Optimal Multilevel Monte Carlo Algorithms for Parametric Integration and Initial Value Problems

Thomas Daun

Vom Fachbereich Informatik der Technischen Universität Kaiserslautern
zur Verleihung des akademischen Grades
Doktor der Naturwissenschaften (Dr. rer. nat.)
genehmigte Dissertation

Datum der Einreichung:	30.04.2014
Datum der wissenschaftlichen Aussprache:	05.12.2014

Dekan:	Prof. Dr. Klaus Schneider
1. Gutachter:	Prof. Dr. Stefan Heinrich
2. Gutachter:	Prof. Dr. Klaus Ritter
Kommissionsvorsitzender:	Prof. Dr. Klaus Schneider

D 386

Danksagung

Mein ganz besonderer Dank gilt meinem Doktorvater Prof. Dr. Stefan Heinrich, sowohl für seine Unterstützung, seine Geduld, als auch für die Freiheiten, die er mir während meiner wissenschaftlichen Arbeit zugestanden hat. Auch, dass er mich umgehend in die internationale Wissenschaftsgemeinde einführte, weiß ich sehr zu schätzen. Die gemeinsame Arbeit mit ihm war sehr lehrreich und ich bin dankbar, dass ich die Möglichkeit hatte auf seine Erfahrung und sein beeindruckendes Wissen zurückgreifen zu können.
Weiterhin möchte ich meinem Zweitgutachter Prof. Dr. Klaus Ritter und dem Kommissionsvorsitzenden Prof. Dr. Klaus Schneider herzlich danken.
Ein besonderer Dank gilt auch meinem Kollegen Bernhard Milla, der mir immer zur Seite stand und auch unserer Sekretärin Mady Gruys möchte ich recht herzlich für ihre Unterstützung danken. Nicht unerwähnt bleiben dürfen meine Freunde und Kollegen aus den Fachbereichen Informatik und Mathematik, sowie diejenigen, die meine Arbeit zur Korrektur gelesen haben.
Insbesondere möchte ich meinen Eltern und meinen Geschwistern für ihre große Unterstützung danken.

Abstract

We intend to find optimal deterministic and randomized algorithms for three related problems: multivariate integration, parametric multivariate integration, and parametric initial value problems. The main interest is concentrated on the question, in how far randomization affects the precision of an approximation. We want to understand when and to which extent randomized algorithms are superior to deterministic ones.
All problems are studied for Banach space valued input functions. The analysis of Banach space valued problems is motivated by the investigation of scalar parametric problems; these can be understood as particular cases of Banach space valued problems. The gain achieved by randomization depends on the underlying Banach space.
For each problem, we introduce deterministic and randomized algorithms and provide the corresponding convergence analysis. Moreover, we also provide lower bounds for the general Banach space valued settings, and thus, determine the complexity of the problems. It turns out that the obtained algorithms are order optimal in the deterministic setting. In the randomized setting, they are order optimal for certain classes of Banach spaces, which includes the L_p spaces and any finite dimensional Banach space. For general Banach spaces, they are optimal up to an arbitrarily small gap in the order of convergence.

Contents

Chapter 1

Introduction

In this thesis, we analyze the complexity of three related numerical problems: multivariate integration, parametric multivariate integration and parametric initial value problems. These problems are relevant in multiple applications when modeling natural phenomena in physics, engineering or biology as well as economics and finance. Analytical solutions are often not accessible, therefore, numerical approximations are needed. Our aim is to find optimal algorithms for the mentioned problems. We also discuss the impact of randomization on the error of the approximation, and we want to understand when and to which extent randomized algorithms are superior to deterministic ones.

In the setting of information-based complexity theory, the complexity of a numerical problem is described in terms of the n-th minimal error. This is the best possible approximation error that can be achieved given a fixed amount of information (about the problem). In the present thesis, the admissible information is based on point evaluations of the input functions. The n-th minimal error is usually estimated by providing upper and lower bounds. Here, all upper bounds are proven due to defining and analyzing an explicit algorithm. The lower bound can be used as a measure of quality for these algorithms. If the algorithm reaches the lower bound, it is optimal for this setting. If not, we may either find a sharper lower bound or need to find a better algorithm. In case of matching bounds the complexity is precisely determined and we found an optimal algorithm for the considered problem. In this manner, for instance, the multilevel Monte Carlo method has been established in [17] and [25], which is, due to [14], a common used algorithm in finance nowadays.

The complexity of multivariate integration is well-studied for scalar valued input functions. It is well-known for definite integration and the indefinite case has recently been considered in [24]. However, all these results are valid for scalar valued input functions only. In this thesis, we consider the problem for Banach space valued input functions and analyze deterministic and randomized algorithms for the definite and the indefinite case. To determine the complexity, we also provide lower bounds. This is the first time that the complexity of Banach space valued problems is studied. It turns out that the algorithms are order optimal in the deterministic setting. In the randomized setting, however, the gain achieved due to randomization depends on the geometry of the underlying Banach space. In this setting, the algorithms are order optimal only for certain classes of Banach spaces. For general Banach spaces, an arbitrarily small gap in the exponent remains.

The analysis of Banach space valued problems is motivated by the study of parametric problems. These can be understood as particular cases of Banach space valued problems. Thus, if the parametric problem can be expressed as a Banach space valued problem, we can apply the results of the general Banach space valued analysis. It turns out that this technique is crucial for the analysis of multilevel algorithms for parametric problems.

The complexity of definite parametric integration was studied in [25] and [18] and later for quantum algorithms in [45]. The indefinite case has not been studied so far. We fill this gap and provide the analysis of indefinite parametric integration. Moreover, we further extend the considered classes of input functions to spaces of dominating mixed derivatives and other types of non-isotropic smoothness. In contrast to the classes that have been studied before, these classes allow to treat different types of smoothness for the parameter dependence and for the basic (nonparametric) integration problem, separately. For this more general setting, we provide and analyze certain multilevel Monte Carlo algorithms for both, the definite and the indefinite case. Beyond that, all results are stated for Banach space valued input functions. If we fix the random parameter, we obtain deterministic algorithms, which turn out to be order optimal for the deterministic setting. In the randomized case, the gain achieved by randomization depends again on the geometry of the underlying Banach space. As for Banach space valued multivariate integration, the algorithm is optimal for certain Banach spaces and an arbitrarily small gap in the order of convergence remains if general Banach spaces are considered. However, for certain Banach spaces, we show optimality even up to logarithmic factors and only in some limit cases a logarithmic gap remains.

Indefinite integration is a particular case of an initial value problem if the integration domain is one-dimensional. Thus, as a next step, the complexity of parametric initial value problems is studied. This problem has not been investigated before. The complexity of initial value problems for ordinary differential equations (ODEs) without dependence on a parameter s was studied in [28, 29, 30, 23, 8] for scalar systems, and in [21] for the Banach space valued case. We use the Banach space valued algorithm of [21] to define a multilevel Monte Carlo algorithm for the parametric problem. The problem is again studied for Banach space valued input functions. Therefore, systems of ordinary differential equation are included if we choose the Banach space to be $\mathbb{R}^d$. We prove lower bounds and this way we determine the complexity of the problem. Since parametric integration is a particular case of an initial value problem, we cannot assume better rates than established for parametric indefinite integration. Moreover, due to the non-linearity of the problem, the passing from non-parametric to the parametric problem is more involved. Nevertheless, we obtain similar results as for parametric integration. This means that the considered multilevel Monte Carlo algorithm is order optimal in the deterministic setting, and that the dependence on the geometry of the Banach space is as before. Furthermore, in certain cases, the considered algorithms are optimal even up to logarithmic factors and as before, only in some limit cases a logarithmic gap remains.

Finally, we restrict ourselves to Hilbert space valued initial value problems, which allows the further extension of the considered class of input functions to certain local classes. We carry over the previous results for Banach space valued initial value problems to these more general classes, i.e., we obtain algorithms that are optimal even up to logarithmic factors (as before, only in some limit cases a logarithmic gap remains). This consideration still includes systems of parameter dependent ordinary differential equation.

The thesis is organized as follows: First, preliminaries, needed in the sequel, are introduced in Chapter 2. Afterwards, related and new results are briefly summarized in an informal way in Chapter 3, called 'The Thesis in a Nutshell'. Having provided an overview of the entire thesis, we will start with the complexity analysis of Banach space valued multivariate integration in Chapter 4. A multilevel algorithm is applied to analyze the complexity of parametric Banach space valued multivariate integration in Chapter 5. In Chapter 6, a similar approach is chosen to study the approximate solution of parametric Banach space valued initial value problems, and in Chapter 7, we finally restrict ourselves to parametric Hilbert space valued initial value problems, which allows us to extend the class of considered input functions to certain local classes.

Chapter 2

Preliminaries

This chapter provides an overview of conventions used in this thesis as well as results required from certain mathematical fields. However, it is not exhaustive. We assume a basic knowledge in functional analysis and probability theory. The monographs [1],[43],[47] and [27],[32] are recommended for an introduction into these fields.

2.1 General Conventions and Notions

We introduce some notations needed in the sequel. We use $\mathbb{N} = \{1, 2, \ldots\}, \mathbb{N}_0 = \{0, 1, 2, \ldots\}$. Throughout the thesis, log always means $\log_2$; $\wedge$ and $\vee$ mean logical conjunction and disjunction, respectively. For a Banach space Z, the closed unit ball is denoted by B_Z, the open unit ball by B_Z^0, the identity mapping on Z by I_Z and the dual space by Z^*. The norm of Z is $\|\cdot\|$, other norms are distinguished by subscripts. We assume that all considered Banach spaces are defined over the scalar fields $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. Given $k \in \mathbb{N}$, Banach spaces X_i $(i = 1, \ldots, k)$ and Y, we let $\mathscr{L}(X_1, \ldots, X_k, Y)$ be the space of bounded multilinear mappings $T : X_1 \times \cdots \times X_k \to Y$ endowed with the canonical norm

$$\|T\|_{\mathscr{L}(X_1,\ldots X_k,Y)} = \sup_{x_1 \in B_{X_1},\ldots,x_k \in B_{X_k}} \|T(x_1, \ldots, x_k)\|_Y.$$

If $X_1 = \cdots = X_k = X$, we write $\mathscr{L}_k(X, Y)$. Similarly, if $k = k_1 + k_2$ with $k_1, k_2 \geq 0$, $X_1 = \cdots = X_{k_1} = X$, and $X_{k_1+1} = \cdots = X_{k_1+k_2} = Z$, we use the notation $\mathscr{L}_{k_1,k_2}(X, Z, Y)$. For convenience, we extend the notation to $k = 0$ by setting $\mathscr{L}_0(X, Y) = \mathscr{L}_{0,0}(X, Z, Y) = Y$. If $k = 1$, $\mathscr{L}_1(X, Y)$ is the space of bounded linear operators, for which we also write $\mathscr{L}(X, Y)$. If $Y = X$, we write $\mathscr{L}(X)$ instead of $\mathscr{L}(X, X)$.

If M is a nonempty set, we let $B(M, Z)$ be the space of all Z-valued functions that are bounded on M, and equip this space with the supremum norm

$$\|g\|_{B(M,Z)} = \sup_{t \in M} \|g(t)\|.$$

If $Z = \mathbb{K}$, we only write $B(M)$.

Moreover, $c, c_0, c_1, \ldots$ denote constants, which may depend on the problem parameters, such as d, r, κ, L, X, Y, Z, but depend neither on the algorithm parameters $n, l_0, l_1, \ldots$ nor on the input. The same symbol may also denote different constants, even in a sequence of relations.

To state complexity results, we use asymptotic notations such as $a_n \preceq b_n$, which means that there exists a constant $c > 0$ and a natural number $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, we have $a_n \leq c\, b_n$. If $a_n \preceq b_n$ and $b_n \preceq a_n$, we write $a_n \asymp b_n$. We also use the notation $a_n \asymp_{\log} b_n$ if there are constants $c_1, c_2 > 0$, $n_0 \in \mathbb{N}$, and $\theta_1, \theta_2 \in \mathbb{R}$ with $\theta_1 \leq \theta_2$ such that for all $n \geq n_0$,

$$c_1 b_n (\log(n+1))^{\theta_1} \leq a_n \leq c_2 b_n (\log(n+1))^{\theta_2}.$$

2.2 Introduction to Information-Based Complexity Theory

We deal with the complexity of certain integration problems. For this purpose it is necessary to introduce a measure of difficulty for numerical problems.

In contrast to discrete complexity theory, where discrete problems, such as graph theoretic problems, are considered, we are interested in the complexity of continuous problems. Examples are multivariate integration, the solution of differential equations as well as matrix multiplication and the solution of systems of linear equations. These examples can be further divided into problems where information is partial and problems where the problem constellation can be described entirely. Concerning the solution of systems of linear equations or matrix multiplication, we obtain an exact solution up to machine precision, since the input can be described by a finite number of parameters.

In information-based complexity theory, we concentrate on problems, where information is only partial. These problems, in general, cannot be solved exactly. This holds in particular for multivariate integration and the solution of differential equations. Here the inputs are elements of infinite dimensional spaces, usually function spaces. Unfortunately, there is no way to describe these functions entirely using discrete sets so that a computer can handle it. However, given some information about the problem, we are usually able to propose an approximation to the exact solution. The accuracy that can be achieved by reasonable algorithms depends in most cases directly on the given information. Thus, the

question arises: What is the best approximation we can achieve, given a certain amount of information about a problem constellation?
This is the basis of information-based complexity theory (IBC), which will be briefly introduced in a formal way. We refer to [42] and [39] for further details.

An abstract numerical problem is described by a tuple

$$\mathcal{P} = (F, G, S, K, \Lambda).$$

The set F is the set of input data, G is a normed linear space, and K is an arbitrary set, in the sequel always a Banach space. Moreover,

$$S : F \to G$$

is an arbitrary mapping, usually called the solution operator, which maps the input $f \in F$ to the exact solution Sf. Λ is a set of mappings from F to K, which is called the class of admissible information functionals $\lambda \in \Lambda$. Thus, Λ describes the admitted information for the calculation of an approximation. In this thesis we primarily consider point evaluations, e.g., for continuous input functions f, we obtain

$$\Lambda = \{\delta_t \ : \ t \in [0,1]^d\},$$

where $\delta_t(f) := f(t)$. This type of information is usually called standard information.

Example 2.2.1. We take the classical integration problem as an introductory example. Here we choose

- $F = \{f \in C^2([0,1]) \ : \ ||f||_{C^2([0,1])} \leq 1\}$,
- $G = \mathbb{R}$,
- $S(f) = \int_0^1 f(\tau)\ d\tau$,
- $K = \mathbb{R}$,
- $\Lambda = \{\delta_t \ : \ t \in [0,1]\}$,

where $C^2([0,1])$ denotes the space of 2-times continuously differentiable functions endowed with the norm

$$||f||_{C^2([0,1])} = \max_{j=0,1,2} \sup_{t\in[0,1]} \left| f^{(j)}(t) \right|.$$

The complexity of a given problem depends highly on the admissible information. For instance, we can allow linear information. In this case, Λ consists of all linear functionals $\lambda \in F_1^*$, where F_1 is a linear space with $F \subseteq F_1$. Since the integration operator itself is a linear functional, it follows that the integration problem is trivial. The exact solution is already given by a single information call, i.e., by

$$\lambda_1(f) = \int_0^1 f(\tau)\, d\tau.$$

However, there are also problems, e.g., the approximation of smooth functions in suitable norms, for which we know that the complexity is the same, no matter if we use linear information or standard information. Thus, even linear information does not help in some cases. Notice that all results of the thesis are obtained using only standard information.

Next, we introduce the concept of information operators, which is crucial for the definition of abstract numerical algorithms. We set

$$K^\infty := \bigcup_{k=1}^\infty K^k. \qquad \text{(disjoint union)}.$$

Definition 2.2.2. An information operator $N : F \to K^\infty$ is given by two sequences of functions

$$\begin{aligned} &\lambda_i : F \times K^{i-1} \to K && (i = 1, 2, \ldots), && \text{[information functions]} \\ &\tau_i : K^i \to \{0, 1\} && (i = 0, 1, 2, \ldots) && \text{[termination functions]} \end{aligned}$$

such that

$$\begin{aligned} &\lambda_1 \in \Lambda, \\ &\lambda_i(\cdot, \lambda_1(f), \ldots, \lambda_{i-1}(f)) \in \Lambda \qquad (i = 2, 3, \ldots). \end{aligned}$$

Given $f \in F$, $N(f)$ is determined as

$$N(f) = \begin{cases} 0 & \text{if } \tau_0 = 1 \\ (\lambda_1(f), \ldots, \lambda_{n(f)}(f)) & \text{else,} \end{cases}$$

where

$$n(f) := \min\{i \ : \tau_i(\lambda_1(f), \ldots, \lambda_i(f)) = 1\}.$$

If no such i exists, we set $n(f) = \infty$.

Remark 2.2.3. If λ_i depends on the already calculated values $\lambda_1(f), \ldots, \lambda_{i-1}(f)$, we call the information operator N adaptive. Otherwise, i.e., if $n(f)$ is fixed and λ_i are just mappings from F to K, we call N non-adaptive.

Definition 2.2.4. A mapping $A : F \to G$ is called an algorithm if it can be represented as $A = \varphi \circ N$, where $N : F \to K^\infty$ is an information operator and $\varphi : K^\infty \to G$ is an arbitrary mapping.

Remark 2.2.5. An algorithm A is called adaptive and non-adaptive if the associated information operator is adaptive and non-adaptive, respectively.

Example 2.2.1 (continued)**.** A standard algorithm for solving integration problems is the composite trapezoidal rule. For $m \in \mathbb{N}$ and an equidistant discretization $t_i := i/m \quad (i = 0, 1, \ldots, m)$, it is defined as

$$A_m^{\text{trapez}}(f) = \frac{1}{2m}\left(f(t_0) + 2\sum_{i=1}^{m-1} f(t_i) + f(t_m)\right).$$

In this case

$$\lambda_1 = \delta_0, \lambda_2 = \delta_{\frac{1}{m}}, \lambda_3 = \delta_{\frac{2}{m}} \ldots, \lambda_{m+1} = \delta_1$$

and

$$\tau_0 = \cdots = \tau_m = 0, \tau_{m+1} = 1.$$

$n(f) = m + 1$ is fixed for all $f \in F$ and N is given by

$$N(f) = (\lambda_1(f), \ldots, \lambda_{m+1}(f)) = (f(t_0), f(t_1), \ldots, f(t_m)).$$

Choosing φ in such a way that φ maps $N(f)$ to

$$\frac{1}{2n}\left(\lambda_1(f) + 2\sum_{i=1}^{m-1} \lambda_{i+1}(f) + \lambda_{m+1}(f)\right)$$

we obtain

$$A_m^{\text{trapez}}(f) = (\varphi \circ N)(f).$$

It is easy to see that the algorithm above is non-adaptive and, as we shall see in the next paragraph, that it is deterministic.

Deterministic Setting

A deterministic algorithm A for $\mathcal{P}$ is defined as a mapping $A : F \to G$, as above. Let $\operatorname{card}(A, f)$ be the number of information functionals used by A at input f, i.e., for $A = \varphi \circ N$, we have $\operatorname{card}(A, f) = n(f)$. Then we set

$$\operatorname{card}^{\det}(A, F) = \sup_{f \in F} \operatorname{card}(A, f).$$

If $\mathrm{card}^{\mathrm{det}}(A, F) < \infty$, the error of A is given by

$$e^{\mathrm{det}}(S, A, F) = \sup_{f \in F} \|Sf - Af\|_G,$$

and $e^{\mathrm{det}}(S, A, F) = \infty$ otherwise. The central notion of IBC is the n-th minimal error, which is defined for $n \in \mathbb{N}_0$ by

$$e_n^{\mathrm{det}}(S, F) = \inf_{\mathrm{card}(A,F) \leq n} e^{\mathrm{det}}(S, A, F).$$

So, $e_n^{\mathrm{det}}(S, F)$ is the minimal possible error that can be achieved among all deterministic algorithms that use at most n information functionals.

Randomized Setting

A randomized (or Monte Carlo) algorithm for $\mathcal{P}$ is a family $A = (A_\omega)_{\omega \in \Omega} = (\varphi_\omega \circ N_\omega)_{\omega \in \Omega}$, where $(\Omega, \Sigma, \mathbb{P})$ is a certain probability space and each A_ω is a mapping $A_\omega : F \to G$. For $\omega \in \Omega$ fixed, $A_\omega : F \to G$ is a deterministic algorithm as defined above.

Let $\mathrm{Dom}(A)$ be the set of all $f \in F$ such that $\mathrm{card}(A_\omega, f)$ is a measurable function of ω,

$$\mathrm{card}(A_\omega, f) < \infty \text{ for almost all } \omega \in \Omega,$$

and $A_\omega(f)$ is a G-valued random variable, that is, $A_\omega(f)$ is Borel measurable and there is a separable subspace G_0 of G (which may depend on f) such that

$$A_\omega(f) \in G_0 \text{ for almost all } \omega \in \Omega.$$

For $f \in F$, let

$$\mathrm{card}^{\mathrm{ran}}(A, f) = \mathbb{E}\mathrm{card}(A_\omega, f)$$

if $f \in \mathrm{Dom}(A)$, and $\mathrm{card}^{\mathrm{ran}}(A, f) = +\infty$ otherwise. We set

$$\mathrm{card}^{\mathrm{ran}}(A, F) = \sup_{f \in F} \mathrm{card}^{\mathrm{ran}}(A, f).$$

Now, the error of A is defined as

$$e^{\mathrm{ran}}(S, A, F) = \sup_{f \in F} \mathbb{E}\|S(f) - A_\omega(f)\|_G$$

if $F \subseteq \mathrm{Dom}(A)$, and $e^{\mathrm{ran}}(S, A, F) = +\infty$ otherwise. For $n \in \mathbb{N}_0$ the n-th minimal randomized error is defined as

$$e_n^{\mathrm{ran}}(S, F) = \inf\{e(S, A, F) : A \in \mathcal{A}^{\mathrm{ran}},\ \mathrm{card}(A, F) \leq n\}.$$

Similar to the deterministic case, $e_n^{\mathrm{ran}}(S, F)$ is the minimal possible error among all randomized algorithms that use at most n information functionals (in expectation).

Remark 2.2.6.

- The deterministic setting can be understood as a special case of the randomized setting by considering only one-point probability spaces $\Omega = \{\omega_0\}$; this means the algorithm is not random. Thus, by fixing the random parameter $\omega \in \Omega$ every randomized algorithm becomes a deterministic one. This approach is used in this thesis. We define only randomized algorithms and analyze the deterministic setting by restricting the algorithm to an arbitrary realization.

- If deterministic and randomized cardinality coincide, we omit the superscript 'det' and 'ran'. This occurs for instance if the cardinality of the randomized algorithm does not depend on randomness at all, which is the case for all algorithms considered here.

Example 2.2.7. The standard example of a randomized algorithm is the classical Monte Carlo method. First, we define the considered probability space $(\Omega, \Sigma, \mathbb{P})$. We choose

$$\Omega = [0,1]^{\mathbb{N}} = \prod_{i=1}^{\infty} [0,1] \qquad \text{(Cartesian product)}$$

and $\omega \in \Omega$ has the form $\omega = (x_1, x_2, \ldots)$, where $x_1, x_2, \ldots \in [0,1]$. Σ is the smallest σ-algebra that contains all sets

$$\prod_{i=1}^{\infty} B_i \qquad (B_i \in \mathcal{B}([0,1]),$$

where only for finitely many i the sets $B_i \neq [0,1]$. Then there exists a unique measure $\mathbb{P}$ such that for all B_i as above

$$\mathbb{P}\left(\prod_{i=1}^{\infty} B_i\right) = \prod_{i=1}^{\infty} \lambda(B_i),$$

where λ is the Lebesgue measure on $[0,1]$. We define for $\omega = (x_1, x_2, \ldots) \in \Omega$

$$\xi_k(\omega) = x_k \qquad (k = 1, 2, \ldots).$$

Thus, ξ_k is a sequence of independent random variables, uniformly distributed on $[0,1]$. Given $f \in C([0,1])$ the classical Monte Carlo method is defined for $m \in \mathbb{N}$ and $\omega \in \Omega$ as

$$A_{m,\omega}(f) = \sum_{k=1}^{m} f(\xi_k(\omega))$$

and $A_m^{\mathrm{MC}} = (A_{m,\omega})_{\omega\in\Omega}$. Thus, N_ω is given by

$$N_\omega(f) = (\lambda_{\xi_1(\omega)}(f), \dots, \lambda_{\xi_m(\omega)}(f)) = (f(\xi_1(\omega)), f(\xi_2(\omega)), \dots, f(\xi_m(\omega)))$$

and φ_ω maps $N_\omega(f)$ to

$$\sum_{k=1}^{m} \lambda_{\xi_k(\omega)}(f).$$

This means, φ_ω does not depend on $\omega \in \Omega$ at all and

$$A_m^{\mathrm{MC}} = (A_{m,\omega})_{\omega\in\Omega} = (\varphi_\omega \circ N_\omega)_{\omega\in\Omega}.$$

Model of Computation

Our standard model is the real number model. This means, we assume the ability to perform arithmetic operations in $\mathbb{R}$; this avoids roundoff issues. For theoretical interest and as a key technique for the analysis of parameter dependent problems, we will also consider Banach space valued problems. Therefore, we also assume the ability to perform vector operations in K if K is a general Banach space.

Considering the cardinality of an algorithm also means neglecting the total arithmetic cost. There are models of computation that further refine the dependence on arithmetic operations, and thus, give a better estimate of the real computational cost. We introduce the following cost model as an example.

Definition 2.2.8. For $f \in F$ the $\mathrm{cost}(A, f)$ of an algorithm A given f is determined as follows:

- every arithmetic operation $+, -, *, /$ has cost 1,
- if K is a general Banach space, vector operation in K have cost $c_0 > 0$,
- every comparison $\geq, >$ has cost 1,
- vector operations in G have cost $c_1 \geq 1$,
- information operations $\lambda(f)$ have cost $c_2 \geq 1$.

If A can be represented as a composition of such elementary operations we define the cost as follows. Given a certain decomposition, the overall cost is the sum of the cost of each elementary operation. Now, the cost of A is defined as the minimal cost of all possible decompositions. If A cannot be represented as such a decomposition, the cost is set to infinity.

It is clear by definition that

$$\mathrm{card}(A, f) \leq \mathrm{cost}(A, f).$$

But for all algorithms used in the sequel, there exists a constant $c > 0$ such that

$$\mathrm{card}(A, f) \leq \mathrm{cost}(A, f) \leq c\, \mathrm{card}(A, f).$$

Thus, for the sake of simplicity, we will focus on the cardinality, having in mind that this is also proportional to the cost.

Upper and Lower Bounds

To estimate the n-th minimal error in the deterministic and the randomized setting, we usually provide upper and lower bounds. The goal is to find matching upper and lower bounds. In this thesis, the proof of an upper bound is always given in a constructive way; this means, we state an explicit algorithm and provide the convergence analysis, which immediately gives the upper bound of the n-th minimal error. If the upper bound matches the lower bound up to some constant factor, we call the algorithm optimal (up to constants). If a logarithmic gap with respect to n remains, we say it is optimal up to logarithmic factors. In both cases, the algorithm is order optimal; thus, the upper and lower bound coincides with respect to the convergence order.

Having bounds for the n-th minimal error, we can easily determine the ε-complexity of a numerical problem, which can be understood as the inverse of the n-th minimal error.

Definition 2.2.9. For set $\in \{\mathrm{det}, \mathrm{ran}\}$ and $\varepsilon > 0$, we define the ε-complexity of a numerical problem S given the input class F as

$$\mathrm{comp}_{\varepsilon}^{\mathrm{set}}(S, F) = \inf\{n \in \mathbb{N}_0 \; : \; e_n^{\mathrm{set}}(S, F) \leq \varepsilon\}.$$

Thus, the n-th minimal error immediately settles the ε-complexity of a numerical problem. For this reason, we concentrate on n-th minimal errors.

Example 2.2.1 (continued)**.** It is well-known that the approximation error of A_m^{trapez} from above satisfies

$$e^{\mathrm{det}}(S, A_m^{\mathrm{trapez}}, F) = \sup_{f \in F} |S(f) - A_m^{\mathrm{trapez}}(f)| \leq \frac{1}{12m^2}.$$

We conclude

$$e_n^{\mathrm{det}}(S, F) \leq \frac{1}{12(n-1)^2} \preceq n^{-2}.$$

Moreover, it is possible to show that $n^{-2} \preceq e_n^{\det}(S,F)$, thus

$$e_n^{\det}(S,F) \asymp n^{-2}$$

and the composite trapezoidal rule is optimal for the given input class. Furthermore, in terms of the ε-complexity, we obtain

$$\mathrm{comp}_{\varepsilon}^{\det}(S,F) \asymp \left(\frac{1}{\varepsilon}\right)^{\frac{1}{2}}.$$

Toolbox from Information-Based Complexity Theory

We will have recourse to certain auxiliary results for subsequent lower bound proofs. In the randomized setting, we use the following lemma due to [2], which describes a relationship between randomized and average case setting. This enables us to concentrate on the average case setting, where lower bounds are easier to prove. Before we state the lemma, we briefly introduce the average case setting.

Given F, we additionally assume that there is a probability measure ν on F. The probability space is thus defined by (F, Σ_0, ν), where Σ denotes a σ_0-algebra of subsets of F, usually the Borel sigma algebra. Then the average cardinality of a deterministic algorithm A is defined as

$$\mathrm{card}^{\mathrm{avg}}(A,F,\nu) = \int_F \mathrm{card}(A,f)\ d\nu(f),$$

and the average case error as

$$e^{\mathrm{avg}}(S,A,F,\nu) = \int_F ||Sf - Af||_G\ d\nu(f).$$

Moreover, the n-th minimal average case error is defined as

$$e_n^{\mathrm{avg}}(S,F,\nu) = \inf_{\mathrm{card}^{\mathrm{avg}}(A,F,\nu)\le n} e^{\mathrm{avg}}(S,A,F,\nu).$$

Lemma 2.2.10. *For each probability measure ν on F of finite support and each $n \in \mathbb{N}$,*

$$e_n^{\mathrm{ran}}(S,F) \ge \frac{1}{2} e_{2n}^{\mathrm{avg}}(S,F,\nu).$$

Proof. see [19], Lemma 5. □

The next lemma is restricted to problems $\mathcal{P}$ that are linear. This means the following:

Definition 2.2.11. The tuple $\mathcal{P} = (F, G, S, K, \Lambda)$ defines a linear problem if

- F is a convex, balanced subset of a linear space F_1,
- S is the restriction to F of a linear operator from F_1 to G,
- all mappings $\lambda \in \Lambda$ are restrictions to F of linear mappings from F_1 to K.

Lemma 2.2.12. *Let $n, \bar{n} \in \mathbb{N}$ with $\bar{n} > 2n$, assume that there are $(f_i)_{i=1}^{\bar{n}} \subset F$ such that the sets $\{\lambda \in \Lambda : \lambda(f_i) \neq 0\} \quad (i = 1, \ldots, \bar{n})$ are mutually disjoint. Then*

$$e_n^{\text{det}}(S, F) \geq \min_{\mathscr{I}} \big|\big| \sum_{i \in \mathscr{I}} S(f_i) \big|\big|_G,$$

where the minimum is taken over all subsets $\mathscr{I}$ of $\{1, \ldots, \bar{n}\}$ with $|\mathscr{I}| \geq \bar{n} - 2n$.

Proof. This is a standard result from IBC, see [42], Ch. 4.5. □

Lemma 2.2.13. *Let $n, \bar{n} \in \mathbb{N}$ with $\bar{n} > 2n$, assume that there are $(f_i)_{i=1}^{\bar{n}} \subset F$ such that the sets $\{\lambda \in \Lambda : \lambda(f_i) \neq 0\} \quad (i = 1, \ldots, \bar{n})$ are mutually disjoint, and for all sequences $(\alpha_i)_{i=1}^{\bar{n}} \in \{-1, 1\}^{\bar{n}}$ we have $\sum_{i=1}^{\bar{n}} \alpha_i f_i \in F$. Define the measure ν on F to be the distribution of $\sum_{i=1}^{\bar{n}} \varepsilon_i f_i$, where ε_i are independent Bernoulli random variables with $\mathbb{P}\{\varepsilon_i = 1\} = \mathbb{P}\{\varepsilon_i = -1\} = 1/2$. Then*

$$e_n^{\text{avg}}(S, \nu) \geq \frac{1}{2} \min_{\mathscr{I}} \mathbb{E} \big|\big| \sum_{i \in \mathscr{I}} \varepsilon_i S(f_i) \big|\big|_G,$$

where the minimum is taken over all subsets $\mathscr{I}$ of $\{1, \ldots, \bar{n}\}$ with $|\mathscr{I}| \geq \bar{n} - 2n$.

Proof. see [19], Lemma 6. The lemma is formulated for $K = \mathbb{K}$, but immediately carries over also to the Banach space valued case. □

The lemmas above are needed to prove lower bounds for linear problems in a direct way. However, it is sometimes easier to reduce a numerical problem to another numerical problem than proving a direct lower bound. This way, we can utilize already proven results. This is especially the case for non-linear problems. For example, the next lemma is applied in the lower bound proof of Theorem 6.6.1.

Assume that $\tilde{\mathcal{P}} = (\tilde{F}, \tilde{G}, \tilde{S}, \tilde{K}, \tilde{\Lambda})$ is another numerical problem. Furthermore, $V_1 : F \to \tilde{F}$ is a mapping that maps input $f \in F$ of S to input $V_1 f \in \tilde{F}$ of $\tilde{S}$ and each information about $V_1 f$ can be obtained from k suitable informations about f and the application of a certain mapping. Let $V_2 : \tilde{G} \to G$ be another mapping such that

$$S = V_2 \circ \tilde{S} \circ V_1,$$

where $V_2 : \tilde{G} \to G$ is Lipschitz. This means that there is a constant $c > 0$ such that

$$||V_2(x) - V_2(y)||_G \leq c||x - y||_{\tilde{G}} \qquad \text{for all } x, y \in \tilde{G}.$$

The Lipschitz constant $||V_2||_{\mathrm{Lip}}$ is the smallest constant c such that the relation above holds. In this situation, P reduces to $\tilde{P}$ and the following Lemma is applicable.

Lemma 2.2.14. *Assume that $V_1 : F \to \tilde{F}$ is a mapping such that there exist a $k \in \mathbb{N}$, mappings*

$$\eta_j : \tilde{\Lambda} \to \Lambda \qquad (j = 1, \ldots, k),$$

and $\varrho : \tilde{\Lambda} \times K^k \to \tilde{K}$ with

$$(V_1 f)(\tilde{\lambda}) = \varrho(\tilde{\lambda}, f(\eta_1(\tilde{\lambda})), \ldots, f(\eta_k(\tilde{\lambda})))$$

for all $f \in F$ and $\tilde{\lambda} \in \tilde{\Lambda}$. Then for all $n \in \mathbb{N}$

$$\begin{aligned} e_{kn}^{\mathrm{det}}(S, F) &\leq ||V_2||_{\mathrm{Lip}} e_n^{\mathrm{det}}(\tilde{S}, \tilde{F}), \\ e_{kn}^{\mathrm{ran}}(S, F) &\leq ||V_2||_{\mathrm{Lip}} e_n^{\mathrm{ran}}(\tilde{S}, \tilde{F}). \end{aligned} \tag{2.1}$$

Proof. See [20], Section 3. □

Corollary 2.2.15. *If S is a linear operator, then for all $c \in \mathbb{K}$*

$$\begin{aligned} e_n^{\mathrm{det}}(S, cF) &= |c| e_n^{\mathrm{det}}(S, F), \\ e_n^{\mathrm{ran}}(S, cF) &= |c| e_n^{\mathrm{ran}}(S, F). \end{aligned}$$

2.3 Mathematical Foundations

2.3.1 Differential Calculus in Banach Spaces

We introduce spaces of continuously differentiable functions with values in a Banach space. Notice, if the domain is a subset of $\mathbb{R}$, the definition of differentiability coincides with the usual definition.

Let Z be an arbitrary Banach space. For $r \in \mathbb{N}_0, d \in \mathbb{N}$, let $Q = [0,1]^d$ and $C^r(Q, Z)$ denote the space of all r-times continuously differentiable functions $f : Q \to Z$ equipped with the norm

$$\|f\|_{C^r(Q,Z)} = \max_{\alpha \in \mathbb{N}_0^d,\, |\alpha| \leq r} \sup_{t \in Q} \left\| \frac{\partial^{|\alpha|} f(t)}{\partial t^\alpha} \right\| = \max_{\alpha \in \mathbb{N}_0^d,\, |\alpha| \leq r} \sup_{t \in Q} \left\| \frac{\partial^{|\alpha|} f(t)}{\partial t^{\alpha_1} \cdots \partial t^{\alpha_d}} \right\|,$$

where $\alpha = (\alpha_1, \dots \alpha_d) \in \mathbb{N}_0^d$ and $|\alpha| = \alpha_1 + \cdots + \alpha_d$. For $r = 0$ we, write $C^0(Q, Z) = C(Q, Z)$, which is the space of continuous Z-valued functions on Q, and if $Z = \mathbb{K}$, we write $C^r(Q)$ and $C(Q)$, respectively.

In Chapter 6, we also consider input functions where the domain is an arbitrary Banach space. In this case, the definition of differentiability has a more general fashion. Referring to [47], we briefly introduce the Fréchet and the Gâteaux derivative. The Fréchet derivative is a generalization of the total derivative and the Gâteaux derivative can be understood as a generalization of the concept of directional derivatives.

Given Banach spaces X, Z, let $V \subseteq X$ be a neighborhood of 0. For $g : V \to Z$, we write $g \in \mathrm{o}(V, Z)$ iff

$$\frac{g(v)}{||v||_X} \to 0 \text{ as } (v \to 0, v \in V).$$

Definition 2.3.1. For $x \in X$, let $U \subseteq X$ be a neighborhood of x and let $f : U \to Z$ be given.

(i) f is Fréchet-differentiable at x iff there exists a linear operator $T \in \mathscr{L}(X, Z)$, a neighborhood $V \subseteq X$ of 0, and a mapping $g \in \mathrm{o}(V, Z)$ such that for all $h \in V$ it follows that $x + h \in U$ and

$$f(x + h) - f(x) = Th + g(h) \qquad (h \to 0, h \in V).$$

If such a T exists, we define $f'(x) = T$, which is the Fréchet derivative of f at x.

(ii) f is Gâteaux-differentiable at x iff there exists a linear operator $T \in \mathscr{L}(X, Z)$, a neighborhood $V \subseteq \mathbb{R}$ of 0, and a mapping $g \in \mathrm{o}(V, \mathbb{R})$ such that for all $k \in B_X, t \in V$ it follows that $x + tk \in U$ and

$$f(x + tk) - f(x) = tTk + g(t) \qquad (t \to 0, t \in V).$$

If such a T exists, we define $f'(x) = T$, which is the Gâteaux derivative of f at x.

(iii) If the Fréchet and Gâteaux derivatives $f'(x)$ exist for all $x \in A \subseteq X$, then the mapping

$$\begin{aligned} f' : A &\to \mathscr{L}(X, Z) \\ x &\mapsto f'(x) \end{aligned}$$

is called the Fréchet and Gâteaux derivative of f on A, respectively.

(iv) Higher derivatives are defined successively. Thus, $f''(x)$ is the derivative of f' at x.

Notice that the Gâteaux derivative $f'(x)$ can be defined equivalently by

$$f'(x)k = \lim_{t\to 0} \frac{f(x+tk) - f(x)}{t} \qquad (k \in B_X). \tag{2.2}$$

In the next proposition we summarize known results for the relationship of Fréchet and Gâteaux derivative.

Proposition 2.3.2. *Let $x \in X$.*

(i) Every Fréchet derivative at x is also a Gâteaux derivative at x.

(ii) A Gâteaux derivative at x for which the passage to the limit in (2.2) is uniform for all $k \in B_X$ is also a Fréchet derivative at x.

(iii) If f' exists as a Gâteaux derivative in some neighborhood of x, and if f' is continuous at x, then $f'(x)$ is also a Fréchet derivative at x.

(iv) If $f'(x)$ exists as an Fréchet derivative at x, then f is continuous at x.

Proof. The proof can be found in [47], Proposition 4.8. □

2.3.2 Tensor Product Representation

This section introduces some notations and facts on tensor products of Banach spaces. For details and proofs we refer to [6] and [34].

Let X and Y be arbitrary Banach spaces. For $n \in \mathbb{N}, (x_i)_{i=1}^n \subset X, (y_i)_{i=1}^n \subset Y$, the formal expression

$$\sum_{i=1}^n x_i \otimes y_i$$

can be identified with an operator $\Phi : X^* \to Y$, given by

$$\Phi(\varphi) := \sum_{i=1}^n \varphi(x_i) y_i \qquad (\varphi \in X^*).$$

For $(u_i)_{i=1}^n \subset X, (v_i)_{i=1}^n \subset Y$, we introduce an equivalence relation

$$\sum_{i=1}^n x_i \otimes y_i \sim \sum_{i=1}^n u_i \otimes v_i$$

if both expressions define the same operator from X^* to Y. The algebraic tensor product $X \otimes Y$ is then defined to be the set of all such equivalence classes. We abuse notation by referring to the expression $\sum_{i=1}^n x_i \otimes y_i$ as a member of $X \otimes Y$ when we intend to refer to the equivalence class of expressions containing $\sum_{i=1}^n x_i \otimes y_i$. For $\lambda \in \mathbb{K}$, we define multiplication of scalars as

$$\lambda \sum_{i=1}^n x_i \otimes y_i = \sum_{i=1}^n \lambda x_i \otimes y_i = \sum_{i=1}^n \lambda x_i \otimes y_i$$

and addition as

$$\sum_{i=1}^n x_i \otimes y_i + \sum_{i=n+1}^m x_i \otimes y_i = \sum_{i=1}^m x_i \otimes y_i.$$

By definition, it is easily checked that for $x, x_1, x_2 \in X, y, y_1, y_2 \in Y$, and $\lambda \in \mathbb{K}$

$$\begin{aligned} x \otimes (y_1 + y_2) &= x \otimes y_1 + x \otimes y_2, \\ (x_1 + x_2) \otimes y &= x_1 \otimes y + x_2 \otimes y, \\ (\lambda x) \otimes y &= x \otimes (\lambda y). \end{aligned}$$

For $z = \sum_{i=1}^n x_i \otimes y_i \in X \otimes Y$, the injective tensor norm is defined as

$$\lambda(z) = \sup_{u \in B_{X^*}, v \in B_{Y^*}} \Big| \sum_{i=1}^n \langle x_i, u \rangle \langle y_i, v \rangle \Big|.$$

The injective tensor product $X \otimes_\lambda Y$ is defined as the completion of $X \otimes Y$ with respect to the norm λ. We will use the canonical isometric identification

$$C(Q, Z) = Z \otimes_\lambda C(Q), \tag{2.3}$$

which is valid for arbitrary Banach spaces Z (see [34], Theorem 1.13). In particular, for $d > 1$, we obtain

$$C([0,1]^d) = C([0,1]) \otimes_\lambda C([0,1]^{d-1}) = C([0,1]) \otimes_\lambda \cdots \otimes_\lambda C([0,1]). \tag{2.4}$$

Given Banach spaces X, Y, X_1, Y_1 and operators $T_1 \in \mathscr{L}(X, X_1)$, $T_2 \in \mathscr{L}(Y, Y_1)$, the algebraic tensor product $T_1 \otimes T_2 : X \otimes Y \to X_1 \otimes Y_1$ is defined as

$$(T_1 \otimes T_2) \sum_{i=1}^n x_i \otimes y_i = \sum_{i=1}^n T_1 x_i \otimes T_2 y_i,$$

where $(T_1 \otimes T_2)(z)$ is independent of the representation of z. We see that $(T_1 \otimes T_2)$ is bilinear and for further Banach spaces X_2, Y_2, and operators $S_1 \in \mathscr{L}(X_1, X_2)$, $S_2 \in \mathscr{L}(Y_1, Y_2)$

$$(S_1 \otimes S_2) \circ (T_1 \otimes T_2) = (S_1 \circ T_1) \otimes (S_2 \circ T_2).$$

Moreover, $(T_1 \otimes T_2)$ extends to a bounded linear operator (we use the same symbol for the extension)

$$T_1 \otimes T_2 \in \mathscr{L}(X \otimes_\lambda Y, X_1 \otimes_\lambda Y_1)$$

with

$$\|T_1 \otimes T_2\|_{\mathscr{L}(X \otimes_\lambda Y, X_1 \otimes_\lambda Y_1)} = \|T_1\|_{\mathscr{L}(X,X_1)} \|T_2\|_{\mathscr{L}(Y,Y_1)}. \tag{2.5}$$

2.3.3 Lagrange Interpolation

For $r, m \in \mathbb{N}_0$, let $P_m^{r,1,\mathbb{K}} \in \mathscr{L}(C([0,1]))$ be the operator of composite Lagrange interpolation of degree r, with respect to the partition of $[0,1]$ into m intervals of length m^{-1}. Let

$$P_m^{r,d,\mathbb{K}} = \otimes^d P_m^{r,1,\mathbb{K}} \in \mathscr{L}(C([0,1]^d)) \tag{2.6}$$

be its d-dimensional version. Setting $\Gamma_m^{r,d} = \left\{\frac{i}{rm} : 0 \le i \le rm\right\}^d$ for $r, m \in \mathbb{N}_0$, it follows that $P_m^{r,d,\mathbb{K}}$ interpolates on $\Gamma_m^{r,d}$. Given a Banach space Z, the Z-valued versions of the operators above are defined in the sense of identification (2.3) as

$$P_m^{r,d,Z} = I_Z \otimes P_m^{r,d,\mathbb{K}}. \tag{2.7}$$

This means, if $P_m^{r,d,\mathbb{K}}$ is represented as

$$P_m^{r,d,\mathbb{K}} f = \sum_{s \in \Gamma_m^{r,d}} f(s)\varphi_s \quad (f \in C(Q))$$

for some $\varphi_s \in C(Q)$, then $P_m^{r,d,Z}$ has the representation

$$P_m^{r,d,Z} f = \sum_{s \in \Gamma_m^{r,d}} f(s)\varphi_s \quad (f \in C(Q,Z)).$$

We can obviously consider $P_m^{r,d}$ also as an operator from $\ell_\infty(\Gamma_m^{r,d}, Z)$ to $C(Q,Z)$. Notice, if there is no ambiguity, we omit the superscript Z.

The next lemma settles the approximation error of the Lagrangian interpolation operators for Banach space valued functions.

Lemma 2.3.3. *Given $r \in \mathbb{N}_0$ and $d \in \mathbb{N}$, there are constants $c_1, c_2 > 0$ such that for all Banach spaces Z and all $m \in \mathbb{N}_0$*

$$\|P_m^{r,d}\|_{\mathscr{L}(C(Q,Z))} \le c_1, \tag{2.8}$$

$$\|J - P_m^{r,d} J\|_{\mathscr{L}(C^r(Q,Z),C(Q,Z))} \le c_2 m^{-r}, \tag{2.9}$$

where $J : C^r(Q,Z) \to C(Q,Z)$ is the canonical embedding.

Proof. The statement is well-known in the scalar case and can easily be extended to the Banach space case in the following way. Denote by $J_{\mathbb{K}} : C^r(Q) \to C(Q)$ the respective scalar embedding. Then

$$\begin{aligned} \|P_m^{r,d} f\|_{C(Q,Z)} &= \sup_{z^* \in B_{Z^*}} \| \left(P_m^{r,d,\mathbb{K}} f, z^* \right) \|_{C(Q)} \\ &= \sup_{z^* \in B_{Z^*}} \left\| P_m^{r,d,\mathbb{K}} (f, z^*) \right\|_{C(Q)} \\ &\le c_1 \sup_{z^* \in B_{Z^*}} \| (f, z^*) \|_{C(Q)} \\ &= c_1 \|f\|_{C(Q,Z)}, \end{aligned} \tag{2.10}$$

and similarly,

$$\begin{aligned} \|(J - P_m^{r,d} J) f\|_{C(Q,Z)} &= \sup_{z^* \in B_{Z^*}} \left\| (J_{\mathbb{K}} - P_m^{r,d,\mathbb{K}} J_{\mathbb{K}}) (f, z^*) \right\|_{C(Q)} \\ &\le c_2 m^{-r} \sup_{z^* \in B_{Z^*}} \| (f, z^*) \|_{C^r(Q)} \\ &= c_2 m^{-r} \max_{\alpha \in \mathbb{N}_0^d, |\alpha| \le r} \sup_{z^* \in B_{Z^*}} \left\| \left(\frac{\partial^{|\alpha|} f}{\partial s^\alpha}, z^* \right) \right\|_{C(Q)} \\ &= c_2 m^{-r} \|f\|_{C^r(Q,Z)}. \end{aligned} \tag{2.11}$$

□

2.3.4 Smolyak Interpolation

Except for the definite integration problem, we use multilevel algorithms to derive upper bounds for all remaining problems. The idea of these algorithms goes back to Smolyak in [41]. He proposed a certain tensor product structure, which we use here to illustrate the underlying idea of these methods. For $n \in \mathbb{N}_0$, the composite Lagrangian tensor product operator $P_n^{r,d}$ samples on the regular grid

$$\Gamma_n^{r,d} = \underbrace{\Gamma_n^{r,1} \times \cdots \times \Gamma_n^{r,1}}_{d \text{ times}},$$

see Figure 2.1. Altogether, cn^d function evaluations are needed for the interpolation. Thus, the number of sample points grows exponentially in d.

The idea of Smolyak's method is to use interpolation operators on several levels. On each level, the number of sample points is fixed. The structure of the grid, thereby, changes for every level. For $L, m \in \mathbb{N}_0, m \ge 2$, every level has the grid structure

$$\Gamma_{m^{l_1}}^{r,d} \times \cdots \times \Gamma_{m^{l_d}}^{r,d}, \tag{2.12}$$

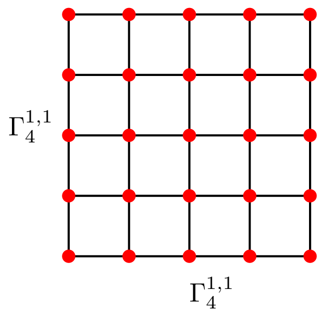

Figure 2.1: Regular grid for $m = 4$,$r = 1$, and $d = 2$.

where $l = (l_1, \ldots, l_d) \in \mathbb{N}_0^d$ and $|l| = l_1 + \cdots + l_d = L$. Often, dyadic settings are chosen, i.e., $m = 2$. By construction, every grid has only cm^L sample points and these are balanced in such a way that the highest precision is obtained at most for one single dimension, see Figure 2.2.

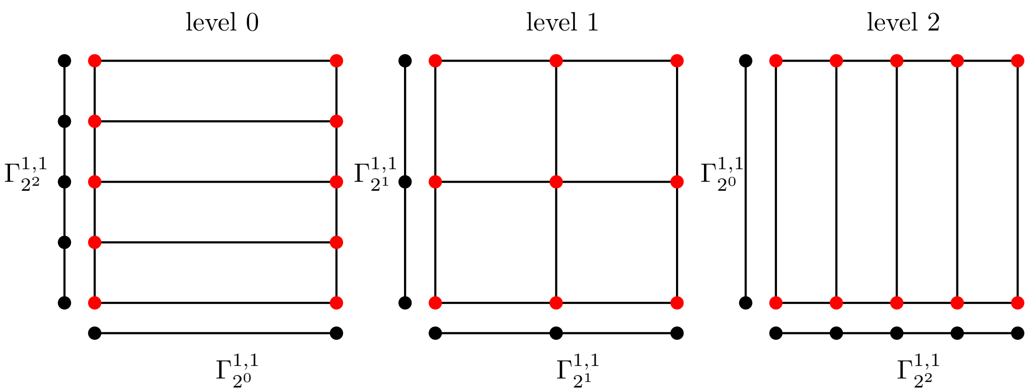

Figure 2.2: Grids of different levels for $L = d = m = 2$ and $r = 1$.

Smolyak's method connects each level via difference operators. We obtain the following structure for the Smolyak interpolation operator

$$V_L = \sum_{l \in \mathbb{N}, |l| = L}^{L} (P_{m^{l_1}}^{r,d} - P_{m^{l_1 - 1}}^{r,d}) \otimes \cdots \otimes (P_{m^{l_{d-1}}}^{r,d} - P_{m^{l_{d-1}-1}}^{r,d}) \otimes P_{m^{l_d}}^{1,1}, \tag{2.13}$$

where $P_{m^{-1}}^{r,d} \equiv 0$. Due to

$$\Gamma_{m^{l-1}}^{r,d} \subset \Gamma_{m^l}^{r,d} \qquad (l \in \mathbb{N}),$$

it is easily seen that every element of the sum is an interpolation operator which samples on a grid of the form (2.12). Moreover, due to the summation, the entire interpolation operator samples on a sparse grid, see Figure 2.3. If we choose $L = \lceil \log_m n \rceil$, Smolyak's method only needs $cn(\log n)^{d-1}$ sample points. Of course, this construction works well for special classes of functions only.

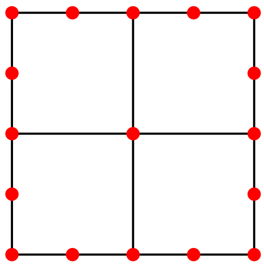

Figure 2.3: Sparse grid for $L = d = m = 2$ and $r = 1$.

For $r \in \mathbb{N}_0, d \in \mathbb{N}$, let

$$C^{\overbrace{r,\ldots,r}^{d}}(Q, Z)$$

be the space of functions $f : Q \to Z$ having continuous partial derivative

$$\frac{\partial^{|\alpha|} f}{\partial t^\alpha}$$

for all $\alpha \in \mathbb{N}_0^d$ with $|\alpha| \leq dr$ and $\alpha_i \leq r \quad (i = 1, \ldots, d)$. The norm of $C^{r,\ldots,r}(Q, Z)$ is defined as

$$||f||_{C^{r,\ldots,r}(Q,Z)} = \max_{\substack{\alpha \in \mathbb{N}_0^d \\ \alpha_i \leq r \ (i=1,\ldots,d)}} \sup_{t \in Q} \left\| \frac{\partial^{|\alpha|} f(t)}{\partial t^\alpha} \right\|.$$

This type of smoothness is an example for dominating mixed smoothness. On the other hand, the class $C^r(Q, Z)$, mentioned before, is an example of a class with isotropic smoothness.

Lemma 2.3.4. *Given $r \in \mathbb{N}_0$ and $d \in \mathbb{N}$, there is a constant $c > 0$ such that for all Banach spaces Z and all $L, m \in \mathbb{N}_0, m \geq 2$*

$$\|J - V_L J\|_{\mathscr{L}(C^{r,\ldots,r}(Q,Z),C(Q,Z))} \;\leq\; c(L+1)^{(d-1)} m^{-rL}.$$

Proof. The result is only given for clarification, thus we do not prove it. It is well-known in the scalar case and carries over to the Banach space valued case in a similar way as before. See also Lemma 4.2.4 and (4.30), where a comparable setting is considered. □

Setting $L = \lceil \log_m n \rceil$, Smolyak's algorithm has an error $\leq c(\log n)^{d-1} n^{-r}$ using only $cn(\log n)^{d-1}$ sample points. But the considered input class is of course much smaller than $C^r(Q, Z)$.

Notice that the multilevel algorithms that will be used for the approximation of subsequent parameter dependent problems also utilizes this balancing of precisions in a similar way. But instead of Lagrangian operators on the right hand side of (2.13), we use Banach space valued randomized algorithms and the error estimates get more involved.

2.3.5 Probability Theory in Banach Spaces

Definition 2.3.5. Let $1 \leq p \leq 2$. A Banach space Z is said to be of (Rademacher) type p if there is a constant $c > 0$ such that for all $n \in \mathbb{N}$ and $z_1, \ldots, z_n \in Z$

$$\mathbb{E} \left\| \sum_{i=1}^{n} \varepsilon_i z_i \right\|^p \leq c^p \sum_{k=1}^{n} \|z_i\|^p, \tag{2.14}$$

where $(\varepsilon_i)_{i=1}^n$ is a sequence of independent Bernoulli random variables with $\mathbb{P}\{\varepsilon_i = -1\} = \mathbb{P}\{\varepsilon_i = +1\} = 1/2$. The smallest constant satisfying (2.14) is called the type p constant of Z and is denoted by $\tau_p(Z)$. If there is no such $c > 0$, we set $\tau_p(Z) = \infty$. We also refer to [37, 33] for related facts.

Remark 2.3.6. As an example, we choose $L_p(\mathcal{N}, \nu)$, where $(\mathcal{N}, \nu)$ is an arbitrary measure space and $1 \leq p < \infty$. For $2 \leq p < \infty$, the L_p spaces satisfy the type 2 property. For $1 \leq p < 2$, they only satisfy the type p property. Certain subspaces such as the Sobolev spaces also satisfy this properties. Moreover, due to orthogonality, all Hilbert spaces H satisfy $\tau_2(H) = 1$. However, if $p = \infty$, we have $\tau_2(L_p(Q)) = \infty$, and also $\tau_2(C^r(Q)) = \infty$ for all $r \in \mathbb{N}_0$.

We need the following results for the analysis of Banach space valued algorithms.

