

Aspects and Applications of the Wilkie Investment Model

Vom Fachbereich Mathematik der
Technischen Universität Kaiserslautern
zur Verleihung des akademischen Grades
Doktor der Naturwissenschaften
(Doctor rerum naturalium, Dr. rer. nat.)
genehmigte

Dissertation

von

Norizarina Ishak

Gutachter:
Prof. Dr. Ralf Korn
Juniorprof. Dr. Noriszura Ismail

Datum der Disputation 30.07.2015

D386

Contents

Acknowledgement	v
Abstract	vi
1. Introduction	1
1.1. Background of the Research	1
1.2. Development of the Wilkie Model	2
1.3. Research Problems, Research Issues and Contributions	3
1.4. Outline of Dissertation	5
2. The Wilkie Model: The Basics in Discrete Time	7
2.1. Introduction	7
2.2. The Wilkie Model in Discrete Time	7
2.2.1. The Retail Prices Index Model	11
2.2.2. The Share Dividend Yield Model	12
2.2.3. The Share Dividend Index Model	13
2.2.4. The Consols Yield Model	14
2.3. Comments and Criticisms about the Wilkie Model	15
3. Stochastic Asset Liability Modelling: A Case of Malaysia	19
3.1. Introduction	19
3.2. Box-Jenkins Models	19
3.2.1. The Autoregressive Model	19
3.2.2. The Moving Average Model	21
3.2.3. The Autoregressive Moving Average Model	21
3.2.4. The Autoregressive Integrated Moving Average Model	22
3.2.5. The Seasonal Autoregressive Moving Average Integrated Model	22
3.3. Box-Jenkins Methodology	23
3.4. Malaysian Stochastic Asset Liability Model	28
3.4.1. Outline of the Approach	28
3.4.2. The Inflation Model	29
3.4.3. The FTSE Bursa Malaysia KLCI Yield Model	37
3.4.4. The FTSE Bursa Malaysia KLCI Model	45
3.4.5. The 10-Year MGS Yield Model	52
3.4.6. Summary of the Malaysian Stochastic Asset Liability Model	59

Contents

4. The Continuous-Time Model: A Description	61
4.1. The Ornstein-Uhlenbeck Process	61
4.2. The Continuous-Time Retail Prices Index Model	62
4.3. The Continuous-Time Share Dividend Yield Model	63
4.4. The Continuous-Time Share Dividend Index Model	64
4.5. The Continuous-Time Consols Yield Model	65
4.6. The Share Price in the Wilkie Model	67
5. Portfolio Optimisation in the Continuous-Time Wilkie Model	70
5.1. Introduction	70
5.2. Controlled Stochastic Differential Equations	70
5.3. Formulation of the Optimisation Problem	72
5.4. The Hamilton-Jacobi-Bellman-Equation	73
5.5. Algorithm to the Solution of the HJB-Equation	75
5.6. The Optimal Self-Financing Portfolio	75
5.6.1. States of the Control Process	77
5.6.2. Stochastic Control Methods	77
5.7. The Growth-Optimal Constant Portfolio	89
5.8. The Optimal Buy-and-Hold Portfolio	91
6. Conclusions	95
Appendices	98
A. Augmented Dickey-Fuller unit root test	99
B. Autocorrelation function	100
C. Partial autocorrelation function	101
D. Akaike information criterion	102
E. Bayesian information criterion	103
F. Ljung-box test	104
G. Multi-dimensional Itô formula	105

List of Figures

2.1.	The Wilkie model [Wilkie, 1984]	8
2.2.	The Wilkie model [Wilkie, 1995]	9
2.3.	The non-linear Wilkie model [Whitten and Thomas, 1999]	10
3.1.	The Box-Jenkins modelling approach	27
3.2.	Annual force of inflation, I_t : 1961 - 2012 together with its correlogram	30
3.3.	ACF plot for the first differenced inflation	32
3.4.	PACF plot for the first differenced inflation	33
3.5.	Output from residuals analysis for inflation	35
3.6.	ARIMA(0,1,2) model for inflation	37
3.7.	Monthly FTSE Bursa Malaysia KLCI yield, $Y(t)$: July 2009 - September 2013 together with its correlogram	38
3.8.	ACF plot for the first differenced FTSE Bursa Malaysia KLCI yield	40
3.9.	PACF plot for the first differenced FTSE Bursa Malaysia KLCI yield	41
3.10.	Output from residuals analysis for FTSE Bursa Malaysia KLCI yield	43
3.11.	ARIMA(1,1,0) model for FTSE Bursa Malaysia KLCI yield	45
3.12.	Monthly FTSE Bursa Malaysia KLCI, $D(t)$: January 1994 - December 2013 together with its correlogram	46
3.13.	Seasonally differenced FTSE Bursa Malaysia KLCI	48
3.14.	Output from residuals analysis for FTSE Bursa Malaysia KLCI	50
3.15.	ARIMA(2,1,1)(2,0,2)[12] model for FTSE Bursa Malaysia KLCI	52
3.16.	Monthly 10-Year MGS yield, $C(t)$: January 1996 - January 2014 together with its correlogram	53
3.17.	Seasonal differenced 10-Year MGS yield	55
3.18.	Output from residuals analysis for 10-Year MGS yield	57
3.19.	ARIMA(4,1,3)(0,1,1)[12] model for 10-Year MGS yield	59
5.1.	The relationship between deterministic Consols yield and time	86

List of Tables

2.1.	Estimated parameters of the retail prices index model	16
2.2.	Estimated parameters of the share dividend yield model	16
2.3.	Estimated parameters of the share dividend index model	17
2.4.	Estimated parameters of the Consols yield model	17
3.1.	ACF and PACF trends of the non-seasonal Box-Jenkins model	24
3.2.	Stationary and invertible conditions of the non-seasonal Box-Jenkins model	25
3.3.	Summary statistics for inflation	31
3.4.	AIC and BIC values of possible ARIMA models for inflation	34
3.5.	Forecast values of inflation for year 2013-2042	36
3.6.	Summary statistics for FTSE Bursa Malaysia KLCI yield	39
3.7.	AIC and BIC values of possible ARIMA models for FTSE Bursa Malaysia KLCI yield	42
3.8.	Forecast values of FTSE Bursa Malaysia KLCI yield for Oct 2013-Mac 2016	44
3.9.	Summary statistics for FTSE Bursa Malaysia KLCI	47
3.10.	Forecast values of FTSE Bursa Malaysia KLCI for Jan 2014-June 2016 . .	51
3.11.	Summary of the statistics for the 10-Year MGS yield model	54
3.12.	Forecast values of 10-Year MGS yield for Feb 2014-July 2016	58

Acknowledgement

First and foremost, I would like to express my sincere gratitude to my advisor, Professor Doctor Ralf Korn for his continuous support, patience, motivation and enthusiasm as well as sharing his enormous knowledge towards completing my Ph.D. His guidance has turned my Ph.D journey into the most rewarding experience in my entire life. It is indeed a great honour to be one of his Ph.D students.

Besides my advisor, I would also like to thank the members of the Stochastic Control and Financial Mathematics Group who had significantly contributed to my professional time at the University of Kaiserslautern. I really appreciate their encouragement and insightful comments.

My sincere thanks also goes to both of my sponsors, Universiti Sains Islam Malaysia and the Ministry of Education Malaysia for believing in me to pursue my Ph.D and easing my path in fulfilling my dreams.

Last but not least, I would like to thank my family and friends for supporting me spiritually throughout my life. From the bottom of my heart, I would like to dedicate this study especially for my late parents, Haji Ishak Hassan and Hajah Nursiah Abdul Samad. Their memories have been a source of inspiration to me. Their determination to do everything in their power to ensure that their children received adequate education although they were uneducated themselves have proven fruitful.

I also pray that this research will be a source of inspiration to my nephews and nieces so that they too can achieve their dreams particularly in academic if they are up the challenge.

Abstract

The Wilkie model is a stochastic asset model, developed by A.D. Wilkie in 1984 with the purpose to explore the behaviour of investment factors of insurers within the United Kingdom. Even so, thus far, there is still no analysis that studies the Wilkie model in a portfolio optimisation framework. Originally, the Wilkie model was considered as a discrete-time horizon and we applied the concept from the Wilkie model to develop a suitable ARIMA model for Malaysian data by using the Box-Jenkins methodology. We obtained the estimated parameters for each sub model within the Wilkie model that suited the cases in Malaysia, and consequently permitted us to analyse the result based on statistics and economics. We then reviewed the continuous time case which was initially introduced by Terence Chan in 1998. The continuous-time Wilkie model inspired framework was then employed to develop the wealth equation of a portfolio that consisted of a bond and a stock. We are interested in building portfolios based on three well-known trading strategies, a self-financing strategy, a constant growth optimal strategy as well as a buy-and-hold strategy. In dealing with the portfolio optimisation problems, we used the stochastic control technique consisting of the maximisation problem itself, the Hamilton-Jacobi-equation, the solution to the Hamilton-Jacobi-equation and finally the verification theorem. In finding the optimal portfolio, we obtained the specific solution of the Hamilton-Jacobi-equation and proved the validity of the solution via the verification theorem. For a simple buy-and-hold strategy, we used the mean-variance analysis to solve the portfolio optimisation problem.

1. Introduction

1.1. Background of the Research

According to the Cambridge Dictionary, the meaning of investment is described as "the act of putting money, effort, time, etc. into something, to make a profit or get an advantage, or the money, effort, time etc.". In finance according to Investopedia, investment simply means that an investor will buy assets and sell them in the future to gain profit. The expectation is that the price of the asset will increase later in the future. The investor expects to gain from the investment even though somehow there is a possibility of losing. The possibility of losing is the risk that the investor has to bear with. All types of investment involve some forms of risk; for example, equities investment, fixed interest securities and property are open to inflation risk. Financial assets range from low risk ones, such as government bonds, to the high risk such as international stocks.

Economics and investment are often interlinked, where for example, fixed interest loans and securities are considered as low risk financial assets and have become the main investment choice for insurance companies when the economic status is at low yields. Despite the rise in the need for that kind of investment since the middle of the 20th century, life offices and pension funds are taking a step forward by investing more in ordinary shares offering a higher risk. The ordinary share is an equity share that allows ownership privilege in a company, depending on the percentage of the shares in the company. Ordinary shares are affected by price inflation in the market. One factor that may contribute to this is the economic behaviour at a certain time, which in turn is influenced by other factors such as management decision making which might come from a central bank that rapidly increases the supply of money. Therefore, the demand for goods and services in the economy rises more rapidly than the economic productive capacity. Another factor is the increase of the production process input. Rapid wage increments or rising raw material prices are common causes of this type of inflation. Thus, the inflation of retail prices become an important growing feature and fixed interest rates have continued to increase. Investment decision making and management have become a serious matter to these type of wealth institutions. Therefore, a basic investment model should require at least an investigation about inflation, ordinary shares as well as fixed interest securities.

Investment modelling can be divided into two categories; single-asset and multi-asset models. Single-asset models may include interest rates, term structure, stock price and inflation models. The interest rate model is designed to model the price of fixed income asset. This is achieved by looking at the relation between interest rates and fixed income asset. Examples of this type of modelling are the famous Cox-Ingersoll-Ross model, the

1. Introduction

Ho-Lee model, the Hull-White model and the Vasicek model. These four models are one-factor interest rate models which acknowledge only one common factor, usually market returns. The term structure model is nothing more or less than a model of zero-coupon bond prices. It is particularly used to determine the spot rate based on one bond data. The LIBOR market model is an example of the term structure model. On the other hand, the stock price model is similar to binomial and Black-Scholes models where the Black-Scholes model focuses on the geometric Brownian motion whereas, the model adapted in this study which is the multi-asset model, compares two or more factors and analyse relationships between variables and the security's resulting performance. Examples of this type of model are the Cairns model, the Whitten & Thomas model and the Wilkie model.

In recent years, the stochastic investment modelling has become a great concern among actuaries and financial experts around the world. At this moment, the stochastic modelling has been used substantially in modelling investment returns in order to obtain the distribution for the variable of interest whereas the ordinary (deterministic) models are only able to give the results of a single expected return. In addition, this type of investment model provides a range of possible investment returns and can also be a powerful tool to forecast investment returns in the long run. The stochastic model uses prior data and combines it with present data to forecast the future of investment returns. Data from the past gives us information about the overall works of economy, such as

- how the different economic factors, i.e. the inflation rates gave impact on the different assets class,
- volatility,
- how much extra returns are required for extra risks, i.e. the equity risk premium,
- how frequent market shocks happened, i.e. the equity crashes.

This is why the Wilkie model was built, to consider the stochastic aspect for multi factors of investment.

1.2. Development of the Wilkie Model

The Wilkie model was designed by A.D Wilkie in 1984 and was presented to the Faculty of Actuaries [Wilkie, 1984]. The Wilkie model is an investment model to facilitate the factors influencing the returns of an investment. The factors studied were inflation, share dividend index, share dividend yield and Consols yield. Consols is a type of government bond in Britain. The method of building the Wilkie model was fundamentally derived from the idea Box and Jenkins developed in 1976. Most of the parameters were derived from a least square estimation technique calculated by a non-liner optimisation method or in practice it is referred to as the Nelder-Mead simplex method. Most models (the factors) associated with the Wilkie model are considered as stationary. Some models in

1. Introduction

the Wilkie model are co-integrated. As an example, Wilkie [Wilkie, 1984][Wilkie, 1995] represented the share dividend yield as the share dividend index divided by the price index and logarithm was used for the operation.

The original Wilkie model was built from the United Kingdom investment data over the interval 1919-1982. The Financial Management Group (FIMAG) working party later suggested that the Wilkie model should be examined using the post-1945 data and should include recent data as much as possible. This is because the fundamental changes happened before and after World War II. In fact, A.D Wilkie had updated his research using the year interval 1945-1982 [Wilkie, 1995]. The reason behind the annual data is that it showed long-term investment performance. As mentioned by [Wilkie, 1992]; "where the discrete models are equivalent to a continuous diffusion process, it is often the case that they are indistinguishable from a random walk when the observation period is sufficiently short. The "noise" overwhelms the signal". This may explain why observations over too short a period have not observed the longer term stabilities represented in the models of [Wilkie, 1984] and [Tilley, 1990].

There are many studies related to the Wilkie model after its inception. The Wilkie model has become a huge reference to life offices to evaluate their investment performance. Many applications of the Wilkie model have also been studied in the areas of actuarial work, specific asset and liability management, pension funds, life assurance, investment management and general insurance.

1.3. Research Problems, Research Issues and Contributions

Insurers profit in two ways, first by investing the premium they obtained from the policy holders, and secondly via underwriting, which is the process of selecting the risk to insure and deciding the suitable premium to be paid by the policy holders in order to bear the risk. Apart from that, modelling the investment of life offices is important to ensure the continuing profit of the life offices.

Life offices have used a range of tools available to manage the risk and also to model its investment. Currently, it is common practice to use computer packages to generate scenarios using a stochastic model. Using the stochastic investment model, we can simulate the possible returns for many years in the future. Initially, the ideas were generated by the Maturity Guarantees Working Party (MGWP) which were presented in 1980. Then the ideas were continuously developed by A.D Wilkie in 1981 [Wilkie, 1981].

The stochastic investment model can be applied to deterministic time as well as continuous time. This research fully utilises the Wilkie model and focuses on both time frameworks. In continuous time setting there are no restrictions in the selection of unit of time and we are able to model the various investment variables at any time. [Wilkie, 1984] introduced a stochastic investment model based on time series and this model was later updated in Wilkie [Wilkie, 1995]. The Wilkie model used a discrete time

1. Introduction

setting but a new approach was taken by Terence Chan which transformed the discrete time Wilkie model to a continuous time setting [Chan, 1998a].

The stochastic investment model can be used by actuaries in many applications including portfolio selection. The first portfolio selection model was developed by [Markowitz, 1952] which is the simplest model of an investment with a single time period and a set of possible investments. In his model, expected returns, variances and covariances are all assumed to be known. The simulated returns lead to the method of selecting the optimum portfolio over a period of time. This is where the portfolio optimisation plays its role which will be one of the scope in this study.

To conclude this section, we enlist the aims as follows:

1. To study the development of the Wilkie model.
2. To explore the discrete-time framework of the Wilkie model.
3. To apply the concept of the discrete-time Wilkie model in modelling the Malaysian investment data.
4. To discover a continuous-time framework of the Wilkie model.
5. To apply the continuous-time Wilkie model in portfolio optimisation problems.

Concurrently, the following are the objectives of this study:

1. To investigate the transformation of the Wilkie model from discrete-time to continuous-time.
2. To develop a suitable Autoregressive Integrated Moving Average model (ARIMA) according to Malaysian data.
3. To analyse the new ARIMA model for Malaysian data.
4. To construct a wealth equation based on the continuous-time Wilkie model and a self-financing trading strategy.
5. To construct a wealth equation based on the continuous-time Wilkie model and a constant growth portfolio.
6. To construct a wealth equation based on the continuous-time Wilkie model and a buy-and-hold trading strategy.
7. To build a Hamilton-Jacobi-Bellman equation for the self-financing wealth equation.
8. To solve the portfolio optimisation problem with respect to the self-financing trading strategy using stochastic control method.
9. To solve the portfolio optimisation problem with respect to the constant growth portfolio using the stochastic control method.

10. To solve the portfolio optimisation problem with respect to the buy-and-hold trading strategy using the mean-variance analysis.

1.4. Outline of Dissertation

This dissertation uses the basic Wilkie model developed in 1984 as a benchmark to other enhanced settings. Therefore, we begin this dissertation with a general introduction to the research which we named as Chapter One, which also includes the background of the research. In the sub section, we explain the importance of investment management to any financial institution or to be exact, to life offices. We also relate the economics and financial factors that contribute to investment decision. Next, we highlight the use of the investment model in a modern world especially the stochastic investment model which utilises random data. The introduction chapter further explains the development of the Wilkie model where we focus on the Wilkie model itself as well as keep track of almost all studies related to it. Then, we explain our aims and objectives as well as research problems that motivated us to do this research. This chapter ends with an outline of the dissertation.

In Chapter Two, we explain in detail the structure of the Wilkie model itself, paying special attention to the discrete time framework. This covers the cascade structure of the Wilkie model built in 1984 and also some extra parameters added by A.D Wilkie in 1995. Overall, Chapter Two provides the explanation of each sub models in the Wilkie model such as the retail price index model, the share dividend index model, the share dividend yield model and the Consols yield model.

The methods employed in this research are classified into two categories, one dealing with the discrete time framework while the other deals with continuous time framework. Thus in Chapter Three, we will use the concept of the Wilkie model to build the ARIMA models for Malaysian investment data. The data are obtained from financial websites hosted by the Malaysian government as well as international bodies. The data used for simulation were Consumer Prices Index (CPI), FTSE Bursa Malaysia KLCI and 10-YR Malaysian Government Security (MGS). We then construct an ARIMA model for each sub models in the Wilkie model based on these data. We compare the new ARIMA model with the original Wilkie model which was developed based on the UK investment data. We also analyse the results of our simulation.

Chapter Four focuses on the continuous-time Wilkie model introduced by Terence Chan. We comprehensively explain all four sub models in the Wilkie model but in a continuous-time framework. We make a comparison with the content in Chapter Two in order to see clearly the transformation of each variable to a new time setting. We simply list down the variables representing the discrete and continuous time. Prior to that, we also discuss the famous Ornstein-Uhlenbeck process which will be used to develop the continuous-time Wilkie model.

We continue this dissertation with the application of the continuous-time Wilkie model to portfolio optimisation, which is discussed at length in Chapter Five. We use the

1. Introduction

Wilkie model to establish a wealth equation corresponding to three cases, a self-financing trading strategy, a constant growth portfolio and a buy-and-hold trading strategy. For these three strategies, we solve them by considering some generalisations. We will also attempt to prove the solution and introduce new theorems towards our solutions. We end this dissertation with Chapter Six by concluding all the works involved throughout the research.

2. The Wilkie Model: The Basics in Discrete Time

2.1. Introduction

Our aim in this chapter is to give a review of the Wilkie model and previous works on refining the model. The review contains two major sections where in the first section we discuss the four basic models in the Wilkie model, the retail prices index model, the share dividend yield model, the share dividend index model and the Consols yield model. We also illustrate the correlation between each models. In the second section, we identify and list out comments and criticisms about the Wilkie model from previous researches. This section will also cover the statistical and economical aspects of the Wilkie model itself.

2.2. The Wilkie Model in Discrete Time

[Wilkie, 1984] had proposed a linear stochastic asset model and it consisted of four sub models as follows:

- A retail prices index model (also known as an inflation model).
- A share dividend yield model.
- A share dividend index model.
- A Consols yield model (also known as a long term interest rate model).

Although the 1984 Wilkie model consisted of four models or factors, it did not work with a full multivariate structure where each factor could affect each other. The Wilkie model used a cascade structure where price inflation influenced other asset returns. As we go through the model in detail in the next section, we can see that the Wilkie model is a partial cascade model because each model had different random variables. As demonstrated in figure 2.1 below, the arrows represent the direction of influence between each variable. We can visibly see that the other models are driven by the retail prices index model.

2. The Wilkie Model: The Basics in Discrete Time

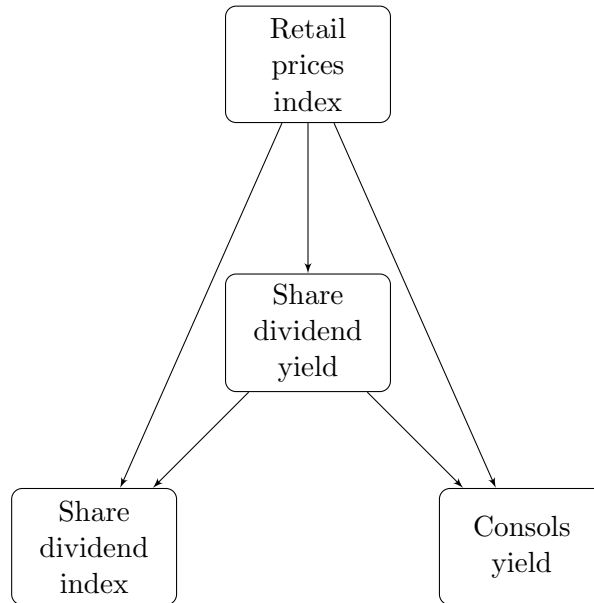


Figure 2.1.: The Wilkie model [Wilkie, 1984]

A new improved Wilkie model was introduced in 1995 by A.D Wilkie himself [Wilkie, 1995]. This new reformed model applied an Autoregressive Conditional Heteroscedastic (ARCH) to the inflation model and included other variables which had an impact to the investment return. The new variables that were introduced are as follows:

- A wages index model (also known as a wages inflation model).
- A short-term interest rate model.
- A property yield and income model.
- An index-linked yield model.
- An exchange rate model.

The 1995 Wilkie model was built to analyse data from the year 1923 until 1994 and skipped the data between the periods of 1919-1923 since it involved a very high inflation rate. For both developments, he used the annual data which were taken at the end of June of each year. Figure 2.2 illustrates the cascade structure of the 1995 Wilkie model which included the newly introduced variables.

2. The Wilkie Model: The Basics in Discrete Time

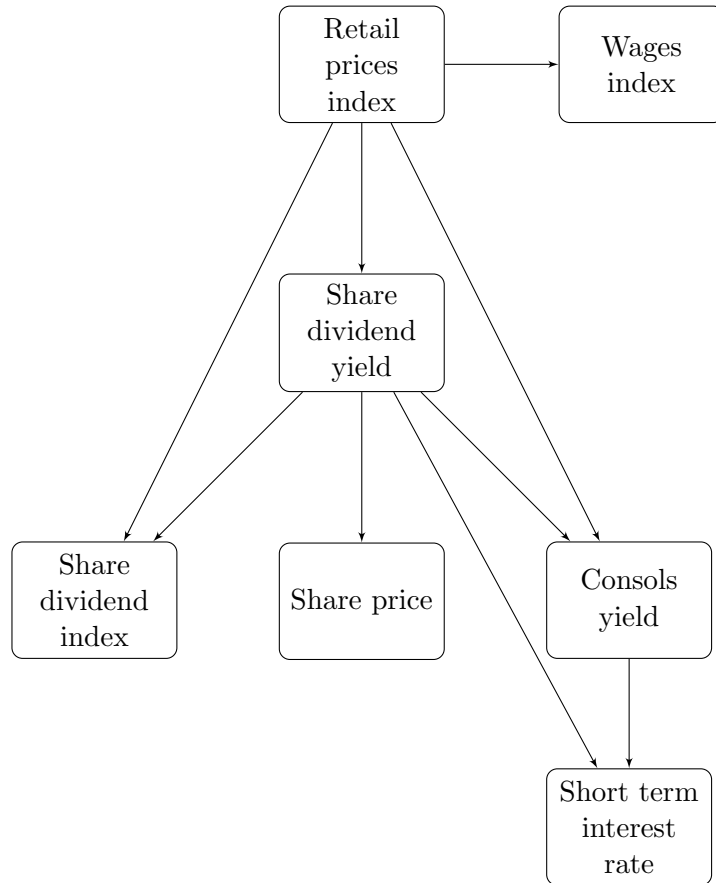


Figure 2.2.: The Wilkie model [Wilkie, 1995]

From figure 2.2, it is noted that the 1995 Wilkie model had also applied the cascade structure, the same as in the 1984 model. The retail prices model or specifically the price inflation model influenced the salary inflation that was obtained from the wages index model. Other classes of assets such as the share dividend yield, the share dividend index, the share price and the Consols yield have been influenced by the retail prices index. Meanwhile, a different treatment was applied to the short term interest rates model where it was dependent directly on the Consols yield, also known as long-term interest rate. The retail prices index was selected as the main driving force because of its strength in assessing the real asset returns. The designation of the cascade structure was basically based on the statistical thought, the economic point of view and of course by investment considerations.

However, a slight variation in the structure of the Wilkie model was made by [Whitten and Thomas, 1999], where the share dividend yield model was dependent on the Consols yield model rather than the opposite. Besides that, the model was built based on the threshold autoregressive (TAR) method and was an extension from the [Wilkie, 1995] model. [Whitten and Thomas, 1999]

2. The Wilkie Model: The Basics in Discrete Time

considered a non-linear stochastic asset modelling where initially they wanted to include other types of asset classes such as property and index-linked bonds. However, due to the lack of investment data, they only managed to include price inflation, wage inflation, share dividends, share yields, Consols yields and base rates in their model. The structure of the Wilkie built by [Whitten and Thomas, 1999] is shown in figure 2.3.

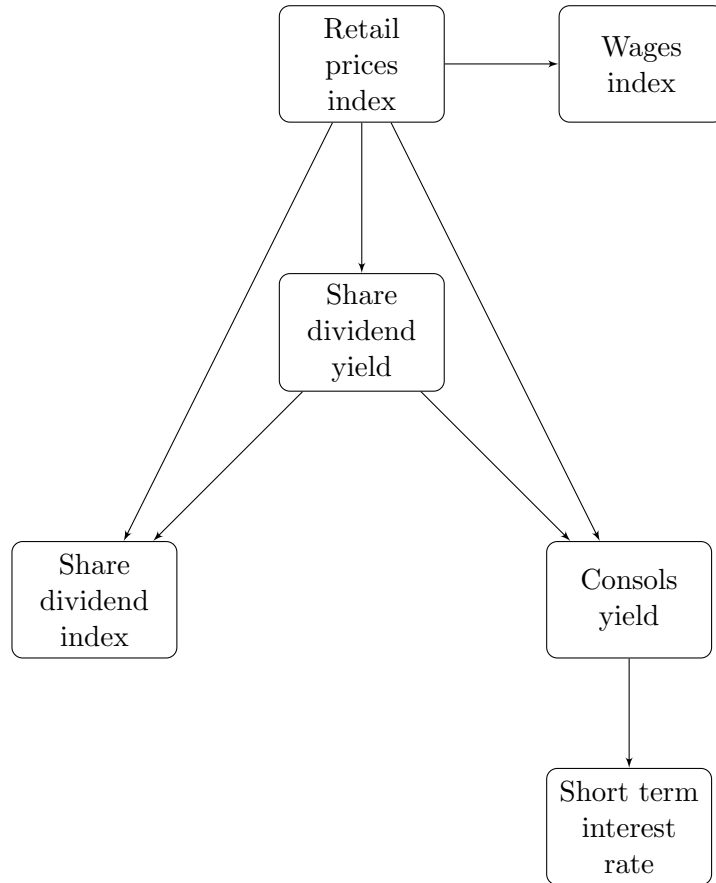


Figure 2.3.: The non-linear Wilkie model [Whitten and Thomas, 1999]

In 2008, the Wilkie model was once again enhanced by extending the parameters up to the year 2007 besides testing the retail price with the ARCH effects[Sahin et al., 2008]. The estimation of parameters became the premier aim of this study. In 2011, the Wilkie model was refitted from the model developed in 1995, by extending the parameters until June 2009 [Wilkie et al., 2011]. Similar to the study in 2008, the Wilkie model in 2011 also aimed to study the parameters extension and the stability of the confidence interval. The interval of this study was from 1994 to 2009 which is one year updated from the previous research. The result of this research showed that the residuals of many models were fatter-tailed rather than the normal distribution. New issues like stochastic and

parameter uncertainty were also observed.

2.2.1. The Retail Prices Index Model

The retail prices index model was developed based on the Retail Prices Index (RPI) for United Kingdom. RPI measures the consumer inflation where it tracks changes in the cost of a fixed basket of retail goods and services over time. Some countries refer to a Consumer Prices Index (CPI) to reflect their countries' inflation. The retail prices index at time t is denoted by Q_t . The difference of the natural logarithm of the RPI between time t and time $t - 1$ is written as

$$\nabla \ln Q_t = QMU + QA \cdot (\nabla \ln Q_{t-1} - QMU) + QSD \cdot QZ_t. \quad (2.1)$$

The difference of the natural logarithm of the RPI can also be called as the force of inflation I_t over year $t - 1$ to t with the backward difference operator ∇ defined by

$$\nabla = Q_t - Q_{t-1}.$$

From (2.1), we can see that the inflation depends on its past value and it can reflect the economic instability well.

The mean of the model which is denoted as QMU is fixed to take the value of 0.05 [Wilkie, 1984]. A constant QA is an autoregressive parameter and QSD is a standard deviation while QZ_t is a sequence of independent identically distributed standard normal random variables, i.e. those with a mean of 0 and variance of 1. This model is indeed an autoregressive model of order 1 (AR(1)) because of the dependency of $\nabla \ln Q_t$ towards $\nabla \ln Q_{t-1}$.

[Hürlimann, 1992] carried out a study about the moments generated from the Wilkie inflation model. He suggested that the mean and variance were calculated from the average force of inflation. As shown by [Hürlimann, 1992] and [Huber, 1997], the future value of the force of inflation has a log normal distribution. Thus, the future value of the logarithm of the force of inflation conditioned on knowing its value at time t is normally distributed is shown as the following:

$$\nabla \ln Q(t+k|t) \sim N\left(QMU + QA^k \cdot (\nabla \ln Q_t - QMU), \frac{QSD^2 \cdot (1 - QA^{2k})}{(1 - QA^2)}\right) \quad (2.2)$$

for $t > 0$, $k > 0$ and $QA \neq \pm 1$ while for $QA = 1$,

$$\nabla \ln Q(t+k|t) \sim N\left(\nabla \ln Q_t, k \cdot QSD^2\right). \quad (2.3)$$

2.2.2. The Share Dividend Yield Model

Share dividend yield is a measure of how much cash flow you are getting for each dollar invested in an equity position (stock). It shows how much a company pays out in dividends each year relative to its share price. Therefore, to obtain the share dividend yield, the share dividend index need to be divided with the price index. Since 1962, the index referred to the FTSE-Actuaries All-Shares Index but a slight change was made in 1997 in which they used actual dividends instead of gross dividends to evaluate the share index. Let Y_t be the share dividend yield value at time t which has the following equation:

$$\ln Y_t = YW \cdot \nabla \ln Q_t + YN_t \quad (2.4)$$

where

$$YN_t = \ln YMU + YA \cdot (YN_{t-1} - \ln YMU) + YSD \cdot YZ_t.$$

A constant YMU is the mean for this model, YA and YW are autoregressive parameters and YSD is a standard deviation whereas YZ_t is a sequence of independent identically distributed standard normal random variables. From (2.4), we can see that the share dividend yield is correlated directly to the retail prices index with the term of $\nabla \ln Q_t$. Both models are seen to have a mean reversion effect, that is the past years inflation would have to be deducted from its mean rate, $\nabla \ln Q_{t-1} - QMU$, the same as $YN_{t-1} - \ln YMU$ in this model. This model is indeed an AR(1) model because the dependency of YN_t towards YN_{t-1} .

The future value of the share dividend yield also has the same distribution as the previous model which is the log normal distribution based on a study by [Huber, 1997]. Thus, the future value of the logarithm of share dividend yield conditioned on knowing its value at time t is normally distributed, is shown as the following (for $QA \neq \pm 1, YA \neq \pm 1$):

$$\ln Y(t+k|t) \sim N\left(E[\ln Y(t+k)|t], Var[\ln Y(t+k)|t]\right). \quad (2.5)$$

The mean and variance of the logarithm of share dividend yield are conditional upon knowing the underlying processes at time t , are shown as follows:

$$\begin{aligned} E[\ln Y(t+k|t)] &= \ln YMU + YW \cdot QMU + QA^k \cdot YW \cdot (\nabla \ln Q_t - QMU) \\ &\quad + YA^k \cdot (\ln Y_t - \ln YMU - YW \cdot \nabla \ln Q_t), \\ Var[\ln Y(t+k|t)] &= \frac{YSD^2 \cdot (1 - YA^{2k})}{(1 - YA^2)} + \frac{(YW \cdot QSD)^2 \cdot (1 - QA^{2k})}{(1 - QA^2)}. \end{aligned}$$

2.2.3. The Share Dividend Index Model

Share dividend is payment made by a corporation to its shareholders, as a portion of its profit. A dividend is allocated as a fixed amount per share. Therefore, a shareholder receives a dividend in proportion to their shareholding. This is where the share dividend index is calculated. This model refers to the same source of index as in the share dividend yield model. We let D_t be the share dividend index at time t . Unlike the previous two models, D_t is a moving average model of order 1 (MA(1)) since the dividend index depends on the residuals DZ_t . The share dividend index has the following relationship:

$$\begin{aligned} \nabla \ln D_t = & DW \cdot DM_t + DX \cdot \nabla \ln Q_t + DMU + DY \cdot YSD \cdot YZ_{t-1} \\ & + DB \cdot DSD \cdot DZ_{t-1} + DSD \cdot DZ_t \end{aligned} \quad (2.6)$$

where $\nabla \ln D_t$ is the logarithm of the increase in the share dividend index from year $t-1$ to t and DM_t is

$$DM_t = DD \cdot \nabla \ln Q_t + (1 - DD) \cdot DM_{t-1}.$$

Constants DW, DX, DB, DY are the parameters in this model. The mean and standard deviation are DMU and DSD respectively. The model's residual is a sequence of independent identically distributed standard normal random variables which is denoted as DZ_t . Equation (2.6) obviously shows that the share dividend index is correlated directly to the retail prices index and share dividend yield by looking at the terms $\nabla \ln Q_t$ and $YSD \cdot YZ_{t-1}$ respectively.

