Worst-Case Portfolio Optimization: Transaction Costs and Bubbles

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Abstract

In this thesis we extend the worst-case modeling approach as first introduced by Hua and Wilmott [48] (option pricing in discrete time) and Korn and Wilmott [71] (portfolio optimization in continuous time) in various directions.

In the continuous-time worst-case portfolio optimization model (as first introduced by Korn and Wilmott [71]), the financial market is assumed to be under the threat of a crash in the sense that the stock price may crash by an unknown fraction at an unknown time. It is assumed that only an upper bound on the size of the crash is known and that the investor prepares for the worst-possible crash scenario. That is, the investor aims to find the strategy maximizing her objective function in the worst-case crash scenario.

In the first part of this thesis, we consider the model of Korn and Wilmott [71] in the presence of proportional transaction costs. First, we treat the problem without crashes and show that the value function is the unique viscosity solution of a dynamic programming equation (DPE) and then construct the optimal strategies. We then consider the problem in the presence of crash threats, derive the corresponding DPE and characterize the value function as the unique viscosity solution of this DPE.

In the last part, we consider the worst-case problem with a random number of crashes by proposing a regime switching model in which each state corresponds to a different crash regime. We interpret each of the crash-threatened regimes of the market as states in which a financial bubble has formed which may lead to a crash. In this model, we prove that the value function is a classical solution of a system of DPEs and derive the optimal strategies.

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Contents

1. Introduction

One of the classical questions addressed in financial mathematics is the following: How should an investor optimally allocate her wealth to maximize her expected utility at some terminal time $T > 0$ in the future? The objective of this thesis is to answer this question for market models in which the risk-bearing assets (which in the sequel will always be assumed to be stocks) are under the threat of crashes. In particular, we focus on investors with an extremely high level of risk aversion towards the impact of these market crashes. To be more precise, we assume that our investor bases her decisions on the worst-possible crash scenario.

This assumption translates to the following optimization approach: For each admissible trading strategy, the investor first determines which crash scenario is the worst in the sense that her performance criterion is minimized. Among all these worst-case scenarios, the investor then picks the strategy which maximizes her performance criterion in the worst-case scenario. Note that the worst-case crash scenario may be different depending on the trading strategy under consideration. Typically, if the investor follows a trading strategy which invests a large fraction of her total wealth in the crash-threatened assets, the worst-case scenario may be an immediate crash of large size, whereas if the investor takes a short position in these assets the worst-case scenario may be no crash at all.

These considerations show that the worst-case optimization approach leads to robust optimal strategies: The optimal strategy protects the investor against the worst-possible outcome, and she may even benefit from a particular crash scenario if it is not the worst-case scenario for the optimal strategy.

In this thesis we consider two types of worst-case portfolio optimization problems. First, we consider an investor facing proportional transaction costs in the case when only a fixed (and known) number of crashes can occur within the investment period. Afterwards, we consider the portfolio problem in the absence of transaction costs, but with an unknown and possibly unbounded number of market crashes. In both cases we consider an investor aiming to maximize her expected utility of terminal wealth.

In what follows we give a brief overview of the existing literature and methods which can be used to tackle these problems. We begin in Section 1.1 with the portfolio optimization problem in the absence of crashes. In Section 1.2 we give an overview of the existing literature on the worst-case approach.

1.1. Portfolio optimization

The objective of this section is to present the literature on and the solutions of the portfolio optimization problem in three particular situations. We first consider the so-called Merton problem in which the investor aims to maximize her expected utility from terminal wealth in a Black-Scholes market. Then we consider an extension of the Merton model which takes transaction costs into account and finally we consider the portfolio optimization problem in a regime-switching model, i.e. a model in which the market parameters differ depending on the state of the economy. These three models are the fundamental market models which we extend in the main body of this thesis by allowing for market crashes.

1.1.1. The Merton problem

Fix a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a filtration $\mathbb{F} = (\mathcal{F}(t))_{t\geq 0}$ satisfying the usual assumptions of completeness and right-continuity. We furthermore assume that $(\Omega, \mathcal{F}, \mathbb{P})$ supports a one-dimensional standard Brownian motion $W = (W(t))_{t\geq 0}$ with respect to $\mathbb{F}$. Let us moreover fix a finite time horizon $T > 0$. In what follows we consider the optimal terminal wealth problem in a Black-Scholes market which was first solved in Merton [82, 83].

We consider a financial market consisting of two assets. One of the assets is assumed to be risk-free (this could e.g. be a default-free bond or a money market account), whereas the other asset is assumed to be risk-bearing (this could e.g. be a stock). We assume that the prices of these two assets, denoted by

$P^0 = (P^0(t))_{t\geq 0}$ and $P^1 = (P^1(t))_{t\geq 0}$, respectively, are given by

$$\begin{aligned} dP^0(t) &= rP^0(t)\,dt, && P^0(0) = 1, \\ dP^1(t) &= \alpha P^1(t)\,dt + \sigma P^1(t)\,dW(t), && P^1(0) = 1. \end{aligned}$$

Here $r \geq 0$ denotes the interest rate of the bond, $\alpha > r$ the drift and $\sigma > 0$ the volatility of the stock. Taking P^0 to be the numéraire we may without loss of generality assume $r = 0$ and we interpret α as the excess return of the stock over the bond.

We let $t \in [0, T)$ denote the beginning of the investment period and we let $x > 0$ be the investor's initial wealth. At any time $u \in [t, T]$ the investor is allowed to choose which fraction $\pi(u)$ of her total wealth $X(u)$ she invests in the stock. We assume that $\pi = (\pi(u))_{u\in[t,T]}$ is a progressively measurable and bounded process. Assuming that π is a self-financing trading strategy, the investor's wealth $X = X^\pi_{t,x} = (X^\pi_{t,x}(u))_{u\in[t,T]}$ evolves as

$$dX(u) = \alpha\pi(u)X(u)\,du + \sigma\pi(u)X(u)\,dW(u), \qquad X(t) = x.$$

We say that a strategy π is admissible for initial time t and initial wealth x if the corresponding wealth process $X^\pi_{t,x}(u)$ is almost surely positive for all $u \in [t, T]$. We denote the set of all admissible strategies of this form by $\mathcal{A}_M(t, x)$. With this, the investor aims to maximize her expected utility of terminal wealth, i.e.

$$\mathcal{V}_M(t, x) := \sup_{\pi\in\mathcal{A}_M(t,x)} \mathbb{E}\left[U_p\left(X^\pi_{t,x}(T)\right)\right],$$

where $U_p : (0, \infty) \to \mathbb{R}$ is given by

$$U_p(x) := \begin{cases} \frac{1}{p}x^p, & \text{if } p < 1, p \neq 0, \\ \log(x), & \text{if } p = 0, \end{cases} \tag{1.1}$$

and extended to $[0, \infty)$ by setting $U_p(0) = \lim_{x\downarrow 0} U_p(x)$. One can then show (see e.g. Merton [83] or Pham [90]) that the optimal strategy is given by

$$\pi_M := \frac{\alpha}{(1-p)\sigma^2}$$

and $\mathcal{V}_M$ is given explicitly as

$$\mathcal{V}_M(t, x) = \frac{1}{p}x^p \exp\left(\frac{1}{2}\frac{p}{(1-p)}\frac{\alpha^2}{\sigma^2}(T-t)\right), \qquad \text{if } p < 1, p \neq 0, \tag{1.2}$$

and

$$\mathcal{V}_M(t,x) = \log(x) + \frac{1}{2}\frac{\alpha^2}{\sigma^2}(T-t), \qquad \text{if } p = 0, \tag{1.3}$$

respectively. We refer to π_M as the Merton fraction and we call $\mathcal{V}_M$ the Merton value function.

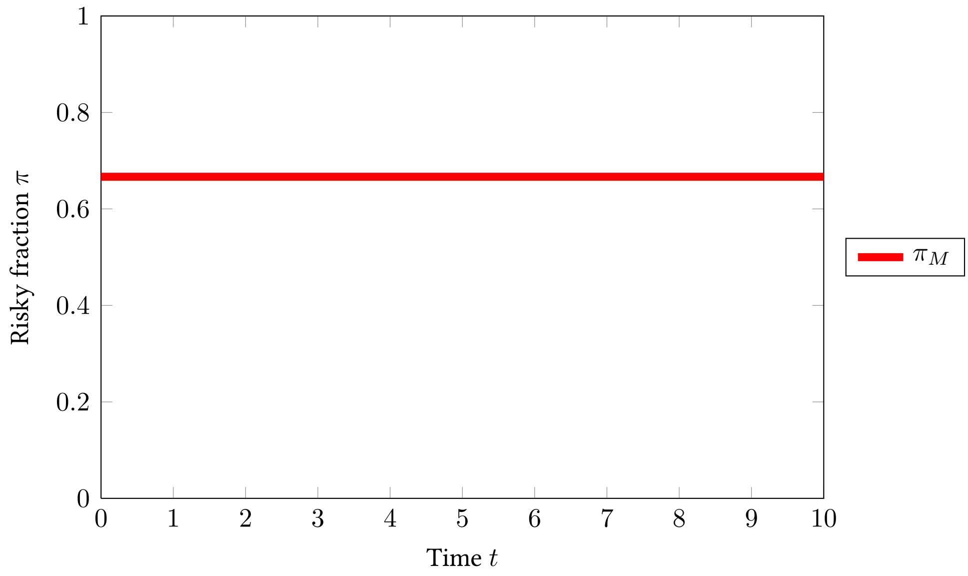

Figure 1.1. The Merton fraction over time.

Figure 1.1 depicts the Merton fraction over time. Note that since π_M is constant, the investor has to trade continuously and the trading volume is of infinite variation on any strictly positive time interval.

1.1.2. Portfolio optimization with transaction costs

In the presence of transaction costs, the strategy π_M is no longer feasible since it would lead to immediate bankruptcy of the investor. It is therefore of interest to consider the Merton problem in the presence of transaction costs. In the literature, one typically finds combinations of the following cost structures:

1. Constant costs: The investor pays a constant fee for each transaction.

2. Management costs: For each transaction, the investor pays a fee proportional to her total wealth.

3. Proportional costs: The investor pays fees proportional to the size of the transaction.

Constant and management costs punish the frequency of trading whereas proportional costs punish the volume of the transaction.

Constant costs (e.g. treated in Eastham and Hastings [35], Korn [62] and Øksendal and Sulem [88], among others) turn out to be the most difficult to handle from a mathematical point of view and results in this direction are still scarce compared to the other cost structures. While management costs appear to be too unrealistic for practical applications, the models are usually more tractable and the resulting optimal strategies resemble the optimal strategies for constant costs. Hence, a lot of effort has been put into the analysis of optimal trading under this cost structure. We refer to Morton and Pliska [85], Irle and Sass [49], Tamura [106] and Korn [64] and the references therein for an overview of the results in this direction.

In this thesis, however, we focus exclusively on proportional transaction costs. Suppose that the investor is endowed with a capital of b units of money in the bond, s units of money in the stock and wants to buy shares of the stock worth Δ units of money. It is then assumed that the investor has to pay a fee of $\lambda\Delta$ for this transaction, where we refer to $\lambda > 0$ as the proportional cost component for buying shares of the stock. After this transaction the investor's wealth hence changes to

$$\bar{b} = b - (1+\lambda)\Delta, \qquad \bar{s} = s + \Delta.$$

In a similar fashion we suppose that the investor has to pay a fee of $\mu\Delta$ whenever she sells shares of the stock worth Δ units of money. We refer to $\mu \in (0,1)$ as the proportional cost component for selling shares of the stock. Under such a transaction the wealth now changes to

$$\bar{b} = b + (1-\mu)\Delta, \qquad \bar{s} = s - \Delta.$$

A detailed description of the model and the formulation of the optimization problem will be given in Chapter 2.

Magill and Constantinides [77] were the first to consider this type of costs in a continuous-time portfolio optimization context. More precisely, the authors considered an optimal consumption problem over an infinite time horizon and heuristically derived insights into the structure of the optimal strategy. Davis and Norman [26] solved the problem rigorously by means of a stochastic control approach and Shreve and Soner [97] solved the same problem under weaker assumptions by employing the theory of viscosity solutions. The model was then extended by Akian et al. [3] and Kabanov and Klüppelberg [56] to a multidimensional setting and by de Vallière and Kabanov [28] to Lévy-driven price processes. Moreover, starting with Kallsen and Muhle-Karbe [59] and followed by Herczegh and Prokaj [45] and Choi et al. [19], the optimal lifetime consumption problem was solved by means of a dual approach.

The infinite-horizon optimal consumption problem does not admit closed form solutions and it is hence of interest to consider a more tractable problem formulation. Dumas and Luciano [32] considered the problem of maximizing the expected utility of terminal wealth in the limit as the time horizon goes to infinity. Similar results for the asymptotic growth rate were obtained by Akian et al. [5], Gerhold et al. [43], and Gerhold et al. [41]. A different approach which also leads to explicit solutions consists of analyzing the problem asymptotically for vanishing transaction costs. This has been done in Janeček and Shreve [50], Gerhold et al. [42], Soner and Touzi [102], Possamaï et al. [91], Kallsen and Muhle-Karbe [60] and Kallsen and Li [58].

It is by now understood that the optimal strategies in these models are so-called constant-boundary strategies. That is, the optimal strategies in these models can be represented by two constants $\underline{\pi}^0 < \overline{\pi}^0$, such that the investor sells shares of the stock whenever her risky fraction is above $\overline{\pi}^0$, buys shares of the stock whenever her risky fraction is below $\underline{\pi}^0$ and refrains from trading whenever her risky fraction is in between $\underline{\pi}^0$ and $\overline{\pi}^0$. In particular, the optimally controlled risky fraction is a diffusion reflected at the boundary of the interval $[\underline{\pi}^0, \overline{\pi}^0]$. For the optimal terminal wealth problem the situation becomes more involved: Since the optimization problem is now time-dependent one expects that the boundaries $\underline{\pi}^0$ and $\overline{\pi}^0$ which characterize the optimal strategy are time-dependent as well. Figure 1.2 below exemplifies the location of buy and sell boundaries in the optimal terminal wealth problem.

The optimal terminal wealth problem in the presence of transaction costs was introduced in Akian et al. [4]. Davis et al. [27] studied the same dynamic programming equation (DPE) in the context of utility indifference pricing. The authors proved in particular that the value function is a viscosity solution of the

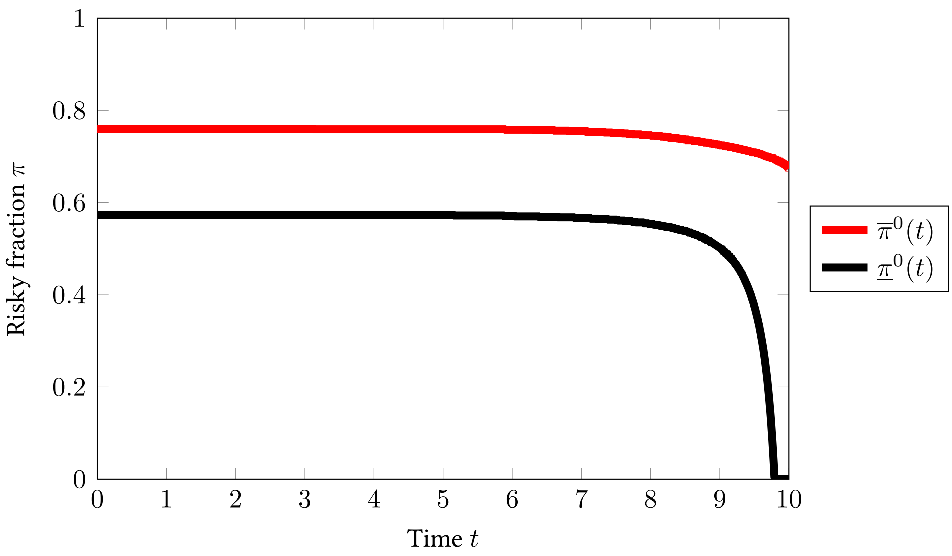

Figure 1.2. Location of the buy and sell boundaries under transaction costs.

DPE and that it is unique within the class of continuous and bounded solutions. Dai and Yi [24] studied the same DPE and showed that there exists a classical solution for terminal conditions corresponding to logarithmic and power utility at least as long as one restricts the state space to strictly positive stock holdings. Dai et al. [23] extended this result to include consumption and Chen et al. [18] considered the case of exponential utility. Moreover, Kunisch and Sass [72], Dai and Zhong [25] and Herzog et al. [46] proposed algorithms to approximate the value function and the optimal strategies numerically. Liu and Loewenstein [74] obtained a closed form solution under the assumption that the terminal time is random and Bichuch [15] studied the finite-horizon problem by means of an asymptotic analysis.

Despite this wealth of papers on the optimal terminal wealth problem there are still some open questions which remain to be solved. In particular, there does not appear to be any proof that the value function for this problem is continuous. This result is needed to ensure that the value function satisfies the DPE in the sense of viscosity solutions (all existing proofs of the viscosity property assume continuity of the value function). Moreover, the uniqueness of solutions of the DPE is only known to hold in the class of bounded and continuous functions, which does not include the case in which the investor's utility function is given by U_p as defined in (1.1). Finally, the existence of an optimal strategy is yet to be established and it needs to be verified that the classical solution of the DPE (constructed in Dai and Yi [24]) coincides with the value function. All of these issues will be addressed in Chapter 2. In Chapter 3 we then extend this model to allow for crashes in the stock price.

1.1.3. Portfolio optimization in regime-switching models

Another possible extension of the Merton problem is to allow for more general coefficients of the bond and stock. In order to capture long-term macroeconomic influences one can make the market coefficients dependent on an exogenously given finite-state Markov process which drives the state of the economy. This approach has been studied in Bäuerle and Rieder [7], Sotomayor and Cadenillas [104] and Escobar et al. [37]. We follow Bäuerle and Rieder [7] in what follows.

We let $Z = (Z(t))_{t\geq 0}$ denote a time-homogeneous finite-state Markov process with state space $E = \{0, \dots, d\}$, $d > 0$. We assume that the transition rate matrix of Z is given by $Q = (q_{i,j})_{0\leq i,j\leq d}$ and we assume that Z is independent of the Brownian motion W. We interpret each state $i \in E$ as a different state of the economy. Moreover, we assume that the coefficients of the bond and stock price, denoted by $P^0 = (P^0(t))_{t\geq 0}$ and $P^1 = (P^1(t))_{t\geq 0}$, respectively, depend on the state of the economy. That is, we assume that the dynamics of P^0 and P^1 are given by

$$\begin{aligned} dP^0(t) &= 0, && P^0(0) = 1, \\ dP^1(t) &= \alpha(Z(t))P^1(t)\,dt + \sigma(Z(t))P^1(t)\,dW(t), && P^1(0) = 1. \end{aligned}$$

Here $\alpha(i) = \alpha_i, \sigma(i) = \sigma_i$ for constants $\alpha_i, \sigma_i > 0$ for all $i \in E$.

The investor is allowed to choose a risky fraction process $\pi^i = (\pi^i(u))_{u\in[t,T]}$ for each state $i \in E$, so that the dynamics of the wealth process $X = X^\pi_{t,x,i} = (X^\pi_{t,x,i}(u))_{u\in[t,T]}$ with $\pi = (\pi^0, \dots, \pi^d)$ can be written as

$$dX(u) = \alpha_j \pi^j(u)X(u)\,du + \sigma_j \pi^j(u)X(u)\,dW(u), \qquad \text{on } \{Z(u) = j\}.$$

Here $X^\pi_{t,x,i}$ denotes the wealth process started at time t with initial wealth x and where the initial state of the economy is $Z(t) = i$. This leads to a value function given by

$$\mathcal{V}_{RS}(t,x,i) := \sup_{\pi \in \mathcal{A}_{RS}(t,x)} \mathbb{E}\left[U_p\left(X^{\pi}_{t,x,i}(T)\right)\right],$$

where the set of admissible strategies $\mathcal{A}_{RS}(t,x)$ consists of all π which are progressively measurable, bounded and $X^{\pi}_{t,x,i}(u) > 0$ almost surely for all $u \in [t,T]$ and all $i \in E$.

Since Z is assumed to be independent of W it is not surprising that the optimal strategy in state i is given by

$$\pi^i_M := \frac{\alpha_i}{(1-p)\sigma_i^2},$$

i.e. in each state the investor applies the Merton strategy which corresponds to the market coefficients in the current state. For $p < 1, p \neq 0$ the value function can be shown to be of the form

$$\mathcal{V}_{RS}(t,x,i) = \frac{1}{p} x^p f_i(t), \tag{1.4}$$

where the family $(f_i)_{i\in E}$ solves the system of ordinary differential equations

$$\frac{\partial}{\partial t} f_i(t) = -\frac{1}{2}\frac{p}{1-p}\frac{\alpha_i^2}{\sigma_i^2} f_i(t) - \sum_{j=0}^{d} q_{i,j} f_j(t), \qquad f_i(T) = 1.$$

In the case of $p = 0$ the value function can be written as

$$\mathcal{V}_{RS}(t,x,i) = \log(x) + f_i(t),$$

where the family $(f_i)_{i\in E}$ solves

$$\frac{\partial}{\partial t} f_i(t) = -\frac{1}{2}\frac{\alpha_i^2}{\sigma_i^2} - \sum_{j=0}^{d} q_{i,j} f_j(t), \qquad f_i(T) = 0.$$

Figure 1.3 visualizes the optimal strategies.

In Chapter 4 we generalize the regime-switching model outlined above. In addition to state-dependent market coefficients we allow for crashes in the stock in the states $i = 1, \ldots, d$, and assume that a crash sends the state of the economy back to state zero. With this, we can interpret the states $i = 1, \ldots, d$ as market states in which a financial bubble is present. Once the bubble bursts (i.e. a crash

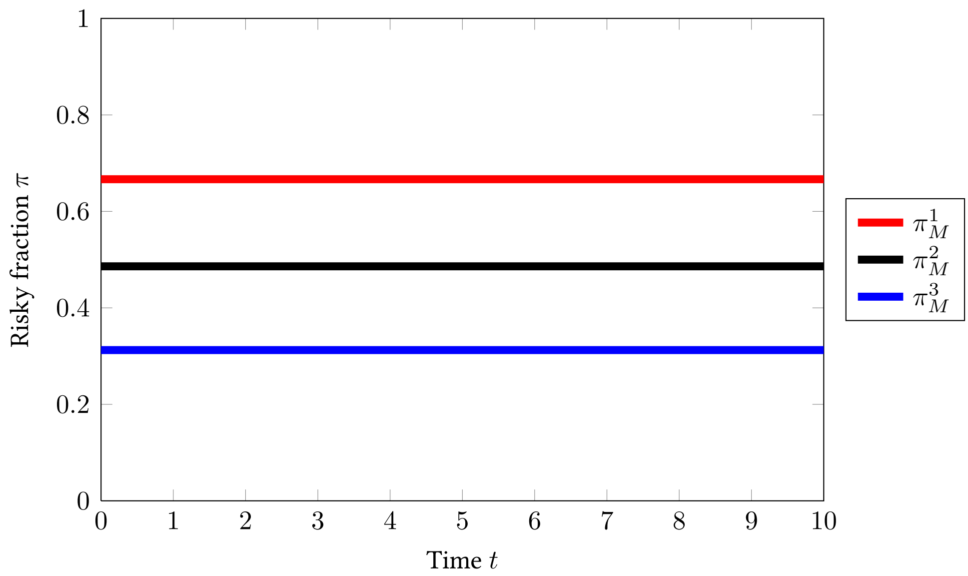

Figure 1.3. The optimal strategies in a regime-switching model.

occurs) the economy jumps back into the bubble-free state $i = 0$. Note that this implies that we may potentially observe an unbounded number of crashes since with each jump of the Markov chain Z a new crash threat arises.

The existence, formation and modeling of financial bubbles has been studied extensively over the last decades. The formation of bubbles on a microeconomic level has been studied e.g. in Tirole [108], Scheinkman and Xiong [94], and Abreu and Brunnermeier [2]. There is also a growing literature on bubbles from a pricing point of view, see e.g. Loewenstein and Willard [76, 75], Cox and Hobson [21], Jarrow et al. [54, 55], Heston et al. [47], Jarrow and Protter [52, 53], Jarrow et al. [51] and Biagini et al. [14].

1.2. Worst-case portfolio optimization

Chapters 3 and 4 are devoted to extending the models introduced in Section 1.1 to allow for crashes in the stock prices. The objective of this section is to introduce the crash model, give an overview of the existing literature and motivate the extensions considered in the following chapters.

The market models in Section 1.1 share one common feature: The stock price process has continuous sample paths and hence these asset-price models are not able to explain extreme jumps in the stock price (in particular large downward jumps, i.e. crashes). One possible remedy is to consider price processes which are driven by Lévy processes (see e.g. Aase [1] or Kallsen [57]) or, more generally, to consider semimartingale models for the prices (see e.g. Goll and Kallsen [44]). Since in these models the distribution of the jump times and sizes is known to the investor, this leads to optimal strategies which hedge the risk coming from the jump possibility on average over the investment period. In particular, at the moment a large downward jump occurs the investor may still suffer substantial losses, especially if the crash occurs just before the end of the investment period at time T.

One way to deal with this effect is to take a robust approach to modeling the jumps of the stock price. That is, one assumes that the investor knows an upper bound on the number and the size of the jumps but the true distribution of the jumps remains unknown. Moreover, the investor is assumed to expect the worst-possible jump scenario to occur. More precisely, given a fixed trading strategy, the worst-case jump scenario for this particular strategy is determined (in the sense that the investor's optimization criterion is minimized) and the investor aims to find the strategy which maximizes the optimization criterion in the worst-case scenario. If the objective is to maximize expected utility of terminal wealth, this leads to a stochastic differential game of the form

$$\sup_{\pi} \inf_{\vartheta} \mathbb{E}\left[U_p(X^{\pi,\vartheta}(T))\right],$$

where π denotes a trading strategy and where ϑ denotes a jump scenario. The optimization problem is hence a version of Wald's maximin model (see Wald [109]) in the sense that the investor first chooses a strategy and presents this strategy to her opponent (which is usually assumed to be the market), who in turn chooses a jump scenario. We refer to Korn and Menkens [66] and Korn and Seifried [69] for an overview of different approaches to this type of problem.

The worst-case portfolio optimization problem outlined above has first been studied in Korn and Wilmott [71] for logarithmic utility. In this paper the authors consider asset price dynamics as in the Merton model (see Section 1.1.1), but additionally assume that the stock price P^1 may drop once by an unknown fraction of $\tilde{\beta}$ at an unknown time τ, i.e.

$$P^1(\tau) = (1 - \tilde{\beta})P^1(\tau-).$$

The optimization problem in this model is formulated as

$$\mathcal{V}_1(t,x) := \sup_{\pi \in \mathcal{A}(t,x)} \inf_{\substack{\tau \in [t,T] \\ \tilde{\beta} \in [0,\beta]}} \mathbb{E}\left[\log\left((1-\pi(\tau)\tilde{\beta})X^{\pi}_{t,x}(T)\right)\right],$$

where $\beta \in (0,1)$ denotes the maximum crash size and where $\mathcal{A}(t,x)$ denotes the set of all trading strategies π for which $(1-\pi(u)\tilde{\beta})X^{\pi}_{t,x}(u) > 0$ for all $\tilde{\beta} \in [0,\beta]$ and all $u \in [t,T]$ (and $X^{\pi}_{t,x}$ is defined as in the Merton model). Note that since the investor's utility function is strictly increasing one can without loss of generality assume that if a crash occurs it is of maximum size β since the worst-case scenario is either a crash of maximum size or no crash at all.

Since at most one crash can occur it is straightforward to argue that after the occurrence of a crash the investor should invest according to the Merton fraction π_M (since after the crash the market coincides with the market in the Merton model). The authors use this insight to derive a trading strategy π^* which renders the investor indifferent between an immediate crash and no crash at all. This can be done by solving the equation

$$\mathbb{E}\left[\log\left(X^{\pi^*}_{t,x}(T)\right)\right] = \mathcal{V}_M(t,(1-\pi^*(t)\beta)x) \tag{1.5}$$

for π^*. Notice that the left-hand side of (1.5) corresponds to the no-crash scenario whereas the right-hand side corresponds to an immediate crash of maximum size. Solving (1.5) leads to the following differential equation for π^*:

$$\begin{aligned} \frac{\partial}{\partial t}\pi^*(t) &= \frac{1}{\beta}(1-\pi^*(t)\beta)\left[-\frac{1}{2}\sigma^2(\pi^*(t)-\pi_M)^2\right], \\ \pi^*(T) &= 0. \end{aligned} \tag{1.6}$$

From here it is easy to argue that the strategy π^* must indeed be optimal: First, since this strategy renders the investor indifferent between no crash at all and a crash of maximum size, it suffices to show that for any other strategy π there exists one crash scenario in which π^* performs better. If this π is such that $\pi(\tau) > \pi^*(\tau)$ for some $\tau \in [t,T]$, the strategy π^* performs better in the crash scenario in which a crash of maximum size β occurs at time τ. On the other hand if π is dominated by π^* the strategy π^* outperforms π in the no-crash scenario and hence π^* must be optimal. We note however that for this argument to be true we require that $\pi^* < \pi_M$ since otherwise π_M outperforms π^* in the no-crash scenario.

Figure 1.4 exemplifies π^* and π_M. Note that π^* is decreasing with decreasing time to maturity $T - t$, i.e. the investor decreases her relative position in the stock as she approaches the investment horizon (to protect against losses due to a crash), but since $\pi^*(t) > 0$ for all $t \in [0, T)$ she will always keep a long position in the stock (i.e. despite the threat of a crash a risky investment still outperforms the pure bond strategy). Only at terminal time T does the investor close her position entirely. It is furthermore possible to show that $\pi^*(0) < \pi_M$, but $\pi^*(0) \to \pi_M$ as $T \to \infty$, i.e. the worst-case investor's position in the stock is strictly less than the Merton investor's position in the stock, but the difference vanishes as the investment period becomes infinitely large.

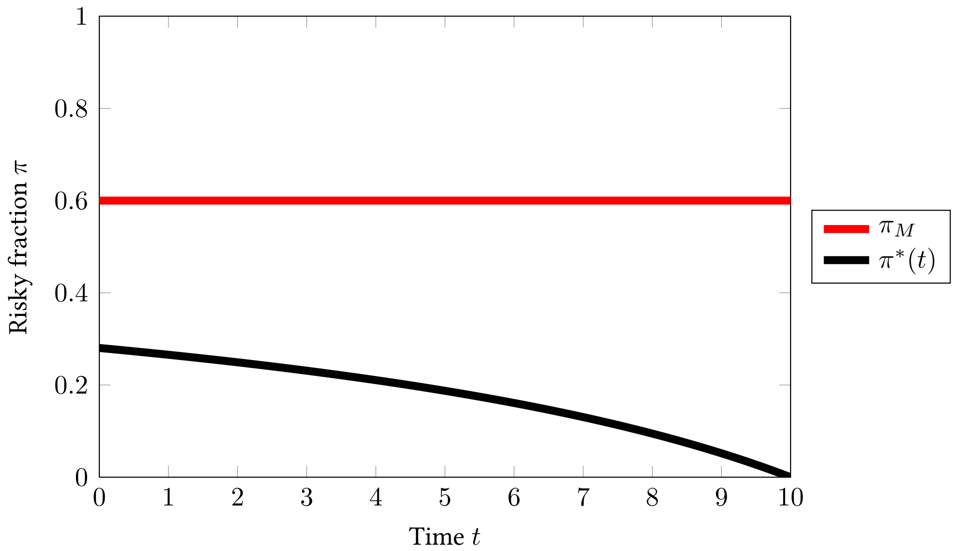

Figure 1.4. The worst-case optimal strategy for logarithmic utility.

It is also possible to extend the results of Korn and Wilmott [71] to $n > 1$ crashes. It can be shown that the optimal strategy $\pi^{n,*}$ in the presence of n crash possibilities is given as the solution of

$$\frac{\partial}{\partial t}\pi^{n,*}(t) = \frac{1}{\beta}(1 - \pi^{n,*}(t)\beta)\left[-\frac{1}{2}\sigma^2(\pi^{n,*}(t) - \pi^{n-1,*}(t))^2\right], \qquad (1.7)$$
$$\pi^{n,*}(T) = 0.$$

Korn and Menkens [67] extend these results to power utility and changing market coefficients after the occurrence of a crash by deriving a dynamic programming equation for the value function and hence embedding the worst-case portfolio problem into the stochastic control framework. The authors show that if the market coefficients do not change after a crash, the strategy π^* solving

$$\frac{\partial}{\partial t}\pi^*(t) = \frac{1}{\beta}(1-\pi^*(t)\beta)\Big[-\frac{1}{2}(1-p)\sigma^2(\pi^*(t)-\pi_M)^2\Big], \tag{1.8}$$

with terminal condition $\pi^*(T) = 0$ is optimal in the class of deterministic strategies. Note that this equation reduces to (1.6) for $p = 0$ and hence covers the logarithmic utility case as well.

If the market coefficients change after the occurrence of a crash the situation becomes slightly more complicated. Assume that the market coefficients before a crash are given by α_1 and σ_1, and by α_0 and σ_0 after the occurrence of a crash. Similarly, we denote by π_M^1 and π_M^0 the Merton fraction with respect to the parameters before and after a crash, respectively. Finally, we denote by

$$\Psi_1 := \frac{1}{2}\frac{\alpha_1^2}{(1-p)\sigma_1^2} \qquad \text{and} \qquad \Psi_0 := \frac{1}{2}\frac{\alpha_0^2}{(1-p)\sigma_0^2}$$

the utility growth potentials in the respective markets (these are the utility-adjusted growth factors of the portfolio under the Merton strategy in the absence of crashes, see (1.2) and (1.3)). With this, it is possible to show that the strategy π^{ind} which is given as the solution of

$$\frac{\partial}{\partial t}\pi^{\text{ind}}(t) = \frac{1}{\beta}(1-\pi^{\text{ind}}(t)\beta)\Big[\Psi_1 - \Psi_0 - \frac{1}{2}(1-p)\sigma^2(\pi^{\text{ind}}(t)-\pi_M^1)^2\Big], \tag{1.9}$$

with $\pi^{\text{ind}}(T) = 0$ renders the investor indifferent between an immediate crash of maximum size and no crash at all. Again, note that (1.9) reduces to (1.8) if the market coefficients do not change after a crash. The strategy π^{ind} obtained like this need not necessarily be optimal. Indeed, if the market coefficients after a crash are strictly better than before a crash (in the sense that $\Psi_1 > \Psi_0$) it may occur that $\pi^{\text{ind}}(t) > \pi_M^1$ for some t, in which case π_M^1 outperforms $\pi^{\text{ind}}(t)$ in the no-crash scenario. Therefore, the optimal pre-crash strategy in this situation is

$$\pi^*(t) = \min\{\pi_M^1, \pi^{\text{ind}}(t)\}, \qquad t \in [0,T].$$

Figure 1.5 illustrates π^*, π^{ind}, π_M^1 and π_M^0. A more detailed discussion of the effects of changing market coefficients and some further extensions of the results in Korn and Menkens [67] can also be found in Menkens [78].

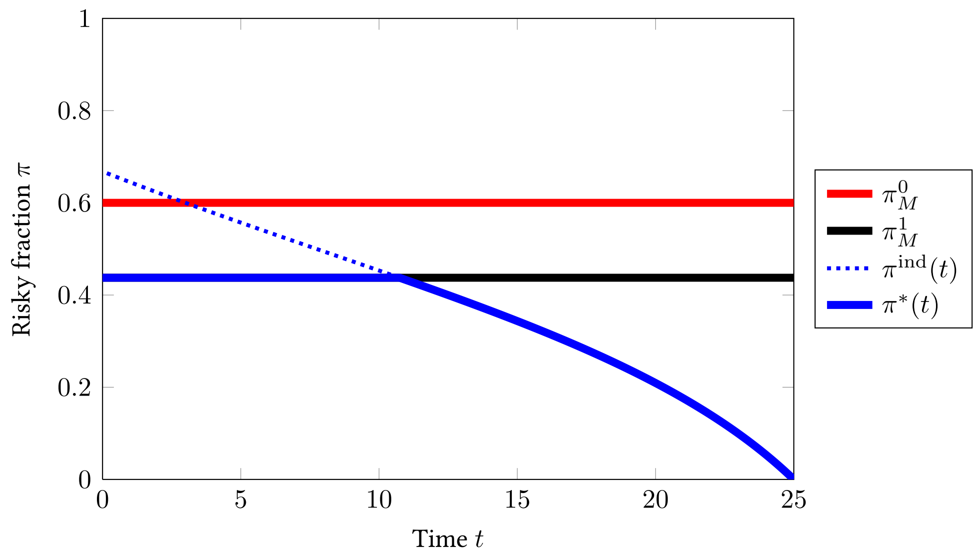

Figure 1.5. The worst-case optimal strategy in the case of changing market coefficients.

Korn and Steffensen [70] consider the situation without changing market coefficients and show that the value function can be found by solving a system of dynamic programming equations as follows. Consider the differential operator $\mathcal{L}^\pi$ defined as

$$\mathcal{L}^\pi := \frac{\partial}{\partial t} + \alpha x \pi \frac{\partial}{\partial x} + \frac{1}{2}\sigma^2 x^2 \pi^2 \frac{\partial^2}{\partial x^2}.$$

Assuming that $\mathcal{V}_1 \in C^{1,2}([0,T) \times (0,\infty))$, we set

$$\begin{aligned} \mathcal{A}'(t,x) &:= \{\pi : \mathcal{L}^\pi \mathcal{V}_1(t,x) \geq 0\}, \\ \mathcal{A}''(t,x) &:= \{\pi : \mathcal{V}_1(t,x) \leq \mathcal{V}_M(t,(1-\pi\beta)x)\}. \end{aligned}$$

Under some technical conditions one can then show that $\mathcal{V}_1$ solves

$$\begin{aligned} 0 = \min\Bigg\{ & \sup_{\pi \in \mathcal{A}''(t,x)} \Big\{\mathcal{L}^\pi \mathcal{V}_1(t,x)\Big\}, \\ & \sup_{\pi \in \mathcal{A}'(t,x)} \Big\{\mathcal{V}_1(t,x) - \mathcal{V}_M(t,(1-\pi\beta)x)\Big\}\Bigg\} \end{aligned} \tag{1.10}$$

and that the strategy obtained from the maximizer of

$$\sup_{\pi \in \mathcal{A}''(t,x)} \Big\{\mathcal{L}^\pi \mathcal{V}_1(t,x)\Big\}$$

is pre-crash optimal. With this, it is possible to show that the optimal strategy obtained by Korn and Menkens [67] in the power utility case (see (1.8)) is not only optimal in the class of deterministic strategies, but also in the bigger class of right-continuous strategies with existing moments of all orders.

Seifried [95] generalizes the existing results by recasting the worst-case portfolio problem as a controller-vs-stopper game. The idea is to find a strategy π^{ind} which turns the process

$$\left(\mathcal{V}_M(t,(1-\pi^{\text{ind}}(t)\beta)X^{\pi^{\text{ind}}}(t))\right)_{t\in[0,T]}$$

into a martingale and then show that this is sufficient for π^{ind} to be an indifference strategy. Moreover, it is possible to show that no strategy which exceeds π^{ind} at some point in time can be optimal. Hence, by the indifference property of π^{ind} it suffices to find the strategy which performs best in the no-crash scenario among all strategies which are bounded from above by π^{ind}. The optimal strategy is then found to be $\pi^*(t) = \min\{\pi_M^1, \pi^{\text{ind}}(t)\}$ as before. We note that the approach in Seifried [95] extends to multidimensional jump-diffusion models for the asset prices.

The worst-case portfolio problem has also been considered in other situations: Hua and Wilmott [48] considered worst-case option pricing in a discrete-time setting. Menkens [79, 80] analyzed the properties of indifference strategies in more detail and Menkens [81] considered the worst-case problem given the probability of the crash. Korn [65] and Korn et al. [68] applied the worst-case modeling approach in an insurance context. Desmettre et al. [30] analyzed the robustness of the problem with respect to the choice of the maximum crash size. Desmettre et al. [31] considered the case of optimal consumption over an infinite time horizon and Mönnig [84] considered a more abstract combined stochastic control and impulse control game as well as a stochastic target problem.

1.3. Outline of this thesis

Let us conclude this introduction with an outlook on the topics covered in this thesis. First, we consider the optimal terminal wealth problem under transaction costs and then we extend this model to allow for crashes. Finally, we consider the regime-switching model in the presence of crash threats.

In the first part of this thesis we consider the optimal terminal wealth problem under transaction costs in a crash-free setting. As pointed out in Section 1.1.2, despite the wealth of papers on this model there are still some open problems which have not been addressed yet. The objective of Chapter 2 is to provide the missing results. To be more precise, we show that there exists at most one solution of the dynamic programming equation and we construct the optimal strategy. From the results in Chapter 3 it furthermore follows that the value function is continuous, the dynamic programming principle holds and that the value function is a viscosity solution of the dynamic programming equation. Finally, we show that the classical solution constructed in Dai and Yi [24] coincides with the value function on the domain on which the classical solution is defined.

In Chapter 3 we extend the transaction costs model to allow for crashes in the stock price. Our aim is to show that the value function is the unique viscosity solution of the corresponding dynamic programming equation and then to rely on numerical methods to approximate the optimal strategies. We proceed as follows: First, we show that the value function in this model is continuous and use this to prove a version of the dynamic programming principle. The dynamic programming principle in turn allows us to show that the value function is a viscosity solution of the dynamic programming equation and uniqueness follows from a straightforward extension of the uniqueness result in the crash-free model.

In Chapter 4 we turn our focus to a regime-switching model with crashes, hence extending the model in Section 1.1.3. We first consider a simplified model in which there are only two regimes: One in which crashes are possible and one crash-free regime. We assume that the market switches from the crash-free to the crash-threatened regime at exponential stopping times, whereas the market switches from the crash-threatened to the crash-free regime only after the occurrence of a crash. Classical indifference arguments allow us to compute and verify the optimal strategies directly. We then turn to a more general setting in which the switching of regimes occurs at jump times of a continuous time Markov chain. We derive a system of dynamic programming equations in the spirit of Korn and Steffensen [70] which allows us to construct and verify the optimal strategies as a system of ordinary differential equations. Finally, we analyze some of the features of the optimal strategies numerically.

2. Portfolio optimization with transaction costs

In this chapter we study the problem of maximizing expected utility of the liquidation value of terminal wealth in the presence of proportional transaction costs. The results obtained here build the basis for studying the same problem in the presence of market crashes. As can be seen from the discussions in Section 1.2, in order to analyze a financial market under the threat of crashes from a worst-case perspective, it is crucial to have a good understanding of the corresponding market in the absence of crashes.

The problem considered in this chapter has received considerable interest over the last decades (cf. Section 1.1.2). Nevertheless, a careful inspection of the existing literature shows that not all aspects have yet been treated in sufficient generality so that they can be readily applied for our subsequent worst-case analysis. Our objective is hence to extend some of the known results and, in addition, provide some new previously unknown results.

The first main result of this chapter is a comparison principle for the dynamic programming equation which allows us to prove uniqueness of the value function. From the literature it is known that the value function is the unique continuous viscosity solution of the dynamic programming equation as long as the investor's utility function is bounded (see Davis et al. [27]) and a straightforward adaptation of the results for the infinite-horizon optimal consumption problem can be used to show that uniqueness holds also if the absolute value of the utility function is bounded by $C(1 + |x|^p)$ with $p \in (0, 1)$ and $C > 0$ (cf. Akian et al. [3] and Kabanov and Klüppelberg [56]), which excludes e.g. logarithmic utility and power utility with a negative power. While the value function corresponding to logarithmic utility and negative power utility has a nice behavior at infinity, it tends to negative infinity at the boundary of the state space. We adapt an idea from Soner and Vukelja [103] to deal with this problem.

In the second part of this chapter we construct and verify the optimal strategies. While Dai and Yi [24] show that there exists a classical solution of the dynamic programming equation if the state space is reduced to positive holdings in the stock, it is still an open question whether this classical solution coincides with the value function. The main problem of the proof is the existence of the optimal strategies which are expected to be finite-variation processes which turn the optimally controlled asset holdings process into an obliquely reflected diffusion at the boundary of some time-dependent region within the state space. We provide a simple method to construct these strategies and then verify their optimality. In particular this result shows that the classical solution of the dynamic programming equation obtained by Dai and Yi [24] coincides with the value function on the reduced state space. Moreover, we study the regularity of the value function in more detail.

The results of this chapter correspond in large parts to the following preprint and working paper:

1. C. Belak, O. Menkens, J. Sass (2013): On the uniqueness of unbounded viscosity solutions arising in an optimal terminal wealth problem with transaction costs [11].

2. C. Belak, J. Sass (2014): Finite-horizon optimal investment with transaction costs: Construction of the optimal strategies [13].

2.1. The market model and problem formulation

We assume that $W = (W(t))_{t\geq 0}$ is a standard Brownian motion defined on the canonical Wiener space $(\Omega, \mathcal{F}, \mathbb{P})$, where $\Omega = C_0([0,\infty))$ denotes the set of continuous functions $\omega : [0,\infty) \to \mathbb{R}$ satisfying $\omega(0) = 0$ and where $\mathbb{P}$ denotes the Wiener measure. We denote the augmented filtration generated by W by $\mathbb{F} = \mathbb{F}^0 = (\mathcal{F}(t))_{t\geq 0}$ and similarly, given $t > 0$, we denote by $\mathbb{F}^t = (\mathcal{F}^t(u))_{u\geq t}$ the augmented filtration generated by $(W(u) - W(t))_{u\geq t}$. Moreover, we fix some terminal time $T > 0$ as well as some initial time $t \in [0, T)$.

We consider a market consisting of two assets, namely a risk-free asset (called bond) with price process $P^0 = (P^0(u))_{u\in[t,T]}$ and a risk-bearing asset (called stock) with price process $P^1 = (P^1(u))_{u\in[t,T]}$. We assume that the prices of the two assets evolve as

$$\begin{aligned} dP^0(u) &= 0, && u \in [t,T], && P^0(t) = 1, \\ dP^1(u) &= \alpha P^1(u)du + \sigma P^1(u)dW(u), && u \in [t,T], && P^1(t) = 1. \end{aligned}$$

We refer to $\alpha > 0$ as the excess return and $\sigma > 0$ as the volatility of the stock.

We assume that the investor faces proportional transaction costs. That is, we assume that the investor buys shares of the stock at the ask price $(1+\lambda)P^1$ where $\lambda > 0$ denotes the proportional costs for purchases of the stock. Similarly, we assume that the stock is sold at the bid price $(1-\mu)P^0$, where $\mu \in (0,1)$ denotes the proportional costs for sales of the stock.

To model trading strategies in the presence of the proportional transaction costs (λ, μ) we take $L = (L(u))_{u\in[t,T]}$ and $M = (M(u))_{u\in[t,T]}$ to be two $\mathbb{F}^t$-adapted, non-decreasing, càdlàg processes such that $L(t-) = M(t-) = 0$. We assume that L and M represent the cumulative units of money used for purchases and sales of the stock, respectively. Let us denote by $B = B^{L,M}_{t,b} = (B^{L,M}_{t,b}(u))_{u\in[t,T]}$ the investor's wealth invested in the bond and let $S = S^{L,M}_{t,s} = (S^{L,M}_{t,s}(u))_{u\in[t,T]}$ denote the investor's wealth invested in the stock. Under a self-financing condition on the trading strategy (L, M) the evolution of B and S can be written as

$$dB(u) = -(1+\lambda)dL(u) + (1-\mu)dM(u), \qquad u \in [t,T], \tag{2.1}$$

$$dS(u) = \alpha S(u)du + \sigma S(u)dW(u) + dL(u) - dM(u), \qquad u \in [t,T], \tag{2.2}$$

where the initial values are given by $B(t-) = b$ and $S(t-) = s$, respectively. The net wealth $X = X^{L,M}_{t,b,s} = (X^{L,M}_{t,b,s}(u))_{u\in[t,T]}$ of the investor after liquidation of the stock position is then given by

$$X(u) := \begin{cases} B(u) + (1-\mu)S(u), & \text{if } S(u) > 0, \\ B(u) + (1+\lambda)S(u), & \text{if } S(u) \leq 0, \end{cases} \qquad u \in [t,T].$$

We say that such a trading strategy is admissible if it leads to a non-negative net wealth. For this, we define the following solvency cone:

$$\mathcal{S}^0 := \left\{ (b,s) \in \mathbb{R}^2 \,\middle|\, b + (1+\lambda)s > 0, b + (1-\mu)s > 0 \right\}.$$

So, whenever $(B, S) \in \overline{\mathcal{S}}^0$ the investor can liquidate her stock holdings to end up with non-negative wealth. A trading strategy (L, M) is called admissible for an initial position $(b, s) \in \overline{\mathcal{S}}^0$ if the corresponding pair $(B^{L,M}_{t,b}, S^{L,M}_{t,s})$ takes values in $\overline{\mathcal{S}}^0$ for all $u \in [t, T]$. The set of all admissible trading strategies of this form is denoted by $\mathcal{A}_0(t, b, s)$. Figure 2.1 provides a sketch of the solvency cone.

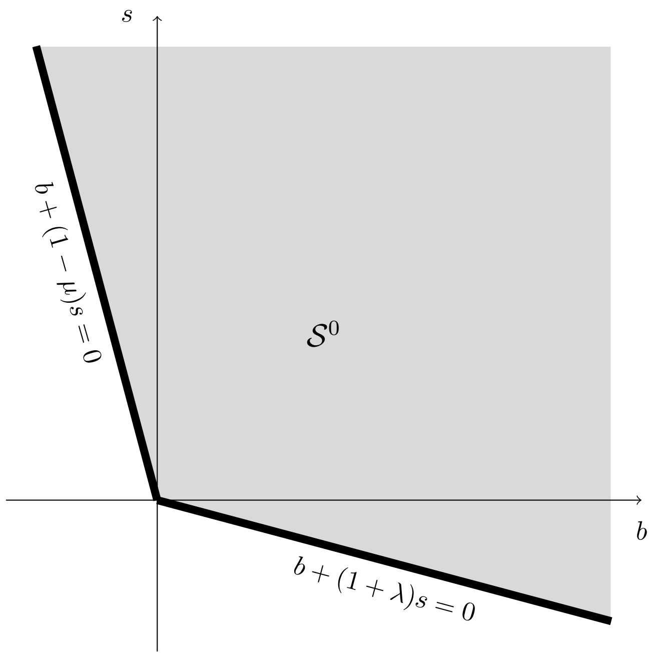

Figure 2.1. A sketch of the solvency cone $\mathcal{S}^0$.

With this, the objective of the investor is to maximize the expected utility of the liquidation value of terminal wealth, i.e. she faces the optimization problem

$$\mathcal{V}_0(t, b, s) := \sup_{(L,M) \in \mathcal{A}_0(t,b,s)} \mathbb{E}\left[U_p\left(X^{L,M}_{t,b,s}(T) \right) \right], \tag{2.3}$$

where the utility function U_p is defined in (1.1) for $p < 1$ (i.e. we restrict our attention to power utility and logarithmic utility).

2.2. Heuristics

Before we solve the optimization problem (2.3), let us first build up some intuition by treating the problem heuristically.

We first note that the investor faces a balancing problem: On the one hand, one expects that the investor would like her risky fraction to be close to the Merton fraction since this is the optimal strategy in the absence of costs. On the other hand, the transaction costs punish large transactions so that the investor cannot keep her risky fraction constant (such a strategy results in immediate bankruptcy since the trading volume is of infinite variation within any strictly positive time period). It is therefore reasonable to expect that the investor refrains from trading as long as her risky fraction is close to the Merton fraction and that she makes "minimal" transactions to keep the risky fraction from moving too far away from the Merton fraction.

In addition, the investor has to take into account the finite time horizon. Since the investor has to liquidate her stock holdings at terminal time T she should decrease her stock holdings before the investment period comes to an end as to bound the fees upon liquidation. In a similar fashion, buying shares of the stock becomes less and less desirable as the investment horizon approaches since there is no longer enough time left to recoup the losses due to transaction fees incurred during purchases. This leads to strategies which look like the strategy $\pi(t)$ illustrated in Figure 2.2.

To formalize this mathematically, let us recall that the martingale optimality principle of optimal stochastic control (see Korn [63]) suggests that the process

$$\Big(\mathcal{V}_0(u, B_{t,b}^{L,M}(u), S_{t,s}^{L,M}(u))\Big)_{u\in[t,T]}$$

is a supermartingale for every $(L, M) \in \mathcal{A}_0(t, b, s)$ and a martingale for the optimal (L^*, M^*). In order to find $\mathcal{V}_0$ and (L^*, M^*), let us therefore take an arbitrary $(L, M) \in \mathcal{A}_0(t, b, s)$ and assume that $\mathcal{V}_0 \in C^{1,2,2}([0, T] \times \overline{\mathcal{S}})$. Let us furthermore assume for simplicity that L and M are continuous (we can do so since we expect optimal trading to be "minimal"). By Itô's formula we then obtain

$$\mathcal{V}_0(u, B(u), S(u)) = \mathcal{V}_0(t, b, s) - \int_t^u \mathcal{L}^{nt}\mathcal{V}_0(r, B(r), S(r))\, dr$$

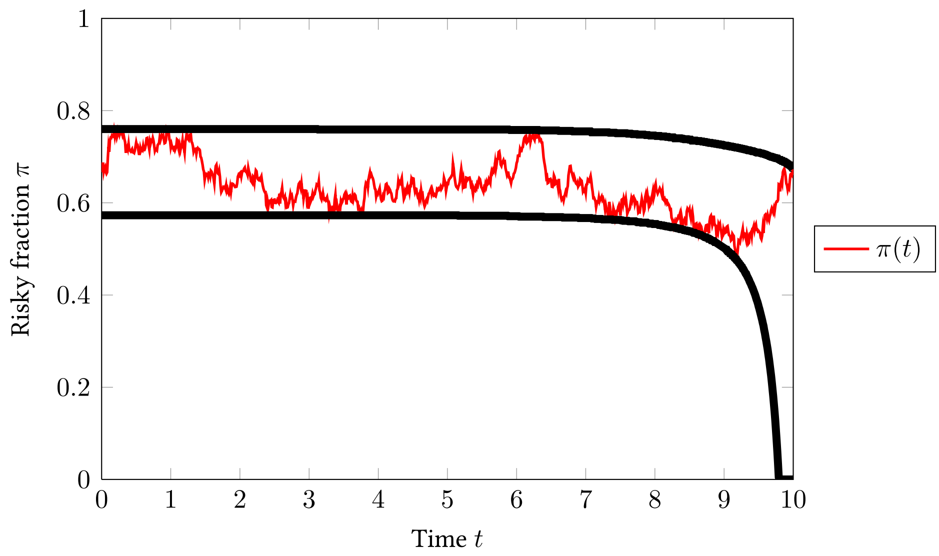

Figure 2.2. Optimal trading under transaction costs.

$$
\begin{aligned}
&- \int_t^u \mathcal{L}^{buy} \mathcal{V}_0(r, B(r), S(r))\, dL(r) \\
&\quad - \int_t^u \mathcal{L}^{sell} \mathcal{V}_0(r, B(r), S(r))\, dM(r) \\
&\quad\quad + \int_t^u \sigma S(r) \frac{\partial}{\partial s} \mathcal{V}_0(r, B(r), S(r))\, dW(r),
\end{aligned}
$$

where the differential operators $\mathcal{L}^{nt}$, $\mathcal{L}^{buy}$ and $\mathcal{L}^{sell}$ are given by

$$
\mathcal{L}^{nt} = -\frac{\partial}{\partial t} - \alpha s \frac{\partial}{\partial s} - \frac{1}{2}\sigma^2 s^2 \frac{\partial^2}{\partial s^2}, \tag{2.4}
$$

$$
\mathcal{L}^{buy} = (1+\lambda)\frac{\partial}{\partial b} - \frac{\partial}{\partial s}, \tag{2.5}
$$

$$
\mathcal{L}^{sell} = -(1-\mu)\frac{\partial}{\partial b} + \frac{\partial}{\partial s}, \tag{2.6}
$$

respectively. Assuming that the stochastic integral process

$$
\left(\int_t^u \sigma S(r) \frac{\partial}{\partial s} \mathcal{V}_0(r, B(r), S(r))\, dW(r) \right)_{u \in [t,T]}
$$

is a martingale, we hence need that

$$
\mathcal{L}^{nt} \mathcal{V}_0(t, b, s) \geq 0, \quad \mathcal{L}^{buy} \mathcal{V}_0(t, b, s) \geq 0, \quad \mathcal{L}^{sell} \mathcal{V}_0(t, b, s) \geq 0,
$$

for all $(t,b,s) \in [0,T)\times\mathcal{S}^0$ in order for $\mathcal{V}_0(u,B(u),S(u))$ to be a supermartingale for every (L,M).

For the optimal strategy (L^*,M^*) we expect $\mathcal{V}_0(u,B(u),S(u))$ to be an honest martingale. For this to be true we need that the two integrals

$$\int_t^u \mathcal{L}^{buy}\mathcal{V}_0(r,B(r),S(r))\,dL^*(r) \quad \text{and} \quad \int_t^u \mathcal{L}^{sell}\mathcal{V}_0(r,B(r),S(r))\,dM^*(r)$$

vanish. This is certainly the case if $dL^*(r) > 0$ only if $\mathcal{L}^{buy}\mathcal{V}_0(r,B(r),S(r)) = 0$ and similarly $dM(r) > 0$ only if $\mathcal{L}^{sell}\mathcal{V}_0(r,B(r),S(r)) = 0$. Moreover, for the integral

$$\int_t^u \mathcal{L}^{nt}\mathcal{V}_0(r,B(r),S(r))\,dr$$

to disappear we need that $\mathcal{L}^{nt}\mathcal{V}_0(r,B(r),S(r)) = 0$ along the paths of the optimally controlled $(r,B(r),S(r))$. This together with the discussion at the beginning of this section suggests that the state space $[0,T)\times\mathcal{S}^0$ can be partitioned into three regions

$$\mathcal{R}_0^{nt} := \left\{(t,b,s)\in[0,T)\times\mathcal{S}^0 : \mathcal{L}^{nt}\mathcal{V}_0(t,b,s) = 0\right\}, \tag{2.7}$$
$$\mathcal{R}_0^{buy} := \left\{(t,b,s)\in[0,T)\times\mathcal{S}^0 : \mathcal{L}^{buy}\mathcal{V}_0(t,b,s) = 0\right\}, \tag{2.8}$$
$$\mathcal{R}_0^{sell} := \left\{(t,b,s)\in[0,T)\times\mathcal{S}^0 : \mathcal{L}^{sell}\mathcal{V}_0(t,b,s) = 0\right\}, \tag{2.9}$$

and that it is optimal to buy shares of the stock in $\mathcal{R}_0^{buy}$, sell shares of the stock in $\mathcal{R}_0^{sell}$ and refrain from trading in $\mathcal{R}_0^{nt}$. Moreover, the value function $\mathcal{V}_0$ can be determined by solving the dynamic programming equation

$$0 = \min\left\{\mathcal{L}^{nt}\mathcal{V}_0(t,b,s), \mathcal{L}^{buy}\mathcal{V}_0(t,b,s), \mathcal{L}^{sell}\mathcal{V}_0(t,b,s)\right\} \tag{2.10}$$

with suitable boundary conditions. Indeed, Proposition 2.4 and Corollary 2.6 show that $\mathcal{V}_0$ is the unique viscosity solution of the DPE (2.10).

Consider the strategy $L \equiv M \equiv 0$ and denote the corresponding wealth process by (B^0,S^0). For every classical supersolution φ of the DPE (2.10) we have $\mathcal{L}^{nt}\varphi(t,b,s) \geq 0$, and Itô's formula together with a suitable localization procedure shows that

$$\varphi(t,b,s) \geq \mathbb{E}\left[\varphi(\tau, B^0_{t,b}(\tau), S^0_{t,s}(\tau))\right]$$

for any stopping time $\tau \in [t,T]$. That is, φ is space-time superharmonic with respect to the uncontrolled wealth process (B^0,S^0). Moreover, the inequalities

$\mathcal{L}^{buy}\varphi(t,b,s) \geq 0$ and $\mathcal{L}^{sell}\varphi(t,b,s) \geq 0$ show that for every $l, m \geq 0$ we have

$$\varphi(t,b,s) \geq \varphi(t, b-(1+\lambda)l + (1-\mu)m, s+l-m),$$

i.e. φ is non-increasing in the direction of transactions. Indeed, by the gradient theorem we have

$$\begin{aligned}\varphi(t,b,s) &- \varphi(t, b-(1+\lambda)l + (1-\mu)m, s+l-m) \\ &= \int_0^l \mathcal{L}^{buy}\varphi(t, b-(1+\lambda)x, s+x)\,dx \\ &\quad + \int_0^m \mathcal{L}^{sell}\varphi(t, b-(1+\lambda)l+(1-\mu)x, s+l-x)\,dx \\ &\geq 0. \end{aligned} \tag{2.11}$$

Since $\mathcal{V}_0$ is a viscosity solution of the DPE we expect that the same properties hold for $\mathcal{V}_0$ as well and hence the comparison theorem for the DPE (Theorem 2.5) suggests that $\mathcal{V}_0$ is the smallest superharmonic function which is non-increasing in the direction of transactions. We make use of this idea in Section 2.6 to verify the optimality of the candidate optimal strategy.

2.3. Some preliminary properties

The aim of this section is to gather some preliminary properties of $\mathcal{V}_0$ and the DPE (2.10). We start by constructing a parametrized family of smooth functions which dominate $\mathcal{V}_0$. For this, recall that $p < 1$ denotes the parameter associated with the utility function U_p, fix constants $K \geq 1$ and $\gamma \in [1-\mu, 1+\lambda]$ and define a function $\varphi_{\gamma,p,K} : [0,T] \times \overline{\mathcal{S}}^0 \to [0,\infty)$ by

$$\varphi_{\gamma,p,K}(t,b,s) := U_p\left((b+\gamma s) f_{p,K}(t)\right) \tag{2.12}$$

with $f_{p,K} : [0,T] \to \mathbb{R}_+$ given by

$$f_{p,K}(t) := \exp\left(K \frac{1}{2(1-p)} \frac{\alpha^2}{\sigma^2}(T-t)\right).$$

Note that $\varphi_{\gamma,p,1}(t,b,s) = \mathcal{V}_M(t, b+\gamma s)$, where $\mathcal{V}_M$ is the Merton value function defined in (1.2) and (1.3). Hence, we can expect $\varphi_{\gamma,p,K} \geq \mathcal{V}_0$. Indeed, the next lemma shows that $\varphi_{\gamma,p,K}$ is a supersolution of the DPE (2.10) and a classical verification argument shows that $\varphi_{\gamma,p,K} \geq \mathcal{V}_0$.

Lemma 2.1. *1. The function $\varphi_{\gamma,p,K}$ is a supersolution of the DPE* (2.10) *and a strict supersolution if $\gamma \in (1-\mu, 1+\lambda)$ and $K > 1$.*

2. $\varphi_{\gamma,p,K}$ dominates the value function $\mathcal{V}_0$. In particular, $\mathcal{V}_0(t,b,s) < +\infty$ for all $(t,b,s) \in [0,T] \times \overline{\mathcal{S}}^0$.

Proof. 1. Direct computations reveal that

$$\begin{aligned}\mathcal{L}^{nt}\varphi_{\gamma,p,K}(t,b,s) &= \frac{(b+\gamma s)^p}{2(1-p)\sigma^2}(f_{p,K}(t))^p\Bigg[\left(\alpha - \frac{\gamma\sigma^2 s}{b+\gamma s}\right)^2 \\ &\qquad\qquad + (K-1)\alpha^2\Bigg] \geq 0, \\ \mathcal{L}^{buy}\varphi_{\gamma,p,K}(t,b,s) &= (b+\gamma s)^{p-1}(f_{p,K}(t))^p(1+\lambda-\gamma) \geq 0, \\ \mathcal{L}^{sell}\varphi_{\gamma,p,K}(t,b,s) &= (b+\gamma s)^{p-1}(f_{p,K}(t))^p(-(1-\mu)+\gamma) \geq 0\end{aligned}$$

where the inequalities are strict if $\gamma \in (1-\mu, 1+\lambda)$ and $K > 1$.

2. Fix $(t,b,s) \in [0,T] \times \overline{\mathcal{S}}^0$, $\varepsilon > 0$ and let $(L,M) \in \mathcal{A}_0(t,b,s)$ so that $(L,M) \in \mathcal{A}_0(t,b+\varepsilon,s)$. Let $(K_j)_{j\in\mathbb{N}}$ be a sequence of compact sets containing (b,s) and $(b+\varepsilon,s)$ such that the K_j increase to $\overline{\mathcal{S}}^0$ as $j \to \infty$. For each $j \in \mathbb{N}$ we define a stopping time

$$\tau_j := \inf\left\{u \geq t : \left(B^{L,M}_{t,b}(u) + \varepsilon, S^{L,M}_{t,s}(u)\right) \notin K_j\right\} \wedge T$$

and note that $\tau_j \to T$ as $j \to \infty$.

Note that $B^{L,M}_{t,b+\varepsilon} = B^{L,M}_{t,b} + \varepsilon$ and write $B^\varepsilon := B^{L,M}_{t,b+\varepsilon}$ as well as $S := S^{L,M}_{t,s}$. Itô's formula for càdlàg semimartingales (see Protter [92, Theorem II.32]) shows that

$$\begin{aligned}&\varphi_{\gamma,p,K}(\tau_j, B^\varepsilon(\tau_j), S(\tau_j)) \\ &= \varphi_{\gamma,p,K}(t, b+\varepsilon, s) - \int_t^{\tau_j} \mathcal{L}^{nt}\varphi_{\gamma,p,K}(u, B^\varepsilon(u), S(u))\, du \\ &\quad - \int_t^{\tau_j} \mathcal{L}^{buy}\varphi_{\gamma,p,K}(u, B^\varepsilon(u), S(u))\, dL^c(u) \\ &\quad - \int_t^{\tau_j} \mathcal{L}^{sell}\varphi_{\gamma,p,K}(u, B^\varepsilon(u), S(u))\, dM^c(u)\end{aligned}$$

$$+\int_t^{\tau_j} \sigma S(u) \frac{\partial}{\partial s} \varphi_{\gamma,p,K}(u, B^\varepsilon(u), S(u))\, dW(u)$$
$$+\sum_{t \le u \le \tau_j} \left[\varphi_{\gamma,p,K}(u, B^\varepsilon(u), S(u)) - \varphi_{\gamma,p,K}(u-, B^\varepsilon(u-), S(u-))\right],$$

where L^c and M^c denote the continuous parts of L and M, respectively. Since $\mathcal{L}^{buy}\varphi_{\gamma,p,K}, \mathcal{L}^{sell}\varphi_{\gamma,p,K} \ge 0$ we see that $\varphi_{\gamma,p,K}$ is non-increasing in the directions of the jumps of $(u, B^\varepsilon(u), S(u))$ (by the gradient theorem, see also (2.11)) and hence

$$\sum_{t \le u \le \tau_j} \left[\varphi_{\gamma,p,K}(u, B^\varepsilon(u), S(u)) - \varphi_{\gamma,p,K}(u-, B^\varepsilon(u-), S(u-))\right] \le 0.$$

Moreover, since $\varphi_{\gamma,p,K}$ is a supersolution of the DPE (2.10) it follows that

$$0 \le \int_t^{\tau_j} \mathcal{L}^{nt}\varphi_{\gamma,p,K}(u, B^\varepsilon(u), S(u))\, du,$$
$$0 \le \int_t^{\tau_j} \mathcal{L}^{buy}\varphi_{\gamma,p,K}(u, B^\varepsilon(u), S(u))\, dL^c(u),$$
$$0 \le \int_t^{\tau_j} \mathcal{L}^{sell}\varphi_{\gamma,p,K}(u, B^\varepsilon(u), S(u))\, dM^c(u).$$

We therefore obtain

$$\varphi_{\gamma,p,K}(\tau_j, B^\varepsilon(\tau_j), S(\tau_j)) \le \varphi_{\gamma,p,K}(t, b+\varepsilon, s) + \int_t^{\tau_j} \sigma S(u) \frac{\partial}{\partial s} \varphi_{\gamma,p,K}(u, B^\varepsilon(u), S(u))\, dW(u)$$

and by taking expectations on both sides

$$\varphi_{\gamma,p,K}(t, b+\varepsilon, s) \ge \mathbb{E}\left[\varphi_{\gamma,p,K}(\tau_j, B^\varepsilon(\tau_j), S(\tau_j))\right]$$

for all $j \in \mathbb{N}$. Since

$$\varphi_{\gamma,p,K}(\tau_j, B^\varepsilon(\tau_j), S(\tau_j)) \ge U_p\left(B^\varepsilon(\tau_j) + \gamma S(\tau_j)\right) \ge U_p(\varepsilon)$$

we can send $j \to \infty$ and use Fatou's lemma to see that

$$\begin{aligned}\varphi_{\gamma,p,K}(t, b+\varepsilon, s) &\ge \mathbb{E}\left[\varphi_{\gamma,p,K}(T, B^\varepsilon(T), S(T))\right] \\ &= \mathbb{E}\left[U_p(B^\varepsilon(T) + \gamma S(T))\right].\end{aligned}$$

Next, observe that since $\gamma \in [1-\mu, 1+\lambda]$ we have $B^{\varepsilon}(T) + \gamma S(T) \geq X^{L,M}_{t,b+\varepsilon,s}(T)$. This implies that

$$\varphi_{\gamma,p,K}(t, b+\varepsilon, s) \geq \mathbb{E}\left[U_p\left(X^{L,M}_{t,b+\varepsilon,s}(T)\right)\right] = \mathbb{E}\left[U_p\left(X^{L,M}_{t,b,s}(T) + \varepsilon\right)\right].$$

Now send $\varepsilon \downarrow 0$ and use monotone convergence to obtain

$$\varphi_{\gamma,p,K}(t, b, s) \geq \mathbb{E}\left[U_p\left(X^{L,M}_{t,b,s}(T)\right)\right]$$

and we conclude since (L, M) was chosen arbitrarily. □

We now turn our focus to the value function $\mathcal{V}_0$. The following results are straightforward and have already been observed in Shreve and Soner [97] in a similar context.

Lemma 2.2. *1. Let $(b, s) \in \partial\mathcal{S}^0$. Then the only admissible strategy is to instantly jump to the position $(0, 0)$ and remain there.*

2. For $(b, s) \in \overline{\mathcal{S}}^0$, the trading strategy of instantly closing the stock position and no trading afterwards is an admissible strategy. Furthermore, for every $(b, s) \in \overline{\mathcal{S}}^0$, we have

$$\mathcal{V}_0(t, b, s) \geq \begin{cases} U_p\left(b + (1-\mu)s\right), & \text{if } s > 0, \\ U_p\left(b + (1+\lambda)s\right), & \text{if } s \leq 0. \end{cases}$$

Proof. 1. This is proved in Shreve and Soner [97, Remark 2.1]. The idea is obvious: If the investor were not to close the stock position immediately, then the state process would leave the solvency cone with positive probability.

2. This is obvious. See also Shreve and Soner [97, Remark 2.2]. □

Note that Lemma 2.2.2 allows us to restrict the admissible strategies in $\mathcal{A}_0(t, b, s)$ to those strategies (L, M) which satisfy

$$\mathbb{E}\left[U_p\left(X^{L,M}_{t,b,s}(T)\right)\right] \geq \begin{cases} U_p\left(b + (1-\mu)s\right), & \text{if } s > 0, \\ U_p\left(b + (1+\lambda)s\right), & \text{if } s \leq 0, \end{cases}$$

which we will assume in the sequel. With a slight abuse of notation we denote the restricted set of strategies again by $\mathcal{A}_0(t, b, s)$.

Lemma 2.3. *1. $\mathcal{V}_0$ is homogeneous of order p. That is, for every $\kappa > 0$, $\mathcal{V}_0$ satisfies*

$$\mathcal{V}_0(t, \kappa b, \kappa s) = \begin{cases} \kappa^p \mathcal{V}_0(t, b, s) & \text{if } p < 1, p \neq 0, \\ \log(\kappa) + \mathcal{V}_0(t, b, s) & \text{if } p = 0. \end{cases}$$

for every $(t, b, s) \in [0, T] \times \overline{\mathcal{S}}^0$.

2. Let $l, m \geq 0$. Then

$$\mathcal{V}_0(t, b, s) \geq \mathcal{V}_0(t, b - (1 + \lambda)l + (1 - \mu)m, s + l - m)$$

whenever $(b, s), (b - (1 + \lambda)l + (1 - \mu)m, s + l - m) \in \overline{\mathcal{S}}^0$.

Proof. 1. This is a well-known fact, see for example Shreve and Soner [97, Proposition 3.5] or Bichuch [15, Equation (2.11)] for a justification.

2. This is again obvious since $(b - (1 + \lambda)l + (1 - \mu)m, s + l - m)$ can be reached by an immediate transaction from (b, s) by buying l and selling m shares of the stock. See also Shreve and Soner [97, Proposition 3.5]. □

The following proposition establishes the link between the value function $\mathcal{V}_0$ and the DPE (2.10) by showing that the value function is a viscosity solution thereof. We refer to Appendix B for a brief introduction to viscosity solutions. The proof of the following proposition can be found in Davis et al. [27] in a slightly different context under the assumption that $\mathcal{V}_0$ is continuous or can be established along the lines of Shreve and Soner [97, Theorem 7.7] (also under the continuity assumption). For a rigorous proof we refer to Chapter 3 in a more general setting, see Corollary 3.12 and Corollary 3.23.

Proposition 2.4. *The value function $\mathcal{V}_0$ is continuous and a viscosity solution of the DPE* (2.10) *with boundary condition*

$$\mathcal{V}_0(t, b, s) = U_p(0), \qquad \text{if } (b, s) \in \partial\mathcal{S}^0,\ t \in [0, T],$$

and terminal condition

$$\mathcal{V}_0(T, b, s) = \begin{cases} U_p(b + (1 - \mu)s), & \text{if } s > 0, \\ U_p(b + (1 + \lambda)s), & \text{if } s \leq 0. \end{cases}$$

In what follows, we show that the value function is unique within a suitable class of functions.

2.4. The comparison principle

Before we present the comparison principle for the dynamic programming equation let us first introduce some notation. We define

$$\begin{aligned}F^{nt}(s, q, p_s, X) &:= -q - \alpha s p_s - \frac{1}{2}\sigma^2 s^2 X_{22},\\ F^{buy}(p_b, p_s) &:= (1+\lambda)p_b - p_s,\\ F^{sell}(p_b, p_s) &:= -(1-\mu)p_b + p_s,\end{aligned}$$

where $(b, s) \in \mathcal{S}^0$, $q \in \mathbb{R}$, $p = (p_b, p_s) \in \mathbb{R}^2$ and $X = (X_{ij})_{i,j=1,2} \in \mathbb{S}^2$. Moreover, set

$$F(s, q, p, X) := \min\Big\{F^{nt}(s, q, p_s, X), F^{buy}(p_b, p_s), F^{sell}(p_b, p_s)\Big\}. \tag{2.13}$$

Then

$$F(s, D_t\mathcal{V}_0(t, b, s), D_{(b,s)}\mathcal{V}_0(t, b, s), D^2_{(b,s)}\mathcal{V}_0(t, b, s)) = 0$$

corresponds to the DPE (2.10).

