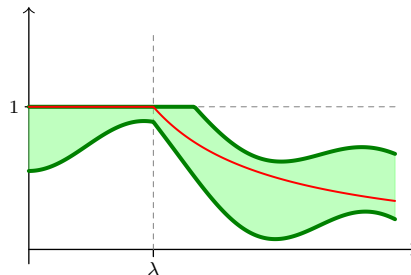


Portfolio Optimization and Stochastic Control under Transaction Costs

Yaroslav Melnyk



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1. Gutachter: Prof. Dr. Ralf Korn
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Abstract

This thesis is concerned with stochastic control problems under transaction costs. In particular, we consider a generalized menu cost problem with partially controlled regime switching, general multidimensional running cost problems and the maximization of long-term growth rates in incomplete markets. The first two problems are considered under a general cost structure that includes a fixed cost component, whereas the latter is analyzed under proportional and Morton-Pliska transaction costs.

For the menu cost problem and the running cost problem we provide an equivalent characterization of the value function by means of a generalized version of the Itô-Dynkin formula instead of the more restrictive, traditional approach via the use of quasi-variational inequalities (QVIs). Based on the finite element method and weak solutions of QVIs in suitable Sobolev spaces, the value function is constructed iteratively. In addition to the analytical results, we study a novel application of the menu cost problem in management science. We consider a company that aims to implement an optimal investment and marketing strategy and must decide when to issue a new version of a product and when and how much to invest into marketing.

For the long-term growth rate problem we provide a rigorous asymptotic analysis under both proportional and Morton-Pliska transaction costs in a general incomplete market that includes, for instance, the Heston stochastic volatility model and the Kim-Omberg stochastic excess return model as special cases. By means of a dynamic programming approach leading-order optimal strategies are constructed and the leading-order coefficients in the expansions of the long-term growth rates are determined. Moreover, we analyze the asymptotic performance of Morton-Pliska strategies in settings with proportional transaction costs. Finally, pathwise optimality of the constructed strategies is established.

Zusammenfassung

Diese Arbeit befasst sich mit stochastischen Kontrollproblemen unter Transaktionskosten. Insbesondere werden eine Verallgemeinerung des Menu-Cost-Problems mit teilweise kontrolliertem Regimewechsel, allgemeine Running-Cost-Probleme und die Maximierung von Wachstumsraten in unvollständigen Märkten untersucht. Die ersten beiden Probleme werden unter einer allgemeinen Kostenstruktur mit einer fixen Kostenkomponente betrachtet, wohingegen das letzte Problem unter proportionalen und Morton-Pliska Kosten untersucht wird.

Für das Menu-Cost-Problem and das Running-Cost-Problem beweisen wir eine äquivalente Charakterisierung der Wertfunktion durch eine verallgemeinerte Version der Itô-Dynkin Formel anstatt des restriktiveren traditionellen Zugangs mittels quasi-variationeller Ungleichungen (QVIs). Basierend auf der Finite-Elemente-Methode und schwachen Lösungen von QVIs in geeigneten Sobolev-Räumen wird die Wertfunktion iterativ konstruiert. Zusätzlich zu unseren analytischen Resultaten wird eine neuartige Anwendung des Menu-Cost-Problems in Operations Research untersucht. Wir betrachten ein Unternehmen, das eine optimale Marketing- und Investmentstrategie umsetzt und dabei entscheiden muss, wann eine neue Version eines Produkts einzuführen ist und wann und wieviel in das Marketing investiert werden muss.

Das Wachstumsratenproblem untersuchen wir asymptotisch für geringe proportionale Transaktionskosten und kleine Kosten vom Morton-Pliska Typ in einem unvollständigen Finanzmarktmodell. Dieses beinhaltet unter anderem das stochastische Volatilitätsmodell von Heston und das Modell von Kim und Omberg für stochastische Überschussrenditen. Mittels dynamischer Programmierung werden Strategien konstruiert, die in führender Ordnung optimal sind, und die entsprechenden Koeffizienten in den Entwicklungen der Wachstumsraten bestimmt. Darüber hinaus wird die Effizienz von Morton-Pliska Strategien in Märkten mit proportionalen Kosten analysiert. Schließlich zeigen wir, dass die zuvor bestimmten Strategien pfadweise optimal sind.

1 Introduction

Motivation. There are many everyday occurrences that are naturally modeled by means of non-deterministic systems. Counteracting and/or taking advantage of possible random outcomes of these systems gives rise to stochastic optimization problems that play a crucial role in many aspects of modern life and sciences including economics, operations research, financial mathematics etc. Formulating these problems always includes a trade-off between tractability and realism of the underlying models. For this reason we often obtain solutions to stochastic optimization problems such that the associated optimal strategies satisfy some undesirable properties such as e.g. infinite variation over finite intervals, and are therefore impossible to implement. A natural question in this context is: What kind of strategies would we like to obtain and what are the corresponding optimization problems?

One possible option are the so-called *impulse control strategies* $\mathcal{S} = \{\tau_k, a_k\}_{k \geq 1}$ consisting of a sequence of intervention times $\{\tau_k\}_{k \geq 1}$ and corresponding actions $\{a_k\}_{k \geq 1}$ at those intervention times. This type of strategies are appealing for two reasons: First, they can be implemented relatively easily; secondly, these strategies arise naturally in stochastic optimization problems with transaction costs that include a *fixed cost component* and such problems are often discovered in practice, see e.g. Bensoussan and Lions [1982], Dixit and Pindyck [1994], Stokey [2008].

Mathematical Methods. A complete solution to a stochastic optimization problem with fixed costs consists of an optimal strategy and the value function that is achieved if the controller implements the optimal strategy. A classical approach consists of the following three steps: First, one relates the impulse control problem to a system of (*quasi*)-*variational inequalities* (QVIs) by means of a verification result, see e.g. Theorems 2.4, 3.6, 4.2 and 4.5. QVIs are given by a set of inequalities such that for every state at least one of them holds with equality. Intuitively, each inequality corresponds either to inaction of the controller or to a certain action, thus naturally subdividing the state space into an *inaction* or *no-trading*

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region and an *action* or *intervention region*. Second, one intends to solve the quasi-variational inequalities in a suitable sense. Finally, it has to be demonstrated that the obtained solution is indeed the value function of the optimization problem. Further, an optimal strategy has to be constructed.

There are four main methods to analyze quasi-variational inequalities:

- (i) *Smooth-pasting technique*. In general, it is notoriously difficult to find an explicit solution to variational inequalities. However, for specific one-dimensional models (or models that can be reduced to a one-dimensional case) solutions can be constructed by solving each inequality separately. Sufficient regularity for Itô's formula at points of intersection of the inaction and intervention regions is insured via imposing smooth-fit conditions, see e.g. Dixit and Pindyck [1994].
- (ii) *Viscosity approach*. A priori there is no evidence about regularity of the value function. Nevertheless, it is often the case that the value function satisfies the quasi-variational inequalities associated with the optimization problem in a viscosity sense, see Fleming and Soner [2006]. This approach, however, does not immediately yield an optimal strategy, as additional regularity is required.
- (iii) *Sobolev space approach*. A solution to the quasi-variational inequalities is constructed by an iteration scheme in a suitable Sobolev space. The verification follows by Itô's formula for weakly differentiable functions. This scheme can also provide the basis for a numerical method using, e.g., finite element methods. However, the analysis of the QVIs in Sobolev spaces requires additional regularity properties of the model, see Bensoussan and Lions [1978] and Bensoussan and Lions [1982].
- (iv) Via *asymptotic analysis* of the quasi-variational inequalities with respect to a small cost parameter one can often demonstrate that the value function admits an asymptotic expansion. Furthermore, the expansion of the value function typically yields an intervention strategy that is asymptotically optimal, see, for instance, Atkinson and Wilmott [1995], Janeček and Shreve [2004].

Contribution. In this thesis we address several stochastic optimization problems and their applications in finance and economics. A key characteristic of the problems within the scope of the thesis is a state process that can be only partially

influenced by the controller of the system. Thus, in Chapter 2 we consider a menu cost problem with regime shifts that can be triggered either by an underlying Markov chain or an intervention of the controller. In Chapter 3 we consider a multidimensional running cost problem where the controller is limited in her actions, i.e. it is not allowed to shift the state process by an arbitrary value. This setting subsumes, for instance, diffusion models that include unspanned factor processes. Finally, in Chapter 4 we maximize long-term growth rates in a general incomplete two-dimensional Markov factor model.

As we consider general models and work either in multiple dimensions or do not know a priori the location of the inaction and intervention regions (as in the case of Chapter 2, see Section 2.5), the smooth-pasting method (i) is inapplicable. For this reason, in Chapters 2 and 3 we generalize the Sobolev method (iii). Instead of characterizing the value function by means of quasi-variational inequalities, we provide necessary and sufficient conditions in terms of a generalized Itô-Dynkin inequality for the value function (see Theorems 2.4 and 3.6). Thus, rather than solving variational inequalities in a weak sense, we iteratively construct a function satisfying the conditions in the verification results. This approach allows us to weaken the assumptions imposed in similar problems in Bensoussan and Lions [1978] and Bensoussan and Lions [1982]. Furthermore, in the setting of Chapter 2 we provide a numerical method for variational inequalities that results in an approximation scheme for the value function.

Our analysis in Chapter 4 is based on the asymptotic approach (iv). We investigate the maximization of long-term growth rates under both proportional and Morton-Pliska transaction costs in a general Markov factor model. For both types of costs we establish rigorous asymptotic expansions and provide leading-order strategies: For proportional costs it is a combination of singular control and discrete trades, whereas for Morton-Pliska costs it is optimal to use an impulse control strategy. In addition, we investigate the performance of certain impulse control strategies under proportional transaction costs. Finally, under a slightly stronger ergodicity assumption we show that the constructed strategies maximize long-term growth rates pathwise.

2 Stochastic Impulse Control with Regime-Switching Dynamics

In this chapter we formulate and solve a general version of the menu cost problem with partially controlled regime shifts. Problems of this type originate, for instance, in the optimal product management decision of a company that aims to maximize expected profits by choosing an optimal investment and marketing strategy. We formulate the corresponding system of quasi-variational inequalities (QVIs) and construct a generalized solution to the QVIs by means of a modification of the finite element method. Thus we obtain a computational method for solving the relevant class of QVIs and for deriving the associated optimal strategies.

This chapter is based on joint work with Ralf Korn and Frank Thomas Seifried, see Korn et al. [2015].

2.1 Introduction

In this chapter we solve a generalized version the menu cost problem, see e.g. Chapter 7 in Stokey [2008]. The generalization includes the following key aspects:

- (i) The dynamics of the underlying state process Y are modulated by a Markov chain I tracking the regime of the system. The scope of the manager's control extends over the state process Y and, in certain regimes, also over the Markov chain I , thus allowing for partial management of the system.
- (ii) Instead of considering a particular model for the dynamics, we provide an analytical method for constructing the solution in the context of a generic model of the state process dynamics. Our approach also provides the basis for a numerical computation of the solution.
- (iii) We consider the problem under a general cost structure that may depend on both the pre-intervention state of the system and the controller's action.

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Thus, the generalized menu cost model under consideration can be formalized as follows:

$$\sup_{\mathcal{S}} \mathbb{E} \left[\int_0^{\infty} e^{-rt} P(I_t^{\mathcal{S}}, Y_t^{\mathcal{S}}) dt - \sum_{k=1}^{\infty} e^{-r\tau_k} \mathcal{C}(a_k) \right] \quad (2.1)$$

where the supremum extends over all admissible impulse control strategies $\mathcal{S} = \{\tau_k, a_k\}_{k \geq 1}$ consisting of an increasing sequence of intervention times $\{\tau_k\}_{k \geq 1}$ and a sequence of corresponding management decisions $\{a_k\}_{k \geq 1}$. The integral term represents total discounted profits generated by the regime process $I^{\mathcal{S}}$ and the state process $Y^{\mathcal{S}}$, whereas the sum denotes the costs of the strategy \mathcal{S} . Here $\mathcal{C}(a_k)$ are the costs that are due upon undertaking the action a_k at τ_k .

Contribution. First, by means of a verification result (Theorem 2.4) we establish sufficient conditions for a function to be the value function of the optimization problem (2.1). Instead of conventional quasi-variational inequalities these conditions include, among others, an Itô-Dynkin inequality that generalizes the associated Bellman programming principle, cf. (7.2) in Stokey [2008]. The Itô-Dynkin inequality, in turn, relates the optimization problem (2.1) to a system of variational inequalities, which also give rise to our approximating method.

Second, we construct a sequence that converges to the value function. For this purpose we iteratively solve suitable systems of quasi-variational inequalities on compact intervals in a similar fashion to Bensoussan and Lions [1982]¹ by means of a version of the finite elements method, see Theorem 2.10. We conclude the construction by establishing convergence of the constructed sequence to the value function of (2.1). Besides the described analytical approach, we pay particular attention to the numerical procedure of solving variational inequalities, see Theorem 2.6 and Figure 2.1. In this way a numerical method is established.

Finally, we provide an application of our model in management science. More precisely, we consider a company that focuses on a single product. The product can be in one of two regimes: “old” and “new”. Transitions from “new” to “old” are triggered by shifts in demand patterns that are beyond the management’s control. By contrast, a transition from “old” to “new” can only take place as a result of an explicit management investment decision to issue the next version of the product. The sales of the product generate instantaneous profits that are subject to random changes in consumer tastes, financial risks, etc. In addition, instantaneous profits depend on the state of the product and can be influenced by

¹Bensoussan and Lions [1982] do not consider systems with regime switching.

the management's marketing decisions. Transaction costs with a fixed component are associated with every intervention (investment or marketing). The company aims to maximize the net present value of total profits, which can be represented in the form (2.1).

Related Literature. Besides providing both an analytical and a numerical method for an impulse control problem with regime shifts, this chapter fills the gap between previous studies where regime switches represent an exogenous source and are beyond any kind of control, and the literature on fully controlled regime switching. The former was originally proposed by Hamilton [1989] for modeling stock return times series and the idea of modulating non-deterministic systems by Markov regime processes has become increasingly popular ever since. In particular, there are many studies of stochastic optimization problems with regime shifts. Thus, optimal investment and consumption problems in frictionless Geometric Brownian motion markets with regime switching are studied, among others, by Bäuerle and Rieder [2004], Sotomayor and Cadenillas [2009], Zhang et al. [2010], Capponi and Figueroa-López [2014]; we refer to Sotomayor and Cadenillas [2009] for an overview. Shen and Siu [2012] provide a numerical solution to a similar problem in a market with a stochastic interest rate modulated by regime switching. Gassiat et al. [2014] analyze a utility maximization problem in a Black-Scholes market with regime switching under liquidity constraints. Guo et al. [2005] consider an irreversible investment problem with regime shifts in a Geometric Brownian motion model that is modulated by a Markov chain.

Several authors have combined regime switching with different types of frictions. Zariphopoulou [1992] maximizes utility of consumption under proportional transaction costs in a market where stock returns are determined by a continuous-time Markov chain and establishes a viscosity property of the value function. Similar results for a Black-Scholes market with regime switching and proportional transaction costs are obtained by Liu [2014]. Sotomayor and Cadenillas [2013] solve a dividend optimization problem in a Black-Scholes market with regime switching and fixed transaction costs using the smooth-pasting approach.

The analysis of this chapter is also related to the literature on so-called *reversible investment* or *optimal switching* problems that were originally introduced by Brennan and Schwartz [1985] in the context of resource exploitation. In such problems, the investor decides when to open or close production, i.e. regime shifts occur only as a result of management's decisions. Since each intervention is costly, admissible policies are restricted to impulse control strategies. There is a vast amount of

2 Stochastic Impulse Control with Regime-Switching Dynamics

articles studying problems of this type and we refer to Chapter 5 in Pham [2009] and Zervos et al. [2013] for an overview. In some specific models it is possible to construct solutions by the smooth-pasting technique based on explicit solutions (as e.g. in Chapter 7 in Dixit and Pindyck [1994], Duckworth and Zervos [2001], Johnson and Zervos [2010], Lumley and Zervos [2001], Zervos et al. [2013]; Ly Vath and Pham [2007] combine the smooth-fit approach with viscosity properties). Hamadéne and Jeanblanc [2007] solve a reversible investment problem with finite horizon using BSDE methods in a general market model. Tang and Yong [1993] modify the problem by allowing for additional continuous and impulse control and establish a viscosity property for the value function. Finally, we note that the proposed application of the optimization problem (2.1) can be considered as an iterated “goodwill” problem, see e.g. Lon and Zervos [2011], under the assumption that the marketing actions take effect immediately.

In summary, (2.1) is a hybrid optimization problem that includes a) models with completely controlled regimes switches and b) models with Markov-governed observable regime switches. By providing a complete analysis of (2.1) we add to the literature on stochastic optimization problems with regime shifts and transaction costs and study a novel application in management science.

Outline. The chapter is structured as follows: In Section 2.2 we rigorously formulate the problem (2.1). The verification result for (2.1) and its relation to certain quasi-variational inequalities are discussed in Section 2.3. The main results of the chapter are contained in Section 2.4. First, all the necessary results for one-dimensional variational inequalities are established in Section 2.4.1. In Section 2.4.2 quasi-variational inequalities on compact intervals are solved and, finally, in Section 2.4.3 we construct a sequence of functions and establish its convergence towards the value function of (2.1). All proofs are delegated to the appendix.

The application of the model together with numerical illustrations is presented in Section 2.5. In particular, by means of the numerical examples we demonstrate why the classical smooth-pasting methods are inapplicable in the model under consideration.

2.2 Mathematical Formulation

In this section we provide a general rigorous formulation of the optimization problem (2.1) described above. In particular, we detail the construction of the relevant

state and regime processes Y and I as a controlled switching diffusion.

2.2.1 Probabilistic Setting

Throughout the chapter let $(\Omega, \mathcal{A}, \mathbb{P})$ be a complete probability space that carries a standard Wiener process W and $N^2 - N$ independent Poisson processes $N^{i,j}$ with intensities $\lambda^{i,j} \geq 0$ for $i, j \in \mathcal{E}$, $i \neq j$. We set

$$\begin{aligned}\mathcal{F}_t^o &\triangleq \sigma(W_s, \{N_s^{i,j}\}_{i,j \in \mathcal{E}, i \neq j} : s \in [0, t]), \\ \mathcal{F}_\infty^o &\triangleq \sigma(W_t, \{N_t^{i,j}\}_{i,j \in \mathcal{E}, i \neq j} : t \in [0, \infty)).\end{aligned}$$

Blumenthal's 0-1 law implies that the completed filtration $\mathcal{F} = \{\mathcal{F}_t\}_{t \geq 0}$ is right-continuous; see, e.g., [Sato, 2007, Proposition 40.3]. Hence the filtered probability space $(\Omega, \mathcal{A}, \mathcal{F}, \mathbb{P})$ satisfies the usual conditions. Moreover, let²

$$\mathcal{J} \subset \mathbb{R} \text{ be a closed and unbounded interval and } \mathcal{E} \triangleq \{1, \dots, N\}.$$

For each regime $i \in \mathcal{E}$ we are given functions $d_i, S_i : \mathcal{J}^0 \rightarrow \mathbb{R}$ that satisfy the following regularity conditions:

- (A1)** d_i is continuous, S_i is continuously differentiable and $S_i > 0$ on \mathcal{J}^0 .
- (A2)** For every \mathcal{J}^0 -valued initial value Υ there exists a unique strong solution of the stochastic differential equation (SDE)

$$\xi_0 = \Upsilon, \quad d\xi_t = d_i(\xi_t) dt + S_i(\xi_t) dW_t \quad (2.2)$$

on $[0, \tau_\infty]$ with $\tau_\infty \triangleq \inf\{t \geq 0 : \xi_t \in \partial\mathcal{J}\}$.

In Section 2.4 we further require the following technical condition on the behavior of the coefficients near the boundary $\partial\mathcal{J}$ of \mathcal{J} :

- (A3)** There exist $p \in (1, 2]$ and a neighborhood \mathcal{B} of $\partial\mathcal{J}$ such that

$$(-d_i + S_i S_i') \in L_{\frac{p}{p-1}}(\mathcal{B}), \quad S_i^2 \in L_{\frac{p}{p-1}}(\mathcal{B}) \quad \text{and} \quad \frac{1}{S_i^2} \in L_{\frac{p}{2-p}}(\mathcal{B})$$

for every $i \in \mathcal{E}$.

²The corresponding analysis for a bounded interval is analogous, but simpler; see Section 2.4. We therefore focus on the unbounded case, and leave it to the reader to modify our results for the bounded case.

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Clearly, assumptions (A2) and (A3) are satisfied in the standard case where d_i and S_i can be extended to locally Lipschitz functions of linear growth on \mathcal{J} with $S_i > 0$ on \mathcal{J} . In addition, (A3) also subsumes models where the coefficients d_i and S_i degenerate on the boundary. In this context, note that by Theorem 51.2 in [Rogers and Williams, 2000, Chapter V] condition (A3) implies that the unique solution to (2.2) reaches $\partial\mathcal{J}$ a.s. Finally, let

$$\mathcal{L}_i \triangleq d_i \frac{\partial}{\partial y} + \frac{1}{2} S_i^2 \frac{\partial^2}{\partial y^2}$$

denote the infinitesimal generator associated with (2.2).

Notation. If f is a function defined on $\mathcal{E} \times \mathcal{J}$, we use the notation $f_i(y) = f(i, y)$ interchangeably for the value of f at (i, y) .

2.2.2 Construction of Switching Diffusions

Our first goal is to construct an uncontrolled **switching diffusion**, i.e. a diffusion process Y that starts at a stopping time and whose dynamics are modulated by an exogenous regime shift process I taking values in \mathcal{E} . Let us fix a stopping time τ and an \mathcal{F}_τ -measurable pair (ι, Υ) taking values in $\mathcal{E} \times \mathcal{J}$ as initial data.

In a first step, define an n -variate point process $\{\rho_k, \iota_k\}_{k \geq 0}$, in the sense of Brémaud [1981], by setting $(\rho_0, \iota_0) \triangleq (\tau, \iota)$ and

$$\rho_k \triangleq \min_{j \in \mathcal{E} \setminus \{\iota_{k-1}\}} \inf\{t > \rho_{k-1} : \Delta N_t^{\iota_{k-1}, j} = 1\}$$

with

$$\iota_k \triangleq \arg \min_{j \in \mathcal{E} \setminus \{\iota_{k-1}\}} \inf\{t > \rho_{k-1} : \Delta N_t^{\iota_{k-1}, j} = 1\}$$

denoting the associated minimizer. By independence of $\{W_t\}_{t \geq 0}$ and $\{N_t^{i,j}\}_{t \geq 0}$ and Theorem 7 from [Brémaud, 1981, Chapter III], the $(\mathbb{P}, \mathcal{F})$ -intensity of $\{\rho_k, \iota_k\}_{k \geq 0}$ is given by

$$\Lambda_t(\omega, \{j\}) = \sum_{k=0}^{\infty} \lambda^{\iota_k, j} \mathbb{I}_{\rho_k \leq t < \rho_{k+1}} \quad \text{for } t \geq 0 \quad \text{and } j \in \mathcal{E}$$

where by convention $\lambda^{i,i} \triangleq 0$ for all $i \in \mathcal{E}$.

Second, we define the associated **regime process** I via

$$I_t \triangleq \sum_{k \geq 0} \iota_k \mathbb{I}_{[\rho_k, \rho_{k+1})}(t), \quad t \geq 0 \tag{2.3}$$

and the corresponding **state process** Y as the unique solution of the SDE

$$Y_t = \Upsilon + \int_{\tau}^t dI_s(Y_s) ds + \int_{\tau}^t S_{I_s}(Y_s) dW_s \quad (2.4)$$

on $[\tau, \tau_{\infty}]$, where $\tau_{\infty} \triangleq \inf\{t \geq \tau : Y_t \in \partial\mathcal{J}\}$ denotes the first exit time of Y from \mathcal{J}^0 .

Definition 2.1. *The switching diffusion starting at time τ in (ι, Υ) is given by*

$$Z_t^{\tau, \iota, \Upsilon} \triangleq (I_t^{\tau, \iota, \Upsilon}, Y_t^{\tau, \iota, \Upsilon}) \triangleq (I_{t \wedge \tau_{\infty}}, Y_{t \wedge \tau_{\infty}}), \quad t \geq \tau. \quad (2.5)$$

We refer to $I^{\tau, \iota, \Upsilon}$ as the associated regime process and to $Y^{\tau, \iota, \Upsilon}$ as the corresponding state process. The stopping times $\{\rho_k\}_{k \geq 1}$ are called switching times.

Note that the switching diffusion is stopped if the state process reaches the boundary of \mathcal{J} . In order to construct controlled switching diffusions, we next address interventions and intervention costs.

2.2.3 Intervention Costs and Intervention Operator

Intervention costs are captured by a **cost function**

$$\mathcal{C} : \mathcal{E}^2 \times \mathcal{J}^2 \rightarrow [K, +\infty)$$

Here $\mathcal{C}(i, i', y, y')$ represents the costs of a regime shift from i to i' and a simultaneous shift of the state process Y from y to y' , and $K > 0$. The cost function is assumed to satisfy the following conditions:

(A4) \mathcal{C} is continuous, and there is a function $\mathfrak{d} \in C^1([0, \infty))$ with $\mathfrak{d}(0) = 0$ such that

$$|\mathcal{C}(i, i', y_1, y) - \mathcal{C}(i, i', y_2, y)| \leq \mathfrak{d}(|y_1 - y_2|)$$

for all $i, i' \in \mathcal{E}$ and $y, y_1, y_2 \in \mathcal{J}$.

(A5) For every compact set $\mathcal{K} \subset \mathcal{J}$ we have

$$\max_{i, i' \in \mathcal{E}, y \in \mathcal{K}} \mathcal{C}(i, i', y, y') \rightarrow +\infty \quad \text{as } y' \rightarrow \infty.$$

Condition (A5) implies that large interventions become arbitrarily expensive. (A4) is a regularity condition on the dependence of intervention costs on the pre-intervention state. Conditions (A4) and (A5) are satisfied for a large number of relevant cost specifications, including the following standard case:

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Example. Conditions (A4) and (A5) hold for fixed-plus-proportional costs of the form

$$\mathcal{C}(i, i', y, y') \triangleq c_0 + p|y - y'| + c_1 \mathbb{I}_{i \neq i'} \quad (2.6)$$

where c_0, p, c_1 are positive constants.

To model state-dependent constraints on interventions, we further fix a set-valued function

$$\mathcal{A} : \mathcal{E} \rightarrow 2^{\mathcal{E}} \quad \text{such that} \quad i \in \mathcal{A}(i) \quad \text{for all} \quad i \in \mathcal{E}.$$

Thus $\mathcal{A}(i)$ represents the set of regimes that are attainable with finite costs by an intervention in regime i .³ We next define the intervention operator, which represents the state process after an immediate, optimal transaction.

Definition 2.2. For an interval $\mathcal{I} \subset \mathcal{J}$ and a bounded continuous function $\phi : \mathcal{E} \times \mathcal{I} \rightarrow \mathbb{R}$, the **intervention operator** $\mathcal{M}^{\mathcal{I}}$ is defined via

$$\mathcal{M}^{\mathcal{I}}\phi(i, y) \triangleq \sup_{i' \in \mathcal{A}(i), y' \in \mathcal{I}} (\phi(i', y') - \mathcal{C}(i, i', y, y')). \quad (2.7)$$

For ease of notation we also write $\mathcal{M}_i^{\mathcal{I}}\phi(y) \triangleq \mathcal{M}^{\mathcal{I}}\phi(i, y)$ and skip the upper index if $\mathcal{I} = \mathcal{J}$, i.e. $\mathcal{M}\phi \triangleq \mathcal{M}^{\mathcal{J}}\phi$. Note that condition (A4) implies that for all $y_1, y_2 \in \mathcal{I}$

$$\mathcal{M}^{\mathcal{I}}\phi(i, y_1) \leq \mathcal{M}^{\mathcal{I}}\phi(i, y_2) + \mathfrak{d}(|y_1 - y_2|). \quad (2.8)$$

In particular, it follows that

$$\mathcal{M}^{\mathcal{I}}(C_b(\mathcal{E} \times \mathcal{I})) \subset C_b(\mathcal{E} \times \mathcal{I})$$

where $C_b(\mathcal{E} \times \mathcal{I})$ denotes the set of bounded continuous functions on $\mathcal{E} \times \mathcal{I}$. Property (2.8) is important for our theoretical analysis in Section 2.4 since it allows us to use test functions from Sobolev spaces with bounded derivatives (see Theorem 2.10 and Proposition 2.11 below).

Remark. The preceding assumptions on the cost function \mathcal{C} and the set of admissible decisions \mathcal{A} are rather weak. In particular, they do not necessarily rule out multiple simultaneous interventions. Simultaneous interventions do not occur if for any $\phi \in C_b(\mathcal{E} \times \mathcal{I})$

$$\phi(i', y') > \mathcal{M}\phi(i', y') \quad \text{where} \quad (i', y') \quad \text{is a maximizer of} \quad \mathcal{M}\phi(i, y).$$

³By setting $\mathcal{A}(i) \triangleq \{i\}$ for every $i \in \mathcal{E}$ any influence of the controller on the regime process is prohibited. Thus our setting subsumes models with uncontrolled regime switching.

2.2 Mathematical Formulation

The following additional conditions are sufficient to ensure this property:⁴

- (i) $\mathcal{A}(j) \subset \mathcal{A}(i)$ for all $j \in \mathcal{A}(i)$
- (ii) $\mathcal{C}(i, i'', y, y'') < \mathcal{C}(i, i', y, y') + \mathcal{C}(i', i'', y', y'')$.

We next construct the controlled switching diffusion and provide a rigorous formulation of the impulse control problem (2.1) investigated in this chapter.

2.2.4 Impulse Control Problem

Let the initial regime $i \in \mathcal{E}$ and the initial value of the state process $y \in \mathcal{J}$ be given. A **control strategy** \mathcal{S} consists of a non-decreasing sequence of stopping times $\{\tau_k\}_{k \geq 1}$ and a sequence of actions $\{(\iota_k, \Upsilon_k)\}_{k \geq 1}$ where (ι_k, Υ_k) is an $\mathcal{E} \times \mathcal{J}$ -valued \mathcal{F}_{τ_k} -measurable random variable for each $k \geq 1$. Given a strategy $\mathcal{S} = \{\tau_k, \iota_k, \Upsilon_k\}_{k \geq 1}$ we construct the corresponding controlled regime process $I^{\mathcal{S}}$ and the controlled state process $Y^{\mathcal{S}}$ iteratively: We set $\tau_0 \triangleq 0$, $\iota_0 \triangleq i$ and $\Upsilon_0 \triangleq y$ and define the **controlled switching diffusion** $Z^{\mathcal{S}} \triangleq (I^{\mathcal{S}}, Y^{\mathcal{S}})$ for $t \in [\tau_k, \tau_{k+1})$ by

$$Z_t^{\mathcal{S}} = (I_t^{\mathcal{S}}, Y_t^{\mathcal{S}}) \triangleq Z_t^k = (I_t^k, Y_t^k) \triangleq Z_t^{\tau_k, \iota_k, \Upsilon_k} = (I_t^{\tau_k, \iota_k, \Upsilon_k}, Y_t^{\tau_k, \iota_k, \Upsilon_k}) \quad (2.9)$$

where $Z^{\tau_k, \iota_k, \Upsilon_k}$ is the switching diffusion defined by (2.5). This construction is feasible whenever the strategy \mathcal{S} is admissible in the sense of the following definition:

Definition 2.3. A strategy $\mathcal{S} = \{\tau_k, \iota_k, \Upsilon_k\}_{k \geq 1}$ is **admissible** if

- (i) $0 \leq \tau_k \leq \tau_{k+1} \rightarrow \infty$ a.s. as $k \rightarrow \infty$;
- (ii) $\iota_k \in \mathcal{A}(I_{\tau_k}^{k-1})$ for $k \geq 1$ on $\{\tau_k < \infty\}$.

Our goal is to find the optimal policy and the value function for the **impulse control problem**

$$U(i, y) \triangleq \sup_{\mathcal{S}} J(\mathcal{S}; i, y) \quad \text{where} \quad (P)$$

$$J(\mathcal{S}; i, y) \triangleq \mathbb{E} \left[\int_0^\infty e^{-rt} P(I_t^{\mathcal{S}}, Y_t^{\mathcal{S}}) dt - \sum_{k=1}^\infty e^{-r\tau_k} \mathcal{C}(I_{\tau_k}^{k-1}, \iota_k, Y_{\tau_k}^{k-1}, \Upsilon_k) \right]$$

⁴(ii) is not implied by (A4) and (A5). A counterexample is given by the cost function $\mathcal{C}(y, z) \triangleq \epsilon + (|y - z| - 1) \mathbb{I}_{|y - z| > 1}$, which satisfies $\mathcal{C}(0, 2) = \epsilon + 1 > 2\epsilon = \mathcal{C}(0, 1) + \mathcal{C}(1, 2)$ for sufficiently small $\epsilon > 0$.

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and where the supremum extends over all admissible strategies. Here $P(i, y)$ represents profits in regime i if the state process is at y , where the **profit function**

(A6) $P : \mathcal{E} \times \mathcal{J} \rightarrow [0, \infty)$ is bounded and continuous.

Due to this assumption the integral term in (P) is always finite and therefore J is well-defined even for admissible strategies with infinite costs.

Standing Assumption. *Throughout this chapter we assume (A1)–(A6) to hold true without any further mentioning.*

2.3 Quasi-Variational Inequalities and Verification

In this section we demonstrate how the value function of the impulse control problem (P) is related to a corresponding system of quasi-variational inequalities (QVIs). The key link is established by the following result:

Theorem 2.4 (Verification Theorem). *Suppose that a function $U : \mathcal{E} \times \mathcal{J} \rightarrow \mathbb{R}$ is such that*

- (i) U is non-negative, bounded and continuous;
- (ii) $U \geq \mathcal{M}U$ on \mathcal{J} ;
- (iii) $\min\{U_i - \frac{P_i}{r}, U_i - \mathcal{M}_i U\} = 0$ on $\partial\mathcal{J}$ for each $i \in \mathcal{E}$;
- (iv) U satisfies the Itô-Dynkin Inequality (\star) (see below).

Then U is the value function of the impulse control problem (P). The control strategy $\hat{\mathcal{S}}$, constructed as follows, is optimal:

- (a) $\hat{\tau}_0 \triangleq 0$ and $(\hat{i}_0, \hat{Y}_0) \triangleq (i, y)$;
- (b) *Given the controlled switching diffusion $\hat{Z} = (\hat{I}, \hat{Y})$ on $[0, \hat{\tau}_k)$ and an intervention (\hat{i}_k, \hat{Y}_k) at $\hat{\tau}_k$, set*

$$\hat{\tau}_{k+1} \triangleq \inf \{t \geq \hat{\tau}_k : U(\hat{Z}_t^k) = \mathcal{M}U(\hat{Z}_t^k)\}$$

where $\hat{Z}^k = (\hat{I}^k, \hat{Y}^k) \triangleq Z^{\hat{\tau}_k, \hat{i}_k, \hat{Y}_k}$. Then the process $\hat{Z} = (\hat{I}, \hat{Y})$ is defined as \hat{Z}_t^k on $[\hat{\tau}_k, \hat{\tau}_{k+1})$. If $\hat{\tau}_{k+1} < \infty$, we set

$$(\hat{i}_{k+1}, \hat{Y}_{k+1}) \triangleq \Xi^U(\hat{Z}_{\hat{\tau}_{k+1}}^k)$$

where Ξ^U is a measurable maximizer of $\mathcal{M}U$.

2.3 Quasi-Variational Inequalities and Verification

Proof. See Appendix 2.A. □

Note that no regularity assumptions are imposed on the function U in Theorem 2.4; in particular, U need not be twice continuously differentiable. Instead, we directly assume the key property required to verify optimality, i.e. the following generalized version of Itô's formula:⁵

Itô-Dynkin Inequality (\star). *We say that a function $U : \mathcal{E} \times \mathcal{J} \rightarrow \mathbb{R}$ satisfies the Itô-Dynkin Inequality (\star) if the following two conditions hold for arbitrary stopping times $\tau \leq \sigma_1 \leq \sigma_2$ and an arbitrary \mathcal{F}_τ -measurable $\mathcal{E} \times \mathcal{J}$ -valued random vector (ι, Υ) :*

- (i) *For the switching diffusion $Z = (I, Y) \triangleq Z^{\tau, \iota, \Upsilon}$ starting at τ in (ι, Υ) , see (2.5), the inequality*

$$e^{-r\sigma_1 \wedge \tau_\infty} U(Z_{\sigma_1 \wedge \tau_\infty}) \mathbb{I}_{\sigma_1 < \infty} \geq \mathbb{E} \left[e^{-r\sigma_2 \wedge \tau_\infty} U(Z_{\sigma_2 \wedge \tau_\infty}) \mathbb{I}_{\sigma_1 < \infty} + \int_{\sigma_1 \wedge \tau_\infty}^{\sigma_2 \wedge \tau_\infty} e^{-rt} P(Z_t) dt \mathbb{I}_{\sigma_1 < \infty} \middle| \mathcal{F}_{\sigma_1} \right] \quad (2.10)$$

holds almost surely. Here τ_∞ is the exit time of Y from \mathcal{J}^0 , i.e.

$$\tau_\infty \triangleq \inf \{ t \geq \tau : Y_t \in \partial \mathcal{J} \}.$$

- (ii) *(2.10) is an equality almost surely if $\sigma_2 \leq \hat{\theta}$ a.s., where $\hat{\theta}$ is defined via*

$$\hat{\theta} \triangleq \inf \{ t \geq \sigma_1 : U(Z_t) = \mathcal{M}U(Z_t) \}.$$

We wish to emphasize here that the Itô-Dynkin Inequality (\star) is not a conclusion, but a condition on U that will have to be verified separately whenever it is used; we will do this for the solution we construct in the subsequent section. The Itô-Dynkin Inequality (\star) apparently holds true for every function U that is twice continuously differentiable, bounded with bounded derivatives, and a strong of the *quasi-variational inequalities* (QVIs)

$$\begin{aligned} \min \left\{ -\mathcal{L}_i U_i + rU_i - \sum_{j \in \mathcal{E}} \lambda^{i,j} (U_j - U_i) - P_i, U_i - \mathcal{M}_i U \right\} &= 0, \quad y \in \mathcal{J}^0 \\ \min \left\{ U_i - \frac{P_i}{r}, U_i - \mathcal{M}_i U \right\} &= 0, \quad y \in \partial \mathcal{J} \end{aligned} \quad (2.11)$$

⁵Related results can be found in [Bensoussan and Lions, 1982, Chapter 6, Theorem 1.2, Lemma 3.6].

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where $i \in \mathcal{E}$ and \mathcal{J}^0 denotes the interior of \mathcal{J} . However, it is well-known that in general we cannot expect the QVI system (2.11) to have a strong solution of class C^2 . Even if it does, it may be difficult to obtain the solution in closed form. Therefore, in the following we directly construct a function U that satisfies the Itô-Dynkin Inequality (\star) together with the other conditions of the Verification Theorem 2.4 and therefore is the value function of the impulse control problem (P). At the same time, our approach also yields a numerical method that is guaranteed to converge to the value function. Thus we provide a complete solution of (P) on both a theoretical and a practical level.

2.4 Solution of the Impulse Control Problem

In this section we construct the value function. We proceed as follows: First, we demonstrate how to solve QVIs of the type (2.11) in a weak sense on compact sets using a finite-element approach, and we establish a suitable version of the Itô-Dynkin Inequality for the relevant solutions. Then we show that these solutions on compacts converge as their domains increase. Finally, we demonstrate that the conditions of the Verification Theorem 2.4 are satisfied for the limiting function, so that using Theorem 2.4 we are able to conclude that this limit coincides with the desired value function of the impulse control problem (P). For ease of exposition, we focus here on the benchmark case

$$\mathcal{J} = [0, \infty)$$

but our results carry over *mutatis mutandis* to other closed subintervals $\mathcal{J} \subset \mathbb{R}$.

Notation. When there is no risk of confusion, we use the following short-hand notation: For two non-negative functions $\xi(\cdot)$ and $\eta(\cdot)$ defined on some set \mathfrak{S} we write

$$\xi(\cdot) \lesssim \eta(\cdot) \text{ if there exists a } C > 0 \text{ such that } \xi(s) \leq C\eta(s) \text{ for all } s \in \mathfrak{S}.$$

Typically, \mathfrak{S} is either a set of functions in a Sobolev space, or a set of indices.

2.4.1 Variational Inequalities on $\mathcal{I} = [a, b]$

Weak solutions to QVIs of the type (2.11) on compact intervals will be constructed by iteratively solving certain *variational inequalities* (VIs). This is in the spirit

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of classical results as in, e.g., Bensoussan and Lions [1982]. As first step, in this subsection we establish relevant results for general one-dimensional variational inequalities that will be applied later in every regime $i \in \mathcal{E}$ and every iteration step. Among others, we prove a version of the Itô-Dynkin Inequality for solutions to the variational inequalities. The results of this section are based on a finite element approach and thus also provide the basis for our numerical approximation method.

Setting

We fix an arbitrary interval $\mathcal{I} \triangleq [a, b]$ and consider continuous functions $d, S, \lambda, f, \Psi : \mathcal{I} \rightarrow \mathbb{R}$. Throughout this subsection we assume that

(A7) $\lambda \geq 0$ and d and S satisfy conditions (A1) and (A2) in Section 2.2.

We further fix $r > 0$ and define the differential operator $\mathcal{L} \triangleq \frac{1}{2}S^2 \frac{\partial^2}{\partial y^2} + d \frac{\partial}{\partial y}$, which under our standing assumption (A2) is strictly elliptic on \mathcal{I} . We denote the associated bilinear form by

$$\mathbf{a}(u, v) \triangleq \int_{\mathcal{I}} \left[\frac{1}{2}S^2 u'v' + (-d + SS')u'v + (r + \lambda)uv \right] dy, \quad u, v \in W^{\mathcal{I}} \quad (2.12)$$

where

$$W^{\mathcal{I}} \triangleq W^{1,2}(\mathcal{I}) \quad \text{is equipped with the canonical norm} \quad \|u\|_{W^{\mathcal{I}}} \triangleq \|u\|_{W^{1,2}(\mathcal{I})}.$$

Note that \mathbf{a} is bounded in $W^{\mathcal{I}}$, i.e. $\mathbf{a}(u, v) \lesssim \|u\|_{W^{\mathcal{I}}}\|v\|_{W^{\mathcal{I}}}$. Throughout Section 2.4.1 we assume in addition that \mathbf{a} is *coercive*, i.e.

(A8) $\mathbf{a}(u, u) \gtrsim \|u\|_{W^{\mathcal{I}}}^2$ for all $u \in W^{\mathcal{I}}$.

Note that under the coercivity assumption the form $\mathbf{a}(\cdot, \cdot)$ is weakly lower semi-continuous, i.e. for every $\{u_n\}_{n \geq 1}$ that converges weakly towards a u in $W^{\mathcal{I}}$

$$\liminf_{n \rightarrow \infty} \mathbf{a}(u_n, u_n) \geq \mathbf{a}(u, u).$$

Indeed, this follows from the following inequality

$$\mathbf{a}(u_n, u_n) \geq \mathbf{a}(u, u) + \mathbf{a}(u_n - u, u) + \mathbf{a}(u, u_n - u) \rightarrow \mathbf{a}(u, u) \quad n \rightarrow \infty.$$

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Our goal is to solve the following system of *variational inequalities* (VIs):

$$\begin{aligned} \min \{ -\mathcal{L}u + (r + \lambda)u - f, u - \Psi \} &= 0, & y \in (a, b), \\ \min \{ u - c, u - \Psi \} &= 0, & y = a, \\ \min \{ u', u - \Psi \} &= 0, & y = b, \end{aligned} \quad (2.13)$$

where c is a constant. The boundary condition at a is motivated by condition (iii) in the Verification Theorem 2.4, and is equivalent to the Dirichlet condition $u(a) = \Psi(a) \vee c$. At the upper boundary b we impose a homogeneous Neumann boundary condition.

Construction of Weak Solutions

Weak Formulation. Typically, we cannot expect to obtain a strong solution to (2.13). Hence we attack (2.13) in a weak sense. For this purpose we use the Sobolev spaces⁶ $W^{\mathcal{I}}$ and

$$W_0^{\mathcal{I}} \triangleq \{v \in W^{1,2}(\mathcal{I}) : v(a) = 0\}.$$

If there exists a solution of (2.13) in $W^{\mathcal{I}}$, we must have

$$\mathbf{(A9)} \quad K^{\mathcal{I}}(\Psi, c) \triangleq \{v \in W^{\mathcal{I}} : v \geq \Psi \text{ and } v(a) = \Psi(a) \vee c\} \neq \emptyset$$

as a necessary condition. We will therefore assume (A9) throughout Section 2.4.1 (this will be verified explicitly in Sections 2.4.2 and 2.4.3 below). Then we obtain the following weak formulation of (2.13):

$$\begin{aligned} \text{Find } u \in K^{\mathcal{I}}(\Psi, c) \text{ such that for all } v \in K^{\mathcal{I}}(\Psi, c) \\ \mathbf{a}(u, v - u) \geq \langle f, v - u \rangle^{\mathcal{I}}. \end{aligned} \quad (\text{VI})$$

Here $\langle \cdot, \cdot \rangle^{\mathcal{I}}$ denotes the inner product in $L^2(\mathcal{I})$, and \mathbf{a} is given by (2.12). Following the classical approach of Bensoussan and Lions [1978], we also introduce the associated penalized problem:

$$\begin{aligned} \text{Find } u \in W^{\mathcal{I}} \text{ such that } u(a) = \Psi(a) \vee c, \text{ and for all } v \in W_0^{\mathcal{I}} \\ \mathbf{a}(u, v) - \langle \frac{1}{\epsilon}(\Psi - u)^+, v \rangle^{\mathcal{I}} = \langle f, v \rangle^{\mathcal{I}}. \end{aligned} \quad (\text{VI}_{\epsilon})$$

⁶We implicitly always use the continuous version of any function in $W^{\mathcal{I}} = W^{1,2}(\mathcal{I})$.

2.4 Solution of the Impulse Control Problem

Finite-Element Approach. We next introduce suitable finite-element subspaces W^h and W_0^h and consider the discretized versions of the problems (VI) and (VI_ϵ) . Thus let $h \in \mathbb{N}$ be arbitrary and define an increasing sequence of partitions of \mathcal{I} by setting $\pi_h \triangleq \{x_0 = a, x_1 = a + \frac{b-a}{2^h}, \dots, x_l = a + l \frac{b-a}{2^h}, \dots, x_{2^h} = b\}$. For each $h \in \mathbb{N}$ consider the finite-dimensional subspaces $W^h \subset W^\mathcal{I}$ spanned by pyramid functions on π_h and the space $W_0^h \subset W^h$ of functions in W^h that are equal to 0 on $\partial\mathcal{I}$; we refer to Appendix 2.B for formal definitions. The discretized versions of (VI) and (VI_ϵ) thus read as follows:

$$\begin{aligned} & \text{Find } u \in W^h \text{ such that } u(a) = \Psi(a) \vee c, \text{ and for all } v \in W_0^h \\ & \quad \mathfrak{a}(u, v) - \langle \frac{1}{\epsilon}(\text{pr}_h \Psi - u)^+, v \rangle^\mathcal{I} = \langle f, v \rangle^\mathcal{I} \end{aligned} \quad (VI_\epsilon^h)$$

and

$$\begin{aligned} & \text{Find } u \in \mathcal{K}^\mathcal{I}(\text{pr}_h \Psi, c) \cap W^h \text{ such that for all } v \in \mathcal{K}^\mathcal{I}(\text{pr}_h \Psi, c) \cap W^h \\ & \quad \mathfrak{a}(u, v - u) \geq \langle f, v - u \rangle^\mathcal{I}. \end{aligned} \quad (VI^h)$$

Here $\text{pr}_h \Psi \in W^h$ is defined as the unique element of W^h that coincides with Ψ in the nodes of π_h .

Remark. (VI^h) is equivalent to a linear complementarity problem. Such problems are well-studied and numerically accessible; see Section 2.5 below.

To provide a rigorous basis for both our theoretical results here and our numerical analysis in Section 2.5, we next establish several crucial convergence statements, which are summarized in Figure 2.1.

$$\begin{array}{ccc} (VI_\epsilon^h) & \xrightarrow{h \rightarrow \infty} & (VI_\epsilon) \\ \epsilon \downarrow 0 \Big\downarrow & & \Big\downarrow \epsilon \downarrow 0 \\ (VI^h) & \xrightarrow{h \rightarrow \infty} & (VI) \end{array}$$

Figure 2.1: Convergence of discretized and penalized VIs.

In particular, $(VI^h) \rightarrow (VI)$ guarantees convergence of our numerical method, and $(VI_\epsilon) \rightarrow (VI)$ is key for the proof of the Itô-Dynkin Inequality for the solution of (VI). The construction of solutions to (VI^h) and (VI_ϵ) is, in turn, based on the other two convergence statements, which are established in the following convergence result.

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Theorem 2.5 ((VI $_{\epsilon}^h$) \rightarrow (VI h) and (VI $_{\epsilon}^h$) \rightarrow (VI $_{\epsilon}$)). *Suppose that (A7)–(A9) are satisfied. Then we have:*

- (i) (Solution to (VI $_{\epsilon}^h$)). *There exists a unique solution u_{ϵ}^h to (VI $_{\epsilon}^h$). Furthermore, the sequence $\{u_{\epsilon}^h\}_{\epsilon>0, h \in \mathbb{N}}$ is bounded in $W^{\mathcal{I}}$.*
- (ii) (Convergence (VI $_{\epsilon}^h$) \rightarrow (VI h)). *There exists a unique solution u^h to (VI h) and $u_{\epsilon}^h \rightarrow u^h$ weakly in $W^{\mathcal{I}}$ and uniformly on \mathcal{I} as $\epsilon \downarrow 0$.*

Moreover, the sequence $\{u^h\}_{h \geq 1}$ is bounded in $W^{\mathcal{I}}$.

- (iii) (Convergence (VI $_{\epsilon}^h$) \rightarrow (VI $_{\epsilon}$)). *There exists a unique C^2 -solution u_{ϵ} to (VI $_{\epsilon}$), and*

$$u_{\epsilon}^h \rightarrow u_{\epsilon} \text{ weakly in } W^{\mathcal{I}} \text{ and uniformly on } \mathcal{I} \text{ as } h \rightarrow \infty.$$

Moreover, the sequence $\{u_{\epsilon}\}_{\epsilon>0}$ is bounded in $W^{\mathcal{I}}$.

Proof. See Appendix 2.B.1. □

We next investigate the limit points of the sequences $\{u^h\}_{h \geq 1}$ and $\{u_{\epsilon}\}_{\epsilon>0}$ in (ii) and (iii) of Theorem 2.5. The Banach-Alaoglu Theorem and compactness of the embedding $W^{\mathcal{I}} \hookrightarrow C(\mathcal{I})$ (see, e.g., Theorem 8.8 in Brezis [2010]) imply that each of the sequences $\{u^h\}_{h \geq 1}$ and $\{u_{\epsilon}\}_{\epsilon>0}$ has an accumulation point with respect to both weak convergence in $W^{\mathcal{I}}$ and uniform convergence on \mathcal{I} . The following first main result of Section 2.4.1 uniquely identifies possible limit points and thus shows that the above sequences in fact converge to the unique solution of (VI).

Theorem 2.6 ((VI $_{\epsilon}$) \rightarrow (VI) and (VI h) \rightarrow (VI)). *Under assumptions (A7)–(A9) there exists a unique solution u to (VI), and we have*

$$u = \lim_{\epsilon \rightarrow 0^+} u_{\epsilon} = \lim_{h \rightarrow \infty} u^h$$

where $\{u^h\}_{h \geq 1}$ and $\{u_{\epsilon}\}_{\epsilon>0}$ converge both weakly in $W^{\mathcal{I}}$ and uniformly on \mathcal{I} .

Proof. See Appendix 2.B.3. □

The key step in the proof of Theorem 2.6 is the following comparison principle for solutions of the limiting equation (VI).

Proposition 2.7 (Comparison Principle for (VI)). *Suppose that $u_i \in K^{\mathcal{I}}(\Psi_i, c_i)$ with*

$$\mathfrak{a}(u_i, v - u_i) \geq \langle f_i, v - u_i \rangle^{\mathcal{I}}$$

for all $v \in K^{\mathcal{I}}(\Psi_i, c_i)$ and $i = 1, 2$. If $\Psi_1 \geq \Psi_2$, $f_1 \geq f_2$ and $c_1 \geq c_2$ then $u_1 \geq u_2$.

Proof. See Appendix 2.B.2. □

Itô-Dynkin Inequality for (VI)

For any stopping time τ and any \mathcal{F}_τ -measurable \mathcal{I} -valued random variable η , let $\xi^{\mathcal{L}, \tau, \eta}$ denote the uniquely determined process such that $\xi_t^{\mathcal{L}, \tau, \eta} = 0$ for $t \in [0, \tau)$ and

$$\xi_t = \eta + \int_\tau^t d(\xi_s) ds + \int_\tau^t S(\xi_s) dW_s \quad \text{for } t \in [\tau, \tau_{\mathcal{I}}]$$

where $\tau_{\mathcal{I}} \triangleq \inf\{t \geq \tau : \xi_t \in \partial\mathcal{I}\}$. The second main result of this subsection establishes a version of the Itô-Dynkin Inequality for the unique solution of (VI).

Theorem 2.8 (Itô-Dynkin Inequality for (VI)). *Let u be the unique solution of (VI), let τ be a stopping time and let η be an \mathcal{F}_τ -measurable random variable. Then for two arbitrary stopping times σ_1 and σ_2 satisfying $\tau \leq \sigma_1 \leq \sigma_2$ we have*

$$\begin{aligned} e^{-r\sigma_1 \wedge \tau_{\mathcal{I}}} u(\xi_{\sigma_1 \wedge \tau_{\mathcal{I}}}) \mathbb{I}_{\sigma_1 < \infty} &\geq \mathbb{E} \left[e^{-r\sigma_2 \wedge \tau_{\mathcal{I}}} u(\xi_{\sigma_2 \wedge \tau_{\mathcal{I}}}) \mathbb{I}_{\sigma_1 < \infty} \right. \\ &\quad \left. + \int_{\sigma_1 \wedge \tau_{\mathcal{I}}}^{\sigma_2 \wedge \tau_{\mathcal{I}}} e^{-rt} [f - \lambda u](\xi_t) dt \mathbb{I}_{\sigma_1 < \infty} \middle| \mathcal{F}_{\sigma_1} \right]. \end{aligned} \quad (2.14)$$

Moreover, if $\sigma_2 \leq \hat{\theta}$ a.s., where

$$\hat{\theta} \triangleq \begin{cases} \inf\{t \geq \sigma_1 : u(\xi_{t \wedge \tau_{\mathcal{I}}}) = \Psi(\xi_{t \wedge \tau_{\mathcal{I}}})\}, & \sigma_1 < \tau_{\mathcal{I}} \\ +\infty, & \sigma_1 \geq \tau_{\mathcal{I}} \end{cases} \quad (2.15)$$

then (2.14) holds with equality almost surely.

Proof. The proof is based on the convergence statement $(VI_\epsilon) \rightarrow (VI)$ in Theorem 2.5. See Appendix 2.B.4. \square

2.4.2 Quasi-Variational Inequalities on $\mathcal{E} \times \mathcal{I}$

In this subsection we use the results of Section 2.4.1 to construct a weak solution of a system of quasi-variational inequalities of the type (2.11), on an arbitrary bounded interval $\mathcal{I} \triangleq [a, b] \subset \mathcal{J}^0$. First, for each $i \in \mathcal{E}$ define the bilinear form $\mathfrak{a}_i^{\mathcal{I}}$ via

$$\mathfrak{a}_i^{\mathcal{I}}(\phi, \psi) \triangleq \int_{\mathcal{I}} \left[\frac{1}{2} S_i^2 \phi' \psi' + (-d_i + S_i S_i') \phi' \psi + r \phi \psi \right] dy, \quad \phi, \psi \in W^{\mathcal{I}}. \quad (2.16)$$

Here, in contrast to Section 2.4.1, we do not assume that $\mathfrak{a}_i^{\mathcal{I}}$ is coercive. However, note that due to (A2) there exists a *coercivity coefficient* $c^{\mathcal{I}} > 0$ such that

$$\mathfrak{a}_i^{\mathcal{I}}(\phi, \phi) + c^{\mathcal{I}} \langle \phi, \phi \rangle^{\mathcal{I}} \gtrsim \|\phi\|_{W^{\mathcal{I}}}^2 \quad \text{for all } \phi \in W^{\mathcal{I}} \quad \text{and } i \in \mathcal{E}. \quad (2.17)$$

2 Stochastic Impulse Control with Regime-Switching Dynamics

The weak formulation of the QVIs (2.11) reads:

$$\begin{aligned} & \text{Find } V : \mathcal{E} \times \mathcal{I} \rightarrow [0, \infty) \text{ such that for every } i \in \mathcal{E} \\ & V_i(\cdot) \in \mathcal{K}^{\mathcal{I}}(\mathcal{M}_i^{\mathcal{I}}V, \frac{P_i(a)}{r}) \text{ and for all } v \in \mathcal{K}^{\mathcal{I}}(\mathcal{M}_i^{\mathcal{I}}V, \frac{P_i(a)}{r}) \\ & \alpha_i^{\mathcal{I}}(V_i, v - V_i) - \langle \sum_{j \in \mathcal{E}} \lambda^{i,j} (V_j - V_i), v - V_i \rangle^{\mathcal{I}} \geq \langle P_i, v - u_i \rangle^{\mathcal{I}} \end{aligned} \quad (2.18)$$

where the intervention operator $\mathcal{M}^{\mathcal{I}}$ is given by (2.7) and $\mathcal{K}^{\mathcal{I}}(\Psi, c)$ is defined in (A9). Note that $\mathcal{K}^{\mathcal{I}}(\mathcal{M}_i^{\mathcal{I}}V, P_i(a)/r) \neq \emptyset$ by (2.8) for an arbitrary continuous function V on $\mathcal{E} \times \mathcal{I}$.

We now iteratively construct a solution to (2.18). We define $V_i^0 \triangleq 0$ for $i \in \mathcal{E}$. For any $m \geq 1$, given continuous functions $V_i^{m-1} : \mathcal{I} \rightarrow \mathbb{R}$ for each $i \in \mathcal{E}$, we set $V^{m-1}(i, \cdot) \triangleq V_i^{m-1}(\cdot)$, $i \in \mathcal{E}$, and note that $\mathcal{M}^{\mathcal{I}}V^{m-1}$ is continuous. Theorem 2.6 implies that for every fixed $i \in \mathcal{E}$ there exists a unique continuous solution $V_i^m : \mathcal{I} \rightarrow \mathbb{R}$ of the variational inequality

$$\begin{aligned} & \text{Find } u \in \mathcal{K}^{\mathcal{I}}(\mathcal{M}_i^{\mathcal{I}}V^{m-1}, \frac{P_i(a)}{r}) \text{ such that for all } v \in \mathcal{K}^{\mathcal{I}}(\mathcal{M}_i^{\mathcal{I}}V^{m-1}, \frac{P_i(a)}{r}) \\ & \alpha_i^{\mathcal{I}}(u, v - u) + c^{\mathcal{I}}\langle u, v - u \rangle^{\mathcal{I}} + \langle \sum_{j \in \mathcal{E}} \lambda^{i,j} u, v - u \rangle^{\mathcal{I}} \\ & \geq \langle P_i + c^{\mathcal{I}}V_i^{m-1} + \sum_{j \in \mathcal{E}} \lambda^{i,j} V_j^{m-1}, v - u \rangle^{\mathcal{I}}. \end{aligned} \quad (2.19)$$

The sequence $\{V^m\}_{m \geq 0}$ thus constructed is monotone:

Proposition 2.9. $\{V^m\}_{m \geq 0}$ is increasing and uniformly bounded above by a constant that can be chosen independently of \mathcal{I} .

Proof. See Appendix 2.C.1. □

Using Proposition 2.9 we can define a pointwise limit of $\{V^m\}_{m \geq 0}$. The following result shows that this limit is, in fact, a solution of the QVIs (2.18), and that the convergence is, in fact, uniform.

Theorem 2.10 (Weak Solution of the QVIs on \mathcal{I}). *For each $i \in \mathcal{E}$ we can define*

$$V_i \triangleq \lim_{m \rightarrow \infty} V_i^m$$

where the limits exist in the sense of both weak convergence in $W^{\mathcal{I}}$ and uniform convergence on \mathcal{I} . Moreover, the function V solves the QVIs (2.18) and is bounded above by a constant independent of \mathcal{I} .

Proof. See Appendix 2.C.2. □

2.4 Solution of the Impulse Control Problem

The function V can be interpreted as the value function of a related impulse control problem on $\mathcal{I} \subset \mathcal{J}$, where the underlying state process is absorbed at the lower and reflected at the upper boundary of \mathcal{I} . However, we will neither prove nor use this in the following.

2.4.3 Construction of the Value Function on $\mathcal{E} \times \mathcal{J}$

The goal of this subsection is to construct a function U on $\mathcal{E} \times \mathcal{J}$ that satisfies the conditions of the Verification Theorem 2.4. For this purpose we consider an increasing sequence of intervals $\{\mathcal{I}_n\}_{n \geq 1}$ such that

$$\mathcal{J}^0 = \cup_{n \geq 1} \mathcal{I}_n = \cup_{n \geq 1} [a_n, b_n].$$

We denote by \bar{U}^n the solutions of the QVIs (2.18) on $\mathcal{E} \times \mathcal{I}_n$, whose existence is guaranteed by Theorem 2.10. Theorem 2.10 also implies that $\{\bar{U}^n\}_{n \geq 1}$ is uniformly bounded, and hence so are the functions defined, for each $n \geq 1$ and $i \in \mathcal{E}$, via

$$U_i^n(y) \triangleq \begin{cases} \bar{U}_i^n(y) & \text{for } y \in \mathcal{I}_n \\ \bar{U}_i^n(a_n) = (\mathcal{M}_i^{\mathcal{I}_n} U^n(a_n)) \vee \frac{P_i(a_n)}{r} & \text{for } y \in [0, a_n). \end{cases} \quad (2.20)$$

The basic idea is to define U as the limit of U^n as $n \rightarrow \infty$. The following result is a first step in this direction and ensures that a suitable subsequence $\{U^{n(l)}\}_{l \geq 1}$ has an accumulation point U with respect to uniform convergence on compacts.

Proposition 2.11. *There exist a subsequence $n(l)$ and a function $U : \mathcal{E} \times \mathcal{J} \rightarrow \mathbb{R}$ such that*

$$U^{n(l)} \rightarrow U \quad \text{and} \quad \mathcal{M}^{\mathcal{I}_{n(l)}} U^{n(l)} \rightarrow \mathcal{M}^{\mathcal{J}} U$$

uniformly on compact sets of $\mathcal{E} \times \mathcal{J}$ for $l \rightarrow \infty$.

