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Construction of a Mittag-Leffler Analysis and its Applications

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I hereby affirm that I wrote the present Dissertation without any inadmissible help by a third party and without using any other means than indicated. Concepts and ideas that were taken directly or indirectly from other sources are cited where necessary. This Dissertation has not been presented to any other examination board in this or a similar form, neither in Germany nor in any other country.

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Contents

Abstract

Motivated by the results of infinite dimensional Gaussian analysis and especially white noise analysis, we construct a Mittag-Leffler analysis. This is an infinite dimensional analysis with respect to non-Gaussian measures of Mittag-Leffler type which we call Mittag-Leffler measures. The results indicate that the Wick ordered polynomials, which play a key role in Gaussian analysis, cannot be generalized to this non-Gaussian case. We provide evidence that a system of biorthogonal polynomials, called generalized Appell system, is applicable to the Mittag-Leffler measures, instead of using Wick ordered polynomials. With the help of an Appell system, we introduce a test function and a distribution space. Furthermore we give characterizations of the distribution space and we characterize the weak integrable functions and the convergent sequences within the distribution space. We construct Donsker's delta in a non-Gaussian setting as an application.

In the second part, we develop a grey noise analysis. This is a special application of the Mittag-Leffler analysis. In this framework, we introduce generalized grey Brownian motion and prove differentiability in a distributional sense and the existence of generalized grey Brownian motion local times. Grey noise analysis is then applied to the time-fractional heat equation and the time-fractional Schrödinger equation. We prove a generalization of the fractional Feynman-Kac formula for distributional initial values. In this way, we find a Green's function for the time-fractional heat equation which coincides with the solutions given in the literature.

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Introduction

This dissertation provides a connection between two branches in analysis - the infinite dimensional analysis on the one hand and the fractional calculus on the other hand. The connection is established by the Mittag-Leffler analysis which is a generalization of Gaussian analysis.

During the last three decades Gaussian analysis and especially white noise analysis have evolved into an infinite dimensional distribution theory with rapid developments in mathematical structure and applications in various domains. For an overview we refer to the monographs [HKPS93, Oba94, Kuo96]. White noise analysis is based on a nuclear triple

$$\mathcal{N} \subset \mathcal{H} \subset \mathcal{N}',$$

where $\mathcal{N}$ is a nuclear space, densely embedded in a real separable Hilbert space $(\mathcal{H}, (\cdot, \cdot))$ and $\mathcal{N}'$ denotes the dual space of $\mathcal{N}$ with respect to $\mathcal{H}$. More details will be given in Chapter 1. We assume that $\mathcal{H}$ is an infinite dimensional Hilbert space and wish to introduce a measure on $\mathcal{H}$. It has been established that the only locally finite and translation invariant Borel measure on $(\mathcal{H}, (\cdot, \cdot))$ is the trivial measure, i.e. a generalization of the Lebesgue measure to infinite dimensions does not exist. The next idea is to equip $\mathcal{H}$ with a Gaussian measure. But due to the fact that $\dim \mathcal{H} = \infty$, also a Gaussian measure (with identity covariance operator) does not exist. For this reason, we consider the larger space $\mathcal{N}'$. By the theorem of Bochner and Minlos, see e.g. [BK95], a Gaussian measure μ can be defined on $\mathcal{N}'$ via the characteristic function

$$\int_{\mathcal{N}'} \mathrm{e}^{\mathrm{i}\langle \omega, \xi \rangle} \, \mathrm{d}\mu(\omega) = \exp\left(-\frac{1}{2} \langle \xi, \xi \rangle \right), \quad \xi \in \mathcal{N}.$$

Further details concerning the σ-algebra of $\mathcal{N}'$ are given in Section 1.3. Denote by $L^2(\mu)$ the complex valued functions on $\mathcal{N}'$ which are square integrable with respect to μ. Various nuclear triples of test function and distribution spaces of the form

$$(\mathcal{N}) \subset L^2(\mu) \subset (\mathcal{N})' \tag{1}$$

are constructed in Gaussian analysis, see for example [KS93]. A basic property of Gaussian analysis used in the construction of the nuclear triple (1) is the existence of so-called Wick-ordered polynomials. This system of polynomials on $\mathcal{N}'$, denoted by $(I_n(\xi))_{n \in \mathbb{N}}$ for $\xi \in \mathcal{N}$, fulfils

(*i*) $I_n(\xi) = p_{n,\xi}(\langle \cdot, \xi \rangle)$ for some $\mathbb{K}$-valued polynomial $p_{n,\xi}$ of degree $n \in \mathbb{N}$.

(*ii*) $I_n(\xi) \perp I_m(\xi)$ in $L^2(\mu)$ for all $n \neq m$ and for all $\xi \in \mathcal{N}$.

(iii) $I_n(\xi) \perp I_n(\eta)$ in $L^2(\mu)$ whenever $\xi \perp \eta$ in $\mathcal{H}$.

The Wick-ordered polynomials on $\mathcal{N}'$ are related to the Hermite polynomials H_n, $n \in \mathbb{N}$, on $\mathbb{R}$. Indeed,

$$I_n(\xi) = \frac{|\xi|^n}{2^{n/2}} H_n\left(\frac{\langle \cdot, \xi\rangle}{\sqrt{2}\,|\xi|}\right), \quad \xi \in \mathcal{N},\ \xi \neq 0.$$

Various characterization theorems provide a deep understanding of the structure of the nuclear triples as in (1), see e.g. [PS91, KLP$^+$96]. The main ingredient is an infinite dimensional analogue of the Laplace transform, called S-transform. The S-transform is a one-to-one correspondence between the distribution space $(\mathcal{N})'$ and spaces of holomorphic functions on locally convex spaces satisfying a certain growth condition. With this theoretical background, many applications in mathematics and physics can be rigorously treated, for example intersection local times of Brownian motion [dFHSW97, AOS01] and fractional Brownian motion [DODS08, OSS11, GOdSS11], Feynman integration [dFPS91, KS92, LLSW93, dFOS05], stochastic partial differential equations [GKU99, GKS00, HØUZ96] and many others. We focus on a particular application of white noise analysis in Section 1.4. Consider the heat equation on $\mathbb{R}$ with potential V

$$\frac{\partial}{\partial t}u(t,x) = \frac{1}{2}\frac{\partial^2}{\partial x^2}u(t,x) + V(x)u(t,x), \quad u(0,x) = u_0(x),$$

for $x \in \mathbb{R}$, $t > 0$ and initial value $u_0\colon \mathbb{R} \to \mathbb{R}$. The solution to the heat equation can be represented as the expectation of some Brownian functional by the Feynman-Kac formula. Indeed, assume that the potential V is continuous and bounded from above. Furthermore let the initial value u_0 be a bounded and continuous function satisfying

$$\int_{\mathbb{R}} \mathrm{e}^{-ax^2}\,|u_0(x)|\,\mathrm{d}x < \infty$$

for some $a > 0$. If a solution u exists such that

$$\max_{0\leq t\leq T} |u(t,x)| \leq K\mathrm{e}^{ax^2}, \quad x \in \mathbb{R},$$

for some constants $K, T > 0$ and $0 < a < \frac{1}{2T}$ then u has the stochastic representation

$$u(t,x) = \mathbb{E}\left(u_0(x+B_t)\exp\left(\int_0^t V(x+B_s)\,\mathrm{d}s\right)\right), \quad t > 0,\ x \in \mathbb{R},$$

see e.g. [KS91]. With techniques from white noise analysis, it is possible to choose as u_0 the Dirac delta distribution. The term $\delta(x+B_t)$ is well-defined and called Donsker's delta. In the theory of regular distributions, the product of Donsker's delta with the exponential function can be defined for suitable potentials with the help of the so-called Wick formula [Vog10]. In this way, a Green's function is obtained by

$$K(t,x,y) = \frac{1}{\sqrt{2\pi t}}\exp\left(-\frac{(x-y)^2}{2t}\right)\mathbb{E}\left(\exp\left(\int_0^t V(BB_s^{x,y,t}\,\mathrm{d}s\right)\right), \qquad (2)$$

where $BB_s^{x,y,t}$ denotes a Brownian bridge starting for $s = 0$ at $x \in \mathbb{R}$ and ending in $y \in \mathbb{R}$ for $s = t$, compare also [HT94].

Because of the powerful techniques of Gaussian analysis and the numerous applications, first attempts were made to introduce a non-Gaussian infinite dimensional analysis by transferring properties of the Gaussian measure to the Poisson measure [Ito88]. This approach can be generalized with the help of a biorthogonal system, which consists of generalized Appell systems with respect to a measure μ, see e.g. [Dal91, ADKS96, KSWY98]. These Appell systems replace the Wick-ordered polynomials in Gaussian analysis. Therefore the constructions and concepts from Gaussian analysis become suitable for a wide class of measures μ, including for example the Poisson measure, see e.g. [KSS97], even if no Wick-ordered polynomials exist. The measure μ has to fulfill two properties:

(A1) The Laplace transform of μ is an analytic function, i.e. the mapping

$$\mathcal{N}_{\mathbb{C}} \ni \xi \mapsto l_\mu(\xi) = \int_{\mathcal{N}'} \exp(\langle x, \xi \rangle) \, \mathrm{d}\mu(x) \in \mathbb{C}$$

is holomorphic in a neighborhood $\mathcal{U} \subset \mathcal{N}_{\mathbb{C}}$ of zero.

(A2) The measure μ is non-degenerate or, more precisely, positive on open sets, i.e.

$$\mu(\mathcal{U}) > 0 \quad \text{for all open and non-empty } \mathcal{U} \subset \mathcal{N}'.$$

It was shown in [KK99] that it is not enough to require that μ is non-degenerate as for example in [KSWY98]. The stronger condition that μ is positive on open sets is needed in order to ensure the embedding of test functions into a function space. Then similar notions and characterizations as in Gaussian analysis can be introduced, see [KSWY98]. In particular a space of test functions and distributions is constructed, leading to the nuclear triple

$$(\mathcal{N})^1 \subset L^2(\mu) \subset (\mathcal{N})^{-1}_\mu.$$

The distribution space $(\mathcal{N})^{-1}_\mu$ is characterized via the S_μ-transform. More details on this are provided in Chapter 2.

Section 2.2 and 2.3 contain new contributions to non-Gaussian analysis. In Section 2.2, we consider functions Φ defined on an arbitrary measure space $(T, \mathcal{B}, \nu)$ with values in the distribution space $(\mathcal{N})^{-1}_\mu$. We define the integral of Φ in $(\mathcal{N})^{-1}_\mu$, denoted by $\int_T \Phi(t) \, \mathrm{d}\nu(t)$. We give sufficient conditions on the S_μ-transform of Φ in order to obtain the existence of $\int_T \Phi(t) \, \mathrm{d}\nu(t)$ in $(\mathcal{N})^{-1}_\mu$ as a Bochner integral and as a weak integral, respectively. In particular, we prove the existence of some $\Psi \in (\mathcal{N})^{-1}_\mu$ such that

$$(S_\mu \Psi)(\xi) = \int_T (S_\mu \Phi(t))(\xi) \, \mathrm{d}\nu(t),$$

where $\xi \in \mathcal{N}_{\mathbb{C}}$ lies in a suitable neighborhood of zero. In Section 2.3 the convergent sequences in $(\mathcal{N})^{-1}_\mu$ are characterized. Let $(\Phi_n)_{n\in\mathbb{N}}$ be a sequence in $(\mathcal{N})^{-1}_\mu$. Necessary and sufficient conditions on the S_μ-transform of Φ_n are given for $(\Phi_n)_{n\in\mathbb{N}}$ to be convergent with respect to the inductive topology of $(\mathcal{N})^{-1}_\mu$. Indeed, we observe that weak convergence already implies strong convergence due to the structure of the nuclear triple.

In Chapter 3, we introduce a class of probability measures μ_β, $0 < \beta < 1$, on $\mathcal{N}'$ equipped with the cylinder σ-algebra. The measures μ_β will be called Mittag-Leffler measures. Using the theorem of Bochner and Minlos μ_β is defined via its characteristic function

$$\int_{\mathcal{N}'} \mathrm{e}^{\mathrm{i}\langle \omega,\xi\rangle}\,\mathrm{d}\mu_\beta(\omega) = \mathrm{E}_\beta\left(-\frac{1}{2}\langle \xi,\xi\rangle\right), \quad \xi \in \mathcal{N}.$$

The Mittag-Leffler function E_β is a generalization of the exponential function and given by its power series

$$\mathrm{E}_\beta(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\beta n+1)}, \quad z \in \mathbb{C}.$$

Obviously the Mittag-Leffler measures are a generalization of the Gaussian measure, in particular for $\beta = 1$ the Mittag-Leffler measure μ_1 is the Gaussian measure. If $\beta \neq 1$ Wick-ordered polynomials with respect to the Mittag-Leffler measures μ_β cannot be defined. Indeed, using the Gram-Schmidt orthogonalization procedure, we obtain the unique polynomials H_n^β on $\mathbb{R}$, $n \in \mathbb{N}$, which are orthogonal with respect to the finite dimensional Mittag-Leffler measure, i.e. $H_n^\beta \perp H_m^\beta$ in $L^2(\mathbb{R},\mu_\beta)$ for $n \neq m$. However, we observe that the system of polynomials on $\mathcal{N}'$, given by $(H_n^\beta(\langle\cdot,\xi\rangle))_{n\in\mathbb{N}}$ for $\xi \in \mathcal{N}$, is not orthogonal with respect to the kernels ξ, i.e. there exist natural numbers $m,n \in \mathbb{N}$ and $\xi,\eta \in \mathcal{N}$ such that $\xi \perp \eta$ in $\mathcal{H}$ but $H_n^\beta(\langle\cdot,\xi\rangle) \not\perp H_m^\beta(\langle\cdot,\eta\rangle)$ in $L^2(\mu_\beta)$. Instead of Wick-ordered polynomials, we make use of generalized Appell systems. We show in detail that the Mittag-Leffler measures μ_β fulfill the assumptions (A1) and (A2) which are sufficient to introduce a generalized Appell system. Moreover we provide an explicit formula for the Appell polynomials. The test function and the distribution space can be introduced following the procedure of Chapter 2. This results in the following nuclear triple

$$(\mathcal{N})^1 \subset L^2(\mu_\beta) \subset (\mathcal{N})^{-1}_{\mu_\beta}.$$

Furthermore, we construct Donsker's delta as an element in the distribution space $(\mathcal{N})^{-1}_{\mu_\beta}$ using the integral representation of Dirac delta

$$\delta(\langle\cdot,\eta\rangle) = \frac{1}{2\pi}\int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}x\langle\cdot,\eta\rangle}\,\mathrm{d}x, \quad \eta \in \mathcal{H}.$$

We substantiate the existence of this integral as a weak integral by applying the characterization results from Chapter 2. Moreover we give an approximation of Donsker's delta with function from $L^2(\mu_\beta)$. Indeed, these functions are integrals in $L^2(\mu_\beta)$ in the sense of Bochner.

In Chapter 4, we choose a concrete realization of the nuclear triple from Chapter 3. We find that Mittag-Leffler analysis includes as a special case the grey noise analysis which was originally introduced by Schneider in [Sch90, Sch92] and further developed in [MM09]. In Section 4.1, we recall this construction. Our setting is slightly different. We choose the nuclear triple

$$\mathcal{S}(\mathbb{R}) \subset L^2(\mathbb{R},\mathrm{d}x) \subset \mathcal{S}'(\mathbb{R})$$

of Schwartz test functions $\mathcal{S}(\mathbb{R})$ and tempered distributions $\mathcal{S}'(\mathbb{R})$. Then the mapping

$$\mathcal{S}'(\mathbb{R}) \ni \omega \mapsto B_t^{\alpha,\beta}(\omega) := \langle \omega, M_-^{\alpha/2}\mathbf{1}_{[0,t)}\rangle, \quad t \geq 0,\ 0<\alpha<2,$$

defines a generalized grey Brownian motion. The operator $M_{\pm}^{\alpha/2}$ denotes the fractional integral in the case $1 < \alpha < 2$ and the fractional derivative in the case $0 < \alpha < 1$, respectively. More details are given in Section 4.2, for the definitions and properties of fractional integrals and derivatives see Appendix B. Although our setting differs from the approach of [Sch92, MM09] the resulting generalized grey Brownian motion coincides with the processes given in [Sch92, MM09], see Proposition 4.2.8. We further analyse generalized grey Brownian motion. Consider the differential quotient

$$\Phi_n = \frac{B_{t+h_n}^{\alpha,\beta} - B_t^{\alpha,\beta}}{h_n} \in (\mathcal{N})_{\mu_\beta}^{-1}, \quad n \in \mathbb{N},$$

for a sequence $(h_n)_{n\in\mathbb{N}}$ in $\mathbb{R}$ with $\lim_{n\to\infty} h_n = 0$. We calculate the S_{μ_β}-transform of Φ_n and we verify the assumption from Section 2.3 in order to ensure convergence of $(\Phi_n)_{n\in\mathbb{N}}$ in $(\mathcal{N})_{\mu_\beta}^{-1}$. In this way, we show differentiability of generalized grey Brownian motion in a distributional sense and construct a grey noise process being the derivative of $B_t^{\alpha,\beta}$. In Section 4.3, we prove that $B_t^{\alpha,\beta}$ admits a local time in the distributional sense. The existence of a local time was already shown in [dSE15]. There, the Berman criterium for the existence of a square integrable local time was verified. We utilize a different approach. We can give sense to the ansatz

$$L = \int_0^T \delta_a(B_t^{\alpha,\beta}) \, \mathrm{d}t,$$

since Donsker's delta $\delta_a(B_t^{\alpha,\beta})$ is now available in Mittag-Leffler analysis. We show that the integral exists in $(\mathcal{N})_{\mu_\beta}^{-1}$. Again, we use the characterization for integrable functions in $(\mathcal{N})_{\mu_\beta}^{-1}$ which was obtained in Section 2.2.

Finally in Chapter 5, we draw the link of infinite dimensional analysis to fractional calculus. Fractional calculus is the theory of integration and differentiation of arbitrary order α, where α can be chosen as an arbitrary real or even complex number. There are numerous applications of fractional calculus. Many models in biology, chemistry, physics and also psychology become more accurate by using fractional differential equations, for an overview see [Mac11] and the references therein. The main advantage of the fractional calculus compared to usual differential calculus is the following: While in the usual calculus the derivative is a local property, it is well-known that the fractional derivative depends on the "history of the function". This means, that fractional differential equations are predestined to model systems with memory. There are several monographs devoted to this subject and presenting solution techniques for a broad class of fractional (partial) differential equations, [OS74, SKM93, MR93, Pod99, EIK04, KST06, Die10]. In this dissertation we will focus on a special fractional differential equation: The time-fractional heat equation, which is given for $x \in \mathbb{R}$ and $t > 0$ by

$$({}^C D_{0+}^{\alpha} u)(t,x) = \frac{\partial^2}{\partial x^2} u(t,x), \quad u(0,x) = u_0(x), \tag{3}$$

where u is mapping from $\mathbb{R}$ to R with initial value u_0. This equation directly generalizes the heat equation by substituting the usual time derivative with the time-fractional derivative of Caputo type denoted by ${}^C D_{0+}^{\alpha}$. [Nig86] introduced the time-fractional heat equation of order $\alpha = 1/2$ as a model for diffusion in a porous media. Later on (3) was studied for example by Mainardi in [Mai95] and also by Kochubei in [Koc89, Koc90] and in a more general form in [EK04, EIK04]. We provide the main ideas of these authors in Chapter 5. Another form of (3) uses the Riemann-Liouville fractional integral:

$$u(t,x) = u_0(x) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \frac{\partial^2}{\partial x^2} u(s,x)\, \mathrm{d}s, \quad x \in \mathbb{R},\ t \geq 0. \tag{4}$$

Schneider showed in [Sch90] a first version of a fractional Feynman-Kac formula. He showed that a solution to (4) for $u_0 \in \mathcal{S}(\mathbb{R})$ is given by

$$u(t,x) = \mathbb{E}\left(u_0(x + B_t^{\alpha,\alpha}\right), \quad x \in \mathbb{R},\ t > 0.$$

We apply the techniques of grey noise analysis to the time-fractional heat equation (4). Since the existence of Dirac delta composed with generalized Brownian motion is obtained in Section 3.3, we are able to show that the fractional Feynman-Kac formula can be extended to the case where we choose Dirac delta as initial value u_0. This gives a Green's function to (4) which is proven to coincide with the Green's functions found by Schneider [Sch90], Kochubei [Koc90] or Mainardi [Mai95].

Furthermore, we consider a more general time-fractional heat equation. In a first step we add an inhomogeneity f and consider for $t > 0$ and $x \in \mathbb{R}$ the equation

$$({}^C D_{0+}^{\alpha} u)(t,x) = \frac{1}{2} \frac{\partial^2}{\partial x^2} u(t,x) + f(t,x), \quad u(0,x) = u_0(x).$$

Under certain conditions on f, we show that a Green's function to the equation is given by

$$K_\alpha^f(t,x) = \mathbb{E}\left(\delta(x - y + B_t^{\alpha,\alpha}) + \left(I_{0+}^{\alpha} f(x + B_\cdot^{\alpha,\alpha})\right)(t)\right), \quad t > 0,\ x \in \mathbb{R},$$

where I_{0+}^{α} denotes the left-sided Riemann-Liouville integral.

In a second step, we consider the time-fractional heat equation for $t > 0$ and $x \in \mathbb{R}$ with a potential V:

$$({}^C D_{0+}^{\alpha} u)(t,x) = \frac{\partial^2}{\partial x^2} u(t,x) + V(x), \quad u(0,x) = u_0(x).$$

We use the results from white noise analysis, especially the Green's function to the heat equation in (2), in order to construct a solution to the time-fractional heat equation. This construction is called subordination. We will show that the Green's function K_α has the representation $K_\alpha = \mathbb{E}(\Psi)$ for some $\Psi \in (\mathcal{N})_{\mu_\alpha}^{-1}$.

Also the time-fractional Schrödinger equation is of interest in physics, see for example [Nab04, Bay13, NAYH13]. We construct a Green's function to the time-fractional Schrödinger equation by using a complex scaled Donsker's delta.

The Green's function to the time-fractional heat equation (3) is expressed in terms of certain special functions. Mainardi introduces the M-Wright function M_β, whereas Schneider and Kochubei make use of the so-called Fox-H-function. We provide the definitions and the main properties of these functions in Appendix A. For the reader's convenience, we present the technical parts of the proofs from Chapter 3, Chapter 4 and Chapter 5 in the Appendix A. In particular, we work out the relations between the H-function, the M-Wright function M_β and the Mittag-Leffler function. In fact, we show the existence of the integral

$$\int_0^\infty M_\beta(r) r^{\rho-1} \mathrm{e}^{-rz}\,\mathrm{d}r$$

for all $z \in \mathbb{C}$ and we prove that this integral equals a certain H-function. The above integral plays a central role in many results of this work, for example in proving the existence of Donsker's delta, the existence of grey Brownian motion local times and for the fractional Feynman-Kac formula. As a corollary, we obtain that for the Laplace transform of M_β

$$\int_0^\infty M_\beta(r) \mathrm{e}^{-rz}\,\mathrm{d}r = \mathrm{E}_\beta(z) \tag{5}$$

holds for all $z \in \mathbb{C}$. To the authors knowledge, this fact was up to now only known for $z \in \mathbb{C}$ such that $\Re(z) > 0$. But for the construction of Donsker's delta, it is necessary to have (5) for all $z \in \mathbb{C}$.

In Appendix B, we present the main definitions and properties from Fractional calculus. For a detailed overview and proofs, we refer to [SKM93]. There are many different definitions for a fractional derivative. Historically the most important approach is due to Riemann and Liouville. We will state the definitions and basic properties of the Riemann-Liouville fractional integral and derivative in Appendix B. In applications, especially in physics, the fractional derivative of Caputo type is used. In general, the Riemann-Liouville fractional derivative is different from the Caputo derivative. But we will show that both derivatives coincide on the space of Schwartz test functions. Another fractional derivative, which is used in the definition of fractional Brownian motion in the framework of white noise analysis, is the Marchaud fractional derivative, see e.g. [Ben03]. As we show in Appendix B, the Marchaud fractional derivative coincides with the Caputo fractional derivative on $\mathcal{S}(\mathbb{R})$.

Chapter 1

Preliminaries

The first chapter describes the theoretical basis for this dissertation. The fundamental setting for the development of a Mittag-Leffler analysis is an arbitrary nuclear triple, consisting of a real separable Hilbert space, a nuclear space and its dual. In Section 1.1 we list some facts about nuclear triples, introducing notations and main results. We will not reproduce all details concerning nuclear triples, but rather focus on those facts which we will need later on in this work. For further details we refer to e.g. [GV64, Sch71]. As an example of a nuclear triple we give the standard setting of white noise analysis, see e.g. [HKPS93]. The nuclear space is chosen to be the space of Schwartz test functions which is densely embedded in the space of all square integrable functions. Moreover we introduce spaces of holomorphic functions on a nuclear space, see [Din81]. These spaces become important later on in the characterization theorems.

In Section 1.2, we introduce the notion of integrability of functions with values in a nuclear space (or its dual). Three kinds of integrals from literature are presented: The Bochner integral, the integral in the sense of Pettis and the so-called Gelfand integral. For more details, see [DU77, Kuo96]. In addition, we introduce a new kind of integral which we call the weak integral. This integral is defined similar to the Gelfand integral, but with weaker conditions. Therefore it is applicable later on in order to define Donsker's delta.

Section 1.3 contains the main concepts and results from Gaussian analysis, including construction and characterization of certain distribution spaces. For further details see e.g. [HKPS93, Oba94, Kuo96]. A special case of Gaussian analysis is the white noise analysis. White noise analysis has numerous applications. We work out the application of white noise analysis to the Feynman-Kac formula in Section 1.4.

1.1 Facts on nuclear triples and holomorphy

1.1.1 Nuclear triples

Consider a real separable Hilbert space $\mathcal{H}$ with scalar product $(\cdot,\cdot)$ and corresponding norm $|\cdot|$. Let $\mathcal{N}$ be a nuclear Fréchet space, densely and topologically embedded in $\mathcal{H}$ and let $\mathcal{N}'$ be the dual space of $\mathcal{N}$. Then we get the nuclear triple

$$\mathcal{N} \subset \mathcal{H} \subset \mathcal{N}',$$

where we identify $\mathcal{H}$ with its dual space $\mathcal{H}'$ via the Riesz isomorphism

$$J\colon \mathcal{H} \to \mathcal{H}', \quad f \mapsto (f, \cdot). \tag{1.1}$$

In this convention, the scalar product $(\cdot, \cdot)$ on $\mathcal{H}$ is assumed to be linear in the second component and conjugate linear in the first component if $\mathcal{H}$ is assumed to be a Hilbert space over $\mathbb{C}$. The dual pairing of $\mathcal{N}'$ and $\mathcal{N}$ is denoted by $\langle \cdot, \cdot \rangle$ and is an extension of the scalar product in $\mathcal{H}$, i.e.

$$\langle f, \xi \rangle = (f, \xi), \quad f \in \mathcal{H},\ \xi \in \mathcal{N}.$$

Remark 1.1.1. (*i*) The mathematical convention in Functional Analysis for the definition of the dual pairing is $\langle \cdot, \cdot \rangle \colon \mathcal{N} \times \mathcal{N}' \to \mathbb{K}$. Here, $\mathbb{K}$ denotes either $\mathbb{R}$ or $\mathbb{C}$. In this thesis we will follow the standard notation of white noise analysis and define the dual pairing by $\langle \cdot, \cdot \rangle \colon \mathcal{N}' \times \mathcal{N} \to \mathbb{K}$, compare [HKPS93, Oba94, Kuo96].

(*ii*) The reason for working with nuclear triples is as follows. If $\dim \mathcal{H} = \infty$, then the only translation invariant measure on $\mathcal{H}$ is the trivial measure. Thus it is not possible to define a Lebesgue measure on infinite dimensional spaces. The second choice is to define a Gaussian measure. But also a Gaussian measure with covariance operator $\mathbf{1}$ does not exist if $\dim \mathcal{H} = \infty$, see e.g. [Kuo96]. Nevertheless, such a Gaussian measure can be defined on the bigger space $\mathcal{N}'$ by the theorem of Bochner and Minlos, see Section 1.3 below.

We do not give the abstract definition of a nuclear space, since we work throughout this thesis with a more convenient characterization describing $\mathcal{N}$ as the projective limit of Hilbert spaces, see e.g. Section III.7.3 in [Sch71]:

Theorem 1.1.2. *The nuclear Fréchet space $\mathcal{N}$ can be represented as*

$$\mathcal{N} = \operatorname*{pr\,lim}_{p \in \mathbb{N}} \mathcal{H}_p,$$

where $\{\mathcal{H}_p \mid p \in \mathbb{N}\}$ is a family of Hilbert spaces such that for all $q \in \mathbb{N}$ there exists $p \in \mathbb{N}$ such that the embedding $\mathcal{H}_p \hookrightarrow \mathcal{H}_q$ is of Hilbert-Schmidt class. This means that $\mathcal{N}$ as a set is given by

$$\mathcal{N} = \bigcap_{p \in \mathbb{N}} \mathcal{H}_p$$

and the topology of $\mathcal{N}$ is the projective limit topology, i.e. the coarsest topology such that the canonical embeddings $\mathcal{N} \hookrightarrow \mathcal{H}_p$ are continuous for all $p \in \mathbb{N}$.

The norms on $\mathcal{H}_p$ are denoted by $|\cdot|_p$ and without loss of generality we may assume that the system of norms $\left(|\cdot|_p\right)_{p \in \mathbb{N}}$ is ordered, i.e. $|\cdot|_p \le |\cdot|_q$ whenever $p \le q$ and that $\mathcal{H} = \mathcal{H}_0$.

We denote the dual space of $\mathcal{H}_p$ by $\mathcal{H}_p' = \mathcal{H}_{-p}$. $\mathcal{H}_{-p}$ is a Hilbert space with norm denoted by $|\cdot|_{-p}$ and the dual pairing between $\mathcal{H}_{-p}$ and $\mathcal{H}_p$ is again denoted by $\langle \cdot, \cdot \rangle$. Note that in this construction, we do not identify $\mathcal{H}_p$ and $\mathcal{H}_{-p}$, but using the Riesz isomorphism in (1.1) we get

$$\mathcal{H}_p \subset \mathcal{H} \cong \mathcal{H}' \subset \mathcal{H}_{-p}, \quad p \in \mathbb{N}.$$

General duality theory then shows that

$$\mathcal{N}' = \operatorname{ind}\lim_{p\in\mathbb{N}} \mathcal{H}_{-p},$$

see e.g. Section I.4.3 in [GS68]. This means that $\mathcal{N}'$ as a set is obtained by

$$\mathcal{N}' = \bigcup_{p\in\mathbb{N}} \mathcal{H}_{-p}$$

and $\mathcal{N}'$ is equipped with the inductive limit topology, i.e. the finest topology such that all inclusions $\mathcal{H}_{-p} \hookrightarrow \mathcal{N}'$ are continuous. Thus we arrive at the following chain of spaces:

$$\mathcal{N} \subset \mathcal{H}_q \subset \mathcal{H}_p \subset \mathcal{H} \subset \mathcal{H}_{-p} \subset \mathcal{H}_{-q} \subset \mathcal{N}', \quad p \leq q.$$

We also use the corresponding tensor products of these spaces. Let $\mathcal{H}_p^{\otimes n}$ denote the n-fold Hilbert space tensor product of $\mathcal{H}_p$. Again we denote the norm of $\mathcal{H}_p^{\otimes n}$ by $|\cdot|_p$. By $\mathcal{H}_p^{\hat{\otimes} n}$ we denote the subspace of all symmetric elements from $\mathcal{H}_p^{\otimes n}$. We use the same notations for the spaces $\mathcal{H}_{-p}$. Then $\mathcal{H}_{-p}^{\otimes n}$ is the dual space of $\mathcal{H}_p^{\otimes n}$ and the dual pairing is again denoted by $\langle \cdot, \cdot \rangle$. The tensor powers of $\mathcal{N}$ are denoted by $\mathcal{N}^{\otimes n}$ and defined as the projective limit of the spaces $\left(\mathcal{H}_p^{\otimes n}\right)_{p\in\mathbb{N}}$, i.e.

$$\mathcal{N}^{\otimes n} := \operatorname{pr}\lim_{p\in\mathbb{N}} \mathcal{H}_p^{\otimes n}.$$

Then $(\mathcal{N}^{\otimes n})'$ is the inductive limit of the spaces $\left(\mathcal{H}_{-p}^{\otimes n}\right)_{p\in\mathbb{N}}$, i.e.

$$(\mathcal{N}^{\otimes n})' = \operatorname{ind}\lim_{p\in\mathbb{N}} \mathcal{H}_{-p}^{\otimes n}.$$

In addition to all the real spaces above, we also consider their complexified version using the subscript $\mathbb{C}$. An element $f \in \mathcal{H}_{p,\mathbb{C}}$, given as $f = [f_1, f_2]$ for $f_1, f_2 \in \mathcal{H}_p$, is in the following always identified with $f = f_1 + \mathrm{i} f_2$. The element $[F, G] \in \mathcal{H}_{-p,\mathbb{C}}$ is identified with $F + \mathrm{i}G$ in the sense

$$\langle F + \mathrm{i}G, f + \mathrm{i}g\rangle = \langle F, f\rangle - \langle G, g\rangle + \mathrm{i}(\langle F, g\rangle + \langle G, f\rangle),$$

where $f + \mathrm{i}g \in \mathcal{H}_{p,\mathbb{C}}$. Thus the dual pairing is a bilinear extension of the scalar product $(\cdot, \cdot)$, i.e. for $F + \mathrm{i}G \in \mathcal{H}_{\mathbb{C}}$ and $f + \mathrm{i}g \in \mathcal{H}_{p,\mathbb{C}}$ we have

$$\langle F + \mathrm{i}G, f + \mathrm{i}g\rangle = (F, f) - (G, g) + \mathrm{i}\left((F, g) + (G, f)\right) = (\overline{F + \mathrm{i}G}, f + \mathrm{i}g).$$

Remark 1.1.3. Theorem 1.1.2 shows that $\mathcal{N}$ is a countably Hilbert space. Thus the following properties are valid:

(i) The projective topology on $\mathcal{N}$ coincides with the topology which is induced by the family of norms $\left(|\cdot|_p\right)_{p\in\mathbb{N}}$. Thus an open neighborhood base at $0 \in \mathcal{N}$ is given by the sets $U_{\varepsilon,p} = \left\{\xi \in \mathcal{N} \,\middle|\, |\xi|_p < \varepsilon\right\}$, for $p \in \mathbb{N}$ and $\varepsilon > 0$. Consider a mapping $F\colon \mathcal{N} \to \mathcal{G}$, where $\mathcal{G}$ is a topological space. Then F is continuous if and only if F is continuous on $\mathcal{H}_p$ for some $p \in \mathbb{N}$.

(*ii*) The strong topology on $\mathcal{N}'$ is defined by the neighborhood base

$$\left\{ \Phi \in \mathcal{N}' \,\middle|\, \sup_{\xi \in A} |\langle \Phi, \xi \rangle| < \varepsilon \right\},$$

where $\varepsilon > 0$ and $A \subset \mathcal{N}$ bounded, i.e. $\sup_{\xi \in A} |\xi|_p < \infty$ for all $p \in \mathbb{N}$. It turns out that the strong topology and the inductive topology on $\mathcal{N}'$ coincide, see e.g. A.5 in [HKPS93].

(*iii*) $\mathcal{N}$ is a perfect space, i.e. every bounded and closed set in $\mathcal{N}$ is compact. As a consequence, strong and weak convergence coincide in both $\mathcal{N}$ and $\mathcal{N}'$, see page 73 in [GV64] and Section I.6.3 and I.6.4 in [GS68].

(*iv*) Considering $\mathcal{N}''$ together with the strong topology it can be shown that $\mathcal{N}''$ is isomorphic to $\mathcal{N}$, see page 61 in [GV64]. Thus $\mathcal{N}$ is reflexive.

(*v*) An important result for nuclear spaces is the kernel theorem, see e.g. [GV64]. Let $F \colon \mathcal{N}^{\otimes n} \to \mathbb{R}$ be n-linear and continuous in each of the arguments, i.e. there exists $C < \infty$ and $p \in \mathbb{N}$ such that

$$|F(\xi_1, \dots, \xi_n)| \leq C \, |\xi_1|_p \cdot \ldots \cdot |\xi_n|_p$$

for all $\xi_1, \dots, \xi_n \in \mathcal{N}$. Then there exists a unique $\Phi^{(n)} \in (\mathcal{N}^{\otimes n})'$ such that

$$F(\xi_1, \dots, \xi_n) = \langle \Phi^{(n)}, \xi_1 \otimes \ldots \otimes \xi_n \rangle.$$

Example 1.1.4. A special choice of the nuclear triple is as follows: Choose $\mathcal{N}$ to be the space of Schwartz test functions, denoted by $\mathcal{S}(\mathbb{R})$, and let $\mathcal{H} = L^2(\mathbb{R}, \mathrm{d}x)$ be the space of square integrable functions on $\mathbb{R}$ with respect to the Lebesgue measure $\mathrm{d}x$. It is known that $\mathcal{S}(\mathbb{R}) \subset L^2(\mathbb{R}, \mathrm{d}x)$ is dense. $\mathcal{N}' = \mathcal{S}'(\mathbb{R})$ is the space of tempered distributions and we obtain

$$\mathcal{S}(\mathbb{R}) \subset L^2(\mathbb{R}, \mathrm{d}x) \subset \mathcal{S}'(\mathbb{R}).$$

A function f is in $\mathcal{S}(\mathbb{R})$ if and only if f is infinitely often continuously differentiable and f together with all its derivatives goes to zero for $x \to \pm\infty$ faster than any polynomial. By $|\cdot|_{r,s}$, $r, s \in \mathbb{N}$, we denote the seminorms on $\mathcal{S}(\mathbb{R})$ given by

$$|\xi|_{r,s} = \sup_{x \in \mathbb{R}} \left| x^r \frac{\mathrm{d}^s}{\mathrm{d}x^s} \xi(x) \right|, \quad \xi \in \mathcal{S}(\mathbb{R}).$$

For $p \in \mathbb{N}$ define the norm

$$|\xi|_p := |A^p \xi|_{L^2(\mathbb{R}, \mathrm{d}x)}, \quad \xi \in \mathcal{S}(\mathbb{R}),$$

where A denotes the Hamiltonian of the harmonic oscillator, given by

$$A\xi(x) = \left(\left(\frac{1}{2}x - \frac{\mathrm{d}}{\mathrm{d}x} \right) \left(\frac{1}{2}x + \frac{\mathrm{d}}{\mathrm{d}x} \right) + 2 \right) \xi(x).$$

Then the families $\left(|\cdot|_p \right)_{p \in \mathbb{N}}$ and $\left(|\cdot|_{r,s} \right)_{r,s \in \mathbb{N}}$ are equivalent. Moreover it holds that $\mathcal{S}(\mathbb{R}) = \bigcap_{p \in \mathbb{N}} \mathcal{H}_p$, where the Hilbert spaces $\mathcal{H}_p$, $p \in \mathbb{N}$, are given by

$$\mathcal{H}_p = \left\{ f \in L^2(\mathbb{R}, \mathrm{d}x) \mid |f|_p < \infty \right\}.$$

Thus $\mathcal{S}(\mathbb{R})$ is a countably Hilbert space. It can also be shown that for each $p \in \mathbb{N}$ the injection $i_{p+1,p} \colon \mathcal{H}_{p+1} \to \mathcal{H}_p$ is a Hilbert-Schmidt operator. Hence $\mathcal{S}(\mathbb{R})$ is a nuclear space. For further details and proofs see e.g. [RS72].

1.1.2 Holomorphy on locally convex spaces

Spaces of holomorphic functions on the locally convex space $\mathcal{N}_{\mathbb{C}}$ are used later on for the characterization of test functions and distributions. We give the main definitions and results, for more details we refer to [Din81].

Let $U \subset \mathcal{N}_{\mathbb{C}}$ be open. A function $f\colon U \to \mathbb{C}$ is called *Gâteaux-holomorphic* (or G-holomorphic) if for each $\xi \in U$ and $\eta \in \mathcal{N}_{\mathbb{C}}$ the mapping

$$\mathbb{C} \ni \lambda \mapsto f(\xi + \lambda\eta) \in \mathbb{C}$$

is holomorphic in some neighborhood of zero. If f is G-holomorphic then for each $\xi \in U$ there exists a unique sequence of homogeneous polynomials from $\mathcal{N}_{\mathbb{C}}$ into $\mathbb{C}$, denoted by

$$\left(\frac{\hat{\mathrm{d}}^n f(\xi)}{n!}\right)_{n\in\mathbb{N}},$$

such that f admits the Taylor series

$$f(\xi + \eta) = \sum_{n=0}^{\infty} \frac{\hat{\mathrm{d}}^n f(\xi)}{n!}(\eta)$$

for all $\eta \in \mathcal{N}_{\mathbb{C}}$ in some open neighborhood of zero. A function $f\colon U \to \mathbb{C}$ is called *holomorphic*, if f is G-holomorphic and for each $\xi \in U$ there is an open neighborhood $V \subset \mathcal{N}_{\mathbb{C}}$ of zero such that the function

$$V \ni \eta \mapsto \sum_{n=0}^{\infty} \frac{\hat{\mathrm{d}}^n f(\xi)}{n!}(\eta)$$

converges and defines a continuous function. [Din81] proves a very convenient result which gives two necessary and sufficient conditions for a function f to be holomorphic:

Proposition 1.1.5. Let $U \subset \mathcal{N}_{\mathbb{C}}$ be open and $f\colon U \to \mathbb{C}$. Then f is holomorphic, if and only if f is G-holomorphic and locally bounded, i.e. each point $\xi \in U$ has a neighborhood whose image under f is bounded.

Remark 1.1.6. Taking into account the topological properties of $\mathcal{N}_{\mathbb{C}}$, see Remark 1.1.3, then we can conclude from Proposition 1.1.5 that a function $f\colon \mathcal{N}_{\mathbb{C}} \to \mathbb{C}$ is holomorphic at $0 \in \mathcal{N}_{\mathbb{C}}$ if and only if there exist $p \in \mathbb{N}$ and $\varepsilon > 0$ such that the following holds:

(*i*) *(locally bounded)* There exists $c > 0$ such that $|f(\xi)| \leq c$ for all $\xi \in \mathcal{N}_{\mathbb{C}}$ with $|\xi|_p \leq \varepsilon$.

(*ii*) *(G-holomorphic)* For all $\xi_0 \in \mathcal{N}_{\mathbb{C}}$ with $|\xi_0|_p \leq \varepsilon$ and for all $\xi \in \mathcal{N}_{\mathbb{C}}$ the function

$$\mathbb{C} \ni \lambda \mapsto f(\lambda\xi + \xi_0) \in \mathbb{C}$$

is holomorphic at $0 \in \mathbb{C}$.

By $\mathrm{Hol}_0(\mathcal{N}_{\mathbb{C}})$ we denote the space of all holomorphic functions at zero. Let f and g be holomorphic on a neighborhood $\mathcal{V}, \mathcal{U} \subset \mathcal{N}_{\mathbb{C}}$ of zero, respectively. We identify

f and g if there is a neighborhood $\mathcal{W} \subset \mathcal{V}$ and $\mathcal{W} \subset \mathcal{U}$ such that $f(\xi) = g(\xi)$ for all $\xi \in \mathcal{W}$. $\mathrm{Hol}_0(\mathcal{N}_\mathbb{C})$ is the union of the spaces

$$\left\{ f \in \mathrm{Hol}_0(\mathcal{N}_\mathbb{C}) \;\middle|\; n_{p,l,\infty}(f) := \sup_{|\xi|_p \leq 2^{-l}} |f(\xi)| < \infty \right\}, \quad p, l \in \mathbb{N},$$

and carries the inductive limit topology.

1.2 Integration in vector spaces

In the following let X be a vector space and consider a function $f\colon T \to X$ where $(T, \mathcal{B}, \nu)$ is a measure space. There are essentially two ways to define the integral of f over T with respect to ν. The first possibility characterizes the integral using the dual space of X, denoted by X'. Therefore we have to assure that X' separates points in X, i.e. for all $0 \neq x \in X$ there exists $x' \in X'$ such that $\langle x', x \rangle \neq 0$. If we assume that X is locally convex then this condition is fulfilled by the Hahn-Banach theorem, see e.g. Section II.4 in [Sch71]. In this way, we introduce the Pettis and the Gelfand integral, see e.g. [DU77]. We also define a slight modification of the Gelfand integral which we call the weak integral. This integral will later on be applied in Mittag-Leffler analysis, see Section 3.3.

The second type of integral which we describe in this section is the so-called Bochner integral. It is defined via approximation by simple functions. Therefore we need that the space X is complete, i.e. we will require in the corresponding section that X is a Banach space.

1.2.1 The weak integral

In this subsection X is a locally convex space and let $f\colon T \to X$ where $(T, \mathcal{B}, \nu)$ is a measure space.

Definition 1.2.1. The mapping $f\colon T \to X$ is called *Pettis integrable* if the following conditions are fulfilled:

(*i*) The mapping $T \ni t \mapsto \langle x', f(t) \rangle$ is in $L^1(T, \nu)$ for all $x' \in X'$.

(*ii*) For all $B \in \mathcal{B}$ there exists $x_B \in X$ such that for all $x' \in X'$

$$\langle x', x_B \rangle = \int_B \langle x', f(t) \rangle \, \mathrm{d}\nu(t).$$

Then x_B is called the *Pettis integral* and denoted by $x_B = (P) \int_B f(t) \, \mathrm{d}\nu(t)$.

Note that due to the Hahn-Banach theorem $x_B \in X$ is unique, provided it exists. If X is a Banach space and if *(i)* is fulfilled then there always exists an element $x_B \in X''$ such that for $x' \in X'$

$$\langle x_B, x' \rangle = \int_B \langle x', f(t) \rangle \, \mathrm{d}\nu(t),$$

compare [DU77]. Indeed, the linearity of the mapping $X' \ni x' \mapsto \int_B \langle x', f(t) \rangle \, \mathrm{d}\nu(t)$ is obvious. Moreover we use the closed graph theorem to obtain continuity of the mapping

$$X' \ni x' \mapsto \langle x', f(\cdot) \rangle \in L^1(T, \nu).$$

To see this, let $(x'_n)_{n\in\mathbb{N}}$ be a sequence in X' such that $\lim_{n\to\infty} x'_n = x'$ for some $x' \in X'$ and assume that $(\langle x'_n, f(\cdot)\rangle)_{n\in\mathbb{N}}$ converges to some $g \in L^1(T,\nu)$. The L^1-convergence implies that there exists a subsequence $(n_k)_{k\in\mathbb{N}}$ such that for ν-almost all $t \in T$

$$g(t) = \lim_{k\to\infty} \langle x'_{n_k}, f(t)\rangle.$$

Since $\left(x'_{n_k}\right)_{k\in\mathbb{N}}$ converges in X' to x', it follows that $g(t) = \langle x', f(t)\rangle$ and hence the graph is closed.

The integral for a mapping $f\colon T \to X'$ is defined similar to the Pettis integral. This integral is called Gelfand integral, see [DU77]:

Definition 1.2.2. The mapping $f\colon T \to X'$ is called *Gelfand integrable* if the following conditions are fulfilled:

(i) The mapping $T \ni t \mapsto \langle f(t), x\rangle$ is in $L^1(T,\nu)$ for all $x \in X$.

(ii) For all $B \in \mathcal{B}$ there exists $x_B \in X'$ such that for all $x \in X$

$$\langle x_B, x\rangle = \int_B \langle f(t), x\rangle \,\mathrm{d}\nu(t).$$

Then we call x_B the *Gelfand integral* of f and it is denoted by $x_B = (G)\int_B f(t)\,\mathrm{d}\nu(t)$.

If X is a Banach space and *(i)* is fulfilled then $x_B \in X'$ as in (ii) always exists. Similar as before we show that the mapping

$$X \ni x \mapsto \langle f(\cdot), x\rangle \in L^1(T,\nu)$$

is continuous. Indeed, let $x \in X$ and $(x_n)_{n\in\mathbb{N}}$ a sequence in X with $\lim_{n\to\infty} x_n = x$ and assume that $(\langle f(\cdot), x_n\rangle)_{n\in\mathbb{N}}$ converges to $g \in L^1(T,\nu)$. Then there is a subsequence such that for ν-almost all $t \in T$

$$g(t) = \lim_{n\to\infty} \langle f(t), x_{n_k}\rangle = \langle f(t), x\rangle.$$

The closed graph theorem implies the desired continuity.

Remark 1.2.3. If X is reflexive then Gelfand and Pettis integrability on X' are equivalent. More precisely, let $J_X\colon X \to X''$ denote the isomorphism defined by

$$\langle J_X x, x'\rangle = \langle x', x\rangle, \quad x' \in X',$$

and let $f\colon T \to X'$. Since J_X is bijective, we have that for all $x'' \in X''$ there is some $x \in X$ such that $\langle x'', f(t)\rangle = \langle J_X x, f(t)\rangle = \langle f(t), x\rangle$ for all $t \in T$. Thus we have that

$$\langle x'', f(\cdot)\rangle \in L^1(T,\nu)\ \forall x'' \in X'' \iff \langle f(\cdot), x\rangle \in L^1(T,\nu)\ \forall x \in X.$$

Moreover for $B \in \mathcal{B}$ we have

$$\begin{aligned}
&\exists x_B \in X' :\ \langle x'', x_B\rangle = \int_B \langle x'', f(t)\rangle \,\mathrm{d}\nu(t) \quad \forall x'' \in X'' \\
\Leftrightarrow\ &\exists x_B \in X' :\ \langle J_X x, x_B\rangle = \int_B \langle J_X x, f(t)\rangle \,\mathrm{d}\nu(t) \quad \forall x \in X \\
\Leftrightarrow\ &\exists x_B \in X' :\ \langle x_B, x\rangle = \int_B \langle f(t), x\rangle \,\mathrm{d}\nu(t) \quad \forall x \in X.
\end{aligned}$$

This proves the equivalence of Pettis and Gelfand integral.

For later purpose, we introduce a modification of the Gelfand integral. The conditions (i) and (ii) are weakened by requiring that they only hold on a dense subset $D \subset X$. More precisely:

Definition 1.2.4. Let $D \subset X$ be dense and let $f\colon T \to X'$ fulfill:

(i) The mapping $T \ni t \mapsto \langle f(t), x\rangle$ is in $L^1(T,\nu)$ for all $x \in D$.

(ii) For all $B \in \mathcal{B}$ there exists $x_B \in X'$ such that for all $x \in D$

$$\langle x_B, x\rangle = \int_B \langle f(t), x\rangle \,\mathrm{d}\nu(t).$$

Then we call f *weakly integrable* in X' and by $x_B = (W)\int_B f(t)\,\mathrm{d}\nu(t)$ we denote the *weak integral* of f.

Proposition 1.2.5. Let $D \subset X$ be dense. Let $f\colon T \to X'$ be weakly integrable in X' and let the mapping $F\colon D \to L^1(T,\nu)$, $x \mapsto \langle f(\cdot), x\rangle \in L^1(T,\nu)$ be continuous. Then f is Gelfand integrable and the Gelfand integral coincides with the weak integral.

Proof. We first observe that $\langle f(\cdot), x\rangle \in L^1(T,\nu)$ for all $x \in X$ by extending F by the BLT theorem. The extension is again denoted by F. For $x \in X$ let $(x_n)_{n\in\mathbb{N}}$ be a sequence in D which converges in X to x. Due to continuity of F there exists $p \in \mathbb{N}$ such that $\|F(x)\|_{L^1(T,\nu)} \le C\|x\|_p$ for some $p \in \mathbb{N}$, see Remark 1.1.3. Here the family $(\|\cdot\|_p)_{p\in\mathbb{N}}$ denotes the seminorms which define the topology of the locally convex space X. Then:

$$\begin{aligned}
&\left|\int_B \langle f(\cdot), x_n\rangle \,\mathrm{d}\nu(t) - \int_B \langle f(\cdot), x\rangle \,\mathrm{d}\nu(t)\right| \\
&\le \|\langle f(\cdot), x_n - x\rangle\|_{L^1(T,\nu)} = \|F(x_n - x)\|_{L^1(T,\nu)} \\
&\le C\|x_n - x\|_p \to 0, \quad n \to \infty.
\end{aligned}$$

Let $x_B \in X'$ denote the weak integral, i.e. $\langle x_B, y\rangle = \int \langle f(t), y\rangle \,\mathrm{d}\nu(t)$ for all $y \in D$. Then:

$$\langle x_B, x\rangle = \lim_{n\to\infty} \langle x_B, x_n\rangle = \lim_{n\to\infty} \int_B \langle f(t), x_n\rangle \,\mathrm{d}\nu(t) = \int_B \langle f(t), x\rangle \,\mathrm{d}\nu(t).$$

This shows that the weak integral x_B coincides with the Gelfand integral. □

1.2.2 The Bochner integral

Now we introduce the Bochner integral. For proofs and details we refer to [Kuo96]. Let X be a Banach space with norm $\|\cdot\|$ and $(T,\mathcal{B},\nu)$ a measure space. We call a function $f\colon T \to X$ *countably-valued*, if there exist $(x_k)_{k\in\mathbb{N}}$, $x_k \in X$ for all $k \in \mathbb{N}$ and $(E_k)_{k\in\mathbb{N}}$, $E_k \in \mathcal{B}$ disjoint for all $k \in \mathbb{N}$ and

$$f = \sum_{k=1}^{\infty} x_k \mathbf{1}_{E_k}.$$

A countably-valued function f is called *Bochner integrable* if

$$\sum_{k=1}^{\infty} \|x_k\| \nu(E_k) < \infty.$$

Then for any $B \in \mathcal{B}$ the *Bochner integral* of f on B is given by

$$\int_B f(t)\,\mathrm{d}\nu(t) := \sum_{k=1}^{\infty} x_k \nu(B \cap E_k),$$

and it holds that $\| \int_B f(t)\,\mathrm{d}\nu(t)\| \leq \int_B \|f(t)\|\,\mathrm{d}\nu(t)$.

Definition 1.2.6. A function $f\colon T \to X$ is *Bochner integrable* if there exists a sequence of countably-valued Bochner integrable functions $(f_n)_{n\in\mathbb{N}}$ such that

(*i*) $\lim_{n\to\infty} f_n = f$ ν-almost everywhere.

(*ii*) $\lim_{n\to\infty} \int_T \|f(t) - f_n(t)\|\,\mathrm{d}\nu(t) = 0$.

Then for each $B \in \mathcal{B}$ the Bochner integral of f is defined by

$$(B)\int_B f(t)\,\mathrm{d}\nu(t) := \lim_{n\to\infty} \int_B f_n(t)\,\mathrm{d}\nu(t).$$

Remark 1.2.7. (*i*) The Bochner integral is well-defined since *(ii)* ensures that $\int_B f_n(t)\,\mathrm{d}\nu(t)$, $n \in \mathbb{N}$, defines a Cauchy sequence in the Banach space X. Moreover the definition is independent of the choice of $(f_n)_{n\in\mathbb{N}}$. Indeed, let $(g_n)_{n\in\mathbb{N}}$ be another sequence of countably-valued functions fulfilling (*i*) and (*ii*). Then

$$\begin{aligned}
&\| \int_B f_n(t)\,\mathrm{d}\nu(t) - \int_B g_n(t)\,\mathrm{d}\nu(t)\| \leq \int_B \|f_n(t) - g_n(t)\|\,\mathrm{d}\nu(t) \\
&\leq \int_B \|f_n(t) - f(t)\|\,\mathrm{d}\nu(t) + \int_B \|f(t) - g_n(t)\|\,\mathrm{d}\nu(t) \to 0, \quad n \to \infty.
\end{aligned}$$

(*ii*) If f is Bochner integrable then f is also Pettis integrable and the Bochner integral coincides with the Pettis integral, see e.g. [Kuo96].

(*iii*) $f\colon T \to X$ is Bochner integrable if and only if

- (*i*) The mapping $T \ni t \mapsto \langle x', f(t)\rangle$ is measurable for all $x' \in X'$.
- (*ii*) f is almost separably-valued, i.e. there exists $B_0 \in \mathcal{B}$ with $\nu(B_0) = 0$ and the set $f(T \setminus B_0)$ is separable.
- (*iii*) $\int_T \|f(t)\|\,\mathrm{d}\nu(t) < \infty$.

For the proof we refer to [DU77]. Note that if X is a separable Banach space then (*ii*) is automatically fulfilled.

1.3 Gaussian Analysis

Back to the arbitrary nuclear triple $\mathcal{N} \subset \mathcal{H} \subset \mathcal{N}'$ described in Section 1.1. In the following we explain how to introduce a Gaussian measure on $\mathcal{N}'$ and we give its basic properties. We recall the construction of Wick-ordered polynomials on $\mathcal{N}'$ which leads to the Wiener-Itô-Segal-Chaos decomposition. This decomposition is then used to introduce various spaces of test functions and distributions and we state the main characterization theorems.

1.3.1 The Gaussian measure

Let $\mathcal{B}(\mathbb{R}^n)$ denote the Borel σ-algebra on $\mathbb{R}^n$, $n \in \mathbb{N}$. Consider for given $F \in \mathcal{B}(\mathbb{R}^n)$ and $\xi_1, \ldots, \xi_n \in \mathcal{N}$ the cylinder sets

$$C_F^{\xi_1,\ldots,\xi_n} = \left\{ x \in \mathcal{N}' \mid (\langle x, \xi_1 \rangle, \ldots, \langle x, \xi_n \rangle) \in F \right\}.$$

The σ-algebra on $\mathcal{N}'$ generated by the cylinder sets is called the *cylinder σ-algebra* and denoted by $\mathcal{C}_\sigma(\mathcal{N}')$. The cylinder σ-algebra $\mathcal{C}_\sigma(\mathcal{N}')$ and the σ-algebra generated by the weak and strong topology in $\mathcal{N}'$ coincide, see e.g. [BK95].

The theorem of Bochner and Minlos, see e.g. [BK95], states that there is a one-to-one correspondence between probability measures on $(\mathcal{N}', \mathcal{C}_\sigma(\mathcal{N}'))$ and characteristic functions f on $\mathcal{N}$. These are continuous functions $f \colon \mathcal{N} \to \mathbb{C}$ with $f(0) = 1$ and for any $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$ and $\xi_1, \ldots, \xi_n \in \mathcal{N}$ it holds

$$\sum_{k,i=1}^{n} \lambda_k \bar{\lambda}_i f(\xi_k - \xi_i) \geq 0,$$

i.e. f is positive definite. Lemma 2.1.1 in [Oba94] shows that the mapping

$$\mathcal{N} \ni \xi \mapsto \exp\left(-\frac{1}{2} |\xi|^2 \right)$$

is a characteristic function. The probability measure which corresponds to this characteristic function is denoted by μ_1:

Definition 1.3.1. Define the *Gaussian measure* μ_1 on $(\mathcal{N}', \mathcal{C}_\sigma(\mathcal{N}'))$ to be the unique probability measure fulfilling

$$\int_{\mathcal{N}'} \mathrm{e}^{\mathrm{i}\langle x, \xi \rangle} \, \mathrm{d}\mu_1(x) = \mathrm{e}^{-\frac{1}{2}\langle \xi, \xi \rangle}, \quad \xi \in \mathcal{N}.$$

The corresponding L^p spaces of complex-valued functions are denoted by

$$L^p(\mu_1) := L^p(\mathcal{N}', \mu_1; \mathbb{C}), \quad p \geq 1,$$

with corresponding norms $\|\cdot\|_{L^p(\mu_1)}$. By $\mathbb{E}_{\mu_1}(f) := \int_{\mathcal{N}'} f(x) \, \mathrm{d}\mu_1(x)$ we define the expectation of $f \in L^p(\mu_1)$.