In what follows we prove a comparison theorem for the DPE, which typically takes the following form: If u and v are viscosity sub- and supersolutions of the DPE, respectively, and $u \leq v$ on the boundary of the state space, then $u \leq v$ everywhere. The main difficulty in our setting is to control the behavior of the viscosity solutions near $\partial\mathcal{S}^0$ since a boundary value of negative infinity is possible ($\mathcal{V}_0 = -\infty$ on $\partial\mathcal{S}^0$ for $p \leq 0$). More precisely, we cannot guarantee that $u - v \leq 0$ near $\partial\mathcal{S}^0$.

We adopt an idea from Soner and Vukelja [103] to deal with this problem: Instead of merely specifying an upper bound on the growth of the solutions we add an additional lower bound. Moreover, we shift the supersolution v by ε in the b direction to ensure that v is finite on $\partial\mathcal{S}^0$. We can then prove a comparison principle for the shifted v and conclude the classical comparison result by sending $\varepsilon \to 0$. We note that it is crucial for the following proof that the operators $\mathcal{L}^{nt}$, $\mathcal{L}^{buy}$ and $\mathcal{L}^{sell}$ are independent of b, which in our problem setting is achieved since the bond is chosen as the numéraire.

Theorem 2.5. *Let $u, v : [0,T] \times \overline{\mathcal{S}}^0 \to \mathbb{R}$ and fix $\varepsilon > 0$. Assume that u is an upper semi-continuous viscosity subsolution of* (2.10) *and v is a lower semi-continuous viscosity supersolution of* (2.10) *such that*

$$U_p(b + \min\{(1-\mu)s, (1+\lambda)s\}) \leq u(t,b,s), v(t,b,s) \leq \varphi_{\gamma,p,K}(t,b,s) \quad (2.14)$$

for some $p < 1$, $\gamma \in (1-\mu, 1+\lambda)$ and $K > 1$. If $u(T,b,s) \leq v(T, b+\varepsilon, s)$ and $u(t,b,s) \leq U_p(0)$ for every $(b,s) \in \partial\mathcal{S}^0$, then $u(t,b,s) \leq v(t, b+\varepsilon, s)$ on $[0,T] \times \overline{\mathcal{S}}^0$.

Proof. Step 1: Suppose that there exists some $(t^*, b^*, s^*) \in [0,T) \times \overline{\mathcal{S}}^0$ such that

$$u(t, b^*, s^*) - v(t^*, b^* + \varepsilon, s^*) > 0.$$

Let us note that by the growth condition (2.14) we have

$$-v(t, b+\varepsilon, s) \leq -U_p(b + \varepsilon + \min\{(1-\mu)s, (1+\lambda)s\}) \leq -U_p(\varepsilon) < \infty.$$

We therefore have $(b^*, s^*) \notin \partial\mathcal{S}^0$ since otherwise

$$u(t^*, b^*, s^*) - v(t^*, b^* + \varepsilon, s^*) \leq U_p(0) - U_p(\varepsilon) < 0$$

is a contradiction.

Step 2: Define the set

$$\mathcal{D}_\varepsilon := \left\{(t,b,s,\bar{t},\bar{b},\bar{s}) : (t,b,s) \in [0,T) \times \mathcal{S}^0, (\bar{t}, \bar{b} - \varepsilon, \bar{s}) \in [0,T) \times \mathcal{S}^0\right\}.$$

Now, for some $p' \in (p, 1)$ with $p' > 0$, for some $\delta_0 > 0$ to be fixed later and every $n \in \mathbb{N}$ we consider the upper semi-continuous functions $\phi_n : \overline{\mathcal{D}}_\varepsilon \to \mathbb{R}$ defined as

$$\begin{aligned}\phi_n(t,b,s,\bar{t},\bar{b},\bar{s}) := u(t,b,s) - v(\bar{t},\bar{b},\bar{s}) - \delta_0 \varphi_{\gamma,p',K}(t,b,s) \\ - \frac{n}{2}\left(|t-\bar{t}|^2 + |b - \bar{b} + \varepsilon|^2 + |s - \bar{s}|^2\right)\end{aligned}$$

as well as $\phi_\infty : [0,T] \times \overline{\mathcal{S}}^0 \to \mathbb{R}$ given by

$$\phi_\infty(t,b,s) := u(t,b,s) - v(t, b+\varepsilon, s) - \delta_0 \varphi_{\gamma,p',K}(t,b,s).$$

Note that if $(t,b,s,\bar{t},\bar{b},\bar{s}) \in \overline{\mathcal{D}}_\varepsilon$, then

$$\bar{b} + \min\{(1-\mu)\bar{s}, (1+\lambda)\bar{s}\} \geq \varepsilon$$

and hence

$$-v(\bar{t}, \bar{b}, \bar{s}) \leq -U_p(\varepsilon) < \infty.$$

Moreover, since $u \leq \varphi_{\gamma,p,K} \leq \varphi_{\gamma,p',K}$ we have

$$\lim_{|b|,|s| \to \infty} u(t, b, s) - \delta_0 \varphi_{\gamma,p',K}(t, b, s) = -\infty$$

which implies that the supremum in

$$M_n := \sup_{\overline{\mathcal{D}}_\varepsilon} \phi_n(t, b, s, \bar{t}, \bar{b}, \bar{s})$$

is attained at some point $(t_n, b_n, s_n, \bar{t}_n, \bar{b}_n, \bar{s}_n) \in \overline{\mathcal{D}}_\varepsilon$. Also, note that the sequence $(t_n, b_n, s_n, \bar{t}_n, \bar{b}_n, \bar{s}_n)_{n \in \mathbb{N}}$ is bounded and $M_n < +\infty$. Similarly, we have

$$M_\infty := \sup_{[0,T] \times \overline{\mathcal{S}}^0} \phi_\infty(t, b, s) < +\infty$$

and the supremum is attained at some point $(t_\infty, b_\infty, s_\infty) \in [0, T] \times \overline{\mathcal{S}}^0$. Let us now choose

$$\delta_0 < \frac{u(t^*, b^*, s^*) - v(t^*, b^* + \varepsilon, s^*)}{\varphi_{\gamma,p',K}(t^*, b^*, s^*)}$$

so that we have

$$M_n \geq M_\infty \geq u(t^*, b^*, s^*) - v(t^*, b^* + \varepsilon, s^*) - \delta_0 \varphi_{\gamma,p',K}(t^*, b^*, s^*) > 0.$$

Step 3: We want to show that (up to a subsequence)

$$(t_n, b_n, s_n, \bar{t}_n, \bar{b}_n, \bar{s}_n) \to (t_\infty, b_\infty, s_\infty, t_\infty, b_\infty + \varepsilon, s_\infty), \qquad M_n \to M_\infty \tag{2.15}$$

and

$$n \left(|t_n - \bar{t}_n|^2 + |b_n - \bar{b}_n + \varepsilon|^2 + |s_n - \bar{s}_n|^2 \right) \to 0. \tag{2.16}$$

First, let us recall that the sequence $(t_n, b_n, s_n, \bar{t}_n, \bar{b}_n, \bar{s}_n)_{n \in \mathbb{N}}$ is bounded and hence so is the sequence

$$\left(u(t_n, b_n, s_n) - v(\bar{t}_n, \bar{b}_n, \bar{s}_n) - \delta_0 \varphi_{\gamma,p',K}(t_n, b_n, s_n) \right)_{n \in \mathbb{N}}$$

because $u - v - \delta_0 \varphi_{\gamma,p',K}$ is upper semi-continuous. Now since $M_n \geq M_\infty$ we have

$$0 \leq \frac{n}{2} \left(|t_n - \bar{t}_n|^2 + |b_n - \bar{b}_n + \varepsilon|^2 + |s_n - \bar{s}_n|^2 \right)$$

$$
\begin{aligned}
&= u(t_n, b_n, s_n) - v(\bar{t}_n, \bar{b}_n, \bar{s}_n) - \delta_0 \varphi_{\gamma,p',K}(t_n, b_n, s_n) - M_n \\
&\leq u(t_n, b_n, s_n) - v(\bar{t}_n, \bar{b}_n, \bar{s}_n) - \delta_0 \varphi_{\gamma,p',K}(t_n, b_n, s_n) - M_\infty,
\end{aligned} \tag{2.17}
$$

which implies that the sequence

$$
\left(\frac{n}{2} \left(|t_n - \bar{t}_n|^2 + |b_n - \bar{b}_n + \varepsilon|^2 + |s_n - \bar{s}_n|^2 \right) \right)_{n \in \mathbb{N}}
$$

is bounded. We can hence find a subsequence of $(t_n, b_n, s_n, \bar{t}_n, \bar{b}_n, \bar{s}_n)_{n\in\mathbb{N}}$ (which we again denote by $(t_n, b_n, s_n, \bar{t}_n, \bar{b}_n, \bar{s}_n)_{n\in\mathbb{N}}$ for simplicity) such that

$$
(t_n, b_n, s_n, \bar{t}_n, \bar{b}_n, \bar{s}_n) \to (\hat{t}, \hat{b}, \hat{s}, \hat{t}, \hat{b} + \varepsilon, \hat{s}) \in \overline{\mathcal{D}}_\varepsilon.
$$

Passing to the limit in (2.17) now implies that

$$
\begin{aligned}
0 &\leq \limsup_{n\to\infty} \frac{n}{2} \left(|t_n - \bar{t}_n|^2 + |b_n - \bar{b}_n + \varepsilon|^2 + |s_n - \bar{s}_n|^2 \right) \\
&\leq \limsup_{n\to\infty} u(t_n, b_n, s_n) - v(\bar{t}_n, \bar{b}_n, \bar{s}_n) - \delta_0 \varphi_{\gamma,p',K}(t_n, b_n, s_n) - M_\infty \\
&\leq u(\hat{t}, \hat{b}, \hat{s}) - v(\hat{t}, \hat{b} + \varepsilon, \hat{s}) - \delta_0 \varphi_{\gamma,p',K}(\hat{t}, \hat{b}, \hat{s}) - M_\infty \leq 0
\end{aligned}
$$

which proves (2.15) and (2.16).

Step 4: Next we show that $t_\infty \neq T$ and $(b_\infty, s_\infty) \notin \partial\mathcal{S}^0$. Suppose that on the contrary we have $t_\infty = T$. Then

$$
\begin{aligned}
0 < M_\infty = u(T, b_\infty, s_\infty) - v(T, b_\infty + \varepsilon, s_\infty) - \delta_0 \varphi_{\gamma,p',K}(T, b_\infty, s_\infty) \\
\leq u(T, b_\infty, s_\infty) - v(T, b_\infty + \varepsilon, s_\infty) \leq 0
\end{aligned}
$$

which is a contradiction. Similarly, assuming that $(b_\infty, s_\infty) \in \partial\mathcal{S}^0$ leads to a contradiction since

$$
\begin{aligned}
0 < M_\infty = u(t_\infty, b_\infty, s_\infty) - v(t_\infty, b_\infty + \varepsilon, s_\infty) - \delta_0 \varphi_{\gamma,p',K}(t_\infty, b_\infty, s_\infty) \\
\leq U_p(0) - U_p(\varepsilon) < 0.
\end{aligned}
$$

Hence $t_\infty \neq T$ and $(b_\infty, s_\infty) \notin \partial\mathcal{S}^0$ and since $t_n, \bar{t}_n \to t_\infty$, $b_n \to b_\infty$, $\bar{b}_n \to b_\infty + \varepsilon$ and $s_n, \bar{s}_n \to s_\infty$, we furthermore have $(t_n, b_n, s_n, \bar{t}_n, \bar{b}_n, \bar{s}_n) \in \mathcal{D}_\varepsilon$ for n sufficiently large.

Step 5: Let n be large enough such that $(t_n, b_n, s_n, \bar{t}_n, \bar{b}_n, \bar{s}_n) \in \mathcal{D}_\varepsilon$. Then we can apply Theorem B.4 (Ishii's lemma) to the upper semi-continuous function

$u - \delta_0 \varphi_{\gamma,p',K}$ and the lower semi-continuous function v to obtain the existence of $X, Y \in \mathbb{S}^2$ such that

$$\begin{aligned}
&(n(t_n - \bar{t}_n), n(b_n - \bar{b}_n + \varepsilon), n(s_n - \bar{s}_n), X) \in \overline{J}^{2,+}[u - \delta_0 \varphi_{\gamma,p',K}](t_n, b_n, s_n),\\
&(n(t_n - \bar{t}_n), n(b_n - \bar{b}_n + \varepsilon), n(s_n - \bar{s}_n), Y) \in \overline{J}^{2,-} v(\bar{t}_n, \bar{b}_n, \bar{s}_n),
\end{aligned}$$

and such that

$$\begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leq 3n \begin{pmatrix} I & -I \\ -I & I \end{pmatrix}. \tag{2.18}$$

Since $\varphi_{\gamma,p',K}$ is smooth, it follows that

$$\begin{aligned}
\Big(n(t_n - \bar{t}_n) &+ \delta_0 \frac{\partial}{\partial t} \varphi_{\gamma,p',K}(t_n, b_n, s_n),\\
&n(b_n - \bar{b}_n + \varepsilon) + \delta_0 \frac{\partial}{\partial b} \varphi_{\gamma,p',K}(t_n, b_n, s_n),\\
&\quad n(s_n - \bar{s}_n) + \delta_0 \frac{\partial}{\partial s} \varphi_{\gamma,p',K}(t_n, b_n, s_n),\\
&\qquad X + \delta_0 D^2_{(b,s)} \varphi_{\gamma,p',K}(t_n, b_n, s_n)\Big) \in \overline{J}^{2,+} u(t_n, b_n, s_n).
\end{aligned}$$

To ease notation, let us define

$$p_t^n := n(t_n - \bar{t}_n), \quad p_b^n := n(b_n - \bar{b}_n + \varepsilon), \quad p_s^n := n(s_n - \bar{s}_n).$$

Step 6: Since u is a viscosity subsolution of (2.10) and by the linearity of the operators $\mathcal{L}^{nt}$, $\mathcal{L}^{buy}$ and $\mathcal{L}^{sell}$ we have

$$\begin{aligned}
\min\Big\{ F^{nt}(s_n, p_t^n, p_s^n, X) &+ \delta_0 \mathcal{L}^{nt} \varphi_{\gamma,p',K}(t_n, b_n, s_n),\\
&F^{buy}(p_b^n, p_s^n) + \delta_0 \mathcal{L}^{buy} \varphi_{\gamma,p',K}(t_n, b_n, s_n),\\
&\quad F^{sell}(p_b^n, p_s^n) + \delta_0 \mathcal{L}^{sell} \varphi_{\gamma,p',K}(t_n, b_n, s_n)\Big\} \leq 0
\end{aligned} \tag{2.19}$$

and since v is a viscosity supersolution we have

$$\min\Big\{ F^{nt}(\bar{s}_n, p_t^n, p_s^n, Y), F^{buy}(p_b^n, p_s^n), F^{sell}(p_b^n, p_s^n)\Big\} \geq 0. \tag{2.20}$$

Our aim is to show that (2.19) and (2.20) lead to a contradiction.

Suppose first that in (2.19) we have

$$F^{buy}(p_b^n, p_s^n) + \delta_0 \mathcal{L}^{buy} \varphi_{\gamma,p',K}(t_n, b_n, s_n) \leq 0.$$

But since by (2.20) we have $F^{buy}(p_b^n, p_s^n) \geq 0$ it follows that

$$\delta_0 \mathcal{L}^{buy} \varphi_{\gamma,p',K}(t_n, b_n, s_n) \leq 0$$

which is a contradiction since $\varphi_{\gamma,p',K}$ is a strict supersolution of the DPE by Lemma 2.1.1 and since $\gamma \in (1-\mu, 1+\lambda)$ and $K > 1$. In a similar fashion assuming that

$$F^{sell}(p_b^n, p_s^n) + \delta_0 \mathcal{L}^{sell} \varphi_{\gamma,p',K}(t_n, b_n, s_n) \leq 0$$

leads to a contradiction. We must therefore have

$$F^{nt}(s_n, p_t^n, p_s^n, X) + \delta_0 \mathcal{L}^{nt} \varphi_{\gamma,p',K}(t_n, b_n, s_n) \leq 0.$$

Thus (2.20) implies that

$$F^{nt}(s_n, p_t^n, p_s^n, X) - F^{nt}(\bar{s}_n, p_t^n, p_s^n, Y) + \delta_0 \mathcal{L}^{nt} \varphi_{\gamma,p',K}(t_n, b_n, s_n) \leq 0.$$

Direct computations show that

$$\begin{aligned} &F^{nt}(s_n, p_t^n, p_s^n, X) - F^{nt}(\bar{s}_n, p_t^n, p_s^n, Y) \\ &\quad = -p_t^n - \alpha s_n p_s^n - \frac{1}{2}\sigma^2 s_n^2 X_{22} + p_t^n + \alpha \bar{s}_n p_s^n + \frac{1}{2}\sigma^2 \bar{s}_n^2 Y_{22} \\ &\quad = -\alpha n |s_n - \bar{s}_n|^2 - \frac{1}{2}\sigma^2 [s_n^2 X_{22} - \bar{s}_n^2 Y_{22}]. \end{aligned}$$

Define

$$\tilde{\sigma}(s) := \begin{pmatrix} 0 & 0 \\ 0 & s \end{pmatrix}, \qquad \Sigma := \begin{pmatrix} \tilde{\sigma}(s_n)\tilde{\sigma}(s_n) & \tilde{\sigma}(s_n)\tilde{\sigma}(\bar{s}_n) \\ \tilde{\sigma}(s_n)\tilde{\sigma}(\bar{s}_n) & \tilde{\sigma}(\bar{s}_n)\tilde{\sigma}(\bar{s}_n) \end{pmatrix}.$$

Then (2.18) implies that

$$\begin{aligned} s_n^2 X_{22} - \bar{s}_n^2 Y_{22} &= \operatorname{tr}\left[\tilde{\sigma}(s_n)\tilde{\sigma}(s_n) X - \tilde{\sigma}(\bar{s}_n)\tilde{\sigma}(\bar{s}_n) Y\right] \\ &= \operatorname{tr}\left[\Sigma \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix}\right] \\ &\leq 3n \operatorname{tr}\left[\Sigma \begin{pmatrix} I & -I \\ -I & I \end{pmatrix}\right] \\ &= 3n\left(s_n^2 - 2 s_n \bar{s}_n + \bar{s}_n^2\right) \\ &= 3n |s_n - \bar{s}_n|^2. \end{aligned}$$

Therefore,

$$\begin{aligned}
F^{nt}(s_n, p_t^n, p_s^n, X) - F^{nt}(\bar{s}_n, p_t^n, p_s^n, Y) \\
&= -\alpha n|s_n - \bar{s}_n|^2 - \frac{1}{2}\sigma^2[s_n^2 X_{22} - \bar{s}_n^2 Y_{22}] \\
&\geq -\alpha n|s_n - \bar{s}_n|^2 - \frac{3}{2}\sigma^2 n|s_n - \bar{s}_n|^2 \\
&\geq -\max\{\alpha, \frac{3}{2}\sigma^2\} n|s_n - \bar{s}_n|^2.
\end{aligned}$$

We therefore have

$$\begin{aligned}
0 &\geq F^{nt}(s_n, p_t^n, p_s^n, X) - F^{nt}(\bar{s}_n, p_t^n, p_s^n, Y) + \delta_0 \mathcal{L}^{nt}\varphi_{\gamma,p',K}(t_n, b_n, s_n) \\
&\geq -\max\{\alpha, \frac{3}{2}\sigma^2\} n|s_n - \bar{s}_n|^2 + \delta_0 \mathcal{L}^{nt}\varphi_{\gamma,p',K}(t_n, b_n, s_n)
\end{aligned}$$

and since $n|s_n - \bar{s}_n|^2 \to 0$ as $n \to \infty$ we obtain

$$0 \geq \delta_0 \mathcal{L}^{nt}\varphi_{\gamma,p',K}(t_\infty, b_\infty, s_\infty) > 0$$

which is again a contradiction and hence finishes the proof. □

The comparison theorem implies the following uniqueness result. In particular, the value function $\mathcal{V}_0$ is the unique viscosity solution of the DPE.

Corollary 2.6. *Let u, v be upper semi-continuous viscosity solutions of the DPE satisfying*

$$U_p(b + \min\{(1-\mu)s, (1+\lambda)s\}) \leq u(t,b,s), v(t,b,s) \leq \varphi_{\gamma,p,K}(t,b,s)$$

with $u(t,b,s) = v(t,b,s) = U_p(0)$ on $\partial\mathcal{S}^0$ and

$$u^*(T,b,s) = u_*(T,b,s) = \mathcal{V}_0(T,b,s) = v^*(T,b,s) = v_*(T,b,s). \tag{2.21}$$

Then $u = v$.

Proof. Let $\varepsilon > 0$ be arbitrary. Since u and v are viscosity solutions of the DPE, u_* is a viscosity supersolution and $v^* = v$ is a viscosity subsolution. Moreover, by (2.21),

$$v(T,b,s) = U_p(b + \min\{(1-\mu)s, (1+\lambda)s\})$$

$$\begin{aligned}&\le U_p(b+\varepsilon+\min\{(1-\mu)s,(1+\lambda)s\})\\&=u(T,b+\varepsilon,s)\\&=u_*(T,b+\varepsilon,s).\end{aligned}$$

Hence $v(t,b,s) \le u_*(t,b+\varepsilon,s) \le u(t,b+\varepsilon,s)$ everywhere by Theorem 2.5. Sending ε to zero shows that $v \le u$ by the upper semi-continuity of u. Switching the roles of v and u shows the reverse inequality. □

2.5. Construction of the optimal strategies

As pointed out in Section 2.2 we expect the operators $\mathcal{L}^{nt}$, $\mathcal{L}^{buy}$ and $\mathcal{L}^{sell}$ defined in (2.4)-(2.6) to determine the optimal strategy in the sense that the operators give rise to the trading regions $\mathcal{R}_0^{nt}$, $\mathcal{R}_0^{buy}$ and $\mathcal{R}_0^{sell}$ defined in (2.7)-(2.9). Note however that in order for the definitions of the trading regions to make sense we need that $\mathcal{V}_0 \in C^{1,2,2}([0,T)\times\mathcal{S}^0)$. It is therefore necessary to study the DPE in more detail for the existence of a sufficiently regular solution and hope that this regular solution coincides with the value function.

The fact that $\mathcal{V}_0$ and U_p are homogeneous (cf. Lemma 2.3.1) can be used to reduce the dimension of the DPE (2.10). For this, let us temporarily assume that $p \neq 0$ and consider the reduced solvency cone

$$\tilde{\mathcal{S}}^0 := \left\{(b,s)\in\mathcal{S}^0 : s>0\right\}.$$

For every $(b,s)\in\tilde{\mathcal{S}}^0$ consider the transformation

$$\mathcal{V}_0(t,b,s) =: s^p\tilde{\mathcal{V}}_0(t,b/s), \qquad \tilde{\mathcal{V}}_0(t,x) := \mathcal{V}_0(t,x,1). \tag{2.22}$$

Then, writing $x=b/s$, the DPE reduces to

$$0=\min\left\{\tilde{\mathcal{L}}_p^{nt}\tilde{\mathcal{V}}_0(t,x),\tilde{\mathcal{L}}_p^{buy}\tilde{\mathcal{V}}_0(t,x),\tilde{\mathcal{L}}_p^{sell}\tilde{\mathcal{V}}_0(t,x)\right\}, \tag{2.23}$$

for $(t,x)\in[0,T)\times(-(1-\mu),\infty)$. The terminal condition is now given as $\tilde{\mathcal{V}}_0(T,x)=U_p(x+1-\mu)$ and differential operators $\tilde{\mathcal{L}}_p^{nt}$, $\tilde{\mathcal{L}}_p^{buy}$ and $\tilde{\mathcal{L}}_p^{sell}$ are defined as

$$\tilde{\mathcal{L}}_p^{nt}\tilde{\mathcal{V}}_0 := -\frac{\partial}{\partial t}\tilde{\mathcal{V}}_0-\left(\alpha-\frac{1}{2}(1-p)\sigma^2\right)p\tilde{\mathcal{V}}_0$$

$$
\begin{aligned}
&\qquad\qquad + \left(\alpha - (1-p)\sigma^2\right) x \frac{\partial}{\partial x}\tilde{\mathcal{V}}_0 - \frac{1}{2}\sigma^2 x^2 \frac{\partial^2}{\partial x^2}\tilde{\mathcal{V}}_0,\\
&\tilde{\mathcal{L}}_p^{buy}\tilde{\mathcal{V}}_0 := -(x+1-\mu)\frac{\partial}{\partial x}\tilde{\mathcal{V}}_0 + p\tilde{\mathcal{V}}_0,\\
&\tilde{\mathcal{L}}_p^{sell}\tilde{\mathcal{V}}_0 := (x+1+\lambda)\frac{\partial}{\partial x}\tilde{\mathcal{V}}_0 - p\tilde{\mathcal{V}}_0.
\end{aligned}
$$

For the case $p = 0$ we can set

$$
\mathcal{V}_0(t,b,s) =: \log(s) + \tilde{\mathcal{V}}_0(t, b/s), \qquad \tilde{\mathcal{V}}_0(t,x) := \mathcal{V}_0(t,x,1).
$$

to obtain the same reduced DPE (2.23), but with the differential operators $\tilde{\mathcal{L}}_0^{nt}$, $\tilde{\mathcal{L}}_0^{buy}$ and $\tilde{\mathcal{L}}_0^{sell}$ given by

$$
\begin{aligned}
&\tilde{\mathcal{L}}_0^{nt}\tilde{\mathcal{V}}_0 := -\frac{\partial}{\partial t}\tilde{\mathcal{V}}_0 - \left(\alpha - \frac{1}{2}\sigma^2\right) + \left(\alpha - \sigma^2\right) x \frac{\partial}{\partial x}\tilde{\mathcal{V}}_0 - \frac{1}{2}\sigma^2 x^2 \frac{\partial^2}{\partial x^2}\tilde{\mathcal{V}}_0,\\
&\tilde{\mathcal{L}}_0^{buy}\tilde{\mathcal{V}}_0 := -(x+1-\mu)\frac{\partial}{\partial x}\tilde{\mathcal{V}}_0 + 1,\\
&\tilde{\mathcal{L}}_0^{sell}\tilde{\mathcal{V}}_0 := (x+1+\lambda)\frac{\partial}{\partial x}\tilde{\mathcal{V}}_0 - 1.
\end{aligned}
$$

Dai and Yi [24] prove the existence of a regular solution of the reduced DPE (2.23). By inverting the above transformation it follows that there exists a classical solution to the original DPE (2.10) on the reduced state space $[0,T) \times \tilde{\mathcal{S}}^0$. The hope is of course that this classical solution coincides with the value function on $[0,T) \times \tilde{\mathcal{S}}^0$ and that the so-obtained regularity is sufficient to construct the optimal strategy. The following Theorem sums up the results of Dai and Yi [24, Theorem 5.1, Proposition 3.2].

Theorem 2.7. *There exists a function* $\tilde{V}_0(t,x) \in C^{1,2}(([0,T)\times(-(1-\mu),\infty))\setminus F)$ *with* $(\partial/\partial t)\tilde{V}_0(t,x) \leq 0$ *which solves the reduced DPE* (2.23) *in the classical sense. Here, the set* F *is given by*

$$
F := \begin{cases} \emptyset, & \text{if } \pi_M < 1,\\ \{(t,0) : t \in [0,T)\}, & \text{if } \pi_M = 1,\\ \{(t^{up},0)\}, & \text{if } \pi_M > 1, \end{cases} \tag{2.24}
$$

with $\pi_M = \alpha/(1-p)\sigma^2$ *as defined in Section 1.1 and*

$$
t^{up} = T - \frac{\log(1+\lambda) - \log(1-\mu)}{\alpha - (1-p)\sigma^2}. \tag{2.25}
$$

The classical solution obtained in Theorem 2.7 allows us to define the trading regions as follows:

$$\begin{aligned}
\tilde{\mathcal{R}}_0^{buy} &:= \left\{(t,x) \in [0,T) \times (-(1-\mu),\infty) : \tilde{\mathcal{L}}_p^{buy}\tilde{V}_0(t,x) = 0\right\}, \\
\tilde{\mathcal{R}}_0^{sell} &:= \left\{(t,x) \in [0,T) \times (-(1-\mu),\infty) : \tilde{\mathcal{L}}_p^{sell}\tilde{V}_0(t,x) = 0\right\}, \\
\tilde{\mathcal{R}}_0^{nt} &:= \big([0,T) \times (-(1-\mu),\infty)\big) \setminus (\tilde{\mathcal{R}}^{buy} \cup \tilde{\mathcal{R}}^{sell}).
\end{aligned}$$

Note that we must necessarily have $\tilde{\mathcal{L}}_p^{nt}\tilde{V}_0(t,x) = 0$ for all $(t,x) \in \tilde{\mathcal{R}}_0^{nt}$. In order to construct the optimal strategy it is important to determine the geometry of these sets and the location of the boundaries between them. Dai and Yi [24, Theorems 4.3, 4.5 and 4.7] provide the following characterization of these free boundaries.

Theorem 2.8. 1. *There exist two non-decreasing functions*

$$\tilde{x}_b, \tilde{x}_s : [0,T) \to [-(1-\mu),\infty],$$

with $\tilde{x}_b(t) > \tilde{x}_s(t)$ for all $t \in [0,T)$ such that

$$\begin{aligned}
\tilde{\mathcal{R}}_0^{nt} &= \{(t,x) \in [0,T) \times (-(1-\mu),\infty) : \tilde{x}_s(t) < x < \tilde{x}_b(t)\}, \\
\tilde{\mathcal{R}}_0^{buy} &= \{(t,x) \in [0,T) \times (-(1-\mu),\infty) : x \geq \tilde{x}_b(t)\}, \\
\tilde{\mathcal{R}}_0^{sell} &= \{(t,x) \in [0,T) \times (-(1-\mu),\infty) : x \leq \tilde{x}_s(t)\}.
\end{aligned}$$

Moreover, $\tilde{V}_0 \in C^\infty(\tilde{\mathcal{R}}_0^{nt})$.

2. *The function $\tilde{x}_b$ is continuous and satisfies*

$$\tilde{x}_b(t) \begin{cases} > 0, & \text{if } \pi_M \leq 1, \\ < 0, & \text{if } \pi_M > 1, t < t^{up}, \\ = 0, & \text{if } \pi_M > 1, t = t^{up}, \\ > 0, & \text{if } \pi_M > 1, t > t^{up}, \end{cases}$$

where t^{up} is defined in (2.25). Furthermore, we have $\tilde{x}_b(t) = \infty$ for $t \in [t^{down}, T)$ where

$$t^{down} := T - \frac{\log(1+\lambda) - \log(1-\mu)}{\alpha}. \tag{2.26}$$

3. *We have* $\tilde{x}_s \in C^\infty([0,T))$ *and*

$$\tilde{x}_s(t) \begin{cases} > 0, & \text{if } \pi_M < 1, \\ = 0, & \text{if } \pi_M = 1, \\ < 0, & \text{if } \pi_M > 1. \end{cases}$$

Remark 2.9. A close inspection of the results of Dai and Yi [24] implies the slightly stronger result

$$\inf_{t\in[0,T)} |\tilde{x}_b(t) - \tilde{x}_s(t)| > 0.$$

This can be seen by looking at the double obstacle formulation of the problem in Equation (3.4) and the discussion in the first paragraph in Section 3.2 in Dai and Yi [24]. ◇

Remark 2.10. The point t^{up} turns out to be the time point at which the buy boundary $\tilde{x}_b$ crosses the line $x = 0$ (i.e. all wealth invested in the stock). The point t^{down} defined in (2.26) is exactly the point in time from which onwards it is no longer optimal to buy shares of the stock. ◇

Note that since $x = b/s$ we have $b = s\tilde{x}_b(t)$ and $b = s\tilde{x}_s(t)$ along the free boundaries. This shows that for every t the free boundaries define a cone in the original variables. See also Figure 2.3.

Figures 2.4-2.6 visualize the different scenarios for the location of the free boundaries. Note that we parametrize the free boundaries in terms of risky fractions here, i.e.

$$\underline{\pi}^0(t) := \frac{1}{1+\tilde{x}_b(t)}, \qquad \overline{\pi}^0(t) := \frac{1}{1+\tilde{x}_s(t)}.$$

Note that t^{up} is the time point at which the free boundary in terms of the risky fractions is equal to one, i.e. $\underline{\pi}^0(t^{up}) = 1$ (this may only happen if $\pi_M > 1$) and t^{down} is the time point from which onwards the buy boundary in terms of risky fractions is equal to zero, i.e. $\underline{\pi}^0(t) = 0$ for all $t \in [t^{down}, T]$.

For obvious reasons we refer to $\tilde{x}_b$ and $\tilde{x}_s$ as the buy and sell boundary, respectively. If our conjecture that the buy and sell boundaries characterize the optimal strategies is indeed correct (and which will be rigorously proved in Section 2.6) then Theorem 2.8 has the following implications:

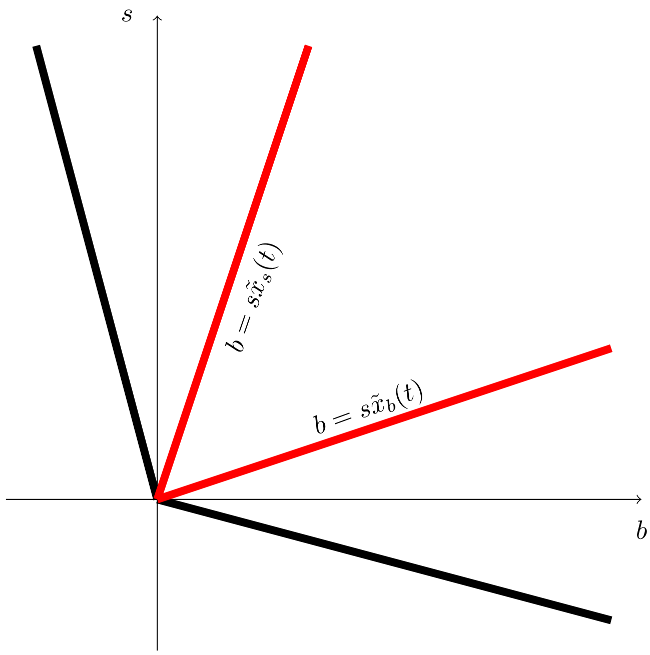

Figure 2.3. The free boundaries in the original variables for a fixed t.

1. If $\pi_M < 1$ (c.f. Figure 2.4), i.e. if borrowing is not optimal in the absence of transaction costs, then it is also not optimal in the presence of transaction costs. This is because both $\tilde{x}_b > 0$ and $\tilde{x}_s > 0$, implying that $\underline{\pi}^0, \overline{\pi}^0 \in (0, 1)$.

2. If $\pi_M = 1$ (c.f. Figure 2.5), i.e. if it is optimal to invest all money in the stock in the absence of transaction costs, then two cases must be distinguished in the presence of transaction costs: If the initial position of the investor is such that $x \leq 0$ (i.e. $b \leq 0$) then the bond position is closed and all money is kept in the stock (since $\tilde{x}_s = 0$, i.e. $\overline{\pi}^0 = 1$). However, if the initial position is such that $x > 0$ (which implies in particular $b > 0$), then it is not optimal to close the bond position. This is because we force the investor to close the stock position at terminal time T and hence it is too expensive to first buy shares of the stock at initial time just to liquidate the stock position once the investment horizon is reached.

3. If $\pi_M > 1$ (c.f. Figure 2.6), i.e. if borrowing is optimal in the absence of costs, we need to distinguish three cases. Since after the initial transaction the investor never switches from borrowing to no-borrowing or vice versa

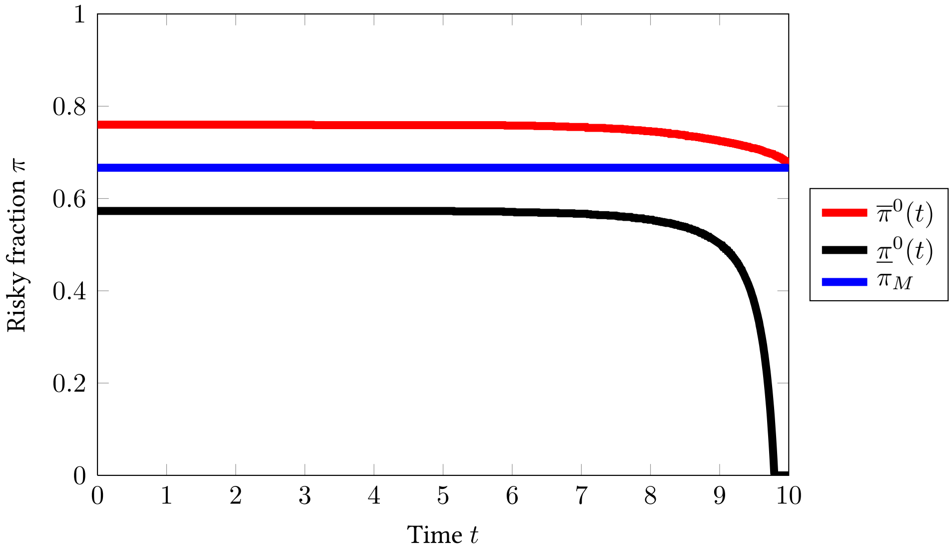

Figure 2.4. The trading regions for $\pi_M < 1$.

($\tilde{x}_s < 0$ so that $\overline{\pi}^0 > 1$, $\tilde{x}_b$ non-decreasing so that $\underline{\pi}^0$ is non-increasing), the initial transaction determines whether borrowing or no-borrowing is optimal.

(a) $t^{up} > 0$:
In this case borrowing is optimal since $\tilde{x}_b(0) < 0$, i.e. $\underline{\pi}^0(0) > 1$.

(b) $t^{up} = 0$:
If the initial position is such that $x < 0$ (i.e. $b < 0$) then borrowing is optimal, otherwise the investor invests all of her wealth in the stock (since $\tilde{x}_b(0) = 0$ and hence $\underline{\pi}^0(0) = 1$).

(c) $t^{up} < 0$: In this case borrowing is optimal if $x < 0$ (i.e. $b < 0$) and no-borrowing is optimal if $x \geq 0$ (i.e. $b \geq 0$). This is because $\tilde{x}_s(t) < 0 < \tilde{x}_b(t)$ and hence $\underline{\pi}^0(t) < 1 < \overline{\pi}^0(t)$ for all $t \in [0, T)$.

4. In any case, as soon as $t \geq t^{down}$, the investor refrains from buying shares of the stock since $\tilde{x}_b(t) = \infty$, i.e. $\underline{\pi}^0(t) = 0$, see also Remark 2.10.

5. If the initial position (b, s) in the original variables is such that $s \leq 0$ then it is not immediately clear what the optimal strategy is. We expect

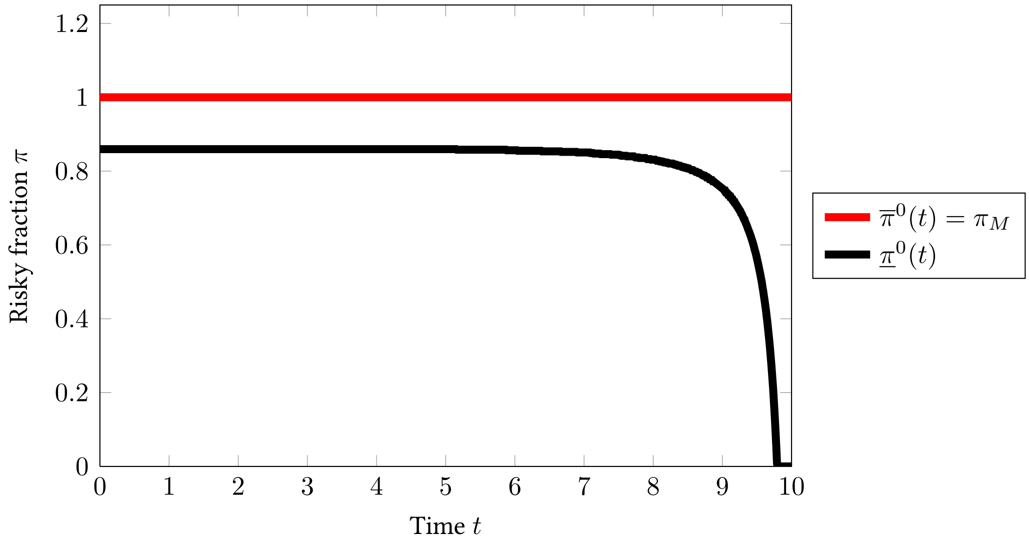

Figure 2.5. The trading regions for $\pi_M = 1$.

that whenever $\tilde{x}_b(t) = \infty$ (i.e. $\underline{\pi}^0(t) = 0$) then it is optimal to liquidate to stock position and refrain from further trading. Whenever $\tilde{x}_b(t) < \infty$ (i.e. $\underline{\pi}^0(t) > 0$) we expect that the investor performs an initial transaction which takes her position on the boundary of the no-trading region. This conjecture is proved in Section 2.6, but intuitively this behavior is clear: Since the excess return α is positive and since the investor has to liquidate her stock holdings at time T it should never be optimal to have a short position in the stock before time T.

Remark 2.11. Note that, given the Merton fraction π_M, the terminal time T and the investor's initial position (b, s), we know whether borrowing is optimal or not. That is, the investor's position never changes from borrowing to no-borrowing (or vice versa) after the initial transaction. With this, Theorem 2.8 and the discussion preceding this remark indicate that we have to distinguish between the borrowing, the no-borrowing, and the liquidation case (i.e. liquidation of either the bond or stock position at initial time and no trading afterwards) for the construction of the optimal strategies. ◇

For the construction of the optimal strategy we need to prove the existence of finite-variation processes $L^* = (L^*(u))_{u\in[t,T]}$ and $M^* = (M^*(u))_{u\in[t,T]}$ which turn the controlled wealth process $(B_{t,b}^{L^*,M^*}, S_{t,s}^{L^*,M^*})$ into a diffusion reflected

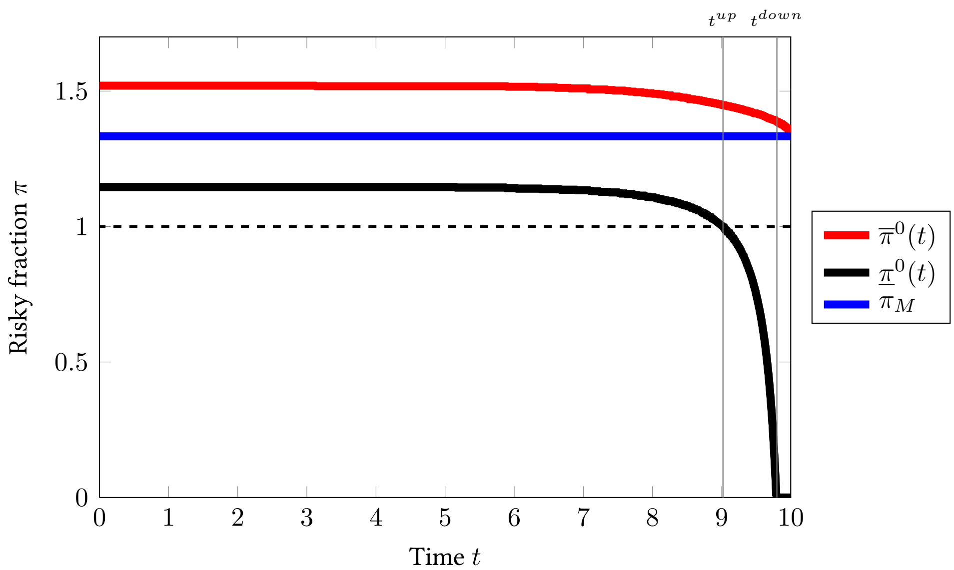

Figure 2.6. The trading regions for $\pi_M > 1$.

at the boundary of $\mathcal{R}_0^{nt}$. The main difficulty is the geometry of $\mathcal{R}_0^{nt}$: It is an unbounded cone changing over time (see Figure 2.3). There is a vast literature on the construction of reflected diffusions, most notably Skorohod [98], Strook and Varadhan [105], Tanaka [107], Lions and Sznitman [73], Dupuis and Ishii [33, 34] and recently Nyström and Önskog [87], but the assumptions are usually very technical and are difficult to verify in our situation. However, Theorem 2.7 and Theorem 2.8 suggest to first reduce the dimension of the problem so that we only have to deal with reflection in a time-dependent interval. This problem is by now well-understood, see e.g. Burdzy et al. [17], Slaby [99] and Słomiński and Wojciechowski [101, 100].

As pointed out in Remark 2.11 we consider the borrowing, no-borrowing and the liquidation cases separately. For this, let us fix an initial datum $(t_0, b_0, s_0) \in [0, T) \times \overline{\mathcal{S}}^0$. For $(b_0, s_0) \in \partial\mathcal{S}^0$ the optimal strategies are known (Lemma 2.2.1), so let us assume further that $(b_0, s_0) \in \mathcal{S}^0$. Next, note that we can without loss of generality assume that the initial position is inside the closure of the no-trading region $\mathcal{R}_0^{nt}$. Indeed, if $(t_0, b_0, s_0) \notin \mathcal{R}_0^{nt}$, then we can find (b^*, s^*) and (minimal) $l, m \geq 0$ such that $(t_0, b^*, s^*) \in \partial\mathcal{R}_0^{nt}$ and

$$b^* = b_0 - (1 + \lambda)l + (1 - \mu)m, \qquad s^* = s_0 + l - m.$$

If (L^*, M^*) is now the candidate optimal strategy for (t_0, b^*, s^*) then (L^*+l, M^*+m) is the candidate optimal strategy for (t_0, b_0, s_0). In other words: By a suitable initial transaction we can always ensure that we start within the closure of the no-trading region.

Comparing with Figure 2.4, we see that in the case of $\pi_M < 1$ we have to distinguish the following cases (after the initial transaction):

Cases for $\pi_M < 1$	
t_0 arbitrary, $s_0 > 0$, $b_0 > 0$	Case (e): No borrowing
$t_0 \geq t^{down}$, $s_0 = 0$, $b_0 > 0$	Case (d): Liquidation (stock)

In the case of $\pi_M = 1$, the following cases may occur after the initial transaction (compare with Figure 2.5):

Cases for $\pi_M = 1$	
t_0 arbitrary, $s_0 > 0$, $b_0 = 0$	Case (a): Liquidation (bond)
$t_0 \geq t^{down}$, $s_0 = 0$, $b_0 > 0$	Case (d): Liquidation (stock)
$t < t^{down}$, $s_0 > 0$, $b_0 > 0$	Case (f): No borrowing

Finally, if $\pi_M > 1$ we have to distinguish the following cases (see Figure 2.6):

Cases for $\pi_M > 1$	
$t_0 < t^{up}$, $s_0 > 0$, $b_0 < 0$	Case (h): Borrowing
$t_0 = t^{up}$, $s_0 > 0$, $b_0 = 0$	Case (b): Liquidation (bond)
$t_0 = t^{up}$, $s_0 > 0$, $b_0 < 0$	Case (h): Borrowing
$t^{up} < t_0 < t^{down}$, $s_0 > 0$, $b_0 < 0$	Case (h): Borrowing
$t^{up} < t_0 < t^{down}$, $s_0 > 0$, $b_0 = 0$	Case (b): Liquidation (bond)
$t^{up} < t_0 < t^{down}$, $s_0 > 0$, $b_0 > 0$	Case (g): No borrowing
$t_0 \geq t^{down}$, $s_0 = 0$, $b_0 > 0$	Case (d): Liquidation (stock)
$t_0 \geq t^{down}$, $s_0 > 0$, $b_0 < 0$	Case (h): Borrowing
$t_0 \geq t^{down}$, $s_0 > 0$, $b_0 = 0$	Case (b): Liquidation (bond)
$t_0 \geq t^{down}$, $s_0 > 0$, $b_0 > 0$	Case (g): No borrowing

The cases (a)-(d) are liquidation cases and can be summed up as follows:

(a) $\pi_M = 1$ with $s_0 > 0$ and $b_0 = 0$.

(b) $\pi_M > 1$ with $s_0 > 0$, $b_0 = 0$ and $t_0 \geq t^{up}$.

(c) $\pi_M > 1$ with $s_0 = 0$, $b_0 > 0$ and $t_0 = t^{up}$.

(d) π_M arbitrary, $t_0 \geq t^{down}$ and $s_0 = 0$.

It is clear that we can exclude these cases in the following since the investor refrains from further trading after the initial transaction. We are hence left with the cases

(e) $\pi_M < 1$ with $s_0, b_0 > 0$,

(f) $\pi_M = 1$ with $s_0, b_0 > 0$,

(g) $\pi_M > 1$ with $s_0, b_0 > 0$ and $t_0 > t^{up}$,

(h) $\pi_M > 1$ with $s_0 > 0, b_0 < 0$.

The cases (e)-(g) are no-borrowing cases whereas we expect borrowing to be optimal in the case (h). It turns out that for the construction of the reflected diffusions it is advantageous to consider the change of variables s/b in the no-borrowing case and $s/(-b)$ in the borrowing case (as opposed to the transformation b/s as considered for the construction of the classical solution of the reduced DPE (2.23)).

By Theorem 2.7 we see that by reversing the transformation in (2.22) we can construct a function V_0 on $[0,T] \times \tilde{\mathcal{S}}^0$ from $\tilde{V}_0$, such that V_0 solves the original DPE given in (2.10) on the reduced state space $[0,T] \times \tilde{\mathcal{S}}^0$. Note that, with this and Theorem 2.8, V_0 is of class $C^{1,2,2}$ except for the points $(t,b,s) \in [0,T] \times \tilde{\mathcal{S}}^0$ on which the boundaries of the no-trading region intersect with the set $\{(t,0,s) : t \in [0,T], (0,s) \in \tilde{\mathcal{S}}^0\}$ (compare with the set F defined in Theorem 2.7). Next, let us define

$$\mathcal{S}^0_+ := \left\{(b,s) \in \mathcal{S}^0 : b > 0, s > 0\right\}, \qquad \mathcal{S}^0_- := \left\{(b,s) \in \mathcal{S}^0 : b < 0, s > 0\right\}.$$

In the sequel we work on the reduced state space $[0,T]\times\mathcal{S}_+^0$ in the no-borrowing cases (e)-(g) and $[0,T]\times\mathcal{S}_-^0$ in the borrowing case (h).

2.5.1. Construction in the no-borrowing case (e)

The main idea for the construction of the optimal strategy is to find a suitable transformation of the state space so that the problem of constructing an obliquely reflected diffusion in an unbounded and time-dependent cone simplifies to normal reflection in a time-dependent interval. The transformation is based on ideas from Gerhold et al. [41]. We restrict ourselves to the case $p<1$, $p\neq 0$ and remark that the construction for the case $p=0$ follows similarly.

Let us first assume that we are in case (e), i.e. $\pi_M<1$ with $s_0,b_0>0$. In particular, $\tilde{V}_0$ is of class $C^{1,2}$ and hence V_0 is of class $C^{1,2,2}$ everywhere. We define

$$l(t):=\begin{cases}1/\tilde{x}_b(t), & \text{if } \tilde{x}_b(t)<\infty,\\ 0, & \text{if } \tilde{x}_b(t)=\infty,\end{cases}\qquad u(t):=\frac{1}{\tilde{x}_s(t)}. \tag{2.27}$$

By Theorem 2.8 we see that $l(t)<u(t)$, $l(t)\in C([0,T))$ and $u(t)\in C^\infty([0,T))$.

On the set $[0,T]\times\mathcal{S}_+^0$ we consider the transformation

$$V_0(t,b,s)=b^p\exp\left(-p\int_{\log(s/(bu(t)))}^0 w(t,y)\,dy\right).$$

Then, setting $x=\log(s/(bu(t)))$,

$$\begin{aligned}\frac{\partial}{\partial t}V_0(t,b,s)&=-pV_0(t,b,s)\left(\int_x^0 w_t(t,y)\,dy+\frac{u'(t)}{u(t)}w(t,x)\right),\\ \frac{\partial}{\partial b}V_0(t,b,s)&=pV_0(t,b,s)\frac{1}{b}\left[1-w(t,x)\right],\\ \frac{\partial}{\partial s}V_0(t,b,s)&=pV_0(t,b,s)\frac{w(t,x)}{s},\\ \frac{\partial^2}{\partial s^2}V_0(t,b,s)&=pV_0(t,b,s)\frac{1}{s^2}\left(w_x(t,x)+pw(t,x)^2-w(t,x)\right).\end{aligned}$$

With this and using that V_0 satisfies $\mathcal{L}^{buy}V_0\geq 0$ and $\mathcal{L}^{sell}V_0\geq 0$ we see that w satisfies

$$1-\mu\leq\frac{w(t,x)}{u(t)(1-w(t,x))e^x}\leq 1+\lambda, \tag{2.28}$$

and where equality holds if $\mathcal{L}^{sell}V_0 = 0$ or $\mathcal{L}^{buy}V_0 = 0$, respectively. Moreover, since $\mathcal{L}^{nt}V_0 = 0$ whenever $\mathcal{L}^{buy}V_0 > 0$ and $\mathcal{L}^{buy}V_0 > 0$, we see that

$$\int_x^0 w_t(t,y)\,dy - \Big(\alpha - \frac{1}{2}\sigma^2 - \frac{u'(t)}{u(t)}\Big)w(t,x) - \frac{1}{2}p\sigma^2 w(t,x)^2 - \frac{1}{2}\sigma^2 w_x(t,x) = 0,$$

whenever $w/u(1-w)e^x \notin \{1-\mu, 1+\lambda\}$. Using that V_0 is C^∞ in the interior of the no-trading region (Theorem 2.8) we can take the derivative with respect to x in the last equation to obtain

$$\frac{1}{2}\sigma^2 w_{xx}(t,x) \\ = -w_t(t,x) - \Big(\alpha - \frac{1}{2}\sigma^2 - \frac{u'(t)}{u(t)}\Big)w_x(t,x) - p\sigma^2 w(t,x)w_x(t,x). \quad (2.29)$$

Consider again the fraction in (2.28), i.e.

$$f(t,x) = \frac{w(t,x)}{u(t)(1-w(t,x))e^x}.$$

Note that, since the points $x = 0$ and $x = \log(l(t)/u(t))$ constitute the boundary points of the no-trading region in the new variables by (2.27), we must have

$$f\big(t,x\big) = 1-\mu, \qquad \text{if } x \geq 0, \quad (2.30)$$
$$f(t,x) = 1+\lambda, \qquad \text{if } x \leq \log(l(t)/u(t)). \quad (2.31)$$

Note that for the point $x = \log(l(t)/u(t))$ these considerations are only valid for $t \in [0, t^{down})$ since otherwise $\log(l(t)/u(t)) = -\infty$.

Remark 2.12. We have

$$f(t,x) = \frac{w(t,x)}{u(t)(1-w(t,x)e^x)} \in [1-\mu, 1+\lambda]$$

and $f(t,x) \in \{1-\mu, 1+\lambda\}$ inside the buy and sell regions. This suggests that

$$f(t, X^*(t))P^1(t)$$

with $X^*(t) = \log(S^*(t)/B^*(t)u(t))$ (and where (B^*, S^*) denotes the optimally controlled wealth process) is the shadow price corresponding to our problem. This is confirmed in Gerhold et al. [41]. ◇

The next step is to construct a reflected diffusion in the time-dependent interval $[log(l(t)/u(t)), 0]$.

Lemma 2.13. *There exists a process* $\Psi = (\Psi(t))_{t\in[t_0,T)}$ *and finite-variation processes* $L = (L(t))_{t\in[t_0,T)}$ *and* $M = (M(t))_{t\in[t_0,T)}$ *such that*

$$d\Psi(t) = \left[\alpha - \frac{1}{2}\sigma^2 - \frac{u'(t)}{u(t)}\right] dt + \sigma\, dW(t) + \mathbb{1}_{[t_0,t^{down})} dL(t) - dM(t), \quad t \in [t_0, T), \tag{2.32}$$

with

$$\Psi(t_0) = \log\left(\frac{s_0}{b_0 u(t_0)}\right),$$

and such that Ψ *is a diffusion reflected on the boundaries of the time-dependent interval* $[\log(l(t)/u(t)), 0]$.

Proof. This follows from Słomiński and Wojciechowski [101, Theorem 3.3] together with Remark 2.9. □

Let us now define a process $N = (N(t))_{t\in[t_0,T)}$ through

$$\begin{aligned} dN(t) = N(t)\left(1 - w\big(t, log(l(t)/u(t))\big)\right) \mathbb{1}_{[t_0,t^{down})}\, dL(t) \\ - N(t)\left(1 - w(t, 0)\right) dM(t), \qquad t \in [t_0, T), \end{aligned}$$

with $N(t_0) = s_0/P^1(t_0)$. Note that since $dL(t) = 0$ for every $t \in [t^{down}, T)$ this can equivalently be written as

$$\begin{aligned} dN(t) = N(t)\left(1 - w\big(t, log(l(t)/u(t))\big)\right) dL(t) \\ - N(t)\left(1 - w(t, 0)\right) dM(t), \qquad t \in [t_0, T). \end{aligned} \tag{2.33}$$

Remark 2.14. Comparing with Gerhold et al. [41], we interpret Ψ as the optimal stock to bond ratio and N as the optimal cumulative number of shares of the stock bought up to time t. Furthermore, again as in Gerhold et al. [41], the function w can be interpreted as the optimal risky fraction and hence $1 - w$ coincides with the optimal fraction of wealth invested into the bond which gives a nice interpretation for all the terms occurring in (2.33). ◇

With this, we now have all the necessary tools at hand to construct the optimal strategies. Let us define a process $S^* = (S^*(t))_{t\in[t_0,T)}$ through $S^*(t) := N(t)P^1(t)$. Then $S^*(t_0) = N(t_0)P^1(t_0) = s_0$ and

$$\begin{aligned} dS^*(t) = \alpha S^*(t)\,dt + \sigma S^*(t)\,dW(t) \\ + S^*(t)\left(1 - w\big(t, log(l(t)/u(t))\big)\right)\,dL(t) \\ - S^*(t)\left(1 - w(t,0)\right)\,dM(t), \qquad t \in [t_0, T). \end{aligned}$$

Furthermore, define $B^* = (B^*(t))_{t\in[t_0,T)}$ by $B^*(t) := S^*(t)e^{-\Psi(t)}/u(t)$ to obtain

$$B^*(t_0) = \frac{S^*(t_0)}{u(t)e^{\Psi(t_0)}} = b_0$$

and

$$\begin{aligned} dB^*(t) = -w\big(t, log(l(t)/u(t))\big)B^*(t)\,dL(t) \\ + w(t,0)B^*(t)\,dM(t), \qquad t \in [t_0, T). \end{aligned}$$

Using the definition of B^*, (2.30) and (2.31) we see that

$$w(t,0)B^*(t) = \frac{w(t,0)S^*(t)}{u(t)e^0} = (1-\mu)S^*(t)(1 - w(t,0)),$$

and similarly

$$\begin{aligned} w\big(t, log(l(t)/u(t))\big)B^*(t) &= \frac{w\big(t, log(l(t)/u(t))\big)S^*(t)}{u(t)e^{\log(l(t)/u(t))}} \\ &= (1+\lambda)S(t)^*\left(1 - w\big(t, log(l(t)/u(t))\big)\right). \end{aligned}$$

So, in total, the dynamics of B^* simplify to

$$\begin{aligned} dB^*(t) = -(1+\lambda)S^*(t)\left(1 - w\big(t, log(l(t)/u(t))\big)\right)\,dL(t) \\ + (1-\mu)S^*(t)\left(1 - w(t,0)\right)\,dM(t), \qquad t \in [t_0, T). \end{aligned}$$

Hence, if we define

$$\begin{aligned} dL^*(t) &= N(t)P^1(t)\left(1 - w\big(t, log(l(t)/u(t))\big)\right)\,dL(t), \qquad && t \in [t_0, T), \\ dM^*(t) &= N(t)P^1(t)\left(1 - w(t,0)\right)\,dM(t), && t \in [t_0, T), \end{aligned}$$

with $L^*(t_0) = 0$, $M^*(t_0) = 0$ and if we set (liquidation at terminal time)

$$\begin{aligned} L^*(T) &= L^*(T-) + S^*(T-)\mathbb{1}_{\{S^*(T-)<0\}}, \\ M^*(T) &= M^*(T-) + S^*(T-)\mathbb{1}_{\{S^*(T-)>0\}}, \\ S^*(T) &= 0, \\ B^*(T) &= B^*(T-) + \min\{(1-\mu)S^*(T-), (1+\lambda)S^*(T-)\}, \end{aligned}$$

then $(L^*, M^*) \in \mathcal{A}_0(t_0, b_0, s_0)$, and (B^*, S^*) is a diffusion reflected on the boundary of $\mathcal{R}_0^{nt}$.

2.5.2. Construction in the other cases

The construction in the other cases (f)-(h) follows in a similar way as the construction in the case (e). We outline some of the details in the sequel.

Let us first assume that we are in one of the no-borrowing cases (f) or (g). That is, we either have $\pi_M = 1$ with $s_0, b_0 > 0$, or $\pi_M > 1$ with $s_0, b_0 > 0$ and $t_0 > t^{up}$. While the construction here is similar to the case (e) we have to be more careful since the upper boundary in terms of the transformation s/b is now equal to infinity which does not allow us to consider the transformation $x = \log(s/bu(t))$. However, since the upper boundary is now equal to infinity, we deal with one-sided reflection which simplifies matters again (we never have to sell shares of the stock!). As before, we define the lower boundary $l(t)$ to be given by

$$l(t) := \begin{cases} 1/\tilde{x}_b(t), & \text{if } \tilde{x}_b(t) < \infty, \\ 0, & \text{if } \tilde{x}_b(t) = \infty, \end{cases}$$

and consider the slightly different transformation

$$V_0(t, b, s) = b^p \exp\left(-p \int_{\log(s/b)}^{0} w(t, y)\, dy\right).$$

Setting $x = \log(s/b)$ and arguing in a similar fashion as before the existence of the candidate optimal strategy follows. Note, however, that the process Ψ now has to be be constructed without the $u'(t)/u(t)$ term in its drift (see (2.32)).

Let us now turn to the borrowing case (h). That is, assume $\pi_M > 1$, $s_0 > 0$, $b_0 < 0$ and $t_0 \in [0, T)$. Since $b_0 < 0$ and since we want the optimally controlled bond wealth B^* to satisfy $B^* < 0$, we have to consider a different transformation. More precisely, we consider the transformation $s/(-b)$ instead. We first define the trading boundaries to be

$$l(t) = -\frac{1}{\tilde{x}_s(t)}, \qquad u(t) = \begin{cases} -1/\tilde{x}_b(t), & \text{if } t < t^{up} \\ \infty, & \text{if } t \geq t^{up}. \end{cases}$$

Theorem 2.8 implies $0 < l(t) < u(t) \leq \infty$, $l(t) \in C^\infty([0, T))$ and $u(t) \in C([0, T))$ (note, however, that in this case the lower boundary l is defined by means of $\tilde{x}_s$ instead of $\tilde{x}_b$). The correct transformation of the function V_0 has then to be chosen to be

$$V_0(t,b,s) = (-b)^p \exp\left(-p \int_0^{\log(s/(-bl(t)))} w(t,y)\,dy\right),$$

where we restrict V_0 to the set $\mathcal{S}^0_-$. This leads to similar calculations as in the case (e) but with the lower boundary l in place of the upper boundary u in the drift term of the process Ψ (see (2.32)).

2.6. Verification and value function regularity

We now proceed by verifying that the strategies constructed in the previous section are indeed optimal. Since V_0 is only defined on $[0,T] \times \tilde{\mathcal{S}}^0$ classical verification arguments are difficult (it will turn out that $\mathcal{V}_0$ is not sufficiently regular everywhere, see Theorem 2.28). Instead, we adapt the approach introduced in Christensen [20] for impulse control problems to our setting: The idea is to show that the value function is the point-wise minimum within a suitable class of superharmonic functions.

More precisely, let us denote by $\mathbb{H}$ the set of all continuous functions $h : [0,T] \times \overline{\mathcal{S}}^0 \to \mathbb{R}$ satisfying the following properties:

(i) $h(T,b,s) \geq \mathcal{V}_0(T,b,s)$ on $\{T\} \times \overline{\mathcal{S}}^0$.

(ii) h is non-increasing in the direction of transactions, i.e. whenever $(t,b,s) \in [0,T] \times \overline{\mathcal{S}}^0$ and if $l,m \geq 0$ are such that $(b-(1+\lambda)l+(1-\mu)m, s+l-m) \in \overline{\mathcal{S}}^0$, then

$$h(t,b,s) \geq h(t, b-(1+\lambda)l+(1-\mu)m, s+l-m).$$

(iii) h is space-time superharmonic with respect to the uncontrolled wealth process. More precisely, denote by $(B^0, S^0) = (B^0_{t,b}, S^0_{t,s})$ the wealth process corresponding to the strategy $L \equiv M \equiv 0$ and let ϑ be the first hitting time of $\partial\mathcal{S}^0$. Then h is called space-time superharmonic (or superharmonic for short) if and only if

$$h(t,b,s) \geq \mathbb{E}\left[h\big(\tau \wedge \vartheta, B^0_{t,b}(\tau \wedge \vartheta), S^0_{t,s}(\tau \wedge \vartheta)\big)\right]$$

for every $[t, T]$-valued stopping time τ.

(iv) There exists $\gamma \in (1-\mu, 1+\lambda)$ and $K > 1$ such that

$$U_p(b + \min\{(1-\mu)s, (1+\lambda)s\}) \leq h(t,b,s) \leq \varphi_{\gamma,p,K}(t,b,s).$$

We expect that $\mathcal{V}_0$ is the point-wise minimum of the elements of $\mathbb{H}$. If this is true we can prove the optimality of (L^*, M^*) as follows:

1. Show that every $h \in \mathbb{H}$ dominates $\mathcal{V}_0$.

2. Define the function

$$h_0(t,b,s) := \mathbb{E}\left[U_p\left(X_{t,b,s}^{L^*,M^*}(T)\right)\right]$$

   and show that $h_0 \in \mathbb{H}$.

It follows that $\mathcal{V}_0 \leq h_0$, but $h_0 \leq \mathcal{V}_0$ since (L^*, M^*) is admissible. Hence $h_0 = \mathcal{V}_0$ and (L^*, M^*) is optimal.

In Lemma 2.15 below we show that every $h \in \mathbb{H}$ dominates $\mathcal{V}_0$. Then we proceed by analyzing the regularity of h_0 and use these results to show that h_0 is superharmonic in Proposition 2.25 and show that h_0 is non-increasing in the direction of transactions in Proposition 2.26. The optimality of (L^*, M^*) then follows in Corollary 2.27.

Lemma 2.15. *Let $h \in \mathbb{H}$. Then $\mathcal{V}_0 \leq h$.*

Proof. We show that h is a viscosity supersolution of the DPE (2.10). By Theorem 2.5 (comparison principle) it then follows that for every $\varepsilon > 0$ we have $\mathcal{V}_0(t,b,s) \leq h(t, b+\varepsilon, s)$ everywhere and by the continuity of h we can send $\varepsilon \downarrow 0$ to conclude.

Let us therefore fix $(t_0, b_0, s_0) \in [0,T) \times \mathcal{S}^0$ and let $\varphi \in C^{1,2,2}([0,T) \times \mathcal{S}^0)$ be such that $\varphi \leq h$ and $\varphi(t_0,b_0,s_0) = h(t_0,b_0,s_0)$. We have to show that

$$\min\left\{\mathcal{L}^{nt}\varphi(t_0,b_0,s_0), \mathcal{L}^{buy}\varphi(t_0,b_0,s_0), \mathcal{L}^{sell}\varphi(t_0,b_0,s_0)\right\} \geq 0.$$

Let $l > 0$ be such that $(b_0 - (1+\lambda)s, s+l) \in \mathcal{S}^0$. Then

$$\begin{aligned}\varphi(t_0, b_0, s_0) - \varphi(t_0, b_0 - (1+\lambda)l, s+l) \\ \geq h(t_0, b_0, s_0) - h(t_0, b_0 - (1+\lambda)l, s+l) \geq 0\end{aligned}$$

since $\varphi(t_0, b_0, s_0) = h(t_0, b_0, s_0)$, $\varphi \leq h$ and since h is non-increasing in the direction of transactions. Now divide by l end send $l \downarrow 0$ to obtain

$$\mathcal{L}^{buy}\varphi(t_0, b_0, s_0) \geq 0.$$

By similar arguments we can show that

$$\mathcal{L}^{sell}\varphi(t_0, b_0, s_0) \geq 0$$

and hence it only remains to show that

$$\mathcal{L}^{nt}\varphi(t_0, b_0, s_0) \geq 0.$$

Suppose that on the contrary we have

$$\mathcal{L}^{nt}\varphi(t_0, b_0, s_0) < 0.$$

Then there exist $\varepsilon, \delta > 0$ such that $t_0 + \varepsilon < T$, $\overline{B}_\varepsilon(b_0, s_0) \subset \mathcal{S}^0$ and

$$\mathcal{L}^{nt}\varphi(t, b, s) < -\delta$$

for all $(t, b, s) \in [t_0, t_0 + \varepsilon] \times \overline{B}_\varepsilon(b_0, s_0)$. Now define the stopping time

$$\tau_\varepsilon := \inf\left\{u \geq t_0 : (B^0_{t_0,b_0}(u), S^0_{t_0,s_0}(u)) \notin \overline{B_\varepsilon}(b_0, s_0)\right\} \wedge (t_0 + \varepsilon).$$

Since h is space-time superharmonic and by Itô's formula we have

$$\begin{aligned}\varphi(t_0, b_0, s_0) &= h(t_0, b_0, s_0) \\ &\geq \mathbb{E}\left[h(\tau_\varepsilon, B^0_{t_0,b_0}(\tau_\varepsilon), S^0_{t_0,s_0}(\tau_\varepsilon))\right] \\ &\geq \mathbb{E}\left[\varphi(\tau_\varepsilon, B^0_{t_0,b_0}(\tau_\varepsilon), S^0_{t_0,s_0}(\tau_\varepsilon))\right] \\ &= \varphi(t_0, b_0, s_0) - \mathbb{E}\left[\int_{t_0}^{\tau_\varepsilon} \mathcal{L}^{nt}\varphi(u, B^0_{t_0,b_0}(u), S^0_{t_0,s_0}(u))\, du\right],\end{aligned}$$

i.e.

$$\mathbb{E}\left[\int_{t_0}^{\tau_\varepsilon} \mathcal{L}^{nt}\varphi(u, B^0_{t_0,b_0}(u), S^0_{t_0,s_0}(u))\, du\right] \geq 0.$$

This must however imply that

$$\max_{\substack{u\in[t_0,t_0+\varepsilon]\\(b,s)\in\overline{B}_\varepsilon(b_0,s_0)}} \mathcal{L}^{nt}\varphi(u,b,s)\geq 0.$$

Sending $\varepsilon\downarrow 0$ hence implies that

$$\mathcal{L}^{nt}\varphi(t_0,b_0,s_0)\geq 0$$

which is a contradiction. □

In Section 2.5 and Lemma 2.2.1 we have constructed the candidate optimal strategies $(L^*,M^*)=(L^*_{t,b,s}(u),M^*_{t,b,s}(u))_{u\in[t,T]}$ for every $(t,b,s)\in[0,T)\times\overline{\mathcal{S}}^0$. Moreover, it is obvious that the candidate optimal strategy $(L^*_{T,b,s}(u),M^*_{T,b,s}(u))$ is the strategy which merely liquidates the stock position s. This allows us to define the function

$$h_0(t,b,s):=\mathbb{E}\left[U_p\left(X^{L^*,M^*}_{t,b,s}(T)\right)\right],\qquad (t,b,s)\in[0,T]\times\overline{\mathcal{S}}^0. \tag{2.34}$$

Our next aim is to show that $h_0\in\mathbb{H}$ and hence $h_0=\mathcal{V}_0$ and (L^*,M^*) is optimal. As a first step, we show that h_0 coincides with V_0 on $[0,T]\times\tilde{\mathcal{S}}^0$.

Proposition 2.16. *The function h_0 defined in* (2.34) *coincides with the classical solution V_0 of the DPE on the reduced state space $[0,T]\times\tilde{\mathcal{S}}^0$.*

Proof. Let $(t,b,s)\in[0,T)\times\tilde{\mathcal{S}}^0$. If (t,b,s) is such that we are in one of the liquidation cases (a), (b), or (c), then direct computations reveal that $h_0(t,b,s)=V_0(t,b,s)$ since V_0 is explicitly known at these points (cf. Dai and Yi [24, Proposition 3.2]). For example, assume that $p=0$, $\pi_M>1$ and $(t,b,s)=(t^{up},0,s)$. Then Dai and Yi [24, Proposition 3.2] show that

$$\begin{aligned}
V_0(t,b,s)&=\log(s)+\tilde{V}_0(t,0)\\
&=\log(s)+\log(1-\mu)+\Big(\alpha-\frac{1}{2}\sigma^2\Big)(T-t)\\
&=\mathbb{E}\Big[\log((1-\mu)S^{L^*,M^*}_{t,s}(T))\Big]\\
&=\mathbb{E}\Big[U_0(b+(1-\mu)S^{L^*,M^*}_{t,s}(T))\Big]\\
&=h_0(t,b,s).
\end{aligned}$$

We therefore exclude these cases in the sequel. For ease of notation, we denote the controlled processes $(B^{L^*,M^*}_{t,b,s}, S^{L^*,M^*}_{t,b,s})$ by (B^*, S^*).

