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Freeness of hyperplane arrangements with multiplicities

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1 Introduction

An *arrangement of hyperplanes* $\mathcal{A}$ is a finite collection of *hyperplanes* in a vector space V over a field $\mathbb{K}$. Arrangements can be studied with methods from combinatorics, topology or algebraic geometry, providing an interesting opportunity for interdisciplinary research. This is best exemplified in the following quote by Orlik and Terao:

> "Arrangements are easily defined and may be enjoyed at levels ranging from the recreational to the expert, yet these simple objects lead to deep and beautiful results. Their study combines methods from many areas of mathematics and reveals unexpected connections" [OT92].

In this thesis we will predominantly examine an arrangement $\mathcal{A}$ algebraically via its module of *logarithmic derivations* (vector fields) $D(\mathcal{A})$. This approach was inspired by singularity theory and initiated by Saito [Sai80]. Saito studied general hypersurfaces and defined so-called free divisors in the analytic category. Subsequently, Terao showed that in the special case of an arrangement of hyperplanes, one can pass to algebraic considerations [Ter80], leading to the developmen of a rich theory of *free arrangments*. The open question of whether freeness can be decided combinatorially, namely the *Terao conjecture*, is one of the most important open conjectures in the theory of arrangements.

Ziegler introduced the notion of a *multiarrangement* $(\mathcal{A}, m)$ by giving an arrangement $\mathcal{A}$ a multiplicity function $m : \mathcal{A} \to \mathbb{Z}_{\geq 0}$ [Zie89b]. Multiarrangements arise naturally as restrictions with multiplicities: for an arrangement $\mathcal{A}$ and a distinguished hyperplane $H_0 \in \mathcal{A}$, the restricted arrangement $\mathcal{A}''$ is defined to be

$$\mathcal{A}'' := \{H \cap H_0 \mid H \in \mathcal{A} \setminus \{H_0\}\}.$$

Furthermore, Ziegler introduced a multiplicity m'' on $\mathcal{A}''$ by setting for $X \in \mathcal{A}''$

$$m''(X) := \sharp\{H \in \mathcal{A} \setminus \{H_0\} \mid X = H \cap H_0\}.$$

Multiarrangements can also be examined in terms of their freeness, as Ziegler demonstrated in the following theorem:

Theorem 1.1 *[Zie89b, Theorem 11] For a free arrangement $\mathcal{A}$, the restricted multiarrangement $(\mathcal{A}'', m'')$ is free for any hyperplane $H_0 \in \mathcal{A}$.*

Inversely, Yoshinaga showed that the freeness of arrangements is closely related to the freeness of multiarrangements using the multirestriction as in Theorem 1.1 from above [Yos04] [Yos05]. This approach was refined in the recent articles by Schulze [Sch12] and Abe and Yoshinaga [AY13]. As a special case, one can decide the freeness of a simple arrangement $\mathcal{A}$ in dimension three just by comparing the degrees of a basis of $D(\mathcal{A}'', m'')$ of the restricted multiarrangement $(\mathcal{A}'', m'')$ in dimension two with the characteristic polynomial of $\mathcal{A}$. Note that Ziegler proved that a multiarrangement $(\mathcal{A}, m)$ in dimension two is always free (see Theorem 2.18), and thus $D(\mathcal{A}, m)$

always has a basis. Since the characteristic polynomial can be determined combinatorially (see Definition 2.12), the difficulty of Terao's conjecture is, in this case, equivalent to the difficulty of determining the degrees of a basis of $D(\mathcal{A}, m)$ for a multiarrangement in dimension two. However, Ziegler showed that these degrees are not combinatorial in general [Zie89b]. Nevertheless, Wakefield and Yuzvinsky determined these degrees generically (i.e. in almost all cases) [WY07] and Wakamiko constructed an explicit basis if $(\mathcal{A}, m)$ consist of three hyperplanes [Wak07], as quoted in Theorem 5.6 of this thesis.

Therefore, the natural question arises to determine which multiplicities m for an arrangement $\mathcal{A}$ are free and which are not. In general, this question turns out to be rather difficult and not well-understood. It is known, however, that for a Boolean arrangement (see Example 2.4) any multiplicity is free (see Example 2.24). Conversely, for generic arrangements (see Definition 2.6) the following theorem holds:

Theorem 1.2 *Let V be a vector space with $\ell = \dim(V) \geq 3$ and $\mathcal{A}$ be a generic arrangement in V consisting of more than ℓ hyperplanes. Then, any multiplicity for $\mathcal{A}$ is non-free, i.e. $\mathcal{A}$ is totally non-free (see Definition 4.1).*

This result was first proved by Ziegler by giving an explicit minimal generating set for $D(\mathcal{A}, m)$ which is not a basis [Zie89a]. Wiens showed, with methods from homological algebra, that any generic arrangement $\mathcal{A}$ itself is non-free [Wie01]. Yoshinaga proved Theorem 1.2 for arrangements $\mathcal{A}$ in a vector space V over a field $\mathbb{K}$ with characteristic 0 using a converse statement of Theorem 1.1 [Yos10]. We will give a different proof for this result by applying a non-freeness criterion recently found by Abe, Terao and Wakefield [ATW07].

The first complete classification of free multiplicities on an arrangement $\mathcal{A}$ which admits both free and non-free multiplicities was found by Abe:

Theorem 1.3 (Theorem 0.2 [Abe07]) *Let $\mathcal{A}$ be the deleted A_3 arrangement defined by*

$$\mathcal{Q}(\mathcal{A}, m) = (y - z)^a y^b (x - y)^c x^d (x - z)^e.$$

Then $(\mathcal{A}, m)$ is free if and only if $c \geq a + e - 1$ or $c \geq b + d - 1$.

Abe posed the question of whether the set of free multiplicities on an arrangement $\mathcal{A}$ consists of chambers of a hyperplane arrangement in $\mathbb{Z}_{>0}^{|\mathcal{A}|}$. This question is still unanswered. In this thesis we examine the property of being free for an arrangement asymptotically via the following definition:

Definition 1.4 Let $(\mathcal{A}, m)$ be a multiarrangement with a fixed hyperplane $H_0 \in \mathcal{A}$. A new multiplicity function $m_k : \mathcal{A} \to \mathbb{Z}_{>0}$ may be defined for $k \in \mathbb{Z}_{\geq 0}$ by setting

$$m_k(H) = \begin{cases} m(H) & \text{if } H \neq H_0, \\ m(H) + k & \text{if } H = H_0. \end{cases}$$

We show in the following theorem that the the multiarrangements $(\mathcal{A}, m_k)$ are either constantly free or non-free for $k \gg 0$. This could hint towards a positive answer to Abe's above question.

Theorem 1.5 *Let $(\mathcal{A}, m)$ be a multiarrangement with a fixed hyperplane $H_0 \in \mathcal{A}$ and $m_0 = m(H_0)$. Assume $(\mathcal{A}, m_{k_0})$ is free for one $k_0 > \frac{1}{2}|m| - m_0$ where $|m|$ is the sum over all multiplicities of $\mathcal{A}$. Then $(\mathcal{A}, m_k)$ is free for all $k \geq \frac{1}{2}|m| - m_0$. Conversely if $(\mathcal{A}, m_{k_0})$ is non-free for one $k_0 > \frac{1}{2}|m| - m_0$, then $(\mathcal{A}, m_k)$ is non-free for all $k > \frac{1}{2}|m| - m_0$.*

To prove Theorem 1.5, we examine multiarrangements with one large multiplicity and we generalize Ziegler's Theorem 1.1 to such multiarrangements:

Theorem 1.6 *Suppose* $(\mathcal{A}, m)$ *is a free multiarrangement with* $H_0 \in \mathcal{A}$ *such that* $2m_0 > |m|$ *where* $m_0 = m(H_0)$*. Then, the restricted arrangement* $\mathcal{A}''$ *has a natural multiplicity function for* $X \in \mathcal{A}''$

$$m''(X) = \sum_{\substack{X \subset H \\ H \neq H_0}} m(H)$$

and the multiarrangment $(\mathcal{A}'', m'')$ *is free.*

Furthermore, we give a counter-example of a free multiarrangement which does not have a relatively large multiplicity and whose restriction with multiplicities is non-free (see Example 5.7). Theorem 1.6 enables us to apply the addition-deletion theorems developed by Abe, Terao and Wakefield in their recent article [ATW08] to prove Theorem 1.5.

2 Basics of arrangements

2.1 Definitions and examples of arrangements of hyperplanes

We begin by giving the basic definitions of a hyperplane arrangements. Our notation follows mainly the book "Arrangements of hyperplanes" by Orlik and Terao [OT92].

Definition 2.1 Let V be a vector space over a field $\mathbb{K}$ of dimension ℓ. A *hyperplane* H is an affine subspace of V of dimension $\ell - 1$. A *hyperplane arrangement* $\mathcal{A}$ in V is a finite set of hyperplanes of V.

In order to emphasize that an arrangement $\mathcal{A}$ is defined in an ℓ-dimensional vector space, $\mathcal{A}$ is called an ℓ-arrangement. Φ_ℓ is defined to be the empty ℓ-arrangement. The cardinality of $\mathcal{A}$ is denoted with $|\mathcal{A}|$. Let V^* be the dual space of V and $S = S(V^*)$ the symmetric algebra of V^*. Suppose $\{e_1, ..., e_\ell\}$ is a basis of V, then there exists the dual basis $\{x_1, ..., x_\ell\}$ of V^* such that $x_i(e_j) = \delta_{ij}$ holds for all $1 \leq i, j \leq \ell$. Therefore, we are able to identify $S = \mathbb{K}[x_1, ..., x_\ell]$. Each hyperplane H in V is the kernel of a polynomial α_H of degree 1, which is only unique up to a constant $c \in \mathbb{K}^*$.

Definition 2.2 The product of the defining polynomials

$$\mathcal{Q}(\mathcal{A}) = \prod_{H \in \mathcal{A}} \alpha_H$$

is called a *defining polynomial* of $\mathcal{A}$. The defining polynomial of the empty arrangement is defined to be $\mathcal{Q}(\Phi_\ell) = 1$.

Definition 2.3 An arrangement $\mathcal{A}$ is called *central* if each hyperplane contains the origin $0 \in V$. In this case each hyperplane is the kernel of a linear form of V and $\mathcal{Q}(\mathcal{A})$ is a homogeneous polynomial of degree $|\mathcal{A}|$.

Throughout this thesis we assume all arrangements to be central. To finish this section, a number of examples of hyperplane arrangements will be examined.

Example 2.4 The arrangement $\mathcal{A}$ defined by

$$\mathcal{Q}(\mathcal{A}) = x_1 x_2 \cdots x_l$$

is called a *Boolean* arrangement. In $\mathbb{R}^n$ it consists of the ℓ coordinate planes. It can be noted that any ℓ-arrangement of ℓ linearly independent hyperplanes is Boolean after a suitable change of coordinates.

Example 2.5 Consider in $\mathbb{R}^3$ with its canonical basis the cube with vertices $\{\pm 1, \pm 1, \pm 1\}$. It has 9 planes of symmetry, which form a central 3-arrangement $\mathcal{A}$ with defining polynomial

$$\mathcal{Q}(\mathcal{A}) = xyz(x-y)(x-z)(y-z)(x+y)(x+z)(y+z).$$

The intersections of two of these hyperplanes are rotational axes of symmetry of the cube. This arrangement is called a B_3-arrangement and is an example for a *reflection arrangement.*

Definition 2.6 An arrangement $\mathcal{B}$ such that $\mathcal{B}$ is a subset of $\mathcal{A}$ is called a *subarrangement* of $\mathcal{A}$. Let $\mathcal{A}$ be a central ℓ-arrangement with $\ell \geq 2$, then $\mathcal{A}$ is called *generic* if the hyperplanes of each subarrangement $\mathcal{B} \subset \mathcal{A}$ with $|\mathcal{B}| = \ell$ are linearly independent.

Example 2.7 The 3-arrangement defined by

$$\mathcal{Q}(\mathcal{A}) = xyz(x+y+z)$$

is a generic arrangement.

2.2 Combinatorial aspects

In arrangements of hyperplanes one can often notice an interesting interplay of algebraic and combinatorial aspects, of which we focus on the combinatorial in this section. To this end, an important combinatorial invariant of an arrangement, its intersection lattice, will be defined:

Definition 2.8 For an arrangement $\mathcal{A}$ the set of all non-empty intersections of elements of $\mathcal{A}$ is defined to be $L(\mathcal{A})$. It can be agreed that V is in $L(\mathcal{A})$ as the intersection of the empty collection of hyperplanes. $L(\mathcal{A})$ is ordered by *reverse* inclusion, that is

$$X \leq Y \Leftrightarrow X \supseteq Y.$$

Furthermore, a rank function is defined on $L(\mathcal{A})$ as $r(X) = \operatorname{codim}(X)$. Let $L(\mathcal{A})_k = \{X \in L(\mathcal{A}) \mid r(X) = k\}$. $L(\mathcal{A})$ is called the *intersection lattice* of $\mathcal{A}$. Lastly, the *rank* of an arrangement $r(\mathcal{A})$ is

$$r(\mathcal{A}) := r\left(\bigcap_{H \in \mathcal{A}} H\right).$$

Note, that this definition is well-defined, since it was assumed that all arrangements are central.

The intersection lattice $L(\mathcal{A})$ carries already a relatively large part of the information of a hyperplane arrangement. A feature of an arrangement is called *combinatorial* if it can be determined by its intersection lattice and the dimension of the vector space V alone. Next, some basic constructions of arrangements are introduced.

Definition 2.9 Let $\mathcal{A}$ be an arrangement and $X \in L(\mathcal{A})$. The *localization* of $\mathcal{A}$ at X is defined to be

$$\mathcal{A}_X = \{H \in \mathcal{A} \mid X \subseteq H\}.$$

The *restriction* of $\mathcal{A}$ to X is defined to be

$$\mathcal{A}^X = \{X \cap H \mid H \in \mathcal{A} \setminus \mathcal{A}_X \text{ and } X \cap H \neq \emptyset\}.$$

Let $H_0 \in \mathcal{A}$. Then, $\mathcal{A}' := \mathcal{A} \setminus \{H_0\}$ and $\mathcal{A}'' := \mathcal{A}^{H_0}$ are respectively called the *deleted* and *restricted* arrangement of $\mathcal{A}$ with respect to H_0.

