An Application of Klop's Counterexample to a Higher-Order Rewrite System

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Abstract

In 1978, Klop demonstrated that a rewrite system constructed by adding the *untyped lambda calculus*, which has the Church-Rosser property, to a Church-Rosser first-order algebraic rewrite system may not be Church-Rosser. In contrast, Breazu-Tannen recently showed that augmenting any Church-Rosser first-order algebraic rewrite system with the *simply-typed lambda calculus* results in a Church-Rosser rewrite system. In addition, Breazu-Tannen and Gallier have shown that the *second-order polymorphic lambda calculus* can be added to such rewrite systems without compromising the Church-Rosser property (for terms which can be provably typed).

There are other systems for which a Church-Rosser result would be desirable, among them being $\lambda^\tau \oplus SP \oplus FIX$, the simply-typed lambda calculus extended with surjective pairing and fixed points. This paper will show that Klop's untyped counterexample can be lifted to a typed system to demonstrate that $\lambda^\tau \oplus SP \oplus FIX$ is not Church-Rosser.

1 Introduction

As is well known, the *untyped lambda calculus* (λ) is Church-Rosser (CR) yet, as [Klop 80] has shown, the result of adding a CR rewrite system to λ may not be CR (a simpler example may be found in [Breazu-Tannen 88a]). However, the *simply-typed lambda calculus* (λ^τ), also CR, has the property that adding it to any CR first-order rewrite system results in a CR system [Breazu-Tannen 88a]. In addition, in [Breazu-Tannen 88b] it is shown that the *second-order polymorphic lambda calculus* can be added to a CR first-order rewrite system without losing the Church-Rosser property (for terms that have a provable type).

These results are quite useful. For example, they imply that strongly-typed functional programming languages using first-order rewrite rules may be implemented in a parallel manner: the results of a program's execution are independent of the order in which the reductions are made.

One higher-order rewrite system which seems to be a natural extension of λ^τ is $\lambda^\tau \oplus SP \oplus FIX$, which is λ^τ extended with surjective pairing at all types and fixed points at all types. We will

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show that this system is not CR. Our approach is to first use Klop's untyped counterexample as a model for a higher-order CR rewrite system to which adding λ^τ results in a non-CR system, then to show that this higher-order system can be embedded in $\lambda^\tau \oplus SP \oplus FIX$.

2 Preliminaries

We follow the notation used in [Barendregt 84]. We assume that the reader is familiar with λ and λ^τ. We use $M \equiv N$ to denote that two terms M and N are identical up to renaming of bound variables.

By *first-order rewrite systems* we refer to what are commonly called *term rewriting systems*. See [Huet 80] for details.

Higher-order rewrite systems arise naturally in the study of program transformation schemes, e.g., [Huet 78], which discusses second-order rewrite systems. There does not appear to be a natural characterization of such systems. Our concept of such systems allows the formation of any types by using $\rightarrow$, and of any terms which are allowed by the type discipline.

In the examples given below, we will be informal about the presentation of the rewrite systems with which we deal.

3 Klop's counterexample

In this section, we present Klop's counterexample[1], which shows that an algebraic rewrite system that is CR need not remain CR when λ is added. For clarity we will write constants as uppercase boldface letters (e.g., D, X), variables as lowercase italic letters (e.g., x,y), and meta-variables as uppercase italic letters (e.g., Z).

The construction followed in this section can be found in more detail in [Klop 80].

Definition 1 *Let $\mathcal{K}$ be the first-order rewrite system containing the constants* D *and* E *along with the reduction rule:*[2]

$$\mathsf{D}ZZ \underset{\mathcal{K}}{\longrightarrow} Z \tag{1}$$

$\mathcal{K}$ is clearly Church-Rosser. Now we let $\lambda\mathcal{K}$ denote the rewrite system derived by adding λ to the system $\mathcal{K}$, i.e., the terms of $\lambda\mathcal{K}$ are all λ-terms generated by the variables of λ along with the constants D and E of $\mathcal{K}$, and the reduction relation $\underset{\lambda\mathcal{K}}{\longrightarrow}$ denotes the union of the relations $\underset{\lambda}{\longrightarrow}$ and $\underset{\mathcal{K}}{\longrightarrow}$. The counterexample we will construct involves the following $\lambda\mathcal{K}$-terms.

