C})} \\
= \; & \left\langle \theta\Lambda^{-1}Af, \Lambda^{-1}Ag\right\rangle_{H^{3/2}(\mathcal{C})} - \left\langle \Lambda^{-1}Af, \theta\Lambda^{-1}Ag\right\rangle_{H^{3/2}(\mathcal{C})} \\
= \; & \left\langle 2\Pi\left(\mathrm{Id} - P_\alpha A\right) f, \Lambda^{-1}Ag\right\rangle_{H^{3/2}(\mathcal{C})} - \left\langle \Lambda^{-1}Af, 2\Pi\left(\mathrm{Id} - P_\alpha A\right) g\right\rangle_{H^{3/2}(\mathcal{C})} \\
= \; & 2\left(\left\langle (\mathrm{Id} - P_\alpha A) f, \Pi\Lambda^{-1}Ag\right\rangle_{H^{3/2}(\mathcal{C})} - \left\langle \Pi\Lambda^{-1}Af, (\mathrm{Id} - P_\alpha A) g\right\rangle_{H^{3/2}(\mathcal{C})}\right) \\
= \; & 2\left(\langle (\mathrm{Id} - P_\alpha A) f, Ag\rangle_{L^2(\mathcal{C})} - \langle Af, (\mathrm{Id} - P_\alpha A) g\rangle_{L^2(\mathcal{C})}\right) \\
= \; & 2\left(\langle f, Ag\rangle_{L^2(\mathcal{C})} - \langle Af, g\rangle_{L^2(\mathcal{C})} - \langle P_\alpha Af, Ag\rangle_{L^2(\mathcal{C})} + \langle Af, P_\alpha Ag\rangle_{L^2(\mathcal{C})}\right).
\end{aligned}$$

P_α is $L^2(\mathcal{C})$-symmetric, hence

$$\begin{aligned} & \langle f, Ag\rangle_{L^2(\mathcal{C})} - \langle Af, g\rangle_{L^2(\mathcal{C})} - \langle P_\alpha Af, Ag\rangle_{L^2(\mathcal{C})} + \langle Af, P_\alpha Ag\rangle_{L^2(\mathcal{C})} \\ = \ & \langle f, Ag\rangle_{L^2(\mathcal{C})} - \langle Af, g\rangle_{L^2(\mathcal{C})} \\ = \ & 0 \end{aligned}$$

since A is symmetric and we are done. The last step is to show that

$$\{f \in D(T^*) \,:\, \kappa f = A\tilde{\tau} f\} = \left\{f \in D(T^*) \,:\, \Lambda^{-1}\kappa f \in D(\theta), 2\Pi\tau_\alpha f = \theta\Lambda^{-1}\kappa f\right\}.$$

To do this take $f \in D(T^*)$ such that $\kappa f = A\tilde{\tau} f$. Then we have

$$\Lambda^{-1}\kappa f = \Lambda^{-1}A\tilde{\tau} f \in \mathcal{R}\left(\Lambda^{-1}A\right) = D(\theta)$$

and

$$2\Pi\tau_\alpha f = 2\Pi(\tilde{\tau} f - P_\alpha \kappa f) = 2\Pi(\mathrm{Id} - P_\alpha A)\tilde{\tau} f = \theta\Lambda^{-1}A\tilde{\tau} f = \theta\Lambda^{-1}\kappa f.$$

For the inclusion in the other direction take $f \in D(T^*)$ such that $\Lambda^{-1}\kappa f \in D(\theta) = \mathcal{R}\left(\Lambda^{-1}A\right)$. This implies the existence of some $g \in D(A)$ with $\kappa f = Ag$. The condition $2\Pi\tau_\alpha f = \theta\Lambda^{-1}\kappa f$ is then equivalent to

$$2\Pi(\tilde{\tau} f - P_\alpha Ag) = \theta\Lambda^{-1}Ag.$$

But by the definition of θ given in (2.10) we have

$$\theta\Lambda^{-1}Ag = 2\Pi(\mathrm{Id} - P_\alpha A) g$$

and hence

$$2\Pi(\tilde{\tau} f - g) = 0.$$

Proof of part (b) Since Π is the orthogonal projection into $\mathcal{N}(A)^\perp$ we have $\tilde{\tau} f = g + u$ for some $u \in \mathcal{N}(A)$. Hence $\tilde{\tau} f \in D(A)$ and

$$A\tilde{\tau} f = Ag = \kappa f.$$

Thus we can obtain that

$$\{f \in D(T^*) \,:\, \kappa f = A\tilde{\tau} f\} = \left\{f \in D(T^*) \,:\, \Lambda^{-1}\kappa f \in D(\theta), 2\Pi\tau_\alpha f = \theta\Lambda^{-1}\kappa f\right\}.$$

But in theorem 9 it was shown that

$$\left\{f \in D(T^*) \,:\, \Lambda^{-1}\kappa f \in D(\theta), 2\Pi\tau_\alpha f = \theta\Lambda^{-1}\kappa f\right\}$$

gives us a self-adjoint extension of T if and only if θ is self-adjoint on $\mathcal{R}(\Pi)$. □

The next theorem will answer the question which conditions on the operator A guarantees that the symmetric operator defined in theorem 13 is self-adjoint.

Theorem 14 Take the operators A, Π and θ as in theorem 13 (a). Let us additionally assume that

$$A\,:\,D\left(A\right)\subseteq H^{3/2}\left(\mathcal{C}\right)\to H^{1/2}\left(\mathcal{C}\right)$$

with $D(A)$ dense in $H^{3/2}\left(\mathcal{C}\right)$. Denote by

$$\tilde{A}\,:\,D\left(\tilde{A}\right)\subseteq H^{-1/2}\left(\mathcal{C}\right)\to H^{-3/2}\left(\mathcal{C}\right)$$

the extension of A to $H^{-1/2}\left(\mathcal{C}\right)$ given by the adjoint of A w.r.t. the $L^2\left(\mathcal{C}\right)$-dual pairing. (For further information on this construction of $\tilde{A}$ see [18], Ch.9)
If $\mathcal{R}\left(P_\alpha\Lambda\Pi\right)\subseteq D\left(\tilde{A}\right)$ then the restriction of T^* to $\{f\in D\left(T^*\right)\,:\,\kappa f=A\tilde{\tau}f\}$ is a self-adjoint extension of T if and only if for every $g\in H^{3/2}\left(\mathcal{C}\right)$ and $v\in H^{3/2}\left(\mathcal{C}\right)\cap D\left(\tilde{A}\right)$

$$\left(\mathrm{Id}-\tilde{A}P_\alpha\right)\Lambda\Pi g=\tilde{A}v\Rightarrow\Pi g\in\mathcal{R}\left(\Lambda^{-1}A\right)=D\left(\theta\right). \tag{2.11}$$

Remark 15 The condition $\mathcal{R}\left(P_\alpha\Lambda\Pi\right)\subseteq D\left(\tilde{A}\right)$ implies that $\left(\mathrm{Id}-\tilde{A}P_\alpha\right)\Lambda\Pi$ is a well-defined operator on $H^{3/2}\left(\mathcal{C}\right)$.

Proof. Due to theorem 13 (b) we have to check whether the operator θ defined in (2.10) is self-adjoint on $\mathcal{N}\left(A\right)^\perp$. For this reason take $f\in D(A)$ and $g\in H^{3/2}\left(\mathcal{C}\right)$, then we have

$$\left\langle\theta\Lambda^{-1}Af,g\right\rangle_{H^{3/2}(\mathcal{C})}=\left\langle 2\Pi\left(\mathrm{Id}-P_\alpha A\right)f,g\right\rangle_{H^{3/2}(\mathcal{C})}$$

and if $\Pi g\in D\left(\theta^*\right)$ we obtain that

$$\left\langle\theta\Lambda^{-1}Af,g\right\rangle_{H^{3/2}(\mathcal{C})}=\left\langle f,\Lambda^{-1}\tilde{A}\theta^*\Pi g\right\rangle_{H^{3/2}(\mathcal{C})}. \tag{2.12}$$

Lemma 16 The operator

$$\Lambda^{-1}\left(\mathrm{Id}-\tilde{A}P_\alpha\right)\Lambda\Pi\,:\,H^{3/2}\left(\mathcal{C}\right)\to H^{3/2}\left(\mathcal{C}\right)$$

coincides with

$$\left(\Pi\left(\mathrm{Id}-P_\alpha A\right)\right)^*\,:\,H^{3/2}\left(\mathcal{C}\right)\to H^{3/2}\left(\mathcal{C}\right).$$

on the set $\mathcal{R}\left(\Lambda^{-1}A\right)\oplus\mathcal{N}\left(A\right)$.

Proof. Since $D(A)$ is dense in $H^{3/2}\left(\mathcal{C}\right)$ we could talk about $\left(2\Pi\left(\mathrm{Id}-P_\alpha A\right)\right)^*$. Take $u\in D(A)$ and $v\in\mathcal{N}\left(A\right)$. Define $g=\Lambda^{-1}Au+v$. Then we get

$$\left(\mathrm{Id}-\tilde{A}P_\alpha\right)\Lambda\Pi g=\left(\mathrm{Id}-\tilde{A}P_\alpha\right)Au.$$

Since $\tilde{A} = A$ on $D(A)$ we have

$$\left(\mathrm{Id} - \tilde{A}P_\alpha\right) Au = \tilde{A}\left(\mathrm{Id} - P_\alpha A\right) u.$$

By the definition of $\tilde{A}$ and the symmetry of P_α we have for $f \in D(A)$

$$\begin{aligned} \left\langle f, \Lambda^{-1}\tilde{A}\left(\mathrm{Id} - P_\alpha A\right) u\right\rangle_{H^{3/2}(\mathcal{C})} &= \left\langle Af, \left(\mathrm{Id} - P_\alpha A\right) u\right\rangle_{L^2(\mathcal{C})} \\ &= \left\langle \left(\mathrm{Id} - P_\alpha A\right) f, Au\right\rangle_{L^2(\mathcal{C})} \\ &= \left\langle \left(\mathrm{Id} - P_\alpha A\right) f, \Lambda^{-1}Au\right\rangle_{H^{3/2}(\mathcal{C})} \\ &= \left\langle \Pi\left(\mathrm{Id} - P_\alpha A\right) f, g\right\rangle_{H^{3/2}(\mathcal{C})} \end{aligned}$$

and we are done. □

By combining now lemma 16 and (2.12) we can conclude that

$$2\Lambda^{-1}\left(\mathrm{Id} - \tilde{A}P_\alpha\right)\Lambda\Pi = \Lambda^{-1}\tilde{A}\theta^*\Pi \tag{2.13}$$

on the set $\mathcal{R}\left(\Lambda^{-1}A\right) \oplus \mathcal{N}(A)$. But we have the following observations:

Lemma 17 The set $\mathcal{R}\left(\Lambda^{-1}A\right) \oplus \mathcal{N}(A)$ is dense in $H^{3/2}(\mathcal{C})$ and the operator

$$\Lambda^{-1}\left(\mathrm{Id} - \tilde{A}P_\alpha\right)\Lambda\Pi\ : H^{3/2}(\mathcal{C}) \to H^{3/2}(\mathcal{C})$$

is bounded.

Proof. The statement that $\mathcal{R}\left(\Lambda^{-1}A\right) \oplus \mathcal{N}(A)$ is dense in $H^{3/2}(\mathcal{C})$ follows immediately by our assumption (2.9), i.e. $\mathcal{R}\left(\Lambda^{-1}A\right)^{\perp} = \mathcal{N}(A)$. For the boundedness of $\left(\mathrm{Id} - \tilde{A}P_\alpha\right)\Lambda\Pi$ we use that $\Lambda\Pi$ is a bounded operator from $H^{3/2}(\mathcal{C})$ to $H^{-3/2}(\mathcal{C})$, that due to (2.6) P_α is bounded from $H^{-3/2}(\mathcal{C})$ to $H^{-1/2}(\mathcal{C})$ and that $\tilde{A}$ is as adjoint operator a closed operator from $H^{-1/2}(\mathcal{C})$ to $H^{-3/2}(\mathcal{C})$. Hence $\left(\mathrm{Id} - \tilde{A}P_\alpha\right)\Lambda\Pi$ is a closed operator defined on the full space $H^{3/2}(\mathcal{C})$ and therefore by the closed graph theorem $\left(\mathrm{Id} - \tilde{A}P_\alpha\right)\Lambda\Pi$ is a bounded operator from $H^{3/2}(\mathcal{C})$ to $H^{-3/2}(\mathcal{C})$. Since $\Lambda^{-1}\ :\ H^{-3/2}(\mathcal{C}) \to H^{3/2}(\mathcal{C})$ is bounded the statement follows. □

By combining now lemma 17 and (2.13) we can conclude that

$$2\Lambda^{-1}\left(\mathrm{Id} - \tilde{A}P_\alpha\right)\Lambda\Pi = \Lambda^{-1}\tilde{A}\theta^*\Pi$$

on the set $D(\theta^*) \oplus \mathcal{N}(A)$.

Hence if θ is not self-adjoint we can find an element of $D\left(\tilde{A}\right)$ such that (2.11) does not hold and if (2.11) does not hold we can extend θ^* to a larger set than $\mathcal{R}\left(\Lambda^{-1}A\right)$. This implies that θ is self-adjoint if and only if (2.11) holds. □

Example 1 Closed range If $\mathcal{R}\left(\Lambda^{-1}A\right) = V$ i.e. $\mathcal{R}\left(\Lambda^{-1}A\right)$ is closed in $H^{3/2}(\mathcal{C})$ then θ is obviously self-adjoint as symmetric operator on the full set V.

Proposition 18 If $\mathcal{R}(A)$ is closed in $H^{-3/2}(\mathcal{C})$ then θ is self-adjoint.

Proof. Since Λ^{-1} is unitary mapping we have that $\mathcal{R}(A)$ is closed in $H^{-3/2}(\mathcal{C})$ if and only if $\mathcal{R}\left(\Lambda^{-1}A\right)$ is closed in $H^{3/2}(\mathcal{C})$. □

An easy example of an operator having a closed range in $H^{-3/2}(\mathcal{C})$ is a finite rank operator. For the next examples the following observation is very useful.

Proposition 19 Under the chosen assumptions we have

$$\mathcal{R}\left(\Lambda^{-1}A\right)^{\perp} = \mathcal{R}(A)^{\perp} \cap H^{3/2}(\mathcal{C})$$

where the orthogonal complement on the left side is taken w.r.t. the $H^{3/2}(\mathcal{C})$-scalar product and the one on the right side w.r.t the $L^2(\mathcal{C})$-scalar product.

Proof. Take $u \in \mathcal{R}\left(\Lambda^{-1}A\right)^{\perp}$, then

$$0 = \left\langle u, \Lambda^{-1}Av\right\rangle_{H^{3/2}(\mathcal{C})} = \langle u, Av\rangle_{L^2(\mathcal{C})}$$

for every $v \in D(A)$. Hence $u \in \mathcal{R}(A)^{\perp} \cap H^{3/2}(\mathcal{C})$.
For $u \in \mathcal{R}(A)^{\perp} \cap H^{3/2}(\mathcal{C})$ we get

$$0 = \langle u, Av\rangle_{L^2(\mathcal{C})} = \left\langle u, \Lambda^{-1}Av\right\rangle_{H^{3/2}(\mathcal{C})}$$

for every $v \in D(A)$ and we are done. □

Example 2 Multiplication operator Our next example will be the multiplication by a regular function.

Theorem 20 Take a bounded and measurable $\eta : \mathcal{C} \to \mathbb{R}$ such that the multiplication operator given by η maps for some $0 < \varepsilon \leq 1$ $H^s(\mathcal{C})$ to $H^{s-1+\varepsilon}(\mathcal{C})$ for any $s \in \left[-\frac{1}{2}, \frac{3}{2}\right]$. Here the multiplication operator given by η is defined as $M_\eta : L^2(\mathcal{C}) \to L^2(\mathcal{C})$ with $M_\eta f = \eta f$. The multiplication operator for $H^s(\mathcal{C})$ with $s < 0$ is then given by the adjoint of M_η w.r.t. the $L^2(\mathcal{C})$-dual pairing. Define now A as restriction of M_η to $H^{3/2}(\mathcal{C})$. Then the restriction T_η of T^* to $D(T_\eta) = \left\{f \in D(T^*) \,:\, \tilde{\tau}f \in H^{3/2}(\mathcal{C}) \text{ and } \kappa f = \eta\tilde{\tau}f\right\}$ is a self-adjoint extension of T.

Remark 21 It is mentioned in [26] that one can choose $\eta \in C^{0,1/2+\varepsilon}(\mathcal{C})$ to get a sufficiently regular multiplication operator.

Proof. We prove

$$\left\{f \in D(M_\eta) \,:\, (\mathrm{Id} - P_\alpha M_\eta) f \in H^{3/2}(\mathcal{C})\right\} = H^{3/2}(\mathcal{C}). \tag{2.14}$$

If this holds true then the operator A is the restriction of M_η to

$$\left\{f \in D(M_\eta) \,:\, (\mathrm{Id} - P_\alpha M_\eta) f \in H^{3/2}(\mathcal{C})\right\}$$

and hence A is of the form which we need in theorem 13. We start the proof with

Lemma 22 $\{f \in D(M_\eta) \,:\, (\mathrm{Id} - P_\alpha M_\eta) f \in H^{3/2}(\mathcal{C})\} = H^{3/2}(\mathcal{C})$

Proof. Take $f \in H^{3/2}(\mathcal{C})$. Then $M_\eta f \in H^{1/2+\varepsilon}(\mathcal{C})$ by the mapping properties of M_η and hence $P_\alpha M_\eta f \in H^{3/2+\varepsilon}(\mathcal{C})$ by (2.6). Therefore $f - P_\alpha M_\eta f \in H^{3/2}(\mathcal{C})$.
Take now $f \in D(M_\eta)$ such that $(\mathrm{Id} - P_\alpha M_\eta) f \in H^{3/2}(\mathcal{C})$. Then we can write $f = u + P_\alpha M_\eta f$ for some $u \in H^{3/2}(\mathcal{C})$. For $f \in D(M_\eta) = L^2(\mathcal{C})$ we get $P_\alpha M_\eta f \in H^\varepsilon(\mathcal{C})$ since M_η maps $L^2(\mathcal{C})$ to $H^{-1+\varepsilon}(\mathcal{C})$ and P_α maps $H^{-1+\varepsilon}(\mathcal{C})$ to $H^\varepsilon(\mathcal{C})$ by (2.6). Hence $f \in H^\varepsilon(\mathcal{C})$ as sum of $u \in H^{3/2}(\mathcal{C})$ and $P_\alpha M_\eta f \in H^\varepsilon(\mathcal{C})$. But for $f \in H^\varepsilon(\mathcal{C})$ we get $P_\alpha M_\eta f \in H^{2\varepsilon}(\mathcal{C})$ and hence $f \in H^{2\varepsilon}(\mathcal{C})$. By an iteration of this argument we will end up with the fact that $f \in H^{3/2}(\mathcal{C})$ and (2.14) is proven. □

Now we want to show that the operator A fulfills the conditions of theorem 14, i.e.

(a) A is symmetric,

(b) $D(A)$ is dense in $H^{3/2}(\mathcal{C})$,

(c) $\mathrm{Id} - P_\alpha A : D(A) \subseteq L^2(\mathcal{C}) \to H^{3/2}(\mathcal{C})$,

(d) $\mathcal{R}\left(\Lambda^{-1}A\right)^\perp = \mathcal{N}(A)$,

(e) $\mathcal{R}(P_\alpha \Lambda \Pi) \subseteq D\left(\tilde{A}\right)$

and that (2.11) holds.
The conditions (a) and (b) are obviously fulfilled by the definition of A and property (c) follows immediately by lemma 22.
Condition (d) we will get in the following way: Take $\phi \in L^2(\mathcal{C})$ and a sequence $\phi_n \in H^{3/2}(\mathcal{C})$ such that $\phi_n \to \phi$ in $L^2(\mathcal{C})$. Then since η is bounded we have $\eta\phi_n \to \eta\phi$ in $L^2(\mathcal{C})$. Therefore $\mathcal{R}(A)$ is dense in $\mathcal{R}(M_\eta)$. Due to proposition 14 we have that

$$\mathcal{R}\left(\Lambda^{-1}A\right)^\perp = \mathcal{R}(A)^\perp \cap H^{3/2}(\mathcal{C})$$

and hence

$$\mathcal{R}\left(\Lambda^{-1}A\right)^\perp = \mathcal{R}(M_\eta)^\perp \cap H^{3/2}(\mathcal{C}).$$

But since η is bounded and real-valued M_η is self-adjoint on $L^2(\mathcal{C})$ and therefore $\mathcal{R}(M_\eta)^\perp = \mathcal{N}(M_\eta)$. So we have

$$\mathcal{R}\left(\Lambda^{-1}A\right)^\perp = \mathcal{N}(M_\eta) \cap H^{3/2}(\mathcal{C}) = \mathcal{N}(A)$$

by the definition of A.

Since A is just the restriction of M_η to $H^{3/2}(\mathcal{C})$ we obviously have that the adjoint of A is just M_η and by the mapping properties of M_η we have that $H^{-1/2}(\mathcal{C}) = D\left(\tilde{M}_\eta\right)$. But we also have that $\Lambda\Pi$ maps $H^{3/2}(\mathcal{C})$ into $H^{-3/2}(\mathcal{C})$ and hence by (2.6) we get $\mathcal{R}(P_\alpha\Lambda\Pi) \subseteq H^{-1/2}(\mathcal{C})$ and property (e) is proven.

Finally, in order to apply theorem 14 we need to check if (2.11) holds. Take $g, v \in H^{3/2}(\mathcal{C})$ such that

$$\left(\mathrm{Id} - \tilde{M}_\eta P_\alpha\right)\Lambda\Pi g = \tilde{M}_\eta v.$$

Since $\tilde{M}_\eta v = M_\eta v \in H^{1/2}(\mathcal{C})$ we get $\left(\mathrm{Id} - \tilde{M}_\eta P_\alpha\right)\Lambda\Pi g \in H^{1/2}(\mathcal{C})$. $\Lambda\Pi g \in H^{-3/2}(\mathcal{C})$ by the definition of Λ. But P_α maps $H^{-3/2}(\mathcal{C})$ to $H^{-1/2}(\mathcal{C})$ by (2.6) and $\tilde{M}_\eta$ maps $H^{-1/2}(\mathcal{C})$ to $H^{-3/2+\varepsilon}(\mathcal{C})$ and hence $\Lambda\Pi g \in H^{-3/2+\varepsilon}(\mathcal{C})$. By an iteration of this argument we get $\Lambda\Pi g \in H^{1/2}(\mathcal{C})$ and hence $P_\alpha\Lambda\Pi g \in H^{3/2}(\mathcal{C})$. This implies that $\Lambda\Pi g = M_\eta(v + P_\alpha\Lambda\Pi g)$ with $v + P_\alpha\Lambda\Pi g \in H^{3/2}(\mathcal{C})$. Hence $\Pi g \in \mathcal{R}\left(\Lambda^{-1}A\right)$ and we are done. □

In subsequent chapters we will need some properties of the constructed T_η. Indeed we have that $D(T_\eta)$ consists of continuous functions and we can represent T_η with the help of some Dirac-delta distributions on $\mathcal{C}$.

Corollary 23 The set $D(T_\eta)$ is contained in the continuous functions over $\mathbb{R}^2$ and for any $f \in D(T_\eta)$ we have

$$T_\eta f = \frac{1}{2}\Delta f + \int_{\mathcal{C}} \eta(y) f(y) \delta_0(\cdot - y)\, \mathrm{d}S(y) \tag{2.15}$$

where $\delta_0(\cdot)$ is the Dirac-delta distribution in $\mathbb{R}^2$ at 0.

Proof. We will use the representation of elements of $D(T^*)$ shown in corollary 6, p.15. There it is proven that for every $f \in D(T_\eta)$ we can find a $v \in H^{2,2}\left(\mathbb{R}^2\right)$ such that $f = v + S_z\kappa f$ where S_z is the acoustic single layer potential for some fixed z used in (2.2). For further details see pp.12. But for $f \in D(T_\eta)$ we get that $\kappa f = \eta\tilde{\tau} f$ with $\tilde{\tau} f \in H^{3/2}(\mathcal{C})$. And hence by the mapping properties of η we have $\kappa f \in H^{1/2+\varepsilon}(\mathcal{C})$. By the usual Sobolev embeddings, see e.g. [13], pp.275 we get $\kappa f \in C^0(\mathcal{C})$. But it is shown in [30],Ch.201, pp.528 that $S_z g \in C^0\left(\mathbb{R}^2\right)$ for any $g \in C^0(\mathcal{C})$. Therefore we have that $S_z\kappa f \in C^0\left(\mathbb{R}^2\right)$ and since $H^{2,2}\left(\mathbb{R}^2\right) \subseteq C^0\left(\mathbb{R}^2\right)$, see again [13], pp.275, we can obtain that f is continuous.

Write again $f = v + S_z \kappa f$. It is shown, again in [30], Ch.201, pp.528, that in a distributional sense we get

$$\left(-\frac{1}{2}\Delta + z\right) S_\alpha \kappa f = \int_{\mathcal{C}} \kappa f(y) \delta_0 (\cdot - y) \, \mathrm{d}S(y).$$

But we assumed that $f \in D(T_\eta)$. Hence $\kappa f = \eta \tilde{\tau} f$. From the first part of the proof we can deduce that f is continuous and thus we can obtain that

$$\left(-\frac{1}{2}\Delta + z\right) S_z \kappa f = \int_{\mathcal{C}} \eta(y) f(y) \delta_0 (\cdot - y) \, \mathrm{d}S(y).$$

therefore we get again in a distributional sense that

$$\left(-\frac{1}{2}\Delta + z\right) f = \left(-\frac{1}{2}\Delta + z\right) v + \int_{\mathcal{C}} \eta(y) f(y) \delta_0 (\cdot - y) \, \mathrm{d}S(y). \tag{2.16}$$

But due to the definition of the self-adjoint extensions given in (2.2) we get the equality $(-T_\eta + z) f = \left(-\frac{1}{2}\Delta + z\right) v$. Thus (2.15) follows immediately by (2.16). □

Example 3 Laplace-Beltrami operator Our last example will be the Laplace-Beltrami operator on a suitable domain.

Theorem 24 Denote by Δ_{LB} the self-adjoint Laplace-Beltrami operator from $H^2(\mathcal{C}) \subseteq L^2(\mathcal{C})$ to $L^2(\mathcal{C})$. We will denote by D_{LB} the distributional Laplace-Beltrami which has the properties that

$$D_{LB} \, : \, H^s(\mathcal{C}) \to H^{s-2}(\mathcal{C}) \text{ for any } s \in \mathbb{R}, \tag{2.17}$$

$$\text{If } f \in H^s(\mathcal{C}) \text{ and } D_{LB} f \in H^{s-1}(\mathcal{C}) \text{ then } f \in H^{s+1}(\mathcal{C}) \text{ for any } s \in \mathbb{R}. \tag{2.18}$$

For further details see e.g. [23], Ch.2, Section 3.
Define now A as restriction of Δ_{LB} to the set $H^{5/2}(\mathcal{C})$. Then the restriction of T^* to $\left\{f \in D(T^*) \, : \, \tilde{\tau} f \in H^{5/2}(\mathcal{C}) \text{ and } \kappa f = \Delta_{LB} \tilde{\tau} f\right\}$ is a self-adjoint extension of T.

Proof. We prove

$$\left\{f \in D(\Delta_{LB}) \, : \, (\mathrm{Id} - P_\alpha \Delta_{LB}) f \in H^{3/2}(\mathcal{C})\right\} = H^{5/2}(\mathcal{C}). \tag{2.19}$$

If this holds true then the operator A is the restriction of Δ_{LB} to

$$\left\{f \in D(\Delta_{LB}) \, : \, (\mathrm{Id} - P_\alpha \Delta_{LB}) f \in H^{3/2}(\mathcal{C})\right\}$$

and hence A is of the form which we need in theorem 13 (a). We start the proof with

Lemma 25 $\left\{f \in D(\Delta_{LB}) \, : \, (\mathrm{Id} - P_\alpha \Delta_{LB}) f \in H^{3/2}(\mathcal{C})\right\} = H^{5/2}(\mathcal{C})$

Proof. Take $f \in H^{5/2}(\mathcal{C})$. Then $\Delta_{LB} f \in H^{1/2}(\mathcal{C})$ by (2.17) and thus we can obtain by (2.6) that $P_\alpha \Delta_{LB} f \in H^{3/2}(\mathcal{C})$. Therefore $f - P_\alpha \Delta_{LB} f \in H^{3/2}(\mathcal{C})$.
Take now $f \in D(\Delta_{LB})$ such that $(\mathrm{Id} - P_\alpha \Delta_{LB}) f = u \in H^{3/2}(\mathcal{C})$. For any $f \in D(\Delta_{LB}) = H^2(\mathcal{C})$ we get $P_\alpha \Delta_{LB} f \in H^{3/2}(\mathcal{C})$ as sum of $u \in H^{3/2}(\mathcal{C})$ and $f \in H^2(\mathcal{C})$. Hence $\Delta_{LB} f \in H^{1/2}(\mathcal{C})$ by (2.7). Thus $f \in H^{3/2}(\mathcal{C})$ and $\Delta_{LB} f \in H^{1/2}(\mathcal{C})$ hence $f \in H^{5/2}(\mathcal{C})$ by (2.18) and (2.19) is proven. □

As in the example before we have to check whether

(a) A is symmetric,

(b) $D(A)$ is dense in $H^{3/2}(\mathcal{C})$,

(c) $\mathrm{Id} - P_\alpha A : D(A) \subseteq L^2(\mathcal{C}) \to H^{3/2}(\mathcal{C})$,

(d) $\mathcal{R}(\Lambda^{-1} A)^\perp = \mathcal{N}(A)$,

(e) $\mathcal{R}(P_\alpha \Lambda \Pi) \subseteq D\left(\tilde{A}\right)$

and that (2.11) holds.
The conditions (a) and (b) are obviously fulfilled by the definition of A and property (c) follows immediately by lemma 25.
Condition (d) follows by the fact that the closure of A w.r.t. $L^2(\mathcal{C})$ is Δ_{LB} which is shown in [28], p.160, example 4. Therefore the image of A is dense in the range of Δ_{LB} w.r.t. $L^2(\mathcal{C})$. Combining now the self-adjointness of Δ_{LB} with proposition 14 we get as in example 2

$$\mathcal{R}(\Lambda^{-1} A)^\perp = \mathcal{N}(\Delta_{LB}) \cap H^{3/2}(\mathcal{C}) = \mathcal{N}(\Delta_{LB}) \tag{2.20}$$

by the definition of Δ_{LB}. In [28], p.160, example 4 it is also shown that $\mathcal{N}(\Delta_{LB}) \subseteq C^\infty(\mathcal{C})$. Since $C^\infty(\mathcal{C}) \subseteq H^{5/2}(\mathcal{C})$ we obtain that $\mathcal{N}(\Delta_{LB}) = \mathcal{N}(A)$. This implies by (2.20) that $\mathcal{R}(\Lambda^{-1} A)^\perp = \mathcal{N}(A)$ and condition (d) is proven.
Since $\tilde{A} = D_{LB}$, see [28], p.160, example 4 , we have that $D\left(\tilde{A}\right) = H^{-1/2}(\mathcal{C})$. By (2.6) $P_\alpha \Lambda \Pi$ is a mapping from $H^{3/2}(\mathcal{C})$ to $H^{-1/2}(\mathcal{C})$. Hence $\mathcal{R}(P_\alpha \Lambda \Pi) \subseteq D\left(\tilde{A}\right)$ which is condition (e).
Since $\tilde{A} = D_{LB}$ (2.11) is equivalent to

$$(\mathrm{Id} - D_{LB} P_\alpha) \Lambda \Pi g = D_{LB} v \Rightarrow \Pi g \in \mathcal{R}(\Lambda^{-1} A). \tag{2.21}$$

for every $g,\, v \in H^{3/2}(\mathcal{C})$.
To proof that (2.21) holds true take $g,\, v \in H^{3/2}(\mathcal{C})$ such that

$$(\mathrm{Id} - D_{LB} P_\alpha) \Lambda \Pi g = D_{LB} v.$$

Since $\Lambda\Pi g \in H^{-3/2}(\mathcal{C})$ we get by (2.6) that $P_\alpha\Lambda\Pi g \in H^{-1/2}(\mathcal{C})$. But we also have that $D_{LB}P_\alpha\Lambda\Pi g = \Lambda\Pi g - D_{LB}v$. Since $v \in H^{3/2}(\mathcal{C})$ we get by (2.17) that $D_{LB}v \in H^{-1/2}(\mathcal{C})$ and therefore $D_{LB}P_\alpha\Lambda\Pi g \in H^{-3/2}(\mathcal{C})$. This implies by (2.18) that $P_\alpha\Lambda\Pi g \in H^{1/2}(\mathcal{C})$. Hence due to (2.7) we have $\Lambda\Pi g \in H^{-1/2}(\mathcal{C})$. Following the same argumentation line now starting with $\Lambda\Pi g \in H^{-1/2}(\mathcal{C})$ we get $\Lambda\Pi g \in H^{1/2}(\mathcal{C})$. By (2.21) we can write $\Lambda\Pi g = D_{LB}(v + P_\alpha\Lambda\Pi g)$. But $v + P_\alpha\Lambda\Pi g \in H^{3/2}(\mathcal{C})$ by (2.6) and since $\Lambda\Pi g \in H^{1/2}(\mathcal{C})$ we get by (2.18) that $v + P_\alpha\Lambda\Pi g \in H^{5/2}(\mathcal{C}) = D(A)$ and hence $\Pi g \in \mathcal{R}\left(\Lambda^{-1}A\right)$ and we are done. □

Open problems It is still an open question if the multiplication by an indicator function of some subset U of $\mathcal{C}$ gives us a self-adjoint extension of T. The difficulty arises in the question whether $\mathcal{R}(P\Lambda\Pi) \subseteq D\left(\tilde{A}\right)$.

