Tropical Geometry in SINGULAR

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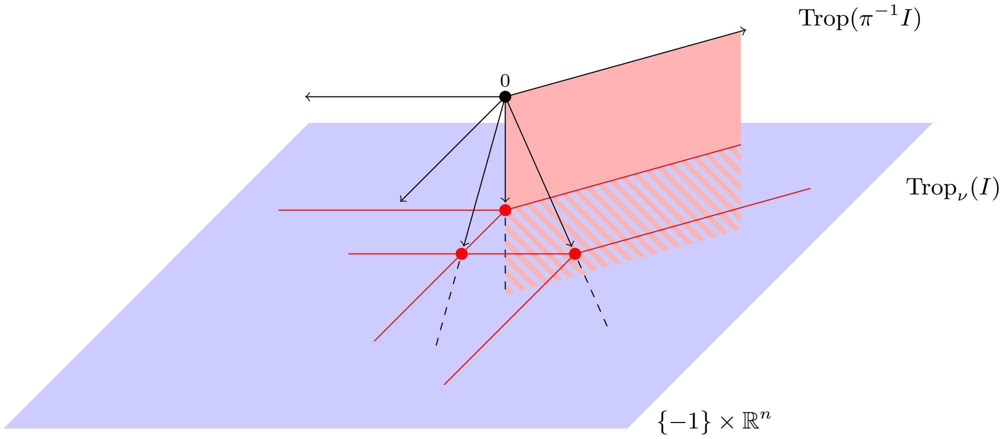

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Preface and acknowledgements

This thesis is divided into two independent parts. As such, it contains two different introductions, one for each part respectively.

The first part contains the main focus of my studies in tropical geometry, which I have worked upon for the vast majority of my time together with my advisor apl. Prof. Thomas Markwig. It presents an approach for computing tropical varieties over fields with valuation using classical standard basis techniques building upon the techniques developed by Bogart, Jensen, Speyer, Sturmfels, Thomas [**BJS+07**]. All algorithms have been implemented in the computer algebra system Singular [**DGPS14**] using its native standard basis engine.

The second part represents one of several side-projects I pursued with other collaborators outside of tropical geometry. It is a project in toric geometry with Prof. Michael Cuntz from Hanover and Prof. Günther Trautmann from Kaiserslautern and revolves around classifying smooth toric varieties that arise from hyperplane arrangements.

While the two parts are independent, it should be noted that the practical foundation that was laid for the main topic, namely support for convex geometry inside Singular, proved to be of great help for me and my collaborators in studying the second topic.

Two other projects that I would have loved to include in this thesis, however it was impossible to do so without overcrowding it: one is on exploiting symmetries in the computation of GIT-fans in collaboration with Janko Böhm and Simon Keicher; the other is an algorithmic approach to localizations of affine coordinate rings at prime ideals in collaboration with Magdaleen Marais.

I would like to thank the following people who greatly contributed to the wonderful experience that were the past few years:

- my advisor Thomas Markwig for the great support and always making time for me, even if it meant staying at the university till late in the evening. I would also like to thank him and his wife, Hannah Markwig, for their repeated efforts to improve the atmosphere in our working group.
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- all my other colleagues and former colleagues who are or were part of this great working group: Anna-Lena Birkmeyer, Michael Cuntz, Christian Eder, Andreas Gross, Tommy Hofmann, Martin Lee, Adi Popescu, Gerhard Pfister, Matthias Schulze.
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Part 1

Computing tropical varieties over fields with valuation

CHAPTER 0

Introduction

0.1. Motivation

Tropical geometry is a field of mathematics which can be studied from numerous perspectives. And while this does not come surprising for an area rich in combinatorics, the wide range of the perspectives is.

One perspective comes from the study of max-plus or min-plus semirings, which have caught the interest of theoretical computer scientists long time ago [**Sim78**]. It was this context, in which the adjective *tropical* was coined by French mathematicians honoring the pioneering work of the Brazilian mathematician Imre Simon [**Sim88**] [**Pin98**]. Other perspectives and applications of tropical geometry come from problems arising in biology [**PS04**] [**PS07**], physics [**CGQS05**] and economics [**BK14**].

This thesis takes a perspective originating from Bergman, who made the surprising observation that complex algebraic sets in logarithmic scale resemble polyhedral complexes [**Ber71**]. Bieri and Groves later expanded that observation to general fields with valuation [**BG84**]. Decades later, these polyhedral complexes are referred to as *tropical varieties*, and they are widely regarded as combinatorial shadows of algebraic varieties. Given an affine algebraic variety X in a torus $(K^*)^n$, its tropical variety $\operatorname{Trop}(X)$ in $\mathbb{R}^n$ carries enough information to shed some light on its classical counterpart, yet at the same time it is light enough to be reduced to an almost purely combinatorial structure. One of the more intriguing pieces of information it carries is described in "Mikhalkin's Correspondence Theorem" [**Mik05**], which established tropical geometry as a powerful tool to study enumerative geometry. Since then, tropical geometers have developed new combinatorial approaches to classical theories [**GM07**] [**GM08**] and pushed their knowledge beyond what was known classically [**IKS13a**] [**IKS13b**].

A centerpiece in studying tropical geometry with a view towards algebraic geometry and the main topic of this thesis is the *tropicalization*, a process which maps an affine algebraic variety X to its tropical variety $\operatorname{Trop}(X)$. Computing the tropical variety of an ideal $I \trianglelefteq K[x^{\pm 1}]$, i.e. that of its affine variety $V(I) \subseteq (K^*)^n$, can be surprisingly simple in some cases. If I happens to be generated by a single polynomial f, its tropical variety $\operatorname{Trop}(I)$ is, at least combinatorially, completely determined by the subdivision of the Newton polytope of f induced by the valuation of the coefficients (see [**MS14**] Chapter 1.2). Generally however, it is a computationally hard problem to solve.

Bogart, Jensen, Speyer, Sturmfels and Thomas were the first to describe an algorithm for computing tropical varieties [**BJS$^+$07**], which is then further expanded upon in Jensen's thesis [**Jen07**]. It covers ideals over the complex numbers $\mathbb{C}$ as well as the Puiseux series $\mathbb{C}\{\{t\}\}$ thanks to a trick described in Chapter 6.3 of the aforementioned thesis. The algorithms are implemented in GFAN [**Jen11**], a software package for computing Gröbner fans and tropical varieties by Anders Jensen.

In order to apply their techniques to other fields with valuation however, say the p-adic numbers $\mathbb{Q}_p$, one seemingly reaches the limit of classical Gröbner basis theory, as it ignores the valuation on the ground field. This urged Chan and Maclagan to extend the classical Gröbner basis theory by taking valuations on the ground field into account [**CM13**], which in turn inspired other people to apply new experimental ideas for Gröbner basis computations to the newly founded framework [**Vac14**]. Algorithms for computing Gröbner complexes based on this can also be found in GFAN, though none exist for tropical varieties yet.

This thesis takes a step back to the seeming limit and explores a small detour around the dead end. The detour will eventually lead to the tropical variety $\mathrm{Trop}(I)$ of an ideal $I \trianglelefteq K[x]$ over a valued field K, and it will be accessible with the classical notions of computer algebra. Albeit not of better complexity than an approach using Chan and Maclagan's idea at first glance, this approach enables the usage of the newest cutting-edge techniques in Gröbner bases computation, which is a highly active field of research [**EF14**], [**Neu13**].

Before continuing with the summary, it is also noteworthy that the fascination of tropical varieties did not take long to convince mathematicians to study them on a firm axiomatic footing. Definitions of tropical varieties surfaced that were detached from any algebraic variety they might arise from. While the axiomatic footing provided tropical geometers with a solid foundation to work on, examples of tropical varieties emerged that were not realisable as tropicalizations of algebraic varieties [**Spe07**]. Since then, driven by the advances in tropical geometry, the realizability of tropical varieties has developed into a worthwhile and challenging problem of its own. While the computation of tropical varieties is not of immediate theoretical interest for the realizability of tropical varieties, it is central for experiments while hunting for new insights.

0.2. Summary of Results

Chapter 1. At the end of this chapter, in Definition 1.2.11, we introduce tropical varieties over rings and we will show how tropical varieties over valued fields can be traced back to them, despite the rings not carrying any non-trivial valuation themselves.

By restricting our ground field to its valuation ring, forgetting the valuation in the process, and introducing a power series variable t responsible for tracking a fixed uniformizing parameter as in Definition 1.2.5, it is easy to imagine how the old valuations of the coefficients reflect in the new degrees of t when working with monomial orderings such that $t < 1$. Proposition 1.2.9 essentially captures this observation in the terminology of initial ideals, its proof is fairly straight forward, once broken down into digestible cases. Theorem 1.2.13 then establishes the crucial connection between tropical varieties over valued fields and tropical varieties over their valuation rings without the valuation.

On a side note, it is refreshing to see these tropical varieties play a central role in our approach. Though a little bit neglected in tropical geometry when studied with a view towards algebraic geometry, tropical varieties over the integers have actually been one of the focal points in the origin of tropical geometry. Studied by Bieri, Groves, Neumann and Strebel [**BS80**] [**BG84**] [**BNS87**], these objects yielded surprising insights in a wide class of discrete groups. A short exposition on their collaborate work can be found in Chapter 1.6 of [**MS14**], which concludes with the words: The beautiful group theory results by Bieri, Groves, Neumann and Strebel suggest that further research on this topic is desirable.

Chapter 2. There is little to no doubt that standard basis theory, respectively Gröbner basis theory, is a major driving force behind any computational algebra that studies anything remotely similar to ideals in polynomial rings. It should therefore be no surprise that our first priority is to delve into it.

The idea of standard bases goes back to Gordan [**Gor99**] in 1899. Later, monomial orderings were used by Macaulay and Gröbner to study Hilbert functions of graded algebras and, more generally, to find K-bases of zero dimensional quotient rings. Formally, the notion of standard bases was introduced independently by Hironaka [**Hir64**] and Grauert [**Gra72**], though only for special local orderings. Around the same time, the name Gröbner basis was coined by Buchberger [**Buc65**] for global orderings, who introduced them together with the famous algorithm bearing his name, acknowledging the influence of his adviser Gröbner in his work.

In this chapter, we combine techniques from Markwig [**Mar08**] and Wienand [**Wie11**] to extend the classical standard basis theory to a mixed polynomial and power series ring $R[\![t]\!][x]$, where R is a noetherian ring, and $t = (t_1, \ldots, t_m)$, $x = (x_1, \ldots, x_n)$ stand for a tuples of variables. Note that while in theory R is only required to be noetherian, for our algorithm we additionally require in Convention 2.1.1 that some specific computational problems can be solved in it.

[**Mar08**] unifies the division theorems from Grauert [**Gra72**] and Mora [**Mor82**] to $K[\![t]\!][x]$, where K is a field and t, x are as before, and establishes a standard basis theory over it, following the guideline of the classical theory. Also of great importance are its statements about weighted orderings and $\langle t\rangle$-adic convergence, which we will use one-to-one as Lemma 2.1.13 and Lemma 2.1.14.

[**Wie11**] generalizes Adams and Loustaunau's work on Gröbner bases over rings [**AL94**] and Mora's work on standard bases over fields [**Mor82**]. Establishing a standard basis theory for polynomial rings over a ground rings, it analyses which computational problems are required to be solvable in the ground ring and provides several optimizations for special cases of ground rings. The previously mentioned conditions in Convention 2.1.1 were taken from it.

Structurally and logically, introducing the division with remainder in Section 2.1 follows [**Mar08**], applying ideas from [**Wie11**] whenever the theory needs to be adjusted to ground rings. There is some slight divergence from the source material in Algorithm 2.1.16 for the homogeneous division with remainder which is explained in Example 2.1.19. Introducing standard bases in Section 2.2 in turn borrows heavily from [**Wie11**]. Therefore, this chapter is more to be regarded as a fitting together of two existing theories rather than a purely original work.

It should be pointed out that, similar to [**Mar08**], we put an emphasis on termination in the case our input data is polynomial, i.e. bounded in degree of t. While one might think of this as the triviality it is at times, we observe in Example 2.2.24 that reducing a standard basis is no finite process. No surprising side effect when working with orderings under which the monomials are not well ordered, this is nevertheless an essential problem that needs to be worked around.

Chapter 3. In this chapter, we introduce Gröbner fans of x-homogeneous ideals in $R[\![t]\!][x] = R[\![t]\!][x_1, \dots, x_n]$ and describe an algorithm for their computation by applying techniques developed by Fukuda, Jensen and Thomas [**FJT07**].

Originally introduced as an invariant for homogeneous ideals I in polynomial rings $K[x]$ by Mora and Robbiano [**MR88**], the Gröbner fan of I describes the variation of initial ideals $\mathrm{in}_w(I)$ as the weight vector w ranges over the whole weight space $\mathbb{R}^n$. Therefore Gröbner fans naturally appear in theories in which the monomial ordering is not fixed, for example in Gröbner walks or in dynamical Gröbner basis computations [**CP12**]. For tropical geometry, Gröbner fans are important because they provide a natural polyhedral structure compatible with the tropical variety.

The biggest challenge when trying to apply the techniques of Fukuda, Jensen and Thomas is to find a suitable replacement for the reduced Gröbner bases that play a central role in their theory. As we have seen in Example 2.2.24, their canonical counterparts in our setting, reduced standard bases, are useless for practical purposes. Hence we begin Section 3.1 by introducing initially reduced standard bases in Definition 3.1.7, and we will show over the course of the chapter that initially reduced standard bases are weak enough to be computed in finite time, yet strong enough to do justice to their role in the Gröbner fan construction.

In this chapter, we restrict ourselves to ideals in $R[\![t]\!][x]$ which are homogeneous in x, see convention 3.1.1, and Convention 2.1.1 on the ground ring is also assumed to be true. Structurally, the construction of the Gröbner fan follows **[FJT07]**, substituting Gröbner bases with standard bases and reduced Gröbner bases with initially reduced standard bases. It becomes apparent that the proofs only require minimal adjustments, as the notion of initially reduced standard bases were specifically designed with them in mind. One additional ingredient that needs to be provided, however, is the finiteness of leading ideals. A well known result in the classical Gröbner basis theory, it is proven similarly for ideals in $R[\![t]\!][x]$ in 3.1.4 by abusing the t-local property of our monomial orderings.

In the end, we obtain a generalized Gröbner fan theory, which specializes to the classical theory if R is a field and t does not exist. It also goes beyond the classical theory as we weaken the classical homogeneity constraint which is too restrictive on our ideals. This is crucial because the ideals arising from Theorem 1.2.13 are never homogeneous, unless we are dealing with examples in which the valuation can be neglected in the first place.

We then continue in Section 3.2 by developing an algorithm to reduce a standard basis initially in finite time. The proof of correctness is, as one might expect, a little bit technical, while the proof of termination uses a standard exhaustion argument that has, in its essence, appeared in several works before, for example **[CM13]** Lemma 2.6.

Finally, we conclude the chapter with Section 3.3 by transcribing the algorithms in **[FJT07]** for computing the Gröbner fans, which only requires very little adjustments thanks to the groundwork done in the previous sections.

Chapter 4. In this chapter, we apply the ideas of Bogart, Jensen, Speyer, Sturmfels and Thomas **[BJS$^+$07]** to describe an algorithm for computing the tropical varieties over rings for ideals that arose from Theorem 1.2.13. Combined with the theorem, we thus obtain an algorithm for computing tropical varieties over valued fields requiring only classical standard bases techniques, i.e. without taking valuations on coefficients into account.

As a caveat, note that Theorem 1.2.13 imposes several nice properties on the tropical varieties, which we exploit in our algorithm. In particular, we rely on the fact that our tropical varieties are pure and connected in codimension 1, which in general tropical varieties over rings are not.

Lastly, we introduce some important optimizations specifically tailored to our class of ideals. These optimizations imply that, amongst others, all but one standard basis computation over the ground ring can be reduced to Gröbner basis computations over the residue field. Moreover, these Gröbner basis computations will run on essentially homogeneous data. This is especially nice when working with the p-adic valuation, as it allows us to work over a finite field $\mathbb{F}_p$. It also allows us to use monomial orderings under which the monomials are well-ordered.

Apart from the obvious decrease in complexity of the setting our computations run in, it gains special importance due to the vast majority of software libraries for Gröbner basis computation which specialize in this case, for example [**Neu13**]. This is because the finiteness of the coefficients represents a hard bound on its internal size and the well-ordering yields a hard bound on the amount of monomials appearing during the computation.

Chapter 5. Chapter 5 contains information about the implementation of the algorithms in SINGULAR, as well as the interface to GFANLIB [**Jen11**] and POLYMAKE [**GJ00**], which were a necessary prerequisite.

Moreover, we benchmark our software to GFAN by looking at a selection of determinantal ideals, Plücker ideals and tropical linear spaces.

CHAPTER 1

Tropical Varieties

1.1. Tropical varieties over fields with valuation

In this section, we recall some fundamental notions of tropical geometry from the book of Maclagan and Sturmfels [**MS14**]. We will only cover the concepts that are of immediate relevance and refer to the said book for a wider exposition of the theory.

Recap 1.1.1 (basic concepts of valuation theory) Let K be a field. A *valuation* on K is a function $\nu : K \to \mathbb{R} \cup \{\infty\}$ such that

(1) $\nu(a) = \infty$ if and only if $a = 0$,
(2) $\nu(a \cdot b) = \nu(a) + \nu(b)$ for all $a, b \in K^*$,
(3) $\nu(a + b) \geq \min(\nu(a), \nu(b))$ for all $a, b \in K^*$.

We call a valuation ν *trivial*, if $\nu(a) = 0$ for all $a \in K^*$.

Given a field K with valuation ν, its *valuation ring* is defined to be

$$R_\nu := \{a \in K \mid \nu(a) \geq 0\}.$$

It is a local ring with maximal ideal $\mathfrak{m} := \{a \in R_\nu \mid \nu(a) > 0\}$, and $\mathfrak{K} := R_\nu/\mathfrak{m}$ is referred to as its *residue field*.

We call R_ν a *discrete valuation ring*, if it is not a field and satisfies one of the following equivalent conditions:

(1) R_ν is noetherian,
(2) $(\nu(K), +) \leq (\mathbb{R}, +)$ is isomorphic to $(\mathbb{Z}, +)$,
(3) R_ν is a principal ideal domain.

In that case, its maximal ideal $\mathfrak{m}$ is generated by a single element $p \in R_\nu$, which we refer to as a *uniformizing parameter*.

Example 1.1.2 (Puiseux series, Laurent series) The field of Puiseux series is defined to be the set

$$\mathbb{C}\{\{t\}\} = \left\{ \sum_{k=k_0}^{\infty} a_k t^{k/n} \;\middle|\; k_0 \in \mathbb{Z},\ n \in \mathbb{N}_{>0},\ a_k \in \mathbb{C} \right\}$$

equipped with the natural addition and multiplication, which extend the operations in the ring of power series. It is the algebraic closure of the field of Laurent series

$$\mathbb{C}(\!(t)\!) = \left\{ \sum_{k=k_0}^{\infty} a_k t^k \;\middle|\; k_0 \in \mathbb{Z},\ a_k \in \mathbb{C} \right\}.$$

The order function in the ring of power series extends naturally to a valuation on both fields:

$$\nu\Big(\sum_{k=k_0}^{\infty} a_k t^{k/n}\Big) = \min\big(\{k/n \mid k \geq k_0, a_k \neq 0\} \cup \{\infty\}\big),$$
$$\nu\Big(\sum_{k=k_0}^{\infty} a_k t^{k}\Big) = \min\big(\{k \mid k \geq k_0, a_k \neq 0\} \cup \{\infty\}\big).$$

The valuation ring of the Laurent series is the ring of power series $\mathbb{C}[\![t]\!]$, which is local with maximal ideal generated by a uniformizing parameter t. Its residue field is therefore $\mathbb{C}$.

Example 1.1.3 (p-adic numbers, p-adic integers) Let $p \in \mathbb{N}$ be prime. Then the field of p-adic numbers can be thought of as the set of formal Laurent series in p

$$\mathbb{Q}_p = \left\{ \sum_{k=k_0}^{\infty} a_k p^k \;\middle|\; k_0 \in \mathbb{Z},\ a_k \in \mathbb{N},\ 0 \leq a_k \leq p-1 \right\}$$

with the same operations, except the minor addition that coefficients above $p-1$ need to be carried over to the next power in p. And similarly to the Laurent series, the valuation on it is defined to be

$$\nu\Big(\sum_{k=k_0}^{\infty} a_k p^{k}\Big) = \min\big(\{k \mid k \geq k_0,\ a_k \neq 0\} \cup \{\infty\}\big).$$

Its valuation ring is the ring of p-adic integers

$$\mathbb{Z}_p = \left\{ \sum_{k=0}^{\infty} a_k p^k \;\middle|\; a_k \in \mathbb{N},\ 0 \leq a_k \leq p-1 \right\}$$

with a maximal ideal generated by the uniformizing parameter $p \in \mathbb{Z}_p$. Consequently, its residue field is $\mathbb{F}_p$.

Convention 1.1.4 For the remainder of the section, fix a field K with valuation ν and all its associated objects. Consider the ring of multivariate Laurent polynomials $K[x^{\pm 1}] = K[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$.

Definition 1.1.5 (initial form, initial ideal) For a Laurent polynomial $f = \sum_{\alpha \in \mathbb{Z}^n} c_\alpha \cdot x^\alpha \in K[x^{\pm 1}]$ and a weight vector $w \in \mathbb{R}^n$, we define the *valued initial form* of f with respect to w to be:

$$\mathrm{in}_{\nu,w}(f) := \sum_{\substack{w\cdot\alpha - \nu(c_\alpha) \\ \text{maximal}}} \overline{c_\alpha \cdot p^{-\nu(c_\alpha)}} \cdot x^\alpha \in \mathfrak{K}[x^{\pm 1}].$$

Similarly, for an ideal $I \trianglelefteq K[x^{\pm 1}]$ and a weight vector $w \in \mathbb{R}^n$, we define the *valued initial ideal* of I with respect to w to be:

$$\mathrm{in}_{\nu,w}(I) := \langle \mathrm{in}_{\nu,w}(f) \mid f \in I \rangle \trianglelefteq \mathfrak{K}[x^{\pm 1}].$$

Should the valuation be trivial, then we will omit the subscript ν.

Example 1.1.6 For $f_1 := x + y + 1$, $f_2 := t^2 x + y + t \in \mathbb{C}\{\{t\}\}[x^{\pm 1}, y^{\pm 1}]$ and weight vector $w := (w_x, w_y) \in \mathbb{R}^2$ we have, amongst other cases,

$$\mathbb{C}[x^{\pm 1}, y^{\pm 1}] \ni \mathrm{in}_{\nu,w}(f_1) = \begin{cases} x, & \text{if } w_x > w_y \text{ and } w_x > 0, \\ y, & \text{if } w_y > w_x \text{ and } w_y > 0, \\ 1, & \text{if } 0 > w_x \text{ and } 0 > w_y, \end{cases}$$

$$\mathbb{C}[x^{\pm 1}, y^{\pm 1}] \ni \mathrm{in}_{\nu,w}(f_2) = \begin{cases} x, & \text{if } w_x - 2 > w_y \text{ and } w_x - 2 > -1, \\ y, & \text{if } w_y > w_x - 2 \text{ and } w_y > -1, \\ 1, & \text{if } -1 > w_x - 2 \text{ and } -1 > w_y. \end{cases}$$

In particular, for $I := \langle f_1, f_2 \rangle$ this implies

$$\mathrm{in}_w(I) = \langle 1 \rangle, \quad \text{if } (w_x < 0 \text{ and } w_y < 0) \text{ or } (w_x < 1 \text{ and } w_y < -1).$$

In fact, it can be shown that the condition above is not only sufficient, but also necessary, which means that the set of weight vectors $w \in \mathbb{R}^2$ for which $\mathrm{in}_w(I) = \langle 1 \rangle$ is not convex. However, note that I is not homogeneous.

Definition 1.1.7 (tropical variety) Let $I \trianglelefteq K[x^{\pm 1}]$ be an ideal. Then we refer to the following set of weight vectors as the *tropical variety* of I:

$$\mathrm{Trop}_\nu(I) := \{w \in \mathbb{R}^n \mid \mathrm{in}_{\nu,w}(I) \neq \langle 1 \rangle\}.$$

As before, we will omit the subscript ν should the valuation be trivial.

Example 1.1.8 (tropical varieties of polynomials) Note that for a principal ideal $I = \langle f \rangle \trianglelefteq K[x^{\pm 1}]$ we have

$$\mathrm{Trop}_\nu(I) = \{w \in \mathbb{R}^n \mid \mathrm{in}_{\nu,w}(f) \text{ no term}\} =: \mathrm{Trop}_\nu(f),$$

in which case we will refer to $\mathrm{Trop}_\nu(I)$ also as the tropical variety of f.

Figure 1 shows the tropical varieties of the two polynomials of the previous Example 1.1.6,

$$f_1 = x + y + 1, \quad f_2 = t^2 x + y + t \in \mathbb{C}\{\{t\}\}[x^{\pm 1}, y^{\pm 1}],$$

as well as that of their product $f_1 \cdot f_2$, which turns out to be the union of their tropical varieties.

It can be shown that the weight vector $(1, -1)$ highlighted in red is the tropical variety of $\mathrm{Trop}_\nu(I)$ with $I = \langle f_1, f_2 \rangle$. It is the intersection of $\mathrm{Trop}_\nu(f_1)$ and $\mathrm{Trop}_\nu(f_2)$, so that f_1, f_2 form a *tropical basis* of I.

Tropical varieties such as those of f_1 and f_2 are commonly refered to as *tropical lines*. The vertex in the intersection of the three edges is also called the *apex*.

Next comes a theorem that states that tropical varieties behave nicely under field extensions. This is essential, because computationally we will generally be restricted to practially manageable fields, see Convention 1.2.2, while geometrically we are traditionally more interested in algebraically closed fields, see Theorem 1.1.12.

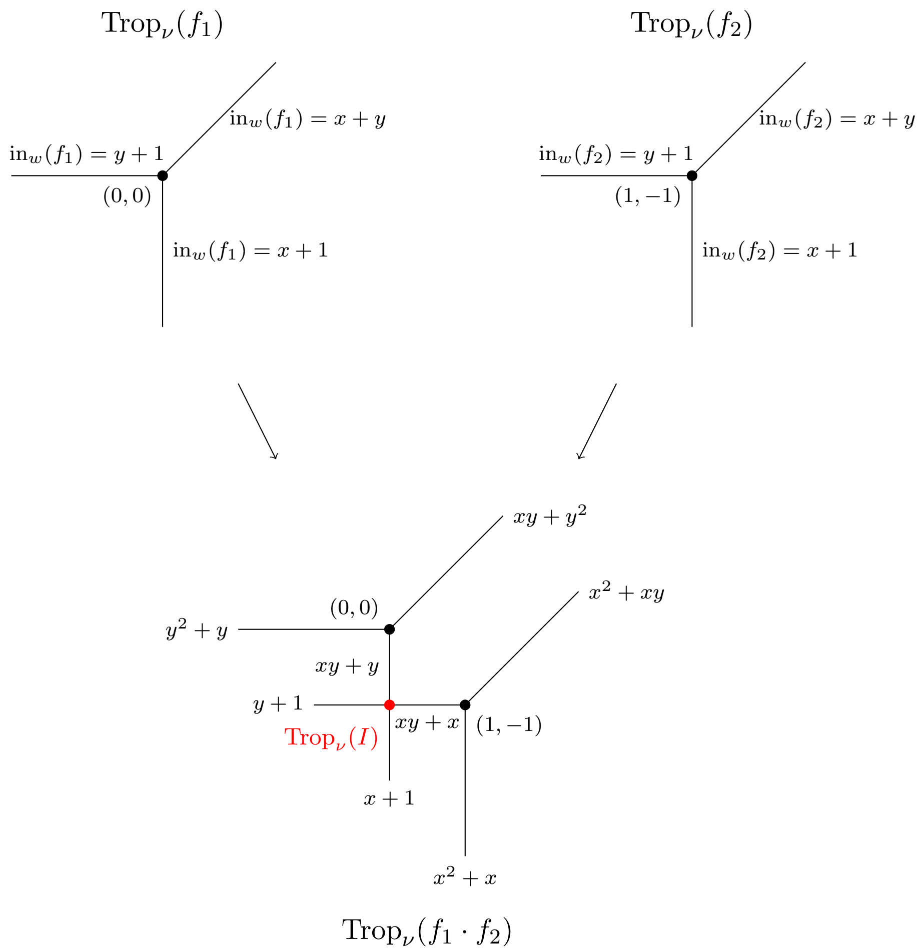


FIGURE 1. $\mathrm{Trop}_\nu(f_1), \mathrm{Trop}_\nu(f_2)$ and $\mathrm{Trop}_\nu(f_1 \cdot f_2)$

Recap 1.1.9 (valued field extentions) Let K and L be two fields with valuations $\nu_K : K \to \mathbb{R} \cup \{\infty\}$ and $\nu_L : L \to \mathbb{R} \cup \{\infty\}$. We say $L|K$ is a *valued field extension*, if $L|K$ is a field extension and $\nu_L|_K = \nu_K$.

Theorem 1.1.10 ([**MS14**] Theorem 3.2.4) *Let $I \trianglelefteq K[x^{\pm 1}]$ be an ideal, and let $L|K$ be a valued field extension with valuation ν_L on L extending our valuation ν on K. Then*

$$\mathrm{Trop}_\nu(I) = \mathrm{Trop}_{\nu_L}(I \cdot L[x^{\pm 1}]).$$

In particular, if $L := \overline{K}$ is the algebraic closure of K, then

$$\mathrm{Trop}_\nu(I) = \mathrm{Trop}_{\nu_{\overline{K}}}(I \cdot \overline{K}[x^{\pm 1}]).$$

Example 1.1.11 Consider the ideal

$$I := \langle x_1 - 2x_2 + 3x_3, 3x_2 - 4x_3 + 5x_4 \rangle \trianglelefteq \mathbb{Q}_2[x^{\pm 1}] := \mathbb{Q}_2[x_1^{\pm 1}, \dots, x_4^{\pm 1}].$$

Since it can clearly be generated over the subfield $\mathbb{Q} \subseteq \mathbb{Q}_2$, we restrict ourselves to $\mathbb{Q}$ (but equipped with the 2-adic valuation) while computing its tropical variety, and we will also obtain its tropical variety over the algebraic closure $\overline{\mathbb{Q}}_p$ at the same time,

$$\mathrm{Trop}_{\nu_2|\mathbb{Q}}(I \cap \mathbb{Q}[x^{\pm 1}]) = \mathrm{Trop}_{\nu_2}(I) = \mathrm{Trop}_{\nu_2}(I \cdot \overline{\mathbb{Q}_p}[x^{\pm 1}]).$$

With this out of the way, we now introduce two well-known theorems of tropical geometry. The first is the Fundamental Theorem of tropical algebraic geometry, which connects tropical geometry with algebraic geometry, and the latter is the Structure Theorem, which states that tropical varieties are the support of nicely structured polyhedral complexes.

Theorem 1.1.12 (Fundamental Theorem of Tropical Algebraic Geometry, [**MS14**] Theorem 3.2.3) *Let $I \trianglelefteq K[x^{\pm 1}]$, and let K be algebraically closed. Let $X = V(I)$ be the affine variety of I in the torus $(K^*)^n$. Then, if the valuation ν is non-trivial,*

$$\mathrm{Trop}_\nu(I) = \overline{\{(\nu(z_1), \ldots, \nu(z_n)) \in \mathbb{R}^n \mid (z_1, \ldots, z_n) \in X\}} \subseteq \mathbb{R}^n,$$

where $\overline{(\cdot)}$ denotes the closure in the euclidean topology of $\mathbb{R}^n$.

Recap 1.1.13 (polyhedral complexes) A *polyhedral complex* is a finite collection Σ of closed polytopes, also referred to as its *cells*, in a common real vector space $\mathbb{R}^n$, such that

(1) for each $\sigma \in \Sigma$ and each face $\tau \leq \sigma$ we have $\tau \in \Sigma$,
(2) for two $\sigma_1, \sigma_2 \in \Sigma$ the intersection $\sigma_1 \cap \sigma_2$ is a face of both.

Naturally, it is partially ordered by the relation

$$\tau \leq \sigma \quad :\Longleftrightarrow \quad \tau \text{ is a face of } \sigma,$$

and we call Σ *pure*, if all its maximal cells are of the same dimension.

The *support* of Σ is the union of all its cells in $\mathbb{R}^n$, and we say a pure Σ is *connected in codimension* 1, if its support minus the cells of codimension 2 and higher remains connected in $\mathbb{R}^n$. Visually, this means that one can reach any maximal cell from any maximal cell by crossing only the relative interiors of maximal and one-codimensional cells.

The *lineality space* of Σ is the maximal linear subspace, that is, up to translation, included in every cell of Σ. It represents a redundancy which can be divided out without changing the combinatorial structure of a polyhedral complex.

Note that we have a distinguished lattice $\mathbb{Z}^n \leq \mathbb{R}^n$. We say that Σ has *rational slopes*, if every $\sigma \in \Sigma$ is of the form

$$\sigma = \{w \in \mathbb{R}^n \mid A \cdot w \geq b\}, \text{ for some } A \in \mathrm{Mat}(m \times n, \mathbb{Z}) \text{ and } b \in \mathbb{R}^m,$$

where $m \in \mathbb{N}$ is the number of defining inequalities for σ. We call a vector $w \in \mathbb{Z}^n$ *primitive*, if for $v \in \mathbb{Z}$ and $k \in \mathbb{Z}$, $k \neq 0$, the inclusion $k \cdot v \in w \cdot \mathbb{Z}$ implies $v \in w \cdot \mathbb{Z}$.

A polyhedral complex Σ is *weighted*, if its maximal cones $\sigma \in \Sigma$ are endowed with multiplicities $\mathrm{mult}(\sigma) \in \mathbb{Z}$, also referred to as weights. And we say a pure, weighted polyhedral complex is *balanced*, if for all $\tau \in \Sigma$ of codimension one we have

$$\sum_{\tau \leq \sigma \in \Sigma} \operatorname{mult}(\sigma) \cdot \overline{u}_{\sigma,\tau} = 0 \in \mathbb{R}^n / \mathbb{R}(\tau),$$

where $u_{\sigma,\tau} \in \mathbb{Z}^n$ is a primitive inner normal vector of τ in σ, and $\mathbb{R}(\tau)$ is the subspace spanned by the translation of τ to the origin.

Example 1.1.14 (polyhedral fans) A very important class of polyhedral complexes are the so-called *polyhedral fans*, which consist solely of convex polyhedral cones. In fact, every polyhedral complex can be represented as a polyhedral fan, intersected by an affine hypersurface, as pictured in Figure 2 showing the tropical variety of Example 1.1.8. The cone generated by the two highlighted rays in the polyhedral fan yield the highlighted unbounded one-dimensional cell in the polyhedral complex when intersected with the affine hypersurface in blue.

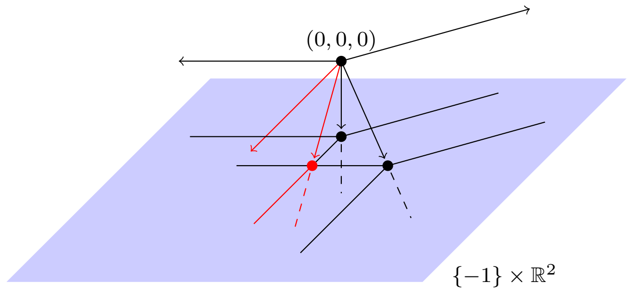


FIGURE 2. fans giving rise to complexes and vice versa

Software systems that deal with convex geometry such as `polymake` [**GJ00**] or `gfan` [**Jen11**] use this to safe the work of implementing separate frameworks for polyhedral fans and polyhedral complexes. These two have a framework for polyhedral fans and represent polyhedral complexes in the way explained above.

Recap 1.1.15 (primary decomoposition) An ideal $Q \trianglelefteq \mathfrak{K}[x^{\pm 1}]$ over the residue field is called *primary*, if $f \cdot g \in Q$ implies $f \in Q$ or $g^m \in Q$ for some $m > 0$, and any ideal $I \trianglelefteq \mathfrak{K}[x^{\pm 1}]$ can be written as an intersection of primary ideals, $I = Q_1 \cap \ldots \cap Q_k$, the so-called *primary decomposition*. The decomposition is called *irredundant*, if no Q_i can be omitted in the decomposition and $\sqrt{Q_i} \neq \sqrt{Q_j}$ for $i \neq j$.

The set of *associated primes* of I is defined to be the set

$$\operatorname{Ass}(I) = \{P \trianglelefteq \mathfrak{K}[x] \mid P \text{ prime and } P = I : \langle b \rangle \text{ for some } b \in \mathfrak{K}[x]\},$$

and it is an important theorem in commutative algebra that, given an irredundant primary decomposition as above, we have

$$\operatorname{Ass}(I) = \left\{\sqrt{Q_1}, \ldots, \sqrt{Q_k}\right\},$$

which proves that the radicals of an irredundant primary decomposition are unique.

The set $\mathrm{Ass}(I)$ is partially ordered by inclusion and its minimal elements are of special interest to us. Amongst other properties, for a minimal $P \in \mathrm{Ass}(I)$, the primary ideal Q in the irredundant decomposition of I with $\sqrt{Q} = P$ does not depend on the choice of the decomposition. In particular, we can define its *multiplicity* to be

$$\mathrm{mult}(P, I) = \mathrm{length}_{\mathfrak{K}[x^{\pm 1}]_P}(\mathfrak{K}[x^{\pm 1}]/Q)_P.$$

Example 1.1.16 We will anticipate some calculations of Example 1.1.19 in the ring $\mathbb{F}_2[x^{\pm 1}, y^{\pm 1}]$. Over $\mathbb{F}_2$, the ideal $\langle y^2+1\rangle$ has an irredundant primary decomposition $\langle y^2+1\rangle = \langle y+1\rangle^2$, and therefore $\mathrm{Ass}(\langle y^2+1\rangle) = \{\langle y+1\rangle\}$. For the multiplicity $\mathrm{mult}(\langle y+1\rangle, \langle y^2+1\rangle)$, note that

$$0 \subsetneq \langle y+1\rangle \cdot (\mathbb{F}_2[x^{\pm 1}, y^{\pm 1}]/\langle y^2+1\rangle)_{\langle y+1\rangle} \subsetneq (\mathbb{F}_2[x^{\pm 1}, y^{\pm 1}]/\langle y^2+1\rangle)_{\langle y+1\rangle}$$

is a composition series of $\mathbb{F}_2[x^{\pm 1}, y^{\pm 1}]_{\langle y+1\rangle}$-modules. Hence the length of the last module and the wanted multiplicity equals 2.