The method of deletion and restriction is a very useful construction which will be used in Chapter 5. It allows one to apply the method of induction on the cardinality $|\mathcal{A}|$ and the dimension of $\mathcal{A}$.

Definition 2.10 Let $\mathcal{A}$ be an arrangement and $L(\mathcal{A})$ its intersection lattice. The *Möbius function* $\mu : L(\mathcal{A}) \times L(\mathcal{A}) \to \mathbb{Z}$ is defined as:

$$\begin{aligned} \mu(X, X) &= 1 && \text{if } X \in L, \\ \sum_{X \leq Z \leq Y} \mu(X, Z) &= 0 && \text{if } X, Z, Y \in L \text{ and } X < Y, \\ \mu(X, Y) &= 0 && \text{otherwise.} \end{aligned}$$

Note that from the above definition one can recursively compute $\mu(X, Y) = -\sum_{X \leq Z < Y} \mu(X, Z)$. This function is closely related to the Möbius function in elementary number theory. In both fields the Möbius function turns out to be useful due to the existence of an inversion formula. In the case of arrangements the formula can be stated as follows:

Proposition 2.11 [OT92, Proposition 2.39] Let f, g be functions on $L(\mathcal{A})$ with values in any abelian group. It follows that

$$g(Y) = \sum_{X \leq Y} f(X) \Leftrightarrow f(Y) = \sum_{X \leq Y} \mu(X, Y) g(X)$$
$$g(X) = \sum_{X \leq Y} f(Y) \Leftrightarrow f(X) = \sum_{X \leq Y} \mu(X, Y) g(Y)$$

Using this Möbius function one can associate a polynomial to an arrangement $\mathcal{A}$ which turns out to be combinatorial by definition.

Definition 2.12 The *characteristic polynomial* of an arrangement $\mathcal{A}$ is defined to be

$$\chi(\mathcal{A}, t) = \sum_{X \in L} \mu(V, X) t^{\dim X}.$$

The characteristic polynomial can be characterized by the recursive relations of the following Proposition 2.13. This proposition is the first example of an application of the technique of deletion and restriction.

Proposition 2.13 [OT92, Corollary 2.57] Let $\mathcal{A}$ be an arrangement and $H_0 \in \mathcal{A}$ a distinguished hyperplane. It follows that

(1) $\chi(\Phi_\ell, t) = t^\ell$

(2) $\chi(\mathcal{A}, t) = \chi(\mathcal{A}', t) - \chi(\mathcal{A}'', t)$.

2.3 Multiarrangements and their freeness

An arrangement $\mathcal{A}$ can also be studied algebraically. For this, a module of logarithmic derivations $D(\mathcal{A})$ will be defined and examined. Ziegler generalized arrangements of hyperplanes by introducing multiarrangements [Zie89b]. As this thesis is mainly concerned with the freeness of multiarrangements, the theory of logarithmic derivations will be developed for multiarrangements.

Definition 2.14 An arrangement $\mathcal{A}$ together with a multiplicity function $m : \mathcal{A} \to \mathbb{Z}_{>0}$ is called a *multiarrangement*. It is denoted by $(\mathcal{A}, m)$. If an order on $\mathcal{A} = \{H_1, ..., H_n\}$ is fixed, one sometimes writes $m = (b_1, ..., b_n)$ with $b_i = m(H_i)$ for all $i = 1, ..., n$. If in addition $b_1 \leq b_2 \leq ... \leq b_n$ holds, one writes $m = (b_1, ..., b_n)_{\leq}$.

If $m(H) = 1$ for all $H \in \mathcal{A}$, then $(\mathcal{A}, m)$ can be identified with $\mathcal{A}$ and $\mathcal{A}$ is called a *simple* arrangement. The *defining polynomial* of an multiarrangement $(\mathcal{A}, m)$ is defined to be

$$\mathcal{Q}(\mathcal{A}, m) = \prod_{H \in \mathcal{A}} \alpha_H^{m(H)}.$$

The cardinality of $(\mathcal{A}, m)$ is defined by

$$|m| = \sum_{H \in \mathcal{A}} m(H).$$

Recall that $S = S(V^*)$ and one can identify $S = \mathbb{K}[x_1, ..., x_\ell]$.

Definition 2.15 A $\mathbb{K}$-linear map $\theta : S \to S$ that satisfies the product rule

$$\theta(fg) = \theta(f)g + f\theta(g)$$

for all $f, g \in S$ is called *derivation* of S over $\mathbb{K}$. For $p \geq 1$, $\mathrm{Der}^p(S)$ is the set of all alternating p-linear functions $\theta : S^p \to S$ such that θ is a derivation in each variable. For $p = 0$ one defines $\mathrm{Der}^0(S) := S$ and one also writes $\mathrm{Der}(S) := \mathrm{Der}^1(S)$.

By setting $\theta_1 + \theta_2 \in \mathrm{Der}^p(S)$ and $f\theta_1 \in \mathrm{Der}^p(S)$ for $f \in S$ and $\theta_1, \theta_2 \in \mathrm{Der}^p(S)$ in the canonical way, $\mathrm{Der}^p(S)$ becomes an S-module. $\mathrm{Der}(S)$ has a basis $\partial_1, ..., \partial_\ell$, where $\partial_i(f) = \frac{\partial f}{\partial x_i}$ for $f \in S$ is the usual partial derivation with respect to x_i. Note that one can identify $\mathrm{Der}^p(S)$ with $\wedge_{i=1}^p \mathrm{Der}(S)$ as these modules are naturally isomorphic. So $\{\partial_{i_1} \wedge ... \wedge \partial_{i_p} \mid 1 \leq i_1 < ... < i_p \leq \ell\}$ is a basis of $\mathrm{Der}^p(S)$ for $p \geq 1$.

Definition 2.16 A Derivation $\theta \in \mathrm{Der}^p$ is called *homogeneous of polynomial degree p* if $\theta(f_1, ..., f_p)$ is zero or a polynomial of degree p for all $f_1, ..., f_p \in V^*$. It is denoted by $\mathrm{pdeg}\, \theta = p$. A derivation $\theta \in \mathrm{Der}(S)$ with $\theta = \sum_{i=1}^{\ell} f_i \partial_i$ has $\mathrm{pdeg}\, \theta = p$ if and only if $f_1, ..., f_\ell$ are all zero or homogeneous polynomials of degree p in S.

Thus, $\mathrm{Der}^p(S)$ is a free graded S-module for any $0 \leq p$. Now one can define the module of logarithmic derivations of a multiarrangment.

Definition 2.17 Let $(\mathcal{A}, m)$ be a multiarrangement in V. For $p \geq 1$ one defines

$$D^p(\mathcal{A}, m) = \{\theta \in \mathrm{Der}^p(S) \mid \theta(\alpha_H, f_2, ..., f_p) \in \alpha_H^{m(H)} S \text{ for all } H \in \mathcal{A} \text{ and } f_2, ..., f_p \in S\}.$$

The module $D(\mathcal{A}, m) := D^1(\mathcal{A}, m)$ is called the *module of $\mathcal{A}$-derivations* whereas elements of $D(\mathcal{A})$ are called derivations *tangent* to $\mathcal{A}$. If $D(\mathcal{A}, m)$ is a free S-module $(\mathcal{A}, m)$ is called a *free* multiarrangement.

The presence of the module $\Omega^p(\mathcal{A}, m)$ of logarithmic p-forms with poles along $(\mathcal{A}, m)$ and the dual nature of the modules $D(\mathcal{A}, m)$ and $\Omega^1(\mathcal{A}, m)$ (as shown in [OT92, Theorem 4.75]) implies that $D(\mathcal{A}, m)$ is a reflexive module. This fact allows one to prove the following theorem, first shown by Ziegler [Zie89b]:

Theorem 2.18 *Any* 2*-multiarrangement* $(\mathcal{A}, m)$ *is free.*

Proof: The module $D(\mathcal{A}, m)$ is a finitely generated module over $S = \mathbb{K}[x_1, x_2]$. Due to Hilbert's syzygy theorem ([Eis95, Theorem 1.13]), S has a global dimension of 2, i.e. any finitely generated module over S has a projective dimension of at most 2. We denote the S-dual by $-^* := \operatorname{Hom}(-, S)$. Let

$$F_1 \xrightarrow{\varphi} F_0 \longrightarrow D(\mathcal{A}, m)^* \longrightarrow 0$$

be a free presentation of $D(\mathcal{A}, m)^*$. Dualizing this exact sequence, and the fact that $D(\mathcal{A}, m)^{**} = D(\mathcal{A}, m)$, yields the exact sequence

$$0 \longrightarrow D(\mathcal{A}, m) \longrightarrow F_0^* \xrightarrow{\varphi^*} F_1^* \longrightarrow \operatorname{Coker}(\varphi^*) \longrightarrow 0. \tag{2.1}$$

The module $\operatorname{Coker}(\varphi^*)$ has a projective dimension of at most 2, because S has a global dimension of 2. The exact sequence (2.1) can be extended to a projective resolution of $\operatorname{Coker}(\varphi^*)$. Thus, $D(\mathcal{A}, m)$ is a 1-st syzygy module of $\operatorname{Coker}(\varphi^*)$. Then $D(\mathcal{A}, m)$ is a projective module by [Rot79, Theorem 9.5]. In the graded case, the notions of free and projective modules coincide by [Eis95, Theorem 19.2] which is an easy consequence of Nakayama's Lemma. Therefore, $D(\mathcal{A}, m)$ is a free module. □

Proposition 2.19 [OT92, Proposition 4.18] If $(\mathcal{A}, m)$ is a free ℓ-mulitarrangement, $D(\mathcal{A}, m)$ has a basis consisting of ℓ homogeneous elements.

Proof: Assume $D(\mathcal{A}, m)$ is a free module of rank r. By the definition of $D(\mathcal{A}, m)$ it holds that

$$\mathcal{Q}(\mathcal{A}, m) \operatorname{Der}(S) \subseteq D(\mathcal{A}, m) \subseteq \operatorname{Der}(S). \tag{2.2}$$

As stated above, $\operatorname{Der}(S)$ has the basis $\partial_1, ..., \partial_\ell$; therefore, $\mathcal{Q}(\mathcal{A}, m)\partial_1, ..., \mathcal{Q}(\mathcal{A}, m)\partial_\ell$ is a basis of $\mathcal{Q}(\mathcal{A}, m) \operatorname{Der}(S)$. Thus, $\operatorname{Der}(S)$ and $\mathcal{Q}(\mathcal{A}, m) \operatorname{Der}(S)$ are both free of rank ℓ. Combining this with (2.2) implies $r = \ell$. By [OT92, Theorem A.20] any free graded S-module of rank ℓ has a homogeneous basis $\{m_1, ..., m_\ell\}$. □

The *Hilbert series* of an $\mathbb{Z}_{\geq 0}$-graded, finitely generated module $M = \oplus M_q$ is

$$H(M, q) = \sum_{p=0}^{\infty} \dim_{\mathbb{K}}(M_p) q^p.$$

Suppose $(\mathcal{A}, m)$ is a free multiarrangement and $\{\theta_1, ..., \theta_\ell\}$ is a homogeneous basis of $D(\mathcal{A}, m)$. Let $d_i = \operatorname{pdeg} \theta_i$ for $1 \leq i \leq \ell$ denote the degree of the basis elements. Since it holds that $D(\mathcal{A}, m) \cong \oplus_{i=1}^{\ell} S(-d_i)$ where $S(d)$ denotes the ring S with a grading shifted by d, one finds the following equation for the Hilbert series of $D(\mathcal{A}, m)$:

$$H(D(\mathcal{A}, m), q) = (q^{d_1} + \cdots + q^{d_\ell}) H(S, q). \tag{2.3}$$

This fact implies that the degrees of a basis of $D(\mathcal{A}, m)$ only depend on $(\mathcal{A}, m)$ and not on a chosen basis, because the Hilbert series is independent of a chosen basis. Therefore, the following Definition 2.20 is well-defined.

Definition 2.20 Let $(\mathcal{A}, m)$ be a free multiarrangement and suppose $\{\theta_1, ..., \theta_\ell\}$ is a homogeneous basis of $D(\mathcal{A}, m)$. The degrees $\operatorname{pdeg}\theta_1, ..., \operatorname{pdeg}\theta_\ell$ are called the *exponents* of $(\mathcal{A}, m)$ and one writes

$$\exp(\mathcal{A}, m) = (\operatorname{pdeg}\theta_1, ..., \operatorname{pdeg}\theta_\ell).$$

If the degrees are ordered (i.e. $\operatorname{pdeg}\theta_1 \le ... \le \operatorname{pdeg}\theta_\ell$), one sometimes writes $\exp(\mathcal{A}, m) = (\operatorname{pdeg}\theta_1, ..., \operatorname{pdeg}\theta_\ell)_\le$.

For $\theta_1, ..., \theta_\ell \in D(\mathcal{A}, m)$ a $(\ell \times \ell)$-matrix $M(\theta_1, ..., \theta_\ell)$ can be defined by setting the (i, j)-th entry to be $\theta_j(x_i)$. The determinant of this matrix is a useful criterion to decide the freeness of a multiarrangement via the following Theorem 2.22. It was first proved by Saito for simple arrangements [Sai80, Theorem 1.8] and by Ziegler for multiarrangements [Zie89b, Theorem 8]. We give a proof which is based on the proof for simple arrangments from [OT92, Proposition 4.12].

Proposition 2.21 [OT92, Proposition 4.12] If $\theta_1, ..., \theta_\ell \in D(\mathcal{A}, m)$, then it holds that $\det M(\theta_1, ..., \theta_\ell) \in \mathcal{Q}(\mathcal{A}, m)S$.