As shown by [Huber, 1997], the future value of the logarithm of share dividend index conditioned upon knowing its value at time t is normally distributed, is shown as follows (for $t, k > 0$):

$$\nabla \ln D(t+k|t) \sim N\left(E[\ln D(t+k|t)], Var[\ln D(t+k|t)]\right). \quad (2.7)$$

For $k=1$, the mean and variance of the logarithm of share dividend index are conditional upon knowing the underlying processes at time t , are shown as follows:

$$\begin{aligned} E[\ln D(t+1|t)] = & (DW \cdot DD + DX) \cdot (QMU + QA \cdot (\nabla \ln Q_t - QMU)) \\ & + DM_t \cdot (1 - DD) + DMU + DY \cdot YSD \cdot YZ_t + DB \cdot DSD \cdot DZ_t, \\ Var[\ln D(t+1|t)] = & DSD^2 + QSD^2 \cdot (DW \cdot DD + DX)^2. \end{aligned}$$

For $k > 1$, we have $(1-DD) \neq \pm 1, QA \cdot (1-DD) \neq 1, QA - (1-DD) \neq 0$ and $QA \neq \pm 1$. Therefore,

$$\begin{aligned} E[\ln D(t+k|t)] = & DMU + QMU \cdot (DX + DW) + (DM_t - DW \cdot QMU) \cdot (1 - DD)^k \\ & + (\nabla \ln Q_t - QMU) \cdot (DX \cdot QA^k + (\alpha - DX) \cdot (QA^k - (1 - DD)^k)), \\ Var[\ln D(t+k|t)] = & DSD^2 \cdot (1 + DB^2) + YSD^2 \cdot DY^2 + QSD^2 \left[\alpha^2 \cdot \left(\frac{1 - QA^{2k}}{1 - QA^2} \right) \right. \\ & \left. - 2\alpha \cdot \beta \cdot \left(\frac{1 - (QA \cdot (1 - DD))^k}{1 - QA \cdot (1 - DD)} \right) + \beta^2 \cdot \left(\frac{1 - (1 - DD)^{2k}}{1 - (1 - DD)^2} \right) \right], \end{aligned}$$

with

$$\alpha = \frac{DW \cdot DD \cdot QA}{QA - (1 - DD)} + DX,$$

$$\beta = \frac{DW \cdot DD \cdot (1 - DD)}{QA - (1 - DD)}.$$

2.2.4. The Consols Yield Model

Consols is a form of British government bond which is also known as the perpetual bond with no maturity date, non-redeemable but pays a steady stream of interest forever. Consequently, the Consols yield is an income earned from the Consols. A.D Wilkie used Consols as the source for the long term bond yield. The original value for this model was derived based on the yield of 2^{1/2}% Consols. This index was chosen because it can be redeemed if the market yield decreases beyond the limit, i.e. the authority has an option to redeem the bond at par value but since the coupon rate is at 2^{1/2}% , the authority has a choice not to redeem unless it can be refinanced at less than 2^{1/2}% of the coupon rate. Later on, the yield was based on the FTSE-Actuaries BGS Indices. 3^{1/2}% War Stock (War Loan) represented the FTSE-Actuaries BGS Indices. The indices are not dependable on the redemption or coupon rate and have a longer past value. Although the indices are relatively small in the market, they are the best index for this model so far because of its non-dependency on the redemption. The Consols yield at time t is denoted as C_t encloses two parts; the future inflation CM_t and the Consols real yield CN_t , where C_t is in the form

$$C_t = CW \cdot CM_t + CN_t \quad (2.8)$$

as well as,

$$CM_t = CD \cdot \nabla \ln Q_t + (1 - CD) \cdot CM_{t-1},$$

$$\ln CN_t = \ln CMU + CA \cdot (\ln CN_{t-1} - \ln CMU) + CY \cdot YSD \cdot YZ_t + CSD \cdot CZ_t.$$

The model consists of CZ_t which is a sequence of independent identically distributed standard normal random variables, parameters CW, CD, CA , the mean CMU and the standard deviation CSD . The expression $YSD \cdot YZ_t$ shows the correlation of the Consols yield towards the share dividend yield while the term $\nabla \ln Q_t$ shows the correlation with the retail prices index. Earlier, C_t was modelled as an autoregressive model of order 3 (AR(3))[Wilkie, 1984] but in 1995 it was changed to AR(1)[Wilkie, 1995]. From (2.8), we can see that CM_t depends on CM_{t-1} and $\ln CN_t$ depends on $\ln CN_{t-1}$.

As shown by [Huber, 1997], the value of the future inflation conditioned on knowing its value at time t is normally distributed, is shown as follows:

$$CM(t+k|t) \sim N\left(E[CM(t+k|k)], Var[CM(t+k|t)]\right) \quad (2.9)$$

2. The Wilkie Model: The Basics in Discrete Time

for $t, k > 0, (1 - CD) \neq \pm 1, QA \cdot (1 - CD) \neq 1, QA - (1 - CD) \neq 0$ and for $QA \neq \pm 1$, the mean is

$$E[CM(t + k|t)] = CW \cdot QMU + (CM_t - CW \cdot QMU) \cdot (1 - CD)^k \\ + (\nabla \ln Q_t - QMU) \cdot CD \cdot CW \cdot QA \cdot \left(\frac{QA^k - (1 - CD)^2}{QA - (1 - CD)} \right)$$

and the variance is

$$Var[CM(t + k|t)] = QSD^2 \cdot \left(\frac{CW \cdot CD}{QA - (1 - CD)} \right)^2 \cdot \left[QA^2 \cdot \left(\frac{1 - QA^{2k}}{1 - QA^2} \right) \right. \\ \left. - 2QA \cdot (1 - CD) \cdot \left(\frac{1 - (QA \cdot (1 - CD))^k}{1 - QA \cdot (1 - CD)} \right) + (1 - CD)^2 \cdot \left(\frac{1 - (1 - CD)^{2k}}{1 - (1 - CD)^2} \right) \right].$$

To complete this section, we present the future value of the logarithm of Consols real yield conditioned on knowing its value at time t as follows:

$$\ln CN(t + k|t) \sim N\left(E[\ln CN(t + k|t)], Var[\ln CN(t + k|t)]\right)$$

for $t, k > 0$, where the complete form of mean and variance can be seen from [Huber, 1997].

2.3. Comments and Criticisms about the Wilkie Model

This section will provide an in-depth analysis on the Wilkie model, based on a detailed study of other researches as well as our own observations. Tests were conducted on the Wilkie model to analyse the residuals, independency and the normality and we were also able to conclusively decide on data period selection for parameters estimation. [Wilkie, 1995] noticed that the non-constant variance of residuals, the existence of random shock effect and the residuals showed a non-normal distribution in the retail prices index model. This observation was corroborated by Kitts (1990) who declared that the time intervals consisting of extreme inflations and deflations had an impact on the non- independence of residuals and assumed it to have a non-normal distribution. [Wilkie, 1995] solved the non-constant variance of residuals by applying ARCH into the inflation series since [Engle, 1982] had applied ARCH models to generalise the non-constant variance.

To overcome the random shock effects, one could use two or more distributions, known as mixture distribution, for the residuals. However, despite this solution, a few problems still exist such as the identification of the distribution or the appropriate time periods between the shocks. These issues are still a part of controversial debate. Models involving random shocks are suitable for the short time period because they lead to a slight difference in the mean squared error in the medium and long-term models. However, the random shock effect is acceptable for medium-term modelling with some extreme values as discussed by the FIMAG working party.

2. The Wilkie Model: The Basics in Discrete Time

[Wilkie, 1984] had assumed that residuals were normally distributed even though sometimes it showed a negative skew and a definite fat-tailed distribution. To solve this matter, Wilkie had a larger standard deviation for the residuals, keeping the same model. As a result, some extreme values of residuals appeared. One of the approaches towards this matter is to have an empirical distribution of the actual residuals from the fitted model. However with this approach, it will be difficult to modify the distribution. Another approach suggested is to use many kinds of distributions for residuals, i.e. Pearson Type IV, t-distribution or Stable Paretian but again it is hard to identify the most suitable distribution. Thus, this suggestion will not help in trying to overcome this issue.

Next, this section will focus on the outcomes of the Wilkie model to different time intervals. The individual time intervals will be discussed according to each type of the original Wilkie model. This discussion will compare the parameters of each Wilkie model for three different time intervals which were studied by [Wilkie, 1984], [Wilkie, 1995] and [Sahin et al., 2008].

Below are the estimated parameters of the retail prices index according to the three time intervals:

I_t	1919-1982	1923-1994	1923-2007
QA	0.6	0.5773	0.5794
QMU	0.05	0.0473	0.0446
QSD	0.05	0.0427	0.0396

Table 2.1.: Estimated parameters of the retail prices index model

As can be seen from table 2.1, there is no huge difference in the values of parameters for the three periods. QA has a slightly increased value while QMU and QSD have slightly decreased for the two latter periods [Sahin et al., 2008].

Table 2.2 below shows the estimated parameters of the share dividend yield model according to the three time intervals.

$\ln Y_t$	1919-1982	1923-1994	1923-2007
YW	1.35	1.794	1.6473
YA	0.6	0.5492	0.6354
YMU	0.04	0.0377	0.0364
YSD	0.175	0.1552	0.1529

Table 2.2.: Estimated parameters of the share dividend yield model

As seen in table 2.2, all parameters are almost equal for all three periods of simulation, except for the value of YW that changed from 1.35 for the period of 1919-1982 to 1.794 for the period of 1923-1994. By considering the major results, one can assume that there is no strong evidence for the changes in parameters after updating the data.

2. The Wilkie Model: The Basics in Discrete Time

Table 2.3 lists the estimated parameters of the share dividend index model according to the three time intervals.

$\ln D_t$	1919-1982	1923-1994	1923-2007
DW	0.8	0.5793	0.5779
DD	0.2	0.1344	0.1441
DY	-0.2	-0.1761	-0.15
DB	0.375	0.5734	0.6070
DMU	0	0.0157	0.0142
DSD	0.075	0.0671	0.0654

Table 2.3.: Estimated parameters of the share dividend index model

The smoothing parameters, DD and DW reflect the effects of inflation on the share yield. [Wilkie, 1984] found that it was economically necessary to keep both parameters in the model after taking into consideration the direct transfer from retail prices to dividends. There was only a slight difference between the values of parameters for the 1923-1994 and 1923-2007 intervals, but there are significant differences observed when compared to the earlier period. Negative derivations were also discovered from year 1999 onwards in the analysis of $\ln D_t$ and A.D Wilkie clarified that the reason for this issue was that the inflation rate in the last 15 years was much lower than [Wilkie, 1995] had expected.

Below are the estimated parameters of the Consols yield model according to the three time intervals.

C_t	1919-1982	1923-1994	1923-2007
CW	1	1	1
CD	0.045	0.045	0.045
CA	CA1 = 1.20, CA2 = -0.48, CA3 = 0.20	0.8974	0.8954
CY	0.06	0.3371	0.4690
CMU	0.035	0.0305	0.0233
CSD	0.14	0.1853	0.2568

Table 2.4.: Estimated parameters of the Consols yield model

For the time interval 1919-1982, the value of CA was divided into three values; CA1, CA2 and CA3. This is because a different form of equation was used in the original Wilkie model but it still worked the same. Other parameters with the exception of CY and CSD, had almost the same values. The values of CY and CSD had increased with the extended data. [Wilkie, 1984] fixed the value of CW and CD to become 1 and 0.045 respectively. This action led to negative real interest rates for the year 1999, 2000, 2003, 2005 and 2006.

Despite the close values of estimated parameters for the UK data, we find that it is unnecessary and impractical to simulate the UK data just by using different time

2. The Wilkie Model: The Basics in Discrete Time

intervals. Therefore, in this study, we have decided to use the concept of the Wilkie model with new data which are the Malaysian data and analyse the results after building a suitable Box-Jenkins model for each variable. This is discussed in the next chapter after we describe the Box-Jenkins models as well as its methodology.

3. Stochastic Asset Liability Modelling: A Case of Malaysia

3.1. Introduction

In this chapter, we will apply a methodology derived from [Wilkie, 1984] to a Malaysian investment data. The same concepts and variables in the Wilkie model will be used with several modifications. The adjustments are necessary as we believe that modelling the asset liability is more appropriate than simply simulating the original Wilkie model to Malaysian data because the Wilkie model is basically built based on UK data. This chapter consists of two main sections. In the first section, we discuss thoroughly the concept of Box-Jenkins models, which is fundamental to the development of the Wilkie model. We explain the four types of Box-Jenkins models comprising of an autoregressive model, a moving average model, an autoregressive moving average model and a Box-Jenkins model for a non-stationary series, an autoregressive integrated moving average model. We fit the Box-Jenkins model to investment data of Malaysia. We evaluate the ability of the Wilkie model to analyse and also predict the investment in Malaysia. These procedures will be conducted using the Box-Jenkins modelling methods. We run some statistical analysis to each investment factor and include the appropriate economics theories behind them.

3.2. Box-Jenkins Models

Fundamentally, the Wilkie investment model was constructed based on the Box-Jenkins methods. For instance, the Box-Jenkins model expresses a process y_t as a function of observations of past processes $y_{t-1}, y_{t-2}, \dots, y_1$. The autoregressive moving average (ARMA) model deals with stationary time series while the autoregressive integrated moving average (ARIMA) model deals with non-stationary time series and these two models are the Box-Jenkins model. One of the objective of the model building is for forecasting.

3.2.1. The Autoregressive Model

The autoregressive model of order 1 or AR(1) is in the form of

$$y_t = \delta + \phi_1 y_{t-1} + \varepsilon_t \tag{3.1}$$

where y_t is a time series, δ is a constant and ϕ_1 is an autoregressive coefficient and ε_t is a series of errors at time t with zero-mean and variance σ_ε^2 . The AR(1) model is a simple

3. Stochastic Asset Liability Modelling: A Case of Malaysia

linear regression model where y_t denotes the dependent variable while y_{t-1} denotes the independent variable.

By taking the expectation to (3.1), we obtained

$$E[y_t] = E[\delta] + \phi_1 E[y_{t-1}] + E[\varepsilon_t] \quad (3.2)$$

with $E[\varepsilon_t] = 0$. When we assumed stationary conditions which are $|\delta| < 1$ and $E[y_t] = E[y_{t-1}] = \mu$, it produces a result of

$$\mu = \delta + \phi_1 \mu.$$

Thus,

$$\mu = \frac{\delta}{1 - \phi_1}$$

is the mean of the AR(1) model. We can see that the constant δ is related to the mean μ . This relation implies that the mean only exists if $\phi_1 \neq 1$ and the mean is zero if and only if $\delta = 0$. Therefore, δ can be expressed as

$$\delta = (1 - \phi_1)\mu.$$

We substituted δ into (3.1) and obtained

$$y_t - \mu = \phi_1(y_{t-1} - \mu) + \varepsilon_t.$$

If we repeatedly substitute the prior equations, we will achieve

$$\begin{aligned} y_t - \mu &= \varepsilon_t + \phi_1 \varepsilon_{t-1} + \phi_1^2 \varepsilon_{t-2} + \dots \\ &= \sum_{i=0}^{\infty} \phi_1^i \varepsilon_{t-i}. \end{aligned} \quad (3.3)$$

Equation (3.3) shows that $y_t - \mu$ is linearly dependent on ε_{t-i} when $i \geq 0$. By taking the square and expectation to (3.3), we will obtain the variance of this series as follows:

$$Var[y_t] = \phi_1^2 Var[y_{t-1}] + \sigma_\varepsilon^2 \quad (3.4)$$

where σ_ε^2 is a variance of ε_t . We know that $Cov[y_{t-1}, \varepsilon_t] = 0$ and by stationary condition, we will have $Var[y_t] = Var[y_{t-1}]$. Therefore, the variance of y_t can be written as

$$Var[y_t] = \frac{\sigma_\varepsilon^2}{1 - \phi_1^2}$$

where $\phi_1^2 < 1$. Under the generalisation of the AR(1) model, we have the autoregressive model of order p , or simply written as AR(p), with non-negative integer p . The AR(p) model satisfies the following equation:

$$y_t = \delta + \phi_1 y_{t-1} + \phi_2 y_{t-2} + \dots + \phi_p y_{t-p} + \varepsilon_t. \quad (3.5)$$

The process y_t is a linear function of the p th past values of itself with some errors ε_t which states any information left by the past values. We may assume that ε_t is independent of the process $y_{t-1}, y_{t-2}, \dots, y_{t-p}$.

3.2.2. The Moving Average Model

We now move to another type of Box-Jenkins model which is the moving average (MA) model. The MA model is a simple extension of white noise series (the errors). The terminology of building the MA model exists by multiplying the weights $1, -\theta_1, -\theta_2, \dots, -\theta_q$ to error terms $\varepsilon_t, \varepsilon_{t-1}, \varepsilon_{t-2}, \dots, \varepsilon_{t-q}$ and further, moving the weights to $\varepsilon_{t+1}, \varepsilon_t, \varepsilon_{t-1}, \dots, \varepsilon_{t-q+1}$ to get y_{t+1} process and this concept will continue for the rest. The moving average model of order 1, MA(1) is in the form of

$$y_t = \delta + \varepsilon_t - \theta_1 \varepsilon_{t-1} \quad (3.6)$$

where ε_{t-1} and ε_t are errors of the series at time $t-1$ and t respectively. The coefficient θ_1 is the first order moving average parameter and δ is a constant. By taking the variance in equation (3.6), we obtained

$$Var[y_t] = \sigma_\varepsilon^2 + \theta_1^2 \sigma_\varepsilon^2 = \sigma_\varepsilon^2(1 + \theta_1^2)$$

with σ_ε representing a standard deviation of ε_t . From equation (3.6), we can see that a large value of θ_1 shows that the process y_t is influenced strongly by the previous values of the error. Furthermore, we presented the moving average model of order q which is known as MA(q)

$$y_t = \delta + \varepsilon_t - \theta_1 \varepsilon_{t-1} - \theta_2 \varepsilon_{t-2} - \dots - \theta_q \varepsilon_{t-q}. \quad (3.7)$$

Clearly, the series y_t is considered to be a linear process of its past and present errors and does not depend on its past values.

3.2.3. The Autoregressive Moving Average Model

The idea of the development of the ARMA model is to prevent a high number of parameters that AR or MA models may have. Therefore, the ARMA model combines the AR and MA terms into a compact form so that the number of parameters is kept small. The ARMA(p, q) or to be understood as ARMA model of order p and q , is in the form of

$$y_t = \delta + \phi_1 y_{t-1} + \dots + \phi_p y_{t-p} + \varepsilon_t - \theta_1 \varepsilon_{t-1} - \dots - \theta_q \varepsilon_{t-q} \quad (3.8)$$

where δ is a constant, ϕ_1, \dots, ϕ_p are autoregressive parameters and $\theta_1, \dots, \theta_q$ are moving average parameters while $\{\varepsilon_t\}$ is a series of errors. The unconditional mean of the ARMA (p, q) is given by

$$E[y_t] = \frac{\delta}{1 - \phi_1 - \dots - \phi_p}.$$

3.2.4. The Autoregressive Integrated Moving Average Model

In the previous subtopics, we discussed the Box-Jenkins model that fits the stationary series but now, we want to study non-stationary series and the suitable Box-Jenkins model to treat this kind of series. The Box-Jenkins model that fits the non-stationary series is an autoregressive integrated moving average, also called as the ARIMA model. The non-stationary series after the first difference can be written as the following:

$$\tilde{y}_t = y_t - y_{t-1} \quad (3.9)$$

or in other notation as

$$\tilde{y}_t = \nabla^d y_t. \quad (3.10)$$

In this section, we only discuss the ARIMA($p, 1, q$) model, also called the ARIMA model of order p and q with the difference d as the following equation:

$$\tilde{y}_t = \phi_1 \tilde{y}_{t-1} + \phi_2 \tilde{y}_{t-2} + \dots + \phi_p \tilde{y}_{t-p} + \varepsilon_t - \theta_1 \varepsilon_{t-1} - \theta_2 \varepsilon_{t-2} - \dots - \theta_q \varepsilon_{t-q}. \quad (3.11)$$

By the substitution of (3.9) into (3.11), we obtained

$$\begin{aligned} y_t - y_{t-1} &= \phi_1(y_{t-1} - y_{t-2}) + \phi_2(y_{t-2} - y_{t-3}) + \dots + \phi_p(y_{t-p} - y_{t-p-1}) + \varepsilon_t \\ &\quad - \theta_1 \varepsilon_{t-1} - \theta_2 \varepsilon_{t-2} - \dots - \theta_q \varepsilon_{t-q} \end{aligned} \quad (3.12)$$

and it can be written as

$$\begin{aligned} y_t &= (1 + \phi_1)y_{t-1} + (\phi_2 - \phi_1)y_{t-2} + (\phi_3 - \phi_2)y_{t-3} + \dots + (\phi_p - \phi_{p-1})y_{t-p} - \phi_p y_{t-p} + \varepsilon_t \\ &\quad - \theta_1 \varepsilon_{t-1} - \theta_2 \varepsilon_{t-2} - \dots - \theta_q \varepsilon_{t-q}. \end{aligned} \quad (3.13)$$

3.2.5. The Seasonal Autoregressive Moving Average Integrated Model

Previously, we discussed the seasonal Box-Jenkins model which included AR, MA, ARMA as well as ARIMA. Now, we want to expand the seasonal Box-Jenkins model, to the seasonal autoregressive moving average integrated (SARIMA) model. Seasonality is a pattern that repeats over S time period. In the case of monthly data, $S = 12$ whereas for quarterly data, $S = 4$. The objective of seasonal differencing is to remove the seasonal trend in a time series. Hence, for

- $S = 12$, the seasonal difference is $(1 - B^{12})y_t = y_t - y_{t-12}$,
- $S = 4$, the seasonal difference is $(1 - B^4)y_t = y_t - y_{t-4}$.

For this model, we used a back shift operator B in order to build the seasonal ARIMA model, where B satisfies

$$\begin{aligned} (B)y_t &= y_{t-1}, \\ (B^j)y_t &= y_{t-j}. \end{aligned}$$

3. Stochastic Asset Liability Modelling: A Case of Malaysia

The notation for the model is ARIMA $(p, d, q)(P, D, Q)[S]$ where P is a seasonal AR order, D is a seasonal differencing and Q is a seasonal MA order. Basically, the ARIMA $(p, d, q)(P, D, Q)[S]$ could be written as

$$\Phi(B^S)\phi(B)\nabla_S^D\nabla^d y_t = \Theta(B^S)\theta(B)\varepsilon_t \quad (3.14)$$

with ε_t errors. The non-seasonal components can be expressed as

$$\begin{aligned} AR : \phi(B) &= 1 - \phi_1 B - \dots - \phi_p B^p, \\ MA : \theta(B) &= 1 + \theta_1 B + \dots + \theta_q B^q. \end{aligned}$$

Whereas the seasonal components are

$$\begin{aligned} SAR : \Phi(B^S) &= 1 - \Phi_1 B^S - \dots - \Phi_p B^{pS}, \\ SMA : \Theta(B^S) &= 1 + \Theta_1 B^S + \dots + \Theta_Q B^{QS}. \end{aligned}$$

As an example, ARIMA $(0, 1, 1)(0, 1, 1)[12]$ satisfies the following form:

$$(1 - B^{12})(1 - B)y_t = (1 + \Theta B^{12})(1 + \theta B)\varepsilon_t.$$

The previous equation can be expanded to form

$$(1 - B - B^{12} + B^{13})y_t = (1 + \theta B + \Theta B^{12} + \Theta\theta B^{13})\varepsilon_t$$

By referring to back shift operator, the equation can be written as

$$y_t = y_{t-1} + y_{t-12} - y_{t-13} + \varepsilon_t + \theta\varepsilon_{t-1} + \Theta\varepsilon_{t-12} + \Theta\theta\varepsilon_{t-13}. \quad (3.15)$$

3.3. Box-Jenkins Methodology

The Box-Jenkins methodology involves a four-step iterative routine as follows:

Step 1: Tentative identification.

In order to model time series according to the Box-Jenkins methodology, the series must be in stationary state. The series is stationary if its mean and variance do not fluctuate over time systematically. We can see the stationarity of the series from its plot. The plot is vital to show up important features of the series such as trend, seasonality, outlier and others. Additionally, there are many tests to check for stationarity of time series and in this study, we used an Augmented Dickey-Fuller unit root test (ADF test). The unit root test has been used widely for testing stationarity over the past few years [Gujarati, 2012]. The description of this test can be found in appendix A. If the series is not stationary, we can differentiate the series because differencing helps to stabilise the mean of the series by removing changes in the level of the series, and so eliminating trend and seasonality. Practically, the series is stationary at most at the second difference. In the case where the series is not stationary, the differentiated series follows the ARIMA model. ARIMA (p, d, q) has the same form as ARMA (p, q) , except for the existence of the number of

3. Stochastic Asset Liability Modelling: A Case of Malaysia

difference d in ARIMA. Apart from that, we are required to find the suitable provisional values for p and q by analysing a plot of autocorrelation function (ACF) and partial autocorrelation function (PACF) of the series (refer to appendix B and C). We analysed the plots according to certain trends as stated in table 3.1 below. This led to model identification for the series.

Model	ACF	PACF
MA(q) $y_t = \delta + \varepsilon_t - \theta_1\varepsilon_{t-1}$ $- \theta_2\varepsilon_{t-2} - \dots - \theta_q\varepsilon_{t-q}$	Cuts off after lags q	Dies down
AR(p) $y_t = \delta + \varepsilon_t + \phi_1y_{t-1}$ $+ \phi_2y_{t-2} + \dots + \phi_py_{t-p}$	Dies down	Cuts off after lags p
ARMA (p, q) $y_t = \delta + \phi_1y_{t-1} + \dots + \phi_py_{t-p}$ $+ \varepsilon_t - \theta_1\varepsilon_{t-1} - \dots - \theta_q\varepsilon_{t-q}$	Dies down	Dies down

Table 3.1.: ACF and PACF trends of the non-seasonal Box-Jenkins model

Step 2: Estimation.

Historical data were used to generate the values of parameters $\delta, \phi_1, \dots, \phi_p$ and $\theta_1, \dots, \theta_q$ that we have in the model. Basically, we will have a few ARMA/ARIMA models that could possibly fit the series. In order to determine the most fitted ARMA/ARIMA model, we used an Akaike information criterion (AIC) and a Bayesian information criterion (BIC) which are explained thoroughly in appendix D and E. AIC and BIC are used for choosing the best order of p of an AR model which leads to a selection of a lower AR model when data are large [Tsay, 2005]. We chose the ARMA/ARIMA model with the lowest value of AIC and BIC. In addition, it is optional to check the stationary and invertible of each parameters as stated in table 3.2. The stationary and invertible conditions imply that the parameters used in the model are reasonable.

3. Stochastic Asset Liability Modelling: A Case of Malaysia

Model	Stationary conditions	Invertible conditions
MA(1) $y_t = \delta + \varepsilon_t - \theta_1\varepsilon_{t-1}$	None	$ \theta_1 < 1$
MA(2) $y_t = \delta + \varepsilon_t - \theta_1\varepsilon_{t-1} - \theta_2\varepsilon_{t-2}$	None	$\theta_1 + \theta_2 < 1$ $\theta_1 - \theta_2 < 1$] $ \theta_2 < 1$
AR(1) $y_t = \delta + \phi_1y_{t-1} + \varepsilon_t$	$ \phi_1 < 1$	None
AR(2) $y_t = \delta + \phi_1y_{t-1} + \phi_2y_{t-2} + \varepsilon_t$	$\phi_1 + \phi_2 < 1$ $\phi_1 - \phi_2 < 1$ $ \phi_2 < 1$	None
ARMA (1,1) $y_t = \delta + \phi_1y_{t-1} + \varepsilon_t - \theta_1\varepsilon_{t-1}$	$ \phi_1 < 1$	$ \theta_1 < 1$

Table 3.2.: Stationary and invertible conditions of the non-seasonal Box-Jenkins model

Step 3: Diagnostic checking.

This step is executed by checking the adequacy of the estimated model, and if needed, to suggest an improved model. The best way to check the adequacy of an overall Box-Jenkins model is to examine its residuals. The residuals are calculated as the difference between the actual values and the fitted values and it is unpredictable in every observation. Firstly, the plot of residuals must show no pattern. If the plot shows a pattern, then the relationship may be non linear and the model will need to be modified accordingly. Secondly, we looked for no serial correlation between residuals. If there is no serial correlation, the autocorrelations at all lags should be nearly zero, which is approximately a white noise. Additionally, the autocorrelations must all be within the 95% zero-bound. Thirdly, we referred to a plot of p -values for Ljung-Box statistics which is supposed to show significant values. The Ljung-Box statistics is explained in appendix F. After tremendous checking we found that the chosen model was inadequate and therefore we are expected to reformulate the model.

Step 4: Forecasting.

Once the final model is achieved, it can be executed to forecast future time series values. Basically, the point prediction of

$$y_t = \delta + \phi_1y_{t-1} + \dots + \phi_py_{t-p} + \varepsilon_t - \theta_1\varepsilon_{t-1} - \dots - \theta_q\varepsilon_{t-q}$$

is

$$\hat{y}_t = \delta + \hat{\phi}_1y_{t-1} + \dots + \hat{\phi}_py_{t-p} + \hat{\varepsilon}_t - \hat{\theta}_1\hat{\varepsilon}_{t-1} - \dots - \hat{\theta}_q\hat{\varepsilon}_{t-q}$$

where

- The point prediction $\hat{\varepsilon}_t$ of the future random shock ε_t is zero.
- The point prediction $\hat{\varepsilon}_t$ of the future random shock ε_{t-1} is the $(t-1)$ st residual $(y_{t-1} - \hat{y}_{t-1})$ if we can calculate \hat{y}_{t-1} , and zero if we cannot calculate \hat{y}_{t-1} .

3. Stochastic Asset Liability Modelling: A Case of Malaysia

- The $100(1 - \alpha)\%$ prediction interval calculated at time origin n for the time series value in time period $n + \tau$ is

$$\hat{y}_{n+\tau}(n) \pm t_{[\alpha/2]}^{(n-n_p)} SE_{n+\tau}(n)$$

where SE is the estimated standard error of the series and $t_{[\alpha/2]}^{(n-n_p)}$ is the t -multiplier which has $n - n_p$ degrees of freedom. It is common to consider a degree of freedom of $n - 2$ with 95% prediction interval.

To conclude the Box-Jenkins methodology, we illustrate the steps required in the following diagram:

3. Stochastic Asset Liability Modelling: A Case of Malaysia

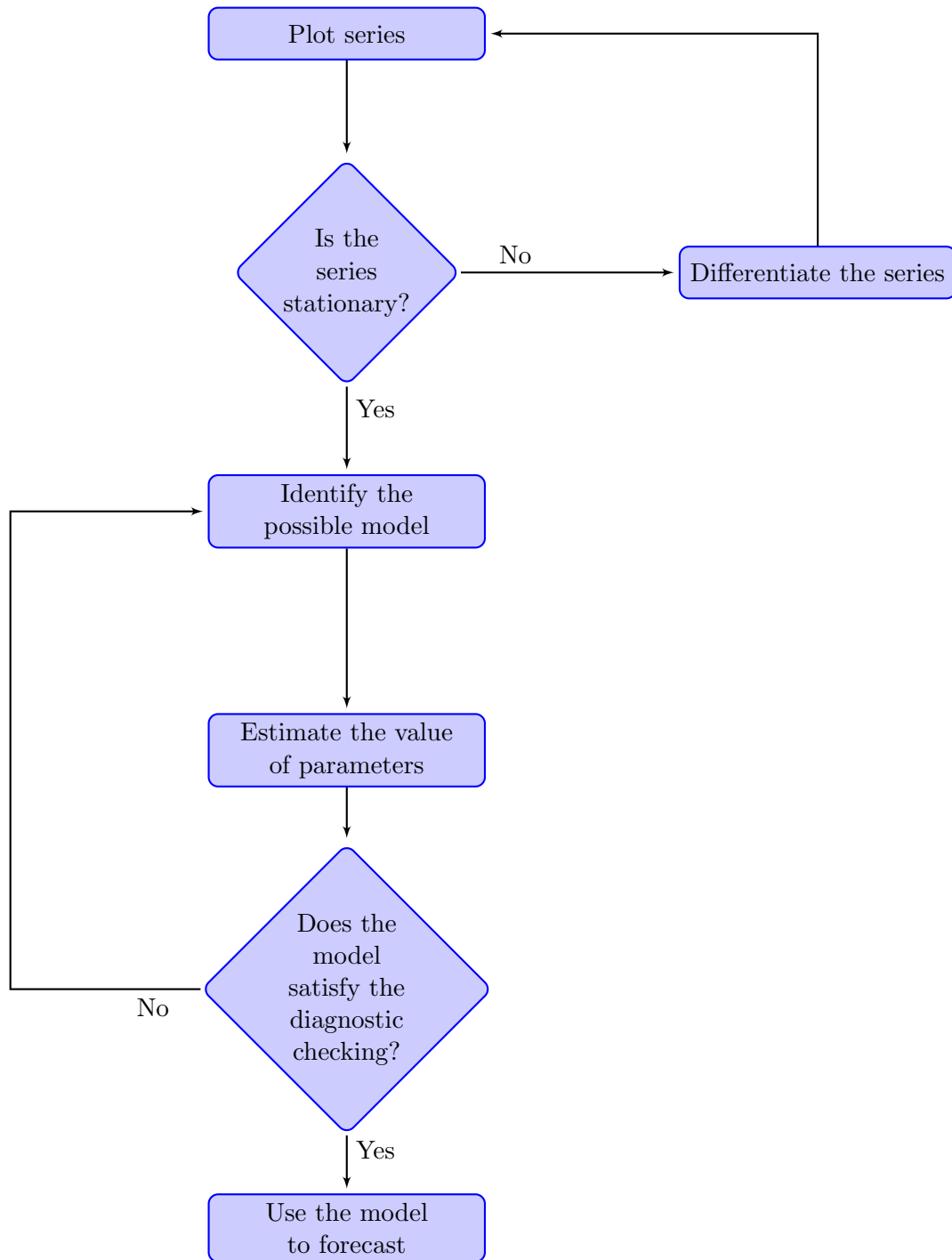


Figure 3.1.: The Box-Jenkins modelling approach

3.4. Malaysian Stochastic Asset Liability Model

This section suggests the same methodology of stochastic asset liability modelling derived from [Wilkie, 1984] and [Thomson, 1996]. A similar study was done by [Metz and Ort, 1993] who developed a consumer price index model for Switzerland following the ARIMA process. The model was then used to modify the individual pension fund. [Thomson, 1996] focused on the model development and of course, some analysis towards the model. The stochastic models in his study were developed for inflation rates, short term and long term interest rates, dividend rates and its yield, rental rates and its yield, for South Africa. Other related studies found were by [Sherris et al., 1996] for Australia, [Frees et al., 1997] for United States and [Chan, 1998b] for four developing countries; United Kingdom, United states, Canada and Australia. Therefore in this study, we will use Malaysia as our scope.