Next consider the ideal $\langle xy+y^2\rangle = \langle x+y\rangle$. Because it is prime itself, its primary decomposition is rather trivial and we get $\mathrm{Ass}(\langle x+y\rangle) = \{\langle x+y\rangle\}$. It follows that $\mathrm{mult}(\langle x+y\rangle, \langle x+y\rangle) = 1$. The same holds true for the ideals $\langle x^2+xy\rangle = \langle x+y\rangle$, $\langle x^2+x\rangle = \langle x+1\rangle$ and $\langle xy+x\rangle = \langle y+1\rangle$.

Theorem 1.1.17 (Structure Theorem for Tropical Varieties, [**MS14**] Theorem 3.3.5) *Let $I \trianglelefteq K[x^{\pm 1}]$ be a prime ideal of dimension d. Then $\mathrm{Trop}_\nu(I)$ is the support of a pure polyhedral complex of dimension d with rational slopes that is connected in codimension 1.*

Moreover, let Δ be a polyhedral complex Δ with support $\mathrm{Trop}_\nu(I)$ such that for any cell $\sigma \in \Delta$ we have $\mathrm{in}_{\nu,u}(I) = \mathrm{in}_{\nu,w}(I)$ for all $u, w \in \mathrm{relint}(\sigma)$. Then Δ is balanced when each maximal cone $\sigma \in \Delta$ is equipped with the following multiplicity:

$$\mathrm{mult}(\sigma) = \sum_{\substack{P \in \mathrm{Ass}(\mathrm{in}_{\nu,w}(I)) \\ \textit{minimal}}} \mathrm{mult}(P, \mathrm{in}_{\nu,w}(I)), \ \textit{for any } w \in \mathrm{relint}(\sigma).$$

Next, we illustrate the Structure Theorem on some examples that stem from some of applications mentioned in the introduction.

Example 1.1.18 Consider the polynomial

$$f = t^3x^3 + tx^2y + txy^2 + t^3y^3 + tx^2 + xy + ty^2 + tx + ty + t^3 \in \mathbb{C}\{\{t\}\}[x^{\pm 1}, y^{\pm 1}].$$

The structure of $\mathrm{Trop}_\nu(f)$ is shown in Figure 3. The x and the y for example are representing that the edge between them is the closure of all weights $w \in \mathbb{R}^2$ such that $\mathrm{in}_w(f) = x + y$. All edges have weight 1.

Since the discovery of Mikhalkin's Correspondence Theorem in [**Mik05**], tropical geometry has gained a lot of positive attention from the enumerative geometers, and mathematicians began translating established concepts of

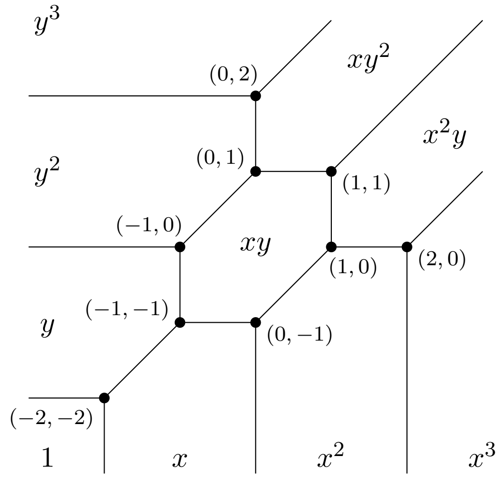


FIGURE 3. a tropical cubic

algebraic geometry to the newly discovered tropical world. The tropical variety above is an example of a smooth tropical curve of degree 3 with genus 1. You can find it as Example 2.2 in Block's survey on counting curves with tropical geometry [**Blo12**].

Example 1.1.19 Let $f = 1+64x+16y+128x^2+32xy+256y^2 \in \mathbb{Q}_2[x^{\pm 1}, y^{\pm 1}]$, then $\mathrm{Trop}_{\nu_2}(f)$ is of the form shown in Figure 4. The bold 2 indicates the weight of the edge next to it, while the remaining edges all have weight 1, see Example 1.1.16 for the computation.

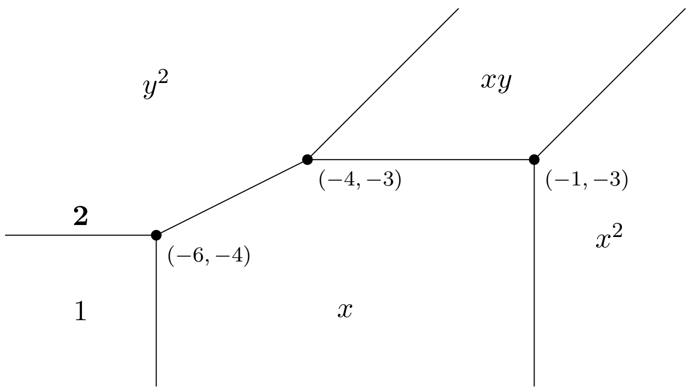


FIGURE 4. $\mathrm{Trop}_{\nu_2}(1 + 64x + 16y + 128x^2 + 32xy + 256y^2)$

In [**BK14**], Baldwin and Klemperer demonstrate how tropical Geometry can be used to analyze demand. In so called product-mix auctions, in which bidders offer prices for alternative bundles of goods, each bidder has a well-defined set of vectors in price space for which there is no unique bundle of interest. These sets are tropical varieties, and the tropical variety above is the negative of Example 2.9 in the cited paper. It represents a bidder, in an auction offering following bundles of goods:

$$\{(0,0),(1,0),(0,1),(2,0),(1,1),(0,2)\}.$$

The bidder puts the bundles $(0,0),(1,0),(0,1),(2,0),(1,1),(0,2)$ at values $0,6,4,7,9,8$ respectively, and the tropical variety is the set of price vectors with respect to which there is no unique bundle to buy for maximizing his profit.

Example 1.1.20 Consider the ideal

$$I = \langle x_1 - 2x_2 + 3x_3, 3x_2 - 4x_3 + 5x_4\rangle \trianglelefteq \mathbb{Q}_2[x_1,\ldots,x_4].$$

Because the ideal is homogeneous, we have

$$\mathrm{in}_w(I) = \mathrm{in}_{w+\lambda\cdot(1,1,1,1)}(I) \text{ for all } \lambda \in \mathbb{R}.$$

Therefore, its tropical variety is invariant under translation by scalar multiples $(1,1,1,1)$. Hence $\mathrm{Trop}_{\nu_2}(I)$ can be covered by polyhedral complexes with lineality space generated by that vector.

Figure 5 shows the combinatorial structure of one possible polyhedral complex covering $\mathrm{Trop}_{\nu_2}(I)$. The Figure has two vertices, each representing an one-dimensional polytope generated by the weight vector next to it and the lineality space. It has a bounded edge, which represents a two-dimensional polytope generated by the two adjacent vertices and the lineality space. It also has four unbounded edges, each representing a two-dimensional pohedron generated by the adjacent vertices and unbounded in the direction of the arrows. All edges have weight 1.

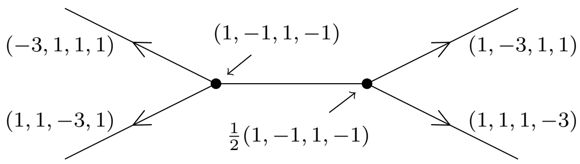


FIGURE 5. $\mathrm{Trop}_\nu(\langle x_1 - 2x_2 + 3x_3, 3x_2 - 4x_3 + 5x_4\rangle)$

Definition 1.1.21 For a weight vector $w \in \mathbb{R}^n$ we define its *Gröbner polytope* or *Gröbner cell* to be

$$C_{\nu,w}(I) := \overline{\{v \in \mathbb{R}^n \mid \mathrm{in}_{\nu,v}(I) = \mathrm{in}_{\nu,w}(I)\}} \subseteq \mathbb{R}^n,$$

where $\overline{(\cdot)}$ denotes the closure in the euclidean topology.

We will refer to the collection $\Sigma(I) := \{C_{\nu,w}(I) \mid w \in \mathbb{R}^n\}$ as the *Gröbner complex* of I.

Theorem 1.1.22 (Gröbner complex, [**MS14**] Theorem 2.5.3) *Let $I \trianglelefteq K[x^{\pm 1}]$ be a homogeneous ideal. Then the $C_{\nu,w}(I)$ are convex polytopes and $\Sigma(I)$ is a complete polyhedral complex.*

Example 1.1.23 Note that for the guaranteed convexity of the Gröbner cells, I needs to be homogeneous. Consider for example the inhomogeneous ideal $I = \langle x+1, y+1\rangle \trianglelefteq \mathbb{Q}[x^{\pm 1}, y^{\pm 1}]$, where $\mathbb{Q}$ is endowed with the trivial valuation. Figure 6 shows the decomposition of the weight space $\mathbb{R}^2$ into two Gröbner cells, one of which is clearly not convex.

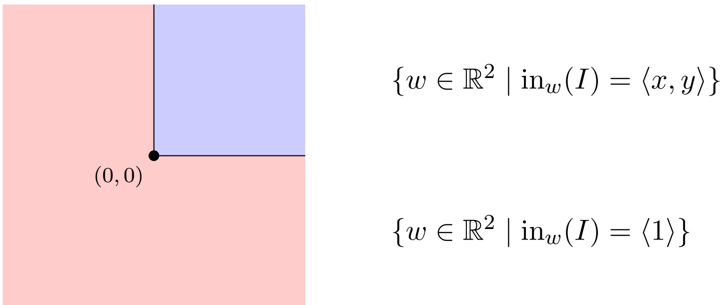


FIGURE 6. Gröbner cells of $\langle x+1, y+1\rangle$

However suppose the valuation is trivial. Then the Gröbner cells of the positive weights are guaranteed to be convex polytopes even in the inhomogeneous case (see **[Jen07]** Chapter 3).

The ideal $J = \langle x+z, y+z\rangle \trianglelefteq \mathbb{Q}[x^{\pm 1}, y^{\pm 1}, z^{\pm 1}]$, where $\mathbb{Q}$ is again endowed with the trivial valuation, is the homogenization of I. Its Gröbner complex therefore has a lineality space generated by $(1,1,1)$ and a section through it has the form shown in Figure 7, cf. Example 3.3.8.

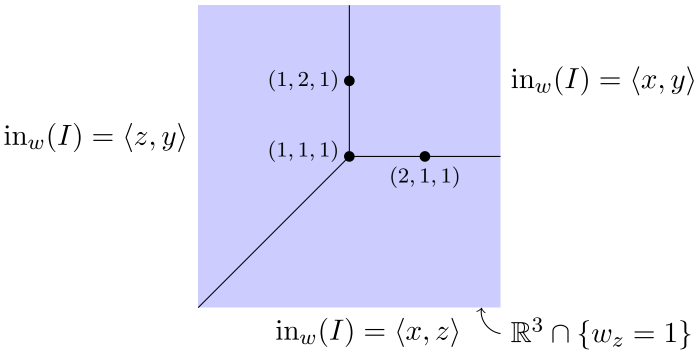


FIGURE 7. Gröbner cells of $\langle x+z, y+z\rangle$

The Gröbner complex $\Sigma(I)$ plays a central role in the computation of tropical varieties $\mathrm{Trop}_\nu(I)$. In fact, computing tropical varieties is not possible unless there is a warrant that $\Sigma(I)$ is a well-defined polyhedral complex, which can then be naturally restricted to a subcomplex on $\mathrm{Trop}_\nu(I)$ satisfying the conditions in the Structure Theorem 1.1.17. Naturally, these subcomplexes are the coarsest polyhedral complexes satisfying the conditions in the Structure Theorem 1.1.17.

The go-to solution for computing tropical varieties of inhomogeneous ideals I is to compute that of their homogenization I^h. One can show that

$$\mathrm{Trop}_\nu(I^h) \cap \{w_z = 0\} = \mathrm{Trop}_\nu(I) \times \{0\} \subset \mathbb{R}^{n+1},$$

where z denotes the homogenization variable. In our example above, this simply means

$$\begin{array}{ccccc} \mathrm{Trop}_\nu(I^h) & \cap\ \{w_z = 0\} = & \mathrm{Trop}_\nu(I) & \times\ \{0\}. \\ \| & & \| & \\ (1,1,1)\cdot\mathbb{R} & & \{(0,0)\} & \end{array}$$

Also note that while everything up to know was in the ring of Laurent polynomials $K[x^{\pm 1}]$, we may easily restrict ourselves to the polynomial ring $K[x]$ for computational purposes, for which there exist similar notions. This is important, as the majority of prominent computer algebra systems specialize in polynomial rings.

Definition 1.1.24 Given a weight vector $w \in \mathbb{R}^n$, a polynomial $f = \sum_{\alpha\in\mathbb{N}^n} c_\alpha \cdot x^\alpha \in K[x]$ and an ideal $I \trianglelefteq K[x]$, we define the *valued initial form* of f and the *valued initial ideal* of I with respect to w to be respectively:

$$\mathrm{in}_{\nu,w}(f) := \sum_{\substack{w\cdot\alpha-\nu(c_\alpha)\\ \text{maximal}}} \overline{c_\alpha \cdot p^{-\nu(c_\alpha)}} \cdot x^\alpha \in \mathfrak{K}[x].$$

and

$$\mathrm{in}_{\nu,w}(I) := \langle \mathrm{in}_{\nu,w}(f) \mid f \in I\rangle \trianglelefteq \mathfrak{K}[x].$$

Moreover, the *tropical variety* of I is defined to be

$$\mathrm{Trop}_\nu(I) := \{w \in \mathbb{R}^n \mid \mathrm{in}_{\nu,w}(I) \text{ monomial free}\}.$$

Should the valuation be trivial, then we will omit the subscript ν.

Lemma 1.1.25 *Let $I \trianglelefteq K[x^{\pm 1}]$ be an ideal, then we have*

$$\mathrm{Trop}_\nu(I) = \mathrm{Trop}_\nu(I \cap K[x]).$$

Proof. Follows from the equivalence

$$\mathrm{in}_{\nu,w}(I) = \langle 1\rangle \iff \mathrm{in}_{\nu,w}(I\cap K[x]) \text{ monomial free}$$

for all weight vectors $w \in \mathbb{R}^n$. □

1.2. Reduction to a trivial valuation

In this section, we will show how the case of non-trivial valuations can be traced back to the case of trivial valuations, ultimately allowing us to apply classical Gröbner basis techniques to compute tropical varieties over fields with valuation.

Recap 1.2.1 (topological rings and completion) If the valuation ν is non-trivial, then R_ν is a local ring with maximal ideal $\mathfrak{m} \trianglelefteq R_\nu$, giving it the natural structure of a *topological ring*, where the subsets $\mathfrak{m}^k$, $k \in \mathbb{N}$, form a basis of open neighbourhoods around 0. If R_ν is additionally noetherian, then Krull's intersection theorem implies $\bigcap_{k\in\mathbb{N}} \mathfrak{m}^k = 0$ and, by the Artin-Rees Lemma, the topology is Hausdorff.

The *completion* of R_ν is defined to be the inverse limit $\varprojlim R_\nu/\mathfrak{m}^k$ and we say R_ν is *complete*, if $R_\nu = \varprojlim R_\nu/\mathfrak{m}^k$. A subring $R \leq R_\nu$ naturally inherits a topology from R_ν, a topology induced by $\mathfrak{m} \cap R$, and we call $R \leq R_\nu$ *dense*, if R_ν is contained in the completion of R with respect to it.

Let $p \in R$ be a *uniformizing parameter*, that is a generator of the maximal ideal in R_ν. Should R_ν be complete and noetherian, elements of R_ν can be thought of as formal power series in a uniformizing parameter p and coefficients in R.

Note that the valuation ν induces a norm on K by $\|x\| := a^{-\nu(x)}$ for a fixed $a \in \mathbb{R}$, $a > 1$. If K is complete with respect to the metric induced by its norm, then R_ν is complete with respect to its topology.

Convention 1.2.2 In this section, we require the valuation on K to be non-trivial and discrete, making R_ν a discrete valuation ring and hence in particular noetherian. Moreover, we assume K and hence R_ν to be complete. Let $R \leq R_\nu$ be a dense, noetherian subring, and let $p \in R$ be a uniformizing parameter of R_ν.

From Theorem 8.13 in **[Mat89]** we obtain two exact sequences

$$\begin{tikzcd}
0 \arrow[r] & \langle p-t\rangle \cdot R[\![t]\!]_{\langle p-t\rangle} \arrow[r] & R[\![t]\!]_{\langle p-t\rangle} \arrow[r] & K \arrow[r] & 0, \\
0 \arrow[r] & \langle p-t\rangle \cdot R[\![t]\!] \arrow[r] \arrow[u] & R[\![t]\!] \arrow[r, "t \longmapsto p"'] \arrow[u] & R_\nu \arrow[r] \arrow[u] & 0.
\end{tikzcd}$$

and it immediately follows that

$$R/\langle p\rangle = R[\![t]\!] \Big/ \langle p,t\rangle = R[\![t]\!]/\langle t-p\rangle \Big/ \langle p\rangle = R_\nu \Big/ \langle p\rangle = \mathfrak{K},$$

where $\mathfrak{K}$ is the residue field of R_ν.

Remark 1.2.3 For all practical applications, requiring the valuation on K to be discrete is a sensible assumption. As mentioned in Theorem 1.1.10, given an ideal $I \trianglelefteq K[x]$, we are free to restrict ourselves to any subfield of K over which I can be generated. And examples for which we cannot restrict ourselves to a discretely valued subfield are very likely to be computationally unmaneagable in the first place, see Example 1.2.6.

Also, assuming K to be complete is merely done for the sake of convenience and poses no restriction on the examples we can study. Were K not complete, Theorem 1.1.10 would allows us to pass to its completion without changing the tropical variety.

Example 1.2.4 Here are some examples which satisfy Convention 1.2.2:

(1) $K = \mathbb{Q}_p$ the field of p-adic numbers, $R_\nu = \mathbb{Z}_p$ the ring of p-adic integers, $R = \mathbb{Z}$ and the prime number p as uniformizing parameter.
(2) $K = \mathbb{Q}(\!(t)\!)$ the field of Laurent series, $R_\nu = \mathbb{Q}[\![t]\!]$ the ring of power series, $R = \mathbb{Q}[t]$ and $p = t$.
(3) R_ν the completion of $\mathbb{Z}[i]_{(1+i)}$ at the ideal generated by $p = i + 1$ and $R = \mathbb{Z}[i]$.
(4) R_ν the completion of $R = \mathbb{Q}[x]_{(f)}$ at the ideal generated by $f \in \mathbb{Q}[x]$ for some $p = f \in \mathbb{Q}[x]$ irreducible.
(5) R_ν the completion of $R = S^{-1}\mathbb{Q}[x, y]$ with respect to $\langle x\rangle$, where $S = \mathbb{Q}[x, y] \setminus (\langle x-1, y\rangle \cup \langle x\rangle)$, which is multiplicatively closed as the complement of a union of two prime ideals, and let $p = x$. Note that R is not catenarian. Of course replacing R by $\mathbb{Q}(y)[x]$ would lead to the same R_ν. This example merely shows that the assumption on R does allow some odd rings as well.
(6) R_ν any completion of a localization of a Dedekind domain R at a prime ideal $P \trianglelefteq R$, $p \in P$ a suitable element. Note that p does not need to generate P, e.g. $R = \mathbb{Z}[\sqrt{-5}]$, $P = \langle 2, 1 + \sqrt{-5}\rangle$ and $p = 2$.

Definition 1.2.5 Let π denote the map below:

$$\begin{tikzcd}
0 \arrow[r] & \langle p-t\rangle \cdot R[\![t]\!]_{\langle p-t\rangle}[x] \arrow[r] & R[\![t]\!]_{\langle p-t\rangle}[x] \arrow[r] & K[x] \arrow[r] & 0, \\
0 \arrow[r] & \langle p-t\rangle \cdot R[\![t]\!][x] \arrow[r] \arrow[u] & R[\![t]\!][x] \arrow[r, "\pi", "t \longmapsto p"'] \arrow[u] & R_\nu[x] \arrow[r] \arrow[u] & 0.
\end{tikzcd}$$

Since $R_\nu[x] \subseteq K[x]$, we will abuse the notation and use π to refer to both the map $R[\![t]\!][x] \to R_\nu[x]$ as well as the composition $R[\![t]\!][x] \to R_\nu[x] \hookrightarrow K[x]$.

Example 1.2.6 (Puiseux series) Consider the ideal

$$I = \langle t^3x^3 + tx^2y + txy^2, t^5x^2y + t^3xy^2 + t^5y^3\rangle \in \mathbb{C}\{\{t\}\}[x, y].$$

Since the coefficients of the generators only have integral powers of t, we may restrict ourselves to the subfield $K := \mathbb{Q}(\!(t)\!)$.

More generally, given any generators whose coefficients in $\mathbb{C}\{\{t\}\}$ have a finite amount of terms, we may restrict ourselves to the subfield $\mathbb{Q}(\!(t^{1/N})\!)$, where N is the lowest common denominator of all exponents. And should the coefficients of our generators have an infinite amount of terms, then we are well beyond the limit of the practically manageable.

Now suppose $K := \mathbb{Q}((t))$ as above. In this case, $R_\nu = \mathbb{Q}[\![t]\!]$, and $R := \mathbb{Q}[t]$ and $p := t$ would be a natural choice for a dense subring and a uniformizing parameter that are easy to handle. Changing the power series variable in Convention 1.2.2 from t to u, as the first is already taken, we obtain:

$$\begin{tikzcd}
0 \arrow[r] & \langle t-u\rangle \cdot \mathbb{Q}[t][\![u]\!]_{\langle t-u\rangle}[x] \arrow[r] & \mathbb{Q}[t][\![u]\!]_{\langle t-u\rangle}[x] \arrow[r] & \mathbb{Q}((t))[x] \arrow[r] & 0 \\
0 \arrow[r] & \langle t-u\rangle \cdot \mathbb{Q}[t][\![u]\!][x] \arrow[r] \arrow[u] & \mathbb{Q}[t][\![u]\!][x] \arrow[r, "\pi"] \arrow[u] & \mathbb{Q}[\![t]\!][x] \arrow[r] \arrow[u] & 0 \\
 & & \mathbb{Q}[t][\![u]\!][x]/(u-t) \cong \mathbb{Q}[\![u]\!][x] \arrow[ur, "t \longmapsto t'"'] & &
\end{tikzcd}$$

This diagram reflects a known trick for computing tropical varieties over the field of Puiseux series, see Chapter 6.3 in [**Jen07**], in which we stop thinking of t as a uniformizing parameter in our ground field, and start thinking of it as a variable in our polynomial ring.

Definition 1.2.7 (initial form, initial ideal) Given an element $f = \sum_{\beta,\alpha} c_{\alpha,\beta} \cdot t^\beta x^\alpha \trianglelefteq R[\![t]\!][x]$ and a weight vector $w' \in \mathbb{R}_{<0} \times \mathbb{R}^n$, we define the *initial form* of f with respect to w' to be

$$\mathrm{in}_{w'}(f) := \sum_{w' \cdot (\beta,\alpha) \text{ maximal}} c_\alpha t^\beta x^\alpha \in R[t,x].$$

And given an ideal $J \trianglelefteq R[\![t]\!][x]$ and a weight vector $w' \in \mathbb{R}_{<0} \times \mathbb{R}^n$, we define the *initial ideal* of J with respect to w' to be:

$$\mathrm{in}_{w'}(J) := \langle \mathrm{in}_{w'}(f) \mid f \in J \rangle \trianglelefteq R[t,x].$$

This can be thought of as a natural extension of Definition 1.1.24 in the case that the valuation on R is trivial. Note that we only allow weight vectors with negative weight in t, so that our result lies in a polynomial ring.

Example 1.2.8 (p-adic numbers) Let $K := \mathbb{Q}_p$ be the field of p-adic numbers, $R_\nu := \mathbb{Z}_p$ the ring of p-adic integers. Then $\mathbb{Z} \leq \mathbb{Z}_p$ is a natural dense subset to choose and we obtain:

$$\begin{tikzcd}
0 \arrow[r] & \langle p-t\rangle \cdot \mathbb{Z}[\![t]\!]_{\langle p-t\rangle}[x] \arrow[r] & \mathbb{Z}[\![t]\!]_{\langle p-t\rangle}[x] \arrow[r] & \mathbb{Q}_p[x] \arrow[r] & 0, \\
0 \arrow[r] & \langle p-t\rangle \cdot \mathbb{Z}[\![t]\!][x] \arrow[r] \arrow[u] & \mathbb{Z}[\![t]\!][x] \arrow[r, "\pi", "t \longmapsto p"'] \arrow[u] & \mathbb{Z}_p[x] \arrow[r] \arrow[u] & 0.
\end{tikzcd}$$

This diagram merely reflects our definition of the p-adic integers as $\mathbb{Z}_p$ as power series in p.

Now consider one of the unnumbered examples in Chapter 3.6 of [**Cha13**], namely the ideal

$$I = \langle 2x_1^2 + 3x_1x_2 + 24x_3x_4, 8x_1^3 + x_2x_3x_4 + 18x_3^2x_4 \rangle \trianglelefteq \mathbb{Q}_3[x_1,\ldots,x_4]$$

and a monomial ordering $>_w$ with weight vector $w = (1, 11, 3, 19) \in \mathbb{R}^4$. Then

$$\pi^{-1}I = \langle 3-t, 2x_1^2 + 3x_1x_2 + 24x_3x_4, 8x_1^3 + x_2x_3x_4 + 18x_3^2x_4\rangle \trianglelefteq \mathbb{Z}[\![t]\!][x]$$

and, anticipating Chapter 2 and Chapter 3, a standard basis computation of $\pi^{-1}I$ with respect to a weighted monomial ordering on $\mathrm{Mon}(t, x)$ with weight vector $(-1, w) \in \mathbb{R}_{<0} \times \mathbb{R}^4$ yields the abomination

$$\begin{aligned} G = \{3-t, \quad & x_1^2 - tx_1^2 - 3x_1x_2 - 24x_3x_4, \\ & tx_1x_3x_4 + 2t^2x_1x_2^2 - 11t^2x_1x_3x_4 + 18x_3^2x_4 + 145x_2x_3x_4, \\ & t^3x_1x_2^2x_3 - 22t^3x_1x_3^2x_4 + 36tx_3^3x_4 + t^4x_1x_2^2x_3 + 27x_1x_2^3 - 289x_1x_2x_3x_4 \\ & \quad + 290tx_2x_3^2x_4 + 216x_2^2x_3x_4 - 1152x_3^2x_4^2, \\ & t^4x_1x_2^4 + 56t^4x_1x_2^2x_3x_4 - 18t^2x_2^2x_3^2x_4 - 242t^4x_1x_3^2x_4^2 + 408t^2x_3^3x_4^2 \\ & \quad - t^5x_1x_2^4 + 27x_1x_2^3x_4 - 289x_1x_2x_3x_4^2 + 290tx_2x_3^2x_4^2 - 145t^2x_2^3x_3x_4 \\ & \quad + 3190t^2x_2x_3^2x_4^2 + 216x_2^2x_3x_4^2 - 1152x_3^2x_4^3, \\ & t^3x_3^4x_4^2 - t^4x_3^4x_4^2 - 81x_1x_2^5 + 1299x_1x_2^3x_3x_4 - 4624x_1x_2x_3^2x_4^2 \\ & \quad - 435x_2x_3^3x_4^2 - 648x_2^4x_3x_4 + 6912x_2^2x_3^2x_4^2 - 18432x_3^3x_4^3\}, \end{aligned}$$

from which we can deduct that

$$\mathrm{in}_{(-1,w)}(\pi^{-1}I) = \langle 3, x_1^2, tx_1x_3x_4, t^3x_1x_2^2x_3, t^4x_1x_2^4, t^3x_3^4x_4^2\rangle,$$

The similarity to the following result of Chan result is no coincidence:

$$\mathrm{in}_{\nu,w}(I) = \langle x_1^2, x_1x_3x_4, x_1x_2^2x_3, x_1x_2^4, x_3^4x_4^2\rangle \trianglelefteq \mathbb{F}_3[x_1, \dots, x_4].$$

Proposition 1.2.9 *For any ideal $I \trianglelefteq R_\nu[x]$ and any weight vector $w \in \mathbb{Q}^n$, we have:*

$$\overline{\mathrm{in}_{(-1,w)}(\pi^{-1}I)|}_{t=1} = \mathrm{in}_{\nu,w}(I),$$

where $\overline{(\cdot)}$ denotes the canonical projection $\overline{(\cdot)} : R[x] \to \mathfrak{K}[x]$.

PROOF. $\supseteq$: Any term $s \in R_\nu[x]$ is of the form $s = (\sum_\beta c_\beta p^\beta) \cdot x^\alpha$ with $p \nmid c_\beta$ for all $\beta \in \mathbb{N}$. Then the element $s' := (\sum_\beta c_\beta t^\beta) \cdot x^\alpha \in R[\![t]\!][x]$ is a natural preimage of it under π for which we have

$$\mathrm{in}_{\nu,w}(s) = \bar{c}_{\beta_0} \cdot x^\alpha = \overline{\mathrm{in}_{(-1,w)}(s')|}_{t=1}, \text{ where } \beta_0 = \min\{\beta \in \mathbb{N} \mid c_\beta \neq 0\}.$$

And because the valued weighted degree in $R_\nu[x]$ and the weighted degree in $R[\![t]\!][x]$ coincide,

$$\deg_w(x^\alpha) - \mathrm{val}(\textstyle\sum_\beta c_\beta p^\beta) = \deg_{(-1,w)}(\textstyle\sum_\beta c_\beta \cdot t^\beta x^\alpha),$$

this implies any $f \in R_\nu[x]$ has a preimage $f' \in R[\![t]\!][x]$ under π such that

$$\mathrm{in}_{\nu,w}(f) = \overline{\mathrm{in}_{(-1,w)}(f')|}_{t=1},$$

simply by applying the above argument to each of its terms.

$\subseteq$: Once again consider a term $s = \sum_\beta c_\beta p^\beta \cdot x^\alpha \in R_\nu[x]$ with $p \nmid c_\beta$ for all $\beta \in \mathbb{N}$. Then any preimage of it under π is of the form $s' = \sum_\beta c_\beta t^\beta x^\alpha + r$ for some $r \in \langle t - p\rangle$.

If $\deg_{(-1,w)}(r) > \deg_{(-1,w)}(\sum_\beta c_\beta t^\beta x^\alpha)$, we would have

$$\overline{\mathrm{in}_{(-1,w)}(s')|_{t=1}} = \overline{\mathrm{in}_{(-1,w)}(r)|_{t=1}} = 0,$$

since $\mathrm{in}_{(-1,w)}(r) \in \mathrm{in}_{(-1,w)}\langle p - t\rangle = \langle p\rangle$.

And if $\deg_{(-1,w)}(r) < \deg_{(-1,w)}(\sum_\beta c_\beta t^\beta x^\alpha)$, we would have

$$\begin{aligned}\overline{\mathrm{in}_{(-1,w)}(s')|_{t=1}} &= \overline{\mathrm{in}_{(-1,w)}(\textstyle\sum_\beta c_\beta t^\beta x^\alpha)|_{t=1}} = \overline{c}_{\beta_0} \cdot x^\alpha \\ &= \mathrm{in}_{\nu,w}(\textstyle\sum_\beta c_\beta p^\beta \cdot x^\alpha) = \mathrm{in}_{\nu,w}(s),\end{aligned}$$

where $\beta_0 := \min\{\beta \in \mathbb{N} \mid c_\beta \neq 0\}$.

Now suppose $\deg_{(-1,w)}(r) = \deg_{(-1,w)}(\sum_\beta c_\beta t^\beta x^\alpha)$. First observe that because t is weighted negatively, there can be no cancellation amongst the highest weighted terms of r and the terms of $\sum_\beta c_\beta t^\beta x^\alpha$, as the terms of $\sum_\beta c_\beta t^\beta x^\alpha$ are not divisible by p, unlike the terms of the highest weighted terms of r. Therefore, we have

$$\overline{\mathrm{in}_{(-1,w)}(s')|_{t=1}} = \underbrace{\overline{\mathrm{in}_{(-1,w)}(\textstyle\sum_\beta c_\beta t^\beta x^\alpha)|_{t=1}}}_{=\mathrm{in}_{\nu,w}(\sum_\beta c_\beta p^\beta \cdot x^\alpha)} + \underbrace{\overline{\mathrm{in}_{(-1,w)}(r)|_{t=1}}}_{=\overline{0}} = \mathrm{in}_{\nu,w}(s).$$

Either way, we always have $\overline{\mathrm{in}_{(-1,w)}(s')|_{t=1}} \in \langle \mathrm{in}_{\nu,w}(s)\rangle$ for any arbitrary preimage $s' \in \pi^{-1}(s)$, and, as before, the same hence holds true for any arbitrary element $f \in R_\nu[x]$. □

Corollary 1.2.10 *Let $I \trianglelefteq K[x]$ be an ideal. Then for any weight vector $w \in \mathbb{Q}^n$ we have*

$$\overline{\mathrm{in}_{(-1,w)}(\pi^{-1}I)|_{t=1}} = \mathrm{in}_{\nu,w}(I).$$

Proof. Follows from $\mathrm{in}_{\nu,w}(I) = \mathrm{in}_{\nu,w}(I \cap R_\nu[x])$. □

Definition 1.2.11 (tropical variety) For an ideal $J \trianglelefteq R[\![t]\!][x]$ we define its *tropical variety* to be

$$\mathrm{Trop}(J) = \overline{\{w \in \mathbb{R}_{<0} \times \mathbb{R}^n \mid \mathrm{in}_w(J) \text{ monomial free}\}} \subseteq \mathbb{R}_{\leq 0} \times \mathbb{R}^n,$$

where $\overline{(\cdot)}$ denotes the closure in the euclidean topology.

Example 1.2.12 Note that, unlike when working over fields, we have over rings

$$\mathrm{in}_w(J) \text{ monomial free} \quad \not\Longleftrightarrow \quad \mathrm{in}_w(J) \text{ term free}.$$

In fact, this is the main reason why tropical varieties over rings are not as nicely structured as their pendants over fields, see Theorem 1.1.17 and Example 1.1.23.

For an easy example, consider the principal ideal I generated by the polynomial $g := x + y + 2z \in \mathbb{Z}[\![t]\!][x, y, z]$, which is both homogeneous and prime as required in the Theorem. Because g is homogeneous as a polynomial in x, y, z, its tropical variety is invariant under translation by $(0, 1, 1, 1)$, and since no variable t is occuring in g, it is also closed under translation by $(-1, 0, 0, 0)$. Note that it is not invariant under the last translation because we are restricted to the lower halfspace $\mathbb{R}_{\leq 0} \times \mathbb{R}^n$.

Figure 8 shows the intersection of the tropical variety with an affine subspace of codimension 2, the remaining points are then uniquely determined due to symmetry. Since $\mathrm{in}_{(-1,-1,-1,1)}(g) = 2z$ does not count as a monomial, the lower left quadrant is included in our tropical variety, while the two other maximal cones are not. And because, $\mathrm{in}_{(-1,1,1,0)}(g) = x + y$ is no monomial either, the edge containing it is also part of our tropical variety. Therefore, the tropical variety cannot be the support of a pure polyhedral complex.

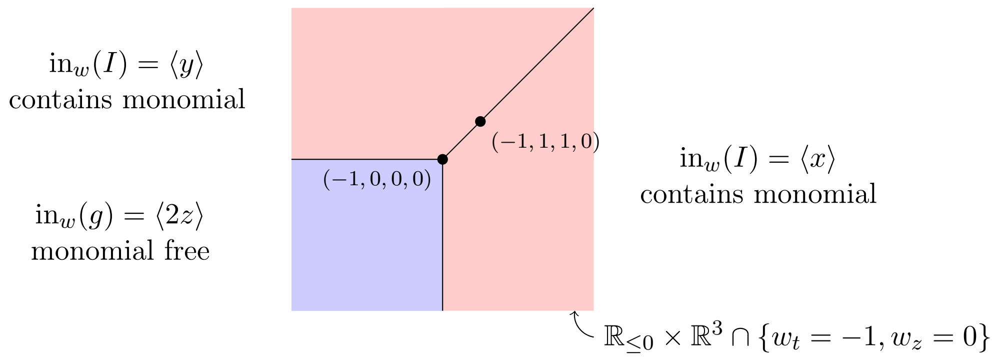


FIGURE 8. section of $\mathrm{Trop}(\langle x + y + z2 \rangle)$

Now, with our previous considerations, we can show that these tropical varieties over R can be used to compute tropical varieties over the valued field K.

Theorem 1.2.13 *Let $I \trianglelefteq K[x]$ be an ideal. Then the linear map*

$$\mathbb{R}_{\leq 0} \times \mathbb{R}^n \longrightarrow \mathbb{R}^n, \quad (w_0, \ldots, w_n) \longmapsto (w_1, \ldots, w_n)$$

induces a bijection

$$\mathrm{Trop}(\pi^{-1}I) \cap (\{-1\} \times \mathbb{R}^n) \xrightarrow{\sim} \mathrm{Trop}_\nu(I).$$

PROOF. For the bijection, we show that

$$\mathrm{in}_{\nu,w}(I) \text{ monomial free} \quad \Longleftrightarrow \quad \mathrm{in}_{(-1,w)}(\pi^{-1}I) \text{ monomial free.}$$

$\Rightarrow$: Assume that $\mathrm{in}_{(-1,w)}(\pi^{-1}I)$ contains some monomial $t^\beta x^\alpha \in R[\![t]\!][x]$. Then, by Corollary 1.2.10, we have $\mathrm{in}_{\nu,w}(I) = \overline{\mathrm{in}_{(-1,w)}(\pi^{-1}I)|_{t=1}}$, which means $\mathrm{in}_{\nu,w}(I)$ must contain the monomial $x^\alpha \in \mathfrak{K}[x]$.