Chapter 3

A Tanaka type formula

Assume we have a function $g \in C^0(\mathbb{R}^2)$ and for $\alpha > 0$ a closed $C^{2,\alpha}$-curve $\mathcal{C} \subseteq \mathbb{R}^2$, such that g is harmonic in the complement of $\mathcal{C}$ in $\mathbb{R}^2$, i.e. $\Delta g = 0$ in $\mathbb{R}^2 \backslash \mathcal{C}$. But $\kappa g \neq 0$, i.e. the normal derivative has a jump along $\mathcal{C}$, and therefore $g \notin C^1(\mathbb{R}^2)$. What can we say about $g(B_t)$, where B_t is a Brownian motion in $\mathbb{R}^2$?

The question of dealing with functions having a jump in the normal derivatives along $\mathcal{C}$ naturally arises in the context of the self-adjoint extensions constructed in the chapter before. Remember that we have shown that the extensions are given via a jump relation. Thus if we want to do some stochastic with such functions we have to know how we can gain control of these jumps in connection with stochastic differential equations. For a detailed study how we can use the results of chapter 3 in the case of the self-adjoint extensions constructed in chapter 2 see chapter 4, especially theorem 57 for the explicit use of a single-layer potential.

The idea behind this chapter is to generalize Tanaka's formula for the absolute value function, which obviously fulfills the same properties as the function g in two dimensions. For the absolute value function one gets

$$|B_t - x| = |B_0 - x| + \int_0^t \operatorname{sgn}(B_s - x)\,\mathrm{d}B_s + L_t^x$$

where L_t^x is a local time of the Brownian motion in the point x, namely the Radon-Nikodyn derivative of the Brownian occupation measure, i.e.

$$\mu(a,b) = \int_0^t \mathbb{1}_{(a,b)}(B_s)\,\mathrm{d}s = \int_a^b L_t^x \mathrm{d}x \tag{3.1}$$

for $a, b \in \mathbb{R}$, $a \leq b$. The aim of this section is the proof of the following theorem:

Theorem 26 Let $\mathcal{C}$ be chosen as before, $\mathrm{d}S$ the usual volume measure on $\mathcal{C}$ and $\eta \in C^0(\mathcal{C})$.

Define

$$g\left(x\right)=-\frac{1}{2\pi}\int_{\mathcal{C}}\eta\left(y\right)\ln\left(\left|x-y\right|\right)\mathrm{d}S(y). \tag{3.2}$$

The function g is called harmonic single-layer potential with layer function η.
We have

$$g\left(B_t\right)=g\left(B_0\right)+\int\limits_0^t\nabla g\left(B_s\right)\mathrm{d}B_s+\int\limits_0^t\eta\left(B_s\right)\mathrm{d}L\left(s,\mathcal{C}\right) \tag{3.3}$$

where $L\left(t,\mathcal{C}\right)$ is the invariant local time of the Brownian motion on $\mathcal{C}$ in the sense of [4], i.e. $L\left(t,\mathcal{C}\right)$ is a non-decreasing, time-continuous and adapted process which only increases if B_t is on $\mathcal{C}$, see [4], pp.216.

In addition, as already mentioned in the introduction we are able to prove that

$$\int\limits_0^t\mathbb{1}_{\mathcal{C}_\varepsilon}\left(B_s\right)\mathrm{d}s=\int\limits_{-\varepsilon}^{\varepsilon}\frac{1}{2}L\left(t,\mathcal{C}^r\right)\mathrm{d}r \tag{3.4}$$

where $\mathcal{C}_\varepsilon$ is just the set of points in $\mathbb{R}^2$ with distance less than ε to $\mathcal{C}$ and $L\left(t,\mathcal{C}^r\right)$ is the local time process for parallel curves $\mathcal{C}^r$. The proof is based on our choice of f_ε which allows us to generalize the proof of (3.1) in [5]. For more details see proposition 51.
The chapter is organized in the following way: First of all we will collect basic facts on g and $\mathcal{C}$. Afterwards we will prove theorem 26. This proof is split into several parts:

- **Approximation procedure** At first we want to construct a sequence $f_\varepsilon\in C^1\left(\mathbb{R}^2\right)$ with $\Delta f_\varepsilon\in L^\infty\left(\mathbb{R}^2\right)$ such that $f_\varepsilon\to g$ uniformly in $\mathbb{R}^2$ and $\nabla f_\varepsilon\to\nabla g$ point-wise in $\mathbb{R}^2\backslash\mathcal{C}$ if ε tends to zero. What we have in mind is to generalize the approximation of the absolute value function used in [5]. There it is shown that

$$h_\varepsilon(x)=\frac{1}{2\varepsilon}\int\limits_{-\varepsilon}^{\varepsilon}|x-y|\mathrm{d}y$$

admits an uniform approximation of the absolute value with

$$h'_\varepsilon=\frac{1}{2\varepsilon}\int\limits_{-\varepsilon}^{\varepsilon}\mathrm{sgn}(x-y)\mathrm{d}y. \tag{3.5}$$

This proof is not really hard because one can calculate h_ε explicitly. But then (3.5) is used to proof (3.1). We want to perform a similar approximation using single-layer potentials g_r for parallel curves of $\mathcal{C}$. We will define

$$f_\varepsilon(x)=\frac{1}{2\varepsilon}\int\limits_{-\varepsilon}^{\varepsilon}g_r(x)\mathrm{d}r$$

and show that

$$\nabla f_\varepsilon(x) = \frac{1}{2\varepsilon}\int\limits_{-\varepsilon}^{\varepsilon} \nabla g_r(x)\mathrm{d}r.$$

The proof of the uniform convergence of f_ε to g and the explicit formula for ∇f_ε is much harder than the proofs for h_ε because we cannot use an explicit calculation of f_ε.

- **Ito formula** The next step is to show that $\int\limits_0^t \nabla f_\varepsilon(B_s)\,\mathrm{d}B_s \to \int\limits_0^t \nabla g(B_s)\,\mathrm{d}B_s$ and then deduce by use of the Ito formula for f_ε that we have

$$g(B_t) = g(B_0) + \int\limits_0^t \nabla g(B_s)\,\mathrm{d}B_s + \lim_{n\to\infty}\int\limits_0^t \frac{1}{2}\Delta f_\varepsilon(B_s)\,\mathrm{d}s.$$

- **Identifying the limit point** The last step is to identify $\lim\limits_{\varepsilon\to 0}\int\limits_0^t \frac{1}{2}\Delta f_\varepsilon(B_s)\,\mathrm{d}s$ with $\int\limits_0^t \eta(B_s)\,\mathrm{d}L(s,\mathcal{C})$.

At the end of the chapter we will give the proof of (3.4).

3.1 Notations and basic facts

Before we start with the approximation procedure let us fix some notation and collect properties of g and some geometric facts on $\mathcal{C}$ which we will need later on.

Notation For any $x \in \mathbb{R}^2$ and $\delta > 0$ $B_\delta(x)$ is the closed ball around x with radius δ. By $\|\cdot\|_{\infty,U}$ we will denote the $L^\infty(U)$-norm and for $z \in \mathcal{C}$ n_z is the outer normal of $\mathcal{C}$ in z. For $r > 0$ denote by $\mathcal{C}_r$ the set

$$\left\{x \in \mathbb{R}^2 \,:\, d(x,\mathcal{C}) < r\right\}$$

where $d(x,\mathcal{C})$ is the usual distance of x to $\mathcal{C}$. By $\mathcal{C}^r$ we will denote a parallel surface of $\mathcal{C}$, i.e.

$$\mathcal{C}^r = \left\{x \in \mathbb{R}^2 \,:\, x = z + rn_z \text{ for some } z \in \mathcal{C}\right\}$$

for fixed $r \in \mathbb{R}$. Such a parallel surface is called a Runge parallel surface if it has the same regularity properties as $\mathcal{C}$, see e.g. [30], Ch.200, pp.513.

Lemma 27 For a $C^{2,\alpha}$-surface $\mathcal{C}$ it exists an $\beta > 0$ such that for every $x \in \mathcal{C}_\beta$ it exists a unique $z \in \mathcal{C}$ with $x = z + rn_z$ for some $r \in (-\beta,\beta)$ and such that $\mathcal{C}^r$ is a Runge parallel surface of $\mathcal{C}$ for every $r \in (-\beta,\beta)$.

Proof. A proof of this statement can be found in [30], Ch.200, pp.513. □

Notation Fix $\beta > 0$ such that $\mathcal{C}^r$ is a Runge parallel surface of $\mathcal{C}$ for every $r \in (-\beta, \beta)$ and denote for every $x \in \mathcal{C}_\beta$ by $\Pi_{\mathcal{C}}(x)$ the unique $z \in \mathcal{C}$ such that $x = z + rn_z$ for some $r \in (-\beta, \beta)$.

Lemma 28 For an integrable function f on $\mathcal{C}_\beta$ one has

$$\int_{\mathcal{C}_\beta} f(x)\,\mathrm{d}x = \int_{-\beta}^{\beta} \int_{\mathcal{C}^r} f(y)\,\mathrm{d}S_r(y)\,\mathrm{d}r \tag{3.6}$$

and it exists an interval $(-\alpha, \alpha) \subseteq (-\beta, \beta)$ such that for any $r \in (-\alpha, \alpha)$

$$\int_{\mathcal{C}^r} f(y)\,\mathrm{d}S_r(y) = \int_{\mathcal{C}} f(y + rn_y)(1 + r\kappa(y))\,\mathrm{d}S(y) \tag{3.7}$$

where $\kappa(y)$ is the curvature of $\mathcal{C}$ in y.

Proof. A proof of these statements can be found in [30], Ch.200, pp.513.
A short proof for the existence of the interval $(-\alpha, \alpha)$ can be given in the following way: Take $\psi : [a, b] \to \mathbb{R}^2$ to be a parametrization of $\mathcal{C}$. Denote by $n_{\psi(t)}$ the unit outer normal vector in $\psi(t)$. Then $\psi_r : [a, b] \to \mathbb{R}^2$ with $\psi_r(t) = \psi(t) + rn_{\psi(t)}$ is a parametrization of $\mathcal{C}^r$ for any $r \in (-\beta, \beta)$. Hence

$$\int_{\mathcal{C}^r} f(y)\,\mathrm{d}S_r(y) = \int_a^b f(\psi_r(t)) \left\|\dot{\psi}_r(t)\right\| \mathrm{d}t.$$

The derivative $\dot{\psi}_r(t)$ is obviously given by $\dot{\psi}(t) + r\dot{n}_{\psi(t)}$. But it is well known, see e.g. [13], appendix B, that $\dot{n}_{\psi(t)} = \kappa(\psi(t))\dot{\psi}(t)$ where $\kappa(\psi(t))$ is the curvature of $\mathcal{C}$ in $\psi(t)$. Therefore $\left\|\dot{\psi}_r(t)\right\| = |1 + r\kappa(\psi(t))| \left\|\dot{\psi}(t)\right\|$ and thus we obtain

$$\begin{aligned} \int_{\mathcal{C}^r} f(y)\,\mathrm{d}S_r(y) &= \int_a^b f(\psi_r(t)) \left\|\dot{\psi}_r(t)\right\| \mathrm{d}t \\ &= \int_a^b f\left(\psi(t) + rn_{\psi(t)}\right) |1 + r\kappa(\psi(t))| \left\|\dot{\psi}(t)\right\| \mathrm{d}t. \end{aligned}$$

Since $\mathcal{C}$ is a closed $C^{2,\alpha}$-curve we have that $\kappa(y)$ is a continuous and therefore bounded function, see again [13], appendix B. If we define $\alpha = \frac{1}{\sup\limits_{t\in[a,b]} |\kappa(\psi(t))|}$ then $\alpha > 0$ since $\kappa(y)$ is bounded and for every $r \in (-\alpha, \alpha)$ we have that $|1 + r\kappa(\psi(t))| = 1 + r\kappa(y)$ and (3.7) is shown. By the continuity of $\kappa(y)$ we get that $|1 + r\kappa(y)|$ is a non-negative, continuous function on $(-\alpha, \alpha) \times \mathcal{C}$. □

Notations For the rest of this chapter we fix $\beta = \alpha$ and we define $\phi(r, y) = 1 + r\kappa(y)$. Now we want to give some bounds on the scalar product of elements of $\mathcal{C}$ with the normal field.

Lemma 29 There is a positive constant M such that for any $r \in (-\beta, \beta)$ and every x, $y \in \mathcal{C}^r$

$$|\langle x - y, n_y \rangle| \leq M \, |x - y|^2 \, .$$

Proof. A proof of this statement can be found in [15], Ch.3, pp.151. $\square$

This lemma allows us to deduce the following.

Corollary 30 It exists an $R > 0$ and a $K > 0$ such that for every r_1, $r_2 \in (-R, R)$ and every x, $y \in \mathcal{C}^r$

$$|x + r_1 n_x - y - r_2 n_y| \geq K \, |x - y| \, . \tag{3.8}$$

Proof. We get

$$\begin{aligned} |x + r_1 n_x - y - r_2 n_y|^2 &= |x - y|^2 + 2 \langle x - y, r_1 n_x - r_2 n_y \rangle + |r_1 n_x - r_2 n_y|^2 \\ &\geq |x - y|^2 - 2 \, |r_1| \, |\langle x - y, n_x \rangle| - 2 \, |r_2| \, |\langle x - y, n_y \rangle| \\ &\geq |x - y|^2 - 2 \, |r_1| \, C_y \, |x - y|^2 - 2 \, |r_2| \, C_x \, |x - y|^2 \\ &\geq |x - y|^2 \left(1 - 2M \left(|r_1| + |r_2|\right)\right) \end{aligned}$$

by the lemma before. Fix now some $0 < R < \frac{1}{4M}$. Then for any r_1, $r_2 \in (-R, R)$ we have

$$1 - 2M \left(|r_1| + |r_2|\right) \geq 1 - 4MR > 0.$$

Fix now $K = \sqrt{1 - 4MR}$ and we are done. $\square$

Notations For the rest of this chapter fix $K > 0$ such that (3.8) holds.
The corollary 30 gives us the following.

Lemma 31 It exists a constant $W > 0$ and for any $0 < \delta < \frac{1}{2}$ an $R > 0$ such that for all $r \in (-R, R)$ and all x, $y \in \mathcal{C}^r$ with $|x - y| < \delta$ we have

$$|\ln \left(|x - y - r n_y|\right)| \leq W + |\ln \left(|x - y|\right)| \, . \tag{3.9}$$

Proof. Fix $0 < \delta < \frac{1}{2}$. Choose $R_1 > 0$ such that (3.8) holds and define $R = \min\{\frac{1}{2}, R_1\}$. Then for $r \in (-R, R)$ we get by the monotonicity of the logarithm that

$$\ln \left(|x - y - r n_y|\right) \geq \ln \left(K \, |x - y|\right) .$$

But $|x-y-rn_y| \leq |x-y| + |r| \leq \delta + \frac{1}{2} \leq 1$ by our choice of R. Hence

$$\begin{aligned} |\ln(|x-y-rn_y|)| &= -\ln(|x-y-rn_y|) \\ &\leq -\ln(K|x-y|) \\ &= |\ln(K|x-y|)| \\ &\leq |\ln(K)| + |\ln(|x-y|)|. \end{aligned}$$

Define $W = |\ln(K)|$ and we are done. □

Definitions Take $x \in \mathcal{C}_\beta$. For η continuous on $\mathcal{C}$ define

$$\tilde{\eta}(x) = \eta(\Pi_{\mathcal{C}}(x)) \tag{3.10}$$

and for $x \in \mathbb{R}^2$ we define

$$g_r(x) = -\frac{1}{2\pi} \int_{\mathcal{C}^r} \tilde{\eta}(y) \ln(|x-y|) \,\mathrm{d}S_r(y). \tag{3.11}$$

We obviously have that $g_0 = g$.

Proposition 32 For any $r \in (-\beta, \beta)$ the function g_r fulfills

- g_r is continuous in $\mathbb{R}^2$
- $\kappa g_r = -\tilde{\eta}$
- $\Delta g_r = 0$ in $\mathbb{R}^2 \backslash \mathcal{C}$.

For any $x \in \mathcal{C}^r$ define

$$H_r(y) = \int_{\mathcal{C}^r} \frac{\cos(x-y, n_x)}{|x-y|} \mathrm{d}S_r(y)$$

where $(x-y, n_x)$ is the angle between $x-y$ and n_x. One has

- H_r is continuous on $\mathcal{C}^r$
- It exist positive constants K_1 and K_2 such that for every $r \in (-\beta, \beta)$

$$\|\nabla g_r\|_{\infty,\mathbb{R}^2} \leq K_1 + \|\eta\|_{\infty,\mathcal{C}} \left(K_2 + \|H_r\|_{\infty,\mathcal{C}^r}\right). \tag{3.12}$$

Remark 33 $\nabla g_r(x)$ is not well-defined on $\mathcal{C}^r$ but $\nabla g_r(x)$ can be continuously continued to $\mathcal{C}^r$ from the inner domain and from the outer domain, respectively. Both continuations are bounded on $\mathcal{C}$ and hence we could talk about $\|\nabla g_r\|_{\infty,\mathbb{R}^2}$ and if we denote by $\mathcal{C}^r_{int} \subseteq \mathbb{R}^2$ the set bounded by $\mathcal{C}^r$ and by $\mathcal{C}^r_{ext} = \mathbb{R}^2 \backslash \overline{\mathcal{C}^r_{int}}$ then the classical mean value theorem holds on $\overline{\mathcal{C}^r_{int}}$ and on $\overline{\mathcal{C}^r_{ext}}$ with the related continuous extensions of ∇g_r to $\mathcal{C}^r$, respectively. For further information see [30], Ch.192ff, pp.489.

Proof. A proof of these statements can be found in [30], Ch.192ff, pp.489. □

We have even more:

Lemma 34 $g_r(x)$ is continuous in r for fixed x.

Proof. Fix $x \in \mathbb{R}^2$ and choose $r_m,\ r \in (-\beta, \beta)$ such that $r_m \to r$. By (3.7) and the definition of $\tilde{\eta}$ we have

$$\begin{aligned} g_{r_m}(x) &= -\frac{1}{2\pi} \int\limits_{\mathcal{C}^{r_m}} \tilde{\eta}(y) \ln(|x-y|)\, \mathrm{d}S_{r_m}(y) \\ &= -\frac{1}{2\pi} \int\limits_{\mathcal{C}} \eta(y) \ln(|x-y-r_m n_y|)\, \phi(r_m, y)\, \mathrm{d}S(y). \end{aligned}$$

By the continuity of ϕ we get

$$\eta(y) \ln(|x-y-r_m n_y|)\, \phi(r_m, y) \to \eta(y) \ln(|x-y-r n_y|)\, \phi(r, y)$$

for every $y - r n_y \neq x$. If $x \notin \mathcal{C}^r$ one has $0 < m \leq |x-y-r_m n_y| \leq M < \infty$ for any $|r_m| < d(x, \mathcal{C}^r)$. Hence

$$|\eta(y) \ln(|x-y-r_m n_y|)\, \phi(r_m, y)| \leq K < \infty$$

and therefore by dominated convergence

$$\begin{aligned} g_{r_m}(x) = -\frac{1}{2\pi} \int\limits_{\mathcal{C}} \eta(y) \ln(|x-y-r_m n_y|)\, \phi(r_m, y)\, \mathrm{d}S(y) \\ \to -\frac{1}{2\pi} \int\limits_{\mathcal{C}} \eta(y) \ln(|x-y-r n_y|)\, \phi(r, y)\, \mathrm{d}S(y) = g_r(x). \end{aligned}$$

If $x \in \mathcal{C}^r$ split $\mathcal{C}$ into $\mathcal{C} \backslash B_\delta(\Pi_{\mathcal{C}}(x))$ and $\mathcal{C} \cap B_\delta(\Pi_{\mathcal{C}}(x))$ for some $0 < \delta < \frac{1}{2}$. For $y \in \mathcal{C} \backslash B_\delta(\Pi_{\mathcal{C}}(x))$ one has again $0 < m \leq |x-y-r_m n_y| \leq M < \infty$ and hence by dominated convergence

$$\begin{aligned} &-\frac{1}{2\pi} \int\limits_{\mathcal{C} \backslash B_\delta(\Pi_{\mathcal{C}}(x))} \eta(y) \ln(|x-y-r_m n_y|)\, \phi(r_m, y)\, \mathrm{d}S(y) \\ \to &-\frac{1}{2\pi} \int\limits_{\mathcal{C} \backslash B_\delta(\Pi_{\mathcal{C}}(x))} \eta(y) \ln(|x-y-r n_y|)\, \phi(r, y)\, \mathrm{d}S(y). \end{aligned}$$

For $|r_m - r|$ small enough we have that (3.9) holds on $\mathcal{C} \cap B_\delta(\Pi_{\mathcal{C}}(x))$. Since $\ln(|x-y|)$ is an integrable function along $\mathcal{C}$, see [30], Ch.192ff, pp.489, we get again by dominated convergence

$$\begin{aligned} &-\frac{1}{2\pi} \int\limits_{\mathcal{C} \cap B_\delta(\Pi_{\mathcal{C}}(x))} \eta(y) \ln(|x-y-r_m n_y|)\, \phi(r_m, y)\, \mathrm{d}S(y) \\ \to &-\frac{1}{2\pi} \int\limits_{\mathcal{C} \cap B_\delta(\Pi_{\mathcal{C}}(x))} \eta(y) \ln(|x-y-r n_y|)\, \phi(r, y)\, \mathrm{d}S(y). \end{aligned}$$

Hence

$$\begin{aligned}
g_{r_m}(x) &= -\frac{1}{2\pi}\int\limits_{\mathcal{C}} \eta(y)\ln(|x-y-r_m n_y|)\,\phi(r_m,y)\,\mathrm{d}S(y) \\
&= -\frac{1}{2\pi}\int\limits_{\mathcal{C}\cap B_\delta(\Pi_{\mathcal{C}}(x))} \eta(y)\ln(|x-y-r_m n_y|)\,\phi(r_m,y)\,\mathrm{d}S(y) \\
&\quad + -\frac{1}{2\pi}\int\limits_{\mathcal{C}\setminus B_\delta(\Pi_{\mathcal{C}}(x))} \eta(y)\ln(|x-y-r_m n_y|)\,\phi(r_m,y)\,\mathrm{d}S(y) \\
&\to -\frac{1}{2\pi}\int\limits_{\mathcal{C}\cap B_\delta(\Pi_{\mathcal{C}}(x))} \eta(y)\ln(|x-y-r n_y|)\,\phi(r,y)\,\mathrm{d}S(y) \\
&\quad + -\frac{1}{2\pi}\int\limits_{\mathcal{C}\setminus B_\delta(\Pi_{\mathcal{C}})(x)} \eta(y)\ln(|x-y-r n_y|)\,\phi(r,y)\,\mathrm{d}S(y) \\
&= -\frac{1}{2\pi}\int\limits_{\mathcal{C}} \eta(y)\ln(|x-y-r n_y|)\,\phi(r,y)\,\mathrm{d}S(y) = g_r(x)
\end{aligned}$$

and we are done. □

Additionally we get continuity of $g_r(x)$ in x and r. To proof these joint continuity we need the following:

Lemma 35 The function $\|H_r\|_{\infty,\mathcal{C}^r}$ is bounded in r for $r \in (-\beta, \beta)$.

Proof. Take $x \in \mathcal{C}^r$ then

$$\begin{aligned}
|H_r(x)| &\le \int\limits_{\mathcal{C}^r} \frac{|\cos(x-y,n_x)|}{|x-y|}\mathrm{d}S(y) \\
&= \int\limits_{\mathcal{C}^r} \frac{|\langle x-y,n_x\rangle|}{|x-y|^2}\mathrm{d}S(y)
\end{aligned}$$

by the definition of the scalar product. By lemma 22 we have

$$|\langle x-y,n_y\rangle| \le M\,|x-y|^2$$

and hence

$$|H_r(x)| \le ML(\mathcal{C}^r)$$

where $L(\mathcal{C}^r)$ is the length of $\mathcal{C}^r$. This implies that

$$\|H_r\|_{\infty,\mathcal{C}^r} \le ML(\mathcal{C}^r)$$

and the lemma is proven since $L(\mathcal{C}^r)$ is obviously bounded in r for $r \in (-\beta, \beta)$. □

By this lemma we can conclude the boundedness of ∇g_r.

Corollary 36 The function $\nabla g_r(x)$ is bounded in (r, x) on $(-\beta, \beta) \times \mathbb{R}^2$.

Proof. The statements follows immediately by (3.12) and the lemma before. □

Now we are able to proof the continuity property of g_r which we are interested in.

Proposition 37 $g_r(x)$ is continuous on $(-\beta,\beta)\times\mathbb{R}^2$

Proof. Take r_n and x_n such that $r_n \to r$ and $x_n \to x$. Then

$$|g_{r_n}(x_n) - g_r(x)| \leq |g_{r_n}(x_n) - g_{r_n}(x)| + |g_{r_n}(x) - g_r(x)|$$

Take a closer look at $|g_{r_n}(x_n) - g_{r_n}(x)|$. Let us separate into two cases: $x \in \mathcal{C}^r$ and $x \notin \mathcal{C}^r$. At first take $x \notin \mathcal{C}^r$. W.l.o.g. assume that $x \in \mathcal{C}^r_{ext}$ with $d(x,\mathcal{C}^r) \geq \delta > 0$. By the convergence of r_n to r and x_n to x we obtain an $N \in \mathcal{N}$ such that $|x_n - x| \leq \frac{\delta}{4}$ and $|r_n - r| \leq \frac{\delta}{4}$ for $n \geq N$. But then x_n and x are both elements of $\overline{\mathcal{C}^{r_n}_{ext}}$ and we get by the mean value theorem and the boundedness of ∇g_r that

$$|g_{r_n}(x_n) - g_{r_n}(x)| \leq M\,|x_n - x|$$

for some positive constant M.
Take now $x \in \mathcal{C}^r$. If x_n and x are both elements of $\mathcal{C}^{r_n}_{ext}$ or $\mathcal{C}^{r_n}_{int}$, respectively we get again by the mean value theorem and the boundedness of ∇g_r that

$$|g_{r_n}(x_n) - g_{r_n}(x)| \leq M\,|x_n - x|$$

for some positive constant M. Let us assume now that $x \in \mathcal{C}^{r_n}_{ext}$ and $x_n \in \mathcal{C}^{r_n}_{int}$. Define for $c \in [0,1]$ $T(c) = (1-c)x_n + cx$ and denote by $z_n \in \mathcal{C}^{r_n}$ the point where $T(c)$ hits $\mathcal{C}^{r_n}$. Then

$$|g_{r_n}(x_n) - g_{r_n}(x)| \leq |g_{r_n}(x_n) - g_{r_n}(z_n)| + |g_{r_n}(z_n) - g_{r_n}(x)|\,.$$

Now again by the mean value theorem and the boundedness of ∇g_r we have that

$$|g_{r_n}(x_n) - g_{r_n}(x)| \leq M\left(|x_n - z_n| + |z_n - x|\right)$$

for some positive constant M. Since $x_n \to x$ and $r_n \to r$ we obtain that $z_n \to x$. Hence $|x_n - x|$, $|x_n - z_n|$ and $|z_n - x|$ tends to zero. By the continuity of $g_r(x)$ in r for fixed x we get

$$|g_{r_n}(x_n) - g_r(x)| \leq |g_{r_n}(x_n) - g_{r_n}(x)| + |g_{r_n}(x) - g_r(x)| \to 0$$

as $n \to \infty$ and we are done. □

This finishes our collection of properties of g_r and we will start with the proof of theorem 26.

3.2 Approximation procedure

First of all let us fix the definition of the approximation function f_ε:

$$f_\varepsilon(x) = \frac{1}{2\varepsilon}\int\limits_{-\varepsilon}^{\varepsilon} g_r(x)\mathrm{d}r \tag{3.13}$$

for $x \in \mathbb{R}^2$ and $0 < \varepsilon < \beta$.