First, let us remark that by the gradient theorem

$$V_0(t,b,s) = V_0\Big(t, b-(1+\lambda)L^*(t)+(1-\mu)M^*(t), s+L^*(t)-M^*(t)\Big)$$

since $L^*(t) \neq 0$ only if $(t,b,s) \in \mathcal{R}_0^{buy}$ and $M^*(t) \neq 0$ only if $(t,b,s) \in \mathcal{R}_0^{sell}$. For $n \in \mathbb{N}$ define the stopping times

$$\tau_n := \inf\left\{u \geq t : \int_t^u \left[\sigma S^*(r)\frac{\partial}{\partial s}V_0(r, B^*(r), S^*(r))\right]^2 dr \geq n\right\} \wedge T.$$

Since (B^*, S^*) has continuous paths after the initial transaction and V_0 is of class $C^{1,2,2}$ along the paths of (u, B^*, S^*) we can apply Itô's formula to obtain

$$\begin{aligned}
V_0(t,b,s) &= V_0\Big(t, b-(1+\lambda)L^*(t)+(1-\mu)M^*(t), s+L^*(t)-M^*(t)\Big)\\
&= V_0(\tau_n, B^*(\tau_n), S^*(\tau_n)) + \int_t^{\tau_n} \mathcal{L}^{nt}V_0(u, B^*(u), S^*(u))\,du\\
&\qquad + \int_t^{\tau_n} \mathcal{L}^{buy}V_0(u, B^*(u), S^*(u))\,dL^*(u)\\
&\qquad + \int_t^{\tau_n} \mathcal{L}^{sell}V_0(u, B^*(u), S^*(u))\,dM^*(u)\\
&\qquad - \int_t^{\tau_n} \sigma S^*(u)\frac{\partial}{\partial s}V_0(u, B^*(u), S^*(u))\,dW(u)\\
&= V_0(\tau_n, B^*(\tau_n), S^*(\tau_n)) - \int_t^{\tau_n} \sigma S^*(u)\frac{\partial}{\partial s}V_0(u, B^*(u), S^*(u))\,dW(u)
\end{aligned}$$

since V_0 is a classical solution of the DPE and by the construction of (L^*, M^*). Taking expectations on both sides shows that

$$V_0(t,b,s) = \mathbb{E}\left[V_0(\tau_n, B^*(\tau_n), S^*(\tau_n))\right].$$

We are left with showing that

$$\lim_{n\to\infty} \mathbb{E}\left[V_0(\tau_n, B^*(\tau_n), S^*(\tau_n))\right] = \mathbb{E}\left[U_p\left(X^{L^*,M^*}_{t,b,s}(T)\right)\right] = h_0(t,b,s). \tag{2.35}$$

We first note that there exists a constant $C > 0$ such that

$$V_0(t,b,s) = C\mathcal{V}_0(t,b,s) \tag{2.36}$$

for all $(t,b,s) \in \overline{\mathcal{R}}_0^{nt}$. Indeed, by Lemma 2.1.2 and Lemma 2.2.2 we have

$$U_p(b + (1-\mu)s) \leq \mathcal{V}_0(t,b,s) \leq \varphi_{1,p,1}(t,b,s)$$

and since $(\partial/\partial t)V_0 \geq 0$ and by the classical maximum principle (see Evans [38, Theorem 7.11]) we similarly have

$$U_p(b + (1-\mu)s) \leq V_0(t,b,s) \leq \varphi_{1,p,1}(t,b,s).$$

Now since V_0 and $\mathcal{V}_0$ are both homogeneous of order p we can write (for $p \neq 0$, the case $p = 0$ is similar)

$$V_0(t,b,s) = (b+s)^p \bar{V}_0(t, 1-\pi, \pi) \quad \text{and} \quad \mathcal{V}_0(t,b,s) = (b+s)^p \bar{\mathcal{V}}_0(t, 1-\pi, \pi)$$

where $\pi := s/(b+s)$. Since U_p and $\varphi_{1,p,1}$ are homogeneous as well it follows that $\bar{V}_0(t, 1-\pi, \pi)$ and $\bar{\mathcal{V}}_0(t, 1-\pi, \pi)$ are bounded in the no-trading region and hence (2.36) follows.

Let us now turn to (2.35). We consider the cases $p \in (0,1)$, $p = 0$ and $p < 0$ separately.

Case 1: $p \in (0,1)$.
We claim that the sequence $(V_0(\tau_n, B^*(\tau_n), S^*(\tau_n)))_{n\in\mathbb{N}}$ is uniformly integrable, in which case (2.35) clearly holds. Let $\varepsilon > 0$ such that $p(1+\varepsilon) < 1$. Then

$$\begin{aligned} 0 &\leq \mathbb{E}\left[|V_0(\tau_n, B^*(\tau_n), S^*(\tau_n))|^{1+\varepsilon}\right] \\ &\leq \mathbb{E}\left[|\varphi_{1,p,1}(\tau_n, B^*(\tau_n), S^*(\tau_n))|^{1+\varepsilon}\right] \\ &\leq \frac{1+\varepsilon}{p^\varepsilon}\mathbb{E}\left[\varphi_{1,p(1+\varepsilon),1}(\tau_n, B^*(\tau_n), S^*(\tau_n))\right] \\ &\leq \frac{1+\varepsilon}{p^\varepsilon}\varphi_{1,p(1+\varepsilon),1}(t,b,s), \end{aligned}$$

where the last inequality follows from the proof of Lemma 2.1.2

Case 2: $p < 0$.
We write

$$\begin{aligned} \mathbb{E}\left[V_0(\tau_n, B^*(\tau_n), S^*(\tau_n))\right] = {} & \mathbb{E}\left[V_0(\tau_n, B^*(\tau_n), S^*(\tau_n))\mathbb{1}_{\{\tau_n < T\}}\right] \\ & + \mathbb{E}\left[V_0(\tau_n, B^*(\tau_n), S^*(\tau_n))\mathbb{1}_{\{\tau_n = T\}}\right]. \end{aligned}$$

Since by monotone convergence

$$\begin{aligned}\lim_{n\to\infty} \mathbb{E}\left[V_0(\tau_n, B^*(\tau_n), S^*(\tau_n))\mathbb{1}_{\{\tau_n=T\}}\right] &= \lim_{n\to\infty} \mathbb{E}\left[U_p\left(X_{t,b,s}^{L^*,M^*}(T)\right)\mathbb{1}_{\{\tau_n=T\}}\right] \\ &= \mathbb{E}\left[U_p\left(X_{t,b,s}^{L^*,M^*}(T)\right)\right]\end{aligned}$$

we only have to show that

$$\lim_{n\to\infty} \mathbb{E}\left[V_0(\tau_n, B^*(\tau_n), S^*(\tau_n))\mathbb{1}_{\{\tau_n<T\}}\right] = 0.$$

We have

$$\begin{aligned}0 &\geq \lim_{n\to\infty} \mathbb{E}\left[V_0(\tau_n, B^*(\tau_n), S^*(\tau_n))\mathbb{1}_{\{\tau_n<T\}}\right] \\ &\geq C \lim_{n\to\infty} \mathbb{E}\left[\mathcal{V}_0(\tau_n, B^*(\tau_n), S^*(\tau_n))\mathbb{1}_{\{\tau_n<T\}}\right] \\ &\geq C \lim_{n\to\infty} \mathbb{E}\left[U_p\left(X_{t,b,s}^{L^*,M^*}(T)\right)\mathbb{1}_{\{\tau_n<T\}}\right] \\ &= 0\end{aligned}$$

by monotone convergence.

Case 3: $p = 0$.
This follows from case 1 and case 2 by splitting up $V_0(\tau_n, B^*(\tau_n), S^*(\tau_n))$ into its positive and negative part and using that

$$\frac{1}{-p}x^{-p} \leq \log(x) \leq \frac{1}{p}x^p$$

for every $p \in (0,1)$. □

Remark 2.17. Note that the proof of the previous theorem shows that the process

$$\left(V_0(u, B_{t,b}^{L^*,M^*}(u), S_{t,s}^{L^*,M^*}(u))\right)_{u\in[t,T]}$$

is an $\mathbb{F}^t$-martingale whenever $(b,s) \in \tilde{\mathcal{S}}^0$. Indeed, we just have to replace the sequence of stopping times $(\tau_n)_{n\in\mathbb{N}}$ by the sequence $(\tau_n \wedge u)_{n\in\mathbb{N}}$ to obtain

$$\begin{aligned}V_0(t,b,s) &= \lim_{n\to\infty} \mathbb{E}\left[V_0(\tau_n\wedge u, B_{t,b}^{L^*,M^*}(\tau_n\wedge u), S_{t,s}^{L^*,M^*}(\tau_n\wedge u))\middle|\mathcal{F}^t(t)\right] \\ &= \mathbb{E}\left[V_0(u, B_{t,b}^{L^*,M^*}(u), S_{t,s}^{L^*,M^*}(u))\middle|\mathcal{F}^t(t)\right].\end{aligned}$$

This indicates that we really may expect $V_0 = \mathcal{V}_0$ by the martingale optimality principle. ◇

Proposition 2.16 proves the regularity of h_0 on $[0,T)\times\tilde{\mathcal{S}}^0$. The following lemmas investigate the regularity of h_0 for $s\leq 0$.

Lemma 2.18. *$h_0 \in C^{1,2,2}([0,t^{down})\times\mathcal{S}^0\setminus\tilde{\mathcal{S}}^0)$. Moreover, for every $(t,b,s)\in[0,t^{down})\times\mathcal{S}^0\setminus\tilde{\mathcal{S}}^0$ and every $l^*>0$ such that $(t,b-(1+\lambda)l^*,s+l^*)\in\mathcal{R}_0^{buy}$ and $s+l^*>0$ we have*

$$\begin{aligned}
\frac{\partial}{\partial t}h_0(t,b,s) &= \frac{\partial}{\partial t}V_0(t,b,s)|_{(b,s)=(b-(1+\lambda)l^*,s+l^*)},\\
\frac{\partial}{\partial b}h_0(t,b,s) &= \frac{\partial}{\partial b}V_0(t,b,s)|_{(b,s)=(b-(1+\lambda)l^*,s+l^*)},\\
\frac{\partial}{\partial s}h_0(t,b,s) &= \frac{\partial}{\partial s}V_0(t,b,s)|_{(b,s)=(b-(1+\lambda)l^*,s+l^*)},\\
\frac{\partial^2}{\partial s^2}h_0(t,b,s) &= \frac{\partial^2}{\partial s^2}V_0(t,b,s)|_{(b,s)=(b-(1+\lambda)l^*,s+l^*)}.
\end{aligned}$$

Proof. The idea is to bootstrap the regularity of h_0 for non-positive s from the regularity of $h_0=V_0$ for positive s inside the buy region $\mathcal{R}_0^{buy}$. See Figure 2.7 for a sketch.

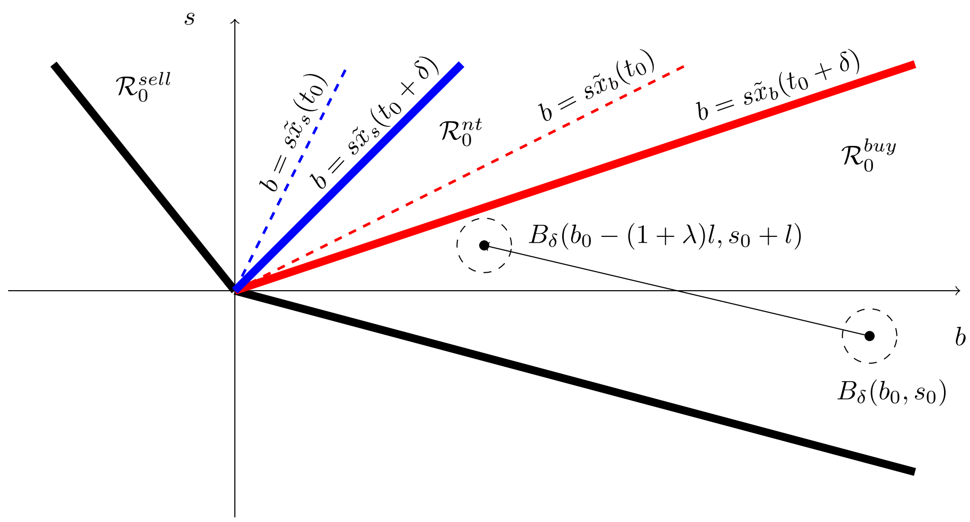

Figure 2.7. Bootstrapping the regularity of h_0.

Fix $(t_0,b_0,s_0)\in[0,T]\times\mathcal{S}^0$ with $t_0<t^{down}$ and $s_0\leq 0$ and let $\delta>0$ be small enough such that $t_0+\delta<t^{down}$ and $B_\delta(b_0,s_0)\subset\mathcal{S}^0$. By making δ smaller if necessary we may furthermore assume that $[t_0,t_0+\delta)\times B_\delta(b_0,s_0)\subset\mathcal{R}_0^{buy}$.

Now, for every $(t, b, s) \in [t_0, t_0 + \delta) \times B_\delta(b_0, s_0)$ there exists some $l_0 > 0$ such that $(b - (1 + \lambda)l_0, s + l_0) \in \partial\mathcal{R}_0^{nt} \cap \partial\mathcal{R}_0^{buy}$ and $s + l_0 > 0$. Moreover, by the construction of h_0 we have

$$h_0(t, b, s) = h_0(t, b - (1 + \lambda)l, s + l)$$

for every $l \in [0, l_0]$. By the monotonicity of the buy boundary $\tilde{x}_b$, making δ even smaller if necessary, we can therefore find some $l^* \in (0, l_0)$ such that $[t_0, t_0 + \delta) \times B_\delta(b_0 - (1 + \lambda)l^*, s_0 + l^*)$ is contained in the interior of $\mathcal{R}_0^{buy}$ and $s > 0$ for all $(b, s) \in B_\delta(b_0 - (1 + \lambda)l^*, s_0 + l^*)$. Note that by construction and Proposition 2.16 we have

$$h_0(t, b, s) = h_0(t, b - (1 + \lambda)l^*, s + l^*) = V_0(t, b - (1 + \lambda)l^*, s + l^*)$$

for all $(t, b, s) \in [t_0, t_0 + \delta) \times B_\delta(b_0, s_0)$. Since V_0 is $C^{1,2,2}$ in $[t_0, t_0 + \delta) \times B_\delta(b_0 - (1 + \lambda)l^*, s_0 + l^*)$ the result follows. □

Proposition 2.19. *h_0 satisfies*

$$\mathcal{L}^{nt} h_0(t, b, s) \geq 0, \quad \mathcal{L}^{buy} h_0(t, b, s) = 0 \quad \textit{and} \quad \mathcal{L}^{sell} h_0(t, b, s) > 0$$

in the classical sense on $[0, t^{down}) \times \mathcal{S}^0 \setminus \tilde{\mathcal{S}}^0$.

Proof. It follows immediately from Lemma 2.18 that

$$\mathcal{L}^{buy} h_0(t, b, s) = \mathcal{L}^{buy} V_0(t, b - (1 + \lambda)l^*, s + l^*) = 0, \tag{2.37}$$

$$\mathcal{L}^{sell} h_0(t, b, s) = \mathcal{L}^{sell} V_0(t, b - (1 + \lambda)l^*, s + l^*) > 0, \tag{2.38}$$

for a suitable choice of l^*. From (2.37) we obtain

$$\frac{\partial}{\partial s} h_0(t, b, s) = (1 + \lambda) \frac{\partial}{\partial b} h_0(t, b, s).$$

Plugging this into (2.38) yields

$$(1 - \mu) \frac{\partial}{\partial b} h_0(t, b, s) < \frac{\partial}{\partial s} h_0(t, b, s) = (1 + \lambda) \frac{\partial}{\partial b} h_0(t, b, s)$$

which implies that $(\partial/\partial b)h_0(t, b, s) > 0$ and hence $(\partial/\partial s)h_0(t, b, s) > 0$ for all $(b, s) \in \mathcal{S}^0 \setminus \tilde{\mathcal{S}}^0$. It only remains to show that

$$\mathcal{L}^{nt} h_0(t, b, s) > 0.$$

Consider the case $s = 0$. Fix some $l^* > 0$ such that $(t, b, l^*) \in \mathcal{R}_0^{buy}$ so that

$$\frac{\partial}{\partial t} h_0(t, b, 0) = \frac{\partial}{\partial t} V_0(t, b - (1+\lambda)l^*, l^*) \leq 0$$

by Theorem 2.7. Therefore,

$$\mathcal{L}^{nt} h_0(t, b, 0) = -\frac{\partial}{\partial t} h_0(t, b, 0) \geq 0. \tag{2.39}$$

Suppose now that $s < 0$. Then for some suitable $l^* > 0$ we have

$$\begin{aligned}
\frac{\partial}{\partial t} h_0(t, b, s) &= \frac{\partial}{\partial t} V_0(t, b, s)|_{(b,s)=(b-(1+\lambda)l^*, s+l^*)} = \frac{\partial}{\partial t} h_0(t, b-(1+\lambda)s, 0), \\
\frac{\partial}{\partial s} h_0(t, b, s) &= \frac{\partial}{\partial s} V_0(t, b, s)|_{(b,s)=(b-(1+\lambda)l^*, s+l^*)} = \frac{\partial}{\partial s} h_0(t, b-(1+\lambda)s, 0), \\
\frac{\partial^2}{\partial s^2} h_0(t, b, s) &= \frac{\partial^2}{\partial s^2} V_0(t, b, s)|_{(b,s)=(b-(1+\lambda)l^*, s+l^*)} = \frac{\partial^2}{\partial s^2} h_0(t, b-(1+\lambda)s, 0).
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\mathcal{L}^{nt} h_0(t, b, s) \\
&\quad = -\frac{\partial}{\partial t} h_0(t, b, s) - \alpha s \frac{\partial}{\partial s} h_0(t, b, s) - \frac{1}{2}\sigma^2 s^2 \frac{\partial^2}{\partial s^2} h_0(t, b, s) \\
&\quad = -\frac{\partial}{\partial t} h_0(t, b-(1+\lambda)s, 0) - \alpha s \frac{\partial}{\partial s} h_0(t, b-(1+\lambda)s, 0) \\
&\qquad\qquad - \frac{1}{2}\sigma^2 s^2 \frac{\partial^2}{\partial s^2} h_0(t, b-(1+\lambda)s, 0)
\end{aligned}$$

By (2.39) we have

$$-\frac{\partial}{\partial t} h_0(t, b-(1+\lambda)s, 0) = \mathcal{L}^{nt} h_0(t, b-(1+\lambda)s, 0) \geq 0.$$

Moreover, since $s < 0$ and $(\partial/\partial s) h_0(t, b-(1+\lambda)s, 0) > 0$ we have

$$-\alpha s \frac{\partial}{\partial s} h_0(t, b-(1+\lambda)s, 0) > 0$$

and since $(\partial^2/\partial s^2) h_0(t, b-(1+\lambda)s, 0) \leq 0$ (since V_0 is concave by Dai and Yi [24, Remark 4.2]) we see that

$$-\frac{1}{2}\sigma^2 s^2 \frac{\partial^2}{\partial s^2} h_0(t, b-(1+\lambda)s, 0) \geq 0.$$

Putting the pieces together we obtain

$$\mathcal{L}^{nt} h_0(t,b,s) > 0,$$

which completes the proof. □

We have similar statements for the case $t \geq t^{down}$ with $s < 0$, the proofs are however significantly easier.

Lemma 2.20. *We have $h_0 \in C^\infty([t^{down}, T) \times \mathcal{S}^0 \setminus \{(b,s) \in \mathcal{S}^0 : s \geq 0\})$ and h_0 is given explicitly as*

$$h_0(t,b,s) = U_p(b + (1+\lambda)s)$$

on $[t^{down}, T) \times \mathcal{S}^0 \setminus \{(b,s) \in \mathcal{S}^0 : s > 0\}$.

Proof. This is an immediate consequence of the definition of h_0 and (L^*, M^*). Indeed, if $(t,b,s) \in [t^{down}, T) \times \mathcal{S}^0 \setminus \{(b,s) \in \mathcal{S}^0 : s > 0\}$, then (L^*, M^*) is such that the stock position is immediately liquidated and the investor refrains from further trading. That is

$$\begin{aligned} h_0(t,b,s) &= h_0(t, b + (1+\lambda)s, 0) \\ &= \mathbb{E}\left[U_p\left(B^{L^*,M^*}_{t,b+(1+\lambda)s}(T) + 0\right)\right] = U_p(b + (1+\lambda)s) \end{aligned}$$

from which the assertion of the lemma follows. □

Proposition 2.21. *h_0 satisfies*

$$\mathcal{L}^{nt} h_0(t,b,s) > 0, \quad \mathcal{L}^{buy} h_0(t,b,s) = 0 \quad \text{and} \quad \mathcal{L}^{sell} h_0(t,b,s) > 0$$

in the classical sense on $[t^{down}, T) \times \mathcal{S}^0 \setminus \{(b,s) \in \mathcal{S}^0 : s \geq 0\}$.

Proof. We consider the case $p < 1, p \neq 0$. The case $p = 0$ follows similarly. We have

$$\begin{aligned} &\frac{\partial}{\partial t} h_0(t,b,s) = 0, \\ &\frac{\partial}{\partial b} h_0(t,b,s) = (b + (1+\lambda)s)^{p-1}, \\ &\frac{\partial}{\partial s} h_0(t,b,s) = (1+\lambda)(b + (1+\lambda)s)^{p-1}, \end{aligned}$$

$$\frac{\partial}{\partial s} h_0(t,b,s) = -(1-p)(1+\lambda)^2(b+(1+\lambda)s)^{p-2},$$

and hence direct calculations show that

$$\begin{aligned}&\mathcal{L}^{buy} h_0(t,b,s) = 0,\\ &\mathcal{L}^{sell} h_0(t,b,s) = (\lambda+\mu)(b+(1+\lambda)s)^{p-1} > 0.\end{aligned}$$

Finally, we calculate

$$\begin{aligned}&\mathcal{L}^{nt} h_0(t,b,s)\\ &\quad = (b+(1+\lambda)s)^p \left[-\alpha \frac{(1+\lambda)s}{b+(1+\lambda)s} + \frac{1}{2}(1-p)\sigma^2 \frac{(1+\lambda)^2 s^2}{(b+(1+\lambda)s)^2}\right] > 0\end{aligned}$$

since $b+(1+\lambda)s > 0$ and $s < 0$. □

Corollary 2.22. *For every $t \in [t^{down}, T)$ we have $h_0(t,\cdot,\cdot) \in C(\overline{\mathcal{S}}^0)$.*

Proof. We only need to show that $h_0(t,b,s)$ is continuous at the point $s=0$. By Lemma 2.20 $h_0(t,b,s)$ is given explicitly as $h_0(t,b,s) = U_p(b+(1+\lambda)s)$ if $s \le 0$. By Theorem 2.7 and Theorem 2.8 $h_0 = V_0$ is $C^{1,2,2}$ and solves

$$0 = -\frac{\partial}{\partial t} h_0(t,b,s) - \alpha s \frac{\partial}{\partial s} h_0(t,b,s) - \frac{1}{2}\sigma^2 s^2 \frac{\partial^2}{\partial s^2} h_0(t,b,s) \tag{2.40}$$

for $s > 0$ sufficiently small. Sending $s \downarrow 0$ shows that $h_0(t,b,0+)$ solves the ordinary differential equation

$$\frac{\partial}{\partial t} h_0(t,b,0) = 0, \qquad h_0(T,b,0) = U_p(b),$$

i.e. $h_0(t,b,0+) = U_p(b)$ which shows that h_0 is continuous at $s=0$. □

Remark 2.23. We cannot expect more regularity of h_0 at $s=0$. Indeed, taking the derivative with respect to s in (2.40) and sending $s \downarrow 0$ shows that $(\partial/\partial s) h_0(t,b,0+)$ solves

$$0 = -\frac{\partial}{\partial t}\frac{\partial}{\partial s} h_0(t,b,0) - \alpha \frac{\partial}{\partial s} h_0(t,b,0), \quad h_0(T,b,0) = \frac{\partial}{\partial s} U_p(b+(1+\lambda)s)|_{s=0},$$

i.e.

$$\frac{\partial}{\partial s} h_0(t,b,0+) = e^{\alpha(T-t)} \frac{\partial}{\partial s} U_p(b+(1+\lambda)s)|_{s=0}.$$

This is in contrast to

$$h_0(t,b,0-) = \frac{\partial}{\partial s} U_p(b+(1+\lambda)s)|_{s=0},$$

showing that the partial derivative of h_0 with respect to s is not continuous at $s=0$. ◇

We are now ready to prove that $h_0 \in \mathbb{H}$. By construction we have $h_0 \le \mathcal{V}_0$ and hence we can find $\gamma \in (1-\mu, 1+\lambda)$ and $K > 1$ such that $h_0 \le \mathcal{V}_0 \le \varphi_{\gamma,p,K}$. Moreover, it is clear from the above analysis that

$$h_0(t,b,s) \ge U_p\left(b + \min\{(1-\mu)s, (1+\lambda)s\}\right)$$

since this is clearly satisfied for $t = T$ and h_0 is non-increasing in t.

We proceed in three steps: First we show that h_0 is continuous, then we show that h_0 is superharmonic and finally we show that h_0 is non-increasing in the direction of transactions.

Proposition 2.24. h_0 *is continuous on* $[0,T] \times \overline{\mathcal{S}}^0$.

Proof. By Proposition 2.16, Lemma 2.18, Lemma 2.20 and Corollary 2.22 it only remains to prove that h_0 is continuous in $(t^{down}, b, 0)$ for every $b \ge 0$. Let us therefore take a sequence $(t_n, b_n, s_n)_{n\in\mathbb{N}}$ converging to $(t^{down}, b, 0)$. We note that by Lemma 2.20 and Corollary 2.22 we may without loss of generality assume that $t_n < t^{down}$ for all $n \in \mathbb{N}$ and by the construction of h_0 and since $(\partial/\partial t)V_0 \le 0$ we may assume that $(t_n, b_n, s_n) \in \overline{\mathcal{R}}_0^{buy} \cap \overline{\mathcal{R}}_0^{nt}$. In particular, this implies that

$$0 = -\frac{\partial}{\partial t} h_0(t, b_n, s_n) - \alpha s_n \frac{\partial}{\partial s} h_0(t, b_n, s_n) - \frac{1}{2}\sigma^2 s_n^2 \frac{\partial^2}{\partial s_n^2} h_0(t, b_n, s_n)$$

in the classical sense for every $t \in [t_n, T]$ and $n \in \mathbb{N}$. Sending $n \to \infty$ hence shows that $h_0^*(t^{down}, b, 0) := \limsup_{n\to\infty} h_0(t_n, b_n, s_n)$ can be found by solving the ODE

$$0 = -\frac{\partial}{\partial t} h_0^*(t, b, 0), \qquad h_0^*(T, b, 0) = U_p(b),$$

on $[t^{down}, T]$. We therefore have $h_0^*(t^{down}, b, 0) = U_p(b) = h_0(t^{down}, b, 0)$. On the other hand, we have $h_0(t_n, b_n, s_n) \ge h_0(t^{down}, b, 0)$ and hence

$$\lim_{n\to\infty} h_0(t_n, b_n, s_n) = h_0(t^{down}, b, 0).$$

□

Proposition 2.25. *The function h_0 is superharmonic.*

Proof. Fix $(t, b, s) \in [0, T) \times \mathcal{S}^0$, let τ be a $[t, T]$-valued stopping time and denote by ϑ the first exit time of the uncontrolled wealth process $(B^0_{t,b}, S^0_{t,s})$ from $\mathcal{S}^0$.

Suppose first that $s > 0$. Then $S^0_{t,s}(u) > 0$ for all $u \in [t, \tau \wedge \vartheta]$ and hence $h_0 = V_0$ is $C^{1,2,2}$ along the paths of $(u, B^0(u), S^0(u))$ by Proposition 2.16. Let $\varepsilon > 0$ and denote by ϑ^ε the first exit time of $(B^0_{t,b} + \varepsilon, S^0_{t,s})$ from $\mathcal{S}^0$. Then clearly $\vartheta^\varepsilon \geq \vartheta$. For every $n \in \mathbb{N}$ let us define a stopping time

$$\tau_n := \inf \left\{ u \geq t : \int_t^u \left[\sigma S^0_{t,s}(r) \frac{\partial}{\partial s} h_0(r, B^0_{t,b}(r) + \varepsilon, S^0_{t,s}(r)) \right]^2 dr \geq n \right\} \wedge \tau \wedge \vartheta.$$

Application of Itô's formula shows that

$$\begin{aligned} h_0(t, b + \varepsilon, s) = h_0(\tau_n, B^0_{t,b}(\tau_n) + \varepsilon, S^0_{t,s}(\tau_n)) \\ + \int_t^{\tau_n} \mathcal{L}^{nt} h_0(u, B^0_{t,b}(u) + \varepsilon, S^0_{t,s}(u))\, du \\ + \int_t^{\tau_n} \sigma S^0_{t,s}(u) \frac{\partial}{\partial s} h_0(u, B^0_{t,b}(u) + \varepsilon, S^0_{t,s}(u))\, dW(u). \end{aligned}$$

Taking expectations reveals that

$$\begin{aligned} h_0(t, b + \varepsilon, s) = \mathbb{E}\Bigg[h_0(\tau_n, B^0_{t,b}(\tau_n) + \varepsilon, S^0_{t,s}(\tau_n)) \\ + \int_t^{\tau_n} \mathcal{L}^{nt} h_0(u, B^0_{t,b}(u) + \varepsilon, S^0_{t,s}(u))\, du \Bigg]. \end{aligned}$$

Since by Proposition 2.16 h_0 is a classical solution of the DPE on $[0, T] \times \tilde{\mathcal{S}}^0$ we see that

$$\int_t^{\tau_n} \mathcal{L}^{nt} h_0(u, B^0_{t,b}(u), S^0_{t,s}(u))\, du \geq 0$$

and hence

$$h_0(t, b + \varepsilon, s) \geq \mathbb{E}\left[h_0(\tau_n, B^0_{t,b}(\tau_n) + \varepsilon, S^0_{t,s}(\tau_n)) \right]. \tag{2.41}$$

Next, since $(\partial/\partial t) h_0 \leq 0$ and since $h_0(T, b, s) = V_0(T, b, s) = U_p(b + (1 - \mu)s)$, we have $h_0(t, b, s) \geq U_p(b + (1 - \mu)s)$ and hence

$$h_0(\tau_n, B^0_{t,b}(\tau_n) + \varepsilon, S^0_{t,s}(\tau_n)) \geq U_p(\varepsilon).$$

We can therefore send $n \to \infty$ in (2.41) and use Fatou's lemma to obtain

$$h_0(t, b+\varepsilon, s) \geq \mathbb{E}\left[h_0(\tau \wedge \vartheta, B^0_{t,b}(\tau \wedge \vartheta) + \varepsilon, S^0_{t,s}(\tau \wedge \vartheta))\right].$$

Now since $(\partial/\partial b)h_0 \geq 0$ we can send $\varepsilon \downarrow 0$ and use monotone convergence to obtain

$$h_0(t, b, s) \geq \mathbb{E}\left[h_0(\tau \wedge \vartheta, B^0_{t,b}(\tau \wedge \vartheta), S^0_{t,s}(\tau \wedge \vartheta))\right],$$

i.e. h_0 is superharmonic.

Consider now that case $s \leq 0$. For simplicity, let us denote $\tilde{\tau} := \tau \wedge \vartheta$. If $t \geq t^{down}$, then (L^*, M^*) performs an initial transaction from (b, s) to $(b + (1+\lambda)s, 0)$ so that

$$h_0(t, b, s) = h(t, b + (1+\lambda)s, 0) = U_p(b + (1+\lambda)s). \tag{2.42}$$

On the other hand, by the same arguments and using Jensen's inequality we see that

$$\begin{aligned}
\mathbb{E}\left[h_0(\tilde{\tau}, B^0_{t,b}(\tilde{\tau}), S^0_{t,s}(\tilde{\tau}))\right] &= \mathbb{E}\left[h_0(\tilde{\tau}, B^0_{t,b}(\tilde{\tau}) + (1+\lambda)S^0_{t,s}(\tilde{\tau}), 0)\right] \\
&= \mathbb{E}\left[U_p\left(B^0_{t,b}(\tilde{\tau}) + (1+\lambda)S^0_{t,s}(\tilde{\tau})\right)\right] \\
&\leq U_p\left(\mathbb{E}\left[B^0_{t,b}(\tilde{\tau}) + (1+\lambda)S^0_{t,s}(\tilde{\tau})\right]\right) \\
&= U_p\left(b + (1+\lambda)\mathbb{E}\left[S^0_{t,s}(\tilde{\tau})\right]\right).
\end{aligned}$$

Now since $S^0_{t,s}$ is a supermartingale for every $s \leq 0$ it follows that

$$\mathbb{E}[S^0_{t,s}(\tilde{\tau})] \leq s$$

and hence

$$\mathbb{E}\left[h_0(\tilde{\tau}, B^0_{t,b}(\tilde{\tau}), S^0_{t,s}(\tilde{\tau}))\right] \leq U_p\left(b + (1+\lambda)s\right) = h_0(t, b, s)$$

by the monotonicity of U_p and (2.42).

Finally, let us assume that $s \leq 0$ and $t < t^{down}$. We have

$$\begin{aligned}
&h_0(\tilde{\tau}, B^0_{t,b}(\tilde{\tau}), S^0_{t,s}(\tilde{\tau})) \\
&\quad = h_0(\tilde{\tau}, B^0_{t,b}(\tilde{\tau}), S^0_{t,s}(\tilde{\tau}))\mathbb{1}_{\{\tilde{\tau} < t^{down}\}} + h_0(\tilde{\tau}, B^0_{t,b}(\tilde{\tau}), S^0_{t,s}(\tilde{\tau}))\mathbb{1}_{\{\tilde{\tau} \geq t^{down}\}}
\end{aligned}$$

On $\{\tilde{\tau} \geq t^{down}\}$ we have as before

$$\mathbb{E}\left[h_0(\tilde{\tau}, B^0_{t,b}(\tilde{\tau}), S^0_{t,s}(\tilde{\tau}))\right] \leq U_p(b + (1+\lambda)s) = h_0(t, b, s),$$

so we may without loss of generality assume that $\tilde{\tau} < t^{down}$. However, since we know by Lemma 2.18 and Proposition 2.19 that h_0 is $C^{1,2,2}$ and satisfies the DPE in the classical sense we obtain

$$h(t,b,s) \geq \mathbb{E}\left[h_0(\tilde{\tau}, B^0_{t,b}(\tilde{\tau}), S^0_{t,s}(\tilde{\tau}))\right]$$

as in the case $s > 0$. □

Proposition 2.26. *The function h_0 is non-increasing in the directions of transactions.*

Proof. Fix $(t,b,s) \in [0,T] \times \overline{\mathcal{S}}^0$ and let $l, m \geq 0$ be such that $(b-(1+\lambda)l+(1-\mu)m, s+l-m) \in \overline{\mathcal{S}}^0$. We have to show that

$$h_0(t,b,s) \geq h_0(t, b-(1+\lambda)l+(1-\mu)m, s+l-m).$$

However, since by Proposition 2.24 $h(t,\cdot,\cdot)$ is continuous and satisfies

$$\mathcal{L}^{buy} h_0(t,b,s), \mathcal{L}^{sell} h_0(t,b,s) \geq 0$$

in the classical sense for every $(t,b,s) \in ([0,T) \times \mathcal{S}^0) \setminus \{(t,b,s) \in [0,T) \times \mathcal{S}^0 : t \geq t^{down}, s = 0\}$ and $(t, b/s) \notin F$ (defined in (2.24)) by Proposition 2.16, Proposition 2.19 and Proposition 2.21. Therefore, by the gradient theorem, we immediately obtain the claim. □

Combining Propositions 2.24, 2.25 and 2.26 proves the optimality of (L^*, M^*).

Corollary 2.27. *We have $h_0 \in \mathbb{H}$. In particular, $h_0 = \mathcal{V}_0$ and (L^*, M^*) is optimal.*

Since $h_0 = \mathcal{V}_0$ we furthermore have the following regularity result.

Theorem 2.28. *The value function $\mathcal{V}_0$ is continuous everywhere and (at least) of class $C^{1,2,2}$ except for possibly the points (t,b,s) for which one of the following statements is true:*

1. *$b = 0$ and (t,b,s) is on the buy boundary.*
2. *$\pi_M = 1$ and $b = 0$.*

3. $t = t^{down}$ *and* $s \le 0$. *However,* $\mathcal{V}_0(t^{down}, \cdot, \cdot) \in C^2(\mathcal{S}^0 \setminus \{(b,s) \in \mathcal{S}^0 : s = 0\})$.

4. $t \ge t^{down}$ *and* $s = 0$. *However,* $\mathcal{V}_0(\cdot, b, 0) \in C^\infty((t^{down}, T))$ *for every* $b \ge 0$.

Moreover, $\mathcal{V}_0 \in C^\infty(\mathcal{R}_0^{nt})$.

2.7. Numerical results

We conclude this chapter with numerical examples. By the homogeneity of the value function (Lemma 2.3.1) we first reduce the dimension of the problem by expressing the value function in terms of risky fractions. Then we use the algorithm developed in Kunisch and Sass [72] to simulate the value function and the optimal trading regions. We fix the following parameters:

$$\alpha := 0.096, \qquad \sigma := 0.4, \qquad T := 10$$
$$p := 0.1, \qquad \mu := 0.01, \qquad \lambda := 0.01.$$

This implies in particular that

$$\pi_M = \frac{\alpha}{(1-p)\sigma^2} = \frac{2}{3} \in (0,1)$$

and hence $\mathcal{V}_0 \in C^{1,2,2}([0,T) \times \tilde{\mathcal{S}})$ by Theorem 2.7. Hence, if we restrict simulation of $\mathcal{V}_0$ to $[0,T) \times \tilde{\mathcal{S}}$, then the regularity assumptions in Kunisch and Sass [72] are satisfied so that we can be assured that our numerical approximation converges to the value function.

2.7.1. Outline of the algorithm

Let us first give a quick outline of the algorithm introduced in Kunisch and Sass [72]. As a preliminary step, we need to reduce the dimension of the problem by introducing the following transformation of the value function:

$$\mathcal{V}_0(t,b,s) =: (b+s)^p \bar{\mathcal{V}}_0(t, s/(b+s)), \qquad \bar{\mathcal{V}}_0(t,\pi) := \mathcal{V}_0(t, 1-\pi, \pi).$$

By formally expressing the derivatives of $\mathcal{V}_0$ in terms of the derivatives of $\bar{\mathcal{V}}_0$ one can show (as in Shreve and Soner [97, Proposition 8.1]) that $\bar{\mathcal{V}}_0$ is the unique viscosity solution of

$$0 = \min\big\{\bar{\mathcal{L}}^{nt}\bar{\mathcal{V}}_0(t,\pi), \bar{\mathcal{L}}^{buy}\bar{\mathcal{V}}_0(t,\pi), \bar{\mathcal{L}}^{sell}\bar{\mathcal{V}}_0(t,\pi)\big\}, \tag{2.43}$$

on $[0,T) \times (-1/\lambda, 1/\mu)$ with terminal condition

$$\bar{\mathcal{V}}_0(T,\pi) = \mathcal{V}_0(T, 1-\pi, \pi) = \begin{cases} \frac{1}{p}(1-\mu\pi)^p, & \text{if } \pi > 0, \\ \frac{1}{p}(1+\lambda\pi)^p, & \text{if } \pi \le 0, \end{cases}$$

and where the operators $\bar{\mathcal{L}}^{nt}$, $\bar{\mathcal{L}}^{buy}$ and $\bar{\mathcal{L}}^{sell}$ are given by

$$\begin{aligned} \bar{\mathcal{L}}^{nt}\bar{\mathcal{V}}_0 := & -\frac{\partial}{\partial t}\bar{\mathcal{V}}_0 - \big(\alpha\pi - \frac{1}{2}(1-p)\sigma^2\pi^2\big)p\bar{\mathcal{V}}_0 - \frac{1}{2}\sigma^2\pi^2(1-\pi)^2\frac{\partial^2}{\partial\pi^2}\bar{\mathcal{V}}_0 \\ & - \big(\alpha(1-\pi) - (1-p)\sigma^2\pi(1-\pi)\big)\pi\frac{\partial}{\partial\pi}\bar{\mathcal{V}}_0, \end{aligned} \tag{2.44}$$

$$\bar{\mathcal{L}}^{buy}\bar{\mathcal{V}}_0 := p\lambda\bar{\mathcal{V}}_0 - (1+\lambda\pi)\frac{\partial}{\partial\pi}\bar{\mathcal{V}}_0, \tag{2.45}$$

$$\bar{\mathcal{L}}^{sell}\bar{\mathcal{V}}_0 := p\mu\bar{\mathcal{V}}_0 + (1-\mu\pi)\frac{\partial}{\partial\pi}\bar{\mathcal{V}}_0. \tag{2.46}$$

Note that since $\mathcal{V}_0 \in C^{1,2,2}([0,T) \times \tilde{\mathcal{S}})$ we have $\bar{\mathcal{V}}_0 \in C^{1,2}([0,T) \times (0,1/\mu))$ and hence $\bar{\mathcal{V}}_0$ is a classical solution of (2.43) on $[0,T) \times (0,1/\mu)$. This is the requirement for the algorithm in Kunisch and Sass [72] to converge.

Let us now give a brief outline of the algorithm. First, we restrict the approximation of $\bar{\mathcal{V}}_0$ to $[0,1] \subset (-1/\lambda, 1/\mu)$. We discretize $[0,T]$ using an equidistant grid with mesh size Δt. Similarly, we discretize $[0,1]$ with an equidistant grid with mesh size Δx. The derivatives in the operators $\bar{\mathcal{L}}^{nt}$, $\bar{\mathcal{L}}^{buy}$ and $\bar{\mathcal{L}}^{sell}$ are approximated using a central finite-difference scheme. We solve the differential equation (2.43) backwards in time. In every time step (say, we are at time $t < T$), we make an initial guess $N_0 := [a_0, b_0]$ for the no-trading region. On $[a_0, b_0]$ we solve $\bar{\mathcal{L}}^{nt}\bar{v}_0^1(t,\pi) = 0$ for $\bar{v}_0^1$. Since $\bar{\mathcal{L}}^{buy}\bar{v}_0^1(t,\pi) = 0$ and $\bar{\mathcal{L}}^{sell}\bar{v}_0^1(t,\pi) = 0$ can be solved explicitly on $[0, a_0)$ and $(b_0, 1]$, respectively, we can extend $\bar{v}_0^1$ to $[0,1]$ using a smooth pasting condition on a_0 and b_0. For every $\pi \in [0,1]$ we then define

$$\lambda_1^B(t,\pi) := -\bar{\mathcal{L}}^{nt}\bar{v}_0^1(t,\pi)\mathbb{1}_{[0,a_0]}, \qquad \lambda_1^S(t,\pi) := -\bar{\mathcal{L}}^{nt}\bar{v}_0^1(t,\pi)\mathbb{1}_{[b_0,1]},$$

and we introduce the sets

$$B_1 := \left\{\pi \in [0,1] : \lambda_1^B(t,\pi) + \bar{\mathcal{L}}^{buy}\bar{v}_0^1(t,\pi) < 0\right\}, \tag{2.47}$$

$$S_1 := \left\{\pi \in [0,1] : \lambda_1^S(t,\pi) + \bar{\mathcal{L}}^{sell}\bar{v}_0^1(t,\pi) < 0\right\}. \tag{2.48}$$

We set $N_1 = [0,1] \setminus (B_1 \cup S_1)$ to be the new guess for the no-trading region and repeat the procedure until $N_k \approx N_{k-1}$ for some $k > 0$. Once the no-trading region converges we proceed with the next time step $t - \Delta t$.

2.7.2. A numerical example

The algorithm outlined above allows us to simulate $\bar{\mathcal{V}}_0$. Figure 2.8 depicts the resulting free boundaries. Whenever the investor holds a position which is below the buy boundary $\underline{\pi}^0(t)$ it is optimal to buy shares of the stock and if the position is above the sell boundary $\overline{\pi}^0(t)$ it is optimal to sell shares of the stock. If the position is in between the buy and sell boundary the optimal action of the investor is not to trade at all. Moreover, the optimally controlled risky fraction process is a diffusion reflected at $\underline{\pi}^0(t)$ and $\overline{\pi}^0(t)$. We emphasize that the optimal strategy in the absence of costs π_M is located in the no-trading region. This feature is always observed whenever $\pi_M \in (0,1)$.

Note that if the investment horizon is sufficiently large, the boundaries become stationary. When the investment horizon becomes smaller we have two effects. First, the sell boundary decreases. This is because we optimize the total wealth after liquidation of the stock, i.e. the investor has to close the risky position at terminal time. Since she has to pay transaction costs in the process of liquidation a lower risky position at terminal time is preferable. On the other hand the buy boundary also decreases. This shows that as the investor approaches the investment horizon, the less she wants to engage in transactions since there is not enough time left to gain the transaction costs back.

Figure 2.9 illustrates the simulated value function $\bar{\mathcal{V}}_0$. In order to visualize the qualitative properties of $\bar{\mathcal{V}}_0$ more clearly we restrict ourselves to the smaller investment horizon $T = 1$. Note that for a fixed time t the value function is relatively flat along different values for the risky fraction. Furthermore, it can be seen that the value function is a concave function of π, a property inherited from the concavity of $\mathcal{V}_0$. The solid black lines highlight the location of the buy and the sell boundary.

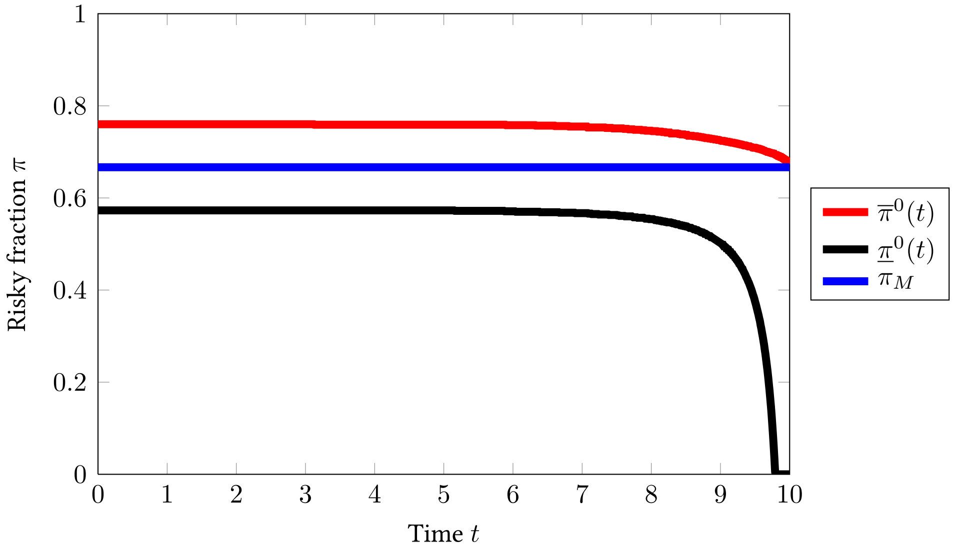

Figure 2.8. Optimal trading regions under transaction costs.

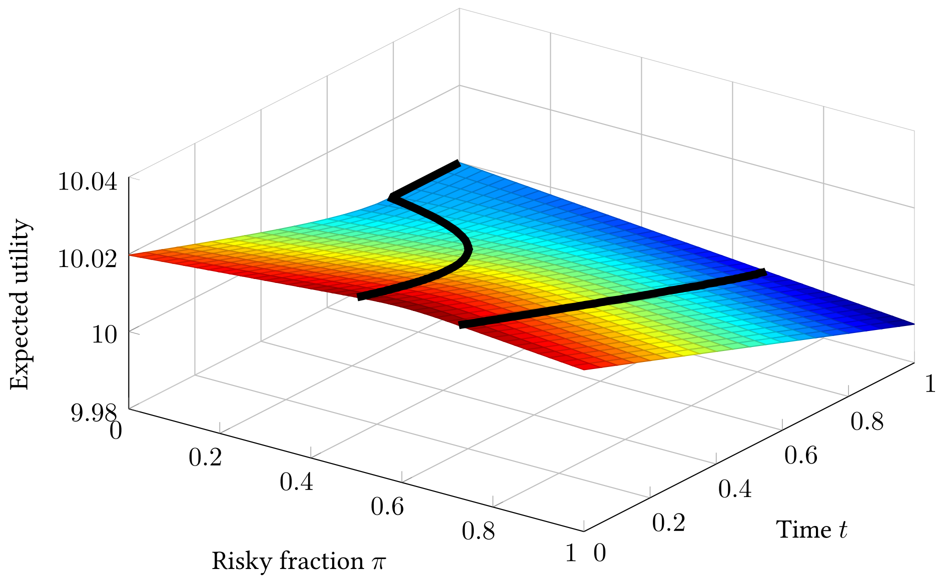

Figure 2.9. The value function $\bar{\mathcal{V}}_0$.

3. Worst-case portfolio optimization with transaction costs

In this chapter we extend the model considered in Chapter 2 to allow for crashes in the stock. We assume that the price of the stock drops at some unknown stopping time by an unknown fraction which is bounded from above by a maximum crash size known to the investor. We do not specify any distribution on the time and size of the crash, but assume that the investor takes a worst-case perspective towards the impact of a crash. That is, for each admissible strategy we determine the worst-case crash in the sense that expected utility of terminal wealth is minimized. We call a strategy worst-case optimal if the corresponding expected terminal utility in its worst-case scenario dominates the expected terminal utility in the worst-case scenario of any other strategy.

Similar to the first part of Chapter 2 it is our objective to characterize the value function as the unique viscosity solution of the corresponding dynamic programming equation. More precisely, we prove the continuity of the value function, we establish a version of the dynamic programming principle and use this to show that the value function is a viscosity solution of the DPE. Then we extend the comparison principle obtained in the previous chapter to prove uniqueness of the value function and we conclude with numerical examples.

The numerical results suggest that some of the features of the optimal strategy in the presence of crashes change compared to the optimal crash-free and/or zero-costs strategies. For example, we always observe that the sell boundary of the no-trading region falls below the optimal strategy in the case without transaction costs close to terminal time, which in the absence of crashes can only occur when leverage is optimal. Moreover, not only is the buy boundary (parametrized in terms of risky fractions) zero before the investment horizon T is reached (as in the no-crash case), but the sell boundary is also zero strictly before the investment horizon T (unlike in the no-crash case). In other words, the worst-case optimal strategy in the presence of transaction costs for short investment periods is the pure bond strategy.

The results of this chapter correspond in large parts to the following article:

1. C. Belak, O. Menkens, J. Sass (2013): Worst-case portfolio optimization with proportional transaction costs [12].

The problem considered in this chapter was treated before in the diploma thesis Belak [8], the methods and results are however different. First, let us mention that the general solution approach is the same in the sense that the main goal is to characterize the value function as the unique continuous viscosity solution of the DPE. The setup of our model here and in Belak [8] is different in the sense that we consider the strong formulation of the control problem on the canonical Wiener space, whereas in Belak [8] we consider the weak formulation (in the spirit of Yong and Zhou [110]) in which an admissible control consists not only of the trading strategy, but of the underlying probability space as well. Regarding the continuity of the value function we prove stronger growth estimates and stronger convergence results and fix several gaps in the proofs in Belak [8]. The proof of the dynamic programming principle is different since we use a different approach to construct the ε-optimal strategies which does not require continuity of the worst-case bound for arbitrary trading strategies. Our proof of the viscosity property of the value function is different as well, but essentially boils down to the same ideas. Finally, we obtain several additional results here, most notably the uniqueness of the value function and the numerical results.

3.1. The market model and problem formulation

In this section we specify the market model and formulate the optimization problem. The market is an extension of the model considered in Chapter 2 which allows for a crash in the stock price. Let us hence assume that $W = (W(t))_{t \geq 0}$ is a standard Brownian motion on the canonical Wiener space $(\Omega, \mathcal{F}, \mathbb{P})$. We let $\mathbb{F} = \mathbb{F}^0 = (\mathcal{F}(u))_{u \geq 0}$ be the augmented filtration generated by W and for every $t > 0$ we denote by $\mathbb{F}^t = (\mathcal{F}^t(u))_{u \geq t}$ the augmented filtration generated by $(W(u) - W(t))_{u \geq t}$. Moreover, we fix some terminal time $T > 0$ as well as some initial time $t \in [0, T)$.

We consider a market consisting of a risk-free bond $P^0 = (P^0(u))_{u \in [t,T]}$ and one risky stock $P^1 = (P^1(u))_{u \in [t,T]}$. We assume that in normal times (i.e. in crash-free times) the prices of the two assets evolve according to

$$\begin{aligned} dP^0(u) &= 0, & u &\in [t, T], & P^0(t) &= 1, \\ dP^1(u) &= \alpha P^1(u) du + \sigma P^1(u) dW(u), & u &\in [t, T], & P^1(t) &= 1. \end{aligned}$$

As before, we refer to $\alpha > 0$ as the excess return and $\sigma > 0$ as the volatility of the stock.

A crash is modeled as a pair $(\tau, \beta(\tau))$ consisting of a crash time τ and an $\mathcal{F}^t(\tau-)$-measurable crash size $\beta(\tau) \in [0, \beta]$, where $\beta \in (0, 1)$ denotes some maximum deterministic crash size. τ is assumed to be a $[t, T] \cup \{\infty\}$-valued $\mathbb{F}^t$-stopping time. On $\{\tau \leq T\}$, we assume that the price of the stock drops by the fraction $\beta(\tau)$ at time τ, i.e.

$$P^1(\tau) = \big(1 - \beta(\tau)\big) P^1(\tau-), \qquad \text{on } \{\tau < \infty\}.$$

We interpret the event $\{\tau = \infty\}$ as the crash $(\tau, \beta(\tau))$ not occurring within the investment period $[t, T]$. We assume for simplicity that the crash size is constant and equal to the maximum crash size β. In light of Korn and Steffensen [70, Remark 1(a)] this does not pose any restriction on our model since the monotonicity of the utility function U_p implies that the worst-case optimal crash scenario is either a crash of maximum size β or no crash at all. We denote the set of all crash times of the above form by $\mathcal{B}(t)$.

Throughout this chapter we assume that at most one crash can occur within the investment period $[t, T]$. However, our results can be extended to the general case of at most n crashes by an iterative procedure. We outline some of the details in Section 3.8.

A crucial point about the worst-case model (see Section 1.2) for a crash (τ, β) is that we do not assume that it has a pre-specified distribution. Instead, a crash is regarded as a control variable which can be chosen as to minimize expected utility of terminal wealth. We make this more precise in the problem formulation below.

In the presence of crashes, the model for the trading strategies becomes more involved since we want to allow the investor to observe crashes and to be able to switch to a different strategy afterwards. First, the investor chooses a strategy $(L_1, M_1) = (L_1(u), M_1(u))_{u\in[t,T]}$ which is used as long as no crash has been observed. We assume that L_1 and M_1 are $\mathbb{F}^t$-adapted, non-decreasing càdlàg processes with $L_1(t-) = M_1(t-) = 0$. We refer to such a strategy as a pre-crash strategy.

After the crash time τ, the investor is allowed to switch to a different strategy $(L_0^\tau, M_0^\tau) = (L_0^\tau(u), M_0^\tau(u))_{u\in[\tau,T]}$ possibly depending on τ. Since the investor does not know τ a priori this implies that the investor chooses a whole family of post-crash strategies $(L_0^\tau, M_0^\tau)_{\tau\in\mathcal{B}(t)}$ and is hence prepared to react on every crash scenario $\tau \in \mathcal{B}(t)$. As usual, we assume that the pair (L_0^τ, M_0^τ) is $\mathbb{F}^\tau$-adapted, non-decreasing and càdlàg and we set $L_0^\tau(\tau-) = M_0^\tau(\tau-) = 0$. With this setup, the investor is able to observe crashes and react on the new information made available to her. Note that this approach is the same as in Seifried [95].

In order to simplify notations we write $\pi_1 = (L_1, M_1)$ for pre-crash strategies and $\pi_0 = (L_0^\tau, M_0^\tau)_{\tau\in\mathcal{B}(t)}$ for a family of post-crash strategies. More generally, we make the following convention: We denote pre-crash quantities by $\aleph_1$ and we denote the corresponding post-crash quantities by $\aleph_0$.

Given a pre-crash trading strategy $\pi_1 = (L_1, M_1)$, a family of post-crash strategies $\pi_0 = (L_0^\tau, M_0^\tau)_{\tau\in\mathcal{B}(t)}$ and a crash time $\tau \in \mathcal{B}(t)$, the investor's wealth $B = B_{t,b}^{\pi_1,\pi_0,\tau} = (B_{t,b}^{\pi_1,\pi_0,\tau}(u))_{u\in[t,T]}$ invested in the risk-free bond is given by

$$\begin{aligned} dB(u) &= -(1+\lambda)dL_1(u) + (1-\mu)dM_1(u), && u \in [t,\tau) \cap [t,T], && (3.1)\\ B(\tau) &= B(\tau-) - (1+\lambda)L_0^\tau(\tau) + (1-\mu)M_0^\tau(\tau), && \text{on } \{\tau < \infty\}, && (3.2)\\ dB(u) &= -(1+\lambda)dL_0^\tau(u) + (1-\mu)dM_0^\tau(u), && u \in (\tau, T], && (3.3) \end{aligned}$$

with initial wealth $B(t-) = b$. Similarly, the investor's wealth $S = S_{t,s}^{\pi_1,\pi_0,\tau} = (S_{t,s}^{\pi_1,\pi_0,\tau}(u))_{u\in[t,T]}$ invested in the stock is given by

$$\begin{aligned} dS(u) &= S(u)(\alpha du + \sigma dW(u)) + dL_1(u) - dM_1(u), && u \in [t,\tau) \cap [t,T], && (3.4)\\ S(\tau) &= (1-\beta)S(\tau-) + L_0^\tau(\tau) - M_0^\tau(\tau), && \text{on } \{\tau < \infty\}, && (3.5)\\ dS(u) &= S(u)(\alpha du + \sigma dW(u)) + dL_0^\tau(u) - dM_0^\tau(u), && u \in (\tau, T], && (3.6) \end{aligned}$$

with initial wealth $S(t-) = s$.

Remark 3.1. Observe that (3.5) is set up such that the crash is executed first, since it is applied to $S(\tau-)$. The control of the investor (L_0^τ, M_0^τ) is applied only thereafter. Thus, the investor can only react to a crash, but she cannot prevent being negatively affected by a crash at time τ by selling all risky holdings at time τ since her transaction is executed after the crash. $\diamond$

The net wealth $X = X_{t,b,s}^{\pi_1,\pi_0,\tau} = (X_{t,b,s}^{\pi_1,\pi_0,\tau}(u))_{u\in[t,T]}$ of the investor after liquidation of the stock position is given by

$$X(u) := \begin{cases} B(u) + (1-\mu)S(u), & \text{if } S(u) > 0, \\ B(u) + (1+\lambda)S(u), & \text{if } S(u) \leq 0. \end{cases} \qquad u \in [t,T].$$

Let us now turn to the question of solvency and admissibility of strategies. Taking into account that in case of a positive stock position a crash decreases the net wealth and that in case of a negative stock position a crash increases the net wealth, the following open solvency cones can be defined:

$$\mathcal{S}^1 := \left\{ (b,s) \in \mathbb{R}^2 \,\middle|\, b + (1+\lambda)s > 0, b + (1-\mu)(1-\beta)s > 0 \right\},$$
$$\mathcal{S}^0 := \left\{ (b,s) \in \mathbb{R}^2 \,\middle|\, b + (1+\lambda)s > 0, b + (1-\mu)s > 0 \right\}.$$

So, whenever $(b,s) \in \overline{\mathcal{S}}^1$, the investor can liquidate the stock holdings and end up with non-negative wealth even if a crash occurs momentarily. The boundaries of the solvency regions are parametrized as follows:

$$\partial\mathcal{S}_-^1 := \partial\mathcal{S}_-^0 := \left\{ (b,s) \in \mathbb{R}^2 \middle| s \leq 0, b + (1+\lambda)s = 0 \right\},$$
$$\partial\mathcal{S}_+^1 := \left\{ (b,s) \in \mathbb{R}^2 \middle| s > 0, b + (1-\mu)(1-\beta)s = 0 \right\},$$
$$\partial\mathcal{S}_+^0 := \left\{ (b,s) \in \mathbb{R}^2 \middle| s > 0, b + (1-\mu)s = 0 \right\}.$$

Figure 3.1 sketches the location of the boundaries of the solvency cones.

With this, we say that a pre-crash trading strategy π_1 is admissible for initial positions $(b,s) \in \overline{\mathcal{S}}^1$ if the corresponding pair (B,S) given by Equations (3.1) and (3.4) with initial values $B(t-) = b$ and $S(t-) = s$ and for $\tau \equiv \infty$ takes values in $\overline{\mathcal{S}}^1$ for all $u \in [t,T]$. The set of all admissible pre-crash trading strategies of this form is denoted by $\mathcal{A}_1(t,b,s)$.

A family of post-crash strategies $\pi_0 = (L_0^\tau, M_0^\tau)_{\tau\in\mathcal{B}(t)}$ corresponding to a pre-crash strategy $\pi_1 \in \mathcal{A}_1(t,b,s)$ is called admissible if for every $\tau \in \mathcal{B}(t)$ and for

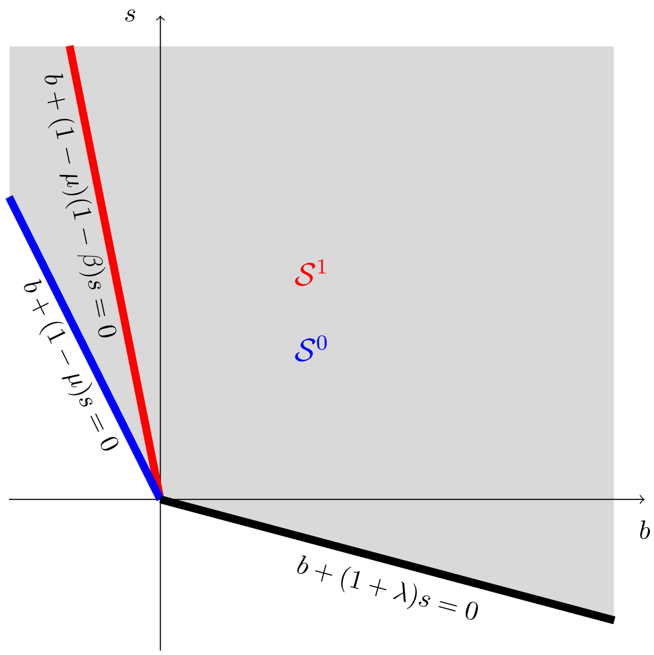

Figure 3.1. Sketch of the solvency cones.

every $u \in [\tau, T]$ the corresponding pair (B, S) given by Equations (3.1) to (3.6) takes values in $\overline{\mathcal{S}}^0$. The set of all admissible families of post-crash trading strategies of this form is denoted by $\mathcal{A}_0(\pi_1)$. Note that this implies that

$$(L_0^\tau, M_0^\tau) \in \mathcal{A}_0\left(\tau, B_{t,b}^{\pi_1,\pi_0,\tau}(\tau-), (1-\beta)S_{t,s}^{\pi_1,\pi_0,\tau}(\tau-)\right),$$

i.e. (L_0^τ, M_0^τ) is admissible in the corresponding crash-free market.

Fix $p < 1$, $(b, s) \in \overline{\mathcal{S}}^1$ and let $\pi_1 \in \mathcal{A}_1(t, b, s)$, $\pi_0 \in \mathcal{A}_0(\pi_1)$ and $\tau \in \mathcal{B}(t)$. We define the performance criterion of π_1, π_0 and τ by

$$\mathcal{J}_1(\pi_1, \pi_0, \tau, t, b, s) := \mathbb{E}\left[U_p\left(X_{t,b,s}^{\pi_1,\pi_0,\tau}(T)\right)\right].$$

The worst-case bound of π_1 and π_0 is defined as

$$\mathcal{W}_1(\pi_1, \pi_0, t, b, s) := \inf_{\tau \in \mathcal{B}(t)} \mathcal{J}_1(\pi_1, \pi_0, \tau, t, b, s). \tag{3.7}$$

Finally, the value function is defined as

$$\mathcal{V}_1(t, b, s) := \sup_{\substack{\pi_1 \in \mathcal{A}_1(t,b,s) \\ \pi_0 \in \mathcal{A}_0(\pi_1)}} \mathcal{W}_1(\pi_1, \pi_0, t, b, s).$$

It is sometimes helpful to interpret the optimization problem as a game between the investor and the market. The investor decides on a trading strategy and aims to maximize expected utility of terminal wealth, whereas the market decides on a crash scenario with the objective of minimizing the investor's expected utility of terminal wealth. In this spirit we refer to a crash $\tau^* \in \mathcal{B}(t)$ as optimal for given strategies $\pi_1 \in \mathcal{A}_1(t,b,s)$ and $\pi_0 \in \mathcal{A}_0(\pi_1)$, if it attains the infimum in (3.7).

3.2. Heuristics

Before we start proving some of the basic properties of $\mathcal{V}_1$ let us first gather some insights into the nature of the optimal trading strategies and the corresponding optimal crash time. First of all, note that given some arbitrary $\pi_1 \in \mathcal{A}_1(t,b,s)$, $\pi_0 \in \mathcal{A}_0(\pi_1)$ and $\tau \in \mathcal{B}(t)$ we have

$$\mathcal{J}_1(\pi_1, \pi_0, \tau, t, b, s) \leq \mathcal{J}_1(\pi_1, \pi_0^*, \tau, t, b, s), \tag{3.8}$$

where $\pi_0^* := (L_0^{\tau,*}, M_0^{\tau,*})_{\tau \in \mathcal{B}(t)}$ and where $(L_0^{\tau,*}, M_0^{\tau,*})$ is the optimal strategy in the crash-free market corresponding to initial values

$$\left(\tau, B_{t,b}^{\pi_1,\pi_0^*,\tau}(\tau-), (1-\beta) S_{t,s}^{\pi_1,\pi_0^*,\tau}(\tau-)\right)$$

whenever this makes sense (i.e. on $\{\tau \leq T\}$). Indeed, a simple calculation shows that

$$\begin{aligned}
&\mathcal{J}_1(\pi_1, \pi_0, \tau, t, b, s) \\
&= \mathbb{E}\left[U_p\left(X_{t,b,s}^{\pi_1,\pi_0,\tau}(T)\right)\right] \\
&= \mathbb{E}\left[U_p\left(X^{\pi_1,\pi_0,\tau}_{\tau, B_{t,b}^{\pi_1}(\tau-), (1-\beta) S_{t,s}^{\pi_1}(\tau-)}(T)\right) \mathbb{1}_{\{\tau \leq T\}} + U_p\left(X_{t,b,s}^{\pi_1,\pi_0,\tau}(T)\right) \mathbb{1}_{\{\tau = \infty\}}\right] \\
&\leq \mathbb{E}\left[\mathcal{V}_0\left(\tau, B_{t,b}^{\pi_1}(\tau-), (1-\beta) S_{t,s}^{\pi_1}(\tau-)\right) \mathbb{1}_{\{\tau \leq T\}} + U_p\left(X_{t,b,s}^{\pi_1,\pi_0,\tau}(T)\right) \mathbb{1}_{\{\tau = \infty\}}\right] \\
&= \mathbb{E}\left[U_p\left(X_{t,b,s}^{\pi_1,\pi_0^*,\tau}(T)\right) \mathbb{1}_{\{\tau \leq T\}} + U_p\left(X_{t,b,s}^{\pi_1,\pi_0,\tau}(T)\right) \mathbb{1}_{\{\tau = \infty\}}\right] \\
&= \mathcal{J}_1(\pi_1, \pi_0^*, \tau, t, b, s).
\end{aligned}$$

Therefore, taking the infimum over all $\tau \in \mathcal{B}(t)$ in (3.8) we see that

$$\mathcal{W}_1(\pi_1, \pi_0, t, b, s) \leq \mathcal{W}_1(\pi_1, \pi_0^*, t, b, s)$$

and since π_0 was chosen arbitrarily we have

$$\sup_{\pi_0 \in \mathcal{A}(\pi_1)} \mathcal{W}_1(\pi_1, \pi_0, t, b, s) \leq \mathcal{W}_1(\pi_1, \pi_0^*, t, b, s)$$

and hence

$$\mathcal{V}_1(t, b, s) \leq \sup_{\pi_1 \in \mathcal{A}(t,b,s)} \mathcal{W}_1(\pi_1, \pi_0^*, t, b, s).$$

On the other hand we clearly have

$$\mathcal{V}_1(t, b, s) = \sup_{\substack{\pi_1 \in \mathcal{A}_1(t,b,s) \\ \pi_0 \in \mathcal{A}_0(\pi_1)}} \mathcal{W}_1(\pi_1, \pi_0, t, b, s) \geq \sup_{\pi_1 \in \mathcal{A}(t,b,s)} \mathcal{W}_1(\pi_1, \pi_0^*, t, b, s),$$

and hence the family π_0^* is post-crash optimal for all $\pi_1 \in \mathcal{A}_1(t, b, s)$ and all $\tau \in \mathcal{B}(t)$.

We would like to point out that the results of this chapter are not contingent on the existence of the optimal strategies in the crash-free market. This allows us to extend our results to more than one crash in Section 3.8.

Regarding the optimal pre-crash strategy π_1^*, we expect that it behaves similarly to π_0^* in the sense that trading occurs continuously at two time dependent boundaries, but the optimal trading boundaries $\overline{\pi}^1(t)$ and $\underline{\pi}^1(t)$ in terms of risky fractions should be distinctly lower than the optimal trading boundaries $\overline{\pi}^0(t)$ and $\underline{\pi}^0(t)$ in the absence of crashes. In particular, we expect $\underline{\pi}^1(T-) = \overline{\pi}^1(T-) = 0$ since otherwise the investor would make significant losses if a crash occurs just before the investment horizon is reached. Figure 3.2 below illustrates how the optimal trading regions in the presence of crash threats might look like.

Suppose that the optimal pre-crash trading strategy π_1^* exists and is continuous. Classical dynamic programming arguments suggest that

$$\mathcal{V}_1(t, b, s) = \mathcal{W}_1(\pi_1^*, \pi_0^*, t, b, s) = \inf_{\tau \in \mathcal{B}(t)} \mathbb{E}\left[\mathcal{V}_0\left(\tau, B_{t,b}^{\pi_1^*}(\tau-), (1-\beta) S_{t,s}^{\pi_1^*}(\tau-)\right)\right].$$

By the general theory of optimal stopping (see e.g. Peskir and Shiryaev [89]) this suggests that $\mathcal{V}_1(t, b, s) \leq \mathcal{V}_0(t, b, (1-\beta)s)$ and that the optimal crash time τ^* corresponding to π_1^* is given as

$$\tau^* := \inf\left\{u \geq t : \mathcal{V}_1\Big(u, B_{t,b}^{\pi_1^*}(u), S_{t,s}^{\pi_1^*}(u)\Big) = \mathcal{V}_0\Big(u, B_{t,b}^{\pi_1^*}(u), (1-\beta) S_{t,s}^{\pi_1^*}(u)\Big)\right\}.$$

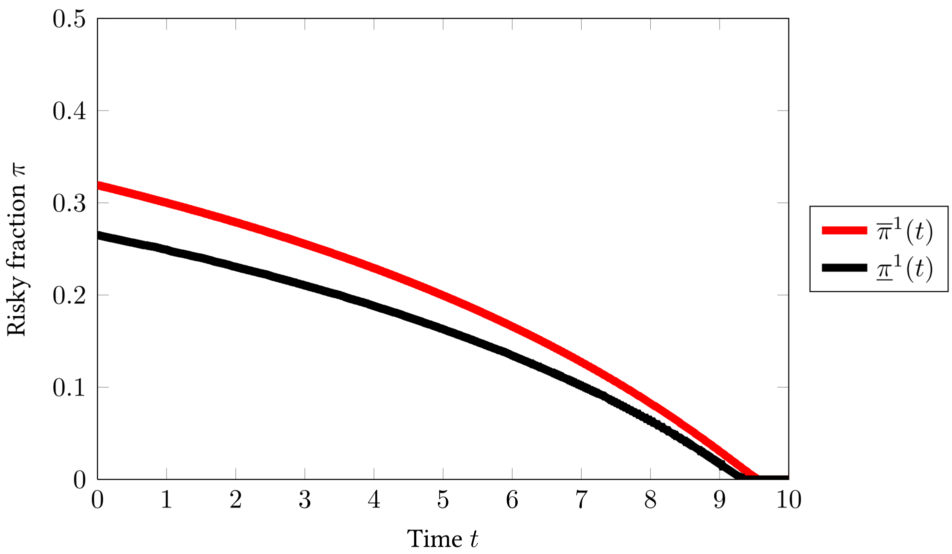

Figure 3.2. Optimal trading regions under transaction costs and crash threats.

This implies in particular that the investor can ignore the threat of crashes whenever $\mathcal{V}_1(t,b,s) < \mathcal{V}_0(t,b,(1-\beta)s)$ and hence the value function can be expected to be given as the solution of

$$\min\left\{\mathcal{L}^{nt}\mathcal{V}_1(t,b,s), \mathcal{L}^{buy}\mathcal{V}_1(t,b,s), \mathcal{L}^{sell}\mathcal{V}_1(t,b,s)\right\} = 0$$

as in the crash-free case. On the other hand, if $\mathcal{V}_1(t,b,s) = \mathcal{V}_0(t,b,(1-\beta)s)$ the optimality of τ^* suggests that an optimally trading investor may benefit if no crash occurs. In other words, we expect that whenever $\mathcal{V}_1(t,b,s) = \mathcal{V}_0(t,b,(1-\beta)s)$, then at least one of the operators $\mathcal{L}^{nt}$, $\mathcal{L}^{buy}$ or $\mathcal{L}^{sell}$ applied to $\mathcal{V}_1$ should be non-positive. Putting the pieces together this implies that we expect that $\mathcal{V}_1(t,b,s)$ solves the dynamic programming equation

$$\begin{aligned} 0 = \max\Big\{&\mathcal{V}_1(t,b,s) - \mathcal{V}_0(t,b,(1-\beta)s), \\ &\min\left\{\mathcal{L}^{nt}\mathcal{V}_1(t,b,s), \mathcal{L}^{buy}\mathcal{V}_1(t,b,s), \mathcal{L}^{sell}\mathcal{V}_1(t,b,s)\right\}\Big\} \end{aligned} \tag{3.9}$$

if we specify the correct boundary conditions. Indeed, we show in Theorem 3.19 that $\mathcal{V}_1$ is a viscosity solution of this DPE.

3.3. Some preliminary properties

Let us now analyze some of the basic properties of the value function $\mathcal{V}_1$. Note that most of the results in this section are very similar to those obtained in Chapter 2 and Shreve and Soner [97]. We therefore keep the exposition to a minimum.

Since the investor can always liquidate the stock position immediately at initial time t and stop trading afterwards we naturally obtain a lower bound on the value function. Furthermore, this strategy is the only admissible (and hence optimal) strategy on the boundary of the solvency region. The following lemma (which is the worst-case equivalent of Lemma 2.2) makes this statement precise.

Lemma 3.2. *1. Let $(b, s) \in \partial\mathcal{S}^1$. Then the only admissible strategy is to instantly jump to the position $(0, 0)$ and remain there.*

2. For $(b, s) \in \overline{\mathcal{S}}^1$ the trading strategy of instantly closing the stock position and no trading afterwards is an admissible strategy. Furthermore, for every $(b, s) \in \overline{\mathcal{S}}^1$, we have

$$\mathcal{V}_1(t, b, s) \geq \begin{cases} U_p\left(b + (1-\mu)(1-\beta)s\right), & \text{if } s > 0, \\ U_p\left(b + (1+\lambda)s\right), & \text{if } s \leq 0. \end{cases} \tag{3.10}$$

Proof. 1. The proof is very similar to the proof of [97, Remark 2.1] by Shreve and Soner and will thus not be reproduced here. The only additional difficulty arises due to the presence of crashes. This can be handled as follows:

a) If $(b, s) \in \partial\mathcal{S}^1_-$, then $s \leq 0$. In this case a crash would be beneficial for the investor in the sense that the net wealth increases. Thus, it cannot be optimal from the market's point of view to trigger a crash. At this point the proof follows exactly as in Shreve and Soner [97, Remark 2.1].

b) If $(b, s) \in \partial\mathcal{S}^1_+$, then it must be optimal for the market to crash immediately. To see this, note that in this case the investor's position after the crash at time t is given by $(b, (1-\beta)s) \in \partial\mathcal{S}^0_+$. Since we are in the crash-free market at this point, following Shreve and Soner [97, Remark 2.1], we can conclude that the only admissible strategy is to close the position in the stock and that the crash is indeed optimal since it leads to a terminal net wealth of $X(T) = 0$.