Proof: The statement of Proposition 2.21 is clear for $\mathcal{A} = \Phi_\ell$ since $\mathcal{Q}(\mathcal{A}, m) = 1$. So let $H \in \mathcal{A}$ and let $H = \ker(\alpha_H)$ where $\alpha_H = \sum_{i=1}^{\ell} c_i x_i$. It may be assumed that $c_i = 1$ for some i. It then holds that

$$\det M(\theta_1, ..., \theta_\ell) = \begin{bmatrix} \theta_1(x_1) & \cdots & \theta_\ell(x_1) \\ \vdots & & \vdots \\ \theta_1(x_i) & \cdots & \theta_\ell(x_i) \\ \vdots & & \vdots \\ \theta_1(x_\ell) & \cdots & \theta_\ell(x_\ell) \end{bmatrix} = \begin{bmatrix} \theta_1(x_1) & \cdots & \theta_\ell(x_1) \\ \vdots & & \vdots \\ \sum_{i=1}^{\ell} c_i\theta_1(x_i) & \cdots & \sum_{i=1}^{\ell} c_i\theta_\ell(x_i) \\ \vdots & & \vdots \\ \theta_1(x_\ell) & \cdots & \theta_\ell(x_\ell) \end{bmatrix}$$

$$= \begin{bmatrix} \theta_1(x_1) & \cdots & \theta_\ell(x_1) \\ \vdots & & \vdots \\ \theta_1(\alpha_H) & \cdots & \theta_\ell(\alpha_H) \\ \vdots & & \vdots \\ \theta_1(x_\ell) & \cdots & \theta_\ell(x_\ell) \end{bmatrix} \in \alpha_H^{m(H)} S.$$

Since H is arbitrary, $\det M(\theta_1, ..., \theta_\ell)$ is divisible by all $\alpha_H^{m(H)}$, and thus also by $\mathcal{Q}(\mathcal{A}, m)$. □

Theorem 2.22 *[OT92, Theorem 4.19]* ***Saito-Ziegler's criterion*** *Let $\theta_1, ..., \theta_\ell$ be derivations in $D(\mathcal{A}, m)$. Then the following two statements are equivalent:*

(1) $\det M(\theta_1, ..., \theta_\ell) \in \mathbb{K}^*\mathcal{Q}(\mathcal{A}, m)$.

(2) $\theta_1, ..., \theta_\ell$ form a basis for $D(\mathcal{A}, m)$ over S.

Proof: (1) ⇒ (2) The derivations $\theta_1, ..., \theta_\ell$ are linearly independent over S because $\det M(\theta_1, ..., \theta_\ell) \neq 0$. One may assume that $\det M(\theta_1, ..., \theta_\ell) = \mathcal{Q}(\mathcal{A}, m)$. Therefore, it suffices to show that $\theta_1, ..., \theta_\ell$ generate $D(\mathcal{A}, m)$ over S. Let $\eta \in D(\mathcal{A}, m)$. We show $\eta \in S\theta_1 + \cdots + S\theta_\ell$. Since

$$\theta_i = \sum_{j=1}^{\ell} \theta_i(x_j)\partial_j,$$

the system of linear equations $(y_1 \cdots y_\ell)M(\theta_1, ..., \theta_\ell) = (\theta_1 \cdots \theta_\ell)$ in y_i has the unique solution $(\partial_1 \cdots \partial_\ell)$. When Cramer's rule is applied to this system of linear equations, it yields

$$\mathcal{Q}(\mathcal{A}, m)\partial_j = \det \begin{bmatrix} \theta_1(x_1) & \cdots & \theta_\ell(x_1) \\ \vdots & & \vdots \\ \theta_1 & \cdots & \theta_\ell \\ \vdots & & \vdots \\ \theta_1(x_\ell) & \cdots & \theta_\ell(x_\ell) \end{bmatrix}.$$

The Laplace expansion along the j-th row shows that

$$\mathcal{Q}(\mathcal{A}, m)\partial_j \in S\theta_1 + \cdots + S\theta_\ell.$$

Again, since $\eta = \sum_{j=1}^{\ell} \eta(x_j)\partial_j$, one may write

$$\mathcal{Q}(\mathcal{A}, m)\eta = \sum_{j=1}^{\ell} f_j \theta_j \tag{2.4}$$

for some $f_j \in S$. Due to Proposition 2.21, $\det M(\theta_1, ..., \theta_{i-1}, \eta, \theta_{i+1}, ..., \theta_\ell) \in \mathcal{Q}(\mathcal{A}, m)S$. Thus,

$$\begin{aligned} f_i \mathcal{Q}(\mathcal{A}, m) &= f_i \det M(\theta_1, ..., \theta_\ell) \\ &= \det M(\theta_1, ..., \theta_{i-1}, f_i\theta_i, \theta_{i+1}, ..., \theta_\ell) \\ &= \det M(\theta_1, ..., \theta_{i-1}, \mathcal{Q}(\mathcal{A}, m)\eta, \theta_{i+1}, ..., \theta_\ell) \\ &= \mathcal{Q}(\mathcal{A}, m) \det M(\theta_1, ..., \theta_{i-1}, \eta, \theta_{i+1}, ..., \theta_\ell) \in \mathcal{Q}(\mathcal{A}, m)^2 S. \end{aligned}$$

Therefore, $f_i \in \mathcal{Q}(\mathcal{A}, m)S$ for $i = 1, ..., \ell$. This fact combined with Equation (2.4) shows that

$$\eta = \sum_{i=1}^{\ell} \frac{f_i}{\mathcal{Q}(\mathcal{A}, m)} \theta_i \in S\theta_1 + \cdots + S\theta_\ell.$$

(2) ⇒ (1) Due to Proposition 2.21, one can write

$$\det M(\theta_1, ..., \theta_\ell) = f\mathcal{Q}(\mathcal{A}, m) \tag{2.5}$$

for some $f \in S$. Fix a hyperplane $H \in \mathcal{A}$. It may be assumed that $H = \ker(x_1)$ by choosing coordinates in a suitable way. Then $\mathcal{Q}_H := \mathcal{Q}(\mathcal{A}, m)/x_1^{m(H)}$ is a defining polynomial for $(\mathcal{A}, m) \setminus \{H\}$. Define $\eta_1 := \mathcal{Q}(\mathcal{A}, m)\partial_1$ and for $i = 2, ..., \ell$ $\eta_i := \mathcal{Q}_H \partial_i$. It is easy to see that these derivations are in $D(\mathcal{A}, m)$. Since $\theta_1, ..., \theta_\ell$ generate $D(\mathcal{A}, m)$, each η_i is an S-linear combination of $\theta_1, ..., \theta_\ell$. Thus, there exists an $\ell \times \ell$ matrix N with entries in S such that $M(\eta_1, ..., \eta_\ell) = M(\theta_1, ..., \theta_\ell)N$. Therefore, Equation (2.5) implies that

$$\mathcal{Q}(\mathcal{A}, m)\mathcal{Q}_H^{\ell-1} = \det M(\eta_1, ..., \eta_\ell) = \det M(\theta_1, ..., \theta_\ell) \det N \in f\mathcal{Q}(\mathcal{A}, m)S.$$

Therefore, $\mathcal{Q}_H^{\ell-1}$ is divisible by f. This is true for all $H \in \mathcal{A}$. Then $f \in \mathbb{K}^*$ must hold, because the polynomials $\{\mathcal{Q}_H^{\ell-1}\}_{H \in \mathcal{A}}$ have no common factor. □

Corollary 2.23 [OT92, Theorem 4.23] If $\theta_1, ..., \theta_\ell \in D(\mathcal{A}, m)$ are all homogeneous and linearly independent over S, then $\theta_1, ..., \theta_\ell$ form a basis for $D(\mathcal{A}, m)$ if and only if

$$\sum_{i=1}^{\ell} \operatorname{pdeg} \theta_i = \sum_{H \in \mathcal{A}} m(H) = |m|.$$

Proof: Since $\theta_1, ..., \theta_\ell$ are linearly independent, $\det M(\theta_1, ..., \theta_\ell) \neq 0$. As before, due to Proposition 2.21, one may write $\det M(\theta_1, ..., \theta_\ell) = f\mathcal{Q}(\mathcal{A}, m)$ with some homogeneous polynomial $f \in S$ since the $\theta_1, ..., \theta_\ell$ are homogeneous. It then holds that

$$\sum_{i=1}^{\ell} \operatorname{pdeg} \theta_i = \deg \det M(\theta_1, ..., \theta_\ell)$$
$$= \deg \mathcal{Q}(\mathcal{A}, m) + \deg f = |m| + \deg f.$$

The conclusion follows directly from the equivalence in Theorem 2.22. □

Example 2.24 Let $\mathcal{A}$ be the Boolean ℓ-arrangement from Example 2.4 with multiplicities $m = (b_1, ..., b_\ell)$. Thus, the defining polynomial of $(\mathcal{A}, m)$ is

$$\mathcal{Q}(\mathcal{A}, m) = x_1^{b_1} \cdots x_\ell^{b_\ell}.$$

Consider the derivations $\theta_i := x_i^{b_i}\partial_i$ for $1 \leq i \leq \ell$. It holds for $1 \leq i, j \leq \ell$ that

$$\theta_i(\alpha_{H_j}) = x_i^{b_i}\partial_i(x_j) = \delta_{ij}x_i^{b_i} \in x_i^{b_i}S.$$

Therefore, all these derivations are in $D(\mathcal{A}, m)$. They are clearly linearly independent and we have $\sum_{i=1}^{\ell} \operatorname{pdeg} \theta_i = \sum_{i=1}^{\ell} b_i = |m|$. From Corollary 2.23, it follows that $(\mathcal{A}, m)$ is a free multiarrangement with exponents $\exp(\mathcal{A}, m) = (b_1, ..., b_\ell)$.

Next we introduce the product of two multiarrangements. This construction turns out to be helpful when the multiarrangement is not of full rank (i.e. $r(\mathcal{A}) < \dim V$). In this case, one can decompose $(\mathcal{A}, m)$ into two smaller multiarrangements. Lemma 2.26 describes the decomposition of the derivation modules of a product-multiarrangement.

Definition 2.25 Let V_i be vector spaces over $\mathbb{K}$ and $(\mathcal{A}_i, m_i)$ be multiarrangements in V_i for $i = 1, 2$. Then, the *product arrangement* $(\mathcal{A}_1 \times \mathcal{A}_2, m_1 \times m_2)$ in the vector space $V_1 \oplus V_2$ is defined in the following way:

$$\mathcal{A}_1 \times \mathcal{A}_2 := \{H_1 \oplus V_2 \mid H_1 \in \mathcal{A}_1\} \cup \{V_1 \oplus H_2 \mid H_2 \in \mathcal{A}_2\},$$
$$(m_1 \times m_2)(H_1 \oplus V_2) := m_1(H_1),$$
$$(m_1 \times m_2)(V_1 \oplus H_2) := m_2(H_2).$$

Lemma 2.26 [ATW07, Lemma 1.4] For two multiarrangements $(\mathcal{A}_1, m_1)$ and $(\mathcal{A}_2, m_2)$ and $0 \leq k \leq \ell$ we have that

$$D^k(\mathcal{A}_1 \times \mathcal{A}_2, m_1 \times m_2) \cong \bigoplus_{i+j=k} D^i(\mathcal{A}_1, m_1) \otimes_{\mathbb{K}} D^j(\mathcal{A}_2, m_2).$$

In particular

$$D(\mathcal{A}_1 \times \mathcal{A}_2, m_1 \times m_2) \cong (S \otimes_{\mathbb{K}} D(\mathcal{A}_1, m_1)) \oplus (S \otimes_{\mathbb{K}} D(\mathcal{A}_2, m_2))$$

where $S = S((V_1 \oplus V_2)^*)$.

Suppose $(\mathcal{A}, m)$ is a ℓ-multiarrangement of rank k with $0 \leq k \leq \ell$. One can choose a coordinate system for S in such a way that the maximal element $X := \bigcap_{H \in \mathcal{A}}$ in $L(\mathcal{A})$ is defined by $X = \{x_1 = ... = x_k = 0\}$. Then, $(\mathcal{A}, m)$ can be decomposed in a multiarrangement in $\mathbb{K}^k$ and the empty arrangement $\Phi_{\ell-k}$ in $\mathbb{K}^{\ell-k} \cong X$. From this construction one immediately obtains the following corollary by applying Theorem 2.18 and Lemma 2.26:

Corollary 2.27 A multiarrangement $(\mathcal{A}, m)$ with $r(\mathcal{A}) \leq 2$ is free.

3 The characteristic polynomial of a multiarrangement

3.1 Definiton of $\chi((\mathcal{A}, m), t)$

In Definition 2.12 the notion of a characteristic polynomial of a simple arrangement was introduced in a combinatorial way. This definition does not generalize directly to multiarrangements, since the intersection lattice contains no information about the multiplicities of the hyperplanes. However, Solomon and Terao proved the following algebraic formula for the characteristic polynomial of a simple arrangement, which can be generalized.

Theorem 3.1 *[ST87, Theorem 1.2] For any simple ℓ-arrangement $\mathcal{A}$ in V it holds that*

$$\chi(\mathcal{A}, t) = (-1)^\ell \lim_{q \to 1} \sum_{p=0}^{\ell} H(D^p(\mathcal{A}), q)(t(q-1)-1)^p \tag{3.1}$$

where $H(D^p(\mathcal{A}), q)$ is the Hilbert series of $D^p(\mathcal{A})$.

Abe, Terao and Wakefield generalized this Formula (3.1) to multiarrangements as follows [ATW07]. Since each $D^p(\mathcal{A}, m)$ is $\mathbb{Z}_{\geq 0}$-graded by polynomial degree, one may define a function

$$\psi(\mathcal{A}, m; t, q) := \sum_{p=0}^{\ell} H(D^p(\mathcal{A}, m), q)(t(q-1)-1)^p$$

in t and q where, $H(D^p(\mathcal{A}, m), x)$ is the Hilbert series of $D^p(\mathcal{A}, m)$. A priori, $\psi(\mathcal{A}, m; t, q)$ is a rational function in q, but it is shown in [ATW07, Theorem 2.5] that this series is a polynomial in q and t. Therefore, the following definition is justified.

Definition 3.2 [ATW07, Definition 2.6] The *characteristic polynomial* of any multiarrangement $(\mathcal{A}, m)$ is the polynomial

$$\chi((\mathcal{A}, m), t) = (-1)^\ell \psi(\mathcal{A}, m; t, 1)$$

and the *Poincaré polynomial* is

$$\pi((\mathcal{A}, m), t) = (-t)^\ell \chi((\mathcal{A}, m), -t^{-1}).$$

Due to the formula for $\chi(\mathcal{A}, t)$ as earlier discussed in Theorem 3.1 this definition generalizes the characteristic polynomial of a simple arrangement.