Lemma 38 The function f_ε defined as in (3.13) is an element of $C^1\left(\mathbb{R}^2\right)$ with

$$\Delta f_\varepsilon(x) = \frac{1}{2\varepsilon}\tilde{\eta}(x)\mathbb{1}_{\mathcal{C}_\varepsilon}(x) \tag{3.14}$$

and

$$\nabla f_\varepsilon(x) = \frac{1}{2\varepsilon}\int\limits_{-\varepsilon}^{\varepsilon} \nabla g_r(x)\mathrm{d}r. \tag{3.15}$$

Proof. Since the logarithm is an integrable function on $\mathcal{C}_\beta$ we get by use of (3.6) that

$$\begin{aligned} -\frac{1}{2\pi}\int\limits_{\mathbb{R}^2}\frac{1}{2\varepsilon}\mathbb{1}_{\mathcal{C}_\varepsilon}(y)\tilde{\eta}(y)\ln\left(|x-y|\right)\mathrm{d}y &= -\frac{1}{2\pi}\int\limits_{\mathcal{C}_\varepsilon}\frac{1}{2\varepsilon}\tilde{\eta}(y)\ln\left(|x-y|\right)\mathrm{d}y \\ &= \frac{1}{2\varepsilon}\int\limits_{-\varepsilon}^{\varepsilon} -\frac{1}{2\pi}\int\limits_{\mathcal{C}^r}\tilde{\eta}(y)\ln\left(y\right)ln\left(|x-y|\right)\mathrm{d}S_r(y)\mathrm{d}r \\ &= f_\varepsilon(x). \end{aligned}$$

But

$$\tilde{f}(x) = -\frac{1}{2\pi}\int\limits_{\mathbb{R}^2}\frac{1}{2\varepsilon}\mathbb{1}_{\mathcal{C}_\varepsilon}(y)\tilde{\eta}(y)\ln\left(|x-y|\right)\mathrm{d}y$$

is an element of $C^1\left(\mathbb{R}^2\right)$ and (3.14) holds as it is shown in [30], Ch.192ff., pp.489.

For the proof of (3.15) we have to look at two different cases:

If $x \notin \mathcal{C}_\varepsilon$ we could just interchange differentiation and integration because $\nabla g_r(x)$ is bounded by $\frac{1}{d(x,\mathcal{C}_\varepsilon)}$.

For $x \in \mathcal{C}^\varepsilon$ take r_x such that $x \in \mathcal{C}^{r_x}$. Define $x_t = x + te_i$ for some $t \in \mathbb{R}$ and a standard basis vector e_i. Then for small t we have $x_t \in \mathcal{C}^{r_{x_t}}$ for $r_{x_t} \in (-\varepsilon, \varepsilon)$. Define $\delta_t = |r_{x_t} - r_x|$. Then we have

$$\delta_t = |r_{x_t} - r_x| \leq |x_t - x| = |t|\,. \tag{3.16}$$

And we have

$$\begin{aligned}
f_\varepsilon(x_t) - f_\varepsilon(x) =& \frac{1}{2\varepsilon} \underbrace{\int_{-\varepsilon}^{r_x-\delta_t} g_r(x_t) - g_r(x)\mathrm{d}r}_{\text{(I)}} \\
&+ \frac{1}{2\varepsilon} \underbrace{\int_{r_x-\delta_t}^{r_x+\delta_t} g_r(x_t) - g_r(x)\mathrm{d}r}_{\text{(II)}} \\
&+ \frac{1}{2\varepsilon} \underbrace{\int_{r_x+\delta_t}^{\varepsilon} g_r(x_t) - g_r(x)\mathrm{d}r}_{\text{(III)}}.
\end{aligned}$$

In (III) x_t and x are elements of $\mathcal{C}^r_{int}$ hence

$$\text{(III)} = \int_{r_x+\delta_t}^{\varepsilon} \frac{\partial}{\partial x_i} g_r(x)t + o(t)\mathrm{d}r.$$

Therefore we can can obtain by skipping terms of order $o(t)$ that

$$\lim_{t\to 0} \frac{\text{(III)}}{t} = \lim_{t\to 0} \int_{r_x-\delta_t}^{\varepsilon} \frac{\partial}{\partial x_i} g_r(x)\mathrm{d}r.$$

Now as mentioned in remark 26 we can $\frac{\partial}{\partial x_i} g_{r_x}$ continuously continue from $\mathcal{C}^{r_x}_{int}$ to $\mathcal{C}^{r_x}$ and hence the integral

$$\int_{r_x}^{\varepsilon} \frac{\partial}{\partial x_i} g_r(x)\mathrm{d}r$$

is well-defined. This implies that

$$\begin{aligned}
\lim_{t\to 0} \frac{\text{(III)}}{t} &= \lim_{t\to 0} \int_{r_x-\delta_t}^{\varepsilon} \frac{\partial}{\partial x_i} g_r(x)\mathrm{d}r \\
&= \int_{r_x}^{\varepsilon} \frac{\partial}{\partial x_i} g_r(x)\mathrm{d}r - \lim_{t\to 0} \int_{r_x}^{r_x-\delta_t} \frac{\partial}{\partial x_i} g_r(x)\mathrm{d}r.
\end{aligned}$$

But is shown in corollary 29 that $\nabla g_r(x)$ is uniformly bounded in r and x and therefore

$$\lim_{t\to 0} \int_{r_x}^{r_x-\delta_t} \frac{\partial}{\partial x_i} g_r(x)\mathrm{d}r = 0.$$

Thus we can conclude that

$$\lim_{t\to 0} \frac{\text{(III)}}{t} = \int_{r_x}^{\varepsilon} \frac{\partial}{\partial x_i} g_r(x)\mathrm{d}r.$$

By repeating now these arguments using the continuous continuation of $\frac{\partial}{\partial x_i} g_{r_x}$ from $\mathcal{C}^{r_x}_{ext}$ to $\mathcal{C}^{r_x}$ as mentioned in remark 26 we are able to show that

$$\lim_{t\to 0} \frac{(\mathrm{I})}{t} = \int\limits_{-\varepsilon}^{r_x} \frac{\partial}{\partial x_i} g_r(x) \mathrm{d}r.$$

For (II) we have by the continuity of g_r

$$(\mathrm{II}) = 2g_{r_1}(x_t)\,\delta_t - 2g_{r_2}(x)\delta_t$$

for some r_1 and $r_2 \in (r_{x_t}, r_x)$. By (3.16) and since $g_r(x)$ is continuous in r and x as shown in proposition 30 we get

$$\lim_{t\to 0} \frac{(\mathrm{II})}{t} = \lim_{t\to 0} \frac{2\delta_t}{t} \left(g_{r_1}(x_t) - g_{r_2}(x)\right) = 0.$$

These calculations imply

$$\begin{aligned} \lim_{t\to 0} \frac{f_\varepsilon(x_t) - f_\varepsilon(x)}{t} &= \frac{1}{2\varepsilon} \int\limits_{-\varepsilon}^{r_x} \frac{\partial}{\partial x_i} g_r(x)\mathrm{d}r + \int\limits_{r_x}^{\varepsilon} \frac{\partial}{\partial x_i} g_r(x)\mathrm{d}r \\ &= \frac{1}{2\varepsilon} \int\limits_{-\varepsilon}^{\varepsilon} \frac{\partial}{\partial x_i} g_r(x)\mathrm{d}r \end{aligned}$$

and we are done. □

The representation of ∇f_ε allows us to deduce boundedness of ∇f_ε.

Corollary 39 The function $\nabla f_\varepsilon(x)$ is bounded in (ε, x) on $(-\beta, \beta) \times \mathbb{R}^2$.

Proof. ∇g_r is uniformly bounded by corollary 36 and hence we get that ∇f_ε is uniformly bounded. □

Our next aim is to show convergence results for f_ε.

Proposition 40 f_ε converges to g uniformly in $\mathbb{R}^2$.

Proof. First of all we want to mention that

Lemma 41 f_ε converges to g point-wise in $\mathbb{R}^2$.

Proof. Since $g_r(x)$ is continuous on r for fixed x as shown in lemma 34 we have

$$\begin{aligned} \lim_{\varepsilon\to 0} f_\varepsilon(x) &= \lim_{\varepsilon\to 0} \frac{1}{2\varepsilon} \int\limits_{-\varepsilon}^{\varepsilon} g_r(x)\mathrm{d}r \\ &= \lim_{\varepsilon\to 0} g_{\tilde r}(x) = g(x) \end{aligned}$$

where $\tilde r \in (-\varepsilon, \varepsilon)$. □

Now we want to prove the uniform convergence of f_ε by splitting $\mathbb{R}^2$ into a compact set K containing $\mathcal{C}_\beta$ and $\mathbb{R}^2\backslash\{K\}$.

Since ∇f_ε is uniformly bounded on $(-\beta,\beta)\times\mathbb{R}^2$, i.e. there exists a positive constant C such that $|\nabla f_\varepsilon(x)| \leq C$ for any $x\in\mathbb{R}^2$ and any $\varepsilon\in(-\beta,\beta)$, we get by the mean value theorem that $|f_\varepsilon(x)-f_\varepsilon(y)| \leq C\,|x-y|$ for any possible choice of x and $y\in\mathbb{R}^2$. Hence the family f_ε is equicontinuous on every compact subset K of $\mathbb{R}^2$ and converges point-wise to the continuous function g. Therefore f_ε converges to g uniformly on K, see [13], p.718, theorem C.8. Take now an $R>0$ such that $\mathcal{C}_\beta\subseteq B_R(0)=K$. Then we have for any $x\notin B_R(0)$ that

$$\begin{aligned}|f_\varepsilon(x)-g(x)| &= \left|\frac{1}{2\varepsilon}\int\limits_{-\varepsilon}^{\varepsilon} g_r(x)\mathrm{d}r - g_0(x)\right| \\ &= \left|\int\limits_{\mathcal{C}}\eta(y)\left(\frac{1}{2\varepsilon}\int\limits_{-\varepsilon}^{\varepsilon}\phi(r,y)\ln\left(|x-y-rn_y|\right)\mathrm{d}r - \ln\left(|x-y|\right)\right)\mathrm{d}S(y)\right|.\end{aligned}$$

Since $x\notin K$ we have that $x-y-rn_y\neq 0$ for any possible choice of y and r hence we can make a Taylor expansion of $\ln\left(|x-y-rn_y|\right)$ around $|x-y|$ w.r.t. r. We get a $\zeta\in(0,r)$ such that

$$\ln\left(|x-y-rn_y|\right) = \ln\left(|x-y|\right) + r\frac{\partial}{\partial r}\ln\left(|x-y-\zeta n_y|\right).$$

Since $\frac{\partial}{\partial r}\ln\left(|x-y-rn_y|\right)$ is continuous in $(-\beta,\beta)\times\mathbb{R}^2\backslash B_R(0)$ and behaves like $\frac{1}{d(x,\mathcal{C}_\beta)}$ for large x we get a positive constant M such that

$$\sup_{(r,x)\in(-\beta,\beta)\times\mathbb{R}^2\backslash B_R(0)}\left|\frac{\partial}{\partial r}\ln\left(|x-y-rn_y|\right)\right| \leq M < \infty.$$

Hence

$$\begin{aligned} &\left|\frac{1}{2\varepsilon}\int\limits_{-\varepsilon}^{\varepsilon}\phi(r,y)\ln\left(|x-y-rn_y|\right)\mathrm{d}r - \ln\left(|x-y|\right)\right| \\ \leq\ &\left|\ln\left(|x-y|\right)\left[\frac{1}{2\varepsilon}\int\limits_{-\varepsilon}^{\varepsilon}\phi(r,y)\mathrm{d}r - 1\right]\right| + \varepsilon M \\ =\ &\left|\ln\left(|x-y|\right)\left[\frac{1}{2\varepsilon}\int\limits_{-\varepsilon}^{\varepsilon}\left(1+r\kappa(y)\right)\mathrm{d}r - 1\right]\right| + \varepsilon M \\ =\ &\varepsilon M.\end{aligned}$$

Therefore $|f_\varepsilon(x)-g(x)| \leq \|\eta\|_{\infty,\mathcal{C}}\,M\varepsilon$ and this implies that f_ε converges uniformly to g on $\mathbb{R}^2\backslash B_R(0)$. Hence we have uniform convergence on $\mathbb{R}^2\backslash B_R(0)$ and on $B_R(0)$ and therefore we get uniform convergence in $\mathbb{R}^2$. □

The last step in the approximation procedure is to show that

Lemma 42 $\nabla f_\varepsilon \to \nabla g$ point-wise in $\mathbb{R}^2\backslash\mathcal{C}$.

Proof. Take $x \in \mathbb{R}^2 \backslash \mathcal{C}$ and choose $0 < \varepsilon < d(x, \mathcal{C})$. Then $\nabla g_r(x)$ is continuous in $r \in (-\varepsilon, \varepsilon)$ for fixed x which can be seen as follows: Since $\varepsilon < d(x, \mathcal{C})$ and $r \in (-\varepsilon, \varepsilon)$ we have

$$\begin{aligned}\nabla g_r(x) &= -\frac{1}{2\pi} \int\limits_{\mathcal{C}^r} \tilde{\eta}(y) \nabla \ln(|x - y|) \, \mathrm{d}S_r(y) \\ &= -\frac{1}{2\pi} \int\limits_{\mathcal{C}} \eta(y) \phi(r, y) \nabla \ln(|x - y - r n_y|) \, \mathrm{d}S(y).\end{aligned}$$

If we take now a sequence $r_m \to r$ then we see that the integrand

$$\eta(y) \phi(r_m, y) \nabla \ln(|x - y - r_m n_y|)$$

converges to

$$\eta(y) \phi(r, y) \nabla \ln(|x - y - r n_y|)$$

and is bounded by some constant since x has a positive distance to $\mathcal{C}_\varepsilon$. Hence by dominated convergence we get $\nabla g_{r_m}(x) \to \nabla g_r(x)$ and this implies the point-wise convergence of $\nabla f_\varepsilon(x) = \frac{1}{2\varepsilon} \int\limits_{-\varepsilon}^{\varepsilon} \nabla g_r(x) \mathrm{d}r$ to $\nabla g(x)$. □

3.3 Ito formula

In this section we want to calculate $g(B_t)$ by use of Ito's formula for f_ε. Let us first mention some conclusion of Ito's isometry which will use later on.

Proposition 43 For any $f : \mathbb{R}^2 \to \mathbb{R}^2$ such that the Ito integral $\int\limits_0^t f(B_s) \mathrm{d}B_s$ exists we have

$$\mathbb{E}\left(\int\limits_0^t f(B_s) \, \mathrm{d}B_s{}^2 \right) \leq 2\mathbb{E}\left(\int\limits_0^t |f(B_s)|^2 \, \mathrm{d}s \right). \tag{3.17}$$

Proof. We obtain by the inequality $(a+b)^2 \leq 2\left(a^2 + b^2\right)$ for any a and $b \in \mathbb{R}$ that almost surely

$$\begin{aligned}\int\limits_0^t f(B_s) \, \mathrm{d}B_s{}^2 &= \left(\sum_{i=1}^2 \int\limits_0^t f_i(B_s) \, \mathrm{d}B_s^i \right)^2 \\ &\leq 2 \sum_{i=1}^2 \left(\int\limits_0^t f_i(B_s) \, \mathrm{d}B_s^i \right)^2.\end{aligned}$$

Hence we have

$$\mathbb{E}\left(\int\limits_0^t f(B_s) \, \mathrm{d}B_s{}^2 \right) \leq 2 \sum_{i=1}^2 \mathbb{E}\left(\left(\int\limits_0^t f_i(B_s) \, \mathrm{d}B_s^i \right)^2 \right).$$

By Ito's isometry we obtain

$$\begin{aligned}\sum_{i=1}^{2}\mathbb{E}\left(\left(\int_0^t f_i\left(B_s\right)\mathrm{d}B_s^i\right)^2\right) &= \sum_{i=1}^{2}\mathbb{E}\left(\int_0^t f_i\left(B_s\right)^2\mathrm{d}s\right)\\ &= \mathbb{E}\left(\int_0^t\sum_{i=1}^{2} f_i\left(B_s\right)^2\mathrm{d}s\right)\\ &= \mathbb{E}\left(\int_0^t\left|f\left(B_s\right)\right|^2\mathrm{d}s\right)\end{aligned}$$

and (3.17) is shown. □

Now we are able to proof the following:

Theorem 44 For each $t\in\mathbb{R}_+$ we have almost surely

$$g\left(B_t\right)=g\left(B_0\right)+\int_0^t\nabla g\left(B_s\right)\mathrm{d}B_s+\lim_{\varepsilon\to 0}\frac{1}{4\varepsilon}\int_0^t\tilde{\eta}\left(B_s\right)\mathbb{1}_{\mathcal{C}_\varepsilon}\left(B_s\right)\mathrm{d}s \tag{3.18}$$

where the limit is taken w.r.t. $L^2\left(\Omega,\mathbb{P}\right)$.

Proof. It is shown in [4], pp.210 that for any closed C^2-curve $\mathcal{C}$ the set $\{t\in\mathbb{R}\,:\,B_t\in\mathcal{C}\}$ has Lebesgue measure 0 almost surely. Thus we can apply Ito's formula to any $h\in C^1\left(\mathbb{R}^2\right)$ with $\Delta h\in L^\infty\left(\mathbb{R}^2\right)\cap C^0\left(\mathbb{R}^2\backslash\{M\}\right)$ where M is a finite union of C^2-curves, i.e.

$$h\left(B_t\right)=h\left(B_0\right)+\int_0^t\nabla h\left(B_s\right)\mathrm{d}B_s+\frac{1}{2}\int_0^t\Delta h\left(B_s\right)\mathrm{d}s. \tag{3.19}$$

But we have already shown that $f_\varepsilon\in C^1\left(\mathbb{R}^2\right)$ and that $\Delta f_\varepsilon=\frac{1}{2\varepsilon}\tilde{\eta}(x)\mathbb{1}_{\mathcal{C}_\varepsilon}(x)$, see lemma 31. Hence since $\mathcal{C}^\varepsilon\cup\mathcal{C}^{-\varepsilon}$ is a union of C^2-curves we have for every $0<\varepsilon<\beta$:

$$f_\varepsilon\left(B_t\right)=f_\varepsilon\left(B_t\right)+\int_0^t\nabla f_\varepsilon\left(B_s\right)\mathrm{d}B_s+\frac{1}{4\varepsilon}\int_0^t\tilde{\eta}\left(B_s\right)\mathbb{1}_{\mathcal{C}_\varepsilon}\left(B_s\right)\mathrm{d}s. \tag{3.20}$$

Since f_ε converges uniformly to g we get

$$\mathbb{E}\left(\left|f_\varepsilon\left(B_t\right)-f_\varepsilon\left(B_0\right)-g\left(B_t\right)+g\left(B_0\right)\right|^2\right)\to 0.$$

We also have that $\nabla f_\varepsilon(x)$ converges to $\nabla g(x)$ for $x\in\mathbb{R}^2\backslash\{\mathcal{C}\}$ and ∇f_ε and ∇g are bounded by some constant M, see corollary 33. Hence, since $\mathbb{1}_{\mathcal{C}}\left(B_s\right)=0$ almost surely for every s we get

$$\mathbb{E}\left(\int_0^t\left|\nabla f_\varepsilon\left(B_s\right)-\nabla g\left(B_s\right)\right|^2\mathrm{d}s\right)\to 0$$

by dominated convergence and Fubini. By (3.17) we obtain

$$\mathbb{E}\left(\int_0^t \nabla f_\varepsilon(B_s) - \nabla g(B_s)\,\mathrm{d}B_s{}^2\right) \to 0$$

and the theorem is proven. □

We get even a more general result

Remark 45 Take $\mathcal{C}^r$ to be a Runge parallel surface of $\mathcal{C}$. Then we have

$$g_r(B_t) = g_r(B_0) + \int_0^t \nabla g_r(B_s)\,\mathrm{d}B_s + \lim_{\varepsilon\to 0}\frac{1}{4\varepsilon}\int_0^t \tilde{\eta}(B_s)\,\mathbb{1}_{\mathcal{C}^r_\varepsilon}(B_s)\,\mathrm{d}s. \tag{3.21}$$

Proof. Follows exactly in the same way as in the case of $\mathcal{C}$. □

3.4 Tanaka type formula

The last step is to identify the limit point of (3.18), i.e $\lim_{\varepsilon\to 0}\frac{1}{4\varepsilon}\int_0^t \tilde{\eta}(B_s)\,\mathbb{1}_{\mathcal{C}_\varepsilon}(B_s)\,\mathrm{d}s$. The aim of this section is to show the following theorem

Theorem 46 For every $r \in (-\beta, \beta)$ and for any $\phi \in C^0(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$ we have

$$\lim_{\varepsilon\to 0}\frac{1}{4\varepsilon}\int_0^t \phi(B_s)\,\mathbb{1}_{(\mathcal{C}^r)_\varepsilon}(B_s)\,\mathrm{d}s = \int_0^t \phi(B_s)\,\mathrm{d}L(s,\mathcal{C}^r) \tag{3.22}$$

where the right hand side is defined as Riemann-Stieltjes integral for the continuous and non-decreasing function $L(\cdot,\mathcal{C}^r)$. Here $L(t,\mathcal{C})$ is the invariant local time of the Brownian motion on $\mathcal{C}$ in the sense of [4], i.e. $L(t,\mathcal{C})$ is a non-decreasing, time-continuous and adapted process which only increases if B_t is on $\mathcal{C}$. Additionally we have for every open subset $U \subseteq \mathbb{R}^2$ and every $f \in C^\infty(\mathbb{R}^2)$ with $f^{-1}(\{0\}) \cap U = \mathcal{C} \cap U$ and $\frac{\partial}{\partial_n} f > 0$ on $\mathcal{C} \cap U$ that

$$\int_0^t \mathbb{1}_U(B_s)\,\mathrm{d}l_s = \int_0^t \mathbb{1}_U(B_s)\,|\nabla f(B_s)|\,\mathrm{d}L(s,\mathcal{C})$$

where l is the local time of $f(B)$ at 0. For further details see [4], pp.216.

Remark 47 It is shown in [29], Lemma 2.3 that $L(t,\mathcal{C}^r)$ has a version which is jointly continuous in r and t.

Proof. For the proof of theorem 46 we will start with the following observation.

Lemma 48 For any $r \in (-\beta, \beta)$ and every $t > 0$ we obtain

$$\lim_{\varepsilon \to 0} \frac{1}{4\varepsilon} \int_0^t \mathbb{1}_{(\mathcal{C}^r)_\varepsilon} (B_s) \, \mathrm{d}s = L(t, \mathcal{C}^r) \tag{3.23}$$

Proof. We obviously have

$$\frac{1}{4\varepsilon} \int_0^t \mathbb{1}_{\mathcal{C}^r_\varepsilon} (B_s) \, \mathrm{d}s = \frac{1}{2} \frac{1}{2\varepsilon} \int_0^t \mathbb{1}_{(0,\varepsilon)} (d(B_s, \mathcal{C}^r)) \, \mathrm{d}s.$$

But it is mentioned in [22], Theorem 4.1 and its corollary, that the local time of the distance functional, i.e. $\lim_{\varepsilon \to 0} \frac{1}{2\varepsilon} \int_0^t \mathbb{1}_{(0,\varepsilon)} (d(B_s, \mathcal{C}^r)) \, \mathrm{d}s$ is equal to $2L(t, \mathcal{C}^r)$. $\square$

We conclude by use of theorem 44 and lemma 48 that for $\phi \equiv 1$ we have for any $r \in (-\beta, \beta)$ and every $t > 0$

$$g_r(B_t) = g_r(B_0) + \int_0^t \nabla g_r(B_s) \, \mathrm{d}B_s + L(t, \mathcal{C}^r) \tag{3.24}$$

Now we want to show that (3.22) holds for an arbitrary continuous ϕ. We will give the proof in the case $r = 0$, i.e. for $\mathcal{C}$. If $r \neq 0$ we will get the result in the same way but to clarify the notation we only proof the special case $r = 0$. Let us start with the following.

Lemma 49 Take $\phi \in C^0(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$ and denote by $\eta \in C^0(\mathcal{C})$ the restriction of ϕ to $\mathcal{C}$. Define $\tilde{\eta}$ as in (3.10). Then we have

$$\lim_{\varepsilon \to 0} \frac{1}{4\varepsilon} \int_0^t \tilde{\eta}(B_s) \mathbb{1}_{\mathcal{C}_\varepsilon} (B_s) \, \mathrm{d}s = \int_0^t \eta(B_s) \, \mathrm{d}L(s, \mathcal{C}).$$

Proof. Denote by $f_{\varepsilon,1}$ and $g_{0,1}$ the approximation function and the single layer in the case $\eta \equiv 1$, respectively. Define $X_t = g_{0,1}(B_t) - f_{\varepsilon,1}(B_t)$. The proof is split into two steps:
1. Step: If $\eta \in C^2(\mathcal{C})$ then $\tilde{\eta}$ is obviously an element of $C^2(\mathcal{C}_\beta)$. Now we want to give an extension $\overline{\eta}$ of $\tilde{\eta}$ which is in $C_0^2(\mathbb{R}^2)$. Fix $0 < \delta < \beta$ and denote by ϕ a cut-off function of $\mathcal{C}_\delta$ in $\mathcal{C}_\beta$, i.e. $\phi \in C_0^2(\mathbb{R}^2)$ with $\phi \equiv 1$ on $\mathcal{C}_\delta$ and $\phi \equiv 0$ on $\mathbb{R}^2 \backslash \{\overline{\mathcal{C}_\beta}\}$, for further details see e.g. [13], p.328. Define $\overline{\eta} = \tilde{\eta}(x)\phi(x)$. Then $\overline{\eta} \in C_0^2(\mathbb{R}^2)$ and $\overline{\eta} = \tilde{\eta}$ on $\mathcal{C}_\delta$.
Choose $0 < \varepsilon < \delta$ and take a look at $\overline{\eta}(B_t) X_t$. By stochastic product rule we obtain

$$\overline{\eta}(B_t) X_t = \overline{\eta}(B_0) X_0 - \int_0^t \overline{\eta}(B_s) \, \mathrm{d}X_s - \int_0^t X_s \mathrm{d}\overline{\eta}(B_s) - \int_0^t \mathrm{d}[X, \overline{\eta}(B.)] \, s. \tag{3.25}$$

By (3.24) and (3.34) we get

$$\mathrm{d}X = \nabla(g_{0,1} - f_{\varepsilon,1})(B) \, \mathrm{d}B + \mathrm{d}L(\cdot, \mathcal{C}) - \frac{1}{4\varepsilon} \mathbb{1}_{\mathcal{C}_\varepsilon}(B) \, \mathrm{d}B.$$

By substituting this in (3.25) we get

$$\frac{1}{4\varepsilon}\int_0^t \tilde{\eta}(B_s)\,\mathbb{1}_{\mathcal{C}_\varepsilon}(B_s)\,\mathrm{d}B_s - \int_0^t \eta(B_s)\,\mathrm{d}L(s,\mathcal{C}) = \tag{3.26}$$

$$\underbrace{\overline{\eta}(B_t)X_t - \overline{\eta}(B_0)X_0}_{\text{(I)}} + \underbrace{\int_0^t \overline{\eta}(B_s)(\nabla g_{0,1} - f_{\varepsilon,1})(B_s)\,\mathrm{d}B_s}_{\text{(II)}} + \underbrace{\int_0^t X_s \mathrm{d}\overline{\eta}(B_s) + \int_0^t \mathrm{d}[X,\overline{\eta}(B.)]\,s}_{\text{(III)}}$$

We want to show that the left hand side of (3.26) tends to 0 if $\varepsilon \to 0$. Since $f_{\varepsilon,1}$ converges uniformly to $g_{0,1}$ and $\overline{\eta}$ is bounded we get $\mathbb{E}\left((\mathrm{I})^2\right) \to 0$. By dominated convergence we have

$$\mathbb{E}\left(\int_0^t \overline{\eta}(B_s)\left|(\nabla g_{0,1} - f_{\varepsilon,1})(B_s)\right|^2 \mathrm{d}s\right) \to 0$$

and therefore we obtain by (3.17) that $\mathbb{E}\left((\mathrm{II})^2\right) \to 0$. We can conclude by use of Ito's formula for $\overline{\eta}$, (3.24) and (3.34) that

$$(\mathrm{III}) = \int_0^t X_s \nabla\overline{\eta}(B_s)\,\mathrm{d}B_s + \int_0^t X_s \frac{1}{2}\Delta\overline{\eta}(B_s) + (\nabla g_{0,1} - f_{\varepsilon,1})(B_s)\,\nabla\overline{\eta}(B_s)\,\mathrm{d}s. \tag{3.27}$$

Now since $\nabla\overline{\eta}$ and $\Delta\overline{\eta}$ are bounded we get by repeating the arguments for (I) and (II) that $\mathbb{E}\,(\text{r.h.s. of (3.27)})^2 \to 0$ and hence $\mathbb{E}\left((\mathrm{III})^2\right) \to 0$. Thus we have

$$\lim_{\varepsilon\to 0}\frac{1}{4\varepsilon}\int_0^t \tilde{\eta}(B_s)\,\mathbb{1}_{\mathcal{C}_\varepsilon}(B_s)\,\mathrm{d}s = \int_0^t \eta(B_s)\,\mathrm{d}L(s,\mathcal{C}). \tag{3.28}$$

2. Step: Take $\eta \in C^0(\mathcal{C})$. Choose a sequence $\eta_n \in C^2(\mathcal{C})$ such that $\|\eta_n - \eta\|_{\infty,\mathcal{C}} \to 0$ as $n \to \infty$. We obviously have that

$$\left|\frac{1}{4\varepsilon}\int_0^t (\tilde{\eta_n} - \tilde{\eta})(B_s)\,\mathbb{1}_{\mathcal{C}_\varepsilon}(B_s)\,\mathrm{d}s\right| \le \|\eta_n - \eta\|_{\infty,\mathcal{C}}\,\frac{1}{4\varepsilon}\int_0^t \mathbb{1}_{\mathcal{C}_\varepsilon}(B_s)\,\mathrm{d}s. \tag{3.29}$$

Since $L(\cdot,\mathcal{C})$ is non-negative and non-decreasing we get

$$\left|\int_0^t (\eta_n - \eta)(B_s)\,\mathrm{d}L(s,\mathcal{C})\right| \le \|\eta_n - \eta\|_{\infty,\mathcal{C}}\,L(t,\mathcal{C}). \tag{3.30}$$

Furthermore we obtain by (3.24) that

$$\begin{aligned}
\mathbb{E}\left(L(t,\mathcal{C})^2\right) &= \mathbb{E}\left(\left(g_{0,1}(B_t) - g_{0,1}(B_0) - \int_0^t \nabla g_{0,1}(B_s)\,\mathrm{d}B_s\right)^2\right) \\
&\le 2\mathbb{E}\left((g_{0,1}(B_t) - g_{0,1}(B_0))^2\right) + 2\mathbb{E}\left(\left(\int_0^t \nabla g_{0,1}(B_s)\,\mathrm{d}B_s\right)^2\right).
\end{aligned}$$

Since $\mathbb{E}\left(\int\limits_0^t |\nabla g_{0,1}(B_s)|^2 \,\mathrm{d}s\right) \leq Mt$ for some positive constant M we conclude by (3.17) that $\mathbb{E}\left(\left(\int\limits_0^t \nabla g_{0,1}(B_s)\,\mathrm{d}B_s\right)^2\right)$ is bounded. Since $g_{0,1}$ is a continuous function which behaves like $\ln(d(x,\mathcal{C}))$ for large x we get by the exponential decay of the Brownian transition kernel that $2\mathbb{E}\left((g_{0,1}(B_t) - g_{0,1}(B_0))^2\right)$ is bounded. Hence $\mathbb{E}\left(L(t,\mathcal{C})^2\right)$ is bounded. By the L^2-convergence of $\frac{1}{4\varepsilon}\int\limits_0^t \mathbb{1}_{\mathcal{C}_\varepsilon}(B_s)\,\mathrm{d}s$ to $L(t,\mathcal{C})$ we obtain that

$$\mathbb{E}\left(\frac{1}{4\varepsilon}\int_0^t \mathbb{1}_{\mathcal{C}_\varepsilon}(B_s)\,\mathrm{d}s^2\right) \leq M < \infty \tag{3.31}$$

for some positive M independent of ε. By this boundedness, (3.29) and (3.30) we have that

$$\begin{aligned} &\mathbb{E}\left(\left(\frac{1}{4\varepsilon}\int_0^t \tilde{\eta}(B_s)\,\mathbb{1}_{\mathcal{C}_\varepsilon}(B_s)\,\mathrm{d}s - \int_0^t \eta(B_s)\,\mathrm{d}L(s,\mathcal{C})\right)^2\right) \\ \leq\ & C\,\|\eta_n - \eta\|_{\infty,\mathcal{C}} + \mathbb{E}\left(\left(\frac{1}{4\varepsilon}\int_0^t \tilde{\eta}_n(B_s)\,\mathbb{1}_{\mathcal{C}_\varepsilon}(B_s)\,\mathrm{d}s - \int_0^t \eta_n(B_s)\,\mathrm{d}L(s,\mathcal{C})\right)^2\right) \end{aligned}$$

for some positive constant independent of ε and n. Fix $\delta > 0$. Choose $n \in \mathbb{N}$ such that $\|\eta_n - \eta\|_{\infty,\mathcal{C}} \leq \frac{\delta}{2C}$. For this n take an $\alpha > 0$ due to (3.28) such that for every $\varepsilon \leq \alpha$ we have

$$\mathbb{E}\left(\left(\frac{1}{4\varepsilon}\int_0^t \tilde{\eta}_n(B_s)\,\mathbb{1}_{\mathcal{C}_\varepsilon}(B_s)\,\mathrm{d}s - \int_0^t \eta_n(B_s)\,\mathrm{d}L(s,\mathcal{C})\right)^2\right) \leq \frac{\delta}{2}.$$

Hence

$$\mathbb{E}\left(\left(\frac{1}{4\varepsilon}\int_0^t \tilde{\eta}(B_s)\,\mathbb{1}_{\mathcal{C}_\varepsilon}(B_s)\,\mathrm{d}s - \int_0^t \eta(B_s)\,\mathrm{d}L(s,\mathcal{C})\right)^2\right) \leq \delta$$

for every $\varepsilon \leq \alpha$ and the lemma is proven. □

The last step for proving theorem 46 is to show that (3.22) holds not only in the special case of $\tilde{\eta}$.

