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New aspects of optimal investment in continuous time

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Preface

The work of Merton (see [1], [2]) is often viewed as the starting point of continuous-time portfolio theory. His approach is based on the idea that the investment problem can also be considered as a stochastic control problem. By applying the dynamic programming principle he was able to reduce the optimal portfolio problem to that of solving the Hamilton-Jacobi-Bellman (HJB) equation. Although the HJB equation often leads to a highly non-linear partial differential equation, there are some special cases where explicit solutions were obtained.

There is another approach -the so-called martingale method which is based on elegant results of stochastic calculus and convex optimization. It was developed by Karatzas e.a. ([3]) or Karatzas ([4]) and Cox and Huang ([5]). The martingale method to solve portfolio optimization problems involves two steps. First, we derive a closed form expression for the optimal terminal wealth. The next step is to determine a portfolio strategy which generates the optimal terminal wealth. The martingale method normally requires the completeness of the market. For an incomplete market the stochastic control approach is more appropriate.

A lot of researches are devoted to tackle optimal investment problems by the stochastic control method in more realistic frameworks. For example, Korn and Kraft ([6]) obtained the explicit solutions of portfolio problems with stochastic interest rates. The portfolio problem for Heston's stochastic volatility model was solved by Kraft ([7]). In both papers ([6],[7]), the optimal portfolio strategies are deterministic functions of time. The stochastic control method can also be applied to portfolio problems containing Markov-switching parameters. Bäuerle and Rieder ([8]) investigated an investment problem where assets' parameters depend on an external finite state Markov chain. They also obtained the explicit solutions of the portfolio optimization problems in which the drift rate of the stock is Markov modulated and the driving factors cannot be observed by the investor

([9]). The portfolio optimization problems for large investors also received attention from researchers. For instance, Busch, Korn and Seifried ([10]) considered a model in which the investment decisions and consumption of the large investor can influence the shifting intensity of the market.

The results from the continuous-time portfolio optimization theory can be applied to the asset liability management (ALM) problems where investors try to ensure that a given liability is met. One of the most popular dynamic trading strategies in ALM is the constant proportion portfolio insurance (CPPI) strategy which was first studied by Perold ([11]) and by Black and Jones ([12]). The justification of the CPPI strategy can be achieved by modifying the classical expected utility maximizing problem. The CPPI strategies are optimal for an investor who is interested in maximizing the expected utility from the difference between the terminal strategy value and a given subsistence level. In particular, the classical Merton problem implies the optimality of the original CPPI strategy ([13]). In his thesis([14]), Horsky explained the variable-multiple CPPI strategies in the framework where the interest rate or the stock volatility is stochastic. His work is based on the previous studies in continuos-time portfolio optimization of Korn and Kraft ([6]) and Kraft ([7]).

This thesis focuses on dealing with some aspects of continuous time portfolio optimization by using the stochastic control method. Chapter 1 gives a short introduction to the portfolio optimization for a small investor, which includes the Merton investment problem and the Bäuerle-Rieder investment problem. In Chapter 2, we extend the Busch-Korn-Seifried model for a large investor ([10]), and that problem is solved in Chapter 3 for two types of intensity functions. Chapter 4 introduces the CPPI strategy in the Black-Scholes framework and provides a justification for it. A similar overview can be found in the paper [15]. In Chapter 5, we extend results from Horsky's thesis ([14]) to include the Markov switching parameters. The generalization is based on the Bäuerle-Rieder investment problem ([8]). Finally, Chapter 6 applies the method used in [10] to solve the portfolio optimization with a stochastic benchmark.

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Chapter 1

Portfolio optimization for a small investor

This chapter covers optimal investment problems for a small investor who is interested in maximizing his/her utility of final wealth with respect to his/her investment strategy in a finite-time horizon. The classical Merton portfolio optimization is presented in Section 1.1, while the Bäuerle-Rieder investment problem is summarized in Section 1.2. In each section, we first describe the mathematical framework and then formulate the optimal investment problem. Next, we will derive the Hamilton-Jacobi-Bellman equation. Finally, we present the Verification Theorem and the optimal investment strategies without giving proofs. All the relevant proofs can be found in the references.

1.1 The Merton investment problem

This section presents a simple framework of the Merton investment problem. More general settings can be found in Merton's paper [2] or in the textbook [16].

1.1.1 Mathematical framework

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with a filtration $(\mathcal{F})_{t\geq 0}$ satisfying the usual conditions. We fix a time horizon $T > 0$. Let W be a standard Brownian motion on $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}, P)$. We suppose that the investor has two investment opportunities.

- The investor can invest in the money market account B which satisfies

$$dB(t) = rB(t)dt, \quad B(0) = B_0 > 0,$$

  where $r > 0$ is the constant short rate.

- The investor can invest in a risky asset with a price process given by the Black-Scholes model

$$dS(t) = S(t)(\mu dt + \sigma dW(t)), \quad S(0) = S_0 > 0,$$

  where $\mu > r$ and $\sigma > 0$.

We denote the fraction of wealth invested in the stock at time t by $\pi(t)$. The process $\pi = (\pi(t))_{t\in[0,T]}$ is called the portfolio strategy. If the strategy is self-financing, the wealth process satisfies

$$\begin{aligned} dX(t) &= \pi(t)X(t)\frac{dS(t)}{S(t)} + (1-\pi(t))X(t)\frac{dB(t)}{B(t)} \\ &= X(t)\Big[\big(r + \pi(t)(\mu - r)\big)dt + \sigma\pi(t)dW(t)\Big] \end{aligned}$$

with $X(0) = x$ being the initial wealth.

To compare different portfolio strategies, we need a criterion to judge the benefit of the final wealth in a convincing way. Often the monetary scale is not good enough to measure the usefulness of money. The following definition of a utility function is provided to serve our purposes.

Definition 1.1. *A function $U : (0,\infty) \to \mathbb{R}$ such that $U \in C^1$ is strictly concave and satisfies*

$$U'(0) := \lim_{x\to 0} U'(x) = +\infty, \quad U'(\infty) := \lim_{x\to\infty} U'(x) = 0$$

will be called a utility function.

The definition implies that $U(x)$ is an increasing function of x on $(0, \infty)$. It indicates the fact that for any investor more wealth is preferred to less wealth. In addition, the decreasing property of the marginal utility $U'(x)$ is well-suited for the investor's behaviours. Indeed, we can observe that conclusion by considering marginal utility obtained by the the acquisition of one additional dollar. For someone having no money, obtaining one dollar is very important. On the other hand, if you already have a lot of money, an additional dollar more does not change your well-being dramatically. The more money you have, the less important it is to earn one extra dollar.

Example 1.2. *In the finance literature, people intensively use the following constant relative risk aversion (CRRA) utility functions:*

- *the power utility function:* $U(x) = \frac{x^{1-\eta}}{1-\eta}, \quad 0 < \eta < 1,$
- *the logarithm utility function:* $U(x) = \log(x)$.

We denote by

$$\mathcal{A} := \left\{ \pi | \pi \text{ bounded, } \mathbb{F}\text{-progressively measurable and } \mathbb{E}(U(X^\pi(T))^-) < \infty \right\}$$

the class of admissible strategies where $U(.)$ is a utility function.

1.1.2 The optimal investment problem

Given an initial capital x and a utility function U, the investor tries to maximize the expected utility

$$\mathbb{E}[U(X^\pi(T))]$$

over the class of admissible strategies $\mathcal{A}$.

Two major approaches have been used to solve the portfolio problem in the continuous-time setting. The first approach is the stochastic control method introduced by Merton (see [1] and [2]). The portfolio problem can be understood as a stochastic control problem. This interpretation motivated him to apply standard methods of stochastic control theory to solve the portfolio problem. The continuous time portfolio problem can also be worked out by the martingale method which involves two steps. First, we derive a closed form expression for the optimal terminal wealth. The next step is to determine a portfolio strategy which generates the optimal terminal wealth. The martingale method normally requires the completeness of the market. For an incomplete market, the stochastic control approach is more appropriate.

1.1.3 Hamilton-Jacobi-Bellman equation and Verification Theorem

In this section, we introduce the basic tools of stochastic control. For this, we first define the value function and then heuristically derive a partial differential equation for it. The obtained equation is called Hamilton-Jacobi-Bellman (HJB) equation for the value function. Finally, we give a theorem to verify that the solution of the HJB equation indeed coincides with the value function.

We define the value function $J : [0, T] \times (0, \infty) \to \mathbb{R}$ as

$$J(t, x) := \sup_{\pi \in \mathcal{A}_t} \mathbb{E}[U(X^\pi(T)) | X^\pi(t) = x].$$

Here, $\mathcal{A}_t$ denotes the class of admissible strategies starting at time t. Assume that from a given (but arbitrary time) θ onwards, we already know an optimal strategy π^*. For a given $(t, x) \in [0, T] \times (0, \infty)$ and $\theta \in [t, T]$, we define a new strategy π^1 on $[0, T]$ as the following:

$$\pi^1(s) = \begin{cases} \pi(s), & \text{if } s \in [t, \theta], \\ \pi^*(s), & \text{if } s \in (\theta, T], \end{cases}$$

where π is an arbitrary admissible strategy.

Since π^* is the optimal strategy, the definition of the value function gives

$$\mathbb{E}[U(X^{\pi^*}(T)) | X^{\pi^*}(t) = x] = J(t, x).$$

With the strategy π^1 we obtain

$$\begin{aligned}\mathbb{E}[U(X^{\pi^1}(T))|X^{\pi^1}(t)=x] &= \mathbb{E}\big[\mathbb{E}[U(X^{\pi^1}(T))|X^{\pi^1}(\theta)]|X^{\pi^1}(t)=x\big] \\ &= \mathbb{E}\big[\mathbb{E}[U(X^{\pi^1}(T))|X^{\pi^1}(\theta)]|X^{\pi}(t)=x\big] \\ &= \mathbb{E}[J(\theta, X^{\pi}(\theta))|X^{\pi}(t)=x] \end{aligned} \tag{1.1}$$

for any admissible strategy π.

Since the strategy π^1 might not be optimal, we have

$$\mathbb{E}[U(X^{\pi^1}(T))|X^{\pi^1}(t)=x] \leq J(t,x), \tag{1.2}$$

where the equality holds if $\pi^1 = \pi^*$.

We now assume that J is smooth enough to apply the Ito formula, i.e. $J \in C^{1,2}([0,T) \times (0,\infty)) \cap C^1([0,T] \times (0,\infty))$ yields

$$\begin{aligned}J(\theta, X^{\pi}(\theta)) &= J(t,x) + \int_t^\theta J_t ds + \int_t^\theta J_{X^\pi} dX^\pi + \frac{1}{2}\int_t^\theta J_{X^\pi X^\pi} dX^\pi dX^\pi \\ &= J(t,x) + \int_t^\theta H(s, X^\pi(s), \pi(s)) ds \\ &\quad + \int_t^\theta J_{X^\pi}(s, X^\pi(s)) \sigma \pi(s) dW(s), \end{aligned} \tag{1.3}$$

where

$$H(s,x,\pi) = J_t + J_x x(r + \pi(\mu - r)) + \frac{1}{2} J_{xx}\sigma^2\pi^2x^2. \tag{1.4}$$

Notice that the local martingale term in (1.3) is in fact a martingale under suitable conditions. Thus, taking the expectation on both sides of the equation (1.3) conditioning on $X^\pi(t) = x$ yields

$$\mathbb{E}[J(\theta, X^\pi(\theta))|X^\pi(t)=x] = J(t,x) + \mathbb{E}\Big[\int_t^\theta H(s, X^\pi(s), \pi(s))ds|X^\pi(t)=x\Big]. \tag{1.5}$$

Combining (1.1), (1.2) and (1.5), we obtain

$$\mathbb{E}\Big[\int_t^\theta H(s, X^\pi(s), \pi)ds|X^\pi(t)=x\Big] \leq 0.$$

Finally, let $\theta \searrow t$ in the above inequality, the following holds

$$H(t, X^\pi(t), \pi(t)) \leq 0.$$

Since the above inequality becomes an equality when $\pi = \pi^*$, we obtain

$$\sup_{\pi(t)\in\mathbb{R}} H(t, X^\pi(t), \pi(t)) = 0.$$

Finally, we thus formally arrive at the following partial differential equations for the value function

$$\sup_{\pi\in\mathbb{R}} \left\{ J_t + J_x x(r + \pi(\mu - r)) + \frac{1}{2} J_{xx}\sigma^2\pi^2 x^2 \right\} = 0 \tag{1.6}$$

for $(t, x) \in [0, T) \times (0, \infty)$ subject to the boundary condition

$$J(T, x) = U(x), \;\; x \in (0, \infty). \tag{1.7}$$

The equation (1.6) with the boundary (1.7) is called the Hamilton-Jacobi-Bellman (HJB) equation.

To solve the investment problem, we first find the solution of the HJB equation. Next, we have to check that the obtained solution satisfies all conditions that we used to derive the HJB equation. When all conditions have been met, the solution of the HJB equation is indeed the optimal strategy. The next theorem states conditions when indeed the solution of the HJB equation coincides with the value function and is called a verifcation theorem.

Theorem 1.3. *(Verification Theorem)*

Suppose that $G \in C^{1,2}([0, T) \times (0, \infty)) \cap C^1([0, T] \times (0, \infty))$ *is a solution to the HJB equation* (1.6) *subject to* (1.7)*, and assume that* $G(t, x) \leq K(1 + x^k)$ *for suitable* $K > 0$ *and* $k \in \mathbb{N}$*. Then*

i) $G(t, x) \geq J(t, x)$ *for any* $(t, x) \in [0, T] \times (0, \infty)$*.*

ii) If there is an admissible strategy π^* *such that*

$$\pi^*(t) \in \arg\max_{\pi\in\mathbb{R}} H(t, x, \pi) \;\; \text{for } (t, x) \in [0, T] \times (0, \infty)$$

where H *is given by* (1.4)*, then*

$$G(t, x) = J(t, x) = J_{\pi^*}(t, x).$$

In particular, π^* is an optimal strategy.

1.1.4 Solution for CRRA investors

This section gives the optimal strategies for investors with power utility functions and logarithmic utility function.

Theorem 1.4. *(Solution of the Merton investment problem with power utilities) Given the utility function*

$$U(x) = \frac{x^{1-\eta}}{1-\eta}, \quad 0 < \eta < 1,$$

the optimal strategy π^* of the Merton investment problem is

$$\pi^*(t) = \frac{\mu - r}{\eta\sigma^2}.$$

Further the value function reads

$$J(t,x) = \frac{x^{1-\eta}}{1-\eta} \exp\{a(T-t)\},$$

where

$$a = r(1-\eta) + \frac{1}{2}\frac{1-\eta}{\eta}(\frac{\mu - r}{\sigma})^2.$$

Theorem 1.5. *(Solution of the Merton investment problem with logarithmic utility) In the case of logarithmic utility*

$$U(x) = log(x),$$

the optimal strategy π^* of the Merton investment problem is given by

$$\pi^*(t) = \frac{\mu - r}{\sigma^2}$$

and the optimal value is given by

$$J(t,x) = log(x) + \big(r + \frac{1}{2}(\frac{\mu - r}{\sigma})^2\big)(T-t).$$

1.2 The Bäuerle-Rieder investment problem

In last section, all parameters of the risk free asset and the risky asset are constant; nonetheless, they might be functions of time and other random variables. In their paper ([8]), Bäuerle and Rieder investigated an investment problem where assets' parameters depend on an external finite state Markov chain. By stochastic control methods, they obtained explicit optimal strategies for different utility functions. This section presents their model and important results from their paper.

1.2.1 Mathematical framework

Let $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}, P)$ be a filtered probability space with time horizon $T > 0$ such that the filtration $\mathbb{F}$ satisfies the usual conditions and $\mathcal{F} = F_T$. The asset prices are driven by the Brownian motion W and the continuous-time Markov chain Y. We assume Y takes values in a finite state space E and has the intensity matrix $Q = (q_{ij})_{i,j\in E}$. In addition, we suppose that Y is càdlàg.

The investor has two investment opportunities.

- The investor can invest money in a bond B whose price process evolves according to
$$dB(t) = r(Y(t))B(t)dt, \quad B(0) = B_0 > 0.$$
- The investor can invest in a risky asset with price processes are given by
$$dS(t) = S(t)(\mu(Y(t))dt + \sigma(Y(t))dW(t)), \quad S(0) = S_0 > 0.$$

Here, $\mu, r, \sigma : E \to \mathbb{R}_+$ and $\mu(i) > r(i) > 0$ for all $i \in E$.

We denote by $\pi(t)$ the fraction of wealth invested in the stock at time t. The process $\pi = (\pi(t))_{t\in[0,T]}$ is called portfolio strategy. Furthermore, we assume the portfolio strategy π is self-financing. Then the wealth process satisfies

$$\begin{aligned} dX^\pi(t) &= X^\pi(t)\pi(t)\frac{dS(t)}{S(t)} + X^\pi(t)(1-\pi(t))\frac{dB(t)}{B(t)} \\ &= X^\pi(t)\Big[\big(r(Y(t)) + \pi(t)(\mu(Y(t)) - r(Y(t)))dt + \pi(t)\sigma(Y(t))dW(t)\big)\Big] \end{aligned} \tag{1.8}$$

with $X^\pi(0) = x > 0$ being the initial wealth.

We define the set of admissible strategies over the planning period [t,T] by

$$\mathcal{A}(t,\mathbb{F}) := \left\{ \pi : [t,T] \to \mathbb{R} | \pi \text{ is progressively measurable }, \int_t^T \pi(s)^2 ds < \infty \ a.s. \right\}$$

1.2.2 The investment problem

As in the classical Merton problem, our aim is to solve the problem

$$\sup_{\pi \in \mathcal{A}[0,T]} \mathbb{E}[U(X^\pi(T)) | X^\pi(0) = x].$$

We suppose that the processes W and Y are observable. By this assumption, we can also assume that the filtration $\mathbb{F}$ is generated by W and Y, i.e. $\mathcal{F}_t = \sigma(W(s), Y(s), s \le t)$.

1.2.3 Hamilton-Jacobi-Bellman equation and Verification Theorem

We denote the conditional expectation given $X(t) = x$ and $Y(t) = i$ by $\mathbb{E}^{t,x,i}$. The optimization problem is now

$$\sup_{\pi \in \mathcal{A}[0,T]} \mathbb{E}^{0,x,i}[U(X^\pi(T))].$$

We define the value function of the investment problem over period $[t,T]$ by

$$J(t,x,i) = \sup_{\pi \in \mathcal{A}[t,T]} \mathbb{E}^{t,x,i}[U(X^\pi(T))].$$

Then the corresponding Hamilton-Jacobi-Bellman equation formally reads as

$$\begin{aligned} \sup_{u \in \mathbb{R}} \quad & \Big\{ J_t + x\big[r(i) + u(\mu(i) - r(i))\big] J_x + \frac{1}{2} x^2 u^2 \sigma^2(i) J_{xx} \\ & + \sum_{j \in E} q_{ij} [J(t,x,j) - J(t,x,i)] \Big\} = 0 \end{aligned} \tag{1.9}$$

with the boundary condition $J(T,x,i) = U(x)$. For convenience, we abbreviate the term appearing in the brackets of the HJB equation (1.9) by $\mathcal{G}^u(t,x,i)$.

Before going to the Verification Theorem, we make an assumption for admissible portfolio strategies. In particular, we require $\pi \in [-M, M]$ for some $M \in \mathbb{R}_+$. This assumption, however, does not restrict the application of the Verification Theorem in most cases we consider later.

Theorem 1.6. *([8])(Verification Theorem) Suppose $G \in C^{1,2}([0,T] \times (0,\infty))$ is a solution of the HJB-equation* (1.9) *with the boundary condition $G(T,x,i) = U(x)$ and $|G(t,x,i)| \leq K(1+|x|^k)$ for constants $K > 0$ and $k \in \mathbb{N}$ and for all $x \in \mathbb{R}_+$, $i \in E$. Then*

i) $G(t,x,i) \geq J(t,x,i)$ for all $0 \leq t \leq T, x \in \mathbb{R}_+$ and $i \in E$.

ii) If $\pi^* = (\pi^*(t))$ is a maximizer of the HJB-equation and an admissible strategy, i.e. $\pi^*(s)$ maximizes

$$u \longmapsto \mathcal{G}^u(s, X^*(s), Y(s))$$

for all $s \in [t,T]$, where X^*, π^* and Y solve (1.8) *then $G(t,x,i) = J(t,x,i) = J_{\pi^*}(t,x,i)$ for all $x \in \mathbb{R}_+, i \in E$. In particular, π^* is an optimal portfolio strategy.*

1.2.4 Solution of the investment problem

Theorem 1.7. *([8])(Solution of the Bäuerle-Rieder investment problem with power utilities) In case of power utility, the optimal portfolio strategy is given by*

$$\pi^*(t,x,i) = \frac{1}{\eta}\frac{\mu(i) - r(i)}{\sigma^2(i)}$$

and the optimal value is given by

$$J(t,x,i) = \frac{1}{1-\eta} x^{1-\eta} g(t,i)$$

where $g(t,i)$ is the unique solution of the following system of linear differential equations

$$g_t(t,i) + a(i)g(t,i) + \sum_{j \in E} q_{ij}[g(t,j) - g(t,i)] = 0$$

with boundary condition $g(T,i) = 1$ *and*

$$a(i) = r(i)(1-\eta) + \frac{1}{2}\frac{1-\eta}{\eta}\left(\frac{\mu(i)-r(i)}{\sigma(i)}\right)^2 \tag{1.10}$$

for $i \in E$.

The following lemma implies the positiveness of the value function.

Lemma 1.8. *([8]) The function* $g(t,i)$ *appearing in the value function of the previous theorem can be written as*

$$g(t,i) = \mathbb{E}^{t,i}\Big[\exp\Big\{\int_t^T a(Y(s))ds\Big\}\Big]$$

where $a(.)$ *is also defined by* (1.10).

Theorem 1.9. *([8]) In the case of logarithmic utility, the optimal portfolio strategy is given by*

$$\pi^*(t,x,i) = \frac{\mu(i)-r(i)}{\sigma^2(i)}$$

and the optimal value is given by

$$J(t,x,i) = log(x) + g(t,i)$$

where $g(t,i)$ *is the unique solution of the following system of linear differential equations*

$$g_t(t,i) + r(i) + \frac{1}{2}\left(\frac{\mu(i)-r(i)}{\sigma(i)}\right)^2 + \sum_{j\in E} q_{ij}[g(t,j)-g(t,i)] = 0$$

with boundary condition $g(T,i) = 0$ *for all* $i \in E$.

The following lemma gives a probabilistic representation for $g(.)$.

Lemma 1.10. *([8]) The function* $g(t,i)$ *appearing in the value function of Theorem 1.9 can be written as*

$$g(t,i) = \mathbb{E}^{t,i}\Big[\int_t^T r(Y(s)) + \frac{1}{2}\big(\frac{\mu(Y(s))-r(Y(s))}{\sigma(Y(s))}\big)^2 ds\Big].$$

Chapter 2

Continuous-time portfolio optimization for a large investor

This chapter extends the Busch-Korn-Seifried model for a large investor (see [10] or [17]) by using the Vasicek model for the short rate. In their model, the investment decisions and consumption of the large investor can influence the shifting intensity of the market. We first present the mathematical framework and then formulate the optimal investment problem. Finally, we introduce the Hamilton-Jacobi-Bellman equation and deliver a verification theorem. The optimal investment strategy for a large investor with power utility will be presented in the next chapter.

2.1 Mathematical framework

Let $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}, P)$ be a filtered probability space with time horizon $T > 0$ such that the filtration $\mathbb{F}$ satisfies the usual conditions and $\mathcal{F} = F_T$. We assume that the asset prices are driven by two Brownian motions W_1, W_2 and two Poisson processes $N^{0,1}, N^{1,0}$ with intensity 1. All random processes $W_1, W_2, N^{0,1}$ and $N^{1,0}$ are supposed to be independent.

Analogous to the Busch-Korn-Seifried model, the market in our framework has two states: **a normal state** (i=0) and an **alerted state** (i=1). The state of the market I is assumed to satisfy

$$dI = 1_{\{I_-=0\}}dN^{0,1} - 1_{\{I_-=1\}}dN^{1,0}, \;\; I(0) = 0.$$

We suppose further that the investor has two investment opportunities: a zero-coupon bond B and a stock S. Let the short rate r follow

$$dr(t) = a(b - r(t) - q(t)\frac{\sigma_r}{a})dt + \sigma_r dW_1(t),$$

where a, b and σ_r are positive constants, and $-q$ is the market price of risk. Then the zero-coupon bond prices with maturity T satisfy

$$dB(t,T) = B(t,T)\left((r(t) + q(t)\sigma_B(t,T))dt - \sigma_B(t,T)dW_1(t)\right), \tag{2.1}$$

where $\sigma_B(t,T) = \frac{\sigma_r}{a}(1 - e^{a(t-T)})$ (see [18]). Moreover, the stock prices follow the stochastic differential equation

$$dS(t) = S(t)\left[(r(t) + \mu(I(t)))\,dt + \sigma_S(\rho dW_1(t) + \bar{\rho}dW_2(t))\right], \;\; S(0) = S_0, \tag{2.2}$$

where $\mu(I(.))$ denotes the excess return on the stock, $\sigma_S > 0$ and $\bar{\rho} = \sqrt{1-\rho^2}$. The stock price model implies that the excess return on the stock takes two distinct values corresponding to the two states of the market. In the normal state $\mu = \mu_0$, while $\mu = \mu_1$ in the alerted state. It is reasonable to make the following assumption: $\mu_0 > \mu_1$. We also require that $\mu_1 + \sigma_B^2 - q\sigma_B + \rho\sigma_S\sigma_B \geq 0$.

Portfolio strategy and wealth process. Let π be a self-financing portfolio strategy and π^i denote the portfolio strategy when the market is in state i. Then the wealth process satisfies

$$dX^\pi(t) = \pi X^\pi(t)\frac{dS(t)}{S(t)} + (1-\pi)X^\pi(t)\frac{dB(t,T)}{B(t,T)}. \tag{2.3}$$

Let $M > 0$ be a constant. We define the class of pre-admissible strategies on the planned time interval $[t, T]$ as

$$\begin{aligned} \mathcal{A}_0[t,\mathbb{F}] \;\; = \;\; & \big\{\pi : [t,T] \to \mathbb{R} | \pi \text{ is progressively measurable}, \; -M \leq \pi \leq M, \\ & \pi^i \text{ is predictable for } i = 1,2\big\}. \end{aligned}$$

Remark 2.1. *With regard to the stochastic integration with respect to a Brownian* *motion, we only need the progressively measurable property of integrands. The predictability of π^i is necessary to define the stochastic integration with respect to the compensated Poisson process in the proof of Lemma 2.2 and Proposition 2.3 (see [17]). Later we will see that those requirements do not cause any problem for our application.*

Now we will build a new probability measure P^π which is equivalent to P such that the transition intensity of I corresponding to P^π is a function of the trading strategy π. For each $\pi \in \mathcal{A}_0[0,\mathbb{F}]$, we define

$$\frac{dP^\pi}{dP} = \prod_{i=0,1} \exp\left\{ \int_0^T [1 - \vartheta^{i,1-i}(\pi^i(t))]dt \right\} \prod_{t\in[0,T],\Delta N^{i,1-i}(t)\neq 0} \vartheta^{i,1-i}(\pi^i(t)), \quad (2.4)$$

where

$$\begin{aligned} \vartheta^{i,1-i}: \quad & \mathbb{R} \longrightarrow \mathbb{R}_0^+ \\ & \pi \longrightarrow \vartheta^{i,1-i}(\pi) \end{aligned}$$

is a given function which is deterministic and bounded on any closed subset of $\mathbb{R}$.

Lemma 2.2. *For any $\pi \in \mathcal{A}_0[0,\mathbb{F}]$, there exists a uniquely determined probability measure P^π on $\mathcal{F} = \mathcal{F}_T$ such that (2.4) is satisfied.*

Proof. The proof can be found in [17]. □

The following proposition describes the properties of W_1, W_2, $N^{0,1}$ and $N^{1,0}$ under the new probability measure.

Proposition 2.3. *Given $\pi \in \mathcal{A}_0[0,\mathbb{F}]$, for $i = 0,1$ the process $N^{i,1-i}$ is a counting process with $(\mathcal{F}, P^\pi)$-intensity $\vartheta^{i,1-i}(\pi^i)$. Moreover, W is a Wiener process and*

$$[W, N^{i,1-i}] = [N^{i,1-i}, N^{1-i,i}] = 0.$$

Proof. The proof can be found in [17]. □

In what follows, it will be very convenient to use the zero-coupon bond price as numeraire. We make some abbreviations for convenience: X for $X^{\pi}(t)$, S for $S(t)$, B for $B(t,T)$, σ_B for $\sigma_B(t,T)$, $\tilde{S}$ for $\frac{S}{B}$ and $\tilde{X}$ for $\frac{X}{B}$. An application of Ito's formula gives

$$d\tilde{S} = d(\frac{S}{B}) = \frac{dS}{B} + S(-\frac{1}{B^2}dB + \frac{1}{B^3}dBdB) + (-\frac{1}{B^2}dB + \frac{1}{B^3}dBdB)dS.$$

Using the equations (2.1) and (2.2), we obtain

$$\begin{aligned}
&\frac{dS}{B} := \tilde{S}\left[(r(t) + \mu(I(t)))dt + \sigma_S(\rho dW_1(t) + \bar{\rho} dW_2(t))\right],\\
&S(-\frac{1}{B^2}dB + \frac{1}{B^3}dBdB) = \tilde{S}\left[\left(\sigma_B^2 - r(t) - q\sigma_B\right)dt + \sigma_B dW_1\right],\\
&(-\frac{1}{B^2}dB + \frac{1}{B^3}dBdB)dS = \rho\sigma_S\sigma_B\tilde{S}dt.
\end{aligned}$$

Hence,

$$d\tilde{S} = \tilde{S}\left[\nu(t, I(t))dt + (\sigma_S\rho + \sigma_B)dW_1(t) + \sigma_S\bar{\rho}dW_2(t)\right], \tag{2.5}$$

where $\nu(t, I(t)) = \mu(I(t)) + \sigma_B^2 - q\sigma_B + \rho\sigma_S\sigma_B$.

By using Ito's product rule and the equation (2.3), we obtain

$$\begin{aligned}
dX &= d(B\tilde{X}) = Bd\tilde{X} + \tilde{X}dB + dBd\tilde{X} && (2.6)\\
&= \frac{\pi^I X}{S}d(B\tilde{S}) + \tilde{X}dB - \tilde{X}\pi^I dB. && (2.7)
\end{aligned}$$

Comparing (2.6) and (2.7) yields

$$Bd\tilde{X} + dBd\tilde{X} = \frac{\pi^I X}{S}(Bd\tilde{S} + \tilde{S}dB + d\tilde{S}dB) - \pi^I\tilde{X}dB.$$

It then follows that

$$(B + dB)d\tilde{X} = \frac{\pi^I X}{S}(Bd\tilde{S} + d\tilde{S}dB) = \frac{\pi^I X}{S}d\tilde{S}(B + dB).$$

Therefore,

$$d\tilde{X} = \pi^I\tilde{X}\frac{d\tilde{S}}{\tilde{S}}. \tag{2.8}$$

Lemma 2.4. *The discounted wealth process $\tilde{X}$ satisfies the stochastic differential equation*

$$d\tilde{X}(t) = \pi^I(t)\tilde{X}(t)\left[\nu(t, I(t)))dt + (\sigma_S\rho + \sigma_B)dW_1(t) + \sigma_S\bar{\rho}dW_2(t)\right], \quad (2.9)$$

where $\nu(t, I(t)) = \mu(I(t)) + \sigma_B^2 - q\sigma_B + \rho\sigma_S\sigma_B$. *Furthermore, if* $E\int_0^T \pi^I(t)^2 dt < \infty$, *the equation* (2.9) *has a unique solution.*

Proof. The dynamic of $\tilde{X}$ follows directly from the equations (2.5) and (2.8).

Let $A(s) = \pi^I(s)\nu(s, I(s))$, $S_1(s) = \pi^I(s)(\sigma_S\rho+\sigma_B)$, $S_2(s) = \pi^I(s)\sigma_S\bar{\rho}$ and $a(s) = \sigma_1(s) = \sigma_2(s) = 0$.

Since $E\int_0^T \pi^I(s)^2 ds < \infty$ and $\nu(s, x(s)), \sigma_B(s)$ are bounded, all coefficients $A(s)$, $S_1(s)$, $S_2(s)$, $a(s)$, $\sigma_1(s)$ and $\sigma_2(s)$ satisfy the conditions of Theorem A.3 (see Appendix A). Therefore, the equation (2.9) with initial condition $\tilde{X}(0) = \tilde{x}$ has a unique solution. □

We assume that the filtration $\mathbb{F}$ is generated by W_1, W_2 and I. In particular, for each $t \in [0, T]$, $\mathcal{F}_t = \sigma(W_1(s), W_2(s), I(s), s \leq t)$. The class of admissible portfolio strategies is defined as

$$\mathcal{A}[t, \mathbb{F}] = \left\{\pi \in \mathcal{A}_0[t, \mathbb{F}],\ \mathbb{E}_\pi[U(X^\pi(T))^-] < \infty\right\},$$

where $U(.)$ is a utility function and $\mathbb{E}_\pi$ denotes the expectation under the probability measure P^π.