Lemma 2.3.7 (equivalence of moments)**.** *Let Z be an arbitrary Banach space. Then for any $0 < p, q < \infty$, there is a constant $K_{p,q}$ depending on p, q only such that*

$$\left(\mathbb{E} \left\| \sum_{i \in \mathbb{N}} \varepsilon_i z_i \right\|^p \right)^{\frac{1}{p}} \leq K_{p,q} \left(\mathbb{E} \left\| \sum_{i \in \mathbb{N}} \varepsilon_i z_i \right\|^q \right)^{\frac{1}{q}},$$

where $(\varepsilon_i)_{i \in \mathbb{N}}$ is a sequence of independent Bernoulli random variables with

$$\mathbb{P}\{\varepsilon_i = -1\} = \mathbb{P}\{\varepsilon_i = +1\} = \frac{1}{2},$$

and $(z_i)_{i \in \mathbb{N}} \subseteq Z$. Furthermore, there is a constant $c > 0$ such that

$$K_{p,2} = c\sqrt{p} \qquad (p \geq 2).$$

Proof. See, e.g., [33], p. 100. □

Lemma 2.3.8. *There are constants $c_1, c_2 > 0$ such that for all $1 \le p \le 2$, $p \le q < \infty$, for all $n \in \mathbb{N}$, for any Banach space Z and measure space $(\mathcal{N}, \nu)$ the following holds:*

$$\begin{aligned} \tau_p(L_q(\mathcal{N}, \nu, Z)) &\le c_1 \sqrt{q}\, \tau_p(Z), && (2.15)\\ \tau_p(\ell_\infty^n(Z)) &\le c_2 \sqrt{\log(n+1)}\, \tau_p(Z). && (2.16) \end{aligned}$$

Proof. We start with (2.15). Let $(\vartheta_i)_{i=1}^m \subset L_q(\mathcal{N}, \nu, Z)$. Then we obtain

$$\begin{aligned} \left(\mathbb{E} \Big\| \sum_{i=1}^m \varepsilon_i \vartheta_i \Big\|_{L_q(\mathcal{N},\nu,Z)}^p \right)^{\frac{q}{p}} &\le \mathbb{E} \Big\| \sum_{i=1}^m \varepsilon_i \vartheta_i \Big\|_{L_q(\mathcal{N},\nu,Z)}^q \\ &= \int_{\mathcal{N}} \mathbb{E} \Big\| \sum_{i=1}^m \varepsilon_i \vartheta_i(t) \Big\|_Z^q d\nu(t), \qquad (2.17) \end{aligned}$$

where $(\varepsilon_i)_{i=0}^m$ is a sequence of independent centered Bernoulli random variables. We apply Lemma 2.3.7 and (2.14) to obtain

$$\begin{aligned} &\int_{\mathcal{N}} \mathbb{E} \Big\| \sum_{i=1}^m \varepsilon_i \vartheta_i(t) \Big\|_Z^q d\nu(t) && (2.18)\\ &\le (c\sqrt{q})^q \int_{\mathcal{N}} \left(\mathbb{E} \Big\| \sum_{i=1}^m \varepsilon_i \vartheta_i(t) \Big\|_Z^p \right)^{\frac{q}{p}} d\nu(t) && (2.19)\\ &\le (c\sqrt{q}\, \tau_p(Z))^q \int_{\mathcal{N}} \left(\sum_{i=1}^m \|\vartheta_i(t)\|_Z^p \right)^{\frac{q}{p}} d\nu(t) && (2.20) \end{aligned}$$

with a constant $c > 0$ independent of p and q. Using the triangle inequality in $L_{q/p}(\mathcal{N}, \nu)$, we get

$$\begin{aligned} \int_{\mathcal{N}} \left(\sum_{i=1}^m \|\vartheta_i(t)\|_Z^p \right)^{\frac{q}{p}} d\nu(t) &\le \left(\sum_{i=1}^m \left(\int_{\mathcal{N}} \|\vartheta_i(t)\|_Z^q d\nu(t) \right)^{\frac{p}{q}} \right)^{\frac{q}{p}} \\ &= \left(\sum_{i=1}^m \|\vartheta_i\|_{L_q(\mathcal{N},\nu,Z)}^p \right)^{\frac{q}{p}}. \qquad (2.21) \end{aligned}$$

Joining (2.17), (2.20), and (2.21) yields (2.15). To show (2.16), we observe that the identity map $I_Z : \ell_q^n(Z) \to \ell_\infty^n(Z)$ satisfies

$$\|I_Z\| = 1, \quad \|I_Z^{-1}\| = n^{\frac{1}{q}}. \qquad (2.22)$$

If $n \ge 4$, we set $q = \log n$, so $q \ge 2 \ge p$ and $n^{1/q} \le 2$. For $n < 4$, we put $q = 2$. Now, (2.16) follows from (2.15) and (2.22). □

Taking account of the type of a certain Banach space, we will need the following result for the upper bound proof of the Banach space valued multivariate integration problem.

Lemma 2.3.9. *Let $1 \leq p \leq 2$, Z be a Banach space, $n \in \mathbb{N}$, let $(\vartheta)_{i=1}^n$ be a sequence of independent Z-valued random variables with $\mathbb{E}||\vartheta_i||^p < \infty$, and $\mathbb{E}\vartheta_i = 0$ for all $i = 1, \ldots, n$. Then*

$$\left(\mathbb{E}\left\|\sum_{i=1}^n \vartheta_i\right\|^p\right)^{\frac{1}{p}} \leq 2\tau_p(Z)\left(\sum_{k=1}^n \mathbb{E}\left\|\vartheta_i\right\|^p\right)^{\frac{1}{p}}. \tag{2.23}$$

Proof. The proof can be found in [33], Proposition 9.11. □

The next statements will be useful to prove lower bounds for Banach space valued problems.

Theorem 2.3.10 (Contraction principle)**.** *Let $g : [0, \infty) \to [0, \infty)$ be convex and Z be an arbitrary Banach space. For every $n \in \mathbb{N}$, any sequence $(z_i)_{i=1}^n \subset Z$ and any sequence $(\alpha_i)_{i=1}^n \subset B_{\mathbb{R}}$, we have*

$$\mathbb{E}g\left(\left|\left|\sum_{i=1}^n \alpha_i\varepsilon_i x_i\right|\right|\right) \leq \mathbb{E}g\left(\left|\left|\sum_{i=1}^n \varepsilon_i x_i\right|\right|\right),$$

where $(\varepsilon_i)_{i=1}^n$ is again a sequence of independent Bernoulli random variables.

Proof. See [33], Theorem 4.4. □

Definition 2.3.11. Let $1 \leq p \leq 2, n \in \mathbb{N}$ and $\varepsilon > 0$. A Banach space Z is said to contain a subspace which is $(1+\varepsilon)-$isomorphic to ℓ_p^n if there exist $z_1, \ldots, z_n \in Z$ such that for all $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$

$$\left(\sum_{i=1}^n |\alpha_i|^p\right)^{\frac{1}{p}} \leq \left|\left|\sum_{i=1}^n \alpha_i z_i\right|\right| \leq (1+\varepsilon)\left(\sum_{i=1}^n |\alpha_i|^p\right)^{\frac{1}{p}}.$$

Moreover, Z contains almost isometric copies of ℓ_p^n if it contains subspaces $(1+\varepsilon)$-isomorphic to ℓ_p^n for all $n \in \mathbb{N}$ and $\varepsilon > 0$.

Theorem 2.3.12 (Maurey-Pisier)**.** *For a Banach space Z, let p_Z denote the supremum of all $1 \leq p_1 \leq 2$ such that Z is of type p_1. The set of all $1 \leq p \leq 2$ for which an infinite dimensional Banach space Z contains almost isometric copies of ℓ_p^n is equal to $[p_Z, 2]$.*

Proof. See [33], Chap. 9 or [37], Th. 2.3. □

Chapter 3

The Thesis in a Nutshell

The thesis deals with the complexity analysis of certain integration problems, where complexity is meant in the sense of IBC as introduced in the previous chapter. As a main interest, we want to understand the typical behavior of the complexity in relation of the deterministic and randomized setting. In this manner, we clarify in which cases and to which extent randomized methods are superior to deterministic ones and compare the corresponding optimal rates.

In the present chapter, we summarize the new results of the thesis and discuss previous results on related problems. Sometimes, only simplified versions of subsequent statements are given. Moreover, since certain definitions of later chapters are needed as well, some definitions may occur twice in the thesis.

3.1 Banach Space Valued Integration

The complexity of Banach space valued multivariate integration is studied in Chapter 4.

Problem formulation. Let $r \in \mathbb{N}_0, d \in \mathbb{N}$, and $Q = [0,1]^d$. The definite integration operator $\mathcal{S}_0 : C(Q,Z) \to Z$ is defined by

$$\mathcal{S}_0 f = \int_Q f(t)dt \tag{3.1}$$

in the sense of a Bochner integral. The Bochner integral is a natural generalization of the classical Lebesgue integral for functions $f : Q \to Z$ with values in a Banach space. The corresponding rules of calculation are similar to these obtained for the scalar valued case. As a reference, see [1] or [47].

The indefinite integration operator $\mathcal{S}_1 : C(Q,Z) \to C(Q,Z)$ is given by

$$(\mathcal{S}_1 f)(t) = \int_{[0,t]} f(u)du \qquad (t \in Q), \tag{3.2}$$

where $[0,t] = \prod_{i=1}^{d}[0,t_i]$ for $t = (t_i)_{i=1}^{d} \in Q$. Thus, $\mathcal{S}_1 f$ is again a function, the anti-derivative of f.

To define the abstract numerical problem, we set $F = B_{C^r(Q,Z)}$, $G = Z$, $K = Z$, and

$$\Lambda = \Lambda(Q,Z) = \{\delta_t : \ t \in Q\},$$

for the definite integration problem, where $\delta_t(f) = f(t)$ for $f \in F$. So we consider Z-valued information functionals and the definite integration problem is described by

$$\mathcal{P}_0 = (B_{C^r(Q,Z)}, Z, \mathcal{S}_0, Z, \Lambda(Q,Z)).$$

Moreover, we take $G = C(Q,Z)$ for the indefinite integration problem, while F,K, and Λ are the same as above. Thus, the indefinite integration problem is defined by

$$\mathcal{P}_1 = (B_{C^r(Q,Z)}, C(Q,Z), \mathcal{S}_1, Z, \Lambda(Q,Z)).$$

Previous results. While the complexity of these Banach space valued problems have not been studied so far, the scalar valued integration problems have been studied very well. In the deterministic setting, it is well-known that

$$n^{-\frac{r}{d}} \preceq e_n^{\mathrm{det}}(\mathcal{S}_\iota, B_{C^r(Q)}) \preceq n^{-\frac{r}{d}} \qquad (\iota \in \{0,1\}). \tag{3.3}$$

Moreover, if we allow randomized algorithms, it is known due to [2] that

$$n^{-\frac{r}{d}-\frac{1}{2}} \preceq e_n^{\mathrm{ran}}(\mathcal{S}_0, B_{C^r(Q)}) \preceq n^{-\frac{r}{d}-\frac{1}{2}} \tag{3.4}$$

in the definite case. The right hand inequality is, for instance, satisfied by Monte Carlo integration with separation of the main part. Finding an order optimal algorithm for the indefinite problem remained open for a long time. However, it was shown recently in [24] that the same order as for the definite case can be reached. Thus,

$$n^{-\frac{r}{d}-\frac{1}{2}} \preceq e_n^{\mathrm{ran}}(\mathcal{S}_1, B_{C^r(Q)}) \preceq n^{-\frac{r}{d}-\frac{1}{2}}. \tag{3.5}$$

New results. Chapter 4 contains a generalization of the previous results to Banach space valued integrands. In the deterministic setting, we obtain the same rates as above, i.e.,

$$n^{-\frac{r}{d}} \preceq e_n^{\mathrm{det}}(\mathcal{S}_\iota, B_{C^r(Q,Z)}) \preceq n^{-\frac{r}{d}} \qquad (\iota \in \{0,1\}).$$

In the randomized setting, the complexity depends on the type of the underlying Banach space. Let p_Z be the supremum of all p_1 such that Z is of type p_1. Then we obtain, if Z is of type p,

$$n^{-\frac{r}{d}-1+\frac{1}{p_Z}} \preceq e_n^{\mathrm{ran}}(\mathcal{S}_\iota, B_{C^r(Q,Z)}) \preceq \tau_p(Z) n^{-\frac{r}{d}-1+\frac{1}{p}} \qquad (\iota \in \{0,1\}).$$

Since $1 \le p \le p_Z \le 2$, it follows that the gain, we achieve due to randomization, is between 0 and 1/2. Furthermore, there remains an arbitrarily small gap in the exponent and the gap is closed if $p_Z = p$, which holds, for instance, for type 2 Banach spaces and the L_p spaces with $1 \le p < \infty$.

Notice that the upper bounds are achieved by providing and analyzing randomized algorithms. In the definite case, the algorithm is a generalization of the Monte Carlo method with separation of the main part. In the indefinite case, we use a general version of the Smolyak multilevel Monte Carlo method from [24]. The deterministic algorithms are obtained by fixing any random parameter $\omega \in \Omega$. This way, we prove that each realization of these randomized algorithms satisfies at least the corresponding optimal deterministic order. Moreover, the result above yields that for Banach spaces, satisfying $p_Z = p$, the considered algorithms are order optimal in the randomized setting. For general Banach spaces these algorithms are order optimal up to an arbitrarily small gap in the exponent.

3.2 Parametric Banach Space Valued Integration

Parametric integration is a problem intermediate between integration and approximation. It is known that randomized algorithms are superior to deterministic algorithms if pure integration problems are considered. On the other hand, for certain approximation problems we know that randomization does not improve the complexity, see [42] and [39]. We will investigate the behavior of this problem in Chapter 5.

Problem formulation. For $d_0 \in \mathbb{N}$, let $Q_0 = [0,1]^{d_0}$ and for $r_0, r \in \mathbb{N}_0$, let $C^{r_0,r}(Q_0 \times Q, Z)$ be the space of continuous functions $f : Q_0 \times Q \to Z$ having for $\alpha = (\alpha_0, \alpha_1)$, $\alpha_0 \in \mathbb{N}_0^{d_0}$, $\alpha_1 \in \mathbb{N}_0^d$ with $|\alpha_0| \le r_0$, $|\alpha_1| \le r$ continuous partial derivatives $\frac{\partial^{|\alpha|} f(s,t)}{\partial s^{\alpha_0} \partial t^{\alpha_1}}$, endowed with the norm

$$\|f\|_{C^{r_0,r}(Q_0 \times Q, Z)} = \max_{|\alpha_0| \le r_0, |\alpha_1| \le r} \sup_{s \in Q_0, t \in Q} \left\| \frac{\partial^{|\alpha|} f(s,t)}{\partial s^{\alpha_0} \partial t^{\alpha_1}} \right\|.$$

For $r_1 \in \mathbb{N}_0$, we then define the input set by

$$\mathscr{F}_0 = B_{C^{0,r}(Q_0 \times Q, Z)} \cap B_{C^{r_0,r_1}(Q_0 \times Q, Z)}. \tag{3.6}$$

The definite parametric integration operator $\mathscr{S}_0 : C(Q_0 \times Q, Z) \to C(Q_0, Z)$ is given by

$$(\mathscr{S}_0 f)(s) = \int_Q f(s,t) dt \quad (s \in Q_0),$$

and the indefinite parametric integration operator $\mathscr{S}_1 : C(Q_0 \times Q, Z) \to C(Q_0 \times Q, Z)$ by

$$(\mathscr{S}_1 f)(s,t) = \int_{[0,t]} f(s,u)du \quad (s \in Q_0, t \in Q).$$

Remark 3.2.1. For $r < r_1$, we obtain

$$\begin{aligned} B_{C^{0,r}(Q_0\times Q,Z)} \cap B_{C^{r_0,r_1}(Q_0\times Q,Z)} &= B_{C^{r_0,r_1}(Q_0\times Q,Z)} \\ &= B_{C^{0,r_1}(Q_0\times Q,Z)} \cap B_{C^{r_0,r_1}(Q_0\times Q,Z)}. \end{aligned} \tag{3.7}$$

Hence, we can assume $r \geq r_1$ without loss of generality.

The admissible information in consideration is standard information, thus the class of information functionals is defined by

$$\Lambda(Q_0 \times Q, Z) = \{\delta_{s,t} : \ s \in Q_0, t \in Q\},$$

where $\delta_{s,t}(f) = f(s,t)$ and $K = Z$. In terminology of Section 2.2, the definite parametric integration problem is now described by the tuple

$$\Pi_0 = (B_{C^{0,r}(Q_0\times Q,Z)} \cap B_{C^{r_0,r_1}(Q_0\times Q,Z)}, C(Q_0,Z), \mathscr{S}_0, Z, \Lambda(Q_0 \times Q, Z)),$$

and the indefinite parametric integration problem by

$$\Pi_1 = (B_{C^{0,r}(Q_0\times Q,Z)} \cap B_{C^{r_0,r_1}(Q_0\times Q,Z)}, C(Q_0 \times Q,Z), \mathscr{S}_1, Z, \Lambda(Q_0 \times Q, Z)).$$

Previous results. The complexity of scalar valued definite parametric integration was first studied in [25],[18] and later for the case of quantum algorithms in [45]. It is known that

$$n^{-\frac{r}{d_0+d}} \preceq e_n^{\det}(\mathscr{S}_0, B_{C^r([0,1]^{d_0}\times[0,1]^d)}) \preceq n^{-\frac{r}{d_0+d}}.$$

Moreover, due to [25], we know for the randomized setting that

$$e_n^{\mathrm{ran}}(\mathscr{S}_0, B_{C^r([0,1]^{d_0}\times[0,1]^d)}) \asymp \begin{cases} n^{-\frac{2r+d}{2(d_0+d)}}(\log_2 n)^{\frac{1}{2}} & \text{if } r > \frac{d_0}{2} \\ n^{\frac{r}{d_0}}(\log_2 n)^{\frac{r}{d_0}} & \text{if } r < \frac{d_0}{2} \end{cases}$$

and

$$n^{\frac{1}{2}}(\log_2 n)^{\frac{1}{2}} \preceq e_n^{\mathrm{ran}}(\mathscr{S}_0, B_{C^r([0,1]^{d_0}\times[0,1]^d)}) \preceq n^{\frac{1}{2}}(\log_2 n)^{\frac{3}{2}} \quad \text{if } r = \frac{d_0}{2}.$$

New results. The indefinite problem has not been considered so far. In Chapter 5, we give an exhaustive investigation of the Banach space valued setting defined above. This means the following: The randomized setting as well as the deterministic setting for both, definite and indefinite parametric Banach space valued integration is studied. In contrast to previous investigations, we consider the more general input class $\mathscr{F}_0$, which includes dominating mixed smoothness and other types of non-isotropic smoothness. These classes allow us to treat different types of smoothness for the parameter dependence and for the basic (non-parametric) integration problem, separately.

In the deterministic setting, we obtain

$$n^{-\upsilon_2(1)} \preceq e_n^{\det}(\mathscr{S}_\iota, \mathscr{F}_0) \preceq_{\log} n^{-\upsilon_2(1)} \qquad (\iota \in \{0,1\}),$$

where for $1 \le p \le 2$

$$\upsilon_2(p) \quad = \quad \begin{cases} \frac{\frac{r_0}{d_0}}{\frac{r_0}{d_0}+\frac{r}{d}-\frac{r_1}{d}} \left(\frac{r}{d}+1-\frac{1}{p}\right) & \text{if} \quad \frac{r_0}{d_0} > \frac{r_1}{d}+1-\frac{1}{p} \\ \frac{r_0}{d_0} & \text{if} \quad \frac{r_0}{d_0} \le \frac{r_1}{d}+1-\frac{1}{p} \end{cases}.$$

Notice that logarithmic factors are neglected in the whole chapter for the sake of simplicity. For certain cases, the corresponding estimates in Chapter 5 are even sharp up to logarithmic factors.

In the randomized setting the situation is similar as for the non-parametric integration problem. This means that the bounds depend on the type of the Banach space in a similar way, but the rates are different. If Z is a type p Banach space, we obtain

$$n^{-\upsilon_2(p_Z)} \preceq e_n^{\mathrm{ran}}(\mathscr{S}_\iota, \mathscr{F}_0) \preceq_{\log} n^{-\upsilon_2(p)} \qquad (\iota \in \{0,1\}).$$

In case of $p_Z = p$, we therefore get matching upper and lower bounds. However, for general Banach spaces, an arbitrarily small gap in the exponent remains again.

Remark 3.2.2. The upper bounds are reached by certain multilevel Monte Carlo algorithms for the definite and the indefinite case, respectively. The corresponding deterministic algorithms are obtained by fixing the random parameter. We present two versions of these algorithms. In the first version, the algorithm parameters depend on the problem setting, which provides sharp bounds even up to logarithmic factors. For the second version, we use the same choice of parameters for the deterministic and the randomized setting. In this case, additional logarithmic factors occur, but the algorithms are still order optimal, and by construction, every realization of these randomized algorithms satisfies the respective optimal deterministic error.

At the end of Chapter 5, we present some applications of the general results to various smoothness classes. As an example, let $r_1 = r$, then $\mathscr{F}_0 = B_{C^{r_0,r}(Q_0 \times Q, Z)}$;

this is a class of dominating mixed smoothness. For type 2 Banach spaces, which includes the case $Z = \mathbb{R}^d$, it follows that

$$\begin{aligned} e_n^{\mathrm{det}}(\mathscr{S}_\iota, \mathscr{F}_0) &\asymp_{\log} n^{-\min\left(\frac{r}{d}, \frac{r_0}{d_0}\right)}, \\ e_n^{\mathrm{ran}}(\mathscr{S}_\iota, \mathscr{F}_0) &\asymp_{\log} n^{-\min\left(\frac{r}{d}+\frac{1}{2}, \frac{r_0}{d_0}\right)}. \end{aligned}$$

If $\frac{r_0}{d_0}$ is large enough, r/d and $r/d + 1/2$ are the limiting factors and we reach the same orders as for (non-parametric) integration of functions from $C^r(Q, Z)$. This means that calculating the integral for all $s \in Q$ at once has almost the same cost as calculating just one single integral. This benefit is achieved due to the multilevel structure and the smoothness of the underlying class.

3.3 Parametric Banach Space Valued Initial Value Problems

The third problem in consideration is the approximation of parametric initial value problems with values in an arbitrary Banach space. The problem is related to the problem of parametric indefinite integration, which will be motivated after a formal formulation of the problem.

Problem formulation. Let Z be an arbitrary Banach space, $-\infty < a < b < \infty$, and $Q_0 = [0, 1]^{d_0}$. Then the Z valued parametric initial value problem depending on a parameter $s \in Q_0$ is defined by

$$\frac{d}{dt}u(s,t) = f(s,t,u(s,t)) \quad (s \in Q_0, t \in [a,b]), \tag{3.8}$$

$$u(s,a) = u_0(s) \quad (s \in Q_0), \tag{3.9}$$

where $f \in C(Q_0 \times [a,b] \times Z, Z)$ and $u_0 \in C(Q_0, Z)$. A function $u : Q_0 \times [a,b] \to Z$ is called a solution if for each $s \in Q_0$, $u(s,t)$ is continuously differentiable as a function of t and (3.8-3.9) is satisfied. As mentioned above, the problem is related to the problem of indefinite parametric integration; choosing f independent with respect to the third variable and $u_0(s) \equiv 0, a = 0, b = 1$, we obtain

$$u(s,t) = \int_0^t f(s,\tau)\, d\tau \qquad (s \in Q_0, t \in [0,1]).$$

Thus, the parametric indefinite integration problem with $d = 1$ can be understood as a particular case of an initial value problem. For this reason, we cannot expect better rates than those obtained for parametric integration.

In the setting above, existence and uniqueness of a solution are not even guaranteed. Moreover, we need further smoothness assumptions for the numerical analysis. For a precise introduction of the considered class $\mathscr{F}$ of input functions, we refer to Definition 6.4.2. Here, we only mention that, for $r, r_0, r_1 \in \mathbb{N}_0, 0 \le \varrho, \varrho_1 \le 1$, and $\sigma > 0$, $\mathscr{F}$ contains all tuples $(f, u_0) \in C(Q_0 \times [a,b] \times Z, Z) \times \sigma B_{C^{r_0}(Q_0,Z)}$, where f considered as a function independent of the third variable z satisfies

$$f \in B_{C^{0,r+\varrho}(Q_0 \times [a,b],Z)} \cap B_{C^{r_0,r_1+\varrho_1}(Q_0 \times [a,b],Z)}. \tag{3.10}$$

In contrast to the previous setting (3.6), the functions additionally satisfy a ϱ and ϱ_1 Hölder condition with respect to the second variable, respectively. For convenience, we can choose $\varrho = \varrho_1 = 0$ and we arrive at (3.6), where Q is replaced by $[a, b]$. Without loss of generality, we assume $r + \varrho \ge r_1 + \varrho_1$. The justification is similar to the one in Remark 3.2.1, see also Remark 6.5.5. In Chapter 6, we prove that the convergence order only mildly depends on the third variable. Thus, the smoothness with respect to z is chosen in an appropriate way, which guarantees the respective optimal convergence rate.

In terminology of Section 2.2, we set $K = Z$ and define the set of information functionals Λ_{ivp} by

$$\Lambda_{\text{ivp}} = \{\delta_{s,t,z} : \; s \in Q_0, t \in [a,b], z \in Z\} \cup \{\delta_s : \; s \in Q_0\}, \tag{3.11}$$

where, for $(f, u_0) \in \mathscr{F}$,

$$\delta_{s,t,z}(f, u_0) = f(s,t,z), \quad \delta_s(f, u_0) = u_0(s). \tag{3.12}$$

So, the admissible information is Z-valued and consists of values of f and u_0. Setting $F = \mathscr{F}$, $G = B(Q_0 \times [a,b], Z)$ and $S = \mathscr{S}$, where $\mathscr{S}(f, u_0) = u$ is the exact solution of the initial value problem defined by (3.8-3.9). The corresponding numerical problem Π is defined by

$$\Pi = (\mathscr{F}, B(Q_0 \times [a,b], Z), \mathscr{S}, Z, \Lambda_{\text{ivp}}).$$

Previous results. The complexity of initial value problems for ordinary differential equations (ODEs) without dependence on a parameter s was studied in [28, 29, 30, 23, 8] for scalar systems. It turned out that, given certain smoothness with respect to the input functions f, randomized algorithms are superior to deterministic algorithms by a factor $n^{-\frac{1}{2}}$. This shows that the typical speedup of randomized algorithms for classical integration carries over to the situation of initial value problems. In [21], the Banach space valued case is considered and similar results to those for Banach space valued integration are obtained. Regularity and approximation properties of the solution of parameter dependent initial value problems for ODEs have recently been considered in [16], however, with linear dependence on the parameters and an infinite dimensional parameter space.

New results. The complexity of general parametric initial value problems has not been studied so far, not even for the case $Z = \mathbb{R}$. This is catched up on here. We study the complexity in the deterministic and the randomized setting for various smoothness classes. These classes are closely related to those considered in Chapter 5 and include cases of isotropic and dominating mixed smoothness as well.

We develop a multilevel Monte Carlo algorithm and establish its convergence rate. The deterministic version, which is obtained from the randomized one by fixing the random parameters in an arbitrary way, is also studied. The algorithmic approach is a nonlinear analogue of the method used in Chapter 5. We use the Banach space valued generalizations [21] of the scalar results in [23] and [8] for the analysis. We also present lower bounds and settle the complexity in this way.

In the deterministic setting, we obtain

$$n^{-\tilde{\upsilon}_2(1)} \preceq e_n^{\det}(\mathscr{S}, \mathscr{F}) \preceq_{\log} n^{-\tilde{\upsilon}_2(1)},$$

where for $1 \leq p \leq 2$

$$\tilde{\upsilon}_2(p) \;=\; \begin{cases} \frac{\frac{r_0}{d_0}}{\frac{r_0}{d_0}+r+\varrho-r_1-\varrho_1}\left(r+\varrho+1-\frac{1}{p}\right) & \text{if} \quad \frac{r_0}{d_0} > r_1+\varrho_1+1-\frac{1}{p} \\ \frac{r_0}{d_0} & \text{if} \quad \frac{r_0}{d_0} \leq r_1+\varrho_1+1-\frac{1}{p} \end{cases}.$$

In the randomized setting, we obtain

$$n^{-\tilde{\upsilon}_2(p_Z)} \preceq e_n^{\det}(\mathscr{S}, \mathscr{F}) \preceq_{\log} n^{-\tilde{\upsilon}_2(p)} \tag{3.13}$$

if Z is a type p Banach space. Moreover, in case of $p_Z = p$, we obtain matching upper and lower bounds and we will see in Chapter 6 that the bounds are matching even up to logarithmic factors for type 2 Banach spaces (except for some limit cases). Since the considered multilevel algorithms are quite similar to those for parametric integration, Remark 3.2.2 is also valid here.

3.4 Parametric Hilbert Space Valued Initial Value Problems

In Chapter 7, we restrict ourselves to parametric initial value problems with values in a Hilbert space. This is not a strong restriction, since it still enables us to investigate systems of parametric ordinary differential equations by setting $H = \mathbb{R}^d$. On the other hand, this restriction allows us to consider more local input classes. In Chapter 6, we demand smoothness assumptions for functions over the whole domain $Q_0 \times [a,b] \times Z$. If $Z = H$ is an arbitrary Hilbert space, we use a localization technique, which allows us to drop this stronger assumption; i.e., we consider functions with smoothness over $Q_0 \times [a,b] \times \lambda_1 B_H^0$ only. This localization technique cannot be applied to the general Banach space valued case.

Considering these more general local input classes, we prove that the same rates are obtained as before for type 2 Banach spaces. This means that the considered multilevel algorithm is order optimal even up to logarithmic factors and that only in some limit cases a small logarithmic gap remains.

3.5 Basis of the Thesis

The thesis is based on the papers [9, 10, 11, 12]. The first two papers deal with Banach space valued integration and parametric integration, the remaining ones with parametric ordinary differential equations having values in Banach or Hilbert spaces. This structure is kept throughout the thesis. Chapter 4 corresponds to [9], Chapter 5 to [10]. Moreover, Chapter 6 corresponds to [11] and [12] is summarized in Chapter 7.

However, this work also goes beyond [9]–[12] and contains new, more general results. Here we consider general Banach space valued parametric integration in contrast to the scalar valued case covered in [10]. Furthermore, we present sharper rates for the complexity analysis of the Banach space valued initial value problem than those obtained in [11]; in some cases the bounds are sharp even up to logarithmic factors.

Chapter 4

Banach Space Valued Integration

The present chapter contains the analysis of deterministic and randomized algorithms for definite and indefinite Banach space valued multivariate integration. We also prove lower bounds and estimate the complexity of the problems. Finally, a general multilevel scheme is introduced, which serves as a bridge between the non-parametric and the parametric problem.

The chapter is organized as follows: Section 1 provides the formulation of the considered problem. In Section 2, we introduce the algorithms and establish the convergence analysis. Section 3 deals with the complexity analysis, including lower bound proofs, and in Section 4, we present the general multilevel approach mentioned above.

4.1 Problem Formulation

Let Z be an arbitrary Banach space, $r \in \mathbb{N}_0, d \in \mathbb{N}$, and set $Q := [0,1]^d$. The definite integration operator $\mathcal{S}_0 : C(Q,Z) \to Z$ is defined by

$$\mathcal{S}_0 f = \int_Q f(t)dt \tag{4.1}$$

in the sense of a Bochner integral.
The indefinite integration operator $\mathcal{S}_1 : C(Q,Z) \to C(Q,Z)$ is given by

$$(\mathcal{S}_1 f)(t) = \int_{[0,t]} f(u)du \qquad (t \in Q), \tag{4.2}$$

where $[0,t] = \prod_{i=1}^d [0,t_i]$ for $t = (t_i)_{i=1}^d \in Q$. Notice that using identification (2.3) we obtain

$$\mathcal{S}_\iota = I_Z \otimes \mathcal{S}_\iota^{\mathbb{K}} \qquad (\iota = 0,1), \tag{4.3}$$

where I_Z is the identity operator in Z and $\mathcal{S}_\iota^{\mathbb{K}}$ denotes the scalar valued version of $\mathcal{S}_\iota$.

4.2 Algorithms and Convergence Analysis

We present algorithms for the two integration problems (4.1) and (4.2). We start with the definite problem.

Algorithm 4.2.1. Let $n \in \mathbb{N}$ and let $\xi_i : \Omega \to Q$ $(i = 1, \dots, n)$ be independent random variables, uniformly distributed on Q, defined on some complete probability space $(\Omega, \Sigma, \mathbb{P})$. If $r = 0$, we set for $f \in C(Q, Z)$

$$\mathcal{A}^{0,0}_{n,\omega} f = \frac{1}{n} \sum_{i=1}^{n} f(\xi_i(\omega)). \tag{4.4}$$

Moreover, if $r \geq 1$, let $k = \lceil n^{1/d} \rceil$ and

$$\mathcal{A}^{0,r}_{n,\omega} f = \mathcal{S}_0(P^{r,d}_k f) + \mathcal{A}^{0,0}_{n,\omega}(f - P^{r,d}_k f). \tag{4.5}$$

Finally, we set $\mathcal{A}^{0,r}_n = \left(\mathcal{A}^{0,r}_{n,\omega}\right)_{\omega \in \Omega}$.

In the scalar case for $r = 0$ this is just the standard Monte Carlo method and for $r \geq 1$ the Monte Carlo method with separation of the main part. Notice that for $r \in \mathbb{N}_0$, $n \in \mathbb{N}$, $\omega \in \Omega$

$$\mathcal{A}^{0,r}_{n,\omega} = I_Z \otimes \mathcal{A}^{0,r,\mathbb{K}}_{n,\omega}. \tag{4.6}$$

As before, $\mathcal{A}^{0,r,\mathbb{K}}_{n,\omega}$ denotes the scalar version of $\mathcal{A}^{0,r}_{n,\omega}$.

Let us turn to the error analysis for this algorithm. Remember, fixing the random parameter $\omega \in \Omega$ means that we obtain a deterministic method. This way we consider both, the randomized and the deterministic case.

Proposition 4.2.2. *Let $r \in \mathbb{N}_0$ and $1 \leq p \leq 2$. Then there are constants $c_{1-3} > 0$ such that for all Banach spaces Z, $n \in \mathbb{N}$, $\omega \in \Omega$, we have*

$$\operatorname{card}\left(\mathcal{A}^{0,r}_{n,\omega}\right) \leq c_1 n,$$

and for all $f \in C^r(Q, Z)$

$$\|\mathcal{S}_0 f - \mathcal{A}^{0,r}_{n,\omega} f\| \;\leq\; c_2 n^{-\frac{r}{d}} \|f\|_{C^r(Q,Z)}, \tag{4.7}$$

$$\left(\mathbb{E}\, \|\mathcal{S}_0 f - \mathcal{A}^{0,r}_{n,\omega} f\|^p\right)^{\frac{1}{p}} \;\leq\; c_3 \tau_p(Z) n^{-\frac{r}{d}-1+\frac{1}{p}} \|f\|_{C^r(Q,Z)}. \tag{4.8}$$

Proof. Let $r = 0$ and $f \in C(Q, Z)$. With

$$\eta_i(\omega) = \int_Q f(t)dt - f(\xi_i(\omega)),$$

we get $\mathbb{E}\,\eta_i(\omega) = 0$ as well as

$$\mathcal{S}_0 f - \mathcal{A}^{0,0}_{n,\omega} f = \frac{1}{n} \sum_{i=1}^{n} \eta_i(\omega), \tag{4.9}$$

and

$$\|\eta_i(\omega)\| \le 2\|f\|_{C(Q,Z)}. \tag{4.10}$$

Thus,

$$\|\mathcal{S}_0 f - \mathcal{A}^{0,r}_{n,\omega} f\| \le 2||f||_{C(Q,Z)}, \tag{4.11}$$

which yields (4.7). Furthermore, using (2.14), (4.9), and (4.10), we obtain

$$\begin{aligned}\left(\mathbb{E}\,\|\mathcal{S}_0 f - \mathcal{A}^{0,0}_{n,\omega} f\|^p\right)^{\frac{1}{p}} &= \left(\mathbb{E}\,\|\frac{1}{n}\sum_{i=1}^{n} \eta_i(\omega)\|^p\right)^{\frac{1}{p}} \\ &\le 2\tau_p(Z) n^{-1} \left(\sum_{k=1}^{n} \mathbb{E}\,\|\eta_i\|^p\right)^{\frac{1}{p}} \\ &\le c\tau_p(Z) n^{-1+\frac{1}{p}} ||f||_{C(Q,Z)},\end{aligned} \tag{4.12}$$

which is (4.8). In case of $r \ge 1$, we obtain

$$\begin{aligned}\mathcal{S}_0 f - \mathcal{A}^{0,r}_{n,\omega} f &= \mathcal{S}_0(f - P^{r,d}_k f) - \mathcal{A}^{0,0}_{n,\omega}(f - P^{r,d}_k f) \\ &= (\mathcal{S}_0 - \mathcal{A}^{0,0}_{n,\omega})(f - P^{r,d}_k f).\end{aligned} \tag{4.13}$$

Thus, using (4.11) and relation (2.9), we get

$$\begin{aligned}\|\mathcal{S}_0 f - \mathcal{A}^{0,r}_{n,\omega} f\| &\le c||f - P^{r,d}_k f||_{C(Q,Z)} \\ &\le cn^{-\frac{r}{d}} ||f||_{C^r(Q,Z)},\end{aligned}$$

and accordingly, using (4.12),

$$\begin{aligned}\left(\mathbb{E}\,\|\mathcal{S}_0 f - \mathcal{A}^{0,r}_{n,\omega} f\|^p\right)^{\frac{1}{p}} &\le c\tau_p(Z) n^{-1+\frac{1}{p}} ||f - P^{r,d}_k f||_{C(Q,Z)} \\ &\le c\tau_p(Z) n^{-1+\frac{1}{p}} n^{-\frac{r}{d}} ||f||_{C^r(Q,Z)},\end{aligned}$$

which yields the statement also for $r \ge 1$. □

Next we consider indefinite integration. First, we assume $r = 0$ and present the Banach space version of the algorithm from Section 4 of [24]. It is a combination of Smolyak's algorithm from Section 2.3.4 and the Monte Carlo method.

Algorithm 4.2.3. Let $r = 0$ and fix any $m \in \mathbb{N}$, $m \geq 2$ and $L \in \mathbb{N}_0$. For $\bar{l} = (l_1, \ldots, l_d) \in \mathbb{N}_0^d$, we set $|\bar{l}| = l_1 + \cdots + l_d$ and define $U_{\bar{l}}^{\mathbb{K}}, V_L^{\mathbb{K}} \in \mathscr{L}(C(Q))$ by

$$U_{\bar{l}}^{\mathbb{K}} = (P_{m^{l_1}}^{1,1,\mathbb{K}} - P_{m^{l_1-1}}^{1,1,\mathbb{K}}) \otimes \cdots \otimes (P_{m^{l_{d-1}}}^{1,1,\mathbb{K}} - P_{m^{l_{d-1}-1}}^{1,1,\mathbb{K}}) \otimes P_{m^{l_d}}^{1,1,\mathbb{K}}, \tag{4.14}$$

with the meaning that $P_{m^{-1}}^{1,1,\mathbb{K}} := 0$. Furthermore, put

$$V_L^{\mathbb{K}} = \sum_{\bar{l} \in \mathbb{N}_0^d,\, |\bar{l}| = L} U_{\bar{l}}^{\mathbb{K}} \tag{4.15}$$

and let

$$U_{\bar{l}} = I_Z \otimes U_{\bar{l}}^{\mathbb{K}}, \quad V_L = I_Z \otimes V_L^{\mathbb{K}} \tag{4.16}$$

be the respective Banach space versions. Notice that V_L coincides with Smolyak's algorithm in Section 2.3.4 for $r = 1$. Set

$$\bar{1} = (\underbrace{1, \ldots, 1}_{d}), \quad m^{\bar{l}} = (m^{l_1}, \ldots, m^{l_d}), \quad \Gamma_{m^{\bar{l}}}^{1,d} = \Gamma_{m^{l_1}}^{1,1} \times \cdots \times \Gamma_{m^{l_d}}^{1,1},$$

and for $\bar{i} = (i_1, \ldots, i_d) \in \mathbb{N}^d$ with $\bar{1} \leq \bar{i} \leq m^{\bar{l}}$ (component-wise inequalities)

$$Q_{\bar{l},\bar{i}} = \left[\frac{i_1 - 1}{m^{l_1}}, \frac{i_1}{m^{l_1}}\right] \times \cdots \times \left[\frac{i_d - 1}{m^{l_d}}, \frac{i_d}{m^{l_d}}\right].$$

So $(Q_{\bar{l},\bar{i}})_{\bar{1} \leq \bar{i} \leq m^{\bar{l}}}$ is the partition of Q corresponding to the grid $\Gamma_{m^{\bar{l}}}^{1,d}$. Let

$$\xi_{\bar{l},\bar{i}} : \Omega \to Q_{\bar{l},\bar{i}} \qquad (|\bar{l}| = L,\, \bar{1} \leq \bar{i} \leq m^{\bar{l}})$$

be independent random variables on a complete probability space $(\Omega, \Sigma, \mathbb{P})$ such that $\xi_{\bar{l},\bar{i}}$ is uniformly distributed on $Q_{\bar{l},\bar{i}}$. Define $g_{\bar{l},\omega} \in \ell_\infty(\Gamma_{m^{\bar{l}}}^{1,d}, Z)$ by

$$g_{\bar{l},\omega}(t) = \sum_{\bar{j} : Q_{\bar{l},\bar{j}} \subseteq [0,t]} |Q_{\bar{l},\bar{j}}| f(\xi_{\bar{l},\bar{j}}(\omega)) \quad (t \in \Gamma_{m^{\bar{l}}}^{1,d}) \tag{4.17}$$

with the convention that $g_{\bar{l},\omega}(t) = 0$ if there is no $\bar{j}$ with $Q_{\bar{l},\bar{j}} \subseteq [0,t]$ (that is, if some component of t is zero). Finally, let

$$L = 2d - 1 \tag{4.18}$$

and, given $n \in \mathbb{N}$,

$$m = \left\lceil (n+1)^{\frac{1}{L}} \right\rceil. \tag{4.19}$$

If $r = 0$, we define

$$\mathcal{A}_{n,\omega}^{1,0} f := \sum_{\bar{l} \in \mathbb{N}_0^d,\, |\bar{l}| = L} U_{\bar{l}} g_{\bar{l},\omega}. \tag{4.20}$$

In case of $r \geq 1$, we set $k = \lceil n^{1/d} \rceil$ and

$$\mathcal{A}^{1,r}_{n,\omega} f = \mathcal{S}_1(P^{r,d}_k f) + \mathcal{A}^{1,0}_{n,\omega}(f - P^{r,d}_k f). \tag{4.21}$$

Finally, set $\mathcal{A}^{1,r}_n = \left(\mathcal{A}^{1,r}_{n,\omega}\right)_{\omega\in\Omega}$, and in a similar way to (4.6), we obtain for $r \in \mathbb{N}_0$, $n \in \mathbb{N}$, $\omega \in \Omega$

$$\mathcal{A}^{1,r}_{n,\omega} = I_Z \otimes \mathcal{A}^{1,r,\mathbb{K}}_{n,\omega,}. \tag{4.22}$$

Before we proceed with the analysis of the algorithm, we need additional lemmas. The first one is proven in [24] for the scalar case. It is stated for $L_p(Q)$ spaces, but literally carries over to the $C(Q)$ case. For the sake of completeness, we recall the lemma and also the proof.

Lemma 4.2.4. *There is a constant $c > 0$ such that for all $m, L \in \mathbb{N}_0$ with $m \geq 2$*

$$||\mathcal{S}^{\mathbb{K}}_1 - V^{\mathbb{K}}_L \mathcal{S}^{\mathbb{K}}_1||_{\mathscr{L}(C(Q))} \leq c(L+1)^{d-1} m^{-L+d-1}. \tag{4.23}$$

Proof. Let $I^{(d)}_{\mathbb{K}}$ be the identity operator on $C([0,1]^d)$ and we explicitly identify by $\mathcal{S}^{d,\mathbb{K}}_1$ the indefinite integration operator $\mathcal{S}^{\mathbb{K}}_1$ from above for dimension d. First, notice that for $d = 1$

$$\mathcal{S}^{1,\mathbb{K}}_1 \in \mathscr{L}(C([0,1]), C^1([0,1])),$$

which, by (2.9), implies

$$||(I^{(1)}_{\mathbb{K}} - P^{1,1,\mathbb{K}}_{m^l})\mathcal{S}^{1,\mathbb{K}}_1||_{\mathscr{L}(C([0,1]))} \leq cm^{-l} \tag{4.24}$$

and hence

$$||(P^{1,1,\mathbb{K}}_{m^l} - P^{1,1,\mathbb{K}}_{m^{l-1}})\mathcal{S}^{1,\mathbb{K}}_1||_{\mathscr{L}(C([0,1]))} \leq cm^{-(l-1)}. \tag{4.25}$$

To prove (4.23), we argue by induction over the dimension d. For $d = 1$, the result is just (4.24). Now let $d > 1$ and assume that (4.23) holds for $d - 1$. By the triangle inequality, we obtain

$$\begin{aligned} ||\mathcal{S}^{d,\mathbb{K}}_1 - V^{\mathbb{K}}_L \mathcal{S}^{d,\mathbb{K}}_1||_{\mathscr{L}(C([0,1]^d))} \leq{}& ||\mathcal{S}^{d,\mathbb{K}}_1 - (P^{1,1,\mathbb{K}}_{m^L} \otimes I^{(d-1)}_{\mathbb{K}})\mathcal{S}^{d,\mathbb{K}}_1||_{\mathscr{L}(C([0,1]^d))} \\ &+ ||(P^{1,1,\mathbb{K}}_{m^L} \otimes I^{(d-1)}_{\mathbb{K}})\mathcal{S}^{d,\mathbb{K}}_1 - V^{\mathbb{K}}_L \mathcal{S}^{d,\mathbb{K}}_1||_{\mathscr{L}(C([0,1]^d))}. \end{aligned}$$

Using (4.24), (2.4), (2.5), and the fact that $\mathcal{S}^{d,\mathbb{K}}_1 = \mathcal{S}^{1,\mathbb{K}}_1 \otimes \mathcal{S}^{d-1,\mathbb{K}}_1$, the first term is estimated as

$$\begin{aligned} &||\mathcal{S}^{d,\mathbb{K}}_1 - (P^{1,1,\mathbb{K}}_{m^L} \otimes I^{(d-1)}_{\mathbb{K}})\mathcal{S}^{d,\mathbb{K}}_1||_{\mathscr{L}(C([0,1]^d)} \\ &\qquad = ||(I^{(1)}_{\mathbb{K}} - P^{1,1,\mathbb{K})}_{m^L}) \otimes I^{(d-1)}_{\mathbb{K}})(\mathcal{S}^{1,\mathbb{K}}_1 \otimes \mathcal{S}^{d-1,\mathbb{K}}_1)||_{C([0,1]^d)} \\ &\qquad = ||(I^{(1)}_{\mathbb{K}} - P^{1,1,\mathbb{K})}_{m^L})\mathcal{S}^{1,\mathbb{K}}_1||_{C([0,1])} ||\mathcal{S}^{d-1,\mathbb{K}}_1||_{C([0,1]^{d-1})} \\ &\qquad \leq cm^{-L}, \end{aligned} \tag{4.26}$$

since $||\mathcal{S}_1^{d-1,\mathbb{K}}||_{C([0,1]^{d-1})} = 1$. The second term can be estimated as follows:

$$
\begin{aligned}
&||(P_{m^L}^{1,1,\mathbb{K}} \otimes I_{\mathbb{K}}^{(d-1)})\mathcal{S}_1^{d,\mathbb{K}} - V_L^{\mathbb{K}}\mathcal{S}_1^{d,\mathbb{K}}||_{\mathscr{L}(C([0,1]^d))} \\
&= \Bigg|\Bigg| \sum_{l=0}^{L}((P_{m^l}^{1,1,\mathbb{K}} - P_{m^{l-1}}^{1,1,\mathbb{K}}) \otimes I_{\mathbb{K}}^{(d-1)})\mathcal{S}_1^{d,\mathbb{K}} \\
&\qquad\qquad - \sum_{l=0}^{L}((P_{m^l}^{1,1,\mathbb{K}} - P_{m^{l-1}}^{1,1,\mathbb{K}}) \otimes P_{m^{L-l}}^{1,d-1,\mathbb{K}})\mathcal{S}_1^{d,\mathbb{K}}\Bigg|\Bigg|_{\mathscr{L}(C([0,1]^d))} \\
&= \Bigg|\Bigg| \sum_{l=0}^{L}((P_{m^l}^{1,1,\mathbb{K}} - P_{m^{l-1}}^{1,1,\mathbb{K}})\mathcal{S}_1^{1,\mathbb{K}}) \otimes ((I_{\mathbb{K}}^{(d-1)} - P_{m^{L-l}}^{1,d-1,\mathbb{K}})\mathcal{S}_1^{d-1,\mathbb{K}})\Bigg|\Bigg|_{\mathscr{L}(C([0,1]^d))} \\
&\le \sum_{l=0}^{L} ||(P_{m^l}^{1,1,\mathbb{K}} - P_{m^{l-1}}^{1,1,\mathbb{K}})\mathcal{S}_1^{1,\mathbb{K}}||_{\mathscr{L}(C([0,1]))}||(I_{\mathbb{K}}^{(d-1)} - P_{m^{L-l}}^{1,d-1,\mathbb{K}})\mathcal{S}_1^{d-1,\mathbb{K}}||_{\mathscr{L}(C([0,1]^{d-1})} \\
&\le c\sum_{l=0}^{L} m^{-(l-1)}(L-l+1)^{d-2}m^{-(L-l-d+2)} \\
&\le c(L+1)^{d-1}m^{-(L-d+1)},
\end{aligned}
$$

where we used (4.25), (2.4), (2.5), and the induction hypothesis. □

The next lemma is a direct consequence of the Kolmogorov-Doob inequality.

Lemma 4.2.5. *Let $1 < p < \infty, \bar{k} \in \mathbb{N}^d$ and let $\{\vartheta_{\bar{i}} : \bar{1} \le \bar{i} \le \bar{k}\}$ be independent, mean zero, Z-valued random variables with $\mathbb{E}||\vartheta_{\bar{i}}||^p < \infty$ for all $\bar{1} \le \bar{i} \le \bar{k}$. Then*

$$
\left(\mathbb{E}\max_{\bar{1}\le\bar{i}\le\bar{k}}\Bigg|\Bigg|\sum_{\bar{1}\le\bar{j}\le\bar{i}}\vartheta_{\bar{j}}\Bigg|\Bigg|^p\right)^{\frac{1}{p}} \le c_1^d\left(\mathbb{E}\Bigg|\Bigg|\sum_{\bar{1}\le\bar{j}\le\bar{k}}\vartheta_{\bar{j}}\Bigg|\Bigg|^p\right)^{\frac{1}{p}},
$$

where $c_1 = \frac{p}{(p-1)}$.

Proof. The proof can be found in [24] for the scalar case. The Banach space valued case literally carries over. □

Proposition 4.2.6. *Let $r \in \mathbb{N}_0$, $1 \le p \le 2$. Then there are constants $c_{1-3} > 0$ such that for all Banach spaces Z, $n \in \mathbb{N}$, $\omega \in \Omega$, we have*

$$
\operatorname{card}\big(\mathcal{A}_{n,\omega}^{1,r}\big) \le c_1 n,
$$

and for all $f \in C^r(Q,Z)$

$$
\|\mathcal{S}_1 f - \mathcal{A}_{n,\omega}^{1,r} f\|_{C(Q,Z)} \le c_2 n^{-\frac{r}{d}}\|f\|_{C^r(Q,Z)}, \tag{4.27}
$$

$$
(\mathbb{E}\,\|\mathcal{S}_1 f - \mathcal{A}_{n,\omega}^{1,r} f\|_{C(Q,Z)}^p)^{\frac{1}{p}} \le c_3\tau_p(Z) n^{-\frac{r}{d}-1+\frac{1}{p}}\|f\|_{C^r(Q,Z)}. \tag{4.28}
$$

Proof. We start with the case $r = 0$. We have

$$\|\mathcal{S}_1 f - \mathcal{A}^{1,0}_{n,\omega} f\|_{C(Q)} \le \|\mathcal{S}_1 f - V_L \mathcal{S}_1 f\|_{C(Q,Z)} + \|V_L \mathcal{S}_1 f - \mathcal{A}^{1,0}_{n,\omega} f\|_{C(Q,Z)}. \quad (4.29)$$

The first term is estimated for the scalar case in Lemma 4.23. The Banach space case follows by taking tensor products and using (2.3),(4.3),(4.16) as follows:

$$\begin{aligned}\|\mathcal{S}_1 - V_L \mathcal{S}_1\|_{\mathscr{L}(C(Q,Z))} &= \|I_Z \otimes \mathcal{S}_1^{\mathbb{K}} - I_Z \otimes V_L^{\mathbb{K}} \mathcal{S}_1^{\mathbb{K}}\|_{\mathscr{L}(Z \otimes_\lambda C(Q))} \\ &= \|I_Z \otimes (\mathcal{S}_1^{\mathbb{K}} - V_L^{\mathbb{K}} \mathcal{S}_1^{\mathbb{K}})\|_{\mathscr{L}(Z \otimes_\lambda C(Q))} \\ &= ||I_Z||_{\mathscr{L}(Z)} \|\mathcal{S}_1^{\mathbb{K}} - V_L^{\mathbb{K}} \mathcal{S}_1^{\mathbb{K}}\|_{\mathscr{L}(C(Q))} \\ &\le c m^{-L+d-1}, \end{aligned} \quad (4.30)$$

where we have in mind that L is fixed, thus, $(L+1)^{d-1}$ is a constant factor. Now we consider the second term. We get

$$\|V_L \mathcal{S}_1 f - \mathcal{A}^{1,0}_{n,\omega} f\|_{C(Q,Z)} \le \sum_{\bar{l} \in \mathbb{N}_0^d,\, |\bar{l}| = L} \|U_{\bar{l}} \mathcal{S}_1 f - U_{\bar{l}} g_{\bar{l},\omega}\|_{C(Q,Z)} \quad (4.31)$$

and

$$\begin{aligned}\|U_{\bar{l}} \mathcal{S}_1 f - U_{\bar{l}} g_{\bar{l},\omega}\|_{C(Q,Z))} &\le \|U_{\bar{l}}\|_{\mathscr{L}\left(\ell_\infty(\Gamma^{1,d}_{m^{\bar{l}}}, Z), C(Q,Z)\right)} \left\| (\mathcal{S}_1 f)|_{\Gamma^{1,d}_{m^{\bar{l}}}} - g_{\bar{l},\omega} \right\|_{\ell_\infty(\Gamma^{1,d}_{m^{\bar{l}}}, Z)} \\ &\le c \max_{t \in \Gamma^{1,d}_{m^{\bar{l}}}} \left\| \int_{[0,t]} f(t) dt - \sum_{\bar{j} : Q_{\bar{l},\bar{j}} \subseteq [0,t]} |Q_{\bar{l},\bar{j}}| f(\xi_{\bar{l},\bar{j}}) \right\| \\ &= c \max_{\bar{1} \le \bar{i} \le m^{\bar{l}}} \left\| \sum_{\bar{1} \le \bar{j} \le \bar{i}} \eta_{\bar{l},\bar{j}} \right\|, \end{aligned} \quad (4.32)$$

where

$$\eta_{\bar{l},\bar{j}} = \int_{Q_{\bar{l},\bar{j}}} f(t) dt - |Q_{\bar{l},\bar{j}}| f(\xi_{\bar{l},\bar{j}}) \quad (\bar{1} \le \bar{j} \le m^{\bar{l}}). \quad (4.33)$$

The random variables $\{\eta_{\bar{l},\bar{j}} : \bar{1} \le \bar{j} \le m^{\bar{l}}\}$ are independent, of mean zero, and satisfy

$$\|\eta_{\bar{l},\bar{j}}\| \le 2 |Q_{\bar{l},\bar{j}}| \|f\|_{C(Q,Z)} = 2 m^{-L} \|f\|_{C(Q,Z)}. \quad (4.34)$$

Combining (4.18–4.19) and (4.29–4.34), we obtain

$$\begin{aligned}\|\mathcal{S}_1 f - \mathcal{A}^{1,r}_{n,\omega} f\|_{C(Q,Z)} &\le c m^{-L+d-1} ||f||_{C(Q,Z)} + c \sum_{\bar{l} \in \mathbb{N}_0^d,\, |\bar{l}| = L} m^L m^{-L} ||f||_{C(Q,Z)} \\ &\le c ||f||_{C(Q,Z)}, \end{aligned} \quad (4.35)$$

which proves (4.27) for $r = 0$.