On top of that, there was a study conducted by [Chong, 2007] who elaborated the required methods to prepare a stochastic asset liability model for a Malaysian participated annuity fund. The asset classes that were investigated in the study included cash, short-term and long-term bond, property and equity. The output of the simulated stochastic models was then used to produce a balance sheet, profit and loss statement and mean portfolio investment return.

3.4.1. Outline of the Approach

The variable selection for this study are based on the basic variables contained in the Wilkie model [Wilkie, 1984]. This is because these variables are also important to major investment of asset classes that are categorised in Malaysia. The assets that we considered in this study were; shares and long-term security which is bond. Nevertheless, we also studied inflation rates because it has a great impact on investment, i.e. when the inflation rates are high, investors will lose their purchasing (investment) power. After considering the factors described, it was concluded that all the four basic Wilkie models will be employed in this study.

The data used to model the four variables are as follows:

i. Data representing the force of inflation is the Consumer Prices Index (*CPI*). The annual rate of inflation is measured as

$$\nabla \ln Q(t) = \ln \left(\frac{CPI_t}{CPI_{t-1}} \right).$$

The data were downloaded from the world bank website and we analysed the data for the years 1960-2013.

ii. Data representing the share dividend yield is the FTSE Bursa Malaysia KLCI yield. The data were provided by the FTSE Group by contacting them personally. Unfortunately, the data available were for average monthly basis only, starting from July 2009 to September 2013.

iii. Data representing the share dividend index were also the FTSE Bursa Malaysia

3. Stochastic Asset Liability Modelling: A Case of Malaysia

KLCI but for this model, we analysed the index not the yield as we did in ii. The data were available from January 1994 to December 2013. On the other hand, to find the dividend index D_t from the yield, one can use the following formula:

$$D_t = S_t \times \frac{Y_t}{100}$$

where Y_t denotes the share yield and S_t is a share price.

iv. Data representing the bond yield was the 10-year Malaysian Government Securities (MGS) yield which indicates the long-term interest bearing securities for Malaysia. [Chong, 2007] also used the same data in his study to represent the long-term bond. The monthly MGS data were available from January 1996 to January 2014 and the data were downloaded from the Bursa Malaysia website.

In the next section, we will discuss the four Wilkie sub models in detail.

3.4.2. The Inflation Model

For this model, the 2005 was used as the base index. The annual force of inflation is shown in figure 3.2 as it follows condition i in the data description.

3. Stochastic Asset Liability Modelling: A Case of Malaysia

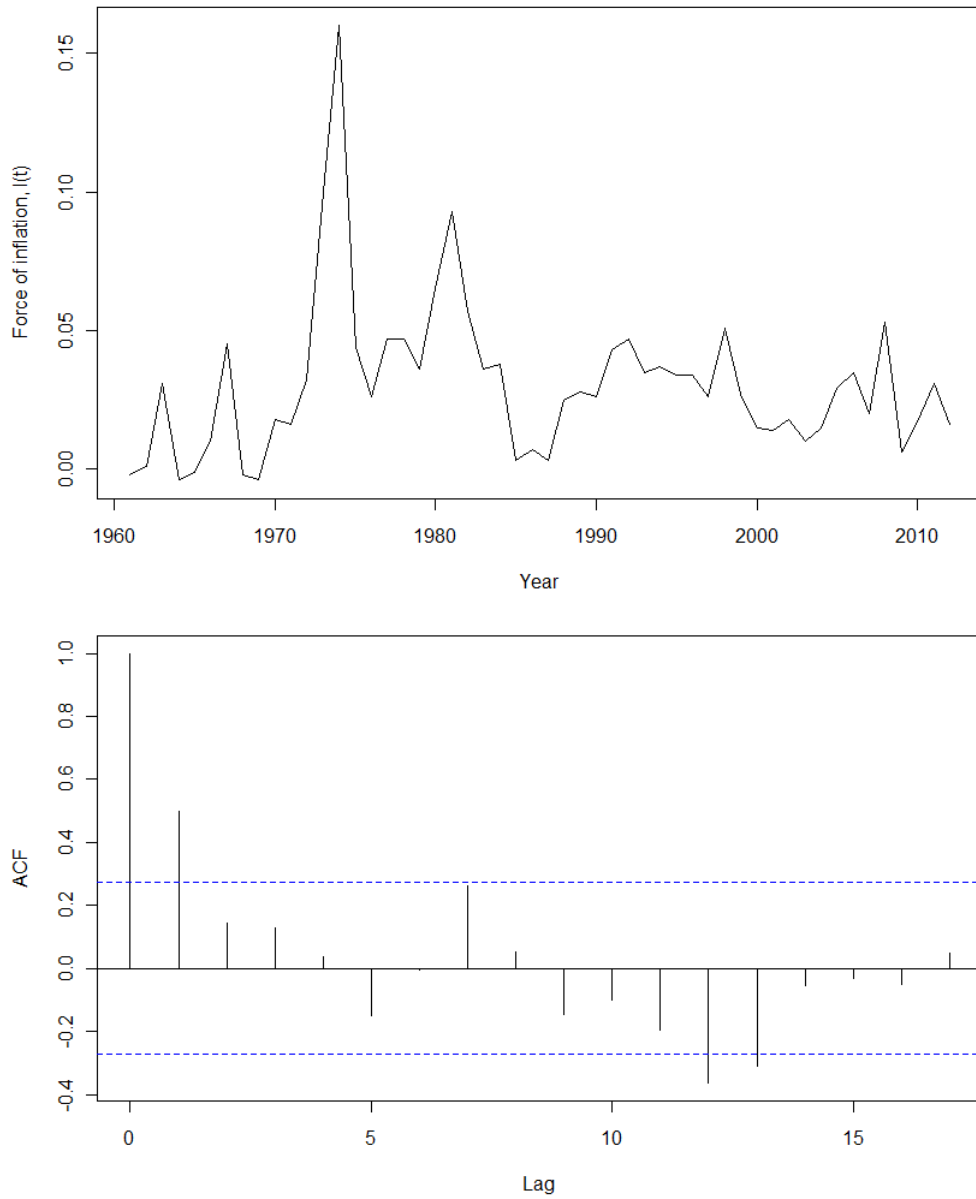


Figure 3.2.: Annual force of inflation, I_t : 1961 - 2012 together with its correlogram

Figure 3.2 shows that the force of inflation remained positive in most of the experimental years except in 1961, 1964, 1965, 1968 and 1969, where the rates are negative. Determinants of inflation in Malaysia include food, transport and communication, gross rent and power and others. In the 1960s, a large portion of the household expenditure

3. Stochastic Asset Liability Modelling: A Case of Malaysia

was allocated to food and consequently, the item had a higher weight in the CPI basket. This explains the small value of inflation during that period. The inflation had an extreme value in the mid 1970s due to the "oil shock" effect the world experienced during 1974-1975. At that period of time, Malaysia experienced a 16 per cent increase of the inflation rate. Malaysia faced a second rise in inflation in 1980 due to the same reason. For the rest of the times, the inflation rate remained below 5 per cent. Even after suffering from the Asian financial crisis which occurred in 1997 and 1998, Malaysia had succeeded in maintaining its inflation rate at a low level.

The descriptive statistics for inflation are summarised in table 3.3. The inflation ($n=52$) averaged by 3 per cent from 1961 to 2012. The 0.03 standard deviation shows the inflation response at 3 per cent away from its average value. The coefficient of skewness is greater than zero which means the distribution of the inflation is positively skewed. The inflation has a coefficient of kurtosis of 6.52 indicating a high degree of peakedness or what might be characterised as a leptokurtic distribution.

Mean	0.03
Standard deviation	0.03
Skewness	2.07
Kurtosis	6.52

Table 3.3.: Summary statistics for inflation

Next, we proceeded with the first step in the Box-Jenkins methodology which is the tentative identification. Figure 3.2 shows that the inflation has a short-term autocorrelation. As mentioned by [Chatfield, 2013], if a time series has a trend where the autocorrelation values are high and goes down to zero as the lags are increasing, the inflation might not be in a stationary state and we believe that the ARIMA model is more suitable. However, it would be prudent to test the series stationarity with the ADF test. We obtained a p -value of the ADF test of 0.119 which means we have no presumption to reject the null hypothesis. Therefore, we can say that the inflation contains a unit root. To resolve this, we need to differentiate the series and check again its p -value. After a single differentiation, we found that the inflation has become stationary with the p -value = 0.01397 and thus enabling us to identify the possible ARIMA model for inflation.

We continued the process by analysing the ACF and the PACF plots of the first differenced inflation. The autocorrelation plots appear in figure 3.3 and 3.4 respectively.

3. Stochastic Asset Liability Modelling: A Case of Malaysia

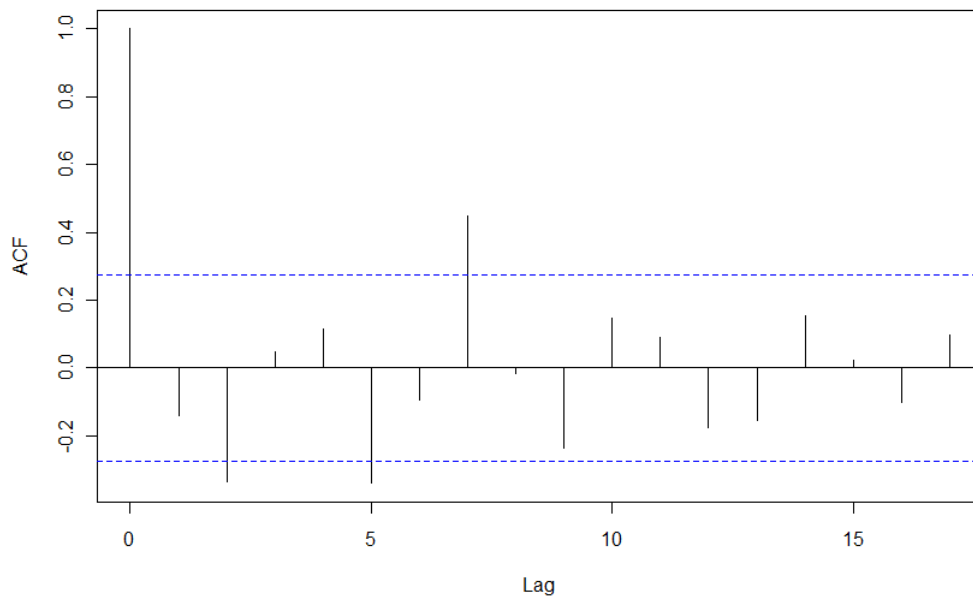


Figure 3.3.: ACF plot for the first differenced inflation

From figure 3.3, we can see that the autocorrelations at lags 2, 5, and 7 exceeded the significance bound, but the other autocorrelation remained significant. However, we have to analyse the PACF plot before deciding the type of ARIMA model.

3. Stochastic Asset Liability Modelling: A Case of Malaysia

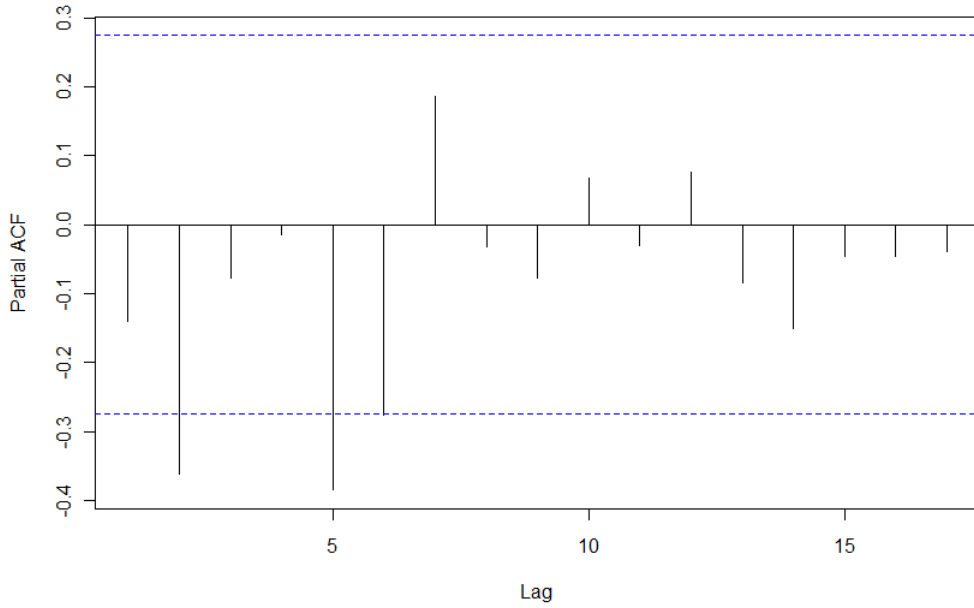


Figure 3.4.: PACF plot for the first differenced inflation

From figure 3.4, we noticed that the partial autocorrelations at lag 2 and 5 exceeded the significant boundary negatively and the magnitude gradually decreased after lag 5 as the lag increase. By considering the patterns of the autocorrelations, we can estimate the reasonable ARIMA models of inflation as follows:

- An ARIMA(2,1,0) model.

It is an autoregressive model of order $p=2$ with the first difference $d=1$. This is because we believe that the partial autocorrelogram is almost zero after lag 2 while the autocorrelogram tails off to zero.

- An ARIMA(0,1,2) model.

It is a moving average model of order $q=2$ with the first difference $d=1$. This is because we believe that the autocorrelogram is zero after lag 2 and the partial autocorrelogram tails off to zero.

- An ARIMA(2,1,2) model.

It is a mixed model, $p = 2$ and $d = 2$ with the first difference $d=1$. This is when we believe that the autocorrelogram and the partial autocorrelogram both tail off to zero after lag 2.

Then, we checked the values of AIC and BIC of the three possible models where we observed the following results:

3. Stochastic Asset Liability Modelling: A Case of Malaysia

ARIMA	AIC	BIC
ARIMA(2,1,0)	-221.5	-215.71
ARIMA (0,1,2)	-224.95	-219.16
ARIMA (2,1,2)	-221.9	-212.25

Table 3.4.: AIC and BIC values of possible ARIMA models for inflation

ARIMA (0,1,2) has the smallest AIC and BIC values. So far, we have decided that the inflation fitted well with the ARIMA (0,1,2) model but we still tested the model output in the next step. The results contradicted with the original retail prices index model developed by A.D Wilkie which was modelled as an AR(1) model.

Next, we continued with the second step in the Box-Jenkins methodology. We have to estimate the values of parameters for ARIMA(0,1,2). Below are the estimated parameters MA1 and MA2 with its standard errors in brackets:

$$\text{MA1} = -0.3478 (0.1366), \quad \text{MA2} = -0.4344 (0.1514).$$

We checked the significance of the parameters. For each parameter, we calculated $z = \text{estimated parameter} / \text{standard error of parameter}$. If $|z| > 1.96$, the estimated parameter is significantly different from zero and is approved for use in the model. In this case, both parameters are significantly different from zero. Henceforth, we let the inflation series as I_1, I_2, \dots, I_t and the inflation series after the first difference as $\tilde{I}_1, \tilde{I}_2, \dots, \tilde{I}_t$ with $\tilde{I}_t = \nabla I_t$. Thereby, the fitted force of inflation was modelled as

$$\tilde{I}_t = \varepsilon_t + 0.3478\varepsilon_{t-1} + 0.4344\varepsilon_{t-2}, \quad (3.16)$$

or can be written as

$$I_t = I_{t-1} + \varepsilon_t + 0.3478\varepsilon_{t-1} + 0.4344\varepsilon_{t-2} \quad (3.17)$$

with ε_{t-i} , $i = 1, 2$ as the errors of this series.

As for the third step in the Box-Jenkins methodology, we analysed the outputs of the residuals. This included a plot of residuals, an ACF plot of residuals and a plot of p -values of the Ljung-Box statistics for the first 10 lags. These plots are demonstrated in figure 3.5.

3. Stochastic Asset Liability Modelling: A Case of Malaysia

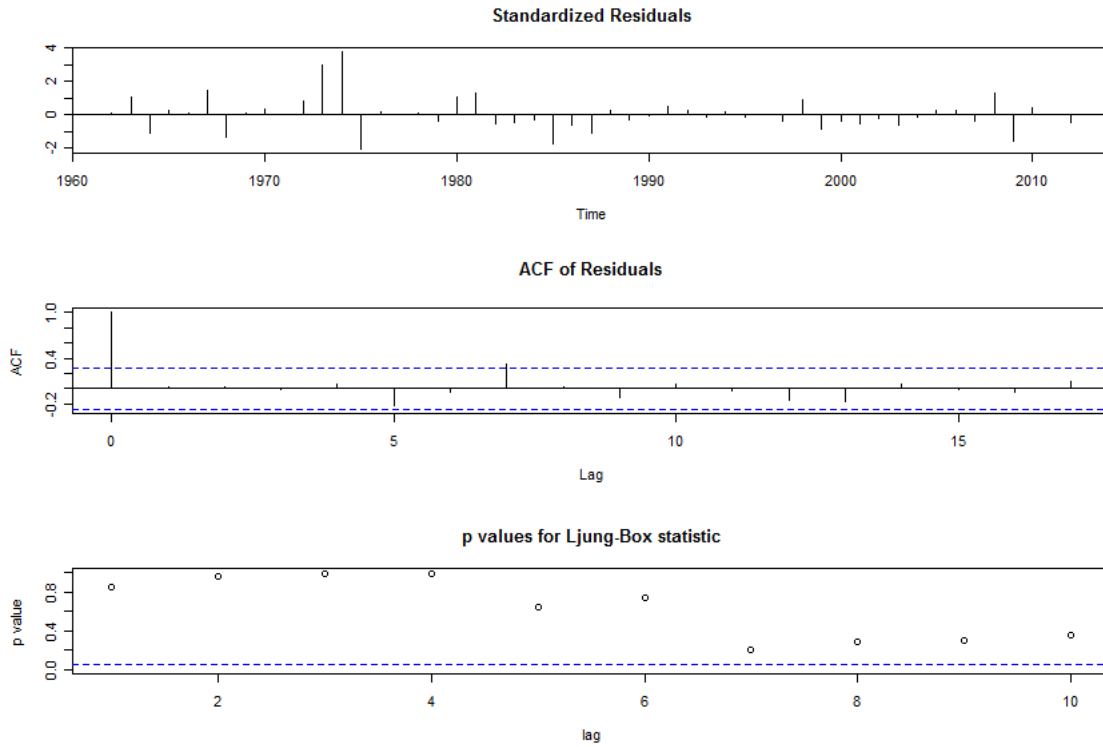


Figure 3.5.: Output from residuals analysis for inflation

Referring to figure 3.5, the top plot is the plot of standardised residuals. The plot revealed no particular pattern or trend. The middle plot is the plot of ACF of residuals. The plot shows that at lag-2 onward, the residuals are significant. Even though there is a spike of correlation at lag-5, we believe that it will not affect our analysis significantly. The bottom plot is a plot of p -values for Ljung-Box statistic. It shows that the p -values are all greater than 0.05 which means that we may accept the null hypothesis (see appendix F) at a 95 % significance level. Thus, it is concluded that the residuals are independent and identically distributed with a mean of 0 and variance of σ^2 . Hence, the residuals are to be called white noise.

Furthermore, the estimated values of parameter MA1 and MA2 must meet the stationary and invertible conditions as stated in table 3.2. We found the estimated parameters satisfied the stationary and invertible conditions. By considering all procedures that were conducted earlier, we concluded that the ARIMA(0,1,2) is the best fitted model for inflation.

Since the model diagnostic tests showed that all parameters were significant and the residuals were white noise, the estimation and diagnostic checking stage is now completed. Therefore, we can now forecast the inflation by using the fitted ARIMA(0,1,2)

3. Stochastic Asset Liability Modelling: A Case of Malaysia

model. We aim to forecast thirty years ahead from the latest inflation value. The output of the forecast is shown in table 3.5 and figure 3.6.

Year	Forecast	Lo 80	Hi 80	Lo 95	Hi 95
2013	0.01909425	-0.01284660	0.05103510	-0.02975506	0.06794356
2014	0.02334453	-0.01478982	0.06147888	-0.03497693	0.08166599
2015	0.02334453	-0.01541945	0.06210852	-0.03593987	0.08262893
2016	0.02334453	-0.01603902	0.06272808	-0.03688741	0.08357648
2017	0.02334453	-0.01664899	0.06333806	-0.03782028	0.08450935
2018	0.02334453	-0.01724980	0.06393886	-0.03873914	0.08542820
2019	0.02334453	-0.01784184	0.06453091	-0.03964459	0.08633366
2020	0.02334453	-0.01842549	0.06511456	-0.04053721	0.08722628
2021	0.02334453	-0.01900110	0.06569017	-0.04141753	0.08810659
2022	0.02334453	-0.01956899	0.06625806	-0.04228604	0.08897510
2023	0.02334453	-0.02012946	0.06681853	-0.04314321	0.08983227
2024	0.02334453	-0.02068280	0.06737187	-0.04398946	0.09067853
2025	0.02334453	-0.02122927	0.06791833	-0.04482521	0.09151428
2026	0.02334453	-0.02176912	0.06845818	-0.04565084	0.09233991
2027	0.02334453	-0.02230258	0.06899165	-0.04646671	0.09315577
2028	0.02334453	-0.02282989	0.06951895	-0.04727315	0.09396221
2029	0.02334453	-0.02335124	0.07004030	-0.04807048	0.09475955
2030	0.02334453	-0.02386683	0.07055589	-0.04885901	0.09554808
2031	0.02334453	-0.02437685	0.07106591	-0.04963902	0.09632809
2032	0.02334453	-0.02488148	0.07157054	-0.05041078	0.09709985
2033	0.02334453	-0.02538088	0.07206994	-0.05117455	0.09786362
2034	0.02334453	-0.02587521	0.07256428	-0.05193057	0.09861964
2035	0.02334453	-0.02636463	0.07305370	-0.05267908	0.09936814
2036	0.02334453	-0.02684928	0.07353834	-0.05342028	0.10010934
2037	0.02334453	-0.02732929	0.07401836	-0.05415440	0.10084346
2038	0.02334453	-0.02780480	0.07449387	-0.05488162	0.10157069
2039	0.02334453	-0.02827593	0.07496499	-0.05560215	0.10229121
2040	0.02334453	-0.02874280	0.07543186	-0.05631616	0.10300523
2041	0.02334453	-0.02920552	0.07589458	-0.05702383	0.10371289
2042	0.02334453	-0.02966420	0.07635326	-0.05772532	0.10441439

Table 3.5.: Forecast values of inflation for year 2013-2042

3. Stochastic Asset Liability Modelling: A Case of Malaysia

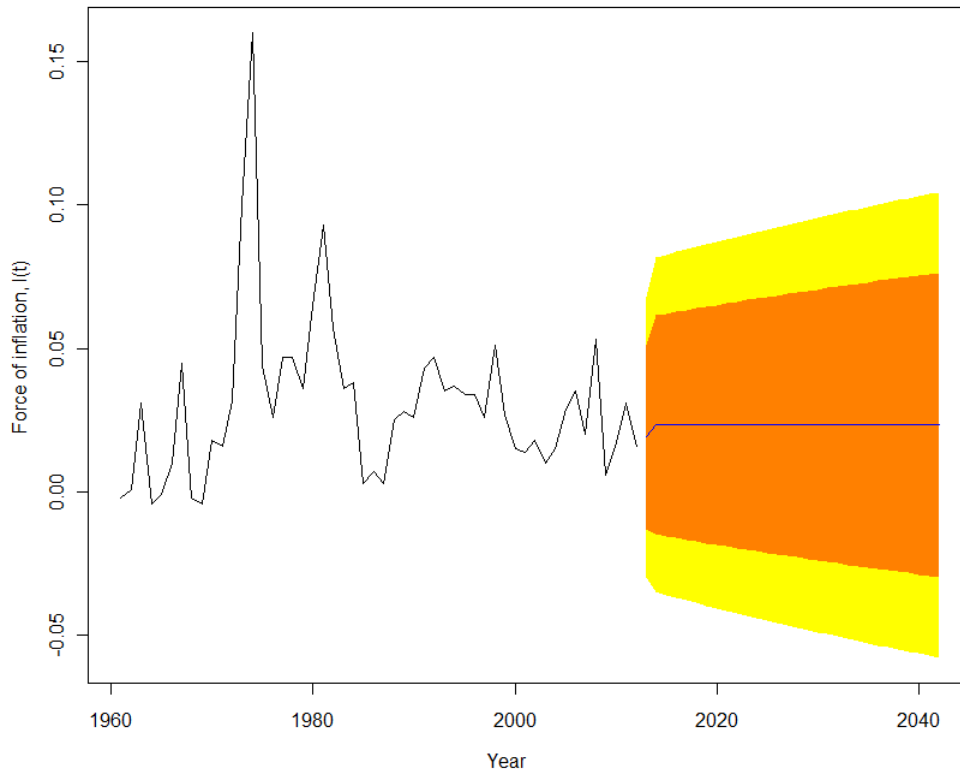


Figure 3.6.: ARIMA(0,1,2) model for inflation

Table 3.5 shows the minimum and maximum number of point prediction according to 80% and 95% prediction intervals. As an example, the inflation rate in Malaysia is forecasted to be 2.33 per cent in 2015 which is in between -3.59 per cent as the lowest percentage to 8.26 per cent as the highest percentage, in within the 95% prediction interval. In addition, the inflation rate in Malaysia is forecasted to be 1.9 per cent in 2013 and remains stable at 2.3 per cent in 2014 until 2042. This rate indicates that the Malaysian market will be in good condition for the next 30 years and this will be driven by lower consumer price. On the other hand, figure 3.6 displays the forecast inflation rate in Malaysia for the period 2013-2042 which is plotted in a blue line whereas the 80% prediction interval is in the orange shaded area and the 95% prediction interval is in the yellow shaded area.

3.4.3. The FTSE Bursa Malaysia KLCI Yield Model

KLCI was introduced in 1986 comprising of 30 largest companies enrolled in the Malaysian main market. KLCI is the acronym for the Kuala Lumpur Composite Index and the name was then changed to FTSE Bursa Malaysia KLCI in July 2006. The monthly FTSE Bursa Malaysia KLCI yield from July 2009 until September 2013 is plotted in

3. Stochastic Asset Liability Modelling: A Case of Malaysia

figure 3.7.

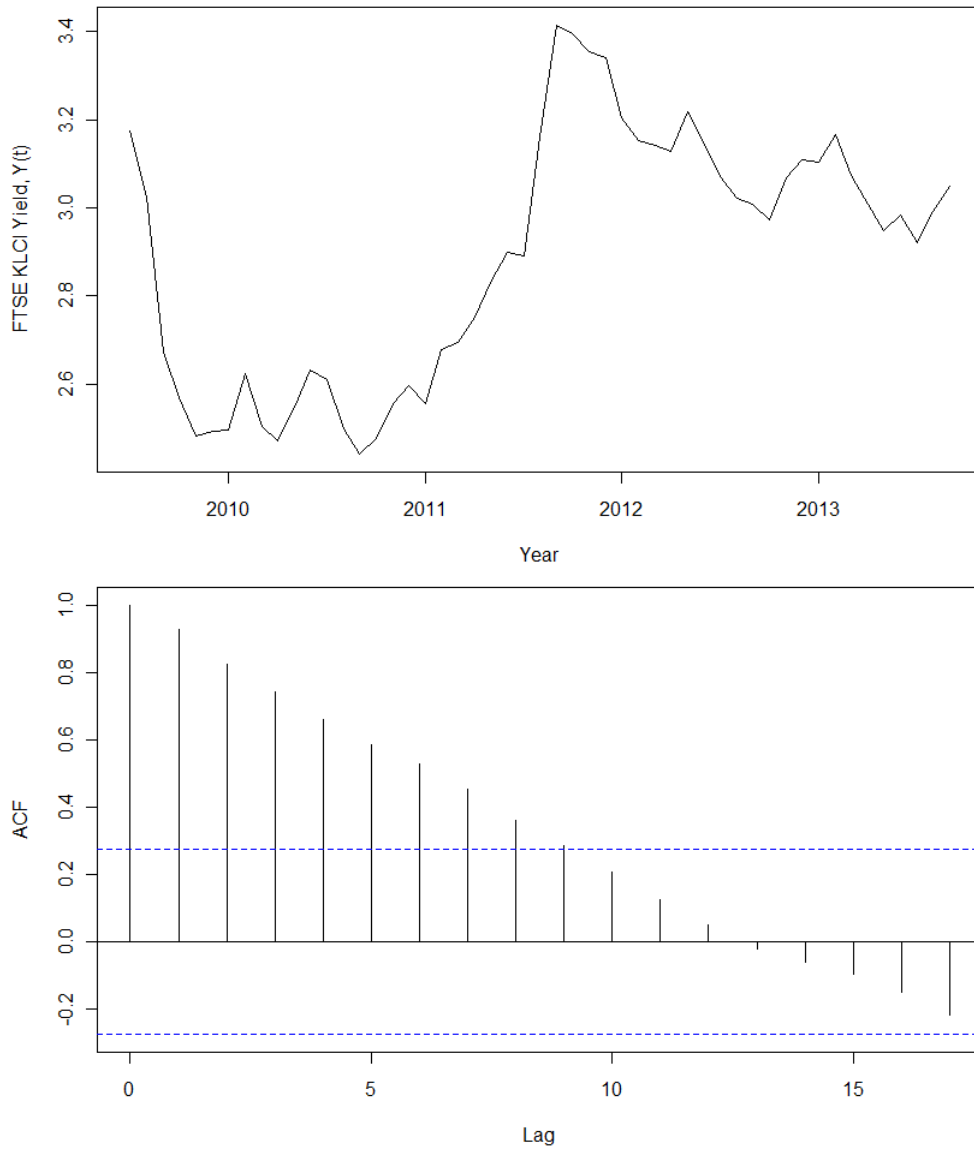


Figure 3.7.: Monthly FTSE Bursa Malaysia KLCI yield, $Y(t)$: July 2009 - September 2013 together with its correlogram

As we can see from figure 3.7, the FTSE Bursa Malaysia KLCI yield was positive for the entire testing period. The maximum FTSE Bursa Malaysia KLCI yield was 3.416

3. Stochastic Asset Liability Modelling: A Case of Malaysia

per cent which occurred at the end of 2012. This high yield offered a growth that was expected from 2012 onwards despite marked low yields in most of 2010. The decay of the yield in 2010 was suspected as a reflection of the European financial problems, weak economic performance of the United States, as well as the rising global inflation. Fortunately, Malaysia still showed a positive growth in the economy and the domestic interest rates seemed to remain stable.

Descriptive statistics for FTSE Bursa Malaysia KLCI yield are summarised in table 3.6. The FTSE Bursa Malaysia KLCI yield ($n=51$) averaged 2.89 per cent from July 2009 to September 2013. The FTSE Bursa Malaysia KLCI yield was 0.29 per cent away from the average, which we can say it is quite closely spread. Meanwhile, the coefficient of skewness has taken a negative value which means most probably the yield constructs a negative skew distribution. A negative kurtosis shows the distribution of this series is more flat to the left.

Mean	2.89
Standard deviation	0.29
Skewness	-0.06
Kurtosis	-1.33

Table 3.6.: Summary statistics for FTSE Bursa Malaysia KLCI yield

From the plot in figure 3.7, we noticed that the FTSE Bursa Malaysia KLCI yield is not stationary because the autocorrelation values were high and then dipped to zero when the lags are large. We also saw a short-term autocorrelation in this series. Therefore, the ARIMA model would be suitable for the FTSE Bursa Malaysia KLCI yield. On the other hand, it is proven by the ADF test that the FTSE Bursa Malaysia KLCI yield was only stationary at the first difference. Thus, we can use the first difference of this series to build a suitable ARIMA model.

Again, we referred to ACF and PACF of the first difference of this series to decide the order of the ARIMA (p, d, q) . The plots of autocorrelations are shown in figure 3.8 and 3.9 respectively.