2.2 Investment problem of a large investor

The large investor is interested in maximizing the expected utility of his final wealth. More specifically, the problem is

$$\sup_{\pi \in \mathcal{A}[0,\mathbb{F}]} \mathbb{E}_\pi[U(X^\pi(T))].$$

Since $X^\pi(T) = B(T,T)\tilde{X}^\pi(T) = \tilde{X}^\pi(T)$, the above problem can be written in following equivalent form

$$\sup_{\pi \in \mathcal{A}[0,\mathbb{F}]} \mathbb{E}_\pi \left[U(\tilde{X}^\pi(T)) \right].$$

2.3 HJB equation and Verification Theorem

Given $\tilde{X}(t) = \tilde{X}$ and $I(t) = i$, the optimization problem is now

$$\sup_{\pi \in \mathcal{A}(t,\mathbb{F})} \mathbb{E}_\pi^{t,\tilde{X},i} \left[U(\tilde{X}_T^\pi) \right],$$

where $\mathbb{E}_\pi^{t,\tilde{X},i}$ denotes the conditional expectation provided $\tilde{X}(t) = \tilde{X}$ and $I(t) = i$. We define the value function as

$$J(t,\tilde{X},i) = \sup_{\pi \in \mathcal{A}(t,\mathbb{F})} \mathbb{E}_\pi^{t,\tilde{X},i} \left[U(\tilde{X}_T^\pi) \right].$$

Then the corresponding HJB equation is

$$\sup_{\pi \in [-M,M]} \left\{ \begin{aligned} & J_t + J_{\tilde{X}} \nu(t,i) \pi \tilde{X} + \frac{1}{2} J_{\tilde{X}\tilde{X}} \left((\sigma_S \rho + \sigma_B)^2 + (\sigma_S \bar{\rho})^2 \right) \pi^2 \tilde{X}^2 \\ & + \vartheta^{i,1-i}(\pi) [J(t,\tilde{X},1-i) - J(t,\tilde{X},i)] \end{aligned} \right\} = 0 \tag{2.10}$$

for $(t,\tilde{X}) \in [0,T) \times \mathbb{R}_+$ subject to the boundary condition

$$J(T,\tilde{X},i) = U(\tilde{X}) \tag{2.11}$$

for all $\tilde{X} \in \mathbb{R}_+$ and $i = 0, 1$. We will denote the term in the brackets of the HJB equation (2.10) by $\mathcal{G}^\pi(t,\tilde{X},i)$ for convenience.

Remark 2.5. *The HJB equation* (2.10) *with the boundary condition* (2.11) *does not directly contain the stochastic interest rate* r*. However, it depends on the coefficients of the short rate equation.*

Theorem 2.6. *(Verification Theorem) Suppose* $G \in C^{1,2}([0,T] \times (0,\infty))$ *is a solution of the HJB-equation* (2.10) *with the boundary condition* (2.11)*, and* *assume that* $|G(t,x,i)| \le K(1+\frac{1}{x})^k$, $x \in (0,\delta)$ *and* $|G(t,x,i)| \le K(1+x)^k$, $x > \frac{1}{\delta}$ *for* $i = 0,1$ *for some constants* $K > 0, k, \delta > 0$. *Then*

i) $G(t,x,i) \ge J(t,x,i)$ *for all* $0 \le t \le T, x \in \mathbb{R}_+$ *and* $i = 0,1$.

ii) If $\pi^* = (\pi^*(t))$ *is an admissible strategy and a maximizer of the HJB-equation, i.e.* $\pi^*(s)$ *maximizes*

$$\pi \longmapsto \mathcal{G}^\pi(s, \tilde{X}^*(s), I(s))$$

for all $s \in [t,T]$, *where* $\tilde{X}^*, \pi^*$ *and* I *solve* (2.9) *then* $G(t,x,i) = J(t,x,i) = J_{\pi^*}(t,x,i)$ *for all* $x \in \mathbb{R}_+, i \in \{0,1\}$. *In particular,* π^* *is an optimal portfolio strategy.*

Proof. Let $0 \le t < T$, $I(t) = i$ and $\tilde{X}(t) = \tilde{X}$. Assume that $t < \theta \le T$ is a stopping time with respect to the filtration $\mathbb{F}$, and $\pi \in \mathcal{A}(t,\mathbb{F})$ is an arbitrary admissible portfolio strategy. Let us denote by $\tilde{N}^{i,1-i}$ the compensated Poisson process associated to $N^{i,1-i}$ under P^π.

Ito's formula and (2.9) yield

$$\begin{aligned}
& G(\theta, \tilde{X}(\theta), I(\theta)) \\
= \ & G(t, \tilde{X}, i) + \int_t^\theta \mathcal{G}^\pi(s, \tilde{X}(s), I(s))ds \\
& + \int_t^\theta G_{\tilde{X}}(s, \tilde{X}(s), I(s))\pi^I(s)\tilde{X}(s)(\sigma_S\rho + \sigma_B)dW_1(s) \\
& + \int_t^\theta G_{\tilde{X}}(s, \tilde{X}(s), I(s))\pi^I(s)\tilde{X}(s)\sigma_S\bar{\rho}dW_2(s) \\
& + \sum_{i=0}^{1} \int_t^\theta 1_{\{I(s-)=i\}} \left[G(s, \tilde{X}(s), 1-i) - G(s, \tilde{X}(s), i) \right] d\tilde{N}^{i,1-i}(s).
\end{aligned} \tag{2.12}$$

Notice that all the stochastic integrals in (2.12) are local martingales. In order to get martingale processes, we consider the following localizing trick. Set

$$O_q = \{x \in (0,\infty) : \frac{1}{q} < x < q\} \text{ and } Q_q = [0, T - \frac{1}{q}) \times O_q$$

for $q > \frac{1}{T}$. Let us denote τ_q the exit time of $\{(t, \tilde{X}(t))\}$ from Q_q and $\theta_q = \min\{\tau_q, T\}$. Then the integrands of the local martingale terms in (2.12) are bounded on $[0, \theta_q]$, $q > \frac{1}{T}$. Taking the expectation on both sides of (2.12) and noticing that G is the solution of the HJB equation, we obtain

$$\mathbb{E}_{\pi}^{t,\tilde{X},i}[G(\theta, \tilde{X}(\theta_q), I(\theta_q))] \leq G(t, \tilde{X}, i) \text{ for all } q > \frac{1}{T}. \tag{2.13}$$

It is clear that $\theta_q \to T$ as $q \to \infty$ almost surely. Due to the continuity of G and $G(T, \tilde{X}(T), 0) = G(T, \tilde{X}(T), 1) = U(X(\tilde{T}))$, the following holds

$$\lim_{q\to\infty} G(\theta_q, \tilde{X}(\theta_q), I(\theta_q)) = G(T, \tilde{X}(T), I(T)) \ a.s. \tag{2.14}$$

Based on the polynomial bounded assumptions of G, we can show that $|G(t, x, i)| \leq \bar{K}\max\{x^{-\bar{k}}, x^{\bar{k}}\}$ for $i = 0, 1$ and some constants $\bar{K}, \bar{k} > 0$. For an arbitrary $l \in \mathbb{R}$, we consider $\bar{G}^l(x, i) = \bar{K}x^l$. An application of Ito's formula gives

$$d\bar{G}^l(t, \tilde{X}(t), I(t)) = \bar{G}^l(t, \tilde{X}(t), I(t))[A(t)dt + B(t)dW_1(t) + C(t)dW_2(t)],$$

where

$$\begin{aligned} A(t) &= l\pi^I(t)\nu(t, I(t)) + l(l-1)(\pi^I(t))^2[(\sigma_S\rho + \sigma_B)^2 + (\sigma_S\bar{\rho})^2], \\ B(t) &= l\pi^I(t)(\sigma_S\rho + \sigma_B), \\ C(t) &= l\pi^I(t)\sigma_S\bar{\rho}. \end{aligned}$$

Hence ,

$$\bar{G}^l(t, \tilde{X}(t), I(t)) = \bar{K}\tilde{X}(0)^l e^{\int_0^t A(s)ds} M(t),$$

where

$$M(t) = \exp\{-\frac{1}{2}\int_0^t B^2(s)ds + \int_0^t B(s)dW_1(s) - \frac{1}{2}\int_0^t C^2(s)ds + \int_0^t C(s)dW_2(s)\}$$

is a martingale process.

Applying Holder's inequality and Doob's inequality yields

$$\begin{aligned} \mathbb{E}_{\pi}^{t,\tilde{X},i}[\sup_{t\in[0,T]} \bar{G}^l(t, \tilde{X}(t), I(t))] &\leq \mathcal{K}\mathbb{E}_{\pi}^{t,\tilde{X},i}[\sup_{t\in[0,T]} M(t)] \\ &\leq \mathcal{K}\{\mathbb{E}_{\pi}^{t,\tilde{X},i}[\sup_{t\in[0,T]} M(t)^2]\}^{\frac{1}{2}} \\ &\leq 4\mathcal{K}\{\mathbb{E}_{\pi}^{t,\tilde{X},i}[M_T^2]\}^{\frac{1}{2}} < \infty \end{aligned}$$

for a suitable constant $\mathcal{K} > 0$. Since $|G(t, x, i)| \leq \max\{\bar{G}^{-\bar{k}}(t, x, i), \bar{G}^{\bar{k}}(t, x, i)\}$ for

$i = 0, 1$, $\{G(\theta_q, \tilde{X}(\theta_q), I(\theta_q))\}_{q > \frac{1}{T}}$ is a uniformly integrable family (see Theorem 7.6 in [19]). This property together with (2.14) imply

$$\mathbb{E}_{\pi}^{t,\tilde{X},i}[G(\theta_q, \tilde{X}(\theta_q), I(\theta_q))] \to \mathbb{E}_{\pi}^{t,\tilde{X},i}[G(T, \tilde{X}(T), I(T))] \text{ as } q \to \infty. \tag{2.15}$$

The equations (2.13) and (2.15) yield

$$\mathbb{E}_{\pi}^{t,\tilde{X},i}[G(T, \tilde{X}(T), I(T))] \leq G(t, \tilde{X}, i).$$

However, due to the boundary condition $G(T, \tilde{X}(T), I(T)) = U(\tilde{X}(T))$ we obtain

$$\mathbb{E}_{\pi}^{t,\tilde{X},i}[U(\tilde{X}(T))] \leq G(t, \tilde{X}, i). \tag{2.16}$$

Taking the supremum over the set of admissible controls $\mathcal{A}(t, \mathbb{F})$ gives

$$J(t, \tilde{X}, i) \leq G(t, \tilde{X}, i). \tag{2.17}$$

If

$$\pi^*(s) \in \underset{\pi \in [-M,M]}{\arg\max} \left(\mathcal{G}^{\pi}(s, \tilde{X}^*(s), I(s)) \right)$$

for all $s \in [t, T]$, where $\tilde{X}^*(s)$ is the controlled process with respect to $\pi^*(s)$ via (2.9). Then equality in (2.13) holds; thus, equality in (2.17) holds too. Therefore, π^* is an optimal portfolio strategy. □

Chapter 3

Optimal investment for a large investor with power utility

This chapter solves the investment problem formulated in the last chapter for two different types of intensity functions. The large investor is assumed to have a power utility function

$$U(x) = \frac{x^{1-\eta}}{1-\eta}, \;\; 0 < \eta < 1.$$

We first simplify the HJB system (2.10) and (2.11) and then consider three sections. The constant intensity is studied in Section 3.1. Section 3.2 is devoted to investigate the investment problem for a step intensity function. Finally, we provide further aspects.

It is important to notice that the arguments in [10] and [17] are still valid in our framework with some minor modifications. Nonetheless, we present them here for completeness.

We make the following ansatz for the value functions

$$J(t, \tilde{X}, 0) = \frac{(\tilde{X} e^{g(t)})^{1-\eta}}{1-\eta},$$
$$J(t, \tilde{X}, 1) = \frac{(\tilde{X} e^{g(t)-h(t)})^{1-\eta}}{1-\eta}$$

for $(t, \tilde{X}) \in [0, T] \times (0, \infty)$ with C^1-functions g and h on $[0, T]$.

Remark 3.1. *In [10] and [17], they argued that h represents the difference between two states of the market. In particular, a large investor with a wealth $\tilde{X}_0^0$ when the* *market is in the normal state at time t has the same utility as another investor with a wealth $\tilde{X}_0^1 = \tilde{X}_0^0 e^{h(t)}$ if at time t the market is in the alerted state. Notice that the normal state is better than the alerted state for the large investor; thus, we expect h to be non-negative. This is indeed the content of Lemma 3.2.*

Plugging the above representations of $J(t, \tilde{X}, 0)$ and $J(t, \tilde{X}, 1)$ into the HJB equation (2.10) yields the following reduced-HJB equation

$$\sup_{\pi \in [-M,M]} \Bigg\{ g'(t) - 1_{\{i=1\}} h'(t) + \nu(t,i)\pi - \frac{1}{2}\eta \left((\sigma_S \rho + \sigma_B)^2 + (\sigma_S \bar{\rho})^2 \right) \pi^2 + \frac{\vartheta^{i,1-i}(\pi)}{1-\eta} \left(e^{(-1)^{1-i}(1-\eta)h(t)} - 1 \right) \Bigg\} = 0 \tag{3.1}$$

for $t \in [0, T)$ with the boundary conditions

$$g(T) = 0, h(T) = 0. \tag{3.2}$$

Denote

$$\pi^{i,M}(t) = \arg\max_{\pi \in \mathbb{R}} \left\{ \nu(t,i)\pi - \frac{1}{2}\eta \left((\sigma_S \rho + \sigma_B)^2 + (\sigma_S \bar{\rho})^2 \right) \pi^2 \right\}$$

for $t \in [0, T]$ and $i = 0, 1$. Then

$$\pi^{i,M}(t) = \frac{\nu(t,i)}{\eta \left((\sigma_S \rho + \sigma_B)^2 + (\sigma_S \bar{\rho})^2 \right)} \tag{3.3}$$

and the corresponding maxima are

$$\psi^i(t) = \frac{1}{2} \frac{\nu(t,i)^2}{\eta \left((\sigma_S \rho + \sigma_B)^2 + (\sigma_S \bar{\rho})^2 \right)}$$

for $t \in [0, T]$ and $i = 0, 1$. We can suppose that M is big enough such that $\pi^{i,M}(t) \in [-M, M]$ for all $t \in [0, T]$ and $i = 0, 1$.

By the previous assumptions, we obtain

$$\nu(t, 0) = \mu^0 + \sigma_B^2 - q\sigma_B + \rho\sigma_S\sigma_B > \mu^1 + \sigma_B^2 - q\sigma_B + \rho\sigma_S\sigma_B = \nu(t, 1) > 0.$$

Therefore, the following inequality holds

$$\psi^0(t) > \psi^1(t) \text{ for all } t \in [0, T].$$

Lemma 3.2. *(Non-negativity of h) If $g, h \in C^1[0,T]$ are solutions of (3.1) subject to the boundary conditions (3.2), then*

$$h(t) \geq 0$$

for every $t \in [0,T]$.

Proof. First, we prove that the equation (3.1) is also valid with $t = T$. In particular, we have to verify

$$\sup_{\pi\in[-M,M]} \left\{ g'(T) - 1_{\{i=1\}}h'(T) + \nu(T,i)\pi - \frac{1}{2}\eta\left((\sigma_S\rho + \sigma_B)^2 + (\sigma_S\bar{\rho})^2\right)\pi^2 \right\} = 0.$$

Notice that the last term in (3.1) does not appear due to $h(T) = 0$. For $t \in [0,T)$ and $i = 0, 1$ let us denote

$$\begin{aligned}\pi^{i,*}(t) = \arg\max_{\pi\in[-M,M]} \Bigg\{ & \nu(t,i)\pi - \frac{1}{2}\eta\left((\sigma_S\rho + \sigma_B)^2 + (\sigma_S\bar{\rho})^2\right)\pi^2 \\ & + \frac{\vartheta^{i,1-i}(\pi)}{1-\eta}\left(e^{(-1)^{1-i}(1-\eta)h(t)} - 1\right) \Bigg\}. \end{aligned} \tag{3.4}$$

In what follows, we will prove that

$$\pi^{i,*}(t) \to \pi^{i,M}(T) \text{ as } t \to T. \tag{3.5}$$

Let us assume

$$\pi^{i,*}(t) = \pi^{i,M}(t) + \epsilon(t) \text{ for } t \in [0,T).$$

In order to prove (3.5), we only need to show that

$$\epsilon(t) \to 0 \text{ as } t \to T.$$

For each $t \in [0,T)$ and $\pi \in [-M,M]$, (3.4) implies

$$\begin{aligned}&\nu(t,i)(\pi^{i,M}(t) + \epsilon(t)) - \frac{1}{2}\eta\left((\sigma_S\rho + \sigma_B)^2 + (\sigma_S\bar{\rho})^2\right)(\pi^{i,M}(t) + \epsilon(t))^2 \\ &+\frac{\vartheta^{i,1-i}(\pi^{i,M}(t) + \epsilon(t))}{1-\eta}\left(e^{(-1)^{1-i}(1-\eta)h(t)} - 1\right) \\ &\geq \nu(t,i)\pi - \frac{1}{2}\eta\left((\sigma_S\rho + \sigma_B)^2 + (\sigma_S\bar{\rho})^2\right)\pi^2 + \frac{\vartheta^{i,1-i}(\pi)}{1-\eta}\left(e^{(-1)^{1-i}(1-\eta)h(t)} - 1\right).\end{aligned}$$

Using (3.3), the above inequality simplifies to

$$\frac{1}{2}\eta\left((\sigma_S\rho+\sigma_B)^2+(\sigma_S\bar{\rho})^2\right)\epsilon^2(t)\leq\frac{1}{2}\frac{\nu^2(t,i)}{\eta\left((\sigma_S\rho+\sigma_B)^2+(\sigma_S\bar{\rho})^2\right)}-\nu(t,i)\pi$$
$$+\frac{1}{2}\eta\left((\sigma_S\rho+\sigma_B)^2+(\sigma_S\bar{\rho})^2\right)\pi^2+\frac{(\vartheta^{i,1-i}(\pi^{i,M}(t)+\epsilon(t))-\vartheta^{i,1-i}(\pi))}{1-\eta}$$
$$\times\left(e^{(-1)^{1-i}(1-\eta)h(t)}-1\right) \tag{3.6}$$

for all $\pi\in[-M,M]$.

By the boundedness assumption of the $\vartheta^{i,1-i}(\pi)$ on every bounded subset of $\mathbb{R}$, there exists $L>0$ such that $\frac{\vartheta^{i,1-i}(\pi)}{1-\eta}\leq L$ for all $\pi\in[-M,M]$. Hence, the inequality (3.6) implies

$$\frac{1}{2}\eta\left((\sigma_S\rho+\sigma_B)^2+(\sigma_S\bar{\rho})^2\right)\epsilon^2(t)\leq\frac{1}{2}\frac{\nu^2(t,i)}{\eta\left((\sigma_S\rho+\sigma_B)^2+(\sigma_S\bar{\rho})^2\right)}-\nu(t,i)\pi$$
$$+\frac{1}{2}\eta\left((\sigma_S\rho+\sigma_B)^2+(\sigma_S\bar{\rho})^2\right)\pi^2+2L\left|e^{(-1)^{1-i}(1-\eta)h(t)}-1\right|. \tag{3.7}$$

Choosing $\pi=\pi^{i,M}(t)=\frac{\nu(t,i)}{\eta((\sigma_S\rho+\sigma_B)^2+(\sigma_S\bar{\rho})^2)}$, we obtain

$$\frac{1}{2}\frac{\nu^2(t,i)}{\eta\left((\sigma_S\rho+\sigma_B)^2+(\sigma_S\bar{\rho})^2\right)}-\nu(t,i)\pi+\frac{1}{2}\eta\left((\sigma_S\rho+\sigma_B)^2+(\sigma_S\bar{\rho})^2\right)\pi^2=0.$$

From (3.7), we then get

$$0<\epsilon^2(t)\leq\frac{4L\left|e^{(-1)^{1-i}(1-\eta)h(t)}-1\right|}{\eta\left((\sigma_S\rho+\sigma_B)^2+(\sigma_S\bar{\rho})^2\right)}. \tag{3.8}$$

Let $t\to T$ then the right hand side of (3.8) tends to 0. Therefore,

$$\epsilon(t)\to 0 \text{ as } t\to T.$$

By the definition of the $\pi^{i,*}(t)$, we obtain

$$g'(t)-1_{\{i=1\}}h'(t)+\nu(t,i)\pi^{i,*}(t)-\frac{1}{2}\eta\left((\sigma_S\rho+\sigma_B)^2+(\sigma_S\bar{\rho})^2\right)(\pi^{i,*}(t))^2$$
$$+\frac{\vartheta^{i,1-i}(\pi^{i,*}(t))}{1-\eta}\left(e^{(-1)^{1-i}(1-\eta)h(t)}-1\right)=0. \tag{3.9}$$

When $t\to T$, the equation (3.9) becomes

$$g'(T)-1_{\{i=1\}}h'(T)+\nu(T,i)\pi^{i,M}(T)-\frac{1}{2}\eta\left((\sigma_S\rho+\sigma_B)^2+(\sigma_S\bar{\rho})^2\right)(\pi^{i,M}(T))^2=0.$$

Since $\pi^{i,M}(T)$ maximizes the left hand side of the above equation, we conclude that

$$\sup_{\pi\in[-M,M]}\left\{g'(T)-1_{\{i=1\}}h'(T)+\nu(T,i)\pi-\frac{1}{2}\eta\left((\sigma_S\rho+\sigma_B)^2+(\sigma_S\bar{\rho})^2\right)\pi^2\right\}=0. \tag{3.10}$$

If $h(t)=0$ for some $t\in[0,T]$, then equations (3.1) and (3.10) imply

$$\begin{aligned}0 &= -\sup_{\pi\in[-M,M]}\left\{g'(t)-h'(t)+\nu(t,1)\pi-\frac{1}{2}\eta\left((\sigma_S\rho+\sigma_B)^2+(\sigma_S\bar{\rho})^2\right)\pi^2\right\}\\ &\quad+\sup_{\pi\in[-M,M]}\left\{g'(t)+\nu(t,0)\pi-\frac{1}{2}\eta\left((\sigma_S\rho+\sigma_B)^2+(\sigma_S\bar{\rho})^2\right)\pi^2\right\}\\ &= h'(t)+\psi^0(t)-\psi^1(t).\end{aligned} \tag{3.11}$$

Since $\psi^0(t)>\psi^1(t)$ for all $t\in[0,T]$, the equation (3.11) implies $h'(t)<0$.

In what follows, we will use the contradiction to prove our statement. We suppose that there exists $t_0\in[0,T]$ such that $h(t_0)<0$. Let us denote

$$t_1=\inf\{t:t_0\le t\le T,h(t)=0\}.$$

Notice that the above t_1 exists as $h(T)=0$. Since h is continuous,

$$h(t)<0 \text{ for all } t_0\le t<t_1.$$

However, from the previous argument, $h'(t_1)<0$ which implies a contradiction. Therefore, we can conclude that $h(t)\ge 0$ for all $t\in[0,T]$. □

3.1 Constant intensity functions

Suppose that

$$\vartheta^{i,1-i}(\pi)=C^i \text{ with } C^i>0$$

for $i=0,1$.

Remark 3.3. *The constant intensity assumption implies that investing decisions of the large investor do not affect the transition between the two states of the market.*

In this case, the reduced HJB system (3.1) simplifies to

$$\sup_{\pi\in[-M,M]} \left\{ g'(t) - 1_{\{i=1\}}h'(t) + \nu(t,i)\pi - \frac{1}{2}\eta\left((\sigma_S\rho+\sigma_B)^2 + (\sigma_S\bar{\rho})^2\right)\pi^2 + \frac{C^i}{1-\eta}\left(e^{(-1)^{1-i}(1-\eta)h(t)} - 1\right) \right\} = 0 \tag{3.12}$$

for $t \in [0,T)$ and $i = 0,1$ subject to the boundary conditions

$$g(T) = 0, h(T) = 0.$$

For $t \in [0,T]$ and $i = 0,1$ the maximizer of the term in the brackets in (3.12) is

$$\pi^{i,*}(t) = \pi^{i,M}(t) = \frac{\nu(t,i)}{\eta\left((\sigma_S\rho+\sigma_B)^2 + (\sigma_S\bar{\rho})^2\right)}.$$

Plugging $\pi^{i,*}(t)$ into the equation (3.12) gives

$$g'(t) = -\psi^0(t) - C^0\frac{1}{1-\eta}(e^{-(1-\eta)h(t)} - 1), \tag{3.13}$$

$$\begin{aligned} h'(t) &= g'(t) + \psi^1(t) + C^1\frac{1}{1-\eta}(e^{(1-\eta)h(t)} - 1) \\ &= \psi^1(t) - \psi^0(t) - C^0\frac{1}{1-\eta}(e^{-(1-\eta)h(t)} - 1) \\ &\quad + C^1\frac{1}{1-\eta}(e^{(1-\eta)h(t)} - 1), \end{aligned} \tag{3.14}$$

subject to the boundary conditions

$$g(T) = 0, h(T) = 0. \tag{3.15}$$

Lemma 3.4. *(Uniqueness of the solution) The system of ordinary differential equations* (3.13) *and* (3.14) *subject to the boundary conditions* (3.15) *has a unique solution.*

Proof. From the equation (3.13) and the boundary condition $g(T) = 0$, we obtain

$$g(t) = \int_t^T \psi^0(s) + C^0\frac{1}{1-\eta}(e^{-(1-\eta)h(s)} - 1)ds.$$

Therefore, we only have to prove that the following ODE

$$h'(t) = \psi^1(t) - \psi^0(t) - C^0 \frac{1}{1-\eta}(e^{-(1-\eta)h(t)} - 1) + C^1 \frac{1}{1-\eta}(e^{(1-\eta)h(t)} - 1)$$

subject to the condition $h(T) = 0$ has a unique solution.

For $i = 0, 1$ we define functions $\chi^i(y) : [0, \infty) \to \mathbb{R}$ as

$$\chi^i(y) = (-1)^{1-i} C^i \frac{1}{1-\eta}(e^{(-1)^{1-i}(1-\eta)y} - 1)$$

and a function $F : [0, T] \times \mathbb{R}_0^+ \to \mathbb{R}$ as

$$F(t, y) = -(\psi^0(t) - \psi^1(t)) + \chi^0(y) + \chi^1(y).$$

Then the above ODE can be written as

$$h'(t) = F(t, h(t))$$

subject to $h(T) = 0$.

Existence of a unique local solution. Notice that $-(\psi^0(t) - \psi^1(t))$ is continuous in t. Furthermore, $\chi^i(y)$ is continuously differentiable in y for $i = 0, 1$. Thus, they are locally Lipschitz in y. The Picard-Lindelöf theorem implies the existence of a unique local solution.

Boundedness of the ODE. We observe that for all $y > 0$,

$$\chi^0(y) = -C^0 \frac{1}{1-\eta}(e^{-(1-\eta)y} - 1) > 0,$$
$$\chi^1(y) = C^1 \frac{1}{1-\eta}(e^{(1-\eta)y} - 1) > 0.$$

Hence,

$$h'(t) \geq -(\psi^0(t) - \psi^1(t)) \geq -K, \tag{3.16}$$

where $K = \max_{t\in[0,T]} (\psi^0(t) - \psi^1(t)) > 0$. Integrating (3.16) from t to T gives

$$h(t) \leq K(T - t) \text{ for every } t \in [0, T]. \tag{3.17}$$

The inequality (3.17) and the positivity of h (Lemma 3.2) indicate that h is linearly bounded in t. Therefore, we can conclude that the ODE has a unique solution. $\square$

After establishing the previous lemma, we obtain the following theorem.

Theorem 3.5. *(Optimal investment strategy) For the constant intensity functions, the optimal portfolio strategy is given as*

$$\pi^{i,*}(t) = \frac{\nu(t,i)}{\eta\left((\sigma_S\rho+\sigma_B)^2+(\sigma_S\bar{\rho})^2\right)}, \quad i=0,1$$

for all $t \in [0,T]$. The value functions are given as

$$J(t,\tilde{X},0) = \frac{(\tilde{X}e^{g(t)})^{1-\eta}}{1-\eta},$$
$$J(t,\tilde{X},1) = \frac{(\tilde{X}e^{g(t)-h(t)})^{1-\eta}}{1-\eta},$$

where $(g(t),h(t))$ is the unique solution of the system (3.13) *and* (3.14) *subject to the boundary conditions* (3.15).

Proof. The theorem is a consequence of Verification Theorem 2.6 and Lemma 3.4. □

3.2 Step intensity functions

In this section, we consider the intensity function of the following form

$$\vartheta^{i,1-i}(\pi) = C_1^i 1_{\{A^i+\pi B_\pi^i \le C^i\}} + C_2^i 1_{\{A^i+\pi B_\pi^i > C^i\}}, \quad \pi \in \mathbb{R} \tag{3.18}$$

with $A^i, B_\pi^i, C^i \in \mathbb{R}, C_j^i \in \mathbb{R}_0^+$ for $i=0,1$ and $j=1,2$ where $C_2^0 > C_1^0$ and $C_1^1 > C_2^1$.

Remark 3.6. *The parameters A^i, B^i and C^i specify a barrier for portfolio strategies of the large investor. This barrier together with π decide whether the intensity is equal to C_1^i or C_2^i . While C_1^i is advantageous for the large investor, C_2^i is not. For example, when the market is in the normal state($i=0$), the large investor is interested in staying in this state as long as possible. That means he prefers the small intensity C_1^0 to the larger intensity C_2^0. Conversely, when the market is in the alerted state ($i=1$), the large investor wants to get out of this state as soon as possible.*

It is important to mention that the step intensity function can explain reactions of the market toward investing decisions of the large investor. If B^i_π is positive, the larger the proportion of the stock in the portfolio, the bigger ($i = 0$), resp. smaller ($i = 1$), transition intensities. This indicates an interesting situation where market participations do not believe that the market will go the same way as the stock position. More specifically, they think that the market is unlikely to shift to the adverse state soon (if $i = 0$) or to stay in the alerted state longer (if $i = 1$), although the large investor increases his stock position. Similarly, the negativity of B^i_π represents the situation where the other market participations agree with the large investor's holding portfolio.

Plugging the step intensity function (3.18) into the reduced-HJB system (3.1) yields

$$\begin{aligned} &\sup_{\pi\in[-M,M]} \Big\{ g'(t) - 1_{\{i=1\}} h'(t) + \nu(t,i)\pi - \frac{1}{2}\eta\left((\sigma_S\rho+\sigma_B)^2 + (\sigma_S\bar{\rho})^2\right)\pi^2 \\ + &\frac{1}{1-\eta}(C^i_1 1_{\{A_i+\pi B^i_\pi \le C_i\}} + C^i_2 1_{\{A_i+\pi B^i_\pi > C_i\}})\big(e^{(-1)^{1-i}(1-\eta)h(t)} - 1\big)\Big\} = 0 \end{aligned} \quad (3.19)$$

for $t \in [0,T)$ with the boundary conditions

$$g(T) = 0, h(T) = 0. \quad (3.20)$$

Denote

$$\mathcal{F}^{\pi,i} = \{\pi \in \mathbb{R} | A_i + \pi B^i_\pi \le C_i\}.$$

The critical value corresponding to $A_i + \pi B^i_\pi = C_i$ is

$$\tilde{\pi}^{i,crit} = \frac{C^i - A^i}{B^i_\pi}.$$

For $i = 0, 1$ we define functions $H^{\pi,i}(t,\pi,y) : [0,T] \times [-M,M] \times \mathbb{R}^+_0 \to \mathbb{R}$ as

$$\begin{aligned} H^{\pi,i}(t,\pi,y) &= \nu(t,i)\pi - \frac{1}{2}\eta\left((\sigma_S\rho+\sigma_B)^2 + (\sigma_S\bar{\rho})^2\right)\pi^2 \\ &\quad +(C^i_1 1_{\{A_i+\pi B^i_\pi \le C_i\}} + C^i_2 1_{\{A_i+\pi B^i_\pi > C_i\}})\frac{1}{1-\eta}\big(e^{(-1)^{1-i}(1-\eta)y} - 1\big). \end{aligned}$$

Then the equations (3.19) and (3.20) can be rewritten as

$$\sup_{\pi\in[-M,M]} \Big\{ g'(t) - 1_{\{i=1\}} h'(t) + H^{\pi,i}(t,\pi,y) \Big\} = 0$$

for $t \in [0,T)$ and $i = 0,1$ with the boundary conditions $g(T) = 0, h(T) = 0$.

For $j = 1,2$ and $i = 0,1$, we define functions $H_j^{\pi,i}(t,\pi,y) : [0,T]\times[-M,M]\times\mathbb{R}_0^+ \to \mathbb{R}$ as

$$H_j^{\pi,i}(t,\pi,y) = \nu(t,i)\pi - \frac{1}{2}\eta\left((\sigma_S\rho+\sigma_B)^2 + (\sigma_S\bar{\rho})^2\right)\pi^2 + C_j^i\frac{1}{1-\eta}\left(e^{(-1)^{1-i}(1-\eta)y} - 1\right).$$

Then it is obvious that

$$H^{\pi,i}(t,\pi,y) = H_1^{\pi,i}(t,\pi,y)1_{\{\pi\in\mathcal{F}^{\pi,i}\}} + H_2^{\pi,i}(t,\pi,y)1_{\{\pi\notin\mathcal{F}^{\pi,i}\}}.$$

Notice that $H_j^{\pi,i}(t,\pi,y)$ is a quadratic function of π and takes

$$\pi^{i,M}(t) = \frac{\nu(t,i)}{\eta\left((\sigma_S\rho+\sigma_B)^2 + (\sigma_S\bar{\rho})^2\right)}$$

as its maximizer. Therefore, $\pi^{i,M}(t)$ and $\tilde{\pi}^{i,crit}$ are candidates for the maximizers of $H^{\pi,i}(t,\pi,y)$.