Remark 1.3.2. The non-standard notation μ_1 for the Gaussian measure is chosen since we anticipate the Mittag-Leffler measures μ_β, $0 < \beta \leq 1$, from Chapter 3. The choice $\beta = 1$ will give exactly the Gaussian measure as defined above. Moreover the index 1 distinguishes the Gaussian measure μ_1 from the more general, possibly non-Gaussian, measure μ considered in Chapter 2.

Remark 1.3.3. The measure μ_1 has some useful properties which we collect in this remark. For details we refer to [Oba94]:

(*i*) The image measure of μ_1 under the mapping

$$\mathcal{N}' \ni x \mapsto (\langle x, \xi_1 \rangle, \ldots, \langle x, \xi_n \rangle) \in \mathbb{R}^n,$$

where $\xi_1, \ldots, \xi_n \in \mathcal{N}$ is an orthonormal system with respect to $(\cdot, \cdot)$, is the standard Gaussian measure on $\mathbb{R}^n$, i.e. the measure with density

$$\frac{1}{\sqrt{2\pi}^n} \exp\left(-\frac{1}{2}(x_1^2 + \ldots + x_n^2)\right).$$

(*ii*) Let $f_1, \ldots, f_n$ be integrable functions on $\mathbb{R}$ with respect to the 1-dimensional Gaussian measure and let $\xi_1, \ldots, \xi_n \in \mathcal{N}$ be an orthogonal system. Then

$$\int_{\mathcal{N}'} f_1(\langle x, \xi_1 \rangle) \cdot \ldots \cdot f_n(\langle x, \xi_n \rangle) \, \mathrm{d}\mu_1(x) = \prod_{i=1}^{n} \int_{\mathcal{N}'} f_i(\langle x, \xi_i \rangle) \, \mathrm{d}\mu_1(x).$$

(*iii*) The mapping

$$\mathcal{N}_{\mathbb{C}} \ni \xi \mapsto \langle \cdot, \xi \rangle \in L^2(\mu_1)$$

is an isometry between $\mathcal{H}$ and $L^2(\mu_1)$, i.e.

$$\int_{\mathcal{N}'} |\langle x, \xi \rangle|^2 \, \mathrm{d}\mu_1(x) = |\xi|^2, \quad \xi \in \mathcal{N}_{\mathbb{C}}.$$

(*iv*) Using the preceding isometry it is possible to extend the dual pairing $\langle \cdot, \cdot \rangle$ to $\mathcal{N}' \times \mathcal{H}$. Indeed let $\eta \in \mathcal{H}$ and choose a sequence $(\xi_n)_{n \in \mathbb{N}}$, $\xi_n \in \mathcal{N}$ for all $n \in \mathbb{N}$, such that $\lim_{n\to\infty} |\eta - \xi_n| = 0$. Then

$$\|\langle \cdot, \xi_n \rangle - \langle \cdot, \xi_m \rangle\|_{L^2(\mu_1)} = |\xi_n - \xi_m|, \quad n, m \in \mathbb{N}.$$

Hence $(\langle \cdot, \xi_n \rangle)_{n \in \mathbb{N}}$ is a Cauchy sequence in $L^2(\mu_1)$ and thus convergent. We define $\langle \cdot, \eta \rangle := \lim_{n\to\infty} \langle \cdot, \xi_n \rangle$ in $L^2(\mu_1)$.

(*v*) For all $\eta, \theta \in \mathcal{H}_{\mathbb{C}}$ and for all $n \in \mathbb{N}$ it holds

$$\int_{\mathcal{N}'} \langle x, \eta \rangle^{2n} \, \mathrm{d}\mu_1(x) = \frac{(2n)!}{2^n n!} \langle \eta, \eta \rangle^2,$$

$$\int_{\mathcal{N}'} \langle x, \eta \rangle^{2n+1} \, \mathrm{d}\mu_1(x) = 0,$$

$$\int_{\mathcal{N}'} \mathrm{e}^{\langle x, \eta \rangle} \, \mathrm{d}\mu_1(x) = \exp\left(\frac{1}{2}\langle \eta, \eta \rangle\right),$$

$$\int_{\mathcal{N}'} \langle x, \eta \rangle \langle x, \theta \rangle \, \mathrm{d}\mu_1(x) = \langle \eta, \theta \rangle.$$

Example 1.3.4. Choose the nuclear triple

$$\mathcal{S}(\mathbb{R}) \subset L^2(\mathbb{R}, \mathrm{d}x) \subset \mathcal{S}'(\mathbb{R})$$

as in Example 1.1.4. We define the stochastic process $B_t := \langle \cdot, \mathbf{1}_{[0,t)} \rangle \in L^2(\mu_1)$ for $t > 0$ and $B_0 = 0$. Then using Remark 1.3.3 we get that $\mathbb{E}_{\mu_1}(B_t) = 0$ and the covariance of B_t is given by

$$\int_{\mathcal{S}'(\mathbb{R})} B_t(\omega) B_s(\omega) \, \mathrm{d}\mu_1(\omega) = \int_{\mathbb{R}} \mathbf{1}_{[0,t)}(x) \mathbf{1}_{[0,s)}(x) \, \mathrm{d}x = \min(t, s), \quad t, s > 0.$$

Note that the variable of integration $x \in \mathcal{N}'$ is denoted in the white noise case by $\omega \in \mathcal{S}'(\mathbb{R})$. Moreover for $A \in \mathcal{B}(\mathbb{R})$:

$$\mu_1(B_t \in A) = \frac{1}{\sqrt{2\pi t}} \int_A \exp\left(-\frac{1}{2t} s^2\right) \mathrm{d}s,$$

This shows that B_t is a centered Gaussian process with covariance $\min(t, s)$. Thus a continuous version of B_t is a Brownian motion.

1.3.2 Polynomials on $\mathcal{N}'$

Now we introduce the smooth polynomials on $\mathcal{N}'$. These are an easy but important example for square integrable functions on $\mathcal{N}'$. They are later on used in order to construct more convenient subspaces of $L^2(\mu_1)$.

Definition 1.3.5. The space of *smooth polynomials* $\mathcal{P}(\mathcal{N}')$ on $\mathcal{N}'$ is the space consisting of finite linear combinations of functions of the form $\langle \cdot, \xi \rangle^n$, where $\xi \in \mathcal{N}_{\mathbb{C}}$ and $n \in \mathbb{N}$. Every smooth polynomial φ has a representation

$$\varphi(x) = \sum_{n=0}^{N} \langle x^{\otimes n}, \varphi^{(n)} \rangle, \quad x \in \mathcal{N}',$$

where $N \in \mathbb{N}$ and $\varphi^{(n)}$ is a finite sum of elements of the form $\xi^{\otimes n}$, $\xi \in \mathcal{N}_{\mathbb{C}}$.

$\mathcal{P}(\mathcal{N}')$ is an elementary set of functions in $L^2(\mu_1)$, but polynomials are not useful since they do not satisfy an orthogonality relation. Thus using an orthogonalization procedure we find the so-called *Wick-ordered polynomials*, denoted by $\langle : \cdot^{\otimes n} :, \xi^{\otimes n} \rangle$, $\xi \in \mathcal{N}_{\mathbb{C}}$. Instead of repeating the construction of Wick-ordered polynomials, we give a descriptive definition using the Hermite polynomials H_n, $n \in \mathbb{N}$, on $\mathbb{R}$. Then $\langle : \cdot^{\otimes n} :, \xi^{\otimes n} \rangle$ is given by

$$\langle : x^{\otimes n} :, \xi^{\otimes n} \rangle = \frac{|\xi|^n}{2^{n/2}} H_n\left(\frac{\langle x, \xi \rangle}{\sqrt{2}\,|\xi|}\right), \quad \xi \in \mathcal{N}_{\mathbb{C}}, \ \xi \neq 0, \ x \in \mathcal{N}'.$$

For details on Hermite polynomials we refer to Appendix A.1 in [HKPS93].

Remark 1.3.6. The following properties of smooth and Wick-ordered polynomials are valid. For proofs see e.g. [Oba94].

(*i*) Each smooth polynomial in $\mathcal{P}(\mathcal{N}')$ can be expressed as a Wick-ordered polynomial, i.e. for all $\varphi \in \mathcal{P}(\mathcal{N}')$ there exist $\varphi^{(n)} \in \mathcal{N}_{\mathbb{C}}^{\hat{\otimes} n}$, $n = 0, \ldots, N$, such that

$$\varphi(x) = \sum_{n=0}^{N} \langle : x^{\otimes n} :, \varphi^{(n)} \rangle, \quad x \in \mathcal{N}'.$$

(*ii*) The Wick-ordered polynomials satisfy the following orthogonality relation: For $\xi, \eta \in \mathcal{N}_{\mathbb{C}}$ it holds

$$\int_{\mathcal{N}'} \langle :x^{\otimes n}:, \xi^{\otimes n}\rangle \langle :x^{\otimes m}:, \eta^{\otimes m}\rangle \, \mathrm{d}\mu_1(x) = n! \langle \xi, \eta\rangle^n \delta_{n,m}, \quad n, m \in \mathbb{N}.$$

(*iii*) Both, the smooth polynomials and the Wick-ordered polynomials are a dense subset in $L^2(\mu_1)$.

Using that $\mathcal{P}(\mathcal{N}') \subset L^2(\mu_1)$ is dense, the Wiener-Itô-Segal-Chaos decomposition can be proven:

Theorem 1.3.7. *Let $F \in L^2(\mu_1)$. Then there exists $f^{(n)} \in \mathcal{H}_{\mathbb{C}}^{\widehat{\otimes} n}$, $n \in \mathbb{N}$, such that*

$$F = \sum_{n=0}^{\infty} \langle :\cdot^{\otimes n}:, f^{(n)}\rangle$$

in the $L^2(\mu_1)$-sense. In particular:

$$\|F\|^2_{L^2(\mu_1)} = \sum_{n=0}^{\infty} n! \left| f^{(n)} \right|^2.$$

An important example of a square integrable function is the so-called Wick exponential:

Example 1.3.8. The Wick exponential $:\exp(\langle \cdot, \xi\rangle):$ for $\xi \in \mathcal{N}_{\mathbb{C}}$ is defined by

$$:\exp(\langle x, \xi\rangle): := \frac{\exp(\langle x, \xi\rangle)}{\mathbb{E}_{\mu_1}(\exp(\langle \cdot, \xi\rangle))} = \exp\left(\langle x, \xi\rangle - \frac{1}{2}\langle \xi, \xi\rangle\right), \quad x \in \mathcal{N}'.$$

The Wiener-Itô-Chaos decomposition of the Wick exponential is given as

$$:\exp(\langle \cdot, \xi\rangle):= \sum_{n=0}^{\infty} \frac{1}{n!} \langle :\cdot^{\otimes n}:, \xi^{\otimes n}\rangle, \quad \xi \in \mathcal{N}_{\mathbb{C}}.$$

1.3.3 Test functions and distributions

In order to introduce the topological dual space of the smooth polynomials we equip the space $\mathcal{P}(\mathcal{N}')$ with the natural topology such that the mapping

$$\varphi = \sum_{n=0}^{\infty} \langle \cdot^{\otimes n}, \varphi^{(n)}\rangle \longleftrightarrow \vec{\varphi} = \left\{\varphi^{(n)} : n \in \mathbb{N}\right\}$$

becomes a topological isomorphism from $\mathcal{P}(\mathcal{N}')$ to the topological direct sum of tensor powers $\mathcal{N}_{\mathbb{C}}^{\widehat{\otimes} n}$, i.e.

$$\mathcal{P}(\mathcal{N}') \simeq \bigoplus_{n=0}^{\infty} \mathcal{N}_{\mathbb{C}}^{\widehat{\otimes} n}$$

(note that $\varphi^{(n)} \neq 0$ only for finitely many $n \in \mathbb{N}$). By the choice of this topology the convergence in $\mathcal{P}(\mathcal{N}')$ is given as follows: A sequence $(\varphi_k)_{k \in \mathbb{N}}$ in $\mathcal{P}(\mathcal{N}')$ converges to $\varphi \in \mathcal{P}(\mathcal{N}')$ if and only if

(i) $\varphi_k = \sum_{n=0}^{N(k)} \langle \cdot^{\otimes n}, \varphi_k^{(n)} \rangle$, $k \in \mathbb{N}$, and $\varphi = \sum_{n=0}^{N} \langle \cdot^{\otimes n}, \varphi^{(n)} \rangle$.

(ii) The sequence $(N(k))_{k\in\mathbb{N}}$ is bounded.

(iii) $\varphi_k^{(n)} \to \varphi^{(n)}$ in $\mathcal{N}_{\mathbb{C}}^{\otimes n}$ for all $n \in \mathbb{N}$ if $k \to \infty$.

Then we introduce the dual space of $\mathcal{P}(\mathcal{N}')$ with respect to the above described topology and denote it by $\mathcal{P}'_{\mu_1}(\mathcal{N}')$. We obtain the chain of spaces

$$\mathcal{P}(\mathcal{N}') \subset L^2(\mu_1) \subset \mathcal{P}'_{\mu_1}(\mathcal{N}')$$

by identifying $L^2(\mu_1)$ with its dual space. The dual pairing between $\mathcal{P}'_{\mu_1}(\mathcal{N}')$ and $\mathcal{P}(\mathcal{N}')$ is denoted by $\langle\!\langle \cdot, \cdot \rangle\!\rangle_{\mu_1}$ and for $F \in L^2(\mu_1)$ and $\varphi \in \mathcal{P}(\mathcal{N}')$ it holds

$$\langle\!\langle F, \varphi \rangle\!\rangle_{\mu_1} = (\bar{F}, \varphi)_{L^2(\mu_1)}.$$

Formally we give a chaos decomposition for $\Phi \in \mathcal{P}'_{\mu_1}(\mathcal{N}')$. Consider $\Phi^{(n)} \in \left(\mathcal{N}_{\mathbb{C}}^{\hat{\otimes} n}\right)'$ and let $I(\Phi^{(n)})$ act on $\mathcal{P}(\mathcal{N}')$ as

$$I(\Phi^{(n)})(\varphi) = n! \langle \Phi^{(n)}, \varphi^{(n)} \rangle,$$

where $\varphi = \sum_{n=0}^{N} \langle :\cdot^{\otimes n}:, \varphi^{(n)} \rangle \in \mathcal{P}(\mathcal{N}')$ with $\varphi^{(n)} \in \mathcal{N}_{\mathbb{C}}^{\hat{\otimes} n}$. Obviously $I(\Phi^{(n)})$ is linear. Let now $\varphi_k \to \varphi$ in $\mathcal{P}(\mathcal{N}')$ as k tends to infinity. Then $\varphi_k^{(n)} \to \varphi^{(n)}$ in $\mathcal{N}_{\mathbb{C}}^{\otimes n}$ for $k \to \infty$ and thus

$$\left| I(\Phi^{(n)})(\varphi_k) - I(\Phi^{(n)})(\varphi) \right| = n! \left| \langle \Phi^{(n)}, \varphi_k^{(n)} - \varphi^{(n)} \rangle \right| \to 0$$

as $k \to \infty$. Hence we obtain that $I(\Phi^{(n)}) \in \mathcal{P}'_{\mu_1}(\mathcal{N}')$. Formally we denote $I(\Phi^{(n)})$ by $\langle :\cdot^{\otimes n}:, \Phi^{(n)} \rangle$. Denote $\Phi = \sum_{n=0}^{\infty} \langle :\cdot^{\otimes n}:, \Phi^{(n)} \rangle$, acting on $\mathcal{P}(\mathcal{N}')$ via

$$\Phi(\varphi) = \sum_{n=0}^{N} n! \langle \Phi^{(n)}, \varphi^{(n)} \rangle$$

for $\varphi = \sum_{n=0}^{N} \langle :\cdot^{\otimes n}:, \varphi^{(n)} \rangle \in \mathcal{P}(\mathcal{N}')$. Continuity of Φ follows from the continuity of $I(\Phi^{(n)})$ and thus $\Phi \in \mathcal{P}'_{\mu_1}(\mathcal{N}')$.

Conversely let $\Phi \in \mathcal{P}'_{\mu_1}(\mathcal{N}')$ be given and let $\varphi = \sum_{n=0}^{N} \langle :\cdot^{\otimes n}:, \varphi^{(n)} \rangle \in \mathcal{P}(\mathcal{N}')$ with $\varphi^{(n)} \in \mathcal{N}_{\mathbb{C}}^{\hat{\otimes} n}$. Then the mapping

$$\mathcal{P}(\mathcal{N}') \ni \varphi \mapsto \langle\!\langle \Phi, \langle :\cdot^{\otimes n}:, \varphi^{(n)} \rangle \rangle\!\rangle_{\mu_1} \in \mathbb{C}$$

is continuous for all $n \in \mathbb{N}$ and applying the kernel theorem, see Remark 1.1.3, we obtain the existence of $\Phi^{(n)} \in \left(\mathcal{N}_{\mathbb{C}}^{\hat{\otimes} n}\right)'$ such that

$$\langle\!\langle \Phi, \langle :\cdot^{\otimes n}:, \varphi^{(n)} \rangle \rangle\!\rangle_{\mu_1} = n! \langle \Phi^{(n)}, \varphi^{(n)} \rangle.$$

Then the mapping $\mathcal{P}(\mathcal{N}') \ni \varphi \mapsto \sum_{n=0}^{\infty} n! \langle \Phi^{(n)}, \varphi^{(n)} \rangle$ is linear and continuous and defines an element in $\mathcal{P}'_{\mu_1}(\mathcal{N}')$ which we denote by $\sum_{n=0}^{\infty} \langle :\cdot^{\otimes n}:, \Phi^{(n)} \rangle$. By construction we see that

$$\langle\!\langle \Phi, \varphi \rangle\!\rangle_{\mu_1} = \sum_{n=0}^{\infty} n! \langle \Phi^{(n)}, \varphi^{(n)} \rangle = \langle\!\langle \sum_{n=0}^{\infty} \langle :\cdot^{\otimes n}:, \Phi^{(n)} \rangle, \varphi \rangle\!\rangle_{\mu_1}$$

and thus $\Phi = \sum_{n=0}^{\infty}\langle :\cdot^{\otimes n}:, \Phi^{(n)}\rangle$. This shows that $\Phi \in \mathcal{P}'_{\mu_1}(\mathcal{N}')$ if and only if there exists $\Phi^{(n)} \in \left(\mathcal{N}_{\mathbb{C}}^{\widehat{\otimes} n}\right)'$ for each $n \in \mathbb{N}$ and $\Phi = \sum_{n=0}^{\infty}\langle :\cdot^{\otimes n}:, \Phi^{(n)}\rangle$, i.e.

$$\mathcal{P}'_{\mu_1}(\mathcal{N}') = \left\{ \Phi = \sum_{n=0}^{\infty}\langle :\cdot^{\otimes n}:, \Phi^{(n)}\rangle \;\middle|\; \Phi^{(n)} \in \left(\mathcal{N}_{\mathbb{C}}^{\widehat{\otimes} n}\right)' \right\}. \tag{1.2}$$

The distribution space $\mathcal{P}'_{\mu_1}(\mathcal{N}')$ is not appropriate for applications. In literature there are many ways to construct smaller and more convenient distribution spaces. One possibility is as follows, see e.g. [KS93]:

For each $p, q \in \mathbb{N}$ and $0 \le \rho \le 1$ define the norm

$$\|F\|^2_{p,q,\rho,\mu_1} := \sum_{n=0}^{\infty}(n!)^{(1+\rho)}2^{nq}\left|f^{(n)}\right|_p^2, \quad F \in L^2(\mu_1),$$

for uniquely determined $f^{(n)} \in \mathcal{H}_{\mathbb{C}}^{\widehat{\otimes} n}$ coming from the Chaos decomposition, see Theorem 1.3.7. Then the space

$$(\mathcal{H}_p)^{\rho}_{q,\mu_1} := \left\{F \in L^2(\mu_1) \;\middle|\; \|F\|_{p,q,\rho,\mu_1} < \infty\right\}$$

is a Hilbert space and it coincides with the completion of $\mathcal{P}(\mathcal{N}')$ with respect to $\|\cdot\|_{p,q,\rho,\mu_1}$. The space of test functions $(\mathcal{N})^{\rho}_{\mu_1}$ is then given as

$$(\mathcal{N})^{\rho}_{\mu_1} := \operatorname*{pr\,lim}_{p,q\in\mathbb{N}}(\mathcal{H}_p)^{\rho}_{q,\mu_1}.$$

Remark 1.3.9. The common notation in literature for the spaces of test functions and distributions uses β instead of ρ, see e.g. [KS93, Kuo96]. The notation is changed here since the parameter β is reserved for the Mittag-Leffler measures μ_β in Chapter 3.

The dual space of $(\mathcal{H}_p)^{\rho}_{q,\mu_1}$ with respect to $L^2(\mu_1)$ is denoted by $(\mathcal{H}_{-p})^{-\rho}_{-q,\mu_1}$. This space is given by

$$(\mathcal{H}_{-p})^{-\rho}_{-q,\mu_1} := \left\{ \Phi = \sum_{n=0}^{\infty}\langle :\cdot^{\otimes n}:, \Phi^{(n)}\rangle \in \mathcal{P}'(\mathcal{N}') \;\middle|\; \sum_{n=0}^{\infty}(n!)^{(1-\rho)}2^{-nq}\left|\Phi^{(n)}\right|_{-p}^2 < \infty \right\},$$

where we denote

$$\|\Phi\|^2_{-p,-q,-\rho,\mu_1} := \sum_{n=0}^{\infty}(n!)^{(1-\rho)}2^{-nq}\left|\Phi^{(n)}\right|_{-p}^2.$$

It can be shown that $\|\cdot\|_{-p,-q,-\rho,\mu_1}$ coincides with the operator norm in $(\mathcal{H}_{-p})^{-\rho}_{-q,\mu_1}$. We define the distribution spaces $(\mathcal{N})^{-\rho}_{\mu_1}$ by

$$(\mathcal{N})^{-\rho}_{\mu_1} := \operatorname*{ind\,lim}_{p,q\in\mathbb{N}}(\mathcal{H}_{-p})^{-\rho}_{-q,\mu_1}.$$

By construction $(\mathcal{N})^{\rho}_{\mu_1}$ is a countably Hilbert space. Moreover $(\mathcal{N})^{\rho}_{\mu_1}$ is nuclear, see e.g. [Oba94]. Thus all properties from Remark 1.1.3 apply also to $(\mathcal{N})^{\rho}_{\mu_1}$ and $(\mathcal{N})^{-\rho}_{\mu_1}$. In particular, $(\mathcal{N})^{-\rho}_{\mu_1}$ is the dual space of $(\mathcal{N})^{\rho}_{\mu_1}$. Finally we have constructed the chain of spaces

$$(\mathcal{N})^{\rho}_{\mu_1} \subset (\mathcal{H}_p)^{\rho}_{q,\mu_1} \subset L^2(\mu_1) \subset (\mathcal{H}_{-p})^{-\rho}_{-q,\mu_1} \subset (\mathcal{N})^{-\rho}_{\mu_1},$$

where $0 \le \rho \le 1$. For different $\rho \in [0,1]$ we get

$$(\mathcal{N})^1_{\mu_1} \subset (\mathcal{N})^\rho_{\mu_1} \subset (\mathcal{N})_{\mu_1} \subset L^2(\mu_1) \subset (\mathcal{N})'_{\mu_1} \subset (\mathcal{N})^{-\rho}_{\mu_1} \subset (\mathcal{N})^{-1}_{\mu_1}.$$

Here $(\mathcal{N})$ and $(\mathcal{N})'$ denote the spaces $(\mathcal{N})^\rho_{\mu_1}$ and $(\mathcal{N})^{-\rho}_{\mu_1}$ for $\rho = 0$, respectively.

Example 1.3.10. In the white noise case, see Example 1.1.4, the spaces $(\mathcal{N})_{\mu_1}$ and $(\mathcal{N})'_{\mu_1}$ are usually referred to as Hida spaces of test functions and distributions and denoted by $(\mathcal{S})$ and $(\mathcal{S})'$, respectively.

1.3.4 S_{μ_1}-transform and characterization

This subsection is concerned with the characterization of the above given distribution spaces. Indeed it is possible to give a one-to-one correspondence between the spaces $(\mathcal{N})^{-\rho}_{\mu_1}$ and spaces of holomorphic functions on $\mathcal{N}_{\mathbb{C}}$. The isomorphism realizing this correspondence is called S_{μ_1}-transform.

Recall the definition of the Wick exponential from Example 1.3.8. We calculate its norm for $p, q, \in \mathbb{N}$:

$$\| :\exp(\langle\cdot,\xi\rangle): \|^2_{p,q,\rho,\mu_1} = \sum_{n=0}^\infty (n!)^{1+\rho} 2^{nq} \left| \frac{1}{n!}\xi^{\otimes n}\right|_p^2 = \sum_{n=0}^\infty (n!)^{\rho-1} 2^{nq} \left|\xi\right|_p^{2n}, \quad \xi \in \mathcal{N}_{\mathbb{C}}.$$

This norm is finite for all $\xi \in \mathcal{N}_{\mathbb{C}}$ and for all $p, q \in \mathbb{N}$ if $\rho < 1$. Thus the Wick exponential $:\exp(\langle\cdot,\xi\rangle): \in (\mathcal{N})^\rho_{\mu_1}$ and we can define the S_{μ_1}*-transform* of $\Phi \in (\mathcal{N})^{-\rho}_{\mu_1}$ by:

$$(S_{\mu_1}\Phi)(\xi) := \langle\!\langle \Phi, :\exp(\langle\cdot,\xi\rangle): \rangle\!\rangle_{\mu_1}, \quad \xi \in \mathcal{N}_{\mathbb{C}}.$$

Let Φ be given via the generalized chaos decomposition by

$$\Phi = \sum_{n=0}^\infty \langle :\cdot^{\otimes n}:, \Phi^{(n)}\rangle$$

for $\Phi^{(n)} \in \mathcal{N}'^{\otimes n}_{\mathbb{C}}$, see (1.2). Then its S_{μ_1}-transform is calculated to be

$$(S_{\mu_1}\Phi)(\xi) := \sum_{n=0}^\infty \langle \Phi^{(n)}, \xi^{\otimes n}\rangle. \tag{1.3}$$

Remark 1.3.11. (i) In the case $\rho = 1$ the situation is a bit different. In general, $:\exp(\langle\cdot,\xi\rangle): \notin (\mathcal{N})^1_{\mu_1}$. But if $\xi \in \mathcal{N}_{\mathbb{C}}$ is chosen from a neighborhood of zero, i.e.

$$\xi \in U_{p,q} = \left\{ \xi \in \mathcal{N}_{\mathbb{C}} \mid 2^q \left|\xi\right|_p^2 < 1 \right\},$$

then $:\exp(\langle\cdot,\xi\rangle): \in (\mathcal{H}_p)^1_{q,\mu_1}$. Now let $\Phi \in (\mathcal{N})^{-1}_{\mu_1} = \cup_{p,q\in\mathbb{N}} (\mathcal{H}_{-p})^{-1}_{-q,\mu_1}$. Then there exists $p, q \in \mathbb{N}$ such that $\Phi \in (\mathcal{H}_{-p})^{-1}_{-q,\mu_1}$ and we define

$$(S_{\mu_1}\Phi)(\xi) := \langle\!\langle \Phi, :\exp(\langle\cdot,\xi\rangle): \rangle\!\rangle_{\mu_1}, \quad \xi \in U_{p,q}.$$

For the case $\rho = 1$ the S_{μ_1}-transform of Φ is defined only locally in a neighborhood of zero, whereas the S_{μ_1}-transform of $\Phi \in (\mathcal{N})^{-\rho}_{\mu_1}$ for $\rho \neq 1$ is defined on whole $\mathcal{N}_{\mathbb{C}}$. We will meet this fact again in the characterization theorems. Indeed the case $\rho = 1$ will require less conditions than the case $\rho \neq 1$.

(ii) Similar to the S_{μ_1}-transform we define the T_{μ_1}-*transform* for $\Phi \in (\mathcal{N})^{-\rho}_{\mu_1}$, $\rho < 1$, by

$$(T_{\mu_1}\Phi)(\xi) := \langle\langle \Phi, \exp(\mathrm{i}\langle \cdot, \xi\rangle)\rangle\rangle_{\mu_1}, \quad \xi \in \mathcal{N}_{\mathbb{C}}.$$

By the definition of the Wick exponential, see Example 1.3.8, the T_{μ_1}-transform and the S_{μ_1}-transform are related for all $\xi \in \mathcal{N}_{\mathbb{C}}$ via

$$(S_{\mu_1}\Phi)(\mathrm{i}\xi) = \langle\langle \Phi, :\exp(\langle \cdot, \mathrm{i}\xi\rangle):\rangle\rangle_{\mu_1} = \langle\langle \Phi, \exp(\mathrm{i}\langle \cdot, \xi\rangle)\rangle\rangle_{\mu_1} \exp\left(\frac{1}{2}|\xi|^2\right).$$

Thus

$$(T_{\mu_1}\Phi)(\xi) = (S_{\mu_1}\Phi)(\mathrm{i}\xi) \exp\left(-\frac{1}{2}|\xi|^2\right), \quad \xi \in \mathcal{N}_{\mathbb{C}}. \tag{1.4}$$

In the case $\rho = 1$ let $\Phi \in (\mathcal{H}_{-p})^{-1}_{-q,\mu_1}$. Then $(T_{\mu_1}\Phi)(\xi)$ is well defined for all $\xi \in U_{p,q}$ and (1.4) holds.

(iii) If $F \in L^2(\mu_1)$ then $\mathbb{E}_{\mu_1}(F) = \int_{\mathcal{N}'} F(x)\,\mathrm{d}\mu_1(x) = \langle\langle F, 1\rangle\rangle_{\mu_1}$, where 1 denotes the function which is constantly 1. Note that $\int_{\mathcal{N}'} F(x)\,\mathrm{d}\mu_1(x) < \infty$ by the Hölder inequality. This gives rise to the definition of the generalized expectation: For $\Phi \in (\mathcal{N})^{-\rho}_{\mu_1}$, $\rho \in [0,1]$, we define the *generalized expectation* of Φ by

$$\mathbb{E}_{\mu_1}(\Phi) := (T_{\mu_1}\Phi)(0).$$

The importance of the S_{μ_1}-transform can be seen from the next theorem. In fact, a distribution is in $(\mathcal{N})^{-\rho}_{\mu_1}$ if and only if its S_{μ_1}-transform is ray-analytic and exponentially bounded. This result has two important corollaries concerning convergence and integration in $(\mathcal{N})^{-\rho}_{\mu_1}$. For the proofs we refer to [Kuo96], Theorem 8.2 for the characterization theorem, Theorem 8.6 concerning the convergence and Theorem 13.4 and 13.5 for the integration in the sense of Pettis and Bochner, respectively.

Theorem 1.3.12. *Let $0 \le \rho < 1$ and $\Phi \in (\mathcal{N})^{-\rho}_{\mu_1}$. Then its S_{μ_1}-transform satisfies the following:*

(i) For all $\xi, \eta \in \mathcal{N}_{\mathbb{C}}$ the function $\mathbb{C} \ni z \mapsto (S_{\mu_1}\Phi)(z\xi + \eta) \in \mathbb{C}$ is an entire function.

(ii) There exist constants $K, a > 0$ and $p \in \mathbb{N}$ such that

$$|(S_{\mu_1}\Phi)(\xi)| \le K \exp\left(a\,|\xi|_p^{\frac{2}{1-\rho}}\right), \quad \xi \in \mathcal{N}_{\mathbb{C}}.$$

Conversely, if $F\colon \mathcal{N}_{\mathbb{C}} \to \mathbb{C}$ satisfies (i) and (ii) then there exists a unique $\Phi \in (\mathcal{N})^{-\rho}_{\mu_1}$ such that $F = S_{\mu_1}\Phi$.

Remark 1.3.13. Because of (1.4) $S_{\mu_1}\Phi$ is holomorphic and exponentially bounded if and only if $T_{\mu_1}\Phi$ is holomorphic and exponentially bounded. Thus the above characterization theorem and all succeeding corollaries are still valid if the S_{μ_1}-transform is replaced by the T_{μ_1}-transform.

The characterization theorems have useful corollaries characterizing the convergent sequences in $(\mathcal{N})^{-\rho}_{\mu_1}$.

Corollary 1.3.14. Let $(\Phi_n)_{n\in\mathbb{N}}$ be a sequence in $(\mathcal{N})^{-\rho}_{\mu_1}$, $0 \le \rho < 1$. Then $(\Phi_n)_{n\in\mathbb{N}}$ converges strongly in $(\mathcal{N})^{-\rho}_{\mu_1}$ if and only if (*i*) For each $\xi \in \mathcal{N}_{\mathbb{C}}$ the sequence $((S_{\mu_1}\Phi_n)(\xi))_{n\in\mathbb{N}}$ is a Cauchy sequence.

(*ii*) There exist constants $K, a < \infty$ and $p \in \mathbb{N}$ such that for all $n \in \mathbb{N}$ and for all $\xi \in \mathcal{N}_{\mathbb{C}}$

$$|(S_{\mu_1}\Phi_n)(\xi)| \leq K \exp\left(a\,|\xi|_p^{\frac{2}{1-\rho}}\right).$$

Example 1.3.15. Consider a Brownian motion $B_t = \langle \cdot, \mathbf{1}_{[0,t)}\rangle$, $t \geq 0$, from Example 1.3.4. For calculating the S_{μ_1}-transform of B_t apply (1.3). Then

$$(S_{\mu_1}B_t)(\xi) = \langle \mathbf{1}_{[0,t)}, \xi\rangle = \int_0^t \xi(s)\,\mathrm{d}s, \quad \xi \in \mathcal{S}_{\mathbb{C}}(\mathbb{R}).$$

It is easy to see that a Brownian motion is differentiable. Denote

$$\Phi_n = \frac{B_{t+h_n} - B_t}{h_n} \in L^2(\mu_1),$$

where $(h_n)_{n\in\mathbb{N}}$ is a sequence in $[0,\infty)$ and $\lim_{n\to\infty} h_n = 0$. Corollary 1.3.14 yields that $(\Phi_n)_{n\in\mathbb{N}}$ converges in $(\mathcal{S})_{\mu_1}^{-\rho}$ for all $\rho \in [0,1)$. Indeed, the continuity of ξ implies that

$$(S_{\mu_1}\Phi_n)(\xi) = \frac{1}{h_n}\int_t^{t+h_n} \xi(s)\,\mathrm{d}s \to \xi(t), \quad n \to \infty.$$

This shows in particular that $(S_{\mu_1}\Phi_n)_{n\in\mathbb{N}}$ is a Cauchy sequence. Moreover there holds the estimate

$$|(S_{\mu_1}\Phi_n)(\xi)| \leq \mathrm{e}^{(1+\rho)/2}\exp\left(C\frac{1-\rho}{2}\,|\xi|_p^{2/(1-\rho)}\right), \quad \xi \in \mathcal{S}_{\mathbb{C}}(\mathbb{R}),$$

for some $C > 0$ and $p \in \mathbb{N}$. This proves the convergence of $(S_{\mu_1}\Phi_n)_{n\in\mathbb{N}}$ in $(\mathcal{S})_{\mu_1}^{-\rho}$.

The next two corollaries of the characterization theorem give sufficient conditions for functions $f\colon T \to (\mathcal{N})_{\mu_1}^{-\rho}$ to be Bochner integrable or Pettis integrable:

Corollary 1.3.16. Let $(T, \mathcal{B}, \nu)$ be a measure space and $\Phi.\colon T \to (\mathcal{N})_{\mu_1}^{-\rho}$, $0 \leq \rho < 1$. Suppose that Φ fulfils

(*i*) $(S_{\mu_1}\Phi.)(\xi)$ is measurable for all $\xi \in \mathcal{N}_{\mathbb{C}}$.

(*ii*) There exist $p \in \mathbb{N}$ and non-negative functions $K \in L^1(T,\nu)$ and $b \in L^\infty(T,\nu)$ such that for ν-almost all $t \in T$

$$|(S_{\mu_1}\Phi_t)(\xi)| \leq K(t)\exp\left(b(t)\,|\xi|_p^{\frac{2}{1-\rho}}\right), \quad \xi \in \mathcal{N}_{\mathbb{C}}.$$

Then there exist $p, q \in \mathbb{N}$ such that $\Phi_t \in (\mathcal{H}_{-p})_{-q,\mu_1}^{-\rho}$ for ν-almost all $t \in T$ and Φ is Bochner integrable with Bochner integral $\int_T \Phi_t\,\mathrm{d}\nu(t) \in (\mathcal{H}_{-p})_{-q,\mu_1}^{-\rho}$ satisfying

$$\left(S_{\mu_1}\int_T \Phi_t\,\mathrm{d}\nu(t)\right)(\xi) = \int_T (S_{\mu_1}\Phi_t)(\xi)\,\mathrm{d}\nu(t), \quad \xi \in \mathcal{N}.$$

Corollary 1.3.17. Let $(T, \mathcal{B}, \nu)$ be a measure space and $\Phi_\cdot : T \to (\mathcal{N})_{\mu_1}^{-\rho}$, $0 \leq \rho < 1$. Suppose that Φ fulfils

(*i*) $(S_{\mu_1}\Phi_\cdot)(\xi)$ is measurable for all $\xi \in \mathcal{N}_\mathbb{C}$.

(*ii*) There exist $p \in \mathbb{N}$ and constants $K, a < \infty$ such that

$$\int_T |(S_{\mu_1}\Phi_t)(\xi)| \, \mathrm{d}\nu(t) \leq K \exp\left(a\, |\xi|_p^{\frac{2}{1-\rho}}\right), \quad \xi \in \mathcal{N}_\mathbb{C}.$$

Then Φ is Pettis integrable with Pettis integral $\int_T \Phi_t \, \mathrm{d}\nu(t) \in (\mathcal{N})_{\mu_1}^{-\rho}$ satisfying

$$\left(S_{\mu_1} \int_T \Phi_t \, \mathrm{d}\nu(t)\right)(\xi) = \int_T (S_{\mu_1}\Phi_t)(\xi) \, \mathrm{d}\nu(t), \quad \xi \in \mathcal{N}.$$

Example 1.3.18. For $\eta \in \mathcal{H}$ and $a \in \mathbb{R}$ the mapping

$$\mathbb{R} \ni \lambda \mapsto \Phi(\lambda) = \frac{1}{2\pi} \int_\mathbb{R} \mathrm{e}^{\mathrm{i}\lambda(\langle \cdot, \eta \rangle - a)} \, \mathrm{d}\lambda \in (\mathcal{N})'_{\mu_1}$$

satisfies the assumptions of Corollary 1.3.16. This shows that Φ is Bochner integrable and we denote the Bochner integral by

$$\delta_a(\langle \cdot, \eta \rangle) = \frac{1}{2\pi} \int_\mathbb{R} \mathrm{e}^{\mathrm{i}\lambda(\langle \cdot, \eta \rangle - a)} \, \mathrm{d}\lambda.$$

The T_{μ_1}-transform of $\delta_a(\langle \cdot, \eta \rangle)$ can be calculated to be

$$\left(T_{\mu_1}\delta_a\langle \cdot, \eta \rangle\right)(\xi) = \frac{1}{\sqrt{2\pi\, |\eta|^2}} \exp\left(-\frac{1}{2}\, |\xi|^2\right) \exp\left(-\frac{1}{2\, |\eta|^2}(a - \mathrm{i}\langle \eta, \xi \rangle)^2\right).$$

In the white noise case, see Example 1.1.4, we choose $\eta = \mathbf{1}_{[0,t)}$ and obtain Donsker's delta $\delta_a(B_t) \in (\mathcal{S})'$ which is the composition of Dirac delta and Brownian motion. In particular, since $\left|\mathbf{1}_{[0,t)}\right|^2_{L^2(\mathbb{R}, \mathrm{d}x)} = t$, we get for the T_{μ_1}-transform of Donsker's delta

$$\left(T_{\mu_1}\delta_a B_t\right)(\xi) = \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{1}{2}\, |\xi|^2\right) \exp\left(-\frac{1}{2t}(a - \mathrm{i}\langle \mathbf{1}_{[0,t)}, \xi \rangle)^2\right).$$

The characterization for the case $\rho = 1$ is a bit different since the S_{μ_1}-transform is only defined on some neighborhood of zero, see Remark 1.3.11. Condition (*i*) in the characterization theorem 1.3.12 for $\xi, \eta \in \mathcal{U} \subset \mathcal{N}_\mathbb{C}$ is then exactly the definition of G-holomorphy, see Section 1.1.2, and condition (*ii*) means that the S_{μ_1}-transform of Φ is locally bounded. This yields the following theorem, see [KLS96] for a proof of the following three results.

Theorem 1.3.19. *For every $\Phi \in (\mathcal{N})_{\mu_1}^{-1}$ it holds that $S_{\mu_1}\Phi \in \mathrm{Hol}_0(\mathcal{N}_\mathbb{C})$. Conversely if $F \in \mathrm{Hol}_0(\mathcal{N}_\mathbb{C})$ then there is a unique $\Phi \in (\mathcal{N})_{\mu_1}^{-1}$ such that $F = S_{\mu_1}\Phi$. Moreover if F is holomorphic for all $\xi \in U_{p,l}$ and if $p' > p$ with $\|i_{p',p}\|_{HS} < \infty$ and $q \in \mathbb{N}$ satisfying $\kappa = 2^{2l-q}e^2\|i_{p',p}\|^2_{HS} < 1$ then $\Phi \in (\mathcal{H}_{-p'})_{-q,\mu_1}^{-1}$ and*

$$\|\Phi\|_{-p',-q,\mu_1} \leq n_{p,l,\infty}(F)(1-\kappa)^{-1/2}.$$

Corollary 1.3.20. Let $(\Phi_n)_{n\in\mathbb{N}}$ be a sequence in $(\mathcal{N})^{-1}_{\mu_1}$. Then $(\Phi_n)_{n\in\mathbb{N}}$ converges strongly in $(\mathcal{N})^{-1}_{\mu_1}$ if and only if there exist $p, q \in \mathbb{N}$ such that

(*i*) For each $\xi \in U_{p,q}$ the sequence $((S_{\mu_1}\Phi_n)(\xi))_{n\in\mathbb{N}}$ is a Cauchy sequence.

(*ii*) There exist a constant $K < \infty$ such that for all $n \in \mathbb{N}$ and for all $\xi \in U_{p,q}$

$$|(S_{\mu_1}\Phi_n)(\xi)| \leq K.$$

Corollary 1.3.21. Let $(T, \mathcal{B}, \nu)$ be a measure space and $\Phi. \colon T \to (\mathcal{N})^{-1}_{\mu_1}$. Suppose that there exist $p, q \in \mathbb{N}$ such that Φ fulfils

(*i*) $(S_{\mu_1}\Phi.)(\xi)$ is measurable for all $\xi \in U_{p,q}$.

(*ii*) There exist a non-negative function $L \in L^1(T, \nu)$ such that for ν-almost all $t \in T$

$$|(S_{\mu_1}\Phi_t)(\xi)| \leq L(t), \quad \xi \in U_{p,q}.$$

Then there exist $p', q' \in \mathbb{N}$ such that $\Phi_t \in (\mathcal{H}_{-p'})^{-1}_{-q',\mu_1}$ for ν-almost all $t \in T$ and Φ is Bochner integrable with Bochner integral $\int_T \Phi_t \, \mathrm{d}\nu(t) \in (\mathcal{H}_{-p'})^{-1}_{-q',\mu_1}$ satisfying

$$\left(S_{\mu_1} \int_T \Phi_t \, \mathrm{d}\nu(t)\right)(\xi) = \int_T (S_{\mu_1}\Phi_t)(\xi) \, \mathrm{d}\nu(t), \quad \xi \in U_{p',q'}.$$

1.4 Application to the Feynman-Kac formula

In this section, we give a short idea on the interplay between analysis and probability theory. Our focus lies on the so-called Feynman-Kac formula [Kac49] which expresses the solution of the heat equation in terms of Brownian motion. With methods from white noise analysis, we obtain in this way a Green's function to the heat equation:

$$\begin{aligned} \frac{\partial}{\partial t} u(t,x) &= \frac{1}{2}\frac{\partial^2}{\partial x^2} u(t,x) \\ u(0,x) &= u_0(x) \end{aligned} \tag{1.5}$$

for $t > 0$, $x \in \mathbb{R}$ and some initial value $u_0 \colon \mathbb{R} \to \mathbb{R}$. A function K is called a *Green's function* to (1.5) if the function u defined by

$$u(t,x) = \int_{\mathbb{R}} K(t,x,y) u_0(y) \, \mathrm{d}y, \quad t > 0,\ x \in \mathbb{R},$$

is a solution of the heat equation (1.5). Formally the Green's function K is a solution to (1.5) with initial value $u_0(x) = \delta_y(x)$. Assume that u_0 is bounded and continuous and satisfies

$$\int_{\mathbb{R}} \mathrm{e}^{-ax^2} |u_0(x)| \, \mathrm{d}x < \infty$$

for some $a > 0$. Then it is well known that a Green's function to (1.5) is given by the so-called heat kernel

$$K(t,x,y) = \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{1}{2t}(x-y)^2\right), \quad 0 < t < 1/(2a),\ x, y \in \mathbb{R},$$

i.e. the solution u reads as

$$u(t,x) = \frac{1}{\sqrt{2\pi t}} \int_{\mathbb{R}} u_0(y) \exp\left(-\frac{1}{2t}(x-y)^2\right) \,\mathrm{d}y = \mathbb{E}_{\mu_1}\left(u_0(x+B_t)\right)$$

where B_t denotes a Brownian motion starting at $B_0 = 0$. Using Example 1.3.18 the expression $\delta(x + B_t)$ makes sense in $(\mathcal{S})'_{\mu_1}$ and its expectation is given by

$$\mathbb{E}_{\mu_1}\left(\delta(x+B_t)\right) = \left(T_{\mu_1}\delta(x+B_t)\right)(0) = \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{x^2}{2t}\right).$$

Thus we see that $\mathbb{E}_{\mu_1}(\delta_y(x + B_t))$ coincides with the heat kernel and is therefore a Green's function to (1.5).

The inhomogeneous heat equation

Consider for an inhomogeneity $f\colon (0,\infty) \times \mathbb{R} \to \mathbb{R}$ the equation

$$\frac{\partial}{\partial t} u(t,x) = \frac{1}{2}\frac{\partial^2}{\partial x^2} u(t,x) + f(t,x), \quad u(0,x) = u_0(x),$$

for $t > 0$ and $x \in \mathbb{R}$ and suitable initial value u_0. It is shown for example in [KS92] that a solution u admits the stochastic representation

$$u(t,x) = \mathbb{E}_{\mu_1}\left(u_0(x+B_t) + \int_0^t f(t-\theta, x+B_t)\,\mathrm{d}\theta\right), \quad t > 0,\ x \in \mathbb{R}, \tag{1.6}$$

provided that the following conditions are fulfilled: u_0 and f are continuous and there exist constants $K, T > 0$ and $0 < a < 1/(2T)$ such that the solution u satisfies the estimate

$$\max_{0\le t\le T} |u(t,x)| + \max_{0\le t\le T} |f(t,x)| \le K\mathrm{e}^{ax^2}.$$

With the help of white noise analysis we may choose $u_0 = \delta_y$ for $y \in \mathbb{R}$ and obtain the Green's function to the inhomogeneous heat equation by

$$K^f(t,x,y) = \mathbb{E}_{\mu_1}\left(\delta_y(x+B_t) + \int_0^t f(t-\theta, x+B_t)\,\mathrm{d}\theta\right), \quad t > 0,\ x \in \mathbb{R}.$$

The heat equation with potential

Now let $u\colon [0,\infty) \times \mathbb{R} \to \mathbb{R}$ be continuously differentiable on $(0,\infty)$ such that $\frac{\partial}{\partial t}u$ can be continuously extended to 0 and assume that u is two times continuously differentiable on $\mathbb{R}$ fulfilling

$$\frac{\partial}{\partial t} u(t,x) = \frac{1}{2}\frac{\partial^2}{\partial x^2} u(t,x) + V(x)u(t,x), \quad u(0,x) = u_0(x),$$

with continuous initial value $u_0\colon \mathbb{R} \to \mathbb{R}$ and bounded and continuous potential $V\colon \mathbb{R} \to (-\infty, 0]$. Suppose that there exist $K > 0$ and $T > 0$ and $0 < a < \frac{1}{2T}$ with

$$\max_{0\le t\le T} |u(t,x)| \le K\mathrm{e}^{ax^2}$$

then u has the stochastic representation

$$u(t,x) = \mathbb{E}\left(u_0(x+B_t)\exp\left(\int_0^t V(x+B_s)\,\mathrm{d}s\right)\right), \quad 0 \le t \le T,\ x \in \mathbb{R},$$

see e.g. Chapter 4, Theorem 4.2 in [KS91]. The Green's function K_V can be found for example in [HT94] and it is given by

$$K_V(t,x,y) = \frac{1}{\sqrt{2\pi t}}\exp\left(-\frac{(x-y)^2}{2t}\right)\mathbb{E}_{\mu_1}\left(\exp\left(\int_0^t V(BB_s^{x,y,t})\,\mathrm{d}s\right)\right).$$

Here $BB^{x,y,t}$ denotes a Brownian Bridge starting for $s=0$ at x and ending in y for $s=t$, given by

$$BB_s^{x,y,t} = x + \frac{s}{t}(y-x) + B_s - \frac{s}{t}B_t.$$

With methods from white noise analysis it is possible to choose u_0 to be Dirac delta δ_y and interpret the expectation as

$$\mathbb{E}_{\mu_1}\left(\delta_y(x+B_t)\exp\left(\int_0^t V(x+B_s)\,\mathrm{d}s\right)\right) = \langle\langle \delta_y(x+B_t), \exp\left(\int_0^t V(x+B_s)\,\mathrm{d}s\right)\rangle\rangle_{\mu_1}.$$

This dual pairing can be defined rigorously using regular test functions and distributions and a so-called Wick formula [Vog10]. Then for all measurable potentials V such that $\exp\left(\int_0^t V(x+B_s)\,\mathrm{d}s\right)$ is a regular test function it holds that

$$K_V(t,x,y) = \mathbb{E}_{\mu_1}\left(\delta_y(x+B_t)\exp\left(\int_0^t V(x+B_s)\,\mathrm{d}s\right)\right).$$

Chapter 2

Non-Gaussian Analysis

In Gaussian analysis, an important role is played by the Wick-ordered polynomials since they form an orthogonal system with respect to the L^2-scalar product, see Section 1.3 and in particular Remark 1.3.6. Thereby one obtains the Wiener-Itô-Chaos decomposition. In non-Gaussian analysis it might happen that Wick-ordered polynomials do not exist, for an example see Theorem 3.1.13 below. Instead of Wick-ordered polynomials, one uses in non-Gaussian analysis the so-called Appell systems which are biorthogonal systems consisting of Appell polynomials and the corresponding distributions. Such systems were first proposed by Daletskii in [Dal91] for probability measures with smooth logarithmic derivative. More details and results, including characterization theorems, were obtained in [ADKS96]. In the following we give the construction from [KSWY98] where the conditions on the measure μ could be weakened. We also refer to [KK99] for further results.

2.1 Generalized Appell systems

Again we work on a nuclear triple $\mathcal{N} \subset \mathcal{H} \subset \mathcal{N}'$ as described in Section 1.1 and we consider a probability measure μ defined on the cylinder σ-algebra of $\mathcal{N}'$. For the definition of the cylinder σ-algebra see Section 1.3. This measure shall fulfill the following two properties:

(A1) The Laplace transform of μ is an analytic function, i.e. the mapping

$$\mathcal{N}_{\mathbb{C}} \ni \xi \mapsto l_\mu(\xi) = \int_{\mathcal{N}'} \mathrm{e}^{\langle x,\xi\rangle}\, \mathrm{d}\mu(x) \in \mathbb{C}$$

is holomorphic in a neighborhood $\mathcal{U} \subset \mathcal{N}_{\mathbb{C}}$ of zero.

(A2) The measure μ is non-degenerate or, more precisely, positive on open sets, i.e.

$$\mu(\mathcal{U}) > 0 \quad \text{for all open and non-empty} \quad \mathcal{U} \subset \mathcal{N}'.$$

Instead of (A2) [KSWY98] require μ to be non-degenerate, which is a weaker assumption. But in [KK99] it is shown that this assumption is not sufficient to guarantee the embedding of test functions into the space of square integrable functions.

Example 2.1.1. The Gaussian measure μ_1 from Section 1.3 satisfies

$$l_{\mu_1}(\xi) = \int_{\mathcal{N}'} \mathrm{e}^{\langle x,\xi\rangle}\, \mathrm{d}\mu_1(x) = \exp\left(\frac{1}{2}\langle \xi,\xi\rangle\right), \quad \xi \in \mathcal{N}_{\mathbb{C}}.$$

Thus l_{μ_1} is holomorphic in a neighborhood $\mathcal{U} \subset \mathcal{N}_{\mathbb{C}}$ of zero and μ_1 fulfils assumption (A1). By Remark 1.3.3 the image measure of μ_1 is the standard Gaussian measure. Hence also (A2) is satisfied, see for example Theorem 2.2 in [Kuo96].

Remark 2.1.2. For a measure μ on $\mathcal{N}'$ the following are equivalent:

(*i*) μ satisfies (A1).

(*ii*) There exist $p \in \mathbb{N}$ and $C > 0$ such that

$$\left| \int_{\mathcal{N}'} \langle x, \xi \rangle^n \, \mathrm{d}\mu(x) \right| \leq n! C^n \left| \xi \right|_p^n, \quad \xi \in \mathcal{H}_{p,\mathbb{C}}.$$

(*iii*) There exist $p \in \mathbb{N}$ and $\varepsilon > 0$ such that for all $n \in \mathbb{N}$

$$\int_{\mathcal{N}'} \mathrm{e}^{\varepsilon |x|_{-p}} \, \mathrm{d}\mu(x) < \infty.$$

For the proof see [KSWY98].

2.1.1 The biorthogonal system

We repeat in the following the approach of [KSWY98] for the construction of the biorthogonal system with respect to the measure μ. For more details and proofs the reader is referred to [KSWY98] and the references therein.

To give a representation for smooth polynomials $\varphi \in \mathcal{P}(\mathcal{N}')$, see Definition 1.3.5, in terms of Appell polynomials we first introduce the *μ-exponential* by

$$e_\mu(\xi; x) := \frac{\mathrm{e}^{\langle x, \xi \rangle}}{l_\mu(\xi)}, \quad x \in \mathcal{N}', \; \xi \in \mathcal{N}_{\mathbb{C}}. \tag{2.1}$$

This expression is well defined if and only if $l_\mu(\xi) \neq 0$. Note that $l_\mu(0) = 1$ and l_μ is holomorphic. This implies that there exists a neighborhood $\mathcal{U}_0 \subset \mathcal{N}_{\mathbb{C}}$ of zero such that $l_\mu(\xi) \neq 0$ for all $\xi \in \mathcal{U}_0$. For $\xi \in \mathcal{U}_0$ the μ-exponential is thus well defined and holomorphic. Moreover e_μ can be expanded in a power series, i.e.

$$e_\mu(\xi; x) = \sum_{n=0}^{\infty} \frac{1}{n!} \langle P_n^\mu(x), \xi^{\otimes n} \rangle, \quad x \in \mathcal{N}', \; \xi \in \mathcal{U}_0, \tag{2.2}$$

for suitable $P_n^\mu(x) \in \left(\mathcal{N}_{\mathbb{C}}^{\hat{\otimes} n} \right)'$. By

$$\mathbb{P}^\mu = \left\{ \langle P_n^\mu(\cdot), \xi^{(n)} \rangle \mid \xi^{(n)} \in \mathcal{N}_{\mathbb{C}}^{\hat{\otimes} n}, \; n \in \mathbb{N} \right\}$$

we denote the so-called *$\mathbb{P}^\mu$-system* of polynomials on $\mathcal{N}'$. The elements in $\mathbb{P}^\mu$ are called *Appell polynomials*. It is known that $\mathbb{P}^\mu$ coincides with $\mathcal{P}(\mathcal{N}')$.

Remark 2.1.3. If μ_1 is the Gaussian measure from Section 1.3 then it holds that $l_{\mu_1}(\xi) = \exp\left(\frac{1}{2}\langle \xi, \xi \rangle\right)$ for $\xi \in \mathcal{N}_{\mathbb{C}}$. In this case the μ_1-exponential is given by

$$e_{\mu_1}(\xi, x) = \exp\left(\langle x, \xi \rangle - \frac{1}{2} \langle \xi, \xi \rangle \right), \quad x \in \mathcal{N}', \; \xi \in \mathcal{N}_{\mathbb{C}}.$$

A comparison with Example 1.3.8 shows that in this case the μ_1-exponential coincides with the Wick exponential. The series expansion of the Wick exponential is obtained via the Wick-ordered polynomials. Thus the Appell polynomials in the Gaussian case are the Wick-ordered polynomials.

