Edgeworth expansions for lattice triangular arrays

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Abstract: Edgeworth expansions have been introduced as a generalization of the central limit theorem and allow to investigate the convergence properties of sums of i.i.d. random variables. We consider triangular arrays of lattice random vectors and obtain a valid Edgeworth expansion for this case. The presented results can be used, for example, to study the convergence behavior of lattice models.

Keywords: Fourier-Stieltjes transform, Edgeworth expansion, lattice triangular arrays.

1 Introduction

In financial mathematics, statistics and probability theory problems often arise that deal with sums of random variables. The CLT plays an important role here in investigating the limiting distribution of such sums. However, in many cases we are interested in a more detailed analysis of the convergence behavior and a better approximation than simply the normal distribution function.

The idea of Edgeworth expansions goes back to P.L. Chebyshev (see e.g. [GK54]), F.Y. Edgeworth [Edge1905], C.V.L. Charlier [Ch1906], and others. They propose to expand the distribution of a sum of random variables in a series of the normal distribution function and its derivatives. These series and their convergence properties have been thoroughly studied by H. Cramer. In [BR76] R.N. Bhattacharya and R.R. Rao present numerous results on the topic, offer a generalization to higher dimensions and a detailed exposition for lattice random vectors. However, the results available in the literature do not cover the case of lattice triangular arrays, which occurs, for example, in the theory of binomial trees. Therefore, the purpose of this work is to justify the Edgeworth expansion also for this setting. We will be using the notation and following the proofs given in [BR76].

Edgeworth expansions are based on a taylor series of the corresponding characteristic function. This series is then inverted to recover the necessary result for the distribution function itself. Therefore, we start with a brief introduction of the basic definitions and properties of characteristic functions. Then, following the main literature on the topic, we give a heuristic explanation of the approach for a better understanding of the intuition behind it. In the last section we present the main theorems.

2 The characteristic function: basic definitions and properties

We will follow the definitions in [BR76]. We will need the following notation. For a nonnegative integer vector $\alpha = (\alpha_1, \dots, \alpha_d)$ and $x \in \mathbb{R}^d$ let

$$x^\alpha = x^{\alpha_1} \cdots x^{\alpha_d},$$
$$D^\alpha = D_1^{\alpha_1} \cdots D_d^{\alpha_d}, \quad \text{with } D_j = \frac{\partial}{\partial x_j}.$$

In addition, let $\langle \cdot, \cdot \rangle$ be the usual inner product in $\mathbb{C}^d$, and for $t \in \mathbb{C}^d$

$$\|t\| = \sqrt{\sum_{j=1}^{d} |t_j|^2}, \quad |t| = \sum_{j=1}^{k} |t_j|.$$

Definition 2.1. The *Fourier-Stieltjes transform* of a finite signed measure μ is the function $\psi_\mu : \mathbb{R}^d \to \mathbb{C}$ defined by

$$\psi_\mu(t) = \int_{\mathbb{R}^d} e^{i\langle t,x\rangle} \mu(dx), \quad t \in \mathbb{R}^d.$$

If μ has density f with respect to the Lebesgue measure, the $\psi_\mu = \psi_f$, where ψ_f is the Fourier transform of f. If μ is a probability measure, ψ_μ is usually referred to as the *characteristic function* of μ.

Remark 2.2. We will also use the notation ψ_X to denote the characteristic function of the probability measure corresponding to the random vector X.

Example 2.3. Consider the multidimensional normal distribution $\Phi_{m,\Sigma}$ with mean m and covariance matrix Σ. The density function is given by

$$\phi_{m,\Sigma}(x) = \frac{1}{(2\pi)^{d/2}\sqrt{\det \Sigma}} e^{-\frac{1}{2}\langle x, \Sigma^{-1}x\rangle}, \quad x \in \mathbb{R}^d.$$

In this case the characteristic function is equal to

$$\psi_{\Phi_{m,\Sigma}}(t) = e^{i\langle t,m\rangle - \frac{1}{2}\langle t, \Sigma t\rangle}, \quad t \in \mathbb{R}^d.$$

(See e.g. [BR76], Chapter 5.)

There exists a one-to-one correspondence between finite signed measures and Fourier-Stieltjes transforms (Uniqueness Theorem, see e.g. [BR76] Theorem 5.1 (i)). If μ is absolutely continuous with respect to the Lebesgue measure, then its density f can be recovered from ψ_μ using the following theorem (see [BR76], Theorems 4.1 and 5.1).

Theorem 2.4 (Fourier Inversion Formula). *i) If $\psi_\mu \in L^1(\mathbb{R}^d)$, then μ is absolutely continuous with respect to the Lebesgue measure and has a uniformly continuous, bounded density f,*

$$f(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-i\langle t,x\rangle} \psi_\mu(t)\, dt, \quad x \in \mathbb{R}^k.$$

ii) Suppose $f \in L^1(\mathbb{R}^d)$. Let $x \in \mathbb{R}^d$ and $\alpha = (\alpha_1, \dots, \alpha_d)$ be a nonnegative integer vector. If $x^\alpha f(x) \in L^1(\mathbb{R}^d)$, then $D^\alpha \psi_f$ exists and

$$x^\alpha f(x) = \frac{(-1)^{|\alpha|}}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-i\langle t,x\rangle} D^\alpha \psi_f(t)\, dt.$$

This suggests that the asymptotic expansions for distribution functions could be derived from the corresponding results for characteristic functions.

Let P be a probability measure on $\mathbb{R}^d$. To obtain an expansion for the characteristic function we will be using the following results.

Definition 2.5. (i) Let $\nu = (\nu_1, \dots, \nu_k)$ be a nonnegative integer vector such that

$$\int_{\mathbb{R}^d} |x^\nu| P(dx) < \infty.$$

The *moment of order* ν of P is defined as

$$\mu_\nu = \int_{\mathbb{R}^d} x^\nu P(dx).$$

(ii) For $s \in \mathbb{R}$, $s \geq 0$ the *absolute moment of order* s of P is defined as

$$\rho_s = \int_{\mathbb{R}^d} \|x\|^s P(dx).$$

The following theorem characterizes the derivatives of ψ_P.

Theorem 2.6. *If $\rho_s < \infty$ for some positive integer s, then $D^\alpha \psi_P$ exists for every nonnegative integer vector $\nu \in \mathbb{R}^d$, $|\nu| \leq s$, and is equal to*

$$(D^\nu \psi_P)(t) = (i)^{|\nu|} \int_{\mathbb{R}^d} x^\nu e^{i\langle t,x\rangle} P(dx), \quad t \in \mathbb{R}^d.$$

Moreover,

$$i^{|\nu|} \mu_\nu = (D^\nu \psi_P)(0).$$

(see e.g. [BR76] Section 5).

Using Theorem 2.6 and a Taylor expansion for complex-valued functions (see e.g. [BR76], Corollary 8.2) we get

$$\psi_P(t) = 1 + \sum_{|\nu| \le s} \frac{\mu_\nu}{\nu!} (it)^\nu + o\left(\|t\|^s\right), \quad t \to 0, \tag{1}$$

if $\rho_s < \infty$ for some positive integer s. Therefore, we have an expansion for the characteristic function in terms of the moments of the distribution. However, in the theory of Fourier transforms it is usually more convenient to work with cumulants, also referred to as semi-invariants (e.g. [GK54], [P75]). Instead of (1) consider the Taylor expansion of the logarithm of the characteristic function.

$$\log \psi_P(t) = \sum_{|\nu| \le s} \frac{\kappa_\nu}{\nu!} (it)^\nu + o\left(\|t\|^s\right), \quad t \to 0. \tag{2}$$

Here the coefficients κ_ν are the *cumulants of order* ν of P, and are given by

$$i^{|\nu|}\kappa_\nu = \left(D^\nu \log \psi_P\right)(0).$$

The cumulants κ_ν, $|\nu| \le s$, as well as expansion (2) exist if $\rho_s < \infty$. Here $\log \psi_P(t)$ is the principal branch of the complex logarithm. Note that

$$\kappa_0 = 0, \quad \text{and} \quad \kappa_\nu = \mu_\nu,\ |\nu| = 1.$$

Example 2.7. Consider once again the normal distribution function $\Phi_{m,\Sigma}$. Since $\log \psi_{\Phi_{m,\Sigma}}(t)$ is quadratic in t all cumulants for $|\nu| > 2$ are equal to zero. For $|\nu| = 1, 2$ we have

$$\kappa_{e_i} = m_i, \quad i = 1, \ldots, d,$$

and

$$\kappa_{e_i + e_j} = \Sigma(i, j), \quad i, j = 1, \ldots, d$$

where e_i is the i-th unit vector (see [BR76], pp. 50-51).