Proof. The key step in the proof is to construct a bounded sequence $\{v_i^n\}_{n \geq 1}$ of test functions, which is possible by (2.8). See Appendix 2.D.1. \square

As an immediate consequence, we get

Corollary 2.12. *The function U constructed in Proposition 2.11 is non-negative and satisfies*

$$U(0) = (\mathcal{M}U(0)) \vee \frac{P_i(0)}{r} \quad \text{as well as} \quad U \geq \mathcal{M}U.$$

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Since each function U_i^n solves (2.18) on $\mathcal{E} \times \mathcal{I}_n$, it satisfies the Itô-Dynkin inequality stated in Theorem 2.8. Uniform convergence on compacts allows us to carry this over to \mathcal{J} and establish the Itô-Dynkin Inequality (\star) for U :

Proposition 2.13. *The Itô-Dynkin Inequality (\star) holds for the function U constructed in Proposition 2.11.*

Proof. See Appendix 2.D.2. □

Using Proposition 2.11, Corollary 2.12 and Proposition 2.13 we are in a position to apply the Verification Theorem 2.4 to the function U constructed in Proposition 2.11. Theorem 2.4 shows that U is the value function of the impulse control problem (P). In particular, U is uniquely determined. Since Proposition 2.11 applies to an arbitrarily selected subsequence as well, this argument implies that $\{U^n\}_{n \geq 1}$ converges to U uniformly on compacts. In summary, we have established the following main result of this section:

Theorem 2.14 (Solution of the Impulse Control Problem (P)). *We have*

$$U^n \rightarrow U \quad \text{as } n \rightarrow \infty \quad \text{uniformly on compact subsets of } \mathcal{E} \times \mathcal{J}$$

and U is the value function of the impulse control problem (P). In particular, the sufficient conditions in Theorem 2.4 are necessary and the impulse control strategy \hat{S} constructed in the verification theorem is optimal.

2.5 Application: Optimal Product Management

In this section we develop a showcase application of the framework discussed in the current chapter in management science. For this purpose we reinterpret the quantities used in the previous sections in economic terms and state a version of the problem (P). We then conclude the chapter by providing results of some numerical experiments for the version of (P).

2.5.1 An Optimal Product Marketing and Investment Problem.

Let us first attach an economic meaning to the model considered in Section 2.2. For notational convenience and better understanding, in this subsection we do not overcomplicate the notations and describe the problem rather informally.

We consider a company that produces a single product. The sales of this product generate *instantaneous profits* at a rate X_t . Thus, in the absence of marketing or

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product improvement investments, the expected net present value of total profits is

$$\mathbb{E} \left[\int_0^\infty e^{-rt} X_t dt \right]$$

where $r > 0$ is a suitable discount rate. Instantaneous profits X_t are to a certain extent determined by the current regime I_t and state process Y_t describing current performance of the product (for a more precise interpretation see below). However, X_t is also subject to random changes in consumers' tastes, financial risks, etc. For this reason we assume that instantaneous profits evolve in accordance with

$$dX_t = -\varkappa [X_t - \bar{P}(I_t, Y_t)] dt + \sigma(I_t, Y_t, X_t) d\bar{W}_t \quad (2.21)$$

i.e. they are *mean-reverting* to a level $\bar{P}(I_t, Y_t)$ that is a function of I_t and Y_t . The noise term $\sigma(I_t, Y_t, X_t) d\bar{W}_t$ represents stochastic fluctuations in profits. Let us specify the nature of the processes I and Y :

“Old” vs. “New”: The Regime Process I_t . To model consumer tastes and other persistent shifts in demand for the product, we assume there are two *regimes*: In regime 1 the product is considered as “new” and in great demand, and in regime 2 it is “old” and therefore ceteris paribus in lower demand. The regime process $I_t \in \mathcal{E} = \{1, 2\}$ is only partly under the control of the company's management: Transitions from “new” to “old” are triggered by shifts in demand patterns beyond the management's control and occur with intensity $\lambda^{1,2} > 0$ (black arrow in Figure 2.2). By contrast, a transition from “old” to “new” can occur only as a result of a *management decision* to issue a new version of the product (red arrow), i.e. $\lambda^{2,1} = 0$.

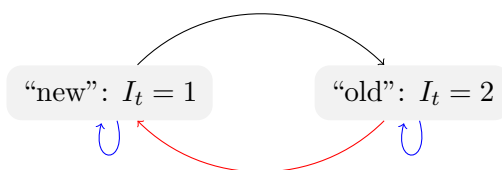


Figure 2.2: **Transitions between Different Regimes.** Transitions from “old” to “new” (red) occur on management interventions, whereas transitions from “new” to “old” (black) are exogenous.

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Cashflow Process Y_t . The profit rate X_t is driven by an underlying *cashflow process* Y_t that may represent, for instance, infinitesimal turnover or revenues. This interpretation of Y justifies the assumption

$$\mathcal{J} = [0, \infty).$$

The dynamics of the cashflow process Y_t depend on the current regime (“new” vs. “old”), and on the management’s product policy. In the absence of interventions, Y_t satisfies (2.4), i.e.

$$dY_t = d_{I_t}(Y_t) dt + S_{I_t}(Y_t) dW_t. \quad (2.22)$$

Note that here we make the simplifying assumption that there is no market for the product after the first time when Y hits 0, which corresponds to the stopping of the switching diffusion in (2.5). In such situations the company can counteract via appropriate management decisions.

Investment and Marketing Decisions. To maximize its profits the company runs a trading strategy $\mathcal{S} = \{\tau_k, \iota_k, \Upsilon_k\}_{k \geq 1}$ consisting of intervention times τ_k and the corresponding management decisions (ι_k, Υ_k) . Every management decision may include an *investment decision* ι_k and a *marketing decision* Υ_k . Within the model of Section 2.2 we focus exclusively on the impact of these interventions on the regime process I and the cashflow process Y . We thus identify an *investment* action with the decision to issue the next version of the product; this action is represented by the red arrow in Figure 2.2. This is, of course, only possible in the “old” product regime (when $I = 2$), i.e. the set of admissible actions \mathcal{A} in the case under consideration is given by

$$\mathcal{A}(1) = \{1\} \quad \text{and} \quad \mathcal{A}(2) = \mathcal{E} = \{1, 2\}.$$

By contrast, a *marketing* action is represented by a shift of the cashflow process Y , but does not alter the regime of the product; marketing actions may be taken independently of the current regime, i.e. for every $k \geq 1$ the marketing decision Υ_k is, as in Section 2.2, a $[0, \infty)$ -valued random variable. In Figure 2.2 the marketing decisions are depicted as blue arrow.

Formally, the impact of management actions on the dynamics of the regime process I and the state process Y is modeled as in Section 2.2.4: The decision (ι_k, Υ_k) is the initial value for the both the regime and the cashflow processes upon intervening at τ_k . Of course, any investment or marketing action results in costs $\mathcal{C}(I_{\tau_k-}, \iota_k, Y_{\tau_k-}, \Upsilon_k)$ that depend on ι_k , Υ_k and the pre-action level of both the regime process I_{τ_k-} and the turnover process Y_{τ_k-} .

Summary and Goal. In summary, the model can be visualized in the following diagram:

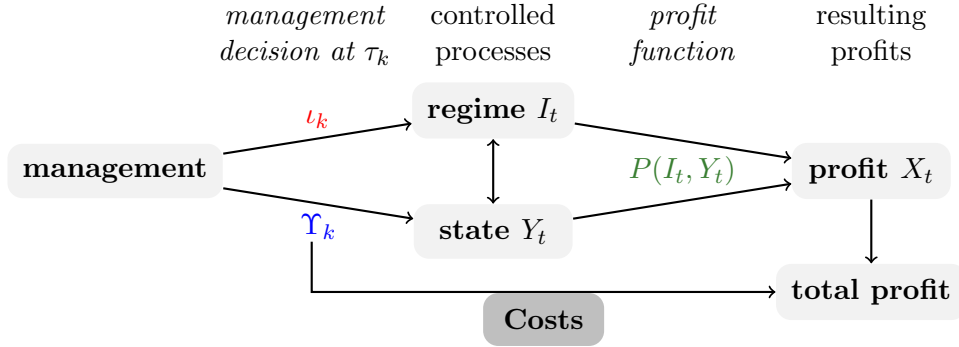


Figure 2.3: **Optimal Marketing and Investment Model.** Management decisions are costly and influence both the regime process I_t and the cashflow process Y_t . Profits are a function of the current regime and state, and the management aims to maximize expected profits less costs.

In this setting, the management's objective is to *maximize total profits*

$$\bar{J}(\mathcal{S}; i, y, x) \triangleq \mathbb{E} \left[\int_0^\infty e^{-rt} X_t dt - \sum_{k=1}^\infty e^{-r\tau_k} \mathcal{C}(I_{\tau_k-}, \iota_k, Y_{\tau_k-}, \Upsilon_k) \right] \quad (2.23)$$

by choosing an optimal product management strategy $\mathcal{S} = \{\tau_k, \iota_k, \Upsilon_k\}_{k \geq 1}$, consisting of intervention times τ_k and investment-marketing decisions (ι_k, Υ_k) , subject to the above dynamics of X_t , Y_t and I_t . Using the dynamics of X_t , see (2.21), it is not hard to see that maximizing (2.23) is equivalent to a problem of the type (P):

$$U(i, y) \triangleq \sup_{\mathcal{S}} J(\mathcal{S}; i, y) \quad \text{where}$$

$$J(\mathcal{S}; i, y) \triangleq \mathbb{E} \left[\int_0^\infty e^{-rt} \frac{\kappa}{\kappa+r} \bar{P}(I_t^{\mathcal{S}}, Y_t^{\mathcal{S}}) dt - \sum_{k=1}^\infty e^{-r\tau_k} \mathcal{C}(I_{\tau_k-}, \iota_k, Y_{\tau_k-}, \Upsilon_k) \right].$$

2.5.2 Numerical Examples

In this section we provide numerical illustrations for the optimal product management problem discussed above. In the examples we assume that the diffusion

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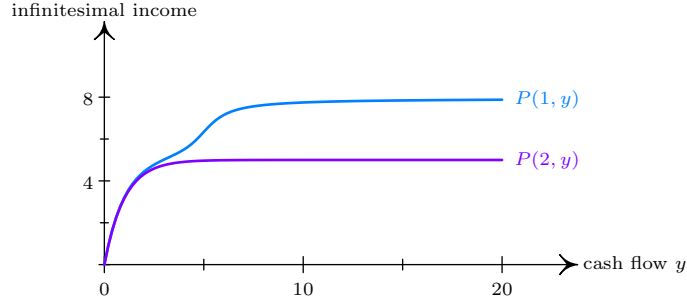


Figure 2.4: Profit functions.

coefficients in (2.22) depend only on the regime process I . Thus the assumptions from Section 2.2 are satisfied. We consider the following values for the SDE coefficients and switching intensities:

$$S_1 = 1, \quad S_2 = 0.2, \quad d_1 = -1, \quad \lambda^{1,2} = 0.5 \quad \text{and} \quad \lambda^{2,1} = 0.$$

Here we assume that $S_2 < S_1$, as in the “new” regime we expect more variability in the price of the product. The drift coefficient d_2 is one of the two parameters varied in the numerical illustrations. The second varying value in the examples below is the switching cost parameter c_1 in the following cost function, see (2.6):

$$\mathcal{C}(i, i', y, y') = 0.5 + 0.1|y - y'| + \mathbb{I}_{i \neq i'} c_1.$$

Finally, it remains to specify the profit function $P(i, y) \triangleq \frac{\kappa}{\kappa+r} \bar{P}(i, y)$ (see Figure 2.4):

$$P(i, y) = 5(1 - e^{-y}) + \mathbb{I}_{i=1} (\arctan(y - 5) + \arctan(5)).$$

Motivated by the dependence of the profits on the cash flow, the first summand is chosen to be increasing and concave. The company obtains an additional premium for the novelty of the product which is given by the second summand. Concavity at infinity of the profit function is explained by the market saturation effect.

We can readily apply the procedure developed in Section 2.4 to solve the discretized version of quasi-variational inequalities (2.18) by iteratively solving the finite-dimensional version of the inequalities (2.19) on a fixed interval $[0, b] \subset \mathcal{J}$. Then, by Theorems 2.6 and 2.10 we obtain an approximate solution of (2.18) and therefore by Theorem 2.14 an approximation of the value function. As noted in

2.5 Application: Optimal Product Management

Section 2.4.1, discretized variational inequalities are equivalent to linear complementarity problems. Therefore, to solve discretized versions of (2.19), we employ the projected Gauss-Seidel method. For details, definitions and convergence results we refer to Cottle et al. [2009]. In particular, convergence of this method in our case follows from Corollary 5.3.16 in Cottle et al. [2009].

The numerical results also provide an approximation for the inaction and intervention regions. An optimal strategy can be extracted from the definition of the intervention operator \mathcal{M} . In regime 1 the optimal strategy is to wait until the cashflow process hits the *shifting region* given by $\{y \in \mathcal{J} : U_1(y) = \mathcal{M}_1 U(y)\}$ and the *optimal shifting decision* is given by a maximizer of $\mathcal{M}_1 U(y)$. In contrast, in regime 2 the optimal strategy is slightly more complex as it possibly includes launching a new version of the product. Let us separate possible actions into those that involve regime change and the ones that do not by transforming the intervention operator: Immediately from the definition of \mathcal{M} and the cost function \mathcal{C} we obtain that

$$\mathcal{M}_2 U(y) = \max\{\mathcal{M}_1 U(y) - c_1, \tilde{\mathcal{M}}_2 U(y)\}$$

where the shift-only operator $\tilde{\mathcal{M}}_2$ is given by

$$\tilde{\mathcal{M}}_2 U(y) \triangleq \sup_{y' \in \mathcal{J}} (U_2(y') - \mathcal{C}(2, 2, y, y')).$$

Thus, in the “old” product regime it is optimal to only shift the cash flow process Y whenever Y hits the *shifting region* given by $\{y \in \mathcal{J} : U_2(y) = \tilde{\mathcal{M}}_2 U(y)\}$. The optimal action in this case is given by a maximizer of $\tilde{\mathcal{M}}_2 U$. Regime switching, on the other hand, is optimal when Y is in the *switching region* $\{y \in \mathcal{J} : U_2(y) = \mathcal{M}_1 U(y) - c_1\}$. The associated optimal change of the cashflow process is given, as previously, by a maximizer of $\mathcal{M}_1 U(y)$. On the graphs below these regions together with optimal shifting decisions will be depicted as in Figure 2.5.

⌈—⌋ shifting region
 ⌈—⌋ switching region
 ● optimal shifting decision

Figure 2.5: Legend

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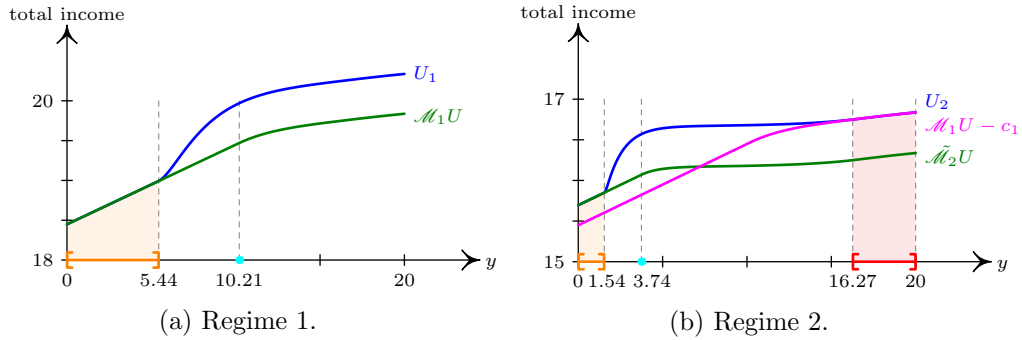


Figure 2.6: Switching region of the form $[\alpha, \infty)$ for $c_1 = 3$, $d_2 = 1$.

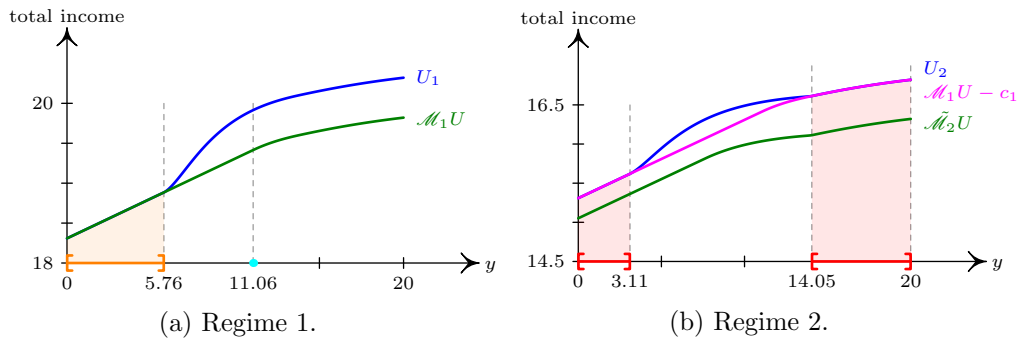


Figure 2.7: Switching region of the form $[0, \alpha] \cup [\beta, \infty)$ for $c_1 = 3$, $d_2 = -1$.

From the interpretation of the model we expect that it is optimal to intervene for low values of the cashflow process Y . Our intuition is substantiated by the numerical results presented in Figures 2.6–2.9. Thus, in Figures 2.6 and 2.8 it is optimal to only increase the cashflow process for small values of Y . In contrast, in the cases of Figures 2.7 and 2.9 the optimal strategy in regime 2 for small values of Y consists of switching between regimes and investing into marketing according to the rules of regime 1. Figures 2.6 and 2.7 share another interesting property: It is optimal to switch to regime 1 if the market is “too good”, i.e. for high values of Y . Finally, we point out that the case of Figure 2.6 is the only example that combines both switching and shifting in regime 2.

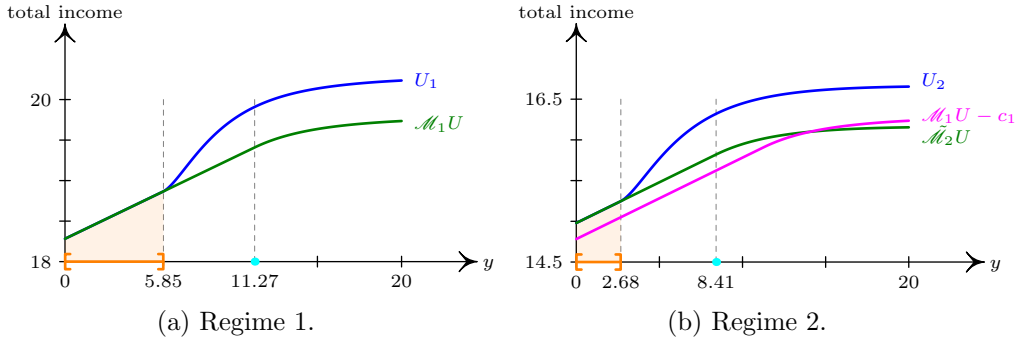


Figure 2.8: An empty switching region for $c_1 = 3.5$, $d_2 = -1$.

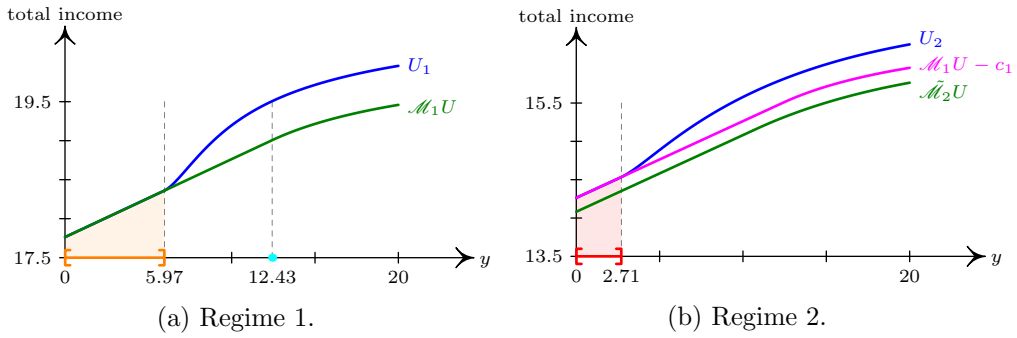


Figure 2.9: Switching region of the form $[0, \alpha]$ for $c_1 = 3.5$, $d_2 = -3$.

In summary, only by varying the drift parameter d_2 and the switching costs c_1 we obtained completely different locations of shifting and switching regions. Thus, without additional analysis the smooth-pasting method is inapplicable in the model under consideration.

2.6 Conclusion and Outlook

The main focus of this chapter was on an impulse control problem that relates models modulated by a Markov chain and reversible investment problems where regime shifts are under control of the manager. For a generic model we have developed an analytic approach that allows us to construct the value function and optimal strategies. An application of the developed theory in management science has been discussed.

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As noted above, our setting subsumes models with uncontrolled switching if $\mathcal{A}(i) = \{i\}$ for every $i \in \mathcal{E}$, whereas models with fully controlled regime changes and an uncontrolled state process Y are not covered by our setting. This is due to the fact that the proofs of the main convergence results of this chapter (Theorem 2.10 and Proposition 2.11) rely on the smoothing property (2.8) of the intervention operator \mathcal{M} . Therefore it is of great interest to extend the approach of this chapter to reversible investment problems with an uncontrolled state process.

Appendix to Chapter 2

2.A Proof of the Verification Theorem 2.4

The proof consists of two steps. In the first step, we prove that U is an upper bound for the value of the impulse control problem (P). Second, we demonstrate that the upper bound U is actually achieved by the strategy $\hat{\mathcal{S}}$ defined in Theorem 2.4. Recall that for this purpose we imposed the following assumptions on U :

- (i) U is non-negative, bounded and continuous;
- (ii) U satisfies $U \geq \mathcal{M}U$;
- (iii) $\min\{U_i - \frac{P_i}{r}, U_i - \mathcal{M}_i U\} = 0$ on $\partial\mathcal{J}$ for each $i \in \mathcal{E}$;
- (iv) U satisfies the Itô-Dynkin Inequality (\star).

Step 1: Upper Bound. Let $\mathcal{S} \triangleq \{\tau_k, \iota_k, \Upsilon_k\}_{k \geq 1}$ be an arbitrary admissible strategy and let $Z^{\mathcal{S}} = (I^{\mathcal{S}}, Y^{\mathcal{S}})$ denote the corresponding controlled switching diffusion, see (2.9). In particular, $Z^k = (I^k, Y^k)$ denotes the switching diffusion starting at time τ_k in (ι_k, Υ_k) for each $k \geq 0$ where $\tau_0 \triangleq 0$, $\iota_0 \triangleq i$ and $\Upsilon_0 \triangleq y$. Further, we define the stopping times

$$\tau_{k,\infty} \triangleq \inf\{t \geq \tau_k : Y_t^k \in \partial\mathcal{J}\}, \quad k \geq 0$$

when Y^k reaches $\partial\mathcal{J}$. Note that by definition Z^k is stopped in $\tau_{k,\infty}$. Bearing this in mind, for each $k \geq 0$ by the Itô-Dynkin Inequality (\star) and (iii) we obtain

$$\begin{aligned} & \mathbb{E} \left[\int_{\tau_k}^{\tau_{k+1}} e^{-rt} P(Z_t^{\mathcal{S}}) dt \mathbb{I}_{\tau_k < \infty} \right] \\ &= \mathbb{E} \left[\int_{\tau_k}^{\tau_{k+1} \wedge \tau_{k,\infty}} e^{-rt} P(Z_t^{\mathcal{S}}) dt \mathbb{I}_{\tau_k < \infty} + \int_{\tau_{k,\infty}}^{\tau_{k+1}} e^{-rt} P(Z_t^{\mathcal{S}}) dt \mathbb{I}_{\tau_{k,\infty} < \tau_{k+1}} \right] \\ &\leq \mathbb{E} \left[e^{-r\tau_k} U(Z_{\tau_k}^k) \mathbb{I}_{\tau_k < \infty} - e^{-r\tau_{k+1} \wedge \tau_{k,\infty}} U(Z_{\tau_{k+1} \wedge \tau_{k,\infty}}^k) \mathbb{I}_{\tau_k < \infty} \right. \\ &\quad \left. + U(Z_{\tau_{k,\infty}}^k) (e^{-r\tau_{k,\infty}} - e^{-r\tau_{k+1}}) \mathbb{I}_{\tau_{k,\infty} < \tau_{k+1}} \right] \\ &= \mathbb{E} \left[e^{-r\tau_k} U(Z_{\tau_k}^k) \mathbb{I}_{\tau_k < \infty} - e^{-r\tau_{k+1}} U(Z_{\tau_{k+1}}^k) \mathbb{I}_{\tau_k < \infty} \right]. \end{aligned} \tag{2.24}$$

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Further, by (ii) we have for every $k \geq 0$

$$\begin{aligned} & e^{-r\tau_{k+1}} \mathcal{C}(I_{\tau_{k+1}}^k, \iota_{k+1}, Y_{\tau_{k+1}}^k, \Upsilon_{k+1}) \mathbb{I}_{\tau_{k+1} < \infty} \\ & \geq e^{-r\tau_{k+1}} U(Z_{\tau_{k+1}}^{k+1}) \mathbb{I}_{\tau_{k+1} < \infty} - e^{-r\tau_{k+1}} U(Z_{\tau_{k+1}}^k) \mathbb{I}_{\tau_{k+1} < \infty}. \end{aligned} \quad (2.25)$$

Sequentially applying (2.24) and (2.25), we can estimate the profit generated by the strategy \mathcal{S} on $[0, \tau_m]$ for an arbitrary $m \geq 1$ via a telescopic sum:

$$\begin{aligned} & \mathbb{E} \left[\int_0^{\tau_m} e^{-rt} P(Z_t^{\mathcal{S}}) dt - \sum_{k=1}^m e^{-r\tau_k} \mathcal{C}(I_{\tau_k}^{k-1}, \iota_k, Y_{\tau_k}^{k-1}, \Upsilon_k) \right] \\ & \leq U(i, y) - \mathbb{E} \left[e^{-r\tau_m} U(Z_{\tau_m}^{\mathcal{S}}) \right]. \end{aligned} \quad (2.26)$$

Letting $m \rightarrow \infty$ in (2.26) yields that $J(\mathcal{S}; i, y) \leq U(i, y)$ and, as the strategy \mathcal{S} was chosen arbitrarily, we conclude that $\sup_{\mathcal{S}} J(\mathcal{S}; i, y) \leq U(i, y)$.

Step 2: Exact Upper Bound. Let us first recall the definition of $\hat{\mathcal{S}}$: We set $\hat{\tau}_0 \triangleq 0$ and $(\hat{\iota}_0, \hat{\Upsilon}_0) \triangleq (i, y)$. For any $k \geq 0$, if the action at $\hat{\tau}_k$ is $(\hat{\iota}_k, \hat{\Upsilon}_k)$, then define

$$\begin{aligned} \hat{\tau}_{k+1} & \triangleq \inf \{ t \geq \hat{\tau}_k : U(\hat{Z}_t^k) = \mathcal{M}U(\hat{Z}_t^k) \} \\ & = \inf \{ t \geq \hat{\tau}_k : U(\hat{Z}_{t \wedge \hat{\tau}_{k, \infty}}^k) = \mathcal{M}U(\hat{Z}_{t \wedge \hat{\tau}_{k, \infty}}^k) \} \end{aligned} \quad (2.27)$$

where

$$\hat{Z}^k = (\hat{I}^k, \hat{Y}^k) \triangleq Z^{\hat{\tau}_k, \hat{\iota}_k, \hat{\Upsilon}_k} \quad \text{and} \quad \hat{\tau}_{k, \infty} \triangleq \inf \{ t \geq \hat{\tau}_k : \hat{Y}_t^k \in \partial \mathcal{J} \}.$$

On $\{\hat{\tau}_{k+1} < \infty\}$ we define

$$(\hat{\iota}_{k+1}, \hat{\Upsilon}_{k+1}) \triangleq \Xi^U(\hat{Z}_{\hat{\tau}_{k+1}}^k)$$

where Ξ^U is a measurable maximizer of $\mathcal{M}U$. Existence of such a measurable selection is implied by Corollary 4 in Schael [1974] due to boundedness of U and (A4), (A5); we refer to Proposition 3.4 in Chapter 2 where existence of a measurable maximizer is discussed in great detail.

We now verify that the strategy $\hat{\mathcal{S}} \triangleq \{\hat{\tau}_k, \hat{\iota}_k, \hat{\Upsilon}_k\}_{k \geq 1}$ is indeed optimal. For this purpose let $\hat{Z} = (\hat{I}, \hat{Y})$ denote the associated controlled switching diffusion. The definition (2.27) implies that either $\hat{\tau}_{k+1} \leq \hat{\tau}_{k, \infty}$ or $\hat{\tau}_{k+1} = \infty$. Therefore by (iii) for every $k \geq 0$

$$\begin{aligned} \mathbb{E} \left[\int_{\hat{\tau}_k, \infty}^{\hat{\tau}_{k+1}} e^{-rt} P(\hat{Z}_t) dt \mathbb{I}_{\hat{\tau}_{k, \infty} < \hat{\tau}_{k+1}} \right] & = \mathbb{E} \left[\int_{\hat{\tau}_{k, \infty}}^{\infty} e^{-rt} P(\hat{Z}_{\hat{\tau}_{k, \infty}}) dt \mathbb{I}_{\hat{\tau}_{k, \infty} < \hat{\tau}_{k+1}} \right] \\ & = \mathbb{E} \left[e^{-r\hat{\tau}_{k, \infty}} U(\hat{Z}_{\hat{\tau}_{k, \infty}}) \mathbb{I}_{\hat{\tau}_{k, \infty} < \hat{\tau}_{k+1}} \right]. \end{aligned}$$

This, combined with the Itô-Dynkin Inequality (\star), yields that (2.24) holds with equality for the strategy $\hat{\mathcal{S}}$ for each $k \geq 0$. So does (2.25) as by definition the action $(\hat{l}_{k+1}, \hat{Y}_{k+1})$ at $\hat{\tau}_{k+1}$ maximizes $\mathcal{M}U(\hat{Z}_{\hat{\tau}_{k+1}}^k)$ on $\{\hat{\tau}_{x+1} < \infty\}$. Therefore, for an arbitrary $m \geq 1$ (2.26) is an equality for $\hat{\mathcal{S}}$. This, in particular, implies that

$$\mathbb{E}\left[\sum_{k=1}^{\infty} e^{-r\hat{\tau}_k}\right] < \infty.$$

Thus $\hat{\tau}_k \rightarrow \infty$ a.s. and $\hat{\mathcal{S}}$ is admissible. By letting $m \rightarrow \infty$ in (2.26) we conclude that

$$U(i, y) = J(\hat{\mathcal{S}}; i, y) \quad \text{for all } i \in \mathcal{E}, y \in \mathcal{J}$$

and the proof is complete. \square

2.B Proofs for Section 2.4.1

We first provide formal definitions of W^h and W_0^h . Recall that $\pi_h = \{x_0 = a, x_1 = a + \frac{b-a}{2^h}, \dots, x_l = a + l \frac{b-a}{2^h}, \dots, x_{2^h} = b\}$, $h \geq 1$. The pyramid functions v_l^h over π_h are defined by

$$v_l^h \triangleq \begin{cases} \frac{2^h}{b-a}(x - x_{l-1}), & x \in [x_{l-1}, x_l] \\ -\frac{2^h}{b-a}(x - x_{l+1}), & x \in [x_l, x_{l+1}] \\ 0, & \text{otherwise} \end{cases} \quad \text{for } l = 1, \dots, 2^h - 1$$

and

$$v_0^h \triangleq \begin{cases} -\frac{2^h}{b-a}(x - x_1), & x \in [a, x_1] \\ 0, & \text{otherwise} \end{cases} \quad \text{and} \quad v_{2^h}^h \triangleq \begin{cases} \frac{2^h}{b-a}(x - x_{2^h-1}), & x \in [x_{2^h-1}, b] \\ 0, & \text{otherwise.} \end{cases}$$

Then the finite-dimensional subspaces $W^h \subset W^{\mathcal{I}}$ and $W_0^h \subset W_0^{\mathcal{I}}$ are generated by $v_0^h, \dots, v_{2^h}^h$ and $v_1^h, \dots, v_{2^h}^h$, respectively. For every $v \in W^{\mathcal{I}}$ we denote by $\text{pr}_h v$ the unique function in W^h that coincides with v in the nodes of π_h . Note that the weak derivatives v' and $(\text{pr}_h v)'$ of v and $\text{pr}_h v$ satisfy

$$\|(\text{pr}_h v)'\|_{L^2} \leq \|v'\|_{L^2}.$$

Moreover, standard Hilbert space theory (see, e.g., [Galántai, 2004, Theorem 7.87]) implies that

$$\text{pr}_h v \rightarrow v \quad \text{in } W^{\mathcal{I}} \text{ as } h \rightarrow \infty.$$

2.B.1 Proof of Theorem 2.5

Proof of Theorem 2.5 (i): Solution to (VI_ϵ^h)

Step 1: Existence. First, define a continuous function $\mathfrak{P} : \mathbb{R}^{2h} \rightarrow \mathbb{R}^{2h}$ by the requirement that for all $z, \bar{z} \in \mathbb{R}^{2h}$

$$(\mathfrak{P}(z), \bar{z}) = \mathbf{a}(\Psi(a) \vee c + u_z, u_{\bar{z}}) - \langle \frac{1}{\epsilon}(\text{pr}_h \Psi - \Psi(a) \vee c - u_z)^+ + f, u_{\bar{z}} \rangle^{\mathcal{I}}$$

where $u_z = \sum_{l=1}^{2h} z_l v_l^h$. Coercivity of \mathbf{a} implies that $\mathbf{a}(u_z, u_z) \gtrsim \|z\|^2$. In combination with the inequality

$$\langle \frac{1}{\epsilon}(\text{pr}_h \Psi - \Psi(a) \vee c - u_z)^+, \text{pr}_h \Psi - \Psi(a) \vee c - u_z \rangle^{\mathcal{I}} \geq 0$$

this implies that $(\mathfrak{P}(z), z) \rightarrow \infty$ as $z \rightarrow \infty$. Hence, by Brouwer's fixed point theorem (see, e.g., Lemma 4.3 in [Lions, 1969, Chapter 1]) there exists $z^* \in \mathbb{R}^{2h}$ such that $\mathfrak{P}(z^*) = 0$ or, equivalently,

$$\mathbf{a}(\Psi(a) \vee c + u_{z^*}, v) - \langle \frac{1}{\epsilon}(\text{pr}_h \Psi - \Psi(a) \vee c - u_{z^*})^+, v \rangle^{\mathcal{I}} = \langle f, v \rangle^{\mathcal{I}}$$

for all $v \in W_0^h$. Thus the function $\Psi(a) \vee c + u_{z^*}$ solves (VI_ϵ^h) .

Step 2: Uniqueness. Let u_1 and u_2 be two solutions to (VI_ϵ^h) . Adding the equations for u_i , $i = 1, 2$, with $u_1 - u_2$ and $u_2 - u_1$ as test functions, respectively, yields

$$\mathbf{a}(u_1 - u_2, u_1 - u_2) + \frac{1}{\epsilon} \langle (\text{pr}_h \Psi - u_1)^+ - (\text{pr}_h \Psi - u_2)^+, u_2 - u_1 \rangle^{\mathcal{I}} = 0.$$

Here each summand on the left-hand side of the equality is non-negative, so the coercivity assumption (A8) implies that $u_1 = u_2$.

Step 3: Boundedness. Let u_ϵ^h denote the unique solution of (VI_ϵ^h) . By assumption (A9) there exists some $v \in \mathcal{K}^{\mathcal{I}}(\Psi, c)$. Then we have $\text{pr}_h v \in \mathcal{K}^{\mathcal{I}}(\text{pr}_h \Psi, c)$. Moreover, $v^h \triangleq \text{pr}_h v - \Psi(a) \vee c$ and $v_\epsilon^h \triangleq u_\epsilon^h - \Psi(a) \vee c$ are elements of W_0^h and can thus be used as test functions in (VI_ϵ^h) . Using the coercivity assumption (A8), we then obtain

$$\begin{aligned} \|u_\epsilon^h\|_{W^{\mathcal{I}}}^2 &\lesssim \mathbf{a}(u_\epsilon^h, u_\epsilon^h) \\ &= \mathbf{a}(u_\epsilon^h, \Psi(a) \vee c) + \langle f, v_\epsilon^h \rangle^{\mathcal{I}} + \langle \frac{1}{\epsilon}(\text{pr}_h \Psi - u_\epsilon^h)^+, v_\epsilon^h \rangle^{\mathcal{I}} \\ &\leq \mathbf{a}(u_\epsilon^h, \Psi(a) \vee c) + \langle f, v_\epsilon^h \rangle^{\mathcal{I}} + \langle \frac{1}{\epsilon}(\text{pr}_h \Psi - u_\epsilon^h)^+, v^h \rangle^{\mathcal{I}} \\ &= \mathbf{a}(u_\epsilon^h, \Psi(a) \vee c) + \langle f, v_\epsilon^h \rangle^{\mathcal{I}} + \mathbf{a}(u_\epsilon^h, v^h) - \langle f, v^h \rangle^{\mathcal{I}} \\ &\leq C_1 \|u_\epsilon^h\|_{W^{\mathcal{I}}} + C_2 \end{aligned}$$

where $C_1, C_2 > 0$ are constants that depend neither on h nor on ϵ , and where the last inequality follows from compactness of the embedding $W^{\mathcal{I}} \hookrightarrow C(\mathcal{I})$. \square

Proof of Theorem 2.5 (ii): $(\mathbf{VI}_\epsilon^h) \rightarrow (\mathbf{VI}^h)$

Step 1: Existence. For every fixed $h \geq 1$, consider the sequence $\{u_\epsilon^h\}_{\epsilon>0}$, which by part (i) is bounded in $W^\mathcal{I}$. Since the embedding $W^\mathcal{I} \hookrightarrow C(\mathcal{I})$ is compact, we conclude that

$u_\epsilon^h \rightarrow u^h$ weakly in $W^\mathcal{I}$ and uniformly on \mathcal{I} as $\epsilon \downarrow 0$ along a subsequence

for some $u^h \in W^h \subset W^\mathcal{I}$. Now fix a function $v \in \mathcal{K}^\mathcal{I}(\text{pr}_h \Psi, c) \cap W^h$. Then the inequality

$$\begin{aligned} 0 &\leq \langle \frac{1}{\epsilon}(\text{pr}_h \Psi - u_\epsilon^h)^+, \text{pr}_h \Psi - u_\epsilon^h \rangle^\mathcal{I} \\ &\leq \langle \frac{1}{\epsilon}(\text{pr}_h \Psi - u_\epsilon^h)^+, v - u_\epsilon^h \rangle^\mathcal{I} = \mathbf{a}(u_\epsilon^h, v - u_\epsilon^h) - \langle f, v - u_\epsilon^h \rangle^\mathcal{I} \lesssim 1 \end{aligned}$$

implies that $\|(\text{pr}_h \Psi - u_\epsilon^h)^+\|_{L^2}^2 \lesssim \epsilon$ and thus $u^h \geq \text{pr}_h \Psi$. Since $u^h(a) = \Psi(a) \vee c$ it follows that $u^h \in \mathcal{K}^\mathcal{I}(\text{pr}_h \Psi, c)$, and it remains only to verify that u^h satisfies the inequality in (\mathbf{VI}^h) .

Note that since $\frac{1}{\epsilon}(\text{pr}_h \Psi - \cdot)^+$ is monotone in $L^2(\mathcal{I})$, i.e. for an arbitrary $v \in \mathcal{K}^\mathcal{I}(\text{pr}_h \Psi, c) \cap W^h$ we have $\langle -\frac{1}{\epsilon}(\text{pr}_h \Psi - v)^+ + \frac{1}{\epsilon}(\text{pr}_h \Psi - u_\epsilon^h)^+, v - u_\epsilon^h \rangle^\mathcal{I} \geq 0$, it follows that

$$\begin{aligned} \mathbf{a}(u_\epsilon^h, v - u_\epsilon^h) &\geq \mathbf{a}(u_\epsilon^h, v - u_\epsilon^h) - \langle -\frac{1}{\epsilon}(\text{pr}_h \Psi - v)^+ + \frac{1}{\epsilon}(\text{pr}_h \Psi - u_\epsilon^h)^+, v - u_\epsilon^h \rangle^\mathcal{I} \\ &= \langle f, v - u_\epsilon^h \rangle^\mathcal{I}. \end{aligned}$$

Since \mathbf{a} is weakly lower semi-continuous, letting $\epsilon \downarrow 0$ implies that u^h solves (\mathbf{VI}^h) .

Step 2: Uniqueness. As in the proof of uniqueness in part (i), for any two solutions u_1 and u_2 of (\mathbf{VI}^h) we have

$$\mathbf{a}(u_1 - u_2, u_1 - u_2) \leq 0$$

and therefore $u_1 = u_2$ by coercivity of \mathbf{a} . Uniqueness implies that $u_\epsilon^h \rightarrow u^h$ weakly in $W^\mathcal{I}$ and uniformly on \mathcal{I} as $\epsilon \downarrow 0$.

Step 3: Boundedness. This follows immediately since $u_\epsilon^h \rightarrow u^h$ weakly in $W^\mathcal{I}$ and $\{u_\epsilon^h\}_{h \geq 1, \epsilon > 0}$ is uniformly bounded in $W^\mathcal{I}$. \square

Proof of Theorem 2.5 (iii): $(\mathbf{VI}_\epsilon^h) \rightarrow (\mathbf{VI}_\epsilon)$

Step 1: Existence. For every fixed $\epsilon > 0$ the sequence $\{u_\epsilon^h\}_{h \geq 1}$ is bounded in $W^\mathcal{I}$. As in the proof of part (ii), compactness of the embedding $W^\mathcal{I} \hookrightarrow C(\mathcal{I})$ yields $u_\epsilon \in W^\mathcal{I}$ such that

$$u_\epsilon^h \rightarrow u_\epsilon \text{ weakly in } W^\mathcal{I} \text{ and uniformly on } \mathcal{I} \text{ as } h \rightarrow \infty \quad (2.28)$$

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along a subsequence. Uniform convergence implies that $u_\epsilon(a) = \Psi(a) \vee c$. Furthermore, for an arbitrary $v \in W_0^\mathcal{I}$ we use $v_h \triangleq \text{pr}_h v \in W_0^h$ as a test function in (VI_ϵ^h) . Then upon letting $h \uparrow \infty$ we conclude that u_ϵ satisfies (VI_ϵ) .

Step 2: Uniqueness. This is shown in exactly the same way as in *Step 2* of the proof of (i). Uniqueness implies in particular that (2.28) holds for the entire sequence.

Step 3: Regularity. We now show that the weak solution u_ϵ of (VI_ϵ) is indeed a classical solution. For an arbitrary $\phi \in C_c^1(a, b)$ take $2\frac{\phi}{S^2} \in W_0^{1,2}([a, b])$ as a test function in (VI_ϵ) to get

$$\int_a^b u'_\epsilon \phi' dy = \int_a^b (d u'_\epsilon - (r + \lambda) u_\epsilon) 2\frac{\phi}{S^2} dy + \langle \frac{1}{\epsilon} (\Psi - u_\epsilon)^+, 2\frac{\phi}{S^2} \rangle^\mathcal{I} + \langle f, 2\frac{\phi}{S^2} \rangle^\mathcal{I}$$

and hence $|\int_a^b u'_\epsilon \phi' dy| \leq C(\epsilon, u_\epsilon) \|\phi\|_{L^2(\mathcal{I})}$ for some constant $C = C(\epsilon, u_\epsilon)$. Therefore by Proposition 8.3 in Brezis [2010] we have $u'_\epsilon \in W^{1,2}(\mathcal{I})$, i.e. u_ϵ is twice weakly differentiable. Integration by parts yields that for all $v \in L^2(\mathcal{I})$

$$\int_a^b (-\frac{1}{2} S^2) u''_\epsilon v dy = \int_a^b (d u'_\epsilon - (r + \lambda) u_\epsilon) v dy + \langle \frac{1}{\epsilon} (\Psi - u_\epsilon)^+, v \rangle^\mathcal{I} + \langle f, v \rangle^\mathcal{I}.$$

Since Ψ , f , d , r , S and u_ϵ are continuous, we can choose a continuous version of u''_ϵ and thus $u_\epsilon \in C^2(\mathcal{I})$.

Step 4: Boundedness. This follows exactly as in the proof of (ii) above. \square

2.B.2 Proof of Proposition 2.7: Comparison Principle for (VI)

First note that $u_2(0) = \Psi_2(a) \vee c_2 \leq \Psi_1(a) \vee c_1 = u_1(a)$, so $(u_2 - u_1)^+ \in W_0^\mathcal{I}$. Hence we can take $v_1 \triangleq u_1 + (u_2 - u_1)^+$ and $v_2 \triangleq u_2 - (u_2 - u_1)^+$ as test functions in the variational inequalities for u_1 and u_2 , respectively. Adding these inequalities, we obtain

$$\mathbf{a}(u_1 - u_2, (u_2 - u_1)^+) \geq \langle f_1 - f_2, (u_2 - u_1)^+ \rangle^\mathcal{I} \geq 0$$

and the result follows from the coercivity property (A8) of \mathbf{a} . \square

2.B.3 Proof of Theorem 2.6: Solution to (VI)

Step 1: Uniqueness. Uniqueness of the solution to (VI) follows immediately from the comparison principle for (VI), see Proposition 2.7.

Step 2: Existence. Since $\{u_\epsilon\}_{\epsilon > 0}$ and $\{u^h\}_{h \geq 1}$ are bounded in $W^\mathcal{I}$, both have accumulation points with respect to uniform convergence on \mathcal{I} and with respect

to weak convergence in $W^{\mathcal{I}}$. By *Step 1*, the proof will be complete if we can show that every accumulation point u of $\{u_\epsilon\}_{\epsilon>0}$ and every accumulation point \bar{u} of $\{u^h\}_{h\geq 1}$ solve the variational inequalities (VI).

Step 2a: u solves (VI). This can be seen analogously as in *Step 1* of the proof of Theorem 2.5 (ii) in Appendix 2.B.1: Uniform convergence and the inequality $\|(\Psi - u_\epsilon)^+\|^2 \lesssim \epsilon$ imply that $u \in \mathcal{K}^{\mathcal{I}}(\Psi, c)$. On the other hand, monotonicity of $-\frac{1}{\epsilon}(\Psi - \cdot)^+$ in $L^2(\mathcal{I})$ and weak lower semi-continuity of \mathbf{a} imply that u satisfies the variational inequality (VI).

Step 2b: \bar{u} solves (VI). Fix an arbitrary $v \in \mathcal{K}^{\mathcal{I}}(\Psi, c)$ and define $v_h \triangleq \text{pr}_h v \in W^h$. Then $v_h \in \mathcal{K}^{\mathcal{I}}(\text{pr}_h \Psi, c)$ and $v_h \rightarrow v$ in $W^{\mathcal{I}}$ as $h \rightarrow \infty$. By inserting v_h as a test function in (VI^h) and letting $h \rightarrow \infty$ along a subsequence that converges to \bar{u} , it follows that \bar{u} solves the variational inequality (VI). \square

2.B.4 Proof of Theorem 2.8: Itô-Dynkin Inequality for (VI)

Step 1: Inequality. By Theorem 2.6 the classical solutions $\{u_\epsilon\}_{\epsilon>0}$ of the penalized problems (VI _{ϵ}) converge to u uniformly on \mathcal{I} . Itô's formula for u_ϵ implies

$$\begin{aligned} & e^{-r\sigma_1 \wedge \tau_{\mathcal{I}}} u_\epsilon(\xi_{\sigma_1 \wedge \tau_{\mathcal{I}}}) \mathbb{I}_{\sigma_1 < \infty} - e^{-r\sigma_2 \wedge \tau_{\mathcal{I}}} u_\epsilon(\xi_{\sigma_2 \wedge \tau_{\mathcal{I}}}) \mathbb{I}_{\sigma_1 < \infty} \\ &= \int_{\sigma_1 \wedge \tau_{\mathcal{I}}}^{\sigma_2 \wedge \tau_{\mathcal{I}}} e^{-rt} [(-\mathcal{L}u_\epsilon + ru_\epsilon)](\xi_t) dt \mathbb{I}_{\sigma_1 < \infty} \\ &\quad - \int_{\sigma_1 \wedge \tau_{\mathcal{I}}}^{\sigma_2 \wedge \tau_{\mathcal{I}}} e^{-rt} [Su'_\epsilon](\xi_t) dW_t \mathbb{I}_{\sigma_1 < \infty} \\ &= \int_{\sigma_1 \wedge \tau_{\mathcal{I}}}^{\sigma_2 \wedge \tau_{\mathcal{I}}} e^{-rt} [f + \frac{1}{\epsilon}(\Psi - u_\epsilon)^+ - \lambda u_\epsilon](\xi_t) dt \mathbb{I}_{\sigma_1 < \infty} \\ &\quad - \int_{\sigma_1 \wedge \tau_{\mathcal{I}}}^{\sigma_2 \wedge \tau_{\mathcal{I}}} e^{-rt} [Su'_\epsilon](\xi_t) dW_t \mathbb{I}_{\sigma_1 < \infty} \\ &\geq \int_{\sigma_1 \wedge \tau_{\mathcal{I}}}^{\sigma_2 \wedge \tau_{\mathcal{I}}} e^{-rt} [f - \lambda u_\epsilon](\xi_t) dt \mathbb{I}_{\sigma_1 < \infty} \\ &\quad - \int_{\sigma_1 \wedge \tau_{\mathcal{I}}}^{\sigma_2 \wedge \tau_{\mathcal{I}}} e^{-rt} [Su'_\epsilon](\xi_t) dW_t \mathbb{I}_{\sigma_1 < \infty}. \end{aligned}$$

Rearranging and taking conditional expectations with respect to \mathcal{F}_{σ_1} yields

$$\begin{aligned} e^{-r\sigma_1 \wedge \tau_{\mathcal{I}}} u_\epsilon(\xi_{\sigma_1 \wedge \tau_{\mathcal{I}}}) \mathbb{I}_{\sigma_1 < \infty} &\geq \mathbb{E}[e^{-r\sigma_2 \wedge \tau_{\mathcal{I}}} u_\epsilon(\xi_{\sigma_2 \wedge \tau_{\mathcal{I}}}) \mathbb{I}_{\sigma_1 < \infty} \\ &\quad + \int_{\sigma_1 \wedge \tau_{\mathcal{I}}}^{\sigma_2 \wedge \tau_{\mathcal{I}}} e^{-rt} [f - \lambda u_\epsilon](\xi_t) dt \mathbb{I}_{\sigma_1 < \infty} | \mathcal{F}_{\sigma_1}]. \end{aligned}$$

Now the first part of the claim follows upon letting $\epsilon \downarrow 0$.

Step 2: Equality. To establish the second part, let $\sigma_2 \leq \hat{\theta}$ and define, for each $\epsilon > 0$, the stopping time

$$\hat{\theta}_\epsilon \triangleq \begin{cases} \sigma_2 \wedge \inf \{t \geq \sigma_1 : u_\epsilon(\xi_{t \wedge \tau_{\mathcal{I}}}) \leq \Psi(\xi_{t \wedge \tau_{\mathcal{I}}})\}, & \sigma_1 < \tau_{\mathcal{I}} \\ \sigma_2, & \sigma_1 \geq \tau_{\mathcal{I}}. \end{cases}$$

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Since $u_\epsilon \rightarrow u$ uniformly on \mathcal{I} , it follows that $\hat{\theta}_\epsilon \rightarrow \sigma_2$ as $\epsilon \downarrow 0$: Indeed, if we had $\hat{\theta}_\epsilon \rightarrow \hat{\theta} < \sigma_2$ as $\epsilon \downarrow 0$, then uniform convergence of $\{u_\epsilon\}_{\epsilon>0}$ would imply that $u(\xi_{\hat{\theta} \wedge \tau_{\mathcal{I}}}) = \Psi(\xi_{\hat{\theta} \wedge \tau_{\mathcal{I}}})$, contradicting the definition of $\hat{\theta}$. Using Itô's formula for u_ϵ once again, we obtain

$$\begin{aligned} e^{-r\sigma_1 \wedge \tau_{\mathcal{I}}} u_\epsilon(\xi_{\sigma_1 \wedge \tau_{\mathcal{I}}}) \mathbb{I}_{\sigma_1 < \infty} &= \mathbb{E} \left[e^{-r\hat{\theta}_\epsilon \wedge \tau_{\mathcal{I}}} u_\epsilon(\xi_{\hat{\theta}_\epsilon \wedge \tau_{\mathcal{I}}}) \mathbb{I}_{\sigma_1 < \infty} \right. \\ &\quad \left. + \int_{\sigma_1 \wedge \tau_{\mathcal{I}}}^{\hat{\theta}_\epsilon \wedge \tau_{\mathcal{I}}} e^{-rt} [f - \lambda u_\epsilon](\xi_t) dt \mathbb{I}_{\sigma_1 < \infty} \middle| \mathcal{F}_{\sigma_1} \right] \end{aligned}$$

and letting $\epsilon \downarrow 0$ yields the second part of the assertion. \square

2.C Proofs for Section 2.4.2

2.C.1 Proof of Proposition 2.9

Step 1: Monotonicity. First, using $v \triangleq V_i^1 + (-V_i^1)^+$ as a test function in (2.19) we find that

$$\begin{aligned} \|(-V_i^1)^+\|_{\mathbb{W}^{\mathcal{I}}}^2 &\lesssim \mathbf{a}_i^{\mathcal{I}}(-V_i^1, (-V_i^1)^+) + \langle (c^{\mathcal{I}} + \sum_{j \in \mathcal{E}} \lambda^{i,j})(-V_i^1), (-V_i^1)^+ \rangle^{\mathcal{I}} \\ &\leq -\langle P_i, (-V_i^1)^+ \rangle^{\mathcal{I}} \leq 0 \end{aligned}$$

so $V^1 \geq 0$. Now the first part of the claim follows by induction from the comparison principle in Proposition 2.7.

Step 2: Boundedness. Select $C > 0$ such that $P_i \leq rC$ on $[0, \infty)$ for each $i \in \mathcal{E}$. By definition we have $0 = V^0 \leq C$; assume by induction that $V_j^{m-1} \leq C$, $j \in \mathcal{E}$, set $v_i^+ \triangleq (V_i^m - C)^+$ and take $v \triangleq V_i^m - v_i^+$ as a test function in (2.19). Then for each $i \in \mathcal{E}$ we obtain

$$\begin{aligned} \|(V_i^m - C)^+\|_{\mathbb{W}^{\mathcal{I}}}^2 &\lesssim \mathbf{a}_i^{\mathcal{I}}(V_i^m - C, v_i^+) + \langle (c^{\mathcal{I}} + \sum_{j \in \mathcal{E}} \lambda^{i,j})(V_i^m - C), v_i^+ \rangle^{\mathcal{I}} \\ &\leq \langle (P_i - rC) + c^{\mathcal{I}}(V_i^{m-1} - C) + \sum_{j \in \mathcal{E}} \lambda^{i,j}(V_j^{m-1} - C), v_i^+ \rangle^{\mathcal{I}} \\ &\leq 0 \end{aligned}$$

so $V_i^m \leq C$. This completes the induction and demonstrates that V_i^m is bounded above by the universal constant $C > 0$ for $i \in \mathcal{E}$, $m \geq 1$. \square

2.C.2 Proof of Theorem 2.10

Step 1: Uniform Convergence on \mathcal{I} and Convergence on $\mathbb{W}^{\mathcal{I}}$. Proposition 2.9 implies that the pointwise limit $V \triangleq \lim_{m \rightarrow \infty} V^m$ is well-defined and that the

2.C Proofs for Section 2.4.2

sequence $\{V_i^m\}_{m \geq 1}$ is bounded above by a universal constant, and in particular bounded in $L^2(\mathcal{I})$, for each $i \in \mathcal{E}$.

Next, we demonstrate that $\{V_i^m\}_{m \geq 1}$ is also bounded in $W^{\mathcal{I}}$. Note that by (2.8) we have $v_i^m \in \mathcal{K}^{\mathcal{I}}(\mathcal{M}_i^{\mathcal{I}}V^{m-1}, P_i(a)/r)$ for

$$v_i^m(y) \triangleq (\mathcal{M}_i^{\mathcal{I}}V^{m-1}(a)) \vee \frac{P_i(a)}{r} + \mathfrak{d}(y - a), \quad y \in \mathcal{I}.$$

Further note that $\{v_i^m\}_{m \geq 1}$ is bounded in $W^{\mathcal{I}}$ by (A4) and since $\{V_i^m\}_{m \geq 1}$ is bounded in $L^2(\mathcal{I})$. Hence using (2.19) we obtain

$$\begin{aligned} \|V_i^m\|_{W^{\mathcal{I}}}^2 &\lesssim \mathfrak{a}_i^{\mathcal{I}}(V_i^m, V_i^m) + \langle (c^{\mathcal{I}} + \sum_{j \in \mathcal{E}} \lambda^{i,j})V_i^m, V_i^m \rangle^{\mathcal{I}} \\ &\leq \mathfrak{a}_i^{\mathcal{I}}(V_i^m, v_i^m) + \langle (c^{\mathcal{I}} + \sum_{j \in \mathcal{E}} \lambda^{i,j})V_i^m, v_i^m \rangle^{\mathcal{I}} \\ &\quad - \langle P_i + c^{\mathcal{I}}V_i^{m-1} + \sum_{j \in \mathcal{E}} \lambda^{i,j}V_j^{m-1}, v_i^m - V_i^m \rangle^{\mathcal{I}} \end{aligned}$$

so that, with suitable constants $C_1, C_2 > 0$ that are independent of $m \geq 1$,

$$\|V_i^m\|_{W^{\mathcal{I}}}^2 \leq C_1 \|V_i^m\|_{W^{\mathcal{I}}} + C_2.$$

It follows that $\{V_i^m\}_{m \geq 1}$ is bounded in $W^{\mathcal{I}}$. Hence by compactness and a subsequence argument we conclude that

$$V_i^m \rightarrow V_i \text{ weakly in } W^{\mathcal{I}} \text{ and uniformly on } \mathcal{I} \text{ as } m \rightarrow \infty \text{ for each } i \in \mathcal{E}.$$

Moreover, uniform convergence immediately yields $\mathcal{M}^{\mathcal{I}}V^m \uparrow \mathcal{M}^{\mathcal{I}}V$ for $m \rightarrow \infty$ uniformly on $\mathcal{E} \times \mathcal{I}$.

Step 2: V solves (2.18). Fix an arbitrary $i \in \mathcal{E}$ and note that by construction $V_i \geq 0$ and $V_i \in \mathcal{K}^{\mathcal{I}}(\mathcal{M}_i^{\mathcal{I}}V, P_i(a)/r)$. To demonstrate that V_i satisfies the quasi-variational inequality (2.18), consider an arbitrary test function $v \in \mathcal{K}^{\mathcal{I}}(\mathcal{M}_i^{\mathcal{I}}V, P_i(a)/r)$. We first construct suitable functions w_δ^m that we can use as test functions in (2.19) for V^m . For each $\delta > 0$ and $m \geq 1$ set

$$\begin{aligned} w_\delta^m(y) &\triangleq (\mathcal{M}_i^{\mathcal{I}}V^{m-1}(a)) \vee \frac{P_i(a)}{r} + \mathfrak{d}(y - a) \\ &\quad + \frac{1}{\delta} \left(v(a + \delta) - (\mathcal{M}_i^{\mathcal{I}}V^{m-1}(a)) \vee \frac{P_i(a)}{r} \right)^+ (y - a) \end{aligned}$$

and define

$$v_\delta^m \triangleq \begin{cases} w_\delta^m \wedge v & \text{on } [a, a + \delta] \\ v & \text{on } (a + \delta, b]. \end{cases}$$

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Then for all sufficiently small $\delta > 0$ and $m \geq 1$ we have $v_\delta^m \in \mathcal{K}^\mathcal{I}(\mathcal{M}_i^\mathcal{I} V^{m-1}, \frac{P_i(a)}{r})$, so

$$\begin{aligned} & \mathfrak{a}_i^\mathcal{I}(V_i^m, v_\delta^m - V_i^m) + c^\mathcal{I} \langle V_i^m, v_\delta^m - V_i^m \rangle^\mathcal{I} + \langle \sum_{j \in \mathcal{E}} \lambda^{i,j} V_j^m, v_\delta^m - V_i^m \rangle^\mathcal{I} \\ & \geq \langle P_i + c^\mathcal{I} V_i^{m-1} + \sum_{j \in \mathcal{E}} \lambda^{i,j} V_j^{m-1}, v_\delta^m - V_i^m \rangle^\mathcal{I}. \end{aligned}$$

Using lower semi-continuity of $\mathfrak{a}_i^\mathcal{I}(\cdot, \cdot) + c^\mathcal{I} \langle \cdot, \cdot \rangle^\mathcal{I}$ and the fact that

$$\lim_{\delta \downarrow 0} \lim_{m \rightarrow \infty} v_\delta^m = v \quad \text{in } W^\mathcal{I}$$

it follows that V_i solves (2.18), as asserted. \square

2.D Proofs for Section 2.4.3

2.D.1 Proof of Proposition 2.11

Step 1: Uniform Convergence of U^n on Compact Subsets of $\mathcal{E} \times \mathcal{J}^0$. Theorem 2.10 implies that $0 \leq U^n \leq C$ for all $n \geq 1$, where $C > 0$ is a universal constant. Choose $n_0 \geq 1$ and set $\mathcal{K} \triangleq [a, b] \triangleq \mathcal{I}_{n_0}$. For each $i \in \mathcal{E}$ and $n \geq n_0$ define the function $v_i^n : \mathcal{I}_n \rightarrow \mathbb{R}$ by

$$\begin{aligned} v_i^n(y) & \triangleq \min\{U_i^n(a) + \mathfrak{d}(y - a) + C \frac{y-a}{b-a}, U_i^n(b) + \mathfrak{d}(b - y) + C \frac{b-y}{b-a}\} \mathbb{I}_{\mathcal{K}}(y) \\ & + U_i^n(y) \mathbb{I}_{\mathcal{I}_n \setminus \mathcal{K}}(y) \end{aligned}$$

and note that $\{v_i^n\}_{n \geq n_0}$ restricted to \mathcal{K} is bounded in $W^\mathcal{K} \triangleq W^{1,2}(\mathcal{K})$.