Now consider the triple consisting of smooth polynomials $\mathcal{P}(\mathcal{N}')$, the square integrable functions on $\mathcal{N}'$ denoted by $L^2(\mu)$ and the dual space of the polynomials

$$\mathcal{P}(\mathcal{N}') \subset L^2(\mu) \subset \mathcal{P}'_\mu(\mathcal{N}'),$$

see Section 1.3 for further details. Note that if μ is a measure on a Hilbert space then assumption (A1) ensures that $\mathcal{P}(\mathcal{N}') \subset L^2(\mu)$ is dense, see [Sko74]. The next theorem shows that this property also holds for measures on $\mathcal{N}'$:

Theorem 2.1.4. *The set of all smooth polynomials $\mathcal{P}(\mathcal{N}')$ is dense in $L^2(\mu)$.*

Proof. Assume that $F \in L^2(\mathcal{N}', \mu; \mathbb{R})$ with $(F, \varphi)_{L^2(\mu)} = 0$ for all $\varphi \in \mathcal{P}(\mathcal{N}')$. We will show that $F(x) = 0$ for μ-almost all $x \in \mathcal{N}'$. First note that by Remark 2.1.2 there exist ε_μ and $p_\mu \in \mathbb{N}$ such that

$$\int_{\mathcal{N}'} \mathrm{e}^{\varepsilon_\mu |x|_{-p_\mu}} \,\mathrm{d}\mu(x) < \infty.$$

Let $0 \neq \xi \in \mathcal{N}$ and choose $\varepsilon := \varepsilon_\mu / (4\, |\xi|_{p_\mu})$. Then for $z \in B_\varepsilon(0) = \{z \in \mathbb{C} \mid |z| < \varepsilon\}$ it holds:

$$|\exp(\mathrm{i}z\langle x, \xi\rangle)|^2 \leq \exp\left(2\,|z|\,|x|_{-p_\mu}\,|\xi|_{p_\mu}\right) \leq \exp\left(\frac{1}{2}\varepsilon_\mu\,|x|_{-p_\mu}\right).$$

From this we conclude that $\exp(\mathrm{i}z\langle \cdot, \xi\rangle) \in L^2(\mu)$ for all $z \in B_\varepsilon(0)$ and we may define

$$h_\xi(z) := \int_{\mathcal{N}'} \mathrm{e}^{\mathrm{i}z\langle x,\xi\rangle} F(x) \,\mathrm{d}\mu(x), \quad z \in B_\varepsilon(0).$$

First we show that h_ξ is continuous. Indeed, let $z_n \to z$ as $n \to \infty$ and $z_n, z \in B_\varepsilon(0)$ for all $n \in \mathbb{N}$. Then the integrand

$$\left|F(x)\mathrm{e}^{\mathrm{i}z_n\langle x,\xi\rangle}\right| \leq |F(x)| \exp\left(\frac{1}{4}\varepsilon_\mu\,|x|_{-p_\mu}\right)$$

is dominated by an integrable function for all $n \in \mathbb{N}$. We conclude that h_ξ is continuous by Lebesgue's dominated convergence. Next we verify that h_ξ is holomorphic in $B_\varepsilon(0)$. Let γ be any closed curve in $B_\varepsilon(0)$. Since γ is compact it holds

$$\begin{aligned}
&\int_\gamma \int_{\mathcal{N}'} \left|F(x)\mathrm{e}^{\mathrm{i}z\langle x,\xi\rangle}\right| \,\mathrm{d}\mu(x)\,\mathrm{d}z \\
&\leq \int_\gamma \int_{\mathcal{N}'} |F(x)| \exp\left(\frac{1}{4}\varepsilon_\mu\,|x|_{-p_\mu}\right) \mathrm{d}\mu(x)\,\mathrm{d}z \\
&\leq \int_\gamma \left(\int_{\mathcal{N}'} |F(x)|^2 \,\mathrm{d}\mu(x)\right)^{1/2} \left(\int_{\mathcal{N}'} \exp\left(\frac{1}{2}\varepsilon_\mu\,|x|_{-p_\mu}\right) \mathrm{d}\mu(x)\right)^{1/2} \mathrm{d}z \\
&< \infty.
\end{aligned}$$

Applying Fubini's theorem we see that $\int_\gamma h_\xi(z)\,\mathrm{d}z = 0$ and hence h_ξ is holomorphic by Morera. In fact, h_ξ is holomorphic on a bigger domain. Consider the complex set $D := \{z \in \mathbb{C} \mid |\Im(z)| < \varepsilon\}$. Then it holds for all $z \in D$ that

$$|\exp(\mathrm{i}z\langle\cdot,\xi\rangle)|^2 \leq \exp(|2\Im(z)|\,|\langle\cdot,\xi\rangle|) \leq \exp\left(\frac{1}{2}\varepsilon_\mu\,|\cdot|_{-p_\mu}\right) \in L^1(\mu)$$

and thus h_ξ is well defined for all $z \in D$. The same arguments as above show that h_ξ is holomorphic on D. Now we calculate $\frac{\mathrm{d}^n}{\mathrm{d}z^n}h_\xi(0)$. Let $z \in B_\varepsilon(0)$. Then the integrand

$$\frac{\mathrm{d}^{n-1}}{\mathrm{d}z^{n-1}}\mathrm{e}^{\mathrm{i}z\langle x,\xi\rangle}F(x) = (\mathrm{i}\langle x,\xi\rangle)^{n-1}\mathrm{e}^{\mathrm{i}z\langle x,\xi\rangle}F(x)$$

is integrable for all $z \in B_\varepsilon(0)$. Indeed, it holds that

$$\begin{aligned}&\left|\frac{\mathrm{d}^{n-1}}{\mathrm{d}z^{n-1}}\mathrm{e}^{\mathrm{i}z\langle x,\xi\rangle}F(x)\right| \leq \varepsilon^{1-n}\,|\varepsilon\langle x,\xi\rangle|^{n-1}\,\mathrm{e}^{|z||\langle x,\xi\rangle|}\,|F(x)|\\ &\leq \varepsilon^{1-n}\mathrm{e}^{\varepsilon|\langle x,\xi\rangle|}\mathrm{e}^{\varepsilon|\langle x,\xi\rangle|}\,|F(x)| \leq \varepsilon^{1-n}\left(\mathrm{e}^{2\varepsilon\langle x,\xi\rangle} + \mathrm{e}^{-2\varepsilon\langle x,\xi\rangle}\right)|F(x)| =: g(x).\end{aligned}$$

g is integrable by Hölder's inequality since $F \in L^2(\mu)$ and $\exp(2\varepsilon\langle\cdot,\xi\rangle) \in L^2(\mu)$. Furthermore the n-th derivative is given by

$$\frac{\mathrm{d}^n}{\mathrm{d}z^n}\mathrm{e}^{\mathrm{i}z\langle x,\xi\rangle}F(x) = (\mathrm{i}\langle x,\xi\rangle)^n\mathrm{e}^{\mathrm{i}z\langle x,\xi\rangle}F(x)$$

and also dominated by $g \in L^1(\mu)$. An application of Lebesgue's dominated convergence theorem then yields

$$\int_{\mathcal{N}'}\frac{\mathrm{d}^n}{\mathrm{d}z^n}\mathrm{e}^{\mathrm{i}z\langle x,\xi\rangle}F(x)\,\mathrm{d}\mu(x) = \frac{\mathrm{d}}{\mathrm{d}z}\int_{\mathcal{N}'}\frac{\mathrm{d}^{n-1}}{\mathrm{d}z^{n-1}}\mathrm{e}^{\mathrm{i}z\langle x,\xi\rangle}F(x)\,\mathrm{d}\mu(x).$$

Recursively we obtain for all $n \in \mathbb{N}$

$$\frac{\mathrm{d}^n}{\mathrm{d}z^n}h_\xi(z) = \int_{\mathcal{N}'}(\mathrm{i}\langle x,\xi\rangle)^n\mathrm{e}^{\mathrm{i}z\langle x,\xi\rangle}F(x)\,\mathrm{d}\mu(x), \quad z \in B_\varepsilon(0),$$

and in particular $\frac{\mathrm{d}^n}{\mathrm{d}z^n}h_\xi(0) = \mathrm{i}^n\int_{\mathcal{N}'}\langle x,\xi\rangle^n F(x)\,\mathrm{d}\mu(x) = 0$ because $\langle\cdot,\xi\rangle^n \in \mathcal{P}(\mathcal{N}')$. From $h_\xi(0) = 0$ we conclude that $h_\xi(z) = 0$ for all $z \in B_\varepsilon(0)$. By the identity principle $h_\xi(z) = 0$ for all $z \in D$ and in particular it holds

$$C_F(\xi) := h_\xi(1) = \int_{\mathcal{N}'}\mathrm{e}^{\mathrm{i}\langle x,\xi\rangle}F(x)\,\mathrm{d}\mu(x) = 0.$$

Assume now that $F \neq 0$. Without loss of generality the positive part F^+ of F satisfies $\int_{\mathcal{N}'}F^+(x)\,\mathrm{d}\mu(x) = 1$. Define two measures on $\mathcal{N}'$ by

$$\mu_{F^+}(A) = \int_A F^+(x)\,\mathrm{d}\mu(x), \quad \mu_{F^-}(A) = \int_A F^-(x)\,\mathrm{d}\mu(x), \quad A \in \mathcal{C}_\sigma(\mathcal{N}').$$

Calculating the characteristic functions of $\mu_{F^\pm}$ we find

$$\begin{aligned}C_{\mu_{F^+}}(\xi) &= \int_{\mathcal{N}'}\mathrm{e}^{\mathrm{i}\langle x,\xi\rangle}\,\mathrm{d}\mu_{F^+}(x) = \int_{\mathcal{N}'}\mathrm{e}^{\mathrm{i}\langle x,\xi\rangle}F^+(x)\,\mathrm{d}\mu(x)\\ &= \int_{\mathcal{N}'}\mathrm{e}^{\mathrm{i}\langle x,\xi\rangle}(F(x) + F^-(x))\,\mathrm{d}\mu(x)\\ &= C_F(\xi) + \int_{\mathcal{N}'}\mathrm{e}^{\mathrm{i}\langle x,\xi\rangle}\,\mathrm{d}\mu_{F^-}(x) = 0 + C_{\mu_{F^-}}(\xi), \quad \xi \in \mathcal{N}.\end{aligned}$$

Hence $\mu_{F^+} = \mu_{F^-}$ by the Theorem of Bochner and Minlos. Finally we obtain

$$\begin{aligned}|F|^2_{L^2(\mu)} &= \int_{\mathcal{N}'} (F^+(x) - F^-(x))^2 \,\mathrm{d}\mu(x)\\ &= \int_{\mathcal{N}'} F^+F^+ \,\mathrm{d}\mu - \int_{\mathcal{N}'} F^+F^- \,\mathrm{d}\mu - \int_{\mathcal{N}'} F^-F^+ \,\mathrm{d}\mu + \int_{\mathcal{N}'} F^-F^- \,\mathrm{d}\mu\\ &= \int_{\mathcal{N}'} F^+ \,\mathrm{d}\mu_{F^+} - \int_{\mathcal{N}'} F^+ \,\mathrm{d}\mu_{F^-} - \int_{\mathcal{N}'} F^- \,\mathrm{d}\mu_{F^+} + \int_{\mathcal{N}'} F^- \,\mathrm{d}\mu_{F^-} = 0.\end{aligned}$$

This is a contradiction to the assumption that $F \neq 0$ and thus $F(x) = 0$ for μ-almost all $x \in \mathcal{N}'$. By the choice of $F \in L^2(\mathcal{N}', \mu; \mathbb{R})$ this implies that $\mathcal{P}(\mathcal{N}') \subset L^2(\mathcal{N}', \mu; \mathbb{R})$ is dense. For complex valued $F \in L^2(\mu)$ apply the real valued result to $\Re(F)$ and $\Im(F)$. This ends the proof. □

The aim is now to describe the distributions in $\mathcal{P}'_\mu(\mathcal{N}')$ in a similar way as the smooth polynomials, i.e. we find elements $\Phi^{(n)} \in \left(\mathcal{N}_{\mathbb{C}}^{\widehat{\otimes} n}\right)'$ and an operator Q^μ_n on $\left(\mathcal{N}_{\mathbb{C}}^{\widehat{\otimes} n}\right)'$, such that

$$\Phi = \sum_{n=0}^{\infty} Q^\mu_n \left(\Phi^{(n)}\right)$$

and moreover a certain biorthogonality relation should hold, see Theorem 2.1.5 below. To find this required Q^μ_n, consider a differential operator on $\mathcal{P}(\mathcal{N}')$ depending on $\Phi^{(n)} \in \left(\mathcal{N}_{\mathbb{C}}^{\widehat{\otimes} n}\right)'$ by

$$D\left(\Phi^{(n)}\right) \langle x^{\otimes m}, \xi^{(m)} \rangle := \begin{cases} \frac{m!}{(m-n)!} \langle x^{\otimes (m-n)} \hat{\otimes} \Phi^{(n)}, \xi^{(m)} \rangle, & m \geq n \\ 0, & m < n \end{cases}$$

for a monomial $\mathcal{N}' \ni x \mapsto \langle x^{\otimes m}, \xi^{(m)} \rangle$ with $\xi^{(m)} \in \mathcal{N}_{\mathbb{C}}^{\widehat{\otimes} m}$. If $\mathcal{N}$ is the space of Schwartz test functions $\mathcal{S}(\mathbb{R})$ and $\mathcal{H} = L^2(\mathbb{R}, \mathrm{d}x)$, then for $n = 1$ and $\Phi^{(1)} = \delta_t \in \mathcal{N}'_{\mathbb{C}}$ this differential operator coincides with the Hida derivative, see [HKPS93]. For each $\Phi^{(n)} \in \left(\mathcal{N}_{\mathbb{C}}^{\widehat{\otimes} n}\right)'$ the operator $D(\Phi^{(n)})$ is continuous from $\mathcal{P}(\mathcal{N}')$ to $\mathcal{P}(\mathcal{N}')$ and this enables us to define the dual operator $D(\Phi^{(n)})^* \colon \mathcal{P}'_\mu(\mathcal{N}') \to \mathcal{P}'_\mu(\mathcal{N}')$. We set $Q^\mu_n(\Phi^{(n)}) = D(\Phi^{(n)})^* \mathbf{1}$ for $\Phi^{(n)} \in \left(\mathcal{N}_{\mathbb{C}}^{\widehat{\otimes} n}\right)'$ and denote the so-called $\mathbb{Q}^\mu$-*system* in $\mathcal{P}'_\mu(\mathcal{N}')$ by

$$\mathbb{Q}^\mu = \left\{ Q^\mu_n(\Phi^{(n)}) \,\middle|\, \Phi^{(n)} \in \left(\mathcal{N}_{\mathbb{C}}^{\widehat{\otimes} n}\right)', n \in \mathbb{N} \right\}.$$

For each $\Phi \in \mathcal{P}'_\mu(\mathcal{N}')$ there is a unique sequence of kernels $\left(\Phi^{(n)}\right)_{n\in\mathbb{N}}$ in $\left(\mathcal{N}_{\mathbb{C}}^{\widehat{\otimes} n}\right)'$, such that

$$\Phi = \sum_{n=0}^{\infty} Q^\mu_n(\Phi^{(n)}). \tag{2.3}$$

Vice versa, every such sum is a generalized function in $\mathcal{P}'_\mu(\mathcal{N}')$. The pair $(\mathbb{P}^\mu, \mathbb{Q}^\mu)$ is called *Appell system* $\mathbb{A}^\mu$ generated by the measure μ. By the next theorem, the main achievement of Appell systems becomes apparent since it proves a biorthogonality relation between $\mathbb{P}^\mu$ and $\mathbb{Q}^\mu$.

Theorem 2.1.5. *For* $\Phi^{(n)} \in \left(\mathcal{N}_{\mathbb{C}}^{\hat{\otimes} n}\right)'$ *and* $\xi^{(m)} \in \mathcal{N}_{\mathbb{C}}^{\hat{\otimes} m}$ *we have*

$$\langle\langle Q_n^\mu\left(\Phi^{(n)}\right), \langle P_m^\mu(\cdot), \xi^{(m)}\rangle\rangle\rangle_\mu = \delta_{m,n} n! \langle \Phi^{(n)}, \xi^{(n)}\rangle, \quad n, m \in \mathbb{N}_0.$$

Remark 2.1.6. Theorem 2.1.5 shows in particular, that the $\mathbb{Q}^\mu$-system in non-Gaussian analysis corresponds to the generalized chaos decomposition in Gaussian analysis, see (1.2). Comparing the biorthogonality relation with Remark 1.3.6 we see directly that the Appell system takes the place of the Wick-ordered polynomials.

2.1.2 Test functions and distributions

With the help of the Appell system $\mathbb{A}^\mu$, a test function and a distribution space can now be constructed. The procedure is similar to the construction of test functions and distributions in Section 1.3. There the kernels of the Wiener-Itô-Chaos decomposition were used to define new norms. In the non-Gaussian case, we use instead the decomposition in Appell polynomials. For $\varphi = \sum_{n=0}^{N} \langle P_n^\mu, \varphi^{(n)}\rangle \in \mathcal{P}(\mathcal{N}')$ we define for any $p, q \in \mathbb{N}_0$ the norm

$$\|\varphi\|_{p,q,\mu}^2 := \sum_{n=0}^{N} (n!)^2 2^{nq} \left|\varphi^{(n)}\right|_p^2.$$

By $(\mathcal{H}_p)_{q,\mu}^1$ we denote the completion of $\mathcal{P}(\mathcal{N}')$ with respect to $\|\cdot\|_{p,q,\mu}$. Under the condition (A2), [KK99] shows that there are $p', q' \in \mathbb{N}$, such that $(\mathcal{H}_p)_{q,\mu}^1$ can be topologically embedded in $L^2(\mu)$ for all $p > p', q > q'$. The test function space $(\mathcal{N})_\mu^1$ is defined as the projective limit of the spaces $\left((\mathcal{H}_p)_{q,\mu}^1\right)_{p,q\in\mathbb{N}}$, i.e.

$$(\mathcal{N})_\mu^1 := \underset{p,q\in\mathbb{N}}{\operatorname{pr\,lim}} \, (\mathcal{H}_p)_{q,\mu}^1.$$

This is a nuclear space, which is continuously embedded in $L^2(\mu)$ and it turns out that the test function space $(\mathcal{N})_\mu^1$ is the same for all measures μ satisfying (A1) and (A2), thus we will just use the notation $(\mathcal{N})^1$.

Example 2.1.7. For the μ-exponential $e_\mu(\xi;\cdot)$ defined in (2.2), its norm is given by $\|e_\mu(\xi;\cdot)\|_{p,q,\mu}^2 = \sum_{n=0}^{\infty} 2^{nq} |\xi|_p^{2n}$. Hence we get that the μ-exponential is not in $(\mathcal{N})^1$ if $\xi \neq 0$, but $e_\mu(\xi;\cdot) \in (\mathcal{H}_p)_{q,\mu}^1$ if $\xi \in U_{p,q} := \left\{\xi \in \mathcal{N}_{\mathbb{C}} \;\middle|\; 2^q |\xi|_p^2 < 1\right\}$. Moreover, the set

$$\left\{e_\mu(\xi;\cdot) \;\middle|\; 2^q |\xi|_p^2 < 1, \xi \in \mathcal{N}_{\mathbb{C}}\right\}$$

is total in $(\mathcal{H}_p)_{q,\mu}^1$.

The representation $\Phi = \sum_{n=0}^{\infty} Q_n^\mu(\Phi^{(n)}) \in \mathcal{P}_\mu'(\mathcal{N}')$ is used in order to define the norms

$$\|\Phi\|_{-p,-q,\mu}^2 := \sum_{n=0}^{\infty} 2^{-qn} \left|\Phi^{(n)}\right|_{-p}^2, \quad p, q \in \mathbb{N}_0.$$

By $(\mathcal{H}_{-p})_{-q,\mu}^{-1}$ we denote the set of all $\Phi \in \mathcal{P}_\mu'(\mathcal{N}')$ for which $\|\Phi\|_{-p,-q,\mu}$ is finite. It holds that $(\mathcal{H}_{-p})_{-q,\mu}^{-1}$ is the dual of $(\mathcal{H}_p)_{q,\mu}^1$. The space of distributions $(\mathcal{N})_\mu^{-1}$ is defined as the inductive limit

$$(\mathcal{N})_\mu^{-1} := \underset{p,q\in\mathbb{N}}{\operatorname{ind\,lim}} (\mathcal{H}_{-p})_{-q,\mu}^{-1}.$$

$(\mathcal{N})^{-1}_{\mu}$ is the dual of $(\mathcal{N})^1$ with respect to $L^2(\mu)$ and the dual pairing between a distribution $\Phi = \sum_{n=0}^{\infty} Q_n^{\mu}\left(\Phi^{(n)}\right)$ and a test function $\varphi = \sum_{n=0}^{\infty}\langle P_n^{\mu}, \varphi^{(n)}\rangle$ is given by Theorem 2.1.5 as

$$\langle\!\langle \Phi, \varphi \rangle\!\rangle_{\mu} = \sum_{n=0}^{\infty} n! \langle \Phi^{(n)}, \varphi^{(n)} \rangle.$$

We shall use the same notation for the dual pairing between $(\mathcal{H}_{-p})^{-1}_{-q,\mu}$ and $(\mathcal{H}_p)^1_{q,\mu}$. Finally we have constructed the nuclear triple

$$(\mathcal{N})^1 \subset L^2(\mu) \subset (\mathcal{N})^{-1}_{\mu}$$

which has the same properties as described in Remark 1.1.3.

2.1.3 Integral transforms and characterization

As in Gaussian analysis, we introduce the S_μ- and T_μ-transform on $(\mathcal{N})^{-1}_{\mu}$. By Remark 2.1.2 there is $p'_\mu \in \mathbb{N}$ and $\varepsilon_\mu > 0$ such that

$$\int_{\mathcal{N}'} \mathrm{e}^{\varepsilon_\mu |x|_{-p'_\mu}} \,\mathrm{d}\mu(x) < \infty.$$

Thus if $\xi \in \mathcal{V}_0 = \left\{\xi \in \mathcal{N}_{\mathbb{C}} \;\middle|\; 2\,|\xi|_{p'_\mu} \leq \varepsilon_\mu \right\}$ we have that $\mathrm{e}^{\langle \cdot, \xi\rangle} \in L^2(\mu)$. We define the Laplace transform of $f \in L^2(\mu)$ by

$$L_\mu f(\xi) := \int_{\mathcal{N}'} f(x) \mathrm{e}^{\langle x, \xi\rangle} \,\mathrm{d}\mu(x), \quad \xi \in \mathcal{V}_0.$$

Moreover we introduce for all $\xi \in \mathcal{U}_0 \cap \mathcal{V}_0$ the *S_μ-transform* of $f \in L^2(\mu)$ by

$$S_\mu f(\xi) := \frac{L_\mu f(\xi)}{l_\mu(\xi)} = \int_{\mathcal{N}'} f(x) e_\mu(\xi; x) \,\mathrm{d}\mu(x).$$

We would like to extend the definition of the S_μ-transform to $(\mathcal{N})^{-1}_{\mu}$. Note that for $\Phi \in (\mathcal{N})^{-1}_{\mu}$ there are $p, q \in \mathbb{N}$, such that $\Phi \in (\mathcal{H}_{-p})^{-1}_{-q,\mu}$. Moreover for $\xi \in U_{p,q}$ as in Example 2.1.7 we have that $e_\mu(\xi; \cdot) \in (\mathcal{H}_p)^1_{q,\mu}$. Thus we can define:

$$S_\mu \Phi(\xi) := \langle\!\langle \Phi, e_\mu(\xi; \cdot) \rangle\!\rangle_\mu, \quad \xi \in U_{p,q}. \tag{2.4}$$

If $\Phi = \sum_{n=0}^{\infty} Q_n^{\mu}(\Phi^{(n)})$ is given as in (2.3), we have by biorthogonality relation in Theorem 2.1.5

$$S_\mu \Phi(\xi) = \sum_{n=0}^{\infty} \langle \Phi^{(n)}, \xi^{\otimes n}\rangle, \quad \xi \in U_{p,q}. \tag{2.5}$$

Using the S_μ-transform and holomorphic functions on $\mathcal{N}_{\mathbb{C}}$, see Section 1.1, the space $(\mathcal{N})^{-1}_{\mu}$ can be characterized. The following theorem is proved in [KSWY98]:

Theorem 2.1.8. *The S_μ-transform is a topological isomorphism from $(\mathcal{N})^{-1}_{\mu}$ to* $\mathrm{Hol}_0(\mathcal{N}_{\mathbb{C}})$. *Moreover, if $F \in \mathrm{Hol}_0(\mathcal{N}_{\mathbb{C}})$, $F(\xi) = \sum_{n=0}^{\infty}\langle \Phi^{(n)}, \xi^{\otimes n}\rangle$ for all $\xi \in \mathcal{N}_{\mathbb{C}}$ with $|\xi|_p \leq 2^{-l}$ for some $l \in \mathbb{N}$ and if $p' > p$ with $\|i_{p',p}\|_{HS} < \infty$ and $q \in \mathbb{N}$ such that $\kappa := 2^{2l-q}\mathrm{e}^2 \|i_{p',p}\|^2_{HS} < 1$ then $\Phi = \sum_{n=0}^{\infty} Q_n^{\mu}(\Phi^{(n)}) \in (\mathcal{H}_{-p'})^{-1}_{-q}$ and*

$$\|\Phi\|_{-p',-q,\mu} \leq n_{p,l,\infty}(F)(1-\kappa)^{-1/2}.$$

We introduce now the T_μ-transform on $(\mathcal{N})_\mu^{-1}$ by the same means as in Gaussian analysis.

Lemma 2.1.9. Let $\Phi \in (\mathcal{N})_\mu^{-1}$ and $p, q \in \mathbb{N}$ such that $\Phi \in (\mathcal{H}_{-p})_{-q,\mu}^{-1}$. Then the T_μ*-transform* given by

$$T_\mu \Phi(\xi) := \langle\!\langle \Phi, \exp(\mathrm{i}\langle \cdot, \xi\rangle)\rangle\!\rangle_\mu$$

is well-defined for $\xi \in U_{p,q}$ as in Example 2.1.7 and we have:

$$T_\mu \Phi(\xi) = l_\mu(\mathrm{i}\xi) S_\mu \Phi(\mathrm{i}\xi).$$

In particular, $T_\mu \Phi \in \mathrm{Hol}_0(\mathcal{N}_\mathbb{C})$ if and only if $S_\mu \Phi \in \mathrm{Hol}_0(\mathcal{N}_\mathbb{C})$ and Theorem 2.1.8 also holds if the S_μ-transform is replaced by the T_μ-transform.

Proof. For $\Phi \in (\mathcal{H}_{-p})_{-q,\mu}^{-1}$ and $\xi \in U_{p,q}$ the S_μ-transform as in (2.4) is well-defined and we have:

$$S_\mu \Phi(\mathrm{i}\xi) = \langle\!\langle \Phi, e_\mu(\mathrm{i}\xi; \cdot)\rangle\!\rangle_\mu = \frac{1}{l_\mu(\mathrm{i}\xi)} \langle\!\langle \Phi, \exp(\mathrm{i}\langle \cdot, \xi\rangle)\rangle\!\rangle_\mu.$$

Since the S_μ- and the T_μ-transform differ only by a holomorphic factor from $\mathrm{Hol}_0(\mathcal{N}_\mathbb{C})$ and since $\mathrm{Hol}_0(\mathcal{N}_\mathbb{C})$ is an algebra, the second assertion follows immediately. □

Remark 2.1.10. As in Gaussian analysis it is possible to introduce subspaces of the distribution space $(\mathcal{N})_\mu^{-1}$, indexed with a parameter $\rho \in [0, 1)$. For $p, q \in \mathbb{N}$ let

$$(\mathcal{H}_{-p})_{-q,\mu}^{-\rho} = \left\{ \Phi = \sum_{n=0}^{\infty} Q_n^\mu(\Phi^{(n)}) \;\middle|\; \|\Phi\|_{-p,-q,-\rho,\mu}^2 = \sum_{n=0}^{\infty} (n!)^{1-\rho} 2^{-nq} \left|\Phi^{(n)}\right|_{-p}^2 < \infty \right\}$$

and $(\mathcal{N})_\mu^{-\rho} = \operatorname{ind\,lim}_{p,q\in\mathbb{N}} (\mathcal{H}_{-p})_{-q,\mu}^{-\rho}$. We end up with the chain of spaces

$$L^2(\mu) \subset (\mathcal{N})_\mu^{-0} \subset (\mathcal{N})_\mu^{-\rho} \subset (\mathcal{N})_\mu^{-1}, \quad 0 < \rho < 1.$$

We focus in the following on the largest distribution space $(\mathcal{N})_\mu^{-1}$. The developed analysis also works in the smaller spaces $(\mathcal{N})_\mu^{-\rho}$ with slight changes.

2.2 Characterization of integrable functions

In this section, we present several corollaries of the characterization theorem 2.1.8. These results are known in Gaussian analysis, see Section 1.3.4. We prove that the results can be carried over to the case where the measure μ might not be Gaussian but fulfils (A1) and (A2).

2.2.1 The Bochner integral

Theorem 2.2.1. *Let $(T, \mathcal{B}, \nu)$ be a measure space and $\Phi_\cdot \colon T \to (\mathcal{N})_\mu^{-1}$ such that there is a neighborhood $U_{p,q} \subset \mathcal{N}_\mathbb{C}$ of zero such that the following holds:*

(i) $S_\mu \Phi_\cdot(\xi) \colon T \to \mathbb{C}$ is measurable for all $\xi \in U_{p,q}$.

(ii) There exists $C \in L^1(T, \nu)$ such that $|S_\mu \Phi_t(\xi)| \le C(t)$ for all $\xi \in U_{p,q}$ and for ν-almost all $t \in T$.

Then there exist $p', q' \in \mathbb{N}$ such that

$$\int_T \Phi_t \, \mathrm{d}\nu(t) \text{ exists as a Bochner integral in } (\mathcal{H}_{-p'})^{-1}_{-q',\mu}$$

and

$$\left(S_\mu \int_T \Phi_t \, \mathrm{d}\nu(t) \right)(\xi) = \int_T (S_\mu \Phi_t)(\xi) \, \mathrm{d}\nu(t), \quad \xi \in U_{p',q'}.$$

Proof. Let $\Phi_t = \sum_{n=0}^\infty Q_n^\mu(\Phi_t^{(n)})$ be given as in (2.3). By assumption the S_μ-transform of Φ exists for all $\xi \in U_{p,q}$ and is given by

$$(S_\mu \Phi_t)(\xi) = \sum_{n=0}^\infty \langle \Phi_t^{(n)}, \xi^{\otimes n} \rangle,$$

see (2.5). From the characterization theorem 2.1.8 we get the existence of $p', q' \in \mathbb{N}$ such that the following estimate holds:

$$\begin{aligned} &\|\Phi_t\|_{-p',-q',\mu} \le n_{p,q,\infty}(S_\mu \Phi_t)(1-\kappa)^{-1/2} \\ &= \sup_{|\xi|_p \le 2^{-q}} |S_\mu \Phi_t(\xi)| \, (1-\kappa)^{-1/2} \le C(t)(1-\kappa)^{-1/2}. \end{aligned}$$

This shows the integrability of $\|\Phi_t\|_{-p',-q',\mu}$. In view of Remark 1.2.7 it is left to show that the mapping $T \ni t \mapsto \langle\!\langle \Phi_t, \varphi \rangle\!\rangle_\mu$ is measurable for all $\varphi \in (\mathcal{H}_{p'})^1_{q',\mu}$. In fact, we can use that the μ-exponentials are a total set in each $(\mathcal{H}_p)^1_{q,\mu}$. Let $(e_n)_{n\in\mathbb{N}}$ be a sequence in the linear span of the μ-exponentials $e_\mu(\xi; \cdot)$ with $\xi \in U_{p',q'}$ which converges to φ in $(\mathcal{H}_{p'})^1_{q',\mu}$. Then

$$\langle\!\langle \Phi_t, \varphi \rangle\!\rangle_\mu = \lim_{n\to\infty} \langle\!\langle \Phi_t, e_n \rangle\!\rangle_\mu, \quad t \in T.$$

Since $U_{p',q'} \subset U_{p,q}$ it holds by assumption (i) that $T \ni t \mapsto \langle\!\langle \Phi_t, e_n \rangle\!\rangle_\mu$ is measurable and thus we obtain that $t \mapsto \langle\!\langle \Phi_t, \varphi \rangle\!\rangle_\mu$ is measurable for all $\varphi \in (\mathcal{H}_{p'})^1_{q',\mu}$ as limit of measurable functions. □

2.2.2 The weak integral

The conditions for the Bochner integral are comparatively hard to verify in the application to Donsker's Delta, see Section 3.3. However, it suffices in many cases to ensure the existence of $\Psi \in (\mathcal{N})^{-1}_\mu$ such that the S_μ-transform of Ψ is the integral of the S_μ-transform of Φ_t. This property is of course fulfilled by the Bochner integral, see the Theorem above, but also by the weak integral from Section 1.2. This is the motivation to present now a result which characterizes the integrable mappings with values in $(\mathcal{N})^{-1}_\mu$ in a weak sense. Again this result is a corollary from the characterization theorem 2.1.8.

Theorem 2.2.2. *Let $(T, \mathcal{B}, \nu)$ be a measure space and $\Phi_t \in (\mathcal{N})^{-1}_\mu$ for all $t \in T$. Assume that there exist $p, q \in \mathbb{N}$ and $C < \infty$, such that:*

(i) $S_\mu \Phi_\cdot(\xi) \colon T \to \mathbb{C}$ is measurable for all $\xi \in U_{p,q}$.

(ii) $\int_T |S_\mu \Phi_t(\xi)| \, \mathrm{d}\nu(t) \le C$ for all $\xi \in U_{p,q}$.

Then there exists $\Psi \in (\mathcal{N})_\mu^{-1}$ *such that for all* $\xi \in U_{p,q}$

$$S_\mu \Psi(\xi) = \int_T S_\mu \Phi_t(\xi)\, \mathrm{d}\nu(t).$$

In particular Φ *is weakly integrable and* Ψ *is the weak integral of* Φ *denoted by* $(W)\int_T \Phi_t \, \mathrm{d}\nu(t)$.

Remark 2.2.3. If in addition the mapping

$$(\mathcal{H}_{p'})^1_{q',\mu} \supset \operatorname{span}\left\{ e_\mu(\xi;\cdot) \;\middle|\; \xi \in U_{p',q'} \right\} \ni \varphi \mapsto \langle\!\langle \Phi_\cdot, \varphi \rangle\!\rangle_\mu \in L^1(T,\nu)$$

is continuous for large enough $p', q' \in \mathbb{N}$, then Ψ coincides with the Gelfand integral, compare Proposition 1.2.5. In this case the weak integral also coincides with the Pettis integral since $(\mathcal{N})^1$ is a reflexive space, see Remark 1.2.3.

Proof. First we show that the mapping

$$U_{p,q} \ni \xi \mapsto F(\xi) := \int_T S_\mu \Phi_t(\xi)\, \mathrm{d}\nu(t)$$

is holomorphic on $U_{p,q}$. Let $\xi \in \mathcal{N}_\mathbb{C}$, $\xi_0 \in U_{p,q}$ and $U \subset \mathbb{C}$ open and small enough, such that $z\xi + \xi_0 \in U_{p,q}$ for all $z \in \overline{U}$. Furthermore let $(z_n)_{n\in\mathbb{N}}$ be a sequence in U, $\lim_{n\to\infty} z_n = z \in \overline{U}$. Since $S_\mu\Phi_t$ is holomorphic on $U_{p,q}$ we find $z^* \in \overline{U}$ such that for all $n \in \mathbb{N}$

$$|S_\mu \Phi_t(\xi_0 + z_n\xi)| \le |S_\mu \Phi_t(\xi_0 + z^*\xi)|\,.$$

Hence we have by dominated convergence that $U \ni z \mapsto F(\xi_0 + z\xi)$ is continuous. Choose now any closed and bounded curve γ in U. Since γ is compact, we have that $\int_\gamma \int_T |S_\mu \Phi_t(\xi_0 + z\xi)|\, \mathrm{d}\nu(t)\, \mathrm{d}z < \infty$ and by Fubini we obtain

$$\int_\gamma F(\xi_0 + z\xi)\, \mathrm{d}z = \int_T \int_\gamma S_\mu \Phi_t(\xi_0 + z\xi)\, \mathrm{d}z\, \mathrm{d}\nu(t) = 0,$$

since the S_μ-transform is holomorphic. This shows that F is a G-holomorphic function. Note that $|F(\xi)| \le \int_T |S_\mu \Phi_t(\xi)|\, \mathrm{d}\nu(t) \le C$ by assumption. Hence F is also locally bounded and thus holomorphic on $U_{p,q}$, see Proposition 1.1.5. By the characterization theorem 2.1.8 there exists $\Psi \in (\mathcal{N})_\mu^{-1}$ such that

$$S_\mu \Psi(\xi) = \int_T S_\mu \Phi_t(\xi)\, \mathrm{d}\nu(t)$$

for all $\xi \in U_{p,q}$. By Theorem 2.1.8 Ψ is even in $(\mathcal{H}_{-p'})^{-1}_{-q',\mu}$ for some $p', q' \in \mathbb{N}$.

Now we verify the conditions for the weak integrability, see Definition 1.2.4. Define $D := \operatorname{span}\{e_\mu(\xi;\cdot) \mid \xi \in U_{p',q'}\}$. Then D is dense in $(\mathcal{H}_{p'})^1_{q',\mu}$. For $e \in D$ there exist $N \in \mathbb{N}$ and $a_1, \dots, a_N \in \mathbb{C}$ and $\xi_1, \dots, \xi_N$ in $U_{p',q'}$ such that

$$e = \sum_{n=1}^N a_n e_\mu(\xi_n;\cdot).$$

Moreover the mapping

$$T \ni t \mapsto \langle\!\langle \Phi_t, e \rangle\!\rangle_\mu = \sum_{n=1}^N a_n (S_\mu \Phi_t)(\xi_n)$$

is integrable by condition (ii). Hence the first condition in the definition of the weak integral is verified. By the same arguments as above, we prove that for all $B \in \mathcal{B}$ there exists $\Psi_B \in (\mathcal{H}_{-p'})^{-1}_{-q',\mu}$ such that

$$\langle\langle \Psi_B, e\rangle\rangle_\mu = \sum_{n=1}^{N} a_n (S_\mu \Psi_B)(\xi_n) = \sum_{n=1}^{N} a_n \int_B (S\Phi_t)(\xi_n)\,\mathrm{d}\nu(t) = \int_B \langle\langle \Phi_t, e\rangle\rangle_\mu \,\mathrm{d}\nu(t).$$

Then we have shown that $\Phi.$ is weakly integrable in $(\mathcal{H}_{-p'})^{-1}_{-q',\mu}$. □

2.3 Characterization of convergent sequences

In the Gaussian case it was possible to characterize the convergent sequences in the distribution space $(\mathcal{N})^{-1}_{\mu_1}$, see Section 1.3.4. We prove in the following that the same is possible for non-Gaussian measures.

Theorem 2.3.1. *Let $(\Phi_n)_{n\in\mathbb{N}}$ be a sequence in $(\mathcal{N})^{-1}_{\mu}$. Then $(\Phi_n)_{n\in\mathbb{N}}$ converges strongly in $(\mathcal{N})^{-1}_{\mu}$ if and only if there exist $p, q \in \mathbb{N}$ with the following two properties:*

(i) $(S_\mu\Phi_n(\xi))_{n\in\mathbb{N}}$ is a Cauchy sequence for all $\xi \in U_{p,q}$.

(ii) $S_\mu\Phi_n$ is holomorphic on $U_{p,q}$ and there is a constant $C > 0$ such that

$$|S_\mu\Phi_n(\xi)| \le C$$

for all $\xi \in U_{p,q}$ and for all $n \in \mathbb{N}$.

Proof. First assume that (i) and (ii) are fulfilled. The S_μ-transform of Φ_n at $\xi \in U_{p,q}$ is given by (2.5) as $(S_\mu\Phi_n)(\xi) = \sum_{k=0}^{\infty}\langle \Phi_n^{(k)}, \xi^{\otimes k}\rangle$ for suitable kernels $\Phi^{(k)} \in (\mathcal{N}_{\mathbb{C}}^{\otimes k})'$ coming from the decomposition (2.3). By Theorem 2.1.8 it follows that there exist $p', q' \in \mathbb{N}$ such that

$$\|\Phi_n\|_{-p',-q',\mu} \le n_{p,q,\infty}(S_\mu\Phi_n)(1-\kappa)^{-1/2} \le C(1-\kappa)^{-1/2}.$$

Because of *(i)* and since the μ-exponentials are a total set in $(\mathcal{H}_{p'})^{1}_{q',\mu}$ we conclude that the sequence $(\langle\langle \Phi_n, \varphi\rangle\rangle_\mu)_{n\in\mathbb{N}}$ is a Cauchy sequence for all $\varphi \in (\mathcal{H}_{p'})^{1}_{q',\mu}$ by approximating φ with μ-exponentials. In fact, let $\varphi \in (\mathcal{H}_{p'})^{1}_{q',\mu}$ and $\varepsilon > 0$ and choose e in the linear span of the μ-exponentials $e_\mu(\xi;\cdot)$ with $\xi \in U_{p',q'}$, such that $\|\varphi - e\|_{p',q',\mu} < \varepsilon\left(4C(1-\kappa)^{-1/2}\right)^{-1}$. Then it holds

$$\begin{aligned}
&|\langle\langle \Phi_n, \varphi\rangle\rangle_\mu - \langle\langle \Phi_m, \varphi\rangle\rangle_\mu| \le |\langle\langle \Phi_n - \Phi_m, \varphi - e\rangle\rangle| + |\langle\langle \Phi_n - \Phi_m, e\rangle\rangle| \\
&\le \|\Phi_n - \Phi_m\|_{-p',-q',\mu}\|\varphi - e\|_{p',q',\mu} + |\langle\langle \Phi_n - \Phi_m, e\rangle\rangle| \\
&\le 2C(1-\kappa)^{-1/2}\|\varphi - e\|_{p',q,1} + |\langle\langle \Phi_n - \Phi_m, e\rangle\rangle|\,. \qquad (2.6)
\end{aligned}$$

Since $e = \sum_{k=1}^{N} a_k e_\mu(\xi_k;\cdot)$ we choose for $k = 1, \dots, N$ natural numbers $M_k \in \mathbb{N}$ such that

$$|\langle\langle \Phi_n - \Phi_m, e_\mu(\xi_k,\cdot)\rangle\rangle_\mu| = |(S_\mu\Phi_n)(\xi) - (S_\mu\Phi_m)(\xi)| < \varepsilon\left(2\sum_{k=1}^{N}|a_k|\right)^{-1},$$

for all $n, m > M_k$. Let $M = \max\{M_1, \dots, M_N\}$. Then

$$|\langle\!\langle \Phi_n - \Phi_m, e \rangle\!\rangle_\mu| \le \sum_{k=1}^{N} |a_k|\, |\langle\!\langle \Phi_n - \Phi_m, e_\mu(\xi_k; \cdot) \rangle\!\rangle_\mu| < \varepsilon/2$$

for all $n, m > M$. Finally (2.6) shows

$$|\langle\!\langle \Phi_n, \varphi \rangle\!\rangle_\mu - \langle\!\langle \Phi_m, \varphi \rangle\!\rangle_\mu| < \varepsilon.$$

Thus the mapping $\Phi\colon (\mathcal{H}_{p'})^1_{q',\mu} \to \mathbb{C}$, $\Phi(\varphi) := \lim_{n\to\infty} \langle\!\langle \Phi_n, \varphi \rangle\!\rangle_\mu$, is well-defined, linear and continuous since

$$|\Phi(\varphi)| \le \liminf_{n\to\infty} \|\Phi_n\|_{-p',-q',\mu} \|\varphi\|_{p',q',\mu} \le C(1-\kappa)^{-1/2} \|\varphi\|_{p',q',\mu}.$$

Hence $\Phi \in (\mathcal{H}_{-p'})^{-1}_{-q',\mu}$. This shows that $(\Phi_n)_{n\in\mathbb{N}}$ converges weakly to Φ in $(\mathcal{N})^{-1}_\mu$ since $(\mathcal{N})^1$ is reflexive. Finally we use that in the dual space of a nuclear space strong and weak convergence coincide, see Remark 1.1.3. Hence $(\Phi_n)_{n\in\mathbb{N}}$ converges strongly to Φ in $(\mathcal{N})^{-1}_\mu$.

Conversely let $(\Phi_n)_{n\in\mathbb{N}}$ converge strongly to some $\Phi \in (\mathcal{N})^{-1}_\mu$. Strong convergence implies that there exist $p, q \in \mathbb{N}$ such that $\Phi_n \to \Phi$ in $(\mathcal{H}_{-p})^{-1}_{-q,\mu}$ as $n \to \infty$. Thus it obviously holds for all $\xi \in U_{p,q}$ that

$$\lim_{n\to\infty} (S_\mu \Phi_n)(\xi) = \lim_{n\to\infty} \langle\!\langle \Phi_n, e_\mu(\xi; \cdot) \rangle\!\rangle_\mu = \langle\!\langle \Phi, e_\mu(\xi; \cdot) \rangle\!\rangle_\mu = (S_\mu \Phi)(\xi).$$

Hence (i) is verified. The strong convergence also implies that there exists $K < \infty$ such that $\sup_{n\in\mathbb{N}} \|\Phi_n\|_{-p,-q,\mu} < K$. Moreover we have for all $\xi \in U_{p,2q}$ that

$$\|e_\mu(\xi; \cdot)\|^2_{p,q,\mu} = \sum_{n=0}^{\infty} 2^{nq}\, |\xi|_p^{2n} \le \sum_{n=0}^{\infty} (2^{-q})^n = \frac{1}{1-2^{-q}}.$$

Then it follows that

$$|(S\Phi_n)(\xi)| \le \|\Phi_n\|_{-p,-q,\mu} \|e_\mu(\xi; \cdot)\|_{p,q,\mu} \le C$$

for all $\xi \in U_{p,2q}$ and for all $n \in \mathbb{N}$. Here $C = K(1-2^{-q})^{-1/2} < \infty$. Thus (ii) is shown. □

Chapter 3

Mittag-Leffler Analysis

In this chapter, we first recall the Mittag-Leffler measures in the finite dimensional Euclidean space $\mathbb{R}^d$, $d \in \mathbb{N}$. Using the Gram-Schmidt method we compute the first orthogonal polynomials for these measures and show certain key properties of them. Then we introduce the Mittag-Leffler measures μ_β, $0 < \beta < 1$, on the infinite-dimensional space $\mathcal{N}'$ equipped with the cylinder σ-algebra and we show that no Wick-ordered polynomials exist. However, we verify that μ_β satisfies the assumptions (A1) and (A2), see page 31, hence we may use in the following the generalized Appell systems from Chapter 2 instead of Wick-ordered polynomials. Finally we construct Donsker's Delta in this non-Gaussian setting and study some of its properties.

3.1 The Mittag-Leffler measure

3.1.1 The finite-dimensional Mittag-Leffler measure

The Mittag-Leffler function was introduced as a generalization of the exponential function by Gösta Mittag-Leffler in [ML05], see also [Wim05a, Wim05b]. We also consider a generalization first appeared in [Wim05a]:

Definition 3.1.1. For $0 < \beta < \infty$ the *Mittag-Leffler function* is an entire function defined by its power series

$$\mathrm{E}_\beta(z) := \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\beta n + 1)}, \quad z \in \mathbb{C}.$$

Define for $0 < \gamma < \infty$ the following entire function of Mittag-Leffler type, see also [EMOT55],

$$\mathrm{E}_{\beta,\gamma}(z) := \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\beta n + \gamma)}, \quad z \in \mathbb{C}.$$

Remark 3.1.2. Here Γ denotes the *Gamma function* defined by

$$\Gamma(z) := \int_0^\infty x^{z-1} \mathrm{e}^{-x}\, \mathrm{d}x, \quad z \in \mathbb{C},$$

which is an extension of the factorial to complex numbers, such that $\Gamma(n+1) = n!$ for $n \in \mathbb{N}$. This shows that $\mathrm{E}_1(z) = \mathrm{e}^z$ for all $z \in \mathbb{C}$. Moreover the Γ-function fulfils

$$\Gamma(z+1) = z\Gamma(z), \quad z \in \mathbb{C}.$$

The power series in Definition 3.1.1 converge absolutely which can be proved by using the Stirling formula, see e.g. [EMOT53]

$$\Gamma(z) = \mathrm{e}^{-z} z^{z-1/2} (2\pi)^{1/2} \left(1 + \mathcal{O}(z^{-1})\right), \quad z \to \infty.$$

Moreover it is well known that

$$\Gamma\left(\frac{1}{2} - n\right) = \frac{n!(-4)^n \sqrt{\pi}}{(2n)!}, \quad n \in \mathbb{N}. \tag{3.1}$$

From this we deduce that

$$\Gamma\left(-\frac{1}{2} - n\right) = \Gamma\left(\frac{1}{2} - (n+1)\right) = \frac{(n+1)!(-4)^{n+1}\sqrt{\pi}}{(2n+2)!}, \quad n \in \mathbb{N}. \tag{3.2}$$

Lemma 3.1.3. For the derivative of the Mittag-Leffler function E_β, $0 < \beta < \infty$, it holds

$$\frac{\mathrm{d}}{\mathrm{d}z} \mathrm{E}_\beta(z) = \frac{\mathrm{E}_{\beta,\beta}(z)}{\beta}, \quad z \in \mathbb{C}.$$

Proof. The Mittag-Leffler function is analytic and thus absolutely convergent. Hence we may interchange sum and derivative and calculate the derivative of E_β:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}z} \mathrm{E}_\beta(z) &= \frac{\mathrm{d}}{\mathrm{d}z} \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\beta n + 1)} = \sum_{n=1}^{\infty} \frac{n}{\Gamma(\beta n + 1)} z^{n-1} \\
&= \sum_{n=0}^{\infty} \frac{n+1}{\Gamma(\beta(n+1)+1)} z^n = \sum_{n=0}^{\infty} \frac{n+1}{\Gamma(\beta(n+1))\beta(n+1)} z^n \\
&= \frac{1}{\beta} \sum_{n=0}^{\infty} \frac{1}{\Gamma(\beta n + \beta)} z^n = \frac{\mathrm{E}_{\beta,\beta}(z)}{\beta}.
\end{aligned}$$

This ends the proof. □

In [Pol48, Fel71] the authors have shown that for all $\beta \in (0, 1]$ the mapping $\{t \in \mathbb{R} \mid t > 0\} \ni t \mapsto \mathrm{E}_\beta(-t) \in \mathbb{R}$ is completely monotonic. This is sufficient to prove the following lemma, see [Pol48]:

Lemma 3.1.4. For $\beta \in (0, 1]$ there exists a unique probability measure ν_β on $(0, \infty)$ such that

$$\mathrm{E}_\beta(-t) = \int_0^\infty \mathrm{e}^{-ts} \, \mathrm{d}\nu_\beta(s)$$

for all $t \geq 0$. For $\beta \in (0, 1)$ the measure ν_β is absolutely continuous with respect to the Lebesgue measure on $(0, \infty)$ and the density g_β of ν_β is given by [Pol48, Sch92]:

$$g_\beta(t) = \frac{1}{\beta} t^{-1-1/\beta} f_\beta(t^{-1/\beta}), \quad t > 0,$$

where f_β denotes the one-sided β-stable probability density which can be characterized by its Laplace transform

$$\int_0^\infty \mathrm{e}^{-tx} f_\beta(x) \, \mathrm{d}x = \mathrm{e}^{-t^\beta}, \quad t \geq 0.$$

The density g_β can also be expressed in terms of the H-function [Sch86, Sch92]:

$$g_\beta(t) = H^{1\,0}_{1\,1}\left(t \,\middle|\, \begin{matrix}(1-\beta,\beta)\\(0,1)\end{matrix}\right), \quad t>0. \tag{3.3}$$

More details on the H-function are given in Appendix A. We only want to mention here that it follows from (3.3) and Lemma A.1.4 below that the probability measure ν_β has the M-Wright function M_β as density. Again we refer to Appendix A for more details. In particular it holds that

$$\mathrm{E}_\beta(-t) = \int_0^\infty M_\beta(s)\mathrm{e}^{-ts}\,\mathrm{d}s \tag{3.4}$$

for all $t \geq 0$. Note that (3.4) even holds for all $z \in \mathbb{C}$, see Corollary A.1.9.

With this result one can show, see [Sch92], that the mapping

$$\mathbb{R}^d \ni p \mapsto \mathrm{E}_\beta\left(-\tfrac{1}{2}(p,p)_{euc}\right) \in \mathbb{R} \tag{3.5}$$

is a characteristic function. Here $(\cdot,\cdot)_{euc}$ denotes the euclidean scalar product on $\mathbb{R}^d$. Via the characteristic function a unique probability measure on $\mathbb{R}^d$ can be defined:

Definition 3.1.5. For $\beta \in (0,1]$ we define μ^d_β to be the unique probability measure on $\mathbb{R}^d$ fulfilling

$$\int_{\mathbb{R}^d} \mathrm{e}^{\mathrm{i}(p,x)_{euc}}\,\mathrm{d}\mu^d_\beta(x) = \mathrm{E}_\beta\Big(-\frac{1}{2}(p,p)_{euc}\Big)$$

for all $p \in \mathbb{R}^d$.

In [Sch92] the moments of μ^d_β are computed:

Lemma 3.1.6. The measure μ^d_β has moments of all order. More precisely, if

$$M^\beta_d(n_1,\dots,n_d) := \int_{\mathbb{R}^d} x_1^{n_1}\cdots x_d^{n_d}\,\mathrm{d}\mu^d_\beta(x)$$

for $n_1,\dots,n_d \in \mathbb{N}$, then $M^\beta_d(n_1,\dots,n_d) = 0$ if any of the n_i is odd, and for the even moments we have

$$M^\beta_d(2n_1,\dots,2n_d) = \frac{(2n_1)!\cdots(2n_d)!(n_1+\cdots+n_d)!}{2^{n_1+\cdots+n_d}n_1!\cdots n_d!\Gamma(\beta(n_1+\cdots+n_d)+1)}.$$

We apply Gram-Schmidt orthogonalization to the monomials x^n, $n \in \mathbb{N}$, to obtain monic polynomials H^β_n, $n \in \mathbb{N}$, with $\deg H^\beta_n = n$, which are orthogonal in $L^2(\mathbb{R},\mu^1_\beta)$. These polynomials are determined by the moments of the measure μ^1_β. The first five of these polynomials are given by

$$H^\beta_0(x) = 1, \qquad H^\beta_1(x) = x,$$

$$H^\beta_2(x) = x^2 - \frac{1}{\Gamma(\beta+1)}, \qquad H^\beta_3(x) = x^3 - \frac{6\Gamma(\beta+1)}{\Gamma(2\beta+1)}x,$$

$$H^\beta_4(x) = x^4 - c(\beta)x^2 + \frac{c(\beta)}{\Gamma(\beta+1)} - \frac{6}{\Gamma(2\beta+1)},$$

where

$$c(\beta) = \frac{90\Gamma(\beta+1)^2\Gamma(2\beta+1) - 6\Gamma(\beta+1)\Gamma(3\beta+1)}{6\Gamma(\beta+1)^2\Gamma(3\beta+1) - \Gamma(2\beta+1)\Gamma(3\beta+1)}.$$

Note that for $\beta = 1$ the measure μ_β^d is the standard Gaussian measure, hence in this case these polynomials reduce to the Hermite polynomials, which are orthogonal with respect to the weighting function $\exp(-\frac{1}{2}x^2)$. In [GJRdS15] the following corollary is proved:

Corollary 3.1.7. For $\beta \in (0,1]$ it holds

$$\int_{\mathbb{R}^2} H_4^\beta(x) H_2^\beta(y)\, \mathrm{d}\mu_\beta^2(x,y) = 0$$

if and only if $\beta = 1$.

Remark 3.1.8. This result can also be used to show that μ_β^{k+l} is the product measure of μ_β^k and μ_β^l for $k, l \geq 1$ if and only if $\beta = 1$.

3.1.2 The infinite-dimensional Mittag-Leffler measure

Similarly to (3.5) one can show that the mapping

$$\mathcal{N} \ni \xi \mapsto \mathrm{E}_\beta\left(-\frac{1}{2}\langle \xi, \xi\rangle\right) \in \mathbb{R}$$

is a characteristic function on $\mathcal{N}$. Using the theorem of Bochner and Minlos, see e.g. [BK95], the following probability measures on $\mathcal{N}'$, equipped with its cylindrical σ-algebra, can be defined:

Definition 3.1.9. For $\beta \in (0,1]$ the *Mittag-Leffler measure* is defined to be the unique probability measure μ_β on $\mathcal{N}'$ such that for all $\xi \in \mathcal{N}$

$$\int_{\mathcal{N}'} \exp(\mathrm{i}\langle x, \xi\rangle)\, \mathrm{d}\mu_\beta(x) = \mathrm{E}_\beta\left(-\frac{1}{2}\langle \xi, \xi\rangle\right).$$

The corresponding L^p spaces of complex-valued functions are denoted by

$$L^p(\mu_\beta) := L^p(\mathcal{N}', \mu_\beta; \mathbb{C}), \quad p \geq 1,$$

with corresponding norms $\|\cdot\|_{L^p(\mu_\beta)}$. By $\mathbb{E}_{\mu_\beta}(f) := \int_{\mathcal{N}'} f(x)\, \mathrm{d}\mu_\beta(x)$ we define the expectation of $f \in L^p(\mu_\beta)$.

Lemma 3.1.10. Let $\xi_1, \ldots, \xi_d \in \mathcal{N}$ be orthonormal in $\mathcal{H}$. Then the image measure of μ_β under the mapping $\mathcal{N}' \ni x \mapsto Tx := (\langle x, \xi_1\rangle, \ldots, \langle x, \xi_d\rangle) \in \mathbb{R}^d$ is the finite dimensional Mittag-Leffler measure μ_β^d.

Proof. Let m denote the image measure of μ_β under T. The characteristic function of m is given by:

$$\int_{\mathbb{R}^d} \mathrm{e}^{\mathrm{i}(p,z)_{euc}}\, \mathrm{d}m(z) = \int_{\mathcal{N}'} \exp\left(\mathrm{i}\sum_{k=1}^d p_k \langle x, \xi_k\rangle\right) \mathrm{d}\mu_\beta(x)$$

$$= \int_{\mathcal{N}'} \exp\left(\mathrm{i}\langle x, \sum_{k=1}^d p_k\xi_k\rangle\right) \mathrm{d}\mu_\beta(x) = \mathrm{E}_\beta\left(-\frac{1}{2}\langle \sum_{k=1}^d p_k\xi_k, \sum_{k=1}^d p_k\xi_k\rangle\right)$$

$$= \mathrm{E}_\beta\left(-\frac{1}{2}\sum_{k=1}^d p_k^2\right) = \mathrm{E}_\beta\left(-\frac{1}{2}(p,p)_{euc}\right).$$

A comparison with Definition 3.1.5 shows that $m = \mu_\beta^d$. □

As a direct consequence, the moments of μ_β can be determined with the help of the moments of the finite-dimensional Mittag-Leffler measure, see Lemma 3.1.6:

Lemma 3.1.11. Let $\xi \in \mathcal{N}$ and $n \in \mathbb{N}$. Then

$$\int_{\mathcal{N}'} \langle x, \xi\rangle^{2n+1} \,\mathrm{d}\mu_\beta(x) = 0$$

and

$$\int_{\mathcal{N}'} \langle x, \xi\rangle^{2n} \,\mathrm{d}\mu_\beta(x) = \frac{(2n)!}{\Gamma(\beta n+1)2^n}\langle \xi, \xi\rangle^n.$$

In particular $\|\langle \cdot, \xi\rangle\|^2_{L^2(\mu_\beta)} = \frac{1}{\Gamma(\beta+1)}\,|\xi|^2$.

Remark 3.1.12. Using Lemma 3.1.11 it is possible to define $\langle \cdot, \eta\rangle$ for $\eta \in \mathcal{H}$ as the $L^2(\mu_\beta)$-limit of $\langle \cdot, \xi_n\rangle$, where $(\xi_n)_{n\in\mathbb{N}}$ is a sequence in $\mathcal{N}$ converging to η in $\mathcal{H}$, compare Remark 1.3.3. Since $(\langle \cdot, \xi_n\rangle)_{n\in\mathbb{N}}$ converges in $L^2(\mu_\beta)$ we choose a subsequence such that $(\langle \cdot, \xi_{n_k}\rangle)_{k\in\mathbb{N}}$ converges μ_β-almost everywhere to $\langle \cdot, \eta\rangle$. Then $\lim_{k\to\infty} \exp(\mathrm{i}t\langle x, \xi_{n_k}\rangle) = \exp(\mathrm{i}t\langle x, \eta\rangle)$ for μ_β-almost all $x \in \mathcal{N}'$ and for all $t \in \mathbb{R}$. Moreover $|\exp(\mathrm{i}t\langle x, \xi_{n_k}\rangle)|$ is dominated by 1. By dominated convergence it holds

$$\int_{\mathcal{N}'} \mathrm{e}^{\mathrm{i}t\langle x,\eta\rangle} \,\mathrm{d}\mu_\beta(x) = \lim_{k\to\infty} \int_{\mathcal{N}'} \mathrm{e}^{\mathrm{i}t\langle x,\xi_{n_k}\rangle} \,\mathrm{d}\mu_\beta(x) = \mathrm{E}_\beta\left(-\frac{1}{2}t^2\langle \xi_{n_k}, \xi_{n_k}\rangle\right) \quad t \in \mathbb{R}.$$

Since $\lim_{n\to\infty}\langle \xi_n, \xi_n\rangle = \langle \eta, \eta\rangle$ we obtain

$$\int_{\mathcal{N}'} \mathrm{e}^{\mathrm{i}t\langle x,\eta\rangle} \,\mathrm{d}\mu_\beta(x) = \mathrm{E}_\beta\left(-\frac{1}{2}t^2\langle \eta, \eta\rangle\right), \quad t \in \mathbb{R}. \tag{3.6}$$

Lemma 3.1.10 can now be extended to all $\eta \in \mathcal{H}$ with $|\eta| = 1$. Thus also Lemma 3.1.11 is valid for all $\eta \in \mathcal{H}$.

3.1.3 Polynomials on $\mathcal{N}'$

In Gaussian analysis, the Wick-ordered polynomials are a system of polynomials $I_n(\xi) \in \mathcal{P}(\mathcal{N}')$ for $\xi \in \mathcal{N}$ such that $I_n(\xi)$ is a monic polynomial in $\langle \cdot, \xi\rangle$ of degree $n \in \mathbb{N}$ with the property that $I_n(\xi) \perp I_m(\zeta)$ whenever $n \neq m$ or $\xi \perp \zeta$ in $\mathcal{H}$. These polynomials give a useful orthogonal decomposition of the space $L^2(\mu_1)$ and allow to introduce spaces of test functions and distributions easily, see e.g. [Kuo96, HKPS93] and Section 1.3. There, the system of polynomials is denoted by $I_n(\xi) = \langle :\cdot^{\otimes n}:, \xi^{\otimes n}\rangle$.