$\Leftarrow$: Assume that $\mathrm{in}_{\nu,w}(I)$ contains some monomial $x^\alpha \in \mathfrak{K}[x]$. Then, by Corollary 1.2.10, $\mathrm{in}_{(-1,w)}(\pi^{-1}I)$ must contain an element of of the form $x^\alpha + (t-1) \cdot r + p \cdot s$, for some $r, s \in R[t, x]$. Recall that p lies in $\mathrm{in}_{(-1,w)}(\pi^{-1}I)$, therefore so does $p \cdot s$, and hence we have $x^\alpha + (t-1) \cdot r \in \mathrm{in}_{(-1,w)}(\pi^{-1}I)$. Let $r = h_l + \ldots + h_1$ be a decomposition of r into its $(-1, w)$-homogeneous layers with $\deg_{(-1,w)}(h_1) < \ldots < \deg_{(-1,w)}(h_l)$. For sake of simplicity, we now distinguish between three cases:

1. $\deg_{(-1,w)}(x^\alpha) \geq \deg_{(-1,w)}(h_l)$: Set $g_1 := r - h_1 = h_l + \ldots + h_2$. Then
$$x^\alpha + (t-1)\cdot r = x^\alpha + (t-1)\cdot(g_1+h_1) = \underbrace{x^\alpha + (t-1)\cdot g_1 - h_1}_{\text{higher weighted degree}} + t\cdot h_1.$$
Hence $t\cdot h_1, x^\alpha + (t-1)\cdot g_1 - h_1 \in \mathrm{in}_{(-1,w)}(\pi^{-1}I)$ and, more importantly,
$$t\cdot(x^\alpha + (t-1)\cdot g_1 - h_1) + t\cdot h_1 = tx^\alpha + (t-1)\cdot t\cdot g_1 \in \mathrm{in}_{(-1,w)}(\pi^{-1}I),$$
effectively shaving off the h_1 layer. We can continue this process by setting $g_2 := g_1 - h_2 = h_l + \ldots + h_3$. Then
$$\begin{aligned} tx^\alpha + (t-1)\cdot t\cdot g_1 &= tx^\alpha + (t-1)\cdot t\cdot(g_2+h_2) \\ &= \underbrace{tx^\alpha + (t-1)\cdot t\cdot g_2 - t\cdot h_2}_{\text{higher weighted degree}} + t^2\cdot h_2. \end{aligned}$$
Hence $t^2\cdot h_2, tx^\alpha + (t-1)\cdot t\cdot g_2 - t\cdot h_2 \in \mathrm{in}_{(-1,w)}(\pi^{-1}I)$ and, as above,
$$\begin{aligned} t\cdot(tx^\alpha + (t-1)\cdot t\cdot g_2 - t\cdot h_2) + t^2\cdot h_2 \\ = t^2x^\alpha + (t-1)\cdot t^2\cdot g_2 \in \mathrm{in}_{(-1,w)}(\pi^{-1}I) \end{aligned}$$
removing the h_2 layer. Eventually, we will obtain $t^l\cdot x^\alpha \in \mathrm{in}_{(-1,w)}(\pi^{-1}I)$.
2. $\deg_{(-1,w)}(x^\alpha) \leq \deg_{(-1,w)}(h_1)$: Set $g_1 := r - h_l = h_{l-1} + \ldots + h_1$. Then
$$x^\alpha + (t-1)\cdot r = x^\alpha + (t-1)\cdot(g_1+h_l) = \underbrace{x^\alpha + (t-1)\cdot g_1 + t\cdot h_l}_{\text{lower weighted degree}} - h_l.$$
Thus $h_l, x^\alpha + (t-1)\cdot g_1 + t\cdot h_l \in \mathrm{in}_{(-1,w)}(\pi^{-1}I)$ and, more importantly,
$$x^\alpha + (t-1)\cdot h_1 + t\cdot g_1 - t\cdot g_1 = x^\alpha + (t-1)\cdot h_1 \in \mathrm{in}_{(-1,w)}(\pi^{-1}I),$$
shaving off the the h_l layer this time. Continuing this pattern eventually yields $x^\alpha \in \mathrm{in}_{(-1,w)}(\pi^{-1}I)$.
3. $\deg_{(-1,w)}(h_1) < \deg_{(-1,w)}(x^\alpha) < \deg_{(-1,w)}(h_l)$: In this case we can use a combination of the steps in the previous cases to see $t^i\cdot x^\alpha \in \mathrm{in}_{(-1,w)}(\pi^{-1}I)$ for the $1 \leq i \leq k$ such that $\deg_{(-1,w)}(h_{i-1}) < \deg_{(-1,w)}(x^\alpha) \leq \deg_{(-1,w)}(h_i)$.

In either case, we see that $\mathrm{in}_{(-1,w)}(\pi^{-1}I)$ contains a monomial. □

As an immediately corollary, we obtain:

Corollary 1.2.14 *Suppose that $I \trianglelefteq K[x]$ is prime and of dimension d. Then $\mathrm{Trop}(\pi^{-1}I)$ is the support of a pure polyhedral fan of dimension $d+1$ with rational slopes that is connected in codimension 1.*

Proof. By Theorem 1.1.17, there exists a pure polyhedral complex Δ in $\mathbb{R}^n$ of dimension d such that
$$\mathrm{Trop}_\nu(I) = \textstyle\bigcup_{\sigma\in\Delta}\sigma.$$
Theorem 1.2.13 now implies for the lower open halfspace,
$$\mathrm{Trop}(I) \cap (\mathbb{R}_{<0}\times\mathbb{R}^n) = \big(\textstyle\bigcup_{\sigma\in\Delta}\mathrm{Cone}(\{-1\}\times\sigma)\big) \cap (\mathbb{R}_{<0}\times\mathbb{R}^n),$$

where $\mathrm{Cone}(\{-1\} \times \sigma)$ stands for the polyhedral cone over the origin which is spanned by all points of $\{-1\} \times \sigma \subseteq \{-1\} \times \mathbb{R}^n$. These cones generate a pure polyhedral fan of dimension $d+1$ in $\mathbb{R}_{\leq 0} \times \mathbb{R}^n$, and taking the closure on both sides now yields

$$\mathrm{Trop}(I) = \bigcup_{\sigma \in \Delta} \mathrm{Cone}(\{-1\} \times \sigma). \qquad \square$$

Example 1.2.15 Let $I \trianglelefteq \mathbb{Q}(\!(t)\!)[x,y]$ be the principal ideal generated by the element $(x+y+1) \cdot (t^2x+y+t)$. In Example 1.1.8, we have seen that $\mathrm{Trop}_\nu(I)$ is the union of two tropical lines, one with vertex at $(0,0)$ and one with vertex at $(1,-1)$.

Its preimage in the diagram of Example 1.2.6 is then given by

$$\begin{aligned}\pi^{-1}I &= \langle (x+y+1)\cdot(t^2x+y+t), t-u\rangle \\ &= \langle (x+y+1)\cdot(u^2x+y+u), t-u\rangle \trianglelefteq \mathbb{Q}[t][\![u]\!][x,y].\end{aligned}$$

Anticipating Chapter 2 and 3, it is not hard to see that the last pair of generators form a standard basis with respect to any t-local monomial ordering. Therefore, by Chapter 3, initial ideals of $\pi^{-1}I$ with respect to weights $w \in \mathbb{R}_{<0} \times \mathbb{R}^2$ will always be of the form

$$\mathrm{in}_w(\pi^{-1}I) = \langle \mathrm{in}_w((x+y+1)\cdot(u^2x+y+u)), t\rangle.$$

Note here that t represents the uniformizing parameter in our ground ring and hence does not yield a monomial. Thus, for any weight vector $w = (w_u, w_x, w_y) \in \mathbb{R}_{<0} \times \mathbb{R}^2$ in the lower open halfspace and $\lambda \in \mathbb{R}$ such that $\lambda \cdot w \in \{-1\} \times \mathbb{R}^2$, it is easy to see that

$$w \in \mathrm{Trop}(\pi^{-1}I) \quad \Longleftrightarrow \quad \lambda \cdot (w_x, w_y) \in \mathrm{Trop}_\nu(I).$$

This implies that $\mathrm{Trop}(\pi^{-1}I)$ is as shown in Figure 9. The polyhedral complex consists of 6 rays and 8 two-dimensional cones in a way that the intersection with the affine hyperplane yields the highlighted polyhedral complex, $\mathrm{Trop}_\nu(I)$ from Example 1.1.8.

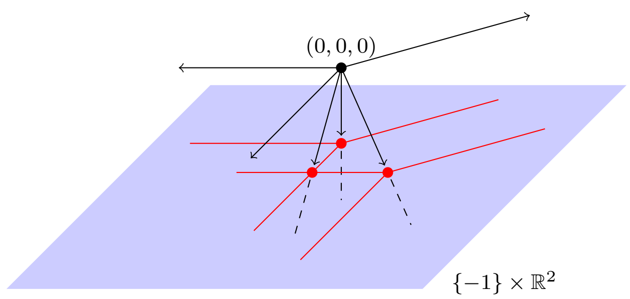


FIGURE 9. $\mathrm{Trop}(\pi^{-1}I)$

Example 1.2.16 Consider again the ideal from Example 1.1.20,

$$I = \langle x_1 - 2x_2 + 3x_3, 3x_2 - 4x_3 + 5x_4 \rangle \trianglelefteq \mathbb{Q}_2[x_1, \ldots, x_4].$$

Its preimage is given by

$$\pi^{-1} I = \langle x_1 - 2x_2 + 3x_3, 3x_2 - 4x_3 + 5x_4, 2 - t \rangle \trianglelefteq \mathbb{Z}[\![t]\!][x_1, \ldots, x_4].$$

and the tropical variety of its preimage can be covered by a polyhedral complex of the combinatorial form as in Figure 10 and a one-dimensional homogeneity space generated by $(0, 1, 1, 1, 1)$, see Definition 4.1.10. Each of the six vertices represents a two-dimensional cone generated by the corresponding weight vector and the homogeneity space. And each of the five edges represents a three-dimensional cone generated by the two adjacent weight vectors and the homogeneity space.

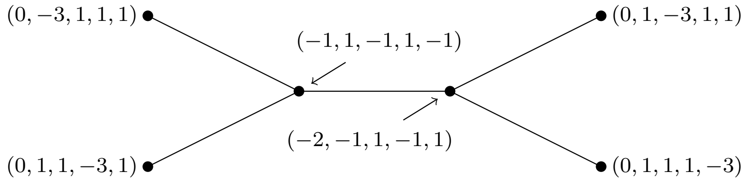


FIGURE 10. $\mathrm{Trop}(\langle x_1 - 2x_2 + 3x_3, 3x_2 - 4x_3 + 5x_4, 2 - t \rangle)$

Intersected with the affine hyperplane $\{-1\} \times \mathbb{R}^4$, we obtain a polyhedral complex of the combinatorial form as in Figure 11, further explained in Example 1.1.20. Note that any maximal cone, that contains a ray in the coordinate hyperplane $\{0\} \times \mathbb{R}^4$, becomes an unbounded polytope in the intersection. Projectively speaking, these rays become points at infinity.

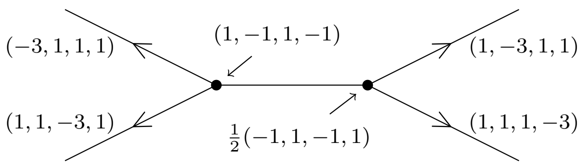


FIGURE 11. $\mathrm{Trop}_{\nu_2}(\langle x_1 - 2x_2 + 3x_3, 3x_2 - 4x_3 + 5x_4 \rangle)$

We see how the problem of computing $\mathrm{Trop}_\nu(I)$ for some $I \trianglelefteq K[x]$ is equivalent to computing $\mathrm{Trop}(\pi^{-1} I)$ with $\pi^{-1} I \trianglelefteq R[\![t]\!][x]$. To see how the latter can be achieved, we will a notion of standard bases in $R[\![t]\!][x]$.

CHAPTER 2

Standard bases in $R[\![t]\!][x]^s$

In this chapter, we combine techniques from [**Mar08**] and [**Wie11**] to construct a division with remainder over $R[\![t]\!][x]$, the ring of multivariate power series and polynomials over a ground ring R, and use them to introduce standard bases theory for free modules over it. While doing that, we will pay special attention to the termination of our algorithms for strictly polynomial input. The standard bases theory will be a foundation for establishing Gröbner fans of ideals in $R[\![t]\!][x]$ in the next chapter.

2.1. Division with remainder

In this section, we construct a division with remainder following the first three chapters of [**Mar08**]. Please mind the assumptions on our ground ring in Convention 2.1.1 for that, which were taken from Definition 1.3.14 in [**Wie11**].

After a quick introduction of the basic terminology, we begin with a division algorithm over the ground ring in the form of Algorithm 2.1.11. We then continue with homogeneous division with remainder in Algorithm 2.1.16, and finally end with a weak division with remainder in Algorithm 2.1.21. See Figure 1 for a rough sketch with numbering.

Convention 2.1.1 For this chapter, let R be a noetherian ring in which linear equations are solvable as in Definition 1.3.14 of [**Wie11**]. The latter means that, given any finite tuple of arbitrary length $(c_1, \dots, c_k)$ with $c_i \in R$, we must be able to:

(1) for $b \in R$ decide whether $b \in \langle c_1, \dots, c_k \rangle$, and, if yes, find $a_1, \dots, a_k \in R$ such that
$$b = a_1 \cdot c_1 + \cdots + a_k \cdot c_k.$$
(2) find a finite generating set $S \subseteq R^k$ of its syzygies as module over R,
$$\mathrm{syz}_R(c_1, \dots, c_k) = \{(a_1, \dots, a_k) \in R^k \mid a_1 \cdot c_1 + \ldots + a_k \cdot c_k = 0\} = \langle S \rangle_R.$$

Moreover, we will use the notion $R[\![t]\!][x] := R[\![t_1, \dots, t_m]\!][x_1, \dots, x_n]$ to denote a mixed power series and polynomial ring over R in several variables $t = (t_1, \dots, t_m)$ and $x = (x_1, \dots, x_n)$, and we fix a free module $R[\![t]\!][x]^s$ over it.

R being noetherian is most notably required for the conditional termination of Algorithm 2.1.21, while linear equations being solvable is required in the instructions of Algorithm 2.1.11 and Algorithm 2.2.17.

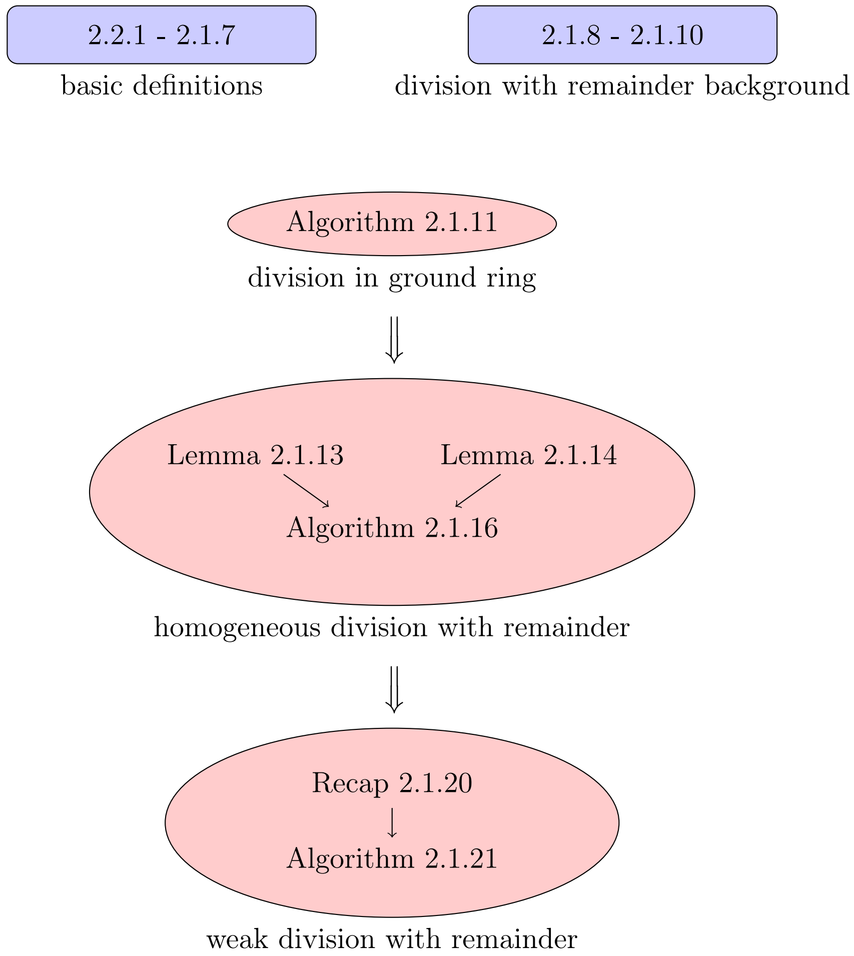


FIGURE 1. outline of Section 2.1

Example 2.1.2 Viable ground rings satisfying Convention 2.1.1 include:

- Obviously any field, assuming we are able to compute inverse elements.
- The ring of integers $\mathbb{Z}$. The division with remainder in $\mathbb{Z}$ allows us to solve the ideal membership problem, while the least common multiple allows us to compute finite generating sets of syzygies, see Theorem 2.2.5 in [**Wie11**] for the latter.
- Also, $\mathbb{Z}/m\mathbb{Z}$ for an arbitrary $m \in \mathbb{Z}$. While it generally is neither euklidean nor factorial like $\mathbb{Z}$, many problems can nonetheless be solved by tracing them back to the integers.
- Similarly, any euklidean ring for which we are able to compute its division with remainder, or, more generally, any factorial ring for which we can compute the unique factorization. Classical examples hereof are $\mathbb{Z}[i]$, $\mathbb{Q}[x]$, $\mathbb{Q}[\![t]\!]$ or multivariate polynomial rings.
- Moreover, thanks to the theory of Gröbner bases, any quotient ring of a polynomial ring, e.g. the ring of Laurent polynomials $K[x_1^{\pm 1}, \ldots, x_n^{\pm 1}] = K[x_0, \ldots, x_n]/(1 - x_0 \cdots x_n)$.

- And, thanks to the theory of standard bases, any localization of a polynomial ring at a prime ideal, as it can be traced back to a quotient of a polynomial ring localized at a mixed ordering, see [**Mor91**].
- Also, Dedeking domains. A solution to the ideal membership problem and the computation of syzygies can be found in [**HKY10**].
- Finally, product rings like $\mathbb{Z} \times \mathbb{Z}$, because any ideal in it is the product of two ideals in $\mathbb{Z}$.

We now begin with introducing some very basic notions of Gröbner basis theory to our ring resp. module, definitions such as monomials, monomial orderings and leading monomials.

Definition 2.1.3 The *set of monomials* of $R[\![t]\!][x]$ is defined to be

$$\mathrm{Mon}(t,x) := \{t^\beta x^\alpha \mid \beta \in \mathbb{N}^m, \alpha \in \mathbb{N}^n\} \subseteq R[\![t]\!][x],$$

and a *monomial ordering* on $\mathrm{Mon}(t,x)$ is an ordering $>$ that is compatible with its natural semigroup structure, i.e.

$$\forall a,b,q \in \mathrm{Mon}(t,x): \quad a > b \quad \Longrightarrow \quad q \cdot a > q \cdot b.$$

We call a monomial ordering $>$ *t-local*, if $1 > t^\beta$ for all $\beta \in \mathbb{N}^m$.

Let $>$ be a t-local monomial ordering on $\mathrm{Mon}(t,x)$, and let $w \in \mathbb{R}^m_{<0} \times \mathbb{R}^n$ be a weight vector. Then the ordering $>_w$ is defined to be:

$$\begin{aligned} t^\beta x^\alpha >_w t^\delta x^\gamma. \quad :\Longleftrightarrow \quad & w \cdot (\beta,\alpha) > w \cdot (\delta,\gamma) \text{ or} \\ & w \cdot (\beta,\alpha) = w \cdot (\delta,\gamma) \text{ and } t^\beta x^\alpha > t^\delta x^\gamma. \end{aligned}$$

We will refer to orderings of the form $>_w$ as a *weighted ordering* with weight vector w and tiebreaker $>$.

Definition 2.1.4 The *set of module monomials* of $R[\![t]\!][x]^s$ is defined to be

$$\mathrm{Mon}^s(t,x) := \{t^\beta x^\alpha \cdot e_i \mid \beta \in \mathbb{N}^m, \alpha \in \mathbb{N}^n, i = 1,\ldots,s\} \subseteq R[\![t]\!][x]^s.$$

A *monomial ordering* on $\mathrm{Mon}^s(t,x)$ is an ordering $>$ that is compatible with the natural $\mathrm{Mon}(t,x)$-action on it, i.e.

$$\forall a,b \in \mathrm{Mon}^s(t,x) \;\; \forall q \in \mathrm{Mon}(t,x): \quad a > b \quad \Longrightarrow \quad q \cdot a > q \cdot b,$$

and that restricts onto the same monomial ordering on $\mathrm{Mon}(t,x)$ in each component, i.e.

$$\forall a,b \in \mathrm{Mon}(t,x) \;\; \forall i,j \in \{1,\ldots,s\}: \quad a \cdot e_i > b \cdot e_i \quad \Longleftrightarrow \quad a \cdot e_j > b \cdot e_j.$$

We call a monomial ordering $>$ *t-local*, if $1 \cdot e_i > t^\beta \cdot e_i$ for all $\beta \in \mathbb{N}^m$ and $i = 1,\ldots,s$.

Let $>$ be a t-local monomial ordering on $\mathrm{Mon}^s(t,x)$, and let $w \in \mathbb{R}^m_{<0} \times \mathbb{R}^n \times \mathbb{R}^s$ be a weight vector. Then the ordering $>_w$ is defined to be:

$$\begin{aligned} & t^\beta x^\alpha \cdot e_i >_w t^\delta x^\gamma \cdot e_j \quad \Longleftrightarrow \\ & \quad w \cdot (\beta,\alpha,e_i) > w \cdot (\delta,\gamma,e_j) \text{ or} \\ & \quad w \cdot (\beta,\alpha,e_i) = w \cdot (\delta,\gamma,e_j) \text{ and } t^\beta x^\alpha \cdot e_i > t^\delta x^\gamma \cdot e_j. \end{aligned}$$

We will refer to orderings of the form $>_w$ as a *weighted ordering* with weight vector w and tiebreaker $>$.

From now on, we will simply refer to module monomials as monomials.

Definition 2.1.5 Given a t-local monomial ordering $>$ on $\mathrm{Mon}^s(t,x)$ and an element $f = \sum_{\alpha,\beta,i} c_{\alpha,\beta,i} \cdot t^\beta x^\alpha \cdot e_i \in R[\![t]\!][x]^s$, we define its *leading monomial*, *leading coefficient*, *leading term* and *tail* to be

$$\begin{aligned} \mathrm{LM}_>(f) &= \max\{t^\beta x^\alpha \cdot e_i \mid c_{\alpha,\beta,i} \neq 0\}, \\ \mathrm{LC}_>(f) &= c_{\alpha,\beta,i}, \text{ where } t^\beta x^\alpha \cdot e_i = \mathrm{LM}_>(f), \\ \mathrm{LT}_>(f) &= c_{\alpha,\beta,i} \cdot t^\beta x^\alpha \cdot e_i, \text{ where } t^\beta x^\alpha \cdot e_i = \mathrm{LM}_>(f), \\ \mathrm{tail}_>(f) &= f - \mathrm{LT}_>(f). \end{aligned}$$

For a submodule $M \leq R[\![t]\!][x]^s$, we set

$$\begin{aligned} \mathrm{LM}_>(M) &= \langle \mathrm{LM}_>(f) \mid f \in M \rangle_{R[t,x]} \leq R[t,x]^s, \\ \mathrm{LT}_>(M) &= \langle \mathrm{LT}_>(f) \mid f \in M \rangle_{R[t,x]} \leq R[t,x]^s. \end{aligned}$$

Note that we regard the two modules above as submodules of $R[t,x]^s$, while the original module lies in $R[\![t]\!][x]^s$. We refer to $\mathrm{LT}_>(M)$ as the *leading module* of M with respect to $>$.

Example 2.1.6 Similar to Example 1.2.12, observe that generally

$$\mathrm{LM}_>(M) \neq \mathrm{LT}_>(M).$$

Consider the ideal

$$I := \langle 1 + t^6 x + t^4 y + t^7 x^2 + t^5 xy + t^8 y^2, 2 - t \rangle \trianglelefteq \mathbb{Z}[\![t]\!][x],$$

which is the preimage of the polynomial giving the tropical cubic in Example 1.1.18 under the map π of Definition 1.2.5. Let $>_w$ be the weighted ordering with weight vector $w = (-1,3,3)$ and any arbitrary tiebreaker. Then by weighted degree alone we have

$$\mathrm{LT}_{>_w}(I) = \langle t^5 xy, 2 \rangle \neq \mathrm{LM}_{>_w}(I) = \langle 1 \rangle, \text{ since } \mathrm{LM}_{>_w}(2-t) = 1.$$

In fact, the last equation holds true for any t-local monomial ordering, while the former varies depending on the ordering. This is why the role of leading monomials in the classical standard basis theory over fields is played by leading terms over rings.

Remark 2.1.7 Note that the t-locality of the monomial ordering $>$ is essential for leading monomials and other associated objects to exist, as elements of $R[\![t]\!][x]$ resp. $R[\![t]\!][x]^s$ may be unbounded in their degrees of t.

However, given a weight vector in $\mathbb{R}^m_{<0} \times \mathbb{R}^n$ resp. $\mathbb{R}^m_{<0} \times \mathbb{R}^n \times \mathbb{R}^s$, a weighted monomial ordering does not need a t-local tiebreaker for leading monomials to be well-defined. But for sake of simplicity, we nevertheless assume all occuring monomial orderings to be t-local.

$\mathrm{Mon}(t, x)$ comes equipped with a natural notion of divisibility and least common multiple. For module monomials, we define:

Definition 2.1.8 For two module monomials $t^\beta x^\alpha \cdot e_i$ and $t^\delta x^\gamma \cdot e_j \in \mathrm{Mon}(t, x)^s$, we say

$$t^\beta x^\alpha \cdot e_i \text{ divides } t^\delta x^\gamma \cdot e_j \quad :\Longleftrightarrow \quad e_i = e_j \text{ and } t^\beta x^\alpha \text{ divides } t^\delta x^\gamma,$$

and in this case we set

$$\frac{t^\beta x^\alpha \cdot e_i}{t^\delta x^\gamma \cdot e_j} := \frac{t^\beta x^\alpha}{t^\delta x^\gamma} \in \mathrm{Mon}(t, x).$$

We define the *least common multiple* of two module monomials $t^\beta x^\alpha \cdot e_i$ and $t^\delta x^\gamma \cdot e_j \in \mathrm{Mon}(t, x)^s$ to be

$$\mathrm{lcm}(t^\beta x^\alpha \cdot e_i, t^\delta x^\gamma \cdot e_j) := \begin{cases} \mathrm{lcm}(t^\beta x^\alpha, t^\delta x^\gamma) \cdot e_j, & \text{if } i = j, \\ 0, & \text{otherwise.} \end{cases}$$

We now devote the remaining section to proving the existence of a division with remainder, starting with its definition.

Definition 2.1.9 Let $>$ be a t-local monomial ordering on $\mathrm{Mon}^s(t, x)$. Given $f \in R[\![t]\!][x]^s$ and $g_1, \ldots, g_k \in R[\![t]\!][x]^s$ we say that a representation

$$f = q_1 \cdot g_1 + \ldots + q_k \cdot g_k + r$$

with $q_1, \ldots, q_k \in R[\![t]\!][x]$ and $r = \sum_{j=1}^s r_j \cdot e_j \in R[\![t]\!][x]^s$ satisfies

(ID1): if $\mathrm{LM}_>(f) \geq \mathrm{LM}_>(q_i \cdot g_i)$ for all $i = 1, \ldots, k$,

(ID2): if $\mathrm{LT}_>(r) \notin \langle \mathrm{LT}_>(g_1), \ldots, \mathrm{LT}_>(g_k) \rangle$, unless $r = 0$,

(DD1): if no term of $q_i \cdot \mathrm{LT}_>(g_i)$ lies in $\langle \mathrm{LT}_>(g_j) \mid j < i \rangle$ for all $i = 1, \ldots, k$,

(DD2): if no term of r lies in $\langle \mathrm{LT}_>(g_1), \ldots, \mathrm{LT}_>(g_k) \rangle$,

(SID2): if $\mathrm{LT}_>(r_j \cdot e_j) \notin \langle \mathrm{LT}_>(g_1), \ldots, \mathrm{LT}_>(g_k) \rangle$, unless $r_j = 0$, for all $j = 1, \ldots, s$.

A representation satisfying (ID1) and (ID2) is called an *(indeterminate) division with remainder*, and a representation satisfying (DD1) and (DD2) is called a *determinate division with remainder*.

A division with remainder of $u \cdot f$ for some $u \in R[\![t]\!][x]$ with $\mathrm{LT}_>(u) = 1$ is also called a *weak division with remainder* of f.

Proposition 2.1.10 *Consider a representation*

$$f = q_1 \cdot g_1 + \ldots + q_k \cdot g_k + r \quad \textit{or} \quad u \cdot f = q_1 \cdot g_1 + \ldots + q_k \cdot g_k + r$$

with $f, g_1, \ldots, g_k, r \in R[\![t]\!][x]^s$*,* $q_1, \ldots, q_k \in R[\![t]\!][x]$ *and* $\mathrm{LT}_>(u) = 1$*. Then:*
(1) if the representation satisfies (DD2)*, then it also satisfies* (SID2)*,*
(2) if the representation satisfies (SID2)*, then it also satisfies* (ID2)*,*
(3) if it satisfies both (DD1) *and* (ID2)*, then it also satisfies* (ID1)*.*
In particular, (DD1) *and* (DD2) *imply* (ID1) *and* (ID2)*.*

Proof. (1) and (2) are obvious, so suppose the representation satisfies both (DD1) and (DD2).

Take the maximal monomial $t^\beta x^\alpha$ occurring in any of the expressions $q_i \cdot g_i$ or r on the right hand side, and assume $t^\beta x^\alpha > \mathrm{LM}_>(f)$. Because of maximality, it has to be the leading monomial of each expression it occurs in. And because it does not occur on the left hand side, the leading terms have to cancel each other out. Let $q_{i_1} \cdot g_{i_1}, \ldots, q_{i_l} \cdot g_{i_l}$ be the $q_i \cdot g_i$ containing $t^\beta x^\alpha$ with $i_1 < \ldots < i_l$.

If r contains a, then $\sum_{j=1}^{l} \mathrm{LT}_>(q_{i_j} \cdot g_{i_j}) + \mathrm{LT}_>(r) = 0$, and hence

$$\mathrm{LT}_>(r) = t^\beta x^\alpha \in \langle \mathrm{LT}_>(g_1), \ldots, \mathrm{LT}_>(g_k) \rangle,$$

contradicting (ID2).

If r does not contain a, then we have $\sum_{j=1}^{l} \mathrm{LT}_>(q_{i_j} \cdot g_{i_j}) = 0$, thus

$$\mathrm{LT}_>(q_{i_l} \cdot g_{i_l}) \in \langle \mathrm{LT}_>(g_j) \mid j < i_l \rangle,$$

contradicting (DD1). □

Next, we pay a little attention to our ground ring. Convention 2.1.1 states that our ring already comes equipped with everything we need to compute representations of members in given ideals, but we still need to make sure that these representations satisfy our needs.

Algorithm 2.1.11 (Div_R, division in the ground ring)
Input: (b, C), where $C = \{c_1, \ldots, c_k\} \subseteq R$ is an indexed set and $b \in \langle C \rangle$.
Output: $\{a_1, \ldots, a_k\} \subseteq R$, an indexed set such that

$$b = a_1 \cdot c_1 + \ldots + a_k \cdot c_k$$

with $a_i \cdot c_i \notin \langle c_j \mid j < i \rangle$ unless $a_i \cdot c_i = 0$, for all $i = 1, \ldots, k$.

Find $a_1, \ldots, a_k \in R$ with $b = a_1 \cdot c_1 + \ldots + a_k \cdot c_k$, which is possible by Convention 2.1.1.
for $i = k, \ldots, 1$ **do**
  if $a_i \cdot c_i \neq 0$ and $a_i \cdot c_i \in \langle c_j \mid j < i \rangle$ **then**
    Find $h_1, \ldots, h_{i-1} \in R$ such that $a_i \cdot c_i = h_1 \cdot c_1 + \ldots + h_{i-1} \cdot c_{i-1}$.
    Set $a_j := a_j + h_j$ for all $j < i$, and $a_i := 0$.
return $(a_1, \ldots, a_k)$

Proof. Termination and correctness are obvious. □

Now we will make use of two Lemmata of [**Mar08**] to construct a homogeneous determinate division with remainder. The first Lemma allows us to restrict ourselves to weighted monomial orderings, while the second guarantees $\langle t\rangle$-adic convergence.

Definition 2.1.12 For an element $f = \sum_{\beta,\alpha,i} c_{\alpha,\beta,i} \cdot t^\beta x^\alpha \cdot e_i \in R[\![t]\!][x]^s$ we define its *x-degree* to be

$$\deg_x(f) := \max\{|\alpha| \mid c_{\alpha,\beta,i} \neq 0\},$$

and we call it *x-homogeneous*, if all its terms are of the same x-degree.

Given a weight vector $w \in \mathbb{R}^m_{<0} \times \mathbb{R}^n \times \mathbb{R}^s$, we define its *weighted degree* with respect to w to be

$$\deg_w(f) := \max\{w \cdot (\beta, \alpha, e_i) \mid c_{\alpha,\beta,i} \neq 0\},$$

and we call it *weighted homogeneous*, if all its terms are of the same weighted degree.

Lemma 2.1.13 ([**Mar08**] Lemma 2.5) *Let $>$ be a t-local monomial ordering on* $\mathrm{Mon}^s(t, x)$*, and let $g_1, \ldots, g_k \in R[\![t]\!][x]^s$ be x-homogeneous. Then there exists a weight vector $w \in \mathbb{R}^m_{<0} \times \mathbb{R}^{n+s}$ such that any t-local weight ordering with weight vector w, say $>_w$, induces the same leading monomials as $>$ on $g_1, \ldots, g_k$, i.e.*

$$\mathrm{LM}_{>_w}(g_i) = \mathrm{LM}_>(g_i) \text{ for all } i = 1, \ldots, k.$$

Lemma 2.1.14 ([**Mar08**] Lemma 2.6) *Let $>_w$ be a t-local monomial ordering on* $\mathrm{Mon}^s(t, x)$ *with weight vector $w \in \mathbb{R}^m_{<0} \times \mathbb{R}^{n+s}$, and let $(f_k)_{k\in\mathbb{N}}$ be a sequence of x-homogeneous elements of fixed x-degree in $R[\![t]\!][x]^s$ such that* $\mathrm{LM}_{>_w}(f_k) > \mathrm{LM}_{>_w}(f_{k+1})$ *for all $k \in \mathbb{N}$. Then $(f_k)_{k\in\mathbb{N}}$ converges to zero in the $\langle t\rangle$-adic topology, i.e.*

$$\forall\, N \in \mathbb{N}\ \exists\, M \in \mathbb{N}: \ \ f_k \in \langle t\rangle^N \cdot R[\![t]\!][x]^s \ \ \forall k \geq M.$$

In particular, the element $\sum_{k=0}^\infty f_k \in R[\![t]\!][x]^s$ exists.

Example 2.1.15 A monomial ordering can always be expressed by an invertible matrix. For example, the lexicographical ordering $>$ on $\mathrm{Mon}(t, x)$ with $x_1 > x_2 > 1 > t$ is given by

$$t^\beta x^\alpha > t^\delta x^\gamma \quad \Longleftrightarrow \quad A \cdot (\beta, \alpha)^t > A \cdot (\delta, \gamma)^t, \text{ where } A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix},$$

where the $>$ on the right hand side denotes the lexicographical ordering on $\mathbb{R}^3$.

Consider the polynomial $g = t^5 x_1 + t^2 x_2$. In order to find a weight vector $w \in \mathbb{R}_{<0} \times \mathbb{R}^2$ such that $\mathrm{LM}_{>_w}(g) = \mathrm{LM}_>(g) = t^5 x_1$, consider the first row vector of A, $a_1 = (0, 1, 0) \in \mathbb{R}^3$. Since $a_1 \notin \mathbb{R}_{<0} \times \mathbb{R}^2$ it represents no viable choice for w. But because $\deg_{a_1}(t^5 x_1) > \deg_{a_1}(t^2 x_2)$, adding a sufficiently small negative weight in t will not break the strict inequality. Hence we obtain $w = (-\frac{1}{5}, 1, 0) \in \mathbb{R}_{<0} \times \mathbb{R}^2$:

$$\begin{array}{ccccccc} \deg_{(0,1,0)}(t^5x_1) & = & 1 & > & 0 & = & \deg_{(0,1,0)}(t^2x_2) \\ \big\downarrow{\scriptstyle -(1/5,0,0)} & & & & & & \big\downarrow{\scriptstyle -(1/5,0,0)} \\ \deg_{(-1/5,1,0)}(t^5x_1) & = & 0 & > & -\frac{2}{5} & = & \deg_{(-1/5,1,0)}(t^2x_2). \end{array}$$

In particular, a determinate division with remainder with respect to $>_w$ will also be a determinate division with remainder with respect to $>$, as (DD1) and (DD2) are only dependant on the leading terms.

With the two Lemmata, we can now construct a division with remainder for the homogeneous case.

Algorithm 2.1.16 (HDDwR, homogeneous determinate division with remainder)

Input: $(f, G, >)$, where $f \in R[\![t]\!][x]^s$ x-homogeneous, $G = \{g_1, \ldots, g_k\} \subseteq R[\![t]\!][x]^s$ an indexed set of x-homogeneous elements and $> = >_w$ a weighted t-local monomial ordering with $w \in \mathbb{R}^m_{<0} \times \mathbb{R}^{n+s}$.

Output: (Q, r), where $Q = \{q_1, \ldots, q_k\} \subseteq R[\![t]\!][x]$ an indexed set and $r \in R[\![t]\!][x]^s$ such that

$$f = q_1 \cdot g_1 + \ldots + q_k \cdot g_k + r$$

satisfies

(DD1): no term of $q_i \cdot \mathrm{LT}_>(g_i)$ lies in $\langle \mathrm{LT}_>(g_j) \mid j < i \rangle$ for all i,
(DD2): no term of r lies in $\langle \mathrm{LT}_>(g_1), \ldots, \mathrm{LT}_>(g_k) \rangle$,
(DDH): the $q_1, \ldots, q_k, r$ are either 0 or x-homogeneous of x-degree $\deg_x(f) - \deg_x(g_1), \ldots, \deg_x(f) - \deg_x(g_k), \deg_x(f)$ respectively.