[1]Actually, Klop constructed several counterexamples of similar form. We have chosen the one that seems best suited to our purposes.

[2]The reduction rule (1) was first suggested by Hindley.

Definition 2 *Define the $\lambda\mathcal{K}$-terms* Y,C,X *as follows:*

$$\begin{aligned} \mathsf{Y} &\equiv (\lambda ab.b(aab))(\lambda ab.b(aab)) \\ \mathsf{C} &\equiv \mathsf{Y}(\lambda cz.\mathsf{E}(\mathsf{D}z(cz))) \\ \mathsf{X} &\equiv \mathsf{YC} \end{aligned}$$

Y is Turing's fixed point combinator.[3]

The following lemma is easy to derive.

Lemma 3 ([Klop 80]) *The reduction system $\lambda\mathcal{K}$ has the following properties:*

$$\mathsf{Y}Z \underset{\lambda\mathcal{K}}{\twoheadrightarrow} Z(\mathsf{Y}Z) \quad (2)$$

$$\mathsf{X} \underset{\lambda\mathcal{K}}{\twoheadrightarrow} \mathsf{E}(\mathsf{CX}) \quad (3)$$

$$\mathsf{X} \underset{\lambda\mathcal{K}}{\twoheadrightarrow} \mathsf{C}(\mathsf{E}(\mathsf{CX})) \quad (4)$$

Proof: (2) is well-known and easy to show. As for (3),

$$\mathsf{X} \underset{\lambda\mathcal{K}}{\twoheadrightarrow} \mathsf{CX} \underset{\lambda\mathcal{K}}{\twoheadrightarrow} \mathsf{E}(\mathsf{DX}(\mathsf{CX})) \underset{\lambda\mathcal{K}}{\twoheadrightarrow} \mathsf{E}(\mathsf{D}(\mathsf{CX})(\mathsf{CX})) \underset{\lambda\mathcal{K}}{\twoheadrightarrow} \mathsf{E}(\mathsf{CX}).$$

Using this result, (4) holds by the reduction sequence

$$\mathsf{X} \underset{\lambda\mathcal{K}}{\twoheadrightarrow} \mathsf{CX} \underset{\lambda\mathcal{K}}{\twoheadrightarrow} \mathsf{C}(\mathsf{E}(\mathsf{CX})). \quad \blacksquare$$

We state the following theorem without proof.

Theorem 4 ([Klop 80]) *The terms* $\mathsf{E}(\mathsf{CX})$ *and* $\mathsf{C}(\mathsf{E}(\mathsf{CX}))$ *have no common $\lambda\mathcal{K}$-reduct.*

The desired result is now easily obtainable.

Corollary 5 ([Klop 80]) *$\lambda\mathcal{K}$ is not CR.*

Proof: By (3) and (4) along with Theorem 4. ■

4 Lifting Klop's Counterexample

In Klop's proof, essential use is made of the fixed point combinator Y, which has the property that $\mathsf{Y}Z \underset{\lambda}{\twoheadrightarrow} Z(\mathsf{Y}Z)$ for any λ-term Z. Such a recursion combinator is, however, not definable in λ^τ. Since our counterexample will require such combinators, we make them a part of the algebraic system.[4]

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[3] [Barendregt 84] refers to this combinator as Θ, using Y for Curry's fixed point combinator. We follow Klop's notation here.

[4] Note that such a recursion operator is often provided even in strongly-typed programming languages such as ML.

In devising the counterexample, we determined principal types for the terms which appear in Klop's counterexample. We require analogues for the constants D and E of $\mathcal{K}$, as well as the fixed point combinator which we were able to define in λ.