Lemma 3.3 For two multiarrangements $(\mathcal{A}_1, m_1)$ and $(\mathcal{A}_2, m_2)$ we have that

$$\chi((\mathcal{A}_1 \times \mathcal{A}_2, m_1 \times m_2), t) = \chi((\mathcal{A}_1, m_1), t)\chi((\mathcal{A}_2, m_2), t).$$

Proof: By definition of $\psi(\mathcal{A}, m; t, q)$ it holds that

$$\psi(\mathcal{A}_1 \times \mathcal{A}_2, m_1 \times m_2; t, q) = \sum_{p=0}^{\ell} H(D^p(\mathcal{A}_1 \times \mathcal{A}_2, m_1 \times m_2), q)(t(q-1)-1)^p.$$

Therefore, Lemma 2.26 yields

$$\begin{aligned}
&\psi(\mathcal{A}_1 \times \mathcal{A}_2, m_1 \times m_2; t, q) \\
&= \sum_{p=0}^{\ell} H\left(\left(\oplus_{i+j=p} D^i(\mathcal{A}_1, m_1) \otimes_{\mathbb{K}} D^j(\mathcal{A}_2, m_2)\right), q\right)(t(q-1)-1)^p \\
&= \sum_{p=0}^{\ell} \left(\sum_{i+j=p} H(D^i(\mathcal{A}_1, m_1), q) H(D^j(\mathcal{A}_2, m_2), q) \right) (t(q-1)-1)^p \\
&= \left(\sum_{i=0}^{\ell} H(D^i(\mathcal{A}_1, m_1), q)(t(q-1)-1)^i \right) \left(\sum_{j=0}^{\ell} H(D^j(\mathcal{A}_2, m_2), q)(t(q-1)-1)^j \right) \\
&= \psi(\mathcal{A}_1, m_1; t, q)\psi(\mathcal{A}_2, m_2; t, q).
\end{aligned}$$

If one now sets $q = 1$ and uses the fact that $\dim V = \dim V_1 + \dim V_2$, the claim follows directly.□

3.2 Local-global formula for $\chi((\mathcal{A}, m), t)$

Let $(\mathcal{A}, m)$ be a multiarrangement. One writes $(\mathcal{A}, m) \subseteq (\mathcal{B}, m')$ if $\mathcal{A} \subseteq \mathcal{B}$ and for all $H \in \mathcal{A}$ $0 < m(H) \leq m'(H)$ holds.

Lemma 3.4 [ATW07, Lemma 1.1] If $(\mathcal{A}, m) \subseteq (\mathcal{B}, m')$ then $D^p(\mathcal{A}, m) \supseteq D^p(\mathcal{B}, m')$.

Proof: Let $\theta \in D^p(\mathcal{B}, m')$ and let $H \in \mathcal{A}$ with $H = \ker \alpha_H$. It follows that $\theta(\alpha_H, f_2, ..., f_p) \in \alpha_H^{m'(H)} S \subseteq \alpha_H^{m(H)} S$ for all $f_2, ..., f_p \in S$. Thus, $\theta \in D^p(\mathcal{A}, m)$. □

In the previous chapter, we introduced for an arrangement $\mathcal{A}$ and $X \in L(\mathcal{A})$ the localized arrangement $\mathcal{A}_X = \{H \in \mathcal{A} \mid H \supseteq X\}$. This definition can be extended to multiarrangements by setting $m_X := m|_{\mathcal{A}_X}$. Let $(S-\mathrm{Mod})$ be the category of finitely generated S-modules. Regard $L(\mathcal{A})$ as a category with morphisms $\leq$. Then, $D^p(-)$ is a contravariant functor

$$D^p : L(\mathcal{A}) \to (S - \mathrm{Mod})$$

where $D^p(X) := D^p(\mathcal{A}_X, m_X)$ and $D^p(\leq)$ is the inclusion from Lemma 3.4, since for $X \leq Y \in L(\mathcal{A})$ it holds that $(\mathcal{A}_X, m_X) \subseteq (\mathcal{A}_Y, m_y)$. Note that the modules $D^p(\mathcal{A}, m)$ over the Noetherian ring S are submodules of the finitely generated modules $\mathrm{Der}^p(S)$. Therefore, they are finitely generated as well.

Definition 3.5 [ST87, Definition 6.4] Let $P \subset S$ be a prime ideal and $X \in L(\mathcal{A})$. Set $X(P) := \cap H$ where the intersection is over all $H \in \mathcal{A}$ such that $X \subseteq H$ and $\alpha_H \in P$. A contravariant functor $F : L(\mathcal{A}) \to (S - \mathrm{Mod})$ is called *local* if the localization of $F(X) \to F(X(P))$ at P is an isomorphism for every $X \in L(\mathcal{A})$ and every prime ideal P.

Proposition 3.6 [ATW07, Proposition 1.6] For every $0 \le p \le \ell$ the functors D^p are local.

Proof: Let P be a prime ideal of S, $X \in L(\mathcal{A})$ and let $0 \le p \le \ell$. Since $X(P) \le X$, one has the inclusion $D^p(\mathcal{A}_{X(P)}, m_{X(P)}) \supseteq D^p(\mathcal{A}_X, m_X)$ due to Lemma 3.4. The localization of this inclusion is still injective. Conversely, let

$$\frac{\theta}{f} \in D^p(\mathcal{A}_{X(P)}, m_{X(P)})_P$$

where $\theta \in D^p(\mathcal{A}_{X(P)}, m_{X(P)})$ and $f \in S \setminus P$. Define the polynomial

$$g = \prod_{H \in \mathcal{A}_X \setminus \mathcal{A}_{X(P)}} \alpha_H^{m(H)}.$$

Then, by definition of $X(P)$ one has $g \in S \setminus P$ and $g\theta \in D^p(\mathcal{A}_X, m_X)$ which implies

$$\frac{\theta}{f} = \frac{g\theta}{gf} \in D^p(\mathcal{A}_X, m_X)_P.$$

Hence, $D^p(\mathcal{A}_X, m_X)_P \cong D^p(\mathcal{A}_{X(P)}, m_{X(P)})_P$. □

Corollary 3.7 If $(\mathcal{A}, m)$ is a free multiarrangement, $(\mathcal{A}_X, m_X)$ is free for any $X \in L(\mathcal{A})$.

Proof: Let $X \in L(\mathcal{A})$ and set $O := \cap_{H \in \mathcal{A}} H$. Note that $O \in L(\mathcal{A})$ since all arrangements are assumed to be central. Hence, $(\mathcal{A}_O, m_O) = (\mathcal{A}, m)$. Let P_X be the prime ideal corresponding to X, i.e. X is the vanishing locus of P_X. Therefore, $X = X(P_X)$. By the assumption that $(\mathcal{A}, m)$ is free, $D(\mathcal{A}, m)$ is a free module. Therefore, the S_{P_X}-module $D(\mathcal{A}, m)_{P_X}$ is free as well, since the localization of a free module is free in general. Thus, Proposition 3.6 implies

$$D(\mathcal{A}, m)_{P_X} \cong D(\mathcal{A}_X, m_X)_{P_X}$$

which means that the module $D(\mathcal{A}_X, m_X)_{P_X}$ is free. By choosing suitable coordinates $x_1, ..., x_\ell$ of S one may assume that $P_X = \langle x_1, ..., x_{r(X)} \rangle$ and $(\mathcal{A}_X, m_X)$ has a decomposition

$$(\mathcal{A}_X, m_X) = (\mathcal{B} \times \Phi_{\dim X}, m_{\mathcal{B}} \times m_\emptyset)$$

where $(\mathcal{B}, m_{\mathcal{B}})$ is a $r(X)$-multiarrangement. Then, Lemma 2.26 implies

$$D(\mathcal{A}_X, m_X) \cong S \otimes_{\mathbb{K}} D(\mathcal{B}, m_{\mathcal{B}}) \oplus S \otimes_{\mathbb{K}} D(\Phi_{\dim X}, m_\emptyset), \tag{3.2}$$

$$D(\mathcal{A}_X, m_X)_{P_X} \cong (S \otimes_{\mathbb{K}} D(\mathcal{B}, m_{\mathcal{B}}))_{P_X} \oplus (S \otimes_{\mathbb{K}} D(\Phi_{\dim X}, m_\emptyset))_{P_X}. \tag{3.3}$$

Since $D(\Phi_{\dim X}, m_\emptyset)$ is trivially free,

$$(S \otimes_{\mathbb{K}} D(\Phi_{\dim X}, m_\emptyset))_{P_X} = \mathbb{K}(x_{r(X)+1}, ..., x_\ell) \otimes_{\mathbb{K}} D(\Phi_{\dim X}, m_\emptyset)_{P_X}$$

is free as well. As $D(\mathcal{A}_X, m_X)_{P_X}$ is free, the isomorphism (3.3) implies that $(S \otimes_{\mathbb{K}} D(\mathcal{B}, m_{\mathcal{B}}))_{P_X}$ is free. We can compute

$$\begin{aligned} &(S \otimes_{\mathbb{K}} D(\mathcal{B}, m_{\mathcal{B}}))_{P_X} \\ \cong &(\mathbb{K}[x_{r(X)+1}, ..., x_\ell] \otimes_{\mathbb{K}} D(\mathcal{B}, m_{\mathcal{B}}))_{P_X} \\ \cong &\mathbb{K}(x_{r(X)+1}, ..., x_\ell) \otimes_{\mathbb{K}} (D(\mathcal{B}, m_{\mathcal{B}}))_{P_X} \end{aligned}$$

which is a free $\mathbb{K}(y) \otimes_{\mathbb{K}} \mathbb{K}[x_1, ..., x_{r(X)}]_{P_X}$-modules. Thus, $D(\mathcal{B}, m_{\mathcal{B}})_{P_X}$ is a free $\mathbb{K}[x_1, ..., x_{r(X)}]_{P_X}$-module and hence $D(\mathcal{B}, m_{\mathcal{B}})$ is a free $\mathbb{K}[x_1, ..., x_{r(X)}]$-module. Therefore, $S \otimes_{\mathbb{K}} D(\mathcal{B}, m_{\mathcal{B}})$ is finally a free S-module. This implies that $D(\mathcal{A}_X, m_X)$ is free by the isomorphism (3.2). □

The following theorem is a crucial ingredient in the proof of Theorem 2.5 in [ATW07].

Theorem 3.8 *[ATW07, Theorem 1.7] A local functor $F : L(\mathcal{A}) \to (S - \mathrm{Mod})$ is cumulative which means that for any $X \in L(\mathcal{A})$*

$$\sum_{Y \leq X} \mu(Y, X) H(F(Y), q)$$

has a pole of order at most $\dim X$ at $q = 1$.

Definition 3.9 [ATW07, Definition 3.1] For $p \in \{0, ..., \ell\}$ the functions $C_p : L(\mathcal{A}) \to \mathbb{Z}$ are defined by setting $C_p(X)$ equal to the coefficient of t^p in the polynomial $\pi((\mathcal{A}_X, m_X), t)$ or equivalently to the coefficient of $t^{\ell-p}$ in the polynomial $(-1)^\ell \chi((\mathcal{A}_X, m_X), -t)$.

It follows from Definition 3.2 that one can write

$$\psi(\mathcal{A}_X, m_X; -t, 1) = \sum_{p=0}^{\ell} C_p(X) t^{\ell-p} \tag{3.4}$$

for all $X \in L(\mathcal{A})$. Suppose $X \in L(\mathcal{A})$. As previously described, one is able to choose coordinates in such a way, that $(\mathcal{A}_X, m_X)$ has a decomposition $(\mathcal{A}_X, m_X) = (\mathcal{B} \times \Phi_{\dim X}, m_{\mathcal{B}} \times m_\emptyset)$ where $(\mathcal{B}, m_{\mathcal{B}})$ is a $r(\mathcal{A}_X)$-multiarrangement and $(\Phi_k, m_\emptyset)$ is the empty multiarrangement in dimension k. Note that $r(\mathcal{A}_X) = r(X)$ holds by the definitions of the rank and the localized arrangement. Theoem 3.12 from the following section implies that $\chi((\Phi_k, m_\emptyset), t) = t^k$, since $(\Phi_k, m_\emptyset)$ is trivially free with $\exp(\Phi_k, m_\emptyset) = (0, ..., 0)$. It follows from Lemma 3.3 that $\chi((\mathcal{A}_X, m_X), t)$ is divisible by $t^{\dim X}$. Therefore, by the definition of $C_p(X)$

$$C_p(X) = 0 \tag{3.5}$$

for all p such that $\ell - p < \dim X$ or equivalently $r(X) < p$.

Now we are able to give the theorem, which relates the local data to the global data of derivations on a multiarrangement.

Theorem 3.10 *[ATW07, Theorem 3.3] Let $(\mathcal{A}, m)$ be a multiarrangement. Then it holds for an arbitrary $X \in L(\mathcal{A})$ and p such that $0 \leq p \leq r(X)$ that*

$$C_p(X) = \sum_{Y \in L(\mathcal{A}_X)_p} C_p(Y).$$

Proof: Let

$$\psi_X(\mathcal{A}, m; t, q) := \sum_{p=0}^{\ell} \sum_{Y \leq X} \mu(Y, X) H(D^p(\mathcal{A}_Y, m_Y), q)(t(1-q) - 1)^p. \tag{3.6}$$

By interchanging the sums and the definition of $\psi(\mathcal{A}, m; t, q)$ we get

$$\psi_X(\mathcal{A}, m; t, q) = \sum_{Y \leq X} \mu(Y, X) \psi(\mathcal{A}_Y, m_Y; -t, q). \tag{3.7}$$

Thus, setting $q = 1$ and using Equation (3.4) yields

$$\psi_X(\mathcal{A}, m; t, 1) = \sum_{Y \leq X} \mu(Y, X) \sum_{p=0}^{\ell} C_p(Y) t^{\ell-p}. \tag{3.8}$$

Now consider the series

$$M_p(q) := \sum_{Y \le X} \mu(Y,X) H(D^p(\mathcal{A}_Y, m_Y), q).$$

Theorem 3.8 and Proposition 3.6 show that $(1-q)^{\dim X} M_p(q)$ does not have a pole at $q=1$. Therefore, $(1-q)^n M_p(q)$ is divisible by $(1-q)$ for $n > \dim X$. Hence, using the equation

$$M_p(q)(t(1-q)-1)^p = \sum_{k=0}^{p} \binom{p}{k} (-1)^{p-k} (1-q)^k M_p(q) t^k$$

the coefficient of t^n in $M_p(q)(t(1-q)-1)^p$ is divisible by $(1-q)$ for $n > \dim X$. If one now sets $q=1$ and uses Equation (3.6), one obtains that the coefficient of t^n in $\psi_X(\mathcal{A}, m; t, 1)$ is zero for $n > \dim X$.