Set $q_i := 0$ for $i = 1, \ldots, k$, $r := 0$, $\nu := 0$, $f_\nu := f$.
while $f_\nu \neq 0$ **do**
  if $\mathrm{LT}_>(f_\nu) \in \langle \mathrm{LT}_>(g_1), \ldots, \mathrm{LT}_>(g_k) \rangle$ **then**
    Let $D_\nu := \{g_i \in G \mid \mathrm{LM}_>(g_i) \text{ divides } \mathrm{LM}_>(f_\nu)\}\{g_{i_1}, \ldots, g_{i_l}\}$.
    Compute $\{a_{i_1}, \ldots, a_{i_l}\} = \mathrm{Div}_R(\mathrm{LC}_>(f_\nu), \{\mathrm{LC}_>(g_{i_1}), \ldots, \mathrm{LC}_>(g_{i_l})\})$.
    Set

$$q_{i,\nu} := \begin{cases} a_i \cdot \frac{\mathrm{LM}_>(f_\nu)}{\mathrm{LM}_>(g_i)} & \text{, if } g_i \in D_\nu, \\ 0 & \text{, otherwise,} \end{cases}$$

    for $i = 1, \ldots, k$, and $r_\nu := 0$.
  else
    Set $q_{i,\nu} := 0$, for $i = 1, \ldots, k$, and $r_\nu := \mathrm{LT}_>(f_\nu)$.
  Set $q_i := q_i + q_{i,\nu}$ for $i = 1, \ldots, k$ and $r := r + r_\nu$.
  Set $f_{\nu+1} := f_\nu - (q_{1,\nu} \cdot g_1 + \ldots + q_{k,\nu} \cdot g_k + r_\nu)$ and $\nu := \nu + 1$.
return $(\{q_1, \ldots, q_k\}, r)$

PROOF. Note that we have a descending chain of terms to be eliminated

$$\mathrm{LM}_>(f_0) > \mathrm{LM}_>(f_1) > \mathrm{LM}_>(f_2) > \ldots,$$

which implies that, except the terms that are zero, we have $k+1$ descending chains of factors and remainders

$$\begin{array}{ccccccccc} \mathrm{LM}_>(q_{i,0}) & > & \mathrm{LM}_>(q_{i,1}) & > & \mathrm{LM}_>(q_{i,2}) & > & \dots & , \\ \mathrm{LM}_>(r_0) & > & \mathrm{LM}_>(r_1) & > & \mathrm{LM}_>(r_2) & > & \dots & . \end{array}$$

By construction, each $q_{i,\nu}$, $i = 1, \dots, k$, is x-homogeneous of x-degree $\deg_x(f) - \deg_x(g_i)$, and each r_ν is x-homogeneous of x-degree $\deg_x(f)$, unless they are zero. Thus, by Lemma 2.1.14, the $q_{i,\nu}$ and r_ν converge to zero in the $\langle t\rangle$-adic topology, so that

$$q_i := \sum_{\nu=0}^{\infty} q_{i,\nu} \in R[\![t]\!][x] \text{ and } r := \sum_{\nu=0}^{\infty} r_\nu \in R[\![t]\!][x]^s$$

exist and the following representation satisfies (DDH):

$$f = q_1 \cdot g_1 + \dots + q_k \cdot g_k + r. \tag{1}$$

Observe that, because all $q_{i,\nu}$ and r_ν are terms with distinct monomials, each non-zero term of $q_i \cdot \mathrm{LT}_>(g_i)$ or r equals $q_{i,\nu} \cdot \mathrm{LT}_>(g_i)$ or r_ν respectively, for some $\nu \in \mathbb{N}$.

So first, let p be a non-zero term of $q_i \cdot \mathrm{LT}_>(g_i)$, say $p = q_{i,\nu} \cdot \mathrm{LT}_>(g_i)$ for some $\nu \in \mathbb{N}$. Then $\mathrm{LC}_>(q_{i,\nu}) \neq 0$ implies that $\mathrm{LC}_>(q_{i,\nu} \cdot g_i) \notin \langle \mathrm{LC}_>(g_j) \mid j < i \text{ with } g_j \in D_\nu\rangle_R$. In particular, we have $\mathrm{LT}_>(q_{i,\nu} \cdot g_i) = q_{i,\nu} \cdot \mathrm{LT}_>(g_i) \notin \langle \mathrm{LT}_>(g_j) \mid j < i \text{ with } g_j \in D_\nu\rangle$. Therefore we also get $q_{i,\nu} \cdot \mathrm{LT}_>(g_i) \notin \langle \mathrm{LT}_>(g_j) \mid j < i\rangle$, since the leading monomials of all $g_j \notin D_\nu$ do not divide $\mathrm{LM}_>(f_\nu) = \mathrm{LM}_>(q_{i,\nu} \cdot g_i)$. Thus (1) satisfies (DD1).

Lastly, let p be a non-zero term of r, i.e. $p = r_\nu$ for a suitable ν. But because $r_\nu \neq 0$, we have $r_\nu = \mathrm{LT}_>(f_\nu) \notin \langle \mathrm{LT}_>(g_1), \dots, \mathrm{LT}_>(g_k)\rangle$ by default. Therefore, our representation (1) also satisfies (DD2). □

Remark 2.1.17 (polynomial input) In case $m = 0$, i.e. $R[\![t]\!][x]^s = R[x]^s$, all $f, g_1, \dots, g_k \in R[x]^s$ are homogeneous and so is any polynomial appearing in our algorithm. Moreover, all f_ν, unless $f_\nu = 0$, have the same x-degree as f. And since there are only finitely many monomials of a given degree, there cannot exist an infinite sequence of decreasing leading monomials

$$\mathrm{LM}_>(f_0) > \mathrm{LM}_>(f_1) > \mathrm{LM}_>(f_2) > \dots ,$$

and Algorithm 2.1.16 has to terminate.

Remark 2.1.18 (weighted homogeneous input) Similar to how the output is x-homogeneous because the input is x-homogeneous, note that if the input is weighted homogeneous with respect to a certain weight vector $w \in \mathbb{R}^m_{<0} \times \mathbb{R}^n$, then so is the output. This will be essential for the proof of Lemma 4.1.9.

Example 2.1.19 Over a ground field, as in the proof of Theorem 2.1 in [**Mar08**], all the terms of f_ν can be simultaneously checked for containment in $\langle \mathrm{LT}_>(g_1), \ldots, \mathrm{LT}_>(g_k)\rangle$, eliminating the terms which lie in the ideal using $g_1, \ldots, g_k$ and discarding the terms which are outside the ideal to the remainder. However, this is not possible if R is no field.

Let $f = 2x, g = 2x + 2tx + t^2x + 3t^3x \in \mathbb{Z}[\![t]\!][x]$ and consider a weighted ordering $>=>_w$ with weight vector $w = (-1, 1) \in \mathbb{R}_{<0} \times \mathbb{R}$. Then Figure 2 illustrates a division algorithm, which discards any term of f_ν not divisible by $\mathrm{LT}_>(g)$ directly to the remainder. The underlined term marks the respective leading term.

$$
\begin{array}{rcl c}
f_0 & = & \underline{2x} & r \\
 & {\scriptstyle -g}\Big\downarrow & & \| \\
 & & & -t^2x \\
 & & & - \\
f_1 & = & -\underline{2tx} - \overbrace{t^2x - 3t^3x}^{\text{to remainder}} & 3t^3x \\
 & {\scriptstyle +tg}\Big\downarrow & & + \\
 & & & t^3x \\
 & & & + \\
f_2 & = & \underline{2t^2x} + \overbrace{t^3x + 3t^4x}^{\text{to remainder}} & 3t^4x \\
 & {\scriptstyle -t^2g}\Big\downarrow & & - \\
 & & & t^4x \\
 & & & - \\
f_3 & = & -\underline{2t^3x} - \overbrace{t^4x - 3t^5x}^{\text{to remainder}} & 3t^5x \\
 & \vdots & & \vdots
\end{array}
$$

FIGURE 2. division slice by slice

Not only would this process continue indefinitely, every term in our remainder but the first would actually be divisible by $\mathrm{LT}_>(g)$:

$$r = -t^2x - 3t^3x + t^3x + 3t^4x - t^4x - \ldots = -xt^2 - 2xt^3 + 2xt^4 - 2xt^5 + \ldots .$$

As we see, it is important to know when terms can be safely discarded to the remainder, and the only way to guarantee that is by proceeding term by term instead of slice by slice. And in order to guarantee that our result converges in the $\langle t\rangle$-adic topology, the order needs to be compatible with a weighted monomial order $>_w$ with $w \in \mathbb{R}^m_{<0} \times \mathbb{R}^{n+s}$. Figure 3 shows the same example in our algorithm.

We obtain a representation satisfying (DD1), (DD2) and (DDH):

$$f = (\underbrace{1 - t - t^3}_{=q}) \cdot g + (\underbrace{xt^2 + 5xt^4 + xt^5 + 3xt^6}_{=r}).$$

Having constructed a homogeneous determinate division with remainder, we will now introduce homogenization, dehomogenization and the ecart to continue with a weak division with remainder.

$$
\begin{array}{rcllc}
 & f_0 & = & \underline{2x} & r \\
{\scriptstyle -g} & \downarrow & & & \| \\
 & f_1 & = & -\underline{2tx} - \overbrace{t^2x - 3t^3x}^{\text{to be processed}} & \\
{\scriptstyle +tg} & \downarrow & & & \\
 & f_2 & = & \underline{t^2x} - 2t^3x + 3t^4x & t^2x \\
{\scriptstyle \text{to remainder}} & \downarrow & & & \\
 & f_3 & = & -\underline{2t^3x} + 3t^4x & \\
{\scriptstyle +t^2g} & \downarrow & & & + \\
 & f_4 & = & \underline{5t^4x} + t^5x + 3t^6x & 5t^4x \\
{\scriptstyle \text{to remainder}} & \downarrow & & & + \\
 & \vdots & & & t^5x \\
 & & & & + \\
{\scriptstyle \text{to remainder}} & \downarrow & & & 3t^6x \\
 & f_7 & = & 0 &
\end{array}
$$

FIGURE 3. division term by term

Recap 2.1.20 (Homogenization and dehomogenization) For an element $f = \sum_{\beta,\alpha,i} c_{\alpha,\beta,i} \cdot t^\beta x^\alpha \cdot e_i \in R[\![t]\!][x]^s$ we define its *homogenization* to be

$$f^h := \sum_{\alpha,\beta,i} c_{\alpha,\beta,i} \cdot t^\beta x_0^{\deg_x(f)-|\alpha|} x^\alpha \cdot e_i \in R[\![t]\!][x_h]^s := R[\![t]\!][x_0, x]^s.$$

And for an element $F \in R[\![t]\!][x_h]^s$ we define its *dehomogenization* to be $F|_{x_0=1} \in R[\![t]\!][x]^s$.

Any monomial ordering $>$ on $\mathrm{Mon}^s(t,x)$, can be naturally extended to an ordering $>_h$ on $\mathrm{Mon}^s(t, x_0, x)$ through

$$
\begin{aligned}
a >_h b \quad :\Longleftrightarrow \quad & \deg_{x_h}(a) > \deg_{x_h}(b) \text{ or} \\
& \deg_{x_h}(a) = \deg_{x_h}(b) \text{ and } a|_{x_0=1} > b|_{x_0=1}.
\end{aligned}
$$

Defining the *ecart* of an element $f \in R[\![t]\!][x]^s$ with respect to $>$ to be

$$\mathrm{ecart}_>(f) := \deg_x(f) - \deg_x(\mathrm{LM}_>(f)) \in \mathbb{N},$$

one can show that for any elements $g, f \in R[\![t]\!][x]^s$ and any x_h-homogeneous $F \in R[\![t]\!][x_h]$:

(1) $f = (f^h)^d$,
(2) $F = x_0^{\deg_{x_h}(F) - \deg_x(F^d)} \cdot (F^d)^h$,
(3) $\mathrm{LT}_{>_h}(f^h) = x_0^{\mathrm{ecart}_>(f)} \cdot \mathrm{LT}_>(f)$,

(4) $\mathrm{LT}_{>_h}(F) = x_0^{\mathrm{ecart}_>(F^d)+\deg_{x_h}(F)-\deg_x(F^d)} \cdot \mathrm{LT}_>(F^d)$,

(5) $\mathrm{LM}_{>_h}(g^h) \,|\, \mathrm{LM}_{>_h}(f^h) \iff$
$\mathrm{LM}_>(g) \,|\, \mathrm{LM}_>(f)$ and $\mathrm{ecart}_>(g) \le \mathrm{ecart}_>(f)$,

(6) $\mathrm{LM}_{>_h}(g^h) \mid \mathrm{LM}_{>_h}(F) \impliedby$
$\mathrm{LM}_>(g) \,|\, \mathrm{LM}_>(F^d)$ and $\mathrm{ecart}_>(g) \le \mathrm{ecart}_>(F^d)$.

Algorithm 2.1.21 (DwR, weak division with remainder)

Input: $(f, G, >)$, where $f \in R[\![t]\!][x]^s$ and $G = \{g_1, \ldots, g_k\} \subseteq R[\![t]\!][x]^s$ an indexed set and $> = >_w$ a weighted t-local monomial ordering with $w \in \mathbb{R}^m_{<0} \times \mathbb{R}^{n+s}$.

Output: (u, Q, r), where $u \in R[\![t]\!][x]$ with $\mathrm{LT}_>(u) = 1$, $Q = \{q_1, \ldots, q_k\} \subseteq R[\![t]\!][x]$ an indexed subset and $r \in R[\![t]\!][x]^s$ such that

$$u \cdot f = q_1 \cdot g_1 + \ldots + q_k \cdot g_k + r$$

satisfies

(ID1): $\mathrm{LM}_>(f) \ge \mathrm{LM}_>(q_i \cdot g_i)$ for $i = 1, \ldots, k$ and

(ID2): $\mathrm{LT}_>(r) \notin \langle \mathrm{LT}_>(g_1), \ldots, \mathrm{LT}_>(g_k) \rangle$, unless $r = 0$.

Moreover, the algorithm requires only a finite number of recursions.

if $f \neq 0$ and $\mathrm{LT}_>(f) \in \langle \mathrm{LT}_>(g_1), \ldots, \mathrm{LT}_>(g_k) \rangle$ **then**
  Set $D := \{g_i \in G \mid \mathrm{LM}_>(g_i) \text{ divides } \mathrm{LM}_>(f)\}$ and $D' := \emptyset$.
  while $\mathrm{LT}_>(f) \notin \langle \mathrm{LT}_>(g_i) \mid g_i \in D' \rangle$ **do**
    Pick $g \in D$ with minimal ecart.
    Set $D' := D' \cup \{g\}$ and $D := D \setminus \{g\}$.
  if $e := \max\{\mathrm{ecart}_>(g) \mid g \in D'\} - \mathrm{ecart}_>(f) > 0$ **then**
    Compute

$$(\{Q'_1, \ldots, Q'_k\}, R') := \mathrm{HDDwR}(x_0^e \cdot f^h, \{\mathrm{LT}_>(g_1^h), \ldots, \mathrm{LT}_>(g_k^h)\}, >_h).$$

    Set $f' := (x_0^e \cdot f^h - \sum_{i=1}^k Q'_i \cdot g_i^h)^d$.
    Run

$$(u'', \{q''_1, \ldots, q''_{k+1}\}, r) := \mathrm{DwR}(f', \{g_1, \ldots, g_k, f\}, >).$$

    Set $q_i := q''_i + u'' \cdot Q'^d_i$, $i = 1, \ldots, k$.
    Set $u := u'' - q''_{k+1}$.
  else
    Compute

$$(\{Q'_1, \ldots, Q'_k\}, R') := \mathrm{HDDwR}(f^h, \{g_1^h, \ldots, g_k^h\}, >_h).$$

    Run

$$(u, \{q''_1, \ldots, q''_k\}, r) := \mathrm{DwR}((R')^d, \{g_1, \ldots, g_k\}, >).$$

    Set $q_i := q''_i + u \cdot Q'^d_i$, $i = 1, \ldots, k$.
else
  Set $(u, \{q_1, \ldots, q_k\}, r) := (1, \{0, \ldots, 0\}, f)$.
return $(u, \{q_1, \ldots, q_k\}, r)$.

PROOF. *Finiteness of recursions:* For sake of clarity, label all the objects appearing in the ν-th recursion step by a subscript ν. For example the ecart $e_\nu \in \mathbb{N}$, the element $f_\nu \in R[\![t]\!][x]^s$ and the subset $G_\nu \subseteq R[\![t]\!][x]^s$.

Since $G_1^h \subseteq G_2^h \subseteq G_3^h \subseteq \ldots$, we have an ascending chain of leading ideals in $R[\![t]\!][x_h]^s$, which eventually stabilizes unless the algorithm terminates beforehand

$$\mathrm{LT}_{>_h}(G_1^h) \subseteq \mathrm{LT}_{>_h}(G_2^h) \subseteq \ldots \subseteq \mathrm{LT}_{>_h}(G_N^h) = \mathrm{LT}_{>_h}(G_{N+1}^h) = \ldots.$$

Assume $e_N > 0$. Then we'd have $f_N \in G_{N+1}$, and thus

$$\mathrm{LT}_{>_h}(f_N^h) \in \mathrm{LT}_{>_h}(G_{N+1}^h) = \mathrm{LT}_{>_h}(G_N^h).$$

To put it differently, we'd have

$$\mathrm{LT}_{>_h}(f_N^h) \in \langle \mathrm{LT}_{>_h}(g^h) \mid g^h \in G_N^h \text{ with } \mathrm{LM}_{>_h}(g^h) \text{ divides } \mathrm{LM}_{>_h}(f_N^h)\rangle,$$

which by Recap 2.1.20 (5) would imply that

$$\begin{aligned}\mathrm{LT}_>(f_N) \in \langle \mathrm{LT}_>(g) \mid g \in G_N \text{ with } &\mathrm{LM}_>(g) \text{ divides } \mathrm{LM}_>(f_N),\\ &\text{and } \mathrm{ecart}_>(g) \leq \mathrm{ecart}_>(f_N)\rangle.\end{aligned}$$

Consequently, we'd get

$$D'_N \subseteq \{g \in G_N \mid \mathrm{LM}_>(g) \text{ divides } \mathrm{LM}_>(f_N) \text{ and } \mathrm{ecart}_>(g) \leq \mathrm{ecart}_>(f_N)\},$$

contradicting our assumption

$$e_N = \max\{\mathrm{ecart}_>(g) \mid g \in D'_N\} - \mathrm{ecart}_>(f_N) \overset{!}{>} 0. \quad \lightning$$

Therefore we have $e_N \leq 0$. By induction we conclude that $e_\nu \leq 0$ for all $\nu \geq N$, i.e. that we will exclusively run through steps 14-16 of the "else" case from the N-th recursion step onwards.

By the properties of HDDwR we know that in particular

$$\mathrm{LT}_{>_h}(R'_N) \notin \mathrm{LT}_>(G_N^h).$$

Now assume that the recursions would not stop with the next recursion. That means there exists a $D'_{N+1} \subseteq D_{N+1}$ with

$$\mathrm{LT}_>((R'_N)^d) = \mathrm{LT}_>(f_{N+1}) \in \langle \mathrm{LT}_>(g) \mid g \in D'_{N+1}\rangle$$

such that $e_{N+1} = \max\{\mathrm{ecart}_>(g) \mid g \in D'_{N+1}\} - \mathrm{ecart}_>((R'_N)^d) \leq 0$. From Recap 2.1.20 (6), this immediately implies the following contradiction

$$\mathrm{LT}_{>_h}(R'_N) \in \mathrm{LT}_{>_h}(G_{N+1}^h) = \mathrm{LT}_{>_h}(G_N^h). \quad \lightning$$

Hence the algorithm terminates after the $N+1$-th recursion step.

Correctness: We make an induction on the number of recursions, say $N \in \mathbb{N}$. If $N = 1$ then either $f = 0$ or $\mathrm{LT}_>(f) \notin \langle \mathrm{LT}_>(g_1), \ldots, \mathrm{LT}_>(g_k)\rangle$, and in both cases

$$1 \cdot f = 0 \cdot g_1 + \ldots + 0 \cdot g_k + f$$

satisfies (ID1) and (ID2).

So suppose $N > 1$ and consider the first recursion step. If $e \leq 0$, then by the properties of HDDwR the representation

$$f^h = Q'_1 \cdot g_1^h + \ldots + Q'_k \cdot g_k^h + R'$$

satisfies (DD1), (DD2) and (DDH). (DD1) and (DD2) imply (ID1), which means that for each $i = 1, \ldots, k$ we have

$$x_0^{\mathrm{ecart}_>(f)} \cdot \mathrm{LM}_>(f) = \mathrm{LM}_{>_h}(f^h) \overset{\text{(ID1)}}{\geq_h} \mathrm{LM}_{>_h}(Q'_i) \cdot \mathrm{LM}_{>_h}(g_i^h) = \ldots$$

$$\ldots = x_0^{a_i + \mathrm{ecart}_>(g_i)} \cdot \mathrm{LM}_>(Q_i'^d) \cdot \mathrm{LM}_>(g_i)$$

for some $a_i \geq 0$. Since f^h and $Q'_i \cdot g_i^h$ are both x_h-homogeneous of the same x_h-degree by (DDH), the definition of the homogenized ordering $>_h$ implies

$$\mathrm{LM}_>(f) \geq \mathrm{LM}_>(Q_i'^d) \cdot \mathrm{LM}_>(g_i) \text{ for all } i = 1, \ldots, k. \tag{2}$$

Moreover, by induction the representation $u \cdot R'^d = q''_1 \cdot g_1 + \ldots + q''_k \cdot g_k + r$ satisfies (ID1), (ID2) and $\mathrm{LT}_>(u) = 1$, the first implying that

$$\mathrm{LM}_>(f) \overset{(2)}{\geq} \mathrm{LM}_> \underbrace{\left(f - \sum_{i=1}^{k} Q_i'^d \cdot g_i \right)}_{=R'^d} \overset{\text{(ID1)}}{\geq} \mathrm{LM}_>(q''_i \cdot g_i). \tag{3}$$

Therefore, the representation

$$u \cdot f = \sum_{i=1}^{k} (q''_i + u'' \cdot Q_i'^d) \cdot g_i + r$$

satisfies (ID1) by (2), (3), $\mathrm{LT}_>(u) = 1$ and (ID2) by induction.

Similarly, if $e > 0$, then by the properties of HDDwR the representation

$$x_0^e \cdot f^h = Q'_1 \cdot \mathrm{LT}_{>_h}(g_1^h) + \ldots + Q'_k \cdot \mathrm{LT}_{>_h}(g_k^h) + R'$$

satisfies (DD1), (DD2) and (DDH). (DD1) and (DD2) imply (ID1), which means that for each $i = 1, \ldots, k$ we have

$$x_0^{e + \mathrm{ecart}_>(f)} \cdot \mathrm{LM}_>(f) = \mathrm{LM}_{>_h}(x_0^e \cdot f^h) \geq \ldots$$

$$\ldots \geq \mathrm{LM}_{>_h}(Q'_i) \cdot \mathrm{LM}_{>_h}(\mathrm{LT}_{>_h}(g_i^h)) = x_0^{a_i + \mathrm{ecart}_>(g_i)} \cdot \mathrm{LM}_>(Q_i'^d) \cdot \mathrm{LM}_>(g_i),$$

for some $a_i \geq 0$. Since $x_0^e \cdot f^h$ and $Q'_i \cdot \mathrm{LT}_{>_h}(g_i^h)$ are both x_h-homogeneous of the same x_h-degree by (DDH), the definition of the homogenized ordering $>_h$ implies

$$\mathrm{LM}_>(f) \geq \mathrm{LM}_>(Q_i'^d) \cdot \mathrm{LM}_>(g_i). \tag{4}$$

Moreover, by induction the representation $u'' \cdot f' = \sum_{i=1}^{k} q''_i \cdot g_i + q''_{k+1} \cdot f + r$ satisfies (ID1), (ID2) and $\mathrm{LT}_>(u'') = 1$ with the first implying that

$$\mathrm{LM}_>(f) \overset{(4)}{\geq} \underbrace{\mathrm{LM}_> \left(f - \sum_{i=1}^{k} Q_i'^d \cdot g_i \right)}_{=\mathrm{LM}_>(R'^d)} \overset{\text{(ID1)}}{\geq} \mathrm{LM}_>(q''_i \cdot g_i). \tag{5}$$

Therefore, the representation

$$u \cdot f = \sum_{i=1}^{k} (q_i'' + u'' \cdot Q_i'^d) \cdot g_i + r, \text{ with } u = u'' - q_{k+1}''$$

satisfies (ID1) by (4), (5), $\mathrm{LT}_>(u'') = 1$ and (ID2) by induction.

To see that $\mathrm{LT}_>(u) = 1$, observe that

$$\mathrm{LT}_{>_h}(x_0^e \cdot f^h) \in \langle \mathrm{LT}_>(g_1^h), \ldots, \mathrm{LT}_>(g_k^h) \rangle,$$

which is why

$$\mathrm{LM}_>(f) = \mathrm{LM}_{>_h}(x_0^e \cdot f^h)^d > \mathrm{LM}_{>_h}\left(x_0^e \cdot f^h - \sum_{i=1}^{k} Q_i' \cdot g_i^h \right)^d = \mathrm{LM}_>(f').$$

Thus $\mathrm{LM}_>(f) > \mathrm{LM}_>(f') \geq \mathrm{LM}_>(q_{k+1}'') \cdot \mathrm{LM}_>(f)$, which necessarily implies $\mathrm{LM}(q_{k+1}'') < 1$. By induction we get $\mathrm{LT}_>(u) = \mathrm{LT}_>(u'') = 1$. □

Remark 2.1.22 (polynomial input) If the input is polynomial, $f, g_1, \ldots, g_k \in R[t,x]^s$, then we can regard them as elements of $R[\![t']\!][x'] = R[t,x]$ with $t' = ()$ and $x' = (t,x)$. In that case, our homogeneous determinate divisions with remainder terminates by Remark 2.1.17, and hence so does our weak division with remainder. In particular, the output $q_1, \ldots, q_k, r$ will be polynomial as well.

The next corollary will prove to be very useful in Theorem 2.2.15, though not for elements in $R[\![t]\!][x]^s$, but for elements in $R[\![t]\!][x]^k$ under the Schreyer ordering.

Corollary 2.1.23 *Let $>$ be a t-local monomial ordering and $g_1, \ldots, g_k \in R[\![t]\!][x]^s$. Then any $f \in R[\![t]\!][x]^s$ has a weak division with remainder*

$$u \cdot f = q_1 \cdot g_1 + \ldots + q_k \cdot g_k + r$$

with $r = \sum_{j=1}^{s} r_j e_j \in R[\![t]\!][x]^s$ satisfying

(SID2): $\mathrm{LT}_>(r_j \cdot e_j) \notin \langle \mathrm{LT}_>(g_1), \ldots, \mathrm{LT}_>(g_k) \rangle$, *unless $r_j = 0$, for $j = 1, \ldots, s$.*

Proof. We make an induction on s, in which the base case $s = 1$ follows from Algorithm 2.1.21, as condition (SID2) coincides with (ID2).

Suppose $s > 1$. By Algorithm 2.1.21 there exists a weak division with remainder

$$u \cdot f = q_i \cdot g_1 + \ldots + q_k \cdot g_k + r. \tag{6}$$

If $r = 0$, then the representation satisfies (SID2) and we're done. If $r \neq 0$, there is a unique $j \in \{1, \ldots, s\}$ such that $\mathrm{LT}_>(r) \in R[\![t]\!][x] \cdot e_j$. For sake of simplicity, suppose that $j = s$ and that $g_1, \ldots, g_k$ are ordered in such that

$$\underbrace{\mathrm{LT}_>(g_1), \ldots, \mathrm{LT}_>(g_l)}_{\notin R[\![t]\!][x] \cdot e_s}, \quad \underbrace{\mathrm{LT}_>(g_{l+1}), \ldots, \mathrm{LT}_>(g_s)}_{\in R[\![t]\!][x] \cdot e_s} \quad \text{for some } 1 \leq l < s.$$

Consider the projection

$$\sigma : R[\![t]\!][x]^s \longrightarrow R[\![t]\!][x]^{s-1}, \quad (p_1, \dots, p_s) \longmapsto (p_1, \dots, p_{s-1}),$$

the inclusion

$$\iota : R[\![t]\!][x]^{s-1} \longrightarrow R[\![t]\!][x]^s, \quad (p_1, \dots, p_{s-1}) \longmapsto (p_1, \dots, p_{s-1}, 0),$$

and let $>_*$ denote the restriction of $>$ on $\mathrm{Mon}(t, x)^{s-1}$. Note that we have

(1) for $h \in R[\![t]\!][x]^{s-1}$: $\mathrm{LM}_>(\iota(h)) = \iota(\mathrm{LM}_{>_*}(h))$,
(2) for $i = 1, \dots, l$: $\mathrm{LM}_>(g_i) = \mathrm{LM}_>(\iota(\sigma(g_i)))$.

By induction, there exists a weak division with remainder of $\sigma(r) \in R[\![t]\!][x]^{s-1}$ satisfying (SID2), say

$$u' \cdot \sigma(r) = q'_1 \cdot \sigma(g_1) + \ldots + q'_l \cdot \sigma(g_l) + r'. \tag{7}$$

Writing $r = \sum_{j=1}^{s} r_j \cdot e_j$ and $r' = \sum_{j=1}^{s-1} r'_j \cdot e_j$, we want to show that the following constructed representation

$$u \cdot u' \cdot f = \sum_{i=1}^{l} (u' \cdot q_i + q'_i) \cdot g_i + \sum_{i=l+1}^{k} u' \cdot q_i \cdot g_i + r'' \text{ with } r'' = \sum_{j=1}^{s-1} r'_j \cdot e_j + r_s \cdot e_s$$

is a weak division with remainder satisfying (SID2).

As (6) satisfies (ID2), (7) satisfies (ID1), and $\mathrm{LT}_>(r) \in R[\![t]\!][x]_> \cdot e_s$, we obtain for $i = 1, \dots, l$

$$\mathrm{LM}_>(f) \geq \mathrm{LM}_>(r) > \mathrm{LM}_>(\iota(\sigma(r))) \geq \mathrm{LM}_>(\iota(q'_i \cdot \sigma(g_i))) = \ldots$$
$$\ldots = \mathrm{LM}_>(q'_i \cdot \iota(\sigma(g_i))) = \mathrm{LM}_>(q'_i \cdot g_i).$$

Now since (6) satisfies (ID1) and $\mathrm{LT}_>(u) = 1 = \mathrm{LT}_>(u')$, we have for $i \leq l$

$$\mathrm{LM}_>(u \cdot u' \cdot f) = \mathrm{LM}_>(f) \geq \mathrm{LM}_>((u' \cdot q_i + q'_i) \cdot g_i)$$

and for $i > l$

$$\mathrm{LM}_>(u \cdot u' \cdot f) = \mathrm{LM}_>(f) \geq \mathrm{LM}_>(q_i \cdot g_i) = \mathrm{LM}_>(u' \cdot q_i \cdot g_i),$$

proving that our constructed representation satisfies (ID1).

Moreover, (SID2) of (7) tells us that for $j = 1, \dots, s-1$

$$\mathrm{LT}_{>_*}(r'_j \cdot e_j) \notin \langle \mathrm{LT}_{>_*}(\sigma(g_1)), \dots, \mathrm{LT}_{>_*}(\sigma(g_l)) \rangle, \text{ unless } r'_j = 0,$$

And because $\mathrm{LT}_>(g_i) \in R[\![t]\!][x] \cdot e_s$ for $i > l$, we get for $j = 1, \dots, s-1$

$$\mathrm{LT}_>(r'_j \cdot e_j) \notin \langle \mathrm{LT}_>(g_1), \dots, \mathrm{LT}_>(g_s) \rangle, \text{ unless } r'_j = 0.$$

In addition, by (ID2) of (6), we have

$$\mathrm{LT}_>(r'_s \cdot e_s) = \mathrm{LM}_>(r) \notin \langle \mathrm{LT}_>(g_1), \dots, \mathrm{LT}_>(g_s) \rangle,$$

which completes the proof that our constructed representation satisfies (SID2). By Proposition 2.1.10 this implies (ID2). $\square$

We will now introduce localizations at monomial orderings. More than just a convenience to get rid of the u with $\mathrm{LM}_>(u) = 1$ in our weak division with remainder, localization at monomial orderings allows geometers to compute in localization at ideals generated by variables. It is a technique that has been applied in the study of isolated singularities to great success.

Recap 2.1.24 (Localization at monomial orderings) For a t-local monomial ordering $>$ on $\mathrm{Mon}(t, x)$, we define

$$S_> := \{u \in R[\![t]\!][x] \mid \mathrm{LT}_>(u) = 1\} \text{ and } R[\![t]\!][x]_> := S_>^{-1} R[\![t]\!][x].$$

We will refer to $R[\![t]\!][x]_>$ as $R[\![t]\!][x]$ *localized at the monomial ordering* $>$.

Let $>$ be a module monomial ordering on $\mathrm{Mon}^s(t, x)$. Recall that it restricts to the same monomial ordering on $\mathrm{Mon}(t, x)$ in each component by Definition 2.1.4, which we will denote by $>_{R[\![t]\!][x]}$. We then define for any $k \in \mathbb{N}$

$$R[\![t]\!][x]^s_> := S^{-1}_{>_{R[\![t]\!][x]}} \left(R[\![t]\!][x]^s\right).$$

We will refer to $R[\![t]\!][x]^s_>$ as $R[\![t]\!][x]^s$ *localized at the monomial ordering* $>$. For $s = 1$, it coincides with the first definition.

Our definitions on $R[\![t]\!][x]^s$ extend naturally to $R[\![t]\!][x]^s_>$, since for any element $f \in R[\![t]\!][x]^s_>$ there exists an element $u \in S_>$ such that $u \cdot f \in R[\![t]\!][x]^s$. We define the *leading monomial, leading coefficient* and *leading term* of f with respect to $>$ to be that of $u \cdot f \in R[\![t]\!][x]^s$. The *leading module* of a submodule $M \leq R[\![t]\!][x]^s_>$ is again the module generated by the leading terms of its elements.

And given $f, g_1, \ldots, g_k, r = \sum_{j=1}^s r_j \cdot e_j \in R[\![t]\!][x]^s_>$, we say a representation

$$f = q_1 \cdot g_1 + \ldots + q_k \cdot g_k + r$$

satisfies

(ID1): if $\mathrm{LM}_>(f) \geq \mathrm{LM}_>(q_i \cdot g_i)$ for all $i = 1, \ldots, k$,

(ID2): if $\mathrm{LT}_>(r) \notin \langle \mathrm{LT}_>(g_1), \ldots, \mathrm{LT}_>(g_k)\rangle_{R[\![t]\!][x]}$, unless $r = 0$,

(DD1): if no term of $q_i \cdot \mathrm{LT}_>(g_i)$ lies in $\langle \mathrm{LT}_>(g_j) \mid j < i\rangle_{R[\![t]\!][x]}$ for all $i = 1, \ldots, k$,

(DD2): if no term of r lies in $\langle \mathrm{LT}_>(g_1), \ldots, \mathrm{LT}_>(g_k)\rangle$,

(SID2): if $\mathrm{LT}_>(r_j \cdot e_j)$ does not lie in $\langle \mathrm{LT}_>(g_1), \ldots, \mathrm{LT}_>(g_k)\rangle_{R[\![t]\!][x]}$, unless $r_j = 0$, for $j = 1, \ldots, s$.

We will refer to a representation satisfying (ID1) and (ID2) as *(indeterminate) division with remainder*, and we will refer to a representation satisfying (DD1) and (DD2) as *determinate division with remainder*.

With these notions, Corollary 2.1.23 then implies:

Corollary 2.1.25 *Let $>$ be a monomial ordering and $g_1, \ldots, g_k \in R[\![t]\!][x]^s_>$. Then any $f \in R[\![t]\!][x]^s_>$ has a division with remainder with respect to $g_1, \ldots, g_k$ satisfying (SID2).*

2.2. Standard bases and syzygies

In this section, we introduce standard bases following [**Wie11**] closely, correcting some of the slight inaccuracies. We also incorporate some remarks on possible optimizations for R being a principal ideal domain. Similar to the classical theory, it opens with introducing the Schreyer ordering and syzygies, and finishes with proving Buchberger's criterion. Moreover, we recall the notion of reduced standard bases and illustrate the problem we run into with them. See Figure 4 for a rough outline with numbering.

Recall that by Convention 2.1.1, we assumed R to be noetherian and linear equations in R to be solvable.

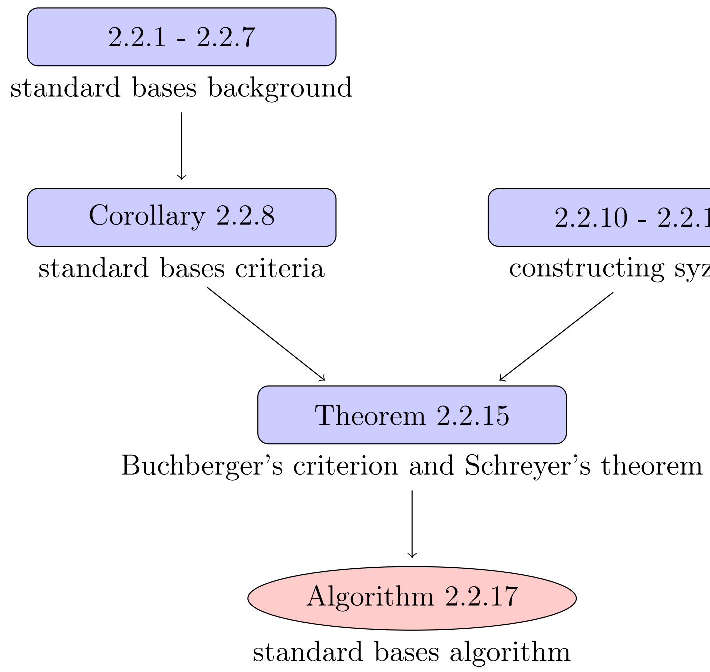


FIGURE 4. outline of Section 2.2

Definition 2.2.1 Given a t-local monomial ordering $>$ on $\mathrm{Mon}(t,x)^s$ and a module $M \leq R[\![t]\!][x]^s$ or $M \leq R[\![t]\!][x]^s_>$, a finite set $G \subseteq M$ with

$$\mathrm{LT}_>(G) = \mathrm{LT}_>(M), \text{ where } \mathrm{LT}_>(G) := \langle \mathrm{LT}_>(g) \mid g \in G\rangle$$

is called *standard basis* of M with respect to $>$.