For the measure μ_β, $0 < \beta < 1$, the system of polynomials given by

$$I_n(\xi) = |\xi|^n\, H_n^\beta\left(\frac{\langle \cdot, \xi\rangle}{|\xi|}\right), \quad n \in \mathbb{N},\ \mathcal{N} \ni \xi \neq 0, \tag{3.7}$$

is the unique system $I_n(\xi) \in \mathcal{P}(\mathcal{N}')$, $n \in \mathbb{N}$, such that $I_n(\xi) = p_{n,\xi}(\langle \cdot, \xi\rangle)$ for some monic polynomial $p_{n,\xi}$ on $\mathbb{R}$ or $\mathbb{C}$ of degree n with the property $I_n(\xi) \perp I_m(\xi)$ in $L^2(\mu_\beta)$ for $n \neq m$, see Lemma 3.1.10. Unfortunately, this system is not an analogon to the Wick polynomials in Gaussian analysis. The next theorem is a consequence of Corollary 3.1.7, for further details we refer to [GJRdS15]:

Theorem 3.1.13. *Let $\beta \in (0, 1)$ be given and assume $\dim \mathcal{H} \geq 2$. Then, for the system $I_n(\xi)$ as in (3.7), for each orthonormal $\xi, \zeta \in \mathcal{N}$ there exist $n \neq m$ such that $I_n(\xi) \not\perp I_m(\zeta)$ in $L^2(\mu_\beta)$.*

3.1.4 Appell systems are applicable

Lemma 3.1.14. For $\xi \in \mathcal{N}$ and $\lambda \in \mathbb{R}$ the exponential function $\mathcal{N}' \ni x \mapsto \mathrm{e}^{|\lambda\langle x,\xi\rangle|}$ is integrable and:

$$\int_{\mathcal{N}'} \mathrm{e}^{\lambda\langle x,\xi\rangle}\,\mathrm{d}\mu_\beta(x) = \mathrm{E}_\beta\left(\frac{1}{2}\lambda^2\langle\xi,\xi\rangle\right).$$

Proof. Obviously, the mapping is measurable. Define the monotonely increasing sequence $g_N(x) = \sum_{n=0}^N \frac{1}{n!}\left|\langle x,\lambda\xi\rangle\right|^n$. Then we have by Lemma 3.1.11 that g_N is integrable and

$$\begin{aligned}
&\int_{\mathcal{N}'} g_N(x)\,\mathrm{d}\mu_\beta(x)\\
&= \sum_{m=0}^{\left\lfloor\frac{N}{2}\right\rfloor}\int_{\mathcal{N}'}\frac{1}{(2m)!}\left|\langle x,\lambda\xi\rangle\right|^{2m}\mathrm{d}\mu_\beta(x) + \sum_{m=0}^{\left\lceil\frac{N}{2}\right\rceil-1}\int_{\mathcal{N}'}\frac{1}{(2m+1)!}\left|\langle x,\lambda\xi\rangle\right|^{2m+1}\mathrm{d}\mu_\beta(x)\\
&= \sum_{m=0}^{\left\lfloor\frac{N}{2}\right\rfloor}\frac{1}{(2m)!}\frac{(2m)!}{\Gamma(\beta m+1)2^m}\lambda^{2m}\langle\xi,\xi\rangle^m\\
&\quad + \sum_{m=0}^{\left\lceil\frac{N}{2}\right\rceil-1}\frac{1}{(2m+1)!}\int_{\mathcal{N}'}\left|\langle x,\lambda\xi\rangle\right|^{m}\left|\langle x,\lambda\xi\rangle\right|^{m+1}\mathrm{d}\mu_\beta(x)\\
&= A + B.
\end{aligned}$$

The first sum A converges to $\mathrm{E}_\beta(\frac{\lambda^2}{2}\langle\xi,\xi\rangle)$ for $N\to\infty$. The sum B is estimated by the Cauchy-Schwarz inequality:

$$B \le \sum_{m=0}^{\left\lceil\frac{N}{2}\right\rceil-1}\frac{1}{(2m+1)!}\left(\int_{\mathcal{N}'}\left|\langle x,\lambda\xi\rangle\right|^{2m}\mathrm{d}\mu_\beta(x)\right)^{1/2}\left(\int_{\mathcal{N}'}\left|\langle x,\lambda\xi\rangle\right|^{2m+2}\mathrm{d}\mu_\beta(x)\right)^{1/2}.$$

Now we use that for all $a,b\in\mathbb{R}$ it holds $ab \le 1/2(a^2+b^2)$. This yields

$$\begin{aligned}
B &\le \sum_{m=0}^{\left\lceil\frac{N}{2}\right\rceil-1}\frac{1}{(2m+1)!}\left(\frac{1}{2}\int_{\mathcal{N}'}\left|\langle x,\lambda\xi\rangle\right|^{2m}\mathrm{d}\mu_\beta(x) + \frac{1}{2}\int_{\mathcal{N}'}\left|\langle x,\lambda\xi\rangle\right|^{2m+2}\mathrm{d}\mu_\beta(x)\right)\\
&= \sum_{m=0}^{\left\lceil\frac{N}{2}\right\rceil-1}\frac{1}{(2m+1)!}\frac{1}{2}\frac{(2m)!}{\Gamma(\beta m+1)2^m}\lambda^{2m}\langle\xi,\xi\rangle^m\\
&\quad + \sum_{m=0}^{\left\lceil\frac{N}{2}\right\rceil-1}\frac{1}{(2m+1)!}\frac{1}{2}\frac{(2m+2)!}{\Gamma(\beta(m+1)+1)2^{m+1}}\lambda^{2m+2}\langle\xi,\xi\rangle^{m+1} = C + D.
\end{aligned}$$

The first summand C is less than $\sum\frac{1}{2\Gamma(\beta m+1)}\left(\frac{\lambda^2}{2}\langle\xi,\xi\rangle\right)^m$ and thus converging to

$\frac{1}{2}\mathrm{E}_\beta(\frac{\lambda^2}{2}\langle\xi,\xi\rangle)$. For the second summand D it holds (denoting $m+1=n$)

$$D = \frac{1}{2}\sum_{n=1}^{\left\lceil\frac{N}{2}\right\rceil} \frac{2n}{\Gamma(\beta n+1)2^n}\left(\lambda^2\langle\xi,\xi\rangle\right)^n \leq \sum_{n=1}^{\left\lceil\frac{N}{2}\right\rceil} \frac{1}{\Gamma(\beta n+1)}(\lambda^2\langle\xi,\xi\rangle)^n.$$

D converges to $\mathrm{E}_\beta\left(\lambda^2\langle\xi,\xi\rangle\right)$. Hence also the sum B is convergent and we finally get that $\lim_{N\to\infty}\int_{\mathcal{N}'} g_N(x)\,\mathrm{d}\mu_\beta(x)$ exists. Thus we have by monotone convergence that

$$\int_{\mathcal{N}'} \mathrm{e}^{|\lambda\langle x,\xi\rangle|}\,\mathrm{d}\mu_\beta(x) = \lim_{N\to\infty}\int_{\mathcal{N}'} g_N(x)\,\mathrm{d}\mu_\beta(x) < \infty.$$

This shows the integrability. Moreover, we are allowed by dominated convergence to do the following calculation, where we use again Lemma 3.1.11:

$$\begin{aligned}\int_{\mathcal{N}'} \mathrm{e}^{\lambda\langle x,\xi\rangle}\,\mathrm{d}\mu_\beta(x) &= \sum_{n=0}^{\infty}\frac{\lambda^n}{n!}\int_{\mathcal{N}'}\langle x,\xi\rangle^n\,\mathrm{d}\mu_\beta(x) = \sum_{n=0}^{\infty}\frac{\lambda^{2n}}{(2n)!}\int_{\mathcal{N}'}\langle x,\xi\rangle^{2n}\,\mathrm{d}\mu_\beta(x)\\ &= \sum_{n=0}^{\infty}\frac{\lambda^{2n}}{(2n)!}\frac{(2n)!}{2^n\Gamma(\beta n+1)}\langle\xi,\xi\rangle^n = \mathrm{E}_\beta\left(\frac{1}{2}\lambda^2\langle\xi,\xi\rangle\right).\end{aligned}$$

□

Now we show that the Mittag-Leffler measure fulfils (A1):

Theorem 3.1.15. *The mapping*

$$\mathcal{N}_{\mathbb{C}} \ni \xi \mapsto l_{\mu_\beta}(\xi) = \int_{\mathcal{N}'} \mathrm{e}^{\langle x,\xi\rangle}\,\mathrm{d}\mu_\beta(x)$$

is an holomorphic mapping from $\mathcal{N}_{\mathbb{C}}$ to $\mathbb{C}$.

Proof. By Lemma 3.1.14 we get for $\xi\in\mathcal{N}_{\mathbb{C}}$, where $\xi=\xi_1+i\xi_2$ with $\xi_1,\xi_2\in\mathcal{N}$

$$\begin{aligned}\left|l_{\mu_\beta}(\xi)\right| &\leq \int_{\mathcal{N}'}\left|\mathrm{e}^{\langle x,\xi_1\rangle}\mathrm{e}^{\mathrm{i}\langle x,\xi_2\rangle}\right|\,\mathrm{d}\mu_\beta(x)\\ &= \int_{\mathcal{N}'}\mathrm{e}^{\langle x,\xi_1\rangle}\,\mathrm{d}\mu_\beta(x) = \mathrm{E}_\beta\left(\frac{1}{2}\langle\xi_1,\xi_1\rangle\right) < \infty.\end{aligned}$$

This shows that the mapping $\mathcal{N}_{\mathbb{C}}\ni\xi\mapsto l_{\mu_\beta}(\xi)$ is locally bounded.

Consider now the mapping $\mathbb{C}\ni z\mapsto f(z) := l_{\mu_\beta}(\xi_0+z\xi)$, $\xi_0,\xi\in\mathcal{N}_{\mathbb{C}}$. We show that f is analytic in z. First we remark that f is continuous. To see this, choose $z\in\mathbb{C}$ and a sequence $(z_n)_{n\in\mathbb{N}}$ in $\mathbb{C}$ with $z_n\to z$ as $n\to\infty$. Then it holds that for sufficiently large $n\in\mathbb{N}$:

$$\left|\mathrm{e}^{\langle x,\xi_0+z_n\xi\rangle}\right| \leq \mathrm{e}^{|\langle x,\xi_0\rangle|}\mathrm{e}^{(|z|+1)|\langle x,\xi\rangle|} =: g(x).$$

Since $g\in L^1(\mu_\beta)$ by Lemma 3.1.14 we may apply dominated convergence and yield

$$\lim_{n\to\infty} f(z_n) = \int_{\mathcal{N}'}\mathrm{e}^{\langle x,\xi_0+z\xi\rangle}\,\mathrm{d}\mu_\beta(x) = f(z).$$

Thus f is indeed continuous.

Let now γ be a closed and bounded curve in $\mathbb{C}$. Since γ is compact we can use Fubini and get

$$\int_\gamma \int_{\mathcal{N}'} \mathrm{e}^{\langle x, \xi_0 + z\xi \rangle} \, \mathrm{d}\mu_\beta(x) \, \mathrm{d}z = \int_{\mathcal{N}'} \int_\gamma \mathrm{e}^{\langle x, \xi_0 + z\xi \rangle} \, \mathrm{d}z \, \mathrm{d}\mu_\beta(x) = 0,$$

because the exponential function is holomorphic. Hence we have by Morera's theorem that f is holomorphic in $\mathbb{C}$. This shows that l_{μ_β} is G-holomorphic, which implies holomorphy, see Proposition 1.1.5. □

We obtain the following corollary by applying the identity principle from complex analysis:

Corollary 3.1.16. For each $z \in \mathbb{C}$ and for each $\xi \in \mathcal{N}_\mathbb{C}$ we have that

$$\int_{\mathcal{N}'} \mathrm{e}^{z\langle x, \xi \rangle} \, \mathrm{d}\mu_\beta(x) = \mathrm{E}_\beta \left(\frac{1}{2} z^2 \langle \xi, \xi \rangle \right).$$

Proof. The last proof has shown that

$$\mathbb{C} \ni z \mapsto f(z) := \int_{\mathcal{N}'} \mathrm{e}^{\langle x, \xi_1 + z\xi_2 \rangle} \, \mathrm{d}\mu_\beta(x)$$

is an holomorphic mapping for each $\xi_1, \xi_2 \in \mathcal{N}$ and by Lemma 3.1.14

$$f(s) = \mathrm{E}_\beta \left(\frac{1}{2} \langle \xi_1 + s\xi_2, \xi_1 + s\xi_2 \rangle \right), \quad s \in \mathbb{R}.$$

This equality extends by the identity principle to all $z \in \mathbb{C}$. In particular, for all $\xi_1, \xi_2 \in \mathcal{N}$ it holds that

$$\int_{\mathcal{N}'} \mathrm{e}^{\langle x, \xi_1 + \mathrm{i}\xi_2 \rangle} \, \mathrm{d}\mu_\beta(x) = f(i) = \mathrm{E}_\beta \left(\frac{1}{2} \langle \xi_1 + \mathrm{i}\xi_2, \xi_1 + \mathrm{i}\xi_2 \rangle \right).$$

Finally for $\xi = \xi_1 + \mathrm{i}\xi_2 \in \mathcal{N}_\mathbb{C}$ and $z = a + ib$, it holds that $z\xi = a\xi_1 - b\xi_2 + \mathrm{i}(b\xi_1 + a\xi_2)$ and, denoting $z\xi = \tilde{\xi_1} + \mathrm{i}\tilde{\xi_2}$ for $\tilde{\xi_1}, \tilde{\xi_2} \in \mathcal{N}$, we obtain

$$\begin{aligned} \int_{\mathcal{N}'} \mathrm{e}^{z\langle x, \xi \rangle} \, \mathrm{d}\mu_\beta(x) &= \int_{\mathcal{N}'} \mathrm{e}^{\langle x, \tilde{\xi_1} + \mathrm{i}\tilde{\xi_2} \rangle} \, \mathrm{d}\mu_\beta(x) \\ &= \mathrm{E}_\beta \left(\frac{1}{2} \langle \tilde{\xi_1} + \mathrm{i}\tilde{\xi_2}, \tilde{\xi_1} + \mathrm{i}\tilde{\xi_2} \rangle \right) = \mathrm{E}_\beta \left(\frac{1}{2} z^2 \langle \xi, \xi \rangle \right). \end{aligned}$$

□

Remark 3.1.17. Corollary 3.1.16 shows that the Laplace transform of μ_β is given by

$$l_{\mu_\beta}(\xi) = \int_{\mathcal{N}'} \mathrm{e}^{\langle x, \xi \rangle} \, \mathrm{d}\mu_\beta(x) = \mathrm{E}_\beta \left(\frac{1}{2} \langle \xi, \xi \rangle \right), \quad \xi \in \mathcal{N}_\mathbb{C}.$$

Note that E_β is holomorphic and $\mathrm{E}_\beta(0) = 1$. Thus there exists $\varepsilon_\beta > 0$ such that $\mathrm{E}_\beta(z) > 0$ for all $z \in \mathbb{C}$, $|z| \leq \varepsilon_\beta$. Furthermore set

$$\mathcal{U}_\beta := \left\{ \xi \in \mathcal{N}_\mathbb{C} \,\middle|\, \frac{1}{2} |\langle \xi, \xi \rangle| < \varepsilon_\beta \right\} \subset \mathcal{N}_\mathbb{C}.$$

Then it holds that $l_{\mu_\beta}(\xi) > 0$ for all $\xi \in \mathcal{U}_\beta$. $\mathcal{U}_\beta$ is the suitable neighborhood of $0 \in \mathcal{N}_\mathbb{C}$ in order to define the μ_β-exponential later on, compare (2.1).

Lemma 3.1.14, Theorem 3.1.15 and Corollary 3.1.16 require that ξ is a test function. Since $\mathcal{N}_{\mathbb{C}} \subset \mathcal{H}_{\mathbb{C}}$ is dense we are able to extend the above results for $\eta \in \mathcal{H}_{\mathbb{C}}$ by approximation with elements from $\mathcal{N}_{\mathbb{C}}$. Taking into account the consideration in Remark 3.1.12 the following Corollary is valid:

Corollary 3.1.18. For $\eta \in \mathcal{H}_{\mathbb{C}}$ and $z \in \mathbb{C}$ we have that

$$\int_{\mathcal{N}'} \mathrm{e}^{z\langle x,\eta\rangle}\, \mathrm{d}\mu_\beta(x) = \mathrm{E}_\beta\left(\frac{1}{2}z^2\langle \eta,\eta\rangle\right).$$

Proof. Let first $\eta \in \mathcal{H}$ and $\lambda \in \mathbb{R}$. As shown in Remark 3.1.12 we may apply Lemma 3.1.11 for $\eta \in \mathcal{H}$. With the same ideas as in the proof of Lemma 3.1.14 we get that the mapping $\mathcal{N}' \ni x \mapsto \mathrm{e}^{|\lambda\langle x,\eta\rangle|}$ is integrable and that

$$\int_{\mathcal{N}'} \mathrm{e}^{\lambda\langle x,\eta\rangle}\, \mathrm{d}\mu_\beta(x) = \mathrm{E}_\beta\left(\frac{1}{2}\langle \eta,\eta\rangle\right).$$

For $\eta \in \mathcal{H}_{\mathbb{C}}$ and $z \in \mathbb{C}$ use the arguments given in Theorem 3.1.15 and Corollary 3.1.16 to prove the assertion. ☐

Corollary 3.1.19. For $\xi, \eta \in \mathcal{H}_{\mathbb{C}}$ it holds that

$$\int_{\mathcal{N}'} \langle x,\xi\rangle\langle x,\eta\rangle\, \mathrm{d}\mu_\beta(x) = \frac{1}{\Gamma(\beta+1)}\langle \xi,\eta\rangle.$$

Proof. By Corollary 3.1.18 we have for each $a_1, a_2 \in (-1,1) \subset \mathbb{R}$

$$\int_{\mathcal{N}'} \mathrm{e}^{\langle x, a_1\xi + a_2\eta\rangle}\, \mathrm{d}\mu_\beta(x) = \mathrm{E}_\beta\left(\frac{1}{2}\langle a_1\xi + a_2\eta, a_1\xi + a_2\eta\rangle\right).$$

Applying the operator $\frac{\partial^2}{\partial a_1 \partial a_2}$ to both sides and evaluate at $a_1 = a_2 = 0$ we get the result. Indeed, by Lemma 3.1.3 it follows that

$$\begin{aligned}
&\frac{\partial}{\partial a_2}\mathrm{E}_\beta\left(\frac{1}{2}\langle a_1\xi + a_2\eta, a_1\xi + a_2\eta\rangle\right)\\
&= \frac{1}{\beta}\mathrm{E}_{\beta,\beta}\left(\frac{1}{2}\langle a_1\xi + a_2\eta, a_1\xi + a_2\eta\rangle\right)(a_2\langle \eta,\eta\rangle + a_1\langle \xi,\eta\rangle)\\
&= \frac{1}{\beta}\left(\sum_{n=0}^{\infty}\frac{1}{2^n\Gamma(\beta n+\beta)}\langle a_1\xi + a_2\eta, a_1\xi + a_2\eta\rangle^n\right)(a_2\langle \eta,\eta\rangle + a_1\langle \xi,\eta\rangle).
\end{aligned}$$

We may interchange sum and derivative since the Mittag-Leffler function is entire and get

$$\begin{aligned}
&\frac{\partial^2}{\partial a_1 \partial a_2}\mathrm{E}_\beta\left(\frac{1}{2}\langle a_1\xi + a_2\eta, a_1\xi + a_2\eta\rangle\right)\\
&= \frac{1}{\beta}\left(\sum_{n=1}^{\infty}\frac{n}{2^n\Gamma(\beta n+\beta)}\langle a_1\xi + a_2\eta, a_1\xi + a_2\eta\rangle^{n-1}\right)(a_1\langle \xi,\xi\rangle + a_2\langle \xi,\eta\rangle)\\
&\quad \times (a_2\langle \eta,\eta\rangle + a_1\langle \xi,\eta\rangle) + \frac{1}{\beta}\left(\sum_{n=0}^{\infty}\frac{1}{2^n\Gamma(\beta n+\beta)}\langle a_1\xi + a_2\eta, a_1\xi + a_2\eta\rangle^n\right)\langle \xi,\eta\rangle.
\end{aligned}$$

Evaluation at $a_1 = a_2 = 0$ yields

$$\frac{\partial^2}{\partial a_1 \partial a_2} \mathrm{E}_\beta \left(\frac{1}{2} \langle a_1\xi + a_2\eta, a_1\xi + a_2\eta \rangle \right) = \frac{1}{\beta} \frac{1}{\Gamma(\beta)} \langle \xi, \eta \rangle = \frac{1}{\Gamma(\beta+1)} \langle \xi, \eta \rangle. \tag{3.8}$$

For the change of derivatives and integral consider

$$\int_{\mathcal{N}'} \frac{\partial^2}{\partial a_1 \partial a_2} \mathrm{e}^{\langle x, a_1\xi + a_2\eta \rangle} \, \mathrm{d}\mu_\beta(x) = \int_{\mathcal{N}'} \frac{\partial}{\partial a_1} \langle x, \eta \rangle \mathrm{e}^{\langle x, a_1\xi + a_2\eta \rangle} \, \mathrm{d}\mu_\beta(x).$$

The function $\mathcal{N}' \ni x \mapsto \langle x, \eta \rangle \exp(\langle x, a_1\xi + a_2\eta \rangle)$ is integrable for all $a_1 \in (-1, 1)$ by Lemma 3.1.11 since $|\langle \cdot, \eta \rangle| \leq \exp(|\langle \cdot, \eta \rangle|) \in L^2(\mu_\beta)$. Moreover it obviously holds that the mapping $(-1, 1) \ni a_1 \mapsto \langle x, \eta \rangle \exp(\langle x, a_1\xi + a_2\eta \rangle)$ is differentiable and we have for the derivative that

$$\begin{aligned} \left| \frac{\partial}{\partial a_1} \langle x, \eta \rangle \mathrm{e}^{\langle x, a_1\xi + a_2\eta \rangle} \right| &= |\langle x, \eta \rangle| \, |\langle x, \xi \rangle| \, \mathrm{e}^{|a_1||\langle x, \xi \rangle|} \mathrm{e}^{|a_2||\langle x, \eta \rangle|} \\ &\leq \mathrm{e}^{2|\langle x, \xi \rangle|} \mathrm{e}^{2|\langle x, \xi \rangle|} =: g(x) \end{aligned}$$

for all $a_1, a_2 \in (-1, 1)$. $g \in L^1(\mu_\beta)$ by Lemma 3.1.14 and by Lebesgue's dominated convergence we may interchange integral and $\frac{\partial}{\partial a_1}$. Similar arguments show that also $\frac{\partial}{\partial a_2}$ interchanges with the integral. This finally yields

$$\begin{aligned} \frac{\partial^2}{\partial a_1 \partial a_2} \int_{\mathcal{N}'} \mathrm{e}^{\langle x, a_1\xi + a_2\eta \rangle} \, \mathrm{d}\mu_\beta(x) &= \int_{\mathcal{N}'} \frac{\partial^2}{\partial a_1 \partial a_2} \mathrm{e}^{\langle x, a_1\xi + a_2\eta \rangle} \, \mathrm{d}\mu_\beta(x) \\ &= \int_{\mathcal{N}'} \langle x, \xi \rangle \langle x, \eta \rangle \mathrm{e}^{\langle x, a_1\xi + a_2\eta \rangle} \, \mathrm{d}\mu_\beta(x). \end{aligned}$$

Evaluate at $a_1 = a_2 = 0$ and compare to (3.8) to show the desired result. □

Now we show that μ_β satisfies (A2). In fact, we proof a statement, which is a little bit stronger. For this, we need a refined result of Lemma 3.1.4 which shows that the Mittag-Leffler measure is a mixed measure of Gaussian measures $\mu^{(s)}$. Here, $\mu^{(s)}$ denotes the centered Gaussian measure on $\mathcal{N}'$ with variance $s > 0$, i.e.

$$\int_{\mathcal{N}'} \mathrm{e}^{\mathrm{i}\langle x, \xi \rangle} \, \mathrm{d}\mu^{(s)}(x) = \exp\left(-\frac{s}{2} \langle \xi, \xi \rangle \right), \quad \xi \in \mathcal{N}.$$

Then it can be shown that the mapping

$$(0, \infty) \ni s \mapsto \mu^{(s)}(E), \quad E \in \mathcal{C}_\sigma(\mathcal{N}'),$$

is continuous and therefore measurable. For details we refer to [GJRdS15]. This suffices to show that $\int_0^\infty \mu^{(s)} \, \mathrm{d}\nu_\beta(s)$ defines a probability measure on $\mathcal{N}'$, compare Lemma 452A in [Fre06]. Here, ν_β is the same measure as in Lemma 3.1.4. By comparing the characteristic functions it follows

$$\mu_\beta = \int_0^\infty \mu^{(s)} \, \mathrm{d}\nu_\beta(s), \tag{3.9}$$

see also [KS93].

Remark 3.1.20. Equation (3.9) shows that the Mittag-Leffler measure μ_β is a so-called elliptically contoured measure, see for example Theorem 2 in [Mis84]. These are exactly those measures σ which are a "mixture" of Gaussian measures $\mu^{(s)}$ with respect to some probability measure. Elliptically contoured measures are characterized by the fact that the characteristic function has the form

$$\int_{\mathcal{N}'} \mathrm{e}^{\mathrm{i}\langle x,\xi\rangle}\,\mathrm{d}\sigma(x) = \Psi(\langle \xi,\xi\rangle), \quad \xi \in \mathcal{N}.$$

In the case of the Mittag-Leffler measures μ_β the function Ψ is given by the Mittag-Leffler function $\mathrm{E}_\beta(-\frac{1}{2}\cdot)$.

Now let $U \subset \mathcal{N}'$ be open and non-empty. It is well-known that for the Gaussian measures $\mu^{(s)}$ it holds $\mu^{(s)}(U) > 0$ for each $s \in (0,\infty)$, hence

$$\mu_\beta(U) = \int_0^\infty \mu^{(s)}(U)\,\mathrm{d}\nu_\beta(s) > 0.$$

This shows the following

Theorem 3.1.21. *μ_β satisfies assumption (A2) for all* $0 < \beta < 1$.

3.2 Generalized Appell system for μ_β

The two Theorems 3.1.15 and 3.1.21 show that the Mittag-Leffler measures μ_β, $0 < \beta < 1$, fulfil the assumption (A1) and (A2) from page 31. Thus we can now introduce the μ_β-exponential. Note that $l_{\mu_\beta}(\xi) = \mathrm{E}_\beta\left(\frac{1}{2}\langle\xi,\xi\rangle\right) > 0$ for all $\xi \in \mathcal{U}_\beta$, see Remark 3.1.17. Then

$$e_{\mu_\beta}(\xi;x) = \frac{1}{l_{\mu_\beta}(\xi)}\mathrm{e}^{\langle x,\xi\rangle} = \frac{1}{\mathrm{E}_\beta\left(\frac{1}{2}\langle\xi,\xi\rangle\right)}\mathrm{e}^{\langle x,\xi\rangle}, \quad x \in \mathcal{N}',\ \xi \in \mathcal{U}_\beta, \tag{3.10}$$

A generalized Appell system $\mathbb{A}^{\mu_\beta} = (\mathbb{P}^{\mu_\beta}, \mathbb{Q}^{\mu_\beta})$ with respect to μ_β consisting of Appell polynomials

$$\mathbb{P}^{\mu_\beta} = \left\{\langle P_n^{\mu_\beta}(\cdot), \xi^{(n)}\rangle \;\middle|\; \xi^{(n)} \in \mathcal{N}_{\mathbb{C}}^{\hat{\otimes} n},\ n \in \mathbb{N}\right\} = \mathcal{P}(\mathcal{N}')$$

and the $\mathbb{Q}^{\mu_\beta}$-system

$$\mathbb{Q}^{\mu_\beta} = \left\{Q_n^{\mu_\beta}(\Phi^{(n)}) \;\middle|\; \Phi^{(n)} \in \left(\mathcal{N}_{\mathbb{C}}^{\hat{\otimes} n}\right)',\ n \in \mathbb{N}\right\} = \mathcal{P}'_{\mu_\beta}(\mathcal{N}')$$

can be constructed. Following Chapter 2 we introduce the spaces of test functions and distributions $(\mathcal{N})^1$ and $(\mathcal{N})^{-1}_{\mu_\beta}$, respectively. The integral transform S_{μ_β} of a distribution $\Phi \in (\mathcal{H}_{-p})^{-1}_{-q,\mu_\beta}$ is given by

$$\left(S_{\mu_\beta}\Phi\right)(\xi) = \frac{1}{\mathrm{E}_\beta\left(\frac{1}{2}\langle\xi,\xi\rangle\right)}\langle\!\langle \Phi, \mathrm{e}^{\langle\cdot,\xi\rangle}\rangle\!\rangle_{\mu_\beta}, \quad \xi \in U_{p,q},$$

and the relation between S_{μ_β}- and T_{μ_β}-transform, Lemma 2.1.9, reads

$$\left(T_{\mu_\beta}\Phi\right)(\xi) = \mathrm{E}_\beta\left(\frac{1}{2}\langle\xi,\xi\rangle\right)\left(S_{\mu_\beta}\Phi\right)(\mathrm{i}\xi), \quad \xi \in U_{p,q}. \tag{3.11}$$

In particular, we want to point out that all the characterization theorems in Section 2.2 and 2.3 concerning the characterization of integrable functions and convergence sequences in $(\mathcal{N})^{-1}_{\mu_\beta}$ apply to the Mittag-Leffler measure μ_β.

3.2.1 Appell polynomials

In the following, we give an explicit formula for the Appell polynomials with respect to the measure μ_β. Note that the Appell polynomials are defined by the Taylor series of the μ_β-exponential, i.e.

$$e_{\mu_\beta}(\xi;x) = \sum_{n=0}^{\infty} \langle \frac{1}{n!} P_n^{\mu_\beta}(x), \xi^{\otimes n}\rangle, \quad x \in \mathcal{N}', \tag{3.12}$$

for $\xi \in \mathcal{U}_\beta$, see (2.2). As in Remark 3.1.17 we find $\varepsilon_\beta > 0$ such that $\mathrm{E}_\beta(z) \neq 0$ for all $|z| < \varepsilon_\beta$. Hence the function $\mathcal{E} := 1/\mathrm{E}_\beta$ is holomorphic on $B_{\varepsilon_\beta}(0)$. Assume that $\mathcal{E}$ has the series expansion

$$\mathcal{E}(z) = \sum_{n=0}^{\infty} b_n z^n, \quad |z| < \varepsilon_\beta,$$

for some coefficients $b_n \in \mathbb{C}$ which are determined later on in Theorem 3.2.4. Assume that $\xi \in \mathcal{U}_\beta$ then

$$\mathcal{E}\left(\frac{1}{2}\langle \xi, \xi\rangle\right) = \sum_{n=0}^{\infty} \frac{b_n}{2^n} \langle \tau, \xi^{\otimes 2}\rangle^n = \sum_{n=0}^{\infty} \frac{b_n}{2^n} \langle \tau^{\otimes n}, \xi^{\otimes 2n}\rangle. \tag{3.13}$$

Here $\tau \in \left(\mathcal{N}_{\mathbb{C}}^{\otimes 2}\right)'$ denotes the *trace operator* which is uniquely defined by

$$\langle \tau, \xi \otimes \eta\rangle = \langle \xi, \eta\rangle, \quad \xi, \eta \in \mathcal{N}_{\mathbb{C}}.$$

We insert the series expansion (3.13) in the definition of the μ_β-exponential (3.10). Then we get for $x \in \mathcal{N}'$ and $\xi \in \mathcal{U}_\beta$:

$$\begin{aligned}
e_{\mu_\beta}(\xi;x) &= \left(\sum_{n=0}^{\infty} \frac{1}{n!} \langle x^{\otimes n}, \xi^{\otimes n}\rangle\right) \left(\sum_{n=0}^{\infty} \frac{b_n}{2^n} \langle \tau^{\otimes n}, \xi^{\otimes 2n}\rangle\right) \\
&= \sum_{n=0}^{\infty} \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{b_k}{2^k (n-2k)!} \langle \tau^{\otimes k}, \xi^{\otimes 2k}\rangle \langle x^{\otimes (n-2k)}, \xi^{\otimes (n-2k)}\rangle \\
&= \sum_{n=0}^{\infty} \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{b_k}{2^k (n-2k)!} \langle \tau^{\otimes k} \otimes x^{\otimes (n-2k)}, \xi^{\otimes n}\rangle \\
&= \sum_{n=0}^{\infty} \frac{1}{n!} \langle \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{b_k n!}{2^k (n-2k)!} \tau^{\otimes k} \otimes x^{\otimes (n-2k)}, \xi^{\otimes n}\rangle.
\end{aligned}$$

A comparison with the expansion of the μ_β-exponential in terms of Appell polynomials (3.12) yields

$$P_n^{\mu_\beta}(x) = \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{b_k n!}{2^k (n-2k)!} \tau^{\otimes k} \otimes x^{\otimes (n-2k)}, \quad x \in \mathcal{N}'. \tag{3.14}$$

We determine now the coefficients b_k, $k \in \mathbb{N}$, with the help of the following formula for $z \in B_{\varepsilon_\beta}(0)$:

$$1 = E_\beta(z)\mathcal{E}(z) = \left(\sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\beta n + 1)}\right) \left(\sum_{n=0}^{\infty} b_n z^n\right) = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{b_{n-k}}{\Gamma(\beta k + 1)} z^n.$$

Hence we can deduce that $b_0 = 1$ and

$$\sum_{k=0}^{n} \frac{b_{n-k}}{\Gamma(\beta k+1)} = 0, \quad n > 0.$$

Then the coefficients b_n for $n > 0$ are given recursively by

$$b_n = -\sum_{k=1}^{n} \frac{b_{n-k}}{\Gamma(\beta k+1)}, \quad n > 0. \tag{3.15}$$

We use the recursion in order to calculate the first two coefficients. This yields

$$b_1 = -\frac{1}{\Gamma(\beta+1)}, \qquad b_2 = \frac{1}{\Gamma(\beta+1)^2} - \frac{1}{\Gamma(2\beta+1)}. \tag{3.16}$$

Inserting the coefficients in (3.14) we obtain for $x \in \mathcal{N}'$ the first five Appell polynomials:

$$\begin{aligned}
P_0^{\mu_\beta}(x) &= 1,\\
P_1^{\mu_\beta}(x) &= x,\\
P_2^{\mu_\beta}(x) &= x^{\otimes 2} - \frac{1}{\Gamma(\beta+1)}\tau,\\
P_3^{\mu_\beta}(x) &= x^{\otimes 3} - \frac{3}{\Gamma(\beta+1)}\tau \otimes x,\\
P_4^{\mu_\beta}(x) &= x^{\otimes 4} - \frac{6}{\Gamma(\beta+1)}\tau \otimes x^{\otimes 2} + 6\left(\frac{1}{\Gamma(\beta+1)^2} - \frac{1}{\Gamma(2\beta+1)}\right)\tau^{\otimes 2}.
\end{aligned}$$

Remark 3.2.1. Up to now we have constructed two systems of polynomials on $\mathcal{N}'$. First, via the Gram-Schmidt orthogonalization procedure, we calculated the orthogonal polynomials H_n^β, $n \in \mathbb{N}$, with respect to the one-dimensional Mittag-Leffler measure μ_β^1. With the help of H_n^β we defined the system of polynomials $(I_n(\xi))_{n\in\mathbb{N}}$, $\xi \in \mathcal{N}$, $\xi \neq 0$, by

$$I_n(\xi) = |\xi|^n \, H_n^\beta\left(\frac{\langle \cdot, \xi\rangle}{|\xi|}\right),$$

see (3.7). The second system of polynomials, denoted by $\langle P_n^{\mu_\beta}, \xi^{\otimes n}\rangle$, arises in the series expansion of the μ_β-exponential in (3.12). It is a natural question whether these systems coincide. In the case $\beta \neq 1$ this is obviously not true since

$$\begin{aligned}
\langle P_3^{\mu_\beta}(\cdot), \xi^{\otimes 3}\rangle &= \langle \cdot, \xi\rangle^3 - \frac{3}{\Gamma(\beta+1)}\langle \xi, \xi\rangle\langle \cdot, \xi\rangle\\
&\neq I_3(\xi) = \langle \cdot, \xi\rangle^3 - \frac{6\Gamma(\beta+1)}{\Gamma(2\beta+1)}\langle \xi, \xi\rangle\langle \cdot, \xi\rangle.
\end{aligned}$$

Thus the systems are different and due to uniqueness of the orthogonal system $(I_n(\xi))_{n\in\mathbb{N}}$, the $\mathbb{P}^{\mu_\beta}$-system is not orthogonal if $\beta \neq 1$.

More general, let ν be a "nice" measure on $\mathbb{R}$, i.e. we assume that ν has moments of all orders and that the exponential function is integrable. Then we obtain via orthogonalization the orthogonal polynomials denoted by H_n^ν with respect to ν. On the other hand consider the normalized exponential e_ν and assume that

$$e_\nu(t,x) = \frac{\mathrm{e}^{tx}}{\int_{\mathbb{R}} \mathrm{e}^{tx}\,\mathrm{d}\nu(x)} = \sum_{n=0}^{\infty} \frac{1}{n!} P_n^\nu(x) t^n, \quad x,t \in \mathbb{R},$$

i.e. e_ν is the generating function of the polynomials P_n^ν, $n \in \mathbb{N}$. The question is again whether $H_n^\nu = P_n^\nu$. In the case that ν is a mean zero Gaussian measure with variance 1 we obtain for H_n^ν the Hermite polynomials. On the other hand

$$e_\nu(t,x) = \exp\left(-\frac{1}{2}t^2 + tx\right)$$

is the generating function of the Hermite polynomials. In this case it holds $H_n^\nu = P_n^\nu$. In [Mei34] all those generating functions are given which generate a system of orthogonal polynomials.

We end this section by giving an explicit formula for the coefficients b_k in the series expansion of $\mathcal{E} = 1/\mathrm{E}_\beta$. For this purpose we define for each $k \in \mathbb{N}$ the sets

$$A(k) = \left\{ a = \left(a_1^{(k)}, \dots, a_{l_a^{(k)}}^{(k)}\right) \;\middle|\; 1 \le l_a^{(k)} \le k,\ a_i^{(k)} \ge 1,\ \sum_{i=0}^{l_a^{(k)}} a_i^{(k)} = k \right\} \tag{3.17}$$

consisting of all possible compositions a of the number k with length $l_a^{(k)}$. For example, $k = 4$ has the compositions

$$4 = 1+1+1+1 = 3+1 = 1+3 = 2+2 = 2+1+1 = 1+2+1 = 1+1+2,$$

thus

$$A(4) = \{(1,1,1,1),(3,1),(1,3),(2,2),(1,1,2),(2,1,1),(1,2,1),(4)\}.$$

For the proof of the next theorem we need two lemmas. The first one is well known in combinatorics:

Lemma 3.2.2. For each $k \in \mathbb{N}$ it holds that $|A(k)| = 2^{k-1}$.

Proof. Each element in $A(k)$ can be constructed by starting from $(\underbrace{1,1,\dots,1}_{k-times})$ by replacing the comma by $+$ or not. For the $k-1$ places we have thus 2^{k-1} choices. □

Lemma 3.2.3. For each $n \in \mathbb{N}$ it holds that

$$\sum_{k=1}^{n} \sum_{a \in A(k)} 1 = 2^n - 1.$$

Proof. For $n = 1$ the assertion is obviously true. Using induction and the previous lemma we see that

$$\sum_{k=1}^{n+1} \sum_{a \in A(k)} 1 = \sum_{k=1}^{n+1} 2^{k-1} = 2^n - 1 + 2^{n+1-1} = 2^{n+1} - 1.$$

□

Theorem 3.2.4. *Let $A(n)$, $l_a^{(n)}$ and $a_i^{(n)}$, $i = 1, \ldots, l_a^{(n)}$ be defined as in (3.17). Then the coefficients b_n from (3.15) for $n \geq 1$ are given by:*

$$b_n = \sum_{a \in A(n)} (-1)^{l_a^{(n)}} \frac{1}{\prod_{i=1}^{l_a^{(n)}} \Gamma(\beta a_i^{(n)} + 1)}.$$

Proof. For $n = 1$ we see that $A(1) = \{(1)\}$ and thus:

$$\sum_{a \in A(1)} (-1)^{l_a^{(1)}} \frac{1}{\prod_{i=1}^{l_a^{(1)}} \Gamma(\beta a_i^{(1)} + 1)} = -\frac{1}{\Gamma(\beta + 1)}.$$

Comparing with (3.16) we obtain that the assertion is true for $n = 1$. By the recursion formula (3.15) we get that

$$\begin{aligned}
b_{n+1} &= -\sum_{k=0}^{n} \frac{b_k}{\Gamma(\beta(n+1-k)+1)} \\
&= -\frac{1}{\Gamma(\beta(n+1)+1)} - \sum_{k=1}^{n} \frac{b_k}{\Gamma(\beta(n+1-k)+1)} \\
&= -\frac{1}{\Gamma(\beta(n+1)+1)} - \sum_{k=1}^{n} \sum_{a \in A(k)} (-1)^{l_a^{(k)}} \frac{1}{\Gamma(\beta(n+1-k)+1) \prod_{i=1}^{l_a^{(k)}} \Gamma(\beta a_i^{(k)} + 1)}.
\end{aligned}$$

Now we claim that

$$\begin{aligned}
&\sum_{a \in A(n+1)} (-1)^{l_a^{(n+1)}} \frac{1}{\prod_{i=1}^{l_a^{(n+1)}} \Gamma(\beta a_i^{(n+1)} + 1)} \\
&= -\frac{1}{\Gamma(\beta(n+1)+1)} - \sum_{k=1}^{n} \sum_{a \in A(k)} (-1)^{l_a^{(k)}} \frac{1}{\Gamma(\beta(n+1-k)+1) \prod_{i=1}^{l_a^{(k)}} \Gamma(\beta a_i^{(k)} + 1)}.
\end{aligned}$$

First we remark that on both sides we have $1 + 2^n - 1 = 2^n$ summands by Lemma 3.2.2 and 3.2.3. Next we show that each summand from the right hand side occurs also in the left hand side. Indeed, the summand $-1/\Gamma(\beta(n+1)+1)$ from the right hand side belongs to the composition $a = (n+1) \in A(n+1)$ in the left hand side. Now choose an arbitrary $k \in \{1, \ldots, n\}$ and $a \in A(k)$. Define

$$a^* = \left(a_1^{(k)}, \ldots, a_{l_a^{(k)}}^{(k)}, n+1-k\right).$$

We see that $l_{a^*} = l_{a^{(k)}}^{(k)} + 1 \leq k+1 \leq n+1$ and $a_i^* \geq 1$ for each $i = 1, \ldots, l_{a^*}$. Moreover

$$\sum_{i=1}^{l_{a^*}} a_i^* = \sum_{i=1}^{l_a^{(k)}} a_i^{(k)} + (n+1-k) = k + (n+1-k) = n+1.$$

This shows that $a^* \in A(n+1)$. Furthermore $(-1)^{l_{a^*}} = -(-1)^{l_a^{(k)}}$ and

$$\prod_{i=1}^{l_{a^*}} \Gamma(\beta a_i^* + 1) = \Gamma(\beta(n+1-k)+1) \prod_{i=1}^{l_a^{(k)}} \Gamma(\beta a_i^{(k)} + 1).$$

Hence the summand

$$-(-1)^{l_a^{(k)}} \frac{1}{\Gamma(\beta(n+1-k)+1)\prod_{i=1}^{l_a^{(k)}} \Gamma(\beta a_i^{(k)}+1)}$$

for an arbitrary k and $a \in A(k)$ coincides with

$$(-1)^{l_{a^*}} \frac{1}{\prod_{i=1}^{l_{a^*}} \Gamma(\beta a_i^* + 1)}.$$

This ends the proof. □

3.3 Donsker's Delta in Mittag-Leffler analysis

In Gaussian analysis, Donsker's Delta from Example 1.3.18 is an important example for a Hida distribution with many applications in quantum field theory, in the theory of stochastic differential equations and in mathematical finance, see [HKPS93, Wes95, AØU01] and references therein. In this section we introduce a distribution in $(\mathcal{N})_{\mu_\beta}^{-1}$ being the analogon of Donsker's Delta in Gaussian analysis, i.e. we would like to construct a composition of Dirac Delta at $a \in \mathbb{R}$, δ_a, and a random variable $\langle \cdot, \eta \rangle$, $\eta \in \mathcal{H}$. The strategy is as follows:

Use the integral representation for the Dirac Delta distribution δ and give sense to the expression

$$\delta(\langle \cdot, \eta \rangle) = \frac{1}{2\pi} \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}s\langle \cdot, \eta \rangle} \, \mathrm{d}s, \quad \eta \in \mathcal{H},$$

by using the characterization Theorem 2.2.2 for integration. We first remark that $\exp(\mathrm{i}s\langle \cdot, \eta \rangle) \in L^2(\mu_\beta)$ for all $s \in \mathbb{R}$ since the absolute value is $|\exp(\mathrm{i}s\langle x, \eta \rangle)|^2 = 1$.

Now we calculate the T_{μ_β}-transform, see Lemma 2.1.9, of $\exp(\mathrm{i}s\langle \cdot, \eta \rangle) \in L^2(\mu_\beta)$ for $\xi \in \mathcal{N}_{\mathbb{C}}$ with the help of Corollary 3.1.16:

$$\begin{aligned} T_{\mu_\beta} \exp(\mathrm{i}s\langle \cdot, \eta \rangle)(\xi) &= \int_{\mathcal{N}'} \exp(\mathrm{i}\langle x, s\eta + \xi \rangle) \, \mathrm{d}\mu_\beta(x) \\ &= \mathrm{E}_\beta\Big(-\frac{1}{2}\langle s\eta + \xi, s\eta + \xi \rangle\Big) \\ &= \mathrm{E}_\beta\Big(-\frac{1}{2}s^2\langle \eta, \eta \rangle - \frac{1}{2}\langle \xi, \xi \rangle - s\langle \xi, \eta \rangle\Big). \end{aligned} \tag{3.18}$$

In a next step we show that $\int_{\mathbb{R}} \left| T_{\mu_\beta} \exp(\mathrm{i}s\langle \cdot, \eta \rangle)(\xi) \right| \mathrm{d}s$ is bounded for all ξ in a neighborhood of zero of $\mathcal{N}_{\mathbb{C}}$.

Proposition 3.3.1. For $0 \neq \eta \in \mathcal{H}$, $\xi \in \mathcal{N}_{\mathbb{C}}$ and $s \in \mathbb{R}$ we denote $z(s, \eta, \xi) := \frac{1}{2}s^2\langle \eta, \eta \rangle + \frac{1}{2}\langle \xi, \xi \rangle + s\langle \eta, \xi \rangle$. Then there exists a constant $C < \infty$ such that

$$\int_{\mathbb{R}} |\mathrm{E}_\beta(-z(s, \eta, \xi))| \, \mathrm{d}s \leq C, \quad \xi \in \mathcal{U} := \left\{ \xi \in \mathcal{N}_{\mathbb{C}} \;\middle|\; |\xi| < M \right\}$$

for any $0 < M < \infty$.

Proof. First note that $\mathrm{E}_\beta(-z(\cdot, \eta, \xi))$ is obviously measurable. Furthermore it holds for $\xi = \xi_1 + \mathrm{i}\xi_2$ for $\xi_1, \xi_2 \in \mathcal{N}$ that

$$\begin{aligned} \Re\left(z(s, \eta, \xi)\right) &= \frac{1}{2}s^2\langle \eta, \eta \rangle + \frac{1}{2}\left(\langle \xi_1, \xi_1 \rangle - \langle \xi_2, \xi_2 \rangle\right) + s\langle \eta, \xi_1 \rangle \\ &= \frac{1}{2}s^2\langle \eta, \eta \rangle + \frac{1}{2}\left(\left|\xi_1^2\right| - |\xi_2|^2\right) + s\langle \eta, \xi_1 \rangle. \end{aligned}$$

Now use Corollary A.1.9 below which states that E_β is the Laplace transform of M_β. Then we obtain:

$$\begin{aligned}
&\int_{\mathbb{R}} |\mathrm{E}_\beta(-z(s,\eta,\xi))| \,\mathrm{d}s \\
&= \int_{\mathbb{R}} \left| \int_0^\infty M_\beta(r) \exp\left(-rz(s,\eta,\xi)\right) \,\mathrm{d}r \right| \,\mathrm{d}s \\
&\leq \int_0^\infty M_\beta(r) \int_{\mathbb{R}} \exp\left(-r\Re(z(s,\eta,\xi))\right) \,\mathrm{d}s \,\mathrm{d}r.
\end{aligned}$$

The Gaussian integral is calculated as follows

$$\begin{aligned}
\int_{\mathbb{R}} \exp\left(-r\Re(z(s,\eta,\xi))\right) \,\mathrm{d}s &= \mathrm{e}^{-\frac{1}{2}r(|\xi_1|^2-|\xi_2|^2)} \int_{\mathbb{R}} \exp\left(-\frac{1}{2}s^2 r\langle\eta,\eta\rangle - sr\langle\eta,\xi_1\rangle\right) \mathrm{d}s \\
&= \mathrm{e}^{-\frac{1}{2}r(|\xi_1|^2-|\xi_2|^2)} \sqrt{\frac{2\pi}{r\langle\eta,\eta\rangle}} \exp\left(\frac{r^2\langle\eta,\xi_1\rangle^2}{2r\langle\eta,\eta\rangle}\right).
\end{aligned}$$

Thus we obtain

$$\begin{aligned}
&\int_{\mathbb{R}} |\mathrm{E}_\beta(-z(s,\eta,\xi))| \,\mathrm{d}s \\
&\leq \sqrt{\frac{2\pi}{\langle\eta,\eta\rangle}} \int_0^\infty M_\beta(r) r^{-1/2} \exp\left(-r\left(\frac{1}{2}|\xi_1|^2 - \frac{1}{2}|\xi_2|^2 - \frac{\langle\eta,\xi_1\rangle^2}{2\langle\eta,\eta\rangle}\right)\right) \mathrm{d}r.
\end{aligned}$$

Since ξ is bounded, we have by Cauchy-Schwarz inequality that

$$|\xi_1|^2 - |\xi_2|^2 - \frac{\langle\eta,\xi_1\rangle^2}{\langle\eta,\eta\rangle} > |\xi_1|^2 - M^2 - \frac{|\eta|^2|\xi_1|^2}{\langle\eta,\eta\rangle} = -M^2.$$

This yields that

$$\int_{\mathbb{R}} |\mathrm{E}_\beta(-z(s,\eta,\xi))| \,\mathrm{d}s \leq \sqrt{\frac{2\pi}{\langle\eta,\eta\rangle}} \int_0^\infty M_\beta(r) r^{-1/2} \exp\left(\frac{1}{2}M^2 r\right) \mathrm{d}r.$$

But this integral is finite as shown in Lemma A.1.5 below. □

Theorem 3.3.2. *Let $0 \neq \eta \in \mathcal{H}$. Then Donsker's Delta is defined via the integral*

$$\delta(\langle\cdot,\eta\rangle) := \frac{1}{2\pi} \int_{\mathbb{R}} \exp\left(\mathrm{i}s\langle\cdot,\eta\rangle\right) \mathrm{d}s,$$

and exists in the space $(\mathcal{N})^{-1}_{\mu_\beta}$ as a weak integral in the sense of Theorem 2.2.2. Moreover for all $\xi \in \mathcal{U}$, $\mathcal{U}$ as in Proposition 3.3.1, we have

$$\left(T_{\mu_\beta}\delta(\langle\cdot,\eta\rangle)\right)(\xi) = \frac{1}{\sqrt{2\pi\langle\eta,\eta\rangle}} H^{1\,1}_{1\,2}\left(\frac{1}{2}\langle\xi,\xi\rangle - \frac{\langle\eta,\xi\rangle^2}{2\langle\eta,\eta\rangle} \,\middle|\, \begin{matrix}(1/2,1)\\(0,1),(1/2\beta,\beta)\end{matrix}\right),$$

where H denotes the Fox-H-function, see Appendix A.

Proof. Due to Proposition 3.3.1 there exists $C < \infty$ such that

$$\frac{1}{2\pi}\int_{\mathbb{R}} \left| \left(T_{\mu_\beta} \exp(\mathrm{i}s\langle\cdot,\eta\rangle)\right)(\xi)\right| \,\mathrm{d}x < C, \quad \xi \in \mathcal{U}.$$

Thus, by Theorem 2.2.2 the existence of $\delta(\langle\cdot,\eta\rangle) \in (\mathcal{N})^{-1}_{\mu_\beta}$ follows. Finally we calculate the T_{μ_β}-transform of Donsker's Delta using Corollary A.1.9 below:

$$\begin{aligned} &T_{\mu_\beta}\delta(\langle\cdot,\eta\rangle)(\xi) = \frac{1}{2\pi}\int_{\mathbb{R}} \mathrm{E}_\beta(-z(s,\eta,\xi))\,\mathrm{d}s \\ &= \frac{1}{2\pi}\int_0^\infty M_\beta(r)\exp\left(-\frac{1}{2}r\langle\xi,\xi\rangle\right)\int_{\mathbb{R}}\exp\left(-\frac{1}{2}r\langle\eta,\eta\rangle s^2 - r\langle\eta,\xi\rangle s\right)\mathrm{d}s\,\mathrm{d}r \\ &= \frac{1}{\sqrt{2\pi\langle\eta,\eta\rangle}}\int_0^\infty M_\beta(r) r^{-1/2}\exp\left(-r\left(\frac{1}{2}\langle\xi,\xi\rangle - \frac{\langle\eta,\xi\rangle^2}{2\langle\eta,\eta\rangle}\right)\right)\mathrm{d}r. \end{aligned}$$

Using Lemma A.1.8 below we get the desired result. □

Remark 3.3.3. With the help of (A.3) below we can find the series expansion of the T_{μ_β}-transform of Donsker's Delta. In fact we have

$$H^{1\,1}_{1\,2}\left(z\,\middle|\,\begin{matrix}(1/2,1)\\(0,1),(1/2\beta,\beta)\end{matrix}\right) = \sum_{k=0}^\infty \frac{(-1)^k\Gamma(k+1/2)}{k!\Gamma(1+\beta(k-1/2))}z^k, \quad z\in\mathbb{C}\setminus\{0\}.$$

The above H-function can be analytically continued to $z=0$ using the power series and has the value

$$H^{1\,1}_{1\,2}\left(0\,\middle|\,\begin{matrix}(1/2,1)\\(0,1),(1/2\beta,\beta)\end{matrix}\right) = \frac{\Gamma(1/2)}{\Gamma(1-1/2\beta)} = \frac{\sqrt{\pi}}{\Gamma(1-1/2\beta)}.$$

Note that in the case $\beta = 1$:

$$\sum_{k=0}^\infty \frac{(-1)^k\Gamma(k+1/2)}{k!\Gamma(1+k-1/2)}z^k = \mathrm{e}^{-z}.$$

Thus our definition of Donsker's Delta for $\beta=1$ becomes the usual Donsker's Delta as known in Gaussian analysis, see Example 1.3.18.

Corollary 3.3.4. The generalized expectation of Donsker's Delta is given by

$$\mathbb{E}_{\mu_\beta}\left(\delta\left(\langle\cdot,\eta\rangle\right)\right) = \left(T_{\mu_\beta}\delta\left(\langle\cdot,\eta\rangle\right)\right)(0).$$

Using the series expansion from Remark 3.3.3, we get:

$$\mathbb{E}_{\mu_\beta}\left(\delta\left(\langle\cdot,\eta\rangle\right)\right) = \frac{1}{\sqrt{2\pi\langle\eta,\eta\rangle}}\frac{\Gamma(1/2)}{\Gamma(1-1/2\beta)} = \frac{1}{\sqrt{2\langle\eta,\eta\rangle}\Gamma(1-1/2\beta)}.$$

For any arbitrary point $a\in\mathbb{R}$ Donsker's Delta $\delta_a(\langle\cdot,\eta\rangle)$ can be constructed in a similar way:

Theorem 3.3.5. *Let $0 \neq \eta \in \mathcal{H}$ and $a \in \mathbb{R}$ arbitrary. Then*

$$\delta_a(\langle\cdot,\eta\rangle) = \frac{1}{2\pi}\int_{\mathbb{R}}\exp\left(\mathrm{i}s(\langle\cdot,\eta\rangle - a)\right)\mathrm{d}s$$

exists in $(\mathcal{N})^{-1}_{\mu_\beta}$ as a weak integral in the sense of Theorem 2.2.2 and it is called Donsker's Delta in $a\in\mathbb{R}$.

Proof. The T_{μ_β}-transform of the integrand for $\xi \in \mathcal{N}_{\mathbb{C}}$ is given by:

$$\frac{1}{2\pi}\exp(-\mathrm{i}sa)\mathrm{E}_\beta\left(-\frac{1}{2}s^2\langle\eta,\eta\rangle - \frac{1}{2}\langle\xi,\xi\rangle - s\langle\xi,\eta\rangle\right).$$

Hence its absolute value coincides with the T_{μ_β}-transform in (3.18) in the case $a = 0$. Now we can proceed as in the case $a = 0$. □

Remark 3.3.6. Using Corollary A.1.9 the T_{μ_β}-transform of $\delta_a(\langle\cdot,\eta\rangle)$ for $\xi \in \mathcal{U}$ is given by:

$$\begin{aligned}
\left(T_{\mu_\beta}\delta_a(\langle\cdot,\eta\rangle)\right)(\xi) &= \frac{1}{2\pi}\int_{\mathbb{R}} \mathrm{e}^{-\mathrm{i}sa}\mathrm{E}_\beta\left(-\frac{1}{2}s^2\langle\eta,\eta\rangle - \frac{1}{2}\langle\xi,\xi\rangle - s\langle\xi,\eta\rangle\right)\mathrm{d}s \\
&= \frac{1}{2\pi}\int_{\mathbb{R}} \mathrm{e}^{-\mathrm{i}sa}\int_0^\infty M_\beta(r)\exp\left(-\frac{1}{2}rs^2\langle\eta,\eta\rangle - \frac{1}{2}r\langle\xi,\xi\rangle - sr\langle\xi,\eta\rangle\right)\mathrm{d}r\,\mathrm{d}s \\
&= \frac{1}{2\pi}\int_0^\infty M_\beta(r)\exp\left(-\frac{1}{2}r\langle\xi,\xi\rangle\right)\int_{\mathbb{R}}\exp\left(-\frac{1}{2}rs^2\langle\eta,\eta\rangle - s(r\langle\xi,\eta\rangle + \mathrm{i}a)\right)\mathrm{d}s\,\mathrm{d}r \\
&= \frac{1}{2\pi}\int_0^\infty M_\beta(r)\exp\left(-\frac{1}{2}r\langle\xi,\xi\rangle\right)\sqrt{\frac{2\pi}{r\langle\eta,\eta\rangle}}\exp\left(\frac{(r\langle\xi,\eta\rangle + \mathrm{i}a)^2}{2r\langle\eta,\eta\rangle}\right)\mathrm{d}r \\
&= \frac{1}{\sqrt{2\pi\langle\eta,\eta\rangle}}\int_0^\infty M_\beta(r)r^{-1/2}\exp\left(-\frac{1}{2}r\left(\langle\xi,\xi\rangle - \frac{\langle\xi,\eta\rangle^2}{\langle\eta,\eta\rangle}\right) + \frac{\mathrm{i}a\langle\xi,\eta\rangle}{\langle\eta,\eta\rangle} - \frac{a^2}{2r\langle\eta,\eta\rangle}\right)\mathrm{d}r.
\end{aligned}$$

In particular we obtain the generalized expectation of Donsker's Delta:

$$\begin{aligned}
\mathbb{E}_{\mu_\beta}\left(\delta_a(\langle\cdot,\eta\rangle)\right) &= \left(T_{\mu_\beta}\delta_a(\langle\cdot,\eta\rangle)\right)(0) \\
&= \frac{1}{\sqrt{2\pi\langle\eta,\eta\rangle}}\int_0^\infty M_\beta(r)r^{-1/2}\exp\left(-\frac{a^2}{2r\langle\eta,\eta\rangle}\right)\mathrm{d}r.
\end{aligned}$$

This integral can be calculated explicitly:

$$\mathbb{E}_{\mu_\beta}\left(\delta_a(\langle\cdot,\eta\rangle)\right) = \frac{1}{\sqrt{2\pi\langle\eta,\eta\rangle}}H^{2\,0}_{1\,2}\left(\frac{a^2}{2\langle\eta,\eta\rangle}\,\middle|\,\begin{matrix}(1-\beta/2,\beta)\\(1/2,1),(0,1)\end{matrix}\right).$$

For the calculation we refer to Section A.2.