On the other hand, for $\ell - p < \dim X$ one also has $\ell - p < \dim Y$ for all $Y \le X$, since $L(\mathcal{A})$ is ordered by reverse inclusion. Therefore, Equation (3.5) implies that $C_p(Y) = 0$ in Equation (3.8) when $\ell - p < \dim X$. Thus, $\psi_X(\mathcal{A}, m; t, 1)$ is divisible by $t^{\dim X}$. In total, one obtains that $\psi_X(\mathcal{A}, m; t, 1)$ is a monomial of degree $\dim X$. Since $C_{r(X)}(Y) = 0$ for $Y < X$ by Equation (3.5), one obtains from Equation (3.8) by comparing the coefficients

$$\psi_X(\mathcal{A}, m; t, 1) = C_{r(X)}(X) t^{\dim X}.$$

Combining this equation with Equation (3.7) one obtains

$$\sum_{Y \le X} \mu(Y,X) \psi(\mathcal{A}_Y, m_Y; -t, 1) = C_{r(X)}(X) t^{\dim X}.$$

The Möbius inversion formula from Proposition 2.11 converts this equation into

$$\psi(\mathcal{A}_X, m_X; -t, 1) = \sum_{Y \le X} C_{r(Y)}(Y) t^{\dim Y}.$$

Therefore, Equation (3.4) yields

$$\sum_{p=0}^{\ell} C_p(X) t^{\ell - p} = \sum_{Y \le X} C_{r(Y)}(Y) t^{\dim Y}.$$

Thus, a comparison of coefficients of these polynomials completes the proof:

$$C_p(X) = \sum_{\substack{Y \le X \\ \dim Y = \ell - p}} C_{r(Y)}(Y) = \sum_{Y \in L(\mathcal{A}_X)_p} C_p(Y).$$

□

3.3 $\chi((\mathcal{A}, m), t)$ for free multiarrangements

In this section we examine $\chi((\mathcal{A}, m), t)$ for free multiarrangements. Solomon and Terao showed in [ST87] the celebrated "Factorization Theorem" for simple arrangements. First, we will generalize this theorem to multiarrangements. Then we will derive a useful non-freeness criterion using this generalized theorem and Theorem 3.10, .

Lemma 3.11 [ATW07, Lemma 1.3] If $(\mathcal{A}, m)$ is a free multiarrangmenent, then $D^p(\mathcal{A}, m) \cong \wedge^p D^1(\mathcal{A}, m)$ for $1 \leq p \leq \ell$.

Theorem 3.12 *[ATW07, Theorem 4.1] If $D^1(\mathcal{A}, m)$ is free with exponents $\exp(\mathcal{A}, m) = (d_1, ..., d_\ell)$ then*

$$\chi((\mathcal{A}, m), t) = \prod_{i=1}^{\ell} (t - d_i)$$

and

$$\pi((\mathcal{A}, m), t) = \prod_{i=1}^{\ell} (1 + d_i t)$$

Proof: By the geometric series formula it holds that

$$\frac{1}{1 - qx_i} = 1 + qx_i + q^2 x_i^2 + ...$$

for all $i = 1, ..., \ell$. Multiplying theses series yields

$$\begin{aligned} \prod_{i=1}^{\ell} \frac{1}{1 - qx_i} &= (1 + qx_1 + q^2 x_1^2 + ...)(1 + qx_2 + q^2 x_2^2 + ...) \cdots (1 + qx_\ell + q^2 x_\ell^2 + ...) \\ &= \sum_{k \geq 0} \left(\sum_{i_1 + ... + i_\ell = k} x_1^{i_1} \cdots x_\ell^{i_\ell} \right) q^k. \end{aligned}$$

By substituting $x_i = 1$ for all $i = 1, ..., \ell$ it follows

$$\frac{1}{(1 - q)^\ell} = \sum_{k \geq 0} (\dim S_k) q^k = H(S, q)$$

where S_k denotes the k-th graded piece of S. Therefore, one obtains from Equation (2.3)

$$H(D^1(\mathcal{A}, m), q) = \frac{q^{d_1} + \cdots + q^{d_\ell}}{(1 - q)^\ell}. \tag{3.9}$$

If $\{\theta_1, ..., \theta_\ell\}$ with $\operatorname{pdeg} \theta_i = d_i$ forms a basis of $D^1(\mathcal{A}, m)$, Lemma 3.11 shows that $\{\theta_{i_1} \wedge ... \wedge \theta_{i_p} \mid 1 \leq i_1 < ... < i_p \leq \ell\}$ is a basis of $D^p(\mathcal{A}, m)$ for $p \geq 1$. Since these basis elements have degree $d_{i_1} + \cdots + d_{i_p}$, Equation (3.9) generalizes to

$$H(D^p(\mathcal{A}, m), q) = \sum \frac{q^{d_{i_1} + \cdots + d_{i_p}}}{(1 - q)^\ell} \tag{3.10}$$

where the sum is over all p-tuples such that $1 \leq i_1 < ... < i_p \leq \ell$. Thus, by the definition of

$\psi(\mathcal{A}, m; t, q)$ we have

$$\begin{aligned}
\psi(\mathcal{A}, m; t, q) &= \sum_{p=0}^{\ell} H(D^p(\mathcal{A}, m), q)(t(q-1)-1)^p \\
&= \frac{1}{(1-q)^\ell} \sum_{p=0}^{\ell} \sum_{1 \le i_1 < \ldots < i_p \le \ell} (q^{d_{i_1}+\cdots+d_{i_p}})(t(q-1)-1)^p \\
&= \frac{1}{(1-q)^\ell} \sum_{p=0}^{\ell} \sum_{1 \le i_1 < \ldots < i_p \le \ell} \prod_{j=1}^{p} q^{d_{i_j}} (t(q-1)-1) \\
&= \frac{\prod_{i=1}^{\ell} (1 + q^{d_i}(t(q-1)-1))}{(1-q)^\ell} \\
&= \prod_{i=1}^{\ell} \left(\frac{1-q^{d_i}}{1-q} + \frac{q^{d_i} t(q-1)}{1-q} \right) \\
&= \prod_{i=1}^{\ell} \left(1 + q + q^2 + \cdots + q^{d_i - 1} - q^{d_i} t\right)
\end{aligned}$$

where the last equality follows from the geometric sum formula. Now, substituting $q = 1$ yields

$$\psi(\mathcal{A}, m; t, 1) = \prod_{i=1}^{\ell} (d_i - t). \tag{3.11}$$

Since $\chi((\mathcal{A}, m), t) = (-1)^\ell \psi(\mathcal{A}, m; t, 1)$ the formula for the characteristic polynomial follows immediately. For the Poincaré polynomial one finally obtains from Equation (3.11)

$$\begin{aligned}
\pi((\mathcal{A}, m), t) &= (-t)^\ell \chi((\mathcal{A}, m), -t^-1) \\
&= (-t)^\ell \prod_{i=1}^{\ell} (\frac{1}{-t} - d_i) \\
&= \prod_{i=1}^{\ell} (1 + d_i t)
\end{aligned}$$

□

Suppose that $(\mathcal{A}_X, m_X)$ is free with exponents

$$\exp(\mathcal{A}_X, m_X) = (d_1^X, ..., d_\ell^X).$$

Since one can decompose $(\mathcal{A}_X, m_X)$ as discussed previously, it then holds that $\exp(\mathcal{A}_X, m_X) = \exp(\mathcal{B}, m_\mathcal{B}) \cup \exp(\Phi_{\dim X}, m_\Phi)$. Therefore, without loss of generality one may assume that $d_i^X = 0$ for $i > r(X)$. Hence, one can define the following:

Definition 3.13 Let $0 \le p \le \ell$. $(\mathcal{A}, m)$ is called *p-locally free* if for all $0 \le k \le p$ and for any $X \in L(\mathcal{A})_k$ the multiarrangement $(\mathcal{A}_X, m_X)$ is free.

Definition 3.14 Suppose $(\mathcal{A}, m)$ is p-locally free and that $0 \le k \le p$. The k-th *local mixed product* is

$$LMP_k(\mathcal{A}, m) = \sum_{X \in L(\mathcal{A})_k} d_1^X d_2^X \cdots d_k^X.$$

Since Theorem 2.18 implies that every multiarrangment is 2-locally free, $LMP_2(\mathcal{A}, m)$ is always well-defined.

Corollary 3.15 [ATW07, Corollary 4.4] If $(\mathcal{A}, m)$ is p-locally free then for all $0 \leq k \leq p$ the coefficient of t^k in $\pi((\mathcal{A}, m), t)$ is $LMP_k(\mathcal{A}, m)$.

Proof: Let O be the intersection of all hyperplanes in $\mathcal{A}$. Then by Definition 3.9, the coefficient of t^k in $\pi((\mathcal{A}, m), t)$ is $C_k(O)$. Theorem 3.10 then shows that

$$C_k(O) = \sum_{Y \in L(\mathcal{A})_k} C_{r(Y)}(Y). \tag{3.12}$$

On the other hand, let $Y \in L(\mathcal{A})_k$. By assumption, $(\mathcal{A}_Y, m_Y)$ is free with exponents $\exp(\mathcal{A}_Y, m_Y) = (d_1^Y, ..., d_k^X, 0, ..., 0)$ (As above one may assume $d_i^Y = 0$ for $i > r(Y) = k$). Thus, Theorem 3.12 implies

$$\pi((\mathcal{A}_Y, m_Y), t) = \prod_{i=1}^{k} (1 + d_i t).$$

Therefore, again by Definition 3.9 one has

$$C_{r(Y)}(Y) = d_1^Y \cdots d_{r(Y)}^Y.$$

Thus, the claim follows directly from Equation (3.12) and Definition 3.14. □

Definition 3.16 Let $0 \leq k \leq \ell$ and suppose that $(\mathcal{A}, m)$ is a free multiarrangement with $\exp(\mathcal{A}, m) = (d_1, ..., d_\ell)$. The k-th *global mixed product* is

$$GMP_k(\mathcal{A}, m) = \sum d_{i_1} d_{i_2} \cdots d_{i_k}$$

where the sum is over all k-tuples such that $1 \leq i_1 < ... < i_k \leq \ell$.

Corollary 3.17 [ATW07, Corollary 4.6] If $(\mathcal{A}, m)$ is a free multiarrangement with $\exp(\mathcal{A}, m) = (d_1, ..., d_\ell)$ then for all $0 \leq k \leq \ell$

$$GMP_k(\mathcal{A}, m) = LMP_k(\mathcal{A}, m).$$

Proof: By Corollary 3.7 a free multiarrangement is p-locally free for any $0 \leq p \leq \ell$. Thus, Corollary 3.15 shows that the coefficient of t^k in $\pi((\mathcal{A}, m), t)$ is $LMP_k(\mathcal{A}, m)$. On the other hand, Theorem 3.12 implies that

$$\pi((\mathcal{A}, m), t) = \prod_{i=1}^{\ell} (1 + d_i t).$$

By expanding this product and using Definition 3.16, one sees that the coefficient of t^k in $\pi((\mathcal{A}, m), t)$ is $GMP_k(\mathcal{A}, m)$. Combining these two statements completes the proof. □

In the case $k = 2$, Corollary 3.17 is a useful criterion for the determination of non-freeness. It was used effectively by Abe Terao and Yoshinaga in [ATY09] and by Abe in [Abe07]. We apply this criterion in the following chapter to prove Theorem 1.2.

4 Totally non-freeness of generic arrangements

Definition 4.1 For a given simple arrangement $\mathcal{A}$ the multiplicity $m : \mathcal{A} \to \mathbb{Z}_{>0}$ is called a *free multiplicity* for $\mathcal{A}$ if $(\mathcal{A}, m)$ is a free multiarrangement, otherwise it is called a *non-free multiplicity*. If any possible multiplicity function $m : \mathcal{A} \to \mathbb{Z}_{>0}$ is a non-free multiplicity for $\mathcal{A}$, the arrangement $\mathcal{A}$ is called *totally non-free*.

In general, it turns out to be difficult to determine which multiplicities are free on $\mathcal{A}$ and which are not. However, Theorem 1.2 shows that any generic arrangement $\mathcal{A}$ in an ℓ-dimensional vector space with $3 \leq \ell$ and $\ell < |\mathcal{A}|$ is totally non-free. This result was already proven by Ziegler [Zie89a, Corollary 7.6] and by Yoshinaga in [Yos10, Proposition 4.1] for complex arrangements. We will give a different proof below, applying the non-freeness criterion of Corollary 3.17.

Firstly, we state the known arithmetic-quadratic mean inequality:

Proposition 4.2 Suppose $a_1, ..., a_n \in \mathbb{R}_0$, then

$$\frac{a_1 + ... + a_n}{n} \leq \sqrt{\frac{a_1^2 + ... + a_n^2}{n}}.$$

Proof: By the Cauchy-Schwarz inequality, it holds that

$$(1a_1 + ... + 1a_n)^2 \leq (1^2 + ... + 1^2)(a_1^2 + ... + a_n^2).$$

Dividing by n^2 and taking the square root of this inequality yields the desired result. □

Lemma 4.3 Suppose $b_1, \ldots, b_n \in \mathbb{Z}_{>0}$ with $\sum_{i=1}^{n} b_i = m$ and $\ell < n$ such that $b_i \leq \frac{m}{\ell}$ for all $i = 1, \ldots, n$. Then

$$\sum_{i=1}^{n} b_i^2 < \frac{m^2}{\ell}.$$

Proof: From $b_i \leq \frac{m}{\ell}$, we immediately deduce for $i = 1, .., n$

$$(\ell - 1)b_i \leq m - b_i = \sum_{\substack{j=1 \\ j \neq i}}^{n} b_j.$$

Thus, it follows that

$$\begin{aligned} m^2 &= \sum_{i=1}^{n} b_i^2 + \sum_{i=1}^{n} b_i (\sum_{\substack{j=1 \\ j\neq i}}^{n} b_j) \\ &= \sum_{i=1}^{n} b_i^2 + \sum_{i=1}^{n} b_i (m - b_i) \\ &\geq \sum_{i=1}^{n} b_i^2 + \sum_{i=1}^{n} b_i ((\ell-1) b_i) \\ &= \ell \sum_{i=1}^{n} b_i^2 \end{aligned}$$

Let us assume that equality holds in this inequality. This is only possible if $(\ell - 1)b_i = m - b_i$ for all $i = 1, .., n$, since $b_i > 0$ for all $i = 1, .., n$. Therefore, $b_i = \frac{m}{\ell}$ for all $i = 1, .., n$ and hence,

$$m = \sum_{i=1}^{n} b_i = \sum_{i=1}^{n} \frac{m}{\ell} = \frac{mn}{\ell}$$

holds. This equality is equivalent to $n = \ell$ which is a contradiction, as we assumed $\ell < n$. □

Lemma 4.4 Let N, M be free graded modules over a graded commutative ring R with $N \subset M$. Assume $N = \langle a_1, ..., a_n\rangle$ and $M = \langle e_1, ..., e_m\rangle$ are homogeneous bases, then $1 \leq n \leq m$. Suppose the bases are ordered with respect to their degrees, i.e. $\deg(a_1) \leq ... \leq \deg(a_n)$ and $\deg(e_1) \leq ... \leq \deg(e_m)$. Then it holds that $\deg(a_k) \geq \deg(e_k)$ for all $k = 1, .., n$.