Also, given a division with remainder of an element f with respect to $g_1, \dots, g_k \in R[\![t]\!][x]^s$ or $g_1, \dots, g_k \in R[\![t]\!][x]^s_>$,

$$f = q_1 \cdot g_1 + \ldots + q_k \cdot g_k + r,$$

with $q_1, \dots, q_k \in R[\![t]\!][x]$, $r \in R[\![t]\!][x]^s$ or $q_1, \dots, q_k \in R[\![t]\!][x]_>$, $r \in R[\![t]\!][x]^s_>$ respectively, we call r a *normal form* of f with respect to $g_1, \dots, g_k$ over

$R[\![t]\!][x]$ resp. over $R[\![t]\!][x]_>$ and, in case $r = 0$, we will refer to the representation above as a *standard representation* of f with respect to $g_1, \ldots, g_k$ over $R[\![t]\!][x]$ resp. over $R[\![t]\!][x]_>$.

A normal form and standard representation over $R[\![t]\!][x]$ of $u \cdot f$ with $\mathrm{LT}_>(u) = 1$ is commonly referred to as a *weak normal form* and *weak standard representation* of f respectively.

It is easy to see that every ideal in $R[\![t]\!][x]$ has a standard basis assuming R is noetherian. The proof is nearly identical to the classical case of ideals in a polynomial ring $K[x]$ over a field K.

Proposition 2.2.2 *If R is noetherian, any submodule $M \leq R[\![t]\!][x]^s$ has a standard basis for any monomial ordering $>$.*

PROOF. Let $M \leq R[\![t]\!][x]^s$ be a submodule. Since R is noetherian, so is $R[\![t]\!][x]^s$, and $\mathrm{LT}_>(M) \leq R[\![t]\!][x]^s$ has a finite generating set $h_1, \ldots, h_k$. Because

$$\mathrm{LT}_>(I) = \langle \mathrm{LT}_>(g) \mid g \in M \rangle \overset{!}{=} \{\mathrm{LT}_>(g) \mid g \in M\},$$

there exist $g_1, \ldots, g_k$ with $\mathrm{LT}_>(g_i) = h_i$ forming a standard basis of M. $\square$

Computing weak normal forms is essential in the upcoming standard bases algorithm. While it can be essentially done by computing a division with remainder and discarding everything but the remainder, as in the following algorithm, the fact that everything but the remainder is discarded may be used for some optimization in the division algorithm, which we leave out for sake of clarity.

Algorithm 2.2.3 normal form

Input: $(f, G, >)$, where $f \in R[\![t]\!][x]$, $G = \{g_1, \ldots, g_k\} \subseteq R[\![t]\!][x]^s$ an indexed set and $>$ a t-local monomial ordering.

Output: $r = \mathrm{NF}(f, G, >) \in R[\![t]\!][x]$, a normal form of f with respect to G and $>$.

1: Use Algorithm 2.1.21 to compute a division with remainder,

$$(u, \{q_1, \ldots, q_k\}, r) = \mathrm{DwR}(f, G, >).$$

2: **return** r.

Remark 2.2.4 (polynomial input) Should the input be polynomial, i.e. $f \in R[t, x]$ and $G \subseteq R[t, x]$, then by Remark 2.1.22 we automatically obtain a polynomial normal form $\mathrm{NF}(f, G, >) \in R[t, x]$.

Convention 2.2.5 For the remainder of the section, fix a t-local monomial ordering $>$ on $\mathrm{Mon}(t, x)^s$.

Proposition 2.2.6 *Let $M \leq R[\![t]\!][x]^s_>$ be a module and let $G = \{g_1, \ldots, g_k\}$ be a standard basis of M. Then given an element $f \in R[\![t]\!][x]_>$ and a weak division with remainder*

$$u \cdot f = q_1 \cdot g_1 + \ldots + q_k \cdot g_k + r,$$

we have $f \in M$ if and only if $r = 0$. In particular, we see that $M = \langle G \rangle$

Proof. If $r = 0$, then obviously $f \in \langle G \rangle \subseteq J$. Conversely, if $f \in J$, then $r = u \cdot f - q_1 \cdot g_1 + \ldots + q_k \cdot g_k \in J$ and therefore $\mathrm{LT}_>(r) \in \mathrm{LT}_>(J) = \mathrm{LT}_>(G)$. Hence $r = 0$ by (ID2).

We obviously have $M \supseteq \langle G \rangle$. For the converse, note that $u \in R[\![t]\!][x]_>$ with $\mathrm{LT}_>(u) = 1$ is a unit, and hence the weak division with remainder implies $M \subseteq \langle G \rangle$. □

Proposition 2.2.7 *Let $M \leq R[\![t]\!][x]^s_>$ be a module and let $G = \{g_1, \ldots, g_k\} \subseteq M$. Then the following statements are equivalent:*

(a) *G is a standard basis of M.*
(b) *Every normal form of any element in M with respect to G is zero.*
(c) *Every element in M has a standard representation with respect to G.*

Proof. By Proposition 2.2.6 (a) implies (b), and the implication (b) to (c) is true by Corollary 2.1.25. And if any $f \in J$ has a standard representation

$$f = q_1 \cdot g_1 + \ldots + q_k \cdot g_k,$$

then, since $\mathrm{LM}_>(f) \geq \mathrm{LM}_>(q_i \cdot g_i)$ for $i = 1, \ldots, k$, there can be no total cancellation of the leading terms on the right hand side. Hence $\mathrm{LT}_>(f) \in \mathrm{LT}_>(G)$, and (c) implies (a). □

As an immediate consequence, we get:

Corollary 2.2.8 *Let $M \leq R[\![t]\!][x]$ be a module and $G = \{g_1, \ldots, g_k\} \subseteq M$. Then the following statements are equivalent:*

(a) *G is a standard basis of M.*
(b) *Every weak normal form of any element in M with respect to G is zero.*
(c) *Every element in M has a weak standard representation with respect to G.*

Also note that this in particular implies for x-homogeneous modules that being a standard basis only depends on the leading monomials.

Corollary 2.2.9 *Let G be an x-homogeneous standard basis of an x-homogeneous module $M \leq R[\![t]\!][x]$ with respect to $>$. Let $>'$ be another t-local monomial ordering on $\mathrm{Mon}^s(t,x)$ such that*

$$\mathrm{LM}_{>'}(g) = \mathrm{LM}_>(g) \text{ for all } g \in G.$$

Then G is also a standard basis of M with respect to $>'$.

PROOF. By Algorithm 2.1.16, for any $f \in M = \langle G \rangle$ we can compute a determinate division with remainder 0 with respect to $>$,

$$f = q_1 \cdot g_1 + \ldots + q_k \cdot g_k + 0.$$

However, since the conditions (DD1) and (DD2) are only dependant on $\mathrm{LM}_>(g_i) = \mathrm{LM}_{>'}(g_i)$, this is also a valid determinate division with remainder under $>'$. By Proposition 2.1.10, this is in particular a valid division with remainder, proving that G is also a standard basis with respect to $>'$. □

Recap 2.2.10 (Syzygies and Schreyer ordering) Given a finite indexed set $G = \{g_1, \ldots, g_k\} \subseteq R[\![t]\!][x]^s_>$, we define the *Schreyer ordering* $>_S$ on $\mathrm{Mon}^k(t,x)$ associated to G and $>$ to be

$$\begin{aligned} t^\alpha \cdot x^\beta \cdot \varepsilon_i >_S t^{\alpha'} \cdot x^{\beta'} \cdot \varepsilon_j \quad &:\Longleftrightarrow \\ &t^\alpha \cdot x^\beta \cdot \mathrm{LM}_>(g_i) > t^{\alpha'} \cdot x^{\beta'} \cdot \mathrm{LM}_>(g_j) \text{ or} \\ &t^\alpha \cdot x^\beta \cdot \mathrm{LM}_>(g_i) = t^{\alpha'} \cdot x^{\beta'} \cdot \mathrm{LM}_>(g_j) \text{ and } i > j. \end{aligned}$$

Note that we distinguish between the canonical basis elements e_j of the free module $R[\![t]\!][x]^s_>$ and the canonical basis elements ε_i of the free module $R[\![t]\!][x]^k_>$.

Moreover, observe that $>_S$ and $>$ restrict to the same monomial ordering on $\mathrm{Mon}(t,x)$, so that

$$R[\![t]\!][x]^k_{>_S} = S^{-1}_{>_{S,R[\![t]\!][x]}} R[\![t]\!][x]^k = S^{-1}_{>_{R[\![t]\!][x]}} R[\![t]\!][x]^k = R[\![t]\!][x]^k_>.$$

It is therefore not wrong to use $R[\![t]\!][x]^k_>$, even though working with the Schreyer ordering $>_S$. In fact, we will generally do this, as it is more advantageous when the Schreyer ordering varies because of different G while the localization stays the same.

Let φ denote the substitution homomorphism

$$\begin{aligned} \varphi : R[\![t]\!][x]^k_> = \textstyle\bigoplus_{i=1}^k R[\![t]\!][x]_> \cdot \varepsilon_i \quad &\longrightarrow \quad R[\![t]\!][x]^s_> = \textstyle\bigoplus_{j=1}^s R[\![t]\!][x]_> \cdot e_j, \\ \varepsilon_i \quad &\longmapsto \quad g_i. \end{aligned}$$

We call its kernel the *syzygy module* or simply the *syzygies* of G,

$$\mathrm{syz}(G) := \left\{ \sum_{i=1}^k q_i \cdot \varepsilon_i \in R[\![t]\!][x]^k_{>_S} \;\middle|\; \sum_{i=1}^k q_i \cdot g_i = 0 \right\}.$$

The concept of syzygies is one that can be applied to any ring, and one of the conditions on our ground ring R in Convention 2.1.1 states that we assume to be able to compute a finite system of generators for the syzygies of our leading coefficients,

$$\begin{aligned} &\mathrm{syz}_R(\mathrm{LC}_>(g_1), \ldots, \mathrm{LC}_>(g_k)) := \\ &\quad \{(c_1, \ldots, c_k) \in R^k \mid c_1 \cdot \mathrm{LC}_>(g_1) + \ldots + c_k \cdot \mathrm{LC}_>(g_k) = 0\}. \end{aligned}$$

Definition 2.2.11 For an indexed set $G = \{g_1, \ldots, g_k\} \subseteq R[\![t]\!][x]^s$ and a fixed index $1 \leq l \leq k$, we will now introduce several objects which will be of importance in the upcoming theory.

Recall the notions of divisibility and least common multiple of module monomials in Definition 2.1.8. We denote the set of least common multiples of the leading monomials up to and including g_l with

$$C_l := \Big\{ \operatorname{lcm}(\mathrm{LM}_>(g_i) \mid i \in J) \;\Big|\; J \subseteq \{1, ..., k\} \text{ with } \max(J) = l \Big\} \setminus \{0\}.$$

Note that necessarily $C_l \subseteq R[\![t]\!][x] \cdot e_i$ for the index $1 \leq i \leq s$ such that $\mathrm{LT}_>(g_l) \in R[\![t]\!][x] \cdot e_i$.

And for a least common multiple $a \in C_l$, we abbreviate the set of all indices j up to l such that $\mathrm{LM}_>(g_j)$ divides it with

$$J_{l,a} := \Big\{ i \in \{1, \ldots, l\} \;\Big|\; \mathrm{LM}_>(g_i) \text{ divides } a \Big\}.$$

Now given $J_{l,a}$, we can compute a finite generating set for the syzygies of the tuple $(\mathrm{LC}_>(g_i))_{i \in J_{l,a}}$, which we will temporarily denote with S_R. Let $\mathrm{syz}_{R,l,a}$ be the set of elements of S_R with non-trivial entry in l:

$$\begin{aligned} \langle\ S_R\ \rangle_R &= \Big\{ (c_i)_{i \in J_{l,a}} \in R^{|J_{l,a}|} \;\Big|\; \textstyle\sum_{i \in J_{l,a}} c_i \cdot \mathrm{LC}_>(g_i) = 0 \Big\}, \\ &\cup\!\shortmid \\ \mathrm{syz}_{R,l,a} &= \Big\{ (c_i)_{i \in J_{l,a}} \in S_R \;\Big|\; c_l \neq 0 \Big\}. \end{aligned}$$

With this, we can write down a finite set of syzygies of the leading terms of the g_i up to and including $\mathrm{LT}_>(g_l)$ with non-trivial entry in l,

$$\mathrm{syz}_l := \left\{ \sum_{i \in J_{l,a}} \frac{c_i \cdot a}{\mathrm{LM}_>(g_i)} \cdot \varepsilon_i \in R[\![t]\!][x]^k \;\middle|\; a \in C_l \text{ and } c \in \mathrm{syz}_{R,l,a} \right\}.$$

For each $\xi' \in \mathrm{syz}_l$, we can then fix a single weak division with remainder of $\varphi(\xi') \in R[\![t]\!][x]^s$ with respect to $g_1, \ldots, g_l$ to obtain

$$\mathfrak{S}_l := \left\{ u \cdot \xi' - \sum_{i=1}^{k} q_i \cdot \varepsilon_i \;\middle|\; \begin{array}{c} \xi' \in \mathrm{syz}_l \text{ and } u \cdot \varphi(\xi') = q_1 \cdot g_1 + \ldots + q_l \cdot g_l + r \\ \text{the fixed weak division with remainder} \end{array} \right\}.$$

As $\mathfrak{S}_l$ obviously depends on G, we write $\mathfrak{S}_{G,l}$ instead whenever there might be confusion. Moreover, we abbreviate

$$\mathfrak{S}^{(G)} := \mathfrak{S}_{G,|G|}.$$

Also, there is a certain degree of ambiguity in the construction of $\mathfrak{S}_l$ since we are actively choosing generating sets and divisions with remainders. Hence whenever we use $\mathfrak{S}_l$, it will represent any possible outcome of our construction. For example, when we write $\mathfrak{S} \subseteq \mathfrak{S}_l$ for a set $\mathfrak{S} \subseteq R[\![t]\!][x]^k_{>_S}$, it means that the elements of $\mathfrak{S}$ are possible outcomes of our construction of $\mathfrak{S}_l$.

Remark 2.2.12 (factorial ground rings) Should R be a factorial ring in which we have a natural notion of a least common multiple, then the construction above simplifies to extensions of classical techniques.

Suppose $a \in C_l$ is a least common multiple of various leading monomials including $\mathrm{LM}_>(g_l)$. Let $J_{l,a}$ be the set of all indices i for which $\mathrm{LM}_>(g_i)$ divides a. Then the syzygy module of all leading coefficients of g_i with $i \in J_{l,a}$ can be seen to generated by syzygies of the form, see Proposition 2.2.13.

$$\frac{\mathrm{lcm}(\mathrm{LC}_>(g_i), \mathrm{LC}_>(g_j))}{\mathrm{LC}_>(g_i)} \cdot \varepsilon_i - \frac{\mathrm{lcm}(\mathrm{LC}_>(g_i), \mathrm{LC}_>(g_j))}{\mathrm{LC}_>(g_j)} \cdot \varepsilon_j, \text{ with } i,j \in J_{l,a}, i > j.$$

Abbreviating $\lambda_i := \mathrm{LC}_>(g_i)$, we consequently get

$$\mathrm{syz}_{R,l,a} = \left\{ \frac{\mathrm{lcm}(\lambda_l, \lambda_i)}{\lambda_l} \cdot \varepsilon_l - \frac{\mathrm{lcm}(\lambda_l, \lambda_i)}{\lambda_i} \cdot \varepsilon_i \;\middle|\; i \in J_{l,a} \right\}.$$

Hence,

$$\mathrm{syz}_l = \bigcup_{a \in C_l} \left\{ \frac{\mathrm{lcm}(\lambda_l, \lambda_i) \cdot a}{\mathrm{LT}_>(g_l)} \cdot \varepsilon_l - \frac{\mathrm{lcm}(\lambda_l, \lambda_i) \cdot a}{\mathrm{LT}_>(g_i)} \cdot \varepsilon_i \;\middle|\; i \in J_{l,a} \right\}.$$

The definition of the Schreyer ordering $>_S$ now states

$$\mathrm{LT}_{>_S} \left(\frac{\mathrm{lcm}(\lambda_l, \lambda_i) \cdot a}{\mathrm{LT}_>(g_l)} \cdot \varepsilon_l - \frac{\mathrm{lcm}(\lambda_l, \lambda_i) \cdot a}{\mathrm{LT}_>(g_i)} \cdot \varepsilon_i \right) = \frac{\mathrm{lcm}(\lambda_l, \lambda_i) \cdot a}{\mathrm{LT}_>(g_l)} \cdot \varepsilon_l.$$

Therefore, the module generated by the leading terms of syz_l is generated by the leading terms of its elements of the form

$$\frac{\mathrm{lcm}(\mathrm{LT}_>(g_l), \mathrm{LT}_>(g_i))}{\mathrm{LT}_>(g_l)} \cdot \varepsilon_l - \frac{\mathrm{lcm}(\mathrm{LT}_>(g_l), \mathrm{LT}_>(g_i))}{\mathrm{LT}_>(g_i)} \cdot \varepsilon_i \text{ with } l > i \in J_{l,a},$$

which we obtain by setting $a = \mathrm{lcm}(\mathrm{LM}_>(g_l), \mathrm{LM}_>(g_i))$. Note that for $i \notin J_{l,a}$ the expression would just be zero.

The images of these generators under φ are, in the classical case of polynomial rings, commonly known as *s-polynomials*, and the fixed divisions with remainder, which we considered for the definition of $\mathfrak{S}_l$, represent the normal form computations of these s-polynomials that are commonly done in the standard basis algorithm (and also Buchberger's Algorithm).

The elements of $\mathfrak{S}_l$ of the form above also commonly appear in the literature on standard basis algorithms over polynomial rings. They have no name, but they are commonly denoted by s_{ij} (or rather s_{li} for our choice of indices), see [**GP08**], [**Mar08**] and [**Wie11**]. We continue this train of thought in Remark 2.2.16.

We quickly prove a statement we used in Remark 2.2.12 on syzygy modules over factorial ring. The proof is borrowd from Theorem 2.2.5 of [**Wie11**]

Proposition 2.2.13 *Let R be a factorial ring, and let $c_1, \ldots, c_k \in R$. Then*

$$\mathrm{syz}(c_1, \ldots, c_k) = \left\langle \frac{\mathrm{lcm}(c_i, c_j)}{c_i} \cdot \varepsilon_i - \frac{\mathrm{lcm}(c_i, c_j)}{c_j} \cdot \varepsilon_j \;\middle|\; k \geq i > j \geq 1 \right\rangle.$$

PROOF. We make an induction on k with $k = 1, 2$ being clear. Now let $k > 2$ and consider a syzygy $a := a_1 \cdot \varepsilon_1 + \ldots + a_k \cdot \varepsilon_k$. Then

$$a_k \cdot c_k \in \langle c_1, \ldots, c_{k-1} \rangle,$$

from which we can infer

$$a_k \in \langle c_1, \ldots, c_{k-1}\rangle : \langle c_k\rangle = \langle c_1\rangle : \langle c_k\rangle + \ldots + \langle c_{k-1}\rangle : \langle c_k\rangle$$
$$= \left\langle \frac{\operatorname{lcm}(c_1, c_k)}{c_k} \right\rangle + \ldots + \left\langle \frac{\operatorname{lcm}(c_{k-1}, c_k)}{c_k} \right\rangle$$

Setting

$$s_{ij} := \frac{\operatorname{lcm}(c_i, c_j)}{c_i} \cdot \varepsilon_i - \frac{\operatorname{lcm}(c_i, c_j)}{c_j} \cdot \varepsilon_j \quad \text{and} \quad \mu_{ij} := \frac{\operatorname{lcm}(c_i, c_j)}{c_j},$$

we have shown that there are $b_1, \ldots, b_{k-1} \in R$ such that

$$a_k = b_1 \cdot \mu_{k1} + \ldots + b_{k-1} \cdot \mu_{kk-1},$$

so that, by induction,

$$a - b_1 \cdot s_{k1} + \ldots + b_{k-1} \cdot s_{kk-1} \in \operatorname{syz}(c_1, \ldots, c_{k-1})$$
$$= \langle s_{ij} \mid k-1 \geq i > j \geq 1\rangle.$$

Hence,

$$a \in \langle s_{ij} \mid k-1 \geq i > j \geq 1\rangle + \langle s_{k1}, \ldots, s_{kk-1}\rangle. \qquad \square$$

Now back to the general case that R is a general noetherian ring in which linear equations are solvable. For the objects in Definition 2.1.8, following holds:

Lemma 2.2.14 *For any $a \in C_l$ and any $(c_i)_{i\in J_{l,a}} \in syz_{R,l,a}$ there exists a $\xi \in \mathfrak{S}_l$ such that*

$$\operatorname{LT}_{>_S}(\xi) = \frac{c_l \cdot a}{\operatorname{LM}_>(g_l)} \cdot \varepsilon_l.$$

Proof. By construction in Definition 2.2.11, for any $a \in C_l$ and any $(c_i)_{i\in J_{l,a}} \in syz_{R,l,a}$, there exists a $\xi \in \mathfrak{S}_l$ of the form

$$\xi = u \cdot \xi' - \sum_{i=1}^{k} q_i \cdot \varepsilon_i = \sum\nolimits_{i\in J_{l,a}} \frac{c_i \cdot a}{\operatorname{LM}_>(g_i)} \cdot \varepsilon_i - \sum_{i=1}^{l} q_i \cdot \varepsilon_i.$$

First, recall that $J_{l,a}$ is the set of indices i up to l for which $\operatorname{LM}_>(g_i)$ divides a. Hence for all $i, j \in J_{l,a}$ we have

$$\operatorname{LM}_>\Big(\underbrace{\frac{c_i \cdot a}{\operatorname{LM}_>(g_i)}}_{\neq 0} \cdot g_i\Big) = a = \operatorname{LM}_>\Big(\underbrace{\frac{c_j \cdot a}{\operatorname{LM}_>(g_j)}}_{\neq 0} \cdot g_j\Big).$$

As an immediate consequence, we get

$$\operatorname{LT}_{>_S}\Big(\sum\nolimits_{i\in J_{l,a}} \frac{c_i \cdot a}{\operatorname{LM}_>(g_i)} \cdot \varepsilon_i\Big) = \frac{c_l \cdot a}{\operatorname{LM}_>(g_l)} \cdot \varepsilon_l, \tag{8}$$

because the Schreyer ordering prefers the highest component in case of a tie, and $l = \max J_{l,a}$, $c_l \neq 0$ by definition.

Next, recall that $(c_i)_{i\in J_{l,a}} \in \operatorname{syz}_R(\mathrm{LC}_>(g_i) \mid i \in J_{l,a})$, which means that

$$\sum_{i\in J_{l,a}} \frac{c_i \cdot a}{\mathrm{LM}_>(g_i)} \cdot \mathrm{LT}_>(g_i) = \sum_{i\in J_{l,a}} c_i \,\mathrm{LC}_>(g_i) \cdot a \overset{!}{=} 0.$$

Therefore, for all $j \in J_{l,a}$,

$$\mathrm{LM}_> \Big(\frac{c_j \cdot a}{\mathrm{LM}_>(g_j)} \cdot g_j \Big) > \mathrm{LM}_> \Big(\sum_{i\in J_{l,a}} \frac{c_i \cdot a}{\mathrm{LM}_>(g_i)} \cdot g_i \Big) = \mathrm{LM}_>(\varphi(\xi))$$

as all summands have the same leading monomial a and the leading terms in the sum cancel each other out.

Finally, recall that $\varphi(\xi) = q_1 \cdot g_1 + \ldots + q_l \cdot g_l + r$ was a division with remainder, whose (ID1) property implies for all $j \in J_{l,a}$ and $i = 1, \ldots, l$

$$\mathrm{LM}_> \Big(\frac{c_j \cdot a}{\mathrm{LM}_>(g_j)} \cdot g_j \Big) > \mathrm{LM}_>(\varphi(\xi)) \overset{\text{(ID1)}}{\geq} \mathrm{LM}_>(q_i \cdot g_i).$$

Thus we have for all $j \in J_{l,a}$ and $i = 1, \ldots, l$

$$\mathrm{LM}_{>_S} \Big(\frac{c_j \cdot a}{\mathrm{LM}_>(g_j)} \cdot \varepsilon_j \Big) >_S \mathrm{LM}_{>_S}(q_i \cdot \varepsilon_i). \tag{9}$$

Together, we obtain

$$\begin{aligned} \mathrm{LT}_{>_S}(\xi) &= \mathrm{LT}_{>_S} \Big(u \cdot \sum_{j\in J_{l,a}} \frac{c_j \cdot a}{\mathrm{LM}_>(g_j)} \cdot \varepsilon_j - \sum_{i=1}^{l} q_i \cdot \varepsilon_i \Big) \\ &\overset{(9)}{=} \mathrm{LT}_{>_S} \Big(u \cdot \sum_{j\in J_{l,a}} \frac{c_j \cdot a}{\mathrm{LM}_>(g_j)} \cdot \varepsilon_j \Big) \overset{(8)}{=} \frac{c_l \cdot a}{\mathrm{LM}_>(g_l)} \cdot \varepsilon_l. \end{aligned}$$ □

Theorem 2.2.15 *Let $G = \{g_1, \ldots, g_k\} \subseteq R[\![t]\!][x]^s$ be a generating set of $M \leq R[\![t]\!][x]^s_>$ and let $\mathfrak{S}_1, \ldots, \mathfrak{S}_k$ be constructed as above. Suppose there exists an $\mathfrak{S} \subseteq \bigcup_{l=1}^k \mathfrak{S}_l$ such that*

(1) $\mathrm{LT}_{>_S}(\mathfrak{S}) = \mathrm{LT}_{>_S}(\bigcup_{l=1}^k \mathfrak{S}_l)$,
(2) $\varphi(\xi) = 0$ *for all* $\xi \in \mathfrak{S}$.

Then

(1) G is a standard basis of M with respect to $>$,
(2) $\mathfrak{S}$ is a standard basis of $\operatorname{syz}(G)$ *with respect to $>_S$.*

Proof. Let $q_1, \ldots, q_k \in R[\![t]\!][x]_> = R[\![t]\!][x]_{>_S}$ be chosen arbitrarily. We will proof both statements simultaneously via the standard representation criteria in Proposition 2.2.7 (c), by considering

$$\chi := \sum_{i=1}^{k} q_i \cdot \varepsilon_i \quad \text{and} \quad g := \varphi(\chi) = \sum_{i=1}^{k} q_i \cdot g_i.$$

Here g represents an arbitrary element of M, and, in case $g = 0$, χ represents an arbitrary element of $\operatorname{syz}(G)$.

First compute a division with remainder of χ with respect to $\mathfrak{S}$ and the Schreyer ordering,

$$\chi = \sum_{\xi \in \mathfrak{S}} a_\xi \cdot \xi + r.$$

Should r be zero, then the expression above is a standard representation of χ with respect to $>_S$. Moreover, as $\varphi(\xi) = 0$ for all $\xi \in \mathfrak{S}$ by assumption, $g = \varphi(\chi) = 0$ trivially possesses a standard representation. Hence, in case $r = 0$, both g and χ satisfy the standard representation criteria. So suppose $r \neq 0$ for the remainder of the proof.

By Corollary 2.1.25, we may assume that our division with remainder satisfies (SID2), i.e. say

$$r = r_1 \cdot \varepsilon_1 + \ldots + r_k \cdot \varepsilon_k \text{ with } \mathrm{LT}_>(r_i \cdot \varepsilon_i) \notin \mathrm{LT}_{>_S}(\mathfrak{S}) \text{ for all } i = 1, \ldots, k. \quad (10)$$

Since by assumption $\varphi(\xi) = 0$ for all $\xi \in \mathfrak{S}$, we have

$$g = \varphi(\chi) = \varphi(r) = r_1 \cdot g_1 + \ldots + r_k \cdot g_k. \quad (11)$$

To proof the statement for $G \subseteq M$, it suffices to show that the expression above is a standard representation of g. To proof the statement for $\mathfrak{S} \subseteq \mathrm{syz}(G)$, we will show that $r \neq 0$ contradicts $g = 0$. This leaves $r = 0$ as the only viable case, assuming $g = 0$, for which we have already established that χ satisfies the standard representation criteria.

Now assume that $\mathrm{LM}_>(g) < \mathrm{LM}_>(r_i \cdot g_i)$ for some $i = 1, \ldots, k$, and hence for $J := \{i \in \{1, \ldots, k\} \mid \mathrm{LM}_>(r_i \cdot g_i) \text{ maximal}\}$

$$\sum_{i \in J} \mathrm{LT}_>(r_i \cdot g_i) = 0.$$

Set $l := \max(J)$ and $a := \mathrm{lcm}(\mathrm{LM}_>(g_i) \mid i \in J)$, so that obviously $J \subseteq J_{l,a}$. We will now concentrate on $r_l \cdot \varepsilon_l$.

For the leading coefficient of $r_l \cdot \varepsilon_l$, note that the leading coefficients sum up to zero, i.e. $\sum_{i \in J} \mathrm{LC}_>(r_i) \cdot \varepsilon_i \in \mathrm{syz}(\mathrm{LC}_>(g_i) \mid i \in J_{l,a})$. Recall that $\mathrm{syz}_{R,l,a}$ are the elements of a generating system of $\mathrm{syz}(\mathrm{LC}_>(g_i) \mid i \in J_{l,a})$ with non-trivial entry in l. Hence there are suitable $d_{(c_i)} \in R$ such that

$$\mathrm{LC}_>(r_l) \cdot \varepsilon_l = \sum_{(c_i) \in \mathrm{syz}_{l,a}} d_{(c_i)} \cdot c_l \cdot \varepsilon_l. \quad (12)$$

For the leading monomial of $r_l \cdot \varepsilon_l$, note that $\mathrm{LM}_>(r_l \cdot g_l)$ is divisible by $\mathrm{LM}_>(g_i)$ for all $i \in J$. Hence it is divisible by $a = \mathrm{lcm}(\mathrm{LM}_>(g_i) \mid i \in J)$, i.e. there exists a $t^\delta x^\gamma$ such that $\mathrm{LM}_>(r_l \cdot g_l) = t^\delta x^\gamma \cdot a$, or equivalently

$$\mathrm{LM}_>(r_l) = t^\delta x^\gamma \cdot \frac{a}{\mathrm{LM}_>(g_l)}. \quad (13)$$

Now, by the previous Lemma 2.2.14 there exists a $\xi_{(c_i)} \in \mathfrak{S}_l$ for any $(c_i) \in \mathrm{syz}_{R,l,a}$ such that

$$\mathrm{LT}_{>_S}(\xi_{(c_i)}) = \frac{c_l \cdot a}{\mathrm{LM}_>(g_l)} \cdot \varepsilon_l. \quad (14)$$

Piecing everything together, we thus get

$$\begin{aligned}\mathrm{LT}_>(r_l)\cdot\varepsilon_l &\overset{(12)+(13)}{=} t^\delta x^\gamma \sum_{(c_i)\in\mathrm{syz}_{l,a}} d_{(c_i)}\cdot\frac{c_l\cdot a}{\mathrm{LM}_>(g_l)}\cdot\varepsilon_l \\ &\overset{(14)}{=} t^\delta x^\gamma \sum_{(c_i)\in\mathrm{syz}_{l,a}} d_{(c_i)}\cdot\mathrm{LT}_{>_S}(\xi_{(c_i)})\in\mathrm{LT}_{>_S}(\mathfrak{S}_l).\end{aligned}$$

And since $\mathrm{LT}_{>_S}(\mathfrak{S}_l)\subseteq\mathrm{LT}_{>_S}(\mathfrak{S})$ by our first assumption, this contradicts the (SID2) condition in Equation (10). Therefore, Equation (11) has to be a standard representation, implying that G is a standard basis of M with respect to $>$.

Moreover, since $r\neq 0$, Equation (11) being standard representation yields an obvious contradiction if $g=0$. Hence in the case $g=0$, we have $r=0$ and we have already seen how this implies that $\mathfrak{S}$ is a standard basis of $\mathrm{syz}(G)$ with respect to $>_S$. $\square$

Remark 2.2.16 (R factorial continued) Suppose again that R is a factorial ring. As we have seen in Remark 2.2.12, that the leading module of $\bigcup_{l=1}^k \mathfrak{S}_{G,l}$ is generated by the leading terms of elements of the form

$$\frac{\mathrm{lcm}(\mathrm{LT}_>(g_i),\mathrm{LT}_>(g_j))}{\mathrm{LT}_>(g_i)}\cdot\varepsilon_i-\frac{\mathrm{lcm}(\mathrm{LT}_>(g_i),\mathrm{LT}_>(g_j))}{\mathrm{LT}_>(g_j)}\cdot\varepsilon_j,\ i>j,$$

which is why they are the only elements we need to keep track of for Theorem 2.2.15. These elements are obviously characterized by a pairs of distinct elements $(g_i,g_j)\in M\times M$, by elements in a so-called *pair-set*, which commonly appear in the classical standard basis algorithm and in Buchberger's Algorithm.

Algorithm 2.2.17 standard basis algorithm

Input: $(G,>)$, where G an indexed generating set of $M\leq R[\![t]\!][x]^s$ and $>$ a t-local monomial ordering on $\mathrm{Mon}^s(t,x)$.

Output: $G'\subseteq M$ a standard basis of M with respect to $>$.

Suppose $G:=\{g_1,\dots,g_k\}$.

Pick $\mathfrak{S}\subseteq\bigcup_{l=1}^k\mathfrak{S}_{G,l}\subseteq R[\![t]\!][x]^k$ such that

$$\mathrm{LT}_{>_S}(\mathfrak{S})=\mathrm{LT}_{>_S}\Big(\bigcup_{l=1}^k\mathfrak{S}_{G,l}\Big),$$

where $>_S$ is the Schreyer ordering on $\mathrm{Mon}^k(t,x)$ associated to G and $>$.

while $\mathfrak{S}\neq\emptyset$ **do**

  Set $k:=|G|$, so that $G:=\{g_1,\dots,g_k\}$ and $\mathfrak{S}\subseteq R[\![t]\!][x]^k_>$.

  Choose $q=\sum_{i=1}^k q_i\cdot\varepsilon_i\in\mathfrak{S}$.

  Set $\mathfrak{S}:=\mathfrak{S}\setminus\{q\}$.

  Compute a weak normal form r of $q_1\cdot g_1+\ldots+q_k\cdot g_k$ with respect to G

$$r:=\mathrm{NF}_>(q_1\cdot g_1+\ldots+q_k\cdot g_k,G,>).$$

8: **if** $r \neq 0$ **then**
9: Set $g_{k+1} := r$.
10: Set $G := G \cup \{g_{k+1}\}$.
11: Pick $\mathfrak{S}' \subseteq \mathfrak{S}^{(G)} \subseteq R[\![t]\!][x]^{k+1}$ such that

$$\mathrm{LT}_{>_S}(\mathfrak{S}') = \mathrm{LT}_{>_S}\left(\mathfrak{S}^{(G)}\right),$$

where $>_S$ is the Schreyer ordering on $\mathrm{Mon}^{k+1}(t,x)$ induced by the newly extended G and $>$.
12: Set $\mathfrak{S} := (\mathfrak{S} \times \{0\}) \cup \mathfrak{S}'$.
13: **return** G.

Proof. Label all objects in the ν-th iteration of the while loop with a subscript ν. That is, to be more precise,

- G_ν as it exists in Step 4,
- k_ν as it exists in Step 4,
- q_ν as chosen in Step 5
- r_ν as computed in Step 7,
- $\mathfrak{S}_\nu$ as $\mathfrak{S}$ exists in Step 4,
- $\mathfrak{S}'_{\nu+1}$ as $\mathfrak{S}'$ exists in Step 9 if $r_{\nu-1} \neq 0$, $\mathfrak{S}'_{\nu+1} = \emptyset$ otherwise, $\mathfrak{S}'_1 := \mathfrak{S}_1$,

so that

$$G_{\nu+1} = G_\nu \cup \{r_\nu\} \text{ and } \mathfrak{S}_{\nu+1} = (\mathfrak{S}_\nu \times \{0\}) \cup \mathfrak{S}'_{\nu+1}.$$

Termination. Note that we have a nested sequence of modules

$$\mathrm{LT}_>(G_1) \subseteq \mathrm{LT}_>(G_2) \subseteq \mathrm{LT}_>(G_3) \subseteq \ldots \subseteq \mathrm{LT}_>(G_\nu) \subseteq \mathrm{LT}_>(G_{\nu+1}) \subseteq \ldots,$$

which has to stabilize at some point. Because $r_\nu \neq 0$ implies $\mathrm{LT}_>(G_\nu) \subsetneq \mathrm{LT}_>(G_{\nu+1})$, it means that our sets $\mathfrak{S}_\nu$ have to be strictly decreasing in every step beyond the point of stabilization. And since all $\mathfrak{S}_\nu$ are finite, our algorithm terminates eventually.

Correctness. Let N be the total number of iterations, and let G be the return value, $k := |G|$. We will prove that G is a standard basis by constructing a set $\mathfrak{S} \subseteq R[\![t]\!][x]^k$ that satisfies the two conditions in Theorem 2.2.15. For that, consider all $\mathfrak{S}_\nu \subseteq R[\![t]\!][x]^{k_\nu}_>$ canonically embedded in $R[\![t]\!][x]^k_>$ due to $G_\nu \subseteq G$ and $k_\nu \leq k$. Let $\mathfrak{S}$ be the union of all $\mathfrak{S}'_\nu$,

$$\mathfrak{S} := \bigcup_{\nu=1}^{N+1} \mathfrak{S}'_\nu \subseteq R[\![t]\!][x]^k.$$

Note that $\mathfrak{S}'_\nu \subseteq \mathfrak{S}_{G,k_\nu}$, because the construction of $\mathfrak{S}_{G,k_\nu}$ only depends on the first k_ν elements of G, which are exactly the elements of G_ν. Moreover, Step 9 implies that $\mathrm{LT}_{>_S}(\mathfrak{S}_\nu) = \mathrm{LT}_{>_S}(\mathfrak{S}_{G,k_\nu})$, which shows that $\mathfrak{S}$ satisfies the first condition of our theorem,

$$\mathrm{LT}_{>_S}(\mathfrak{S}) = \mathrm{LT}_{>_S}\left(\bigcup_{l=1}^{k} \mathfrak{S}_{G,l}\right).$$

Now for each $\xi \in \mathfrak{S}$ there exists an iteration $1 \leq \nu \leq N$ in which it is chosen in Step 5, $\xi = \sum_{i=1}^{k_\nu} q_{i,\nu} \cdot \varepsilon_i$.