Remark 2.8. The main reason why cumulants are usually the preferred choice is their additivity. Consider the sum $X = X_1 + \cdots + X_n$ of independent random vectors on $\mathbb{R}^d$, then

$$\psi_X(t) = \prod_{i=1}^n \psi_{X_i}(t),$$

and

$$\log \psi_X(t) = \sum_{i=1}^n \log \psi_{X_i}(t).$$

Therefore, if X_i, $i = 1, \ldots, n$ have finite cumulants of order ν, $\kappa_\nu(X_i)$, then

$$\kappa_\nu(X) = \kappa_\nu(X_1) + \cdots + \kappa_\nu(X_n).$$

This is, obviously, not true for moments of higher orders.

The formal identity

$$\log\left(1+\sum_{|\nu|\geq 1}\frac{\mu_\nu}{\nu!}(it)^\nu\right)=\sum_{|\nu|\geq 1}\frac{\kappa_\nu}{\nu!}(it)^\nu,$$

allows to uniquely express the cumulants in terms of the moments and vice versa. As a result we have

$$\begin{aligned} m_\nu &= \sum \frac{1}{q!}\frac{\nu!}{\nu_1!\cdots\nu_q!}\prod_{p=1}^{q}\kappa_{\nu_p} \quad \text{and} \\ \kappa_\nu &= \sum \frac{(-1)^{q-1}}{q}\frac{\nu!}{\nu_1!\cdots\nu_q!}\prod_{p=1}^{q} m_{\nu_p}, \end{aligned} \tag{3}$$

where the summation is over all $q-$tuples of nonnegative integer vectors ν_p, $|\nu_p|>0$, such that

$$\sum_{p=1}^{q}\nu_p=\nu, \quad q=1,\ldots,|\nu|,$$

(see e.g. [Sh84], Theorem 12.8).

Example 2.9. In the one-dimensional case (3) gives

$$\begin{aligned} \mu_2 &= \kappa_2+\kappa_1^2, \\ \mu_3 &= \kappa_3+3\kappa_2\kappa_1+\kappa_1^3, \\ \mu_4 &= \kappa_4+4\kappa_3\kappa_1+3\kappa_2^2+6\kappa_2\kappa_1^2+\kappa_1^4, \end{aligned}$$

and

$$\begin{aligned} \kappa_2 &= \mu_2-\mu_1^2, \\ \kappa_3 &= \mu_3-3\mu_2\mu_1+2\mu_1^3, \\ \kappa_4 &= \mu_4-4\mu_3\mu_1-3\mu_2^2+12\mu_2\mu_1^2-6\mu_1^4, \end{aligned}$$

For ease of reference we state the following well-known properties of norms and moments.

Lemma 2.10. *For any nonnegative integer vector* $\nu\in\mathbb{R}^d$ *and integers* $0\leq m_1\leq m\leq m_2$

(i) $|x^\nu|\leq\|x\|^{|\nu|}, \quad x\in\mathbb{R}^d$,

(ii) $\|x\|^m\leq\|x\|^{m_1}+\|x\|^{m_2}, \quad x\in\mathbb{R}^d$.

For a random vector X *in* $\mathbb{R}^d$ *with* $\rho_s<\infty$, $|\nu|\leq s$

(iii) $|m_\nu|\leq E\,|X^\nu|\leq\rho_{|\nu|}$,

(iv) $|\kappa_\nu|\leq c\rho_{|\nu|}$, *where the constant* c *depends only on* ν.

(Cf. [BR76], Lemma 6.3 and (9.13))

3 Heuristic considerations

Consider the following problem setting. Let $X_1, \ldots, X_n : (\Omega, \mathcal{F}, P) \to \mathbb{R}^d$ be i.i.d. random vectors with $EX_1 = 0$, $Cov\,(X_1) = V$, where V is a positive-definite matrix, and $\rho_s < \infty$ for some $s \geq 3$. We are interested in the asymptotics of the distribution function F_n,

$$F_n\,(x) := P\,(S_n \leq x)\,, \quad x \in \mathbb{R}^d,$$

where

$$S_n = \frac{1}{\sqrt{n}}\,(X_1 + \cdots + X_n)\,.$$

By the properties of characteristic functions (see e.g. [BR76], Theorem 5.1)

$$\psi_{S_n}\,(t) = \left(\psi_{\frac{1}{\sqrt{n}}X_1}\,(t)\right)^n = \left(\psi_{X_1}\left(\frac{t}{\sqrt{n}}\right)\right)^n.$$

Then by equation (2) and taking into account that $\kappa_\nu = \mu_\nu = 0$, for $|\nu| = 1$ we have

$$\begin{aligned}
\log\psi_{S_n}\,(t) &= n\log\psi_{X_1}\left(\frac{t}{\sqrt{n}}\right) \\
&= \sum_{r=1}^{s}\sum_{|\nu|=r}\frac{\kappa_\nu}{\nu!}(it)^\nu n^{-(r-2)/2} + n\cdot o\left(\left\|\frac{t}{\sqrt{n}}\right\|^s\right) \\
&= -\frac{1}{2}\,\langle t, Vt\rangle + \sum_{r=1}^{s-2}\sum_{|\nu|=r+2}\frac{\kappa_\nu}{\nu!}(it)^\nu n^{-r/2} + n\cdot o\left(\left\|\frac{t}{\sqrt{n}}\right\|^s\right), \quad \frac{t}{\sqrt{n}} \to \infty.
\end{aligned}$$

If we now fix t we get

$$\psi_{S_n}\,(t) = e^{-\frac{1}{2}\langle t,Vt\rangle}\exp\left(\sum_{r=1}^{s-2}\sum_{|\nu|=r+2}\frac{\kappa_\nu}{\nu!}(it)^\nu n^{-r/2} + o\left(n^{-(s-2)/2}\right)\right), \quad n \to \infty. \tag{4}$$

Now define the functions $\tilde{P}_r\,(z, \{\kappa_\nu\})$ from the formal identity

$$1 + \sum_{r=1}^{+\infty}\tilde{P}_r\,(z, \{\kappa_\nu\})\,u^r = \exp\left\{\sum_{r=1}^{+\infty}\sum_{|\nu|=r+2}\frac{\kappa_{n,\nu}z^\nu}{\nu!}u^r\right\}.$$

Lemma 3.1. *The functions $\tilde{P}_r$ are polynomials of degree $3r$ with coefficients that depend only on the cumulants κ_ν of order $|\nu| \leq r + 2$. They can be calculated as*

$$\tilde{P}_r\,(z, \{\kappa_\nu\}) = \sum_{m=1}^{r}\frac{1}{m!}\sum_{j_1,\ldots,j_m}\left(\sum_{\nu_1,\ldots,\nu_m}\frac{\kappa_{\nu_1}\cdots\kappa_{\nu_m}}{\nu_1!\cdots\nu_m!}z^{\nu_1+\cdots+\nu_m}\right), \quad r \geq 1,\ z \in \mathbb{R}^d,$$

where $\sum_{j_1,\dots,j_m}$ is the summation over all m-tuples of positive integers $(j_1,\dots,j_m)$ satisfying $\sum_{i=1}^{m} j_i = r$, and $\sum_{\nu_1,\dots,\nu_m}$ is the summation over all m-tuples of nonnegative integral vectors $(\nu_1,\dots,\nu_m)$ s.t. $|\nu_i| = j_i + 2$, $i = 1,\dots,m$.

(See [BR76], Chapter 7).

Example 3.2. The first two polynomials are

$$\tilde{P}_1(z,\{\kappa_\nu\}) = \sum_{|\nu|=3} \frac{\kappa_\nu}{\nu!} z^\nu,$$
$$\tilde{P}_2(z,\{\kappa_\nu\}) = \sum_{|\nu|=4} \frac{\kappa_\nu}{\nu!} z^\nu + \frac{1}{2!} \sum_{|\nu_1|=|\nu_2|=3} \frac{\kappa_{\nu_1}\kappa_{\nu_2}}{\nu_1!\nu_2!} z^{\nu_1+\nu_2}.$$

We can now write (4) as

$$\psi_{S_n}(t) = e^{-\frac{1}{2}\langle t,Vt\rangle} + \sum_{r=1}^{s-2} n^{-r/2} \tilde{P}_r(it,\{\kappa_\nu\}) e^{-\frac{1}{2}\langle t,Vt\rangle} + o\left(n^{-(s-2)/2}\right), \quad n \to \infty, \tag{5}$$

and we have an expansion of the characteristic function of a sum of i.i.d. random vectors in terms of $\tilde{P}_r(it,\{\kappa_\nu\}) e^{-\frac{1}{2}\langle t,Vt\rangle}$. The question now is, how can we use (5) to get the corresponding asymptotics for the distribution function?