For $p > 1$, we get from Lemma 4.2.5

$$\Big(\mathbb{E} \max_{\bar{1} \le \bar{i} \le m^{\bar{l}}} \Big\| \sum_{\bar{1} \le \bar{j} \le \bar{i}} \eta_{\bar{l},\bar{j}} \Big\|^p \Big)^{\frac{1}{p}} \le c \Big(\mathbb{E} \Big\| \sum_{\bar{1} \le \bar{j} \le m^{\bar{l}}} \eta_{\bar{l},\bar{j}} \Big\|^p \Big)^{\frac{1}{p}}. \tag{4.36}$$

Moreover, Lemma 2.3.9 gives

$$\Big(\mathbb{E} \Big\| \sum_{\bar{1} \le \bar{j} \le m^{\bar{l}}} \eta_{\bar{l},\bar{j}} \Big\|^p \Big)^{\frac{1}{p}} \le 2\tau_p(Z) \Big(\sum_{\bar{1} \le \bar{j} \le m^{\bar{l}}} \mathbb{E} \, \| \eta_{\bar{l},\bar{j}} \|^p \Big)^{\frac{1}{p}}, \tag{4.37}$$

and we conclude with (4.36) and (4.37) for $p > 1$

$$\Big(\mathbb{E} \max_{\bar{1} \le \bar{i} \le m^{\bar{l}}} \Big\| \sum_{\bar{1} \le \bar{j} \le \bar{i}} \eta_{\bar{l},\bar{j}} \Big\|^p \Big)^{\frac{1}{p}} \le c\tau_p(Z) \Big(\sum_{\bar{1} \le \bar{j} \le m^{\bar{l}}} \mathbb{E} \, \| \eta_{\bar{l},\bar{j}} \|^p \Big)^{\frac{1}{p}}. \tag{4.38}$$

The same relation also holds for $p = 1$ by the triangle inequality. We obtain from (4.31–4.32), (4.34), and (4.38)

$$\begin{aligned} (\mathbb{E} \, \| V_L \mathcal{S}_1 f - \mathcal{A}^{1,0}_{n,\omega} f \|^p_{C(Q,Z)})^{\frac{1}{p}} &\le c\tau_p(Z) \sum_{\bar{l} \in \mathbb{N}_0^d, \, |\bar{l}| = L} m^{\frac{L}{p}} m^{-L} ||f||_{C(Q,Z)} \\ &\le c\tau_p(Z) m^{-(1-\frac{1}{p})L} \|f\|_{C(Q,Z)}. \end{aligned} \tag{4.39}$$

Combining (4.18–4.19), (4.29–4.30), and (4.39), we conclude

$$\begin{aligned} (\mathbb{E} \| \mathcal{S}_1 f - \mathcal{A}^{1,r}_{n,\omega} f \|^p_{C(Q,Z)})^{\frac{1}{p}} &\le c m^{-L+d-1} ||f||_{C(Q,Z)} + c \sum_{\bar{l} \in \mathbb{N}_0^d, \, |\bar{l}| = L} m^{-(1-\frac{1}{p})L} ||f||_{C(Q,Z)} \\ &\le c n^{-\frac{1}{2}} ||f||_{C(Q,Z)} + c n^{-(1-\frac{1}{p})} ||f||_{C(Q,Z)} \\ &\le c n^{-(1-\frac{1}{p})} ||f||_{C(Q,Z)}, \end{aligned} \tag{4.40}$$

which proves relation (4.28) for $r = 0$.

As in the proof of Proposition 4.2.2, the case $r \ge 1$ follows from the case $r = 0$ and (2.9), since

$$\begin{aligned} \mathcal{S}_1 f - \mathcal{A}^{1,r}_{n,\omega} f &= \mathcal{S}_1 (f - P^{r,d}_k f) - \mathcal{A}^{1,0}_{n,\omega} (f - P^{r,d}_k f) \\ &= (\mathcal{S}_1 - \mathcal{A}^{1,0}_{n,\omega})(f - P^{r,d}_k f). \end{aligned} \tag{4.41}$$

By (4.14–4.15) and (4.17–4.21), the number of function values used by $\mathcal{A}^{1,r}_{n,\omega} f$ is

$$c k^d + c \sum_{|\bar{l}| = L} m^{l_1} \dots m^{l_d} \le cn.$$

□

4.3 Complexity Analysis

For the definite integration problem, we set $F = B_{C^r(Q,Z)}$, $G = Z$, $K = Z$ and $\Lambda = \Lambda(Q, Z) = \{\delta_t : t \in Q\}$ with $\delta_t(f) = f(t)$. As mentioned before, we consider Z-valued information functionals and describe the definite integration problem by

$$\mathcal{P}_0 = (B_{C^r(Q,Z)}, Z, \mathcal{S}_0, Z, \Lambda(Q, Z)).$$

Moreover, for the indefinite integration problem, we take $G = C(Q, Z)$, while F,K, and Λ are the same as above. So the indefinite integration problem is

$$\mathcal{P}_1 = (B_{C^r(Q,Z)}, C(Q, Z), \mathcal{S}_1, Z, \Lambda(Q, Z)).$$

Theorem 4.3.1. *Let $r \in \mathbb{N}_0$, $\iota \in \{0, 1\}$, $1 \leq p \leq 2$. For all Banach spaces Z, the deterministic n-th minimal errors satisfy*

$$n^{-\frac{r}{d}} \preceq e_n^{\det}(\mathcal{S}_\iota, B_{C^r(Q,Z)}) \preceq n^{-\frac{r}{d}}.$$

Moreover, if Z is of type p and p_Z is the supremum of all p_1 such that Z is of type p_1, the randomized n-th minimal errors fulfill

$$n^{-\frac{r}{d}-1+\frac{1}{p_Z}} \preceq e_n^{\mathrm{ran}}(\mathcal{S}_\iota, B_{C^r(Q,Z)}) \preceq \tau_p(Z) n^{-\frac{r}{d}-1+\frac{1}{p}}.$$

Proof. The upper bounds follow from Propositions 4.2.2, Proposition 4.2.6, and Hölders inequality.

Since definite integration is a particular case of indefinite integration in the sense that $\mathcal{S}_0 f = (\mathcal{S}_1 f)(\bar{1})$, it suffices to prove the lower bounds for $\mathcal{S}_0$. The lower bounds for the deterministic setting and for the randomized setting with $p_Z = 2$ follow from the respective scalar cases, see (3.3-3.5), since every Banach space Z over $\mathbb{K}$ contains an isometric copy of $\mathbb{K}$.

It remains to show the lower bound for the randomized setting for Banach spaces with $p_Z < 2$. Let $n \in \mathbb{N}$ and let $m \in \mathbb{N}$ be such that

$$(m-1)^d < 8n \leq m^d. \tag{4.42}$$

Let $\psi \in C^\infty(\mathbb{R}^d)$ be such that $\psi(t) > 0$ for $t \in (0, 1)^d$, $\operatorname{supp}\psi \subset [0, 1]^d$, and $\sup_{t\in[0,1]^d} ||\psi(t)|| = \sigma_0 > 0$. Let $(Q_i)_{i=1}^{m^d}$ be the partition of Q into closed cubes of side length m^{-1} of disjoint interior, let t_i be the point in Q_i with minimal coordinates and define $\psi_i \in C(Q)$ by

$$\psi_i(t) = \psi(m(t - t_i)) \quad (i = 1, \ldots, m^d).$$

We check that there is a constant $c_0 > 0$ such that for all $(\alpha_i)_{i=1}^{m^d} \in [-1,1]^{m^d}$ and all $(z_i)_{i=1}^{m^d} \subset B_Z$

$$c_0 m^{-r} \sum_{i=1}^{m^d} \alpha_i z_i \psi_i \in B_{C^r(Q,Z)}. \tag{4.43}$$

By the chain rule

$$\psi_i^{(r)}(t) = m^r \psi_i(m(t-t_i)),$$

thus,

$$\Big\| \sum_{i=1}^{m^d} \alpha_i z_i \psi_i \Big\|_{C^r(Q)} \le \max_{i=1,\dots,m^d} ||\alpha_i z_i \psi_i||_{C^r(Q)} \le m^r \sigma_0,$$

which yields (4.43). Put $f_i = c_0 m^{-r} z_i \psi_i$ and $\sigma = \int_Q \psi(t)dt$. Then for $(\alpha_i)_{i=1}^{m^d} \in \mathbb{R}^{m^d}$

$$\mathbb{E}\Big\| \sum_{i=1}^{m^d} \alpha_i \mathcal{S}_0 f_i \Big\| = c_0 m^{-r} \mathbb{E}\Big\| \sum_{i=1}^{m^d} \alpha_i z_i \int_Q \psi_i(t)dt \Big\| = c_0 \sigma m^{-r-d} \mathbb{E}\Big\| \sum_{i=1}^{m^d} \alpha_i z_i \Big\|.$$

Since $p_Z < 2$, Z must be infinite dimensional because a finite dimensional space Z always satisfies $p_Z = 2$. By the Maurey-Pisier Theorem 2.3.12, there is a sequence $(w_i)_{i=1}^{m^d} \subset B_Z$ such that for all $(\alpha_i)_{i=1}^{m^d} \subset \mathbb{R}$

$$\frac{1}{2} \left(\sum_{i=1}^{m^d} |\alpha_i|^{p_Z} \right)^{\frac{1}{p_Z}} \le \Big\| \sum_{i=1}^{m^d} \alpha_i w_i \Big\| \le \left(\sum_{i=1}^{m^d} |\alpha_i|^{p_Z} \right)^{\frac{1}{p_Z}}.$$

Setting $z_i = w_i$ $(i = 1, \dots, m^d)$, we get

$$\mathbb{E}\Big\| \sum_{i=1}^{m^d} \alpha_i z_i \Big\| \ge c \left(\sum_{i=1}^{m^d} |\alpha_i|^{p_Z} \right)^{\frac{1}{p_Z}}. \tag{4.44}$$

Next we use Lemma 2.2.10, Lemma 2.2.13 and (4.42–4.44) to conclude

$$\begin{aligned} e_n^{\text{ran}}(\mathcal{S}_0, B_{C^r(Q,Z)}) &\ge \frac{1}{4} \min_{I \subseteq \{1,\dots,m^d\}, |I| \ge m^d - 4n} \mathbb{E} \Big\| \sum_{i \in I} \varepsilon_i \mathcal{S}_0 f_i \Big\| \\ &\ge c m^{-r-(d-\frac{d}{p_Z})} \\ &\ge c n^{-\frac{r}{d}-1+\frac{1}{p_Z}}, \end{aligned}$$

where $(\varepsilon_i)_{i=1}^{m^d}$ is a sequence of independent centered Bernoulli random variables.

□

Notice that the bounds in the randomized case of Theorem 4.3.1 are matching up to an arbitrarily small gap in the exponent. In some cases, they are even of matching order.

Corollary 4.3.2. *Let $r \in \mathbb{N}_0$, $1 \le p \le 2$, $\iota \in \{0,1\}$. Let p_Z be the supremum of all p_1 such that Z is of type p_1. Then for each $\varepsilon > 0$*

$$n^{-\frac{r}{d}-1+\frac{1}{p_Z}} \preceq e_n^{\mathrm{ran}}(\mathcal{S}_\iota, B_{C^r(Q,Z)}) \preceq \tau_p(Z) n^{-\frac{r}{d}-1+\frac{1}{p_Z}+\varepsilon}.$$

If, moreover, the supremum of types is attained that is, Z is of type p_Z, then

$$n^{-\frac{r}{d}-1+\frac{1}{p_Z}} \preceq e_n^{\mathrm{ran}}(\mathcal{S}_\iota, B_{C^r(Q,Z)}) \preceq \tau_{p_Z}(Z) n^{-\frac{r}{d}-1+\frac{1}{p_Z}}.$$

Remark 4.3.3. The latter holds in particular for spaces of type 2 with $p_Z = p = 2$ and, if $1 \le p_1 < \infty$, for spaces $Z = L_{p_1}(\mathcal{N}, \nu)$ with $p_Z = p = \min(p_1, 2)$, where $(\mathcal{N}, \nu)$ is some measure space.

For general Banach spaces Z, upper and lower bounds of matching order for

$$e_n^{\mathrm{ran}}(\mathcal{S}_\iota, B_{C^r(Q,Z)}) \qquad (\iota = 0, 1)$$

remain an open problem. However, there are new results for Banach spaces satisfying equal norm type, see [22]. To state the result, we need further preparations.

Let $1 \le p \le 2$ and Z be an arbitrary Banach space. Given $n \in \mathbb{N}$, let $\sigma_{p,n}(Z)$ be the smallest constant $c > 0$ such that for all $z_1, \dots, z_n \in Z$ with $||z_1|| = \dots = ||z_n||$

$$\mathbb{E}\Big\|\sum_{i=1}^n \varepsilon_i z_i\Big\|^p \le c^p \sum_{k=1}^n \|z_i\|^p.$$

Z is of equal norm type p if there is a constant $c_1 > 0$ such that

$$\sigma_{p,n}(Z) \le c_1 \qquad (n \in \mathbb{N}).$$

It is clear that $\sigma_{p,n}(Z) \le \tau_p(Z)$ and type p implies equal norm type p. The following theorem is the main result of [22].

Theorem 4.3.4. *Let $1 \le p \le 2, \iota \in \{0,1\}$, and $r \in \mathbb{N}_0$. Then there are constants $c_1, c_2 > 0$ such that for all Banach spaces X and all $n \in \mathbb{N}$*

$$c_1 n^{\frac{r}{d}+1-\frac{1}{p}} e_n^{\mathrm{ran}}(\mathcal{S}_\iota, B_{C^r(Q,Z)}) \le \sigma_{p,n}(Z) \le c_2 \max_{1\le k\le n} k^{\frac{r}{d}+1-\frac{1}{p}} e_k^{\mathrm{ran}}(\mathcal{S}_\iota, B_{C^r(Q,Z)}).$$

A question that arises is a sharp characterization of Banach spaces such that the optimal randomized rate $n^{-r/d-1+1/p}$ is obtained. This question is answered as a corollary.

Corollary 4.3.5. *Let $r \in \mathbb{N}_0$ and $1 \le p \le 2$. Then the following are equivalent:*

- *Z is of equal norm type p.*
- *There is a constant $c > 0$ such that for all $n \in \mathbb{N}$*

$$e_n^{\mathrm{ran}}(\mathcal{S}_\iota, B_{C^r(Q,Z)}) \le cn^{-\frac{r}{d}-1+\frac{1}{p}} \qquad (\iota \in \{0,1\}).$$

Remark 4.3.6. Using techniques from [22], it is easy to carry over the results from above to the situation of equal norm type p. Thus, in the preceding estimates, and also in estimates that follow, we could easily replace $\tau_p(Z)$ by $\sigma_{p,n}(Z)$ in the upper bound estimates.

4.4 A Multilevel Algorithm for Banach Space Valued Integration

We develop a scheme, which will serve as a bridge between the parametric and the non-parametric case. It is based on the multilevel Monte Carlo approach from [17, 25]. Assume that a Banach space Y is continuously embedded into the Banach space X and let J be the embedding map. We shall identify elements of Y with their images in X. For $r, r_1 \in \mathbb{N}_0$, we consider integration of functions from the set

$$B_{C^r(Q,X)} \cap B_{C^{r_1}(Q,Y)}.$$

Let $(T_l)_{l=0}^\infty \subset \mathscr{L}(X)$ (this is intended to be a sequence, which approximates the embedding J) and set for $l \in \mathbb{N}_0$

$$R_l \;=\; T_l \otimes I_{C(Q)} \in \mathscr{L}(C(Q,X)). \tag{4.45}$$

The operator R_l is just the pointwise application of T_l in the sense that for $f \in C(Q,X)$ and $t \in Q$, we get $(R_l f)(t) = T_l f(t)$.

Algorithm 4.4.1. Fix any $l_0, l_1 \in \mathbb{N}_0$, $l_0 \le l_1$, $n_{l_0}, \dots, n_{l_1} \in \mathbb{N}$ and define for $\iota \in \{0,1\}$ and $f \in C(Q,X)$ an approximation $\mathfrak{A}_\omega^\iota f$ to $\mathcal{S}_\iota f$ as follows:

$$\mathfrak{A}_\omega^\iota f \;=\; R_{l_0} \mathcal{A}_{n_{l_0},\omega}^{\iota,r} f + \sum_{l=l_0+1}^{l_1} (R_l - R_{l-1}) \mathcal{A}_{n_l,\omega}^{\iota,r_1} f \tag{4.46}$$

and $\mathfrak{A}^\iota = (\mathfrak{A}_\omega^\iota)_{\omega\in\Omega}$. Without loss of generality, we demand that the underlying probability space $(\Omega, \Sigma, \mathbb{P})$ is such that all random variables that are required on the levels $l_0, \dots, l_1$ are defined on it. It follows from (4.6), (4.22), and (4.45) that

$$\mathfrak{A}_\omega^\iota \;=\; T_{l_0} \otimes \mathcal{A}_{n_{l_0},\omega}^{\iota,r,\mathbb{K}} + \sum_{l=l_0+1}^{l_1} (T_l - T_{l-1}) \otimes \mathcal{A}_{n_l,\omega}^{\iota,r_1,\mathbb{K}}. \tag{4.47}$$

In the sequel, we set

$$\begin{aligned} X_l &= \mathrm{cl}_X(T_l(X)) \qquad (l \in \mathbb{N}_0), \\ X_{l-1,l} &= \mathrm{cl}_X((T_l - T_{l-1})(X)) \quad (l \in \mathbb{N}), \end{aligned} \tag{4.48}$$

where cl_X denotes the closure in X. In particular, X_l and $X_{l-1,l}$ are endowed with the norm induced by X. Given a Banach space Z, we introduce the notation

$$G_0(Z) = Z, \tag{4.49}$$

$$G_1(Z) = C(Q, Z). \tag{4.50}$$

Now we estimate the error of $\mathfrak{A}_\omega^\iota$ on

$$B_{C^r(Q,X)} \cap B_{C^{r_1}(Q,Y)}.$$

Proposition 4.4.2. *Let $1 \le p \le 2$, $r, r_1 \in \mathbb{N}_0$, and $\iota \in \{0,1\}$. Then there are constants $c_1, c_2 > 0$ such that for all Banach spaces X,Y, and operators $(T_l)_{l=0}^\infty$ as above, for all $l_0, l_1 \in \mathbb{N}_0$ with $l_0 \le l_1$, and for all $(n_l)_{l=l_0}^{l_1} \subset \mathbb{N}$, the so-defined algorithm $\mathfrak{A}_\omega^\iota$ satisfies*

$$\begin{aligned} &\sup_{f \in B_{C^r(Q,X)} \cap B_{C^{r_1}(Q,Y)}} \|\mathcal{S}_\iota f - \mathfrak{A}_\omega^\iota f\|_{G_\iota(X)} \\ &\quad \le \|J - T_{l_1} J\|_{\mathscr{L}(Y,X)} + c_1 \|T_{l_0}\|_{\mathscr{L}(X)}\, n_{l_0}^{-\frac{r}{d}} \\ &\qquad + c_1 \sum_{l=l_0+1}^{l_1} \|(T_l - T_{l-1})J\|_{\mathscr{L}(Y,X)}\, n_l^{-\frac{r_1}{d}} \quad (\omega \in \Omega), \end{aligned} \tag{4.51}$$

and for all $l^* \in \mathbb{N}_0$ with $l_0 \le l^* \le l_1$

$$\begin{aligned} &\sup_{f \in B_{C^r(Q,X)} \cap B_{C^{r_1}(Q,Y)}} \left(\mathbb{E}\, \|\mathcal{S}_\iota f - \mathfrak{A}_\omega^\iota f\|_{G_\iota(X)}^p\right)^{\frac{1}{p}} \\ &\quad \le \|J - T_{l_1} J\|_{\mathscr{L}(Y,X)} + c_2 \tau_p(X_{l_0}) \|T_{l_0}\|_{\mathscr{L}(X)} n_{l_0}^{-\frac{r}{d}-1+\frac{1}{p}} \\ &\qquad + c_2 \sum_{l=l_0+1}^{l_*} \tau_p(X_{l-1,l}) \|(T_l - T_{l-1})J\|_{\mathscr{L}(Y,X)} n_l^{-\frac{r_1}{d}-1+\frac{1}{p}} \\ &\qquad + c_2 \sum_{l=l^*+1}^{l_1} \|(T_l - T_{l-1})J\|_{\mathscr{L}(Y,X)} n_l^{-\frac{r_1}{d}}. \end{aligned} \tag{4.52}$$

Proof. Let $f \in B_{C^r(Q,X)} \cap B_{C^{r_1}(Q,Y)}$. From (4.46) and the linearity of $\mathcal{S}_\iota$, we get

$$\begin{aligned} &\|\mathcal{S}_\iota f - \mathfrak{A}_\omega^\iota f\|_{G_\iota(X)} \\ &\quad \le \|\mathcal{S}_\iota f - \mathcal{S}_\iota R_{l_1} f\|_{G_\iota(X)} + \|\mathcal{S}_\iota R_{l_0} f - \mathcal{A}_{n_{l_0},\omega}^{\iota,r} R_{l_0} f\|_{G_\iota(X_{l_0})} \\ &\qquad + \sum_{l=l_0+1}^{l_1} \|\mathcal{S}_\iota (R_l - R_{l-1}) f - \mathcal{A}_{n_l,\omega}^{\iota,r_1} (R_l - R_{l-1}) f\|_{G_\iota(X_{l-1,l})}. \end{aligned} \tag{4.53}$$

For the first part, we estimate

$$\begin{aligned}\|\mathcal{S}_\iota f - \mathcal{S}_\iota R_{l_1} f\|_{G_\iota(X)} &\le \|\mathcal{S}_\iota\|_{\mathscr{L}(C(Q,X),G_\iota(X))}\|f - R_{l_1} f\|_{C(Q,X)} \\ &\le \|J - T_{l_1} J\|_{\mathscr{L}(Y,X)}\|f\|_{C(Q,Y)} \\ &\le \|J - T_{l_1} J\|_{\mathscr{L}(Y,X)}. \end{aligned} \tag{4.54}$$

Furthermore, by Propositions 4.2.2 and 4.2.6

$$\begin{aligned}\|\mathcal{S}_\iota R_{l_0} f - \mathcal{A}^{\iota,r}_{n_{l_0},\omega} R_{l_0} f\|_{G_\iota(X_{l_0})} &\le c n_{l_0}^{-\frac{r}{d}} \|R_{l_0} f\|_{C^r(Q,X_{l_0})} \\ &\le c n_{l_0}^{-\frac{r}{d}} \|T_{l_0}\|_{\mathscr{L}(X)}\|f\|_{C^r(Q,X)} \\ &\le c\|T_{l_0}\|_{\mathscr{L}(X)} n_{l_0}^{-\frac{r}{d}}, \end{aligned} \tag{4.55}$$

and similarly,

$$\mathbb{E}\left(\|\mathcal{S}_\iota R_{l_0} f - \mathcal{A}^{\iota,r}_{n_{l_0},\omega} R_{l_0} f\|^p_{G_\iota(X_{l_0})}\right)^{\frac{1}{p}} \le c\tau_p(X_{l_0})\|T_{l_0}\|_{\mathscr{L}(X)} n_{l_0}^{-\frac{r}{d}-1+\frac{1}{p}}. \tag{4.56}$$

For $l_0 < l \le l_1$, we obtain

$$\begin{aligned}\|\mathcal{S}_\iota(R_l - R_{l-1})f - &\mathcal{A}^{\iota,r_1}_{n_l,\omega}(R_l - R_{l-1})f\|_{G_\iota(X_{l-1,l})} \\ &\le c n_l^{-r_1/d}\|(R_l - R_{l-1})f\|_{C^{r_1}(Q,X_{l-1,l})} \\ &\le c n_l^{-r_1/d}\|(T_l - T_{l-1})J\|_{\mathscr{L}(Y,X)}\|f\|_{C^{r_1}(Q,Y)} \\ &\le c\|(T_l - T_{l-1})J\|_{\mathscr{L}(Y,X)} n_l^{-\frac{r_1}{d}} \end{aligned} \tag{4.57}$$

and

$$\begin{aligned}\mathbb{E}&\left(\|\mathcal{S}_\iota(R_l - R_{l-1})f - \mathcal{A}^{\iota,r_1}_{n_l,\omega}(R_l - R_{l-1})f\|^p_{G_\iota(X_{l-1,l})}\right)^{\frac{1}{p}} \\ &\le c\tau_p(X_{l-1,l})\|(T_l - T_{l-1})J\|_{\mathscr{L}(Y,X)} n_l^{-\frac{r_1}{d}-1+\frac{1}{p}}. \end{aligned} \tag{4.58}$$

Combining (4.53),(4.54),(4.55),(4.57) yields (4.51). Moreover, using (4.57) and (4.58), we obtain

$$\begin{aligned}\sum_{l=l_0+1}^{l_1} &\mathbb{E}\left(\|\mathcal{S}_\iota(R_l - R_{l-1})f - \mathcal{A}^{\iota,r_1}_{n_l,\omega}(R_l - R_{l-1})f\|^p_{G_\iota(X_{l-1,l})}\right)^{\frac{1}{p}} \\ &\le \sum_{l=l_0+1}^{l_*} c\tau_p(X_{l-1,l})\|(T_l - T_{l-1})J\|_{\mathscr{L}(Y,X)} n_l^{-\frac{r_1}{d}-1+\frac{1}{p}} \\ &\quad + \sum_{l=l_*+1}^{l_1} c\|(T_l - T_{l-1})J\|_{\mathscr{L}(Y,X)} n_l^{-\frac{r_1}{d}}. \end{aligned} \tag{4.59}$$

Thus, (4.53),(4.54),(4.56),(4.59) yields (4.52). □

Chapter 5

Parametric Banach Space Valued Integration

The chapter is concerned with the complexity analysis of parametric Banach space valued multivariate integration. The considered classes of input functions are chosen in a general way and contain spaces of dominating mixed derivatives as well as other types of non-isotropic smoothness. We apply the multilevel algorithm of the previous chapter, and show how it fits to the parametric problem. As before, the definite and the indefinite cases are included. We state randomized algorithms and analyze both, the deterministic and the randomized setting. Considering the complexity of the problem, we further establish lower bounds for the general Banach space valued setting. Applications to various smoothness classes are given in the last section, together with some comments on the relation between the deterministic and the randomized setting.

The chapter is organized as follows: In Section 5.1, we give the formal definition of the considered problems, and in Section 5.2, the algorithms for the upper bounds are introduced. In Section 5.3, the main complexity results are stated and applications to various smoothness classes are given in the last section.

5.1 Problem Formulation

Let $d_0 \in \mathbb{N}$, $Q_0 = [0,1]^{d_0}$, and Z be an arbitrary Banach space. We study definite and indefinite integration of functions depending on a parameter $s \in Q_0$.

For $r_0, r \in \mathbb{N}_0$, let $C^{r_0,r}(Q_0 \times Q, Z)$ be the space of continuous functions $f : Q_0 \times Q \to Z$ having, for $\alpha = (\alpha_0, \alpha_1)$, $\alpha_0 \in \mathbb{N}_0^{d_0}$, $\alpha_1 \in \mathbb{N}_0^d$ with $|\alpha_0| \le r_0$, $|\alpha_1| \le r$, continuous partial derivatives $\frac{\partial^{|\alpha|} f(s,t)}{\partial s^{\alpha_0} \partial t^{\alpha_1}}$; endowed with the norm

$$\|f\|_{C^{r_0,r}(Q_0 \times Q, Z)} = \max_{|\alpha_0| \le r_0, |\alpha_1| \le r} \sup_{s \in Q_0, t \in Q} \left\| \frac{\partial^{|\alpha|} f(s,t)}{\partial s^{\alpha_0} \partial t^{\alpha_1}} \right\|.$$

Let furthermore $r_1 \in \mathbb{N}_0$ and put

$$\mathscr{F}_0 = B_{C^{0,r}(Q_0\times Q,Z)} \cap B_{C^{r_0,r_1}(Q_0\times Q,Z)}.$$

The definite parametric integration operator $\mathscr{S}_0 : C(Q_0 \times Q, Z) \to C(Q_0, Z)$ is given by

$$(\mathscr{S}_0 f)(s) = \int_Q f(s,t)dt \quad (s \in Q_0),$$

and the indefinite parametric integration operator $\mathscr{S}_1 : C(Q_0 \times Q, Z) \to C(Q_0 \times Q, Z)$ by

$$(\mathscr{S}_1 f)(s,t) = \int_{[0,t]} f(s,u)du \quad (s \in Q_0, t \in Q).$$

Remark 5.1.1. For $r < r_1$, we obtain

$$\begin{aligned} B_{C^{0,r}(Q_0\times Q,Z)} \cap B_{C^{r_0,r_1}(Q_0\times Q,Z)} &= B_{C^{r_0,r_1}(Q_0\times Q,Z)} \\ &= B_{C^{0,r_1}(Q_0\times Q,Z)} \cap B_{C^{r_0,r_1}(Q_0\times Q,Z)}. \end{aligned} \tag{5.1}$$

Hence, we can assume $r \geq r_1$ without loss of generality.

5.2 Algorithms and Convergence Analysis

To connect parametric integration with Banach space valued integration as considered in Chapter 4, we set $X = C(Q_0, Z)$ and $Y = C^{r_0}(Q_0, Z)$. Thus, $C(Q_0 \times Q, Z) = C(Q, X)$ and

$$\mathscr{S}_\iota = \mathcal{S}_\iota^{C(Q_0,Z)} \qquad (\iota = 0, 1).$$

Moreover,

$$\begin{aligned} B_{C^{0,r}(Q_0\times Q,Z)} \cap B_{C^{r_0,r_1}(Q_0\times Q,Z)} &= B_{C^r(Q,C(Q_0,Z))} \cap B_{C^{r_1}(Q,C^{r_0}(Q_0,Z))} \\ &= B_{C^r(Q,X)} \cap B_{C^{r_1}(Q,Y)}. \end{aligned}$$

Let $\hat{r}_0 = \max(r_0, 1)$ and define for $l \in \mathbb{N}_0$

$$\mathscr{P}_l = P_{2^l}^{\hat{r}_0,d_0} \in \mathscr{L}(C(Q_0, Z)) \tag{5.2}$$

and set

$$T_l = \mathscr{P}_l \qquad (l \in \mathbb{N}_0). \tag{5.3}$$

This way the algorithm $\mathfrak{A}_\omega^\iota$ defined in (4.47) becomes

$$\mathscr{A}_\omega^\iota = \mathscr{P}_{l_0} \otimes \mathcal{A}_{n_{l_0},\omega}^{\iota,r,\mathbb{K}} + \sum_{l=l_0+1}^{l_1} (\mathscr{P}_l - \mathscr{P}_{l-1}) \otimes \mathcal{A}_{n_l,\omega}^{\iota,r_1,\mathbb{K}}.$$

For $f \in C(Q_0 \times Q, Z)$ this means

$$\mathscr{A}_\omega^\iota f \;=\; \mathscr{P}_{l_0}\Big(\big(\mathcal{A}_{n_{l_0},\omega}^{\iota,r,\mathbb{K}}(f_s)\big)_{s\in\Gamma_{2^{l_0}}^{\hat r_0,d_0}}\Big) + \sum_{l=l_0+1}^{l_1} (\mathscr{P}_l - \mathscr{P}_{l-1})\Big(\big(\mathcal{A}_{n_l,\omega}^{\iota,r_1,\mathbb{K}}(f_s)\big)_{s\in\Gamma_{2^l}^{\hat r_0,d_0}}\Big),$$

where we use the notation $f_s = f(s,\,\cdot\,)$ for $s \in Q_0$. It is clear, by the definition of $\mathscr{A}_\omega^\iota$, that

$$\operatorname{card}(\mathscr{A}_\omega^\iota) \le c \sum_{l=l_0}^{l_1} n_l 2^{d_0 l} \quad (\omega \in \Omega). \tag{5.4}$$

First, we estimate the error of $\mathscr{A}_\omega^\iota$. Recall the notation $G_0(C(Q_0)) = C(Q_0, Z)$ and $G_1(C(Q_0)) = C(Q_0 \times Q, Z)$.

Theorem 5.2.1. *Let $r_0, r, r_1 \in \mathbb{N}_0$, $r \ge r_1$, $\iota \in \{0,1\}, 1 \le p \le 2$, and let Z be an arbitrary Banach space. Then there are constants $c_1, c_2 > 0$ such that for all $l_0, l_1 \in \mathbb{N}_0$ with $l_0 \le l_1$, and for all $(n_l)_{l=l_0}^{l_1} \subset \mathbb{N}$, we have*

$$\sup_{f\in\mathscr{F}_0} \|\mathscr{S}_\iota f - \mathscr{A}_\omega^\iota f\|_{G_\iota(C(Q_0))}$$
$$\le c_1 2^{-r_0 l_1} + c_1 n_{l_0}^{-\frac{r}{d}} + c_1 \sum_{l=l_0+1}^{l_1} 2^{-r_0 l} n_l^{-\frac{r_1}{d}} \quad (\omega \in \Omega), \tag{5.5}$$

and for $l_0 \le l^* \le l_1$

$$\begin{aligned} &\sup_{f\in\mathscr{F}_0} \Big(\mathbb{E}\,\|\mathscr{S}_\iota f - \mathscr{A}_\omega^\iota f\|_{G_\iota(C(Q_0))}^p\Big)^{\frac{1}{p}} \\ &\le\; c_2 2^{-r_0 l_1} + c_2 \tau_p(Z)(l_0+1)^{\frac{1}{2}} n_{l_0}^{-\frac{r}{d}-1+\frac{1}{p}} \\ &\quad + c_2 \sum_{l=l_0+1}^{l^*} \tau_p(Z)(l+1)^{\frac{1}{2}} 2^{-r_0 l} n_l^{-\frac{r_1}{d}-1+\frac{1}{p}} + c_2 \sum_{l=l^*+1}^{l_1} 2^{-r_0 l} n_l^{-\frac{r_1}{d}}. \end{aligned} \tag{5.6}$$

Proof. By (2.8),(2.9), and (5.2),

$$\|\mathscr{P}_l\|_{\mathscr{L}(C(Q_0,Z))} \le c_1, \tag{5.7}$$
$$\|J - \mathscr{P}_l J\|_{\mathscr{L}(C^{r_0}(Q_0,Z),C(Q_0,Z))} \le c_2 2^{-r_0 l}, \tag{5.8}$$

where $J : C^{r_0}(Q_0, Z) \to C(Q_0, Z)$ is the embedding operator. Moreover, by (4.48) and (5.3),

$$X_l = P_{2^l}^{\hat r_0,d_0}(C(Q_0,Z)) = P_{2^l}^{\hat r_0,d_0}(\ell_\infty(\Gamma_{2^l}^{\hat r_0,d_0}, Z)).$$

Consequently, $X_{l-1} \subseteq X_l$ for $l \ge 1$. Therefore, $X_{l-1,l} \subseteq X_l$ and

$$\tau_p(X_{l-1,l}) \le \tau_p(X_l). \tag{5.9}$$

Moreover, $P_{2^l}^{\hat{r}_0,d_0} : \ell_\infty(\Gamma_{2^l}^{\hat{r}_0,d_0}, Z) \to X_l$ is an isomorphism which satisfies

$$\left\| P_{2^l}^{\hat{r}_0,d_0} \right\| \le c_1, \quad \left\| \left(P_{2^l}^{\hat{r}_0,d_0} \right)^{-1} \right\| = 1.$$

Indeed, the first estimate is just (2.8), the second estimate is a consequence of the fact that the inverse of the interpolation operator is just the restriction of functions in X_l to $\Gamma_{2^l}^{\hat{r}_0,d_0}$. It follows that

$$\tau_p(X_{l-1,l}) \le \tau_p(X_l) \le c\tau_2\Big(\ell_\infty\big(\Gamma_{2^l}^{\hat{r}_0,d_0}, Z\big)\Big) \le c(l+1)^{\frac{1}{2}}\tau_p(Z). \tag{5.10}$$

Now relations (5.5) and (5.6) are a direct consequence of Proposition 4.4.2 together with (5.7-5.10). □

Remark 5.2.2. Notice that the natural case of estimate (5.6) would be $l^* = l_1$. However, the more general approach will lead to sharper estimates, including precise powers of logarithms in several cases.

The further estimate of (5.5) and (5.6) will be covered in an additional lemma. It contains the key estimates for the upper bound proof. It is formulated in a general way, which allows some shortcuts to use these estimates also directly for the analysis of parametric initial value problems, where different but related smoothness classes are considered.

Let $\beta, \beta_0, \beta_1, \beta_2 \in \mathbb{R}$. Given $l_0, l^*, l_1 \in \mathbb{N}_0$ with $l_0 \le l^* \le l_1$ and $(n_l)_{l=l_0}^{l_1} \subset \mathbb{N}$, we define

$$M\big(l_0, l_1, (n_l)_{l=l_0}^{l_1}\big) = 2^{-\beta_0 d_0 l_1} + n_{l_0}^{-\beta} + \sum_{l=l_0+1}^{l_1} 2^{-\beta_0 d_0 l} n_l^{-\beta_1}, \tag{5.11}$$

$$\begin{aligned} E\big(l_0, l^*, l_1, (n_l)_{l=l_0}^{l_1}\big) &= 2^{-\beta_0 d_0 l_1} + (l_0+1)^{\frac{1}{2}} n_{l_0}^{-\beta} + \sum_{l=l_0+1}^{l^*} (l+1)^{\frac{1}{2}} 2^{-\beta_0 d_0 l} n_l^{-\beta_1} \\ &\quad + \sum_{l=l^*+1}^{l_1} 2^{-\beta_0 d_0 l} n_l^{-\beta_2}. \end{aligned} \tag{5.12}$$

The connection of M and E to (5.5) and (5.6) is easily seen. The corresponding choices of algorithm parameters l_0, l^*, l_1 and $(n_l)_{l=l_0}^{l_1}$ can be found in the proof of the lemma. Recall that $\wedge$ and $\vee$ mean logical conjunction and disjunction, respectively.

Lemma 5.2.3. *Let $\beta, \beta_0, \beta_1 \in \mathbb{R}$ with $\beta_0 \ge 0$ and $\beta \ge \beta_1 \ge 0$. Then there are constants $c_{1-3} > 0$ such that for each $n \in \mathbb{N}$ with $n \ge 2$ there is a choice of $l_0, l_1 \in \mathbb{N}_0$, $l_0 \le l_1$, and $(n_l)_{l=l_0}^{l_1} \subset \mathbb{N}$ such that*

$$\sum_{l=l_0}^{l_1} n_l 2^{d_0 l} \le c_1 n \tag{5.13}$$

and

$$M\big(l_0, l_1, (n_l)_{l=l_0}^{l_1}\big) \le c_2 \begin{cases} n^{-\upsilon} & \text{if} \quad \beta_0 > \beta_1 \\ n^{-\beta_0}(\log n)^{\beta_0+1} & \text{if} \quad \beta_0 = \beta_1 > 0 \\ n^{-\beta_0} & \text{if} \quad \beta_0 = \beta_1 = 0 \;\vee\; \beta_0 < \beta_1, \end{cases} \tag{5.14}$$

where

$$\upsilon \;=\; \frac{\beta_0\beta}{\beta_0 + \beta - \beta_1}. \tag{5.15}$$

Moreover, let $1 \le p \le 2$ *and set* $\beta_2 = \beta_1 - 1 + 1/p$*. If* $\beta_1 \ge 1 - 1/p$*, then for each* $n \in \mathbb{N}$ *with* $n > 2$ *there is a choice of* $l_0, l^*, l_1 \in \mathbb{N}_0$*,* $l_0 \le l^* \le l_1$*, and* $(n_l)_{l=l_0}^{l_1} \subset \mathbb{N}$ *satisfying* (5.13) *and*

$$E\big(l_0, l^*, l_1, (n_l)_{l=l_0}^{l_1}\big) \le c_3 \begin{cases} n^{-\beta} & \text{if} \quad \beta_0 > \beta_1 = \beta \\ n^{-\upsilon}(\log n)^{1/2} & \text{if} \quad \beta_0 > \beta_1 \wedge \beta > \beta_1 \\ n^{-\beta_0}(\log n)^{\beta_0+3/2} & \text{if} \quad \beta_0 = \beta_1 \\ n^{-\beta_0}(\log n)^{\frac{p(\beta_0-\beta_2)}{2(p-1)}} & \text{if} \quad \beta_2 < \beta_0 < \beta_1 \\ n^{-\beta_0}(\log\log n)^{\beta_0+1} & \text{if} \quad \beta_0 = \beta_2. \end{cases} \tag{5.16}$$

Proof. In the case $\beta_0 = 0$, the statements trivially follow from (5.11) and (5.12) with $l_0 = l_1 = 0$ and $n_0 = 1$. Therefore, we can assume $\beta_0 > 0$ in the sequel. Let $n \in \mathbb{N}$, $n \ge 2$, and put

$$l_1 = \left\lceil \frac{\log n}{d_0} \right\rceil, \quad l_0 = \left\lfloor \frac{\beta - \beta_1}{\beta_0 + \beta - \beta_1}\, l_1 \right\rfloor \tag{5.17}$$

(recall that log always means $\log_2$). Notice that (5.17) implies

$$l_1 - l_0 \ge \frac{\beta_0 l_1}{\beta_0 + \beta - \beta_1}, \tag{5.18}$$

hence

$$(\beta - \beta_1)(l_1 - l_0) \ge \frac{(\beta - \beta_1)\beta_0 l_1}{\beta_0 + \beta - \beta_1} \ge \beta_0 l_0,$$

and thus,

$$\beta(l_1 - l_0) \ge \beta_0 l_0 + \beta_1(l_1 - l_0). \tag{5.19}$$

Let $\sigma \in \{0, 1\}$, $\delta_0, \delta_1 \ge 0$ to be fixed later on and set

$$n_{l_0} \;=\; 2^{d_0(l_1 - l_0)}, \tag{5.20}$$

$$n_l \;=\; \big\lceil (l_1 + 1)^{-\sigma} 2^{d_0(l_1 - l) - \delta_0(l - l_0) - \delta_1(l_1 - l)} \big\rceil \quad (l = l_0 + 1, \ldots, l_1). \tag{5.21}$$

This gives

$$\sum_{l=l_0}^{l_1} n_l 2^{d_0 l} \quad \leq \quad c2^{d_0 l_1} + (l_1+1)^{-\sigma} \sum_{l=l_0+1}^{l_1} 2^{d_0 l_1 - \delta_0(l-l_0) - \delta_1(l_1-l)} \leq cn, \tag{5.22}$$

provided $\delta_0 > 0$ or $\delta_1 > 0$ or $\sigma = 1$. By (5.19) and (5.20), we have

$$n_{l_0}^{-\beta} = 2^{-\beta d_0(l_1-l_0)} \leq 2^{-\beta_0 d_0 l_0 - \beta_1 d_0(l_1-l_0)} \leq 2^{-\beta_0 d_0 l_0 - \beta_1 d_0(l_1-l_0) + \beta_1 \delta_1 (l_1-l_0)} , \tag{5.23}$$

and, using (5.21), for $l_0 < l \leq l_1$

$$2^{-\beta_0 d_0 l} n_l^{-\beta_1} \quad \leq \quad (l_1+1)^{\sigma\beta_1} 2^{-\beta_0 d_0 l - \beta_1 d_0(l_1-l) + \beta_1\delta_0(l-l_0) + \beta_1\delta_1(l_1-l)}. \tag{5.24}$$

Furthermore,

$$\begin{aligned} &-\beta_0 d_0 l - \beta_1 d_0(l_1-l) + \beta_1\delta_0(l-l_0) + \beta_1\delta_1(l_1-l) \\ &\qquad = -\beta_0 d_0 l_0 - (\beta_0 d_0 - \beta_1\delta_0)\,(l-l_0) \\ &\qquad\qquad - \beta_1(d_0-\delta_1)(l_1-l) \quad (l_0 \leq l \leq l_1). \end{aligned} \tag{5.25}$$

By (5.11) and (5.23–5.25),

$$\begin{aligned} &M\big(l_0, l_1, (n_l)_{l=l_0}^{l_1}\big) \\ &\leq \;\; 2^{-\beta_0 d_0 l_1} + (l_1+1)^{\sigma\beta_1} \sum_{l=l_0}^{l_1} 2^{-\beta_0 d_0 l_0 - (\beta_0 d_0 - \beta_1\delta_0)(l-l_0) - \beta_1(d_0-\delta_1)(l_1-l)}. \end{aligned} \tag{5.26}$$

If $\beta_0 > \beta_1$, we set $\sigma = \delta_1 = 0$ and choose $\delta_0 > 0$ in such a way that $\beta_0 d_0 - \beta_1\delta_0 > \beta_1 d_0$. From (5.26), we obtain

$$\begin{aligned} M\big(l_0, l_1, (n_l)_{l=l_0}^{l_1}\big) \quad &\leq \quad 2^{-\beta_0 d_0 l_1} + \sum_{l=l_0}^{l_1} 2^{-\beta_0 d_0 l_0 - (\beta_0 d_0 - \beta_1\delta_0)(l-l_0) - \beta_1 d_0(l_1-l)} \\ &\leq \quad 2^{-\beta_0 d_0 l_1} + c2^{-\beta_0 d_0 l_0 - \beta_1 d_0(l_1-l_0)}. \end{aligned} \tag{5.27}$$

Notice that by (5.15), (5.17), and (5.18),

$$\begin{aligned} \beta_0 l_0 + \beta_1(l_1-l_0) \quad &\geq \quad \frac{\beta_0(\beta-\beta_1)l_1}{\beta_0+\beta-\beta_1} - \beta_0 + \frac{\beta_1\beta_0 l_1}{\beta_0+\beta-\beta_1} \\ &= \quad \frac{\beta_0\beta l_1}{\beta_0+\beta-\beta_1} - \beta_0 = \upsilon l_1 - \beta_0 \end{aligned} \tag{5.28}$$

and, since $\beta_0 > \beta_1$,

$$\upsilon = \frac{\beta_0\beta}{\beta_0+\beta-\beta_1} < \beta_0. \tag{5.29}$$

It follows from (5.17) and (5.27–5.29) that

$$M\big(l_0, l_1, (n_l)_{l=l_0}^{l_1}\big) \;\leq\; 2^{-\beta_0 d_0 l_1} + c2^{-v d_0 l_1} \leq c2^{-v d_0 l_1} \leq cn^{-v}.$$

This together with (5.22) proves (5.14) for $\beta_0 > \beta_1$.

If $\beta_0 = \beta_1 > 0$, we set $\sigma = 1$, $\delta_0 = \delta_1 = 0$, and get from (5.17) and (5.26)

$$\begin{aligned} M\big(l_0, l_1, (n_l)_{l=l_0}^{l_1}\big) &\leq 2^{-\beta_0 d_0 l_1} + (l_1+1)^{\beta_0} \sum_{l=l_0}^{l_1} 2^{-\beta_0 d_0 l_0 - \beta_0 d_0 (l-l_0) - \beta_0 d_0 (l_1 - l)} \\ &\leq c(l_1+1)^{\beta_0+1} 2^{-\beta_0 d_0 l_1} \leq cn^{-\beta_0} (\log n)^{\beta_0+1}. \end{aligned}$$

Combining this with (5.22) gives the respective estimate of (5.14).

Since we assumed $\beta_0 > 0$, it remains to consider the case $\beta_0 < \beta_1$, where we set $\sigma = \delta_0 = 0$ and choose $\delta_1 > 0$ in such a way that $\beta_1(d_0 - \delta_1) > \beta_0 d_0$. By (5.17) and (5.26)

$$\begin{aligned} M\big(l_0, l_1, (n_l)_{l=l_0}^{l_1}\big) &\leq 2^{-\beta_0 d_0 l_1} + \sum_{l=l_0}^{l_1} 2^{-\beta_0 d_0 l_0 - \beta_0 d_0 (l-l_0) - \beta_1 (d_0 - \delta_1)(l_1 - l)} \\ &\leq 2^{-\beta_0 d_0 l_1} + c2^{-\beta_0 d_0 l_0 - \beta_0 d_0 (l_1 - l_0)} \leq cn^{-\beta_0}. \end{aligned}$$

This together with (5.22) completes the proof of (5.14).

Now, we turn to the proof of (5.16) and assume that $\beta_1 \geq 1 - 1/p$. If $\beta_0 > \beta_1 = \beta$, then we set $l^* = l_1$, $\sigma = \delta_1 = 0$, and choose $\delta_0 > 0$ satisfying $\beta_0 d_0 - \beta_1 \delta_0 > \beta_1 d_0$. It follows from (5.17) that $l_0 = 0$. Then (5.12), (5.23), and (5.24) give

$$\begin{aligned} &E\big(l_0, l_1, l_1, (n_l)_{l=l_0}^{l_1}\big) \\ &\leq 2^{-\beta_0 d_0 l_1} + 2^{-\beta_1 d_0 l_1} + \sum_{l=1}^{l_1} (l+1)^{\frac{1}{2}} 2^{-\beta_0 d_0 l - \beta_1 d_0 (l_1 - l) + \beta_1 \delta_0 l} \\ &\leq 2^{-\beta_0 d_0 l_1} + \sum_{l=0}^{l_1} (l+1)^{\frac{1}{2}} 2^{-(\beta_0 d_0 - \beta_1 \delta_0) l - \beta_1 d_0 (l_1 - l)} \\ &\leq 2^{-\beta_0 d_0 l_1} + c2^{-\beta d_0 l_1} \leq cn^{-\beta}, \end{aligned}$$

which together with (5.22) proves the first case of (5.16).

If $(\beta_0 > \beta_1 \wedge \beta > \beta_1)$ or $\beta_0 = \beta_1$, we choose $l^* = l_1$ and get from (5.11–5.12)

$$E\big(l_0, l_1, l_1, (n_l)_{l=l_0}^{l_1}\big) \;\leq\; (l_1+1)^{\frac{1}{2}} M\big(l_0, l_1, (n_l)_{l=l_0}^{l_1}\big)$$

and the desired results follow from (5.17) and the respective cases of (5.14).

It remains to consider the case

$$\beta_2 \le \beta_0 < \beta_1. \tag{5.30}$$

Here we make another choice of the parameters $(n_l)_{l=l_0}^{l_1}$ (while l_0 and l_1 remain the same, given by (5.17)). Let $\sigma \in \{0,1\}$, $\delta_1, \delta_2 \ge 0$, and $l^* \in \mathbb{N}_0$ with $l_0 \le l^* \le l_1$ to be fixed later on and set

$$\begin{aligned} n_{l_0} &= 2^{d_0(l_1-l_0)}, & (5.31)\\ n_l &= \lceil 2^{d_0(l_1-l)-\delta_1(l^*-l)} \rceil \quad (l = l_0+1,\dots,l^*), & (5.32)\\ n_l &= \lceil (l_1-l^*+1)^{-\sigma} 2^{d_0(l_1-l)-\delta_2(l-l^*)} \rceil \quad (l = l^*+1,\dots,l_1). & (5.33) \end{aligned}$$

In the sequel, we need the following estimate that results from (5.17) and (5.31–5.33):

$$\begin{aligned} \sum_{l=l_0}^{l_1} n_l 2^{d_0 l} &\le c 2^{d_0 l_1} + \sum_{l=l_0+1}^{l^*} 2^{d_0 l_1 - \delta_1(l^*-l)} + (l_1-l^*+1)^{-\sigma} \sum_{l=l^*+1}^{l_1} 2^{d_0 l_1 - \delta_2(l-l^*)} \\ &\le cn \end{aligned} \tag{5.34}$$

whenever $(\delta_1 > 0 \wedge \delta_2 > 0)$ or $(\sigma = 1 \wedge \delta_1 > 0)$. Using (5.19) and (5.31), we obtain

$$n_{l_0}^{-\beta} = 2^{-\beta d_0(l_1-l_0)} \le 2^{-\beta_0 d_0 l_0 - \beta_1 d_0(l_1-l_0)} \le 2^{-\beta_0 d_0 l - \beta_1 d_0(l_1-l_0) + \beta_1\delta_1(l^*-l_0)}. \tag{5.35}$$

From (5.32–5.33), we get

$$\begin{aligned} 2^{-\beta_0 d_0 l} n_l^{-\beta_1} &\le 2^{-\beta_0 d_0 l - \beta_1 d_0(l_1-l) + \beta_1\delta_1(l^*-l)} \quad (l_0 < l \le l^*), & (5.36)\\ 2^{-\beta_0 d_0 l} n_l^{-\beta_2} &\le (l_1-l^*+1)^{\sigma\beta_2} 2^{-\beta_0 d_0 l - \beta_2 d_0(l_1-l) + \beta_2\delta_2(l-l^*)} \quad (l^* < l \le l_1). & (5.37) \end{aligned}$$

Moreover, for $l_0 \le l \le l^*$

$$\begin{aligned} &-\beta_0 d_0 l - \beta_1 d_0(l_1-l) + \beta_1\delta_1(l^*-l) \\ &\qquad = -\beta_0 d_0 l_0 - \beta_1 d_0(l_1-l^*) - \beta_0 d_0(l-l_0) - \beta_1(d_0-\delta_1)(l^*-l), \end{aligned} \tag{5.38}$$

and for $l^*+1 \le l \le l_1$

$$\begin{aligned} &-\beta_0 d_0 l - \beta_2 d_0(l_1-l) + \beta_2\delta_2(l-l^*) \\ &\qquad = -\beta_0 d_0 l^* - \beta_2 d_0(l_1-l) - (\beta_0 d_0 - \beta_2\delta_2)(l-l^*). \end{aligned} \tag{5.39}$$

Now, (5.12) and (5.35–5.39) imply

$$E\big(l_0, l^*, l_1, (n_l)_{l=l_0}^{l_1}\big) \le 2^{-\beta_0 d_0 l_1} + E_1 + E_2, \tag{5.40}$$

where

$$E_1 = \sum_{l=l_0}^{l^*} (l+1)^{\frac{1}{2}} 2^{-\beta_0 d_0 l_0 - \beta_1 d_0 (l_1 - l^*) - \beta_0 d_0 (l - l_0) - \beta_1 (d_0 - \delta_1)(l^* - l)}, \quad (5.41)$$

$$E_2 = (l_1 - l^* + 1)^{\sigma \beta_2} \sum_{l=l^*+1}^{l_1} 2^{-\beta_0 d_0 l^* - \beta_2 d_0 (l_1 - l) - (\beta_0 d_0 - \beta_2 \delta_2)(l - l^*)}. \quad (5.42)$$

By (5.30), $p > 1$. We put

$$l^* = l_1 - \left\lceil \frac{p \log(l_1 + 1)}{2(p-1) d_0} \right\rceil \quad (5.43)$$

and observe that the assumption $\beta_0 > 0$, (5.17), and (5.43) imply that there is a constant $c_0 \in \mathbb{N}$ such that for $n \geq c_0$

$$l_0 < l^* \leq l_1. \quad (5.44)$$

Since for $n < c_0$ the estimate (5.16) trivially follows from (5.40–5.42) by a suitable choice of the constant, we can assume $n \geq c_0$, and thus (5.44). We choose $\delta_1 > 0$ in such a way that $\beta_0 d_0 < \beta_1 (d_0 - \delta_1)$. Then by (5.41), (5.43), and (5.17

$$\begin{aligned} E_1 &\leq c(l_1+1)^{\frac{1}{2}} 2^{-\beta_0 d_0 l_0 - \beta_1 d_0 (l_1 - l^*) - \beta_0 d_0 (l^* - l_0)} \\ &= c(l_1+1)^{\frac{1}{2}} 2^{-\beta_0 d_0 l_1 + (\beta_0 d_0 - \beta_1 d_0)(l_1 - l^*)} \\ &\leq c(l_1+1)^{\frac{1}{2} + \frac{p(\beta_0 - \beta_1)}{2(p-1)}} 2^{-\beta_0 d_0 l_1} \\ &\leq c n^{-\beta_0} (\log n)^{\frac{p-1+p(\beta_0-\beta_1)}{2(p-1)}} \\ &= c n^{-\beta_0} (\log n)^{\frac{p(\beta_0-\beta_2)}{2(p-1)}}. \end{aligned} \quad (5.45)$$

Now we deal with E_2 and distinguish between two sub-cases of (5.30). If $\beta_2 < \beta_0$, we set $\sigma = 0$ and choose $\delta_2 > 0$ in such a way that $\beta_2 d_0 < \beta_0 d_0 - \beta_2 \delta_2$. Then, using (5.17),(5.42), and (5.43),

$$\begin{aligned} E_2 &\leq c 2^{-\beta_0 d_0 l^* - \beta_2 d_0 (l_1 - l^*)} \\ &= c 2^{-\beta_0 d_0 l_1 + (\beta_0 d_0 - \beta_2 d_0)(l_1 - l^*)} \\ &\leq c 2^{-\beta_0 d_0 l_1} (l_1 + 1)^{\frac{p(\beta_0-\beta_2)}{2(p-1)}} \\ &\leq c n^{-\beta_0} (\log n)^{\frac{p(\beta_0-\beta_2)}{2(p-1)}}. \end{aligned} \quad (5.46)$$

Combining (5.40),(5.45),(5.46), and taking into account (5.34), we obtain the fourth case of (5.16). If $\beta_2 = \beta_0$, we set $\sigma = 1$ and $\delta_2 = 0$. Here we have

$$E_2 \leq c(l_1 - l^* + 1)^{\beta_2 + 1} 2^{-\beta_0 d_0 l_1} \leq c n^{-\beta_0} (\log \log n)^{\beta_0 + 1}. \quad (5.47)$$

The last case of (5.16) is now a consequence of (5.40),(5.45),(5.47), and (5.34).

□

With help of the previous lemma we are now ready to estimate the error of $\mathscr{A}_\omega^\iota$. For this purpose we only have to connect the setting from Theorem 5.2.1 to the general setting estimated in Lemma 5.2.3.