3.3.1 Approximation of Donsker's Delta

We constructed Donsker's Delta in Theorem 3.3.2 and 3.3.5 as a weak integral in $(\mathcal{N})^{-1}_{\mu_\beta}$. In the following theorem we prove that Donsker's Delta can be approximated by a sequence of square integrable functions. In fact, the sequence consists of Bochner integrals in $L^2(\mu_\beta)$:

Theorem 3.3.7. *For $0 \neq \eta \in \mathcal{H}$ and $a \in \mathbb{R}$ it holds that*

$$\delta_a(\langle\cdot,\eta\rangle) = \lim_{n\to\infty}\frac{1}{2\pi}\int_{-n}^{n}\mathrm{e}^{\mathrm{i}s(\langle\cdot,\eta\rangle - a)}\,\mathrm{d}s \text{ in } (\mathcal{N})^{-1}_{\mu_\beta}.$$

Proof. For $n \in \mathbb{N}$ denote $\Phi_n := (2\pi)^{-1}\int_{-n}^{n}\mathrm{e}^{\mathrm{i}s(\langle\cdot,\eta\rangle - a)}\,\mathrm{d}s$. We verify that Φ_n exists in $L^2(\mu_\beta)$ as a Bochner integral. Let the mapping $f\colon[-n,n] \to L^2(\mu_\beta)$ be defined by $f(s) = \exp(\mathrm{i}s(\langle\cdot,\eta\rangle - a))$. First note that the mapping

$$[-n,n] \ni s \mapsto (f(s),g)_{L^2(\mu_\beta)}$$

is continuous for all $g \in L^2(\mu_\beta)$ and therefore measurable. Next, since $L^2(\mu_\beta)$ is separable it is clear that f is almost separably-valued. Furthermore for each $s \in [-n, n]$ it holds that

$$\|f(s)\|^2_{L^2(\mu_\beta)} = \int_{\mathcal{N}'} \left|\mathrm{e}^{\mathrm{i}s(\langle x,\eta\rangle - a)}\right|^2 \,\mathrm{d}\mu_\beta(x) = 1.$$

This yields

$$\int_{-n}^{n} \|f(s)\|_{L^2(\mu_\beta)} \,\mathrm{d}s = 2n < \infty.$$

Remark 1.2.7 shows that Φ_n exists indeed as a Bochner integral in $L^2(\mu_\beta)$.

Next we show that $(\Phi_n)_{n\in\mathbb{N}}$ converges in $(\mathcal{N})^{-1}_{\mu_\beta}$ to Donsker's Delta as $n \to \infty$. We have for the T_{μ_β}-transform of Φ_n at $\xi \in \mathcal{N}_\mathbb{C}$:

$$\begin{aligned}\left(T_{\mu_\beta}\Phi_n\right)(\xi) &= \frac{1}{2\pi}\int_{-n}^{n} \left(T_{\mu_\beta}\mathrm{e}^{\mathrm{i}s(\langle\cdot,\eta\rangle - a)}\right)(\xi)\,\mathrm{d}s \\ &= \frac{1}{2\pi}\int_{-n}^{n} \mathrm{e}^{-\mathrm{i}sa}\mathrm{E}_\beta\Big(-\frac{1}{2}s^2\langle\eta,\eta\rangle - \frac{1}{2}\langle\xi,\xi\rangle - s\langle\xi,\eta\rangle\Big)\,\mathrm{d}s,\end{aligned}$$

compare (3.18). Now it holds that the integrand

$$\mathbf{1}_{[-n,n]}(s)\mathrm{e}^{-\mathrm{i}sa}\mathrm{E}_\beta\Big(-\frac{1}{2}s^2\langle\eta,\eta\rangle - \frac{1}{2}\langle\xi,\xi\rangle - s\langle\xi,\eta\rangle\Big)$$

converges pointwisely for each $s \in \mathbb{R}$ to

$$\mathrm{e}^{-\mathrm{i}sa}\mathrm{E}_\beta\Big(-\frac{1}{2}s^2\langle\eta,\eta\rangle - \frac{1}{2}\langle\xi,\xi\rangle - s\langle\xi,\eta\rangle\Big)$$

as $n \to \infty$ and it is bounded by $\left|\mathrm{E}_\beta\Big(-\frac{1}{2}s^2\langle\eta,\eta\rangle - \frac{1}{2}\langle\xi,\xi\rangle - s\langle\xi,\eta\rangle\Big)\right|$. It was shown in Proposition 3.3.1 that there is a neighborhood $\mathcal{U} \subset \mathcal{N}_\mathbb{C}$ of zero and a constant $C > 0$ such that

$$\int_\mathbb{R} \left|\mathrm{E}_\beta\Big(-\frac{1}{2}s^2\langle\eta,\eta\rangle - \frac{1}{2}\langle\xi,\xi\rangle - s\langle\xi,\eta\rangle\Big)\right| \,\mathrm{d}s < C, \quad \xi \in \mathcal{U}.$$

Applying dominated convergence we see that $\left(T_{\mu_\beta}\Phi_n\right)(\xi)$ converges as $n \to \infty$ to

$$\frac{1}{2\pi}\int_\mathbb{R} \mathrm{e}^{-\mathrm{i}sa}\mathrm{E}_\beta\Big(-\frac{1}{2}s^2\langle\eta,\eta\rangle - \frac{1}{2}\langle\xi,\xi\rangle - s\langle\xi,\eta\rangle\Big)\,\mathrm{d}s.$$

This shows in particular that $\big((T_{\mu_\beta}\Phi_n)(\xi)\big)_{n\in\mathbb{N}}$ is a Cauchy sequence. Moreover

$$\left|\left(T_{\mu_\beta}\Phi_n\right)(\xi)\right| \le \frac{1}{2\pi}\int_\mathbb{R} \left|\mathrm{E}_\beta\Big(-\frac{1}{2}s^2\langle\eta,\eta\rangle - \frac{1}{2}\langle\xi,\xi\rangle - s\langle\xi,\eta\rangle\Big)\right| \,\mathrm{d}s < C$$

for all $n \in \mathbb{N}$ and for all $\xi \in \mathcal{U}$. The assertion follows by applying Theorem 2.3.1. □

Chapter 4

Grey Noise Analysis

The main ideas of grey noise analysis go back to Schneider in [Sch90]. He constructed grey Brownian motion on a concrete probability space. Further details were given amongst others by Mura and Mainardi in [MM09] and also shortly in [Kuo96, KS93]. In this chapter, we give some more details to the construction of [Sch90, MM09] and we describe the arising difficulties. We prefer to introduce a different but equivalent construction of grey noise analysis as a special case of Mittag-Leffler analysis, where the spaces $\mathcal{N}$ and $\mathcal{H}$ are chosen in a suitable way. We introduce generalized grey Brownian motion (ggBm for short) and prove differentiability using the results of Chapter 3. In this way we construct a grey noise process as derivative of ggBm. Moreover we prove the existence of local times of ggBm using Donsker's delta from Section 3.3.

4.1 Schneider's and Mura's construction

Consider the space $\mathcal{S}(\mathbb{R})$ of Schwartz test functions, see Example 1.1.4, equipped with the following scalar product

$$(\xi,\eta)_\alpha = C(\alpha)\int_{\mathbb{R}} |x|^{1-\alpha}\,\overline{\tilde{\xi}(x)}\tilde{\eta}(x)\,\mathrm{d}x, \quad \xi,\eta\in\mathcal{S}(\mathbb{R}).$$

Here, $0<\alpha<2$ and $C(\alpha)=\Gamma(\alpha+1)\sin(\frac{\pi\alpha}{2})$. The notation $\tilde{\eta}$ stands for the Fourier transform of $\eta\in\mathcal{S}(\mathbb{R})$ which is defined by

$$\tilde{\eta}(x) = (\mathcal{F}\eta)(x) := \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \eta(t)\mathrm{e}^{\mathrm{i}tx}\,\mathrm{d}t, \quad x\in\mathbb{R}. \tag{4.1}$$

By $\|\cdot\|_\alpha$ we denote the norm generated by $(\cdot,\cdot)_\alpha$. Define the Hilbert space $\mathcal{H}_\alpha$ to be the abstract completion of $\mathcal{S}(\mathbb{R})$ with respect to $(\cdot,\cdot)_\alpha$. By $I_\alpha\colon\mathcal{S}(\mathbb{R})\hookrightarrow\mathcal{H}_\alpha$ we denote the embedding $\xi\mapsto\vec{\xi}:=(\xi)_{n\in\mathbb{N}}\in\mathcal{H}_\alpha$. The following can be shown:

Proposition 4.1.1. $\mathcal{S}(\mathbb{R})$ is densely and topologically embedded in $\mathcal{H}_\alpha$, i.e. the embedding $I_\alpha\colon\mathcal{S}(\mathbb{R})\hookrightarrow\mathcal{H}_\alpha$ is continuous and $I_\alpha(\mathcal{S}(\mathbb{R}))\subset\mathcal{H}_\alpha$ is dense.

Proof. From general theory it follows that every metric space X is densely embedded in its completion, see e.g. [Alt06]. Now let $\xi\in\mathcal{S}(\mathbb{R})$ and $0<\alpha<2$. By construction

$\|I_\alpha(\xi)\|_\alpha = \|\vec{\xi}\|_\alpha = \|\xi\|_\alpha$. Furthermore:

$$\begin{aligned}\|\xi\|_\alpha^2 &= C(\alpha)\int_{|x|\leq 1}\left|\tilde{\xi}(x)\right|^2 |x|^{1-\alpha}\,\mathrm{d}x + C(\alpha)\int_{|x|>1}|x|^2\left|\tilde{\xi}(x)\right|^2\frac{|x|^{1-\alpha}}{|x|^2}\,\mathrm{d}x\\ &\leq C(\alpha)\sup_{|x|\leq 1}\left|\tilde{\xi}(x)\right|^2\int_{|x|\leq 1}|x|^{1-\alpha}\,\mathrm{d}x + C(\alpha)\sup_{|x|>1}\left|x\tilde{\xi}(x)\right|^2\int_{|x|>1}|x|^{-1-\alpha}\,\mathrm{d}x\\ &\leq C(\alpha)\left|\tilde{\xi}\right|_{0,0}^2\frac{2}{2-\alpha} + C(\alpha)\left|\tilde{\xi}\right|_{1,0}^2\frac{2}{\alpha},\end{aligned}$$

where $|\cdot|_{r,s}$ denotes the seminorm from Example 1.1.4. Since the Fourier transform is continuous on $\mathcal{S}(\mathbb{R})$ we find $m,n\in\mathbb{N}$ and $K<\infty$ such that

$$\left|\tilde{\xi}\right|_{0,0}^2 + \left|\tilde{\xi}\right|_{1,0}^2 \leq K\,|\xi|_{m,n}^2.$$

Then $\|\xi\|_\alpha^2 \leq C\,|\xi|_{m,n}^2$, where $C = KC(\alpha)\max(2/(2-\alpha), 2/\alpha)$. Thus the continuity of I_α is shown. □

Schneider gives in [Sch92] an orthonormal system $(h_n^\alpha)_{n\in\mathbb{N}}$ in $\mathcal{S}(\mathbb{R})$ with respect to $(\cdot,\cdot)_\alpha$. Indeed, consider the family of functions $(h_n^\alpha)_{n\in\mathbb{N}}$ defined via the Fourier transform by

$$\begin{aligned}\widetilde{h_{2n}^\alpha}(x) &= a_{n,\alpha}\exp(-\tfrac{x^2}{2})L_n^{-\frac{\alpha}{2}}(x^2),\\ \widetilde{h_{2n+1}^\alpha}(x) &= b_{n,\alpha}\exp(-\tfrac{x^2}{2})xL_n^{1-\alpha/2}(x^2),\quad x\in\mathbb{R},\end{aligned}$$

where $a_{n,\alpha} = \left(\frac{\Gamma(n+1)}{C(\alpha)\Gamma(n+1-\alpha/2)}\right)^{-1/2}$ and $b_{n,\alpha} = \left(\frac{\Gamma(n+1)}{C(\alpha)\Gamma(n+2-\alpha/2)}\right)^{-1/2}$ are normalization constants and L_n^γ, $\gamma>-1$, are the generalized Laguerre polynomials. Then for each $n\in\mathbb{N}$ it holds that $h_n^\alpha\in\mathcal{S}(\mathbb{R})$ and that $(h_n^\alpha, h_m^\alpha)_\alpha = \delta_{n,m}$ for all $n,m\in\mathbb{N}$.

[MM09] constructed an operator A^α on $\mathcal{H}_\alpha$ such that $A^\alpha h_n^\alpha = (2n+2+1-\alpha)h_n^\alpha$. Indeed, first define the two differential operators $A^\alpha_{\text{even}} = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{\alpha-1}{x}\frac{\mathrm{d}}{\mathrm{d}x} + x^2$ and $A^\alpha_{\text{odd}} = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{\alpha-1}{x}\frac{\mathrm{d}}{\mathrm{d}x} + x^2 - \frac{\alpha-1}{x^2}$. Then it can be calculated, that

$$\begin{aligned}A^\alpha_{\text{even}}\widetilde{h_{2n}^\alpha}(x) &= (4n+2-\alpha)\widetilde{h_{2n}^\alpha}(x),\\ A^\alpha_{\text{odd}}\widetilde{h_{2n+1}^\alpha}(x) &= (4n+4-\alpha)\widetilde{h_{2n+1}^\alpha}(x),\quad x\in\mathbb{R}.\end{aligned}$$

Denote now by $\tilde{A}^\alpha_{\text{even}} = \mathcal{F}^{-1}A^\alpha_{\text{even}}\mathcal{F}$ and the same in the odd case and decompose the space $\mathcal{H}_\alpha = L\oplus L^\perp$, where $L = \operatorname{span}\{h_{2n},\ n\in\mathbb{N}\}$ and $L^\perp = \operatorname{span}\{h_{2n+1},\ n\in\mathbb{N}\}$. Finally the operator

$$A^\alpha = \tilde{A}^\alpha_{\text{even}}|_L \oplus \tilde{A}^\alpha_{\text{odd}}|_{L^\perp} + 1$$

gives the desired operator. The properties of this operator allow to define a chain of spaces, similar to the case in white noise analysis, see Example 1.1.4. For $p\in\mathbb{N}$ let

$$\mathcal{H}_{\alpha,p} = \left\{\xi\in\mathcal{H}_\alpha \;\middle|\; \|(A^\alpha)^p\xi\|_\alpha < \infty\right\}.$$

Then we obtain

$$\mathcal{S}_\alpha \subset \mathcal{H}_{\alpha,p} \subset \mathcal{H}_\alpha \subset \mathcal{H}_{\alpha,-p} \subset \mathcal{S}'_\alpha$$

where $\mathcal{H}_{\alpha,-p}$ is the dual of $\mathcal{H}_{\alpha,p}$ and $\mathcal{S}_\alpha$ denotes the projective limit of $\mathcal{H}_{\alpha,p}$ with dual space $\mathcal{S}'_\alpha$ which is the inductive limit of $\mathcal{H}_{\alpha,-p}$.

Remark 4.1.2. The above considerations show that $\mathcal{S}_\alpha \subset \mathcal{H}_\alpha \subset \mathcal{S}'_\alpha$ is a nuclear triple as in Section 1.1. Thus we may introduce a Mittag-Leffler measure μ_β on $\mathcal{S}'_\alpha$ as demonstrated in Chapter 3. Nevertheless we propose a slightly different choice of the nuclear triple $\mathcal{N} \subset \mathcal{H} \subset \mathcal{N}'$ for two reasons.

(*i*) The space $\mathcal{H}_\alpha$ is defined as an abstract completion. Therefore its elements are Cauchy sequences. For the definition of ggBm later on it is necessary to have $\mathbf{1}_{[0,t)} \in \mathcal{H}_\alpha$. The question arises in which sense the indicator functions $\mathbf{1}_{[a,b)}$, $a, b \in \mathbb{R}$, are in $\mathcal{H}_\alpha$. A way out would be to identify the indicator functions with a sequence of Schwartz test functions $(\xi_n)_{n\in\mathbb{N}}$, where $(\xi_n)_{n\in\mathbb{N}}$ approximates the indicator function in $L^2(\mathbb{R}, \mathrm{d}x)$. But then the later analysis will become more complicated.

(*ii*) The nature of the test function space $\mathcal{S}_\alpha$ is not obvious. It is not clear if $\mathcal{S}_\alpha$ coincides with $\mathcal{S}(\mathbb{R})$.

Therefore we find it useful to develop in the following an alternative approach to a grey noise analysis. We will see that ggBm can be introduced in a convenient way starting with the nuclear triple $\mathcal{S}(\mathbb{R}) \subset L^2(\mathbb{R}, \mathrm{d}x) \subset \mathcal{S}'(\mathbb{R})$.

4.2 Grey noise analysis as special case of Mittag-Leffler analysis

We consider the usual nuclear triple from white noise analysis, see Example 1.1.4,

$$\mathcal{S}(\mathbb{R}) \subset L^2(\mathbb{R}, \mathrm{d}x) \subset \mathcal{S}'(\mathbb{R}).$$

Define the operator $M^H_\pm$ for $H \in (0, 1)$ on $\mathcal{S}(\mathbb{R})$ by

$$M^H_\pm f := \begin{cases} K_H D_\pm^{-(H-\frac{1}{2})} f, & H \in (0, \frac{1}{2}), \\ f, & H = \frac{1}{2}, \\ K_H I_\pm^{H-\frac{1}{2}} f, & H \in (\frac{1}{2}, 1), \end{cases}$$

with normalization constant

$$K_H := \Gamma\left(H + \frac{1}{2}\right) \left(\int_0^\infty (1+s)^{H-\frac{1}{2}} - s^{H-\frac{1}{2}} \,\mathrm{d}s + \frac{1}{2H}\right)^{-\frac{1}{2}}.$$

Remark 4.2.1. (*i*) The choice of the normalization constant K_H is common in literature, see e.g. Proposition 7.2.6 in [ST94] and will get important in Lemma 4.2.2 below. In fact, K_H is connected to $C(\alpha)$ via $K^2_{\alpha/2} = C(\alpha)$.

(*ii*) $D^\alpha_\pm$ and $I^\alpha_\pm$ denote the fractional derivative and fractional integral of order α, respectively. More details are given in Appendix B. Note that it is not important for the definition of $M^H_\pm$, whether the fractional derivative is in the sense of Riemann-Liouville, Caputo or Marchaud, since all these fractional derivatives coincide on $\mathcal{S}(\mathbb{R})$, see Theorem B.4.1. A necessary condition for f to be in the domain of the Caputo derivative is that f is differentiable. Hence the Caputo derivative of indicator functions is not defined. Therefore the operator $M^H_\pm$ denotes in the following the Riemann-Liouville fractional derivative or the Marchaud fractional derivative.

(*iii*) Although defined on $\mathcal{S}(\mathbb{R})$, the domain of $M_\pm^H$ is larger. In particular, $M_\pm^H$ can be applied to indicator functions $\mathbf{1}_{[a,b)}$, $-\infty < a < b < \infty$, and it holds that

$$\left(M_\pm^H \mathbf{1}_{[a,b)}\right)(t) = \frac{1}{\Gamma(1+H-1/2)} \left(\mp(b-t)_\mp^{H-1/2} \pm (a-t)_\mp^{H-1/2}\right), \quad t \in \mathbb{R},$$

see Example B.1.7. In the case $H \in (1/2, 1)$ note that $\mathbf{1}_{[a,b)} \in L^{1/H}(\mathbb{R}, \mathrm{d}x)$ and $1 < 1/H < 1/(H-1/2)$. Then Theorem B.1.4 yields that $M_\pm^H \mathbf{1}_{[a,b)} \in L^2(\mathbb{R}, \mathrm{d}x)$. For $H \in (0, 1/2)$ Corollary B.1.9 shows that $M_\pm^H \mathbf{1}_{[a,b)} \in L^2(\mathbb{R}, \mathrm{d}x)$.

The following result for the Fourier transform of a fractional integral or a fractional derivative of $\xi \in \mathcal{S}(\mathbb{R})$ is known, see e.g. [SKM93]:

$$\mathcal{F} I_\pm^\alpha \xi(x) = \tilde{\xi}(x)(\mp \mathrm{i}x)^{-\alpha} \tag{4.2}$$

$$\mathcal{F} D_\pm^\alpha \xi(x) = \tilde{\xi}(x)(\mp \mathrm{i}x)^{\alpha}, \quad x \in \mathbb{R}, \tag{4.3}$$

where $0 < \alpha < 1$, $(\mp \mathrm{i}x)^\alpha = |x|^\alpha \exp(\mp \frac{\alpha\pi\mathrm{i}}{2} \operatorname{sign}(x))$. With the help of this result we give a relation between the α-scalar product and the usual L^2-scalar product, compare [LV14].

Lemma 4.2.2. The α-scalar product and the $L^2(\mathbb{R}, \mathrm{d}x)$-scalar product are related by the operator $M_\pm^H$. In fact for all $\xi, \eta \in \mathcal{S}(\mathbb{R})$ it holds

$$(\xi, \eta)_\alpha = (M_\pm^{\alpha/2}\xi, M_\pm^{\alpha/2}\eta)_{L^2(\mathbb{R},\mathrm{d}x)}.$$

Proof. From [Mis08] we know that the constant K_H can be calculated to be $K_H = \sqrt{2H \sin(\pi H)\Gamma(2H)}$ and thus $K_{\alpha/2}^2 = C(\alpha)$. Consider first the case $1 < \alpha < 2$. Let $\xi \in \mathcal{S}(\mathbb{R})$, then $\xi \in L^p(\mathbb{R}, \mathrm{d}x)$ for $p = 2/\alpha$. Note that $1 < 2/\alpha < 2/(\alpha-1)$. It follows from Theorem B.1.4 that $I_\pm^{(\alpha-1)/2}\xi \in L^q(\mathbb{R}, \mathrm{d}x)$ for $q = \frac{p}{1-p(\alpha-1)/2} = 2$. Thus it holds that $M_\pm^{\alpha/2}\xi \in L^2(\mathbb{R}, \mathrm{d}x)$. Hence we can do the following computation using (4.2) and Plancherel's theorem:

$$\begin{aligned}
(\xi, \eta)_\alpha &= C(\alpha) \int_{\mathbb{R}} |x|^{1-\alpha}\, \overline{\tilde{\xi}(x)} \tilde{\eta}(x)\, \mathrm{d}x \\
&= C(\alpha) \int_{\mathbb{R}} \overline{|x|^{-(\alpha-1)/2}\, \tilde{\xi}(x) \exp\left(\mp \tfrac{(\alpha-1)\pi\mathrm{i}}{4} \operatorname{sign}(x)\right)} \\
&\qquad |x|^{-(\alpha-1)/2}\, \tilde{\eta}(x) \exp\left(\mp \tfrac{(\alpha-1)\pi\mathrm{i}}{4} \operatorname{sign}(x)\right) \mathrm{d}x \\
&= C(\alpha) \int_{\mathbb{R}} \overline{\widetilde{I_\pm^{(\alpha-1)/2}\xi}(x)}\, \widetilde{I_\pm^{(\alpha-1)/2}\eta}(x)\, \mathrm{d}x \\
&= \left(\widetilde{M_\pm^{\alpha/2}\xi}, \widetilde{M_\pm^{\alpha/2}\eta}\right)_{L^2(\mathbb{R},\mathrm{d}x)} = \left(M_\pm^{\alpha/2}\xi, M_\pm^{\alpha/2}\eta\right)_{L^2(\mathbb{R},\mathrm{d}x)}.
\end{aligned}$$

In the case $0 < \alpha < 1$ note that $\left(\widetilde{D_\pm^{(1-\alpha)/2}\xi}\right)(x) = \tilde{\xi}(x)(\mp\mathrm{i}x)^{(1-\alpha)/2}$. This defines a square integrable function and hence $D_\pm^{(1-\alpha)/2}\xi \in L^2(\mathbb{R}, \mathrm{d}x)$ and we can do an analog calculation as above. □

In the following we need that Lemma 4.2.2 not only holds on $\mathcal{S}(\mathbb{R})$ but also for indicator functions. This will be established by the next lemma.

Lemma 4.2.3. Let $0 < \alpha < 2$. Then for every $a, b, c, d \in \mathbb{R}$ it holds

$$\left(\mathbf{1}_{[a,b)}, \mathbf{1}_{[c,d)}\right)_\alpha = \left(M_\pm^{\alpha/2}\mathbf{1}_{[a,b)}, M_\pm^{\alpha/2}\mathbf{1}_{[c,d)}\right)_{L^2(\mathbb{R},\mathrm{d}x)}.$$

Proof. Consider first the case $1 < \alpha < 2$. It is calculated in [SKM93] that for all $\xi \in L^1(\mathbb{R}, \mathrm{d}x)$ it holds

$$\mathcal{F}M_\pm^{\alpha/2}\xi(x) = \mathcal{F}I_\pm^{(\alpha-1)/2}\xi(x) = \tilde{\xi}(x)(\mp\mathrm{i}x)^{(1-\alpha)/2}, \quad x \in \mathbb{R}.$$

From Remark 4.2.1 we already know that $M_\pm^{\alpha/2}\mathbf{1}_{[a,b)} \in L^2(\mathbb{R}, \mathrm{d}x)$. Hence we may apply Plancherel's theorem. The assertion follows now by the same calculation as in the proof of Lemma 4.2.2.

For the case $0 < \alpha < 1$ note that it is known from [GS64] that a Fourier transform pair is given in the distributional sense by

$$t_\pm^{\alpha-1} \quad \overset{\mathcal{F}}{\longleftrightarrow} \quad \frac{\Gamma(\alpha)}{\sqrt{2\pi}}(\mp\mathrm{i}x)^{-\alpha}, \quad 0 < \alpha < 1.$$

Here,

$$t_+ := \begin{cases} t, & t > 0, \\ 0, & t < 0, \end{cases} \quad \text{and} \quad t_- := \begin{cases} 0, & t > 0, \\ -t, & t < 0. \end{cases}$$

This means that for each $\xi \in \mathcal{S}_\mathbb{C}(\mathbb{R})$ it holds that

$$\langle \mathcal{F}(\cdot)_\pm^{-\alpha}, \xi\rangle = \langle (\cdot)_\pm^{-\alpha}, \mathcal{F}\xi\rangle = \frac{\Gamma(1-\alpha)}{\sqrt{2\pi}}\langle (\mp\mathrm{i}\cdot)^{\alpha-1}, \xi\rangle,$$

i.e.

$$\int_\mathbb{R} t_\pm^{-\alpha}(\mathcal{F}\xi)(t)\,\mathrm{d}t = \frac{\Gamma(1-\alpha)}{\sqrt{2\pi}}\int_\mathbb{R}(\mp\mathrm{i}x)^{\alpha-1}\xi(x)\,\mathrm{d}x.$$

Denote by $f_+(x) := -(b-x)_-^{-\alpha} + (a-x)_-^{-\alpha}$ for $x \in \mathbb{R}$. Then Corollary B.1.9 shows that $f_+ \in L^1(\mathbb{R}, \mathrm{d}x) \cap L^2(\mathbb{R}, \mathrm{d}x)$ for $0 < \alpha < 1/2$. Furthermore it holds for each $\xi \in \mathcal{S}_\mathbb{C}(\mathbb{R})$:

$$\begin{aligned}
\langle \mathcal{F}f_+, \xi\rangle &= \langle f_+, \mathcal{F}\xi\rangle = \int_\mathbb{R} f_+(x)(\mathcal{F}\xi)(x)\,\mathrm{d}x \\
&= -\int_\mathbb{R}(b-x)_-^{-\alpha}(\mathcal{F}\xi)(x)\,\mathrm{d}x + \int_\mathbb{R}(a-x)_-^{-\alpha}(\mathcal{F}\xi)(x)\,\mathrm{d}x \\
&= -\int_b^\infty (x-b)^{-\alpha}(\mathcal{F}\xi)(x)\,\mathrm{d}x + \int_a^\infty (x-a)^{-\alpha}(\mathcal{F}\xi)(x)\,\mathrm{d}x \\
&= -\int_0^\infty t^{-\alpha}(\mathcal{F}\xi)(b+t)\,\mathrm{d}t + \int_0^\infty t^{-\alpha}(\mathcal{F}\xi)(a+t)\,\mathrm{d}t \\
&= \int_\mathbb{R} t_+^{-\alpha}\left((\mathcal{F}\xi)(a+t) - (\mathcal{F}\xi)(b+t)\right)\mathrm{d}t \\
&= \frac{1}{\sqrt{2\pi}}\int_\mathbb{R}\int_\mathbb{R} t_+^{-\alpha}\xi(s)(\mathrm{e}^{\mathrm{i}as} - \mathrm{e}^{\mathrm{i}bs})\mathrm{e}^{\mathrm{i}ts}\,\mathrm{d}s\,\mathrm{d}t = \int_\mathbb{R} t_+^{-\alpha}\mathcal{F}\left((\mathrm{e}^{\mathrm{i}a\cdot} - \mathrm{e}^{\mathrm{i}b\cdot})\xi\right)(t)\,\mathrm{d}t \\
&= \frac{\Gamma(1-\alpha)}{\sqrt{2\pi}}\int_\mathbb{R}(-\mathrm{i}x)^{\alpha-1}(\mathrm{e}^{\mathrm{i}ax} - \mathrm{e}^{\mathrm{i}bx})\xi(x)\,\mathrm{d}x \\
&= \frac{\Gamma(1-\alpha)}{\sqrt{2\pi}}\langle(-\mathrm{i}\cdot)^{\alpha-1}(\mathrm{e}^{\mathrm{i}a\cdot} - \mathrm{e}^{\mathrm{i}b\cdot}), \xi\rangle.
\end{aligned}$$

Since $D_+^\alpha \mathbf{1}_{[a,b)}(x) = \frac{1}{\Gamma(1-\alpha)} f_+(x)$, see Example B.1.7, we conclude that

$$\widetilde{D_+^\alpha \mathbf{1}_{[a,b]}}(x) = -\frac{1}{\sqrt{2\pi}}(-\mathrm{i}x)^{\alpha-1}(\mathrm{e}^{\mathrm{i}bx} - \mathrm{e}^{\mathrm{i}ax}), \quad x \in \mathbb{R}.$$

Since it is well known that

$$\widetilde{\mathbf{1}_{[a,b)}}(x) = \frac{1}{\sqrt{2\pi}} \frac{\mathrm{e}^{ibx} - \mathrm{e}^{\mathrm{i}ax}}{\mathrm{i}x}, \quad x \in \mathbb{R},$$

we obtain that for all $0 < \alpha < 1/2$ and for all $x \in \mathbb{R}$ it holds

$$\widetilde{D_+^\alpha \mathbf{1}_{[a,b)}}(x) = \widetilde{\mathbf{1}_{[a,b)}}(x)(-\mathrm{i}x)(-\mathrm{i}x)^{\alpha-1} = \widetilde{\mathbf{1}_{[a,b)}}(x)(-\mathrm{i}x)^\alpha.$$

A similar calculation for $f_-(x) = (b-x)_+^{-\alpha} - (a-x)_+^{-\alpha}$ shows that

$$\widetilde{D_-^\alpha \mathbf{1}_{[a,b)}}(x) = \widetilde{\mathbf{1}_{[a,b)}}(x)(\mathrm{i}x)^\alpha.$$

Thus (4.3) also holds for indicator functions if $0 < \alpha < 1/2$. Note that for $0 < \alpha < 1$ we have $M_\pm^{\alpha/2} = K_{\alpha/2} D_\pm^{(1-\alpha)/2}$. For the order of the fractional derivative it holds $0 < (1-\alpha)/2 < 1/2$. Therefore we may apply (4.3) and the assertion follows by the same arguments as in Lemma 4.2.2. □

Corollary 4.2.4. For all $s, t \geq 0$ and for all $0 < \alpha < 2$ it holds that

$$\left(M_-^{\alpha/2}\mathbf{1}_{[0,t)}, M_-^{\alpha/2}\mathbf{1}_{[0,s)}\right)_{L^2(\mathbb{R},\mathrm{d}x)} = \frac{1}{2}(t^\alpha + s^\alpha - |t-s|^\alpha).$$

Proof. The result follows from the previous lemma and Example 3.1 in [MM09]. □

4.2.1 General grey noise measure and generalized grey Brownian motion

As in Definition 3.1.9 we define the corresponding Mittag-Leffler measures μ_β for $0 < \beta < 1$ on $\mathcal{S}'(\mathbb{R})$ by

$$\int_{\mathcal{S}'(\mathbb{R})} \mathrm{e}^{\mathrm{i}\langle\omega,\xi\rangle}\, \mathrm{d}\mu_\beta(\omega) = \mathrm{E}_\beta\left(-\frac{1}{2}\langle\xi,\xi\rangle\right), \quad \xi \in \mathcal{S}(\mathbb{R}).$$

The Mittag-Leffler measures μ_β on $\mathcal{S}'(\mathbb{R})$ are in the following referred to as *general grey noise measures*. For $\beta = 1$ the general grey noise measure is the Gaussian white noise measure, see Example 1.3.4. Lemma 3.1.11 gives all moments of μ_β and Remark 3.1.12 allows the extension of the dual pairing $\langle\cdot,\cdot\rangle$ to $\mathcal{S}'(\mathbb{R}) \times L^2(\mathbb{R}, \mathrm{d}x)$. Furthermore note that $M_\pm^{\alpha/2}\mathbf{1}_{[0,t)} \in L^2(\mathbb{R}, \mathrm{d}x)$ for all $0 < \alpha < 2$ by Remark 4.2.1. Thus the following definition makes sense:

Definition 4.2.5. For $0 < \alpha < 2$ we define the process

$$\mathcal{S}'(\mathbb{R}) \ni \omega \mapsto B_t^{\alpha,\beta}(\omega) := \langle\omega, M_-^{\alpha/2}\mathbf{1}_{[0,t)}\rangle, \quad t \geq 0,$$

and call this process *generalized grey Brownian motion*. In the case $\alpha = \beta$ we write B_t^α instead of $B_t^{\alpha,\alpha}$. B_t^α is called *grey Brownian motion*.

Remark 4.2.6. In the approach of [MM09] the grey noise measure is defined via the characteristic function $\mathrm{E}_\beta(-(\cdot,\cdot)_\alpha)$ and denoted by $\mu_{\alpha,\beta}$. This means that first the parameters $0 < \alpha < 2$ and $0 < \beta < 1$ are fixed and then generalized grey Brownian motion $B_t^{\alpha,\beta}$ is constructed in $L^2(\mu_{\alpha,\beta})$. The measure μ_β as defined above is named general grey noise measure since for fixed $0 < \beta < 1$ all generalized grey Brownian motions $B_t^{\alpha,\beta}$ for $0 < \alpha < 2$ can be constructed in the single space $L^2(\mu_\beta)$. The general grey noise measure μ_β is also considered in Example 8.5 in [Kuo96]. It is denoted by ν_λ, $0 < \lambda \leq 1$ and defined via the characteristic function

$$\int_{\mathcal{N}'} \mathrm{e}^{\mathrm{i}\langle x,\xi\rangle}\,\mathrm{d}\nu_\lambda(x) = \mathrm{E}_\lambda\left(-\langle \xi,\xi\rangle\right).$$

[Kuo96] refers to ν_λ as grey noise measure.

Proposition 4.2.7. Let $0 < \alpha < 2$ and $0 < \beta < 1$. Then for all $p \in \mathbb{N}$ there exists $K < \infty$ such that

$$\mathbb{E}_{\mu_\beta}\left(\left|B_t^{\alpha,\beta} - B_s^{\alpha,\beta}\right|^{2p}\right) \leq K\,|t-s|^{\alpha p}, \quad t,s \geq 0.$$

Proof. Without loss of generality let $s < t$. First note that $\|\mathbf{1}_{[s,t)}\|_\alpha = \|\mathbf{1}_{[0,t-s)}\|_\alpha$. Indeed, it holds for the Fourier transform of the indicator function

$$(\mathcal{F}\mathbf{1}_{[s,t)})(x) = \frac{1}{\sqrt{2\pi}}\frac{\mathrm{e}^{\mathrm{i}tx} - \mathrm{e}^{\mathrm{i}sx}}{\mathrm{i}x} = \frac{\mathrm{e}^{\mathrm{i}sx}}{\sqrt{2\pi}}\frac{\mathrm{e}^{\mathrm{i}(t-s)x} - 1}{\mathrm{i}x} = \mathrm{e}^{\mathrm{i}sx}(\mathcal{F}\mathbf{1}_{[0,t-s)})(x), \quad x \in \mathbb{R}.$$

Then by the definition of $\|\cdot\|_\alpha$ we get

$$\begin{aligned}
\|\mathbf{1}_{[s,t)}\|_\alpha^2 &= C(\alpha)\int_{\mathbb{R}} |x|^{1-\alpha}\left|(\mathcal{F}\mathbf{1}_{[s,t)})(x)\right|^2\,\mathrm{d}x \\
&= C(\alpha)\int_{\mathbb{R}} |x|^{1-\alpha}\left|(\mathcal{F}\mathbf{1}_{[0,t-s)})(x)\right|^2\,\mathrm{d}x \\
&= \|\mathbf{1}_{[0,t-s)}\|_\alpha^2.
\end{aligned}$$

Corollary 4.2.4 shows that $\|\mathbf{1}_{[s,t)}\|_\alpha^2 = (t-s)^\alpha$. Using the moments of μ_β, see Lemma 3.1.11, we find

$$\begin{aligned}
\mathbb{E}_{\mu_\beta}\left(\left|B_t^{\alpha,\beta} - B_s^{\alpha,\beta}\right|^{2p}\right) &= \int_{\mathcal{S}'(\mathbb{R})} \langle \omega, M_-^{\alpha/2}\mathbf{1}_{[s,t)}\rangle^{2p}\,\mathrm{d}\mu_\beta(\omega) \\
&= \frac{(2p)!}{2^p\Gamma(\beta p+1)}\langle M_-^{\alpha/2}\mathbf{1}_{[s,t)}, M_-^{\alpha/2}\mathbf{1}_{[s,t)}\rangle^p \\
&= \frac{(2p)!}{2^p\Gamma(\beta p+1)}\|\mathbf{1}_{[s,t)}\|_\alpha^{2p} \\
&= \frac{(2p)!}{2^p\Gamma(\beta p+1)}(t-s)^{\alpha p}.
\end{aligned}$$

□

The last proposition ensures that generalized grey Brownian motion has a continuous version. Indeed, choose $p \in \mathbb{N}$ such that $\alpha p > 1$ then the previous proposition provides the estimate $\mathbb{E}_{\mu_\beta}((B_t^{\alpha,\beta} - B_s^{\alpha,\beta})^{2p}) \leq K\,|t-s|^{1+q}$ with $q = \alpha p - 1 > 0$. This estimate is sufficient to apply Kolmogorov's continuity theorem.

Proposition 4.2.8. $B_t^{\alpha,\beta}$ from Definition 4.2.5 has the following properties:

(i) $B_t^{\alpha,\beta}$ has covariance

$$\mathbb{E}_{\mu_\beta}\left(B_t^{\alpha,\beta}B_s^{\alpha,\beta}\right) = \frac{1}{2\Gamma(\beta+1)}\left(t^\alpha + s^\alpha - |t-s|^\alpha\right) =: \frac{\gamma_{\alpha,\beta}(t,s)}{2}, \quad t,s \geq 0,$$

and $\mathbb{E}_{\mu_\beta}((B_t^{\alpha,\beta})^2) = \frac{1}{\Gamma(\beta+1)}t^\alpha$ for $t \geq 0$.

(ii) The finite dimensional distributions of $B_t^{\alpha,\beta}$ are given by

$$f_{\alpha,\beta}(x) = \frac{(2\pi)^{-n/2}}{\sqrt{\Gamma(1+\beta)^n \det \gamma_{\alpha,\beta}}}\int_0^\infty \frac{M_\beta(\tau)}{\tau^{n/2}} \exp\left(-\frac{1}{2}\frac{x^T\gamma_{\alpha,\beta}^{-1}x}{\tau\Gamma(1+\beta)}\right) \mathrm{d}\tau,$$

where $x = (x_1,\dots,x_n) \in \mathbb{R}^n$ and $\gamma_{\alpha,\beta} = \left(\gamma_{\alpha,\beta}(t_i,t_j)\right)_{i,j=1,\dots,n} \in \mathbb{R}^{n\times n}$. For the definition of the M-function M_β see Chapter A.

(iii) $B_t^{\alpha,\beta}$ has stationary increments.

Proof. For $s,t \geq 0$ it holds

$$\mathbb{E}_{\mu_\beta}\left(B_t^{\alpha,\beta}B_s^{\alpha,\beta}\right) = \int_{\mathcal{N}'} \langle \omega, M_-^{\alpha/2}\mathbf{1}_{[0,t)}\rangle\langle \omega, M_-^{\alpha/2}\mathbf{1}_{[0,s)}\rangle \, \mathrm{d}\mu_\beta(\omega).$$

With Corollary 3.1.19 we obtain

$$\begin{aligned}\mathbb{E}_{\mu_\beta}\left(B_t^{\alpha,\beta}B_s^{\alpha,\beta}\right) &= \frac{1}{\Gamma(\beta+1)}\langle M_-^{\alpha/2}\mathbf{1}_{[0,t)}, M_-^{\alpha/2}\mathbf{1}_{[0,s)}\rangle \\ &= \frac{1}{\Gamma(\beta+1)}\left(M_-^{\alpha/2}\mathbf{1}_{[0,t)}, M_-^{\alpha/2}\mathbf{1}_{[0,s)}\right)_{L^2(\mathbb{R},\mathrm{d}x)}.\end{aligned}$$

Corollary 4.2.4 yields

$$\mathbb{E}_{\mu_\beta}\left(B_t^{\alpha,\beta}B_s^{\alpha,\beta}\right) = \frac{1}{2\Gamma(\beta+1)}\left(t^\alpha + s^\alpha - |t-s|^\alpha\right).$$

Obviously we can deduce that $\mathbb{E}_{\mu_\beta}\left((B_t^{\alpha,\beta})^2\right) = \frac{1}{\Gamma(\beta+1)}t^\alpha$ and *(i)* is proved.

For proving *(ii)* let $0 \leq t_1 < \ldots < t_n$ and $w \in \mathbb{R}^n$. We calculate the characteristic function

$$\mathbb{E}_{\mu_\beta}\left(\exp\left(\mathrm{i}\sum_{j=1}^n w_j B_{t_j}^{\alpha,\beta}\right)\right) = \mathbb{E}_{\mu_\beta}\left(\exp\left(\mathrm{i}\langle\cdot, \sum_{j=1}^n w_j M_-^{\alpha/2}\mathbf{1}_{[0,t_j)}\rangle\right)\right).$$

By definition of the general grey noise measure we obtain

$$\mathbb{E}_{\mu_\beta}\left(\exp\left(\mathrm{i}\sum_{j=1}^n w_j B_{t_j}^{\alpha,\beta}\right)\right) = \mathrm{E}_\beta\left(-\frac{1}{2}\sum_{i,j=1}^n w_i w_j (M_-^{\alpha/2}\mathbf{1}_{[0,t_i)}, M_-^{\alpha/2}\mathbf{1}_{[0,t_j)})_{L^2(\mathbb{R},\mathrm{d}x)}\right).$$

Using Corollary 4.2.4 we end up with

$$\mathbb{E}_{\mu_\beta}\left(\exp\left(\mathrm{i}\sum_{j=1}^n w_j B_{t_j}^{\alpha,\beta}\right)\right) = \mathrm{E}_\beta\left(-\frac{1}{2}\Gamma(\beta+1)\sum_{i,j} w_i w_j \gamma_{\alpha,\beta}(t_i,t_j)\right).$$

For the Fourier transform of $f_{\alpha,\beta}$ we find:

$$\begin{aligned}
&\int_{\mathbb{R}^n} \mathrm{e}^{\mathrm{i}w^T x} f_{\alpha,\beta}(x)\, \mathrm{d}^n x \\
&= \frac{(2\pi)^{-n/2}}{\sqrt{\Gamma(\beta+1)^n \det \gamma_{\alpha,\beta}}} \int_0^\infty \frac{M_\beta(\tau)}{\tau^{n/2}} \int_{\mathbb{R}^n} \mathrm{e}^{\mathrm{i}w^T x} \exp\left(-\frac{1}{2}\frac{x^T \gamma_{\alpha,\beta}^{-1} x}{\Gamma(\beta+1)\tau}\right) \mathrm{d}^n x\, \mathrm{d}\tau.
\end{aligned}$$

Making the change of variables $x = y\sqrt{\Gamma(\beta+1)\tau}$ we get

$$\begin{aligned}
&\int_{\mathbb{R}^n} \mathrm{e}^{\mathrm{i}w^T x} f_{\alpha,\beta}(x)\, \mathrm{d}^n x \\
&= \frac{(2\pi)^{-n/2}}{\sqrt{\det \gamma_{\alpha,\beta}}} \int_0^\infty M_\beta(\tau) \int_{\mathbb{R}^n} \exp\left(-\frac{1}{2} y^T \gamma_{\alpha,\beta}^{-1} y + \mathrm{i}\sqrt{\Gamma(\beta+1)\tau}\, w^T y\right) \mathrm{d}^n y\, \mathrm{d}\tau \\
&= \int_0^\infty M_\beta(\tau) \exp\left(-\tau \frac{1}{2}\Gamma(\beta+1) w^T \gamma_{\alpha,\beta} w\right) \mathrm{d}\tau = \mathrm{E}_\beta\left(-\frac{1}{2}\Gamma(\beta+1) w^T \gamma_{\alpha,\beta} w\right).
\end{aligned}$$

The last equality is a consequence of Corollary A.1.9. By uniqueness of the characteristic function the second assertion follows.

(*iii*): For all $x \in \mathbb{R}$ it holds

$$\begin{aligned}
\mathbb{E}_{\mu_\beta}\left(\mathrm{e}^{\mathrm{i}x(B_{t+h}^{\alpha,\beta} - B_t^{\alpha,\beta})}\right) &= \int_{\mathcal{N}'} \mathrm{e}^{\mathrm{i}\langle \omega, x M_-^{\alpha/2} \mathbf{1}_{[t,t+h)}\rangle}\, \mathrm{d}\mu_\beta(\omega) \\
&= \mathrm{E}_\beta\left(-\frac{1}{2}\langle x M_-^{\alpha/2} \mathbf{1}_{[t,t+h)}, x M_-^{\alpha/2} \mathbf{1}_{[t,t+h)}\rangle\right).
\end{aligned}$$

By Lemma 4.2.2 it follows that

$$\mathbb{E}_{\mu_\beta}\left(\mathrm{e}^{\mathrm{i}x(B_{t+h}^{\alpha,\beta} - B_t^{\alpha,\beta})}\right) = \mathrm{E}_\beta\left(-\frac{1}{2}x^2 \|1_{[t,t+h)}\|_\alpha^2\right).$$

Note that for each $t, h \geq 0$ it holds $\|\mathbf{1}_{[t,t+h)}\|_\alpha = \|\mathbf{1}_{[0,h)}\|_\alpha$. Thus

$$\begin{aligned}
\mathbb{E}_{\mu_\beta}\left(\mathrm{e}^{\mathrm{i}x(B_{t+h}^{\alpha,\beta} - B_t^{\alpha,\beta})}\right) &= \mathrm{E}_\beta\left(-\frac{1}{2}x^2 \|1_{[0,h)}\|_\alpha^2\right) \\
&= \int_{\mathcal{N}'} \mathrm{e}^{\mathrm{i}\langle \omega, x M_-^{\alpha/2} \mathbf{1}_{[0,h)}\rangle}\, \mathrm{d}\mu_\beta(\omega) \\
&= \mathbb{E}_{\mu_\beta}\left(\mathrm{e}^{\mathrm{i}x B_h^{\alpha,\beta}}\right).
\end{aligned}$$

Hence the statement follows. □

Remark 4.2.9. Consider the scaled ggBm $\hat{B}_t^{\alpha,\beta} := B_{2^{1/\alpha}t}^{\alpha,\beta}$. Then a comparison of Proposition 4.2.8 with Proposition 2 in [MP08] shows that the scaled version of Definition 4.2.5 coincides with the definitions of [MP08, MM09]. Consequently $\hat{B}^{\beta,\beta}$ is a grey Brownian motion as in [Sch92].

Remark 4.2.10. Generalized grey Brownian motion is a generalization of well known stochastic processes. Choosing $\beta = 1$ then Proposition 4.2.8 shows that $\mathbb{E}_{\mu_1}(B_t^{\alpha,1} B_s^{\alpha,1}) = 1/2((t^\alpha + s^\alpha - |t-s|^\alpha)$ and the finite dimensional distributions of $B_t^{\alpha,1}$ are given by

$$f_{\alpha,1}(x) = \frac{1}{\sqrt{2\pi \det \gamma_{\alpha,1}}^n} \exp\left(-\frac{1}{2}x^T \gamma_{\alpha,1}^{-1} x\right).$$

Thus $B_t^{\alpha,1}$ is a fractional Brownian motion with Hurst parameter $H = \alpha/2$. Consequently the process $B_t^{1,1}$ is a Brownian motion.

Remark 4.2.11. A random vector $X = (X_1, \dots, X_n)^T$, $n \in \mathbb{N}$, is said to have an elliptical distribution if its characteristic function is of the form

$$\mathbb{E}(\mathrm{e}^{\mathrm{i}\theta^T X}) = \mathrm{e}^{\mathrm{i}\theta^T \mu}\phi(\theta^T \Sigma \theta), \quad \theta \in \mathbb{R}^n,$$

with parameter $\mu \in \mathbb{R}^n$, symmetric and positive definite $\Sigma \in \mathbb{R}^{n\times n}$ and some characteristic function $\phi\colon \mathbb{R} \to \mathbb{R}$, see e.g. equation (2.11) in [FKN90]. For short, the distribution of X is denoted by $X \sim \mathrm{EC}_n(\mu, \Sigma, \phi)$. The proof of Proposition 4.2.8 shows that $X = (B_{t_1}^{\alpha,\beta}, \dots, B_{t_n}^{\alpha,\beta})$ has an elliptical distribution and $X \sim \mathrm{EC}_n(0, \Gamma(\beta+1)\gamma_{\alpha,\beta}, \mathrm{E}_\beta(-\cdot))$, see also [dSE15].

4.2.2 Grey noise process

Applying the results of Chapter 2 and 3 we construct the corresponding spaces of test functions $(\mathcal{N})^1$ and distributions $(\mathcal{N})^{-1}_{\mu_\beta}$, denoted by $(\mathcal{S})^1$ and $(\mathcal{S})^{-1}_{\mu_\beta}$, respectively.

In the following we calculate the S_{μ_β}-transform of a generalized grey Brownian motion $B_t^{\alpha,\beta} = \langle \cdot, M_-^{\alpha/2}\mathbf{1}_{[0,t)}\rangle \in L^2(\mu_\beta)$ for $\xi = \xi_1 + i\xi_2 \in \mathcal{U}_\beta \subset \mathcal{S}_{\mathbb{C}}(\mathbb{R})$, compare Remark 3.1.17:

$$\begin{aligned}\left(S_{\mu_\beta} B_t^{\alpha,\beta}\right)(\xi) &= \frac{1}{\mathrm{E}_\beta\left(\frac{1}{2}\langle \xi,\xi\rangle\right)} \int_{\mathcal{S}'(\mathbb{R})} \langle \omega, M_-^{\alpha/2}\mathbf{1}_{[0,t)}\rangle \mathrm{e}^{\langle \omega,\xi\rangle}\, \mathrm{d}\mu_\beta(\omega)\\ &= \frac{1}{\mathrm{E}_\beta\left(\frac{1}{2}\langle \xi,\xi\rangle\right)} \int_{\mathcal{S}'(\mathbb{R})} \frac{\mathrm{d}}{\mathrm{d}s} \mathrm{e}^{\langle \omega,\xi\rangle + s\langle \omega, M_-^{\alpha/2}\mathbf{1}_{[0,t)}\rangle}\Big|_{s=0}\, \mathrm{d}\mu_\beta(\omega).\end{aligned}$$

We may interchange integral and derivative for the following reason: For $\omega \in \mathcal{S}'(\mathbb{R})$ and $s \in [-1,1]$ denote $f(\omega,s) := \exp(\langle \omega, \xi + sM_-^{\alpha/2}\mathbf{1}_{[0,t)}\rangle)$. Then f is integrable with respect to ω by Corollary 3.1.18 and differentiable with respect to s and

$$\frac{\mathrm{d}}{\mathrm{d}s} f(\omega,s) = \langle \omega, M_-^{\alpha/2}\mathbf{1}_{[0,t)}\rangle \mathrm{e}^{\langle \omega, \xi + sM_-^{\alpha/2}\mathbf{1}_{[0,t)}\rangle}.$$

Moreover we have for all $s \in [-1,1]$ and for all $\omega \in \mathcal{S}'(\mathbb{R})$ the estimate

$$\begin{aligned}\left|\frac{\mathrm{d}}{\mathrm{d}s} f(\omega,s)\right| &\le \left|\langle \omega, M_-^{\alpha/2}\mathbf{1}_{[0,t)}\rangle\right| \left|\mathrm{e}^{\langle \omega, \xi + sM_-^{\alpha/2}\mathbf{1}_{[0,t)}\rangle}\right|\\ &\le \left|\langle \omega, M_-^{\alpha/2}\mathbf{1}_{[0,t)}\rangle\right| \mathrm{e}^{\langle \omega,\xi_1\rangle} \mathrm{e}^{\left|\langle \omega, M_-^{\alpha/2}\mathbf{1}_{[0,t)}\rangle\right|}\\ &\le \mathrm{e}^{\langle \omega,\xi_1\rangle}\mathrm{e}^{2\left|\langle \omega, M_-^{\alpha/2}\mathbf{1}_{[0,t)}\rangle\right|} =: g(\omega).\end{aligned}$$

In Lemma 3.1.14 and Corollary 3.1.18 it is shown that both mappings $\exp(\langle \cdot, \xi_1\rangle)$ and $\exp\left(2\left|\langle \cdot, M_-^{\alpha/2}\mathbf{1}_{[0,t)}\rangle\right|\right)$ are in $L^2(\mu_\beta)$. Hence we have that $g \in L^1(\mu_\beta)$ by the Hölder inequality. Thus interchanging derivative and integral is possible and Corollary 3.1.16 shows that

$$\begin{aligned}&\left(S_{\mu_\beta} B_t^{\alpha,\beta}\right)(\xi)\\ &= \frac{1}{\mathrm{E}_\beta\left(\frac{1}{2}\langle \xi,\xi\rangle\right)} \frac{\mathrm{d}}{\mathrm{d}s} \mathrm{E}_\beta\left(\frac{1}{2}\langle \xi + sM_-^{\alpha/2}\mathbf{1}_{[0,t)}, \xi + sM_-^{\alpha/2}\mathbf{1}_{[0,t)}\rangle\right)\Big|_{s=0}.\end{aligned}$$

Using Lemma 3.1.3 we have

$$\frac{\mathrm{d}}{\mathrm{d}s}\mathrm{E}_\beta\left(\frac{1}{2}\langle\xi+sM_-^{\alpha/2}\mathbf{1}_{[0,t)},\xi+sM_-^{\alpha/2}\mathbf{1}_{[0,t)}\rangle\right)$$
$$=\frac{1}{\beta}\mathrm{E}_{\beta,\beta}\left(\frac{1}{2}\langle\xi+sM_-^{\alpha/2}\mathbf{1}_{[0,t)},\xi+sM_-^{\alpha/2}\mathbf{1}_{[0,t)}\rangle\right)$$
$$\left(s\langle M_-^{\alpha/2}\mathbf{1}_{[0,t)},M_-^{\alpha/2}\mathbf{1}_{[0,t)}\rangle+\langle\xi,M_-^{\alpha/2}\mathbf{1}_{[0,t)}\rangle\right).$$

We evaluate at $s=0$ and obtain

$$\left(S_{\mu_\beta}B_t^{\alpha,\beta}\right)(\xi)=\langle\xi,M_-^{\alpha/2}\mathbf{1}_{[0,t)}\rangle\frac{\mathrm{E}_{\beta,\beta}\left(\frac{1}{2}\langle\xi,\xi\rangle\right)}{\beta\mathrm{E}_\beta(\frac{1}{2}\langle\xi,\xi\rangle)}.$$

We may apply the integration by-parts-formula (B.5) in the case $1<\alpha<2$ since ξ and $\mathbf{1}_{[0,t)}$ are in every $L^p(\mathbb{R},\mathrm{d}x)$, $p\geq 1$. For the case $0<\alpha<1$ note that $D_\pm^{(1-\alpha)/2}\mathbf{1}_{[0,t)}\in L^p(\mathbb{R},\mathrm{d}x)$ for $p=2/(1+2\alpha)$ by Corollary B.1.9, since $\frac{1-\alpha}{2}\frac{2}{1+2\alpha}<1$. Moreover $D_\pm^{(1-\alpha)/2}\xi\in L^2(\mathbb{R},\mathrm{d}x)$ and $1/p+1/2=1+\alpha$. Thus we are allowed to use the integration by-parts-formula (B.6) and we get

$$\left(S_{\mu_\beta}B_t^{\alpha,\beta}\right)(\xi)=\langle M_+^{\alpha/2}\xi,\mathbf{1}_{[0,t)}\rangle\frac{\mathrm{E}_{\beta,\beta}\left(\frac{1}{2}\langle\xi,\xi\rangle\right)}{\beta\mathrm{E}_\beta(\frac{1}{2}\langle\xi,\xi\rangle)}$$
$$=\frac{\mathrm{E}_{\beta,\beta}\left(\frac{1}{2}\langle\xi,\xi\rangle\right)}{\beta\mathrm{E}_\beta(\frac{1}{2}\langle\xi,\xi\rangle)}\int_0^t\left(M_+^{\alpha/2}\xi\right)(x)\,\mathrm{d}x. \tag{4.4}$$

Remark 4.2.12. Remember the series expansion of the factor $\mathcal{E}(z)=1/\mathrm{E}_\beta(z)=\sum_{n=0}^\infty b_nz^n$ if $|z|<\varepsilon_\beta$, see (3.13). For $\xi\in\mathcal{U}_\beta$ we then get:

$$\frac{\mathrm{E}_{\beta,\beta}\left(\frac{1}{2}\langle\xi,\xi\rangle\right)}{\beta\mathrm{E}_\beta(\frac{1}{2}\langle\xi,\xi\rangle)}=\frac{1}{\beta}\left(\sum_{n=0}^\infty\frac{1}{2^n\Gamma(\beta n+\beta)}\langle\tau^{\otimes n},\xi^{\otimes 2n}\rangle\right)\left(\sum_{n=0}^\infty\frac{b_n}{2^n}\langle\tau^{\otimes n},\xi^{\otimes 2n}\rangle\right)$$
$$=\frac{1}{\beta}\sum_{n=0}^\infty\sum_{k=0}^n\frac{b_{n-k}}{2^{n-k}}\langle\tau^{\otimes n-k},\xi^{\otimes 2n-2k}\rangle\frac{1}{2^k\Gamma(\beta k+\beta)}\langle\tau^{\otimes k},\xi^{\otimes 2k}\rangle$$
$$=\frac{1}{\beta}\sum_{n=0}^\infty\frac{1}{2^n}\langle\tau^{\otimes n},\xi^{\otimes 2n}\rangle\sum_{k=0}^n\frac{b_{n-k}}{\Gamma(\beta k+\beta)}.$$

From this calculation we can deduce the chaos decomposition of generalized grey Brownian motion. Indeed, it holds for every $t\geq 0$ that

$$\left(S_{\mu_\beta}B_t^{\alpha,\beta}\right)(\xi)=\sum_{n=0}^\infty\frac{1}{2^n\beta}\sum_{k=0}^n\frac{b_{n-k}}{\Gamma(\beta k+\beta)}\langle\tau^{\otimes n}\otimes M_-^{\alpha/2}\mathbf{1}_{[0,t)},\xi^{\otimes 2n+1}\rangle. \tag{4.5}$$

Assume that $B_t^{\alpha,\beta}=\sum_{n=0}^\infty Q_n^{\mu_\beta}(\Phi^{(n)})$ for suitable kernels $\Phi^{(n)}\in\mathcal{S}'_{\mathbb{C}}(\mathbb{R}^n)$, see (2.3). Then it holds for the S_{μ_β}-transform:

$$\left(S_{\mu_\beta}B_t^{\alpha,\beta}\right)(\xi)=\sum_{n=0}^\infty\langle\Phi^{(n)},\xi^{\otimes n}\rangle.$$

By comparison with (4.5) we get for all $n\in\mathbb{N}$ that $\Phi^{(2n)}=0$ and

$$\Phi^{(2n+1)}=\frac{1}{2^n\beta}\sum_{k=0}^n\frac{b_{n-k}}{\Gamma(\beta k+\beta)}\tau^{\otimes n}\otimes M_-^{\alpha/2}\mathbf{1}_{[0,t)}.$$

Remark 4.2.13. In the special case $\alpha = \beta = 1$ we obtain

$$\left(S_{\mu_1} B_t^{1,1}\right)(\xi) = \langle \xi, \mathbf{1}_{[0,t)} \rangle = \int_0^t \xi(x)\,\mathrm{d}x$$

which is the S_{μ_1}-transform of Brownian motion, see Example 1.3.15. Moreover, the chaos decomposition of generalized grey Brownian motion reduces in the case $\alpha = \beta = 1$ to the chaos decomposition of Brownian motion. Note that for $\beta = 1$ it holds $b_n = (-1)^n/n!$. Then

$$\sum_{k=0}^{n} \frac{b_{n-k}}{\Gamma(\beta k + \beta)} = \frac{1}{n!} \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} = \frac{1}{n!}(1-1)^n = 0,$$

for $n > 0$ and the only remaining kernel is $\Phi^{(1)} = \mathbf{1}_{[0,t)}$.