3. Stochastic Asset Liability Modelling: A Case of Malaysia

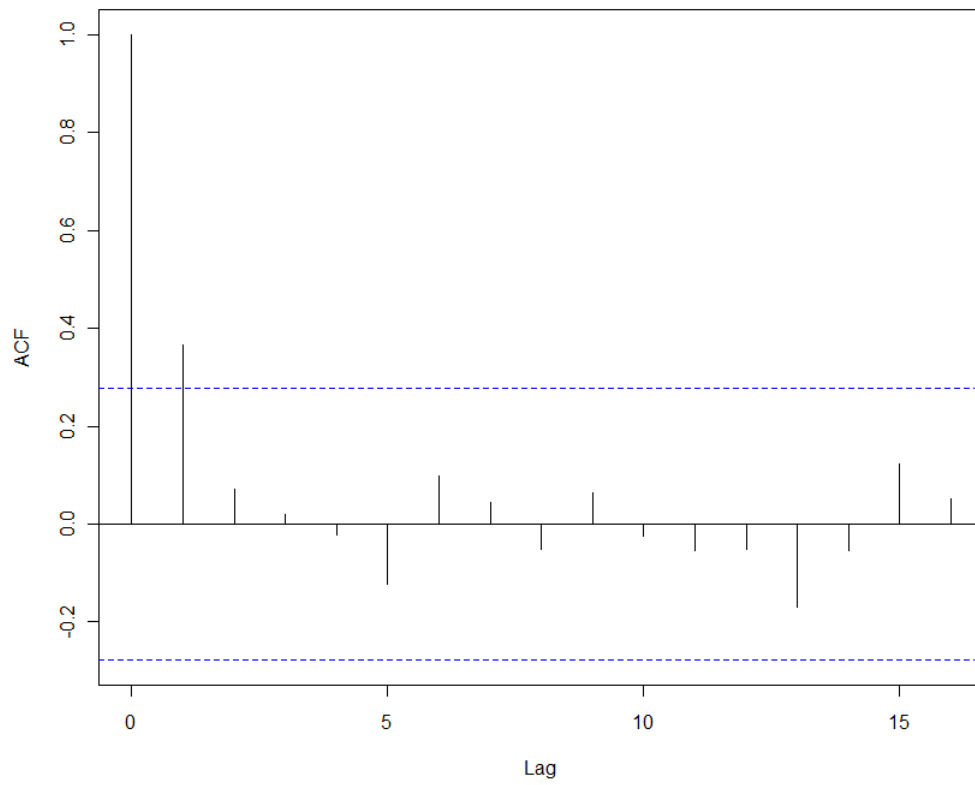


Figure 3.8.: ACF plot for the first differenced FTSE Bursa Malaysia KLCI yield

3. Stochastic Asset Liability Modelling: A Case of Malaysia

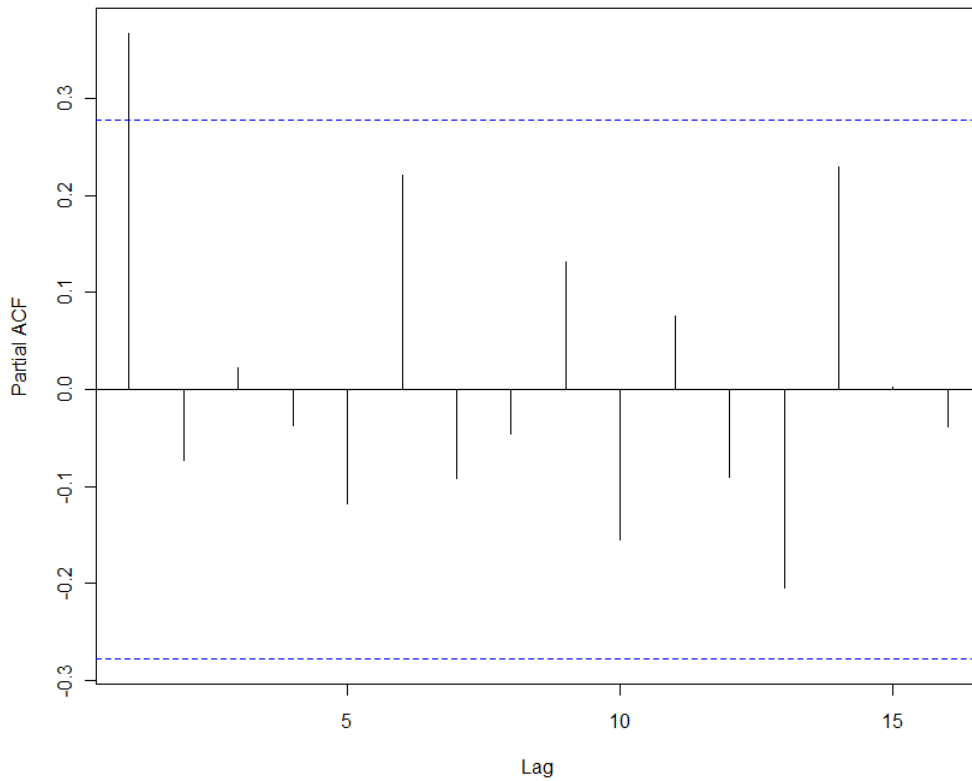


Figure 3.9.: PACF plot for the first differenced FTSE Bursa Malaysia KLCI yield

The correlogram in figure 3.8 shows that the autocorrelation at lag 1 exceeded the significant boundary positively whereas the other autocorrelations between lags 2-17, although significant, continued to decrease in slow motion. We can say that the autocorrelations tail off to zero after lag 1. We then analysed the partial autocorrelation in figure 3.9.

The partial autocorrelations at lag 1 exceeded the significant boundary positively. By considering the patterns of autocorrelations, we list down the possible ARIMA models for FTSE Bursa Malaysia KLCI yield as follows:

- An ARIMA(1,1,0) model.

It is an autoregressive model of order $p=1$ with the first difference $d=1$. We came to this conclusion due to the fact that the partial autocorrelogram is almost zero after lag 1 and the autocorrelogram tails off to zero.

- An ARIMA(0,1,1) model.

It is a moving average model of order $q=1$ with the first difference $d=1$. This is

3. Stochastic Asset Liability Modelling: A Case of Malaysia

due to the fact that the autocorrelogram is almost zero after lag 1 and the partial autocorrelogram tails off to zero.

- An ARIMA(1,1,1) model.

It is a mixed model, $p=1$ and $q=1$ with the first difference $d=1$. It is because we believe that the autocorrelogram and partial correlogram both tail off to zero after lag 1.

We then compared the AIC and BIC values of the three possible ARIMA models in order to select the lowest value. The values are shown in table 3.7.

ARIMA	AIC	BIC
ARIMA(1,1,0)	-92.23	-88.4
ARIMA (0,1,1)	-91.84	-88.02
ARIMA (1,1,1)	-90.3	-84.57

Table 3.7.: AIC and BIC values of possible ARIMA models for FTSE Bursa Malaysia KLCI yield

ARIMA(1,1,0) had the lowest AIC and BIC values among all possible fitted model for the FTSE Bursa Malaysia KLCI yield. This is slightly parallel to the share dividend yield model developed by A.D Wilkie [Wilkie, 1984], which was modelled as an AR(1). Then, we obtained the estimated parameter AR1 for this model

$$\text{AR1} = 0.3788 (0.1327)$$

where the value in the bracket is the standard error. The estimated value of AR1 was significantly different from zero. Henceforth, we denoted the FTSE Bursa Malaysia KLCI yield series as Y_1, Y_2, \dots, Y_t , then the FTSE Bursa Malaysia KLCI yield series after the first difference as $\tilde{Y}_1, \tilde{Y}_2, \dots, \tilde{Y}_t$ where $\tilde{Y}_t = \nabla Y_t$. Thereby, the fitted FTSE Bursa Malaysia KLCI yield is assumed to follow

$$\tilde{Y}_t = 0.3788\tilde{Y}_{t-1} + \varepsilon_t \quad (3.18)$$

or may be written as

$$Y_t = 1.3788Y_{t-1} - 0.3788Y_{t-2} + \varepsilon_t. \quad (3.19)$$

with ε_t as the error of this series.

With regards to the adequacy of the Box-Jenkins model, we are required to analyse the residual. The result is given in figure 3.10.

3. Stochastic Asset Liability Modelling: A Case of Malaysia

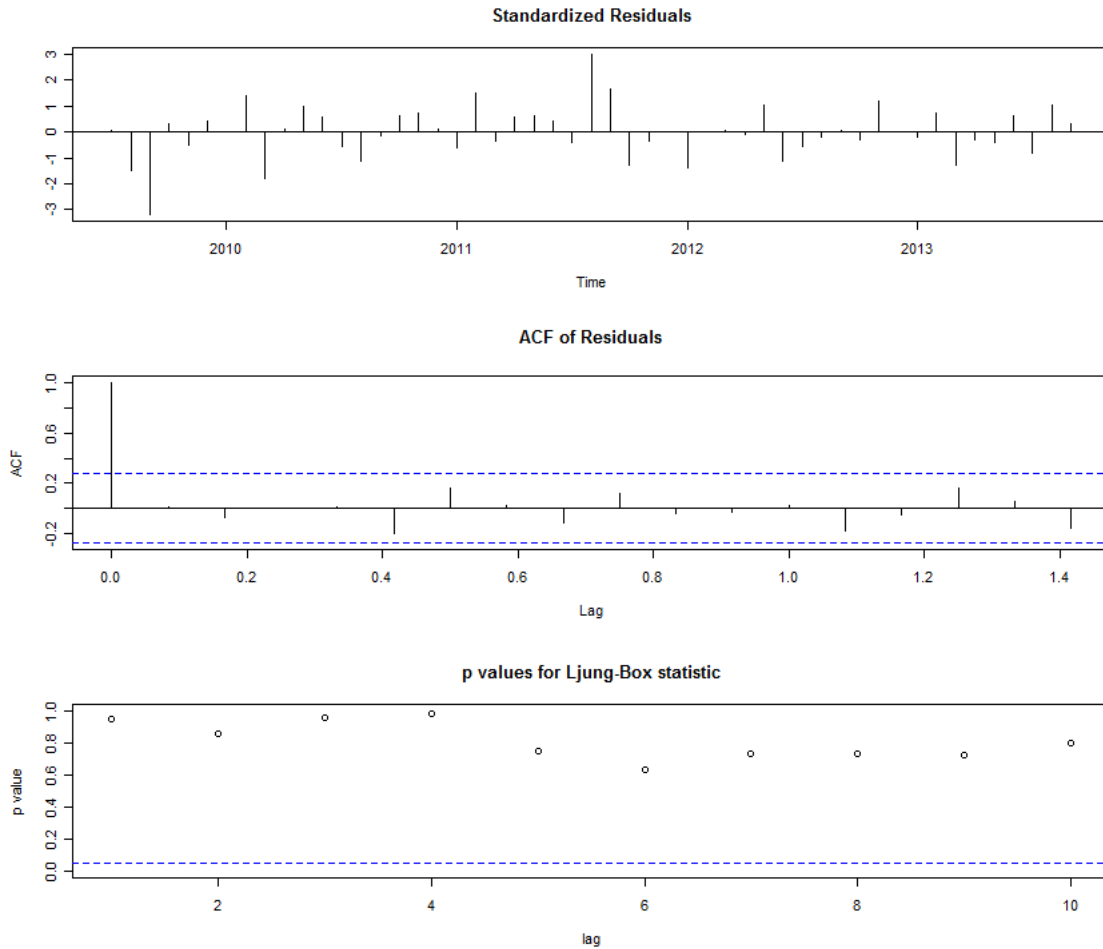


Figure 3.10.: Output from residuals analysis for FTSE Bursa Malaysia KLCI yield

In figure 3.10, the top plot showed no particular pattern in the residuals. The middle plot showed that the residual autocorrelations were significant while the bottom plot showed that the p -values for Ljung-Box statistic were all greater than 0.05. The results led us to the conclusion that the residuals are independently distributed with zero-mean and variance σ^2 and yet to be called as white noise.

Just as for the inflation model, we needed to check the stationary and invertible conditions. According to table 3.2, we referred to AR(1) condition since ARIMA (1,1,0) is equivalent to the AR(1) model if we took out the difference. We have checked that the parameters satisfied the stationary and invertible conditions. Thus, it strengthened the evidence that ARIMA(1,1,0) is the most suitable model for FTSE Bursa Malaysia KLCI yield.

We then used the corresponding fitted model for forecasting. We would like to forecast

3. Stochastic Asset Liability Modelling: A Case of Malaysia

30 months ahead of the FTSE Bursa Malaysia KLCI yield. The forecasting output is shown in table 3.8 and figure 3.11.

Month	Forecast	Lo 80	Hi 80	Lo 95	Hi 95
Oct 2013	3.071589	2.953308	3.189870	2.890694	3.252484
Nov 2013	3.079766	2.878307	3.281226	2.771661	3.387872
Dec 2013	3.082864	2.812672	3.353055	2.669641	3.496086
Jan 2014	3.084037	2.755743	3.412331	2.581954	3.586119
Feb 2014	3.084481	2.705714	3.463248	2.505207	3.663755
Mar 2014	3.084649	2.660974	3.508325	2.436694	3.732605
Apr 2014	3.084713	2.620310	3.549116	2.374470	3.794956
May 2014	3.084737	2.582851	3.586623	2.317169	3.852306
Jun 2014	3.084746	2.547971	3.621522	2.263819	3.905674
Jul 2014	3.084750	2.515211	3.654288	2.213716	3.955784
Aug 2014	3.084751	2.484233	3.685270	2.166337	4.003165
Sep 2014	3.084752	2.454774	3.714730	2.121284	4.048220
Oct 2014	3.084752	2.426632	3.742872	2.078244	4.091260
Nov 2014	3.084752	2.399644	3.769859	2.036970	4.132533
Dec 2014	3.084752	2.373681	3.795823	1.997262	4.172242
Jan 2015	3.084752	2.348632	3.820872	1.958954	4.210550
Feb 2015	3.084752	2.324408	3.845096	1.921906	4.247598
Mar 2015	3.084752	2.300932	3.868572	1.886003	4.283500
Apr 2015	3.084752	2.278140	3.891364	1.851145	4.318359
May 2015	3.084752	2.255974	3.913530	1.817245	4.352259
Jun 2015	3.084752	2.234385	3.935119	1.784228	4.385276
Jul 2015	3.084752	2.213331	3.956173	1.752029	4.417475
Aug 2015	3.084752	2.192774	3.976730	1.720590	4.448914
Sep 2015	3.084752	2.172680	3.996823	1.689859	4.479645
Oct 2015	3.084752	2.153020	4.016484	1.659791	4.509713
Nov 2015	3.084752	2.133766	4.035738	1.630344	4.539160
Dec 2015	3.084752	2.114894	4.054610	1.601482	4.568022
Jan 2016	3.084752	2.096382	4.073122	1.573171	4.596333
Feb 2016	3.084752	2.078211	4.091293	1.545380	4.624124
Mar 2016	3.084752	2.060362	4.109142	1.518082	4.651422

Table 3.8.: Forecast values of FTSE Bursa Malaysia KLCI yield for Oct 2013-Mac 2016

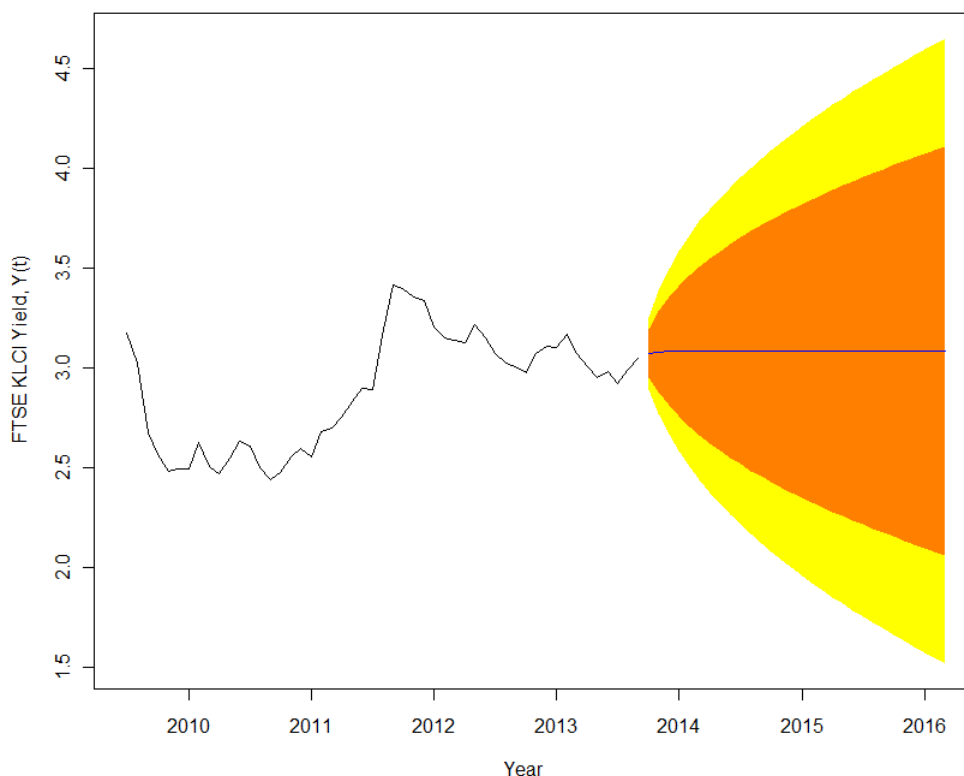


Figure 3.11.: ARIMA(1,1,0) model for FTSE Bursa Malaysia KLCI yield

Table 3.8 displays the minimum and maximum number of point prediction according to 80% and 95% prediction intervals respectively. The number in table 3.8 demonstrated that the FTSE Bursa Malaysia KLCI yield will increase from 3.07 per cent in October 2013 to 3.08 per cent in November 2013 until March 2016. The yield continued to grow with no sign of decay for the next 30 months. On the other hand, figure 3.11 shows graphically the forecast values of the FTSE Bursa Malaysia KLCI yield for 30 months ahead. The orange grey shaded area highlights the 80% prediction interval while the yellow shaded area highlights the 95% prediction interval. The blue line represents the forecast values.

3.4.4. The FTSE Bursa Malaysia KLCI Model

In this model, we studied the index of the FTSE Bursa Malaysia instead of the yield as in the previous model. We wanted to build a Box-Jenkins model for the FTSE Bursa Malaysia index. We used monthly data and its plot from January 1994 to December 2013 as shown in figure 3.12.

3. Stochastic Asset Liability Modelling: A Case of Malaysia

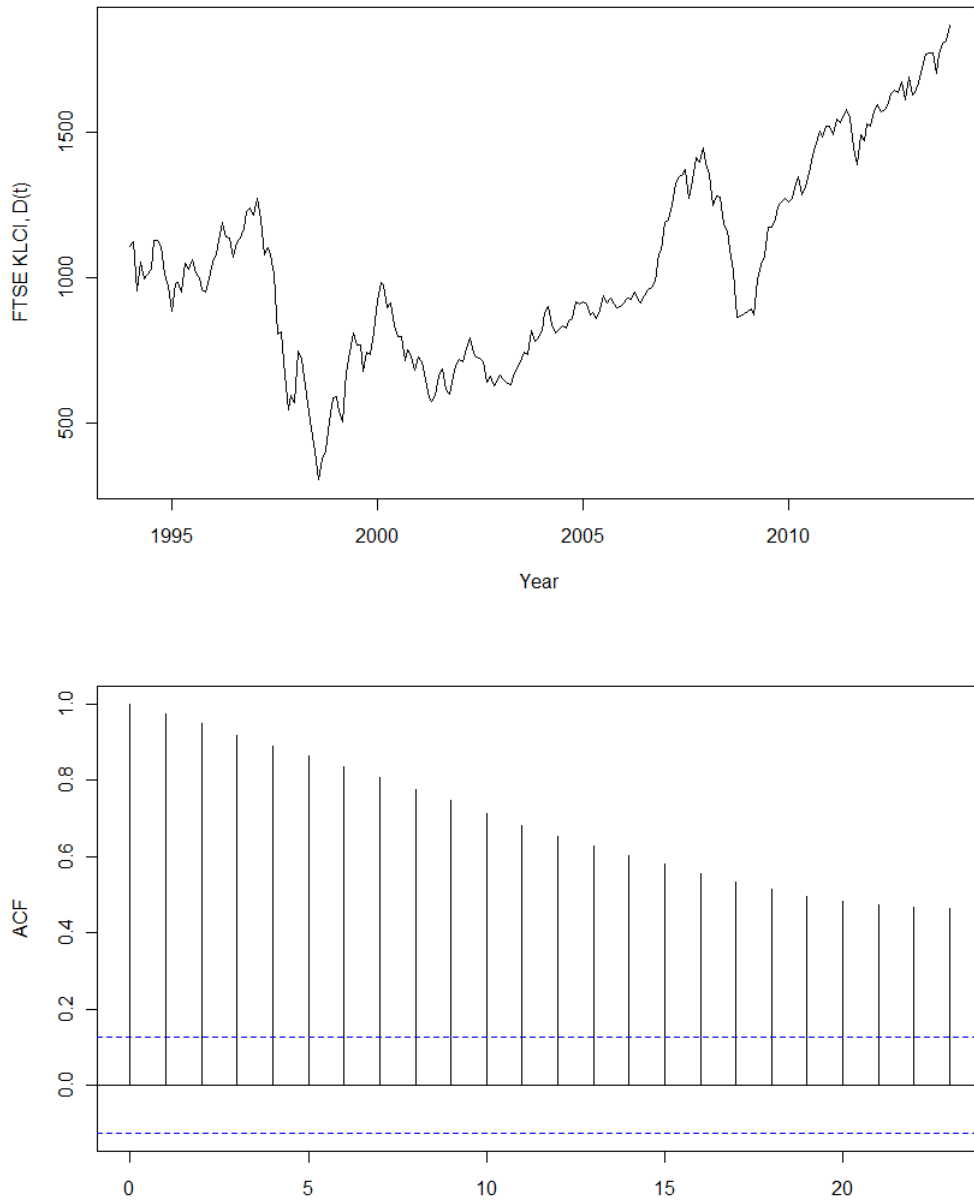


Figure 3.12.: Monthly FTSE Bursa Malaysia KLCI, $D(t)$: January 1994 - December 2013 together with its correlogram

It is obvious that FTSE Bursa Malaysia KLCI maintained a positive value throughout the years of experiment and in fact showed some increment starting 2009 onwards. We summarise the descriptive statistics for this series in table 3.9. With the total number

3. Stochastic Asset Liability Modelling: A Case of Malaysia

of data $n=240$, the FTSE Bursa Malaysia KLCI averaged 1037.75 points from January 1994 to December 2013. The standard deviation reveals that the index is 342.55 points away from the average. The positive skew shows that the series has a positive or right skew distribution and the negative kurtosis -0.62 indicates that the distribution of the series is more flat to the right.

Mean	1037.75
Standard deviation	342.55
Skewness	0.43
Kurtosis	-0.62

Table 3.9.: Summary statistics for FTSE Bursa Malaysia KLCI

Statistics presented in Figure 3.12 indicated that the series are not stationary with some seasonality through the slow decay in the ACF. The ACF plot also showed that the series is facing a short-term autocorrelation with a strong ACF coefficients at small lag. After conducting the ADF test to this series, we found that the series was not stationary but becomes stationary at the first difference. By considering the non-stationarity as well as the seasonality of the series, we can build the ARIMA model for seasonal series. We continued the process by plotting the first differenced series as well as the ACF and PACF as seen in figure 3.13.

3. Stochastic Asset Liability Modelling: A Case of Malaysia

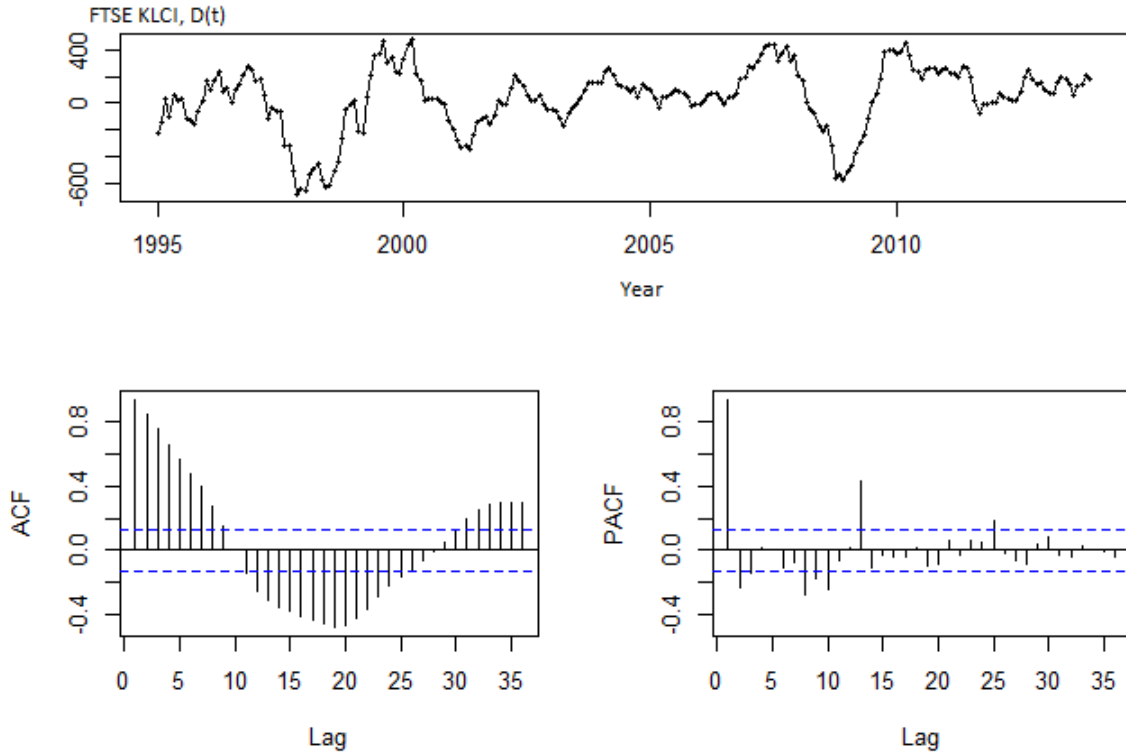


Figure 3.13.: Seasonally differenced FTSE Bursa Malaysia KLCI

Our aim now is to find an appropriate ARIMA model based on the ACF and PACF plots. As seen in figure 3.13, both the ACF and PACF showed significant spikes at lag 2 which suggested seasonal AR(2) and MA(2) components. However, indeed it is difficult to guess the possible seasonal ARIMA models and because of that, we used a function "auto.arima" in R which will give the best model with the lowest AIC and BIC values. We obtained the possible model as

$$\text{ARIMA}(2,1,1)(2,0,2)[12].$$

The result is slightly inconsistent with the share dividend index model by [Wilkie, 1984], which developed to be a MA(1) model. We list out the estimated parameters for

3. Stochastic Asset Liability Modelling: A Case of Malaysia

ARIMA(2,1,1)(2,0,2)[12] as below:

$$\begin{aligned}
 AR1 &= 0.6797(0.2123), \\
 AR2 &= 0.0542(0.0769), \\
 MA1 &= -0.5562(0.2124), \\
 SAR1 &= -1.1731(0.0720), \\
 SAR2 &= -0.3618(0.0580), \\
 SMA1 &= 1.2531(0.0963), \\
 SMA2 &= 0.4485(0.0836),
 \end{aligned}$$

where the values in the bracket are the standard errors. We then denoted the observed FTSE Bursa Malaysia KLCI as D_1, D_2, \dots, D_t . Therefore, the fitted FTSE Bursa Malaysia KLCI model can be written as the following:

$$\begin{aligned}
 (1 - SAR1 \cdot B^{12} - SAR2 \cdot B^{24})(1 - AR1 \cdot B - AR2 \cdot B^2)(1 - B)D_t \\
 = (1 + SMA1 \cdot B^{12} + SMA2 \cdot B^{24})(1 + MA1 \cdot B)\varepsilon_t.
 \end{aligned}$$

We expanded the previous equation to form

$$\begin{aligned}
 (1 - B - AR1 \cdot B + AR1 \cdot B^2 - AR2 \cdot B^2 + AR2 \cdot B^3 - SAR1 \cdot B^{12} + SAR1 \cdot B^{13} \\
 + SAR1 \cdot AR1 \cdot B^{13} - SAR1 \cdot AR1 B^{14} + SAR1 \cdot AR2 \cdot B^{14} - SAR1 \cdot AR2 B^{15} - AR2 \cdot B^{24} \\
 + AR2 \cdot B^{25} + AR2 \cdot AR1 \cdot B^{25} - AR2 \cdot AR1 \cdot B^{26} + SAR2 \cdot AR2 B^{26} - SAR2 \cdot AR2 \cdot B^{27})D_t \\
 = (1 + MA1 \cdot B + SMA1 \cdot B^{12} + SMA1 \cdot MA1 B^{13} + SMA2 \cdot B^{24} + SMA2 \cdot MA1 \cdot B^{25})\varepsilon_t.
 \end{aligned}$$

By substituting the back shift operator B to previous equation, we obtained

$$\begin{aligned}
 D_t &= D_{t-1} + AR1 \cdot D_{t-1} - AR1 \cdot D_{t-2} + AR2 \cdot D_{t-2} - AR2 \cdot D_{t-3} + SAR1 \cdot D_{t-12} \\
 &\quad - SAR1 \cdot D_{t-13} - SAR1 \cdot AR1 D_{t-13} + SAR1 \cdot AR1 D_{t-14} - SAR1 \cdot AR2 \cdot D_{t-14} \\
 &\quad + SAR1 \cdot AR2 \cdot D_{t-15} + SAR2 \cdot D_{t-24} - SAR2 \cdot D_{t-25} - SAR2 \cdot AR1 \cdot D_{t-25} \\
 &\quad + SAR2 \cdot AR1 D_{t-26} - SAR2 \cdot AR2 \cdot D_{t-26} + SAR2 \cdot AR2 \cdot D_{t-27} + \varepsilon_t + MA1 \cdot \varepsilon_{t-1} \\
 &\quad + SMA1 \cdot \varepsilon_{t-12} + SMA1 \cdot MA1 \cdot \varepsilon_{t-13} + SMA2 \cdot \varepsilon_{t-24} + SMA2 \cdot MA1 \cdot \varepsilon_{t-25}.
 \end{aligned} \tag{3.20}$$

Then, we substituted the estimated parameter values to (3.20) as the following:

$$\begin{aligned}
 D_t &= D_{t-1} + 0.6797D_{t-1} - 0.6797D_{t-2} + 0.0542D_{t-2} - 0.0542D_{t-3} - 1.1731D_{t-12} \\
 &\quad + 1.1731D_{t-13} + 0.7974D_{t-13} - 0.7974D_{t-14} + 0.0635D_{t-14} - 0.0635D_{t-15} \\
 &\quad - 0.3618D_{t-24} + 0.3618D_{t-25} + 0.2459D_{t-25} - 0.2459D_{t-26} + 0.0196D_{t-26} \\
 &\quad - 0.0196D_{t-27} + \varepsilon_t - 0.5562\varepsilon_{t-1} + 1.2531\varepsilon_{t-12} - 0.6970\varepsilon_{t-13} + 0.4485\varepsilon_{t-24} \\
 &\quad - 0.2496\varepsilon_{t-25}.
 \end{aligned} \tag{3.21}$$

3. Stochastic Asset Liability Modelling: A Case of Malaysia

Then, we plotted the residuals as well as the p -values for Ljung-Box statistic as shown in figure 3.14.

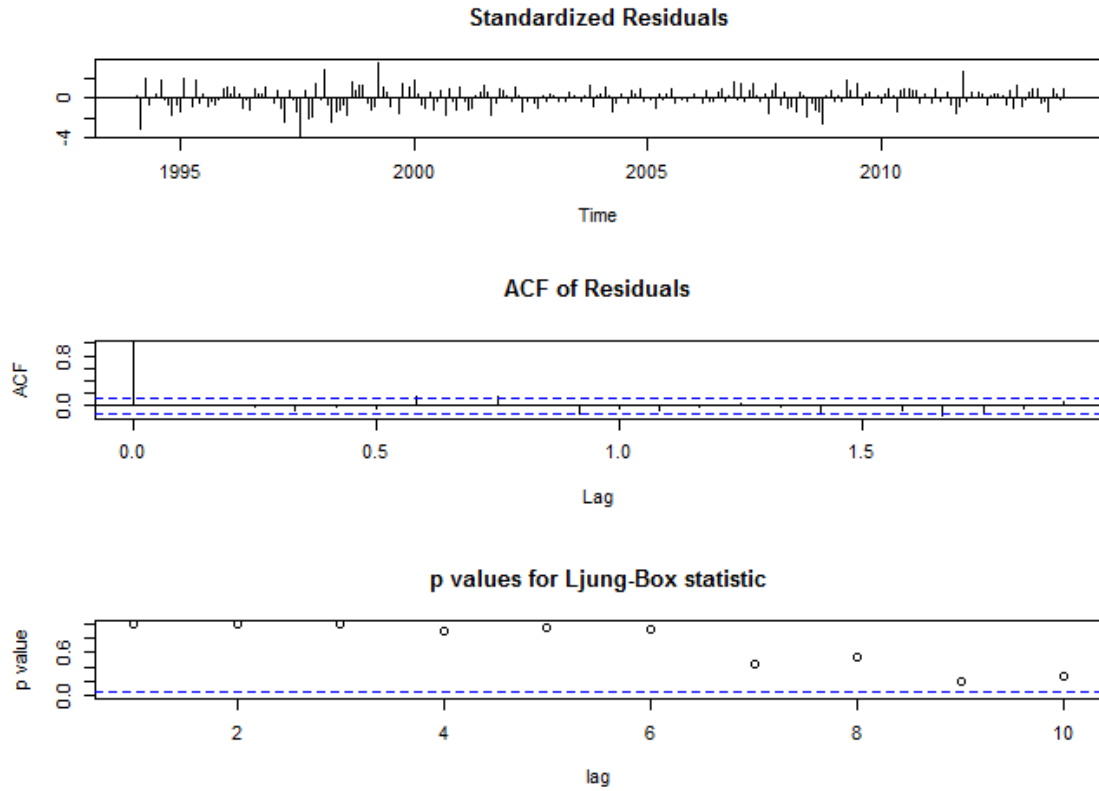


Figure 3.14.: Output from residuals analysis for FTSE Bursa Malaysia KLCI

From figure 3.14, we can see that the plot of the standardised residuals showed no particular pattern while the ACF plot of residuals showed significant autocorrelations. The plot of the p -values for each lags based on the Lyung-Box statistics showed that all the p -values are greater than 0.05. This indicates that the residuals are independently distributed with zero-mean and variance of σ^2 and the residuals are to be called white noise.