Lemma 50 Take $\phi \in C^0\left(\mathbb{R}^2\right) \cap L^\infty\left(\mathbb{R}^2\right)$ and denote by $\eta \in C^0(\mathcal{C})$ the restriction of ϕ to $\mathcal{C}$. Define $\tilde{\eta}$ as in (3.10). Then we have

$$\lim_{\varepsilon\to 0}\frac{1}{4\varepsilon}\int_0^t \phi(B_s)\,\mathbb{1}_{\mathcal{C}_\varepsilon}(B_s)\,\mathrm{d}s = \int_0^t \eta(B_s)\,\mathrm{d}L(s,\mathcal{C}).$$

Proof. For the proof it is enough to show that

$$\lim_{\varepsilon\to 0}\mathbb{E}\left(\frac{1}{4\varepsilon}\int_0^t (\phi - \tilde{\eta})(B_s)\,\mathbb{1}_{\mathcal{C}_\varepsilon}(B_s)\,\mathrm{d}s\right)^2 = 0.$$

By use of (3.31) we obtain that

$$\mathbb{E}\left(\frac{1}{4\varepsilon}\int_0^t (\phi-\tilde{\eta})(B_s)\,\mathbb{1}_{\mathcal{C}_\varepsilon}(B_s)\,\mathrm{d}s\right)^2 \leq M\,\|\phi-\tilde{\eta}\|_{\infty,\mathcal{C}_\varepsilon} \to 0$$

by the continuity of $\phi-\tilde{\eta}$ and because $\phi-\tilde{\eta}\equiv 0$ on $\mathcal{C}$. □

Thus theorem 46 is proven and this was the last step to prove theorem 26. □

Now we want to show the "Radon-Nikodyn"- property of $L(t,\mathcal{C}^r)$.

Proposition 51 For every $(\alpha,\gamma)\subseteq(-\beta,\beta)$ we have

$$\int_\alpha^\gamma L(t,\mathcal{C}^r)\,\mathrm{d}r = 2\int_0^t \mathbb{1}_{\mathcal{C}_\alpha^\gamma}(B_s)\,\mathrm{d}s. \tag{3.32}$$

Remark 52 The proof for (3.32) which is given here seems to be new in the case of an arbitrary closed $C^{2,\alpha}$-curve in $\mathbb{R}^2$. In the special case of a sphere one gets the result by writing the distance functional of $\mathcal{C}$ in terms of the radial part of the polar coordinates. Then one can identify the radial part of the Brownian motion by a Bessel process and then for example the results of Blumenthal and Getoor , see [3], Corollary 3.4, gives us (3.32). There an explicit structure of the resolvent of the Bessel process is used which is unknown in the case of an arbitrary $C^{2,\alpha}$-curve.

Proof. Since $g_r(x)$ is jointly continuous in r and x we find a version of $g_r(B_t)$ which is jointly continuous in r and t. Since $L(t,\mathcal{C}^r)$ is also jointly continuous in r and t we found by (3.24) a jointly continuous version of $\int_0^t \nabla g_r(B_s)\,\mathrm{d}B_s$. Define

$$f_\varepsilon^r(x) = \frac{1}{2\varepsilon}\int_{r-\varepsilon}^{r+\varepsilon} g_z(x)\mathrm{d}z$$

for any $r\in(-\beta,\beta)$. The rest of the proof follows the ideas of the proof for the one-dimensional case, which can be found in [5], Theorem 7.3. Since

$$\nabla f_\varepsilon^r(x) = \frac{1}{2\varepsilon}\int_{r-\varepsilon}^{r+\varepsilon} \nabla g_z(x)\mathrm{d}z$$

as shown in lemma 31 we get almost surely that

$$\begin{aligned}\int_0^t \nabla f_\varepsilon^r(B_s)\,\mathrm{d}B_s =& \frac{1}{2\varepsilon}\int_0^t\int_{r-\varepsilon}^{r+\varepsilon} \nabla g_z(B_s)\,\mathrm{d}z\mathrm{d}B_s\\ =& \frac{1}{2\varepsilon}\int_{r-\varepsilon}^{r+\varepsilon}\int_0^t \nabla g_z(B_s)\,\mathrm{d}B_s\mathrm{d}z\end{aligned} \tag{3.33}$$

if the change of integration is allowed. We cannot take Fubini's theorem here since there is a stochastic integral involved. To obtain the result we do some approximation via Riemann sums. Let

$$\phi_n(x) = \sum_k \frac{1}{2^n} \nabla g_{(r-\varepsilon+k2-n)}(x)$$

where the sum is over all k such that $k2^{-n} \in (r-\varepsilon, r+\varepsilon)$. Since $\int_0^t \nabla g_\cdot (B_s)\, \mathrm{d}B_s$ is continuous we have almost surely that

$$\begin{aligned} \int_{r-\varepsilon}^{r+\varepsilon} \int_0^t \nabla g_z (B_s)\, \mathrm{d}B_s \mathrm{d}z &= \lim_{n\to\infty} \sum_k \frac{1}{2^n} \int_0^t \nabla g_{(r-\varepsilon+k2-n)} (B_s)\, \mathrm{d}B_s \\ &= \lim_{n\to\infty} \int_0^t \phi_n (B_s)\, \mathrm{d}B_s. \end{aligned}$$

We also have that $\phi_n(x)$ converges point-wise to $\int_{r-\varepsilon}^{r+\varepsilon} \nabla g_y(x)\mathrm{d}y$ on $\mathbb{R}^2$. Since $\nabla g_z(x)$ is uniformly bounded on $(-\beta,\beta)\times\mathbb{R}^2$ we obtain by dominated convergence that

$$\mathbb{E}\left(\int_0^t \left|\phi_n (B_s) - 2\varepsilon \nabla f_\varepsilon (B_s)\right|^2 \mathrm{d}s \right) \to 0.$$

Hence by (3.17) $\lim_{n\to\infty} \int_0^t \phi_n (B_s)\, \mathrm{d}B_s$ is equal almost surely to $\int_0^t 2\varepsilon \nabla f_\varepsilon (B_s)\, \mathrm{d}B_s$ and the change of integration is verified. Putting (3.33) into (3.20) we obtain almost surely that

$$f_\varepsilon^r (B_t) - f_\varepsilon^r (B_t) - \frac{1}{2\varepsilon} \int_{r-\varepsilon}^{r+\varepsilon} \int_0^t \nabla g_z (B_s)\, \mathrm{d}B_s \mathrm{d}z = \frac{1}{4\varepsilon} \int_0^t \mathbb{1}_{(\mathcal{C}^r)_\varepsilon} (B_s)\, \mathrm{d}s. \tag{3.34}$$

We want to integrate this over r, but (3.34) holds only for each fixed r hence we again approximate by Riemann sums to do this integration. Denote by $F(r)$ the left member of (3.34). Since F is continuous we have

$$\int_\alpha^\gamma F(r)\mathrm{d}r = \lim_{n\to\infty} \sum_k 2^{-n} F\left(k2^{-n}\right)$$

where the sum is over all k such that $k2^{-n} \in (\alpha,\gamma)$. By (3.34) this integral is equal almost surely to the limit of the following as $n\to\infty$:

$$\frac{1}{4\varepsilon} \int_0^t \sum_k 2^{-n} \mathbb{1}_{\left(\mathcal{C}^{k2^{-n}}\right)_\varepsilon} (B_s)\, \mathrm{d}s.$$

For $B_s \in \mathcal{C}_{\alpha-\varepsilon}^{\gamma+\varepsilon}$ take r_{B_s} such that $B_s \in \mathcal{C}^{r_{B_s}}$ then

$$\mathbb{1}_{\left(\mathcal{C}^{k2^{-n}}\right)_\varepsilon} (B_s) = \mathbb{1}_{(r_{B_s}-\varepsilon, r_{B_s}+\varepsilon)} \left(k2^{-n}\right).$$

Hence

$$\sum_k 2^{-n}\mathbb{1}_{\left(\mathcal{C}^{k2^{-n}}\right)_\varepsilon}(B_s) = \sum_k 2^{-n}\mathbb{1}_{(r_{B_s}-\varepsilon, r_{B_s}+\varepsilon)}\left(k2^{-n}\right).$$

The sum on the right hand side is bounded by $2\varepsilon+1$ and converges as $n\to\infty$ to

$$\int_\alpha^\gamma \mathbb{1}_{(r_{B_s}-\varepsilon, r_{B_s}+\varepsilon)}(r)\mathrm{d}(r).$$

It follows that almost surely

$$\int_\alpha^\gamma \left(f_\varepsilon^r(B_t) - f_\varepsilon^r(B_t) - \frac{1}{2\varepsilon}\int_{r-\varepsilon}^{r+\varepsilon}\int_0^t \nabla g_z(B_s)\,\mathrm{d}B_s\mathrm{d}z \right)\mathrm{d}r = \frac{1}{4\varepsilon}\int_0^t\int_\alpha^\gamma \mathbb{1}_{(\mathcal{C}^\triangledown)_\varepsilon}(B_s)\,\mathrm{d}r\mathrm{d}s. \quad (3.35)$$

For $x\in\mathbb{R}^2$

$$\lim_{\varepsilon\to 0}\frac{1}{2\varepsilon}\int_\alpha^\gamma \mathbb{1}_{(\mathcal{C}^r)_\varepsilon}(x)\mathrm{d}r = \mathbb{1}_{\mathcal{C}_\alpha^\gamma}(x) + \frac{1}{2}\left(\mathbb{1}_{\mathcal{C}^\gamma}(x) + \mathbb{1}_{\mathcal{C}^\alpha}(x)\right). \quad (3.36)$$

By letting $\varepsilon\to 0$ in (3.35), using the continuity of $\int_0^t \nabla g_\cdot(B_s)\,\mathrm{d}B_s$ and (3.36) we obtain almost surely

$$\int_\alpha^\gamma \left(g_r(B_t) - g_r(B_0) - \int_0^t \nabla g_r(B_s)\,\mathrm{d}B_s \right)\mathrm{d}r = \frac{1}{2}\int_0^t \mathbb{1}_{\mathcal{C}_\alpha^\gamma}(B_s)\,\mathrm{d}s$$

since $\mathbb{1}_{\mathcal{C}^r}(B_s)=0$ almost surely for each r and s. But this is nothing else but (3.32) in the view of (3.24). □

Chapter 4

Stochastic processes related to extensions of T

The goal of this chapter is to construct stochastic processes whose generators are given by self-adjoint extensions of T, where T is given via

$$T : D(T) \subseteq L^2\left(\mathbb{R}^2\right) \to L^2\left(\mathbb{R}^2\right)$$
$$Tf = \frac{1}{2}\Delta f$$

with

$$D(T) = \left\{f \in C_0^\infty\left(\mathbb{R}^2\right) : \tau_{\mathcal{C}}(f) = 0\right\}.$$

The self-adjoint extensions which will be investigated are the extensions given by a multiplication operator, see chapter 2, example 2, and the extension given by the Laplace-Beltrami operator, see chapter 2, example 3.
We will show that the self-adjoint extension T_η of T given by the multiplication operator

$$M_\eta : D(M_\eta) \subseteq L^2(\mathcal{C}) \to L^2(\mathcal{C})$$
$$M_\eta f = \eta f$$

for a non-negative η generates a process X which has a killing rate depending on the integral $\int_0^t \eta(B_s)\,\mathrm{d}L(s,\mathcal{C})$. To be more detailed: The semigroup of X is given by a Feynman-Kac type formula, i.e. $\mathbb{E}^x[f(X_t)] = \mathbb{E}^x\left[f(B_t)\exp-\left(\int_0^t \eta(B_s)\,\mathrm{d}L(s,\mathcal{C})\right)\right]$. This result is not surprising since in one dimension we already know that the boundary condition $f'(0^+) - f'(0^-) = \kappa f(0)$ generates a sticky Brownian motion, i.e. a stochastic process X_t with semigroup $\mathbb{E}^x[f(X_t)] = \mathbb{E}^x[f(B_t)\exp-(L(t,0))]$, see e.g. the work of Ito and McKean jr., [20]. Thus the extensions related to multiplication operators just generalize the case of one point

interaction in one dimension to the case of a closed curve in two dimensions.
For the extension $T_{\Delta_{LB}}$ given by the Laplace-Beltrami operator we get a different picture. We will get processes behaving like a Brownian motions with additional movement along $\mathcal{C}$ in the time scale of $L(t, \mathcal{C})$. This process shows a difference between point and curve interactions. The geometry of the curve allows us to move along the curve in a local time scale whereas in the case of one point there is no possibility to move since the geometry is to simple. So in the case of curve interactions we have a richer situation than in the one point case.
The chapter is organized as follows: At first we want to give basic results of classical L^2-semigroup theory which can be found in the book of Ma and Röckner, [24]. Afterwards we will show how we can use the harmonic single-layer defined in chapter 3 to get rid of the jump in the normal-derivative of special elements of $D(T^*)$ and to obtain a function where we can easily apply an Ito formula. Later on we will use this construction to calculate the Ito formula for elements of $D(T_\eta)$ and $D(T_{\Delta_{LB}})$, respectively. After this spadework we construct the processes for the multiplication case and then for the case of the Laplace-Beltrami operator.

4.1 Basic results

Before we start with our calculations we want to state the main results on semigroups which we will need later on and we want to show how we can deal with the jump in the normal derivative of elements of $D(T^*)$ with the help of the single-layer g given in (3.2).

General results on semigroups

In this section we will deal with strongly continuous contraction semigroups or short s.c.c.s. on $L^2(\mathbb{R}^2)$ and their generators.

Definition 53 Denote by $B\left(L^2\left(\mathbb{R}^2\right)\right)$ the set of bounded operators in $L^2\left(\mathbb{R}^2\right)$. A map $P : \mathbb{R}_+ \to B\left(L^2\left(\mathbb{R}^2\right)\right)$ is called an $L^2\left(\mathbb{R}^2\right)$- strongly continuous contraction semigroup if

- $\|P_t\| \leq 1$ for every $t > 0$, i.e. every P_t is a contraction,
- $P_0 = \mathrm{Id}$,
- for every t, $s \in \mathbb{R}_+$ we have $P_{t+s} = P_t P_s$ (semigroup property),
- for every $f \in L^2\left(\mathbb{R}^2\right)$ we have $\lim_{t \downarrow 0} \|P_t f - f\| = 0$ (strong continuity).

Here $\|\cdot\|$ denotes the strong operator topology. The generator A of such a semigroup $(P_t)_{t>0}$ is defined by

$$Af = \lim_{t \downarrow 0} \frac{1}{t} (V_t f - f)$$

whenever the limit exists.

For further details on semigroups and their generators we refer to the book of Ma and Röckner, [24]. Later on we want to apply two general results on semigroups and its generators to prove our statements. Both statements are proven in [24], chapter 2. The first one is an immediate consequence of the Hille-Yosida-theorem.

Theorem 54 Every negative definite self-adjoint operator in $L^2(\mathbb{R}^2)$ generates a strongly continuous contraction semigroup in $L^2(\mathbb{R}^2)$.

The second statement allows us to restrict our calculations to nice subsets.

Theorem 55 If an operator B generates a strongly continuous contraction semigroup $(P_t)_{t>0}$ in $L^2(\mathbb{R}^2)$ and if for a Markov process X_t

$$\lim_{t \downarrow 0} \frac{1}{t} (\mathbb{E}^x (f(X_t)) - f(x)) = Af(x) \tag{4.1}$$

in $L^2(\mathbb{R}^2)$ for a core of A, then X_t is the stochastic process associated to $(P_t)_{t>0}$, i.e. $\mathbb{E}^x(f(X_t)) = V_t f(x)$ for every $f \in L^2(\mathbb{R}^2)$.

In the case of a self-adjoint operator there is a nice way of characterizing a core with the help of essentially self-adjointness.

Remark 56 Take B to be a self-adjoint operator. If the restriction of B to a subset $V \subseteq D(B)$ is essentially self-adjoint, then V is a core of B.

Proof. Denote by A the restriction of B to V. Then we have have that $V = D(A) \subseteq D(B)$. Since B is self-adjoint we can conclude that

$$D(B) \subseteq D(A^*) \tag{4.2}$$

and that

$$D(\overline{A}) \subseteq D(B) \tag{4.3}$$

since B is a closed operator. Here $\overline{A}$ is the closure of A.We assume A to be essentially self-adjoint, i.e. $D(\overline{A}) = D(A^*)$. Hence we can conclude by combining (4.2) and ((4.3) that

$$D(\overline{A}) \subseteq D(B) \subseteq D(A^*) = D(\overline{A}).$$

Hence $D(\overline{A}) = D(B)$ and this is equivalent to our statement that $V = D(A)$ is a core of $D(B)$.

□

A differentiability theorem

In this part of the work we want to show a theorem which gives us the possibility to apply Ito's formula to elements of $D(T^*)$. We will use this result later on for the calculation of $\mathbb{E}(f(X_t))$ where X_t is some given stochastic process.

Theorem 57 Take $f \in D(T^*) \cap C^2\left(\overline{\mathcal{C}_{int}}\right) \oplus C^2\left(\overline{\mathcal{C}_{ext}}\right)$. Denote by $g_{\kappa f}$ the single layer potential with layer function κf as it is defined in (3.2). Then $F = f + g_{\kappa f} \in C^1\left(\mathbb{R}^2\right)$ with $\Delta F = \Delta f$ on $\mathbb{R}^2 \backslash \{\mathcal{C}\}$.

Remark 58 By this result we are able to apply Ito's formula to f. We have that $F \in C^1\left(\mathbb{R}^2\right)$ and by construction $F \in C^2(\mathcal{C}_{int}) \oplus C^2(\mathcal{C}_{ext})$ and hence we can just apply an Ito formula to F, see the remarks to (3.19) in chapter 3. For $g_{\kappa f}$ we have shown a Tanaka type formula in the chapter before. By putting these two formulas together we obtain the Ito formula for f.

Proof. First of all we want to mention that by our choice of f $\kappa f \in C^0(\mathcal{C})$.
We will start the proof by showing that $F \in C^0\left(\mathbb{R}^2\right)$. Since $\kappa f \in C^0(\mathcal{C})$ we have by proposition 32, p.34, that $g_{\kappa f}$ is continuous. Since $f \in C^2\left(\overline{\mathcal{C}_{int}}\right) \oplus C^2\left(\overline{\mathcal{C}_{ext}}\right)$ it is continuous up to the boundary for $\mathcal{C}_{int}$ and $\mathcal{C}_{ext}$, respectively. But it was shown in lemma 3, p.12, that $f \in D(T^*)$ implies that the restriction of f to $\mathcal{C}$ coming from the inner domain is the same as passing from the outer domain. Hence $f \in D(T^*) \cap \left\{C^2\left(\overline{\mathcal{C}_{int}}\right) \oplus C^2\left(\overline{\mathcal{C}_{ext}}\right)\right\}$ is continuous up to $\mathcal{C}$ from the outer and the inner domain, respectively, with the same value on $\mathcal{C}$. This implies that f and therefore F is continuous.
To show that $F \in C^1\left(\mathbb{R}^2\right)$ we have to look what happens at $\mathcal{C}$ since we obviously have that $F \in C^1\left(\mathbb{R}^2 \backslash \{\mathcal{C}\}\right)$. By proposition 32, p.34, we obtain that $\kappa g_{\kappa f} = -\kappa f$ and hence $\kappa F = 0$. Therefore we have no jump in the normal derivatives of F along $\mathcal{C}$ and since the normal derivatives of f and $g_{\kappa f}$ can be continuously continued to $\mathcal{C}$, see remark 33, p.34, we get the continuity of the normal derivatives of F by passing through $\mathcal{C}$.
To prove that also the tangential derivative of F along $\mathcal{C}$ is continuous by passing through $\mathcal{C}$ we will use the representation of elements of $D(T^*)$ shown in corollary 6, p.15. There it is shown that for every $f \in D(T^*) \cap \left\{C^2\left(\overline{\mathcal{C}_{int}}\right) \oplus C^2\left(\overline{\mathcal{C}_{ext}}\right)\right\}$ we can find a $v \in H^{2,2}\left(\mathbb{R}^2\right)$ such that $f = v + S_\alpha \kappa f$ where S_α is the acoustic single layer potential for some fixed $\alpha \in \mathbb{R}$. For further details see chapter 2, pp.12. Hence we have that $F = v + S_\alpha \kappa f + g_{\kappa f}$. But it is shown in [30], Ch.192ff, pp.489, that for continuous layer functions the tangential derivative of the acoustic single layer potential S_α and the tangential derivative of the single layer potential g are continuous by passing through $\mathcal{C}$. Furthermore it is shown in [25] that the tangential derivative of elements of $H^{2,2}\left(\mathbb{R}^2\right) \cap \left\{C^2\left(\overline{\mathcal{C}_{int}}\right) \oplus C^2\left(\overline{\mathcal{C}_{ext}}\right)\right\}$ is continuous by passing through $\mathcal{C}$. Hence the tangential derivative of F is continuous and combining this

with the fact that the normal derivative is also continuous along $\mathcal{C}$ we get that $F \in C^1\left(\mathbb{R}^2\right)$. The equality $\Delta F = \Delta f$ on $\mathbb{R}^2 \backslash \{\mathcal{C}\}$ follows immediately by the fact that $g_{\kappa f}$ is harmonic in $\mathbb{R}^2 \backslash \{\mathcal{C}\}$, see proposition 32, p.34. □

4.2 The extensions given by multiplication operators

The results of this subsection are based on [20], pp.198, where some similar questions in the case of a half-line are answered.

For this section fix $\phi \notin \mathbb{R}^2$ and take the compactification of $\mathbb{R}^2 \cup \phi$ with the convention $f(\phi) = 0$ for all $f \in C^0\left(\mathbb{R}^2\right)$. The detailed statement on the processes related to self-adjoint extensions is the following:

Theorem 59 Choose a real-valued and non-negative $\eta \in L^\infty(\mathcal{C})$ such that the multiplication operator M_η given by η fulfills the requirements of chapter 2, example 2, i.e. it exists an $\varepsilon > 0$ such that $M_\eta : H^s(\mathcal{C}) \to H^{s-1+\varepsilon}(\mathcal{C})$ for every $s \in \left[-\frac{1}{2}, \frac{3}{2}\right]$. Denote by $T_{-\eta}$ the self-adjoint extension of T given by $D(T_{-\eta}) = \left\{f \in D(T^*) : \tilde{\tau} f \in H^{3/2}(\mathcal{C}) \text{ and } \kappa f = -\eta \tilde{\tau} f\right\}$.[1] Let B be a Brownian motion in $\mathbb{R}^2$ and define an exponential holding time c independent of B with law

$$\mathbb{P}^{\cdot}(c > t) = e^{-t}. \tag{4.4}$$

Define $\tau = \inf\{t : \int_0^t \eta(B_s)\, \mathrm{d}L(s, \mathcal{C}) \geq c\}$. Then the generator of the s.c.c.s. associated to the process X_t given by

$$\begin{aligned} X_t &= B_t \text{ if } t < \tau \\ X_t &= \phi \text{ if } t \geq \tau \end{aligned}$$

is just $T_{-\eta}$.

Proof. If $T_{-\eta}$ and X_t fulfill the requirements of theorem 55, i.e. $T_{-\eta}$ generates a s.c.c.s. and we have a core of $T_{-\eta}$ such that (4.1) holds for X_t on this core then the application of theorem 55 to $T_{-\eta}$ and X_t will give the statement of theorem 59. We will start the proof with the following proposition.

Proposition 60 The operator $T_{-\eta}$ generates a s.c.c.s.

Proof. We want to apply theorem 54 hence we have to show that $T_{-\eta}$ is a negative definite operator. To do this take $f \in D(T_{-\eta})$. By use of Green's formula for elements of $D(T^*)$

[1] See chapter 2, example 2 for a detailed study of such extensions.

mentioned in chapter 2, p.9, we reach at

$$\begin{aligned}\langle T_{-\eta} f, f\rangle_{L^2(\mathbb{R}^2)} &= -\frac{1}{2}\langle \nabla f, \nabla f\rangle_{L^2(\mathbb{R}^2)} + \frac{1}{2}\langle \kappa f, \tilde{\tau} f\rangle_{L^2(\mathcal{C})} \\ &= -\frac{1}{2}\langle \nabla f, \nabla f\rangle_{L^2(\mathbb{R}^2)} - \frac{1}{2}\langle \eta\tilde{\tau} f, \tilde{\tau} f\rangle_{L^2(\mathcal{C})} \\ &\leq 0.\end{aligned}$$

This implies that $T_{-\eta}$ is a negative definite operator. Since $T_{-\eta}$ is self-adjoint by construction we will get by theorem 54 that $T_{-\eta}$ generates a s.c.c.s. □

The next step is to find a suitable core V of $T_{-\eta}$. By remark 56 it is enough to show that $T_{-\eta}$ restricted to the set V is essentially self-adjoint.

Proposition 61 Define the set $V = D(T_{-\eta}) \cap C^2\left(\overline{\mathcal{C}_{int}}\right) \oplus C^2\left(\overline{\mathcal{C}_{ext}}\right)$. Then the restriction of $T_{-\eta}$ to V is essentially self-adjoint.

Proof. Denote by A the restriction $T_{-\eta}$ to V. Due to [28], p.141 A is essentially self-adjoint if and only if $\mathcal{N}(A^* \pm i) = \{0\}$. We will show that

$$\mathcal{N}(A^* \pm i) \subseteq \mathcal{N}(T_{-\eta} \pm i). \tag{4.5}$$

Since $T_{-\eta}$ is self-adjoint we obtain that $\mathcal{N}(T_{-\eta} \pm i) = \{0\}$ and hence (4.5) implies the essentially self-adjointness of A and therefore the core property of V.
The proof of (4.5) can be given in the following way: Since $D(T) \subseteq V$ we obviously have that $D(A^*) \subseteq D(T^*)$. Take now $f \in V$. Since $V \subseteq D(T_{-\eta})$ we have that $\kappa f = -\eta\tilde{\tau} f$. Using this boundary condition and Green's formula, see chapter 2, p.9 we get for any $g \in D(T^*)$ that

$$\begin{aligned}\left\langle \frac{1}{2}\Delta f, g\right\rangle_{L^2(\mathbb{R}^2)} - \left\langle f, \frac{1}{2}\Delta g\right\rangle_{L^2(\mathbb{R}^2)} &= \frac{1}{2}\langle \kappa f, \tilde{\tau} g\rangle_{L^2(\mathcal{C})} - \frac{1}{2}\langle \tilde{\tau} f, \kappa g\rangle_{L^2(\mathcal{C})} \\ &= -\frac{1}{2}\langle \tilde{\tau} f, \eta\tilde{\tau} g + \kappa g\rangle_{L^2(\mathcal{C})}.\end{aligned}$$

Here we used that η is real-valued and hence $\langle \eta\tilde{\tau} f, \tilde{\tau} g\rangle_{L^2(\mathcal{C})} = \langle \tilde{\tau} f, \eta\tilde{\tau} g\rangle_{L^2(\mathcal{C})}$. Thus we obtain that

$$D(A^*) = \{f \in D(T^*) \,:\, \kappa f = -\eta\tilde{\tau} f\}.$$

By the results of [27], Theorem 2.1 we can represent every element f of $\mathcal{N}(T^* \pm i)$ as $G(\pm i)\, w$ for some $w \in H^{3/2}(\mathcal{C})$ where the operator $G(\pm i)$ is defined in chapter 2, p.10. Due to the results of chapter 2, pp.12 we have that $G(\pm i)\, w = S_{\pm i} g$ where $g = \Lambda w \in H^{-3/2}(\mathcal{C})$ and $S_{\pm i}$ is the so-called acoustic single-layer potential. By chapter 2, lemma 2 we have that $\kappa f = 2g$ and $\tilde{\tau} f = 2P_{\pm i} g$ with the operator $P_{\pm i}$ defined as in corollary 12, p.18. This implies that for $f \in D(A^*)$ we have that

$$2g = \kappa f = -\eta\tilde{\tau} f = -2\eta P_{\pm i} g. \tag{4.6}$$

It is mentioned in proposition 12, p.18, that $P_{\pm i}$ maps $H^s(\mathcal{C})$ to $H^{s+1}(\mathcal{C})$ for any $s \in [-2, 2]$. Hence if we take $g \in H^{-3/2}(\mathcal{C})$ then $-\eta P_{\pm i} g \in H^{-3/2+\varepsilon}(\mathcal{C})$ by the mapping properties of η. If we now assume that (4.6) holds then we get that $g \in H^{-3/2+\varepsilon}(\mathcal{C})$. By repeating this argument we will come up with the property that $g \in H^{1/2}(\mathcal{C})$ and hence $P_{\pm i} g \in H^{3/2}(\mathcal{C})$. This is equivalent to $\tilde{\tau} f \in H^{3/2}(\mathcal{C})$ and therefore we get that $f \in \mathcal{N}(A^* \pm i)$ implies that $f \in D(T^*)$, $\tilde{\tau} f \in H^{3/2}(\mathcal{C})$ and $\kappa f = -\eta \tilde{\tau} f$. But this is nothing else but $f \in D(T_{-\eta})$. Hence $\mathcal{N}(A^* \pm i) \subseteq \mathcal{N}(T_{-\eta} \pm i)$. Thus (4.5) is proven and this shows proposition 61. □

The last step in the proof of theorem 59 is to show that (4.1) holds on V.