Definition 6 *Let $\mathcal{L}$ be the higher-order algebraic rewrite system defined as follows. The set of base types of $\mathcal{L}$ is $\{0\}$; $\mathcal{L}$ contains the constants $\underline{\mathsf{Y}}_1^{(\sigma\to\sigma)\to\sigma}$, $\underline{\mathsf{Y}}_2^{((\sigma\to\sigma)\to\sigma\to\sigma)\to\sigma\to\sigma}$, $\underline{\mathsf{D}}^{\sigma\to\sigma\to\sigma}$, and $\underline{\mathsf{E}}^{\sigma\to\sigma}$ for some type σ; $Term(\mathcal{L})$, the set of terms of $\mathcal{L}$, is defined inductively by:*

1. *$\underline{\mathsf{Y}}_1$, $\underline{\mathsf{Y}}_2$, $\underline{\mathsf{E}}$ are in $Term(\mathcal{L})$;*
2. *If $M^{\tau\to\pi} \in Term(\mathcal{L})$, and $N^{\tau} \in Term(\mathcal{L})$, then $(MN) \in Term(\mathcal{L})$;*
3. *If $M^{\sigma} \in Term(\mathcal{L})$ and $N^{\sigma} \in Term(\mathcal{L})$, then $(\underline{\mathsf{D}}MN) \in Term(\mathcal{L})$.*

The reduction rules of $\mathcal{L}$ are

$$\underline{\mathsf{Y}}_1 Z^{\sigma\to\sigma} \underset{\mathcal{L}}{\longrightarrow} Z(\underline{\mathsf{Y}}_1 Z) \tag{5}$$

$$\underline{\mathsf{Y}}_2 Z^{(\sigma\to\sigma)\to\sigma\to\sigma} \underset{\mathcal{L}}{\longrightarrow} Z(\underline{\mathsf{Y}}_2 Z) \tag{6}$$

$$\underline{\mathsf{D}} Z^{\sigma} Z \underset{\mathcal{L}}{\longrightarrow} Z \tag{7}$$

The reader may be wondering why we have restricted $Term(\mathcal{L})$ in the above definition, so that $\underline{\mathsf{D}}$ only appears with two arguments. We wish to show that $\mathcal{L}$ is CR, and this restriction allows us to apply a theorem of Klop.

Note that though the weak Church-Rosser property holds for $\mathcal{L}$, $\mathcal{L}$ does not have the finite termination property. Thus we cannot directly conclude that $\mathcal{L}$ is CR.

Theorem 7 *$\mathcal{L}$ is CR.*

Proof: $\mathcal{L}$ can be considered as a substructure of the direct sum of two reduction systems: $\mathcal{L}'$ which contains just the constants $\underline{\mathsf{Y}}_1$ and $\underline{\mathsf{Y}}_2$ and the reduction rules (5) and (6); and the reduction system $\mathcal{L}''$, which contains the constants $\underline{\mathsf{D}}$ and $\underline{\mathsf{E}}$, and the reduction rule (7). $\mathcal{L}$, as a substructure of $\mathcal{L}' \oplus \mathcal{L}''$, is restricted to terms in which $\underline{\mathsf{D}}$ only appears with two arguments. We follow the proof given for the analogous untyped system in [Klop 80, Theorem III.5.6], which relies upon the "black box" lemma. It is clear that $\mathcal{L}'$ is CR. A straightforward adaptation of Klop's proof shows that $\mathcal{L}$ is a black box extension of $\mathcal{L}'$, and that the resulting system is CR. ■

Now we will add λ^{τ} to $\mathcal{L}$.

Definition 8 *Let $\lambda^{\tau}\mathcal{L}$ be the rewrite system obtained by adding λ^{τ} to $\mathcal{L}$. For each $\lambda^{\tau}\mathcal{L}$-term M^{σ}, define $|M^{\sigma}| \in \lambda\mathcal{K}$ inductively as follows:*

(i) $|x^{\sigma}| \equiv x$ *for every variable* $x \in \lambda^{\tau}\mathcal{L}$;

(ii) $|\underline{\mathsf{Y}}_2| \equiv |\underline{\mathsf{Y}}_1| \equiv \mathsf{Y}$;

(iii) $|\underline{\mathsf{D}}| \equiv \mathsf{D}$;

(iv) $|\underline{\mathsf{E}}| \equiv \mathsf{E}$;

(v) $|\lambda x^\sigma.N^\tau| \equiv \lambda x.|N^\tau|$;

(vi) $|(M_1^{\tau\to\sigma} M_2^\tau)| \equiv (|M_1^{\tau\to\sigma}|\,|M_2^\tau|)$;

Intuitively, $|M|$ is the result of "erasing" the type information from the term M.