If $\varphi(\xi) = r_\nu = 0$, then ξ satisfies the second condition of our theorem. However if $\varphi(\xi) = r_\nu \neq 0$, then $g_{\nu+1} = r_\nu$ and ξ can be replaced with $\xi - \varepsilon_{\nu+1}$ so that $\varphi(\xi - \varepsilon_{\nu+1}) = 0$. Note that this does not change the leading term, since by construction the maximal leading terms of $q_1 \cdot g_1, \ldots, q_{l_\nu} \cdot g_{l_\nu}$ cancel each other out, which implies that $q_{i,\nu} \cdot \varepsilon_i >_S \varepsilon_{\nu+1}$ for any $1 \leq i \leq \nu$ with $q_{i,\nu} \neq 0$. Hence we obtain a set $\mathfrak{S}$ completely satisfying the second condition of our theorem. □

Remark 2.2.18 (polynomial input) Should our input be polynomial, $G \subseteq R[t,x]$, then all normal form computations terminate and yield polynomial outputs as noted in 2.2.4. In particular, our standard basis algorithm will terminate and the output will be polynomial as well.

Moreover, if our input is x-homogeneous, then so is the resulting standard basis.

Should R be a factorial ring, Algorithm 2.2.17 can be simplified to:

Algorithm 2.2.19 standard basis algorithm for factorial rings

Input: $(G, >)$, where G an indexed generating set of $M \leq R[\![t]\!][x]^s$ and $>$ a t-local monomial ordering on $\mathrm{Mon}^s(t,x)$, R a principal ideal domain.

Output: $G' \subseteq M$ a standard basis of M with respect to $>$.

Suppose $G := \{g_1, \ldots, g_k\}$.
Initialize a pair-set, $P := \{(g_i, g_j) \mid i < j\}$.
while $P \neq \emptyset$ **do**
  Pick $(g_i, g_j) \in P$.
  Set $P := P \setminus \{(g_i, g_j)\}$.
  Compute a weak normal form

$$r := \mathrm{NF}_>(\mathrm{spoly}(g_i, g_j), G, >),$$

  where

$$\mathrm{spoly}(g_i, g_j) = \frac{\mathrm{lcm}(\mathrm{LT}_>(g_i), \mathrm{LT}_>(g_j))}{\mathrm{LT}_>(g_i)} \cdot g_i - \frac{\mathrm{lcm}(\mathrm{LT}_>(g_i), \mathrm{LT}_>(g_j))}{\mathrm{LT}_>(g_j)} \cdot g_j$$

  and

$$\mathrm{lcm}(\mathrm{LT}_>(g_i), \mathrm{LT}_>(g_j)) = \mathrm{lcm}(\mathrm{LC}_>(g_i), \mathrm{LC}_>(g_j)) \cdot \mathrm{lcm}(\mathrm{LM}_>(g_i), \mathrm{LM}_>(g_j)).$$

  if $r \neq 0$ **then**
    Extend the pair-set, $P := P \cup \{(g, r) \mid g \in G\}$.
    Set $G := G \cup \{r\}$.
return $G' := G$.

Next, we will introduce an important type of standard basis for the classical theory of Gröbner fans, and show why we run into problems with them in our situation.

Definition 2.2.20 Let $G, H \subseteq R[\![t]\!][x]^s$ be two finite subsets. Given a t-local monomial ordering $>$ on $\mathrm{Mon}^s(t,x)$, we call G *reduced* with respect to H, if, for all $g \in G$, no term of $\mathrm{tail}_>(g)$ lies in $\mathrm{LT}_>(H)$.

And we simply call G *reduced*, if it is reduced with respect to itself and minimal in the sense that no proper subset $G' \subsetneq G$ is sufficient to generate its leading module, i.e. $\mathrm{LT}_>(G') \subsetneq \mathrm{LT}_>(G)$.

Observe that we forego any kind of normalization of the leading coefficients that is normally done in polynomial rings over ground fields.

If our module is generated by x-homogeneous elements, it is not hard to show that reduced standard bases exist. Given an x-homogeneous standard basis, one can persue a strategy similar to the classical reduction algorithm based on repeated tail reduction. Lemma 2.1.14 guarantees its convergence in the $\langle t\rangle$-adic topology.

Algorithm 2.2.21 reduction algorithm

Input: $(G,>)$, where $G = \{g_1, \ldots, g_k\}$ is a minimal x-homogeneous standard basis of $M \leq R[\![t]\!][x]^s$ with respect to the weighted ordering $> = >_w$ with $w \in \mathbb{R}^m_{<0} \times \mathbb{R}^{n+s}$.

Output: $G' = \{g'_1, \ldots, g'_k\}$ an x-homogeneous reduced standard basis of M with respect to $>$ such that $\mathrm{LM}_>(g'_i) = \mathrm{LM}_>(g_i)$.

for $i = 1, \ldots, k$ **do**
  Set $g'_i := g_i$.
  Create a working list

$$L := \{p \in R[\![t]\!][x]^s \mid p \text{ term of } g'_i,\ \mathrm{LM}_>(g'_i) > p\},$$

  while $L \neq \emptyset$ **do**
    Pick $p \in L$ with $\mathrm{LM}_>(p)$ maximal.
    Set $L := L \setminus \{p\}$.
    if $p \in \mathrm{LT}_>(M)$ **then**
      Compute homogeneous division with remainder

$$(\{q_1, \ldots, q_k\}, r) = \mathrm{HDDwR}(p, G, >).$$

      Set $g'_i := g'_i - (q_1 \cdot g_1 + \ldots + q_k \cdot g_k)$.
      Update the working list

$$L := \{p' \in R[\![t]\!][x]^s \mid p' \text{ term of } g_i,\ \mathrm{LM}_>(p) > \mathrm{LM}_>(p')\}.$$

return $\{g'_1, \ldots, g'_k\}$

Proof. Pick an $i = 1, \ldots, k$. Labelling all objects occurring in the ν-the recurring step by a subscript ν, we have a strictly decreasing sequence

$$\mathrm{LM}_>(p_1) > \mathrm{LM}_>(p_2) > \mathrm{LM}_>(p_3) > \ldots .$$

And since $\mathrm{LM}_>(p_\nu) \geq \mathrm{LM}_>(q_{j,\nu} \cdot g_j)$ for all $j = 1, \ldots, k$, the sequence $(q_{j,\nu} \cdot g_j)_{\nu\in\mathbb{N}}$ must also converge in the $\langle t\rangle$-adic topology together with $(p_\nu)_{\nu\in\mathbb{N}}$. In particular, the element $g_i' = g_i - \sum_{\nu=0}^{\infty} \sum_{j=1}^{k} q_{j,\nu} \cdot g_j$ in our output exists.

Also, while setting $g_{i,\nu+1}' = g_{i,\nu}' - (q_{1,\nu} \cdot g_1 + \ldots + q_{k,\nu} \cdot g_k)$ apart from the term p_ν cancelling, the terms changed are all strictly smaller than p. Hence for any term p of g_i', $p \neq \mathrm{LT}_>(g_i)$, there is a recursion step in which it is picked. Because p is not cancelled during the step, we have $p \notin \mathrm{LT}_>(M)$. Therefore no term of g_i' apart from its leading term lies in $\mathrm{LT}_>(M)$. □

One nice property of reduced standard bases, that is repeatedly used in the established theory of Gröbner fans of polynomial ideals over a ground field, is their uniqueness up to multiplication by units. In fact, this property does not change even if we add power series into the mix.

Lemma 2.2.22 *Let R be a field and let $M \leq R[\![t]\!][x]^s$ or $M \leq R[\![t]\!][x]^s_>$ be a module generated by x-homogeneous elements. Then M has a unique monic, reduced standard basis.*

Proof. Because R is a field, we have $\mathrm{LT}_>(M) = \mathrm{LM}_>(M)$ and since $\mathrm{LM}_>(M)$ has a unique minimal generating system consisting of monomials, let's call it A, so does $\mathrm{LT}_>(M)$.

Let $G = \{g_1, \ldots, g_k\}$ be a monic, reduced standard bases of M. Observe that the leading terms of G form a standard basis of the leading module of M. That means each $a \in A \subseteq \mathrm{LT}_>(M)$ can be expressed with a standard representation of the leading terms of G,

$$a = q_1 \cdot \mathrm{LT}_>(g_1) + \ldots + q_k \cdot \mathrm{LT}_>(g_k).$$

Since there is no cancellation of higher terms in the standard representation, there must exist an $i = 1 \ldots, k$ with $a = \mathrm{LM}_>(q_i \cdot g_i)$. This implies $\mathrm{LM}_>(g_i) = a$ because a wouldn't be a minimal generator of $\mathrm{LM}_>(M)$ otherwise. And because G is monic, $\mathrm{LT}_>(g_i) = a$.

Therefore, given a reduced standard basis G, we see that for any minimal generator $a \in A$ there exists an element $g \in G$ with $\mathrm{LM}_>(g) = a$. And since reduced standard bases are minimal themselves, it means that there is exactly one element $g \in G$ per minimal generator $a \in A$.

Now let G and H be two different reduced standard basis of M. Let $a \in A$ and let $g \in G$, $h \in H$ be the basis element with leading monomial a. If $g - h \neq 0$, then $g - h \in M$ must have a non-zero leading monomial which lies in $\mathrm{LM}_>(M)$. However, that monomial also has to occur in either g and h, and since R is a field the term with that monomial has to lie in $\mathrm{LT}_>(M) = \mathrm{LM}_>(M)$, contradicting that G and H were reduced. □

However, it can easily be seen that this does not hold over rings.

Example 2.2.23 Consider the ring $\mathbb{Z}[x,y]$ and the degree lexicographical ordering $>$, i.e.

$$\begin{aligned} x^{a_1}y^{a_2} > x^{b_1}y^{b_2} \quad &:\Longleftrightarrow \\ &a_1 + a_2 > b_1 + b_2 \text{ or} \\ &a_1 + a_2 = b_1 + b_2 \text{ and } (a_1, a_2) > (b_1, b_2) \text{ lexicographically in } \mathbb{R}^2. \end{aligned}$$

Consider the following ideal and its leading ideal:

$$I := \langle 2x^2y + 1, 3xy^2 + 1 \rangle \text{ and } \mathrm{LT}_>(I) = \langle 2x, 9y^3, xy^2 \rangle.$$

Two possible standard bases, both reduced, are

$$\begin{array}{cccccc} G_1 = \{ & \underline{2x} - 3y, & \underline{9y^3} + 2, & \underline{xy^2} + 3y^3 + 1 & \}, \\ & \| & \| & \nparallel & \\ G_2 = \{ & \underline{2x} - 3y, & \underline{9y^3} + 2, & \underline{xy^2} - 6y^3 - 1 & \}. \end{array}$$

Hence, unlike their classical counterparts over ground fields, reduced standard bases over ground rings are not unique up to multiplication with units. The key problem is that leading modules are not necessarily saturated with respect to the ground ring. This allowed the third basis element to have terms with monomials in $\mathrm{LM}_>(M)$, to which we could add a constant multiple of the second basis element without changing it being reduced.

Additionally, even if our coefficient ring R were a field and reduced standard bases are unique, they might not be feasable for practical computations.

Example 2.2.24 Consider the principal ideal generated by the element $g = x + y + tx \in \mathbb{Q}[\![t]\!][x,y]$ and the monomial ordering $>_w$ with weight vector $w = (-1,1,1)$ and $>$ the lexicographical ordering with $x > y > 1 > t$ as tiebreaker. Then $\{g\}$ is a standard basis and one can show that it converges to $g' = x + \sum_{i=0}^{\infty}(-1)^i \cdot t^i y$ in its reduction process.

Since the reduced standard bases is unique, this implies that I has no reduced standard basis consisting of polynomials, even though I is generated by a polynomial itself. Consequently, this means that the reduced standard bases which play a central role in the established Gröbner fan theory are useless in our case from a practical perspective.

$$
\begin{array}{r}
x + y + \ \underline{tx} \\
\Big\downarrow{\scriptstyle -t \cdot g} \\
x + y - ty - \underline{t^2 x} \\
\Big\downarrow{\scriptstyle +t^2 \cdot g} \\
x + y - ty + t^2 y + \underline{t^3 x} \\
\Big\downarrow \\
\vdots \\
\Big\downarrow \\
x + \sum_{i=0}^{\infty} (-1)^i \cdot t^i y
\end{array}
$$

FIGURE 5. reduction of $tx + t^2x + y$

CHAPTER 3

Gröbner fans in $R[\![t]\!][x]$

In this chapter, we introduce the notion of initially reduced standard bases and construct the Gröbner fan of an x-homogeneous ideal in $R[\![t]\!][x]$, the latter following the lines of [**FJT07**]. We present algorithms to reduce standard bases initially in finite time, and algorithms to compute Gröbner fans of x-homogeneous ideals. For necessary conditions on our ground ring, please note Convention 3.1.1.

Based on the standard bases introduced in the last chapter, the Gröbner fans introduced in this chapter will provide us with a natural polyhedral structure on our tropical varieties. This will prove very useful in the computation of tropical varieties in the next chapter.

3.1. Construction

In this section, we construct the Gröbner fan of an x-homogeneous ideal $I \trianglelefteq R[\![t]\!][x]$ satisfying Convention 3.1.1. The construction is similar to Section 2 of Fukuda, Jensen and Thomas' work on computing Gröbner fans of polynomial ideals over a ground field [**FJT07**] with some changes to encompass initially reduced standard bases. Also, caution is warranted since we only allow weight vectors with strictly negative weight in t, while the Gröbner fan is defined on the closed half space $\mathbb{R}_{\leq 0} \times \mathbb{R}$. The Gröbner fan was originally introduced as early as in Mora and Robbiano's work [**MR88**], and many ideas in the construction originate from there.

The crucial step is proving that standard bases of initial ideals can be derived from standard bases of the original ideal. From this, one can show that the Gröbner cones satisfy all properties to form a polyhedral fan, as defined in Definition 1.1.13 and Example 1.1.14:

(1) every Gröbner cone is a polyhedral cone,
(2) the intersection of two Gröbner cones is a face of each,
(3) there are only finitely many Gröbner cones.

See Figure 1 for a rough outline of the chapter with numbering.

Convention 3.1.1 For the entirety of the chapter, let R continue to satisfy Convention 2.1.1, i.e. R is noetherian and linear equations in R are solvable, so that, fixing the ring $R[\![t]\!][x] := R[\![t]\!][x_1, \ldots, x_n]$ with a single variable t, standard bases exist and are computable (in finite time and with polynomial output, if the ideal is generated by polynomials).

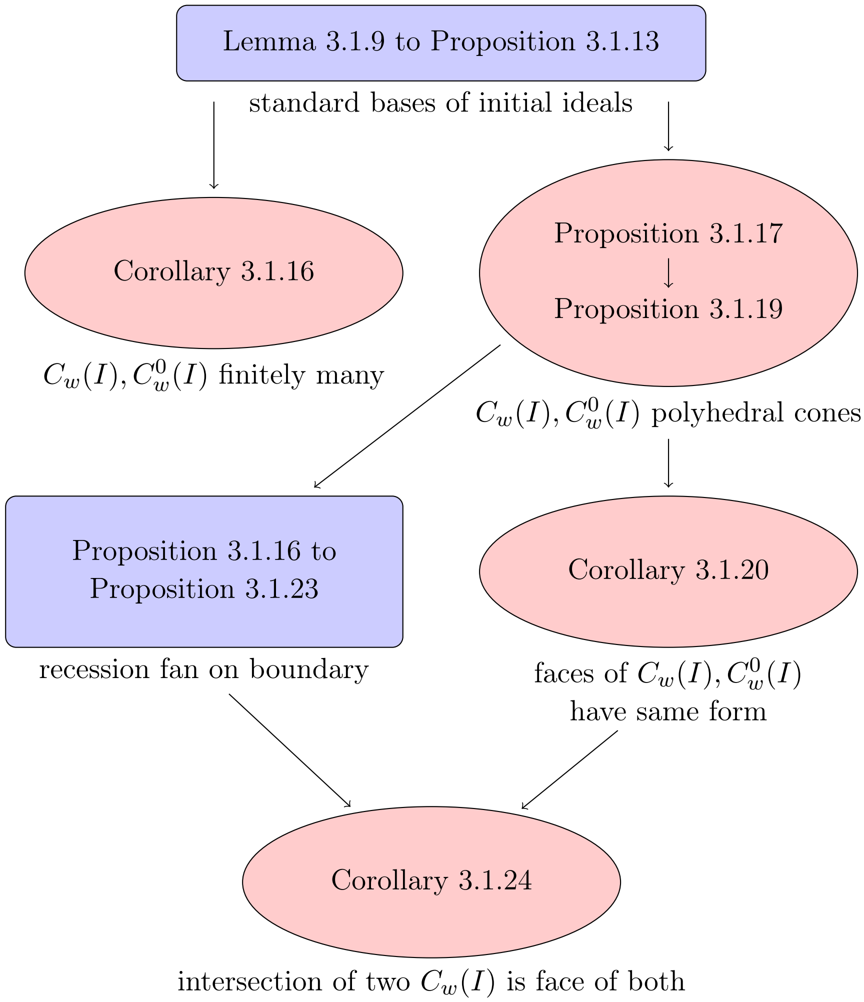


FIGURE 1. outline of Section 3.1

Before we begin with the actual constructing the Gröbner fan, let us quickly prove some necessary statements about polynomial ideals over a ground ring. The first is an easy generalization of a well-known fact about monomial ideals, in fact so is its proof.

Lemma 3.1.2 *Let $J \trianglelefteq R[t,x]$ be an ideal generated by terms and let $f \in J$. Then each term of f is again contained in J.*

PROOF. Let $>$ be a t-local monomial ordering on $\mathrm{Mon}(t,x)$, and let $p_1,\ldots,p_k \in R[t,x]$ be terms generating J. Because R is noetherian, so is $R[t,x]$, and hence there exist $q_1,\ldots,q_k \in R[t,x]$ such that

$$f = q_1 \cdot p_1 + \ldots + q_k \cdot p_k,$$

and we may assume that $\mathrm{LM}_>(f) \geq \mathrm{LM}_>(q_i \cdot p_i)$, because we may drop all terms s_i of q_i with $\mathrm{LM}_>(f) < \mathrm{LM}_>(s_i \cdot p_i)$ and still retain the equality. Hence there are suitable $1 \leq i_1 < \ldots < i_l \leq k$ contributing to the leading term such that

$$\begin{aligned} \mathrm{LT}_>(f) &= \mathrm{LT}_>(q_{i_1} \cdot p_{i_1}) + \ldots + \mathrm{LT}_>(q_{i_l} \cdot p_{i_l}) \\ &= \mathrm{LT}_>(q_{i_1}) \cdot p_{i_1} + \ldots + \mathrm{LT}_>(q_{i_l}) \cdot p_{i_l} \in J. \end{aligned}$$

Moreover, this also means $f - \mathrm{LT}_>(f) \in J$ and we can continue this process leading term by leading term to see that every term of f lies in J. □

As an immediate consequence, we get:

Lemma 3.1.3 *Let $J \trianglelefteq R[t,x]$ be an ideal generated by terms and let $>$ be a t-local monomial ordering on $\mathrm{Mon}(t,x)$. Let $f \in R[t,x]$ such that $u \cdot f \in J$ for some $u \in R[t,x]$ with $\mathrm{LT}_>(u) = 1$. Then $f \in J$.*

Proof. Lemma 3.1.2 implies $\mathrm{LT}_>(u \cdot f) = \mathrm{LT}_>(f) \in J$. Moreover, this implies $u \cdot \mathrm{LT}_>(f) \in J$ and therefore we also obtain

$$\mathrm{LT}_>(f - \mathrm{LT}_>(f)) = \mathrm{LT}_>(u \cdot (f - \mathrm{LT}_>(f))) = \mathrm{LT}_>(u \cdot f - u \cdot \mathrm{LT}_>(f)) \in J.$$

We can again continue this process leading term by leading term to see that every term of f lies in J. In particular, because f consists of only finitely many terms, $f \in J$. □

The next is another generalization of a statement from the classical Gröbner basis theory. The proof of is an adaptation of the proof of Theorem 4.1 in [**CLO05**].

Proposition 3.1.4 *Any x-homogeneous ideal $I \trianglelefteq R[\![t]\!][x]$ has only finitely many possible leading ideals.*

Proof. Observe that any element $g \in R[\![t]\!][x]$ has only finitely many possible leading terms, since there are only finitely many distinct monomials in x and a leading term with respect to a t-local monomial ordering has to have minimal power in t.

Now assume there are infinitely many leading ideals. For each leading ideal J, let $>_J$ be a t-local monomial ordering such that $\mathrm{LT}_{>_J}(I) = J$. Set $\Delta_0 := \{>_J \mid J \text{ leading ideal of } I\}$, so that different orderings in Δ_0 yield different leading ideals. By our assumption, Δ_0 is infinite.

Let $G_1 \subseteq I$ be a finite x-homogeneous generating set of I and set Σ_1 to be the union of all potential leading terms of elements of G_1. Then Σ_1 is finite and hence, by the pigeonhole principle, there must be infinitely many monomial orderings $\Delta_1 \subseteq \Delta_0$ which agree on Σ_1. Corollary 2.2.9 now implies that if $G_1 \subseteq I$ were a standard basis for one of them, it would be a standard basis for all of them. As this cannot be the case, given an ordering $>_1 \in \Delta_1$ there must be an element $g_2 \in I$ such that $\mathrm{LT}_{>_1}(g_2) \notin J_1 := \langle \mathrm{LT}_{>_1}(g) \mid g \in G_1 \rangle$ with J_1 being independent from the ordering chosen.

Since I is x-homogeneous, we may choose g_2 to be x-homogeneous. Moreover, by computing a determinate division with remainder with respect to

G_1 and $>_1$, we may assume that no term of g_2 lies in J_1, see condition (DD2) in Algorithm 2.1.16. In particular,

$$\mathrm{LT}_>(g_2) \notin J_1 := \langle \mathrm{LT}_>(g) \mid g \in G_1 \rangle \text{ for any ordering } > \in \Delta_1.$$

Setting $G_2 := G_1 \cup \{g_2\}$, we can repeat the entire process, and find an infinite subset of monomial orderings $\Delta_2 \subseteq \Delta_1$ such that G_2 is either a standard basis for all of them or for none of them. Consequently, there is a $g_3 \in I$ such that $\mathrm{LT}_>(g_3) \notin J_2 := \langle \mathrm{LT}_>(g) \mid g \in G_2 \rangle$ for all monomial orderings $> \in \Delta_2$. We thus obtain an infinite chain of strictly ascending ideals $J_1 \subsetneq J_2 \subsetneq \ldots$, which contradicts the ascending chain condition of our noetherian ring $R[\![t]\!][x]$. □

With this out of the way, the remaining section is now dedicated to the construction of the Gröbner fan, as roughly outlined in Figure 1.

Definition 3.1.5 Recalling the definition of initial ideals in Definition 1.2.7, the initial ideals of I define an equivalence relation on the space of weight vectors $\mathbb{R}_{<0} \times \mathbb{R}^n$:

$$w \sim v \quad :\Longleftrightarrow \quad \mathrm{in}_w(I) = \mathrm{in}_v(I).$$

We denote the closure the equivalence class of a weight vector $w \in \mathbb{R}_{<0} \times \mathbb{R}^n$ in the euclidean topology with

$$C_w(I) := \overline{\{v \in \mathbb{R}_{<0} \times \mathbb{R}^n \mid \mathrm{in}_v(I) = \mathrm{in}_w(I)\}} \subseteq \mathbb{R}_{\leq 0} \times \mathbb{R}^n,$$

and call it an *interior Gröbner cone* of I. We then call the intersection of the closure with the boundary,

$$C_w^0(I) := C_w(I) \cap (\{0\} \times \mathbb{R}^n),$$

a *boundary Gröbner cone* of I, and we refer to the collection

$$\Sigma(I) := \{C_w(I) \mid w \in \mathbb{R}_{<0} \times \mathbb{R}^n\} \cup \{C_w^0(I) \mid w \in \mathbb{R}_{<0} \times \mathbb{R}^n\},$$

of all cones as the *Gröbner fan* of I.

Moreover, given any t-local monomial ordering $>$, we set

$$C_>(I) := \overline{\{v \in \mathbb{R}_{<0} \times \mathbb{R}^n \mid \mathrm{in}_v(I) = \mathrm{LT}_>(I)\}} \subseteq \mathbb{R}_{\leq 0} \times \mathbb{R}^n.$$

Example 3.1.6 Consider the principal ideal

$$I = \langle \underbrace{tx^2 + xy + ty^2}_{=:g} \rangle \trianglelefteq \mathbb{Z}[\![t]\!][x,y].$$

Because $\mathrm{in}_w(I) = \langle \mathrm{in}_w(g) \rangle$ for any $w \in \mathbb{R}_{<0} \times \mathbb{R}^2$ and g is (x,y)-homogenous, it is easy to see that every Gröbner cone of I is invariant under translation by $(0,1,1)$. Its Gröbner fan divides the weight space $\mathbb{R}_{\leq 0} \times \mathbb{R}^2$ into three distinct maximal Gröbner cones, see Figure 2. Note that two red maximal cones intersect each other solely in the boundary $\{0\} \times \mathbb{R}^2$, while the third maximal cone intersects the boundary in codimension 2.

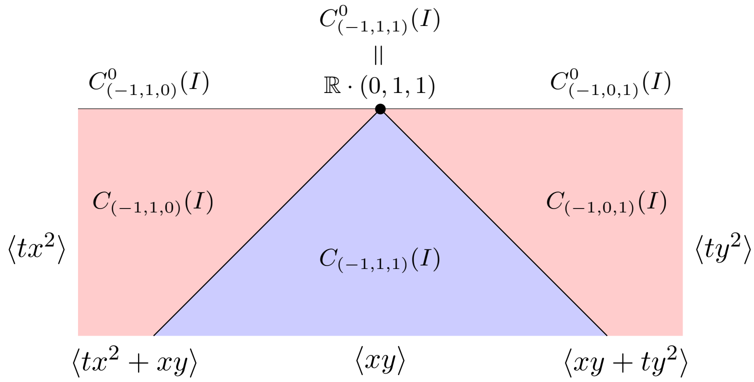


FIGURE 2. $\Sigma(\langle tx^2+xy+ty^2\rangle)$

Definition 3.1.7 Let $G,H\subseteq R[\![t]\!][x]$ be two finite subsets, and suppose $G=\{g_1,\dots,g_k\}$ with $g_i=\sum_{\alpha\in\mathbb{N}^n}g_{i,\alpha}\cdot x^\alpha$, $g_{i,\alpha}\in R[\![t]\!]$.

Given a t-local monomial ordering $>$, we call G *initially reduced* with respect to H, if the following set is reduced with respect to H:

$$G':=\Big\{\underbrace{\sum_{\alpha\in\mathbb{N}}\mathrm{LT}_>(g_{i,\alpha})\cdot x^\alpha}_{=:g_i'}\;\Big|\;i=1,\dots,k\Big\},$$

i.e. no term of $\mathrm{tail}_>(g_i')$ lies in $\mathrm{LT}_>(H)$.

We call a standard basis G *initially reduced*, if it is minimal and initially reduced with respect to itself.

For now, note that every x-homogeneous ideal has an initially reduced standard basis. We will show how it can be computed in finite time, provided polynomial input, in the next section.

Proposition 3.1.8 *Let $I\trianglelefteq R[\![t]\!][x]$ be an x-homogeneous ideal. Then I has an initially reduced standard basis.*

PROOF. Since $I\trianglelefteq R[\![t]\!][x]$ is x-homogeneous, there exists a reduced standard basis G by Algorithm 2.2.21. In particular, G is initially reduced. $\square$

We will now use initially reduced standard bases to construct the Gröbner fan of an x-homogeneous ideal, or rather show that the definition above indeed yields a polyhedral fan.

Lemma 3.1.9 *Let $>$ be a t-local monomial ordering, G an initially reduced standard basis of I with respect to it. Then for all $w\in\mathbb{R}_{<0}\times\mathbb{R}^n$ we have*

$$\mathrm{in}_w(I)=\mathrm{LT}_>(I)\iff\forall g\in G:\mathrm{in}_w(g)=\mathrm{LT}_>(g).$$

PROOF. $\Rightarrow$: Let $g\in G$. Then $\mathrm{in}_w(g)\in\mathrm{in}_w(I)=\mathrm{LT}_>(I)$. Writing $g=\sum_{\alpha\in\mathbb{N}^n}g_\alpha\cdot x^\alpha$ with $g_\alpha\in R[\![t]\!]$, note that the only terms of g which can occur in $\mathrm{in}_w(g)$ are of the form $\mathrm{LT}_>(g_\alpha)\cdot x^\alpha$ for some $\alpha\in\mathbb{N}^n$. And since our leading ideal is naturally generated by terms, these terms of $\mathrm{in}_w(g)$ also lie

in $\mathrm{LT}_>(I)$ by Lemma 3.1.2. Because G is initially reduced, we see that the only term of g which can occur in $\mathrm{in}_w(g)$ is $\mathrm{LT}_>(g)$, i.e. $\mathrm{in}_w(g) = \mathrm{LT}_>(g)$.
$\Leftarrow$: It is clear that $\mathrm{in}_w(I) \supseteq \mathrm{LT}_>(I)$. For the converse, it suffices to show $\mathrm{in}_w(f) \in \mathrm{LT}_>(I)$ for all $f \in I$. For that, consider the weighted ordering $>_w$ with weight vector w and tiebreaker $>$, and note that G is also a standard basis with respect to that ordering. Hence any $f \in I$ will have a weak division with remainder 0 with respect to G and $>_w$:

$$u \cdot f = q_1 \cdot g_1 + \ldots + q_k \cdot g_k.$$

The weighted monomial ordering ensures, that there is no cancellation of highest weighted degree terms on the right hand side, and that 1 is amongst the highest weighted degree terms in u. Taking the initial form with respect to w on both sides then yields:

$$\begin{aligned}\mathrm{in}_w(u) \cdot \mathrm{in}_w(f) &= \mathrm{in}_w(q_{i_1}) \cdot \mathrm{in}_w(g_{i_1}) + \ldots + \mathrm{in}_w(q_{i_l}) \cdot \mathrm{in}_w(g_{i_l}) \\ &= \mathrm{in}_w(q_{i_1}) \cdot \mathrm{LT}_>(g_{i_1}) + \ldots + \mathrm{in}_w(q_{i_l}) \cdot \mathrm{LT}_>(g_{i_l}) \in \mathrm{LT}_>(I)\end{aligned}$$

for the $1 \leq i_1 < \ldots < i_l \leq k$ whose terms contribute to the highest weighted degree. Now since $\mathrm{LT}_>(I)$ is generated by terms, any term of $\mathrm{in}_w(u) \cdot \mathrm{in}_w(f)$ is contained in it. In particular, that means $\mathrm{in}_w(f) \in \mathrm{LT}_>(I)$ by Lemma 3.1.3. □

Example 3.1.10 Consider the ideal

$$\langle \underbrace{x - t^3x + t^3z - t^4z}_{=:g_1}, \underbrace{y - t^3y + t^2z - t^4z}_{=:g_2} \rangle \trianglelefteq \mathbb{Z}[\![t]\!][x, y, z]$$

and the weighted ordering $> = >_v$ on $\mathrm{Mon}(t, x, y, z)$ with weight vector $v = (-1, 3, 3, 3) \in \mathbb{R}_{<0} \times \mathbb{R}^3$ and t-local lexicographical ordering $x > y > 1 > t$ as tiebreaker.

Since g_1 and g_2 already form an initially reduced standard basis, the set whose euclidean closure yields $C_>(I)$ is, by our previous Lemma 3.1.9, given by

$$\begin{aligned}&\{w \in \mathbb{R}_{<0} \times \mathbb{R}^3 \mid \mathrm{in}_w(I) = \mathrm{LT}_>(I)\} = \\ &\qquad \{w \in \mathbb{R}_{<0} \times \mathbb{R}^3 \mid \mathrm{in}_w(g_1) = \mathrm{LT}_>(g_1) = x \text{ and } \mathrm{in}_w(g_2) = \mathrm{LT}_>(g_2) = y\}.\end{aligned}$$

Hence it is cut out by the following two systems of inequalities:

$$\mathrm{in}_w(g_1) = x \iff \begin{cases} \deg_w(x) > \deg_w(t^3x) \\ \deg_w(x) > \deg_w(t^3z) \\ \deg_w(x) > \deg_w(t^4z) \end{cases} \iff \begin{cases} 0 > w_0 \\ w_1 > 3w_0 + w_3 \\ w_1 > 4w_0 + w_3 \end{cases}$$

and

$$\mathrm{in}_w(g_2) = y \iff \begin{cases} \deg_w(y) > \deg_w(t^3y) \\ \deg_w(y) > \deg_w(t^2z) \\ \deg_w(y) > \deg_w(t^4z) \end{cases} \iff \begin{cases} 0 > w_0 \\ w_2 > 2w_0 + w_3 \\ w_2 > 4w_0 + w_3. \end{cases}$$

Note that the third inequalities stemming from the terms t^4z in g_1 and g_2 are made redundant by the second inequalities because $w_0 < 0$ always. Similarly, the first inequalities coming from the terms t^3x in g_1 and t^3y in g_2 become redundant in the euclidean closure. This is why these terms may be ignored for our purposes in the initial reduction.

Observe that the set above is invariant under translation by $(0, 1, 1, 1)$. Figure 3 shows an image in which we restrict ourselves to the affine subspace $\{w_0 = -1, w_3 = 1\}$. Due to the invariance under translation, no information is lost by doing so.

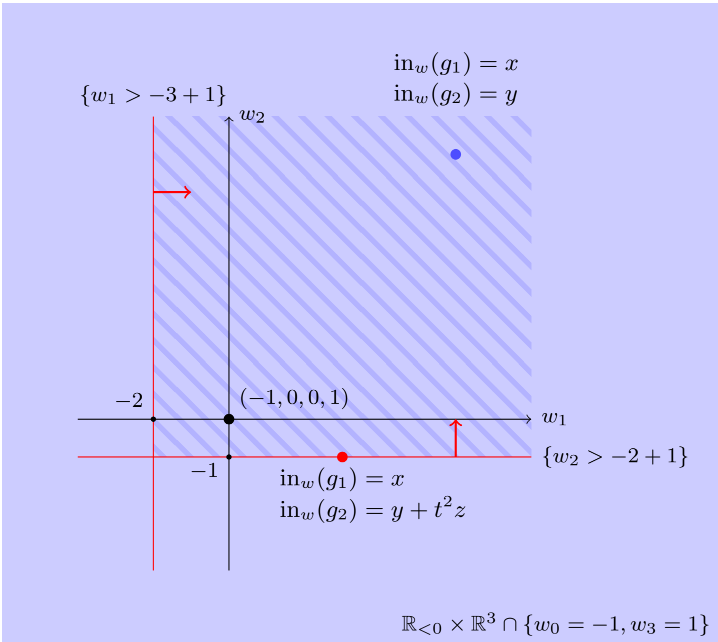


FIGURE 3. $C_>(I)$ having the structure of a polyhedral cone

Also note that while the weight vectors on the euclidean boundary may not induce initial forms of g_1 and g_2 coinciding to the leading terms, the initial forms still contain the leading terms. This is a direct consequence of our last lemma.

Lemma 3.1.11 *Let $>$ be a t-local monomial ordering and G an initially reduced standard basis of I with respect to it. Then for all $w \in \mathbb{R}_{<0} \times \mathbb{R}^n$ we have*

$$w \in C_>(I) \quad \Longleftrightarrow \quad \forall g \in G : \mathrm{LT}_>(\mathrm{in}_w(g)) = \mathrm{LT}_>(g).$$

PROOF. Suppose $G = \{g_1, \ldots, g_k\}$. As explained in Example 3.1.10, Lemma 3.1.9 implies that the set $\{w \in \mathbb{R}_{<0} \times \mathbb{R}^n \mid \mathrm{in}_w(I) = \mathrm{LT}_>(I)\}$ is cut out by a system of strict inequalities of the form:

$$\begin{array}{rcl} \deg_w(\mathrm{LT}_>(g_1)) & > & \deg_w(\mathrm{tail}_>(g_1)), \\ \deg_w(\mathrm{LT}_>(g_2)) & > & \deg_w(\mathrm{tail}_>(g_2)), \\ & \vdots & \\ \deg_w(\mathrm{LT}_>(g_k)) & > & \deg_w(\mathrm{tail}_>(g_k)). \end{array}$$

Note that each line, despite $g_i \in R[\![t]\!][x]$, only yields a finite amount of minimal inequalities, since higher degrees of t yield redundant inequalities. Therefore, its euclidean closure $C_>(I)$ is given by a system of inequalities of the form

$$\begin{array}{rcl} \deg_w(\mathrm{LT}_>(g_1)) & \geq & \deg_w(\mathrm{tail}_>(g_1)) \\ \deg_w(\mathrm{LT}_>(g_2)) & \geq & \deg_w(\mathrm{tail}_>(g_2)) \\ & \vdots & \\ \deg_w(\mathrm{LT}_>(g_k)) & \geq & \deg_w(\mathrm{tail}_>(g_k)) \end{array}$$

which is equivalent to $\mathrm{LT}_>(g_i)$ occuring in $\mathrm{in}_w(g)$ and translates to the condition in the claim. $\square$

This allows us to prove the final lemma necessary for our statement about the standard bases of initial ideals.