Theorem 4.2.14. *A generalized grey Brownian motion $B_t^{\alpha,\beta}$ is differentiable in $(\mathcal{S})_{\mu_\beta}^{-1}$ for all $t \geq 0$, i.e.*

$$\lim_{h\to 0} \frac{B_{t+h}^{\alpha,\beta} - B_t^{\alpha,\beta}}{h}$$

exists and converges to some element in $(\mathcal{S})_{\mu_\beta}^{-1}$ denoted by $N_t^{\alpha,\beta}$ fulfilling

$$\left(S_{\mu_\beta} N_t^{\alpha,\beta}\right)(\xi) = \left(M_+^{\alpha/2}\xi\right)(t) \frac{\mathrm{E}_{\beta,\beta}\left(\frac{1}{2}\langle \xi, \xi \rangle\right)}{\beta \mathrm{E}_\beta(\frac{1}{2}\langle \xi, \xi \rangle)},$$

for every ξ from a suitable neighborhood $\mathcal{U} \subset \mathcal{S}_{\mathbb{C}}(\mathbb{R})$ of zero.

Proof. Consider for $n \in \mathbb{N}$ and $t \geq 0$

$$\Phi_n := \frac{B_{t+h_n}^{\alpha,\beta} - B_t^{\alpha,\beta}}{h_n} \in (\mathcal{S})_{\mu_\beta}^{-1},$$

for a sequence $(h_n)_{n\in\mathbb{N}}$ such that $h_n \to 0$ as $n \to \infty$. By (4.4)

$$S_{\mu_\beta}\Phi_n(\xi) = \frac{1}{h_n} \int_t^{t+h_n} \left(M_+^{\alpha/2}\xi\right)(x)\,\mathrm{d}x \frac{\mathrm{E}_{\beta,\beta}\left(\frac{1}{2}\langle \xi, \xi \rangle\right)}{\beta \mathrm{E}_\beta(\frac{1}{2}\langle \xi, \xi \rangle)},$$

for every $\xi \in \mathcal{U}_\beta \subset \mathcal{S}_{\mathbb{C}}(\mathbb{R})$. Since $M_+^{\alpha/2}\xi$ is continuous, see [Ben03], we have

$$\lim_{n\to\infty} \frac{1}{h_n} \int_t^{t+h_n} \left(M_+^{\alpha/2}\xi\right)(x)\,\mathrm{d}x \frac{\mathrm{E}_{\beta,\beta}\left(\frac{1}{2}\langle \xi, \xi \rangle\right)}{\beta \mathrm{E}_\beta(\frac{1}{2}\langle \xi, \xi \rangle)} = \left(M_+^{\alpha/2}\xi\right)(t) \frac{\mathrm{E}_{\beta,\beta}\left(\frac{1}{2}\langle \xi, \xi \rangle\right)}{\beta \mathrm{E}_\beta(\frac{1}{2}\langle \xi, \xi \rangle)}.$$

Thus $(S_{\mu_\beta}\Phi_n(\xi))_{n\in\mathbb{N}}$ is a Cauchy sequence. Since the Mittag-Leffler functions are holomorphic there are $p, q \in \mathbb{N}$ and $K < \infty$ such that

$$\left| \frac{\mathrm{E}_{\beta,\beta}\left(\frac{1}{2}\langle \xi, \xi \rangle\right)}{\beta \mathrm{E}_\beta(\frac{1}{2}\langle \xi, \xi \rangle)} \right| \leq K, \quad \text{for all } \xi \in U_{p,q}.$$

Furthermore it follows from Theorem 2.3 in [Ben03] that there is $p' \in \mathbb{N}$ such that for all $x \in \mathbb{R}$

$$\left| \left(M_+^{\alpha/2}\xi\right)(x) \right| \leq C\,|\xi|_{p'}, \quad \xi \in \mathcal{S}_{\mathbb{C}}(\mathbb{R}).$$

Here, $(|\cdot|_p)_{p\in\mathbb{N}}$ denotes the family of norms which gives the topology of $\mathcal{S}(\mathbb{R})$, see Example 1.1.4. Choose $p^* > \max(p,p')$. By the mean value theorem we find $s \in [t,t+h]$ such that for all $\xi \in U_{p^*,q}$ and for all $n \in \mathbb{N}$

$$\left|(S_{\mu_\beta}\Phi_n)(\xi)\right| \le \left|\frac{\mathrm{E}_{\beta,\beta}\left(\frac{1}{2}\langle\xi,\xi\rangle\right)}{\beta\mathrm{E}_\beta(\frac{1}{2}\langle\xi,\xi\rangle)}\right| \left|\left(M_+^{\alpha/2}\xi\right)(s)\right| \le K\,C\,|\xi|_{p'} \le K\,C 2^{-q/2} < \infty.$$

Applying Theorem 2.3.1, $(\Phi_n)_{n\in\mathbb{N}}$ converges to some distribution $N_t^{\alpha,\beta}$ in $(\mathcal{S})^{-1}_{\mu_\beta}$ fulfilling

$$\left(S_{\mu_\beta}N_t^{\alpha,\beta}\right)(\xi) = \lim_{n\to\infty} S_{\mu_\beta}\Phi_n(\xi), \quad \xi \in U_{p^*,q}.$$

□

4.3 Local times of generalized grey Brownian motion

The ggBm local time, denoted by $L_{\alpha,\beta}(a,T)$, measures the time for $B_t^{\alpha,\beta}$ spending up to time $T > 0$ in $a \in \mathbb{R}$. It is already shown in [dSE15] that generalized grey Brownian motion admits a square integrable local time by checking Berman's criteria. In the paper [dSE15] the local time is introduced as the Radon-Nikodym derivative of the occupation measure. In detail: Let I be a measurable set in the interval $[0,T]$, $T > 0$, and $f\colon I \to \mathbb{R}$ measurable. The occupation measure μ_f is defined as

$$\mu_f(B) = \int_I \mathbf{1}_B(f(s))\,\mathrm{d}s, \quad B \in \mathcal{B}(\mathbb{R}).$$

If μ_f is absolutely continuous with respect to the Lebesgue measure $\mathrm{d}x$ we denote the corresponding density by $L^f(\cdot,I)$, i.e.

$$\int_I \mathbf{1}_B(f(s))\,\mathrm{d}s = \mu_f(B) = \int_B L^f(x,I)\,\mathrm{d}x, \quad B \in \mathcal{B}(\mathbb{R}).$$

$L^f(\cdot,I)$ is called *local time* of f. Formally the local time can be expressed with the Dirac delta distribution. In fact it holds that

$$L^f(x,I) = \int_I \delta_x(f(s))\,\mathrm{d}s, \tag{4.6}$$

see also [Kuo96], page 273. This can be seen by the following considerations. First note that for each $B \in \mathcal{B}(\mathbb{R})$ we have

$$\int_B \delta_x\,\mathrm{d}x = \mathbf{1}_B. \tag{4.7}$$

Here, the integral is in the sense of Pettis, see Section 1.2. Indeed, let $\xi \in \mathcal{S}(\mathbb{R})$. Then the mapping $\mathbb{R} \ni x \mapsto \langle\delta_x,\xi\rangle = \xi(x)$ is measurable and integrable. Moreover

$$\int_B \langle\delta_x,\xi\rangle\,\mathrm{d}x = \int_B \xi(x)\,\mathrm{d}x = \int_{\mathbb{R}} \mathbf{1}_B(x)\xi(x)\,\mathrm{d}x = \langle\mathbf{1}_B,\xi\rangle.$$

This shows that for all $B \in \mathcal{B}$ it holds $\langle \mathbf{1}_B, \xi\rangle = \int_B \langle \delta_x, \xi\rangle \, \mathrm{d}x$. Note that $\mathbf{1}_B \in \mathcal{S}'(\mathbb{R})$ since

$$\int_{\mathbb{R}} \mathbf{1}_B(x) \, |\xi(x)| \, \mathrm{d}x \leq \int_{\mathbb{R}} \left|(1+x)^2 \xi(x)\right| (1+x)^{-2} \, \mathrm{d}x \leq \pi \, |\xi|_{2,0} \, .$$

Thus (4.7) is established. Now we see that (4.6) indeed gives the density of the occupation measure μ_f:

$$\int_B L^f(x, I) \, \mathrm{d}x = \int_B \int_I \delta_x(f(s)) \, \mathrm{d}s \, \mathrm{d}x = \int_I \int_B \delta_x(f(s)) \, \mathrm{d}x \, \mathrm{d}s.$$

Using (4.7) we obtain

$$\int_B \int_I \delta_x(f(s)) \, \mathrm{d}s \, \mathrm{d}x = \int_I \mathbf{1}_B(f(s)) \, \mathrm{d}s = \mu_f(B),$$

and (4.6) is shown.

Motivated by (4.6) we give a second approach for constructing the ggBm local time $L_{\alpha,\beta}(a,T)$. For $T > 0$ and $a \in \mathbb{R}$ let

$$L_{\alpha,\beta}(a,T) := \int_0^T \delta_a(B_t^{\alpha,\beta}) \, \mathrm{d}t.$$

Theorem 4.3.1. *The generalized grey Brownian motion local time $L_{\alpha,\beta}(a,T)$ with $0 < \alpha < 2$, $0 < \beta < 1$, $T > 0$ and $a \in \mathbb{R}$ exists as a weak integral in $(\mathcal{S})^{-1}_{\mu_\beta}$ with T_{μ_β}-transform*

$$\left(T_{\mu_\beta} L_{\alpha,\beta}(a,T)\right)(\xi) = \int_0^T \left(T_{\mu_\beta} \delta_a(B_t^{\alpha,\beta})\right)(\xi) \, \mathrm{d}t$$

for all ξ from a suitable neighborhood $\mathcal{U} \subset \mathcal{S}_{\mathbb{C}}(\mathbb{R})$ of zero.

Proof. Let $\xi \in \mathcal{S}_{\mathbb{C}}(\mathbb{R})$ with $|\xi|_{L^2(\mathbb{R},\mathrm{d}x)} < M$ for some $M < \infty$. According to Theorem 3.3.2 with $\eta = M_-^{\alpha/2} \mathbf{1}_{[0,t)} \in L^2(\mathbb{R}, \mathrm{d}x)$ we have

$$\begin{aligned}
&\left(T_{\mu_\beta} \delta_a(B_t^{\alpha,\beta})\right)(\xi) \\
&= \frac{1}{2\pi} \int_{\mathbb{R}} \exp(-\mathrm{i}xa) \mathrm{E}_\beta \left(-\frac{1}{2} x^2 t^\alpha - \frac{1}{2}\langle \xi, \xi\rangle - x \langle \xi, M_-^{\alpha/2} \mathbf{1}_{[0,t)}\rangle\right) \mathrm{d}x.
\end{aligned}$$

The same calculation as in Proposition 3.3.1 shows that

$$\begin{aligned}
&\int_0^T \left|T_{\mu_\beta} \delta_a(B_t^{\alpha,\beta})(\xi)\right| \, \mathrm{d}t \\
&\leq \frac{1}{\sqrt{2\pi}} \int_0^T \int_0^\infty M_\beta(r) r^{-1/2} t^{-\alpha/2} \exp\left(-\frac{1}{2} r z(t,\xi)\right) \mathrm{d}r \, \mathrm{d}t,
\end{aligned}$$

with $z(t,\xi) = \langle \xi_1, \xi_1\rangle - \langle \xi_2, \xi_2\rangle - \frac{\langle \xi_1, M_-^{\alpha/2} \mathbf{1}_{[0,t)}\rangle^2}{t^\alpha} > -M^2$ since $|\xi|_{L^2(\mathbb{R},\mathrm{d}x)} < M$ and thus

$$\begin{aligned}
&\int_0^T \left|T_{\mu_\beta} \delta_a(B_t^{\alpha,\beta})(\xi)\right| \, \mathrm{d}t \\
&\leq \frac{1}{\sqrt{2\pi}} \frac{2}{2-\alpha} T^{1-\alpha/2} \int_0^\infty M_\beta(r) r^{-1/2} \exp\left(\frac{1}{2} M^2 r\right) \mathrm{d}r.
\end{aligned}$$

By Lemma A.1.5 we get

$$\int_0^T \left| T_{\mu_\beta} \delta_a(B_t^{\alpha,\beta})(\xi) \right| \mathrm{d}t \leq \frac{K}{\sqrt{2\pi}} \frac{2}{2-\alpha} T^{1-\alpha/2}$$

for some constant $K < \infty$. The assertion follows by Theorem 2.2.2. □

Remark 4.3.2. The expectation of $L_{\alpha,\beta}(0,T)$ can be calculated explicitly, compare Corollary 3.3.4:

$$\mathbb{E}_{\mu_\beta}(L_{\alpha,\beta}(0,T)) = \int_0^T \mathbb{E}_{\mu_\beta}(\delta_0(B_t^{\alpha,\beta})) \,\mathrm{d}t$$
$$= \int_0^T \frac{1}{\sqrt{2\pi t^\alpha}} \frac{1}{\Gamma(1-\beta/2)} \,\mathrm{d}t = \frac{1}{\Gamma(1-\beta/2)\sqrt{2\pi}} \frac{2}{2-\alpha} T^{1-\alpha/2}.$$

Chapter 5

Application to Fractional Differential Equations

In this chapter, we study the relation of generalized grey Brownian motion from Chapter 4 and the time-fractional heat equation. In a first part, we give an overview over the existing literature and results, especially for the work of Schneider and Wyss [SW89, Sch92], Mainardi [Mai95] and Kochubei [Koc89, Koc90, EK04]. Moreover we explain, how a solution to the time-fractional equation can be obtained via subordination from the solution to the usual heat equation (1.5). Afterwards we establish a fractional version of the Feynman-Kac formula for grey Brownian motion and generalized grey Brownian motion, respectively, extending the work of [Sch90]. Moreover we give an outlook on a more general fractional Feynman-Kac formula by adding a potential and an inhomogeneity to the time-fractional heat equation. We end this chapter with a few remarks to the time-fractional Schrödinger equation.

5.1 The time-fractional heat equation

The well-known heat equation, presented in Section 1.4, can be generalized to a time-fractional heat equation in two different ways. In the first approach the time derivative $\partial/\partial t$ in the heat equation (1.5) is replaced by a Caputo fractional derivative of order α, denoted by ${}^C D^\alpha_{0+}$. More details on the Caputo derivative are given in Appendix B. This approach was pursued for example in [Mai95] and also in [Koc89, Koc90] and [EIK04, EK04] and leads to the following equation:

$$({}^C D^\alpha_{0+} u)(t,x) = \frac{1}{2}\frac{\partial^2}{\partial x^2} u(t,x), \quad u(0,x) = u_0(x), \tag{5.1}$$

for $0 < \alpha < 1$, $t \geq 0$, $x \in \mathbb{R}$ and initial value $u_0\colon \mathbb{R} \to \mathbb{R}$. Here ${}^C D^\alpha_{0+}$ denotes the Caputo fractional derivative with respect to the variable t.

For the second approach note that the heat equation (1.5) is equivalent to

$$u(t,x) = u_0(x) + \frac{1}{2}\int_0^t \frac{\partial^2}{\partial x^2} u(s,x)\,\mathrm{d}s, \quad x \in \mathbb{R},\ t \geq 0. \tag{5.2}$$

Now the time-fractional generalization of the heat equation is obtained by substituting the integral in (5.2) with a Riemann-Liouville fractional integral of order α, denoted by I^α_{0+}. Definitions and Properties of the fractional integral can be found in

Appendix B. Then we obtain the following time-fractional heat equation in integral form:

$$\begin{aligned} u(t,x) &= u_0(x) + \frac{1}{2}\left(I^{\alpha}_{0+}\frac{\partial^2}{\partial x^2}u(\cdot,x)\right)(t) \\ &= u_0(x) + \frac{1}{2\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\frac{\partial^2}{\partial x^2}u(s,x)\,\mathrm{d}s, \end{aligned} \tag{5.3}$$

with $0 < \alpha < 1$, $t \geq 0$, $x \in \mathbb{R}$ and initial value $u_0 \colon \mathbb{R} \to \mathbb{R}$. This approach was studied for example in [SW89, Sch90].

5.1.1 Comparison of existing results

Mainardi's result

Mainardi considers the following fractional differential equation involving the Caputo derivative of order 2α:

$$({}^{C}D^{2\alpha}_{0+}u)(t,x) = D\frac{\partial^2}{\partial x^2}u(t,x), \quad x \in \mathbb{R},\ t \geq 0, \tag{5.4}$$

where $D > 0$ and $0 < \alpha \leq 1$. In [Mai95] the Laplace transform is used to find a Green's function G_M to (5.4). In fact, the Laplace transform of G_M is given by

$$\widehat{G_M}(s,x) = \frac{1}{2s^{1-\alpha}\sqrt{D}}\mathrm{e}^{-\frac{|x|}{\sqrt{D}}s^{\alpha}}, \quad x \in \mathbb{R},\ s > 0.$$

[Mai95] introduces the entire M-function M_α as inverse Laplace transform of $\widehat{G_M}$, see Appendix A for details. Here we only want to mention that M_α has the series expansion

$$M_\alpha(z) = \sum_{n=0}^{\infty}\frac{(-1)^n}{n!\Gamma(1-\alpha n-\alpha)}z^n, \quad z \in \mathbb{C}.$$

Then the Green's function is given by

$$G_M(t,x,\alpha) = \frac{1}{2\sqrt{D}}t^{-\alpha}M_\alpha\left(\frac{|x|}{t^{\alpha}\sqrt{D}}\right), \quad x \in \mathbb{R},\ t > 0.$$

Let us calculate the series expansion of G_M:

$$\begin{aligned} G_M(t,x,\alpha) &= \frac{1}{2t^{\alpha}\sqrt{D}}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!\Gamma(1-\alpha n-\alpha)}\left(\frac{|x|}{t^{\alpha}\sqrt{D}}\right)^n \\ &= \frac{1}{2t^{\alpha}\sqrt{D}}\sum_{n=0}^{\infty}\frac{1}{(2n)!\Gamma(1-2\alpha n-\alpha)}\left(\frac{|x|^2}{t^{2\alpha}D}\right)^n \\ &\quad + \frac{|x|}{2Dt^{2\alpha}}\sum_{n=0}^{\infty}\frac{-1}{(2n+1)!\Gamma(1-2\alpha n-2\alpha)}\left(\frac{|x|^2}{t^{2\alpha}D}\right)^n, \quad x \in \mathbb{R},\ t > 0. \end{aligned}$$

Applying know (3.1) and (3.2) we obtain for all $x \in \mathbb{R}$ and $t > 0$ the following expression for G_M:

$$\begin{aligned} G_M(t,x,\alpha) &= \frac{1}{2t^{\alpha}\sqrt{D\pi}}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\frac{\Gamma(\frac{1}{2}-n)}{\Gamma(1-2\alpha n-\alpha)}\left(\frac{|x|^2}{4t^{2\alpha}D}\right)^n \\ &\quad + \frac{|x|}{4Dt^{2\alpha}\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\frac{\Gamma(-\frac{1}{2}-n)}{\Gamma(1-2\alpha n-2\alpha)}\left(\frac{|x|^2}{4Dt^{2\alpha}}\right)^n. \end{aligned} \tag{5.5}$$

For $\alpha \to 1/2$, i.e. in the case of the usual heat equation, we see that the second sum converges to zero because of the singularity of the Gamma function and hence

$$G_M(t,x,1/2) = \frac{1}{2\sqrt{D\pi t}} \exp\left(-\frac{|x|^2}{4tD}\right), \quad x \in \mathbb{R},\ t > 0,$$

which is the expected heat kernel for $D = 1/2$, see Section 1.4.

Schneider's result

Let us now examine the Green's function which is given in [Sch90]. He starts with the time-fractional heat equation in integral form on $\mathbb{R}^n$ for $0 < \alpha < 2$:

$$u(t,x) = u_0(x) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \frac{\partial^2}{\partial x^2} u(s,x)\,\mathrm{d}s, \quad t \geq 0,\ x \in \mathbb{R}^n. \tag{5.6}$$

Applying the Fourier and the Laplace transform to the equation we end up with

$$\mathcal{LF}u(p,k) = \frac{1}{p}\mathcal{F}u_0(k) - k^2 p^{-\alpha} \mathcal{LF}u(p,k), \quad k \in \mathbb{R}^n,\ p > 0,$$

see Remark B.1.6 for the Laplace transform of the fractional integral. From this we conclude that the Green's function G_S satisfies

$$\left(\mathcal{LF}G_S\right)(p,k) = \left(\mathcal{F}u_0\right)(k) \frac{1}{p(1+k^2p^{-\alpha})} = \left(\mathcal{F}u_0\right)(k) \frac{p^{\alpha-1}}{p^\alpha + k^2}.$$

The inversion of the Laplace transform yields:

$$\left(\mathcal{F}G_S\right)(t,k) = \left(\mathcal{F}u_0\right)(k) \mathrm{E}_\alpha\left(-k^2 t^\alpha\right).$$

The inversion of the Fourier transform is executed in [Sch90] and leads to the Green's function

$$G_S(t,x,\alpha) = \frac{1}{\sqrt{4\pi t^\alpha}^n} H^{2\,0}_{1\,2}\left(\frac{|x|^2}{4t^\alpha} \middle| \begin{matrix} (1-\alpha n/2, \alpha) \\ (0,1), (1-n/2, 1) \end{matrix}\right). \tag{5.7}$$

By (A.3) the series expansion of G_S is given by

$$\frac{1}{\sqrt{4\pi t^\alpha}} \sum_{k=0}^\infty \frac{(-1)^k}{k!} \frac{\Gamma(1/2-k)}{\Gamma(1-1/2\alpha-\alpha k)} z^k + \frac{1}{\sqrt{4\pi t^\alpha}} \sum_{k=0}^\infty \frac{(-1)^k}{k!} \frac{\Gamma(-1/2-k)}{\Gamma(1-\alpha-\alpha k)} z^{k+1/2}, \tag{5.8}$$

where we denote $z = |x|^2/(4t^\alpha)$. Comparing Schneider's Green's function with Mainardi's Green's function (5.5) in the case $D = 1$ we see that both solutions coincide.

Remark 5.1.1. In general, fractional differential equations with Caputo derivative are equivalent to the fractional integral equations using the Riemann-Liouville fractional integral if the solution is absolutely continuous. Indeed, it holds in this case that $(I^\alpha_{0+}\,{}^C D^\alpha_{0+} u)(t,x) = u(t,x) - u(0,x)$, see Theorem B.2.2. Thus applying the fractional integral I^α_{0+} to the fractional heat equation (5.1) with Caputo derivative we obtain the fractional heat equation (5.3) with Riemann-Liouville fractional integral:

$$u(t,x) - u(0,x) = \frac{1}{2}\left(I^\alpha_{0+} \frac{\partial^2}{\partial x^2} u(\cdot,x)\right)(t).$$

On the other hand, it holds that the Caputo fractional derivative of u_0 vanishes since u_0 is independent of t. Moreover for continuous u we have $({}^C D^\alpha_{0+} I^\alpha_{0+} u)(t,x) = u(t,x)$, see again Theorem B.2.2. Thus by applying the Caputo derivative ${}^C D^\alpha_{0+}$ to (5.3) the equation is transformed to (5.1).

Kochubei's result

A more general fractional differential equation is treated in [EK04, EIK04]. Let B be a uniformly elliptic second order differential operator with bounded and continuous real-valued coefficients $a_{ij}, b_j, c\colon \mathbb{R}^n \to \mathbb{R}$ given by

$$B = \sum_{i,j=1}^n a_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{j=1}^n b_j(x) \frac{\partial}{\partial x_j} + c(x).$$

Uniformly elliptic here means that there exists a constant $0 < C < \infty$ such that for any $x, y \in \mathbb{R}^n$:

$$\sum_{i,j=1}^n a_{ij}(x) y_i y_j \geq C \left|y\right|^2.$$

Then consider the following fractional differential equation

$$({}^C D^\alpha_{0+} u)(t,x) - Bu(t,x) = f(t,x), \quad t \in (0,T],\ x \in \mathbb{R}^n, \tag{5.9}$$

with the Caputo fractional derivative ${}^C D^\alpha_{0+}$, $0 < \alpha < 1$, and u_0 is the bounded and (Hölder) continuous initial value and f is assumed to be bounded, jointly continuous in (t,x) and locally Hölder continuous in x. Under this condition [EK04] constructs a Green matrix, i.e. a pair

$$\{Z(t,x,y), Y(t,x,y)\}, \quad t \in (0,T],\ x, y \in \mathbb{R}^n,$$

such that a solution is of the form

$$u(t,x) = \int_{\mathbb{R}^n} Z(t,x,y) u_0(y)\, \mathrm{d}y + \int_0^t \int_{\mathbb{R}^n} Y(t-\lambda, x, y) f(\lambda, y)\, \mathrm{d}y\, \mathrm{d}\lambda.$$

Since $B = \Delta$ is uniformly elliptic, the time-fractional heat equation (5.1) is a special case of (5.9). In this case, [EIK04] shows that the Green's function has the form $Z(t,x,y) = Z_0(t, x-y)$ with

$$Z_0(t,x) = \pi^{-n/2} \left|x\right|^{-n} H^{2\,0}_{1\,2}\left(\frac{|x|^2}{4t^\alpha} \middle| \begin{matrix} (1,\alpha) \\ (n/2,1),(1,1) \end{matrix} \right), \quad x \in \mathbb{R}^n,\ t \in (0,T].$$

Using the multiplication formula for the H-function, see Remark A.1.2, we get

$$\begin{aligned} Z_0(t,x) &= \frac{1}{(4\pi t^\alpha)^{n/2}} \left(\frac{|x|^2}{4t^\alpha} \right)^{-n/2} H^{2\,0}_{1\,2}\left(\frac{|x|^2}{4t^\alpha} \middle| \begin{matrix} (1,\alpha) \\ (n/2,1),(1,1) \end{matrix} \right) \\ &= \frac{1}{(4\pi t^\alpha)^{n/2}} H^{2\,0}_{1\,2}\left(\frac{|x|^2}{4t^\alpha} \middle| \begin{matrix} (1-\alpha n/2,\alpha) \\ (0,1),(1-n/2,1) \end{matrix} \right). \end{aligned}$$

Thus $Z_0(t,x) = G_S(t,x)$, see (5.7), and the solutions of Kochubei and Schneider coincide.

Now consider $B = \sum_{i,j=1}^n a_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j}$. In the following we correct a mistake in [EK04] and [EIK04]. The formulas eq. (5.2.28) in [EIK04] and eq. (2.3) in [EK04] are wrong which can be seen by the following Lemma.

Lemma 5.1.2. Consider the equation

$$\left({}^{C}D^{\alpha}_{0+}u\right)(t,x) = \sum_{i,j=1}^{n} a_{ij}\frac{\partial^2}{\partial x_i \partial x_j}u(t,x), \quad u(0,x) = u_0(x), \quad t>0,\ x\in\mathbb{R}^n, \quad (5.10)$$

with a positive definite and symmetric matrix $A = (a_{ij})_{ij} \in \mathbb{R}^{n\times n}$. Then a Green's function is given by

$$V_0(t,x) = \frac{\pi^{-n/2}}{(\det A)^{1/2}} \left(\sum_{i,j=1}^{n} (A^{-1})_{ij} x_i x_j\right)^{-n/2} H^{2\,0}_{1\,2}\left(\frac{\sum_{i,j=1}^{n}(A^{-1})_{ij}x_ix_j}{4t^\alpha}\,\middle|\,\begin{matrix}(1,\alpha)\\(n/2,1),(1,1)\end{matrix}\right).$$

Remark 5.1.3. The difference to the solution in [EIK04, EK04] lies within the prefactor $\pi^{-n/2}/(\det A)^{1/2}$. In [EIK04, EK04] A does not denote the matrix $(a_{ij})_{ij} \in \mathbb{R}^{n\times n}$ but its inverse. Thus the prefactor in eq. (5.2.28) in [EIK04] and eq. (2.3) in [EK04] has to be changed to $\pi^{-n/2}/(\det A)^{-1/2}$.

Proof. Using a coordinate transformation we find a solution in terms of the Green's function Z_0. Indeed, let $C \in \mathbb{R}^{n\times n}$ such that $A^{-1} = CC^T$. Then it follows by the chain rule that

$$\sum_{i,j=1}^{n} a_{ij}\frac{\partial^2}{\partial x_i\partial x_j}u(t,Cx) = (\Delta u)(t,Cx), \quad t\geq 0,\ x\in\mathbb{R}^n.$$

Moreover

$$\int_{\mathbb{R}} u_0(y)Z_0(t,C(x-y))\,\mathrm{d}y = \frac{1}{|\det C|}\int_{\mathbb{R}} u_0(y)Z_0(t,x-y)\,\mathrm{d}y \to \frac{1}{|\det C|}u_0(x), \quad t\to 0.$$

Let $V_0(t,x) = |\det C|\,Z_0(t,Cx) = (\det A)^{-1/2}Z_0(t,Cx)$. Then we have shown that if Z_0 is the Green's function to (5.9) with $B=\Delta$ then V_0 is a Green's function to (5.10). Thus the correct Green's function to (5.10) is given by:

$$V_0(t,x) = \frac{\pi^{-n/2}}{(\det A)^{1/2}}\,|Cx|^{-n}\,H^{2\,0}_{1\,2}\left(\frac{|Cx|^2}{4t^\alpha}\,\middle|\,\begin{matrix}(1,\alpha)\\(n/2,1),(1,1)\end{matrix}\right), \quad t>0,\ x\in\mathbb{R}^n.$$

Since A is symmetric and $CC^T = A^{-1}$ we also have $C^TC = A^{-1}$, i.e.

$$(A^{-1})_{ij} = \sum_{k=1}^{n} c_{ki}c_{kj}.$$

From this we conclude that

$$\begin{aligned}|Cx|^2 &= \sum_{k=1}^{n}\left(\sum_{i=1}^{n} c_{ki}x_i\right)^2 = \sum_{k=1}^{n}\sum_{i,j=1}^{n} c_{ki}c_{kj}x_ix_j\\ &= \sum_{i,j=1}^{n} x_ix_j\sum_{k=1}^{n}c_{ki}c_{kj} = \sum_{i,j=1}^{n}(A^{-1})_{ij}x_ix_j.\end{aligned}$$

Finally:

$$V_0(t,x) = \frac{\pi^{-n/2}}{(\det A)^{1/2}} \left(\sum_{i,j=1}^{n} (A^{-1})_{ij} x_i x_j \right)^{-n/2} H^{2\,0}_{1\,2} \left(\frac{\sum_{i,j=1}^{n} (A^{-1})_{ij} x_i x_j}{4t^\alpha} \middle| \begin{matrix} (1,\alpha) \\ (n/2,1),(1,1) \end{matrix} \right).$$

□

Remark 5.1.4. Then also the formulas 5.2.29 in [EIK04] and 2.4 in [EK04] have to be adjusted in the following way. The function Y_0 is given by the fractional derivative of order $1-\alpha$ of Z_0. Thus we have

$$\begin{aligned} &\left({}^C D_{0+}^{1-\alpha} V_0(\cdot,x)\right)(t) \\ &= \frac{\pi^{-n/2}}{(\det A)^{1/2}} \left(\sum_{i,j=1}^{n} (A^{-1})_{ij} x_i x_j \right)^{-n/2} t^{\alpha-1} H^{2\,0}_{1\,2} \left(\frac{\sum_{i,j=1}^{n} (A^{-1})_{ij} x_i x_j}{4t^\alpha} \middle| \begin{matrix} (\alpha,\alpha) \\ (n/2,1),(1,1) \end{matrix} \right), \end{aligned}$$

where we used (3.5) in [EK04]. Here, A denotes the matrix $(a_{ij})_{ij} \in \mathbb{R}^{n\times n}$ and not its inverse as in [EK04, EIK04].

Remark 5.1.5. With the help of Remark A.1.2 an easier form of V_0 and Y_0 is given by

$$\begin{aligned} &V_0(t,x) \\ &= \frac{(4\pi t^\alpha)^{-n/2}}{(\det A)^{1/2}} \left(\frac{\sum_{i,j=1}^{n} (A^{-1})_{ij} x_i x_j}{4t^\alpha} \right)^{-n/2} H^{2\,0}_{1\,2} \left(\frac{\sum_{i,j=1}^{n} (A^{-1})_{ij} x_i x_j}{4t^\alpha} \middle| \begin{matrix} (1,\alpha) \\ (n/2,1),(1,1) \end{matrix} \right) \\ &= \frac{1}{\sqrt{(4t^\alpha \pi)^n \det A}} H^{2\,0}_{1\,2} \left(\frac{\sum_{i,j=1}^{n} (A^{-1})_{ij} x_i x_j}{4t^\alpha} \middle| \begin{matrix} (1-\alpha n/2,\alpha) \\ (0,1),(1-n/2,1) \end{matrix} \right). \end{aligned}$$

Then, for $A = Id$, Schneider's Green's function G_S in (5.7) coincides with V_0. Similarly it holds:

$$\begin{aligned} &Y_0(t,x,y) \\ &= \frac{1}{\sqrt{(4\pi t^\alpha)^n \det A}} t^{\alpha-1} H^{2\,0}_{1\,2} \left(\frac{\sum_{i,j=1}^{n} (A^{-1})_{ij} x_i x_j}{4t^\alpha} \middle| \begin{matrix} (\alpha-\alpha n/2,\alpha) \\ (0,1),(1-n/2,1) \end{matrix} \right). \end{aligned}$$

5.1.2 A subordination approach

The method of subordination is used to obtain a solution to a fractional differential equation, given that the corresponding non-fractional equation has a known solution. This method is for example described in [WW01] and [BM01] from two different standpoints.

In [WW01] the equation

$$\frac{\partial}{\partial t} u(t,x) = (L(x)u)(t,x), \quad u(0,x) = u_0(x), \quad x \in \mathbb{R}^n,\ t \geq 0, \tag{5.11}$$

is considered, where L denotes a linear operator acting on the x-variable of u. It is assumed that a solution u to (5.11) has been found. The fractional equation is then given in integral form as

$$u_\alpha(t,x) = u_0(x) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left(L(x)u_\alpha\right)(s,x)\,\mathrm{d}s, \tag{5.12}$$

where $0 < \alpha < 1$. Then the solution u_α to (5.12) is related to u via

$$u_\alpha(t,x) = t^{-\alpha} \int_0^\infty g_\alpha(t^{-\alpha}s)u(s,x)\,\mathrm{d}s = \int_0^\infty g_\alpha(s)u(t^\alpha s,x)\,\mathrm{d}s. \tag{5.13}$$

Here g_α is the same probability density as in Lemma 3.1.4. Recall that

$$g_\alpha(t) = \frac{1}{\alpha} t^{-1-1/\alpha} f_\alpha(t^{-1/\alpha}) = M_\alpha(t), \quad t > 0,$$

where f_α denotes the one-sided α-stable probability density. The same relation also holds for the corresponding Green's functions. Indeed, denote the Green's function to (5.11) and (5.12) by G and G_α, respectively. Then

$$G_\alpha(t,x) = \int_0^\infty g_\alpha(s)G(t^\alpha s,x)\,\mathrm{d}s, \quad t > 0.$$

In [BM01] the method of subordination is treated in the context of semigroups. Consider a Banach space X with norm $\|\cdot\|$ and let L be the generator of a uniformly bounded, strongly continuous semigroup $(T_t)_{t\geq 0}$. Define for $0 < \alpha < 1$ and $t \geq 0$

$$S_t f = \int_0^\infty f_\alpha(s) \left(T_{(t/s)^\alpha} f\right)\,\mathrm{d}s, \quad f \in X,\ t > 0. \tag{5.14}$$

Here, the integration is in the sense of Bochner and f_α is the one-sided α-stable probability density as in Lemma 3.1.4. Then the family $(S_t)_{t\geq 0}$ is uniformly bounded, strongly continuous and strongly analytic in a sectorial region. Furthermore if we set $g(t) = S_t f$ then g is a solution to

$$\frac{\partial^\alpha}{\partial t^\alpha} u(t) = Lu(t) + u_0 \frac{t^{-\alpha}}{\Gamma(1-\alpha)}, \quad t > 0, \tag{5.15}$$

where $u(0) = u_0 \in X$ is some initial value.

Remark 5.1.6. (*i*) The fractional derivative in (5.15) which is denoted by $\partial^\alpha/\partial t^\alpha$ is defined via its Laplace transform. Indeed, it holds for suitable functions u that

$$\mathcal{L}\left(\frac{\partial^\alpha}{\partial t^\alpha} u\right)(s) = s^\alpha (\mathcal{L}u)(s), \quad s \geq 0.$$

Hence this fractional derivative is different to the Riemann-Liouville derivative and the Caputo fractional derivative which are usually considered in literature and also in this work. But taking into account the initial value it follows

$$\mathcal{L}\left(\frac{\partial^\alpha}{\partial t^\alpha} u - \frac{u_0}{\Gamma(1-\alpha)} (\cdot)^{-\alpha}\right)(s) = s^\alpha(\mathcal{L}u)(s) - u(0)s^{\alpha-1}.$$

But this is exactly the Laplace transform of the Caputo fractional derivative of u, see Remark B.2.3. Thus $g(t) = S_t f$ is a solution to the equation

$${}^{C}D^\alpha_{0+} u(t) = Lu(t), \quad u(0) = u_0.$$

(*ii*) On the first glance the approach of [BM01] in (5.14) is different to (5.13) in [WW01]. But consider in (5.14) the coordinate transform $s = u^{-1/\alpha}$. Then the solution to the time fractional equation reads

$$g(t) = S_t f = \int_0^\infty f_\alpha(u^{-1/\alpha}) \frac{1}{\alpha} u^{-1/\alpha - 1} \left(T_{t^\alpha u} f\right) \mathrm{d}u,$$

where f_α denotes the density of a stable law. From [Sch92] we know that

$$f_\alpha(u^{-1/\alpha}) \frac{1}{\alpha} u^{-1/\alpha-1} = H^{1\,0}_{1\,1}\left(u \,\middle|\, \begin{matrix}(1-\alpha,\alpha)\\(0,1)\end{matrix}\right),$$

see also Lemma 3.1.4. Thus

$$g(t) = \int_0^\infty g_\alpha(u) \left(T_{t^\alpha u} f\right) \mathrm{d}u$$

with the same g_α as in [WW01]. Furthermore it is shown in Lemma A.1.4 that $g_\alpha = M_\alpha$.

5.2 The fractional Feynman-Kac formula

We recall the stochastic representation for the solution of the heat equation, see Section 1.4. Suppose that u is a solution to the heat equation (1.5) with suitable initial value u_0. Then u is given by

$$u(t,x) = \mathbb{E}_{\mu_1}\left(u_0(x + B_t)\right), \quad x \in \mathbb{R},\ t \geq 0,$$

where B is a Brownian motion starting at $B_0 = 0$. [Sch92] showed a first extension of the Feynman-Kac formula to the fractional heat equation. Let $u_0 \in \mathcal{S}(\mathbb{R})$ and suppose that u is given by

$$u(t,x) = \mathbb{E}_{\mu_\alpha}\left(u_0(x + B_t^{\alpha,\alpha})\right), \quad x \in \mathbb{R},\ t \geq 0.$$

Then Schneider proved that u solves the fractional heat equation

$$u(t,x) = u_0(x) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \frac{\partial^2}{\partial x^2} u(s,x)\, \mathrm{d}s, \quad x \in \mathbb{R},\ t \geq 0,$$

Here $B_t^{\alpha,\alpha}$ is a generalized grey Brownian motion, $0 < \alpha < 1$. For notational convenience we just write in the following B_t^α instead of $B_t^{\alpha,\alpha}$.

In the sequel we extend the result of [Sch92]. Using the results from Chapter 3 and especially from Section 3.3 we are able to deal with distributions. Thus we can prove in the following that, replacing u_0 by Dirac delta, a Green's function to (5.6) is obtained. For the proof we need the following lemma, see also [Sch92]:

Lemma 5.2.1. For all $\lambda \in \mathbb{C}$ and for all $\beta \in (0,1)$ the Mittag-Leffler function E_β satisfies the following equation:

$$\mathrm{E}_\beta\left(-\frac{1}{2}\lambda^2 t^\beta\right) = 1 - \frac{1}{2}\lambda^2 \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} \mathrm{E}_\beta\left(-\frac{1}{2}\lambda^2 s^\beta\right) \mathrm{d}s, \quad t \geq 0.$$

Furthermore for $\alpha \in (0,2)$ it holds that

$$\mathrm{E}_\beta\left(-\frac{1}{2}\lambda^2 t^\alpha\right) = 1 - \frac{\lambda^2}{2} \frac{1}{\Gamma(\beta)} \int_0^t (t^{\alpha/\beta} - s^{\alpha/\beta})^{\beta-1} \frac{\alpha}{\beta} s^{\alpha/\beta-1} \mathrm{E}_\beta\left(-\frac{\lambda^2}{2} s^\alpha\right) \mathrm{d}s.$$

Proof. Let $\lambda \in \mathbb{C}$, $\beta \in (0,1)$ and $t \geq 0$. By definition of the Mittag-Leffler function the right hand side of the equation gives

$$\begin{aligned}
&1 - \frac{1}{2}\lambda^2 \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} \mathrm{E}_\beta\left(-\frac{1}{2}\lambda^2 s^\beta\right) \mathrm{d}s \\
&= 1 - \frac{1}{2}\lambda^2 \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} \sum_{n=0}^{\infty} \frac{1}{\Gamma(\beta n+1)} \left(-\frac{1}{2}\lambda^2 s^\beta\right)^n \mathrm{d}s \\
&= 1 + \frac{1}{\Gamma(\beta)} \sum_{n=0}^{\infty} \frac{1}{\Gamma(\beta n+1)} \left(-\frac{1}{2}\lambda^2\right)^{n+1} \int_0^t (t-s)^{\beta-1} s^{n\beta}\, \mathrm{d}s.
\end{aligned}$$

Here we may interchange sum and integral due to the uniform convergence of the series. Next we use the following well-known integral of the Beta-distribution:

$$\int_a^b (u-a)^{p-1}(b-u)^{q-1}\, \mathrm{d}u = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}(b-a)^{p+q-1}, \quad a, b \in \mathbb{R},\ p, q > 0.$$

This gives

$$\int_0^t (t-s)^{\beta-1} s^{n\beta}\, \mathrm{d}s = \frac{\Gamma(\beta n+1)\Gamma(\beta)}{\Gamma(\beta n+1+\beta)} t^{\beta n+1+\beta-1}.$$

Thus

$$\begin{aligned}
&= 1 + \frac{1}{\Gamma(\beta)} \sum_{n=0}^{\infty} \frac{1}{\Gamma(\beta n+1)} \left(-\frac{1}{2}\lambda^2\right)^{n+1} \int_0^t (t-s)^{\beta-1} s^{n\beta}\, \mathrm{d}s \\
&= 1 + \frac{1}{\Gamma(\beta)} \sum_{n=0}^{\infty} \frac{1}{\Gamma(\beta n+1)} \left(-\frac{1}{2}\lambda^2\right)^{n+1} \frac{\Gamma(\beta n+1)\Gamma(\beta)}{\Gamma(\beta(n+1)+1)} t^{\beta(n+1)} \\
&= 1 + \sum_{n=0}^{\infty} \frac{1}{\Gamma(\beta(n+1)+1)} \left(-\frac{1}{2}\lambda^2\right)^{n+1} t^{\beta(n+1)} \\
&= 1 + \sum_{n=1}^{\infty} \frac{1}{\Gamma(\beta n+1)} \left(-\frac{1}{2}\lambda^2\right)^{n} t^{\beta n} \\
&= \sum_{n=0}^{\infty} \frac{1}{\Gamma(\beta n+1)} \left(-\frac{1}{2}\lambda^2 t^\beta\right)^{n} \\
&= \mathrm{E}_\beta\left(-\frac{1}{2}\lambda^2 t^\beta\right).
\end{aligned}$$

Hence the first assertion is shown. To prove the second assertion let $\alpha \in (0,2)$. Then:

$$\begin{aligned}
\mathrm{E}_\beta\left(-\frac{1}{2}\lambda^2 t^\alpha\right) &= \mathrm{E}_\beta\left(-\frac{1}{2}\lambda^2 (t^{\alpha/\beta})^\beta\right) \\
&= 1 - \frac{1}{2}\lambda^2 \frac{1}{\Gamma(\beta)} \int_0^{t^{\alpha/\beta}} (t^{\alpha/\beta} - s)^{\beta-1} \mathrm{E}_\beta\left(-\frac{1}{2}\lambda^2 s^\beta\right) \mathrm{d}s.
\end{aligned}$$

Let now $s = u^{\alpha/\beta}$ then:

$$\mathrm{E}_\beta\left(-\frac{1}{2}\lambda^2 t^\alpha\right) = 1 - \frac{1}{2}\lambda^2 \frac{1}{\Gamma(\beta)} \frac{\alpha}{\beta} \int_0^t (t^{\alpha/\beta} - u^{\alpha/\beta})^{\beta-1} u^{\alpha/\beta-1} \mathrm{E}_\beta\left(-\frac{1}{2}\lambda^2 u^\alpha\right) \mathrm{d}u.$$

□

Remark 5.2.2. The result of the last lemma is known in literature as follows. For $\alpha > 0$ and $\lambda \in \mathbb{R}$ consider the function $y(x) = \mathrm{E}_\alpha(\lambda x^\alpha)$ for $x \in \mathbb{R}$. Then

$$\left({}^C D^\alpha_{0+} y\right)(x) = \lambda y(x), \quad x \in \mathbb{R},$$

see e.g. Theorem 4.3 in [Die10].

Theorem 5.2.3. *Let $0 < \alpha < 1$. Define $K(t,x,y) := \mathbb{E}_{\mu_\alpha}(\delta_y(x + B^\alpha_t))$ for $x, y \in \mathbb{R}$ and $t > 0$. Then K is a Green's function to the equation*

$$u(t,x) = u_0(x) + \frac{1}{2}\left(I^\alpha_{0+}\frac{\partial^2}{\partial x^2}u(\cdot,x)\right)(t), \quad t > 0,\ x \in \mathbb{R},$$

with initial value $u_0 \in L^1(\mathbb{R}, \mathrm{d}x) \cap L^2(\mathbb{R}, \mathrm{d}x)$ satisfying

$$\int_{\mathbb{R}} \left|(\mathcal{F}^{-1}u_0)(\lambda)\lambda\right|^{1+\varepsilon} \mathrm{d}\lambda < \infty \quad \text{and} \quad \int_{\mathbb{R}} \left|(\mathcal{F}^{-1}u_0)(\lambda)\lambda^2\right|^{1+\varepsilon} \mathrm{d}\lambda < \infty,$$

for some $\varepsilon > 0$.

Proof. Define $u(t,x) = \int_{\mathbb{R}} u_0(y) K(t,x,y)\,\mathrm{d}y$ for $x \in \mathbb{R}$ and $t > 0$. Note that

$$\mathbb{E}_{\mu_\alpha}\left(\delta_y(x + B^\alpha_t)\right) = \frac{1}{2\pi}\int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}\lambda(x-y)} \mathrm{E}_\alpha\left(-\frac{1}{2}\lambda^2 t^\alpha\right)\mathrm{d}\lambda,$$

see Remark 3.3.6. Corollary A.1.9 shows that E_β is the Laplace transform of M_β and thus we obtain

$$\begin{aligned}
&\int_{\mathbb{R}} \left|u_0(y)\mathbb{E}_{\mu_\alpha}\left(\delta_y(x + B^\alpha_t)\right)\right| \mathrm{d}y \le \frac{1}{2\pi}\int_{\mathbb{R}}\int_{\mathbb{R}} |u_0(y)| \left|\mathrm{E}_\alpha\left(-\frac{1}{2}\lambda^2 t^\alpha\right)\right| \mathrm{d}\lambda\,\mathrm{d}y \\
&\le \frac{1}{2\pi}\int_{\mathbb{R}}\int_{\mathbb{R}}\int_0^\infty |u_0(y)|\, M_\alpha(r) \exp\left(-\frac{1}{2}\lambda^2 t^\alpha r\right) \mathrm{d}r\,\mathrm{d}\lambda\,\mathrm{d}y \\
&= \frac{1}{2\pi}\int_{\mathbb{R}} |u_0(y)| \int_0^\infty M_\alpha(r)\sqrt{\frac{2\pi}{t^\alpha r}}\,\mathrm{d}r\,\mathrm{d}y \\
&= \frac{1}{\sqrt{2\pi t^\alpha}}\int_{\mathbb{R}} |u_0(y)| \int_0^\infty M_\alpha(r) r^{-1/2}\,\mathrm{d}r\,\mathrm{d}y.
\end{aligned}$$

Since $\int_0^\infty M_\alpha(r) r^{-1/2}\,\mathrm{d}r = \pi/\Gamma(1-\alpha/2)$, see e.g. [MMP10], we obtain

$$\int_{\mathbb{R}} \left|u_0(y)\mathbb{E}_{\mu_\alpha}\left(\delta_y(x + B^\alpha_t)\right)\right| \mathrm{d}y \le \frac{\sqrt{\pi}}{\sqrt{2t^\alpha}\,\Gamma(1-\alpha/2)}\int_{\mathbb{R}} |u_0(y)|\,\mathrm{d}y < \infty.$$

Hence u is well defined and it holds

$$\begin{aligned}
u(t,x) &= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}\lambda x}\mathrm{E}_\alpha\left(-\frac{1}{2}\lambda^2 t^\alpha\right)\left(\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} u_0(r)\mathrm{e}^{-\mathrm{i}\lambda y}\,\mathrm{d}y\right)\mathrm{d}\lambda \\
&= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}\lambda x}\mathrm{E}_\alpha\left(-\frac{1}{2}\lambda^2 t^\alpha\right)\left(\mathcal{F}^{-1}u_0\right)(\lambda)\,\mathrm{d}\lambda. \qquad (5.16)
\end{aligned}$$

Denote the integrand by $f(x,\lambda)$, $x, \lambda \in \mathbb{R}$. Then $f(x,\cdot) \in L^1(\mathbb{R})$ for all $x \in \mathbb{R}$ and $f(\cdot,\lambda)$ is differentiable for all $\lambda \in \mathbb{R}$ with

$$\left|\frac{\partial}{\partial x} f(x,\lambda)\right| = |\lambda| \left|\mathrm{E}_\alpha\left(-\frac{1}{2}\lambda^2 t^\alpha\right)\left(\mathcal{F}^{-1}u_0\right)(\lambda)\right| := g(\lambda).$$

We show that $g \in L^1(\mathbb{R}, \mathrm{d}x)$. By Corollary A.1.9 we estimate g as follows:

$$\int_{\mathbb{R}} g(\lambda)\,\mathrm{d}\lambda \le \int_0^\infty M_\alpha(r) \int_{\mathbb{R}} \left|(\mathcal{F}^{-1}u_0)(\lambda)\lambda\right| \exp\left(-\frac{1}{2}\lambda^2 t^\alpha r\right)\,\mathrm{d}\lambda\,\mathrm{d}r.$$

Next we use Hölder's inequality. Choose $p = 1 + 1/\varepsilon$ and $q = p/(p-1) = 1 + \varepsilon$. Then $1/p + 1/q = 1$ and

$$\begin{aligned}
&\int_{\mathbb{R}} g(\lambda)\,\mathrm{d}\lambda \\
&\le \int_0^\infty M_\alpha(r) \left(\int_{\mathbb{R}} \left|(\mathcal{F}^{-1}u_0)(\lambda)\lambda\right|^{1+\varepsilon}\,\mathrm{d}\lambda\right)^{1/(1+\varepsilon)} \left(\int_{\mathbb{R}} \exp\left(-\frac{1}{2}p\lambda^2 t^\alpha r\right)\,\mathrm{d}\lambda\right)^{1/p}\,\mathrm{d}r \\
&\le \left(\int_{\mathbb{R}} \left|(\mathcal{F}^{-1}u_0)(\lambda)\lambda\right|^{1+\varepsilon}\,\mathrm{d}\lambda\right)^{1/(1+\varepsilon)} \sqrt{\frac{2\pi}{pt^\alpha}}^{1/p} \int_0^\infty M_\alpha(r) r^{-1/(2p)}\,\mathrm{d}r < \infty.
\end{aligned}$$

The first integral is finite due to the condition on u_0. Since $-1/2 < -1/(2p)$ the second integral is finite by Lemma A.1.5. Hence $g \in L^1(\mathbb{R}, \mathrm{d}x)$ and we may interchange derivative and integral and obtain

$$\frac{\partial}{\partial x} \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}\lambda x} \mathrm{E}_\alpha\left(-\frac{1}{2}\lambda^2 t^\alpha\right)\left(\mathcal{F}^{-1}u_0\right)(\lambda)\,\mathrm{d}\lambda = \int_{\mathbb{R}} \mathrm{i}\lambda \mathrm{e}^{\mathrm{i}\lambda x} \mathrm{E}_\alpha\left(-\frac{1}{2}\lambda^2 t^\alpha\right)\left(\mathcal{F}^{-1}u_0\right)(\lambda)\,\mathrm{d}\lambda.$$

With similar arguments we can show that also the second derivative interchanges with the integral. Thus

$$\frac{\partial^2}{\partial x^2} u(t,x) = -\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \lambda^2 \mathrm{e}^{\mathrm{i}\lambda x} \mathrm{E}_\alpha\left(-\frac{1}{2}\lambda^2 t^\alpha\right)(\mathcal{F}^{-1}u_0)(\lambda)\,\mathrm{d}\lambda.$$

Hence it holds using Lemma 5.2.1 that

$$\begin{aligned}
&\frac{1}{2}\left(I^\alpha_{0+} \frac{\partial^2}{\partial x^2} u(\cdot, x)\right)(t) \\
&= -\frac{1}{2}\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \lambda^2 \mathrm{e}^{\mathrm{i}\lambda x} \mathrm{E}_\alpha\left(-\frac{1}{2}\lambda^2 s^\alpha\right)(\mathcal{F}^{-1}u_0)(\lambda)\,\mathrm{d}\lambda\,\mathrm{d}s \\
&= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}\lambda x}(\mathcal{F}^{-1}u_0)(\lambda)\left(-\frac{1}{2}\frac{1}{\Gamma(\alpha)}\lambda^2 \int_0^t (t-s)^{\alpha-1} \mathrm{E}_\alpha\left(-\frac{1}{2}\lambda^2 s^\alpha\right)\mathrm{d}s\right)\mathrm{d}\lambda \\
&= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}\lambda x}(\mathcal{F}^{-1}u_0)(\lambda)\left(\mathrm{E}_\alpha\left(-\frac{1}{2}\lambda^2 t^\alpha\right) - 1\right)\mathrm{d}\lambda \\
&= -u_0(x) + u(t,x).
\end{aligned}$$

This ends the proof. □

Remark 5.2.4. Define for $t > 0$ and $x \in \mathbb{R}$ the function $u(t,x) = \mathbb{E}_{\mu_\alpha}(\delta(x + B^\alpha_t))$. Then it can be shown that in a distributional sense u solves the equation

$$u(t,x) = \delta_0(x) + \frac{1}{2}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \frac{\partial^2}{\partial x^2} u(s,x)\,\mathrm{d}s.$$

Note that

$$u(t,x) = \frac{1}{2\pi}\int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}\lambda x} \mathrm{E}_\alpha\left(-\frac{1}{2}\lambda^2 t^\alpha\right)\mathrm{d}\lambda.$$

Inserting the result from Lemma 5.2.1 we achieve:

$$\begin{aligned}u(t,x) &= \frac{1}{2\pi}\int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}\lambda x}\,\mathrm{d}\lambda - \frac{1}{2\pi}\int_{\mathbb{R}} \frac{\lambda^2 \mathrm{e}^{\mathrm{i}\lambda x}}{2}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\mathrm{E}_\alpha\left(-\frac{1}{2}\lambda^2 s^\alpha\right)\mathrm{d}s\,\mathrm{d}\lambda \\ &= \frac{1}{2\pi}\int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}\lambda x}\,\mathrm{d}\lambda \\ &\quad + \frac{1}{2}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\frac{\partial^2}{\partial x^2}\left(\frac{1}{2\pi}\int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}\lambda x}\mathrm{E}_\alpha\left(-\frac{1}{2}\lambda^2 s^\alpha\right)\mathrm{d}\lambda\right)\mathrm{d}s \\ &= \delta_0(x) + \frac{1}{2}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\frac{\partial^2}{\partial x^2}u(s,x)\,\mathrm{d}s.\end{aligned}$$

The calculation and the involving divergent integrals can be treated rigorously by applying them to a test function $\xi \in \mathcal{S}(\mathbb{R})$.