Proposition 62 For $f \in V$ we have that

$$\lim_{t \downarrow 0} \frac{1}{t} \left(\mathbb{E}^x \left(f \left(X_t \right) \right) - f(x) \right) = T_{-\eta} f(x)$$

in $L^2\left(\mathbb{R}^2\right)$.

Proof. We want to start with the representation of $\mathbb{E}^x(f(X_t))$ by a Feynman-Kac type formula. Afterwards we will use theorem 57 to calculate $f(B_t)$ for functions in V. Then the combination of these two results will finish the proof.

Lemma 63 For every $f \in L^2\left(\mathbb{R}^2\right)$ we have

$$\mathbb{E}^x \left(f \left(X_t \right) \right) = \mathbb{E}^x \left(f \left(B_t \right) \exp \left(- \int_0^t \eta \left(B_s \right) \mathrm{d}L \left(s, \mathcal{C} \right) \right) \right). \tag{4.7}$$

Proof. With the help of the law (4.4) and since $\eta \geq 0$ we get

$$\mathbb{P}(\tau > t) = \mathbb{P} \left(c > \int_0^t \eta \left(B_s \right) \mathrm{d}L \left(s, \mathcal{C} \right) \right) = \exp \left(- \int_0^t \eta \left(B_s \right) \mathrm{d}L \left(s, \mathcal{C} \right) \right). \tag{4.8}$$

Take now an arbitrary $f \in L^2\left(\mathbb{R}^2\right)$. Then by the definition of X_t we obtain that

$$\mathbb{E}^x \left(f \left(X_t \right) \right) = \mathbb{E}^x \left(f \left(B_t \right) \mathbb{1}_{\{\tau > t\}} \right).$$

By (4.8) we get that

$$\begin{aligned} \mathbb{E}^x \left(f \left(B_t \right) \mathbb{1}_{\{\tau > t\}} \right) &= \mathbb{E}^x \left(\mathbb{E} \left[f \left(B_t \right) \mathbb{1}_{\{\tau > t\}} | B \right] \right) \\ &= \mathbb{E}^x \left(f \left(B_t \right) \mathbb{P} \left(\tau > t \right) \right) \\ &= \mathbb{E}^x \left(f \left(B_t \right) \exp \left(- \int_0^t \eta \left(B_s \right) \mathrm{d}L \left(s, \mathcal{C} \right) \right) \right) \end{aligned}$$

which proves the lemma. □

Take $f \in V$. Then as already mentioned in theorem 57 we have that $f \in C^0\left(\mathbb{R}^2\right)$. This implies that $\tilde{\tau} f$ is just the restriction of f to $\mathcal{C}$. Additionally we get by $f \in V$ that $\kappa f \in C^0(\mathcal{C})$. For an $f \in V$ define $F = f + g_{\kappa f}$ where $g_{\kappa f}$ is the single layer potential with layer function κf defined in (3.2). Then by theorem 57 we can apply Ito's formula to F and obtain that

$$F(B_t) = F(B_0) + \int_0^t \nabla F(B_s)\, \mathrm{d}B_s + \frac{1}{2}\int_0^t \Delta F(B_s)\, \mathrm{d}s.$$

Together with the Tanaka type formula for $g_{\kappa f}$ shown in theorem 26, p.29, i.e.

$$g_{\kappa f}(B_t) = g_{\kappa f}(B_0) + \int_0^t \nabla g_{\kappa f}(B_s)\, \mathrm{d}B_s + \int_0^t \kappa f(B_s)\, \mathrm{d}L(s, \mathcal{C})$$

we are able to deduce that

$$f(B_t) = f(B_0) + \int_0^t \nabla f(B_s)\, \mathrm{d}B_s + \frac{1}{2}\int_0^t \Delta F(B_s)\, \mathrm{d}s - \int_0^t \kappa f(B_s)\, \mathrm{d}L(s, \mathcal{C}). \tag{4.9}$$

By use of (4.9) and stochastic product rule we get that

$$\begin{aligned} &\mathbb{E}^x\left(f(B_t)\exp\left(-\int_0^t \eta(B_s)\, \mathrm{d}L(s, \mathcal{C})\right)\right) \\ =& f(x) + \mathbb{E}^x\left(\frac{1}{2}\int_0^t \Delta F(B_s)\exp\left(-\int_0^s \eta(B_u)\, \mathrm{d}L(u, \mathcal{C})\right)\mathrm{d}s\right) \\ &+ \mathbb{E}^x\left(\int_0^t \left(-\kappa f(B_s) - f(B_s)\eta(B_s)\right)\exp\left(-\int_0^s \eta(B_u)\, \mathrm{d}L(u, \mathcal{C})\right)\mathrm{d}L(s, \mathcal{C})\right). \end{aligned} \tag{4.10}$$

But we assume that $\kappa f = -\eta\tilde{\tau} f$ and this implies that

$$\int_0^t \left(-\kappa f(B_s) - f(B_s)\eta(B_s)\right)\exp\left(-\int_0^s \eta(B_u)\, \mathrm{d}L(u, \mathcal{C})\right)\mathrm{d}L(s, \mathcal{C}) = 0. \tag{4.11}$$

Hence by combining (4.10) and (4.11) we obtain with the help of lemma 63 that

$$\mathbb{E}^x\left(f(X_t)\right) = f(x) + \mathbb{E}^x\left(\frac{1}{2}\int_0^t \Delta F(B_s)\exp\left(-\int_0^s \eta(B_u)\, \mathrm{d}L(u, \mathcal{C})\right)\mathrm{d}s\right)$$

and thus we obtain that point-wise

$$\lim_{t\downarrow 0}\frac{1}{t}\left(\mathbb{E}^x\left(f(X_t)\right) - f(x)\right) = \frac{1}{2}\Delta F(x).$$

But we obviously have that

$$\left|\frac{1}{t}\left(\mathbb{E}^x\left(f(X_t)\right) - f(x)\right)\right| \leq \frac{1}{t}\mathbb{E}^x\left(\frac{1}{2}\int_0^t \left|\Delta F(B_s)\right| \mathrm{d}s\right).$$

But it is well known, see e.g. [24], Ch.3, Example 2, that

$$\lim_{t\downarrow 0}\frac{1}{t}\mathbb{E}^x\left(\frac{1}{2}\int_0^t |\Delta F(B_s)|\,\mathrm{d}s\right)=|\Delta F(x)|$$

in $L^2(\mathbb{R}^2)$. Therefore we can conclude by dominated convergence that also in $L^2(\mathbb{R}^2)$

$$\lim_{t\downarrow 0}\frac{1}{t}\left(\mathbb{E}^x\left(f\left(X_t\right)\right)-f(x)\right)=\frac{1}{2}\Delta F(x).$$

Since $\Delta F=\Delta f$ on $\mathbb{R}^2\backslash\{\mathcal{C}\}$ as shown in theorem 57 and $f\in D(T_{-\eta})$ we get that $\frac{1}{2}\Delta F=T_{-\eta}f$ and the proposition is proven. □

To put it in a nutshell: $T_{-\eta}$ generates a s.c.c.s. and on a core of $T_{-\eta}$ we have that $\lim_{t\downarrow 0}\frac{1}{t}\left(\mathbb{E}^x\left(f\left(X_t\right)\right)-f(x)\right)=T_{-\eta}f$. Hence by theorem 55 we get the statement of theorem 59. □

4.3 The extension given by the Laplace-Beltrami operator

In this section we want to construct the stochastic process related to the self-adjoint extension of T given by the Laplace-Beltrami operator as given in chapter 2, example 3. We will see that the process behaves like a Brownian motion in $\mathbb{R}^2\backslash\{\mathcal{C}\}$ whereas on $\mathcal{C}$ we get an additional movement in the the time scale of $L(t,\mathcal{C})$. The additional movement is just a Brownian motion on $\mathcal{C}$. The generator of the $\mathcal{C}$-Brownian motion is given by the Laplace-Beltrami operator, see [19], Ch.V, Theorem 1.1.

Theorem 64 Take Δ_{LB} to be the self-adjoint Laplace-Beltrami operator from $H^2(\mathcal{C})\subseteq L^2(\mathcal{C})$ to $L^2(\mathcal{C})$.Denote by $T_{\frac{1}{2}\Delta_{LB}}$ the self-adjoint extension of T given by $D\left(T_{\frac{1}{2}\Delta_{LB}}\right)=\left\{f\in D(T^*)\,:\,\tilde{\tau}f\in H^{5/2}(\mathcal{C})\text{ and }\kappa f=\frac{1}{2}\Delta_{LB}\tilde{\tau}f\right\}$.[2]
Take now a Brownian motion W in $\mathbb{R}$ and define $Z=\psi(W)$, where ψ is a parametrization of $\mathcal{C}$with respect to the arc length. Then the generator of the process $X_t=B_t+Z_{L(t,\mathcal{C})}$ is $T_{\frac{1}{2}\Delta_{LB}}$. Here B_t is a Brownian motion in $\mathbb{R}^2$ independent of W.

Proof. As in the proof of theorem 59 we want to show that $T_{\frac{1}{2}\Delta_{LB}}$ and X_t fulfill the requirements of 55, i.e. $T_{\frac{1}{2}\Delta_{LB}}$ generates a s.c.c.s. and we have a core of $T_{\frac{1}{2}\Delta_{LB}}$ such that (4.1) holds for X_t on this core. Then the application of theorem 55 to $T_{\frac{1}{2}\Delta_{LB}}$ and X_t will give the statement of theorem 64. We will start the proof with the following proposition.

Proposition 65 The operator $T_{\frac{1}{2}\Delta_{LB}}$ generates a s.c.c.s.

[2] For further details see chapter 2, example 3.

Proof. As in the proof of proposition 60 we want to show that $T_{\frac{1}{2}\Delta_{LB}}$ is a negative definite operator. Then an application of theorem 54 will give the result. By the same calculations as in the proof of proposition 60 we get for any $f \in D\left(T_{\frac{1}{2}\Delta_{LB}}\right)$ that

$$\left\langle T_{\frac{1}{2}\Delta_{LB}} f, f\right\rangle_{L^2(\mathbb{R}^2)} = -\frac{1}{2}\left\langle \nabla f, \nabla f\right\rangle_{L^2(\mathbb{R}^2)} + \frac{1}{2}\left\langle \frac{1}{2}\Delta_{LB}\tilde{\tau} f, \tilde{\tau} f\right\rangle_{L^2(\mathcal{C})}.$$

Since Δ_{LB} is a negative definite operator, see [28], p.160, example 4, we get that $T_{\frac{1}{2}\Delta_{LB}}$ is a negative definite and self-adjoint operator. Hence we will get by theorem 54 that $T_{\frac{1}{2}\Delta_{LB}}$ generates a s.c.c.s. □

Now we will find a core of $T_{\frac{1}{2}\Delta_{LB}}$.

Proposition 66 The restriction of $T_{\frac{1}{2}\Delta_{LB}}$ to $V = D\left(T_{\frac{1}{2}\Delta_{LB}}\right) \cap C^2\left(\overline{\mathcal{C}_{int}}\right) \oplus C^2\left(\overline{\mathcal{C}_{ext}}\right)$ is essentially self-adjoint.

Proof. In correspondence to the proof of proposition 61 it is enough to show that A, i.e. the restriction of $T_{\frac{1}{2}\Delta_{LB}}$ to V, is essentially self-adjoint. Then remark 56 shows that V is a core of $T_{\frac{1}{2}\Delta_{LB}}$.
By the same argumentation as in proposition 61 it is enough to show that

$$\mathcal{N}\left(A^* \pm i\right) \subseteq \mathcal{N}\left(T_{\frac{1}{2}\Delta_{LB}} \pm i\right).$$

By copying the arguments of proposition 61 we get for any $f \in \mathcal{N}\left(A^* \pm i\right)$ the existence of a function $g \in H^{-3/2}(\mathcal{C})$ such that

$$g = \kappa f = \frac{1}{2} D_{LB}\tilde{\tau} f = \frac{1}{2} D_{LB} P_{\pm i} g. \tag{4.12}$$

Here D_{LB} denotes the distributional Laplace-Beltrami which has the property that

$$\text{If } f \in H^s(\mathcal{C}) \text{ and } D_{LB} f \in H^{s-1}(\mathcal{C}) \text{ then } f \in H^{s+1}(\mathcal{C}) \text{ for any } s \in \mathbb{R}, \tag{4.13}$$

see chapter 2, example 3. If we start with $g \in H^{-3/2}(\mathcal{C})$ then we get that $P_{\pm i} g \in H^{-1/2}(\mathcal{C})$ since $P_{\pm i}$ maps $H^s(\mathcal{C})$ to $H^{s+1}(\mathcal{C})$ for any $s \in [-2, 2]$ as mentioned in proposition 12, p.18. If we now assume that (4.12) holds we get by (4.13) that $P_{\pm i} g \in H^{1/2}(\mathcal{C})$. But it is also mentioned in proposition 12, p.18, that for any $s \in \left[-\frac{1}{2}, \frac{3}{2}\right]$ and for all $\sigma \geq 0$ we can obtain if $\phi \in H^s(\mathcal{C})$ and $P\phi \in H^{s+1+\sigma}(\mathcal{C})$ then $\phi \in H^{s+\sigma}(\mathcal{C})$. Hence $g \in H^{-1/2}(\mathcal{C})$. By repeating now this argument we end up with the fact that $P_{\pm i} g = \tilde{\tau} f \in H^{5/2}(\mathcal{C})$. Since we also have that $\kappa f = \frac{1}{2} D_{LB}\tilde{\tau} f$ we conclude that $f \in D\left(T_{\frac{1}{2}\Delta_{LB}}\right)$. This finishes the proof of proposition 66. □

Hence $V = D\left(T_{\frac{1}{2}\Delta_{LB}}\right) \cap C^2\left(\overline{\mathcal{C}_{int}}\right) \oplus C^2\left(\overline{\mathcal{C}_{ext}}\right)$ is a core of $T_{\frac{1}{2}\Delta_{LB}}$. The last step in the proof of theorem 64 is to show that (4.1) holds on V.

Proposition 67 For $f \in V$ we have that

$$\lim_{t \downarrow 0} \frac{1}{t} \left(\mathbb{E}^x \left(f \left(X_t \right) \right) - f(x) \right) = T_{\frac{1}{2}\Delta_{LB}} f(x)$$

in $L^2 \left(\mathbb{R}^2 \right)$.

Proof. We will start the proof by collecting some useful facts on Z_t. Afterwards we will use again the Tanaka type formula for the single-layer potential proven in chapter 3 to determine the generator of X_t.

Lemma 68 It exists a $\sigma : \mathbb{R}^2 \to \mathbb{R}^{2\times 2}$ and a $b : \mathbb{R}^2 \to \mathbb{R}^2$ such that

$$\mathrm{d}Z = \sigma(Z)\mathrm{d}W + b(Z)\mathrm{d}s.$$

Moreover the generator of Z is equal to $\frac{1}{2}\Delta_{LB}$ on $\mathcal{C}$.

Proof. A proof of these statements can be found in [19], Ch.V, Theorem 1.1 and Theorem 1.2. □

Take $f \in V$ and define $F = f + g_{\kappa f}$ where $g_{\kappa f}$ is the single layer potential with layer function κf defined in (3.2). Then by theorem 57 we can apply Ito's formula to F and obtain that

$$\begin{aligned} F \left(X_t \right) =& F \left(X_0 \right) + \int_0^t \nabla F \left(X_s \right) \mathrm{d}B_s + \int_0^t \nabla F \left(X_s \right) \sigma \left(Z_s \right) \mathrm{d}W_{L(s,\mathcal{C})} + \frac{1}{2} \int_0^t \Delta F \left(X_s \right) \mathrm{d}s \\ &+ \frac{1}{2} \int_0^t \Delta_{LB} F \left(X_s \right) \mathrm{d}L \left(s, \mathcal{C} \right). \end{aligned}$$

With the help of the Tanaka type formula for the single layer potential given in (3.3) we get

$$\begin{aligned} g_{\kappa f} \left(X_t \right) =& g_{\kappa f} \left(X_0 \right) + \int_0^t \nabla g_{\kappa f} \left(X_s \right) \mathrm{d}B_s + \int_0^t \nabla g_{\kappa f} \left(X_s \right) \sigma \left(Z_s \right) \mathrm{d}W_{L(s,\mathcal{C})} \\ &+ \int_0^t \kappa f \left(X_s \right)) \mathrm{d}L \left(s, \mathcal{C} \right) + \frac{1}{2} \int_0^t \Delta_{LB} g_{\kappa f} \left(X_s \right) \mathrm{d}L \left(s, \mathcal{C} \right). \end{aligned}$$

By combining these two equalities we get

$$\begin{aligned} f \left(X_t \right) =& f \left(X_0 \right) + \int_0^t \nabla f \left(X_s \right) \mathrm{d}B_s + \int_0^t \nabla f \left(X_s \right) \sigma \left(Z_s \right) \mathrm{d}W_{L(s,\mathcal{C})} \\ &+ \frac{1}{2} \int_0^t \Delta F \left(X_s \right) \mathrm{d}s \\ &+ \int_0^t \frac{1}{2} \Delta_{LB} f \left(X_s \right) - \kappa f \left(X_s \right) \mathrm{d}L \left(s, \mathcal{C} \right). \end{aligned}$$

But we assume that $\kappa f = \frac{1}{2}\Delta_{LB} f$ and hence

$$\mathbb{E}^x \left(f \left(X_t \right) \right) = f(x) + \mathbb{E}^x \left(\frac{1}{2} \int_0^t \Delta F \left(X_s \right) \mathrm{d}s \right).$$

Thus we obtain that point-wise

$$\lim_{t \downarrow 0} \frac{1}{t} \left(\mathbb{E}^x \left(f \left(X_t \right) \right) - f(x) \right) = \frac{1}{2} \Delta F \left(x \right).$$

By our construction we have that ΔF is a bounded function and hence

$$\begin{aligned}\lim_{t\downarrow 0}\frac{1}{t}\left(\mathbb{E}^x\left(f\left(X_t\right)\right)-f(x)\right) &= \lim_{t\downarrow 0}\frac{1}{t}\mathbb{E}^x\left(\frac{1}{2}\int_0^t \Delta F\left(X_s\right)\mathrm{d}s\right)\\ &\leq \leq \frac{1}{2}\left\|\Delta F\right\|_\infty .\end{aligned}$$

Thus we can obtain by dominated convergence that for every bounded $K\subseteq\mathbb{R}^2$ we have

$$\lim_{t\downarrow 0}\frac{1}{t}\left(\mathbb{E}^x\left(f\left(X_t\right)\right)-f(x)\right)=\frac{1}{2}\Delta F\left(x\right)$$

in $L^2\left(K\right)$. Take now K such that there exists a positive constant M with $d\left(x,\mathcal{C}\right)\geq M$ for any $x\in\mathbb{R}^2\backslash K$. If we denote by $S_r(x)$ the ball with radius r around x and by $\tau_{r,x}$ the first exit time of $S_r(x)$ for the process X_t starting in x we see immediately from the definition of X_t that for any $x\in\mathbb{R}^2\backslash K$ and every $r\leq\frac{M}{2}$ the process X_t behaves like a Brownian motion up to $\tau_{r,x}$. Therefore we get by Dynkin's representation of the generator, see [21], Chapter 7, that

$$\begin{aligned}\lim_{t\downarrow 0}\frac{1}{t}\left(\mathbb{E}^x\left(f\left(X_t\right)\right)-f(x)\right) &= \lim_{r\downarrow 0}\frac{\mathbb{E}^x\left(f\left(X_{\tau_{r,x}}\right)\right)-f(x)}{\mathbb{E}^x\left(\tau_{r,x}\right)}\\ &= \lim_{r\downarrow 0}\frac{\mathbb{E}^x\left(f\left(B_{\tau_{r,x}}\right)\right)-f(x)}{\mathbb{E}^x\left(\tau_{r,x}\right)}\\ &= \lim_{t\downarrow 0}\frac{1}{t}\left(\mathbb{E}^x\left(f\left(B_t\right)\right)-f(x)\right).\end{aligned}$$

But $\lim\limits_{t\downarrow 0}\frac{1}{t}\left(\mathbb{E}^x\left(f\left(B_t\right)\right)-f(x)\right)=\frac{1}{2}\Delta f(x)$ in $L^2\left(\mathbb{R}^2\right)$ and since $\Delta F=\Delta f$ on $\mathbb{R}^2\backslash\{\mathcal{C}\}$ as shown in theorem 57 we have that

$$\lim_{t\downarrow 0}\frac{1}{t}\left(\mathbb{E}^x\left(f\left(X_t\right)\right)-f(x)\right)=\frac{1}{2}\Delta F(x)$$

in $L^2\left(\mathbb{R}^2\backslash K\right)$. Thus we obtain that

$$\lim_{t\downarrow 0}\frac{1}{t}\left(\mathbb{E}^x\left(f\left(X_t\right)\right)-f(x)\right)=\frac{1}{2}\Delta F\left(x\right)$$

in $L^2\left(\mathbb{R}^2\right)$. Since $\Delta F=\Delta f$ on $\mathbb{R}^2\backslash\{\mathcal{C}\}$ as shown in theorem 57 and $f\in D\left(T_{\frac{1}{2}\Delta_{LB}}\right)$ we get that $\frac{1}{2}\Delta F=T_{\frac{1}{2}\Delta_{LB}}f$ and the proposition is proven. $\square$

To put it in a nutshell: $T_{\frac{1}{2}\Delta_{LB}}$ generates a s.c.c.s. and on a core of $T_{\frac{1}{2}\Delta_{LB}}$ we have that $\frac{d}{dt}\mathbb{E}^x\left(f\left(X_t\right)\right)=T_{\frac{1}{2}\Delta_{LB}}f$. Hence by theorem 55 we get the statement of theorem 64. $\square$

Chapter 5

Superprocesses

In the chapter before we restrict ourselves to non-negative functions η to construct stochastic processes related to the extensions T_η of T. In this section we get rid of the restriction on the sign of η. But we have to pay a price for forgetting the sign of η. Instead of classical stochastic processes as in chapter 4 we have to deal with measure valued processes, so-called superprocesses.

An overview on the theory of superprocesses can be found in [12]. One main interesting fact is that the semigroup of the generator of such a measure valued process does not have to be sub-Markovian anymore, i.e. $P_t 1 \nleq 1$. Hence we can construct superprocesses for a larger class of operators than classical stochastic processes.

The strategy of this chapter is to show the existence of a superprocess related to T_η with the help of a similar approximation procedure as in chapter 3. At first we will give the general definition of a superprocess with the help of a partial differential equation. Then we will give the description of the superprocess we are interested in with the help of some integral equation. The next step is to show the connection between the integral equation in our special case, the partial differential equation in the general case and the operators T_η. Afterwards we will use an approximation procedure to show the unique solvability of the defining integral equation. Furthermore we will use the existence of a superprocess for any approximation step to show the existence of the superprocess in the limiting case. The existence of the approximating superprocesses is shown by J. Engländer and R.G.Pinsky, see [10].

Preliminaries As in [9], let $\mathcal{M}$ denote the set of finite measures on $\mathbb{R}^2$ equipped with the topology of weak convergence and denote by $\langle \mu, g \rangle$ the integral $\int_{\mathbb{R}^2} g \, \mathrm{d}\mu$. Let α, γ be Hölder continuous functions on $\mathbb{R}^2$ with $\sup_{x \in \mathbb{R}^2} \gamma(x) < \infty$ and $\alpha > 0$.

Then we have a unique $\mathcal{M}$-valued time continuous Markov process X satisfying

$$\mathbb{E}_{\mu}\left(\exp\left\langle X_t, -g\right\rangle\right) = \exp\left(-\int_{\mathbb{R}^2} u\left(t,x\right)\mu(dx)\right)$$

for any $\mu \in \mathcal{M}$ and any bounded, continuous $g : \mathbb{R}^2 \to \mathbb{R}$, where u is the minimal non-negative solution to

$$\frac{\partial}{\partial t}u = \frac{1}{2}\Delta u + \gamma u - \alpha u^2 \text{ on } \mathbb{R}^2 \times (0,\infty) \tag{5.1}$$

with the boundary condition

$$\lim_{t\to 0^+} u\left(t,\cdot\right) = g\left(\cdot\right).$$

This process is called a superdiffusion and the process $(X, \mathbb{P}_{\mu}, \mu \in \mathcal{M})$ is sometimes called the $\left(\frac{1}{2}\Delta, \alpha, \gamma, \mathbb{R}^2\right)$-superprocess.

5.1 Main statement and basic notations

We want to proof the existence of a superprocess X under a slight modification of (5.1).

Theorem 69 For any $\eta \in C^{0,1/2+\varepsilon}\left(\mathcal{C}\right)$ with $\varepsilon \in (0,1)$ it exists a unique $\mathcal{M}$-valued time continuous Markov process X such that

$$\mathbb{E}_{\mu}\left(\exp\left\langle X_t, -g\right\rangle\right) = \exp\left(-\int_{\mathbb{R}^2} u\left(t,x\right)\mu(dx)\right) \tag{5.2}$$

for any $\mu \in \mathcal{M}$ and any bounded, continuous $g : \mathbb{R}^2 \to \mathbb{R}$, where u is the minimal non-negative solution to

$$\begin{aligned} u\left(t,\cdot\right) = &\int_{\mathbb{R}^2} g\left(y\right) p\left(t,\cdot,y\right) \mathrm{d}y \\ &+ \int_0^t \int_{\mathcal{C}} \eta\left(y\right) u\left(s,y\right) p\left(t-s,\cdot,y\right) \mathrm{d}S(y)\,\mathrm{d}s \\ &- \int_0^t \int_{\mathbb{R}^2} \alpha\left(y\right) u^2\left(s,y\right) p\left(t-s,\cdot,y\right) \mathrm{d}y\,\mathrm{d}s. \end{aligned} \tag{5.3}$$

Here α is a positive, bounded and Hölder-continuous function on $\mathbb{R}^2$ and $p\left(t,\cdot,y\right)$ is the usual transition kernel of the Brownian motion in $\mathbb{R}^2$.
Before we start with the proof of theorem 69 we give some integral equation approximation result and fix some notation.

Approximation result and notations

For establishing the existence of a solution of equation (5.3) we will use the following integral equation approximation theorem which can be found in [11], Ch. 5.

Theorem 70 Take a $k(t,x,y) : \mathbb{R}_+ \times \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}_+$ which is for any $t > 0$ continuous in x and y and it exists a constant C such that for any $t \geq 0$ and every $x \in \mathbb{R}^2$ $\int\limits_{\mathbb{R}^2} k(t,x,y)\,\mathrm{d}y = C$. Furthermore take a sequence $\psi_\varepsilon \in C^0(\mathbb{R}^2) \cap L^1(\mathbb{R}^2)$ such that ψ_ε converges weakly to a distribution Ψ, i.e. for every bounded and continuous f we have

$$\lim_{\varepsilon\to 0} \int\limits_{\mathbb{R}^2} \psi_\varepsilon(x) f(x)\mathrm{d}x = \Psi(f)$$

with $|\Psi(f)| \leq \|f\|_\infty$. Define the integral equation

$$\begin{aligned} u_\varepsilon(t,\cdot) = & \int\limits_{\mathbb{R}^2} g(y)\,k(t,\cdot,y)\,\mathrm{d}y \\ & + \int\limits_0^t \int\limits_{\mathbb{R}^2} \psi_\varepsilon(y) u_\varepsilon(s,y)\,k(t-s,\cdot,y)\,\mathrm{d}y\,\mathrm{d}s \\ & - \int\limits_0^t \int\limits_{\mathbb{R}^2} \alpha(y)\,u_\varepsilon^2(s,y)\,k(t-s,\cdot,y)\,\mathrm{d}y\,\mathrm{d}s \end{aligned} \tag{5.4}$$

for some non-negative and bounded α.

If there exists a $\delta > 0$ such that the equation (5.4) has a solution $u_\varepsilon(t,x)$ which is uniformly bounded in (ε,t,x) on $[0,\delta] \times [0,T] \times \mathbb{R}^2$ for any finite $T > 0$ and if additionally there exists a function $h(t)$ which is integrable on every finite interval $[0,T]$ such that for every $x \in \mathbb{R}^2$ and every $0 < \varepsilon \leq \delta$

$$\int\limits_{\mathbb{R}^2} |\psi_\varepsilon(y)|\,k(t,x,y)\,\mathrm{d}y \leq h(t) \tag{5.5}$$

then $u_\varepsilon(t,x)$ (maybe by dropping to a subsequence) converges uniformly to $u(t,x)$ as ε tends to zero where $u(t,x)$ is a solution of

$$\begin{aligned} u(t,x) = & \int\limits_{\mathbb{R}^2} g(y)\,k(t,x,y)\,\mathrm{d}y \\ & + \int\limits_0^t \Psi(u(s,\cdot)\,k(t-s,x,\cdot))\,\mathrm{d}s \\ & - \int\limits_0^t \int\limits_{\mathbb{R}^2} \alpha(y)\,u^2(s,y)\,k(t-s,x,y)\,\mathrm{d}y\,\mathrm{d}s. \end{aligned} \tag{5.6}$$

The main tool for the prove of theorem 70 is the application of the theorem of Arzela and Ascoli .

Later on we will identify $k(t,x,y)$ with $p(t,x,y)$ and construct a weak convergent sequence ψ_ε such that (5.5) holds.

Notations Denote as in chapter 2 by T_η the extension of T related to the multiplication operator given by η as in (2.5). Fix $\beta > 0$ such that for every $r \in (-\beta, \beta)$ $\mathcal{C}^r$ is a Runge parallel surface of $\mathcal{C}$ and (3.8) holds. Define $\tilde{\eta}$ as in (3.10) and take $\phi(r,y)$ as in (3.7).