Therefore, we can now use ARIMA(2,1,1)(2,0,2)[12] model to forecast the FTSE Bursa Malaysia KLCI index for 30 months ahead and the results of the forecast are presented in table 3.10 as well as illustrated in figure 3.15.

3. Stochastic Asset Liability Modelling: A Case of Malaysia

Month	Forecast	Lo 80	Hi 80	Lo 95	Hi 95
Jan 2014	1877.345	1808.892	1945.798	1772.655	1982.035
Feb 2014	1892.537	1792.215	1992.858	1739.108	2045.965
Mar 2014	1899.790	1768.142	2031.437	1698.453	2101.127
Apr 2014	1894.239	1736.972	2051.506	1653.719	2134.759
May 2014	1889.170	1708.903	2069.436	1613.476	2164.864
Jun 2014	1905.995	1705.293	2106.697	1599.048	2212.942
Jul 2014	1906.712	1687.326	2126.098	1571.190	2242.234
Aug 2014	1920.412	1683.806	2157.018	1558.554	2282.270
Sep 2014	1913.740	1661.062	2166.417	1527.303	2300.177
Oct 2014	1948.957	1681.171	2216.743	1539.414	2358.501
Nov 2014	1947.789	1665.700	2229.878	1516.371	2379.207
Dec 2014	1953.766	1658.071	2249.461	1501.540	2405.993
Jan 2015	1961.446	1653.716	2269.177	1490.813	2432.080
Feb 2015	1960.704	1641.509	2279.899	1472.538	2448.871
Mar 2015	1964.480	1634.377	2294.583	1459.631	2469.329
Apr 2015	1968.444	1627.793	2309.096	1447.463	2489.426
May 2015	1979.773	1628.916	2330.630	1443.183	2516.362
Jun 2015	1979.484	1618.711	2340.257	1427.730	2531.238
Jul 2015	1979.277	1608.858	2349.696	1412.771	2545.783
Aug 2015	1980.954	1601.135	2360.774	1400.071	2561.838
Sep 2015	1969.141	1580.148	2358.133	1374.228	2564.053
Oct 2015	1983.317	1585.363	2381.271	1374.699	2591.935
Nov 2015	1979.884	1573.166	2386.601	1357.863	2601.905
Dec 2015	1990.794	1575.497	2406.090	1355.653	2625.935
Jan 2016	1993.231	1568.169	2418.292	1343.156	2643.305
Feb 2016	1988.876	1554.176	2423.576	1324.060	2653.692
Mar 2016	1987.075	1542.709	2431.441	1307.476	2666.674
Apr 2016	1996.639	1542.797	2450.481	1302.548	2690.730
May 2016	2011.108	1547.944	2474.271	1302.760	2719.455
Jun 2016	1997.543	1525.239	2469.847	1275.217	2719.870

Table 3.10.: Forecast values of FTSE Bursa Malaysia KLCI for Jan 2014-June 2016

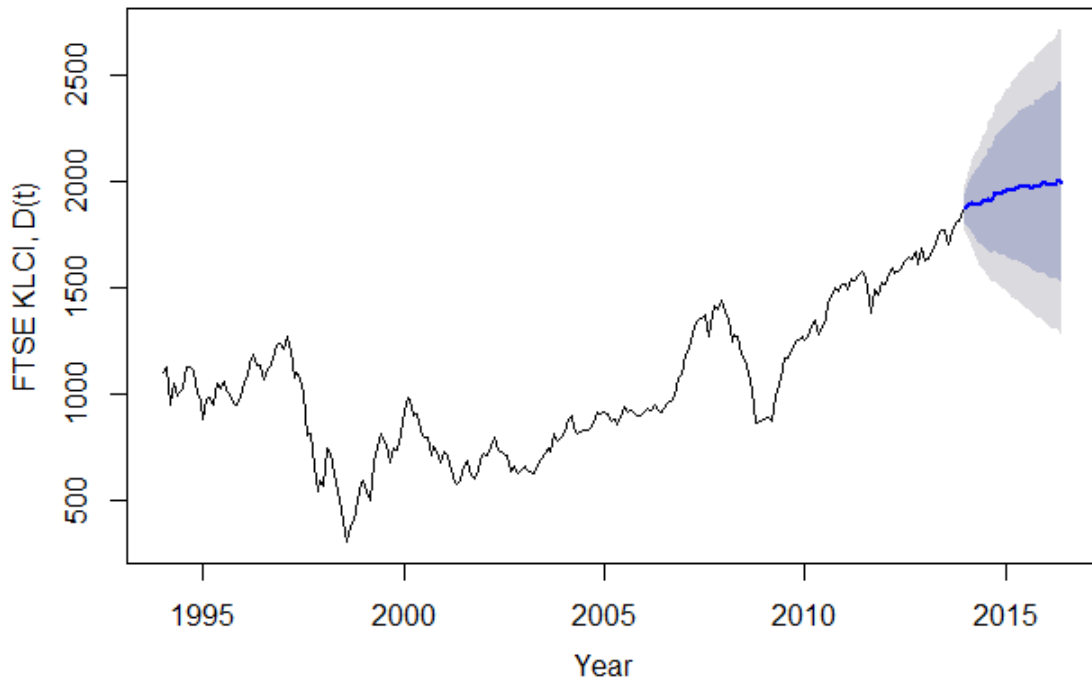


Figure 3.15.: ARIMA(2,1,1)(2,0,2)[12] model for FTSE Bursa Malaysia KLCI

Table 3.10 displays the minimum and maximum number of point prediction based on 80% and 95% prediction interval. The number in table 3.10 demonstrates that the FTSE Bursa Malaysia KLCI is forecasted to be 1877.35 points in January 2014. In between this time interval, there were fluctuations in the index. In the last month of forecasting, which is June 2016, the index is forecasted to be 1997.54 points.

3.4.5. The 10-Year MGS Yield Model

The 10-Year MGS yield model represents the long-term bond yield in the context of Malaysia. The monthly data starting from January 1996 to January 2014 are plotted in figure 3.16.

3. Stochastic Asset Liability Modelling: A Case of Malaysia

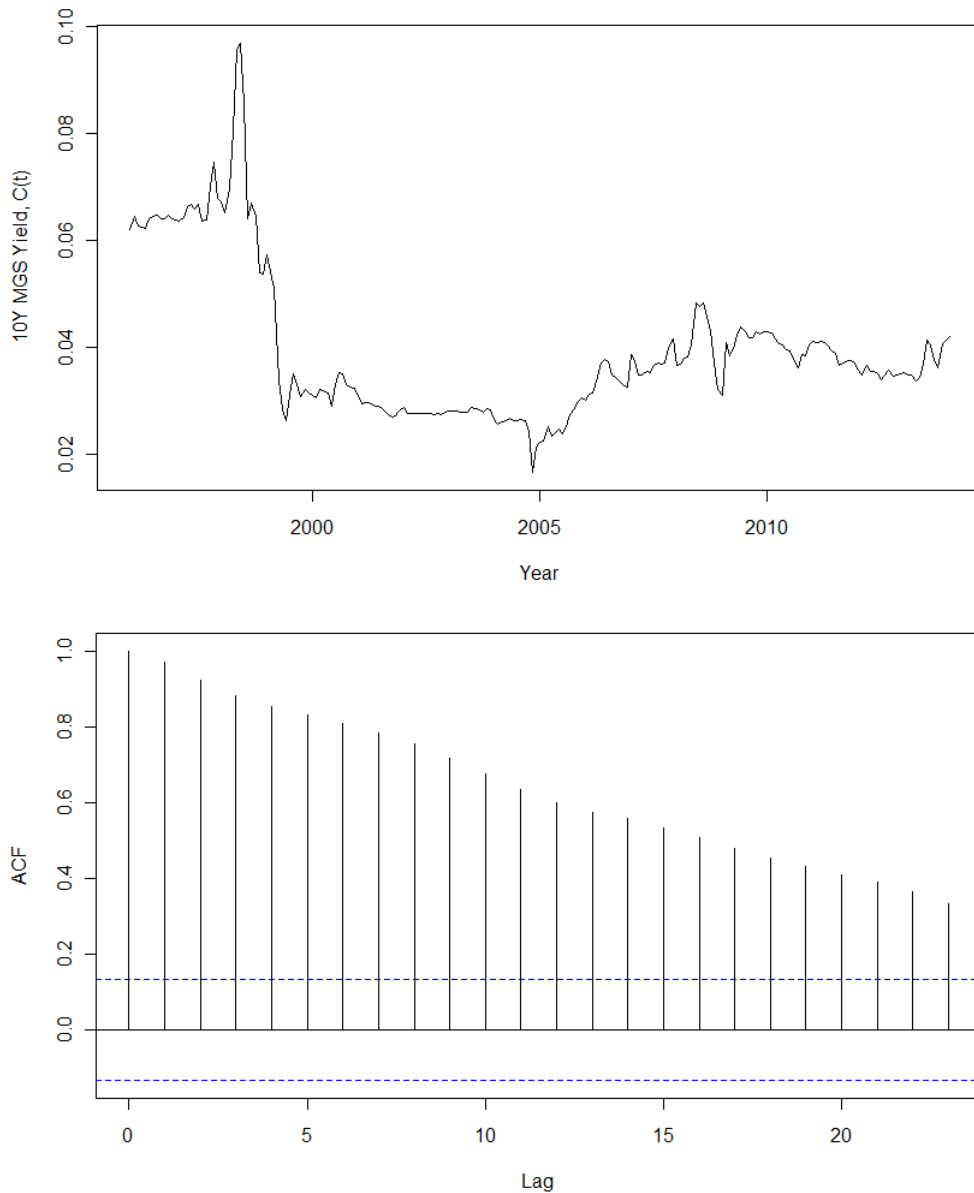


Figure 3.16.: Monthly 10-Year MGS yield, $C(t)$: January 1996 - January 2014 together with its correlogram

The 10-Year MGS yield remained positive throughout the period and had the highest value at 9.7 per cent in June 1998 whereas the lowest value was at 1.65 per cent which

3. Stochastic Asset Liability Modelling: A Case of Malaysia

occurred in November 2004. The yields were high before 1999 but a drastic decrease can be seen after that.

On the top of that, we performed descriptive statistics to this series and the result is shown in table 3.12. The descriptive statistics for the 10-Year MGS yield (n=217) had an average of 3.948 per cent in between January 1996 to January 2014 with 1 per cent away from the average. The skewness is at 1.48 points showing that the distribution of this series is positively skewed. Meanwhile, with 1.94 points for the coefficient of kurtosis, it shows that the distribution of this series is more peaked than a Gaussian distribution.

Mean	0.03948
Standard deviation	0.01
Skewness	1.48
Kurtosis	1.94

Table 3.11.: Summary of the statistics for the 10-Year MGS yield model

As for the first step in the Box-Jenkins methodology, we needed to figure out the stationarity of this series. From the plot of ACF in figure 3.16, we can see that there was a short-term autocorrelation in the early lags. In addition, the series also showed some seasonality. We then ran the ADF test and found that the series was not stationary. We then differentiated the series once and re-ran the ADF test. After the first difference, the series became stationary and therefore we can proceed with the development of the ARIMA model for seasonal series.

Again, we are required to analyse the plot of ACF and PACF as well as the plot of the first differenced series, as shown in figure 3.17.

3. Stochastic Asset Liability Modelling: A Case of Malaysia

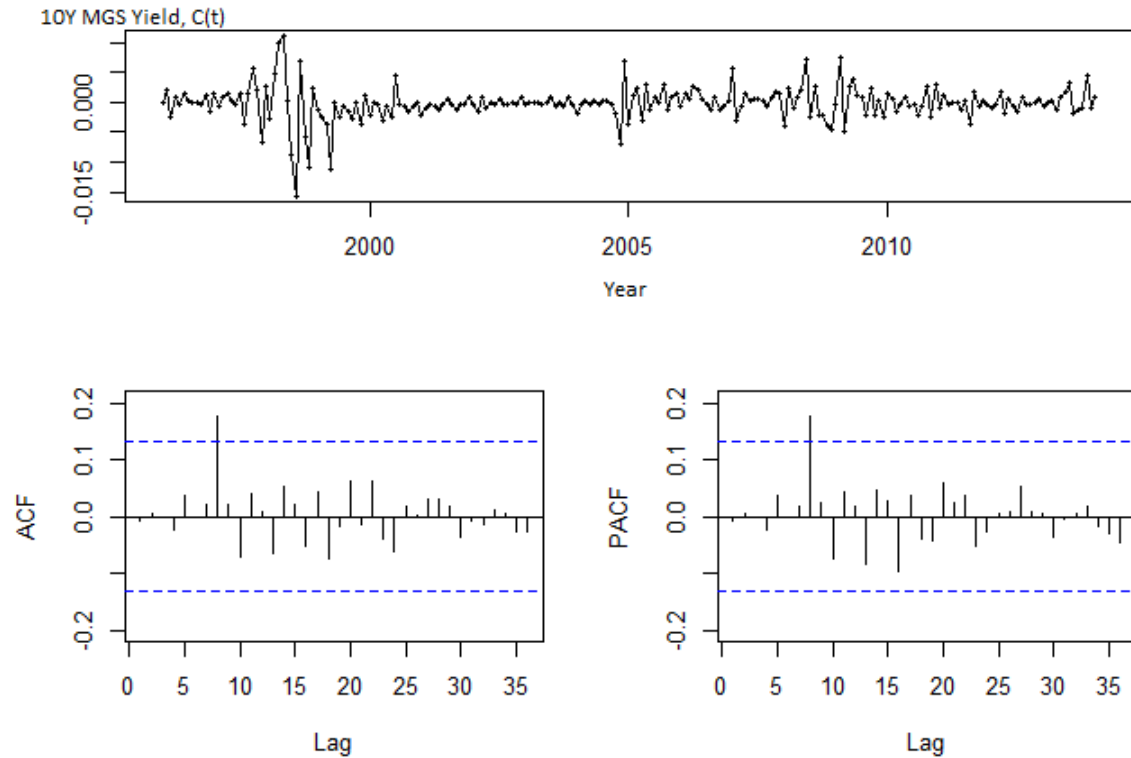


Figure 3.17.: Seasonal differenced 10-Year MGS yield

As can be seen from figure 3.17, both the ACF and PACF showed significant spikes at lag 8. However, the other autocorrelations were also found within the significant boundary. Again' by using "auto.arima" function in R, we obtained the following model as the best fitted model for this series:

$$\text{ARIMA}(4,1,3)(0,0,1)[12]$$

where,

$$\begin{aligned} AR1 &= -1.0217(0.2715), \\ AR2 &= 0.1413(0.3531), \\ AR3 &= 0.0158(0.1539), \\ AR4 &= -0.1924(0.0903), \\ MA1 &= 1.3871(0.2671), \\ MA2 &= 0.0884(0.4400), \\ MA3 &= -0.3803(0.2046), \\ SMA1 &= -0.2708(0.0764) \end{aligned}$$

3. Stochastic Asset Liability Modelling: A Case of Malaysia

with the standard errors in the bracket. Furthermore, we let C_1, C_2, \dots, C_t denote the 10-Year MGS yield. The fitted 10-Year MGS yield model can be written as the following:

$$\begin{aligned} & (1 - AR1 \cdot B - AR2 \cdot B^2 - AR3 \cdot B^3 - AR4 \cdot B^4)(1 - B)C_t \\ & = (1 + SMA1 \cdot B^{12})(1 + MA1 \cdot B + MA2 \cdot B^2 + MA3 \cdot B^3)\varepsilon_t. \end{aligned}$$

We expanded the previous equation to form

$$\begin{aligned} & (1 - AR1 \cdot B - AR2 \cdot B^2 - AR3 \cdot B^3 - AR4 \cdot B^4 - B + AR1 \cdot B^2 + AR2 \cdot B^3 + AR3 \cdot B^4 + AR4 \cdot B^5)C_t \\ & = (1 + MA1 \cdot B + MA2 \cdot B^2 + MA3 \cdot B^3 + SMA1 \cdot B^{12} + SMA1 \cdot MA1 \cdot B^{13} \\ & \quad + SMA1 \cdot MA2 \cdot B^{14} + SMA1 \cdot MA3 \cdot B^{15})\varepsilon_t. \end{aligned}$$

We substituted the back shift operator B into previous equation and obtained

$$\begin{aligned} C_t & = AR1 \cdot C_{t-1} + AR2C_{t-2} + AR3 \cot C_{t-3} - AR4 \cdot C_{t-4} + C_{t-1} - AR1 \cdot C_{t-2} \\ & \quad - AR2 \cdot C_{t-3} - AR3 \cdot C_{t-4} - AR4 \cdot C_{t-5} + \varepsilon_t + MA1 \cdot \varepsilon_{t-1} + MA2 \cdot \varepsilon_{t-2} \\ & \quad + MA3 \cdot \varepsilon_{t-3} + SMA1 \cdot \varepsilon_{t-12} + SMA1 \cdot MA1\varepsilon_{t-13} + SMA1 \cdot MA2\varepsilon_{t-14} \\ & \quad + SMA1 \cdot MA3\varepsilon_{t-15}. \end{aligned} \tag{3.22}$$

Then, we plotted the residuals as well as the p -values for Ljung-ox statistic which can be seen in figure 3.18.

3. Stochastic Asset Liability Modelling: A Case of Malaysia

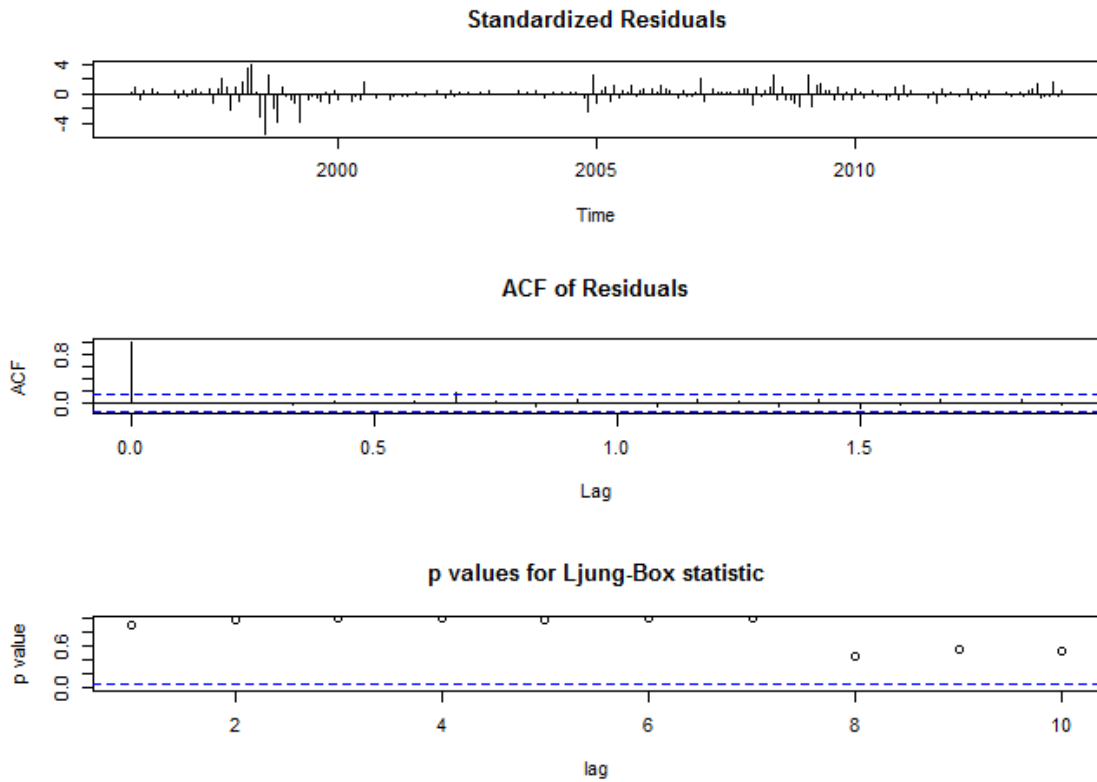


Figure 3.18.: Output from residuals analysis for 10-Year MGS yield

The plot of standardised residuals in figure 3.18 showed that the fitted model had no particular pattern while the plot of ACF of residuals showed that the residuals were significant. In addition, the plot of the p -values for Ljung-Box statistic showed positivity where the p -values are bigger than 0.05. Therefore, these results proved that the residuals are white noise and we can use the fitted model for forecasting.

We are now interested to forecast the 10-Year MGS yield using the selected $ARIMA(4,1,3)(0,0,1)[12]$ model. We would like to forecast 30 months into the future which will start on February 2014 and will end on July 2016. The results of the forecast can be seen numerically in table 3.12 and graphically in figure 3.19.

3. Stochastic Asset Liability Modelling: A Case of Malaysia

Month	Forecast	Lo 80	Hi 80	Lo 95	Hi 95
Feb 2014	0.04155514	0.03791706	0.04519322	0.03599118	0.04711910
Mar 2014	0.04163872	0.03548162	0.04779583	0.03222224	0.05105521
Apr 2014	0.04123098	0.03363713	0.04882484	0.02961719	0.05284478
May 2014	0.04177094	0.03326121	0.05028066	0.02875644	0.05478544
Jun 2014	0.04042932	0.03114796	0.04971068	0.02623470	0.05462393
Jul 2014	0.04008006	0.03016283	0.04999729	0.02491297	0.05524716
Aug 2014	0.03958280	0.02896903	0.05019658	0.02335044	0.05581517
Sep 2014	0.04099670	0.02979797	0.05219543	0.02386972	0.05812368
Oct 2014	0.04076965	0.02892776	0.05261154	0.02265904	0.05888026
Nov 2014	0.04017468	0.02780302	0.05254634	0.02125386	0.05909549
Dec 2014	0.03935128	0.02639859	0.05230397	0.01954185	0.05916071
Jan 2015	0.03980191	0.02636283	0.05324099	0.01924861	0.06035521
Feb 2015	0.03941703	0.02568229	0.05315178	0.01841155	0.06042252
Mar 2015	0.03984390	0.02592571	0.05376209	0.01855787	0.06112993
Apr 2015	0.03961104	0.02545184	0.05377025	0.01795641	0.06126568
May 2015	0.03981652	0.02542526	0.05420778	0.01780699	0.06182605
Jun 2015	0.03965445	0.02499598	0.05431292	0.01723625	0.06207264
Jul 2015	0.03976328	0.02486130	0.05466527	0.01697266	0.06255390
Aug 2015	0.03967722	0.02452348	0.05483095	0.01650158	0.06285286
Sep 2015	0.03973844	0.02435032	0.05512657	0.01620434	0.06327255
Oct 2015	0.03969662	0.02407004	0.05532320	0.01579783	0.06359541
Nov 2015	0.03972571	0.02387015	0.05558127	0.01547673	0.06397469
Dec 2015	0.03970760	0.02362174	0.05579346	0.01510640	0.06430880
Jan 2016	0.03971777	0.02340779	0.05602776	0.01477381	0.06466174
Feb 2016	0.03971333	0.02318015	0.05624651	0.01442801	0.06499865
Mar 2016	0.03971343	0.02296141	0.05646544	0.01409343	0.06533342
Apr 2016	0.03971634	0.02274750	0.05668518	0.01376475	0.06566794
May 2016	0.03971135	0.02252887	0.05689383	0.01343301	0.06598969
Jun 2016	0.03971772	0.02232407	0.05711137	0.01311643	0.06631901
Jul 2016	0.03971053	0.02210819	0.05731288	0.01279008	0.06663099

Table 3.12.: Forecast values of 10-Year MGS yield for Feb 2014-July 2016

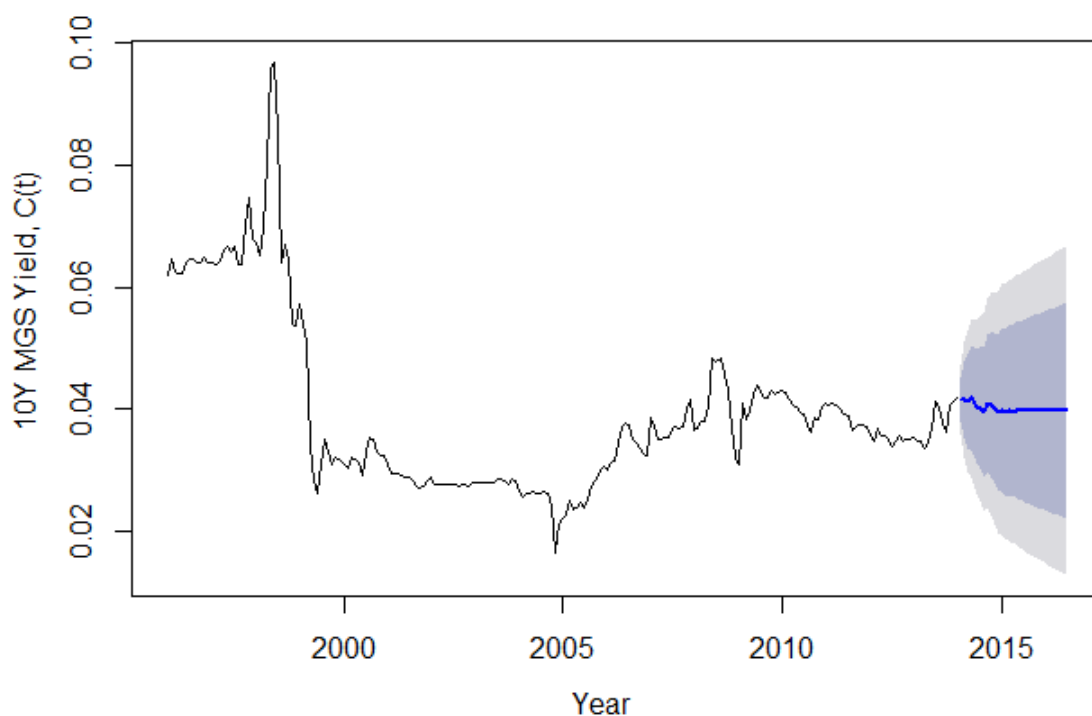


Figure 3.19.: ARIMA(4,1,3)(0,1,1)[12] model for 10-Year MGS yield

The 10-Year MGS yield is forecasted to be stable at 4.16 per cent in February 2014 to May 2014. Then, it fluctuates for the rest of forecasting months and is forecasted to be 3.97 per cent in July 2016.

3.4.6. Summary of the Malaysian Stochastic Asset Liability Model

This section summarises the Malaysian stochastic asset liability model that we built based on Box-Jenkins methodology and applied the concept of the Wilkie model. The model consists of four sub-models; the inflation model, the FTSE Bursa Malaysia KLCI yield model, the FTSE Bursa Malaysia KLCI model, the 10-Year MGS yield model. The inflation rate in Malaysia followed ARIMA(0,1,2) model based on data for the years 1960-2013. The rate was forecasted to be 1.9 per cent in 2013 and stabled at 2.3 per cent in 2014 to 2042. Comparing these results to actual inflation rate in Malaysia, according to Worldbank Data, the actual inflation rate in Malaysia for 2014 was 2.1 per cent as in the total averaged. These percentages were approximated to each other and we can conclude that the ARIMA (0,1,2) model is appropriate to forecast the inflation rate in Malaysia. For the FTSE Bursa Malaysia KLCI yield, it followed ARIMA(1,1,0) model based on data for the period from July 2009 to September 2013. The yield was forecasted to be 3.08 per cent in December 2013 whereas based on the Bursa Malaysia

3. *Stochastic Asset Liability Modelling: A Case of Malaysia*

Annual Report 2013, the actual yield was 6.3 per cent. There was a drastic increment in the yield happened in December 2013, which was increased from 4.3 per cent in December 2012 to 6.3 per cent in December 2013. Therefore, we thought that might be the reason of the insufficient forecast of our model. On the other hand, the FTSE Bursa Malaysia KLCI (the index) followed ARIMA (2,1,1)(2,0,2)[12] model since the data was not stationary throughout the period from January 1994 to December 2013. To decide the most suitable Box-Jenkins model that represent the data, we use the function "auto.arima" in the R software. The index was forecasted to be 1877.35 points in January 2014 and increased to 1892.54 points in February 2014. The index fluctuated from 1894 points as the minimum points to 2011 points as the maximum points in March 2014 to June 2016. For the 10-Year MGS yield, it followed ARIMA (4,1,3)(0,1,1)[12] model since the data was not stationary as the previous model. The yield was forecasted to be 4.16 per cent in February 2014 and also fluctuated for the rest of the months. The yield was forecasted to be 3.97 per cent in July 2016.

4. The Continuous-Time Model: A Description

The continuous-time model inspired by the Wilkie model was introduced by Terence Chan in 1998 [Chan, 1998a]. This version of the Wilkie model was constructed using the stochastic differential equation (SDE) which was driven by Brownian motion. This model was constructed because the investment variables are mostly handled in continuous time. The study suggested possible ways to transform the discrete time stochastic model to a continuous time setting although they may not be directly considered as the "right" continuous time models. In this study, we will use the model developed by Terence Chan as a reference but we will also show in detail the transformation of each variable in the Wilkie model. In the last section, we will explain the relationship between the share price, the share dividend index and the share dividend yield. We mentioned in chapter two that the retail prices index model, the share dividend yield model and the share dividend index model were AR(1) models. Therefore, we will also attempt to describe a continuous-time AR(1) model which is known as an Ornstein-Uhlenbeck process before we continue with all four models in the Wilkie model.

4.1. The Ornstein-Uhlenbeck Process

The Ornstein-Uhlenbeck process represents the continuous time AR(1) process. This process is considered to be stationary, Gaussian, Markovian, has a mean-reverting criteria and has a bounded variance where it is

- stationary, for all $t_1 < t_2 < \dots < t_n$ and $h > 0$, the random n-vectors $(X_{t_1}, X_{t_2}, \dots, X_{t_n})$ and $(X_{t_1+h}, X_{t_2+h}, \dots, X_{t_n+h})$ are identically distributed; that is, time shifts leave joint probabilities unchanged
- Gaussian, for all $t_1 < t_2 < \dots < t_n$, the n-vector $(X_{t_1}, X_{t_2}, \dots, X_{t_n})$ is multivariate normally distributed
- Markovian, for all $t_1 < t_2 < \dots < t_n$, $P(X_{t_n} \leq x | X_{t_1}, X_{t_2}, \dots, X_{t_{n-1}}) = P(X_{t_n} \leq x | X_{t_{n-1}})$; that is, the future is determined only by the present and not the past
- the mean-reverting criteria means that the random walk always has the tendency to move back towards a central location, which is the mean.

Definition 4.1(The Ornstein-Uhlenbeck process)

4. The Continuous-Time Model: A Description

The unique solution to the SDE

$$dX_t = \theta(\mu - X_t)dt + \sigma dW_t, \quad X_0 = x \quad (4.1)$$

is called an Ornstein-Uhlenbeck process with $\theta, \mu, \sigma > 0$ and W_t denoting the Wiener process.

With the help of variation of constants (theorem 5.2), we obtained the following proposition:

Proposition 4.1

The unique solution X_t to the SDE (4.1) is given by

$$X_t = xe^{-\theta t} + \mu(1 - e^{-\theta t}) + \int_0^t \sigma e^{\theta(s-t)} dW_s. \quad (4.2)$$

Moreover, the Ornstein-Uhlenbeck process is normally distributed according to

$$X_t \sim N\left(xe^{-\theta t} + \mu(1 - e^{-\theta t}), \int_0^t \sigma^2 e^{2\theta(s-t)} ds\right).$$

If the initial value x has a normal distribution with zero-mean and variance equal to $\frac{\sigma^2}{2\theta}$, then we say that the process X_t is stationary and is a Gaussian process with zero-mean and covariance function $\rho(s, t) = \frac{\sigma^2}{2\theta} e^{-\theta|t-s|}$.

The Ornstein-Uhlenbeck process is used to model a lot of financial processes such as interest rates, currency exchange and others, stochastically. The parameter μ denotes the mean of the process while the parameter σ denotes the volatility occurrence caused by any random shocks. In addition, the parameter θ is called the mean-reversion speed. One example of the application of the Ornstein-Uhlenbeck process is the Vasicek model for modelling short-term interest rates. The Vasicek model incorporates the mean reversion criteria which help to prevent the interest rates from deviating far from a long-term norm. It is a one factor model which describes interest rate movements driven by only one source of market risk.

4.2. The Continuous-Time Retail Prices Index Model

The discrete version of this model was presented in Section 2.2.1. There, $\nabla \ln Q_t$ represents the force of inflation at time t . In the continuous time, we let $\nabla \ln Q_t \equiv R_t$ and then we constructed the model via the Ornstein-Uhlenbeck type of process. Therefore, R_t is given by the following SDE:

$$dR_t = a_1 \left(\frac{\phi}{a_1} - R_t \right) dt + \sigma_1 dZ_1(t), \quad R_0 = r \quad (4.3)$$

4. The Continuous-Time Model: A Description

where $a_1 > 0, \sigma_1 \in \mathbb{R}, \phi$ are some constants and Z_1 is the Brownian motion. Next, by comparing the discrete time retail prices index model, equation (2.1) with (4.3) we have the following correspondences:

$$\begin{aligned} a_1 &\equiv 1 - QA, \\ \phi &\equiv QMU \cdot (1 - QA), \\ \sigma_1 &\equiv QSD, \\ \Delta Z_1 &\equiv QZ \end{aligned}$$

with the increment in Z_1 , $\Delta Z_1 = Z_1(t+1) - Z_1(t)$.