From example 2.3 we know that the first term $e^{-\frac{1}{2}\langle t,Vt\rangle}$ is the characteristic function of the normal distribution $\Phi_{0,V}$. Now define $P_r(-\phi_{0,V},\{\kappa_\nu\})$ as the function whose Fourier transform is $\tilde{P}_r(it,\{\kappa_\nu\}) e^{-\frac{1}{2}\langle t,Vt\rangle}$, i.e.

$$\int_{\mathbb{R}^d} e^{i\langle t,x\rangle} P_r(-\phi_{0,V},\{\kappa_\nu\})(x)\,dx = \tilde{P}_r(it,\{\kappa_\nu\}) e^{-\frac{1}{2}\langle t,Vt\rangle}. \tag{6}$$

By taking the ν-th derivative with respect to x on both sides of the Fourier inversion formula in Theorem 2.4 *i)* we get

$$\int_{\mathbb{R}^d} e^{i\langle t,x\rangle} D^\nu \phi_{0,V}(x)\,dx = (-it)^\nu \psi_{\Phi_{0,V}}(t) = (-it)^\nu e^{-\frac{1}{2}\langle t,Vt\rangle}.$$

Therefore, (6) will be satisfied if we set $P_r(-\phi_{0,V},\{\kappa_\nu\})$ equal to $\tilde{P}_r(it,\{\kappa_\nu\})$ after substituting $(-1)^{|\nu|} D^\nu \phi_{0,V}$ for each power $(it)^\nu$, i.e.

$$P_r(-\phi_{0,V},\{\kappa_\nu\}) = \tilde{P}_r(-D,\{\kappa_\nu\})\,\phi_{0,V}.$$

As a result, using Lemma 3.1 we get

Lemma 3.3. *$P_r(-\phi_{0,V},\{\kappa_\nu\})$ is a polynomial multiple of $\phi_{0,V}$ and can be written as*

$$P_r(-\phi_{0,V},\{\kappa_\nu\}) = \sum_{m=1}^{r} \frac{1}{m!} \sum_{j_1,\dots,j_m} \left(\sum_{\nu_1,\dots,\nu_m} \frac{\kappa_{\nu_1}\cdots\kappa_{\nu_m}}{\nu_1!\cdots\nu_m!} (-1)^{r+2m} D^{\nu_1+\dots+\nu_m} \phi_{0,V} \right),$$

where the summation is as in Lemma 3.1.

Example 3.4. If $d = 1$, then $V = \sigma^2$ and

$$P_1\left(-\phi_{0,\sigma^2}, \{\kappa_\nu\}\right)(x) = \frac{\kappa_3}{6\sigma^3}\left(\frac{x^3}{\sigma^3} - \frac{3x}{\sigma}\right)\phi_{0,\sigma^2}(x).$$

Finally, define $P_r\left(-\Phi_{0,V_n}, \kappa_{n,\nu}\right)$ as the finite signed measure on $\mathbb{R}^d$ whose density is $P_r\left(-\phi_{0,V_n}, \kappa_{n,\nu}\right)$. By the Lebesgue dominated convergence theorem we have

$$P_r\left(-\Phi_{0,V}, \{\kappa_\nu\}\right) = \tilde{P}_r\left(-D, \{\kappa_\nu\}\right)\Phi_{0,V}.$$

For a detailed discussion see [BR76], Chapter 7.

We can now state the desired expansion for the distribution function, known as the *Edgeworth expansion*

$$F_n(x) = \Phi_{0,V}(x) + \sum_{r=1}^{s-2} n^{-r/2} P_r\left(-\Phi_{0,V}, \{\kappa_\nu\}\right)(x) + o\left(n^{-(s-2)/2}\right), \quad n \to \infty. \tag{7}$$

For a more detailed argumentation see e.g. [BR76], Theorem 20.1 or [GK54], Chapter 45 for the one-dimensional case. Note that expansion (7) is only valid under an additional assumption on the distribution of the vectors X_i, $i = 1, \dots, n$, the so-called *Cramér condition*

$$\lim \sup_{\|t\| \to \infty} \left|\psi_{X_1}(t)\right| < 1, \tag{8}$$

which is satisfied, for example, for absolutely continuous distributions. However, for lattice distributions (8) does not hold, and, therefore, the Edgeworth expansion in the form (7) is not valid, and additional terms are required.

4 Edgeworth expansions for lattice triangular arrays

We now turn to lattice random vectors in more detail. First we present the main definitions and properties of lattice distributions. We then discuss the exact form of the Edgeworth expansion for lattice triangular arrays.

4.1 Lattice random vectors

Definition 4.1. The discrete subgroup L of $\mathbb{R}^d$ is called a *lattice* if there exist linearly independent vectors $h_1, \dots, h_d$ in L such that

$$L = \left\{ m_1 h_1 + \dots + m_d h_d \middle| \; m_i \in \mathbb{Z}, \; i = 1, \dots, d \right\}.$$

The set of vectors $\{h_1, \dots, h_d\}$ is the *basis* of L. The *volume* of the lattice is defined as

$$D(L) = \left|Det\left(h_1, \dots, h_d\right)\right|.$$

The volume is independent of the choice of the basis and is uniquely determined for each lattice.

Definition 4.2. Let $(\Omega, \mathcal{F}, P)$ be a probability space. The distribution of a random vector X on $\mathbb{R}^d$ is said to be a *lattice distribution* if there exists a $x_0 \in \mathbb{R}^d$ and a lattice L such that

$$P\left(X \in x_0 + L\right) = 1. \tag{9}$$

It is clear that one can find various vectors x_0 and lattices such that (9) holds. Just consider the possible representations for the values $0, 1$ of the Bernoulli distribution. Therefore, lattice distributions are often characterized in terms of the unique minimal lattice.

Definition 4.3. The lattice L is called the *minimal lattice* of X in $\mathbb{R}^d$ if L satisfies (9) with some $x_0 \in \mathbb{R}^d$, and for every sublattice $L' \subset L$

$$P\left(X \in y_0 + L'\right) < 1, \quad \forall y_0 \in \mathbb{R}^d.$$

Note that the minimal lattice has the maximal volume $D\left(L\right)$ out of all lattices satisfying (9).

Definition 4.4. A random vector X is called *degenerate* if there exists a hyperplane $H = \{x : \langle a, x\rangle = c\}$, $a \in \mathbb{R}^d$, $c \in R$, such that $P\left(X \in H\right) = 1$.

From now on we assume that X is a nondegenerate lattice random vector, and focus on the properties of the characteristic function of X

$$\psi_X(t) = \sum_{\alpha \in L} P\left(X = x_0 + \alpha\right) e^{i\langle t, x_0 + \alpha\rangle}.$$

Note that $|\psi_X|$ is a periodic function, therefore, the Cramér condition (8) is indeed not satisfied. Consider the set L^* of periods of $|\psi_X|$. Let $\{h_1, \ldots, h_d\}$ be a basis of L and $\{\hat{h}_1, \ldots, \hat{h}_d\}$ its dual basis, i.e.

$$\left\langle h_i, \hat{h}_j \right\rangle = \delta_{i,j}, \quad i, j = 1, \ldots, d,$$

where $\delta_{i,j}$ is Kronecker's delta. Then L^* is the lattice defined as

$$L^* = \{m_1 2\pi \hat{h}_1 + \cdots + m_d 2\pi \hat{h}_d |\ m_i \in \mathbb{Z},\ i = 1, \ldots, d\}.$$

By the properties of a dual basis the volume of L^* is given by

$$D\left(L^*\right) = \frac{(2\pi)^d}{D\left(L\right)}.$$

Note that L^* can also be characterized in the following way (see e.g. [BR76], Lemma 21.6)

$$L^* = \left\{t : \left|\psi_X\left(t\right)\right| = 1\right\}.$$

We now introduce the *fundamental domain* $\mathcal{F}^*$ of L^*

$$\mathcal{F}^* = \{t_1\hat{h}_1 + \cdots + t_k\hat{h}_d \mid |t_i| < \pi \; \forall j\}.$$

$$vol\mathcal{F}^* = D\left(L^*\right).$$

The fundamental domain allows to partition the space $\mathbb{R}^d$ in the following way

$$\mathbb{R}^d = \bigcup_{\hat{\alpha}\in L^*} Cl\left(\mathcal{F}^* + \hat{\alpha}\right). \tag{10}$$

In addition $\mathcal{F}^* \cap L^* = \{0\}$, i.e. $\mathcal{F}^*$ doesn't contain any periods of $|\psi_X|$, other than 0, and, therefore,

$$|\psi_X(t)| < 1, \quad t \in \mathcal{F}^*, t \neq 0.$$

This is an important property that we will often make use of in the proofs below. Finally, we state the Fourier inversion formula for lattice random vectors.