Lemma 3.7. *(Maximizer of $H^{\pi,i}(t,.,y)$) Given $(t,y) \in [0,T]\times\mathbb{R}_0^+$, the maximizer of*

$$H^{\pi,i}(t,\pi,y), i = 0,1,$$

is

$$\pi^{i,*}(t,y) = \begin{cases} \pi^{i,M}(t), & \text{if } y < h^{i,crit}(t), \\ \pi^{i,crit}(t), & \text{if } y \geq h^{i,crit}(t), \end{cases}$$

where

$$h^{i,crit}(t) = (-1)^{1-i}\frac{1}{1-\eta}\log\left((1-\eta)\frac{\zeta^{i,crit}(t)}{C_2^i - C_1^i} + 1\right)$$

with

$$\zeta^{i,crit}(t) = -\frac{1}{2}\left(\sqrt{\eta\left((\sigma_S\rho+\sigma_B)^2 + (\sigma_S\bar{\rho})^2\right)^2}\left(\pi^{i,M}(t) - \frac{C^i - A^i}{B_\pi^i}\right)^+\right)^2.$$

and

$$\pi^{i,crit}(t) = \frac{1}{\eta\left((\sigma_S\rho+\sigma_B)^2 + (\sigma_S\bar{\rho})^2\right)}\left[\nu(t,i) - \frac{(A^i + \pi^{i,M}(t)B_\pi^i - C^i)^+}{\frac{1}{\eta((\sigma_S\rho+\sigma_B)^2+(\sigma_S\bar{\rho})^2)}B_\pi^i}\right].$$

Proof. First, we make the following abbreviations for convenience

$$H^{\pi,i}_{1,crit}(t,y) = H^{\pi,i}_1(t,\pi^{i,crit}(t),y) \text{ and } H^{\pi,i}_{2,M}(t,y) = H^{\pi,i}_2(t,\pi^{i,M}(t),y).$$

Given $(t,y) \in [0,T] \times \mathbb{R}^+_0$, we observe that

$$H^{\pi,i}_1(t,\pi,y) > H^{\pi,i}_2(t,\pi,y) \tag{3.21}$$

for all $\pi \in [-M,M]$. The inequality (3.21) is valid based on the fact that $C^0_1 < C^0_2$ and $C^1_1 > C^1_2$.

We consider the two following cases:

Case 1: $\pi^{i,M}(t) \in \mathcal{F}^{\pi,i}$ then

$$\begin{aligned} H^{\pi,i}(t,\pi^{i,M}(t),y) &= H^{\pi,i}_1(t,\pi^{i,M}(t),y) \geq H^{\pi,i}_1(t,\pi,y) \\ &> H^{\pi,i}_2(t,\pi,y) \end{aligned}$$

for all $\pi \in [-M,M]$. Hence, $\pi^{i,*}(t,y) = \pi^{i,M}(t)$.

Case 2: $\pi^{i,M}(t) \notin \mathcal{F}^{\pi,i}$. Notice that if $H^{\pi,i}_{2,M}(t,y) > H^{\pi,i}_{1,crit}(t,y)$, then (3.21) implies $\pi^{i,M}(t)$ is the maximizer of $H^{\pi,i}(t,.,y)$. Otherwise, $\pi^{i,crit}(t)$ maximizes $H^{\pi,i}(t,.,y)$. Therefore, we have to examine the condition $H^{\pi,i}_{2,M}(t,y) \leq H^{\pi,i}_{1,crit}(t,y)$. It is equivalent to

$$\begin{aligned} & \frac{\frac{1}{2}\eta\left((\sigma_S\rho+\sigma_B)^2+(\sigma_S\bar{\rho})^2\right)\nu(t,i)^2}{\eta^2\left((\sigma_S\rho+\sigma_B)^2+(\sigma_S\bar{\rho})^2\right)^2} + C^i_2\frac{1}{1-\eta}(e^{(-1)^{1-i}(1-\eta)y}-1) \\ \leq\ & \nu(t,i)\frac{C^i-A^i}{B^i_\pi} - \frac{1}{2}\eta\left((\sigma_S\rho+\sigma_B)^2+(\sigma_S\bar{\rho})^2\right)(\frac{C^i-A^i}{B^i_\pi})^2 \\ & + C^i_1\frac{1}{1-\eta}(e^{(-1)^{1-i}(1-\eta)y}-1). \end{aligned}$$

After simplifying, we obtain

$$(C^i_2 - C^i_1)\frac{1}{1-\eta}(e^{(-1)^{1-i}(1-\eta)y}-1) \leq \zeta^{i,crit}(t), \tag{3.22}$$

where

$$\zeta^{i,crit}(t) = -\frac{1}{2}\Big(\sqrt{\eta\left((\sigma_S\rho+\sigma_B)^2+(\sigma_S\bar{\rho})^2\right)^2}\Big(\pi^{i,M}(t) - \frac{C^i-A^i}{B^i_\pi}\Big)^+\Big)^2.$$

Notice that the inequality (3.22) is equivalent to

$$y \geq (-1)^{1-i}\frac{1}{1-\eta}\log\left((1-\eta)\frac{\zeta^{i,crit}(t)}{C_2^i - C_1^i} + 1\right) = h^{i,crit}(t).$$

Thus, the following holds

$$\pi^{i,M}(t) \notin \mathcal{F}^{\pi,i} \Rightarrow \pi^{i,*}(t,y) = \begin{cases} \pi^{i,M}(t), & \text{if } y < h^{i,crit}(t), \\ \pi^{i,crit}(t), & \text{if } y \geq h^{i,crit}(t). \end{cases} \tag{3.23}$$

If $\pi^{i,M}(t) \in \mathcal{F}^{\pi,i}$, then $h^{i,crit}(t) = 0$ and $\pi^{i,crit} = \pi^{i,M}(t)$. Therefore, the formula (3.23) includes the case $\pi^{i,M}(t) \in \mathcal{F}^{\pi,i}$.

In conclusion, we obtain

$$\pi^{i,*}(t,y) = \begin{cases} \pi^{i,M}(t), & \text{if } y < h^{i,crit}(t), \\ \pi^{i,crit}(t), & \text{if } y \geq h^{i,crit}(t). \end{cases}$$

□

Remark 3.8. *Later we will replace y by h in order to find the functions h and g. Thus, y can also be explained as the difference between the two states of the market. This interpretation implies $h^{i,crit}$ is a critical barrier for the large investor to make his investing decisions.*

Inserting the maximizers $\pi^{i,*}(t,h(t))$ into the HJB system (3.19), we obtain

$$\begin{aligned} g'(t) &= -\psi^0(t) - C_2^0\frac{1}{1-\eta}(e^{(-1)^{1-i}(1-\eta)h(t)} - 1) \\ &\quad -\left[(C_1^0 - C_2^0)\frac{1}{1-\eta}(e^{-(1-\eta)h(t)} - 1) + \zeta^{0,crit}(t)\right]^+, \end{aligned} \tag{3.24}$$

$$\begin{aligned} h'(t) &= -(\psi^0(t) - \psi^1(t)) - C_2^0\frac{1}{1-\eta}(e^{-(1-\eta)h(t)} - 1) + C_2^1\frac{1}{1-\eta}(e^{(1-\eta)h(t)} - 1) \\ &\quad +\left[(C_1^1 - C_2^1)\frac{1}{1-\eta}(e^{(1-\eta)h(t)} - 1) + \zeta^{1,crit}(t)\right]^+ \\ &\quad -\left[(C_1^0 - C_2^0)\frac{1}{1-\eta}(e^{-(1-\eta)h(t)} - 1) + \zeta^{0,crit}(t)\right]^+, \end{aligned} \tag{3.25}$$

subject to the boundary conditions

$$g(T) = 0, h(T) = 0. \tag{3.26}$$

Lemma 3.9. *(Uniqueness of the solution) The system of ordinary differential equations* (3.24) *and* (3.25) *with the boundary conditions* (3.26) *has a unique solution.*

Proof. From the equation (3.24) and the boundary condition $g(T) = 0$, we obtain

$$\begin{aligned} g(t) = \int_t^T \ & \psi^0(s) + C_2^0 \frac{1}{1-\eta}(e^{(-1)^{1-i}(1-\eta)h(t)} - 1) \\ & + \big[(C_1^0 - C_2^0)\frac{1}{1-\eta}(e^{-(1-\eta)h(t)} - 1) + \zeta^{0,crit}(s)\big]^+ ds. \end{aligned}$$

Therefore, we only have to prove the following ODE

$$\begin{aligned} h'(t) &= -(\psi^0(t) - \psi^1(t)) - C_2^0 \frac{1}{1-\eta}(e^{-(1-\eta)h(t)} - 1) + C_2^1 \frac{1}{1-\eta}(e^{(1-\eta)h(t)} - 1) \\ &\quad + \big[(C_1^1 - C_2^1)\frac{1}{1-\eta}(e^{(1-\eta)h(t)} - 1) + \zeta^{1,crit}(t)\big]^+ \\ &\quad - \big[(C_1^0 - C_2^0)\frac{1}{1-\eta}(e^{-(1-\eta)h(t)} - 1) + \zeta^{0,crit}(t)\big]^+ \end{aligned}$$

subject to the boundary condition $h(T) = 0$ has a unique solution. For $i = 0, 1$ we define functions $\chi^i(t, y) : [0, T] \times \mathbb{R}_0^+ \to \mathbb{R}$ as

$$\begin{aligned} \chi^i(t, y) &= (-1)^{1-i}\Big(C^i \frac{1}{1-\eta}(e^{(-1)^{1-i}(1-\eta)y} - 1) \\ &\quad + \big[(C_1^i - C_2^i)\frac{1}{1-\eta}(e^{(-1)^{1-i}(1-\eta)y} - 1) + \zeta^{i,crit}(t)\big]^+\Big) \end{aligned}$$

and a function $F : [0, T] \times \mathbb{R}_0^+ \to \mathbb{R}$ as

$$F(t, y) = -(\psi^0(t) - \psi^1(t)) + \chi^0(t, y) + \chi^1(t, y).$$

Then the above ODE can be written as

$$h'(t) = F(t, h(t))$$

subject to $h(T) = 0$.

Existence of a unique local solution. Notice that $-(\psi^0(t) - \psi^1(t))$ and $\chi^i(t, y)$ for $i = 0, 1$ are continuous in t. Furthermore, $\chi^i(t, y)$ is composition of continuously differentiable functions of y and $[.]^+$ function for $i = 0, 1$. Thus they are locally

Lipschitz continuous in y. The Picard-Lindelöf theorem implies the existence of a unique local solution.

Boundedness of the ODE. We only need to show the non-negativity of $\chi^i(t,y)$ for $i = 0,1$, then the remaining is the same as in Lemma 3.4.

Since $C_1^0 < C_2^0$ and $\zeta^{0,crit}(t) < 0$, for $t \in [0,T]$ we get

$$(C_1^0 - C_2^0)\frac{1}{1-\eta}(e^{-(1-\eta)y} - 1) \geq \left[(C_1^0 - C_2^0)\frac{1}{1-\eta}(e^{-(1-\eta)y} - 1) + \zeta^{0,crit}(t)\right]^+.$$

Therefore, for every $y \in \mathbb{R}_0^+$,

$$\chi^0(t,y) \geq -C_1^0\frac{1}{1-\eta}(e^{-(1-\eta)y} - 1) \geq 0.$$

The non-negativity of $\chi^1(t,y)$ is obvious. □

Having Lemma 3.7 and Lemma 3.9, we obtain the following theorem.

Theorem 3.10. *(Optimal investment strategy) For the step intensity functions* (3.18)*, the optimal portfolio strategy is given as*

$$\pi^{i,*}(t) = \pi^{i,*}(t,h(t)), \;\; i = 0,1 \text{ and } t \in [0,T],$$

where $\pi^{i,*}(t,y)$ is defined in Lemma 3.7. The value functions are given as

$$J(t,\tilde{X},0) = \frac{(\tilde{X}e^{g(t)})^{1-\eta}}{1-\eta},$$
$$J(t,\tilde{X},1) = \frac{(\tilde{X}e^{g(t)-h(t)})^{1-\eta}}{1-\eta},$$

where $(g(t),h(t))$ is the unique solution of the system (3.24) *and* (3.25) *subject to the boundary condition* (3.26)*.*

Remark 3.11. *We can separate the optimal strategy into the Merton strategy (constant intensity) and an additional hedging component as in the proof of Lemma 3.2. In particular,*

$$\pi^{i,*}(t) = \pi^{i,M}(t) + \pi^{i,H}(t), \;\; i = 0,1$$

where

$$\begin{aligned}\pi^{i,H}(t) &= -\big(\pi^{i,M}(t) - \frac{C^i - A^i}{B^i_\pi}\big)^+ 1_{\{h(t)\geq h^{i,crit}(t)\}}\\ &= -\frac{(-C^i + A^i + \pi^{i,M}(t)B^i_\pi)^+}{B^i_\pi} 1_{\{h(t)\geq h^{i,crit}(t)\}}.\end{aligned}$$

Notice that

$$\pi^{i,H}(t)B^i_\pi \leq 0 \text{ for } t \in [0,T]. \tag{3.27}$$

Lemma 3.12. *The optimal strategy* $\pi^{i,*}$ *satisfies*

$$\vartheta^{0,1}(\pi^{0,*}(t)) \leq \vartheta^{0,1}(\pi^{0,M}(t))$$

and

$$\vartheta^{1,0}(\pi^{1,*}(t)) \geq \vartheta^{1,0}(\pi^{1,M}(t))$$

for every $t \in [0,T]$.

Proof. We have

$$\begin{aligned}\vartheta^{0,1}(\pi^{0,*}(t)) &= C^0_1 1_{\{A^0+\pi^{0,*}(t)B^0_\pi \leq C^0\}} + C^0_2 1_{\{A^0+\pi^{0,*}(t)B^0_\pi > C^0\}}\\ &= C^0_1 1_{\{A^0+\pi^{0,M}(t)B^0_\pi+\pi^{i,H}(t)B^0_\pi \leq C^0\}} + C^0_2 1_{\{A^0+\pi^{0,M}(t)B^0_\pi+\pi^{i,H}(t)B^0_\pi > C^0\}}\\ &\leq C^0_1 1_{\{A^0+\pi^{0,M}(t)B^0_\pi \leq C^0\}} + C^0_2 1_{\{A^0+\pi^{0,M}(t)B^0_\pi > C^0\}}\\ &= \vartheta^{0,1}(\pi^{0,M}(t)).\end{aligned}$$

Notice that in above derivation, we have used the property (3.27) and $C^0_2 > C^0_1$. Similarly, we obtain

$$\begin{aligned}\vartheta^{1,0}(\pi^{1,*}(t)) &= C^1_1 1_{\{A^1+\pi^{1,*}(t)B^1_\pi \leq C^1\}} + C^1_2 1_{\{A^1+\pi^{1,*}(t)B^1_\pi > C^1\}}\\ &= C^1_1 1_{\{A^1+\pi^{1,M}(t)B^1_\pi+\pi^{i,H}(t)B^1_\pi \leq C^1\}} + C^1_2 1_{\{A^1+\pi^{1,M}(t)B^1_\pi+\pi^{i,H}(t)B^1_\pi > C^1\}}\\ &\geq C^1_1 1_{\{A^1+\pi^{1,M}(t)B^1_\pi \leq C^1\}} + C^1_2 1_{\{A^1+\pi^{1,M}(t)B^1_\pi > C^1\}}\\ &= \vartheta^{1,0}(\pi^{1,M}(t)).\end{aligned}$$

□

Remark 3.13. *As explained in [17], the optimal investment strategy of the large investor is a compromise strategy. The utility goal leads to the Merton strategy,* *while the intensity criterion leads to strategies in the favourable half space $\mathcal{F}^{\pi,i}$. The intensity criterion implies that the investor invests in a way such that shifting intensities are in favour of his decision.*

3.3 Summary and further aspects

The Busch-Korn-Seifried model ([10],[17]) for a large investor is generalized by allowing the interest rate to follow the Vasicek model. However, the portfolio in our investment problem only contains two assets, the zero-coupon bond and the stock. Furthermore, consumption is not allowed in our model. The investment problems with respect to the constant intensity function and the step intensity function, which are introduced in [10] or [17], are considered. We were able to obtain the explicit solutions for those problems for the large investor with power utility.

By the same method as in [10] or [17], we can also solve the investment problem with respect to the following intensity functions

$$\vartheta^{i,1-i}(\pi) = \max\{A^i + \pi B^i_\pi, C^i\}, \;\; \pi \in \mathbb{R}$$

with $A^i, B^i_\pi \in \mathbb{R}$ and $C^i \geq 0$ for $i = 0, 1$. Since it does not provide more insight, we do not present it here.

Chapter 4

CPPI in the Black-Scholes framework

In this chapter, the Constant Propotion Portfolio Insurance strategy is presented in the Black-Scholes setting. We begin the chapter by describing the mathematical framework and the CPPI principle. Next, the basic properties of the strategy will be stated and proved. To end the chapter, we verify the existence of the CPPI strategy by using a utility argument.

4.1 Mathematical framework and CPPI's principle

Let $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}, P)$ be a filtered probability space with time horizon $T > 0$ such that the filtration $\mathbb{F}$ satisfies the usual conditions. The asset prices are assumed to be driven by the Brownian motion W. The investment opportunity set includes a riskless asset and a risky asset which satisfy the following equations:

$$dB(t) = rB(t)dt, \quad B(0) = b > 0 \tag{4.1}$$

and

$$dS(t) = S(t)(\mu dt + \sigma dW(t)), \quad S(0) = s > 0, \tag{4.2}$$

where μ, r and σ are positive constants and $\mu > r$.

CPPI is a dynamic trading strategy for investors who have zero tolerance for risk below a specific floor. Assume that V is the portfolio process of an investor who wants at least F at time T. To ensure that F can be reached, we can invest an amount equals to the current value of the guarantee (the floor $F(.)$) in the riskless asset. Notice that the floor $F(.)$ satisfies

$$dF(t) = rF(t)dt = F(t)\frac{dB(t)}{B(t)}$$

with $F(T) = F$. Therefore, $F(t) = e^{-r(T-t)}F$ for $0 \leq t \leq T$.

The CPPI's principle: The cushion C, which is defined as the gap between the portfolio value V and the floor F, is calculated at every time t. The CPPI principle states that an amount

$$E = mC(t)$$

is invested in the risky asset S and the remaining

$$V(t) - mC(t)$$

is invested in the riskless asset B at time t.

The constant $m \geq 0$ is called the multiple. It determines the performance of the portfolio while the floor takes care of the guarantee. The amount of money $E = mC$ invested in the risky asset is called the exposure.

4.2 Basic properties of the CPPI strategy

In what follows, we will derive an equation for the cushion, and then provide the explicit formula for it. Next, we determine the portfolio value together with its mean and variance.

Lemma 4.1. *The cushion $C(t)_{0\leq t\leq T}$ satisfies*

$$dC(t) = C(t)\Big(\big(r + m(\mu - r)\big)dt + \sigma m dW(t)\Big). \tag{4.3}$$

Proof. By the definition of the cushion $C(t) = V(t) - F(t)$, we obtain

$$dC(t) = dV(t) - dF(t). \tag{4.4}$$

The principle of the CPPI strategy gives

$$dV(t) = mC(t)\frac{dS(t)}{S(t)} + (V(t) - mC(t))\frac{dB(t)}{B(t)}. \tag{4.5}$$

From the equations (4.4) and (4.5), the following holds

$$\begin{aligned} dC(t) &= mC(t)\frac{dS(t)}{S(t)} + (V(t) - mC(t))\frac{dB(t)}{B(t)} - F(t)\frac{dB(t)}{B(t)} \\ &= mC(t)\frac{dS(t)}{S(t)} + (1-m)C(t)\frac{dB(t)}{B(t)}. \end{aligned}$$

Finally, plugging the stochastic equations of $S(t)$ and $B(t)$ in (4.1) and (4.2) into the above equation yields the proof. □

Remark 4.2. *A consequence of the above lemma is*

$$C(t) = C(0)\exp\Big(\big(r + m(\mu - r) - \frac{1}{2}\sigma^2 m^2\big)t + \sigma m W(t)\Big). \tag{4.6}$$

Then if $C(0) \geq 0$, $C(t) \geq 0$ for all $t \in [0,T]$. Therefore, the portfolio value is always larger than the floor, which implies that the guarantee can be reached at time T.

Proposition 4.3. *The portfolio value at time $t \in [0,T]$ equals*

$$V(t) = Fe^{-r(T-t)} + \frac{V(0) - Fe^{-rT}}{S(0)^m}\exp\Big(\big(r - m(r - \frac{1}{2}\sigma^2) - m^2\frac{\sigma^2}{2}\big)t\Big)S(t)^m. \tag{4.7}$$

Proof. First, the stock price equation (4.2) yields

$$S(t) = S(0)\exp\big((\mu - \frac{1}{2}\sigma^2)t + \sigma W(t)\big).$$

Hence, we have

$$S(t)^m = S(0)^m \exp\big(m(\mu - \frac{1}{2}\sigma^2)t + m\sigma W(t)\big). \tag{4.8}$$

Recall from the equation (4.6) that

$$C(t) = C(0)\exp\Big(\big(r + m(\mu - r) - \frac{1}{2}\sigma^2 m^2\big)t + \sigma m W(t)\Big), \tag{4.9}$$

where $C(0) = V(0) - F(0) = V(0) - e^{-rT}F$.

From the equations (4.8) and (4.9), we obtain

$$C(t) = \frac{V(0) - Fe^{-rT}}{S(0)^m}\exp\Big(\big(r - m(r - \frac{1}{2}\sigma^2) - m^2\frac{\sigma^2}{2}\big)t\Big)S(t)^m.$$

Remember that $V(t) = F(t) + C(t)$ and $F(t) = e^{-rT}F$. Therefore,

$$V(t) = Fe^{-r(T-t)} + \frac{V(0) - Fe^{-rT}}{S(0)^m}\exp\Big(\big(r - m(r - \frac{1}{2}\sigma^2) - m^2\frac{\sigma^2}{2}\big)t\Big)S(t)^m.$$

□

Remark 4.4. *The portfolio value in the equation* (4.7) *only depends on the current value of the risky asset, but it does not depend on the path of the risky asset prices. Therefore, we can say that the portfolio is path independent.*

Proposition 4.5. *The mean and the variance of the portfolio value at time t are*

$$\begin{aligned}\mathbb{E}(t)] &= F(t) + (V(0) - F(0))\exp\Big\{(r + m(\mu - r))t\Big\},\\ \mathrm{Var}[V(t)] &= (V(0) - F(0))^2\exp\Big(2(r + m(\mu - r))t\Big)\Big(\exp\{m^2\sigma^2 t\} - 1\Big).\end{aligned}$$

Proof. Recall from Proposition 4.3 that

$$V(t) = F(t) + C(t), \tag{4.10}$$

where

$$\begin{aligned}C(t) &= C(0)\exp\Big(\big(r + m(\mu - r) - \frac{1}{2}\sigma^2 m^2\big)t + \sigma m W(t)\Big).\\ &=: C(0)\exp\big(X\big).\end{aligned}$$

Then X is normally distributed with

$$\mu_X = r + m(\mu - r) - \frac{1}{2}\sigma^2 m^2 t$$

and

$$\sigma_X^2 = \sigma^2 m^2 t.$$

Notice that if $Y \sim N(\mu_Y, \sigma_Y^2)$, then

$$\mathbb{E}\big[e^Y\big] = e^{\mu_Y + \frac{1}{2}\sigma_Y^2} \text{ and } \mathrm{Var}\big[e^Y\big] = e^{2\mu_Y + \sigma_Y^2}(e^{\sigma_Y^2} - 1).$$

Therefore,

$$\mathbb{E}[C(t)] = \exp\Big\{\mu_X + \frac{1}{2}\sigma_X^2\Big\} = C(0)\exp\Big\{(r + m(\mu - r))t\Big\} \tag{4.11}$$

and

$$\begin{aligned} \mathrm{Var}[C(T)] &= \exp\left\{2\mu_X + \sigma_X^2\right\}\Big(\exp\left\{\sigma_X^2\right\} - 1\Big) \\ &= \exp\Big(2(r + m(\mu - r))t\Big)\Big(\exp\left\{m^2\sigma^2 t\right\} - 1\Big). \end{aligned} \tag{4.12}$$

From the equations (4.10) to (4.12), we obtain the proof. □

Remark 4.6. *Proposition 4.5 indicates that the mean of the portfolio value does not depend on the volatility σ of the risky asset. However, the standard deviation grows exponentially with respect to σ.*

4.3 Optimality of the CPPI strategy

In their paper [20], Døskeland and Nordahl considered four different pension contracts. They concluded that assuming CRRA utility, we cannot explain the existence of any form of guarantees. However, the demand for products with guarantees may be explained through behaviour models.

The justification of the CPPI strategy can be achieved by modifying the classical utility maximizing problem (see [13]). The CPPI strategies are optimal for an investor who is interested in maximizing utility from the difference between the terminal strategy value and a given subsistence level. More specifically, given the

CPPI investor as in the two previous sections and his/her utility function

$$U(x) = \frac{x^{1-\eta}}{1-\eta}, \quad 0 < \eta < 1, \tag{4.13}$$

we will show that the optimal multiple maximizes the expected utility

$$\mathbb{E}[U(C^m(T))] \tag{4.14}$$

over the class of admissible multiples.

Recall from Lemma 4.1 that the cushion $C(t)_{0 \le t \le T}$ satisfies

$$dC(t) = C(t)\Big(\big(r + m(\mu - r)\big)dt + \sigma m dW(t)\Big).$$

Notice that the above equation of C is similar to the the dynamic of the wealth process in the classical Merton problem with X is replaced by C and π is substituted by m. Therefore, the optimal multiple m^* can be obtained from the Theorem 1.4 as

$$m^* = \frac{\mu - r}{\eta \sigma^2}.$$

Remark 4.7. *The optimal multiple is an increasing function of μ, but it decreases when either η, r or σ increases.*

Horsky ([14]) studied problems of maximizing (4.14) with Vasicek's model for the interest rate. If there are only a zero-coupon bond and a stock in the portfolio, then the optimal multiple m^* is deterministic. When the portfolio contains also a money market account (MMA), then the exposure at time t is

$$\begin{aligned} E(t) &= \text{Exposure in the stock} + \text{Exposure in the MMA} \\ &= m(t)C(t) + \tilde{m}(t)C(t). \end{aligned}$$

The optimal multiples m^* (stock) and $\tilde{m}^*$ (money market account) are deterministic too provided certain conditions on the excess returns are met otherwise singularities occur.

He also considered Heston's model for the stock, but the interest rate is still assumed to be constant. In this case, the optimal multiple m^* is also deterministic.

Chapter 5

CPPI strategy in Markov-switching models

In this chapter, we justify the existence of the CPPI strategy in more general framework than what is introduced in the last chapter. The Vasicek model for the short rate and the Markov switching parameters will be taken into the consideration. We want to investigate how these changes impact asset allocation strategies. The effect of Vasicek short rate on the CPPI strategy has been studied by Roman Horsky (see [14]). This chapter will extend his research to include the Markov switching parameters. The generalization is based on the Bäuerle and Rieder [8] investment problem, which is summarized in Chapter 1.

5.1 Merton problem

This section limits the investment opportunities set to include only a zero-coupon bond and a stock. The money market account will take part in the next section in a different problem. In what follows, we first present the mathematical framework and then determine the equation for the discounted cushion. Based on that framework, we formulate the investment problem for a CPPI investor. Next, we solve the optimization problem and then deliver a verification theorem. When the solution has been found, we provide the basic properties of the optimal solutions. Furthermore, an average-data model will be studied in order to compare with our model. Finally, we consider a special case of our model and provide numerical examples.

5.1.1 Mathematical framework

Let $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}, P)$ be a filtered probability space with time horizon $T > 0$ such that the filtration $\mathbb{F}$ satisfies the usual conditions and $\mathcal{F} = F_T$. Assume that the short rate $r(t)$ follows the Vasicek model

$$dr(t) = a(b - r(t))dt + \sigma_r d\bar{W}_1(t),$$

where a, b and σ_r are positive constants, and $\bar{W}_1$ is a Brownian motion under the risk-neutral measure. Then under real-world measure P, the zero-bond price satisfies

$$dB(t,T) = B(t,T)\left((r(t) + q(t)\sigma_B(t,T))dt - \sigma_B(t,T)dW_1(t)\right), \tag{5.1}$$

where $\sigma_B(t,T) = \frac{\sigma_r}{a}(1 - e^{a(t-T)})$ and $q : [0,T] \to \mathbb{R}$ is the market price of risk (see [18]). Moreover, W_1 is a Brownian motion under P with respect to the filtration $\mathbb{F}$.

Suppose that $Y = (Y(t))_{t\in[0,T]}$ is a continuous-time Markov chain with finite state-space E and intensity matrix $Q = (q_{ij})_{i,j\in E}$ under P with respect to $\mathbb{F}$. In this section, we assume that the stock price evolves as

$$\frac{dS(t)}{S(t)} = (r(t) + \mu(Y(t)))\,dt + \sigma_S(\rho dW_1(t) + \bar{\rho} dW_2(t)), \tag{5.2}$$

where $\bar{\rho} = \sqrt{1-\rho^2}$, and W_2 is another Brownian motion under P with respect to $\mathbb{F}$. The three processes W_1, W_2 and Y are supposed to be mutually independent. Furthermore, $\mu(Y(.))$ takes positive finite values in $\mathbb{R}$.

5.1.2 Discounted cushion

In what follows, the bond price will be used as numeraire to simplify future calculations and proofs. In particular, we consider $\tilde{S}(t) = \frac{S(t)}{B(t,T)}$ and $\tilde{C}(t) = \frac{C(t)}{B(t,T)}$ for $t \in [0,T]$. Furthermore, we use short notations S for $S(t)$, B for $B(t,T)$ and σ_B for $\sigma_B(t,T)$.

An application of Ito's formula gives

$$d\tilde{S} = d(\frac{S}{B}) = \frac{dS}{B} + S(-\frac{1}{B^2}dB + \frac{1}{B^3}dBdB) + (-\frac{1}{B^2}dB + \frac{1}{B^3}dBdB)dS.$$

Using the equations (5.1) and (5.2), we obtain

$$\begin{aligned}
\frac{dS}{B} &= \tilde{S}\left[(r(t)+\mu(Y(t)))dt+\sigma_S(\rho dW_1(t)+\bar{\rho}dW_2(t))\right],\\
S(-\frac{1}{B^2}dB+\frac{1}{B^3}dBdB) &= \tilde{S}\left[\left(\sigma_B^2-r(t)-q\sigma_B\right)dt+\sigma_B dW_1\right],\\
(-\frac{1}{B^2}dB+\frac{1}{B^3}dBdB)dS &= \rho\sigma_S\sigma_B\tilde{S}dt.
\end{aligned}$$

Hence, the following equation holds

$$d\tilde{S}=\tilde{S}\left[\nu(t,Y(t))dt+(\sigma_S\rho+\sigma_B)dW_1(t)+\sigma_S\bar{\rho}dW_2(t)\right], \tag{5.3}$$

where $\nu(t,Y(t))=\mu(Y(t))+\sigma_B^2-q\sigma_B+\rho\sigma_S\sigma_B$.

Since the investor follows the CPPI strategy, the equation (4.5) is still valid

$$dC=\frac{mC}{S}dS+\frac{C(1-m)}{B}dB.$$

We also have

$$\begin{aligned}
dC &= d(B\tilde{C})=Bd\tilde{C}+\tilde{C}dB+dBd\tilde{C} &(5.4)\\
&= \frac{mC}{S}d(B\tilde{S})+\tilde{C}dB-\tilde{C}mdB &(5.5)
\end{aligned}$$

Comparing (5.4) and (5.5), we obtain

$$Bd\tilde{C}+dBd\tilde{C}=\frac{mC}{S}(Bd\tilde{S}+\tilde{S}dB+d\tilde{S}dB)-m\tilde{C}dB.$$

It then follows that

$$(B+dB)d\tilde{C}=\frac{mC}{S}(Bd\tilde{S}+d\tilde{S}dB)=\frac{mC}{S}d\tilde{S}(B+dB).$$

Therefore,

$$d\tilde{C}=m\tilde{C}\frac{d\tilde{S}}{\tilde{S}}. \tag{5.6}$$

Lemma 5.1. *The discounted cushion $\tilde{C}$ follows the stochastic differential equation*

$$d\tilde{C}(t)=m(t)\tilde{C}(t)\left[\nu(t,Y(t)))dt+(\sigma_S\rho+\sigma_B)dW_1(t)+\sigma_S\bar{\rho}dW_2(t)\right], \tag{5.7}$$

where $\nu(t, Y(t)) = \mu(Y(t)) + \sigma_B^2 - q\sigma_B + \rho\sigma_S\sigma_B$. *Furthermore, if* $\mathbb{E}\int_0^T m(t)^2 dt < \infty$, *the equation* (5.7) *has a unique solution.*

Proof. The dynamic of $\tilde{C}$ follows directly from equations (5.3) and (5.6).