2. See Shreve and Soner [97, Remark 2.2]. Note that the worst-case crash scenario for $s > 0$ is an immediate crash at time t, since once the stock position is closed, crashes do not affect the net wealth. This explains the factor $(1-\beta)$ in the first case of (3.10). $\square$

If $(b,s) \in \overline{\mathcal{S}}^1$, Lemma 3.2 allows us to restrict the sets of admissible strategies $\mathcal{A}_1(t,b,s)$ and $\mathcal{A}_0(\pi_1)$ to those strategies π_1 and π_0 which have a worst-case bound satisfying

$$\mathcal{W}_1(\pi_1,\pi_0,t,b,s) \geq \begin{cases} U_p\left(b+(1-\mu)(1-\beta)s\right), & \text{if } s>0, \\ U_p\left(b+(1+\lambda)s\right), & \text{if } s\leq 0. \end{cases}$$

Abusing notations, we denote the sets of such strategies again by $\mathcal{A}_1(t,b,s)$ and $\mathcal{A}_0(\pi_1)$, respectively. We therefore have

$$\mathcal{W}_1(\pi_1,\pi_0,t,b,s) = -\infty \qquad \text{if and only if} \qquad (b,s) \in \partial\mathcal{S}^1 \text{ and } p \leq 0$$

for all pre-crash strategies $\pi_1 \in \mathcal{A}_1(t,b,s)$ and post-crash strategies $\pi_0 \in \mathcal{A}_0(\pi_1)$.

The next lemma gathers some further properties of the value function $\mathcal{V}_1$. Compare also with Lemma 2.1 and Lemma 2.3 for the equivalent results in the crash-free case and recall the function $\varphi_{\gamma,p,K}$ defined in (2.12).

Lemma 3.3. *1. Let $\gamma \in [1-\mu, 1+\lambda]$, $K \geq 1$. Then $\mathcal{V}_1 \leq \mathcal{V}_0 \leq \varphi_{\gamma,p,K} < +\infty$.*

2. For every $(t,b,s) \in [0,T] \times \overline{\mathcal{S}}^1$ we have

$$\mathcal{V}_1(t,b,s) \leq \mathcal{V}_0(t,b,(1-\beta)s). \tag{3.11}$$

3. Let $t \in [0,T]$. Then $\mathcal{V}_1(t,\cdot,\cdot)$ is concave on $\overline{\mathcal{S}}^1$. In particular, $\mathcal{V}_1(t,\cdot,\cdot)$ is locally Lipschitz-continuous on $\mathcal{S}^1$.

4. $\mathcal{V}_1$ is homogeneous of order p, i.e. for every $\kappa > 0$ and $(b,s) \in \overline{\mathcal{S}}^1$ we have

$$\mathcal{V}_1(t,\kappa b,\kappa s) = \begin{cases} \kappa^p \mathcal{V}_1(t,b,s) & \text{if } p<1, p\neq 0, \\ \log(\kappa) + \mathcal{V}_1(t,b,s) & \text{if } p=0. \end{cases}$$

Proof. 1. The relation $\mathcal{V}_1 \leq \mathcal{V}_0$ is obvious. The inequality $\mathcal{V}_0 \leq \varphi_{\gamma,p,K}$ is proved in Lemma 2.1.2.

2. Consider the crash time $\tau^* \equiv t$. Then

$$\mathcal{V}_1(t,b,s) \leq \sup_{\substack{\pi_1 \in \mathcal{A}_1(t,b,s) \\ \pi_0 \in \mathcal{A}_0(\pi_1)}} \mathbb{E}\left[U_p\left(X^{\pi_1,\pi_0,\tau^*}_{t,b,s}(T)\right)\right]$$
$$= \sup_{\pi_0 \in \mathcal{A}_0(t,b,(1-\beta)s)} \mathbb{E}\left[U_p\left(X^{\pi_0}_{t,b,(1-\beta)s}(T)\right)\right]$$
$$= \mathcal{V}_0\big(t,b,(1-\beta)s\big).$$

3. The concavity is inherited from the utility function U_p. The details can be found in Shreve and Soner [97, Proposition 3.1]. Note that every concave function is locally Lipschitz-continuous in the interior of its domain.

4. The result follows from the linearity of the dynamics of the wealth invested in the bond and stock, respectively, and using the homogeneity of U_p. See also Shreve and Soner [97, Proposition 3.3]. □

Remark 3.4. The proof of Lemma 3.3 shows that a crash at time t cannot be optimal if $\mathcal{V}_1(t,b,s) < \mathcal{V}_0(t,b,(1-\beta)s)$. ◇

3.4. Continuity of the value function

The aim of this section is to prove that $\mathcal{V}_1$ is continuous. We note that the same line of arguments (with some obvious adaptations) can be used to prove the continuity of $\mathcal{V}_0$ and hence establishes the first claim of Proposition 2.4.

We start by proving a time-shifting property of $\mathcal{V}_1$ so that we can prove the time-continuity by varying the terminal time T instead of the initial time t.

Lemma 3.5. *Denote the value function corresponding to terminal time T by $\mathcal{V}_1^T$. Let $t \in [0,T]$ and $h \geq -t$. Then*

$$\mathcal{V}_1^T(t,b,s) = \mathcal{V}_1^{T+h}(t+h,b,s).$$

Proof. We denote by $\mathcal{A}_1^T(t,b,s)$, $\mathcal{A}_0^T(\pi_1)$ and $\mathcal{B}^T(t)$ the respective sets of admissible strategies corresponding to terminal time T. Let $\pi_1 \in \mathcal{A}_1^T(t,b,s)$. Since π_1 is $\mathbb{F}^t$-adapted it follows that there exists a measurable function g_1 such that we can write

$$\pi_1(u,\omega) = g_1\big(u, (\omega(r))_{r\in[t,u]} - \omega(t)\big), \qquad (u,\omega) \in [t,T]\times\Omega.$$

Given $\omega \in \Omega$ we set $\tilde{\omega}(r) := \omega(r+h)$ for all $r \geq 0$ and define

$$\pi_1^h(u+h,\omega) := g_1(u, (\tilde{\omega}(r))_{r\in[t,u]} - \tilde{\omega}(t)), \qquad (u,\omega) \in [t,T]\times\Omega.$$

Then $\pi_1^h \in \mathcal{A}_1^{T+h}(t+h,b,s)$. In a similar fashion we can construct strategies $\pi_0^h \in \mathcal{A}_0^{T+h}(\pi_1^h)$ from every $\pi_0 \in \mathcal{A}_0^T(\pi_1)$ and crash times $\tau^h \in \mathcal{B}^{T+h}(t+h)$ from every $\tau \in \mathcal{B}^T(t)$. Then

$$\begin{aligned}
\mathbb{E}\left[U_p\left(X_{t,b,s}^{\pi_1,\pi_0,\tau}(T)\right)\right] &= \int_\Omega U_p\left(X_{t,b,s}^{\pi_1,\pi_0,\tau}(T,\omega)\right)\,\mathbb{P}(d\omega) \\
&= \int_\Omega U_p\left(X_{t+h,b,s}^{\pi_1^h,\pi_0^h,\tau^h}(T+h,\tilde{\omega})\right)\,\mathbb{P}(d\tilde{\omega}) \\
&= \mathbb{E}\left[U_p\left(X_{t,b,s}^{\pi_1^h,\pi_0^h,\tau^h}(T+h)\right)\right].
\end{aligned}$$

Since we can similarly construct π_1, π_0 and τ from given π_1^h, π_0^h and τ^h the claim follows. □

The next lemma provides growth estimates on the investor's wealth processes.

Lemma 3.6. *Let $\pi_1 \in \mathcal{A}_1(t,b,s)$, $\pi_0 \in \mathcal{A}_0(\pi_1)$ and $\tau \in \mathcal{B}(t)$ and denote*

$$B(u) := B_{t,b}^{\pi_1,\pi_0,\tau}(u), \qquad S(u) := S_{t,s}^{\pi_1,\pi_0,\tau}(u).$$

Assume that τ is such that $S(\tau) \leq S(\tau-)$.

1. *There exists a constant $C_0 > 0$ independent of (π_1,π_0) such that*

$$\mathbb{E}\left[B(T)+S(T)\right] = \mathbb{E}\left[|B(T)+S(T)|\right] \leq C_0(b+s).$$

2. *There exists a constant $C_1 > 0$ independent of (π_1,π_0) such that*

$$\mathbb{E}\left[(B(T)+S(T))^2\right] \leq C_1(1+b^2+s^2).$$

Proof. We frequently make use of the fact that on $\mathcal{S}^1$ and $\mathcal{S}^0$ we have

$$s \leq |s| \leq C(b+s), \qquad b \leq |b| \leq C(b+s)$$

for $C = 1 + \max\{1/\mu, 1/\lambda\}$. We note further that by (3.1)-(3.6) and by the condition on τ it can easily be verified that for every stopping time $\theta \leq T$ we have

$$B(\theta) + S(\theta) \leq b + s + \int_t^\theta \alpha S(u)\,du + \int_t^\theta \sigma S(u)\,dW(u).$$

1. Let $\tau_n := \inf\{u \geq t : |S(u)| \geq n\} \wedge T$. Setting $K := \alpha C$ we have

$$\begin{aligned} B(\tau_n) + S(\tau_n) &\leq b + s + \int_t^{\tau_n} \alpha S(u)\,du + \int_t^u \sigma S(u)\,dW(u) \\ &\leq b + s + K\int_t^{\tau_n} B(u) + S(u)\,du + \sigma \int_t^{\tau_n} S(u)\,dW(u). \end{aligned}$$

   Taking expectations on both sides implies that

$$\mathbb{E}\left[B(\tau_n) + S(\tau_n)\right] \leq b + s + K\mathbb{E}\left[\int_t^{\tau_n} B(u) + S(u)\,du\right].$$

   Since $B(u) + S(u) \geq 0$ we have

$$\mathbb{E}\left[B(\tau_n) + S(\tau_n)\right] \leq b + s + K\mathbb{E}\left[\int_t^{T} B(u) + S(u)\,du\right].$$

   Taking the limit $n \to \infty$ together with Fatou's lemma and using that $\tau_n \to T$ this implies that

$$\mathbb{E}\left[B(T) + S(T)\right] \leq b + s + K\int_t^T \mathbb{E}\left[B(u) + S(u)\right]\,du$$

   and we conclude by Gronwall's inequality.

2. We have

$$\begin{aligned} (B(T) + S(T))^2 &\leq (1 + B(T) + S(T))^2 \\ &\leq \left(1 + b + s + K\int_t^T B(u) + S(u)\,du + \sigma\int_t^T S(u)\,dW(u)\right)^2. \end{aligned}$$

Using the fact that $(a+b)^2 \leq 2a^2+2b^2$ and Hölder's inequality, this implies that there exists a constant $L > 0$ such that

$$\begin{aligned}(B(T)+S(T))^2 \\ &\leq L\biggl(1+b^2+s^2+K^2(T-t)\int_t^T (B(u)+S(u))^2\,du \\ &\qquad + \sigma^2\left[\int_t^T S(u)\,dW(u)\right]^2\biggr). \qquad (3.12)\end{aligned}$$

Note that

$$\begin{aligned}\mathbb{E}\left[\left(\int_t^T S(u)\,dW(u)\right)^2\right] &= \mathbb{E}\left[\int_t^T S(u)^2\,du\right] \\ &\leq \mathbb{E}\left[\int_t^T (B(u)+S(u))^2\,du\right].\end{aligned}$$

Hence, taking expectations in (3.12) we see that

$$\begin{aligned}\mathbb{E}\left[(B(T)+S(T))^2\right] \leq L\biggl(1+b^2+s^2 \\ &+ K^2(T-t)\int_t^T \mathbb{E}\left[(B(u)+S(u))^2\right]\,du \\ &+ \sigma^2\int_t^T \mathbb{E}\left[(B(u)+S(u))^2\right]\,du\biggr)\end{aligned}$$

and we can again conclude by Gronwall's inequality. □

Remark 3.7. Since we take a worst-case perspective the condition on the crash time τ in Lemma 3.6 poses no restriction to our subsequent analysis since clearly an optimal crash should never increase the net wealth. We therefore assume from now on that this condition always holds. ◇

We can now prove the continuity in time. As a first step we prove the result in the case $p \in (0,1)$ and then extend the result to $p \leq 0$ by means of an approximation procedure.

Proposition 3.8. *Assume that $p \in (0,1)$ and let $(b,s) \in \mathcal{S}^1$ be fixed. Then $\mathcal{V}^1(\cdot,b,s)$ is uniformly continuous on $[0,T]$.*

Proof. By Lemma 3.5 we have

$$\left|\mathcal{V}_1^T(t,b,s) - \mathcal{V}_1^T(t+h,b,s)\right| = \left|\mathcal{V}_1^T(t,b,s) - \mathcal{V}_1^{T-h}(t,b,s)\right|$$

for every $h \geq -t$ and hence in order to prove continuity in t it suffices to prove continuity in T.

1. We first show that $\mathcal{V}_1^T$ is increasing in T. For this, let $T_- < T_+$ and fix $t \in [0, T_-]$. Let $\pi_1^- \in \mathcal{A}_1^{T_-}(t,b,s)$, $\pi_0^- \in \mathcal{A}_0^{T_-}(\pi_1^-)$ and define π_1^+ and π_0^+ such that $\pi_1^+ = \pi_1^-$ and $\pi_0^+ = \pi_0^-$ (componentwise) on $[t, T_0)$ and such that $S^{\pi_1^+,\pi_0^+}(u) = 0$ on $[T_-, T_+]$ (i.e. liquidation of the stock position at T_- and no trading afterwards). Then $\pi_1^+ \in \mathcal{A}_1^{T_+}(t,b,s)$, $\pi_0^+ \in \mathcal{A}_0^{T_+}(\pi_1^+)$ and noticing that every crash time for time horizon T_- is also admissible for horizon T_+ it follows that

$$X_{t,b,s}^{\pi_1^+,\pi_0^+,\tau}(T_+) = X_{t,b,s}^{\pi_1^-,\pi_0^-,\tau}(T_-)$$

for every $\tau \in \mathcal{B}^{T_-}(t)$. Thus, since the position in the stock is closed on $(T_-, T_+]$ and hence the worst-case bound of (π_1^+, π_0^+) is not attained for stopping times with values in this interval, we get

$$\mathcal{W}_1^{T_+}(\pi_1^+, \pi_0^+, t, b, s) = \mathcal{W}_1^{T_-}(\pi_1^-, \pi_0^-, t, b, s)$$

and since π_1^- and π_-^0 were chosen arbitrarily it follows that

$$\mathcal{V}_1^{T_+}(t,b,s) \geq \mathcal{V}_1^{T_-}(t,b,s).$$

2. Let $\varepsilon > 0$. We are left with showing that

$$\mathcal{V}_1^{T_+}(t,b,s) - \mathcal{V}_1^{T_-}(t,b,s) \leq \varepsilon,$$

if $T_+ - T_-$ is sufficiently small. Choose $\pi_1^+ \in \mathcal{A}_1^{T_+}(t,b,s)$ and $\pi_0^+ \in \mathcal{A}_0^{T_+}(\pi_1^+)$ to be ε-optimal, i.e.

$$\mathcal{W}_1^{T_+}(\pi_1^+, \pi_0^+, t, b, s) + \varepsilon \geq \mathcal{V}_1^{T_+}(t,b,s).$$

Denote by π_1^- and π_0^- the restrictions of the strategies π_1^+ and π_0^+ to $[t, T_-]$. Then $\pi_1^- \in \mathcal{A}_1^{T_-}(t,b,s)$ and $\pi_0^- \in \mathcal{A}_0^{T_-}(\pi_1^-)$. Furthermore, there exists a crash time $\tau^\varepsilon \in \mathcal{B}^{T_-}(t)$ which is ε-optimal in the sense that

$$\mathcal{W}_1^{T_-}(\pi_1^-, \pi_0^-, t, b, s) + \varepsilon \geq \mathcal{J}_1^{T_-}(\pi_1^-, \pi_0^-, \tau^\varepsilon, t, b, s)$$

and since it is possible to consider $\mathcal{B}^{T_-}(t) \subset \mathcal{B}^{T_+}(t)$, τ^ε also defines an admissible crash time for time horizon T_+. Then the sub-additivity of U_p and Jensen's inequality show that

$$\begin{aligned}
&\mathcal{V}_1^{T_+}(t,b,s) - \mathcal{V}_1^{T_-}(t,b,s) \\
&\quad \le \mathcal{W}_1^{T_+}(\pi_1^+,\pi_0^+,t,b,s) - \mathcal{W}_1^{T_-}(\pi_1^-,\pi_0^-,t,b,s) + \varepsilon \\
&\quad \le \mathcal{J}_1^{T_+}(\pi_1^+,\pi_0^+,\tau^\varepsilon,t,b,s) - \mathcal{J}_1^{T_-}(\pi_1^-,\pi_0^-,\tau^\varepsilon,t,b,s) + 2\varepsilon \\
&\quad \le U_p\left(\mathbb{E}\left[\left(X_{t,b,s}^{\pi_1^+,\pi_0^+,\tau^\varepsilon}(T_+) - X_{t,b,s}^{\pi_1^-,\pi_0^-,\tau^\varepsilon}(T_-)\right)\mathbb{1}_A\right]\right) + 2\varepsilon, \qquad (3.13)
\end{aligned}$$

where

$$A := \left\{X_{t,b,s}^{\pi_1^+,\pi_0^+,\tau^\varepsilon}(T_+) - X_{t,b,s}^{\pi_1^-,\pi_0^-,\tau^\varepsilon}(T_-) > 0\right\}.$$

Next, since $(\pi_1^+,\pi_0^+) = (\pi_1^-,\pi_0^-)$ on $[t,T_-]$ and since τ^ε is $[t,T_-]\cup\{+\infty\}$-valued it is not hard to see that

$$\begin{aligned}
&\left(X_{t,b,s}^{\pi_1^+,\pi_0^+,\tau^\varepsilon}(T_+) - X_{t,b,s}^{\pi_1^-,\pi_0^-,\tau^\varepsilon}(T_-)\right)\mathbb{1}_A \\
&\qquad\qquad \le (1+\lambda)\left|\int_{T_-}^{T_+} \alpha S(u)\,du\right| + (1+\lambda)\left|\int_{T_-}^{T_+} \sigma S(u)\,dW(u)\right|
\end{aligned}$$

with

$$B(u) := B_{t,b}^{\pi_1^+,\pi_0^+,\tau^\varepsilon}(u), \qquad S(u) := S_{t,s}^{\pi_1^+,\pi_0^+,\tau^\varepsilon}(u).$$

With this, we see that there exists a constant $C>0$ such that

$$\begin{aligned}
&\mathbb{E}\left[\left(X_{t,b,s}^{\pi_1^+,\pi_0^+,\tau^\varepsilon}(T_+) - X_{t,b,s}^{\pi_1^-,\pi_0^-,\tau^\varepsilon}(T_-)\right)\mathbb{1}_A\right] \\
&\quad \le (1+\lambda)\mathbb{E}\left[\left|\int_{T_-}^{T_+} \alpha S(u)\,du\right| + \left|\int_{T_-}^{T_+} \sigma S(u)\,dW(u)\right|\right] \\
&\quad \le C\mathbb{E}\left[\int_{T_-}^{T_+} |B(u)+S(u)|\,du\right] + C\mathbb{E}\left[\int_{T_-}^{T^+} S(u)^2\,du\right]^{1/2} \\
&\quad \le C\int_{T_-}^{T_+} \mathbb{E}[B(u)+S(u)]\,du + C\left[\int_{T_-}^{T^+} \mathbb{E}[(B(u)+S(u))^2]\,du\right]^{1/2}.
\end{aligned}$$

By Lemma 3.6 we can hence find a constant $K>0$ independent of π_1^+,π_0^+ and τ^ε such that

$$\mathbb{E}\left[\left(X_{t,b,s}^{\pi_1^+,\pi_0^+,\tau^\varepsilon}(T_+) - X_{t,b,s}^{\pi_1^-,\pi_0^-,\tau^\varepsilon}(T_-)\right)\mathbb{1}_A\right]$$

$$\leq K(b+s)(T_+ - T_-) + K(1+b^2+s^2)^{1/2}(T_+ - T_-)^{1/2}.$$

Combining this with (3.13) yields the desired result. □

Note that the only reason why the proof of Proposition 3.8 does not work for $p \leq 0$ is because U_p is not sub-additive and hence we cannot derive (3.13). Nevertheless, we can define

$$U_p^j(x) := U_p(x + 1/j), \qquad \tilde{U}_p^j(x) = U_p^j(x) - U_p^j(0), \qquad x \in [0, \infty),$$

where $j \in \mathbb{N}$. Note that with this $\tilde{U}_p^j(0) = 0$ and hence $\tilde{U}_p^j$ is sub-additive. We denote by $\mathcal{V}_1^j$ the value function corresponding to $U_p^j(x)$. It can then be verified that $\mathcal{V}_1^j(\cdot, b, s)$ is also uniformly continuous on $[0, T]$ for all (b, s) fixed. Indeed, in the proof of Proposition 3.8 we only need to replace U_p by $\tilde{U}_p^j$ in (3.13) to make the same proof work.

Lemma 3.9. *Let $p \leq 0$ and fix $(b, s) \in \mathcal{S}^1$. Then*

$$\lim_{j \to \infty} \mathcal{V}_1^j(t, b, s) = \mathcal{V}_1(t, b, s)$$

uniformly in t.

Proof. We consider the case $p < 0$ only. The case $p = 0$ follows similarly. First, note that the family

$$\left\{ U_p\left(X_{t,b,s}^{\pi_1,\pi_0,\tau}(T)\right) \right\}_{t \in [0,T], \pi_1 \in \mathcal{A}_1(t,b,s), \pi_0 \in \mathcal{A}_0(\pi_1), \tau \in \mathcal{B}(t)} \tag{3.14}$$

is uniformly integrable. Indeed, choose $q > 1$ arbitrary. Then

$$\mathbb{E}\left[\left| U_p\left(X_{t,b,s}^{\pi_1,\pi_0,\tau}(T)\right) \right|^q \right] = \frac{pq}{|p|^q} \mathbb{E}\left[U_{pq}\left(X_{t,b,s}^{\pi_1,\pi_0,\tau}(T)\right) \right],$$

and since

$$\begin{aligned} &U_{pq}\big(b + \min\{(1-\mu)(1-\beta)s, (1+\lambda)s\}\big) \\ &\qquad \leq \mathbb{E}\left[U_{pq}\left(X_{t,b,s}^{\pi_1,\pi_0,\tau}(T)\right) \right] \leq \varphi_{1,pq,1}(0, b, s) \end{aligned}$$

by Lemma 3.2.2 and Lemma 3.3.1 the uniform integrability follows.

Let us now fix some $j \in \mathbb{N}$, $(t,b,s) \in [0,T] \times \mathcal{S}^1$, $\pi_1 \in \mathcal{A}_1(t,b,s)$, $\pi_0 \in \mathcal{A}_0(\pi_1)$ and $\tau \in \mathcal{B}(t)$. Let furthermore $\delta > 0$. We calculate

$$\begin{aligned}
0 &\le \mathbb{E}\left[U_p^j\big(X_{t,b,s}^{\pi_1,\pi_0,\tau}(T)\big)\right] - \mathbb{E}\left[U_p\big(X_{t,b,s}^{\pi_1,\pi_0,\tau}(T)\big)\right] \\
&= \mathbb{E}\left[\left(U_p^j\big(X_{t,b,s}^{\pi_1,\pi_0,\tau}(T)\big) - U_p\big(X_{t,b,s}^{\pi_1,\pi_0,\tau}(T)\big)\right)\mathbb{1}_{\{X_{t,b,s}^{\pi_1,\pi_0,\tau}(T)>\delta\}}\right] \\
&\qquad + \mathbb{E}\left[\left(U_p^j\big(X_{t,b,s}^{\pi_1,\pi_0,\tau}(T)\big) - U_p\big(X_{t,b,s}^{\pi_1,\pi_0,\tau}(T)\big)\right)\mathbb{1}_{\{X_{t,b,s}^{\pi_1,\pi_0,\tau}(T)\le\delta\}}\right] \\
&\le U_p^j(\delta) - U_p(\delta) - \mathbb{E}\left[U_p\big(X_{t,b,s}^{\pi_1,\pi_0,\tau}(T)\big)\mathbb{1}_{\{X_{t,b,s}^{\pi_1,\pi_0,\tau}(T)\le\delta\}}\right],
\end{aligned}$$

where the last inequality follows from the fact that the difference $U_p^j(x) - U_p(x)$ on $[\delta, \infty)$ is maximal at δ and since $U_p^j \le 0$. Let now $\varepsilon > 0$. By the uniform integrability of (3.14) it follows that if δ is small enough, then

$$\mathbb{E}\left[\left|U_p\big(X_{t,b,s}^{\pi_1,\pi_0,\tau}(T)\big)\mathbb{1}_{\{X_{t,b,s}^{\pi_1,\pi_0,\tau}(T)\le\delta\}}\right|\right] \le \varepsilon/2,$$

uniformly in t, π_1, π_0 and τ. Next, for this choice of δ there exists $J \in \mathbb{N}$ large enough such that

$$U_p^j(\delta) - U_p(\delta) \le \varepsilon/2$$

for all $j \ge J$. In total, this implies that

$$\sup_{t\in[0,T]} \sup_{\substack{\pi_1\in\mathcal{A}_1(t,b,s)\\ \pi_0\in\mathcal{A}_0(\pi_1)}} \inf_{\tau\in\mathcal{B}(t)} \left|\mathbb{E}\left[U_p^j\big(X_{t,b,s}^{\pi_1,\pi_0,\tau}(T)\big)\right] - \mathbb{E}\left[U_p\big(X_{t,b,s}^{\pi_1,\pi_0,\tau}(T)\big)\right]\right| \le \varepsilon$$

for all $j \ge J$. □

Proposition 3.10. *Assume that $p \le 0$ and let $(b,s) \in \mathcal{S}^1$ be fixed. Then $\mathcal{V}_1(\cdot, b, s)$ is uniformly continuous on $[0,T]$.*

Proof. Let $\varepsilon > 0$, $t \in [0,T]$ and let $(t_n)_{n\in\mathbb{N}}$ be a sequence in $[0,T]$ converging to t. By Lemma 3.9 there exists $j \in \mathbb{N}$ such that

$$\sup_{t\in[0,T]} |\mathcal{V}_1(t,b,s) - \mathcal{V}_1^j(t,b,s)| \le \varepsilon/3$$

and by the continuity of $\mathcal{V}_1^j$ there exists some $N \in \mathbb{N}$ such that

$$|\mathcal{V}_1^j(t_n,b,s) - \mathcal{V}_1^j(t,b,s)| \le \varepsilon/3$$

for all $n \ge N$. Hence

$$
\begin{aligned}
&|\mathcal{V}_1(t_n, b, s) - \mathcal{V}_1(t, b, s)| \\
&\qquad \le |\mathcal{V}_1(t_n, b, s) - \mathcal{V}_1^j(t_n, b, s)| + |\mathcal{V}_1^j(t_n, b, s) - \mathcal{V}_1^j(t, b, s)| \\
&\qquad\qquad + |\mathcal{V}_1^j(t, b, s) - \mathcal{V}_1(t, b, s)| \le \varepsilon
\end{aligned}
$$

for all $n \ge N$. □

Putting the pieces together we can prove the joint continuity of $\mathcal{V}_1$.

Theorem 3.11. *The value function $\mathcal{V}_1$ is continuous.*

Proof. Since $\mathcal{V}_1(t, b, s)$ is locally bounded in a small neighborhood of (b, s) uniformly in t, the local Lipschitz continuity (Lemma 3.3.3) of $\mathcal{V}_1$ holds uniformly in t. With this, it is easy to prove the joint continuity on $[0, T] \times \mathcal{S}^1$. Indeed, let $t \in [0, T]$ and $(b, s) \in \mathcal{S}^1$ and choose a sequence (t_n, b_n, s_n) converging to (t, b, s). Note that (b_n, s_n) is contained in a compact subset K of $\mathcal{S}^1$. By the local Lipschitz continuity of $\mathcal{V}_1$ there exists a constant $L > 0$ such that

$$
|\mathcal{V}_1(u, b_n, s_n) - \mathcal{V}_1(u, b, s)| \le L(|b_n - b| + |s_n - s|)
$$

for all $u \in [0, T]$ and all n. Hence

$$
\begin{aligned}
&\lim_{n\to\infty} |\mathcal{V}_1(t_n, b_n, s_n) - \mathcal{V}_1(t, b, s)| \\
&\qquad \le \lim_{n\to\infty} |\mathcal{V}_1(t_n, b_n, s_n) - \mathcal{V}_1(t_n, b, s)| + |\mathcal{V}_1(t_n, b, s) - \mathcal{V}_1(t, b, s)| \\
&\qquad = \lim_{n\to\infty} L(|b_n - b| + |s_n - s|) + |\mathcal{V}_1(t_n, b, s) - \mathcal{V}_1(t, b, s)| = 0.
\end{aligned}
$$

In order to show that the continuity of $\mathcal{V}_1$ extends to the boundary of $\mathcal{S}^1$ we let $(t, b, s) \in [0, T] \times \partial\mathcal{S}^1$ and let $(t_n, b_n, s_n)_{n\in\mathbb{N}}$ be a sequence converging to (t, b, s). If $s \le 0$ we have

$$
\lim_{n\to\infty} \mathcal{V}_1(t_n, b_n, s_n) \le \lim_{n\to\infty} \varphi_{1+\lambda,p,1}(t_n, b_n, s_n) = U_p(0),
$$

and if $s > 0$ we have

$$
\begin{aligned}
\lim_{n\to\infty} \mathcal{V}_1(t_n, b_n, s_n) &\le \lim_{n\to\infty} \mathcal{V}_0(t_n, b_n, (1-\beta)s_n) \\
&\le \lim_{n\to\infty} \varphi_{1-\mu,p,1}(t_n, b_n, (1-\beta)s_n) = U_p(0).
\end{aligned}
$$

Since $\mathcal{V}_1(t, b, s) = U_p(0)$ this concludes the proof. □

We note that the same arguments presented in this section can also be used to prove the continuity of $\mathcal{V}_0$.

Corollary 3.12. *The value function $\mathcal{V}_0$ is continuous.*

3.5. The dynamic programming principle

Equipped with the continuity of the value function we are now in the position to prove the dynamic programming principle. The main problem arising in the proof is the construction of strategies $\pi_1^\varepsilon \in \mathcal{A}_1(t,b,s)$ and $\pi_0^\varepsilon \in \mathcal{A}_0(\pi_1^\varepsilon)$ which are ε-optimal in the sense that

$$\mathcal{V}_1(t,b,s) \leq \mathcal{W}_1(\pi_1^\varepsilon, \pi_0^\varepsilon, t, b, s) + \varepsilon$$

and similarly crash times $\tau^\varepsilon \in \mathcal{B}(t)$ which are ε-optimal in the sense that

$$\mathcal{J}_1(\pi_1, \pi_0, \tau^\varepsilon, t, b, s) \leq \mathcal{W}_1(\pi_1, \pi_0, t, b, s) - \varepsilon$$

for a given pair of trading strategies $\pi_1 \in \mathcal{A}_1(t,b,s)$ and $\pi_0 \in \mathcal{A}_0(\pi_1)$. The existence of such controls is clear if the initial time t as well as the initial holdings b and s are deterministic. The first aim is the construction of such strategies for random t, b and s.

3.5.1. Existence of ε-optimal strategies

The problem with the construction of ε-optimal strategies is the following: Denote by θ a random initial time and by (B,S) a random initial position of the investor. Then for every $\omega \in \Omega$ we can find a strategy $\pi_1^{\varepsilon,\omega} \in \mathcal{A}_1(\theta(\omega), B(\omega), S(\omega))$ which is ε-optimal. However, if we compose such strategies $\pi_1^{\varepsilon,\omega}$ into a single strategy π_1^ε then it is not clear if $\pi^\varepsilon \in \mathcal{A}_1(\theta, B, S)$ since it is not clear if the mapping $\omega \mapsto \pi_1^{\varepsilon,\omega}$ is measurable.

Since we can construct ε-optimal strategies for deterministic initial data (t,b,s) the idea of the construction of ε-optimal strategies for random initial data is to find a suitable sequence of points $(t_n, b_n, s_n)_{n\in\mathbb{N}}$ in $(t,T] \times \mathcal{S}^1$, construct an ε-optimal strategy $\pi_1^{\varepsilon,n}$ for every such (t_n, b_n, s_n) and then construct the ε-optimal strategy for the random initial datum (θ, B, S) by setting it equal to a strategy closely related to $\pi_1^{\varepsilon,n}$ whenever (θ, B, S) is close to (t_n, b_n, s_n). We note that a crucial component for this construction is the continuity of $\mathcal{V}_1$.

We start by constructing a suitable decomposition of $(t,T]\times\mathcal{S}^1$. Given $(b,s) \in \mathcal{S}^1$ and $r > 0$ we denote by $K(b,s;r)$ the set of all $(\bar{b},\bar{s})$ such that $|(b,s)-(\bar{b},\bar{s})| < r$ and such that there exist $l, m \geq 0$ with

$$b = \bar{b} - (1+\lambda)l + (1-\mu)m, \qquad s = \bar{s} + l - m,$$

i.e. (b,s) can be reached by the transaction (l,m) from $(\bar{b},\bar{s})$. See also Figure 3.3 for a sketch of the set $K(b,s;r)$.

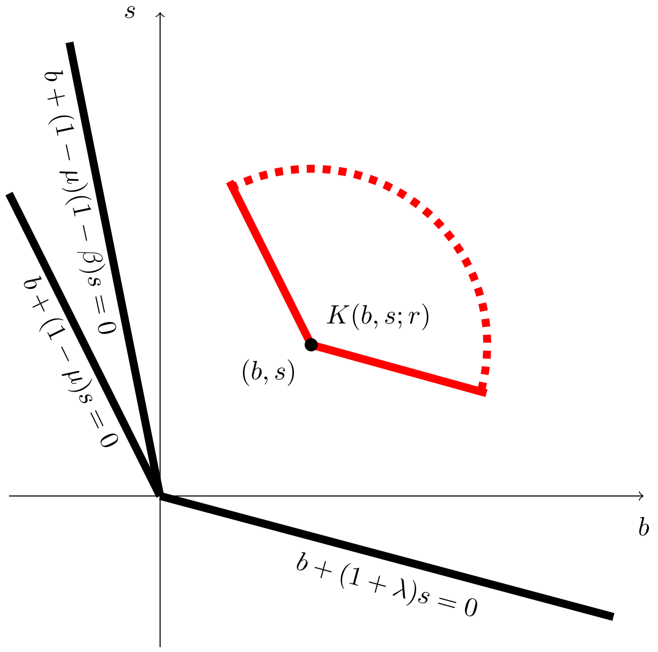

Figure 3.3. A sketch of the set $K(b,s;r)$.

Let us now fix $\varepsilon > 0$ and $(u,b,s) \in (t,T] \times \mathcal{S}^1$. By the continuity of $\mathcal{V}_1$ there exists $r(u,b,s) > 0$ such that $u - r(u,b,s) \geq t$, $K(b,s;r(u,b,s)) \subset \mathcal{S}^1$ and

$$|\mathcal{V}_1(\bar{u},\bar{b},\bar{s}) - \mathcal{V}_1(\tilde{u},\tilde{b},\tilde{s})| \leq \varepsilon$$

for all $(\bar{u},\bar{b},\bar{s}), (\tilde{u},\tilde{b},\tilde{s}) \in (u - r(u,b,s), u] \times K(b,s;r(u,b,s))$. The family

$$\Big((u - r(u,b,s), u] \times K(b,s;r(u,b,s)) \Big)_{(u,b,s)\in(t,T]\times\mathcal{S}^1}$$

forms an open covering of $(t,T] \times \mathcal{S}^1$ (in the topology induced by the sets of the form $(v,u] \times K(b,s;r)$) and hence there exists a countable sub-covering

$$\Big((t_i - r(t_i,b_i,s_i), t_i] \times K(b_i,s_i;r(t_i,b_i,s_i))\Big)_{i\in\mathbb{N}}.$$

We furthermore set $K_i := (t_i - r(t_i,b_i,s_i), t_i] \times K(b_i,s_i;r(t_i,b_i,s_i))$ as short-hand notation.

Lemma 3.13. *Let $\varepsilon > 0$, let θ be a $[t,T]$-valued stopping time and fix an arbitrary pre-crash trading strategy $\pi_1 \in \mathcal{A}_1(t,b,s)$. For every crash time $\tau \in \mathcal{B}(t)$ there exists $\pi_1^\varepsilon \in \mathcal{A}_1(t,b,s)$ which coincides with π_1 on $[t,\tau\wedge\theta)$ and a family of post-crash strategies $\pi_0^\varepsilon = (\pi_0^{\varepsilon,\tau})_{\tau\in\mathcal{B}(t)}$ such that $\pi_0^\varepsilon \in \mathcal{A}_0(\pi_1^\varepsilon)$ and*

$$\begin{aligned}\mathbb{E}\Big[U_p\Big(X_{t,b,s}^{\pi_1^\varepsilon,\pi_0^{\varepsilon},\tau}(T)\Big)\Big|\mathcal{F}^t(\tau\wedge\theta)\Big] + \varepsilon \\ \geq \mathcal{V}_0\big(\tau, B(\tau-), (1-\beta)S(\tau-)\big)\mathbb{1}_{\{\tau\leq\theta\}} + \mathcal{V}_1\big(\theta, B(\theta), S(\theta)\big)\mathbb{1}_{\{\tau>\theta\}}.\end{aligned}$$

Proof. Let $\varepsilon > 0$ and let $(K_i)_{i\in\mathbb{N}}$ and $(t_i,b_i,s_i)_{i\in\mathbb{N}}$ be the sequences constructed in the beginning of this section. For every $i \in \mathbb{N}$ there exists some $\pi_1^{\varepsilon,i} \in \mathcal{A}_1(t_i,b_i,s_i)$ and $\pi_0^{\varepsilon,i} \in \mathcal{A}_0(\pi_1^{\varepsilon,i})$ such that

$$\inf_{\tau\in\mathcal{B}(t_i)} \mathbb{E}\Big[U_p\Big(X_{t_i,b_i,s_i}^{\pi_1^{\varepsilon,i},\pi_0^{\varepsilon,i},\tau}\Big)\Big] \geq \mathcal{V}_1(t_i,b_i,s_i) - \varepsilon$$

so that in particular

$$\mathcal{J}_1(\pi_1^{\varepsilon,i}, \pi_0^{\varepsilon,i}, \tau, t_i, b_i, s_i) \geq \mathcal{V}_1(t_i,b_i,s_i) - \varepsilon$$

for every $\tau \in \mathcal{B}(t_i)$ and every $i \in \mathbb{N}$.

Given any $[t,T]$-valued stopping time θ and $\pi_1 \in \mathcal{A}_1(t,b,s)$ we define

$$\iota(\omega) := \min\Big\{i \in \mathbb{N} : \Big(\theta(\omega), B^{\pi_1}(\theta(\omega),\omega), S^{\pi_1}(\theta(\omega),\omega)\Big) \in K_i\Big\}.$$

We denote by $D_i = \{\tau > \theta\} \cap \{\iota = i\}$ so that

$$\Big(\theta(\omega), B^{\pi_1}(\theta(\omega),\omega), S^{\pi_1}(\theta(\omega),\omega)\Big) \in K_i$$

on D_i and $D_i \cap D_j = \emptyset$ if $i \neq j$. Finally, we set

$$A_1 := \{\tau > \theta\} \cap \{\theta = t\},$$

$$A_2 := \{\tau > \theta\} \cap \{(B^{\pi_1}(\theta), S^{\pi_1}(\theta)) \in \partial\mathcal{S}_1\}.$$

Then the sets $\{\tau \leq \theta\}$, A_1, A_2 and $(D_i)_{i\in\mathbb{N}}$ form a partition of Ω.

Let us now define

$$\pi_1^\varepsilon := \pi_1 \mathbb{1}_{[t,\tau\wedge\theta]} + \mathbb{1}_{[\tau\wedge\theta,T]} \left[\pi_1^{A_1} \mathbb{1}_{A_1} + \pi_1^{A_2} \mathbb{1}_{A_2} + \sum_{i=1}^{\infty} \tilde{\pi}_1^{\varepsilon,i} \mathbb{1}_{D_i} \right], \tag{3.15}$$

where $\pi_1^{A_1}$, $\pi_1^{A_2}$ and $\tilde{\pi}_1^{\varepsilon,i}$ are defined below. Note that since $\pi_1^{A_1}$, $\pi_1^{A_2}$ and $\tilde{\pi}_1^{\varepsilon,i}$ only enter into the definition of π_1^ε on the sets A_1, A_2 and D_i, respectively, it does not matter how the strategies are defined outside these sets. Moreover, since the investor switches to a different strategy at time τ we see that the performance of π_1^ε does not depend on the choice of τ. Similarly, we set

$$\pi_0^{\varepsilon,\tau} := \pi_0^\tau \mathbb{1}_{\{\tau\leq\theta\}} + \mathbb{1}_{\{\tau>\theta\}} \left[\pi_0^{A_1} \mathbb{1}_{A_1} + \pi_0^{A_2} \mathbb{1}_{A_2} + \sum_{i=1}^{\infty} \tilde{\pi}_0^{\varepsilon,i} \mathbb{1}_{D_i} \right], \tag{3.16}$$

for π_0^τ, $\pi_0^{A_1}$, $\pi_0^{A_2}$ and $\tilde{\pi}_0^{\varepsilon,i}$ to be defined later. Our objective is to choose the strategies in such a way that π_1^ε and $\pi_0^\varepsilon := (\pi_0^{\varepsilon,\tau})_{\tau\in\mathcal{B}(t)}$ are ε-optimal from time $\tau \wedge \theta$ onwards.

We choose π_0^τ in (3.16) to be ε-optimal in the crash-free market corresponding to initial values $(\tau, B^{\pi_1}(\tau-), (1-\beta)S^{\pi_1}(\tau-))$. On $\{\tau \leq \theta\}$ we then have

$$\begin{aligned}
\mathbb{E}\left[U_p\left(X_{t,b,s}^{\pi_1^\varepsilon,\pi_0^{\varepsilon,\tau}}(T)\right)\middle|\mathcal{F}^t(\tau\wedge\theta)\right] &= \mathbb{E}\left[U_p\left(X_{t,b,s}^{\pi_1^\varepsilon,\pi_0^{\varepsilon,\tau}}(T)\right)\middle|\mathcal{F}^t(\tau)\right] \\
&= \mathbb{E}\left[U_p\left(X_{\tau,B^{\pi_1}(\tau-),(1-\beta)S^{\pi_1}(\tau-)}^{\pi_1^\varepsilon,\pi_0^{\varepsilon,\tau}}(T)\right)\middle|\mathcal{F}^t(\tau)\right] \\
&= \mathbb{E}\left[U_p\left(X_{\tau,B^{\pi_1}(\tau-),(1-\beta)S^{\pi_1}(\tau-)}^{\pi_0^{\varepsilon,\tau}}(T)\right)\right] \\
&\geq \mathcal{V}_0\big(\tau, B^{\pi_1}(\tau-), (1-\beta)S^{\pi_1}(\tau-)\big) - \varepsilon.
\end{aligned}$$

On A_1 we have $\theta = t$ and therefore $B^{\pi_1}(\theta-) = b$ and $S^{\pi_1}(\theta-) = s$ so that $B^{\pi_1}(\theta) = b-(1+\lambda)l+(1-\mu)m$ and $S^{\pi_1}(\theta) = s+l-m$ for some $l, m \geq 0$ such that l and m are deterministic almost everywhere on A_1. Hence if we choose $\pi_1^{A_1}$ and $\pi_0^{A_1}$ in (3.15) and (3.16) to be any ε-optimal strategy for initial values $(t, B(\theta), S(\theta))$ we have

$$\mathbb{E}\left[U_p\left(X_{t,b,s}^{\pi_1^\varepsilon,\pi_0^{\varepsilon,\tau}}(T)\right)\middle|\mathcal{F}^t(\tau\wedge\theta)\right] = \mathbb{E}\left[U_p\left(X_{\theta,B(\theta),S(\theta)}^{\pi_1^\varepsilon,\pi_0^{\varepsilon,\tau}}(T)\right)\right]$$

$$\begin{aligned} &= \mathcal{J}_1(\pi_1^\varepsilon, \pi_0^\varepsilon, \tau, \theta, B(\theta), S(\theta)) \\ &\geq \mathcal{V}_1(\theta, B(\theta), S(\theta)) - \varepsilon. \end{aligned}$$

On A_2 we have $(B^{\pi_1}(\theta), S^{\pi_1}(\theta)) \in \partial\mathcal{S}^1$ and hence if we choose $\pi_1^{A_2}$ and $\pi_0^{A_2}$ to be the strategies which immediately liquidate the stock position and refrain from further trading we obtain

$$\begin{aligned} \mathbb{E}\left[U_p\left(X_{t,b,s}^{\pi_1^\varepsilon,\pi_0^\varepsilon,\tau}(T)\right)\middle|\mathcal{F}^t(\tau\wedge\theta)\right] &= \mathbb{E}\left[U_p\left(X_{t,b,s}^{\pi_1^\varepsilon,\pi_0^\varepsilon,\tau}(T)\right)\middle|\mathcal{F}^t(\theta)\right] \\ &= \mathbb{E}\left[U_p\left(X_{\theta,B^{\pi_1}(\theta),S^{\pi_1}(\theta)}^{\pi_1^{A_2},\pi_0^{A_2},\tau}(T)\right)\right] \\ &\geq \inf_{\tau\in\mathcal{B}(\theta)} \mathbb{E}\left[U_p\left(X_{\theta,B^{\pi_1}(\theta),S^{\pi_1}(\theta)}^{\pi_1^{A_2},\pi_0^{A_2},\tau}(T)\right)\right] \\ &= \mathcal{V}_1\big(\theta, B^{\pi_1}(\theta), S^{\pi_1}(\theta)\big) \end{aligned}$$

on A_2. The last equality follows from Lemma 3.2.1 (optimality of $\pi_1^{A_2}$ and $\pi_0^{A_2}$) since $(B^{\pi_1}(\theta), S^{\pi_1}(\theta)) \in \partial\mathcal{S}^1$ on A_2.

Let us now turn to the definition of the strategies $\tilde{\pi}_1^{\varepsilon,i}$. On the set D_i we have $(\theta, B^{\pi_1}(\theta), S^{\pi_1}(\theta)) \in K_i$ and hence $\theta \leq t_i$. Now let $\bar{\pi}_1^{\varepsilon,i}$ be arbitrary off D_i and on D_i be equal to the strategy $\pi_1^{\varepsilon,i}$ shifted from $[t_i, T]$ to $[\theta, T-(t_i-\theta)]$ (as in Lemma 3.5) and extended to $[\theta, T]$ through the convention that $\bar{\pi}_1^{\varepsilon,i}$ liquidates the stock position at time $T-(t_i-\theta)$ and refrains from further trading on $(T-(t_i-\theta), T]$. Similarly, we let $\bar{\pi}_0^{\varepsilon,i}$ be the time-shifted version of $\pi_0^{\varepsilon,i}$. Since the time-shift operation is measurable we see that $\bar{\pi}_1^{\varepsilon,i} \in \mathcal{A}_1(\theta, b_i, s_i)$, $\tau + (t_i - \theta) \in \mathcal{B}(t_i)$ and $\bar{\pi}_0^{\varepsilon,i} \in \mathcal{A}_0(\bar{\pi}_1^{\varepsilon,i})$. We therefore have

$$\mathcal{J}_1(\bar{\pi}_1^{\varepsilon,n}, \pi_0^{\varepsilon,i}, \tau, \theta, b_i, s_i) = \mathcal{J}_1(\pi_1^{\varepsilon,i}, \pi_0^{\varepsilon,i}, \tau + (t_i - \theta), t_i, b_i, s_i) \tag{3.17}$$

as in Lemma 3.5. By the definition of D_i and K_i we know that (b_i, s_i) can be reached by a transaction from $(B^{\pi_1}(\theta), S^{\pi_1}(\theta))$ on D_i. That is, there exist $l, m \geq 0$ such that

$$\begin{aligned} b_i &= B^{\pi_1}(\theta(\omega), \omega) - (1+\lambda)l(\omega) + (1-\mu)m(\omega), \\ s_i &= S^{\pi_1}(\theta(\omega), \omega) + l(\omega) - m(\omega). \end{aligned}$$

Hence, if we define $\tilde{\pi}_1^{\varepsilon,i}$ to be arbitrary off D_i and equal to $\bar{\pi}_1^{\varepsilon,i} + (l, m)$ on D_i then $\tilde{\pi}_1^{\varepsilon,i} \in \mathcal{A}_1(\theta, B^{\pi_1}(\theta), S^{\pi_1}(\theta))$. Using that $\tau > \theta$ on D_i it follows that $\tilde{\pi}_0^{\varepsilon,i} := \bar{\pi}_0^{\varepsilon,i} \in \mathcal{A}_0(\tilde{\pi}_1^{\varepsilon,i})$ and

$$\mathcal{J}_1\big(\tilde{\pi}_1^{\varepsilon,i}, \tilde{\pi}_0^{\varepsilon,i}, \tau, \theta, B^{\pi_1}(\theta), S^{\pi_1}(\theta)\big) = \mathcal{J}_1(\bar{\pi}_1^{\varepsilon,i}, \bar{\pi}_0^{\varepsilon,i}, \tau, \theta, b_i, s_i)$$

$$= \mathcal{J}_1(\pi_1^{\varepsilon,i}, \pi_0^{\varepsilon,i}, \tau + (t_i - \theta), t_i, b_i, s_i)$$

by (3.17). With this and the ε-optimality of $\pi_1^{\varepsilon,i}$ and $\pi_0^{\varepsilon,i}$ we therefore have

$$\begin{aligned}
\mathbb{E}\left[U_p\left(X_{t,b,s}^{\pi_1^\varepsilon,\pi_0^\varepsilon,\tau}(T)\right)\middle|\mathcal{F}^t(\tau\wedge\theta)\right] &= \mathbb{E}\left[U_p\left(X_{t,b,s}^{\pi_1^\varepsilon,\pi_0^\varepsilon,\tau}(T)\right)\middle|\mathcal{F}^t(\theta)\right] \\
&= \mathbb{E}\left[U_p\left(X_{\theta,B^{\pi_1}(\theta),S^{\pi_1}(\theta)}^{\tilde{\pi}_1^{\varepsilon,i},\tilde{\pi}_0^{\varepsilon,i},\tau}(T)\right)\right] \\
&= \mathcal{J}_1\big(\tilde{\pi}_1^{\varepsilon,i}, \tilde{\pi}_0^{\varepsilon,i}, \tau, \theta, B^{\pi_1}(\theta), S^{\pi_1}(\theta)\big) \\
&= \mathcal{J}_1(\pi_1^{\varepsilon,i}, \pi_0^{\varepsilon,i}, \tau + (t_i - \theta), t_i, b_i, s_i) \\
&\geq \mathcal{V}_1(t_i, b_i, s_i) - \varepsilon \\
&\geq \mathcal{V}_1\big(\theta, B^{\pi_1}(\theta), S^{\pi_1}(\theta)\big) - 2\varepsilon
\end{aligned}$$

on D_i. Here the last inequality follows from $(\theta, B^{\pi_1}(\theta), S^{\pi_1}(\theta)) \in K_i$ on D_i and the construction of K_i. □

In a similar fashion we need to construct ε-optimal crash times τ^ε to prove the dynamic programming principle. More precisely, given $\pi_1 \in \mathcal{A}_1(t,b,s)$, $\pi_0 \in \mathcal{A}_0(\pi_1)$ and $\bar{\tau} \in \mathcal{B}(t)$ we need to find $\tau^\varepsilon \geq \bar{\tau} \wedge \theta$ such that

$$\begin{aligned}
&\mathbb{E}\left[U_p\left(X_{t,b,s}^{\pi_1,\pi_0,\tau_\varepsilon}(T)\right)\middle|\mathcal{F}^t(\bar{\tau}\wedge\theta)\right] - \varepsilon \\
&\quad \leq \mathcal{V}_0\big(\bar{\tau}, B(\bar{\tau}-), (1-\beta)S(\bar{\tau}-)\big)\mathbb{1}_{\{\bar{\tau}\leq\theta\}} + \mathcal{V}_1\big(\theta, B(\theta), S(\theta)\big)\mathbb{1}_{\{\bar{\tau}>\theta\}}. \qquad (3.18)
\end{aligned}$$

Let us comment on some of the issues involved here. The first big issue is that we have to construct τ_ε for arbitrary pre-crash strategies $\pi_1 \in \mathcal{A}_1(t,b,s)$. Since we cannot guarantee that the controlled wealth process (B^{π_1}, S^{π_1}) runs through the grid points (t_i, b_i, s_i) we need a different method as compared to the construction of the ε-optimal trading strategies.

Our construction of τ^ε is based on the existence of the Snell envelope of a suitable optimal stopping problem. In the literature, the existence of the Snell envelope is typically proved under the assumption that the process to be stopped is right-continuous. Even though in our setting the wealth process (B,S) is right-continuous, the general theory of optimal stopping does not apply without modification since a crash acts on the left limits of the process S, i.e. on $(S(u-))_{u\in[t,T]}$.

The remedy is to first consider the situation in which a crash acts on $S(u)$ instead of $S(u-)$ to establish the existence of the Snell envelope. The construction of an ε-optimal stopping time for the right-continuous problem is then classical, but this stopping time need not be ε-optimal for the left-continuous problem. Nevertheless, it is intuitively clear how to construct such a stopping time for the left-continuous case: Simply stop a little bit later. However, since this can only be done on an event of probability arbitrarily close to one, we are not able to prove ε-optimality ω-wise as in (3.18), but can only show that

$$\begin{aligned}&\mathbb{E}\left[U_p\left(X_{t,b,s}^{\pi_1,\pi_0,\tau_\varepsilon}(T)\right)\right]-\varepsilon\\&\qquad\le\mathbb{E}\left[\mathcal{V}_0\big(\bar\tau,B(\bar\tau-),(1-\beta)S(\bar\tau-)\big)\mathbb{1}_{\{\bar\tau\le\theta\}}+\mathcal{V}_1\big(\theta,B(\theta),S(\theta)\big)\mathbb{1}_{\{\bar\tau>\theta\}}\right].\end{aligned}$$

The details of the construction of the Snell envelope for the left-continuous problem can be found in Appendix C.

Let us first formulate the correct (right-continuous) optimal stopping problem for our situation and prove the existence of the Snell envelope.

Lemma 3.14. *Fix $(t,b,s)\in[0,T]\times\mathcal{S}^1$. Let $\pi_1\in\mathcal{A}_1(t,b,s)$ be an arbitrary pre-crash strategy and let $\pi_0\in\mathcal{A}_0(\pi_1)$ be an arbitrary family of post-crash strategies corresponding to π_1. Define a process $Y=(Y(u))_{u\in[t,T]\cup\{\infty\}}$ through*

$$Y(u):=\begin{cases}\mathcal{V}_0(u,B_{t,b}^{\pi_1,\pi_0,\infty}(u),(1-\beta)S_{t,s}^{\pi_1,\pi_0,\infty}(u)), & \text{if } u\in[t,T),\\ \mathcal{V}_0(u,B_{t,b}^{\pi_1,\pi_0,\infty}(u-),(1-\beta)S_{t,s}^{\pi_1,\pi_0,\infty}(u-)), & \text{if } u=T,\\ U_p\left(X_{t,b,s}^{\pi_1,\pi_0,\infty}(T)\right), & \text{if } u=\infty.\end{cases}$$

Then there exists a process $Z=(Z(u))_{u\in[t,T]\cup\{+\infty\}}$ which is càdlàg on $[t,T]$ and such that

$$Z(\theta)=\operatorname*{ess\,inf}_{\tau\in\mathcal{B}(\theta)}\mathbb{E}\left[Y(\tau)\middle|\mathcal{F}^t(\theta)\right]$$

for every $\mathbb{F}^t$-stopping time θ with values in $[t,T]$. Moreover, Z is the smallest submartingale which is càdlàg on $[t,T]$ and which is dominated by Y. Finally, Z satisfies

$$\mathbb{E}[Z(t)]=\inf_{\tau\in\mathcal{B}(t)}\mathbb{E}[Y(\tau)].$$

Proof. We note that the process Y is càdlàg by definition and left-continuous at time T. The existence of Z is proved in Theorem C.6 in the appendix under the assumption that Y is uniformly integrable over all stopping times $\tau\in\mathcal{B}(t)$. To show this we proceed in three steps.

Step 1: We show that the positive part of Y is uniformly integrable. The result is clear for $p < 0$ and since $\log(x) \leq x^p/p$ for every $p \in (0,1)$ we can restrict ourselves to the case $p \in (0,1)$. Let us fix some $\varepsilon > 0$ such that $p(1+\varepsilon) < 1$ and let $\tau \in \mathcal{B}(t)$ be given.

We first note that on $\{\tau < T\}$ there exists a constant $K \geq 1$ such that

$$\begin{aligned}\mathcal{V}_0\big(\tau, B_{t,b}^{\pi_1,\pi_0,\infty}(\tau), (1-\beta)S_{t,s}^{\pi_1,\pi_0,\infty}(\tau)\big)&\\ \leq K\mathcal{V}_0\big(\tau, B_{t,b}^{\pi_1,\pi_0,\infty}(\tau), S_{t,s}^{\pi_1,\pi_0,\infty}(\tau)\big).& \qquad (3.19)\end{aligned}$$

On $\{S_{t,s}^{\pi_1,\pi_0,\infty}(\tau) \geq 0\}$ this is trivially satisfied for any choice of $K \geq 1$ since $\mathcal{V}_0$ is non-decreasing in its last argument. By the definition of the function $\varphi_{1+\lambda,p,1}$ we can find a constant $K > 1$ such that

$$\varphi_{1+\lambda,p,1}(t,b,s) \leq KU_p(b + (1+\lambda)s)$$

for every $t \in [0,T]$ and $(b,s) \in \mathcal{S}^1$ with $s < 0$. On $\{S_{t,s}^{\pi_1,\pi_0,\infty}(\tau) < 0\}$ this implies that

$$\begin{aligned}&\mathcal{V}_0\big(\tau, B_{t,b}^{\pi_1,\pi_0,\infty}(\tau), (1-\beta)S_{t,s}^{\pi_1,\pi_0,\infty}(\tau)\big)\\ &\leq \varphi_{1+\lambda,p,1}\big(\tau, B_{t,b}^{\pi_1,\pi_0,\infty}(\tau), (1-\beta)S_{t,s}^{\pi_1,\pi_0,\infty}(\tau)\big)\\ &\leq KU_p\big(B_{t,b}^{\pi_1,\pi_0,\infty}(\tau) + (1+\lambda)(1-\beta)S_{t,s}^{\pi_1,\pi_0,\infty}(\tau)\big)\\ &\leq K\mathcal{V}_1\big(\tau, B_{t,b}^{\pi_1,\pi_0,\infty}(\tau), (1-\beta)S_{t,s}^{\pi_1,\pi_0,\infty}(\tau)\big)\\ &\leq K\mathcal{V}_0\big(\tau, B_{t,b}^{\pi_1,\pi_0,\infty}(\tau), S_{t,s}^{\pi_1,\pi_0,\infty}(\tau)\big),\end{aligned}$$

proving (3.19). By the left-continuity of Y at T we furthermore have

$$\begin{aligned}\mathcal{V}_0\big(\tau, B_{t,b}^{\pi_1,\pi_0,\infty}(\tau-), (1-\beta)S_{t,s}^{\pi_1,\pi_0,\infty}(\tau-)\big)&\\ \leq K\mathcal{V}_0\big(\tau, B_{t,b}^{\pi_1,\pi_0,\infty}(\tau-), S_{t,s}^{\pi_1,\pi_0,\infty}(\tau-)\big).&\end{aligned}$$

Let us now prove the uniform integrability of the positive part of Y. Using (3.19) and the elementary fact that $(a+b)^{1+\varepsilon} \leq 2^{1+\varepsilon}(a^{1+\varepsilon} + b^{1+\varepsilon})$ whenever $a, b \geq 0$ we calculate

$$\begin{aligned}&\mathbb{E}\left[|Y(\tau)|^{1+\varepsilon}\right]\\ &\leq 4^{1+\varepsilon}\mathbb{E}\Big[\mathcal{V}_0\left(\tau, B_{t,b}^{\pi_1,\pi_0,\infty}(\tau), (1-\beta)S_{t,s}^{\pi_1,\pi_0,\infty}(\tau)\right)^{1+\varepsilon}\mathbb{1}_{\{\tau<T\}}\end{aligned}$$

$$
\begin{aligned}
&\quad + \mathcal{V}_0\left(\tau, B_{t,b}^{\pi_1,\pi_0,\infty}(\tau-), (1-\beta)S_{t,s}^{\pi_1,\pi_0,\infty}(\tau-)\right)^{1+\varepsilon} \mathbb{1}_{\{\tau=T\}}\Bigg] \\
&\qquad\qquad + U_p\left(X_{t,b,s}^{\pi_1,\pi_0,\infty}(T)\right)^{1+\varepsilon} \mathbb{1}_{\{\tau=\infty\}}\Bigg] \\
&\le 4^{1+\varepsilon}\mathbb{E}\Bigg[K^{1+\varepsilon}\mathcal{V}_0\left(\tau, B_{t,b}^{\pi_1,\pi_0,\infty}(\tau), S_{t,s}^{\pi_1,\pi_0,\infty}(\tau)\right)^{1+\varepsilon} \mathbb{1}_{\{\tau<T\}} \\
&\quad + K^{1+\varepsilon}\mathcal{V}_0\left(\tau, B_{t,b}^{\pi_1,\pi_0,\infty}(\tau-), S_{t,s}^{\pi_1,\pi_0,\infty}(\tau-)\right)^{1+\varepsilon} \mathbb{1}_{\{\tau=T\}}\Bigg] \\
&\qquad\qquad + U_p\left(X_{t,b,s}^{\pi_1,\pi_0,\infty}(T)\right)^{1+\varepsilon} \mathbb{1}_{\{\tau=\infty\}}\Bigg] \\
&\le 4^{1+\varepsilon}K^{1+\varepsilon}\mathbb{E}\left[\varphi_{1,p,1}\left(\tau\wedge T, B_{t,b}^{\pi_1,\pi_0,\infty}(\tau\wedge T), S_{t,s}^{\pi_1,\pi_0,\infty}(\tau\wedge T)\right)^{1+\varepsilon}\right] \\
&= 4^{1+\varepsilon}K^{1+\varepsilon}\frac{1+\varepsilon}{p^\varepsilon}\mathbb{E}\left[\varphi_{1,p(1+\varepsilon),1}\left(\tau\wedge T, B_{t,b}^{\pi_1,\pi_0,\infty}(\tau\wedge T), S_{t,s}^{\pi_1,\pi_0,\infty}(\tau\wedge T)\right)\right] \\
&\le 4^{1+\varepsilon}K^{1+\varepsilon}\frac{1+\varepsilon}{p^\varepsilon}\varphi_{1,p(1+\varepsilon),1}(t,b,s),
\end{aligned}
$$

where the last inequality follows from the fact that $\pi_1 \in \mathcal{A}_0(t,b,s)$ and since by the proof of Lemma 2.1.2 the process

$$
\Big(\varphi_{1,p(1+\varepsilon),1}\big(u, B_{t,b}^{\pi_1,\pi_0,\infty}(u), S_{t,s}^{\pi_1,\pi_0,\infty}(u)\big)\Big)_{u\in[t,T]}
$$

is a supermartingale.

Step 2: Consider the process $\tilde{Y} = (\tilde{Y}(u))_{u\in[t,T]\cup\{\infty\}}$ defined through

$$
\begin{aligned}
\tilde{Y}(u) := \mathcal{V}_0\left(u, B_{t,b}^{\pi_1,\pi_0,u}(u-), (1-\beta)S_{t,s}^{\pi_1,\pi_0,u}(u-)\right)\mathbb{1}_{\{u\le T\}} \\
+ U_p\left(X_{t,b,s}^{\pi_1,\pi_0,u}(T)\right)\mathbb{1}_{\{u=\infty\}}.
\end{aligned} \tag{3.20}
$$

We show that the negative part of $\tilde{Y}$ is uniformly integrable over all $\tau \in \mathcal{B}(t)$. The result is obvious for $p \in (0,1)$ and since $\log(x) \ge x^p/p$ for all $p < 0$ it suffices to consider the case $p < 0$ only. We fix $\tau \in \mathcal{B}(t)$. Since $\pi_0^\tau \in \mathcal{A}_0(\tau, B_{t,b}^{\pi_1,\pi_0,\tau}(\tau-), (1-\beta)S_{t,s}^{\pi_1,\pi_0,\tau}(\tau-))$ we have

$$
\begin{aligned}
&\mathcal{V}_0\big(\tau, B_{t,b}^{\pi_1,\pi_0,\tau}(\tau-), (1-\beta)S_{t,s}^{\pi_1,\pi_0,\tau}(\tau-)\big)\mathbb{1}_{\{\tau\le T\}} \\
&\qquad \ge \mathbb{E}\left[U_p\left(X^{\pi_0^\tau}_{\tau, B_{t,b}^{\pi_1,\pi_0,\tau}(\tau-), (1-\beta)S_{t,s}^{\pi_1,\pi_0,\tau}(\tau-)}(T)\right)\right]\mathbb{1}_{\{\tau\le T\}} \\
&\qquad = \mathbb{E}\left[U_p\left(X_{t,b,s}^{\pi_1,\pi_0,\tau}(T)\right)\right]\mathbb{1}_{\{\tau\le T\}}.
\end{aligned}
$$

It therefore follows that

$$\begin{aligned}
\mathbb{E}\left[|\tilde{Y}(\tau)|^{1+\varepsilon}\right] &= \mathbb{E}\Bigg[\Big|\mathcal{V}_0\left(\tau, B_{t,b}^{\pi_1,\pi_0,\tau}(\tau-), (1-\beta)S_{t,s}^{\pi_1,\pi_0,\tau}(\tau-)\right)\mathbb{1}_{\{\tau\le T\}} \\
&\qquad\qquad + U_p\left(X_{t,b,s}^{\pi_1,\pi_0,\tau}(T)\right)\mathbb{1}_{\{\tau=\infty\}}\Big|^{1+\varepsilon}\Bigg] \\
&\le 2^{1+\varepsilon}\mathbb{E}\Bigg[\Big|\mathcal{V}_0\left(\tau, B_{t,b}^{\pi_1,\pi_0,\tau}(\tau-), (1-\beta)S_{t,s}^{\pi_1,\pi_0,\tau}(\tau-)\right)\Big|^{1+\varepsilon}\mathbb{1}_{\{\tau\le T\}} \\
&\qquad\qquad + \Big|U_p\left(X_{t,b,s}^{\pi_1,\pi_0,\tau}(T)\right)\mathbb{1}_{\{\tau=\infty\}}\Big|^{1+\varepsilon}\Bigg] \\
&\le 2^{1+\varepsilon}\mathbb{E}\Bigg[\mathbb{E}\left[\left|U_p\left(X_{t,b,s}^{\pi_1,\pi_0,\tau}(T)\right)\right|^{1+\varepsilon}\right]\mathbb{1}_{\{\tau\le T\}} \\
&\qquad\qquad + \Big|U_p\left(X_{t,b,s}^{\pi_1,\pi_0,\tau}(T)\right)\Big|^{1+\varepsilon}\mathbb{1}_{\{\tau=\infty\}}\Bigg] \\
&\le 2^{2+\varepsilon}\mathbb{E}\left[\left|U_p\left(X_{t,b,s}^{\pi_1,\pi_0,\tau}(T)\right)\right|^{1+\varepsilon}\right] \\
&= \frac{p(1+\varepsilon)2^{2+\varepsilon}}{|p|^{1+\varepsilon}}\mathbb{E}\left[U_{p(1+\varepsilon)}\left(X_{t,b,s}^{\pi_1,\pi_0,\tau}(T)\right)\right] \\
&\le \frac{p(1+\varepsilon)2^{2+\varepsilon}}{|p|^{1+\varepsilon}}\inf_{\tau\in\mathcal{B}(t)}\mathbb{E}\left[U_{p(1+\varepsilon)}\left(X_{t,b,s}^{\pi_1,\pi_0,\tau}(T)\right)\right].
\end{aligned}$$

We conclude since

$$\inf_{\tau\in\mathcal{B}(t)}\mathbb{E}\left[U_{p(1+\varepsilon)}\left(X_{t,b,s}^{\pi_1,\pi_0,\tau}(T)\right)\right] = \mathcal{W}_1(\pi_1,\pi_0,t,b,s) > -\infty.$$

Step 3: We now show that the uniform integrability of $\tilde{Y}$ implies the uniform integrability of Y. For this, we first show that

$$\tilde{Y}(u) = Y(u-)$$

for every $u \in [t,T]$. Indeed, for $u = t$ the result is immediate. For $u \in (t,T]$ we take a sequence $(u_n)_{n\in\mathbb{N}}$ such that $u_n \uparrow u$. Then

$$\begin{aligned}
Y(u-) = \lim_{n\to\infty} Y(u_n) &= \lim_{n\to\infty}\mathcal{V}_0(u_n, B_{t,b}^{\pi_1,\pi_0,\infty}(u_n), (1-\beta)S_{t,s}^{\pi_1,\pi_0,\infty}(u_n)) \\
&= \lim_{n\to\infty}\mathcal{V}_0(u_n, B_{t,b}^{\pi_1,\pi_0,u}(u_n), (1-\beta)S_{t,s}^{\pi_1,\pi_0,u}(u_n))
\end{aligned}$$

$$
\begin{aligned}
&= \mathcal{V}_0(u, B_{t,b}^{\pi_1,\pi_0,u}(u-), (1-\beta)S_{t,s}^{\pi_1,\pi_0,u}(u-)) \\
&= \tilde{Y}(u).
\end{aligned}
$$

We make the convention that $Y(\tau-) = Y(\tau)$ on $\{\tau = \infty\}$. Then

$$
\mathbb{E}\left[|Y(\infty)|^{1+\varepsilon}\right] = \mathbb{E}\left[|\tilde{Y}(\infty)|^{1+\varepsilon}\right] < +\infty.
$$

and

$$
\begin{aligned}
\mathbb{E}\left[|Y(T)|^{1+\varepsilon}\right] &= \mathbb{E}\left[\left|\mathcal{V}_0(T, B_{t,b}^{\pi_1,\pi_0,\infty}(T-), (1-\beta)S_{t,s}^{\pi_1,\pi_0,\infty}(T-))\right|^{1+\varepsilon}\right] \\
&= \mathbb{E}\left[\left|U_p\left(X_{t,b,s}^{\pi_1,\pi_0,T}(T)\right)\right|^{1+\varepsilon}\right] \\
&\leq \frac{p(1+\varepsilon)}{|p|^{1+\varepsilon}}\mathbb{E}\left[U_{p(1+\varepsilon)}\left(U_p\left(X_{t,b,s}^{\pi_1,\pi_0,T}(T)\right)\right)\right] \\
&< +\infty,
\end{aligned}
$$

where the finiteness follows from the admissibility of π_1 and π_0. Now given $\tau \in \mathcal{B}(t)$ and $n \in \mathbb{N}$ we define

$$
\tau_n := \begin{cases} (\tau + 1/n) \wedge T, & \text{on } \{\tau < T\}, \\ \infty, & \text{otherwise.} \end{cases}
$$

Then $\tau_n \downarrow \tau$ on $\{\tau < T\}$ and $\tilde{Y}(\tau_n) \to Y(\tau)$ on $\{\tau < T\}$ as $n \to \infty$. An application of Fatou's lemma then shows that

$$
\begin{aligned}
&\mathbb{E}\left[|Y(\tau)|^{1+\varepsilon}\right] \\
&= \mathbb{E}\left[|Y(\tau)|^{1+\varepsilon}\mathbb{1}_{\{\tau<T\}} + |Y(T)|^{1+\varepsilon}\mathbb{1}_{\{\tau=T\}} + |Y(\infty)|^{1+\varepsilon}\mathbb{1}_{\{\tau=\infty\}}\right] \\
&\leq \mathbb{E}\left[\lim_{n\to\infty}|\tilde{Y}(\tau_n)|^{1+\varepsilon}\mathbb{1}_{\{\tau<T\}}\right] + \mathbb{E}\left[|Y(T)|^{1+\varepsilon}\right] + \mathbb{E}\left[|Y(\infty)|^{1+\varepsilon}\right] \\
&\leq \liminf_{n\to\infty}\mathbb{E}\left[|\tilde{Y}(\tau_n)|^{1+\varepsilon}\mathbb{1}_{\{\tau<T\}}\right] + \mathbb{E}\left[|Y(T)|^{1+\varepsilon}\right] + \mathbb{E}\left[|Y(\infty)|^{1+\varepsilon}\right]
\end{aligned}
$$

and we conclude since $\tilde{Y}$ is uniformly integrable. □

We refer to Z as the Snell envelope of Y. We note that we are actually not interested in optimally stopping the process Y but want to stop the process $\tilde{Y}$ defined in (3.20) instead. Nevertheless, the existence of Z is sufficient to prove the existence of ε-optimal stopping times. We refer to Proposition C.9 in the appendix for the proof.

Lemma 3.15. *Let $\varepsilon, \delta > 0$ and let θ be a $[t,T]$-valued $\mathbb{F}^t$-stopping time. Then there exists $\tau_\delta^* \in \mathcal{B}(\theta)$ such that*

$$\tilde{Z}(\theta) := \operatorname*{ess\,inf}_{\tau\in\mathcal{B}(\theta)} \mathbb{E}\left[\tilde{Y}(\tau)\middle|\mathcal{F}^t(\theta)\right] \geq \mathbb{E}\left[\tilde{Y}(\tau_\delta^*)\middle|\mathcal{F}^t(\theta)\right] - \varepsilon,$$

on a set $A \subset \Omega$ with $\mathbb{P}[A] \geq 1-\delta$ and $\tau_\delta^* = \infty$ on A^c.

A crucial point which will become very important later is that we can bound $\tilde{Z}(\theta)$ from above by $\mathcal{V}_1(\theta, B(\theta), S(\theta))$ which essentially boils down to the question of whether or not we are allowed to replace the essential infimum in the definition of $\tilde{Z}$ by the infimum.