Theorem 4.5 (Fourier inversion formula for lattice distributions)**.** *Let X be a nondegenerate lattice random vector in $\mathbb{R}^d$ with lattice L and $x_0 \in \mathbb{R}^d$, such that $P(X \in x_0 + L) = 1$. Then*

$$P(X = x_0 + \alpha) = \frac{D(L)}{(2\pi)^d} \int_{\mathcal{F}^*} e^{-i\langle t, x_0+\alpha\rangle} \psi_X(t)\, dt,$$

and

$$(x_0 + \alpha)^\nu P(X = x_0 + \alpha) = \frac{D(L)}{(2\pi)^d} (-i)^{|\nu|} \int_{\mathcal{F}^*} e^{-i\langle t, x_0+\alpha\rangle} D^\nu \psi_X(t)\, dt, \quad \alpha \in L.$$

For details see [BR76], Chapter 5.21.

4.2 Local expansions

We now derive Edgeworth expansions for the point masses of a sum of triangular array lattice random vectors. These results are used in section 4.3 to obtain expansions for the distribution function. Supplementary lemmas, that are used in the proofs below are presented in section 4.4.

Consider a triangular array of lattice random vectors $X_{n,1}, X_{n,2}, \ldots, X_{n,n}$ defined on the probability spaces $(\Omega_n, \mathcal{F}_n, P_n)$, with common minimal lattice $\mathbb{Z}^d$ s.t.

$$\begin{gathered} E\left(X_{n,1}\right) = \mu_n, \quad Cov\left(X_{n,1}\right) = V_n, \quad P\left(X_{n,1} \in \mathbb{Z}^d\right) = 1 \quad \text{and} \\ \rho_{n,s+1} = E\left\|X_{n,1} - \mu_n\right\|^{s+1} = O(1), \quad \text{for some integer } s \geq 2, \end{gathered} \tag{11}$$

where the sequence of positive-definite covariance matrices $\{V_n\}$ converges to a positive-definite limit matrix V. For each $n \in \mathbb{N}$ let S_n be the normalized sum

$$S_n = \frac{X_{n,1} + \cdots + X_{n,n} - n\mu_n}{\sqrt{n}}.$$

Note that S_n is also a lattice random vector with minimal lattice $L = n^{-1/2}\mathbb{Z}^d$ and

$$P\left(S_n \in -\sqrt{n}\mu_n + L\right) = 1.$$

Following the notation in [BR76] we define the values attained by S_n as $x_{\alpha,n} := \frac{1}{\sqrt{n}}\left(\alpha - n\mu_n\right),\ n \in \mathbb{N},\ \alpha \in \mathbb{Z}^d$. Set

$$p_n\left(x_{\alpha,n}\right) = P\left(X_{n,1} + \cdots + X_{n,n} = \alpha\right) = P\left(S_n = x_{\alpha,n}\right),$$
$$q_{n,s} = n^{-d/2}\sum_{r=0}^{s-2} n^{-r/2} P_r\left(-\phi_{0,V_n}, \{\kappa_{n,\nu}\}\right),$$

where $\kappa_{n,\nu}$ is the $\nu - th$ cumulant of $X_{n,1}$.

Theorem 4.6. *Let $E(C) = \left\{t \in \mathbb{R}^d : \|t\| \leq C\right\}$, and $\mathcal{F}^*$ be a fundamental domain of $\left(\mathbb{Z}^d\right)^*$. Under conditions (11), if for all constants $C > 0$, s.t. $\mathcal{F}^* \setminus E(C)$ is non-empty, the characteristic functions $\psi_{X_{n,1}}$ satisfy the condition*

$$N_C := \sup\left\{\left|\psi_{X_{n,1}}(t)\right| : t \in \mathcal{F}^* \setminus E(C),\ n \in \mathbb{N}\right\} < 1, \tag{12}$$

then

$$\sup_{\alpha\in\mathbb{Z}^d}\left(1 + \|x_{\alpha,n}\|^s\right)\left|p_n\left(x_{\alpha,n}\right) - q_{n,s}\left(x_{\alpha,n}\right)\right| = O\left(n^{-(d+s-1)/2}\right), \quad n \to \infty, \tag{13}$$

and

$$\sum_{\alpha\in\mathbb{Z}^d}\left|p_n\left(x_{\alpha,n}\right) - q_{n,s}\left(x_{\alpha,n}\right)\right| = O\left(n^{-(s-1)/2}\right), \quad n \to \infty. \tag{14}$$

Proof of Theorem 4.6. We will follow the proof of Theorem 22.1 in [BR76].

For each $n \in \mathbb{N}$ set $Y_{i,n} := X_{i,n} - \mu_n,\ i = 1, \ldots, n$. Then $\psi_{Y_{1,n}}(t) = \psi_{X_{1,n}}(t)\, e^{-i\langle t,\mu_n\rangle}$ and the characteristic function of S_n can be determined as

$$\psi_{S_n}(t) = \left(\psi_{Y_{1,n}}\left(\frac{t}{\sqrt{n}}\right)\right)^n.$$

Applying the inversion formulas in Theorem 4.5 we get

$$p_n\left(x_{\alpha,n}\right) = \frac{1}{(2\pi)^d n^{d/2}}\int_{\sqrt{n}\mathcal{F}^*}\psi_{S_n}(t)\, e^{-i\langle t,x_{\alpha,n}\rangle}dt,$$

and

$$x_{\alpha,n}^{\beta} p_n\left(x_{\alpha,n}\right) = \frac{1}{(2\pi)^d n^{d/2}}(-1)^{|\beta|}\int_{\sqrt{n}\mathcal{F}^*} e^{-i\langle t,x_{\alpha,n}\rangle} D^{\beta}\psi_{S_n}(t)\, dt, \tag{15}$$

where β is a non-negative integer vector with $|\beta| \leq s$. By Theorem 2.4 and the definition of the functions P_r (6) we have

$$x_{\alpha,n}^{\beta} q_{n,s}\left(x_{\alpha,n}\right) = \frac{1}{(2\pi)^d n^{d/2}}(-1)^{|\beta|}\int_{\mathbb{R}^d} e^{-i\langle t,x_{\alpha,n}\rangle} D^{\beta}\left(\sum_{r=0}^{s-2} n^{-r/2}\tilde{P}_r\left(it, \{\kappa_{n,\nu}\}\right) e^{-\frac{1}{2}\langle t,V_n t\rangle}\right) dt. \tag{16}$$

Since $\rho_{n,s+1} = O(1)$, by Lemma 4.12 (ii), there exist a constant $\hat{c}_1$ and $n_0 \in \mathbb{N}$, such that

$$c_1 \Lambda_n^{-1/2} \lambda_n^{s/(2(s-2))} \rho_{n,s}^{-1/(s-2)} \geq \hat{c}_1, \quad \forall n \geq n_0.$$

Set

$$E(\hat{c}_1) = \left\{ t \in \mathbb{R}^k : \|t\| \leq \hat{c}_1 \right\}.$$

Note that for all $t \in \sqrt{n} E(\hat{c}_1)$ the assumptions of Lemma 4.11 hold for all $n \geq n_0$.

Now consider the difference $\left| x_{\alpha,n}^{\beta} \left(p_n(x_{\alpha,n}) - q_{n,s}(x_{\alpha,n}) \right) \right|$. Due to equations (15) and (16) we have

$$\begin{aligned}
\left| x_{\alpha,n}^{\beta} \left(p_n(x_{\alpha,n}) - q_{n,s}(x_{\alpha,n}) \right) \right| &= \frac{1}{(2\pi)^d n^{d/2}} \Bigg| \int_{\sqrt{n}\mathcal{F}^*} e^{-i\langle t, x_{\alpha,n} \rangle} D^{\beta} \psi_{S_n}(t)\, dt \\
&\quad - \int_{\mathbb{R}^d} e^{-i\langle t, x_{\alpha,n} \rangle} D^{\beta} \left(\sum_{r=0}^{s-2} n^{-r/2} \tilde{P}_r(it, \{\kappa_{n,\nu}\}) e^{-\frac{1}{2}\langle t, V_n t \rangle} \right) dt \Bigg| \\
&\leq \frac{1}{(2\pi)^d n^{d/2}} (I_1 + I_2 + I_3),
\end{aligned} \tag{17}$$

where

$$I_1 := \int_{\sqrt{n}E(\hat{c}_1)} \left| D^{\beta} \left[\psi_{S_n}(t) - e^{-\frac{1}{2}\langle t, V_n t \rangle} \sum_{r=0}^{s-2} n^{-r/2} \tilde{P}_r(it, \{\kappa_{n,\nu}\}) \right] \right| dt$$

$$I_2 := \int_{\sqrt{n}\mathcal{F}^* \setminus \sqrt{n}E(\hat{c}_1)} \left| D^{\beta} \psi_{S_n}(t) \right| dt$$

$$I_3 := \int_{\mathbb{R}^d \setminus \sqrt{n}E(\hat{c}_1)} \left| D^{\beta} \left[e^{-\frac{1}{2}\langle t, V_n t \rangle} \sum_{r=0}^{s-2} n^{-r/2} \tilde{P}_r(it, \{\kappa_{n,\nu}\}) \right] \right| dt$$

We will now estimate each of these integrals separately.