We next show that the sequence $\{U_i^n\}_{n \geq n_0}$ is bounded in $W^\mathcal{K}$ for each $i \in \mathcal{E}$. It follows from (2.8) that $v_i^n \in \mathcal{K}^{\mathcal{I}_n}(\mathcal{M}_i^{\mathcal{I}_n} U^n, P_i(a_n)/r)$ for each $n \geq n_0$, so the quasi-variational inequality (2.18) yields

$$\mathfrak{a}_i^\mathcal{K}(U_i^n, v_i^n - U_i^n) - \langle \sum_{j \in \mathcal{E}} \lambda^{i,j} (U_j^n - U_i^n), v_i^n - U_i^n \rangle^\mathcal{K} \geq \langle P_i, v_i^n - U_i^n \rangle^\mathcal{K} \quad (2.29)$$

where $\mathfrak{a}_i^\mathcal{K}$ is defined by (2.16) and $\langle \cdot, \cdot \rangle^\mathcal{K}$ denotes the scalar product in $L^2(\mathcal{K})$. Let $c^\mathcal{K}$ denote the coercivity coefficient of $\mathfrak{a}_i^\mathcal{K}$. Then $\mathfrak{a}_i^\mathcal{K}(\cdot, \cdot) + c^\mathcal{K} \langle \cdot, \cdot \rangle^\mathcal{K}$ is coercive, so (2.29) yields

$$\begin{aligned} \|U_i^n\|_{W^\mathcal{K}}^2 & \lesssim \mathfrak{a}_i^\mathcal{K}(U_i^n, U_i^n) + c^\mathcal{K} \langle U_i^n, U_i^n \rangle^\mathcal{K} \\ & \leq \mathfrak{a}_i^\mathcal{K}(U_i^n, v_i^n) - \langle \sum_{j \in \mathcal{E}} \lambda^{i,j} (U_j^n - U_i^n), v_i^n - U_i^n \rangle^\mathcal{K} \\ & \quad + c^\mathcal{K} \langle U_i^n, U_i^n \rangle^\mathcal{K} - \langle P_i, v_i^n - U_i^n \rangle^\mathcal{K}. \end{aligned}$$

Since $\{U_i^n\}_{n \geq n_0}$ is bounded in $L^2(\mathcal{K})$ and $\{v_i^n\}_{n \geq n_0}$ is bounded in $W^{\mathcal{K}}$, it follows that $\{U_i^n\}_{n \geq n_0}$ is bounded in $W^{\mathcal{K}}$. Thus we can find a subsequence of $\{U^n\}_{n \geq n_0}$ that converges uniformly on $\mathcal{E} \times \mathcal{K}$. Since $\mathcal{I}_n \subset \mathcal{I}_{n+1}$ for $n \geq 1$ and $\cup_{n \geq 1} \mathcal{I}_n = \mathcal{J}^0$, we can use a diagonal argument to construct a subsequence $n = n(l)$ and a function $U : \mathcal{E} \times \mathcal{J}^0 \rightarrow \mathbb{R}$ such that

$$U^{n(l)} \rightarrow U \quad \text{uniformly on compact subsets of } \mathcal{E} \times \mathcal{J}^0 \text{ as } l \rightarrow \infty.$$

Step 2: Uniform Convergence of U^n on Compact Subsets of $\mathcal{E} \times \mathcal{J}$. By assumption (A3) there exist $p \in (1, 2]$ and $\delta > 0$ such that

$$(-d_i + S_i S'_i) \in L_{\frac{p}{p-1}}([0, \delta]), \quad S_i^2 \in L_{\frac{p}{p-1}}([0, \delta]) \quad \text{and} \quad \frac{1}{S_i^2} \in L_{\frac{p}{2-p}}([0, \delta]). \quad (2.30)$$

It suffices to show that there exists a subsequence of $\{U^{n(l)}\}_{l \geq 1}$ that converges uniformly on $\mathcal{E} \times [0, \delta]$. For simplicity of notation, we relabel the sequence and replace $\{U^{n(l)}\}_{l \geq 1}$ by $\{U^n\}_{n \geq 1}$ in the following. We also fix $i \in \mathcal{E}$ and fix $n_0 \geq 1$ such that $[0, \delta] \cap \mathcal{I}_n = [a_n, \delta] \neq \emptyset$ for all $n \geq n_0$ and, similarly to *Step 1*, define the test functions $\{v_i^n\}_{n \geq n_0}$ by setting

$$\begin{aligned} v_i^n(y) &\triangleq \min\{U_i^n(a_n) + \mathfrak{d}(y - a_n) + C \frac{y - a_n}{\delta - a_n}, U_i^n(\delta) + \mathfrak{d}(\delta - y) + C \frac{\delta - y}{\delta - a_n}\} \mathbb{1}_{[a_n, \delta]}(y) \\ &\quad + U_i^n(y) \mathbb{1}_{\mathcal{I}_n \setminus [a_n, \delta]}(y) \end{aligned}$$

As in *Step 1* the functions $\{v_i^n\}_{n \geq n_0}$ are uniformly bounded and have uniformly bounded derivatives on $[a_n, \delta]$. Then, rewriting (2.29) with $[a_n, \delta]$ replacing \mathcal{K} , and using the definition of \mathbf{a}_i (see (2.12)) we get

$$\begin{aligned} \int_{a_n}^{\delta} \frac{1}{2} S_i^2 ((U_i^n)')^2 d\lambda &\leq \int_{a_n}^{\delta} \frac{1}{2} S_i^2 (U_i^n)' (v_i^n)' d\lambda + \int_{a_n}^{\delta} (-d_i + S_i S'_i) (U_i^n)' (v_i^n - U_i^n) d\lambda \\ &\quad + \langle r U_i^n, v_i^n - U_i^n \rangle_{\mathcal{I}_n} - \langle \sum_{j \in \mathcal{E}} \lambda^{i,j} (U_j^n - U_i^n), v_i^n - U_i^n \rangle_{\mathcal{I}_n} \\ &\quad - \langle P_i, v_i^n - U_i^n \rangle_{\mathcal{I}_n}. \end{aligned}$$

Thus, by Theorem 2.10, Hölder's inequality and the definition of U_i^n in (2.20) we find a constant $C > 0$ such that

$$\|(U_i^n)'\|_p^2 \leq C \left\| \frac{1}{S_i^2} \right\|_{\frac{p}{2-p}} \left[1 + (\|S_i^2\|_{\frac{p}{p-1}} + \|d_i - S_i S'_i\|_{\frac{p}{p-1}}) \|(U_i^n)'\|_p \right]$$

where all norms are taken over the interval $[0, \delta]$. Thus, taking into account (2.30), we obtain boundedness of $\{(U_i^n)'\}_{n \geq n_0}$ in $L_p([0, \delta])$ and therefore of $\{U_i^n\}_{n \geq n_0}$ in

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$W^{1,p}([0, \delta])$ for some $p > 1$. This implies that the family $\{U_i^n\}_{n \geq n_0}$ is equicontinuous on $[0, \delta]$, see e.g. Theorem 8.8 in Brezis [2010], so the Arzelà-Ascoli theorem yields a subsequence $\{n(l)\}_{l \geq 1}$ such that $U_i^{n(l)} \rightarrow U$ uniformly⁷ on $[0, \delta]$ as $l \rightarrow \infty$. In combination with the result of *Step 1*, this shows that

$$U^{n(l)} \rightarrow U \quad \text{uniformly on compact subsets of } \mathcal{E} \times \mathcal{J} \text{ as } l \rightarrow \infty.$$

Step 3: Uniform Convergence of $\mathcal{M}^{\mathcal{I}_n} U^n$ along a Subsequence. Let $\mathcal{K} \subset \mathcal{J}$ be compact. Then by (A5) and uniform boundedness of U^n and U there exists a compact set $\mathcal{K}' \subset \mathcal{J}$ such that for all $y \in \mathcal{K}$ and $n \geq 1$ we have

$$\mathcal{M}^{\mathcal{I}_n} U^n(i, y) = \sup_{y' \in \mathcal{K}' \cap \mathcal{I}_n, i' \in \mathcal{A}(i)} [U^n(i', y') - \mathcal{C}(i, i', y, y')]$$

and

$$\mathcal{M}U(i, y) = \sup_{y' \in \mathcal{K}', i' \in \mathcal{A}(i)} [U(i', y') - \mathcal{C}(i, i', y, y')].$$

This implies

$$\sup_{\mathcal{E} \times \mathcal{K}} |\mathcal{M}U - \mathcal{M}^{\mathcal{I}_{n(l)}} U^{n(l)}| \leq \sup_{\mathcal{E} \times \mathcal{K}'} |U - U^{n(l)}| \rightarrow 0 \quad \text{as } l \rightarrow \infty$$

as asserted. \square

2.D.2 Proof of Proposition 2.13: Itô-Dynkin Inequality (\star)

The function U defined in Proposition 2.11 was constructed as an accumulation point of the sequence $\{U^n\}_{n \geq 1}$. For notational simplicity we denote the subsequence of $\{U^n\}_{n \geq 1}$ approaching U by $\{U^n\}_{n \geq 1}$ in this proof.

Let $Z = (I, Y) = Z^{\tau, \iota, \Upsilon} = (I^{\tau, \iota, \Upsilon}, Y^{\tau, \iota, \Upsilon})$ denote the switching diffusion starting at τ in (ι, Υ) , and let $\{\rho_k, \iota_k\}_{k \geq 0}$ denote the associated marked point process. To establish (2.10) it suffices to verify that, for all stopping times σ_1 and σ_2 with $\tau \leq \sigma_1 \leq \sigma_2$,

$$\begin{aligned} e^{-r\sigma_1} U(Z_{\sigma_1}) \mathbb{I}_{\sigma_1 < \sigma_2 \wedge \tau_\infty} &\geq \mathbb{E} \left[e^{-r\sigma_2 \wedge \tau_\infty} U(Z_{\sigma_2 \wedge \tau_\infty}) \mathbb{I}_{\sigma_1 < \sigma_2 \wedge \tau_\infty} \right. \\ &\quad \left. + \int_{\sigma_1}^{\sigma_2 \wedge \tau_\infty} e^{-rt} P(Z_t) dt \mathbb{I}_{\sigma_1 < \sigma_2 \wedge \tau_\infty} \middle| \mathcal{F}_{\sigma_1} \right] \end{aligned}$$

⁷In particular, U defined in *Step 1* on $\mathcal{E} \times \mathcal{J}^0$ can be continuously extended to $\mathcal{E} \times \mathcal{J}$.

with equality if $\sigma_2 \leq \hat{\theta}$. This is equivalent to the assertion that, for all stopping times σ_1 and σ_2 with $\tau \leq \sigma_1 \leq \sigma_2 \leq \tau_\infty$ and $\sigma_1 < \sigma_2$, we have

$$\mathbb{E} \left[\int_{\sigma_1}^{\sigma_2} e^{-rt} P(Z_t) dt \middle| \mathcal{F}_{\sigma_1} \right] \leq \mathbb{E} \left[e^{-r\sigma_1} U(Z_{\sigma_1}) - e^{-r\sigma_2} U(Z_{\sigma_2}) \middle| \mathcal{F}_{\sigma_1} \right]; \quad (2.31)$$

and that (2.31) holds with equality if $\sigma_2 \leq \hat{\theta}$. First, we subdivide the stochastic interval $(\sigma_1, \sigma_2]$ into subintervals where the regime process is constant via

$$(\sigma_1, \sigma_2] = \bigcup_{k=0}^{\infty} (\zeta_k, \zeta_{k+1}] \quad \text{where} \quad \zeta_k \triangleq (\sigma_1 \vee \rho_k) \wedge \sigma_2, \quad k \geq 0.$$

The proof is now divided into steps. In *Step 1* we prove the inequality in (2.31) on each interval $(\zeta_k, \zeta_{k+1}]$. In *Step 2*, we show that equality holds if $\sigma_2 \leq \hat{\theta}$. Finally, in *Step 3* we combine these results to obtain (2.31).

Step 1: Itô-Dynkin Inequality on $(\zeta_k, \zeta_{k+1}]$. First, for each $k \geq 0$ and $n \geq 1$ define

$$\zeta_k^n \triangleq \zeta_k \wedge \tau_n \quad \text{where} \quad \tau_n \triangleq \inf\{t \geq \tau : Y_t \notin \mathcal{I}_n\}$$

and note that $\tau_n \uparrow \tau_\infty$, so $\zeta_k^n \uparrow \zeta_k$ as $n \rightarrow \infty$. Recall that for each $n \geq 1$ the function U^n solves (2.18) on $\mathcal{I}_n = [a_n, b_n]$, so U_i^n is the unique solution of the following variational inequalities for every $i \in \mathcal{E}$

$$\begin{aligned} \text{Find } u \in \mathcal{K}^{\mathcal{I}_n}(\mathcal{M}_i^{\mathcal{I}_n} U^n, \frac{P_i(a_n)}{r}) \text{ such that for all } v \in \mathcal{K}^{\mathcal{I}_n}(\mathcal{M}_i^{\mathcal{I}_n} U^n, \frac{P_i(a_n)}{r}) \\ \mathfrak{a}_i^{\mathcal{I}_n}(u, v - u) + c^{\mathcal{I}_n}(u, v - u)^{\mathcal{I}_n} \\ \geq \langle P_i + \sum_{j \in \mathcal{E}} \lambda^{i,j} (U_j^n - U_i^n) + c^{\mathcal{I}_n} U_i^n, v - u \rangle^{\mathcal{I}_n}. \end{aligned}$$

Since $\mathfrak{a}_i^{\mathcal{I}_n}(\cdot, \cdot) + c^{\mathcal{I}_n}(\cdot, \cdot)^{\mathcal{I}_n}$ is coercive, this problem is a special case of (VI). Therefore, Theorem 2.8 applies to show that for every $k \geq 0$

$$\begin{aligned} \mathbb{E} \left[\int_{\zeta_k}^{\zeta_{k+1}^n} e^{-rt} P(Z_t) dt \middle| \mathcal{F}_{\zeta_k} \right] \\ = \mathbb{E} \left[\int_{\zeta_k}^{\zeta_{k+1}^n} e^{-rt} P_{\iota_k}(Y_t) dt \middle| \mathcal{F}_{\zeta_k} \right] \\ \leq \mathbb{E} \left[e^{-r\zeta_k} U_{\iota_k}^n(Y_{\zeta_k}) - e^{-r\zeta_{k+1}^n} U_{\iota_k}^n(Y_{\zeta_{k+1}^n}) \right. \\ \left. - \int_{\zeta_k}^{\zeta_{k+1}^n} e^{-rt} \sum_{j \in \mathcal{E}} \Lambda_t(\{j\}) (U_j^n(Y_t) - U_{\iota_k}^n(Y_t)) dt \middle| \mathcal{F}_{\zeta_k} \right]. \end{aligned}$$

Next recall that by the projection theorem for marked point processes [Brémaud, 1981, Chapter VIII, Theorem 3, Corollary 4], the process

$$\sum_{k=1}^{\infty} \mathbb{1}_{\rho_k \leq t} H(\rho_k, \iota_k) - \int_0^t \sum_{j \in \mathcal{E}} H(s, j) \Lambda_s(\{j\}) ds \quad (2.32)$$

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is a $(\mathbb{P}, \mathcal{F})$ -martingale for every \mathcal{F} -predictable \mathcal{E} -indexed bounded process H . Since $U^n \rightarrow U$ as $n \rightarrow \infty$ uniformly on compacts, upon letting $n \rightarrow \infty$ and using the martingale property (2.32) we obtain

$$\begin{aligned}
& \mathbb{E} \left[\int_{\zeta_k}^{\zeta_{k+1}} e^{-rt} P(Z_t) dt \middle| \mathcal{F}_{\zeta_k} \right] \\
& \leq \mathbb{E} \left[e^{-r\zeta_k} U_{\iota_k}(Y_{\zeta_k}) - e^{-r\zeta_{k+1}} U_{\iota_k}(Y_{\zeta_{k+1}}) \right. \\
& \quad \left. - \int_{\zeta_k}^{\zeta_{k+1}} e^{-rt} \sum_{j \in \mathcal{E}} \Lambda_t(\{j\})(U_j(Y_t) - U_{\iota_k}(Y_t)) dt \middle| \mathcal{F}_{\zeta_k} \right] \\
& = \mathbb{E} \left[e^{-r\zeta_k} U_{\iota_k}(Y_{\zeta_k}) - e^{-r\zeta_{k+1}} U_{\iota_k}(Y_{\zeta_{k+1}}) \right. \\
& \quad \left. - e^{-r\rho_{k+1}} (U_{\iota_{k+1}}(Y_{\rho_{k+1}}) - U_{\iota_k}(Y_{\rho_{k+1}})) \mathbb{1}_{\sigma_1 < \rho_{k+1} \leq \sigma_2} \middle| \mathcal{F}_{\zeta_k} \right] \\
& = \mathbb{E} \left[e^{-r\zeta_k} U(Z_{\zeta_k}) - e^{-r\zeta_{k+1}} U(Z_{\zeta_{k+1}}) \middle| \mathcal{F}_{\zeta_k} \right]
\end{aligned} \tag{2.33}$$

so (2.31) holds on $(\zeta_k, \zeta_{k+1}]$.

Step 2: Equality in (2.33). Next suppose that $\sigma_2 \leq \hat{\theta}$ and define, for all $k, n \geq 1$,

$$\theta^{k,n} \triangleq \zeta_{k+1}^n \wedge \hat{\theta}^{k,n}$$

where

$$\hat{\theta}^{k,n} \triangleq \begin{cases} \inf\{t \geq \zeta_k : U_{\iota_k}^n(Y_{t \wedge \tau_n}) = \mathcal{M}_{\iota_k}^{\mathcal{I}_n} U^n(Y_{t \wedge \tau_n})\} & \text{on } \{\zeta_k < \tau_n\} \\ +\infty & \text{otherwise.} \end{cases}$$

Then it follows that

$$\theta^{k,n} \uparrow \zeta_{k+1} \text{ as } n \rightarrow \infty.$$

Indeed, assume by contradiction that, for some $\omega \in \Omega$ and some subsequence $\{n(l)\}_{l \geq 1}$,

$$\tilde{\theta} \triangleq \lim_{l \rightarrow \infty} \theta^{k,n(l)}(\omega) < \zeta_{k+1}(\omega).$$

Then there exists $l_0 = l_0(\omega)$ such that for all $l \geq l_0$, and for that fixed ω ,

$$\theta^{k,n(l)} = \hat{\theta}^{k,n(l)} < \zeta_{k+1}^{n(l)}.$$

By continuity we have $U_{\iota_k}^{n(l)}(Y_{\theta^{k,n(l)}}) = \mathcal{M}_{\iota_k}^{\mathcal{I}_m} U^{n(l)}(Y_{\theta^{k,n(l)}})$, and since $U^n \rightarrow U$ locally uniformly, this would imply, again for the above ω ,

$$U(Z_{\tilde{\theta}}) = U_{\iota_k}(Y_{\tilde{\theta}}) = \mathcal{M}_{\iota_k} U(Y_{\tilde{\theta}}) = \mathcal{M}U(Z_{\tilde{\theta}})$$

contradicting the assumption $\sigma_2 \leq \hat{\theta}$.

Since $\theta^{k,n} \leq \hat{\theta}^{k,n}$ the second part of Theorem 2.8 yields

$$\begin{aligned} & \mathbb{E} \left[\int_{\zeta_k}^{\theta^{k,n}} e^{-rt} P(Z_t) dt \middle| \mathcal{F}_{\zeta_k} \right] \\ &= \mathbb{E} \left[e^{-r\zeta_k} U_{\iota_k}^n(Y_{\zeta_k}) - e^{-r\theta^{k,n}} U_{\iota_k}^n(Y_{\theta^{k,n}}) \right. \\ & \quad \left. - \int_{\zeta_k}^{\theta^{k,n}} e^{-rt} \sum_{j \in \mathcal{E}} \Lambda_t(\{j\}) (U_j^n(Y_t) - U_{\iota_k}^n(Y_t)) dt \middle| \mathcal{F}_{\zeta_k} \right]. \end{aligned}$$

Letting $n \rightarrow \infty$ and using the martingale property (2.32) as in (2.33), we obtain, as desired,

$$\mathbb{E} \left[\int_{\zeta_k}^{\zeta_{k+1}} e^{-rt} P(Z_t) dt \middle| \mathcal{F}_{\zeta_k} \right] = \mathbb{E} \left[e^{-r\zeta_k} U(Z_{\zeta_k}) - e^{-r\zeta_{k+1}} U(Z_{\zeta_{k+1}}) \middle| \mathcal{F}_{\zeta_k} \right]. \quad (2.34)$$

Step 3: Itô-Dynkin on $(\sigma_1, \sigma_2]$. Summing (2.33) over $k \geq 0$ we obtain (2.31), i.e. the first part of the claim, and doing the same with (2.34) yields the second part. \square

3 General Running Cost Problems

In this chapter we consider a general multidimensional running cost problem in \mathbb{R}^N with possibly partially controlled state process and a cost structure that includes a fixed cost component. For this problem we establish a verification result where, as in Chapter 2, we replace the conventional quasi-variational inequalities by a generalized dynamic programming principle which is referred to as the Itô-Dynkin Inequality (*). Finally, we iteratively construct a function that satisfies all the sufficient conditions, thus providing a solution to the underlying optimization problem.

3.1 Introduction

In this chapter we study a generalization of the running cost problem considered by Bensoussan and Lions [1982] in Chapter 6. We assume that the system is characterized by a non-degenerate diffusion process Z that takes values in \mathbb{R}^N . As costs with a fixed cost component apply, the controller's goal is to minimize the running costs

$$\mathbb{E} \left[\int_0^\infty e^{-\int_0^t r(Z_s^S) ds} P(Z_t^S) dt + \sum_{k=1}^\infty e^{-\int_0^{\tau_k} r(Z_s^S) ds} \mathcal{C}(a_k) \right] \quad (3.1)$$

over all admissible impulse control strategies $\mathcal{S} = \{\tau_k, a_k\}_{k \geq 1}$. Here the process Z^S is defined by the strategy \mathcal{S} , P is a running cost function and $\mathcal{C}(a_k)$ denotes the costs for the transaction a_k .

Applications of this type problem range from optimal electricity production and resource control to applications in finance and medicine, see e.g. Bensoussan and Lions [1982], Stokey [2008]. The cost minimization problem is closely related to the menu cost problem considered in Chapter 2. Furthermore, based on solutions to running cost problems of the form (3.1) quasi-variational inequalities of ergodic type (such as, for instance, (4.13) and (4.16) in Chapter 4) can be solved, as e.g. by Akian et al. [2001] and Tamura [2008].

In comparison to Bensoussan and Lions [1982], the generalization considered in this chapter includes the following aspects:

3 General Running Cost Problems

- (i) We do not impose boundedness on the coefficients describing the evolution of the diffusion Z .
- (ii) Admissible actions in every state $z \in \mathbb{R}^N$ are constrained by a set-valued function $\mathcal{A} : \mathbb{R}^N \rightarrow 2^{\mathbb{R}^N}$. In particular, this formulation subsumes models with unspanned factor processes, such as e.g. stochastic excess return in the Kim-Omberg model (see Kim and Omberg [1996]).

Thus, in contrast to Chapter 2, the unspanned randomness in the model under consideration is provided by a diffusion process.

Contribution. Similarly to Theorem 2.4 in Chapter 2, we first establish a verification result (Theorem 3.6) for the control problem (3.1). As previously, characterization via the associated quasi-variational inequalities is replaced by a version of the Itô-Dynkin formula.

The remainder of the chapter is devoted to constructing a function, that satisfies all sufficient conditions in the Verification Theorem 3.6. This is conducted in a similar manner to Bensoussan and Lions [1982] by means of iteratively solving optimal stopping problems. We solve each stopping problem in \mathbb{R}^N by means of the finite element method in a suitable weighted Sobolev space. For this purpose we extend the penalization routine used by Bensoussan and Lions [1978] for bounded domains to \mathbb{R}^N . The penalization method allows us to verify the relevant version of the Itô-Dynkin inequality on every iteration step. Finally, we verify that there is a limit of the iteratively constructed sequence that complies with the conditions of Theorem 3.6. In particular, as a by-product of our analysis it follows that the conditions required in the verification theorem are in fact necessary for the true value function.

In comparison to Bensoussan and Lions [1982], we use probabilistic methods in many proofs. In this way some assumptions that are required for the Sobolev-space approach may be weakened.

Outline. In Section 3.2 we rigorously formulate the running cost problem (3.1). The associated verification result is established in Section 3.3. As the running cost problem can be interpreted as a sequence of optimal stopping problems, we first construct value functions of relevant optimal stopping problems in Section 3.4 (Section 3.4.1 is devoted to the case with bounded coefficients whereas the unbounded case is treated in Section 3.4.1). Finally, in Section 3.5 the value function of (3.1) is iteratively constructed.

3.2 Mathematical Formulation

Probabilistic Setting. To define (3.1) in a rigorous way, we consider a probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$ with an N -dimensional Brownian motion W . Let the diffusion coefficients $d : \mathbb{R}^N \rightarrow \mathbb{R}^N$ and $S : \mathbb{R}^N \rightarrow \mathbb{R}^{N \times N}$ governing the state process be given and satisfy the following assumptions:

(A1) d and S are Lipschitz continuous;

(A2) there exists a $\delta > 0$ such that¹ for every $w \in \mathbb{R}^N$ and $z \in \mathbb{R}^N$

$$w^\top S(z)S(z)^\top w \geq \delta|w|^2.$$

Assumption (A2) ensures that the operator²

$$\mathcal{L} \triangleq \nabla^\top S S^\top \nabla + d^\top \nabla$$

is strictly elliptic in \mathbb{R}^N . Furthermore, we assume that

(A3) for every $R > 0$ there are bounded functions $d^R : \mathbb{R}^N \rightarrow \mathbb{R}^N$, $S^R : \mathbb{R}^N \rightarrow \mathbb{R}^{N \times N}$ that satisfy (A1), (A2) and additionally

$$d^R = d \quad \text{and} \quad S^R = S \quad \text{on} \quad \mathcal{B}_R$$

where $\mathcal{B}_R \triangleq \overline{B_R(0)}$ is a closed ball around 0 of radius R in \mathbb{R}^N .

Remark. Note that if $S(z)$ is symmetric for every $z \in \mathbb{R}^N$, i.e. $S = (SS)^\top = (SS^\top)^\top$, then S^R can be easily constructed: For every $R > 0$ define

$$S^R(z) \triangleq ((1 - \gamma^R(z)) S(z)S(z) + \delta \gamma^R(z) \text{Id})^{1/2}$$

with $\gamma^R(z) \triangleq \mathbb{I}_{|z| > R} (1 \wedge (|z| - R))$ and Id denoting the identity matrix. Then S^R obviously satisfies (A2), is bounded by definition and Lipschitz continuous by Theorem 5.2.2 in Stroock and Varadhan [2006].

By assumption (A1) for every stopping time τ and every \mathcal{F}_τ -measurable random variable ζ there exists a unique \mathbb{R}^N -valued process $\{Z_t^{\tau, \zeta}\}_{t \geq 0}$ that solves the SDE

$$Z_t = 0, \quad t \in [0, \tau), \quad Z_t = \zeta + \int_\tau^t d(Z_s) ds + \int_\tau^t S(Z_s) dW_s. \quad (3.2)$$

¹In what follows, $|\cdot|$ denotes the Euclidean norm.

²Here $\nabla \triangleq (\partial_{z_1}, \dots, \partial_{z_N})^\top$.

3 General Running Cost Problems

Without loss of generality we assume that $\{Z_t^{\tau, \zeta}\}_{t \geq \tau}$ is an element of the N -dimensional Wiener space \mathcal{W}^N for every ω in Ω . Condition (A2) ensures that $Z^{\tau, \zeta}$ is non-degenerate. We will refer to $Z^{\tau, \zeta}$ as the **diffusion process starting at time τ in ζ** . Note that the distribution of $Z^{\tau, \zeta}$ depends on S only through the matrix SS^\top , see e.g. [Stroock and Varadhan, 2006, Chapter 5]. Throughout this chapter we will make use of the following two properties of $Z^{\tau, \zeta}$:

Proposition 3.1 (Strong Markov Property.). *For an arbitrary stopping time $\theta \geq \tau$ and a bounded and measurable functional $h : \mathcal{W}^N \rightarrow \mathbb{R}$ on the N -dimensional Wiener space \mathcal{W}^N*

$$\mathbb{E}[h(Z_{\theta+}^{\tau, \zeta}) | \mathcal{F}_\theta] = \mathbb{E}[h(Z^{0, z})] \Big|_{z=Z_\theta^{\tau, \zeta}} \quad \text{a.s. on } \{\theta < \infty\}.$$

Proof. The assertion is implied by Blagovešćensky-Freidlin Theorem (see Theorem 6.26 in Hackenbroch and Thalmaier [1994]) in a similar manner as the strong Markov property for diffusion processes, see Theorem 1.25 in Diesinger [2009] for a similar result. \square

Proposition 3.2 (Convergence of Exit Times from \mathcal{B}_R). *Let $\mathcal{K} \subset \mathbb{R}^N$ be an arbitrary compact set and $\alpha > 0$ be an arbitrary positive number. Then*

$$\sup_{z \in \mathcal{K}} \mathbb{E}[e^{-\alpha \tau_R(z)}] \rightarrow 0 \quad \text{as } R \rightarrow 0$$

with $\tau_R(z) \triangleq \inf\{t \geq 0 : Z_t^{0, z} \notin \mathcal{B}_R\}$ and $\mathcal{B}_R = \overline{B_R(0)}$.

Proof. See Section 3.A. \square

Intervention Costs and Intervention Operator. In the previous paragraph we defined a model that governs the behaviour of the state process between intervention times. Actions of the trader at intervention times will be determined by the **cost function**

$$\mathcal{C} : \mathbb{R}^N \times \mathbb{R}^N \rightarrow [K, \infty) \quad \text{with some } K > 0$$

and the **set of admissible actions**

$$\mathcal{A}(z) : \mathbb{R}^N \rightarrow 2^{\mathbb{R}^N}$$

that satisfy the following conditions:

3.2 Mathematical Formulation

(A4) \mathcal{A} is closed-valued with $z \in \mathcal{A}(z)$ for every $z \in \mathbb{R}^N$ and there exists an increasing sequence of compact sets $\{\mathcal{K}_L\}_{L \geq 1}$ in \mathbb{R}^N such that $\cup_{L \geq 1} \mathcal{K}_L = \mathbb{R}^N$ and for every $L \geq 1$ the mapping $\mathbb{R}^N \ni z \mapsto \mathcal{K}_L \cap \mathcal{A}(z)$ is continuous;³

(A5) \mathcal{C} is continuous and such that for every compact set $\mathcal{K} \subset \mathbb{R}^N$

$$\min_{z \in \mathcal{K}} \mathcal{C}(z, z') \rightarrow +\infty \quad \text{as } z' \rightarrow \infty.$$

$\mathcal{C}(z, z')$ defines the costs for transaction from state z to z' . Note that, as in Chapter 2, the assumptions (A4) and (A5) do not ensure suboptimality of strategies with multiple interventions at the same time instant.

Example. Let us assume that the first N_1 coordinates of the state process Z , here denoted by X , are beyond management's control, whereas the remaining $N_2 \triangleq N - N_1$ can be arbitrarily changed. In this case $\mathcal{A}(z) = \mathcal{A}(x, y) = \{x\} \times \mathbb{R}^{N_2}$, $x \in \mathbb{R}^{N_1}$, $y \in \mathbb{R}^{N_2}$, satisfies (A4) for $\mathcal{K}_L \triangleq B_L(0) \subset \mathbb{R}^N$, $L \geq 1$.

Definition 3.3. For every bounded and continuous function $\phi : \mathbb{R}^N \rightarrow \mathbb{R}$, the **intervention operator** \mathcal{M} is defined by

$$\mathcal{M}\phi(z) \triangleq \inf_{z' \in \mathcal{A}(z)} (\phi(z') + \mathcal{C}(z, z')).$$

Proposition 3.4. Conditions (A4) and (A5) imply the following properties of \mathcal{M} :

(i) For every $\phi \in C_b(\mathbb{R}^N)$ there exists a measurable minimizer $\Xi^\phi : \mathbb{R}^N \rightarrow \mathbb{R}^N$ such that

$$\mathcal{M}\phi(z) = \phi(\Xi^\phi(z)) + \mathcal{C}(z, \Xi^\phi(z)).$$

(ii) \mathcal{M} applied to continuous bounded functions returns continuous bounded functions:

$$\mathcal{M}(C_b(\mathbb{R}^N)) \subset C_b(\mathbb{R}^N).$$

Proof. See Section 3.A. □

³As in Aubin and Frankowska [1990], a set-valued mapping $\mathbb{R}^N \ni z \rightarrow \mathcal{M}(z) \in 2^{\mathbb{R}^N}$ is called continuous if \mathcal{M} is both lower and upper semi-continuous at every $z \in \text{Dom}(\mathcal{M}) \triangleq \{z \in \mathbb{R}^N : \mathcal{M}(z) \neq \emptyset\}$:

(i) \mathcal{M} is called upper semi-continuous at $z \in \text{Dom}(\mathcal{M})$ if and only if for any neighborhood \mathcal{U} of $\mathcal{M}(z)$ there exists a neighborhood \mathcal{V} of z such that $\mathcal{M}(\zeta) \subset \mathcal{U}$ for every $\zeta \in \mathcal{V}$.

(ii) \mathcal{M} is called lower semi-continuous at $z \in \text{Dom}(\mathcal{M})$ if and only if for any $z' \in \mathcal{M}(z)$ and for every sequence $\{z_n\}_{n \geq 1} \subset \text{Dom}(\mathcal{M})$ that approximates z there exists a sequence of elements $z'_n \in \mathcal{M}(z_n)$ that converges to z' .

3 General Running Cost Problems

Admissible Strategies and the Impulse Control Problem. For every initial value $z \in \mathbb{R}^N$ we define an *impulse control strategy* \mathcal{S} as a sequence $\{\tau_k, \zeta_k\}_{k \geq 0}$ where $\tau_0 \triangleq 0$, $\zeta_0 \triangleq z$, $\{\tau_k\}_{k \geq 0}$ is a non-decreasing sequence of stopping times and for every $k \geq 1$ the \mathbb{R}^N -valued random vector ζ_k is \mathcal{F}_{τ_k} -measurable. Thus, for every impulse control strategy $\mathcal{S} = \{\tau_k, \zeta_k\}_{k \geq 0}$ for an arbitrary $k \geq 0$, by $Z^k \triangleq Z^{\tau_k, \zeta_k}$ we denote the solutions to (3.2) with initial time τ_k and initial value ζ_k . Then, every impulse control strategy \mathcal{S} defines the corresponding controlled state process that is denoted by $Z^{\mathcal{S}}$ and defined by

$$Z_t^{\mathcal{S}} \triangleq Z_t^k, \quad t \in [\tau_k, \tau_{k+1}), \quad k \geq 0.$$

Similarly to Definition 2.3 we introduce the notion of admissibility:

Definition 3.5 (Admissible Strategies). *An impulse control strategy $\mathcal{S} \triangleq \{\tau_k, \zeta_k\}_{k \geq 0}$ is called **admissible** whenever*

- (i) $\tau_k \rightarrow \infty$ for $k \rightarrow \infty$ almost surely;
- (ii) for each $k \geq 1$ the action $\zeta_k \in \mathcal{A}(Z_{\tau_k}^{k-1})$ on $\{\tau_k < \infty\}$.

Our goal is to compute the value function and determine an optimal strategy for the impulse control problem (3.1) that is specified by

$$U(z) \triangleq \inf_{\mathcal{S}} I(\mathcal{S}; z) \quad \text{where} \quad (P)$$

$$I(\mathcal{S}; z) \triangleq \mathbb{E} \left[\int_0^\infty e^{-\int_0^t r(Z_s^{\mathcal{S}}) ds} P(Z_t^{\mathcal{S}}) dt + \sum_{k=1}^\infty e^{-\int_0^{\tau_k} r(Z_s^{\mathcal{S}}) ds} \mathcal{C}(Z_{\tau_k}^{k-1}, \zeta_k) \right]$$

where the infimum in (P) is taken over all admissible strategies. In the definition the **discount factor**

$$(A6) \quad r : \mathbb{R}^N \rightarrow [r_0, \infty) \quad \text{is bounded and continuous}$$

where $r_0 > 0$ is some constant. $P(z)$ represents now the infinitesimal running costs in state z and will be referred to as the **running cost function**. We assume that

$$(A7) \quad P : \mathbb{R}^N \rightarrow [0, \infty) \quad \text{is bounded and continuous.}$$

Standing Assumption. *Throughout this chapter we impose without further mentioning assumptions (A1)–(A7).*

3.3 Quasi-Variational Inequalities and Verification

As in Chapter 2, prior to constructing a solution to (P), we establish sufficient conditions for a bounded continuous function to be the value function of (P) in the following verification theorem (cf. Theorem 2.4):

Theorem 3.6 (Verification Theorem). *Assume that a function $U : \mathbb{R}^N \rightarrow \mathbb{R}$ is such that*

- (i) U is non-negative, bounded and continuous;
- (ii) $U \leq \mathcal{M}U$ on \mathbb{R}^N ;
- (iii) U satisfies the Itô-Dynkin Inequality (*) (see below).

Then U is the value function of the impulse control problem (P) and an optimal impulse control strategy $\hat{\mathcal{S}}$ is given by:

- (a) $\hat{\tau}_0 \triangleq 0$ and $\hat{\zeta}_0 \triangleq z$;
- (b) *Given the controlled diffusion \hat{Z} on $[0, \hat{\tau}_k)$ and an intervention $\hat{\zeta}_k$ at $\hat{\tau}_k$, we define*

$$\hat{\tau}_{k+1} \triangleq \inf \{t \geq \hat{\tau}_k : U(\hat{Z}_t^k) = \mathcal{M}U(\hat{Z}_t^k)\}$$

where $\hat{Z}^k \triangleq Z^{\hat{\tau}_k, \hat{\zeta}_k}$. Then the process \hat{Z} is defined as \hat{Z}_t^k on $[\hat{\tau}_k, \hat{\tau}_{k+1})$. If $\hat{\tau}_{k+1} < \infty$, we set

$$\hat{\zeta}_{k+1} \triangleq \Xi^U(\hat{Z}_{\hat{\tau}_{k+1}}^k)$$

where Ξ^U is a measurable minimizer of $\mathcal{M}U$.

Proof. See Appendix 3.B. □

As previously, we do not require any differentiability properties of U in Theorem 3.6 and assume instead the following reformulation of the Itô-Dynkin Inequality (*) from Chapter 2 for the case under consideration:

Itô-Dynkin Inequality (*). *We say that a function $U : \mathbb{R}^N \rightarrow \mathbb{R}$ satisfies the Itô-Dynkin Inequality (*) if the following two conditions hold for arbitrary stopping times $\tau \leq \sigma_1 \leq \sigma_2$ and an arbitrary \mathcal{F}_τ -measurable \mathbb{R}^N -valued random vector ζ :*

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(i) For the diffusion process $Z \triangleq Z^{\tau, \zeta}$ starting at τ in ζ and defined by (3.2)

$$U(Z_{\sigma_1}) \mathbb{I}_{\sigma_1 < \infty} \leq \mathbb{E} \left[e^{-\int_{\sigma_1}^{\sigma_2} r(Z_s) ds} U(Z_{\sigma_2}) \mathbb{I}_{\sigma_1 < \infty} + \int_{\sigma_1}^{\sigma_2} e^{-\int_{\sigma_1}^t r(Z_s) ds} P(Z_t) dt \mathbb{I}_{\sigma_1 < \infty} \middle| \mathcal{F}_{\sigma_1} \right] \quad (3.3)$$

almost surely and

(ii) (3.3) holds with equality if $\sigma_2 \leq \hat{\theta}$ a.s., where $\hat{\theta}$ is defined via

$$\hat{\theta} \triangleq \inf \{t \geq \sigma_1 : U(Z_t) = \mathcal{M}U(Z_t)\}.$$

Note that if U is twice continuously differentiable and bounded together with its derivatives, and solves the following system of quasi-variational inequalities

$$\max \{ -\mathcal{L}U + rU - P, U - \mathcal{M}U \} = 0 \quad \text{on } \mathbb{R}^N \quad (3.4)$$

then the Itô-Dynkin Inequality (*) can be easily verified for U . This fact, however, is of limited use as the discussion following the Itô-Dynkin Inequality (★) in Section 2.3 applies mutatis mutandis to the case under consideration.

In the following sections we construct a sequence that approximates the value function U of the problem (P). This, on the one hand, allows us to verify necessity of the conditions in Theorem 3.6; on the other hand, the approximation procedure can serve as a basis for a numerical method.

In contrast to Chapter 2, we do not solve quasi-variational inequalities (3.4) in a weak sense on bounded domains. Instead, we extend the technique of Bensoussan and Lions [1978] and Bensoussan and Lions [1982] for unbounded domains.

3.4 Optimal Stopping Problem

The value function U of the problem (P) will be approximated by a sequence consisting of value functions that correspond to stopping problems of the type

$$V(z) \triangleq \inf_{\theta} J(\theta; z) \quad \text{with} \quad (3.5)$$

$$J(\theta; z) \triangleq \mathbb{E} \left[\int_0^{\theta} e^{-\int_0^t r(Z_s^{0,z}) ds} P(Z_t^{0,z}) dt + e^{-\int_0^{\theta} r(Z_s^{0,z}) ds} \Psi(Z_{\theta}^{0,z}) \right]$$

where the infimum is taken over all stopping times θ . The function Ψ will be called **obstacle** and in terms of the running cost problem (P) Ψ represents the

value of the running cost problem upon the first optimal trade. Throughout this chapter we assume that

$$(A8) \quad \Psi : \mathbb{R}^N \rightarrow [0, \infty) \text{ is bounded and continuous.}$$

3.4.1 Optimal Stopping for Bounded Coefficients

In this section we suppose that, additionally to the assumptions of Section 3.2, the SDE coefficients

$$(A9) \quad d \text{ and } S \text{ are bounded on } \mathbb{R}^N.$$

The value function V of (3.5) is constructed as a weak solution to variational inequalities (cf. (3.4))

$$\max \{ -\mathcal{L}V + rV - P, V - \Psi \} = 0 \text{ on } \mathbb{R}^N. \quad (3.6)$$

The relation between these variational inequalities and the stopping problem (3.5) can be established similarly to Theorem 3.6 by replacing $\mathcal{M}U$ with the obstacle Ψ .

Note that (A8) together with (A6) and (A7) implies that the value function V is bounded and therefore does not necessarily belong to $W^{1,2}(\mathbb{R}^N)$. Therefore a weak version of the variational inequalities (3.6) will be defined in the weighted spaces

$$\begin{aligned} \mathbf{L} &\triangleq \{ f : \mathbb{R}^N \rightarrow \mathbb{R} : f \text{ is measurable and } \int_{\mathbb{R}^N} |f|^2 \pi \, d\lambda < \infty \} \\ \mathbf{W} &\triangleq \{ f \in \mathbf{L} : \text{for every } i = 1, \dots, N \text{ there exists } D_i f \in \mathbf{L} \} \end{aligned}$$

that are equipped with the scalar products

$$\langle f, g \rangle_{\mathbf{L}} \triangleq \int_{\mathbb{R}^N} fg \, \pi \, d\lambda \quad \text{and} \quad \langle f, g \rangle_{\mathbf{W}} \triangleq \langle f, g \rangle_{\mathbf{L}} + \sum_{i=1}^N \langle D_i f, D_i g \rangle_{\mathbf{L}}.$$

Here the weight function⁴ π is defined by $\pi \triangleq \exp\{-\sqrt{1+|z|^2}\}$, D_i denotes the weak derivative in the direction of z_i , $i \in \{1, \dots, N\}$, and λ is the Lebesgue measure on \mathbb{R}^N . Note that both \mathbf{L} and \mathbf{W} are separable Hilbert spaces. Therefore,

⁴ The weight function π can be specified in a different way, for instance $\pi \triangleq \frac{1}{(1+|x|^2)^\gamma}$ for $\gamma > 0$ large enough, as in Bensoussan and Lions [1978]. In general, the following conditions are sufficient for π to be a weight function : (i) $\pi > 0$ and is bounded away from 0 on every compact set; (ii) $\pi \in C^\infty(\mathbb{R}^N) \cap L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$; (iii) $\frac{D_i \pi}{\pi}$ is bounded; (iv) π is Lipschitz on every compact set.

3 General Running Cost Problems

by the Banach-Alaoglu theorem, every bounded sequence in the weighted Sobolev space W has a weakly converging subsequence. In what follows, the dual space of W is denoted by W^* .

To state (3.6) in weak form, we first define for every i and j in $\{1, \dots, N\}$

$$A_{ij} \triangleq \frac{1}{2}(SS^\top)_{ij} \quad \text{and} \quad A_i \triangleq -d_i + \sum_{j=1}^N D_j A_{ij} + \sum_{j=1}^N A_{ij} \frac{D_j \pi}{\pi}.$$

Note that the weak derivative of A_{ij} exists and is bounded by Lipschitz continuity of S and the product rule (see e.g. Propositions 9.3 and 9.4 in Brezis [2010]). Thus both A_{ij} and A_i belong to $L^\infty(\mathbb{R}^N)$. Therefore we can correctly define a bilinear form $\mathfrak{a} : W \times W \rightarrow \mathbb{R}$ by setting

$$\mathfrak{a}(u, v) \triangleq \int_{\mathbb{R}^N} \left(\sum_{i,j=1}^N A_{ij} D_i u D_j v + \sum_{i=1}^N A_i D_i u v + r u v \right) \pi \, d\lambda.$$

Note that by definition for every $u \in W \cap C^2(\mathbb{R}^N)$ that has bounded derivatives

$$\mathfrak{a}(u, v) = \langle -\mathcal{L}u + ru, v \rangle_{\mathbb{L}} \quad \text{for every } v \in W.$$

Then the weak form of the variational inequalities (3.6) in W reads:

$$\text{Find } u \in \mathcal{K}(\Psi) \text{ such that for all } v \in \mathcal{K}(\Psi) \quad \mathfrak{a}(u, v - u) \geq \langle P, v - u \rangle_{\mathbb{L}}. \quad (3.7)$$

Here

$$\mathcal{K}(\Psi) \triangleq \{v \in W : v \leq \Psi \text{ a.e.}\}.$$

Similarly to Chapter 2, we construct a solution by considering the associated penalized problem for each $\epsilon > 0$:

$$\text{Find } u \in W \text{ such that for all } v \in W \quad \mathfrak{a}(u, v) + \langle \beta_\epsilon(u), v \rangle_{\mathbb{L}} = \langle P, v \rangle_{\mathbb{L}} \quad (3.8)$$

where the **penalizing operator** β_ϵ is defined by

$$\beta_\epsilon(v) \triangleq \frac{1}{\epsilon}(v - \Psi)^+.$$

Note that β_ϵ is monotone, i.e. $\langle \beta_\epsilon(u) - \beta_\epsilon(v), u - v \rangle_{\mathbb{L}} \geq 0$ for all $u, v \in \mathbb{L}$.

As before, by solving the penalized problems (3.8) for every $\epsilon > 0$ and investigating properties of the limit of the solutions for $\epsilon \downarrow 0$ we are able to construct a bounded solution to (3.7) and verify a relevant version of the Itô-Dynkin Inequality (*) for the constructed solution. In particular, in this way we establish that the bounded solution to (3.7) coincides with the value function of the stopping problem (3.5). Throughout the proofs we make use of the following key properties of \mathfrak{a} :

3.4 Optimal Stopping Problem

(P1) Coercivity. By (A2) there exists a coercivity coefficient $c > 0$ such that

$$\mathbf{a}(u, u) + c\langle u, u \rangle_{\mathbb{L}} \gtrsim \|u\|_{\mathbb{W}}^2. \quad (3.9)$$

(P2) Weak lower semi-continuity of $\mathbf{a}(\cdot, \cdot) + c\langle \cdot, \cdot \rangle_{\mathbb{L}}$ is verified in the same fashion as in Section 2.4.1.

(P3) Boundedness: For all $u, v \in \mathbb{W}$

$$|\mathbf{a}(u, v)| \lesssim \|u\|_{\mathbb{W}} \|v\|_{\mathbb{W}}.$$

Here we keep the notation introduced in Section 2.4: If for some functions $\xi, \eta : \mathfrak{S} \rightarrow [0, \infty)$ we can find a constant $C > 0$ such that $\xi \leq C\eta$ on \mathfrak{S} , we write $\xi(\cdot) \lesssim \eta(\cdot)$.

Let us first establish existence and uniqueness of bounded solutions to the penalized problem (3.8):

Theorem 3.7 (Existence, Uniqueness and Comparison Principle for (3.8)). *Let the additional assumptions (A8) and (A9) hold true. Then*

- (i) *For every $\epsilon \in (0, \infty]$ there exists a unique bounded solution u_ϵ to (3.8). Furthermore, there exists a constant $C > 0$ such that for every $\epsilon > 0$*

$$0 \leq u_\epsilon \leq C \quad \text{almost everywhere in } \mathbb{R}^N.$$

- (ii) *If $u_i \in \mathbb{W}$ are such that*

$$\mathbf{a}(u_i, v) + \langle \frac{1}{\epsilon_i}(u_i - \Psi_i)^+, v \rangle_{\mathbb{L}} = \langle P_i, v \rangle_{\mathbb{L}} \quad \text{for all } v \in \mathbb{W}, \quad i = 1, 2 \quad (3.10)$$

with $\infty \geq \epsilon_1 \geq \epsilon_2 > 0$, $\Psi_1 \geq \Psi_2 \geq 0$, $P_1 \geq P_2 \geq 0$ and Ψ_i, P_i are bounded, then $u_1 \geq u_2$.

- (iii) *For every $\epsilon > 0$ the solution u_ϵ to (3.8) is continuous. Furthermore, $u_\epsilon \in W_{loc}^{2,2}(\mathbb{R}^N)$ and*

$$-\mathcal{L}u_\epsilon + r u_\epsilon = P - \beta_\epsilon(u_\epsilon) \quad \text{almost everywhere in } \mathbb{R}^N.$$

Proof. The statement of the theorem is first verified under the assumption that the form \mathbf{a} is coercive, i.e. (3.9) holds with $c = 0$. Then the result is extended to the general case. Note that in the multidimensional case we cannot make use of the compact inclusion $W^{1,2} \hookrightarrow C$ as in Chapter 2. The proof is therefore modified in the spirit of Bensoussan and Lions [1978] and Bensoussan and Lions [1982]. For details see Section 3.C.1. \square

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Next, we are interested in a stochastic representation of u_ϵ and in a version of Itô-Dynkin formula for u_ϵ . In contrast to Chapter 2, we do not have the relevant regularity of u_ϵ to apply Itô's lemma. Nevertheless, by means of the standard mollification routine we are able to establish the following version of Itô-Dynkin formula and a stochastic representation for u_ϵ :

Theorem 3.8 (Itô-Dynkin for u_ϵ). *Let the additional assumptions (A8) and (A9) hold true. Then for every fixed $\epsilon > 0$:*

- (i) *for arbitrary stopping times $\tau \leq \sigma_1 \leq \sigma_2$ and an \mathcal{F}_τ -measurable random vector ζ we find that*

$$u_\epsilon(Z_{\sigma_1}^{\tau, \zeta}) \mathbb{I}_{\sigma_1 < \infty} = \mathbb{E} \left[u_\epsilon(Z_{\sigma_2}^{\tau, \zeta}) e^{-\int_{\sigma_1}^{\sigma_2} r(Z_s^{\tau, \zeta}) ds} \mathbb{I}_{\sigma_1 < \infty} \right. \\ \left. + \int_{\sigma_1}^{\sigma_2} e^{-\int_{\sigma_1}^t r(Z_s^{\tau, \zeta}) ds} [P - \beta_\epsilon(u_\epsilon)](Z_t^{\tau, \zeta}) dt \mathbb{I}_{\sigma_1 < \infty} \middle| \mathcal{F}_{\sigma_1} \right]$$

where $Z^{\tau, \zeta}$ is a diffusion process starting at τ in ζ defined by (3.2).

- (ii) $u_\epsilon(z) = \inf_\gamma J_\epsilon(\gamma; z)$ where

$$J_\epsilon(\gamma; z) \triangleq \mathbb{E} \left[\int_0^\infty \exp \left\{ -\int_0^t (r(Z_s^{0, z}) + \frac{\gamma_s}{\epsilon}) ds \right\} \left[P + \frac{1}{\epsilon} \Psi \right](Z_t^{0, z}) dt \right]$$

and the infimum is taken over all adapted processes $\{\gamma_t\}_{t \geq 0}$ with values in $[0, 1]$. The optimal control is given by $\hat{\gamma}_t \triangleq \mathbb{I}_{u_\epsilon(Z_t^{0, z}) \geq \Psi(Z_t^{0, z})}$, $t \geq 0$. In particular, for $\epsilon = \infty$

$$u_\infty(z) \triangleq \mathbb{E} \left[\int_0^\infty \exp \left\{ -\int_0^t r(Z_s^{0, z}) ds \right\} P(Z_t^{0, z}) dt \right].$$

Proof. See Section 3.C.2. □

An immediate consequence of (i) is the fact that u_ϵ is an upper bound for the optimal stopping problem (3.5) for every $\epsilon > 0$. Indeed, by taking $\sigma_1 = \tau = 0$, $\zeta = z$ and $\sigma_2 = \inf\{t \geq 0 : u_\epsilon(Z_t^{0, z}) \geq \Psi(Z_t^{0, z})\}$ we immediately see that

$$u_\epsilon(z) = J(\sigma_2; z) \geq V(z) \tag{3.11}$$

where J is given by (3.5). Further, note that by Theorem 3.7 we obtain that the sequence $\{u_\epsilon\}_{\epsilon > 0}$ of solutions to (3.8) is bounded and monotonically decreasing as $\epsilon \downarrow 0$. Therefore the limit

$$u \triangleq \lim_{\epsilon \downarrow 0} u_\epsilon \tag{3.12}$$

is well-defined in the pointwise sense. In the following theorem we verify that u is indeed a solution to (3.7).

3.4 Optimal Stopping Problem

Theorem 3.9. *Let the additional assumptions (A8) and (A9) be true.*

- (i) u defined by (3.12) is the unique bounded solution to (3.7).
- (ii) Let $u_i \in \mathcal{K}(\Psi_i)$ be the bounded solution to the variational inequality

$$\mathfrak{a}(u, v - u) \geq \langle P_i, v - u \rangle_{\mathbb{L}} \quad \text{for all } v \in \mathcal{K}(\Psi_i), \quad i = 1, 2 \quad (3.13)$$

with $P_1 \geq P_2 \geq 0$ and $\Psi_1 \geq \Psi_2 \geq 0$ satisfying (A7) and (A8) respectively. Then $u_1 \geq u_2$.

Proof. See Section 3.C.3. □

Our next goal is to establish a version of the Itô-Dynkin Inequality (*) for the bounded solution u of (3.7). To deduce this from the Itô-Dynkin formula for u_ϵ (see Theorem 3.8 (i)) we need to verify that $u_\epsilon \rightarrow u$ uniformly on compact sets. This is verified indirectly by demonstrating the following result:

Theorem 3.10. *Under the additional assumptions (A8) and (A9)*

$$u_\epsilon \rightarrow V \quad \text{uniformly on compact sets in } \mathbb{R}^N \quad \text{as } \epsilon \rightarrow 0$$

where V is the value function of the stopping problem (3.5). In particular, the unique bounded solution u of (3.7) is continuous and coincides with V .

Proof. The proof of this theorem is inspired by Theorem 3.7 in [Bensoussan and Lions, 1978, Chapter 3], where a similar result is shown for a bounded state space. See Section 3.C.4 for details. □

Remark. *Taking into account the stochastic interpretation of u established in Theorem 3.10, the comparison principle in Theorem 3.9 (ii) is an immediate consequence of the definition of V in (3.5).*

Having verified uniform convergence on compacts we can readily prove the relevant version of the Itô-Dynkin Inequality (*) for the solution $u = V$ to (3.7) in a similar way to Theorem 2.8:

Theorem 3.11 (Itô-Dynkin Inequality for $u = V$). *Let the additional assumptions (A8), (A9) hold true and u be the unique bounded solution to (3.7). Then for arbitrary stopping times $\tau \leq \sigma_1 \leq \sigma_2$ and an \mathcal{F}_τ -measurable random vector ζ*

$$\begin{aligned} u(Z_{\sigma_1}^{\tau, \zeta}) \mathbb{I}_{\sigma_1 < \infty} &\leq \mathbb{E} \left[u(Z_{\sigma_2}^{\tau, \zeta}) e^{-\int_{\sigma_1}^{\sigma_2} r(Z_s^{\tau, \zeta}) ds} \mathbb{I}_{\sigma_1 < \infty} \right. \\ &\quad \left. + \int_{\sigma_1}^{\sigma_2} e^{-\int_{\sigma_1}^t r(Z_s^{\tau, \zeta}) ds} P(Z_t^{\tau, \zeta}) dt \mathbb{I}_{\sigma_1 < \infty} \middle| \mathcal{F}_{\sigma_1} \right] \quad a.s. \end{aligned} \quad (3.14)$$

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where $Z^{\tau, \zeta}$ is the diffusion process starting at τ in ζ defined by (3.2). Further, if $\sigma_2 \leq \hat{\theta}$ a.s., where $\hat{\theta}$ is defined by

$$\hat{\theta} \triangleq \inf\{t \geq \tau : u(Z_t^{\tau, \zeta}) = \Psi(Z_t^{\tau, \zeta})\}$$

then (3.14) holds with equality.

Proof. See Section 3.C.5. □

Remark. Theorem 3.11 provides an optimal stopping time for the stopping problem (3.5):

$$\hat{\theta} \triangleq \inf\{t \geq 0 : V(Z_t^{0, z}) = \Psi(Z_t^{0, z})\}.$$

3.4.2 Optimal Stopping for General Coefficients

In this section we extend the main result of the previous section, namely the Itô-Dynkin inequality for the value function V of the stopping problem (3.5), to the case without the boundedness assumption (A9) on the SDE coefficients. To approximate V we consider the sequence $\{V^R\}_{R>0}$ consisting of bounded solutions to (3.7) with bilinear forms $\mathbf{a} = \mathbf{a}^R$ defined by d^R and S^R from assumption (A3). Note that for every $R > 0$ by Theorem 3.10 V^R coincides with the value function of (3.5) with the underlying state process driven by d^R and S^R . We use this representation of V^R to demonstrate the following convergence result:

Lemma 3.12. *Under assumption (A8) $V^R \rightarrow V$ uniformly on compact sets for $R \rightarrow \infty$.*

Proof. See Section 3.C.6. □

Local uniform convergence of $\{V^R\}_{R>0}$ allows us to verify the Itô-Dynkin inequality from Theorem 3.11 for the value function V associated with the stopping problem (3.5):

Theorem 3.13 (Itô-Dynkin Inequality for V). *Under the additional assumption (A8) the statement of Theorem 3.11 holds for V given by (3.5).*

Proof. See Section 3.C.7. □

3.5 Solution of the Impulse Control Problem

In this section we conclude our analysis of the impulse control problem (P) and prove necessity of the conditions in the verification theorem (Theorem 3.6). For this purpose we construct two sequences of functions by iteratively solving optimal stopping problems of the type (3.5) and prove the following two facts: first, the constructed sequences converge towards the value function of (P) uniformly and, second, the Itô-Dynkin Inequality (*) holds for the limit. The approximating sequences will be constructed via iterating the following operator, that is defined on the set of all non-negative bounded functions $C_b^+(\mathbb{R}^N)$ on \mathbb{R}^N :

$$\mathcal{T} : C_b^+(\mathbb{R}^N) \rightarrow C_b^+(\mathbb{R}^N) \quad \text{with} \quad \Psi \mapsto V^\Psi$$

where V^Ψ is the value of the optimal stopping problem (3.5) for the obstacle function Ψ . Note that by definition for every $\Psi \in C_b^+(\mathbb{R}^N)$

$$0 \leq \mathcal{T}\Psi(z) \leq U^0(z) \triangleq \mathbb{E} \left[\int_0^\infty e^{-\int_0^t r(Z_s^{0,z}) ds} P(Z_t^{0,z}) dt \right]. \quad (3.15)$$

Further, if $\Psi_i \in C_b^+(\mathbb{R}^N)$, $i = 1, 2$, are such that $\Psi_1 \leq \Psi_2$, then immediately by (3.5)

$$\mathcal{T}\Psi_1 \leq \mathcal{T}\Psi_2.$$

The value function will be approximated by the following sequences:

$$U_m \triangleq \mathcal{T}(\mathcal{M}U_{m-1}) \quad \text{and} \quad U^m \triangleq \mathcal{T}(\mathcal{M}U^{m-1}), \quad m \geq 1$$

with $U_0 \triangleq 0$ and U^0 given by (3.15). Note that by the properties of \mathcal{T} both $\{U_m\}_{m \geq 0}$ and $\{U^m\}_{m \geq 0}$ are monotone and

$$0 \leq U_1 \leq \dots \leq U_m \leq U^m \leq \dots \leq U^0 < \infty. \quad (3.16)$$

Theorem 3.14 (Stochastic Representation of U_m and U^m). *For an arbitrary $m \geq 0$*

$$U_m(z) = \inf_{\mathcal{S}_m} I_m(\mathcal{S}_m; z) \quad \text{and} \quad U^m(z) = \inf_{\mathcal{S}_m} I(\mathcal{S}_m; z) \quad (3.17)$$

where the infima are taken over all admissible impulse control strategies $\mathcal{S}_m = \{\tau_k, \zeta_k\}_{0 \leq k \leq m}$ for $m \geq 0$ (\mathcal{S}_0 denotes the no-action strategy) with at most m

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interventions⁵ and

$$I_m(\mathcal{S}_m; z) \triangleq \mathbb{E} \left[\int_0^{\tau_m} e^{-\int_0^t r(Z_s^{\mathcal{S}_m}) ds} P(Z_t^{\mathcal{S}_m}) dt + \sum_{k=1}^m e^{-\int_0^{\tau_k} r(Z_s^{\mathcal{S}_m}) ds} \mathcal{C}(Z_{\tau_k}^{k-1}, \zeta_k) \right].$$

Here $\{Z_t^{\mathcal{S}_m}\}_{t \geq 0}$ and the processes $\{Z_t^k\}_{t \geq 0}$, $k \geq 0$, are determined by the strategy \mathcal{S}_m in the usual way, see in Section 3.2.

Proof. See Section 3.D.1. □

The stochastic representations were established with the purpose of proving the announced convergence of both $\{U_m\}_{m \geq 0}$ and $\{U^m\}_{m \geq 0}$ towards the value function U of the impulse control problem (P). First, note that by the definition of U and Theorem 3.14 we immediately obtain the missing part in the sequence of inequalities (3.16), namely that

$$U_m \leq U \leq U^m \quad \text{for every } m \geq 0.$$

This observation is used in the proof of the following result which implies uniform convergence and is demonstrated in a similar manner as Lemmas 1.29 and 1.36 in Diesinger [2009].

Theorem 3.15. *For the approximation sequences $\{U_m\}_{m \geq 0}$ and $\{U^m\}_{m \geq 0}$*

$$\|U_m - U\|_\infty \leq \frac{\|U^0\|_\infty^2}{Km} \quad \text{and} \quad \|U^m - U\|_\infty \leq \frac{\|U^0\|_\infty^2}{Km} \quad (3.18)$$

Proof. See Section 3.D.2. □

This result implies uniform convergence of $\{U_m\}_{m \geq 0}$ and $\{U^m\}_{m \geq 0}$ towards U and therefore continuity of U . Moreover, the inequality $\|\mathcal{M}\phi - \mathcal{M}\psi\|_\infty \leq \|\phi - \psi\|_\infty$, which holds for arbitrary continuous bounded functions ϕ, ψ , implies

Corollary 3.16. *Both $\{\mathcal{M}U_m\}_{m \geq 0}$ and $\{\mathcal{M}U^m\}_{m \geq 0}$ converge to $\mathcal{M}U$ uniformly on \mathbb{R}^N as $m \rightarrow \infty$. In particular, by construction we obtain that $0 \leq U \leq \mathcal{M}U$.*

⁵An admissible strategy $\mathcal{S} = \{\tau_k, \zeta_k\}_{k \geq 0}$ consists of at most m interventions if $\tau_k = \infty$ for all $k > m$. To emphasize this for an admissible strategy \mathcal{S}_m we write $\mathcal{S}_m = \{\tau_k, \zeta_k\}_{0 \leq k \leq m}$. The state process governed by an impulse control strategy $\mathcal{S}_m = \{\tau_k, \zeta_k\}_{0 \leq k \leq m}$ is defined by the standard routine presented in Section 3.2.