Remark 9 *If $\mathcal{L}$ were a first-order system, we would have the property [Breazu-Tannen 88a]:*

$$X \underset{\mathcal{L}}{\longrightarrow} Y \Rightarrow \beta nf(X) \underset{\mathcal{L}}{\twoheadrightarrow} \beta nf(Y)$$

Note, however, that if $X \equiv \underline{\mathsf{Y}}_1(\lambda x^\sigma.y^\sigma)$ and $Y \equiv (\lambda x^\sigma.y^\sigma)(\underline{\mathsf{Y}}_1(\lambda x^\sigma.y^\sigma))$, then $\beta nf(X) \equiv X$ and $\beta nf(Y) \equiv y$, but $X \underset{\mathcal{L}}{\not\twoheadrightarrow} y$.

Now we show that the systems $\lambda^\tau\mathcal{L}$ and $\lambda\mathcal{K}$ have a nice relationship which can be exploited. This is a very natural extension of the relationship between λ^τ and λ, for which we have

$$A \underset{\lambda^\tau}{\longrightarrow} B \Rightarrow |A| \underset{\lambda}{\longrightarrow} |B|.$$

Lemma 10 *If M and N are $\lambda^\tau\mathcal{L}$ terms such that $M \underset{\lambda^\tau\mathcal{L}}{\twoheadrightarrow} N$, then $|M| \underset{\lambda\mathcal{K}}{\twoheadrightarrow} |N|$.*

Proof: We will show that if $M \underset{\lambda^\tau\mathcal{L}}{\longrightarrow} N$ (a single reduction step), then $|M| \underset{\lambda\mathcal{K}}{\twoheadrightarrow} |N|$. Then by induction on the number of reduction steps from M to N, the theorem will hold.

Suppose that $M \underset{\lambda^\tau\mathcal{L}}{\longrightarrow} N$. Let P be the redex which is contracted in going from M to N, let Q be the term to which P reduces, and write $M \equiv \mathbb{C}[P]$, $N \equiv \mathbb{C}[Q]$, where $\mathbb{C}$ indicates a context. Clearly, $|P| \underset{\lambda\mathcal{K}}{\twoheadrightarrow} |Q|$ implies that $|M| \underset{\lambda\mathcal{K}}{\twoheadrightarrow} |N|$. P could be one of four types of redexes:

1. P is a β-redex: $|P| \equiv |(\lambda x M_1)M_2| \equiv (\lambda x.|M_1|)|M_2| \underset{\lambda\mathcal{K}}{\longrightarrow} |M_1|[x := |M_2|] \equiv |Q|$.
2. $P \equiv \underline{\mathsf{D}} M_1 M_1$, $Q \equiv M_1$: $|P| \equiv \mathsf{D}|M_1||M_1| \underset{\lambda\mathcal{K}}{\longrightarrow} |Q|$.
3. $P \equiv \underline{\mathsf{Y}}_1 M_1$, $Q \equiv M_1(\underline{\mathsf{Y}}_1 M_1)$: $|P| \equiv \mathsf{Y}|M_1| \underset{\lambda\mathcal{K}}{\twoheadrightarrow} |M_1|(\mathsf{Y}|M_1|) \equiv |Q|$.
4. $P \equiv \underline{\mathsf{Y}}_2 M_1$, $Q \equiv M_1(\underline{\mathsf{Y}}_2 M_1)$: Same as the previous case.

In each case we have that $|P| \underset{\lambda\mathcal{K}}{\twoheadrightarrow} |Q|$, and thus $|M| \underset{\lambda\mathcal{K}}{\twoheadrightarrow} |N|$. ■

Now we mimic Klop's counterexample, using the constants of $\mathcal{L}$.

Definition 11 *Define the $\lambda^\tau\mathcal{L}$-terms $\underline{\mathsf{C}}^{\sigma\to\sigma}$ and $\underline{\mathsf{X}}^\sigma$ as follows:*

$$\begin{aligned} \underline{\mathsf{C}}^{\sigma\to\sigma} &\equiv \underline{\mathsf{Y}}_2(\lambda c^{\sigma\to\sigma} z^\sigma.\underline{\mathsf{E}}(\underline{\mathsf{D}} z(cz))) \\ \underline{\mathsf{X}}^\sigma &\equiv \underline{\mathsf{Y}}_1\underline{\mathsf{C}} \end{aligned}$$

Remark 12 *Note that* $|\underline{\mathsf{C}}^{\sigma\to\sigma}| \equiv \mathsf{C}$ *and that* $|\underline{\mathsf{X}}^{\sigma}| \equiv \mathsf{X}$.