By Lemmas 4.11 and 4.12 (i), there exist constants $\hat{c}_2$ ans c_3, such that for all nonnegative integer vectors β, $0 \leq |\beta| \leq s$, for all $t \in \sqrt{n} E(\hat{c}_1)$

$$\begin{aligned}
&\left| D^{\beta} \left[\psi_{S_n}(t) - e^{-\frac{1}{2}\langle t, V_n t \rangle} \sum_{r=0}^{s-2} n^{-r/2} \tilde{P}_r(it, \kappa_{n,\nu}) \right] \right| \\
&\leq \hat{c}_2 n^{-(s-1)/2} \left(\|t\|^{s+1-|\beta|} + \|t\|^{3(s-1)+|\beta|} \right) e^{-\frac{c_3 \|t\|^2}{4}},
\end{aligned} \tag{18}$$

Then

$$
\begin{aligned}
I_1 &\leq \hat{c}_2 n^{-(s-1)/2} \int_{\sqrt{n}E(\hat{c}_1)} \left(\|t\|^{s+1-|\beta|} + \|t\|^{3(s-1)+|\beta|} \right) e^{-\frac{c_3\|t\|^2}{4}} dt \\
&\leq c_{11} n^{-(s-1)/2} \int_{\mathbb{R}^d} e^{-\frac{c_3\|t\|^2}{8}} dt
\end{aligned}
$$

where $c_{11} = \hat{c}_2 \cdot \sup_{t\in\mathbb{R}^d} \left\{ \left(\|t\|^{s+1-|\beta|} + \|t\|^{3(s-1)+|\beta|} \right) e^{-\frac{c_3\|t\|^2}{8}} \right\}$. Therefore,

$$
I_1 = O\left(n^{-(s-1)/2}\right), \quad n \to \infty. \tag{19}
$$

Now let us consider I_3. By Lemma 4.13

$$
\begin{aligned}
I_3 &\leq c_5 \int_{\mathbb{R}^d \setminus \sqrt{n}E(\hat{c}_1)} e^{-c_3\|t\|^2/4} dt \\
&\leq c_5 \left(\hat{c}_1\sqrt{n}\right)^{-(s-1)} \int_{\mathbb{R}^d} \|t\|^{s-1} e^{-c_3\|t\|^2/4} dt,
\end{aligned}
$$

where the last step holds due to the Chebyshev-Markov inequality (see e.g. [Bau92], Lemma 20.1). Therefore,

$$
I_3 = O\left(n^{-(s-1)/2}\right), \quad n \to \infty. \tag{20}
$$

Finally, we estimate I_2. We assume that $\sqrt{n}E\left(\hat{c}_1\right) \subseteq \sqrt{n}\mathcal{F}^*$, otherwise, no further calculations are necessary. By Assumption (12)

$$
\sup\left\{ \left|\psi_{X_{1,n}}(t)\right| : t \in \mathcal{F}^* \setminus E\left(\hat{c}_1\right),\ n \in \mathbb{N} \right\} = N_{\hat{c}_1} < 1, \tag{21}
$$

Since $\left|\psi_{Y_{n,1}}\right| = \left|\psi_{X_{n,1}}\right|$, by applying Lemma 4.14 for each $n \in \mathbb{N}$ we get

$$
\begin{aligned}
I_2 &\leq \rho_{n,|\beta|} n^{|\beta|/2} \int_{\sqrt{n}\mathcal{F}^* \setminus \sqrt{n}E(\hat{c}_1)} \left| \psi_{X_{1,n}}\left(\frac{t}{\sqrt{n}}\right) \right|^{n-|\beta|} dt \\
&= \rho_{n,|\beta|} n^{|\beta|/2} n^{d/2} \int_{\mathcal{F}^* \setminus E(\hat{c}_1)} \left|\psi_{X_{1,n}}(t)\right|^{n-|\beta|} dt \\
&\leq c_{12} N_{\hat{c}_1}^{n-|\beta|} n^{|\beta|/2+d/2}, \quad \forall n \geq n_0.
\end{aligned} \tag{22}
$$

Since $N_{\hat{c}_1} < 1$, $N_{\hat{c}_1}^{n-|\beta|}$ tends to zero faster than any power of $1/n$, as $n \to \infty$. Therefore,

$$
I_2 = O\left(n^{-(s-1)/2}\right). \tag{23}
$$

From equations (17), (19), (20) and (23) we get

$$\sup_{\alpha\in L}\left|x_{\alpha,n}^{\beta}\left(p_n\left(x_{\alpha,n}\right)-q_{n,s}\left(x_{\alpha,n}\right)\right)\right|=O\left(n^{-(d+s-1)/2}\right),\quad n\to\infty,\quad |\beta|\le s. \tag{24}$$

By Jensen's inequality

$$\|x\|^s\le d^{s/2-1}\left(|x_1|^s+\cdots+|x_d|^s\right)=d^{s/2-1}\sum_{\beta=se_i,1\le i\le d}\left|x^{\beta}\right|,$$

where e_i is the i-th unit vector. Hence, by setting $\beta=(0,\ldots,0)$, $\beta=(s,0\ldots,0),\ldots,\beta=(0,0\ldots,s)$ in (24) we get the necessary relation (13).

We now consider equation (14).

$$\begin{aligned}&\sum_{\alpha\in\mathbb{Z}^d}\left|p_n\left(x_{\alpha,n}\right)-q_{n,s}\left(x_{\alpha,n}\right)\right|\\&\le\sup_{\alpha\in\mathbb{Z}^d}\left[\left(1+\|x_{\alpha,n}\|^s\right)\left|p_n\left(x_{\alpha,n}\right)-q_{n,s}\left(x_{\alpha,n}\right)\right|\right]\sum_{\alpha\in\mathbb{Z}^d}\frac{1}{\left(1+\|x_{\alpha,n}\|^s\right)}.\end{aligned}$$

By applying (13) we get

$$\begin{aligned}n^{(s-1)/2}\sum_{\alpha\in\mathbb{Z}^d}\left|p_n\left(x_{\alpha,n}\right)-q_{n,s}\left(x_{\alpha,n}\right)\right|&\le c_{13}n^{-d/2}\sum_{\alpha\in\mathbb{Z}^d}\frac{1}{\left(1+\|x_{\alpha,n}\|^s\right)}\\&=c_{13}n^{-d/2}\sum_{\alpha\in\mathbb{Z}^d-n\mu_n}\frac{1}{\left(1+\left\|\frac{\alpha}{\sqrt{n}}\right\|^s\right)}\\&\overset{(10)}{\le}c_{13}n^{-d/2}\sup_{\|\xi\|\le\frac{1}{\sqrt{n}}vol\mathcal{F}^*}\sum_{\alpha\in\mathbb{Z}^d}\frac{1}{\left(1+\left\|\frac{\alpha}{\sqrt{n}}+\xi\right\|^s\right)}\\&\to\int_{\mathbb{R}^d}\frac{1}{1+x^s}dx,\quad n\to\infty,\end{aligned}$$

where the last integral converges if $s\ge d+1$. Therefore, in this case, (14) follows directly from (13). The proof of (14) in the general case follows that of Bhattacharya and Rao pp. 233-236, dealing with the n-dependent parameters in (11) as presented above. □

Remark 4.7. The main difference from the i.i.d. case in the theorem above is the uniform condition (12). It is necessary to ensure that $\left|\psi_{X_{1,n}}\right|^n\to 0$, $n\to\infty$ on the fundamental domain. The question now is how restrictive this condition really is, and how applicable the theorem is in practice. In general, its verification is not straightforward. However, following the idea in [KorKu07], we can state a sufficient condition for it, which holds for multidimensional and multinomial trees.