5.2 Construction of superprocesses

Before we start with the actual proof of theorem 69 we want to establish the connection of (5.3) to T_η. Remember that we have shown in corollary 23 that

$$T_\eta u(\cdot) = \frac{1}{2}\Delta u(\cdot) + \int\limits_{\mathcal{C}} \eta(y)u(y)\delta_0(\cdot - y)\,\mathrm{d}S(y) \tag{5.7}$$

for any $u \in D(T_\eta)$. Define now

$$u(t,\cdot) = \int\limits_{\mathbb{R}^2} g(y)\,p(t,\cdot,y)\,\mathrm{d}y + \int\limits_0^t \int\limits_{\mathcal{C}} \eta(y)\,u(s,y)\,p(t-s,\cdot,y)\,\mathrm{d}S(y)\,\mathrm{d}s.$$

Then formally we have

$$\frac{d}{dt}u(t,\cdot) = \frac{1}{2}\Delta u(t,\cdot) + \int\limits_{\mathcal{C}} \eta(y)u(y)\delta_0(\cdot - y)\,\mathrm{d}s(y)$$

and thus (5.3) is nothing else but a weak formulation of (5.1) in the case of T_η.
As already mentioned in the beginning of this chapter we have for any Hölder continuous functions α and γ a $\left(\frac{1}{2}\Delta, \alpha, \gamma, \mathbb{R}^2\right)$-superprocess, see [10], Appendix A. Now we want to approximate (5.3) by use of Hölder continuous functions γ_ε and then use the existence of an $\left(\frac{1}{2}\Delta, \alpha, \gamma_\varepsilon, \mathbb{R}^2\right)$-superprocess to construct the superprocess we are interested in.

Proof. We will start the proof with the construction of an approximation of (5.3).

Lemma 71 Choose a Dirac-sequence $m_\varepsilon \in C^1(\mathbb{R})$ with $\operatorname{supp}(m_\varepsilon) \subseteq (-\varepsilon, \varepsilon)$. Take

$$\begin{aligned} D(x) &= -\,d(x,\mathcal{C}) \ \text{ if } x \in \overline{\mathcal{C}_{int}} \\ &= d(x,\mathcal{C}) \ \text{ if } x \in \overline{\mathcal{C}_{ext}} \end{aligned}$$

and define

$$\gamma_\varepsilon(x) = \tilde{\eta}(x) m_\varepsilon(D(x)). \tag{5.8}$$

for any $\varepsilon \leq \frac{\beta}{2}$. Then γ_ε is a Hölder continuous function in $\mathbb{R}^2$ for every $\varepsilon < \frac{\beta}{2}$.

Proof. It is shown in [30], Ch.201, pp.528, that $D \in C^1(\mathcal{C}_\beta)$. Hence we get by the definition of m_ε that $\gamma_\varepsilon \in C^0(\mathbb{R}^2)$. Furthermore γ_ε is Hölder-continuous in $\mathcal{C}_\beta$ and $\gamma_\varepsilon \equiv 0$ on $\mathbb{R}^2 \backslash \mathcal{C}_\varepsilon$. Thus γ_ε is Hölder continuous in $\mathbb{R}^2$. □

Integral equation

First of all we want to show the unique solvability of (5.3) with some approximation via γ_ε. The existence of a solution will be established with the help of theorem 70. The uniqueness of the solution is then just a little calculation.

Proposition 72 The integral equation (5.3) has a unique non-negative solution.

Proof. Take γ_ε as in (5.8). Then by [10], Appendix A and the references therein we have that

$$\begin{aligned} u_\varepsilon(t,\cdot) &= \int_{\mathbb{R}^2} g(y)\, p(t,\cdot,y)\,\mathrm{d}y \\ &\quad + \int_0^t \int_{\mathbb{R}^2} \gamma_\varepsilon(y) u_\varepsilon(s,y)\, p(t-s,\cdot,y)\,\mathrm{d}y\,\mathrm{d}s \\ &\quad - \int_0^t \int_{\mathbb{R}^2} \alpha(y)\, u_\varepsilon^2(s,y)\, p(t-s,\cdot,y)\,\mathrm{d}y\,\mathrm{d}s, \end{aligned} \tag{5.9}$$

has a unique non-negative solution continuous in t and x with $\sup_{0\le s\le t} \|u_\varepsilon(s,\cdot)\|_\infty < \infty$. The next step is to proof that $u_\varepsilon \to u$ and that u is indeed a solution of (5.3). Afterwards we show that (5.3) has a unique solution.

Existence of solutions For the existence of a solution we want to apply theorem 70. If we identify $k(t,x,y)$ with $p(t,x,y)$ and ψ_ε with γ_ε then we easily see that for any bounded and continuous function f we have

$$\lim_{\varepsilon\downarrow 0} \int_{\mathbb{R}^2} \gamma_\varepsilon(y) f(y)\mathrm{d}y = \int_{\mathcal{C}} \eta(y) f(y) \mathrm{d}S(y).$$

Hence we only have to check the required uniform boundedness of u_ε and (5.5).

Lemma 73 It exist constants K_1 and K_2 such that for every $x \in \mathbb{R}^2$ and every $\varepsilon < \frac{\beta}{2}$

$$\int_{\mathbb{R}^2} |\gamma_\varepsilon(y)|\, p(t,x,y)\,\mathrm{d}y \le \|\eta\|_\infty \left(K_1 + \frac{K_2}{\sqrt{2\pi t}}\right). \tag{5.10}$$

Remark 74 The function $\frac{1}{\sqrt{t}}$ is obviously integrable on every finite interval $[0,T]$ and thus we obtain by lemma 73 the needed estimate of (5.5).

Proof. By the definition of γ_ε given in (5.8) we immediately see that

$$\int_{\mathbb{R}^2} |\gamma_\varepsilon(y)|\, p(t,x,y)\,\mathrm{d}y \le \|\eta\|_\infty \int_{\mathbb{R}^2} m_\varepsilon(D(y))\, p(t,x,y)\,\mathrm{d}y. \tag{5.11}$$

Take a closer look at

$$\begin{aligned}
\int\limits_{\mathbb{R}^2} m_\varepsilon\left(D(y)\right) p\left(t,x,y\right) \mathrm{d}y &= \int\limits_{\mathcal{C}_\varepsilon} m_\varepsilon\left(D(y)\right) p\left(t,x,y\right) \mathrm{d}y \\
&= \int\limits_{-\varepsilon}^{\varepsilon} \int\limits_{\mathcal{C}^r} p\left(t,x,y\right) \mathrm{d}S_r\left(y\right) m_\varepsilon\left(r\right) \mathrm{d}r \\
&= \int\limits_{-\varepsilon}^{\varepsilon} \int\limits_{\mathcal{C}} p\left(t,x,y+rn_y\right) \phi\left(r,y\right) \mathrm{d}S\left(y\right) m_\varepsilon\left(r\right) \mathrm{d}r.
\end{aligned}$$

If $x \in \mathcal{C}_\beta$ then there exists a $z \in \mathcal{C}$ and an $\tilde{r} \in (-\beta, \beta)$ such that $x = z + \tilde{r}n_z$. Hence

$$\begin{aligned}
\int\limits_{\mathcal{C}} \phi\left(r,y\right) p\left(t,x,y+rn_y\right) \mathrm{d}S(y) &= \int\limits_{\mathcal{C}} \phi\left(r,y\right) p\left(t,z+\tilde{r}n_z,y+rn_y\right) \mathrm{d}S(y) \\
&= \int\limits_{\mathcal{C}\setminus B_\delta(z)} \phi\left(r,y\right) p\left(t,z+\tilde{r}n_z,y+rn_y\right) \mathrm{d}S(y) \qquad (5.12) \\
&\quad + \int\limits_{\mathcal{C}\cap B_\delta(z)} \phi\left(r,y\right) p\left(t,z+\tilde{r}n_z,y+rn_y\right) \mathrm{d}S(y). \qquad (5.13)
\end{aligned}$$

for some $0 < \delta < \frac{1}{2}$. For (5.12) we have $|\left(z+\tilde{r}n_z\right) - \left(y+rn_y\right)| \geq \delta$ and therefore

$$\int\limits_{\mathcal{C}\setminus B_\delta(z)} \phi\left(r,y\right) p\left(t,z+\tilde{r}n_z,y+rn_y\right) \mathrm{d}S(y) \leq N\frac{1}{2\pi t}\exp\left(-\frac{\delta^2}{2t}\right) L\left(\mathcal{C}\right) \qquad (5.14)$$

where $L\left(\mathcal{C}\right)$ represents the length of $\mathcal{C}$ and N is the upper bound of ϕ. For (5.13) we get by (3.8) a positive constant K such that

$$|\left(z+\tilde{r}n_z\right) - \left(y+rn_y\right)|^2 \geq K\left|z-y\right|^2$$

Hence

$$\begin{aligned}
&\int\limits_{\mathcal{C}\cap B_\delta(z)} \phi\left(r,y\right) p\left(t,z+\tilde{r}n_z,y+rn_y\right) \mathrm{d}S(y) \\
= &\int\limits_{\mathcal{C}\cap B_\delta(z)} \phi\left(r,y\right) \frac{1}{2\pi t}\exp\left(-\frac{|\left(z+rn_z\right) - \left(y+\overline{r}n_y\right)|^2}{2t}\right) \mathrm{d}S(y) \\
\leq \; & N \int\limits_{\mathcal{C}\cap B_\delta(z)} \frac{1}{2\pi t}\exp\left(-\frac{K|z-y|^2}{2t}\right) \mathrm{d}S(y).
\end{aligned}$$

Take now a parametrization ϕ of $\mathcal{C}$ with $\phi(0) = z$. Then there exist v, $w \in \mathbb{R}$ such that

$$\begin{aligned}
&\int\limits_{\mathcal{C}\cap B_\delta(z)} \frac{1}{2\pi t}\exp\left(-\frac{K|z-y|^2}{2t}\right) \mathrm{d}S(y) \\
= &\int\limits_{v}^{w} \frac{1}{2\pi t}\exp\left(-\frac{K\left|\phi(0)-\phi(u)\right|^2}{2t}\right) \left|\left(\dot{\phi}\right)(u)\right| \mathrm{d}u.
\end{aligned}$$

By a Taylor expansion of ϕ we get

$$\phi(u) = \phi(0) + u\dot{\phi}(\zeta)$$

for some ζ between 0 and u. And ϕ is $C^{2,\alpha}$ hence $\left|\dot{\phi}\right|$ is bounded by a positive constant M. Therefore we have

$$\begin{aligned}
&\int_v^w \frac{1}{2\pi t} \exp\left(-\frac{K\left|\phi(0)-\phi(u)\right|^2}{2t}\right)\left|\left(\dot{\phi}\right)(u)\right| \mathrm{d}u \\
=& \int_v^w \frac{1}{2\pi t} \exp\left(-\frac{K\left|u\dot{\phi}(\zeta)\right|^2}{2t}\right)\left|\left(\dot{\phi}\right)(u)\right| \mathrm{d}u \\
\leq& \; M\int_v^w \frac{1}{2\pi t} \exp\left(-\frac{u^2\left(K\left|\dot{\phi}(\zeta)\right|\right)^2}{2t}\right)\mathrm{d}u.
\end{aligned}$$

Since ϕ is $C^{2,\alpha}$ we have that $\left|\dot{\phi}\right| \geq C > 0$, see [30], Ch.200, pp.513. Thus we obtain

$$\begin{aligned}
&\int_v^w \frac{1}{2\pi t} \exp\left(-\frac{u^2\left(K\left|\dot{\phi}(\zeta)\right|\right)^2}{2t}\right)\mathrm{d}u \\
\leq& \int_v^w \frac{1}{2\pi t} \exp\left(-\frac{u^2\left(KC\right)^2}{2t}\right)\mathrm{d}u.
\end{aligned}$$

By performing the substitution $x\sqrt{t} = uKC$ we get

$$\begin{aligned}
&\int_v^w \frac{1}{2\pi t} \exp\left(-\frac{u^2\left(KC\right)^2}{2t}\right)\mathrm{d}u \\
\leq& \frac{1}{\sqrt{2\pi t}}\frac{1}{\sqrt{2\pi}KC}\int_v^w \exp\left(-\frac{u^2}{2}\right)\mathrm{d}u \\
\leq& \frac{1}{\sqrt{2\pi t}}\frac{1}{KC} \qquad (5.15)
\end{aligned}$$

which is obviously independent of x.

If $x \notin \mathcal{C}_\beta$ we obviously have that $p(t,x,y)$ is bounded by

$$\frac{1}{2\pi t}\exp\left(-\frac{d\left(x, \operatorname{supp}\left(m_\varepsilon\right)\right)}{2t}\right) \leq \frac{1}{2\pi t}\exp\left(-\frac{d\left(x,\mathcal{C}_\varepsilon\right)}{2t}\right).$$

But $d\left(x,\mathcal{C}_\varepsilon\right) \geq \beta - \varepsilon$ and since we have chosen $\varepsilon \leq \frac{\beta}{2}$ we can conclude that

$$\frac{1}{2\pi t}\exp\left(-\frac{d\left(x,\mathcal{C}_\varepsilon\right)}{2t}\right) \leq \frac{1}{2\pi t}\exp\left(-\frac{\beta}{2t}\right). \qquad (5.16)$$

Since $\int_{-\beta}^{\beta} m_\varepsilon(r)\mathrm{d}r = 1$ for any ε we obtain by combining (5.14), (5.15) and (5.16) the existence of positive constants L and M such that

$$\int_{\mathbb{R}^2} m_\varepsilon\left(D\left(y\right)\right)p\left(t,\cdot,y\right)\mathrm{d}y \leq \frac{1}{2\pi t}\exp\left(-\frac{L}{2t}\right) + \frac{M}{\sqrt{2\pi t}}. \qquad (5.17)$$

Since $\frac{1}{2\pi t}\exp\left(-\frac{L}{2t}\right)$ is bounded on every $[0,T]$ we get by combining (5.11) and (5.17) positive constants K_1 and K_2 such that

$$\int_{\mathbb{R}^2} |\gamma_\varepsilon(y)|\, p(t,x,y)\,\mathrm{d}y \le \|\eta\|_\infty \left(K_1 + \frac{K_2}{\sqrt{2\pi t}}\right)$$

which is just the statement of our lemma. □

The last step to establish a solution of (5.3) is to show the uniform boundedness of u_ε.

Lemma 75 For any finite $T \ge 0$ we have that the unique non-negative bounded solution $u_\varepsilon(t,x)$ of (5.9) is uniformly bounded in (ε,t,x) on $\left[0,\frac{\beta}{2}\right]\times[0,T]\times\mathbb{R}^2$.

Proof. Take u_ε given by (5.9). Since $\alpha \ge 0$ we get

$$\begin{aligned} 0 \le u_\varepsilon(t,x) \le& \int_{\mathbb{R}^2} g(y)\,p(t,\cdot,y)\,\mathrm{d}y + \int_0^t\int_{\mathbb{R}^2} \gamma_\varepsilon(y)\,u_\varepsilon(s,y)\,p(t-s,\cdot,y)\,\mathrm{d}y\,\mathrm{d}s \\ \le& \|g\|_{\infty,\mathbb{R}^2} + \|\eta\|_{\infty,\mathcal{C}} \int_0^t \|u_\varepsilon(s,\cdot)\|_{\infty,\mathbb{R}^2}\left(K_1 + \frac{K_2}{\sqrt{2\pi(t-s)}}\right)\mathrm{d}s \end{aligned}$$

where the last inequality is an immediate consequence of lemma 73. Hence

$$\|u_\varepsilon(t,\cdot)\|_{\infty,\mathbb{R}^2} \le \|g\|_{\infty,\mathbb{R}^2} + \|\eta\|_{\infty,\mathcal{C}} \int_0^t \|u_\varepsilon(s,\cdot)\|_{\infty,\mathbb{R}^2}\left(K_1 + \frac{K_2}{\sqrt{2\pi(t-s)}}\right)\mathrm{d}s$$

and thus we can obtain by Gronwall's inequality, see [13], Appendix B, pp.706, that

$$\|u_\varepsilon(t,\cdot)\|_{\infty,\mathbb{R}^2} \le \|g\|_{\infty,\mathbb{R}^2}\exp\left(\|\eta\|_{\infty,\mathcal{C}}\int_0^t\left(K_1 + \frac{K_2}{\sqrt{2\pi(t-s)}}\right)\mathrm{d}s\right). \tag{5.18}$$

But the right hand side of (5.18) is obviously independent of ε and bounded on every finite interval $[0,T]$ which proves the lemma. □

Now the lemmata 73 and 75 imply with the help of theorem 70 the existence of a solution u of equation (5.3). Since u_ε is non-negative we have u is non-negative.

The uniqueness of the solution It is left to show that (5.3) has a unique solution.

Lemma 76 The integral equation (5.3) has a unique non-negative solution.

Proof. Assume u and v are non-negative solutions of (5.3). Then

$$\begin{aligned} u(t,\cdot) - v(t,\cdot) =& \int_0^t\int_{\mathcal{C}} \eta(y)\,(u(s,y)-v(s,y))\,p(t-s,\cdot,y)\,\mathrm{d}S(y)\,\mathrm{d}s \\ &- \int_0^t\int_{\mathbb{R}^2} \alpha(y)\left(u^2(s,y)-v^2(s,y)\right)p(t-s,\cdot,y)\,\mathrm{d}y\,\mathrm{d}s. \end{aligned}$$

With the help of a binomial formula and since u and v are uniformly bounded on $[0,T]\times\mathbb{R}^2$ which can be shown as in lemma 75 we get the existence of a positive constant M such that

$$\int_0^t\int_{\mathbb{R}^2}\alpha(y)\left|u^2(s,y)-v^2(s,y)\right|p(t-s,\cdot,y)\,\mathrm{d}y\,\mathrm{d}s\leq\int_0^t M\left\|u(s,\cdot)-v(s,\cdot)\right\|_{\infty,\mathbb{R}^2}\mathrm{d}s.$$

Hence with the help of lemma 73 we get that

$$\|u(t,\cdot)-v(t,\cdot)\|_{\infty,\mathbb{R}^2}\leq\int_0^t\|u(s,\cdot)-v(s,\cdot)\|_{\infty,\mathbb{R}^2}\left(M+\|\eta\|_{\infty,\mathcal{C}}\left(K_1+\frac{K_2}{\sqrt{2\pi(t-s)}}\right)\right)\mathrm{d}s.$$

Thus by an easy application of Gronwall's inequality we get that $\|u(t,\cdot)-v(t,\cdot)\|_{\infty,\mathbb{R}^2}=0$ which proves the lemma. □

The lemmata 73, 75 and 76 together just shows proposition 72. □

Construction of the superprocess

The next step is to construct the superprocesses which will be done by a result of [10]. There it is shown that for every $\varepsilon>0$ in our approximating procedure we have a superprocess $X^{(\varepsilon)}$ such that

$$\mathbb{E}_\mu\left(\exp\left(\left\langle X_t^{(\varepsilon)},-g\right\rangle\right)\right)=\exp\left(-\int_{\mathbb{R}^2}u_\varepsilon(t,x)\,\mu(dx)\right)\tag{5.19}$$

for every bounded and continuous g and $\mu\in\mathcal{M}$.

Proposition 77 For any fixed t there is a random measure X_t such that

$$\mathbb{E}_\mu\left(\exp\left(\langle X_t,-g\rangle\right)\right)=\exp\left(-\int_{\mathbb{R}^2}u(t,x)\,\mu(dx)\right).$$

.

Proof. By the uniform convergence of $u_\varepsilon(t,\cdot)$ we obtain that the right hand side of (5.19) converges to

$$\exp\left(-\int_{\mathbb{R}^2}u(t,x)\,\mu(dx)\right)$$

where u solves (5.3). Therefore also the left hand side converges. Due to [8], sect.3.3.4, pp 50-51, it is enough to show that $\int_{\mathbb{R}^2}u(t,x)\,\mu(dx)$ tends to zero from above as g tends uniformly to zero to get the limit of the left hand side to be a Laplace transform of a random measure, which we will call X_t. But by (5.18) and since μ is a finite measure we find a positive constant L such that

$$\begin{aligned}0\leq\int_{\mathbb{R}^2}u(t,x)\,\mu(dx)&\leq\|u(t,\cdot)\|_{\infty,\mathbb{R}^2}\mu(\mathbb{R}^2)\\&\leq L\|g\|_{\infty,\mathbb{R}^2}\to 0\end{aligned}$$

and hence there exists for every fixed t a random measure X_t such that

$$\mathbb{E}_\mu \left(\exp\left(\langle X_t, -g\rangle\right)\right) = \exp\left(-\int_{\mathbb{R}^2} u\left(t,x\right)\mu(dx)\right).$$

□

The last step is to establish the Markovian character of X. Due to [14], Theorem 4.4 it is enough to show that $u(t,x)$ fulfills the semigroup property, i.e. if we write $u\left(t,x\right) = \left(V_t g\right)\left(x\right)$ then $V_{s+t}\left(g\right) = V_s\left(V_t\left(g\right)\right)$. So for finishing the proof of theorem 69 we have to establish that $V_{s+t}\left(g\right) = V_s\left(V_t g\right)$.

To do this look at

$$\begin{aligned} u\left(s+t,\cdot\right) = &\int_{\mathbb{R}^2} g\left(y\right) p\left(s+t,\cdot,y\right) \mathrm{d}y \\ &+ \int_0^{s+t}\int_{\mathcal{C}} \eta\left(y\right) u\left(\omega,y\right) p\left(s+t-\omega,\cdot,y\right) \mathrm{d}S(y)\,\mathrm{d}\omega \\ &- \int_0^{s+t}\int_{\mathbb{R}^2} \alpha\left(y\right) u^2\left(\omega,y\right) p\left(s+t-\omega,\cdot,y\right) \mathrm{d}y\,\mathrm{d}\omega. \end{aligned}$$

Since $p\left(\cdot,x,y\right)$ fulfills Chapman-Kolmogorov we can write this in the following form

$$\begin{aligned} u\left(s+t,\cdot\right) = &\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} g\left(z\right) p\left(t,y,z\right) \mathrm{d}z\, p\left(s,\cdot,y\right) \mathrm{d}y \\ &+ \int_{\mathbb{R}^2}\int_0^{t}\int_{\mathcal{C}} \eta\left(y\right) u\left(\omega,z\right) p\left(t-\omega,y,z\right) \mathrm{d}S(z)\,\mathrm{d}\omega\, p\left(s,\cdot,y\right) \mathrm{d}y \\ &- \int_{\mathbb{R}^2}\int_0^{t}\int_{\mathbb{R}^2} \alpha\left(z\right) u^2\left(\omega,z\right) p\left(t-\omega,y,z\right) \mathrm{d}z\,\mathrm{d}\omega\, p\left(s,\cdot,y\right) \mathrm{d}y \\ &+ \int_t^{s+t}\int_{\mathcal{C}} \eta\left(y\right) u\left(\omega,y\right) p\left(s+t-\omega,\cdot,y\right) \mathrm{d}S(y)\,\mathrm{d}\omega \\ &- \int_t^{s+t}\int_{\mathbb{R}^2} \alpha\left(y\right) u^2\left(\omega,y\right) p\left(s+t-\omega,\cdot,y\right) \mathrm{d}y\,\mathrm{d}\omega \end{aligned}$$

and this is by the notation above nothing else but

$$\begin{aligned} u\left(s+t,\cdot\right) = &\int_{\mathbb{R}^2} \left(V_t g\right)\left(y\right) p\left(s,\cdot,y\right) \mathrm{d}y \\ &+ \int_t^{s+t}\int_{\mathcal{C}} \eta\left(y\right) \left(V_\omega g\right)\left(y\right) p\left(s+t-\omega,\cdot,y\right) \mathrm{d}S(y)\,\mathrm{d}\omega \\ &- \int_t^{s+t}\int_{\mathbb{R}^2} \alpha\left(y\right) \left(V_\omega g\right)^2\left(y\right) p\left(s+t-\omega,\cdot,y\right) \mathrm{d}y\,\mathrm{d}\omega \end{aligned}$$

and therefore with the help of the substitution $\omega = \omega + t$ we regard

$$\begin{aligned}
&(V_{s+t}g)(\cdot) - V_s(V_t g)(\cdot) \\
&= \int_0^s \int_{\mathcal{C}} \eta(y)\left((V_{t+\omega}g)(y) - V_\omega(V_t g)(y)\right) p(s-\omega,\cdot,y)\,\mathrm{d}S(y)\,\mathrm{d}\omega \\
&\quad - \int_0^s \int_{\mathbb{R}^2} \alpha(y)\left((V_{t+\omega}g)^2(y) - (V_\omega(V_t g))^2(y)\right) p(s-\omega,\cdot,y)\,\mathrm{d}y\,\mathrm{d}\omega.
\end{aligned}$$

Now exactly the same calculation as in the uniqueness proof of the solution $u(t,\cdot)$ shows that

$$(V_{s+t}g)(\cdot) = V_s(V_t g)(\cdot)$$

and we have established the Markovian character of X.

Thus for any Hölder-continuous function $\eta \in C^0(\mathcal{C})$ we have constructed a superprocess given by (5.2) and (5.3). □

Remark 78 The integral $\int_{\mathcal{C}} \eta(y)\, u(s,y)\, p(t-s,\cdot,y)\, dS(y)$ shows somehow the kind of interaction related to T_η.

It is no point interaction in the classical sense because there we would not have the integral along $\mathcal{C}$, we would just have $\eta(y)\, u(s,y)\, p(t-s,\cdot,y)$ for some fixed point y. Remember that we have shown that $\int_{\mathcal{C}} \eta(y)\, u(s,y)\, p(t-s,\cdot,y)\, dS(y)$ has a singularity, namely $\frac{1}{\sqrt{t-s}}$, which is integrable on $[0,t]$ whereas $p(t-s,\cdot,y)$ has a non-integrable singularity, namely $\frac{1}{t-s}$. Hence the interaction described here is somehow weaker than a point interaction.

Chapter 6

Approximation by branching processes

It seems to be interesting to find branching processes, i.e. a system of particles, which approximates our superprocesses constructed in the chapter before. This approximation could give a better understanding of these processes, because one should see there the dynamics behind. We will see that the superprocesses can be understand as limit of branching processes which admit a special performance in mass creation. For every approximation step we will get an area around the interaction point in one dimension, around $\mathcal{C}$ in two dimensions, respectively, such that outside this area we have a mass creation with expectation value whereas inside the area we have an additional mass creation with expected value $1 + \alpha$ in the one-dimensional case and $1 + \eta(y)$ in the two dimensional case, respectively. Here α is a non-negative real number and η is a continuous function in the area around $\mathcal{C}$. In the next approximation step we will get the same mass creation but on a smaller area. We will also scale the lifetime of the branching particles in the right manner, i.e. find the correct convergence speed to 0. Then we can show that the approximation with given additional mass creation on an area which tends to the interaction point, $\mathcal{C}$, respectively, and with the right convergence speed of the lifetime parameter will have as limit point just the superprocesses we are interested in. Our idea behind is as follows: The additional mass creation happens only in the interesting area. Thus we have to take the right ratio of the enhancement of the area and the lifetime of the particles. It should be possible for particles created in the area to leave the area but not too many since otherwise the effect of the additional mass creation will drop out. On the other hand we do not want to have too many particles to die in the area since then we get a mass explosion.

Our results here are divided into two parts. At first we want to show how to approximate the one-dimensional case, the so called super-Brownian motion with single point source, which is generated via a Dirac-delta perturbation of the Laplace operator in one dimension. In the second part we want to use this result for finding the approximation for our superprocesses of chapter 5 in the case of $\eta \geq 0$.

Definition 79 (Branching process) The definition of a branching process given here can be found in [12],p.1. A branching process is given by three ingredients

- The spatial motion: During her lifetime, each individual in the population moves around in $\mathbb{R}^d$ (independently of each other) according to a stochastic process in our case this will be a Brownian motion.

- The branching rate, V: Each individual has an exponentially distributed lifetime with parameter V

- The branching mechanism Φ: When she dies an individual leaves behind (at the location where she died) a random number of offspring with probability generating function Φ. Conditional on their lifetime and place of birth offspring evolve independently of each other (in the same way as their parent)

Tightness Later on we will show that a sequence of branching processes is tight in a suitable path space which implies that the sequence has at least one weakly convergent subsequence. For a better understanding of tightness we will give a short introduction here. For a detailed discussion we refer to [12], pp.8.
Define $D([0,\infty),\mathcal{M})$ to be the space of cadlag paths in $\mathcal{M}$. We call a sequence, $\left\{X_\cdot^{(n)}\right\}_{n\geq 1}$, of processes taking values in the space $D([0,\infty),\mathcal{M})$ tight if their distributions are tight. And a family of probability measures M on a metric space S is said to be tight if for each $\varepsilon > 0$ there exists a compact $K \subseteq S$ with such that for all $\mu \in M$, $\mu(K) > 1-\varepsilon$. Since checking tightness of $\left\{X_\cdot^{(n)}\right\}_{n\geq 1}$ in $D([0,\infty),\mathcal{M})$ directly can be very difficult one uses the following equivalence: The tightness of $\left\{X_\cdot^{(n)}\right\}_{n\geq 1}$ in $D([0,\infty),\mathcal{M})$ is equivalent to the tightness of $\left\langle X_\cdot^{(n)},\phi\right\rangle$ as a family of processes in $D([0,\infty),\mathbb{R})$ for every non-negative, bounded and continuous function ϕ.

6.1 The one dimensional case

For the one dimensional case we have to understand first what we have in mind by talking about super-Brownian motion with single point source.

Definition 80 Denote by $\mathcal{M}$ the set of finite measures on $\mathbb{R}$. A super-Brownian motion with single point source at the origin is the continuous $\mathcal{M}$-valued process X given by the log-Laplace equation

$$\mathbb{E}_\mu \left(\exp \langle X_t, -g\rangle\right) = \exp\left(-\int_{\mathbb{R}} u\left(t,x\right)\mu(dx)\right) \tag{6.1}$$

and the integral equation

$$u\left(t,\cdot\right) = \int_{\mathbb{R}} g\left(y\right) p\left(t,\cdot,y\right) \mathrm{d}y + \int_0^t \gamma u\left(s,\cdot\right) p\left(t-s,\cdot,0\right) \mathrm{d}s \tag{6.2}$$

$$-\int_0^t \int_{\mathbb{R}} \alpha\left(y\right) u^2\left(s,y\right) p\left(t-s,\cdot,y\right) \mathrm{d}y\, \mathrm{d}s \tag{6.3}$$

where $p\left(t,x,y\right) = \frac{1}{\sqrt{2\pi t}} \exp -\frac{(x-y)^2}{2t}$ is the usual transition kernel of the 1-dimensional Brownian motion. For more details see [9], Paragraph 1.4, p.42 and the references therein.

Remark 81 All of the results presented in this section also holds in the case of a single point source not in the origin, i.e instead of (6.2) we have

$$u\left(t,\cdot\right) = \int_{\mathbb{R}} g\left(y\right) p\left(t,\cdot,y\right) \mathrm{d}y + \int_0^t \gamma u\left(s,\cdot\right) p\left(t-s,\cdot,\omega\right) \mathrm{d}s$$
$$-\int_0^t \int_{\mathbb{R}} \alpha\left(y\right) u^2\left(s,y\right) p\left(t-s,\cdot,y\right) \mathrm{d}y\, \mathrm{d}s$$

for some $\omega \in \mathbb{R}$. The only reason for restricting ourselves to the special case of the origin is a simplification of notation.