Lemma 13 *The reduction system* $\lambda^{\tau}\mathcal{L}$ *has the following properties:*

$$\underline{\mathsf{C}}\,\underline{\mathsf{X}} \underset{\lambda^{\tau}\mathcal{L}}{\twoheadrightarrow} \underline{\mathsf{E}}(\underline{\mathsf{C}}\,\underline{\mathsf{X}}) \quad (8)$$

$$\underline{\mathsf{C}}\,\underline{\mathsf{X}} \underset{\lambda^{\tau}\mathcal{L}}{\twoheadrightarrow} \underline{\mathsf{C}}(\underline{\mathsf{E}}(\underline{\mathsf{C}}\,\underline{\mathsf{X}})) \quad (9)$$

Proof: Analogous to the derivations in Lemma 3 for $\lambda\mathcal{K}$. ■

Theorem 14 *The terms* $\underline{\mathsf{E}}(\underline{\mathsf{C}}\,\underline{\mathsf{X}})$ *and* $\underline{\mathsf{C}}(\underline{\mathsf{E}}(\underline{\mathsf{C}}\,\underline{\mathsf{X}}))$ *have no common* $\lambda^{\tau}\mathcal{L}$*-reduct.*

Proof: Suppose that $\underline{\mathsf{E}}(\underline{\mathsf{C}}\,\underline{\mathsf{X}})$ and $\underline{\mathsf{C}}(\underline{\mathsf{E}}(\underline{\mathsf{C}}\,\underline{\mathsf{X}}))$ have a common reduct M. If so, then by Lemma 10, $|\underline{\mathsf{E}}(\underline{\mathsf{C}}\,\underline{\mathsf{X}})| \equiv \mathsf{E}(\mathsf{CX}) \underset{\lambda\mathcal{K}}{\twoheadrightarrow} |M|$ and $|\underline{\mathsf{C}}(\underline{\mathsf{E}}(\underline{\mathsf{C}}\,\underline{\mathsf{X}}))| \equiv \mathsf{C}(\mathsf{E}(\mathsf{CX})) \underset{\lambda\mathcal{K}}{\twoheadrightarrow} |M|$. This contradicts Theorem 4. ■

Corollary 15 $\lambda^{\tau}\mathcal{L}$ *is not CR.*

Proof: By (8) and (9), along with Theorem 14. ■

5 Surjective Pairing

Using the counterexample constructed in the previous section, we will show that $\lambda^{\tau} \oplus SP \oplus FIX$, i.e., λ^{τ} extended with surjective pairing and fixed points at all types, is not CR.

First we define $\lambda^{\tau} \oplus SP \oplus FIX$. For brevity, we will denote this system by $\mathcal{M}$.

Definition 16 *By* $\mathcal{M}$ *we denote the extension of* λ^{τ} *which has the following properties:*

1. *Surjective pairing at all types:*
   (a) *For all types* σ, τ *in* $\mathcal{M}$, $\sigma \times \tau$ *is a type;*
   (b) *For all types* σ, τ *in* $\mathcal{M}$, *there exist unique constants* $\mathsf{P}^{\sigma\to\tau\to\sigma\times\tau}$, $\mathsf{L}^{\sigma\times\tau\to\sigma}$, *and* $\mathsf{R}^{\sigma\times\tau\to\tau}$, *such that the following reduction rules hold:*
      i. $\mathsf{L}(\mathsf{P}X^{\sigma}Y^{\tau}) \underset{\mathcal{M}}{\to} X$ *(*L*-reduction)*
      ii. $\mathsf{R}(\mathsf{P}X^{\sigma}Y^{\tau}) \underset{\mathcal{M}}{\to} Y$ *(*R*-reduction)*
      iii. $\mathsf{P}(\mathsf{L}\,X^{\sigma\times\tau})(\mathsf{R}\,X) \underset{\mathcal{M}}{\to} X$ *(*P*-reduction)*
2. *Fixed points at all types: For each type* σ *in* $\mathcal{M}$, *there exists a unique constant* $\overline{\mathsf{Y}}^{(\sigma\to\sigma)\to\sigma}$ *with the reduction rule (*$\overline{\mathsf{Y}}$*-reduction)*

$$\overline{\mathsf{Y}}Z \underset{\mathcal{M}}{\to} Z(\overline{\mathsf{Y}}Z).$$