3.5 Solution of the Impulse Control Problem

This corollary together with Theorem 3.15 yields an alternative stochastic representation of U : By the definition of U_m and monotonicity of \mathcal{M} we obtain the following estimate

$$\begin{aligned} 0 &\leq \inf_{\theta} \mathbb{E} \left[\int_0^{\theta} e^{-\int_0^t r(Z_s^{0,z}) ds} P(Z_t^{0,z}) dt + e^{-\int_0^{\theta} r(Z_s^{0,z}) ds} \mathcal{M}U(Z_{\theta}^{0,z}) \right] - U_m(Z_{\theta}^{0,z}) \\ &\leq \mathbb{E} \left[e^{-\int_0^{\hat{\theta}_m} r(Z_s^{0,z}) ds} (\mathcal{M}U(Z_{\hat{\theta}_m}^{0,z}) - \mathcal{M}U_{m-1}(Z_{\hat{\theta}_m}^{0,z})) \right] \\ &\leq \|U - U_{m-1}\|_{\infty} \end{aligned}$$

where $\hat{\theta}_m$ is the optimal stopping time in the stopping problem (3.5) for U_m given by Theorem 3.13. By the uniform convergence (3.18) we conclude that the value function U of the impulse control problem (P) is a fixed point of the operator \mathcal{T} , i.e. U solves the following optimal stopping problem:

$$U(z) \triangleq \inf_{\theta} \mathbb{E} \left[\int_0^{\theta} e^{-\int_0^t r(Z_s^{0,z}) ds} P(Z_t^{0,z}) dt + e^{-\int_0^{\theta} r(Z_s^{0,z}) ds} \mathcal{M}U(Z_{\theta}^{0,z}) \right].$$

This fact together with the Itô-Dynkin inequality for optimal stopping problems of the type (3.5) proved in Theorem 3.13 immediately yields the Itô-Dynkin Inequality (*) for U .

Corollary 3.17. *Itô-Dynkin Inequality (*) holds for the value function U of the stopping problem (P).*

We conclude the chapter by noting, that Corollaries 3.16 and 3.17 yield the following characterization result:

Theorem 3.18. *A function $U : \mathbb{R}^N \rightarrow \mathbb{R}$ is the value function of the impulse control problem (P) if and only if it satisfies the following conditions:*

- (i) U is non-negative, bounded and continuous;
- (ii) $U \leq \mathcal{M}U$;
- (iii) U satisfies the Itô-Dynkin Inequality (*).

An optimal strategy is given by Theorem 3.6.

3.6 Conclusion and Outlook

The focus of this chapter was on a general version of the running cost problem. We have verified an equivalent characterization of the value function via the Itô-Dynkin Inequality (*) and iteratively constructed the value function. The iterative procedure may serve as a basis for a numerical scheme if additional results on convergence of finite elements in the spirit of Chapter 2 are established. Another possible extension of our results was announced in the introduction to this chapter: The running cost problem (P) can be used to construct solutions to variational inequalities of ergodic type. This is an interesting problem to look at in the context of Chapter 4 of the thesis.

Appendix to Chapter 3

3.A Proofs for Section 3.2

Proof of Proposition 3.2

By the inequality

$$\sup_{z \in \mathcal{K}} \mathbb{E}[e^{-\alpha \tau_R(z)}] \leq \int_0^1 \sup_{z \in \mathcal{K}} \mathbb{P}(\tau_R(z) < -\frac{\ln t}{\alpha}) dt$$

it suffices to verify that for an arbitrary fixed $T > 0$

$$\sup_{z \in \mathcal{K}} \mathbb{P}(\tau_R(z) < T) \rightarrow 0, \quad R \rightarrow \infty. \quad (3.19)$$

For this purpose we fix a $z_0 \in \mathcal{K}$ and denote by $Z^0 \triangleq Z^{0, z_0}$ the diffusion starting at 0 in z_0 defined by (3.2). By, for instance, Corollary 11.7 in [Rogers and Williams, 2000, Chapter 5] and Gronwall's inequality there exists a constant $D > 0$ such that for every $z \in \mathcal{K}$ we have

$$\mathbb{E}[\sup_{t \in [0, T]} |Z_t^{0, z} - Z_t^0|^2] \leq D|z - z_0|^2.$$

Therefore for a fixed $\Delta > 0$ and an arbitrary $R > \Delta$

$$\begin{aligned} \mathbb{P}(\tau_R(z) < T) &\leq \mathbb{P}(\{\tau_R(z) < T\} \cap \{\sup_{t \in [0, T]} |Z_t^{0, z} - Z_t^0| \leq \Delta\}) \\ &\quad + \mathbb{P}(\{\sup_{t \in [0, T]} |Z_t^{0, z} - Z_t^0| > \Delta\}) \\ &\leq \mathbb{P}(\sup_{t \in [0, T]} |Z_t^0| > R - \Delta) + \frac{D}{\Delta^2} |z - z_0|^2 \\ &\leq \frac{1}{(R - \Delta)^2} \mathbb{E}[\sup_{t \in [0, T]} |Z_t^0|^2] + \frac{D}{\Delta^2} (\text{diam } \mathcal{K})^2. \end{aligned}$$

Thus

$$\limsup_{R \rightarrow \infty} \sup_{z \in \mathcal{K}} \mathbb{P}(\tau_R(z) < T) \leq \frac{D}{\Delta^2} (\text{diam } \mathcal{K})^2$$

and letting $\Delta \rightarrow \infty$ yields the assertion (3.19). \square

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Proof of Proposition 3.4

Step 1: Proof of (i) It suffices to verify existence of a measurable minimizer Ξ_R^ϕ of $\mathcal{M}\phi(z)$ for $z \in \mathcal{B}_R = \overline{B_R(0)}$ for an arbitrary but fixed $R > 0$. Indeed, then Ξ^ϕ can be defined as $\mathbb{R}^N \ni z \mapsto \Xi_1^\phi(z) \mathbb{I}_{|z| \leq 1} + \sum_{n=2}^\infty \Xi_n^\phi(z) \mathbb{I}_{n-1 < |z| \leq n}$.

Therefore let $R > 0$ be fixed. By (A4) and (A5) there exists an L such that

$$\inf_{z' \in \mathcal{A}(z)} (\phi(z') + \mathcal{C}(z, z')) = \inf_{z' \in \mathcal{K}_L \cap \mathcal{A}(z)} (\phi(z') + \mathcal{C}(z, z')) \quad \text{for all } z \in \mathcal{B}_R. \quad (3.20)$$

Further, by (A4) the mapping $\mathcal{B}_R \ni z \mapsto \mathcal{K}_L \cap \mathcal{A}(z)$ is compact-valued and upper semi-continuous. Thus, existence of a measurable minimizer on \mathcal{B}_R is implied by Corollary 4 in Schael [1974].

Step 2: Proof of (ii) We fix an arbitrary bounded and continuous function ϕ in \mathbb{R}^N and a $z_0 \in \mathbb{R}^N$. It suffices to show that $\mathcal{M}\phi(z_n) \rightarrow \mathcal{M}\phi(z_0)$ as $n \rightarrow \infty$ for an arbitrary sequence $\{z_n\}_{n \geq 1}$ approximating z_0 . This, in turn, is equivalent to the following statement: For every subsequence $\{n(k)\}_{k \geq 1}$ of \mathbb{N} there exists a sub-subsequence $\{\bar{n}(k)\}_{k \geq 1} \subset \{n(k)\}_{k \geq 1}$ such that

$$\mathcal{M}\phi(z_{\bar{n}(k)}) - \mathcal{M}\phi(z_0) \rightarrow 0 \quad \text{as } k \rightarrow \infty. \quad (3.21)$$

First note, that by (A5) there exists an L such that (3.20) holds for every z_n , $n \geq 0$, i.e. the sequence of minimizers $\{z'_n\}_{n \geq 0}$ with z'_n minimizing $\mathcal{M}\phi(z_n)$ lies in \mathcal{K}_L . In particular, $z_n \in \text{Dom}(\mathcal{K}_L \cap \mathcal{A}) = \{z : \mathcal{K}_L \cap \mathcal{A}(z) \neq \emptyset\}$ for all $n \geq 0$. Thus, for an arbitrary fixed subsequence $\{n(k)\}_{k \geq 1}$ there exists a sub-subsequence $\{\bar{n}(k)\}_{k \geq 1} \subset \{n(k)\}_{k \geq 1}$ and a $z''_0 \in \mathcal{K}_L$ such that $z'_{\bar{n}(k)} \rightarrow z''_0$ for $k \rightarrow \infty$. By contradiction we easily derive from the definition of upper semi-continuity and compactness of $\mathcal{K}_L \cap \mathcal{A}(z_0)$ that $z''_0 \in \mathcal{K}_L \cap \mathcal{A}(z_0)$ and thus by continuity of ϕ and \mathcal{C}

$$\begin{aligned} & \liminf_{k \rightarrow \infty} (\mathcal{M}\phi(z_{\bar{n}(k)}) - \mathcal{M}\phi(z_0)) \\ & \geq \lim_{k \rightarrow \infty} (\phi(z'_{\bar{n}(k)}) + \mathcal{C}(z_{\bar{n}(k)}, z'_{\bar{n}(k)}) - \phi(z''_0) - \mathcal{C}(z_0, z''_0)) = 0. \end{aligned}$$

On the other hand, by definition of the lower semi-continuity we find a sequence of elements $\{z''_{\bar{n}(k)}\}_{k \geq 1}$ such that $z''_{\bar{n}(k)} \in \mathcal{K}_L \cap \mathcal{A}(z_{\bar{n}(k)})$ for every $k \geq 1$ and $z''_{\bar{n}(k)} \rightarrow z'_0$ as $k \rightarrow \infty$. Then

$$\begin{aligned} & \limsup_{k \rightarrow \infty} (\mathcal{M}\phi(z_{\bar{n}(k)}) - \mathcal{M}\phi(z_0)) \\ & \leq \lim_{k \rightarrow \infty} (\phi(z''_{\bar{n}(k)}) + \mathcal{C}(z_{\bar{n}(k)}, z''_{\bar{n}(k)}) - \phi(z'_0) - \mathcal{C}(z_0, z'_0)) = 0 \end{aligned}$$

and (3.21) follows. \square

3.B Proof of the Verification Theorem 3.6

As in the verification theorem from Chapter 2, we separate the proof in two steps: first, we show, that every function U satisfying the conditions (i)–(iii) is an upper bound for the value function of the problem (P); second, we verify that this upper bound is achieved if the trader follows the strategy $\hat{\mathcal{S}}$.

Step 1: Upper Bound. Let $\mathcal{S} = \{\tau_k, \zeta_k\}_{k \geq 0}$ be an admissible trading strategy and $Z^{\mathcal{S}}$ the corresponding state process, i.e.

$$Z_t^{\mathcal{S}} = Z_t^k \quad \text{for } t \in [\tau_k, \tau_{k+1}) \quad \text{with } Z^k \triangleq Z^{\tau_k, \zeta_k}.$$

For every $k \geq 0$ by the Itô-Dynkin Inequality (*) we obtain

$$\begin{aligned} & \mathbb{E} \left[\int_{\tau_k}^{\tau_{k+1}} e^{-\int_0^t r(Z_s^{\mathcal{S}}) ds} P(Z_t^k) dt \mathbb{I}_{\tau_k < \infty} \right] \\ & \geq \mathbb{E} \left[e^{-\int_0^{\tau_k} r(Z_s^{\mathcal{S}}) ds} U(Z_{\tau_k}^k) \mathbb{I}_{\tau_k < \infty} - e^{-\int_0^{\tau_{k+1}} r(Z_s^{\mathcal{S}}) ds} U(Z_{\tau_{k+1}}^k) \mathbb{I}_{\tau_k < \infty} \right]. \end{aligned} \quad (3.22)$$

On the other hand, the inequality $U \leq \mathcal{M}U$ yields that for each $k \geq 0$

$$\mathcal{C}(Z_{\tau_{k+1}}^k, \zeta_{k+1}) \geq U(Z_{\tau_{k+1}}^k) - U(\zeta_{k+1}) \quad \text{on } \{\tau_{k+1} < \infty\}. \quad (3.23)$$

Combining (3.22) and (3.23) we find that for every $n \geq 1$

$$\begin{aligned} & \mathbb{E} \left[\int_0^{\tau_n} e^{-\int_0^t r(Z_s^{\mathcal{S}}) ds} P(Z_t^{\mathcal{S}}) dt + \sum_{k=1}^n e^{-\int_0^{\tau_k} r(Z_s^{\mathcal{S}}) ds} \mathcal{C}(Z_{\tau_k}^{k-1}, \zeta_k) \mathbb{I}_{\tau_k < \infty} \right] \\ & \geq U(z) - \mathbb{E} \left[e^{-\int_0^{\tau_n} r(Z_s^{\mathcal{S}}) ds} U(\zeta_n) \mathbb{I}_{\tau_n < \infty} \right] \end{aligned} \quad (3.24)$$

Letting $n \rightarrow \infty$ and taking infimum over all admissible strategies \mathcal{S} yields that $\inf_{\mathcal{S}} I(\mathcal{S}; z) \geq U(z)$, i.e. U is indeed a lower bound for the minimization problem (P).

Step 2: Exact Lower Bound. To verify the assertion, let us first recall the definition of $\hat{\mathcal{S}} = \{\hat{\tau}_k, \hat{\zeta}_k\}_{k \geq 0}$: $\hat{\tau}_0 \triangleq 0$ and $\hat{\zeta}_0 \triangleq z$ and for any $k \geq 0$, assumed that the action at $\hat{\tau}_k$ is $\hat{\zeta}_k$, the stopping time $\hat{\tau}_{k+1}$ was defined by

$$\hat{\tau}_{k+1} \triangleq \inf \{t \geq \hat{\tau}_k : U(\hat{Z}_t^k) = \mathcal{M}U(\hat{Z}_t^k)\}$$

with $\hat{Z}^k \triangleq Z^{\hat{\tau}_k, \hat{\zeta}_k}$. On $\{\hat{\tau}_{k+1} < \infty\}$ the intervention

$$\hat{\zeta}_{k+1} \triangleq \Xi^U(\hat{Z}_{\hat{\tau}_{k+1}}^k)$$

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with Ξ^U denoting a measurable minimizer $\mathcal{M}U$ (see Proposition 3.4). Note, that to verify admissibility of $\hat{\mathcal{S}}$ it suffices to show that $\hat{\tau}_k \rightarrow \infty$ for $k \rightarrow \infty$ almost surely. By the Itô-Dynkin Inequality (*) and by definition both (3.22) and (3.23), and hence (3.24), hold with equality for the strategy $\hat{\mathcal{S}}$ for every $n \geq 1$. In particular, for every $n \geq 1$

$$\mathbb{E}\left[\sum_{k=1}^n e^{-r_0 \hat{\tau}_k}\right] < \infty$$

and therefore $\hat{\tau}_k \rightarrow \infty$ almost surely as $k \rightarrow \infty$. Thus the strategy $\hat{\mathcal{S}}$ is admissible and the proof is concluded by letting $n \rightarrow \infty$ in (3.24) for $\hat{\mathcal{S}}$ which implies that

$$I(\hat{\mathcal{S}}; z) = U(z) \quad \square$$

3.C Proofs for Section 3.4

3.C.1 Proof of Theorem 3.7

We proceed as follows: First, we establish the assertions of Theorem 3.7 under the assumption that

$$\mathbf{a}(\cdot, \cdot) \text{ is coercive, i.e. (3.9) holds for } c = 0 \quad (3.25)$$

and then extend them to the general case.

Proof of (ii) under Coercivity Assumption (3.25)

This result is shown similarly to the proof of uniqueness in Theorem 2.5 (i) and (iii): First, we note that

$$\left\langle \frac{1}{\epsilon_1}(u_1 - \Psi_1)^+ - \frac{1}{\epsilon_2}(u_2 - \Psi_2)^+, (u_1 - u_2)^- \right\rangle_{\mathbf{L}} \leq 0.$$

Upon taking $v = -(u_1 - u_2)^-$ as a test function, we subtract (3.10) for $i = 2$ from (3.10) for $i = 1$ to obtain by coercivity of \mathbf{a} the following estimate:

$$\begin{aligned} \|(u_1 - u_2)^-\|_{\mathbf{W}}^2 &\lesssim \mathbf{a}(u_1 - u_2, -(u_1 - u_2)^-) \\ &\quad + \left\langle \frac{1}{\epsilon_1}(u_1 - \Psi_1)^+ - \frac{1}{\epsilon_2}(u_2 - \Psi_2)^+, -(u_1 - u_2)^- \right\rangle_{\mathbf{L}} \\ &= \langle P_1 - P_2, -(u_1 - u_2)^- \rangle_{\mathbf{L}} \leq 0. \end{aligned}$$

Thus $u_1 \geq u_2$ almost everywhere in \mathbb{R}^N and the assertion follows. \square

Proof of (i) under Coercivity Assumption (3.25)

If assumption (3.25) holds true, uniqueness of the solution to (3.8) follows immediately from the comparison principle in part (ii) demonstrated above for the coercive case. Let us verify existence of the solution. The proof of this fact is conducted in a similar fashion as the proof of Theorem 1.2 in [Bensoussan and Lions, 1978, Chapter 3] and Theorem 2.5: We first solve discretized versions of (3.8) and then construct a sequence converging toward the solution.

Step 1: Solution to (3.8) in Finite Dimensions. Due to separability of W there exists a linearly independent sequence $\{v_k\}_{k \geq 1} \subset W$ such that the set of all (finite) linear combinations $V \triangleq \mathcal{L}(\{v_k\}_{k \geq 1})$ is dense in W . For every $m \geq 1$ we consider the finite-dimensional version of (3.8) in $V_m \triangleq \mathcal{L}(v_1, \dots, v_m)$:

$$\text{Find } u \in V_m \text{ such that for all } v \in V_m \quad \mathbf{a}(u, v) + \langle \beta_\epsilon(u), v \rangle_L = \langle P, v \rangle_L. \quad (3.26)$$

We can define a continuous function $\mathfrak{P} : \mathbb{R}^m \rightarrow \mathbb{R}^m$ such that

$$(\mathfrak{P}(\eta), \bar{\eta}) = \mathbf{a}(u_\eta, u_{\bar{\eta}}) + \langle \beta_\epsilon(u_\eta), u_{\bar{\eta}} \rangle_L - \langle P, u_{\bar{\eta}} \rangle_L \quad \text{for all } \eta, \bar{\eta} \in \mathbb{R}^m \quad (3.27)$$

where $u_\eta = \sum_{k=1}^m \eta_k v_k$. Coercivity of \mathbf{a} yields that $\mathbf{a}(u_\eta, u_\eta) \gtrsim \|\eta\|^2$ and therefore, taking into the fact that $\langle \frac{1}{\epsilon}(u - \Psi)^+, u - \Psi \rangle_L \geq 0$, from (3.27) we deduce that $(\mathfrak{P}(\eta), \eta) \rightarrow \infty$ as $\eta \rightarrow \infty$. Thus, by Brouwer's fixed point theorem (see Lemma 4.3 in [Lions, 1969, Chapter 1]) there exists an $\eta^* \in \mathbb{R}^m$ such that $\mathfrak{P}(\eta^*) = 0$, i.e. u_{η^*} solves (3.26). On the other hand, similarly to the proof of the comparison principle in (ii) under the coercivity assumption, we demonstrate that u_{η^*} is the unique solution to (3.26) in V_m . For every $m \geq 1$ this unique solution will be denoted by u_m .

Step 2: Convergence of $\{u_m\}_{m \geq 1}$. Let us consider the sequence $\{u_m\}_{m \geq 1}$ of solutions to (3.26). Then coercivity of \mathbf{a} implies that $\{u_m\}_{m \geq 1}$ is bounded in W :

$$\|u_m\|_W^2 \lesssim \mathbf{a}(u_m, u_m) + \langle \beta_\epsilon(u_m), u_m - \Psi \rangle_L \leq \langle P, u_m \rangle_L \leq \|P\|_L \|u_m\|_W.$$

Therefore, w.l.o.g. we may assume that $\{u_m\}_{m \geq 1}$ converges weakly to some $u_\epsilon \in W$. Let us verify that u_ϵ solves the penalized problem (3.8). For this purpose we fix a test function $v \in W$ and a sequence $\{v_m\}_{m \geq 1}$ such that $v_m \in V_m$ for each $m \geq 1$ and $v_m \rightarrow v$ in W for $m \rightarrow \infty$. Then $\langle P, v_m \rangle_L \rightarrow \langle P, v \rangle_L$ and $\mathbf{a}(u_m, v_m) \rightarrow \mathbf{a}(u_\epsilon, v)$ by boundedness of \mathbf{a} as $m \rightarrow \infty$. Moreover, by taking u_m as a test function in (3.26), we obtain boundedness of $\{\beta_\epsilon(u_m)\}_{m \geq 1}$ in L from the following inequality

$$\epsilon \langle \beta_\epsilon(u_m), \beta_\epsilon(u_m) \rangle_L + \langle \beta_\epsilon(u_m), \Psi \rangle_L = \langle \beta_\epsilon(u_m), u_m \rangle_L \leq \langle P, u_m \rangle_L.$$

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Again, w.l.o.g. we assume that $\{\beta_\epsilon(u_m)\}_{m \geq 1}$ converges weakly to some ξ in \mathbf{L} for $m \rightarrow \infty$ and therefore $\langle \beta_\epsilon(u_m), v_m \rangle_{\mathbf{L}} \rightarrow \langle \xi, v \rangle_{\mathbf{L}}$ as $m \rightarrow \infty$. Hence for an arbitrary $v \in \mathbf{W}$ we obtain that

$$\mathbf{a}(u_\epsilon, v) + \langle \xi, v \rangle_{\mathbf{L}} = \langle P, v \rangle_{\mathbf{L}}$$

and it remains to verify that $\xi = \beta_\epsilon(u_\epsilon)$. This is demonstrated by applying the monotonicity argument from the proof of Theorem 1.2 in [Bensoussan and Lions, 1978, Chapter 3] without any modifications. \square

Proof of (i) in the Non-Coercive Case

The idea of the proof of the results for the penalized problem (3.8) is inspired by similar results for variational inequalities on bounded domains presented in Section 1.7 in [Bensoussan and Lions, 1978, Chapter 3].

Step 1: Uniqueness among Bounded Non-Negative Solutions. Let us assume that there exists a number $p_0 > 0$ such that $P \geq p_0$ and that there are two different non-negative bounded solutions to (3.8) denoted by u_1 and u_2 . Then we define

$$\alpha^* \triangleq \sup\{\alpha \geq 0 : \alpha u_1 \leq u_2 \text{ a.e.}\}$$

W.l.o.g. we can assume that $\alpha^* \in [0, 1)$. Indeed, by definition $\alpha^* \geq 0$ and if $\alpha^* \geq 1$ then the pair (u_2, u_1) possesses the desired property. Further, we find a number $\xi \in (\alpha^*, 1)$ such that

$$c(\xi - \alpha^*) \sup u_1 \leq p_0(1 - \xi),$$

where c is the coercivity coefficient from (3.9). Then $c(\xi - \alpha^*) u_1 \leq P(1 - \xi)$ and by definition of α^*

$$\xi(P + c u_1) \leq P + c u_2.$$

On the other hand, note that for an arbitrary $v \in V$

$$\begin{aligned} \mathbf{a}(\xi u_1, v) + c \langle \xi u_1, v \rangle_{\mathbf{L}} + \langle \frac{1}{\epsilon}(\xi u_1 - \xi \Psi)^+, v \rangle_{\mathbf{L}} &= \langle \xi(P + c u_1), v \rangle_{\mathbf{L}} \\ \mathbf{a}(u_2, v) + c \langle u_2, v \rangle_{\mathbf{L}} + \langle \frac{1}{\epsilon}(u_2 - \Psi)^+, v \rangle_{\mathbf{L}} &= \langle P + c u_2, v \rangle_{\mathbf{L}} \end{aligned}$$

and therefore by (ii) in the coercive case we obtain that $\xi u_1 \leq u_2$ a.e., what contradicts the definition of α^* and the choice of $\xi > \alpha^*$.

Step 2: Uniqueness for Bounded Solutions. Assume that u_1 and u_2 are two bounded solutions to (3.8). Let us choose a $p \triangleq \inf u_1 \wedge \inf u_2 \wedge (-1) < 0$. Then $\bar{u}_i \triangleq u_i - p$ is non-negative and for an arbitrary $v \in \mathbf{W}$

$$\mathbf{a}(\bar{u}_i, v) \bar{n} + \langle \frac{1}{\epsilon}(\bar{u}_i - (\Psi - p))^+, v \rangle_{\mathbf{L}} = \langle P - p r, v \rangle_{\mathbf{L}}.$$

Thus the assertion is implied by *Step 1* as $p < 0$ and $r \geq r_0 > 0$.

Step 3: Existence of a Bounded Solution. A solution to (3.8) is constructed iteratively: $u_0 \triangleq 0$ and for every $n \geq 1$ u_n is uniquely determined by

$$\mathbf{a}(u_n, v) + c \langle u_n, v \rangle_{\mathbf{L}} + \langle \beta_\epsilon(u_n), v \rangle_{\mathbf{L}} = \langle P + c u_{n-1}, v \rangle_{\mathbf{L}}, \quad v \in \mathbf{W}. \quad (3.28)$$

Note that $\{u_n\}_{n \geq 1}$ is non-decreasing by the comparison principle from (ii) for the coercive case. Furthermore, $\{u_n\}_{n \geq 1}$ is bounded from above: Choose $C \geq 0$ such that

$$\Psi \leq C \quad \text{and} \quad P \leq C r_0. \quad (3.29)$$

Then assuming that $u_{n-1} \leq C$ we obtain that $u_n \leq C$ in the following way:

$$\begin{aligned} \|(u_n - C)^+\|_{\mathbf{W}}^2 &\lesssim \mathbf{a}(u_n - C, (u_n - C)^+) + \langle c(u_n - C) + \beta_\epsilon(u_n), (u_n - C)^+ \rangle_{\mathbf{L}} \\ &= \langle P - C r + c(u_{n-1} - C), (u_n - C)^+ \rangle_{\mathbf{L}} \leq 0 \end{aligned}$$

Thus, by monotonicity and boundedness the pointwise limit $u_\epsilon \triangleq \lim_{n \rightarrow \infty} u_n$ is well-defined. On the other hand, $\{u_n\}_{n \geq 1}$ is bounded in \mathbf{W} :

$$\|u_n\|_{\mathbf{W}}^2 \lesssim \mathbf{a}(u_n, u_n) + \langle c u_n + \beta_\epsilon(u_n), u_n \rangle_{\mathbf{L}} = \langle P + c u_{n-1}, u_n \rangle_{\mathbf{L}} \leq C < \infty$$

for some constant $C > 0$ that is independent of n . Therefore we can choose a subsequence $\{u_{n_k}\}_{k \geq 1}$ converging weakly in \mathbf{W} to u_ϵ . Thus, by letting $n_k \rightarrow \infty$ in (3.28) we deduce that u_ϵ solves (3.8).

Step 4: An Upper Bound. By *Step 3* every constant $C > 0$ satisfying (3.29) can be chosen as a universal upper bound. \square

Proof of (ii) in the Non-Coercive Case

Let $\{u_{i,n}\}_{n \geq 1}$ be the two sequences from the existence proof approximating u_i , $i = 1, 2$. Then for each $n \geq 1$ by induction from part (ii) for the coercive case we obtain that $u_{1,n} \geq u_{2,n}$. Hence $u_1 \geq u_2$. \square

Proof of (iii)

Let $\mathcal{D} \subset \mathbb{R}^N$ be an arbitrary bounded domain. Then

$$\int_{\mathcal{D}} \left\{ \sum_{i,j=1}^N \hat{A}_{ij} D_i u_\epsilon D_j v + \sum_{i=1}^N \hat{A}_i D_i u_\epsilon v + \hat{r} u_\epsilon v \right\} d\lambda = \int_{\mathcal{D}} \hat{P}_\epsilon v d\lambda$$

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for every $v \in C^1(\mathcal{D})$ with compact support. Here for $i, j \in \{1, \dots, N\}$

$$\hat{A}_{ij} \triangleq A_{ij} \pi, \quad \hat{A}_i \triangleq A_i \pi, \quad \hat{r} \triangleq r \pi, \quad \hat{P}_\epsilon \triangleq (P - \beta_\epsilon(u_\epsilon))\pi \in L^\infty(\mathbb{R}^N).$$

Theorem 8.22 in Gilbarg and Trudinger [1998] yields local Hölder continuity of u in \mathcal{D} and therefore continuity of u_ϵ in \mathbb{R}^N . On the other hand, the coefficients \hat{A}_{ij} are Lipschitz continuous in \mathcal{D} and hence by Theorem 8.8 in Gilbarg and Trudinger [1998] $u_\epsilon \in W^{2,2}(\mathcal{D}')$ for every subdomain $\mathcal{D}' \subset\subset \mathcal{D}$. This yields that

$$\begin{aligned} \hat{P} &= - \sum_{i,j=1}^N \hat{A}_{ij} D_{ij} u_\epsilon + \sum_{i=1}^N (\hat{A}_i - \sum_{j=1}^N D_j \hat{A}_{ij}) D_i u_\epsilon + \hat{r} u_\epsilon \\ &= \pi(-\mathcal{L}u_\epsilon + r u_\epsilon) \end{aligned}$$

almost everywhere in \mathcal{D}' and the assertion follows. \square

3.C.2 Proof of Theorem 3.8

Proof of (i)

The Itô's formula for u_ϵ asserted in (i) is verified by constructing an approximating sequence $\{u_\epsilon^\delta\}_{\delta>0}$ consisting of smooth functions that approximate u_ϵ uniformly on compact sets.

Step 1: Construction of the Approximating Sequence. For every $\delta > 0$ we defined

$$u_\epsilon^\delta \triangleq \rho^\delta * u_\epsilon \quad \text{on } \mathbb{R}^N$$

with $\rho^\delta(z) \triangleq \frac{1}{\delta^N} \rho(\frac{z}{\delta})$ and $\rho(z) \triangleq C \exp\{\frac{1}{|z|^2-1}\} \mathbb{I}_{|z|<1}$ denoting the standard mollifier. The constant C is defined by the condition $\int_{\mathbb{R}^N} \rho \, d\lambda = 1$. By properties of mollifiers, see [Evans, 2010, Appendix C.3] for details, $u_\epsilon^\delta \in C^\infty(\mathbb{R}^N)$ and for every $R > 0$

$$u_\epsilon^\delta \rightarrow u_\epsilon \quad \text{pointwise on } \mathbb{R}^N \quad \text{for } \delta \rightarrow 0$$

as u_ϵ is continuous. On the other hand, it is known that $D_i u_\epsilon^\delta = \rho^\delta * (D_i u_\epsilon)$ and therefore

$$\mathcal{L}u_\epsilon^\delta = \rho^\delta * (\mathcal{L}u_\epsilon) = -\rho^\delta * (P - \beta_\epsilon(u_\epsilon) - r u_\epsilon)$$

where the last relation is due to the property of u_ϵ established in Theorem 3.7 (iii), namely that $u_\epsilon \in W_{loc}^{2,2}(\mathbb{R}^N)$ and $-\mathcal{L}u_\epsilon + r u_\epsilon = P - \beta_\epsilon(u_\epsilon)$ almost everywhere. Therefore by continuity if P, Ψ, r and u_ϵ

$$-\mathcal{L}u_\epsilon^\delta \rightarrow P - \beta_\epsilon(u_\epsilon) - r u_\epsilon \quad \text{pointwise on } \mathbb{R}^N \quad \text{for } \delta \rightarrow 0.$$

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Step 2: Itô's Formula for u_ϵ . Let us consider the diffusion $Z^{\tau, \zeta}$ starting at τ in ζ . For an arbitrary but fixed $R > 0$ we set $\tau_R \triangleq \inf\{t \geq \tau : Z_t^{\tau, \zeta} \notin B_R(0)\}$ and apply Itô's formula to $u_\epsilon^\delta(Z^{\tau, \zeta}) e^{-\int_{\sigma_1 \wedge \tau_R}^{\cdot} r(Z_s^{\tau, \zeta}) ds}$ on $[\sigma_1 \wedge \tau_R, \sigma_2 \wedge \tau_R]$:

$$\begin{aligned} u_\epsilon^\delta(Z_{\sigma_1 \wedge \tau_R}^{\tau, \zeta}) &= u_\epsilon^\delta(Z_{\sigma_2 \wedge \tau_R}^{\tau, \zeta}) e^{-\int_{\sigma_1 \wedge \tau_R}^{\sigma_2 \wedge \tau_R} r(Z_s^{\tau, \zeta}) ds} \\ &\quad + \int_{\sigma_1 \wedge \tau_R}^{\sigma_2 \wedge \tau_R} e^{-\int_{\sigma_1 \wedge \tau_R}^t r(Z_s^{\tau, \zeta}) ds} [-\mathcal{L}u_\epsilon^\delta + ru_\epsilon^\delta](Z_t^{\tau, \zeta}) dt \\ &\quad + \int_{\sigma_1 \wedge \tau_R}^{\sigma_2 \wedge \tau_R} e^{-\int_{\sigma_1 \wedge \tau_R}^t r(Z_s^{\tau, \zeta}) ds} [(\nabla u_\epsilon^\delta)^\top S](Z_t^{\tau, \zeta}) dW_t. \end{aligned}$$

Note that the conditional expectation of the stochastic integrals w.r.t. \mathcal{F}_{σ_1} is 0 as $(\nabla u_\epsilon^\delta)^\top S$ is bounded on $B_R(0)$ and $r(z) \geq r_0 > 0$. Therefore, upon conditioning on \mathcal{F}_{σ_1} and letting $\delta \rightarrow 0$ by dominated convergence we obtain

$$\begin{aligned} \mathbb{E}[u_\epsilon(Z_{\sigma_1 \wedge \tau_R}^{\tau, \zeta}) | \mathcal{F}_{\sigma_1}] &= \mathbb{E}\left[u_\epsilon(Z_{\sigma_2 \wedge \tau_R}^{\tau, \zeta}) e^{-\int_{\sigma_1 \wedge \tau_R}^{\sigma_2 \wedge \tau_R} r(Z_s^{\tau, \zeta}) ds} \right. \\ &\quad \left. + \int_{\sigma_1 \wedge \tau_R}^{\sigma_2 \wedge \tau_R} e^{-\int_{\sigma_1 \wedge \tau_R}^t r(Z_s^{\tau, \zeta}) ds} [P - \beta_\epsilon(u_\epsilon)](Z_t^{\tau, \zeta}) dt \middle| \mathcal{F}_{\sigma_1}\right] \end{aligned}$$

We conclude the proof by letting $R \rightarrow \infty$ and using boundedness of u_ϵ , Ψ , P and the fact that r is bounded away from 0. \square

Proof of (ii)

The proof is inspired by a similar result for bounded domains (see Theorem 3.6 in [Bensoussan and Lions, 1978, Chapter 3]):

Let $\{u_\epsilon^\delta\}_{\delta > 0}$ be the approximating sequence constructed in *Step 1* of the proof of (i) and $\{\gamma_t\}_{t \geq 0}$ an arbitrary adapted process taking values in $[0, 1]$. Applying Itô's formula to $u_\epsilon^\delta(Z^{0, z}) \exp\{-\int_0^{\cdot} (r(Z_s^{0, z}) + \frac{\gamma_s}{\epsilon}) ds\}$ on $[0, \tau_R]$ for $\tau_R \triangleq \inf\{t \geq 0 : Z_t^{0, z} \notin B_R(0)\}$ and letting $\delta \rightarrow 0$, as in the proof of (i), by dominated convergence we obtain that

$$\begin{aligned} u_\epsilon(z) &= \mathbb{E}\left[u_\epsilon(Z_{\tau_R}^{0, z}) e^{-\int_0^{\tau_R} (r(Z_s^{0, z}) + \frac{\gamma_s}{\epsilon}) ds} \right. \\ &\quad \left. + \int_0^{\tau_R} e^{-\int_0^t (r(Z_s^{0, z}) + \frac{\gamma_s}{\epsilon}) ds} [P - \beta_\epsilon(u_\epsilon) + \frac{\gamma_t}{\epsilon} u_\epsilon](Z_t^{0, z}) dt \right] \end{aligned}$$

The proof is concluded by letting $R \rightarrow \infty$ and taking into account the inequality

$$-\beta_\epsilon(u_\epsilon) + \frac{\gamma_t}{\epsilon} u_\epsilon \leq \frac{1}{\epsilon} \Psi$$

which holds with equality for $\hat{\gamma}_t$. \square

3.C.3 Proof of Theorem 3.9

The proof is structured as similarly to the proof of the existence and uniqueness result for the penalized problem (3.8), see Theorem 3.7: We first prove the comparison principle stated in (ii) in the coercive case, i.e. under assumption (3.25); employing this result, we are able to verify uniqueness and existence of bounded solutions to (3.7); finally, the comparison principle in (ii) in the general case is deduced from an analogous result for penalized problems (Theorem 3.7 (ii)).

Proof of Theorem 3.9 (ii) under Coercivity Assumption (3.25)

The assertion is verified similarly to the comparison principle for penalized problems under the coercivity assumption (3.25) (cf. Section 3.C.1): in (3.13) we take the following test functions: $v = u_1 + (u_1 - u_2)^- \in \mathcal{K}(\Psi_1)$ for $i = 1$ and $v = u_2 - (u_1 - u_2)^- \in \mathcal{K}(\Psi_2)$ for $i = 2$. Next, adding the inequalities together with the coercivity assumption yields the assertion via the following inequality:

$$\|(u_1 - u_2)^-\|_{\mathbb{W}}^2 \lesssim \mathbf{a}(u_1 - u_2, -(u_1 - u_2)^-) \leq \langle P_1 - P_2, -(u_1 - u_2)^- \rangle_{\mathbb{L}} \leq 0 \quad \square$$

Proof of Theorem 3.9 (i)

Step 1: Uniqueness for Bounded Non-Negative Solutions is demonstrated in a similar fashion as in [Bensoussan and Lions, 1978, Chapter 3, Section 1.7]: First, assuming that $P \geq p_0 > 0$ for some constant p_0 , we repeat the considerations of *Step 1* in the proof of Theorem 3.7 (i) for the non-coercive case to obtain that for two non-negative solutions u_1, u_2 of (3.7)

$$\begin{aligned} \mathbf{a}(\xi u_1, v - \xi u_1) + c \langle \xi u_1, v - \xi u_1 \rangle_{\mathbb{L}} &\geq \langle \xi(P + c u_1), v - \xi u_1 \rangle_{\mathbb{L}} \quad \text{for all } v \in \mathcal{K}(\xi \Psi) \\ \mathbf{a}(u_2, v - u_2) + c \langle u_2, v - u_2 \rangle_{\mathbb{L}} &\geq \langle P + c u_2, v - u_2 \rangle_{\mathbb{L}} \quad \text{for all } v \in \mathcal{K}(\Psi) \end{aligned}$$

for some $\xi \in (\alpha^*, 1)$ where $\alpha^* \triangleq \sup\{\alpha \geq 0 : \alpha u_1 \leq u_2 \text{ a.e.}\}$. By the comparison principle for the coercive case verified above we deduce that $\xi u_1 \leq u_2$ and obtain a contradiction as in the proof of Theorem 3.7 (i).

Step 2: Uniqueness in the Class of Bounded Solutions is verified in the same way as in Theorem 3.7 (i): For two bounded solutions u_1 and u_2 of (3.7) we find that $\bar{u}_i \triangleq u_i - p$, $i = 1, 2$, for $p \triangleq \inf u_1 \wedge \inf u_2 \wedge (-1) < 0$ are non-negative and solve the variational inequality

$$\text{Find } u \in \mathcal{K}(\bar{\Psi}) \text{ such that for all } v \in \mathcal{K}(\bar{\Psi}) \quad \mathbf{a}(u, v - u) \geq \langle \bar{P}, v - u \rangle_{\mathbb{L}}$$

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with $\bar{\Psi} \triangleq \Psi - p$ and $\bar{P} \triangleq P - pr \geq -pr_0 > 0$. Thus uniqueness in the class of bounded solutions follows from *Step 1*.

Step 3: Existence. By Theorem 3.7 there exists a constant $C > 0$ such that $0 \leq u_\epsilon \leq C$ for all $\epsilon > 0$. Therefore $\{u_\epsilon\}_{\epsilon>0}$ is bounded in \mathbf{L} . Then we can derive boundedness of $\{u_\epsilon\}_{\epsilon>0}$ in \mathbf{W} by a coercivity argument similar to the one in the proof of Theorem 2.5:

$$\|u_\epsilon\|_{\mathbf{W}}^2 \lesssim \mathbf{a}(u_\epsilon, u_\epsilon) + c \langle u_\epsilon, u_\epsilon \rangle_{\mathbf{L}} + \langle \beta_\epsilon(u_\epsilon), u_\epsilon \rangle_{\mathbf{L}} = c \langle u_\epsilon, u_\epsilon \rangle_{\mathbf{L}} + \langle P, u_\epsilon \rangle_{\mathbf{L}} \leq C_1 < \infty$$

for some $C_1 > 0$ that does not depend on ϵ . Hence, $\{u_\epsilon\}_{\epsilon>0}$ converges weakly to u along a subsequence in \mathbf{W} . Furthermore, u_ϵ solves (3.8) and therefore

$$\langle \beta_\epsilon(u_\epsilon), u_\epsilon - \Psi \rangle_{\mathbf{L}} \leq \langle P, u_\epsilon \rangle_{\mathbf{L}} - \mathbf{a}(u_\epsilon, u_\epsilon).$$

Thus $\|(u_\epsilon - \Psi)^+\|_{\mathbf{L}}^2 \leq \epsilon C_2$ for some constant $C_2 > 0$. Hence, taking into account the pointwise convergence $u_\epsilon \rightarrow u$, we deduce that $u \in \mathbf{K}(\Psi)$. Similarly to Theorem 2.6, we demonstrate that u solves the variational inequality in (3.7): For an arbitrary fixed $v \in \mathbf{K}(\Psi)$ monotonicity of the penalizing operator β_ϵ yields

$$\mathbf{a}(u_\epsilon, v - u_\epsilon) \geq \mathbf{a}(u_\epsilon, v - u_\epsilon) + \langle \beta_\epsilon(u_\epsilon) - \beta_\epsilon(v), v - u_\epsilon \rangle_{\mathbf{L}} = \langle P, v - u_\epsilon \rangle_{\mathbf{L}}.$$

The assertion is thus obtained by letting $\epsilon \rightarrow \infty$ along the subsequence from above and employing weak lower semi-continuity of $\mathbf{a}(\cdot, \cdot) + c \langle \cdot, \cdot \rangle_{\mathbf{L}}$. \square

Proof of Theorem 3.9 (ii)

The comparison principle in the general case is an immediate consequence of the comparison principle for penalized problems (Theorem 3.7 (ii)) and the fact that solutions of (3.8) converge pointwise to the unique bounded solution of (3.7). \square

3.C.4 Proof of Theorem 3.10

As u_ϵ is an upper bound for the optimal stopping problem (3.5) for every $\epsilon > 0$ (see (3.11)), it suffices to verify that

$$\limsup_{\epsilon \downarrow 0} \sup_{z \in \mathcal{K}} (u_\epsilon(z) - V(z)) \leq 0 \tag{3.30}$$

for an arbitrary compact set \mathcal{K} in \mathbb{R}^N . For this purpose we fix \mathcal{K} and estimate the difference $J_\epsilon(\gamma^\theta; z) - J(\theta; z)$ for an arbitrary stopping time θ and $z \in \mathcal{K}$. Here $\{\gamma_t^\theta\}_{t \geq 0} \triangleq \{\mathbb{I}_{t \geq \theta}\}_{t \geq 0}$ is an admissible control in the stochastic representation of

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u_ϵ (see Theorem 3.8 (ii)) and J_ϵ is the corresponding functional. J , on the other hand, determines the optimal stopping problem (3.5). To simplify notation we denote the discounting factor by

$$\xi_t^z \triangleq \exp \left\{ - \int_0^t r(Z_s^{0,z}) ds \right\}, \quad t \geq 0.$$

Then for arbitrary $\delta \in (0, 1)$, $z \in \mathcal{K}$ and $R > 0$ we set $\tau_R(z) \triangleq \inf \{ t \geq 0 : Z_t^{0,z} \notin \mathcal{B}_R \}$ with $\mathcal{B}_R \triangleq \overline{B_R(0)}$ denoting the closed ball of radius R around 0 and obtain

$$\begin{aligned} & J_\epsilon(\gamma^\theta, z) - J(\theta; z) \\ &= \mathbb{E} \left[\int_\theta^\infty P(Z_t^{0,z}) \xi_t^z e^{-\frac{t-\theta}{\epsilon}} dt + \int_\theta^\infty \frac{1}{\epsilon} \Psi(Z_t^{0,z}) \xi_t^z e^{-\frac{t-\theta}{\epsilon}} dt - \Psi(Z_\theta^{0,z}) \xi_\theta^z \right] \\ &= \mathbb{E} \left[\int_\theta^\infty P(Z_t^{0,z}) \xi_t^z e^{-\frac{t-\theta}{\epsilon}} dt + \int_\theta^\infty \frac{1}{\epsilon} \Psi(Z_t^{0,z}) \xi_t^z e^{-\frac{t-\theta}{\epsilon}} dt \mathbb{I}_{\theta \geq \tau_R(z)} \right. \\ &\quad + \int_{(\theta+\delta) \wedge \tau_R(z)}^\infty \frac{1}{\epsilon} \Psi(Z_t^{0,z}) \xi_t^z e^{-\frac{t-\theta}{\epsilon}} dt \mathbb{I}_{\theta < \tau_R(z)} \\ &\quad \left. + \int_\theta^{(\theta+\delta) \wedge \tau_R(z)} \frac{1}{\epsilon} \Psi(Z_t^{0,z}) \xi_t^z e^{-\frac{t-\theta}{\epsilon}} dt \mathbb{I}_{\theta < \tau_R(z)} - \Psi(Z_\theta^{0,z}) \xi_\theta^z \right]. \end{aligned} \quad (3.31)$$

To estimate $J_\epsilon(\gamma^\theta; z) - J(\theta; z)$ we consider each addend in the last equality separately. Taking into account that $\xi_t^z \leq e^{-r_0 t} \leq 1$, by straight-forward integration we obtain that

$$\begin{aligned} & \mathbb{E} \left[\int_\theta^\infty P(Z_t^{0,z}) \xi_t^z e^{-\frac{t-\theta}{\epsilon}} dt \right] \leq \|P\|_\infty \epsilon \\ & \mathbb{E} \left[\int_\theta^\infty \frac{1}{\epsilon} \Psi(Z_t^{0,z}) \xi_t^z e^{-\frac{t-\theta}{\epsilon}} dt \mathbb{I}_{\theta \geq \tau_R(z)} \right] \leq \|\Psi\|_\infty \sup_{z \in \mathcal{K}} \mathbb{E} [e^{-r_0 \tau_R(z)}]. \end{aligned}$$

An estimate for the third summand is given by

$$\begin{aligned} & \mathbb{E} \left[\int_{(\theta+\delta) \wedge \tau_R(z)}^\infty \frac{1}{\epsilon} \Psi(Z_t^{0,z}) \xi_t^z e^{-\frac{t-\theta}{\epsilon}} dt \mathbb{I}_{\theta < \tau_R(z)} \right] \\ & \leq \|\Psi\|_\infty \mathbb{E} \left[\int_{\tau_R(z)}^\infty \frac{1}{\epsilon} \xi_t^z e^{-\frac{t-\theta}{\epsilon}} dt \mathbb{I}_{\theta < \tau_R(z) < \theta+\delta} + \int_{\theta+\delta}^\infty \frac{1}{\epsilon} \xi_t^z e^{-\frac{t-\theta}{\epsilon}} dt \mathbb{I}_{\theta+\delta < \tau_R(z)} \right] \\ & \leq \|\Psi\|_\infty \left(\sup_{z \in \mathcal{K}} \mathbb{E} [e^{-r_0 \tau_R(z)}] + e^{-\frac{\delta}{\epsilon}} \right). \end{aligned}$$

Finally, the last line in (3.31) can be bounded from above as follows:

$$\begin{aligned} & \mathbb{E} \left[\int_\theta^{(\theta+\delta) \wedge \tau_R(z)} \frac{1}{\epsilon} \Psi(Z_t^{0,z}) \xi_t^z e^{-\frac{t-\theta}{\epsilon}} dt \mathbb{I}_{\theta < \tau_R(z)} - \Psi(Z_\theta^{0,z}) \xi_\theta^z \right] \\ & \leq \mathbb{E} \left[\int_\theta^{(\theta+\delta) \wedge \tau_R(z)} \frac{1}{\epsilon} (\Psi(Z_t^{0,z}) \xi_t^z - \Psi(Z_\theta^{0,z}) \xi_\theta^z) e^{-\frac{t-\theta}{\epsilon}} dt \mathbb{I}_{\theta < \tau_R(z)} \right] \\ & \leq \mathbb{E} \left[\int_\theta^{(\theta+\delta) \wedge \tau_R(z)} \frac{1}{\epsilon} (\Psi(Z_t^{0,z}) \xi_t^z - \Psi(Z_\theta^{0,z}) \xi_t^z) e^{-\frac{t-\theta}{\epsilon}} dt \mathbb{I}_{\theta < \tau_R(z)} \right] \\ & \leq \mathbb{E} \left[\sup_{t \in \mathcal{I}_R^z} |\Psi(Z_t^{0,z}) - \Psi(Z_\theta^{0,z})| \mathbb{I}_{\theta < \tau_R(z)} \right] \end{aligned}$$

with $\mathcal{I}_R^z \triangleq [\theta, (\theta + \delta) \wedge \tau_R(z)]$. By uniform continuity of Ψ on \mathcal{B}_R

$$\rho_R(\lambda) \triangleq \sup_{z, z' \in \mathcal{B}_R: |z-z'| \leq \lambda} |\Psi(z) - \Psi(z')| \rightarrow 0 \quad \text{as } \lambda \downarrow 0.$$

Therefore, employing the strong Markov property of $Z^{0,z}$ and Doob's maximal inequality, we deduce that

$$\begin{aligned} & \mathbb{E} \left[\sup_{t \in \mathcal{I}_R^z} |\Psi(Z_t^{0,z}) - \Psi(Z_\theta^{0,z})| \mathbb{I}_{\theta < \tau_R(z)} \right] \\ & \leq \rho_R(\lambda) \mathbb{P}(\sup_{t \in [\theta, \theta + \delta]} |Z_t^{0,z} - Z_\theta^{0,z}| \mathbb{I}_{\theta < \infty} \leq \lambda) \\ & \quad + 2\|\Psi\|_\infty \mathbb{P}(\sup_{t \in [\theta, \theta + \delta]} |Z_t^{0,z} - Z_\theta^{0,z}| \mathbb{I}_{\theta < \infty} > \lambda) \\ & \leq \rho_R(\lambda) + \frac{2\|\Psi\|_\infty}{\lambda^2} C_N (\max_i \|d_i\|_\infty^2 \delta^2 + \max_{i,j} \|S_{ij}\|_\infty^2 \delta) \end{aligned}$$

for some non-negative constant $C_N \geq 0$. Therefore, taking into account all estimates for the summands in (3.31), we find a constant $C > 0$ such that

$$\begin{aligned} u_\epsilon(z) - J(\theta; z) & \leq J_\epsilon(\gamma^\theta; z) - J(\theta; z) \\ & \leq C(\epsilon + \sup_{z \in \mathcal{K}} \mathbb{E}[e^{-r_0 \tau_R(z)}]) + e^{-\frac{\delta}{\epsilon}} + \rho_R(\lambda) + \frac{\delta}{\lambda^2}. \end{aligned}$$

As the right-hand side depends neither on θ nor on z , we obtain the following estimate:

$$\sup_{z \in \mathcal{K}} (u_\epsilon(z) - V(z)) \leq C(\epsilon + \sup_{z \in \mathcal{K}} \mathbb{E}[e^{-r_0 \tau_R(z)}]) + e^{-\frac{\delta}{\epsilon}} + \rho_R(\lambda) + \frac{\delta}{\lambda^2}. \quad (3.32)$$

To finalize the proof we have to show that the right-hand side of (3.32) can be chosen arbitrarily small. By Proposition 3.2

$$\sup_{z \in \mathcal{K}} \mathbb{E}[e^{-r_0 \tau_R(z)}] \rightarrow 0 \quad \text{as } R \rightarrow 0.$$

and therefore we conclude the proof by noting that the right-hand side of (3.32) can be made arbitrarily small by first choosing R large enough, then fixing a $\lambda = \lambda(R) > 0$ and $\delta = \delta(\lambda) \in (0, 1)$ small enough and finally taking ϵ such that $\epsilon + e^{-\frac{\delta}{\epsilon}}$ remains small. \square

3.C.5 Proof of Theorem 3.11

Let stopping times $\tau \leq \sigma_1 \leq \sigma_2$ and an \mathcal{F}_τ -measurable random variable ζ be fixed. As previously, $Z^{\tau, \zeta}$ denotes the diffusion starting at τ in ζ defined by (3.2).

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Step 1: Proof of the Inequality. Let $\{u_\epsilon\}_{\epsilon>0}$ be the sequence of solutions to penalized problems (3.8) that approximates u . Recalling non-negativity of β_ϵ , we deduce from the Itô-Dynkin formula for u_ϵ established in Theorem 3.8 (i) that

$$\begin{aligned} u_\epsilon(Z_{\sigma_1}^{\tau,\zeta}) \mathbb{I}_{\sigma_1<\infty} &\leq \mathbb{E} \left[u_\epsilon(Z_{\sigma_2}^{\tau,\zeta}) e^{-\int_{\sigma_1}^{\sigma_2} r(Z_s^{\tau,\zeta}) ds} \mathbb{I}_{\sigma_1<\infty} \right. \\ &\quad \left. + \int_{\sigma_1}^{\sigma_2} e^{-\int_{\sigma_1}^t r(Z_s^{\tau,\zeta}) ds} P(Z_t^{\tau,\zeta}) dt \mathbb{I}_{\sigma_1<\infty} \middle| \mathcal{F}_{\sigma_1} \right]. \end{aligned}$$

The assertion is obtained by dominated convergence upon $\epsilon \downarrow 0$.

Step 2: Proof of the Equality. To verify the equality result let σ_2 be such that $\sigma_2 \leq \hat{\theta} = \inf\{t \geq 0 : u(Z_t^{\tau,\zeta}) = \Psi(Z_t^{\tau,\zeta})\}$. Then for every $\epsilon > 0$ we set

$$\sigma_2^\epsilon \triangleq \sigma_2 \wedge \inf\{t \geq \sigma_1 : u_\epsilon(Z_t^{\tau,\zeta}) \geq \Psi(Z_t^{\tau,\zeta})\}.$$

As $\{u_\epsilon\}_{\epsilon>0}$ is monotonically decreasing towards u as $\epsilon \downarrow 0$, the stopping times $\{\sigma_2^\epsilon\}_{\epsilon>0}$ are non-decreasing for $\epsilon \downarrow 0$. Furthermore, $\lim_{\epsilon \downarrow 0} \sigma_2^\epsilon = \sigma_2$ almost surely. Indeed, assume the contrary: let $\omega \in \Omega$ be such that $\sigma_2^\epsilon \rightarrow \tilde{\theta} < \sigma_2$, $\epsilon \downarrow 0$. Then by continuity we have that $u_\epsilon(Z_{\sigma_2^\epsilon}^{\tau,\zeta}) = \Psi(Z_{\sigma_2^\epsilon}^{\tau,\zeta})$ and by local uniform convergence of $\{u_\epsilon\}_{\epsilon>0}$ (see Theorem 3.10) we obtain for the ω from above $u(Z_{\tilde{\theta}}^{\tau,\zeta}) = \Psi(Z_{\tilde{\theta}}^{\tau,\zeta})$. This contradicts the assumption $\sigma_2 \leq \hat{\theta}$.

By Itô's formula for u_ϵ on the interval $[\sigma_1, \sigma_2^\epsilon]$ we immediately obtain that

$$\begin{aligned} u_\epsilon(Z_{\sigma_1}^{\tau,\zeta}) \mathbb{I}_{\sigma_1<\infty} &= \mathbb{E} \left[u_\epsilon(Z_{\sigma_2^\epsilon}^{\tau,\zeta}) e^{-\int_{\sigma_1}^{\sigma_2^\epsilon} r(Z_s^{\tau,\zeta}) ds} \mathbb{I}_{\sigma_1<\infty} \right. \\ &\quad \left. + \int_{\sigma_1}^{\sigma_2^\epsilon} e^{-\int_{\sigma_1}^t r(Z_s^{\tau,\zeta}) ds} P(Z_t^{\tau,\zeta}) dt \mathbb{I}_{\sigma_1<\infty} \middle| \mathcal{F}_{\sigma_1} \right]. \end{aligned}$$

The proof is finalized by letting $\epsilon \downarrow 0$ and employing convergence of $\{\sigma_2^\epsilon\}_{\epsilon>0}$, continuity of the paths and local uniform convergence of $\{u_\epsilon\}_{\epsilon>0}$ from Theorem 3.10. \square

3.C.6 Proof of Lemma 3.12

For an arbitrary $z \in \mathbb{R}^N$ by $Z^{0,z}$ we denote the diffusion process starting at 0 in z that is uniquely determined by (3.2). Further, for every $R > 0$ let $Z^{0,z,R}$ be the unique solution to

$$Z_t = z + \int_0^t d^R(Z_s) ds + \int_0^t S^R(Z_s) dW_s, \quad t \geq 0 \quad (3.33)$$

where d^R and S^R are the approximating sequences from assumption (A3). Then

$$Z_t^{0,z} = Z_t^{0,z,R} \quad \text{for } t \in [0, \tau_R(z)] \quad \text{with } \tau_R(z) \triangleq \inf\{t \geq 0 : Z_t^{0,z} \notin \mathcal{B}_R\}$$

with $\mathcal{B}_R \triangleq \overline{B_R(0)}$. Hence for an arbitrary stopping time θ

$$|J(\theta; z) - J^R(\theta; z)| \leq \mathbb{E} \left[2 \left(\frac{\|P\|_\infty}{r_0} + \|\Psi\|_\infty \right) e^{-r_0 \tau_R(z)} \mathbb{I}_{\theta < \tau_R(z)} \right]$$

where J and J^R are the functionals defining the optimal stopping problem (3.5) and the upper index R in J^R indicates that the underlying state process is determined by (3.33) instead of (3.2). Therefore we find a constant $C > 0$ such that for an arbitrary compact set $\mathcal{K} \subset \mathbb{R}^N$

$$\sup_{z \in \mathcal{K}} |V(z) - V^R(z)| \leq C \sup_{z \in \mathcal{K}} \mathbb{E} \left[e^{-r_0 \tau_R(z)} \right]$$

and the assertion follows by Proposition 3.2. \square

3.C.7 Proof of Theorem 3.13

Let τ , σ_1 and σ_2 be arbitrary stopping times such that $\tau \leq \sigma_1 \leq \sigma_2$. Further, we fix an \mathcal{F}_τ -measurable random vector ζ . To simplify notation we set $\{Z_t\}_{t \geq 0} \triangleq \{Z_t^{\tau, \zeta}\}_{t \geq 0}$ and let $\{Z_t^R\}_{t \geq 0}$ denote the unique solution of

$$Z_t = 0, \quad t \in [0, \tau), \quad Z_t = \zeta + \int_\tau^t d^R(Z_s) ds + \int_\tau^t S^R(Z_s) dW_s, \quad t \geq \tau$$

for an arbitrary $R > 0$. As previously, d^R and S^R are given by assumption (A3). Note that

$$Z_t = Z_t^R \quad \text{for } t \in [\tau, \tau_R] \quad \text{with } \tau_R \triangleq \inf\{t \geq \tau : Z_t \notin \mathcal{B}_R\}$$

with $\mathcal{B}_R = \overline{B_R(0)}$ and $\tau_R \rightarrow \infty$ almost surely as $R \rightarrow \infty$.

Step 1: Itô-Dynkin Inequality. For every $R > 0$ Theorem 3.11 applied to V^R yields

$$\begin{aligned} V^R(Z_{\sigma_1}^R) \mathbb{I}_{\sigma_1 < \infty} &\leq \mathbb{E} \left[V^R(Z_{\sigma_2}^R) e^{-\int_{\sigma_1}^{\sigma_2} r(Z_s^R) ds} \mathbb{I}_{\sigma_1 < \infty} \right. \\ &\quad \left. + \int_{\sigma_1}^{\sigma_2} e^{-\int_{\sigma_1}^t r(Z_s^R) ds} P(Z_t^R) dt \mathbb{I}_{\sigma_1 < \infty} \middle| \mathcal{F}_{\sigma_1} \right]. \end{aligned}$$

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Thus, by letting $R \rightarrow \infty$ due to Lemma 3.12 we obtain the Itô-Dynkin inequality (3.14) for the value function V :

$$\begin{aligned} V(Z_{\sigma_1}) \mathbb{I}_{\sigma_1 < \infty} &\leq \mathbb{E} \left[V(Z_{\sigma_2}) e^{-\int_{\sigma_1}^{\sigma_2} r(Z_s) ds} \mathbb{I}_{\sigma_1 < \infty} \right. \\ &\quad \left. + \int_{\sigma_1}^{\sigma_2} e^{-\int_{\sigma_1}^t r(Z_s) ds} P(Z_t) dt \mathbb{I}_{\sigma_1 < \infty} \middle| \mathcal{F}_{\sigma_1} \right]. \end{aligned} \quad (3.34)$$

Step 2: Equality in (3.34). Let us now demonstrate that (3.34) holds with equality if $\sigma_2 \leq \hat{\theta}$ for

$$\hat{\theta} \triangleq \inf\{t \geq \tau : V(Z_t^{\tau, \zeta}) = \Psi(Z_t^{\tau, \zeta})\}.$$

For this purpose we define

$$\sigma_2^R \triangleq \sigma_2 \wedge \hat{\theta}^R \quad \text{for} \quad \hat{\theta}^R \triangleq \inf\{t \geq \tau : V^R(Z_t^R) = \Psi(Z_t^R)\}.$$

Then $\sigma_2^R \rightarrow \sigma_2$ almost surely as $R \rightarrow \infty$. To verify this convergence we apply a similar pointwise argument as in Theorem 3.11: Let $\omega \in \Omega$ be such that there exists a subsequence $\{R(l)\}_{l \geq 1}$ converging to infinity

$$\tilde{\theta} \triangleq \lim_{l \rightarrow \infty} \sigma_2^{R(l)}(\omega) < \sigma_2(\omega).$$

Then, we can find an l_0 such that $\sigma_2^{R(l)} = \hat{\theta}^{R(l)} < \sigma_2$ for all $l \geq l_0$. By continuity of the state process and continuity of V we find that $V^{R(l)}(Z_{\hat{\theta}^{R(l)}}^{R(l)}) = \Psi(Z_{\hat{\theta}^{R(l)}}^{R(l)})$. Then for $l \rightarrow \infty$ by means of local uniform convergence of $\{V^R\}_{R>0}$ we conclude that

$$V(Z_{\tilde{\theta}}) = \Psi(Z_{\tilde{\theta}}).$$

for the ω fixed above. This contradicts the assumption that $\sigma_2 \leq \hat{\theta}$.

Next, Itô's formula (3.14) applied to V^R on the interval $[\sigma_1, \sigma_2^R]$ immediately yields

$$\begin{aligned} V^R(Z_{\sigma_1}^R) \mathbb{I}_{\sigma_1 < \infty} &= \mathbb{E} \left[V^R(Z_{\sigma_2^R}^R) e^{-\int_{\sigma_1}^{\sigma_2^R} r(Z_s^R) ds} \mathbb{I}_{\sigma_1 < \infty} \right. \\ &\quad \left. + \int_{\sigma_1}^{\sigma_2^R} e^{-\int_{\sigma_1}^t r(Z_s^R) ds} P(Z_t^R) dt \mathbb{I}_{\sigma_1 < \infty} \middle| \mathcal{F}_{\sigma_1} \right]. \end{aligned}$$

The proof is concluded by letting $R \rightarrow \infty$ and employing convergence of $\{\sigma_2^R\}_{R>0}$, continuity of paths of state process and local uniform convergence of $\{V^R\}_{R>0}$ towards V that we established in Lemma 3.12. \square

3.D Proofs for Section 3.5

3.D.1 Proof of Theorem 3.14

We verify the assertion of the theorem only for the increasing sequence $\{U_m\}_{m \geq 0}$. The result for $\{U^m\}_{m \geq 0}$ is obtained in a similar manner.