Then we applied theorem 5.1 to solve equation (4.3) and the solution is

$$R_t = re^{-a_1 t} + \frac{\phi}{a_1}(1 - e^{-a_1 t}) + \int_0^t \sigma_1 e^{-a_1(t-s)} dZ_1(s). \quad (4.4)$$

As equation (4.2), the solution of the continuous-time retail prices index model also has a normal distribution with mean

$$E[R_t] = re^{-a_1 t} + \frac{\phi}{a_1}(1 - e^{-a_1 t}),$$

and variance

$$Var[R_t] = \int_0^t \sigma_1^2 e^{-2a_1(t-s)} ds.$$

4.3. The Continuous-Time Share Dividend Yield Model

In the continuous time setting, we used the same notation as in discrete time, where we let Y_t denote the share dividend yield and the continuous time share dividend yield is modelled as

$$Y_t = Y_* \cdot e^{(\zeta R_t + K_t)} \quad (4.5)$$

with Y_* as the "modified" initial value for Y_t , $Y_* = Y_0 \cdot e^{-(\zeta R_0 + K_0)}$. The variable K_t is the continuous version of the variable YN_t in the discrete-time share dividend model. Therefore, K_t is assumed to satisfy the following SDE:

$$dK_t = -a_2 K_t dt + b_1 dt + \sigma_2 dZ_2(t), \quad K_0 = k \quad (4.6)$$

where $a_2 > 0, \sigma_2 \in \mathbb{R}, b_1$ are some constants and Z_2 is a Brownian motion which is independent of Z_1 . Then, we compared equation (2.4) with (4.5) and (4.6), and we have the following correspondences:

$$\begin{aligned} a_2 &\equiv 1 - YA, \\ b_1 &\equiv \ln YMU \cdot (1 - YA), \\ \sigma_2 &\equiv YSD, \\ \Delta Z_2 &\equiv YZ. \end{aligned}$$

4. The Continuous-Time Model: A Description

The next step is the execution of theorem 5.1 to solve (4.6) and the solution is

$$K_t = ke^{-a_2t} + \frac{b_1}{a_2}(1 - e^{-a_2t}) + \int_0^t \sigma_2 e^{-a_2(t-s)} dZ_2(s).$$

Then we are ready to write the full form of Y_t where

$$\begin{aligned} Y_t = Y_* \cdot \exp\left(\zeta r e^{-a_1t} + \frac{\zeta \phi}{a_1}(1 - e^{-a_1t}) + \zeta \int_0^t \sigma_1 e^{-a_1(t-s)} dZ_1(s) \right. \\ \left. + ke^{-a_2t} + \frac{b_1}{a_2}(1 - e^{-a_2t}) + \int_0^t \sigma_2 e^{-a_2(t-s)} dZ_2(s)\right). \end{aligned} \quad (4.7)$$

Since Y_t is log-normally distributed, $\ln Y_t$ has a normal distribution with mean and variance as the following, respectively:

$$E[\ln Y_t] = \ln Y_* + ke^{-a_2t} + \frac{b_1}{a_2}(1 - e^{-a_2t}) + \zeta \cdot E[R_t],$$

and

$$Var[\ln Y_t] = \int_0^t \sigma_2^2 e^{-2a_2(t-s)} ds + \zeta^2 \cdot Var[R_t]$$

where $E[R_t]$ and $Var[R_t]$ were explicitly given in section 4.2.

4.4. The Continuous-Time Share Dividend Index Model

This model follows the Wilkie model in using an exponentially discounted "sum of inflation effects". Let $\ln D_t$ denote the share dividend index at time t . The variable $\ln D_t$ is assumed to satisfy the following SDE:

$$d(\ln D_t) = \left(b_2 + \beta \lambda \int_0^t e^{-\lambda s} R_{t-s} ds + \gamma R_t\right) dt + \eta_2 dZ_2(t) + \eta_3 dZ_3(t) \quad (4.8)$$

where $\beta, \lambda, \gamma > 0, \eta_2, \eta_3 \in \mathbb{R}, b_2$ are some constants, Z_2 and Z_3 are Brownian motions with Z_3 independent of Z_1 and Z_2 . Then, we compared the discrete time share dividend index model, (2.6) with (4.8), and we achieved the following correspondences:

$$\begin{aligned} b_2 &\equiv DSD, \\ \beta &\equiv DW, \\ \lambda &\equiv 1 - DD, \\ \gamma &\equiv DX, \\ \eta_2 &\equiv DY.YSD, \\ \eta_3 &\equiv (DB.DSD, DSD)', \\ \Delta Z_3 &\equiv DZ. \end{aligned}$$

4. The Continuous-Time Model: A Description

By interchanging the order of integration, we reached at

$$\beta\lambda \int_0^t e^{-\lambda s} R_{t-s} ds = \int_0^t \beta(1 - e^{-\lambda(t-s)}) R_s ds.$$

Thus, we have

$$d(\ln D_t) = \left(b_2 + \int_0^t \beta(1 - e^{-\lambda(t-s)}) R_s ds + \gamma R_t \right) dt + \eta_2 dZ_2(t) + \eta_3 dZ_3(t). \quad (4.9)$$

Then, we solved (4.9) and got the solution as

$$\ln D_t = \ln D_* + \eta_2 Z_2(t) + \eta_3 Z_3(t) + \beta\lambda \int_0^t e^{-\lambda(t-s)} R_s ds + \gamma \int_0^t R_s ds + b_2 t \quad (4.10)$$

where D_* has the same pattern as we had in Y_* . By simplification and multiplication of (4.10) with exponent, we obtained

$$D_t = D_* \cdot \exp\left(\eta_2 Z_2(t) + \eta_3 Z_3(t) + \int_0^t (\beta + \gamma - \beta e^{-\lambda(t-s)}) R_s ds + b_2 t\right). \quad (4.11)$$

Hence $\ln D_t$ has a normal distribution with mean and variance of

$$E[\ln D_t] = \ln D_* + b_2 t + \int_0^t (\beta + \gamma - \beta e^{-\lambda(t-s)}) \cdot E[R_s] ds,$$

$$\text{Var}[\ln D_t] = (\eta_2^2 + \eta_3^2)t + 2 \int_0^t (\beta + \gamma - \beta e^{-\lambda(t-s)}) \cdot \text{Var}[R_s] ds$$

respectively.

4.5. The Continuous-Time Consols Yield Model

[Chan, 1998a] modelled the yield on Consols as

$$C_t = \xi\rho \int_0^t e^{-\rho s} R_{t-s} ds + C_* e^{M_t}, \quad (4.12)$$

By letting M_t be the same as e^{CN_t} in the discrete-time Consols yield model, the SDE of M_t is in the form of

$$dM_t = -a_4 M_t dt + \sigma_4 dZ_4(t), \quad M_0 = m. \quad (4.13)$$

Here we have $\xi, \rho, a_4 > 0, \sigma_4 \in \mathbb{R}$ as some constants and Z_4 as a Brownian motion which is independent of $Z_i, i = 1, 2, 3$ while C_* has the same pattern as Y_* and D_* . By comparing

4. The Continuous-Time Model: A Description

equation (2.8) with (4.12) and (4.13), we obtained the following correspondences:

$$\begin{aligned} a_4 &\equiv 1 - CA, \\ \xi &\equiv CW, \\ \rho &\equiv 1 - CD, \\ C_* &\equiv CMU, \\ \sigma_4 &\equiv (CY \cdot YSD, CSD)', \\ \Delta Z_4 &\equiv CZ. \end{aligned}$$

Via theorem 5.1, we were able to solve equation (4.12) and (4.13). The solution is as follows:

$$M_t = me^{-a_4 t} + \int_0^t \sigma_4 e^{-a_4(t-s)} dZ_4(s).$$

We then reached the explicit form of C_t which is

$$C_t = \xi \rho \int_0^t e^{-\rho s} R_{t-s} ds + C_* e^{\{me^{-a_4 t} + \int_0^t \sigma_4 e^{-a_4(t-s)} dZ_4(s)\}}. \quad (4.14)$$

C_t has a combination of normal and log-normal distribution with the mean

$$E[C_t] = \xi \rho \int_0^t \{e^{-\rho s} \cdot E[R_{t-s}]\} ds + C_* e^{E[M_t]},$$

and variance

$$Var[C_t] = 2\xi^2 \rho^2 \int_0^t e^{-2\rho s} \cdot Var[R_{t-s}] ds + C_*^2 e^{2 \cdot Var[M_t]},$$

where M_t has a normal distribution with the mean

$$E[M_t] = me^{-a_1 t},$$

and variance,

$$Var[M_t] = \int_0^t \sigma_4^2 e^{-2a_4(t-s)} ds.$$

Remark 1 *The relation between the processes from Section 4.2-4.5 is illustrated via their quadratic covariations;*

$$\begin{aligned} \langle R, \ln Y \rangle_t &= \zeta \int_0^t \sigma_1^2 e^{-2a_1(t-s)} ds = \frac{\zeta \sigma_1^2}{2a_1} (1 - e^{-2a_1 t}), \\ \langle \ln \left(\frac{Y}{Y_*} \right), \ln D \rangle_t &= \eta_2 \int_0^t \sigma_2 e^{-a_2(t-s)} ds = \frac{\eta_2 \sigma_2}{a_2} (1 - e^{-a_2 t}), \\ \langle R, \ln D \rangle_t &= 0, \\ \langle R, C \rangle_t &= 0, \\ \langle C, Y \rangle_t &= 0, \\ \langle C, D \rangle_t &= 0. \end{aligned}$$

4. The Continuous-Time Model: A Description

The presence of R in Y, D and C resembles the cascade structure of the discrete-time Wilkie model. However, we could also revert it by choosing another parametrisation of the underlying Brownian motion. Note that not all the dependency induced by this cascade structure is mirrored in the quadratic covariation.

4.6. The Share Price in the Wilkie Model

Our aim in this section is to explain the relationship between the share price and the other two variables in the Wilkie model; the share dividend index and the share dividend yield. As mentioned earlier, the share price S_t is related to the share dividend index and yield by this formula

$$S_t = \frac{D_t}{Y_t}. \quad (4.15)$$

In this particular Wilkie model application, we used D_t and Y_t from the continuous-time Wilkie model. We are going to show that S_t satisfies the SDE of the form

$$dS_t = S_t \left(a_t dt + \sum_{j=1}^3 \delta_j dZ_j(t) \right), \quad S_0 = s. \quad (4.16)$$

From (4.7) and (4.11) and based on (4.15), we obtained the following equation:

$$\begin{aligned} S_t = s \cdot \exp & \left(-\zeta e^{-a_1 t} \int_0^t \sigma_1 e^{a_1 s} dZ_1(s) + \eta_2 Z_2(t) - e^{-a_2 t} \int_0^t \sigma_2 e^{a_2 s} dZ_2(s) + \eta_3 Z_3(t) \right. \\ & + \beta \int_0^t (1 - e^{-\lambda(t-s)}) R_s ds + \gamma \int_0^t R_s ds + b_2 t - \zeta r e^{-a_1 t} + \frac{\zeta \phi}{a_1} (e^{-a_1 t} - 1) \\ & \left. - k e^{-a_2 t} + \frac{b_1}{a_2} (e^{-a_2 t} - 1) \right). \end{aligned} \quad (4.17)$$

with $s = \frac{D^*}{Y^*}$. Now we let

$$\begin{aligned} \kappa_t = & -\zeta e^{-a_1 t} \int_0^t \sigma_1 e^{a_1 s} dZ_1(s) + \eta_2 Z_2(t) - e^{-a_2 t} \int_0^t \sigma_2 e^{a_2 s} dZ_2(s) + \eta_3 Z_3(t) \\ & + \beta \int_0^t (1 - e^{-\lambda(t-s)}) R_s ds + \gamma \int_0^t R_s ds + b_2 t - \zeta r e^{-a_1 t} + \frac{\zeta \phi}{a_1} (e^{-a_1 t} - 1) \\ & - k e^{-a_2 t} + \frac{b_1}{a_2} (e^{-a_2 t} - 1) \end{aligned} \quad (4.18)$$

with $S_t = \exp(\kappa_t)$ we obtained

$$dS_t = d(\exp(\kappa_t)) = \exp(\kappa_t) d\kappa_t + \frac{1}{2} \exp(\kappa_t) d \langle \kappa \rangle_t, \quad (4.19)$$

i.e.

$$dS_t = S_t d\kappa_t + \frac{1}{2} S_t d \langle \kappa \rangle_t. \quad (4.20)$$

4. The Continuous-Time Model: A Description

Now we differentiate all terms in (4.18) and have

$$\begin{aligned}
d\kappa_t &= a_1\zeta e^{-a_1t} \int_0^t \sigma_1 e^{a_1s} dZ_1(s)dt - \zeta\sigma_1 dZ_1(t) + \eta_2 dZ_2(t) \\
&+ a_2 e^{-a_2t} \int_0^t \sigma_2 e^{a_2s} dZ_2(s)dt - \sigma_2 dZ_2(t) + \eta_3 dZ_3(t) + \beta\lambda \int_0^t e^{-\lambda(t-s)} R_s ds dt \\
&+ \gamma R_t dt + b_2 dt + \zeta a_1 r e^{-a_1t} dt - \zeta \phi e^{-a_1t} dt + k a_2 e^{-a_2t} dt - b_1 e^{a_2t} dt. \tag{4.21}
\end{aligned}$$

By using $d\langle \kappa \rangle_t = d\kappa_t \cdot d\kappa_t$, we achieved

$$d\langle \kappa \rangle_t = [\zeta^2 \sigma_1^2 + (\eta_2 - \sigma_2)^2 + \eta_3^2] dt. \tag{4.22}$$

We then substituted (4.21) and (4.22) into (4.20) and acquired

$$\begin{aligned}
dS_t &= S_t \left[a_1\zeta e^{-a_1t} \int_0^t \sigma_1 e^{a_1s} dZ_1(s)dt - \zeta\sigma_1 dZ_1(t) + \eta_2 dZ_2(t) \right. \\
&+ a_2 e^{-a_2t} \int_0^t \sigma_2 e^{a_2s} dZ_2(s)dt - \sigma_2 dZ_2(t) + \eta_3 dZ_3(t) + \beta\lambda \int_0^t e^{-\lambda(t-s)} R_s ds dt \\
&+ \gamma R_t dt + b_2 dt + \zeta a_1 r e^{-a_1t} dt - \zeta \phi e^{-a_1t} dt + k a_2 e^{-a_2t} dt - b_1 e^{a_2t} dt \\
&\left. + \frac{1}{2} \left(\zeta^2 \sigma_1^2 + (\eta_2 - \sigma_2)^2 + \eta_3^2 \right) dt \right]. \tag{4.23}
\end{aligned}$$

We rearranged (4.23) as

$$\begin{aligned}
dS_t &= S_t \left[\left(a_1\zeta e^{-a_1t} \int_0^t \sigma_1 e^{a_1s} dZ_1(s) + a_2 e^{-a_2t} \int_0^t \sigma_2 e^{a_2s} dZ_2(s) + \beta\lambda \int_0^t e^{-\lambda(t-s)} R_s ds \right. \right. \\
&+ \gamma R_t + b_2 + \zeta a_1 r e^{-a_1t} - \zeta \phi e^{-a_1t} + k a_2 e^{-a_2t} - b_1 e^{a_2t} + \frac{1}{2} \zeta^2 \sigma_1^2 + \frac{1}{2} (\eta_2 - \sigma_2)^2 + \frac{1}{2} \eta_3^2 \left. \right) dt \\
&\left. + (-\zeta\sigma_1 dZ_1(t)) + (\eta_2 - \sigma_2) dZ_2(t) + \eta_3 dZ_3(t) \right] \tag{4.24}
\end{aligned}$$

and this led us to a simpler form of a_t as

$$\begin{aligned}
a_t &= a_1\zeta e^{-a_1t} \int_0^t \sigma_1 e^{a_1s} dZ_1(s) + a_2 e^{-a_2t} \int_0^t \sigma_2 e^{a_2s} dZ_2(s) + \beta\lambda \int_0^t e^{-\lambda(t-s)} R_s ds + \gamma R_t \\
&+ b_2 + \zeta a_1 r e^{-a_1t} - \zeta \phi e^{-a_1t} + k a_2 e^{-a_2t} - b_1 e^{a_2t} + \frac{1}{2} \zeta^2 \sigma_1^2 + \frac{1}{2} (\eta_2 - \sigma_2)^2 + \frac{1}{2} \eta_3^2. \tag{4.25}
\end{aligned}$$

With the appearance of the parameter δ in (4.16), the diffusion part in (4.24) can be written explicitly as

$$\delta dZ(t) = -\zeta\sigma_1 dZ_1(t) + (\eta_2 - \sigma_2) dZ_2(t) + \eta_3 dZ_3(t) \tag{4.26}$$

4. The Continuous-Time Model: A Description

where we have a new Brownian motion Z which is given by

$$Z(t) = \frac{1}{\sqrt{\zeta^2\sigma_1^2 + (\eta_2 - \sigma_2)^2 + \eta_3^2}}(-\zeta\sigma_1 Z_1(t) + (\eta_2 - \sigma_2)Z_2(t) + \eta_3 Z_3(t)) \quad (4.27)$$

with

$$\delta = \sqrt{\zeta^2\sigma_1^2 + (\eta_2 - \sigma_2)^2 + \eta_3^2}. \quad (4.28)$$

To conclude, we rewrote equation (4.16) to obtain the dynamics of the share price in a simpler form as

$$dS_t = S_t(a_t dt + \delta dZ(t)) \quad (4.29)$$

with a_t as in the form of (4.25), Z_t as in the form of (4.27) and δ as in the form of (4.28).

5. Portfolio Optimisation in the Continuous-Time Wilkie Model

5.1. Introduction

This chapter introduces the classic mathematical technique for obtaining an optimal portfolio, the stochastic control approach. There are five main points which will be discussed in this chapter. We begin this chapter by giving an introduction to a linear controlled stochastic differential equation by stating down one important definition admissible control and then we move on to discuss a theorem of variation of constants. The next topic that we discuss in this chapter is the formulation of the maximisation problem. Then, we show the derivation of the Hamilton-Jacobi-equation known as the HJB-equation, heuristically. In this discussion, we apply the famous Itô formula to a value function of the HJB-equation in order to get the solution. After getting the solution to the HJB-equation, we used a verification theorem to ensure that the solution is true. The most interesting part in this discussion is that we state a corollary which named "to the verification theorem's corollary", that can be used before we apply the verification theorem and make this process easier. Then, we list out all algorithms required in solving the HJB-equation and automatically solve the maximisation problem. Lastly, we present our own portfolio problem based on the continuous-time Wilkie model and then use the stochastic control approach to solve the portfolio.

5.2. Controlled Stochastic Differential Equations

The main ideas presented in this chapter are mainly based on the book "Option Pricing and Portfolio Optimization: Modern Methods of Financial Mathematics" by Ralf Korn and Elke Korn [Korn and Korn, 2001]. The aim of this method is to find the optimal controlled stochastic process with respect to a certain cost function. In a portfolio optimisation problem, the stochastic control method is used to find the optimal strategy that will maximise the portfolio. This approach was introduced by Robert Merton in the 1960s [Merton, 1969][Merton, 1971].

Let (Ω, \mathcal{F}, P) be a complete probability space equipped with an m -dimensional Brownian motion W_t and a Brownian filtration of $\{\mathcal{F}\}_{t \in [0, \infty)}$. Then we let X_t be an n -dimensional Ito process with its stochastic differential equation (SDE)

$$dX_t = \mu(t, X_t, u_t)dt + \sigma(t, X_t, u_t)dW_t \quad (5.1)$$

where its initial values $X_{t_0} = x$, and u_t are a d -dimensional control process that we can choose. Further, we chose the time interval $[t_0, t_1]$ with $0 < t_0 < t_1 < \infty$. We let $u_t \in U$

5. Portfolio Optimisation in the Continuous-Time Wilkie Model

be a progressively measurable process for all $t \in [t_0, t_1]$. Then, as $U \subset \mathbb{R}^d, d \in \mathbb{N}$ is a closed set, we let $Q_0 := [t_0, t_1] \times \mathbb{R}^n, n \in \mathbb{N}, m \in \mathbb{N}$ and all variables are considered to be continuous. Thus, the corresponding coefficient functions in equation (5.1) would have these forms

$$\begin{aligned}\mu &: \bar{Q}_0 \times U \rightarrow \mathbb{R}^n, \\ \sigma &: \bar{Q}_0 \times U \rightarrow \mathbb{R}^{n,m}.\end{aligned}$$

Definition 5.1 (Admissible control)

A U -valued progressively measurable process $u_t, t \in [t_0, t_1]$ will be called an admissible control, if

- (i) for all values $x \in \mathbb{R}^n$ the stochastic differential equation (5.1) with initial condition $X_{t_0} = x$ is a unique solution $\{X_t^u\}_{t \in [t_0, t_1]}$,
- (ii) and if we have

$$E\left(\int_{t_0}^{t_1} |u_s|^k ds\right) < \infty$$

the integrability condition for all $k \in \mathbb{N}$ is satisfied

- (iii) and the following state process X^u satisfies

$$E^{t_0, x}\left(\sup_{t \in [t_0, t_1]} |X_t^u|^k\right) < \infty.$$

To find the unique solution claimed in property (i) in definition 5.1, we need the following theorem, which is called the variation of constants. According to remark 2, the following theorem that focuses on a linear SDE is sufficient for the biggest part of our application.

Theorem 5.1 (Variation of constants)

Let $\{(W_t, \mathcal{F}_t)\}_{t \in [0, \infty)}$ be an m -dimensional Brownian motion. Let $x \in \mathbb{R}$ and A, a, S_j, σ_j be progressively measurable, real-valued processes with

$$\begin{aligned}\int_0^t (|A_s| + |a_s|) ds &< \infty \text{ for all } t \geq 0 \text{ a.s. } P, \\ \int_0^t (|S_j^2(s)| + |\sigma_j^2(s)|) ds &< \infty \text{ for all } t \geq 0 \text{ a.s. } P,\end{aligned}$$

Then the SDE

$$\begin{aligned}dX_t &= (A_t \cdot X_t + a_t)dt + \sum_{j=1}^m (S_j(t)X(t) + \sigma_j(t))dW_j(t) \\ X_0 &= x\end{aligned}\tag{5.2}$$

possesses the unique solution $\{(X_t, \mathcal{F}_t)\}_{t \in [0, \infty)}$ with respect to $\lambda \otimes P$ given by

$$X_t = Z_t \cdot \left(x \cdot + \int_0^t \frac{1}{Z(u)} \left(a(u) - \sum_{j=1}^m S_j(u)\sigma_j(u) \right) du + \sum_{j=1}^m \int_0^t \frac{\sigma_j(u)}{Z(u)} dW_j(u) \right)$$

5. Portfolio Optimisation in the Continuous-Time Wilkie Model

where

$$Z_t = \exp\left(\int_0^t \left(A(u) - \frac{1}{2} \|S(u)\|^2\right) du + \int_0^t S(u) dW(u)\right)$$

is the unique solution of the homogeneous equation

$$\begin{aligned} dZ_t &= Z_t(A_t dt + S'_t dW_t), \\ Z_0 &= 1. \end{aligned}$$

The process $\{(X_t, \mathcal{F}_t)\}_{t \in [0, \infty)}$ solves the SDE (5.2) in the sense that X_t satisfies

$$X_t = x + \int_0^t (A_s \cdot X_s + a_s) ds + \sum_{j=1}^m \int_0^t (S_j(s)X(s) + \sigma_j(s)) dW_j(s)$$

for all $t \geq 0$ P -almost surely. For the proof of this theorem, it is given comprehensively in [Korn and Korn, 2001].

5.3. Formulation of the Optimisation Problem

We are now considering X_t as only factor to be controlled in an open set $O \subseteq \mathbb{R}^n$. Here we focus on $O = \mathbb{R}^n$ or an open set which its boundary δO forms a compact, $(n - 1)$ -dimensional C^3 -manifold. We let

$$\begin{aligned} Q &:= [t_0, t_1] \times O \\ \bar{Q} &:= [t_0, t_1] \times \bar{O} \\ \tau &:= \inf\{t \geq t_0 \mid (t, X_t) \notin Q\} \end{aligned}$$

Next, we come across the cost functional that we want to minimise as mentioned earlier, as follows:

$$J(t_0, x; u) = E^{t_0, x} \left(\int_{t_0}^{\tau} L(s, X_s^u, u_s) ds + \Psi(\tau, X_\tau^u) \right)$$

where L is a continuous function which satisfies

$$|L(t, x, u)| \leq C(1 + |x|^k + |u|^k)$$

and Ψ satisfies the polynomial growth condition

$$|\Psi(t, x)| = C(1 + |x|^k)$$

with $k \in \mathbb{N}$. Hence, our aim is to solve the problem

$$\max_{u \in \mathcal{A}(t_0, x)} J(t_0, x; u). \tag{5.3}$$

Here the $\mathcal{A}(t_0, x)$ is the set of all admissible controls $u(\cdot)$ with $(t_0, x) \in Q$. The value function of the maximisation problem is

$$V(t, x) = \sup_{u \in \mathcal{A}(t, x)} J(t, x; u), \quad (t, x) \in Q.$$

5.4. The Hamilton-Jacobi-Bellman-Equation

The Hamilton-Jacobi-Bellman-equation, called the HJB-equation, is a partial differential equation. Under certain conditions the solution to the HJB-equation is the value function which is the optimal solution to any dynamical system, either a maximisation or minimisation problem. Therefore, the HJB-equation is used to solve the stochastic control problem (5.3). The HJB-equation can be derived using the following Bellman principle:

$$V(t, x) = \sup_{u \in \mathcal{A}(t, x)} \left(E^{t, x} \left(\int_t^\theta L(s, X_s, u_s) ds + V(\theta, X_\theta) \right) \right)$$

with $V(t_1, x) = \Psi(t_1, x)$ for all $x \in \mathbb{R}^n$, $t \in [t_0, t_1]$ and $\theta \in [t, t_1]$. Note that we did not attempt to prove the Bellman principle. We only used it as a heuristic motivation. We are able to get the minimum cost $V(t, x)$ by selecting the control $u(\cdot)$ on $[t, \theta]$ that can be found optimal on $[\theta, t_1]$. This principle brings us to the cost of $V(\theta, X_\theta)$.

By applying the multi-dimensional Itô formula in the appendix G to the value function $V(\theta, X(\theta))$, on the right-hand side, we now have heuristically derived the formula of the HJB-equation

$$V(t, x) = \sup_{u \in \mathcal{A}(t, x)} E^{t, x} \left(\int_t^\theta L(s, X(s), u(s)) ds + V(t, x) + \int_t^\theta \left[V_t(s, X(s)) + V_x(s, X(s)) \mu(s, X(s), u(s)) + \frac{1}{2} \cdot \text{tr}(\sigma(s, X(s), u(s)) \sigma(s, X(s), u(s))' V_{xx}(s, X(s))) \right] ds \right).$$

We now use $\sigma^* := \sigma \sigma'$, subtracted $V(t, x)$ on the left and right side, then divided them by $(\theta - t)$, considered the limit $\theta \downarrow t$, and obtained

$$\begin{aligned} 0 &= \sup_{u \in \mathcal{A}(t, x)} E^{t, x} \left(\lim_{\theta \downarrow t} \frac{1}{\theta - t} \int_t^\theta \left[L(s, X(s), u(s)) + V_t(s, X(s)) \right. \right. \\ &\quad \left. \left. + \frac{1}{2} \cdot \text{tr}(\sigma^*(s, X(s), u(s)) \cdot V_{xx}(s, X(s))) + V_x(s, X(s)) \cdot \mu(s, X(s), u(s)) \right] ds \right) \\ &= \sup_{u \in \mathcal{A}(t, x)} E^{t, x} \left(L(t, X(t), u(t)) + V_t(t, X(t)) \right. \\ &\quad \left. + \frac{1}{2} \cdot \text{tr}(\sigma^*(t, X(t), u(t)) \cdot V_{xx}(t, X(t))) + V_x(t, X(t)) \cdot \mu(t, X(t), u(t)) \right). \end{aligned}$$

After that, we removed the expectation from the equation because we know the value of $X(t)$ and $u(t)$ at time t , and arrived at

$$0 = \sup_{u \in \mathcal{U}} \left(L(t, x, u) + V_t(t, x) + \frac{1}{2} \cdot \text{tr}((\sigma^*(t, x, u) \cdot V_{xx}(t, x)) + V_x(t, x) \cdot \mu(t, x, u)) \right)$$

5. Portfolio Optimisation in the Continuous-Time Wilkie Model

which is the HJB-equation for our problem (5.3). Hence, we can get the value function $V(t, x)$ by performing the maximisation in the HJB-equation by substituting the maximiser u^* into the HJB-equation, then cutting out the supremum operator and solving the partial differential equation with $V(T, x) = \Psi(x)$ for all $x \in \mathbb{R}^n$. We would achieve the optimum strategy $u^*(\cdot)$ but the solution $V(t, x)$ for the corresponding HJB-equation is only valid if it satisfies the so-called verification theorem.

Theorem 5.2 (Verification theorem for solutions of the HJB-equation)

For this theorem, we used the notation as follows:

(1) For $G \in C^{1,2}(Q)$, $(t, x) \in Q$, $\sigma_{ij}^* := \sigma\sigma'$, $u \in U$, we considered

$$A^u G(t, x) := G_t(t, x) + \frac{1}{2} \sum_{i,j=1}^n \sigma_{ij}^*(t, x, u) \cdot G_{x_i x_j} + \sum_{i=1}^n \mu_i(t, x, u) \cdot G_{x_i}(t, x).$$

(2) $\partial^* Q := ([t_0, t_1] \times \partial O) \cup (\{t_1\} \times \bar{O})$.

We then considered any $K > 0$, $k \in \mathbb{N}$, let $G \in C^{1,2}(Q) \cup C(\bar{Q})$ with $|G(t, x)| \leq K(1 + |x|^k)$ to be a solution to the HJB-equation

$$\sup_{u \in U} (A^u G(t, x) + L(t, x, u)) = 0, \quad (t, x) \in Q, \quad (5.4)$$

$$G(t, x) = \Psi(t, x), \quad (t, x) \in \partial^* Q. \quad (5.5)$$

Therefore,

(1) $G(t, x) \leq J(t, x; u)$ for all $(t, x) \in Q$ and $u(\cdot) \in \mathcal{A}(t, x)$.

(2) If for all $(t, x) \in Q$ there exists a $u^*(\cdot) \in \mathcal{A}(t, x)$ with

$$u^*(s) \in \arg \max_{u \in U} (A^u G(s, X^*(s)) + L(s, X^*(s), u))$$

for all $s \in [t, \tau)$. while $X^*(s)$ is the controlled stochastic process with respect to the control $u^*(\cdot)$. We get

$$G(t, x) = V(t, x) = J(t, x; u^*).$$

The proof for theorem 5.2 can be seen in our major reference for this chapter [Korn and Korn, 2001]. The next corollary that we want to introduce is basically the same corollary in [Korn and Kraft, 2002] known as corollary 3.2. The proof for this corollary is provided in the reference as well.

Corollary 5.1 (To the verification theorem)

In this corollary we consider the linear controlled SDE (5.2), where L and Ψ are continuous, real valued functions satisfying the polynomial growth condition such that

$$|L(t, x, u)| \leq C(1 + |x|^k + |u|^k)$$

and

$$|\Psi(t, x)| = C(1 + |x|^k)$$

5. Portfolio Optimisation in the Continuous-Time Wilkie Model

in space $\bar{Q} \times U$ and \bar{Q} with constants $k \in \mathbb{N}$ and $C > 0$. While G , the solution to the HJB-equation (5.4) and its condition (5.5), admits the condition (1) in theorem 5.2. For all $(t, x) \in Q$, $u(\cdot) \in \mathcal{A}(t, x)$ and $\rho > 1$, then we have

$$E \left(\sup_{s \in [t, t_1]} |G(s, X_s)|^\rho \right) < \infty. \quad (5.6)$$

The condition (1) and (2) in theorem 5.2 are satisfied as consequences from the above corollary.

5.5. Algorithm to the Solution of the HJB-Equation

Here, we provide the steps to be taken in order to solve the corresponding HJB-equation. There are three steps involved:

Step 1

Solve the minimisation problem within the HJB-equation reckoning on the unknown function G with its partial derivatives.

Step 2

Using the solution of the minimisation problem which is u^* ,

$$u^*(s) := u^*(s, x, G(s, x), G_t(s, x), G_x(s, x), G_{xx}(s, x))$$

we solved the partial differential equation

$$A^{u^*(t)}G(t, x) + L(t, x, u^*(t)) = 0$$

for $(t, x) \in Q$, and

$$G(t, x) = \Psi(t, x)$$

for $(t, x) \in \partial^*Q$.

Step 3

All assumptions in theorem 5.2 must be satisfied.

5.6. The Optimal Self-Financing Portfolio

In this section we used the continuous-time Wilkie model to develop a wealth equation for a portfolio consisting of one bond and stock. We then find the portfolio's optimum wealth corresponding to a certain optimal control process. The process will be divided into several sections. First, we considered the optimal portfolio as mentioned earlier. The price process of a stock is modelled from the Wilkie model as in Section 4.6. Second, we presented the objective function with respect to five states. Third, we showed the big picture of the portfolio problem and divided them into special cases from the simplest 1-dimensional-case; the basic case, to the first and the second generalisation of the

5. Portfolio Optimisation in the Continuous-Time Wilkie Model

basic case, and we find the solutions for the corresponding HJB-equations. Finally, we presented the theorems that we developed as a result of solving the HJB-equations.