Theorem 17 *$\mathcal{M}$ is not CR.*

Proof: Let σ denote the type 0×0. Define the term $\overline{\mathsf{D}}^{\sigma\to\sigma\to\sigma}$ ([Klop 80]) of $\mathcal{M}$ by

$$\overline{\mathsf{D}} \equiv \lambda x^{\sigma} y^{\sigma}.\mathsf{P}^{0\to 0\to\sigma}(\mathsf{L}^{\sigma\to 0}x)(\mathsf{R}^{\sigma\to 0}y)$$

It is trivial to see that for all terms Z^{σ},

$$\overline{\mathsf{D}}ZZ \twoheadrightarrow_{\mathcal{M}} Z \tag{10}$$

Let $\overline{\mathsf{Y}}_1$ be the fixed point at type σ, and let $\overline{\mathsf{Y}}_2$ be the fixed point at type $\sigma\to\sigma$. Thus we also have the properties that

$$\overline{\mathsf{Y}}_1 Z^{\sigma\to\sigma} \rightarrow_{\mathcal{M}} Z(\overline{\mathsf{Y}}_1 Z) \tag{11}$$

$$\overline{\mathsf{Y}}_2 Z^{((\sigma\to\sigma)\to\sigma\to\sigma)\to\sigma\to\sigma} \rightarrow_{\mathcal{M}} Z(\overline{\mathsf{Y}}_2 Z) \tag{12}$$

Note that the rewrite rules (11), (12), (10) in $\mathcal{M}$ directly correspond to the rules (5), (6), and (7) of the system $\mathcal{L}$. Thus $\lambda^{\tau}\mathcal{L}$ is "embeddable" in $\mathcal{M}$. $\mathcal{M}$ contains additional rewrite relations that $\lambda^{\tau}\mathcal{L}$ does not, however, so we must show that these do not affect the correspondence between the two systems.

Let $\overline{\mathsf{E}}$ be a new variable of type $\sigma\to\sigma$. Thus we can construct:

$$\overline{\mathsf{C}} \equiv \overline{\mathsf{Y}}_2(\lambda c^{\sigma\to\sigma} z^{\sigma}.\overline{\mathsf{E}}(\mathsf{P}(\mathsf{L}z)(\mathsf{R}(cz))))$$

$$\overline{\mathsf{X}} \equiv \overline{\mathsf{Y}}_1 \overline{\mathsf{C}}$$

Now we compare the two terms $\underline{\mathsf{C}}\,\underline{\mathsf{X}}$ and $\overline{\mathsf{C}}\,\overline{\mathsf{X}}$. To do this, we define the following type of reduction system.

By $\mathcal{G}(\underline{\mathsf{C}}\,\underline{\mathsf{X}})$ we denote the *reduction graph* of the term $\underline{\mathsf{C}}\,\underline{\mathsf{X}}$, i.e., the reduction system whose terms consist of all terms M in $\lambda^{\tau}\mathcal{L}$ such that $\underline{\mathsf{C}}\,\underline{\mathsf{X}} \twoheadrightarrow_{\lambda^{\tau}\mathcal{L}} M$, and whose reduction relation is $\twoheadrightarrow_{\lambda^{\tau}\mathcal{L}}$ restricted to such terms. Likewise, we denote the reduction graph of the term $\overline{\mathsf{C}}\,\overline{\mathsf{X}}$ by $\mathcal{G}(\overline{\mathsf{C}}\,\overline{\mathsf{X}})$.