Lemma 4.8. *Let $\xi_{n,1},\ldots,\xi_{n,n}\in\mathbb{R}^d$, $n\in\mathbb{N}$ be a triangular array of lattice random vectors with a common minimal lattice L and support $S=\left\{x\in\mathbb{R}^d \;\middle|\; p_{x,n}:=P(\xi_{n,1}=x)>0\right\}$, $|S|=m$. If for each $x\in S$ there exists a constant $K_x>0$ such that*

$$p_{x,n}\geq K_x,\quad n\in\mathbb{N},$$

then for all constants $C>0$, s.t. $\mathcal{F}^*\setminus E(C)$ is non-empty

$$N_C:=\sup\left\{\left|\psi_{\xi_{n,1}}(t)\right|: t\in\mathcal{F}^*\setminus E(C),\ n\in\mathbb{N}\right\}<1, \tag{25}$$

Here $\mathcal{F}^*$ is the fundamental domain of L^* and $E(C)$ is defined as in Theorem 4.6.

Proof. Since S has a finite number of elements, define $K:=\min\{K_x, x\in S\}$. Then

$$\begin{aligned}\psi_{\xi_{n,1}}(t)=\sum_{x\in S}e^{i\langle t,x\rangle}p_{x,n}&=\sum_{x\in S}e^{i\langle t,x\rangle}(p_{x,n}-K)+\sum_{x\in S}e^{i\langle t,x\rangle}K\\&=\sum_{x\in S}e^{i\langle t,x\rangle}(p_{x,n}-K)+Km\sum_{x\in S}e^{i\langle t,x\rangle}\frac{1}{m}.\end{aligned}$$

Set

$$\psi(t):=\sum_{x\in S}e^{i\langle t,x\rangle}\frac{1}{m},$$

which is the characteristic function of a m-nomial random vector that has the same support S, but assigns an equal probability $\frac{1}{m}$ to each attainable value. Note that $\psi(t)$ is independent of n and for any constant $C>0$, $\delta(C):=|\psi(t)|<1$, for $t\in\mathcal{F}^*\setminus E(C)$. Then, since $p_{x,n}\geq K_x\geq K$, for all $x\in S$,

$$\begin{aligned}\left|\psi_{\xi_{n,1}}(t)\right|&\leq\sum_{x\in S}(p_{x,n}-K)+Km\,|\psi(t)|\,.\\&=\sum_{x\in S}p_{x,n}+Km\,(\delta(C)-1)=1-Km\,(1-\delta(C)):=\varepsilon(C)\,.\end{aligned}$$

Since $\delta(C)<1$, we have $\left|\psi_{\xi_{n,1}}(t)\right|\leq\varepsilon(C)<1$ for all $n\in\mathbb{N}$ and $t\in\mathcal{F}^*\setminus E(C)$. Therefore, we have shown (25). ☐

4.3 Expansions for distribution functions

We now follow the proof of Theorem 23.1 in [BR76] to obtain an Edgeworth expansion for the distribution function of a sum of triangular array lattice random vectors. However, we will first need some additional notation.

Consider the sequence of functions S_j, $j\geq 0$, $S_j:\mathbb{R}\to\mathbb{R}$ defined by the Fourier series

$$S_j(x)=\begin{cases}(-1)^{j/2-1}\sum_{n=1}^{\infty}\frac{2\cos(2n\pi x)}{(2n\pi)^j}, & j \text{ even },j>0\\[2ex](-1)^{(j-1)/2}\sum_{n=1}^{\infty}\frac{2\sin(2n\pi x)}{(2n\pi)^j}, & j \text{ odd},\end{cases}$$

for non-integer $x \in \mathbb{R}$. These functions are periodic with period 1. S_j is continuous for $j \geq 2$, right-continuous for $j = 1$ and $S'_{j+1}(x) = S_j(x)$, $j \geq 0$ for all non-integer values x. S_j can be determined by the j-th Bernoulli polynomial, for example

$$S_1(x) = x - \frac{1}{2},$$
$$S_2(x) = \frac{1}{2}\left(x^2 - x + \frac{1}{6}\right), \quad 0 \leq x < 1.$$

Due to periodicity, for all non-integer x

$$S_1(x) = x - \lfloor x \rfloor - \frac{1}{2}.$$

Analogously, expressions can be found for the other S_j, $j \geq 0$.

In the following let $\alpha \in \mathbb{R}^d$ be a nonnegative integer vector and $x \in \mathbb{R}^d$. Define $S_\alpha(x) : \mathbb{R}^d \to \mathbb{R}$ as

$$S_\alpha(x) = S_{\alpha_1}(x_1) \cdots S_{\alpha_k}(x_d). \tag{26}$$

For more details see [BR76], Chapter A.4.

Theorem 4.9. *Under the conditions of Theorem 4.6, the distribution function of S_n satisfies*

$$\begin{aligned}\sup_{x \in \mathbb{R}^d} \Bigg| & P(S_n \leq x) - \sum_{|\alpha| \leq s-2} n^{-|\alpha|/2} (-1)^{|\alpha|} S_\alpha\left(n\mu_n + \sqrt{n}x\right) \left(D^\alpha \Phi_{0,V_n}\right)(x) \\ & - n^{-1/2} \sum_{|\alpha| \leq s-3} n^{-|\alpha|/2} (-1)^{|\alpha|} S_\alpha\left(n\mu_n + \sqrt{n}x\right) \left(D^\alpha P_1\left(-\Phi_{0,V_n}, \{\kappa_{n,\nu}\}\right)\right)(x) \\ & - \cdots - n^{-(s-2)/2} P_{s-2}\left(-\Phi_{0,V_n}, \{\kappa_{n,\nu}\}\right)\Bigg| = O\left(n^{-(s-1)/2}\right),\end{aligned} \tag{27}$$

where the functions S_α are defined as in (26).

Proof. Let

$$P(S_n \leq x) = \sum_{\alpha \in \mathbb{Z}^d:\, x_{\alpha,n} \leq x} p_n(x_{\alpha,n})$$

and

$$Q_{n,s}(x) := \sum_{\alpha \in \mathbb{Z}^d:\, x_{\alpha,n} \leq x} q_{n,s}(x_{\alpha,n})$$

Since by equation (14)

$$\begin{aligned}\left|P(S_n \leq x) - Q_{n,s}(x)\right| &\leq \sum_{\alpha \in \mathbb{Z}^d:\, x_{\alpha,n} \leq x} \left|p_n(x_{\alpha,n}) - q_{n,s}(x_{\alpha,n})\right| \\ &\leq \sum_{\alpha \in \mathbb{Z}^d} \left|p_n(x_{\alpha,n}) - q_{n,s}(x_{\alpha,n})\right| \\ &= O\left(n^{-(s-1)/2}\right),\end{aligned}$$

it remains to show (27) with $P\left(S_n \le x\right)$ replaced by $Q_{n,s}\left(x\right)$. We will apply the generalized Euler-Maclaurin summation formula (Theorem 4.15) to prove the latter statement. In Theorem 4.15 set $r = s-1$, $h = \frac{1}{\sqrt{n}}$, $v_n = -\sqrt{n}\mu_n$ and $f_n = \sum_{r=0}^{s-2} n^{-r/2} P_r\left(-\phi_{0,V_n}, \{\kappa_{n,\nu}\}\right)$. Due to the representation of $P_r\left(-\phi_{0,V_n}, \{\kappa_{n,\nu}\}\right)$ in Lemma 3.3, f_n is a Schwartz function (see (33)) for each $n \in \mathbb{N}$.