Lemma 3.16. *We have $\tilde{Z}(\theta) \leq \mathcal{V}_1(\theta, B(\theta), S(\theta))$.*

Proof. We first show that for every $\varepsilon > 0$ there exists $\pi_0^\varepsilon \in \mathcal{A}_0(\pi_1)$ such that

$$\tilde{Z}(\theta) \leq \operatorname*{ess\,inf}_{\tau\in\mathcal{B}(\theta)} \mathbb{E}\left[U_p\left(X_{\theta,B(\theta),S(\theta)}^{\pi_1,\pi_0^\varepsilon,\tau}\right)\middle|\mathcal{F}^t(\theta)\right] + \varepsilon.$$

Indeed, for every $\tau \in \mathcal{B}(t)$ we let $\pi_0^{\tau,\varepsilon} \in \mathcal{A}_0(\tau, B(\tau-), (1-\beta)S(\tau-))$ be an ε-optimal strategy in the crash-free market and write $\pi_0^\varepsilon = (\pi_0^{\tau,\varepsilon})_{\tau\in\mathcal{B}(t)}$. Then

$$\begin{aligned}
\tilde{Z}(\theta) &= \operatorname*{ess\,inf}_{\tau\in\mathcal{B}(\theta)} \mathbb{E}\left[\tilde{Y}(\tau)\middle|\mathcal{F}^t(\theta)\right] \\
&= \operatorname*{ess\,inf}_{\tau\in\mathcal{B}(\theta)} \mathbb{E}\Bigg[\mathcal{V}_0\left(\tau, B_{t,b}^{\pi_1,\pi_0,\tau}(\tau-), (1-\beta)S_{t,s}^{\pi_1,\pi_0,\tau}(\tau-)\right)\mathbb{1}_{\{\tau\leq T\}} \\
&\qquad\qquad + U_p\left(X_{t,b,s}^{\pi_1,\pi_0,\tau}(T)\right)\mathbb{1}_{\{\tau=\infty\}}\Bigg|\mathcal{F}^t(\theta)\Bigg] \\
&\leq \operatorname*{ess\,inf}_{\tau\in\mathcal{B}(\theta)} \mathbb{E}\Bigg[\mathbb{E}\left[U_p\left(X_{\tau,B(\tau-),(1-\beta)S(\tau-)}^{\pi_0^{\tau,\varepsilon}}(T)\right)\right]\mathbb{1}_{\{\tau\leq T\}} \\
&\qquad\qquad + U_p\left(X_{\theta,B(\theta),S(\theta)}^{\pi_1,\pi_0,\tau}(T)\right)\mathbb{1}_{\{\tau=\infty\}}\Bigg|\mathcal{F}^t(\theta)\Bigg] + \varepsilon \\
&= \operatorname*{ess\,inf}_{\tau\in\mathcal{B}(\theta)} \mathbb{E}\Bigg[\mathbb{E}\left[U_p\left(X_{\tau,B(\tau-),(1-\beta)S(\tau-)}^{\pi_0^{\tau,\varepsilon}}(T)\right)\mathbb{1}_{\{\tau\leq T\}}\right] \\
&\qquad\qquad + U_p\left(X_{\theta,B(\theta),S(\theta)}^{\pi_1,\pi_0,\tau}(T)\right)\mathbb{1}_{\{\tau=\infty\}}\Bigg|\mathcal{F}^t(\theta)\Bigg] + \varepsilon \\
&= \operatorname*{ess\,inf}_{\tau\in\mathcal{B}(\theta)} \mathbb{E}\Bigg[U_p\left(X_{\theta,B(\theta),S(\theta)}^{\pi_1,\pi_0^\varepsilon,\tau}(T)\right)\mathbb{1}_{\{\tau\leq T\}}
\end{aligned}$$

$$+ U_p\left(X^{\pi_1,\pi_0^\varepsilon,\tau}_{\theta,B(\theta),S(\theta)}(T)\right)\mathbb{1}_{\{\tau=\infty\}}\middle|\mathcal{F}^t(\theta)\right] + \varepsilon$$

$$= \operatorname*{ess\,inf}_{\tau\in\mathcal{B}(\theta)} \mathbb{E}\left[U_p\left(X^{\pi_1,\pi_0,\tau}_{\theta,B(\theta),S(\theta)}(T)\right)\middle|\mathcal{F}^t(\theta)\right] + \varepsilon.$$

Let us now fix some $\omega \in \Omega$. For every $s \geq 0$ we denote the path stopped at s by $\omega^s_\cdot := \omega(\cdot \wedge s)$. Moreover, we define the shift operator

$$\boldsymbol{T}_s(\omega)(\cdot) = \omega(\cdot \vee s) - \omega(s).$$

We note that this implies that $\omega(\cdot) = \omega^s_\cdot + \boldsymbol{T}_s(\omega)(\cdot)$. We can therefore write

$$\pi_1(\omega) = \pi_1(\omega^s + \boldsymbol{T}_s(\omega)), \qquad \pi_0(\omega) = \pi_0^\varepsilon(\omega^s + \boldsymbol{T}_s(\omega)),$$

and therefore

$$\begin{aligned}\pi_1^\omega(\tilde{\omega}) &:= \pi_1(\omega^s + \boldsymbol{T}_s(\tilde{\omega})) \in \mathcal{A}_1(\theta(\omega), B(\theta(\omega),\omega), S(\theta(\omega),\omega)),\\ \pi_0^{\omega,\varepsilon}(\tilde{\omega}) &:= \pi_0^\varepsilon(\omega^s + \boldsymbol{T}_s(\tilde{\omega})) \in \mathcal{A}_0(\pi_1^\omega).\end{aligned}$$

Then, for almost all $\omega \in \Omega$,

$$\begin{aligned}&\mathbb{E}\left[U_p\left(X^{\pi_1,\pi_0^\varepsilon,\tau}_{\theta,B(\theta),S(\theta)}(T)\right)\middle|\mathcal{F}^t(\theta)\right](\omega)\\ &= \int U_p\left(X^{\pi_1(\omega^s+\boldsymbol{T}_{\theta(\omega)}(\omega)),\pi_0^\varepsilon(\omega^s+\boldsymbol{T}_{\theta(\omega)}(\omega)),\tau(\boldsymbol{T}_{\theta(\omega)}(\omega))}_{\theta(\omega),B(\theta)(\omega),S(\theta)(\omega)}(T)(\boldsymbol{T}_{\theta(\omega)}(\omega))\right) d\mathbb{P}(\boldsymbol{T}_{\theta(\omega)}(\omega))\\ &= \int U_p\left(X^{\pi_1(\omega^s+\boldsymbol{T}_{\theta(\omega)}(\tilde{\omega})),\pi_0^\varepsilon(\omega^s+\boldsymbol{T}_{\theta(\omega)}(\tilde{\omega})),\tau(\boldsymbol{T}_{\theta(\omega)}(\tilde{\omega}))}_{\theta(\omega),B(\theta)(\omega),S(\theta)(\omega)}(T)(\boldsymbol{T}_{\theta(\omega)}(\tilde{\omega}))\right) d\mathbb{P}(\tilde{\omega})\\ &= \mathcal{J}_1(\pi_1^\omega, \pi_0^{\omega,\varepsilon}, \tau, \theta(\omega), B(\theta)(\omega), S(\theta)(\omega)).\end{aligned}$$

Now $\mathcal{J}_1(\pi_1^\omega, \pi_0^{\omega,\varepsilon}, \tau, \theta(\omega), B(\theta)(\omega), S(\theta)(\omega))$ is constant for ω fixed and hence

$$\begin{aligned}\tilde{Z}(\theta)(\omega) &\leq \operatorname*{ess\,inf}_{\tau\in\mathcal{B}(\theta(\omega))} \mathcal{J}_1(\pi_1^\omega, \pi_0^{\omega,\varepsilon}, \tau, \theta(\omega), B(\theta)(\omega), S(\theta)(\omega)) + \varepsilon\\ &= \inf_{\tau\in\mathcal{B}(\theta(\omega))} \mathcal{J}_1(\pi_1^\omega, \pi_0^{\omega,\varepsilon}, \tau, \theta(\omega), B(\theta)(\omega), S(\theta)(\omega)) + \varepsilon\\ &\leq \mathcal{V}_1(\theta(\omega), B(\theta)(\omega), S(\theta)(\omega)) + \varepsilon\end{aligned}$$

for almost all $\omega \in \Omega$ and we conclude since ε was chosen arbitrarily. □

We can now put the pieces together to prove the existence of an ε-optimal crash time.

Lemma 3.17. *Let $\pi_1 \in \mathcal{A}_1(t,b,s)$ be an arbitrary pre-crash strategy and let $\pi_0 \in \mathcal{A}_0(\pi_1)$ be an arbitrary family of post-crash strategies corresponding to π_1. Let $\varepsilon > 0$ and let θ be a $[t,T]$-valued stopping time. Then for every $\bar\tau \in \mathcal{B}(t)$ there exists a crash time $\tau^\varepsilon \in \mathcal{B}(t)$ such that $\tau^\varepsilon \geq \bar\tau \wedge \theta$ and such that*

$$\mathbb{E}\left[U_p\left(X^{\pi_1,\pi_0,\tau^\varepsilon}_{t,b,s}(T)\right)\right] \leq \mathbb{E}\Big[\mathcal{V}_0\big(\bar\tau, B(\bar\tau-), (1-\beta)S(\bar\tau-)\big)\mathbb{1}_{\{\bar\tau\leq\theta\}} + \mathcal{V}_1\big(\theta, B(\theta), S(\theta)\big)\mathbb{1}_{\{\bar\tau>\theta\}}\Big] + \varepsilon.$$

Proof. We define

$$\tau^\varepsilon := \bar\tau \mathbb{1}_{\{\bar\tau\leq\theta\}} + \tau^*_\delta \mathbb{1}_{\{\bar\tau>\theta\}},$$

where τ^*_δ is an $\varepsilon/2$-optimal stopping time from Lemma 3.15 satisfying

$$\mathbb{E}\left[\tilde Y(\tau^*_\delta)\middle|\mathcal{F}^t(\theta)\right] \leq \tilde Z(\theta) + \frac{1}{2}\varepsilon$$

on a set $A \subset \Omega$ with $\mathbb{P}[A] \geq 1-\delta$ and $\tau^*_\delta = \infty$ on A^c. We choose $\delta > 0$ such that

$$\mathbb{P}[A^c] \leq \frac{\varepsilon}{2\mathbb{E}[|\tilde Z(\theta)|^{1+\kappa} + |Y(+\infty)|^{1+\kappa}]}$$

and where $\kappa > 0$ is a constant such that $p(1+\kappa) < 1$.

It is clear that $\tau^\varepsilon \in \mathcal{B}(t)$ and $\tau^\varepsilon \geq \bar\tau \wedge \theta$. Then

$$\begin{aligned}
&\mathbb{E}\left[U_p\left(X^{\pi_1,\pi_0,\tau^\varepsilon}_{t,b,s}\right)\middle|\mathcal{F}^t(\bar\tau\wedge\theta)\right]\\
&= \mathbb{E}\left[U_p\left(X^{\pi_1,\pi_0,\tau^\varepsilon}_{t,b,s}\right)\middle|\mathcal{F}^t(\bar\tau)\right]\mathbb{1}_{\{\bar\tau\leq\theta\}} + \mathbb{E}\left[U_p\left(X^{\pi_1,\pi_0,\tau^\varepsilon}_{t,b,s}\right)\middle|\mathcal{F}^t(\theta)\right]\mathbb{1}_{\{\bar\tau>\theta\}}\\
&= \mathbb{E}\left[U_p\left(X^{\pi_1,\pi_0,\bar\tau}_{\bar\tau,B(\bar\tau-),(1-\beta)S(\bar\tau-)}\right)\middle|\mathcal{F}^t(\bar\tau)\right]\mathbb{1}_{\{\bar\tau\leq\theta\}}\\
&\qquad + \mathbb{E}\left[U_p\left(X^{\pi_1,\pi_0,\tau^*_\delta}_{\tau^*_\delta,B(\tau^*_\delta-),(1-\beta)S(\tau^*_\delta-)}\right)\middle|\mathcal{F}^t(\theta)\right]\mathbb{1}_{\{\bar\tau>\theta\}}\\
&\leq \mathcal{V}_0\big(\bar\tau, B(\bar\tau-), (1-\beta)S(\bar\tau-)\big)\mathbb{1}_{\{\bar\tau\leq\theta\}} + \mathbb{E}\left[\tilde Y(\tau^*_\delta)\middle|\mathcal{F}^t(\theta)\right]\mathbb{1}_{\{\bar\tau>\theta\}}\\
&\leq \mathcal{V}_0\big(\bar\tau, B(\bar\tau-), (1-\beta)S(\bar\tau-)\big)\mathbb{1}_{\{\bar\tau\leq\theta\}} + \big(\tilde Z(\theta) + \frac{1}{2}\varepsilon\big)\mathbb{1}_{\{\bar\tau>\theta\}}\mathbb{1}_A\\
&\qquad + \mathbb{E}\left[\tilde Y(\infty)\middle|\mathcal{F}^t(\theta)\right]\mathbb{1}_{\{\bar\tau>\theta\}}\mathbb{1}_{A^c}\\
&\leq \mathcal{V}_0\big(\bar\tau, B(\bar\tau-), (1-\beta)S(\bar\tau-)\big)\mathbb{1}_{\{\bar\tau\leq\theta\}} + \tilde Z(\theta)\mathbb{1}_{\{\bar\tau>\theta\}} + \frac{1}{2}\varepsilon
\end{aligned}$$

$$+\left(\mathbb{E}\left[\tilde{Y}(\infty)\middle|\mathcal{F}^t(\theta)\right]-\tilde{Z}(\theta)\right)\mathbb{1}_{\{\bar{\tau}>\theta\}}\mathbb{1}_{A^c}.$$

Taking expectations on both sides hence shows that

$$\begin{aligned}\mathbb{E}\left[U_p\left(X_{t,b,s}^{\pi_1,\pi_0,\tau^\varepsilon}\right)\right]\le \mathbb{E}\left[\mathcal{V}_0\big(\bar{\tau},B(\bar{\tau}-),(1-\beta)S(\bar{\tau}-)\big)\mathbb{1}_{\{\bar{\tau}\le\theta\}}\right]&\\ +\mathbb{E}\left[\tilde{Z}(\theta)\mathbb{1}_{\{\bar{\tau}>\theta\}}\right]+\frac{1}{2}\varepsilon&\\ +\mathbb{E}\left[\left(\mathbb{E}\left[\tilde{Y}(\infty)\middle|\mathcal{F}^t(\theta)\right]-\tilde{Z}(\theta)\right)\mathbb{1}_{\{\bar{\tau}>\theta\}}\mathbb{1}_{A^c}\right]&.\end{aligned}$$

We now show that

$$\mathbb{E}\left[\left(\mathbb{E}\left[\tilde{Y}(\infty)\middle|\mathcal{F}^t(\theta)\right]-\tilde{Z}(\theta)\right)\mathbb{1}_{\{\bar{\tau}>\theta\}}\mathbb{1}_{A^c}\right]\le\frac{1}{2}\varepsilon.$$

To this end, let us note that the uniform integrability of Y implies the uniform integrability of $\tilde{Z}$ (see the proof of Proposition C.9 in the appendix). We conclude from Hölder's and Jensen's inequality that

$$\begin{aligned}&\mathbb{E}\left[\left(\mathbb{E}\left[\tilde{Y}(\infty)\middle|\mathcal{F}^t(\theta)\right]-\tilde{Z}(\theta)\right)\mathbb{1}_{\{\bar{\tau}>\theta\}}\mathbb{1}_{A^c}\right]\\ &\le\mathbb{E}\left[\mathbb{E}\left[|\tilde{Y}(\infty)|\middle|\mathcal{F}^t(\theta)\right]\mathbb{1}_{A^c}\right]+\mathbb{E}\left[|\tilde{Z}(\theta)|\mathbb{1}_{A^c}\right]\\ &\le\mathbb{E}\left[|\tilde{Y}(\infty)|^{1+\kappa}\right]\mathbb{P}[A^c]+\mathbb{E}\left[|\tilde{Z}(\theta)|^{1+\kappa}\right]\mathbb{P}[A^c]\\ &\le\frac{1}{2}\varepsilon\end{aligned}$$

by our choice of δ. Putting the pieces together hence shows that

$$\begin{aligned}&\mathbb{E}\left[U_p\left(X_{t,b,s}^{\pi_1,\pi_0,\tau^\varepsilon}\right)\right]\\ &\le\mathbb{E}\left[\mathcal{V}_0\big(\bar{\tau},B(\bar{\tau}-),(1-\beta)S(\bar{\tau}-)\big)\mathbb{1}_{\{\bar{\tau}\le\theta\}}+\tilde{Z}(\theta)\mathbb{1}_{\{\bar{\tau}>\theta\}}\right]+\varepsilon\\ &\le\mathbb{E}\left[\mathcal{V}_0\big(\bar{\tau},B(\bar{\tau}-),(1-\beta)S(\bar{\tau}-)\big)\mathbb{1}_{\{\bar{\tau}\le\theta\}}+\mathcal{V}_1(\theta,B(\theta),S(\theta))\mathbb{1}_{\{\bar{\tau}>\theta\}}\right]+\varepsilon,\end{aligned}$$

where the last inequality is a consequence of Lemma 3.16. □

3.5.2. The dynamic programming principle

With the help of Lemma 3.13 and Lemma 3.17 we can prove the dynamic programming principle.

Theorem 3.18. *Let* $(t, b, s) \in [0, T) \times \mathcal{S}^1$ *and let* $(\theta_{\pi_1})_{\pi_1 \in \mathcal{A}_1(t,b,s)}$ *be a family of* $[t, T]$*-valued stopping times. Then*

$$\mathcal{V}_1(t, b, s) = \sup_{\pi_1 \in \mathcal{A}_1(t,b,s)} \inf_{\tau \in \mathcal{B}(t)} \mathbb{E}\Big[\mathcal{V}_1\big(\theta_{\pi_1}, B(\theta_{\pi_1}), S(\theta_{\pi_1})\big)\mathbb{1}_{\{\theta_{\pi_1} < \tau\}} + \mathcal{V}_0\big(\tau, B(\tau-), (1-\beta)S(\tau-)\big)\mathbb{1}_{\{\theta_{\pi_1} \geq \tau\}}\Big].$$

Proof. 1. Let $\varepsilon > 0$ and fix an arbitrary pre-crash strategy $\pi_1 \in \mathcal{A}_1(t, b, s)$ and a crash time $\tau \in \mathcal{B}(t)$. By Lemma 3.13 we can find $\pi^\varepsilon \in \mathcal{A}_1(t, b, s)$ which coincides with π_1 on $[t, \tau \wedge \theta_{\pi_1})$ and a family of post-crash strategies $\pi_0^\varepsilon \in \mathcal{A}_0(\pi_1^\varepsilon)$ such that

$$\begin{aligned}\mathbb{E}\left[U_p\left(X_{t,b,s}^{\pi_1^\varepsilon, \pi_0^\varepsilon, \tau}(T)\right)\Big|\mathcal{F}^t(\tau \wedge \theta_{\pi_1})\right] + \varepsilon &\geq \mathcal{V}_0\big(\tau, B(\tau-), (1-\beta)S(\tau-)\big)\mathbb{1}_{\{\tau \leq \theta_{\pi_1}\}} \\ &\quad + \mathcal{V}_1\big(\theta_{\pi_1}, B(\theta_{\pi_1}), S(\theta_{\pi_1})\big)\mathbb{1}_{\{\tau > \theta_{\pi_1}\}}.\end{aligned}$$

We therefore have

$$\begin{aligned}\mathcal{V}_1(t, b, s) &\geq \inf_{\tau \in \mathcal{B}(t)} \mathbb{E}\left[U_p\left(X_{t,b,s}^{\pi_1^\varepsilon, \pi_0^\varepsilon, \tau}(T)\right)\right] \\ &= \inf_{\tau \in \mathcal{B}(t)} \mathbb{E}\left[\mathbb{E}\left[U_p\left(X_{t,b,s}^{\pi_1^\varepsilon, \pi_0^\varepsilon, \tau}(T)\right)\Big|\mathcal{F}^t(\tau \wedge \theta_{\pi_1})\right]\right] \\ &\geq \inf_{\tau \in \mathcal{B}(t)} \mathbb{E}\Big[\mathcal{V}_0\big(\tau, B(\tau-), (1-\beta)S(\tau-)\big)\mathbb{1}_{\{\tau \leq \theta_{\pi_1}\}} \\ &\qquad + \mathcal{V}_1\big(\theta_{\pi_1}, B(\theta_{\pi_1}), S(\theta_{\pi_1})\big)\mathbb{1}_{\{\tau > \theta_{\pi_1}\}}\Big] - \varepsilon.\end{aligned}$$

Since we can find corresponding π_1^ε and π_0^ε for every $\pi_1 \in \mathcal{A}_1(t, b, s)$ and θ_{π_1} and since ε was chosen arbitrarily this implies

$$\mathcal{V}_1(t, b, s) \geq \sup_{\pi_1 \in \mathcal{A}_1(t,b,s)} \inf_{\tau \in \mathcal{B}(t)} \mathbb{E}\Big[\mathcal{V}_0\big(\tau, B(\tau-), (1-\beta)S(\tau-)\big)\mathbb{1}_{\{\tau \leq \theta_{\pi_1}\}} + \mathcal{V}_1\big(\theta_{\pi_1}, B(\theta_{\pi_1}), S(\theta_{\pi_1})\big)\mathbb{1}_{\{\tau > \theta_{\pi_1}\}}\Big].$$

2. Let $\pi_1 \in \mathcal{A}_1(t, b, s)$ be an arbitrary pre-crash strategy and let $\pi_0 \in \mathcal{A}_0(\pi_1)$ be an arbitrary family of post-crash strategies corresponding to π_1. Fix an

arbitrary $\bar{\tau} \in \mathcal{B}(t)$. By Lemma 3.17 there exists $\tau^\varepsilon \in \mathcal{B}(t)$ with $\tau^\varepsilon \geq \bar{\tau} \wedge \theta_{\pi_1}$ such that

$$\mathbb{E}\left[U_p\left(X_{t,b,s}^{\pi_1,\pi_0,\tau^\varepsilon}(T)\right)\right] \leq \mathbb{E}\Big[\mathcal{V}_0\big(\bar{\tau}, B(\bar{\tau}-), (1-\beta)S(\bar{\tau}-)\big)\mathbb{1}_{\{\bar{\tau}\leq\theta_{\pi_1}\}} + \mathcal{V}_1\big(\theta_{\pi_1}, B(\theta_{\pi_1}), S(\theta_{\pi_1})\big)\mathbb{1}_{\{\bar{\tau}>\theta_{\pi_1}\}}\Big] + \varepsilon.$$

It follows that

$$\begin{aligned}&\inf_{\tau\in\mathcal{B}(t)} \mathbb{E}\left[U_p\left(X_{t,b,s}^{\pi_1,\pi_0,\tau}(T)\right)\right]\\ &\leq \mathbb{E}\left[U_p\left(X_{t,b,s}^{\pi_1,\pi_0,\tau^\varepsilon}(T)\right)\right]\\ &\leq \mathbb{E}\Big[\mathcal{V}_0\big(\bar{\tau}, B(\bar{\tau}-), (1-\beta)S(\bar{\tau}-)\big)\mathbb{1}_{\{\bar{\tau}\leq\theta_{\pi_1}\}} + \mathcal{V}_1\big(\theta_{\pi_1}, B(\theta_{\pi_1}), S(\theta_{\pi_1})\big)\mathbb{1}_{\{\bar{\tau}>\theta_{\pi_1}\}}\Big] + \varepsilon.\end{aligned}$$

Since $\bar{\tau}$ was chosen arbitrarily this implies that

$$\inf_{\tau\in\mathcal{B}(t)} \mathbb{E}\left[U_p\left(X_{t,b,s}^{\pi_1,\pi_0,\tau}(T)\right)\right] \leq \inf_{\tau\in\mathcal{B}(t)} \mathbb{E}\Big[\mathcal{V}_0\big(\tau, B(\tau-), (1-\beta)S(\tau-)\big)\mathbb{1}_{\{\tau\leq\theta_{\pi_1}\}} + \mathcal{V}_1\big(\theta_{\pi_1}, B(\theta_{\pi_1}), S(\theta_{\pi_1})\big)\mathbb{1}_{\{\tau>\theta_{\pi_1}\}}\Big] + \varepsilon.$$

Using that π_1, π_0 and ε were chosen arbitrarily hence shows that

$$\mathcal{V}_1(t,b,s) \leq \sup_{\pi_1\in\mathcal{A}_1(t,b,s)} \inf_{\tau\in\mathcal{B}(t)} \mathbb{E}\Big[\mathcal{V}_0\big(\tau, B(\tau-), (1-\beta)S(\tau-)\big)\mathbb{1}_{\{\tau\leq\theta_{\pi_1}\}} + \mathcal{V}_1\big(\theta_{\pi_1}, B(\theta_{\pi_1}), S(\theta_{\pi_1})\big)\mathbb{1}_{\{\tau>\theta_{\pi_1}\}}\Big]$$

which is the desired inequality. □

3.6. The viscosity property

In Chapter 2 we saw that the value function $\mathcal{V}_0$ in the crash-free market is the unique viscosity solution of the DPE

$$0 = \min\{\mathcal{L}^{nt}\mathcal{V}_0(t,b,s), \mathcal{L}^{buy}\mathcal{V}_0(t,b,s), \mathcal{L}^{sell}\mathcal{V}_0(t,b,s)\},$$

where the differential operators $\mathcal{L}^{nt}$, $\mathcal{L}^{buy}$ and $\mathcal{L}^{sell}$ are given by

$$\begin{aligned}
\mathcal{L}^{nt} &= -\frac{\partial}{\partial t} - \alpha s\frac{\partial}{\partial s} - \frac{1}{2}\sigma^2 s^2 \frac{\partial^2}{\partial s^2}, \\
\mathcal{L}^{buy} &= (1+\lambda)\frac{\partial}{\partial b} - \frac{\partial}{\partial s}, \\
\mathcal{L}^{sell} &= -(1-\mu)\frac{\partial}{\partial b} + \frac{\partial}{\partial s},
\end{aligned}$$

respectively. The aim of this section is to show that $\mathcal{V}_1$ is the unique viscosity solution of

$$\begin{aligned}
0 = \max\big\{&\mathcal{V}_1(t,b,s) - \mathcal{V}_0(t,b,(1-\beta)s), \\
&\min\{\mathcal{L}^{nt}\mathcal{V}_1(t,b,s), \mathcal{L}^{buy}\mathcal{V}_1(t,b,s), \mathcal{L}^{sell}\mathcal{V}_1(t,b,s)\}\big\}. \qquad (3.21)
\end{aligned}$$

Let us first address the viscosity property of $\mathcal{V}_1$ before we turn to the uniqueness of solutions of (3.21). We summarize the result in the next theorem.

Theorem 3.19. *$\mathcal{V}_1$ is a viscosity solution of the DPE* (3.21) *on* $[0,T) \times \mathcal{S}^1$ *with boundary condition*

$$\mathcal{V}_1(t,b,s) = U_p(0), \qquad \text{if } (b,s) \in \partial\mathcal{S}^1,\ t \in [0,T],$$

and terminal condition

$$\mathcal{V}_1(T,b,s) = \begin{cases} U_p(b+(1-\mu)(1-\beta)s), & \text{if } s > 0, \\ U_p(b+(1+\lambda)s), & \text{if } s \le 0. \end{cases}$$

Remark 3.20. To see that $\mathcal{V}_1$ satisfies the terminal condition note that from the market's point of view a crash at terminal time must be optimal whenever the investor's stock position is positive. ◇

We split the proof in two cases, the supersolution and the subsolution property.

Proposition 3.21. *$\mathcal{V}_1$ is a viscosity supersolution of the DPE* (3.21).

Proof. Let $(t_0, b_0, s_0) \in [0,T) \times \mathcal{S}^1$ and let $\varphi \in C^{1,2,2}([0,T) \times \mathcal{S}^1)$ with $\varphi \leq \mathcal{V}_1$ and $\varphi(t_0, b_0, s_0) = \mathcal{V}_1(t_0, b_0, s_0)$. We want to show that

$$0 \leq \max\big\{\varphi(t_0, b_0, s_0) - \mathcal{V}_0(t_0, b_0, (1-\beta)s_0), \\ \min\{\mathcal{L}^{nt}\varphi(t_0, b_0, s_0), \mathcal{L}^{buy}\varphi(t_0, b_0, s_0), \mathcal{L}^{sell}\varphi(t_0, b_0, s_0)\}\big\}.$$

Step 1: By (3.11) we have

$$\varphi(t_0, b_0, s_0) = \mathcal{V}_1(t_0, b_0, s_0) \leq \mathcal{V}_0\big(t_0, b_0, (1-\beta)s_0\big).$$

If equality holds we are done. We are therefore left with showing that

$$0 \leq \mathcal{L}^{nt}\varphi(t_0, b_0, s_0), \quad 0 \leq \mathcal{L}^{buy}\varphi(t_0, b_0, s_0), \quad 0 \leq \mathcal{L}^{sell}\varphi(t_0, b_0, s_0),$$

under the assumption that

$$\mathcal{V}_1(t_0, b_0, s_0) < \mathcal{V}_0\big(t_0, b_0, (1-\beta)s_0\big). \tag{3.22}$$

Step 2: Let $l_0 > 0$ be small enough such that $(b_0 - (1+\lambda)l_0, s_0 + l_0) \in \mathcal{S}^1$. By the continuity of $\mathcal{V}_1$ and $\mathcal{V}_0$ and making l_0 smaller if necessary we can assume that

$$\mathcal{V}_1(t_0, b_0 - (1+\lambda)l, s_0 + l) < \mathcal{V}_0(t_0, b_0 - (1+\lambda)l, (1-\beta)s_0 + l) \tag{3.23}$$

for all $l \in [0, l_0]$. For any such l we have

$$\mathcal{V}_1(t_0, b_0, s_0) \geq \mathcal{V}_1(t_0, b_0 - (1+\lambda)l, s_0 + l). \tag{3.24}$$

Indeed, by Remark 3.4 together with (3.22) and (3.23) we know that a crash at time t_0 cannot be optimal (neither for initial holdings (b_0, s_0) nor for $(b_0 - (1+\lambda)l, s_0 + l)$) and since $(b_0 - (1+\lambda)l, s_0 + l)$ can be reached by a transaction from (b_0, s_0) the claim follows. Using that $\varphi \leq \mathcal{V}_1$ and $\varphi(t_0, b_0, s_0) = \mathcal{V}_1(t_0, b_0, s_0)$ it follows from (3.24) that

$$0 \leq \mathcal{V}_1(t_0, b_0, s_0) - \mathcal{V}_1(t_0, b_0 - (1+\lambda)l, s_0 + l) \\ \leq \varphi(t_0, b_0, s_0) - \varphi(t_0, b_0 - (1+\lambda)l, s_0 + l).$$

Now divide both sides by l and send $l \downarrow 0$ to obtain

$$0 \le \mathcal{L}^{buy}\varphi(t_0, b_0, s_0)$$

and a similar argument shows that

$$0 \le \mathcal{L}^{sell}\varphi(t_0, b_0, s_0).$$

Step 3: Suppose that on the contrary we have

$$\mathcal{L}^{nt}\varphi(t_0, b_0, s_0) < 0.$$

Then there exists some $\varepsilon > 0$ and $\delta > 0$ such that $t_0 + \varepsilon < T$, $\overline{B}_\varepsilon(b_0, s_0) \subset \mathcal{S}^1$ and

$$\mathcal{L}^{nt}\varphi(t, b, s) < -\delta$$

for all $(t, b, s) \in [t_0, t_0 + \varepsilon] \times \overline{B}_\varepsilon(b_0, s_0)$.

Consider a strategy $\pi_1 = (L_1, M_1) \in \mathcal{A}_1(t_0, b_0, s_0)$ such that $L_1 \equiv M_1 \equiv 0$ on $[t_0, \tau_\varepsilon]$, where τ_ε is given by

$$\tau_\varepsilon := \inf\{u \ge t_0 : (B(u), S(u)) \notin B_\varepsilon(b_0, s_0)\} \wedge (t_0 + \varepsilon).$$

By Theorem 3.18 with $\theta_{\pi_1} = \tau_\varepsilon$ we have

$$\begin{aligned}\mathcal{V}_1(t_0, b_0, s_0) \ge \inf_{\tau \in \mathcal{B}(t_0)} \mathbb{E}\Big[&\mathcal{V}_1(\tau_\varepsilon, B(\tau_\varepsilon), S(\tau_\varepsilon))\mathbb{1}_{\{\tau_\varepsilon < \tau\}} \\ &+ \mathcal{V}_0(\tau, B(\tau-), (1-\beta)S(\tau-))\mathbb{1}_{\{\tau_\varepsilon \ge \tau\}}\Big].\end{aligned}$$

For every $n \in \mathbb{N}$ we can find a stopping time $\tau_n \in \mathcal{B}(t_0)$ such that

$$\begin{aligned}\mathcal{V}_1(t_0, b_0, s_0) \ge \mathbb{E}\Big[&\mathcal{V}_1(\tau_\varepsilon, B(\tau_\varepsilon), S(\tau_\varepsilon))\mathbb{1}_{\{\tau_\varepsilon < \tau_n\}} \\ &+ \mathcal{V}_0(\tau_n, B(\tau_n-), (1-\beta)S(\tau_n-))\mathbb{1}_{\{\tau_\varepsilon \ge \tau_n\}}\Big] - \frac{1}{n}. \quad (3.25)\end{aligned}$$

Using $\mathcal{V}_1(t, b, s) \le \mathcal{V}_0(t, b, (1-\beta)s)$ this implies that

$$\mathcal{V}_1(t_0, b_0, s_0) \ge \mathbb{E}\Big[\mathcal{V}_1\big(\tau_\varepsilon \wedge \tau_n, B(\tau_\varepsilon \wedge (\tau_n-)), S(\tau_\varepsilon \wedge (\tau_n-))\big)\Big] - \frac{1}{n}.$$

Set $\tilde{\tau}_n := \tau_\varepsilon \wedge (\tau_n -)$ and use $\varphi \le \mathcal{V}_1$ to conclude that

$$\begin{aligned}\mathcal{V}_1(t_0, b_0, s_0) &\ge \mathbb{E}\left[\mathcal{V}_1(\tilde{\tau}_n, B(\tilde{\tau}_n), S(\tilde{\tau}_n))\right] - \frac{1}{n} \\ &\ge \mathbb{E}\left[\varphi(\tilde{\tau}_n, B(\tilde{\tau}_n), S(\tilde{\tau}_n))\right] - \frac{1}{n}.\end{aligned}$$

Using $\varphi(t_0, b_0, s_0) = \mathcal{V}_1(t_0, b_0, s_0)$ and Itô's formula now shows that

$$\begin{aligned}\varphi(t_0, b_0, s_0) &= \mathcal{V}_1(t_0, b_0, s_0) \\ &\ge \varphi(t_0, b_0, s_0) - \mathbb{E}\left[\int_{t_0}^{\tilde{\tau}_n} \mathcal{L}^{nt}\varphi(u, B(u), S(u))\, du\right] - \frac{1}{n},\end{aligned}$$

i.e.

$$\mathbb{E}\left[\int_{t_0}^{\tilde{\tau}_n} \mathcal{L}^{nt}\varphi(u, B(u), S(u))\, du\right] \ge -\frac{1}{n}.$$

On the other hand we have

$$\mathcal{L}^{nt}\varphi(t, b, s) < -\delta$$

on $[t_0, t_0 + \varepsilon] \times \overline{B}_\varepsilon(b_0, s_0)$ and hence

$$\mathbb{E}\left[\int_{t_0}^{\tilde{\tau}_n} \mathcal{L}^{nt}\varphi(u, B(u), S(u))\, du\right] \le -\delta\mathbb{E}[\tilde{\tau}_n - t_0].$$

Therefore

$$0 \le \mathbb{E}[\tilde{\tau}_n - t_0] \le \frac{1}{\delta n}.$$

Sending $n \to \infty$ shows that there exists a subsequence of $(\tau_n)_{n\in\mathbb{N}}$ which we again denote by $(\tau_n)_{n\in\mathbb{N}}$ such that $\lim_{n\to\infty} \tilde{\tau}_n = t_0$ and therefore $\lim_{n\to\infty} \tau_n = t_0$.

Since $\mathcal{V}_1$ and $\mathcal{V}_0$ are bounded on $[t_0, t_0 + \varepsilon] \times \overline{B}_\varepsilon(b_0, s_0)$ we can send $n \to \infty$ in (3.25) and use dominated convergence to see that

$$\begin{aligned}\mathcal{V}_1(t_0, b_0, s_0) &\ge \lim_{n\to\infty} \mathbb{E}\Big[\mathcal{V}_1(\tau_\varepsilon, B(\tau_\varepsilon), S(\tau_\varepsilon))\mathbb{1}_{\{\tau_\varepsilon < \tau_n\}} \\ &\qquad\qquad + \mathcal{V}_0(\tau_n, B(\tau_n -), (1-\beta)S(\tau_n -))\mathbb{1}_{\{\tau_\varepsilon \ge \tau_n\}}\Big] - \frac{1}{n} \\ &= \mathbb{E}\Big[\mathcal{V}_1(\tau_\varepsilon, B(\tau_\varepsilon), S(\tau_\varepsilon))\mathbb{1}_{\{\tau_\varepsilon < t_0\}} \\ &\qquad\qquad + \mathcal{V}_0(t_0, b_0, (1-\beta)s_0)\mathbb{1}_{\{\tau_\varepsilon \ge t_0\}}\Big] \\ &= \mathcal{V}_0(t_0, b_0, (1-\beta)s_0)\end{aligned}$$

which contradicts (3.22). □

Proposition 3.22. *$\mathcal{V}_1$ is a viscosity subsolution of* (3.21).

Proof. Let $(t_0, b_0, s_0) \in [0,T) \times \mathcal{S}^1$ and let $\varphi \in C^{1,2,2}([0,T) \times \overline{\mathcal{S}}^1)$ with $\varphi \geq \mathcal{V}_1$ and $\varphi(t_0, b_0, s_0) = \mathcal{V}_1(t_0, b_0, s_0)$. We want to show that

$$0 \geq \max\big\{\varphi(t_0, b_0, s_0) - \mathcal{V}_0(t_0, b_0, (1-\beta)s_0),\\ \min\{\mathcal{L}^{nt}\varphi(t_0, b_0, s_0), \mathcal{L}^{buy}\varphi(t_0, b_0, s_0), \mathcal{L}^{sell}\varphi(t_0, b_0, s_0)\}\big\}.$$

As before we have

$$\varphi(t_0, b_0, s_0) = \mathcal{V}_1(t_0, b_0, s_0) \leq \mathcal{V}_0(t_0, b_0, (1-\beta)s_0),$$

i.e. we only have to show that

$$\min\{\mathcal{L}^{nt}\varphi(t_0, b_0, s_0), \mathcal{L}^{buy}\varphi(t_0, b_0, s_0), \mathcal{L}^{sell}\varphi(t_0, b_0, s_0)\} \leq 0.$$

Assume that on the contrary we have

$$\min\{\mathcal{L}^{nt}\varphi(t_0, b_0, s_0), \mathcal{L}^{buy}\varphi(t_0, b_0, s_0), \mathcal{L}^{sell}\varphi(t_0, b_0, s_0)\} > 0.$$

Define

$$\phi(t, b, s) = \varphi(t, b, s) + |t_0 - t|^2 + |b_0 - b|^2 + |s_0 - s|^4.$$

Then $\phi(t_0, b_0, s_0) = \varphi(t_0, b_0, s_0)$ and the relevant partial derivatives of ϕ and φ at (t_0, b_0, s_0) coincide so that

$$\min\{\mathcal{L}^{nt}\phi(t_0, b_0, s_0), \mathcal{L}^{buy}\phi(t_0, b_0, s_0), \mathcal{L}^{sell}\phi(t_0, b_0, s_0)\} > 0.$$

We can therefore find some $\varepsilon > 0$ with $t_0 + \varepsilon < T$, $\overline{B}_\varepsilon(b_0, s_0) \subset \mathcal{S}^1$ and

$$\mathcal{L}^{nt}\phi(t, b, s) > 0, \quad \mathcal{L}^{buy}\phi(t, b, s) > 0, \quad \mathcal{L}^{sell}\phi(t, b, s) > 0 \tag{3.26}$$

on $[t_0, t_0 + \varepsilon] \times \overline{B}_\varepsilon(b_0, s_0)$. Moreover, there exists some $\delta > 0$ such that

$$\phi(t, b, s) \geq \varphi(t, b, s) + \delta \tag{3.27}$$

for every $(t, b, s) \notin [t_0, t_0 + \varepsilon) \times B_\varepsilon(b_0, s_0)$ with $t \geq t_0$.

Let $\pi_1 = (L_1, M_1) \in \mathcal{A}_1(t_0, b_0, s_0)$ and $\pi_0 \in \mathcal{A}_0(\pi_1)$. Choose $\tau^* \in \mathcal{B}(t)$ such that $\tau^* \equiv +\infty$. Now define the stopping time

$$\theta_{\pi_1} := \inf\{u \geq t_0 : (B(u), S(u)) \notin B_\varepsilon(b_0, s_0)\} \wedge (t_0 + \varepsilon).$$

Itô's formula shows that

$$\begin{aligned}\phi(t_0,b_0,s_0) = \phi(\theta_{\pi_1},B(\theta_{\pi_1}),S(\theta_{\pi_1})) &+ \int_{t_0}^{\theta_{\pi_1}} \mathcal{L}^{nt}\phi(u,B(u),S(u))\,du \\ &+ \int_{t_0}^{\theta_{\pi_1}} \mathcal{L}^{buy}\phi(u,B(u),S(u))\,dL_1^c(u) \\ &+ \int_{t_0}^{\theta_{\pi_1}} \mathcal{L}^{sell}\phi(u,B(u),S(u))\,dM_1^c(u) \\ &- \int_{t_0}^{\theta_{\pi_1}} \sigma S(u)\frac{\partial}{\partial s}\phi(u,B(u),S(u))\,dW(u) \\ &- \sum_{t_0\le u\le\theta_{\pi_1}} \Big[\phi\big(u,B(u),S(u)\big) - \phi\big(u-,B(u-),S(u-)\big)\Big].\end{aligned}$$

By (3.26) it follows that ϕ is non-increasing in the direction of transactions and hence again by (3.26) and by taking expectations on both sides we see that

$$\phi(t_0,b_0,s_0) \ge \mathbb{E}\left[\phi(\theta_{\pi_1},B(\theta_{\pi_1}),S(\theta_{\pi_1}))\right]. \tag{3.28}$$

Note that $(\theta_{\pi_1},B(\theta_{\pi_1}),S(\theta_{\pi_1})) \notin [t_0,t_0+\varepsilon)\times B_\varepsilon(b_0,s_0)$ and $\theta_{\pi_1}\ge t_0$. Therefore, by (3.27) and since $\varphi \ge \mathcal{V}_1$ we have

$$\begin{aligned}\mathbb{E}\left[\phi(\theta_{\pi_1},B(\theta_{\pi_1}),S(\theta_{\pi_1}))\right] &\ge \mathbb{E}\left[\varphi(\theta_{\pi_1},B(\theta_{\pi_1}),S(\theta_{\pi_1}))\right]+\delta \\ &\ge \mathbb{E}\left[\mathcal{V}_1(\theta_{\pi_1},B(\theta_{\pi_1}),S(\theta_{\pi_1}))\right]+\delta. \end{aligned}\tag{3.29}$$

We furthermore have

$$\phi(t_0,b_0,s_0) = \varphi(t_0,b_0,s_0) = \mathcal{V}_1(t_0,b_0,s_0)$$

and hence combining (3.28) and (3.29) yields

$$\mathcal{V}_1(t_0,b_0,s_0) \ge \mathbb{E}\left[\mathcal{V}_1(\theta_{\pi_1},B(\theta_{\pi_1}),S(\theta_{\pi_1}))\right]+\delta.$$

Since π_1 was chosen arbitrarily this implies that

$$\mathcal{V}_1(t_0,b_0,s_0) \ge \sup_{\pi_1\in\mathcal{A}_1(t_0,b_0,s_0)} \mathbb{E}\left[\mathcal{V}_1(\theta_{\pi_1},B(\theta_{\pi_1}),S(\theta_{\pi_1}))\right]+\delta. \tag{3.30}$$

On the other hand Theorem 3.18 shows that

$$\mathcal{V}_1(t_0,b_0,s_0) = \sup_{\pi_1\in\mathcal{A}_1(t_0,b_0,s_0)}\ \inf_{\tau\in\mathcal{B}(t_0)} \mathbb{E}\Big[\mathcal{V}_1(\theta_{\pi_1},B(\theta_{\pi_1}),S(\theta_{\pi_1}))\mathbb{1}_{\{\theta_{\pi_1}<\tau\}}$$

$$
\begin{aligned}
&\qquad + \mathcal{V}_0(\tau, B(\tau-), (1-\beta)S(\tau-))\mathbb{1}_{\{\theta_{\pi_1} \geq \tau\}}\Big] \\
&\leq \sup_{\pi_1 \in \mathcal{A}_1(t_0,b_0,s_0)} \mathbb{E}\Big[\mathcal{V}_1(\theta_{\pi_1}, B(\theta_{\pi_1}), S(\theta_{\pi_1}))\mathbb{1}_{\{\theta_{\pi_1} < \tau^*\}} \\
&\qquad + \mathcal{V}_0(\tau^*, B(\tau^*-), (1-\beta)S(\tau^*-))\mathbb{1}_{\{\theta_{\pi_1} \geq \tau^*\}}\Big] \\
&= \sup_{\pi_1 \in \mathcal{A}_1(t_0,b_0,s_0)} \mathbb{E}\left[\mathcal{V}_1(\theta_{\pi_1}, B(\theta_{\pi_1}), S(\theta_{\pi_1}))\right]
\end{aligned}
$$

which contradicts (3.30). □

Clearly, combining Proposition 3.21 and Proposition 3.22 proves Theorem 3.19. Moreover, simple adaptations of the proofs of the two propositions can be used to show that $\mathcal{V}_0$ is a viscosity solution of the DPE (2.10) and hence proves the second claim of Proposition 2.4.

Corollary 3.23. *The value function $\mathcal{V}_0$ is a viscosity solution of the DPE* (2.10).

We now turn our focus to the uniqueness of viscosity solutions of (3.21). Since the DPE (3.21) in the crash setting and the DPE (2.10) in the crash-free setting are very similar it is not surprising that we can recycle the comparison principle for the crash-free DPE.

Theorem 3.24. *Let $u, v : [0,T]\times\overline{\mathcal{S}}^1 \to \mathbb{R}$ and fix $\varepsilon > 0$. Assume that u is an upper semi-continuous viscosity subsolution of* (3.21) *and v is a lower semi-continuous viscosity supersolution of* (3.21) *such that*

$$U_p(b + \min\{(1-\mu)(1-\beta)s, (1+\lambda)s\}) \leq u(t,b,s), v(t,b,s) \leq \varphi_{\gamma,p,K}(t,b,s)$$

for some $\gamma \in (1-\mu, 1+\lambda)$ and $K > 1$. If $u(T,b,s) \leq v(T, b+\varepsilon, s)$ and $u(t,b,s) \leq U_p(0)$ for every $(b,s) \in \partial\mathcal{S}^1$, then $u(t,b,s) \leq v(t, b+\varepsilon, s)$ on $[0,T] \times \overline{\mathcal{S}}^1$.

Proof. Since u is a viscosity subsolution it follows that u is also a subsolution of the crash-free DPE (2.10) and satisfies

$$u(t,b,s) \leq \mathcal{V}_0(t,b,(1-\beta)s), \qquad \text{for all } (t,b,s) \in [0,T) \times \mathcal{S}^1.$$

Since v is a viscosity supersolution we need to distinguish two cases. Given $(t,b,s) \in [0,T) \times \mathcal{S}^1$ we either have

$$v(t, b+\varepsilon, s) \geq \mathcal{V}_0(t, b+\varepsilon, (1-\beta)s)$$

and we are done, or v is a viscosity supersolution of the crash-free DPE and we can conclude as in the proof of Theorem 2.5. □

Remark 3.25. Note that the lower bound on u and v in Theorem 3.24 has the additional factor $(1-\beta)$ as compared to Theorem 2.5. Nevertheless we can still apply the same technique as in the proof of Theorem 2.5 since we only need the lower bound function to show that $-v(t, b+\varepsilon, s)$ is bounded away from infinity near $\partial\mathcal{S}^1$. ◇

Theorem 3.24 implies that the DPE (3.21) characterizes the value function $\mathcal{V}_1$ uniquely.

Corollary 3.26. *Let u, v be upper semi-continuous viscosity solutions of* (3.21) *satisfying*

$$U_p(b+\min\{(1-\mu)(1-\beta)s, (1+\lambda)s\}) \leq u(t,b,s), v(t,b,s) \leq \varphi_{\gamma,p,K}(t,b,s)$$

with $u(t,b,s) = v(t,b,s) = U_p(0)$ on $\partial\mathcal{S}^1$ and

$$u^*(T,b,s) = u_*(T,b,s) = \mathcal{V}_1(T,b,s) = v^*(T,b,s) = v_*(T,b,s).$$

Then $u = v$.

Proof. The proof is the same as the proof of Corollary 2.6. □

3.7. Numerical results

The objective of this section is to study the DPE (3.21) numerically to determine the optimal trading regions and compare them to the optimal trading regions in the crash-free case. We fix the same parameters as in Chapter 2, i.e.

$$\alpha := 0.096, \qquad \sigma := 0.4, \qquad T := 10$$
$$p := 0.1, \qquad \mu := 0.01, \qquad \lambda := 0.01.$$

Moreover, we assume the maximum crash size to be given by $\beta = 0.5$.

3.7.1. The candidate optimal strategies

As in Chapter 2 we can use the homogeneity of $\mathcal{V}_1$ to reduce the dimension of the problem. We define

$$\mathcal{V}_1(t,b,s) =: (b+s)^p \bar{\mathcal{V}}_1(t, s/(b+s)), \qquad \bar{\mathcal{V}}_1(t,\pi) := \mathcal{V}_1(t, 1-\pi, \pi).$$

It then follows that $\bar{\mathcal{V}}_1$ is the unique viscosity solution of

$$\begin{aligned} 0 = \max\Big\{ &\bar{\mathcal{V}}_1(t,\pi) - (1-\beta\pi)^p \bar{\mathcal{V}}_0\left(t, \frac{(1-\beta)\pi}{1-\beta\pi}\right), \\ &\min\big\{ \bar{\mathcal{L}}^{nt}\bar{\mathcal{V}}_1(t,\pi), \bar{\mathcal{L}}^{buy}\bar{\mathcal{V}}_1(t,\pi), \bar{\mathcal{L}}^{sell}\bar{\mathcal{V}}_1(t,\pi) \big\} \Big\} \end{aligned} \tag{3.31}$$

on $[0,T) \times \mathcal{S}^1_\pi$ with $\mathcal{S}^1_\pi := (-1/\lambda, 1/[1-(1-\beta)(1-\mu)])$. The terminal condition is

$$\bar{\mathcal{V}}_1(T,\pi) = \mathcal{V}_1(T, 1-\pi, \pi) = \begin{cases} \frac{1}{p}(1-\pi+(1-\mu)(1-\beta)\pi)^p, & \text{if } \pi > 0, \\ \frac{1}{p}(1+\lambda\pi)^p, & \text{if } \pi \le 0. \end{cases}$$

The differential operators appearing in (3.31) are the same operators as in the crash-free case, see (2.44)-(2.46). In what follows we assume that $\mathcal{V}_1$ is regular enough to be a classical solution of the DPE.

As a next step, let us take a look at the optimal strategies in the presence of crashes. Since compared to the crash-free case we have an additional operator in the DPE (3.31) we have to adjust the definition of the three trading regions slightly. We define the crash region in the reduced variables to be

$$\bar{\mathcal{R}}_1^{crash} := \left\{ (t,\pi) \in [0,T) \times \mathcal{S}^1_\pi : \bar{\mathcal{V}}_1(t,\pi) = (1-\beta\pi)^p \bar{\mathcal{V}}_0\left(t, \frac{(1-\beta)\pi}{1-\beta\pi}\right) \right\}.$$

Note that $\bar{\mathcal{R}}_1^{crash}$ is a closed set. The trading regions are defined to be

$$\begin{aligned} \bar{\mathcal{R}}_1^{nt} &:= \left\{ (t,\pi) \in \left([0,T) \times \mathcal{S}^1_\pi\right) \setminus \bar{\mathcal{R}}_1^{crash} : \bar{\mathcal{L}}^{nt}\bar{\mathcal{V}}_1(t,\pi) = 0 \right\}, \\ \bar{\mathcal{R}}_1^{buy} &:= \left\{ (t,\pi) \in \left([0,T) \times \mathcal{S}^1_\pi\right) \setminus \bar{\mathcal{R}}_1^{crash} : \bar{\mathcal{L}}^{buy}\bar{\mathcal{V}}_1(t,\pi) = 0 \right\}, \\ \bar{\mathcal{R}}_1^{sell} &:= \left\{ (t,\pi) \in \left([0,T) \times \mathcal{S}^1_\pi\right) \setminus \bar{\mathcal{R}}_1^{crash} : \bar{\mathcal{L}}^{sell}\bar{\mathcal{V}}_1(t,\pi) = 0 \right\}. \end{aligned}$$

We conjecture that these sets form a partition of $([0,T) \times \mathcal{S}^1_\pi) \setminus \bar{\mathcal{R}}_1^{crash}$ and that the optimal trading strategy on these sets is determined in the same way as in the crash-free case, i.e. buying is optimal in $\bar{\mathcal{R}}_1^{buy}$, selling is optimal in $\bar{\mathcal{R}}_1^{sell}$ and no-trading is optimal in $\bar{\mathcal{R}}_1^{nt}$. We furthermore conjecture that selling is optimal within the crash region $\bar{\mathcal{R}}_1^{crash}$. To see this we show that if $\mathcal{V}_1(t,b,s) = \mathcal{V}_0(t,b,(1-\beta)s)$ and (t,b,s) is not on the boundary of the crash region then

$$\mathcal{L}^{sell}\mathcal{V}_1(t,b,s) \leq 0$$

whenever

$$\beta \geq \frac{\mu+\lambda}{1+\lambda}. \tag{3.32}$$

Indeed, set $\tilde{s} = (1-\beta)s$ and calculate

$$\begin{aligned}\mathcal{L}^{sell}\mathcal{V}_1(t,b,s) &= \mathcal{L}^{sell}\mathcal{V}_0(t,b,\tilde{s})\\ &= -(1-\mu)\frac{\partial}{\partial b}\mathcal{V}_0(t,b,\tilde{s}) + \frac{\partial}{\partial s}\mathcal{V}_0(t,b,\tilde{s})\\ &= -(1-\mu)\frac{\partial}{\partial b}\mathcal{V}_0(t,b,\tilde{s}) + (1-\beta)\frac{\partial}{\partial \tilde{s}}\mathcal{V}_0(t,b,\tilde{s}).\end{aligned}$$

Now add and subtract $(1-\beta)(1+\lambda)(\partial/\partial b)\mathcal{V}_0(t,b,\tilde{s})$ and rearrange to obtain

$$\begin{aligned}\mathcal{L}^{sell}\mathcal{V}_1(t,b,s) = -(1-\beta)&\left[(1+\lambda)\frac{\partial}{\partial b}\mathcal{V}_0(t,b,\tilde{s}) - \frac{\partial}{\partial \tilde{s}}\mathcal{V}_0(t,b,\tilde{s})\right]\\ &+ \Big[(1-\beta)(1+\lambda) - (1-\mu)\Big]\frac{\partial}{\partial b}\mathcal{V}_0(t,b,\tilde{s}).\end{aligned}$$

Since

$$(1+\lambda)\frac{\partial}{\partial b}\mathcal{V}_0(t,b,\tilde{s}) - \frac{\partial}{\partial \tilde{s}}\mathcal{V}_0(t,b,\tilde{s}) = \mathcal{L}^{buy}\mathcal{V}_0(t,b,\tilde{s}) \geq 0$$

and $(\partial/\partial b)\mathcal{V}_0 \geq 0$ we see that

$$\mathcal{L}^{sell}\mathcal{V}_1(t,b,s) \leq 0$$

if $(1-\beta)(1+\lambda) \leq (1-\mu)$ which is equivalent to (3.32).

We therefore have $\bar{\mathcal{L}}^{sell}\bar{\mathcal{V}}_1 \leq 0$ in the interior of $\bar{\mathcal{R}}_1^{crash}$ and $\bar{\mathcal{L}}^{sell}\bar{\mathcal{V}}_1 \geq 0$ outside of $\bar{\mathcal{R}}_1^{crash}$. Since $\bar{\mathcal{V}}_1$ is assumed to be of class $C^{1,2}$ this implies $\bar{\mathcal{L}}^{sell}\bar{\mathcal{V}}_1 = 0$ on the boundary of $\bar{\mathcal{R}}_1^{crash}$. Hence, we expect that the investor should sell whenever she reaches the crash region.

3.7.2. A numerical example

Let us now consider a numerical example. First, we need to adjust the algorithm in Kunisch and Sass [72] to work with the DPE (3.31) in the crash-threatened case. This is done as follows. Assume that we want to approximate the value function at time $t < T$. For the k-th iteration we first solve (as in the crash-free case)

$$0 = \bar{\mathcal{L}}^{nt}\bar{v}_1^k(t,\pi)$$

inside our guess for the no-trading region $[a_{k-1}, b_{k-1}] \subset [0,1]$ and extend $\bar{v}_1^k$ to $[0,1]$ using the explicit solutions of $\bar{\mathcal{L}}^{buy}\bar{v}_1^k(t,\pi) = 0$ and $\bar{\mathcal{L}}^{sell}\bar{v}_1^k(t,\pi) = 0$. Also, we construct the active sets B_k and S_k as before (see (2.47) and (2.48)). Then we check if the crash constraint is satisfied for all $\pi \in [0,1]$. For this, define

$$C_k := \left\{ \pi \in [0,1] : \bar{v}_1^k(t,\pi) - (1-\beta\pi)^p \bar{\mathcal{V}}_0\left(t, \frac{(1-\beta)\pi}{1-\beta\pi}\right) \geq 0 \right\}.$$

On C_k we set

$$\bar{v}_1^k(t,\pi) = (1-\beta\pi)^p \bar{\mathcal{V}}_0\left(t, \frac{(1-\beta)\pi}{1-\beta\pi}\right).$$

Now set $N_k = [0,1] \setminus (B_k \cup S_k \cup C_k)$ and proceed with the next iteration.

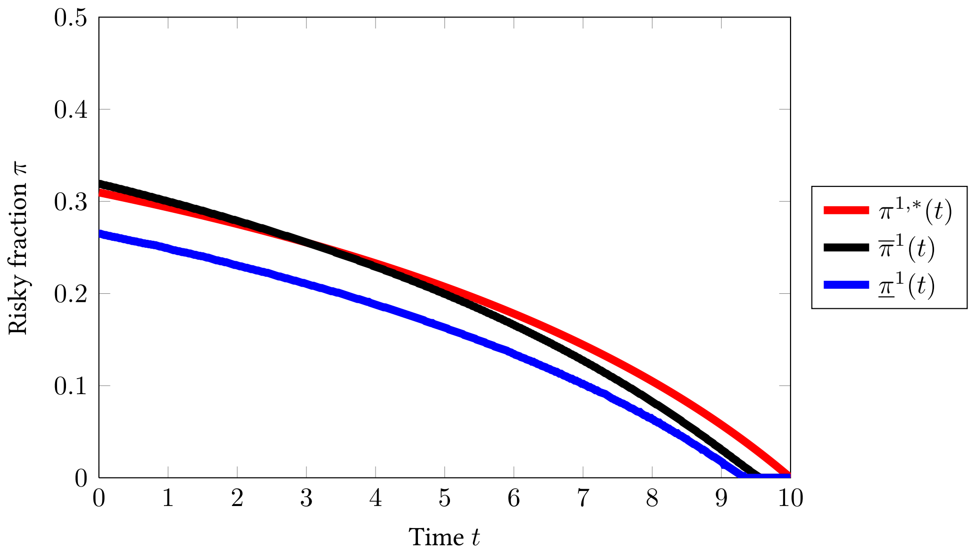

Figure 3.4. Optimal trading regions in the presence of crashes.

The resulting buy boundary $\underline{\pi}^1(t)$ and sell boundary $\overline{\pi}^1(t)$ are illustrated in Figure 3.4. For comparison the figure also depicts the optimal strategy $\pi^{1,*}(t)$ in the case without costs, see (1.8) in Section 1.2. We observe a striking feature of the sell boundary: If the time to maturity becomes small the sell boundary crosses the optimal strategy without costs. Even more, the sell boundary hits zero strictly before terminal time T. In the absence of crashes one can also observe that the sell boundary falls below the optimal strategy without costs (see e.g. Shreve and Soner [97, Equation (11.4)] for the infinite-horizon model and Liu and Loewenstein [74, Equation (22)] for the finite-horizon case). However, in these models this behavior can only be observed in special cases, e.g. if the Merton fraction is sufficiently high (in particular $\pi_M > 1$, i.e. borrowing is optimal). See also Gerhold et al. [41] for a discussion of this effect. In our model this behavior can be observed as soon as $\beta > 0$, i.e. as soon as we allow for crashes.

Remark 3.27. Let us consider a (fairly heuristic) example to explain why the sell boundary reaches zero strictly before terminal time. Intuitively, we expect that the sell boundary reaches zero in the crash case at least as soon as the buy boundary in crash-free case reaches zero, that is $t_0 \leq t^{down}$ (where t^{down} is defined in (2.26)). This is because the investor cannot recoup any losses made in the stock by buying more shares after the crash. Indeed, assume that $t_0 \in [0, T]$ is such that $t_0 \geq t^{down}$. Assume moreover that at time t_0 the investor has a positive stock position $s > 0$ and a positive bond position $b > 0$ sufficiently large such that $s/(b + s)$ is close to 0. Assume now that we are in the crash-threatened case and assume that at time t_0 a crash of size β occurs, leaving the investor with $(1 - \beta)s$ units of money invested in the stock. After the crash we are in the crash-free setting and since $t_0 \geq t^{down}$ we see that buying is no longer optimal. Heuristically, the wealth invested in the stock follows approximately a geometric Brownian motion starting in $(1-\beta)s$ (since $s/(b+s)$ is assumed to be close to 0). The expected terminal utility of wealth is then approximately

$$U^* := \mathbb{E}\left[U_p\left(b + (1-\mu)(1-\beta)se^{(\alpha-\sigma^2/2)(T-t_0)+\sigma(W(T)-W(t_0))}\right)\right].$$

Using Jensen's inequality and the definition of t^{down} we can estimate

$$\begin{aligned}
U^* &\leq U_p\left(\mathbb{E}\left[b + (1-\mu)(1-\beta)se^{(\alpha-\sigma^2/2)(T-t_0)+\sigma(W(T)-W(t_0))}\right]\right) \\
&= U_p\left(b + (1-\mu)(1-\beta)se^{\alpha(T-t_0)}\right) \\
&\leq U_p\left(b + (1-\mu)(1-\beta)se^{\alpha(T-t^{down})}\right) \\
&= U_p(b + (1+\lambda)(1-\beta)s).
\end{aligned}$$

On the other hand, the expected terminal utility in the case of immediate liquidation is

$$U_p\left(b+(1-\mu)s\right)$$

which is greater or equal than $U_p(b+(1+\lambda)(1-\beta)s)$ if and only if (3.32) holds. Hence, for small investment periods it is not optimal to invest any money in the stock at all! ◇

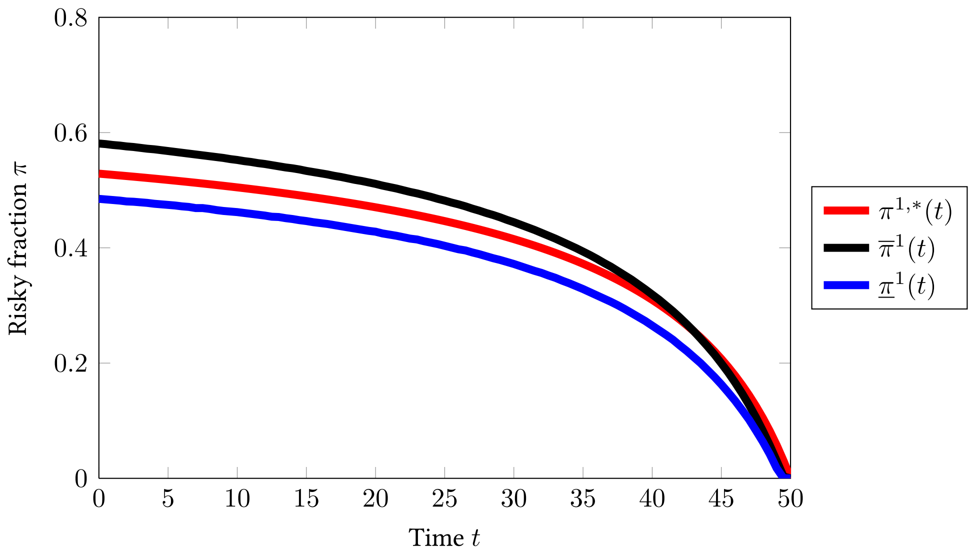

Figure 3.5. Long term behavior of the optimal trading regions.

Let us now turn to the long term behavior of the trading boundaries. Figure 3.5 shows the optimal trading regions for a time horizon of $T = 50$. In the case without crashes (see Figure 2.8) the difference between the sell and the buy boundary stabilizes quickly as $T - t$ becomes large (see also Gerhold et al. [41] for a rigorous justification). In the presence of crashes this effect can no longer be observed: The presence of a crash threat has a significant influence on the optimal trading strategy even if the investment horizon is far into the future.

Figure 3.6 shows this effect in more detail. It shows the difference between the sell and the buy boundary over a time horizon of $T = 100$ for both the case with and without crashes. Without crashes, the difference appears to be stable for maturities greater than approximately two to three years, meaning that the time-influence of the transaction costs on the optimal strategies is only significant

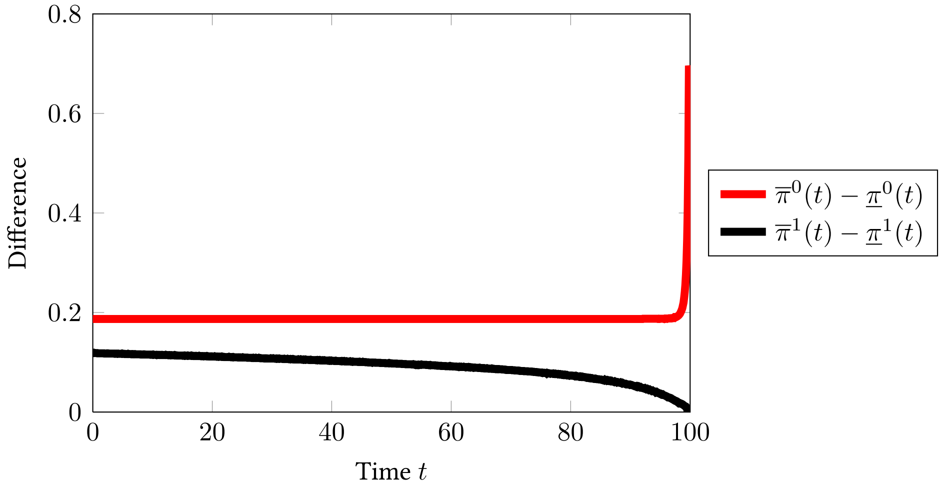

Figure 3.6. Difference of sell and buy boundaries with and without crashes.

for small investment periods. On the other hand, in the presence of crashes the difference is increasing with increasing time to maturity even for large time horizons, indicating that the sensitivity of the optimal strategies with respect to time is significantly higher in the presence of crashes. Note however that the difference is always smaller for the case with a potential crash than for the case with no crash threat.

Let us now turn our focus to the value function $\bar{\mathcal{V}}_1$ which is shown in Figure 3.7 for an investment horizon of $T = 10$. It is interesting to see that $\bar{\mathcal{V}}_1$ is decreasing at a much faster rate than $\bar{\mathcal{V}}_0$ for risky fractions above the sell boundary (compare with Figure 2.9). That is, the kink at the sell boundary is more pronounced in the crash-threatened case. The reason for this lies in the crash constraint.

We would also like to point out that the sell region $\bar{\mathcal{R}}_1^{sell}$ is empty in our example and that the crash region $\bar{\mathcal{R}}_1^{crash}$ coincides with the whole region above the sell boundary. That is, the investor sells shares of the stock because she is afraid of a crash and not because of the risk caused by the fluctuations coming from the stock dynamics in normal times.

Another implication of the above considerations concerns the characterization of the optimal crash time τ^* which is conjectured to be the first hitting time of $\bar{\mathcal{R}}_1^{crash}$, see Section 3.2. The numerical example suggests that τ^* is the first hitting

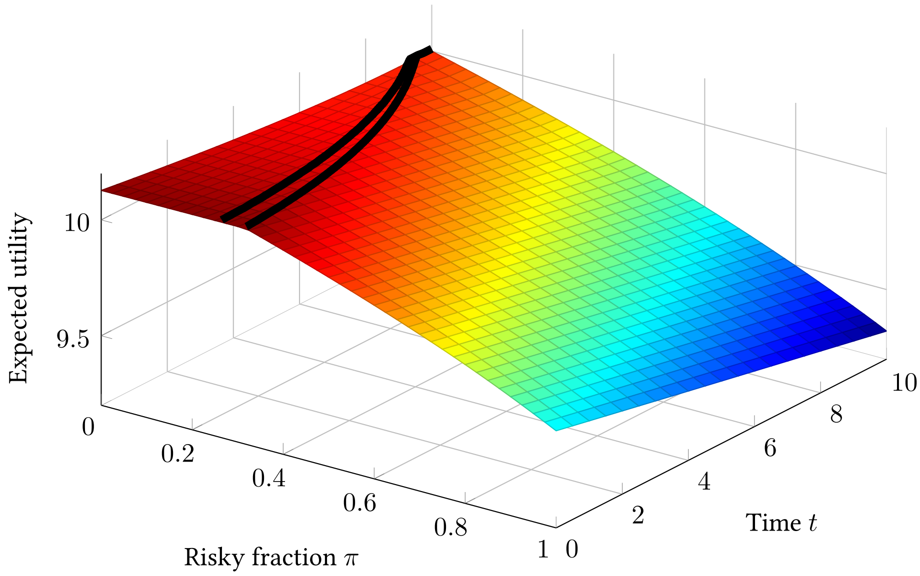

Figure 3.7. The value function in the presence of a crash.

time of the sell boundary. Moreover, the investor benefits from a crash if her risky fraction is below the sell boundary since then $\mathcal{V}_1(t,b,s) < \mathcal{V}_0(t,b,(1-\beta)s)$, incurs losses at the moment a crash occurs if her risky fraction is above the sell boundary since $\mathcal{V}_1(t,b,s) = \mathcal{V}_0(t,b,(1-\beta)s)$ and $\mathcal{L}^{sell}\mathcal{V}_1(t,b,s) < 0$ and is indifferent about the occurrence of a crash on the sell boundary since $\mathcal{V}_1(t,b,s) = \mathcal{V}_0(t,b,(1-\beta)s)$ and $\mathcal{L}^{sell}\mathcal{V}_1(t,b,s) = 0$.

Clearly, if no crash occurs an investor following the optimal strategy in the presence of crashes achieves less expected utility compared to the investor who follows the optimal strategy for zero crashes. To estimate the trade-off we plot in Figure 3.8 the relative loss of utility given by

$$\frac{\bar{\mathcal{V}}_0(t,\pi) - \bar{\mathcal{V}}_1(t,\pi)}{\bar{\mathcal{V}}_0(t,\pi)}, \qquad (t,\pi) \in [0,T] \times [0,1].$$

One can see that the relative loss of utility for protection against a 50% crash is at most 7.5% and as long as the initial risky fraction is small (meaning that at time $t = 0$ it is inside the no-trading region) the relative loss is at most around 3% even for investment periods of 50 years.

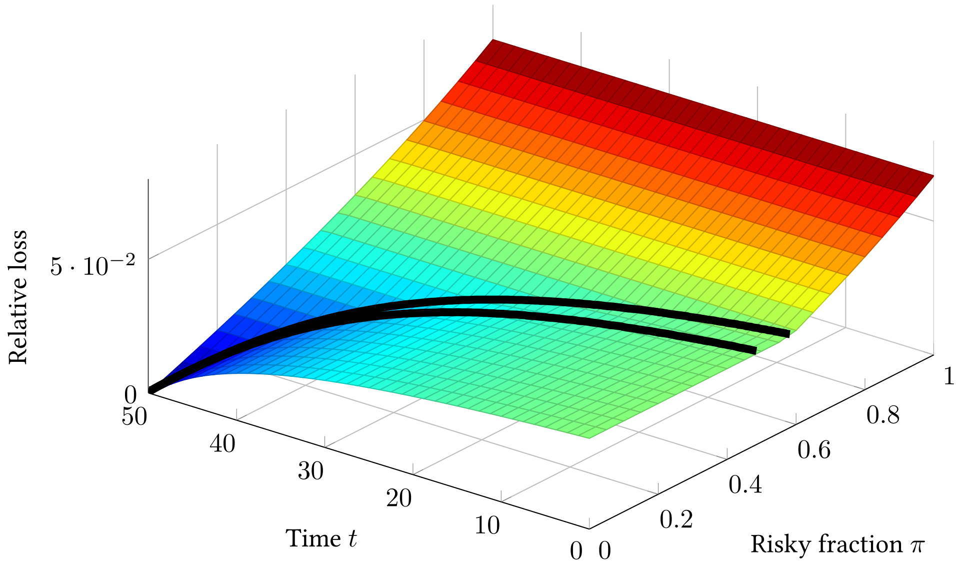

Figure 3.8. Relative loss of utility.

3.8. Extension to multiple crashes

We conclude this chapter with a short outline on how to extend our results to the case of an arbitrary but fixed maximal number of crashes. The generalizations of the previous results are mostly straightforward and can be obtained by iterating through the number of crashes, but one needs to be careful to setup the model correctly.

Let us first explain how we define admissible crash strategies. A crash is modeled as a pair $(\tau, \beta(\tau))$ consisting of an $\mathbb{F}^t$-stopping time τ and an $\mathcal{F}^t(\tau-)$-measurable crash size $\beta(\tau)$. The crash size $\beta(\tau)$ is assumed to be of the form

$$\beta(\tau) = \beta_j := 1 - (1 - \beta)^j,$$

where $\beta \in (0, 1)$ is fixed and where $j \in \mathbb{N}$. Note that with this, whenever a crash occurs we have

$$P^1(\tau) = (1 - \beta_j)P^1(\tau-) = (1 - \beta)^j P^1(\tau-). \tag{3.33}$$

We can therefore interpret the crash (τ, β_j) as j crashes of size β occurring at the same time τ but worked off one after another. Moreover the investor cannot react to the crashes in between. In this spirit we call a crash of size β_j a crash of order j.

Remark 3.28. The form of the crash size $\beta(\tau) = \beta_j$ is a technical tool. In the classical worst-case models multiple crashes at the same time are typically not allowed (see e.g. Korn and Steffensen [70]). In our setting however it is more convenient to allow for more than one crash to occur at the same time. The reason is the following: If multiple crashes at the same time were not allowed then one would need to be very careful about distinguishing whether or not at some given time t a crash can still occur or not. This would lead to two different value functions – one for the case where a crash can occur at initial time and one for the case where a crash cannot occur at initial time. This would for example cause additional difficulties in the formulation of the dynamic programming principle. It is therefore easier to allow for multiple crashes to occur at once, but we have to assume that they are of the form (3.33).

A different modeling approach for multiple crashes would be to assume that if i crashes occur at the same time, then the investor loses a fraction of $(1-i\beta)$ of the wealth invested in the stock. However, this causes inconsistencies when determining the optimal strategies. More precisely, if the investor expects two crashes to occur at the same time then she would choose a different optimal strategy than if she would expect only one crash to happen. This effect is demonstrated in Korn et al. [68, Section 8].