Let $A(s) = m(s)\nu(s, Y(s))$, $S_1(s) = m(s)(\sigma_S\rho + \sigma_B)$, $S_2(s) = m(s)\sigma_S\bar{\rho}$ and $a(s) = \sigma_1(s) = \sigma_2(s) = 0$. Since $\mathbb{E}\int_0^T m(s)^2 ds < \infty$ and $\nu(s, Y(s)), \sigma_B(s)$ are bounded, all coefficients $A(s)$, $S_1(s)$, $S_2(s)$, $a(s)$, $\sigma_1(s)$ and $\sigma_2(s)$ satisfy the conditions of Theorem A.3. Therefore, the equation (5.7) with initial condition $\tilde{C}(0) = \tilde{C}_0$ has a unique solution. □

5.1.3 Optimal portfolio

Assume that the filtration $\mathbb{F}$ is generated by W_1, W_2 and Y, i.e. for $0 \leq t \leq T$, $\mathcal{F}_t = \sigma(W_1(s), W_2(s), Y(s), s \leq t)$.

From Lemma 5.1, it is reasonable to define the set of admissible controls as

$$\mathcal{A}(t, \mathbb{F}) := \left\{ m : [t, T] \to \mathbb{R} | m \text{ is progressively measurable, } \mathbb{E}\int_t^T m(s)^2 ds < \infty \right\}.$$

The CPPI investor is interested in solving the optimization problem

$$\sup_{m \in \mathcal{A}(0,\mathbb{F})} \mathbb{E}\left[U(C^m(T))\right],$$

where $U(x) = \frac{x^{1-\eta}}{1-\eta}$ with $0 < \eta < 1$.

Since $C^m(T) = B(T, T)\tilde{C}^m(T) = \tilde{C}^m(T)$, the investment problem can be written in the following equivalent form

$$\sup_{m \in \mathcal{A}(0,\mathbb{F})} \mathbb{E}\left[U(\tilde{C}^m(T))\right].$$

Given $\tilde{C}(t) = \tilde{C}$ and $Y(t) = i$, the optimization is now

$$\sup_{m \in \mathcal{A}(t,\mathbb{F})} \mathbb{E}^{t,\tilde{C},i}\left[U(\tilde{C}_T^m)\right],$$

where $\mathbb{E}^{t,\tilde{C},i}$ denotes the conditional expectation providing $\tilde{C}(t) = \tilde{C}$ and $Y(t) = i$.

We define the value function as

$$J(t,\tilde{C},i) = \sup_{m\in\mathcal{A}(t,\mathbb{F})} \mathbb{E}^{t,\tilde{C},i}\left[U(\tilde{C}_T^m)\right].$$

Then the corresponding HJB system is

$$\sup_{m\in\mathcal{A}(t,\mathbb{F})} \left\{ J_t + J_{\tilde{C}}\nu(t,i)m\tilde{C} + \frac{1}{2}J_{\tilde{C}\tilde{C}}\left((\sigma_S\rho+\sigma_B)^2 + (\sigma_S\bar{\rho})^2\right)m^2\tilde{C}^2 + \sum_{j\in E} q_{ij}[J(t,\tilde{C},j) - J(t,\tilde{C},i)] \right\} = 0, \quad i\in E. \tag{5.8}$$

We denote the term in the brackets of the HJB equation (5.8) by $\mathcal{A}^m(t,\tilde{C}(t),i)$. This system stems from the proof of Verification Theorem 5.4. Now we shall find the value function and the optimal multiple m^* by solving the HJB equation (5.8). Next, we shall give the proof of the verification theorem.

Theorem 5.2. *The HJB system* (5.8) *with the boundary condition*

$$J(T,\tilde{C}(T),Y(T)) = \frac{\tilde{C}(T)^{1-\eta}}{1-\eta}, \quad 0<\eta<1$$

for the portfolio problem (5.7) *has the solution* $J(t,\tilde{C},i)\in C^{1,2}([0,T]\times\mathbb{R})$,

$$J(t,\tilde{C},i) = \frac{\tilde{C}(T)^{1-\eta}}{1-\eta}g(t,i)$$

for $i\in E$. *Here,* $g(t,i)$ *is the unique positive solution of the following system of ordinary differential equations with boundary conditions* $g(T,i)=1$ *for all* $i\in E$,

$$g_t(t,i) + a(t,i)g(t,i) + \sum_{j\in E} q_{ij}(g(t,j)-g(t,i)) = 0, \tag{5.9}$$

where $a(t,i) = \frac{1}{2}m^*(t,i)^2\eta(1-\eta)((\sigma_S\rho+\sigma_B)^2 + (\sigma_S\bar{\rho})^2)$. *The optimal multiple is*

$$m^*(t,i) = \frac{\nu(t,i)}{\eta\left((\sigma_S\rho+\sigma_B)^2 + (\sigma_S\bar{\rho})^2\right)}.$$

Proof. Differentiating the term in the brackets of the equation (5.8) with respect to m gives

$$J_{\tilde{C}}\nu(t,i)\tilde{C} + m\tilde{C}^2J_{\tilde{C}\tilde{C}}((\sigma_S\rho+\sigma_B)^2 + (\sigma_S\bar{\rho})^2).$$

The necessary condition for m to be the maximizer is that the above term is equal to zero. This leads to

$$m(t,i) = \frac{-J_{\tilde{C}}\nu(t,i)}{\tilde{C}J_{\tilde{C}\tilde{C}}((\sigma_S\rho+\sigma_B)^2+(\sigma_S\bar{\rho})^2)}. \tag{5.10}$$

Plugging $m(t,i)$ in (5.10) into the HJB equation (5.8), we obtain

$$\begin{aligned} J_t &- \frac{1}{2}J_{\tilde{C}\tilde{C}}\Big((\sigma_S\rho+\sigma_B)^2+(\sigma_S\bar{\rho})^2\Big)m^2\tilde{C}^2 \\ &+ \sum_{j\in E} q_{ij}\Big[J(t,\tilde{C},j)-J(t,\tilde{C},i)\Big] = 0. \end{aligned} \tag{5.11}$$

Consider $J(t,\tilde{C},i) = \frac{\tilde{C}^{1-\eta}}{1-\eta}g(t,i)$ where $g(T,i)=1$. Inserting this formula into the equations (5.10) and (5.11) gives

$$m(t,i) = \frac{\nu(t,i)}{\eta((\sigma_S)\rho+\sigma_B)^2+(\sigma_S\bar{\rho})^2)}$$

and

$$\begin{aligned} g_t(t,i) &+ \frac{1}{2}m^2\eta(1-\eta)((\sigma_S\rho+\sigma_B)^2+(\sigma_S\bar{\rho})^2)g(t,i) \\ &+ \sum_{j\in E} q_{ij}(g(t,j)-g(t,i)) = 0. \end{aligned}$$

Therefore,

$$g_t(t,i)+a(t,i)g(t,i)+\sum_{j\in E} q_{ij}(g(t,j)-g(t,i)) = 0,$$

where $a(t,i) = \frac{1}{2}m^2(t,i)\eta(1-\eta)((\sigma_S\rho+\sigma_B)^2+(\sigma_S\bar{\rho})^2)$. It can be written as

$$g_t(t) = A(t)g(t),$$

where $A(t) = (a_{ij})_{i,j\in E}$ is a matrix satisfying

$$\begin{aligned} a_{ii}(t) &= -a(t,i)-q_{ii} \text{ for all } i\in E, \\ a_{ij}(t) &= -q_{ij} \text{ for all } i,j\in E \text{ and } i\neq j. \end{aligned}$$

The boundary condition is $g(T) = 1_E$ where 1_E is a vector of 1. Since $A(t)$ is continuous on $[0,T]$, the above system of ODEs has a solution (see Proposition 2.1 in [22]).

The uniqueness and positiveness of the solution are the consequences of Lemma 5.3. This property then guarantees that the control $m(t,i) = \frac{\nu(t,i)}{\eta\left((\sigma_S\rho+\sigma_B)^2+(\sigma_S\bar{\rho})^2\right)}$ is indeed the maximizer of $\mathcal{A}^m(t,\tilde{C}(t),i)$. □

Lemma 5.3. *The function $g(t,i)$ which solves the equation (5.9) with the boundary condition $g(T,i)=1$ for all $i \in E$ can be written as*

$$g(t,i) = \mathbb{E}^{t,i}\left[\exp\left\{\int_t^T a(s,Y(s))ds\right\}\right],$$

where $a(s,Y(s)) = \frac{1}{2}(m^*(s))^2\eta(1-\eta)((\sigma_S\rho+\sigma_B)^2+(\sigma_S\bar{\rho})^2)$.

Proof. Applying Ito's lemma for $t < s \leq T$ gives

$$g(T,Y(T)) = g(t,i) + \int_t^T g_t(s,Y(s))ds + \sum_{t<s\leq T}[g(s,Y(s)) - g(s,Y(s-))].$$

Replacing $g_t(s,Y(s))$ (using equation (5.9)) by

$$-a(s,Y(s))g(s,Y(s)) - \sum_{j\in E} q_{Y(s)j}(g(s,j) - g(s,Y(s)))$$

yields

$$\begin{aligned} g(T,Y(T)) &= g(t,i) \\ &\quad - \int_t^T \left[a(s,Y(s))g(s,Y(s)) + \sum_{j\in E} q_{Y(s)j}(g(s,j) - g(s,Y(s)))\right]ds \\ &\quad + \sum_{t<s\leq T}\Big[g(s,Y(s)) - g(s,Y(s-))\Big]. \end{aligned} \tag{5.12}$$

Since $g(t,i) \in C^1[0,T]$ is a solution of the system (5.9) on $[0,T]$ with the boundary conditions $g(T,i) = 1$ for all $i \in E$, $g(t,i)$ is bounded on $[0,T]$. Suppose that $|g(t,i)| < L$ for some $L > 0$ and for all $i \in E$.

Let μ be the jump measure of Y and T_n be the successive jump time points. Then

$$\mu([0,t]\times\{j\}) = \sum_{n\in\mathbb{N}} I_{[Y(T_n)=j, T_n\leq t]}.$$

The compensator of μ is defined as

$$\nu([0,t]\times\{j\}) = \int_0^t q_{Y(s-)j}ds = \int_0^t \sum_{i\neq j} q_{ij} I_{[Y(s-)=i]}ds.$$

Using the jump measure instead of summation, we obtain

$$\begin{aligned}
\sum_{t<s\leq T}[g(s,Y(s))-g(s,Y(s-))] &= \int_t^T \sum_{j\in E}\big[g(s,j)-g(s,Y(s-))\big]\mu(ds,j)\\
&= \int_t^T \sum_{j\in E}\big[g(s,j)-g(s,Y(s-))\big](\mu-\nu)(ds,j)\\
&+ \int_t^T \sum_{j\in E}\big[g(s,j)-g(s,Y(s-))\big]\nu(ds,j)\\
&= \int_t^T \sum_{j\in E}\big[g(s,j)-g(s,Y(s-))\big](\mu-\nu)(ds,j)\\
&+ \int_t^T \sum_{j\in E}\big[g(s,j)-g(s,Y(s-))\big]q_{Y(s-)j}ds.
\end{aligned}$$

Since $|g(s,i)|<L$ for all $i\in E$ and $s\in[0,T]$,

$$\begin{aligned}
&\mathbb{E}^{t,i}\int_t^T \sum_{j\in E}|g(s,j)-g(s,Y(s-))|\nu(ds,j) < \mathbb{E}^{t,i}\sum_{t<T_i\leq T} 2Lq_M\\
&= 2Lq_M\mathbb{E}^{t,i}\big[N(T,t)\big],
\end{aligned}$$

where $N(T,t)$ is the number of jumps in $[t,T]$ and $q_M=\max\{q_{ij}: i,j\in E\}$. Using Lemma B.2 (see Appendix B), we obtain $\mathbb{E}^{t,i}\big[N(T,t)\big]<\infty$. Hence,

$$\mathbb{E}^{t,i}\int_t^T \sum_{j\in E}|g(s,j)-g(s,Y(s-))|\nu(ds,j)<\infty.$$

Applying Theorem B.4 (see Appendix B) gives that

$$\int_t^T \sum_{j\in S}\big[g(s,j)-g(s,Y(s-))\big](\mu-\nu)(ds,j)$$

is a martingale with respect to $\mathbb{F}$. It then follows that

$$\mathbb{E}^{t,i}\int_t^T \sum_{j\in S}\big[g(s,j)-g(s,Y(s-))\big](\mu-\nu)(ds,j)=0.$$

This implies

$$\mathbb{E}^{t,i}\sum_{t<s\leq T}[g(s,Y(s))-g(s,Y(s-))]=\mathbb{E}^{t,i}\int_t^T\sum_{j\in S}\big[g(s,j)-g(s,Y(s-))\big]q_{Y(s-)j}ds. \tag{5.13}$$

Taking expectation on both sides of (5.12) and using (5.13), we obtain

$$\mathbb{E}^{t,i}[g(T,Y(T))]=g(t,i)-\mathbb{E}^{t,i}\int_t^T a(s,Y(s))g(t,Y(s))ds.$$

Since $g(T,Y(T))$ =1,

$$g(t,i)=1+\mathbb{E}^{t,i}\int_t^T a(s,Y(s))g(t,Y(s))ds. \tag{5.14}$$

We claim that the equation (5.14) has a unique solution in $C^1[0,T]$. To prove the uniqueness of the solution of (5.14), first we notice that $g(t,i)$ is bounded on $[0,T]$.

The equation (5.14) implies

$$g(t,Y(t))=1+\mathbb{E}\left[\int_t^T a(s,Y(s))g(s,Y(s))ds|\mathcal{F}_t\right]. \tag{5.15}$$

Suppose there is another $\tilde{g}\in C^1[0,T]$ which satisfies the equation (5.15). That means

$$\tilde{g}(t,Y(t))=1+\mathbb{E}\left[\int_t^T a(s,Y(s))\tilde{g}(s,Y(s))ds|\mathcal{F}_t\right]. \tag{5.16}$$

Subtracting (5.15) by (5.16) gives

$$g(t,Y(t))-\tilde{g}(t,Y(t))=\mathbb{E}\left[\int_t^T a(s,Y(s))[g(s,Y(s))-\tilde{g}(s,Y(s))]ds|\mathcal{F}_t\right].$$

Therefore, we obtain

$$\begin{aligned}|g(t,Y(t)) \quad - \quad & \tilde{g}(t,Y(t))| \\ &=\left|\mathbb{E}\left[\int_t^T a(s,Y(s))[g(s,Y(s))-\tilde{g}(s,Y(s))]ds\Big|\mathcal{F}_t\right]\right| \\ &\leq \mathbb{E}\left[\int_t^T \Big|a(s,Y(s))g(s,Y(s))-\tilde{g}(s,Y(s))\Big|ds\Big|\mathcal{F}_t\right] \\ &=\mathbb{E}\left[\int_t^T a(s,Y(s))\Big|g(s,Y(s))-\tilde{g}(s,Y(s))\Big|ds\Big|\mathcal{F}_t\right].\end{aligned}$$

Notice that $a(s, Y(s)), g(s, Y(s)), \tilde{g}(s, Y(s))$ are bounded, so we can use Fubini's theorem

$$
\begin{aligned}
&\mathbb{E}|g(t, Y(t)) - \tilde{g}(t, Y(t))| \\
&\leq \mathbb{E}\left[\mathbb{E}\left[\int_t^T a(s, Y(s))|g(s, Y(s)) - \tilde{g}(s, Y(s))|ds \Big| \mathcal{F}_t\right]\right] \\
&= \mathbb{E}\left[\int_t^T a(s, Y(s))|g(s, Y(s)) - \tilde{g}(s, Y(s))|ds\right] \\
&= \int_t^T \mathbb{E}\Big[a(s, Y(s))|g(s, Y(s)) - \tilde{g}(s, Y(s))|\Big] ds \\
&\leq \int_t^T R\mathbb{E}|g(s, Y(s)) - \tilde{g}(s, Y(s))|ds, \qquad (5.17)
\end{aligned}
$$

where R is a constant such that $a(s, Y(s)) < R$ for all $s \in [0, T]$.

Application of Grønwall's inequality to $u(t) = \mathbb{E}|g(t, Y(t)) - \tilde{g}(t, Y(t))|$ yields $u(t) = 0$ due to (5.17). This implies $g(t, Y(t)) = \tilde{g}(t, Y(t))$ a.s. Now we consider

$$
\hat{g}(t, i) = \mathbb{E}^{t,i}\left[\exp\left\{\int_t^T a(s, Y(s))ds\right\}\right]. \qquad (5.18)
$$

We then obtain

$$
\begin{aligned}
&1 + \mathbb{E}^{t,i}\int_t^T a(\tau, Y(\tau))\hat{g}(\tau, Y(\tau))d\tau \\
&= 1 + \mathbb{E}^{t,i}\left[\int_t^T \mathbb{E}\left[\exp\left\{\int_\tau^T a(s, Y(s))ds\right\}|\mathcal{F}_\tau\right] a(\tau, Y(\tau))d\tau\right] \\
&= 1 + \mathbb{E}^{t,i}\left[\int_t^T \mathbb{E}\left[a(\tau, Y(\tau))\exp\left\{\int_\tau^T a(s, Y(s))ds\right\}\Big|\mathcal{F}_\tau\right] d\tau\right].
\end{aligned}
$$

Since $a(s, Y(s))$ is bounded on $[0, T]$, Fubini's theorem can be applied

$$
\begin{aligned}
&1 + \mathbb{E}^{t,i}\left[\int_t^T E\left[a(\tau, Y(\tau))\exp\left\{\int_\tau^T a(s, Y(s))ds\right\}\Big|\mathcal{F}_\tau\right] d\tau\right] \\
&= 1 + \int_t^T \mathbb{E}^{t,i}\left[\mathbb{E}\left[a(\tau, Y(\tau))\exp\left\{\int_\tau^T a(s, Y(s))ds\right\}\Big|\mathcal{F}_\tau\right]\right] d\tau \\
&= 1 + \int_t^T \mathbb{E}^{t,i}\left[a(\tau, Y(\tau))\exp\left\{\int_\tau^T a(s, Y(s))ds\right\}\right] d\tau \\
&= 1 + \mathbb{E}^{t,i}\int_t^T \left[a(\tau, Y(\tau))\exp\left\{\int_\tau^T a(s, Y(s))ds\right\}\right] d\tau \\
&= \mathbb{E}^{t,i}\left[\exp\left\{\int_t^T a(\tau, Y(\tau))ds\right\}\right].
\end{aligned}
$$

The last equality follows from the fact that

$$\exp\left\{\int_t^T a(\tau, Y(\tau))ds\right\} = 1 + \int_t^T a(\tau, Y(\tau))\exp\left\{\int_\tau^T a(s, Y(s))ds\right\} d\tau.$$

Therefore,

$$\hat{g}(t,i) = 1 + E^{t,i}\int_t^T a(\tau, Y(\tau))\hat{g}(\tau, Y(\tau))ds.$$

This means $\hat{g}(t,i)$ is a solution of the equation (5.14), then $\hat{g}(t, Y(t))$ is the unique solution of (5.15). Thus, $\hat{g}(t,i)$ is the unique solution of (5.14). $\square$

We assume for the moment that we only consider $m(t)$ which is in $[-M, M]$ with $M > 0$ big enough such that m^* is in $[-M, M]$. This restriction does not cause any problem for our application since Theorem 5.2 implies that $m^*(t, Y(t))$ is bounded on $[0, T]$ (see Section 5.4 in [16]).

Theorem 5.4. *(Verification Theorem) Let $Q := [0,T] \times \mathbb{R}$. Suppose $G \in C^{1,2}(Q)$ with $|G(t,\tilde{C},i)| \leq K(1 + |\tilde{C}|^k)$ for constants $K > 0$, $k \in \mathbb{N}$ and for all $i \in E$, is a solution of the HJB equation (5.8). Then we have*

a) $G(t,\tilde{C},i) \geq J(t,\tilde{C},i)$ for all $0 \leq t \leq T, \tilde{C} \in \mathbb{R}_+$ and $i \in E$.

b) If for all $(t,\tilde{C}) \in Q$ there exists $m^* \in \mathcal{A}(t,\mathbb{F})$ with

$$m^*(s) \in \operatorname*{arg\,max}_{m\in[-M,M]}\left(\mathcal{A}^m(s, \tilde{C}^*(s), Y(s))\right)$$

for all $s \in [t,T]$, where $\tilde{C}^*(s)$ is the controlled process with respect to $m^*(s)$ via (5.7), then we obtain $G(t,\tilde{C},i) = J(t,\tilde{C},i) = J_{m^*}(t,\tilde{C},i)$.

Proof. Let $0 \leq t < T$, $Y(t) = i$ and $\tilde{C}(t) = \tilde{C}$. Assume that O is a bounded subset of $\mathbb{R}$ and $t < \theta \leq T$ is a stopping time with respect to the filtration $\mathbb{F}$ such that $\tilde{C}(s) \in O$ for all $t \leq s \leq \theta$. Let $m \in \mathcal{A}(t,\mathbb{F})$ be an arbitrary admissible multiple. An application of Ito's lemma for semi-martingales (see [23]) gives

$$\begin{aligned} G(\theta, \tilde{C}(\theta), Y(\theta)) &= G(t,\tilde{C},i) + \int_t^\theta G_t(s,\tilde{C}(s),Y(s))ds + \int_t^\theta G_{\tilde{C}}(s,\tilde{C}(s),Y(s))d\tilde{C} \\ &\quad + \int_t^\theta G_{\tilde{C}\tilde{C}}(s,\tilde{C}(s),Y(s))d\tilde{C}d\tilde{C} \\ &\quad + \sum_{t<s\leq\theta}\left[G(s,\tilde{C}(s),Y(s)) - G(s,\tilde{C}(s),Y(s-))\right]. \end{aligned} \tag{5.19}$$

By the discounted cushion equation (5.7), we obtain

$$\begin{aligned} G(\theta, \tilde{C}(\theta), Y(\theta)) &= G(t, C, i) \\ &+ \int_t^\theta [G_t(s, \tilde{C}(s), Y(s)) + G_{\tilde{C}}(s, \tilde{C}(s), Y(s)) m(s)\nu(s)\tilde{C}(s) \\ &+ \frac{1}{2} G_{\tilde{C}\tilde{C}}((\sigma_S\rho + \sigma_B)^2 + (\sigma_S\bar{\rho})^2) m^2 \tilde{C}^2(s)] ds \\ &+ \int_t^\theta G_{\tilde{C}}(s, \tilde{C}(s), Y(s)) m(s)\tilde{C}(s)(\sigma_S\rho + \sigma_B) dW_1(s) \\ &+ \int_t^\theta G_{\tilde{C}}(s, \tilde{C}(s), Y(s)) m(s)\tilde{C}(s)\sigma_S\bar{\rho} dW_2(s) \\ &+ \sum_{t<s\le\theta} \Big[G(s, \tilde{C}(s), Y(s)) - G(s, \tilde{C}(s), Y(s-)) \Big]. \end{aligned}$$

Recall that the dynamics of $\tilde{C}$ is

$$d\tilde{C}(t) = m(t)\tilde{C}\left[\nu(t, Y(t))dt + (\sigma_S\rho + \sigma_B)dW_1(t) + \sigma_S\bar{\rho}dW_2(t)\right].$$

Since $m(t) \in [0, M], \sigma_S$ and $\nu(t, Y(t))$ are bounded, by Proposition A.2 (see Appendix A), there exists a constant N such that for all $q \ge 0$ and $t \in [0, T]$,

$$\mathbb{E}[\sup_{s\le t} |\tilde{C}(s)|^q] \le N e^{Nt}(1 + |\tilde{C}_0|)^q. \tag{5.20}$$

Since O is bounded and $G(t, \tilde{C}, i) \in C^{1,2}([0, \theta] \times O)$, the boundedness of $G_{\tilde{C}}$ on $[0, \theta] \times O$ is clear. Furthermore, by our assumption, $\tilde{C}^m(s)$ only takes values in a bounded set O. Thus it is clear that

$$\mathbb{E}^{t,\tilde{C},i} \int_t^\theta G_{\tilde{C}}(s, \tilde{C}(s), Y(s)) m(s)\tilde{C}(s)(\sigma_S\rho + \sigma_B) dW_1(s) = 0$$

and

$$\mathbb{E}^{t,\tilde{C},i} \int_t^\theta G_{\tilde{C}}(s, \tilde{C}(s), Y(s)) m(s)\tilde{C}(s)\sigma_S\bar{\rho} dW_2(s) = 0.$$

By the same argument in Lemma 5.3 and the bounded property of $G(s, \tilde{C}^m(s), Y(s))$ on $[0, \theta] \times O$, we obtain

$$\begin{aligned} &\mathbb{E}^{t,\tilde{C},i} \sum_{t<s\le T} [G(s, \tilde{C}(s), Y(s)) - G(s, \tilde{C}(s), Y(s-))] \\ &= \mathbb{E}^{t,\tilde{C},i} \int_t^T \sum_{j\in E} \left[G(s, \tilde{C}(s), j) - G(s, \tilde{C}(s), Y(s-))\right] q_{Y(s-)j} ds. \end{aligned}$$

Taking the expectation on both sides of (5.19) and noticing that G satisfies the HJB equation (5.8) yield

$$\mathbb{E}^{t,\tilde{C},i} G(\theta, \tilde{C}(\theta), Y(\theta)) \leq G(t, \tilde{C}, i). \tag{5.21}$$

In the general case, for $0 < \rho^{-1} < T$ let

$$O_\rho = \{\frac{1}{\rho} < \tilde{C} < \rho\}, Q_\rho = [0, T - \frac{1}{\rho}) \times O_\rho.$$

Let τ_ρ be the exist time of $(t, \tilde{C}(t))$ from Q_ρ, and $\theta_\rho = \min(T, \tau_\rho)$. We then obtain

$$\mathbb{E}^{t,\tilde{C},i}[G(\theta_\rho, \tilde{C}(\theta_\rho), Y(\theta_\rho))] \leq G(t, \tilde{C}, i). \tag{5.22}$$

First, notice that $\theta_\rho \to T$ a.s. Furthermore, due to the continuity of G on $[0, T]$ and $G(T, \tilde{C}(T), 0) = G(T, \tilde{C}(T), 1) = U(\tilde{C(T)})$, we obtain

$$G(\theta_\rho, \tilde{C}(\theta_\rho), Y(\theta_\rho)) \to G(T, \tilde{C}(T), Y(T)) \text{ a.s.}$$

when $\rho \to \infty$. An application of the polynomial growth condition and the inequality (5.20) yields

$$\begin{aligned} &\mathbb{E}^{t,\tilde{C},i}\big[G(\theta_\rho, \tilde{C}(\theta_\rho), Y(\theta_\rho))\big] \leq \mathbb{E}^{t,\tilde{C},i}\big[K(1 + |\tilde{C}(\theta_\rho)|^k)\big] \\ &\leq \mathbb{E}^{t,\tilde{C},i}\big[K(1 + \sup_{s \leq T} |\tilde{C}(s)|^k)\big] < \infty. \end{aligned}$$

Therefore, by the Dominated Convergence Theorem

$$\mathbb{E}^{t,\tilde{C},i}\big[G(\theta_\rho, \tilde{C}(\theta_\rho), Y(\theta_\rho))\big] \to \mathbb{E}^{t,\tilde{C},i} G(T, \tilde{C}(T), Y(T)) \text{ as } \rho \to \infty.$$

However, due to the boundary condition $G(T, \tilde{C}(T), Y(T)) = U(\tilde{C}(T))$ and (5.22), we obtain

$$\mathbb{E}^{t,\tilde{C},i} U(\tilde{C}(T)) \leq G(t, \tilde{C}, i).$$

Taking the supremum over the set of admissible controls $\mathcal{A}(t, \mathbb{F})$ gives

$$J(t, \tilde{C}, i) \leq G(t, \tilde{C}, i). \tag{5.23}$$

If

$$m^*(s) \in \underset{m \in [-M, M]}{\arg\max} \Big(\mathcal{A}^m(s, \tilde{C}^*(s), Y(s))\Big)$$

for all $s \in [t,T]$, where $\tilde{C}^*(s)$ is the controlled process with respect to $m^*(s)$ via (5.8). Then equality in (5.21) holds, thus equality in (5.23) holds too. This means $G(t,\tilde{C},i) = J(t,\tilde{C},i) = J_{m^*}(t,\tilde{C},i)$. □

5.1.4 Mean and variance of the variable-multiple CPPI

From the stochastic differential equation (5.7), we obtain

$$\begin{aligned} C(T) &= \tilde{C}(T) \\ &= \tilde{C}(0)\exp\Big\{ \int_0^T \Big[m(s)\nu(s) - \frac{1}{2}m^2(s)\big((\sigma_S\rho+\sigma_B)^2 + \sigma_S^2(1-\rho^2)\big)\Big] ds \\ &\quad + \int_0^T m(s)(\sigma_S\rho+\sigma_B)dW_1(s) + \int_0^T m(s)\sigma_S\bar{\rho}dW_2(s)\Big\} =: \tilde{C}(0)\exp\{X\}. \end{aligned}$$

Conditioning on knowing the Markov process Y, Theorem A.4 (see Appendix A) concludes that $X|\{Y(t) : t \in [0,T]\}$ is a normally distributed random variable with

$$\mu_{X|Y} = \int_0^T \Big[m(s|Y(s))\nu(s|Y(s)) - \frac{1}{2}m^2(s|Y(s))\big((\sigma_S\rho+\sigma_B)^2 + \sigma_S^2(1-\rho^2)\big)\Big] ds$$

and

$$\sigma^2_{X|Y} = \int_0^T m^2(s|Y(s))\big((\sigma_S\rho+\sigma_B)^2 + \sigma_S^2(1-\rho^2)\big)ds.$$

Here, $m(s|Y(s))$ has the meaning that m is considered as a function of the known value of $Y(s)$. The same explanation is applied to the notation $\nu(s|Y(s))$.

The expected value and variance of the variable-CPPI are calculated as follow.

Proposition 5.5.

$$\begin{aligned} \mathbb{E}[C(T)] &= \tilde{C}(0)\mathbb{E}\Big[e^{\int_0^T m(s|Y(s))\nu(s|Y(s))ds}\Big], \\ \mathrm{Var}[C(T)] &= \tilde{C}^2(0)\mathbb{E}\Big[e^{\sigma^2_{X|Y}+2\int_0^T m(s|Y(s))\nu(s|Y(s))ds}\Big] - \big(\mathbb{E}[C(T)]\big)^2. \end{aligned}$$

Proof. First, we notice that if $Z \sim N(\mu_Z, \sigma_Z^2)$, then

$$\mathbb{E}\big[e^Z\big] = e^{\mu_Z + \frac{1}{2}\sigma_Z^2}$$

and

$$\mathrm{Var}(Z) = e^{2\mu_Z} e^{\sigma_Z^2}\Big(e^{\sigma_Z^2} - 1\Big).$$

From the above notice and the tower property of condition expectation, we obtain

$$\mathbb{E}[C(T)] = \mathbb{E}\big[\mathbb{E}[C(T)|Y]\big] = \tilde{C}(0)\mathbb{E}\big[e^{\mu_{X|Y}+\frac{1}{2}\sigma_{X|Y}^2}\big] \tag{5.24}$$

and

$$\begin{aligned}\mathrm{Var}[C(T)] &= \tilde{C}^2(0)\mathrm{Var}\big[e^X\big] = \tilde{C}^2(0)\Bigg\{\mathbb{E}\Big[\mathrm{Var}[e^X|Y]\Big] + \mathrm{Var}\Big[\mathbb{E}[e^X|Y]\Big]\Bigg\} \\ &= \tilde{C}^2(0)\Bigg\{\mathbb{E}\Big[e^{2\mu_{X|Y}} e^{\sigma_{X|Y}^2}\Big(e^{\sigma_{X|Y}^2} - 1\Big)\Big] + \mathrm{Var}\Big[e^{\mu_{X|Y}+\frac{1}{2}\sigma_{X|Y}^2}\Big]\Bigg\}.\end{aligned} \tag{5.25}$$

Plugging the formulas of $\mu_{X|Y}$ and $\sigma_{X|Y}^2$ into the equations (5.24) and (5.25) and simplifying, we have the proof. □

If m is constant, then σ_X is independent of the Markov chain Y. Therefore, we have the following corollary for the constant-multiple CPPI strategy.

Corollary 5.6.

$$\mathbb{E}[C(T)] = \tilde{C}(0)\mathbb{E}\Big[e^{m\int_0^T \nu(s|Y(s))ds}\Big],$$
$$\mathrm{Var}[C(T)] = \tilde{C}^2(0)e^{\sigma_X^2}\mathbb{E}\Big[e^{2m\int_0^T \nu(s|Y(s))ds}\Big] - \big(\mathbb{E}[C(T)]\big)^2.$$

Proof. The corollary follows directly from Proposition 5.5. □

5.1.5 Average-data model

We assume that the Markov chain Y has a unique stationary distribution $p = (p_j, j \in E)$, and let

$$\bar{\mu} = \sum_{j\in E} \mu(j)p_j.$$

In the model (5.2), we use $\mu(Y(t))$, a function of a continuous time finite state Markov chain, to represent the excess return on the stock. Now it is replaced by

$\bar{\mu}$ in the new model. In particular, the stock price satisfies the following stochastic differential equation

$$\frac{dS(t)}{S(t)} = (r(t) + \bar{\mu})\, dt + \sigma_S(\rho dW_1(t) + \bar{\rho} dW_2(t)).$$

We shall call that average-data model to distinguish from the Markov-switching model. The value function $\bar{J}(t, \tilde{C})$ obtained in the model with average-data is then compared with the value function $\mathbb{E}^p[J(t, \tilde{C}, Y(t))]$ in the Markov-parameter framework with $Y(t) \overset{d}{=} p$.