Remark 5.2.5. The series expansion of $u(t,x)$ for $t>0$ and $x\in\mathbb{R}$ is calculated in Appendix A:

$$\begin{aligned}u(t,x) &= \frac{1}{\sqrt{2\pi t^\alpha}}\sum_{k=0}^\infty \frac{(-1)^k}{k!}\frac{\Gamma(1/2-k)}{\Gamma(1-1/2\alpha-\alpha k)}\left(\frac{x^2}{2t^\alpha}\right)^k \\ &\quad + \frac{x}{2t^\alpha\sqrt{\pi}}\sum_{k=0}^\infty \frac{(-1)^k}{k!}\frac{\Gamma(-1/2-k)}{\Gamma(1-\alpha-\alpha k)}\left(\frac{x^2}{2t^\alpha}\right)^k.\end{aligned}$$

Comparing the series expansion to (5.5) and (5.8) we see that the solution from Theorem 5.2.3 $K(t,x,y) = \mathbb{E}_{\mu_\alpha}(\delta_y(x+B_t^\alpha))$ coincides with the Green's functions given in literature.

Remark 5.2.6. The result of Theorem 5.2.3 can also be verified by the subordination method. From Section 1.4 we know already that a Green's function K to the non-fractional heat equation is obtained by

$$K(t,x,y) = \mathbb{E}_{\mu_1}\left(\delta(x-y+B_t)\right), \quad x,y\in\mathbb{R},\ t>0.$$

Then the subordination method provides a Green's function K_α to the fractional equation via

$$K_\alpha(t,x,y) = \int_0^\infty M_\alpha(r)K(t^\alpha r,x,y)\,\mathrm{d}r.$$

We calculate this integral, see Example 1.3.18 for the expectation of Donsker's delta:

$$\begin{aligned}K_\alpha(t,x,y) &= \int_0^\infty M_\alpha(r)K(t^\alpha r,x,y)\,\mathrm{d}r \\ &= \int_0^\infty M_\alpha(r)\mathbb{E}_{\mu_1}\left(\delta_y(x+B_{t^\alpha r})\right)\,\mathrm{d}r \\ &= \int_0^\infty M_\alpha(r)\frac{1}{\sqrt{2\pi t^\alpha r}}\exp\left(-\frac{(x-y)^2}{2t^\alpha s}\right)\,\mathrm{d}r.\end{aligned}$$

Comparing with the generalized expectation in Remark 3.3.6 we see that it holds

$$\int_0^\infty M_\alpha(r)\mathbb{E}_{\mu_1}\left(\delta_a(B_{t^\alpha s})\right)\,\mathrm{d}r = \mathbb{E}_{\mu_\alpha}\left(\delta_a(B_t^\alpha)\right), \quad a\in\mathbb{R}.$$

We conclude that $\mathbb{E}_{\mu_\alpha}(\delta_y(x+B_t^\alpha))$ coincides with the Green's function K_α obtained via subordination.

The result from Theorem 5.2.3 has shown that a solution to the fractional heat equation can be constructed via grey Brownian motion. Recall that grey Brownian motion B_t^α is a special case of generalized grey Brownian motion $B_t^{\alpha,\beta}$ where we set $\alpha = \beta$. Define the function K by

$$K(t,x,y) = \mathbb{E}_{\mu_\beta}\left(\delta_y(x + B_t^{\alpha,\beta})\right), \quad x, y \in \mathbb{R},\ t > 0.$$

Then K is a Green's function to the fractional heat equation if and only if $\alpha = \beta$. The next theorem gives a fractional differential equation whose solution can be constructed with the help of a generalized grey Brownian motion.

Theorem 5.2.7. *Let $0 < \beta < 1$ and $0 < \alpha < 2$. For $x, y \in \mathbb{R}$ and $t > 0$ define $K(t,x,y) := \mathbb{E}_{\mu_\beta}\left(\delta_y(x + B_t^{\alpha,\beta})\right)$. Then K is a Green's function to the equation*

$$u(t,x) = u_0(x) + \frac{1}{2}\left(I_{0+}^\beta \frac{\partial^2}{\partial x^2} u((\cdot)^{\beta/\alpha}, x)\right)(t^{\alpha/\beta}), \quad t > 0,\ x \in \mathbb{R},$$

where u_0 satisfies the same condition as in Theorem 5.2.3.

Proof. The proof is similar to Theorem 5.2.3. Define $u(t,x) = \int_{\mathbb{R}} u_0(y) K(t,x,y)\, \mathrm{d}y$ for $t > 0$ and $x \in \mathbb{R}$. All estimates remain true when replacing E_α and M_α by E_β and M_β, respectively. Hence it holds that

$$u(t,x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}\lambda x} \mathrm{E}_\beta\left(-\frac{1}{2}\lambda^2 t^\alpha\right)\left(\mathcal{F}^{-1} u_0\right)(\lambda)\, \mathrm{d}\lambda,$$

is well-defined and twice differentiable and using Lemma 5.2.1 we get

$$\begin{aligned}
&\frac{1}{2}\left(I_{0+}^\beta \frac{\partial^2}{\partial x^2} u((\cdot)^{\beta/\alpha}, x)\right)(t^{\alpha/\beta}) \\
&= -\frac{1}{2}\frac{1}{\Gamma(\beta)} \int_0^{t^{\alpha/\beta}} (t^{\alpha/\beta} - s)^{\beta-1} \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \lambda^2 \mathrm{e}^{\mathrm{i}\lambda x} \mathrm{E}_\beta\left(-\frac{1}{2}\lambda^2 s^\beta\right)(\mathcal{F}^{-1}u_0)(\lambda)\, \mathrm{d}\lambda\, \mathrm{d}s \\
&= \int_{\mathbb{R}} \frac{\mathrm{e}^{\mathrm{i}\lambda x}}{\sqrt{2\pi}} (\mathcal{F}^{-1}u_0)(\lambda) \left(-\frac{\alpha\lambda^2}{2\beta\Gamma(\beta)} \int_0^t (t^{\alpha/\beta} - s^{\alpha/\beta})^{\beta-1} s^{\alpha/\beta - 1} \mathrm{E}_\beta\left(-\frac{1}{2}\lambda^2 s^\alpha\right) \mathrm{d}s\right) \mathrm{d}\lambda \\
&= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}\lambda x} (\mathcal{F}^{-1}u_0)(\lambda) \left(\mathrm{E}_\beta\left(-\frac{1}{2}\lambda^2 t^\alpha\right) - 1\right) \mathrm{d}\lambda \\
&= -u_0(x) + u(t,x).
\end{aligned}$$

□

Remark 5.2.8. For the special case of fractional Brownian motion we set $\beta = 1$. Then we obtain that

$$K(t,x,y) = \mathbb{E}_{\mu_1}\left(\delta_y(x + B_t^{\alpha,1})\right), \quad t > 0,\ x, y \in \mathbb{R},$$

is a solution to

$$\begin{aligned}
u(t,x) &= u_0(x) + \frac{1}{2}\int_0^{t^\alpha} \frac{\partial^2}{\partial x^2} u(s^{1/\alpha}, x)\, \mathrm{d}s \\
&= u_0(x) + \frac{1}{2}\int_0^t \alpha s^{\alpha-1} \frac{\partial^2}{\partial x^2} u(s,x)\, \mathrm{d}s,
\end{aligned}$$

or equivalently

$$\frac{\partial}{\partial t}u(t,x) = \frac{1}{2}\alpha t^{\alpha-1}\frac{\partial^2}{\partial x^2}u(t,x). \tag{5.17}$$

In the case of the Gaussian measure μ_1 the T_{μ_1}-transform of Donsker's delta can be calculated explicitly:

$$\begin{aligned}
K(t,x,y) &= \left(T_{\mu_1}\delta_y(x+B_t^{\alpha,1})\right)(0)\\
&= \frac{1}{2\pi}\int_{\mathbb{R}} \mathrm{e}^{i\lambda(x-y)}\left(T_{\mu_1}\mathrm{e}^{i\lambda B_t^{\alpha,1}}\right)(0)\,\mathrm{d}\lambda\\
&= \frac{1}{2\pi}\int_{\mathbb{R}} \mathrm{e}^{i\lambda(x-y)}\exp\left(-\frac{1}{2}\lambda^2 t^\alpha\right)\mathrm{d}\lambda\\
&= \frac{1}{\sqrt{2\pi t^\alpha}}\exp\left(-\frac{(x-y)^2}{2t^\alpha}\right).
\end{aligned}$$

It can be easily verified that K is indeed a solution to (5.17). In fact consider

$$\begin{aligned}
\frac{\partial}{\partial t}K(t,x,y) = &-\frac{\alpha}{2\sqrt{2\pi}}t^{-\alpha/2-1}\exp\left(-\frac{(x-y)^2}{2t^\alpha}\right)\\
&+\frac{1}{\sqrt{2\pi t^\alpha}}\frac{1}{2}(x-y)^2\alpha t^{-\alpha-1}\exp\left(-\frac{(x-y)^2}{2t^\alpha}\right).
\end{aligned}$$

Moreover

$$\frac{\partial}{\partial x}K(t,x,y) = -\frac{x-y}{t^\alpha\sqrt{2\pi t^\alpha}}\exp\left(-\frac{(x-y)^2}{2t^\alpha}\right)$$

and

$$\frac{\partial^2}{\partial x^2}K(t,x,y) = \frac{(x-y)^2}{t^{2\alpha}\sqrt{2\pi t^\alpha}}\exp\left(-\frac{(x-y)^2}{2t^\alpha}\right) - \frac{1}{t^\alpha\sqrt{2\pi t^\alpha}}\exp\left(-\frac{(x-y)^2}{2t^\alpha}\right).$$

This verifies that K satisfies (5.17).

5.2.1 Inhomogeneous fractional heat equation

We consider the fractional heat equation with inhomogeneity $f\colon \mathbb{R}\to\mathbb{R}$

$$({}^C D^\alpha_{0+}u)(t,x) = \frac{1}{2}\frac{\partial^2}{\partial x^2}u(t,x) + f(x), \quad u(0,x) = u_0(x), \tag{5.18}$$

for $t > 0$ and $x \in \mathbb{R}$. The aim is to find a fractional Feynman-Kac type formula, expressing the solution as expectation of some stochastic process. We first consider the non-fractional inhomogeneous heat equation. Let K^f denote the Green's function to the equation

$$\frac{\partial}{\partial t}u(t,x) = \frac{1}{2}\frac{\partial^2}{\partial x^2}u(t,x) + f(x), \quad u(0,x) = u_0(x).$$

From Section 1.4 we know that under suitable conditions K^f is given by

$$K^f(t,x,y) = \mathbb{E}_{\mu_1}\left(\int_0^t f(x+B_s)\,\mathrm{d}s + \delta_y(x+B_t)\right), \quad t>0,\ x,y\in\mathbb{R}, \tag{5.19}$$

compare equation (1.6). Next we use the subordination method and obtain that a Green's function to the fractional equation (5.18) is given by

$$K_\alpha^f(t,x,y) = \int_0^\infty M_\alpha(r) K^f(t^\alpha r, x, y)\,\mathrm{d}r.$$

We insert K^f from (5.19) and get for K_α^f:

$$\begin{aligned} K_\alpha^f(t,x,y) &= \int_0^\infty M_\alpha(r)\mathbb{E}_{\mu_1}\left(\int_0^{t^\alpha r} f(x+B_s)\,\mathrm{d}s + \delta_y(x+B_{t^\alpha r})\right)\,\mathrm{d}r \\ &= \int_0^\infty M_\alpha(r)\mathbb{E}_{\mu_1}\left(\int_0^{t^\alpha r} f(x+B_s)\,\mathrm{d}s\right)\,\mathrm{d}r \\ &\quad + \int_0^\infty M_\alpha(r)\mathbb{E}_{\mu_1}\left(\delta_y(x+B_{t^\alpha r})\right)\,\mathrm{d}r. \end{aligned}$$

Using the result from Remark 5.2.6 K_α^f is given by

$$K_\alpha^f(t,x,y) = \int_0^\infty M_\alpha(r)\mathbb{E}_{\mu_1}\left(\int_0^{t^\alpha r} f(x+B_s)\,\mathrm{d}s\right)\,\mathrm{d}r + \mathbb{E}_{\mu_\alpha}\left(\delta_y(x+B_t^\alpha)\right). \quad (5.20)$$

A first idea for generalizing the Feynman-Kac formula in (5.19) is to replace the Brownian motion by a grey Brownian motion. Then the Green's function K_α^f has the form

$$K_\alpha^f(t,x,y) = \mathbb{E}_{\mu_\alpha}\left(\int_0^t f(x+B_s^\alpha)\,\mathrm{d}s + \delta_y(x+B_t^\alpha)\right).$$

Compared to the subordination approach (5.20) we have to show that

$$\int_0^\infty M_\alpha(r)\mathbb{E}_{\mu_1}\left(\int_0^{t^\alpha r} f(x+B_s)\,\mathrm{d}s\right)\,\mathrm{d}r = \mathbb{E}_{\mu_\alpha}\left(\int_0^t f(x+B_s^\alpha)\,\mathrm{d}s\right).$$

This is a wrong assertion, which can be seen by calculating for example the easy case $f(x) = x$. The correct statement is contained in the following theorem:

Theorem 5.2.9. *Let f be an analytic function on $\mathbb{R}$ such that f itself and all derivatives are bounded, i.e. there exists $M < \infty$ such that for all $k \in \mathbb{N}$ we have $\left|f^{(k)}(x)\right| \leq M$ for all $x \in \mathbb{R}$. Then the Green's function to the inhomogeneous time-fractional heat equation*

$$({}^C D_{0+}^\alpha u)(t,x) = \frac{1}{2}\frac{\partial^2}{\partial x^2} u(t,x) + f(x), \quad t > 0,\ x \in \mathbb{R},$$

where $0 < \alpha < 1$ is given by

$$K_\alpha^f(t,x,y) = \mathbb{E}_{\mu_\alpha}\left(\left(I_{0+}^\alpha f(x+B_\cdot^\alpha)\right)(t) + \delta_y(x+B_t^\alpha)\right),$$

where B^α is a grey Brownian motion.

Proof. Consider first for $t > 0$ and $x \in \mathbb{R}$ the expression

$$\mathbb{E}_{\mu_\alpha}\left(\left(I_{0+}^\alpha f(x+B_\cdot^\alpha)\right)(t)\right) = \frac{1}{\Gamma(\alpha)}\int_{\mathcal{S}'(\mathbb{R})}\int_0^t (t-s)^{\alpha-1} f(x+B_s^\alpha(\omega))\,\mathrm{d}s\,\mathrm{d}\mu_\alpha(\omega).$$

Due to the analyticity of f we can expand f in a Taylor series around x and obtain:

$$\begin{aligned}&\mathbb{E}_{\mu_\alpha}\left(\left(I^\alpha_{0+}f(x+B^\alpha_\cdot)\right)(t)\right)\\&=\frac{1}{\Gamma(\alpha)}\int_{\mathcal{S}'(\mathbb{R})}\int_0^t(t-s)^{\alpha-1}\sum_{k=0}^\infty\frac{1}{k!}f^{(k)}(x)(B^\alpha_s(\omega))^k\,\mathrm{d}s\,\mathrm{d}\mu_\alpha(\omega).\end{aligned}$$

Since f and all derivatives are bounded we can show that the above expression is well defined. Indeed, note that $|B^\alpha_s(\omega))|^k\leq\exp\left(|B^\alpha_s(\omega)|\right)$ for all $\omega\in\mathcal{S}'(\mathbb{R})$, $k\in\mathbb{N}$ and $s\geq 0$. Then

$$\begin{aligned}&\frac{1}{\Gamma(\alpha)}\int_{\mathcal{S}'(\mathbb{R})}\int_0^t(t-s)^{\alpha-1}\sum_{k=0}^\infty\frac{1}{k!}\left|f^{(k)}(x)\right||(B^\alpha_s(\omega))|^k\,\mathrm{d}s\,\mathrm{d}\mu_\alpha(\omega)\\&\leq\frac{1}{\Gamma(\alpha)}\sum_{k=0}^\infty\frac{M}{k!}\int_0^t(t-s)^{\alpha-1}\int_{\mathcal{S}'(\mathbb{R})}\exp\left(\left|\langle\omega,M^{\alpha/2}_-\mathbf{1}_{[0,s)}\rangle\right|\right)\,\mathrm{d}\mu_\alpha(\omega)\,\mathrm{d}s.\end{aligned}$$

We estimate $\mathrm{e}^{|x|}$ by $\mathrm{e}^x+\mathrm{e}^{-x}$ and get

$$\begin{aligned}\int_{\mathcal{S}'(\mathbb{R})}\exp\left(\left|\langle\omega,M^{\alpha/2}_-\mathbf{1}_{[0,s)}\rangle\right|\right)\,\mathrm{d}\mu_\alpha(\omega)&\leq 2\mathrm{E}_\alpha\left(\frac{1}{2}\langle M^{\alpha/2}_-\mathbf{1}_{[0,s)},M^{\alpha/2}_-\mathbf{1}_{[0,s)}\rangle\right)\\&=2\mathrm{E}_\alpha\left(\frac{1}{2}s^\alpha\right),\end{aligned}$$

where we used Corollary 3.1.18 and Corollary 4.2.4. This yields

$$\begin{aligned}&\frac{1}{\Gamma(\alpha)}\sum_{k=0}^\infty\frac{M}{k!}\int_0^t(t-s)^{\alpha-1}\int_{\mathcal{S}'(\mathbb{R})}\exp\left(\left|\langle\omega,M^{\alpha/2}_-\mathbf{1}_{[0,s)}\rangle\right|\right)\,\mathrm{d}\mu_\alpha(\omega)\,\mathrm{d}s\\&\leq 2\sum_{k=0}^\infty\frac{M}{k!}\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\mathrm{E}_\alpha\left(\frac{1}{2}s^\alpha\right)\,\mathrm{d}s.\end{aligned}$$

An application of Lemma 5.2.1 shows that

$$2\sum_{k=0}^\infty\frac{M}{k!}\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\mathrm{E}_\alpha\left(\frac{1}{2}s^\alpha\right)\,\mathrm{d}s=2\sum_{k=0}^\infty\frac{M}{k!}\left(2\mathrm{E}_\alpha\left(\frac{1}{2}t^\alpha\right)-2\right)<\infty.$$

Now we may apply Fubini and calculate the expression $\mathbb{E}_{\mu_\alpha}\left(\left(I^\alpha_{0+}f(x+B^\alpha_\cdot)\right)(t)\right)$:

$$\begin{aligned}&\mathbb{E}_{\mu_\alpha}\left(\left(I^\alpha_{0+}f(x+B^\alpha_\cdot)\right)(t)\right)=\mathbb{E}_{\mu_\alpha}\left(\left(I^\alpha_{0+}\sum_{k=0}^\infty\frac{1}{k!}f^{(k)}(x)(B^\alpha_\cdot)^k\right)(t)\right)\\&=\sum_{k=0}^\infty\frac{1}{k!}f^{(k)}(x)\left(I^\alpha_{0+}\mathbb{E}_{\mu_\alpha}\left((B^\alpha_\cdot)^k\right)\right)(t).\end{aligned}$$

The moments of grey Brownian motion are given by Lemma 3.1.11 and Corollary 4.2.4. We obtain

$$\mathbb{E}_{\mu_\alpha}\left(\left(I^\alpha_{0+}f(x+B^\alpha_\cdot)\right)(t)\right)=\sum_{k=0}^\infty\frac{1}{(2k)!}f^{(2k)}(x)\frac{(2k)!}{2^k\Gamma(\alpha k+1)}\left(I^\alpha_{0+}(\cdot)^{\alpha k}\right)(t).$$

The fractional integral of order α of the mapping $\mathbb{R} \ni s \mapsto g(s) = s^{\alpha k}$ for $k \in \mathbb{N}$ is given by

$$\left(I_{0+}^{\alpha} g\right)(t) = \frac{\Gamma(\alpha k + 1)}{\Gamma(\alpha + \alpha k + 1)} t^{\alpha + \alpha k},$$

see e.g. Example 2.1 in [Die10]. Hence

$$\begin{aligned}\mathbb{E}_{\mu_\alpha}\left(\left(I_{0+}^{\alpha} f(x + B_{\cdot}^{\alpha})\right)(t)\right) &= \sum_{k=0}^{\infty} f^{(2k)}(x) \frac{1}{2^k \Gamma(\alpha k + 1)} \frac{\Gamma(\alpha k + 1)}{\Gamma(\alpha + \alpha k + 1)} t^{\alpha + \alpha k} \\ &= \sum_{k=0}^{\infty} f^{(2k)}(x) \frac{1}{2^k \Gamma(\alpha(k+1) + 1)} t^{\alpha(k+1)}. \qquad (5.21)\end{aligned}$$

In the following we compare this result with the Green's function coming from the subordination approach, see (5.20). It suffices to show that:

$$\int_0^{\infty} M_\alpha(r) \mathbb{E}_{\mu_1}\left(\int_0^{t^\alpha r} f(x + B_s)\,\mathrm{d}s\right)\mathrm{d}r = \mathbb{E}_{\mu_\alpha}\left(\left(I_{0+}^{\alpha} f(x + B_{\cdot}^{\alpha})\right)(t)\right).$$

Indeed, note that the moments of M_α are given by, see [MMP10]:

$$\int_0^{\infty} M_\alpha(r) r^{k+1}\,\mathrm{d}r = \frac{\Gamma(k+2)}{\Gamma(\alpha(k+1)+1)} = \frac{(k+1)!}{\Gamma(\alpha(k+1)+1)}, \quad k \in \mathbb{N}.$$

This shows that (5.21) equals to

$$\mathbb{E}_{\mu_\alpha}\left(\left(I_{0+}^{\alpha} f(x + B_{\cdot}^{\alpha})\right)(t)\right) = \sum_{k=0}^{\infty} \frac{t^{\alpha(k+1)}}{2^k (k+1)!} f^{(2k)}(x) \int_0^{\infty} M_\alpha(r) r^{k+1}\,\mathrm{d}r.$$

Since $\int_0^{t^\alpha r} s^k\,\mathrm{d}s = 1/(k+1) t^{\alpha(k+1)} r^{k+1}$ we get

$$\mathbb{E}_{\mu_\alpha}\left(\left(I_{0+}^{\alpha} f(x + B_{\cdot}^{\alpha})\right)(t)\right) = \int_0^{\infty} M_\alpha(r) \sum_{k=0}^{\infty} \frac{1}{2^k k!} f^{(2k)}(x) \int_0^{t^\alpha r} s^k\,\mathrm{d}s\,\mathrm{d}r.$$

The moments of Brownian motion with respect to the Gaussian measure μ_1 are given by

$$\int_{\mathcal{S}'(\mathbb{R})} (B_s(\omega))^{2k}\,\mathrm{d}\mu_1(\omega) = \frac{(2k)!}{2^k k!} s^k, \quad k \in \mathbb{N},$$

and the odd moments vanishes. Hence

$$\begin{aligned}&\mathbb{E}_{\mu_\alpha}\left(\left(I_{0+}^{\alpha} f(x + B_{\cdot}^{\alpha})\right)(t)\right) \\ &= \int_0^{\infty} M_\alpha(r) \sum_{k=0}^{\infty} \frac{1}{(2k)!} f^{(2k)}(x) \int_0^{t^\alpha r} \int_{\mathcal{S}'(\mathbb{R})} (B_s(\omega))^{2k}\,\mathrm{d}\mu_1(\omega)\,\mathrm{d}s\,\mathrm{d}r \\ &= \int_0^{\infty} M_\alpha(r) \mathbb{E}_{\mu_1}\left(\int_0^{t^\alpha r} \sum_{k=0}^{\infty} \frac{1}{k!} f^{(k)}(x) (B_s(\omega))^k\,\mathrm{d}s\right)\mathrm{d}r \\ &= \int_0^{\infty} M_\alpha(r) \mathbb{E}_{\mu_1}\left(\int_0^{t^\alpha r} f(x + B_s)\,\mathrm{d}s\right)\mathrm{d}r.\end{aligned}$$

This ends the proof. □

5.2.2 Fractional heat equation with potential

We consider the following equation:

$$({}^C D_{0+}^{\alpha} u)(t,x) = \frac{1}{2}\frac{\partial^2}{\partial x^2} u(t,x) + V(x)u(t,x), \quad u(0,x) = u_0(x) \tag{5.22}$$

for $0 < t < T$ and $x \in \mathbb{R}$ and some potential V. Under some conditions on u_0 and the potential V a Feynman-Kac formula has been established in the case $\alpha = 1$, i.e. a solution is given by

$$u(t,x) = \mathbb{E}_{\mu_1}\left(u_0(x+B_t)\exp\left(\int_0^t V(x+B_s)\,\mathrm{d}s\right)\right),$$

with a Brownian motion B_t, see Section 1.4. With means of white noise analysis the Green's function K for $t > 0$ and $x, y \in \mathbb{R}$ is given by

$$K(t,x,y) = \frac{1}{\sqrt{2\pi t}}\exp\left(-\frac{(x-y)^2}{2t}\right)\mathbb{E}_{\mu_1}\left(\exp\left(\int_0^t V(BB_s^{x,y,t})\,\mathrm{d}s\right)\right),$$

where $BB^{x,y,t}$ denotes a Brownian Bridge starting for $s = 0$ at x and ending in y for $s = t$, given by

$$BB_s^{x,y,t} = x + \frac{s}{t}(y-x) + B_s - \frac{s}{t}B_t.$$

From the subordination approach in [WW01] we know that a Green's function K_α to the fractional equation

$$u_\alpha(t,x) = u_0(x) + \frac{1}{2}\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left(\frac{\partial^2}{\partial x^2}u_\alpha(s,x) + V(x)u_\alpha(s,x)\right)\mathrm{d}s$$

is given by

$$K_\alpha(t,x,y) = \int_0^\infty M_\alpha(r)K(t^\alpha r, x, y)\,\mathrm{d}r, \quad t > 0,\ x, y \in \mathbb{R}.$$

Lemma 5.2.10. If the potential V is bounded from above, then K_α is well-defined.

Proof. Inserting the expression for K yields for $t > 0$ and $x \in \mathbb{R}$:

$$\begin{aligned} K_\alpha(t,x,y) = \frac{1}{\sqrt{2\pi t^\alpha}}\int_{\mathcal{S}'(\mathbb{R})}\int_0^\infty & M_\alpha(r)r^{-1/2}\exp\left(-\frac{(x-y)^2}{2t^\alpha r}\right) \\ & \exp\left(\int_0^{t^\alpha r} V(BB_s^{x,y,t^\alpha r}(\omega))\,\mathrm{d}s\right)\mathrm{d}r\,\mathrm{d}\mu_1(\omega). \end{aligned}$$

There exists a constant $C < \infty$ such that $V(x) \le C$ for all $x \in \mathbb{R}$. Then it holds

$$\begin{aligned} |K_\alpha(t,x,y)| &\le \frac{1}{\sqrt{2\pi t^\alpha}}\int_0^\infty M_\alpha(r)r^{-1/2}\exp\left(-\frac{(x-y)^2}{2t^\alpha r}\right)\exp(Ct^\alpha r)\,\mathrm{d}r \\ &\le \frac{1}{\sqrt{2\pi t^\alpha}}\int_0^\infty M_\alpha(r)r^{-1/2}\exp(Ct^\alpha r)\,\mathrm{d}r. \end{aligned}$$

By Lemma A.1.5 it follows that $|K_\alpha(t,x,y)| < \infty$. □

The subordination approach and Lemma 5.2.10 show that for potentials V which are bounded from above there exists a Green's function K_α such that u defined by

$$u(t,x) = \int_{\mathbb{R}} K_\alpha(t,x,y)u_0(y)\,\mathrm{d}y, \quad t>0,\ x\in\mathbb{R},$$

is a solution to the fractional heat equation with bounded potential (5.22). The aim is now to relate the Green's function K_α to a distribution $\Psi \in (\mathcal{S})^{-1}_{\mu_\alpha}$ such that

$$K_\alpha(t,x,y) = \mathbb{E}_{\mu_\alpha}\left(\Psi(t,x,y)\right), \quad t>0,\ x,y\in\mathbb{R}.$$

In the following we will call such a $\Psi \in (\mathcal{S})^{-1}_{\mu_\alpha}$ a *Green's distribution*. Note that in the case $\alpha = 1$ a Green's distribution $\Psi \in (\mathcal{S})^{-1}_{\mu_1}$ is known from Section 1.4. In this case

$$\Psi(t,x,y) = \delta(x-y+B_t)\exp\left(-\int_0^t V(x+B_s)\,\mathrm{d}s\right).$$

The first idea is now to use the subordination procedure in order to show a formula of the type:

$$K_\alpha(t,x,y) = \mathbb{E}_{\mu_\alpha}\left(\delta_y(x+B^\alpha_t)\exp\left(\int_0^t V(x+B^\alpha_s)\,\mathrm{d}s\right)\right).$$

We start first with the easiest potential $V=0$. In Remark 5.2.6 we have already shown that

$$K_\alpha(t,x,y) = \mathbb{E}_{\mu_\alpha}\left(\delta(x-y+B^\alpha_t)\right).$$

In this case the Green's distribution Ψ is thus given by

$$\Psi(t,x,y) = \delta(x-y+B^\alpha_t).$$

In a first result for $\alpha \neq 1$ and a potential $V \neq 0$ we prove the existence of a Green's distribution $\Psi \in (\mathcal{S})^{-1}_{\mu_\alpha}$ using subordination:

Theorem 5.2.11. *Consider for* $0<\alpha<1$ *the fractional heat equation*

$$u(t,x) = u_0(x) + \frac{1}{2}\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left(\frac{\partial^2}{\partial x^2}u(s,x) + V(x)u(s,x)\right)\mathrm{d}x$$

for $t>0$, $x\in\mathbb{R}$ *and a continuous potential* $V\colon\mathbb{R}\to\mathbb{R}$ *bounded from above. Then a Green's function is given by*

$$K_\alpha(t,x,y) = \mathbb{E}_{\mu_\beta}(\Psi(t,x,y)), \quad t>0,\ x,y\in\mathbb{R},$$

for some distribution $\Psi(t,x,y)\in(\mathcal{N})^{-1}_{\mu_\alpha}$.

Proof. By the subordination approach from Section 5.1.2 and Lemma 5.2.10 we obtain the Green's function for $t>0$ and $x,y\in\mathbb{R}$ by

$$K_\alpha(t,x,y) = \int_0^\infty M_\alpha(r)K(t^\alpha r,x,y)\,\mathrm{d}r, \quad t>0,\ x,y\in\mathbb{R}.$$

We know from Section 1.4 that

$$K(t,x,y) = \frac{1}{\sqrt{2\pi t}}\exp\left(-\frac{(x-y)^2}{2t}\right)\mathbb{E}_{\mu_1}\left(\exp\left(\int_0^t V(BB^{x,y,t}_s)\,\mathrm{d}s\right)\right).$$

Thus we obtain

$$K_\alpha(t,x,y) = \frac{1}{\sqrt{2\pi t^\alpha}} \int_0^\infty M_\alpha(r) r^{-1/2} \exp\left(-\frac{(x-y)^2}{2t^\alpha r}\right)$$
$$\mathbb{E}_{\mu_1}\left(\exp\left(\int_0^{t^\alpha r} V(BB_s^{x,y,t^\alpha r})\,\mathrm{d}s\right)\right)\mathrm{d}r.$$

We define a mapping $F\colon \mathcal{S}_\mathbb{C}(\mathbb{R}) \to \mathbb{C}$ by

$$F(\xi) = \frac{1}{\sqrt{2\pi t^\alpha}} \int_0^\infty M_\alpha(r) r^{-1/2} \exp\left(-\frac{(x-y)^2}{2t^\alpha r}\right)$$
$$\left(T_{\mu_1} \exp\left(\int_0^{t^\alpha r} V(BB_s^{x,y,t^\alpha r})\,\mathrm{d}s\right)\right)(\xi)\,\mathrm{d}r.$$

Since the potential V is bounded from above we find $v \in \mathbb{R}$ such that

$$\left|\exp\left(\int_0^{t^\alpha r} V(BB_s^{x,y,t^\alpha r})\,\mathrm{d}s\right)\right| \leq \mathrm{e}^{vt^\alpha r}.$$

Hence $\exp\left(\int_0^{t^\alpha r} V(BB_s^{x,y,t^\alpha r})\,\mathrm{d}s\right) \in L^2(\mu_1)$ and we can estimate the T_{μ_1}-transform at $\xi \in \mathcal{S}_\mathbb{C}(\mathbb{R})$, $\xi = \xi_1 + \mathrm{i}\xi_2$ for $\xi_1, \xi_2 \in \mathcal{S}(\mathbb{R})$ by

$$\begin{aligned}
&\left|\left(T_{\mu_1} \exp\left(\int_0^{t^\alpha r} V(BB_s^{x,y,t^\alpha r})\,\mathrm{d}s\right)\right)(\xi)\right| \\
&\leq \int_{\mathcal{S}'(\mathbb{R})} \left|\exp\left(\int_0^{t^\alpha r} V(BB_s^{x,y,t^\alpha r}(\omega))\,\mathrm{d}s\right)\mathrm{e}^{\mathrm{i}\langle\omega,\xi\rangle}\right| \mathrm{d}\mu_1(\omega) \\
&\leq \mathrm{e}^{vt^\alpha r} \int_{\mathcal{S}'(\mathbb{R})} \mathrm{e}^{-\langle\omega,\xi_2\rangle}\,\mathrm{d}\mu_1(\omega) \\
&= \mathrm{e}^{vt^\alpha r} \exp\left(\frac{1}{2}\langle\xi_2,\xi_2\rangle\right).
\end{aligned}$$

This yields the following estimate:

$$\begin{aligned}
|F(\xi)| &\leq \frac{1}{\sqrt{2\pi t^\alpha}} \int_0^\infty M_\alpha(r) r^{-1/2} \left|\left(T_{\mu_1} \exp\left(\int_0^{t^\alpha r} V(BB_s^{x,y,t^\alpha r})\,\mathrm{d}s\right)\right)(\xi)\right| \mathrm{d}r \\
&\leq \frac{1}{\sqrt{2\pi t^\alpha}} \int_0^\infty M_\alpha(r) r^{-1/2} e^{vt^\alpha r} \exp\left(\frac{1}{2}\langle\xi_2,\xi_2\rangle\right) \mathrm{d}r.
\end{aligned}$$

Now choose a neighborhood $\mathcal{U} = \{\xi \in \mathcal{S}_\mathbb{C}(\mathbb{R}) \mid |\xi|_{L^2(\mathbb{R},\mathrm{d}x)} < M\} \subset \mathcal{S}_\mathbb{C}(\mathbb{R})$ of zero for arbitrary $M < \infty$. Then $\langle\xi_2,\xi_2\rangle < M^2$. Applying Lemma A.1.5 we see that F is locally bounded for any choice of M. To show that F is G-holomorphic, let $\xi_0 \in \mathcal{U}$ and $\xi \in \mathcal{S}_\mathbb{C}(\mathbb{R})$. Let $V \subset \mathbb{C}$ be an open neighborhood of 0 such that $\xi_0 + z\xi \in \mathcal{U}$ for all $z \in V$. Choosing any closed curve γ in V we get

$$\begin{aligned}
&\int_\gamma F(\xi_0 + z\xi)\mathrm{d}z \\
&= \frac{1}{\sqrt{2\pi t^\alpha}} \int_0^\infty M_\beta(r) r^{-1/2} \exp\left(-\frac{(x-y)^2}{2t^\alpha r}\right) \\
&\quad \int_\gamma \left(T_{\mu_1} \exp\left(\int_0^{t^\alpha r} V(BB_s^{x,y,t^\alpha r})\,\mathrm{d}s\right)\right)(\xi_0 + z\xi)\,\mathrm{d}z\,\mathrm{d}r = 0,
\end{aligned}$$

since the T_{μ_1}-transform is holomorphic. Fubini's theorem is applicable, since γ is compact and F locally bounded. Using Morera's theorem we have proven that F is G-holomorphic and thus holomorphic by Proposition 1.1.5. By the characterization theorem 2.1.8 we find for each $t > 0$ and $x, y \in \mathbb{R}$ a distribution $\Psi(t,x,y) \in (\mathcal{N})^{-1}_{\mu_\alpha}$ such that $(T_{\mu_\alpha}\Psi(t,x,y))(\xi) = F(\xi)$. Finally

$$\mathbb{E}_{\mu_\alpha}(\Psi(t,x,y)) = F(0) = K_\alpha(t,x,y)$$

and the proof is finished. □

We proved the existence of some Green's distribution Ψ in $(\mathcal{S})^{-1}_{\mu_\beta}$. In the following we want to find the explicit form of Ψ. The idea is to replace the Brownian motion in the well known Feynman-Kac formula by a grey Brownian motion:

$$K_\alpha(t,x,y) = \mathbb{E}_{\mu_\alpha}\left(\delta_y(x+B_t^\alpha)\exp\left(\int_0^t V(x+B_t^\alpha)\,\mathrm{d}s\right)\right), \quad t>0,\ x,y\in\mathbb{R}. \tag{5.23}$$

In the following we will see that this first guess is not correct, even in the easiest case of a constant potential $V \neq 0$.

Constant potential

We consider the case of a constant potential $V(x) = v$ for all $x \in \mathbb{R}$ and some $v \in \mathbb{R}$. Assuming that (5.23) gives the right Green's distribution we get

$$\begin{aligned}
&\mathbb{E}_{\mu_\alpha}\left(\delta_y(x+B_t^\alpha)\exp\left(\int_0^t V(x+B_s^\alpha)\,\mathrm{d}s\right)\right)\\
&= \mathrm{e}^{vt}\mathbb{E}_{\mu_\alpha}\left(\delta_y(x+B_t^\alpha)\right)\\
&= \mathrm{e}^{vt}\int_0^\infty M_\alpha(r)\mathbb{E}_{\mu_1}\left(\delta_y(x+B_{t^\alpha r})\right)\mathrm{d}r\\
&= \frac{1}{\sqrt{2\pi t^\alpha}}\int_0^\infty M_\alpha(r)r^{-1/2}\mathrm{e}^{vt}\exp\left(-\frac{(x-y)^2}{2t^\alpha r}\right)\mathrm{d}r.
\end{aligned}$$

On the other hand the subordination approach gives

$$K_\alpha(t,x,y) = \frac{1}{\sqrt{2\pi t^\alpha}}\int_0^\infty M_\alpha(r)r^{-1/2}\mathrm{e}^{vt^\alpha r}\exp\left(-\frac{(x-y)^2}{2t^\alpha r}\right)\mathrm{d}r.$$

From this we guess that

$$K_\alpha(t,x,y) \neq \mathbb{E}_{\mu_\alpha}\left(\delta_y(x+B_t^\alpha)\exp\left(\int_0^t V(x+B_s^\alpha)\,\mathrm{d}s\right)\right).$$

Indeed, assume that

$$K_\alpha(t,x,y) = \mathbb{E}_{\mu_\alpha}\left(\delta_y(x+B_t^\alpha)\exp\left(\int_0^t V(x+B_s^\alpha)\,\mathrm{d}s\right)\right).$$

Then it holds for all $x, y \in \mathbb{R}$ and $t > 0$:

$$0 = \int_0^\infty M_\alpha(r)r^{-1/2}\left(\mathrm{e}^{vt}-\mathrm{e}^{vt^\alpha r}\right)\exp\left(-\frac{(x-y)^2}{2t^\alpha}r^{-1}\right)\mathrm{d}r.$$

Using the coordinate transformation $u = 1/(2t^\alpha r)$ and the notation $(x-y)^2 = s$ we obtain

$$0 = -\frac{1}{\sqrt{2t^\alpha}} \int_0^\infty M_\alpha((2t^\alpha u)^{-1}) u^{-3/2} \left(\mathrm{e}^{vt} - \mathrm{e}^{1/2vu^{-1}} \right) \mathrm{e}^{-su} \, \mathrm{d}u, \tag{5.24}$$

for all $s \in \mathbb{R}$, $s > 0$. Denote for $u > 0$

$$f(u) = M_\alpha \left(\frac{1}{2t^\alpha u} \right) u^{-3/2} \left(\mathrm{e}^{vt} - \mathrm{e}^{1/2vu^{-1}} \right).$$

The asymptotic behaviour of M_α ensures that $f \in L^1([0,\infty), \mathrm{d}x)$. Indeed,

$$\begin{aligned}
&\int_0^\infty |f(u)| \, \mathrm{d}u \\
&\leq \int_0^\infty M_\alpha \left(\frac{1}{2t^\alpha u} \right) u^{-3/2} \mathrm{e}^{vt} \, \mathrm{d}u + \int_0^\infty M_\alpha \left(\frac{1}{2t^\alpha u} \right) u^{-3/2} \mathrm{e}^{1/2vu^{-1}} \, \mathrm{d}u \\
&= -\frac{\alpha}{2t^\alpha} \int_0^\infty M_\alpha(r/\alpha) \left(\frac{\alpha}{2t^\alpha r} \right)^{-3/2} \mathrm{e}^{vt} \frac{1}{r^2} \, \mathrm{d}r \\
&\quad - \frac{\alpha}{2t^\alpha} \int_0^\infty M_\alpha(r/\alpha) \left(\frac{\alpha}{2t^\alpha r} \right)^{-3/2} \exp \left(\frac{2t^\alpha v r}{2\alpha} \right) \frac{1}{r^2} \, \mathrm{d}r \\
&= -\sqrt{\frac{2t^\alpha}{\alpha}} \left(\mathrm{e}^{vt} \int_0^\infty M_\alpha(r/\alpha) r^{-1/2} \, \mathrm{d}r + \int_0^\infty M_\alpha(r/\alpha) r^{-1/2} \exp \left(\frac{2t^\alpha v}{2\alpha} r \right) \, \mathrm{d}r \right).
\end{aligned}$$

Lemma A.1.5 shows then that $\int_0^\infty |f(u)| \, \mathrm{d}u$ is finite. Now (5.24) reads

$$0 = \int_0^\infty f(u) \mathrm{e}^{-su} \, \mathrm{d}u = (\mathcal{L}f)(s), \quad s > 0.$$

The injectivity of the Laplace transform yields that $f = 0$ in $L^1([0,\infty), \mathrm{d}x)$. In fact, [Doe70], Theorem 5.3, proves that if $\mathcal{L}f$ vanishes on a sequence of equidistant points which are parallel to the real axis, then f is a null function, i.e. $\int_0^t f(\tau) \, \mathrm{d}\tau = 0$ for all $t > 0$. From this we can conclude that $f = 0$ in $L^1([0,\infty))$: Assume that there exist $t_1, t_2 > 0$ such that $f(\tau) \neq 0$ for all $\tau \in (t_1, t_2)$. Then

$$0 = \int_0^{t_2} f(\tau) \, \mathrm{d}\tau = \int_0^{t_1} f(\tau) \, \mathrm{d}\tau + \int_{t_1}^{t_2} f(\tau) \, \mathrm{d}\tau = 0 + \int_{t_1}^{t_2} f(\tau) \, \mathrm{d}\tau \neq 0.$$

This is a contradiction and $f(u) = 0$ for almost all $u > 0$. Thus

$$\mathrm{e}^{vt} = \mathrm{e}^{1/2vu^{-1}}$$

for almost all $u > 0$. But this is a contradiction and we conclude that

$$K_\alpha(t,x,y) \neq \mathbb{E}_{\mu_\alpha} \left(\delta_y(x + B_t^\alpha) \exp \left(\int_0^t V(x + B_s^\alpha) \, \mathrm{d}s \right) \right).$$

5.3 Application to the time-fractional Schrödinger equation

In this section, we give some remarks to the time-fractional Schrödinger equation. Note that there is an ambiguity in literature concerning the correct formulation of the fractional Schrödinger equation. [Nab04] develops a differential equation which generalizes the usual Schrödinger equation. In this approach, the time derivative is replaced by a Caputo fractional derivative of order $0 < \alpha < 1$. This yields the following equation

$$(\mathrm{i}T_p)^{\alpha\,C}D^{\alpha}_{0+}u(t,x) = -\frac{L_p^2}{2N_m}\frac{\partial^2}{\partial x^2}u(t,x) + N_V u(t,x), \quad u(0,x) = u_0(x),$$

for $t > 0$, $x \in \mathbb{R}$ and some initial value $u_0\colon \mathbb{R} \to \mathbb{R}$. The constants T_p, L_p, N_m and N_V are introduced in order to preserve the units of the wave function when fractionalizing the Schrödinger equation. [Nab04] also gives arguments why the imaginary unit i should also be raised to the power α. In [Bay13] the time-fractional Schrödinger equation is obtained via a Wick rotation and has the form

$${}^{C}D^{\alpha}_{0+}u(t,x) = \mathrm{i}^{\alpha}\hbar D\frac{\partial^2}{\partial x^2}u(t,x) - \frac{\mathrm{i}^{\alpha}}{\hbar}V(x)u(t,x), \quad t > 0,\ x \in \mathbb{R},$$

for some diffusion constant $D > 0$ and potential V. Note that these two approaches are not equivalent, since $-\mathrm{i}^{-\alpha} \neq \mathrm{i}^{\alpha}$ if $0 < \alpha < 1$. For further formulations of the time-fractional Schrödinger equation we refer to [NAYH13] and the references therein.

In the following, we consider the time-fractional Schrödinger equation in integral form which is equivalent to [Bay13]. We choose all constants to be 1 and omit the potential. Then:

$$u(t,x) = u_0(x) + \frac{1}{2}\mathrm{i}^{\alpha}I^{\alpha}_{0+}\frac{\partial^2}{\partial x^2}u(t,x), \quad t > 0,\ x \in \mathbb{R}.$$

In order to prove a Feynman-Kac type formula for the fractional Schrödinger equation we first prove that Donsker's delta from Section 3.3 can be constructed in a complex setting.

Theorem 5.3.1. *Let $0 \neq \eta \in \mathcal{H}$ and $a \in \mathbb{R}$. Then the complex scaled Donsker's delta given by*

$$\delta_a(\mathrm{i}^{\alpha/2}\langle\cdot,\eta\rangle) := \frac{1}{2\pi}\int_{\mathbb{R}} \exp\left(\mathrm{i}s(\mathrm{i}^{\alpha/2}\langle\cdot,\eta\rangle - a)\right)\,\mathrm{d}s$$

for $0 < \alpha < 1$ is well defined as a weak integral in $(\mathcal{N})^{-1}_{\mu_\beta}$ in the sense of Theorem 2.2.2.

Proof. Denote the integrand by $\Phi_s = \exp\left(\mathrm{i}s(\mathrm{i}^{\alpha/2}\langle\cdot,\eta\rangle - a)\right)$ for $s \in \mathbb{R}$. Then $\Phi_s \in L^2(\mu_\beta)$ for all $s \in \mathbb{R}$. Indeed,

$$\begin{aligned}|\Phi_s(x)|^2 &= \exp\left(2\Re(\mathrm{i}s\mathrm{i}^{\alpha/2}\langle x,\eta\rangle)\right)\\ &= \exp\left(2\Re(\mathrm{i}s(\cos(\alpha\pi/4) + \mathrm{i}\sin(\alpha\pi/4))\langle x,\eta\rangle\right)\\ &= \exp\left(-2s\sin(\alpha\pi/4)\langle x,\eta\rangle\right), \quad x \in \mathcal{N}'.\end{aligned}$$

Thus $\int_{\mathcal{N}'}|\Phi_s(x)|^2\,\mathrm{d}\mu_\beta(x) = \int_{\mathcal{N}'}\exp\left(\langle x, -2s\sin(\alpha\pi/4)\eta\rangle\right)\mathrm{d}\mu_\beta(x) < \infty$ for all $s \in \mathbb{R}$ by Corollary 3.1.18.

The T_{μ_β}-transform of Φ_s is calculated for all $\xi \in \mathcal{N}_{\mathbb{C}}$, $\xi = \xi_1 + \mathrm{i}\xi_2$ for $\xi_1, \xi_2 \in \mathcal{N}$, with the help of Corollary 3.1.18:

$$\begin{aligned}(T_{\mu_\beta}\Phi_s)(\xi) &= \mathrm{e}^{-\mathrm{i}sa}\int_{\mathcal{N}'} \exp\left(\mathrm{i}\langle x, \xi + \mathrm{i}^{\alpha/2}s\eta\rangle\right)\,\mathrm{d}\mu_\beta(x)\\ &= \mathrm{e}^{-\mathrm{i}sa}\mathrm{E}_\beta\left(-\frac{1}{2}\langle \xi + \mathrm{i}^{\alpha/2}s\eta, \xi + \mathrm{i}^{\alpha/2}s\eta\rangle\right)\\ &= \mathrm{e}^{-\mathrm{i}sa}\mathrm{E}_\beta\left(-\frac{1}{2}\mathrm{i}^{\alpha}s^2\langle\eta,\eta\rangle - \frac{1}{2}\langle\xi,\xi\rangle - \mathrm{i}^{\alpha/2}s\langle\xi,\eta\rangle\right).\end{aligned}$$

We use that E_α is the Laplace transform of M_α, see Corollary A.1.9, in order to estimate the T_{μ_β}-transform:

$$\begin{aligned}&\int_{\mathbb{R}} \left|(T_{\mu_\beta}\Phi_s)(\xi)\right|\,\mathrm{d}s\\ &\leq \int_0^\infty M_\beta(r)\int_{\mathbb{R}}\left|\exp\left(-\frac{1}{2}r\mathrm{i}^{\alpha}s^2\langle\eta,\eta\rangle - \frac{1}{2}r\langle\xi,\xi\rangle - \mathrm{i}^{\alpha/2}sr\langle\xi,\eta\rangle\right)\right|\,\mathrm{d}s\,\mathrm{d}r\\ &= \int_0^\infty M_\beta(r)\exp\left(-\frac{1}{2}r(\langle\xi_1,\xi_1\rangle - \langle\xi_2,\xi_2\rangle)\right)\\ &\quad\times\int_{\mathbb{R}}\exp\left(-\frac{1}{2}\cos(\frac{\alpha\pi}{2})s^2r\langle\eta,\eta\rangle - sr(\cos(\frac{\alpha\pi}{4})\langle\xi_1,\eta\rangle - \sin(\frac{\alpha\pi}{4})\langle\xi_2,\eta\rangle)\right)\,\mathrm{d}s\,\mathrm{d}r\\ &= \sqrt{\frac{2\pi}{\cos(\frac{\alpha\pi}{2})\langle\eta,\eta\rangle}}\int_0^\infty M_\beta(r)r^{-1/2}\exp\left(-\frac{1}{2}r(\langle\xi_1,\xi_1\rangle - \langle\xi_2,\xi_2\rangle)\right)\\ &\quad\times\exp\left(\frac{r^2(\cos(\frac{\alpha\pi}{4})\langle\xi_1,\eta\rangle - \sin(\frac{\alpha\pi}{4})\langle\xi_2,\eta\rangle)^2}{2\cos(\frac{\alpha\pi}{2})r\langle\eta,\eta\rangle}\right)\,\mathrm{d}r\\ &= \sqrt{\frac{2\pi}{\cos(\frac{\alpha\pi}{2})\langle\eta,\eta\rangle}}\int_0^\infty M_\beta(r)r^{-1/2}\exp\left(-\frac{1}{2}r\left(|\xi_1|^2 - |\xi_2|^2\right)\right)\\ &\times\exp\left(r\frac{\cos^2(\frac{\alpha\pi}{4})\langle\xi_1,\eta\rangle^2}{2\cos(\frac{\alpha\pi}{2})\langle\eta,\eta\rangle} - r\frac{\cos(\frac{\alpha\pi}{4})\sin(\frac{\alpha\pi}{4})}{\cos(\frac{\alpha\pi}{2})}\frac{\langle\xi_1,\eta\rangle\langle\xi_2,\eta\rangle}{\langle\eta,\eta\rangle} + r\frac{\sin^2(\frac{\alpha\pi}{4})\langle\xi_2,\eta\rangle}{2\cos(\frac{\alpha\pi}{2})\langle\eta,\eta\rangle}\right)\,\mathrm{d}r.\end{aligned}$$

We estimate the exponentials from above. Let $M < \infty$ be an arbitrary constant and $\xi \in \mathcal{U} := \left\{\xi \in \mathcal{N}_{\mathbb{C}} \;\middle|\; |\xi|^2 < M\right\}$. Then $|\xi_1|^2 - |\xi_2|^2 > -M$ and thus

$$\exp\left(-\frac{1}{2}r\left(|\xi_1|^2 - |\xi_2|^2\right)\right) \leq \exp\left(\frac{1}{2}Mr\right).$$

Note that for $0 < \alpha < 1$ all appearing values of sin and cos are positive. By Cauchy-Schwarz we get

$$\frac{\langle\xi_2,\eta\rangle^2}{\langle\eta,\eta\rangle} \leq |\xi_2|^2 \leq M, \quad \text{and} \quad \frac{\langle\xi_1,\eta\rangle^2}{\langle\eta,\eta\rangle} \leq |\xi_1|^2 \leq M.$$

Hence

$$\frac{\cos^2(\frac{\alpha\pi}{4})\langle\xi_1,\eta\rangle^2}{2\cos(\frac{\alpha\pi}{2})\langle\eta,\eta\rangle} + \frac{\sin^2(\frac{\alpha\pi}{4})\langle\xi_2,\eta\rangle}{2\cos(\frac{\alpha\pi}{2})\langle\eta,\eta\rangle} \leq \frac{M}{2\cos(\frac{\alpha\pi}{2})}.$$

Moreover

$$\frac{|\langle\xi_1,\eta\rangle\langle\xi_2,\eta\rangle|}{\langle\eta,\eta\rangle} \leq |\xi_1|\,|\xi_2| \leq M$$

and since $\sin(x)\cos(x) = 1/2\sin(2x)$ for all $x \in \mathbb{R}$ it holds

$$\frac{\cos(\frac{\alpha\pi}{4})\sin(\frac{\alpha\pi}{4})}{\cos(\frac{\alpha\pi}{2})} = \frac{1}{2}\tan(\frac{\alpha\pi}{2}).$$

This yields

$$-\frac{\cos(\frac{\alpha\pi}{4})\sin(\frac{\alpha\pi}{4})}{\cos(\frac{\alpha\pi}{2})}\frac{\langle\xi_1,\eta\rangle\langle\xi_2,\eta\rangle}{\langle\eta,\eta\rangle} \leq \frac{1}{2}\tan(\frac{\alpha\pi}{2})M.$$

Combining all estimates, we finally arrive at

$$\begin{aligned}&\int_{\mathbb{R}} \left|(T_{\mu_\beta}\Phi_s)(\xi)\right| \,\mathrm{d}s\\ &\leq \sqrt{\frac{2\pi}{\cos(\frac{\alpha\pi}{2})\langle\eta,\eta\rangle}}\int_0^\infty M_\beta(r)r^{-1/2}\exp\left(\frac{1}{2}Mr + \frac{1}{2}\frac{M}{\cos(\frac{\alpha\pi}{2})}r + \frac{1}{2}\tan(\frac{\alpha\pi}{2})Mr\right)\,\mathrm{d}r\\ &\leq \sqrt{\frac{2\pi}{\cos(\frac{\alpha\pi}{2})\langle\eta,\eta\rangle}}\int_0^\infty M_\beta(r)r^{-1/2}\exp\left(Cr\right)\,\mathrm{d}r < \infty\end{aligned}$$

for some constant $C < \infty$ and using Lemma A.1.5. Theorem 2.2.2 proves the existence of $\delta_a(\mathrm{i}^{\alpha/2}\langle\cdot,\eta\rangle)$ in $(\mathcal{N})^{-1}_{\mu_\beta}$. □

Remark 5.3.2. We calculate the T_{μ_β}-transform of the complex scaled Donsker's delta for $\xi \in \mathcal{U}$, $\mathcal{U}$ as in the above proof:

$$\begin{aligned}&(T_{\mu_\beta}\delta_a(\mathrm{i}^{\alpha/2}\langle\cdot,\eta\rangle))(\xi) = \frac{1}{2\pi}\int_{\mathbb{R}}(T_{\mu_\beta}\Phi_s)(\xi)\,\mathrm{d}s\\ &= \frac{1}{2\pi}\int_0^\infty M_\beta(r)\int_{\mathbb{R}}\exp\left(-\frac{1}{2}\mathrm{i}^\alpha s^2 r\langle\eta,\eta\rangle - \frac{1}{2}r\langle\xi,\xi\rangle - s(\mathrm{i}^{\alpha/2}r\langle\xi,\eta\rangle + \mathrm{i}a)\right)\,\mathrm{d}s\,\mathrm{d}r\\ &= \frac{1}{\sqrt{2\pi\mathrm{i}^\alpha\langle\eta,\eta\rangle}}\int_0^\infty M_\beta(r)r^{-1/2}\exp\left(-\frac{1}{2}r\langle\xi,\xi\rangle\right)\exp\left(\frac{(\mathrm{i}^{\alpha/2}r\langle\xi,\eta\rangle+\mathrm{i}a)^2}{2\mathrm{i}^\alpha r\langle\eta,\eta\rangle}\right)\,\mathrm{d}r\\ &= \frac{1}{\sqrt{2\pi\mathrm{i}^\alpha\langle\eta,\eta\rangle}}\int_0^\infty M_\beta(r)r^{-1/2}\\ &\quad\times\exp\left(-\frac{1}{2}r\left(\langle\xi,\xi\rangle - \frac{\langle\xi,\eta\rangle^2}{\langle\eta,\eta\rangle}\right) + \frac{\mathrm{i}^{1-\alpha/2}a\langle\xi,\eta\rangle}{\langle\eta,\eta\rangle} - \frac{a^2}{2\mathrm{i}^\alpha r\langle\eta,\eta\rangle}\right)\,\mathrm{d}r.\end{aligned}$$

It follows for the generalized expectation of the complex scaled Donsker's delta:

$$\mathbb{E}_{\mu_\beta}\left(\delta_a(\mathrm{i}^{\alpha/2}\langle\cdot,\eta\rangle)\right) = \frac{1}{\sqrt{2\pi\mathrm{i}^\alpha\langle\eta,\eta\rangle}}\int_0^\infty M_\beta(r)r^{-1/2}\exp\left(-\frac{a^2}{2\mathrm{i}^\alpha r\langle\eta,\eta\rangle}\right)\,\mathrm{d}r.$$

The integral can be calculated by the same means as in Section A.2 and we obtain

$$\mathbb{E}_{\mu_\beta}\left(\delta_a(\mathrm{i}^{\alpha/2}\langle\cdot,\eta\rangle)\right) = \frac{1}{\sqrt{2\pi\mathrm{i}^\alpha\langle\eta,\eta\rangle}}H^{2\,0}_{1\,2}\left(\frac{a^2}{2\mathrm{i}^\alpha\langle\eta,\eta\rangle}\,\middle|\,\begin{matrix}(1-\beta/2,\beta)\\(1/2,1),(0,1)\end{matrix}\right).$$

Having established the existence of the complex scaled Donsker's delta, we construct now a Green's function to the time-fractional Schrödinger equation using generalized grey Brownian motion:

Theorem 5.3.3. *Let $0 < \alpha < 1$. Define $K(t,x,y) := \mathbb{E}_{\mu_\alpha}(\delta_y(x + \mathrm{i}^{\alpha/2} B_t^\alpha))$ for $t > 0$ and $x, y \in \mathbb{R}$. Then K is a Green's function to the equation*

$$u(t,x) = u_0(x) + \frac{1}{2}\mathrm{i}^\alpha \left(I_{0+}^\alpha \frac{\partial^2}{\partial x^2} u(\cdot, x) \right)(t), \quad t > 0,\ x \in \mathbb{R}.$$

Here the initial value u_0 satisfies the same conditions as in Theorem 5.2.3.