Lemma 3.1.12 *Let $>$ be a t-local monomial ordering. Then for all $w \in C_>(I)$, $w \in \mathbb{R}_{<0} \times \mathbb{R}^n$, we have*

$$\mathrm{LT}_>(\mathrm{in}_w(I)) = \mathrm{LT}_>(I).$$

PROOF. Let G be an initially reduced standard basis of I with respect to $>$. Since $\mathrm{LT}_>(\mathrm{in}_w(g)) = \mathrm{LT}_>(g)$ for all $g \in G$ by Lemma 3.1.11, we have

$$\mathrm{LT}_>(I) = \langle \mathrm{LT}_>(g) \mid g \in G \rangle \underset{3.1.11}{\overset{\text{Lem.}}{=}} \langle \mathrm{LT}_>(\mathrm{in}_w(g)) \mid g \in G \rangle \subseteq \mathrm{LT}_>(\mathrm{in}_w(I)).$$

For the opposite inclusion, we can again consider the weighted ordering $>_w$. Any $h \in \mathrm{in}_w(I)$ with $h = \mathrm{in}_w(f)$ for some $f \in I$ has a weak division with remainder 0 with respect to $G = \{g_1, \ldots, g_k\}$ and that ordering:

$$u \cdot f = q_1 \cdot g_1 + \ldots + q_k \cdot g_k.$$

Because no cancellation of highest weighted degree terms occurs on the right, taking the initial forms on both sides yields:

$$\mathrm{in}_w(u) \cdot \mathrm{in}_w(f) = \mathrm{in}_w(q_{i_1}) \cdot \mathrm{in}_w(g_{i_1}) + \ldots + \mathrm{in}_w(q_{i_l}) \cdot \mathrm{in}_w(g_{i_l})$$

for the $1 \leq i_1 < \ldots < i_l \leq k$ whose terms contribute to the highest weighted degree. Moreover, $\mathrm{LT}_>(\mathrm{in}_w(u)) = \mathrm{LT}_{>_w}(u) = 1$. Therefore taking the leading terms on both sides produces:

$$\begin{array}{rcl} \mathrm{LT}_>(\mathrm{in}_w(f)) & = & q'_{i_1} \cdot \mathrm{LT}_>(\mathrm{in}_w(g_{i_1})) + \ldots + q'_{i_l} \cdot \mathrm{LT}_>(\mathrm{in}_w(g_{i_l})) \\ & \underset{3.1.11}{\overset{\text{Lem.}}{=}} & q'_{i_1} \cdot \mathrm{LT}_>(g_{i_1}) + \ldots + q'_{i_l} \cdot \mathrm{LT}_>(g_{i_l}) \in \mathrm{LT}_>(I), \end{array}$$

where we abbreviated $q'_{i_j} := \mathrm{LT}_>(\mathrm{in}_w(q_{i_j}))$ for $j = 1, \ldots, l$. $\square$

Proposition 3.1.13 *Let $>$ be a t-local monomial ordering and let G be an initially reduced standard basis of I with respect to it. Then for all $w \in C_>(I)$, $w \in \mathbb{R}_{<0} \times \mathbb{R}^n$, the set*

$$H := \{\mathrm{in}_w(g) \mid g \in G\}$$

is an initially reduced standard basis of $\mathrm{in}_w(I)$ with respect to the same ordering.

Proof. By the previous Lemmata, we have

$$\mathrm{LT}_>(\mathrm{in}_w(I)) \underset{3.1.12}{\overset{\text{Lem.}}{=}} \mathrm{LT}_>(I) = \langle \mathrm{LT}_>(g) \mid g \in G\rangle \underset{3.1.11}{\overset{\text{Lem.}}{=}} \langle \mathrm{LT}_>(\mathrm{in}_w(g)) \mid g \in G\rangle,$$

and therefore H is a standard basis of $\mathrm{in}_w(I)$. Moreover, because G was initially reduced, so is H. $\square$

Example 3.1.14 Given the same ideal and ordering as in Example 3.1.10, $g_1 = x - t^3x + t^3z - t^4z$ and $g_2 = y - t^3y + t^2z - t^4z$ form an initially reduced standard basis. Because $w := (-1, 2, -1, 1) \in C_>(I)$, Proposition 3.1.13 implies that the initial ideal with respect to it has the initially reduced standard bases

$$\{\mathrm{in}_w(g_1), \mathrm{in}_w(g_2)\} = \{x, y + t^2z\}.$$

As we go over all weight vectors in $C_>(I)$ in the affine subspace, we obtain four distinct initial ideals as illustrated in Figure 4.

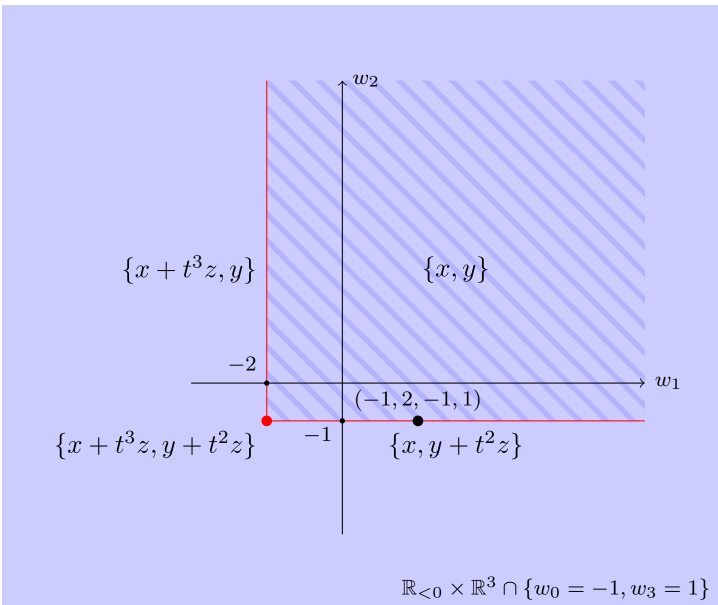


Figure 4. standard bases of initial ideals with various weights

The finiteness of distinct leading ideals holds in general and it can be stated as an easy corollary.

Remark 3.1.15 Let $J \trianglelefteq R[t,x]$ be an x-homogeneous ideal in the polynomial ring $R[t,x]$, and consider the ideal $I := J \cdot R[\![t]\!][x] \trianglelefteq R[\![t]\!][x]$ it generates in the mixed power series and polynomial ring $R[\![t]\!][x]$.

Any generating system $F \subseteq J$ of J also represents a generating system of $F \subseteq I$ of I, and if we use F to compute an initially reduced standard basis of I, no step of the calculation will leave $J \subseteq R[t,x]$, cf. Section 2.2. Because F is polynomial, we hence obtain a polynomial standard basis of I that is in particular still contained in J. Due to Proposition 3.1.13, this shows

$$\mathrm{in}_w(I) = \langle \mathrm{in}_w(g) \mid g \in G \rangle = \langle \mathrm{in}_w(f) \mid f \in J \rangle =: \mathrm{in}_w(J),$$

i.e. the initial ideal of I is generated by the initial forms of elements in J. We will make frequent use of this fact hereinafter, when we consider initial ideals of the form $\mathrm{in}_v(\mathrm{in}_w(I))$, as in Proposition 3.3.4 and Definition 4.1.6. Note also, that we can compute initial forms of polynomials with respect to arbitrary weight vectors in R^{n+1}.

Corollary 3.1.16 *There are only finitely many distinct initial ideals of I. In particular, there are only finitely many Gröbner cones of I.*

Proof. Note that an arbitrary element $g = \sum_{\alpha \in \mathbb{N}^n} g_\alpha x^\alpha \in R[\![t]\!][x]$ with $g_\alpha \in R[\![t]\!]$ has only finitely many distinct initial forms. Consider a weight vector $w = (w_0, w_1, \ldots, w_n) \in \mathbb{R}_{<0} \times \mathbb{R}^n$, and let $>$ be a t-local monomial ordering. The initial forms of g with respect to it are of the form

$$\mathrm{in}_w(g) = \sum_{\alpha \in \Lambda} \mathrm{LT}_>(g_\alpha) \cdot x^\alpha$$

for the finite set $\Lambda \subseteq \{\alpha \in \mathbb{N}^n \mid g_\alpha \neq 0\}$.

Now since there are only finitely many leading ideals by Proposition 3.1.4, Proposition 3.1.13 thus implies that there can only be finitely many initial ideals of I. Thus the number of cones in $\Sigma(I)$ of the form $C_w(I)$ and hence also of the form $C_w^0(I)$ is finite. □

The next proposition allows us to read off the inequalities and equations of the Gröbner cones, from which we can derive the remaining properties needed to show that they form a polyhedral fan.

Proposition 3.1.17 *Let $>$ be a t-local monomial ordering, G an initially reduced standard basis of I with respect to it and $w \in C_>(I)$, $w \in \mathbb{R}_{<0} \times \mathbb{R}^n$. Then for all $v \in \mathbb{R}_{<0} \times \mathbb{R}^n$ we have*

$$\mathrm{in}_v(I) = \mathrm{in}_w(I) \quad \Longleftrightarrow \quad \forall g \in G : \mathrm{in}_v(g) = \mathrm{in}_w(g).$$

Proof. $\Leftarrow$: Note that

$$\mathrm{LT}_>(\mathrm{in}_v(g)) = \mathrm{LT}_>(\mathrm{in}_w(g)) \overset{\text{Lem.}}{\underset{3.1.11}{=}} \mathrm{LT}_>(g),$$

thus $v \in C_>(I)$, again by Lemma 3.1.11. This allows us to use Proposition 3.1.13, which says that $\mathrm{in}_w(I)$ and $\mathrm{in}_v(I)$ share a common standard basis, therefore they must coincide.

$\Rightarrow$: Let $g \in G$. On the one hand, Lemma 3.1.11 implies that $\mathrm{LT}_>(g)$ is a term of $\mathrm{in}_w(g)$. On the other hand,

$$\mathrm{LT}_>(\mathrm{in}_v(g)) \in \mathrm{LT}_>(\mathrm{in}_v(I)) = \mathrm{LT}_>(\mathrm{in}_w(I)) \overset{\text{Lem.}}{\underset{3.1.12}{=}} \mathrm{LT}_>(I).$$

But because G is initially reduced, the only term of g occurring in $\mathrm{in}_v(g)$ and $\mathrm{LT}_>(I)$ is $\mathrm{LT}_>(g)$. Thus $\mathrm{LT}_>(g)$ is also a term of $\mathrm{in}_v(g)$.

Now consider $\mathrm{in}_w(g) - \mathrm{in}_v(g) \in \mathrm{in}_w(I) = \mathrm{in}_v(I)$. Our previous arguments show that $\mathrm{LT}_>(\mathrm{in}_w(g) - \mathrm{in}_v(g)) \neq \mathrm{LT}_>(g)$. However, because

$$\mathrm{LT}_>(\mathrm{in}_w(g) - \mathrm{in}_v(g)) \in \mathrm{LT}_>(\mathrm{in}_w(I)) \overset{\text{Lem.}}{\underset{3.1.12}{=}} \mathrm{LT}_>(I),$$

it represents another term of $\mathrm{in}_w(g)$ or $\mathrm{in}_v(g)$ in $\mathrm{LT}_>(I)$, which can only be 0. $\square$

Example 3.1.18 Consider the same ideal and ordering as in Example 3.1.10 and Example 3.1.14, where $g_1 = x - t^3x + t^3z - t^4z$ and $g_2 = y - t^3y + t^2z - t^4z$ form an initially reduced standard basis.

For $w = (-1, 2, -1, 1) \in C_>(I) \cap \mathbb{R}_{<0} \times \mathbb{R}^3$ we have by Proposition 3.1.17:

$$\mathrm{in}_{w'}(I) = \langle x, y + t^2z \rangle \quad \Longleftrightarrow \quad \begin{cases} \mathrm{in}_{w'}(g_1) = x, \\ \mathrm{in}_{w'}(g_2) = y + t^2z. \end{cases}$$

Therefore, its equivalence class of weight vectors $w' \in \mathbb{R}_{<0} \times \mathbb{R}^3$ such that $\mathrm{in}_{w'}(I) = \mathrm{in}_w(I)$ is determined by the following system of inequalities and equations:

$$\mathrm{in}_{w'}(g_1) = x \quad \Longleftrightarrow \quad \begin{cases} \deg_{w'}(x) > \deg_{w'}(t^3x) \\ \deg_{w'}(x) > \deg_{w'}(t^3z) \\ \deg_{w'}(x) > \deg_{w'}(t^4z) \end{cases} \quad \Longleftrightarrow \quad \begin{cases} 0 > w'_0 \\ w'_1 > 3w'_0 + w'_3 \\ w'_1 > 4w'_0 + w'_3 \end{cases}$$

$$\mathrm{in}_w(g_2) = y + t^2z \quad \Longleftrightarrow \quad \begin{cases} \deg_{w'}(y) > \deg_{w'}(t^3y) \\ \deg_{w'}(y) = \deg_{w'}(t^2z) \\ \deg_{w'}(y) > \deg_{w'}(t^4z) \end{cases} \quad \Longleftrightarrow \quad \begin{cases} 0 > w'_0 \\ w'_2 = 2w'_0 + w'_3 \\ w'_2 > 4w'_0 + w'_3 \end{cases}$$

In particular, its euclidean closure, the Gröbner cone $C_w(I)$, is the face of $C_>(I)$ cut out by the hyperplane $\{w'_2 = 2w'_0 + w'_3\}$.

In fact, Proposition 3.1.17 implies that $C_>(I)$ is stratified by equivalence classes of weight vectors as Figure 4 already suggested. Each equivalence class is an open polyhedral cone whose euclidean closure yields a face of $C_>(I)$.

Proposition 3.1.19 *The Gröbner cones $C_w(I)$ and $C_w^0(I)$, $w \in \mathbb{R}_{<0} \times \mathbb{R}^n$, are closed polyhedral cones.*

Proof. Let $>$ be a t-local weighted monomial ordering with respect to weight vector w, and let G be an initially reduced standard basis of I with respect to it.

Suppose $G = \{g_1, \ldots, g_k\}$ with $g_i = \sum_{\beta,\alpha} c_{\alpha,\beta,i} \cdot t^\beta x^\alpha$. Let Λ_i be the finite set of exponent vectors with minimal entry in t,

$$\Lambda_i := \left\{ (\beta, \alpha) \in \mathbb{N} \times \mathbb{N}^n \;\middle|\; \alpha \in \mathbb{N}^n, \beta = \min\{\beta' \in \mathbb{N} \mid c_{\alpha,\beta',i} \neq 0\} \right\}.$$

As demonstrated in Example 3.1.18, Proposition 3.1.17 implies that the equivalence class of w, $\{v \in \mathbb{R}_{<0} \times \mathbb{R}^n \mid \mathrm{in}_v(I) = \mathrm{in}_w(I)\}$, is cut out by a system of inequalities and equations

$$\begin{aligned} v \cdot (\beta, \alpha) > v \cdot (\delta, \gamma), &\qquad \text{for all } (\beta, \alpha), (\delta, \gamma) \in \Lambda_i \text{ with } w \cdot (\beta, \alpha) > w \cdot (\delta, \gamma), \\ v \cdot (\beta, \alpha) = v \cdot (\delta, \gamma), &\qquad \text{for all } (\beta, \alpha), (\delta, \gamma) \in \Lambda_i \text{ with } w \cdot (\beta, \alpha) = w \cdot (\delta, \gamma). \end{aligned}$$

Therefore, the equivalence class forms a relative open polyhedral cone contained in the open lower half space $\mathbb{R}_{<0} \times \mathbb{R}^n$ and its closure $C_w(I)$ yields a closed polyhedral cone in the closed lower half space $\mathbb{R}_{\leq 0} \times \mathbb{R}^n$. In particular, $C_w^0(I) = C_w(I) \cap (\{0\} \times \mathbb{R}^n)$ is also a closed polyhedral cone. $\square$

Corollary 3.1.20 *Let $w \in \mathbb{R}_{<0} \times \mathbb{R}^n$. Then any face $\tau \leq C_w(I)$ with $\tau \not\subseteq \{0\} \times \mathbb{R}^n$ coincides with the closure of the equivalence class of any weight vector in its relative interior.*

In particular, each face $\tau \leq C_w(I)$ is of the form $\tau = C_v(I)$ or $\tau = C_v^0(I)$ for some $v \in C_w(I)$ and each face $\tau \leq C_w^0(I)$ is of the form $\tau = C_v^0(I)$ for some $v \in C_w(I)$.

Proof. Consider again the system of inequalities and equations that are determined by $\Lambda_1, \ldots, \Lambda_k$ and cut out $C_w(I)$ in the proof of the previous Proposition 3.1.19, which we obtained from the exponent vectors of an initially reduced standard basis with respect to a weighted ordering $>_w$.

A face $\tau \leq C_w(I)$ is cut out by supporting hyperplanes, on which some of the inequalities above become equations. Assuming that there are weights $v \in \tau \cap (\mathbb{R}_{<0} \times \mathbb{R}^n)$, all weight vectors in the relative interior yield the same initial forms on $g_1, \ldots, g_k \in G$, since they satisfy the same equations and inequalities determined by $\Lambda_1, \ldots, \Lambda_k$. This implies that they belong to the same equivalence class whose closure is then τ. In particular, $\tau = C_v(I)$.

And any face $\tau \leq C_w^0(I) \leq C_w(I)$ can be cut out by a supporting hyperplane which also cuts out a face $C_v(I) \leq C_w(I)$. It is then clear that $\tau = C_v^0(I)$. $\square$

Proposition 3.1.21 *Let $C_u(I)$ and $C_v(I)$ be two interior Gröbner cones such that $C_u(I) \cap C_v(I) \not\subseteq \{0\} \times \mathbb{R}^n$. Then $C_u(I) \cap C_v(I) = C_w(I)$ for some $w \in \mathbb{R}_{<0} \times \mathbb{R}^n$, and $C_w(I)$ is a face of both $C_u(I)$ and $C_v(I)$.*

PROOF. By Proposition 3.1.19, both $C_u(I) \cap (\mathbb{R}_{<0} \times \mathbb{R}^n)$ and $C_v(I) \cap (\mathbb{R}_{<0} \times \mathbb{R}^n)$ can be decomposed into a union of equivalence classes, and hence so can $(C_u(I) \cap C_v(I)) \cap (\mathbb{R}_{<0} \times \mathbb{R}^n) \neq \emptyset$.

Suppose $\dim(C_u(I) \cap C_v(I)) = k$. Then the intersection contains exactly one equivalence class of dimension k. If there were none, then the intersection would be covered by a collection of lower dimensional open cones of which there are, however, only finitely many by Corollary 3.1.16. If there were more than one, then that would contradict Proposition 3.1.19, which states that the closure of each equivalence class yields a distinct face of both $C_u(I)$ and $C_v(I)$, and no two k-dimensional faces of a polyhedral cone may be cut out by the same k-dimensional supporting hylperplane.

So let w be in the maximal equivalence class in $C_u(I) \cap C_v(I)$. Taking the euclidean closure, we necessarily have $C_w(I) = C_u(I) \cap C_v(I)$, and, by Corollary 3.1.19, it is a face of both $C_u(I)$ and $C_v(I)$. $\square$

Note that the proposition above falls a bit short in proving that the intersection of two Gröbner cones yields a face of both, as it only covers Gröbner cones with an intersection in the open part of the lower halfspace. Nevertheless, it is has big implications on the interior of the Gröbner fan, which is the relevant part for the tropical varieties, as well as the boundary of it.

Definition 3.1.22 For an interior Gröbner cone $C_w(I)$, $w \in \mathbb{R}_{<0} \times \mathbb{R}^n$, let $C_w^{-1}(I)$ denote the intersection

$$C_w^{-1}(I) := C_w(I) \cap (\{-1\} \times \mathbb{R}^n).$$

It is a polytope whose *recession cone* is defined to be the set of all weight vectors in $\mathbb{R}_{\leq 0} \times \mathbb{R}^n$ under whose translation it is closed,

$$\operatorname{rec}(C_w^{-1}(I)) := \{v \in \mathbb{R}_{\leq 0} \times \mathbb{R}^n \mid v + C_w^{-1}(I) \subseteq C_w^{-1}(I)\}.$$

Note that $C_w^{-1}(I) \subseteq \{-1\} \times \mathbb{R}^n$ necessarily implies $\operatorname{rec}(C_w^{-1}(I)) \subseteq \{0\} \times \mathbb{R}^n$.

Proposition 3.1.23

(1) The collection

$$\Sigma(I) \cap (\{-1\} \times \mathbb{R}^n) = \{C_w^{-1}(I) \mid w \in \mathbb{R}_{<0} \times \mathbb{R}^n\}$$

is a polyhedral complex whose support is the affine hyperplane $\{-1\} \times \mathbb{R}^n$.

(2) For any weight vector $w \in \mathbb{R}_{<0} \times \mathbb{R}^n$, $C_w^0(I) = \operatorname{rec}(C_w^{-1}(I))$.

(3) The collection

$$\Sigma(I) \cap (\{0\} \times \mathbb{R}^n) = \{C_w^0(I) \mid w \in \mathbb{R}_{<0} \times \mathbb{R}^n\}$$

is a polyhedral fan whose support is the boundary hylperplane $\{0\} \times \mathbb{R}^n$.

PROOF. (1) follows from Proposition 3.1.19, Corollary 3.1.20 and Proposition 3.1.21. (2) is clear, and (3) follows from Corollary 3.10 in [**BGS11**]. $\square$

We can now complete the missing intersections in Proposition 3.1.21.

Corollary 3.1.24 *Consider two weight vectors $u, v \in \mathbb{R}_{<0} \times \mathbb{R}^n$. Then*

$$C_u^0(I) \cap C_v^0(I) = C_w^0(I) \textit{ for some } w \in \mathbb{R}_{<0} \times \mathbb{R}^n,$$
$$C_u^0(I) \cap C_v(I) = C_w^0(I) \textit{ for some } w \in \mathbb{R}_{<0} \times \mathbb{R}^n.$$

Proof. Since the number boundary Gröbner cones form a polyhedral fan by Proposition 3.1.23, the intersection $C_u^0(I) \cap C_v^0(I)$ is a face of both. In particular, by Corollary 3.1.20, there is a weight vector $w \in \mathbb{R}_{<0} \times \mathbb{R}^n$ with

$$C_u^0(I) \cap C_v^0(I) = C_w^0(I).$$

And for the intersection of a boundary Gröbner cone and an interior Gröbner cone, note that

$$C_u^0(I) \cap C_v(I) = C_u^0(I) \cap C_v^0(I) = C_w^0(I). \qquad \square$$

We therefore have shown:

Theorem 3.1.25 *Let $I \trianglelefteq R[\![t]\!][x]$ be an x-homogeneous ideal, then the Gröbner fan*

$$\Sigma(I) = \{C_w(I) \mid w \in \mathbb{R}_{<0} \times \mathbb{R}^n\} \cup \{C_w^0(I) \mid w \in \mathbb{R}_{<0} \times \mathbb{R}^n\}$$

is a polyhedral fan with support $\mathbb{R}_{\leq 0} \times \mathbb{R}^n$.

Proof. Proposition 3.1.19 shows that each Gröbner cone is a polyhedral cone, while Corollary 3.1.20 proves that the face of a Gröbner cone is again a Gröbner cone. Proposition 3.1.21 and Corollary 3.1.24 infer that the intersection of two Gröbner cones is again a Gröbner cone, and Corollary 3.1.16 shows that there are only finitely many of them. $\square$

Example 3.1.26 Consider the following ideal generated by polynomials

$$\langle 2x + 2y, t + 2 \rangle \trianglelefteq \mathbb{Z}[\![t]\!][x, y].$$

Now because the ideal is generated by elements in $\mathbb{Z}[t, x, y]$, one might be tempted to believe that restricting ourselves to the polynomial ideal

$$\langle 2x + 2y, t + 2 \rangle \trianglelefteq \mathbb{Z}[t, x, y],$$

might allow us to work with weight vectors $\mathbb{R}_{\geq 0} \times \mathbb{R}^2$ with positive weight in t, obtain similar results about the existance of a Gröbner fan there and patch the two Gröbner fans in $\mathbb{R}_{\leq 0} \times \mathbb{R}^2$ and in $\mathbb{R}_{\geq 0} \times \mathbb{R}^2$ together.

While the existence of a Gröbner fan in the positive halfspace is true for our specific example, note that the two Gröbner fans cannot be glued together to a polyhedral fan on $\mathbb{R} \times \mathbb{R}^2$, as illustrated in Figure 5.

As demonstrated in Example 3.1.18 and used in the proof of Proposition 3.1.19, Proposition 3.1.17 allows us to read of the inequalities and equations of a Gröbner cone from an initially reduced standard basis. This can be done as described in the following algorithm.

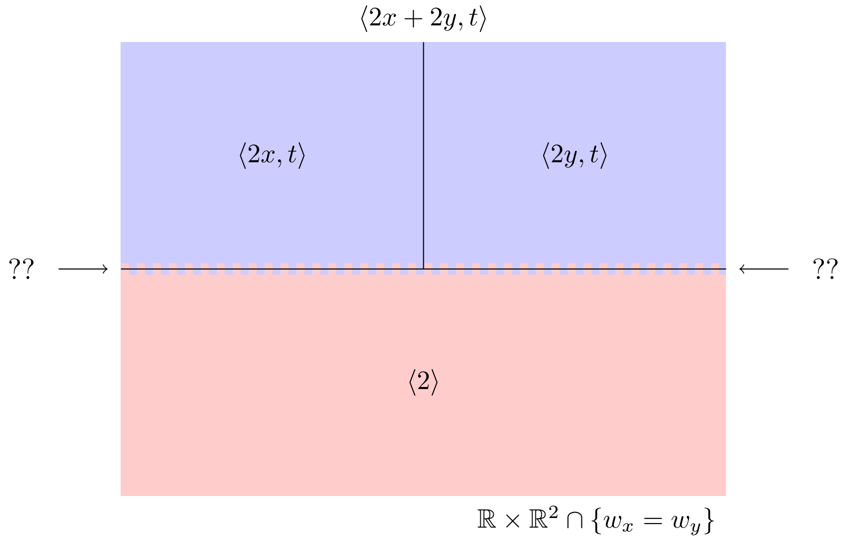


FIGURE 5. $\Sigma(\langle 2x+2y, t+2\rangle)$ on $\mathbb{R}\times\mathbb{R}^2$...?

Algorithm 3.1.27 Inequalities and equations of a Gröbner cone

Input: (G, w), where $G = \{g_1, \dots, g_k\} \subseteq R[\![t]\!][x]$ an initially reduced standard basis of an x-homogeneous ideal I with respect to a t-local monomial ordering $>$ such that $w \in C_>(I) \cap (\mathbb{R}_{<0} \times \mathbb{R}^n)$.

Output: (A, B), a pair of matrices

$$A \in \mathrm{Mat}(l_A \times (n+1), \mathbb{R}), \quad B \in \mathrm{Mat}(l_B \times (n+1), \mathbb{R})$$

such that

$$C_w(I) = \{v \in \mathbb{R}_{\leq 0} \times \mathbb{R}^n \mid A \cdot v \in (\mathbb{R}_{\geq 0})^{l_A} \text{ and } B \cdot v = 0 \in \mathbb{R}^{l_B}\}.$$

for $i = 1, \dots, k$ **do**
  Suppose $g_i = \sum_{\beta,\alpha} c_{\alpha,\beta,i} \cdot t^\beta x^\alpha$.
  Construct the set of exponent vectors with minimal entry in t,

$$\Lambda_i := \left\{(\beta, \alpha) \in \mathbb{N} \times \mathbb{N}^n \;\middle|\; \alpha \in \mathbb{N}^n, \beta = \min\{\beta' \in \mathbb{N} \mid c_{\alpha,\beta',i} \neq 0\}\right\}.$$

  Construct a set of vectors that will yield the inequalities,

$$\Omega_i := \{a - b \in \mathbb{R} \times \mathbb{R}^n \mid a, b \in \Lambda_i, \;\; a \cdot w > b \cdot w\}.$$

  Construct a set of vectors that will yield the equations,

$$\Theta_i := \{a - b \in \mathbb{R} \times \mathbb{R}^n \mid a, b \in \Lambda_i, \;\; a \cdot w = b \cdot w\}.$$

Let A be a matrix whose row vectors consists of $\bigcup_{i=1}^k \Omega_i$.
Let B be a matrix whose row vectors consists of $\bigcup_{i=1}^k \Theta_i$.
return (A, B).

Alternatively, it is possible to compute a Gröbner cone without knowing a predetermined weight w in it. This is important for computing initial ideals with respect to generic weights in Chapter 4, where we compute the initial ideal without computing a weight beforehand.

Algorithm 3.1.28 Inequalities and equations of a Gröbner cone

Input: $(H, G, >)$, where for an x-homogeneous ideal I and an undetermined weight vector $w \in \mathbb{R}_{<0} \times \mathbb{R}^n$

(1) $>$ a t-local monomial ordering such that $w \in C_>(I)$,
(2) $G = \{g_1, \ldots, g_k\}$ an initially reduced standard basis of I w.r.t. $>$,
(3) $H = \{h_1, \ldots, h_k\}$ with $h_i = \mathrm{in}_w(g_i)$.

Output: (A, B), a pair of matrices

$$A \in \mathrm{Mat}(l_A \times (n+1), \mathbb{R}), \quad B \in \mathrm{Mat}(l_B \times (n+1), \mathbb{R})$$

such that

$$C_w(I) = \{v \in \mathbb{R}_{\leq 0} \times \mathbb{R}^n \mid A \cdot v \in (\mathbb{R}_{\geq 0})^{l_A} \text{ and } B \cdot v = 0 \in \mathbb{R}^{l_B}\}.$$

for $i = 1, \ldots, k$ **do**
  Suppose $g_i = \sum_{\beta,\alpha} c_{\alpha,\beta,i} \cdot t^\beta x^\alpha$ and $\mathrm{LM}_>(g) = t^\delta x^\gamma$.
  Construct the set of exponent vectors with minimal entry in t,

$$\Lambda_i := \left\{(\beta, \alpha) \in \mathbb{N} \times \mathbb{N}^n \;\middle|\; \alpha \in \mathbb{N}^n, \beta = \min\{\beta' \in \mathbb{N} \mid c_{\alpha,\beta',i} \neq 0\}\right\}.$$

  Construct a set of vectors that will yield the inequalities,

$$\Omega_i := \{(\delta, \gamma) - (\alpha, \beta) \in \mathbb{R} \times \mathbb{R}^n \mid (\alpha, \beta) \in \Lambda_i, \;\; (\alpha, \beta) \neq (\delta, \gamma)\}.$$

Let A be a matrix whose row vectors consists of $\bigcup_{i=1}^k \Omega_i$.
for $i = 1, \ldots, k$ **do**
  Suppose $h_i = \sum_{\beta,\alpha} d_{\alpha,\beta,i} \cdot t^\beta x^\alpha$.
  Construct the set of exponent vectors with minimal entry in t,

$$\Lambda_i' := \left\{(\beta, \alpha) \in \mathbb{N} \times \mathbb{N}^n \;\middle|\; \alpha \in \mathbb{N}^n, \beta = \min\{\beta' \in \mathbb{N} \mid d_{\alpha,\beta',i} \neq 0\}\right\}.$$

  Construct a set of vectors that will yield the equations,

$$\Theta_i := \left\{a - b \in \mathbb{R} \times \mathbb{R}^n \mid a, b \in \Lambda_i'\right\}.$$

Let B be a matrix whose row vectors consists of $\bigcup_{i=1}^k \Theta_i$.
return (A, B).

In particular, for maximal Gröbner cones we obtain:

Algorithm 3.1.29 Inequalities of a maximal Gröbner cone

Input: $(G, >)$, where $G = \{g_1, \ldots, g_k\} \subseteq R[\![t]\!][x]$ an initially reduced standard basis of an x-homogeneous ideal I with respect to a t-local monomial ordering $>$.

Output: $A \in \mathrm{Mat}(l \times (n+1), \mathbb{R})$, a matrix such that

$$C_>(I) = \{v \in \mathbb{R}_{\leq 0} \times \mathbb{R}^n \mid A \cdot v \in (\mathbb{R}_{\geq 0})^l\}.$$

for $i = 1, \ldots, k$ **do**
  Suppose $g_i = \sum_{\beta,\alpha} c_{\alpha,\beta,i} \cdot t^\beta x^\alpha$ and $\mathrm{LM}_>(g) = t^\delta x^\gamma$.
  Construct the set of exponent vectors with minimal entry in t,

$$\Lambda_i := \left\{(\beta, \alpha) \in \mathbb{N} \times \mathbb{N}^n \;\middle|\; \alpha \in \mathbb{N}^n, \beta = \min\{\beta' \in \mathbb{N} \mid c_{\alpha,\beta',i} \neq 0\}\right\}.$$

Construct a set of vectors that will yield the inequalities,

$$\Omega_i := \{(\delta, \gamma) - (\alpha, \beta) \in \mathbb{R} \times \mathbb{R}^n \mid (\alpha, \beta) \in \Lambda_i, \ \ (\alpha, \beta) \neq (\delta, \gamma)\}.$$

Let A be a matrix whose row vectors consists of $\bigcup_{i=1}^{k} \Omega_i$.

return A

3.2. Initial Reduction

In this section, we present an algorithm for the initial reduction of an polynomial x-homogeneous standard basis in finite time. For the sake of simplicity, we will restrict ourselves to ideals obtained via Theorem 1.2.13, see Convention 3.2.1, though the basic ideas behind the algorithm should also apply to the general case as well.

This section has a simple monolithic structure. Because our ideals are all x-homogeneous, the problems that commonly arise when lacking a well-ordering actually root in the inhomogeneity in t alone. It turns out that these problems can be circumvented by reducing with respect to $p-t$ diligently. Hence we begin with a dedicated algorithm for tha.t

Next, we continue with an algorithm for reducing a set of elements of same x-degree with respect to themselves and $p-t$. Having all elements sharing the same x-degree makes the inhomogeneity in t easy to handle. Using it, we construct an algorithm for reducing a set of elements of same x-degree with respect to themselves, $p-t$ and another set of elements of strictly lower x-degree. This is the part in which the difficulty of our lack of well-ordering becomes apparent. With that algorithm, we conclude the section with an 3.2.9 for reducing a standard bases with respect to itself.

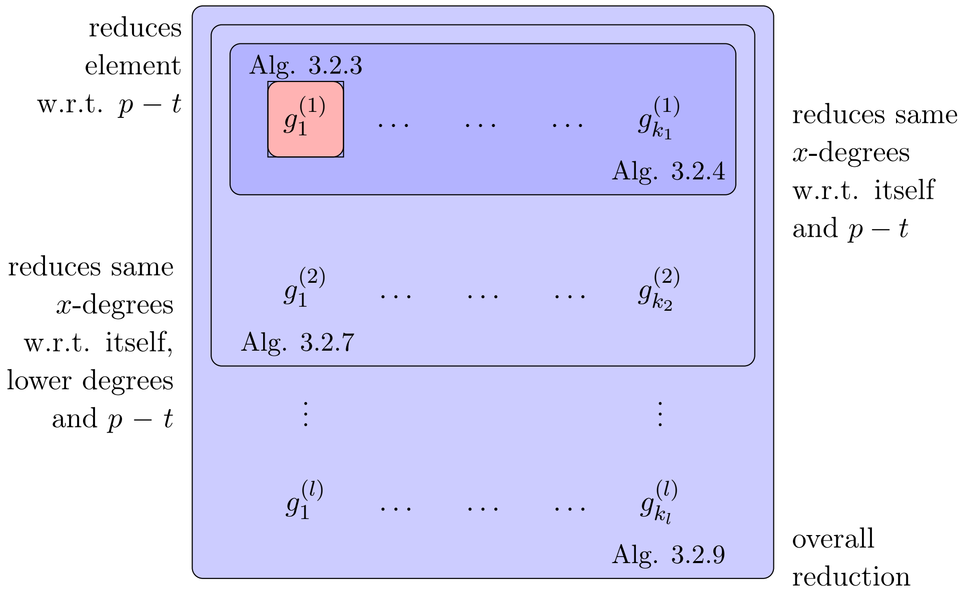


FIGURE 6. reduction algorithms in Section 3.2, given a set $\{g_1^{(1)},\ldots,g_{k_1}^{(1)}\}\cup\ldots\{g_1^{(l)},\ldots,g_{k_l}^{(l)}\}$ of x-homogeneous elements split up by x-degree

Convention 3.2.1 For this section, we adopt Convention 1.2.2, in which we have: K a field with non-trivial discrete valuation, $\mathfrak{K}$ its residue field, R_ν its discrete valuation ring, $p \in R_\nu$ a uniformizing parameter and $R \subset R_\nu$ a dense noetherian subring with $p \in R$. Both K and R_ν are assumed to be complete, so that we have exact sequences

$$\begin{tikzcd}
0 \arrow[r] & \langle p-t\rangle \cdot R[\![t]\!]_{\langle p-t\rangle}[x] \arrow[r] & R[\![t]\!]_{\langle p-t\rangle}[x] \arrow[r] & K[x] \arrow[r] & 0, \\
0 \arrow[r] & \langle p-t\rangle \cdot R[\![t]\!][x] \arrow[r] \arrow[u] & R[\![t]\!][x] \arrow[r, "t \longmapsto p"'] \arrow[u] & R_\nu[x] \arrow[r] \arrow[u] & 0.
\end{tikzcd}$$

and $R/\langle p\rangle = \mathfrak{K}$.

As stated in the beginning of the last section, we still require that R continues to satisfy Convention 2.1.1, i.e. R is noetherian and linear equations in R are solvable, so that standard bases in $R[\![t]\!][x]$ exist and are computable.

Fix a preimage $I \trianglelefteq R[\![t]\!][x]$ of a homogeneous ideal in $K[x]$, which in particular implies that I is x-homogeneous and $p - t \in I$. Moreover, fix a t-local monomial ordering $>$ on $\mathrm{Mon}(t, x)$.

Before we begin with the reduction algorithms, let us recall again why initially reduced standard bases are important.

Example 3.2.2 Let $R[\![t]\!][x] = \mathbb{Z}[\![t]\!][x, y, z]$, and let $>=>_v$ be a weighted ordering with weight vector $v = (-1, 1, 1, 1) \in \mathbb{R}_{<0} \times \mathbb{R}^3$ and the t-local lexicographical ordering $x > y > 1 > t$ as tiebreaker. Then given the following ideal I a possible standard basis G would be, leading terms highlighted:

$$\begin{array}{rcl}
I &=& \langle \underline{2} - t, \underline{3x} - tx + t^2y + t^3z, \underline{5y} + tx - t^2y + t^2z\rangle \;\trianglelefteq\; \mathbb{Z}[\![t]\!][x, y, z] \\
\cup\!\mid & & \\
G &=& \{\underline{2} - t, \underline{x} + t^2y + t^3z, \underline{y} + tx + t^2z\}.
\end{array}$$

In the previous section, we have seen how initially reduced standard bases play a central role in determining the inequalities and equations of Gröbner cones, see for example Algorithm 3.1.27. To illustrate that initially reduced standard bases are indeed necessary, let g_0, g_1 and g_2 denote the standard basis elements above.