The procedure now is to define the martingale problem related to the super-Brownian motion with single point source and then define branching processes such that the coefficients of their martingale problems converge to the coefficients of the super-Brownian motion with single point source martingale problem. But first of all we want to show the connection to the one dimensional point interactions defined by Albeverio et al. in [2].

Definition 82 Take $\gamma \geq 0$ and define the operator T_γ by

$$D\left(T_\gamma\right) = \{f \in H^{1,2}\left(\mathbb{R}\right) \cap H^{2,2}\left(-\infty,0\right) \oplus H^{2,2}\left(0,\infty\right) : f'\left(0^+\right) - f'\left(0^-\right) = -2\gamma f\left(0\right)\}$$
$$T_\gamma f = \frac{1}{2} f''.$$

The operator T_γ is called a point interaction with strength γ.

The connection of the point interactions T_γ and the super-Brownian motion with single point source can be seen as follows.

Lemma 83 The integral equation (6.2) is an equivalent formulation of

$$\frac{d}{dt} u\left(t,\cdot\right) = T_\gamma u\left(t,\cdot\right) - \alpha\left(\cdot\right) u^2\left(t,\cdot\right). \tag{6.4}$$

Proof. It is mentioned for example in [9], Paragraph 1.4, p.42, that (6.2) is an equivalent formulation to

$$\frac{d}{dt}u(t,\cdot) = \frac{1}{2}u''(t,\cdot) + \gamma u(t,0)\,\delta_0 - \alpha(\cdot)\,u^2(t,\cdot)\,. \tag{6.5}$$

where δ_0 represents the Dirac-delta distribution in 0. In [2], Ch.1.3, p.75, it is shown that $\frac{1}{2}u''(t,\cdot) + \gamma u(t,0)\,\delta_0$ is nothing else but $T_\gamma u(t,\cdot)$ and we are done. □

The interpretation of $T_\gamma u$ as $\frac{1}{2}u''(t,\cdot) + \gamma u(t,0)\,\delta_0$ let us think about an approximation of T_γ by a suitable approximation of δ_0. The rigorous statement is as follows.

Proposition 84 For any $\phi \in D(T_\gamma)$ we find a sequence $\phi_n \in C^1(\mathbb{R})$ such that

$$\phi_n \to \phi \text{ uniformly}, \tag{6.6}$$

$$\frac{1}{2}\phi_n'' = T_\gamma\phi - \frac{n}{2}\gamma\phi(0)\mathbb{1}_{\left(-\frac{1}{n},\frac{1}{n}\right)} \tag{6.7}$$

and it exists a constant K such that for all $n \in \mathbb{N}$

$$\|\phi_n'\|_{\infty,(-1,1)} \le K. \tag{6.8}$$

Proof. Take $\phi \in D(T_\gamma)$. Define $F_\phi = \phi + \gamma\phi(0)\,|x|$. Then F_ϕ is obviously an element of $C^1(\mathbb{R})$ since the jump in the first derivative just cancels. Moreover we have

$$\frac{1}{2}F_\phi'' = \frac{1}{2}\phi'' + \gamma\phi(0)\,\delta_0 = T_\gamma\phi.$$

Take $w_n = \frac{n}{2}\int_{-\frac{1}{n}}^{\frac{1}{n}} |x-y|\,\mathrm{d}y$. Then direct calculations show that

$$\|w_n - |x|\|_\infty \le \frac{1}{2n}, \tag{6.9}$$

$$\|w_n'\|_{\infty,\mathbb{R}} = 1 \tag{6.10}$$

and that

$$w_n'' = n\mathbb{1}_{\left(-\frac{1}{n},\frac{1}{n}\right)}. \tag{6.11}$$

Define now $\phi_n = F_\phi - \gamma\phi(0)w_n$. Then $\phi_n \in C^1(\mathbb{R})$ and with the help of (6.11) we obtain that

$$\frac{1}{2}\phi_n'' = T_\gamma\phi - \frac{n}{2}\gamma\phi(0)\mathbb{1}_{\left(-\frac{1}{n},\frac{1}{n}\right)}.$$

Since $\phi = F_\phi - \gamma\phi(0)\,|x|$ we have

$$\|\phi_n - \phi\|_\infty = |\gamma\phi(0)|\,\|w_n - |x|\|_\infty \le \frac{1}{2n}\,|\gamma\phi(0)|$$

by (6.9). Hence ϕ_n converges uniformly to ϕ and and (6.8) is an immediate consequence of (6.10). □

Later on we will use this approximation to determine the right parameters of the approximating branching processes.

Martingale problem Our next goal is to give two equivalent descriptions of the martingale problem related to the super-Brownian motion with single point source.

Theorem 85 The martingale problem for the super-Brownian motion with single point source is given in the following way: A distribution $\mathbb{P}_\mu$ solves the martingale problem if

$$\mathbb{P}_\mu (X_0 = \mu) = 1$$

and if

$$\exp(\langle X_t, -\phi\rangle) - \exp(\langle X_0, -\phi\rangle) - \int_0^t \left\langle X_s, -T_\gamma \phi + \alpha\phi^2 \right\rangle \exp(\langle X_s, -\phi\rangle)\, \mathrm{d}s$$

is a $\mathbb{P}_\mu$ martingale for every non-negative and bounded $\phi \in D(T_\gamma)$ such that the integrand is bounded. An equivalent formulation is as follows: A distribution $\mathbb{P}$ satisfies the martingale problem if for all $\phi \in D(T_\gamma)$

$$[M_t](\phi) = \langle X_t, \phi\rangle - \langle X_0, \phi\rangle - \int_0^t \langle X_s, T_\gamma \phi\rangle\, \mathrm{d}s$$

is a $\mathbb{P}$-martingale with quadratic variation process

$$[M.\phi]_t = \int_0^t \left\langle X_s, \alpha\phi^2 \right\rangle \mathrm{d}s.$$

The solution of the martingale problem is unique.

Proof. The statements follow immediately by the general results of [12], pp.15. □

Branching processes Now we want to give a sequence of branching processes $X_t^{(n)}$ which should converge to X_t.

Theorem 86 For $n \in \mathbb{N}$ define $X_t^{(n)}$ to be a branching process with

- n individuals each of mass $\frac{1}{n}$ moving around in $\mathbb{R}$ (independently of each other) according to a Brownian motion,
- the life-time parameter is given by $V_n = \frac{n}{2}$,
- the sample mechanism fulfills

$$\Phi_x(1) = 1, \quad \Phi_x'(1) = 1 + \gamma \mathbb{1}_{\left(-\frac{1}{n}, \frac{1}{n}\right)}(x), \quad \Phi_x'' = 4\alpha(x).$$

The sequence $X_t^{(n)}$ is tight in $D([0,\infty), \mathcal{M})$ and converges to X_t.

Remark 87 The last point in the definition of the sample mechanism means nothing else but different performance of the mass creation depending on the place of death of an individual. If she dies outside of $\left(-\frac{1}{n},\frac{1}{n}\right)$ we expect 1 new individual ($\Phi'_x(1)=1$) with variance $4\alpha(x)$, if she dies inside of $\left(-\frac{1}{n},\frac{1}{n}\right)$ we expect $1+\gamma$ new individuals again with variance 4α. This means nothing else but additional mass creation in $\left(-\frac{1}{n},\frac{1}{n}\right)$.

Before proving the theorem, we need a lemma.

Lemma 88 For any stopping time $\tau\le t$ we have

$$\sup_{x\in\mathbb{R}}\mathbb{E}_{\delta_x}\left(\left\langle X_\tau^{(n)}.1\right\rangle\right)\le\exp\left(\gamma\sqrt{\frac{2t}{\pi}}\right)\tag{6.12}$$

and

$$\sup_{x\in\mathbb{R}}\mathbb{E}_{\delta_x}\left(\frac{\gamma n}{2}\int_0^\tau\left\langle X_s^{(n)},\mathbb{1}_{\left(-\frac{1}{n},\frac{1}{n}\right)}\right\rangle\mathrm{d}s\right)\le\exp\left(\gamma\sqrt{\frac{2t}{\pi}}\right)-1.\tag{6.13}$$

Proof. Since

$$\mathbb{E}_{\delta_x}\left(\left\langle X_\tau^{(n)},1\right\rangle\right)\le\mathbb{E}_{\delta_x}\left(\left\langle X_t^{(n)},1\right\rangle\right)$$

and

$$\mathbb{E}_{\delta_x}\left(\frac{\gamma n}{2}\int_0^\tau\left\langle X_s^{(n)},\mathbb{1}_{\left(-\frac{1}{n},\frac{1}{n}\right)}\right\rangle\mathrm{d}s\right)\le\mathbb{E}_{\delta_x}\left(\frac{\gamma n}{2}\int_0^t\left\langle X_s^{(n)},\mathbb{1}_{\left(-\frac{1}{n},\frac{1}{n}\right)}\right\rangle\mathrm{d}s\right)$$

it is enough to show (6.12) and (6.13) for $\tau=t$. Furthermore it is shown for example in [12], p.13, that

$$\left\langle X_t^{(n)},1\right\rangle-\left\langle X_0^{(n)},1\right\rangle-\left(\frac{\gamma n}{2}\int_0^t\left\langle X_s^{(n)},\mathbb{1}_{\left(-\frac{1}{n},\frac{1}{n}\right)}\right\rangle\mathrm{d}s\right)$$

is a martingale and thus

$$\mathbb{E}_{\delta_x}\left(\frac{\gamma n}{2}\int_0^t\left\langle X_s^{(n)},\mathbb{1}_{\left(-\frac{1}{n},\frac{1}{n}\right)}\right\rangle\mathrm{d}s\right)=\mathbb{E}_{\delta_x}\left(\left\langle X_t^{(n)},1\right\rangle\right)-1.$$

Therefore for $\tau=t$ (6.13) follows by (6.12). It is mentioned in [8], section I.3.1 that $v_n(t,x)=\mathbb{E}_{\delta_x}\left(\left\langle X_t^{(n)},1\right\rangle\right)$ solves

$$\frac{\partial}{\partial t}v_n(t,x)=\frac{1}{2}v_n''(t,x)+\frac{\gamma n}{2}\mathbb{1}_{-\frac{1}{n},\frac{1}{n}}v_n(t,x)\tag{6.14}$$

with boundary condition $v_n(0,x)=1$. But equality (6.14) with the boundary condition $v_n(0,x)=1$ is obviously equivalent to

$$v_n(t,x)=1+\int_0^t\int_{\mathbb{R}}\frac{\gamma n}{2}\mathbb{1}_{\left(-\frac{1}{n},\frac{1}{n}\right)}v_n(s,y)p(t-s,x,y)\mathrm{d}y\,\mathrm{d}s$$

where $p(t,x,y) = \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{|x-y|^2}{2t}\right)$ is the usual transition kernel of the Brownian motion. Since $\exp\left(-\frac{|x-y|^2}{2t}\right) \leq 1$ for all x, $y \in \mathbb{R}$ we have

$$\|v_n(t,\cdot)\|_\infty \leq 1 + \int_0^t \|v_n(s,\cdot)\|_\infty \, \gamma \frac{1}{\sqrt{2\pi(t-s)}} \mathrm{d}s.$$

But it is also mentioned in [8], section I.3.2 that $\sup_{0\leq s\leq t} \|v_n(s,\cdot)\|_\infty < \infty$ and thus we can obtain by Gronwall's inequality that

$$\|v_n(t,\cdot)\|_\infty \leq \exp\left(\gamma\sqrt{\frac{2t}{\pi}}\right)$$

which proves the lemma. □

Now we are able to proof theorem 86.

Proof. We will denote by μ_n the starting measure and we assume that the sequence μ_n is tight in the space $\mathcal{M}(\mathbb{R})$, i.e. especially that the sequence $\langle \mu_n, 1\rangle$ is bounded. Furthermore define $\beta_n = \frac{\gamma n}{2} \mathbb{1}_{\left(-\frac{1}{n},\frac{1}{n}\right)}$.
For proving tightness of the sequence $X_t^{(n)}$ we have to check the following properties (for details see [10], Theorem A.2)

(a) For $T > 0$ fixed and $\varepsilon > 0$ given it exists $K > 0$ such that

$$\mathbb{P}_{\mu_n}\left(\sup_{0\leq t\leq T} \left\langle X_t^{(n)}, 1\right\rangle \leq K\right) \geq 1-\varepsilon$$

(b) For $\psi \in C^2(\mathbb{R})$ such that ψ and ψ'' are bounded one can define a semimartingale

$$Y_{t,\psi}^{(n)} = \left\langle X_t^{(n)}, \psi\right\rangle$$

Denote by $V_{t,\psi}^{(n)}$ the predictable finite variation process and by $M_{t,\psi}^{(n)}$ the quadratic variation related to $Y_{t,\psi}^{(n)}$, respectively. Then we have to check that for each $n \in \mathbb{N}$, $\varepsilon > 0$ and $K \geq 0$ it exists a $\delta = \delta(n,\varepsilon,K,\psi) > 0$ such that for each positive stopping time τ_n bounded by n

$$\limsup_{n\to\infty} \sup_{0\leq\theta\leq\delta} \mathbb{P}_{\mu_n}\left(\left|V_{\tau_n+\theta,\psi}^{(n)} - V_{\tau_n,\psi}^{(n)}\right| \geq K\right) \leq \varepsilon \tag{6.15}$$

and

$$\limsup_{n\to\infty} \sup_{0\leq\theta\leq\delta} \mathbb{P}_{\mu_n}\left(\left|M_{\tau_n+\theta,\psi}^{(n)} - M_{\tau_n,\psi}^{(n)}\right| \geq K\right) \leq \varepsilon. \tag{6.16}$$

For proving (a) choose K_1 such that

$$\frac{1}{K_1}\left(2\exp\left(\gamma\sqrt{\frac{2T}{\pi}}\right) - 1\right)\langle \mu_n, 1\rangle < \varepsilon/2$$

and K_2 with

$$\frac{1}{K_2}\left(\exp\left(\gamma\sqrt{\frac{2T}{\pi}}\right)-1\right)\langle\mu_n,1\rangle<\varepsilon/2.$$

Define $K=K_1+K_2$. Then we can obtain that

$$\begin{aligned}
&\mathbb{P}_{\mu_n}\left(\sup_{0\le t\le T}\left\langle X_t^{(n)},1\right\rangle>K\right)\\
=\;&\mathbb{P}_{\mu_n}\left(\sup_{0\le t\le T}\left(\left\langle X_t^{(n)},1\right\rangle-\int_0^t\left\langle X_s^{(n)},\beta_n\right\rangle\mathrm{d}s+\int_0^t\left\langle X_s^{(n)},\beta_n\right\rangle\mathrm{d}s\right)>K\right)\\
\le\;&\mathbb{P}_{\mu_n}\left(\sup_{0\le t\le T}\left(\left\langle X_t^{(n)},1\right\rangle-\int_0^t\left\langle X_s^{(n)},\beta_n\right\rangle\mathrm{d}s\right)+\sup_{0\le t\le T}\left(\int_0^t\left\langle X_s^{(n)},\beta_n\right\rangle\mathrm{d}s\right)>K\right)\\
=\;&\mathbb{P}_{\mu_n}\left(\sup_{0\le t\le T}\left(\left\langle X_t^{(n)},1\right\rangle-\int_0^t\left\langle X_s^{(n)},\beta_n\right\rangle\mathrm{d}s\right)+\int_0^T\left\langle X_s^{(n)},\beta_n\right\rangle\mathrm{d}s>K_1+K_2\right)
\end{aligned}$$

since $\beta_n\ge 0$. Furthermore we have

$$\begin{aligned}
&\mathbb{P}_{\mu_n}\left(\sup_{0\le t\le T}\left(\left\langle X_t^{(n)},1\right\rangle-\int_0^t\left\langle X_s^{(n)},\beta_n\right\rangle\mathrm{d}s\right)+\int_0^T\left\langle X_s^{(n)},\beta_n\right\rangle\mathrm{d}s>K_1+K_2\right)\\
=\;&\mathbb{P}_{\mu_n}\left(\sup_{0\le t\le T}\left(\left\langle X_t^{(n)},1\right\rangle-\int_0^t\left\langle X_s^{(n)},\beta_n\right\rangle\mathrm{d}s\right)+\int_0^T\left\langle X_s^{(n)},\beta_n\right\rangle\mathrm{d}s>K_1+K_2\cap\right.\\
&\qquad\left.\int_0^T\left\langle X_s^{(n)},\beta_n\right\rangle\mathrm{d}s\ge K_2\right)\\
&+\mathbb{P}_{\mu_n}\left(\sup_{0\le t\le T}\left(\left\langle X_t^{(n)},1\right\rangle-\int_0^t\left\langle X_s^{(n)},\beta_n\right\rangle\mathrm{d}s\right)+\int_0^T\left\langle X_s^{(n)},\beta_n\right\rangle\mathrm{d}s>K_1+K_2\cap\right.\\
&\qquad\left.\int_0^T\left\langle X_s^{(n)},\beta_n\right\rangle\mathrm{d}s<K_2\right)\\
=\;&\mathbb{P}_{\mu_n}\left(\sup_{0\le t\le T}\left(\left\langle X_t^{(n)},1\right\rangle-\int_0^t\left\langle X_s^{(n)},\beta_n\right\rangle\mathrm{d}s\right)+\int_0^T\left\langle X_s^{(n)},\beta_n\right\rangle\mathrm{d}s>K_1+K_2\cap\right.\\
&\qquad\left.\int_0^T\left\langle X_s^{(n)},\beta_n\right\rangle\mathrm{d}s\ge K_2\right)\\
&+\mathbb{P}_{\mu_n}\left(\sup_{0\le t\le T}\left(\left\langle X_t^{(n)},1\right\rangle-\int_0^t\left\langle X_s^{(n)},\beta_n\right\rangle\mathrm{d}s\right)>K_1+K_2-\int_0^T\left\langle X_s^{(n)},\beta_n\right\rangle\mathrm{d}s\cap\right.\\
&\qquad\left.\int_0^T\left\langle X_s^{(n)},\beta_n\right\rangle\mathrm{d}s<K_2\right)\\
\le\;&\mathbb{P}_{\mu_n}\left(\sup_{0\le t\le T}\left(\left\langle X_t^{(n)},1\right\rangle-\int_0^t\left\langle X_s^{(n)},\beta_n\right\rangle\mathrm{d}s\right)\ge K_1\right)
\end{aligned}$$

$$+\mathbb{P}_{\mu_n}\left(\int_0^T \left\langle X_s^{(n)}, \beta_n \right\rangle \mathrm{d}s \geq K_2\right)$$

$$\leq \frac{1}{K_1}\mathbb{E}_{\mu_n}\left(\left\langle 1, X_T^{(n)}\right\rangle - \int_0^T \left\langle X_s^{(n)}, \beta_n \right\rangle \mathrm{d}s\right) + \frac{1}{K_2}\mathbb{E}_{\mu_n}\left(\int_0^T \left\langle X_s^{(n)}, \beta_n \right\rangle \mathrm{d}s\right).$$

The last inequality is an immediate consequence of Markov's inequality and the martingale property of $\left\langle X_t^{(n)}, 1\right\rangle - \left\langle X_0^{(n)}, 1\right\rangle - \left(\frac{\gamma n}{2}\int_0^t \left\langle X_s^{(n)}, \mathbb{1}_{\left(-\frac{1}{n},\frac{1}{n}\right)}\right\rangle \mathrm{d}s\right)$. By lemma 88 we can conclude that

$$\frac{1}{K_1}\mathbb{E}_{\mu_n}\left(\left\langle X_T^{(n)}, 1\right\rangle - \int_0^T \left\langle X_s^{(n)}, \beta_n \right\rangle \mathrm{d}s\right) + \frac{1}{K_2}\mathbb{E}_{\mu_n}\left(\int_0^T \left\langle X_s^{(n)}, \beta_n \right\rangle \mathrm{d}s\right)$$

$$\leq \frac{1}{K_1}\left(2\exp\left(\gamma\sqrt{\frac{2T}{\pi}}\right) - 1\right)\langle \mu_n, 1\rangle + \frac{1}{K_2}\left(\exp\left(\gamma\sqrt{\frac{2T}{\pi}}\right) - 1\right)\langle \mu_n, 1\rangle.$$

Thus we obtain by our choice of K_1 and K_2 that

$$\mathbb{P}_{\mu_n}\left(\sup_{0\leq t\leq T}\left\langle X_t^{(n)}, 1\right\rangle \geq K\right) < \varepsilon$$

and (a) is proven.

For checking part (b) we follow the calculations of [12], p.6 and get

$$\begin{aligned} V_{t,\psi}^{(n)} &= \left\langle X_t^{(n)}, \frac{1}{2}\psi'' + V_n\left(\Phi_x'(1) - 1\right)\right\rangle \\ &= \left\langle X_t^{(n)}, \frac{1}{2}\psi'' + \frac{n}{2}\gamma\mathbb{1}_{\left(-\frac{1}{n},\frac{1}{n}\right)}\psi\right\rangle \end{aligned}$$

and

$$M_{t,\psi}^{(n)} = \left\langle X_t^{(n)}, \left(\psi^2\right)'' - 2\psi\psi'' + n\gamma\mathbb{1}_{\left(-\frac{1}{n},\frac{1}{n}\right)}\psi + n\left(\frac{\Phi\left(e^{-\phi}\right) - e^{-\phi}}{e^{-\phi}}\right)\right\rangle.$$

To prove (6.15) take a look at

$$\mathbb{E}_{\mu_n}\left(\int_{\tau_n}^{\tau_n+\delta}\left|\left\langle X_s^{(n)}, \frac{1}{2}\psi'' + \beta_n\psi\right\rangle\right| \mathrm{d}s\right) = \mathbb{E}_{\mu_n}\mathbb{E}_{X_{\tau_n}^{(n)}}\left(\int_0^\delta\left|\left\langle X_s^{(n)}, \frac{1}{2}\psi'' + \beta_n\psi\right\rangle\right| \mathrm{d}s\right)$$

$$\leq \frac{1}{2}\left\|\psi''\right\|_\infty \mathbb{E}_{\mu_n}\mathbb{E}_{X_{\tau_n}^{(n)}}\left(\int_0^\delta\left|\left\langle X_s^{(n)}, 1\right\rangle\right| \mathrm{d}s\right) + \|\psi\|_\infty \mathbb{E}_{\mu_n}\mathbb{E}_{X_{\tau_n}^{(n)}}\left(\int_0^\delta\left|\left\langle X_s^{(n)}, \beta_n\right\rangle\right| \mathrm{d}s\right).$$

Again with the help of lemma 88 we obtain that

$$\frac{1}{2}\left\|\psi''\right\|_\infty \mathbb{E}_{\mu_n}\mathbb{E}_{X_{\tau_n}^{(n)}}\left(\int_0^\delta\left|\left\langle X_s^{(n)}, 1\right\rangle\right| \mathrm{d}s\right) + \|\psi\|_\infty \mathbb{E}_{\mu_n}\mathbb{E}_{X_{\tau_n}^{(n)}}\left(\int_0^\delta\left|\left\langle X_s^{(n)}, \beta_n\right\rangle\right| \mathrm{d}s\right)$$

$$\leq \frac{1}{2}\left\|\psi''\right\|_\infty \mathbb{E}_{\mu_n}\left(\left\langle X_{\tau_n}^{(n)}, 1\right\rangle\right)\delta\exp\left(\sqrt{\frac{2\delta}{\pi}}\right)$$

$$+\|\psi\|_\infty \mathbb{E}_{\mu_n}\left(\left\langle X_{\tau_n}^{(n)}, 1\right\rangle\right)\left(\exp\left(\sqrt{\frac{2\delta}{\pi}}\right) - 1\right)$$

$$\leq \langle \mu_n, 1\rangle\exp\left(\sqrt{\frac{2n}{\pi}}\right)\left(\frac{1}{2}\left\|\psi''\right\|_\infty\delta\exp\left(\sqrt{\frac{2\delta}{\pi}}\right) + \|\psi\|_\infty\exp\left(\sqrt{\frac{2\delta}{\pi}} - 1\right)\right).$$

Thus we have

$$\mathbb{P}_{\mu_n}\left(\int\limits_{\tau_n}^{\tau_n+\delta}\left|\left\langle X_s^{(n)},\frac{1}{2}\psi''+\beta_n\psi\right\rangle\right|\mathrm{d}s\geq K\right)$$
$$\leq\ \frac{1}{K}\left\langle\mu_n,1\right\rangle\exp\left(\sqrt{\frac{2n}{\pi}}\right)\left(\frac{1}{2}\left\|\psi''\right\|_\infty\delta\exp\left(\sqrt{\frac{2\delta}{\pi}}\right)+\left\|\psi\right\|_\infty\exp\left(\sqrt{\frac{2\delta}{\pi}}-1\right)\right)$$

which proves (6.15). The calculation for (6.16) is essentially the same. Thus tightness is shown.

The last step in our proof is to find for every non-negative and bounded $\psi\in D\left(T_\gamma\right)$ with $T_\gamma\psi$ bounded a sequence of test functions f_n such that the martingale description of the branching process applied to f_n converge to the martingale description of the super-Brownian motion with single point source applied to ψ. In the following every convergence which is not specified is understood in the sense of weak convergence.

Take a non-negative and bounded $\psi\in D\left(T_\gamma\right)$ and denote by ψ_n the sequence constructed in proposition 84. Define $f_n=1-\frac{\psi_n}{n}$. Then due to the results of proposition 84 we can conclude that f_n is bounded, $f_n\leq 1$, $f_n\in C^1\left(\mathbb{R}\right)$ and f_n'' is bounded. Thus since $nX_t^{(n)}$ is just a process which counts the number of particles moving around we get due to [12],p.13 that

$$\exp\left(\left\langle nX_t^{(n)},\ln\left(f_n\right)\right\rangle\right)-\exp\left(\left\langle nX_0^{(n)},\ln\left(f_n\right)\right\rangle\right)$$
$$-\int_0^t\left\langle nX_s^{(n)},\frac{\frac{1}{2}f_n''+V_n\left(\Phi_{n,x}\left(f_n\right)-f_n\right)}{f_n}\right\rangle\exp\left(\left\langle nX_s^{(n)},\ln\left(f_n\right)\right\rangle\right)\mathrm{d}s$$

is a martingale. By a simple Taylor expansion we obtain that

$$\Phi_{n,x}\left(f_n\right)=\Phi_{n,x}\left(1-\frac{\psi_n}{n}\right)=1-\frac{\psi}{n}\Phi_{n,x}'(1)+\frac{\psi_n^2}{2n^2}\Phi_{n,x}''(1)+o\left(\frac{1}{n^2}\right).$$

By the definition of $\Phi_{n,x}$ we get

$$\Phi_{n,x}\left(f_n\right)-f_n=-\frac{\psi_n}{n}\gamma\mathbb{1}_{\left(-\frac{1}{n},\frac{1}{n}\right)}\left(x\right)+2\frac{\psi_n^2}{n^2}\alpha\left(x\right).$$

By plugging this expansion into the martingale description we get that

$$\exp\left(\left\langle X_t^{(n)},n\ln\left(1-\frac{\psi_n}{n}\right)\right\rangle\right)-\exp\left(\left\langle X_0^{(n)},n\ln\left(1-\frac{\psi_n}{n}\right)\right\rangle\right)+o\left(\frac{1}{n^2}\right)$$
$$-\int_0^t\left\langle X_s^{(n)},\frac{-\frac{1}{2}\psi_n''-\frac{n^2}{2}\left(\frac{\psi_n}{n}\gamma\mathbb{1}_{\left(-\frac{1}{n},\frac{1}{n}\right)}\right)+\psi_n^2\alpha}{1-\frac{\psi}{n}}\right\rangle\exp\left(\left\langle X_s^{(n)},n\ln\left(1-\frac{\psi_n}{n}\right)\right\rangle\right)\mathrm{d}s$$
$$=\exp\left(\left\langle X_t^{(n)},\ln\left(1-\frac{\psi_n}{n}\right)^n\right\rangle\right)-\exp\left(\left\langle X_0^{(n)},\ln\left(1-\frac{\psi_n}{n}\right)^n\right\rangle\right)+o\left(\frac{1}{n^2}\right)$$
$$-\int_0^t\left\langle X_s^{(n)},\frac{-\frac{1}{2}\psi_n''-\frac{n}{2}\psi_n\gamma\mathbb{1}_{\left(-\frac{1}{n},\frac{1}{n}\right)}+\psi_n^2\alpha}{1-\frac{\psi_n}{n}}\right\rangle\exp\left(\left\langle X_s^{(n)},\ln\left(1-\frac{\psi_n}{n}\right)^n\right\rangle\right)\mathrm{d}s.$$

In proposition 84 it is shown that ψ_n converges uniformly to ψ. Thus we get that

$$\left(1-\frac{\psi_n}{n}\right)^n = \left(1-\frac{\psi}{n}+\frac{\psi-\psi_n}{n}\right)^n \to e^{-\psi}$$

uniformly in $\mathbb{R}$. Hence we can conclude with the tightness of $X_t^{(n)}$ in the space of finite measures that we have

$$\left|\left\langle X_t^{(n)}, \ln\left(1-\frac{\psi_n}{n}\right)^n + \psi\right\rangle\right| \leq \left\|\ln\left(1-\frac{\psi_n}{n}\right)^n + \psi\right\|_\infty \left\langle X_t^{(n)}, 1\right\rangle \to 0$$

for $n\to\infty$ and for every $t\geq 0$. Therefore we obtain that for $n\to\infty$ we have

$$\left\langle X_t^{(n)}, \ln\left(1-\frac{\psi_n}{n}\right)^n\right\rangle = \left\langle X_t^{(n)}, \ln\left(1-\frac{\psi_n}{n}\right)^n + \psi - \psi\right\rangle \to \langle X_t, -\psi\rangle .$$

This implies that

$$\begin{aligned}&\exp\left(\left\langle X_t^{(n)}, \ln\left(1-\frac{\psi_n}{n}\right)^n\right\rangle\right) - \exp\left(\left\langle X_0^{(n)}, \ln\left(1-\frac{\psi_n}{n}\right)^n\right\rangle\right)\\ \to &\exp\left(\langle X_t, -\psi\rangle\right) - \exp\left(\langle X_0, -\psi\rangle\right).\end{aligned}$$

With essentially the same calculations we can show that

$$\left\langle X_s^{(n)}, \frac{\psi_n^2\alpha}{1-\frac{\psi_n}{n}}\right\rangle \exp\left(\left\langle X_s^{(n)}, \ln\left(1-\frac{\psi_n}{n}\right)^n\right\rangle\right) \to \left\langle X_s, \alpha\psi^2\right\rangle \exp\left(\langle X_s, -\psi\rangle\right)$$

and thus by the tightness of $X_t^{(n)}$ in $D([0,\infty),\mathcal{M})$ we get

$$\begin{aligned}&\int_0^t \left\langle X_s^{(n)}, \frac{\psi_n^2\alpha}{1-\frac{\psi_n}{n}}\right\rangle \exp\left(\left\langle X_s^{(n)}, \ln\left(1-\frac{\psi_n}{n}\right)^n\right\rangle\right) \mathrm{d}s\\ \to &\int_0^t \left\langle X_s, \alpha\psi^2\right\rangle \exp\left(\langle X_s, -\psi\rangle\right) \mathrm{d}s.\end{aligned}$$