Step 1: Lower Bound. Let us first verify that for every $m \geq 1$ (for $m = 0$ the assertion obviously holds)

$$U_m(z) \leq \inf_{\mathcal{S}_m} I_m(\mathcal{S}_m; z). \quad (3.35)$$

For this purpose we consider an arbitrary admissible strategy $\mathcal{S} = \{\tau_k, \zeta_k\}_{1 \leq k \leq m}$ with at most m interventions. Let $\{Z_t^{\mathcal{S}}\}_{t \geq 0}$ be the corresponding state process, i.e. $Z^k \triangleq Z^{\tau_k, \zeta_k}$ on $[\tau_k, \tau_{k+1})$ for every $k \in \{0, \dots, m\}$ (here $\tau_{m+1} = \infty$). By the definition of \mathcal{M} from the inequality $U_{m-k+1} \leq \mathcal{M}U_{m-k}$ we derive that

$$\mathcal{C}(z, z') \geq U_{m-k+1}(z) - U_{m-k}(z'), \quad k \in \{1, \dots, m\}.$$

Then, applying the Itô-Dynkin inequality to U_{m-k+1} on $[\tau_{k-1}, \tau_k)$ (Theorem 3.13), we obtain that

$$\begin{aligned} & \mathbb{E} \left[\int_0^{\tau_m} e^{-\int_0^t r(Z_s^{\mathcal{S}}) ds} P(Z_t^{\mathcal{S}}) dt + \sum_{k=1}^m e^{-\int_0^{\tau_k} r(Z_s^{\mathcal{S}}) ds} \mathcal{C}(Z_{\tau_k}^{k-1}, \zeta_k) \right] \\ &= \mathbb{E} \left[\sum_{k=1}^m \int_{\tau_{k-1}}^{\tau_k} e^{-\int_0^t r(Z_s^{\mathcal{S}}) ds} P(Z_t^{k-1}) dt \mathbb{I}_{\tau_{k-1} < \infty} \right. \\ & \quad \left. + \sum_{k=1}^m e^{-\int_0^{\tau_k} r(Z_s^{\mathcal{S}}) ds} \mathcal{C}(Z_{\tau_k}^{k-1}, \zeta_k) \right] \\ & \geq \mathbb{E} \left[\sum_{k=1}^m \left(U_{m-k+1}(Z_{\tau_{k-1}}^{k-1}) e^{-\int_0^{\tau_{k-1}} r(Z_s^{\mathcal{S}}) ds} \mathbb{I}_{\tau_{k-1} < \infty} \right. \right. \\ & \quad \left. \left. - U_{m-k+1}(Z_{\tau_k}^{k-1}) e^{-\int_0^{\tau_k} r(Z_s^{\mathcal{S}}) ds} \mathbb{I}_{\tau_k < \infty} \right) \right. \\ & \quad \left. + \sum_{k=1}^m \left(U_{m-k+1}(Z_{\tau_k}^{k-1}) e^{-\int_0^{\tau_k} r(Z_s^{\mathcal{S}}) ds} \mathbb{I}_{\tau_k < \infty} \right. \right. \\ & \quad \left. \left. - U_{m-k}(Z_{\tau_k}^k) e^{-\int_0^{\tau_k} r(Z_s^{\mathcal{S}}) ds} \mathbb{I}_{\tau_k < \infty} \right) \right] \\ &= U_m(z) \end{aligned} \quad (3.36)$$

as $U_0 = 0$ and $\tau_0 = 0$. As the strategy \mathcal{S} was chosen arbitrarily, we deduce (3.35).

Step 2: Exact Lower Bound. To prove that $U_m(z)$ is the exact lower bound for $\inf_{\mathcal{S}_m} I_m(\mathcal{S}_m; z)$, we define an impulse control strategy inductively and in the same fashion as in the Verification Theorem 3.6 in Section 3.3:

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- (a) $\hat{\tau}_0 \triangleq 0$ and $\hat{\zeta}_0 \triangleq z$.
- (b) For every $k \in \{0, \dots, m-1\}$, assumed that the controlled process is already defined on $[0, \hat{\tau}_k)$ and an action $\hat{\zeta}_k$ at $\hat{\tau}_k$ is determined, we set

$$\hat{\tau}_{k+1} \triangleq \inf \{t \geq \hat{\tau}_k : U_{m-k}(\hat{Z}_t^k) = \mathcal{M}U_{m-k-1}(\hat{Z}_t^k)\}$$

where $\hat{Z}^k \triangleq Z^{\hat{\tau}_k, \hat{\zeta}_k}$. Then the state process $\hat{Z}_t \triangleq \hat{Z}_t^k$ for $t \in [\hat{\tau}_k, \hat{\tau}_{k+1})$. If $\hat{\tau}_{k+1} < \infty$, we set

$$\hat{\zeta}_{k+1} \triangleq \Xi^{U_{m-k-1}}(\hat{Z}_{\hat{\tau}_{k+1}}^k).$$

Existence of a measurable minimizer $\Xi^{U_{m-k-1}}$ of $\mathcal{M}U_{m-k-1}$ is ensured by Proposition 3.4.

- (c) On $\{\hat{\tau}_m < \infty\}$ the state process \hat{Z}_t coincides with $\hat{Z}^{\hat{\tau}_m, \hat{\zeta}_m}$ for $t \in [\hat{\tau}_m, \infty)$.

Let $\hat{\mathcal{S}} \triangleq \{\hat{\tau}_k, \hat{\zeta}_k\}_{1 \leq k \leq m}$ denote the constructed strategy. $\hat{\mathcal{S}}$ is obviously admissible. Further, by construction we obtain for each $k \in \{1, \dots, m\}$

$$\mathcal{C}(\hat{Z}_{\hat{\tau}_k}^{k-1}; \hat{Z}_{\hat{\tau}_k}^k) = U_{m-k+1}(\hat{Z}_{\hat{\tau}_k}^{k-1}) - U_{m-k}(\hat{Z}_{\hat{\tau}_k}^k)$$

and hence by the Itô-Dynkin Formula from Theorem 3.13 the inequality (3.36) holds with equality for $\hat{\mathcal{S}}$. This concludes the proof. \square

3.D.2 Proof of Theorem 3.15

First, we verify the statement for the non-decreasing sequence $\{U_m\}_{m \geq 1}$. Let $\hat{\mathcal{S}}_m = \{\hat{\tau}_k, \hat{\zeta}_k\}_{1 \leq k \leq m}$ be the optimal strategy defined in the proof of Theorem 3.14 and $\{\hat{Z}_t\}_{t \geq 0}$ the corresponding state process. Note that $\hat{Z}_t = Z_t^{\hat{\tau}_m, \hat{\zeta}_m}$ for $t \geq \hat{\tau}_m$ and therefore by the strong Markov property from Proposition 3.1 we obtain

$$\begin{aligned} 0 &\leq U(z) - U_m(z) \\ &\leq I(\hat{\mathcal{S}}_m; z) - I_m(\hat{\mathcal{S}}_m; z) \\ &= \mathbb{E} \left[\int_{\hat{\tau}_m}^{\infty} e^{-\int_0^t r(\hat{Z}_s) ds} P(\hat{Z}_t) dt \mathbb{I}_{\hat{\tau}_m < \infty} \right] \\ &= \mathbb{E} \left[e^{-\int_0^{\hat{\tau}_m} r(\hat{Z}_s) ds} U^0(\hat{Z}_{\hat{\tau}_m}) \mathbb{I}_{\hat{\tau}_m < \infty} \right]. \end{aligned}$$

Boundedness of U^0 yields that

$$\begin{aligned} \mathbb{E} \left[e^{-\int_0^{\hat{\tau}_m} r(\hat{Z}_s) ds} U^0(\hat{Z}_{\hat{\tau}_m}) \mathbb{I}_{\hat{\tau}_m < \infty} \right] &\leq \frac{\|U^0\|_\infty}{Km} \mathbb{E} \left[\sum_{k=1}^m e^{-\int_0^{\hat{\tau}_k} r(\hat{Z}_s) ds} \mathcal{C}(\hat{Z}_{\hat{\tau}_k}^{k-1}, \hat{\zeta}_k) \right] \\ &\leq \frac{\|U^0\|_\infty}{Km} U_m(z) \\ &\leq \frac{\|U^0\|_\infty^2}{Km}. \end{aligned}$$

An estimate for $\{U^m\}_{m \geq 1}$ is obtained in a similar fashion by noting that

$$0 \leq U^m(z) - U(z) \leq U^m(z) - U_m(z) \leq \mathbb{E} \left[\int_{\hat{\tau}_m}^\infty e^{-\int_0^t r(\hat{Z}_s) ds} P(\hat{Z}_t) dt \mathbb{I}_{\hat{\tau}_m < \infty} \right]$$

and applying the strong Markov property. \square

4 Small-Cost Asymptotics for Growth Rates in Incomplete Markets

In Chapters 2 and 3 we have analyzed methods for solving problems with transaction costs that are based on finite element methods and convergence results in appropriate Sobolev spaces. This approach requires additional regularity conditions (cf. assumptions (A3) in Chapter 2 or (A2) in Chapter 3) that do not necessarily hold for some important benchmark examples, as e.g. the Heston model. The key characteristic that gives rise to the alternative method discussed in this chapter is the fact that we often encounter cases of very small transaction fees in practice. Mathematically, this means that one considers a financial market with transaction costs of order ϵ and investigates the asymptotic behavior of optimal strategies and indirect utilities as $\epsilon \downarrow 0$.

This chapter is based on joint work with Frank Thomas Seifried and is an extended version of Melnyk and Seifried [2014].

4.1 Introduction

In this chapter, we provide an asymptotic analysis for long-term growth rates under both proportional and Morton-Pliska transaction costs in an incomplete financial market with an unspanned Markov factor process. This framework includes the Heston stochastic volatility model Heston [1993] and the Kim-Omberg Kim and Omberg [1996] stochastic excess return model as special cases. Our optimization criterion is the long-term growth rate of the investor's wealth,

$$R^\epsilon = \sup_{\mathcal{S}} \liminf_{T \rightarrow \infty} \frac{\mathbb{E} \ln Z_T^{\mathcal{S}}}{T}$$

where the supremum extends over all admissible trading strategies \mathcal{S} .

Contribution. First, we demonstrate that the optimal long-term growth rate satisfies

$$R^\epsilon = R^0 + \epsilon^\alpha Q + O(\epsilon^\beta)$$

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where R^0 denotes the optimal growth rate in the absence of frictions, $\alpha = 2/3$, $\beta = 1$ for proportional transaction costs, and $\alpha = 1/2$, $\beta = 3/4$ for Morton-Pliska costs;¹ see Theorems 4.9 and 4.15. In both cases, we are able to identify the leading-order coefficient Q as an average of a local correction term $Q_{\text{loc}}(v)$ with respect to the stationary distribution of the factor process v ,

$$Q = \mathbb{E}[Q_{\text{loc}}(v_\infty) \mathbb{I}_{v_\infty \in \mathcal{I}^0}].$$

Moreover, we explicitly construct strategies that achieve the optimal long-term growth rate at the leading order. The associated no-trading regions are given by truncated, skewed tubular neighborhoods around the frictionless optimizer, with widths of order $\epsilon^{1/3}$ for proportional costs, and of order $\epsilon^{1/4}$ for Morton-Pliska costs; see, e.g., Figures 4.1 and 4.4.

Second, we analyze the performance of Morton-Pliska investment strategies² in settings with proportional transaction costs. We find that the optimal Morton-Pliska strategy achieves the leading-order optimal growth rate $\epsilon^{2/3}$, with a leading-order coefficient reduced by 26%. Equivalently, the optimal Morton-Pliska growth rate is the same as the optimal one with costs increased by a factor $\sqrt{2}$.

Third, we extend a classical result of Breiman [1961] and verify that the leading-order optimal strategies are in fact pathwise optimal: They maximize the long-run growth rate path by path, at the leading order.

The approach taken in this chapter is close in spirit to that of Janeček and Shreve [2004] and Bichuch [2012]. However, while their analysis involves exact sub- and supersolutions, in this chapter we directly use asymptotic expansions of the value functions and provide explicit bounds for the relevant higher-order terms.

Related Literature. This chapter adds to a growing literature in the area of asset allocation under transaction costs by providing a rigorous asymptotic analysis of long-run optimal portfolio choice in the presence of both transaction costs and an unspanned factor process. While several alternative optimization criteria have been used in the literature, including finite and infinite horizons, utility from

¹This is in line with the formal results of Atkinson and Wilmott [1995] and Kallsen and Muhle-Karbe [2013].

²Briefly, a Morton-Pliska strategy dictates that the investor does nothing as long as her risky fraction stays within a no-trading region, and trades to a fixed point in the interior as soon as it hits the boundary; see Section 4.5 for a rigorous definition.

terminal wealth and/or consumption, risk-sensitized growth rates, etc., in this chapter we focus on the optimal long-term growth rate.

Asset allocation under transaction costs has been studied intensively since the seminal work of Magill and Constantinides [1976]. Among many other important contributions, Taksar et al. [1988] and Davis and Norman [1990] investigate optimal portfolio strategies in single-factor Black-Scholes markets under proportional transaction costs, and Korn [1998] extends their analysis to costs with a fixed component. Morton and Pliska [1995] introduce and analyze transaction costs proportional to the investor's wealth. Bielecki and Pliska [2000] analyze optimal long-term growth rates with a general transaction cost structure.

Atkinson and Wilmott [1995] are the first to provide formal small-cost asymptotics in a classical Black-Scholes market with Morton-Pliska costs. Kallsen and Muhle-Karbe [2013, 2014] formally derive leading-order optimal trading policies for general market models in the presence of proportional transaction costs. Bichuch and Sircar [2014] use a perturbation approach to analyze portfolio problems with fast- or slow-moving stochastic volatility and proportional transaction costs. These heuristic approaches have been complemented by a number of papers that provide rigorous asymptotic expansions. Thus Janeček and Shreve [2004] and Bichuch [2012] base their analysis on an exact sub-/supersolution approach in one-dimensional Black-Scholes models. Gerhold et al. [2012, 2013] and Gerhold et al. [2014] and Guasoni and Muhle-Karbe [2014] use a shadow price approach to obtain asymptotic expansions in one-dimensional Black-Scholes markets under small proportional transaction costs for finite and infinite time horizons and constant absolute and relative risk aversion, and Kallsen and Ahrens [2014] verify the main results of Kallsen and Muhle-Karbe [2014]. Soner and Touzi [2013] and Possamaï et al. [2013] attack portfolio problems with general utility functions and asset dynamics using viscosity and homogenization techniques. Akian et al. [2001] analyze optimal long-term growth rates using viscosity methods, but focus on a Black-Scholes market model. Finally, Altarovici et al. [2015] investigate the case of small fixed transaction costs.

Finally, there is a classical related literature on optimal strategies for long-term growth rates in frictionless markets, including Kelly [1956], Breiman [1961], and Bell and Cover [1980]; we refer to MacLean et al. [2011] for an overview. In contrast to this literature, in our setting returns are not independent and transaction costs apply. Nevertheless, we find that the leading-order optimal strategy remains both myopic and pathwise leading-order optimal as in Breiman [1961] and Kelly

[1956].

Outline. The remainder of this chapter is structured as follows: In Section 4.2 we introduce the financial market model, the long-run growth rate criterion, and the portfolio optimization problem without transaction costs, with proportional transaction costs, and with Morton-Pliska costs. We also report some classical verification results on the corresponding variational inequalities (VIs). Section 4.3 provides heuristic intuition for the asymptotic expansions of optimal growth rates using both stochastic and PDE-based arguments. In particular, we define the leading-order VIs, which play a key role for the rigorous analysis of Sections 4.4–4.7. Sections 4.4 and 4.5 establish the first two main results of the chapter: Theorems 4.9 and 4.15 identify the asymptotic expansions of the optimal long-term growth rates under proportional and Morton-Pliska costs, respectively, together with detailed discussions of the proofs and explicit constructions of leading-order optimal trading strategies. In Section 4.6 we analyze the performance of Morton-Pliska strategies under proportional transaction costs and quantify the loss compared to the globally optimal strategy (Theorem 4.21). Finally, Section 4.7 extends the main results of this chapter to pathwise, rather than expected, long-run growth rates (Theorems 4.25, 4.26, 4.27). The Appendix contains proofs omitted from the main text.

4.2 Financial Market Model and Variational Inequalities

This section introduces our mathematical framework. We define the long-term growth rate and discuss the portfolio dynamics in the frictionless case and with proportional or Morton-Pliska costs. We also state two general verification results that apply if a sufficiently smooth solution of the associated dynamic programming equations (variational inequalities), see (4.13) and (4.16), is available. The asymptotic analysis of Sections 4.4 and 4.5 is based on these variational inequalities.

4.2.1 Asset Price Dynamics and Long-Term Growth Rates

Financial Market. Throughout this chapter we fix a filtered probability space $(\Omega, \mathcal{A}, \mathcal{F}, \mathbb{P})$ where the filtration $\mathcal{F} = \{\mathcal{F}_t\}_{t \geq 0}$ satisfies the usual conditions. We consider a general Markov factor financial market model with a cash account B

4.2 Financial Market Model and Variational Inequalities

that satisfies

$$dB_t = B_t r(v_t) dt \tag{4.1}$$

and a stock (stock index) S with dynamics

$$dS_t = S_t [(r(v_t) + \lambda(v_t)) dt + \sigma(v_t) dW_t]. \tag{4.2}$$

Here v is an underlying factor process that takes values in an open interval $\mathcal{I} \subset \mathbb{R}$ and whose dynamics are given by

$$dv_t = \alpha(v_t) dt + \beta(v_t) d\bar{W}_t. \tag{4.3}$$

The functions $r, \lambda, \sigma, \alpha, \beta : \mathcal{I} \rightarrow \mathbb{R}$ are Borel measurable and polynomially bounded, and β and σ are strictly positive and of class C^2 . \bar{W} and W are Brownian motions with

$$dW_t d\bar{W}_t = \rho(v_t) dt$$

where $\rho : \mathcal{I} \rightarrow [-1, 1]$ is Borel measurable. We assume that the system (4.1), (4.2), (4.3) has a pathwise unique strong solution for all $t \geq 0$ and every initial data $(v_0, B_0, S_0) \in \mathcal{I} \times (0, \infty) \times (0, \infty)$.³ This abstract model subsumes two important benchmark examples as special cases:

Example 1: Heston Model. *In the Heston stochastic volatility model (see Heston [1993]) the asset price dynamics are given by*

$$\begin{aligned} dB_t &= rB_t dt \\ dS_t &= S_t [(r + \lambda) dt + \sqrt{v_t} dW_t] \\ dv_t &= -\theta(v_t - \eta) dt + \beta\sqrt{v_t} d\bar{W}_t \end{aligned} \tag{4.4}$$

where $r, \lambda, \theta, \eta, \beta$ are constants, $2\theta\eta > \beta^2$ and $\rho \in (-1, 1)$ is constant.⁴ The factor process v represents squared stochastic volatility and is modeled by a square-root diffusion with values in $\mathcal{I} = (0, \infty)$.

³This assumption is satisfied, e.g., under suitable Lipschitz conditions. We do not, however, make any such assumption here.

⁴Alternative specifications of the excess return subsumed by our framework include, e.g., $\lambda\sqrt{v_t}$, which implies a constant market price of risk, and λv_t . Our definition (4.4) coincides with Heston's original one and yields a non-constant market price of risk of $\lambda/\sqrt{v_t}$.

Example 2: Kim-Omberg Model. *The Kim-Omberg stochastic excess return model introduced in Kim and Omberg [1996] specifies the price dynamics*

$$\begin{aligned} dB_t &= rB_t dt \\ dS_t &= S_t[(r + v_t) dt + \sigma dW_t] \\ dv_t &= -\theta(v_t - \eta) dt + \beta d\bar{W}_t. \end{aligned} \tag{4.5}$$

with constants $r, \sigma, \theta, \eta, \beta, \rho$. Thus the Ornstein-Uhlenbeck process v models a stochastic, mean-reverting excess return in $\mathcal{I} = \mathbb{R}$.

Remark. *Our general framework also subsumes the 3/2 model of Chacko and Viceira [2005], the Brownian version of the COGARCH model in Klüppelberg et al. [2004], and the Schwartz commodity model in Schwartz [1997], among others. With some additional care concerning integrability conditions (see assumptions (A1) and (A2)), our analysis can also be accommodated for these models.*

Throughout this chapter, we assume that the factor process v satisfies the following assumption:

(A1) *There exists a random variable v_∞ such that*

- (i) *For every Borel measurable function $h : \mathcal{I} \rightarrow \mathbb{R}$ with $\mathbb{E}[|h(v_\infty)|] < \infty$ we have*

$$\lim_{T \rightarrow \infty} \frac{1}{T} \mathbb{E} \left[\int_0^T h(v_s) ds \right] = \mathbb{E}[h(v_\infty)].$$

- (ii) *The distribution of v_∞ is absolutely continuous with a locally bounded density and satisfies $\mathbb{E}[|v_\infty|^m] < \infty$ for all $m \geq 1$.*
- (iii) *For all initial values $v_0 \in \mathcal{I}$ and all $a \in \mathcal{I}$ we have*

$$\mathbb{E}[H^a] < \infty \quad \text{where} \quad H^a \triangleq \inf\{t \geq 0 : v_t = a\}$$

and, moreover,

$$\mathbb{E} \left[\int_0^{H^a} v_s^m ds \right] < \infty \quad \text{for every } m \geq 1.$$

Assumption (A1) ensures the long-term ergodic behavior of the state process that is required for our subsequent analysis. The above conditions are easily verified for both the Heston and the Kim-Omberg model.⁵

Notation. *In the following, v_∞ always denotes the random variable in (A1).*

⁵ The stationary distribution of v in the Heston model is a Gamma distribution with shape parameter $\frac{2\eta\theta}{\beta^2}$ and scale parameter $\frac{\beta^2}{2\theta}$. In the Kim-Omberg model the stationary distribution is $\mathcal{N}(\eta, \frac{\beta^2}{2\theta})$.

4.2 Financial Market Model and Variational Inequalities

Long-Term Growth Rate. The investor focuses on her portfolio performance in the long run. Her goal is to maximize the long-term growth rate of her wealth,

$$R \triangleq \sup_{\mathcal{S}} \liminf_{T \rightarrow \infty} \frac{\mathbb{E} \ln Z_T^{\mathcal{S}}}{T} \quad (4.6)$$

over all admissible trading strategies⁶ \mathcal{S} . Here $Z^{\mathcal{S}}$ denotes the investor's *paper wealth* process, or, equivalently for the purposes of (4.6), the *liquidation value* of her portfolio; see the discussion of assumption (A2) below. In general, the criterion (4.6) is also known as the Kelly criterion, see Kelly [1956]. Besides maximizing the growth rate of the investor's wealth (see, e.g., Algoet and Cover [1988], Breiman [1961], Jamshidian [1992] and Karatzas [1989]), Kelly strategies are also competitively optimal, see Bell and Cover [1980], and closely related to the numéraire portfolio theory and the benchmark approach to finance (see, among others, Becherer [2001], Karatzas and Kardaras [2007], Platen [2011]). We refer to MacLean et al. [2011] for an overview and further properties of the Kelly criterion.

(A2) *Short positions in either cash or stock are prohibited. The fraction of the investor's wealth invested in stocks at time $t = 0$ is given by $b^0(v_0)$, where*

$$b^0(v) \triangleq \left(\frac{\lambda(v)}{\sigma(v)^2} \vee 0 \right) \wedge 1.$$

Some observations concerning the criterion (4.6) and assumption (A2) are in order:

First, note that the long-term growth rate does not depend on the level of initial wealth since

$$\liminf_{T \rightarrow \infty} \frac{\mathbb{E} \ln Z_T^{\mathcal{S}}}{T} = \liminf_{T \rightarrow \infty} \frac{\mathbb{E} \ln(Z_T^{\mathcal{S}}/Z_0)}{T}.$$

Second, for both proportional and Morton-Pliska costs (see below), the growth rate is not affected by any single additional trade (e.g., at the beginning or at T), since the post-trade wealth is always in the interval $[(1 - \epsilon)Z_T, Z_T]$. In particular, the long-term growth rate is the same whether we define it using paper wealth or liquidation values. It is for the same reason that the second part of the preceding assumption, concerning the initial stock position, is without loss of generality.

⁶In the sequel we consider different types of costs, and the class of admissible strategies \mathcal{S} will be defined accordingly.

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Third, in what follows we will focus on the case $b^0(v_0) \in (0, 1)$. The proofs for $b^0(v_0) \in \{0, 1\}$ are entirely analogous with minor notational modifications.

Fourth, it is intuitively clear that the growth rate is also independent of the initial value v_0 of the volatility process. In Theorems 4.9 and 4.15 below, we demonstrate rigorously that, up to its leading order, the optimal long-term growth rate does indeed not depend on v_0 .

To see heuristically why the initial volatility v_0 is immaterial for the long-term growth rate, let $a, b \in \mathcal{I}$ and consider an arbitrary trading strategy \mathcal{S}^b that is admissible for $v_0 = b$. Now define an admissible strategy \mathcal{S}^a for initial volatility $v_0 = a$ as follows: Keep all wealth in stock until the factor process hits level b , i.e. until $H^b = \inf\{t \geq 0 : v_t = b\}$, and apply the strategy \mathcal{S}^b on $[H^b, \infty)$. Then for all $T > 0$ it follows that

$$\mathbb{E}^a [\ln Z_{T+H^b}^{\mathcal{S}^a} - \ln Z_{H^b}^{\mathcal{S}^a}] = \mathbb{E}^b [\ln Z_T^{\mathcal{S}^b} - \ln Z_0]$$

so the long-term growth rates of \mathcal{S}^a and \mathcal{S}^b satisfy

$$\begin{aligned} & \liminf_{T \rightarrow \infty} \frac{1}{T} \mathbb{E}^a [\ln Z_{H^b+T}^{\mathcal{S}^a}] \\ &= \liminf_{T \rightarrow \infty} \frac{1}{T} \mathbb{E}^a \left[\ln \frac{Z_{H^b+T}^{\mathcal{S}^a}}{Z_{H^b}^{\mathcal{S}^a}} + \ln Z_{H^b}^{\mathcal{S}^a} \right] \\ &= \liminf_{T \rightarrow \infty} \frac{1}{T} \left(\mathbb{E}^b \left[\ln \frac{Z_T^{\mathcal{S}^b}}{Z_0} \right] + \mathbb{E}^a \left[\int_0^{H^b} r(v_s) ds + \ln Z_0 - \text{cost}(0) - \text{cost}(H^b) \right] \right) \\ &= \liminf_{T \rightarrow \infty} \frac{1}{T} \mathbb{E}^b [\ln Z_T^{\mathcal{S}^b}] \end{aligned}$$

where $\text{cost}(\cdot)$ denotes potential transaction costs. In addition we have

$$\begin{aligned} & \liminf_{T \rightarrow \infty} \frac{1}{T} \mathbb{E}^a [\ln Z_{H^b+T}^{\mathcal{S}^a}] \\ &= \liminf_{T \rightarrow \infty} \frac{1}{T} \mathbb{E}^a \left[\ln Z_T^{\mathcal{S}^a} + \ln \frac{Z_{H^b+T}^{\mathcal{S}^a}}{Z_T^{\mathcal{S}^a}} \right] \\ &\leq \liminf_{T \rightarrow \infty} \frac{1}{T} \left(\mathbb{E}^a [\ln Z_T^{\mathcal{S}^a}] + \mathbb{E}^a \left[\int_T^{T+H^b} (|r(v_s)| + |\lambda(v_s)|) ds \right] \right) \\ &= \liminf_{T \rightarrow \infty} \frac{1}{T} \mathbb{E}^a [\ln Z_T^{\mathcal{S}^a}]. \end{aligned}$$

Hence the long-term growth rate of \mathcal{S}^a exceeds that of \mathcal{S}^b . By interchanging the roles of a and b it follows that the long-term growth rate is independent of initial volatility.

4.2.2 Optimal Growth Rates without Transaction Costs

Having set up our mathematical framework, we first briefly address the frictionless case. In the absence of transaction costs, the investor can freely specify the desired portfolio allocation and implement it without losses. Thus the set of admissible trading strategies \mathcal{A}^0 consists of all progressively measurable processes $\mathcal{S} \triangleq \{b_t\}_{t \geq 0}$ that take values in $[0, 1]$ where b_t represents the fraction of the investor's wealth invested in stocks at time t . The corresponding wealth process $Z^{\mathcal{S}}$ satisfies

$$dZ_t^{\mathcal{S}} = Z_t^{\mathcal{S}}(r(v_t) + \lambda(v_t) b_t) dt + Z_t^{\mathcal{S}} b_t \sigma(v_t) dW_t$$

and remains positive almost surely. The long-term growth rate attained by the strategy \mathcal{S} is given by

$$\liminf_{T \rightarrow \infty} \frac{\mathbb{E} \ln Z_T^{\mathcal{S}}}{T} = \liminf_{T \rightarrow \infty} \frac{1}{T} \mathbb{E} \left[\int_0^T (r(v_t) + \lambda(v_t) b_t - \frac{1}{2} \sigma(v_t)^2 b_t^2) dt \right].$$

Thus we obtain the optimal long-term growth rate by maximizing the deterministic function

$$\begin{aligned} f(v, b) &\triangleq r(v) + \lambda(v) b - \frac{1}{2} \sigma(v)^2 b^2 \\ &= r(v) + \frac{1}{2} \frac{\lambda(v)^2}{\sigma(v)^2} - \frac{1}{2} \sigma(v)^2 \left(b - \frac{\lambda(v)}{\sigma(v)^2} \right)^2. \end{aligned} \quad (4.7)$$

It follows that the optimal strategy is given by the Merton proportion

$$b^0(v) = \left(\frac{\lambda(v)}{\sigma(v)^2} \vee 0 \right) \wedge 1.$$

The associated optimal long-term growth rate without transaction costs can now be calculated directly with the help of assumption (A1) (i):

$$R^0 \triangleq \lim_{T \rightarrow \infty} \frac{1}{T} \mathbb{E} \left[\int_0^T f(v_s, b^0(v_s)) ds \right] = \mathbb{E}[f(v_\infty, b^0(v_\infty))].$$

Equivalently, if we define the *local growth rate* for the frictionless market via

$$R_{loc}^0(v) \triangleq f(v, b^0(v)) \quad (4.8)$$

then the optimal long-term growth rate rewrites as

$$R^0 = \mathbb{E}[R_{loc}^0(v_\infty)]. \quad (4.9)$$

We will show in Sections 4.4 and 4.5 below that a representation of the form (4.9) continues to hold, with a suitably modified local growth rate, under both proportional and Morton-Pliska transaction costs.

In the following we impose a weak regularity condition on the frictionless optimal strategy.

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(A3) *The set*

$$\mathcal{I}^0 \triangleq \{v \in \mathcal{I} : 0 \leq \frac{\lambda(v)}{\sigma(v)^2} \leq 1\}$$

is an interval, and the function λ/σ^2 is of class C^3 with $(\lambda/\sigma^2)'(v) > 0$ or $(\lambda/\sigma^2)'(v) < 0$ on \mathcal{I}^0 . To rule out trivial cases, we further assume that $\mathcal{I}^0 \neq \emptyset$ and $\mathcal{I}^0 \neq \mathcal{I}$.

This assumption, as the previous ones, is satisfied for both the Heston model (with $\lambda(v)/\sigma(v)^2 = \lambda/v$) and the Kim-Omberg model (where $\lambda(v)/\sigma(v)^2 = v/\sigma^2$). It implies that there is a bounded monotone function b^* on \mathcal{I} that is of class $C^3(\mathcal{I})$ with bounded derivatives and that satisfies

$$b^*(v) = \frac{\lambda(v)}{\sigma(v)^2} \quad \text{on } \mathcal{I}^0.$$

We select and fix such a function b^* for all that follows.

4.2.3 Wealth Dynamics with Proportional Costs

In the presence of proportional transaction costs, a trading strategy is defined as a pair of \mathcal{F} -adapted non-decreasing càdlàg processes $\{(L_t, M_t)\}_{t \geq 0}$ with $L_{0-} = M_{0-} = 0$ that represent the cumulative \$-amounts transferred from the cash account into stocks and from stocks into the cash account, respectively, by time t . X_t and Y_t denote the \$-holdings in cash and stock at time t , respectively. Then, similarly to Davis and Norman [1990], the portfolio dynamics are given by

$$\begin{aligned} dX_t &= r(v_t)X_t dt - (1 + \hat{\epsilon}) dL_t + (1 - \hat{\epsilon}) dM_t, \\ dY_t &= Y_t((r(v_t) + \lambda(v_t)) dt + \sigma(v_t) dW_t) + dL_t - dM_t \end{aligned} \tag{4.10}$$

where $\hat{\epsilon}$ represents the size of the proportional transaction costs.

Notation. *As a general rule, to avoid confusion we write \hat{z} to designate a quantity z in the proportional costs setting; in the case of Morton-Pliska below, we will write \check{z} .*

Definition 4.1. *The set $\hat{\mathcal{A}}(x, y)$ of admissible trading strategies in the market with proportional transaction costs consists of all pairs of \mathcal{F} -adapted non-decreasing càdlàg processes (L, M) with $L_{0-} = M_{0-} = 0$ such that (4.10) initiated in (x, y) has a unique solution in $[0, \infty)^2 \setminus \{(0, 0)\}$.*

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Note that this definition rules out short positions in either bond or stock, so solvency is ensured at all times.

The paper wealth process associated with an admissible strategy $\{(L_t, M_t)\}_{t \geq 0}$ is defined by $Z_t \triangleq X_t + Y_t$ and satisfies

$$dZ_t = Z_t(r(v_t) + \lambda(v_t) b_t) dt + Z_t b_t \sigma(v_t) dW_t - \hat{\epsilon} dL_t - \hat{\epsilon} dM_t.$$

Here $b_t \triangleq Y_t/Z_t$ is the *fraction process*. Itô's formula yields the dynamics of b via

$$\begin{aligned} b_t = & b_0 + \int_0^t b_s(1-b_s)(\lambda(v_s) - \sigma(v_s)^2 b_s) ds + \int_0^t \sigma(v_s) b_s(1-b_s) dW_s \\ & + \int_0^t (1+b_s \hat{\epsilon}) \frac{dL_s^c}{Z_s} + \int_0^t (-1+b_s \hat{\epsilon}) \frac{dM_s^c}{Z_s} + \sum_{0 < s \leq t} (b_s - b_{s-}) \end{aligned} \quad (4.11)$$

where L^c and M^c are the continuous parts of L and M . For simplicity, we denote the infinitesimal generators of $\{v_t\}_{t \geq 0}$ and of the pair $\{(v_t, b_t)\}_{t \geq 0}$, respectively, by

$$\begin{aligned} \mathcal{L}^v & \triangleq \alpha(v) \frac{\partial}{\partial v} + \frac{1}{2} \beta(v)^2 \frac{\partial^2}{\partial v^2} \\ \mathcal{L} & \triangleq \mathcal{L}^v + b(1-b)(\lambda(v) - \sigma(v)^2 b) \frac{\partial}{\partial b} \\ & \quad + \frac{1}{2} \sigma(v)^2 b^2 (1-b)^2 \frac{\partial^2}{\partial v^2} + \rho(v) \beta(v) \sigma(v) b(1-b) \frac{\partial^2}{\partial b \partial v}. \end{aligned}$$

Given an arbitrary admissible strategy $\mathcal{S} = (L, M) \in \hat{\mathcal{A}}(x, y)$, we can now represent the long-term growth rate as

$$\begin{aligned} \liminf_{T \rightarrow \infty} \frac{\mathbb{E} \ln Z_T^{\mathcal{S}}}{T} = & \liminf_{T \rightarrow \infty} \frac{1}{T} \mathbb{E} \left[\int_0^T f(v_s, b_s) ds \right. \\ & \left. - \int_0^T \hat{\epsilon} \frac{dL_s^c}{Z_s} - \int_0^T \hat{\epsilon} \frac{dM_s^c}{Z_s} + \sum_{0 < s \leq T} (\ln Z_s - \ln Z_{s-}) \right] \end{aligned}$$

where the function f is defined in (4.7). In particular, the optimal long-term growth rate $\hat{R}^{\hat{\epsilon}}$ with proportional transaction costs is given by

$$\begin{aligned} \hat{R}^{\hat{\epsilon}} \triangleq & \sup_{(L, M) \in \hat{\mathcal{A}}} \liminf_{T \rightarrow \infty} \frac{1}{T} \mathbb{E} \left[\int_0^T f(v_s, b_s) ds \right. \\ & \left. - \int_0^T \hat{\epsilon} \frac{dL_s^c}{Z_s} - \int_0^T \hat{\epsilon} \frac{dM_s^c}{Z_s} + \sum_{0 < s \leq T} (\ln Z_s - \ln Z_{s-}) \right] \end{aligned} \quad (4.12)$$

where the dependence of b and Z on (L, M) is implicit.

It is well-known from singular control theory (see, e.g., Fleming and Soner [2006]) that a sufficiently regular solution of the variational inequalities corresponding to the stochastic control problem (4.12) yields an optimal strategy. In the context of this chapter, a classical result in this spirit is as follows:

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Theorem 4.2 (Classical Verification). *Suppose there are measurable functions $\hat{\phi} : \mathcal{I} \times [0, 1] \rightarrow \mathbb{R}$ and $\hat{R} : \mathcal{I} \rightarrow \mathbb{R}$ such that*

- (i) $\hat{\phi}$ is of class C^2 and bounded together with its derivatives;
- (ii) $\mathbb{E}[|\hat{R}(v_\infty)|] < \infty$;
- (iii) $\hat{\phi}$ and \hat{R} solve the variational inequalities (VIs) for $(v, b) \in \mathcal{I} \times [0, 1]$

$$\min \left\{ -\mathcal{L}\hat{\phi}(v, b) + \hat{R}(v) - f(v, b), \right. \\ \left. -\phi_b(v, b)(1 + \hat{\epsilon}b) + \hat{\epsilon}, -\phi_b(v, b)(-1 + \hat{\epsilon}b) + \hat{\epsilon} \right\} = 0. \quad (4.13)$$

Then

$$\hat{R}^{\hat{\epsilon}} \leq \liminf_{T \rightarrow \infty} \frac{1}{T} \mathbb{E} \left[\int_0^T \hat{R}(v_s) ds \right] = \mathbb{E}[\hat{R}(v_\infty)]. \quad (4.14)$$

Furthermore, if there exists an admissible strategy (\hat{L}, \hat{M}) such that

- (a) both \hat{L} and \hat{M} are continuous,
- (b) the corresponding state process (v, \hat{b}) stays inside the no-trading region

$$\{(v, b) \in \mathcal{I} \times [0, 1] : -\mathcal{L}\hat{\phi}(v, b) + \hat{R}(v) - f(v, b) = 0\}$$

- (c) the process \hat{L} (\hat{M}) increases only at times when (v, \hat{b}) is in the

$$\begin{aligned} &\text{buying region } \{(v, b) \in \mathcal{I} \times [0, 1] : -\phi_b(v_t, b_t)(1 + \hat{\epsilon}b_t) + \hat{\epsilon} = 0\} \\ &\text{(selling region } \{(v, b) \in \mathcal{I} \times [0, 1] : -\phi_b(v_t, b_t)(-1 + \hat{\epsilon}b_t) + \hat{\epsilon} = 0\}) \end{aligned}$$

i.e. trading occurs only on the boundary of the no-trading region,

then (4.14) holds with equality, and (\hat{L}, \hat{M}) is an optimal trading strategy.

Proof. See Section 4.A.1. □

Definition 4.3. *The function \hat{R} in Theorem 4.2 will be called the local growth rate and denoted by $\hat{R}_{loc}^{\hat{\epsilon}}$. The optimal long-term growth rate can thus be represented as*

$$\hat{R}^{\hat{\epsilon}} = \mathbb{E}[\hat{R}_{loc}^{\hat{\epsilon}}(v_\infty)].$$

To the best of our knowledge, closed-form solutions to the VIs (4.13) are not available, and numerical methods are difficult to apply since the operator \mathcal{L} degenerates on the relevant boundaries. Thus, in concrete applications the abstract result of Theorem 4.2 may be of limited use. Hence, in this chapter, we take an alternative approach: We focus on analytical results for *leading-order* optimal strategies. This will be done in Sections 4.3, 4.4 and 4.6. In the remainder of Section 4.2, we first provide a similar classical verification result for Morton-Pliska costs.

4.2.4 Wealth Dynamics with Morton-Pliska Costs

Morton-Pliska costs, as introduced in Morton and Pliska [1995], stipulate that for each trade the investor must pay a constant proportion $\check{\epsilon}$ of her wealth. This type of cost may appear unrealistic, but has benefits from both a mathematical point of view (increased tractability), and from a practical point of view (attractive optimal strategies). In general, Morton-Pliska costs do not allow for infinitely many trades in finite time, for otherwise the investor would go bankrupt. Therefore only impulse control strategies are feasible. The following definition is tailored to the case under consideration and therefore differs from the definitions used in the previous chapters:

Definition 4.4. *The set of admissible trading policies $\check{\mathcal{A}}$ under Morton-Pliska costs consists of all sequences $\{\tau_k, \pi_k\}_{k \geq 0}$ such that*

- (i) $\tau_0 = 0$ and $\pi_0 = b_0$;
- (ii) $\{\tau_k\}_{k \geq 1}$ is a sequence of \mathcal{F} -stopping times with $\tau_k < \tau_{k+1}$ on $\{\tau_k < \infty\}$ and $\tau_k \rightarrow \infty$ almost surely;
- (iii) π_k is a $[0, 1]$ -valued \mathcal{F}_{τ_k} -measurable random variable for all $k \geq 1$.

As previously, the time τ_k represents the k^{th} intervention time, and π_k is the updated value of the fraction process after trading at τ_k .

Remark. *In view of the discussion following the long-term growth rate criterion (4.6), Condition (i) is merely a notational convention. Note that Condition (ii) requires the sequence of intervention times $\{\tau_k\}$ to be strictly increasing, thus ruling out double interventions;⁷ this is done to avoid unnecessary technical complica-*

⁷ It is intuitively obvious that double interventions are suboptimal in the case under consideration.

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tions. We will demonstrate that the leading-order optimal strategy we construct satisfies this property.

Proceeding similarly as in the case of proportional transaction costs, consider an arbitrary admissible trading strategy $\mathcal{S} = \{\tau_k, \pi_k\}_{k \geq 0}$. Itô's formula shows that for each $k \geq 0$ the associated fraction process $b = b^{\mathcal{S}}$ satisfies

$$b_t = \pi_k + \int_{\tau_k}^t b_s(1 - b_s)(\lambda(v_s) - \sigma(v_s)^2 b_s) ds + \int_{\tau_k}^t \sigma(v_s) b_s(1 - b_s) dW_s \quad (4.15)$$

for $t \in [\tau_k, \tau_{k+1})$. The wealth process $Z = Z^{\mathcal{S}}$ evolves on $[\tau_k, \tau_{k+1})$ according to

$$Z_t = Z_{\tau_k} + \int_{\tau_k}^t Z_s(r(v_s) + \lambda(v_s)b_s) ds + \int_{\tau_k}^t Z_s b_s \sigma(v_s) dW_s$$

with $Z_{\tau_{k+1}} = (1 - \epsilon)Z_{\tau_{k+1}-}$. Then using Itô's formula once again we can represent the long-term growth rate as

$$\check{R}^\epsilon \triangleq \sup_{\mathcal{S} \in \mathcal{A}} \liminf_{T \rightarrow \infty} \frac{1}{T} \mathbb{E} \left[\int_0^T f(v_s, b_s^{\mathcal{S}}) ds + \ln(1 - \epsilon) \sum_{k=1}^{\infty} \mathbb{I}_{\tau_k \leq T} \right].$$

Similarly as in the setting with proportional costs, the following classical result can be proved along the lines of Bielecki and Pliska [2000]:

Theorem 4.5 (Classical Verification). *Let $\check{\phi} : \mathcal{I} \times [0, 1] \rightarrow \mathbb{R}$ and $\check{R} : \mathcal{I} \rightarrow \mathbb{R}$ be measurable functions such that*

- (i) $\check{\phi}$ is of class C^2 and bounded together with its derivatives;
- (ii) $\mathbb{E}[|\check{R}(v_\infty)|] < \infty$;
- (iii) $\check{\phi}$ and \check{R} satisfy for $(v, b) \in \mathcal{I} \times [0, 1]$

$$\begin{aligned} \min \left\{ -\mathcal{L}\phi(v, b) + R(v) - f(v, b), \phi(v, b) \right\} &= 0, \\ \sup_{b \in [0, 1]} \phi(v, b) + \ln(1 - \epsilon) &= 0 \end{aligned} \quad (4.16)$$

Then the optimal long-term growth rate is given by

$$\check{R}^\epsilon = \liminf_{T \rightarrow \infty} \frac{1}{T} \mathbb{E} \left[\int_0^T \check{R}(v_s) ds \right] = \mathbb{E}[\check{R}(v_\infty)]$$

and is attained by the strategy $\mathcal{S} \triangleq \{\check{\tau}_k, \check{\pi}_k\}_{k \geq 0}$ that is defined in a similar way as in Theorem 3.6: $\check{\tau}_0 \triangleq 0$ and $\check{\pi}_0 \triangleq b_0$ and for ever $k \geq 1$

4.2 Financial Market Model and Variational Inequalities

(a) $\check{\tau}_k \triangleq \inf\{t \geq \tau_{k-1} : (v_t, \check{b}_t) \notin \mathcal{NT}\}$ with the no-trading region⁸

$$\mathcal{NT} \triangleq \{(v, b) \in \mathcal{I} \times [0, 1] : \check{\phi}(v, b) > 0\}$$

(b) $\check{\pi}_k$ is a (measurable) maximizer of $\check{\phi}(v_{\check{\tau}_k}, \cdot)$ on $\check{\tau}_k < \infty$ and 0 otherwise.

Proof. See Section 4.A.2. □

Definition 4.6. The function \check{R} in Theorem 4.5 will be called the **local growth rate** and is denoted $\check{R}_{loc}^\check{\epsilon}$. In terms of this local growth rate, we have

$$\check{R}^\check{\epsilon} = \mathbb{E}[\check{R}_{loc}^\check{\epsilon}(v_\infty)].$$

Remark. Note that (4.16) differs somewhat from the standard formulation of variational inequalities that we saw in the previous chapters and which in the case considered here would read (see, e.g., Bielecki and Pliska [2000], Korn [1998])

$$\min\{-\mathcal{L}\phi(v, b) + R(v) - f(v, b), \phi(v, b) - \mathcal{M}\phi(v)\} = 0 \quad (4.17)$$

where the intervention operator is given by

$$\mathcal{M}\phi(v) \triangleq \sup_{b \in [0, 1]} \phi(v, b) + \ln(1 - \check{\epsilon}).$$

Note that (4.16) immediately implies (4.17). Conversely, it is not straightforward to derive (4.16) from (4.17) since it is not a priori clear that $\mathcal{M}\phi$ belongs to the kernel of \mathcal{L} . In the context of asymptotic expansions for long-term growth rates as studied in this chapter, the formulation (4.16) is more convenient.

The discussion following Theorem 4.2 applies mutatis mutandis to Theorem 4.5: Since closed-form solutions to the VIs (4.16) are difficult to come by, we take an asymptotic approach for small transaction costs. In the next section we first give a heuristic derivation of the leading-order VIs corresponding to (4.13) and (4.16), respectively.

⁸Note that $\mathcal{NT} \subset \{(v, b) \in \mathcal{I} \times [0, 1] : -\mathcal{L}\check{\phi}(v, b) + \check{R}(v) - f(v, b) = 0\}$ by the variational inequalities (4.16)

4.3 Heuristics for Leading-Order Variational Inequalities

Transaction costs alter tradeoffs in optimal portfolio allocation fundamentally. Before we rigorously address optimal strategies, let us briefly recall the underlying mechanisms on an intuitive level. In general, utility losses caused by transaction costs are due to two effects:

- (i) *direct costs* incurred in market transactions, and
- (ii) indirect costs caused by *misallocations* in the investor's portfolio.

At the optimum, the investor balances these two effects by a suitable choice of the no-trading region. The resulting tradeoff between (i) and (ii) can be studied analytically. Thus it has been shown for different optimization problems that the correct scaling of the no-trading region and the leading-order loss are characterized by elementary minimization problems: For proportional transaction costs, this reads

$$\min_q [C q^2 + \frac{\hat{\epsilon}}{q}] \quad (4.18)$$

(see Janeček and Shreve [2004], Rogers [2004]); for fixed transaction costs as in, for instance, Altarovici et al. [2015], it becomes

$$\min_q [C q^2 + \frac{\hat{\epsilon}}{q^2}]. \quad (4.19)$$

Here q represents the half-width of the no-trading region. The summand $C q^2$ is proportional to the losses due to misallocation, and the second terms are proportional to the respective direct transaction costs.

In this section we provide two alternative heuristic methods to determine the leading orders for the expansions of the relevant VIs (4.13) and (4.16). First, using stochastic arguments tailored to the problem under consideration, we derive elementary optimization problems similar to (4.18) and (4.19). These allow us not only to determine the correct scaling, but also yield the correct width of the no-trading region and the leading-order coefficients. Second, we confirm these results analytically by inserting general expansions into the VIs (4.13) and (4.16) and comparing the leading orders. In addition, that approach also yields variational inequalities for the leading-order coefficients; see (4.35) and (4.36). These will be the basis for the analysis in Sections 4.4 and 4.5.

The arguments in Section 4.3 are heuristic throughout; their merit is in the intuition they provide. The conclusions in Section 4.3 are substantiated by the main

results of this chapter, which are contained in the rigorous analysis of Sections 4.4, 4.5 and 4.6.

4.3.1 Stochastic Approach

In the following we determine the leading orders and the leading-order coefficients implied by proportional and Morton-Pliska transaction costs, as well as the widths of the associated no-trading regions. To fix ideas, we focus on a simplified model with frozen factor $v = \bar{v}$ and assume as above that $\bar{b} \triangleq \lambda(\bar{v})/\sigma(\bar{v})^2 \in (0, 1)$. Moreover we suppose that $b_0 = \bar{b}$.

First, we simplify the dynamics of the fraction process. Note that by (4.11) and (4.15) the fraction process is mean-reverting towards \bar{b} . Since the optimal fraction process is typically close to \bar{b} , we may ignore the drift term, so the dynamics of $\{b_t\}_{t \geq 0}$ are approximatively given by

$$db_t = \sigma(\bar{v})\bar{b}(1 - \bar{b}) dW_t. \quad (4.20)$$

In particular we expect the leading-order approximation of the no-trading region $\mathcal{B}_q \triangleq (\bar{b} - q, \bar{b} + q)$ to be symmetric around the Merton fraction \bar{b} .⁹

Proportional Transaction Costs. Motivated by the classical verification result of Theorem 4.2, we expect the optimal fraction process to be reflected on the boundary of \mathcal{B}_q . Thus the optimal fraction process $\{b_t\}_{t \geq 0}$ is essentially a reflected Brownian motion in \mathcal{B}_q . By ergodicity, $\{b_t\}_{t \geq 0}$ converges to its stationary distribution on \mathcal{B}_q , which is uniform. This puts us into a position to quantify the effects of transaction costs: The local times of $\{b_t\}_{t \geq 0}$ on the lower and upper boundary of \mathcal{B}_q are given, respectively, by

$$\int_0^t (1 + (\bar{b} - q)\hat{\epsilon}) \frac{dL_s}{Z_s} \quad \text{and} \quad \int_0^t (1 - (\bar{b} + q)\hat{\epsilon}) \frac{dM_s}{Z_s}.$$

⁹ Motivated by Gerhold et al. [2014] and Gerhold et al. [2013], for proportional transaction costs we expect the no-trading region to be non-symmetric in the third order, whereas the second-order coefficient in the expansions of the boundaries of the no-trading region should equal 0 as there is no consumption involved. In the case of Morton-Pliska costs the numerical studies of Morton and Pliska [1995] indicate that the optimal rebalance point differs from the Merton fraction \bar{b} and that the no-trading region is asymmetric around both \bar{b} and the optimal rebalance point. To the best of our knowledge, there are no papers that provide higher-order expansions for the Morton-Pliska case.

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Hence the direct costs due to stock purchases are

$$\begin{aligned}
& \lim_{T \rightarrow \infty} \frac{1}{T} \mathbb{E} \left[\int_0^T \hat{\epsilon} \frac{dL_s}{Z_s} \right] \\
&= \frac{\hat{\epsilon}}{1+(b-q)\hat{\epsilon}} \lim_{T \rightarrow \infty} \frac{1}{T} \mathbb{E} \left[\int_0^T (1 + (\bar{b} - q)\hat{\epsilon}) \frac{dL_s}{Z_s} \right] \\
&= \frac{\hat{\epsilon}}{1+(b-q)\hat{\epsilon}} \lim_{T \rightarrow \infty} \lim_{\delta \downarrow 0} \frac{1}{T} \frac{1}{2\delta} \mathbb{E} \left[\int_0^T \sigma(\bar{v})^2 \bar{b}^2 (1 - \bar{b})^2 \mathbb{I}_{b_s \in [\bar{b}-q, \bar{b}-q+\delta)} ds \right] \\
&= \frac{\hat{\epsilon}}{1+(b-q)\hat{\epsilon}} \lim_{\delta \downarrow 0} \frac{1}{2\delta} \sigma(\bar{v})^2 \bar{b}^2 (1 - \bar{b})^2 \frac{1}{2q} \int_{\mathcal{B}_q} \mathbb{I}_{b \in [\bar{b}-q, \bar{b}-q+\delta)} db \\
&= \frac{\hat{\epsilon}}{1+(b-q)\hat{\epsilon}} \frac{1}{4q} \sigma(\bar{v})^2 \bar{b}^2 (1 - \bar{b})^2.
\end{aligned}$$

Analogously we obtain for the direct costs of asset sales

$$\lim_{T \rightarrow \infty} \frac{1}{T} \mathbb{E} \left[\int_0^T \hat{\epsilon} \frac{dM_s}{Z_s} \right] = \frac{\hat{\epsilon}}{1-(b+q)\hat{\epsilon}} \frac{1}{4q} \sigma(\bar{v})^2 \bar{b}^2 (1 - \bar{b})^2.$$

The indirect costs can be calculated using the fact that reflected Brownian motion is ergodic with a uniform stationary distribution. We obtain

$$\lim_{T \rightarrow \infty} \frac{1}{T} \mathbb{E} \left[\int_0^T f(\bar{v}, b_s) ds \right] = f(\bar{v}, \bar{b}) - \frac{1}{2q} \int_{\mathcal{B}_q} \frac{1}{2} \sigma(\bar{v})^2 (b - \bar{b})^2 db = R^0 - \sigma(\bar{v})^2 \frac{1}{6} q^2$$

where f is defined in (4.7). In total, the long-term growth rate corresponding to the no-trading region \mathcal{B}_q is given by

$$R^0 - \sigma(\bar{v})^2 \left(\frac{1}{6} q^2 + \frac{\hat{\epsilon} D}{q} \right) + o(\hat{\epsilon}) \quad \text{where} \quad D \triangleq \frac{1}{2} \bar{b}^2 (1 - \bar{b})^2. \quad (4.21)$$

Thus the optimal half-width q of the no-trading region is the solution of

$$\min_q \left[\frac{1}{6} q^2 + \frac{\hat{\epsilon} D}{q} \right] \quad (4.22)$$

and the optimal strategy is given by

$$\hat{q} = \hat{\epsilon}^{1/3} (3D)^{1/3}. \quad (4.23)$$

The implied optimal long-term growth rate is given by

$$\hat{R}^{\hat{\epsilon}} = R^0 + \hat{\epsilon}^{2/3} \left(-\frac{1}{2} \sigma(\bar{v})^2 (3D)^{2/3} \right) + o(\hat{\epsilon}^{2/3}). \quad (4.24)$$

This is in line with the insights of Kallsen and Muhle-Karbe [2014] who point out that, for proportional transaction costs, in general 1/3 of the leading-order coefficient results from indirect costs, while the remaining 2/3 are due to direct trading costs. The expansion (4.24) and the width of the optimal no-trading region \hat{q} are also consistent with the findings in [Gerhold et al., 2013, Sections 6.2 and 6.3], after a suitable rescaling of the cost parameter.

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Remark. Note that (4.22) can be rewritten equivalently as

$$\min_q \left[\frac{1}{2} \text{Var}^u(q) + \frac{D}{3} \frac{\hat{\epsilon}q}{\text{Var}^u(q)} \right] \quad (4.25)$$

where $\text{Var}^u(q)$ denotes the variance of the uniform distribution on $[-q, q]$. The term $\hat{\epsilon}q$ represents the cost of trading to the frictionless optimizer.

Morton-Pliska Costs. Similarly as for proportional costs, the classical verification result in Theorem 4.5 yields the following candidate for a leading-order optimal trading strategy: Whenever b hits the boundary of the no-trading region \mathcal{B}_q , the investor shifts her fraction process back to the frictionless optimizer \bar{b} . Strategies of this type will be called *Morton-Pliska strategies*. The stationary distribution of the fraction process $\{b_t\}_{t \geq 0}$ corresponding to the Morton-Pliska strategy on \mathcal{B}_q is therefore given by the triangular distribution with density

$$d(b) \triangleq \frac{1}{q^2} \begin{cases} -b + (\bar{b} + q), & b \in (\bar{b}, \bar{b} + q) \\ b - (\bar{b} - q), & b \in (\bar{b} - q, \bar{b}). \end{cases} \quad (4.26)$$

Thus we have

$$\begin{aligned} \lim_{T \rightarrow \infty} \frac{1}{T} \mathbb{E} \left[\int_0^T f(\bar{v}, b_s) ds \right] &= f(\bar{v}, \bar{b}) - \int_{\mathcal{B}_q} \frac{1}{2} \sigma(\bar{v})^2 (b - \bar{b})^2 d(b) db \\ &= R^0 - \frac{\sigma(\bar{v})^2}{12} q^2. \end{aligned} \quad (4.27)$$

To quantify the direct trading costs, we consider the sequence of exit times $\{\tau_n\}_{n \geq 1}$ of the fraction process from the interval \mathcal{B}_q and define $\sigma_i \triangleq \tau_i - \tau_{i-1}$, $i \geq 1$ (here $\tau_0 \triangleq 0$). Then $\{\sigma_i\}_{i \geq 1}$ are i.i.d. and hence

$$\frac{1}{\#(T)} \tau_{\#(T)} = \frac{1}{\#(T)} \sum_{i=1}^{\#(T)} \sigma_i \rightarrow \mathbb{E}[\sigma_1] = \frac{q^2}{\sigma(\bar{v})^2 \bar{b}^2 (1-\bar{b})^2} \quad \text{and} \quad \frac{\tau_{\#(T)}}{T} \rightarrow 1$$

almost surely as $T \rightarrow \infty$. Here $\#(T)$ denotes the number of trades before time T . Therefore

$$\liminf_{T \rightarrow \infty} \frac{\mathbb{E}[\#(T)]}{T} = \frac{\sigma(\bar{v})^2 \bar{b}^2 (1-\bar{b})^2}{q^2}. \quad (4.28)$$

Hence, after combining (4.27) and (4.28), we obtain

$$\begin{aligned} 0 \leq R^0 - \check{R}^\epsilon &\leq \frac{\sigma(\bar{v})^2}{12} q^2 - \ln(1 - \epsilon) \frac{\sigma(\bar{v})^2 \bar{b}^2 (1-\bar{b})^2}{q^2} \\ &= \sigma(\bar{v})^2 \left(\frac{1}{12} q^2 + \check{\epsilon} \frac{2D}{q^2} \right) + o(\check{\epsilon}). \end{aligned} \quad (4.29)$$

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Thus the appropriate scaling and width for the no-trading region under Morton-Pliska costs are determined by the minimization problem

$$\min_q \left[\frac{1}{12} q^2 + \check{\epsilon} \frac{2D}{q^2} \right]. \quad (4.30)$$

The optimal half-width of the no-trading region is

$$\check{q} \triangleq \check{\epsilon}^{1/4} (24D)^{1/4}$$

and by (4.29) the corresponding optimal long-term growth rate is given by

$$\check{R}^\epsilon = R^0 + \check{\epsilon}^{1/2} \left(-\frac{1}{6} \sigma(\bar{v})^2 (24D)^{1/2} \right) + o(\check{\epsilon}^{1/2}).$$

This expansion and the width of the no-trading region \check{q} coincide with those obtained by Atkinson and Wilmott [1995]. In contrast to the case of proportional costs, with Morton-Pliska costs direct and indirect costs contribute equally to the leading-order coefficient.

Remark. *Similarly as above, we can represent the minimization problem (4.30) via*

$$\min_q \left[\frac{1}{2} \text{Var}^{tr}(q) + \frac{D}{3} \frac{\check{\epsilon}}{\text{Var}^{tr}(q)} \right]. \quad (4.31)$$

Here $\text{Var}^{tr}(q)$ is the variance of the triangular distribution on $[-q, q]$. The numerator in the second summand again represents the cost of trading to the Merton fraction.

4.3.2 PDE Approach and Leading-Order VIs

The rigorous analysis of Sections 4.4, 4.5 and 4.6 will be based on the leading-order variational inequalities (VIs) corresponding to (4.13) and (4.16). In this section we use a PDE approach to heuristically derive these leading-order VIs. For this purpose consider the following formal expansions and rescalings:

$$\begin{aligned} \phi^\epsilon(v, b) &= \epsilon^\lambda \psi(v, \frac{b-b^*(v)}{\epsilon^\alpha}) + o(\epsilon^\lambda) \\ R_{loc}^\epsilon(v) &= R_{loc}^0(v) + \epsilon^\gamma Q_{loc}(v) + o(\epsilon^\gamma) \quad \text{and} \quad \xi \triangleq \frac{b-b^*(v)}{\epsilon^\alpha} \end{aligned} \quad (4.32)$$

where $(\phi^\epsilon, R^\epsilon)$ is a solution to (4.13) or (4.16), respectively, with cost parameter ϵ . The first parts of the VIs (4.13) and (4.16) are identical and of the form

$$-\mathcal{L}\phi(v, b) + R(v) - f(v, b).$$

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Their expansions with respect to ϵ can be calculated explicitly. For $v \in \mathcal{I}^0$ they read¹⁰

$$\begin{aligned} -\mathcal{L}\phi^\epsilon(v, b) &= \epsilon^{\lambda-2\alpha}\sigma(v)^2 D(v) \psi_{\xi\xi}(v, \xi) + o(\epsilon^{\lambda-2\alpha}) \\ R_{loc}^\epsilon(v) - f(v, b) &= \epsilon^\gamma Q_{loc}(v) + \epsilon^{2\alpha} \frac{1}{2} \sigma(v)^2 \xi^2 + o(\epsilon^\gamma) \end{aligned}$$

where

$$D(v) \triangleq \frac{1}{2} b_v^*(v)^2 \frac{\beta(v)^2}{\sigma(v)^2} - \rho b^*(v)(1 - b^*(v)) b_v^*(v) \frac{\beta(v)}{\sigma(v)} + \frac{1}{2} b^*(v)^2 (1 - b^*(v))^2 \quad (4.33)$$

for $v \in \mathcal{I}$. Note that for the case of a constant factor v (and, therefore, constant b^*) the definitions of D in (4.21) and (4.33) coincide. Further, note that the definition (4.8) of $R_{loc}^0(v)$ implies that the 0th-order coefficient in the expansion of $R_{loc}^\epsilon(v) - f(v, b)$ vanishes. Thus, equating the leading orders in these expansions, we obtain

$$\lambda - 2\alpha = \gamma = 2\alpha. \quad (4.34)$$

These terms represent the indirect losses due to portfolio misallocations, which are of the same order of magnitude for all types of transaction costs.