Corollary 5.2.4. *Let $r_0, r, r_1 \in \mathbb{N}_0$, $r \geq r_1$, $d, d_0 \in \mathbb{N}$, $\iota \in \{0,1\}, 1 \leq p \leq 2$, and let Z be an arbitrary Banach space. Then there are constants c_{1-4} such that for each $n \in \mathbb{N}$ with $n \geq 2$ there is a choice of $l_0, l_1 \in \mathbb{N}_0$, and $(n_l)_{l=l_0}^{l_1} \subset \mathbb{N}$ such that $l_0 \leq l_1$,*

$$\operatorname{card}(\mathscr{A}_\omega^\iota) \leq c_1 n \qquad (\omega \in \Omega),$$

and

$$\sup_{f \in \mathscr{F}_0} \|\mathscr{S}_\iota f - \mathscr{A}_\omega^\iota f\|_{G_\iota(C(Q_0))} \leq c_2 \begin{cases} n^{-\upsilon_1} & \text{if } \frac{r_0}{d_0} > \frac{r_1}{d} \\ n^{-\frac{r_0}{d_0}} (\log n)^{\frac{r_0}{d_0}+1} & \text{if } \frac{r_0}{d_0} = \frac{r_1}{d} > 0 \\ n^{-\frac{r_0}{d_0}} & \text{if } \frac{r_0}{d_0} = \frac{r_1}{d} = 0 \vee \frac{r_0}{d_0} < \frac{r_1}{d}, \end{cases} \qquad (\omega \in \Omega) \quad (5.48)$$

where

$$\upsilon_1 = \frac{\frac{r_0}{d_0}}{\frac{r_0}{d_0} + \frac{r}{d} - \frac{r_1}{d}} \frac{r}{d}. \qquad (5.49)$$

Moreover, for all Banach spaces Z with $\tau_p(Z) < \infty$ and each $n \in \mathbb{N}$ with $n > 2$, there is a choice of $l_0, l^*, l_1 \in \mathbb{N}_0$, and $(n_l)_{l=l_0}^{l_1} \subset \mathbb{N}$ such that $l_0 \leq l^* \leq l_1$,

$$\operatorname{card}(\mathscr{A}_\omega^\iota) \leq c_3 n \qquad (\omega \in \Omega),$$

and

$$\sup_{f \in \mathscr{F}_0} \left(\mathbb{E} \|\mathscr{S}_\iota f - \mathscr{A}_\omega^\iota f\|_{G_\iota(C(Q_0))}^p \right)^{\frac{1}{p}} \leq c_4 \begin{cases} n^{-\frac{r}{d}-1+\frac{1}{p}} & \text{if } \frac{r_0}{d_0} > \frac{r_1}{d} + 1 - \frac{1}{p} \wedge r = r_1 \\ n^{-\upsilon_2(p)} (\log n)^{\frac{1}{2}} & \text{if } \frac{r_0}{d_0} > \frac{r_1}{d} + 1 - \frac{1}{p} \wedge r > r_1 \\ n^{-\frac{r_0}{d_0}} (\log n)^{\frac{r_0}{d_0}+\frac{3}{2}} & \text{if } \frac{r_0}{d_0} = \frac{r_1}{d} + 1 - \frac{1}{p} \\ n^{-\frac{r_0}{d_0}} (\log n)^{\frac{p}{2(p-1)}\left(\frac{r_0}{d_0} - \frac{r_1}{d}\right)} & \text{if } \frac{r_1}{d} < \frac{r_0}{d_0} < \frac{r_1}{d} + 1 - \frac{1}{p} \\ n^{-\frac{r_0}{d_0}} (\log \log n)^{\frac{r_0}{d_0}+1} & \text{if } \frac{r_0}{d_0} = \frac{r_1}{d} > 0 \\ n^{-\frac{r_0}{d_0}} & \text{if } \frac{r_0}{d_0} < \frac{r_1}{d} \vee \frac{r_0}{d_0} = \frac{r_1}{d} = 0, \end{cases} \qquad (5.50)$$

where

$$\upsilon_2(p) \quad = \quad \begin{cases} \frac{\frac{r_0}{d_0}}{\frac{r_0}{d_0}+\frac{r}{d}-\frac{r_1}{d}}\left(\frac{r}{d}+1-\frac{1}{p}\right) & \textit{if} \quad \frac{r_0}{d_0} > \frac{r_1}{d}+1-\frac{1}{p} \\ \frac{r_0}{d_0} & \textit{if} \quad \frac{r_0}{d_0} \leq \frac{r_1}{d}+1-\frac{1}{p}. \end{cases} \tag{5.51}$$

Proof. We derive the upper bounds in (5.48) and (5.50) from (5.5), (5.6) of Theorem 5.2.1 and Lemma 5.2.3. To deal with (5.48), we set

$$\beta = \frac{r}{d}, \quad \beta_0 = \frac{r_0}{d_0}, \quad \beta_1 = \frac{r_1}{d}, \tag{5.52}$$

which together with (5.15) and (5.49) gives for $r_0/d_0 > r_1/d$ (thus $\beta_0 > \beta_1$)

$$\upsilon = \frac{\beta_0\beta}{\beta_0+\beta-\beta_1} = \frac{\frac{r_0}{d_0}\cdot\frac{r}{d}}{\frac{r_0}{d_0}+\frac{r}{d}-\frac{r_1}{d}} = \upsilon_1. \tag{5.53}$$

Furthermore, notice that (5.5) and (5.11) imply

$$\sup_{f\in\mathscr{F}_0} \|\mathscr{S}_\iota f - \mathscr{A}_\omega^\iota f\|_{G_\iota(C(Q_0))} \leq c\, M\big(l_0, l_1, (n_l)_{l=l_0}^{l_1}\big). \tag{5.54}$$

Now the upper bounds in (5.48) follow from (5.13–5.14) and (5.52–5.54). Finally, we consider (5.50) and choose

$$\beta = \frac{r}{d}+1-\frac{1}{p}, \quad \beta_0 = \frac{r_0}{d_0}, \quad \beta_1 = \frac{r_1}{d}+1-\frac{1}{p}, \quad \beta_2 = \frac{r_1}{d}, \tag{5.55}$$

which, using (5.15) and (5.51), gives for $r_0/d_0 > r_1/d + 1 - 1/p$ (thus $\beta_0 > \beta_1$)

$$\upsilon = \frac{\beta_0\beta}{\beta_0+\beta-\beta_1} = \frac{\frac{r_0}{d_0}\left(\frac{r}{d}+1-\frac{1}{p}\right)}{\frac{r_0}{d_0}+\frac{r}{d}-\frac{r_1}{d}} = \upsilon_2(p). \tag{5.56}$$

We conclude from (5.6) and (5.12) that for any l^* with $l_0 \leq l^* \leq l_1$

$$\sup_{f\in\mathscr{F}_0} \Big(\mathbb{E}\,\|\mathscr{S}_\iota f - \mathscr{A}_\omega^\iota f\|_{G_\iota(C(Q_0))}^p\Big)^{\frac{1}{p}} \leq c\, E\big(l_0, l^*, l_1, (n_l)_{l=l_0}^{l_1}\big). \tag{5.57}$$

The upper estimates in (5.50) are now a consequence of (5.13),(5.16), and (5.55–5.57); except for the last case of (5.50), which follows directly from the respective case of the deterministic setting (5.48). □

It is also possible to find a choice of $l_0, l^*, l_1 \in \mathbb{N}_0$ and $(n_l)_{l=l_0}^{l_1} \subset \mathbb{N}$ simultaneously for the deterministic and the randomized case, having the same order of convergence as above. This way we show that every realization of the randomized algorithm at least satisfies the deterministic optimal convergence order. However, in contrast to the result above, additional logarithmic factors occur.

Corollary 5.2.5. *Assume that the conditions of Corollary 5.2.4 are satisfied. Then there are constants $c_{1-3} > 0$ such that for all $n \in \mathbb{N}$ with $n \geq 2$ the following holds. Setting*

$$\begin{aligned} l_1 &= \left\lceil \frac{\log n}{d_0} \right\rceil, \quad l_0 = \left\lfloor \frac{\frac{r}{d} - \frac{r_1}{d}}{\frac{r_0}{d_0} + \frac{r}{d} - \frac{r_1}{d}} \, l_1 \right\rfloor, \\ n_l &= \left\lceil 2^{d_0(l_1 - l)} \right\rceil \quad (l_0 \leq l \leq l_1), \end{aligned}$$

the so-defined algorithm $(\mathscr{A}_\omega^\iota)_{\omega\in\Omega}$ fulfills

$$\operatorname{card}(\mathscr{A}_\omega^\iota) \leq c_1 n \log n \quad (\omega \in \Omega).$$

Moreover,

$$\sup_{f \in \mathscr{F}_0} \|\mathscr{S}_\iota(f) - \mathscr{A}_\omega^\iota(f, u_0)\|_{G_\iota(C(Q_0))}$$
$$\leq c_2 \begin{cases} n^{-\upsilon_1} & \text{if } \frac{r_0}{d_0} > \frac{r_1}{d} \\ n^{-\frac{r_0}{d_0}}(\log n) & \text{if } \frac{r_0}{d_0} = \frac{r_1}{d} > 0 \\ n^{-\frac{r_0}{d_0}} & \text{if } \frac{r_0}{d_0} = \frac{r_1}{d} = 0 \vee \frac{r_0}{d_0} < \frac{r_1}{d} \end{cases}$$

and finally, if $\tau_p(Z) < \infty$,

$$\sup_{f \in \mathscr{F}_0} \left(\mathbb{E} \, \|\mathscr{S}(f) - \mathscr{A}_\omega^\iota(f)\|_{G_\iota(C(Q_0))}^p \right)^{\frac{1}{p}}$$
$$\leq c_3 \begin{cases} n^{-\upsilon_2(p)}(\log n)^{\frac{1}{2}} & \text{if } \frac{r_0}{d_0} > \frac{r_1}{d} + 1 - \frac{1}{p} \\ n^{-\frac{r_0}{d_0}}(\log n)^{\frac{3}{2}} & \text{if } \frac{r_0}{d_0} = \frac{r_1}{d} + 1 - \frac{1}{p} \\ n^{-\frac{r_0}{d_0}}(\log n)^{\frac{1}{2}} & \text{if } \frac{r_0}{d_0} < \frac{r_1}{d} + 1 - \frac{1}{p}. \end{cases}$$

Proof. The estimates are similar to these in Lemma 5.2.3, we omit them. See also Corollary 4.3 in [11] as a further example. □

5.3 Complexity Analysis

We consider standard information consisting of values of f, so the class of information functionals is

$$\Lambda(Q_0 \times Q, Z) = \{\delta_{s,t} : \ s \in Q_0, t \in Q\},$$

where $\delta_{s,t}(f) = f(s,t)$ and $K = Z$. In terminology of Section 2.2, the definite parametric integration problem is described by the tuple

$$\Pi_0 = (B_{C^{0,r}(Q_0 \times Q, Z)} \cap B_{C^{r_0,r_1}(Q_0 \times Q, Z)}, C(Q_0, Z), \mathscr{S}_0, Z, \Lambda(Q_0 \times Q, Z))$$

and the indefinite parametric integration problem by

$$\Pi_1 = (B_{C^{0,r}(Q_0\times Q,Z)} \cap B_{C^{r_0,r_1}(Q_0\times Q,Z)}, C(Q_0 \times Q, Z), \mathscr{S}_1, Z, \Lambda(Q_0 \times Q, Z)).$$

What follows is the main complexity result for parametric Banach space valued definite and indefinite integration in the deterministic and the randomized setting.

Theorem 5.3.1. *Let $r_0, r, r_1 \in \mathbb{N}_0$, with $r \geq r_1$ and let Z be an arbitrary Banach space. Then in the deterministic setting,*

$$\begin{aligned} &e_n^{\det}(\mathscr{S}_\iota, \mathscr{F}_0) \asymp n^{-\upsilon_1} && \text{if } \tfrac{r_0}{d_0} > \tfrac{r_1}{d} \\ &n^{-\frac{r_0}{d_0}} \preceq e_n^{\det}(\mathscr{S}_\iota, \mathscr{F}_0) \preceq n^{-\frac{r_0}{d_0}} (\log n)^{\frac{r_0}{d_0}+1} && \text{if } \tfrac{r_0}{d_0} = \tfrac{r_1}{d} > 0 \\ &e_n^{\det}(\mathscr{S}_\iota, \mathscr{F}_0) \asymp n^{-\frac{r_0}{d_0}} && \text{if } \tfrac{r_0}{d_0} = \tfrac{r_1}{d} = 0 \vee \tfrac{r_0}{d_0} < \tfrac{r_1}{d}, \end{aligned} \tag{5.58}$$

where υ_1 was given in (5.49) by

$$\upsilon_1 \quad = \quad \frac{\frac{r_0}{d_0}}{\frac{r_0}{d_0} + \frac{r}{d} - \frac{r_1}{d}} \, \frac{r}{d} \, . \tag{5.59}$$

Moreover, let $1 \leq p \leq 2$ and assume that Z is of type p. Let p_Z denote the supremum of all p_1 such that Z is of type p_1. Then in the randomized setting,

$$n^{-\upsilon_2(p_Z)} \preceq e_n^{\det}(\mathscr{S}_\iota, \mathscr{F}_0) \preceq_{\log} n^{-\upsilon_2(p)}, \tag{5.60}$$

with $\upsilon_2(p)$ given in (5.51) by

$$\upsilon_2(p) \quad = \quad \begin{cases} \frac{\frac{r_0}{d_0}}{\frac{r_0}{d_0}+\frac{r}{d}-\frac{r_1}{d}} \left(\frac{r}{d} + 1 - \frac{1}{p}\right) & \text{if } \frac{r_0}{d_0} > \frac{r_1}{d} + 1 - \frac{1}{p} \\ \frac{r_0}{d_0} & \text{if } \frac{r_0}{d_0} \leq \frac{r_1}{d} + 1 - \frac{1}{p}. \end{cases} \tag{5.61}$$

It is easily seen from (5.61) that $\upsilon_2(p)$ is a continuous, monotonically increasing function of $p \in [1,2]$. It follows that the bounds in (5.60) are matching up to an arbitrarily small gap in the exponent.

Corollary 5.3.2. *Assume that the conditions of Theorem 5.3.1 hold. Then for each $\varepsilon > 0$*

$$n^{-\upsilon_2(p_Z)} \preceq e_n^{\text{ran}}(\mathscr{S}, \mathscr{F}) \preceq n^{-\upsilon_2(p_Z)+\varepsilon}.$$

If Z is such that p_Z is attained, and in particular if Z is of type 2, more precise estimates can be given. This is the content of the following theorem.

Theorem 5.3.3. *Assume that the conditions of Theorem 5.3.1 hold. If the supremum of types is attained, that is, Z is of type p_Z, then*

$$\begin{array}{ll}
e_n^{\mathrm{ran}}(\mathscr{S}_\iota,\mathscr{F}_0) \asymp n^{-\frac{r}{d}-1+\frac{1}{p_Z}} & \textit{if } \frac{r_0}{d_0} > \frac{r_1}{d}+1-\frac{1}{p_Z} \wedge r=r_1 \\
n^{-\upsilon_2(p_Z)} \preceq e_n^{\mathrm{ran}}(\mathscr{S}_\iota,\mathscr{F}_0) \preceq n^{-\upsilon_2(p_Z)}(\log n)^{\frac{1}{2}} & \textit{if } \frac{r_0}{d_0} > \frac{r_1}{d}+1-\frac{1}{p_Z} \wedge r>r_1 \\
n^{-\frac{r_0}{d_0}} \preceq e_n^{\mathrm{ran}}(\mathscr{S}_\iota,\mathscr{F}_0) \preceq n^{-\frac{r_0}{d_0}}(\log n)^{\frac{r_0}{d_0}+\frac{3}{2}} & \textit{if } \frac{r_0}{d_0} = \frac{r_1}{d}+1-\frac{1}{p_Z} \\
n^{-\frac{r_0}{d_0}} \preceq e_n^{\mathrm{ran}}(\mathscr{S}_\iota,\mathscr{F}_0) \preceq n^{-\frac{r_0}{d_0}}(\log n)^{\frac{p}{2(p-1)}\left(\frac{r_0}{d_0}-\frac{r_1}{d}\right)} & \textit{if } \frac{r_1}{d} < \frac{r_0}{d_0} < \frac{r_1}{d}+1-\frac{1}{p_Z} \\
n^{-\frac{r_0}{d_0}} \preceq e_n^{\mathrm{ran}}(\mathscr{S}_\iota,\mathscr{F}_0) \preceq n^{-\frac{r_0}{d_0}}(\log\log n)^{\frac{r_0}{d_0}+1} & \textit{if } \frac{r_0}{d_0} = \frac{r_1}{d} > 0 \\
e_n^{\mathrm{ran}}(\mathscr{S}_\iota,\mathscr{F}_0) \asymp n^{-\frac{r_0}{d_0}} & \textit{if } \frac{r_0}{d_0} = \frac{r_1}{d} = 0 \vee \frac{r_0}{d_0} < \frac{r_1}{d}.
\end{array}$$

Furthermore, if Z is of type 2,

$$\begin{array}{ll}
e_n^{\mathrm{ran}}(\mathscr{S}_\iota,\mathscr{F}_0) \asymp n^{-\frac{r}{d}-\frac{1}{2}} & \textit{if } \frac{r_0}{d_0} > \frac{r_1}{d}+\frac{1}{2} \wedge r=r_1 \\
e_n^{\mathrm{ran}}(\mathscr{S}_\iota,\mathscr{F}_0) \asymp n^{-\upsilon_2(2)}(\log n)^{\frac{1}{2}} & \textit{if } \frac{r_0}{d_0} > \frac{r_1}{d}+\frac{1}{2} \wedge r>r_1 \\
n^{-\frac{r_0}{d_0}}(\log n)^{\frac{1}{2}} \preceq e_n^{\mathrm{ran}}(\mathscr{S}_\iota,\mathscr{F}_0) \preceq n^{-\frac{r_0}{d_0}}(\log n)^{\frac{r_0}{d_0}+\frac{3}{2}} & \textit{if } \frac{r_0}{d_0} = \frac{r_1}{d}+\frac{1}{2} \\
e_n^{\mathrm{ran}}(\mathscr{S}_\iota,\mathscr{F}_0) \asymp n^{-\frac{r_0}{d_0}}(\log n)^{\frac{r_0}{d_0}-\frac{r_1}{d}} & \textit{if } \frac{r_1}{d} < \frac{r_0}{d_0} < \frac{r_1}{d}+\frac{1}{2} \\
n^{-\frac{r_0}{d_0}} \preceq e_n^{\mathrm{ran}}(\mathscr{S}_\iota,\mathscr{F}_0) \preceq n^{-\frac{r_0}{d_0}}(\log\log n)^{\frac{r_0}{d_0}+1} & \textit{if } \frac{r_0}{d_0} = \frac{r_1}{d} > 0 \\
e_n^{\mathrm{ran}}(\mathscr{S}_\iota,\mathscr{F}_0) \asymp n^{-\frac{r_0}{d_0}} & \textit{if } \frac{r_0}{d_0} = \frac{r_1}{d} = 0 \vee \frac{r_0}{d_0} < \frac{r_1}{d}.
\end{array} \tag{5.62}$$

Remember that for $1 \le p < 2$ the $L_p(\mathcal{N},\nu)$ spaces satisfy $p_Z = p$, where $(\mathcal{N},\nu)$ is an arbitrary measure space. Classical examples of Banach spaces satisfying the type 2 property are finite dimensional Banach spaces (thus also $\mathbb{K}^d$ for $d \in \mathbb{N}$ is included) and the $L_p(\mathcal{N},\nu)$ spaces for $2 \le p < \infty$.

For the proof of Theorem 5.3.1 and Theorem 5.3.3, we need further preparations. Let $\varphi_0 \not\equiv 0$ be a C^∞ function on $\mathbb{R}^{d_0}$ with support in Q_0,

$$\sup_{s\in Q_0} |\varphi_0(s)| = \sigma_0 > 0,$$

and

$$\varphi_0\left(\frac{1}{2},\dots,\frac{1}{2}\right) = 1. \tag{5.63}$$

Moreover, let φ be a C^∞ function on $\mathbb{R}^d$ with support in Q and

$$\left|\int_Q \varphi(t)dt\right| = \sigma > 0.$$

For $m_0, m \in \mathbb{N}$, let $Q_{0,i}$ $(i = 1,\dots,m_0^{d_0})$ be the subdivision of Q_0 into $m_0^{d_0}$ cubes of disjoint interior of side length m_0^{-1}, and let Q_j $(j = 1,\dots,m^d)$ be the respective subdivision of Q. Let s_i and t_j be the points in $Q_{0,i}$ and Q_j, respectively, with minimal coordinates. Define for $s \in Q_0$, $t \in Q$, $i = 1, \ldots, m_0^{d_0}$, $j = 1, \ldots, m^d$

$$\varphi_{0,i}(s) = \varphi_0(m_0(s - s_i)),$$
$$\varphi_j(t) = \varphi(m(t - t_j)).$$

Finally, define

$$\psi_{ij}(s,t) = \varphi_{0,i}(s)\varphi_j(t),$$

and using $\mathscr{I}_{m_0,m} = \{1, \ldots, m_0^{d_0}\} \times \{1, \ldots, m^d\}$, we set for $\mathscr{Z} = (z_j)_{j=1}^{m^d} \subseteq B_Z$

$$\Psi^0_{m_0,m}(\mathscr{Z}) = \left\{ \sum_{(i,j)\in\mathscr{I}_{m_0,m}} \delta_{ij}\psi_{ij}z_j : \delta_{ij} \in [-1,1],\ (i,j) \in \mathscr{I}_{m_0,m} \right\}. \tag{5.64}$$

Let set $\in \{\text{det}, \text{ran}\}$. As before, the definite problem is a particular case of the indefinite problem, thus $\mathscr{S}_0$ reduces to $\mathscr{S}_1$ and we conclude

$$e_n^{\text{set}}(\mathscr{S}_0, \Psi^0_{m_0,m}(\mathscr{Z})) \preceq e_n^{\text{set}}(\mathscr{S}_1, \Psi^0_{m_0,m}(\mathscr{Z})). \tag{5.65}$$

For this reason, it suffices to concentrate on the definite problem $e_n^{\text{set}}(\mathscr{S}_0, \Psi^0_{m_0,m}(\mathscr{Z}))$ for the lower bound proofs. We first recapitulate a lemma from [25].

Lemma 5.3.4. *Let $n_1, n_2 \in \mathbb{N}$ and let $\varepsilon_{i,j}, (i = 1, \ldots, n_1, j = 1, \ldots, n_2)$ be independent symmetric $\{-1,1\}$-valued Bernoulli random variables. Then*

$$\mathbb{E}\left(\max_{1\le i\le n_1} \left| \sum_{j=1}^{n_2} \varepsilon_{i,j} \right| \right) \asymp \big(n_2 \min(n_2, \log(n_1+1))\big)^{\frac{1}{2}}. \tag{5.66}$$

Proof. The proof and further remarks can be found in [25], Lemma 5.3. □

Lemma 5.3.5. *There are constants $c_1, c_2, c_3 > 0$ such that for all $m_0, m, n \in \mathbb{N}$ with*

$$m_0^{d_0} m^d \ge 8n \tag{5.67}$$

there is a choice $\mathscr{Z} \subseteq B_Z$ with

$$e_n^{\text{det}}(\mathscr{S}_0, \Psi^0_{m_0,m}(\mathscr{Z})) \ge c_1, \tag{5.68}$$

and, if $p_Z = 2$,

$$e_n^{\text{ran}}(\mathscr{S}_0, \Psi^0_{m_0,m}(\mathscr{Z})) \ge c_2 m^{-\frac{d}{2}} \min\left(m^d, \log(m_0+1)\right)^{\frac{1}{2}}. \tag{5.69}$$

Moreover, in case of $p_Z < 2$, there is a choice $\mathscr{Z} \subset B_Z$ with

$$e_n^{\text{ran}}(\mathscr{S}_0, \Psi^0_{m_0,m}(\mathscr{Z})) \ge c_3 m^{-d+\frac{d}{p_Z}}. \tag{5.70}$$

Proof. For all $\delta_{ij} \in \mathbb{R}$ $((i,j) \in \mathscr{I}_{m_0,m})$ we have

$$\begin{aligned}\left\|\mathscr{S}_0 \sum_{(i,j)\in\mathscr{I}_{m_0,m}} \delta_{ij}\psi_{ij}z_j\right\|_{C(Q,Z)} &= \left\|\sum_{(i,j)\in\mathscr{I}_{m_0,m}} \delta_{ij}\varphi_{0,i}z_j \int_Q \varphi_j(t)dt\right\|_{C(Q,Z)} \\ &= \sigma_0\sigma m^{-d} \max_{1\le i\le m_0^{d_0}} \left\|\sum_{\{j:(i,j)\in\mathscr{I}_{m_0,m}\}} \delta_{ij}z_j\right\|. \quad (5.71)\end{aligned}$$

For $1 \le i \le m_0^{d_0}$ and $\mathscr{I} \subseteq \mathscr{I}_{m_0,m}$ with

$$|\mathscr{I}| \ge m_0^{d_0}m^d - 4n, \quad (5.72)$$

let

$$\mathscr{I}_i = \{j \,:\, (i,j) \in \mathscr{I}\}$$

and

$$\mathscr{I}^0 = \{i \in \mathbb{N}_0 \,:\, 1 \le i \le m_0^{d_0}, |\mathscr{I}_i| \ge m^d/4\}. \quad (5.73)$$

Then

$$|\mathscr{I}^0| \ge \frac{m_0^{d_0}}{4}, \quad (5.74)$$

which follows by contradiction. Assume that $|\mathscr{I}^0| < m_0^{d_0}/4$. Then

$$|\mathscr{I}| = \sum_{i\in\mathscr{I}^0} |\mathscr{I}_i| + \sum_{i\notin\mathscr{I}^0} |\mathscr{I}_i| < \frac{1}{2}m_0^{d_0}m^d,$$

which is a contradiction to (5.67) and (5.72).

We first prove the lower bound in the deterministic setting. Here we take any $w_0 \in Z$ with $\|w_0\| = 1$ and set $\mathscr{Z}_m^0 = (w_0)_{j=1}^{m^d}$. Using Lemma 2.2.12 and (5.71), we obtain

$$\begin{aligned}e_n^{\det}(\mathscr{S}_0, \Psi_{m_0,m}^0(\mathscr{Z}_m^0)) &\ge \min_{\mathscr{I}\subseteq\mathscr{I}_{m_0,m},\,|\mathscr{I}|\ge m_0^{d_0}m^d-n} \left\|\mathscr{S}_0 \sum_{(i,j)\in\mathscr{I}} \psi_{ij}z_j\right\| \\ &\ge \sigma_0\sigma m^{-d} \min_{\mathscr{I}\subseteq\mathscr{I}_{m_0,m},\,|\mathscr{I}|\ge m_0^{d_0}m^d-n} \max_{1\le i\le m_0^{d_0}} \left\|\sum_{j\in\mathscr{I}_i} z_j\right\| \\ &= \sigma_0\sigma m^{-d} \min_{\mathscr{I}\subseteq\mathscr{I}_{m_0,m},\,|\mathscr{I}|\ge m_0^{d_0}m^d-n} \max_{1\le i\le m_0^{d_0}} |\mathscr{I}_i| \\ &\ge c, \quad (5.75)\end{aligned}$$

where the last step follows from (5.73) and (5.74).

Next we consider the randomized setting. Lemma 2.2.10 and Lemma 2.2.13 give for $\mathscr{Z} = (z_j)_{j=1}^{m^d} \subseteq B_Z$

$$e_n^{\mathrm{ran}}(\mathscr{S}_0, \Psi^0_{m_0,m}(\mathscr{Z})) \;\geq\; \frac{1}{4} \min_{\mathscr{I}\subseteq \mathscr{I}_{m_0,m},\, |\mathscr{I}|\geq m_0^{d_0} m^d - 4n} \mathbb{E} \left\| \mathscr{S}_0 \sum_{(i,j)\in\mathscr{I}} \varepsilon_{ij}\psi_{ij} z_j \right\|,$$

where $\{\varepsilon_{ij} : (i,j) \in \mathscr{I}_{m_0,m}\}$ are independent Bernoulli random variables with $\mathbb{P}\{\varepsilon_{ij} = -1\} = \mathbb{P}\{\varepsilon_{ij} = +1\} = 1/2$. Using (5.71), we conclude for $m_0^{d_0} m^d \geq 8n$

$$e_n^{\mathrm{ran}}(\mathscr{S}_0, \Psi^0_{m_0,m}(\mathscr{Z})) \geq c\frac{\sigma_0\sigma}{m^d} \min_{\mathscr{I}\subseteq \mathscr{I}_{m_0,m},\, |\mathscr{I}|\geq m_0^{d_0} m^d - 4n} \mathbb{E} \max_{1\leq i\leq m_0^{d_0}} \left\| \sum_{j\in\mathscr{I}_i} \varepsilon_{ij} z_j \right\|. \quad (5.76)$$

Now we distinguish between two cases. If $p_Z = 2$, we use the same choice $\mathscr{Z}_m^0$ as in the deterministic setting. We set

$$n_1 = \left\lceil \frac{m_0^{d_0}}{4} \right\rceil \quad \text{and} \quad n_2 = \left\lceil \frac{m^d}{4} \right\rceil.$$

Using (5.66),(5.73),(5.74), and the contraction principle (see Theorem 2.3.10), we obtain

$$\begin{aligned}
\mathbb{E} \max_{1\leq i\leq m_0^{d_0}} \left\| \sum_{j\in\mathscr{I}_i} \varepsilon_{ij} z_j \right\| &= \mathbb{E} \max_{1\leq i\leq m_0^{d_0}} \left| \sum_{j\in\mathscr{I}_i} \varepsilon_{ij} \right| \\
&\geq \mathbb{E} \max_{1\leq i\leq n_1} \left| \sum_{j=1}^{n_2} \tilde{\varepsilon}_{ij} \right| \\
&\geq c\big(m^d \min(m^d, \log(m_0+1))\big)^{\frac{1}{2}}, \qquad (5.77)
\end{aligned}$$

where $\tilde{\varepsilon}_{ij}$ $(i = 1,\ldots,n_1, j = 1,\ldots,n_2)$ are again independent symmetric Bernoulli random variables. Together with (5.76) and (5.67), we get

$$e_n^{\mathrm{ran}}(\mathscr{S}_0, \Psi^0_{m_0,m}(\mathscr{Z}_m^0)) \geq c m^{-\frac{d}{2}} \min(m^d, \log(m_0+1))^{\frac{1}{2}}.$$

If $p_Z < 2$, we estimate

$$\mathbb{E} \max_{1\leq i\leq m_0^{d_0}} \left\| \sum_{j\in\mathscr{I}_i} \varepsilon_{ij} z_j \right\| \geq \max_{1\leq i\leq m_0^{d_0}} \mathbb{E} \left\| \sum_{j\in\mathscr{I}_i} \varepsilon_{ij} z_j \right\|.$$

Since $p_Z < 2$, Z must be infinite dimensional, because finite dimensional spaces Z always satisfy $p_Z = 2$. By the Maurey-Pisier Theorem 2.3.12, there is a sequence $(w_j)_{j=1}^{m^d} \subset B_Z$ such that for all $(\alpha_j)_{j=1}^{m^d} \subset \mathbb{R}$

$$\frac{1}{2}\left(\sum_{j=1}^{m^d} |\alpha_j|^{p_Z} \right)^{\frac{1}{p_Z}} \leq \left\| \sum_{j=1}^{m^d} \alpha_j w_j \right\| \leq \left(\sum_{j=1}^{m^d} |\alpha_j|^{p_Z} \right)^{\frac{1}{p_Z}}.$$

Setting $\mathscr{Z}_m^1 = (z_j)_{j=1}^{m^d} = (w_j)_{j=1}^{m^d}$, we get for $1 \le i \le m_0^{d_0}$

$$\mathbb{E}\Big\| \sum_{j\in\mathscr{I}_i} \varepsilon_{ij} z_j \Big\| \ge c\,|\mathscr{I}_i|^{\frac{1}{p_Z}}\,. \tag{5.78}$$

Moreover, since $m_0^{d_0} m^d \ge 8n$, we obtain, using (5.73) and (5.74),

$$\begin{aligned} e_n^{\mathrm{ran}}(\mathscr{S}_0, \Psi^0_{m_0,m}(\mathscr{Z}_m^1)) &\ge cm^{-d} \min_{\mathscr{I}\subseteq\mathscr{I}_{m_0,m},\,|\mathscr{I}|\ge m_0^{d_0}m^d-4n}\ \max_{1\le i\le m_0^{d_0}} |\mathscr{I}_i|^{\frac{1}{p_Z}} \\ &\ge cm^{-d+\frac{d}{p_Z}}, \end{aligned} \tag{5.79}$$

which concludes the proof. □

For $\gamma,\gamma_0,\gamma_1\in\mathbb{R}$, and $\mathscr{Z}=(z_j)_{j=1}^{m^d}\subseteq B_Z$, let

$$\Psi^{\gamma,\gamma_0,\gamma_1}_{m_0,m}(\mathscr{Z}) = \min\left(m^{-\gamma}, m_0^{-\gamma_0}m^{-\gamma_1}\right)\Psi^0_{m_0,m}(\mathscr{Z}). \tag{5.80}$$

As a consequence of the previous lemma, we prove the next lemma, which is also stated in a more general way. This enables us to use the results for the complexity analysis of parametric initial value problems in Chapter 6 as well.

Lemma 5.3.6. *Let Z be a Banach space. Let $\iota\in\{0,1\}$ and $\gamma,\gamma_0,\gamma_1\in\mathbb{R}$ with $\gamma_0\ge 0$ and $\gamma\ge\gamma_1\ge 0$. Then there are constants $c_1,c_2,c_3>0$ such that for each $n\in\mathbb{N}$ with $n\ge 2$ there is a choice of $m_0,m\in\mathbb{N}_0$ and $\mathscr{Z}\subseteq B_Z$ fulfilling* (5.67) *and*

$$e_n^{\det}(\mathscr{S}_\iota, \Psi^{\gamma,\gamma_0,\gamma_1}_{m_0,m}(\mathscr{Z})) \ \ge\ c_1 \begin{cases} n^{-\upsilon_3} & \text{if } \frac{\gamma_0}{d_0} > \frac{\gamma_1}{d} \\ n^{-\frac{\gamma_0}{d_0}} & \text{if } \frac{\gamma_0}{d_0} \le \frac{\gamma_1}{d}, \end{cases}$$

where υ_3 is defined by

$$\upsilon_3 \ =\ \frac{\frac{\gamma_0}{d_0}}{\frac{\gamma_0}{d_0}+\frac{\gamma}{d}-\frac{\gamma_1}{d}}\,\frac{\gamma}{d}. \tag{5.81}$$

Moreover, let $1\le p\le 2$ and assume that Z is of type p. Then in the randomized setting for each $n\in\mathbb{N}$ with $n\ge 2$ there is a choice of $m_0,m\in\mathbb{N}_0$ and $\mathscr{Z}\subseteq B_Z$ fulfilling (5.67) *and*

$$e_n^{\mathrm{ran}}(\mathscr{S}_\iota, \Psi^{\gamma,\gamma_0,\gamma_1}_{m_0,m}(\mathscr{Z})) \ge cn^{-\upsilon_4(p_Z)}, \tag{5.82}$$

with $\upsilon_4(p)$ given by

$$\upsilon_4(p) \ =\ \begin{cases} \frac{\frac{\gamma_0}{d_0}}{\frac{\gamma_0}{d_0}+\frac{\gamma}{d}-\frac{\gamma_1}{d}}\left(\frac{\gamma}{d}+1-\frac{1}{p}\right) & \text{if } \frac{\gamma_0}{d_0} > \frac{\gamma_1}{d}+1-\frac{1}{p} \\ \frac{\gamma_0}{d_0} & \text{if } \frac{\gamma_0}{d_0} \le \frac{\gamma_1}{d}+1-\frac{1}{p}. \end{cases} \tag{5.83}$$

Furthermore, in case of $p_Z = 2$, for each $n \in \mathbb{N}$ with $n \geq 2$ there is a choice of $m_0, m \in \mathbb{N}_0$ and $\mathscr{Z} \subseteq B_Z$ such that (5.67) *holds and*

$$e_n^{\mathrm{ran}}(\mathscr{S}_\iota, \Psi_{m_0,m}^{\gamma,\gamma_0,\gamma_1}(\mathscr{Z})) \quad \geq \quad c_2 \begin{cases} n^{-\frac{\gamma}{d}-\frac{1}{2}} & \text{if } \frac{\gamma_0}{d_0} > \frac{\gamma_1}{d} + \frac{1}{2} \wedge \gamma = \gamma_1 \\ n^{-\upsilon_4(2)}(\log n)^{\frac{1}{2}} & \text{if } \frac{\gamma_0}{d_0} > \frac{\gamma_1}{d} + \frac{1}{2} \wedge \gamma > \gamma_1 \\ n^{-\frac{\gamma_0}{d_0}}(\log n)^{\frac{\gamma_0}{d_0}-\frac{\gamma_1}{d}} & \text{if } \frac{\gamma_1}{d} < \frac{\gamma_0}{d_0} \leq \frac{\gamma_1}{d} + \frac{1}{2} \\ n^{-\frac{\gamma_0}{d_0}} & \text{if } \frac{\gamma_0}{d_0} \leq \frac{\gamma_1}{d}. \end{cases}$$

Proof. Let $n \in \mathbb{N}$, $n \geq 2$. We start with the deterministic setting. If $m_0^{d_0} m^d \geq 8n$, it follows from Lemma 5.3.5, Corollary 2.2.15, and (5.65),(5.80) that

$$e_n^{\mathrm{det}}(\mathscr{S}_\iota, \Psi_{m_0,m}^{\gamma,\gamma_0,\gamma_1}(\mathscr{Z}_m^0)) \geq c \min\left(m^{-\gamma}, m_0^{-\gamma_0} m^{-\gamma_1}\right). \tag{5.84}$$

First consider the case $\gamma_0/d_0 > \gamma_1/d$ and put

$$m_0 = 4\left\lceil n^{\frac{\gamma-\gamma_1}{\gamma_0 d+(\gamma-\gamma_1)d_0}}\right\rceil, \quad m = 4\left\lceil n^{\frac{\gamma_0}{\gamma_0 d+(\gamma-\gamma_1)d_0}}\right\rceil. \tag{5.85}$$

It follows that $m_0^{d_0} m^d \geq 8n$ and

$$\min\left(m^{-\gamma}, m_0^{-\gamma_0} m^{-\gamma_1}\right) \geq cn^{-\frac{\gamma_0\gamma}{\gamma_0 d+(\gamma-\gamma_1)d_0}} = cn^{-\frac{\frac{\gamma_0}{d_0}}{\frac{\gamma_0}{d_0}+\frac{\gamma}{d}-\frac{\gamma_1}{d}}\frac{\gamma}{d}} = cn^{-\upsilon_3}. \tag{5.86}$$

This together with (5.84) yields

$$e_n^{\mathrm{det}}(\mathscr{S}_\iota, \Psi_{m_0,m}^{\gamma,\gamma_0,\gamma_1}(\mathscr{Z}_m^0)) \geq cn^{-\upsilon_3}.$$

Next suppose $\gamma_0/d_0 \leq \gamma_1/d$. Here we put

$$m_0 = 8\left\lceil n^{\frac{1}{d_0}}\right\rceil, \quad m = 1. \tag{5.87}$$

Clearly, $m_0^{d_0} m^d \geq 8n$ and

$$\min\left(m^{-\gamma}, m_0^{-\gamma_0} m^{-\gamma_1}\right) \geq cn^{-\frac{\gamma_0}{d_0}}, \tag{5.88}$$

therefore, by (5.84),

$$e_n^{\mathrm{det}}(\mathscr{S}_\iota, \Psi_{m_0,m}^{\gamma,\gamma_0,\gamma_1}(\mathscr{Z}_m^0)) \geq cn^{-\frac{\gamma_0}{d_0}}.$$

Next we consider the randomized case. First let $p_Z < 2$. If $m_0^{d_0} m^d \geq 8n$, it follows from Lemma 5.3.5, Corollary 2.2.15, and (5.65),(5.80) that

$$e_n^{\mathrm{ran}}(\mathscr{S}_\iota, \Psi_{m_0,m}^{\gamma,\gamma_0,\gamma_1}(\mathscr{Z}_m^1)) \geq cm^{-d+\frac{d}{p_Z}} \min\left(m^{-\gamma}, m_0^{-\gamma_0} m^{-\gamma_1}\right). \tag{5.89}$$

If $\gamma_0/d_0 > \gamma_1/d + 1 - 1/p_Z$, we take the choice (5.85), which together with (5.86),(5.89), and (5.83) gives

$$e_n^{\mathrm{ran}}(\mathscr{S}_\iota, \Psi_{m_0,m}^{\gamma,\gamma_0,\gamma_1}(\mathscr{Z}_m^1)) \geq cn^{-\frac{\frac{\gamma_0}{d_0}}{\frac{\gamma_0}{d_0}+\frac{\gamma}{d}-\frac{\gamma_1}{d}}\left(\frac{\gamma}{d}+1-\frac{1}{p_Z}\right)} = cn^{-\upsilon_4(p_Z)}.$$

If $\gamma_0/d_0 \leq \gamma_1/d + 1 - 1/p_Z$, we set $m_0 = \lceil 8n^{1/d_0} \rceil$, $m = 1$ and get from (5.88), (5.89), and (5.83)

$$e_n^{\mathrm{ran}}(\mathscr{S}_\iota, \Psi_{m_0,m}^{\gamma,\gamma_0,\gamma_1}(\mathscr{Z}_m^1)) \geq cn^{-\frac{\gamma_0}{d_0}} = cn^{-\upsilon_4(p_Z)},$$

which shows the lower bound in (5.82).

Now, we turn to the case $p_Z = 2$. If $m_0^{d_0} m^d \geq 8n$, it follows from Lemma 5.3.5, Corollary 2.2.15, and (5.65),(5.80) that

$$\begin{aligned} & e_n^{\mathrm{ran}}(\mathscr{S}_\iota, \Psi_{m_0,m}^{\gamma,\gamma_0,\gamma_1}(\mathscr{Z}_m^0)) \\ & \qquad \geq cm^{-\frac{d}{2}} \min\left(m^d, \log(m_0+1)\right)^{\frac{1}{2}} \min\left(m^{-\gamma}, m_0^{-\gamma_0} m^{-\gamma_1}\right). \end{aligned} \tag{5.90}$$

First we consider the case $\gamma_0/d_0 > \gamma_1/d + 1/2$. Define m_0, m as in (5.85). Then we have

$$\begin{aligned} \min(m^d, \log(m_0+1))^{\frac{1}{2}} & \geq \begin{cases} c & \text{if} \quad \gamma = \gamma_1 \\ c(\log n)^{\frac{1}{2}} & \text{if} \quad \gamma > \gamma_1 \end{cases}, \\ m^{-\frac{d}{2}} & \geq cn^{-\frac{d}{2}\frac{\gamma_0}{\gamma_0 d + (\gamma-\gamma_1)d_0}}, \end{aligned}$$

thus, using (5.90) and estimates as above,

$$\begin{aligned} e_n^{\mathrm{ran}}(\mathscr{S}_\iota, \Psi_{m_0,m}^{\gamma,\gamma_0,\gamma_1}(\mathscr{Z}_m^0)) & \geq \begin{cases} cn^{-\frac{\gamma_0(\gamma+d/2)}{\gamma_0 d + (\gamma-\gamma_1)d_0}} & \text{if} \quad \gamma = \gamma_1 \\ cn^{-\frac{\gamma_0(\gamma+d/2)}{\gamma_0 d + (\gamma-\gamma_1)d_0}} (\log n)^{\frac{1}{2}} & \text{if} \quad \gamma > \gamma_1 \end{cases} \\ & = \begin{cases} cn^{-\frac{\gamma}{d}-\frac{1}{2}} & \text{if} \quad \gamma = \gamma_1 \\ cn^{-\upsilon_4} (\log n)^{\frac{1}{2}} & \text{if} \quad \gamma > \gamma_1 \end{cases}. \end{aligned}$$

Next we consider the case $\gamma_1/d < \gamma_0/d_0 \leq \gamma_1/d + 1/2$. Here we put

$$m_0 = 4\left\lceil \left(\frac{n}{\log n}\right)^{\frac{1}{d_0}} \right\rceil, \quad m = 4\left\lceil (\log n)^{\frac{1}{d}} \right\rceil,$$

which again implies $m_0^{d_0} m^d \geq 8n$. Furthermore, we have

$$\begin{aligned} \min(m^d, \log(m_0+1))^{\frac{1}{2}} & \geq c(\log n)^{\frac{1}{2}}, \\ m^{-\frac{d}{2}} & \geq c(\log n)^{-\frac{1}{2}}, \\ \min\left(m^{-\gamma}, m_0^{-\gamma_0} m^{-\gamma_1}\right) & \geq cn^{-\frac{\gamma_0}{d_0}} (\log n)^{\frac{\gamma_0}{d_0}-\frac{\gamma_1}{d}}. \end{aligned}$$

Combining this with (5.90) gives

$$e_n^{\mathrm{ran}}(\mathscr{S}_\iota, \Psi_{m_0,m}^{\gamma,\gamma_0,\gamma_1}(\mathscr{Z}_m^0)) \geq n^{-\frac{\gamma_0}{d_0}}(\log n)^{\frac{\gamma_0}{d_0}-\frac{\gamma_1}{d}}.$$

Finally, let $\gamma_0/d_0 \leq \gamma_1/d$. Here we use the choice (5.87) and obtain

$$\min(m^d, \log(m_0+1))^{\frac{1}{2}} \geq c.$$

This together with (5.88) yields

$$e_n^{\mathrm{ran}}(\mathscr{S}_\iota, \Psi_{m_0,m}^{\gamma,\gamma_0,\gamma_1}(\mathscr{Z}_m^0)) \geq n^{-\frac{\gamma_0}{d_0}}.$$

□

To prove Theorem 5.3.1 and Theorem 5.3.3 we only have to adapt the setting for the parametric integration problem to the general cases in the previous lemma.

Proof of Theorem 5.3.1 and Theorem 5.3.3. The upper bounds are consequences of Corollary 5.2.4 and Hölder's inequality. Thus, let us focus on the lower bounds. The factor

$$c \min\left(m^{-r}, m_0^{-r_0} m^{-r_1}\right)$$

is a correction factor, due to differentiation of $\varphi_{0,i}, \varphi_j$, similar to the one in the lower bound proof for Banach space valued integration. Thus, for each $\mathscr{Z} = (z_j)_{j=1}^{m^d} \subseteq B_Z$ there is a constant $c > 0$ such that for all $m_0, m \in \mathbb{N}$

$$c \min\left(m^{-r}, m_0^{-r_0} m^{-r_1}\right) \Psi_{m_0,m}^0(\mathscr{Z}) \subseteq \mathscr{F}_0.$$

Consequently, by (5.80),

$$e_n^{\mathrm{set}}(\mathscr{S}_\iota, \mathscr{F}_0) \geq c\, e_n^{\mathrm{set}}(\mathscr{S}_\iota, \Psi_{m_0,m}^{r,r_0,r_1}(\mathscr{Z})) \qquad (m_0, m \in \mathbb{N}).$$

Setting $\gamma = r$, $\gamma_0 = r_0$, $\gamma_1 = r_1$, we obtain by (5.49) and (5.81) for $r_0/d_0 > r_1/d$

$$\upsilon_3 \;=\; \frac{\frac{r_0}{d_0}}{\frac{r_0}{d_0}+\frac{r}{d}-\frac{r_1}{d}}\,\frac{r}{d} = \upsilon_1,$$

and by (5.51) and (5.83) for $r_0/d_0 > r_1/d + 1 - 1/p$,

$$\upsilon_4(p) \;=\; \frac{\frac{r_0}{d_0}}{\frac{r_0}{d_0}+\frac{r}{d}-\frac{r_1}{d}}\left(\frac{r}{d}+1-\frac{1}{p}\right) = \upsilon_2(p).$$

Now Lemma 5.3.6 yields the lower bounds. □

5.4 Some Particular Classes of Functions

If $r_1 = r$, then $\mathscr{F}_0 = B_{C^{r_0,r}(Q_0\times Q,Z)}$, which is a class of dominating mixed smoothness. More precisely, the smoothness with respect to the parameter variables s and the smoothness with respect to the variables t are combined in such a way. For the sake of simplicity, we concentrate on Banach spaces in this paragraph that satisfy the type 2 property.

Corollary 5.4.1. *Let $r_0, r \in \mathbb{N}_0$, $r_1 = r$, $d, d_0 \in \mathbb{N}$, $\iota \in \{0,1\}$, and let Z be an arbitrary Banach space with $\tau_2(Z) < \infty$. Then*

$$\begin{aligned} e_n^{\mathrm{det}}(\mathscr{S}_\iota, \mathscr{F}_0) &\asymp_{\log} n^{-\min\left(\frac{r}{d}, \frac{r_0}{d_0}\right)} \\ e_n^{\mathrm{ran}}(\mathscr{S}_\iota, \mathscr{F}_0) &\asymp_{\log} n^{-\min\left(\frac{r}{d}+\frac{1}{2}, \frac{r_0}{d_0}\right)}. \end{aligned}$$

Let us compare the order of the deterministic and randomized minimal errors neglecting logarithmic factors. If the smoothness r_0 with respect to the parameter satisfies $r_0/d_0 \geq r/d + 1/2$, then the order of $e_n^{\mathrm{ran}}(\mathscr{S}_\iota, \mathscr{F}_0)$ is the same as that of the randomized minimal errors for (nonparametric) integration of functions from $C^r(Q, Z)$ and is by $n^{-1/2}$ faster than parametric integration in the deterministic setting. If $r/d < r_0/d_0 < r/d + 1/2$, the randomized rate is still superior, but the gap becomes smaller and reaches zero when $r_0/d_0 \leq r/d$. The behavior is illustrated in Figure 5.1.

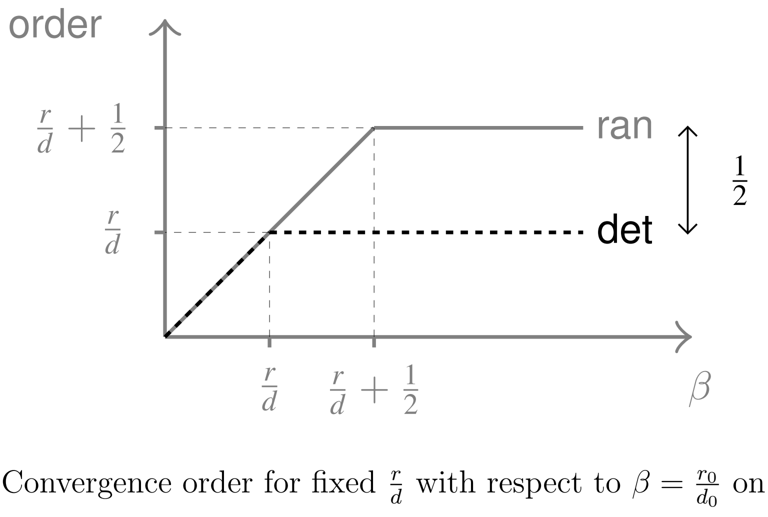

Figure 5.1: Convergence order for fixed $\frac{r}{d}$ with respect to $\beta = \frac{r_0}{d_0}$ on the x-axis.

Next we consider the case $r_1 = 0$. This leads to the class

$$C^{(r_0,0)\wedge(0,r)}(Q_0 \times Q, Z) := C^{r_0,0}(Q_0 \times Q, Z) \cap C^{0,r}(Q_0 \times Q, Z)$$

of continuous functions $f : Q_0 \times Q \to Z$ having, for $\alpha_0 \in \mathbb{N}_0^{d_0}$ with $|\alpha_0| \le r_0$ and for $\alpha_1 \in \mathbb{N}_0^d$ with $|\alpha_1| \le r$, continuous partial derivatives $\frac{\partial^{|\alpha_0|} f(s,t)}{\partial s^{\alpha_0}}$ and $\frac{\partial^{|\alpha_1|} f(s,t)}{\partial t^{\alpha_1}}$; endowed with the norm

$$\begin{aligned}
&\|f\|_{C^{(r_0,0)\wedge(0,r)}(Q_0\times Q,Z)} \\
&= \max\left(\|f\|_{C^{r_0,0}(Q_0\times Q,Z)}, \|f\|_{C^{0,r}(Q_0\times Q,Z)}\right) \\
&= \max\left(\max_{|\alpha_0|\le r_0} \sup_{s\in Q_0, t\in Q} \left\|\frac{\partial^{|\alpha|} f(s,t)}{\partial s^{\alpha_0}}\right\|_Z, \max_{|\alpha_1|\le r} \sup_{s\in Q_0, t\in Q} \left\|\frac{\partial^{|\alpha|} f(s,t)}{\partial t^{\alpha_1}}\right\|_Z\right).
\end{aligned}$$

Thus, here we consider separate differentiability with respect to the s- and t-variables. Before we state the result, we want to mention a closely related subclass. Let $C^{[r_0,r]}(Q_0 \times Q, Z)$ denote the class of continuous functions having continuous partial derivatives $\frac{\partial^{|\alpha|} f(s,t)}{\partial s^{\alpha_0} \partial t^{\alpha_1}}$ for all $\alpha_0 \in \mathbb{N}_0^{d_0}$, $\alpha_1 \in \mathbb{N}_0^d$ satisfying $\frac{|\alpha_0|}{r_0} + \frac{|\alpha_1|}{r} \le 1$ (with the convention $\frac{0}{0} = 0$ and $\frac{c}{0} = +\infty$ for $c > 0$), equipped with the norm

$$\|f\|_{C^{[r_0,r]}(Q_0\times Q,Z)} = \max_{\frac{|\alpha_0|}{r_0}+\frac{|\alpha_1|}{r}\le 1} \sup_{s\in Q_0, t\in Q} \left\|\frac{\partial^{|\alpha|} f(s,t)}{\partial s^{\alpha_0}\partial t^{\alpha_1}}\right\|_Z.$$

For $r_0 = r$ this is just the class $C^r(Q_0 \times Q, Z)$. Clearly, we have

$$C^{[r_0,r]}(Q_0 \times Q, Z) \subseteq C^{(r_0,0)\wedge(0,r)}(Q_0 \times Q, Z) \tag{5.91}$$

and

$$\|f\|_{C^{[r_0,r]}(Q_0\times Q,Z)} \ge \|f\|_{C^{(r_0,0)\wedge(0,r)}(Q_0\times Q,Z)}.$$

In general, the inclusion in (5.91) is strict, see [38, 3].

Corollary 5.4.2. *Let Z be an arbitrary Banach space with $\tau_2(Z) < \infty$. Furthermore, let $r_0, r \in \mathbb{N}_0$, $d, d_0 \in \mathbb{N}$, $\iota \in \{0, 1\}$ and let $\mathscr{F}_1$ be any set with*

$$B_{C^{[r_0,r]}(Q_0\times Q,Z)} \subseteq \mathscr{F}_1 \subseteq B_{C^{(r_0,0)\wedge(0,r)}(Q_0\times Q,Z)}. \tag{5.92}$$

Then

$$\begin{aligned}
e_n^{\mathrm{det}}(\mathscr{S}_\iota, \mathscr{F}_1) &\asymp_{\log} n^{-\upsilon_5}, \\
e_n^{\mathrm{ran}}(\mathscr{S}_\iota, \mathscr{F}_1) &\asymp_{\log} n^{-\upsilon_6},
\end{aligned}$$

where

$$\upsilon_5 = \begin{cases} \frac{\frac{r_0}{d_0}}{\frac{r_0}{d_0}+\frac{r}{d}} \frac{r}{d} & \text{if} \quad r_0 > 0 \\ 0 & \text{if} \quad r_0 = 0, \end{cases} \tag{5.93}$$

$$\upsilon_6 = \begin{cases} \frac{\frac{r_0}{d_0}}{\frac{r_0}{d_0}+\frac{r}{d},} \left(\frac{r}{d} + \frac{1}{2}\right) & \text{if} \quad \frac{r_0}{d_0} > \frac{1}{2} \\ \frac{r_0}{d_0} & \text{if} \quad \frac{r_0}{d_0} \le \frac{1}{2}. \end{cases} \tag{5.94}$$

Proof. The upper bounds follow from (5.92) and Corollary 5.2.4. For the proof of the lower bounds, we observe that for each $\mathscr{Z} = (z_j)_{j=1}^{m^d} \subseteq B_Z$ there is a constant $c > 0$ such that for all $m_0, m_1 \in \mathbb{N}$ and $\psi \in \Psi^0_{m_0,m_1}(\mathscr{Z})$

$$\begin{aligned}
&\|\psi\|_{C^{[r_0,r]}(Q_0\times Q,Z)} \\
&\leq c \max\left\{ m_0^{|\alpha_0|} m_1^{|\alpha_1|} : \alpha_0 \in \mathbb{N}_0^{d_0}, \alpha_1 \in \mathbb{N}_0^d, \frac{|\alpha_0|}{r_0} + \frac{|\alpha_1|}{r} \leq 1 \right\} \\
&= c \max\left\{ \left(m_0^{r_0}\right)^{\frac{|\alpha_0|}{r_0}} \left(m_1^{r}\right)^{\frac{|\alpha_1|}{r}} : \alpha_0 \in \mathbb{N}_0^{d_0}, \alpha_1 \in \mathbb{N}_0^d, \frac{|\alpha_0|}{r_0} + \frac{|\alpha_1|}{r} \leq 1 \right\} \\
&\leq c \max\left(m_0^{r_0}, m_1^{r}\right),
\end{aligned}$$

and therefore,

$$c \min(m_1^{-r}, m_0^{-r_0})\, \Psi^0_{m_0,m_1}(\mathscr{Z}) \subseteq B_{C^{[r_0,r]}(Q_0\times Q,Z)}.$$

Arguing as in the proof of the lower bounds for Theorem 5.3.1 gives the desired result. □

Now, let us compare the exponents υ_5 of the deterministic setting (5.93) and υ_6 of the randomized setting (5.94). We assume $r_0 > 0$, otherwise both exponents are zero. First consider the case $r_0/d_0 > 1/2$. If $r = 0$, then $\upsilon_5 = 0$, $\upsilon_6 = 1/2$, so the randomized rate is by the exponent $1/2$ superior to the (trivial) deterministic one, see also Figure 5.2.