From Theorem 5.2 and Lemma 5.3, we obtain

$$\mathbb{E}^p[J(t, \tilde{C}, Y(t))] = \frac{\tilde{C}^{1-\eta}}{1-\eta} \mathbb{E}^p \Big[\exp \Big(\int_t^T a(s, Y(s)) ds \Big) \Big],$$

where $a(s, Y(s)) = \frac{1}{2}(m^*(s))^2 \eta(1-\eta)((\sigma_S \rho + \sigma_B)^2 + (\sigma_S \bar{\rho})^2)$ with

$$m^*(s) = \frac{\nu(s, Y(s))}{\eta((\sigma_S \rho + \sigma_B)^2 + (\sigma_S \bar{\rho})^2)}$$

and

$$\nu(s, Y(s)) = \mu(Y(s)) + \sigma_B^2 - q\sigma_B + \rho \sigma_S \sigma_B.$$

Therefore,

$$\mathbb{E}^p[J(t, \tilde{C}, Y(t))] = \frac{\tilde{C}^{1-\eta}}{1-\eta} \mathbb{E}^p \Big[\exp \Big(\frac{1-\eta}{2\eta} \int_t^T \frac{(\mu(Y(s)) + \sigma_B^2 - q\sigma_B + \rho \sigma_S \sigma_B)^2}{(\sigma_S \rho + \sigma_b)^2 + (\sigma_S \bar{\rho})^2} ds \Big) \Big].$$

Based on the fact that the functions e^x and $(x+k)^2$ are convex, Jensen's inequality gives

$$\begin{aligned}
\mathbb{E}^p[J(t, \tilde{C}, Y(t))] &= \frac{\tilde{C}^{1-\eta}}{1-\eta} \mathbb{E}^p \Big[\exp \Big(\frac{1-\eta}{2\eta} \int_t^T \frac{\big(\mu(Y(s) + \sigma_B^2 - q\sigma_B + \rho \sigma_S \sigma_B\big)^2}{(\sigma_S \rho + \sigma_b)^2 + (\sigma_S \bar{\rho})^2} ds \Big) \Big] \\
&> \frac{\tilde{C}^{1-\eta}}{1-\eta} \exp \Big(\frac{1-\eta}{2\eta} \int_t^T \frac{\Big(\mathbb{E}^p\big(\mu(Y(s)) + \sigma_B^2 - q\sigma_B + \rho \sigma_S \sigma_B\big)\Big)^2}{(\sigma_S \rho + \sigma_b)^2 + (\sigma_S \bar{\rho})^2} ds \Big) \\
&= \frac{\tilde{C}^{1-\eta}}{1-\eta} \exp \Big(\frac{1-\eta}{2\eta} \int_t^T \frac{(\bar{\mu} + \sigma_B^2 - q\sigma_B + \rho \sigma_S \sigma_B)^2}{(\sigma_S \rho + \sigma_b)^2 + (\sigma_S \bar{\rho})^2} ds \Big) \\
&= \bar{J}(t, \tilde{C}).
\end{aligned}$$

The above inequality implies that this particular Markov-switching model achieves larger expected utility than the corresponding average-data model.

5.1.6 A special case

In this section, we consider a simple situation where the market only has two states corresponding to two different values of the excess return μ:

- **normal state**: $\mu = \mu_0$,
- **recessive state**: $\mu = \mu_1$,

where $\mu_0 > \mu_1 > 0$. Furthermore, we assume the following form of the intensity matrix,

$$Q = \begin{pmatrix} -\lambda & \lambda \\ 0 & 0 \end{pmatrix},$$

where $\lambda > 0$. Therefore, if at the initial time the stock is in normal period, then after a random time with exponential distribution Exp(λ) the stock falls into recession and stays there until the horizon T. Inversely, the stock is in recessive period for a random time Exp(λ) from the beginning and then switches to normal conditions and stays there. The smaller the value of λ the longer the stock is likely to stay in the initial regime.

In what follows, we assume that the market is in the normal state at time $t = 0$, i.e. $\mu(Y(0)) = \mu_0$.

Optimal constant problem. In order to have a benchmark to compare with the variable-multiple CPPI strategy, we make an assumption on the set of possible multiples. Instead of considering the huge set of admissible controls $\mathcal{A}(0, \mathbb{F})$, we are only interested in constant multiples. Therefore, our optimization problem is now

$$\max_{m \in \mathbb{R}} \mathbb{E}\Big[\frac{C(T)^{1-\eta}}{1-\eta}\Big].$$

From the equation (5.7), we obtain

$$\begin{aligned} C(T) &= \tilde{C}(0) \exp\Big\{ \int_0^T \Big[m\nu(s, Y(s)) - \frac{1}{2} m^2 \big((\sigma_S \rho + \sigma_B)^2 + \sigma_S^2 (1-\rho^2) \big) \Big] ds \\ &+ \int_0^T m(\sigma_S \rho + \sigma_B) dW_1(s) + \int_0^T m \sigma_S \bar{\rho} dW_2(s) \Big\}. \end{aligned}$$

Then the following holds

$$\frac{C(T)^{1-\eta}}{1-\eta} = \frac{\tilde{C}^{1-\eta}(0)}{1-\eta} \exp\{X\},$$

where

$$\begin{aligned} X &= (1-\eta)\int_0^T \Big[m\nu(s,Y(s)) - \frac{1}{2}m^2\big((\sigma_S\rho+\sigma_B)^2+\sigma_S^2(1-\rho^2)\big)\Big]ds \\ &\quad + \int_0^T m(1-\eta)(\sigma_S\rho+\sigma_B)dW_1(s) + \int_0^T m(1-\eta)\sigma_S\bar{\rho}dW_2(s). \end{aligned}$$

Conditioning on knowing the Markov process Y, Theorem A.4 (see Appendix A) shows that $X|\{Y(t) : t\in[0,T]\}$ is a normally distributed random variable with

$$\mu_{X|Y} = m(1-\eta)\int_0^T \nu(s|Y(s))ds - \frac{1}{2}m^2(1-\eta)\int_0^T \Big((\sigma_S\rho+\sigma_B)^2+\sigma_S^2(1-\rho^2)\Big)ds \tag{5.26}$$

and

$$\sigma_{X|Y}^2 = m^2(1-\eta)^2\int_0^T \Big((\sigma_S\rho+\sigma_B)^2+\sigma_S^2(1-\rho^2)\Big)ds. \tag{5.27}$$

Here, $\nu(s|Y(s))$ is understood as a function of the known value of $Y(s)$. Hence,

$$\mathbb{E}\big[e^X|Y\big] = e^{\mu_{X|Y}+\frac{1}{2}\sigma_{X|Y}^2}.$$

The tower property of conditional expectation gives

$$\mathbb{E}\Big[\frac{C(T)^{1-\eta}}{1-\eta}\Big] = \mathbb{E}\Big[\mathbb{E}\big[\frac{C(T)^{1-\eta}}{1-\eta}|Y\big]\Big] = \frac{\tilde{C}^{1-\eta}(0)}{1-\eta}\mathbb{E}\Big[e^{\mu_{X|Y}+\frac{1}{2}\sigma_{X|Y}^2}\Big]. \tag{5.28}$$

From the equations (5.26) and (5.27), we get

$$\begin{aligned} \mu_{X|Y}+\frac{1}{2}\sigma_{X|Y}^2 &= -\frac{1}{2}\eta(1-\eta)m^2\int_0^T \Big((\sigma_S\rho+\sigma_B)^2+\sigma_S^2(1-\rho^2)\Big)ds \\ &\quad +(1-\eta)m\int_0^T \nu(s|Y(s))ds. \end{aligned} \tag{5.29}$$

Since $\nu(s|Y(s)) = \mu(s|Y(s)) + \sigma_B^2 - q\sigma_B + \rho\sigma_S\sigma_B$, we obtain

$$\begin{aligned} &\mathbb{E}\Big[\exp\Big\{(1-\eta)m\int_0^T \nu(s|Y(s))ds\Big\}\Big] \\ =\ &\mathbb{E}\Big[\exp\Big\{(1-\eta)m\int_0^T \big(\mu(s|Y(s))+\sigma_B^2-q\sigma_B+\rho\sigma_S\sigma_B\big)ds\Big\}\Big] \\ =\ &\exp\Big\{(1-\eta)m\int_0^T \big(\sigma_B^2-q\sigma_B+\rho\sigma_S\sigma_B\big)ds\Big\} \\ &\times\mathbb{E}\Big[\exp\Big\{(1-\eta)m\int_0^T \mu(s|Y(s)))ds\Big\}\Big]. \end{aligned} \tag{5.30}$$

Notice that the market is in the normal state for a random time with distribution Exp(λ). In the remaining time, it stays in the recessive period. It then follows that

$$
\begin{aligned}
&\mathbb{E}\Big[\exp\Big\{(1-\eta)m\int_0^T \mu(s|Y(s))ds\Big\}\Big]\\
=\ &\int_0^T \exp\Big\{(1-\eta)m\big(\mu_0 t+\mu_1(T-t)\big)\Big\}\lambda e^{-\lambda t}dt+\int_T^\infty \exp\{(1-\eta)m\mu_0 T\}\lambda e^{-\lambda t}dt\\
=\ &\lambda\exp\big\{(1-\eta)m\mu_1 T\big\}\int_0^T \exp\Big\{t\big((1-\eta)m(\mu_0-\mu_1)-\lambda\big)\Big\}dt\\
&+\exp\{(1-\eta)m\mu_0 T\}e^{-\lambda T}\\
=\ &\lambda\exp\big\{(1-\eta)m\mu_1 T\big\}\frac{\exp\Big\{t\big((1-\eta)m(\mu_0-\mu_1)-\lambda\big)\Big\}}{(1-\eta)m(\mu_0-\mu_1)-\lambda}\Big|_0^T\\
&+\exp\{(1-\eta)m\mu_0 T\}e^{-\lambda T}\\
=\ &\lambda\frac{\exp\big\{(1-\eta)m\mu_0 T-\lambda T\big\}-\exp\big\{(1-\eta)m\mu_1 T\big\}}{(1-\eta)m(\mu_0-\mu_1)-\lambda}\\
&+\exp\{(1-\eta)m\mu_0 T\}e^{-\lambda T}. \qquad (5.31)
\end{aligned}
$$

In summary, the equations (5.28) to (5.31) yield

$$
\begin{aligned}
\mathbb{E}\Big[\frac{C(T)^{1-\eta}}{1-\eta}\Big] =\ &\frac{\tilde{C}^{1-\eta}(0)}{1-\eta}\exp\Big\{-\frac{1}{2}\eta(1-\eta)m^2\int_0^T\Big((\sigma_S\rho+\sigma_B)^2+\sigma_S^2(1-\rho^2)\Big)ds\Big\}\\
&\times\exp\Big\{(1-\eta)m\int_0^T\big(\sigma_B^2-q\sigma_B+\rho\sigma_S\sigma_B\big)ds\Big\}\\
&\times\Big(\lambda\frac{\exp\big\{(1-\eta)m\mu_0 T-\lambda T\big\}-\exp\big\{(1-\eta)m\mu_1 T\big\}}{(1-\eta)m(\mu_0-\mu_1)-\lambda}\\
&\quad+\exp\{(1-\eta)m\mu_0 T\}e^{-\lambda T}\Big). \qquad (5.32)
\end{aligned}
$$

Therefore, m is the optimal constant multiple if and only if it maximizes the right hand side of the equation (5.32).

Average-data model. In analogy to Section 5.1.5, we want to compare the Markov-switching model with an "equivalent" average-data model. Suppose that the stock prices of the average-data framework satisfy

$$\frac{dS(t)}{S(t)}=(r(t)+\bar{\mu})\,dt+\sigma_S(\rho dW_1(t)+\bar{\rho}dW_2(t)).$$

The question is what is the "equivalent" $\bar{\mu}$ in this situation. We shall choose $\bar{\mu}$ such that the total return on the stock over the investment period is the same as that in the Markov-parameter model. More specifically, the following holds

$$\mathbb{E}\int_0^T \mu(Y(t))dt = \int_0^T \bar{\mu}dt = \bar{\mu}T. \tag{5.33}$$

The left hand side of (5.33) is

$$\begin{aligned}\mathbb{E}\int_0^T \mu(Y(t))dt &= \int_0^T \Big(\int_0^t \mu_0 ds + \int_t^T \mu_1 ds\Big)\lambda e^{-\lambda t}dt \\ &\quad + \int_T^\infty \Big(\int_0^T \mu_0 ds\Big)\lambda e^{-\lambda t}dt \\ &= \lambda\int_0^T \big(\mu_1 T + (\mu_0-\mu_1)t\big)e^{-\lambda t}dt + \mu_0 T e^{-\lambda T} \\ &= \mu_1 T(1-e^{-\lambda T}) + (\mu_0-\mu_1)\Big(-Te^{-\lambda T} - \frac{e^{-\lambda T}}{\lambda} + \frac{1}{\lambda}\Big) \\ &\quad + \mu_0 T e^{-\lambda T}.\end{aligned}$$

Thus, we obtain

$$\bar{\mu} = \frac{\mu_1 T(1-e^{-\lambda T}) + (\mu_0-\mu_1)\big(-Te^{-\lambda T} - \frac{e^{-\lambda T}}{\lambda} + \frac{1}{\lambda}\big) + \mu_0 T e^{-\lambda T}}{T}.$$

In what follows, we will derive the expected utility of the final cushion in the average-data framework. It then will be used to compare with the Markov-switching model in later section.

From Section 5.1.4, we obtain

$$\frac{C(T)^{1-\eta}}{1-\eta} = \frac{\tilde{C}^{1-\eta}(0)}{1-\eta}\exp\big\{X\big\},$$

where

$$\begin{aligned}X &= \int_0^T (1-\eta)\Big[m(s)\nu(s) - \frac{1}{2}m^2(s)\big((\sigma_S\rho+\sigma_B)^2 + \sigma_S^2(1-\rho^2)\big)\Big]ds \\ &\quad + \int_0^T (1-\eta)m(s)(\sigma_S\rho+\sigma_B)dW_1(s) + \int_0^T (1-\eta)m(s)\sigma_S\bar{\rho}dW_2(s).\end{aligned}$$

Here,

$$m(s) = \frac{\nu(s)}{\eta\big((\sigma_S\rho+\sigma_B)^2 + (\sigma_S\bar{\rho})^2\big)}$$

and

$$\nu(s) = \bar{\mu} + \sigma_B^2 - q\sigma_B + \rho\sigma_S\sigma_B.$$

An application of Theorem A.4 (see Appendix A) shows that X is normally distributed with

$$\mu_X = (1-\eta)\int_0^T \Big[m(s)\nu(s) - \frac{1}{2}m^2(s)\big((\sigma_S\rho+\sigma_B)^2+\sigma_S^2(1-\rho^2)\big)\Big]ds$$

and

$$\sigma_X^2 = (1-\eta)^2\int_0^T m^2(s)\big((\sigma_S\rho+\sigma_B)^2+\sigma_S^2(1-\rho^2)\big)ds.$$

Therefore,

$$\mathbb{E}[\frac{C(T)^{1-\eta}}{1-\eta}] = \frac{\tilde{C}(0)^{1-\eta}}{1-\eta}e^{\mu_X+\frac{1}{2}\sigma_X^2}.$$

5.1.7 Numerical examples and discussion

In this section, we simulate the model that is described in last section. Here, the market has two states:

- **normal state**: $\mu = \mu_0 = 0.04$,
- **recessive state**: $\mu = \mu_1 = 0.01$.

It is assumed that the market is in normal state at $t = 0$. In addition, the Markov-chain is characterized by the intensity matrix,

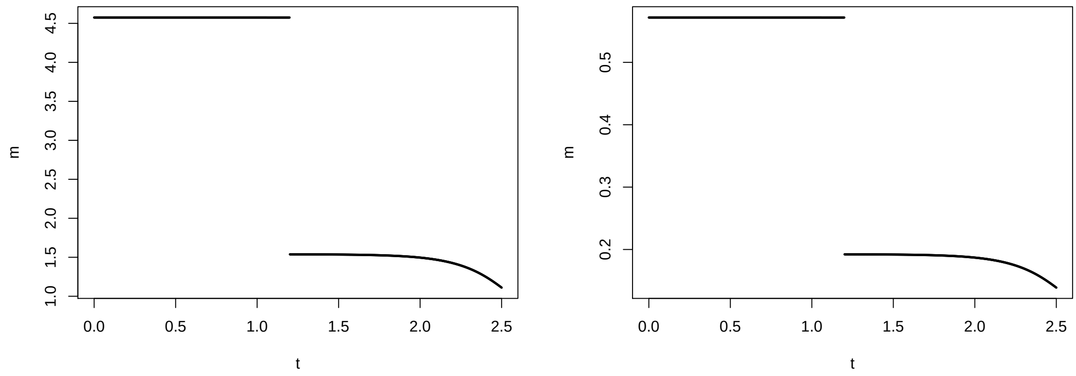
$$Q = \begin{pmatrix} -\lambda & \lambda \\ 0 & 0 \end{pmatrix},$$

where $\lambda > 0$.

We consider two types of investors characterized by their risk appetites:

- **risk-seeking investor** ($\eta = 0.1$),
- **risk averse investor** ($\eta = 0.8$).

The other parameters are chosen as follows: $\sigma_S = 0.3, \sigma_r = 0.2, a = 5, T = 2.5, \rho = 0.3, q = 0$, $\tilde{C}(0) = 2M$ and the guarantee $K = 6M$[EUR] with $M = 1,000,000$. Notice that $\tilde{C}(0)$, $\tilde{K}$ are unit-free and $\tilde{K}(t) = 6M, 0 \leq t \leq T$.



(a) **(b)**

Figure 5.1: m as a function of time for the risk-seeking investor (left) and the risk-adverse investor (right)

We first examine the behaviour of the optimal multiple m for both types of investors. Figure 5.1 shows typical paths of m as a function of time. Both Figure 5.1a and Figure 5.1b show the discontinuity of m. This of course can be explained by the occurrence of the regime switching.

With regard to the risk-seeking investor, the multiple m is nearly 4.6 until $t = 1.2$, then it jumps down to about 1.5 and slightly reduces to 1 when time approaches the horizon T. Thus, all the time that we are considering, the multiple is always higher than 1, which is typical for a risk-seeking investor. Figure 5.1b displays a multiple's path of the risk averse investor. We can observe that the multiple is about 0.57 before having a jump at $t = 1.2$. In the remaining time, it takes values less than 0.2. For the risk averse investor, the multiple is below 1 all the time, which can be anticipated by the high risk aversion of the investor.

In the remaining part of the section, we want to investigate the following main points through simulation:

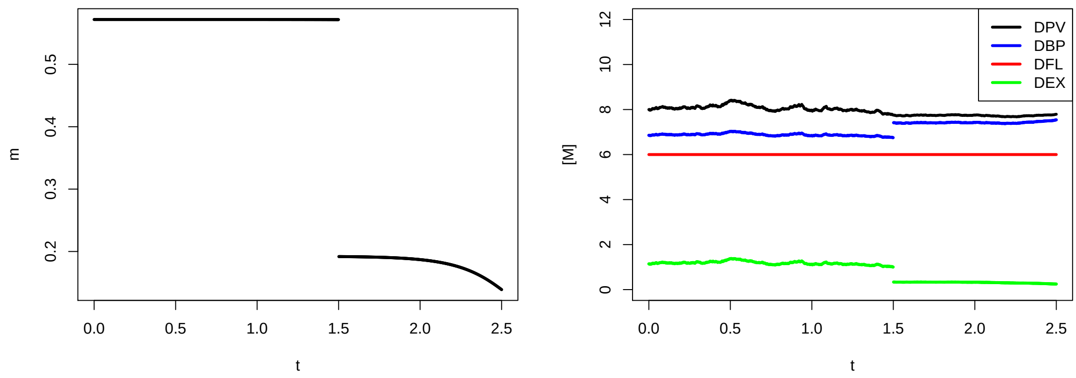
- Behaviours of the optimal portfolio and its components.
- The optimal variable-multiple strategy versus the optimal constant-multiple strategy.
- The Markov-switching model versus the average-data model.

To be convenient and be consistent with previous derivations, we will use discounted values instead of actual values. For instance, the discounted portfolio values shall be used in place of the portfolio values. We also make the following abbreviations for convenience:

- DPV : the discounted portfolio value,
- DBP : the discounted bond position,
- DEX : the discounted exposure,
- DFL : the discounted floor,
- DCP : the discounted constant-multiple portfolio value.

We begin to investigate the portfolio of the risk averse investor. Figure 5.2 presents simulated paths for the multiple, the discounted portfolio, the discounted bond position, the discounted exposure and the discounted floor. We observe from Figure 5.2a that the switching happens at $t = 1.5$. Before this time, the multiple is nearly 0.57. After that, it takes lower values. Also from Figure 5.2b, we see that the discounted bond position is always above the discounted floor. Moreover, there is a jump to a higher value in the discounted bond position at $t = 1.5$. However, we can see that the discounted exposure is quite small compared to the discounted bond position and it even takes smaller values after $t = 1.5$. Those events can be explained as follows. When the market falls into recession, the stock makes less profit than it does before. Therefore, the investor shifts an amount of money into another asset which in our case is the zero-coupon bond. Observing Figure 5.2b, we see that the discounted portfolio value is more fluctuated than the discounted bond position from the beginning to the jump time $t = 1.5$. This is caused by the fact that the stock is riskier than the bond and that the multiple is not too small during this time. In the remaining time, the discounted portfolio value behaves similarly as the discounted bond position, and it shows much less variation than before. Due to the domination of the bond in the portfolio and the definition of the numeraire, the less variation of the discounted portfolio and bond position is understandable.

In order to compare the optimal variable-multiple strategy and the optimal constant-multiple strategy, we have to determine the optimal constant multiple. This mission can be accomplished by finding $m \in \mathbb{R}$ which maximizes the right hand side



(a) **(b)**

Figure 5.2: m (left) and portfolio (right) ($\eta = 0.8$)

of the equation (5.32). Figure 5.3 illustrates how the optimal constant-multiple depends on the parameter λ. We can observe that the bigger the value of λ the smaller the value of the multiple. That can be explained by the following arguments. When the value of λ increases, the stock is more likely to switch to another state with lower excess return. It also means that the stock tends to spend more time in the recessive period. Therefore, the investor will prefer to use smaller multiples in order to protect his/her money.

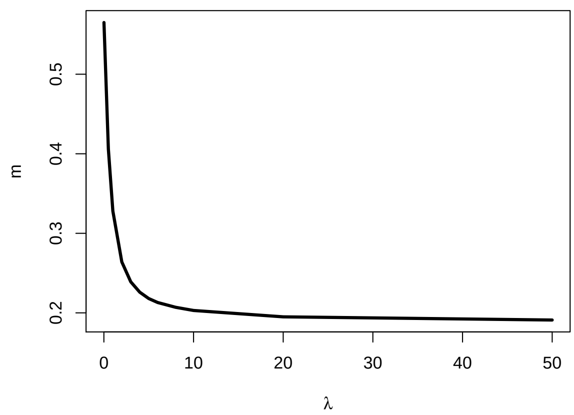
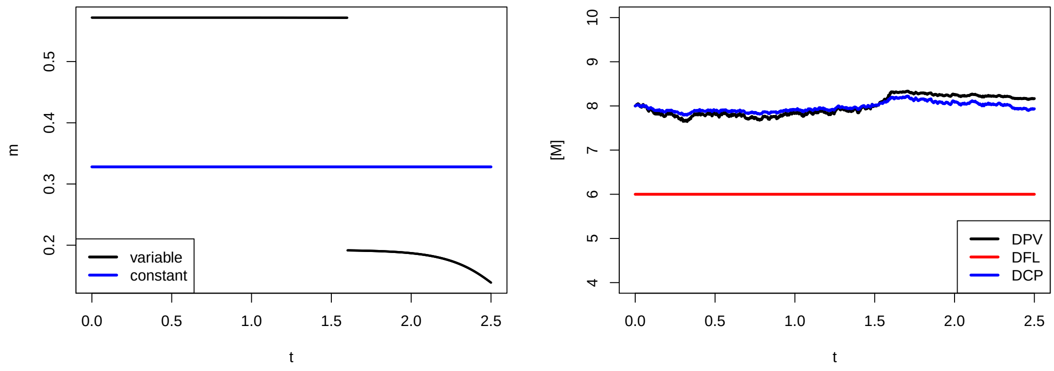


Figure 5.3: Optimal constant-multiple as a function of λ ($\eta = 0.8$)

Fix $\lambda = 1$ and the other parameters as above, then the optimal constant multiple is $m^* \approx 0.328$. It is the blue line in Figure 5.4a, while the discontinuous black curve is the optimal variable-multiples. We can observe that the regime switches at

$t = 1.6$. Figure 5.4b compares the discounted portfolio generated by the optimal

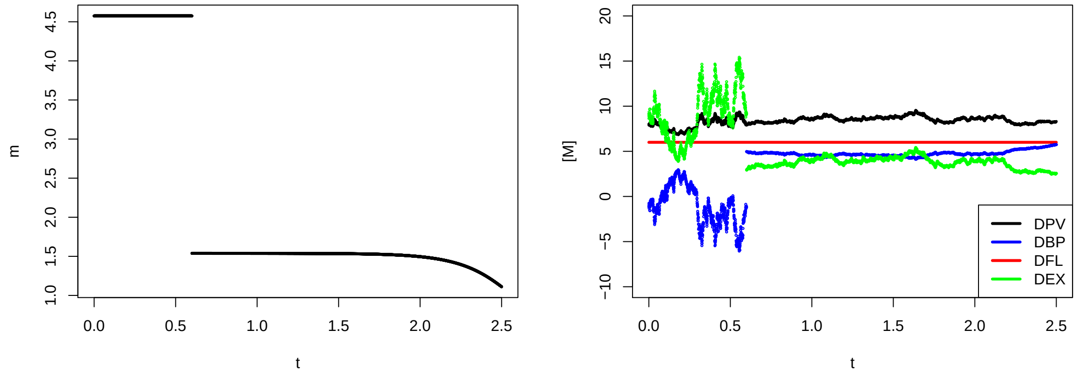


(a) **(b)**

Figure 5.4: Constant-multiple strategy versus variable-multiple strategy ($\eta = 0.8, \lambda = 1$)

variable-multiple strategy and that generated by the optimal constant-multiple strategy. At the first glance, both curves are always above the discounted floor curve. This is the consequence of utilizing CPPI strategy. We can see that there are not too much differences between two discounted portfolio paths. From the beginning to the jump time $t = 1.6$, the optimal variable multiple m is nearly 0.57, and the optimal constant multiple is 0.328. Both values are quite small; thus, less money is invested in the stock and more money is spent on the zero-coupon bond. It results in less variation of the discounted portfolio value. However, due to $m > m^*$, the discounted value of the optimal variable-multiple portfolio fluctuates more than that of the optimal constant-multiple strategy. After the jump, the variable-multiple m is less than the optimal constant-multiple; therefore, we can observe the opposite behaviour. In particular, the optimal variable-multiple curve does not change too much compared to the optimal constant-multiple curve. At the horizon, the optimal variable-multiple strategy is better than the optimal constant-multiple strategy. This is not the case in general, but it agrees with the theory derived above.

Now we investigate the portfolio and its components of the risk-seeking investor. Figure 5.5 contains simulated paths of the multiple, the discounted portfolio value, the discounted exposure, the discounted bond position and the discounted floor. We observe from Figure 5.5a that the multiple keeps high values(≈ 4.57) until the time of switching $t = 0.6$. In the remaining time, the investor lowers his/her multiple to about 1.5 and finally reduces to approximate 1 when time approaches the horizon T. Figure 5.5b clearly shows that the discounted portfolio value is



(a) **(b)**

Figure 5.5: m (left) and portfolio (right) of the risk-seeking investor

always above the discounted floor, but the discounted bond position is usually below. It stems from the high value of the multiple, which results in more money in stock and less money in bond. In the normal period, the discounted bond position is very volatile, and sometime it even takes negative values. The big values of the multiple are responsible for this situation. When the state changes, the bond position jumps to a higher value and remains more stable. The exposure reduces the same amount as the bond position, and it is also steadier. This situation results from small values of the multiple.

With this risk-seeking investor, we also find the optimal constant-multiple and then investigate the difference between two strategies. Figure 5.6 illustrates the dependence of the optimal constant-multiple m on the parameter λ. It is clear that m is a decreasing function of λ.

Fix $\lambda = 1$ and the other parameters as above, then the optimal constant-multiple is $m^* \approx 2.678$. It is the blue line in Figure 5.7a, while the discontinuous black curve is the optimal variable-multiple. We can observe that the regime switches at $t = 0.5$. Figure 5.7b compares the optimal variable-multiple strategy with the optimal constant-strategy. We can see that both discounted portfolio curves are quite volatile in the normal period. This is because the multiple takes high values which leads to the domination of the stock in the portfolio. In the recessive period, the variable-multiple investor reduces his multiple since the stock price

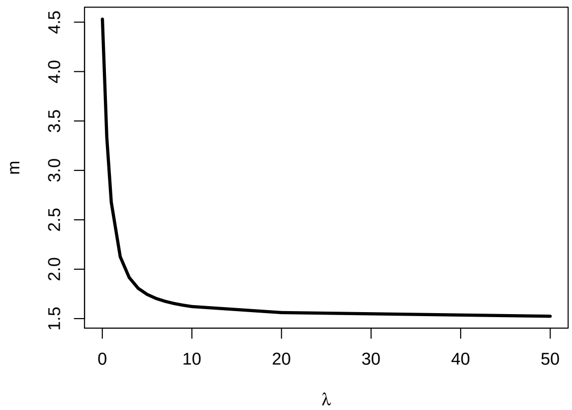
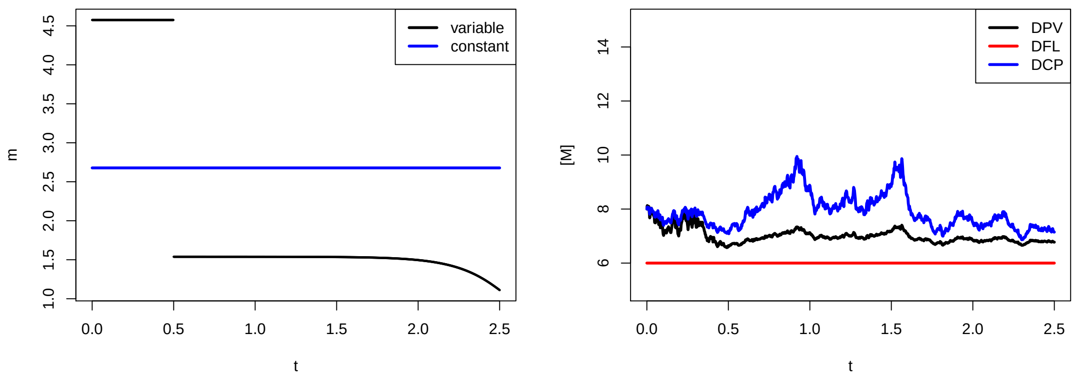


Figure 5.6: Optimal constant-multiple as a function of λ ($\eta = 0.1$)



(a) **(b)**

Figure 5.7: Constant-multiple strategy versus variable-multiple strategy ($\eta = 0.1, \lambda = 1$)

makes less profit in average. However, the constant-multiple investor still keeps his fairly high multiple which results in high volatility in his portfolio value. We can observe this situation in Figure 5.7b. It is obvious that both discounted portfolio values are above the discounted floor which is a direct consequence of utilizing the CPPI strategy. At the horizon T, the variable-multiple portfolio is worse than the constant-multiple portfolio. However, the theory confirms that in average the variable-multiple strategy is better.

Now we compare the optimal variable-multiple strategy and the optimal constant-multiple strategy through their means and variances of final cushions. In order to see clearly the difference, we consider only the risk-seeking investor. Let $mean\_V$,

var_V be the mean and variance of the final cushion generated by the optimal variable-multiple strategy and $mean\_C$, var_C be the corresponding quantities for the optimal constant-multiple strategy.

λ	1	2	3	4	5	6	7	8
$mean\_V[\tilde{C}(0)]$	1.263	1.163	1.126	1.108	1.097	1.090	1.085	1.082
$mean\_C[\tilde{C}(0)]$	1.191	1.117	1.094	1.083	1.078	1.074	1.071	1.069
$var\_V[\tilde{C}^2(0)]$	66.322	14.432	5.186	3.075	2.273	1.883	1.663	1.513
$var\_C[\tilde{C}^2(0)]$	6.836	2.533	1.747	1.440	1.288	1.196	1.135	1.091

Table 5.1: Means and variances ($\eta = 0.1$)

Table 5.1 clearly indicates that λ has an impact on the mean and variance of the terminal cushion whether the investor utilizes the variable-multiple strategy or the constant-multiple one. They are all decreasing functions of λ which is understandable. Notice that the bigger the value of λ is, the longer the stock is likely to stay in the recessive period. Hence, the investor tends to use a small multiple which leads to the reduction of mean and variance. We also observe that the optimal variable-multiple strategy generates more money in average and creates more variance than the optimal constant-multiple strategy does.