We next expand the relevant boundary conditions. For proportional costs we obtain

$$\begin{aligned} -\phi_b^\epsilon(v, b)(1 + \epsilon b) + \epsilon &= -\epsilon^{\lambda-\alpha} \psi_\xi(v, \xi) + o(\epsilon^{\lambda-\alpha}) + \epsilon \\ -\phi_b^\epsilon(v, b)(-1 + \epsilon b) + \epsilon &= \epsilon^{\lambda-\alpha} \psi_\xi(v, \xi) + o(\epsilon^{\lambda-\alpha}) + \epsilon \end{aligned}$$

and therefore $\lambda - \alpha = 1$. In the case of Morton-Pliska costs the boundary condition becomes

$$0 \leq \epsilon^\lambda \psi(v, \xi) \leq \epsilon + o(\epsilon)$$

and we obtain $\lambda = 1$. Combining these results with (4.34) we obtain the same orders that were identified with the stochastic arguments of the preceding subsection: For proportional transaction costs,

$$\lambda = \frac{4}{3}, \quad \alpha = \frac{1}{3} \quad \text{and} \quad \gamma = \frac{2}{3}$$

and for Morton-Pliska costs,

$$\lambda = 1, \quad \alpha = \frac{1}{4} \quad \text{and} \quad \gamma = \frac{1}{2}.$$

¹⁰ Note that in the following heuristic arguments we follow a widespread convention and presume implicitly that all derivatives of a function of order ϵ^λ are of order ϵ^λ , too.

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More importantly, at the same time the above arguments allow us to determine the variational inequalities for the leading-order coefficients ψ and Q_{loc} in the expansions (4.32). Following our notational convention, we denote these coefficients by $\hat{\psi}$ and \hat{Q}_{loc} in a setting with proportional transaction costs, and by \check{Q}_{loc} and $\check{\psi}$ in the case with Morton-Pliska transaction costs. Let us define the leading-order variational inequalities (VIs) for these coefficients:

Definition 4.7. *The leading-order VIs under proportional costs are given by*

$$\begin{aligned} \min\{-\sigma(v)^2 D(v) \hat{\psi}_{\xi\xi}(v, \xi) + \hat{Q}_{loc}(v) + \frac{1}{2}\sigma(v)^2 \xi^2, \\ -\hat{\psi}_{\xi}(v, \xi) + 1, \hat{\psi}_{\xi}(v, \xi) + 1\} = 0, \quad v \in \mathcal{I}. \end{aligned} \quad (4.35)$$

Definition 4.8. *The leading-order VIs under Morton-Pliska costs are*

$$\begin{aligned} \min\{-\sigma(v)^2 D(v) \check{\psi}_{\xi\xi}(v, \xi) + \check{Q}_{loc}(v) + \frac{1}{2}\sigma(v)^2 \xi^2, \check{\psi}(v, \xi)\} = 0, \\ \sup_{\xi \in \mathbb{R}} \check{\psi}(v, \xi) - 1 = 0, \quad v \in \mathcal{I}. \end{aligned} \quad (4.36)$$

The leading-order VIs (4.35) and (4.36) can be derived formally by equating the leading-order terms in the asymptotic expansions of (4.13) and (4.16), respectively. Note that both (4.35) and (4.36) can be solved for each fixed $v \in \mathcal{I}$ separately.

Remark. *In the terminology of homogenization theory, the variable ξ in (4.32) is the “fast variable”, and (4.35), (4.36) are called corrector equations (see, e.g., Soner and Touzi [2013] and the references therein).*

Before we proceed, let us state a final assumption on the coefficients of the dynamics (4.1), (4.2), and (4.3):

(A4) *Let the function $D : \mathcal{I} \rightarrow \mathbb{R}$ be defined as in (4.33). We assume that*

$$D, 1/D, D_v \text{ and } D_{vv} \text{ are polynomially bounded on } \mathcal{I}^0.$$

By a suitable choice of the function b^* , we can further ensure that D is bounded and strictly positive on $\mathcal{I} \setminus \mathcal{I}^0$. This will always be assumed in the following.

Assumption (A4) is satisfied in both our benchmark applications:

Example 1: Heston Model. *We have $\mathcal{I}^0 = [\lambda, \infty)$ and*

$$D(v) = \frac{(\mu-r)^2}{2v^4} (v^2 - 2v(\mu - r - \beta\rho) + (\mu - r)^2 + \beta^2 - 2\rho\beta(\mu - r)), \quad v \geq \lambda.$$

D, D_v and D_{vv} are polynomially bounded on $[\lambda, \infty)$. Polynomial boundedness of $1/D$ on \mathcal{I}^0 follows from the fact that the numerator of D is bounded away from 0.

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Example 2: Kim-Omberg Model. Here $\mathcal{I}^0 = [0, \sigma^2]$ and D is given by

$$D(v) = \frac{1}{2} \frac{\beta^2}{\sigma^6} + \frac{1}{2} \frac{1}{\sigma^8} v^2 (\sigma^2 - v)^2 - \rho \frac{\beta}{\sigma^7} v (\sigma^2 - v).$$

Hence D , D_v and D_{vv} are polynomially bounded and D is bounded away from 0.

Standing Assumption. In what follows we impose assumptions (A1)–(A4) without any further mentioning.

4.4 Asymptotic Optimality for Proportional Costs

In this section and the next, we substantiate the heuristic arguments of Section 4.3 by a rigorous mathematical analysis. This section focuses on proportional transaction costs. More precisely, we determine the exact asymptotic expansion of the optimal long-term growth rate and identify a leading-order optimal strategy. This will be achieved with the help of the leading-order VI (4.35). For the reader's convenience, we give detailed discussions of all relevant proofs and ideas, whereas technical proofs are delegated to the Appendix.

Before we state the main result of this section, define $\hat{\psi} \in C^2(\mathcal{I} \times \mathbb{R})$ via

$$\hat{\psi}(v, \xi) \triangleq \begin{cases} \xi, & \xi < -(3D(v))^{1/3} \\ -\xi, & \xi > (3D(v))^{1/3} \\ \frac{1}{24} \frac{1}{D(v)} \xi^4 - \frac{3}{4} \frac{1}{(3D(v))^{1/3}} \xi^2 - \frac{3}{8} (3D(v))^{1/3}, & \text{otherwise} \end{cases} \quad (4.37)$$

and set

$$\hat{Q}_{loc}(v) \triangleq -\frac{1}{2} \sigma(v)^2 (3D(v))^{2/3}, \quad v \in \mathcal{I}. \quad (4.38)$$

It is not difficult to verify that $\hat{\psi}$ and \hat{Q}_{loc} solve the leading-order VI (4.35). We are now in a position to state the first main result of this chapter:

Theorem 4.9 (Asymptotic Optimality for Proportional Transaction Costs). *Let R^0 be the long-term growth rate in the frictionless market, see (4.9), and define*

$$\hat{Q} \triangleq \mathbb{E}[\hat{Q}_{loc}(v_\infty) \mathbb{1}_{v_\infty \in \mathcal{I}^0}].$$

Then there exist $\hat{\epsilon}_0 > 0$ and constants C_1, C_2 such that for every $\hat{\epsilon} < \hat{\epsilon}_0$

$$\hat{R}^{\hat{\epsilon}} \leq R^0 + \hat{\epsilon}^{2/3} \hat{Q} + \hat{\epsilon} C_1 \quad (4.39)$$

and

$$\hat{R}^{\hat{\epsilon}} \geq R^0 + \hat{\epsilon}^{2/3} \hat{Q} - \hat{\epsilon} C_2. \quad (4.40)$$

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The idea of the proof is inspired by classical dynamic programming methodology: We apply Itô's formula to the state process corresponding to an arbitrary trading strategy to obtain the upper bound (4.39); and to the trading strategy that is determined by the no-trading region implied by $\hat{\psi}$ as defined in (4.37) to establish the lower bound (4.40). In contrast to the standard case, however, here the exact form of the value function is not known. We therefore have to rely on the solutions of the leading-order VIs to prove Theorem 4.9. This means in particular that we have to establish suitable bounds for all relevant error terms. More precisely, we use different auxiliary functions to establish the upper and lower bounds, and we separately estimate the higher-order terms that have been neglected in the heuristic derivations of the leading-order VIs.

In the following we provide a discussion of the corresponding leading-order optimal strategies and the auxiliary results needed in the proof of Theorem 4.9. The proofs of these results, and a complete proof of Theorem 4.9, are delegated to Appendices 4.B.4 and 4.B.5.

Approximate Value Functions. As an exact solution to (4.13) is not available, our analysis is based on auxiliary functions that can be used to establish the leading-order estimates (4.39) and (4.40) for the long-term growth rate. We construct these functions as follows:

$$\hat{\Psi}_{\pm}^{\hat{\epsilon}}(v, b) \triangleq \frac{\hat{\epsilon}^{4/3}}{1 \pm \hat{\epsilon}} \hat{\psi}\left(v, \frac{b - b^*(v)}{\hat{\epsilon}^{1/3}}\right), \quad v \in \mathcal{I}, \quad b \in [0, 1].$$

$\hat{\Psi}_{+}^{\hat{\epsilon}}$ will be used to prove the upper bound (4.39), and $\hat{\Psi}_{-}^{\hat{\epsilon}}$ yields the lower bound (4.40). Note that by construction

$$\hat{\Psi}_{\pm}^{\hat{\epsilon}} \in C^2(\mathcal{I} \times [0, 1]), \quad \hat{\Psi}_{\pm}^{\hat{\epsilon}} \text{ is bounded,} \quad \left| \frac{\partial \hat{\Psi}_{\pm}^{\hat{\epsilon}}}{\partial b} \right| \leq \frac{\hat{\epsilon}}{1 \pm \hat{\epsilon}}. \quad (4.41)$$

Upper Bound. In the following we outline the proof of the upper bound (4.39). Thus let (L, M) be an arbitrary admissible strategy. With $\{(v_t, b_t)\}_{t \geq 0}$ denoting the associated state process, we apply Itô's formula to $\{\hat{\Psi}_{\pm}^{\hat{\epsilon}}(v_t, b_t)\}_{t \geq 0}$ and obtain

$$\begin{aligned} \frac{\mathbb{E} \ln Z_T^S}{T} &= \frac{1}{T} \mathbb{E} \left[\int_0^T f(v_s, b_s) ds - \int_0^T \hat{\epsilon} \frac{dL_s^c}{Z_s} - \int_0^T \hat{\epsilon} \frac{dM_s^c}{Z_s} \right. \\ &\quad \left. + \sum_{s \leq T} (\ln Z_s - \ln Z_{s-}) \mathbb{I}_{s \leq T} \right] \\ &\leq \frac{1}{T} \mathbb{E} \left[\int_0^T R_{loc}^0(v_s) ds + \hat{\epsilon}^{2/3} \int_0^T Q_{loc}^q(v_s) \mathbb{I}_{v_s \in \mathcal{I}^0} ds \right. \\ &\quad \left. + \int_0^T \left(\mathcal{L} \hat{\Psi}_{\pm}^{\hat{\epsilon}}(v_s, b_s) - \hat{\epsilon}^{2/3} \sigma(v_s)^2 D(v_s) \hat{\psi}_{\xi\xi}(v_s, \frac{b_s - b^*(v_s)}{\hat{\epsilon}^{1/3}}) \right) \mathbb{I}_{v_s \in \mathcal{I}^0} ds \right] \end{aligned}$$

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$$\begin{aligned}
& + \hat{\Psi}_{\pm}^{\hat{\epsilon}}(v_0, b_0) - \hat{\Psi}_{\pm}^{\hat{\epsilon}}(v_T, b_T) \\
& + \int_0^T \left((1 + \hat{\epsilon}b_s) \frac{\partial \hat{\Psi}_{\pm}^{\hat{\epsilon}}}{\partial b}(v_s, b_s) - \hat{\epsilon} \right) \frac{dL_s^c}{Z_s} \\
& + \int_0^T \left((-1 + \hat{\epsilon}b_s) \frac{\partial \hat{\Psi}_{\pm}^{\hat{\epsilon}}}{\partial b}(v_s, b_s) - \hat{\epsilon} \right) \frac{dM_s^c}{Z_s} \\
& + \sum_{s \leq T} \left(\hat{\Psi}_{\pm}^{\hat{\epsilon}}(v_s, b_s) - \hat{\Psi}_{\pm}^{\hat{\epsilon}}(v_s, b_{s-}) + \ln Z_s - \ln Z_{s-} \right) \mathbb{I}_{s \leq T} \Big] \quad (4.42)
\end{aligned}$$

where we have used the definitions (4.7) and (4.8) and the leading-order VIs (4.35) for $\hat{\psi}$. Upon letting $T \rightarrow \infty$ in (4.42) we find for the first two summands (*line 3*)

$$\frac{1}{T} \mathbb{E} \left[\int_0^T R_{loc}^0(v_s) ds \right] \rightarrow \mathbb{E}[R_{loc}^0(v_{\infty})] = R^0$$

and

$$\frac{1}{T} \mathbb{E} \left[\int_0^T Q_{loc}^q(v_s) \mathbb{I}_{v_s \in \mathcal{I}^0} ds \right] \rightarrow \mathbb{E}[\hat{Q}_{loc}(v_{\infty}) \mathbb{I}_{v_{\infty} \in \mathcal{I}^0}] = \hat{Q}.$$

As stated above, the proof of the upper bound (4.39) is based on $\hat{\Psi}_{+}^{\hat{\epsilon}}$. Thus, provided the remaining terms in (4.42) for $\hat{\Psi}_{+}^{\hat{\epsilon}}$ are of higher order, we obtain the expansion (4.39) asserted in Theorem 4.9. To show that the quantities in *lines 4-8* of (4.42) are indeed negligible at the leading order, we consider each of them in turn.

Lines 6, 7 and 8 do not contribute to the leading order of the upper bound (4.39) since they are non-positive for $\hat{\Psi}_{+}^{\hat{\epsilon}}$. *Line 5* does not contribute because $\hat{\Psi}_{+}^{\hat{\epsilon}}$ is bounded. Finally, to show that the terms in *line 4* are negligible at the leading order, we need to estimate

$$\begin{aligned}
\mathcal{L} \hat{\Psi}_{+}^{\hat{\epsilon}}(v, b) - \hat{\epsilon}^{2/3} \sigma(v)^2 D(v) \hat{\psi}_{\xi\xi}(v, \frac{b-b^*(v)}{\hat{\epsilon}^{1/3}}) & \quad \text{on } \mathcal{I}^0 \times [0, 1] \\
\mathcal{L} \hat{\Psi}_{+}^{\hat{\epsilon}}(v, b) & \quad \text{on } \mathcal{I} \setminus \mathcal{I}^0 \times [0, 1].
\end{aligned} \quad (4.43)$$

For this purpose we split $\mathcal{I} \setminus \mathcal{I}^0$ into two disjoint sets $\mathcal{J}^{\hat{\epsilon}}$ and $\mathcal{K}^{\hat{\epsilon}}$ defined by

$$\mathcal{J}^{\hat{\epsilon}} \triangleq \{v \in \mathcal{I} \setminus \mathcal{I}^0 : \hat{\Psi}_{\pm}^{\hat{\epsilon}}(v, b) = \frac{\hat{\epsilon}}{1 \pm \hat{\epsilon}} (b - b^*(v)) \text{ for all } b \in [0, 1]\}$$

and

$$\mathcal{K}^{\hat{\epsilon}} \triangleq \mathcal{I} \setminus (\mathcal{I}^0 \cup \mathcal{J}^{\hat{\epsilon}})$$

(see Figures 4.1 and 4.2 for illustration) and use the following auxiliary results:

Lemma 4.10. *There exist constants $\hat{\epsilon}_0 > 0$ and $C > 0$ such that for all $\hat{\epsilon} < \hat{\epsilon}_0$*

$$\lambda(\mathcal{K}^{\hat{\epsilon}}) \leq \hat{\epsilon}^{1/3} C$$

where λ denotes Lebesgue measure.

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Proof. See Appendix 4.B.2. □

Together with assumption (A1) (ii) this immediately yields

Corollary 4.11. *Let $\hat{\epsilon}_0$ be as in Lemma 4.10. Then there exist constants $C > 0$ such that for all $\hat{\epsilon} < \hat{\epsilon}_0$*

$$\mathbb{P}(v_\infty \in \mathcal{K}^{\hat{\epsilon}}) \leq \hat{\epsilon}^{1/3} C.$$

Lemma 4.12. *Let $\hat{\epsilon}_0$ be as in Lemma 4.10. Then there exist constants $C > 0$ and $m \in \mathbb{N}$ such that for all $\hat{\epsilon} < \hat{\epsilon}_0$*

$$\begin{aligned} |\mathcal{L}\hat{\Psi}_\pm^{\hat{\epsilon}}(v, b) - \hat{\epsilon}^{2/3} \sigma(v)^2 D(v) \hat{\psi}_{\xi\xi}(v, \frac{b-b^*(v)}{\hat{\epsilon}^{1/3}})| &\leq \hat{\epsilon} (1 + v^m) C && \text{on } \mathcal{I}^0 \times [0, 1] \\ |\mathcal{L}\hat{\Psi}_\pm^{\hat{\epsilon}}(v, b)| &\leq \hat{\epsilon} (1 + v^m) C && \text{on } \mathcal{J}^{\hat{\epsilon}} \times [0, 1] \\ |\mathcal{L}\hat{\Psi}_\pm^{\hat{\epsilon}}(v, b)| &\leq \hat{\epsilon}^{2/3} C && \text{on } \mathcal{K}^{\hat{\epsilon}} \times [0, 1]. \end{aligned}$$

Proof. See Appendix 4.B.3. □

It follows from Lemmas 4.10 and 4.12 that the terms in *line 4* of (4.42), i.e. in (4.43), do not contribute to the leading order. Put together, the preceding arguments show that only the first two terms in (4.42) are relevant for the long-term growth rate at the leading order. Hence we conclude that the asserted estimate (4.39) holds.

Lower Bound and a Leading-Order Optimal Strategy. To establish the lower bound (4.40) of the optimal long-term growth rate, we explicitly construct a candidate for the leading-order optimal trading strategy. Our candidate is motivated by the solution of the leading-order VIs (4.35): The boundaries of the no-trading regions implied by (4.37) are determined by the functions

$$g_\pm^{\hat{\epsilon}}(v) \triangleq b^*(v) \pm \hat{\epsilon}^{1/3} (3D(v))^{1/3}, \quad v \in \mathcal{I}. \quad (4.44)$$

Note that since $b_t \in [0, 1]$ for every admissible strategy, the no-trading region (4.44) is equivalent to $[(g_-^{\hat{\epsilon}} \vee 0) \wedge 1, (g_+^{\hat{\epsilon}} \vee 0) \wedge 1]$. Note also that the half-width of the no-trading region implied by (4.44) coincides with the half-width (4.23) obtained by heuristic arguments, modulated by the volatility process; and with the half-width obtained in (4.1) in Kallsen and Muhle-Karbe [2013]. The same applies to the corresponding local and optimal long-term growth rates, see (4.39), (4.40) and (4.24).

4.4 Asymptotic Optimality for Proportional Costs

To construct a leading-order optimal strategy, we consider the no-trading region

$$\{(v, b) : v \in \mathcal{I}^0 \text{ and } b \in [g_-^\hat{\epsilon}(v), g_+^\hat{\epsilon}(v)]\} \cup \{v \in \mathcal{I} : b^*(v) \geq 1\} \times \{1\} \\ \cup \{v \in \mathcal{I} : b^*(v) \leq 0\} \times \{0\}$$

with reflection on the boundary for $v \in \mathcal{I}^0$, see Figures 4.1 and 4.2.

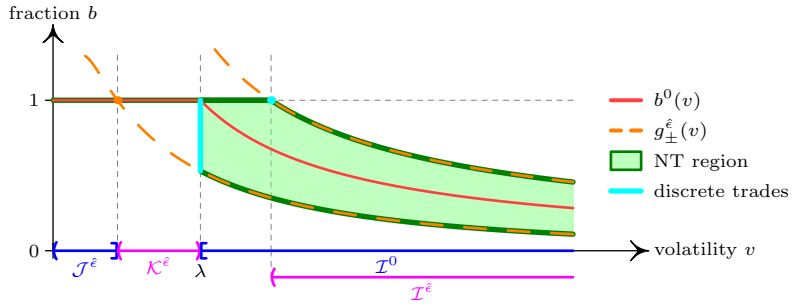


Figure 4.1: Leading-order optimal no-trading region in the Heston model.

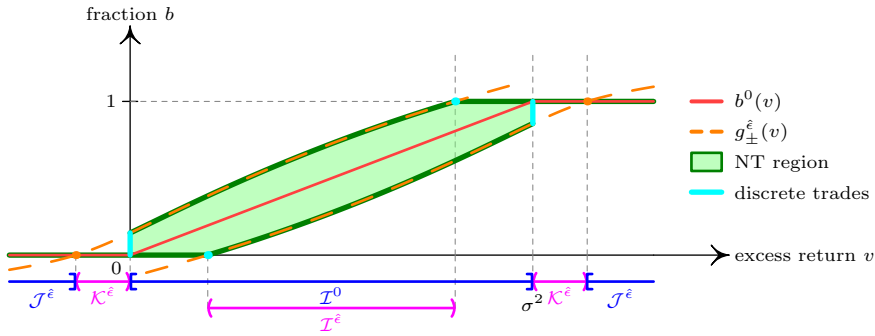


Figure 4.2: Leading-order optimal no-trading region in the Kim-Omberg model.

To avoid reflection on the part of the boundary where the fraction process degenerates, we enforce discrete trades at the stopping times $\{\rho_k\}_{k \geq 1}$, where

$$\rho_{2k+1} \triangleq \inf\{t \geq \rho_{2k} : v_t \in \mathcal{I} \setminus \mathcal{I}^0\} \quad \text{and} \quad \rho_{2k+2} \triangleq \inf\{t \geq \rho_{2k+1} : v_t \in \mathcal{I}^\hat{\epsilon}\}$$

for every $k \geq 0$ with $\rho_0 \triangleq 0$ and $\mathcal{I}^\hat{\epsilon} \triangleq \{v \in \mathcal{I}^0 : g_+^\hat{\epsilon}(v) \in (0, 1) \text{ and } g_-^\hat{\epsilon}(v) \in (0, 1)\} \neq \emptyset$ for $\hat{\epsilon} < \hat{\epsilon}_0$ (see Figures 4.1 and 4.2 for illustration). Then our candidate for the leading-order optimal strategy is defined as follows:

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1. For $t \in [\rho_{2k}, \rho_{2k+1})$ the controlled process $\{(v_t, b_t)\}_{t \geq 0}$ is a diffusion in the no-trading region that is reflected parallel to the b -axis. At ρ_{2k+1} the fraction process is shifted to $b^*(v_{\rho_{2k+1}})$.
2. For $t \in [\rho_{2k+1}, \rho_{2k+2})$ the state process satisfies $(v_t, b_t) = (v_t, 1)$ or $(v_t, b_t) = (v_t, 0)$. As soon as $\{v_t\}_{t \geq \rho_{2k+1}}$ hits $\mathcal{I}^{\hat{\epsilon}}$, the fraction process is shifted to the Merton proportion, i.e. $b_{\rho_{2k+2}} \triangleq b^*(v_{\rho_{2k+2}})$ and we are back in Step 1.

A rigorous construction of this strategy and the associated state process, as well as a proof of its admissibility, can be found in Appendix 4.B.5.

To establish the asserted lower bound, let $\{(v_t, b_t)\}_{t \geq 0}$ denote the corresponding state process. Applying Itô's formula to $\{\hat{\Psi}_-^{\hat{\epsilon}}(v_t, b_t)\}_{t \geq 0}$ yields equality in (4.42). As in the proof of the upper bound, the limit of the first two summands in (4.42) is $R^0 + \hat{\epsilon}^{2/3} \hat{Q}$, as asserted in (4.40). Due to Lemmas 4.10 and 4.12, boundedness of $\hat{\Psi}_-^{\hat{\epsilon}}$ and non-negativity of the terms in *lines 6-7* of (4.42) for $\hat{\Psi}_-^{\hat{\epsilon}}$, none of the terms in *lines 4-7* contributes to the leading order in the lower bound. It remains to estimate the jump terms in *line 8*, which represent the effects of discrete trades. Thus it remains to show that, for our candidate strategy:

Discrete Trading Costs are of Higher Order. Our candidate for the leading-order optimal strategy involves discrete transactions, and these are known not to be optimal in the presence of proportional costs in general. Hence it is not a priori clear that the losses implied by discrete trades are negligible at the leading order. It is shown in Appendix 4.B.5 that the term

$$(\hat{\Psi}_-^{\hat{\epsilon}}(v_{\rho_k}, b_{\rho_k}) - \hat{\Psi}_-^{\hat{\epsilon}}(v_{\rho_k}, b_{\rho_k-}) + \ln Z_{\rho_k} - \ln Z_{\rho_k-})$$

is at most of order $\hat{\epsilon}^{4/3}$. Moreover, Lemmas 4.13 and 4.14 below imply that the expected number of discrete interventions before time T , $\mathbb{E}[\sum_{k=1}^{\infty} \mathbb{1}_{\rho_k \leq T}]$, is of order $\hat{\epsilon}^{-1/3}$. Therefore, the terms in *line 8* of (4.42) are at most of order $\hat{\epsilon}$.

Lemma 4.13. *Let $\hat{\epsilon}_0$ be as in Lemma 4.10. There is a constant $C > 0$ such that for all $\hat{\epsilon} < \hat{\epsilon}_0$*

$$d(\mathcal{I} \setminus \mathcal{I}^0, \mathcal{I}^{\hat{\epsilon}}) \triangleq \inf\{|v_1 - v_2| : v_1 \in \mathcal{I} \setminus \mathcal{I}^0, v_2 \in \mathcal{I}^{\hat{\epsilon}}\} \geq \hat{\epsilon}^{1/3} C.$$

Proof. See Appendix 4.B.2. □

4.4 Asymptotic Optimality for Proportional Costs

Lemma 4.14 (Up- and Downcrossings of Diffusion Processes). *Let A and B be \mathbb{R} -valued progressively measurable processes, and let W be a Brownian motion on the filtered probability space $(\Omega, \mathcal{A}, \mathcal{F}, \mathbb{P})$. Suppose that $\int_0^T B_s^2 ds < \infty$ for every $T > 0$, and define the process ζ by*

$$\zeta_t \triangleq \zeta_0 + \int_0^t A_s ds + \int_0^t B_s dW_s$$

for some arbitrary initial value ζ_0 . Fix $a, b \in \mathbb{R}$ with $a \neq b$ and define $\rho_0 \triangleq 0$,

$$\rho_{2k-1} \triangleq \inf\{t \geq \rho_{2k-2} : \zeta_t = a\} \quad \text{and} \quad \rho_{2k} \triangleq \inf\{t \geq \rho_{2k-1} : \zeta_t = b\}, \quad k \geq 1.$$

Then for all $T > 0$ the number of up- and downcrossings of ζ through $[a, b]$ satisfies

$$\mathbb{E}\left[\sum_{k=2}^{\infty} \mathbb{I}_{\rho_k \leq T}\right] \leq \frac{1}{|b-a|} \left(\mathbb{E}\left[\int_0^T (|A_s| + 4|B_s|^2) ds\right] + 1 \right).$$

Proof. See Appendix 4.B.1. □

In summary, as in the proof of the upper bound, only the first two terms in (4.42) are relevant for the long-term growth rate at the leading order. Hence the candidate strategy constructed above attains the upper bound at the leading order and is therefore leading-order optimal. In particular, the asserted asymptotic expansion holds. This completes our discussion of the proof of Theorem 4.9.

Remark. Finally, we wish to point out that there also exist alternative leading-order optimal strategies. To illustrate, in the Heston model another no-trading region that defines a leading-order optimal strategy is given in Figure 4.3. The

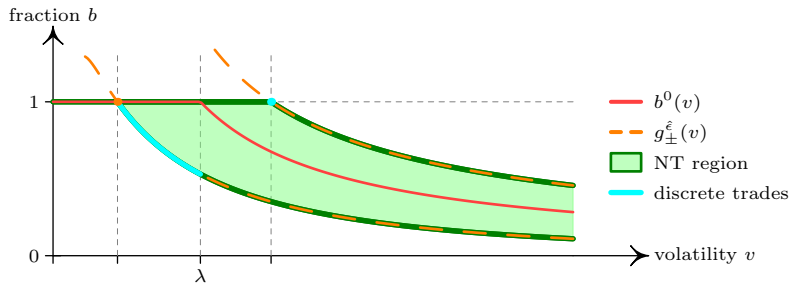


Figure 4.3: Alternative leading-order optimal no-trading region in the Heston Model

trading strategy thus defined dictates discrete trades at the part of the boundary

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where $v \leq \lambda$ and $b < 1$, and at $(\inf \mathcal{I}^{\hat{\epsilon}}, 1)$. The equivalence of this strategy with our candidate for the long-term growth rate criterion follows from Lemmas 4.10 and 4.12: By (4.42) the difference in the performance of the two strategies is bounded above by

$$\lim_{T \rightarrow \infty} \frac{1}{T} \mathbb{E} \left[\int_0^T \hat{\epsilon}^{2/3} C(1 + v_s^m) \mathbb{I}_{v_s \in \mathcal{K}^{\hat{\epsilon}}} ds \right] = \hat{\epsilon}^{2/3} \mathbb{E} \left[C(1 + v_\infty^m) \mathbb{I}_{v_\infty \in \mathcal{K}^{\hat{\epsilon}}} \right]$$

with constants $C > 0$ and $m \in \mathbb{N}$. Hence the long-term growth rates achieved by the two strategies differ only at order $\hat{\epsilon}$.

4.5 Asymptotic Optimality for Morton-Pliska Costs

In the following we carry the results of Section 4.4 over to the setting with Morton-Pliska transaction costs. The arguments that parallel those of Section 4.4 are deliberately kept brief. As a first step, similarly as in Atkinson and Wilmott [1995] we provide a closed-form solution of the leading-order VI (4.36):

$$\check{\psi}(v, \xi) \triangleq \begin{cases} \left(\frac{\xi^2}{\sqrt{24D(v)}} - 1 \right)^2, & v \in \mathcal{I} \quad \text{and} \quad \xi \in (-\sqrt[4]{24D(v)}, \sqrt[4]{24D(v)}) \\ 0, & \text{otherwise} \end{cases}$$

where D is defined by (4.33), and we set

$$\check{Q}_{loc}(v) \triangleq -\frac{1}{6} \sigma(v)^2 \sqrt{24D(v)}, \quad v \in \mathcal{I}. \quad (4.45)$$

Then we have $\check{\psi} \in C^{1,1}(\mathcal{I} \times \mathbb{R})$ and $\check{\psi} \in W_{loc}^{2,p}(\mathcal{I} \times \mathbb{R})$.

Given these definitions, we can state our second main result:

Theorem 4.15 (Asymptotic Optimality with Morton-Pliska Costs). *Let R^0 be the optimal long-term growth rate in the frictionless case as defined in (4.9) and set*

$$\check{Q} \triangleq \mathbb{E} \left[\check{Q}_{loc}(v_\infty) \mathbb{I}_{v_\infty \in \mathcal{I}^0} \right].$$

Then there exist $\check{\epsilon}_0 > 0$ and constants C_1, C_2 such that for every $\check{\epsilon} < \check{\epsilon}_0$

$$\check{R}^{\check{\epsilon}} \leq R^0 + \check{\epsilon}^{1/2} \check{Q} + \check{\epsilon}^{3/4} C_1 \quad (4.46)$$

and

$$\check{R}^{\check{\epsilon}} \geq R^0 + \check{\epsilon}^{1/2} \check{Q} - \check{\epsilon}^{3/4} C_2. \quad (4.47)$$

The proof of this result follows along the same lines as that of Theorem 4.9 and can be found in Appendices 4.C.3 and 4.C.4. In the following we discuss the associated leading-order optimal strategies and state the auxiliary results required in the proof of Theorem 4.15.

4.5 Asymptotic Optimality for Morton-Pliska Costs

Approximate Value Function. In contrast to the case of proportional costs, in the Morton-Pliska setting we are able to prove both the lower and the upper bound in Theorem 4.15 using the same approximation of the value function. This function is defined by

$$\check{\Psi}^\epsilon \triangleq -\ln(1 - \epsilon) \check{\psi}\left(v, \frac{b-b^*(v)}{\epsilon^{1/4}}\right), \quad v \in \mathcal{I}, \quad b \in [0, 1].$$

By construction we have

$$\check{\Psi}^\epsilon \in C^1 \cap W_{loc}^{2,p} \quad \text{on } \mathcal{I} \times [0, 1] \quad \text{and} \quad 0 \leq \check{\Psi}^\epsilon \leq -\ln(1 - \epsilon). \quad (4.48)$$

In particular, by Theorem 8.5 in Bensoussan and Lions [1978] these conditions are sufficient to apply a generalized version of Itô's formula to $\check{\Psi}^\epsilon$.

Upper Bound. As in the proportional costs setting, the leading-order VIs (4.35) and Itô's formula applied to $\{\check{\Psi}^\epsilon(v_t, b_t)\}_{t \geq 0}$ for $\{(v_t, b_t)\}_{t \geq 0}$ defined by an arbitrary admissible trading strategy $\{\tau_k, \pi_k\}_{k \geq 0}$ imply

$$\begin{aligned} \frac{\mathbb{E} \ln Z_T^S}{T} &= \frac{1}{T} \mathbb{E} \left[\int_0^T f(v_s, b_s) ds + \sum_{k=1}^{\infty} \ln(1 - \epsilon) \mathbb{I}_{\tau_k \leq T} \right] \\ &\leq \frac{1}{T} \mathbb{E} \left[\int_0^T R_{loc}^0(v_s) ds + \epsilon^{1/2} \int_0^T \check{Q}_{loc}(v_s) \mathbb{I}_{v_s \in \mathcal{I}^0} ds \right. \\ &\quad + \int_0^T \left(\mathcal{L} \check{\Psi}^\epsilon(v_s, b_s) - \epsilon^{1/2} \sigma(v_s)^2 D(v_s) \check{\psi}_{\xi\xi}(v_s, \frac{b_s-b^*(v_s)}{\epsilon^{1/4}}) \mathbb{I}_{v_s \in \mathcal{I}^0} \right) ds \\ &\quad + \check{\Psi}^\epsilon(v_0, b_0) - \check{\Psi}^\epsilon(v_T, b_T) \\ &\quad \left. + \sum_{k=1}^{\infty} \left(\check{\Psi}^\epsilon(v_{\tau_k}, b_{\tau_k}) - \check{\Psi}^\epsilon(v_{\tau_k-}, b_{\tau_k-}) + \ln(1 - \epsilon) \right) \mathbb{I}_{\tau_k \leq T} \right]. \quad (4.49) \end{aligned}$$

Again, the first two terms in *line 2* yield the asserted leading-order expansion. To estimate the integral in *line 3*, we need bounds for

$$\begin{aligned} \mathcal{L} \check{\Psi}^\epsilon(v, b) - \epsilon^{1/2} \sigma(v)^2 D(v) \check{\psi}_{\xi\xi}(v, \frac{b-b^*(v)}{\epsilon^{1/4}}) &\quad \text{on } \mathcal{I}^0 \times [0, 1] \\ \mathcal{L} \check{\Psi}^\epsilon(v, b) &\quad \text{on } \mathcal{I} \setminus \mathcal{I}^0 \times [0, 1]. \end{aligned}$$

Following the same approach as for proportional transaction costs, we split $\mathcal{I} \setminus \mathcal{I}^0$ into two disjoint sets (see Figures 4.4 and 4.5):

$$\mathcal{J}^\epsilon \triangleq \{v \in \mathcal{I} \setminus \mathcal{I}^0 : \check{\Psi}^\epsilon(v, b) = 0 \text{ for all } b \in [0, 1]\} \quad \text{and} \quad \mathcal{K}^\epsilon \triangleq (\mathcal{I} \setminus \mathcal{I}^0) \setminus \mathcal{J}^\epsilon.$$

The following results show that *line 3* in (4.49) does not contribute to the leading order:

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Lemma 4.16. *There exist $\check{\epsilon}_0 > 0$ and a constant $C > 0$ such that for all $\check{\epsilon} < \check{\epsilon}_0$*

$$\lambda(\mathcal{K}^{\check{\epsilon}}) \leq \check{\epsilon}^{1/4} C.$$

Proof. See Appendix 4.C.1. □

Corollary 4.17. *Let $\check{\epsilon}_0$ be as in Lemma 4.16. Then there exist constants $C > 0$ such that for all $\check{\epsilon} < \check{\epsilon}_0$*

$$\mathbb{P}(v_\infty \in \mathcal{K}^{\check{\epsilon}}) \leq \check{\epsilon}^{1/4} C.$$

Lemma 4.18. *Let $\check{\epsilon}_0$ be as in Lemma 4.16. There are constants $C > 0$ and $m \in \mathbb{N}$ such that for all $\check{\epsilon} < \check{\epsilon}_0$*

$$\begin{aligned} \left| \mathcal{L}\check{\Psi}^{\check{\epsilon}}(v, b) - \check{\epsilon}^{2/3} \sigma(v)^2 D(v) \check{\psi}_{\xi\xi}\left(v, \frac{b-b^*(v)}{\check{\epsilon}^{1/3}}\right) \right| &\leq \check{\epsilon}^{3/4} (1 + v^m) C \quad \text{on } \mathcal{I}^0 \times [0, 1] \\ \left| \mathcal{L}\check{\Psi}^{\check{\epsilon}}(v, b) \right| &\leq \check{\epsilon}^{1/2} C \quad \text{on } \mathcal{K}^{\check{\epsilon}} \times [0, 1]. \end{aligned}$$

Proof. See Appendix 4.C.2. □

Finally, *line 4* in (4.49) has no effect on the leading-order expansion since $\check{\Psi}^{\check{\epsilon}}$ is bounded. The summands in *line 5* are non-positive by (4.48). Hence all terms except those in *line 2* are negligible at the leading order. This concludes our discussion of the proof of the upper bound (4.46).

Lower Bound and a Leading-Order Optimal Strategy. To establish the lower bound (4.47) we define a candidate for a leading-order optimal strategy. This candidate will be defined by the following no-trading region, which is motivated by the VIs (4.36):

$$\begin{aligned} \{(v, b) : v \in \mathcal{I}^0 \text{ and } b \in [g_-^{\check{\epsilon}}(v), g_+^{\check{\epsilon}}(v)]\} \cup \{v \in \mathcal{I} : b^*(v) \geq 1\} \times \{1\} \\ \cup \{v \in \mathcal{I} : b^*(v) \leq 0\} \times \{0\} \end{aligned} \quad (4.50)$$

where

$$g_{\pm}^{\check{\epsilon}}(v) \triangleq b^*(v) \pm \check{\epsilon}^{1/4} (24 D(v))^{1/4}, \quad v \in \mathcal{I}.$$

We refer to Figures 4.4 and 4.5 for graphical illustrations in the Heston and the Kim-Omberg models. The strategy corresponding to the no-trading region (4.50) is then fully specified by the simple rule that, as soon as the fraction process hits the boundary, the investor shifts her portfolio to the optimal frictionless fraction $b^*(v)$. Strategies of this type are Morton-Pliska strategies in the sense of the following definition:

4.5 Asymptotic Optimality for Morton-Pliska Costs

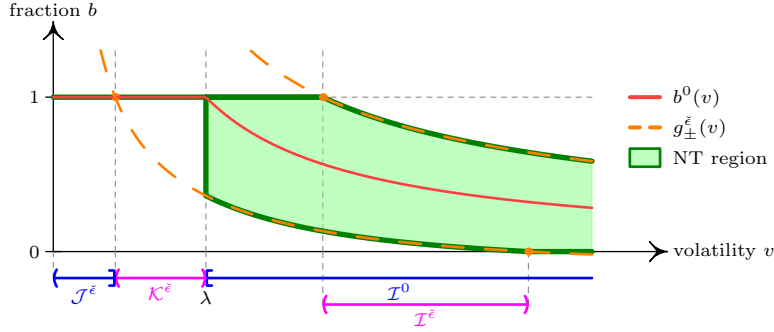


Figure 4.4: Leading-order optimal no-trading region in the Heston model.

Definition 4.19 (Morton-Pliska Strategy). *For an arbitrary strictly positive continuous function $q : \mathcal{I}^0 \rightarrow (0, \infty)$ we define the no-trading region*

$$\begin{aligned} \mathcal{NT}^q \triangleq & \{(v, b) : v \in \mathcal{I}^0 \text{ and } b \in [g_-^q(v), g_+^q(v)]\} \cup \{v \in \mathcal{I} : b^*(v) \geq 1\} \times \{1\} \\ & \cup \{v \in \mathcal{I} : b^*(v) \leq 0\} \times \{0\} \end{aligned}$$

where the boundaries of the no-trading region are given by

$$g_{\pm}^q(v) \triangleq b^*(v) \pm q(v), \quad v \in \mathcal{I}^0.$$

The Morton-Pliska strategy corresponding to \mathcal{NT}^q is formally defined as $\{\tau_k, \pi_k\}_{k \geq 0}$, where

- (a) $\tau_0 \triangleq 0$ and $\pi_0 \triangleq b^*(v_0)$;
- (b) $\tau_{k+1} \triangleq \inf\{t > \tau_k : (v_t, b_t) \notin \mathcal{NT}^q\}$ and $\pi_{k+1} \triangleq b^*(v_{\tau_k})$, $k \geq 0$,

where v is the factor process and b is the diffusion process that starts at time τ_k in π_k and follows the dynamics (4.15).

Note that by definition $\tau_k < \tau_{k+1}$ a.s. on $\{\tau_k < \infty\}$ since the state processes v and b have continuous paths between trading times. Further, if $\{\tau_k\}_{k \geq 0}$ had a finite accumulation point, then by assumption (A1) (iii) entrance and exit times of $\{v_t\}_{t \geq 0}$ into and out of the set $\{(v, b) : b \in \{0, 1\}\}$ would have the same finite accumulation point. This would contradict continuity of $\{v_t\}_{t \geq 0}$, and therefore every Morton-Pliska strategy is admissible.

4 Small-Cost Asymptotics for Growth Rates in Incomplete Markets

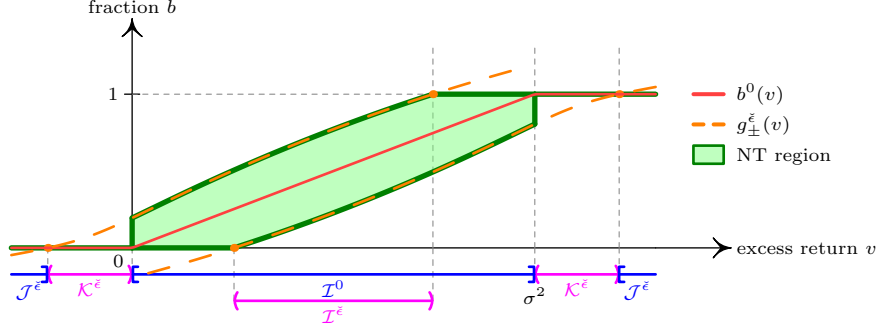


Figure 4.5: Leading-order optimal no-trading region in the Kim-Omberg model.

To prove (4.47) we apply Itô's formula to $\{\check{\Psi}^\epsilon(v_t, b_t)\}_{t \geq 0}$. Here $\{(v_t, b_t)\}_{t \geq 0}$ is the state process corresponding to the Morton-Pliska strategy defined by the no-trading region (4.50). Then by construction (4.49) holds as an equality. Due to boundedness of $\check{\Psi}^\epsilon$ and Lemmas 4.16 and 4.18, the terms in *lines 3-4* in (4.49) do not contribute to the leading order of the lower bound. Let us now consider the sum in *line 5* of (4.49): For the Morton-Pliska strategy constructed above all summands vanish, except those where the factor process v_{τ_k} at time τ_k is in $\partial \mathcal{I}^0$. This case is treated separately and can be dealt with by means of the up- and downcrossing lemma (Lemma 4.14) and the following analog of Lemma 4.13:

Lemma 4.20. *Let $\check{\epsilon}_0$ be as in Lemma 4.16. Then there is a constant $C > 0$ such that for all $\check{\epsilon} < \check{\epsilon}_0$*

$$d(\mathcal{I} \setminus \mathcal{I}^0, \mathcal{I}^\epsilon) \triangleq \inf\{|v_1 - v_2| : v_1 \in \mathcal{I} \setminus \mathcal{I}^0, v_2 \in \mathcal{I}^\epsilon\} \geq \check{\epsilon}^{1/4} C$$

where $\mathcal{I}^\epsilon \triangleq \{v \in \mathcal{I}^0 : g_+^\epsilon(v) \in (0, 1) \text{ and } g_-^\epsilon(v) \in (0, 1)\} \neq \emptyset$.

Proof. See Appendix 4.C.1. □

Hence the number of summands that do not vanish is of order $\frac{1}{\check{\epsilon}^{1/4}}$ and the lower bound (4.47) now follows from the fact that

$$\check{\Psi}^\epsilon(v_{\tau_k}, b_{\tau_k}) - \check{\Psi}^\epsilon(v_{\tau_k-}, b_{\tau_k-}) + \ln(1 - \check{\epsilon}) \geq \ln(1 - \check{\epsilon})$$

where $\ln(1 - \check{\epsilon})$ is of order $\check{\epsilon}$.

4.6 Morton-Pliska Strategies under Proportional Costs

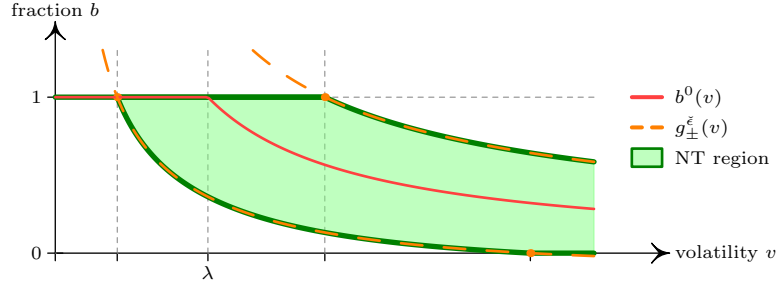


Figure 4.6: Alternative asymptotically optimal no-trading region in the Heston model

Remark. *As in the setting with proportional costs, there exist alternative strategies that are asymptotically optimal. Consider, for instance, the trading strategy given by the no-trading region in Figure 4.6. The difference in long-term growth rates between this strategy and the one constructed in (4.50) is given by*

$$\pm \lim_{T \rightarrow \infty} \frac{1}{T} \mathbb{E} \left[\int_0^T \tilde{\epsilon}^{1/2} C(1 + v_s^m) \mathbb{I}_{v_s \in \mathcal{K}^{\tilde{\epsilon}}} ds \right] = \pm \tilde{\epsilon}^{1/2} \mathbb{E} \left[C(1 + v_{\infty}^m) \mathbb{I}_{v_{\infty} \in \mathcal{K}^{\tilde{\epsilon}}} \right]$$

and hence of higher order by Lemma 4.16.

4.6 Morton-Pliska Strategies under Proportional Costs

So far, we have investigated two types of transaction costs separately: proportional costs and Morton-Pliska costs. For each type of costs we have identified leading-order optimal strategies and derived asymptotic expansions for long-term growth rates; see Sections 4.4 and 4.5. In this section we combine these two threads: We determine the asymptotic expansions of the long-term growth rates achieved by Morton-Pliska strategies under proportional transaction costs, derive an asymptotically optimal Morton-Pliska strategy, and quantify its loss in the long-term growth rate. First, in Section 4.6.1 we adopt an approach similar to that of Section 4.3.1 to calculate the optimal width of the no-trading region heuristically. Section 4.6.2 then provides the corresponding rigorous arguments that justify the heuristic findings.

4.6.1 Heuristics for Optimal Morton-Pliska Strategies

Throughout this subsection we work in the heuristic setting of Section 4.3.1. Thus the volatility process is constant equal to \bar{v} , the corresponding Merton fraction $\bar{b} \in (0, 1)$ and $b_0 = \bar{b}$. As in Section 4.3.1, the candidate for the leading-order optimal no-trading region is $\mathcal{B}_q = (\bar{b} - q, \bar{b} + q)$. Recall further that the intervention times of the Morton-Pliska strategy are given by the exit times of b from the interval \mathcal{B}_q , and that at each such time the investor rebalances her portfolio so that the fraction process is shifted to \bar{b} . Between trading times, the fraction process has the dynamics specified in (4.20),

$$db_t = \sigma(\bar{v}) \bar{b}(1 - \bar{b}) dW_t$$

and its stationary distribution is the triangular law on \mathcal{B}_q , see (4.26). In particular, the losses due to misallocation remain unchanged and are thus given by $\frac{\sigma(\bar{v})^2}{12} q^2$. We next address the direct losses induced by the proportional transaction costs. One easily confirms that

$$\Delta Z_t = -Z_{t-} \frac{\hat{\epsilon} \Delta b_t}{1 + \hat{\epsilon} b_t} \quad \text{for } \Delta b_t > 0 \quad \text{and} \quad \Delta Z_t = Z_{t-} \frac{\hat{\epsilon} \Delta b_t}{1 - \hat{\epsilon} b_t} \quad \text{for } \Delta b_t < 0. \quad (4.51)$$

Thus for the Morton-Pliska strategy we have

$$\ln \frac{Z_t}{Z_{t-}} = \ln \left(1 - \frac{\hat{\epsilon} q}{1 \pm \hat{\epsilon} b} \right) = -\hat{\epsilon} q + o(\hat{\epsilon})$$

and using (4.27) and (4.28) we obtain the following expression for the long-term growth rate:

$$\begin{aligned} \liminf_{T \rightarrow \infty} \frac{1}{T} \mathbb{E} \left[\int_0^T f(\bar{v}, b_s) ds + \sum_{k=1}^{\infty} \ln \frac{Z_{\tau_k}}{Z_{\tau_k-}} \mathbb{I}_{\tau_k \leq T} \right] \\ = R^0 - \frac{\sigma(\bar{v})^2}{12} q^2 - \hat{\epsilon} q \frac{\sigma(\bar{v})^2 \bar{b}^2 (1 - \bar{b})^2}{q^2} + o(\hat{\epsilon}) \\ = R^0 - \sigma(\bar{v})^2 \left(\frac{1}{12} q^2 + \hat{\epsilon} \frac{2D}{q} \right) + o(\hat{\epsilon}). \end{aligned}$$

Hence the optimal half-width of the no-trading region for Morton-Pliska strategies in the market with proportional transaction costs is

$$\arg \min_q \left[\frac{1}{12} q^2 + \hat{\epsilon} \frac{2D}{q} \right] = \hat{\epsilon}^{1/3} (12D)^{1/3} \quad (4.52)$$

and the optimal Morton-Pliska growth rate is given by

$$R^0 + \hat{\epsilon}^{2/3} \left(-\frac{1}{2^{2/3}} \sigma(\bar{v})^2 (3D)^{2/3} \right) + o(\hat{\epsilon}). \quad (4.53)$$

4.6 Morton-Pliska Strategies under Proportional Costs

Similarly to the case with proportional transaction costs, 1/3 of the leading-order coefficient is due to misallocation, whereas the remaining 2/3 are due to direct trading costs.

Remark. Note that, as in Section 4.3, we can equivalently rewrite the minimization problem (4.52) in terms of the variance $\text{Var}^{tr}(q)$ of the triangular distribution via

$$\min_q \left[\frac{1}{2} \text{Var}^{tr}(q) + \frac{D}{3} \frac{\hat{\epsilon}q}{\text{Var}^{tr}(q)} \right].$$

This coincides with (4.22), except for the fact that the underlying distribution is triangular rather than uniform. The above expression differs from (4.31) only in the second summand, which here represents the proportional, rather than Morton-Pliska, costs of trading to the frictionless optimizer.

Remark. A lower bound for the long-term growth rate under proportional costs using Morton-Pliska strategies has previously been identified by Korn [2004]. His analysis differs from ours since he focuses on a pathwise lower bound, based on the fact that suitably chosen Morton-Pliska costs dominate proportional costs. In the notation of this chapter, the lower bound in Korn [2004] is given by

$$R^0 + \hat{\epsilon}^{2/3} \left(-\frac{2}{3} \sigma(\bar{v})^2 (3D)^{2/3} \right) + o(\hat{\epsilon})$$

and corresponds to a no-trading region with half-width $\hat{\epsilon}^{1/3}(24D)^{1/3}$. Note that in contrast to (4.52), the width obtained in Korn [2004] has a typical Morton-Pliska structure: 1/2 of the leading order is due to misallocation, the other 1/2 is due to proportional costs. Our strategy, by contrast, achieves the optimal leading order by using a more specific approximation in terms of Morton-Pliska strategies.

4.6.2 Asymptotically Optimal Morton-Pliska Strategies

We now underpin the heuristic arguments of Section 4.6.1 by a rigorous mathematical analysis. Given any continuous function $q : \mathcal{I} \rightarrow (0, \infty)$ we define the no-trading region

$$\begin{aligned} \mathcal{NT}^{q, \hat{\epsilon}} \triangleq & \{(v, b) : v \in \mathcal{I}^0, b \in [g_-^{q, \hat{\epsilon}}(v), g_+^{q, \hat{\epsilon}}(v)]\} \cup \{v \in \mathcal{I} : b^*(v) \geq 1\} \times \{1\} \\ & \cup \{v \in \mathcal{I} : b^*(v) \leq 0\} \times \{0\} \end{aligned} \quad (4.54)$$

where the boundaries are specified in terms of the functions

$$g_{\pm}^{q, \hat{\epsilon}}(v) \triangleq b^*(v) \pm \hat{\epsilon}^{1/3} q(v), \quad v \in \mathcal{I}.$$

4 Small-Cost Asymptotics for Growth Rates in Incomplete Markets

Let $\{\tau_k, \pi_k\}_{k \geq 0}$ be the Morton-Pliska strategy implied by the no-trading region $\mathcal{NT}^{q, \hat{\epsilon}}$, see Definition 4.19. Further, let $R^{q, \hat{\epsilon}}$ denote the corresponding long-term growth rate. Our goal is to compute the asymptotic expansion of $R^{q, \hat{\epsilon}}$ with respect to the cost parameter $\hat{\epsilon}$, together with the relevant leading-order coefficients, and compare it to the leading-order coefficient \hat{Q} for the asymptotically optimal strategy under proportional costs:

Theorem 4.21 (Performance of Morton-Pliska Strategies under Proportional Costs). *Set*

$$\mathcal{Q} \triangleq \{q : \mathcal{I} \rightarrow (0, \infty) : q \in C^2(\mathcal{I}), q, q_v, q_{vv}, \frac{1}{q} \text{ are polynomially bounded on } \mathcal{I}^0\}$$

and let $q \in \mathcal{Q}$ be arbitrary. Then there exist constants $\hat{\epsilon}_0 > 0$ and $C > 0$ such that for all $\hat{\epsilon} < \hat{\epsilon}_0$ the long-term growth rate $R^{q, \hat{\epsilon}}$ of the Morton-Pliska strategy defined by $\mathcal{NT}^{q, \hat{\epsilon}}$ satisfies

$$|R^{q, \hat{\epsilon}} - R^0 - \hat{\epsilon}^{2/3} \mathbb{E}[Q_{loc}^q(v_\infty) \mathbb{I}_{v_\infty \in \mathcal{I}^0}]| \leq \hat{\epsilon} C$$

where

$$Q_{loc}^q(v) \triangleq -\sigma(v)^2 \left(\frac{q(v)^2}{12} + \frac{2D(v)}{q(v)} \right)$$

and the function D is given by (4.33).

Proof. See Appendix 4.D. □

Note that

$$\max_q Q_{loc}^q(v) = -\frac{1}{2^{2/3}} \sigma(v)^2 (3D(v))^{2/3}$$

and the maximizer is

$$\hat{q}(v) \triangleq (12D(v))^{1/3}.$$

Hence we obtain the following optimality result in the class \mathcal{Q} :

Corollary 4.22 (Asymptotically Optimal Morton-Pliska Strategy). *An asymptotically optimal strategy for proportional transaction costs, in the class of Morton-Pliska strategies defined in terms of no-trading regions with $q \in \mathcal{Q}$, is given by*

$$\hat{q}(v) = (12D(v))^{1/3}.$$

This is wider by a universal, model-independent constant $2^{2/3} \approx 1.59$ than the leading-order optimal no-trading region (see Section 4.4). The associated long-term growth rate satisfies

$$\begin{aligned} R^{\hat{q}, \hat{\epsilon}} &= R^0 + \hat{\epsilon}^{2/3} \left(-\frac{1}{2^{2/3}} \mathbb{E}[\sigma(v_\infty)^2 (3D(v_\infty))^{2/3} \mathbb{I}_{v_\infty \in \mathcal{I}^0}] \right) + o(\hat{\epsilon}^{2/3}) \\ &= R^0 + (\sqrt{2}\hat{\epsilon})^{2/3} \hat{Q} + o(\hat{\epsilon}^{2/3}) \end{aligned}$$

4.6 Morton-Pliska Strategies under Proportional Costs

where \hat{Q} is the leading-order coefficient in the expansion of the long-term growth rate under proportional frictions defined in Theorem 4.9.

In particular, the long-term growth rate achieved by the optimal Morton-Pliska strategy under proportional costs $\hat{\epsilon}$ is the same as the optimal growth rate under increased proportional costs of size $\sqrt{2}\hat{\epsilon}$.

The leading-order loss of the optimal Morton-Pliska strategy, as compared with the leading-order optimal strategy, can thus be calculated in closed form as

$$\hat{\epsilon}^{2/3} \left(\frac{1}{2^{2/3}} - \frac{1}{2} \right) \mathbb{E} \left[\sigma(v_\infty)^2 (3D(v_\infty))^{2/3} \mathbb{1}_{v_\infty \in \mathcal{I}^0} \right].$$

In particular, the optimal Morton-Pliska strategy achieves the same order as the asymptotically optimal strategy, and the relative loss in the leading-order coefficient is $2 \left(\frac{1}{2^{2/3}} - \frac{1}{2} \right) \approx 26.0\%$, independent of $\hat{\epsilon}$ and all market coefficients. To get an idea of the significance of that loss, we illustrate our results in two realistic calibrations of the Heston and the Kim-Omberg model.

4.6.3 Example 1: Heston Model

We adopt the parameter calibration in Example 1 of Andersen et al. [2002]. After rescaling to yearly units (see Table III in Andersen et al. [2002]) we get

$$\begin{aligned} r &= 0.051, & \lambda &= 0.0246, & \theta &= 3.2508, \\ \eta &= 0.0134736, & \beta &= 0.1850, & \rho &= -0.5877. \end{aligned}$$

The optimal growth rate (4.9) and the leading-order coefficient identified in Theorem 4.9 are given by

$$R^0 = 0.0689734, \quad \hat{Q} = -0.0170046.$$

To illustrate the performance of the leading-order optimal Morton-Pliska strategy under proportional costs, Figure 4.7a displays the expansions of both the asymptotically optimal long-term growth rate and that achieved by the optimal Morton-Pliska strategy. It is apparent that for realistic values of the cost parameter, the difference between the two growth rates is small in both absolute and relative terms (e.g., approximately 0.02% of the optimal growth rate 6.82% are lost for transaction costs of 1%). As a second illustration, we compute the implied liquidity premium. Following Constantinides [1986] we define the liquidity premium as the additional excess return that is required to make the investor

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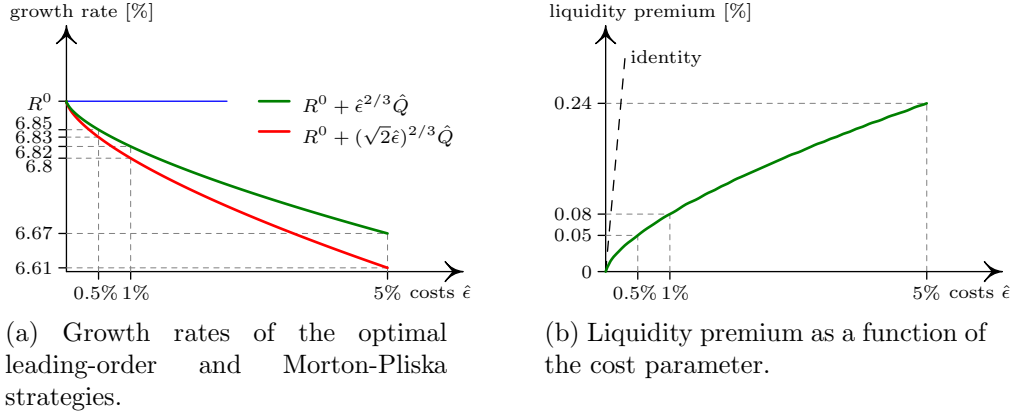


Figure 4.7: Heston model.

indifferent between the market with transaction costs and a hypothetical frictionless, but otherwise identical, reference market. More formally, we ask for the value of $\tilde{\lambda} = \tilde{\lambda}(\epsilon)$ in (4.4) that equates the frictionless growth rate (4.9) to the the long-term growth rate with transaction costs $R^0 + \epsilon^{2/3} \hat{Q}$ derived in Theorem 4.9. The liquidity premium is then given as the difference $\lambda - \tilde{\lambda}(\cdot)$ and is shown in Figure 4.7b. We conclude that the behavior of the liquidity premium is similar to that documented in Constantinides [1986]: For realistic transaction costs, the liquidity premium is an order of magnitude smaller than the cost rate.

4.6.4 Example 2: Kim-Omberg Model

We next consider the Kim-Omberg model of Example 2. We use the following values for the model coefficients, which are derived from Table II in Barberis [2000]:

$$\begin{aligned} r &= 0.0432, & \sigma &= 0.142829, & \beta &= 0.0368496, \\ \theta &= 0.2712, & \eta &= 0.0560018, & \rho &= -0.9351. \end{aligned}$$

Using (4.9) and Theorem 4.9 we obtain

$$R^0 = 0.121944, \quad \hat{Q} = -0.0424355.$$

Figure 4.8a shows the growth rates achieved by the optimal and the optimal Morton-Pliska strategies. As in the Heston model, the difference between them is small for realistic cost parameters (e.g., 0.05% lost growth rate for transaction costs of 1%). Second, Figure 4.8b illustrates the associated liquidity premium

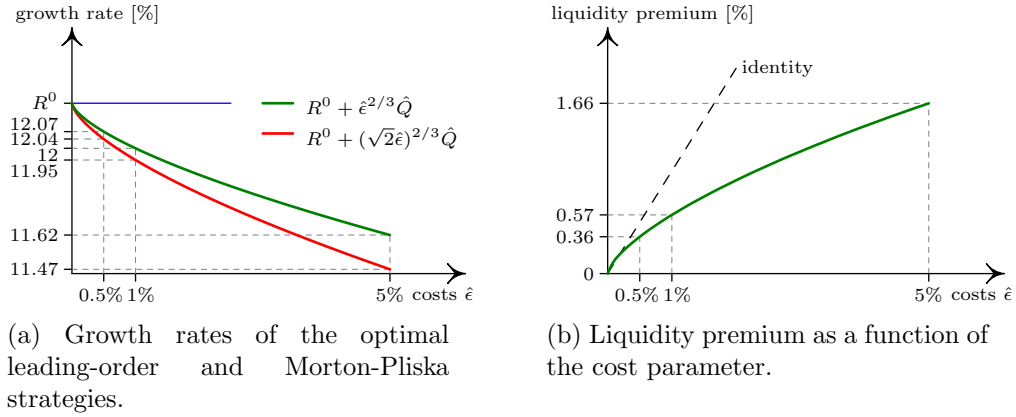


Figure 4.8: Kim-Omberg model.

as a function of the transaction cost parameter. In contrast to our findings in the Heston model, Figure 4.8b demonstrates that the liquidity premium may be sizable in the Kim-Omberg model. This confirms the empirical findings of Lynch and Tan [2011], who show that models with stochastic excess returns are able to produce significant liquidity premia for reasonable cost parameters.