Proof: The proof is obtained by induction on $m = \operatorname{rk} M$. For $n = 1$ we have $\langle a_1\rangle \subset \langle e_1\rangle$ and so $\deg(a_1) \geq \deg(e_1)$.

Now assume $n > 1$ and set $M_i := \langle e_1, ..., e_i\rangle$ for $i = 1, ..., m$. Let j be minimal with the property that $N \subset M_j$. Such a j exists, as $N \subset M = M_m$. Then $n \leq j$ holds since both modules N and M_j are free of rank n and j, respectively. Let π be the canonical projection

$$\pi : M_j \to M_j/M_{j-1}.$$

The image of N, $\pi(N) =: \overline{N}$ is non-zero in M_j/M_{j-1} since j is minimal such that $N \subset M_j$, i.e. $N \not\subseteq M_{j-1}$. Therefore, we can decompose $N = N_1 \oplus N_2$ in such a way that N_1 and N_2 are respectively generated by all basis elements a_i of N with $\pi(a_i) = 0$ and $\pi(a_i) \neq 0$ respectively. Hence, $\pi(N_1) = 0$ and $\operatorname{rk} \pi(N_2) = \operatorname{rk} \pi(N)$. Thus $\deg(a_i) \geq \deg(e_j) \geq \deg(e_n) \geq \deg(e_i)$ for all basis elements $a_i \in N_2$, because $0 \neq \pi(a_i) \in M_j/M_{j-1} \cong \langle e_j\rangle$. Now we have $N_1 \subset M_{j-1}$ and the induction hypothesis can be applied since these modules are both free of smaller rank. □

Corollary 4.5 Let $(\mathcal{A}, m)$ be a free multiarrangement with $\exp(\mathcal{A}, m) = (d_1, \ldots, d_\ell)_\leq$. Suppose $(\mathcal{B}, m')$ is a free subarrangement of $(\mathcal{A}, m)$ (i.e. $m'(H) \leq m(H)$ for all $H \in \mathcal{B}$), with $\exp(\mathcal{B}, m') = (b_1, \ldots, b_\ell)_\leq$. Then for $i = 1, \ldots, \ell$ it holds that $d_i \geq b_i$.

Proof: Since $(\mathcal{B}, m')$ is a subarrangement of $(\mathcal{A}, m)$ it follows from Lemma 3.4 that

$$D(\mathcal{B}, m') \supseteq D(\mathcal{A}, m).$$

These modules are free and graded. Thus, we can apply Lemma 4.4 to obtain $d_i \geq b_i$ for $i = 1, \ldots, \ell$. □

Definition 4.6 Let $(\mathcal{A}, m)$ be a multiarrangement in an ℓ-dimensional vector space V with $m(H_i) = b_i$. Suppose the b_i are ordered such that $b_1 \leq \ldots \leq b_n$ In this case, we call the multiplicity b_i of H_i *unbalanced* for $n - \ell < i$, if

$$b_i > \frac{|m| - \sum_{k=i+1}^{n} b_k}{\ell - (n-i)} = \frac{\sum_{k=1}^{i} b_k}{\ell + i - n} \tag{4.1}$$

holds. Otherwise it will be called *balanced*.

Remark 4.7 If we again assume $b_1 \leq \ldots \leq b_n$, the following holds:

a) If b_i is an unbalanced multiplicity, then inductively the multiplicities b_j are unbalanced for $j = i, \ldots, n$ as well. Or equivalently if b_i is balanced then inductively the multiplicities b_j are balanced for $j = 1, ..., i$.

b) By definition there are at most $\ell - 1$ unbalanced multiplicities in a multiarrangement.

Definition 4.8 Let $(\mathcal{A}, m)$ be a multiarrangement with the notations as in Definition 4.6. In this case, we call b_k the *first unbalanced* hyperplane, if b_k is a unbalanced and b_{k-1} a balanced multiplicity.

Now, we are able to prove the main result of this chapter.

Proof (of Theorem 1.2): Suppose $(\mathcal{A}, m)$ is a free multiarrangement with $\exp(\mathcal{A}, m) = (d_1, \ldots, d_\ell)_\leq$. Furthermore, let $m = (b_1, \ldots, b_n)_\leq$ be its multiplicity-vector such that b_k is the last balanced multiplicity.

It follows from Corollary 2.23 that

$$\sum_{i=1}^{\ell} d_i = |m| = \sum_{i=1}^{n} b_i. \tag{4.2}$$

We compute the second *local mixed product* $LMP_2(\mathcal{A}, m)$ of $(\mathcal{A}, m)$ and the second *global mixed product* $GMP_2(\mathcal{A}, m)$ of $(\mathcal{A}, m)$. We show that $GMP_2(\mathcal{A}, m) < LMP_2(\mathcal{A}, m)$, which implies that $(\mathcal{A}, m)$ is non-free by Corollary 3.17.

Let $\tilde{m} := \sum_{i=1}^{k} b_i$ and $\tilde{\ell} := \ell - (n-k)$. The multiplicity b_k is balanced; therefore, $b_k \leq \frac{\tilde{m}}{\tilde{\ell}}$ by Definition 4.8. Since the multiplicities b_i are ordered, it follows $b_i \leq \frac{\tilde{m}}{\tilde{\ell}}$ for all $i = 1, ..., k$. We also have $\tilde{\ell} := \ell - (n-k) = k + \ell - n < k$, because we assumed $\ell < n$. Therefore, we can apply Lemma 4.3 to obtain

$$\sum_{i=1}^{k} b_i^2 < \frac{\tilde{m}^2}{\tilde{\ell}}.$$

Thus, we can conclude

$$\sum_{i=1}^{n} b_i^2 < \frac{\tilde{m}^2}{\tilde{\ell}} + \sum_{i=k+1}^{n} b_i^2. \tag{4.3}$$

Let $(\mathcal{B}, m_\mathcal{B})$ be the subarrangement of $(\mathcal{A}, m)$ consisting of its last ℓ hyperplanes with multiplicities $m_\mathcal{B}(H) := m(H)$ for all $H \in \mathcal{B}$, i.e. $m_\mathcal{B} = (b_{n-\ell+1}, ..., b_n)_\leq$. $\mathcal{A}$ is a generic arrangement. The definition of a generic arrangement implies that $\mathcal{B}$ is a Boolean arrangement after a suitable change of coordinates. As previously discussed in Example 2.24, $(\mathcal{B}, m_{\mathcal{B}})$ is a free multiarrangement with $\exp(\mathcal{B}, m_{\mathcal{B}}) = (b_{n-\ell+1}, ..., b_n)_{\leq}$. Due to Remark 4.7, a multiarrangement can only contain at most $\ell - 1$ unbalanced multiplicities. By our assumption of the multiplicities being ordered, all the unbalanced multiplicities are contained in the last ℓ multiplicities. Thus, all unbalanced multiplicities of $(\mathcal{A}, m)$ are already contained in $(\mathcal{B}, m_{\mathcal{B}})$. Therefore, we are able to apply Corollary 4.5 to obtain lower bounds on the exponents of $(\mathcal{A}, m)$ in terms of the unbalanced multiplicities, namely $d_{i+\ell-n} \geq b_i$ for all $i = k+1, ..., n$. Thus, we can choose $s_i \in \mathbb{Z}_{\geq 0}$ such that $d_{i+\ell-n} = s_i + b_i$ holds for all $i = k+1, ..., n$.

Using Equation (4.2), we can compute the following:

$$\begin{aligned} |m| &= \sum_{i=1}^{n} b_i = \sum_{i=1}^{\ell} d_i = \sum_{i=1}^{\tilde{\ell}} d_i + \sum_{i=\tilde{\ell}+1}^{\ell} d_i \\ &= \sum_{i=1}^{\tilde{\ell}} d_i + \sum_{i=k+1}^{n} (b_i + s_i). \end{aligned}$$

This fact implies that

$$\sum_{i=1}^{\tilde{\ell}} d_i = \sum_{i=1}^{k} b_i - \sum_{i=k+1}^{n} s_i = \tilde{m} - \sum_{i=k+1}^{n} s_i. \tag{4.4}$$

From Proposition 4.2 it follows that

$$\sum_{i=1}^{\ell} d_i^2 = \sum_{i=1}^{\tilde{\ell}} d_i^2 + \sum_{i=\tilde{\ell}+1}^{\ell} d_i^2 \geq \frac{1}{\tilde{\ell}} \left(\sum_{i=1}^{\tilde{\ell}} d_i \right)^2 + \sum_{i=\tilde{\ell}+1}^{\ell} d_i^2.$$

We continue by further expanding this inequality and applying Equation (4.4).

$$\begin{aligned} \sum_{i=1}^{\ell} d_i^2 &\geq \frac{\left(\tilde{m} - \sum_{i=k+1}^{n} s_i\right)^2}{\tilde{\ell}} + \sum_{i=k+1}^{n} (b_i + s_i)^2 \\ &= \frac{(\tilde{m})^2}{\tilde{\ell}} - 2\frac{\tilde{m}}{\tilde{\ell}} \sum_{i=k+1}^{n} s_i + \frac{1}{\tilde{\ell}} \left(\sum_{i=k+1}^{n} s_i \right)^2 + \sum_{i=k+1}^{n} b_i^2 + 2 \sum_{i=k+1}^{n} s_i b_i + \sum_{i=k+1}^{n} s_i^2 \\ &= \frac{(\tilde{m})^2}{\tilde{\ell}} + \sum_{i=k+1}^{n} b_i^2 + 2 \sum_{i=k+1}^{n} s_i \left(b_i - \frac{\tilde{m}}{\tilde{\ell}} \right) + \frac{1}{\tilde{\ell}} \left(\sum_{i=k+1}^{n} s_i \right)^2 + \sum_{i=k+1}^{n} s_i^2 \end{aligned} \tag{4.5}$$

The fact that the multiplicity b_{k+1} is unbalanced implies that

$$b_{k+1} > \frac{\tilde{m} + b_{k+1}}{\tilde{\ell} + 1}.$$

By rearranging this inequality, we immediately obtain

$$b_{k+1} > \frac{\tilde{m}}{\tilde{\ell}}.$$

Therefore, the fact that the multiplicities b_i are ordered implies that

$$b_i > \frac{\tilde{m}}{\tilde{\ell}} \text{ for all } i = k+1, ..., n.$$

We can simplify Inequality (4.5), because the other summands appear as squares and hence are non-negative:

$$\sum_{i=1}^{\ell} d_i^2 \geq \frac{(\tilde{m})^2}{\tilde{\ell}} + \sum_{i=k+1}^{n} b_i^2.$$

Combining this inequality with Inequality (4.3), we finally obtain:

$$\sum_{i=1}^{\ell} d_i^2 > \sum_{i=1}^{n} b_i^2. \tag{4.6}$$

Due to the fact that $\mathcal{A}$ is generic, $L(\mathcal{A})_2$ consists of all possible intersections of two pairwise different hyperplanes of $\mathcal{A}$. Let $X \in L(\mathcal{A})_2$ with $X = H_{i_1} \cap H_{i_2}$ where $H_{i_1}, H_{i_2} \in \mathcal{A}$. Then, $\mathcal{A}_X = \{H_{i_1}, H_{i_2}\}$ and $m_X = (b_i, b_j)$ for some $1 \leq i < j \leq n$. Thus, $(\mathcal{A}_X, m_X)$ is free with $\exp(\mathcal{A}_X, m_X) = (b_i, b_j, 0_3, ..., 0_\ell)$, because $(\mathcal{A}_X, m_X)$ can be decomposed in a Boolean 2-arrangement and an empty $(\ell-2)$-arrangement. It follows from the definition of $LMP_2(\mathcal{A}, m)$ that

$$LMP_2(\mathcal{A}, m) = \sum_{1 \leq i < j \leq n} b_i b_j.$$

Using the definition of $GMP_2(\mathcal{A}, m)$, this equation and Equation (4.2), we obtain the following equations:

$$\begin{aligned} GMP(\mathcal{A}, m) &= \sum_{1 \leq i < j \leq \ell} d_i d_j, \\ |m|^2 &= \left(\sum_{i=1}^{\ell} d_i \right)^2 = \sum_{i=1}^{\ell} d_i^2 + 2GMP(\mathcal{A}, m), \\ |m|^2 &= \left(\sum_{i=1}^{n} b_i \right)^2 = \sum_{i=1}^{n} b_i^2 + 2LMP(\mathcal{A}, m). \end{aligned}$$

Therefore, it follows from the Inequality (4.6) that

$$GMP(\mathcal{A}, m) < LMP(\mathcal{A}, m). \tag{4.7}$$

In conclusion, $(\mathcal{A}, m)$ is non-free by Corollary 3.17. □

5 Multiarrangements with one large multiplicity

5.1 The Euler multiplicity and addition-deletion theorems for multiarrangements

The goal of this section is to establish the so-called addition-deletion theorems for multiarrangements, as done by Abe, Terao and Wakefield in their recent article [ATW08]. They enable us to construct new free multiarrangements from the deleted and restricted multiarrangement, under certain conditions on the exponents. In Section 2.2 we already defined the triple of arrangements $(\mathcal{A}, \mathcal{A}', \mathcal{A}'')$ for an arrangement $\mathcal{A}$ and a distinguished hyperplane. In the following, we extend this definition to multiarrangements. Let $(\mathcal{A}, m)$ be a multiarrangement and fix a hyperplane $H_0 \in \mathcal{A}$.

Definition 5.1 The deletion $(\mathcal{A}', m')$ of $(\mathcal{A}, m)$ with respect to H_0 is defined as follows:

(1) If $m(H_0) = 1$, then $\mathcal{A}' := \mathcal{A} \setminus \{H_0\}$ and $m'(H) = m(H)$ for all $H \in \mathcal{A}'$.