λ	1	2	3	4	5	6	7	8
Average-data	1.079	1.050	1.041	1.037	1.034	1.033	1.032	1.031
Markov-switching	1.108	1.067	1.054	1.047	1.042	1.040	1.038	1.036

Table 5.2: Expected utilities in the multiples of $\frac{\tilde{C}(0)^{1-\eta}}{1-\eta}$ ($\eta = 0.1$)

Finally, we make a comparison between the Markov-switching model and the average-data model. That average-data model has been investigated in the last part of Section 5.1.6. Table 5.2 shows the expected utility of the final cushion for both models. We can observe that the expected utility of the Markov-switching model is higher than that of the average-data model. Thus, the Markov-switching model might achieves larger expected utility. Furthermore, when λ increases, the expected utility generated by the two models are decreasing.

5.2 Portfolio with the Money Market Account

In last section, the Merton portfolio only had two assets: the zero-coupon bond and the stock. As in Horsky's thesis ([14]), we now include the money market account into the portfolio and solve the optimization problem. In what follows, we first present the mathematical framework and then determine the equation for the discounted cushion. Based on that framework, we formulate the investment problem for a CPPI investor. Next, we solve the optimization problem and then deliver a verification theorem. When the solution has been found, we provide the basic properties of the optimal solution. Furthermore, an average-data model will be studied in order to compare with our model. Finally, we consider a special case of our model and provide numerical examples.

5.2.1 Mathematical framework

As in the Merton problem, the interest rate is assumed to follow

$$dr(t) = a(b - r(t))dt + \sigma_r d\bar{W}_1(t),$$

where $\bar{W}_1$ is a Brownian motion under the risk-neutral measure. Then the money market account M satisfies the following equation

$$dM(t) = M(t)r(t)dt.$$

Since the short rate is of Vasicek's type, the price of a bond with maturity T has the following dynamic

$$dB(t,T) = B(t,T)\Big(\big(r(t) + q(t)\sigma_B^{\xi+1}(t)\big)dt - \sigma_B(t)dW_1(t)\Big),$$

where $\sigma_B(t) = \frac{\sigma_r}{a}(1-e^{a(t-T)})$, $\xi \geq 1$ and W_1 is a Brownian motion under real-world measure. As in Horsky's thesis ([14]), the risk premium of the bond is assumed to have a different form in compared to the equation (5.2) of the last section. This value together with the particular choice for the risk premium of the stock guarantee that the multiples are not singular. The stock price is defined as the solution of the stochastic differential equation

$$dS(t) = S(t)\Big(\big(\lambda(Y(t))\sigma_B(t) + r(t)\big)dt + \sigma_S\big(\rho dW_1(t) + \bar{\rho}dW_2(t)\big)\Big),$$

where $\bar{\rho} = \sqrt{1-\rho^2}$ and W_2 is another Brownian motion under the real world measure P with respect to the filtration $\mathbb{F}$. The continuous time finite state Markov chain Y takes values in E with the intensity matrix $Q = (q_{ij})$ modelling the regime switching in the stock prices. We suppose further that W_1, W_2, and Y are independent.

Since the interest rate is no longer constant, the money market account $M(t)$ and the bond price $B(t,T)$ are distinct. Indeed, at the horizon T, the money market account cannot guarantee a fixed amount as the bond $B(t,T)$ does. Thus, the money market account also contributes to the total exposure of the portfolio. In particular,

$$E = E_S + E_M,$$

where E_S and E_M are the amounts invested in the stock and the money market account, respectively. To be consistent with the CPPI strategies, we consider

$$E_S(t) = m(t)C(t)$$

and

$$E_M(t) = \tilde{m}(t)C(t),$$

where $C(t)$ is the cushion of the portfolio. That means the exposures in the stock and the money market account are proportional to the cushion.

5.2.2 Discounted cushion

Using the bond price $B(t,T)$ as a numeraire, we obtain

$$\begin{aligned} d\tilde{M} &= d(\frac{M}{B}) \\ &= \frac{dM}{B} + M\Big(-\frac{1}{B^2}dB + \frac{1}{B^3}dBdB\Big) + \Big(-\frac{1}{B^2}dB + \frac{1}{B^3}dBdB\Big)dM \\ &= \tilde{M}rdt - \tilde{M}\Big((r + q\sigma_B^{\xi+1})dt - \sigma_B dW_1\Big) + \sigma_B^2\tilde{M}dt \\ &= \tilde{M}\Big((\sigma_B^2 - q\sigma_B^{\xi+1})dt + \sigma_B dW_1\Big). \end{aligned} \tag{5.34}$$

Similarly, the above derivation with respect to the discounted stock prices leads to the following stochastic differential equation

$$d\tilde{S} = \tilde{S}\Big((\lambda(Y(t))\sigma_B - q\sigma_B^{\xi+1} + \sigma_B^2 + \sigma_B\sigma_S\rho)dt + (\sigma_S\rho + \sigma_B)dW_1 + \bar{\rho}\sigma_S dW_2\Big). \tag{5.35}$$

By the definition of the cushion, we obtain $dC = dV - dF$, where V and F are the portfolio value and the floor.

Hence,

$$d(\tilde{C}B) = \frac{mC}{S}dS + \frac{\tilde{m}C}{M}dM + \frac{C - mC - \tilde{m}C}{B}dB.$$

An application of Ito's product rule yields

$$\begin{aligned}(B+dB)d\tilde{C} + \tilde{C}dB &= \frac{m\tilde{C}}{\tilde{S}}(B+dB)d\tilde{S} + \frac{m\tilde{C}}{\tilde{S}}\tilde{S}dB + \frac{\tilde{m}\tilde{C}}{\tilde{M}}(B+dB)d\tilde{M} \\ &\quad + \frac{\tilde{m}\tilde{C}}{\tilde{M}}\tilde{M}dB + (\tilde{C} - m\tilde{C} - \tilde{m}\tilde{C})dB.\end{aligned}$$

Then

$$(B+dB)d\tilde{C} = \frac{m\tilde{C}}{\tilde{S}}(B+dB)d\tilde{S} + \frac{\tilde{m}\tilde{C}}{\tilde{M}}(B+dB)d\tilde{M}.$$

Therefore,

$$d\tilde{C} = \tilde{C}\Big(m\frac{d\tilde{S}}{\tilde{S}} + \tilde{m}\frac{d\tilde{M}}{\tilde{M}}\Big). \tag{5.36}$$

Lemma 5.7. *The discounted cushion $\tilde{C}$ satisfies the following stochastic differential equation*

$$\begin{aligned}d\tilde{C}(t) &= m\tilde{C}(t)\Big(\nu(t,Y(t))dt + (\sigma_S\rho + \sigma_B)dW_1(t) + \bar{\rho}\sigma_S dW_2(t)\Big) \\ &\quad + \tilde{m}\tilde{C}(t)\Big((\sigma_B^2 - q\sigma_B^{\xi+1})dt + \sigma_B dW_1(t)\Big),\end{aligned} \tag{5.37}$$

where $\nu(t,Y(t)) = \lambda(Y(t))\sigma_B - q\sigma_B^{\xi+1} + \sigma_B^2 + \sigma_B\sigma_S\rho$. Furthermore, it has a unique solution if $\mathbb{E}\int_0^T(m^2(t) + \tilde{m}^2(t))dt < \infty$.

Proof. The dynamic of $\tilde{C}$ follows directly from the equations (5.34), (5.35) and (5.36).

Denote $A(s) = m\nu(s,Y(s)) + \tilde{m}(\sigma_B^2 - q\sigma_B^{\xi+1})$, $S_1(s) = m(\sigma_S\rho + \sigma_B) + \tilde{m}\sigma_B$, $S_2(s) = m\bar{\rho}\sigma_S$ and $a(s) = \sigma_1(s) = \sigma_2(s) = 0$.

Since $\mathbb{E}\int_0^T(m^2(t) + \tilde{m}^2(t))dt < \infty$ and $\nu(s,Y(s)), \sigma_B(s)$ are bounded, all the coefficients $A(s)$, $S_1(s)$, $S_2(s)$, $a(s)$, $\sigma_1(s)$ and $\sigma_2(s)$ satisfy the conditions of Theorem A.3 (see Appendix A). Therefore, the equation (5.37) with the initial condition $\tilde{C}(0) = \tilde{C}_0$ has a unique solution. □

5.2.3 Optimal Portfolio

Assume that the filtration $\mathbb{F}$ is generated by W_1, W_2 and Y, i.e. for $0 \le t \le T$, $\mathcal{F}_t = \sigma(W_1(s), W_2(s), Y(s) : s \le t)$.

From Lemma 5.7, it is reasonable to define the set of admissible controls

$$\begin{aligned}\mathcal{A}(t,\mathbb{F}) \quad := \quad & \Big\{ m, \tilde{m} : [t,T] \to \mathbb{R} | m, \tilde{m} \text{ are progressively measurable,} \\ & \mathbb{E} \int_t^T m(t)^2 + \tilde{m}(t)^2 dt < \infty \Big\}.\end{aligned}$$

The CPPI investor is interested in solving the investment problem

$$\sup_{m,\tilde{m} \in \mathcal{A}(0,\mathbb{F})} \mathbb{E}\left[U(C^{m,\tilde{m}}(T))\right],$$

or the equivalent one

$$\sup_{m,\tilde{m} \in \mathcal{A}(0,\mathbb{F})} \mathbb{E}\left[U(\tilde{C}^{m,\tilde{m}}(T))\right].$$

Given $\tilde{C}(t) = \tilde{C}$ and $Y(t) = i$, the value function is defined as

$$J(t, \tilde{C}, i) = \sup_{m,\tilde{m} \in \mathcal{A}(t,\mathbb{F})} \mathbb{E}^{t,\tilde{C},i}\left[U(\tilde{C}_T^{m,\tilde{m}})\right],$$

where $\mathbb{E}^{t,\tilde{C},i}$ denotes the conditional expectation providing $\tilde{C}(t) = \tilde{C}$ and $Y(t) = i$. Then the corresponding HJB equation is

$$\begin{aligned}\sup_{m,\tilde{m} \in \mathcal{A}(t,\mathbb{F})} \quad \Big\{ \quad & J_t + J_{\tilde{C}}\Big(m\nu(t,i) + \tilde{m}(\sigma_B^2 - q\sigma_B^{\xi+1})\Big)\tilde{C} \\ & + \frac{1}{2} J_{\tilde{C}\tilde{C}} \tilde{C}^2 \Big[\Big(m(\sigma_S \rho + \sigma_B) + \tilde{m}\sigma_B\Big)^2 + (m\bar{\rho}\sigma_S)^2\Big] \\ & + \sum_{j \in E} q_{ij}[J(t,\tilde{C},j) - J(t,\tilde{C},i)] \quad \Big\} = 0. \qquad (5.38)\end{aligned}$$

Denote the term in the brackets of the equation (5.38) as $\mathcal{A}^{m,\tilde{m}}(t, \tilde{C}(t), i)$ for convenience.

Theorem 5.8. *The HJB equation* (5.38) *with the boundary condition*

$$J(T, \tilde{C}(T), Y(T)) = \frac{\tilde{C}(T)^{1-\eta}}{1-\eta}, \quad 0 < \eta < 1$$

for the portfolio problem (5.37) *has the solution* $J(t, C, i) \in C^{1,2}([0, T] \times \mathbb{R})$,

$$J(t, \tilde{C}, i) = \frac{\tilde{C}(T)^{1-\eta}}{1-\eta} g(t, i), \ \ i \in E.$$

Here, $g(t, i)$ *is the unique positive solution of the following system of ordinary differential equations with the boundary condition* $g(T, i) = 1$ *for all* $i \in E$,

$$g_t(t, i) + a(t, i)g(t, i) + \sum_{j \in E} q_{ij}(g(t, j) - g(t, i)) = 0, \tag{5.39}$$

where

$$\begin{aligned} a(t, i) &= \eta(1-\eta)\Big(\tilde{m}^*(t, i)\alpha(t)\sigma_B - \frac{1}{2}(\alpha^2(t) - \bar{\rho}^2\sigma_S^2 m^*(t, i)^2) \\ &\quad + m^*(t, i)\alpha(t)(\sigma_S\rho + \sigma_B)\Big) \end{aligned} \tag{5.40}$$

with $\alpha(t) = m^*(t, i)(\sigma_S\rho + \sigma_B) + \tilde{m}^*(t, i)\sigma_B$. *For* $i \in E$, *the optimal controls* $m^*(t, i)$ *and* $\tilde{m}^*(t, i)$ *are*

$$m^*(t, i) = \sigma_B \frac{\lambda(i) + q\sigma_B^{\xi-1}\rho\sigma_S}{\eta\bar{\rho}^2\sigma_S^2}$$

and

$$\tilde{m}^*(t, i) = \frac{1}{\eta}\Big(1 - \frac{(\sigma_B + \sigma_S\rho)\lambda(i)}{\bar{\rho}^2\sigma_S^2} - q\sigma_B^{\xi-1}\frac{\sigma_S + \rho\sigma_B}{\sigma_S\bar{\rho}^2}\Big).$$

Proof. If $(m, \tilde{m})$ is a maximizer of $\mathcal{A}^{m,\tilde{m}}(t, \tilde{C}(t), i)$ in the equation (5.38), then they are the solution of the following system

$$J_{\tilde{C}}\nu(t, i)\tilde{C} + J_{\tilde{C}\tilde{C}}\tilde{C}^2\Big(\big(m(\sigma_S\rho + \sigma_B) + \tilde{m}\sigma_B\big)(\sigma_S\rho + \sigma_B) + m(\bar{\rho}\sigma_S)^2\Big) = 0$$

and

$$J_{\tilde{C}}(\sigma_B^2 - q\sigma_B^{\xi+1})\tilde{C} + J_{\tilde{C}\tilde{C}}\tilde{C}^2\big(m(\sigma_S\rho + \sigma_B) + \tilde{m}\sigma_B\big)\sigma_B = 0.$$

Simplifying the above systems gives

$$J_{\tilde{C}}\nu(t, i) + J_{\tilde{C}\tilde{C}}\tilde{C}\Big(\big(m(\sigma_S\rho + \sigma_B) + \tilde{m}\sigma_B\big)(\sigma_S\rho + \sigma_B) + m(\bar{\rho}\sigma_S)^2\Big) = 0 \tag{5.41}$$

and

$$J_{\tilde{C}}(\sigma_B - q\sigma_B^{\xi}) + J_{\tilde{C}\tilde{C}}\tilde{C}\big(m(\sigma_S\rho + \sigma_B) + \tilde{m}\sigma_B\big) = 0. \tag{5.42}$$

Using the equations (5.41) and (5.42), the HJB equation reduces to

$$J_t - \big(\tilde{m}\alpha\sigma_B - \frac{1}{2}(\alpha^2 - \bar{\rho}^2\sigma_S^2 m^2) + m\alpha(\sigma_S\rho + \sigma_B)\big)\tilde{C}^2 J_{\tilde{C}\tilde{C}} \qquad (5.43)$$
$$+\sum_{j\in E} q_{ij}\big(J(t,\tilde{C},j) - J(t,\tilde{C},i)\big) = 0,$$

where $\alpha = m(\sigma_S\rho + \sigma_B) + \tilde{m}\sigma_B$. Consider

$$J(t,\tilde{C},i) = \frac{\tilde{C}(T)^{1-\eta}}{1-\eta} g(t,i). \qquad (5.44)$$

The boundary conditions imply $g(T,i) = 1$ for all $i \in E$.

Plugging the representation of $J(t,\tilde{C},i)$ in (5.44) into the system (5.41) and (5.42) then solving for m and $\tilde{m}$ yield

$$\begin{aligned} m^*(t,i) &= \frac{\nu(t,i) - (\sigma_B - q\sigma_B^{\xi})(\rho\sigma_S + \sigma_B)}{\eta\bar{\rho}^2\sigma_S^2} \\ &= \sigma_B \frac{\lambda(i) + q\sigma_B^{\xi-1}\rho\sigma_S}{\eta\bar{\rho}^2\sigma_S^2} \end{aligned}$$

and

$$\begin{aligned} \tilde{m}^*(t,i) &= \frac{1}{\eta\sigma_B}\Big(\sigma_B - q\sigma_B^{\xi} - \frac{(\lambda\sigma_B + q\rho\sigma_S\sigma_B^{\xi})(\rho\sigma_S + \sigma_B)}{\bar{\rho}^2\sigma_S^2}\Big) \\ &= \frac{1}{\eta}\Big(1 - \frac{(\sigma_B + \sigma_S\rho)\lambda(i)}{\bar{\rho}^2\sigma_S^2} - q\sigma_B^{\xi-1}\frac{\sigma_S + \rho\sigma_B}{\sigma_S\bar{\rho}^2}\Big). \end{aligned}$$

Notice that when $t \to T$, then $\sigma_B \to 0$. Therefore, the special form of the risk premium of the stock eliminates the singularity in $\tilde{m}$. Inserting the formula (5.44) into the equation (5.43), we obtain

$$g_t(t,i) + a(t,i)g(t,i) + \sum_{j\in E} q_{ij}(g(t,j) - g(t,i)) = 0,$$

where

$$\begin{aligned} a(t,i) &= \eta(1-\eta)\Big(\tilde{m}^*(t,i)\alpha(t)\sigma_B - \frac{1}{2}(\alpha^2(t) - \bar{\rho}^2\sigma_S^2 m^*(t,i)^2) \\ &\quad + m^*(t,i)\alpha(t)(\sigma_S\rho + \sigma_B)\Big) \end{aligned}$$

with $\alpha(t) = m^*(t,i)(\sigma_S\rho + \sigma_B) + \tilde{m}^*(t,i)\sigma_B$.

It can be written as

$$g_t(t) = A(t)g(t),$$

where $A(t) = (a_{ij})_{i,j\in E}$ is a matrix satisfying

$$\begin{aligned} a_{ii}(t) &= -a(t,i) - q_{ii} \text{ for all } i \in E, \\ a_{ij}(t) &= -q_{ij} \text{ for all } i,j \in E \text{ and } i \neq j. \end{aligned}$$

The boundary condition is $g(T) = 1_E$, where 1_E is a vector of 1. Since $A(t)$ is continuous on $[0,T]$, the above system of ODEs has a solution (see Proposition 2.1 in [22]). The positivity and uniqueness of that solution are the consequences of Lemma 5.9.

Finally, we need to check that $(m^*(t,i), \tilde{m}^*(t,i))$ is the maximizer of the term in the brackets of the HJB equation,

$$\begin{aligned} \mathcal{A}^{m,\tilde{m}}(t,\tilde{C}(t),i) &= J_t + J_{\tilde{C}}\Big(m\nu(t,i) + \tilde{m}(\sigma_B^2 - q\sigma_B^{\xi+1})\Big)\tilde{C} \\ &\quad + \frac{1}{2}J_{\tilde{C}\tilde{C}}\tilde{C}^2\Big[\big(m(\sigma_S\rho + \sigma_B) + \tilde{m}\sigma_B\big)^2 + (m\bar{\rho}\sigma_S)^2\Big] \\ &\quad + \sum_{j\in E} q_{ij}[J(t,\tilde{C},j) - J(t,\tilde{C},i)]. \end{aligned}$$

The second derivatives of H are

$$\begin{aligned} A &= \frac{\partial^2\mathcal{A}}{\partial m^2} = J_{\tilde{C}\tilde{C}}\tilde{C}^2\big((\sigma_S\rho + \sigma_B)^2 + (\bar{\rho}\sigma_S)^2\big), \\ B &= \frac{\partial^2\mathcal{A}}{\partial m\partial\tilde{m}} = J_{\tilde{C}\tilde{C}}\tilde{C}^2(\sigma_S\rho + \sigma_B)\sigma_B, \\ C &= \frac{\partial^2\mathcal{A}}{\partial\tilde{m}^2} = J_{\tilde{C}\tilde{C}}\tilde{C}^2\sigma_B^2. \end{aligned}$$

The next lemma will show that $g(t,i) > 0$ for all $t \in [0,T]$ and $i \in E$. Thus, we can easily observe from the second derivatives of H that

$$AC - B > 0 \text{ and } A < 0.$$

This implies that $(m^*(t,i), \tilde{m}^*(t,i))$ is the maximizer of $\mathcal{A}^{m,\tilde{m}}(t,\tilde{C}(t),i)$. □

Similar to the Merton investment problem, we also have the following lemma.

Lemma 5.9. *The function $g(t,i)$ which solves the equation (5.39) with the boundary conditions $g(T,i) = 1$ for all $i \in E$ can be written as*

$$g(t,i) = \mathbb{E}^{t,i}\left[\exp\left\{\int_t^T a(s,Y(s))ds\right\}\right],$$

where

$$\begin{aligned} a(s,Y(s)) &= \eta(1-\eta)\Big(\tilde{m}^*(s,Y(s))\alpha(s)\sigma_B - \frac{1}{2}(\alpha^2(s) - \bar{\rho}^2\sigma_S^2 m^*(s,Y(s))^2) \\ &\quad + m^*(s,Y(s))\alpha(s)(\sigma_S\rho + \sigma_B)\Big) \end{aligned}$$

with $\alpha(s) = m^*(s,Y(s))(\sigma_S\rho + \sigma_B) + \tilde{m}^*(s,Y(s))\sigma_B$.

Proof. This result can be proved by the same argument of Lemma 5.3. □

As in the Merton problem, we assume for the moment that we only consider $m(t), \tilde{m}(t)$ which are in $[-M, M]$ with M big enough such that $m^*, \tilde{m}^*$ are in $[-M, M]$. This restriction does not cause any problem for our application since Theorem 5.8 implies that $m^*(t,Y(t))$ and $\tilde{m}^*(t,Y(t))$ are bounded on $[0,T]$.

Theorem 5.10. *(Verification Theorem) Let $Q := [0,T] \times \mathbb{R}$. Suppose $G \in C^{1,2}(Q)$ with $|G(t,\tilde{C},i)| \le K(1+|\tilde{C}|^k)$ for some constants $K > 0$, $k \in \mathbb{N}$ and for all $i \in E$ is a solution of the HJB equation (5.38). Then we have*

a) $G(t,\tilde{C},i) \ge J(t,\tilde{C},i)$ for all $0 \le t \le T, \tilde{C} \in \mathbb{R}_+$ and $i \in E$.

b) If for all $(t,\tilde{C}) \in Q$ there exists $(m^*,\tilde{m}^*) \in \mathcal{A}(t,\mathbb{F})$ with

$$(m^*(s),\tilde{m}^*(s)) \in \underset{m,\tilde{m}\in[-M,M]}{\arg\max}\Big(\mathcal{A}^{m,\tilde{m}}(s,\tilde{C}^*(s),x(s))\Big)$$

for all $s \in [t,T]$, where $\tilde{C}^*(s)$ is the controlled process w.r.t $m^*(s),\tilde{m}^*(s)$ via (5.37), then we obtain $G(t,\tilde{C},i) = J(t,\tilde{C},i) = J_{m^*,\tilde{m}^*}(t,\tilde{C},i)$.

Proof. Let $0 \le t < T$, $Y(t) = i$ and $\tilde{C}(t) = \tilde{C}$. Assume that O is a bounded subset of $\mathbb{R}$ and $t < \theta \le T$ is a stopping time with respect to the filtration $\mathbb{F}$ such that $\tilde{C}(s) \in O$ for all $t \le s \le \theta$. Let $(m,\tilde{m}) \in \mathcal{A}(t,\mathbb{F})$ be an arbitrary admissible multiples. An application of Ito's Lemma for a semi-martingale process gives

$$\begin{aligned} G(\theta,\tilde{C}(\theta),Y(\theta)) &= G(t,\tilde{C},i)+\int_t^\theta G_t(s,\tilde{C}(s),Y(s))ds \\ &\quad +\int_t^\theta G_{\tilde{C}}(s,\tilde{C}(s),Y(s))d\tilde{C}+\frac{1}{2}\int_t^\theta G_{\tilde{C}\tilde{C}}(s,\tilde{C}(s),Y(s))d\tilde{C}d\tilde{C} \\ &\quad +\sum_{t<s\le\theta}\Big[G(s,\tilde{C}(s),Y(s))-G(s,\tilde{C}(s),Y(s-))\Big]. \end{aligned}$$

Recall that the dynamics of $\tilde{C}$ is

$$\begin{aligned} d\tilde{C} &= m\tilde{C}\Big(\nu(t)dt+(\sigma_S\rho+\sigma_B)dW_1+\bar{\rho}\sigma_S dW_2\Big) \\ &\quad +\tilde{m}\tilde{C}\Big((\sigma_B^2-q\sigma_B^{\xi+1})dt+\sigma_B dW_1\Big). \end{aligned}$$

Therefore, we obtain

$$\begin{aligned} & G(\theta,\tilde{C}(\theta),Y(\theta)) \\ & = G(t,C,i) \\ & +\int_t^\theta\Bigg[G_t(s,\tilde{C}(s),Y(s))+G_{\tilde{C}}(s,\tilde{C}(s),Y(s))\big(m\tilde{C}(s)\nu(s)+\tilde{m}\tilde{C}(s)(\sigma_B^2-q\sigma_B^{\xi+1})\big) \\ & +\frac{1}{2}G_{\tilde{C}\tilde{C}}(s,\tilde{C},Y(s))\tilde{C}^2\Big((m(\sigma_S\rho+\sigma_B)+\tilde{m}\sigma_B)^2+(m\rho\bar{\sigma}_S)^2\Big)\Bigg]ds \\ & +\int_t^\theta G_{\tilde{C}}(s,\tilde{C}(s),Y(s))\tilde{C}(s)\big(m(\sigma_S\rho+\sigma_B)+\tilde{m}\sigma_B\big)dW_1(s) \\ & +\int_t^\theta G_{\tilde{C}}(s,\tilde{C}(s),Y(s))\tilde{C}(s)m\bar{\rho}\sigma_S dW_2(s) \\ & +\sum_{t<s\le\theta}\Big[G(s,\tilde{C}(s),Y(s))-G(s,\tilde{C}(s),Y(s-))\Big]. \end{aligned}$$

Since $m(s),\tilde{m}(s)\in[0,M],\sigma_S$ and $\nu(s)$ are bounded, by Proposition A.2 (see Appendix A), there exists a constant N such that for all $q\ge 0, t\in[0,T]$,

$$\mathbb{E}[\sup_{s\le t}|\tilde{C}(s)|^q]\le Ne^{Nt}(1+|\tilde{C}_0|)^q. \tag{5.45}$$

By the similar arguments as in the Verification Theorem 5.4, we obtain

$$\mathbb{E}^{t,\tilde{C},i}[G(\theta,\tilde{C}(\theta),x(\theta))]\le G(t,\tilde{C},i). \tag{5.46}$$

The remaining derivations are the same as those in the proof of Theorem 5.4. □

5.2.4 Mean and variance of the variable-multiple CPPI

Lemma 5.7 implies

$$C(T) = \tilde{C}(T) = \tilde{C}(0)\exp\{X\},$$

where

$$\begin{aligned} X &= \int_0^T \Big[m(s)\nu(s,Y(s)) + \tilde{m}(s)(\sigma_B^2 - q\sigma_B^{\xi+1}) \\ &\quad -\frac{1}{2}\big(m(s)(\sigma_S\rho+\sigma_B)+\tilde{m}(s)\sigma_B\big)^2 - \frac{1}{2}m^2(s)\bar{\rho}^2\sigma_S^2\Big]ds \\ &\quad + \int_0^T \big(m(s)(\sigma_S\rho+\sigma_B)+\tilde{m}(s)\sigma_B\big)dW_1(s) + \int_0^T m(s)\bar{\rho}\sigma_S dW_2(s). \end{aligned}$$

Conditioning on knowing the Markov process Y and using the notations and arguments on page 58, we have that $X|\{Y(t) : t \in [0,T]\}$ is a normally distributed random variable with

$$\begin{aligned} \mu_{X|Y} &= \int_0^T \Big[m(s|Y(s))\nu(s|Y(s)) + \tilde{m}(s|Y(s))(\sigma_B^2 - q\sigma_B^{\xi+1}) \\ &\quad -\frac{1}{2}\big(m(s|Y(s))(\sigma_S\rho+\sigma_B)+\tilde{m}(s|Y(s))\sigma_B\big)^2 - \frac{1}{2}m^2(s|Y(s))\bar{\rho}^2\sigma_S^2\Big]ds \end{aligned}$$

and

$$\begin{aligned} \sigma_{X|Y}^2 &= \int_0^T \big(m(s|Y(s))(\sigma_S\rho+\sigma_B)+\tilde{m}(s|Y(s))\sigma_B\big)^2 ds \\ &\quad + \int_0^T \big(m(s|Y(s))\bar{\rho}\sigma_S\big)^2 ds. \end{aligned}$$

The expected value and variance of the variable-CPPI are calculated as follow.

Proposition 5.11.

$$\begin{aligned} \mathbb{E}[C(T)] &= \tilde{C}(0)\mathbb{E}\Big[e^{\int_0^T m(s|Y(s))\nu(s|Y(s))+\tilde{m}(s|Y(s))(\sigma_B^2-q\sigma_B^{\xi+1})ds}\Big], \\ \mathrm{Var}[C(T)] &= \tilde{C}^2(0)\mathbb{E}\Big[e^{\sigma_{X|Y}^2+2\int_0^T m(s|Y(s))\nu(s|Y(s))+\tilde{m}(s|Y(s))(\sigma_B^2-q\sigma_B^{\xi+1})ds}\Big] \\ &\quad -\big(\mathbb{E}[C(T)]\big)^2. \end{aligned}$$

Proof. First, recall that if $Z \sim N(\mu_Z, \sigma_Z^2)$, then

$$\mathbb{E}\big[e^Z\big] = e^{\mu_Z + \frac{1}{2}\sigma_Z^2}$$

and

$$\mathrm{Var}[Z] = e^{2\mu_Z} e^{\sigma_Z^2}\Big(e^{\sigma_Z^2} - 1\Big).$$

From the above notice and the tower property of condition expectation, we obtain

$$\mathbb{E}[C(T)] = \mathbb{E}\big[\mathbb{E}[C(T)|Y]\big] = \tilde{C}(0)\mathbb{E}\big[e^{\mu_{X|Y}+\frac{1}{2}\sigma_{X|Y}^2}\big] \tag{5.47}$$

and

$$\begin{aligned}\mathrm{Var}[C(T)] &= \tilde{C}^2(0)\mathrm{Var}\big[e^X\big] = \tilde{C}^2(0)\Big\{\mathbb{E}\Big[\mathrm{Var}[e^X|Y]\Big] + \mathrm{Var}\Big[\mathbb{E}[e^X|Y]\Big]\Big\}\\ &= \tilde{C}^2(0)\Big\{\mathbb{E}\Big[e^{2\mu_{X|Y}}e^{\sigma_{X|Y}^2}\Big(e^{\sigma_{X|Y}^2}-1\Big)\Big] + \mathrm{Var}\Big(e^{\mu_{X|Y}+\frac{1}{2}\sigma_{X|Y}^2}\Big)\Big\}.\end{aligned} \tag{5.48}$$

Inserting the formulas of $\mu_{X|Y}$ and $\sigma_{X|Y}^2$ into the equations (5.47) and (5.48) and simplifying, we have the proof. □

5.2.5 Average-data model

We assume that the Markov chain Y has a unique stationary distribution $p = (p_j, j \in E)$, and let

$$\bar{\lambda} = \sum_{j\in E} \lambda(j)p_j.$$

In our average-data model, the stock price is assumed to satisfy

$$\frac{dS(t)}{S(t)} = \big(r(t) + \bar{\lambda}\sigma_B\big)\,dt + \sigma_S(\rho dW_1(t) + \bar{\rho} dW_2(t)).$$

We want to compare the value function $\bar{J}(t, \tilde{C})$ obtained in the average-data model with the value function $\mathbb{E}^p[J(t, \tilde{C}, Y(t))]$ in the Markov-switching framework with $Y(t) \stackrel{d}{=} p$. From Theorem 5.8 and Lemma 5.9, we obtain

$$\mathbb{E}^p[J(t, \tilde{C}, Y(t))] = \frac{\tilde{C}^{1-\eta}}{1-\eta}\mathbb{E}^p\Big[\exp\Big(\int_t^T a(s, Y(s))ds\Big)\Big].$$

Here,

$$a(s, Y(s)) = \eta(1-\eta)\Big(\tilde{m}^*(s)\alpha(t)\sigma_B - \frac{1}{2}(\alpha^2(s) - \bar{\rho}^2\sigma_S^2 m^*(s)^2) + m^*(s)\alpha(s)(\sigma_S\rho + \sigma_B)\Big) \tag{5.49}$$

with

$$\alpha(s) = m^*(s)(\sigma_S\rho + \sigma_B) + \tilde{m}^*(s)\sigma_B.$$

Notice that the optimal multiples are

$$m^*(s) = \sigma_B \frac{\lambda(Y(s)) + q\sigma_B^{\xi-1}\rho\sigma_S}{\eta\bar{\rho}^2\sigma_S^2} \tag{5.50}$$

and

$$\tilde{m}^*(s) = \frac{1}{\eta}\Big(1 - \frac{(\sigma_B + \sigma_S\rho)\lambda(Y(s))}{\bar{\rho}^2\sigma_S^2} - q\sigma_B^{\xi-1}\frac{\sigma_S + \rho\sigma_B}{\sigma_S\bar{\rho}^2}\Big). \tag{5.51}$$

From the equations (5.49), (5.50) and (5.51), we observe that $a(s, Y(s))$ is a quadratic function of $\lambda(Y(s))$. To apply the similar argument as in Section 5.1.6, we have to prove that $a(s, Y(s))$ is a convex function of $\lambda(Y(s))$. Hence, we are only interested in the coefficient of $\lambda^2(Y(s))$ in the expression of $a(s, Y(s))$.