In this setting $F_n = \sum_{j=0}^{s-2} n^{-j/2} P_j\left(-\Phi_{0,V_n}, \{\kappa_{n,\nu}\}\right)$ and

$$\begin{aligned} Q_{n,s}\left(x\right) &= \sum_{\alpha \in \mathbb{Z}^d:\, x_{\alpha,n} \le x} q_{n,s}\left(\frac{\alpha - n\mu_n}{\sqrt{n}}\right) \\ &= h^d \sum_{\alpha \in \mathbb{Z}^d:\, h\alpha + v_n \le x} f\left(h\alpha + v_n\right). \end{aligned}$$

Then by Theorem 4.15, for every $m > d/2$

$$\begin{aligned} &\left| Q_{n,s}\left(x\right) - \sum_{j(\alpha) < s-1} \left(-1\right)^{|\alpha|} h^{|\alpha|} S_\alpha\left(\frac{x-v}{h}\right) \left(D^\alpha F_n\right)(x) \right| \\ &= \left| Q_{n,s}\left(x\right) - \sum_{j(\alpha) < s-1} \left(-1\right)^{|\alpha|} n^{-|\alpha|/2} S_\alpha\left(x\sqrt{n} + \mu_n n\right) D^\alpha \left(\sum_{j=0}^{s-2} n^{-j/2} P_j\left(-\Phi_{0,V_n}, \{\kappa_{n,\nu}\}\right)\right)(x) \right| \\ &\le c\left(s-1, m, d\right) \sum_{s-1 \le |\gamma| \le d(s-1)} n^{-|\gamma|/2} \sup_{x \in \mathbb{R}^d} \left(\left(1 + \|x\|^2\right)^{m/2} \left|D^\gamma f_n\left(x\right)\right| \right) \end{aligned} \tag{28}$$

Since the right-hand side of (28) is independent of x by taking the supremum on both sides we get

$$\begin{aligned} &\sup_{x \in \mathbb{R}^d} \left| Q_{n,s}\left(x\right) - \sum_{j(\alpha) < s-1} \left(-1\right)^{|\alpha|} n^{-|\alpha|/2} S_\alpha\left(x\sqrt{n} + \mu_n n\right) \right. \\ &\qquad \left. \times (D^\alpha \sum_{j=0}^{s-2} n^{-j/2} P_j\left(-\Phi_{0,V_n} : \kappa_{n,\nu}\right))\left(x\right) \right| \\ &\qquad \le c\left(s-1, m, d\right) \sum_{s-1 \le |\gamma| \le d(s-1)} n^{-|\gamma|/2} \sup_{x \in \mathbb{R}^d} \left(\left(1 + \|x\|^2\right)^{m/2} \left|D^\gamma f_n\left(x\right)\right| \right) \\ &\qquad \le c\left(s-1, m, d\right) n^{-(s-1)/2} \sum_{s-1 \le |\gamma| \le d(s-1)} \sup_{x \in \mathbb{R}^d} \left(\left(1 + \|x\|^2\right)^{m/2} \left|D^\gamma f_n\left(x\right)\right| \right). \end{aligned} \tag{29}$$

As in the proof of Lemma 4.12, there exists a constant c_{14} s.t.

$$e^{-\frac{1}{2}\left\langle x, V_n^{-1} x\right\rangle} \le e^{-\frac{1}{2} c_{14} \|x\|^2}.$$

In addition $\kappa_{n,\nu} = O(1)$ for $|\nu| \le s$, $Det(V_n) = O(1)$ and, therefore, by Lemma 3.3

$$\sup_{x\in\mathbb{R}^d} \left(\left(1 + \|x\|^2\right)^{m/2} |D^\gamma f_n(x)| \right) = O(1), \quad n \to \infty,$$

for all γ s.t. $s - 1 \le |\gamma| \le d(s-1)$. Relation (29) now becomes

$$\sup_{x\in\mathbb{R}^d} \Bigg| Q_{n,s}(x) - \sum_{j(\alpha)<s-1} (-1)^{|\alpha|} n^{-|\alpha|/2} S_\alpha\left(x\sqrt{n} + \mu_n n\right) \\ \times D^\alpha \left(\sum_{j=0}^{s-2} n^{-j/2} P_j\left(-\Phi_{0,V_n}, \{\kappa_{n\nu}\}\right) \right)(x) \Bigg| \\ = O\left(n^{-(s-1)/2}\right).$$

Since $|\alpha| > j(\alpha)$ for all $\alpha \in \mathbb{R}^d$ we get the necessary relation (27) by omitting from the above expansion all terms of order $n^{-j/2}$, $j \ge s-1$, and keeping in mind that the involved cumulants are bounded. □

Remark 4.10. Note that the Edgeworth expansion for absolutely continuous distributions (7) is an expansion in powers of $n^{-1/2}$ in the classical sense, i.e. with constant coefficients. In the case of lattice distributions, on the other hand, we have expansion with bounded coefficients (see [DD04] or [M09]).

4.4 Supplementary results

We now present some additional lemmas and theorems used in the proofs above.

Lemma 4.11. *Let P be a probability measure on $\mathbb{R}^d$ with zero mean, positive-definite covariance matrix V, and finite s-th absolute moment ρ_s for $s \ge 3$. Then there exist two positive constants c_1, c_2, s.t. for all $t \in \mathbb{R}^d$ satisfying*

$$\|t\| \le c_1 n^{1/2} \Lambda^{-1/2} \lambda^{s/(2(s-2))} \rho_s^{-1/(s-2)},$$

where Λ and λ are the largest and smallest eigenvalue of V, one has

$$\left| D^\alpha \left[\psi_P^n\left(\frac{t}{\sqrt{n}}\right) - e^{-\frac{1}{2}\langle t, Vt\rangle} \sum_{r=0}^{s-3} n^{-r/2} \tilde{P}_r\left(it, \{\kappa_\nu\}\right) \right] \right| \\ \le \frac{c_2 \Lambda^{|\alpha|/2} \lambda^{-s/2} \rho_s}{n^{(s-2)/2}} \left[\langle t, Vt\rangle^{(s-|\alpha|)/2} + \langle t, Vt\rangle^{(3(s-2)+|\alpha|)/2} \right] e^{-\frac{1}{4}\langle t, Vt\rangle}.$$

Proof. Let B be a symmetric positive-definite matrix, such that $B^2 = V^{-1}$. Define

$$\eta_s = \int_{\mathbb{R}^d} \|Bx\|^s \, P(dx).$$

Since $\|B\| = \left\|V^{-1/2}\right\| = \lambda^{-1/2}$, where λ is the smallest eigenvalue of V, and $\|Bx\| \leq \|B\| \cdot \|x\|$, we have

$$\eta_s \leq \lambda^{-s/2} \rho_s.$$

The statement of the lemma follows from [BR76], Theorem 9.10 and the remark on p. 83. The matrix norms above are the induced euclidean norms. □

Lemma 4.12. *Let $\{V_n\}$ be a sequence of matrices that converges to a positive-definite matrix V. Then the following statements hold.*

(i) There exist positive constants c_3, c_4 and $n_0 \in \mathbb{N}$, such that

$$c_3 \|t\|^2 \leq \langle t, V_n t \rangle \leq c_4 \|t\|^2, \quad \forall n \geq n_0, \forall t \in \mathbb{R}^k.$$

(ii) If Λ_n and λ_n are the smallest and largest eigenvalues of V_n, then

$$\Lambda_n, \lambda_n^{-1} = O(1), \quad n \to \infty.$$

Proof. (i) Since V is symmetric,

$$\lambda \|t\|^2 \leq \langle t, Vt \rangle \leq \Lambda \|t\|^2, \quad \forall t \in \mathbb{R}^d,$$

where $\lambda > 0$ and $\Lambda > 0$ are the smallest and largest eigenvalue of V.

$$\begin{aligned} \langle t, V_n t \rangle &= \langle t, V_n t \rangle - \langle t, Vt \rangle + \langle t, Vt \rangle \\ &= \sum_{i,j=1}^{k} t_i t_j \left(v_{ij}^{(n)} - v_{ij} \right) + \langle t, Vt \rangle \end{aligned} \tag{30}$$

Since V_n converges to V, $\forall \varepsilon > 0 \ \exists n_\varepsilon$, s.t. $\forall n \geq n_\varepsilon$

$$-\varepsilon \leq v_{ij}^{(n)} - v_{ij} \leq \varepsilon,$$

and from (30) we have

$$(-\varepsilon + \lambda) \|t\|^2 \leq \langle t, V_n t \rangle \leq (\varepsilon + \Lambda) \|t\|^2.$$

(ii) Since $\|V_n\| = \Lambda_n$ and $\|V_n^{-1}\| = \frac{1}{\lambda_n}$, the sequences $\{\Lambda_n\}$ and $\left\{\frac{1}{\lambda_n}\right\}$ converge to Λ and $\frac{1}{\lambda}$, respectively.

□

Lemma 4.13. *Under the conditions (11), for any nonnegative integral vector β, $0 \leq |\beta| \leq s$, there exist a constant c_5 and $n_0 \in \mathbb{N}$, such that*

$$\left| D^\beta \left[e^{-\frac{1}{2}\langle t, V_n t \rangle} \sum_{r=0}^{s-2} n^{-r/2} \tilde{P}_r \left(it, \{\kappa_{n,\nu}\} \right) \right] \right| \leq c_5 e^{-c_3 \|t\|^2/4}, \quad \forall n \geq n_0,$$

where c_3 is as in Lemma 4.12 (i).