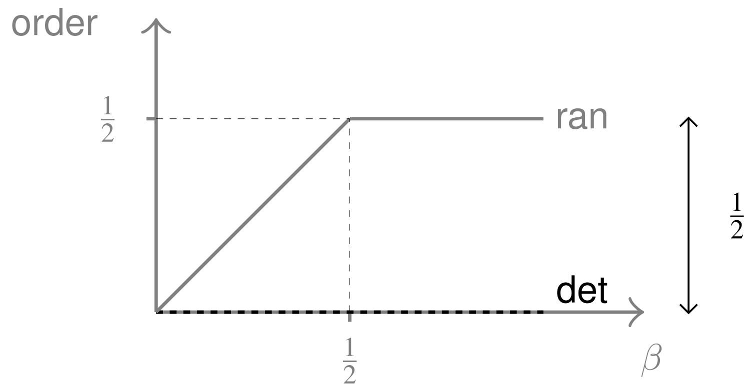

Figure 5.2: Convergence order for fixed $\frac{r}{d} = 0$ with respect to $\beta = \frac{r_0}{d_0}$ on the x-axis.

For $r > 0$, the gap is smaller than $1/2$, but it is never zero. The advantage of randomization can be arbitrarily close to $1/2$ (for large parameter smoothness r_0/d_0 or small t-smoothness r/d), see Figure 5.3 for the case $r/d = 1$.

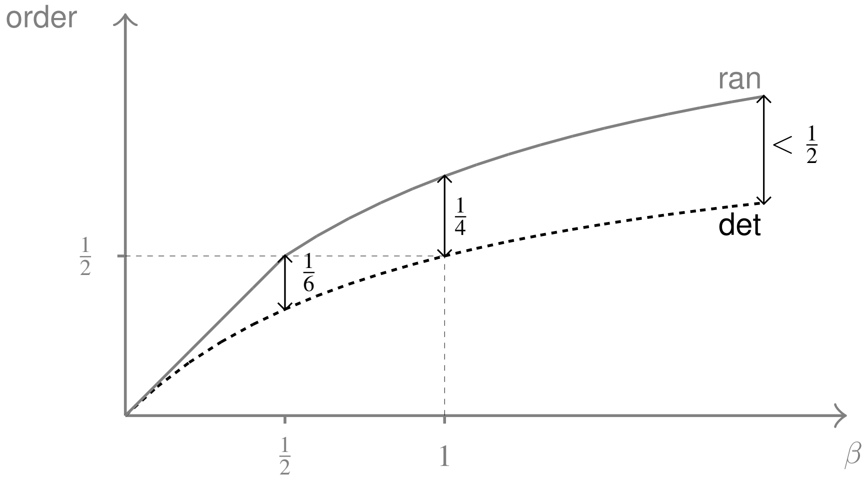

Figure 5.3: Convergence order for fixed $\frac{r}{d} = 1$ with respect to $\beta = \frac{r_0}{d_0}$ on the x-axis.

If $0 < r_0/d_0 \leq 1/2$, we obtain

$$\upsilon_5 = \frac{\frac{r}{d}}{\frac{r_0}{d_0} + \frac{r}{d}} \frac{r_0}{d_0}, \quad \upsilon_6 = \frac{r_0}{d_0}.$$

In this situation, the gain by randomization is also never zero, see Figure 5.3. For small r/d it is close to r_0/d_0, and it reaches this value only for $r = 0$. The latter case is easily seen in Figure 5.2.

Notice that for $r_0 = r$ and $Z = \mathbb{K}$ we recover the results of [9] with the rates

$$\upsilon_5 = \frac{r}{d + d_0}, \quad \upsilon_6 = \begin{cases} \frac{r + \frac{d}{2}}{d + d_0} & \text{if} \quad \frac{r}{d_0} > \frac{1}{2} \\ \frac{r}{d_0} & \text{if} \quad \frac{r}{d_0} \leq \frac{1}{2}. \end{cases}$$

Chapter 6

Parametric Banach Space Valued Initial Value Problems

In the present chapter, we consider the complexity of parametric Banach space valued initial value problems. We study the problem in the deterministic and randomized setting for various classes of smoothness with respect to the input functions. These classes are closely related to those considered in Chapter 5 and include cases of isotropic and of dominating mixed smoothness as well.

We develop a randomized multilevel algorithm and determine its convergence rate. The deterministic version is again obtained from the randomized one. The algorithmic approach is a nonlinear analogue of the approach in Chapter 5. We use the Banach space valued generalizations [21] of the scalar results in [23] and [8]. For the complexity analysis, we also prove lower bounds. To assess the speedup that randomization can bring over deterministic methods, we compare the optimal rates of the deterministic and randomized setting.

The chapter is organized as follows: First, we give a brief introduction into the solution theory of Banach space valued ODEs. In Section 6.2, we consider Banach space valued ODEs and develop a multilevel approach. The parametric problem is formulated in Section 6.4, and we show how it fits into the Banach space valued scheme for a single equation of Section 6.2. In Section 6.5, the algorithm for the parametric problem is described and convergence rates are derived. Section 6.6 contains lower bounds and the complexity is established. Finally, in Section 6.7, we discuss the considered classes and related ones, study special cases of the obtained results, and provide comparisons between deterministic and randomized setting.

6.1 Banach Space Valued ODEs

We briefly summarize results about the solution theory of Banach space valued ordinary differential equations. We refer to the monograph [4] for a more detailed introduction, all results and their proofs can be found there. For further reading on ODEs in Banach spaces, we also refer to the monographs [7, 36, 46, 31, 13].

In order to express the parametric problem as a Banach space valued non-parametric problem, we have to consider local conditions. Demanding global smoothness and global Lipschitz conditions would be too strong, even if we assume global conditions for the parametric input sets. For this reason we only concentrate on local conditions.

Let Z be an arbitrary Banach space and $-\infty < a < b < \infty, \kappa, L \in \mathbb{R}$. We consider initial value problems for ODEs with values in Z of the form

$$u'(t) = f(t, u(t)) \qquad (t \in [a, b]), \tag{6.1}$$

where $f \in C([a, b] \times Z, Z)$. A function $u : [a, b] \to Z$ is called a solution if u is continuously differentiable and (6.1) is satisfied.

Proposition 6.1.1 (local existence). *Let V be a neighborhood of*

$$(t_0, z_0) \in [a, b] \times Z.$$

Let $f \in C(V, Z)$ such that for all $(t, z_1), (t, z_2) \in V$

$$||f(t, z_1) - f(t, z_2)||_Z \leq L||z_1 - z_2||_Z.$$

Then there is a $\delta > 0$ such that the differential equation (6.1) has a unique solution u on $[z_0 - \delta, z_0 + \delta]$, with $u(t_0) = z_0$.

More precisely: Let $\tau_0 > 0, \tau_1 > 0$ be chosen in such a way that $[t_0 - \tau_0, t_0 + \tau_0] \times (z_0 + \tau_1 B_Z)$ is contained in V and

$$||f(t, z)|| \leq \kappa$$

for $|t - t_0| \leq \tau_0, ||z - z_0|| \leq \tau_1$. Then choosing δ as

$$\delta = \inf\left(\tau_0, \frac{\tau_1}{\kappa}\right)$$

yields that $u : [z_0 - \delta, z_0 + \delta] \to (z_0 + \tau_1 B_Z)$.

Proof. See [4], Chap. II, Corollary 1.7.2. □

A function $f : U \to Z$ (with $U \subset [a,b] \times Z$) satisfies a local Lipschitz condition with respect to the second variable if for every $(t_0, z_0) \in U$, there is a neighborhood V of (t_0, z_0) in U and a constant $L > 0$ such that for all $(t, z_1), (t, z_2) \in V$

$$||f(t, z_1) - f(t, z_2)||_Z \leq L||z_1 - z_2||_Z.$$

Corollary 6.1.2. *If $f : U \to Z$ is continuous and satisfies a local Lipschitz condition with respect to the second variable, and if (t_0, z_0) is an inner point of U, then there exists $\delta > 0$ such that* (6.1) *has a unique solution $u : [t_0 - \delta, t_0 + \delta] \to Z$.*

The last corollary only ensures the existence of a unique solution in a region of t_0. But if a global solution exists, the solution is unique on the whole interval $[a, b]$.

Proposition 6.1.3 (global uniqueness)**.** *Let $f : [a,b] \times Z \to Z$ be a function satisfying a local Lipschitz condition with respect to the second variable. Assume that $u_0 : [a,b] \to Z$ and $u_1 : [a,b] \to Z$ are both solutions of the differential equation* (6.1)*. If there is a $t_0 \in [a,b]$ such that $u_0(t_0) = u_1(t_0)$, then*

$$u_0(t) = u_1(t) \qquad (t \in [a,b]).$$

Proof. See [4], Chap. II, Theorem 1.8.2. □

To prove global existence of a solution on the whole interval $[a, b]$, we have to demand global Lipschitz condition. However, even in the local case it is possible to prove existence on a maximal subinterval of $[a, b]$.

Proposition 6.1.4. *Let $f \in C(U, Z)$ satisfy a local Lipschitz condition and let (t_0, z_0) be an inner point of U. Then there exists a maximal interval $\mathfrak{I}$ with $t_0 \in \mathfrak{I}$ such that $u : \mathfrak{I} \to Z$ is a solution of* (6.1) *and $u(t_0) = z_0$. Due to Proposition* 6.1.3*, this solution is unique.*

Proof. See [4], Chap. II, Theorem 1.8.3. □

6.2 Approximation of Banach Space Valued ODEs

Let Z and Z_1 be Banach spaces over the reals. This assumption is made because in the following, we consider only real differentiation. Complex spaces can be included by simply considering them as spaces over the reals. We introduce the Banach space valued setting which will be connected to the parametric setting later on. For the notations used here, we refer to Section 2.1.

Definition 6.2.1. Let $-\infty < a < b < +\infty$, $r \in \mathbb{N}_0$, $0 \le \varrho \le 1$, and let $\kappa, L : (0, +\infty) \to (0, +\infty)$ be any functions. We define the following class

$$\mathcal{C}^{r,\varrho}([a,b] \times Z, Z_1; \kappa) \quad \text{of continuous functions} \quad f : [a,b] \times Z \to Z_1 \tag{6.2}$$

having, for $\alpha = (\alpha_1, \alpha_2) \in \mathbb{N}_0^2$ with $|\alpha| = \alpha_1 + \alpha_2 \le r$, continuous partial (Fréchet-) derivatives

$$\frac{\partial^{|\alpha|} f(t,x)}{\partial t^{\alpha_1} \partial x^{\alpha_2}} \in \mathscr{L}_{\alpha_2}(Z, Z_1)$$

such that for all $R > 0$, $t, t_1, t_2 \in [a,b]$, $x, y \in RB_Z$, $|\alpha| \le r$

$$\left\| \frac{\partial^{|\alpha|} f(t,x)}{\partial t^{\alpha_1} \partial x^{\alpha_2}} \right\|_{\mathscr{L}_{\alpha_2}(Z,Z_1)} \le \kappa(R), \tag{6.3}$$

and, for $|\alpha| = r$,

$$\left\| \frac{\partial^{|\alpha|} f(t_1,x)}{\partial t^{\alpha_1} \partial x^{\alpha_2}} - \frac{\partial^{|\alpha|} f(t_2,y)}{\partial t^{\alpha_1} \partial x^{\alpha_2}} \right\|_{\mathscr{L}_{\alpha_2}(Z,Z_1)} \le \kappa(R)(|t_1 - t_2|^\varrho + \|x - y\|^\varrho). \tag{6.4}$$

Moreover, let $\mathcal{C}^{r,\varrho}_{\mathrm{Lip}}([a,b] \times Z, Z_1; \kappa, L)$ be the class of all $f \in \mathcal{C}^{r,\varrho}([a,b] \times Z, Z_1; \kappa)$ such that for $R > 0$, $t \in [a,b]$, $x, y \in RB_Z$

$$\|f(t,x) - f(t,y)\| \le L(R)\|x - y\|. \tag{6.5}$$

Remark 6.2.2. The classes introduced above have smoothness (and the Lipschitz property) bounded on bounded sets. Notice that all functions $f \in \mathcal{C}^{r,\varrho}_{\mathrm{Lip}}([a,b] \times Z, Z_1; \kappa, L)$ satisfy the local Lipschitz property, but the reverse is not true in general. However, if Z is finite dimensional, there exists a finite cover of RB_Z and the local Lipschitz property is equivalent to the conditions above.

For the further analysis, we consider initial value problems for ODEs with values in Z as follows:

$$u'(t) = f(t, u(t)) \qquad (t \in [a,b]), \tag{6.6}$$

$$u(a) = u_0, \tag{6.7}$$

with $f \in \mathcal{C}^{r,\varrho}_{\mathrm{Lip}}([a,b] \times Z, Z; \kappa, L)$ and $u_0 \in Z$. We demand that the solution u of (6.6) also satisfies (6.7). Then, due to (6.5) and Proposition 6.1.3, the solution is unique.

Next, we introduce the algorithm for Banach space valued (non-parametric) initial value problems developed and studied in [21]. The scalar version has been considered in [8], previously.

Algorithm 6.2.3. Let $m \in \mathbb{N}_0$, $n \in \mathbb{N}$, let $t_k = a + kh \ (k = 0, 1, \ldots, n)$ be the uniform grid on $[a, b]$ of mesh size $h = (b - a)/n$. Moreover, for $0 \le k \le n - 1$ and $1 \le j \le m$ let $P_k^{j,1}$ be, as defined in (2.6), the one dimensional operator of Lagrange interpolation of degree j on the equidistant grid

$$t_{k,j,i} = t_k + i\frac{h}{j} \qquad (i = 0, \ldots, j)$$

on $[t_k, t_{k+1}]$. Let $\xi_1, \ldots, \xi_n$ be independent random variables on some probability space $(\Omega, \Sigma, \mathbb{P})$ such that ξ_k is uniformly distributed on $[t_{k-1}, t_k]$ where $(k = 1, \ldots, n)$. Since we will also consider $\xi_k(\omega)$ for fixed $\omega \in \Omega$, we assume (without loss of generality) that

$$\{(\xi_1(\omega), \ldots, \xi_n(\omega)) : \omega \in \Omega\} = [t_0, t_1] \times \cdots \times [t_{n-1}, t_n]. \tag{6.8}$$

Fix $f \in \mathcal{C}^{r,\varrho}_{\mathrm{Lip}}([a, b] \times Z, Z; \kappa, L)$, $u_0 \in Z$, and define $(u_k)_{k=1}^n \subset Z$ and Z-valued polynomials $p_{k,j}(t)$ for $k = 0, \ldots, n-1$ and $j = 0, \ldots, m$ by induction as follows: Assume that $0 \le k \le n - 1$ and that u_k is already defined. Then we define $p_{k,0}$ by

$$p_{k,0}(t) \;=\; u_k + f(t_k, u_k)(t - t_k) \quad (t \in [t_k, t_{k+1}]). \tag{6.9}$$

Now suppose $m \ge 1$, $0 \le j < m$, and $p_{k,j}$ is already defined. We define $p_{k,j+1}$ by

$$p_{k,j+1}(t) \;=\; u_k + \int_{t_k}^{t} \left(P_k^{j+1,1} q_{k,j}\right)(\tau) d\tau, \tag{6.10}$$

where

$$q_{k,j} \;=\; \left(f(t_{k,j+1,i}, p_{k,j}(t_{k,j+1,i}))\right)_{i=0}^{j+1}. \tag{6.11}$$

Finally, we put

$$u_{k+1} = p_{k,m}(t_{k+1}) + h\left(f(\xi_{k+1}, p_{k,m}(\xi_{k+1})) - p'_{k,m}(\xi_{k+1})\right). \tag{6.12}$$

The result of the algorithm, the approximation $v \in B([a, b], Z)$ to the solution u of (6.6), is now defined by

$$v(t) = \begin{cases} p_{k,m}(t) & \text{if} \quad t \in [t_k, t_{k+1}) \quad \text{and} \quad 0 \le k \le n-1, \\ u_n & \text{if} \quad t = t_n. \end{cases} \tag{6.13}$$

Let

$$\mathcal{A}^m_{n,\omega} : \mathcal{C}^{r,\varrho}_{\mathrm{Lip}}([a, b] \times Z, Z; \kappa, L) \times Z \to B([a, b], Z)$$

denote the resulting mapping for $\omega \in \Omega$ fixed, that is,

$$\mathcal{A}^m_{n,\omega}(f, u_0) = v, \tag{6.14}$$

and let $\mathcal{A}^m_n$ denote the family of mappings $\mathcal{A}^m_n = (\mathcal{A}^m_{n,\omega})_{\omega \in \Omega}$. We write $\mathcal{A}^m_n(f, u_0)$ for the random variable $(\mathcal{A}^m_{n,\omega}(f, u_0))_{\omega \in \Omega}$.

Observe that for $m = 0$

$$\begin{aligned} p_{k,0}(t) &= u_k + f(t_k, u_k)(t - t_k) \qquad (t \in [t_k, t_{k+1}],\ 0 \le k \le n-1), \quad (6.15) \\ u_{k+1} &= u_k + hf(\xi_{k+1}, p_{k,0}(\xi_{k+1})) \qquad (0 \le k \le n-1). \quad (6.16) \end{aligned}$$

Concerning the definition of $\mathcal{A}^m_{n,\omega}$, notice that due to condition (6.8), fixing any $\omega \in \Omega$ is the same as fixing any values of

$$\xi_k \in [t_{k-1}, t_k] \qquad (k = 1, \ldots, n).$$

This way we obtain a deterministic algorithm, where ξ_k are fixed algorithm parameters.

Definition 6.2.4. Given also $\sigma, \lambda > 0$, we let $\mathcal{F}^{r,\varrho}([a,b] \times Z, Z; \kappa, L, \sigma, \lambda)$ be the class of all pairs (f, u_0) with $f \in \mathcal{C}^{r,\varrho}_{\mathrm{Lip}}([a,b] \times Z, Z; \kappa, L)$, $u_0 \in \sigma B_Z$ such that the initial value problem (6.6-6.7) has a solution u (which is unique, due to Proposition 6.1.3 and assumption (6.5)) satisfying

$$\|u\|_{B([a,b],Z)} \le \lambda. \tag{6.17}$$

If $r = \varrho = 0$, we require in addition that (f, u_0) is such that for all $n \in \mathbb{N}$, $\omega \in \Omega$

$$\|\mathcal{A}^0_{n,\omega}(f, u_0)\|_{B([a,b],Z)} \le \lambda. \tag{6.18}$$

The solution operator

$$\mathcal{S} : \mathcal{F}^{r,\varrho}([a,b] \times Z, Z; \kappa, L, \sigma, \lambda) \to B([a,b], Z) \tag{6.19}$$

is defined for $(f, u_0) \in \mathcal{F}^{r,\varrho}([a,b] \times Z, Z; \kappa, L, \sigma, \lambda)$ by $\mathcal{S}(f, u_0) = u$, where u is the solution of the initial value problem (6.6-6.7) as before.

The next proposition gives a convergence analysis for the Banach space valued algorithm from above. It is an immediate consequence of results from [21], adapted to the setting considered here.

Proposition 6.2.5. *Let $r \in \mathbb{N}_0$, $0 \le \varrho \le 1$, $\kappa, L : (0, +\infty) \to (0, +\infty)$, $\sigma, \lambda > 0$, $1 \le p \le 2$, and let $m \in \mathbb{N}_0$ if $r + \varrho > 0$ and $m = 0$ if $r = \varrho = 0$. Then there are constants $c_1, c_2 > 0$ and $\nu_0 \in \mathbb{N}$ such that for all Banach spaces Z and all $n \ge \nu_0$*

$$\sup_{(f,u_0) \in \mathcal{F}^{r,\varrho}([a,b] \times Z, Z; \kappa, L, \sigma, \lambda)} \|\mathcal{S}(f, u_0) - \mathcal{A}^m_{n,\omega}(f, u_0)\|_{B([a,b],Z)} \le c_1 n^{-\min(r+\varrho, m+1)} \qquad (\omega \in \Omega) \tag{6.20}$$

and

$$\sup_{(f,u_0) \in \mathcal{F}^{r,\varrho}([a,b] \times Z, Z; \kappa, L, \sigma, \lambda)} \left(\mathbb{E} \, \|\mathcal{S}(f, u_0) - \mathcal{A}^m_{n,\omega}(f, u_0)\|^p_{B([a,b],Z)} \right)^{1/p} \le c_2 \tau_p(Z) n^{-\min(r+\varrho, m+1) - 1 + 1/p}. \tag{6.21}$$

Proof. We set

$$U = [a,b] \times (\lambda+1)B_Z^0, \quad U_0 = \sigma B_Z, \quad V = [a,b] \times \lambda B_Z.$$

Let $(f, u_0) \in \mathcal{F}^{r,\varrho}([a,b] \times Z, Z; \kappa, L, \sigma, \lambda)$. First we consider the case $r + \varrho > 0$. By (6.17) we have, in the notation of [21],

$$(f|_U, u_0) \in \mathscr{F}^{r,\varrho}(U, \kappa(\lambda+1), L(\lambda+1), U_0, V).$$

Since $\lambda B_Z + \frac{1}{2}B_Z \subset (\lambda+1)B_Z^0$, Theorem 3.3 of [21] gives (6.20–6.21). Now let $r = \varrho = 0$ and put $u = S(f, u_0)$. Then for $t \in [t_k, t_{k+1}]$

$$u(t_k) + \kappa(\lambda+1)(t - t_k)B_Z \subseteq \lambda B_Z + \kappa(\lambda+1)\frac{b-a}{n}B_Z \subseteq (\lambda+1)B_Z^0$$

whenever $n \geq \nu_0 := \lfloor \kappa(\lambda+1)(b-a) \rfloor + 1$. Taking into account (6.17–6.18), we see that in the notation of [21],

$$(f|_U, u_0) \in \mathscr{H}^{0,0}(U, \kappa(\lambda+1), L(\lambda+1), U_0, V, 0, n) \quad (n \geq \nu_0).$$

Therefore, (6.20–6.21) follow for $n \geq \nu_0$ from Proposition 3.4 of [21]. □

6.3 A Multilevel Algorithm for Banach Space Valued ODEs

Lemma 6.3.1. *Let Z and Z_1 be Banach spaces, $f \in \mathcal{C}^{0,0}_{\mathrm{Lip}}([a,b] \times Z, Z; \kappa, L)$, and $T \in \mathscr{L}(Z, Z_1)$. Assume that there are $\kappa_1, L_1 : (0, +\infty) \to (0, +\infty)$ and a function $g \in \mathcal{C}^{0,0}_{\mathrm{Lip}}([a,b] \times Z_1, Z_1; \kappa_1, L_1)$ such that for all $t \in [a,b]$, $z \in Z$*

$$Tf(t,z) = g(t, Tz). \tag{6.22}$$

Then for all $u_0 \in Z$ the following holds. For $m \in \mathbb{N}_0$, $n \in \mathbb{N}$, $\omega \in \Omega$

$$T\mathcal{A}^m_{n,\omega}(f, u_0) = \mathcal{A}^m_{n,\omega}(g, Tu_0). \tag{6.23}$$

Moreover, if u is a solution of (6.6), then Tu is a solution of the ODE in Z_1,

$$w'(t) = g(t, w(t)) \quad (t \in [a,b]), \qquad w(a) = Tu_0. \tag{6.24}$$

Proof. Applying T to (6.6), we get

$$\begin{aligned} (Tu(t))' &= Tu'(t) = Tf(t, u(t)) = g(t, Tu(t)) \quad (t \in [a,b]), \\ Tu(a) &= Tu_0. \end{aligned}$$

Now the second statement follows from uniqueness of the solution of (6.24).

Let u_k, $p_{k,j}$, and $q_{k,j}$ be the resulting sequences (6.9–6.12) when applying $\mathcal{A}_{n,\omega}^m$ to (f, u_0). Furthermore, put $\tilde{u}_0 = Tu_0$ and let $\tilde{u}_k$, $\tilde{p}_{k,j}$, and $\tilde{q}_{k,j}$ be the respective functions from applying $\mathcal{A}_{n,\omega}^m$ to $(g, \tilde{u}_0)$. We show that for $0 \leq k \leq n$

$$Tu_k = \tilde{u}_k, \tag{6.25}$$

and for $0 \leq k \leq n-1$

$$Tp_{k,j} = \tilde{p}_{k,j} \quad (0 \leq j \leq m). \tag{6.26}$$

First we prove that given k with $0 \leq k \leq n-1$, (6.25) implies (6.26). So assume that (6.25) holds. We show (6.26) by induction over j. Let $j = 0$. By (6.22) and (6.25)

$$Tf(t_k, u_k) = g(t_k, Tu_k) = g(t_k, \tilde{u}_k),$$

therefore,

$$Tp_{k,0}(t) = Tu_k + Tf(t_k, u_k)(t - t_k) = \tilde{u}_k + g(t_k, \tilde{u}_k)(t - t_k) = \tilde{p}_{k,0}(t).$$

Now we assume that (6.26) holds for some j with $0 \leq j < m$. Then

$$Tp_{k,j}(t_{k,j+1,i}) = \tilde{p}_{k,j}(t_{k,j+1,i}) \quad (i = 0, \ldots, j+1).$$

It follows that

$$Tf(t_{k,j+1,i}, p_{k,j}(t_{k,j+1,i})) = g(t_{k,j+1,i}, \tilde{p}_{k,j}(t_{k,j+1,i})),$$

and consequently

$$\begin{aligned} Tp_{k,j+1}(t) &= Tu_k + T\int_{t_k}^{t} (P_{k,j+1}q_{k,j+1})(\tau)d\tau \\ &= \tilde{u}_k + \int_{t_k}^{t} (P_{k,j+1}\tilde{q}_{k,j+1})(\tau)d\tau = \tilde{p}_{k,j+1}(t). \end{aligned}$$

This completes the induction over j and the proof that (6.25) implies (6.26).

Next we show (6.25) by induction over k. For $k = 0$ it holds by definition. Now suppose (6.25) and thus (6.26) holds for some k with $0 \leq k \leq n-1$. It follows that

$$\begin{aligned} Tu_{k+1} &= Tp_{k,m}(t_{k+1}) + h\big(Tf(\xi_{k+1}, p_{k,m}(\xi_{k+1})) - Tp'_{k,m}(\xi_{k+1})\big) \\ &= \tilde{p}_{k,m}(t_{k+1}) + h\big(g(\xi_{k+1}, \tilde{p}_{k,m}(\xi_{k+1})) - \tilde{p}'_{k,m}(\xi_{k+1})\big) = \tilde{u}_{k+1}. \end{aligned}$$

This shows (6.25) for $k+1$, completes the induction over k and proves (6.25–6.26). Now, (6.23) follows from (6.25–6.26) and (6.13–6.14).

□

Assume that a Banach space Y is continuously embedded into the Banach space X, and let J be the embedding map. We shall identify elements of Y with their images in X.

Definition 6.3.2. Let $r, r_1 \in \mathbb{N}_0$, $0 \le \varrho, \varrho_1 \le 1$, $\kappa, L : (0, +\infty) \to (0, +\infty)$, $\sigma, \lambda > 0$, then we define

$$\mathcal{K} = \mathcal{F}^{r,\varrho}([a,b] \times X, X; \kappa, L, \sigma, \lambda) \cap \mathcal{F}^{r_1,\varrho_1}([a,b] \times Y, Y; \kappa, L, \sigma, \lambda), \tag{6.27}$$

which is the set of all $(f, u_0) \in \mathcal{F}^{r,\varrho}([a,b] \times X, X; \kappa, L, \sigma, \lambda)$ such that f maps $[a,b] \times Y$ to Y and, if f is considered as such a mapping, (f, u_0) belongs to $\mathcal{F}^{r_1,\varrho_1}([a,b] \times Y, Y; \kappa, L, \sigma, \lambda)$.

Observe that the solution operator $\mathcal{S}$ is correctly defined also on $\mathcal{K}$, since the respective operators on $\mathcal{F}^{r,\varrho}([a,b] \times X, X; \kappa, L, \sigma, \lambda)$ and $\mathcal{F}^{r_1,\varrho_1}([a,b] \times Y, Y; \kappa, L, \sigma, \lambda)$ coincide on the intersection. This follows from Lemma 6.3.1 with $Z = Y$, $Z_1 = X$, $T = J$, and $g = f$.

Next we state the general multilevel algorithm. The construction is similar to the one for parametric integration. However, we will see that the analysis is more involved due to the non-linearity of the problem.

Algorithm 6.3.3. Let $(R_l)_{l=0}^{\infty} \subset \mathscr{L}(X)$ and fix any $l_0, l_1 \in \mathbb{N}_0$, $l_0 \le l_1$, and $(n_l)_{l=l_0}^{l_1} \subset \mathbb{N}$. For $(f, u_0) \in \mathcal{K}$ and $\omega \in \Omega$ we define an approximation $\mathfrak{A}_\omega(f, u_0)$ to $u = \mathcal{S}(f, u_0)$ in the space $B([a,b], X)$ as follows

$$\mathfrak{A}_\omega(f, u_0) \quad = \quad R_{l_0} \mathcal{A}^r_{n_{l_0},\omega}(f, u_0) + \sum_{l=l_0+1}^{l_1} (R_l - R_{l-1}) \mathcal{A}^{r_1}_{n_l,\omega}(f, u_0). \tag{6.28}$$

Without loss of generality, we demand that the underlying probability space $(\Omega, \Sigma, \mathbb{P})$ is such that all random variables required on the levels $l_0, \dots, l_1$ are defined on it.

Moreover, we assume that there is a constant $\gamma_0 > 0$ such that for all $l \in \mathbb{N}_0$

$$\|R_l\|_{\mathscr{L}(X)} \quad \le \quad \gamma_0. \tag{6.29}$$

Furthermore, we assume the existence of a family of operators $(T_l)_{l=0}^{\infty} \subset \mathscr{L}(X)$ with the following properties: There are constants $\gamma_1, \gamma_2 > 0$ such that for $l \in \mathbb{N}_0$

$$\|T_l\|_{\mathscr{L}(X)} \le \gamma_1, \tag{6.30}$$

T_l maps Y to Y,

$$\|T_l\|_{\mathscr{L}(Y)} \le \gamma_2, \tag{6.31}$$

and

$$R_k T_l = R_k \quad (k \le l). \tag{6.32}$$

Definition 6.3.4. Let $\mathcal{K}_0 \subseteq \mathcal{K}$ be a subset with the following property: If f is such that there exists a u_0 with $(f, u_0) \in \mathcal{K}_0$, then

$$T_l f(t,x) = T_l f(t, T_l x) \quad (t \in [a,b], x \in X, l \in \mathbb{N}_0). \tag{6.33}$$

As before we set

$$X_l = \mathrm{cl}_X(T_l(X)), \quad Y_l = \mathrm{cl}_Y(T_l(Y)) \quad (l \in \mathbb{N}_0),$$

where cl denotes the closure in the respective space.

Notice that the T_l do not enter the algorithm definition, they are only needed for the error analysis. Furthermore, (6.30–6.33) hold, in particular, for $\mathcal{K}_0 = \mathcal{K}$ and $T_l \equiv I_X$. In this case, the error estimate (6.35) in the randomized setting of Proposition 6.3.5 below requires some type assumption on the spaces X and Y. However, in Sections 6.4 and 6.5 we consider spaces X and Y which have no nontrivial type, while certain finite dimensional subspaces related to the approximation do have type constants with nontrivial estimates. Therefore, we will also consider other choices of $\mathcal{K}_0$ and T_l, see Section 6.5.

Proposition 6.3.5. *Let $r, r_1 \in \mathbb{N}_0$, $0 \le \varrho, \varrho_1 \le 1$, $\kappa, L : (0,+\infty) \to (0,+\infty)$, $\sigma, \lambda, \gamma_{0-2} > 0$, and $1 \le p \le 2$. Then there are constants $c_1, c_2 > 0$ and $\nu_0 \in \mathbb{N}$ such that the following holds:*

Given Banach spaces X, Y with Y continuously embedded into X, sequences $(R_l)_{l=0}^{\infty}, (T_l)_{l=0}^{\infty} \subset \mathscr{L}(X)$ satisfying (6.29–6.32), let $\mathcal{K}$ be defined by (6.27), and let $\mathcal{K}_0 \subseteq \mathcal{K}$ be such that (6.33) is fulfilled. Then for all $l_0, l_1 \in \mathbb{N}_0$ with $l_0 \le l_1$ and $(n_l)_{l=l_0}^{l_1} \subset \mathbb{N}$ with $n_l \ge \nu_0$ $(l_0 < l \le l_1)$ the so-defined algorithm $(\mathfrak{A}_\omega)$ satisfies

$$\begin{aligned}\sup_{(f,u_0)\in\mathcal{K}_0} &\|\mathcal{S}(f,u_0) - \mathfrak{A}_\omega(f,u_0)\|_{B([a,b],X)} \\ &\le c_1 \|J - R_{l_1} J\|_{\mathscr{L}(Y,X)} + c_1 n_{l_0}^{-r-\varrho} \\ &\quad + c_1 \sum_{l=l_0+1}^{l_1} \|(R_l - R_{l-1})J\|_{\mathscr{L}(Y,X)} n_l^{-r_1-\varrho_1} \qquad (\omega \in \Omega),\end{aligned} \tag{6.34}$$

and for any $l^* \in \mathbb{N}_0$ with $l_0 \le l^* \le l_1$

$$\begin{aligned}\sup_{(f,u_0)\in\mathcal{K}_0} &\left(\mathbb{E}\, \|\mathcal{S}(f,u_0) - \mathfrak{A}_\omega(f,u_0)\|_{B([a,b],X)}^p\right)^{\frac{1}{p}} \\ &\le c_2 \|J - R_{l_1} J\|_{\mathscr{L}(Y,X)} + c_2 \tau_p(X_{l_0}) n_{l_0}^{-r-\varrho-1+\frac{1}{p}} \\ &\quad + c_2 \sum_{l=l_0+1}^{l^*} \tau_p(Y_l) \|(R_l - R_{l-1})J\|_{\mathscr{L}(Y,X)} n_l^{-r_1-\varrho_1-1+\frac{1}{p}} \\ &\quad + c_2 \sum_{l=l^*+1}^{l_1} \|(R_l - R_{l-1})J\|_{\mathscr{L}(Y,X)} n_l^{-r_1-\varrho_1}.\end{aligned} \tag{6.35}$$

Proof. Let $(f, u_0) \in \mathcal{K}_0$. Then by (6.27) and (6.17)

$$\|\mathcal{S}(f, u_0)\|_{B([a,b],Y)} \le \lambda.$$

It follows that

$$\|\mathcal{S}(f, u_0) - R_{l_1}\mathcal{S}(f, u_0)\|_{B([a,b],X)} \le \lambda \|J - R_{l_1} J\|_{\mathscr{L}(Y,X)}. \tag{6.36}$$

We obtain by (6.30) and (6.31)

$$\begin{aligned} T_{l_0} f &\in \mathcal{C}^{r,\varrho}_{\mathrm{Lip}}([a, b] \times X_{l_0}, X_{l_0}; \gamma_1\kappa, \gamma_1 L), && (6.37)\\ T_l f &\in \mathcal{C}^{r_1,\varrho_1}_{\mathrm{Lip}}([a, b] \times Y_l, Y_l; \gamma_2\kappa, \gamma_2 L) \quad (l_0 < l \le l_1), && (6.38) \end{aligned}$$

and therefore, using (6.33) and Lemma 6.3.1 with $g = T_l f$,

$$\begin{aligned} T_l\mathcal{S}(f, u_0) &= \mathcal{S}(T_l f, T_l u_0) \quad (l_0 \le l \le l_1), && (6.39)\\ T_{l_0}\mathcal{A}^r_{n_{l_0},\omega}(f, u_0) &= \mathcal{A}^r_{n_{l_0},\omega}(T_{l_0} f, T_{l_0} u_0) \quad (\omega \in \Omega), && (6.40)\\ T_l\mathcal{A}^{r_1}_{n_l,\omega}(f, u_0) &= \mathcal{A}^{r_1}_{n_l,\omega}(T_l f, T_l u_0) \quad (\omega \in \Omega,\ l_0 < l \le l_1). && (6.41) \end{aligned}$$

This together with (6.30–6.31) and (6.37–6.38) implies

$$\begin{aligned} (T_{l_0} f, T_{l_0} u_0) &\in \mathcal{F}^{r,\varrho}([a, b] \times X_{l_0}, X_{l_0}; \gamma_1\kappa, \gamma_1 L, \gamma_1\sigma, \gamma_1\lambda), && (6.42)\\ (T_l f, T_l u_0) &\in \mathcal{F}^{r_1,\varrho_1}([a, b] \times Y_l, Y_l; \gamma_2\kappa, \gamma_2 L, \gamma_2\sigma, \gamma_2\lambda) \quad (l_0 < l \le l_1). && (6.43) \end{aligned}$$

By (6.28),

$$\begin{aligned} &\|\mathcal{S}(f, u_0) - \mathfrak{A}_\omega(f, u_0)\|_{B([a,b],X)}\\ &\quad \le \|\mathcal{S}(f, u_0) - R_{l_1}\mathcal{S}(f, u_0)\|_{B([a,b],X)}\\ &\qquad + \|R_{l_0}\mathcal{S}(f, u_0) - R_{l_0}\mathcal{A}^r_{n_{l_0},\omega}(f, u_0)\|_{B([a,b],X)}\\ &\qquad + \sum_{l=l_0+1}^{l_1} \|(R_l - R_{l-1})(\mathcal{S}(f, u_0) - \mathcal{A}^{r_1}_{n_l,\omega}(f, u_0))\|_{B([a,b],X)}. && (6.44) \end{aligned}$$

Furthermore, by (6.39), (6.40), and (6.29),

$$\begin{aligned} &\|R_{l_0}\mathcal{S}(f, u_0) - R_{l_0}\mathcal{A}^r_{n_{l_0},\omega}(f, u_0)\|_{B([a,b],X)}\\ &\quad = \|R_{l_0}T_{l_0}\mathcal{S}(f, u_0) - R_{l_0}T_{l_0}\mathcal{A}^r_{n_{l_0},\omega}(f, u_0)\|_{B([a,b],X)}\\ &\quad = \|R_{l_0}\mathcal{S}(T_{l_0} f, T_{l_0} u_0) - R_{l_0}\mathcal{A}^r_{n_{l_0},\omega}(T_{l_0} f, T_{l_0} u_0)\|_{B([a,b],X)}\\ &\quad \le \gamma_0\|\mathcal{S}(T_{l_0} f, T_{l_0} u_0) - \mathcal{A}^r_{n_{l_0},\omega}(T_{l_0} f, T_{l_0} u_0)\|_{B([a,b],X_{l_0})}, && (6.45) \end{aligned}$$

and similarly, by (6.39) and (6.41),

$$\begin{aligned}
&\|(R_l - R_{l-1})(\mathcal{S}(f,u_0) - \mathcal{A}^{r_1}_{n_l,\omega}(f,u_0))\|_{B([a,b],X)} \\
&\quad = \|(R_l - R_{l-1})T_l(\mathcal{S}(f,u_0) - \mathcal{A}^{r_1}_{n_l,\omega}(f,u_0))\|_{B([a,b],X)} \\
&\quad = \|(R_l - R_{l-1})(\mathcal{S}(T_l f, T_l u_0) - \mathcal{A}^{r_1}_{n_l,\omega}(T_l f, T_l u_0))\|_{B([a,b],X)} \\
&\quad \le \|(R_l - R_{l-1})J\|_{\mathscr{L}(Y,X)}\|\mathcal{S}(T_l f, T_l u_0) - \mathcal{A}^{r_1}_{n_l,\omega}(T_l f, T_l u_0)\|_{B([a,b],Y_l)}. \qquad (6.46)
\end{aligned}$$

By (6.42),(6.43), and Proposition 6.2.5, for all $\omega \in \Omega$ and $(n_l)_{l=l_0}^{l_1} \subset \mathbb{N}$ with $n_l \ge \nu_0$ $(l_0 \le l \le l_1)$

$$\|\mathcal{S}(T_{l_0} f, T_{l_0} u_0) - \mathcal{A}^{r}_{n_{l_0},\omega}(T_{l_0} f, T_{l_0} u_0)\|_{B([a,b],X_{l_0})} \le c n_{l_0}^{-r-\varrho}, \qquad (6.47)$$

$$\|\mathcal{S}(T_l f, T_l u_0) - \mathcal{A}^{r_1}_{n_l,\omega}(T_l f, T_l u_0)\|_{B([a,b],Y_l)} \le c n_l^{-r_1-\varrho_1}, \qquad (6.48)$$

and

$$\begin{aligned}
&\left(\mathbb{E}\,\|\mathcal{S}(T_{l_0} f, T_{l_0} u_0) - \mathcal{A}^{r}_{n_{l_0},\omega}(T_{l_0} f, T_{l_0} u_0)\|^p_{B([a,b],X_{l_0})}\right)^{\frac{1}{p}} \\
&\qquad\qquad \le c\tau_p(X_{l_0}) n_{l_0}^{-r-\varrho-1+\frac{1}{p}}, \qquad (6.49) \\
&\left(\mathbb{E}\,\|\mathcal{S}(T_l f, T_l u_0) - \mathcal{A}^{r_1}_{n_l,\omega}(T_l f, T_l u_0)\|^p_{B([a,b],Y_l)}\right)^{\frac{1}{p}} \\
&\qquad\qquad \le c\tau_p(Y_l) n_l^{-r_1-\varrho_1-1+\frac{1}{p}}. \qquad (6.50)
\end{aligned}$$

Combining (6.36) and (6.44–6.48) yields (6.34). Relation (6.35) follows in a similar way from (6.36),(6.44–6.46), and (6.48–6.50).

□

6.4 The Parametric Problem as a Banach Space Valued ODE

In this section, we express the parametric problem in terms of a Banach space valued ODE, which allows us to apply the previous results for the general multilevel algorithm defined above.

Let $d_0 \in \mathbb{N}$, $Q_0 = [0,1]^{d_0}$. To keep notation consistent, instead of considering derivatives with respect to single components of $s \in \mathbb{R}^{d_0}$, we consider derivatives with respect to the vector s, in the sense of calculus on vector spaces as in the previous section. So below $\frac{df}{ds}$ is the Jacobian, $\frac{d^2 f}{ds^2}$ the Hessian, etc. The space $\mathbb{R}^{d_0}$ is equipped with the Euclidean norm. For $r \in \mathbb{N}_0$ and a Banach space Z, we let in the following $C^r(Q_0, Z)$ be the space of Z-valued r-times continuously differentiable functions on Q_0, endowed with the norm

$$\|f\|_{C^r(Q_0,Z)} = \max_{0\le j\le r} \sup_{s\in Q_0} \left\| \frac{d^j f(s)}{ds^j} \right\|_{\mathscr{L}_j(\mathbb{R}^{d_0},Z)}.$$

Notice that for $r \ge 1$ this is not the standard norm on $C^r(Q_0, Z)$ as used before, but it is equivalent to a constant depending only on d_0 and r.

Definition 6.4.1. Given functions $\kappa, L : (0,+\infty) \to (0,+\infty)$, $r_0, r \in \mathbb{N}_0$, $0 \le \varrho \le 1$, and Banach spaces Z, Z_1, we define the following class

$$\mathscr{C}^{r_0,r,\varrho}(Q_0 \times [a,b] \times Z, Z_1; \kappa) \quad \text{of continuous functions} \quad f : Q_0 \times [a,b] \times Z \to Z_1$$

having, for $\alpha = (\alpha_0, \alpha_1, \alpha_2) \in \mathbb{N}_0^3$ with $\alpha_0 \le r_0$, $\alpha_1 \le r$ and $\alpha_0 + \alpha_1 + \alpha_2 \le r_0 + r$, continuous partial derivatives

$$\frac{\partial^{|\alpha|} f(s,t,z)}{\partial s^{\alpha_0} \partial t^{\alpha_1} \partial z^{\alpha_2}} \in \mathscr{L}_{\alpha_0,\alpha_2}(\mathbb{R}^{d_0}, Z, Z_1),$$

satisfying for $R > 0$, $s \in Q_0$, $t \in [a,b]$, $z \in RB_Z$

$$\left\| \frac{\partial^{|\alpha|} f(s,t,z)}{\partial s^{\alpha_0} \partial t^{\alpha_1} \partial z^{\alpha_2}} \right\|_{\mathscr{L}_{\alpha_0,\alpha_2}(\mathbb{R}^{d_0},Z,Z_1)} \le \kappa(R), \tag{6.51}$$

and for $s \in Q_0$, $t_1, t_2 \in [a,b]$, $z_1, z_2 \in RB_Z$

$$\begin{aligned} &\left\| \frac{\partial^{|\alpha|} f(s,t_1,z_1)}{\partial s^{\alpha_0} \partial t^{\alpha_1} \partial z^{\alpha_2}} - \frac{\partial^{|\alpha|} f(s,t_2,z_2)}{\partial s^{\alpha_0} \partial t^{\alpha_1} \partial z^{\alpha_2}} \right\|_{\mathscr{L}_{\alpha_0,\alpha_2}(\mathbb{R}^{d_0},Z,Z_1)} \\ &\qquad \le \kappa(R)|t_1 - t_2|^{\varrho} + \kappa(R)\|z_1 - z_2\|^{\varrho}. \end{aligned} \tag{6.52}$$

Moreover, we let $\mathscr{C}^{r_0,r,\varrho}_{\mathrm{Lip}}(Q_0 \times [a,b] \times Z, Z_1; \kappa, L)$ be the class of all $f \in \mathscr{C}^{r_0,r,\varrho}(Q_0 \times [a,b] \times Z, Z_1; \kappa)$ satisfying for $\alpha = (\alpha_0, 0, \alpha_2)$ with $\alpha_0 + \alpha_2 \le r_0$, $R > 0$, $s \in Q_0$, $t \in [a,b]$, $z_1, z_2 \in RB_Z$

$$\left\| \frac{\partial^{|\alpha|} f(s,t,z_1)}{\partial s^{\alpha_0} \partial z^{\alpha_2}} - \frac{\partial^{|\alpha|} f(s,t,z_2)}{\partial s^{\alpha_0} \partial z^{\alpha_2}} \right\|_{\mathscr{L}_{\alpha_0,\alpha_2}(\mathbb{R}^{d_0},Z,Z_1)} \le L(R)\|z_1 - z_2\|. \tag{6.53}$$

Clearly, if $r_0', r' \in \mathbb{N}_0$ fulfill $r_0' \le r_0$, $r' \le r$, then

$$\mathscr{C}^{r_0,r,\varrho}(Q_0 \times [a,b] \times Z, Z_1; \kappa), \subseteq \mathscr{C}^{r_0',r',\varrho}(Q_0 \times [a,b] \times Z, Z_1; \kappa), \tag{6.54}$$

$$\mathscr{C}^{r_0,r,\varrho}_{\mathrm{Lip}}(Q_0 \times [a,b] \times Z, Z_1; \kappa, L) \subseteq \mathscr{C}^{r_0',r',\varrho}_{\mathrm{Lip}}(Q_0 \times [a,b] \times Z, Z_1; \kappa, L). \tag{6.55}$$

Furthermore, if $\varrho' \le \varrho$, then

$$\mathscr{C}_{\mathrm{Lip}}^{r_0,r,\varrho}(Q_0 \times [a,b] \times Z, Z_1; \kappa, L) \;\subseteq\; \mathscr{C}_{\mathrm{Lip}}^{r_0,r,\varrho'}(Q_0 \times [a,b] \times Z, Z_1; 2\kappa, L), \quad (6.56)$$

where the factor 2 comes from the case $\max(|t_1 - t_2|, \|z_1 - z_2\|) > 1$, in which (6.51) with constant κ trivially implies (6.52) with constant 2κ. Integration yields

$$\mathscr{C}_{\mathrm{Lip}}^{r_0,r+1,0}(Q_0 \times [a,b] \times Z, Z_1; \kappa, L) \;\subseteq\; \mathscr{C}_{\mathrm{Lip}}^{r_0,r,1}(Q_0 \times [a,b] \times Z, Z_1; \kappa, L). \quad (6.57)$$

Finally, notice that it would suffice to require (6.52) and (6.53) for certain subsets of the sets of multiindices α to obtain (up to constants) the same classes – we omit the details, because the definition given above is more convenient for us.

The classes above were introduced for two Banach spaces Z, Z_1. Some of the lemmas below will be formulated in this general form, for technical convenience. However, for the formulation of the problem and later for the main results we have $Z_1 = Z$.

Now we consider the numerical solution of initial value problems for Z-valued ODEs depending on a parameter $s \in Q_0$;

$$\frac{d}{dt}u(s,t) \;=\; f(s,t,u(s,t)) \quad (s \in Q_0, t \in [a,b]), \quad (6.58)$$

$$u(s,a) \;=\; u_0(s) \quad (s \in Q_0), \quad (6.59)$$

with $f \in \mathscr{C}_{\mathrm{Lip}}^{r_0,r,\varrho}(Q_0 \times [a,b] \times Z, Z; \kappa, L)$ and $u_0 \in C^{r_0}(Q_0, Z)$. A function $u : Q_0 \times [a,b] \to Z$ is called a solution if, for each $s \in Q_0$, $u(s,t)$ is continuously differentiable as a function of t and (6.58–6.59) are satisfied.

Similar to the consideration of the parametric integration problem, the class $\mathscr{C}_{\mathrm{Lip}}^{r_0,r,\varrho}(Q_0 \times [a,b] \times Z, Z; \kappa, L)$ introduced above is a certain class of functions with dominating mixed smoothness. As before, we consider the intersection of two such classes. This enables us to exploit the full generality of (6.27) and, in particular, to include also functions with isotropic smoothness.

Definition 6.4.2. Let $r_1 \in \mathbb{N}_0$, $0 \le \varrho_1 \le 1$, $\sigma, \lambda > 0$. Then define $\mathscr{F}$ as the class of all

$$(f, u_0) = \left(\mathscr{C}_{\mathrm{Lip}}^{0,r,\varrho}(Q_0 \times [a,b] \times Z, Z; \kappa, L) \cap \mathscr{C}_{\mathrm{Lip}}^{r_0,r_1,\varrho_1}(Q_0 \times [a,b] \times Z, Z; \kappa, L)\right) \times \sigma B_{C^{r_0}(Q_0,Z)} \quad (6.60)$$

such that the parameter dependent initial value problem (6.58–6.59) has a solution $u(s,t)$ (which is unique due to assumption (6.60) on f) such that

$$\sup_{s \in Q_0,\, t \in [a,b]} \|u(s,t)\| \le \lambda, \quad (6.61)$$

and moreover, if $r = \varrho = r_1 = \varrho_1 = 0$, then for all $n \in \mathbb{N}$, $\omega \in \Omega$

$$\sup_{s \in Q_0} \left\| \mathcal{A}^0_{n,\omega}(f_s, u_0(s)) \right\|_{B([a,b],Z)} \le \lambda, \tag{6.62}$$

where f_s denotes the function $f(s,\cdot,\cdot)$ from $[a,b] \times Z$ to Z for fixed $s \in Q_0$. We define the solution operator

$$\mathscr{S} : \mathscr{F} \to B(Q_0 \times [a,b], Z) \tag{6.63}$$

for $(f, u_0) \in \mathscr{F}$ by $\mathscr{S}(f, u_0) = u$, where $u = u(s,t)$ is the solution of (6.58–6.59).

Let us motivate the choice of the smoothness for the class $\mathscr{F}$ in (6.60–6.62). This is best explained when looking at the subset of those functions f which depend only on s and t. Then the parameters $r_0, r, r_1, \varrho, \varrho_1$ describe the smoothness of $f(s,t)$ and we arrive at classes analogous to those studied in Chapter 5.

The smoothness we impose with respect to z can is chosen in an appropriate way. By this we mean the following: We will show in Section 6.6 that the complexity only mildly depends on the smoothness in z in the sense that increasing this smoothness does not result in a higher rate of the minimal errors. In fact, even if f does not depend on z at all, we get the same rate. Therefore, with the smoothness parameters $r_0, r, r_1, \varrho, \varrho_1$ set for s and t, the smoothness in z has been chosen in such a way that it just guarantees the respective convergence rate. (Of course, a challenging problem is to find minimal smoothness requirements in z that still ensure the same rate. We do not pursue this aspect here.)

The following is the central result of this section. It relates the parametric problem to the problem of a single Banach space valued ODE considered in Section 6.2, with $X = C(Q_0, Z)$ and $Y = C^{r_0}(Q_0, Z)$. As before, for a continuous function $f : Q_0 \times [a,b] \times Z \to Z_1$, we define a function

$$\bar{f} : [a,b] \times C(Q_0, Z) \to C(Q_0, Z_1)$$

by setting for $t \in [a,b]$, $x \in C(Q_0, Z)$

$$(\bar{f}(t,x))(s) = f(s,t,x(s)) \quad (s \in Q_0).$$

Proposition 6.4.3. *Given $r_0, r, r_1 \in \mathbb{N}_0$, $0 \le \varrho, \varrho_1 \le 1$, functions $\kappa, L : (0,+\infty) \to (0,+\infty)$, $\sigma, \lambda > 0$, there are $\lambda_1 > 0$ and $\kappa_1, L_1 : (0,+\infty) \to (0,+\infty)$ such that the following holds: Let Z be a Banach space and let $\mathscr{F}$ be defined by* (6.60)*. Then for all $(f, u_0) \in \mathscr{F}$*

$$(\bar{f}, u_0) \in \mathcal{F}^{r,\varrho}([a,b] \times C(Q_0,Z), C(Q_0,Z); \kappa_1, L_1, \sigma, \lambda_1)$$
$$\cap\ \mathcal{F}^{r_1,\varrho_1}([a,b] \times C^{r_0}(Q_0,Z), C^{r_0}(Q_0,Z); \kappa_1, L_1, \sigma, \lambda_1)$$

and

$$\mathcal{S}(\bar{f}, u_0) = \mathscr{S}(f, u_0). \tag{6.64}$$

Concerning relation (6.64), notice that we identify functions from $B(Q_0 \times [a,b], Z)$, $u = u(s,t)$, with functions from $B([a,b], B(Q_0, Z))$, $u(t) = u(\,\cdot\,, t)$. For the proof of Proposition 6.4.3, we need some additional lemmas. We emphasize that the constants (including the functions κ_1, L_1) in the lemmas of this section do not depend on Z and Z_1.

Lemma 6.4.4. *Given κ, L, there are functions $\kappa_1, L_1 : (0,+\infty) \to (0,+\infty)$ such that the following holds: for all $f \in \mathscr{C}^{r_0,0,\varrho}(Q_0 \times [a,b] \times Z, Z_1; \kappa)$, $\bar{f}$ maps $[a,b] \times C^{r_0}(Q_0, Z)$ to $C^{r_0}(Q_0, Z_1)$ and, considered as such a mapping, satisfies*

$$\bar{f} \in \mathcal{C}^{0,\varrho}([a,b] \times C^{r_0}(Q_0,Z), C^{r_0}(Q_0,Z_1); \kappa_1), \tag{6.65}$$

and if $f \in \mathscr{C}^{r_0,0,\varrho}_{\mathrm{Lip}}(Q_0 \times [a,b] \times Z, Z_1; \kappa, L)$, then

$$\bar{f} \in \mathcal{C}^{0,\varrho}_{\mathrm{Lip}}([a,b] \times C^{r_0}(Q_0,Z), C^{r_0}(Q_0,Z_1); \kappa_1, L_1). \tag{6.66}$$

Proof. To prove the result, we have to show that given $f \in \mathscr{C}^{r_0,0,\varrho}(Q_0 \times [a,b] \times Z, Z_1; \kappa)$, the mapping $\bar{f} : [a,b] \times C^{r_0}(Q_0,Z) \to C^{r_0}(Q_0,Z_1)$ is continuous. Furthermore, we need to show that the boundedness condition as well as the ϱ Hölder condition and the Lipschitz condition is satisfied.

• We argue by induction over $r_0 \in \mathbb{N}_0$. Let $r_0 = 0$.

Continuity of $\bar{f} : [a,b] \times C(Q_0,Z) \to C(Q_0,Z_1)$: We show that if $g : Q_0 \times [a,b] \times Z \to Z_1$ is a continuous function, then $\bar{g}$ is continuous from $[a,b] \times C(Q_0,Z)$ to $C(Q_0,Z_1)$. Let $t, t_n \in [a,b]$, $x, x_n \in C(Q_0,Z)$ $(n \in \mathbb{N})$ be such that

$$\lim_{n\to\infty} |t_n - t| = 0, \quad \lim_{n\to\infty} \|x_n - x\|_{C(Q_0,Z)} = 0.$$

It follows that

$$K = \{x_n(s) : \, s \in Q_0, \, n \in \mathbb{N}\} \cup \{x(s) : \, s \in Q_0\}$$

is a compact subset of Z. Consequently, g is uniformly continuous on $Q_0 \times [a,b] \times K$, and therefore,

$$\lim_{n\to\infty} \sup_{s\in Q_0} \|g(s,t_n,x_n(s)) - g(s,t,x(s))\|_{Z_1} = 0,$$

which is the continuity of $\bar{g}$.