Since the buy and sell boundaries can be expected to be decreasing in the number of crashes we expect that multiple crashes at the same time are only optimal if the investor holds a very risky initial position. Once the position is inside the no trading region multiple crashes are never optimal. ◇

We restrict the maximal number of crashes to be less or equal to $n \in \mathbb{N}$. To be more precise, a crash strategy is a sequence of crashes $\varpi := (\tau_i, \beta_{j_i})_{i=1,\dots,n_0}$, $n_0 \leq n$, such that

$$\sum_{i=1}^{n_0} j_i \leq n.$$

That is, the sum of the orders of the crash sizes is assumed to be bounded by n. In view of Remark 3.28 this setup allows the market to exercise multiple crashes of size β at once but limits the total number of crashes to n. The stopping times $\tau_1, \dots, \tau_{n_0}$ are assumed to satisfy $\tau_i < \tau_{i-1}$ on $\{\tau_i < \infty\}$ for $i = 2, \dots, n_0$. Note that with this convention we count the number of crashes backwards: If a crash occurs at time τ_i then we know that at most $n - \sum_{k=1}^{i} j_k$ crash times may still be realized.

The investor's trading strategies are defined as in the case of at most 1 crash. The investor first chooses a trading strategy π^n which she follows as long as there are at most n crashes left. Then for every possible crash scenario $(\tau_{n_0}, \beta_{j_{n_0}})$ the investor chooses a strategy $\pi^{n-j_{n_0}}$ which she follows after the observed crash. Then for every such strategy $\pi^{n-j_{n_0}}$ with $n_0 \neq n$ and for every crash $(\tau_{n_0-1}, \beta_{j_{n_0-1}})$ the investor chooses a strategy $\pi^{n-j_{n_0}-j_{n_0-1}}$ and so on until no more crashes are possible. This means that for every possible crash strategy $\varpi := (\tau_i, \beta_{j_i})_{i=1,\ldots,n_0}$, $n_0 \leq n$, the investor has to come up with trading strategies which she can apply in between the crash times. For simplicity of notation we denote such families of trading strategies by π and the set of all such trading strategies by $\mathcal{A}_n(t, b, s)$. The crash strategies $\varpi = (\tau_i, \beta_{j_i})_{i=1,\ldots,n_0}$ are called admissible if the $(\tau_i)_{i=1,\ldots,n_0}$ are $\mathbb{F}^t$-stopping times. We denote the set of such crash scenarios by $\mathcal{B}_n(t)$.

Let $\pi \in \mathcal{A}_n(t, b, s)$ and $\varpi = (\tau_i, \beta_{j_i})_{i=1,\ldots,n_0} \in \mathcal{B}_n(t)$. Since the crash strategy is fixed so are the trading strategies the investor follows. We denote these strategies by $\pi_{n_0}, \ldots, \pi_0$ where $\pi_i = (L_i, M_i)$ is the trading strategy applied before the crash at time τ_i if $i = 1, \ldots, n_0$ and $\pi_0 = (L_0, M_0)$ is the strategy which is used when no more crashes can occur. Given an initial position of b units of money, the investor's wealth invested in the bond is given by $B(t-) = b$ at initial time. Then on $[t, \tau_{n_0}) \cap [t, T]$ the wealth follows

$$dB(u) = -(1+\lambda)dL_{n_0}(u) + (1-\mu)dM_{n_0}(u).$$

Similarly, on $[\tau_{i+1}, \tau_i) \cap [t, T]$, $i = 1 \ldots, n_0 - 1$, the wealth follows

$$dB(u) = -(1+\lambda)dL_i(u) + (1-\mu)dM_i(u)$$

and finally on $[\tau_1, T] \cap [t, T]$ the wealth invested in the bond is given by

$$dB(u) = -(1+\lambda)dL_0(u) + (1-\mu)dM_0(u).$$

The equations for the wealth invested in the stock are set up similarly (compare also with the case $n = 1$, see (3.4) to (3.6)) keeping in mind that at the time a crash occurs we have

$$S(\tau_i) = (1-\beta)^{j_i} S(\tau_i-) + L_{i-1}(\tau_i) - M_{i-1}(\tau_i).$$

We can now define the following solvency regions:

$$\mathcal{S}^i := \Big\{(b,s) \in \mathbb{R}^2 : b + (1+\lambda)s > 0, b + (1-\mu)(1-\beta)^i s > 0\Big\}, \quad i = 0, \ldots, n.$$

We restrict the set $\mathcal{A}_n(t,b,s)$ to those families of strategies such that for every admissible crash strategy in $\mathcal{B}_n(t)$ the corresponding pair (B,S) stays in the closure of $\mathcal{S}^i$ whenever there are at most i crashes left up to time T.

The net wealth X of the investor after liquidation of the risky position at time u is given by

$$X(u) := \begin{cases} B(u) + (1-\mu)S(u), & \text{if } S(u) > 0, \\ B(u) + (1+\lambda)S(u), & \text{if } S(u) \leq 0. \end{cases}$$

With this, we can formulate the following optimization problem: For $(t,b,s) \in [0,T) \times \overline{\mathcal{S}}^n$ we are interested in determining

$$\mathcal{V}_n(t,b,s) := \sup_{\pi \in \mathcal{A}_n(t,b,s)} \inf_{\varpi \in \mathcal{B}_n(t)} \mathbb{E}\left[U_p\left(X^{\pi,\varpi}_{t,b,s}(T)\right)\right].$$

We call $\mathcal{V}_n$ the value function in the market with at most n crashes. With this setup, using the techniques developed for the analysis of $\mathcal{V}_1$, the following results are straightforward.

First, one can show that the value functions $\mathcal{V}_i$, $0 \leq i \leq n$, are continuous on their respective domains.

Theorem 3.29. *The value function $\mathcal{V}_n$ is continuous on $[0,T] \times \overline{\mathcal{S}}^n$.*

From the continuity it is possible to deduce the existence of ε-optimal strategies which in turn lead to the dynamic programming principle. The DPP then allows to deduce the viscosity property of the value functions.

Theorem 3.30. *$\mathcal{V}_n$ is a viscosity solution of*

$$0 = \max\Big\{\mathcal{V}_n(t,b,s) - \mathcal{V}_{n-1}(t,b,(1-\beta)s), \\ \min\{\mathcal{L}^{nt}\mathcal{V}_n(t,b,s), \mathcal{L}^{buy}\mathcal{V}_n(t,b,s), \mathcal{L}^{sell}\mathcal{V}_n(t,b,s)\}\Big\},$$

on $[0,T) \times \mathcal{S}^n$ with boundary condition

$$\mathcal{V}_n(t,b,s) = U_p(0), \qquad \text{if } (b,s) \in \partial\mathcal{S}^n,\ t \in [0,T],$$

and terminal condition

$$\mathcal{V}_n(T,b,s) = \begin{cases} U_p(b + (1-\mu)(1-\beta)^n s), & \text{if } s > 0, \\ U_p(b + (1+\lambda)s), & \text{if } s \leq 0. \end{cases}$$

Moreover, $\mathcal{V}_n$ *is unique in the class of upper semi-continuous functions satisfying*

$$U_p(b + \min\{(1-\mu)(1-\beta)^n s, (1+\lambda)s\}) \leq \mathcal{V}_n(t,b,s) \leq \varphi_{\gamma,p,K}(t,b,s).$$

4. Worst-case portfolio optimization in a market with bubbles

We now leave the world of transaction costs behind us and consider a frictionless market underlying the worst-case model. More precisely, the objective of this chapter is to analyze a regime-switching model in the presence of crashes. The motivation of this model is to drop the assumption that the maximum number of crashes is finite and fixed a priori.

In the first part of this chapter we assume that the investor receives warnings about a potential crash at the jump times of an independent Poisson process. That is, whenever the Poisson process jumps one crash in the stock is possible (up to the next jump time of the Poisson process). It turns out that in the case of logarithmic and power utility it is relatively simple to derive a strategy which renders the investor indifferent between an immediate crash and no crash at all and to verify directly that this strategy is optimal.

In the second part of this chapter we extend this simple model by replacing the Poisson process by a finite-state Markov jump process and associate to every state of the Markov process different market coefficients and crash sizes. For this general model we derive a system of dynamic programming equations in the spirit of Korn and Steffensen [70] and use this to construct the optimal strategies, which turn out to be given as the solution of a coupled system of ordinary differential equations.

In contrast to the existing worst-case models with a deterministic maximum number of crashes the optimal strategies in our models exhibit some previously unobserved features. To be more precise, we show the following:

1. In general, the optimal strategies do not converge to the Merton fraction as the investment horizon tends to infinity. This shows that a random number of total crashes introduces an additional long-term effect on the optimal strategies.

2. While in the simple Poisson process model the investor is always indifferent between an immediate crash and no crash at all, this is not necessarily true in our generalized Markov chain model. As is known from Korn and Menkens [67] and Seifried [95], this effect may also occur in the classical models if the market coefficients change after a crash. In our model however this effect may occur also if the market coefficients are independent of the state as soon as we allow for changing crash sizes.

3. Finally, in the generalized model we show that the optimal strategies may not necessarily be monotonically decreasing in time. For example, in the numerical examples at the end of this chapter we construct optimal strategies which oscillating.

The results of this chapter correspond in large parts to the following articles:

1. C. Belak, S. Christensen, O. Menkens (2013): Worst-case optimal investment with a random number of crashes [9].

2. C. Belak, S. Christensen, O. Menkens (2013): Worst-case portfolio optimization in a market with bubbles [10].

4.1. The Poisson market model

As in the previous chapters we consider a financial market consisting of one risk-free bond and one risky stock with price evolutions as in the Black-Scholes model. Fix an investment horizon $T > 0$ and assume that the dynamics of the risk-free asset $P^0 = (P^0(t))_{t\in[0,T]}$ are given by

$$dP^0(t) = 0, \qquad t \in [0, T], \qquad P^0(0) = 1.$$

To model the price of the stock we let $W = (W(t))_{t\geq 0}$ be a standard Brownian motion on a complete probability space $(\Omega, \mathbb{F}, \mathbb{P})$ and let $(T_k)_{k\in\mathbb{N}}$ denote the jump times of an independent Poisson process with parameter λ. Moreover, we set $T_0 = 0$. We denote the augmented filtration generated by W and the Poisson process by $(\mathcal{F}(t))_{t\geq 0}$. As described in the introduction, the sequence $(T_k)_{k\in\mathbb{N}_0}$ models the time points at which the investor receives a warning about a potential market crash. Note that the sequence $(T_k)_{k\in\mathbb{N}_0}$ does not coincide with the crash times in general. The crash times are given by a sequence $(\tau_k)_{k\in\mathbb{N}_0}$ of $[T_k, T] \cup \{\infty\}$-valued stopping times with respect to the filtration $(\mathcal{F}(t))_{t\geq 0}$. We assume that whenever we have

$$T_k \leq \tau_k < T_{k+1}$$

then a crash occurs at time τ_k. This condition means that there is at most one crash between every two warnings. In other words, before each crash the investor receives a warning. We interpret $\{\tau_k \geq T_{k+1}\}$ as the event that no crash occurs between T_k and T_{k+1} so that with this setup at any given time at most one crash warning is active. At each crash time τ_k the stock price drops by a relative amount $0 \leq \beta_k \leq \beta$ where β_k is a $\mathcal{F}(\tau_k-)$-measurable random variable and $\beta \in (0,1)$ denotes the maximum (deterministic) crash size. Note that as before we can assume without loss of generality that $\beta_k = \beta$ since we take a worst-case perspective. We denote the set of all sequences $(\tau_k)_{k\in\mathbb{N}_0}$ fulfilling the above requirements by $\mathcal{T}_1$. The subset of all crash scenarios such that no crash occurs until the first warning (that is $\tau_0 \geq T_1$) is denoted by $\mathcal{T}_0$.

Given a sequence $(\tau_k)_{k\in\mathbb{N}_0} \in \mathcal{T}_1$ of crash scenarios the corresponding stock price process $P^1 = (P^1(t))_{t\in[0,T]}$ is given by

$$dP^1(t) = \alpha P^1(t)dt + \sigma P^1(t)dW(t), \qquad t \in [\tau_k, \tau_{k+1}) \cap [0,T],$$

where $\alpha, \sigma > 0$ denote the excess return and volatility of the stock price process. For the k-th crash time τ_k we assume that

$$\begin{aligned} P^1(\tau_k) &= (1-\beta)P^1(\tau_k-), && \text{on } \{\tau_k < T_{k+1}\} \wedge \{\tau_k \leq T\}, \\ P^1(\tau_k) &= P^1(\tau_k-), && \text{on } \{\tau_k \geq T_{k+1}\} \wedge \{\tau_k \leq T\}. \end{aligned}$$

The investor acts according to the information given by the filtration $(\mathcal{F}(t))_{t\geq 0}$. Furthermore, the investor does not know the crash scenarios $(\tau_k)_{k\in\mathbb{N}_0}$ a priori but can observe each crash whenever it occurs. However, the investor knows β and observes $(T_k)_{k\in\mathbb{N}_0}$ as well. For the investor two different situations must be distinguished: Whenever a crash has already happened and no new crash is announced – that is on an interval $[\tau_k, T_{k+1}) \cap [0,T]$ – the investor does not have to fear a crash and trades according to the strategy $\pi^0 = (\pi^0(t))_{t\in[0,T]}$. At all other times the investor must fear a crash and trades according to the strategy $\pi^1 = (\pi^1(t))_{t\in[0,T]}$. The trading strategies for the investor can therefore be described by a pair $\pi = (\pi^0, \pi^1)$ which is assumed to be a pair of adapted, right-continuous and bounded processes.

Given a crash scenario $\vartheta = (\tau_k)_{k\in\mathbb{N}_0}$, a trading strategy $\pi = (\pi^0, \pi^1)$ and an initial wealth of $x > 0$ the investor's wealth process $X = X^{\pi,\vartheta}_{0,x} = (X^{\pi,\vartheta}_{0,x}(t))_{t\in[0,T]}$ is given by

$$\begin{aligned}
&X(0) = x, \\
&dX(t) = \alpha\pi^1(t)X(t)dt + \sigma\pi^1(t)X(t)dW(t), && \text{on } [T_k, \tau_k) \cap [0,T], \\
&dX(t) = \alpha\pi^0(t)X(t)dt + \sigma\pi^0(t)X(t)dW(t), && \text{on } [\tau_k, T_{k+1}) \cap [0,T], \\
&X(\tau_k) = (1 - \pi^1(\tau_k)\beta)X(\tau_k-), && \text{on } \{\tau_k < T_{k+1}\} \cap \{\tau_k \le T\}, \\
&X(\tau_k) = X(\tau_k-), && \text{on } \{\tau_k \ge T_{k+1}\} \cap \{\tau_k \le T\}.
\end{aligned}$$

If no confusion may occur we drop some of the subscripts or superscripts in the notation of the process $X^{\pi,\vartheta}_{0,x}$. Note that the explicit solution of the above SDE is given by

$$\begin{aligned}
X(t) = x\prod_{k=0}^{\infty} &\left(1 - \pi^1(\tau_k)\beta\mathbb{1}_{\{\tau_k<T_{k+1}\}}\mathbb{1}_{\{\tau_k\le t\}}\right) \\
&\cdot \exp\left(\int_{T_k\wedge t}^{\tau_k\wedge t} \sigma\pi^1(u)\,dW(u) + \int_{\tau_k\wedge t}^{T_{k+1}\wedge t} \sigma\pi^0(u)\,dW(u)\right) \\
&\cdot \exp\left(\int_{T_k\wedge t}^{\tau_k\wedge t} \alpha\pi^1(u) - \frac{1}{2}\sigma^2\left[\pi^1(u)\right]^2 du\right) \\
&\cdot \exp\left(\int_{\tau_k\wedge t}^{T_{k+1}\wedge t} \alpha\pi^0(u) - \frac{1}{2}\sigma^2\left[\pi^0(u)\right]^2 du\right). \qquad (4.1)
\end{aligned}$$

We denote the set of all trading strategies π that correspond to a strictly positive wealth processes $X^{\pi,\vartheta}_{0,x}$ for all $\vartheta \in \mathcal{T}_1$ by $\mathcal{A}(x)$. Note that the strategy π^1 in the crash regime has to satisfy

$$\pi^1(t) < \frac{1}{\beta} \qquad \text{for all } t \in [0,T].$$

If we consider the problem on a sub-interval $[t,T]$, then we denote the corresponding strategies by $\mathcal{A}(t,x)$, $\mathcal{T}_1(t)$ and $\mathcal{T}_0(t)$, respectively.

Fix $p < 1$. We consider the following worst-case optimization problem: The investor optimizes her expected utility under the worst possible crash scenario, i.e.

$$\sup_{\pi\in\mathcal{A}(x)} \inf_{\vartheta\in\mathcal{T}_0} \mathbb{E}\left[U_p\left(X_{0,x}^{\pi,\vartheta}(T)\right)\right] \qquad \text{and} \qquad \sup_{\pi\in\mathcal{A}(x)} \inf_{\vartheta\in\mathcal{T}_1} \mathbb{E}\left[U_p\left(X_{0,x}^{\pi,\vartheta}(T)\right)\right].$$

The first problem corresponds to the case in which we start in a situation without a crash warning at time $t = 0$. In the second problem the first crash may occur immediately. We make the problem time-dependent by introducing the value functions

$$\mathcal{V}(t,x,i) = \sup_{\pi\in\mathcal{A}(t,x)} \inf_{\vartheta\in\mathcal{T}_i(t)} \mathbb{E}\left[U_p\left(X_{t,x}^{\pi,\vartheta}(T)\right)\right], \qquad i = 0,1.$$

Obviously $\mathcal{V}(t,x,1) \le \mathcal{V}(t,x,0)$ since the infimum is taken over a larger set.

4.2. Heuristic derivation of the optimal strategies

In this section we find a candidate optimal solution $\pi^* = (\pi^{0,*}, \pi^{1,*})$ for the logarithmic utility case, i.e. $p = 0$. By the usual pointwise maximization argument (see e.g. Irle and Sass [49, Section 2] in a slightly different context) it is immediately clear that in times with no crash warning it is optimal for the investor to use the Merton strategy. That is

$$\pi^{0,*}(t) := \pi_M = \frac{\alpha}{\sigma^2}, \qquad \text{for all } t \in [0,T].$$

If we start the wealth process at time t with initial wealth x and no crash warning is present, then by the memoryless property of the exponential distribution the next warning arrives at an exponential time e_λ if this time is less than T. It is therefore reasonable to assume that we have the following version of the dynamic programming principle:

$$\mathcal{V}(t,x,0) = \mathbb{E}\Bigg[\mathcal{V}(t+e_\lambda, X_{t,x}^{\pi^{0,*}}(t+e_\lambda), 1)\mathbb{1}_{\{t+e_\lambda<T\}} + \log\left(X_{t,x}^{\pi^{0,*}}(T)\right)\mathbb{1}_{\{t+e_\lambda\ge T\}}\Bigg].$$

Integrating the exponential time we obtain

$$\begin{aligned}\mathcal{V}(t,x,0) &= \int_0^{T-t} \mathbb{E}\big[\mathcal{V}(t+u, X_{t,x}^{\pi^{0,*}}(t+u),1)\big]\lambda e^{-\lambda u}\,du \\ &\qquad\qquad + e^{-\lambda(T-t)}\mathbb{E}\left[\log\left(X_{t,x}^{\pi^{0,*}}(T)\right)\right] \\ &= \int_0^{T-t} \mathbb{E}\big[\mathcal{V}(t+u, X_{t,x}^{\pi^{0,*}}(t+u),1)\big]\lambda e^{-\lambda u}\,du \\ &\qquad\qquad + e^{-\lambda(T-t)}\left(\log(x) + \frac{1}{2}\frac{\alpha^2}{\sigma^2}(T-t)\right). \end{aligned} \tag{4.2}$$

To obtain an expression for $\mathcal{V}(t,x,1)$ we use the indifference approach as described in Korn and Wilmott [71] and formalized in Korn and Menkens [67]. For an initial value x and a time point t such that a crash could happen, we try to find a strategy $\pi^{1,*}$ such that the investor is indifferent between the scenarios *"A crash of size β happens immediately"* and *"No crash happens until T"*. We expect this strategy to be optimal. For the first scenario, after the crash in t, the investor is faced with the problem without an active crash warning discussed above so that

$$\mathcal{V}(t,x,1) = \mathcal{V}(t,(1-\pi^{1,*}(t)\beta)x,0). \tag{4.3}$$

On the other hand, in the second scenario (no crash at all) Itô's formula leads to

$$\begin{aligned}\mathcal{V}(t,x,1) &= \mathbb{E}\left[\log\left(X_{t,x}^{\pi^{1,*}}(T)\right)\right] \\ &= \log(x) + \mathbb{E}\left[\int_t^T \alpha\pi^{1,*}(u) - \frac{1}{2}\sigma^2\big[\pi^{1,*}(u)\big]^2\,du\right], \end{aligned} \tag{4.4}$$

where the stochastic integral vanishes since we assume $\pi^{1,*}$ to be bounded. Combining (4.3) and (4.4) with (4.2) leads to

$$\begin{aligned}&\log(x) + \mathbb{E}\left[\int_t^T \alpha\pi^{1,*}(u) - \frac{1}{2}\sigma^2\big[\pi^{1,*}(u)\big]^2\,du\right] \\ &\quad = \mathcal{V}(t,x,1) \\ &\quad = \mathcal{V}(t,(1-\pi^{1,*}(t)\beta)x,0) \\ &\quad = \int_0^{T-t} \mathbb{E}\left[\mathcal{V}(t+u, X_{t,(1-\pi^{1,*}(t)\beta)x}^{\pi^{0,*}}(t+u),1)\right]\lambda e^{-\lambda u}\,du \\ &\qquad + e^{-\lambda(T-t)}\left(\log(x) + \mathbb{E}\left[\log(1-\pi^{1,*}(t)\beta)\right] + \frac{1}{2}\frac{\alpha^2}{\sigma^2}(T-t)\right). \end{aligned} \tag{4.5}$$

Furthermore, by (4.4) and Itô's formula we have

$$
\begin{aligned}
&\mathbb{E}\big[\mathcal{V}(t+u, X^{\pi^{0,*}}_{t,(1-\pi^{1,*}(t)\beta)x}(t+u), 1)\big] \\
&\quad = \mathbb{E}\left[\log\left(X^{\pi^{0,*}}_{t,(1-\pi^{1,*}(t)\beta)x}(t+u)\right) + \int_{t+u}^{T} \alpha\pi^{1,*}(r) - \frac{1}{2}\sigma^2\big[\pi^{1,*}(r)\big]^2\,dr\right] \\
&\quad = \mathbb{E}\Bigg[\log\left(X^{\pi^{0,*}}_{t,(1-\pi^{1,*}(t)\beta)x}(t)\right) + \int_{t}^{t+u} \alpha\pi^{0,*}(r) - \frac{1}{2}\sigma^2\big[\pi^{0,*}(r)\big]^2\,dr \\
&\qquad\qquad + \int_{t+u}^{T} \alpha\pi^{1,*}(r) - \frac{1}{2}\sigma^2\big[\pi^{1,*}(r)\big]^2\,dr\Bigg] \\
&\quad = \log(x) + \frac{1}{2}\frac{\alpha^2}{\sigma^2}u + \mathbb{E}\Bigg[\log(1-\pi^{1,*}(t)\beta) \\
&\qquad\qquad + \int_{t+u}^{T} \alpha\pi^{1,*}(r) - \frac{1}{2}\sigma^2\big[\pi^{1,*}(r)\big]^2\,dr\Bigg],
\end{aligned}
$$

so that we get from (4.5)

$$
\begin{aligned}
&\mathbb{E}\left[\int_t^T \alpha\pi^{1,*}(u) - \frac{1}{2}\sigma^2\big[\pi^{1,*}(u)\big]^2\,du\right] \\
&= \int_0^{T-t} \mathbb{E}\left[\log(1-\pi^{1,*}(t)\beta) + \int_{t+u}^{T} \alpha\pi^{1,*}(r) - \frac{1}{2}\sigma^2\big[\pi^{1,*}(r)\big]^2\,dr\right]\lambda e^{-\lambda u}\,du \\
&\quad + \int_0^{T-t} \frac{1}{2}\frac{\alpha^2}{\sigma^2}u\lambda e^{-\lambda u}\,du + e^{-\lambda(T-t)}\left(\mathbb{E}\left[\log(1-\pi^{1,*}(t)\beta)\right] + \frac{1}{2}\frac{\alpha^2}{\sigma^2}(T-t)\right) \\
&= \int_0^{T-t} \mathbb{E}\left[\int_{t+u}^{T} \alpha\pi^{1,*}(r) - \frac{1}{2}\sigma^2\big[\pi^{1,*}(r)\big]^2\,dr\right]\lambda e^{-\lambda u}\,du \\
&\qquad + \mathbb{E}\left[\log(1-\pi^{1,*}(t)\beta)\right] + \frac{1}{2}\frac{\alpha^2}{\sigma^2}\frac{1-e^{-\lambda(T-t)}}{\lambda}.
\end{aligned}
$$

Now we make the ansatz that $\pi^{1,*}$ is deterministic and obtain the integral equation

$$
\begin{aligned}
&\int_t^T \alpha\pi^{1,*}(u) - \frac{1}{2}\sigma^2\big[\pi^{1,*}(u)\big]^2\,du \\
&\qquad = \log(1-\pi^{1,*}(t)\beta) + \frac{1}{2}\frac{\alpha^2}{\sigma^2}\frac{1-e^{-\lambda(T-t)}}{\lambda} \\
&\qquad\quad + \int_0^{T-t}\left(\int_{t+u}^{T} \alpha\pi^{1,*}(r) - \frac{1}{2}\sigma^2\big[\pi^{1,*}(r)\big]^2\,dr\right)\lambda e^{-\lambda u}\,du. \qquad (4.6)
\end{aligned}
$$

Integration by parts shows that

$$\begin{aligned}\int_0^{T-t} \left(\int_{t+u}^T \alpha\pi^{1,*}(r) - \frac{1}{2}\sigma^2 \left[\pi^{1,*}(r)\right]^2 dr \right) \lambda e^{-\lambda u}\, du \\ = \int_t^T \alpha\pi^{1,*}(u) - \frac{1}{2}\sigma^2 \left[\pi^{1,*}(u)\right]^2 du \\ - \int_t^T e^{-\lambda u} \left(\alpha\pi^{1,*}(u) - \frac{1}{2}\sigma^2 \left[\pi^{1,*}(u)\right]^2 \right) du\end{aligned}$$

and hence (4.6) simplifies to

$$\begin{aligned}0 = e^{-\lambda t} \log(1 - \pi^{1,*}(t)\beta) + \frac{1}{2}\frac{\alpha^2}{\sigma^2}\frac{e^{-\lambda t} - e^{-\lambda T}}{\lambda} \\ - \int_t^T e^{-\lambda(u-t)} \left(\alpha\pi^{1,*}(u) - \frac{1}{2}\sigma^2 \left[\pi^{1,*}(u)\right]^2 \right) du. \qquad (4.7)\end{aligned}$$

Note that for the degenerate case $\lambda \to 0$ (that is, the time until the second warning can occur is infinite, i.e. only one crash can happen) this equation simplifies to

$$\int_t^T \alpha\pi^{1,*}(u) - \frac{1}{2}\sigma^2 \pi^{1,*}(u)^2\, du = \log(1 - \pi^{1,*}(t)\beta) + \frac{1}{2}\frac{\alpha^2}{\sigma^2}(T - t)$$

which indeed is the equation characterizing the optimal worst case strategy in the problem with only one crash, see Korn and Wilmott [71, Equation (A.5)].

Differentiation with respect to t in (4.7) and rearranging terms yields the following ordinary differential equation for $\pi^{1,*}$:

$$\begin{aligned}\frac{\partial}{\partial t}\pi^{1,*}(t) &= \frac{1}{\beta}(1 - \pi^{1,*}(t)\beta)\Big(\alpha\big(\pi^{1,*}(t) - \pi^{0,*}(t)\big) \\ &\qquad - \frac{1}{2}\sigma^2\big([\pi^{1,*}(t)]^2 - [\pi^{0,*}(t)]^2\big) \\ &\qquad - \lambda \log(1 - \pi^{1,*}(t)\beta)\Big) \\ &= \frac{1}{\beta}(1 - \pi^{1,*}(t)\beta)\Big(-\frac{1}{2}\sigma^2 \left(\pi^{1,*}(t) - \pi^{0,*}(t)\right)^2 \\ &\qquad - \lambda \log(1 - \pi^{1,*}(t)\beta)\Big), \qquad (4.8) \\ \pi^{1,*}(T) &= 0,\end{aligned}$$

where as before $\pi^{0,*}(t) = \pi_M = \alpha/\sigma^2$ for all $t \in [0,T]$. The existence and uniqueness of a solution of this differential equation is assured by arguments similar to the proof in Menkens [78, Theorem 2.5], see also Lemma 4.10 below. In particular we have $\pi^{1,*}(t) < 1/\beta$ and $\pi^{1,*} \leq \pi^{0,*}$.

Again, note that for the degenerate case $\lambda = 0$ the equation for the candidate optimal strategy reduces to

$$\frac{\partial}{\partial t}\pi^{1,*}(t) = \frac{1}{\beta}(1-\pi^{1,*}(t)\beta)\left(-\frac{1}{2}\sigma^2\left(\pi^{1,*}(t)-\pi^{0,*}(t)\right)^2\right)$$

which is the equation obtained in Korn and Wilmott [71], see (1.6) in Chapter 1.

4.3. Direct verification for logarithmic utility

In this section we verify that the indifference strategy $\pi^* = (\pi^{0,*},\pi^{1,*})$ constructed in the previous section is indeed optimal. For this let $\hat{\vartheta} = (\hat{\tau}_k)_{k\in\mathbb{N}_0}$ denote the crash scenario such that no crash occurs at all, i.e. $\hat{\tau}_k \equiv \infty$ for all $k \in \mathbb{N}_0$. The next lemma shows that the strategy π^* leads to the same expected utility no matter which crash scenario occurs.

Lemma 4.1. *For all crash scenarios $\vartheta = (\tau_k)_{k\in\mathbb{N}_0} \in \mathcal{T}_1$ we have*

$$\mathbb{E}\left[\log\left(X_{0,x}^{\pi^*,\vartheta}(T)\right)\right] = \mathbb{E}\left[\log\left(X_{0,x}^{\pi^*,\hat{\vartheta}}(T)\right)\right].$$

Proof. Writing $g(y) = \alpha y - 1/2\sigma^2 y^2$ for short we have by (4.1)

$$\begin{aligned}\mathbb{E}\left[\log\left(X_{0,x}^{\pi^*,\vartheta}(T)\right)\right] &= \log(x) + \mathbb{E}\left[\sum_{k=0}^{\infty}\int_{T_k\wedge T}^{\tau_k\wedge T} g\big(\pi^{1,*}(u)\big)\,du\right]\\ &\quad + \mathbb{E}\left[\sum_{k=0}^{\infty}\int_{\tau_k\wedge T}^{T_{k+1}\wedge T} g\big(\pi^{0,*}(u)\big)\,du\mathbb{1}_{\{\tau_k<T_{k+1}\}}\right]\\ &\quad + \mathbb{E}\left[\sum_{k=0}^{\infty}\log(1-\pi^{1,*}(\tau_k)\beta)\mathbb{1}_{\{\tau_k<T_{k+1}\}}\mathbb{1}_{\{\tau_k\leq T\}}\right]. \quad (4.9)\end{aligned}$$

By construction of the indifference strategy π^* it holds that for each $k \in \mathbb{N}_0$ on the set $\{\tau_k < T_{k+1}\} \cap \{\tau_k \leq T\}$ we have

$$\mathbb{E}\left[\int_{\tau_k \wedge T}^{T} g\big(\pi^{1,*}(u)\big)\,du \,\middle|\, \mathcal{F}(\tau_k)\right] = \mathbb{E}\Bigg[\int_{\tau_k \wedge T}^{T_{k+1}\wedge T} g\big(\pi^{0,*}(u)\big)\,du + \int_{T_{k+1}\wedge T}^{T} g\big(\pi^{1,*}(u)\big)\,du + \log(1-\pi^{1,*}(\tau_k)\beta) \,\Bigg|\, \mathcal{F}(\tau_k)\Bigg],$$

i.e. the investor is indifferent between a crash of size β happening at time τ_k (right-hand side) and no crash happening (left-hand side). This can be rewritten as

$$\mathbb{E}\left[\int_{\tau_k \wedge T}^{T_{k+1}\wedge T} g\big(\pi^{1,*}(u)\big)\,du \,\middle|\, \mathcal{F}(\tau_k)\right] = \mathbb{E}\left[\int_{\tau_k \wedge T}^{T_{k+1}\wedge T} g\big(\pi^{0,*}(u)\big)\,du + \log(1-\pi^{1,*}(\tau_k)\beta) \,\middle|\, \mathcal{F}(\tau_k)\right]. \tag{4.10}$$

Hence, combining (4.9) and (4.10) we see that

$$\begin{aligned}\mathbb{E}\left[\log\left(X_{0,x}^{\pi^*,\vartheta}(T)\right)\right] &= \log(x) + \mathbb{E}\left[\sum_{k=0}^{\infty}\int_{T_k\wedge T}^{\tau_k \wedge T} g\big(\pi^{1,*}(u)\big)\,du\right] \\ &\quad + \mathbb{E}\left[\sum_{k=0}^{\infty}\int_{\tau_k\wedge T}^{T_{k+1} \wedge T} g\big(\pi^{1,*}(u)\big)\,du \mathbb{1}_{\{\tau_k < T_{k+1}\}}\right] \\ &= \log(x) + \mathbb{E}\left[\int_0^T g\big(\pi^{1,*}(u)\big)\,du\right] \\ &= \mathbb{E}\left[\log\left(X_{0,x}^{\pi^*,\hat{\vartheta}}(T)\right)\right].\end{aligned}$$

□

Lemma 4.1 implies that the strategy π^* is indeed an indifference strategy, i.e. if the investor follows this strategy she is indifferent between which crash scenario occurs since each scenario leads to the same expected utility. Note also that since $\mathcal{T}_0 \subset \mathcal{T}_1$ and $\hat{\vartheta} \in \mathcal{T}_0$ the same result also applies for the case in which there is no crash warning at initial time $t = 0$. With this it is easy to prove the optimality of π^* since we now only need to find one crash scenario in which the indifference strategy outperforms any other given strategy.

Proposition 4.2. *Let $\pi = (\pi^0, \pi^1) \in \mathcal{A}(x)$ be arbitrary. Then*

$$\inf_{\vartheta \in \mathcal{T}_0} \mathbb{E}\left[\log\left(X_{0,x}^{\pi,\vartheta}(T)\right)\right] \le \inf_{\vartheta \in \mathcal{T}_0} \mathbb{E}\left[\log\left(X_{0,x}^{\pi^*,\vartheta}(T)\right)\right]$$

and

$$\inf_{\vartheta \in \mathcal{T}_1} \mathbb{E}\left[\log\left(X_{0,x}^{\pi,\vartheta}(T)\right)\right] \le \inf_{\vartheta \in \mathcal{T}_1} \mathbb{E}\left[\log\left(X_{0,x}^{\pi^*,\vartheta}(T)\right)\right].$$

Proof. By Lemma 4.1 the right-hand side is independent of ϑ. In order to prove the first claim it is therefore enough to find at least one $\vartheta \in \mathcal{T}_0$ such that

$$\mathbb{E}\left[\log\left(X_{0,x}^{\pi,\vartheta}(T)\right)\right] \le \mathbb{E}\left[\log\left(X_{0,x}^{\pi^*,\vartheta}(T)\right)\right].$$

For this, set $\tau_0 = T_1$ (since we start without a crash warning) and define

$$\tau_k = \inf\{u \in [T_k, T] : \pi^1(u) \ge \pi^{1,*}(u)\}$$

for all $k \in \mathbb{N}$. Writing $\vartheta = (\tau_k)_{k \in \mathbb{N}_0}$ we have

$$\begin{aligned}
\mathbb{E}\left[\log\left(X_{0,x}^{\pi,\vartheta}(T)\right)\right] &= \log(x) + \mathbb{E}\left[\sum_{k=0}^{\infty} \int_{T_k \wedge T}^{\tau_k \wedge T} g\big(\pi^1(u)\big)\, du\right] \\
&\quad + \mathbb{E}\left[\sum_{k=0}^{\infty} \int_{\tau_k \wedge T}^{T_{k+1} \wedge T} g\big(\pi^0(u)\big)\, du \mathbb{1}_{\{\tau_k < T_{k+1}\}}\right] \\
&\quad + \mathbb{E}\left[\log(1 - \pi^1(\tau_k)\beta) \mathbb{1}_{\{\tau_k < T_{k+1}\}} \mathbb{1}_{\{\tau_k \le T\}}\right]. \qquad (4.11)
\end{aligned}$$

On $[T_k, \tau_k) \cap [0, T]$ we have $\pi^1 \le \pi^{1,*} \le \pi^{0,*}$ and hence $g(\pi^1) \le g(\pi^{1,*})$ since g is quadratic and attains its unique maximum at $\pi^{0,*}$. Therefore

$$\mathbb{E}\left[\sum_{k=0}^{\infty} \int_{T_k \wedge T}^{\tau_k \wedge T} g\big(\pi^1(u)\big)\, du\right] \le \mathbb{E}\left[\sum_{k=0}^{\infty} \int_{T_k \wedge T}^{\tau_k \wedge T} g\big(\pi^{1,*}(u)\big)\, du\right]. \qquad (4.12)$$

Similarly, we have $g(\pi^0) \le g(\pi^{0,*})$ on $[\tau_k, T_{k+1}) \cap [0, T]$ and therefore

$$\begin{aligned}
&\mathbb{E}\left[\sum_{k=0}^{\infty} \int_{\tau_k \wedge T}^{T_{k+1} \wedge T} g\big(\pi^0(u)\big)\, du \mathbb{1}_{\{\tau_k < T_{k+1}\}}\right] \\
&\qquad \le \mathbb{E}\left[\sum_{k=0}^{\infty} \int_{\tau_k \wedge T}^{T_{k+1} \wedge T} g\big(\pi^{0,*}(u)\big)\, du \mathbb{1}_{\{\tau_k < T_{k+1}\}}\right]. \qquad (4.13)
\end{aligned}$$

Finally, by the construction of τ_k and the right-continuity of π^1 and $\pi^{1,*}$ we have $\pi^1(\tau_k) \geq \pi^{1,*}(\tau_k)$ and hence

$$\log(1 - \pi^1(\tau_k)\beta) \leq \log(1 - \pi^{1,*}(\tau_k)\beta)$$

on $\{\tau_k < T_{k+1}\} \cap \{\tau_k \leq T\}$. That is,

$$\begin{aligned} &\mathbb{E}\left[\log(1 - \pi^1(\tau_k)\beta)\mathbb{1}_{\{\tau_k < T_{k+1}\}}\mathbb{1}_{\{\tau_k \leq T\}}\right] \\ &\qquad\qquad \leq \mathbb{E}\left[\log(1 - \pi^{1,*}(\tau_k)\beta)\mathbb{1}_{\{\tau_k < T_{k+1}\}}\mathbb{1}_{\{\tau_k \leq T\}}\right]. \end{aligned} \tag{4.14}$$

Combining (4.12)-(4.14) with (4.11) hence shows that

$$\mathbb{E}\left[\log\left(X_{0,x}^{\pi,\vartheta}(T)\right)\right] \leq \mathbb{E}\left[\log\left(X_{0,x}^{\pi^*,\vartheta}(T)\right)\right].$$

The proof for $\vartheta \in \mathcal{T}_1$ follows similarly. □

4.4. Heuristic derivation in the power utility case

Using a change-of-measure approach, the power-utility case $p < 1$, $p \neq 0$ can be handled similarly to the logarithmic utility case $p = 0$ studied in the previous sections. For the derivation of the candidate optimal strategies we proceed similarly to the indifference arguments in Section 4.2.

More specifically, for each risky fraction process $\pi^{1,*}$ in the scenario without crashes we can decompose the utility process as follows:

$$\begin{aligned} U_p\left(X_{t,x}^{\pi^{1,*}}(T)\right) &= \frac{1}{p}\exp\left(p\log\left(X_{t,x}^{\pi^{1,*}}(T)\right)\right) \\ &= U_p(x)\exp\left(p\int_t^T g\big(\pi^{1,*}(u)\big)\,du + p\int_t^T \sigma\pi^{1,*}(u)\,dW(u)\right) \\ &= U_p(x)\exp\left(p\int_t^T g_p\big(\pi^{1,*}(u)\big)\,du\right)M_t(T), \end{aligned}$$

where

$$g(y) = \alpha y - \frac{1}{2}\sigma^2 y^2, \qquad g_p(y) = \alpha y - \frac{1}{2}(1-p)\sigma^2 y^2,$$

and $M_t = (M_t(u))_{u\in[t,T]}$ given by

$$M_t(u) = \exp\left(-\frac{1}{2}\int_t^u p^2\sigma^2[\pi^{1,*}(r)]^2\,dr + \int_t^u p\sigma\pi^{1,*}(r)\,dW(r)\right)$$

is a local martingale with $M_t(t) = 1$. Since we assume admissible strategies to be bounded we see that M_t is even a martingale which defines a change of measure. We denote the expectation under this measure by $\mathbb{E}^{\pi^{1,*}}$. In the scenario without crashes we therefore obtain

$$\mathbb{E}\left[U_p\left(X_{t,x}^{\pi^{1,*}}(T)\right)\right] = U_p(x)\,\mathbb{E}^{\pi^{1,*}}\left[\exp\left(p\int_t^T g_p\big(\pi^{1,*}(u)\big)\,du\right)\right].$$

If one crash of size β occurs immediately in t, no further crash occurs and we hold the Merton ratio $\pi^{0,*}(u) := \pi_M = \alpha/(1-p)\sigma^2$ until a new crash is possible after an exponential time e_λ the expected performance is

$$\begin{aligned}U_p(x)\,\mathbb{E}^{\pi^{1,*}}&\left[\exp\left(p\int_t^T g_p\big(\pi^{0,*}(u)\big)\,du\right)(1-\pi^{1,*}(t)\beta)^p\mathbb{1}_{\{t+e_\lambda\geq T\}}\right]\\ &+U_p(x)\,\mathbb{E}^{\pi^{1,*}}\Bigg[\exp\Bigg(p\int_t^{t+e_\lambda} g_p\big(\pi^{0,*}(u)\big)\,du\\ &\qquad+p\int_{t+e_\lambda}^T g_p\big(\pi^{1,*}(u)\big)\,du\Bigg)(1-\pi^{1,*}(t)\beta)^p\mathbb{1}_{\{t+e_\lambda<T\}}\Bigg].\end{aligned}$$

Assuming again that $\pi^{1,*}$ is deterministic, the indifference approach suggests that $\pi^{1,*}$ should solve

$$\begin{aligned}&\exp\left(p\int_t^T g_p\big(\pi^{1,*}(u)\big)\,du\right)\\ &\quad=\mathbb{E}^{\pi^{1,*}}\left[\exp\left(p\int_t^T g_p\big(\pi^{0,*}(u)\big)\,du\right)(1-\pi^{1,*}(t)\beta)^p\mathbb{1}_{\{t+e_\lambda\geq T\}}\right]\\ &\qquad+\mathbb{E}^{\pi^{1,*}}\Bigg[\exp\Bigg(p\int_t^{t+e_\lambda} g_p\big(\pi^{0,*}(u)\big)\,du\\ &\qquad\qquad+p\int_{t+e_\lambda}^T g_p\big(\pi^{1,*}(u)\big)\,du\Bigg)(1-\pi^{1,*}(t)\beta)^p\mathbb{1}_{\{t+e_\lambda<T\}}\Bigg].\end{aligned}$$

Integrating the exponential time and some further simplifications show that

$$(1-\pi^{1,*}(t)\beta)^{-p} = \exp\left(-p\int_t^T \overline{g}_p\big(\pi^{1,*}(u)\big)\,du\right) \\ + \lambda\int_0^{T-t}\exp\left(-p\int_t^{t+u}\overline{g}_p\big(\pi^{1,*}(r)\big)\,dr\right)du, \quad (4.15)$$

where $\overline{g}_p(y) = g_p(y) + \frac{\lambda - pc_p}{p}$ and $c_p = g_p\big(\pi^{0,*}\big)$ is the maximum of the function g_p. Differentiating with respect to t in (4.15) and rearranging terms yields the ordinary differential equation

$$\begin{aligned}\frac{\partial}{\partial t}\pi^{1,*}(t) &= \frac{1}{\beta}(1-\pi^{1,*}(t)\beta)\left(\overline{g}_p\big(\pi^{1,*}(t)\big) - \frac{\lambda}{p}(1-\pi^{1,*}(t)\beta)^p\right)\\ &= \frac{1}{\beta}(1-\pi^{1,*}(t)\beta)\bigg(-\frac{1}{2}(1-p)\sigma^2\left(\pi^{1,*}(t)-\pi^{0,*}(t)\right)^2\\ &\qquad\qquad - \frac{\lambda}{p}\left((1-\pi^{1,*}(t)\beta)^p - 1\right)\bigg) \quad (4.16)\end{aligned}$$

with terminal condition $\pi^{1,*}(T) = 0$.

Note that (4.16) converges to the ODE for the indifference strategy in the logarithmic utility case given by (4.8) if we let $p \to 0$. Moreover, sending $\lambda \downarrow 0$ shows that (4.16) converges to the indifference strategy in the model with at most one crash given by (1.8).

At this point one could proceed similarly to the reasoning in Section 4.3 to verify the optimality of the indifference strategy $\pi^* = (\pi^{0,*}, \pi^{1,*})$. We do not work out the details here but develop a different approach in a generalized model based on a system of dynamic programming equations.

4.5. The generalized model

In this section we generalize the model discussed before by introducing a range of maximum crash sizes and state-dependent market coefficients as follows: We let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space which supports a standard Brownian motion $W = (W(t))_{t\geq 0}$ and an independent time-homogeneous continuous-time Markov chain $Z = (Z(t))_{t\geq 0}$ with state space $E := \{0, \ldots, d\}$ for some $d \geq 1$. We denote by $Q = (q_{i,j})_{0\leq i,j\leq d}$ the transition rate matrix of Z such that

$$q_{i,j} \geq 0 \qquad \text{for all } i,j \in E \ \text{ with } \ i \neq j$$

and we set

$$\lambda_i := -q_{i,i} := \sum_{\substack{j=0 \\ j\neq i}}^{d} q_{i,j} \qquad \text{for all } i \in E.$$

We assume moreover that the state 0 cannot be reached from any other point, i.e. $q_{i,0} = 0$ for all $i \in E$. We denote the augmented filtration generated by W and Z by $\mathbb{F} = (\mathcal{F}(t))_{t\geq 0}$.

Let us now fix some investment horizon $T > 0$ as well as some initial time $t \in [0, T)$. We assume that the bond price $P^0 = (P^0(u))_{u\in[t,T]}$ is given as before, that is

$$dP^0(u) = 0, \qquad\qquad u \in [t, T], \qquad\qquad P^0(t) = 1.$$

We denote by $Z_{t,i} = (Z_{t,i}(u))_{u\geq t}$ the process $(Z(u))_{u\geq t}$ conditioned on $Z(t) = i$. We assume that in the absence of crashes the stock price $P^1 = (P^1(u))_{u\in[t,T]}$ has state-dependent excess return and volatility, i.e.

$$dP^1(u) = \alpha_j P^1(u)du + \sigma_j P^1(u)dW(u), \quad \text{on } \{Z(u) = j\}, \quad u \in [t, T],$$

where $\alpha_j, \sigma_j > 0$ for all $j \in E$ and where we set $P^1(t) = 1$.

To each state $i \in E$ we associate a maximum crash size $\beta_i \in [0, 1)$. We assume that $\beta_0 = 0$ (i.e. no crash in state 0) and assume the maximum crash sizes to be ordered:

$$0 = \beta_0 \leq \beta_1 \leq \ldots \leq \beta_d > 0.$$

Moreover, we set $i_{min} = \min\{i \in E : \beta_i > 0\}$ (i.e. if $i < i_{min}$, then no crashes can occur in state i, see also Remark 4.4). We denote the jump times of the Markov chain $Z_{t,i}$ by $(T_k)_{k\in\mathbb{N}}$ and set $T_0 = t$. The crash times are now given by a sequence $(\tau_k)_{k\in\mathbb{N}_0}$ of $\mathbb{F}$-stopping times taking values in $[T_k, T] \cup \{\infty\}$ and we assume that a crash occurs only if $Z_{t,i}(\tau_k-) \geq i_{min}$ and

$$T_k \leq \tau_k < T_{k+1}.$$

The sequence $(\tau_k)_{k\in\mathbb{N}_0}$ now acts as an impulse control strategy for $Z_{t,i}$ and P^1 as follows: Whenever $\tau_k < T_{k+1}$ and $Z_{t,i}(\tau_k-) \geq i_{min}$ the Markov chain $Z_{t,i}$ is sent to the state 0 at time τ_k and the asset crashes in the following sense:

$$P^1(\tau_k) = (1 - \beta_j)P_1(\tau_k-), \qquad \text{on } \{Z_{t,i}(\tau_k-) = j \geq i_{min}\}.$$

We write $\vartheta = (\tau_k)_{k\in\mathbb{N}_0}$ and denote the corresponding controlled Markov chain by $Z^{\vartheta}_{t,i}$. Moreover, we denote by $\mathcal{T}(t,i)$ the set of all sequences of crash times as defined above.

We interpret this market model as follows: Whenever $Z^{\vartheta}_{t,i}$ is in state 0, then the market is in a safe regime in the sense that no crashes may occur. As soon as $Z^{\vartheta}_{t,i}$ jumps into a state $0 < j < i_{min}$ a bubble has formed in the market, and as soon as $Z^{\vartheta}_{t,i}$ jumps into a state $l \geq i_{min}$ this bubble may potentially burst at the unknown time τ_k, leading to a crash in the risky asset and bringing the market back into the crash-free state 0. Since we will allow the investor to observe the process $Z^{\vartheta}_{t,i}$ we can interpret the jump times T_k (which are not caused by ϑ) of $Z^{\vartheta}_{t,i}$ as the times at which warnings are issued to the investor that a bubble has formed in the market.

The investor specifies a strategy $\pi = (\pi^0, \ldots, \pi^d) = (\pi^0(u), \ldots, \pi^d(u))_{u\in[t,T]}$ where π^i denotes the fraction of wealth invested in the stock when the market is in state i. We assume for now that each π^i is adapted, right-continuous and bounded. Given a crash scenario $\vartheta = (\tau_k)_{k\in\mathbb{N}_0} \in \mathcal{T}(t,i)$ and a trading strategy $\pi = (\pi^0, \ldots, \pi^d)$, the investor's wealth process $X = X^{\pi,\vartheta}_{t,x,i} = (X^{\pi,\vartheta}_{t,x,i}(u))_{u\in[t,T]}$ is given by $X(t) = x$ at initial time,

$$dX(u) = \alpha_j \pi^j(u) X(u) du + \sigma_j \pi^j(u) X(u) dW(u), \qquad u \in [t,T], \tag{4.17}$$

on $\{Z^{\vartheta}_{t,i}(u) = j\} \cap \{u \neq \tau_k\}$,

$$X(\tau_k) = \begin{cases} X(\tau_k-), & \text{if } Z^{\vartheta}_{t,i}(\tau_k-) = j < i_{min}, \\ (1-\pi^j(\tau_k)\beta_j)X(\tau_k-), & \text{if } Z^{\vartheta}_{t,i}(\tau_k-) = j \geq i_{min}, \end{cases} \tag{4.18}$$

on $\{\tau_k < T_{k+1}\} \cap \{\tau_k \leq T\}$ and $X(\tau_k) = X(\tau_k-)$ on $\{\tau_k \geq T_{k+1}\} \cap \{\tau_k \leq T\}$. We denote by $\mathcal{A}(t,x)$ the set of all trading strategies which lead to a strictly positive wealth process $X^{\pi,\vartheta}_{t,x,i}$ for every $\vartheta \in \mathcal{T}(t,i)$.

The worst-case optimization problem in this model is given by

$$\mathcal{V}(t,x,i) := \sup_{\pi\in\mathcal{A}(t,x)} \inf_{\vartheta\in\mathcal{T}(t,i)} \mathbb{E}\left[U_p\left(X^{\pi,\vartheta}_{t,x,i}(T)\right)\right]. \tag{4.19}$$

Remark 4.3. The optimization problem (4.19) is to be understood as follows: The investor commits to a trading strategy $\pi \in \mathcal{A}(t,x)$ and only then does the market decide on the crash strategy $\vartheta \in \mathcal{T}(t,i)$. This prohibits the investor to set her risky fraction equal to zero at the moment a crash occurs, i.e. she cannot prevent being negatively affected by a crash. In particular, switching the supremum and the infimum in (4.19) leads to a different value. $\diamond$

Remark 4.4. Note that the case $d = 1$ corresponds to the situation considered in the previous sections if the market coefficients α_i and σ_i are independent of the state i. One can immediately generalize the situation by considering the case

$$0 = \beta_0 = \dots = \beta_{d-1}, \qquad \beta_d = \beta > 0,$$

and the state d is absorbing for Z. Then the time

$$S = \inf\{t \geq 0 : Z(t) = d\}$$

for Z started in 0 is of phase-type, see Asmussen [6, III.4]. Since the distributions of phase-type are dense in all probability distributions on $[0, \infty)$ (with respect to convergence in distribution) we can approximate arbitrary waiting-time distributions between a crash and the next warning. $\diamond$

Remark 4.5. Due to the monotonicity of the utility function U_p we can without loss of generality assume that $\mathcal{T}(t, i)$ contains only those crash strategies $\vartheta = (\tau_k)_{k \in \mathbb{N}_0}$ for which

$$X(\tau_k) \leq X(\tau_k -)$$

for every $k \in \mathbb{N}_0$. $\diamond$

4.6. The verification theorem

In this section we provide a system of dynamic programming equations for the generalized model which is inspired by the system of DPEs introduced in Korn and Steffensen [70], see (1.10). We then present a verification theorem which shows that under some technical assumptions any classical solution of the system of DPEs coincides with the value function. In Section 4.7 we then solve the system of DPEs and derive a coupled system of ordinary differential equations for the optimal strategies.

We fix $K > 0$ and let $\mathcal{A}_K(t, x)$ be the subset of all $\pi = (\pi^0, \dots, \pi^d) \in \mathcal{A}(t, x)$ such that each π^i takes values in $\mathcal{K} := [-K, K]$. We assume K to be large enough such that $\pi_M^i = \alpha_i/(1-p)\sigma_i^2 < K$ and $K > 1/\beta_i$ for all $i \in E$. We denote by

$$\mathcal{V}_K(t,x,i) := \sup_{\pi \in \mathcal{A}_K(t,x)} \inf_{\vartheta \in \mathcal{T}(t,i)} \mathbb{E}\left[U_p\left(X_{t,x,i}^{\pi,\vartheta}(T)\right)\right]$$

the value function which is restricted to admissible strategies taking values in $\mathcal{K}$. This restriction of the set of admissible strategies allows us to prove the following growth estimate.

Lemma 4.6. *Let $\pi \in \mathcal{A}_K(t,x)$ and $\vartheta \in \mathcal{T}(t,i)$. Then there exists a constant $C > 0$ such that*

$$\mathbb{E}\Big[\sup_{u\in[t,T]} |X_{t,x,i}^{\pi,\vartheta}(u)|^2\Big] \le C(1+|x|^2).$$

Proof. Denote by $\hat{\vartheta} \in \mathcal{T}(t,i)$ the no-crash scenario. Then the result for $\hat{\vartheta}$ is classical and follows e.g. from Pham [90, Theorem 1.3.15]. For an arbitrary $\vartheta \in \mathcal{T}(t,i)$ we note that

$$\mathbb{E}\Big[\sup_{u\in[t,T]} |X_{t,x,i}^{\pi,\vartheta}(u)|^2\Big] \le \mathbb{E}\Big[\sup_{u\in[t,T]} |X_{t,x,i}^{\pi,\hat{\vartheta}}(u)|^2\Big]$$

since the wealth decreases at the moment of crashes (see Remark 4.5). □

Assume for now that $\mathcal{V}(\cdot,\cdot,i) \in C^{1,2}([0,T)\times(0,\infty))$ for all $i \in E$. For each $(t,x) \in [0,T)\times(0,\infty)$ we can then define

$$\mathcal{K}_i'(t,x) := \Big\{\pi \in \mathcal{K} : \mathcal{L}_i^\pi \mathcal{V}(t,x,i) + \sum_{j=0}^{d} q_{i,j}\mathcal{V}(t,x,j) \ge 0\Big\},$$

$$\mathcal{K}_i''(t,x) := \Big\{\pi \in \mathcal{K} : \mathcal{V}(t,x,i) \le \mathcal{V}(t,(1-\pi\beta_i)x,0)\Big\},$$

where the operator $\mathcal{L}_i^\pi$ is given by

$$\mathcal{L}_i^\pi := \frac{\partial}{\partial t} + \alpha_i \pi x \frac{\partial}{\partial x} + \frac{1}{2}\sigma_i^2\pi^2x^2\frac{\partial^2}{\partial x^2} \qquad \text{for } i \in E.$$

In any state $i < i_{min}$ the investor does not have to fear the consequences of a possible crash so that she is essentially in the same situation as an investor in a regime switching model as described in Section 1.1.3. It is therefore reasonable to expect that the value function in this state solves

$$0 = \sup_{\pi \in \mathcal{K}} \left\{ \mathcal{L}_i^\pi \mathcal{V}(t,x,i) + \sum_{j=0}^{d} q_{i,j} \mathcal{V}(t,x,j) \right\}. \tag{4.20}$$

On the other hand, if $i \geq i_{\min}$ the investor does fear a crash and hence (up to the possibility of switching to a different state) we are in a situation very similar to Korn and Steffensen [70] and hence we expect that the value function in this state solves

$$0 = \min\left\{ \sup_{\pi \in \mathcal{K}_i''(t,x)} \left\{ \mathcal{L}_i^\pi \mathcal{V}(t,x,i) + \sum_{j=0}^{d} q_{i,j} \mathcal{V}(t,x,j) \right\}, \right. \\ \left. \sup_{\pi \in \mathcal{K}_i'(t,x)} \left\{ \mathcal{V}(t,(1-\pi\beta_i)x,0) - \mathcal{V}(t,x,i) \right\} \right\}. \tag{4.21}$$

This idea is formalized in the following verification theorem.

Theorem 4.7. *Let $V : [0,T] \times (0,\infty) \times E \to \mathbb{R}$ and assume that for each $i \in E$ we have $V(\cdot,\cdot,i) \in C^{1,2}([0,T) \times (0,\infty)) \cap C([0,T] \times (0,\infty))$.*

1. *Assume that for each $i = 0,\dots,i_{min}-1$ the function $V(\cdot,\cdot,i)$ satisfies* (4.20) *with terminal condition $V(T,x,i) = U_p(x)$ and for each $i = i_{min},\dots,d$ the function $V(\cdot,\cdot,i)$ satisfies* (4.21) *with terminal condition $V(T,x,i) = U_p(x)$.*

2. *Assume that V satisfies a quadratic growth condition in x, i.e. there exists a constant $C > 0$ independent of t and i such that*
$$|V(t,x,i)| \leq C\left(1 + |x|^2\right).$$

3. *Suppose moreover that for each $i = 0,\dots,i_{min}-1$ there exists a measurable function $\hat{\pi}^i : [0,T) \times (0,\infty) \to \mathcal{K}$ such that*
$$\hat{\pi}^i(t,x) = \arg\max_{\pi \in \mathcal{K}} \left\{ \mathcal{L}_i^\pi V(t,x,i) + \sum_{j=0}^{d} q_{i,j} V(t,x,j) \right\}$$
and that for each $i = i_{min},\dots,d$ there exists a measurable function $\hat{\pi}^i$: $[0,T) \times (0,\infty) \to \mathcal{K}$ such that
$$\hat{\pi}^i(t,x) = \arg\max_{\pi \in \mathcal{K}_i''(t,x)} \left\{ \mathcal{L}_i^\pi V(t,x,i) + \sum_{j=0}^{d} q_{i,j} V(t,x,j) \right\}.$$

Write $\hat{\pi} = (\hat{\pi}^1, \ldots, \hat{\pi}^d)$ *and suppose moreover that for each* $(t, x, i) \in [0, T) \times (0, \infty) \times E$ *and for every* $\vartheta \in \mathcal{T}(t, i)$ *the SDE* (4.17)-(4.18) *admits a solution* $X^{*,\vartheta} = X^{\pi^*,\vartheta}_{t,x,i}$ *under the trading strategy* $\pi^* := (\hat{\pi}(u, X^*(u-)))_{u\in[t,T]}$ *with* $\pi^*(T) = 0$*. Finally, assume that* $\pi^* \in \mathcal{A}_K(t, x)$*.*

4. *Given any* $(t, x, i) \in [0, T) \times (0, \infty) \times E$ *and* $\pi \in \mathcal{A}_K(t, x)$ *we suppose that we can iteratively define a crash strategy* $\vartheta^*(\pi) = (\tau^*_k)_{k\in\mathbb{N}_0} \in \mathcal{T}(t, i)$ *through*

$$\tau^*_k := \infty$$

on $\{T_k \le T\} \cap \{Z^{\vartheta^*(\pi)}_{t,x,i}(T_k) < i_{min}\}$ *and*

$$\tau^*_k := \inf\Big\{u \in [T_k, T_{k+1} \wedge T] : \\ V(u, X(u-), j) \ge V(u, (1 - \pi^j(u)\beta_j)X(u-), 0)\Big\}$$

on $\{T_k \le T\} \cap \{Z^{\vartheta^*(\pi)}_{t,x,i}(T_k) = j \ge i_{min}\}$*.*

Then $V(t, x, i) = \mathcal{V}_K(t, x, i)$*, the strategy* π^* *is optimal and the corresponding optimal crash strategy is* $\vartheta^*(\pi^*)$*.*

Proof. Step 1: Fix $(t, x, i) \in [0, T) \times (0, \infty) \times E$, let θ be any $[t, T]$-valued stopping time, fix $\pi = (\pi^1, \ldots, \pi^d) \in \mathcal{A}_K(t, x)$ and let $\vartheta = (\tau_k)_{k\in\mathbb{N}_0} \in \mathcal{T}(t, i)$. Write $X = X^{\pi,\vartheta}_{t,x,i}$, $Z = Z^{\vartheta}_{t,i}$ and $\theta_k = \theta \wedge \tau_k$ for short. Then Itô's formula shows that for each $k \in \mathbb{N}_0$ we have

$$\begin{aligned}
&V(\theta_k, X(\theta_k), Z(\theta_k)) \\
&= V(\theta_{k+1}-, X(\theta_{k+1}-), Z(\theta_{k+1}-)) \\
&- \sum_{j=0}^{d} \int_{\theta_k}^{\theta_{k+1}-} \Big[\mathcal{L}^{\pi^j(u)}_j V(u, X(u), j) \\
&\qquad\qquad + \sum_{l=0}^{d} q_{j,l} V(u, X(u), l)\Big] \mathbb{1}_{\{Z(u-)=j\}}\, du \\
&- \sum_{j=0}^{d} \int_{\theta_k}^{\theta_{k+1}-} \sigma_j \pi^j(u) X(u) \frac{\partial}{\partial x} V(u, X(u), j) \mathbb{1}_{\{Z(u-)=j\}}\, dW(u) \\
&- \sum_{j=0}^{d} \sum_{l=0}^{d} \int_{\theta_k}^{\theta_{k+1}-} \Big[V(u, X(u), l)
\end{aligned}$$

$$- V(u, X(u), j)\Big] \mathbb{1}_{\{Z(u-)=j\}} \, \nu_k(du, l), \qquad (4.22)$$

where ν_k denotes the compensated jump measure of the uncontrolled process Z started in state 0 at time τ_k.

Step 2: Consider the strategy $\pi^* \in \mathcal{A}_K(t,x)$ together with an arbitrary $\vartheta \in \mathcal{T}(t,i)$. We write $X^{*,\vartheta} = X^{\pi^*,\vartheta}$. Since $\hat{\pi}^j \in \mathcal{K}''_j(u,y)$ for each $(u,y,j) \in [t,T) \times (0,\infty) \times E$ with $j \geq i_{min}$ this implies that for any $k \in \mathbb{N}_0$ and any $j \in E$ with $j \geq i_{min}$ we have on $\{\tau_k \leq \theta\} \cap \{Z(\tau_k-) = j\}$

$$\begin{aligned} V(\tau_k, X^{*,\vartheta}(\tau_k), Z(\tau_k)) &= V\big(\tau_k, (1 - \pi^{j,*}(\tau_k)\beta_j) X^{*,\vartheta}(\tau_k-), 0\big) \\ &\geq V\big(\tau_k, X^{*,\vartheta}(\tau_k-), Z(\tau_k-)\big) \end{aligned}$$

Using this in (4.22) and then iteratively applying (4.22) hence shows that for each $N \in \mathbb{N}$ we have

$$\begin{aligned} &V(t,x,i) \\ &\leq V(\theta_N-, X^{*,\vartheta}(\theta_N-), Z(\theta_N-)) \\ &- \sum_{k=0}^{N} \sum_{j=0}^{d} \int_{\theta_{k-1}}^{\theta_k-} \Big[\mathcal{L}_j^{\pi^j(u)} V(u, X^{*,\vartheta}(u), j) \\ &\qquad\qquad + \sum_{l=0}^{d} q_{j,l} V(u, X^{*,\vartheta}(u), l) \Big] \mathbb{1}_{\{Z(u-)=j\}} \, du \\ &- \sum_{k=0}^{N} \sum_{j=0}^{d} \int_{\theta_{k-1}}^{\theta_k-} \sigma_j \pi^j(u) X^{*,\vartheta}(u) \frac{\partial}{\partial x} V(u, X^{*,\vartheta}(u), j) \mathbb{1}_{\{Z(u-)=j\}} \, dW(u) \\ &- \sum_{k=0}^{N} \sum_{j=0}^{d} \sum_{l=0}^{d} \int_{\theta_{k-1}}^{\theta_k-} \Big[V(u, X^{*,\vartheta}(u), l) \\ &\qquad\qquad - V(u, X^{*,\vartheta}(u), j) \Big] \mathbb{1}_{\{Z(u-)=j\}} \, \nu_k(du, l), \end{aligned} \qquad (4.23)$$

where we set $\tau_{-1} := t$. Now since for $j = 0, \ldots, i_{min} - 1$ the function $\hat{\pi}^j$ is a pointwise maximizer of

$$\sup_{\pi \in \mathcal{K}} \Big\{ \mathcal{L}_j^{\pi} V(t,x,j) + \sum_{l=0}^{d} q_{j,l} V(t,x,l) \Big\} \geq 0$$

and since for $j = i_{min}, \dots, d$ the function $\hat{\pi}^j$ is a pointwise maximizer of

$$\sup_{\pi \in \mathcal{K}''_j(t,x)} \left\{ \mathcal{L}_j^\pi V(t,x,j) + \sum_{l=0}^{d} q_{j,l} V(t,x,l) \right\} \geq 0,$$

we can estimate the first integral in (4.23) to obtain

$$\begin{aligned}
&V(t,x,i) \\
&\leq V(\theta_N -, X^{*,\vartheta}(\theta_N -), Z(\theta_N -)) \\
&- \sum_{k=0}^{N} \sum_{j=0}^{d} \int_{\theta_{k-1}}^{\theta_k -} \sigma_j \pi^j(u) X^{*,\vartheta}(u) \frac{\partial}{\partial x} V(u, X^{*,\vartheta}(u), j) \mathbb{1}_{\{Z(u-)=j\}} \, dW(u) \\
&- \sum_{k=0}^{N} \sum_{j=0}^{d} \sum_{l=0}^{d} \int_{\theta_{k-1}}^{\theta_k -} \Big[V(u, X^{*,\vartheta}(u), l) \\
&\qquad\qquad - V(u, X^{*,\vartheta}(u), j) \Big] \mathbb{1}_{\{Z(u-)=j\}} \, \nu_k(du, l).
\end{aligned}$$

Now send $N \to \infty$ to obtain

$$\begin{aligned}
&V(t,x,i) \\
&\leq V(\theta -, X^{*,\vartheta}(\theta -), Z(\theta -)) \\
&- \sum_{k=0}^{\infty} \sum_{j=0}^{d} \int_{\theta_{k-1}}^{\theta_k -} \sigma_j \pi^j(u) X^{*,\vartheta}(u) \frac{\partial}{\partial x} V(u, X^{*,\vartheta}(u), j) \mathbb{1}_{\{Z(u-)=j\}} \, dW(u) \\
&- \sum_{k=0}^{\infty} \sum_{j=0}^{d} \sum_{l=0}^{d} \int_{\theta_{k-1}}^{\theta_k -} \Big[V(u, X^{*,\vartheta}(u), l) \\
&\qquad\qquad - V(u, X^{*,\vartheta}(u), j) \Big] \mathbb{1}_{\{Z(u-)=j\}} \, \nu_k(du, l).
\end{aligned} \tag{4.24}$$

Step 3: Note that for any $\pi \in \mathcal{A}_K(t,x)$ and $\vartheta \in \mathcal{T}(t,i)$ the last integral in (4.22) is a martingale. Indeed, by Brémaud [16, Excercise I.E2] we only need to show that

$$\mathbb{E}\left[\sum_{k=0}^{\infty} \sum_{j=0}^{d} \sum_{l=0}^{d} \int_{\theta_{k-1}}^{\theta_k -} \Big| V(u, X(u), l) - V(u, X(u), j) \Big| \, du \right] < +\infty.$$

By the growth condition on V we have for each $j, l \in E$

$$\mathbb{E}\left[\sum_{k=0}^{\infty} \int_{\theta_{k-1}}^{\theta_k -} \Big| V(u, X(u), l) - V(u, X(u), j) \Big| \right] \leq 2CT \left(1 + \mathbb{E}\left[\sup_{u \in [t,T]} |X(u)|^2 \right] \right)$$

which is finite by Lemma 4.6.

Let $n \in \mathbb{N}$ and define θ^n to be the minimum of T and the first time u after t such that

$$\sum_{j=0}^{d} \int_t^u \left| \sigma_j \pi^j(r) X^{*,\vartheta}(r) \frac{\partial}{\partial x} V(r, X^{*,\vartheta}(r), j) \mathbb{1}_{\{Z(r-)=j\}} \right|^2 dr$$

exceeds n. Now replace θ by θ^n in (4.24) and take expectations to obtain

$$V(t,x,i) \leq \mathbb{E}\left[V(\theta^n-, X^{*,\vartheta}(\theta^n-), Z(\theta^n-)) \right]. \tag{4.25}$$

Note that for each $n \in \mathbb{N}$ we have

$$\mathbb{E}\left[|V(\theta^n-, X^{*,\vartheta}(\theta^n-), Z(\theta^n-))| \right] \leq C\left(1 + \mathbb{E}\left[\sup_{u\in[t,T]} |X^{*,\vartheta}(u)|^2 \right] \right) < +\infty$$

by the growth condition on V and by Lemma 4.6. Hence we can send $n \to \infty$ in (4.25) and use dominated convergence to obtain

$$V(t,x,i) \leq \mathbb{E}\left[V(T-, X^{*,\vartheta}(T-), Z(T-)) \right].$$

Since V is continuous, satisfies $V(T,x,\cdot) = U_p(x)$ and since $\pi^*(T) = 0$ it follows that

$$V(t,x,i) \leq \mathbb{E}\left[U_p\left(X^{*,\vartheta}(T) \right) \right].$$

Since ϑ was chosen arbitrarily this implies that

$$V(t,x,i) \leq \inf_{\vartheta \in \mathcal{T}(t,i)} \mathbb{E}\left[U_p\left(X^{*,\vartheta}(T) \right) \right] \leq \mathcal{V}_K(t,x,i). \tag{4.26}$$

Step 4: Consider an arbitrary $\pi \in \mathcal{A}_K(t,x)$ together with the associated crash strategy $\vartheta^*(\pi) \in \mathcal{T}(t,i)$. Write $X = X^{\pi,\vartheta^*(\pi)}$ and $\theta_k^* = \theta \wedge \tau_k^*$. Now for each $k \in \mathbb{N}_0$ and each $j = 0, \ldots, i_{min} - 1$ we have

$$\left[\mathcal{L}_j^{\pi(u)} V(u, X(u), j) + \sum_{l=0}^{d} q_{j,l} V(u, X(u), l) \right] \mathbb{1}_{\{Z(u-)=j\}} \leq 0 \tag{4.27}$$

for all $u \in [t,T)$. Moreover, for each $k \in \mathbb{N}_0$ and each $j = i_{min}, \ldots, d$ we have for each $u \in [\tau_k^*, \tau_{k+1}^*)$ on $\{\tau_{k+1}^* \leq T\} \cap \{Z(u-) = j\}$

$$V(u, X(u-), Z(u-)) < V(u, (1 - \pi^j(u)\beta_j) X(u-), 0) \tag{4.28}$$

by the construction of $\vartheta^*(\pi)$. We must now distinguish two situations: Either we are in case (a) in which

$$\Big[\mathcal{L}_j^{\pi^j(u)} V(u, X(u), j) + \sum_{l=0}^{d} q_{j,l} V(u, X(u), l)\Big] \mathbb{1}_{\{Z(u-)=j\}} < 0$$

or we are in case (b) in which

$$\Big[\mathcal{L}_j^{\pi^j(u)} V(u, X(u), j) + \sum_{l=0}^{d} q_{j,l} V(u, X(u), l)\Big] \mathbb{1}_{\{Z(u-)=j\}} \geq 0.$$

The latter case implies in particular that $\pi^j(u) \in \mathcal{K}'_j(u, X(u-))$ and hence (4.28) shows that

$$\sup_{\pi \in \mathcal{K}'_j(u,X(u-))} \Big[V(u, (1-\pi\beta_j)X(u-), 0) - V(u, X(u-), j)\Big] > 0.$$

Since V solves the system of DPEs and since $j \geq i_{min}$ we therefore have

$$\sup_{\pi \in \mathcal{K}''_j(u,X(u-))} \Big[\mathcal{L}_j^{\pi} V(u, X(u), j) + \sum_{l=0}^{d} q_{j,l} V(u, X(u), l)\Big] = 0.$$

But since by (4.28) we see that $\pi^j(u) \in \mathcal{K}''_j(u, X(u-))$ we conclude that

$$\Big[\mathcal{L}_j^{\pi^j(u)} V(u, X(u), j) + \sum_{l=0}^{d} q_{j,l} V(u, X(u), l)\Big] \mathbb{1}_{\{Z(u-)=j\}} = 0$$

in case (b). Combining case (a) and case (b) hence implies that

$$\Big[\mathcal{L}_j^{\pi^j(u)} V(u, X(u), j) + \sum_{l=0}^{d} q_{j,l} V(u, X(u), l)\Big] \mathbb{1}_{\{Z(u-)=j\}} \leq 0$$

on $\{\tau^*_{k+1} \leq T\}$. Using this with (4.27) in (4.22) hence shows that

$$\begin{aligned} & V(\theta^*_k, X(\theta^*_k), Z(\theta^*_k)) \\ & \geq V(\theta^*_{k+1}-, X(\theta^*_{k+1}-), Z(\theta^*_{k+1}-)) \\ & - \sum_{j=0}^{d} \int_{\theta^*_k}^{\theta^*_{k+1}-} \sigma_j \pi^j(u) X(u) \frac{\partial}{\partial x} V(u, X(u), j) \mathbb{1}_{\{Z(u-)=j\}} \, dW(u) \end{aligned}$$

$$- \sum_{j=0}^{d} \sum_{l=0}^{d} \int_{\theta_k^*}^{\theta_{k+1}^* -} \Big[V(u, X(u), l) - V(u, X(u), j) \Big] \mathbb{1}_{\{Z(u-)=j\}} \, \nu_k(du, l), \quad (4.29)$$

for each $k \in \mathbb{N}_0$. Moreover, by the construction of $\vartheta^*(\pi)$ and the right-continuity of π we have for every $k \in \mathbb{N}_0$ on $\{\tau_k^* \le \theta\} \cap \{Z(\tau_k^*-) \ge i_{min}\}$:

$$\begin{aligned} V(\tau_k^*, X(\tau_k^*), Z(\tau_k^*)) &= V\big(\tau_k^*, (1 - \pi^j(\tau_k^*)\beta_j) X(\tau_k^*-), 0\big) \\ &\le V\big(\tau_k^*, X(\tau_k^*-), Z(\tau_k^*-)\big). \end{aligned}$$

Using this in (4.29) and then inductively applying (4.29) hence shows that

$$\begin{aligned} &V(t, x, i) \\ &\ge V(\theta-, X(\theta-), Z(\theta-)) \\ &- \sum_{k=0}^{\infty} \sum_{j=0}^{d} \int_{\theta_{k-1}^*}^{\theta_k^* -} \sigma_j \pi^j(u) X(u) \frac{\partial}{\partial x} V(u, X(u), j) \mathbb{1}_{\{Z(u-)=j\}} \, dW(u) \\ &- \sum_{k=0}^{\infty} \sum_{j=0}^{d} \sum_{l=0}^{d} \int_{\theta_{k-1}^*}^{\theta_k^* -} \Big[V(u, X(u), l) \\ &\qquad\qquad - V(u, X(u), j) \Big] \mathbb{1}_{\{Z(u-)=j\}} \, \nu_k(du, l). \end{aligned} \quad (4.30)$$

Step 5: Define a sequence of stopping times $(\theta^n)_{n \in \mathbb{N}}$ as in step 3, but with (π^*, ϑ) replaced by $(\pi, \vartheta^*(\pi))$. Taking expectations in (4.30) hence shows that

$$V(t, x, i) \ge \mathbb{E}\left[V(\theta^n-, X(\theta^n-), Z(\theta^n-)) \right]. \quad (4.31)$$

Sending $n \to \infty$ we conclude by dominated convergence that

$$V(t, x, i) \ge \mathbb{E}\left[V(T-, X(T-), Z(T-)) \right].$$

By the definition of $\vartheta^*(\pi)$ it follows moreover that

$$\begin{aligned} \mathbb{E}\left[V(T-, X(T-), Z(T-)) \right] &\ge \mathbb{E}\left[V(T, X(T), Z(T)) \right] \\ &= \mathbb{E}\left[U_p\left(X(T) \right) \right] \\ &\ge \inf_{\vartheta \in \mathcal{T}(t,i)} \mathbb{E}\left[U_p\left(X^{\pi,\vartheta}(T) \right) \right] \end{aligned} \quad (4.32)$$

and since π was chosen arbitrarily this implies

$$V(t, x, i) \ge \mathcal{V}_K(t, x, i).$$

Hence $V(t,x,i) = \mathcal{V}_K(t,x,i)$ by (4.26). Replacing V by $\mathcal{V}_K$ on the right-hand side of (4.26) hence shows that

$$\mathcal{V}_K(t,x,i) \leq \inf_{\vartheta \in \mathcal{T}(t,i)} \mathbb{E}\left[U_p\left(X^{*,\vartheta}(T)\right)\right]$$

which proves the optimality of π^*. The optimality of $\vartheta^*(\pi^*)$ follows similarly by using $V(t,x,i) = \mathcal{V}_K(t,x,i)$ and the optimality of π^* together with (4.31) and (4.32). □

Theorem 4.7 is tailor-made for the case $p \in (0,1)$. For $p \leq 0$ the corresponding V does not satisfy the quadratic growth condition used in steps 3 and 5 of the proof. We will return to this problem in Section 4.7.3 after solving the system of DPEs since it is easier to verify optimality if we have a specific candidate at hand.