We first assumed that there is a market in which a bond and a stock are traded continuously. The bond price process, dB_t is subject to the following ordinary differential equation (ODE):

$$dB_t = B_t C_t dt \tag{5.7}$$

with $B_0 = b = 1$ and for $t \in [0, T]$, where $C_t > 0$ is the Consols yield in the continuous-time Wilkie model. Let (ψ, φ) be a number of bond and stock which the investor decides to hold (in this case, the investor only hold one bond and one stock) without considering the consumption process. In addition, we built the portfolio based on the continuous-time Wilkie model setting in chapter 4 and the dynamics of the share price is following equation (4.29). Hence, we have the wealth equation as the following:

$$\begin{aligned} dX_t &= \psi_t dB_t + \varphi_t dS_t \\ &= \psi_t B_t C_t dt + \varphi_t [a_t S_t dt + \delta S_t dZ(t)] \\ &= X_t [(1 - \pi_t) C_t + a_t \pi_t] dt + X_t \pi_t \delta dZ(t) \end{aligned} \tag{5.8}$$

where, (as in section 4.5)

$$C_t = \xi \rho \int_0^t e^{-\rho s} R_{t-s} ds + C_* e^{M_t}.$$

Due to the form of equation (5.8), $X_t > 0$ for all $t \in [0, t]$ with the initial wealth, $X_0 = x > 0$ and π_t denotes the percentage of the total wealth invest in stock at time t and $(1 - \pi_t)$ denotes the percentage of wealth invest in bond at time t with

$$\pi_t = \frac{\varphi_t \cdot S_t}{X_t}$$

and

$$(1 - \pi_t) = \frac{\psi_t \cdot B_t}{X_t}$$

where φ_t and ψ_t denote the number of stock and bond invested at time t , respectively.

The wealth equation can be treated as a controlled SDE where the control is the portfolio process $\pi(\cdot)$. The aim of the investor is to choose the portfolio process which maximises his utility. We assumed the chosen utility followed the utility function, $U(x) = x^\gamma$, $x \geq 0$, $0 < \gamma < 1$. So now we can formulate his optimisation problem,

$$\max_{\pi(\cdot) \in A^*(0, x)} E(X_T^\pi)^\gamma \tag{5.9}$$

with $A^*(0, x) := \{\pi(\cdot) \in A(0, x) : X_s^\pi \geq 0 \quad P - a.s. \text{ for } s \in [0, T]\}$.

5.6.1. States of the Control Process

Now we come to the list of states of the control process as follows:

$$dX_t = X_t[(1 - \pi_t)C_t + a_t\pi_t]dt + X_t\pi_t\delta dZ(t), \quad X_0 = x \quad (5.10)$$

with, (by referring to Section 4.2 - 4.5 in Chapter 4)

$$\begin{aligned} a_t &= a_1\zeta e^{-a_1t} \int_0^t \sigma_1 e^{a_1s} dZ_1(s) + a_2 e^{-a_2t} \int_0^t \sigma_2 e^{a_2s} dZ_2(s) + \beta\lambda \int_0^t e^{-\lambda(t-s)} R_s ds + \gamma R_t \\ &\quad + b_2 + \zeta a_1 r e^{-a_1t} - \zeta \phi e^{-a_1t} + k a_2 e^{-a_2t} - b_1 e^{a_2t} + \frac{1}{2}\zeta^2 \sigma_1^2 + \frac{1}{2}(\eta_2 - \sigma_2)^2 + \frac{1}{2}\eta_3^2, \\ dC_t &= \left(\xi\rho \left(R_t - \rho e^{-\rho t} \int_0^t e^{\rho s} R_s ds \right) - C_* a_4 M_t e^{M_t} + \frac{1}{2} C_* \sigma_4^2 e^{M_t} \right) dt + C_* \sigma_4 e^{M_t} dZ_4(t), \\ dR_t &= (\phi - a_1 R_t) dt + \sigma_1 dZ_1(t), \\ dM_t &= -a_4 M_t dt + \sigma_4 dZ_4(t). \end{aligned}$$

The Brownian motion $Z(t)$ is correlated with Z_1, Z_2 and Z_3 as explained in Section 4.6. It is obviously shown that the evolution of X_t depended on the following state vector,

$$(a_t, R_t, M_t, C_t)',$$

not only $(a_t, C_t)'$ as C_t related to (R_t, M_t) in (5.10). Besides those four states, X_t also depended on the control process π_t . As the control problem is extremely complicated to handle due to the state dependability, we divided the states of the control process to form simpler cases. To begin with, we looked at the most basic case which is the 1-dimensional-case where only X_t is running stochastically. We employed the stochastic control methods as discussed in the previous chapter to solve this optimisation problem. After that, we tried to generalise the basic case while considering all the states processes with a few adjustments.

5.6.2. Stochastic Control Methods

In this section, we started with the most basic case which is the 1-dimensional case. We solved the corresponding HJB-equations using the stochastic control method that was introduced earlier in this chapter. We applied the algorithm to find the solution of the HJB-equation. There are three steps involved and we will show all the procedures required in each step so that the solution can be understood.

The basic case

In this case, we dealt with the simplest 1-dimensional case. X_t was considered to be the only stochastic process whereas the other three stochastic processes were considered to be constants, i.e. $a_1 = a_4 = \phi = \rho = \xi = \sigma_1 = \sigma_4 = 0$. Note that $R_t = R, M_t = M, C_t = C$,

5. Portfolio Optimisation in the Continuous-Time Wilkie Model

were the consequences of our assumptions. To get rid of the Brownian motions in a_t , we should also assume $a_2 = 0$. Therefore, we rewrote the wealth equation (5.10) as

$$dX_t = X_t[(1 - \pi_t)C + a_t\pi_t]dt + X_t\pi_t\delta dZ(t), \quad X_0 = x. \quad (5.11)$$

Together with a new form of a_t we then have in (5.12)

$$a_t = (\beta(1 - e^{-\lambda t}) + \gamma)R + b_2 - b_1 + \frac{1}{2}(\eta_2 - \sigma_2)^2 + \frac{1}{2}\eta_3^2, \quad (5.12)$$

and

$$\delta = \sqrt{(\eta_2 - \sigma_2)^2 + \eta_3^2}. \quad (5.13)$$

By using the similar notation (please refer to remark 2 below), we reached at

$$\begin{aligned} \mu(t, X, a, \pi) &= X[(1 - \pi)C + a_t\pi], \\ \sigma(t, X, a, \pi) &= X\pi\delta, \\ \sigma^*(t, X, a, \pi) &= X^2\pi^2\delta^2. \end{aligned}$$

Therefore, the operator $A^\pi G(t, X)$ of the HJB-equation for this case is

$$A^\pi G(t, X) = \frac{1}{2}\pi^2\delta^2 X^2 G_{XX}^\pi(t, X) + [(1 - \pi)C + a_t\pi] X G_X^\pi(t, X) + G_t^\pi(t, X). \quad (5.14)$$

Remark 2 We often ignore the dependencies of X and π towards t , i.e. $X_t = X, \pi_t = \pi$ for simplicity in writing.

Solving the basic case

The HJB-equation for this case can be written explicitly as follows:

$$0 = \sup_{\pi \in \mathbb{R}} \{A^\pi G(t, X)\}, \quad G(T, X) = U(X),$$

and with substitution of (5.14), it becomes

$$0 = \sup_{\pi \in \mathbb{R}} \left\{ \frac{1}{2}\pi^2\delta^2 X^2 G_{XX}^\pi(t, X) + [(1 - \pi)C + a_t\pi] X G_X^\pi(t, X) + G_t^\pi(t, X) \right\}, \quad G(T, X) = U(X). \quad (5.15)$$

So the first step is taken by finding the optimal strategy $\pi(\cdot)$. Our candidate by using the first-order optimality conditions in (5.15) is

$$\pi_t^* = -\frac{(a_t - C)}{\delta^2} \frac{G_X}{X G_{XX}} \quad (5.16)$$

5. Portfolio Optimisation in the Continuous-Time Wilkie Model

and (5.16) is only true if $G_{XX} < 0$.

As for the second step, we substituted π_t^* into the corresponding HJB-equation to produce a partial differential equation (PDE) of the form

$$\begin{aligned} 0 &= G_t + \frac{1}{2} \frac{(a_t - C)^2}{\delta^2} \frac{G_X^2}{G_{XX}} - \frac{(a_t - C)^2}{\delta^2} \frac{G_X^2}{G_{XX}} + CXG_X \\ &= G_t - \frac{1}{2} \frac{(a_t - C)^2}{\delta^2} \frac{G_X^2}{G_{XX}} + CXG_X \end{aligned}$$

with the terminal wealth $G(T, X) = X^\gamma$ for all X . Then, we reduced the PDE to form an ODE by the following separation ansatz:

$$G(t, X) = g(t) \cdot X^\gamma \quad g(T) = 1.$$

This leads us to the ODE as follows:

$$0 = X^\gamma \left[g'(t) + \left[\frac{1}{2} \frac{\gamma}{1-\gamma} \frac{(a_t - C)^2}{\delta^2} + \gamma C \right] g(t) \right].$$

We now consider

$$h_1(t) = \frac{1}{2} \frac{\gamma}{1-\gamma} \frac{(a_t - C)^2}{\delta^2} + \gamma C.$$

which brought us to the first order ODE for $g(t)$ as

$$0 = g'(t) + h_1(t)g(t) \tag{5.17}$$

with $g(T) = 1$. As $g'(t) = h_1(t)g(t)$ has a strictly positive and differentiable solution, we have found a $C^{1,2}$ to the HJB-equation with

$$G_{XX} = (\gamma - 1)\gamma X^{\gamma-2} g(t)$$

and $G_{XX} < 0$ as $\gamma < 1$. By separation of variables, we solved for $g(t)$ as follows:

$$g(t) = \exp\left(H_1(t) - H_1(T)\right),$$

where H_1 is a primitive of h_1 . Further, we substituted $g(t)$ into $G(t, X)$, which is the separation ansatz and represented our value function. We then have

$$G(t, X) = X^\gamma \cdot \exp\left(H_1(t) - H_1(T)\right).$$

Next, we inserted the new $G(t, X)$ into the optimal strategy, equation (5.16) and this led to a final form of

$$\pi_t^* = \frac{1}{1-\gamma} \frac{(a_t - C)}{\delta^2}.$$

5. Portfolio Optimisation in the Continuous-Time Wilkie Model

From (5.12), it is obviously shown that π_t^* is deterministic and bounded. Therefore, we can now proceed to step three.

In the third step, we have to prove the relationship between the value function $G(t, X)$ and the HJB-equation, the so-called verification theorem. Bear in mind that the definition of admissible control only states the existence and uniqueness of the solution of the controlled stochastic differential equation without stating that X_t^π must be non negative. Nevertheless, our control $\pi \in A(X)$ is still essential but in this specific case, we are interested in all optimal controls leading to a strictly positive wealth process. Therefore, we needed a special requirement for our control which was

$$\pi_t \in [\alpha_1, \alpha_2]^d, \quad t \in [0, T]. \quad (5.18)$$

We needed this requirement in order to apply theorem 5.2. So now we are required to check all the assumptions needed (the workings are inspired from a book by [Korn and Korn, 2001]).

The assumptions are

- (i) G is a $C^{1,2}$ - solution of the HJB-equation.
- (ii) Condition (5.18) is satisfied.
- (iii) X^{π^*} is positive.

The evidence we found are

- (i) We can clearly see that $G(t, X)$ of the above form is strictly concave since $g(t)$ is strictly positive and this leads to G_{XX} to be less than zero. Hence, we can say that $G(t, X)$ is a classical $C^{1,2}$ -solution that satisfies the polynomial growth conditions.
- (ii) The optimal control that we solved for; π_t^* does not only not depend on X , it does not depend on any random component. Therefore, it is not a random function. We now choose α_1 and α_2 such as

$$\pi_t^* \in \left[\frac{\alpha_1}{2}, \frac{\alpha_2}{2} \right]^d$$

for all $t \in [0, T]$. Automatically, requirement (5.18) is satisfied with the optimal π_t^* .

- (iii) The wealth process correspond to the optimal control, π_t^* for this special case, it satisfies the following SDE:

$$dX_t^{\pi^*} = X_t [(1 - \pi_t^*)C + a_t \pi_t^*] dt + X_t \pi_t^* \delta dZ(t). \quad (5.19)$$

This equation has a unique solution and it is strictly positive. We know this by referring to theorem 5.1 earlier. Besides that, it satisfies the required moment condition (iii) in definition 5.1. The solution to the SDE (5.19) can be written as

$$X_t^{\pi^*} = x \cdot \exp \left[\left\{ (1 - \pi^*)C + a\pi^* - 0.5(\pi^*\delta)^2 \right\} t + \pi^* \delta Z(t) \right]. \quad (5.20)$$

In [Wilkie, 1984], the driving noises $Z(t)$ are considered to be Gaussian (white noise) while [Chan, 1998b] allowed any types of distribution for the driving noises. In this

5. Portfolio Optimisation in the Continuous-Time Wilkie Model

study, we let all process be driven by Brownian motions. Then, we concluded that $\ln X_t^{\pi^*}$ is normally distributed with mean and variance as follows:

$$\begin{aligned} E[\ln X_t^{\pi^*}] &= \ln x + \left\{ (1 - \pi^*)C + a\pi^* - 0.5(\pi^*\delta)^2 \right\}t, \\ \text{Var}[\ln X_t^{\pi^*}] &= (\pi^*\delta)^2t. \end{aligned}$$

The following theorem summarises our results:

Theorem 5.3 (A bond and a share portfolio problem for the basic case)

The optimal portfolio process in a bond and a share portfolio problem for 1-dimensional case is shown by

$$\pi_t^* = \frac{1}{1 - \gamma} \frac{(a_t - C)}{\delta^2}$$

with $0 < \gamma < 1$, C is some constant, a_t is in the form of (5.12) as well as δ is in the form of (5.13). The parameters contained in a_t and δ were explained thoroughly in Section 4.2 - 4.5.

By considering the above optimal position, we want to analyse the effects of variable a_t and constant C which is the bond rate towards this position. We found that the increase of a_t caused the investment in the portfolio to be more attractive and vice versa. As we know, a_t was built from the share dividend yield and its index. Therefore, we are actually looking at the effect of share dividend towards the share price. [Campbell and Shiller, 1988] studied the United States equity (stock) market for 1871-1996 and found that the historical equity market price a good predictor of the present value of future equity market dividend yield. Another study was accomplished by [Wilkie, 1993] proving that there is a strong relation between the share price and the share dividend yield in the United Kingdom data for 1923- 1992. Therefore, it can be concluded that our results are consistent with [Campbell and Shiller, 1988] as well as [Wilkie, 1993]. Besides that, the incline of C led to the decline of the optimal position and vice versa. The investor is most likely to invest in high-risky assets in order to maximise his investment. The term $(a_t - C)$ reflects the consideration of the investor to increase his investment in risky assets such as shares while investing less in lower-risk assets such as bonds. This corroborates the economical theory which states, 'high risk causes high returns'. In addition, the coefficient of the randomness of the share price, δ^2 also showed a negative relation with the optimal position, i.e. as the share price increase, the optimal portfolio value will reduce.

The first generalisation of the basic case

In this case, we are interested in a generalisation of the basic case. This was accomplished by considering a deterministic C_t , i.e. $a_1 = a_4 = \phi = \sigma_1 = \sigma_4 = 0$. Note that $R_t = R, M_t = M$, were consequences of our assumptions. Therefore, we were equipped with the same wealth equation as (5.11). With R, M as constants (let us use $m = 0$)

5. Portfolio Optimisation in the Continuous-Time Wilkie Model

and based on (4.14), we obtained

$$\begin{aligned} C_t &= C_* + \xi \rho R \int_0^t e^{-\rho s} ds \\ &= C_* + \xi R(1 - \rho e^{-\rho t}). \end{aligned} \quad (5.21)$$

Applying Itô formula to (5.21), we got

$$dC_t = \rho \xi R e^{-\rho t} dt. \quad (5.22)$$

As in the basic case, to get rid of the Brownian motions in a_t , we should also assume $a_2 = 0$. Thus, we had the same form of a_t as in (5.12) and δ had the same form as in (5.13).

Using similar notation as the previous case, we reached

$$\begin{aligned} \mu(t, x, a, C, \pi) &= \left(X \left[(1 - \pi) C_t + a_t \pi \right], \right. \\ &\quad \left. \rho \xi R e^{-\rho t} \right), \\ \sigma(t, x, a, C, \pi) &= X \pi \delta, \\ \sigma^*(t, x, a, C, \pi) &= X^2 \pi^2 \delta^2. \end{aligned}$$

Therefore, the operator $A^\pi G(t, X)$ of the HJB-equation for this case can be written as

$$\begin{aligned} A^\pi G(t, X) &= \frac{1}{2} \pi^2 \delta^2 X^2 G_{XX}^\pi(t, X) + [(1 - \pi) C_t + a_t \pi] X G_X^\pi(t, X) \\ &\quad + \rho \xi R e^{-\rho t} G_C^\pi(t, X) + G_t^\pi(t, X). \end{aligned} \quad (5.23)$$

Solving the first generalisation of the basic case

The HJB-equation for this case can be written explicitly as follows:

$$0 = \sup_{\pi \in \mathbb{R}} \{ A^\pi G(t, X) \}, \quad G(T, X) = U(X),$$

and with the substitution of (5.23), it became

$$\begin{aligned} 0 = \sup_{\pi \in \mathbb{R}} \left\{ \frac{1}{2} \pi^2 \delta^2 X^2 G_{XX}^\pi(t, X) + [(1 - \pi) C_t + a_t \pi] X G_X^\pi(t, X) \right. \\ \left. + \rho \xi R e^{-\rho t} G_C^\pi(t, X) + G_t^\pi(t, X) \right\}, \quad G(T, X) = U(X). \end{aligned} \quad (5.24)$$

Therefore, the candidate by using the first-order optimality conditions in (5.24) is

$$\pi_t^* = - \frac{(a_t - C_t)}{\delta^2} \frac{G_X}{X G_{XX}} \quad (5.25)$$

5. Portfolio Optimisation in the Continuous-Time Wilkie Model

and (5.25) is only true if $G_{XX} < 0$. The optimal strategy π_t^* for this case is almost similar with the basic case except for the existence of the deterministic C_t . We then substituted (5.25) into (5.24) to produce a PDE of the form

$$\begin{aligned} 0 &= \rho\xi R e^{-\rho t} G_C + G_t + \frac{1}{2} \frac{(a_t - C_t)^2}{\delta^2} \frac{G_X^2}{G_{XX}} - \frac{(a_t - C_t)^2}{\delta^2} \frac{G_X^2}{G_{XX}} + C_t X G_X \\ &= \rho\xi R e^{-\rho t} G_C + G_t - \frac{1}{2} \frac{(a_t - C_t)^2}{\delta^2} \frac{G_X^2}{G_{XX}} + C_t X G_X \end{aligned}$$

with the terminal wealth condition $G(T, X) = X^\gamma$ for all X . We used the separation ansatz as follows:

$$G(t, X, C) = f(t, C) \cdot X^\gamma \quad (5.26)$$

with the terminal condition for $f(T, C) = 1$, to reduce the PDE as the following:

$$0 = X^\gamma \left[\rho\xi R e^{-\rho t} f_C + f_t + \left[-\frac{1}{2} \frac{(a_t - C_t)^2}{\delta^2} \frac{\gamma}{\gamma - 1} + C_t \gamma \right] f \right].$$

Next, we referred to the following separation ansatz:

$$f(t, C) = g(t) \cdot e^{\beta(t) \cdot C} \quad (5.27)$$

with the terminal condition $\beta(T) = 0$ and $g(T) = 1$, to obtain an ODE. The respective ODE is as follows:

$$0 = X^\gamma \left[g'(t) + \left[\rho\xi R \beta e^{-\rho t} - \frac{1}{2} \frac{(a_t - C_t)^2}{\delta^2} \frac{\gamma}{\gamma - 1} + C_t \gamma \right] g(t) \right].$$

Furthermore, we let

$$h_2(t) = \rho\xi R \beta e^{-\rho t} - \frac{1}{2} \frac{(a_t - C_t)^2}{\delta^2} \frac{\gamma}{\gamma - 1} + C_t \gamma.$$

This led us to the first order ODE for $g(t)$ as follows:

$$0 = g'(t) + h_2(t)g(t) \quad (5.28)$$

where $g(T) = 1$. Now, we are solving $g(t)$ by separation of variables and the solution is

$$g(t) = \exp(H_2(t) - H_2(T))$$

where H_2 is a primitive of h_2 . Next, we plugged in $g(t)$ into (5.27), then used its answer to solve (5.26) which is to obtain

$$G(t, X, C) = X^\gamma \cdot \exp(H_2(t) - H_2(T) + \gamma(T - t)C)$$

by letting $\beta(t) = \gamma(T - t)$. Thus, the optimal control for this case can be written as

$$\pi_t^* = \frac{1}{1 - \gamma} \frac{(a_t - C_t)}{\delta^2}.$$

5. Portfolio Optimisation in the Continuous-Time Wilkie Model

As expected, π_t^* for this case is almost the same as the basic case except for the deterministic C . π_t^* is bounded because it does not contain random process. We can now validate our result by applying the verification theorem.

On top of that, in order to apply corollary 5.1 which leads us to the verification theorem, we need to prove the following assumptions (the workings are inspired from a paper by [Korn and Kraft, 2002]):

- (i) $\pi^*(\cdot)$ is progressively measurable,
- (ii) $\pi^*(\cdot)$ satisfies condition (ii) in definition 5.1,
- (iii) $\pi^*(\cdot)$ satisfies condition (iii) in definition 5.1,
- (iv) G is a $C^{1,2}$ - solution of the HJB-equation,
- (v) condition (5.6) is satisfied,
- (vi) $X^{\pi^*} \geq 0$, for a positive wealth process.

Then, we show the evidence as follows:

- (i) The solution $\pi^*(\cdot)$ is deterministic, thus, it is progressively measurable.
- (ii) The solution $\pi^*(\cdot)$ is bounded, thus, condition (ii) in the admissible control is satisfied.
- (iii) By referring to corollary 5.1, the solution of the SDE for this case is the same as in (5.20). The other conditions in corollary 5.1 is satisfied. Thus, condition (i) in definition 5.1 is satisfied. In addition, the solution $\pi^*(\cdot)$ is obviously bounded, which makes condition (ii) in definition 5.1 also satisfied. Further, through [Aries and Krylov, 2008], we obtained

$$E\left(\max_{0 \leq t \leq T} X_t\right) < +\infty.$$

Upon reflection, condition (iii) in definition 5.1 is satisfied.

(iv) By assuming $G_{XX} < 0$, G is clearly a $C^{1,2}$ - solution of the corresponding HJB-equation.

(v) We can prove that (5.6) satisfied all bounded admissible bond and share positions. We then let $(t', X', a', C') \in [0, T] \times \mathbb{R}_+^2 := \{X \in \mathbb{R}^2 : X > 0\}$ and $t' \leq t \leq T$. By applying [Aries and Krylov, 2008], we found that

$$E\left(\sup_{t \in [t', T]} |G(t, X_t, a_t, C_t)| < \infty\right).$$

Thus, we have proven that (5.6) is satisfied.

(vi) Based on the wealth equation (5.19), we get $X^* \geq 0$.

The following theorem summarises our results.

Theorem 5.4 (A bond and a share portfolio problem for the first generalisation of the basic case)

The optimal portfolio process in a bond and a share portfolio problem with a deterministic bond rate C_t is shown by

$$\pi_t^* = \frac{1}{1 - \gamma} \frac{(a_t - C_t)}{\delta^2}$$

5. Portfolio Optimisation in the Continuous-Time Wilkie Model

with $0 < \gamma < 1$, C_t is in the form of (5.21), a_t is in the form of (5.12) as well as δ is in the form of (5.13). The parameters contained in C_t , a_t and δ were explained thoroughly in Section 4.2 - 4.5.

Considering the optimal position, it is interesting to understand the evolution of the Consols yield. From the explicit form of C_t in equation (5.21), there is an initial term C_* and it is likely to converge to a constant. Therefore, the Consols yield depends on its initial value, on the product of ξ and R and with vanishing effect on ρ . Thus, we should have three typical curves of C_t , i.e. with $C_* = -0.05$, $C_* = 0$, $C_* = 0.05$. We show the plot of C_t versus time in figure 5.1 as the red curve is showing C_t when $C_* = -0.05$, the green curve is showing C_t when $C_* = 0$ and the blue curve is showing C_t when $C_* = 0.05$. For this plot, we chose a fixed value for the parameters, i.e. $\xi = 0.5$, $R = 0.02$, $\rho = 1$ and we plotted from time 0 to 50. Please note that the value of all parameters and the time period were chosen at random. At the beginning of the time, we found that the Consols yield increased (for all curves) as the time increases but immediately at time 5 onward, the Consols yield had a stable curve throughout the end of the time. There is a positive effect of C_* towards the Consols yield, a negative value of C_* produced a negative Consols yield and so on. However, the curves showed that the Consols yield did not have a high impact on our optimal position. It is relevant with our results which found that the optimal position is highly affected by the share dividend as well as the share index, or in short, the high-risk asset and in this case, the share.

5. Portfolio Optimisation in the Continuous-Time Wilkie Model

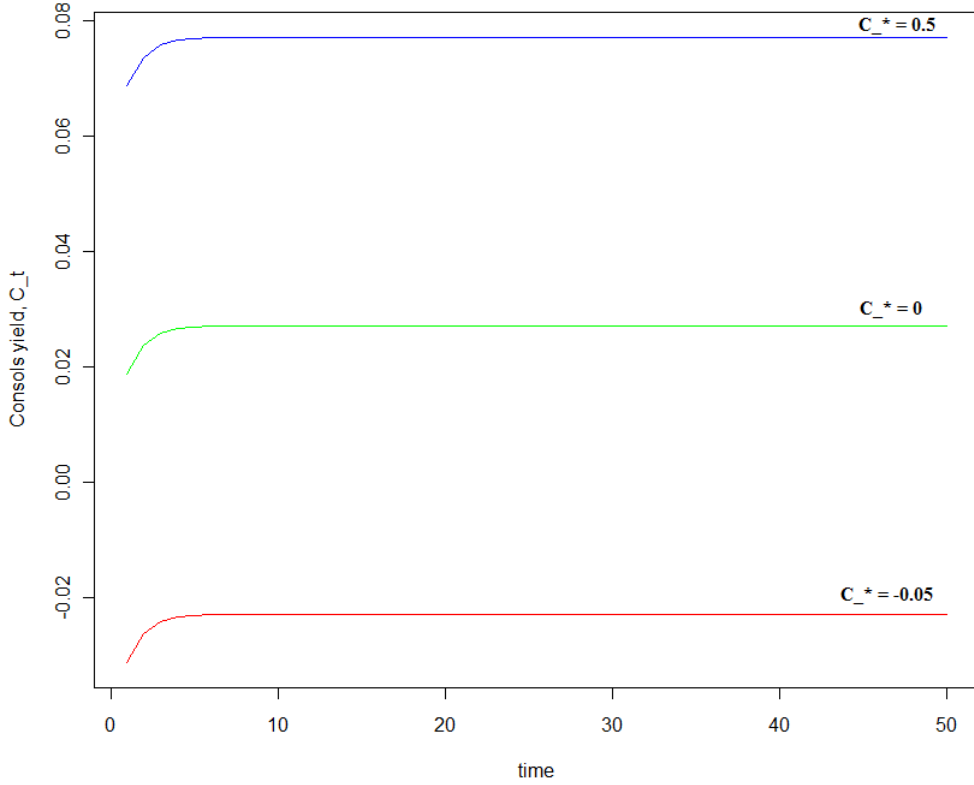


Figure 5.1.: The relationship between deterministic Consols yield and time

In addition to our generalisation, if we consider $m \neq 0$, we will obtain a different equation for the Consols yield and it is

$$C_t = C_* e^m + \xi R(1 - e^{-\rho t}). \quad (5.29)$$

Therefore by applying Itô's to (5.29), we obtained the same dC_t as in the case of $m = 0$. Thus, the optimal solution is also the same as in the case of $m = 0$ except for a slightly different C_t which follows the equation (5.29).

The second generalisation of the basic case

For this case, we are interested to generalise the basic case by considering a deterministic C_t , similar to the first generalisation but with a non-zero parameter a_1 , i.e. only $a_4 = \phi = \sigma_1 = \sigma_4 = 0$. Note that $M_t = M$, is a consequence of our assumptions. In addition, we may assume $a_2 = 0$ to remove the Brownian motion in a_t . The respective wealth equation for this case is the same as we had in the first generalisation but we have the

5. Portfolio Optimisation in the Continuous-Time Wilkie Model

new form of a_t, C_t and dC_t . If we consider $m = 0$, from (4.14) and (4.4) we will get

$$\begin{aligned}
 C_t &= C_* + \xi\rho \int_0^t e^{-\rho s} R_{t-s} ds \\
 &= C_* + \xi\rho \int_0^t e^{-\rho(t-s)} R_s ds \\
 &= C_* + \xi\rho e^{-\rho t} \int_0^t e^{\rho s} R_s ds \\
 &= C_* + \xi\rho e^{-\rho t} \int_0^t e^{\rho s} (re^{-a_1 s}) ds \\
 &= C_* + \xi\rho r t (e^{t(\rho-a_1)-\rho t} - 1),
 \end{aligned} \tag{5.30}$$

and for $m \neq 0$

$$C_t = C_* e^m + \xi\rho r t (e^{t(\rho-a_1)-\rho t} - 1). \tag{5.31}$$

By applying Itô formula to C_t , we obtained the same derivatives for $m = 0$ and $m \neq 0$ as follows:

$$dC_t = \xi\rho r (-e^{-a_1 t})(a_1 t + e^{a_1 t} - 1) dt. \tag{5.32}$$

Thus, we have the same answer to the solutions of SDEs for these two cases. Based on (4.4) and (4.25), we have

$$a_t = \beta\lambda r t (e^{t(\lambda-a_1)-\lambda t} - 1) + \gamma r e^{-a_1 t} + b_2 + \zeta a_1 r e^{-a_1 t} - b_1 + \frac{1}{2}(\eta_2 - \sigma_2)^2 + \frac{1}{2}\eta_3^2. \tag{5.33}$$

Since we considered the non-zero a_1 , we will have the third state process in our problem which is dR_t ,

$$dR_t = -a_1 R dt. \tag{5.34}$$

By using similar notation as the previous cases, we came up with

$$\begin{aligned}
 \mu(t, x, a, C, R, \pi) &= \begin{pmatrix} X[(1-\pi)C_t + a_t\pi] \\ \xi\rho r (-e^{-a_1 t})(a_1 t + e^{a_1 t} - 1) \\ -a_1 R \end{pmatrix}, \\
 \sigma(t, x, a, C, R, \pi) &= X\pi\delta, \\
 \sigma^*(t, x, a, C, R, \pi) &= X^2\pi^2\delta^2.
 \end{aligned}$$

Therefore, the operator $A^\pi G(t, X)$ of the HJB-equation for this case is

$$\begin{aligned}
 A^\pi G(t, X) &= \frac{1}{2}\pi^2\delta^2 X^2 G_{XX}^\pi(t, X) + [(1-\pi)C_t + a_t\pi] X G_X^\pi(t, X) \\
 &\quad + \xi\rho r (-e^{-a_1 t})(a_1 t + e^{a_1 t} - 1) G_C^\pi(t, X) - a_1 R G_R^\pi(t, X) + G_t^\pi(t, X).
 \end{aligned} \tag{5.35}$$

Solving the second generalisation of the basic case

We present the HJB-equation for this case as

$$0 = \sup_{\pi \in \mathbb{R}} \left\{ \frac{1}{2} \pi^2 \delta^2 X^2 G_{XX}^\pi(t, X) + [(1 - \pi)C_t + a_t \pi] X G_X^\pi(t, X) + \xi \rho r (-e^{-a_1 t}) (a_1 t + e^{a_1 t} - 1) G_C^\pi(t, X) - a_1 R G_R^\pi(t, X) + G_t^\pi(t, X) \right\},$$

$$G(T, X) = U(X). \quad (5.36)$$

We have the candidate by using the first-order optimality conditions in (5.36) just as we had in the first generalisation case which was shown in equation (5.25). Therefore, we used the same candidate to solve the corresponding HJB-equation and obtained

$$0 = \xi \rho r (-e^{-a_1 t}) (a_1 t + e^{a_1 t} - 1) G_C - a_1 R G_R + G_t + \frac{1}{2} \frac{(a_t - C_t)^2}{\delta^2} \frac{G_X^2}{G_{XX}}$$

$$- \frac{(a_t - C_t)^2}{\delta^2} \frac{G_X^2}{G_{XX}} + C_t X G_X$$

$$= \xi \rho r (-e^{-a_1 t}) (a_1 t + e^{a_1 t} - 1) G_C - a_1 R G_R + G_t - \frac{1}{2} \frac{(a_t - C_t)^2}{\delta^2} \frac{G_X^2}{G_{XX}} + C_t X G_X$$

with the terminal wealth condition $G(T, X) = X^\gamma$. We used the separation ansatz as follows:

$$G(t, X, C, R) = f(t, C, R) \cdot X^\gamma \quad (5.37)$$

with the terminal condition for $f(T, C, R) = 1$, to reduce the PDE. We then got the new reduced PDE as

$$0 = X^\gamma \left[\xi \rho r (-e^{-a_1 t}) (a_1 t + e^{a_1 t} - 1) f_C - a_1 R f_R + f_t + \left[-\frac{1}{2} \frac{(a_t - C_t)^2}{\delta^2} \frac{\gamma}{\gamma - 1} + C_t \gamma \right] f \right].$$

By using the following separation ansatz:

$$f(t, C, R) = g(t) \cdot e^{\beta(t) \cdot C \cdot R} \quad (5.38)$$

with the terminal condition $\beta(T) = 0$ and $g(T) = 1$, we obtained the ODE as follows:

$$0 = X^\gamma \left[g' + \left\{ \xi \rho \beta R (-e^{-a_1 t}) (a_1 t + e^{a_1 t} - 1) - a_1 \beta C R + \beta' C R - \frac{1}{2} \frac{(a_t - C_t)^2}{\delta^2} \frac{\gamma}{\gamma - 1} + C_t \gamma \right\} g \right].$$

By letting

$$h_3(t) = \xi \rho \beta R (-e^{-a_1 t}) (a_1 t + e^{a_1 t} - 1) - a_1 \beta C R + \beta' C R - \frac{1}{2} \frac{(a_t - C_t)^2}{\delta^2} \frac{\gamma}{\gamma - 1} + C_t \gamma,$$

5. Portfolio Optimisation in the Continuous-Time Wilkie Model

we will have the first order ODE for $g(t)$ as follows:

$$0 = g'(t) + h_3(t)g(t) \quad (5.39)$$

where $g(T) = 1$. Then, we solved for $g(t)$ by separation of variables

$$g(t) = \exp(H_3(t) - H_3(T))$$

where H_3 is a primitive of h_3 . Next, we substituted $g(t)$ into (5.38) and then (5.37) to obtain

$$G(t, X, C) = X^\gamma \cdot \exp(H_3(t) - H_3(T) + \gamma(T - t)C.R)$$

by letting $\beta(t) = \gamma(T - t)$. This led us to the optimal control for this case as

$$\pi_t^* = \frac{1}{1 - \gamma} \frac{(a_t - C_t)}{\delta^2}$$

which is exactly the same solution as in the first generalisation and it is known to be bounded. Therefore, we can simply skip the third step in the stochastic control method and summarise our results in the following theorem:

Theorem 5.5 (A bond and a share portfolio problem for the second generalisation of the basic case)

The optimal portfolio process in a bond and a share portfolio problems with a deterministic bond's rate C_t and a non-zero parameter a_1 is shown by

$$\pi_t^* = \frac{1}{1 - \gamma} \frac{(a_t - C_t)}{\delta^2}$$

with $0 < \gamma < 1$, C_t in the form of (5.30) or (5.31), a_t in the form of (5.33) and δ in the form of (5.13). The parameters contained in C_t, a_t and δ were explained thoroughly in Section 4.2 - 4.5.