By Theorem 14, we know that the reduction system $\mathcal{G}(\underline{\mathsf{C}}\,\underline{\mathsf{X}})$ is not CR. Define the mapping $\epsilon : Term(\mathcal{G}(\underline{\mathsf{C}}\,\underline{\mathsf{X}})) \to Term(\mathcal{G}(\overline{\mathsf{C}}\,\overline{\mathsf{X}}))$ inductively as follows:

- $\epsilon(x) = x$ for all variables x. By this we mean that the type σ in $\lambda^{\tau}\mathcal{L}$ will correspond with the σ that we have defined in $\mathcal{M}$, and we extend this correspondence to higher types. The types of all variables in $Term(\mathcal{G}(\underline{\mathsf{C}}\,\underline{\mathsf{X}}))$ are constructed only from σ, as are the types of the variables in $Term(\mathcal{G}(\overline{\mathsf{C}}\,\overline{\mathsf{X}}))$.
- $\epsilon(\underline{\mathsf{E}}) = \overline{\mathsf{E}}$
- $\epsilon(\underline{\mathsf{Y}}_1) = \overline{\mathsf{Y}}_1$
- $\epsilon(\underline{\mathsf{Y}}_2) = \overline{\mathsf{Y}}_2$
- $\epsilon(\lambda x.M) = \lambda x.\epsilon(M)$

- $\epsilon(\underline{\mathsf{D}}MN) = \mathsf{P}(\mathsf{L}\epsilon(M))(\mathsf{R}\epsilon(N))$
- $\epsilon(AB) = \epsilon(A)\epsilon(B)$ if AB is not of the form $\underline{\mathsf{D}}MN$.

Now it is easy to establish the following facts:

1. $\epsilon(\underline{\mathsf{C}}\,\underline{\mathsf{X}}) = \overline{\mathsf{C}}\,\overline{\mathsf{X}}$
2. For all $M, N \in Term(\mathcal{G}(\underline{\mathsf{C}}\,\underline{\mathsf{X}}))$, the following hold:
   (a) $M \underset{\lambda^\tau}{\longrightarrow} N \iff \epsilon(M) \underset{\lambda^\tau}{\longrightarrow} \epsilon(N)$
   (b) $M \underset{\underline{\mathsf{D}}}{\longrightarrow} N \iff \epsilon(M) \underset{\mathsf{P}}{\longrightarrow} \epsilon(N)$
   (c) $M \underset{\underline{\mathsf{Y}}_1}{\longrightarrow} N \iff \epsilon(M) \underset{\overline{\mathsf{Y}}_1}{\longrightarrow} \epsilon(N)$
   (d) $M \underset{\underline{\mathsf{Y}}_2}{\longrightarrow} N \iff \epsilon(M) \underset{\overline{\mathsf{Y}}_2}{\longrightarrow} \epsilon(N)$
3. L- and R-reductions cannot occur in $\mathcal{G}(\overline{\mathsf{C}}\,\overline{\mathsf{X}})$. This is true because every occurrence of a term of the form $\mathsf{P}MN$ occurs as the argument to a $\overline{\mathsf{E}}$, so subterms of the form $\mathsf{L}(\mathsf{P}MN)$ and $\mathsf{R}(\mathsf{P}MN)$ cannot occur.

Thus the reduction graphs $\mathcal{G}(\underline{\mathsf{C}}\,\underline{\mathsf{X}})$ and $\mathcal{G}(\overline{\mathsf{C}}\,\overline{\mathsf{X}})$ are isomorphic. Since $\mathcal{G}(\underline{\mathsf{C}}\,\underline{\mathsf{X}})$ is not CR, it follows that $\mathcal{G}(\overline{\mathsf{C}}\,\overline{\mathsf{X}})$ is not. Therefore $\mathcal{M} = \lambda^\tau \oplus SP \oplus FIX$ is not CR. ■

This result may be interpreted from a slightly different perspective. Pottinger ([Pottinger 81]) has proved that $\lambda^\tau \oplus SP$, that is, λ^τ with surjective pairing at all types, is CR. The counterexample created above can be viewed as combining the CR higher-order rewrite system represented by (11) and (12) with $\lambda^\tau \oplus SP$, again resulting in a system that is not CR.

6 Conclusion

This paper has shown that, based upon the work of Klop, higher-order CR rewrite systems can be constructed which, when combined with λ^τ are not CR. The crucial component in carrying out the construction is the ability to define a fixed point combinator in a higher-order rewrite system. Because such combinators are "built-into" $\lambda^\tau \oplus SP \oplus FIX$, they can be used to provide a counterexample of the Church-Rosser property for that system.

Note that the constructions required in this paper were imitations of those discovered by Klop, and that the proofs were likewise merely adaptations of Klop's proofs. Thus Klop's work, despite being concerned with untyped systems, provides a useful tool for analyzing the confluence properties of complex higher-order systems.

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