Plugging $m^*(s)$ and $\tilde{m}^*(s)$ in (5.50) and (5.51) into the equation (5.49) and simplifying, we obtain the coefficient of $\lambda^2(Y(s))$ as

$$\frac{1}{2}\eta(1-\eta)\Big(\frac{(\sigma_S\rho + \sigma_B)^2\sigma_B^2}{\eta^2\bar{\rho}^4\sigma_S^4} + \frac{\bar{\rho}^2\sigma_S^2\sigma_B^2}{\eta^2\bar{\rho}^4\sigma_S^4}\Big).$$

It is obvious that the above term is greater than 0. Therefore, the quadratic function $a(s, Y(s))$ of the variable $\lambda(Y(s))$ is convex. Since the functions e^x and x^2 are also convex, Jensen's inequality gives

$$\mathbb{E}^p[J(t, \tilde{C}, Y(t))] \geq \bar{J}(t, \tilde{C}).$$

Therefore, this particular Markov-switching model achieves larger expected utility than the corresponding average-data model.

5.2.6 A special case

Notice that the excess return on the stock $\lambda(Y(t))\sigma_B$ represents the individual opinion of a particular investor about the stock's performance. In this section, we even narrow that assumption by considering the case where the Markov process Y switches between two states. It then follows that λ only takes two values which we shall denote by λ_0 and λ_1. Therefore, the market can be characterized by two regimes:

- **state 0**: $\lambda = \lambda_0$,
- **state 1**: $\lambda = \lambda_1$,

where $\lambda_0 > \lambda_1 > 0$. When the market is in state 0, we can say that the excess return of the stock strongly depends on the bond's volatility σ_B. On the other hand, it weakly depends on the bond's volatility as the market is in state 1. Furthermore, we consider a special intensity matrix

$$Q = \begin{pmatrix} -\gamma & \gamma \\ 0 & 0 \end{pmatrix},$$

where $\gamma > 0$. In what follows, we assume that the market is in state 0 at the beginning. It will stay in that state for a random time with distribution $\mathrm{Exp}(\gamma)$ and then spends the remaining time in state 1.

Average-data model. In analogy to Section 5.2.5, we want to compare the Markov-switching model with an "equivalent" average-data model. Suppose that the stock prices in the average-data framework satisfy

$$\frac{dS(t)}{S(t)} = \big(r(t) + \bar{\lambda}\sigma_B\big)\, dt + \sigma_S(\rho dW_1(t) + \bar{\rho} dW_2(t))$$

such that

$$\mathbb{E}\int_0^T \lambda(x(t))\sigma_B(t)dt = \int_0^T \bar{\lambda}\sigma_B(t)dt = \bar{\lambda}\int_0^T \sigma_B(t)dt. \tag{5.52}$$

Firstly, we have

$$\int_0^T \sigma_B(t)dt = \int_0^T \frac{\sigma_r}{a}(1 - e^{a(t-T)})dt = \frac{\sigma_r}{a}T - \frac{\sigma_r}{a^2}(1 - e^{-aT}). \tag{5.53}$$

The property of the Markov-process Y yields

$$\begin{aligned} \mathbb{E}\int_0^T \lambda(x(t))\sigma_B(t)dt &= \int_0^T \Big(\int_0^t \lambda_0\sigma_B(s)ds + \int_t^T \lambda_1\sigma_B(s)ds\Big)\gamma e^{-\gamma t}dt \\ &\quad + \int_T^\infty \Big(\int_0^T \lambda_0\sigma_B(s)ds\Big)\gamma e^{-\gamma t}dt \\ &= \int_0^T \Big(\int_0^t \lambda_0\sigma_B(s)ds + \int_t^T \lambda_1\sigma_B(s)ds\Big)\gamma e^{-\gamma t}dt \\ &\quad + \lambda_0 e^{-\gamma T}\int_0^T \sigma_B(s)ds. \end{aligned}$$

Using the equation (5.53) gives

$$\begin{aligned}\mathbb{E}\int_0^T \lambda(x(t))\sigma_B(t)dt &= \int_0^T \lambda_0\Big(\frac{\sigma_r}{a}t - \frac{\sigma_r}{a^2}(1-e^{-at})\Big)\gamma e^{-\gamma t}dt \\ &\quad + \int_0^T \lambda_1\Big(\frac{\sigma_r}{a}(T-t) - \frac{\sigma_r}{a^2}(1-e^{-a(T-t)})\Big)\gamma e^{-\gamma t}dt \\ &\quad + \lambda_0 e^{-\gamma T}(\frac{\sigma_r}{a}T - \frac{\sigma_r}{a^2}(1-e^{-aT})). \end{aligned} \tag{5.54}$$

From (5.52) to (5.54), we ontain

$$\bar{\lambda} = \frac{\mathbb{E}\int_0^T \lambda(x(t))\sigma_B(t)dt}{\int_0^T \sigma_B(t)dt}.$$

Finally, we determine the expected utility of the terminal cushion in the average-data framework. Section 5.2.4 gives

$$\frac{C(T)^{1-\eta}}{1-\eta} = \frac{\tilde{C}(0)^{1-\eta}}{1-\eta}\exp\{X\},$$

where

$$\begin{aligned} X &= (1-\eta)\int_0^T \Big[m(s)\nu(s) + \tilde{m}(s)(\sigma_B^2 - q\sigma_B^{\xi+1}) \\ &\quad -\frac{1}{2}\big(m(s)(\sigma_S\rho + \sigma_B) + \tilde{m}(s)\sigma_B\big)^2 - \frac{1}{2}m^2(s)\bar{\rho}^2\sigma_S^2\Big]ds \\ &\quad + \int_0^T (1-\eta)\big(m(s)(\sigma_S\rho+\sigma_B) + \tilde{m}(s)\sigma_B\big)dW_1(s) \\ &\quad + \int_0^T (1-\eta)m(s)\bar{\rho}\sigma_S dW_2(s). \end{aligned}$$

Here,

$$m(s) = \sigma_B\frac{\bar{\lambda} + q\sigma_B^{\xi-1}\rho\sigma_S}{\eta\bar{\rho}^2\sigma_S^2},$$

$$\tilde{m}(s) = \frac{1}{\eta}\Big(1 - \frac{(\sigma_B+\sigma_S\rho)\bar{\lambda}}{\bar{\rho}^2\sigma_S^2} - q\sigma_B^{\xi-1}\frac{\sigma_S+\rho\sigma_B}{\sigma_S\bar{\rho}^2}\Big)$$

and

$$\nu(s) = \bar{\lambda}\sigma_B - q\sigma_B^{\xi+1} + \sigma_B^2 + \sigma_B\sigma_S\rho.$$

Theorem A.4 (see Appendix A) implies that X is normally distributed with

$$\begin{aligned}\mu_X &= (1-\eta)\int_0^T \Big[m(s)\nu(s) + \tilde{m}(s)(\sigma_B^2 - q\sigma_B^{\xi+1}) \\ &\quad -\frac{1}{2}\big(m(s)(\sigma_S\rho + \sigma_B) + \tilde{m}(s)\sigma_B\big)^2 - \frac{1}{2}m^2(s)\bar{\rho}^2\sigma_S^2\Big]ds\end{aligned}$$

and

$$\begin{aligned}\sigma_X^2 &= \int_0^T (1-\eta)^2\big(m(s)(\sigma_S\rho + \sigma_B) + \tilde{m}(s)\sigma_B\big)^2 ds \\ &\quad + \int_0^T (1-\eta)^2 m^2(s)(1-\rho^2)\sigma_S^2 ds.\end{aligned}$$

Therefore, we obtain

$$\mathbb{E}\Big[\frac{C(T)^{1-\eta}}{1-\eta}\Big] = \frac{\tilde{C}(0)^{1-\eta}}{1-\eta}e^{\mu_X + \frac{1}{2}\sigma_X^2}.$$

5.2.7 Numerical examples and discussion

In what follows, we provide numerical examples for the special case which is described in last section. Firstly, we choose values for λ corresponding to two different states of the market:

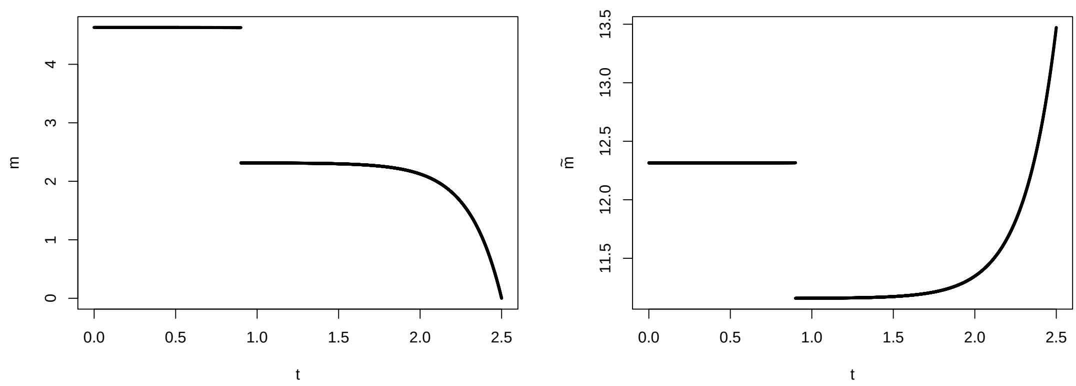
- **state 1**: $\lambda_0 = 1.0$,
- **state 2**: $\lambda_1 = 0.5$.

The stock is in state 1 for a random time with the distribution Exp(γ) and then stays in state 2 for the rest of considering period. In analogy to Section 5.1.7, two types of investor are examined:

- **risky-seeking investor**: $\eta = 0.1$,
- **risk averse investor**: $\eta = 0.8$.

The remaining parameters are chosen as follows: $\sigma_r = 0.2$, $\sigma_S = 0.3$, $\rho = -0.2$, $a = 5$, $q = 0$, $T = 2.5$, $\tilde{C}(0) = 2M$ and $K = 6M$[EUR] with $M = 1,000,000$. Before simulating, we make some abbreviations for convenience:

- DPV : the discounted portfolio value,
- DBP : the discounted bond position,
- DEX : the discounted exposure,
- DES : the discounted exposure of the stock,
- DEM : the discounted exposure of the money market account,
- DFL : the discounted floor.

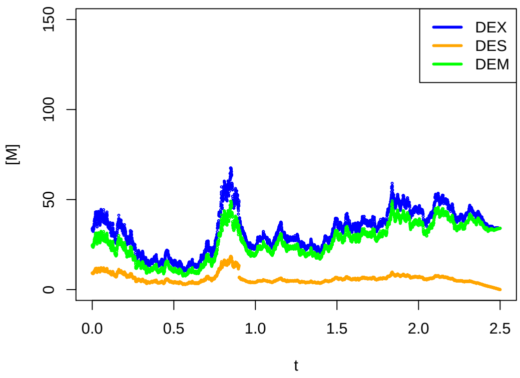


(a) **(b)**

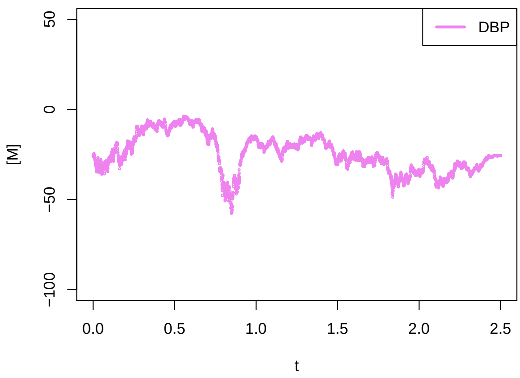
Figure 5.8: m (left) and $\tilde{m}$ (right) ($\eta = 0.1$)

We are interested in examining investment decisions of both investors, i.e. how they distribute their funds during the investment period. Later we compare the Markov-switching model and an average-data model. We choose to start with the risk-seeking investor.

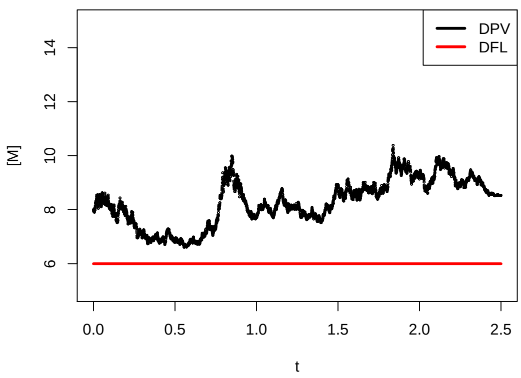
Figure 5.8a and Figure 5.8b show simulated paths for both types of multiple. It is clear that both m and $\tilde{m}$ are discontinuous at $t = 0.9$ when a shift happens. Aslo $m(t)$ is much smaller $\tilde{m}(t)$ at any time $t \in [0, T]$. In Figure 5.8a, the multiple m is nearly 4.6 when the market is in state 0. After the jump time, it reduces to approximate 2.3 and then gradually reduces to 0 since it is proportionately to the bond's volatility σ_B. However, Figure 5.8b indicates an opposite behaviour for $\tilde{m}$. It is obvious that $\tilde{m}$ is about 12.3 during state 0, and it decreases to around 11.2 and then gradually increases to 13.5 after the jump time. The large magnitudes of m and $\tilde{m}$ can be explained by the fact that the investor is less risk averse.



(a)



(b)

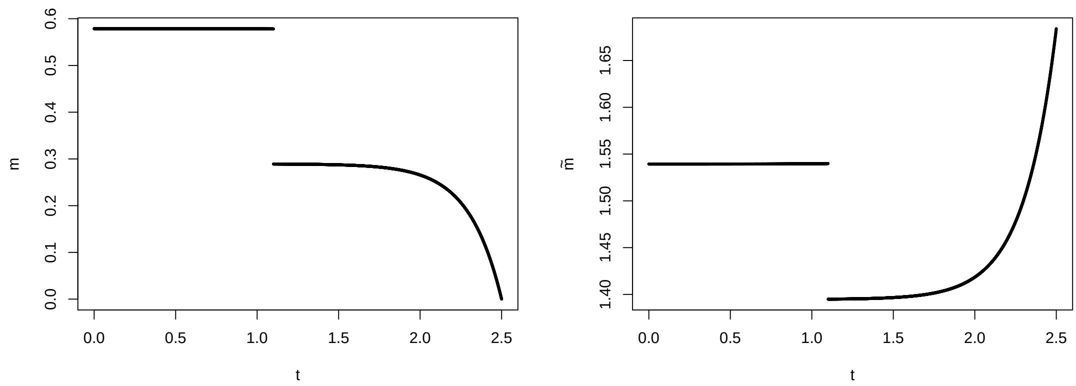


(c)

Figure 5.9: (a) Exposures (b) Bond position (c) Portfolio ($\eta = 0.1$)

We now look more closely at each component of the portfolio. Figure 5.9a demonstrates that the total exposure is extreme and is dominated by the exposure of the money market account. It is caused by the fact that the multiple of the money market account is much larger than that of the stock. In addition, the exposure in state 0 is more volatile than in state 1, and the MMA exposure tends to the total exposure when time approaches the horizon T. The later stems from the vanishing of the stock position at the final time T. In Figure 5.9b, the bond position is extremely negative. That means a big amount of the bond has been shorted in order to obtain money to invest in the money market account and the stock. Before the market shifts to state 1, the discounted bond position looks quite fluctuated; however, it shows less variation when the stock falls into state 1. Those observations are clearly explained by the reduction of both multiples. Finally, the discounted portfolio value in Figure 5.9c is quite volatile; nonetheless, that curve is always above the discounted floor.

We now turn into analyzing the risk averse investor, and we shall deal with the multiples first (see Figure 5.10). For the stock, the multiple is less than 1 and even smaller after jump time $t = 1.1$. The multiple of the money market account is small but still greater than 1.

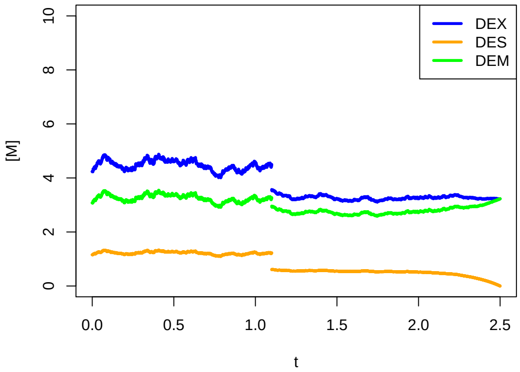


(a) **(b)**

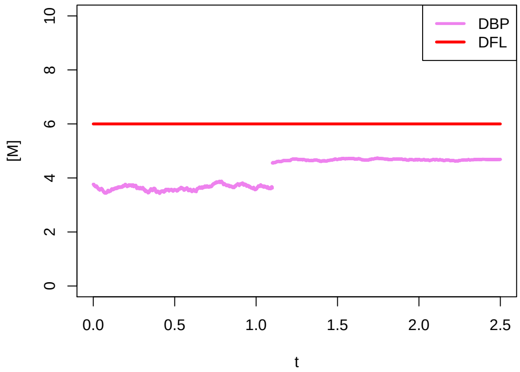
Figure 5.10: m (left) and $\tilde{m}$ (right) ($\eta = 0.8$)

As in Figure 5.11a, the total exposure is not extreme but is still dominated by the money market account. When the regime switches, the behaviour of the exposure is similar to that of the risk-seeking investor. More specifically, the exposure immediately reduces and shows less variation when the market is in state 1. Notice that $\tilde{m} > 1$ and m is positive in the investing period; thus, the bond position is under the floor. That situation is clearly shown in Figure 5.11b. In addition, the discounted bond position jumps up and is stable after the market transfers to state 1. Finally, Figure 5.11c indicates that the discounted portfolio value is quite stable and it is always higher than the discounted floor.

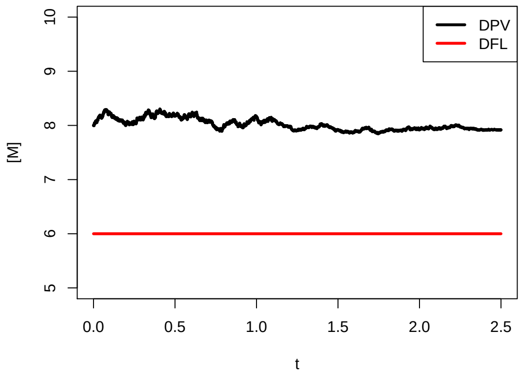
In the final part of this section, we compare the Markov-switching model with the average-data model, which is studied in Section 5.2.6. Table 5.3 shows the expected utility of the final cushion for both kinds of models. We can observe that the expected utility of the Markov-switching model is higher than that of the average-data model for different values of λ.



(a)



(b)



(c)

Figure 5.11: (a) Exposures (b) Bond position (c) Portfolio ($\eta = 0.8$)

γ	1	2	3	4	5	6	7	8
Average-data	1.093	1.073	1.067	1.065	1.064	1.063	1.063	1.062
Markov-switching	1.126	1.097	1.086	1.080	1.077	1.075	1.073	1.072

Table 5.3: Expected utilities in the multiples of $\frac{\tilde{C}(0)^{1-\eta}}{1-\eta}$ ($\eta = 0.1$)

Chapter 6

Optimal investment with a stochastic benchmark

In this chapter, we will solve investment problems for an investor whose performance is assessed with respect to an investment benchmark. That benchmarked investor is interested in maximizing utility of his/her relative consumption and relative final wealth. The benchmark will be modelled as a geometric Brownian motion with a Markov-switching drift. We will first present the mathematical framework then formulate and solve the investment problem. Finally, we will analyze the optimal solution by providing numerical examples.

6.1 Mathematical framework

We assume that the investor can invest in two assets: a bond B and a stock S. Their price processes satisfy

$$dB(t) = rBdt, \quad B(0) = B_0 > 0,$$

and

$$dS(t) = S(t)\big(\mu dt + \sigma_S dW_1\big), \quad S(0) = S_0 > 0,$$

where $\mu > r > 0$ and $\sigma_S > 0$. We assume the benchmark satisfies

$$dI(t) = I(t)\bigg(\lambda(Y(t))dt + \sigma_I(\rho dW_1(t) + \bar{\rho} dW_2(t))\bigg), \quad I(0) = I_0 > 0, \tag{6.1}$$

where $\bar{\rho} = \sqrt{1-\rho^2}$ and Y is a $\{0,1\}$-valued continuous time Markov chain with intensity matrix

$$Q = \begin{pmatrix} -\vartheta^{0,1} & \vartheta^{0,1} \\ \vartheta^{1,0} & -\vartheta^{1,0} \end{pmatrix}.$$

Here, $\vartheta^{1,0}$ and $\vartheta^{0,1}$ are positive, and $\lambda(0) < \lambda(1)$.

Remark 6.1. *The assumption $\lambda(0) < \lambda(1)$ implies that state 1 is better than state 0 for the benchmark. However, for an investor who concerns about his/her relative wealth and consumption, state 0 is preferable. For instance, we choose the benchmark as the consumer price index (CPI). Then relative values with respect to the benchmark are real values. Furthermore, we can understand that state 0 indicates low inflation, while state 1 is corresponding to high inflation. Therefore, the investor would prefer state 0 rather than state 1.*

Let $\pi(t)$ and $C(t)$ be the fraction of stock in the portfolio and the consumption rate at time $t \in [0,T]$, respectively. Then the wealth process X satisfies

$$\begin{aligned} dX(t) &= \pi X(t)\frac{dS(t)}{S(t)} + (1-\pi)X(t)\frac{dB(t)}{B(t)} - C(t)dt \\ &= \pi X(t)(\mu dt + \sigma_S dW_1(t)) + (1-\pi)X(t)rdt - C(t)dt \\ &= X(t)\Big((r + \pi(\mu - r))dt + \pi\sigma_S dW_1(t)\Big) - C(t)dt. \end{aligned}$$

We suppose that the investor consumes proportionally to his/her wealth, i.e. $C(t) = c(t)X(t), 0 \le t \le T$. Therefore, the wealth process equation can be rewritten as

$$dX(t) = X(t)\Big((r - c + \pi(\mu - r))dt + \pi\sigma_S dW_1(t)\Big). \tag{6.2}$$

In what follows, we will work with relative values $\tilde{X} = \frac{X}{I}$. An application of Ito's formula yields

$$d\tilde{X} = d(\frac{X}{I}) = \frac{dX}{I} + X(-\frac{1}{I^2}dI + \frac{1}{I^3}dIdI) + (-\frac{1}{I^2}dI + \frac{1}{I^3}dIdI)dX. \tag{6.3}$$

Inserting the equations (6.2) and (6.1) into (6.3), we obtain

$$\begin{aligned} d\tilde{X}(t) &= \tilde{X}(t)\Big(\big(\pi(\mu - r - \sigma_I\rho\sigma_S) + r + \sigma_I^2 - \lambda(x(t)) - c\big)dt \\ &\quad +(\pi\sigma_S - \sigma_I\rho)dW_1(t) - \sigma_I\bar{\rho}dW_2(t)\Big). \end{aligned}$$

This result is a part of the following lemma.

Lemma 6.2. *The relative wealth process satisfies*

$$\begin{aligned} d\tilde{X}(t) &= \tilde{X}(t)\Big(\big(\pi(\mu - r - \sigma_I\rho\sigma_S) + r + \sigma_I^2 - \lambda(Y(t)) - c\big)dt \\ &\quad +(\pi\sigma_S - \sigma_I\rho)dW_1(t) - \sigma_I\bar{\rho}dW_2(t)\Big). \end{aligned} \tag{6.4}$$

Furthermore, if $\mathbb{E}\int_0^T \pi^2(s) + c^2(s)ds < \infty$*, the equation* (6.4) *with initial condition* $\tilde{X}(0) = \tilde{X}_0$ *has a unique solution.*

Proof. We only have to verify that the equation (6.4) with the initial condition $\tilde{X}(0) = \tilde{X}_0$ has a unique solution.

Let $A(s) = \pi(s)(\mu - r - \sigma_I\rho\sigma_S) + r + \sigma_I^2 - \lambda(Y(s)) - c(s)$, $S_1(s) = \pi(s)\sigma_S - \sigma_I\rho$, $S_2(s) = \sigma_I\bar{\rho}$ and $a(s) = \sigma_1(s) = \sigma_2(s) = 0$. Since $\mathbb{E}\int_0^T \pi(s)^2 + c^2(s)ds < \infty$ and $\lambda(Y(s))$ are bounded, all the coefficients $A(s)$, $S_1(s)$, $S_2(s)$, $a(s)$, $\sigma_1(s)$ and $\sigma_2(s)$ satisfy the conditions of Theorem A.3 (see Appendix A). Therefore, the equation (6.4) with the initial condition $\tilde{X}(0) = \tilde{X}_0$ has a unique solution. □

6.2 Optimal portfolio

Assume that the filtration $\mathbb{F}$ is generated by W_1, W_2 and Y, i.e. for $0 \le t \le T$, $\mathcal{F}_t = \sigma(W_1(s), W_2(s), Y(s), s \le t)$. From Lemma 6.2, it is reasonable to define the set of admissible controls as

$$\begin{aligned} \mathcal{A}(t,\mathbb{F}) \;:=\; &\Big\{\pi : [t,T] \to \mathbb{R}, c : [t,T] \to \mathbb{R}_0^+ | \pi, c \text{ are progressively measurable}, \\ &\mathbb{E}\int_t^T \pi^2(s) + c^2(s)ds < \infty\Big\}. \end{aligned}$$

We assume that the investor is interested in relative values $\tilde{X}$ rather than original values X. In particular, he/she wants to maximize

$$\mathbb{E}\Big[\int_0^T U_1(t, c\tilde{X}(t))dt + U_2(\tilde{X}(T))\Big]$$

over the set of admissible controls $\mathcal{A}(0,\mathbb{F})$. Here, U_1 and U_2 are utility functions defined as

$$U_1(t,z) = e^{-\beta t}\frac{z^{1-\eta}}{1-\eta}$$

and

$$U_2(z) = \frac{z^{1-\eta}}{1-\eta},$$

where $0 < \eta < 1$ and $\beta > 0$.

Given $\tilde{X}(t) = \tilde{X}$ and $Y(t) = i$, the optimization problem becomes

$$\sup_{(\pi,c)\in\mathcal{A}(t,\mathbb{F})} \mathbb{E}^{t,\tilde{X},i}\left[\int_t^T U_1(s, c\tilde{X}(s))ds + U_2(\tilde{X}(T))\right],$$

where $\mathbb{E}^{t,\tilde{X},i}$ denotes the conditional expectation providing $\tilde{X}(t) = \tilde{X}$ and $Y(t) = i$. For $i = 0,1$ we define the value function as

$$J(t,\tilde{X},i) = \sup_{(\pi,c)\in\mathcal{A}(t,\mathbb{F})} \mathbb{E}^{t,\tilde{X},i}\left[\int_t^T U_1(s, c\tilde{X}(s))ds + U_2(\tilde{X}(T))\right].$$

Then the corresponding HJB equation system is

$$\begin{aligned}\sup_{(\pi,c)\in\mathbb{R}\times\mathbb{R}_0^+} \Big\{ & U_1(t,c\tilde{X}) + J_t + J_{\tilde{X}}\tilde{X}[\pi(\mu - r - \sigma_I\rho\sigma_S) + r + \sigma_I^2 - \lambda(i) - c] \\ & + \frac{1}{2}J_{\tilde{X}\tilde{X}}\tilde{X}^2[(\pi\sigma_S - \sigma_I\rho)^2 + \sigma_I^2\bar{\rho}^2] \\ & + \vartheta^{i,1-i}[J(t,\tilde{X},1-i) - J(t,\tilde{X},i)] \Big\} = 0 \end{aligned} \tag{6.5}$$

subject to the boundary conditions $J(T,z,i) = U_2(z)$ for all $z \in \mathbb{R}$ and $i = 0,1$.

We consider the following forms of value functions:

$$J(t,\tilde{X},0) = \frac{(\tilde{X}e^{g(t)})^{1-\eta}}{1-\eta},$$
$$J(t,\tilde{X},1) = \frac{(\tilde{X}e^{g(t)-h(t)})^{1-\eta}}{1-\eta}$$

for $(t,\tilde{X}) \in [0,T]\times(0,\infty)$ with C^1-functions g and h on $[0,T]$.

Remark 6.3. *As in Chapter 2, h represents the difference between two states of the market. For the benchmarked investor, state 0 is better than state 1. Therefore, h is expected to be non-negative, which is indeed the content of Lemma 6.5.*

Plugging the above representations of $J(t,\tilde{X},0)$ and $J(t,\tilde{X},1)$ into the HJB system (6.5) and then dividing the equations by $(\tilde{X}e^{g(t)-1_{\{i=1\}}h(t)})^{1-\eta}$ yield the following reduced-HJB equations

$$\sup_{(\pi,c)\in\mathbb{R}\times\mathbb{R}_0^+}\Bigg\{ \frac{1}{1-\eta}c^{1-\eta}e^{-\beta t-(1-\eta)(g(t)-1_{\{i=1\}}h(t))}-c+g'(t)-1_{\{i=1\}}h'(t)$$
$$+\pi(\mu-r-\sigma_I\rho\sigma_S)-\frac{1}{2}\eta[(\pi\sigma_S-\sigma_I\rho)^2+\sigma_I^2\bar{\rho}^2]+r+\sigma_I^2-\lambda(i)$$
$$+\frac{\vartheta^{i,1-i}}{1-\eta}\big(e^{(-1)^{1-i}(1-\eta)h(t)}-1\big)\Bigg\}=0 \tag{6.6}$$

for $t\in[0,T)$, $i=0,1$, with the boundary conditions

$$g(T)=0, h(T)=0. \tag{6.7}$$

For $i=0,1$ we define functions $H^{c,i}(t,c):[0,T]\times\mathbb{R}_0^+\to\mathbb{R}$ and $H^{\pi}(t,\pi):[0,T]\times\mathbb{R}\to\mathbb{R}$ as

$$H^{c,i}(t,c)=\frac{1}{1-\eta}c^{1-\eta}e^{-\beta t-(1-\eta)(g(t)-1_{\{i=1\}}h(t))}-c$$

and

$$H^{\pi}(t,\pi)=\pi(\mu-r-\sigma_I\rho\sigma_S)-\frac{1}{2}\eta[(\pi\sigma_S-\sigma_I\rho)^2+\sigma_I^2\bar{\rho}^2]+r+\sigma_I^2.$$

Then the equation (6.6) can be rewritten as

$$\sup_{(\pi,c)\in\mathbb{R}\times\mathbb{R}_0^+}\Bigg\{ g'(t)-1_{\{i=1\}}h'(t)+H^{c,i}(t,c)+H^{\pi}(t,\pi)-\lambda(i)$$
$$+\frac{\vartheta^{i,1-i}}{1-\eta}\big(e^{(-1)^{1-i}(1-\eta)h(t)}-1\big)\Bigg\}=0.$$

Lemma 6.4. *Given $t, g(t)$ and $h(t)$, the maximizer $c^{i,*}(t)$ of $H^{c,i}(t,c)$ is*

$$c^{i,*}(t)=\exp\Big\{-\frac{\beta t}{\eta}-\frac{1-\eta}{\eta}(g(t)-1_{\{i=1\}}h(t))\Big\},$$

and the maximizer $\pi^*(t)$ of $H^{\pi}(t,\pi)$ is

$$\pi^*(t)=\frac{\mu-r-\sigma_I\rho\sigma_S}{\eta\sigma_S^2}+\frac{\sigma_I\rho}{\sigma_S}.$$

Proof. Differentiating $H^{c,i}(t,c)$ with respect to c gives

$$c^{-\eta}e^{-\beta t-(1-\eta)(g(t)-1_{\{i=1\}}h(t))}-1.$$

The necessary condition for c to be the maximizer of $H^{c,i}(t,c)$ is that the above term is equal to zero. This leads to

$$c^{i,*}(t)=\exp\Big\{-\frac{\beta t}{\eta}-\frac{1-\eta}{\eta}(g(t)-1_{\{i=1\}}h(t))\Big\}.$$

Since

$$H^{c,i}_{cc}(t,c^{i,*}(t))=-\eta(c^{i,*}(t))^{-\eta-1}e^{-\beta t-(1-\eta)(g(t)-1_{\{i=1\}}h(t))}<0,$$

$c^{i,*}(t)$ is indeed the maximizer of $H^{c,i}(t,c)$.

Finally, since $H^{\pi}(t,\pi)$ is a quadratic function of π,

$$\pi^*(t)=\frac{\mu-r-\sigma_I\rho\sigma_S}{\eta\sigma_S^2}+\frac{\sigma_I\rho}{\sigma_S}$$

is its maximizer. □

Lemma 6.5. *(Non-negativity of h) If $g,h\in C^1[0,T]$ are solutions of* (6.6) *subject to the boundary conditions* (6.7)*, then*

$$h(t)\geq 0$$

for every $t\in[0,T]$.