Remark 4.8. The verification theorem can be extended to the case in which the strategies $\pi \in \mathcal{A}(t,x)$ are merely assumed to be predictable instead of right-continuous. In the proof of Theorem 4.7 the right-continuity of π is only needed to ensure that

$$V(\tau_k^*, X(\tau_k^*), 0) \leq V\big(\tau_k^*, X(\tau_k^*-), j\big) \tag{4.33}$$

on $\{T_k \leq T\} \cap \{Z_{t,x,i}^{\vartheta^*(\pi)}(T_k) = j \geq i_{min}\}$ for all $k \in \mathbb{N}_0$. If π is only assumed to be predictable then clearly (4.33) need not hold. Nevertheless, it is possible to show that by the predictable section theorem (Rogers and Williams [93, Theorem VI.19.1]) we can find stopping times τ_k^l, $l \in \mathbb{N}$, such that

$$\tau_k^* < \tau_k^l \leq \tau_k^* + \frac{1}{l} \text{ and } V(\tau_k^l, X(\tau_k^l), 0) \leq V\big(\tau_k^l, X(\tau_k^l-), j\big) \text{ on } \{\tau_k^l < \infty\}$$

and $\mathbb{P}[\tau_k^l = \infty] \leq 2^{-k}$. Moreover, note that

$$\mathbb{P}[\tau_k^l \geq T_{k+1} | \tau_k^* < T_{k+1}] \leq 1 - e^{-\max_j\{\lambda_j\}/l}.$$

Similarly to the reasoning in Step 5 of the proof of Theorem 4.7 one can then show that

$$\begin{aligned} V(t,x,i) - \mathbb{E}\left[V(T-, X(T-), Z(T-))\right] \\ \geq -\mathbb{E}\Bigg[\sum_{k=0}^{\infty}\Big| V\big(\tau_k^l, X(\tau_k^l-), Z(\tau_k^l-)\big) \\ - V\big(\tau_k^l, X(\tau_k^l), Z(\tau_k^l)\big)\Big| \mathbb{1}_{\{\tau_k^l=\infty\}\cup\{\tau_k^l \geq T_{k+1}\}}\Bigg]. \end{aligned}$$

By the growth condition on V, the boundedness of π and Lemma 4.6 one can further argue that there exists a constant $L > 0$ such that

$$\mathbb{E}\Bigg[\sum_{k=0}^{\infty}\Big|V\big(\tau_k^l, X(\tau_k^l-), Z(\tau_k^l-)\big) - V\big(\tau_k^l, X(\tau_k^l), Z(\tau_k^l)\big)\Big|\mathbb{1}_{\{\tau_k^l=\infty\}\cup\{\tau_k^l\geq T_{k+1}\}}\Bigg] \leq L\left(1 - e^{-\max_j\{\lambda_j\}/l}\right).$$

Since l is arbitrary it follows that

$$V(t,x,i) \geq \mathbb{E}\left[V(T-, X(T-), Z(T-))\right]$$

from where one can conclude as before. ◇

4.7. Derivation of the optimal strategies

Let us now apply Theorem 4.7 to find the value function and determine the optimal strategies. We start with the power utility case $p < 1, p \neq 0$.

4.7.1. Solution of the system of DPEs for power utility

We expect that V takes the form

$$V(t,x,i) = \frac{1}{p}x^p f_i(t), \qquad i \in E. \tag{4.34}$$

Moreover, we assume that f_i is strictly positive on $[0,T]$ for every $i \in \mathbb{E}$. Note that we must have $f_i(T) = 1$ for all $i \in E$. Our first aim is to solve

$$0 = \sup_{\pi\in\mathcal{K}}\left\{\mathcal{L}_i^{\pi}V(t,x,i) + \sum_{j=0}^{d} q_{i,j}V(t,x,j)\right\}$$

for $i < i_{min}$ in order to find $V(t,x,i)$ and $\pi^{i,*}$. Using (4.34) this equation simplifies to

$$0 = \sup_{\pi \in \mathcal{K}} \left[\frac{1}{p} \frac{\partial}{\partial t} f_i(t) + \left(\alpha_i \pi - \frac{1}{2}(1-p)\sigma_i^2 \pi^2 \right) f_i(t) + \frac{1}{p} \sum_{j=0}^{d} q_{i,j} f_j(t) \right].$$

Formally optimizing with respect to π gives the candidate optimal strategy

$$\pi^{i,*}(t) = \frac{\alpha_i}{(1-p)\sigma_i^2} = \pi_M^i$$

which is indeed the maximum if $f_i(t) > 0$ for all $t \in [0,T]$. Plugging the candidate optimal strategy $\pi^{i,*}$ back into the DPE yields the following ODE for f_i:

$$\frac{\partial}{\partial t} f_i(t) = -\frac{1}{2} p \frac{\alpha_i^2}{(1-p)\sigma_i^2} f_i(t) - \sum_{j=0}^{d} q_{i,j} f_j(t). \tag{4.35}$$

Let us now consider the case $i \geq i_{min}$ such that $\beta_i > 0$. We have to solve

$$0 = \min \Bigg\{ \sup_{\pi \in \mathcal{K}_i''(t,x)} \left\{ \mathcal{L}_i^\pi V(t,x,i) + \sum_{j=0}^{d} q_{i,j} V(t,x,j) \right\}, \\ \sup_{\pi \in \mathcal{K}_i'(t,x)} \Big\{ V(t,(1-\pi\beta_i)x,0) - V(t,x,i) \Big\} \Bigg\}.$$

With (4.34) the first equation reduces to

$$0 \leq \sup_{\pi \in \tilde{\mathcal{K}}_i''(t)} \left\{ \frac{1}{p} \frac{\partial}{\partial t} f_i(t) + \left(\alpha_i \pi - \frac{1}{2}(1-p)\sigma_i^2 \pi^2 \right) f_i(t) + \frac{1}{p} \sum_{j=0}^{d} q_{i,j} f_j(t) \right\}$$

where

$$\tilde{\mathcal{K}}_i''(t) := \left\{ \pi \in \mathcal{K} : \frac{1}{p}(1-\pi\beta_i)^p f_0(t) - \frac{1}{p} f_i(t) \geq 0 \right\}.$$

Similarly, the second equation reduces to

$$0 \leq \sup_{\pi \in \tilde{\mathcal{K}}_i'(t)} \left\{ \frac{1}{p}(1-\pi\beta_i)^p f_0(t) - \frac{1}{p} f_i(t) \right\} \tag{4.36}$$

where

$$\tilde{\mathcal{K}}_i'(t) := \Big\{\pi \in \mathcal{K} : \frac{1}{p}\frac{\partial}{\partial t} f_i(t) + \Big(\alpha_i \pi - \frac{1}{2}(1-p)\sigma_i^2\pi^2\Big) f_i(t) + \frac{1}{p}\sum_{j=0}^{d} q_{i,j} f_j(t) \geq 0\Big\}.$$

Let us first consider (4.36). Since f_0 is assumed to be strictly positive and since $(1-\pi\beta_i)^p/p$ is a decreasing function of π, the supremum in (4.36) is attained for the smallest value of π which satisfies the constraint in $\tilde{\mathcal{K}}_i'(t)$, i.e.

$$\frac{1}{p}\frac{\partial}{\partial t} f_i(t) + \Big(\alpha_i \pi - \frac{1}{2}(1-p)\sigma_i^2\pi^2\Big) f_i(t) + \frac{1}{p}\sum_{j=0}^{d} q_{i,j} f_j(t) \geq 0. \tag{4.37}$$

Note that this equation is a quadratic and concave function of π tending to $-\infty$ as $|\pi| \to \infty$. We must therefore have that the supremum in (4.36) is attained for the smallest value of π which satisfies the constraint (4.37) with equality. If the right-hand side of (4.36) is equal to zero we therefore have that $\pi^{i,*}$ and f_i are determined by

$$\begin{aligned} f_i(t) &= (1-\pi^{i,*}(t)\beta_i)^p f_0(t), \\ \frac{\partial}{\partial t} f_i(t) &= -p\Big(\alpha_i\pi^{i,*}(t) - \frac{1}{2}(1-p)\sigma_i^2[\pi^{i,*}(t)]^2\Big) f_i(t) - \sum_{j=0}^{d} q_{i,j} f_j(t). \end{aligned}$$

If the supremum in (4.36) is strictly positive then the complementarity of the two equations in the DPE shows that

$$0 = \sup_{\pi \in \tilde{\mathcal{K}}_i''(t)} \Big\{\frac{1}{p}\frac{\partial}{\partial t} f_i(t) + \Big(\alpha_i \pi - \frac{1}{2}(1-p)\sigma_i^2\pi^2\Big) f_i(t) + \frac{1}{p}\sum_{j=0}^{d} q_{i,j} f_j(t)\Big\}. \tag{4.38}$$

Formally optimizing with respect to π in this equation yields

$$\pi^{i,*}(t) = \frac{\alpha_i}{(1-p)\sigma_i^2} = \pi_M^i.$$

If $\pi_M^i \in \tilde{\mathcal{K}}_i''(t)$ then it is indeed a maximizer of (4.38). Otherwise we have

$$\frac{1}{p}(1-\pi_M^i\beta_i)^p f_0(t) < \frac{1}{p} f_i(t).$$

Since the left-hand side of this equation is decreasing as a function of π and since $\alpha_i\pi - \frac{1}{2}(1-p)\sigma_i^2\pi^2$ is an increasing function of π on $(-\infty, \pi_M^i)$ it follows that if $\pi_M^i \notin \tilde{\mathcal{K}}_i''(t)$ then the supremum in (4.38) is attained for $\pi^{i,*}(t) < \pi_M^i$ which satisfies

$$\frac{1}{p}(1-\pi^{i,*}(t)\beta_i)^p f_0(t) = \frac{1}{p} f_i(t).$$

We have therefore argued that $[0,T)$ can be decomposed into the set $\mathcal{I}_i$ on which $\pi^{i,*}$ and f_i are determined by

$$\begin{aligned} f_i(t) &= (1-\pi^{i,*}(t)\beta_i)^p f_0(t), \\ \frac{\partial}{\partial t} f_i(t) &= -p\Big(\alpha_i \pi^{i,*}(t) - \frac{1}{2}(1-p)\sigma_i^2[\pi^{i,*}(t)]^2\Big) f_i(t) - \sum_{j=0}^{d} q_{i,j} f_j(t). \end{aligned} \tag{4.39}$$

and the set $\mathcal{N}_i$ on which $\pi^{i,*}$ and f_i are determined by

$$\begin{aligned} \pi^{i,*}(t) &= \pi_M^i, \\ \frac{\partial}{\partial t} f_i(t) &= -p\Big(\alpha_i \pi^{i,*}(t) - \frac{1}{2}(1-p)\sigma_i^2[\pi^{i,*}(t)]^2\Big) f_i(t) - \sum_{j=0}^{d} q_{i,j} f_j(t). \end{aligned}$$

Moreover, note that on $\mathcal{I}_i$ we have $\pi^{i,*} < \pi_M^i$ and by solving

$$f_i(t) = (1-\pi^{i,*}(t)\beta_i)^p f_0(t)$$

for $\pi^{i,*}$ we can rewrite the differential equation for f_i as

$$\begin{aligned} \frac{\partial}{\partial t} f_i(t) = -p\frac{\alpha_i}{\beta_i}&\left(1 - \left[\frac{f_i(t)}{f_0(t)}\right]^{1/p}\right) f_i(t) \\ &+ \frac{1}{2}p(1-p)\frac{\sigma_i^2}{\beta_i^2}\left(1 - \left[\frac{f_i(t)}{f_0(t)}\right]^{1/p}\right)^2 f_i(t) - \sum_{j=0}^{d} q_{i,j} f_j(t). \end{aligned}$$

The two differential equations for f_i on $\mathcal{I}_i$ and $\mathcal{N}_i$ can hence by combined to

$$\begin{aligned} \frac{\partial}{\partial t} f_i(t) = -p\alpha_i \min&\left\{\frac{1}{\beta_i}\left(1 - \left[\frac{f_i(t)}{f_0(t)}\right]^{1/p}\right), \pi_M^i\right\} f_i(t) \\ &+ \frac{1}{2}p(1-p)\sigma_i^2\left[\min\left\{\frac{1}{\beta_i}\left(1 - \left[\frac{f_i(t)}{f_0(t)}\right]^{1/p}\right), \pi_M^i\right\}\right]^2 f_i(t) \end{aligned}$$

$$- \sum_{j=0}^{d} q_{i,j} f_j(t). \quad (4.40)$$

We are left with showing that we can solve the system of differential equations for $(f_i)_{i \in E}$ and that $f_i(t) > 0$ for all $(t, i) \in [0, T] \times E$. It then follows that V is indeed a solution of the system of DPEs and that $\pi^{i,*}$ is the candidate optimal strategy.

Lemma 4.9. *The system of ODEs given by* (4.35) *for* $i = 0, \ldots, i_{min} - 1$ *and by* (4.40) *for* $i = i_{min}, \ldots, d$ *with terminal condition* $f_i(T) = 1$ *for all* $i \in E$ *possesses a unique solution on* $[0, T]$*. Moreover, this solution is strictly positive.*

Proof. Note that (4.35) is globally Lipschitz continuous and (4.40) is locally Lipschitz continuous in f_i. Hence, by the Picard-Lindelöf theorem there exists a unique local solution of the system of differential equations. In order to show that there exists a strictly positive solution on $[0, T]$ it hence suffices to show that each f_i is strictly positive on $[0, T]$ and f_i does not explode on $[0, T]$. We only consider the case $p \in (0, 1)$, the case $p < 0$ can be handled similarly.

Let therefore $p \in (0, 1)$. Define

$$g_i(y) = \alpha_i y - \frac{1}{2}(1 - p)\sigma_i^2 y^2$$

and note that g_i attains its maximum at π_M^i. We let

$$M = \max_{i \in E} g_i(\pi_M^i) > 0 \qquad \text{and} \qquad \bar{\lambda} = \max_{i \in E} \lambda_i.$$

Now for any $i \in E$ and any $t \in [0, T]$ with $f_i(t) > 0$ we have

$$\begin{aligned}
\frac{\partial}{\partial t} f_i(t) &= -p g_i(\pi^{i,*}(t)) f_i(t) - \sum_{j=0}^{d} q_{i,j} f_j(t) \\
&\geq -pM f_i(t) - q_{i,i} f_i(t) - \sum_{\substack{j=0 \\ j \neq i}}^{d} q_{i,j} f_j(t) \\
&\geq -pM f_i(t) - \max_{j \in E}\{f_j(t)\} \sum_{\substack{j=0 \\ j \neq i}}^{d} q_{i,j}
\end{aligned}$$

$$\geq -[pM + \bar{\lambda}] \max_{j \in E}\{f_j(t)\}.$$

Hence Gronwall's inequality shows that

$$f_i(t) \leq f_i(T) + \int_t^T (pM + \bar{\lambda}) \max_{j \in E}\{f_j(u)\}\, du$$

for every $i \in E$ and therefore

$$\max_{j \in E}\{f_j(t)\} \leq 1 + \int_t^T (pM + \bar{\lambda}) \max_{j \in E}\{f_j(u)\}\, du.$$

Applying Gronwall's inequality again hence shows that

$$\max_{j \in E}\{f_j(t)\} \leq e^{(pM+\bar{\lambda})(T-t)}. \tag{4.41}$$

For each $i = 0, \dots, i_{min} - 1$ we furthermore have

$$\begin{aligned} \frac{\partial}{\partial t} f_i(t) &= -\frac{1}{2} p \frac{\alpha_i}{(1-p)\sigma_i^2} f_i(t) - \sum_{j=0}^{d} q_{i,j} f_j(t) \\ &\leq \bar{\lambda} f_i(t) \end{aligned} \tag{4.42}$$

as long as $f_j(t) > 0$ for all $j \in E$.

Let us now assume that there exists $t_0 \in [0, T)$ such that

$$\lim_{t \downarrow t_0} f_k(t) = 0$$

for some $k \in E$ and that $f_j(t) > 0$ for all $t \in (t_0, T]$ and all $j \in E$ (Note that $t_0 < T$ is clear from the terminal condition on f_i). It follows from (4.41) and (4.42) that

$$\begin{aligned} f_j(t) &\leq e^{(pM+\bar{\lambda})(T-t)} && \text{for all } (t, j) \in [t_0, T] \times E, \\ f_j(t) &\geq e^{-\bar{\lambda}(T-t)} && \text{for all } (t, j) \in [t_0, T] \times E \text{ with } j < i_{min}. \end{aligned}$$

This implies in particular that $k \geq i_{min}$. Moreover, we have

$$0 \leq \frac{f_k(t)}{f_0(t)} \leq e^{(pM+2\bar{\lambda})(T-t)}, \qquad t \in [t_0, T],$$

and hence

$$\frac{1}{\beta_k}\left(1-e^{\frac{1}{p}(pM+2\bar{\lambda})(T-t)}\right) \leq \frac{1}{\beta_k}\left(1-\left[\frac{f_k(t)}{f_0(t)}\right]^{1/p}\right) \leq \frac{1}{\beta_k}.$$

Define

$$L := \frac{1}{\beta}\left(1-e^{\frac{1}{p}(pM+2\bar{\lambda})T}\right).$$

Since $\pi^{k,*}(t) \leq \pi_M^k$ and since g_k is increasing on $(-\infty, \pi_M^k)$ it follows that

$$g_k(L) \leq g_k(\pi^{k,*}(t)), \qquad t \in [t_0, T].$$

Therefore

$$\begin{aligned}\frac{\partial}{\partial t} f_k(t) &= -p g_k(\pi^{k,*}(t)) f_k(t) - \sum_{j=0}^{d} q_{k,j} f_j(t) \\ &\leq [-p \min\{g_k(L), 0\} + \bar{\lambda}] f_k(t) \end{aligned} \tag{4.43}$$

for every $t \in [t_0, T]$ which shows that

$$f_k(t) \geq e^{(-p\min\{g_k(L),0\}+\bar{\lambda})(T-t)}, \qquad t \in [t_0, T],$$

in contradiction to

$$\lim_{t \downarrow t_0} f_k(t) = 0.$$

Combining this with (4.41) hence shows that $f_i > 0$ on $[0, T]$ and f_i is non-exploding for each $i \in E$. □

The next step is to check if the candidate optimal strategy $\pi^* = (\pi^{0,*}, \ldots, \pi^{d,*})$ is admissible. For every $i \in E$ with $i \geq i_{min}$ we can write

$$\pi^{i,*}(t) = \min\{\pi_M^i, \pi^{i,\text{ind}}(t)\},$$

where $\pi^{i,\text{ind}}$ is given by

$$f_i(t) = (1 - \pi^{i,\text{ind}}(t)\beta_i)^p f_0(t). \tag{4.44}$$

Taking the logarithm and then the derivative with respect to t we arrive at the following differential equation for $\pi^{i,\text{ind}}$:

$$\frac{\partial}{\partial t}\pi^{i,\text{ind}}(t) = \frac{1}{\beta_i}(1-\pi^{i,\text{ind}}(t)\beta_i)\left[\frac{1}{pf_0(t)}\frac{\partial}{\partial t}f_0(t) - \frac{1}{pf_i(t)}\frac{\partial}{\partial t}f_i(t)\right],$$

$$\pi^{i,\text{ind}}(T) = 0.$$

Using the ODE for f_0 in (4.35), the ODE for f_i in $\mathcal{I}_i$ given by (4.39) and using (4.44) shows that

$$\begin{aligned}\frac{\partial}{\partial t}\pi^{i,\text{ind}}(t) = \frac{1}{\beta_i}(1-\pi^{i,\text{ind}}(t)\beta_i)\Bigg[\Psi_i - \Psi_0 - \frac{1}{2}(1-p)\sigma_i^2\left(\pi^{i,\text{ind}}(t)-\pi_M^i\right)^2 \\ + \frac{1}{p}\sum_{j=0}^{d} q_{i,j}\frac{f_j(t)}{f_0(t)}(1-\pi^{i,\text{ind}}(t)\beta_i)^{-p} \\ - \frac{1}{p}\sum_{\substack{j=0\\ j\neq i}}^{d} q_{0,j}\frac{f_j(t)}{f_0(t)} - q_{0,i}\frac{1}{p}(1-\pi^{i,\text{ind}}(t)\beta_i)^{p}\Bigg],\end{aligned}$$

where for each $i \in E$ we denote by

$$\Psi_i = \frac{1}{2}\frac{\alpha_i^2}{(1-p)\sigma_i^2}$$

the utility growth potential in regime i. We now show that the strategy $\pi^{i,\text{ind}}$ is admissible for each $i \in E$ with $i \geq i_{min}$ and hence so is π^*.

Lemma 4.10. *There exists a unique solution of the differential equation*

$$\frac{\partial}{\partial t}y = F(t,y), \qquad y(T) = 0, \qquad (t,y) \in [0,T]\times(-\infty, 1/\beta_i),$$

where

$$\begin{aligned}F(t,y) = \frac{1}{\beta_i}(1-y\beta_i)\Bigg[\Psi_i - \Psi_0 - \frac{1}{2}(1-p)\sigma_i^2\left(y-\pi_M^i\right)^2 \\ + \frac{1}{p}\sum_{j=0}^{d} q_{i,j}\frac{f_j(t)}{f_0(t)}(1-y\beta_i)^{-p} \\ - \frac{1}{p}\sum_{\substack{j=0\\ j\neq i}}^{d} q_{0,j}\frac{f_j(t)}{f_0(t)} - q_{0,i}\frac{1}{p}(1-y\beta_i)^{p}\Bigg].\end{aligned}$$

Proof. Since $F(t,y)$ is continuous in t and globally Lipschitz continuous in y on any closed subinterval of $(-\infty, 1/\beta_i)$ it suffices to show that we can find constants $-\infty < a < b < 1/\beta_i$ such that the solution of the differential equation stays inside the interval $[a, b]$. We only consider the case $p \in (0, 1)$, the case $p \leq 0$ can be proved similarly.

Step 1: We prove the existence of a constant a such that $F(t, y) \leq 0$ whenever $y \leq a$. For this, note that the sign of F only depends on the term

$$\Psi_i - \Psi_0 - \frac{1}{2}(1-p)\sigma_i^2\left(y - \pi_M^i\right)^2 + \frac{1}{p}\sum_{j=0}^{d} q_{i,j}\frac{f_j(t)}{f_0(t)}(1-y\beta_i)^{-p} - \frac{1}{p}\sum_{\substack{j=0\\j\neq i}}^{d} q_{0,j}\frac{f_j(t)}{f_0(t)} - q_{0,i}\frac{1}{p}(1-y\beta_i)^p.$$

Furthermore, note that by the proof of Lemma 4.9 there exist constants $\underline{M}, \overline{M} > 0$ independent of $t \in [0, T]$ and $j \in E$ such that

$$\underline{M} \leq \frac{f_j(t)}{f_0(t)} \leq \overline{M}.$$

Then

$$\begin{aligned}
&\Psi_i - \Psi_0 - \frac{1}{2}(1-p)\sigma_i^2\left(y - \pi_M^i\right)^2 \\
&\quad + \frac{1}{p}\sum_{j=0}^{d} q_{i,j}\frac{f_j(t)}{f_0(t)}(1-y\beta_i)^{-p} - \frac{1}{p}\sum_{\substack{j=0\\j\neq i}}^{d} q_{0,j}\frac{f_j(t)}{f_0(t)} - q_{0,i}\frac{1}{p}(1-y\beta_i)^p \\
&\qquad \leq \Psi_i - \Psi_0 - \frac{1}{2}(1-p)\sigma_i^2\left(y - \pi_M^i\right)^2 \\
&\qquad\qquad + \frac{1}{p}\left(\lambda_0 + \sum_{\substack{j=0\\j\neq i}}^{d} q_{i,j}\overline{M}(1-y\beta_i)^{-p}\right),
\end{aligned}$$

which is less or equal to 0 if and only if

$$\frac{1}{p}\left(\lambda_0 + \sum_{\substack{j=0\\j\neq i}}^{d} q_{i,j}\overline{M}(1-y\beta_i)^{-p}\right) \leq \Psi_0 - \Psi_i + \frac{1}{2}(1-p)\sigma_i^2\left(y - \pi_M^i\right)^2.$$

Since $(1-y\beta_i)^{-p} \to 0$ and $(y - \pi_M^i)^2 \to +\infty$ as $y \to -\infty$ we see that there exists a constant a such that $F(t, y) \leq 0$ whenever $y \leq a$.

Step 2: Next we show that there exists a constant $b < 1/\beta_i$ independent of t such that $F(t,y) \geq 0$ whenever $y \geq b$. We have

$$\begin{aligned}
&\Psi_i - \Psi_0 - \frac{1}{2}(1-p)\sigma_i^2 \left(y - \pi_M^i\right)^2 \\
&\quad + \frac{1}{p}\sum_{j=0}^{d} q_{i,j}\frac{f_j(t)}{f_0(t)}(1-y\beta_i)^{-p} - \frac{1}{p}\sum_{\substack{j=0\\ j\neq i}}^{d} q_{0,j}\frac{f_j(t)}{f_0(t)} - q_{0,i}\frac{1}{p}(1-y\beta_i)^{p} \\
&\qquad \geq \frac{1}{p}\sum_{\substack{j=1\\ j\neq i}}^{d} q_{i,j}\underline{M}(1-y\beta_i)^{-p} - \frac{1}{p}\lambda_i - \frac{1}{p}\sum_{\substack{j=1\\ j\neq i}}^{d} q_{0,j}\overline{M} - q_{0,i}\frac{1}{p}(1-y\beta_i)^{p} \\
&\qquad\qquad + \Psi_i - \Psi_0 - \frac{1}{2}(1-p)\sigma_i^2 \left(y - \pi_M^i\right)^2,
\end{aligned}$$

which is greater or equal than 0 if and only if

$$\begin{aligned}
&\frac{1}{p}\sum_{\substack{j=1\\ j\neq i}}^{d} q_{i,j}\underline{M}(1-y\beta_i)^{-p} - \frac{1}{p}\lambda_i - \frac{1}{p}\sum_{\substack{j=1\\ j\neq i}}^{d} q_{0,j}\overline{M} - q_{0,i}\frac{1}{p}(1-y\beta_i)^{p} \\
&\qquad \geq \Psi_0 - \Psi_i + \frac{1}{2}(1-p)\sigma_i^2 \left(y - \pi_M^i\right)^2.
\end{aligned}$$

Now as $(1-y\beta_i)^{-p}$ approaches $+\infty$ and $(1-y\beta_i)^{p}$ approaches 0 as $y \to 1/\beta_i$ and since $(y - \pi_M^i)^2$ is bounded in y on $[0, 1/\beta_i]$ we see that there exists a constant $b < 1/\beta_i$ such that $F(t,y) \geq 0$ whenever $y \geq b$. □

Remark 4.11. Assume that $d = 1$ and that the excess return and the volatility of the stock are state-independent. Then the differential equation for $\pi^{1,\mathrm{ind}}$ simplifies to

$$\begin{aligned}
\frac{\partial}{\partial t}\pi^{1,\mathrm{ind}}(t) = \frac{1}{\beta_1}(1-\pi^{1,\mathrm{ind}}(t)\beta_1)\Big[&-\frac{1}{2}(1-p)\sigma_1^2\left(\pi^{1,\mathrm{ind}}(t) - \pi^{0,*}(t)\right)^2 \\
&- \frac{\lambda_0}{p}\left((1-\pi^{1,\mathrm{ind}}(t)\beta_1)^p - 1\right)\Big],
\end{aligned}$$

which is exactly the candidate optimal strategy derived in the simplified model in (4.16). ◇

It follows that π^* is an admissible strategy which leads to a strictly positive wealth process in every crash scenario. Moreover, the function $V(t,x,i)$ obviously satisfies a quadratic growth condition in x uniformly in (t,i) as long as $p \in (0,1)$.

Finally, given any trading strategy $\pi = (\pi^0, \dots, \pi^d) \in \mathcal{A}_K(t,x)$ the corresponding optimal crash time $\vartheta^*(\pi)$ is obviously well-defined since it is just the first time at which π^i exceeds $\pi^{i,\text{ind}}$. It follows that $V = \mathcal{V}_K$ and that π^* is optimal. Moreover, since the optimal strategy π^* attains its values in the interior of $\mathcal{K}$ it is immediately clear that π^* is also optimal in the class of all bounded trading strategies $\mathcal{A}(t,x)$ and hence $V = \mathcal{V}$.

Remark 4.12. It is furthermore possible to show that π^* is optimal in the class of all strategies which are not necessarily bounded but satisfy a growth condition of the form

$$\sup_{i\in E} \mathbb{E}\left[\int_t^T |\pi^i(u)|^2\, du\right] < \infty.$$

The idea is to argue by contradiction and assume that there exists a π which performs better than π^*. Then π must be unbounded by our previous discussion, but we can approximate π by bounded strategies π_n simply by cutting off π at n and $-n$. Using that π^* outperforms every π_n and that π_n converges to π one can then lead the optimality of π to a contradiction.

We postpone the verification for $p < 0$ to Section 4.7.3 and solve the system of DPEs in the logarithmic utility case first.

4.7.2. Solution of the system of DPEs for logarithmic utility

Let us now turn to the case $p = 0$. We guess that the value function takes the form

$$V(t,x,i) = \log(x) + f_i(t), \qquad i \in E \tag{4.45}$$

for some functions f_i with $f_i(T) = 0$. We can then proceed as in the power utility case to show that the candidate optimal strategy for $i < i_{min}$ is given by

$$\pi^{i,*}(t) = \frac{\alpha_i}{\sigma_i^2} = \pi_M^i$$

and that f_i solves

$$\frac{\partial}{\partial t} f_i(t) = -\frac{1}{2}\frac{\alpha_i^2}{\sigma_i^2} - \sum_{j=0}^{d} q_{i,j} f_j(t).$$

For $i \geq i_{min}$ the interval $[0,T]$ decomposes into a set $\mathcal{I}_i$ on which $\pi^{i,*}$ and f_i are determined by

$$\begin{aligned} f_i(t) &= \log(1-\pi^{i,*}(t)\beta_i) + f_0(t), \\ \frac{\partial}{\partial t} f_i(t) &= -\alpha_i \pi^{i,*}(t) + \frac{1}{2}\sigma_i^2[\pi^{i,*}(t)]^2 - \sum_{j=0}^{d} q_{i,j} f_j(t), \end{aligned}$$

and a set $\mathcal{N}_i$ on which $\pi^{i,*}$ and f_i are determined by

$$\begin{aligned} \pi^{i,*}(t) &= \pi_M^i, \\ \frac{\partial}{\partial t} f_i(t) &= -\alpha_i \pi^{i,*}(t) + \frac{1}{2}\sigma_i^2[\pi^{i,*}(t)]^2 - \sum_{j=0}^{d} q_{i,j} f_j(t). \end{aligned}$$

The existence of $(f_i)_{i\in E}$ can be proved in a very similar fashion to Lemma 4.9. The candidate optimal strategy is given by

$$\pi^{i,*} = \min\{\pi_M^i, \pi^{i,\text{ind}}(t)\},$$

where $\pi^{i,\text{ind}}$ solves

$$\begin{aligned} \frac{\partial}{\partial t}\pi^{i,\text{ind}}(t) = \frac{1}{\beta_i}(1-\pi^{i,\text{ind}}(t)\beta_i) \Bigg[&\Psi_i - \Psi_0 - \frac{1}{2}\sigma_i^2 \left(\pi^{i,\text{ind}}(t) - \pi_M^i\right)^2 \\ &+ \sum_{\substack{j=0 \\ j\neq i}}^{d} (q_{i,j} - q_{0,j}) f_j(t) \\ &+ (q_{i,i} - q_{0,i}) \left[f_0(t) + \log(1-\pi^{i,\text{ind}}(t)\beta_i)\right] \Bigg] \end{aligned}$$

with terminal condition $\pi^{i,\text{ind}}(T) = 0$. The admissibility of π^* and $\pi^{i,\text{ind}}$ follows by very similar arguments as in Lemma 4.10.

4.7.3. Verification for logarithmic and negative power utility

Let us now verify that the solutions of the system of DPEs constructed in Section 4.7.1 and Section 4.7.2 are indeed the value functions. For $p \in (0,1)$ this is clear by Theorem 4.7. For $p \leq 0$ the function V does not satisfy the quadratic growth condition which was used in step 3 and step 5 of the proof of Theorem 4.7. However, the explicit nature of our solutions allows us to verify these two steps also for $p \leq 0$.

Theorem 4.13. *Let $p = 0$ and let V be the solution of the system of DPEs given in* (4.45)*. Then $V = \mathcal{V}_K = \mathcal{V}$.*

Proof. Let $\pi \in \mathcal{A}(t,x)$ and $\vartheta \in \mathcal{T}(t,i)$ be arbitrary. We show that the two stochastic integrals in (4.22) are martingales and hence we can choose $\theta = T$ and conclude as before. Note that

$$x\frac{\partial}{\partial x}V(t,x,i) = 1$$

for every $i \in E$ and hence the integrand of the Brownian integral is bounded (uniformly in t and i) so that the integral is indeed a martingale. Moreover, for each $i, j \in E$ we have

$$|V(t,x,i) - V(t,x,j)| = |f_i(t) - f_j(t)|$$

which is again bounded (uniformly in t, i and j) and hence the integral with respect to the compensated jump measure is a martingale as well. □

For $p < 0$ we need to rule out some admissible trading strategies first. Note that since the pure bond strategy $\pi \equiv 0$ is admissible we may without loss of generality assume that every $\pi \in \mathcal{A}_K(t,x)$ satisfies

$$U_p(x) \leq \inf_{\vartheta \in \mathcal{T}(t,i)} \mathbb{E}\left[U_p\left(X^{\pi,\vartheta}_{t,x,i}(T)\right)\right] \tag{4.46}$$

Theorem 4.14. *Let $p < 0$ and let V be the solution of the system of DPEs given in* (4.34)*. Then $V = \mathcal{V}_K = \mathcal{V}$.*

Proof. We simply prove step 3 and step 5 of the proof of Theorem 4.7 without relying on the quadratic growth condition.

Step 3: Recall that by (4.24) for π^* and any arbitrary $\vartheta \in \mathcal{T}(t,i)$ we have

$V(t,x,i)$

$$
\begin{aligned}
&\leq V(\theta-, X^{*,\vartheta}(\theta-), Z(\theta-)) \\
&- \sum_{k=0}^{\infty} \sum_{j=0}^{d} \int_{\theta_{k-1}}^{\theta_k-} \sigma_j \pi^j(u) X^{*,\vartheta}(u) \frac{\partial}{\partial x} V(u, X^{*,\vartheta}(u), j) \mathbb{1}_{\{Z(u-)=j\}} \, dW(u) \\
&- \sum_{k=0}^{\infty} \sum_{j=0}^{d} \sum_{l=0}^{d} \int_{\theta_{k-1}}^{\theta_k-} \Big[V(u, X^{*,\vartheta}(u), l) \\
&\qquad\qquad - V(u, X^{*,\vartheta}(u), j) \Big] \mathbb{1}_{\{Z(u-)=j\}} \, \nu_k(du, l). \qquad (4.47)
\end{aligned}
$$

for any $[t,T]$-valued stopping time θ. We have to show that

$$V(t,x,i) \leq \mathcal{V}_K(t,x,i).$$

Using that $V(\theta-, X^{*,\vartheta}(\theta-), Z(\theta-)) \leq 0$ in (4.47) shows that the sum of the stochastic integrals is a local martingale bounded from above by $-V(t,x,i)$ and hence a submartingale. Choosing $\theta = T$ in (4.47) and taking expectations hence shows that

$$V(t,x,i) \leq \mathbb{E}\left[V(T-, X^{*,\vartheta}(T-), Z(T-)) \right].$$

Since V is continuous, satisfies $V(T,x,\cdot) = U_p(x)$ and since $\pi^*(T) = 0$ it follows that

$$V(t,x,i) \leq \mathbb{E}\left[U_p\left(X^{*,\vartheta}(T) \right) \right].$$

Since ϑ was chosen arbitrarily this implies that

$$V(t,x,i) \leq \inf_{\vartheta \in \mathcal{T}(t,i)} \mathbb{E}\left[U_p\left(X^{*,\vartheta}(T) \right) \right] \leq \mathcal{V}_K(t,x,i).$$

Step 5: Let $\pi \in \mathcal{A}_K(t,x)$ and let $\vartheta^*(\pi)$ be the corresponding candidate optimal crash strategy. Recall that by (4.30) we have

$$
\begin{aligned}
&V(t,x,i) \\
&\geq V(\theta-, X(\theta-), Z(\theta-)) \\
&- \sum_{k=0}^{\infty} \sum_{j=0}^{d} \int_{\theta_{k-1}^*}^{\theta_k^*-} \sigma_j \pi^j(u) X(u) \frac{\partial}{\partial x} V(u, X(u), j) \mathbb{1}_{\{Z(u-)=j\}} \, dW(u) \\
&- \sum_{k=0}^{\infty} \sum_{j=0}^{d} \sum_{l=0}^{d} \int_{\theta_{k-1}^*}^{\theta_k^*-} \Big[V(u, X(u), l) \\
&\qquad\qquad - V(u, X(u), j) \Big] \mathbb{1}_{\{Z(u-)=j\}} \, \nu_k(du, l). \qquad (4.48)
\end{aligned}
$$

for any $[t,T]$-valued stopping time θ. We have to show that

$$V(t,x,i) \geq \mathcal{V}_K(t,x,i).$$

For every $n \in \mathbb{N}$ we define

$$\theta^n := \inf\{u \geq t : |V(u, X(u), Z(u))| \geq n\} \wedge T.$$

Note that since

$$x\frac{\partial}{\partial x}V(t,x,i) = pV(t,x,i)$$

this implies that the stochastic integrals in (4.48) stopped at θ^n are martingales and hence replacing θ by θ^n in (4.48) and taking expectations shows that

$$V(t,x,i) \geq \mathbb{E}\left[V(\theta^n-, X(\theta^n-), Z(\theta^n-))\right].$$

If we can show that

$$\lim_{n\to\infty} \mathbb{E}\left[V(\theta^n-, X(\theta^n-), Z(\theta^n-))\right] = \mathbb{E}\left[V(T-, X(T-), Z(T-))\right] \tag{4.49}$$

then we can conclude as in the proof of Theorem 4.7.

First, let us note that

$$\mathbb{E}\left[V(\theta^n-, X(\theta^n-), Z(\theta^n-))\mathbb{1}_{\{\theta^n=T\}}\right] = \mathbb{E}\left[V(T-, X(T-), Z(T-))\mathbb{1}_{\{\theta^n=T\}}\right]$$

and hence

$$\lim_{n\to\infty} \mathbb{E}\left[V(\theta^n-, X(\theta^n-), Z(\theta^n-))\mathbb{1}_{\{\theta^n=T\}}\right] = \mathbb{E}\left[V(T-, X(T-), Z(T-))\right]$$

by monotone convergence. In order to prove (4.49) it is therefore sufficient to prove that

$$\lim_{n\to\infty} \mathbb{E}\left[V(\theta^n-, X(\theta^n-), Z(\theta^n-))\mathbb{1}_{\{\theta^n<T\}}\right] = 0.$$

Let us note that by Lemma 4.9 there exist constants $\underline{M}, \overline{M} > 0$ such that

$$\underline{M}U_p(x) \leq V(t,x,i) \leq \overline{M}U_p(x). \tag{4.50}$$

Next, it is clear that there exists a constant $L > 0$ such that

$$U_p(x) \leq \mathcal{V}_K(t,x,i) \leq LU_p(x). \tag{4.51}$$

Indeed, the first inequality follows from (4.46) and the second inequality follows from considering the no-crash strategy in $\mathcal{V}_K(t,x,i)$ and hence $\mathcal{V}_K \le \mathcal{V}_{RS}$ where $\mathcal{V}_{RS}$ denotes the value function in the regime switching model without crashes, see (1.4). Combining (4.50) and (4.51), we can therefore find a constant $C > 0$ independent of x such that

$$V(t,x,i) \ge C\mathcal{V}_K(t,x,i).$$

Using this we obtain

$$\begin{aligned}
0 &\ge \lim_{n\to\infty} \mathbb{E}\left[V(\theta^n-, X(\theta^n-), Z(\theta^n-))\mathbb{1}_{\{\theta^n<T\}}\right] \\
&\ge \lim_{n\to\infty} C\mathbb{E}\left[\mathcal{V}_K(\theta^n-, X(\theta^n-), Z(\theta^n-))\mathbb{1}_{\{\theta^n<T\}}\right] \\
&\ge \lim_{n\to\infty} C\mathbb{E}\left[\inf_{\vartheta\in\mathcal{T}(\theta^n-,Z(\theta^n-))} \mathbb{E}\left[U_p\left(X^{\pi,\vartheta}_{\theta^n-,X(\theta^n-),Z(\theta^n-)}(T)\right)\right]\mathbb{1}_{\{\theta^n<T\}}\right] \\
&\ge \lim_{n\to\infty} C\mathbb{E}\left[\inf_{\vartheta\in\mathcal{T}(t,i)} \mathbb{E}\left[U_p\left(X^{\pi,\vartheta}_{t,x,i}(T)\right)\right]\mathbb{1}_{\{\theta^n<T\}}\right] \\
&\ge \lim_{n\to\infty} C\mathbb{E}\left[U_p(x)\mathbb{1}_{\{\theta^n<T\}}\right] \\
&= 0. \qquad \square
\end{aligned}$$

4.8. Numerical results

We conclude this chapter with numerical examples. We consider three cases: logarithmic utility in the simple model, power utility in the generalized model and phase-type distributed arrival times of warnings.

4.8.1. Logarithmic utility in the simplified model

Let us first consider the simplified model discussed in the beginning of this chapter for an investor with a logarithmic utility function. We consider the following parameters throughout this section:

$$\alpha = 0.096, \qquad \sigma = 0.4, \qquad \beta = 0.5.$$

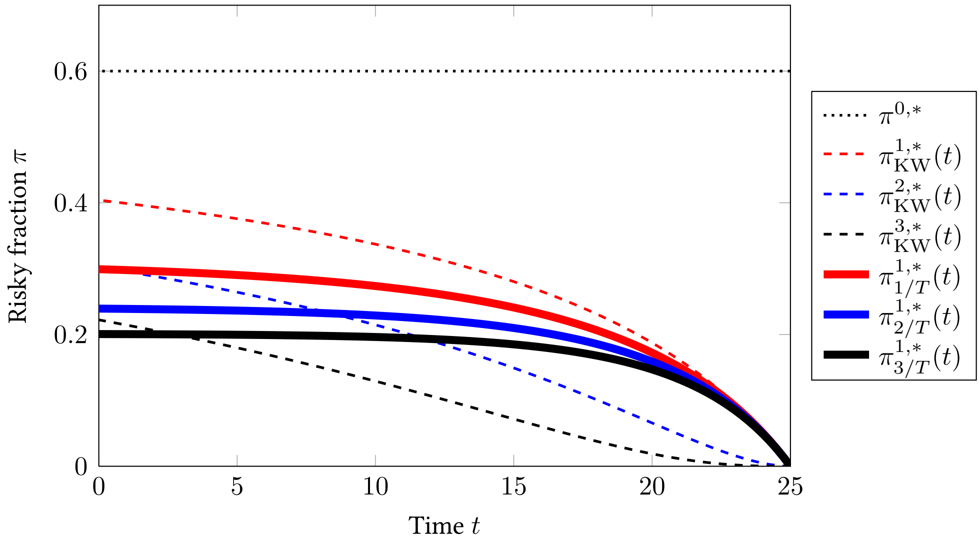

Figure 4.1. Short-term optimal strategies for different λ.

With these parameters the optimal strategy in the absence of crash warnings is given by

$$\pi^{0,*} = \pi_M = 0.6.$$

For the investment horizon we let $T \in \{25, 100\}$ and choose the intensity of the arrival time of crash warnings to be $\lambda \in \{1/T, 2/T, 3/T\}$ such that we receive on average one, two or respectively three warnings during the investment period.

Figure 4.1 shows that the optimal strategy $\pi^{1,*}_{\lambda}$ in the presence of crash warnings with intensity λ exhibits similar qualitative features as the worst-case optimal strategy $\pi^{n,*}_{KW}(t)$ in the Korn/Wilmott model with at most $n \in \{1, 2, 3\}$ crashes. Recall that the Korn/Wilmott optimal strategy can be formally obtained as the limiting case of our model as $\lambda \to 0$ and is given as the solution of (1.7). As can be seen, the optimal strategy in our model is decreasing in time, strictly positive for all $t < T$ and equal to zero if and only if $t = T$. It can also be seen that $\pi^{1,*}_{n/T}$ is more conservative in the long run than the corresponding Korn/Wilmott strategy $\pi^{n,*}_{KW}(t)$. This is due to the possibility of more than n crashes in our model. Clearly, the short term behavior of the Korn/Wilmott strategies changes significantly the higher the maximum number of potential crashes is, while the long term behavior is the same for any n, that is all Korn/Wilmott strategies converge to the optimal Merton fraction $\pi^{0,*} = \pi_M$ as $T - t$ tends to infinity (see Figure 4.2). Note that in Figure 4.2 the intensity λ has been adjusted to have

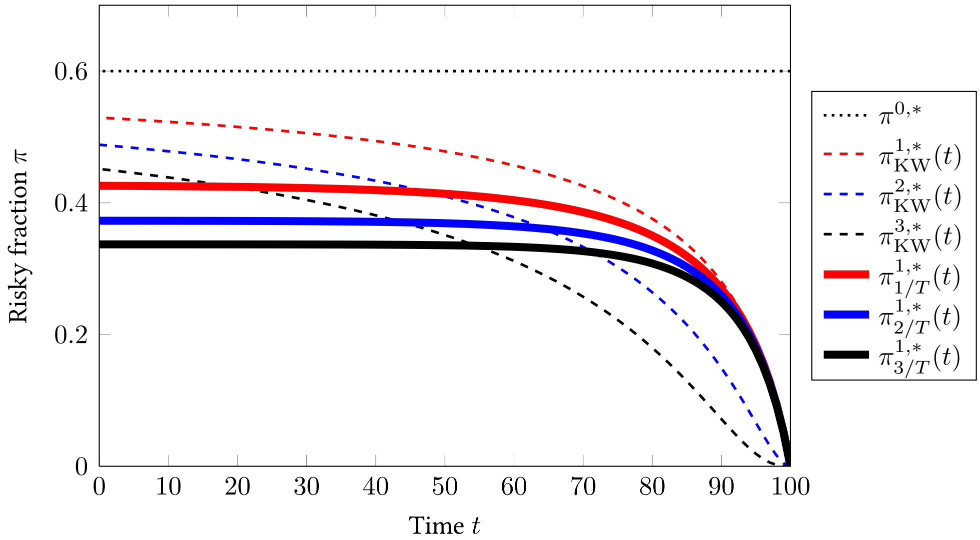

Figure 4.2. Long-term optimal strategies for different λ.

1, 2 and 3 expected crash warnings within an investment horizon of 100 years (instead of 25 years in Figure 4.1). Figure 4.2 shows that more potential crashes impact the Korn/Wilmott strategies only in the short term (which can be rather long...) but not in the long term, while more potential crashes impact $\pi^{1,*}_{\lambda}$ not so much in the short term, but more so in the long term. Randomizing the number of potential crashes has therefore mainly a long term impact while the short term impact is minor if compared to the Korn/Wilmott strategy.

It can be seen that in the long run $\pi^{1,*}_{\lambda}$ is strictly smaller than the corresponding Korn/Wilmott strategy and does not converge to $\pi^{0,*}$ for $\lambda > 0$ as the investment horizon tends to infinity (Figure 4.2). Mathematically, this can be seen by taking a closer look at the differential equation for $\pi^{1,*}$ in (4.8). At terminal time T we have

$$\left.\frac{\partial}{\partial t}\pi^{1,*}(t)\right|_{t=T} = -\frac{\alpha^2}{2\beta\sigma^2} < 0$$

which implies that $\pi^{1,*}$ is increasing with increasing investment horizon until

$$0 = \frac{1}{2}\sigma^2\left(\pi^{1,*}(t) - \pi^{*,0}(t)\right)^2 + \lambda\log\left(1 - \pi^{1,*}(t)\beta\right). \tag{4.52}$$

Since

$$\frac{1}{2}\sigma^2\left(\pi^{1,*}(t) - \pi^{*,0}(t)\right)^2$$

is positive and decreasing in $\pi^{1,*}$ on $[0, \pi^{*,0})$, equal to zero if $\pi^{1,*} = \pi^{*,0}$ and since

$$\lambda \log\left(1 - \pi^{1,*}(t)\beta\right)$$

is negative and increasing in $\pi^{1,*}$ on $(0, \pi^{*,0}]$ we see that $\pi^{1,*}$ is bounded away from $\pi^{0,*}$. This verifies what can be observed in Figure 4.2, i.e. the long term behavior of the optimal strategies is different from the long term behavior of the optimal Korn/Wilmott strategies.

Clearly, for a fixed initial wealth x an investor in our market model obtains less expected utility at terminal time compared to an investor in the classical Merton model if no crash occurs. In order to estimate this trade-off we determine the efficiency η which is the fraction of the initial wealth x a worst-case investor requires at time t to obtain the same expected utility as the Merton investor. More formally, for each $t \in [0, T]$ we want to determine $\eta_\lambda(t)$ such that

$$\mathcal{V}(t, \eta_\lambda(t)x, 1) = \mathcal{V}_M(t, x), \tag{4.53}$$

where $\mathcal{V}_M$ denotes the Merton value function given by (1.3). Plugging the optimal strategy $\pi_\lambda^{1,*}$ and the no-crash scenario into $\mathcal{V}(t, \eta_\lambda(t)x, 1)$ in (4.53) and rearranging terms (following e.g. Menkens [79, p. 601]) yields

$$\eta_\lambda(t) = \exp\left\{\frac{1}{2}\sigma^2 \int_t^T \left[\pi_\lambda^{1,*}(u) - \pi^{0,*}\right]^2 du\right\}.$$

Note that in the Korn/Wilmott model with n crashes the efficiency η_{KW}^n of the worst-case optimal strategy $\pi_{KW}^{n,*}$ is given by the same formula if we replace $\pi_\lambda^{1,*}$ by $\pi_{KW}^{n,*}$.

In Figure 4.3 we see the efficiency for an investment horizon of $T = 25$ years, whereas in Figure 4.4 we consider an investment horizon of $T = 100$ years. Clearly, the efficiency is bounded from below by 1 and is decreasing in t. If the investment horizon is 25 years the Korn/Wilmott investor with at most 1 crash requires about 25.32% of additional initial wealth if she wants to get the same terminal expected utility as a Merton investor who ignores the possibility of a crash. Therefore, we call this 25.32% the cost of worst-case scenario optimal investment (see Menkens [79]). The costs in the case of at most 2 and 3 crashes in the Korn/Wilmott setting are given by 47.67% and 66.14%, respectively. The corresponding costs for the investor in our market model are 32.98% (for $\lambda =$

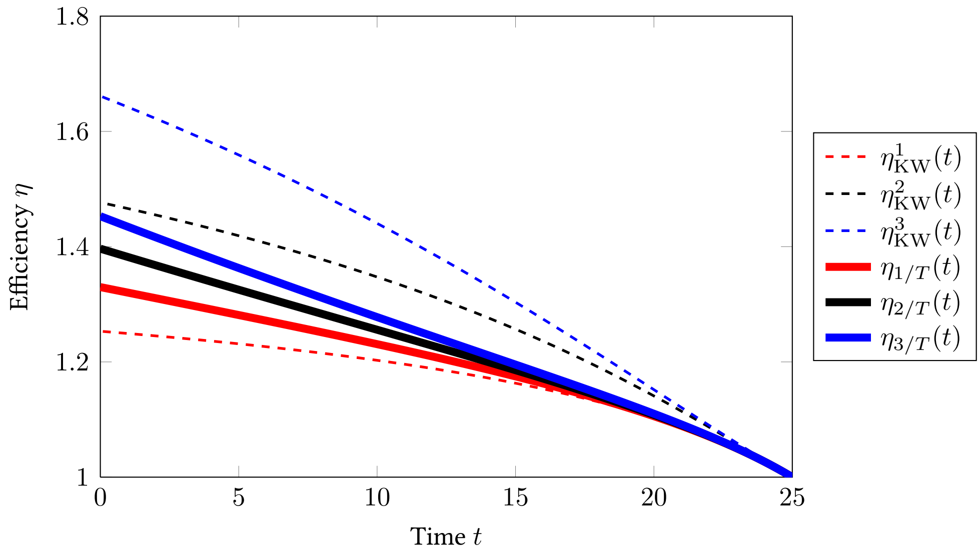

Figure 4.3. The efficiency for terminal time $T = 25$.

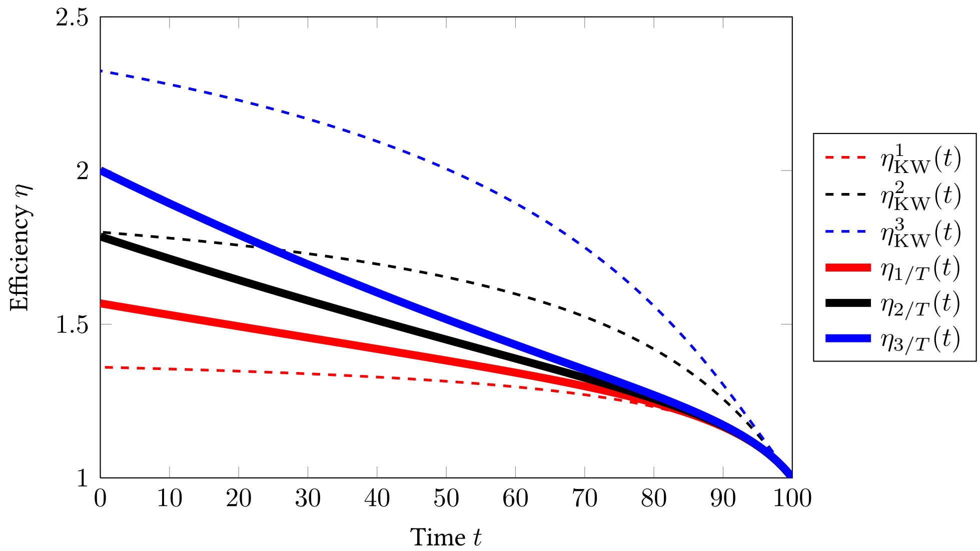

Figure 4.4. The efficiency for terminal time $T = 100$.

$1/T$), 39.65% (for $\lambda = 2/T$) and 45.29% (for $\lambda = 3/T$) on an investment horizon of $T = 25$ years in order to obtain the same expected utility as a Merton investor.

Figure 4.4 shows the corresponding plot for an investment horizon of $T = 100$ years. The costs for an investment horizon of 100 are 36.02%, 79.96% and 132.44% for the Korn/Wilmott investor with at most 1, 2 and 3 crashes, respectively, and 56.78% (for $\lambda = 1/T$), 78.53% (for $\lambda = 2/T$), and 100.08% (for $\lambda = 3/T$) in our model.

Note that the costs of a Korn/Wilmott investor with n crashes have an upper bound since

$$\eta_{KW}^n(t) = \prod_{i=1}^{n} \frac{1}{1 - \pi_{KW}^{i,*}(t)\beta} \to \frac{1}{(1 - \pi^{0,*}\beta)^n} \qquad \text{for } T - t \to \infty.$$

This is because $\pi_{KW}^{i,*}(t) \to \pi^{0,*}$ as $T - t \to \infty$ (cf. Figure 4.2). The asymptotic behavior (that is for $T-t \to \infty$) of the costs for the crash hedging strategy $\pi_\lambda^{1,*}(t)$ with $\lambda > 0$ is different: It is exponential in the investment horizon since $\pi_\lambda^{1,*}(t)$ is bounded away from $\pi^{0,*}$ uniformly in t. Note that this exponential growth takes a long time to become visible and is not to be mistaken by linear growth (see Figure 4.4).

4.8.2. Power utility in the generalized model

Let us now take a closer look at the generalized model. We assume that

$$\alpha_0 = \ldots = \alpha_d = \alpha = 0.096 \qquad \text{and} \qquad \sigma_0 = \ldots = \sigma_d = \sigma = 0.4,$$

and let $T = 25$ and $\lambda = 1/T$. We furthermore choose $d = 5$, $p = 0.1$ and let the generator matrix of Z and the crash sizes β_i be given by

$$Q = \begin{pmatrix} -\lambda & \lambda & 0 & 0 & 0 & 0 \\ 0 & -\lambda & \lambda & 0 & 0 & 0 \\ 0 & 0 & -\lambda & \lambda & 0 & 0 \\ 0 & 0 & 0 & -\lambda & \lambda & 0 \\ 0 & 0 & 0 & 0 & -\lambda & \lambda \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \\ \beta_5 \end{pmatrix} = \begin{pmatrix} 0.1 \\ 0.3 \\ 0.5 \\ 0.7 \\ 0.9 \end{pmatrix}.$$

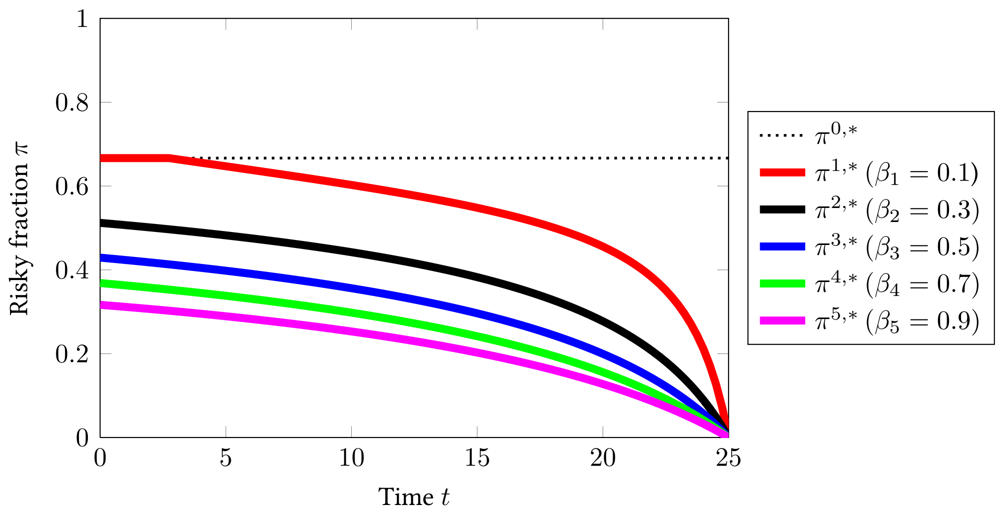

Figure 4.5. Optimal strategies in the generalized model.

So, in particular, the process Z can only jump from state i to state $i+1$ and the last state is absorbing.

The numerical approximations of the optimal strategies can be found in Figure 4.5. As can be seen the optimal strategies are decreasing for an increasing maximum crash size but still display similar qualitative features as the optimal strategy obtained in the simplified model. Note however that the optimal strategy in state 1 is equal to the Merton fraction π_M^i for small t (approximately $t < 2.75$), i.e. for small t we are inside the set $\mathcal{N}_1$. In contrast to the simplified model and the models considered in Korn and Menkens [67] and Seifried [95] this is a new feature. As Korn and Menkens [67] show, the optimal strategy in the presence of crashes is always smaller than the Merton fraction and only if one considers changing market coefficients after a crash it may be optimal for the investor to follow the Merton strategy despite the presence of a crash threat. In our model this phenomenon can already occur without considering state-dependent market coefficients. However this can only be observed in the generalized model with $d > 1$.

In the example considered in Figure 4.5 the market jumps from regimes with lower crash sizes to higher crash sizes from 0.1 to 0.9. Let us now consider the opposite direction, i.e. the market jumps from the safe state 0 to the state with crash size 0.9, from there to the state with crash size 0.7 and so on. This can be

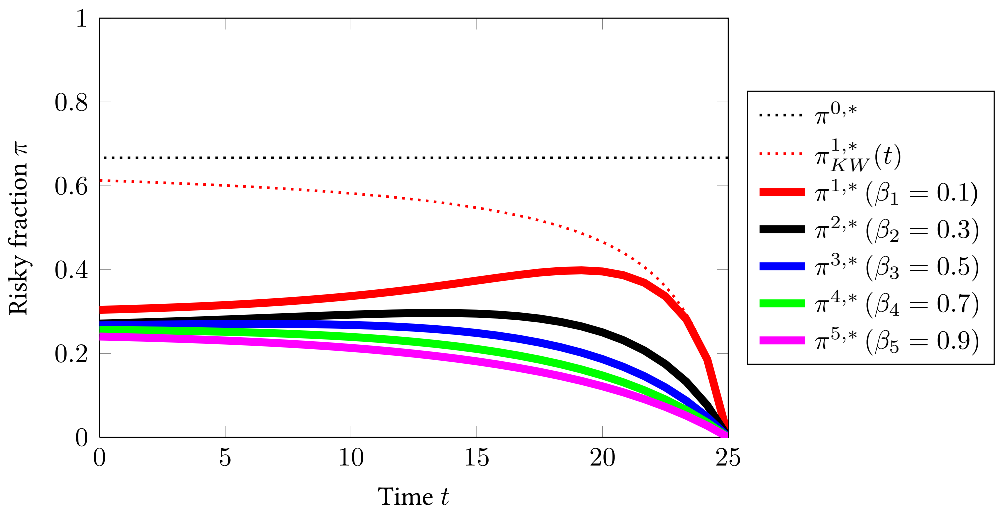

Figure 4.6. Optimal strategies in the generalized model with decreasing maximum crash sizes.

modeled by considering the generator matrix

$$Q = \begin{pmatrix} -\lambda & 0 & 0 & 0 & 0 & \lambda \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda & -\lambda & 0 & 0 & 0 \\ 0 & 0 & \lambda & -\lambda & 0 & 0 \\ 0 & 0 & 0 & \lambda & -\lambda & 0 \\ 0 & 0 & 0 & 0 & \lambda & -\lambda \end{pmatrix},$$

and keeping the remaining parameters as before.

The resulting optimal strategies can be found in Figure 4.6. First, notice that by looking time $t = 0$ the investor is strikingly more conservative since after a crash has occurred the next warning brings the market right back into the most dangerous state 5 with a maximum crash size of 0.9. Also, note that in states 1 and 2 the optimal strategies present a previously unobserved pattern – the strategies are no longer monotone in t but increasing for small values of t and decreasing for larger values of t. The rationale behind this observation is as follows: If a crash occurs at time $t << T$ then the probability of another warning coming in before terminal time T is quite high (as compared to a crash close to T). Hence it is quite likely that the market ends up in the dangerous state 5 again. In order to avoid big losses the investor hence chooses a small risky fraction. On the other hand if t gets closer to T and a crash occurs then the probability of jumping back into state 5 becomes smaller and smaller and hence the investor has to be less and less concerned with this threat as t approaches T. In states 1 and 2 this leads to an increase in the optimal strategy. However as t gets even closer to T the losses due to an immediate crash begin to dominate the threat of jumping back into state 5 and hence the strategies start to decrease again and converge to 0 as $t \to T$. This also explains why in states 3 to 5 the strategies are monotone – because the threat of losing utility due to an immediate crash is bigger than the threat of jumping back into state 5 after this crash.

Observe that the optimal strategy in the case $i = 1$ in Figure 4.6 verifies the findings for the simple model: A fixed number of possible crashes has just a short term (meaning close to the investment horizon) impact while a random (unknown) number of crashes has an additional long term impact. With Figure 4.6 we can make this more precise. In the short term the imminent threat of a crash is dominating and the investor can almost ignore the long term threat of the unknown number of possible crashes (by investing more in the risky asset in the short term than one would in the long term). On the other hand, the unknown number of potential crashes has only a long term impact. This becomes clear by comparing the optimal strategy in state $i = 1$ with the corresponding optimal Korn/Menkens strategy $\pi^{1,*}_{KM}$ with exactly one crash (given by (1.8) with $\beta = \beta_1$): While the behavior close to maturity of the two strategies is very similar, the long-term difference between the two strategies is significant.

We investigate the feature of non-monotone optimal strategies again from a different point of view in the next example where we replace the exponential arrival time distribution of warnings with various phase-type distributions.

4.8.3. Phase-type distributed warning times

We conclude our numerical examples by comparing the optimal strategies which arise for different choices of the distribution of the arrival times of the crash warnings. As pointed out in Remark 4.4, by an appropriate choice of the transition rate matrix Q of the Markov process Z, by making the market coefficients state-independent and by setting $0 = \beta_0 = \beta_1 = \ldots = \beta_{d-1}$, $\beta_d = \beta = 0.5$, the time S which it takes to reach state d from state 0 is phase-type distributed. In this section we consider three different types of phase-type distributions: exponential, Erlang and Coxian.

Remark 4.15. Note that this setup is degenerate in the sense that the investor receives warnings whenever Z jumps to a state $i \geq 0$, but since the maximum crash size is equal to zero for $i < d$ she does not have to fear a crash as long as she is in one of these states. $\diamond$

For our example we choose $T = 50$. In order to normalize the different types of distributions and make them comparable we choose the parameters of the distributions so that we always have

$$\mathbb{E}[S] = 25,$$

i.e. we expect to see 2 warnings if we start in state 0 at time $t = 0$ (Actually, the expectation will be to see $2d$ warnings. However, $2d - 2$ warnings are artificial/degenerate with no potential interpretation).

To obtain an exponential distribution we need to choose the transition matrix Q_{Exp} of the Markov process Z to be

$$Q_{\text{Exp}} := \begin{pmatrix} q_{0,0} & q_{0,1} \\ q_{1,0} & q_{1,1} \end{pmatrix} = \begin{pmatrix} -2/T & 2/T \\ 0 & 0 \end{pmatrix}.$$

We obtain an Erlang distribution by choosing $Q_{\text{Erl}} = (q_{i,j}^{\text{Erl}})_{0 \leq i,j \leq d^{\text{Erl}}}$ such that

$$q_{i,i}^{\text{Erl}} = -\frac{d^{\text{Erl}}}{25}, \qquad q_{i,i+1}^{\text{Erl}} = \frac{d^{\text{Erl}}}{25}, \qquad i = 0, \dots, d^{\text{Erl}} - 1,$$

and $0 = q_{d^{\text{Erl}},0}^{\text{Erl}} = \dots = q_{d^{\text{Erl}},d^{\text{Erl}}}^{\text{Erl}}$. In our example we consider the two cases $d^{\text{Erl}} = 5$ and $d^{\text{Erl}} = 50$.

To obtain the Coxian distribution we need to choose $Q_{\text{Cox}} = (q_{i,j}^{\text{Cox}})_{0 \leq i,j \leq d^{\text{Cox}}}$ such that

$$q_{i,i}^{\text{Cox}} = -\lambda_i, \quad q_{i,i+1}^{\text{Cox}} = p_i \lambda_i, \quad q_{i,d^{\text{Cox}}}^{\text{Cox}} = (1 - p_i)\lambda_i, \quad i = 0, \dots, d^{\text{Cox}} - 2,$$

$q_{d^{\text{Cox}}-1,d^{\text{Cox}}-1}^{\text{Cox}} = -q_{d^{\text{Cox}}-1,d^{\text{Cox}}}^{\text{Cox}} = -\lambda_{d^{\text{Cox}}-1}$ and 0 otherwise. The constants p_i, $i = 0, \dots, d^{\text{Cox}} - 2$, have to be chosen such that $0 < p_i \leq 1$. For our numerical example we consider $d^{\text{Cox}} = 2$ and

$$\lambda_0 = 5, \qquad \lambda_1 = \frac{1}{30}, \qquad p_0 = \frac{124}{150},$$

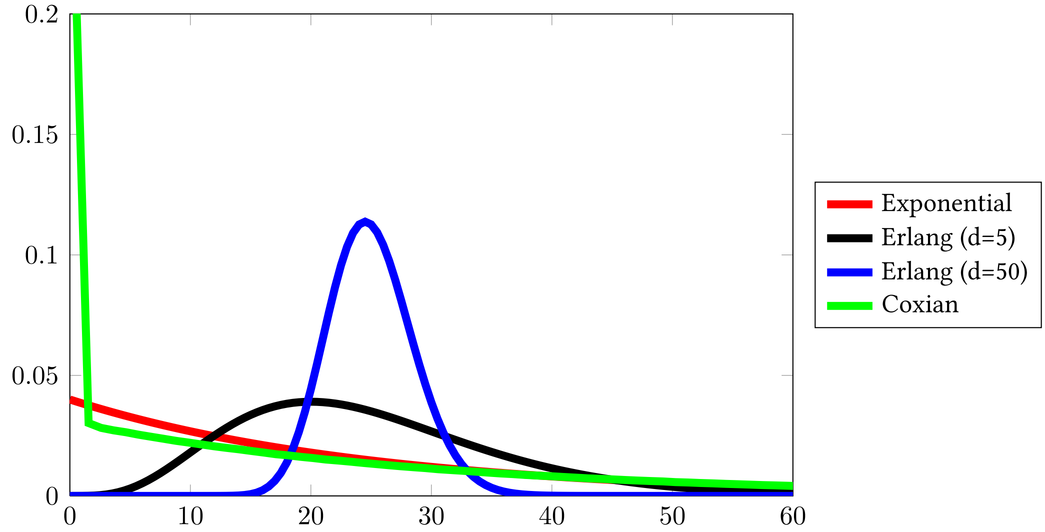

Figure 4.7. Probability density functions of the phase-type distributions.

so that

$$Q_{\text{Cox}} = \begin{pmatrix} -5 & \frac{124}{150} \cdot 5 & \frac{26}{150} \cdot 5 \\ 0 & -\frac{1}{30} & \frac{1}{30} \\ 0 & 0 & 0 \end{pmatrix}.$$

The resulting probability density functions and cumulative distribution functions are depicted in Figure 4.7 and Figure 4.8, respectively. As we can see, the Coxian distribution puts a lot of mass on small values of t, i.e. the probability of jumping into the warning state after a short amount of time is quite high compared to the other distributions. The Erlang distribution with $d^{\text{Erl}} = 50$ on the other hand has a significant peak around the mean arrival time $\mathbb{E}[S] = 25$ and puts almost no weight in the tails. Note also that the Erlang distribution converges to the Dirac measure at $\mathbb{E}[S] = 25$ as $d^{\text{Erl}} \to \infty$.