4.7 Pathwise Optimality

In this section we show that the strategies identified in Sections 4.4, 4.5 and 4.6 are not merely optimal for the expected long-term growth rate, but in fact maximize the long-run growth rate *path by path*. This is a well-known fact in the frictionless case, see, e.g., Algoet and Cover [1988] for a discrete-time setting, [Karatzas and Shreve, 1998, Section 3.10] for a continuous-time case and Goll and Kallsen [2003] for a proof in a general semimartingale setting. Related results are also established in the numeraire portfolio theory or benchmark approach as in, e.g., Karatzas and Kardaras [2007] and Platen [2011]. In a Black-Scholes market with proportional transaction costs, Taksar et al. [1988] show that the expectation can be interchanged with the limit inferior in (4.6) for the optimal strategy. Kallsen and Muhle-Karbe [2013] provide a formal shadow price argument to argue that the growth-optimal portfolio is also pathwise optimal. We complement this by a rigorous analysis for proportional and Morton-Pliska costs in the general setting

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of this chapter. Let us denote the (pathwise) long-term growth rate by

$$R \triangleq \sup_S \liminf_{T \rightarrow \infty} \frac{\ln Z_T^S}{T} \quad (4.55)$$

where, as above, Z^S denotes the investor's wealth process. We continue to impose the global assumptions (A1), (A2), (A3) and (A4). In addition, we suppose that the following stronger version of assumption (A1) (i) holds:

- (i)* For every Borel measurable function $h : \mathcal{I} \rightarrow \mathbb{R}$ with $\mathbb{E}[|h(v_\infty)|] < \infty$ we have

$$\lim_{T \rightarrow \infty} \frac{1}{T} \mathbb{E} \left[\int_0^T h(v_s) ds \right] = \mathbb{E}[h(v_\infty)] \quad \text{and} \quad \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T h(v_s) ds = \mathbb{E}[h(v_\infty)].$$

Both the Heston model and the Kim-Omberg model satisfy (i)* by, for instance, the ergodic theorem from [Rogers and Williams, 2000, V.53]. In the following we provide the leading-order expansions of (4.55) and establish pathwise optimality of the strategies constructed in Sections 4.4, 4.5 and 4.6.

First, we focus on proportional costs and demonstrate that the asymptotic expansion in Theorem 4.9 in fact holds path by path. Arguing as in the proof of optimality in Sections 4.2.3 and 4.4, we choose $\hat{\epsilon}_0$ as in Lemma 4.10 and let $\hat{\epsilon} < \hat{\epsilon}_0$ be arbitrary. Then it follows from (4.42), Lemma 4.12 and Itô's formula that the long-term growth rate satisfies

$$\begin{aligned} \frac{\ln Z_T^S}{T} &\leq \frac{1}{T} \left(\int_0^T R_{loc}^0(v_s) ds + \hat{\epsilon}^{2/3} \int_0^T Q_{loc}^q(v_s) \mathbb{I}_{v_s \in \mathcal{I}^0} ds \right. \\ &\quad \pm \hat{\epsilon}^{2/3} \int_0^T C \mathbb{I}_{v_s \in \mathcal{K}^\epsilon} ds \pm \hat{\epsilon} \int_0^T C(1 + v_s^m) ds \\ &\quad \left. + \hat{\Psi}_\pm^\hat{\epsilon}(v_0, b_0) - \hat{\Psi}_\pm^\hat{\epsilon}(v_T, b_T) \right. \\ &\quad \left. + \int_0^T b_s \sigma(v_s) dW_s + \int_0^T \frac{\partial \hat{\Psi}_\pm^\hat{\epsilon}}{\partial v}(v_s, b_s) \beta(v_s) d\bar{W}_s \right. \\ &\quad \left. + \int_0^T \frac{\partial \hat{\Psi}_\pm^\hat{\epsilon}}{\partial b}(v_s, b_s) b_s (1 - b_s) \sigma(v_s) dW_s \right. \\ &\quad \left. + \sum_{s \leq T} \left(\hat{\Psi}_\pm^\hat{\epsilon}(v_s, b_s) - \hat{\Psi}_\pm^\hat{\epsilon}(v_s, b_{s-}) + \ln Z_s - \ln Z_{s-} \right) \right). \end{aligned} \quad (4.56)$$

Here, as in Section 4.4, the upper bound holds for every admissible strategy, whereas the lower bound applies only to the candidate constructed in Section 4.4. As before, the integrals with respect to the control processes L and M are negligible since they are non-positive for $\hat{\Psi}_+^\hat{\epsilon}$ and non-negative for $\hat{\Psi}_-^\hat{\epsilon}$. Thus the only

4.7 Pathwise Optimality

difference between the pointwise estimate (4.56) and its counterpart (4.42) in Section 4.4 are the stochastic integrals in *lines 4 and 5*. To show that the asymptotic expansion of Theorem 4.9 holds pathwise, we consider the terms in each line of (4.56) separately:

The two summands in *line 1* of (4.56) represent the leading-order expansion in Theorem 4.9. The terms in *line 2* do not contribute to the leading order by our ergodicity assumption and Lemma 4.10, and those in *line 3* do not contribute since $\hat{\Psi}_{\pm}^{\hat{\epsilon}}$ is bounded. The stochastic integrals in *lines 4 and 5* have polynomially bounded integrands in v . The following “law of large numbers”-type result therefore implies that they, too, are negligible at the leading order:

Proposition 4.23. *Let W be a Brownian motion on $(\Omega, \mathcal{A}, \mathcal{F}, \mathbb{P})$ and suppose h is an \mathbb{R} -valued progressively measurable process with $\sup_{T>0} \frac{1}{T} \mathbb{E}[\int_0^T h_s^2 ds] < \infty$. Then*

$$\frac{1}{T} \max_{t \in [0, T]} \left| \int_0^t h_s dW_s \right| \rightarrow 0 \quad \text{a.s. as } T \rightarrow \infty.$$

Proof. Fix an arbitrary $\delta > 0$ and define the sets $A_n \triangleq \{\max_{t \in [0, 2^n]} \left| \int_0^t h_s dW_s \right| > 2^n \delta\}$. By Doob’s martingale inequality we have

$$\sum_{n=1}^{\infty} \mathbb{P}(A_n) \leq \frac{1}{\delta^2} \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\mathbb{E}[\int_0^{2^n} h_s^2 ds]}{2^n} < \infty.$$

Thus the Borel-Cantelli lemma implies that $\mathbb{P}(\limsup_n A_n) = 0$, and the assertion follows. \square

Finally, each summand in *line 6* of (4.56), i.e. in the sum

$$\sum_{s \leq T} (\hat{\Psi}_{\pm}^{\hat{\epsilon}}(v_s, b_s) - \hat{\Psi}_{\pm}^{\hat{\epsilon}}(v_s, b_{s-}) + \ln Z_s - \ln Z_{s-}) \quad (4.57)$$

is non-positive. Therefore it suffices to show that these terms do not contribute to the leading order for the lower bound, i.e. when the trader follows the strategy constructed in Section 4.4. For this strategy the sum can be estimated from below via

$$\begin{aligned} & \sum_{k=1}^{\infty} (\hat{\Psi}_{-}^{\hat{\epsilon}}(v_{\rho_k}, b_{\rho_k}) - \hat{\Psi}_{-}^{\hat{\epsilon}}(v_{\rho_k}, b_{\rho_k-}) + \ln Z_{\rho_k} - \ln Z_{\rho_k-}) \mathbb{I}_{\rho_k \leq T} \\ & \geq \sum_{k=1}^{\infty} (\ln Z_{\rho_k} - \ln Z_{\rho_k-}) \mathbb{I}_{\rho_k \leq T} \geq -\hat{\epsilon}^{4/3} C \sum_{k=1}^{\infty} \mathbb{I}_{\rho_k \leq T} \end{aligned}$$

with a constant $C > 0$, see Appendix 4.B.5. Proceeding similarly as in the proof of (4.40) in Appendix 4.B.5, we define the stopping times for every $k \geq 0$

$$\bar{\rho}_{2k+1} \triangleq \inf\{t \geq \bar{\rho}_{2k} : v_t = \sup \mathcal{I}^0\}, \quad \bar{\rho}_{2k+2} \triangleq \inf\{t \geq \bar{\rho}_{2k+1} : v_t = \sup \mathcal{I}^{\hat{\epsilon}}\}$$

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and

$$\rho_{2k+1} \triangleq \inf\{t \geq \rho_{2k} : v_t = \inf \mathcal{I}^0\}, \quad \rho_{2k+2} \triangleq \inf\{t \geq \rho_{2k+1} : v_t = \inf \mathcal{I}^\varepsilon\}$$

where $\rho_0 \triangleq \bar{\rho}_0 \triangleq 0$. Then it is clear that

$$\sum_{k=1}^{\infty} \mathbb{I}_{\rho_k \leq T} = \sum_{k=1}^{\infty} \mathbb{I}_{\bar{\rho}_k \leq T} + \sum_{k=1}^{\infty} \mathbb{I}_{\rho_k \leq T}$$

and the fact that the relevant terms (4.57) are negligible at the leading order follows from Lemma 4.13 and the following pointwise version of the up- and down-crossing lemma:

Lemma 4.24. *Let $\{v_t\}_{t \geq 0}$ solve (4.3) and $a, b \in \mathcal{I}$ with $a \neq b$. If we define $\rho_0 \triangleq 0$ and*

$$\rho_{2k-1} \triangleq \inf\{t \geq \rho_{2k-2} : v_t = a\} \quad \text{and} \quad \rho_{2k} \triangleq \inf\{t \geq \rho_{2k-1} : v_t = b\}$$

for every $k \geq 1$, then

$$\limsup_{T \rightarrow \infty} \frac{1}{T} \sum_{k=1}^{\infty} \mathbb{I}_{\rho_k \leq T} \leq \frac{1}{|b-a|} \mathbb{E}[|\alpha(v_\infty)|] \quad a.s.$$

Proof. Assume without loss that $a < b$. We have

$$\begin{aligned} (b-a) \sum_{k=2}^{\infty} \mathbb{I}_{\rho_k \leq T} &= \sum_{k=2}^{\infty} (-1)^k (v_{\rho_k \wedge T} - v_{\rho_{k-1} \wedge T}) \mathbb{I}_{\rho_k \leq T} \\ &= \sum_{k=2}^{\infty} (-1)^k \left(\int_{\rho_{k-1} \wedge T}^{\rho_k \wedge T} \alpha(v_s) ds + \int_{\rho_{k-1} \wedge T}^{\rho_k \wedge T} \beta(v_s) d\bar{W}_s \right) \mathbb{I}_{\rho_k \leq T} \\ &\leq \int_0^T |\alpha(v_s)| ds + \max_{t \in [0, T]} \left| \int_0^t h_s d\bar{W}_s \right| \end{aligned}$$

where $h_s \triangleq \beta(v_s) (\mathbb{I}_{s \in \cup_{k \geq 1} [\rho_{2k-1}, \rho_{2k})} - \mathbb{I}_{s \in \cup_{k \geq 1} [\rho_{2k}, \rho_{2k+1})})$. Hence the claim follows from the ergodicity property (i)* of the factor process and Proposition 4.23. \square

We have thus established the following pathwise analog of Theorem 4.9:

Theorem 4.25 (Pathwise Asymptotic Optimality for Proportional Costs). *Let R^0 be the optimal long-term growth rate in the frictionless market and set*

$$\hat{Q} \triangleq \mathbb{E}[\hat{Q}_{loc}(v_\infty) \mathbb{I}_{v_\infty \in \mathcal{I}^0}]$$

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where \hat{Q}_{loc} is given by (4.38). Then there exist $\hat{\epsilon}_0 > 0$ and $C > 0$ such that for all $\hat{\epsilon} < \hat{\epsilon}_0$

$$R^0 + \hat{\epsilon}^{2/3} \hat{Q} - \hat{\epsilon} C \leq \sup_{S \in \hat{\mathcal{A}}} \liminf_{T \rightarrow \infty} \frac{\ln Z_T^S}{T} \leq R^0 + \hat{\epsilon}^{2/3} \hat{Q} + \hat{\epsilon} C \quad a.s.$$

The leading-order optimal strategy is the same as the one constructed in Section 4.4.

Using analogous arguments as above, pathwise optimality can also be shown for Morton-Pliska costs. More precisely, Theorems 4.15 and 4.21 can be reformulated on a path-by-path basis in the following way:

Theorem 4.26 (Pathwise Asymptotic Optimality for Morton-Pliska Costs). *Let R^0 be the optimal long-term growth rate in the frictionless market, see (4.9), and set*

$$\check{Q} \triangleq \mathbb{E}[\check{Q}_{loc}(v_\infty) \mathbb{I}_{v_\infty \in \mathcal{I}^0}]$$

where \check{Q}_{loc} is defined in (4.45). Then there are $\check{\epsilon}_0 > 0$ and $C > 0$ such that for every $\check{\epsilon} < \check{\epsilon}_0$

$$R^0 + \check{\epsilon}^{1/2} \check{Q} - \check{\epsilon}^{3/4} C \leq \sup_{S \in \check{\mathcal{A}}} \liminf_{T \rightarrow \infty} \frac{\ln Z_T^S}{T} \leq R^0 + \check{\epsilon}^{1/2} \check{Q} + \check{\epsilon}^{3/4} C \quad a.s.$$

The leading-order optimal strategy is the same as that constructed in Section 4.5.

Theorem 4.27 (Pathwise Performance of Morton-Pliska Strategies with Proportional Costs). *Let $q \in \mathcal{Q}$ where \mathcal{Q} is defined as in Theorem 4.21. There are $\hat{\epsilon}_0 > 0$ and a constant $C > 0$ such that for all $\hat{\epsilon} < \hat{\epsilon}_0$ the long-term growth rate $R^{q, \hat{\epsilon}}$ of the Morton-Pliska strategy defined by $\mathcal{N}\mathcal{T}^{q, \hat{\epsilon}}$ satisfies*

$$\left| \liminf_{T \rightarrow \infty} \frac{\ln Z_T^{S^q}}{T} - R^0 - \hat{\epsilon}^{2/3} \mathbb{E}[Q_{loc}^q(v_\infty) \mathbb{I}_{v_\infty \in \mathcal{I}^0}] \right| \leq \hat{\epsilon} C \quad a.s.$$

where $Q_{loc}^q(v) \triangleq -\sigma(v)^2 \left(\frac{q(v)^2}{12} + \frac{2D(v)}{q(v)} \right)$ and the function D is given by (4.33).

In particular, the Morton-Pliska strategy defined in Corollary 4.22 also maximizes, at the leading order, the long-term growth rate under proportional costs *path by path*.

In summary, the leading-order expansions for optimal expected long-term growth rates established in Sections 4.4, 4.5 and 4.6 continue to hold on a path-by-path basis. This is an important insight that links our analysis in a financial market with an unspanned Markov factor process and transaction costs to classical results on long-term growth rates, and concludes the chapter.

Appendix to Chapter 4

4.A Proofs for Section 4.1: Classical Verification Results

4.A.1 Proof of Theorem 4.2

It suffices to show that for an arbitrary admissible strategy $\mathcal{S} = (L, M)$

$$\liminf_{T \rightarrow \infty} \frac{\mathbb{E} \ln Z_T^{\mathcal{S}}}{T} \leq \lim_{T \rightarrow \infty} \frac{1}{T} \mathbb{E} \left[\int_0^T \hat{R}_{loc}^{\hat{\epsilon}}(v_s) ds \right] = \mathbb{E}[\hat{R}(v_\infty)] \quad (4.58)$$

where the last relation is implied by assumption (A1) (i). Therefore we fix an admissible strategy $\mathcal{S} = (L, M)$ and let b and Z denote the corresponding fraction and wealth processes, respectively. The result is demonstrated as follows: By means of the variational inequalities (4.13) we transform the right hand-side of (4.12) in such a way that (4.58) immediately follows from Itô's formula.

Without loss of generality we assume that $dL_t dM_t = 0$ for $t \in [0, \infty)$ almost surely, i.e. there is no simultaneous transferring of wealth from the cash account into stocks and vice versa.¹¹ Taking this into account, we note that

$$\Delta Z_t = -\frac{\hat{\epsilon} Z_{t-} \Delta b_t}{1 + \hat{\epsilon} b_t} \text{ for } \Delta b_t > 0 \quad \text{and} \quad \Delta Z_t = -\frac{\hat{\epsilon} Z_{t-} \Delta b_t}{-1 + \hat{\epsilon} b_t} \text{ for } \Delta b_t < 0.$$

Then for $\Delta b_t > 0$ by the second inequality in (4.13)

$$\begin{aligned} \ln Z_t - \ln Z_{t-} &= \int_0^1 \frac{\Delta Z_t}{Z_{t-} + u \Delta Z_t} du = \int_0^1 \frac{-\hat{\epsilon} \Delta b_t}{1 + \hat{\epsilon} b_t - u \hat{\epsilon} \Delta b_t} du \\ &\leq \int_0^1 \frac{-\phi_b(v_t, b_t - u \Delta b_t)(1 + \hat{\epsilon} b_t - u \hat{\epsilon} \Delta b_t) \Delta b_t}{1 + \hat{\epsilon} b_t - u \hat{\epsilon} \Delta b_t} du = -\phi(v_t, b_t) + \phi(v_t, b_{t-}). \end{aligned}$$

¹¹For an arbitrary admissible strategy $\mathcal{S} = (L, M)$ we can define a pair of non-decreasing processes (\bar{L}, \bar{M}) via the Jordan decomposition of $(L - M) = \bar{L} - \bar{M}$. Then (\bar{L}, \bar{M}) is an admissible trading strategy and generates in comparison with \mathcal{S} smaller transaction costs, as $\bar{L}_t + \bar{M}_t \leq L_t + M_t$ for all $t \geq 0$. Furthermore, (\bar{L}, \bar{M}) satisfies the desired property: $d\bar{L}_t d\bar{M}_t = 0$ for all $t \geq 0$.

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To obtain the same estimate for the case $\Delta b_t < 0$ we proceed similarly and employ the third inequality in (4.13):

$$\begin{aligned} \ln Z_t - \ln Z_{t-} &= \int_0^1 \frac{\Delta Z_t}{Z_{t-} + u \Delta Z_t} du = \int_0^1 \frac{-\hat{\epsilon} |\Delta b_t|}{1 - \hat{\epsilon} b_t - u \hat{\epsilon} |\Delta b_t|} du \\ &\leq \int_0^1 \frac{-\phi_b(v_t, b_t - u \Delta b_t) (-1 + \hat{\epsilon} b_t - u \hat{\epsilon} \Delta b_t) |\Delta b_t|}{1 - \hat{\epsilon} b_t - u \hat{\epsilon} |\Delta b_t|} du = -\phi(v_t, b_t) + \phi(v_t, b_{t-}). \end{aligned}$$

Thus, applying (4.13) and Itô's formula to the right hand-side of (4.12) we obtain (4.58):

$$\begin{aligned} &\liminf_{T \rightarrow \infty} \frac{\mathbb{E} \ln Z_T}{T} \\ &\leq \liminf_{T \rightarrow \infty} \frac{1}{T} \mathbb{E} \left[\int_0^T \hat{R}_{loc}^{\hat{\epsilon}}(v_s) ds - \int_0^T \mathcal{L} \phi(v_s, b_s) ds - \int_0^T \phi_b(v_s, b_s) (1 + \hat{\epsilon} b_s) \frac{dL_s^c}{Z_s} \right. \\ &\quad \left. - \int_0^T \phi_b(v_s, b_s) (-1 + \hat{\epsilon} b_s) \frac{dM_s^c}{Z_s} - \sum_{0 < s \leq T} (\phi(v_s, b_s) - \phi(v_s, b_{s-})) \right] \\ &= \lim_{T \rightarrow \infty} \frac{1}{T} \mathbb{E} \left[\int_0^T \hat{R}_{loc}^{\hat{\epsilon}}(v_s) ds + \phi(v_0, b_0) - \phi(v_T, b_T) \right] \\ &= \lim_{T \rightarrow \infty} \frac{1}{T} \mathbb{E} \left[\int_0^T \hat{R}_{loc}^{\hat{\epsilon}}(v_s) ds \right]. \end{aligned} \tag{4.59}$$

We conclude the proof by noting that (4.59) is an equality for (\hat{L}, \hat{M}) by conditions (a), (b) and (c) and therefore the supremum in the definition of the long-term growth rate $\hat{R}^{\hat{\epsilon}}$ (4.12) is achieved along (\hat{L}, \hat{M}) . \square

4.A.2 Proof of Theorem 4.5

Step 1: An Upper Bound. Let $\mathcal{S} = \{\tau_k, \pi_k\}_{k \geq 0}$ be an arbitrary admissible strategy and b and Z the corresponding fraction and wealth processes, respectively. Then, by the variational inequalities (4.16) and Itô's formula

$$\begin{aligned} \liminf_{T \rightarrow \infty} \frac{\mathbb{E} \ln Z_T}{T} &= \liminf_{T \rightarrow \infty} \frac{1}{T} \mathbb{E} \left[\int_0^T f(v_s, b_s) ds + \ln(1 - \hat{\epsilon}) \sum_{k=1}^{\infty} \mathbb{I}_{\tau_k \leq T} \right] \\ &\leq \liminf_{T \rightarrow \infty} \frac{1}{T} \mathbb{E} \left[\int_0^T \check{R}(v_s) ds - \int_0^T \mathcal{L} \check{\phi}(v_s, b_s) ds \right. \\ &\quad \left. - \sum_{k=1}^{\infty} (\check{\phi}(v_{\tau_k}, b_{\tau_k}) - \check{\phi}(v_{\tau_k}, b_{\tau_k-})) \mathbb{I}_{\tau_k \leq T} \right] \\ &= \lim_{T \rightarrow \infty} \frac{1}{T} \mathbb{E} \left[\int_0^T \check{R}(v_s) ds + \check{\phi}(v_0, b_0) - \check{\phi}(v_T, b_T) \right] \\ &= \mathbb{E}[\check{R}(v_{\infty})] \end{aligned} \tag{4.60}$$

4.A Proofs for Section 4.1: Classical Verification Results

where the last equality is due to boundedness of $\check{\phi}$ and assumption (A1) (i). Thus, as the strategy \mathcal{S} was chosen arbitrarily, we infer that

$$\check{R}^\epsilon \leq \mathbb{E}[\check{R}(v_\infty)].$$

Step 2: Exact Upper Bound. Let us first prove that the strategy $\check{\mathcal{S}} = \{\check{\tau}_k, \check{\pi}_k\}_{k \geq 0}$ is admissible. Note that by Corollary 4 in Schael [1974] there exists a measurable function $\check{\Xi} : \mathcal{I} \rightarrow [0, 1]$ that returns a maximizer of $\sup_{b \in [0, 1]} \check{\phi}(v, b)$ for every $v \in \mathcal{I}$. Thus, $\check{\pi}_k = \check{\Xi}(v_{\check{\tau}_k}) \mathbb{1}_{\check{\tau}_k < \infty}$ is a $[0, 1]$ -valued and $\mathcal{F}_{\check{\tau}_k}$ -measurable random variable. Furthermore, $(v, \check{\Xi}(v)) \in \text{int}(\mathcal{NT})$ for every $v \in \mathcal{I}$, as by (4.16) $\sup_{b \in [0, 1]} \check{\phi}(v, b) = -\ln(1 - \epsilon) > 0$. Therefore, by continuity, $\check{\tau}_k < \check{\tau}_{k+1}$ a.s. for every $k \geq 1$ and the fraction process \check{b} controlled by $\{\check{\tau}_k, \check{\pi}_k\}_{k \geq 0}$ is well-defined by (4.15) on $[0, \check{\tau}_\infty)$ with $\check{\tau}_\infty \triangleq \lim_{k \rightarrow \infty} \check{\tau}_k$.

Now, it remains to verify that $\check{\tau}_\infty = +\infty$ almost surely. We prove this by contradiction analogously to Lemma 4.1 in Bielecki and Pliska [2000]. First, by Itô's formula for every $n \geq 1$ on $\{\tau_n < \infty\}$ from the variational inequalities (4.16) we deduce that

$$\begin{aligned} & \ln Z_{\check{\tau}_n}^{\check{\mathcal{S}}} - \ln Z_0 \\ &= \int_0^{\check{\tau}_n} f(v_s, \check{b}_s) ds + \int_0^{\check{\tau}_n} \check{b}_s \sigma(v_s) dW_s + n \ln(1 - \epsilon) \\ &= \int_0^{\check{\tau}_n} \check{R}(v_s) ds + \int_0^{\check{\tau}_n} \check{b}_s \sigma(v_s) dW_s - \int_0^{\check{\tau}_n} \mathcal{L}\check{\phi}(v_s, \check{b}_s) ds + n \ln(1 - \epsilon) \quad (4.61) \\ &= \int_0^{\check{\tau}_n} \check{R}(v_s) ds + \int_0^{\check{\tau}_n} \check{b}_s \sigma(v_s) dW_s + \check{\phi}(v_0, \check{b}_0) - \check{\phi}(v_{\check{\tau}_n}, \check{b}_{\check{\tau}_n}) \\ & \quad + \int_0^{\check{\tau}_n} \check{\phi}_v(v_s, \check{b}_s) \beta(v_s) d\bar{W}_s + \int_0^{\check{\tau}_n} \check{\phi}_b(v_s, \check{b}_s) \check{b}_s (1 - \check{b}_s) \sigma(v_s) dW_s \end{aligned}$$

as by construction $\check{\phi}(v_{\check{\tau}_k}, \check{b}_{\check{\tau}_k}) - \check{\phi}(v_{\check{\tau}_k}, \check{b}_{\check{\tau}_k-}) = -\ln(1 - \epsilon)$ on $\{\check{\tau}_k < \infty\}$ for every $k \geq 1$. Assume that $\check{\tau}_\infty < \infty$ for some $\omega \in \Omega$. Then by letting $n \rightarrow \infty$ in (4.61) for this particular ω we obtain a contradiction as the left hand-side converges to $-\infty$ whereas the right hand-side remains bounded.

Optimality of $\{\check{\tau}_k, \check{\pi}_k\}_{k \geq 0}$ follows from the fact, that by an appropriate modification of (4.61) we verify that (4.60) is an equality for $\check{\mathcal{S}}$. \square

4.B Proofs for Section 4.4

Let us first introduce some notation that is used throughout Appendices 4.B, 4.C and 4.D:

$$\underline{v} \triangleq \inf \mathcal{I}, \quad \underline{v}^0 \triangleq \inf \mathcal{I}^0, \quad \bar{v}^0 \triangleq \sup \mathcal{I}^0 \quad \text{and} \quad \bar{v} \triangleq \sup \mathcal{I}$$

where

$$\mathcal{I}^0 \triangleq \{v \in \mathcal{I} : b^*(v) \in [0, 1]\}.$$

As in the main text, see assumption (A1), v_∞ denotes a random variable with the stationary distribution of the factor process v .

4.B.1 Proof of Lemma 4.14: Up- and Downcrossings for Diffusions

Assume without loss that $a < b$. Then

$$\begin{aligned} (b - a) \sum_{k=2}^{\infty} \mathbb{I}_{\rho_k \leq T} &= \sum_{k=2}^{\infty} (-1)^k (\zeta_{\rho_k \wedge T} - \zeta_{\rho_{k-1} \wedge T}) \mathbb{I}_{\rho_k \leq T} \\ &= \sum_{k=2}^{\infty} (-1)^k \left(\int_{\rho_{k-1} \wedge T}^{\rho_k \wedge T} A_s \, ds + \int_{\rho_{k-1} \wedge T}^{\rho_k \wedge T} B_s \, dW_s \right) \mathbb{I}_{\rho_k \leq T} \\ &\leq \int_0^T |A_s| \, ds + \max_{t \in [0, T]} \left| \int_0^t h_s \, dW_s \right| \end{aligned}$$

where $h_s \triangleq B_s (\mathbb{I}_{s \in \cup_{k \geq 1} [\rho_{2k-1}, \rho_{2k})} - \mathbb{I}_{s \in \cup_{k \geq 1} [\rho_{2k}, \rho_{2k+1})})$. Hence by Doob's maximal inequality

$$(b - a) \mathbb{E} \left[\sum_{k=2}^{\infty} \mathbb{I}_{\rho_k \leq T} \right] = \mathbb{E} \left[\int_0^T (|A_s| + 4|B_s|^2) \, ds \right] + 1$$

so the assertion follows. \square

4.B.2 Proofs of Lemma 4.10 and Lemma 4.13

First recall the following definitions introduced in Section 4.4 (see Lemmas 4.10, 4.12 and 4.13):

$$\begin{aligned} \mathcal{J}^{\hat{\epsilon}} &\triangleq \{v \in \mathcal{I} \setminus \mathcal{I}^0 : \hat{\Psi}_{\pm}^{\hat{\epsilon}}(v, b) = \frac{\hat{\epsilon}}{1 \pm \hat{\epsilon}} (b - b^*(v)) \text{ for all } b \in [0, 1]\} \\ \mathcal{K}^{\hat{\epsilon}} &\triangleq (\mathcal{I} \setminus \mathcal{I}^0) \setminus \mathcal{J}^{\hat{\epsilon}} \\ \mathcal{I}^{\hat{\epsilon}} &\triangleq \{v \in \mathcal{I}^0 : g_+^{\hat{\epsilon}}(v) \in (0, 1) \text{ and } g_-^{\hat{\epsilon}}(v) \in (0, 1)\} \end{aligned}$$

where $g_{\pm}^{\hat{\epsilon}}(v) \triangleq b^*(v) \pm \hat{\epsilon}^{1/3} (3D(v))^{1/3}$, $v \in \mathcal{I}$.

Recall that b^* was chosen to be monotone. In the following we assume b^* to be non-decreasing; the case of a non-increasing b^* is analogous. We further introduce the following notations:

$$\bar{v}_\pm^\hat{\epsilon} \triangleq \sup\{v \in \mathcal{I} : g_\mp^\hat{\epsilon}(v) \in [0, 1]\} \quad \text{and} \quad \underline{v}_\pm^\hat{\epsilon} \triangleq \inf\{v \in \mathcal{I} : g_\mp^\hat{\epsilon}(v) \in [0, 1]\}.$$

Proposition 4.28. *There exist constants $\hat{\epsilon}_0 > 0$, $C_1 > 0$ and $C_2 > 0$ such that*

(i) *If $\bar{v}^0 < \bar{v}$, then for all $\hat{\epsilon} < \hat{\epsilon}_0$ we have $\underline{v} < \bar{v}_-^\hat{\epsilon} < \bar{v}^0 < \bar{v}_+^\hat{\epsilon} < \bar{v}$ and*

$$\hat{\epsilon}^{1/3}C_1 \leq |\bar{v}^0 - \bar{v}_\pm^\hat{\epsilon}| \leq \hat{\epsilon}^{1/3}C_2.$$

Furthermore, for every $\hat{\epsilon} < \hat{\epsilon}_0$, $g_\pm^\hat{\epsilon}$ is strictly increasing on $[\bar{v}_-^\hat{\epsilon}, \bar{v}_+^\hat{\epsilon}]$ and $g_-^\hat{\epsilon}(\bar{v}_-^\hat{\epsilon}) > 0$.

(ii) *If $\underline{v}^0 > \underline{v}$, then for all $\hat{\epsilon} < \hat{\epsilon}_0$ we have $\underline{v} < \underline{v}_-^\hat{\epsilon} < \underline{v}^0 < \underline{v}_+^\hat{\epsilon} < \bar{v}$ and*

$$\hat{\epsilon}^{1/3}C_1 \leq |\underline{v}^0 - \underline{v}_\pm^\hat{\epsilon}| \leq \hat{\epsilon}^{1/3}C_2.$$

Furthermore, for every $\hat{\epsilon} < \hat{\epsilon}_0$, $g_\pm^\hat{\epsilon}$ is strictly increasing on $[\underline{v}_-^\hat{\epsilon}, \underline{v}_+^\hat{\epsilon}]$ and $g_+^\hat{\epsilon}(\underline{v}_+^\hat{\epsilon}) < 1$.

Proof. We prove (i); (ii) is shown analogously.

Since by assumption $\bar{v}^0 < \bar{v}$ and b^* is continuous, we have $\bar{v}^0 \in \mathbb{R}$ and $b^*(\bar{v}^0) = 1$. There exist $\tilde{v}_+ \in (\bar{v}^0, \bar{v})$ and $\tilde{v}_- \in (\underline{v}^0, \bar{v}^0)$ such that $b_v^* > 0$ on $[\tilde{v}_-, \tilde{v}_+]$. Then by regularity of D and D_v (see assumption (A4) and the following discussion) there exist an $\hat{\epsilon}_0 > 0$ and two constants $c_1, c_2 > 0$ such that for all $\hat{\epsilon} < \hat{\epsilon}_0$

$$0 < c_1 \leq \frac{\partial g_\pm^\hat{\epsilon}(v)}{\partial v} \leq c_2 \quad \text{on} \quad [\tilde{v}_-, \tilde{v}_+]$$

and

$$g_\pm^\hat{\epsilon}(v) > 1 \quad \text{on} \quad [\tilde{v}_+, \bar{v}) \quad \text{and} \quad g_-^\hat{\epsilon}(v) > 0 \quad \text{on} \quad [\tilde{v}_-, \tilde{v}_+].$$

Thus the inverse $(g_\mp^\hat{\epsilon})^{-1}$ of $g_\mp^\hat{\epsilon}$ exists on $[\tilde{v}_-, \tilde{v}_+]$ and we have $(g_\mp^\hat{\epsilon})^{-1}(1) = \bar{v}_\pm^\hat{\epsilon}$. By the mean value theorem

$$\frac{1}{c_2}|g_\mp^\hat{\epsilon}(\bar{v}^0) - 1| \leq |\bar{v}_\pm^\hat{\epsilon} - \bar{v}^0| \leq \frac{1}{c_1}|g_\mp^\hat{\epsilon}(\bar{v}^0) - 1|.$$

The assertion (i) follows from the definition of $g_\pm^\hat{\epsilon}(v) = b^*(v) \pm \hat{\epsilon}^{1/3}(3D(v))^{1/3}$. \square

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Proof of Lemma 4.10. Let $\hat{\epsilon}_0$ be as in Proposition 4.28. Then for all $\hat{\epsilon} \in (0, \hat{\epsilon}_0)$

$$\mathcal{J}^{\hat{\epsilon}} = \begin{cases} (\underline{v}, \underline{v}^{\hat{\epsilon}}] \cup [\bar{v}^{\hat{\epsilon}}, \bar{v}) & \text{if } \underline{v} < \underline{v}^0 \text{ and } \bar{v}^0 < \bar{v} \\ (\underline{v}, \underline{v}^{\hat{\epsilon}}] & \text{if } \underline{v} < \underline{v}^0 \text{ and } \bar{v}^0 = \bar{v} \\ [\bar{v}^{\hat{\epsilon}}, \bar{v}) & \text{if } \underline{v} = \underline{v}^0 \text{ and } \bar{v}^0 < \bar{v} \end{cases}$$

and

$$\mathcal{K}^{\hat{\epsilon}} = \begin{cases} (\underline{v}^{\hat{\epsilon}}, \underline{v}^0) \cup (\bar{v}^0, \bar{v}^{\hat{\epsilon}}) & \text{if } \underline{v} < \underline{v}^0 \text{ and } \bar{v}^0 < \bar{v} \\ (\underline{v}^{\hat{\epsilon}}, \underline{v}^0) & \text{if } \underline{v} < \underline{v}^0 \text{ and } \bar{v}^0 = \bar{v} \\ (\bar{v}^0, \bar{v}^{\hat{\epsilon}}) & \text{if } \underline{v} = \underline{v}^0 \text{ and } \bar{v}^0 < \bar{v} \end{cases}$$

Thus the assertion of Lemma 4.10 follows from Proposition 4.28. \square

Proof of Lemma 4.13. Let $\hat{\epsilon}_0$ be as in Proposition 4.28. Then for all $\hat{\epsilon} \in (0, \hat{\epsilon}_0)$ we have $\mathcal{I}^{\hat{\epsilon}} \neq \emptyset$ and we subdivide the proof into three cases:

Case 1: $\underline{v} < \underline{v}^0$ and $\bar{v}^0 < \bar{v}$. In this case Proposition 4.28 implies that $\mathcal{I} \setminus \mathcal{I}^0 = (\underline{v}, \underline{v}^0) \cup (\bar{v}^0, \bar{v})$, $\inf \mathcal{I}^{\hat{\epsilon}} = \underline{v}^{\hat{\epsilon}}$ and $\sup \mathcal{I}^{\hat{\epsilon}} = \bar{v}^{\hat{\epsilon}}$.

Case 2: $\underline{v} < \underline{v}^0$ and $\bar{v}^0 = \bar{v}$. Here Proposition 4.28 yields $\mathcal{I} \setminus \mathcal{I}^0 = (\underline{v}, \underline{v}^0)$ and $\inf \mathcal{I}^{\hat{\epsilon}} = \underline{v}^{\hat{\epsilon}}$.

Case 3: $\underline{v} = \underline{v}^0$ and $\bar{v}^0 < \bar{v}$. Proposition 4.28 implies that $\mathcal{I} \setminus \mathcal{I}^0 = (\bar{v}^0, \bar{v})$ and $\sup \mathcal{I}^{\hat{\epsilon}} = \bar{v}^{\hat{\epsilon}}$.

In each case the claim now follows from Proposition 4.28. \square

4.B.3 Proof of Lemma 4.12: Estimates for the Generators $\mathcal{L}\hat{\Psi}_{\pm}^{\hat{\epsilon}}$

Proof of Lemma 4.12. Let $\hat{\epsilon} < \hat{\epsilon}_0$ be arbitrary. Then for $v \in \mathcal{I} \setminus \mathcal{J}^{\hat{\epsilon}}$ we have

$$\begin{aligned} \mathcal{L}\hat{\Psi}_{\pm}^{\hat{\epsilon}}(v, b) &= \hat{\epsilon}^{2/3} \sigma(v)^2 D(v) \hat{\psi}_{\xi\xi}(v, \frac{b-b^*(v)}{\hat{\epsilon}^{1/3}}) \mp \frac{\hat{\epsilon}^{5/3}}{1 \pm \hat{\epsilon}} \sigma(v)^2 D(v) \hat{\psi}_{\xi\xi}(v, \frac{b-b^*(v)}{\hat{\epsilon}^{1/3}}) \\ &+ \frac{\hat{\epsilon}}{1 \pm \hat{\epsilon}} \hat{\psi}_{\xi\xi}(v, \frac{b-b^*(v)}{\hat{\epsilon}^{1/3}}) \left(-\rho \beta(v) \sigma(v) b_v^*(v) (1 - b^*(v) - b(v)) \frac{b-b^*(v)}{\hat{\epsilon}^{1/3}} \right. \\ &\quad \left. + \frac{1}{2} \sigma(v)^2 (b(1-b) + b^*(v)(1-b^*(v))) (1-b-b^*(v)) \frac{b-b^*(v)}{\hat{\epsilon}^{1/3}} \right) \\ &+ \frac{\hat{\epsilon}}{1 \pm \hat{\epsilon}} \hat{\psi}_{\xi v}(v, \frac{b-b^*(v)}{\hat{\epsilon}^{1/3}}) \left(\rho \beta(v) \sigma(v) b(1-b) - \beta(v)^2 b_v^*(v) \right) \\ &+ \frac{\hat{\epsilon}}{1 \pm \hat{\epsilon}} \hat{\psi}_{\xi}(v, \frac{b-b^*(v)}{\hat{\epsilon}^{1/3}}) \left(-\alpha(v) b_v^*(v) + b(1-b)(\lambda(v) - b \sigma(v)^2) - \frac{1}{2} \beta(v)^2 b_{vv}^*(v) \right) \\ &+ \frac{\hat{\epsilon}^{4/3}}{1 \pm \hat{\epsilon}} \hat{\psi}_{vv}(v, \frac{b-b^*(v)}{\hat{\epsilon}^{1/3}}) \frac{1}{2} \beta(v)^2 \\ &+ \frac{\hat{\epsilon}^{4/3}}{1 \pm \hat{\epsilon}} \hat{\psi}_v(v, \frac{b-b^*(v)}{\hat{\epsilon}^{1/3}}) \alpha(v). \end{aligned} \tag{4.62}$$

In order to estimate the relevant summands, we first consider the derivatives of $\hat{\psi}$ as defined in (4.37). A straightforward calculation shows that

$$\begin{aligned} |\hat{\psi}_\xi(v, \xi)| &\leq 1 \\ |\hat{\psi}_{\xi\xi}(v, \xi)| &\leq K \begin{cases} \frac{1}{(D(v))^{1/3}}, & \xi \in (-(3D(v))^{1/3}, (3D(v))^{1/3}) \\ 0, & \text{otherwise} \end{cases} \\ |\hat{\psi}_{\xi v}(v, \xi)| &\leq \begin{cases} \frac{|D_v(v)|}{D(v)}, & \xi \in (-(3D(v))^{1/3}, (3D(v))^{1/3}) \\ 0, & \text{otherwise} \end{cases} \\ |\hat{\psi}_v(v, \xi)| &\leq K \begin{cases} \frac{|D_v(v)|}{(D(v))^{2/3}}, & \xi \in (-(3D(v))^{1/3}, (3D(v))^{1/3}) \\ 0, & \text{otherwise} \end{cases} \\ |\hat{\psi}_{vv}(v, \xi)| &\leq K \begin{cases} \frac{|D_{vv}(v)|}{(D(v))^{2/3}} + \frac{|D_v(v)|^2}{(D(v))^{5/3}}, & \xi \in (-(3D(v))^{1/3}, (3D(v))^{1/3}) \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

for some constant $K > 0$. Thus all summands in (4.62) are bounded on $\mathcal{I} \setminus \mathcal{J}^{\hat{\epsilon}} \times [0, 1]$ by a polynomial in v whose coefficients do not depend on $\hat{\epsilon}$. On the other hand, for $v \in \mathcal{J}^{\hat{\epsilon}}$ we have

$$\mathcal{L}\hat{\Psi}_\pm^{\hat{\epsilon}}(v, b) = \frac{\hat{\epsilon}}{1 \pm \hat{\epsilon}} \left(-\alpha(v) b_v^*(v) + b(1-b)(\lambda(v) - b\sigma(v)^2) - \frac{1}{2}\beta(v)^2 b_{vv}^*(v) \right).$$

Hence there exist constants C and $m \in \mathbb{N}$ that do not depend on $\hat{\epsilon}$ such that

$$\left| \mathcal{L}\hat{\Psi}_\pm^{\hat{\epsilon}}(v, b) - \hat{\epsilon}^{2/3} \sigma(v)^2 D(v) \hat{\psi}_{\xi\xi}(v, \frac{b-b^*(v)}{\hat{\epsilon}^{1/3}}) \right| \leq \hat{\epsilon} C (1 + v^m) \quad \text{on } \mathcal{I}^0 \times [0, 1]$$

$$\left| \mathcal{L}\hat{\Psi}_\pm^{\hat{\epsilon}}(v, b) \right| \leq \hat{\epsilon}^{2/3} C \quad \text{for all } (v, b) \in \mathcal{K}^{\hat{\epsilon}} \times [0, 1]$$

and

$$\left| \mathcal{L}\hat{\Psi}_\pm^{\hat{\epsilon}}(v, b) \right| \leq \hat{\epsilon} C (1 + v^m) \quad \text{for all } (v, b) \in \mathcal{J}^{\hat{\epsilon}} \times [0, 1].$$

This completes the proof. \square

4.B.4 Proof of Theorem 4.9: Upper Bound (4.39)

Let $\mathcal{S} = \{(L_t, M_t)\}_{t \geq 0} \in \hat{\mathcal{A}}$ be an arbitrary admissible trading strategy and $b \triangleq b^{\mathcal{S}}$ and $Z \triangleq Z^{\mathcal{S}}$ the corresponding fraction and wealth processes, respectively. Without loss of generality we may and do assume that $Z_0 = 1$. Itô's formula yields

$$\mathbb{E}[\ln Z_T] = \mathbb{E} \left[\int_0^T f(v_s, b_s) ds - \int_0^T \hat{\epsilon} \frac{dL_s^c}{Z_s} - \int_0^T \hat{\epsilon} \frac{dM_s^c}{Z_s} + \sum_{s \leq T} (\ln Z_s - \ln Z_{s-}) \right].$$

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In the following we estimate each summand separately.

Step 1: Estimate for $f(v, b)$. By the VIs (4.35) and Lemma 4.12 we have

$$\begin{aligned}
f(v, b) &\leq R_{loc}^0(v) - \hat{\epsilon}^{2/3} \frac{1}{2} \sigma(v)^2 \left(\frac{b-b^*(v)}{\hat{\epsilon}^{1/3}} \right)^2 \mathbb{I}_{v \in \mathcal{I}^0} \\
&\leq R_{loc}^0(v) + \hat{\epsilon}^{2/3} \hat{Q}_{loc}(v) \mathbb{I}_{v \in \mathcal{I}^0} \\
&\quad - \hat{\epsilon}^{2/3} \sigma(v)^2 D(v) \hat{\psi}_{\xi\xi}(v, \frac{b-b^*(v)}{\hat{\epsilon}^{1/3}}) \mathbb{I}_{v \in \mathcal{I}^0} \\
&\leq R_{loc}^0(v) + \hat{\epsilon}^{2/3} \hat{Q}_{loc}(v) \mathbb{I}_{v \in \mathcal{I}^0} \\
&\quad - \mathcal{L} \hat{\Psi}_+^{\hat{\epsilon}}(v, b) \mathbb{I}_{v \in \mathcal{I}^0} + \hat{\epsilon} C (1 + v^m) \mathbb{I}_{v \in \mathcal{I}^0} \\
&\leq R_{loc}^0(v) + \hat{\epsilon}^{2/3} \hat{Q}_{loc}(v) \mathbb{I}_{v \in \mathcal{I}^0} \\
&\quad - \mathcal{L} \hat{\Psi}_+^{\hat{\epsilon}}(v, b) + \hat{\epsilon}^{2/3} C \mathbb{I}_{v \in \mathcal{K}^{\hat{\epsilon}}} + \hat{\epsilon} C (1 + v^m).
\end{aligned} \tag{4.63}$$

Step 2: Estimates for Direct Transaction Costs $\int_0^T \hat{\epsilon} \frac{dL_s^c}{Z_s}$ and $\int_0^T \hat{\epsilon} \frac{dM_s^c}{Z_s}$. The VIs (4.41) imply that $|\frac{\partial \hat{\Psi}_+^{\hat{\epsilon}}}{\partial b}| \leq \frac{\hat{\epsilon}}{1+\hat{\epsilon}}$. Hence we obtain

$$-\hat{\epsilon} \leq -(1 + \hat{\epsilon} b) \frac{\partial \hat{\Psi}_+^{\hat{\epsilon}}}{\partial b} \quad \text{and} \quad -\hat{\epsilon} \leq -(-1 + \hat{\epsilon} b) \frac{\partial \hat{\Psi}_+^{\hat{\epsilon}}}{\partial b}. \tag{4.64}$$

Step 3: Estimates for Jump Terms. To bound the jumps in $\ln Z$ in terms of increments of the fraction process Δb , we proceed similarly as in the proof of the verification theorem (Theorem 4.9). First note that

$$\Delta Z_t = -\frac{\hat{\epsilon} Z_{t-} \Delta b_t}{1 + \hat{\epsilon} b_t} \text{ for } \Delta b_t > 0 \quad \text{and} \quad \Delta Z_t = -\frac{\hat{\epsilon} Z_{t-} \Delta b_t}{-1 + \hat{\epsilon} b_t} \text{ for } \Delta b_t < 0.$$

Hence for $\Delta b_t > 0$ we get

$$\ln Z_t - \ln Z_{t-} = \int_0^1 \frac{\Delta Z_t}{Z_{t-} + u \Delta Z_t} du = \int_0^1 \frac{-\hat{\epsilon} \Delta b_t}{1 + \hat{\epsilon} b_t - u \hat{\epsilon} \Delta b_t} du \leq \frac{-\hat{\epsilon} \Delta b_t}{1 + \hat{\epsilon}}$$

and for $\Delta b_t < 0$ we obtain

$$\ln Z_t - \ln Z_{t-} = \int_0^1 \frac{\Delta Z_t}{Z_{t-} + u \Delta Z_t} du = \int_0^1 \frac{-\hat{\epsilon} |\Delta b_t|}{1 - \hat{\epsilon} b_t - u \hat{\epsilon} |\Delta b_t|} du \leq \frac{-\hat{\epsilon} |\Delta b_t|}{1 + \hat{\epsilon}}.$$

Step 4: Final Estimate. Combining the bounds established above, we obtain

$$\begin{aligned}
\mathbb{E}[\ln Z_T] &\leq \mathbb{E} \left[\int_0^T R_{loc}^0(v_s) ds + \hat{\epsilon}^{2/3} \int_0^T \hat{Q}_{loc}(v_s) \mathbb{I}_{v_s \in \mathcal{I}^0} ds - \int_0^T \mathcal{L} \hat{\Psi}_+^{\hat{\epsilon}}(v_s, b_s) ds \right. \\
&\quad - \int_0^T (1 + \hat{\epsilon} b_s) \frac{\partial \hat{\Psi}_+^{\hat{\epsilon}}}{\partial b}(v_s, b_s) \frac{dL_s^c}{Z_s} - \int_0^T (-1 + \hat{\epsilon} b_s) \frac{\partial \hat{\Psi}_+^{\hat{\epsilon}}}{\partial b}(v_s, b_s) \frac{dM_s^c}{Z_s} \\
&\quad \left. - \sum_{s \leq T} \frac{\hat{\epsilon} |\Delta b_s|}{1 + \hat{\epsilon}} + \hat{\epsilon}^{2/3} \int_0^T C \mathbb{I}_{v_s \in \mathcal{K}^{\hat{\epsilon}}} ds + \hat{\epsilon} \int_0^T C (1 + v_s^m) ds \right]
\end{aligned}$$

$$\begin{aligned}
 &= \mathbb{E} \left[\int_0^T R_{loc}^0(v_s) ds + \hat{\epsilon}^{2/3} \int_0^T \hat{Q}_{loc}(v_s) \mathbb{I}_{v_s \in \mathcal{I}^0} ds \right. \\
 &\quad + \hat{\Psi}_+^{\hat{\epsilon}}(v_0, b_0) - \hat{\Psi}_+^{\hat{\epsilon}}(v_T, b_T) \\
 &\quad + \sum_{s \leq T} \left(\hat{\Psi}_+^{\hat{\epsilon}}(v_s, b_s) - \hat{\Psi}_+^{\hat{\epsilon}}(v_s, b_{s-}) - \frac{\hat{\epsilon} |\Delta b_s|}{1 + \hat{\epsilon}} \right) \\
 &\quad \left. + \hat{\epsilon}^{2/3} \int_0^T C \mathbb{I}_{v_s \in \mathcal{K}^{\hat{\epsilon}}} ds + \hat{\epsilon} \int_0^T C (1 + v_s^m) ds \right] \\
 &\leq \mathbb{E} \left[\int_0^T R_{loc}^0(v_s) ds + \hat{\epsilon}^{2/3} \int_0^T \hat{Q}_{loc}(v_s) \mathbb{I}_{v_s \in \mathcal{I}^0} ds + 2 \hat{\epsilon} K \right. \\
 &\quad \left. + \hat{\epsilon}^{2/3} \int_0^T C \mathbb{I}_{v_s \in \mathcal{K}^{\hat{\epsilon}}} ds + \hat{\epsilon} \int_0^T C (1 + v_s^m) ds \right]
 \end{aligned}$$

where the last inequality follows from boundedness of $\hat{\Psi}_+^{\hat{\epsilon}}$ and its derivative, $|\hat{\Psi}_+^{\hat{\epsilon}}| \leq K$ with a constant $K > 0$ and $|\frac{\partial \hat{\Psi}_+^{\hat{\epsilon}}}{\partial b}| \leq \frac{\hat{\epsilon}}{1 + \hat{\epsilon}}$. Observe that the final estimate in the preceding chain of inequalities does not depend on the strategy \mathcal{S} anymore. Hence we have established a uniform bound, and taking the supremum over all admissible trading strategies and using the ergodicity properties of the factor process, we obtain the following estimate for the optimal long-term growth rate $\hat{R}^{\hat{\epsilon}}$:

$$\hat{R}^{\hat{\epsilon}} \leq \mathbb{E}[R_{loc}^0(v_\infty)] + \hat{\epsilon}^{2/3} \mathbb{E}[\hat{Q}_{loc}(v_\infty) \mathbb{I}_{v_\infty \in \mathcal{I}^0}] + \hat{\epsilon}^{2/3} C \mathbb{E}[\mathbb{I}_{v_\infty \in \mathcal{K}^{\hat{\epsilon}}}] + \hat{\epsilon} C \mathbb{E}[1 + v_\infty^m].$$

Now (4.39) follows from Corollary 4.11. \square

4.B.5 Proof of Theorem 4.9: Lower Bound (4.40)

We establish the lower bound (4.40) for the long-term growth rate using the no-trading region illustrated in Figures 4.1 and 4.2:

$$\begin{aligned}
 \mathcal{NT}^{\hat{\epsilon}} \triangleq & \{(v, b) : v \in \mathcal{I}^0, b \in [g_-^{\hat{\epsilon}}(v), g_+^{\hat{\epsilon}}(v)]\} \cup \{v \in \mathcal{I} : b^*(v) \geq 1\} \times \{1\} \\
 & \cup \{v \in \mathcal{I} : b^*(v) \leq 0\} \times \{0\}.
 \end{aligned} \tag{4.65}$$

Throughout the proof we assume that $\hat{\epsilon} \in (0, \hat{\epsilon}_0)$ with $\hat{\epsilon}_0$ chosen as in Lemma 4.10.

Step 1: Construction of a Leading-Order Optimal Candidate Strategy

Without loss of generality, we assume that the initial value of the factor process satisfies $v_0 \in \text{int}(\mathcal{I}^0)$ (otherwise it suffices to relabel the stopping times ρ_k); and that $b_0 = b^*(v_0)$ (a single trade at time 0 does not influence $\hat{R}^{\hat{\epsilon}}$). First, recall from Section 4.4 the intervention times

$$\rho_{2k+1} \triangleq \inf\{t \geq \rho_{2k} : v_t \in \mathcal{I} \setminus \mathcal{I}^0\} \quad \text{and} \quad \rho_{2k+2} \triangleq \inf\{t \geq \rho_{2k+1} : v_t \in \mathcal{I}^{\hat{\epsilon}}\}$$

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defined $k \geq 0$. Here $\rho_0 \triangleq 0$ and $\mathcal{I}^{\hat{\epsilon}} \triangleq \{v \in \mathcal{I}^0 : g_{\pm}^{\hat{\epsilon}}(v) \in (0, 1)\}$. The candidate for the leading-order trading strategy is constructed iteratively as follows.

Step 1a: Trading Strategy on $[\rho_{2k}, \rho_{2k+1})$. To construct a control that ensures that the fraction process remains inside the no-trading region $\mathcal{NT}^{\hat{\epsilon}}$, we proceed similarly to Bichuch [2012] and make use of the existence result established by Dupuis and Ishii [1993]. Thus we define the no-trading region corresponding to (4.65) in the variables (v, x, y) for the case $v \in \mathcal{I}^0$, where x and y represent the bond and stock holdings, respectively, via

$$\widetilde{\mathcal{NT}}^{\hat{\epsilon}} \triangleq \{(v, x, y) \in \mathcal{I}^0 \times [0, \infty)^2 : x + y \neq 0 \text{ and } (v, \frac{y}{x+y}) \in \mathcal{NT}^{\hat{\epsilon}}\}.$$

We assume that the state process (v, X, Y) is in the interior of $\widetilde{\mathcal{NT}}^{\hat{\epsilon}}$ at time ρ_{2k} ; this holds by assumption for $k = 0$, and will be proved inductively for $k \geq 1$ in *Step 1b* below. Further, let the initial values of the controls $L_{\rho_{2k}}$ and $M_{\rho_{2k}}$ be given; for $k = 0$ we have $L_0 = 0$ and $M_0 = 0$, and $L_{\rho_{2k}}$ and $M_{\rho_{2k}}$ will be defined accordingly for $k \geq 1$ in *Step 1b*. Choose n_0 sufficiently large so that for all $n \geq n_0$

$$(v_{\rho_{2k}}, X_{\rho_{2k}}, Y_{\rho_{2k}}) \in \widetilde{\mathcal{NT}}_n^{\hat{\epsilon}} \triangleq \widetilde{\mathcal{NT}}^{\hat{\epsilon}} \cap \{(v, x, y) : \frac{1}{n} \leq x + y \leq n \text{ and } -n \leq v \leq n\}.$$

We subdivide the boundary of $\widetilde{\mathcal{NT}}_n^{\hat{\epsilon}}$ into two components:

$$\begin{aligned} \mathcal{B}_a^n &\triangleq \{(v, x, y) \in \widetilde{\mathcal{NT}}_n^{\hat{\epsilon}} : v = \underline{v}^0 \vee (-n) \text{ or } v = \bar{v}^0 \wedge n\} \\ &\cup \{(v, x, y) \in \widetilde{\mathcal{NT}}_n^{\hat{\epsilon}} : x + y = n \text{ or } x + y = \frac{1}{n}\} \end{aligned}$$

denotes the absorbing part of the boundary, whereas $\partial \widetilde{\mathcal{NT}}_n^{\hat{\epsilon}} \setminus \mathcal{B}_a^n$ is the reflecting part.¹² By Case 1 of Theorem 4.8 in Dupuis and Ishii [1993] there exists a unique strong solution $\{(v_t, X_t, Y_t)\}_{t \geq \rho_{2k}}$ of (4.3) and (4.10) with initial value $(v_{\rho_{2k}}, X_{\rho_{2k}}, Y_{\rho_{2k}})$ and reflection directions given by

$$\begin{aligned} (0, -(1 + \hat{\epsilon}), 1) &\text{ whenever } \frac{Y_t}{Y_t + X_t} = g_{-}^{\hat{\epsilon}}(v_t) \text{ and} \\ (0, 1 - \hat{\epsilon}, -1) &\text{ whenever } \frac{Y_t}{Y_t + X_t} = g_{+}^{\hat{\epsilon}}(v_t) \end{aligned} \tag{4.66}$$

for $t \in [\rho_{2k}, \tau_n)$, where τ_n is the hitting time of the absorbing component \mathcal{B}_a^n . Thus we can define the process (X, Y) corresponding to the controls (L, M) on $[\rho_{2k}, \tau)$ for $\tau \triangleq \lim_{n \rightarrow \infty} \tau_n$.

¹²Note that there are parts of the reflecting boundary $\partial \widetilde{\mathcal{NT}}_n^{\hat{\epsilon}} \setminus \mathcal{B}_a^n$ that are inaccessible for the state process in the absence of interventions, namely those with $x = 0$ or $y = 0$.