Proof.

$$\left| D^{\beta}\left[e^{-\frac{1}{2}\langle t,V_n t\rangle} \sum_{r=0}^{s-2} n^{-r/2} \tilde{P}_r\left(it,\{\kappa_{n,\nu}\}\right)\right]\right| \le \sum_{r=0}^{s-2} n^{-r/2} \left| D^{\beta}\left[\tilde{P}_r\left(it,\{\kappa_{n,\nu}\}\right) e^{-\frac{1}{2}\langle t,V_n t\rangle}\right]\right|. \quad (31)$$

By applying the product rule, the derivatives in the above equation can be represented as

$$D^{\beta}\left[\tilde{P}_r\left(it,\{\kappa_{n,\nu}\}\right) e^{-\frac{1}{2}\langle t,V_n t\rangle}\right] = \sum_{0\le\alpha\le\beta} c(\alpha)\, D^{\alpha}\tilde{P}_r\left(it,\{\kappa_{n,\nu}\}\right) D^{\beta-\alpha} e^{-\frac{1}{2}\langle t,V_n t\rangle}$$

By Lemmas 2.10 $(i),(ii)$ and 4.12 (i) we get

$$\left| D^{\beta-\alpha} e^{-\frac{1}{2}\langle t,V_n t\rangle}\right| \le c_6\left(1+\|t\|^{|\beta-\alpha|}\right) e^{-c_3\|t\|^2/2}.$$

By Lemma 3.1, $\tilde{P}_r\left(it,\{\kappa_{n,\nu}\}\right)$ is a polynomial of degree $3r$. Since $\rho_{n,s+1}\in O(1)$, by Lemma 9.5 in [BR76] for $0\le|\alpha|\le 3r$, $r=0,\dots,s-2$

$$\begin{aligned}\left| D^{\alpha}\tilde{P}_r\left(it,\{\kappa_{n,\nu}\}\right)\right| &\le c_7\left(1+\rho_{n,2}^{r(s-3)/(s-2)}\right)\left(1+\|t\|^{3r-|\alpha|}\right)\rho_{n,s}^{r/(s-2)}\\ &\le c_8\left(1+\|t\|^{3r-|\alpha|}\right),\end{aligned}$$

starting from some $n\in\mathbb{N}$. Therefore,

$$\begin{aligned}\left| D^{\beta}\left[e^{-\frac{1}{2}\langle t,V_n t\rangle} \sum_{r=0}^{s-2} n^{-r/2}\tilde{P}_r\left(it,\{\kappa_{n,\nu}\}\right)\right]\right| &\le c_9 e^{-c_3\|t\|^2/2}\sum_{r=0}^{s-2} n^{-r/2}\\ &\quad\times\sum_{0\le\alpha\le\min\{\beta,3r\}}\left(1+\|t\|^{|\beta-\alpha|}+\|t\|^{3r-|\alpha|}+\|t\|^{3r-|\alpha|+|\beta-\alpha|}\right)\\ &\le c_{10}e^{-c_3\|t\|^2/4},\quad \forall n\ge n_0,\end{aligned}$$

where the last inequality holds, since any polynomial multiplied by $e^{-c_3\|t\|^2/4}$ is bounded. □

Lemma 4.14. *Let $X_1,\dots,X_n\in\mathbb{R}^d$ be i.i.d. random vectors with zero mean and finite s-th moment ρ_s, and $S_n=\frac{1}{\sqrt{n}}\sum_{i=1}^n X_i$. Then for any nonnegative integral vector $\beta\in\mathbb{R}^d$, $|\beta|\le s$*

$$\left|D^{\beta}\psi_{S_n}(t)\right| \le \rho_{|\beta|} n^{|\beta|/2}\left(\psi_{X_1}\left(\frac{t}{\sqrt{n}}\right)\right)^{n-|\beta|},\quad t\in\mathbb{R}^d.$$

Proof. By the Leibniz' formula for differentiation of a product of n functions $D^{\beta}\psi_{S_n}(t) = D^{\beta}\left(\psi_{X_1}\left(\frac{t}{\sqrt{n}}\right)\right)^n$ can be expressed as a sum of $n^{|\beta|}$ terms of the form

$$\left(\psi_{X_1}\left(\frac{t}{\sqrt{n}}\right)\right)^{n-r}\prod_{j=1}^{r} D^{\beta_j}\left(\psi_{X_1}\left(\frac{t}{\sqrt{n}}\right)\right), \quad (32)$$

where $1 \leq r \leq |\beta|$, $\beta_1, \ldots, \beta_r$ are nonnegative integer vectors s.t. $|\beta_j| \geq 1$ for $1 \leq j \leq r$, $\sum_{j=1}^{r} \beta_j = \beta$. By Theorem 2.6 and Lemma 2.10 (iii) each of the derivatives in the product above can be bounded by

$$\left| D^{\beta_j} \left(\psi_{X_1} \left(\frac{t}{\sqrt{n}} \right) \right) \right| \leq n^{-|\beta_j|/2} E \left| X_1^{\beta_j} \right| \\ \leq n^{-|\beta_j|/2} \rho_{|\beta_j|}.$$

As a result

$$\prod_{j=1}^{r} \left| D^{\beta_j} \left(\psi_{X_1} \left(\frac{t}{\sqrt{n}} \right) \right) \right| \leq n^{-(|\beta_1|+\cdots+|\beta_r|)/2} \rho_{|\beta_1|} \cdots \rho_{|\beta_r|} \\ \leq n^{-|\beta|/2} \rho_{|\beta|}^{(|\beta_1|+\cdots+|\beta_r|)/|\beta|} = n^{-|\beta|/2} \rho_{|\beta|},$$

where the last inequality holds since $\rho_s^{1/s}$ is a non-decreasing function (Lemma 6.2(ii), [BR76]). Therefore, taking into account that $\left| \psi_{X_1} \left(\frac{t}{\sqrt{n}} \right) \right| \leq 1$ and $n - r \geq n - |\beta|$, we get

$$\left| D^{\beta} \psi_{S_n}(t) \right| \leq n^{|\beta|} \left| \psi_{X_1} \left(\frac{t}{\sqrt{n}} \right) \right|^{n-r} \prod_{j=1}^{r} \left| D^{\beta_j} \left(\psi_{X_1} \left(\frac{t}{\sqrt{n}} \right) \right) \right| \\ \leq \rho_{n,|\beta|} n^{|\beta|/2} \left| \psi_{X_1} \left(\frac{t}{\sqrt{n}} \right) \right|^{n-|\beta|}.$$

□

4.4.1 The generalized Euler-Maclaurin summation formula

In the proof of the Edgeworth expansion for distribution functions we make use of the following version of the Euler-Maclaurin formula for an integral representation of multidimensional sums (see [BR76], Theorem A.4.3).

Consider the Schwartz space $\mathcal{S}$ on $\mathbb{R}^d$ of functions all of whose derivatives are rapidly decreasing, i.e $f \in \mathcal{S}$ iff f is infinitely differentiable and

$$\sup_{x \in \mathbb{R}^d} \left| x^{\beta} \left(D^{\alpha} f \right)(x) \right| < \infty \tag{33}$$

for all pairs of nonnegative integer vectors α, β.

Theorem 4.15 (Generalized Euler-Maclaurin Formula)**.** *Let $f \in \mathcal{S}$, $v \in \mathbb{R}^d$, and $h > 0$, and let r be a positive integer. Define*

$$\Lambda_r(x) = \sum_{j(\alpha)<r} (-1)^{|\alpha|} h^{|\alpha|} S_{\alpha} \left(\frac{x-v}{h} \right) \left(D^{\alpha} F \right)(x), \tag{34}$$

where F is defined as

$$F(x) = \int_{-\infty}^{x_1} \cdots \int_{-\infty}^{x_d} f(y)\, dy, \quad x \in \mathbb{R}^d,$$

and for any nonnegative integer vector $\alpha \in \mathbb{R}^d$

$$j(\alpha) = \sum_{\alpha_j \geq 2} (\alpha_j - 1), \quad j(\alpha) = 0, \text{ if } \alpha_j < 2\ \forall j.$$

For every $m > d/2$ there exists a constant $c(r, m, d)$ such that for all Borel sets A

$$\left| h^d \sum_{v+hn \in A} f(v + hn) - \int_A d\Lambda_r \right| \leq c(r, m, d) \sum_{r \leq |\gamma| \leq dr} h^{|\gamma|} \nu_m(D^\gamma f), \tag{35}$$

where ν_m is

$$\nu_m(\phi) = \sup\left\{ \left(1 + \|x\|^2\right)^{m/2} |\phi(x)| : x \in \mathbb{R}^d \right\}.$$

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