(2) If $m(H_0) \geq 2$, then $\mathcal{A}' := \mathcal{A}$ and for $H \in \mathcal{A}' = \mathcal{A}$, we set

$$m'(H) = \begin{cases} m(H) & \text{if } H \neq H_0, \\ m(H) - 1 & \text{if } H = H_0. \end{cases}$$

Recall the definition of the restricted arrangement

$$\mathcal{A}'' = \{H_0 \cap H \mid K \in \mathcal{A} \setminus \{H_0\}\},$$

which is an arrangement on H_0. There is more than one way to define a multiplicity on $\mathcal{A}''$. A purely combinatorial multiplicity would be the *Ziegler multiplicity* m'', defined by

$$m''(X) = \sum_{\substack{H \in \mathcal{A} \setminus \{H_0\} \\ H \cap H_0 = X}} m(H) = |m_X| - m(H_0),$$

where $X \in \mathcal{A}''$. This multiplicity was first defined by Ziegler [Zie89b] and we will use it in the following sections. For the addition-deletion theorems we introduce an algebraic multiplicity m^*, called the *Euler multiplicity*. Before we can give the definition of m^*, we need the following proposition:

Proposition 5.2 [ATW08, Proposition 2.1] Let $(\mathcal{A}, m)$ be a multiarrangement, $H_0 \in \mathcal{A}$ and $\mathcal{A}''$ the restriction of $\mathcal{A}$ with respect to H_0. For a $X \in \mathcal{A}''$ we may assume $H_0 = \ker x_1$ and $X = \{x_1 = x_2 = 0\}$ by choosing coordinates in a suitable way. Then, we may choose a basis

$$\theta_X, \psi_X, \partial_3, ..., \partial_\ell$$

for $D(\mathcal{A}_X, m_X)$ such that $\theta_X \notin x_1 \operatorname{Der}(S)$ and $\psi_X \in x_1 \operatorname{Der}(S)$ where ∂_i are the usual partial derivations with respect to x_i for $i = 3, ..., \ell$.

Definition 5.3 [ATW08, Definition 0.2] The *Euler multiplicity* $m^* : \mathcal{A}'' \to \mathbb{Z}_{>0}$ is defined by $m^*(X) := \deg \theta_X$ for $X \in \mathcal{A}''$ where θ_X is the derivation from Proposition 5.2. We call $(\mathcal{A}'', m^*)$ the *restriction* of $(\mathcal{A}, m)$.

Remark 5.4 If $(\mathcal{A}, m)$ is a simple arrangement, we may choose a basis $\theta_E, \theta_2, ..., \theta_\ell$ of $D(\mathcal{A})$ where θ_E is the *Euler derivation* $\theta_E := x_1\partial_1 + ... + x_\ell\partial_\ell$ [OT92, Proposition 4.27]. Since θ_E is not divisible by any linear polynomial α_0, θ_E can be chosen as θ_X, if $(\mathcal{A}, m)$ is simple. In this case, $m^*(X) = 1$ for all $X \in \mathcal{A}''$ and hence $(\mathcal{A}'', m^*)$ is simple. Thus, m^* generalizes Definiton 2.9.

Now we are able to formulate the addition-deletion theorems, which can be summarized as follows:

Theorem 5.5 (Theorem 0.8 [ATW08]) *Let $(\mathcal{A}, m)$ be a nonempty multiarrangment in an ℓ-dimensional vector space V, $H_0 \in \mathcal{A}$ and let $(\mathcal{A}, m), (\mathcal{A}', m'), (\mathcal{A}'', m^*)$ be the triple with respect to H_0. Then any of the following two statements imply the third:*

(1) $(\mathcal{A}, m)$ is free with $\exp(\mathcal{A}, m) = (d_1, ..., d_\ell)$.

(2) $(\mathcal{A}', m')$ is free with $\exp(\mathcal{A}', m') = (d_1, ..., d_\ell - 1)$.

(3) $(\mathcal{A}'', m^*)$ is free with $\exp(\mathcal{A}'', m^*) = (d_1, ..., d_{\ell-1})$.

5.2 Generalization of Ziegler's multirestriction

Theorem 1.1 shows that the restriction of a simple arrangement with a Ziegler multiplicity is free. The aim of this section is to generalize this result to multiarrangements with one "large" multiplicity. Let $(\mathcal{A}, m)$ be a multiarrangement, $H_0 \in \mathcal{A}$ and $\mathcal{A}''$ the restriction of $\mathcal{A}$ with respect to H_0. In general $(\mathcal{A}'', m'')$ does not need to be a free multiarrangement, if $(\mathcal{A}, m)$ is free, as Example 5.7 shows. Before we can treat this example, we need the following theorem by Wakamiko on the exponents of 2-multiarrangements:

Theorem 5.6 *[Wak07, Theorem 1.5] Let $(\mathcal{A}, m)$ be a 2-multiarrangement with $|\mathcal{A}| = 3$. Assume for $m_i = m(H_i)$, $m_1 \geq m_2 \geq m_3 \geq 0$. If $m_1 \leq m_2 + m_3$, then*

$$\exp(\mathcal{A}, m) = \begin{cases} (k, k) & \text{if } |m| = 2k, \\ (k, k+1) & \text{if } |m| = 2k+1. \end{cases}$$

Example 5.7 Edelman and Reiner presented a counterexample to a conjecture by Orlik which says that the usual restriction of a simple free arrangement $\mathcal{A}$ to any hyperplane H is free [ER93]. They considered an arrangement $\mathcal{ER}$ in $\mathbb{R}^5$ consisting of 21 hyperplanes with the defining equations $x_i = 0$ for i=1,2,3,4,5 and

$$x_1 + \varepsilon_2 x_2 + \varepsilon_3 x_3 + \varepsilon_4 x_4 + \varepsilon_5 x_5 = 0$$

where $\varepsilon_i = \pm 1$ for $i = 2, 3, 4, 5$. They computed that $\mathcal{ER}$ is a free arrangement. Let $(\mathcal{ER}'', m'')$ be the restricted arrangement of $\mathcal{ER}$ with a Ziegler multiplicity to the hyperplane $x_5 = 0$. Thus, $(\mathcal{ER}'', m'')$ consists of the hyperplanes $x_i = 0$ with multiplicity 1 for i=1,2,3,4 and

$$x_1 + \varepsilon_2 x_2 + \varepsilon_3 x_3 + \varepsilon_4 x_4 = 0$$

with multiplicity 2 where $\varepsilon_i = \pm 1$ for $i = 2, 3, 4$. Then Theorem 1.1 shows that $(\mathcal{ER}'', m'')$ is a free multiarrangement. Consider again the restriction of $(\mathcal{ER}'', m'')$ with a Ziegler multiplicity to the hyperplane $x_4 = 0$ and denote it by $(\mathcal{ERR}, m'')$. Therefore, $(\mathcal{ERR}, m'')$ consists of the hyperplanes $x_i = 0$ with multiplicity 1 for i=1,2,3 and

$$x_1 + \varepsilon_2 x_2 + \varepsilon_3 x_3 = 0$$

with multiplicity 4 where $\varepsilon_i = \pm 1$ for $i = 2, 3$. So, $|m''| = 4+4+4+4+1+1+1 = 19$.

Figure 5.1 shows the projectivized picture of $(\mathcal{ERR}, m'')$ with respect to x_3 where the outer circle is the hyperplane at infinitiy (i.e. the hyperplane $x_3 = 0$). The un-circled numbers are the multiplicities of the corresponding lines. The circled numbers are the product of the exponents at the corresponding rank two lattice element as explained below.

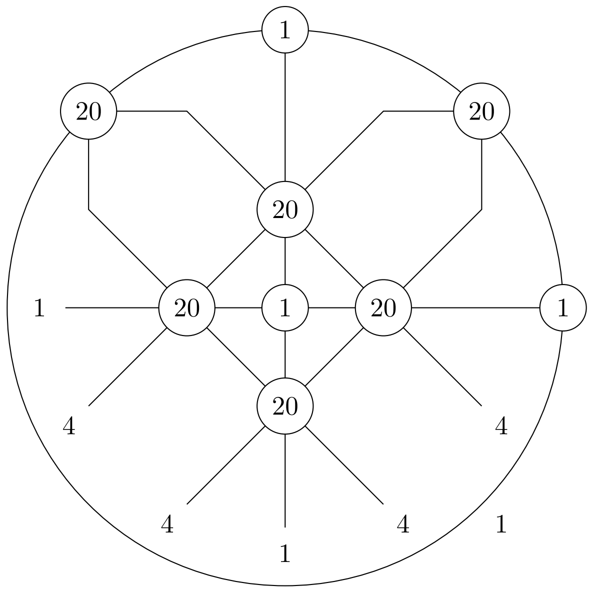

Figure 5.1: The projectivized picture of $(\mathcal{ERR}, m'')$

Now, we can show with the non-freeness criterion of Corollary 3.17 that $(\mathcal{ERR}, m'')$ is non-free. Assume $(\mathcal{ERR}, m'')$ is a free multiarrangement with $\exp(\mathcal{ERR}, m'') = (d_1, d_2, d_3)$. Then, Corollary 2.23 yields

$$19 = |m''| = d_1 + d_2 + d_3.$$

By definition of the global mixed product we have $GMP_2(\mathcal{ERR}, m'') = d_1d_2 + d_1d_3 + d_2d_3$. Suppose (b_1, b_2, b_3) is a set of integers with $b_1 + b_2 + b_3 = 19$ and

$$\sum_{1 \leq i < j \leq 3} (b_i - b_j)^2 \leq \sum_{1 \leq i < j \leq 3} (d_i - d_j)^2.$$

Then it is easy to see that $d_1d_2 + d_1d_3 + d_2d_3 \leq b_1b_2 + b_1b_3 + b_2b_3$. We say that (b_1, b_2, b_3) is "more-balanced" than (d_1, d_2, d_3). $(6, 6, 7)$ is "more-balanced" than any possible exponents (d_1, d_2, d_3) of $(\mathcal{ERR}, m'')$. Thus, in this case

$$GMP_2(\mathcal{ERR}, m'') \leq 6 \cdot 6 + 6 \cdot 7 + 6 \cdot 7 = 120 \tag{5.1}$$

As we see in Figure 5.1 there are three rank two elements in $L(\mathcal{ER})$ formed by the intersection of two hyperplanes with multiplicity 1. Thus, the localized arrangements are Boolean and have exponents $(1,1)$. The other six elements of $L(\mathcal{ER})_2$ are formed by intersections of three hyperplanes with multiplicities $1,4,4$. It follows from Theorem 5.6 that the localized arrangements have exponents $(4,5)$. Thus, summing the products of the exponents of all rank two lattice elements, we obtain with Equation (5.1)

$$LMP_2(\mathcal{ERR}, m'') = 6 \cdot 4 \cdot 5 + 3 \cdot 1 = 123 > 120 \geq GMP_2(\mathcal{ERR}, m'').$$

This is a contradiction to Corollary 3.17 which means that $(\mathcal{ERR}, m'')$ is a non-free multiarrangement. Thus, $(\mathcal{ER}'', m'')$ is an example for a free multiarrangement whose restriction with a Ziegler multiplicity to $x_4 = 0$ is non-free. In the following we will show that $(\mathcal{A}'', m'')$ is free, if $(\mathcal{A}, m)$ is free and $m_0 := m(H_0)$ is large, i.e $2m_0 > |m|$.

Definition 5.8 Fix a hyperplane $H_0 \in (\mathcal{A}, m)$ with $H_0 = \ker \alpha_{H_0}$. Then we define a submodule $D_0(\mathcal{A}, m)$ of $D(\mathcal{A}, m)$ by

$$D_0(\mathcal{A}, m) = \{\delta \in D(\mathcal{A}, m) \mid \delta\alpha_{H_0} = 0\}.$$

Lemma 5.9 [Zie89b, Theorem 11] Let $(\mathcal{A}, m)$ be a multiarrangement and fix a hyperplane $H_0 \in \mathcal{A}$ with $H_0 = \ker \alpha_{H_0}$. If $\delta \in D_0(\mathcal{A}, m)$ then $\delta \mid_{H_0} \in D(\mathcal{A}'', m'')$.

Proof: We may assume $\alpha_{H_0} = x_1$ by choosing suitable coordinates. Let $X \in \mathcal{A}''$ and set

$$\mathcal{A}_X = \{H \in \mathcal{A} \mid H \supset X\} = \{H_0, H_1, ..., H_k\}.$$

Since $H_0 \cap H_i = X$ for all $i = 1, ..., k$, the restriction $\alpha_i \mid_{x_1=0}$ determines the same hyperplane for $H_i = \ker \alpha_i$ and $i = 1, ..., k$. We may assume that the linear forms α_i have the form

$$\alpha_i(x_1, ..., x_\ell) = c_i x_1 + \alpha'(x_2, ..., x_\ell)$$

for $i = 1, ..., k$ and some $c_i \in \mathbb{K}$ which are pairwise different. Let $\delta \in D_0(\mathcal{A}, m)$. By definition of the derivation module it holds for $i = 1, ..., k$

$$\delta(c_i x_1 + \alpha'(x_2, ..., x_\ell)) \in (c_i x_1 + \alpha'(x_2, ..., x_\ell))^{m(H_i)} S.$$

Since $\delta(x_1) = 0$ due to the definition of $D_0(\mathcal{A}, m)$, $(c_i x_1 + \alpha'(x_2, ..., x_\ell))^{m(H_i)}$ is a multiple of $\delta(\alpha'(x_2, ..., x_\ell))$ for all $i = 1, ..., k$. This fact yields

$$\delta(\alpha'(x_2, ..., x_\ell)) \in \prod_{i=1}^{k} (c_i x_1 + \alpha'(x_2, ..., x_\ell))^{m(H_i)} S.$$

Now, the restriction $x_1 = 0$ gives us the claim:

$$\delta \mid_{x_1=0} (\alpha') \in \alpha'^{\sum_{i=1}^{k} m(H_i)} = \alpha'^{m''(X)}.$$

□

Now we are able to prove the following theorem which implies Theorem 1.6 from the introduction:

Theorem 5.10 *Suppose $(\mathcal{A}, m)$ is a free multiarrangement with $H_0 \in \mathcal{A}$ such that $2m_0 > |m|$ where $m_0 = m(H_0)$. Then it holds that*

(1) $D(\mathcal{A}, m) = \theta_0 \cdot S \oplus D_0(\mathcal{A}, m)$ with θ_0 being an element in the basis of $D(\mathcal{A}, m)$ and $\operatorname{pdeg} \theta_0 = m_0$. Thus $\exp(\mathcal{A}, m) = (m_0, d_2, ..., d_\ell)$.