Step 1b: Verification of $\tau = \rho_{2k+1}$ and $Z_{\tau-} \in (0, \infty)$. The next goal is to show that the limits as $n \rightarrow \infty$ of L_{τ_n} , M_{τ_n} , X_{τ_n} and Y_{τ_n} exist and are finite. Further, we will show that the limit of the wealth process satisfies $Z_{\tau-} \in (0, \infty)$ and $\tau = \rho_{2k+1}$. To establish this, we apply Itô's formula to the process $\ln Z$:

$$\begin{aligned} \ln Z_{\tau_n} &= \ln Z_{\rho_{2k}} + \int_{\rho_{2k}}^{\tau_n} f(v_s, b_s) ds \\ &\quad + \int_{\rho_{2k}}^{\tau_n} b_s \sigma(v_s) dW_s - \int_{\rho_{2k}}^{\tau_n} \hat{\epsilon} \frac{dL_s}{Z_s} - \int_{\rho_{2k}}^{\tau_n} \hat{\epsilon} \frac{dM_s}{Z_s}. \end{aligned} \quad (4.67)$$

From the estimates (4.63) and (4.64) we obtain the upper bound

$$\begin{aligned} \ln Z_{\tau_n} &\leq \ln Z_{\rho_{2k}} + \int_{\rho_{2k}}^{\tau_n} R_{loc}^0(v_s) ds + \hat{\epsilon}^{2/3} \int_{\rho_{2k}}^{\tau_n} \hat{Q}_{loc}(v_s) ds + \int_{\rho_{2k}}^{\tau_n} b_s \sigma(v_s) dW_s \\ &\quad + \hat{\Psi}_+^{\hat{\epsilon}}(v_{\rho_{2k}}, b_{\rho_{2k}}) - \hat{\Psi}_+^{\hat{\epsilon}}(v_{\tau_n}, b_{\tau_n}) + \int_{\rho_{2k}}^{\tau_n} \frac{\partial \hat{\Psi}_+^{\hat{\epsilon}}}{\partial v}(v_s, b_s) \beta(v_s) d\bar{W}_s \\ &\quad + \int_{\rho_{2k}}^{\tau_n} \frac{\partial \hat{\Psi}_+^{\hat{\epsilon}}}{\partial b}(v_s, b_s) b_s (1 - b_s) \sigma(v_s) dW_s + \hat{\epsilon} \int_{\rho_{2k}}^{\tau_n} C(1 + v_s^m) ds. \end{aligned} \quad (4.68)$$

On the other hand, for $s \in [\rho_{2k}, \tau_n]$ Lemma 4.12 yields

$$\begin{aligned} f(v_s, b_s) &= f(v_s, b^*(v_s)) + \hat{\epsilon}^{2/3} \left(-\frac{1}{2} \sigma(v_s)^2 \left(\frac{b_s - b^*(v_s)}{\hat{\epsilon}^{1/3}} \right)^2 \right) \\ &= R_{loc}^0(v_s) + \hat{\epsilon}^{2/3} \hat{Q}_{loc}(v_s) - \hat{\epsilon}^{2/3} \sigma(v_s)^2 D(v_s) \hat{\psi}_{\xi\xi}(v_s, \frac{b_s - b^*(v_s)}{\hat{\epsilon}^{1/3}}) \\ &\geq R_{loc}^0(v_s) + \hat{\epsilon}^{2/3} \hat{Q}_{loc}(v_s) - \mathcal{L} \hat{\Psi}_-^{\hat{\epsilon}}(v_s, b_s) - \hat{\epsilon} C(1 + v_s^m). \end{aligned} \quad (4.69)$$

Furthermore, for $\hat{\Psi}_-^{\hat{\epsilon}}$ we have

$$\frac{\partial \hat{\Psi}_-^{\hat{\epsilon}}}{\partial b} = \frac{\hat{\epsilon}}{1 - \hat{\epsilon}} \quad \text{and therefore} \quad \frac{\partial \hat{\Psi}_-^{\hat{\epsilon}}}{\partial b} (1 + b\hat{\epsilon}) = \hat{\epsilon} \frac{1 + b\hat{\epsilon}}{1 - \hat{\epsilon}} \geq \hat{\epsilon} \quad (4.70)$$

on the lower boundary of the no-trading region (where $dL_t > 0$), and

$$\frac{\partial \hat{\Psi}_-^{\hat{\epsilon}}}{\partial b} = -\frac{\hat{\epsilon}}{1 - \hat{\epsilon}} \quad \text{and hence} \quad \frac{\partial \hat{\Psi}_-^{\hat{\epsilon}}}{\partial b} (-1 + b\hat{\epsilon}) = \hat{\epsilon} \frac{1 - b\hat{\epsilon}}{1 - \hat{\epsilon}} \geq \hat{\epsilon} \quad (4.71)$$

on the upper boundary of the no-trading region (where $dM_t > 0$). Thus we get

$$\begin{aligned} \ln Z_{\tau_n} &\geq \ln Z_{\rho_{2k}} + \int_{\rho_{2k}}^{\tau_n} R_{loc}^0(v_s) ds + \hat{\epsilon}^{2/3} \int_{\rho_{2k}}^{\tau_n} \hat{Q}_{loc}(v_s) ds + \int_{\rho_{2k}}^{\tau_n} b_s \sigma(v_s) dW_s \\ &\quad - \int_{\rho_{2k}}^{\tau_n} \mathcal{L} \hat{\Psi}_-^{\hat{\epsilon}}(v_s, b_s) ds - \int_{\rho_{2k}}^{\tau_n} \frac{\partial \hat{\Psi}_-^{\hat{\epsilon}}}{\partial b} (1 + b_s \hat{\epsilon}) \frac{dL_s}{Z_s} \\ &\quad - \int_{\rho_{2k}}^{\tau_n} \frac{\partial \hat{\Psi}_-^{\hat{\epsilon}}}{\partial b} (-1 + b_s \hat{\epsilon}) \frac{dM_s}{Z_s} - \hat{\epsilon} \int_{\rho_{2k}}^{\tau_n} C(1 + v_s^m) ds \\ &\geq \ln Z_{\rho_{2k}} + \int_{\rho_{2k}}^{\tau_n} R_{loc}^0(v_s) ds + \hat{\epsilon}^{2/3} \int_{\rho_{2k}}^{\tau_n} \hat{Q}_{loc}(v_s) ds + \int_{\rho_{2k}}^{\tau_n} b_s \sigma(v_s) dW_s \\ &\quad + \hat{\Psi}_-^{\hat{\epsilon}}(v_{\rho_{2k}}, b_{\rho_{2k}}) - \hat{\Psi}_-^{\hat{\epsilon}}(v_{\tau_n}, b_{\tau_n}) + \int_{\rho_{2k}}^{\tau_n} \frac{\partial \hat{\Psi}_-^{\hat{\epsilon}}}{\partial v}(v_s, b_s) \beta(v_s) d\bar{W}_s \\ &\quad + \int_{\rho_{2k}}^{\tau_n} \frac{\partial \hat{\Psi}_-^{\hat{\epsilon}}}{\partial b}(v_s, b_s) b_s (1 - b_s) \sigma(v_s) dW_s - \hat{\epsilon} \int_{\rho_{2k}}^{\tau_n} C(1 + v_s^m) ds. \end{aligned} \quad (4.72)$$

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By assumption (A1), hitting times of $\{v_t\}_{t \geq 0}$ have finite expectations, so the strong Markov property of $\{v_t\}_{t \geq 0}$ implies that for every $l \geq 1$

$$\mathbb{E}\left[\int_{\rho_{2k}}^{\rho_{2k+1}} v_s^{2l} ds\right] = \mathbb{E}^{v_{\rho_{2k}}}\left[\int_0^H v_s^{2l} ds\right] < \infty \quad \text{for } l \geq 1 \quad \text{and} \quad \mathbb{E}\left[\int_0^{\rho_1} v_s^{2l} ds\right] < \infty$$

where $H \triangleq H_{\mathcal{T} \setminus \mathcal{I}^0}$ is the first hitting time of the set $\mathcal{T} \setminus \mathcal{I}^0$. Hence, as the integrands in the stochastic integrals in (4.72) and (4.68) are bounded by polynomials in $v \in \mathcal{I}^0$, there exist a.s. finite limits of the stochastic integrals as $n \rightarrow \infty$, namely

$$\int_{\rho_{2k}}^{\tau} b_s \sigma(v_s) dW_s, \quad \int_{\rho_{2k}}^{\tau} \frac{\partial \hat{\Psi}_{\pm}^{\epsilon}}{\partial v}(v_s, b_s) \beta(v_s) d\bar{W}_s$$

and

$$\int_{\rho_{2k}}^{\tau} \frac{\partial \hat{\Psi}_{\pm}^{\epsilon}}{\partial b}(v_s, b_s) b_s (1 - b_s) \sigma(v_s) dW_s.$$

Thus we conclude that

$$-\infty < \liminf_{n \rightarrow \infty} \ln Z_{\tau_n} \leq \limsup_{n \rightarrow \infty} \ln Z_{\tau_n} < \infty \quad \text{a.s.}$$

By (4.67) the limits $\lim_{n \rightarrow \infty} L_{\tau_n}$ and $\lim_{n \rightarrow \infty} M_{\tau_n}$ are finite and thus $Z_{\tau-} = \lim_{n \rightarrow \infty} Z_{\tau_n} \in (0, \infty)$. In particular we get $\tau = \rho_{2k+1}$. Furthermore $X_{\tau-}$ and $Y_{\tau-}$ are in $(0, \infty)$.

Step 1c: Trading Strategy for $t \in [\rho_{2k+1}, \rho_{2k+2})$. Having defined the trading strategy (L, M) and the corresponding processes X and Y , the wealth process Z , and the fraction process b on $[0, \rho_{2k+1})$ we proceed as follows: At time ρ_{2k+1} the whole wealth is invested either into stocks or into bonds (i.e. $b_{\rho_{2k+1}} \triangleq b^*(v_{\rho_{2k+1}})$) and no further action is taken until the factor process v hits \mathcal{I}^{ϵ} : $(v_t, b_t) = (v_t, b^*(v_{\rho_{2k+1}}))$ for $t \in [\rho_{2k+1}, \rho_{2k+2})$. At time ρ_{2k+2} the fraction process is shifted to the Merton fraction, i.e. $b_{\rho_{2k+2}} \triangleq b^*(v_{\rho_{2k+2}})$ and we are back in *Step 1a*. Formally, the corresponding controls are specified via

$$\Delta L_{\rho_{2k+1}} \triangleq \frac{1 - b_{\rho_{2k+1}^-}}{1 + \epsilon} Z_{\rho_{2k+1}^-} \quad \text{and} \quad L_t \triangleq L_{\rho_{2k+1}} \quad \text{for } t \in (\rho_{2k+1}, \rho_{2k+2}]$$

and

$$M_t \triangleq M_{\rho_{2k+1}^-} \quad \text{for } t \in [\rho_{2k+1}, \rho_{2k+2}) \quad \text{and} \quad \Delta M_{\rho_{2k+2}} \triangleq \frac{1 - b^*(v_{\rho_{2k+2}})}{1 - \epsilon b^*(v_{\rho_{2k+2}})} Z_{\rho_{2k+2}^-}$$

for the case $b^*(v_{\rho_{2k+1}}) = 1$. For the case $b^*(v_{\rho_{2k+1}}) = 0$ we define the controls as follows:

$$\Delta M_{\rho_{2k+1}} \triangleq b_{\rho_{2k+1}^-} Z_{\rho_{2k+1}^-} \quad \text{and} \quad M_t \triangleq M_{\rho_{2k+1}} \quad \text{for } t \in (\rho_{2k+1}, \rho_{2k+2}]$$

and

$$L_t \triangleq L_{\rho_{2k+1}-} \text{ for } t \in [\rho_{2k+1}, \rho_{2k+2}) \text{ and } \Delta L_{\rho_{2k+2}} \triangleq \frac{b^*(v_{\rho_{2k+2}})}{1 + \hat{\epsilon} b^*(v_{\rho_{2k+2}})} Z_{\rho_{2k+2}-}.$$

Then it develops that

$$\begin{aligned} Z_{\rho_{2k+1}} &= \frac{1 + \hat{\epsilon} b_{\rho_{2k+1}-}}{1 + \hat{\epsilon}} Z_{\rho_{2k+1}-} \quad \text{and} \\ Z_{\rho_{2k+2}} &= \frac{1 - \hat{\epsilon}}{1 - \hat{\epsilon} b^*(v_{\rho_{2k+2}})} Z_{\rho_{2k+2}-} \quad \text{on } \{b^*(v_{\rho_{2k+1}}) = 1\} \end{aligned}$$

and

$$\begin{aligned} Z_{\rho_{2k+1}} &= (1 - \hat{\epsilon} b_{\rho_{2k}-}) Z_{\rho_{2k+1}-} \quad \text{and} \\ Z_{\rho_{2k+2}} &= \frac{1}{1 + \hat{\epsilon} b^*(v_{\rho_{2k+2}})} Z_{\rho_{2k+2}-} \quad \text{on } \{b^*(v_{\rho_{2k+1}}) = 0\} \end{aligned}$$

while for $t \in [\rho_{2k+1}, \rho_{2k+2})$ the wealth process Z_t evolves according to either the bond dynamics (4.1) or the stock dynamics (4.2). Note that the state process $(v_{\rho_{2k+2}}, X_{\rho_{2k+2}}, Y_{\rho_{2k+2}})$ is in the interior of $\widetilde{\mathcal{NT}}^{\hat{\epsilon}}$ at ρ_{2k+2} , and that the wealth process Z remains positive on $[\rho_{2k+1}, \rho_{2k+2}]$.

The control defined in *Steps 1a, 1b* and *1c* is admissible by construction, and will be denoted by (L, M) in what follows. For simplicity of notation we write $\{b_t\}_{t \geq 0}$ for the corresponding fraction process and $\{Z_t\}_{t \geq 0}$ for the associated wealth process.

Step 2: Proof of (4.40)

Step 2a: A Lower Bound. We assume without loss of generality that $Z_0 = 1$. The process $\{Z_t\}_{t \geq 0}$ jumps only at ρ_k , $k \geq 1$. Therefore Itô's formula yields

$$\mathbb{E}[\ln Z_T] = \mathbb{E}\left[\int_0^T f(v_s, b_s) ds - \int_0^T \hat{\epsilon} \frac{dL_s^c}{Z_s} - \int_0^T \hat{\epsilon} \frac{dM_s^c}{Z_s} + \sum_{k=1}^{\infty} (\ln Z_{\rho_k} - \ln Z_{\rho_k-}) \mathbb{I}_{\rho_k \leq T}\right].$$

Next, similarly as in (4.63) and (4.69), from the leading-order VIs (4.35) and Lemma 4.12 we get

$$\begin{aligned} f(v_s, b_s) &= R_{loc}^0(v_s) - \hat{\epsilon}^{2/3} \frac{1}{2} \sigma(v_s)^2 \left(\frac{b_s - b^*(v_s)}{\hat{\epsilon}^{1/3}}\right)^2 \mathbb{I}_{v_s \in \mathcal{I}^0} \\ &= R_{loc}^0(v_s) + \hat{\epsilon}^{2/3} \hat{Q}_{loc}(v_s) \mathbb{I}_{v_s \in \mathcal{I}^0} \\ &\quad - \hat{\epsilon}^{2/3} \sigma(v_s)^2 D(v_s) \hat{\psi}_{\xi\xi}(v_s, \frac{b_s - b^*(v_s)}{\hat{\epsilon}^{1/3}}) \mathbb{I}_{v_s \in \mathcal{I}^0} \\ &\geq R_{loc}^0(v_s) + \hat{\epsilon}^{2/3} \hat{Q}_{loc}(v_s) \mathbb{I}_{v_s \in \mathcal{I}^0} \\ &\quad - \mathcal{L} \hat{\Psi}_-(v_s, b_s) - \hat{\epsilon}^{2/3} C \mathbb{I}_{v_s \in \mathcal{K}^{\hat{\epsilon}}} - \hat{\epsilon} C (1 + v_s^m). \end{aligned}$$

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Thus by Itô's formula for $\hat{\Psi}_-^{\hat{\epsilon}}$

$$\begin{aligned}
\mathbb{E}[\ln Z_T] &\geq \mathbb{E}\left[\int_0^T R_{loc}^0(v_s) ds + \hat{\epsilon}^{2/3} \int_0^T \hat{Q}_{loc}(v_s) \mathbb{I}_{v_s \in \mathcal{I}^0} ds \right. \\
&\quad + \hat{\Psi}_-^{\hat{\epsilon}}(v_0, b_0) - \hat{\Psi}_-^{\hat{\epsilon}}(v_T, b_T) \\
&\quad - \hat{\epsilon}^{2/3} \int_0^T C \mathbb{I}_{v_s \in \mathcal{K}^{\hat{\epsilon}}} ds - \hat{\epsilon} \int_0^T C(1 + v_s^m) ds \\
&\quad + \int_0^T \left(\frac{\partial \hat{\Psi}_-^{\hat{\epsilon}}}{\partial b}(v_s, b_s)(1 + b_s \hat{\epsilon}) - \hat{\epsilon} \right) \frac{dL_s^c}{Z_s} \\
&\quad + \int_0^T \left(\frac{\partial \hat{\Psi}_-^{\hat{\epsilon}}}{\partial b}(v_s, b_s)(-1 + b_s \hat{\epsilon}) - \hat{\epsilon} \right) \frac{dM_s^c}{Z_s} \\
&\quad \left. + \sum_{k=1}^{\infty} (\hat{\Psi}_-^{\hat{\epsilon}}(v_{\rho_k}, b_{\rho_k}) - \hat{\Psi}_-^{\hat{\epsilon}}(v_{\rho_k}, b_{\rho_k-}) + \ln Z_{\rho_k} - \ln Z_{\rho_k-}) \mathbb{I}_{\rho_k \leq T} \right] \\
&\geq \mathbb{E}\left[\int_0^T R_{loc}^0(v_s) ds + \hat{\epsilon}^{2/3} \int_0^T \hat{Q}_{loc}(v_s) \mathbb{I}_{v_s \in \mathcal{I}^0} ds \right. \\
&\quad + \hat{\Psi}_-^{\hat{\epsilon}}(v_0, b_0) - \hat{\Psi}_-^{\hat{\epsilon}}(v_T, b_T) \\
&\quad - \hat{\epsilon}^{2/3} \int_0^T C \mathbb{I}_{v_s \in \mathcal{K}^{\hat{\epsilon}}} ds - \hat{\epsilon} \int_0^T C(1 + v_s^m) ds \\
&\quad \left. + \sum_{k=1}^{\infty} (\ln Z_{\rho_k} - \ln Z_{\rho_k-}) \mathbb{I}_{\rho_k \leq T} \right] \tag{4.73}
\end{aligned}$$

where the last inequality follows from (4.70), (4.71) and the fact that, by construction, $\hat{\Psi}_-^{\hat{\epsilon}}(v_{\rho_k}, b_{\rho_k}) \geq \hat{\Psi}_-^{\hat{\epsilon}}(v_{\rho_k}, b_{\rho_k-})$ for each $k \geq 1$.

Step 2b: Estimate for the Jump Term. It remains only to estimate the jump term in (4.73). First not, that there exists a constant that bounds $(3D(v))^{1/3}$ for v in $\{\underline{v}^0, \inf \mathcal{I}^{\hat{\epsilon}}, \sup \mathcal{I}^{\hat{\epsilon}}, \bar{v}^0\} \setminus \{\underline{v}, \bar{v}\}$ uniformly in $\hat{\epsilon} \in (0, \hat{\epsilon}_0)$. Therefore

$$\ln Z_{\rho_k} - \ln Z_{\rho_k-} = \int_0^1 \frac{-\hat{\epsilon} |\Delta b_{\rho_k}|}{1 \pm \hat{\epsilon} b_{\rho_k} - u \hat{\epsilon} |\Delta b_{\rho_k}|} du \geq -\frac{\hat{\epsilon} |\Delta b_{\rho_k}|}{1 - 2\hat{\epsilon}} \geq -K_1 \hat{\epsilon}^{4/3}$$

where the signs $+$ and $-$ in the denominator correspond to the cases $\Delta b_{\rho_k} > 0$ and $\Delta b_{\rho_k} < 0$, respectively, and $K_1 > 0$ is a constant. Next, we define for every $k \geq 0$

$$\bar{\rho}_{2k+1} \triangleq \inf\{t \geq \bar{\rho}_{2k} : v_t = \bar{v}^0\}, \quad \bar{\rho}_{2k+2} \triangleq \inf\{t \geq \bar{\rho}_{2k+1} : v_t = \sup \mathcal{I}^{\hat{\epsilon}}\}$$

and

$$\rho_{2k+1} \triangleq \inf\{t \geq \rho_{2k} : v_t = \underline{v}^0\}, \quad \rho_{2k+2} \triangleq \inf\{t \geq \rho_{2k+1} : v_t = \inf \mathcal{I}^{\hat{\epsilon}}\}$$

with $\rho_0 \triangleq \bar{\rho}_0 \triangleq 0$. Applying Lemma 4.13 and Lemma 4.14 to the process $\{v_t\}_{t \geq 0}$ we find that

$$\mathbb{E}\left[\sum_{k=1}^{\infty} \mathbb{I}_{\rho_k \leq T}\right] = \mathbb{E}\left[\sum_{k=1}^{\infty} \mathbb{I}_{\bar{\rho}_k \leq T} + \sum_{k=1}^{\infty} \mathbb{I}_{\rho_k \leq T}\right] \leq K_2 \frac{1}{\hat{\epsilon}^{1/3}} \mathbb{E}\left[\int_0^T (1 + v_t^n) dt + 1\right]$$

with a constant $K_2 > 0$ and $n \geq m$. Therefore we obtain the following estimate, uniform in $\hat{\epsilon}$, for the jump term:

$$\mathbb{E}\left[\sum_{k=1}^{\infty}(\ln Z_{\rho_k} - \ln Z_{\rho_k-}) \mathbb{I}_{\rho_k \leq T}\right] \geq -K_1 K_2 \hat{\epsilon} \mathbb{E}\left[\int_0^T (1 + v_t^n) dt + 1\right]. \quad (4.74)$$

Step 2c: Combining the Estimates. Inserting the estimate for the jump term (4.74) into (4.73) we conclude that there exists a constant $K > 0$ such that for all $\hat{\epsilon} < \hat{\epsilon}_0$

$$\begin{aligned} \mathbb{E}[\ln Z_T] \geq & \mathbb{E}\left[\int_0^T R_{loc}^0(v_s) ds + \hat{\epsilon}^{2/3} \int_0^T \hat{Q}_{loc}(v_s) \mathbb{I}_{v_s \in \mathcal{I}^0} ds - K \right. \\ & \left. - \hat{\epsilon}^{2/3} \int_0^T K \mathbb{I}_{v_s \in \mathcal{K}^{\hat{\epsilon}}} ds - \hat{\epsilon} \int_0^T K (1 + v_s^n) ds\right] \end{aligned}$$

as $\hat{\Psi}_{\hat{\epsilon}}$ is bounded. Thus we arrive at the following lower bound for $\hat{R}^{\hat{\epsilon}}$:

$$\hat{R}^{\hat{\epsilon}} \geq R^0 + \hat{\epsilon}^{2/3} \mathbb{E}[\hat{Q}_{loc}(v_{\infty}) \mathbb{I}_{v_{\infty} \in \mathcal{I}^0}] - \hat{\epsilon}^{2/3} K \mathbb{E}[\mathbb{I}_{v_{\infty} \in \mathcal{K}^{\hat{\epsilon}}}] - \hat{\epsilon} K \mathbb{E}[1 + v_{\infty}^n].$$

The desired result now follows from Corollary 4.11. \square

4.C Proofs for Section 4.5

The general structure of this appendix parallels that of Appendix 4.B.

4.C.1 Lemmas 4.16 and 4.20: Bounds for the No-Trading Region

First, we recall the relevant definitions from Section 4.5:

$$\begin{aligned} \mathcal{J}^{\hat{\epsilon}} &\triangleq \{v \in \mathcal{I} \setminus \mathcal{I}^0 : \check{\Psi}^{\hat{\epsilon}}(v, b) = 0 \text{ for all } b \in [0, 1]\} \\ \mathcal{K}^{\hat{\epsilon}} &\triangleq (\mathcal{I} \setminus \mathcal{I}^0) \setminus \mathcal{J}^{\hat{\epsilon}} \\ \mathcal{I}^{\hat{\epsilon}} &\triangleq \{v \in \mathcal{I}^0 : g_+^{\hat{\epsilon}}(v) \in (0, 1) \text{ and } g_-^{\hat{\epsilon}}(v) \in (0, 1)\} \end{aligned}$$

where $g_{\pm}^{\hat{\epsilon}}(v) \triangleq b^*(v) \pm \hat{\epsilon}^{1/4} (24D(v))^{1/4}$, $v \in \mathcal{I}$. Again, without loss of generality, throughout this part of the appendix we assume that b^* is non-decreasing. Moreover we set

$$\bar{v}_{\pm}^{\hat{\epsilon}} \triangleq \sup\{v \in \mathcal{I} : g_{\mp}^{\hat{\epsilon}}(v) \in [0, 1]\} \quad \text{and} \quad \underline{v}_{\pm}^{\hat{\epsilon}} \triangleq \inf\{v \in \mathcal{I} : g_{\mp}^{\hat{\epsilon}}(v) \in [0, 1]\}.$$

Proposition 4.29. *There exist constants $\check{\epsilon}_0 > 0$, $C_1 > 0$ and $C_2 > 0$ such that*

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(i) If $\bar{v}^0 < \bar{v}$ then for all $\check{\epsilon} < \check{\epsilon}_0$ we have $\underline{v} < \bar{v}_-^{\check{\epsilon}} < \bar{v}^0 < \bar{v}_+^{\check{\epsilon}} < \bar{v}$ and

$$\check{\epsilon}^{1/4} C_1 \leq |\bar{v}^0 - \bar{v}_\pm^{\check{\epsilon}}| \leq \check{\epsilon}^{1/4} C_2.$$

Furthermore, for every $\check{\epsilon} < \check{\epsilon}_0$, $g_\pm^{\check{\epsilon}}$ is strictly increasing on $[\bar{v}_-^{\check{\epsilon}}, \bar{v}_+^{\check{\epsilon}}]$ and $g_-^{\check{\epsilon}}(\bar{v}_-^{\check{\epsilon}}) > 0$.

(ii) If $\underline{v}^0 > \underline{v}$ then for all $\check{\epsilon} < \check{\epsilon}_0$ we have $\underline{v} < \underline{v}_-^{\check{\epsilon}} < \underline{v}^0 < \underline{v}_+^{\check{\epsilon}} < \bar{v}$ and

$$\check{\epsilon}^{1/4} C_1 \leq |\underline{v}^0 - \underline{v}_\pm^{\check{\epsilon}}| \leq \check{\epsilon}^{1/4} C_2.$$

Furthermore, for every $\check{\epsilon} < \check{\epsilon}_0$, $g_\pm^{\check{\epsilon}}$ is strictly increasing on $[\underline{v}_-^{\check{\epsilon}}, \underline{v}_+^{\check{\epsilon}}]$ and $g_-^{\check{\epsilon}}(\underline{v}_-^{\check{\epsilon}}) < 1$.

Proof. The proof is analogous to that of Proposition 4.28. \square

Proof of Lemma 4.16. Using Proposition 4.29, the claim can be verified on a case-by-case basis as in the proof of Lemma 4.10. \square

Proof of Lemma 4.20. The assertion follows from Proposition 4.29 in the same way as for proportional costs (see the proof of Lemma 4.13). \square

4.C.2 Proof of Lemma 4.18: Estimates for the Generators $\mathcal{L}\check{\Psi}^{\check{\epsilon}}$

Let $\check{\epsilon} < \check{\epsilon}_0$ be arbitrary. Then for all $v \in \mathcal{I} \setminus \mathcal{J}^{\check{\epsilon}}$ we have

$$\begin{aligned} \mathcal{L}\check{\Psi}^{\check{\epsilon}}(v, b) &= \check{\epsilon}^{1/2} \sigma(v)^2 D(v) \check{\psi}_{\xi\xi}(v, \frac{b-b^*(v)}{\check{\epsilon}^{1/4}}) \\ &+ \left(\frac{-\ln(1-\check{\epsilon})}{\check{\epsilon}^{1/2}} - \check{\epsilon}^{1/2} \right) \sigma(v)^2 D(v) \check{\psi}_{\xi\xi}(v, \frac{b-b^*(v)}{\check{\epsilon}^{1/4}}) \\ &+ \frac{-\ln(1-\check{\epsilon})}{\check{\epsilon}^{1/4}} \check{\psi}_{\xi\xi}(v, \frac{b-b^*(v)}{\check{\epsilon}^{1/4}}) \left(-\rho\beta(v) \sigma(v) b_v^*(v) (1-b^*(v)-b) \frac{b-b^*(v)}{\check{\epsilon}^{1/4}} \right. \\ &\quad \left. + \frac{1}{2} \sigma(v)^2 (b(1-b) + b^*(v)(1-b^*(v))) (1-b-b^*(v)) \frac{b-b^*(v)}{\check{\epsilon}^{1/4}} \right) \\ &+ \frac{-\ln(1-\check{\epsilon})}{\check{\epsilon}^{1/4}} \check{\psi}_{\xi v}(v, \frac{b-b^*(v)}{\check{\epsilon}^{1/4}}) \left(\rho\beta(v) \sigma(v) b(1-b) - \beta(v)^2 b_v^*(v) \right) \\ &+ \frac{-\ln(1-\check{\epsilon})}{\check{\epsilon}^{1/4}} \check{\psi}_{\xi}(v, \frac{b-b^*(v)}{\check{\epsilon}^{1/4}}) \left(-\alpha(v) b_v^*(v) \right. \\ &\quad \left. + b(1-b)(\lambda(v) - b\sigma(v)^2) - \frac{1}{2} \beta(v)^2 b_{vv}^*(v) \right) \\ &+ \frac{-\ln(1-\check{\epsilon})}{\check{\epsilon}^0} \check{\psi}_{vv}(v, \frac{b-b^*(v)}{\check{\epsilon}^{1/4}}) \frac{1}{2} \beta(v)^2 \\ &+ \frac{-\ln(1-\check{\epsilon})}{\check{\epsilon}^0} \check{\psi}_v(v, \frac{b-b^*(v)}{\check{\epsilon}^{1/4}}) \alpha(v). \end{aligned} \tag{4.75}$$

The following estimates follow immediately from the definition of $\check{\psi}$:

$$\begin{aligned}
 |\check{\psi}_\xi(v, \xi)| &\leq K \begin{cases} \frac{1}{(D(v))^{1/4}}, & \xi \in (-(24D(v))^{1/4}, (24D(v))^{1/4}) \\ 0, & \text{otherwise} \end{cases} \\
 |\check{\psi}_{\xi\xi}(v, \xi)| &\leq K \begin{cases} \frac{1}{(D(v))^{1/2}}, & \xi \in (-(24D(v))^{1/4}, (24D(v))^{1/4}) \\ 0, & \text{otherwise} \end{cases} \\
 |\check{\psi}_{\xi v}(v, \xi)| &\leq K \begin{cases} \frac{|D_v(v)|}{(D(v))^{5/4}}, & \xi \in (-(24D(v))^{1/4}, (24D(v))^{1/4}) \\ 0, & \text{otherwise} \end{cases} \\
 |\check{\psi}_v(v, \xi)| &\leq K \begin{cases} \frac{|D_v(v)|}{D(v)}, & \xi \in (-(24D(v))^{1/4}, (24D(v))^{1/4}) \\ 0, & \text{otherwise} \end{cases} \\
 |\check{\psi}_{vv}(v, \xi)| &\leq K \begin{cases} \frac{|D_{vv}(v)|}{D(v)} + \frac{|D_v(v)|^2}{(D(v))^2}, & \xi \in (-(24D(v))^{1/4}, (24D(v))^{1/4}) \\ 0, & \text{otherwise} \end{cases}
 \end{aligned}$$

where $K > 0$ is constant. As in the case of proportional transaction costs, all summands in (4.75) are bounded on $\mathcal{I} \setminus \mathcal{J}^\check{\epsilon} \times [0, 1]$ by a polynomial in v whose coefficients are independent of $\check{\epsilon}$. Hence there exist constants $C > 0$ and $m \in \mathbb{N}$ such that for all $\check{\epsilon} < \check{\epsilon}_0$

$$|\mathcal{L}\check{\Psi}^\check{\epsilon}(v, b) - \check{\epsilon}^{1/2}\sigma(v)^2 D(v) \check{\psi}_{\xi\xi}(v, \frac{b-b^*(v)}{\check{\epsilon}^{1/4}})| \leq \check{\epsilon}^{3/4} C (1 + v^m) \quad \text{on } \mathcal{I}^0 \times [0, 1]$$

and

$$|\mathcal{L}\check{\Psi}^\check{\epsilon}(v, b)| \leq \check{\epsilon}^{1/2} C \quad \text{for } v \in \mathcal{K}^\check{\epsilon} \quad \text{and } b \in [0, 1].$$

This completes the proof. \square

4.C.3 Proof of Theorem 4.15: Upper Bound (4.46)

Let $\check{\epsilon}_0$ be as in Proposition 4.29 and let $\check{\epsilon} \in (0, \check{\epsilon}_0)$ be arbitrary. Using the leading-order VIs (4.36) we obtain by Lemma 4.18 that

$$\begin{aligned}
 f(v, b) &\leq R_{loc}^0(v) - \check{\epsilon}^{1/2} \frac{1}{2} \sigma(v)^2 \left(\frac{b-b^*(v)}{\check{\epsilon}^{1/4}} \right)^2 \mathbb{I}_{v \in \mathcal{I}^0} \\
 &\leq R_{loc}^0(v) + \check{\epsilon}^{1/2} \check{Q}_{loc}(v) \mathbb{I}_{v \in \mathcal{I}^0} \\
 &\quad - \check{\epsilon}^{1/2} \sigma(v)^2 D(v) \check{\psi}_{\xi\xi}(v, \frac{b-b^*(v)}{\check{\epsilon}^{1/4}}) \mathbb{I}_{v \in \mathcal{I}^0} \\
 &\leq R_{loc}^0(v) + \check{\epsilon}^{1/2} \check{Q}_{loc}(v) \mathbb{I}_{v \in \mathcal{I}^0} \\
 &\quad - \mathcal{L}\check{\Psi}^\check{\epsilon}(v, b) + \check{\epsilon}^{1/2} C \mathbb{I}_{v \in \mathcal{K}^\check{\epsilon}} + \check{\epsilon}^{3/4} C (1 + v^m)
 \end{aligned}$$

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for some constants $C > 0$ and $m \in \mathbb{N}$ that do not depend on $\check{\epsilon}$. Next consider an arbitrary admissible impulse control strategy $\mathcal{S} = \{\tau_k, \pi_k\}_{k \geq 1}$ and the corresponding fraction process $b \triangleq b^{\mathcal{S}}$. Itô's formula yields

$$\begin{aligned} & \mathbb{E} \left[\int_0^T f(v_s, b_s) ds + \ln(1 - \check{\epsilon}) \sum_{k=1}^{\infty} \mathbb{I}_{\tau_k \leq T} \right] \\ & \leq \mathbb{E} \left[\int_0^T R_{loc}^0(v_s) ds + \check{\epsilon}^{1/2} \int_0^T \check{Q}_{loc}(v_s) \mathbb{I}_{v_s \in \mathcal{I}^0} ds + \check{\epsilon}^{1/2} \int_0^T C \mathbb{I}_{v_s \in \mathcal{K}^{\check{\epsilon}}} ds \right. \\ & \quad + \check{\epsilon}^{3/4} \int_0^T C (1 + v_s^m) ds + \check{\Psi}^{\check{\epsilon}}(v_0, b_0) - \check{\Psi}^{\check{\epsilon}}(v_T, b_T) \\ & \quad \left. + \sum_{k=1}^{\infty} (\check{\Psi}^{\check{\epsilon}}(v_{\tau_k}, b_{\tau_k}) - \check{\Psi}^{\check{\epsilon}}(v_{\tau_k}, b_{\tau_k-}) + \ln(1 - \check{\epsilon})) \mathbb{I}_{\tau_k \leq T} \right] \\ & \leq \mathbb{E} \left[\int_0^T R_{loc}^0(v_s) ds + \check{\epsilon}^{1/2} \int_0^T \check{Q}_{loc}(v_s) \mathbb{I}_{v_s \in \mathcal{I}^0} ds \right. \\ & \quad \left. + \check{\epsilon}^{1/2} \int_0^T C \mathbb{I}_{v_s \in \mathcal{K}^{\check{\epsilon}}} ds + \check{\epsilon}^{3/4} \int_0^T C (1 + v_s^m) ds - \ln(1 - \check{\epsilon}) \right] \end{aligned}$$

where we have used the fact that $0 \leq \check{\Psi}^{\check{\epsilon}} \leq -\ln(1 - \check{\epsilon})$. Since the upper bound in the last line does not depend on the strategy \mathcal{S} ,

$$\check{R}^{\check{\epsilon}} \leq \mathbb{E}[R_{loc}^0(v_{\infty})] + \check{\epsilon}^{1/2} \mathbb{E}[\check{Q}_{loc}(v_{\infty}) \mathbb{I}_{v_{\infty} \in \mathcal{I}^0}] + \check{\epsilon}^{1/2} C \mathbb{E}[\mathbb{I}_{v_{\infty} \in \mathcal{K}^{\check{\epsilon}}}] + \check{\epsilon}^{3/4} C \mathbb{E}[1 + v_{\infty}^m].$$

Hence the desired inequality (4.46) follows from Corollary 4.17. \square

4.C.4 Proof of Theorem 4.15: Lower Bound (4.47)

As before, we assume without loss that $v_0 \in \text{int}(\mathcal{I}^0)$ and $b_0 = b^*(v_0)$. Let $\mathcal{S} = \{\tau_k, \pi_k\}_{k \geq 0}$ be the Morton-Pliska strategy corresponding to the no-trading region (4.50) constructed in Section 4.5. This strategy is admissible since the paths of the corresponding state process $\{v_t\}_{t \geq 0}$ are continuous. In the following we establish leading-order optimality of this strategy.

Step 1: Lower Bound for $f(v, b)$. The leading-order VIs (4.36) and Lemma 4.18 imply that

$$\begin{aligned} f(v_s, b_s) &= R_{loc}^0(v_s) - \check{\epsilon}^{1/2} \frac{1}{2} \sigma(v_s)^2 \left(\frac{b_s - b^*(v_s)}{\check{\epsilon}^{1/4}} \right)^2 \mathbb{I}_{v_s \in \mathcal{I}^0} \\ &= R_{loc}^0(v_s) + \check{\epsilon}^{1/2} \check{Q}_{loc}(v_s) \mathbb{I}_{v_s \in \mathcal{I}^0} \\ & \quad - \check{\epsilon}^{1/2} \sigma(v_s)^2 D(v_s) \check{\psi}_{\xi\xi}(v_s, \frac{b_s - b^*(v_s)}{\check{\epsilon}^{1/4}}) \mathbb{I}_{v_s \in \mathcal{I}^0} \\ &\geq R_{loc}^0(v_s) + \check{\epsilon}^{1/2} \check{Q}_{loc}(v_s) \mathbb{I}_{v_s \in \mathcal{I}^0} \\ & \quad - \mathcal{L} \check{\Psi}^{\check{\epsilon}}(v_s, b_s) - \check{\epsilon}^{1/2} C \mathbb{I}_{v_s \in \mathcal{K}^{\check{\epsilon}}} - \check{\epsilon}^{3/4} C (1 + v_s^m). \end{aligned}$$

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Step 2: Lower Bound. Using *Step 1* and Itô's formula we obtain

$$\begin{aligned} & \frac{1}{T} \mathbb{E} \left[\int_0^T f(v_s, b_s) ds + \ln(1 - \check{\epsilon}) \sum_{k=1}^{\infty} \mathbb{I}_{\tau_k \leq T} \right] \\ & \geq \frac{1}{T} \mathbb{E} \left[\int_0^T R_{loc}^0(v_s) ds + \check{\epsilon}^{1/2} \int_0^T \check{Q}_{loc}(v_s) \mathbb{I}_{v_s \in \mathcal{I}^0} ds - \check{\epsilon}^{1/2} \int_0^T C \mathbb{I}_{v_s \in \mathcal{K}^{\check{\epsilon}}} ds \right. \\ & \quad - \check{\epsilon}^{3/4} \int_0^T C (1 + v_s^m) ds + \check{\Psi}^{\check{\epsilon}}(v_0, b_0) - \check{\Psi}^{\check{\epsilon}}(v_T, b_T) \\ & \quad \left. + \sum_{k=1}^{\infty} \left(\check{\Psi}^{\check{\epsilon}}(v_{\tau_k}, b_{\tau_k}) - \check{\Psi}^{\check{\epsilon}}(v_{\tau_k}, b_{\tau_k-}) + \ln(1 - \check{\epsilon}) \right) \mathbb{I}_{\tau_k \leq T} \right]. \end{aligned}$$

To obtain the lower bound in (4.47) it remains to estimate the sum in the preceding inequality. Note that by construction of $\check{\Psi}^{\check{\epsilon}}$ all summands in the sum vanish, except those where τ_k coincides with either $\underline{\rho}_{2l+1}$ or $\bar{\rho}_{2l+1}$. Here, similarly as in the proof of (4.40) in Appendix 4.B.5, for every $l \geq 0$

$$\bar{\rho}_{2l+1} \triangleq \inf\{t \geq \bar{\rho}_{2l} : v_t = \bar{v}^0\}, \quad \bar{\rho}_{2l+2} \triangleq \inf\{t \geq \bar{\rho}_{2l+1} : v_t = \sup \mathcal{I}^{\check{\epsilon}}\}$$

and

$$\underline{\rho}_{2l+1} \triangleq \inf\{t \geq \underline{\rho}_{2l} : v_t = \underline{v}^0\}, \quad \underline{\rho}_{2l+2} \triangleq \inf\{t \geq \underline{\rho}_{2l+1} : v_t = \inf \mathcal{I}^{\check{\epsilon}}\}$$

with $\underline{\rho}_0 \triangleq \bar{\rho}_0 \triangleq 0$. Thus, as in the case with proportional transaction costs, by Lemmas 4.14 and 4.20 applied to the intervals $(\underline{v}^0, \inf \mathcal{I}^{\check{\epsilon}})$ and $(\sup \mathcal{I}^{\check{\epsilon}}, \bar{v}^0)$ we find two constants $K > 0$ and $n \geq m$ such that for all $\check{\epsilon} < \check{\epsilon}_0$

$$\begin{aligned} \mathbb{E}[\ln Z_T] & \geq \mathbb{E} \left[\int_0^T R_{loc}^0(v_s) ds + \check{\epsilon}^{1/2} \int_0^T \check{Q}_{loc}(v_s) \mathbb{I}_{v_s \in \mathcal{I}^0} ds \right. \\ & \quad \left. - K - \check{\epsilon}^{1/2} \int_0^T K \mathbb{I}_{v_s \in \mathcal{K}^{\check{\epsilon}}} ds - \check{\epsilon}^{3/4} \int_0^T K (1 + v_s^n) ds \right]. \end{aligned}$$

Hence, using assumption (A1), for every $\check{\epsilon} < \check{\epsilon}_0$ we get

$$\check{R}^{\check{\epsilon}} \geq R^0 + \check{\epsilon}^{1/2} \mathbb{E}[\check{Q}_{loc}(v_{\infty}) \mathbb{I}_{v_{\infty} \in \mathcal{I}^0}] - \check{\epsilon}^{1/2} K \mathbb{E}[\mathbb{I}_{v_{\infty} \in \mathcal{K}^{\check{\epsilon}}}] - \check{\epsilon}^{3/4} K \mathbb{E}[1 + v_{\infty}^n]$$

and the result follows from Corollary 4.17. \square

4.D Proof of Theorem 4.21 in Section 4.6

The proof of Theorem 4.21 is subdivided into several steps.

Step 1: Trading Strategy and Auxiliary Definitions. Let $\{\tau_k, \pi_k\}_{k \geq 0}$ be the Morton-Pliska strategy corresponding to the no-trading region $\mathcal{NT}^{q, \check{\epsilon}}$ defined by (4.54).

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The corresponding state process will be denoted by $\{(v_t, b_t)\}_{t \geq 0}$. As above, we define

$$\underline{v}^{\hat{\epsilon}} \triangleq \begin{cases} \inf\{v \in \mathcal{I} : g_+^{q, \hat{\epsilon}}(v) \in (0, 1) \text{ and } g_-^{q, \hat{\epsilon}}(v) \in (0, 1)\}, & \underline{v}^0 > \underline{v} \\ \underline{v}, & \underline{v}^0 = \underline{v} \end{cases}$$

and

$$\bar{v}^{\hat{\epsilon}} \triangleq \begin{cases} \sup\{v \in \mathcal{I} : g_+^{q, \hat{\epsilon}}(v) \in (0, 1) \text{ and } g_-^{q, \hat{\epsilon}}(v) \in (0, 1)\}, & \bar{v}^0 < \bar{v} \\ \bar{v}, & \bar{v}^0 = \bar{v} \end{cases}$$

where $g_{\pm}^{q, \hat{\epsilon}}(v) \triangleq b^*(v) \pm \hat{\epsilon}^{1/3} q(v)$ for $v \in \mathcal{I}$. Similarly as in Proposition 4.28, the mean value theorem yields an $\hat{\epsilon}_0 > 0$ and constants $C_1 > 0$ and $C_2 > 0$ such that for all $\hat{\epsilon} < \hat{\epsilon}_0$ the set $\{v \in \mathcal{I} : g_+^{q, \hat{\epsilon}}(v) \in (0, 1) \text{ and } g_-^{q, \hat{\epsilon}}(v) \in (0, 1)\}$ is non-empty and

$$\begin{aligned} \hat{\epsilon}^{1/3} C_1 \leq |\underline{v}^0 - \underline{v}^{\hat{\epsilon}}| \leq \hat{\epsilon}^{1/3} C_2 & \quad \text{if } \underline{v}^0 > \underline{v} \quad \text{and} \\ \hat{\epsilon}^{1/3} C_1 \leq |\bar{v}^0 - \bar{v}^{\hat{\epsilon}}| \leq \hat{\epsilon}^{1/3} C_2 & \quad \text{if } \bar{v}^0 < \bar{v}. \end{aligned}$$

This implies that there is a constant $C_3 > 0$ such that for all $\hat{\epsilon} < \hat{\epsilon}_0$

$$\mathbb{P}(v_{\infty} \in \mathcal{K}^{q, \hat{\epsilon}}) \leq \hat{\epsilon}^{1/3} C_3, \quad \text{where } \mathcal{K}^{q, \hat{\epsilon}} \triangleq \text{cl}((\underline{v}^0, \underline{v}^{\hat{\epsilon}}) \cup (\bar{v}^{\hat{\epsilon}}, \bar{v}^0)). \quad (4.76)$$

Step 2: Itô Formula. Next we define a subsequence $\{\rho_k\}_{k \geq 0}$ of $\{\tau_k\}_{k \geq 0}$ via $\rho_0 \triangleq 0$ and

$$\rho_{2k+1} \triangleq \inf\{t \geq \rho_{2k} : v_t \in \{\underline{v}^0, \bar{v}^0\}\}, \quad \rho_{2k+2} \triangleq \inf\{t \geq \rho_{2k+1} : v_t \in \{\underline{v}^{\hat{\epsilon}}, \bar{v}^{\hat{\epsilon}}\}\}$$

for every $k \geq 0$. We assume without loss of generality that $Z_0 = 1$. Itô's formula yields

$$\begin{aligned} & \mathbb{E}[\ln Z_T] \\ &= \mathbb{E}\left[\int_0^T f(v_s, b_s) ds + \sum_{k=1}^{\infty} (\ln Z_{\tau_k} - \ln Z_{\tau_k-}) \mathbb{I}_{\tau_k \leq T}\right] \\ &= \sum_{k=0}^{\infty} \mathbb{E}\left[\int_{T \wedge \rho_{2k}}^{T \wedge \rho_{2k+1}} f(v_s, b_s) ds - \sum_{l=1}^{\infty} (\ln Z_{\tau_l} - \ln Z_{\tau_l-}) \mathbb{I}_{\tau_l \in (T \wedge \rho_{2k}, T \wedge \rho_{2k+1})}\right] \\ & \quad + \sum_{k=0}^{\infty} \mathbb{E}\left[\int_{T \wedge \rho_{2k+1}}^{T \wedge \rho_{2k+2}} f(v_s, b^*(v_{\rho_{2k+1}})) ds\right] \\ & \quad + \sum_{k=1}^{\infty} \mathbb{E}\left[(\ln Z_{\rho_k} - \ln Z_{\rho_k-}) \mathbb{I}_{\rho_k \leq T}\right]. \end{aligned} \quad (4.77)$$

4.D Proof of Theorem 4.21 in Section 4.6

Step 3: Estimates for the Jump Terms. By (4.51) we can estimate the costs incurred by the Morton-Pliska strategy, see *line 3* in (4.77), via

$$\ln Z_{\tau_k} - \ln Z_{\tau_k-} \geq -\frac{\hat{\epsilon}^{4/3}q(v_{\tau_k})}{1-2\hat{\epsilon}} \quad (4.78)$$

Furthermore, for those $k \geq 0$ with $\tau_k \notin \{\rho_k\}_{k \geq 0}$ we also have the upper bound

$$\ln Z_{\tau_k} - \ln Z_{\tau_k-} \leq -\frac{\hat{\epsilon}^{4/3}q(v_{\tau_k})}{1+2\hat{\epsilon}}. \quad (4.79)$$

Step 4: Estimate of Line 3 in (4.77). We consider the function ψ^q defined by

$$\psi^q(v, \xi) \triangleq \frac{1}{24D(v)}\xi^4 - \frac{1}{2}\left(\frac{q(v)^2}{12D(v)} + \frac{2}{q(v)}\right)\xi^2 + q(v), \quad (v, \xi) \in \mathcal{I} \times \mathbb{R}.$$

Note that $\psi^q(v, 0) = q(v)$, $\psi^q(v, \pm q(v)) = 0$ and

$$\hat{\epsilon}^{2/3}\sigma(v)^2 D(v) \psi_{\xi\xi}^q\left(v, \frac{b-b^*(v)}{\hat{\epsilon}^{1/3}}\right) = \frac{1}{2}\sigma(v)^2 (b-b^*(v))^2 + \hat{\epsilon}^{2/3}Q_{loc}^q(v).$$

Hence

$$\begin{aligned} f(v, b) &= R_{loc}^0(v) - \hat{\epsilon}^{2/3}\frac{1}{2}\sigma(v)^2 \left(\frac{b-b^*(v)}{\hat{\epsilon}^{1/3}}\right)^2 \mathbb{I}_{v \in \mathcal{I}^0} \\ &= R_{loc}^0(v) + \hat{\epsilon}^{2/3}Q_{loc}^q(v) \mathbb{I}_{v \in \mathcal{I}^0} \\ &\quad - \hat{\epsilon}^{2/3}\sigma(v)^2 D(v) \psi_{\xi\xi}^q\left(v, \frac{b-b^*(v)}{\hat{\epsilon}^{1/3}}\right) \mathbb{I}_{v \in \mathcal{I}^0}. \end{aligned} \quad (4.80)$$

In addition, if we define

$$\Psi_{\pm}^{q, \hat{\epsilon}}(v, b) \triangleq \frac{\hat{\epsilon}^{4/3}}{1 \pm 2\hat{\epsilon}} \psi^q\left(v, \frac{b-b^*(v)}{\hat{\epsilon}^{1/3}}\right), \quad (v, b) \in \mathcal{I} \times [0, 1]$$

we can represent $\mathcal{L}\Psi_{\pm}^{q, \hat{\epsilon}}$ as follows:

$$\begin{aligned} \mathcal{L}\Psi_{\pm}^{q, \hat{\epsilon}}(v, b) &= \hat{\epsilon}^{2/3}\sigma(v)^2 D(v) \psi_{\xi\xi}^q\left(v, \frac{b-b^*(v)}{\hat{\epsilon}^{1/3}}\right) \mp \frac{\hat{\epsilon}^{5/3}}{1 \pm 2\hat{\epsilon}}\sigma(v)^2 D(v) \psi_{\xi\xi}^q\left(v, \frac{b-b^*(v)}{\hat{\epsilon}^{1/3}}\right) \\ &\quad + \frac{\hat{\epsilon}}{1 \pm 2\hat{\epsilon}} \psi_{\xi\xi}^q\left(v, \frac{b-b^*(v)}{\hat{\epsilon}^{1/3}}\right) \left(-\rho\beta(v)\sigma(v)b_v^*(v)(1-b^*(v)-b)\frac{b-b^*(v)}{\hat{\epsilon}^{1/3}} \right. \\ &\quad \quad \left. + \frac{1}{2}\sigma(v)^2 (b(1-b) + b^*(v)(1-b^*(v)))(1-b-b^*(v))\frac{b-b^*(v)}{\hat{\epsilon}^{1/3}} \right) \\ &\quad + \frac{\hat{\epsilon}}{1 \pm 2\hat{\epsilon}} \psi_{\xi v}^q\left(v, \frac{b-b^*(v)}{\hat{\epsilon}^{1/3}}\right) \left(\rho\beta(v)\sigma(v)b(1-b) - \beta(v)^2 b_v^*(v) \right) \\ &\quad + \frac{\hat{\epsilon}}{1 \pm 2\hat{\epsilon}} \psi_{\xi}^q\left(v, \frac{b-b^*(v)}{\hat{\epsilon}^{1/3}}\right) \left(-\alpha(v)b_v^*(v) + b(1-b)(\lambda(v) - b\sigma(v)^2) - \frac{1}{2}\beta(v)^2 b_{vv}^*(v) \right) \\ &\quad + \frac{\hat{\epsilon}^{4/3}}{1 \pm 2\hat{\epsilon}} \psi_{vv}^q\left(v, \frac{b-b^*(v)}{\hat{\epsilon}^{1/3}}\right) \frac{1}{2}\beta(v)^2 \\ &\quad + \frac{\hat{\epsilon}^{4/3}}{1 \pm 2\hat{\epsilon}} \psi_v^q\left(v, \frac{b-b^*(v)}{\hat{\epsilon}^{1/3}}\right) \alpha(v). \end{aligned}$$

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Thus there exist $m \in \mathbb{N}$ and $C_4 > 0$ such that for all $\hat{\epsilon} < \hat{\epsilon}_0$, $v \in \mathcal{I}^0$ and $b \in (g_-^{q,\hat{\epsilon}}(v), g_+^{q,\hat{\epsilon}}(v))$

$$\left| \mathcal{L} \Psi_{\pm}^{q,\hat{\epsilon}}(v, b) - \hat{\epsilon}^{2/3} \sigma(v)^2 D(v) \psi_{\xi\xi}^q(v, \frac{b-b^*(v)}{\hat{\epsilon}^{1/3}}) \right| \leq \hat{\epsilon} C_4 (1 + v^m). \quad (4.81)$$

Hence, combining (4.78), (4.80) and (4.81) and using Itô's formula, we obtain an estimate for the terms in *line 3* of (4.77):

$$\begin{aligned} & \mathbb{E} \left[\int_{\rho_{2k} \wedge T}^{\rho_{2k+1} \wedge T} f(v_s, b_s) ds + \sum_{l=1}^{\infty} (\ln Z_{\tau_l} - \ln Z_{\tau_{l-}}) \mathbb{I}_{\tau_l \in (\rho_{2k} \wedge T, \rho_{2k+1} \wedge T)} \right] \\ & \geq \mathbb{E} \left[\int_{\rho_{2k} \wedge T}^{\rho_{2k+1} \wedge T} R_{loc}^0(v_s) ds + \hat{\epsilon}^{2/3} \int_{\rho_{2k} \wedge T}^{\rho_{2k+1} \wedge T} Q_{loc}^q(v_s) ds \right. \\ & \quad \left. - \int_{\rho_{2k} \wedge T}^{\rho_{2k+1} \wedge T} \mathcal{L} \Psi_{\mp}^{q,\hat{\epsilon}}(v_s, b_s) ds - \sum_{l=1}^{\infty} \frac{\hat{\epsilon}^{4/3}}{1 \mp \hat{\epsilon}} q(v_{\tau_l}) \mathbb{I}_{\tau_l \in (\rho_{2k} \wedge T, \rho_{2k+1} \wedge T)} \right. \\ & \quad \left. \mp \hat{\epsilon} \int_{\rho_{2k} \wedge T}^{\rho_{2k+1} \wedge T} C_4 (1 + v_s^m) ds \right] \\ & = \mathbb{E} \left[\int_{\rho_{2k} \wedge T}^{\rho_{2k+1} \wedge T} R_{loc}^0(v_s) ds + \hat{\epsilon}^{2/3} \int_{\rho_{2k} \wedge T}^{\rho_{2k+1} \wedge T} Q_{loc}^q(v_s) ds \right. \\ & \quad \left. + [\Psi_{\mp}^{q,\hat{\epsilon}}(v_{\rho_{2k}}, b_{\rho_{2k}}) - \Psi_{\mp}^{q,\hat{\epsilon}}(v_{\rho_{2k+1} \wedge T}, b_{\rho_{2k+1} \wedge T-})] \mathbb{I}_{\rho_{2k} < T} \right. \\ & \quad \left. \mp \hat{\epsilon} \int_{\rho_{2k} \wedge T}^{\rho_{2k+1} \wedge T} C_4 (1 + v_s^m) ds \right]. \end{aligned}$$

Note that by definition $\Psi_{\mp}^{q,\hat{\epsilon}}(v_{\rho_{2k}}, b_{\rho_{2k}})$ equals either $\frac{\hat{\epsilon}^{4/3}}{1 \mp \hat{\epsilon}} q(v^{\hat{\epsilon}})$ or $\frac{\hat{\epsilon}^{4/3}}{1 \mp \hat{\epsilon}} q(\bar{v}^{\hat{\epsilon}})$ for all $k \geq 1$.

Step 5: Estimate of Line 4 in (4.77). By definition of Q_{loc}^q and *Step 1*, there exists a constant $C_5 > 0$ such that for all $\hat{\epsilon} < \hat{\epsilon}_0$

$$\begin{aligned} |Q_{loc}^q(v)| \mathbb{I}_{v \in \mathcal{K}^{q,\hat{\epsilon}}} &+ \left| \frac{1}{2} \sigma(v)^2 \frac{(b^*(v^0) - b^*(v))^2}{\hat{\epsilon}^{2/3}} \right| \mathbb{I}_{v \in [v^0, v^{\hat{\epsilon}}]} \mathbb{I}_{v^0 > v} \\ &+ \left| \frac{1}{2} \sigma(v)^2 \frac{(b^*(\bar{v}^0) - b^*(v))^2}{\hat{\epsilon}^{2/3}} \right| \mathbb{I}_{v \in [\bar{v}^{\hat{\epsilon}}, \bar{v}^0]} \mathbb{I}_{\bar{v}^0 < \bar{v}} \leq C_5 \mathbb{I}_{v \in \mathcal{K}^{q,\hat{\epsilon}}}. \end{aligned}$$

Thus we obtain the following upper and lower bounds for the integral in *line 4* of (4.77):

$$\begin{aligned} & \mathbb{E} \left[\int_{\rho_{2k+1} \wedge T}^{\rho_{2k+2} \wedge T} f(v_s, b_s) ds \right] \\ & = \mathbb{E} \left[\int_{\rho_{2k+1} \wedge T}^{\rho_{2k+2} \wedge T} R_{loc}^0(v_s) ds - \int_{\rho_{2k+1} \wedge T}^{\rho_{2k+2} \wedge T} \frac{1}{2} \sigma(v_s)^2 (b_s - b^*(v_s))^2 \mathbb{I}_{v_s \in \mathcal{K}^{q,\hat{\epsilon}}} ds \right] \\ & \geq \mathbb{E} \left[\int_{\rho_{2k+1} \wedge T}^{\rho_{2k+2} \wedge T} R_{loc}^0(v_s) ds + \hat{\epsilon}^{2/3} \int_{\rho_{2k+1} \wedge T}^{\rho_{2k+2} \wedge T} Q_{loc}^q(v_s) \mathbb{I}_{v_s \in \mathcal{K}^{q,\hat{\epsilon}}} ds \right. \\ & \quad \left. \mp \hat{\epsilon}^{2/3} \int_{\rho_{2k+1} \wedge T}^{\rho_{2k+2} \wedge T} C_5 \mathbb{I}_{v_s \in \mathcal{K}^{q,\hat{\epsilon}}} ds \right]. \end{aligned}$$

4.D Proof of Theorem 4.21 in Section 4.6

Step 6: Combining the Estimates. Putting together the estimates from *Step 3*, *Step 4*, and *Step 5* we conclude that for every $\hat{\epsilon} < \hat{\epsilon}_0$ we have

$$\begin{aligned}
& \mathbb{E}[\ln Z_T] \\
& \stackrel{\geq}{\leq} \mathbb{E} \left[\int_0^T R_{loc}^0(v_s) ds + \hat{\epsilon}^{2/3} \int_0^T Q_{loc}^q(v_s) \mathbb{I}_{v_s \in \mathcal{I}^0} ds \right. \\
& \quad \mp \hat{\epsilon}^{2/3} \int_0^T C_5 \mathbb{I}_{v_s \in \mathcal{K}^{q,\hat{\epsilon}}} ds \mp \hat{\epsilon} \int_0^T C_4(1 + v_s^m) ds \\
& \quad + \sum_{k=0}^{\infty} [\Psi_{\mp}^{q,\hat{\epsilon}}(v_{\rho_{2k}}, b_{\rho_{2k}}) - \Psi_{\mp}^{q,\hat{\epsilon}}(v_{\rho_{2k+1} \wedge T}, b_{\rho_{2k+1} \wedge T-})] \mathbb{I}_{\rho_{2k} < T} \\
& \quad \left. + \sum_{k=1}^{\infty} (\ln Z_{\rho_k} - \ln Z_{\rho_{k-}}) \mathbb{I}_{\rho_k \leq T} \right] \\
& = \mathbb{E} \left[\int_0^T R_{loc}^0(v_s) ds + \hat{\epsilon}^{2/3} \int_0^T Q_{loc}^q(v_s) \mathbb{I}_{v_s \in \mathcal{I}^0} ds \right. \\
& \quad \mp \hat{\epsilon}^{2/3} \int_0^T C_5 \mathbb{I}_{v_s \in \mathcal{K}^{q,\hat{\epsilon}}} ds \mp \hat{\epsilon} \int_0^T C_4(1 + v_s^m) ds \\
& \quad + \Psi_{\mp}^{q,\hat{\epsilon}}(v_0, b_0) - \sum_{k=0}^{\infty} \mathbb{I}_{\rho_{2k} < T \leq \rho_{2k+1}} \Psi_{\mp}^{q,\hat{\epsilon}}(v_T, b_{T-}) \\
& \quad + \sum_{k=0}^{\infty} [-\Psi_{\mp}^{q,\hat{\epsilon}}(v_{\rho_{2k+1}}, b_{\rho_{2k+1}-}) \mathbb{I}_{\rho_{2k+1} < T} + \Psi_{\mp}^{q,\hat{\epsilon}}(v_{\rho_{2k+2}}, b_{\rho_{2k+2}}) \mathbb{I}_{\rho_{2k+2} < T}] \\
& \quad \left. + \sum_{k=1}^{\infty} (\ln Z_{\rho_k} - \ln Z_{\rho_{k-}}) \mathbb{I}_{\rho_k \leq T} \right] \\
& \stackrel{\geq}{\leq} \mathbb{E} \left[\int_0^T R_{loc}^0(v_s) ds + \hat{\epsilon}^{2/3} \int_0^T Q_{loc}^q(v_s) \mathbb{I}_{v_s \in \mathcal{I}^0} ds \right. \\
& \quad \mp \hat{\epsilon}^{2/3} \int_0^T C_5 \mathbb{I}_{v_s \in \mathcal{K}^{q,\hat{\epsilon}}} ds \mp \hat{\epsilon} \int_0^T C_4(1 + v_s^m) ds \\
& \quad \left. + \Psi_{\mp}^{q,\hat{\epsilon}}(v_0, b_0) \mp \frac{\hat{\epsilon}^{4/3}}{1 \mp 2\hat{\epsilon}} p^q(v_T) \mp \frac{\hat{\epsilon}^{4/3}}{1 \mp 2\hat{\epsilon}} C_6 \sum_{k=1}^{\infty} \mathbb{I}_{\rho_k \leq T} \right].
\end{aligned}$$

The last inequality is due to the fact that

$$|\psi^q(v, \xi)| \leq p^q(v) \triangleq \frac{q^4(v)}{12D(v)} + 2q(v) \quad \text{for all } \xi \in (-q(v), q(v)).$$

C_6 is an upper bound for $q(v) + p^q(v)$ for $v \in \{\underline{v}^0, \bar{v}^0, \underline{v}^{\hat{\epsilon}}, \bar{v}^{\hat{\epsilon}}\} \setminus \{\underline{v}, \bar{v}\}$. To finish the proof we use the up- and downcrossing lemma (Lemma 4.14) to estimate the expectation of the sum $\sum_{k=1}^{\infty} \mathbb{I}_{\rho_k \leq T}$ together with the fact that

$$|\mathcal{L}^v p^q(v)| \leq C_7(1 + v^n), \quad v \in \mathcal{I} \tag{4.82}$$

for some $C_7 > 0$ and $n \geq m$. Hence there exists a constants $C > 0$ such that

$$\begin{aligned}
\liminf_{T \rightarrow \infty} \frac{1}{T} \mathbb{E}[\ln Z_T] & \stackrel{\geq}{\leq} R^0 + \hat{\epsilon}^{2/3} \mathbb{E}[Q_{loc}^q(v_{\infty}) \mathbb{I}_{v_{\infty} \in \mathcal{I}^0}] \\
& \mp \hat{\epsilon}^{2/3} C \mathbb{E}[\mathbb{I}_{v_{\infty} \in \mathcal{K}^{q,\hat{\epsilon}}}] \mp \hat{\epsilon} C \mathbb{E}[1 + v_{\infty}^n]
\end{aligned}$$

for all $\hat{\epsilon} \in (0, \hat{\epsilon}_0)$. The claim now follows from (4.76). \square

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Scientific Career

- 06/2005 Ukrainian Higher Education Entrance Qualification
- 09/2005–06/2009 B. Sc. hon. in Mathematics
Kyiv National Taras Shevchenko University, Ukraine
- 10/2009–10/2011 M. Sc. with distinction in Financial Mathematics
University of Kaiserslautern, Germany
- since 10/2011 PhD Student of Prof. Dr. Ralf Korn
University of Kaiserslautern, Germany
- 04/2012-07/2013 Teaching assistant at the University of Kaiserslautern
- since 10/2014 Research fellow at the University of Kaiserslautern

Wissenschaftlicher Werdegang

- 06/2005 Ukrainische Hochschulreife
- 09/2005–06/2009 B. Sc. mit Auszeichnung in Mathematik
Nationale Taras Schevtschenko Universität Kyiv, Ukraine
- 10/2009–10/2011 M. Sc. mit Auszeichnung in Finanzmathematik
Technische Universität Kaiserslautern, Deutschland
- seit 10/2011 Doktorand von Prof. Dr. Ralf Korn
Technische Universität Kaiserslautern, Deutschland
- 04/2012-07/2013 Wissenschaftliche Hilfskraft
an der Technischen Universität Kaiserslautern
- seit 10/2014 Wissenschaftlicher Mitarbeiter
an der Technischen Universität Kaiserslautern