In proposition 84 it is also shown that $\frac{1}{2}\psi_n'' = T_\gamma\psi - \frac{n}{2}\gamma\psi(0)\mathbb{1}_{\left(-\frac{1}{n},\frac{1}{n}\right)}$. Hence we obtain that

$$-\frac{1}{2}\psi_n'' - \frac{n}{2}\psi\mathbb{1}_{\left(-\frac{1}{n},\frac{1}{n}\right)} = -T_\gamma\psi + \frac{n}{2}\gamma\mathbb{1}_{\left(-\frac{1}{n},\frac{1}{n}\right)}\left(\psi_n(x)-\psi(0)\right) \tag{6.17}$$

and this implies that

$$\begin{aligned}&\int_0^t \left\langle X_s^{(n)}, \frac{-\frac{1}{2}\psi_n'' - \frac{n}{2}\psi_n\gamma\mathbb{1}_{\left(-\frac{1}{n},\frac{1}{n}\right)}}{1-\frac{\psi_n}{n}}\right\rangle \exp\left(\left\langle X_s^{(n)}, \ln\left(1-\frac{\psi_n}{n}\right)^n\right\rangle\right) \mathrm{d}s\\ =&\underbrace{\int_0^t \left\langle X_s^{(n)}, \frac{-T_\gamma\psi}{1-\frac{\psi_n}{n}}\right\rangle \exp\left(\left\langle X_s^{(n)}, \ln\left(1-\frac{\psi_n}{n}\right)^n\right\rangle\right) \mathrm{d}s}_{\text{(I)}}\\ &+\underbrace{\int_0^t \frac{n\gamma}{2}\left\langle X_s^{(n)}, \frac{\left(\psi_n-\psi(0)\right)\mathbb{1}_{\left(-\frac{1}{n},\frac{1}{n}\right)}}{1-\frac{\psi_n}{n}}\right\rangle \exp\left(\left\langle X_s^{(n)}, \ln\left(1-\frac{\psi_n}{n}\right)^n\right\rangle\right) \mathrm{d}s}_{\text{(II)}}\end{aligned}$$

For (I) we get with the same arguments as above that it converges to

$$\int_0^t \langle X_s, -T_\gamma \psi \rangle \exp\left(\langle X_s, -\phi \rangle\right) \mathrm{d}s.$$

For (II) taking into account that $\psi_n \in C^1(\mathbb{R})$ which implies for every $x \in \left(-\frac{1}{n}, \frac{1}{n}\right)$ the existence of a $\zeta \in \left(-\frac{1}{n}, \frac{1}{n}\right)$ such that $\psi_n(x) = \psi_n(0) + \psi_n'(\zeta)x$. Therefore $|\psi_n(x) - \psi(0)| \leq |\psi_n(0) - \psi(0)| + |\psi_n'(\zeta)x|$. By proposition 84, especially with the help of (6.9) and (6.10), we find a non-negative constant M such that $|\psi_n(0) - \psi(0)| + |\psi_n'(\zeta)x| \leq \frac{M}{n}$. Therefore we have that

$$\left|\frac{n}{2}\gamma \mathbb{1}_{\left(-\frac{1}{n},\frac{1}{n}\right)}\left(\psi_n(x) - \psi(0)\right)\right| \leq \frac{M}{2}\gamma \mathbb{1}_{\left(-\frac{1}{n},\frac{1}{n}\right)}.$$

Using this inequality and the fact that $\psi_n \geq 0$ we obtain the existence of a constant C such that

$$\left|\int_0^t \frac{n\gamma}{2}\left\langle X_s^{(n)}, \frac{\left(\psi_n - \psi(0)\right)\mathbb{1}_{\left(-\frac{1}{n},\frac{1}{n}\right)}}{1 - \frac{\psi_n}{n}}\right\rangle \exp\left(\left\langle X_s^{(n)}, \ln\left(1 - \frac{\psi_n}{n}\right)^n\right\rangle\right) \mathrm{d}s\right|$$
$$\leq \frac{C\gamma}{2}\int_0^t \left\langle X_s^{(n)}, \mathbb{1}_{\left(-\frac{1}{n},\frac{1}{n}\right)}\right\rangle \mathrm{d}s$$

Thus we can obtain with lemma 88 and Markov's inequality that for every $K > 0$

$$\mathbb{P}_{\mu_n}\left(\left|\int_0^t \frac{n\gamma}{2}\left\langle X_s^{(n)}, \frac{\left(\psi_n - \psi(0)\right)\mathbb{1}_{\left(-\frac{1}{n},\frac{1}{n}\right)}}{1 - \frac{\psi_n}{n}}\right\rangle \exp\left(\left\langle X_s^{(n)}, \ln\left(1 - \frac{\psi_n}{n}\right)^n\right\rangle\right) \mathrm{d}s\right| \geq K\right)$$
$$\leq \mathbb{P}_{\mu_n}\left(\frac{C\gamma}{2}\int_0^t \left\langle X_s^{(n)}, \mathbb{1}_{\left(-\frac{1}{n},\frac{1}{n}\right)}\right\rangle \mathrm{d}s \geq k\right) \leq \frac{C}{K}\frac{1}{n}\langle \mu_n, 1\rangle \exp\left(\gamma\sqrt{\frac{2t}{\pi}}\right) \to 0$$

and hence

$$\int_0^t \frac{n\gamma}{2}\left\langle X_s^{(n)}, \frac{\left(\psi_n - \psi(0)\right)\mathbb{1}_{\left(-\frac{1}{n},\frac{1}{n}\right)}}{1 - \frac{\psi_n}{n}}\right\rangle \exp\left(\left\langle X_s^{(n)}, \ln\left(1 - \frac{\psi_n}{n}\right)^n\right\rangle\right) \mathrm{d}s \to 0.$$

Put all the results together we have shown that

$$\exp\left(\left\langle X_t^{(n)}, \ln\left(1 - \frac{\psi_n}{n}\right)^n\right\rangle\right) - \exp\left(\left\langle X_0^{(n)}, \ln\left(1 - \frac{\psi_n}{n}\right)^n\right\rangle\right)$$
$$- \int_0^t \left\langle X_s^{(n)}, \frac{-\frac{1}{2}\psi_n'' - \frac{n}{2}\psi_n \gamma \mathbb{1}_{\left(-\frac{1}{n},\frac{1}{n}\right)} + \psi_n^2 \alpha}{1 - \frac{\psi_n}{n}}\right\rangle \exp\left(\left\langle X_s^{(n)}, \ln\left(1 - \frac{\psi_n}{n}\right)^n\right\rangle\right) \mathrm{d}s$$
$$\to \exp\left(\langle X_t, -\psi\rangle\right) - \exp\left(\langle X_0, -\psi\rangle\right) - \int_0^t \left\langle X_s, -T_\gamma\psi + \alpha\phi^2\right\rangle \exp\left(\langle X_s, -\psi\rangle\right) \mathrm{d}s$$

which is nothing else but the martingale structure of the super-Brownian with single point source given in theorem 85. Hence the sequence of branching processes $X_t^{(n)}$ converges to the super-Brownian motion with single point source. □

Remark 89 The interesting point in the construction above is the choice of V_n as $\frac{n}{2}$. One can easily see that if V_n is in order less than n, $X_t^{(n)}$ just converges to a super-Brownian motion without a single point source. If we choose V_n in a higher order we get in trouble with the tightness since $\frac{1}{2}\psi'' + V_n\gamma\mathbb{1}_{\left(-\frac{1}{n},\frac{1}{n}\right)}$ would not tend to T_γ. The interpretation of this fact is maybe the following:

Think about the additional mass creation. If an individual starts her life far away from $\left(-\frac{1}{n},\frac{1}{n}\right)$, then the probability that this particle will die in $\left(-\frac{1}{n},\frac{1}{n}\right)$ is very small. Hence most of the effect of additional mass creation will be generated by the individuals start living closed to or in $\left(-\frac{1}{n},\frac{1}{n}\right)$. But the probability for an individual starting in $\left(-\frac{1}{n},\frac{1}{n}\right)$ to die in $\left(-\frac{1}{n},\frac{1}{n}\right)$ is somehow monotone in V_n. Larger V_n gives a higher probability since the life-time is shorter.

Hence if V_n is "'too big"' we have too much mass creation, i.e. too many individuals born in $\left(-\frac{1}{n},\frac{1}{n}\right)$ will die there. The picture behind is that somehow nearly all particles will die before they can leave $\left(-\frac{1}{n},\frac{1}{n}\right)$ and we get a mass explosion.

But if we choose V_n "'too small"' we do not have enough mass creation since too many individuals can leave $\left(-\frac{1}{n},\frac{1}{n}\right)$ before dying. Hence V_n should be the life-time where these two effects are in balance.

A better understanding of this fact seems to be interesting for the question of an approximation by branching processes for super-Brownian motion with a single point source in higher dimensions. The one dimensional case bases on the fact that we find the right approximation for T_γ. This can be easily done by use of stochastic calculus which gives us a description of processes related to T_γ in terms of of the local time L_t. Hence we choose in our approximation something which will approximate L_t. In higher dimensions we do not have this description via a local time in one point and therefore it is not clear which approximation we should use. Therefore one has to find V_n in a different way.

6.2 The two dimensional case

We want to transport our results from one dimension to the case of superprocesses related to interactions along $\mathcal{C}$.The essential tools for the proof in one dimension were the approximation of elements of $D\left(T_\kappa\right)$ by suitable differentiable functions shown in proposition 84 and the upper bound for the expected total mass given in lemma 88. Having similar results in dimension two the tightness and the convergence to the superprocesses of chapter 5 will follow immediately.

Proposition 90 For any $\phi \in D\left(T_\eta\right)$ we find a sequence $\phi_n \in C^1\left(\mathbb{R}^2\right)$ such that

$$\phi_n \to \phi \text{ uniformly,} \tag{6.18}$$

$$\frac{1}{2}\Delta\phi_n' = T_\eta\phi - \frac{n}{2}\tilde{\eta}(x)\mathbb{1}_{\mathcal{C}_{\frac{1}{n}}}(x) \tag{6.19}$$

and it exists a constant K such that for every $n \in \mathbb{N}$

$$\|\nabla \phi_n\|_{\infty, \mathcal{C}_\beta} \leq K. \tag{6.20}$$

Proof. The proof is exactly the same as in proposition 84 by interchanging the absolute value function by a suitable harmonic single-layer potential. Then chapter 4, corollary 23 and the uniform approximation result of chapter 3, proposition 40 gives us the desired result. □

Martingale problem Our next goal is to give two equivalent descriptions of the martingale problem related to the superprocesses constructed in chapter 5.

Theorem 91 The martingale problem for the superprocess X_t constructed in chapter 5, theorem 69 is given in the following way: A distribution $\mathbb{P}_\mu$ solves the martingale problem if

$$\mathbb{P}_\mu \left(X_0 = \mu \right) = 1$$

and if

$$\exp\left(\langle X_t, -\phi\rangle\right) - \exp\left(\langle X_0, -\phi\rangle\right) - \int_0^t \left\langle X_s, -T_{eta}\phi + \alpha\phi^2 \right\rangle \exp\left(\langle X_s, -\phi\rangle\right) \mathrm{d}s$$

is a $\mathbb{P}_\mu$ martingale for every non-negative and bounded $\phi \in D\left(T_\eta\right)$ such that the integrand is bounded. An equivalent formulation is as follows: A distribution $\mathbb{P}$ satisfies the martingale problem if for all $\phi \in D\left(T_\eta\right)$

$$\left[M_t\right](\phi) = \langle X_t, \phi\rangle - \langle X_0, \phi\rangle - \int_0^t \langle X_s, T_\eta \phi\rangle \,\mathrm{d}s$$

is a $\mathbb{P}$-martingale with quadratic variation process

$$[M.\phi]_t = \int_0^t \left\langle X_s, \alpha\phi^2 \right\rangle \mathrm{d}s.$$

The solution of the martingale problem is unique.

Proof. The statements follow immediately by the general results of [12], pp.15. □

Branching processes Now we want to give a sequence of branching processes $X_t^{(n)}$ which should converge to X_t.

Theorem 92 Take a non-negative $\eta \in C^{0,1/2+\varepsilon}$ for a $\varepsilon > 0$. For $n \in \mathbb{N}$ take $X_t^{(n)}$ to be a branching process given with

- n individuals each of mass $\frac{1}{n}$ moving around in $\mathbb{R}$ (independently of each other) according to a Brownian motion,

- the life-time parameter is given by $V_n = \frac{n}{2}$,
- the sample mechanism fulfills

$$\Phi_x(1) = 1, \quad \Phi'_x(1) = 1 + \tilde{\eta}(x)\,\mathbb{1}_{\left\{\mathcal{C}_{\frac{1}{n}}\right\}}(x), \quad \Phi''_x = 4\alpha(x).$$

The sequence $X_t^{(n)}$ is tight and converges to X_t, the superprocess related to T_η constructed in chapter 5.

Before we can prove the theorem we want to give the two dimensional version of lemma 88.

Lemma 93 For any stopping time $\tau \leq t$ we have

$$\sup_{x\in\mathbb{R}^2} \mathbb{E}_{\delta_x}\left(\left\langle X_\tau^{(n)}.1\right\rangle\right) \leq \exp\left(\|\eta\|_{\infty,\mathcal{C}}\left(t+\sqrt{\frac{2t}{\pi}}\right)\right) \tag{6.21}$$

and

$$\sup_{x\in\mathbb{R}^2} \mathbb{E}_{\delta_x}\left(\frac{n}{2}\int_0^\tau \left\langle X_s^{(n)}, \tilde{\eta}\mathbb{1}_{\mathcal{C}_{\frac{1}{n}}}\right\rangle \mathrm{d}s\right) \leq \exp\left(\|\eta\|_{\infty,\mathcal{C}}\left(t+\sqrt{\frac{2t}{\pi}}\right)\right) - 1. \tag{6.22}$$

Proof. Since η is non-negative we see by the definition of $X_t^{(n)}$ that

$$\mathbb{E}_{\delta_x}\left(\left\langle X_\tau^{(n)}, 1\right\rangle\right) \leq \mathbb{E}_{\delta_x}\left(\left\langle X_t^{(n)}, 1\right\rangle\right)$$

and

$$\mathbb{E}_{\delta_x}\left(\frac{n}{2}\int_0^\tau \left\langle X_s^{(n)}, \tilde{\eta}\mathbb{1}_{\mathcal{C}_{\frac{1}{n}}}\right\rangle \mathrm{d}s\right) \leq \mathbb{E}_{\delta_x}\left(\frac{n}{2}\int_0^t \left\langle X_s^{(n)}, \tilde{\eta}\mathbb{1}_{\mathcal{C}_{\frac{1}{n}}}\right\rangle \mathrm{d}s\right).$$

Thus it is enough to show (6.21) and (6.22) for $\tau = t$. Furthermore it is shown for example in [12], p.13, that

$$\left\langle X_t^{(n)}, 1\right\rangle - \left\langle X_0^{(n)}, 1\right\rangle - \left(\frac{n}{2}\int_0^t \left\langle X_s^{(n)}, \tilde{\eta}\mathbb{1}_{\mathcal{C}_{\frac{1}{n}}}\right\rangle \mathrm{d}s\right)$$

is a martingale and thus

$$\mathbb{E}_{\delta_x}\left(\frac{n}{2}\int_0^t \left\langle X_s^{(n)}, \tilde{\eta}\mathbb{1}_{\mathcal{C}_{\frac{1}{n}}}\right\rangle \mathrm{d}s\right) = \mathbb{E}_{\delta_x}\left(\left\langle X_t^{(n)}, 1\right\rangle\right) - 1.$$

Therefore for $\tau = t$ (6.22) follows by (6.21). Now as in the one dimensional case we can use [8], section I.3.1 to see that $v_n(t,x) = \mathbb{E}_{\delta_x}\left(\left\langle X_t^{(n)}, 1\right\rangle\right)$ solves

$$v_n(t,x) = 1 + \int_0^t \int_{\mathbb{R}^2} \frac{n}{2}\tilde{\eta}\mathbb{1}_{\mathcal{C}_{\frac{1}{n}}} v_n(s,y) p(t-s,x,y)\mathrm{d}y\,\mathrm{d}s$$

where $p(t,x,y)$ is the usual transition kernel of the two dimensional Brownian motion. Now we can prove as in chapter 5, lemma 75, that

$$\|v_n(t,\cdot)\|_\infty \leq \exp\left(\|\eta\|_{\infty,\mathcal{C}}\left(t+\sqrt{\frac{2t}{\pi}}\right)\right)$$

which proves the lemma. □

Proof. Now the proof of the tightness and the convergence of $X_t^{(n)}$ is equivalent to the one-dimensional case. □

Chapter 7

The moments of a one-dimensional local time

The aim of this section is to proof the results of L.Takacs on the moments of the local time process in one dimension, see [31], in a new way.

Lemma 94 Let B_t denote a Brownian motion on $\mathbb{R}$ and L_t the local time process of this Brownian motion in 0. Then for every $n \geq 1$ and $x \in \mathbb{R}$ it holds:

$$\mathbb{E}^x\left(L_t^n\right) = n \int_0^\infty u^{(n-1)} \left(1 - p\left(\frac{u + |x|}{\sqrt{2t}}\right)\right) \mathrm{d}u.$$

Especially for $x = 0$ one reaches at

$$\mathbb{E}^0\left(L_t^n\right) = \frac{(2t)^{n/2}}{\sqrt{\pi}} \Gamma\left(\frac{n+1}{2}\right).$$

The function p denotes here the probability integral, i.e.

$$p(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$$

and Γ denotes the Gammafunction. The proof uses the results of Ito and McKean Jr. in [20] and Albeverio et al. in [1]

By the work of Ito and McKean Jr., see [20], p.199 one can easily prove that the generator of the semigroup

$$T_t f(x) = \mathbb{E}^x\left(f(B_t) e^{-\kappa L_t}\right), \kappa \geq 0$$

is given by

$$D(T_\kappa) = \{f \in H^{2,2}(-\infty, 0) \cap H^{2,2}(0, \infty) \cap C^0(\mathbb{R}) : f'\left(0^+\right) - f'\left(0^-\right) = 2\kappa f(0)\}$$

$$T_\kappa (f) = -\frac{1}{2} f''$$

0^+ denotes the right limit of a function at zero, 0^- the left limit, respectively. To be more explicit: In [20], p.199 the resolvent of T is calculated and the equality system (9) states for every $\alpha > 0$

$$\int_0^\infty e^{-\alpha t} T_t f(x) \mathrm{d}x = \int_0^\infty e^{-\alpha t} \mathbb{E}^x \left(f(B_t) e^{-\kappa L_t} \right) \mathrm{d}x$$

and hence by the uniqueness of the Laplace transform we get the desired result.
Albeverio et al. constructed in [1] the heat kernel of the operator

$$\tilde{T}_\kappa f = -f''$$
$$D\left(\tilde{T}_\kappa\right) = \{f \in H^{2,2}(-\infty, 0) \cap H^{2,2}(0,\infty) \cap C^0(\mathbb{R}) : f'\left(0^+\right) - f'\left(0^-\right) = \kappa f(0)\}$$

in the following form:

$$P_t(x,y) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{(x-y)^2}{4t}} - \frac{\kappa}{2} \int_0^\infty e^{-\frac{\kappa}{2}u} \frac{1}{\sqrt{4\pi t}} e^{-\frac{(u+|x|+|y|)^2}{4t}} \mathrm{d}u.$$

Proof. Based on the equation

$$(T_\kappa - \lambda)^{-1} = 2\left(\tilde{T}_{2\kappa} - 2\lambda\right)^{-1}$$

one can calculate the heat kernel of T_κ as

$$P_t(x,y) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{(x-y)^2}{2t}} - \kappa \int_0^\infty e^{-\kappa u} \frac{1}{\sqrt{2\pi t}} e^{-\frac{(u+|x|+|y|)^2}{2t}} \mathrm{d}u.$$

Now applying this formula to the function f, which is constantly 1 on $\mathbb{R}$, we arrive at

$$\begin{aligned}
\mathbb{E}^x\left(e^{-\kappa L_t}\right) &= \mathbb{E}^x\left(f(B_t) e^{-\kappa L_t}\right) \\
&= \int_{-\infty}^\infty \left(\frac{1}{\sqrt{2\pi t}} e^{-\frac{(x-y)^2}{2t}} - \kappa \int_0^\infty e^{-\kappa u} \frac{1}{\sqrt{2\pi t}} e^{-\frac{(u+|x|+|y|)^2}{2t}} \mathrm{d}u \right) \mathrm{d}y \\
&= 1 - \kappa \int_0^\infty e^{-\kappa u} \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^\infty e^{-\frac{(u+|x|+|y|)^2}{2t}} \mathrm{d}y \mathrm{d}u \\
&= 1 - \kappa \int_0^\infty e^{-\kappa u} \frac{2}{\sqrt{2\pi t}} \int_0^\infty e^{-\frac{(u+|x|+y)^2}{2t}} \mathrm{d}y \mathrm{d}u \\
&= 1 - \kappa \int_0^\infty e^{-\kappa u} \left(1 - p\left(\frac{u+|x|}{\sqrt{2t}}\right)\right) \mathrm{d}u
\end{aligned}$$

and therefore we get

$$
\begin{aligned}
\mathbb{E}^x\left(L_t^n\right) &= (-1)^n \lim_{\kappa\downarrow 0} \frac{\partial^n}{\partial \kappa^n} \mathbb{E}^x\left(e^{-\kappa L_t}\right) \\
&= (-1)^n \lim_{\kappa\downarrow 0} \frac{\partial^n}{\partial \kappa^n} \left(1 - \kappa \int_0^\infty e^{-\kappa u}\left(1 - p\left(\frac{u+|x|}{\sqrt{2t}}\right)\right) \mathrm{d}u\right) \\
&= (-1)^n \lim_{\kappa\downarrow 0} \left(-\kappa \int_0^\infty (-u)^n\, e^{-\kappa u}\left(1 - p\left(\frac{u+|x|}{\sqrt{2t}}\right)\right) \mathrm{d}u\right) \\
&\quad - (-1)^n \lim_{\kappa\downarrow 0} \left(n \int_0^\infty (-u)^{n-1}\, e^{-\kappa u}\left(1 - p\left(\frac{u+|x|}{\sqrt{2t}}\right)\right) \mathrm{d}u\right) \\
&= n \int_0^\infty u^{(n-1)} \left(1 - p\left(\frac{u+|x|}{\sqrt{2t}}\right)\right) \mathrm{d}u.
\end{aligned}
$$

The formula for $x = 0$ is just evaluating this integral using formula 6.281 of [17]. □

Chapter 8

List of frequently used symbols

Symbol	Meaning
$\overline{\Omega}$	closure of Ω
$C^k(\Omega)$	set of k-times continuously differentiable functions on Ω
$C^k(\overline{\Omega})$	set of functions with derivatives up to order k has a cont. continuation to $\overline{\Omega}$
D^α	partial derivative of order α, i.e. $D^\alpha = \frac{\partial^\alpha}{\partial^{\alpha_1} .. \partial^{\alpha_n}}$ with $\alpha_1 + ... + \alpha_n = \alpha$
Δ	the Laplace operator
$C^{k,\gamma}(\Omega)$	set of functions on Ω with γ- Hölder-continuous k-th derivative
$L^p(\Omega)$	the usual L^p-space on Ω
$H^{q,p}(\Omega)$	the Sobolev space of order q in $L^p(\Omega)$
$H^q(\Omega)$	the Sobolev space of order q in $L^2(\Omega)$
$\langle \mu, \phi \rangle$	$\int \phi \, d\mu$ for a measure μ and an integrable ϕ
$\langle f, g \rangle_V$	scalar product in V
$\mathcal{N}(\cdot)$	the kernel of an operator
$\mathcal{R}(\cdot)$	the range of an operator
Π	orthogonal projection
$\mathcal{C}$	closed $C^{2,\gamma}$-curve in $\mathbb{R}^2$
n_z	normal vector of $\mathcal{C}$ in $z \in \mathcal{C}$
$\tau_\mathcal{C}$	trace operator along $\mathcal{C}$
Λ	unitary transformation from $H^{3/2}(\mathcal{C})$ to $H^{-3/2}(\mathcal{C})$
$\frac{\partial}{\partial_n}$	a normal derivative
κ	jump operator defined in chapter 2, p.10
τ_α	a special trace operator along $\mathcal{C}$ defined in chapter 2, p.16
S_λ	acoustic single layer to parameter λ defined in chapter 2, p.14
P_λ	a special trace of S_λ defined in chapter 2, p.18

Symbol	Meaning
$\mathcal{C}_\varepsilon$	set of points in $\mathbb{R}^2$ with distance less than ε to $\mathcal{C}$
$\mathcal{C}^\varepsilon$	parallel surface in distance ε to $\mathcal{C}$
$\mathrm{d}S_r$	the usual volume element of $\mathcal{C}^r$
$B_\delta(x)$	closed ball with radius δ around x
$\\|\cdot\\|_{\infty,\Omega}$	the L^∞-norm on Ω
$\mathcal{M}(\Omega)$	set of finite measures on Ω
$\mathbb{E}_\mu$	expectation conditioned on starting measure μ
$\mathbb{P}_\mu$	distribution conditioned on starting measure μ
$\mathbb{E}^x$	expectation conditioned on starting point x
$\mathbb{P}^x$	distribution conditioned on starting point x
B_t	standard Brownian motion
$L(t,\cdot)$	local time of a Brownian motion in $(\cdot)$

Bibliography

[1] S. Albeverio, Z. Brzezniak and L. Dabrowski, Fundamental solution of the heat and Schroedinger equations with point interactions, J.Funct.Anal. 130 (1995), pp. 220-254

[2] S. Albeverio, F. Gesztesy, R. Hoegh-Krohn, H. Holden, Solvable Models in Quantum Mechanics, Springer-Verlag, New York, 1988

[3] R. Blumenthal, R. Getoor, Local times for Markov processes, Z. Wahrscheinlichkeitstheorie verw. Geb. 3 (1964), pp.50-74

[4] R. Blumenthal, R. Getoor, Markov processes and Potential theory, Academic Press, 1968

[5] K-L. Chung, R.J. Williams, Introduction to Stochastic Integration, Progress in Probability and Statistics, Vol.4, Birkhäuser, 1983

[6] M. Costabel Boundary integral operators on Lipschitz domains: Elementary results, SIAM J. Math. Anal. 19 (1988), pp. 613-626

[7] N. Dunford, J.T. Schwartz, Linear Operators Part II: Spectral Theory, Interscience, New York-London, 1963

[8] E.B. Dynkin, An Introduction to Branching Measure-Valued Processes, AMS, Providence, RI, 1994

[9] J. Engländer, K. Fleischmann, Extinction properties of super-Brownian motions with additionally spatially dependent mass production, Stochastic processes and their applications 88 (2000), pp. 37-58

[10] J. Engländer, R.G. Pinsky, On the construction and support properties of measure-valued diffusions on $D \subset \mathbb{R}^d$ with spatially dependent branching, Ann. Probab 27 (2), pp. 684-730

[11] R. Escada, R. Kanwal, Singular Integral Equations, Birkhäuser, 2000

[12] A.M. Etheridge, An introduction to Superprocesses, University Lecture Series, Vol.20, American Mathematical Society, 2000

[13] L. C. Evans, Partial Differential Equations, Graduate Studies in Mathematics, Vol.19, American Mathematical Society, 1998

[14] K. Fleischmann, C. Mueller, Super-Brownian motion with extra birth at one point, SIAM J. Math. Anal. Vol. 36, No. 3, pp. 740-772

[15] G. Folland, Introduction to partial differential equations, Princeton University Press and University of Tokyo Press, 1976

[16] J. Giroire, Integral equation methods for the Helmholtz equation, Integral Equations Operator Theory 5 (1982), pp. 505-517

[17] I.S. Gradshteyn and I.M.Ryzhik, Table of integrals, series and products Corrected and enlarged edition, Academic Press, 1980

[18] G. Grubb, Distributions and operators, Graduate Texts in Mathematics, Springer-Verlag, 2009

[19] N. Ikeda, S. Watanabe, Stochastic Differential Equations and Diffusion Processes Second Edition, North-Holland Mathematical Library, 1989

[20] K. Ito and H.P. McKean Jr., Brownian motion on a half line, IJM 7 (1963), pp. 181-231

[21] K. Ito and H.P. McKean Jr., Diffusion Processes and their Sample Paths, Springer-Verlag, Berlin, Heidelberg, New York, 1979

[22] C. Lederer, Lokalzeiten von Semimartingalen auf Hyperflächen, Diplomarbeit, 1993, LMU München

[23] J. L. Lions and E. Magenes, Non-Homogeneous Boundary Value Problems and Applications, Vol.I, Springer-Verlag, Berlin, Heidelberg, New York, 1972

[24] Z. Ma, M. Röckner, Introduction to the Theory of (Non-Symmetric) Dirichlet Forms, Springer Fachmedien Wiesbaden, 1992

[25] O.V. Besov, V.P. Il'in, S.M. Nikol'skij: Intergral representations of functions and embedding theorems, Vol. 1,2. (1978, 1979), New York

[26] C. Morosi, L. Pizzocchero, New results on multiplication in Sobolev spaces, Advances in applied mathematics, 44 4(2010), pp. 393-432

[27] A. Posiliciano, Self-adjoint extensions of restrictions, Operators and Matrices, Vol.2 (2008), pp. 483-506

[28] Reed, S., Simon, B., Methods of Moderm mathematical Physics II: Fourier Analysis, Self-Adjointness, Academic Press, New York, San Francisco, London, 1975

[29] S. Sheu, Continuity of Multidimensional Brownian Local Times, Proc. Amer. Math. Soc., Vol. 114, 3 (1992), pp. 821-829

[30] W. I. Smirnov, Lehrgang der höheren Mathematik, Teil IV, VEB Deutscher Verlag der Wissenschaften, Berlin 1977

[31] L. Takacs, On the Local Time of the Brownian Motion, Ann.Appl.Prob. Vol.5(3) (1995), pp. 741-756

Wissenschaftlicher Werdegang

12.08.1987	geboren in Oberwesel am Rhein
03/2006	Abitur am Wilhelm-Hofmann-Gymnasium in St. Goarshausen
04/2006 - 03/2010	Studium der Mathematik an der TU Kaiserslautern
03/2010	Diplom in Mathematik, TU Kaiserslautern
Seit 08/2010	Promotion an der TU Kaiserslautern

Scientific curriculum vitae

12.08.1987	born in Oberwesel am Rhein
03/2006	Abitur at the Wilhelm-Hofmann-Gymnasium, St. Goarshausen
04/2006 - 03/2010	Study of mathematics at the TU Kaiserslautern
03/2010	Diploma in mathematics, TU Kaiserslautern
since 08/2010	Ph.d. studies, TU Kaiserslautern