We continued the study with another type of portfolio strategy which is a growth-optimal constant portfolio.

5.7. The Growth-Optimal Constant Portfolio

It is well known that the log has a special role as a utility function. It often allows a direct solution to the corresponding optimal portfolio problem. On the other hand, it has the economic interpretation of maximising the expected growth rate [Cover, 1991] [Luenberger, 1998]. The main idea of this section is to take a constant portfolio π and use it to obtain X_t^π as

$$dX_t = X_t[(1 - \pi)C_t + a_t\pi]dt + X_t\pi\delta dZ(t), \quad X_0 = x \quad (5.40)$$

i.e.

$$X_t^\pi = x \cdot e^{\int_0^t \{(1 - \pi)C_s + \pi a_s\} ds + \pi\delta Z(t) - \frac{1}{2}\delta^2\pi^2 t}. \quad (5.41)$$

5. Portfolio Optimisation in the Continuous-Time Wilkie Model

Then, we looked at the log-optimal portfolio problem which is

$$\max_{\pi} E[\ln(X_T^{\pi})]. \quad (5.42)$$

Please note that

$$\ln(X_t^{\pi}) = \ln(x) + \int_0^t C_s ds + \pi \int_0^t \{a_s - C_s\} ds - \frac{1}{2} \delta^2 \pi^2 t + \pi \delta Z(t).$$

By taking the expectation in the last equation, we obtained the expectation of the terminal wealth as (5.43)

$$E[\ln(X_T^{\pi})] = \ln(x) + E\left[\int_0^T C_s ds\right] + \pi E\left[\int_0^T \{a_s - C_s\} ds\right] - \frac{1}{2} \pi^2 \delta^2 T + E[\pi \delta Z(t)] \quad (5.43)$$

where we knew that $E[\pi \delta Z(t)] = 0$. Therefore, the main tasks for us would be to calculate

$$E\left[\int_0^T C_s ds\right],$$

as well as

$$E\left[\int_0^T a_s ds\right] \quad (5.44)$$

in order to obtain the optimal π . By using the first-order condition to (5.43), we reached

$$\pi = \frac{E\left[\int_0^T \{a_s - C_s\} ds\right]}{\delta^2 T}. \quad (5.45)$$

We believe that it may be possible to even solve the general case for the log-utility function. Therefore, we decided to calculate for a three-dimensional-case, i.e. X_t and C_t are the stochastic processes, a_t is a deterministic process while the other processes remain constant. As a result, we had $\phi = a_1 = a_4 = \sigma_1 = \sigma_4 = 0$ and we kept a_2 as a constant as well to avoid the Brownian motion. The processes $R_t = R, M_t = M$ were the consequences of this decision. Therefore, equation (4.12) in Chapter Four led us to

$$E\left[\int_0^T C_s ds\right] = E\left[\int_0^T \left\{ \xi \rho R \int_0^s e^{-\rho u} du + C_* e^M \right\} ds\right]$$

when we considered $m \neq 0$. We solved the previous equation and got

$$\begin{aligned} E\left[\int_0^T C_s ds\right] &= \int_0^T \xi R (1 - e^{-\rho s}) ds + C_* T e^M \\ &= \xi R T - \frac{\xi R}{\rho} (1 - e^{-\rho T}) + C_* T e^M. \end{aligned} \quad (5.46)$$

5. Portfolio Optimisation in the Continuous-Time Wilkie Model

On the other hand, we had the same form of a_t as for in equation (5.12). We then got

$$\begin{aligned} E \left[\int_0^T a_s ds \right] &= E \left[\int_0^T \left\{ \beta(1 - e^{\lambda s} + \gamma)R + b_2 - b_1 + \frac{1}{2}(\eta_2 - \sigma_2)^2 + \frac{1}{2}\eta_3^2 \right\} ds \right] \\ &= (\beta R + \gamma R + b_2 - b_1 + \frac{1}{2}(\eta_2 - \sigma_2)^2 + \frac{1}{2}\eta_3^2)T - \frac{\beta R}{\lambda}(e^{\lambda T} - 1). \end{aligned} \quad (5.47)$$

Finally, we reached the optimal constant portfolio π^* by substituting (5.46) and (5.47) into (5.45) and the result is summarised in a theorem below (by using the expectation rule, i.e. the expectation of a constant is a constant itself).

Theorem 5.6 (A growth-optimal constant portfolio)

The growth-optimal constant portfolio with a deterministic bond rate C_t is given by

$$\begin{aligned} \pi^* &= \frac{\beta R + \gamma R + b_2 - b_1 + \frac{1}{2}(\eta_2 - \sigma_2)^2 + \frac{1}{2}\eta_3^2 - \xi R - C_* e^M}{\delta^2} \\ &\quad + \frac{\xi R(1 - e^{-\rho T})}{\delta^2 T} - \frac{\beta R \delta^2 T (e^{\lambda T} - 1)}{\lambda} \end{aligned}$$

where the variables $\beta, R, \xi, \rho, \lambda, \gamma, b_2, b_1, \eta_2, \eta_3, \sigma_2$ are constants and were explained thoroughly in Chapter Four. The optimal constant portfolio has a high dependency towards the value of inflation. As we encounter a rise in inflation, the portfolio will also rise. The main objective of this portfolio is to find the maximum value of expected log portfolio wealth, that is why it reacts positively with inflation. In times of high inflation, a company may look like it is prospering, when in actual fact the inflation is the reason behind the growth.

5.8. The Optimal Buy-and-Hold Portfolio

As the general continuous-time portfolio problem is very complicated, an alternative might be to break it into simple one-period problems and follow the approach to perform a forward optimisation, i.e. to determine one-period optimal buy-and-hold portfolios, also called myopic portfolios. Basically, we plan to develop a wealth equation which allows an investor to buy one stock and bond at the beginning of investment time and hold his investment until maturity. In order to address this concern, we applied the continuous-time Wilkie model to assist in building the corresponding wealth equation.

We let (ψ, φ) be a number of bonds and stocks which the investor decides to hold (in this case, the investor can only invest in one bond and one stock) without considering the consumption process. For the dynamics of the bond and stock, we referred to equation (5.7) and (4.29) respectively and the investor does not perform any other trading activity during the investment period $[0, T]$. Hence, we obtain the following wealth equation:

$$\begin{aligned} dX_t &= \psi_0 dB_t + \varphi_0 dS_t \\ &= \psi_0 B_t C_t dt + \varphi_0 [a_t S_t dt + \delta S_t dZ(t)] \end{aligned} \quad (5.48)$$

5. Portfolio Optimisation in the Continuous-Time Wilkie Model

with $X_t > 0$ for all $t \in [0, T]$ as the initial wealth $X_0 = x > 0$. The predetermined proportion of the initial capital invested in the bond at the initial time is presented as $1 - \pi$ and the predetermined proportion of the initial capital invested in the stock at initial time is presented as π . These proportions are also called portfolio processes and they are given by

$$\begin{aligned}\pi &= \frac{\varphi_0 \cdot s}{x}, \\ (1 - \pi) &= \frac{\psi_0 \cdot b}{x}\end{aligned}$$

with $S_0 = s$ and $B_0 = b = 1$. As we keep our holdings constant until T, we can no longer use the HJB-approach, but instead we referred to a Markowitz type mean-variance approach.

For this, we considered

$$\begin{aligned}E[X_T] &= \psi_0 \cdot E[B_T] + \varphi_0 \cdot E[S_T], \\ \text{Var}[X_T] &= \text{Var}[\psi_0 B_T + \varphi_0 S_T] = \varphi_0^2 \text{Var}[S_T]\end{aligned}$$

or in terms of returns with

$$\begin{aligned}R_1(T) &:= \frac{B_T - b}{b}, \\ R_2(T) &:= \frac{S_T - s}{s}, \\ R^\pi(T) &:= \frac{X^\pi(T) - x}{x} = \sum_{i=1}^2 \pi_i R_i(T)\end{aligned}$$

and $\mu_i := E[R_i(T)]$ for $i = 1, 2$, $\sigma_{i,j} := \text{Cov}(R_i(T), R_j(T))$ for $i, j = 1, 2$.

We then obtained the standard Markowitz problem as

$$\max_{\pi_1, \pi_2} E[R^\pi(T)] = \max_{\pi_1, \pi_2} [\mu_1 \cdot \pi_1 + \mu_2 \cdot \pi_2] \tag{5.49}$$

subject to

$$\begin{aligned}\pi_i &\geq 0, \quad i = 1, 2, \\ \pi_1 + \pi_2 &= 1, \\ \text{Var}[R^\pi(T)] &= \begin{pmatrix} \pi_1 \\ \pi_2 \end{pmatrix}^\top \sigma \begin{pmatrix} \pi_1 \\ \pi_2 \end{pmatrix} \leq C\end{aligned}$$

with C an upper bound.

For this, we were required to calculate

$$\begin{aligned}\mu_1 &:= E[R_1(T)], \\ \mu_2 &:= E[R_2(T)],\end{aligned}$$

5. Portfolio Optimisation in the Continuous-Time Wilkie Model

as well as the covariance matrix. As $R_1(T)$ is deterministic, this reduced the calculation of $Var[R_2(T)] = \frac{1}{s^2}Var[S_T]$. We began to assume that the only stochastic process involved in this case was S_t while C_t and a_t were deterministic processes and the other state processes remain constant. This is the same assumption as in the case of constant portfolio. Therefore, we had $\phi = a_1 = a_4 = \sigma_1 = \sigma_4 = 0$ and we kept a_2 as zero as well to avoid the Brownian motion. The processes $R_t = R, M_t = M$ were the consequences of this decision. Bear in mind that $b = 1$. So, we calculated

$$\begin{aligned} E[R_1(T)] &:= E\left[\frac{B_T - b}{b}\right] \\ &= E\left[\frac{B_T}{b}\right] - 1 \\ &= E\left[e^{\int_0^T C_s ds}\right] - 1. \end{aligned}$$

From equation (5.46), we got $E[R_1(T)]$ as

$$E[R_1(T)] = e^{\xi RT - \frac{\xi R}{\rho}(1 - e^{-\rho T}) + C_* T e^M} - 1, \quad m \neq 0. \quad (5.50)$$

Then, we calculated $E[R_2(T)]$ with the form of a_t obtained from equation (5.12). The calculation is as follows:

$$\begin{aligned} E[R_2(T)] &:= E\left[\frac{S_T - s}{s}\right] \\ &= E\left[\frac{s \cdot e^{\int_0^T \{a_s - \frac{1}{2}\delta^2\} ds + \int_0^T \delta dZ(s)} - s}{s}\right] \\ &= E\left[e^{\int_0^T \{a_s - \frac{1}{2}\delta^2\} ds}\right] - E[1] \\ &= e^{\int_0^T \{a_s - \frac{1}{2}\delta^2\} ds} - 1 \\ &= e^{(\beta R + \gamma R + b_2 - b_1 + \frac{1}{2}(\eta_2 - \sigma_2)^2 + \frac{1}{2}\eta_3^2 - \frac{1}{2}\delta^2)T - \frac{\beta R}{\lambda}(e^{\lambda T} - 1)} - 1. \end{aligned} \quad (5.51)$$

What remains is to calculate $Var[S_T]$ as

$$\begin{aligned} Var[S_T] &= e^{2\int_0^T a_s ds} (e^{\delta^2 T} - 1) \\ &= e^{2(\beta R + \gamma R + b_2 - b_1 + \frac{1}{2}(\eta_2 - \eta_3)^2 + \frac{1}{2}\eta_3^2)T - \frac{2\beta R}{\lambda}(e^{\lambda T} - 1)} (e^{\delta^2 T} - 1). \end{aligned} \quad (5.52)$$

In particular,

$$\begin{aligned} Var[R^\pi(T)] &= \pi^2 \cdot Var[R_2(T)] \\ &= \pi^2 \cdot \frac{1}{s^2} Var[S_T] \\ &= \frac{\pi^2}{s^2} e^{2(\beta R + \gamma R + b_2 - b_1 + \frac{1}{2}(\eta_2 - \eta_3)^2 + \frac{1}{2}\eta_3^2)T - \frac{2\beta R}{\lambda}(e^{\lambda T} - 1)} (e^{\delta^2 T} - 1). \end{aligned} \quad (5.53)$$

5. Portfolio Optimisation in the Continuous-Time Wilkie Model

Further, we can simplify the optimisation problem in (5.49) into

$$\begin{aligned}
 \max_{\pi_1, \pi_2; \pi_1 + \pi_2 = 1} E[R^\pi(T)] &= \max_{\pi_1, \pi_2; \pi_1 + \pi_2 = 1} [\mu_1 \cdot \pi_1 + \mu_2 \cdot \pi_2] \\
 &= \max_{\pi \in [0, 1]} [\mu_1 \cdot (1 - \pi) + \mu_2 \cdot \pi] \\
 &= \max_{\pi \in [0, 1]} [(\mu_2 - \mu_1) \cdot \pi + \mu_1]. \tag{5.54}
 \end{aligned}$$

Our problem then is to maximise the linear function (5.54) over an interval $[0, 1]$ or which is determined by the condition $Var[R^\pi(T)] \leq C$. We considered three cases which were $\mu_1 > \mu_2$, $\mu_1 < \mu_2$ and $\mu_1 = \mu_2$ and considered all possible portfolios π with a variance below than the upper bound C . For the case of $\mu_1 > \mu_2$, we obtained an optimal portfolio $\pi^* = \mu_1$ with a downward trend from left to right at gradient $\mu_1 - \mu_2$. The case of $\mu_1 < \mu_2$ led to $\pi^* = \mu_2$ with an upward trend from right to left at gradient $\mu_2 - \mu_1$. In the case where $\mu_1 = \mu_2$, we got $\pi^* = \mu_1$ or $\pi^* = \mu_2$ with no trend and shows the linear function is stable over the interval $[0, 1]$. From these results we noticed that when the expected returns of a stock is greater than the expected returns of a bond, the investor would choose to invest in bonds and vice versa. It shows that the investor is interested to invest in a security that has low risk and hopes that it will be beneficial to him.

6. Conclusions

This study set out to explore the development of the Wilkie model which is widely used in actuarial work and liability management. This study also sought to discover the theoretical concept behind the discrete-time Wilkie model. The general literature in this study was based on the very basic Wilkie model [Wilkie, 1984] which consisted of four investment factors such as inflation, share yield, share dividend and Consols yield. Then the study intended to apply the concept of the discrete-time Wilkie model to the Malaysian investment data. The study continued to investigate the ideas behind the transformation of the discrete-time to the continuous-time Wilkie model. The transformation was first introduced by [Chan, 1998a]. After that, the study examined the use of the continuous-time Wilkie model in a portfolio optimisation problem. To summarise, the study was conducted to reach five main objectives which were divided into five chapters.

The first aim of the study was to learn the development of the Wilkie model and this was explored in Chapter One. The first finding was that investment modelling is important to an investor in order to study his investment so that he can gain profit from it. The second finding was, the fact that inflation, ordinary shares and fixed income securities were the most important factors to an investment has lead us to study the Wilkie model which contained all these three factors. The third finding was the existence of stochastic element in investment modelling. This element is essential in determining the distribution of expected return and the Wilkie model was built with this element. The fourth finding was a comprehensive study on the historical literature involving the development of the Wilkie model besides the other literatures from other related studies which were done from various aspects. It consisted of data updating, model building, application to other countries and many more.

The second aim of the study was to explore the discrete-time framework of the Wilkie model and this was discussed in detail in Chapter Two. The first finding was, there were four important factors to an investment observed in the Wilkie model. They are inflation, share dividend yield, share dividend index and Consols yield and these factors formed the basic variables in the Wilkie model. The second finding was the multivariate structure of the Wilkie model which was driven by inflation. The third finding was the extension of the Wilkie model [Wilkie, 1995] and the updated Wilkie model which included extra factors such as wages index, short-term interest rates, property yield and income, index-linked yield and exchange rates. The fourth finding was the development of the Wilkie model using the Box-Jenkins methodology. We found that the inflation, the share dividend yield and the Consols yield models were set to be autoregressive models of order 1 (AR(1)) whereas the share dividend index was set to be a moving

6. Conclusions

average model of order 1(MA(1))[Wilkie, 1984].

The third aim of the study was to apply the concept of the discrete-time Wilkie model in modelling an investment data for the Malaysian context. The results were presented and discussed in Chapter Three and for the first finding, several types of Box-Jenkins models were explained at length which included the autoregressive, the moving average, the autoregressive moving average, the autoregressive integrated moving average and the seasonal autoregressive integrated moving average models. In addition, we also explained the procedures to establish the Box-Jenkins model. The second finding involved the selection of suitable Malaysian data to be tested in the Wilkie model. The retail price index was chosen to represent the inflation while the FTSE KLCI Bursa Malaysia was chosen to represent the share yield as well as the share index. For the Consols yield, we chose the 10-Y MGS yield as its representative. The third finding was the development of a suitable Box-Jenkins model for investment data in Malaysia. We found inflation followed an autoregressive integrated moving average model (ARIMA(0,1,2)), FTSE Bursa Malaysia KLCI yield followed an ARIMA(1,1,0) model, FTSE Bursa Malaysia KLCI followed a seasonal autoregressive integrated moving average (ARIMA(2,1,1)(2,0,2)[12]) model and the 10-Y MGS yield followed an ARIMA(4,1,3)(0,0,1)[12] model. The fourth finding was the forecast result of the investment factors in Malaysia. The inflation was forecasted to be 1.9 per cent in 2013 and will increase to 2.3 per cent in 2014 to 2042. The FTSE Bursa Malaysia KLCI yield was forecasted to be 3.07 per cent in October 2013 and 3.08 - 3.09 per cent from November 2013 to March 2016. On the other hand, the FTSE Bursa Malaysia KLCI was forecasted to be 1877.35 points in January 2014 but will fluctuate from February 2014 onwards and will have a value of 1997.54 points in June 2016. As for the 10-Y MGS yield, it was forecasted to be 4.16 per cent in February 2014 and will drop to 3.97 per cent in July 2016.

The fourth aim of the study was to discover the continuous-time framework of the Wilkie model which was first introduced by [Chan, 1998b]. This was explained in Chapter Four. The first finding focused on the direct transformation from the discrete-time to the continuous-time variables. We found that the transformation process were based on the Itô formula and the Ornstein-Uhlenbeck process which was presented as an AR model in the continuous-time framework. The second finding was the relationship between the share price with the share dividend and its yield. We found explicit forms of the deterministic as well as the stochastic variables in the share price dynamic.

The last aim of the study was to use the continuous-time Wilkie model [Chan, 1998a] in portfolio optimisation problems. This was explained in Chapter Five. We divided the problems into three categories; a self-financing trading strategy, a growth-optimal constant portfolio and a buy-and-hold trading strategy. For the first finding, we presented an explicit solution to a general case which was basically an implementation of the self-financing trading strategy. We called this case a basic case. Then, we generalised the basic case to form the first and the second generalisation cases. We provided solutions for these two cases as well. For the second finding, we found the solution to the growth-optimal constant portfolio. The third finding was the solution to the optimal buy-and-hold portfolio. All these results were concluded in four new theorems.

6. Conclusions

Recommendations for future research

The following recommendations are offered for related research:

1. Given the importance of the Wilkie model in actuarial work and liability management, an expanded application of this model [Chan, 1998a] to investment data in Malaysia would enhance the investment potential in Malaysia. The expanded application may include the updated Wilkie model by [Wilkie, 1995].
2. Regarding the application of the Wilkie model in portfolio optimisation problems, it would be advantageous to conduct research that considers other trading strategies as well as other types of generalisation.

Appendices

A. Augmented Dickey-Fuller unit root test

The ADF test is used to detect a unit root in the following form of time series:

$$\nabla y_t = \alpha + \beta t + \gamma y_{t-1} + \delta_1 \nabla y_{t-1} + \dots + \delta_{p-1} \nabla y_{t-p+1} + \varepsilon_t \quad (\text{A.1})$$

where α is an intercept constant called a drift, β is a coefficient of a linear time trend, γ is a coefficient presenting process root and p is a p -order autoregressive process. The ADF test has the following hypotheses:

$$\begin{aligned} H_0 : y_t \text{ has a unit root (i.e. } \gamma = 0), \\ H_1 : y_t \text{ is a stationary process (i.e. } \gamma < 0). \end{aligned}$$

For $\gamma = 0$, the time series is called a pure random walk process, for $\gamma = 0$ and $\alpha \neq 0$, the time series is called a random walk with drift process and for $\gamma = 0$ and $\beta \neq 0$, the time series is called a random walk with trend process. The ADF test ensures that the null hypothesis is accepted unless there is strong evidence against it to reject in favour of the alternate stationarity hypothesis. The value for the ADF test statistic is calculated as

$$DF_\tau = \frac{\hat{\gamma}}{SE(\hat{\gamma})} \quad (\text{A.2})$$

where $\hat{\gamma}$ is the estimated value of γ and $SE(\hat{\gamma})$ is a standard error of the estimated value of γ . If the test statistic is less than a selected critical value, the null hypothesis is rejected. In addition, reject the null hypothesis when the p-value of the ADF test is less than or equal to a specified significance level, often 0.05 (5%), or 0.01 (1%) and even 0.1 (10%).

B. Autocorrelation function

The autocorrelation function (ACF) defines the correlations between y_t and y_{t-h} for $t = 1, \dots, n$ and $h = 1, \dots, n - 1$. Let y_t be a stationary time series, then the lag- h autocorrelation is given by

$$\rho(y_h) = \frac{\gamma(y_h)}{\gamma(y_0)} = \text{Corr}(y_t, y_{t-h}), \quad (\text{B.1})$$

where $\gamma(y_h)$ is the autocovariance function of the time series. ACF must be among the time difference t and taken value from -1 to 1 at any time lag h . ACF is used to identify the possible model in Box-Jenkins models.

C. Partial autocorrelation function

The partial autocorrelation function (PACF) defines the autocorrelation between y_t and y_{t-h} for $t = 1, \dots, n$ and $h = 1, \dots, n - 1$ after removing any linear dependence on $y_1, y_2, \dots, y_{t-h+1}$. PACF also used to identify the possible Box-Jenkins models by determining the lag- p in AR(p) model as well as in ARIMA (p, d, q) model.

D. Akaike information criterion

Akaike information criterion (AIC) is a measure of relative quality of a statistical model for a given data set. The model with the smallest AIC value is the most suitable model to represent the given data set. AIC is given as

$$AIC = 2k - 2\ln(L). \tag{D.1}$$

A constant k is the number of parameters and L denotes the maximized value of the likelihood function to the model.

E. Bayesian information criterion

Bayesian information criterion (BIC) is a measure of the likelihood function similar to AIC. In fitting a model, one can adding parameters which possibly cause over fitting. Then the BIC will offer a penalty term and it usually has a bigger value than AIC. BIC is denoted as

$$-2.\ln p(y|M) \approx BIC = -2.\ln \hat{L} + k.(\ln(n) + \ln(2\pi)) \quad (\text{E.1})$$

and for a large n , BIC is approximated to be

$$BIC = -2.\ln \hat{L} + k.\ln(n) \quad (\text{E.2})$$

where,

y : the observed data,

n : the sample size,

k : the number of estimated parameters,

$p(y|M)$: the marginal likelihood of the sample of model M ,

\hat{L} : the maximized value of the likelihood function for model M .

F. Ljung-box test

Ljung-box test is a statistical test that detect any non-zero autocorrelation of a time series. It tests randomness according to a number of lags and based on the following hypothesis

$$\begin{aligned} H_0 &: \text{The data is independently distributed,} \\ H_1 &: \text{The data is not independently distributed.} \end{aligned}$$

Then, the Lyung-box test is formulated as

$$Q = n(n+2) \sum_{k=1}^h \frac{\hat{\rho}_k^2}{n-k} \quad (\text{F.1})$$

where,

n : the sample size,

$\hat{\rho}_k$: the sample autocorrelation at lag k ,

h : the number of lags being tested.

At any significance level α , we reject the null hypothesis if

$$Q > \chi_{1-\alpha, h}^2$$

where $\chi_{1-\alpha, h}^2$ represents an α -quantile of the chi-squared distribution at h degrees of freedom.

G. Multi-dimensional Itô formula

Let $X(t) = (X_1(t), \dots, X_n(t))$ be a n -dimensional Itô process with

$$X_i(t) = X_i(0) + \int_0^t K_i(s) ds + \sum_{j=1}^m \int_0^t H_{ij}(s) dW_j(s), \quad i = 1, \dots, n,$$

where $W(t) = (W_1(t), \dots, W_m(t))$ is a m -dimensional Brownian motion. Then, let $f : [0, \infty] \times \mathbb{R}^n \rightarrow \mathbb{R}$ be a $C^{1,2}$ -function, thus we obtain

$$\begin{aligned} f(t, X_1(t), \dots, X_n(t)) &= f(0, X_1(0), \dots, X_n(0)) + \int_0^t f_t(s, X_1(s), \dots, X_n(s)) ds \\ &\quad + \sum_{i=1}^n \int_0^t f_{x_i}(s, X_1(s), \dots, X_n(s)) dX_i(s) \\ &\quad + \frac{1}{2} \sum_{i,j=1}^n \int_0^t f_{x_i x_j}(s, X_1(s), \dots, X_n(s)) d \langle X_i, X_j \rangle_s \end{aligned} \quad (\text{G.1})$$

Bibliography

- [Aries and Krylov, 2008] Aries, A. and Krylov, N. V. (2008). *Controlled diffusion processes*, volume 14. Springer.
- [Campbell and Shiller, 1988] Campbell, J. Y. and Shiller, R. J. (1988). Stock prices, earnings, and expected dividends. *The Journal of Finance*, 43(3):661–676.
- [Chan, 1998a] Chan, T. (1998a). Some applications of levy processes to stochastic investment models for actuarial use. *ASTIN Bulletin*, 28(1):77–93.
- [Chan, 1998b] Chan, W.-S. (1998b). Outlier analysis of annual retail price inflation: A cross-country study. *Journal of Actuarial Practice*, 6:1.
- [Chatfield, 2013] Chatfield, C. (2013). *The analysis of time series: an introduction*. CRC press.
- [Chong, 2007] Chong, S. F. (2007). Building a simplified stochastic asset liability model (alm) for a malaysian participating annuity und. *14th East Asian Actuarial Conference*.
- [Cover, 1991] Cover, T. M. (1991). *Ja thomas elements of information theory*.
- [Engle, 1982] Engle, R. F. (1982). Autoregressive conditional heteroscedasticity with estimates of the variance of united kingdom inflation. *Econometrica: Journal of the Econometric Society*, pages 987–1007.
- [Frees et al., 1997] Frees, E. W., Kung, Y.-C., Rosenberg, M. A., Young, V. R., and Lai, S.-W. (1997). Forecasting social security actuarial assumptions. *North American Actuarial Journal*, 1(4):49–70.
- [Gujarati, 2012] Gujarati, D. N. (2012). *Basic econometrics*. Tata McGraw-Hill Education.
- [Huber, 1997] Huber, P. (1997). A review of wilkie’s stochastic asset model. *British Actuarial Journal*, 3(01):181–210.
- [Hürlimann, 1992] Hürlimann, W. (1992). Numerical evaluation of the wilkie inflation model. *Insurance: Mathematics and Economics*, 11(4):311–314.
- [Korn and Korn, 2001] Korn, R. and Korn, E. (2001). *Option pricing and portfolio optimization*. American Math. Soc.

Bibliography

- [Korn and Kraft, 2002] Korn, R. and Kraft, H. (2002). A stochastic control approach to portfolio problems with stochastic interest rates. *SIAM Journal on Control and Optimization*, 40(4):1250–1269.
- [Luenberger, 1998] Luenberger, D. G. (1998). *Investment science*, 1998.
- [Markowitz, 1952] Markowitz, H. (1952). Portfolio selection*. *The journal of finance*, 7(1):77–91.
- [Merton, 1969] Merton, R. C. (1969). Lifetime portfolio selection under uncertainty: The continuous-time case. *The review of Economics and Statistics*, 51(3):247–257.
- [Merton, 1971] Merton, R. C. (1971). Optimum consumption and portfolio rules in a continuous-time model. *Journal of economic theory*, 3(4):373–413.
- [Metz and Ort, 1993] Metz, M. and Ort, M. (1993). Stochastic models for the swiss consumers price index and the cost of the adjustment of pensions to inflation for a pension fund. In *Transactions of the 3rd AFIR International Colloquium*, volume 1, pages 789–806.
- [Sahin et al., 2008] Sahin, S., Cairns, A., and Kleinow, T. (2008). Revisiting the wilkie investment model. In *The 18th AFIR Colloquium: Financial Risk in a Changing World, Rome*.
- [Sherris et al., 1996] Sherris, M., Tedesco, L., and Zehnwirth, B. (1996). *Stochastic Investment Models: Unit Roots, Cointegration, State Space and Garch Models for Australia*. Centre for Actuarial Studies, Department of Economics, University of Melbourne.
- [Thomson, 1996] Thomson, R. (1996). Stochastic investment modelling: the case of south africa. *British Actuarial Journal*, pages 765–801.
- [Tilley, 1990] Tilley, J. (1990). A stochastic yield curve model for asset/liability simulations. In *1st AFIR International Colloquium*, volume 2, pages 67–80.
- [Tsay, 2005] Tsay, R. S. (2005). *Analysis of financial time series*, volume 543. John Wiley & Sons.
- [Whitten and Thomas, 1999] Whitten, S. and Thomas, R. (1999). A non-linear stochastic asset model for actuarial use. *British Actuarial Journal*, 5(5):919–954.
- [Wilkie, 1981] Wilkie, A. (1981). Indexing long-term financial contracts. *Journal of the Institute of Actuaries*, 108(3):299–360.
- [Wilkie, 1992] Wilkie, A. (1992). Stochastic investment models for xxist century actuaries. In *Transactions of the 24th International Congress of Actuaries*, volume 5, pages 119–137.

Bibliography

- [Wilkie et al., 2011] Wilkie, A., Şahin, Ş., Cairns, A., and Kleinow, T. (2011). Yet more on a stochastic economic model: Part 1: Updating and refitting, 1995 to 2009. *Annals of Actuarial Science*, 5(01):53–99.
- [Wilkie, 1984] Wilkie, A. D. (1984). A stochastic investment model for actuarial use [with discussion]. *Transactions of the Faculty of Actuaries*, 39:341–403.
- [Wilkie, 1993] Wilkie, A. D. (1993). Can dividend yields predict share price changes. In *Transactions of the 3rd International AFIR Colloquium, Rome*, volume 1, pages 335–347.
- [Wilkie, 1995] Wilkie, A. D. (1995). *More on a stochastic asset model for actuarial use*, volume 24. Cambridge Univ Press.

Curriculum Vitae

Norizarina Ishak

Place of Birth: Kuala Lumpur, Malaysia

2011-2015 Ph.D Student, Department of Financial Mathematics, University of Kaiserslautern, Germany

2009-2015 Research Assistant, Department of Financial Mathematics, Universiti Sains Islam Malaysia

2009-2010 Master of Science in Mathematics Management and Financial Mathematics, Universiti Kebangsaan Malaysia

2008-2009 Research Assistant, Department of Mathematics, Universiti Teknologi Malaysia

2007-2008 Research Assistant, Department of Economics, Universiti Putra Malaysia

2004-2007 Bachelor of Science in Mathematics Management, Universiti Kebangsaan Malaysia

2000-2003 Diploma in Science Computer, Universiti Teknologi MARA Malaysia

1995-1999 Convent Peel Road High School Malaysia

Lebenslauf

Norizarina Ishak

geboren in Kuala Lumpur, Malaysia

2011-2015 Ph.D Studentin, Fachbereich Finanzmathematik, TU Kaiserslautern

2009-2015 Wissenschaftlicher Mitarbeiterin, Fachbereich Finanzmathematik, Universität Sains Islam Malaysia

2009-2010 Master Studentin, Universität Kebangsaan Malaysia

2008-2009 Wissenschaftlicher Mitarbeiterin, Fachbereich Mathematik, Universität Teknologi Malaysia

2007-2008 Wissenschaftlicher Mitarbeiterin, Fachbereich Wirtschaftswissenschaft, Universität Putra Malaysia

2004-2007 Bachelor Studentin, Universität Kebangsaan Malaysia

2000-2003 Diploma Studentin, Universität Teknologi MARA Malaysia

1995-1999 Convent Peel Road Gymnasium, Malaysia