Boundedness: Let $x \in RB_{C(Q_0,Z)}$, hence $\|x(s)\| \le R$, $t \in [a,b]$. Then

$$\left\|\bar{f}(t,x)\right\|_{C(Q_0,Z_1)} = \sup_{s\in Q_0} \|f(s,t,x(s))\|_{Z_1} \le \kappa(R).$$

Hölder condition: Let also $y \in RB_{C(Q_0,Z)}$, $t_1, t_2 \in [a,b]$. Then

$$\begin{aligned} \left\|\bar{f}(t_1,x) - \bar{f}(t_2,y)\right\|_{C(Q_0,Z_1)} &= \sup_{s\in Q_0} \|f(s,t_1,x(s)) - f(s,t_2,y(s))\|_{Z_1} \\ &\le \kappa(R) \sup_{s\in Q_0} (|t_1-t_2|^\varrho + \|x(s)-y(s)\|^\varrho) \\ &= \kappa(R)|t_1-t_2|^\varrho + \|x-y\|^\varrho_{C(Q_0,Z)}. \end{aligned}$$

Lipschitz condition: In a similar way as before we obtain

$$\begin{aligned} \left\|\bar{f}(t,x) - \bar{f}(t,y)\right\|_{C(Q_0,Z_1)} &= \sup_{s\in Q_0} \|f(s,t,x(s)) - f(s,t,y(s))\|_{Z_1} \\ &\le L(R) \sup_{s\in Q_0} \|x(s)-y(s)\| \\ &= L(R)\|x-y\|_{C(Q_0,Z)}. \end{aligned}$$

• Now let $r_0 \ge 1$ and assume that the statements (6.65) and (6.66) hold for $r_0 - 1$. We start with (6.65).
Continuity of $\bar{f} : [a,b]\times C^{r_0}(Q_0,Z) \to C^{r_0}(Q_0,Z_1)$: Let $f \in \mathscr{C}^{r_0,0,\varrho}(Q_0\times[a,b]\times Z, Z_1;\kappa)$. Then by (6.51–6.54)

$$\begin{aligned} f &\in \mathscr{C}^{r_0-1,0,\varrho}(Q_0\times[a,b]\times Z, Z_1;\kappa), \\ g_1 := \frac{\partial f}{\partial s} &\in \mathscr{C}^{r_0-1,0,\varrho}(Q_0\times[a,b]\times Z, \mathscr{L}(\mathbb{R}^{d_0}, Z_1);\kappa), \\ g_2 := \frac{\partial f}{\partial z} &\in \mathscr{C}^{r_0-1,0,\varrho}(Q_0\times[a,b]\times Z, \mathscr{L}(Z, Z_1);\kappa), \end{aligned}$$

therefore, by the induction assumption,

$$\bar{f} \in \mathcal{C}^{0,\varrho}([a,b]\times C^{r_0-1}(Q_0,Z), C^{r_0-1}(Q_0,Z_1);\kappa_1), \tag{6.67}$$
$$\bar{g}_1 \in \mathcal{C}^{0,\varrho}([a,b]\times C^{r_0-1}(Q_0,Z), C^{r_0-1}(Q_0,\mathscr{L}(\mathbb{R}^{d_0},Z_1));\kappa_1), \tag{6.68}$$
$$\bar{g}_2 \in \mathcal{C}^{0,\varrho}([a,b]\times C^{r_0-1}(Q_0,Z), C^{r_0-1}(Q_0,\mathscr{L}(Z,Z_1));\kappa_1). \tag{6.69}$$

Fix $t \in [a,b]$ and $x \in C^{r_0}(Q_0,Z)$. Then

$$\left(\frac{d}{ds}\bar{f}(t,x)\right)(s) = g_1(s,t,x(s)) + g_2(s,t,x(s))\frac{dx(s)}{ds},$$

which means that

$$\frac{d}{ds}\bar{f}(t,x) = \bar{g}_1(t,x) + \bar{g}_2(t,x)\frac{dx}{ds}. \tag{6.70}$$

Next we show that (6.67–6.70) imply that $\bar{f}$ maps $[a,b]\times C^{r_0}(Q_0,Z)$ to $C^{r_0}(Q_0,Z_1)$, and $\bar{f}$ is a continuous function from $[a,b]\times C^{r_0}(Q_0,Z)$ to $C^{r_0}(Q_0,Z_1)$. By (6.67–

6.69) and the assumption on x

$$\begin{aligned}
\bar{f}(t,x) &\in C^{r_0-1}(Q_0, Z_1),\\
\bar{g}_1(t,x) &\in C^{r_0-1}(Q_0, \mathscr{L}(\mathbb{R}^{d_0}, Z_1)),\\
\bar{g}_2(t,x) &\in C^{r_0-1}(Q_0, \mathscr{L}(Z, Z_1)),\\
\frac{dx}{ds} &\in C^{r_0-1}(Q_0, \mathscr{L}(\mathbb{R}^{d_0}, Z)),
\end{aligned}$$

which together with (6.70) gives $\bar{f}(t,x) \in C^{r_0}(Q_0, Z_1)$ and shows that $\bar{f}$ maps $[a,b] \times C^{r_0}(Q_0, Z)$ to $C^{r_0}(Q_0, Z_1)$.

Now let $t_n \in [a,b]$, $x_n \in C^{r_0}(Q_0, Z)$ $(n = 1, 2, \dots)$ be chosen such that

$$\lim_{n\to\infty} |t - t_n| = 0, \quad \lim_{n\to\infty} \|x - x_n\|_{C^{r_0}(Q_0,Z)} = 0.$$

Using again (6.67–6.69), we get

$$\begin{aligned}
\lim_{n\to\infty} \left\|\bar{f}(t,x) - \bar{f}(t_n,x_n)\right\|_{C^{r_0-1}(Q_0,Z_1)} &= 0,\\
\lim_{n\to\infty} \|\bar{g}_1(t,x) - \bar{g}_1(t_n,x_n)\|_{C^{r_0-1}(Q_0,\mathscr{L}(\mathbb{R}^{d_0},Z_1))} &= 0,\\
\lim_{n\to\infty} \|\bar{g}_2(t,x) - \bar{g}_2(t_n,x_n)\|_{C^{r_0-1}(Q_0,\mathscr{L}(Z,Z_1))} &= 0,\\
\lim_{n\to\infty} \left\|\frac{dx}{ds} - \frac{dx_n}{ds}\right\|_{C^{r_0-1}(Q_0,\mathscr{L}(\mathbb{R}^{d_0},Z_1))} &= 0.
\end{aligned}$$

Together with (6.70), this implies

$$\lim_{n\to\infty} \left\|\bar{f}(t,x) - \bar{f}(t_n,x_n)\right\|_{C^{r_0}(Q_0,Z_1)} = 0,$$

so $\bar{f}$ is a continuous function from $[a,b] \times C^{r_0}(Q_0, Z)$ to $C^{r_0}(Q_0, Z_1)$.

Boundedness: We show that $\bar{f}$ satisfies the boundedness for r_0. Let $R > 0$ and

$$x, y \in RB_{C^{r_0}(Q_0,Z)}. \tag{6.71}$$

This implies

$$\left\|\frac{dx}{ds}\right\|_{C^{r_0-1}(Q_0,\mathscr{L}(\mathbb{R}^{d_0},Z))} \leq R, \tag{6.72}$$

and together with (6.67–6.69)

$$\begin{aligned}
\left\|\bar{f}(t,x)\right\|_{C^{r_0-1}(Q_0,Z_1)} &\leq \kappa_1(R), &(6.73)\\
\|\bar{g}_1(t,x)\|_{C^{r_0-1}(Q_0,\mathscr{L}(\mathbb{R}^{d_0},Z_1))} &\leq \kappa_1(R),\\
\|\bar{g}_2(t,x)\|_{C^{r_0-1}(Q_0,\mathscr{L}(Z,Z_1))} &\leq \kappa_1(R), &(6.74)
\end{aligned}$$

so inserting into (6.70) gives

$$\begin{aligned}
&\left\| \frac{d}{ds}\bar{f}(t,x) \right\|_{C^{r_0-1}(Q_0,\mathscr{L}(\mathbb{R}^{d_0},Z_1))} \\
&\qquad \le \|\bar{g}_1(t,x)\|_{C^{r_0-1}(Q_0,\mathscr{L}(\mathbb{R}^{d_0},Z_1))} \\
&\qquad\qquad + c\,\|\bar{g}_2(t,x)\|_{C^{r_0-1}(Q_0,\mathscr{L}(Z,Z_1))} \left\| \frac{dx}{ds} \right\|_{C^{r_0-1}(Q_0,\mathscr{L}(\mathbb{R}^{d_0},Z))} \\
&\qquad \le \kappa_1(R) + c\kappa_1(R)R.
\end{aligned}$$

Combining this with (6.73), we obtain

$$\left\|\bar{f}(t,x)\right\|_{C^{r_0}(Q_0,Z_1)} \le \kappa_1(R)(cR+1).$$

Hölder condition: By (6.71),

$$\left\| \frac{dx}{ds} - \frac{dy}{ds} \right\|_{C^{r_0-1}(Q_0,\mathscr{L}(\mathbb{R}^{d_0},Z))} \le \|x-y\|_{C^{r_0}(Q_0,Z)}. \tag{6.75}$$

Let $t_1, t_2 \in [a,b]$ and set

$$M_0 \;=\; \kappa_1(R)\left(|t_1-t_2|^{\varrho} + \|x-y\|^{\varrho}_{C^{r_0-1}(Q_0,Z)}\right). \tag{6.76}$$

Then (6.67–6.69) imply

$$\begin{aligned}
\left\|\bar{f}(t_1,x)-\bar{f}(t_2,y)\right\|_{C^{r_0-1}(Q_0,Z_1)} &\le M_0, && (6.77)\\
\|\bar{g}_1(t_1,x)-\bar{g}_1(t_2,y)\|_{C^{r_0-1}(Q_0,\mathscr{L}(\mathbb{R}^{d_0},Z_1))} &\le M_0, && (6.78)\\
\|\bar{g}_2(t_1,x)-\bar{g}_2(t_2,y)\|_{C^{r_0-1}(Q_0,\mathscr{L}(Z,Z_1))} &\le M_0. && (6.79)
\end{aligned}$$

Using (6.70),(6.72),(6.74–6.75), and (6.78–6.79) it follows that

$$\begin{aligned}
&\left\| \frac{d}{ds}\bar{f}(t_1,x) - \frac{d}{ds}\bar{f}(t_2,y) \right\|_{C^{r_0-1}(Q_0,\mathscr{L}(\mathbb{R}^{d_0},Z_1))} \\
&\qquad \le \|\bar{g}_1(t_1,x)-\bar{g}_1(t_2,y)\|_{C^{r_0-1}(Q_0,\mathscr{L}(\mathbb{R}^{d_0},Z_1))} \\
&\qquad\qquad + c\,\|\bar{g}_2(t_1,x)-\bar{g}_2(t_2,y)\|_{C^{r_0-1}(Q_0,\mathscr{L}(Z,Z_1))} \left\| \frac{dx}{ds} \right\|_{C^{r_0-1}(Q_0,\mathscr{L}(\mathbb{R}^{d_0},Z))} \\
&\qquad\qquad + c\,\|\bar{g}_2(t_2,y)\|_{C^{r_0-1}(Q_0,\mathscr{L}(Z,Z_1))} \left\| \frac{dx}{ds} - \frac{dy}{ds} \right\|_{C^{r_0-1}(Q_0,\mathscr{L}(\mathbb{R}^{d_0},Z))} \\
&\qquad \le (1+cR)M_0 + c\kappa_1(R)\|x-y\|_{C^{r_0}(Q_0,Z)}.
\end{aligned}$$

Together with (6.76–6.77) this gives

$$\left\|\bar f(t_1,x)-\bar f(t_2,y)\right\|_{C^{r_0}(Q_0,Z_1)} \le (1+cR)\kappa_1(R)\left(|t_1-t_2|^\varrho + \|x-y\|^\varrho_{C^{r_0-1}(Q_0,Z)}\right) + c\kappa_1(R)\|x-y\|_{C^{r_0}(Q_0,Z)}.$$

Taking into account that by (6.71)

$$\|x-y\|_{C^{r_0}(Q_0,Z)} \le (2R)^{1-\varrho}\|x-y\|^\varrho_{C^{r_0}(Q_0,Z)},$$

this proves ϱ-Hölder continuity and thus (6.65).

Lipschitz condition: To prove (6.66) for r_0, it remains to show the Lipschitz property. Let $f \in \mathscr{C}^{r_0,0,\varrho}_{\mathrm{Lip}}(Q_0\times[a,b]\times Z, Z_1;\kappa,L)$. Then

$$\begin{aligned} f &\in \mathscr{C}^{r_0-1,0,\varrho}_{\mathrm{Lip}}(Q_0\times[a,b]\times Z, Z_1;\kappa,L),\\ g_1 := \frac{\partial f}{\partial s} &\in \mathscr{C}^{r_0-1,0,\varrho}_{\mathrm{Lip}}(Q_0\times[a,b]\times Z, \mathscr{L}(\mathbb{R}^{d_0},Z_1);\kappa,L),\\ g_2 := \frac{\partial f}{\partial z} &\in \mathscr{C}^{r_0-1,0,\varrho}_{\mathrm{Lip}}(Q_0\times[a,b]\times Z, \mathscr{L}(Z,Z_1);\kappa,L). \end{aligned}$$

Therefore, again by the induction assumption,

$$\begin{aligned} \bar f &\in \mathcal{C}^{0,\varrho}_{\mathrm{Lip}}([a,b]\times C^{r_0-1}(Q_0,Z), C^{r_0-1}(Q_0,Z_1);\kappa_1,L_1),\\ \bar g_1 &\in \mathcal{C}^{0,\varrho}_{\mathrm{Lip}}([a,b]\times C^{r_0-1}(Q_0,Z), C^{r_0-1}(Q_0,\mathscr{L}(\mathbb{R}^{d_0},Z_1));\kappa_1,L_1),\\ \bar g_2 &\in \mathcal{C}^{0,\varrho}_{\mathrm{Lip}}([a,b]\times C^{r_0-1}(Q_0,Z), C^{r_0-1}(Q_0,\mathscr{L}(Z,Z_1));\kappa_1,L_1). \end{aligned}$$

Consequently,

$$\begin{aligned} \left\|\bar f(t,x)-\bar f(t,y)\right\|_{C^{r_0-1}(Q_0,Z_1)} &\le L_1(R)\|x-y\|_{C^{r_0-1}(Q_0,Z)}, \qquad (6.80)\\ \|\bar g_1(t,x)-\bar g_1(t,y)\|_{C^{r_0-1}(Q_0,\mathscr{L}(\mathbb{R}^{d_0},Z_1))} &\le L_1(R)\|x-y\|_{C^{r_0-1}(Q_0,Z)},\\ \|\bar g_2(t,x)-\bar g_2(t,y)\|_{C^{r_0-1}(Q_0,\mathscr{L}(Z,Z_1))} &\le L_1(R)\|x-y\|_{C^{r_0-1}(Q_0,Z)}. \end{aligned}$$

Using (6.70), (6.72), and (6.74–6.75), it follows that

$$\begin{aligned} &\left\|\frac{d}{ds}\bar f(t,x) - \frac{d}{ds}\bar f(t,y)\right\|_{C^{r_0-1}(Q_0,\mathscr{L}(\mathbb{R}^{d_0},Z_1))}\\ &\le \|\bar g_1(t,x)-\bar g_1(t,y)\|_{C^{r_0-1}(Q_0,\mathscr{L}(\mathbb{R}^{d_0},Z_1))}\\ &\quad + c\,\|\bar g_2(t,x)-\bar g_2(t,y)\|_{C^{r_0-1}(Q_0,\mathscr{L}(Z,Z_1))}\left\|\frac{dx}{ds}\right\|_{C^{r_0-1}(Q_0,\mathscr{L}(\mathbb{R}^{d_0},Z))}\\ &\quad + c\,\|\bar g_2(t,y)\|_{C^{r_0-1}(Q_0,\mathscr{L}(Z,Z_1))}\left\|\frac{dx}{ds}-\frac{dy}{ds}\right\|_{C^{r_0-1}(Q_0,\mathscr{L}(\mathbb{R}^{d_0},Z))}\\ &\le (1+cR)L_1(R)\|x-y\|_{C^{r_0-1}(Q_0,Z)} + c\kappa_1(R)\|x-y\|_{C^{r_0}(Q_0,Z)}. \end{aligned}$$

Together with (6.80), this proves the Lipschitz property. □

It remains to show the previous result for $r > 0$.

Lemma 6.4.5. *Given κ, L, there are $\kappa_1, L_1 : (0, +\infty) \to (0, +\infty)$ such that for all $f \in \mathscr{C}^{r_0,r,\varrho}(Q_0 \times [a,b] \times Z, Z_1; \kappa)$*

$$\bar{f} \in \mathcal{C}^{r,\varrho}([a,b] \times C^{r_0}(Q_0, Z), C^{r_0}(Q_0, Z_1); \kappa_1), \tag{6.81}$$

and for all $f \in \mathscr{C}^{r_0,r,\varrho}_{\mathrm{Lip}}(Q_0 \times [a,b] \times Z, Z_1; \kappa, L)$

$$\bar{f} \in \mathcal{C}^{r,\varrho}_{\mathrm{Lip}}([a,b] \times C^{r_0}(Q_0, Z), C^{r_0}(Q_0, Z_1); \kappa_1, L_1). \tag{6.82}$$

Proof. First we show (6.81). We argue by induction over r. The case $r = 0$ follows from (6.65) of Lemma 6.4.4. Now let $r \geq 1$ and assume that the statement holds for $r - 1$. It follows from (6.51–6.54) that

$$\begin{aligned}
f &\in \mathscr{C}^{r_0,r-1,\varrho}(Q_0 \times [a,b] \times Z, Z_1; \kappa), \\
g_1 := \frac{\partial f}{\partial t} &\in \mathscr{C}^{r_0,r-1,\varrho}(Q_0 \times [a,b] \times Z, Z_1; \kappa), \\
g_2 := \frac{\partial f}{\partial z} &\in \mathscr{C}^{r_0,r-1,\varrho}(Q_0 \times [a,b] \times Z, \mathscr{L}(Z, Z_1); \kappa).
\end{aligned}$$

The induction assumption implies

$$\bar{f} \in \mathcal{C}^{r-1,\varrho}([a,b] \times C^{r_0}(Q_0, Z), C^{r_0}(Q_0, Z_1); \kappa_1), \tag{6.83}$$

$$\bar{g}_1 \in \mathcal{C}^{r-1,\varrho}([a,b] \times C^{r_0}(Q_0, Z), C^{r_0}(Q_0, Z_1); \kappa_1), \tag{6.84}$$

$$\bar{g}_2 \in \mathcal{C}^{r-1,\varrho}([a,b] \times C^{r_0}(Q_0, Z), C^{r_0}(Q_0, \mathscr{L}(Z, Z_1)); \kappa_1). \tag{6.85}$$

Now we study the differentiability of $\bar{f}$ with respect to t and x, as a function from $[a,b] \times C^{r_0}(Q_0, Z)$ to $C^{r_0}(Q_0, Z_1)$. Let $t_1, t_2 \in [a,b]$, $t_1 \neq t_2$, $x \in C^{r_0}(Q_0, Z)$, $s \in Q_0$. Then

$$\begin{aligned}
\frac{\bar{f}(t_2, x)(s) - \bar{f}(t_1, x)(s)}{t_2 - t_1} &= \frac{f(s, t_2, x(s)) - f(s, t_1, x(s))}{t_2 - t_1} \\
&= \int_0^1 g_1(s, t_1 + \tau(t_2 - t_1), x(s)) d\tau \\
&= \int_0^1 \bar{g}_1(t_1 + \tau(t_2 - t_1), x)(s) d\tau.
\end{aligned}$$

By (6.84), $\bar{g}_1$ is a continuous function from $[a,b] \times C^{r_0}(Q_0, Z)$ to $C^{r_0}(Q_0, Z_1)$, therefore, with the integral below considered in $C^{r_0}(Q_0, Z_1)$,

$$\frac{\bar{f}(t_2, x) - \bar{f}(t_1, x)}{t_2 - t_1} = \int_0^1 \bar{g}_1(t_1 + \tau(t_2 - t_1), x) d\tau$$

and

$$\lim_{t_2 \to t_1} \sup_{\tau \in [0,1]} \|\bar{g}_1(t_1 + \tau(t_2 - t_1), x) - \bar{g}_1(t_1, x)\|_{C^{r_0}(Q_0, Z_1)} = 0.$$

Consequently, with differentiation meant in $C^{r_0}(Q_0, Z_1)$,

$$\frac{\partial \bar{f}}{\partial t} = \bar{g}_1. \tag{6.86}$$

Next, we introduce the following mapping

$$V : C^{r_0}(Q_0, \mathscr{L}(Z, Z_1)) \to \mathscr{L}(C^{r_0}(Q_0, Z), C^{r_0}(Q_0, Z_1))$$

given for $w \in C^{r_0}(Q_0, \mathscr{L}(Z, Z_1))$, $x \in C^{r_0}(Q_0, Z)$, and $s \in Q_0$ by

$$((Vw)x)(s) = w(s)x(s).$$

Clearly, V is a bounded linear operator. This together with (6.85) yields

$$V \circ \bar{g}_2 \in \mathcal{C}^{r-1,\varrho}\big([a, b] \times C^{r_0}(Q_0, Z), \mathscr{L}(C^{r_0}(Q_0, Z), C^{r_0}(Q_0, Z_1)); \|V\|\kappa_1\big). \tag{6.87}$$

Next let $t \in [a, b]$, $x, y \in C^{r_0}(Q_0, Z)$, $\theta \in \mathbb{R}$, $\theta \neq 0$, $s \in Q_0$. Then we have

$$\begin{aligned}
\frac{\bar{f}(t, x + \theta y)(s) - \bar{f}(t, x)(s)}{\theta} &= \frac{f(s, t, x(s) + \theta y(s)) - f(s, t, x(s))}{\theta} \\
&= \int_0^1 g_2(s, t, x(s) + \tau\theta y(s))y(s)d\tau \\
&= \int_0^1 \big((V\bar{g}_2(t, x + \tau\theta y))y\big)(s)d\tau.
\end{aligned}$$

Relation (6.87) shows that $V \circ \bar{g}_2$ is a continuous function from $[a, b] \times C^{r_0}(Q_0, Z)$ to $\mathscr{L}(C^{r_0}(Q_0, Z), C^{r_0}(Q_0, Z_1))$. It follows that

$$\frac{\bar{f}(t, x + \theta y) - \bar{f}(t, x)}{\theta} = \int_0^1 (V\bar{g}_2(t, x + \tau\theta y))y \, d\tau, \tag{6.88}$$

moreover,

$$\lim_{\theta \to 0} \sup_{\tau \in [0,1], y \in B_{C^{r_0}(Q_0, Z)}} \|V\bar{g}_2(t, x + \tau\theta y) - V\bar{g}_2(t, x)\|_{\mathscr{L}(C^{r_0}(Q_0, Z), C^{r_0}(Q_0, Z_1))} = 0,$$

and hence

$$\lim_{\theta \to 0} \sup_{\tau \in [0,1], y \in B_{C^{r_0}(Q_0, Z)}} \|(V\bar{g}_2(t, x + \tau\theta y))y - (V\bar{g}_2(t, x))y\|_{C^{r_0}(Q_0, Z_1)} = 0. \tag{6.89}$$

From Proposition 2.3.2,(6.88), and (6.89), we conclude that $\bar{f}$ is Fréchet differentiable with respect to x as a function from $[a,b] \times C^{r_0}(Q_0, Z)$ to $C^{r_0}(Q_0, Z_1)$ and

$$\frac{\partial \bar{f}}{\partial x} = V \circ \bar{g}_2. \tag{6.90}$$

Combining (6.83–6.84),(6.86–6.87), and (6.90) completes the induction and thus the proof of (6.81). By (6.55),

$$\mathscr{C}_{\mathrm{Lip}}^{r_0,r,\varrho}(Q_0 \times [a,b] \times Z, Z_1; \kappa, L) \subseteq \mathscr{C}_{\mathrm{Lip}}^{r_0,0,\varrho}(Q_0 \times [a,b] \times Z, Z_1; \kappa, L).$$

Therefore, relation (6.66) of Lemma 6.4.4 yields the required Lipschitz property, which proves (6.82). □

Given $(f, u_0) \in \mathscr{F}$, we recall that we consider the solution $u = u(s,t)$ of (6.58–6.59) also as a function $u(t) = u(\,\cdot\,, t)$ in $B([a,b], B(Q_0, Z))$, where the required boundedness is a consequence of (6.61).

Lemma 6.4.6. *There is a constant $\lambda_1 > 0$ such that for all $(f, u_0) \in \mathscr{F}$ the following hold: $u(t) \in C^{r_0}(Q_0, Z)$ $(t \in [a,b])$, u is the unique solution of*

$$\frac{du(t)}{dt} = \bar{f}(t, u(t)) \quad (t \in [a,b]), \quad u(a) = u_0, \tag{6.91}$$

considered as an equation in $C^{r_0}(Q_0, Z)$, moreover,

$$\|u\|_{B([a,b],C^{r_0}(Q_0,Z))} \leq \lambda_1, \tag{6.92}$$

$$\|\mathcal{A}^0_{n,\omega}(\bar{f}, u_0)\|_{B([a,b],C^{r_0}(Q_0,Z))} \leq \lambda_1 \quad (n \in \mathbb{N}, \omega \in \Omega). \tag{6.93}$$

Proof. Let $(f, u_0) \in \mathscr{F}$. We start with a preliminary argument. By Lemma 6.4.5,

$$\bar{f} \in \mathcal{C}_{\mathrm{Lip}}^{r_1,\varrho_1}([a,b] \times C^{r_0}(Q_0, Z), C^{r_0}(Q_0, Z); \kappa_1, L_1). \tag{6.94}$$

From Proposition 6.1.4, we conclude that there exists a solution $w(t)$ of

$$\frac{dw(t)}{dt} = \bar{f}(t, w(t)) \quad (t \in [a, b_1)), \qquad w(a) = u_0, \tag{6.95}$$

considered as an ODE in $C^{r_0}(Q_0, Z)$, on a maximal interval $[a, b_1)$ with $a < b_1 \leq b$. Applying δ_s to (6.95), we get

$$\frac{d}{dt}(w(t), \delta_s) = \left(\frac{d}{dt}w(t), \delta_s\right) = (\bar{f}(t, w(t)), \delta_s) = f(s, t, (w(t), \delta_s)) \quad (t \in [a, b_1))$$

and

$$(w(a), \delta_s) = (u_0, \delta_s) = u_0(s).$$

By uniqueness of the solution to (6.58–6.59), we conclude

$$(w(t),\delta_s) = u(s,t) = (u(t),\delta_s) \quad (s\in Q_0, t\in[a,b_1)),$$

hence

$$w(t) = u(t) \quad (t\in[a,b_1)). \tag{6.96}$$

Now assume that

$$\sup_{t\in[a,b_1)} \|w(t)\|_{C^{r_0}(Q_0,Z)} := R_0 < \infty. \tag{6.97}$$

Then (6.94) implies that for all $t\in[a,b]$, $x,y\in(R_0+1)B_{C^{r_0}(Q_0,Z)}$

$$\begin{aligned} \|\bar f(t,x)\|_{C^{r_0}(Q_0,Z)} &\le \kappa_1(R_0+1), \\ \|\bar f(t,x)-\bar f(t,y)\|_{C^{r_0}(Q_0,Z)} &\le L_1(R_0+1)||x-y||_{C^{r_0}(Q_0,Z)}. \end{aligned}$$

Consequently, using Proposition 6.1.1, there is a $\delta>0$ such that for any $b_2\in[a,b_1)$ the solution $w(t)$ of (6.95) on $[a,b_2]$ can be continued to a solution on $[a,\min(b_2+\delta,b)]$. It follows that $b_1=b$ and $w(t)$ can be continued to a solution of (6.95) on $[a,b]$, that is,

$$w\in C^1([a,b],C^{r_0}(Q_0,Z)) \tag{6.98}$$

and

$$\frac{dw(t)}{dt} = \bar f(t,w(t)) \quad (t\in[a,b]), \qquad w(a)=u_0. \tag{6.99}$$

Since $u(s,\,\cdot\,)\in C^1([a,b],Z)$ $(s\in Q_0)$, we use continuity to conclude from (6.96) and (6.98) that

$$w(t)=u(t) \quad (t\in[a,b]) \tag{6.100}$$

and

$$\sup_{t\in[a,b]}\|u(t)\|_{C^{r_0}(Q_0,Z)} \le R_0. \tag{6.101}$$

In summary; so far we have shown that (6.97) implies (6.98–6.101). After this preparation we prove the lemma.

• We argue by induction over r_0. Let $r_0=0$. By (6.61) of the definition of $\mathscr{F}$ we have

$$\sup_{t\in[a,b_1)}\|u(t)\|_{C(Q_0,Z)} = \sup_{s\in Q_0,t\in[a,b_1)}\|u(s,t)\|_Z \le \lambda.$$

Therefore, (6.97) holds with $R_0=\lambda$, so (6.100) and (6.101) imply (6.92) for $r_0=0$. Moreover, if $r=\varrho=r_1=\varrho_1=0$, then (6.93) follows by (6.62), while for $r+\varrho>0$ or $r_1+\varrho_1>0$ we note that by (6.54) and (6.60)

$$\begin{aligned} f &\in \mathscr{C}^{0,r,\varrho}_{\mathrm{Lip}}(Q_0\times[a,b]\times Z,Z;\kappa,L), \\ f &\in \mathscr{C}^{0,r_1,\varrho_1}_{\mathrm{Lip}}(Q_0\times[a,b]\times Z,Z;\kappa,L), \end{aligned}$$

and therefore, by Lemma 6.4.5,

$$\begin{aligned} \bar{f} &\in \mathcal{C}^{r,\varrho}_{\mathrm{Lip}}([a,b] \times C(Q_0,Z), C(Q_0,Z); \kappa_1, L_1), \\ \bar{f} &\in \mathcal{C}^{r_1,\varrho_1}_{\mathrm{Lip}}([a,b] \times C(Q_0,Z), C(Q_0,Z); \kappa_1, L_1). \end{aligned}$$

Now, (6.93) is a consequence of (the already proven) relation (6.92) for $r_0 = 0$ and Proposition 6.2.5 (for $n < \nu_0$ it follows directly from the boundedness properties of f and u_0).

• Next let $r_0 \geq 1$ and assume the statements are true for $r_0 - 1$. Let $(f, u_0) \in \mathscr{F}$ and put

$$g_1 := \frac{\partial f}{\partial s} \in \mathscr{C}^{r_0-1,r,\varrho}(Q_0 \times [a,b] \times Z, \mathscr{L}(\mathbb{R}^{d_0}, Z); \kappa), \tag{6.102}$$

$$g_2 := \frac{\partial f}{\partial z} \in \mathscr{C}^{r_0-1,r,\varrho}(Q_0 \times [a,b] \times Z, \mathscr{L}(Z,Z); \kappa). \tag{6.103}$$

By Lemma 6.4.5,

$$\bar{g}_1 \in \mathcal{C}^{r,\varrho}([a,b] \times C^{r_0-1}(Q_0,Z), C^{r_0-1}(Q_0, \mathscr{L}(\mathbb{R}^{d_0}, Z)); \kappa_1), \tag{6.104}$$

$$\bar{g}_2 \in \mathcal{C}^{r,\varrho}([a,b] \times C^{r_0}(Q_0,Z), C^{r_0}(Q_0, \mathscr{L}(Z,Z)); \kappa_1). \tag{6.105}$$

We start with the proof of (6.92). By the induction assumption, $u(t)$ is the solution of (6.91), considered in $C^{r_0-1}(Q_0,Z))$, and

$$\|u\|_{B([a,b],C^{r_0-1}(Q_0,Z))} \leq c_0. \tag{6.106}$$

From (6.104–6.106) and the assumptions on u_0, we conclude that there is a $c_1 > 0$ such that

$$\sup_{t\in[a,b]} \|\bar{g}_1(t,u(t))\|_{C^{r_0-1}(Q_0,\mathscr{L}(\mathbb{R}^{d_0},Z))} \leq c_1, \tag{6.107}$$

$$\sup_{t\in[a,b]} \|\bar{g}_2(t,u(t))\|_{C^{r_0-1}(Q_0,\mathscr{L}(Z))} \leq c_1, \tag{6.108}$$

$$\left\| \frac{du_0}{ds} \right\|_{C^{r_0-1}(Q_0,\mathscr{L}(\mathbb{R}^{d_0},Z))} \leq \sigma. \tag{6.109}$$

By uniqueness, we obtain $w(t) = u(t)$, where w is the solution of (6.95). Thus, $u(t) \in C^{r_0}(Q_0,Z)$ for all $t \in [a, b_1)$ and $u(t)$ is continuously differentiable as a function from $[a, b_1)$ to $C^{r_0}(Q_0,Z)$. Let D be differentiation $\frac{d}{ds}$, considered as an operator

$$D \in \mathscr{L}(C^{r_0}(Q_0,Z), C^{r_0-1}(Q_0, \mathscr{L}(\mathbb{R}^{d_0}, Z))).$$

By applying D to (6.95) with $w = u$ and inserting (6.102–6.103), we get

$$\begin{aligned} \frac{d(Du(t))}{dt} &= D\frac{du(t)}{dt} = D\bar{f}(t, u(t)) \\ &= \bar{g}_1(t, u(t)) + \bar{g}_2(t, u(t))Du(t) \quad (t \in [a, b_1)) \\ Du(a) &= Du_0. \end{aligned}$$

Integrating with respect to t, we obtain for $t \in [a, b_1)$

$$Du(t) = Du_0 + \int_a^t (\bar{g}_1(\tau, u(\tau)) + \bar{g}_2(\tau, u(\tau))Du(\tau))\, d\tau.$$

Using (6.107–6.109), we conclude for $t \in [a, b_1)$

$$\begin{aligned} &\|Du(t)\|_{C^{r_0-1}(Q_0,\mathscr{L}(\mathbb{R}^{d_0},Z))} \\ &\le \|Du_0\|_{C^{r_0-1}(Q_0,\mathscr{L}(\mathbb{R}^{d_0},Z))} + \int_a^t \|\bar{g}_1(\tau, u(\tau))\|_{C^{r_0-1}(Q_0,\mathscr{L}(\mathbb{R}^{d_0},Z))}\, d\tau \\ &\quad + \int_a^t c_2 \|\bar{g}_2(\tau, u(\tau))\|_{C^{r_0-1}(Q_0,\mathscr{L}(Z))} \|Du(\tau)\|_{C^{r_0-1}(Q_0,\mathscr{L}(\mathbb{R}^{d_0},Z))}\, d\tau \\ &\le \sigma + (b-a)c_1 + c_1c_2 \int_a^t \|Du(\tau)\|_{C^{r_0-1}(Q_0,\mathscr{L}(\mathbb{R}^{d_0},Z))}\, d\tau. \end{aligned}$$

Since $t \to Du(t)$ is a continuous function from $[a, b_1)$ to $C^{r_0-1}(Q_0, \mathscr{L}(\mathbb{R}^{d_0}, Z))$, we can use Gronwall's lemma to get

$$\sup_{t\in[a,b_1)} \|Du(t)\|_{C^{r_0-1}(Q_0,\mathscr{L}(\mathbb{R}^{d_0},Z))} \le (\sigma + (b-a)c_1)e^{c_1c_2(b-a)} := c_3,$$

which together with (6.106) gives

$$\sup_{t\in[a,b_1)} \|w(t)\|_{C^{r_0}(Q_0,Z)} = \sup_{t\in[a,b_1)} \|u(t)\|_{C^{r_0}(Q_0,Z)} \le \max(c_0, c_3) := c_4. \tag{6.110}$$

Consequently, (6.97) holds with $R_0 = c_4$, so (6.100) and (6.101) give (6.92) for r_0. **Now we turn to (6.93).** By (6.13–6.16),

$$\mathcal{A}^0_{n,\omega}(\bar{f}, u_0) = v \in B([a, b], C(Q_0, Z)),$$

where

$$v(t) = \begin{cases} p_{k,0}(t) & \text{if} \quad t \in [t_k, t_{k+1}) \quad \text{and} \quad 0 \le k \le n-1 \\ u_n & \text{if} \quad t = t_n \end{cases},$$

and for $k = 0, \ldots, n-1$

$$p_{k,0}(t) = u_k + \bar{f}(t_k, u_k)(t - t_k) \quad (t \in [t_k, t_{k+1}]), \tag{6.111}$$
$$u_{k+1} = u_k + h\bar{f}(\xi_{k+1}, p_{k,0}(\xi_{k+1})). \tag{6.112}$$

The induction assumption implies

$$\max_{0 \le k \le n-1} \max_{t \in [t_k, t_{k+1}]} \|p_{k,0}(t)\|_{C^{r_0-1}(Q_0,Z)} \le c_0 \tag{6.113}$$

and

$$\max_{0 \le k \le n} \|u_k\|_{C^{r_0-1}(Q_0,Z)} \le c_0. \tag{6.114}$$

Using (6.94) and $u_0 \in C^{r_0}(Q_0, Z)$, it readily follows from (6.111–6.112) that for $0 \le k \le n-1$

$$p_{k,0}(t) \in C^{r_0}(Q_0, Z) \quad (t \in [t_k, t_{k+1}],\ 0 \le k \le n-1), \tag{6.115}$$
$$u_k \in C^{r_0}(Q_0, Z) \quad (0 \le k \le n). \tag{6.116}$$

Differentiating (6.111) and (6.112), we obtain for $0 \le k \le n-1$

$$\frac{dp_{k,0}(\xi_{k+1})}{ds} = \frac{du_k}{ds} + (\xi_{k+1} - t_k)\bar{g}_1(t_k, u_k) + (\xi_{k+1} - t_k)\bar{g}_2(t_k, u_k)\frac{du_k}{ds} \tag{6.117}$$

$$\frac{du_{k+1}}{ds} = \frac{du_k}{ds} + h\bar{g}_1\big(\xi_{k+1}, p_{k,0}(\xi_{k+1})\big) + h\bar{g}_2\big(\xi_{k+1}, p_{k,0}(\xi_{k+1})\big)\frac{dp_{k,0}(\xi_{k+1})}{ds}. \tag{6.118}$$

Inserting (6.117) into (6.118), we get

$$\begin{aligned}\frac{du_{k+1}}{ds} = {} & \frac{du_k}{ds} + h\bar{g}_1\big(\xi_{k+1}, p_{k,0}(\xi_{k+1})\big) + h\bar{g}_2\big(\xi_{k+1}, p_{k,0}(\xi_{k+1})\big)\frac{du_k}{ds} \\ & + h(\xi_{k+1} - t_k)\bar{g}_2\big(\xi_{k+1}, p_{k,0}(\xi_{k+1})\big)\bar{g}_1(t_k, u_k) \\ & + h(\xi_{k+1} - t_k)\bar{g}_2\big(\xi_{k+1}, p_{k,0}(\xi_{k+1})\big)\bar{g}_2(t_k, u_k)\frac{du_k}{ds}.\end{aligned}$$

Hence,

$$\frac{du_{k+1}}{ds} = (I_Z + hv_k)\frac{du_k}{ds} + hw_k, \tag{6.119}$$

where

$$v_k = \bar{g}_2\big(\xi_{k+1}, p_{k,0}(\xi_{k+1})\big) + (\xi_{k+1} - t_k)\bar{g}_2\big(\xi_{k+1}, p_{k,0}(\xi_{k+1})\big)\bar{g}_2(t_k, u_k),$$

$$w_k = \bar{g}_1\big(\xi_{k+1}, p_{k,0}(\xi_{k+1})\big) + (\xi_{k+1} - t_k)\bar{g}_2\big(\xi_{k+1}, p_{k,0}(s, \xi_{k+1})\big)\bar{g}_1(t_k, u_k).$$

By (6.104–6.105) and (6.113–6.114) it follows that

$$\max_{0\le k\le n-1} \|v_k\|_{C^{r_0-1}(Q_0,\mathscr{L}(Z))} \le c_1, \tag{6.120}$$

$$\max_{0\le k\le n-1} \|w_k\|_{C^{r_0-1}(Q_0,\mathscr{L}(\mathbb{R}^{d_0},Z))} \le c_1. \tag{6.121}$$

Moreover, by the assumption on u_0,

$$\left\|\frac{du_0}{ds}\right\|_{C^{r_0-1}(Q_0,\mathscr{L}(\mathbb{R}^{d_0},Z))} \le \sigma. \tag{6.122}$$

We have by (6.119–6.121)

$$\begin{aligned}
&\left\|\frac{du_{k+1}}{ds}\right\|_{C^{r_0-1}(Q_0,\mathscr{L}(\mathbb{R}^{d_0},Z))} \\
&\le \left(1 + c_2 h \|v_k\|_{C^{r_0-1}(Q_0,\mathscr{L}(Z))}\right) \left\|\frac{du_k}{ds}\right\|_{C^{r_0-1}(Q_0,\mathscr{L}(\mathbb{R}^{d_0},Z))} + h\|w_k\|_{C^{r_0-1}(Q_0,\mathscr{L}(\mathbb{R}^{d_0},Z))} \\
&\le (1+c_1c_2h) \left\|\frac{du_k}{ds}\right\|_{C^{r_0-1}(Q_0,\mathscr{L}(\mathbb{R}^{d_0},Z))} + c_1 h.
\end{aligned}$$

From this and (6.122) we conclude for $1 \le k \le n$

$$\begin{aligned}
\left\|\frac{du_k}{ds}\right\|_{C^{r_0-1}(Q_0,\mathscr{L}(\mathbb{R}^{d_0},Z))} &\le \sigma(1+c_1c_2h)^k + c_1 h \sum_{j=0}^{k-1} (1+c_1c_2h)^j \\
&\le \sigma(1+c_1c_2h)^k + c_1 h \frac{(1+c_1c_2h)^k - 1}{c_1c_2h} \\
&\le \sigma(1+1/c_2)(1+c_1c_2h)^n \le \sigma(1+1/c_2)e^{c_1c_2nh} \\
&= \sigma(1+1/c_2)e^{c_1c_2(b-a)} =: c_3.
\end{aligned}$$

In combination with (6.114) this gives

$$\max_{0\le k\le n} \|u_k\|_{C^{r_0}(Q_0,Z)} \le c_4 := \max(c_0, c_3),$$

which, taking into account (6.111) and (6.94), also yields

$$\max_{0\le k\le n-1} \max_{t\in[t_k,t_{k+1}]} \|p_{k,0}(t)\|_{C^{r_0}(Q_0,Z)} \le c_5,$$

and hence the desired result.

□

Proof of Proposition 6.4.3. The result follows from Lemmas 6.4.5–6.4.6, taking into account that (6.92) and (6.93) for $r_0 > 0$ imply the respective estimates also for $r_0 = 0$.

6.5 Approximation of the Parametric Problem

For $r_0 \in \mathbb{N}_0, d_0 \in \mathbb{N}$ we had $\hat{r}_0 = \max(r_0, 1)$. Moreover, for $l \in \mathbb{N}_0$, $\Gamma_{2^l}^{\hat{r}_0,d_0}$ was defined as the equidistant grid on Q_0 with mesh size $\hat{r}_0^{-1}2^{-l}$. Let further $\{Q_{0,li} : i = 1, 2, \dots, 2^{d_0 l}\}$ be the partition of Q_0 into cubes of side length 2^{-l}. Define the following operators E_{li} and R_{li} acting on $\Phi(\mathbb{R}^{d_0}, Z)$, the space of all functions from $\mathbb{R}^{d_0}$ to Z, as follows: For $f \in \Phi(\mathbb{R}^{d_0}, Z)$ and $s \in \mathbb{R}^{d_0}$ put

$$(E_{li}f)(s) = f(s_{li} + 2^{-l}s), \tag{6.123}$$

$$(R_{li}f)(s) = f(2^l(s - s_{li})), \tag{6.124}$$

where s_{li} is the point in $Q_{0,li}$ with minimal coordinates. We also apply these operators to functions which are defined on subsets of $\mathbb{R}^{d_0}$. In this case, we assume that the function is extended to $\mathbb{R}^{d_0}$ by zero. Remember that for $f \in \Phi(\mathbb{R}^{d_0}, Z)$

$$P_1^{\hat{r}_0,d_0} f = \sum_{j=1}^{\nu_1} f(a_j)\varphi_j \tag{6.125}$$

is the Z-valued tensor product Lagrange interpolation operator of degree $\hat{r}_0$, where $(a_j)_{j=1}^{\nu_1}$ are the points of $\Gamma_1^{\hat{r}_0,d_0}$ and $(\varphi_j)_{j=1}^{\nu_1}$ are the respective scalar Lagrange polynomials, considered as functions on $\mathbb{R}^{d_0}$. If $\mathcal{P}_{\hat{r}_0}$ denotes the space of polynomials on $\mathbb{R}^{d_0}$ of degree at most $\hat{r}_0$, with coefficients in Z, we have

$$P_1^{\hat{r}_0,d_0} g = g \quad (g \in \mathcal{P}_{\hat{r}_0}).$$

Define $\mathscr{P}_l : \Phi(Q_0, Z) \to C(Q_0, Z)$ for $l \in \mathbb{N}_0$ by

$$(\mathscr{P}_l f)(s) = (R_{li} P_1^{\hat{r}_0,d_0} E_{li} f)(s) \quad (s \in Q_{0,li}, f \in \Phi(Q_0, Z)),$$

thus, by (6.125),

$$(\mathscr{P}_l f)(s) = \sum_{j=1}^{\nu_1} f(s_{li} + 2^{-l}a_j)\varphi_j(2^l(s - s_{li})) \quad (s \in Q_{0,li}),$$

so $\mathscr{P}_l$ is Z-valued composite tensor product Lagrange interpolation of degree $\hat{r}_0$ with respect to the partition $Q_{0,li}$, and coincides with the previous definition from (5.2) and (2.7). Hence,

$$(\mathscr{P}_l f)(s) = f(s) \quad (s \in \Gamma_{2^l}^{\hat{r}_0,d_0}, f \in \Phi(Q_0, Z))$$

and $\mathscr{P}_l$ is of the form

$$\mathscr{P}_l f = \sum_{s \in \Gamma_{2^l}^{\hat{r}_0,d_0}} f(s)\psi_{ls} \quad (f \in \Phi(Q_0, Z)), \tag{6.126}$$

with $\psi_{ls} \in C(Q_0)$.

In order to apply Proposition 6.3.5, we next construct operators $T_l : C(Q_0, Z) \to C^{r_0}(Q_0, Z)$ with certain suitable boundedness properties. Put

$$W = \left[-\frac{1}{\max(r_0, 1)}, 1 + \frac{1}{\max(r_0, 1)}\right]^{d_0},$$

$\mathcal{I}_l = \{1, 2, \dots, 2^{d_0 l}\}$, and for $l \in \mathbb{N}_0$, $i \in \mathcal{I}_l$

$$W_{li} = s_{li} + 2^{-l} W. \tag{6.127}$$

Let $\eta \in C^\infty(\mathbb{R}^{d_0})$ be such that $\eta \geq 0$, $\eta \equiv 1$ on Q_0, and $\operatorname{supp}(\eta) \subseteq W$. Then

$$\sum_{i \in \mathcal{I}_l} (R_{li}\eta)(s) \geq 1 \qquad (s \in Q_0, l \in \mathbb{N}_0). \tag{6.128}$$

Define functions η_{li} on Q_0 $(i \in \mathcal{I}_l, l \in \mathbb{N}_0)$ by

$$\eta_{li}(s) = \frac{(R_{li}\eta)(s)}{\sum_{j \in \mathcal{I}_l} (R_{lj}\eta)(s)} \qquad (s \in Q_0). \tag{6.129}$$

We define $T_l : \Phi(Q_0, Z) \to C^{r_0}(Q_0, Z)$ for $l \in \mathbb{N}_0$ and $f \in \Phi(Q_0, Z)$ by

$$(T_l f)(s) = \sum_{i \in \mathcal{I}_l} \eta_{li}(s) \, (R_{li} P_1^{\hat{r}_0, d_0} E_{li} f)(s) \quad (s \in Q_0), \tag{6.130}$$

consequently, using (6.123) and (6.125),

$$(T_l f)(s) = \sum_{i \in \mathcal{I}_l} \sum_{j=1}^{\nu_1} f(s_{li} + 2^{-l} a_j) \, \eta_{li}(s) \, R_{li}\varphi_j(s) \quad (s \in Q_0). \tag{6.131}$$

Thus, T_l has the form

$$T_l f = \sum_{s \in \Gamma_{2^l}^{\hat{r}_0, d_0}} f(s)\zeta_{ls} \quad (f \in \Phi(Q_0, Z)), \tag{6.132}$$

with $\zeta_{ls} \in C^{r_0}(Q_0)$.

Remark 6.5.1. Notice that T_l is supposed to be an operator from $C^{r_0}(Q_0, Z)$ to $C^{r_0}(Q_0, Z)$. This is for instance satisfied by Hermitian interpolation operators, too. However, T_l is also supposed to map $C(Q_0, Z)$ to $C(Q_0, Z)$. Both together cannot be satisfied by Hermitian interpolation operators, due to differentiability assumptions. For this reason, we use the approach with partition of unity from above.

For the proof of the next lemma, we denote for $f \in C^m(Q_0)$

$$|f|_{m,Q_0} = \left\| \frac{d^m f}{ds^m} \right\|_{C(Q_0, \mathscr{L}_m(\mathbb{R}^{d_0},\mathbb{R}))}.$$

Lemma 6.5.2. *There are constants $c_1, c_2 > 0$ such that for all Banach spaces Z and $l \in \mathbb{N}_0$*

$$\|T_l\|_{\mathscr{L}(C(Q_0,Z))} \leq c_1, \tag{6.133}$$

$$\|T_l\|_{\mathscr{L}(C^{r_0}(Q_0,Z))} \leq c_2. \tag{6.134}$$

Moreover, for $f \in \Phi(Q_0, Z)$,

$$(T_l f)(s) = f(s) \quad (s \in \Gamma_{2^l}^{\hat{r}_0,d_0}). \tag{6.135}$$

Proof. We first prove the result for $Z = \mathbb{R}$. We have

$$\eta_{li}(s) \geq 0 \quad (s \in Q_0), \tag{6.136}$$

$$\eta_{li}(s) = 0 \quad (s \in Q_0 \setminus W_{li}), \tag{6.137}$$

$$\sum_{i \in \mathcal{I}_l} \eta_{li}(s) = 1 \quad (s \in Q_0). \tag{6.138}$$

Moreover, there are constants $c_1, c_2 > 0$ such that for $m \in \mathbb{N}_0$, $0 \leq m \leq r_0$, $l \in \mathbb{N}_0$

$$\|R_{li}\eta\|_{C^m(\mathbb{R}^{d_0})} \leq c_1 2^{ml} \qquad (i \in \mathcal{I}_l) \tag{6.139}$$

and

$$\Big\| \sum_{i \in \mathcal{I}_l} R_{li}\eta \Big\|_{C^m(\mathbb{R}^{d_0})} \leq c_2 2^{ml}. \tag{6.140}$$

From (6.128–6.129) and (6.139–6.140), we get for $0 \leq m \leq r_0$

$$\|\eta_{li}\|_{C^m(Q_0)} \leq c\, 2^{ml}. \tag{6.141}$$

First we show (6.135). Let $s \in \Gamma_{2^l}^{\hat{r}_0,d_0}$. If $s \in Q_{0,li}$, then $(R_{li} P_1^{\hat{r}_0,d_0} E_{li} f)(s) = f(s)$. On the other hand, by the definition of the support of η, if $s \notin Q_{0,li}$, then $s \notin W_{li}^0$ (the interior of W_{li}), hence $(R_{li}\eta)(s) = 0$, and therefore, $\eta_{li}(s) = 0$. This together with (6.130) and (6.138) implies (6.135).

Relation (6.133) is an immediate consequence of (6.130) and (6.136–6.138). Now we turn to (6.134). Due to (6.133), we can assume that $r_0 > 0$. By (6.125), for $f \in C^{r_0}(W)$ and $0 \leq m \leq r_0$

$$\|P_1^{\hat{r}_0,d_0} f\|_{C^m(W)} \leq c\|f\|_{C^{r_0}(W)},$$

and consequently,

$$\begin{aligned}\|f - P_1^{\hat{r}_0,d_0} f\|_{C^m(W)} &= \inf_{g\in\mathcal{P}_{r_0}} \|(f-g) - P_1^{\hat{r}_0,d_0}(f-g)\|_{C^m(W)} \\ &\le c \inf_{g\in\mathcal{P}_{r_0}} \|f-g\|_{C^{r_0}(W)} \le c|f|_{r_0,W}, \qquad (6.142)\end{aligned}$$

where the latter relation is an application of Theorem 3.1.1 from [5] (this theorem is formulated for Sobolev spaces $W^{r_0}_\infty(W)$, but since $f, P_1^{\hat{r}_0,d_0} f, g \in C^{r_0}(W)$, the corresponding (semi-)norms coincide). Let $f \in C^{r_0}(Q_0)$ and let $\tilde{f} \in C^{r_0}(\mathbb{R}^{d_0})$ be an extension of f with

$$\|\tilde{f}\|_{C^{r_0}(\mathbb{R}^{d_0})} \le c\|f\|_{C^{r_0}(Q_0)},$$

which exists due to the Whitney extension theorem, see [44] or [26], Theorem 2.3.6. From (6.137) and (6.138), we conclude

$$\begin{aligned}\|f - T_l f\|_{C^{r_0}(Q_0)} &= \Big\| \sum_{i\in\mathcal{I}_l} \eta_{li}(f - R_{li}P_1^{\hat{r}_0,d_0} E_{li} f)\Big\|_{C^{r_0}(Q_0)} \\ &\le c \max_{i\in\mathcal{I}_l} \|\eta_{li}(f - R_{li}P_1^{\hat{r}_0,d_0} E_{li} f)\|_{C^{r_0}(Q_0)}. \qquad (6.143)\end{aligned}$$

Furthermore, for $0 \le m \le r_0$

$$\|R_{li} g\|_{C^m(W_{li})} \le c2^{ml}\|g\|_{C^m(W)} \quad (g \in C^m(W)), \qquad (6.144)$$

and, using (6.141),

$$\|\eta_{li} g\|_{C^{r_0}(Q_0\cap W_{li})} \le c\sum_{m=0}^{r_0} 2^{(r_0-m)l}\|g\|_{C^m(Q_0\cap W_{li})} \quad (g \in C^{r_0}(Q_0 \cap W_{li})). \qquad (6.145)$$

Applying (6.144–6.145), and (6.142), we obtain

$$\begin{aligned}\|\eta_{li}(f - R_{li}P_1^{\hat{r}_0,d_0} E_{li} f)\|_{C^{r_0}(Q_0)} &= \|\eta_{li}(f - R_{li}P_1^{\hat{r}_0,d_0} E_{li} f)\|_{C^{r_0}(Q_0\cap W_{li})} \\ &\le c\sum_{m=0}^{r_0} 2^{(r_0-m)l}\|f - R_{li}P_1^{\hat{r}_0,d_0} E_{li} f\|_{C^m(Q_0\cap W_{li})} \\ &\le c\sum_{m=0}^{r_0} 2^{(r_0-m)l}\|\tilde{f} - R_{li}P_1^{\hat{r}_0,d_0} E_{li} \tilde{f}\|_{C^m(W_{li})} \\ &\le c\, 2^{r_0 l}\sum_{m=0}^{r_0} \|E_{li}\tilde{f} - P_1^{\hat{r}_0,d_0} E_{li} \tilde{f}\|_{C^m(W)} \\ &\le c\, 2^{r_0 l}|E_{li}\tilde{f}|_{r_0,W}. \qquad (6.146)\end{aligned}$$

Finally,

$$\max_{i\in\mathcal{I}_l} |E_{li}\tilde{f}|_{r_0,W} = 2^{-r_0 l} \max_{i\in\mathcal{I}_l} |\tilde{f}|_{r_0,W_{li}}$$
$$\le 2^{-r_0 l}|\tilde{f}|_{r_0,\mathbb{R}^{d_0}}$$
$$\le 2^{-r_0 l}\|\tilde{f}\|_{C^{r_0}(\mathbb{R}^{d_0})}$$
$$\le c\, 2^{-r_0 l}\|f\|_{C^{r_0}(Q_0)}. \tag{6.147}$$

Combining (6.143) and (6.146–6.147), we obtain

$$\|T_l f\|_{C^{r_0}(Q_0)} \le \|f\|_{C^{r_0}(Q_0)} + \|f - T_l f\|_{C^{r_0}(Q_0)} \le c\|f\|_{C^{r_0}(Q_0)},$$

which concludes the proof of (6.134) for $Z = \mathbb{R}$.