The resulting optimal strategies can be found in Figure 4.9. The Coxian strategy is the most conservative for $t > 25$ which is due to the high mass on the small time values – since it is more likely to jump back into the crash state shortly after a crash the investor has to take this into account in order to be indifferent. The Erlang strategy with five phases ($d^{\text{Erl}} = 5$) has a similar behavior as the strategies for state 1 and 2 in the previous example. That is, the strategy is first increasing and then decreasing. In the case of $d^{\text{Erl}} = 50$ phases we can even see an oscilla-

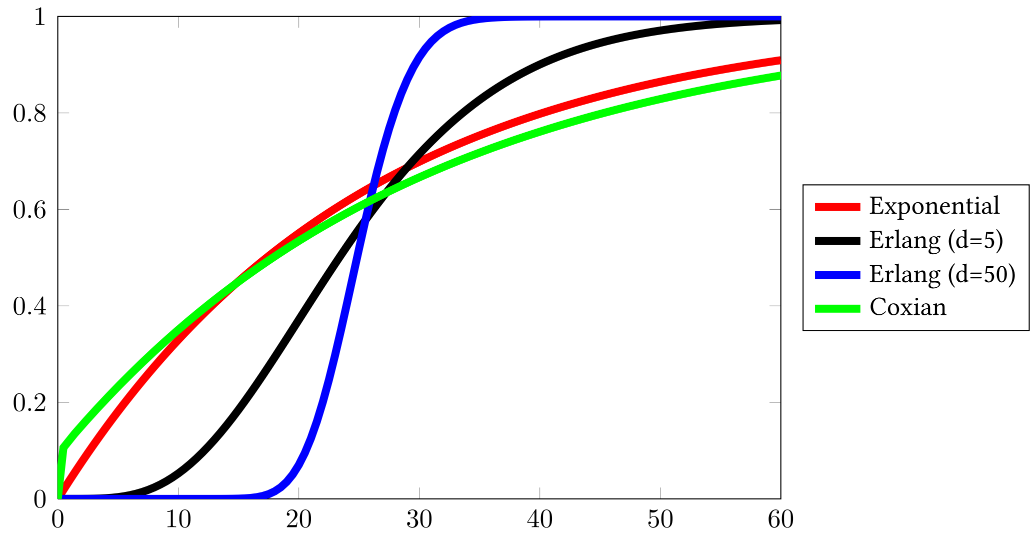

Figure 4.8. Cumulative distribution functions of the phase-type distributions.

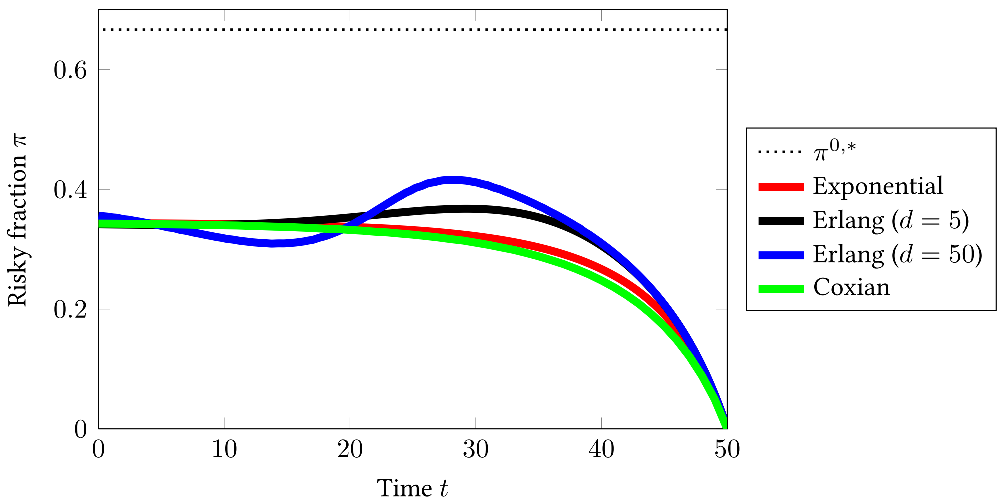

Figure 4.9. Optimal strategies for phase-type distributed arrival times.

tion in the optimal strategy. The reason for this can be found in the density of the Erlang distribution. As $d^{\text{Erl}} \to \infty$ the Erlang distribution converges to the Dirac measure at $t = 25$. That is, the Erlang distribution puts increasingly more mass around the point $t = 25$. This means that after a crash at time t there is a very high probability that the next crash warning will arrive in roughly 25 years and the probability of an earlier warning is small. Hence if t is close to T the investor essentially has to prepare for one more crash – since the likelihood of another warning after a crash is small. However, as $T - t$ increases so does the probability of another crash warning occurring after a crash at time t. Hence around $t = 25$ the investor begins to fear that another warning may arrive before the investment horizon – so she has to be afraid of two more crashes. This explains why the strategy in the $d^{\text{Erl}} = 50$ case is increasing for $t \in [16.5, 29.5]$ (approximately). For even smaller values of t the strategy is again decreasing since the probability of only one more crash warning remains high. The effect becomes more pronounced as d^{Erl} becomes larger due to the convergence property of the Erlang distribution against the Dirac measure – the investor becomes increasingly more certain of how long it will take for another warning to arrive after a crash. Also, note that in the long run all strategies considered in the above example converge to the same level since the stationary distribution of the Markov chain dominates the investor's decisions for large time horizons.

A. Notation and conventions

We use this appendix to introduce some notation and settle on some conventions for the main body of this thesis.

We denote by $\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{Q}$ and $\mathbb{R}$ the sets of natural numbers, non-negative integers, rational numbers and real numbers, respectively. Given $n \in \mathbb{N}$, we denote by $\mathbb{S}^n$ the set of symmetric $n \times n$ matrices with entries in $\mathbb{R}$.

We denote by $|\cdot|$ and $\langle \cdot, \cdot \rangle$ the Euclidean norm and scalar product on $\mathbb{R}^n$. On $\mathbb{S}^n$ we consider the usual partial order by agreeing that $X \geq Y$ whenever $X, Y \in \mathbb{S}^n$ and $X - Y$ is positive semi-definite.

Whenever A is a subset of $\mathbb{R}^n$, we denote by $\overline{A}$, ∂A, A^c the closure of A, the boundary of A and the complement of A, respectively.

Given an open interval $(a, b) \subset \mathbb{R}$, an open set $A \in \mathbb{R}^n$ and a (sufficiently regular) function $f : (a, b) \times A \to \mathbb{R}$, $(t, x) \mapsto f(t, x)$, we denote by $D_t f(t, x)$ the partial derivative of f with respect to t, i.e.

$$D_t f(t, x) = \frac{\partial}{\partial t} f(t, x).$$

We denote by $D_x f(t, x)$ the gradient with respect to the variable x and similarly we denote by $D_x^2 f(t, x)$ the Hessian matrix with respect to x. We denote by $C^k((a, b) \times A \to \mathbb{R})$ the set of all functions which are k-times continuously differentiable. We write $C^{j,k}((a, b) \times A \to \mathbb{R})$ to denote the set of all functions which are j-times continuously differentiable with respect to their first variable and k-times continuously differentiable with respect to there second variable. The set of continuous functions $f : B \to \mathbb{R}$, $B \subset \mathbb{R}^n$, is denoted by $C(B)$.

Given any real number $x \in \mathbb{R}$, we denote by x^+ and x^- its positive and negative part, respectively, i.e.

$$x^+ = \max\{x, 0\}, \qquad x^- = \max\{-x, 0\}.$$

Let $a, b \in \mathbb{R}$ with $a > b$. Then the interval (a, b) is assumed to be the empty set and we assume the obvious analog statements for closed and half-open intervals to hold.

Whenever we take the infimum over the empty set, we make the convention that

$$\inf\{\emptyset\} = +\infty$$

and similarly

$$\sup\{\emptyset\} = -\infty.$$

Inequalities and equalities involving random variables on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ are always to be understood in a $\mathbb{P}$-almost sure sense without explicitly saying so.

Finally, if $\mathbb{T} \subset \mathbb{R}$ is some index set, $t, T \in \mathbb{T}$ and if $X = (X(u))_{u \in \mathbb{T}}$ is a stochastic process, we write

$$\mathbb{E}[X_{t,x}(T)]$$

for the expectation of $X(T)$ conditional on $X(t) = x$.

B. Viscosity solutions

The aim of this appendix is to recall several equivalent definitions of the notion of viscosity solutions of parabolic partial differential equations and state the fundamental results we need in the main body of this thesis. We follow Pham [90] in our exposition.

Let $\mathcal{O} \subset \mathbb{R}^n$ be open, fix some $T > 0$ and let

$$F : [0,T) \times \mathcal{O} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{S}^n \to \mathbb{R}$$

be a continuous function. We assume that F satisfies the following parabolicity condition: For each $t \in [0,T)$, $x \in \mathcal{O}$, $r \in \mathbb{R}$, $q, q' \in \mathbb{R}$, $p \in \mathbb{R}^n$ and $M \in \mathbb{S}^n$, F satisfies

$$F(t,x,r,q,p,M) \geq F(t,x,r,q',p,M)$$

whenever $q \leq q'$. Moreover, we assume that F satisfies the following ellipticity condition: For each $t \in [0,T)$, $x \in \mathcal{O}$, $r \in \mathbb{R}$, $q \in \mathbb{R}$, $p \in \mathbb{R}^n$ and $M, M' \in \mathbb{S}^n$, F satisfies

$$F(t,x,r,q,p,M) \geq F(t,x,r,q,p,M')$$

whenever $M \leq M'$.

Let us now fix a function $w \in C^{1,2}([0,T) \times \mathcal{O})$ and assume that w satisfies the partial differential equation

$$F(t,x,w(t,x),D_t w(t,x),D_x w(t,x),D_x^2 w(t,x)) = 0 \tag{B.1}$$

for each $(t,x) \in [0,T) \times \mathcal{O}$. Now take another function $\varphi \in C^{1,2}([0,T) \times \mathcal{O})$ and assume that $w - \varphi$ attains a local maximum at some point $(t_0,x_0) \in [0,T) \times \mathcal{O}$. By the first- and second-order optimality conditions we have

$$\begin{aligned} &D_t w(t_0,x_0) \leq D_t \varphi(t_0,x_0), && \text{where equality holds if } t_0 > 0, \\ &D_x w(t_0,x_0) = D_x \varphi(t_0,x_0), && \end{aligned}$$

$$D_x^2 w(t_0, x_0) \leq D_x^2 \varphi(t_0, x_0).$$

Replacing the derivatives of w by the derivatives of φ in (B.1) we hence have

$$\begin{aligned} &F(t_0, x_0, w(t_0, x_0), D_t\varphi(t_0, x_0), D_x\varphi(t_0, x_0), D_x^2\varphi(t_0, x_0)) \\ &\qquad \leq F(t_0, x_0, w(t_0, x_0), D_t w(t_0, x_0), D_x w(t_0, x_0), D_x^2 w(t_0, x_0)) = 0 \end{aligned} \tag{B.2}$$

by the parabolicity and the ellipticity condition on F. Note that in order to make sense of the left-hand side of (B.2) we only need w to be upper semi-continuous (so that $w - \varphi$ can attain a local maximum). Similarly, if $w - \varphi$ attains a local minimum at (t_0, x_0), we obtain

$$F(t_0, x_0, w(t_0, x_0), D_t\varphi(t_0, x_0), D_x\varphi(t_0, x_0), D_x^2\varphi(t_0, x_0)) \geq 0, \tag{B.3}$$

and as before we only need that w is lower semi-continuous to make sense of this.

The above discussion hence motivates a notion of weak solutions of the PDE (B.1) by replacing the derivatives of w by the derivatives of smooth test functions φ. In order to ensure that the difference $w - \varphi$ attains extremal values, we assume that w is locally bounded and replace w by its upper and lower semi-continuous envelopes w^* and w_* defined as

$$w^*(t_0, x_0) := \limsup_{(t,x)\to(t_0,x_0)} w(t, x) \quad \text{and} \quad w_*(t_0, x_0) := \liminf_{(t,x)\to(t_0,x_0)} w(t, x),$$

respectively. This leads to the following definition.

Definition B.1. *Let $w : [0, T) \times \mathcal{O}$ be locally bounded.*

1. *We say that w is a viscosity subsolution of* (B.1) *if for each $(t_0, x_0) \in [0, T) \times \mathcal{O}$ and for each $\varphi \in C^{1,2}([0, T) \times \mathcal{O})$ such that $w^* - \varphi$ attains a local maximum at (t_0, x_0) we have*

$$F(t_0, x_0, w^*(t_0, x_0), D_t\varphi(t_0, x_0), D_x\varphi(t_0, x_0), D_x^2\varphi(t_0, x_0)) \leq 0.$$

2. *We say that w is a viscosity supersolution of* (B.1) *if for each $(t_0, x_0) \in [0, T) \times \mathcal{O}$ and for each $\varphi \in C^{1,2}([0, T) \times \mathcal{O})$ such that $w_* - \varphi$ attains a local minimum at (t_0, x_0) we have*

$$F(t_0, x_0, w_*(t_0, x_0), D_t\varphi(t_0, x_0), D_x\varphi(t_0, x_0), D_x^2\varphi(t_0, x_0)) \geq 0.$$

3. *We say that w is a viscosity solution of* (B.1) *if it is a viscosity subsolution as well as a viscosity supersolution.*

Remark B.2. 1. By shifting the test functions in the above definition appropriately we may without loss of generality assume that $w^*(t_0, x_0) = \varphi(t_0, x_0)$ and $w_*(t_0, x_0) = \varphi(t_0, x_0)$, respectively.

2. Since the test functions act only locally we may furthermore assume that the local maximum (or minimum) of $w^* - \varphi$ (or $w_* - \varphi$) is global.

3. By the discussion preceding the definition it is clear that every classical solution of (B.1) is also a viscosity solution. Similarly, if w is a viscosity solution and $w \in C^{1,2}([0,T) \times \mathcal{O})$, then w is also a classical solution: Simply choose the test function φ to be w itself.

4. If the function w is continuous, then $w_* = w^* = w$ and hence the definition simplifies correspondingly.

The above definition of a viscosity solution is typically useful whenever we want to show that a given function is a viscosity solution of a certain PDE. In order to prove uniqueness it is usually easier to work with the equivalent definition we introduce in what follows. Given a locally bounded function w on $[0,T) \times \mathcal{O}$ as before, we define the second-order superjet $J^{2,+}w^*(t_0, x_0)$ of the upper semi-continuous envelope w^* of w at (t_0, x_0) to be the set of all $(q, p, M) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{S}^n$ such that

$$\limsup_{(t,x)\to(t_0,x_0)} \frac{1}{|t_0 - t| + |x_0 - x|} \Big[w^*(t_0, x_0) - w(t, x) - q(t_0 - t) \\ - \langle p, x_0 - x \rangle - \frac{1}{2} \langle M(x_0 - x), x_0 - x \rangle \Big] \leq 0.$$

We define the second-order subjet $J^{2,-}w_*(t_0, x_0)$ of the lower semi-continuous envelope w_* of w by setting

$$J^{2,-}w_*(t_0, x_0) := -J^{2,+}(-w_*)(t_0, x_0).$$

The following definition of viscosity solutions is based on the sub- and superjets. It follows from Fleming and Soner [40, Lemma V.4.1] that this definition is equivalent to Definition B.1.

Definition B.3. *Let $w : [0,T) \times \mathcal{O}$ be locally bounded.*

1. *We say that w is a viscosity subsolution of* (B.1) *if for each $(t_0, x_0) \in [0,T) \times \mathcal{O}$ and for each $(q,p,M) \in J^{2,+}w^*(t_0,x_0)$ we have*

$$F(t_0, x_0, w^*(t_0,x_0), q, p, M) \leq 0.$$

2. *We say that w is a viscosity supersolution of* (B.1) *if for each $(t_0, x_0) \in [0,T) \times \mathcal{O}$ and for each $(q,p,M) \in J^{2,-}w_*(t_0,x_0)$ we have*

$$F(t_0, x_0, w_*(t_0,x_0), q, p, M) \geq 0.$$

3. *We say that w is a viscosity solution of* (B.1) *if it is a viscosity subsolution as well as a viscosity supersolution.*

The closure $\overline{J}^{2,+}w^*(t_0,x_0)$ of the superjet $J^{2,+}w^*(t_0,x_0)$ is defined to be the set of all $(q,p,M) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{S}^n$ for which we can find a sequence $(t_j, x_j, q_j, p_j, M_j)_{j \in \mathbb{N}}$ such that $t_j \in [0,T)$, $x_j \in \mathcal{O}$ and $(q_j, p_j, M_j) \in J^{2,+}w^*(t_j,x_j)$ for all $j \in \mathbb{N}$ and such that

$$\lim_{j \to \infty} (t_j, x_j, w^*(t_j,x_j), q_j, p_j, M_j) = (t_0, x_0, w^*(t_0,x_0), q, p, M).$$

The closure $\overline{J}^{2,-}w_*(t_0,x_0)$ of $J^{2,-}w_*(t_0,x_0)$ is defined analogously.

With this definition and by the continuity of F it follows immediately that w is viscosity subsolution if and only if $F(t_0, x_0, w^*(t_0,x_0), q, p, M) \leq 0$ for all $(t_0,x_0) \in [0,T) \times \mathcal{O}$ and all $(q,p,M) \in \overline{J}^{2,+}w^*(t_0,x_0)$. The analogous statement clearly holds for viscosity supersolutions as well.

The main tool in proving uniqueness of viscosity solutions is the following theorem. We use the formulation in Pham [90, Lemma 4.4.6] and refer to Crandall et al. [22] for the proof.

Theorem B.4 (Ishii's lemma)**.** *Let u be an upper semi-continuous function on $[0,T) \times \mathcal{O}$, let v be a lower semi-continuous function on $[0,T) \times \mathcal{O}$ and let $\phi \in C^{1,1,2,2}([0,T)^2 \times \mathbb{R}^n \times \mathbb{R}^n)$. Suppose that (t_0, s_0, x_0, y_0) is a local maximum of*

$u(t,x) - v(s,y) - \phi(t,s,x,y)$. *Then for each* $\varepsilon > 0$ *there exist* $M, N \in \mathbb{S}^n$ *such that*

$$(D_t\phi(t_0,s_0,x_0,y_0), D_x\phi(t_0,s_0,x_0,y_0), M) \in \overline{J}^{2,+}u(t_0,x_0),$$
$$(-D_s\phi(t_0,s_0,x_0,y_0), -D_y\phi(t_0,s_0,x_0,y_0), N) \in \overline{J}^{2,-}v(t_0,x_0),$$

and

$$\begin{pmatrix} M & 0 \\ 0 & -N \end{pmatrix} \leq D^2_{x,y}\phi(t_0,s_0,x_0,y_0) + \varepsilon(D^2_{x,y}\phi(t_0,s_0,x_0,y_0))^2.$$

We will typically apply Ishii's lemma for $\phi(t,s,x,y) = \frac{k}{2}(|t-s|^2 + |x-y|^2)$ where $k \in \mathbb{N}$. We obtain

$$D_t\phi(t_0,s_0,x_0,y_0) = -D_s\phi(t_0,s_0,x_0,y_0) = k(t_0 - s_0),$$
$$D_x\phi(t_0,s_0,x_0,y_0) = -D_y\phi(t_0,s_0,x_0,y_0) = k(x_0 - y_0)$$

and

$$D^2_{x,y}\phi(t_0,s_0,x_0,y_0) = k\begin{pmatrix} I & -I \\ -I & I \end{pmatrix},$$
$$(D^2_{x,y}\phi(t_0,s_0,x_0,y_0))^2 = 2k^2\begin{pmatrix} I & -I \\ -I & I \end{pmatrix},$$

where I denotes the identity matrix in $\mathbb{S}^n$. Choosing $\varepsilon = 1/k$ in Ishii's lemma implies that

$$(k(t_0 - s_0), k(x_0 - y_0), M) \in \overline{J}^{2,+}u(t_0,x_0),$$
$$(k(t_0 - s_0), k(x_0 - y_0), N) \in \overline{J}^{2,-}v(s_0,y_0)$$

and

$$\begin{pmatrix} M & 0 \\ 0 & -N \end{pmatrix} \leq 3k\begin{pmatrix} I & -I \\ -I & I \end{pmatrix}.$$

C. On the existence of the Snell envelope

In this appendix we prove the existence of the Snell envelope for a càdlàg process $Y = (Y(u))_{u \in [t,T] \cup \{+\infty\}}$ which is uniformly integrable over all $[t,T] \cup \{+\infty\}$-valued stopping times. We then use the Snell envelope to construct ε-optimal stopping times for the problem of optimally stopping the process Y. For existence results on the Snell envelope in slightly different problem settings we refer to Fakeev [39], Shiryaev [96], Dellacherie and Meyer [29, Appendix I], El Karoui [36], Peskir and Shiryaev [89] as well as Karatzas and Shreve [61].

In what follows we fix a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We let $T > 0$, $t \in [0,T)$ and write $\mathcal{T} = [t,T] \cup \{+\infty\}$. We consider a filtration $\mathbb{F}^t = (\mathcal{F}^t(u))_{u \in \mathcal{T}}$ satisfying the usual assumptions. Moreover, we assume that $\mathcal{F}^t(T) = \mathcal{F}^t(\infty)$ and $\mathcal{F}^t(t)$ contains only events of probability zero and one. We let $Y = (Y(u))_{u \in \mathcal{T}}$ be an $\mathbb{F}^t$-adapted process which is càdlàg on $[t,T]$ and we denote by $\mathcal{B}(t)$ the set of all $\mathcal{T}$-valued stopping times. We assume that the family $(Y(\tau))_{\tau \in \mathcal{B}(t)}$ is uniformly integrable in the sense that $(\mathbb{E}[|Y(\tau)|^p])_{\tau \in \mathcal{B}(t)}$ is uniformly bounded for some $p > 1$.

Our aim is to construct the Snell envelope $Z^0 = (Z^0(u))_{u \in \mathcal{T}}$ associated with the problem of optimally stopping the process Y, i.e.

$$\inf_{\rho \in \mathcal{B}(t)} \mathbb{E}\left[Y(\rho)\right].$$

The Snell envelope Z^0 is the largest submartingale which is càdlàg on $[t,T]$, dominated by Y and satisfies

$$\mathbb{E}[Z^0(t)] = \inf_{\rho \in \mathcal{B}(t)} \mathbb{E}\left[Y(\rho)\right].$$

The Snell envelope turns out to be a modification of the process $Z = (Z(u))_{u \in \mathcal{T}}$

defined through

$$Z(u) = \operatorname*{ess\,inf}_{\rho \in \mathcal{B}(u)} \mathbb{E}\left[Y(\rho) \middle| \mathcal{F}^t(u)\right]. \tag{C.1}$$

We note that the uniform integrability assumption on Y implies that

$$Z(t) = \inf_{\rho \in \mathcal{B}(t)} \mathbb{E}\left[Y(\rho)\right] > -\infty. \tag{C.2}$$

We first need the following result on the essential infimum, see Neveu [86] or Karatzas and Shreve [61, Appendix A and Lemma D.1] for the proof.

Lemma C.1. *Let $\tau \in \mathcal{B}(t)$. Then the essential infimum*

$$\operatorname*{ess\,inf}_{\rho \in \mathcal{B}(\tau)} \mathbb{E}\left[Y(\rho) \middle| \mathcal{F}^t(\tau)\right]$$

exists and is unique. Moreover, if for each $\rho_1, \rho_2 \in \mathcal{B}(\tau)$ there exists some $\rho^* \in \mathcal{B}(\tau)$ such that

$$\mathbb{E}\left[Y(\rho^*) \middle| \mathcal{F}^t(\rho)\right] = \min\Big\{\mathbb{E}\left[Y(\rho_1) \middle| \mathcal{F}^t(\tau)\right], \mathbb{E}\left[Y(\rho_2) \middle| \mathcal{F}^t(\tau)\right]\Big\},$$

then there exists a sequence $(\rho_n)_{n\in\mathbb{N}}$ with $\rho_n \in \mathcal{B}(\tau)$ for each $n \in \mathbb{N}$ such that the sequence $(\mathbb{E}[Y(\rho_n)|\mathcal{F}^t(\tau)])_{n\in\mathbb{N}}$ is non-increasing and

$$\lim_{n\to\infty} \mathbb{E}\left[Y(\rho_n) \middle| \mathcal{F}^t(\tau)\right] = \operatorname*{ess\,inf}_{\rho \in \mathcal{B}(\tau)} \mathbb{E}\left[Y(\rho) \middle| \mathcal{F}^t(\tau)\right].$$

The existence of ρ^* in the previous lemma is easily verified in our setting, see Karatzas and Shreve [61, Lemma D.1] for a proof. We can hence always find a sequence $(\rho_n)_{n\in\mathbb{N}}$ such that we can approximate the essential infimum by a monotone sequence.

Let us now turn to some properties of the process Z defined in (C.1). The following lemma is in analogy with Karatzas and Shreve [61, proposition D.2], the proof is the same.

Lemma C.2. *Let $\tau, \sigma \in \mathcal{B}(t)$ and $\nu \in \mathcal{B}(\tau)$. Then*

1. *$Z(\tau) = Z(\sigma)$ on $\{\sigma = \tau\}$,*
2. *$\mathbb{E}\left[Z(\nu)|\mathcal{F}^t(\tau)\right] = \operatorname{ess\,inf}_{\rho \in \mathcal{B}(\nu)} \mathbb{E}\left[Y(\rho)|\mathcal{F}^t(\tau)\right]$,*

3. $\mathbb{E}\left[Z(\nu)|\mathcal{F}^t(\tau)\right] \geq Z(\tau)$,

4. $\mathbb{E}[Z(\nu)] = \inf_{\rho\in\mathcal{B}(\nu)} \mathbb{E}[Y(\rho)] \geq Z(t) > -\infty$.

Proof. 1. We denote $A := \{\sigma = \tau\}$ and note that $A \in \mathcal{F}^t(\sigma \wedge \tau)$. Let $\rho \in \mathcal{B}(\tau)$ and define a new stopping time $\rho_A \in \mathcal{B}(\tau)$ by

$$\rho_A := \begin{cases} \rho, & \text{on } A, \\ +\infty, & \text{on } A^c. \end{cases}$$

It follows that

$$\begin{aligned} \mathbb{1}_A \mathbb{E}\left[Y(\rho)|\mathcal{F}(\tau)\right] &= \mathbb{1}_A \mathbb{E}\left[Y(\rho_A)|\mathcal{F}(\tau)\right] \\ &= \mathbb{1}_A \mathbb{E}\left[Y(\rho_A)|\mathcal{F}(\tau \wedge \sigma)\right] = \mathbb{1}_A \mathbb{E}\left[Y(\rho_A)|\mathcal{F}(\sigma)\right] \geq \mathbb{1}_A Z(\sigma). \end{aligned}$$

Since ρ was chosen arbitrarily it follows that

$$\begin{aligned} \mathbb{1}_A Z(\tau) = \mathbb{1}_A &\operatorname*{ess\,inf}_{\rho\in\mathcal{B}(\tau)} \mathbb{E}\left[Y(\rho)|\mathcal{F}(\tau)\right] \\ &= \operatorname*{ess\,inf}_{\rho\in\mathcal{B}(\tau)} \mathbb{1}_A \mathbb{E}\left[Y(\rho)|\mathcal{F}(\tau)\right] \geq \mathbb{1}_A Z(\sigma). \end{aligned}$$

Reversing the roles of τ and σ yields the desired result.

2. First, note that since $Z(\nu) \leq \mathbb{E}[Y(\rho)|\mathcal{F}^t(\nu)]$ for every $\rho \in \mathcal{B}(\nu)$ we have

$$\mathbb{E}\left[Z(\nu)|\mathcal{F}^t(\tau)\right] \leq \mathbb{E}\left[\mathbb{E}[Y(\rho)|\mathcal{F}^t(\nu)]\middle|\mathcal{F}^t(\tau)\right] = \mathbb{E}\left[Y(\rho)\middle|\mathcal{F}^t(\tau)\right].$$

Since this holds for every $\rho \in \mathcal{B}(\nu)$ we have

$$\mathbb{E}\left[Z(\nu)|\mathcal{F}^t(\tau)\right] \leq \operatorname*{ess\,inf}_{\rho\in\mathcal{B}(\nu)} \mathbb{E}\left[Y(\rho)\middle|\mathcal{F}^t(\tau)\right]. \tag{C.3}$$

For the reverse inequality, we note that by Lemma C.1 we can find a sequence $(\rho_n)_{n\in\mathbb{N}}$ in $\mathcal{B}(\nu)$ such that $(\mathbb{E}[Y(\rho_n)|\mathcal{F}^t(\nu)])_{n\in\mathbb{N}}$ is non-increasing and

$$\operatorname*{ess\,inf}_{\rho\in\mathcal{B}(\nu)} \mathbb{E}\left[Y(\rho)\middle|\mathcal{F}^t(\nu)\right] = \lim_{n\to\infty} \mathbb{E}\left[Y(\rho_n)\middle|\mathcal{F}^t(\nu)\right].$$

The monotone convergence theorem hence shows that

$$\mathbb{E}\left[Z(\nu)|\mathcal{F}^t(\tau)\right] = \mathbb{E}\left[\left[\operatorname*{ess\,inf}_{\rho\in\mathcal{B}(\nu)} \mathbb{E}\left[Y(\rho)|\mathcal{F}^t(\nu)\right]\right]\middle|\mathcal{F}^t(\tau)\right]$$

$$
\begin{aligned}
&= \lim_{n\to\infty} \mathbb{E}\left[\left[\mathbb{E}\left[Y(\rho_n)|\mathcal{F}^t(\nu)\right]\right]\Big|\mathcal{F}^t(\tau)\right] \\
&= \lim_{n\to\infty} \mathbb{E}\left[Y(\rho_n)|\mathcal{F}^t(\tau)\right] \\
&\geq \operatorname*{ess\,inf}_{\rho\in\mathcal{B}(\nu)} \mathbb{E}\left[Y(\rho)\big|\mathcal{F}^t(\tau)\right].
\end{aligned}
$$

Combining this with (C.3) gives the desired equality.

3. This follows immediately from the previous step since

$$
\begin{aligned}
\mathbb{E}\left[Z(\nu)|\mathcal{F}^t(\tau)\right] &= \operatorname*{ess\,inf}_{\rho\in\mathcal{B}(\nu)} \mathbb{E}\left[Y(\rho)\big|\mathcal{F}^t(\tau)\right] \\
&\geq \operatorname*{ess\,inf}_{\rho\in\mathcal{B}(\tau)} \mathbb{E}\left[Y(\rho)\big|\mathcal{F}^t(\tau)\right] = Z(\tau).
\end{aligned}
$$

4. Choosing $\tau = t$ in the second step shows that

$$
\mathbb{E}[Z(\nu)] = \mathbb{E}\left[Z(\nu)|\mathcal{F}^t(t)\right] = \operatorname*{ess\,inf}_{\rho\in\mathcal{B}(\nu)} \mathbb{E}\left[Y(\rho)\big|\mathcal{F}^t(t)\right] = \inf_{\rho\in\mathcal{B}(\nu)} \mathbb{E}\left[Y(\rho)\right].
$$

By the third step we therefore have

$$
\mathbb{E}[Z(\nu)] = \inf_{\rho\in\mathcal{B}(\nu)} \mathbb{E}\left[Y(\rho)\right] \geq Z(t),
$$

and we conclude by (C.2). □

Given any $\tau \in \mathcal{B}(t)$, we denote by $\mathcal{B}^*(\tau)$ the set of all $\rho \in \mathcal{B}(\tau)$ such that $\rho > \tau$ on $\{\tau < T\}$. We can then define

$$
Z^*(\tau) = \operatorname*{ess\,inf}_{\rho\in\mathcal{B}^*(\tau)} \mathbb{E}\left[Y(\rho)\big|\mathcal{F}^t(\tau)\right].
$$

We note that the results of Lemma C.2 remain true if we replace Z by Z^*. The following result is in analogy with Karatzas and Shreve [61, Proposition D.3, Corollary D.4].

Lemma C.3. *Let $\tau \in \mathcal{B}(t)$.*

1. *Let $(\tau_n)_{n\in\mathbb{N}}$ be a decreasing sequence in $\mathcal{B}^*(\tau)$ such that $\tau_n \to \tau$ as $n \to \infty$. Then for all $A \in \mathcal{F}^t(\tau)$ we have*

$$
\mathbb{E}\left[Z^*(\tau)\mathbb{1}_A\right] = \lim_{n\to\infty} \mathbb{E}\left[Z^*(\tau_n)\mathbb{1}_A\right].
$$

2. *We have* $Z^*(\tau) = Z(\tau)$.

Proof. 1. By Lemma C.2.2 for Z^* we have

$$\mathbb{E}\left[Z^*(\tau_n)\middle|\mathcal{F}^t(\tau)\right] = \operatorname*{ess\,inf}_{\rho\in\mathcal{B}^*(\tau_n)} \mathbb{E}\left[Y(\rho)\middle|\mathcal{F}^t(\tau)\right] \geq Z^*(\tau)$$

for each $n \in \mathbb{N}$ and hence the sequence $(E[Z^*(\tau_n)|\mathcal{F}^t(\tau)]\mathbb{1}_A)_{n\in\mathbb{N}}$ is non-increasing and bounded from below by $Z^*(\tau)\mathbb{1}_A$ so that

$$\lim_{n\to\infty} \mathbb{E}\left[Z^*(\tau_n)\mathbb{1}_A\right] = \lim_{n\to\infty} \mathbb{E}\left[\mathbb{E}\left[Z^*(\tau_n)\middle|\mathcal{F}^t(\tau)\right]\mathbb{1}_A\right] \geq \mathbb{E}\left[Z^*(\tau)\mathbb{1}_A\right].$$

For proving the reverse inequality, let us fix $\rho \in \mathcal{B}^*(\tau)$ and define a sequence of stopping times $(\rho_n)_{n\in\mathbb{N}}$ through

$$\rho_n := \begin{cases} \rho, & \text{on } \{\tau_n < \rho\}, \\ +\infty, & \text{on } \{\tau_n \geq \rho\}. \end{cases}$$

Note that $\rho_n \in \mathcal{B}^*(\tau_n)$ for every $n \in \mathbb{N}$ and in particular

$$Z^*(\tau_n) \leq \mathbb{E}\left[Y(\rho_n)\middle|\mathcal{F}^t(\tau_n)\right].$$

We can decompose

$$\begin{aligned} \mathbb{1}_{\{\tau<T\}} &= \mathbb{1}_{\{\tau<T,\tau_n<T,\tau_n<\rho\}} + \mathbb{1}_{\{\tau<T,\tau_n<T,\tau_n\geq\rho\}} + \mathbb{1}_{\{\tau<T,\tau_n=T,\tau_n<\rho\}} \\ &\quad + \mathbb{1}_{\{\tau<T,\tau_n=T,\tau_n\geq\rho\}} + \mathbb{1}_{\{\tau<T,\tau_n=+\infty,\tau_n<\rho\}} + \mathbb{1}_{\{\tau<T,\tau_n=+\infty,\tau_n\geq\rho\}} \\ &= \mathbb{1}_{\{\tau_n<T,\tau_n<\rho\}} + \mathbb{1}_{\{\tau_n<T,\tau_n\geq\rho\}} + \mathbb{1}_{\{\tau<T,\tau_n=T,\tau_n<\rho\}} \\ &\qquad + \mathbb{1}_{\{\tau<T,\tau_n=T,\tau_n\geq\rho\}} + \mathbb{1}_{\{\tau<T,\tau_n=+\infty\}} \end{aligned}$$

and observer that

$$\begin{aligned} \mathbb{1}_{\{\tau<T\}} &= \lim_{n\to\infty} \mathbb{1}_{\{\tau_n<T,\tau_n<\rho\}}, & 0 &= \lim_{n\to\infty} \mathbb{1}_{\{\tau_n<T,\tau_n\geq\rho\}}, \\ 0 &= \lim_{n\to\infty} \mathbb{1}_{\{\tau<T,\tau_n=T,\tau_n<\rho\}}, & 0 &= \lim_{n\to\infty} \mathbb{1}_{\{\tau<T,\tau_n=T,\tau_n\geq\rho\}}, \\ 0 &= \lim_{n\to\infty} \mathbb{1}_{\{\tau<T,\tau_n=+\infty\}}. \end{aligned}$$

Using this and the dominated convergence theorem, it follows that

$$\lim_{n\to\infty} \mathbb{E}\left[Z^*(\tau_n)\mathbb{1}_{\{\tau_n<T,\tau_n<\rho\}}\mathbb{1}_A\right]$$

$$
\begin{aligned}
&\leq \lim_{n\to\infty} \mathbb{E}\left[\mathbb{E}\left[Y(\rho_n)\middle|\mathcal{F}^t(\tau_n)\right] \mathbb{1}_{\{\tau_n<T,\tau_n<\rho\}}\mathbb{1}_A\right] \\
&= \lim_{n\to\infty} \mathbb{E}\left[Y(\rho)\mathbb{1}_{\{\tau_n<T,\tau_n<\rho\}}\mathbb{1}_A\right] \\
&= \mathbb{E}\left[Y(\rho)\mathbb{1}_{\{\tau<T\}}\mathbb{1}_A\right].
\end{aligned}
$$

and by similar arguments we can show that

$$
\begin{aligned}
&\lim_{n\to\infty} \mathbb{E}\left[Z^*(\tau_n)\mathbb{1}_{\{\tau_n<T,\tau_n\geq\rho\}}\mathbb{1}_A\right] \leq 0, \\
&\lim_{n\to\infty} \mathbb{E}\left[Z^*(\tau_n)\mathbb{1}_{\{\tau<T,\tau_n=T,\tau_n<\rho\}}\mathbb{1}_A\right] \leq 0, \\
&\lim_{n\to\infty} \mathbb{E}\left[Z^*(\tau_n)\mathbb{1}_{\{\tau<T,\tau_n=T,\tau_n\geq\rho\}}\mathbb{1}_A\right] \leq 0, \\
&\lim_{n\to\infty} \mathbb{E}\left[Z^*(\tau_n)\mathbb{1}_{\{\tau<T,\tau_n=+\infty\}}\mathbb{1}_A\right] \leq 0.
\end{aligned}
$$

Putting the pieces together we have hence proved that

$$
\lim_{n\to\infty} \mathbb{E}\left[Z^*(\tau_n)\mathbb{1}_{\{\tau<T\}}\mathbb{1}_A\right] \leq \mathbb{E}\left[Y(\rho)\mathbb{1}_{\{\tau<T\}}\mathbb{1}_A\right].
$$

Similar arguments show that

$$
\lim_{n\to\infty} \mathbb{E}\left[Z^*(\tau_n)\mathbb{1}_{\{\tau=T\}}\mathbb{1}_A\right] \leq \mathbb{E}\left[Y(\rho)\mathbb{1}_{\{\tau=T\}}\mathbb{1}_A\right]
$$

and finally it is obvious that

$$
\lim_{n\to\infty} \mathbb{E}\left[Z^*(\tau_n)\mathbb{1}_{\{\tau=+\infty\}}\mathbb{1}_A\right] = \mathbb{E}\left[Y(\rho)\mathbb{1}_{\{\tau=+\infty\}}\mathbb{1}_A\right].
$$

Combining the last three equations shows that

$$
\lim_{n\to\infty} \mathbb{E}\left[Z^*(\tau_n)\mathbb{1}_A\right] \leq \mathbb{E}\left[Y(\rho)\mathbb{1}_A\right]. \tag{C.4}
$$

Now choose a sequence $(\rho_k)_{k\in\mathbb{N}}$ in $\mathcal{B}^*(\tau)$ such that $\mathbb{E}[Y(\rho_k)|\mathcal{F}^t(\tau)]$ converges monotonically down to $Z^*(\tau)$. Then Fatou's lemma and (C.4) show that

$$
\begin{aligned}
\mathbb{E}\left[Z^*(\tau)\mathbb{1}_A\right] &= \mathbb{E}\left[\lim_{k\to\infty} \mathbb{E}\left[Y(\rho_k)\middle|\mathcal{F}^t(\tau)\right] \mathbb{1}_A\right] \\
&\geq \limsup_{k\to\infty} \mathbb{E}\left[Y(\rho_k)\mathbb{1}_A\right] \\
&\geq \limsup_{k\to\infty} \lim_{n\to\infty} \mathbb{E}\left[Z^*(\tau_n)\mathbb{1}_A\right] \\
&= \lim_{n\to\infty} \mathbb{E}\left[Z^*(\tau_n)\mathbb{1}_A\right].
\end{aligned}
$$

2. First, it is clear that $Z(\tau) \le Z^*(\tau) \wedge Y(\tau)$. Now take $\rho \in \mathcal{B}(\tau)$ arbitrary so that

$$\begin{aligned}\mathbb{E}\left[Y(\rho)\middle|\mathcal{F}^t(\tau)\right] &= \mathbb{E}\left[Y(\rho)\middle|\mathcal{F}^t(\tau)\right]\mathbb{1}_{\{\tau=\rho\}} + \mathbb{E}\left[Y(\rho)\middle|\mathcal{F}^t(\tau)\right]\mathbb{1}_{\{\tau<\rho\}} \\ &\ge Y(\tau)\mathbb{1}_{\{\tau=\rho\}} + Z^*(\tau)\mathbb{1}_{\{\tau<\rho\}} \\ &\ge Y(\tau) \wedge Z^*(\tau),\end{aligned}$$

and since ρ was arbitrary we conclude that

$$Z(\tau) \ge Y(\tau) \wedge Z^*(\tau).$$

So, in order to conclude we only need to show that $Z^*(\tau) \le Y(\tau)$.

Let $(\tau_n)_{n\in\mathbb{N}}$ be a monotone sequence of stopping times in $\mathcal{B}^*(\tau)$ converging to τ. Then Fatou's lemma and the right-continuity of Y show that

$$Z^*(\tau) \le \limsup_{n\to\infty} \mathbb{E}\left[Y(\tau_n)\middle|\mathcal{F}^t(\tau)\right] \le \mathbb{E}\left[\lim_{n\to\infty} Y(\tau_n)\middle|\mathcal{F}^t(\tau)\right] = Y(\tau),$$

which is the desired inequality. □

We note that Lemma C.2.3 implies in particular that Z is a submartingale. Moreover, the process Z restricted to $[t,T]$ admits a càdlàg modification. Indeed, by Rogers and Williams [93, Theorem II.67.7] it suffices to show that the mapping $u \mapsto \mathbb{E}[Z(u)]$ is right-continuous on $[t,T]$ which is an immediate consequence of Lemma C.3. We denote this modification by $Z^0 = (Z^0(u))_{u\in\mathcal{T}}$ and remark that Z^0 is still a submartingale which is dominated by Y. Finally, we have

$$\mathbb{E}[Z^0(t)] = \mathbb{E}[Z(t)] = \inf_{\rho\in\mathcal{B}(t)} \mathbb{E}\left[Y(\rho)\right]$$

by Lemma C.2.4. We now show that Z^0 and Z even coincide on stopping times. The proof follows Karatzas and Shreve [61, Theorem D.7].

Lemma C.4. *For any $\tau \in \mathcal{B}(t)$ we have $Z^0(\tau) = Z(\tau)$.*

Proof. Since $Z^0(T) = Z(T)$ and $Z^0(+\infty) = Z(+\infty)$ we may without loss of generality assume that $\tau < T$. Let $(\tau_n)_{n\in\mathbb{N}}$ be a monotone sequence of stopping times taking values in $D([t,T]) \cup \{+\infty\}$ (where $D([t,T])$ denotes the dyadic rationals in $[t,T]$) such that $\tau_n > \tau$ and $\lim_{n\to\infty} \tau_n = \tau$. By the optional sampling theorem we have for every $n, k \in \mathbb{N}$

$$\mathbb{E}\left[Z^0(\tau_n)\middle|\mathcal{F}^t(\tau)\right] \geq \mathbb{E}\left[Z^0(\tau_{n+k})\middle|\mathcal{F}^t(\tau)\right] \geq Z^0(\tau).$$

Let now $A \in \mathcal{F}^t(\tau)$. Then $(\mathbb{E}[Z^0(\tau_n)|\mathcal{F}^t(\tau)]\mathbb{1}_A)_{n\in\mathbb{N}}$ is a non-increasing sequence bounded from below by $Z^0(\tau)\mathbb{1}_A$, i.e.

$$\liminf_{n\to\infty} \mathbb{E}[Z^0(\tau_n)\mathbb{1}_A] = \liminf_{n\to\infty} \mathbb{E}\left[\mathbb{E}\left[Z^0(\tau_n)\middle|\mathcal{F}^t(\tau)\right]\mathbb{1}_A\right] \geq \mathbb{E}[Z^0(\tau)\mathbb{1}_A].$$

On the other hand, the right-continuity of Z^0 and Fatou's lemma (which we are allowed to apply since $Z^0 \leq Y$ implies the uniform integrability of the positive part of Z^0) show that

$$\limsup_{n\to\infty} \mathbb{E}[Z^0(\tau_n)\mathbb{1}_A] \leq \mathbb{E}[Z^0(\tau)\mathbb{1}_A].$$

Moreover, Lemma C.3 shows that

$$\lim_{n\to\infty} \mathbb{E}\left[Z(\tau_n)\mathbb{1}_A\right] = \mathbb{E}\left[Z(\tau)\mathbb{1}_A\right].$$

Finally, since τ_n attains only countably many values we obtain

$$\mathbb{E}\left[Z(\tau_n)\mathbb{1}_A\right] = \mathbb{E}\left[Z^0(\tau_n)\mathbb{1}_A\right]$$

for every $n \in \mathbb{N}$. Combining these observations shows that

$$\mathbb{E}\left[Z(\tau)\mathbb{1}_A\right] = \lim_{n\to\infty} \mathbb{E}\left[Z(\tau_n)\mathbb{1}_A\right] = \lim_{n\to\infty} \mathbb{E}\left[Z^0(\tau_n)\mathbb{1}_A\right] = \mathbb{E}\left[Z(\tau)\mathbb{1}_A\right].$$

Now choosing $A = \{Z(\tau) > Z^0(\tau)\}$ shows that $\mathbb{P}[A] = 0$ so that $Z(\tau) \leq Z^0(\tau)$ almost surely. Interchanging the roles of Z and Z^0 then proves the result. □

We are left with showing that if $\hat{Z} = (\hat{Z}(u))_{u\in\mathcal{T}}$ is another submartingale which is dominated by Y and such that $\hat{Z}$ restricted to $[t,T]$ is càdlàg, then Z^0 dominates $\hat{Z}$. The proof is straightforward an follows as in Karatzas and Shreve [61, Theorem D.7].

Lemma C.5. *Suppose that $\hat{Z} = (\hat{Z}(u))_{u\in\mathcal{T}}$ is a submartingale which is dominated by Y and $\hat{Z}$ restricted to $[t,T]$ is càdlàg. Then*

$$\mathbb{P}\left[Z^0(u) \geq \hat{Z}(u) \text{ for all } u \in \mathcal{T}\right] = 1.$$

Proof. Fix $u \in \mathcal{T}$ and let $\rho \in \mathcal{B}(u)$ be given. By the optional sampling theorem we have
$$\mathbb{E}\left[Y(\rho)\middle|\mathcal{F}^t(u)\right] \geq \mathbb{E}\left[\hat{Z}(\rho)\middle|\mathcal{F}^t(u)\right] \geq \hat{Z}(u)$$
and therefore
$$Z^0(u) = Z(u) = \operatorname*{ess\,inf}_{\rho\in\mathcal{B}(u)} \mathbb{E}\left[Y(\rho)\middle|\mathcal{F}^t(u)\right] \geq \hat{Z}(u).$$
Since Z^0 and $\hat{Z}$ are càdlàg on $[t,T]$ the result follows. □

We have thus proved the existence of the Snell envelope. We gather the results in the following theorem.

Theorem C.6. *There exists a process $Z^0 = (Z^0(u))_{u\in\mathcal{T}}$ which is càdlàg on $[t,T]$ and such that*
$$Z^0(\tau) = \operatorname*{ess\,inf}_{\rho\in\mathcal{B}(\tau)} \mathbb{E}\left[Y(\rho)\middle|\mathcal{F}^t(\tau)\right]$$
for every stopping time $\tau \in \mathcal{B}(t)$. Moreover, Z^0 is the largest submartingale which is càdlàg on $[t,T]$ and which is dominated by Y. Finally, Z^0 satisfies
$$\mathbb{E}[Z^0(t)] = \inf_{\tau\in\mathcal{B}(t)} \mathbb{E}[Y(\tau)].$$

We are now going to construct an ε-optimal stopping time. For this, let θ be an arbitrary $[t,T]$-valued $\mathbb{F}^t$-stopping time and define
$$\tau^\varepsilon := \inf\left\{u \in [\theta,T] : Y(u) \leq Z^0(u) + \varepsilon\right\}. \tag{C.5}$$
We claim that τ^ε is ε-optimal in the sense that
$$\mathbb{E}\left[Y(\tau^\varepsilon)\middle|\mathcal{F}^t(\theta)\right] \leq \operatorname*{ess\,inf}_{\rho\in\mathcal{B}(\theta)} \mathbb{E}\left[Y(\rho)\middle|\mathcal{F}^t(\theta)\right] + \varepsilon = Z^0(\theta) + \varepsilon.$$
For the proof we follow Shiryaev [96, Theorem 3.3.2].

Lemma C.7. *For τ^ε defined in* (C.5) *we have*
$$\mathbb{E}\left[Y(\tau^\varepsilon)\middle|\mathcal{F}^t(\theta)\right] \leq Z^0(\theta) + \varepsilon.$$

Proof. We note that by the right-continuity of Y and Z^0 we have

$$\mathbb{E}\left[Y(\tau^\varepsilon)\middle|\mathcal{F}^t(\theta)\right] \leq \mathbb{E}\left[Z^0(\tau^\varepsilon)\middle|\mathcal{F}^t(\theta)\right] + \varepsilon$$

and hence in order to prove the result it suffices to show that

$$Z^0(\theta) = \mathbb{E}\left[Z^0(\tau^\varepsilon)\middle|\mathcal{F}^t(\theta)\right].$$

For this, we only have to show that

$$\mathbb{E}[Z^0(\theta)] = \mathbb{E}[Z^0(\tau^\varepsilon)] \tag{C.6}$$

since the optional sampling theorem implies that

$$Z^0(\theta) - \mathbb{E}\left[Z^0(\tau^\varepsilon)\middle|\mathcal{F}^t(\theta)\right] \leq 0$$

but assuming that (C.6) holds we obtain

$$\mathbb{E}\left[Z^0(\theta) - \mathbb{E}\left[Z^0(\tau^\varepsilon)\middle|\mathcal{F}^t(\theta)\right]\right] = 0$$

and hence

$$Z^0(\theta) = \mathbb{E}\left[Z^0(\tau^\varepsilon)\middle|\mathcal{F}^t(\theta)\right].$$

For $\delta > 0$ we define $\mathcal{B}(\theta, \delta)$ to be the set of all stopping times $\rho \in \mathcal{B}(\theta)$ such that

$$\mathbb{P}[\rho < \tau^\varepsilon] \leq \delta.$$

Now, let $\rho \in \mathcal{B}(\theta) \setminus \mathcal{B}(\theta, \delta)$. On $\{\rho < \tau^\varepsilon\}$ we have

$$Y(\rho) > Z^0(\rho) + \varepsilon$$

by the definition of τ^ε and by the definition of $\mathcal{B}(\theta, \delta)$ we have

$$\mathbb{P}[\rho < \tau^\varepsilon] > \delta.$$

Therefore,

$$\begin{aligned}
\mathbb{E}\left[Y(\rho)\middle|\mathcal{F}^t(\theta)\right] &= \mathbb{E}\left[Y(\rho)\middle|\mathcal{F}^t(\theta)\right]\mathbb{1}_{\{\rho<\tau^\varepsilon\}} + \mathbb{E}\left[Y(\rho)\middle|\mathcal{F}^t(\theta)\right]\mathbb{1}_{\{\rho\geq\tau^\varepsilon\}} \\
&\geq \mathbb{E}\left[Z^0(\rho) + \varepsilon\middle|\mathcal{F}^t(\theta)\right]\mathbb{1}_{\{\rho<\tau^\varepsilon\}} + \mathbb{E}\left[Z^0(\rho)\middle|\mathcal{F}^t(\theta)\right]\mathbb{1}_{\{\rho\geq\tau^\varepsilon\}} \\
&= \mathbb{E}\left[Z^0(\rho)\middle|\mathcal{F}^t(\theta)\right] + \varepsilon\mathbb{1}_{\{\rho<\tau^\varepsilon\}} \\
&\geq Z^0(\theta) + \varepsilon\mathbb{1}_{\{\rho<\tau^\varepsilon\}}.
\end{aligned}$$

Upon taking expectations on both sides this shows that

$$\mathbb{E}\left[Y(\rho)\right] \geq \mathbb{E}[Z^0(\theta)] + \varepsilon\mathbb{P}[\rho < \tau^\varepsilon] > \mathbb{E}[Z^0(\theta)] + \varepsilon\delta.$$

Taking the infimum over all $\rho \in \mathcal{B}(\theta) \setminus \mathcal{B}(\theta, \delta)$ hence shows that

$$\inf_{\rho \in \mathcal{B}(\theta)\setminus\mathcal{B}(\theta,\delta)} \mathbb{E}\left[Y(\rho)\right] > \mathbb{E}\left[Z^0(\theta)\right] + \varepsilon\delta > \mathbb{E}\left[Z^0(\theta)\right], \tag{C.7}$$

which together with Lemma C.2.4 implies that

$$\mathbb{E}\left[Z^0(\theta)\right] = \inf_{\rho \in \mathcal{B}(\theta,\delta)} \mathbb{E}\left[Y(\rho)\right].$$

Let us now take a sequence $(\rho_n)_{n\in\mathbb{N}}$ such that $\mathbb{E}[Y(\rho_n)]$ converges monotonically down to $\mathbb{E}[Z^0(\theta)]$. In light of (C.7) we can assume without loss of generality that $\rho_n \in \mathcal{B}(\theta, 2^{-n})$. We can now define a new sequence of stopping times $(\tau_n)_{n\in\mathbb{N}}$ by setting

$$\tau_n = \max\{\rho_n, \tau^\varepsilon\}.$$

Then $\tau_n \in \mathcal{B}(\tau^\varepsilon)$ and hence

$$\mathbb{E}[Z^0(\tau^\varepsilon)] \leq \mathbb{E}[\mathbb{E}[Y(\tau_n)|\mathcal{F}^t(\tau^\varepsilon)]] = \mathbb{E}[Y(\tau_n)]$$

for all $n \in \mathbb{N}$ so that

$$\mathbb{E}[Z^0(\tau^\varepsilon)] \leq \limsup_{n\to\infty} \mathbb{E}[Y(\tau_n)].$$

We furthermore have

$$\begin{aligned}\limsup_{n\to\infty} \mathbb{E}[Y(\tau_n)] &= \limsup_{n\to\infty} \mathbb{E}\left[Y(\rho_n) - Y(\rho_n)\mathbb{1}_{\{\rho_n<\tau^\varepsilon\}} + Y(\tau^\varepsilon)\mathbb{1}_{\{\rho_n<\tau^\varepsilon\}}\right] \\ &\leq \limsup_{n\to\infty} \mathbb{E}\left[Y(\rho_n)\right] \\ &\qquad + \limsup_{n\to\infty} \mathbb{E}\left[|Y(\rho_n)|\mathbb{1}_{\{\rho_n<\tau^\varepsilon\}}\right] \\ &\qquad\quad + \limsup_{n\to\infty} \mathbb{E}\left[|Y(\tau^\varepsilon)|\mathbb{1}_{\{\rho_n<\tau^\varepsilon\}}\right].\end{aligned}$$

By the choice of $(\rho_n)_{n\in\mathbb{N}}$ we have

$$\limsup_{n\to\infty} \mathbb{E}\left[Y(\rho_n)\right] = \mathbb{E}\left[Z^0(\theta)\right].$$

Moreover, since Y is uniformly integrable we have by Hölder's inequality

$$\mathbb{E}\left[|Y(\rho_n)|\mathbb{1}_{\{\rho_n<\tau^\varepsilon\}}\right] \leq \mathbb{E}\left[|Y(\rho_n)|^p\right]\mathbb{P}[\rho_n < \tau^\varepsilon] \leq 2^{-n}\mathbb{E}\left[|Y(\rho_n)|^p\right]$$

and hence since $(\mathbb{E}[|Y(\rho_n)|^p])_{n\in\mathbb{N}}$ is bounded we see that

$$\limsup_{n\to\infty}\mathbb{E}\left[|Y(\rho_n)|\mathbb{1}_{\{\rho_n<\tau^\varepsilon\}}\right] \leq 0.$$

Similarly, we can show that

$$\limsup_{n\to\infty}\mathbb{E}\left[|Y(\tau^\varepsilon)|\mathbb{1}_{\{\rho_n<\tau^\varepsilon\}}\right] \leq 0$$

and hence

$$\mathbb{E}[Z^0(\tau^\varepsilon)] \leq \limsup_{n\to\infty}\mathbb{E}[Y(\tau_n)] \leq \mathbb{E}\left[Z^0(\theta)\right].$$

On the other hand, Z^0 is a submartingale and hence an application of the optional sampling theorem shows that

$$\mathbb{E}[Z^0(\tau^\varepsilon)] \geq \mathbb{E}\left[Z^0(\theta)\right],$$

which finishes the proof. □

In Chapter 3 we are concerned with the construction of ε-optimal stopping times for the process $\tilde{Y} = (\tilde{Y}(u))_{u\in\mathcal{T}}$ defined through

$$\tilde{Y}(u) = \begin{cases} y, & \text{if } u = t, \\ Y(u-), & \text{if } u \in (t,T], \\ Y(+\infty), & \text{if } u = +\infty, \end{cases}$$

where $y \in \mathbb{R}$. We assume in addition that $Y(T-) = Y(T)$, i.e. Y is left-continuous at T. Finally, we assume that $\mathbb{F}^t$ is the augmented filtration generated by a standard Brownian motion $W = (W(u) - W(t))_{u\geq t}$. We recall that in this setup every stopping time ρ is predictable.

For every $\theta \in \mathcal{B}(t)$ we define $\tilde{Z} = (\tilde{Z}(u))_{u\in\mathcal{T}}$ by

$$\tilde{Z}(\theta) := \operatorname*{ess\,inf}_{\rho\in\mathcal{B}(\theta)}\mathbb{E}\left[\tilde{Y}(u)\middle|\mathcal{F}^t(\theta)\right].$$

and similarly we define $\tilde{Z}^* = (\tilde{Z}^*(u))_{u\in\mathcal{T}}$ by

$$\tilde{Z}^*(\theta) := \operatorname*{ess\,inf}_{\rho\in\mathcal{B}^*(\theta)} \mathbb{E}\left[\tilde{Y}(u)\middle|\mathcal{F}^t(\theta)\right].$$

We note that the results of Lemma C.2 still remain true if we replace Z by $\tilde{Z}$ or $\tilde{Z}^*$, respectively.

Our first aim is to investigate the relation between $\tilde{Z}$, $\tilde{Z}^*$ and the Snell envelope Z^0 of Y.

Lemma C.8. *Let $\tau \in \mathcal{B}(t)$ and $\varepsilon > 0$. Then*

(i) $\tilde{Z}(\tau) \le Z^0(\tau)$,

(ii) $Z^0(\tau) \le \tilde{Z}^*(\tau)$,

(iii) $Z^0(\tau) = \tilde{Z}(\tau)$ *on* $\{\tilde{Y}(\tau) > \tilde{Z}(\tau) + \varepsilon\}$.

Proof. (i) Let $\rho \in \mathcal{B}(\tau)$. Then we can find a sequence of stopping times $(\rho_n)_{n\in\mathbb{N}}$ in $\mathcal{B}(\tau)$ such that $\rho_n > \rho$ on $\{\rho \in [\tau, T)\}$, $\rho_n = \rho$ on $\{\rho \in \{T, +\infty\}\}$ and such that $\rho_n \downarrow \rho$. It follows that

$$\begin{aligned}
&\mathbb{E}\left[\tilde{Y}(\rho_n)\middle|\mathcal{F}^t(\tau)\right] \\
&= \mathbb{E}\left[\tilde{Y}(\rho_n)\middle|\mathcal{F}^t(\tau)\right] \mathbb{1}_{\{\rho\in[\tau,T)\}} + \mathbb{E}\left[\tilde{Y}(\rho)\middle|\mathcal{F}^t(\tau)\right] \mathbb{1}_{\{\rho\in\{T,+\infty\}\}} \\
&= \mathbb{E}\left[\tilde{Y}(\rho_n)\middle|\mathcal{F}^t(\tau)\right] \mathbb{1}_{\{\rho\in[\tau,T)\}} + \mathbb{E}\left[Y(\rho)\middle|\mathcal{F}^t(\tau)\right] \mathbb{1}_{\{\rho\in\{T,+\infty\}\}}
\end{aligned}$$

Sending $n \to \infty$ we obtain from Fatou's lemma that

$$\begin{aligned}
\tilde{Z}(\tau) &\le \limsup_{n\to\infty} \mathbb{E}\left[\tilde{Y}(\rho_n)\middle|\mathcal{F}^t(\tau)\right] \\
&= \limsup_{n\to\infty} \mathbb{E}\left[\tilde{Y}(\rho_n)\middle|\mathcal{F}^t(\tau)\right] \mathbb{1}_{\{\rho\in[\tau,T)\}} + \mathbb{E}\left[Y(\rho)\middle|\mathcal{F}^t(\tau)\right] \mathbb{1}_{\{\rho\in\{T,+\infty\}\}} \\
&\le \mathbb{E}\left[Y(\rho)\middle|\mathcal{F}^t(\tau)\right].
\end{aligned}$$

Since ρ was chosen arbitrarily this implies $\tilde{Z}(\tau) \le Z^0(\tau)$.

(ii) Let $\rho \in \mathcal{B}^*(\tau)$. Then we can find a sequence of stopping times $(\rho_n)_{n\in\mathbb{N}}$ in $\mathcal{B}(\tau)$ such that $\rho_n < \rho$ on $\{\rho \in (\tau, T]\}$ and $\rho_n = \rho$ on $\{\rho = +\infty\}$. Then

$$\begin{aligned}
&\mathbb{E}\left[Y(\rho_n)\middle|\mathcal{F}^t(\tau)\right] \\
&= \mathbb{E}\left[Y(\rho_n)\middle|\mathcal{F}^t(\tau)\right]\mathbb{1}_{\{\rho\in(\tau,T]\}} + \mathbb{E}\left[Y(\rho)\middle|\mathcal{F}^t(\tau)\right]\mathbb{1}_{\{\rho=+\infty\}} \\
&= \mathbb{E}\left[Y(\rho_n)\middle|\mathcal{F}^t(\tau)\right]\mathbb{1}_{\{\rho\in(\tau,T]\}} + \mathbb{E}\left[\tilde{Y}(\rho)\middle|\mathcal{F}^t(\tau)\right]\mathbb{1}_{\{\rho=+\infty\}}
\end{aligned}$$

Sending $n \to \infty$ we obtain from Fatou's lemma that

$$\begin{aligned}
Z^0(\tau) &\le \limsup_{n\to\infty} \mathbb{E}\left[Y(\rho_n)\middle|\mathcal{F}^t(\tau)\right] \\
&= \limsup_{n\to\infty} \mathbb{E}\left[Y(\rho_n)\middle|\mathcal{F}^t(\tau)\right]\mathbb{1}_{\{\rho\in(\tau,T]\}} + \mathbb{E}\left[\tilde{Y}(\rho)\middle|\mathcal{F}^t(\tau)\right]\mathbb{1}_{\{\rho=+\infty\}} \\
&\le \mathbb{E}\left[\tilde{Y}(\rho)\middle|\mathcal{F}^t(\tau)\right]
\end{aligned}$$

and since ρ was chosen arbitrarily we have $Z^0(\tau) \le \tilde{Z}^*(\tau)$.

(iii) On $\{\tau \in \{T, +\infty\}\}$ it is clear that $Z^0(\tau)$ and $\tilde{Z}(\tau)$ coincide and hence we may without loss of generality assume that $\tau < T$. By (i) and (ii) it suffices to show that $\tilde{Z}^*(\tau) = \tilde{Z}(\tau)$ on $\{\tilde{Y}(\tau) > \tilde{Z}(\tau) + \varepsilon\}$. Let $(\rho_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{B}(\tau)$ such that

$$\tilde{Z}(\tau) = \lim_{n\to\infty} \mathbb{E}\left[\tilde{Y}(\rho_n)\middle|\mathcal{F}^t(\tau)\right].$$

Then

$$\begin{aligned}
\tilde{Z}(\tau) &= \lim_{n\to\infty} \mathbb{E}\left[\tilde{Y}(\rho_n)\middle|\mathcal{F}^t(\tau)\right] \\
&= \lim_{n\to\infty} \mathbb{E}\left[\tilde{Y}(\tau)\middle|\mathcal{F}^t(\tau)\right]\mathbb{1}_{\{\rho_n=\tau\}} + \lim_{n\to\infty} \mathbb{E}\left[\tilde{Y}(\rho_n)\middle|\mathcal{F}^t(\tau)\right]\mathbb{1}_{\{\rho_n>\tau\}} \\
&\ge \lim_{n\to\infty} \mathbb{E}\left[\tilde{Y}(\tau)\middle|\mathcal{F}^t(\tau)\right]\mathbb{1}_{\{\rho_n=\tau\}} + \tilde{Z}(\tau)\lim_{n\to\infty} \mathbb{1}_{\{\rho_n>\tau\}}.
\end{aligned}$$

On $\{\tilde{Y}(\tau) > \tilde{Z}(\tau) + \varepsilon\}$ this implies that

$$\tilde{Z}(\tau) \ge (\tilde{Z}(\tau) + \varepsilon)\lim_{n\to\infty} \mathbb{1}_{\{\rho_n=\tau\}} + \tilde{Z}(\tau)\lim_{n\to\infty} \mathbb{1}_{\{\rho_n>\tau\}}$$

which shows that $\lim_{n\to\infty} \mathbb{1}_{\{\rho_n=\tau\}} = 0$ and hence

$$\tilde{Z}(\tau) = \lim_{n\to\infty} \mathbb{E}\left[\tilde{Y}(\rho_n)\mathbb{1}_{\{\rho_n>\tau\}}\middle|\mathcal{F}^t(\tau)\right]$$

on $\{\tilde{Y}(\tau) > \tilde{Z}(\tau) + \varepsilon\}$. It is then immediate that if we define

$$\tau_n := \begin{cases} \rho_n, & \text{on } \{\rho_n > \tau\}, \\ +\infty, & \text{on } \{\rho_n = \tau\}, \end{cases}$$

that

$$\tilde{Z}(\tau) = \lim_{n\to\infty} \mathbb{E}\left[\tilde{Y}(\tau_n) \middle| \mathcal{F}^t(\tau)\right]$$

on $\{\tilde{Y}(\tau) > \tilde{Z}(\tau) + \varepsilon\}$. Since $\tau_n \in \mathcal{B}^*(\tau)$ this implies that

$$\tilde{Z}(\tau) \geq \tilde{Z}^*(\tau) \geq \tilde{Z}(\tau)$$

on $\{\tilde{Y}(\tau) > \tilde{Z}(\tau) + \varepsilon\}$. ☐

Proposition C.9. *Let θ be a $[t,T]$-valued $\mathbb{F}^t$-stopping time. Then for every $\delta > 0$ there exists a stopping time $\tau^*_\delta \in \mathcal{B}(\theta)$ such that*

$$\mathbb{E}\left[\tilde{Y}(\tau^*_\delta) \middle| \mathcal{F}^t(\theta)\right] \leq \tilde{Z}(\theta) + \varepsilon \tag{C.8}$$

on a set $A \subset \Omega$ with $\mathbb{P}[A] \geq 1 - \delta$ and $\tau^*_\delta = +\infty$ on A^c.

Proof. We first note that the uniform integrability of Y implies the uniform integrability of Z^0. Indeed, fix $\tau \in \mathcal{B}(t)$ and take a sequence $(\rho_n)_{n\in\mathbb{N}}$ in $\mathcal{B}(\tau)$ such that

$$Z^0(\tau) = \lim_{n\to\infty} \mathbb{E}\left[Y(\rho_n) \middle| \mathcal{F}^t(\tau)\right].$$

Then Jensen's inequality and Fatou's lemma show that

$$\begin{aligned} \mathbb{E}\left[|Z^0(\tau)|^p\right] &= \mathbb{E}\left[\left|\lim_{n\to\infty} \mathbb{E}\left[Y(\rho_n) \middle| \mathcal{F}^t(\tau)\right]\right|^p\right] \\ &= \mathbb{E}\left[\lim_{n\to\infty} \left|\mathbb{E}\left[Y(\rho_n) \middle| \mathcal{F}^t(\tau)\right]\right|^p\right] \\ &\leq \mathbb{E}\left[\lim_{n\to\infty} \mathbb{E}\left[|Y(\rho_n)|^p \middle| \mathcal{F}^t(\tau)\right]\right] \\ &\leq \liminf_{n\to\infty} \mathbb{E}\left[|Y(\rho_n)|^p\right] \\ &\leq \sup_{\rho\in\mathcal{B}(t)} \mathbb{E}\left[|Y(\rho)|^p\right]. \end{aligned}$$

It therefore follows that Z^0 admits a Doob-Meyer decomposition on $[t,T]$, i.e.

$$Z^0(u) = M^0(u) + \Lambda^0(u), \qquad u \in [t,T],$$

where $M^0 = (M^0(u))_{u\in[t,T]}$ is a uniformly integrable martingale and $\Lambda^0 = (\Lambda^0(u))_{u\in[t,T]}$ is a non-decreasing càdlàg process with $\Lambda^0(t) = 0$. Consider now the stopping time $\tau^{\varepsilon/2}$ constructed in Lemma C.7. In the proof of the lemma we have seen that

$$Z^0(\theta) = \mathbb{E}\left[Z^0(\tau^{\varepsilon/2})\middle|\mathcal{F}^t(\theta)\right]$$

from which we infer that $\Lambda^0(u) = \Lambda^0(\theta)$ for every $u \in [\theta, \tau^{\varepsilon/2} \wedge T]$.

Let us now define

$$\tau^*_\delta = \begin{cases} \theta, & \text{on } \{\tilde{Y}(\theta) \le \tilde{Z}(\theta) + \varepsilon\}, \\ \tau^{\varepsilon/2}, & \text{on } \{\tilde{Y}(\theta) > \tilde{Z}(\theta) + \varepsilon\} \cap \{\tilde{Y}(\tau^{\varepsilon/2}) = Y(\tau^{\varepsilon/2})\}, \\ \tau^\delta, & \text{on } \{\tilde{Y}(\theta) > \tilde{Z}(\theta) + \varepsilon\} \cap \{\tilde{Y}(\tau^{\varepsilon/2}) \ne Y(\tau^{\varepsilon/2})\}, \end{cases}$$

where τ^δ will be defined below. We note that on $\{\tilde{Y}(\theta) \le \tilde{Z}(\theta) + \varepsilon\}$ the ε-optimality of τ^*_δ in the sense of (C.8) is immediate. On $\{\tilde{Y}(\theta) > \tilde{Z}(\theta) + \varepsilon\} \cap \{\tilde{Y}(\tau^{\varepsilon/2}) = Y(\tau^{\varepsilon/2})\}$ we have $\tilde{Y}(\tau^*_\delta) = \tilde{Y}(\tau^{\varepsilon/2}) = Y(\tau^{\varepsilon/2})$ and hence

$$\mathbb{E}\left[\tilde{Y}(\tau^*_\delta)\middle|\mathcal{F}^t(\theta)\right] = \mathbb{E}\left[Y(\tau^{\varepsilon/2})\middle|\mathcal{F}^t(\theta)\right] \le Z^0(\theta) + \frac{1}{2}\varepsilon < \tilde{Z}(\theta) + \varepsilon$$

by the construction of $\tau^{\varepsilon/2}$ and since $\tilde{Y}(\theta) > \tilde{Z}(\theta) + \varepsilon$ implies that $Z^0(\theta) = \tilde{Z}(\theta)$ by Lemma C.8.

Let us now turn to the construction of τ^δ on $\{\tilde{Y}(\theta) > \tilde{Z}(\theta) + \varepsilon\} \cap \{\tilde{Y}(\tau^{\varepsilon/2}) \ne Y(\tau^{\varepsilon/2})\}$. We note that we must necessarily have $\tau^{\varepsilon/2} < T$. Moreover, since Y jumps at time $\tau^{\varepsilon/2}$ we have

$$\tilde{Y}(\tau^{\varepsilon/2}) > Z^0(\tau^{\varepsilon/2}) + \frac{1}{2}\varepsilon \quad \text{and} \quad Y(\tau^{\varepsilon/2}) \le Z^0(\tau^{\varepsilon/2}) + \frac{1}{2}\varepsilon.$$

We define

$$\tau^\Lambda := \inf\left\{u \in [\tau^{\varepsilon/2}, T] : \Lambda^0(u) > \Lambda^0(\tau^{\varepsilon/2}) + \frac{1}{3}\varepsilon\right\} \wedge T$$

and

$$\hat{\tau} := \inf\left\{u \in (\tau^{\varepsilon/2}, T] : \tilde{Y}(u) > Z^0(u) + \frac{2}{3}\varepsilon\right\} \wedge T.$$

Note that by the right-continuity of Λ^0 we have $\tau^\Lambda > \tau^{\varepsilon/2}$. Moreover, by the right-continuity of Y and Z^0 and since $Y(\tau^{\varepsilon/2}) \le Z^0(\tau^{\varepsilon/2}) + \varepsilon/2$ we see that $\hat{\tau} > \tau^{\varepsilon/2}$.

Since the stochastic interval $(\tau^{\varepsilon/2}, \tau^{\Lambda} \wedge \hat{\tau})$ is a predictable subset of $[t, T] \times \Omega$ the predictable section theorem (Rogers and Williams [93, Theorem VI.19.1]) guarantees that for any $\delta > 0$ there exists a stopping time τ^{δ} such that

$$\mathbb{P}[\tau^{\varepsilon/2} < \tau^{\delta} < \tau^{\Lambda} \wedge \hat{\tau}] \geq 1 - \delta$$

and $\tau^{\delta} = +\infty$ otherwise. We note that by the construction of τ^{Λ} and $\hat{\tau}$ we must have

$$\Lambda^0(\tau^{\delta}) \leq \Lambda^0(\tau^{\varepsilon/2}) + \frac{1}{3}\varepsilon$$

and

$$\tilde{Y}(\tau^{\delta}) \leq Z^0(\tau^{\delta}) + \frac{2}{3}\varepsilon$$

whenever $\tau^{\delta} \in (\tau^{\varepsilon/2}, \tau^{\Lambda} \wedge \hat{\tau})$. On $\{\tilde{Y}(\theta) > \tilde{Z}(\theta) + \varepsilon\} \cap \{\tilde{Y}(\tau^{\varepsilon/2}) \neq Y(\tau^{\varepsilon/2})\}$ and $\tau^{\delta} \in (\tau^{\varepsilon/2}, \tau^{\Lambda} \wedge \hat{\tau})$ it therefore follows that

$$\begin{aligned}
\mathbb{E}\left[\tilde{Y}(\tau^*_{\delta})\middle|\mathcal{F}^t(\theta)\right] &= \mathbb{E}\left[\tilde{Y}(\tau^{\delta})\middle|\mathcal{F}^t(\theta)\right] \\
&\leq \mathbb{E}\left[Z^0(\tau^{\delta})\middle|\mathcal{F}^t(\theta)\right] + \frac{2}{3}\varepsilon \\
&= \mathbb{E}\left[M^0(\tau^{\delta}) + \Lambda^0(\tau^{\delta})\middle|\mathcal{F}^t(\theta)\right] + \frac{2}{3}\varepsilon \\
&\leq \mathbb{E}\left[M^0(\tau^{\delta}) + \Lambda^0(\tau^{\varepsilon/2})\middle|\mathcal{F}^t(\theta)\right] + \varepsilon \\
&= Z^0(\theta) + \varepsilon \\
&= \tilde{Z}(\theta) + \varepsilon,
\end{aligned}$$

which proves the ε-optimality of τ^*_{δ}. □

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