Proof. We will use the contradiction method to prove the lemma. Suppose there exists $t_0\in[0,T]$ such that $h(t_0)<0$. The reduced HJB equation (6.6) at $t=t_0$ gives

$$\begin{aligned}0 \;=\; & \sup_{(\pi,c)\in\mathbb{R}\times\mathbb{R}_0^+}\Big\{g'(t_0)+H^{c,0}(t_0,c)+H^{\pi}(t_0,\pi)-\lambda(0)\\ & +\frac{\vartheta^{0,1}}{1-\eta}\big(e^{-(1-\eta)h(t_0)}-1\big)\Big\}\\ & -\sup_{(\pi,c)\in\mathbb{R}\times\mathbb{R}_0^+}\Big\{g'(t_0)-h'(t_0)+H^{c,1}(t_0,c)+H^{\pi}(t_0,\pi)-\lambda(1)\\ & +\frac{\vartheta^{1,0}}{1-\eta}\big(e^{(1-\eta)h(t_0)}-1\big)\Big\}.\end{aligned}$$

Inserting the maximizers $c^{*,i}(t_0)$ and $\pi^*(t_0)$ in Lemma 6.4 into the above equation yields

$$\begin{aligned}0 &= \Bigg[\frac{\eta}{1-\eta}e^{-\frac{\beta t_0}{\eta}-\frac{1-\eta}{\eta}g(t_0)}(1-e^{\frac{1-\eta}{\eta}h(t_0)})+\frac{1}{1-\eta}\vartheta^{0,1}(e^{-(1-\eta)h(t_0)}-1)\\ &\quad -\frac{1}{1-\eta}\vartheta^{1,0}(e^{(1-\eta)h(t_0)}-1)+\lambda(1)-\lambda(0)\Bigg]+h'(t_0).\end{aligned}$$

Since $h(t_0) < 0$ and $\lambda(1) > \lambda(0)$, the term in [...] is positive. Therefore, we can conclude that for every $t_0 \in [0,T]$ such that $h(t_0) < 0$ then $h'(t_0) < 0$.

Due to the continuity of h and $h(T) = 0$, there exists $t_1 \in [t_0, T]$ such that $h(t) < 0$ for $t_0 < t < t_1$ and $h(t_1) = 0$. By a similar argument as above, we obtain $h'(t_1) < 0$, which implies that h is decreasing in some neighbourhood of t_1. However, $h(t) < 0 = h(t_1)$ for $t_0 < t < t_1$. Thus, we have a contradiction.

In conclusion, we must have $h(t) \geq 0$ for all $t \in [0,T]$. □

Inserting the maximizers $\pi^*(t)$ and $c^{*,i}(t)$ into the reduced-HJB equation (6.6) gives

$$\begin{aligned}g'(t)-h'(t) &= -\frac{\eta}{1-\eta}e^{-\frac{\beta t}{\eta}-\frac{1-\eta}{\eta}(g(t)-h(t))}-H^\pi(t,\pi^*(t))\\ &\quad -\frac{\vartheta^{1,0}}{1-\eta}(e^{(1-\eta)h(t)}-1)+\lambda(1)\end{aligned} \tag{6.8}$$

and

$$\begin{aligned}g'(t) &= -\frac{\eta}{1-\eta}e^{-\frac{\beta t}{\eta}-\frac{1-\eta}{\eta}g(t)}-H^\pi(t,\pi^*(t))\\ &\quad -\frac{\vartheta^{0,1}}{1-\eta}(e^{-(1-\eta)h(t)}-1)+\lambda(0).\end{aligned} \tag{6.9}$$

From the equations (6.8) and (6.9), we obtain

$$\begin{aligned}h'(t) &= \frac{\eta}{1-\eta}e^{-\frac{\beta t}{\eta}-\frac{1-\eta}{\eta}g(t)}(e^{\frac{1-\eta}{\eta}h(t)}-1)+\frac{\vartheta^{1,0}}{1-\eta}(e^{(1-\eta)h(t)}-1)\\ &\quad -\frac{\vartheta^{0,1}}{1-\eta}(e^{-(1-\eta)h(t)}-1)+\lambda(0)-\lambda(1).\end{aligned} \tag{6.10}$$

Notice that the boundary conditions for g and h are

$$g(T)=0, h(T)=0. \tag{6.11}$$

Lemma 6.6. *(Uniqueness of the solution) The system of ordinary differential equations* (6.9) *and* (6.10) *subject to the boundary conditions* (6.11) *have a unique solution.*

Proof. Firstly, for $i = 0, 1$ we define functions $F, F^i : [0,T] \times \mathbb{R} \times \mathbb{R}_0^+$ as

$$F^0(t,x,y) = -\frac{\eta}{1-\eta} e^{-\frac{\beta t}{\eta} - \frac{1-\eta}{\eta} x} - H^\pi(t, \pi^*(t)) - \frac{\vartheta^{0,1}}{1-\eta}(e^{-(1-\eta)y} - 1) + \lambda(0),$$

$$F^1(t,x,y) = -\frac{\eta}{1-\eta} e^{-\frac{\beta t}{\eta} - \frac{1-\eta}{\eta} x} - H^\pi(t, \pi^*(t)) - \frac{\vartheta^{1,0}}{1-\eta}(e^{(1-\eta)y} - 1) + \lambda(1)$$

and

$$\begin{aligned} F(t,x,y) &= \frac{\eta}{1-\eta} e^{-\frac{\beta t}{\eta} - \frac{1-\eta}{\eta} x}(e^{\frac{1-\eta}{\eta} y} - 1) + \frac{\vartheta^{1,0}}{1-\eta}(e^{(1-\eta)y} - 1) \\ &\quad - \frac{\vartheta^{0,1}}{1-\eta}(e^{-(1-\eta)y} - 1) + \lambda(0) - \lambda(1). \end{aligned}$$

Then we have $g'(t) = F^0(t, g(t), h(t))$, $g'(t) - h'(t) = F^1(t, g(t), h(t))$ and $h'(t) = F(t, g(t), h(t))$.

Existence of a unique local solution. Notice that F and $F^i, i = 1, 2$ are continuous in t. Furthermore, they are continuously differentiable in x and y. Thus, they are locally Lipschitz in x and y. The Picard-Lindelöf theorem implies the existence of a unique local solution.

Boundedness of the ODE. Since $y \geq 0$, we have

$$F(t,x,y) \geq -(\lambda(1) - \lambda(0)) \text{ for all } (t,x,y) \in [0,T] \times \mathbb{R} \times \mathbb{R}_0^+. \tag{6.12}$$

Hence, we also have

$$h'(s) = F(s, g(s), h(s)) \geq -(\lambda(1) - \lambda(0)) \text{ for all } s \in [0,T]. \tag{6.13}$$

Integrating from t to T both sides of the inequality (6.13) with respect to s gives

$$h(T) - h(t) \geq -(\lambda(1) - \lambda(0))(T - t).$$

Due to the boundary condition of h, we obtain

$$h(t) \leq (\lambda(1) - \lambda(0))(T - t).$$

Notice that Lemma 6.5 shows that h is positive; therefore, the following holds

$$0 \leq h(t) \leq (\lambda(1) - \lambda(0))(T - t) \text{ for all } t \in [0, T].$$

In order to verify the uniqueness, we only have to prove the boundedness of g. First, we will derive two useful inequalities. Due to the non-negativity of y, we have

$$\begin{aligned} F^0(t, x, y) &= -\frac{\eta}{1-\eta} e^{-\frac{\beta t}{\eta} - \frac{1-\eta}{\eta} x} - H^\pi(t, \pi^*(t)) - \frac{\vartheta^{0,1}}{1-\eta}(e^{-(1-\eta)y} - 1) + \lambda(0) \\ &\geq -\frac{\eta}{1-\eta} e^{-\frac{\beta t}{\eta} - \frac{1-\eta}{\eta} x} - H^\pi(t, \pi^*(t)) + \lambda(0) \geq \xi^0(T) + \lambda(0) \end{aligned} \tag{6.14}$$

for all $(t, x, y) \in [0, T] \times \mathbb{R}_0^+ \times \mathbb{R}_0^+$ with

$$\xi^0(T) = \min_{t \in [0,T]} \Big\{ -\frac{\eta}{1-\eta} e^{-\frac{\beta t}{\eta}} - H^\pi(t, \pi^*(t)) \Big\}.$$

We also have

$$\begin{aligned} F^1(t, x, y) &= -\frac{\eta}{1-\eta} e^{-\frac{\beta t}{\eta} - \frac{1-\eta}{\eta} x} - H^\pi(t, \pi^*(t)) - \frac{\vartheta^{1,0}}{1-\eta}(e^{(1-\eta)y} - 1) + \lambda(1) \\ &\leq \lambda(1) - H^\pi(t, \pi^*(t)) \leq \lambda(1) + \xi^1(T) \end{aligned} \tag{6.15}$$

for all $(t, x, y) \in [0, T] \times \mathbb{R} \times \mathbb{R}_0^+$ with

$$\xi^1(T) = \max_{t \in [0,T]} \Big\{ -H^\pi(t, \pi^*(t)) \Big\}.$$

Now we will show that

$$g(t) \leq (\xi^0(T) + \lambda(0))^-(T - t). \tag{6.16}$$

There are two cases to be considered.

Case 1: $\xi^0(T) + \lambda(0) \leq 0$. Since the inequality (6.14) implies that

$$g'(t) = F^0(t, g(t), h(t)) \geq \xi^0(T) + \lambda(0) \text{ if } g(t) \geq 0,$$

the following holds

$$g(t) \leq -(\xi^0(T) + \lambda(0))(T - t) \text{ for every } t \in [0, T].$$

Case 2: $\xi^0(T) + \lambda(0) > 0$. We suppose that there exists $\hat{t} \in [0, T]$ such that $g(\hat{t}) = 0$. Then the inequality (6.14) implies

$$g'(\hat{t}) = F^0(t, 0, h(\hat{t})) \geq \xi^0(T) + \lambda(0) > 0.$$

However, we know that $g(T) = 0$. Thus, we can conclude that $g(t) \leq 0$ for all $t \in [0, T]$.

Both cases imply that

$$g(t) \leq (\xi^0(T) + \lambda(0))^-(T - t).$$

As the next step, we will verify the following inequality

$$g(t) - h(t) \geq -(\xi^1(T) + \lambda(1))^-(T - t).$$

We also examine the two following cases:

Case 1: $\xi^1(T) + \lambda(1) \geq 0$. Then the inequality (6.15) gives

$$g'(t) - h'(t) = F^1(t, g(t), h(t)) \leq \xi^1(T) + \lambda(1).$$

It then follows that

$$g(t) - h(t) \geq -(\xi^1(T) + \lambda(1))(T - t) \text{ for all } t \in [0, T].$$

Case 2: $\xi^1(T) + \lambda(1) < 0$. Let us suppose that there exists $\hat{t} \in [0, T]$ such that $g(\hat{t}) - h(\hat{t}) = 0$. Applying the inequality (6.15) again for $t = \hat{t}$ yields

$$g'(\hat{t}) - h'(\hat{t}) = F^1(\hat{t}, 0, h(\hat{t})) \leq \xi^1(T) + \lambda(1) < 0.$$

Remember that we have $g(T) - h(T) = 0$. Thus, $g(t) - h(t) \geq 0$ for all $t \in [0, T]$.

Both cases imply that

$$g(t) - h(t) \geq -(\xi^1(T) + \lambda(1))^-(T - t). \tag{6.17}$$

Since h is non-negative, the inequalities (6.16) and (6.17) yield

$$-\left(\xi^1(T)+\lambda(1)\right)^-(T-t) \leq g(t) \leq \left(\xi^0(T)+\lambda(0)\right)^-(T-t).$$

Therefore, we already show that g and h are linearly bounded. This implies the existence and uniqueness of a global solution. □

Now we will state and prove Verification Theorem. As in the previous chapters, we assume for the moment that we only consider π, c which are in $[-M, M]$ with $M > 0$ big enough such that m^* and c^* are in $[-M, M]$. This restriction does not cause any problem for our application since $\pi^*(t)$ and $c^{i,*}(t)$ are bounded on $[0, T]$.

Theorem 6.7. *(Verification Theorem) Let $Q := [0,T] \times \mathbb{R}$. Suppose $G \in C^{1,2}(Q)$, which $|G(t,\tilde{X},i)| \leq K(1+|\tilde{X}|^k)$ for a suitable constant $K > 0$ and $k \in \mathbb{N}$ and for all $i \in E$, is a solution of the HJB equation (6.6). Then we have*

a) $G(t,\tilde{X},i) \geq J(t,\tilde{X},i)$ for all $0 \leq t \leq T, \tilde{X} \in \mathbb{R}_+$ and $i \in E$.

b) If for all $(t,\tilde{X}) \in Q$ there exists $(\pi^*, c^*) \in \mathcal{A}(t,\mathbb{F})$ with

$$(\pi^*(s), c^*(s)) \in \underset{\pi,c\in[-M,M]}{\arg\max} \Big(\mathcal{A}^{\pi,c}(s,\tilde{X}^*(s),x(s))\Big)$$

for all $s \in [t,T]$, where $\tilde{X}^*(s)$ is the controlled process with respect to $\pi^*(s), c^*(s)$ via (6.4), then we obtain $G(t,\tilde{X},i) = J(t,\tilde{X},i) = J_{\pi^*,c^*}(t,\tilde{X},i)$.

Proof. The proof of the theorem is similar to Theorem 5.4. We only need to verify that

$$\lim_{\rho\to\infty} \mathbb{E}\bigg(\int_t^{\theta_\rho} U_1(s,\tilde{X}(s))\bigg) = \mathbb{E}\int_t^T U_1(s,\tilde{X}(s))ds.$$

Notice that

$$U_1(s,\tilde{X}(s)) = e^{-\beta s}\frac{\tilde{X}(s)^{1-\eta}}{1-\eta} \leq C\tilde{X}(s)^{1-\eta}$$

for some constant $C > 0$ and all $s \in [0,T]$. Furthermore,

$$\int_t^{\theta_\rho} U_1(s,\tilde{X}(s))ds \leq \int_0^{\theta_\rho} C\sup|\tilde{X}(s)|^{1-\eta}ds \leq CT\sup|\tilde{X}(s)|^{1-\eta}.$$

Now applying Proposition A.2 (see Appendix A) and the Dominated Convergence Theorem yields the result. □

After establishing the previous results, we obtain the following theorem.

Theorem 6.8. *(Optimal investment strategy) The optimal portfolio strategy is given as*

$$\pi^*(t) = \frac{\mu - r - \sigma_I \rho \sigma_S}{\eta \sigma_S^2} + \frac{\sigma_I \rho}{\sigma_S},$$

and the optimal consumption ratio is given as

$$c^{i,*}(t) = \exp\Big\{ -\frac{\beta t}{\eta} - \frac{1-\eta}{\eta}(g(t) - 1_{\{i=1\}} h(t)) \Big\}$$

for all $t \in [0,T]$. The value functions are given as

$$J(t, \tilde{X}, 0) = \frac{(\tilde{X} e^{g(t)})^{1-\eta}}{1-\eta},$$
$$J(t, \tilde{X}, 1) = \frac{(\tilde{X} e^{g(t)-h(t)})^{1-\eta}}{1-\eta},$$

where $(g(t), h(t))$ is the unique solution of the system (6.9) *and* (6.10) *subject to the boundary conditions* (6.11).

Proof. The theorem is a consequence of Verification Theorem 6.7, Lemma 6.4 and Lemma 6.6. □

Remark 6.9. *Theorem 6.8 indicates that the optimal portfolio strategy is independent of the state. However, the optimal consumption ratio is adjusted to the change of the state. In particular, Theorem 6.8 suggests that the investor should consume more when the benchmark is doing well.*

6.3 Numerical examples

We consider a similar situation as in Section 5.1.6 by assuming that

$$Q = \begin{pmatrix} -\gamma & \gamma \\ 0 & 0 \end{pmatrix}$$

where $\gamma > 0$. Thus, the market permanently switches to another state after staying in one state for an exponential distributed random time with parameter γ.

In this section, we first compute the optimal investment strategy for an investor with respect to a specific set of parameters. Next, we compare the Markov-switching model with an "equivalent" average-data model.

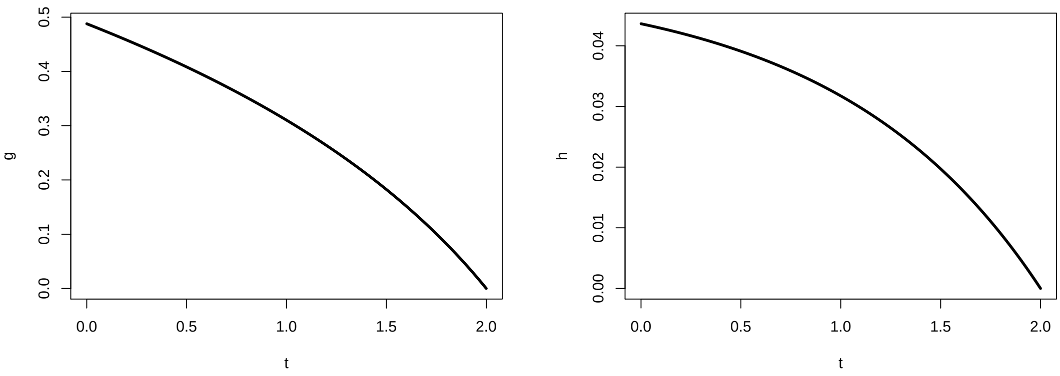


Figure 6.1: g (left) and h (right) as a functions of t ($r = \lambda(0) = \beta = 0.01$, $\mu = 0.06$, $\sigma_S = 0.35$, $\lambda(1) = 0.06$, $\sigma_I = 0.2$, $\rho = 0.3$, $\gamma = 1$, $\eta = 0.3$)

Figure 6.1 shows an example of g and h. Notice that h is non-negative and takes small values on $[0, T]$.

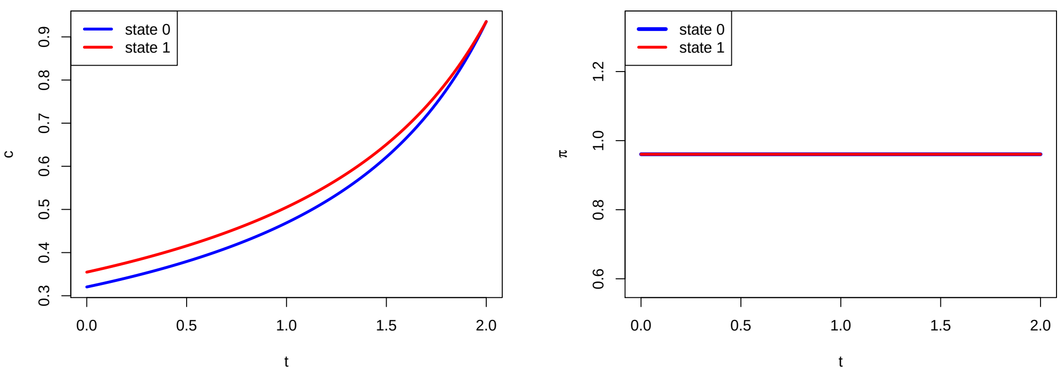


Figure 6.2: c (left) and π (right) as a functions of t ($r = \lambda(0) = \beta = 0.01$, $\mu = 0.06$, $\sigma_S = 0.35$, $\lambda(1) = 0.06$, $\sigma_I = 0.2$, $\rho = 0.3$, $\gamma = 1$, $\eta = 0.3$)

Figure 6.2 shows the optimal investment strategy for an benchmarked investor. It is clear that the state of the market does not affect the portfolio strategy π.

Remember that in our model, the asset prices in the portfolio are independent of the state; thus, there is no reason to adjust the portfolio strategy when the market shifts to another state. However, this is not the case for the optimal consumption ratio of the benchmarked investor. Figure 6.2 indicates that the investor consumes in state 1 more than he/she does in state 0. Notice that state 1 is better than state 0 for the benchmark. Nonetheless, for this benchmarked investor state 0 is preferable. Therefore, this particular investor consumes more when the state is unfavourable and does the opposite when the state is supportive.

Now we want to compare the Markov-switching model with an "equivalent" average-data model whose the benchmark satisfies

$$\frac{dI(t)}{I(t)} = \bar{\lambda}dt + \sigma_I(\rho dW_1(t) + \bar{\rho}dW_2(t)),$$

where $\bar{\lambda}$ is chosen such that the total return on the stock over the investment period is the same as that in the Markov-switching model. More specifically, the following holds

$$\mathbb{E}\int_0^T \lambda(x(t))dt = \int_0^T \bar{\lambda}dt = \bar{\lambda}T.$$

Given $\lambda(0)$, $\lambda(1)$ and T, we will determine γ such that we have the same $\bar{\lambda}$ whether the market starts in state 0 or in state 1. Recall from Section 5.1.6 that if $\lambda(x(0)) = \lambda(0)$, then

$$\bar{\lambda} = \frac{\lambda(1)T(1-e^{-\gamma T}) + (\lambda(0)-\lambda(1))\big(-Te^{-\gamma T} - \frac{e^{-\gamma T}}{\gamma} + \frac{1}{\gamma}\big) + \lambda(0)Te^{-\lambda T}}{T}.$$

Therefore, given $\lambda(0)$, $\lambda(1)$ and T, we have to solve the following equation

$$\begin{aligned}&\lambda(1)T(1-e^{-\gamma T}) + (\lambda(0)-\lambda(1))\big(-Te^{-\gamma T} - \frac{e^{-\gamma T}}{\gamma} + \frac{1}{\gamma}\big) + \lambda(0)Te^{-\gamma T}\\ &= \lambda(0)T(1-e^{-\gamma T}) + (\lambda(1)-\lambda(0))\big(-Te^{-\gamma T} - \frac{e^{-\gamma T}}{\gamma} + \frac{1}{\gamma}\big) + \lambda(1)Te^{-\gamma T},\end{aligned}$$

in which γ is the variable. In order to compare the Markov-switching model and the average-data model, we will utilize the concept of wealth ratio which is defined in [17]. Suppose we want to choose between two investment models A and B. To achieve a given maximal expected utility from terminal wealth and intermediate consumption $J_A(t, \tilde{x}^i_A(t), i)$, we need $\tilde{x}^i_A(t)$ as initial wealth if the market is at state i. $\tilde{x}^i_B(t)$ is defined similarly with respect to the model B. We shall compare $\tilde{x}^i_A(t)$

and $\tilde{x}^i_B(t)$ at every time $t \in [0, T]$ given $J_A(t, \tilde{x}^i_A(t), i) = J_B(t, \tilde{x}^i_B(t), i)$.

Thus, we consider

$$\begin{aligned} & J_A(t, \tilde{x}^i_A(t), i) = J_B(t, \tilde{x}^i_B(t), i) \\ \Leftrightarrow \quad & \frac{(\tilde{x}^i_A(t)e^{g_A(t)-1_{\{i=1\}}h_A(t)})^{1-\eta}}{1-\eta} = \frac{(\tilde{x}^i_B(t)e^{g_B(t)-1_{\{i=1\}}h_B(t)})^{1-\eta}}{1-\eta} \\ \Leftrightarrow \quad & \tilde{x}^i_A(t)e^{g_A(t)-1_{\{i=1\}}h_A(t)} = \tilde{x}^i_B(t)e^{g_B(t)-1_{\{i=1\}}h_B(t)} \\ \Leftrightarrow \quad & \frac{\tilde{x}^i_A(t) - \tilde{x}^i_B(t)}{\tilde{x}^i_A(t)} = 1 - e^{g_A(t)-1_{\{i=1\}}h_A(t)-(g_B(t)-1_{\{i=1\}}h_B(t))}. \end{aligned}$$

The wealth ratio is defined as

$$w^i_{A,B}(t) = 1 - e^{g_A(t)-1_{\{i=1\}}h_A(t)-(g_B(t)-1_{\{i=1\}}h_B(t))}.$$

Here, $w_{A,B}(t) > 0$ indicates the model B is more profitable while $w_{A,B}(t) < 0$ shows the opposite situation. We choose the average-data model as the model A and the Markov-switching model as the model B. Since the model A has only one state, $h_A(t) = 0$ for $t \in [0, T]$. Therefore, the wealth ratio becomes

$$w^i_{A,B}(t) = 1 - e^{g_A(t)-(g_B(t)-1_{\{i=1\}}h_B(t))}.$$

Figure 6.3 presents the behaviour of wealth ratio on $[0, T]$. We can observe that

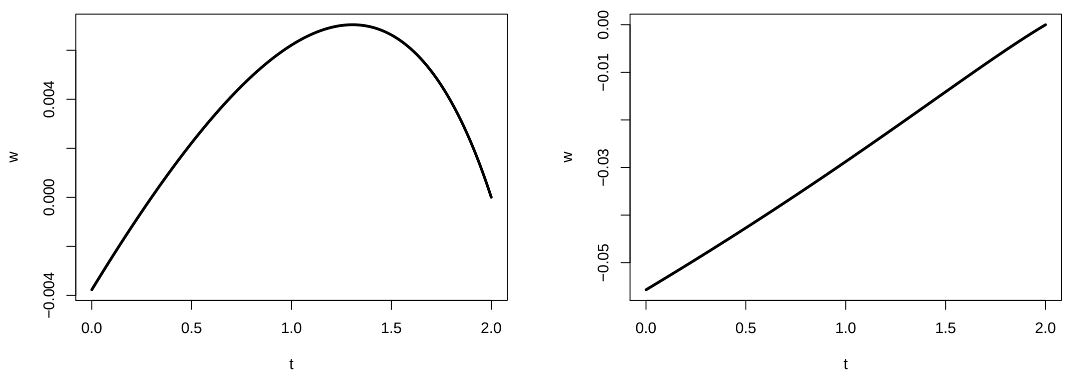


Figure 6.3: $w^0_{A,B}$ (left) and $w^1_{A,B}$ (right) as functions of t ($r = \lambda(0) = \beta = 0.01$, $\mu = 0.06$, $\sigma_S = 0.35$, $\lambda(1) = 0.06$, $\sigma_I = 0.2$, $\rho = 0.3$, $\gamma = 0.797$, $\eta = 0.3$, $\bar{\lambda} = 0.035$)

$w^0_{A,B}$ takes both positive and negative values, while $w^1_{A,B}$ only admits negative values. Therefore, we cannot decide which model is better.

Appendix A

Stochastic Differential Equations

Consider the SDE valued in $\mathbb{R}^n$

$$dX(t) = b(t, X(t))dt + \sigma(t, X(t))dW(t), \tag{A.1}$$

where W is a $d-$ dimensional Brownian motion, and $b(t, x, \omega), \sigma(t, x, \omega)$ take values in $\mathbb{R}^n$ and $\mathbb{R}^{n\times d}$ respectively. Furthermore, it is assumed that for all $x \in \mathbb{R}^n$, the processes $b(., x, .)$ and $\sigma(., x, .)$ are progressively measurable and for all ω, the functions $b(., ., \omega)$ and $\sigma(., ., \omega)$ are Borel measurable on $[0, T] \times \mathbb{R}^n$.

Definition A.1. *([24]) A strong solution of the SDE* (A.1) *starting at a time t is a vectorial progressively measurable process X such that*

$$\int_t^s |b(u, X(u))|du + \int_t^s |\sigma(u, X(u))|^2 du < \infty, \quad a.s., t \leq s \in [0, T]$$

and the following

$$X(s) = X(t) + \int_t^s b(u, X(u))du + \int_t^s \sigma(u, X(u))dW(u), \quad t \leq s \in [0, T]$$

holds.

The following result is used to estimate the moments of $X(t)$.

Proposition A.2. *([25]) Let there exist a constant K_1 such that*

$$\|\sigma(t, x)\| + |b(t, x)| \leq K_1(1 + |x|)$$

for all $t \in [0,T], \omega \in \Omega, x \in \mathbb{R}^n$. *Let* $X(t)$ *be a solution of* (A.1) *with initial condition* $X(0) = x_0$, *where* x_0 *is a fixed point on* $\mathbb{R}^n$. *There exists a constant* $N(q, K_1)$ *such that for all* $q \geq 0, t \in [0,T]$

$$\mathbb{E} \sup_{s \leq t} |X(s) - x_0|^q \leq N t^{q/2} e^{Nt} (1 + |x_0|)^q$$

and

$$\mathbb{E} \sup_{s \leq t} |X(s)|^q \leq N e^{Nt} (1 + |x_0|)^q.$$

Proof. The proof can be found in [25], page 86. □

Theorem A.3. *([16])(Variation of constants) Let* $\{(W(t), \mathcal{F}_t)\}_{t \in [0,\infty)}$ *be an* n*-dimensional Brownian motion. Let* $x \in \mathbb{R}$ *and* A, a, S_j, σ_j *be progressively measurable, real valued processes with*

$$\int_0^t (|A(s)| + |a(s)|)\, ds < \infty \text{ for all } t \geq 0 \text{ a.s. } P,$$
$$\int_0^t \left(S_j^2(s) + \sigma_j^2(s)\right) ds < \infty \text{ for all } t \geq 0, i = 1, ..., n \text{ a.s. } P.$$

Then the stochastic differential equation

$$dX(t) = (A(t)X(t) + a(t))\, dt + \sum_{j=1}^{n} (S_j(t)X(t) + \sigma_j(t))\, dW_j(t)$$
$$X(0) = 0$$

possesses the unique solution $\{(X(t), \mathcal{F}_t)\}_{t \in [0,\infty)}$ *with respect to* $\lambda \otimes P$ *given by*

$$\begin{aligned} X(t) &= Z(s)\left(x + \int_0^t \frac{1}{Z(u)} \left(a(u) - \sum_{j=1}^{n} S_j(u)\sigma_j(u)\right) du \right. \\ &+ \left. \sum_{j=1}^{n} \int_0^t \frac{\sigma_j(u)}{Z(u)} dW_j(u) \right. \end{aligned}$$

where

$$Z(t) = \exp\left(\int_0^t (A(u) - \frac{1}{2}\|S(u)\|^2) du + \int_0^t S(u) dW(u)\right)$$

is the unique solution of the homogeneous equation

$$dZ(t) = Z(s)\left(A(t)dt + S(t)'dW(t)\right)$$
$$Z(0) = 1.$$

Proof. The proof can be found in [16], page 54. □

Theorem A.4. *(Ito integral of a deterministic integrand) Let $\phi(t) : [0,T] \to \mathbb{R}$ be a deterministic function satisfying $\int_0^T \phi^2(t)dt < \infty$. Then the Ito integral*

$$\int_0^T \phi(t)dW(t)$$

is a normally distributed variable with mean zero and variance $\int_0^T \phi^2(t)dt$.

Proof. The proof can be found in [26], page 149. □

Appendix B

Continuous Time Markov Chain

Definition B.1. *([27]) A time-homogeneous continuous time Markov chain with transition rates $\lambda(x, y)$ is a stochastic process $X(t)$ taking values in a finite or countably infinite state space S satisfying*

$$P\{X(t+h) = x | X(t) = x\} = 1 - \lambda(x)h + o(h)$$
$$P\{X(t+h) = y | X(t) = x\} = \lambda(x, y)h + o(h),$$

where $y \neq x$, and $\lambda(x) = \sum_{y \neq x} \lambda(x, y)$.

We can deduce the transition probabilities of the chain via the identity

$$p_{xy} = \frac{\lambda(x, y)}{\lambda(x)} = \frac{\lambda(x, y)}{\sum_{y \neq x} \lambda(x, y)}.$$

Lemma B.2. *Let $(X_t)_{0 \leq t \leq T}$ be a continuous time finite-state Markov chain with transition rates matrix $Q = (q_{ij})$. Then the expected number of jumps $N[0, T]$ in the interval $[0, T]$ is finite.*

Proof. Let $p_i(t) = P(X(t) = i)$ and $N_T(i, j)$ be the number of transitions from i to j on time interval $[0, T]$. Then we obtain (see Proposition 3.2 in [28])

$$\mathbb{E}N_T(i, j) = q_{ij} \int_0^T p_i(u)du < q_{ij}T.$$

Therefore,

$$\mathbb{E}N[0,T] = \sum_{i \neq j} \mathbb{E}N_T(i,j) < \infty.$$

□

Let X be a continuous time Markov chain taking values in a countable state space S. Assume that the transition rate q_{xy} of the process X is finite for all $x, y \in S$, and $\mathcal{S}$ is a σ-algebra on S.

Definition B.3. *([29]) The jump measure μ of X and its compensator ν are random measures on $(0,\infty) \times S$, given by*

$$\mu = \sum_{t: X_t \neq X_{t-}} \delta_{(t, X_t)} \tag{B.1}$$

and

$$\nu(dt, E) = q_{X(t-),E} dt, \quad E \in \mathcal{S}. \tag{B.2}$$

Assume that $\mathcal{P}$ is a predictable $\sigma-$ algebra on $\Omega \times (0,\infty)$. A function defined on $\Omega \times (0,\infty) \times S$ is predictable if it is $\mathcal{P} \otimes \mathcal{S}$ measurable.

Theorem B.4. *([29]) Let H be predictable and assume that, for all $t \geq 0$*

$$f(t) = \mathbb{E} \int_0^t \int_S |H(s,y)| \nu(ds, dy) < \infty$$

Then the following process is a well-defined martingale

$$M_t = \int_{(0,t] \times S} H(s,y)(\mu - \nu)(ds, dy).$$

Proof. The proof can be found in [29]. □

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