Now let Z be an arbitrary Banach space, and let T_l be defined by (6.130) for Z, while $T_l^{\mathbb{R}}$ denotes the respective operator for $\mathbb{R}$. Using the already shown scalar case, the general case follows as in the proof of Lemma 2.3.3. The case (6.133) follows from

$$\|T_l f\|_{C(Q_0,Z)} = \sup_{z^*\in B_{Z^*}} \|\,(T_l f, z^*)\,\|_{C(Q)}$$
$$= \sup_{z^*\in B_{Z^*}} \left\|T_l^{\mathbb{R}}\,(f, z^*)\right\|_{C(Q)}$$
$$\le c_1 \sup_{z^*\in B_{Z^*}} \|\,(f, z^*)\,\|_{C(Q)}$$
$$= c_1\|f\|_{C(Q,Z)}.$$

The Banach space case of (6.134) is derived as

$$\|T_l f\|_{C^{r_0}(Q_0,Z)} = \max_{0\le j\le r_0} \sup_{z^*\in B_{Z^*}} \left\|\left(\frac{d^j(T_l f)}{ds^j}, z^*\right)\right\|_{C(Q_0,\mathscr{L}_j(\mathbb{R}^{d_0},\mathbb{R}))}$$
$$= \sup_{z^*\in B_{Z^*}} \max_{0\le j\le r_0} \left\|\frac{d^j\left(T_l^{\mathbb{R}}\,(f, z^*)\right)}{ds^j}\right\|_{C(Q_0,\mathscr{L}_j(\mathbb{R}^{d_0},\mathbb{R}))}$$
$$= \sup_{z^*\in B_{Z^*}} \left\|T_l^{\mathbb{R}}\,(f, z^*)\right\|_{C^{r_0}(Q_0)}$$
$$\le c_2 \sup_{z^*\in B_{Z^*}} \|\,(f, z^*)\,\|_{C^{r_0}(Q_0)}$$
$$= c_2 \max_{0\le j\le r_0} \sup_{z^*\in B_{Z^*}} \left\|\left(\frac{d^j f}{ds^j}, z^*\right)\right\|_{C(Q_0,\mathscr{L}_j(\mathbb{R}^{d_0},\mathbb{R}))}$$
$$= c_2\|f\|_{C^{r_0}(Q_0,Z)}.$$

□

We define the multilevel algorithm for the approximation of the parametric problem (6.58–6.59). It is similar to the one chosen in Chapter 5.

Algorithm 6.5.3. Let $(f, u_0) \in \mathscr{F}$ and $l_0, l_1 \in \mathbb{N}_0$, $l_0 \le l_1$, $n_{l_0}, \dots, n_{l_1} \in \mathbb{N}$, $\omega \in \Omega$. We set

$$\mathscr{A}_\omega(f, u_0) = \mathscr{P}_{l_0}\left(\left(\mathcal{A}^r_{n_{l_0},\omega}(f_s, u_0(s))\right)_{s\in\Gamma^{\hat{r}_0,d_0}_{2^{l_0}}}\right) + \sum_{l=l_0+1}^{l_1}(\mathscr{P}_l - \mathscr{P}_{l-1})\left(\left(\mathcal{A}^{r_1}_{n_l,\omega}(f_s, u_0(s))\right)_{s\in\Gamma^{\hat{r}_0,d_0}_{2^l}}\right), \quad (6.148)$$

where we use the respective algorithms given by (6.9–6.14).

As before,

$$\operatorname{card}(\mathscr{A}_\omega) \le c\sum_{l=l_0}^{l_1} n_l 2^{d_0 l}, \quad (6.149)$$

and the number of arithmetic operations of $\mathscr{A}_\omega$ (including additions in Z and multiplications of elements of Z by scalars) is bounded from above by $c\,\operatorname{card}(\mathscr{A}_\omega)$ for some $c > 0$. The results of the analysis of this multilevel algorithm are covered in the next statements. The proof of Theorem 6.5.4 will be given later on.

Theorem 6.5.4. *Let $r_0, r, r_1 \in \mathbb{N}_0$, $0 \le \varrho, \varrho_1 \le 1$ with $r + \varrho \ge r_1 + \varrho_1$, let $\kappa, L : (0, +\infty) \to (0, +\infty)$, $\sigma, \lambda > 0$, and $1 \le p \le 2$. Then there are constants $c_1, c_2 > 0$ and $\nu_0 \in \mathbb{N}$ such that the following holds. Let Z be a Banach space and let $\mathscr{F}$ be defined by* (6.60)*. For all $l_0, l_1 \in \mathbb{N}_0$ with $l_0 \le l_1$ and for all $(n_l)_{l=l_0}^{l_1} \subset \mathbb{N}$ with $n_l \ge \nu_0$ $(l_0 \le l \le l_1)$, the so-defined algorithm $(\mathscr{A}_\omega)_{\omega\in\Omega}$ satisfies*

$$\sup_{(f,u_0)\in\mathscr{F}} \|\mathscr{S}(f, u_0) - \mathscr{A}_\omega(f, u_0)\|_{B(Q_0\times[a,b],Z)} \le c_1 2^{-r_0 l_1} + c_1 n_{l_0}^{-r-\varrho} + c_1\sum_{l=l_0+1}^{l_1} 2^{-r_0 l} n_l^{-r_1-\varrho_1} \quad (\omega \in \Omega), \quad (6.150)$$

and for all l^* *with* $l_0 \le l^* \le l_1$

$$\begin{aligned}\sup_{(f,u_0)\in\mathscr{F}} & \left(\mathbb{E}\,\|\mathscr{S}(f,u_0)-\mathscr{A}_\omega(f,u_0)\|^p_{B(Q_0\times[a,b],Z)}\right)^{\frac{1}{p}} \\ & \le c_2 2^{-r_0 l_1} + c_2(l_0+1)^{\frac{1}{2}}\tau_p(Z) n_{l_0}^{-r-\varrho-1+\frac{1}{p}} \\ & \quad + c_2\sum_{l=l_0+1}^{l^*}(l+1)^{\frac{1}{2}}\tau_p(Z)2^{-r_0 l} n_l^{-r_1-\varrho_1-1+\frac{1}{p}} \\ & \quad + c_2\sum_{l=l^*+1}^{l_1} 2^{-r_0 l} n_l^{-r_1-\varrho_1}. \end{aligned} \tag{6.151}$$

Remark 6.5.5. Observe that the restriction $r+\varrho \ge r_1+\varrho_1$ in Theorem 6.5.4 is no loss of generality. Indeed, if $r+\varrho < r_1+\varrho_1$, then either $r<r_1$ or $(r=r_1)\wedge(\varrho<\varrho_1)$. It follows from (6.55–6.57) that in both cases

$$\mathscr{C}^{r_0,r_1,\varrho_1}_{\mathrm{Lip}}(Q_0\times[a,b]\times Z,Z;\kappa,L) \;\subseteq\; \mathscr{C}^{0,r,\varrho}_{\mathrm{Lip}}(Q_0\times[a,b]\times Z,Z;2\kappa,L).$$

Consequently,

$$\begin{aligned} & \mathscr{C}^{r_0,r_1,\varrho_1}_{\mathrm{Lip}}(Q_0\times[a,b]\times Z,Z;\kappa/2,L), \\ \subseteq\; & \mathscr{C}^{0,r,\varrho}_{\mathrm{Lip}}(Q_0\times[a,b]\times Z,Z;\kappa,L)\cap\mathscr{C}^{r_0,r_1,\varrho_1}_{\mathrm{Lip}}(Q_0\times[a,b]\times Z,Z;\kappa,L), \\ \subseteq\; & \mathscr{C}^{r_0,r_1,\varrho_1}_{\mathrm{Lip}}(Q_0\times[a,b]\times Z,Z;\kappa,L), \end{aligned}$$

which by (6.55) and (6.60) means that the case $r+\varrho<r_1+\varrho_1$ is essentially the same as the case $r=r_1$, $\varrho=\varrho_1$.

Corollary 6.5.6. *Assume the conditions of Theorem 6.5.4 are satisfied. Then there are constant* $c_{1-4}>0$ *and* $\nu_0\in\mathbb{N}$ *such that for each* $n\in\mathbb{N}$ *with* $n\ge\nu_0$ *there is a choice of* $l_0,l_1\in\mathbb{N}_0$ *and* $(n_l)_{l=l_0}^{l_1}\subset\mathbb{N}$ *such that* $l_0\le l_1$,

$$\operatorname{card}(\mathscr{A}_\omega)\le c_1 n \qquad (\omega\in\Omega),$$

and

$$\begin{aligned}\sup_{(f,u_0)\in\mathscr{F}} & \|\mathscr{S}(f,u_0)-\mathscr{A}_\omega(f,u_0)\|_{B(Q_0\times[a,b],Z)} \\ & \le c_2\begin{cases} n^{-\tilde{v}_1} & \text{if } \frac{r_0}{d_0}>r_1+\varrho_1 \\ n^{-\frac{r_0}{d_0}}(\log n)^{\frac{r_0}{d_0}+1} & \text{if } \frac{r_0}{d_0}=r_1+\varrho>0 \\ n^{-\frac{r_0}{d_0}} & \text{if } \frac{r_0}{d_0}=r_1+\varrho_1=0 \vee \frac{r_0}{d_0}<r_1+\varrho_1, \end{cases}\end{aligned} \tag{6.152}$$

where

$$\tilde{\upsilon}_1 = \frac{\frac{r_0}{d_0}}{\frac{r_0}{d_0} + r + \varrho - r_1 - \varrho_1} (r + \varrho) . \tag{6.153}$$

Moreover, for all Banach spaces Z with $\tau_p(Z) < \infty$ and each $n \in \mathbb{N}$ with $n \geq \nu_0$ there is a choice of $l_0, l^*, l_1 \in \mathbb{N}_0$ and $(n_l)_{l=l_0}^{l_1} \subset \mathbb{N}$ such that $l_0 \leq l^* \leq l_1$,

$$\operatorname{card}(\mathscr{A}_\omega) \leq c_3 n \qquad (\omega \in \Omega),$$

and

$$\sup_{(f,u_0)\in\mathscr{F}} \left(\mathbb{E}\, \|\mathscr{S}(f,u_0) - \mathscr{A}_\omega(f,u_0)\|^p_{B(Q_0\times[a,b],Z)}\right)^{\frac{1}{p}}$$
$$\leq c_4 \begin{cases} n^{-r-\varrho-1+\frac{1}{p}} & \text{if } \frac{r_0}{d_0} > r_1 + \varrho_1 + 1 - \frac{1}{p} \wedge r + \varrho = r_1 + \varrho_1 \\ n^{-\tilde{\upsilon}_2(p)} (\log n)^{\frac{1}{2}} & \text{if } \frac{r_0}{d_0} > r_1 + \varrho_1 + 1 - \frac{1}{p} \wedge r + \varrho > r_1 + \varrho_1 \\ n^{-\frac{r_0}{d_0}} (\log n)^{\frac{r_0}{d_0}+\frac{3}{2}} & \text{if } \frac{r_0}{d_0} = r_1 + \varrho_1 + 1 - \frac{1}{p} \\ n^{-\frac{r_0}{d_0}} (\log n)^{\frac{p}{2(p-1)}\left(\frac{r_0}{d_0} - r_1 - \varrho_1\right)} & \text{if } r_1 + \varrho_1 < \frac{r_0}{d_0} < r_1 + \varrho_1 + 1 - \frac{1}{p} \\ n^{-\frac{r_0}{d_0}} (\log\log n)^{\frac{r_0}{d_0}+1} & \text{if } \frac{r_0}{d_0} = r_1 + \varrho_1 > 0 \\ n^{-\frac{r_0}{d_0}} & \text{if } \frac{r_0}{d_0} < r_1 + \varrho_1 \vee \frac{r_0}{d_0} = r_1 + \varrho_1 = 0, \end{cases} \tag{6.154}$$

with

$$\tilde{\upsilon}_2(p) = \begin{cases} \frac{\frac{r_0}{d_0}}{\frac{r_0}{d_0} + r + \varrho - r_1 - \varrho_1} \left(r + \varrho + 1 - \frac{1}{p}\right) & \text{if } \frac{r_0}{d_0} > r_1 + \varrho_1 + 1 - \frac{1}{p} \\ \frac{r_0}{d_0} & \text{if } \frac{r_0}{d_0} \leq r_1 + \varrho_1 + 1 - \frac{1}{p}. \end{cases} \tag{6.155}$$

Proof. Similar to the case of parametric integration, we derive the upper bounds in (6.152) and (6.154) from (6.150),(6.151) of Theorem 6.5.4. The deterministic case (6.152) follows analogously to the proof of Corollary 5.2.4, by setting

$$\beta = r + \varrho, \quad \beta_0 = \frac{r_0}{d_0}, \quad \beta_1 = r_1 + \varrho_1,$$

which yields for the case $\frac{r_0}{d_0} > r_1 + \varrho_1$ (thus $\beta_0 > \beta_1$)

$$\upsilon = \frac{\beta_0 \beta}{\beta_0 + \beta - \beta_1} = \frac{\frac{r_0}{d_0}(r + \varrho)}{\frac{r_0}{d_0} + r + \varrho - r_1 - \varrho_1} = \tilde{\upsilon}_1.$$

Using

$$\sup_{(f,u_0)\in\mathscr{F}} \|\mathscr{S}(f,u_0) - \mathscr{A}_\omega(f,u_0)\|_{B(Q_0\times[a,b],Z)} \leq c\, M\big(l_0, l_1, (n_l)_{l=l_0}^{l_1}\big). \tag{6.156}$$

the statement (6.152) follows from Lemma 5.2.3 and Theorem 6.5.4 . Next we consider the randomized case (6.154) and set

$$\beta = r + \varrho + 1 - \frac{1}{p}, \quad \beta_0 = \frac{r_0}{d_0}, \quad \beta_1 = r_1 + \varrho_1 + 1 - \frac{1}{p}, \quad \beta_2 = r_1 + \varrho_1, \quad (6.157)$$

which gives for $r_0/d_0 > r_1 + \varrho_1 + 1 - 1/p$ (thus $\beta_0 > \beta_1$)

$$v = \frac{\beta_0 \beta}{\beta_0 + \beta - \beta_1} = \frac{\frac{r_0}{d_0}\left(r + \varrho + 1 - \frac{1}{p}\right)}{\frac{r_0}{d_0} + r + \varrho - r_1 - \varrho_1} = \tilde{v}_2(p).$$

We conclude from (6.151) and (5.12) that for any l^* with $l_0 \le l^* \le l_1$

$$\sup_{(f,u_0)\in\mathscr{F}} \left(\mathbb{E}\, \|\mathscr{S}(f,u_0) - \mathscr{A}_\omega(f,u_0)\|^p_{B(Q_0\times[a,b],Z)}\right)^{\frac{1}{p}} \le c\, E\big(l_0, l^*, l_1, (n_l)_{l=l_0}^{l_1}\big). \quad (6.158)$$

With this, the upper estimates in (6.154) are consequences of (6.149),(5.13),(5.16), and (6.157)-(6.158); except for the last case of (6.154), which follows directly from the respective case of the deterministic setting (6.152). □

As in Chapter 5, it is also possible to find a choice of $l_0, l^*, l_1 \in \mathbb{N}_0$, $(n_l)_{l=l_0}^{l_1} \subset \mathbb{N}$ in such a way that they depend only on the smoothness class, but not on the setting. This way the randomized algorithm satisfies the (usually stronger) error bound of the randomized setting, while each realization also satisfies the deterministic bound. However, additional logarithmic factors occur in the estimates.

Corollary 6.5.7. *Assume the conditions of Theorem 6.5.4 are satisfied. Then there are constants $c_{1-3} > 0$ such that for all $n \in \mathbb{N}$ with $n \ge 2$ the following holds: Setting*

$$\begin{aligned} l_1 &= \left\lceil \frac{\log n}{d_0} \right\rceil, \quad l_0 = \left\lfloor \frac{r + \varrho - r_1 - \varrho_1}{\frac{r_0}{d_0} + r + \varrho - r_1 - \varrho_1}\, l_1 \right\rfloor, \\ n_l &= \nu_0 \left\lceil 2^{d_0(l_1 - l)} \right\rceil \quad (l_0 \le l \le l_1), \end{aligned}$$

the so-defined algorithm $(\mathscr{A}_\omega)_{\omega\in\Omega}$ fulfills

$$\operatorname{card}(\mathscr{A}_\omega) \le c_1 n \log n \quad (\omega \in \Omega).$$

Moreover,

$$\sup_{(f,u_0)\in\mathscr{F}} \|\mathscr{S}(f,u_0)-\mathscr{A}_\omega(f,u_0)\|_{B(Q_0\times[a,b],Z)}$$
$$\le c_2 \begin{cases} n^{-\upsilon_1} & \text{if } \frac{r_0}{d_0} > r_1+\varrho_1 \\ n^{-\frac{r_0}{d_0}}(\log n) & \text{if } \frac{r_0}{d_0} = r_1+\varrho > 0 \\ n^{-\frac{r_0}{d_0}} & \text{if } \frac{r_0}{d_0} = r_1+\varrho_1 = 0 \vee \frac{r_0}{d_0} < r_1+\varrho_1. \end{cases}$$

Finally, if $\tau_p(Z)<\infty$,

$$\sup_{(f,u_0)\in\mathscr{F}} \left(\mathbb{E}\,\|\mathscr{S}(f,u_0)-\mathscr{A}_\omega(f,u_0)\|^p_{B(Q_0\times[a,b],Z)}\right)^{\frac{1}{p}}$$
$$\le c_3 \begin{cases} n^{-\upsilon_2(p)}(\log n)^{\frac12} & \text{if } \frac{r_0}{d_0} > r_1+\varrho_1+1-\frac1p \\ n^{-\frac{r_0}{d_0}}(\log n)^{\frac32} & \text{if } \frac{r_0}{d_0} = r_1+\varrho_1+1-\frac1p \\ n^{-\frac{r_0}{d_0}}(\log n)^{\frac12} & \text{if } \frac{r_0}{d_0} = r_1+\varrho_1+1-\frac1p. \end{cases}$$

Proof. The estimates are similar to these in Lemma 5.2.3, we omit them. The proof can also be found in [12], Corollary 4.3. □

Proof of Theorem 6.5.4. Our goal is to apply Proposition 6.3.5 with $X=C(Q_0,Z)$ and $Y=C^{r_0}(Q_0,Z)$. Using that $\Gamma^{\hat{r}_0,d_0}_{2^k}\subseteq\Gamma^{\hat{r}_0,d_0}_{2^l}$ for $k\le l$, it follows from (6.126) and (6.135) of Lemma 6.5.2 that

$$\mathscr{P}_k T_l = \mathscr{P}_k \quad (k\le l). \tag{6.159}$$

We put for $l\in\mathbb{N}_0$

$$X_l = T_l(C(Q_0,Z))\subset C(Q_0,Z), \quad Y_l = T_l(C^{r_0}(Q_0,Z))\subset C^{r_0}(Q_0,Z),$$

so $X_l=Y_l$ algebraically, but X_l is endowed with the norm induced by $C(Q_0,Z)$ and Y_l with the norm induced by $C^{r_0}(Q_0,Z)$. Next we derive estimates of $\tau_p(X_l)$ and $\tau_p(Y_l)$. For $i\in\mathcal{I}_l$, we let V_{li} be the linear vector space

$$V_{li} = \operatorname{span}\left\{\eta_{lk}R_{lk}\varphi_j|_{Q_{0,li}} : k\in\mathcal{I}_l \text{ with } W^0_{lk}\cap Q_{0,li}\ne\emptyset,\ j=1,\dots,\kappa\right\}.$$

Furthermore, let

$$\begin{aligned} X_{li} &= \left(V_{li}, \|\,\|_{C(Q_{0,li})}\right), & Y_{li} = \left(V_{li}, \|\,\|_{C^{r_0}(Q_{0,li})}\right), \\ \tilde{X}_{li} &= \left(V_{li}\otimes Z, \|\,\|_{C(Q_{0,li},Z)}\right), & \tilde{Y}_{li} = \left(V_{li}\otimes Z, \|\,\|_{C^{r_0}(Q_{0,li},Z)}\right), \end{aligned}$$

where $\otimes$ denotes the algebraic tensor product. We observe that by (6.131), for $f\in C(Q_0,Z)$

$$T_l f|_{Q_{0,li}} \in \tilde{X}_{li}.$$

Moreover, for $m = 0, \dots, r_0$

$$\|T_l f\|_{C^m(Q_0,Z)} = \max_{i\in\mathcal{I}_l} \left\|T_l f|_{Q_{0,li}}\right\|_{C^m(Q_{0,li},Z)}.$$

Consequently, X_l can be identified isometrically with a subspace of

$$\tilde{X}_l = \left(\bigoplus_{i\in\mathcal{I}_l} \tilde{X}_{li}\right)_\infty \tag{6.160}$$

and Y_l with a subspace of

$$\tilde{Y}_l = \left(\bigoplus_{i\in\mathcal{I}_l} \tilde{Y}_{li}\right)_\infty . \tag{6.161}$$

It follows from (6.127) and (6.137) that there is a constant $c > 0$ such that for all $l \in \mathbb{N}_0$, $i \in \mathcal{I}_l$

$$d_{li} := \dim V_{li} \le c. \tag{6.162}$$

Two Banach spaces Z_1 and Z_2 are called c-isomorphic, where $c \ge 1$, if there is an isomorphism $T : Z_1 \to Z_2$ with $\|T\|\|T^{-1}\| \le c$. The Banach-Mazur distance $d(Z_1, Z_2)$ between Z_1 and Z_2 is defined to be the infimum of all such c. Next we show that there is a constant $c > 0$ such that

$$d(\tilde{X}_{li}, \ell_\infty^{d_{li}}(Z)) \le c, \quad d(\tilde{Y}_{li}, \ell_\infty^{d_{li}}(Z)) \le c \quad (l \in \mathbb{N}_0, i \in \mathcal{I}_l). \tag{6.163}$$

Indeed, it suffices to consider $\tilde{Y}_{li}$, the case $\tilde{X}_{li}$ follows by setting $r_0 = 0$. Let $(g_k)_{k=1}^{d_{li}}$ be an Auerbach basis of Y_{li}, that is,

$$\max_{1\le k\le d_{li}} |\alpha_k| \le \left\|\sum_{k=1}^{d_{li}} \alpha_k g_k\right\|_{Y_{li}} \le \sum_{k=1}^{d_{li}} |\alpha_k| \quad (\alpha_k \in \mathbb{R},\ k = 1, \dots, d_{li}). \tag{6.164}$$

Such bases exist in every finite dimensional Banach space, see [35], Prop. 1.c.3. Now define $T : V_{li} \otimes Z \to \ell_\infty^{d_{li}}(Z)$ for

$$w = \sum_{k=1}^{d_{li}} g_k \otimes z_k \in V_{li} \otimes Z$$

by $Tw = (z_k)_{k=1}^{d_{li}}$. Then

$$\begin{aligned} \|w\|_{\tilde{Y}_{li}} &= \left\|\sum_{k=1}^{d_{li}} g_k \otimes z_k\right\|_{\tilde{Y}_{li}} = \max_{0\le j\le r_0} \max_{z^*\in B_{Z^*}} \left\|\sum_{k=1}^{d_{li}} (z_k, z^*) \frac{d^j g_k}{ds^j}\right\|_{C(Q_{0,li},\mathscr{L}_j(\mathbb{R}^{d_0},\mathbb{R}))} \\ &= \max_{z^*\in B_{Z^*}} \left\|\sum_{k=1}^{d_{li}} (z_k, z^*)\, g_k\right\|_{C^{r_0}(Q_{0,li})} = \max_{z^*\in B_{Z^*}} \left\|\sum_{k=1}^{d_{li}} (z_k, z^*)\, g_k\right\|_{Y_{li}} . \end{aligned}$$

Moreover, using (6.164), it follows that

$$\begin{aligned}\|Tw\|_{\ell_\infty^{d_{li}}(Z)} &= \max_{1\le k\le d_{li}} \|z_k\| = \max_{z^*\in B_{Z^*}} \max_{1\le k\le d_{li}} |\,(z_k, z^*)\,| \\ &\le \max_{z^*\in B_{Z^*}} \left\| \sum_{k=1}^{d_{li}} (z_k, z^*)\, g_k \right\|_{Y_{li}} \\ &\le \max_{z^*\in B_{Z^*}} \sum_{k=1}^{d_{li}} |\,(z_k, z^*)\,| \\ &\le d_{li} \max_{1\le k\le d_{li}} \|z_k\| = d_{li}\|Tw\|_{\ell_\infty^{d_{li}}(Z)},\end{aligned}$$

hence $\|T\|\|T^{-1}\| \le d_{li}$, which together with (6.162), gives the second relation of (6.163).

From (6.162), we get

$$m_l := \sum_{i\in\mathcal{I}_l} d_{li} \le c2^{d_0 l} \quad (l \in \mathbb{N}_0). \tag{6.165}$$

It follows from (6.160),(6.161), and (6.163) that

$$d(\tilde{X}_l, \ell_\infty^{m_l}(Z)) \le c, \quad d(\tilde{Y}_l, \ell_\infty^{m_l}(Z)) \le c \quad (l \in \mathbb{N}_0),$$

and therefore,

$$\begin{aligned}\tau_p(X_l) &\le \tau_p(\tilde{X}_l) \le c\tau_p(\ell_\infty^{m_l}(Z)), \\ \tau_p(Y_l) &\le \tau_p(\tilde{Y}_l) \le c\tau_p(\ell_\infty^{m_l}(Z)).\end{aligned}$$

This together with Lemma 2.3.8 and (6.165) implies that there is a constant $c > 0$ such that for all $l \in \mathbb{N}_0$

$$\tau_p(Y_l) \le c(l+1)^{\frac{1}{2}} \tau_p(Z), \quad \tau_p(X_l) \le c(l+1)^{\frac{1}{2}} \tau_p(Z). \tag{6.166}$$

Furthermore, if $f \in \mathscr{C}_{\mathrm{Lip}}^{0,0,0}(Q_0 \times [a,b] \times Z, Z; \kappa, L)$, we get from (6.132) and (6.135) that for all $l \in \mathbb{N}_0$, $t \in [a,b]$, $x \in C(Q_0, Z)$

$$\begin{aligned}T_l \bar{f}(t,x) &= \sum_{s\in\Gamma_{2^l}^{\hat{r}_0,d_0}} \big(\bar{f}(t,x)\big)(s)\,\zeta_{ls} = \sum_{s\in\Gamma_{2^l}^{\hat{r}_0,d_0}} f(s,t,x(s))\,\zeta_{ls} \\ &= \sum_{s\in\Gamma_{2^l}^{\hat{r}_0,d_0}} f(s,t,(T_l x)(s))\,\zeta_{ls} = \sum_{s\in\Gamma_{2^l}^{\hat{r}_0,d_0}} \big(\bar{f}(t,T_l x)\big)(s)\,\zeta_{ls} \\ &= T_l \bar{f}(t, T_l x).\end{aligned} \tag{6.167}$$

Similarly, for all $x \in C(Q_0, Z)$ and $s \in Q_0$

$$(\bar{f}(t,x), \delta_s) = f(s,t,x(s)) = f(s,t,(x,\delta_s)) = f_s(t,(x,\delta_s)).$$

By Lemma 6.4.4, $\bar{f} \in \mathcal{C}^{0,0}_{\mathrm{Lip}}([a,b] \times C(Q_0,Z), C(Q_0,Z); \kappa_1, L_1)$ for some κ_1, L_1 and by definition, see (6.51–6.53), $f_s \in \mathcal{C}^{0,0}_{\mathrm{Lip}}([a,b] \times Z, Z; \kappa, L)$. Now we apply Lemma 6.3.1 with $T = \delta_s$ and $g = f_s$ and obtain

$$\left(\mathcal{A}^r_{n_{l_0},\omega}(\bar{f}, u_0), \delta_s\right) = \mathcal{A}^r_{n_{l_0},\omega}(f_s, u_0(s)) \quad (s \in Q_0), \tag{6.168}$$
$$(\mathcal{A}^{r_1}_{n_l,\omega}(\bar{f}, u_0), \delta_s) = \mathcal{A}^{r_1}_{n_l,\omega}(f_s, u_0(s)) \quad (s \in Q_0,\ l_0 < l \le l_1). \tag{6.169}$$

As a consequence of (6.28),(6.148),(6.168), and (6.169), we can relate algorithm $\mathscr{A}_\omega$ for the parametric problem to algorithm $\mathfrak{A}_\omega$ for the general Banach space valued problem of Section 6.2 as follows

$$\begin{aligned}\mathscr{A}_\omega(f,u_0) &= \mathscr{P}_{l_0}\left(\left(\mathcal{A}^r_{n_{l_0},\omega}(f_s,u_0(s))\right)_{s\in\Gamma^{\hat{r}_0,d_0}_{2^{l_0}}}\right) \\ &\quad + \sum_{l=l_0+1}^{l_1}(\mathscr{P}_l - \mathscr{P}_{l-1})\left(\left(\mathcal{A}^{r_1}_{n_l,\omega}(f_s,u_0(s))\right)_{s\in\Gamma^{\hat{r}_0,d_0}_{2^l}}\right) \\ &= \mathscr{P}_{l_0}\mathcal{A}^r_{n_{l_0},\omega}(\bar{f},u_0) + \sum_{l=l_0+1}^{l_1}(\mathscr{P}_l - \mathscr{P}_{l-1})\mathcal{A}^{r_1}_{n_l,\omega}(\bar{f},u_0) \\ &= \mathfrak{A}_\omega(\bar{f},u_0).\end{aligned} \tag{6.170}$$

We put

$$\mathcal{K}_0 = \{(\bar{f}, u_0) : (f,u_0) \in \mathscr{F}\}.$$

Then Proposition 6.4.3 gives

$$\begin{aligned}\mathcal{K}_0 \subseteq\ & \mathcal{F}^{r,\varrho}([a,b]\times C(Q_0,Z), C(Q_0,Z); \kappa_1, L_1, \sigma, \lambda_1) \\ &\cap\ \mathcal{F}^{r_1,\varrho_1}([a,b]\times C^{r_0}(Q_0,Z), C^{r_0}(Q_0,Z); \kappa_1, L_1, \sigma, \lambda_1).\end{aligned} \tag{6.171}$$

Furthermore, (2.8),(6.133–6.134),(6.159),(6.167), and (6.171) show that the assumptions of Proposition 6.3.5 are fulfilled. Therefore (6.34) of Proposition 6.3.5 together with (6.64),(2.9), and (6.170) prove (6.150). The estimate (6.151) follows from (6.35) of Proposition 6.3.5 together with (6.64),(2.9),(6.166), and (6.170).

□

6.6 Complexity Analysis

In terminology of 2.2, we set $K = Z$ and the set of information functionals Λ_{ivp} is given by

$$\Lambda_{\mathrm{ivp}} = \{\delta_{s,t,z} : \ s \in Q_0, t \in [a,b], z \in Z\} \cup \{\delta_s : \ s \in Q_0\}, \tag{6.172}$$

where for $(f, u_0) \in \mathscr{F}$

$$\delta_{s,t,z}(f, u_0) = f(s,t,z), \quad \delta_s(f, u_0) = u_0(s). \tag{6.173}$$

So the admissible information is Z-valued and consists of values of f and u_0. Setting $F = \mathscr{F}$, $G = B(Q_0 \times [a,b], Z)$, and $S = \mathscr{S}$, the corresponding numerical problem Π is defined by

$$\Pi = (\mathscr{F}, B(Q_0 \times [a,b], Z), \mathscr{S}, Z, \Lambda_{\mathrm{ivp}}).$$

The following theorem, which is the main result of this chapter, gives almost sharp estimates of the deterministic and randomized minimal errors for arbitrary Banach spaces and hence, of the complexity of the parametric initial value problem. In special cases, the estimates are sharp even up to logarithmic factors.

Moreover, combined with Corollary 6.5.7, it shows that the optimal order is realized by the multilevel algorithm presented before; more precisely, in the deterministic case by $\mathscr{A}_\omega$ for any $\omega \in \Omega$, and in the randomized case by $(\mathscr{A}_\omega)_{\omega\in\Omega}$, with parameters chosen in an appropriate way. Concerning the assumption $r+\varrho \geq r_1+\varrho_1$, we refer to Remark 6.5.5.

Theorem 6.6.1. *Let $r_0, r, r_1 \in \mathbb{N}_0$, $0 \leq \varrho, \varrho_1 \leq 1$, with $r + \varrho \geq r_1 + \varrho_1$, $\kappa, L : (0, +\infty) \to (0, +\infty)$, and $\sigma, \lambda > 0$, where we assume that*

$$\kappa_0 := \inf_{0<R<+\infty} \kappa(R) > 0. \tag{6.174}$$

Let Z be a Banach space and let $\mathscr{F}$ be defined by (6.60–6.62). Then in the deterministic setting,

$$\begin{aligned} & e_n^{\det}(\mathscr{S}, \mathscr{F}) \asymp n^{-\tilde{\upsilon}_1} && \text{if } \tfrac{r_0}{d_0} > \beta_1 \\ & n^{-\frac{r_0}{d_0}} \preceq e_n^{\det}(\mathscr{S}, \mathscr{F}) \preceq n^{-\frac{r_0}{d_0}} (\log n)^{\frac{r_0}{d_0}+1} && \text{if } \tfrac{r_0}{d_0} = \beta_1 > 0 \\ & e_n^{\det}(\mathscr{S}, \mathscr{F}) \asymp n^{-\frac{r_0}{d_0}} && \text{if } \tfrac{r_0}{d_0} = \beta_1 = 0 \ \vee \ \tfrac{r_0}{d_0} < \beta_1, \end{aligned} \tag{6.175}$$

where $\beta_1 = r_1 + \varrho_1$ and $\tilde{\upsilon}_1$ was defined in (6.153) *by*

$$\tilde{\upsilon}_1 \ = \ \frac{\frac{r_0}{d_0}}{\frac{r_0}{d_0} + r + \varrho - r_1 - \varrho_1} (r + \varrho) \ . \tag{6.176}$$

Moreover, let $1 \le p \le 2$ *and assume that* Z *is of type* p*. Let* p_Z *denote the supremum of all* p_1 *such that* Z *is of type* p_1*. Then in the randomized setting,*

$$n^{-\tilde{\upsilon}_2(p_Z)} \preceq e_n^{\mathrm{ran}}(\mathscr{S},\mathscr{F}) \preceq_{\log} n^{-\tilde{\upsilon}_2(p)}, \tag{6.177}$$

with $\tilde{\upsilon}_2(p)$ *given in (6.155) by*

$$\tilde{\upsilon}_2(p) = \begin{cases} \frac{\frac{r_0}{d_0}}{\frac{r_0}{d_0}+r+\varrho-r_1-\varrho_1}\left(r+\varrho+1-\frac{1}{p}\right) & \text{if } \frac{r_0}{d_0} > r_1+\varrho_1+1-\frac{1}{p} \\ \frac{r_0}{d_0} & \text{if } \frac{r_0}{d_0} \le r_1+\varrho_1+1-\frac{1}{p} \end{cases}. \tag{6.178}$$

As discussed for the case of parametric Banach space valued integration, $\tilde{\upsilon}_2(p)$ is a continuous, monotonically increasing function of $p \in [1,2]$. It follows that the bounds in the randomized case of Theorem 6.6.1 are matching up to an arbitrarily small gap in the exponent. Under an additional assumption, upper and lower bounds are of the same order up to logarithmic factors.

Corollary 6.6.2. *Assume that the conditions of Theorem 6.6.1 hold. Then for each* $\varepsilon > 0$

$$n^{-\tilde{\upsilon}_2(p_Z)} \preceq e_n^{\mathrm{ran}}(\mathscr{S},\mathscr{F}) \preceq n^{-\tilde{\upsilon}_2(p_Z)+\varepsilon}.$$

Furthermore, if Z is such that p_Z is attained, and in particular if Z is of type 2, more precise estimates can be given. This is the content of the following theorem.

Theorem 6.6.3. *Assume that the conditions of Theorem 6.6.1 hold. If the supremum of types is attained, that is,* Z *is of type* p_Z*, then*

$$\begin{array}{ll}
e_n^{\mathrm{ran}}(\mathscr{S},\mathscr{F}) \preceq n^{-\frac{r}{d}-1+\frac{1}{p_Z}} & \text{if } \frac{r_0}{d_0} > \beta_1+1-\frac{1}{p_Z} \wedge r+\varrho=\beta_1 \\
n^{-\tilde{\upsilon}_2(p_Z)} \preceq e_n^{\mathrm{ran}}(\mathscr{S},\mathscr{F}) \preceq n^{-\tilde{\upsilon}_2(p_Z)}(\log n)^{\frac{1}{2}} & \text{if } \frac{r_0}{d_0} > \beta_1+1-\frac{1}{p_Z} \wedge r+\varrho>\beta_1 \\
n^{-\frac{r_0}{d_0}} \preceq e_n^{\mathrm{ran}}(\mathscr{S},\mathscr{F}) \preceq n^{-\frac{r_0}{d_0}}(\log n)^{\frac{r_0}{d_0}+\frac{3}{2}} & \text{if } \frac{r_0}{d_0} = \beta_1+1-\frac{1}{p_Z} \\
n^{-\frac{r_0}{d_0}} \preceq e_n^{\mathrm{ran}}(\mathscr{S},\mathscr{F}) \preceq n^{-\frac{r_0}{d_0}}(\log n)^{\frac{p}{2(p-1)}\left(\frac{r_0}{d_0}-\beta_1\right)} & \text{if } \beta_1 < \frac{r_0}{d_0} < \beta_1+1-\frac{1}{p_Z} \\
n^{-\frac{r_0}{d_0}} \preceq e_n^{\mathrm{ran}}(\mathscr{S},\mathscr{F}) \preceq n^{-\frac{r_0}{d_0}}(\log\log n)^{\frac{r_0}{d_0}+1} & \text{if } \frac{r_0}{d_0} = \beta_1 > 0 \\
e_n^{\mathrm{ran}}(\mathscr{S},\mathscr{F}) \asymp n^{-\frac{r_0}{d_0}} & \text{if } \frac{r_0}{d_0} = \beta_1 = 0 \vee \frac{r_0}{d_0} < \beta_1.
\end{array}$$

Furthermore, if Z *is of type* 2*,*

$$\begin{array}{ll}
e_n^{\mathrm{ran}}(\mathscr{S},\mathscr{F}) \asymp n^{-r-\varrho-\frac{1}{2}} & \text{if } \frac{r_0}{d_0} > \beta_1+\frac{1}{2} \wedge r+\varrho=\beta_1 \\
e_n^{\mathrm{ran}}(\mathscr{S},\mathscr{F}) \asymp n^{-\tilde{\upsilon}_2(2)}(\log n)^{\frac{1}{2}} & \text{if } \frac{r_0}{d_0} > \beta_1+\frac{1}{2} \wedge r+\varrho>\beta_1 \\
n^{-\frac{r_0}{d_0}}(\log n)^{\frac{1}{2}} \preceq e_n^{\mathrm{ran}}(\mathscr{S},\mathscr{F}) \preceq n^{-\frac{r_0}{d_0}}(\log n)^{\frac{r_0}{d_0}+\frac{3}{2}} & \text{if } \frac{r_0}{d_0} = \beta_1+\frac{1}{2} \\
e_n^{\mathrm{ran}}(\mathscr{S},\mathscr{F}) \asymp n^{-\frac{r_0}{d_0}}(\log n)^{\frac{r_0}{d_0}-\beta_1} & \text{if } \beta_1 < \frac{r_0}{d_0} < \beta_1+\frac{1}{2} \\
n^{-\frac{r_0}{d_0}} \preceq e_n^{\mathrm{ran}}(\mathscr{S},\mathscr{F}) \preceq n^{-\frac{r_0}{d_0}}(\log\log n)^{\frac{r_0}{d_0}+1} & \text{if } \frac{r_0}{d_0} = \beta_1 > 0 \\
e_n^{\mathrm{ran}}(\mathscr{S},\mathscr{F}) \asymp n^{-\frac{r_0}{d_0}} & \text{if } \frac{r_0}{d_0} = \beta_1 = 0 \vee \frac{r_0}{d_0} < \beta_1.
\end{array}$$

For particular examples of Banach spaces that satisfy the conditions above, we refer to the preceding discussions.

Proof of Theorem 6.6.1 and Theorem 6.6.3. The upper bounds follow from Corollary 6.5.6. To show the lower bounds, let $\mathscr{S}_0 : C(Q_0 \times [0,1], Z) \to C(Q_0, Z)$ be given for $f \in C(Q_0 \times [0,1], Z)$ by

$$(\mathscr{S}_0 f)(s) = \int_0^1 f(s,t)dt \quad (s \in Q_0).$$

This is the operator of Z-valued definite parametric integration, with a one-dimensional integration domain (thus $d = 1$), as defined in the previous chapter. Define

$$V_0 : C(Q_0 \times [0,1], Z) \to C(Q_0 \times [a,b], Z)$$

for $f \in C(Q_0 \times [0,1], Z)$ by

$$(V_0 f)(s,t) = f\left(s, \frac{t-a}{b-a}\right) \qquad (s \in Q_0, t \in [a,b]), \tag{6.179}$$

and

$$V_1 : C(Q_0 \times [a,b], Z) \to C(Q_0 \times [a,b], Z) \times C(Q_0, Z)$$

for $f \in C(Q_0 \times [a,b], Z)$ by

$$V_1 f = (f, 0), \tag{6.180}$$

and

$$V_2 : B(Q_0 \times [a,b], Z) \to B(Q_0, Z)$$

for $g \in B(Q_0 \times [a,b], Z)$ by

$$(V_2 g)(s) = g(s,b) \quad (s \in Q_0).$$

Then we have

$$\|V_2\| = 1. \tag{6.181}$$

For $\mathscr{I}_{m_0,m} = \{1, \ldots, m_0^{d_0}\} \times \{1, \ldots, m\}$ and $\mathscr{Z} = (z_j)_{j=1}^{m^d} \subset B_Z$ set

$$\Psi^0_{m_0,m}(\mathscr{Z}) = \left\{ \sum_{(i,j) \in \mathscr{I}_{m_0,m}} \delta_{ij} \psi_{ij} z_j : \delta_{ij} \in [-1,1],\, (i,j) \in \mathscr{I}_{m_0,m} \right\}. \tag{6.182}$$

with

$$\psi_{ij}(s,t) = \varphi_{0,i}(s)\varphi_j(t)$$

as defined in (5.64) using $d = 1$. Let

$$\eta_i = s_i + \left(\frac{1}{2m_0}, \ldots, \frac{1}{2m_0} \right)$$

be the center of $Q_{0,i}$, and let $T_{m_0} : B(Q_0, Z) \to C(Q_0, Z)$ be given for $v \in B(Q_0, Z)$ by

$$T_{m_0} v = \sum_{i=1}^{m_0^{d_0}} v(\eta_i) \varphi_{0,i}.$$

Then there is a constant $c_1 > 0$ such that for all $m_0 \in \mathbb{N}$

$$\|T_{m_0}\| \le c_1, \tag{6.183}$$

and because of (5.63), we have

$$T_{m_0} v = v \qquad \left(v \in \operatorname{span} \left\{ \varphi_{0,i} z : \ i = 1, \ldots, m_0^{d_0}, z \in Z \right\} \right). \tag{6.184}$$

For $f \in C(Q_0 \times [0,1], Z)$, we set $\tilde{f} := V_0 f$ and consider functions $\tilde{f}$ on $Q_0 \times [a,b]$ as functions on $Q_0 \times [a,b] \times Z$ not depending on $z \in Z$. The solution $u = \mathscr{S}(\tilde{f}, 0)$ of

$$\begin{aligned} \frac{d}{dt} u(s,t) &= \tilde{f}(s,t) \quad (s \in Q_0, t \in [a,b]) \\ u(s,a) &= 0 \quad (s \in Q_0) \end{aligned}$$

is

$$u(s,t) = \int_a^t \tilde{f}(s,\tau) d\tau = (t-a) \int_0^t f(s,\tau) d\tau.$$

Consequently,

$$(b-a)\mathscr{S}_0(f) = (T_{m_0} \circ V_2 \circ \mathscr{S} \circ V_1 \circ V_0)(f) \qquad \left(f \in \operatorname{span} \Psi^0_{m_0,m_1}(\mathscr{Z}) \right). \tag{6.185}$$

Moreover, if $\tilde{f}$ satisfies

$$\|\tilde{f}\|_{C(Q_0 \times [a,b], Z)} \le (b-a)^{-1} \lambda, \tag{6.186}$$

then

$$\sup_{s \in Q_0, t \in [a,b]} \|u(s,t)\| \le \lambda. \tag{6.187}$$

Furthermore, according to (6.14–6.16), for $n \in \mathbb{N}$, $\omega \in \Omega$, $s \in Q_0$, we have $\mathcal{A}^0_{n,\omega}(\tilde{f}_s, 0) = v(s, \cdot)$ with

$$v(s,t) = \begin{cases} p_{k,0}(s,t) & \text{if} \quad t \in [t_k, t_{k+1}), \ 0 \le k \le n-1, \\ u_n(s) & \text{if} \quad t = t_n, \end{cases}$$

$u_0(s) = 0$, and for $0 \le k \le n-1$, $t \in [t_k, t_{k+1}]$,

$$\begin{aligned} p_{k,0}(s,t) &= u_k(s) + (t - t_k) f(s, t_k), \\ u_{k+1}(s) &= u_k(s) + h\tilde{f}(s, \xi_{k+1}). \end{aligned}$$

So (6.186) also implies

$$\sup_{s \in Q_0} \left\| \mathcal{A}^0_{n,\omega}(\tilde{f}_s, 0) \right\|_{B([a,b],Z)} \le \lambda. \tag{6.188}$$

Taking into account (6.174), we obtain (similar to (4.43)) that there is a constant $c_0 > 0$ such that for all $m_0, m \in \mathbb{N}$

$$\begin{aligned} c_0 m^{-r-\varrho} V_0(\Psi^0_{m_0,m}(\mathscr{Z})) &\subseteq \mathscr{C}^{0,r,\varrho}_{\mathrm{Lip}}(Q_0 \times [a,b] \times Z, Z; \kappa, L), &(6.189)\\ c_0 m_0^{-r_0} m^{-r_1-\varrho_1} V_0(\Psi^0_{m_0,m}(\mathscr{Z})) &\subseteq \mathscr{C}^{r_0,r_1,\varrho_1}_{\mathrm{Lip}}(Q_0 \times [a,b] \times Z, Z; \kappa, L), &(6.190)\\ c_0 V_0(\Psi^0_{m_0,m}(\mathscr{Z})) &\subseteq (b-a)^{-1} \lambda B_{C(Q_0 \times [a,b], Z)}. &(6.191) \end{aligned}$$

We set

$$\Psi^{r+\varrho, r_0, r_1+\varrho_1}_{m_0,m}(\mathscr{Z}) = c_0 \min\left(m^{-r-\varrho}, m_0^{-r_0} m^{-r_1-\varrho_1}\right) \Psi^0_{m_0,m}(\mathscr{Z}), \tag{6.192}$$

thus, by (6.189–6.191),

$$\begin{aligned} V_0(\Psi^{r+\varrho, r_0, r_1+\varrho_1}_{m_0,m}(\mathscr{Z})) &\subseteq \mathscr{C}^{0,r,\varrho}_{\mathrm{Lip}}(Q_0 \times [a,b] \times Z, Z; \kappa, L) \\ &\quad \cap \mathscr{C}^{r_0,r_1,\varrho_1}_{\mathrm{Lip}}(Q_0 \times [a,b] \times Z, Z; \kappa, L) &(6.193)\\ V_0(\Psi^{r+\varrho, r_0, r_1+\varrho_1}_{m_0,m}(\mathscr{Z})) &\subseteq (b-a)^{-1} \lambda B_{C(Q_0 \times [a,b], Z)}. &(6.194) \end{aligned}$$

Using (6.193–6.194) and (6.186–6.188), it follows that for all $m_0, m \in \mathbb{N}$

$$V_1 \circ V_0 \left(\Psi^{r+\varrho, r_0, r_1+\varrho_1}_{m_0,m}(\mathscr{Z})\right) \subseteq \mathscr{F}. \tag{6.195}$$

We put $K_0 = Z$ and consider the following class of information functionals on $C(Q_0 \times [0,1], Z)$

$$\Lambda = \{\delta_{s,t} : s \in Q_0, t \in [0,1]\}, \quad \delta_{s,t}(f) = f(s,t). \tag{6.196}$$

We conclude from (6.185) and (6.195) that the problem

$$(\mathscr{S}_0, \Psi^{r+\varrho, r_0, r_1+\varrho_1}_{m_0,m}(\mathscr{Z}), C(Q_0, Z), Z, \Lambda)$$

reduces to

$$(\mathscr{S}, \mathscr{F}, B(Q_0 \times [a,b], Z), Z, \Lambda_{\mathrm{ivp}}).$$

Consequently, by (6.181), (6.185), and Lemma 2.2.14, for all $n, m_0, m \in \mathbb{N}$

$$e^{\mathrm{set}}_n(\mathscr{S}, \mathscr{F}) \ge c e^{\mathrm{set}}_n(\mathscr{S}_0, \Psi^{r+\varrho, r_0, r_1+\varrho_1}_{m_0,m}(\mathscr{Z})), \tag{6.197}$$

where set $\in \{\mathrm{det}, \mathrm{ran}\}$. Moreover, by linearity of $\mathscr{S}_0$, Corollary 2.2.15, and (6.192)

$$e_n^{\mathrm{set}}(\mathscr{S}_0, \Psi_{m_0,m}^{r+\varrho,r_0,r_1+\varrho_1}(\mathscr{Z})) \\ = c_0 \min\left(m^{-r-\varrho}, m_0^{-r_0} m^{-r_1-\varrho_1}\right) e_n^{\mathrm{set}}(\mathscr{S}_0, \Psi_{m_0,m}^0(\mathscr{Z})). \quad (6.198)$$

Thus, it suffices to estimate $e_n^{\mathrm{set}}(\mathscr{S}_0, \Psi_{m_0,m}^0(\mathscr{Z}))$ for certain choices of $\mathscr{Z}$. This can be done, as for parametric integration, using Lemma 5.3.6. In the deterministic case, we apply Lemma 5.3.6 with

$$\gamma := r + \varrho, \qquad \gamma_0 = r_0, \qquad \gamma_1 = r_1 + \varrho_1,$$

and get, using (5.81) and (6.176),

$$\upsilon_3 = \frac{\frac{r_0}{d_0}}{\frac{r_0}{d_0} + r + \varrho - r_1 - \varrho_1}(r+\varrho) = \tilde{\upsilon}_1,$$

which proves the lower bounds in (6.175) using (6.197),(6.198), and Lemma 5.3.6. Next we consider the randomized case (6.154). Using (5.83) and (6.178), we get for $r + \varrho > r_1 + \varrho_1 + 1 - 1/p$

$$\upsilon_4(p) = \frac{\frac{r_0}{d_0}}{\frac{r_0}{d_0} + r + \varrho - r_1 - \varrho_1}\left(r + \varrho + 1 - \frac{1}{p}\right) = \tilde{\upsilon}_2(p).$$

Finally, Lemma 5.3.6,(6.197), and (6.198) yield the lower bounds in Theorem 6.6.1 and Theorem 6.6.3.

□

Remark 6.6.4. Finite systems of d scalar ODEs are included in our analysis by setting $Z = \ell_2^d$. Letting $\mathscr{F}_\infty$ stand for $\mathscr{F}$ with $Z = \ell_2(\mathbb{N})$ and denoting the classes $\mathscr{F}$ for $Z = \ell_2^d$ by $\mathscr{F}_d$ (all with the same dimension of the parameter space d_0 and with the same constants $\kappa, L, \sigma, \lambda$), it is easily shown that $\mathscr{F}_d$ can be embedded into $\mathscr{F}_\infty$ in an uniform way. This shows, in particular, that the error estimates of the algorithm, see Corollary 6.5.6, hold with constants which are independent of the dimension d of the system. Taking into account that an ℓ_2^d-valued information functional is equivalent to d scalar-valued information functionals, it follows that the family $(\mathscr{F}_d)_{d\in\mathbb{N}}$ is polynomially tractable in the randomized setting if $r_0 > 0$ and in the deterministic setting if $r_0 > 0$ and $r + \varrho > 0$. We refer to [40] for the notion of tractability and more on this direction of research.

6.7 Some Particular Classes of Functions

First let us consider the case of globally bounded functions. Here we have $\kappa \equiv \kappa_0$ and $L \equiv L_0$ with $\kappa_0, L_0 \in \mathbb{R}$, $\kappa_0, L_0 > 0$. Then

$$\begin{aligned}\mathscr{F} &= \big(\mathscr{C}^{0,r,\varrho}_{\mathrm{Lip}}(Q_0 \times [a,b] \times Z, Z; \kappa_0, L_0) \cap \mathscr{C}^{r_0,r_1,\varrho_1}_{\mathrm{Lip}}(Q_0 \times [a,b] \times Z, Z; \kappa_0, L_0)\big) \\ &\quad \times \sigma B_{C^{r_0}(Q_0,Z)},\end{aligned}$$

provided the constant λ involved in the definition (6.60–6.62) of $\mathscr{F}$ satisfies

$$\lambda \geq \sigma + \kappa_0(b-a). \tag{6.199}$$

In other words, for globally bounded classes conditions (6.61) and (6.62) are automatically fulfilled whenever (6.199) holds.

Next let us consider the case of linear equations and see how it fits the class $\mathscr{F}$. For $\kappa_0 > 0$ let $C^{r_0,r,\varrho}(Q_0 \times [a,b], Z; \kappa_0)$ denote the subset of all functions in $\mathscr{C}^{r_0,r,\varrho}(Q_0 \times [a,b] \times Z, Z; \kappa_0)$ which do not depend on $z \in Z$. Given $\kappa_0, \kappa_1, \sigma > 0$, let $\mathscr{G}$ be the set of all pairs (f, u_0) with $u_0 \in \sigma B_{C^{r_0}(Q_0,Z)}$ and $f : Q_0 \times [a,b] \times Z \to Z$ of the form

$$f(s,t,z) = g_0(s,t) + g_1(s,t)z, \tag{6.200}$$

with

$$g_0 \in C^{0,r,\varrho}(Q_0 \times [a,b], Z; \kappa_0) \cap C^{r_0,r_1,\varrho_1}(Q_0 \times [a,b], Z; \kappa_0), \tag{6.201}$$

$$g_1 \in C^{0,r,\varrho}(Q_0 \times [a,b], \mathscr{L}(Z); \kappa_1) \cap C^{r_0,r_1,\varrho_1}(Q_0 \times [a,b], \mathscr{L}(Z); \kappa_1). \tag{6.202}$$

This means we consider the linear equation

$$\frac{d}{dt}u(s,t) = g_0(s,t) + g_1(s,t)u(s,t) \tag{6.203}$$

$$u(s,a) = u_0(s). \tag{6.204}$$

Corollary 6.7.1. *Let* $r_0, r, r_1 \in \mathbb{N}_0$, $0 \leq \varrho, \varrho_1 \leq 1$, *with* $r + \varrho \geq r_1 + \varrho_1$, $\kappa_0, \kappa_1, \sigma > 0$. *Then there exist* $\kappa, L : (0, +\infty) \to (0, +\infty)$ *and* $\lambda > 0$ *such that*

$$\mathscr{G} \subseteq \mathscr{F}, \tag{6.205}$$

where $\mathscr{G}$ *is defined in* (6.200–6.202) *and* $\mathscr{F}$ *in* (6.60–6.62), *and the statements of Theorem* 6.6.1 *hold with* $\mathscr{F}$ *replaced by* $\mathscr{G}$.