Proof. For $t > 0$ and $x \in \mathbb{R}$ define $u(t,x) = \int_{\mathbb{R}} K(t,x,y) u_0(y)\, \mathrm{d}y$. Note that

$$\mathbb{E}_{\mu_\alpha}(\delta_y(x + \mathrm{i}^{\alpha/2} B_t^\alpha)) = \frac{1}{2\pi} \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}\lambda(x-y)} \mathrm{E}_\alpha \left(-\frac{1}{2} \lambda^2 \mathrm{i}^\alpha t^\alpha \right) \mathrm{d}\lambda.$$

Since $\Re(\mathrm{i}^\alpha) > 0$ all estimates in the proof of Theorem 5.2.3 remain valid also for the complex scaled Donsker's delta. Hence we may apply Fubini's theorem and obtain

$$u(t,x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}\lambda x} \mathrm{E}_\alpha \left(-\frac{1}{2} \lambda^2 \mathrm{i}^\alpha t^\alpha \right) (\mathcal{F}^{-1} u_0)(\lambda)\, \mathrm{d}\lambda.$$

Moreover we are allowed to interchange derivative and integral. This gives:

$$\begin{aligned}
&\frac{1}{2}\mathrm{i}^\alpha \left(I_{0+}^\alpha \frac{\partial^2}{\partial x^2} u(\cdot, x) \right)(t) \\
&= -\frac{1}{2}\mathrm{i}^\alpha \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \lambda^2 \mathrm{e}^{\mathrm{i}\lambda x} \mathrm{E}_\alpha \left(-\frac{1}{2} \mathrm{i}^\alpha \lambda^2 s^\alpha \right) (\mathcal{F}^{-1} u_0)(\lambda)\, \mathrm{d}\lambda\, \mathrm{d}s \\
&= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}\lambda x} (\mathcal{F}^{-1} u_0)(\lambda) \left(-\frac{1}{2} \mathrm{i}^\alpha \lambda^2 \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \mathrm{E}_\alpha \left(-\frac{1}{2} \lambda^2 \mathrm{i}^\alpha s^\alpha \right) \mathrm{d}s \right) \mathrm{d}\lambda.
\end{aligned}$$

We apply Lemma 5.2.1 and yield

$$\begin{aligned}
\frac{1}{2}\mathrm{i}^\alpha \left(I_{0+}^\alpha \frac{\partial^2}{\partial x^2} u(\cdot, x) \right)(t) &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}\lambda x} (\mathcal{F}^{-1} u_0)(\lambda) \left(\mathrm{E}_\alpha \left(-\frac{1}{2} \lambda^2 \mathrm{i}^\alpha t^\alpha \right) - 1 \right) \mathrm{d}\lambda \\
&= -u_0(x) + u(t,x).
\end{aligned}$$

This ends the proof. □

Appendix A

The H-Function

Several special functions are used throughout this thesis: The Mittag-Leffler function E_β, which is introduced in Definition 3.1.1, the M-Wright function M_β, which is important in the context of the fractional heat equation and finally Fox-H-function, which occurs in the calculation of the T_{μ_β}-transform of Donsker's delta. In this chapter, we provide the main definitions and properties of these functions, see Section A.1. Furthermore we establish the relations between the Mittag-Leffler function E_β, the M-Wright function M_β and Fox-H-function H. In fact we will see that E_β and M_β are special cases of H. The main result in this chapter is the calculation of the Laplace transform of the mapping $(0,\infty) \ni t \mapsto M_\beta(t)t^{\rho-1}$ for $\rho \geq 1/2$. Section A.2 contains the computation of the generalized expectation of Donsker's Delta.

A.1 Definitions, properties and relations

A.1.1 Fox-H-function

The *H-function* was discovered by Charles Fox in 1961 [Fox61] and is a generalization of the G-function of Meijer. The definition is as follows: Let $m,n,p,q \in \mathbb{N}$, $0 \leq n \leq p$ and $1 \leq m \leq q$. Let $A_i, B_j \in \mathbb{R}$ be positive and $a_i, b_j \in \mathbb{R}$ or $\mathbb{C}$ arbitrary for $1 \leq i \leq p, 1 \leq j \leq q$. Then for $z \in \mathbb{C}$

$$H^{m\,n}_{p\,q}\left(z \,\middle|\, \begin{matrix}(a_p, A_p)\\(b_q, B_q)\end{matrix}\right) = H^{m\,n}_{p\,q}\left(z \,\middle|\, \begin{matrix}(a_1, A_1), \dots, (a_p, A_p)\\(b_1, B_1), \dots, (b_q, B_q)\end{matrix}\right) := \frac{1}{2\pi i}\int_{\mathcal{L}} \Theta(s) z^{-s}\,\mathrm{d}s, \tag{A.1}$$

where

$$\Theta(s) = \frac{\left(\prod_{j=1}^m \Gamma(b_j + sB_j)\right)\left(\prod_{j=1}^n \Gamma(1-a_j-sA_j)\right)}{\left(\prod_{j=m+1}^q \Gamma(1-b_j-sB_j)\right)\left(\prod_{j=n+1}^p \Gamma(a_j+sA_j)\right)}.$$

The coefficients a_i, A_i, $1 \leq i \leq p$ and b_j, B_j, $1 \leq j \leq q$ have to satisfy the following restriction: For all $l, k \in \mathbb{N}$ it holds that

$$A_i(b_j + l) \neq B_j(k + 1 - a_i), \quad 1 \leq i \leq n,\ 1 \leq j \leq m.$$

Then the poles of the Γ-function $\Gamma(b_j + sB_j)$ differ from the poles of $\Gamma(1 - a_i - sA_i)$ for all $1 \leq i \leq n$ and $1 \leq j \leq m$. The contour $\mathcal{L}$ is then chosen to separate these poles. For further details concerning the contour $\mathcal{L}$ and existence of H we refer to

[MSH10]. We only want to mention one particular case: Assume that

$$\mu = \sum_{j=1}^{q} B_j - \sum_{i=1}^{p} A_i > 0. \tag{A.2}$$

Then the H-function in (A.1) exists and is analytic for all $z \neq 0$.

The series expansion of H for $|z| > 0$ is given in [PBM90]

$$H_{p\,q}^{m\,n}\left(z \,\middle|\, \begin{matrix}(a_p, A_p)\\(b_q, B_q)\end{matrix}\right) = \sum_{i=1}^{m}\sum_{k=0}^{\infty} \frac{\prod_{j=1,j\neq i}^{m} \Gamma(b_j - (b_i+k)\frac{B_j}{B_i})}{\prod_{j=m+1}^{q} \Gamma(1 - b_j + (b_i+k)\frac{B_j}{B_i})} \times \frac{\prod_{j=1}^{n} \Gamma(1 - a_j + (b_i+k)\frac{A_j}{B_i})}{\prod_{j=n+1}^{p} \Gamma(a_j - (b_i+k)\frac{A_j}{B_i})} \frac{(-1)^k z^{(b_i+k)/B_i}}{k! B_i}, \tag{A.3}$$

under the condition that $\sum_{j=1}^{q} B_j - \sum_{j=1}^{p} A_j > 0$ and $B_k(b_j + l) \neq B_j(b_k + s)$ for $1 \leq j, k \leq m, j \neq k$ and $l, s \in \mathbb{N}_0$.

Example A.1.1. In the following example we set $m = q = 1$ and $n = p = 0$ and we choose the coefficients $b_1 = 0$ and $B_1 = 1$. Then (A.2) yields $\mu = 1 - 0 > 0$ and thus the H-function

$$H_{0\,1}^{1\,0}\left(z \,\middle|\, \begin{matrix}--\\(0,1)\end{matrix}\right)$$

exists for all $z \neq 0$. Furthermore (A.3) gives

$$H_{0\,1}^{1\,0}\left(z \,\middle|\, \begin{matrix}--\\(0,1)\end{matrix}\right) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} z^k.$$

Due to the series expansion the H-function can be extended analytically to $z = 0$ and coincides for all $z \in \mathbb{C}$ with e^{-z}.

Remark A.1.2. A change of variables in the definition (A.1) shows the following properties of the H-function, see also e.g. [KST06]:

(*i*) For all $z \in \mathbb{C} \setminus \{0\}$ such that the H-function is defined the following inversion formula holds:

$$H_{p\,q}^{m\,n}\left(z \,\middle|\, \begin{matrix}(a_p, A_p)\\(b_q, B_q)\end{matrix}\right) = H_{q\,p}^{n\,m}\left(1/z \,\middle|\, \begin{matrix}(1-b_q, B_q)\\(1-a_p, A_p)\end{matrix}\right).$$

(*ii*) For all $\sigma \in \mathbb{C}$ and for all $z \in \mathbb{C}$ such that the H-function is defined there holds the following multiplication formula:

$$z^{\sigma} H_{p\,q}^{m\,n}\left(z \,\middle|\, \begin{matrix}(a_p, A_p)\\(b_q, B_q)\end{matrix}\right) = H_{p\,q}^{m\,n}\left(z \,\middle|\, \begin{matrix}(a_p + \sigma A_p, A_p)\\(b_q + \sigma B_q, B_q)\end{matrix}\right).$$

A.1.2 M-Wright function

The M-Wright function M_β for $0 < \beta < 1$ was introduced by Mainardi as an auxiliary function when finding the Green's function to the time-fractional diffusion-wave equation, see e.g. [Mai96b, Mai96a]. M_β is an entire function and its series expansion is given by

$$M_\beta(z) = \sum_{n=0}^{\infty} \frac{(-z)^n}{n!\Gamma(-\beta n + 1 - \beta)}, \quad z \in \mathbb{C}. \tag{A.4}$$

Remark A.1.3. For more details we refer to [MMP10] and the references therein. We only give those properties of M_β which are essential for this work:

(*i*) For $t \geq 0$ it holds that $M_\beta(t) \geq 0$.

(*ii*) In the case $\beta = 0$ it holds obviously that $M_0(z) = \mathrm{e}^{-z}$ for all $z \in \mathbb{C}$.

(*iii*) For $\beta = 1/2$ all terms for odd indices $2n+1$ vanishes due to the singularities of the Γ-function. Then, using 3.1, we find

$$M_{1/2}(z) = \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!\Gamma(1/2-n)} = \frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{(-z^2)^n}{4^n n!} = \frac{2}{\sqrt{\pi}} \exp\left(-\frac{1}{4}z^2\right).$$

(*iv*) The asymptotic behaviour of M_β for large $r \in \mathbb{R}$ is given by

$$M_\beta(r/\beta) \sim a(\beta) r^{\frac{\beta-1/2}{1-\beta}} \exp\left(-b(\beta) r^{1/(1-\beta)}\right), \quad r \to \infty,$$

where $a(\beta) = (2\pi(1-\beta))^{-1/2} > 0$ and $b(\beta) = (1-\beta)/\beta > 0$.

Lemma A.1.4. The M-Wright function M_β is a special case of Fox-H-function and for all $z \in \mathbb{C}$ it holds that

$$M_\beta(z) = H^{1\,0}_{1\,1}\left(z \,\middle|\, \begin{matrix}(1-\beta,\beta)\\(0,1)\end{matrix}\right), \quad z \in \mathbb{C}.$$

Proof. Applying formula (A.3) we see that

$$H^{1\,0}_{1\,1}\left(z \,\middle|\, \begin{matrix}(1-\beta,\beta)\\(0,1)\end{matrix}\right) = \sum_{k=0}^{\infty} \frac{1}{\Gamma(1-\beta-k\beta)} \frac{(-1)^k z^k}{k!}, \quad z \in \mathbb{C} \setminus \{0\}.$$

The series expansion can be analytically continued to $z = 0$. A comparison with (A.4) finishes the proof. □

Lemma A.1.5. Let $\rho \geq \frac{1}{2}$. Then the integral

$$\int_0^\infty \left| M_\beta(r/\beta) r^{\rho-1} \exp(-rz) \right| \, \mathrm{d}r$$

is finite for all $z \in \mathbb{C}$.

Proof. For the proof we use the asymptotic behaviour of the M-Wright function given in Remark A.1.3. This gives the asymptotic behaviour for the integrand:

$$\begin{aligned} &M_\beta(r/\beta) r^{\rho-1} \exp(-r\Re(z)) \\ &\sim a(\beta) r^{\delta-1} \exp\left(r(-b(\beta) r^{\beta/(1-\beta)} - \Re(z))\right) := g(r), \end{aligned}$$

where $\delta = \frac{\beta-1/2}{1-\beta} + \rho > 0$. This means in particular that there exists $r_1 > 0$, such that for all $r > r_1$

$$\left| M_\beta(r/\beta) r^{\rho-1} \exp(-r\Re(z)) - g(r) \right| \leq |g(r)|.$$

Further we choose $r_2 > 0$ such that for all $r > r_2$

$$-\Re(z) - b(\beta) r^{\beta/(1-\beta)} < 0.$$

We set $r_0 = \max(r_1, r_2)$ and split the integral

$$\begin{aligned}
&\int_0^\infty M_\beta(r/\beta) r^{\rho-1} \exp\left(-r\Re(z)\right) \mathrm{d}r \\
&= \int_0^{r_0} M_\beta(r/\beta) r^{\rho-1} \exp\left(-r\Re(z)\right) \mathrm{d}r + \int_{r_0}^\infty M_\beta(r/\beta) r^{\rho-1} \exp\left(-r\Re(z)\right) \mathrm{d}r.
\end{aligned}$$

For the first integral we use the moments of the M-Wright function (see [MMP10]):

$$\int_0^\infty r^\alpha M_\beta(r) \,\mathrm{d}r = \frac{\Gamma(\alpha+1)}{\Gamma(\beta\alpha+1)}$$

for $\alpha > -1$ and using the coordinate transform $r = s\beta$ we obtain

$$\begin{aligned}
&\int_0^{r_0} M_\beta(r/\beta) r^{\rho-1} \exp\left(-r\Re(z)\right) \mathrm{d}r \le \exp\left(r_0 \left|\Re(z)\right|\right) \beta^\rho \int_0^\infty M_\beta(s) s^{\rho-1} \,\mathrm{d}s \\
&= \exp\left(r_0 \left|\Re(z)\right|\right) \beta^\rho \frac{\Gamma(\rho)}{\Gamma(\beta\rho + 1 - \beta)} < \infty.
\end{aligned}$$

For the second integral we can estimate as follows:

$$\begin{aligned}
&\int_{r_0}^\infty M_\beta(r/\beta) r^{\rho-1} \exp\left(-r\Re(z)\right) \mathrm{d}r \\
&\le \int_{r_0}^\infty \left|M_\beta(r/\beta) r^{\rho-1} \exp\left(-r\Re(z)\right) - g(r)\right| + |g(r)| \;\mathrm{d}r \\
&\le 2a(\beta) \int_{r_0}^\infty r^{\delta-1} \exp\left(r(-\Re(z) - b(\beta) r^{\beta/(1-\beta)})\right) \,\mathrm{d}r \\
&\le 2a(\beta) \exp\left(-r_0\Re(z)\right) \int_{r_0}^\infty r^{\delta-1} \exp\left(-r_0 b(\beta) r^{\beta/(1-\beta)}\right) \,\mathrm{d}r.
\end{aligned}$$

From [GR07] it is known that for $\Re(\mu) > 0$, $\Re(\nu) > 0$ and $p > 0$

$$\int_0^\infty r^{\nu-1} \exp\left(-\mu r^p\right) \,\mathrm{d}r = \frac{1}{p} \mu^{-\nu/p} \Gamma(\nu/p).$$

Thus:

$$\begin{aligned}
&\int_{r_0}^\infty M_\beta(r/\beta) r^{\rho-1} \exp\left(-r\Re(z)\right) \mathrm{d}r \\
&\le 2a(\beta) \exp\left(-r_0\Re(z)\right) \frac{1-\beta}{\beta} \left(r_0 b(\beta)\right)^{-\delta(1-\beta)/\beta} \Gamma\left(\tfrac{\delta(1-\beta)}{\beta}\right) < \infty.
\end{aligned}$$

This finishes the proof. □

Remark A.1.6. The condition $\rho \ge 1/2$ is chosen in order to ensure that $\delta = \frac{\beta-1/2}{1-\beta} + \rho > 0$. A more careful analysis would show that

$$\int_{r_0}^\infty r^{\delta-1} \exp\left(-r_0 b(\beta) r^{\beta/(1-\beta)}\right) \,\mathrm{d}r < \infty$$

also if δ is negative, since the assumption $\delta > 0$ in only needed for the integrability at 0. But $r_0 > 0$. Hence Lemma A.1.5 and also the following results hold for all $\rho > 0$. However, for our purposes $\rho \ge 1/2$ is sufficient.

Lemma A.1.7. The mapping

$$\mathbb{C} \ni z \mapsto \int_0^\infty M_\beta(r) r^{\rho-1} \exp(-rz)\,\mathrm{d}r \in \mathbb{C} \tag{A.5}$$

is holomorphic for $\rho \geq \frac{1}{2}$.

Proof. Let $z \in \mathbb{C}$, $(z_n)_{n\in\mathbb{N}}$ a sequence in $\mathbb{C}$ converging to z. Then for sufficiently large $n \in \mathbb{N}$ the estimate $|z_n| \leq |z| + 1$ holds. Thus

$$\left| M_\beta(r) r^{\rho-1} \exp(-rz_n) \right| \leq M_\beta(r) r^{\rho-1} \exp(r(|z|+1)), \quad r > 0.$$

By Lemma A.1.5 the integral

$$\int_0^\infty M_\beta(r) r^{\rho-1} \exp(r(|z|+1))\,\mathrm{d}r = \int_0^\infty M_\beta(r) r^{\rho-1} \exp(-r(-|z|-1))\,\mathrm{d}r$$

is finite and continuity of (A.5) follows by Lebesgue's dominated convergence. Let now γ be any closed and bounded curve in $\mathbb{C}$. Then:

$$\int_\gamma \int_0^\infty M_\beta(r) r^{\rho-1} \exp(-rz)\,\mathrm{d}r\,\mathrm{d}z = \int_0^\infty M_\beta(r) r^{\rho-1} \int_\gamma \exp(-rz)\,\mathrm{d}z\,\mathrm{d}r = 0,$$

since the exponential is holomorphic on $\mathbb{C}$. Note that Fubini's theorem is applicable since γ is compact and the integrand is bounded on compact sets by Lemma A.1.5. By Morera's theorem (A.5) is holomorphic. □

Lemma A.1.8. We have for all $z \in \mathbb{C}$ and $\rho \geq 1/2$ that

$$\int_0^\infty M_\beta(r) r^{\rho-1} \exp(-rz)\,\mathrm{d}r = H^{1\,1}_{1\,2}\left(z \,\middle|\, \begin{matrix} (1-\rho, 1) \\ (0,1), (\beta(1-\rho), \beta) \end{matrix} \right). \tag{A.6}$$

Proof. Assume first that $\Re(z) > 0$. Then Lemma A.1.4 implies

$$\int_0^\infty M_\beta(r) r^{\rho-1} \exp(-rz)\,\mathrm{d}r = \int_0^\infty H^{1\,0}_{1\,1}\left(r \,\middle|\, \begin{matrix} (1-\beta, \beta) \\ (0,1) \end{matrix} \right) r^{\rho-1} \exp(-rz)\,\mathrm{d}r.$$

Using the formula for the Laplace transform of the H-function in Section 2.2.8 in [MSH10], we obtain (since $\Re(z) > 0$)

$$\begin{aligned} &\int_0^\infty H^{1\,0}_{1\,1}\left(r \,\middle|\, \begin{matrix} (1-\beta, \beta) \\ (0,1) \end{matrix} \right) r^{\rho-1} \exp(-rz)\,\mathrm{d}r \\ &\quad = z^{-\rho} H^{1\,1}_{2\,1}\left(z^{-1} \,\middle|\, \begin{matrix} (1-\rho, 1), (1-\beta, \beta) \\ (0,1) \end{matrix} \right). \end{aligned}$$

With the properties from Remark A.1.2 we transform the H-function in a more suitable form. Indeed, the inversion formula for the H-function yields

$$H^{1\,1}_{2\,1}\left(z^{-1} \,\middle|\, \begin{matrix} (1-\rho, 1), (1-\beta, \beta) \\ (0,1) \end{matrix} \right) = H^{1\,1}_{1\,2}\left(z \,\middle|\, \begin{matrix} (1,1) \\ (\rho, 1), (\beta, \beta) \end{matrix} \right),$$

and the multiplication rule gives

$$z^{-\rho} H^{1\,1}_{1\,2}\left(z \,\middle|\, \begin{matrix} (1,1) \\ (\rho, 1), (\beta, \beta) \end{matrix} \right) = H^{1\,1}_{1\,2}\left(z \,\middle|\, \begin{matrix} (1-\rho, 1) \\ (0,1), (\beta - \rho\beta, \beta) \end{matrix} \right).$$

Thus the statement is shown on the half-plane $\Re(z) > 0$. Since the value μ from (A.2) in this case is $\mu = 1+\beta-\beta = \beta > 0$ the H-function is holomorphic on $\mathbb{C}\setminus\{0\}$. The integral on the left hand side of (A.6) is holomorphic on $\mathbb{C}$ by Lemma A.1.7. Thus the equality extends by the identity principle to $\mathbb{C}\setminus\{0\}$. Using (A.3) we obtain

$$H^{1\,1}_{1\,2}\left(z\,\middle|\,\begin{matrix}(1-\rho,1)\\(0,1),(\beta-\rho\beta,\beta)\end{matrix}\right) = \sum_{k=0}^{\infty}\frac{(-z)^k\Gamma(\rho+k)}{k!\Gamma(1-\beta+\beta\rho+\beta k)}.$$

Hence we can extend the H-function holomorphically to $z=0$ and

$$H^{1\,1}_{1\,2}\left(0\,\middle|\,\begin{matrix}(1-\rho,1)\\(0,1),(\beta-\rho\beta,\beta)\end{matrix}\right) = \frac{\Gamma(\rho)}{\Gamma(1-\beta+\beta\rho)}.$$

On the other hand we know all moments of M_β, see [MMP10], and find

$$\int_0^\infty M_\beta(r)r^{\rho-1}\,\mathrm{d}r = \frac{\Gamma(\rho)}{\Gamma(\beta(\rho-1)+1)}.$$

Thus both sides coincide on $\mathbb{C}$. □

Corollary A.1.9. For all $z \in \mathbb{C}$ the following holds

$$\int_0^\infty M_\beta(r)\exp(-rz)\,\mathrm{d}r = \mathrm{E}_\beta(-z).$$

Proof. Use Lemma A.1.8 with $\rho = 1$ and get

$$\int_0^\infty M_\beta(r)\exp(-rz)\,\mathrm{d}r = \sum_{k=0}^{\infty}\frac{(-z)^k\Gamma(1+k)}{k!\Gamma(1-\beta+\beta+\beta k)} = \mathrm{E}_\beta(-z).$$

□

A.2 Generalized expectation of Donsker's Delta

In this section we calculate the generalized expectation of Donsker's delta $\delta_a(\langle\cdot,\eta\rangle)$ for $\eta \neq 0$ in a Hilbert space $\mathcal{H}$, see Section 3.3. Recall that by Remark 3.3.6 the generalized expectation of Donsker's delta is given by

$$\begin{aligned}\mathbb{E}_{\mu_\beta}\left(\delta_a(\langle\cdot,\eta\rangle)\right) &= \left(T_{\mu_\beta}\delta_a(\langle\cdot,\eta\rangle)\right)(0)\\ &= \frac{1}{\sqrt{2\pi\langle\eta,\eta\rangle}}\int_0^\infty M_\beta(s)s^{-1/2}\exp\left(-\frac{a^2}{2\langle\eta,\eta\rangle}s^{-1}\right)\,\mathrm{d}s.\end{aligned}$$

The strategy of the calculation is as follows. We express the M-function M_β and the exponential function by H-functions. Then we use a known formula for the Mellin-transform of the product of two H-functions. First we have by Example A.1.1 and Lemma A.1.4 that

$$M_\beta(s) = H^{1\,0}_{1\,1}\left(s\,\middle|\,\begin{matrix}(1-\beta,\beta)\\(0,1)\end{matrix}\right) \quad\text{and}\quad \mathrm{e}^{-z} = H^{1\,0}_{0\,1}\left(z\,\middle|\,\begin{matrix}--\\(0,1)\end{matrix}\right).$$

Thus we get

$$\mathbb{E}_{\mu_\beta}\left(\delta_a(\langle\cdot,\eta\rangle)\right) = \frac{1}{\sqrt{2\pi\langle\eta,\eta\rangle}} \int_0^\infty s^{1/2-1} H^{1\,0}_{1\,1}\left(s \,\middle|\, \begin{matrix}(1-\beta,\beta)\\(0,1)\end{matrix}\right) H^{1\,0}_{0\,1}\left(\frac{a^2}{2\langle\eta,\eta\rangle}s^{-1} \,\middle|\, \begin{matrix}--\\(0,1)\end{matrix}\right) \mathrm{d}s.$$

Using the inversion formula for the H-function we end up with

$$\mathbb{E}_{\mu_\beta}\left(\delta_a(\langle\cdot,\eta\rangle)\right) = \frac{1}{\sqrt{2\pi\langle\eta,\eta\rangle}} \int_0^\infty s^{1/2-1} H^{1\,0}_{1\,1}\left(s \,\middle|\, \begin{matrix}(1-\beta,\beta)\\(0,1)\end{matrix}\right) H^{0\,1}_{1\,0}\left(\frac{2\langle\eta,\eta\rangle}{a^2}s \,\middle|\, \begin{matrix}(1,1)\\--\end{matrix}\right) \mathrm{d}s.$$

Lemma 2 in [MS69] shows that

$$\int_0^\infty x^{\sigma-1} H^{m\,n}_{p\,q}\left(\alpha x \,\middle|\, \begin{matrix}(a_p,A_p)\\(b_q,B_q)\end{matrix}\right) H^{k\,l}_{r\,s}\left(\beta x \,\middle|\, \begin{matrix}(c_r,C_r)\\(d_s,D_s)\end{matrix}\right) \mathrm{d}x$$
$$= \alpha^{-\sigma} H^{k+n\,l+m}_{q+r\,p+s}\left(\beta/\alpha \,\middle|\, \begin{matrix}(1-b_q-B_q\sigma,B_q),(c_r,C_r)\\(1-a_p-A_p\sigma,A_p),(d_s,D_s)\end{matrix}\right),$$

under the following conditions

$$\begin{aligned}
&\Re(\sigma + b_j/B_j + d_i/D_i) > 0, \quad j=1,\dots,m,\ i=1,\dots,k,\\
&\Re(\sigma + (a_j-1)/A_j + (c_j-1)/C_j) < 0, \quad j=1,\dots,n,\ i=1,\dots,l,\\
&\lambda_1 = \sum_{j=1}^{m} B_j - \sum_{j=m+1}^{q} B_j + \sum_{j=1}^{n} A_j - \sum_{j=n+1}^{p} A_j > 0,\\
&\lambda_2 = \sum_{j=1}^{k} D_j - \sum_{j=k+1}^{s} D_j + \sum_{j=1}^{l} C_j - \sum_{j=l+1}^{r} C_j > 0,\\
&|\arg\alpha| < \lambda_1\pi/2, \quad |\arg\beta| < \lambda_2\pi/2.
\end{aligned}$$

In our case we have $\sigma = 1/2$ and $m = p = q = 1$, $n = 0$ and $\alpha = 1$ and $k = s = 0$, $l = r = 1$ and $\beta = 2\langle\eta,\eta\rangle/a^2$ and $\lambda_1 = 1-\beta > 0$ and $\lambda_2 = 1 > 0$. Thus all conditions are fulfilled and we obtain

$$\mathbb{E}_{\mu_\beta}\left(\delta_a(\langle\cdot,\eta\rangle)\right) = \frac{1}{\sqrt{2\pi\langle\eta,\eta\rangle}} H^{0\,2}_{2\,1}\left(\frac{2\langle\eta,\eta\rangle}{a^2} \,\middle|\, \begin{matrix}(1/2,1),(1,1)\\(\beta/2,\beta)\end{matrix}\right).$$

By the inversion formula from Remark A.1.2 it holds

$$\mathbb{E}_{\mu_\beta}\left(\delta_a(\langle\cdot,\eta\rangle)\right) = \frac{1}{\sqrt{2\pi\langle\eta,\eta\rangle}} H^{2\,0}_{1\,2}\left(\frac{a^2}{2\langle\eta,\eta\rangle} \,\middle|\, \begin{matrix}(1-\beta/2,\beta)\\(1/2,1),(0,1)\end{matrix}\right).$$

The definition of the H-function (A.1) implies

$$\mathbb{E}_{\mu_\beta}\left(\delta_a(\langle\cdot,\eta\rangle)\right) = \frac{1}{\sqrt{2\pi\langle\eta,\eta\rangle}} H^{2\,0}_{1\,2}\left(\frac{a^2}{2\langle\eta,\eta\rangle} \,\middle|\, \begin{matrix}(1-\beta/2,\beta)\\(0,1),(1/2,1)\end{matrix}\right).$$

Applying (A.3) we finally have

$$\mathbb{E}_{\mu_\beta}\left(\delta_a(\langle\cdot,\eta\rangle)\right) = \frac{1}{\sqrt{2\pi\langle\eta,\eta\rangle}}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\frac{\Gamma(1/2-k)}{\Gamma(1-\beta/2-\beta k)}\left(\frac{a^2}{2\langle\eta,\eta\rangle}\right)^k$$
$$+\frac{a}{2\langle\eta,\eta\rangle\sqrt{\pi}}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\frac{\Gamma(-1/2-k)}{\Gamma(1-\beta-\beta k)}\left(\frac{a^2}{2\langle\eta,\eta\rangle}\right)^k.$$

Appendix B

Fractional Calculus

The fractional calculus has already a long history dating back to the 18th century. From the 20th century on the applied mathematicians discovered the benefit of fractional calculus for example in physics and biology, for an overview we refer to [Mac11, MK00]. In this chapter, we introduce the basic notations of fractional calculus, for a detailed overview and proofs we refer to [SKM93]. First of all we provide the main properties of the Riemann-Liouville fractional integrals and derivatives. Besides the fractional derivative in the sense of Riemann-Liouville there are a couple of other definitions. Mostly common in physical applications is the Caputo fractional derivative. We also consider the fractional derivative in the sense of Marchaud. This derivative appears in the context of fractional Brownian motion in the white noise setting. Although these three definitions of fractional derivatives are different we prove in this chapter that they coincide on the space of Schwartz test functions $\mathcal{S}(\mathbb{R})$ and for indicator functions.

B.1 Riemann-Liouville fractional calculus

We first give the definitions of the fractional integral and derivative in the sense of Riemann-Liouville for functions f which are defined on an interval $[a, b]$ for real numbers $-\infty < a < b < \infty$.

Theorem B.1.1 ([SKM93], Theorem 3.5). *The Riemann-Liouville fractional integrals of order* $0 < \alpha < 1$ *of* $f \in L^p([a, b], \mathrm{d}x)$*, defined by*

$$\left(I_{a+}^{\alpha} f\right)(x) := \frac{1}{\Gamma(\alpha)} \int_a^x f(t)(x-t)^{\alpha-1}\,\mathrm{d}t, \quad x \in [a, b], \tag{B.1}$$

$$\left(I_{b-}^{\alpha} f\right)(x) := \frac{1}{\Gamma(\alpha)} \int_x^b f(t)(t-x)^{\alpha-1}\,\mathrm{d}t, \quad x \in [a, b], \tag{B.1$'$}$$

exist if $1 \leq p < 1/\alpha$*. Moreover* $I_{a+}^{\alpha} f \in L^q([a, b], \mathrm{d}x)$*, where* $1 \leq q < \frac{p}{1-\alpha p}$*. If* $1 < p < 1/\alpha$ *then* $I_{a+}^{\alpha} f \in L^q([a, b], \mathrm{d}x)$ *for* $q = \frac{p}{1-\alpha p}$*.* (B.1) *and* (B.1$'$) *are called left-sided and right-sided Riemann-Liouville fractional integral, respectively.*

Theorem B.1.2 ([SKM93], Lemma 2.2). *Let* $f\colon [a, b] \to \mathbb{R}$ *be absolutely continuous. Then the left-sided (right-sided) Riemann-Liouville fractional derivative of* *order* $0 < \alpha < 1$ *of* f *given by*

$$\left(D^\alpha_{a+} f\right)(x) := \frac{1}{\Gamma(1-\alpha)} \frac{\mathrm{d}}{\mathrm{d}x} \int_a^x f(t)(x-t)^{-\alpha}\,\mathrm{d}t, \quad x \in [a,b], \tag{B.2}$$

$$\left(D^\alpha_{b-} f\right)(x) := \frac{-1}{\Gamma(1-\alpha)} \frac{\mathrm{d}}{\mathrm{d}x} \int_x^b f(t)(t-x)^{-\alpha}\,\mathrm{d}t, \quad x \in [a,b], \tag{B.2$'$}$$

exists. Moreover $D^\alpha_{a+} f, D^\alpha_{b-} f \in L^p([a,b]\,\mathrm{d}x)$ *for* $1 \le p < 1/\alpha$ *and*

$$\left(D^\alpha_{a+} f\right)(x) = \frac{1}{\Gamma(1-\alpha)} \left(\frac{f(a)}{(x-a)^\alpha} + \int_a^x f'(t)(x-t)^{-\alpha}\,\mathrm{d}t \right),$$

$$\left(D^\alpha_{b-} f\right)(x) = \frac{1}{\Gamma(1-\alpha)} \left(\frac{f(b)}{(b-x)^\alpha} - \int_x^b f'(t)(t-x)^{-\alpha}\,\mathrm{d}t \right).$$

Remark B.1.3. For functions f defined on the positive real line, the definitions (B.1) and (B.2) are valid for all $x \in [0,\infty)$.

Next we consider functions f which are defined on the real line.

Theorem B.1.4 ([SKM93], Theorem 5.3)**.** *For all* $f \in L^p(\mathbb{R}, \mathrm{d}x)$ *with* $1 \le p < 1/\alpha$ *the left-sided (right-sided) Riemann-Liouville fractional integral of order* $0 < \alpha < 1$ *is defined as follows:*

$$\left(I^\alpha_{+} f\right)(x) := \frac{1}{\Gamma(\alpha)} \int_{-\infty}^x f(t)(x-t)^{\alpha-1}\,\mathrm{d}t, \quad x \in \mathbb{R}, \tag{B.3}$$

$$\left(I^\alpha_{-} f\right)(x) := \frac{1}{\Gamma(\alpha)} \int_x^{\infty} f(t)(t-x)^{\alpha-1}\,\mathrm{d}t, \quad x \in \mathbb{R}. \tag{B.3$'$}$$

Moreover $I^\alpha_\pm \colon L^p(\mathbb{R}, \mathrm{d}x) \to L^q(\mathbb{R}, \mathrm{d}x)$ *is continuous for* $1 < p < 1/\alpha$ *and* $q = \frac{p}{1-\alpha p}$.

In order to define the fractional derivatives we need the following:

Lemma B.1.5 ([Mis08], Lemma 1.1.2)**.** Let $1 \le p < 1/\alpha$ and $f \in I^\alpha_\pm\big(L^p(\mathbb{R}, \mathrm{d}x)\big)$. Then there is a unique $\varphi \in L^p(\mathbb{R}, \mathrm{d}x)$ such that $f(x) = \left(I^\alpha_\pm \varphi\right)(x)$. This φ is called Riemann-Liouville fractional derivative of f, denoted by $D^\alpha_\pm f$ and given by:

$$\left(D^\alpha_{+} f\right)(x) = \frac{1}{\Gamma(1-\alpha)} \frac{\mathrm{d}}{\mathrm{d}x} \int_{-\infty}^x f(t)(x-t)^{-\alpha}\,\mathrm{d}t, \quad x \in \mathbb{R}, \tag{B.4}$$

$$\left(D^\alpha_{-} f\right)(x) = \frac{-1}{\Gamma(1-\alpha)} \frac{\mathrm{d}}{\mathrm{d}x} \int_x^{\infty} f(t)(t-x)^{-\alpha}\,\mathrm{d}t, \quad x \in \mathbb{R}. \tag{B.4$'$}$$

Remark B.1.6. (*i*) By the definition of the Riemann-Liouville fractional derivative it holds automatically that

$$D^\alpha_\pm I^\alpha_\pm f = f, \quad \text{for all } f \in L^p(\mathbb{R}, \mathrm{d}x),\ 1 \le p < 1/\alpha.$$

Moreover, if $f \in I^\alpha_\pm\big(L^p(\mathbb{R}, \mathrm{d}x)\big)$, $1 \le p < 1/\alpha$, then we have also

$$I^\alpha_\pm D^\alpha_\pm f = f.$$

(*ii*) Equation (5.16) in [SKM93] proves the following integration-by-parts formula: Let $f \in L^p(\mathbb{R}, \mathrm{d}x)$ and $g \in L^q(\mathbb{R}, \mathrm{d}x)$ with $p, q \ge 1$ and $\frac{1}{p} + \frac{1}{q} = 1 + \alpha$. Then it holds for all $0 < \alpha < 1$:

$$\int_{\mathbb{R}} \left(I^\alpha_+ f\right)(x) g(x)\,\mathrm{d}x = \int_{\mathbb{R}} f(x) \left(I^\alpha_- g\right)(x)\,\mathrm{d}x. \tag{B.5}$$

(*iii*) The next formula gives the integration-by-parts formula for the fractional derivative, see (5.17) in [SKM93]: Let $0 < \alpha < 1$ and assume that $D^\alpha_- f \in L^p(\mathbb{R}, \mathrm{d}x)$, $D^\alpha_+ g \in L^q(\mathbb{R}, \mathrm{d}x)$ with $1/p + 1/q = 1 + \alpha$ and $f \in L^r(\mathbb{R}, \mathrm{d}x)$, $g \in L^s(\mathbb{R}, \mathrm{d}x)$ with $1/r = 1/p + \alpha$, $1/s = 1/q - \alpha$. Then,

$$\int_{\mathbb{R}} \left(D^\alpha_- f\right)(x) g(x)\,\mathrm{d}x = \int_{\mathbb{R}} f(x) \left(D^\alpha_+ g\right)(x)\,\mathrm{d}x. \tag{B.6}$$

(*iv*) We state the Laplace transform of the fractional integral and the fractional derivative, respectively, see e.g. [KST06]. Assume that $f \in L^1(\mathbb{R}, \mathrm{d}x)$ such that $|f(x)| \le A\mathrm{e}^{px}$ for some constants $A, p > 0$. Then

$$(\mathcal{L} I^\alpha_{0+} f)(s) = s^{-\alpha} (\mathcal{L} f)(s), \quad s > p.$$

Moreover if f is additionally absolutely continuous then

$$(\mathcal{L} D^\alpha_{0+} f)(s) = s^{\alpha} (\mathcal{L} f)(s), \quad s > p.$$

As an example we state the fractional integral and derivative of the indicator function $\mathbf{1}_{[a,b)}$, see Lemma 1.1.3 in [Mis08]:

Example B.1.7. The indicator function $\mathbf{1}_{(a,b)}$ for $a, b \in \mathbb{R}$ lies in the domain of $I^\alpha_\pm$. Moreover

$$\left(I^\alpha_\pm \mathbf{1}_{(a,b)}\right)(t) = \frac{1}{\Gamma(\alpha+1)} \left(\mp (b-t)^\alpha_\mp \pm (a-t)^\alpha_\mp\right), \quad t \in \mathbb{R}.$$

The Riemann-Liouville fractional derivative of the indicator function $\mathbf{1}_{(a,b)}$ is given by

$$\left(D^\alpha_\pm \mathbf{1}_{(a,b)}\right)(t) = \frac{1}{\Gamma(1-\alpha)} \left(\mp (b-t)^{-\alpha}_\mp \pm (a-t)^{-\alpha}_\mp\right), \quad t \in \mathbb{R}.$$

Proof. For the left-sided integral we can calculate:

$$\begin{aligned}
\left(I^\alpha_+ \mathbf{1}_{(a,b)}\right)(t) &= \frac{1}{\Gamma(\alpha)} \int_{-\infty}^t \mathbf{1}_{(a,b)}(s)(t-s)^{\alpha-1}\,\mathrm{d}s \\
&= \frac{1}{\Gamma(\alpha)} \begin{cases} 0, & t < a \\ \int_a^t (t-s)^{\alpha-1}\,\mathrm{d}s, & t \in (a,b) \\ \int_a^b (t-s)^{\alpha-1}\,\mathrm{d}s, & t > b \end{cases} \\
&= \frac{1}{\Gamma(\alpha)} \begin{cases} 0, & t < a \\ \frac{1}{\alpha}(t-a)^\alpha, & t \in (a,b) \\ \frac{1}{\alpha}\left(-(t-b)^\alpha + (t-a)^\alpha\right), & t > b \end{cases} \\
&= \frac{1}{\Gamma(\alpha+1)} \left(-(b-t)^\alpha_- + (a-t)^\alpha_-\right).
\end{aligned}$$

The right-sided integral is calculated analogously.

Next we prove the second statement. Denote

$$\varphi(t) = \frac{1}{\Gamma(1-\alpha)} \left(-(b-t)^{-\alpha}_- + (a-t)^{-\alpha}_-\right), \quad t \in \mathbb{R}.$$

Note that Lemma B.1.8 below shows that $\varphi \in L^1(\mathbb{R}, \mathrm{d}x)$. Since the Riemann-Liouville fractional derivative of a function $f = I^\alpha_\pm \varphi \in I^\alpha_\pm(L^1(\mathbb{R}, \mathrm{d}x))$ is the unique function $\varphi \in L^1(\mathbb{R}, \mathrm{d}x)$, see Lemma B.1.5, we have to verify that

$$\frac{1}{\Gamma(\alpha)} \int_{-\infty}^t \frac{1}{\Gamma(1-\alpha)} \left(-(b-x)_-^{-\alpha} + (a-x)_-^{-\alpha} \right) (t-x)^{\alpha-1} \,\mathrm{d}x = \mathbf{1}_{(a,b)}(t).$$

This integral is 0 if $t < a$. For the other cases we use the well-known fact of the Beta-distribution for $p, q > 0$

$$\int_a^b (x-a)^{p-1}(b-x)^{q-1} \,\mathrm{d}x = B(a,b,p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}(b-a)^{p+q-1}.$$

Let $a < t < b$, then

$$\begin{aligned}
&\frac{1}{\Gamma(\alpha)} \int_{-\infty}^t \frac{1}{\Gamma(1-\alpha)} \left(-(b-x)_-^{-\alpha} + (a-x)_-^{-\alpha} \right) (t-x)^{\alpha-1} \,\mathrm{d}x \\
&= \frac{1}{\Gamma(\alpha)} \frac{1}{\Gamma(1-\alpha)} \int_a^t (x-a)^{-\alpha}(t-x)^{\alpha-1} \,\mathrm{d}x \\
&= \frac{1}{\Gamma(\alpha)} \frac{1}{\Gamma(1-\alpha)} B(a,t,1-\alpha,\alpha) \\
&= 1.
\end{aligned}$$

If $t > b$, then

$$\begin{aligned}
&\int_{-\infty}^t \left(-(b-x)_-^{-\alpha} + (a-x)_-^{-\alpha} \right) (t-x)^{\alpha-1} \,\mathrm{d}x \\
&= \int_a^t \left(-(b-x)_-^{-\alpha} + (x-a)^{-\alpha} \right) (t-x)^{\alpha-1} \,\mathrm{d}x \\
&= \int_a^b (x-a)^{-\alpha}(t-x)^{\alpha-1} \,\mathrm{d}x + \int_b^t \left((x-a)^{-\alpha} - (x-b)^{-\alpha} \right) (t-x)^{\alpha-1} \,\mathrm{d}x \\
&= \int_a^t (x-a)^{-\alpha}(t-x)^{\alpha-1} \,\mathrm{d}x + \int_t^b (x-a)^{-\alpha}(t-x)^{\alpha-1} \,\mathrm{d}x \\
&\quad + \int_b^t (x-a)^{-\alpha}(t-x)^{\alpha-1} \,\mathrm{d}x - \int_b^t (x-b)^{-\alpha}(t-x)^{\alpha-1} \,\mathrm{d}x \\
&= B(a,t,1-\alpha,\alpha) - B(a,t,1-\alpha,\alpha) \\
&= 0.
\end{aligned}$$

The right-sided derivative is calculated analogously. □

Lemma B.1.8. For $0 < \alpha < 1$ it holds that $D^\alpha_\pm \mathbf{1}_{[a,b)} \in L^1(\mathbb{R}, \mathrm{d}x)$.

Proof. See also [ST94]. For $x < a$ and for some $\xi \in [a,b]$ we have by the mean value theorem

$$\left| (b-x)_+^{-\alpha} - (a-x)_+^{-\alpha} \right| = (b-a)\alpha(\xi - x)^{-\alpha-1} \leq (b-a)\alpha(a-x)^{-\alpha-1}.$$

Thus

$$\int_{-\infty}^{a-1} \left| (b-x)_+^{-\alpha} - (a-x)_+^{-\alpha} \right| \,\mathrm{d}x \leq (b-a)\alpha \int_{-\infty}^{a-1} (a-x)^{-\alpha-1} \,\mathrm{d}x < \infty.$$

For $x \in [a-1, a+\delta]$, $\delta > 0$ small enough such that $a+\delta < b$, there is a constant $C < \infty$ such that $(b-x)^{-\alpha} \leq C$ and we get

$$\int_{a-1}^{a+\delta} \left|(b-x)_+^{-\alpha} - (a-x)_+^{-\alpha}\right| \mathrm{d}x \leq C(1+\delta) + \int_{a-1}^{a} (a-x)^{-\alpha} \,\mathrm{d}x < \infty.$$

For $x \in [a+\delta, b]$, we only have to consider the first term and find that

$$\int_{a+\delta}^{b} (b-x)^{-\alpha} \,\mathrm{d}x < \infty.$$

For $x > b$ the function is zero. This shows that $D_-^\alpha \mathbf{1}_{[a,b)} \in L^1(\mathbb{R}, \mathrm{d}x)$. Use similar arguments to get $D_+^\alpha \mathbf{1}_{[a,b)} \in L^1(\mathbb{R}, \mathrm{d}x)$. □

Corollary B.1.9. A slight modification of the previous proof shows that $D_\pm^\alpha \mathbf{1}_{[a,b)} \in L^p(\mathbb{R}, \mathrm{d}x)$ whenever $\alpha p < 1$.

B.2 Fractional derivative of Caputo-type

In contrast to the Riemann-Liouville fractional derivatives (B.2), (B.2′) and (B.4), (B.4′) we denote the fractional derivative of Caputo-type by ${}^C D^\alpha$.

Theorem B.2.1 ([KST06], Theorem 2.1)**.** *A sufficient condition for the left- and right-sided Caputo fractional derivatives of $f\colon [a,b] \to \mathbb{R}$ of order $0 < \alpha < 1$ is that f is absolutely continuous. Then*

$$\left({}^C D_{a+}^\alpha f\right)(x) := \left(D_{a+}^\alpha f\right)(x) - \frac{f(a)}{\Gamma(1-\alpha)}(x-a)^{-\alpha}, \quad x \in [a,b], \tag{B.7}$$

$$\left({}^C D_{b-}^\alpha f\right)(x) := \left(D_{b-}^\alpha f\right)(x) - \frac{f(b)}{\Gamma(1-\alpha)}(b-x)^{-\alpha}, \quad x \in [a,b]. \tag{B.7′}$$

In this case it holds that (B.7) *and* (B.7′) *coincide with the following expressions:*

$$\left({}^C D_{a+}^\alpha f\right)(x) = \frac{1}{\Gamma(1-\alpha)} \int_a^x f'(t)(x-t)^{-\alpha} \,\mathrm{d}t, \quad x \in [a,b], \tag{B.8}$$

$$\left({}^C D_{b-}^\alpha f\right)(x) = \frac{-1}{\Gamma(1-\alpha)} \int_x^b f'(t)(t-x)^{-\alpha} \,\mathrm{d}t, \quad x \in [a,b]. \tag{B.8′}$$

Theorem B.2.2 ([Die10], Theorem 3.7 and 3.8)**.** *Let f be continuous on $[a,b]$. Then*

$${}^C D_{a+}^\alpha I_{a+}^\alpha f = f.$$

If f is absolutely continuous, then

$$\left(I_{a+}^\alpha \, {}^C D_{a+}^\alpha f\right)(x) = f(x) - f(a), \quad x \in [a,b].$$

For $f\colon \mathbb{R} \to \mathbb{R}$ the Caputo fractional derivatives on the real line are defined as follows, provided they exist:

$$\left({}^C D_+^\alpha f\right)(x) = \frac{1}{\Gamma(1-\alpha)} \int_{-\infty}^x f'(t)(x-t)^{-\alpha} \,\mathrm{d}t, \quad x \in \mathbb{R}, \tag{B.9}$$

$$\left({}^C D_-^\alpha f\right)(x) = \frac{-1}{\Gamma(1-\alpha)} \int_x^\infty f'(t)(t-x)^{-\alpha} \,\mathrm{d}t, \quad x \in \mathbb{R}. \tag{B.9′}$$

A sufficient condition for the existence of ${}^C D_\pm^\alpha f$ is that $f \in \mathcal{S}(\mathbb{R})$, see Theorem B.4.1 below.

Remark B.2.3. The Laplace transform of the Caputo fractional derivative is given for example in [KST06]. Let f be continuously differentiable on $\mathbb{R}$ such that for all $x \in \mathbb{R}$ and for $n \in \{0,1\}$ it holds $\left|f^{(n)}(x)\right| \le A\mathrm{e}^{px}$ for some constants $A, p > 0$. Then

$$(\mathcal{L}\,{}^C D^\alpha_{0+} f)(s) = s^\alpha (\mathcal{L}f)(s) - s^{\alpha-1} f(0), \quad s > p.$$

B.3 Marchaud fractional derivative

Next we define the left- and right-sided Marchaud fractional derivatives on the real line, provided they exist, for $f\colon \mathbb{R} \to \mathbb{R}$ by:

$$\left({}^M D^\alpha_{+} f\right)(x) = \frac{\alpha}{\Gamma(1-\alpha)} \int_0^\infty \frac{f(x) - f(x-\xi)}{\xi^{\alpha+1}}\,\mathrm{d}\xi, \quad x \in \mathbb{R}, \tag{B.10}$$

$$\left({}^M D^\alpha_{-} f\right)(x) = \frac{\alpha}{\Gamma(1-\alpha)} \int_0^\infty \frac{f(x) - f(x+\xi)}{\xi^{\alpha+1}}\,\mathrm{d}\xi, \quad x \in \mathbb{R}. \tag{B.10$'$}$$

We also introduce the truncated Marchaud fractional derivatives on the real line by

$$\left({}^M D^\alpha_{+,\varepsilon} f\right)(x) = \frac{\alpha}{\Gamma(1-\alpha)} \int_\varepsilon^\infty \frac{f(x) - f(x-\xi)}{\xi^{\alpha+1}}\,\mathrm{d}\xi, \quad x \in \mathbb{R}, \tag{B.11}$$

$$\left({}^M D^\alpha_{-,\varepsilon} f\right)(x) = \frac{\alpha}{\Gamma(1-\alpha)} \int_\varepsilon^\infty \frac{f(x) - f(x+\xi)}{\xi^{\alpha+1}}\,\mathrm{d}\xi, \quad x \in \mathbb{R}. \tag{B.11$'$}$$

Then

$$\left({}^M D^\alpha_{\pm} f\right) = \lim_{\varepsilon \to 0} \left({}^M D^\alpha_{\pm,\varepsilon} f\right),$$

where the limit is defined by the problem under consideration. For $f\colon [a,b] \to \mathbb{R}$ the Marchaud fractional derivative on the interval $[a,b]$ is defined for $x \in [a,b]$ by

$$\left({}^M D^\alpha_{a+} f\right)(x) = \frac{f(x)(x-a)^{-\alpha}}{\Gamma(1-\alpha)} + \frac{\alpha}{\Gamma(1-\alpha)} \int_a^x \frac{f(x) - f(x-\xi)}{\xi^{\alpha+1}}\,\mathrm{d}\xi, \tag{B.12}$$

$$\left({}^M D^\alpha_{b-} f\right)(x) = \frac{f(x)(b-x)^{-\alpha}}{\Gamma(1-\alpha)} + \frac{\alpha}{\Gamma(1-\alpha)} \int_x^b \frac{f(x) - f(x-\xi)}{\xi^{\alpha+1}}\,\mathrm{d}\xi. \tag{B.12$'$}$$

Theorem B.3.1 ([SKM93], Theorem 13.1.)**.** *Let $f \in L^p([a,b], \mathrm{d}x)$. Then*

$${}^M D^\alpha_{a+} I^\alpha_{a+} f = f.$$

In particular, the Marchaud fractional derivative and Riemann-Liouville fractional derivative on an interval coincide for $f \in I^\alpha_{a+}\left(L^p([a,b], \mathrm{d}x)\right)$. This is true, if e.g. f is absolutely continuous.

Theorem B.3.2 ([SKM93], Theorem 6.1)**.** *If $f \in L^p(\mathbb{R}, \mathrm{d}x)$ for $1 \le p < 1/\alpha$. Then*

$$\left({}^M D^\alpha_{\pm} I^\alpha_{\pm} f\right)(x) = f(x).$$

Here ${}^M D^\alpha_{\pm} f$ is the L^p-limit of the truncated Marchaud fractional derivative (B.11) *and* (B.11$'$).

Example B.3.3. The Marchaud fractional derivative of the indicator function $\mathbf{1}_{[a,b)}$ is given by:

$$\left({}^M D^\alpha_{\pm} \mathbf{1}_{[a,b)}\right)(t) = \frac{1}{\Gamma(1-\alpha)} \left(\mp(b-t)^{-\alpha}_{\mp} \pm (a-t)^{-\alpha}_{\mp}\right).$$

Proof. If $x > b$, then clearly $\left({}^M D_-^\alpha \mathbf{1}_{[a,b)}\right)(x) = 0$. Let now $x < a$. Then

$$\left({}^M D_-^\alpha \mathbf{1}_{[a,b)}\right)(x) = -\frac{\alpha}{\Gamma(1-\alpha)} \int_0^\infty \frac{\mathbf{1}_{[a,b)}(x+\xi)}{\xi^{\alpha+1}} \,\mathrm{d}\xi$$
$$= -\frac{\alpha}{\Gamma(1-\alpha)} \int_{a-x}^{b-x} \xi^{-\alpha-1} \,\mathrm{d}\xi = \frac{1}{\Gamma(1-\alpha)} \left((b-x)^{-\alpha} - (a-x)^{-\alpha}\right).$$

For $x \in [a,b]$ it holds

$$\left({}^M D_-^\alpha \mathbf{1}_{[a,b)}\right)(x) = \frac{\alpha}{\Gamma(1-\alpha)} \int_0^\infty \frac{1-\mathbf{1}_{[a,b)}(x+\xi)}{\xi^{\alpha+1}} \,\mathrm{d}\xi$$
$$= \frac{\alpha}{\Gamma(1-\alpha)} \int_{b-x}^\infty \xi^{-\alpha-1} \,\mathrm{d}\xi = \frac{1}{\Gamma(1-\alpha)} (b-x)^{-\alpha}.$$

Altogether we obtain

$$\left({}^M D_-^\alpha \mathbf{1}_{[a,b)}\right)(t) = \frac{1}{\Gamma(1-\alpha)} \left((b-t)_+^{-\alpha} - (a-t)_+^{-\alpha}\right).$$

A similar calculation shows

$$\left({}^M D_+^\alpha \mathbf{1}_{[a,b)}\right)(t) = \frac{1}{\Gamma(1-\alpha)} \left(-(b-t)_-^{-\alpha} + (a-t)_-^{-\alpha}\right).$$

This finishes the proof. □

B.4 Fractional derivative of Schwartz test functions

Theorem B.4.1. *Let $0 < \alpha < 1$. Then the Riemann-Liouville fractional derivative coincides with the Marchaud fractional derivative and with the Caputo derivative on $\mathcal{S}(\mathbb{R})$.*

Proof. Let $x \in \mathbb{R}$. The integral $\int_0^\infty f(x \mp s) s^{-\alpha} \,\mathrm{d}s$ for $f \in \mathcal{S}(\mathbb{R})$ exists since

$$\int_0^\infty |f(x \mp s)|\, s^{-\alpha} \,\mathrm{d}s = \int_0^1 |f(x \mp s)|\, s^{-\alpha} \,\mathrm{d}s + \int_1^\infty |f(x \mp s)|\, s^{-\alpha} \,\mathrm{d}s$$
$$\leq \|f\|_\infty \int_0^1 s^{-\alpha} \,\mathrm{d}s + \sup_{s \in \mathbb{R}} |s f(s)| \int_1^\infty s^{-\alpha-1} \,\mathrm{d}s < \infty.$$

The coordinate transformation $s = x - t$ yields

$$\left(D_+^\alpha f\right)(x) = \frac{1}{\Gamma(1-\alpha)} \frac{\mathrm{d}}{\mathrm{d}x} \int_{-\infty}^x f(t)(x-t)^{-\alpha} \,\mathrm{d}t$$
$$= \frac{1}{\Gamma(1-\alpha)} \frac{\mathrm{d}}{\mathrm{d}x} \int_0^\infty f(x-s) s^{-\alpha} \,\mathrm{d}s.$$

Interchange of derivative and integral is justified since $|f'(x \mp s)|\, s^{-\alpha}$ is dominated by the integrable function

$$h(s) = \|f'\|_\infty \mathbf{1}_{[0,1)}(s) s^{-\alpha} + \sup_{s \in \mathbb{R}} |s f'(s)|\, \mathbf{1}_{[1,\infty)}(s) s^{-\alpha-1}.$$

This yields

$$\left(D_{+}^{\alpha} f\right)(x)=\frac{1}{\Gamma(1-\alpha)} \int_{0}^{\infty} f^{\prime}(x-s) s^{-\alpha} \,\mathrm{d} s .$$

For the right-sided derivative use $s=t-x$:

$$\begin{aligned}
\left(D_{-}^{\alpha} f\right)(x) & =\frac{-1}{\Gamma(1-\alpha)} \frac{\mathrm{d}}{\mathrm{d} x} \int_{x}^{\infty} f(t)(t-x)^{-\alpha} \,\mathrm{d} t \\
& =\frac{-1}{\Gamma(1-\alpha)} \frac{\mathrm{d}}{\mathrm{d} x} \int_{0}^{\infty} f(x+s) s^{-\alpha} \,\mathrm{d} s \\
& =\frac{-1}{\Gamma(1-\alpha)} \int_{0}^{\infty} f^{\prime}(x+s) s^{-\alpha} \,\mathrm{d} s .
\end{aligned}$$

Comparing with (B.9) and (B.9′) it follows that for all $f \in \mathcal{S}(\mathbb{R})$

$$D_{\pm}^{\alpha} f={}^{C} D_{\pm}^{\alpha} f .$$

Furthermore since $s^{-\alpha}=\alpha \int_{s}^{\infty} \xi^{-\alpha-1} \,\mathrm{d} \xi$ for all $s \in \mathbb{R}$ we have

$$\begin{aligned}
\left(D_{+}^{\alpha} f\right)(x) & =\frac{\alpha}{\Gamma(1-\alpha)} \int_{0}^{\infty} f^{\prime}(x-s) \int_{s}^{\infty} \xi^{-\alpha-1} \,\mathrm{d} \xi \,\mathrm{d} s \\
& =\frac{\alpha}{\Gamma(1-\alpha)} \int_{0}^{\infty} \int_{0}^{\xi} f^{\prime}(x-s) \xi^{-\alpha-1} \,\mathrm{d} s \,\mathrm{d} \xi \\
& =\frac{\alpha}{\Gamma(1-\alpha)} \int_{0}^{\infty} \frac{f(x)-f(x-\xi)}{\xi^{\alpha+1}} \,\mathrm{d} \xi .
\end{aligned}$$

And analogously

$$\left(D_{+}^{\alpha} f\right)(x)=\frac{\alpha}{\Gamma(1-\alpha)} \int_{0}^{\infty} \frac{f(x)-f(x+\xi)}{\xi^{\alpha+1}} \,\mathrm{d} \xi .$$

(B.10) and (B.10′) show that

$$D_{\pm}^{\alpha} f={}^{M} D_{\pm}^{\alpha} f, \quad f \in \mathcal{S}(\mathbb{R}).$$

□

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