(2) (A'', m'') *is a free multiarrangement with exponents* $\exp(A'', m'') = (d_2, ..., d_\ell)$.

Proof: We may choose coordinates in such a way that $H_0 = \ker x_1$. Let $\theta_1, ..., \theta_\ell$ be a homogeneous basis of $D(\mathcal{A}, m)$. By definition of $D(\mathcal{A}, m)$

$$\theta_i(x_1) \in x_1^{m_0} S \tag{5.2}$$

holds for $i = 1, ..., \ell$.

Assume $\operatorname{pdeg} \theta_i < m_0$ for all $i = 1, ..., \ell$. Thus (5.2) implies $\theta_i(x_1) = 0$ for all $i = 1, ..., \ell$. Therefore,

$$M(\theta_1, ..., \theta_\ell) = \det \begin{bmatrix} 0 & \cdots & 0 \\ \theta_1(x_2) & \cdots & \theta_\ell(x_2) \\ \vdots & & \vdots \\ \theta_1(x_\ell) & \cdots & \theta_\ell(x_\ell) \end{bmatrix} = 0. \tag{5.3}$$

With Theorem 2.22 we may assume

$$\det M(\theta_1, ..., \theta_\ell) = \mathcal{Q}(\mathcal{A}, m). \tag{5.4}$$

In particular $\det M(\theta_1, ..., \theta_\ell) \neq 0$, which is a contradiction to Equation (5.3), and hence we may assume $\operatorname{pdeg} \theta_1 \geq m_0$. Due to Corollary 2.23

$$|m| = \sum_{i=1}^{\ell} \operatorname{pdeg} \theta_i \tag{5.5}$$

holds. Moreover, we assumed $2m_0 > |m|$. The combination of these facts yields $\operatorname{pdeg} \theta_i < m_0$ for all $i = 2, ..., \ell$. Thus, $\theta_i(x_1) = 0$ holds again for all $i = 2, ..., \ell$ by (5.2). This shows already that $\theta_2, ..., \theta_\ell$ form a basis of $D_0(\mathcal{A}, m)$ as well as the desired decomposition of $D(\mathcal{A}, m)$. For (1) it remains to show that $\operatorname{pdeg} \theta_1 = m_0$ holds.

Again by (5.2) and $\operatorname{pdeg} \theta_1 \geq m_0$ we can write $\theta_1(x_1) = p x_1^{m_0}$ for some $p \in S$. Thus, the matrix $M(\theta_1, ..., \theta_\ell)$ has the shape

$$M(\theta_1, ..., \theta_\ell) = \begin{bmatrix} p x_1^{m_0} & 0 & \cdots & 0 \\ \theta_1(x_2) & \theta_2(x_2) & \cdots & \theta_\ell(x_2) \\ \vdots & \vdots & & \vdots \\ \theta_1(x_\ell) & \theta_2(x_\ell) & \cdots & \theta_\ell(x_\ell) \end{bmatrix}.$$

By expanding the determinant of this matrix along the first row and using (5.4) we obtain

$$\mathcal{Q}(\mathcal{A}, m) = \det M(\theta_1, ..., \theta_\ell) = p x_1^{m_0} \det \begin{bmatrix} \theta_2(x_2) & \cdots & \theta_\ell(x_2) \\ \vdots & & \vdots \\ \theta_2(x_\ell) & \cdots & \theta_\ell(x_\ell) \end{bmatrix}. \tag{5.6}$$

Let $\theta_i' := \theta_i \mid_{x_1=0}$ be the restricted derivations for $i = 2, ..., \ell$. Due to Lemma 5.9, $\theta_i' \in D(\mathcal{A}'', m'')$ for all $i = 2, ..., \ell$. Therefore, we can apply Proposition 2.21, to see that

$$\det M(\theta_2', ..., \theta_\ell') \in \mathcal{Q}(\mathcal{A}'', m'') \tag{5.7}$$

holds. Since m_0 is the multiplicity of the hyperplane $H_0 = \ker x_1$, $x_1^{m_0}$ is the highest power of x_1 in $\mathcal{Q}(\mathcal{A}, m)$. Equation (5.6) implies

$$\det M(\theta'_2, ..., \theta'_\ell) = \det \begin{bmatrix} \theta_2(x_2) & \cdots & \theta_\ell(x_2) \\ \vdots & & \vdots \\ \theta_2(x_\ell) & \cdots & \theta_\ell(x_\ell) \end{bmatrix} |_{x_1=0} \neq 0. \tag{5.8}$$

As we have in general $\deg \theta_i = \deg \theta'_i$ or $\theta'_i = 0$, since θ_i are homogeneous, (5.8) implies that $\deg \theta_i = \deg \theta'_i$ for all $i = 2, ..., \ell$. Equation (5.6) yields

$$\deg \det M(\theta'_2, ..., \theta'_\ell) = \deg \mathcal{Q}(\mathcal{A}, m) - \deg p x_1^{m_0} = |m| - m_0 - \deg p. \tag{5.9}$$

Equations (5.7) and (5.8) show that

$$\deg \det M(\theta'_2, ..., \theta'_\ell) \geq \deg \mathcal{Q}(\mathcal{A}'', m'') = |m| - m_0.$$

This inequality, together with (5.9), yields $\deg p = 0$. Thus, $\operatorname{pdeg} \theta_1 = m_0$ and (1) is proved. Using (5.5), we can obtain the following equation on the exponents

$$|m''| = |m| - m_0 = \sum_{i=2}^{\ell} \deg \theta_i = \sum_{i=2}^{\ell} \deg \theta'_i.$$

Furthermore the derivations $\theta'_2, ..., \theta'_\ell$ are linearly independent over S/x_1S due to (5.8). We can conclude from Corollary 2.23 that (A'', m'') is a free multiarrangement. From $\operatorname{pdeg} \theta_i = \operatorname{pdeg} \theta'_i$ for all $i = 2, ..., \ell$ the statement on the exponents of (2) follows directly. □

5.3 Asymptotic behaviour of freeness

In this section, we examine the asymptotic behaviour of freeness, if we increase the multiplicity of one hyperplane. At this time, recall the Definition 1.4 for such multiarrangements. For a multiarrangement $(\mathcal{A}, m)$ and a fixed hyperplane H_0, the multiplicity function $m_k : \mathcal{A} \to \mathbb{Z}_{>0}$ is defined by the same multiplicity on all hyperplanes but H_0 and increasing the multiplicity of H_0 by k.

As an application of the addition-deletion theorems Abe, Terao and Wakefield proved that for a simple free multiarrangement $(\mathcal{A}, m)$ (i.e. $m(H) = 1$ for each $H \in \mathcal{A}$) $(\mathcal{A}, m_k)$ is free for any $k \geq 0$ and any fixed hyperplane $H_0 \in \mathcal{A}$ [ATW08, Proposition 5.2]. In general, one can not expect to obtain the same result for free multiarrangements as Theorem 1.3 shows. However, we show in Theorem 1.5 that the property of $(\mathcal{A}, m_k)$ being free is constant for $k \gg 0$, that means there exists some $0 < K$ such that $(\mathcal{A}, m_k)$ is either free for all $K \leq k$ or non-free for all $K \leq k$. Moreover we prove that in order to decide whether $(\mathcal{A}, m_k)$ is in general free or non-free, it suffices to consider one multiarrangement $(\mathcal{A}, m_{k_0})$ where $k_0 > \frac{1}{2}|m| - m_0$ and $m_0 = m(H_0)$.

Lemma 5.11 [ATW08, Proposition 4.1 (2)] Let $(\mathcal{A}, m)$ be a multiarrangement, $H_0 \in \mathcal{A}$, $m_0 = m(H_0)$ and assume $2m_0 > |m|$. Then we have $m^*(X) = m''(X)$ for all $X \in \mathcal{A}''$ where m^* is the Euler multiplicity and m'' the Ziegler multiplicity defined as $m''(X) = |m_X| - m_0$ (as defined in Section 5.1).

Proof: Abe's, Terao's and Wakefield's proof [ATW07] is based on a result by Wakefield and Yuzvinsky in their publication [WY07]. Here we give an alternative proof which is based on our Theorem 5.10.

Let $X \in \mathcal{A}''$. Since $r(\mathcal{A}_X) = 2$, we can choose coordinates in such a way that $\alpha_{H_0} = x_1$ and $X = \{x_1 = x_2 = 0\}$ such that $(\mathcal{A}_X, m_X)$ is an arrangement in the variables x_1, x_2. This means that the partial derivations $\partial_i \in D(\mathcal{A}_X, m_X)$ for all $i = 3, ..., \ell$. Due to Corollary 2.27, $(\mathcal{A}_X, m_X)$ is free. By our assumption, $2m_0 > |m| \geq |m_X|$. Therefore, we can apply Theorem 5.10 (1) to see that there is an element θ_0 in the basis of $D(\mathcal{A}_X, m_X)$ with $\operatorname{pdeg}\theta_0 = m_0$. Combining these two facts, we see that $(m_0, 0_3, ..., 0_\ell) \subset \exp(\mathcal{A}_X, m_X)$. Thus, Corollary 2.23 implies

$$\exp(\mathcal{A}_X, m_X) = (|m_X| - m_0, m_0, 0_3, ..., 0_\ell). \tag{5.10}$$

Now, we can choose a basis

$$\theta_X, \psi_X, \partial_3, ..., \partial_\ell$$

for $D(\mathcal{A}_X, m_X)$ such that $\theta_X \notin x_1 \operatorname{Der}(S)$ and $\psi_X \in x_1 \operatorname{Der}(S)$ due to Proposition 5.2. Thus, $m^*(X) = \operatorname{pdeg}\theta_X$ from the definition of m^*. Since the degrees of a basis of a free module are independent of a chosen basis, Equation (5.10) implies $\operatorname{pdeg}\theta_X = m_0$ or $\operatorname{pdeg}\theta_X = |m_X| - m_0$. Assume $\operatorname{pdeg}\theta_X = m_0$ and consequently $\operatorname{pdeg}\psi_X = |m_X| - m_0$ holds. By the definition of $D(\mathcal{A}_X, m_X)$, we have

$$\theta_X(x_1), \psi_X(x_1) \in x_1^{m_0} S. \tag{5.11}$$

But since we assumed $m_0 > |m_X| - m_0 = \operatorname{pdeg}\psi_X$, this implies $\psi_X(x_1) = 0$. We can conclude by the choice of ψ_X that

$$\psi_X = x_1 f_2 \partial_2 + \sum_{i=3}^{\ell} f_i \partial_i \tag{5.12}$$

for some $f_i \in S$ and $i = 3, ..., \ell$. Due to Theorem 2.22 we can assume

$$\det M(\theta_X, \psi_X, \partial_3, ..., \partial_\ell) = \mathcal{Q}(\mathcal{A}_X, m_X). \tag{5.13}$$

But (5.12) implies

$$\det M(\theta_X, \psi_X, \partial_3, ..., \partial_\ell) = \theta_X(x_1) x_1 f_2, \tag{5.14}$$

as $M(\theta_X, \psi_X, \partial_3, ..., \partial_\ell)$ is a triangular matrix. Thus, $\theta_X(x_1) \neq 0$. Since we assumed $\operatorname{pdeg}\theta_X = m_0$, (5.11) yields $\theta_X(x_1) = c x_1^{m_0}$ for some $c \in \mathbb{K}^*$. Combining this fact with Equations (5.13) and (5.14) gives

$$x_1^{m_0+1} | \mathcal{Q}(\mathcal{A}_X, m_X).$$

This is a contradiction, since $m_0 = m(H_0)$ is the highest power of x_1 in $\mathcal{Q}(\mathcal{A}_X, m_X)$. Therefore, we can conclude

$$m^*(X) = \operatorname{pdeg}\theta_X = |m_X| - m_0 = m''(X).$$

□

Proof (of Theorem 1.5): Assume that $(\mathcal{A}, m_{k_0})$ is a free multiarrangement with $k_0 > \frac{1}{2}|m| - m_0$ and $\exp(\mathcal{A}, m_{k_0}) = (d_1, ..., d_\ell)$. This means that

$$2m_{k_0}(H_0) = 2k_0 + 2m_0 > |m| - 2m_0 + 2m_0 = |m|. \tag{5.15}$$

Therefore, $(\mathcal{A}, m_{k_0})$ with $H_0 \in \mathcal{A}$ fulfils the assumptions of Theorem 5.10 and Lemma 5.11. Theorem 5.10 implies that $(\mathcal{A}'', m''_{k_0})$ is free with $\exp(\mathcal{A}, m''_{k_0}) = (d_2, ..., d_\ell)$ and $d_1 = m_{k_0}(H_0)$. Furthermore, Lemma 5.11 shows that m''_{k_0} and $m^*_{k_0}$ define the same multiplicity for $\mathcal{A}''$. Thus, we can apply the implication ((2) and (3)) $\Rightarrow$ (1) of Theorem 5.5 to see that $(\mathcal{A}, m_{k_0+1})$ is also a free multiarrangement. Since $k_0 + 1 > k_0 > \frac{1}{2}|m| - m_0$, we inductively obtain that $(\mathcal{A}, m_{k_0+i})$ is a free multiarrangement for all $i \geq 0$.

Due to (5.15), we can also apply the implication ((1) and (3)) $\Rightarrow$ (2) of Theorem 5.5 to see that $(\mathcal{A}, m_{k_0+i-1})$ is free, as long as $k_0 + i > \frac{1}{2}|m| - m_0$ holds. Therefore, we can conclude that $(\mathcal{A}, m_k)$ is a free multiarrangement for all $k \geq \frac{1}{2}|m| - m_0$ under the assumption that $(\mathcal{A}, m_{k_0})$ is free.

Conversely assume that $(\mathcal{A}, m_{k_0})$ is a non-free multiarrangement with $k_0 > \frac{1}{2}|m| - m_0$. Then $(\mathcal{A}, m_k)$ is a non-free multiarrangement for all $k > \frac{1}{2}|m| - m_0$; otherwise we could apply the first part of this theorem to see that $(\mathcal{A}, m_{k_0})$ is free. $\square$

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Hiermit versichere ich, dass ich die Arbeit selbstständig verfasst habe und keine anderen als die angegebenen Quellen als Hilfsmittel benutzt habe.

Kaiserslautern, der 29. Juli 2014 Lukas Kühne