Proof. It is easily checked that

$$\begin{aligned}\mathscr{G} \subseteq \Big(&\mathscr{C}^{0,r,\varrho}_{\mathrm{Lip}}(Q_0 \times [a,b] \times Z, Z; \kappa, L) \\ &\cap \mathscr{C}^{r_0,r_1,\varrho_1}_{\mathrm{Lip}}(Q_0 \times [a,b] \times Z, Z; \kappa, L)\Big) \times \sigma B_{C^{r_0}(Q_0,Z)}\end{aligned}$$

for suitable κ, L. Thus, it remains to verify (6.61–6.62). Since f is Lipschitz continuous with constant κ_1, the solution of (6.203–6.204) exists on $[a,b]$ and is unique. Integrating with respect to t we get

$$u(s,t) \quad = \quad u_0(s) + \int_a^t (g_0(s,\tau) + g_1(s,\tau)u(s,\tau))\, d\tau,$$

consequently for $t \in [a,b]$

$$\|u(\,\cdot\,,t)\|_{B(Q_0,Z)} \quad \leq \quad \sigma + (b-a)\kappa_0 + \kappa_1 \int_a^t \|u(\,\cdot\,,\tau)\|_{B(Q_0,Z)}\, d\tau,$$

which by Gronwall's lemma gives

$$\|u\|_{B(Q_0\times[a,b],Z)} \leq (\sigma + (b-a)\kappa_0)e^{\kappa_1(b-a)}.$$

By (6.14–6.16), we have $\mathcal{A}^0_{n,\omega}(f_s, u_0(s)) = v(s,\,\cdot\,)$, where

$$v(s,t) = \begin{cases} p_{k,0}(s,t) & \text{if} \quad t \in [t_k, t_{k+1}) \quad \text{and} \quad 0 \leq k \leq n-1, \\ u_n(s) & \text{if} \quad t = t_n, \end{cases}$$

and for $0 \leq k \leq n-1$, $t \in [t_k, t_{k+1}]$

$$p_{k,0}(s,t) \quad = \quad u_k(s) + (t-t_k)g_0(s,t_k) + (t-t_k)g_1(s,t_k)u_k(s), \tag{6.206}$$

$$u_{k+1}(s) \quad = \quad u_k(s) + hg_0\big(s,\xi_{k+1}\big) + hg_1\big(s,\xi_{k+1}\big)p_{k,0}(s,\xi_{k+1}). \tag{6.207}$$

Inserting (6.206) with $t = \xi_{k+1}$ into (6.207), we get

$$\begin{aligned} u_{k+1}(s) \quad = \quad & u_k(s) + hg_0\big(s,\xi_{k+1}\big) + h(\xi_{k+1}-t_k)g_1\big(s,\xi_{k+1}\big)g_0(s,t_k), \\ & +hg_1\big(s,\xi_{k+1}\big)u_k(s) + h(\xi_{k+1}-t_k)g_1\big(s,\xi_{k+1}\big)g_1(s,t_k)u_k(s), \end{aligned}$$

thus with $c_0 = \kappa_0(1+h\kappa_1)$, $c_1 = \kappa_1(1+h\kappa_1)$

$$\|u_{k+1}\|_{B(Q_0,Z)} \quad \leq \quad (1+c_1h)\,\|u_k\|_{B(Q_0,Z)} + c_0h.$$

Using $\|u_0\|_{B(Q_0,Z)} \leq \sigma$, we obtain for $1 \leq k \leq n$

$$\begin{aligned} \|u_k\|_{B(Q_0,Z)} \quad & \leq \quad \sigma(1+c_1h)^k + c_0h\sum_{j=0}^{k-1}(1+c_1h)^j \\ & \leq \quad \left(\sigma + \frac{c_0}{c_1}\right)(1+c_1h)^n \leq \left(\sigma + \frac{\kappa_0}{\kappa_1}\right) e^{c_1(b-a)} \\ & \leq \quad \left(\sigma + \frac{\kappa_0}{\kappa_1}\right) e^{\kappa_1(1+(b-a)\kappa_1)(b-a)}. \end{aligned}$$

Together with (6.206), this implies

$$\max_{0\le k\le n-1}\max_{t\in[t_k,t_{k+1}]}\|p_{k,0}(\,\cdot\,,t)\|_{B(Q_0,Z)}$$
$$\le (1+(b-a)\kappa_1)\left(\sigma+\frac{\kappa_0}{\kappa_1}\right)e^{\kappa_1(1+(b-a)\kappa_1)(b-a)}+(b-a)\kappa_0$$

and hence the desired result (6.205), which in turn implies the upper bound.

That the lower bounds of Theorem 6.6.1 also hold for $\mathscr{G}$ follows directly from the proof of Theorem 6.6.1 and the fact that $\mathscr{G}$ contains all pairs $(f,0)$ with

$$f=g_0\in C^{0,r,\varrho}(Q_0\times[a,b],Z;\kappa_0)\cap C^{r_0,r_1,\varrho_1}(Q_0\times[a,b],Z;\kappa_0).$$

□

The class $\mathscr{C}^{r_0,r,\varrho}_{\mathrm{Lip}}(Q_0\times[a,b]\times Z,Z;\kappa,L)$ consists of functions with a certain type of dominating mixed smoothness. We have chosen $\mathscr{F}$ to be given by an intersection of two such classes, because this way we can also include isotropic smoothness and certain anisotropic analogues thereof. Let us look at these special cases in more detail. For the subsequent discussion, we assume, for the sake of simplicity, that Z is of type 2, which includes in particular the case of finite systems of scalar equations $Z=\mathbb{R}^d$.

First we consider the case $r=r_1$, $\varrho=\varrho_1$. Then $\mathscr{F}$ is the set of all

$$(f,u_0)\quad\in\quad\mathscr{C}^{r_0,r,\varrho}_{\mathrm{Lip}}(Q_0\times[a,b]\times Z,Z;\kappa,L)\times\sigma B_{C^{r_0}(Q_0,Z)}$$

satisfying (6.61) and, if $r=\varrho=0$, (6.62). Thus, the involved functions f have dominating mixed smoothness. From Theorem 6.6.1, we obtain

Corollary 6.7.2. *Let $r_0,r\in\mathbb{N}_0$, $0\le\varrho\le 1$, $r=r_1$, $\varrho=\varrho_1$, assume that* (6.174) *holds and that Z is of type* 2*. Then*

$$e_n^{\mathrm{det}}(\mathscr{S},\mathscr{F})\quad\asymp_{\log}\quad n^{-\min\left(r+\varrho,\,\frac{r_0}{d_0}\right)},$$
$$e_n^{\mathrm{ran}}(\mathscr{S},\mathscr{F})\quad\asymp_{\log}\quad n^{-\min\left(r+\varrho+\frac12,\,\frac{r_0}{d_0}\right)}.$$

This is the corresponding case to Corollary 5.4.1 in Chapter 5. Hence, if $r_0/d_0\le r+\varrho$, the rates are the same. If $r_0/d_0>r+\varrho$, the best rate of randomized algorithms is superior to that of deterministic algorithms. If $r_0/d_0\ge r+\varrho+\frac12$, the best randomized algorithms outperform the best deterministic ones by an order of $n^{-1/2}$. This is particularly important if, e.g., $r=0$ and ϱ is small. Then the deterministic rate $n^{-\varrho}$ is slow (for $\varrho=0$, there is no convergence rate at all), while we still have at least $n^{-1/2}$ in the randomized setting. For an illustration, see also Figure 5.1, which coincides if we replace r/d by $r+\varrho$.

Next we assume $r_1 = \varrho_1 = 0$, which means that $\mathscr{F}$ is the set of

$$(f, u_0) \in \Big(\mathscr{C}^{0,r,\varrho}_{\mathrm{Lip}}(Q_0 \times [a,b] \times Z, Z; \kappa, L) \cap \mathscr{C}^{r_0,0,0}_{\mathrm{Lip}}(Q_0 \times [a,b] \times Z, Z; \kappa, L)\Big) \times \sigma B_{C^{r_0}(Q_0,Z)} \tag{6.208}$$

fulfilling (6.61), and if $r = \varrho = 0$, (6.62), so that here the functions f have smoothness in s and t separately. In this case, Theorem 6.6.1 yields

Corollary 6.7.3. *Let $r_0, r \in \mathbb{N}_0$, $0 \le \varrho \le 1$, $r_1 = \varrho_1 = 0$, suppose* (6.174) *holds and Z is of type* 2. *Then*

$$e_n^{\mathrm{det}}(\mathscr{S}, \mathscr{F}) \asymp_{\log} n^{-\tilde{\upsilon}_3}$$
$$e_n^{\mathrm{ran}}(\mathscr{S}, \mathscr{F}) \asymp_{\log} n^{-\tilde{\upsilon}_4},$$

where

$$\tilde{\upsilon}_3 = \begin{cases} 0 & \text{if} \quad r_0 = 0 \\ \frac{\frac{r_0}{d_0}}{\frac{r_0}{d_0}+r+\varrho}\,(r+\varrho) & \text{otherwise} \end{cases} \tag{6.209}$$

$$\tilde{\upsilon}_4 = \begin{cases} \frac{\frac{r_0}{d_0}}{\frac{r_0}{d_0}+r+\varrho}\left(r+\varrho+\frac{1}{2}\right) & \text{if} \quad \frac{r_0}{d_0} \ge \frac{1}{2} \\ \frac{r_0}{d_0} & \text{if} \quad \frac{r_0}{d_0} < \frac{1}{2}. \end{cases} \tag{6.210}$$

Except for the trivial case $r_0 = 0$, the randomized setting is always superior to the deterministic one, although the maximum of improvement $n^{-1/2}$ is only reached if $r = \varrho = 0$ and $r_0/d_0 \ge 1/2$ (this case, in fact, has already been considered above). The rates are similar to these considered in Corollary 5.4.2, therefore, see also Figure 5.2 for the case $r + \varrho = 0$ and Figure 5.3 for the case $r + \varrho = 1$.

Next we keep the restriction $r_1 = \varrho_1 = 0$ and assume also $\varrho = 0$. In this case, we want to identify certain subclasses of $\mathscr{F}$. Let $r_2 \in \mathbb{N}_0$ and let $\mathscr{C}^{[r_0,r,r_2]}(Q_0 \times [a,b] \times Z, Z; \kappa)$ be the space of continuous functions $f : Q_0 \times [a,b] \times Z \to Z$ having for all $\alpha = (\alpha_0, \alpha_1, \alpha_2) \in \mathbb{N}_0^3$ with

$$\frac{\alpha_0}{r_0} + \frac{\alpha_1}{r} + \frac{\alpha_2}{r_2} \le 1 \tag{6.211}$$

(we interpret $\frac{0}{0} = 0$ and $\frac{\tau}{0} = +\infty$ if $\tau > 0$) continuous partial derivatives $\frac{\partial^{|\alpha|} f(s,t,z)}{\partial s^{\alpha_0} \partial t^{\alpha_1} \partial z^{\alpha_2}}$ such that for $R > 0$, $s \in Q_0$, $t \in [a,b]$, $z \in RB_Z$

$$\left\| \frac{\partial^{|\alpha|} f(s,t,z)}{\partial s^{\alpha_0} \partial t^{\alpha_1} \partial z^{\alpha_2}} \right\|_{\mathscr{L}_{\alpha_0,\alpha_2}(\mathbb{R}^{d_0},Z,Z)} \le \kappa(R).$$

If $r_0 = r_2 = r$, then this is just isotropic C^r-smoothness. Furthermore, if $r_2 \ge r_0$, we let $\mathscr{C}^{[r_0,r,r_2]}_{\mathrm{Lip}}(Q_0 \times [a,b] \times Z, Z; \kappa, L)$ be the subset consisting of those $f \in \mathscr{C}^{[r_0,r,r_2]}_{\mathrm{Lip}}(Q_0 \times [a,b] \times Z, Z; \kappa)$ which satisfy the Lipschitz conditions

$$\left\| \frac{\partial^{|\alpha|} f(s,t,z_1)}{\partial s^{\alpha_0} \partial z^{\alpha_2}} - \frac{\partial^{|\alpha|} f(s,t,z_2)}{\partial s^{\alpha_0} \partial z^{\alpha_2}} \right\|_{\mathscr{L}_{\alpha_0,\alpha_2}(\mathbb{R}^{d_0},Z,Z)} \le L(R) \|z_1 - z_2\|$$

for $\alpha_0 + \alpha_2 \le r_0$, $R > 0$, $s \in Q_0$, $t \in [a,b]$, $z_1, z_2 \in RB_Z$. Finally, we let $\mathscr{H}$ be the set of all

$$(f, u_0) \in \mathscr{C}^{[r_0,r,r_2]}_{\mathrm{Lip}}(Q_0 \times [a,b] \times Z, Z; \kappa, L) \times \sigma B_{C^r(Q_0,Z)}$$

satisfying (6.61), and if $r = 0$, (6.62). Considering the class $\mathscr{F}$ with $r_1 = \varrho_1 = \varrho = 0$ and taking into account (6.208), it follows that for all $r_2 \ge \max(r_0, r)$

$$\mathscr{H} \subseteq \mathscr{F}. \tag{6.212}$$

Corollary 6.7.4. *Let $r_0, r, r_2 \in \mathbb{N}_0$, $r_2 \ge \max(r_0, r)$, assume that (6.174) holds and that Z is of type 2. Then for any $\mathscr{M}$ with $\mathscr{H} \subseteq \mathscr{M} \subseteq \mathscr{F}$*

$$\begin{aligned} e_n^{\mathrm{det}}(\mathscr{S}, \mathscr{M}) &\asymp_{\log} n^{-\tilde{v}_3}, \\ e_n^{\mathrm{ran}}(\mathscr{S}, \mathscr{M}) &\asymp_{\log} n^{-\tilde{v}_4}, \end{aligned}$$

where $\tilde{v}_3$ and $\tilde{v}_4$ are as in (6.209–6.210)*, with $\varrho = 0$. In particular, if $r_0 = r > 0$, $\tilde{v}_3$ and $\tilde{v}_4$ take the form*

$$\tilde{v}_3 = \frac{r}{d_0 + 1}, \quad \tilde{v}_4 = \begin{cases} \frac{r + \frac{1}{2}}{d_0 + 1} & \text{if } \frac{r}{d_0} \ge \frac{1}{2} \\ \frac{r}{d_0} & \text{if } \frac{r}{d_0} < \frac{1}{2}. \end{cases} \tag{6.213}$$

Proof. The upper bounds follow from (6.212) and Corollary 6.7.3. Let us show the lower bounds. We use the notation from the proof of Theorem 6.6.1. For each $\mathscr{Z} = (z_j)_{j=1}^{m^d} \subseteq B_Z$ there is a constant $c_0 > 0$ such that for all $m_0, m \in \mathbb{N}$ and $\psi \in \Psi^0_{m_0,m}(\mathscr{Z})$

$$\begin{aligned} &\|V_0 \psi\|_{C^{[r_0,r,r_2]}(Q_0 \times [a,b] \times Z, Z)} \\ &\le c_0 \max \left\{ m_0^{\alpha_0} m^{\alpha_1} : \alpha_0, \alpha_1 \in \mathbb{N}_0,\ \frac{\alpha_0}{r_0} + \frac{\alpha_1}{r} \le 1 \right\} \\ &= c_0 \max \left\{ \left(m_0^{r_0}\right)^{\frac{\alpha_0}{r_0}} \left(m^r\right)^{\frac{\alpha_1}{r}} : \alpha_0, \alpha_1 \in \mathbb{N}_0,\ \frac{\alpha_0}{r_0} + \frac{\alpha_1}{r} \le 1 \right\} \\ &\le c_0 \max\left(m_0^{r_0}, m^r\right), \end{aligned}$$

where V_0 is defined in (6.179). Setting

$$\Psi_{m_0,m}(\mathscr{Z}) = \min\left(\kappa_0, (b-a)^{-1}\lambda\right) c_0^{-1} \min(m_0^{-r_0}, m^{-r})\, \Psi^0_{m_0,m}(\mathscr{Z}),$$

it follows that

$$\begin{aligned} V_0(\Psi_{m_0,m}(\mathscr{Z})) &\subseteq \mathscr{C}^{[r_0,r,r_2]}_{\mathrm{Lip}}(Q_0 \times [a,b] \times Z, Z; \kappa, L), \\ V_0(\Psi_{m_0,m}(\mathscr{Z})) &\subseteq (b-a)^{-1}\lambda B_{C(Q_0\times[a,b],Z)}, \end{aligned}$$

and therefore, by (6.187) and (6.188), for all $m_0, m \in \mathbb{N}$

$$V_1 \circ V_0\left(\Psi_{m_0,m}(\mathscr{Z})\right) \subseteq \mathscr{H} \subseteq \mathscr{M},$$

where V_1 is defined in (6.180). Now the same argument as used in the proof of Theorem 6.6.1 (with $r_1 = \varrho_1 = \varrho = 0$) gives the lower bounds.

□

As we have already discussed before with regard to $\mathscr{F}$, here, the rates do not depend on the smoothness r_2 of f in the variable z as well. We observe that by (6.213), for $r_0 = r$ and in particular in the isotropic case $r_0 = r_2 = r$, the maximal speedup of randomized algorithms over deterministic ones is $n^{-1/4}$, reached for $d_0 = 1$, $r \geq 1$.

Chapter 7

Parametric Hilbert Space Valued Initial Value Problems

The main goal of this chapter is the study of parameter dependent finite systems of scalar ODEs, that is, $Z = \mathbb{R}^d$ for some $d \in \mathbb{N}$. However, we still consider the more general case $Z = H$, where H is any Hilbert space over the reals. This way we also include infinite systems of scalar ODEs.

In Chapter 6, we only considered classes that are defined on the whole space. Choosing Z as an arbitrary Hilbert space enables us to investigate more general local classes. For this purpose, we use standard localization methods that cannot be generalized to the Banach space valued case due to the non-existence of smooth bump functions, see [15]. Although we study a much larger class of input function, we recover the same rates as before for type 2 Banach spaces.

The chapter is organized as follows: Section 7.1 contains the formulation of the problem and the definition of the considered class of input functions. Convergence rates are derived in Section 7.2, and in Section 7.3, we present the complexity analysis.

7.1 Problem Formulation

As before let $d_0 \in \mathbb{N}$, $Q_0 = [0,1]^{d_0}$. Given $r_0, r \in \mathbb{N}_0$, $0 \le \varrho \le 1$, $\lambda_1, \kappa, L > 0$, and a real Hilbert space H, we consider the class $\mathscr{C}_{\mathrm{Lip}}^{r_0,r,\varrho}(Q_0 \times [a,b] \times \lambda_1 B_H^0, H; \kappa, L)$ as defined in Definition 6.4.1, where B_H^0 denotes the open unit ball of H. Thus, concerning the Definition 6.4.1, we set $Z = \lambda_1 B_H^0$, Z_1 is replaced by H and κ, L are only scalar values yet. We write $\mathcal{C}_{\mathrm{Lip}}^{r;\varrho}([a,b] \times \lambda_1 B_H^0, H; \kappa, L)$ for the subclass of $\mathscr{C}_{\mathrm{Lip}}^{0,r,\varrho}(Q_0 \times [a,b] \times \lambda_1 B_H^0, H; \kappa, L)$ consisting of functions not depending on s.

Given $f \in \mathscr{C}_{\mathrm{Lip}}^{r_0,r,\varrho}(Q_0 \times [a,b] \times \lambda_1 B_H^0, H; \kappa, L)$ and $u_0 \in \lambda_1 B_H^0$, we consider the parameter dependent initial value problem

$$\frac{d}{dt}u(s,t) = f(s,t,u(s,t)) \quad (s \in Q_0, t \in [a,b]), \tag{7.1}$$

$$u(s,a) = u_0(s) \quad (s \in Q_0), \tag{7.2}$$

as in Chapter 6. A function $u : Q_0 \times [a,b] \to H$ is called a solution if, for each $s \in Q_0$, $u(s,t)$ is continuously differentiable as a function of t, $u(s,t) \in \lambda_1 B_H^0$ for all $s \in Q_0, t \in [a,b]$, and (7.1–7.2) are satisfied.

Next we present a modification of $\mathcal{A}_n^m$ from Chapter 6. We have to adjust the definition since, in contrast to the previous cases, the algorithm may not always be defined.

Algorithm 7.1.1. Let $m \in \mathbb{N}_0$, $n \in \mathbb{N}$, and put $h = (b-a)/n$, $t_k = a + kh$ $(k = 0, 1, \ldots, n)$. Furthermore, for $0 \le k \le n-1$ and $1 \le j \le m$, let $P_k^{j,1}$ be, as in (2.6), the operator of Lagrange interpolation of degree j on the equidistant grid $t_{k,j,i} = t_k + ih/j$ $(i = 0, \ldots, j)$ on $[t_k, t_{k+1}]$. Let $\xi_1, \ldots, \xi_n$ be independent random variables on some probability space $(\Omega, \Sigma, \mathbb{P})$ such that ξ_k is uniformly distributed on $[t_{k-1}, t_k]$ and

$$\{(\xi_1(\omega), \ldots, \xi_n(\omega)) : \omega \in \Omega\} = [t_0, t_1] \times \cdots \times [t_{n-1}, t_n].$$

We define $(u_k)_{k=1}^n \subset H$ and H-valued polynomials $p_{k,j}(t)$ for $k = 0, \ldots, n-1$, $j = 0, \ldots, m$ by induction. Let $0 \le k \le n-1$, suppose u_k is already defined and

$$u_k \in \lambda_1 B_H^0. \tag{7.3}$$

Then we put

$$p_{k,0}(t) = u_k + f(t_k, u_k)(t - t_k) \quad (t \in [t_k, t_{k+1}]). \tag{7.4}$$

Furthermore, if $m \ge 1$, $0 \le j < m$, $p_{k,j}$ is already defined, and

$$p_{k,j}(t_{k,j+1,i}) \in \lambda_1 B_H^0 \quad (i = 0, \ldots, j+1), \tag{7.5}$$

then we set

$$q_{k,j} = (f(t_{k,j+1,i}, p_{k,j}(t_{k,j+1,i})))_{i=0}^{j+1} \tag{7.6}$$

and

$$p_{k,j+1}(t) = u_k + \int_{t_k}^{t} \left(P_k^{j+1,1} q_{k,j}\right)(\tau) d\tau. \tag{7.7}$$

Finally, if

$$p_{k,m}(t) \in \lambda_1 B_H^0 \quad (t \in [t_k, t_{k+1}]), \tag{7.8}$$

we set

$$u_{k+1} = p_{k,m}(t_{k+1}) + h\left(f(\xi_{k+1}, p_{k,m}(\xi_{k+1})) - p'_{k,m}(\xi_{k+1})\right). \tag{7.9}$$

We define $v \in B([a,b],H)$ by

$$v(t) = \begin{cases} p_{k,m}(t) & \text{if } \ t \in [t_k, t_{k+1}) \ \text{ and } \ 0 \le k \le n-1, \\ u_n & \text{if } \ t = t_n. \end{cases} \tag{7.10}$$

For $\omega \in \Omega$ fixed, let

$$\mathcal{A}^m_{n,\omega} : C^{r,\varrho}_{\mathrm{Lip}}([a,b] \times \lambda_1 B^0_H, H; \kappa, L) \times H \to B([a,b],H)$$

denote the resulting mapping, that is,

$$\mathcal{A}^m_{n,\omega}(f, u_0) = v. \tag{7.11}$$

We say that $\mathcal{A}^m_{n,\omega}(f,u_0)$ is defined on $[a,b] \times \lambda_1 B^0_H$ (or, shortly: defined) if this definition goes through till (7.10), that is, (7.3),(7.5),(7.8) are satisfied for all $0 \le k \le n-1$, and if $m \ge 1$, for $0 \le j \le m-1$. If any of the conditions (7.3),(7.5),(7.8) is violated for some ω and some k, we leave $\mathcal{A}^m_{n,\omega}(f,u_0)$ undefined. We use the same identifier as in Chapter 6, since both operators coincide if $\mathcal{A}^m_{n,\omega}(f,u_0)$ is defined.

Remember that for $m = 0$, we have

$$p_{k,0}(t) = u_k + f(t_k, u_k)(t - t_k) \qquad (t \in [t_k, t_{k+1}],\ 0 \le k \le n-1), \tag{7.12}$$

$$u_{k+1} = u_k + h f(\xi_{k+1}, p_{k,0}(\xi_{k+1})) \qquad (0 \le k \le n-1). \tag{7.13}$$

Next we define the local version $\mathscr{F}_{\mathrm{loc}}$ of the previously considered class of input function $\mathscr{F}$ from Chapter 6.

Definition 7.1.2. Given also $\sigma > 0$ and $\lambda_0 > 0$ with $\lambda_0 < \lambda_1$, let $\mathscr{F}_{\mathrm{loc}}$ be the class of all

$$(f, u_0) \in \Big(\mathscr{C}^{0,r,\varrho}_{\mathrm{Lip}}(Q_0 \times [a,b] \times \lambda_1 B^0_H, H; \kappa, L) \cap \mathscr{C}^{r_0,r_1,\varrho_1}_{\mathrm{Lip}}(Q_0 \times [a,b] \times \lambda_1 B^0_H, H; \kappa, L)\Big) \times \sigma B_{C^{r_0}(Q_0,H)} \tag{7.14}$$

such that the parameter dependent initial value problem (7.1–7.2) has a solution $u(s,t)$ with

$$\sup_{s \in Q_0,\, t \in [a,b]} \|u(s,t)\| \le \lambda_0, \tag{7.15}$$

and moreover, if $r = \varrho = r_1 = \varrho_1 = 0$, then for all $n \in \mathbb{N}$, $\omega \in \Omega$, $s \in Q_0$, $\mathcal{A}^0_{n,\omega}(f_s, u_0(s))$ is defined on $[a,b] \times \lambda_1 B^0_H$ and

$$\sup_{s \in Q_0} \left\| \mathcal{A}^0_{n,\omega}(f_s, u_0(s)) \right\|_{B([a,b],H)} \le \lambda_0. \tag{7.16}$$

The solution operator

$$\mathscr{S} : \mathscr{F}_{\mathrm{loc}} \to B(Q_0 \times [a,b], H) \tag{7.17}$$

is given for $(f, u_0) \in \mathscr{F}_{\mathrm{loc}}$ by $\mathscr{S}(f, u_0) = u$ as before.

7.2 Multilevel Algorithms and Convergence Analysis

The following multilevel algorithm for the approximate solution of the parametric problem (7.1–7.2) has already been introduced in Chapter 6. We recall it here for the more general algorithm $\mathcal{A}^m_{n,\omega}$.

Algorithm 7.2.1. Let $l_0, l_1 \in \mathbb{N}_0$, $l_0 \le l_1$, $n_{l_0}, \dots, n_{l_1} \in \mathbb{N}$, $\omega \in \Omega$, and set

$$\begin{aligned}\mathscr{A}_\omega(f, u_0) = \mathscr{P}_{l_0} &\left(\left(\mathcal{A}^r_{n_{l_0},\omega}(f_s, u_0(s)) \right)_{s \in \Gamma^{\hat{r_0}, d_0}_{2^{l_0}}} \right) \\ &+ \sum_{l=l_0+1}^{l_1} (\mathscr{P}_l - \mathscr{P}_{l-1}) \left(\left(\mathcal{A}^{r_1}_{n_l,\omega}(f_s, u_0(s)) \right)_{s \in \Gamma^{\hat{r}_0, d_0}_{2^l}} \right). \end{aligned} \tag{7.18}$$

The algorithms

$$\mathcal{A}^r_{n_{l_0},\omega}(f_s, u_0(s)) \qquad (s \in \Gamma^{\hat{r}_0, d_0}_{2^{l_0}}), \tag{7.19}$$

$$\mathcal{A}^{r_1}_{n_l,\omega}(f_s, u_0(s)) \qquad (s \in \Gamma^{\hat{r}_0, d_0}_{2^l}, l_0 < l \le l_1) \tag{7.20}$$

are given by (7.3–7.11). We say that $\mathscr{A}_\omega(f, u_0)$ is defined if the algorithms (7.19) and (7.20) are defined. If $\mathscr{A}_\omega(f, u_0)$ is defined, the definition of $\mathscr{A}_\omega$ coincides with the definition of $\mathscr{A}_\omega$ in Chapter 6, which justifies choosing the same symbol as in the previous chapter. Moreover, as before

$$\operatorname{card}(\mathscr{A}_\omega) \le c \sum_{l=l_0}^{l_1} n_l 2^{d_0 l}. \tag{7.21}$$

Theorem 7.2.2. *Let $r_0, r, r_1 \in \mathbb{N}_0$, $d_0 \in \mathbb{N}$, $0 \le \varrho, \varrho_1 \le 1$, with $r + \varrho \ge r_1 + \varrho_1$, $\kappa, L, \sigma > 0$, and $\lambda_1 > \lambda_0 > 0$. There are constants $c_1, c_2 > 0$ and $\nu_0 \in \mathbb{N}$ such that the following holds. Let H be a Hilbert space and let $\mathscr{F}_{\mathrm{loc}}$ be defined by* (7.14–7.16)*. Then for all $l_0, l_1 \in \mathbb{N}_0$ with $l_0 \le l_1$ and for all $(n_l)_{l=l_0}^{l_1} \subset \mathbb{N}$ with $n_l \ge \nu_0$*

$(l_0 \le l \le l_1)$, $\mathscr{A}_\omega(f, u_0)$ is defined for all $(f, u_0) \in \mathscr{F}_{\text{loc}}$, $\omega \in \Omega$

$$\sup_{(f,u_0)\in\mathscr{F}_{\text{loc}}} \|\mathscr{S}(f, u_0) - \mathscr{A}_\omega(f, u_0)\|_{B(Q_0\times[a,b],H)}$$
$$\le c_1 2^{-r_0 l_1} + c_1 n_{l_0}^{-r-\varrho} + c_1 \sum_{l=l_0+1}^{l_1} 2^{-r_0 l} n_l^{-r_1-\varrho_1} \quad (\omega \in \Omega), \tag{7.22}$$

and for all l^* with $l_0 \le l^* \le l_1$

$$\sup_{(f,u_0)\in\mathscr{F}_{\text{loc}}} \left(\mathbb{E}\|\mathscr{S}(f, u_0) - \mathscr{A}_\omega(f, u_0)\|^2_{B(Q_0\times[a,b],H)}\right)^{\frac{1}{2}}$$
$$\le c_2 2^{-r_0 l_1} + c_2 (l_0+1)^{\frac{1}{2}} n_{l_0}^{-r-\varrho-1/2}$$
$$+ c_2 \sum_{l=l_0+1}^{l^*} (l+1)^{\frac{1}{2}} 2^{-r_0 l} n_l^{-r_1-\varrho_1-\frac{1}{2}} + c_2 \sum_{l=l^*+1}^{l_1} 2^{-r_0 l} n_l^{-r_1-\varrho_1}. \tag{7.23}$$

Proof. Let $\delta_0 = (\lambda_1 - \lambda_0)/4 > 0$ and let ψ be a C^∞ function on $[0, +\infty)$ with

$$\begin{aligned} \psi(\tau) = 1 \quad &\text{if} \quad 0 \le \tau \le (\lambda_0 + 2\delta_0)^2, \\ \psi(\tau) = 0 \quad &\text{if} \quad \tau \ge (\lambda_0 + 3\delta_0)^2. \end{aligned}$$

For

$$f \in \mathscr{C}^{0,r,\varrho}_{\text{Lip}}(Q_0\times[a,b]\times\lambda_1 B^0_H, H; \kappa, L)\cap\mathscr{C}^{r_0,r_1,\varrho_1}_{\text{Lip}}(Q_0\times[a,b]\times\lambda_1 B^0_H, H; \kappa, L) \tag{7.24}$$

we put

$$\tilde{f}(s,t,x) = \begin{cases} f(s,t,x)\psi(\|x\|^2) & \text{if} \quad \|x\| < \lambda_1 \\ 0 & \text{otherwise.} \end{cases}$$

It follows that

$$\tilde{f}(s,t,x) = f(s,t,x) \quad (\|x\| \le \lambda_0 + 2\delta_0). \tag{7.25}$$

Moreover, due to the (infinite) differentiability of the scalar product $(x, x) = \|x\|^2$ there are $\kappa_1, L_1 > 0$ (not depending on H) such that for all f satisfying (7.24)

$$\tilde{f} \in \mathscr{C}^{0,r,\varrho}_{\text{Lip}}(Q_0 \times [a,b] \times H, H; \kappa_1, L_1) \cap \mathscr{C}^{r_0,r_1,\varrho_1}_{\text{Lip}}(Q_0 \times [a,b] \times H, H; \kappa_1, L_1). \tag{7.26}$$

Let $u_0 \in \sigma B_{C^r(Q_0,H)}$ and assume that $(f, u_0) \in \mathscr{F}_{\text{loc}}$. Then, by assumption, the solution $u(s,t)$ of (7.1–7.2) exists and fulfills

$$\sup_{s\in Q_0, t\in[a,b]} \|u(s,t)\| \le \lambda_0. \tag{7.27}$$

Consequently,

$$\frac{d}{dt}u(s,t) = f(s,t,u(s,t)) = \tilde{f}(s,t,u(s,t)) \quad (s \in Q_0,\ t \in [a,b]),$$

which implies

$$\mathscr{S}(\tilde{f}, u_0) = \mathscr{S}(f, u_0). \tag{7.28}$$

Let us denote $h_l = (b-a)/n_l$ and

$$r(l) = \begin{cases} r & \text{if} \quad l = l_0 \\ r_1 & \text{if} \quad l_0 < l \leq l_1. \end{cases} \tag{7.29}$$

Now we show that for $(f, u_0) \in \mathscr{F}_{\text{loc}}$ and $\omega \in \Omega$, algorithm $\mathscr{A}_\omega$ is defined and

$$\mathscr{A}_\omega(\tilde{f}, u_0) = \mathscr{A}_\omega(f, u_0). \tag{7.30}$$

First we consider the case $r + \varrho > 0$. It follows from (7.26) and Theorem 3.2 of [21] that there is a $\nu_1 > 0$ such that for all $l_0 \leq l \leq l_1$, $n_l \geq \nu_1$, $\omega \in \Omega$, $s \in Q_0$

$$\|\mathscr{S}(\tilde{f}_s, u_0(s)) - \mathcal{A}^{r(l)}_{n_l,\omega}(\tilde{f}_s, u_0(s))\|_{B([a,b],H)} \leq \delta_0,$$

hence, by (7.27) and (7.28),

$$\|\mathcal{A}^{r(l)}_{n_l,\omega}(\tilde{f}_s, u_0(s))\|_{B([a,b],H)} \leq \lambda_0 + \delta_0. \tag{7.31}$$

Now we fix l with $l_0 \leq l \leq l_1$, $n_l \geq \nu_1$, $\omega \in \Omega$, $s \in Q_0$. Let $\tilde{u}_k(s)$ $(0 \leq k \leq n_l)$, $\tilde{p}_{k,j}(s, \cdot)$, and $\tilde{q}_{k,j}(s)$ $(0 \leq k \leq n_l - 1,\ 0 \leq j \leq r(l))$ be the sequences arising in the definition (7.3–7.9) of $\mathcal{A}^{r(l)}_{n_l,\omega}(\tilde{f}_s, u_0(s))$, and let $u_k(s)$, $p_{k,j}(s, \cdot)$, and $q_{k,j}(s)$ be the corresponding sequences for $(f_s, u_0(s))$, as far as they are defined on $[a,b] \times \lambda_1 B^0_H$ (see (7.3), (7.5), and (7.8)). By (7.31), for $t \in [t_k, t_{k+1}]$

$$\|\tilde{p}_{k,r(l)}(s,t)\| \leq \lambda_0 + \delta_0,$$

and therefore, also

$$\|\tilde{u}_k(s)\| \leq \lambda_0 + \delta_0. \tag{7.32}$$

By (7.4) and (7.6–7.7), for $0 \leq j \leq r(l)$

$$\|\tilde{p}_{k,j}(s,t) - \tilde{u}_k(s)\| \leq c_0(r(l))\kappa_1 h_l,$$

where

$$c_0(0) = 1, \quad c_0(m) = \max_{1 \leq j \leq m} \|P^{j,1}_k\|_{\mathscr{L}(C([t_k,t_{k+1}],H))} \quad (m \geq 1).$$

Notice that $c_0(m)$ is a constant depending only on m. Together with (7.32) this yields

$$\begin{aligned} \|\tilde{p}_{k,j}(s,t)\| &\leq \lambda_0 + \delta_0 + c_0(r(l))\kappa_1 h_l \\ &\leq \lambda_0 + 2\delta_0 \qquad (t \in [t_k, t_{k+1}],\ 0 \leq j \leq r(l)), \end{aligned} \tag{7.33}$$

provided $n_l \geq \nu_0$, with a suitably chosen $\nu_0 \geq \nu_1$.

We prove that for $0 \leq k \leq n_l$ the following holds:

$$u_k(s) \text{ is defined and } u_k(s) = \tilde{u}_k(s), \tag{7.34}$$

and, if $k \leq n_l - 1$, then for all j with $0 \leq j \leq r(l)$

$$p_{k,j}(s,\,\cdot\,) \text{ is defined and } p_{k,j}(s,\,\cdot\,) = \tilde{p}_{k,j}(s,\,\cdot\,). \tag{7.35}$$

First we show that (7.34) implies (7.35). Suppose (7.34) holds for some $0 \leq k \leq n-1$. We argue by induction over j. Let $j = 0$. By (7.25),(7.32), and (7.34)

$$f(s, t_k, u_k(s)) = f(s, t_k, \tilde{u}_k(s)) = \tilde{f}(s, t_k, \tilde{u}_k(s)),$$

$p_{k,0}(s,\,\cdot\,)$ is defined, and

$$\begin{aligned} p_{k,0}(s,t) &= u_k(s) + f(s, t_k, u_k(s))(t - t_k) \\ &= u_k(s) + \tilde{f}(s, t_k, \tilde{u}_k(s))(t - t_k) = \tilde{p}_{k,0}(s,t). \end{aligned}$$

This is (4.39) for $j = 0$. Next suppose (7.35) holds for some j with $0 \leq j < r(l)$. Then

$$p_{k,j}(s, t_{k,j+1,i}) = \tilde{p}_{k,j}(s, t_{k,j+1,i}) \quad (i = 0, \ldots, j+1), \tag{7.36}$$

and therefore, by (7.33),

$$\|p_{k,j}(s, t_{k,j+1,i})\| \leq \lambda_0 + 2\delta_0. \tag{7.37}$$

It follows that $p_{k,j+1}(s,\,\cdot\,)$ is defined. Using (7.25),(7.36), and (7.37), we get

$$f(s, t_{k,j+1,i}, p_{k,j}(s, t_{k,j+1,i})) = \tilde{f}(s, t_{k,j+1,i}, \tilde{p}_{k,j}(s, t_{k,j+1,i})),$$

therefore we also have $q_{k,j}(s) = \tilde{q}_{k,j}(s)$ and

$$\begin{aligned} p_{k,j+1}(s,t) &= u_k(s) + \int_{t_k}^{t} (P_k^{j+1,1} q_{k,j}(s))(\tau)\,d\tau \\ &= \tilde{u}_k(s) + \int_{t_k}^{t} (P_k^{j+1,1} \tilde{q}_{k,j}(s))(\tau)\,d\tau = \tilde{p}_{k,j+1}(s,t). \end{aligned}$$

This completes the induction over j and the proof that (7.34) implies (7.35).

It remains to show (7.34). We use induction over k. The case $k = 0$ holds by definition. Now we assume that (7.34) and therefore also (7.35) hold for some k with $0 \leq k \leq n-1$. From (7.33) and (7.35), we conclude

$$\|p_{k,r(l)}(s,t)\| = \|\tilde{p}_{k,r(l)}(s,t)\| \leq \lambda_0 + 2\delta_0 \quad (t \in [t_k, t_{k+1}]),$$

which shows that $u_{k+1}(s)$ is defined and

$$\begin{aligned} u_{k+1}(s) &= p_{k,r(l)}(s,t_{k+1}) + h_l\Big(f(s,\xi_{k+1},p_{k,r(l)}(s,\xi_{k+1})) - \frac{\partial p_{k,r(l)}}{\partial t}(s,\xi_{k+1})\Big) \\ &= \tilde{p}_{k,r(l)}(s,t_{k+1}) + h_l\Big(\tilde{f}(s,\xi_{k+1},\tilde{p}_{k,r(l)}(s,\xi_{k+1})) - \frac{\partial \tilde{p}_{k,r(l)}}{\partial t}(s,\xi_{k+1})\Big) \\ &= \tilde{u}_{k+1}(s). \end{aligned}$$

This gives (7.34) for $k+1$, completes the induction over k and the proof of (7.34–7.35). It follows that $\mathcal{A}^{r(l)}_{n_l,\omega}(f_s, u_0(s))$ is defined and

$$\mathcal{A}^{r(l)}_{n_l,\omega}(\tilde{f}_s, u_0(s)) = \mathcal{A}^{r(l)}_{n_l,\omega}(f_s, u_0(s)).$$

Consequently, $\mathscr{A}_\omega(f,u_0)$ is defined and (7.30) holds for $r+\varrho>0$.

In the case $r+\varrho=0$, we have by assumption also $r_1=\varrho_1=0$ and therefore, by (7.29),

$$r(l)=0 \qquad (l_0 \le l \le l_1).$$

By definition of $\mathscr{F}_{\rm loc}$, $\mathcal{A}^0_{n_l,\omega}(f_s,u_0(s))$ is defined for $l_0 \le l \le l_1$ and $s \in Q_0$, so $\mathscr{A}_\omega(f,u_0)$ is defined. Fix l with $l_0 \le l \le l_1$, $n_l \in \mathbb{N}$, $\omega \in \Omega$, $s \in Q_0$. Let $\tilde{u}_k(s)$ and $\tilde{p}_{k,0}(s,\,\cdot\,)$ $(0 \le k \le n_l-1)$ be the resulting sequences from $\mathcal{A}^0_{n_l,\omega}(\tilde{f}_s,u_0(s))$, and $u_k(s)$, $p_{k,0}(s,\,\cdot\,)$ the respective sequences from $\mathcal{A}^0_{n_l,\omega}(f_s,u_0(s))$. Then (7.16) implies

$$\begin{aligned} \|u_k(s)\| &\le \lambda_0 \quad (0 \le k \le n_l), && (7.38)\\ \|p_{k,0}(s,t)\| &\le \lambda_0 \quad (t \in [t_k,t_{k+1}],\ 0 \le k \le n_l-1). && (7.39) \end{aligned}$$

For $0 \le k \le n_l$ the following holds:

$$\begin{aligned} u_k(s) &= \tilde{u}_k(s), && (7.40)\\ p_{k,0}(s,\,\cdot\,) &= \tilde{p}_{k,0}(s,\,\cdot\,) \quad (k \le n_l-1). && (7.41) \end{aligned}$$

This follows readily by induction as above. Indeed, the case $k=0$ of (7.40) is clear, and if (7.40) holds for some k, we get, using (7.12–7.13),(7.25), and (7.38–7.39)

$$\begin{aligned} p_{k,0}(s,t) &= u_k(s) + f(s,t_k,u_k(s))(t-t_k) \\ &= \tilde{u}_k(s) + \tilde{f}(s,t_k,\tilde{u}_k(s))(t-t_k) = \tilde{p}_{k,0}(s,t) \end{aligned}$$

and

$$\begin{aligned} u_{k+1}(s) &= u_k(s) + h_l f(s,\xi_{k+1},p_{k,0}(s,\xi_{k+1})) \\ &= \tilde{u}_k(s) + h_l \tilde{f}(s,\xi_{k+1},\tilde{p}_{k,0}(s,\xi_{k+1})) = \tilde{u}_{k+1}(s). \end{aligned}$$

This shows

$$\mathcal{A}^0_{n_l,\omega}(\tilde{f}_s, u_0(s)) = \mathcal{A}^0_{n_l,\omega}(f_s, u_0(s))$$

and consequently (7.30) for $r + \varrho = 0$. Now the statements follow by combining (7.26),(7.28),(7.30), and Theorem 6.5.4.

□

Remark 7.2.3. From Theorem 7.2.2, we readily conclude that the results of Corollary 6.5.6 and Corollary 6.5.7 immediately carry over to the situation here. We don't repeat them here.

7.3 Complexity Analysis

The parametric initial value problem is given by the tuple

$$(\mathscr{S}, \mathscr{F}_{\text{loc}}, B(Q_0 \times [a, b], H), H, \Lambda_{\text{ivp}}),$$

where Λ_{ivp} is defined in the previous chapter.

We state the main result of this chapter, which settles the complexity of the parametric initial value problem. The rates coincide with these of the previous chapter for Banach spaces with type 2, but here for a larger input class. It also shows the optimality (in the limit cases up to logarithmic factors) of the multilevel algorithm (7.18).

Theorem 7.3.1. *Let $r_0, r, r_1 \in \mathbb{N}_0$, $d_0 \in \mathbb{N}$, $0 \le \varrho, \varrho_1 \le 1$, with $r + \varrho \ge r_1 + \varrho_1$, $\kappa, L, \sigma > 0$, and $\lambda_1 > \lambda_0 > 0$. Let H be a Hilbert space, and let $\mathscr{F}_{\text{loc}}$ be defined by* (7.14–7.16). *Then in the deterministic setting,*

$$\begin{array}{ll} e_n^{\det}(\mathscr{S}, \mathscr{F}_{\text{loc}}) \asymp n^{-\tilde{v}_1} & \text{if } \frac{r_0}{d_0} > \beta_1 \\ n^{-\frac{r_0}{d_0}} \preceq e_n^{\det}(\mathscr{S}, \mathscr{F}_{\text{loc}}) \preceq n^{-\frac{r_0}{d_0}} (\log n)^{\frac{r_0}{d_0}+1} & \text{if } \frac{r_0}{d_0} = \beta_1 > 0 \\ e_n^{\det}(\mathscr{S}, \mathscr{F}_{\text{loc}}) \asymp n^{-\frac{r_0}{d_0}} & \text{if } \frac{r_0}{d_0} = \beta_1 = 0 \vee \frac{r_0}{d_0} < \beta_1, \end{array}$$

where $\beta_1 = r_1 + \varrho_1$ and $\tilde{v}_1$ was defined in (6.153) *by*

$$\tilde{v}_1 = \frac{\frac{r_0}{d_0}}{\frac{r_0}{d_0} + r + \varrho - r_1 - \varrho_1} (r + \varrho) \, .$$

Moreover, in the randomized setting,

$$\begin{aligned}
&e_n^{\mathrm{ran}}(\mathscr{S},\mathscr{F}_{\mathrm{loc}}) \asymp n^{-r-\varrho-\frac{1}{2}} && \text{if } \tfrac{r_0}{d_0} > \beta_1 + \tfrac{1}{2} \wedge r+\varrho = \beta_1\\
&e_n^{\mathrm{ran}}(\mathscr{S},\mathscr{F}_{\mathrm{loc}}) \asymp n^{-\tilde{v}_2}(\log n)^{\frac{1}{2}} && \text{if } \tfrac{r_0}{d_0} > \beta_1 + \tfrac{1}{2} \wedge r+\varrho > \beta_1\\
&n^{-\frac{r_0}{d_0}}(\log n)^{\frac{1}{2}} \preceq e_n^{\mathrm{ran}}(\mathscr{S},\mathscr{F}_{\mathrm{loc}}) \preceq n^{-\frac{r_0}{d_0}}(\log n)^{\frac{r_0}{d_0}+\frac{3}{2}} && \text{if } \tfrac{r_0}{d_0} = \beta_1 + \tfrac{1}{2}\\
&e_n^{\mathrm{ran}}(\mathscr{S},\mathscr{F}_{\mathrm{loc}}) \asymp n^{-\frac{r_0}{d_0}}(\log n)^{\frac{r_0}{d_0}-r_1-\varrho_1} && \text{if } \beta_1 < \tfrac{r_0}{d_0} < \beta_1 + \tfrac{1}{2}\\
&n^{-\frac{r_0}{d_0}} \preceq e_n^{\mathrm{ran}}(\mathscr{S},\mathscr{F}_{\mathrm{loc}}) \preceq n^{-\frac{r_0}{d_0}}(\log\log n)^{\frac{r_0}{d_0}+1} && \text{if } \tfrac{r_0}{d_0} = \beta_1 > 0\\
&e_n^{\mathrm{ran}}(\mathscr{S},\mathscr{F}_{\mathrm{loc}}) \asymp n^{-\frac{r_0}{d_0}} && \text{if } \tfrac{r_0}{d_0} = \beta_1 = 0 \vee \tfrac{r_0}{d_0} < \beta_1,
\end{aligned}$$

where

$$\tilde{v}_2 = \frac{\frac{r_0}{d_0}}{\frac{r_0}{d_0} + r + \varrho - r_1 - \varrho_1}\left(r + \varrho + \frac{1}{2}\right).$$

Proof. The upper bounds follow from Theorem 7.2.2 and Corollary 6.5.6. The lower bounds follow in the same way as in the proof of Theorem 6.6.1, case $p_Z = 2$ and the fact that every Hilbert space is of type 2. We only observe that $\|f\|_{C(Q_0\times[a,b])} \le (b-a)^{-1}\lambda_0$ implies

$$\sup_{s\in Q_0, t\in[a,b]} \|u(s,t)\| \le \lambda_0,$$

and, using (7.12) and (7.13), we conclude that, for all $n \in \mathbb{N}$, $\omega \in \Omega$, $s \in Q_0$, $A^0_{n,\omega}(\tilde{f}_s, 0)$ is defined on $[a,b] \times \lambda_1 B^0_H$ and

$$\sup_{s\in Q_0} \left\| \mathcal{A}^0_{n,\omega}(\tilde{f}_s, 0)\right\|_{B([a,b],H)} \le \lambda_0.$$

□

Symbols

Algorithms

$\mathcal{A}^{0,r}_{n,\omega}$ for definite multivariate integration, page 38

$\mathcal{A}^{1,r}_{n,\omega}$ for indefinite multivariate integration, page 41

$\mathfrak{A}^{0}_{\omega}$ general multilevel algorithm for definite multivariate integration, page 48

$\mathfrak{A}^{1}_{\omega}$ general multilevel algorithm for indefinite multivariate integration, page 48

$\mathscr{A}^{0}_{\omega}$ for parametric definite integration, page 52

$\mathscr{A}^{1}_{\omega}$ for parametric indefinite integration, page 52

$\mathcal{A}^{m}_{n,\omega}$ for initial value problems, page 81

$\mathfrak{A}_{\omega}$ general multilevel algorithm for initial value problems, page 85

$\mathscr{A}_{\omega}$ for parametric initial value problems, page 110

Solution operators

$\mathcal{S}_0$ definite multivariate integration, page 37

$\mathcal{S}_1$ indefinite multivariate integration, page 37

$\mathscr{S}_0$ parametric definite multivariate integration, page 52

$\mathscr{S}_1$ parametric indefinite multivariate integration, page 52

$\mathcal{S}$ initial value problems, page 82

$\mathscr{S}$ parametric initial value problems, page 91

Input sets

$\mathscr{F}_0$ parametric multivariate integration, page 52

$\mathcal{F}^{r,\varrho}$ initial value problems, page 82

$\mathscr{F}$ parametric initial value problems, page 90

$\mathscr{F}_{\mathrm{loc}}$ local class for parametric initial value problems, page 133

Function sets

$\mathcal{C}^{r,\varrho}([a,b] \times Z, Z_1; \kappa)$, page 80

$\mathcal{C}^{r,\varrho}_{\mathrm{Lip}}([a,b] \times Z, Z_1; \kappa, L)$, page 80

$\mathcal{K}$, page 85

$\mathcal{K}_0$, page 86

$\mathscr{C}^{r_0,r,\varrho}(Q_0 \times [a,b] \times Z, Z_1; \kappa)$, page 89

$\mathscr{C}^{r_0,r,\varrho}_{\mathrm{Lip}}(Q_0 \times [a,b] \times Z, Z_1; \kappa, L)$, page 89

$\Psi^{0}_{m_0,m_1}(\mathscr{Z})$, page 65

$\Psi^{\gamma,\gamma_0,\gamma_1}_{m_0,m_1}(\mathscr{Z})$, page 68

Bibliography

[1] H. W. Alt, Lineare Funktionalanalysis, Springer, Berlin, 1985.

[2] N. S. Bakhvalov, On approximate computation of integrals, Vestnik MGU, Ser. Math. Mech. Astron. Phys. Chem. 4 (1959), 3-18.

[3] J. Boman, Supremum norm estimates for partial derivatives of functions of several real variables, Illinois J. Math. 16 (1972), 203-216.

[4] H. Cartan, Cours de calcul différentiel, Hermann, Paris, 1977.

[5] P. G. Ciarlet, The Finite Element Method for Elliptic Problems, North-Holland, Amsterdam, 1978.

[6] A. Defant, K. Floret, Tensor Norms and Operator Ideals, North Holland, Amsterdam, 1993.

[7] K. Deimling, Ordinary Differential Equations in Banach Spaces, Lecture Notes in Mathematics 596, Springer, Berlin, 1977.

[8] Th. Daun, On the randomized solution of initial value problems, J. Complexity 27 (2011), 300–311.

[9] Th. Daun, S. Heinrich, Complexity of Banach space valued and parametric integration, in: Monte Carlo and Quasi-Monte Carlo Methods 2012 (J. Dick, F. Y. Kuo, G. W. Peters, I. H. Sloan, eds.), Springer Proceedings in Mathematics and Statistics 65, pp. 297–316, Berlin, 2013.

[10] Th. Daun, S. Heinrich, Complexity of parametric integration in various smoothness classes, J. Complexity, in press, doi:10.1016/j.jco.2014.04.002.

[11] Th. Daun, S. Heinrich, Complexity of parametric initial value problems in Banach spaces, J. Complexity, in press, doi:10.1016/j.jco.2014.01.002.

[12] Th. Daun, S. Heinrich, Complexity of parametric initial value problems for systems of ODEs, submitted.

[13] H. O. Fattorini, Second Order Linear Differential Equations in Banach Spaces, North Holland Mathematical Studies 108, Amsterdam, 1985.

[14] M. B. Giles, Multilevel monte carlo path simulation, Operations Research 56 (2008), 607–617.

[15] G. Godefroy, Renorming of Banach spaces, in: W. B. Johnson, J. Lindenstrauss, eds., Handbook of the Geometry of Banach Spaces, Vol.1, Elsevier, Amsterdam 2001, 781–835.

[16] M. Hansen, C. Schwab, Sparse adaptive approximation of high dimensional parametric initial value problems, Vietnam J. Math. 41 (2013), 181–215.

[17] S. Heinrich, Monte Carlo complexity of global solution of integral equations, Journal of Complexity 14 (1998), 151-175.

[18] S. Heinrich, The multilevel method of dependent tests, Advances in Stochastic Simulation Methods (N. Balakrishnan, V. B. Melas, S. M. Ermakov, editors), 47–62, Birkhäuser, Boston, Basel, Berlin 2000.

[19] S. Heinrich, Monte Carlo approximation of weakly singular integral operators, Journal of Complexity 22 (2006), 192–219.

[20] S. Heinrich, The randomized information complexity of elliptic PDE, Journal of Complexity 22 (2006), 220–249.

[21] S. Heinrich, Complexity of initial value problems in Banach spaces, J. Math. Phys. Anal. Geom. 9 (2013), 73–101.

[22] S. Heinrich, A. Hinrichs, On the randomized complexity of Banach space valued integration, submitted.

[23] S. Heinrich, B. Milla, The randomized complexity of initial value problems, J. Complexity 24 (2008), 77–88.

[24] S. Heinrich, B. Milla, The randomized complexity of indefinite integration, Journal of Complexity 27 (2011), 352–382.

[25] S. Heinrich, E. Sindambiwe, Monte Carlo complexity of parametric integration, Journal of Complexity 15 (1999), 317-341.

[26] L. Hörmander, The Analysis of Linear Partial Differential Operators, I, Springer, Berlin, 1983.

[27] J. Jacod, P. E. Protter, Probability essentials, Springer, 2003.

[28] B. Kacewicz, How to increase the order to get minimal-error algorithms for systems of ODE, Numer. Math. 45 (1984), 93–104.

[29] B. Kacewicz, Randomized and quantum algorithms yield a speed-up for initial-value problems, J. Complexity 20 (2004), 821–834, see also http://arXiv.org/abs/quant-ph/0311148.

[30] B. Kacewicz, Almost optimal solution of initial-value problems by randomized and quantum algorithms, J. Complexity 22 (2006), 676–690, see also http://arXiv.org/abs/quant-ph/0510045.

[31] S. G. Krein, Linear Differential Equations in Banach Spaces, Amer. Math. Soc., Providence, 1971.

[32] U. Krengel, Einführung in die Wahrscheinlichkeitstheorie und Statistik, Vieweg, 1988.

[33] M. Ledoux, M. Talagrand, Probability in Banach Spaces, Springer-Verlag, Berlin, 1991.

[34] W. A. Light, W. Cheney, Approximation Theory in Tensor Product Spaces, Lecture Notes in Mathematics 1169, Springer-Verlag, Berlin, 1985.

[35] J. Lindenstrauss, L. Tzafriri, Classical Banach Spaces, 1: Sequence Spaces, Springer, Berlin, 1977.

[36] R. H. Martin, Nonlinear Operators and Differential Equations in Banach Spaces, John Wiley, New York, 1976.

[37] B. Maurey, G. Pisier, Series de variables aléatoires vectorielles independantes et propriétés geométriques des espaces de Banach, Studia Mathematica 58, 45-90 (1976).

[38] B. S. Mityagin, On the second mixed derivative, Dokl. Akad. Nauk SSSR 123 (1958), 606-609 (in Russian).

[39] E. Novak, Deterministic and Stochastic Error Bounds in Numerical Analysis, Lecture Notes in Mathematics 1349, Springer-Verlag, Berlin, 1988.

[40] E. Novak, H. Woźniakowski, Tractability of Multivariate Problems, Volumes 1-3, European Math. Soc., Zürich, 2008, 2010, 2012.

[41] S. A. Smolyak, Quadrature and interpolation formulas for tensor products of certain classes of functions (in Russian), Dokl. Akad. Nauk SSSR 148, 1963 1042–1045. English transl.: Soviet. Mat. Dokl. 4, 240–243.

[42] J. F. Traub, G. W. Wasilkowski, and H. Woźniakowski, Information-Based Complexity, Academic Press, New York, 1988.

[43] D. Werner, Funktionalanalysis, Springer, Berlin, 2005.

[44] H. Whitney, Analytic extensions of differentiable functions defined in closed sets, Trans. Amer. Math. Soc. 36 (1934), 63–89.

[45] C. Wiegand, Optimal Monte Carlo and Quantum Algorithms for Parametric Integration, PhD Thesis 2005, University of Kaiserslautern; Shaker Verlag Aachen, 2006.

[46] O. A. Zautykov, K. G. Valeev, Infinite Systems of Differential Equations, Nauka, Alma-Ata, 1974.

[47] E. Zeidler, Applied functional analysis, Springer, New York, 1995.

Wissenschaftlicher Werdegang

2002	Abitur am Kurfürst-Balduin Gymnasium in Münstermaifeld
2003	Studium der Informatik mit Nebenfach Mathematik an der TU Kaiserslautern
2009	Diplom in Informatik an der TU Kaiserslautern Diplomarbeit: "Stochastische numerische Lösungsmethoden für gewöhnliche Differentialgleichungen"
2009	Wissenschaftlicher Mitarbeiter an der TU Kaiserslautern in der Arbeitsgruppe "Numerische Algorithmen in der Informatik"