It is easy to see that G is a standard basis, but it is not yet initially reduced as the terms t^2y in g_1 and tx in g_2 still lie in the leading ideal $\mathrm{LT}_>(I) = \langle 2, x, y\rangle$. Consequently, these two terms yield meddling inequalities, so that (the overline denoting the closure in the euclidean topology)

$$\overline{\{w \in \mathbb{R}_{<0} \times \mathbb{R}^3 \mid \mathrm{in}_w(g_i) = \mathrm{in}_v(g_i) \text{ for } i = 0, 1, 2\}} \subsetneq C_v(I).$$

Ignoring g_0, as it yields no non-trivial inequalities in $\mathbb{R}_{\leq 0} \times \mathbb{R}^3$, the former is the euclidean closure of the open polyhedral cone given by the inequalities

$$\mathrm{in}_w(g_1) = x \quad\Longleftrightarrow\quad \begin{cases}\deg_w(x) > \deg_w(t^2y)\\ \deg_w(x) > \deg_w(t^3z)\end{cases} \quad\Longleftrightarrow\quad \begin{cases} w_1 > 2w_0 + w_2\\ w_1 > 3w_0 + w_3\end{cases}$$

and

$$\mathrm{in}_w(g_2) = y \quad\Longleftrightarrow\quad \begin{cases}\deg_w(y) > \deg_w(tx)\\ \deg_w(y) > \deg_w(t^2z)\end{cases} \quad\Longleftrightarrow\quad \begin{cases} w_2 > w_0 + w_1\\ w_2 > 2w_0 + w_3\end{cases}$$

see Figures 7 and 8.

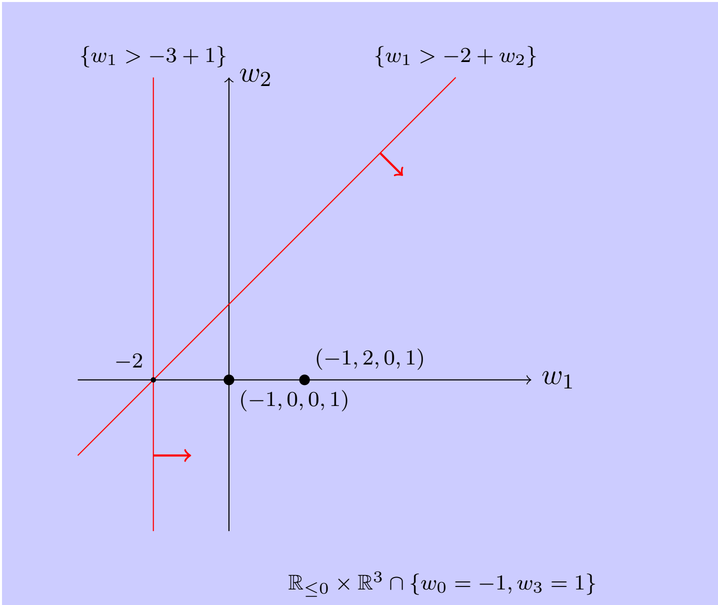


FIGURE 7. inequalities given by $\mathrm{in}_w(g_1) = x$

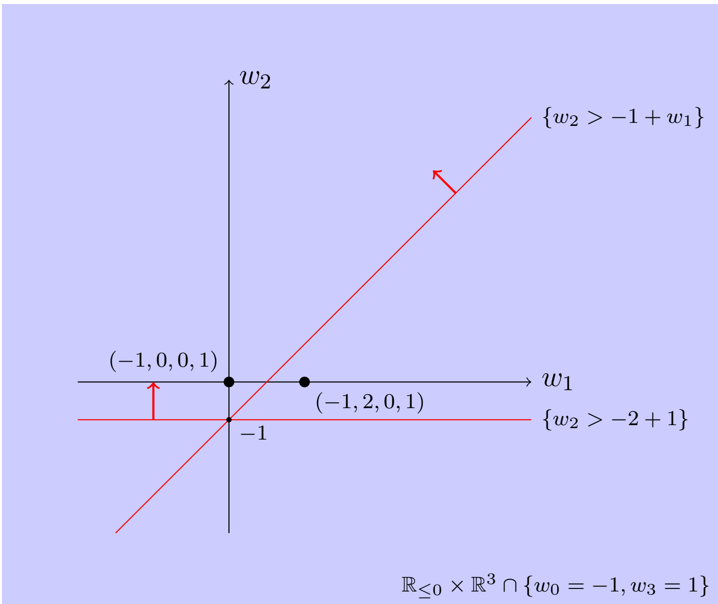


FIGURE 8. inequalities given by $\mathrm{in}_w(g_2) = y$

Clearly, the weight vector $w := (-1,2,0,1)$ lies outside the euclidean closure. However, choosing suitable elements one can see that $\mathrm{in}_w(I) =$

$\mathrm{in}_v(I)$, since

$$\mathrm{in}_w(g_1 - t^2 \cdot g_2) = \mathrm{in}_w(x - t^3x + t^3z - t^4z) = x,$$
$$\mathrm{in}_w(g_2 - t \cdot g_1) = \mathrm{in}_w(y - t^3y + t^2z - t^4z) = y,$$

implying that $w \in C_v(I)$. Replacing $\{g_1, g_2\}$ with the initially reduced standard basis $\{g_1 - t^2 \cdot g_2, g_2 - t \cdot g_1\}$, we see that we are replacing the unnecessary inequalities above, induced by t^2y and tx, with the redundant inequalities of Example 3.1.10, induced by t^3x, t^3y and t^4z.

As we explained in the beginning of the section, the element $p-t$ plays a central role in our reduction process and reductions with it have to be made diligently. Hence, we dedicate an own algorithm to it.

Algorithm 3.2.3 $(p-t)$-Reduce

Input: $(g, >)$, where $>$ a t-local monomial ordering and $g \in I$ x-homogeneous and polynomial, i.e. $g \in R[t, x]$.

Output: $g' \in I$ x-homogeneous and polynomial with $\mathrm{LT}_>(g') = \mathrm{LT}_>(g)$ and initially reduced with respect to $p - t$ under $>$, i.e. no term of $\mathrm{tail}_>(g)$ is divisible by $\mathrm{LT}_>(p - t) = p$.

1: Suppose $g = \sum_\alpha g_\alpha \cdot x^\alpha$ with $g_\alpha \in R[t]$ and $\mathrm{LT}_>(g) = \mathrm{LT}_>(g_\gamma) \cdot x^\gamma$.
2: Set $g' := g_\gamma \cdot x^\gamma$ and $g'' := g - g_\gamma \cdot x^\gamma$, so that $g = g' + g''$.
3: **while** $g'' \neq 0$ **do**
4:     Suppose $g'' = \sum_\alpha g''_\alpha \cdot x^\alpha$ with $g''_\alpha \in R[t]$ and $\mathrm{LT}_>(g'') = \mathrm{LT}_>(g''_\gamma) \cdot x^\gamma$.
5:     **if** $p \mid \mathrm{LT}_>(g''_\gamma)$ **then**
6:         Let $l := \max\{m \in \mathbb{N} \mid p^m \text{ divides } \mathrm{LT}_>(g''_\gamma)\} > 0$.
7:         Set $g'' := g'' - \frac{\mathrm{LT}_>(g''_\gamma)}{p^l} \cdot (p^l - t^l)$.
8:     **else**
9:         Set $g' := g' + g''_\gamma \cdot x^\gamma$ and $g'' := g'' - g''_\gamma \cdot x^\gamma$.
10: **return** g'

Proof. *Termination:* We need to show that $g'' = 0$ eventually. Since all changes to g'' during a single iteration of the `while` loop happen at a distinct monomial in x, namely that of $\mathrm{LM}_>(g'')$, we may assume for our argument that all terms of g'' have the same monomial in x. Suppose, in the beginning of an iteration,

$$g'' = (c_{i_1}t^{i_1} + \ldots + c_{i_j} \cdot t^{i_j}) \cdot x^\gamma \text{ with } i_1 < \ldots < i_j.$$

Now if $p \nmid \mathrm{LT}_>(c_{i_1})$, then g'' will be set to 0 in Step 9 and the algorithm terminates. If $p \mid \mathrm{LT}_>(c_{i_1})$, we substitute the term $c_{i_1} \cdot t^{i_1}x^\gamma$ by the term $c_{i_1}/p^l \cdot t^{i_1+l}x^\gamma$ in Step 7, increasing the minimal t-degree strictly.

Let $\nu_p(c) := \max\{m \in \mathbb{N} \mid p^m \text{ divides } c\}$ denote the p-adic valuation on R, so that $l = \nu_p(c_{i_1})$, and consider the valued degree of g'' defined by

$$\max\{\nu_p(c_{i_1}) + \deg(t^{i_1}), \ldots, \nu_p(c_{i_j}) + \deg(t^{i_j})\}.$$

This is a natural upper bound on the t-degree of our substitute, and hence also for the t-degree of all terms in our new g''.

If the monomial of the substitute, $t^{i_1+l}x^\gamma$, does not occur in the original g'', then this upper bound remains the same for out new g''. If it does occur in the original g'', then this valued degree might increase depending on the sum of the coefficients, however the number of terms in g'' strictly decreases.

Because g'' has only fintely many terms to begin with, this upper bound may therefore only increase a finite number of times. And since the minimal t-degree is strictly increasing, if g'' is not set to 0, our algorithm terminates eventually.

Correctness: It is clear that g' remains polynomial and x-homogeneous. And the only term of g' that might be divisible by $\mathrm{LT}_>(p-t) = p$ is $\mathrm{LT}_>(g') = \mathrm{LT}_>(g)$, since all other terms passed the check in Step 5 negatively. Hence g' is initially reduced with respect to $p-t$ under $>$. □

With this, we can begin formulating an algorithm for initially reducing a set of elements which are x-homogeneous of same degree in x.

Algorithm 3.2.4 initial reduction, same degree in x

Input: $(G, >)$, where $>$ a t-local monomial ordering and $G = \{g_1, \dots, g_k\} \subseteq I$ a finite subset such that

(1) $g_1, \dots, g_k$ x-homogeneous of the same x-degree and polynomial,
(2) $\mathrm{LC}_>(g_i) = 1$ for $i = 1, \dots, k$,
(3) $\mathrm{LM}_>(g_i) \neq \mathrm{LM}_>(g_j)$ for $i \neq j$.

Output: $G' = \{g'_1, \dots, g'_k\} \subseteq I$ such that

(1) $g'_1, \dots, g'_k$ x-homogeneous of the same x-degree and polynomial,
(2) $\mathrm{LT}_>(g'_i) = \mathrm{LT}_>(g_i)$ for $i = 1, \dots, k$,
(3) G' initially reduced with respect to itself and $p - t$.

for $i = 1, \dots, k$ **do**
  Run $g_i := (p-t)$-Reduce$(g_i, >)$.
Reorder $G = \{g_1, \dots, g_k\}$ such that

$$\mathrm{LM}_>(g_1) > \ldots > \mathrm{LM}_>(g_k),$$

and suppose

$$g_i := \sum_{\alpha \in \mathbb{N}} g_{i,\alpha} \cdot x^\alpha \text{ with } g_{i,\alpha} \in R[\![t]\!] \text{ and } \mathrm{LT}_>(g_i) = t^{\beta_i} x^{\alpha_i}.$$

for $i = 1, \dots, k-1$ **do**
  for $j = i+1, \dots, k$ **do**
    if $g_{j,\alpha_i} \neq 0$ **then**
      Set

$$g_j := \frac{g_{i,\alpha_i}}{t^{\beta_i}} \cdot g_j - \frac{g_{j,\alpha_i}}{t^{\beta_i}} \cdot g_i.$$

      Run $g_j := (p-t)$-Reduce$(g_j, >)$.
for $i = 1, \dots, k-1$ **do**
  for $j = i+1, \dots, k$ **do**
    if $t^{\beta_j} \mid g_{i,\alpha_j}$ **then**

12: Set
$$g_i := \frac{g_{j,\alpha_j}}{t^{\beta_j}} \cdot g_i - \frac{g_{i,\alpha_j}}{t^{\beta_j}} \cdot g_j.$$
13: Run $g_i := (p-t)$-Reduce$(g_i, >)$.
14: **return** $G' = \{g_1, \ldots, g_k\}$.

Proof. For the correctness of the instructions note that, by definition and because $>$ is t-local, g_{j,α_j} is divisible by t^{β_j} and g_{i,α_i} is divisible by t^{β_i} in Step 4. From the assumption in Step 8 it follows that g_{i,α_j} in Step 9 will be divisible by t^{β_j}. Observe that due to the reordering in Step 1 and $\mathrm{LM}_>(g_{j,\alpha_i}) \cdot x^{\alpha_i}$ being a monomial in g_j we have for $i < j$:
$$t^{\beta_i} \cdot x^{\alpha_i} = \mathrm{LM}_>(g_i) > \mathrm{LM}_>(g_j) > \mathrm{LM}_>(g_{j,\alpha_i}) \cdot x^{\alpha_i}.$$
Now because $>$ is t-local, this implies that t^{β_i} divides $\mathrm{LM}_>(g_{j,\alpha_i})$, hence also g_{j,α_i}.

It is clear that the algorithm terminates since it only consists of a finite number of steps, and, for the correctness, that the output is x-homogeneous and polynomial.

Next, we show that the leading terms of the g_i are preserved. Observe that in Step 5 we have $\mathrm{LM}_>(\frac{g_{i,\alpha_i}}{t^{\beta_i}}) = 1$ by definition and $\mathrm{LM}_>(\frac{g_{j,\alpha_j}}{t^{\beta_i}}) < 1$ by the previous argument. Due to the assumption that $\mathrm{LC}_>(g_i) = \mathrm{LC}_>(g_{i,\alpha_i}) = 1$ we therefore have
$$\mathrm{LT}_>(g_j) = \mathrm{LT}_>\left(\frac{g_{i,\alpha_i}}{t^{\beta_i}} \cdot g_j\right)$$
and
$$\mathrm{LM}_>(g_j) > \mathrm{LM}_>(g_{j,\alpha_i}) \cdot x^{\alpha_i} = \mathrm{LM}_>\left(\frac{g_{j,\alpha_i}}{t^{\beta_i}} \cdot g_i\right).$$
In Step 9 we similarly have $\mathrm{LM}_>(\frac{g_{j,\alpha_j}}{t^{\beta_j}}) = 1$ and $\mathrm{LM}_>(\frac{g_{i,\alpha_j}}{t^{\beta_j}}) \leq 1$, thus
$$\mathrm{LT}_>(g_i) = \mathrm{LT}_>\left(\frac{g_{j,\alpha_j}}{t^{\beta_j}} \cdot g_i\right)$$
and
$$\mathrm{LM}_>(g_i) > \mathrm{LM}_>(g_{i,\alpha_j}) \cdot x^{\alpha_j} = \mathrm{LM}_>\left(\frac{g_{i,\alpha_j}}{t^{\beta_j}} \cdot g_j\right).$$
On the whole, we see that the leading terms of the $g_1, \ldots, g_k$ remain unchanged.

Consequently, the output is initially reduced with respect to $p - t$. The leading terms do not divide p as every element is monic and the latter terms do not divide p because every element of the output was sent through the Algorithm 3.2.3.

To see that the output G is initially reduced, observe that the first pair of nested `for` loops eliminates all terms in g_j with x^{α_i} for $i < j$. In particular, each g_j is initially reduced with respect to $g_1, \ldots, g_{j-1}$.

Additionally, it will stay reduced with respect to $g_1, \ldots, g_{j-1}$ in the second pair of nested `for` loops, because $g_{j+1}, \ldots, g_k$ contain no monomial x^{α_i}, $i < j$, either.

Moreover, once g_i is reduced initially with respect to g_j for $i < j$ in Step 12, reducing it initially with respect to say g_{j+1} will not change that out of two reasons. First, g_{j+1} contains no term with x^{α_j}, hence adding a multiple of it to g_i is unproblematic. Secondly, $\mathrm{LT}_>(g_{j,\alpha_j}/t^{\beta_j}) = 1$, which means multiplying g_i by it will not change $\mathrm{LT}_>(g_{i,\alpha_j})$. So if t^{β_j} does not divide g_{i,α_j} before, because g_i is initially reduced with respect to g_j, it does not divide g_{i,α_j} after as well.

This shows that the constant changes to g_i in the second pair of nested `for` loops are unproblematic. Once g_i has been initially reduced with respect to g_j, it will stay that way while being reduced initially with respect to $g_{j+1}, \ldots, g_k$. $\square$

Example 3.2.5 Consider the set $G = \{g_1, g_2, g_3\} \subseteq \mathbb{Z}[\![t]\!][x_1, x_2, x_3]$ with

$$\begin{aligned}
g_1 &:= x_1^2 + tx_2^2 - t^2x_3^2,\\
g_2 &:= x_2^2 + tx_1^2 + tx_3^2 + t^2x_3^2 = x_2^2 + tx_1^2 + (t + t^2)x_3^2,\\
g_3 &:= t^3x_3^2 + t^4x_1^2 + t^4x_2^2 + t^5x_2^2 = t^3x_3^2 + t^4x_1^2 + (t^4 + t^5)x_2^2,
\end{aligned}$$

and the weighted ordering $>=>_w$ on $\mathrm{Mon}(t, x)$ with weight vector $(-1, 1, 1, 1) \in \mathbb{R}_{<0} \times \mathbb{R}^3$ and the t-local lexicographical ordering with $x > y > z > 1 > t$ as tiebreaker. The g_i, $i = 1, 2, 3$ as well as their terms have already been ordered above. Suppose $p - t = 2 - t$.

We can illustrate the process with the help of the following 3 by 3 matrix:

$$\begin{pmatrix} 1 & t & -t^2 \\ t & 1 & t + t^2 \\ t^4 & t^4 + t^5 & t^3 \end{pmatrix}.$$

The first row represents g_1. Writing $g_1 = \sum_\alpha g_\alpha x^\alpha$ with $g_\alpha \in R[\![t]\!]$, its first entry is the g_α with x^α being the x-monomial of the leading term of g_1, its second entry is the g_α with x^α being the x-monomial of the leading term of g_2 and its third entry is the g_α with x^α being the x-monomial of the leading term of g_3. The second row represents g_2 and the third row represents g_3. Terms with x-monomial different from all leading monomials, of which there exist none in our example, would be ignored in this illustration.

In the first pass, we begin by taking g_1 and reducing g_2 and g_3 with respect to it. To eliminate the term tx_1^2 in g_2 and $t^4x_1^2$ in g_3 we set

$$\begin{aligned}
g_2 :=& g_2 - t \cdot g_1 = (x_2^2 + tx_1^2 + tx_3^2 + t^2x_3^2) - t \cdot (x_1^2 + tx_2^2 - t^2x_3^2)\\
=& (1 - t^2) \cdot x_2^2 + (t + t^2 + t^3) \cdot x_3^2,\\
g_3 :=& g_3 - t^4 \cdot g_1 = (t^3x_3^2 + t^4x_1^2 + (t^4 + t^5)x_2^2) - t^4 \cdot (x_1^2 + tx_2^2 - t^2x_3^2)\\
=& (t^3 + t^6) \cdot x_3^2 + t^4 \cdot x_2^2.
\end{aligned}$$

Since both g_2 and g_3 remain initially reduced with respect to $2 - t$, there is no need to reduce them with respect to it.

$$\begin{tikzcd} g_1 \arrow[r, bend right] \arrow[rr, bend left] & g_2 & g_3 \end{tikzcd}$$

$$\begin{pmatrix} 1 & t & -t^2 \\ 0 & 1-t^2 & t+t^2+t^3 \\ 0 & t^4 & t^3+t^6 \end{pmatrix}$$

Next, we take g_2 and reduce g_3 with respect to it, i.e.

$$\begin{aligned} g_3 &:= (1-t^2)\cdot g_3 - t^4 \cdot g_2 \\ &= (1-t^2)\cdot((t^3+t^6)x_3^2 + t^4x_2^2) - t^4\cdot((1-t^2)x_2^2 + (t+t^2+t^3)x_3^2) \\ &= (t^3 - 2t^5 - t^7 - t^8)\cdot x_3^2. \end{aligned}$$

And even though g_3 contains a term divisible by 2, it still remains initially reduced with respect to $2-t$.

$$\begin{tikzcd} g_1 & g_2 \arrow[r, bend left] & g_3 \end{tikzcd}$$

$$\begin{pmatrix} 1 & t & -t^2 \\ 0 & 1-t^2 & t+t^2+t^3 \\ 0 & 0 & t^3-2t^5-t^7-t^8 \end{pmatrix}$$

This concludes our first pass. For the second pass, we begin by taking g_1 and reducing it with respect to first g_2 and then g_3. Reducing g_1 with respect to g_2 yields

$$\begin{aligned} g_1 &:= (1-t^2)\cdot g_1 - t\cdot g_2 \\ &= (1-t^2)\cdot(x_1^2 + tx_2^2 - t^2x_3^2) - t\cdot((1-t^2)x_2^2 + (t+t^2+t^3)x_3^2) \\ &= (1-t^2)\cdot x_1^2 + (-2t^2 - t^3)\cdot x_3^2 \end{aligned}$$

and reducing that with respect to $2-t$ we obtain

$$g_1 := g_1 - (-t^2 - t^3)x_3^2 \cdot (2-t) = (1-t^2)\cdot x_1^2 - t^4x_3^2.$$

Reducing g_1 with respect to g_3 yields,

$$\begin{aligned} g_1 &:= (1-2t^2-t^4-t^5)\cdot g_1 - t\cdot g_3 = (1-2t^2-t^4-t^5)(1-t^2)x_1^2 \\ &= (1-3t^2+t^4-t^5+t^6+t^7)\cdot x_1^2, \end{aligned}$$

which is initially reduced with respect to $2-t$.

$$\begin{tikzcd} g_1 & g_2 \arrow[l, bend left] & g_3 \arrow[ll, bend right] \end{tikzcd}$$

$$\begin{pmatrix} 1-3t^2+t^4-t^5+t^6+t^7 & 0 & 0 \\ 0 & 1-t^2 & t+t^2+t^3 \\ 0 & 0 & t^3-t^6-t^7-t^8 \end{pmatrix}$$

Finally, note that while g_2 has a term $t^3x_3^2$ divisible by the leading term t^3x_3 of g_3, it is still initially reduced with respect to g_3. This concludes our second pass and we obtain the initially reduced set

$$\begin{aligned} g_1 &= (1-5t^2+3t^4-t^5+t^6+t^7)\cdot x_1^2, \\ g_2 &= (1-t^2)\cdot x_2^2 + (t+t^2+t^3)\cdot x_3^2, \\ g_3 &= (t^3-2t^5-t^7-t^8)\cdot x_3^2. \end{aligned}$$

Observe that it is possible to reduce the number of terms at the cost of the coefficient size, by substituting p for some of the t. One alternative initially reduced set with the same leading monomials as above would therefore be

$$g_1 := 165 \cdot x_1^2, \quad g_2 := -3 \cdot x_2^2 + 7t \cdot x_3^2 \quad \text{and} \quad g_3 := -55t^3 \cdot x_3^2.$$

Next, we need to discuss how to reduce a set H of x-homogeneous elements of same degree in x with respect to themselves and a set G of x-homogeneous elements of lower degree. There are multiple ways of approaching the problem. Conceptionally, the simplest way would be multiplying the elements of G up to the same degree in x as the elements of H in all possible combinations and using Algorithm 3.2.4 on the resulting set. This resembles a brute force method in which we directly summon the worst case scenario to be resolved. Consequently, it is an algorithm which is good in cases in which the worst case is unavoidable.

Algorithm 3.2.6 initial reduction, all at once

Input: $(G, H, >)$, where $>$ a t-local monomial ordering, $H = \{h_1, \ldots, h_k\}$, $G \subseteq I$ finite subsets such that

(1) $h_1, \ldots, h_k$ x-homogeneous of the same x-degree d and polynomial,
(2) all $g \in G$ x-homogeneous of x-degree less than d and polynomial,
(3) $\mathrm{LC}_>(h_i) = 1$, for $i = 1, \ldots, k$, and $\mathrm{LC}_>(g) = 1$ for all $g \in G$,
(4) $\mathrm{LM}_>(h_i) \neq \mathrm{LM}_>(h_j)$ for $i \neq j$,
(5) $\mathrm{LM}_>(h_i) \notin \langle \mathrm{LM}_>(g) \mid g \in G \rangle$ for $i = 1, \ldots, k$.

Output: $H' = \{h_1', \ldots, h_k'\} \subseteq R[\![t]\!][x]$ such that

(1) $h_1', \ldots, h_k'$ x-homogeneous of the same x-degree d and polynomial,
(2) $\mathrm{LT}_>(h_i') = \mathrm{LT}_>(h_i)$ for $i = 1, \ldots, k$,
(3) H' initially reduced w.r.t. G and itself.

```
Set E := ∅.
for α ∈ ℕ^n, |α| = d do
    if t^β x^α ∈ LT_>(G) for some β ∈ ℕ then
        Pick g ∈ G with LT_>(g) | t^β x^α for some minimal β ∈ ℕ.
        Add E := E ∪ { (t^β x^α / LT_>(g)) · g }.
Reduce H ∪ E initially with Algorithm 3.2.4.
return H
```

Proof. Due to the necessary conditions of this algorithm, $H \cup E$ satisfies the necessary conditions for Algorithm 3.2.4. The correctness of this algorithm now follows from the correctness of Algorithm 3.2.4. $\square$

A more sophisticated method is only multiplying the elements of G up to the same degree in x as the elements of H when they are needed. In the optimal case, we can reduce the complexity drastically with this strategy, in the worst case we are only delaying the inevitable.

Algorithm 3.2.7 initial reduction, step by step

Input: $(G, H, >)$, where $>$ a t-local monomial ordering, $H = \{h_1, \ldots, h_k\}, G \subseteq I$ finite subsets such that

(1) $h_1, \ldots, h_k$ x-homogeneous polynomials of the same x-degree d,
(2) all $g \in G$ x-homogeneous polynomials of x-degree less than d,
(3) $\mathrm{LC}_>(h_i) = 1$, for $i = 1, \ldots, k$, and $\mathrm{LC}_>(g) = 1$ for all $g \in G$,
(4) $\mathrm{LM}_>(h_i) \neq \mathrm{LM}_>(h_j)$ for $i \neq j$,
(5) $\mathrm{LM}_>(h_i) \notin \langle \mathrm{LM}_>(g) \mid g \in G \rangle$ for $i = 1, \ldots, k$.

Output: $H' = \{h'_1, \ldots, h'_k\} \subseteq R[\![t]\!][x]$ such that

(1) $h'_1, \ldots, h'_k$ x-homogeneous polynomials of the same x-degree d,
(2) $\mathrm{LT}_>(h'_i) = \mathrm{LT}_>(h_i)$ for $i = 1, \ldots, k$,
(3) H' initially reduced w.r.t. G and itself.

Reduce H initially using Algorithm 3.2.4 and set $E = \emptyset$.
Suppose $h_i = \sum_{\alpha \in \mathbb{N}^n} h_{i,\alpha} \cdot x^\alpha$ with $h_{i,\alpha} \in R[\![t]\!]$, create the disjoint union

$$T := \{(\mathrm{LT}_>(h_{i,\alpha}) \cdot x^\alpha, i) \mid \alpha \in \mathbb{N}^n \text{ and } \mathrm{LT}_>(h_{i,\alpha}) \cdot x^\alpha < \mathrm{LT}_>(h_i)\},$$

a working list of terms to be checked for potential reduction with respect to G.
while $T \neq \emptyset$ **do**
  Pick $(s, i) \in T$ with $\mathrm{LM}_>(s)$ maximal.
  if $\mathrm{LT}_>(g) \mid s$ for some $g \in G$ **then**
    Pick $g \in G$, $\mathrm{LT}_>(g) \mid s$, and set $E := E \cup \left\{ \frac{\mathrm{LM}_>(s)}{\mathrm{LM}_>(g)} \cdot g \right\}$.
    Reduce $H \cup E$ initially using Algorithm 3.2.4.
    Update the working list:

$$T := \{(\mathrm{LT}_>(h_{i,\alpha}) \cdot x^\alpha, i) \mid \alpha \in \mathbb{N}^n \text{ and } \mathrm{LM}_>(h_{i,\alpha}) \cdot x^\alpha < \mathrm{LM}_>(s)\}.$$

  else
    Set $T := T \setminus \{(h_i, s)\}$.
return H

Proof. For the termination note that in each iteration of the while loop either the set of extra polynomials E increases or the working list T decreases. Also because each s is chosen to be maximal, each other term in the working list T with the same x-monomial must have a higher t-degree and is therefore eliminated alongside s in the initial reduction of $H \cup E$. Because the updated T only includes relevant terms smaller than s, the x-monomial of s is effectively eliminated in all working lists to follow. Hence each elements of E will always have a distinct x-monomial which is of degree d. Thus E has a maximal size after which the algorithm will terminate in a finite number of steps.

For the correctness of the instructions, observe that $H \cup E$ satisfies the conditions for Algorithm 3.2.4 by assumption. For the correctness of the output, it is obvious that the leading terms of H are preserved, that H is initially reduced with respect to itself and that its elements are x-homogeneous as well as polynomial. To show that H is initially reduced with respect to G, observe that, apart from the terms eliminated, any term altered in the initial reduction of $H \cup E$ is strictly smaller than s. Because s was chosen to be maximal, the updated working list therefore contains all relevant terms that have been altered or that have yet to be checked for reduction. Thus in the output any relevant term has been negatively checked for divisibility by an element of G. $\square$

Remark 3.2.8 Note that in Step 6 of Algorithm 3.2.7, we are multiplying g by a power of t even though it is not necessary for its correctness, and the reason is as follows:

Recall Algorithm 3.2.4, which consists of two big nested `for` loops. In the first pass from Step 4 to 8 we take each g_i, $i = 1, \ldots, k-1$, and reduce all g_j, $i < j$, with respect to it. In the second pass from Step 9 to 13 we take each g_i, $i = 1, \ldots, k-1$, and reduce it with respect to all g_j, $i < j$.

Now suppose we enter the Algorithm with a set $H \cup \{g\}$, with $H = \{h_1, \ldots, h_k\}$ already initially reduced with respect to itself and $p-t$. Suppose furthermore

$$\mathrm{LM}_>(h_1) > \ldots > \mathrm{LM}_>(h_l) > \mathrm{LM}_>(g) > \mathrm{LM}_>(h_{l+1}) > \ldots > \mathrm{LM}_>(h_k).$$

By assumption, taking each h_i, $i = 1, \ldots, l$, and reducing all h_j, $i < j$, with respect to it is obsolete. The first necessary action is reducing g with respect to $h_1, \ldots, h_l$.

$$\begin{tikzcd} h_1 \arrow[rrrr, bend left] & h_2 \arrow[rrr, bend left] & \ldots & h_i \arrow[r] & g & h_{i+1} & \ldots & h_{k-1} & h_k \end{tikzcd}$$

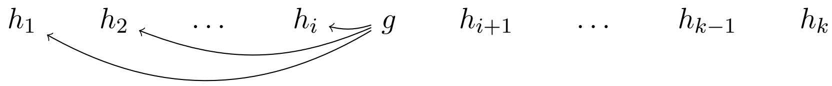
Next, we consider the h_i, $i = l+1, \ldots, k$. Each h_i is already reduced with respect to $h_1, \ldots, h_l$ and remains so after reducing it with respect to g, as the x-monomials of their leading monomials were already completely eliminated in g previously. Hence we may reduce each h_i, $i = l+1, \ldots, k$, with respect to g without inducing the need of reducing them with respect to $h_1, \ldots, h_l$ again.

$$\begin{tikzcd} h_1 & h_2 & \ldots & h_i & g \arrow[r] \arrow[rrr, bend right] \arrow[rrrr, bend right] & h_{i+1} & \ldots & h_{k-1} & h_k \end{tikzcd}$$

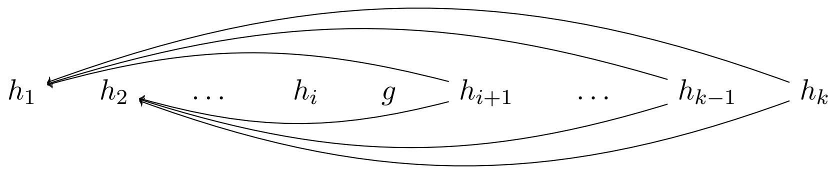
However, g might contain a term with monomial t^2x, which might not be reducible with respect to $\mathrm{LT}_>(h_j) = t^3x$, but if g is multiplied by t while reducing another element with respect to it, we do create a term that is reducible. Thus, we need to reduce each h_i, $i = l+1, \ldots, k$ with respect to h_j, $j = l+1, \ldots, i$ again and this concludes our first pass.

$$\begin{tikzcd} h_1 & h_2 & \ldots & h_i & g & h_{i+1} \arrow[r] \arrow[rr, bend left] \arrow[rrr, bend left] & \ldots \arrow[r] & h_{k-1} \arrow[r] & h_k \end{tikzcd}$$

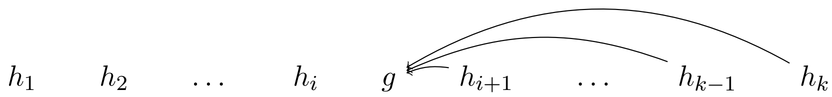
For the second pass, taking each h_i, $i = 1, \ldots, l$, and reducing it with respect to all h_j, $j = i+1, \ldots, l$, is unnecessary. The first necessary step is to take each h_i, $i = 1, \ldots, l$, and reduce it with respect to the newly added g. Similar to a previous step, each h_i remains reduced with respect to all h_j, $j = i+1, \ldots, l$.



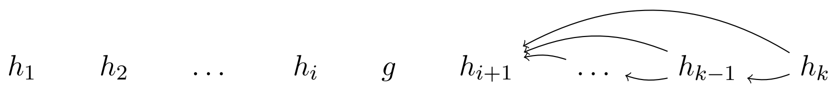
Afterwards, while each h_i remains reduced with respect to all h_j, $j = i+1, \ldots, l$, it nonetheless needs to be reduced with respect to $h_{l+1}, \ldots, h_k$ again.



Next in the second pass, we take g and reduce it with respect to $h_{l+1}, \ldots, h_k$.



And finally, we take each h_i, $i = l+1, \ldots, k-1$ and reduce it with respect to all h_j, $i < j$, as reducing them with respect to g earlier might have broken their reducedness property.



It can be seen that a position of g more to the right minimizes the number of reductions needed. This implies that $\mathrm{LM}_>(g)$ should be as small as possible, and since its monomial in x is fixed, this means that it should have as high a degree in t as possible.

Note that increasing the degree in t to increase performance is not risk-free. For example, suppose we had a $g \in G$ with $\mathrm{LT}_>(g) = x$ and we were to add $t^5 y \cdot g$ to E in order to reduce a term with monomial t^5xy. Then any subsequent term with monomial t^4xy would require adding an additional multiple of g to E. However, since our working list T is worked off in an order induced by a t-local monomial ordering $>$, any later s' picked in Step 4 with the same monomial in x necessarily has to have a higher degree in t. Thus this cannot happen in our algorithm.

With Algorithms 3.2.6 and 3.2.7, writing an algorithm for computing an initially reduced standard basis becomes a straightforward task. All we need to adhere is to proceed x-degree by x-degree while repeatedly applying the previous algorithm.

Algorithm 3.2.9 initially reduced standard basis

Input: $(F, >)$, where $F \subset I$ an x-homogeneous, polynomial generating set of I containing $p - t$.

Output: $G \subseteq I$ an x-homogeneous, polynomial and initially reduced standard basis of I.

Compute an x-homogeneous standard basis G'' of $I = \langle F \rangle$ with Algorithm 2.2.17.
Set $G' := \emptyset$.
for $g \in G''$ with $p \nmid \mathrm{LT}_>(g)$ **do**
    if $\mathrm{LC}_>(g) \neq 1$ **then**
        Since $1 \in \langle \mathrm{LC}_>(g), p \rangle$, find $a, b \in R$ such that

$$1 = a \cdot \mathrm{LC}_>(g) + b \cdot p.$$

        Set

$$g := a \cdot g + b \cdot \mathrm{LM}_>(g) \cdot (p - t),$$

        so that $\mathrm{LC}_>(g) = 1$.
    Set $G' := G' \cup \{g\}$.
Minimize the standard basis G' by gradually removing elements $g \in G$ with $\mathrm{LM}_>(g') \mid \mathrm{LM}_>(g)$ for some $g' \in G$, $g' \neq g$.
Set $G := \emptyset$
while $G' \neq \emptyset$ **do**
    Set

$$\begin{aligned} d &:= \min\{\deg_x(g) \mid g \in G'\}, \\ H' &:= \{g \in G' \mid \deg_x(g) = d\}, \\ G' &:= \{g \in G' \mid \deg_x(g) > d\}. \end{aligned}$$

    Reduce H' initially with respect to G', $p-t$ and itself using Algorithms 3.2.6 or 3.2.7, let H is the output of the initial reduction.
    Set $G := G \cup H$.
return $G \cup \{p - t\}$.

Proof. It is clear that G is a standard basis of I, as we are merely normalizing the leading coefficients of the standard bases G''. It is also obvious that G is polynomial and x-homogeneous. The initial reducedness of G follows from the correctness of Algorithms 3.2.6 or 3.2.7. $\square$

3.3. Computation

In this section, we describe algorithms for computing the Gröbner fan of an ideal $I \trianglelefteq R[\![t]\!][x]$ as in Convention 3.1.1. While computing a Gröbner fan can be as seemingly simple as computing maximal Gröbner cones $C_>(I)$ with respect to random monomial orderings $>$ until the whole weight space $\mathbb{R}_{\leq 0} \times \mathbb{R}^n$ is filled, sophisticated algorithms revolve around avoiding computing initially reduced standard bases of I from scratch. The algorithms in this section are adjusted versions of the algorithms found in Chapter 4 of Jensen's dissertation [**Jen07**], though some of the ideas involved originate in Collart, Kalkbrenner and Mall's work on the Gröbner walk [**CKM97**].

We start with an algorithm for computing witnesses of weighted homogeneous elements in initial ideals, which can then be used to lift standard bases of initial ideals to initially reduced standard bases of the original ideal. Adding in some statements about the perturbation of initial ideals, we obtain an algorithm which allows us to flip initially reduced standard bases of one ordering to initially reduced standard bases of an adjacent ordering. This algorithm can then be used to construct the Gröbner fan, requiring us to compute the standard basis of I from scratch only once. See Figure 9 for a rough outline with numbering.

Importantly, note that all polynomial computations in our algorithms, if given polynomial input, terminate and return polynomial output themselves due to our results in Chapter 2 and Algorithm 3.2.9 from the last section.

Algorithm 3.3.1 Witness
Input: $(h, H, G, >)$, where

- $>$ a weighted t-local monomial ordering on $\mathrm{Mon}(t, x)$,
- $G = \{g_1, \dots, g_k\} \subseteq I$ an initially reduced standard basis of I with respect to $>$,
- $H = \{h_1, \dots, h_k\} \subseteq \mathrm{in}_w(I)$ with $h_i = \mathrm{in}_w(g_i)$ for some $w \in C_>(I) \cap \mathbb{R}_{<0} \times \mathbb{R}^n$,
- $h \in \mathrm{in}_w(I)$ weighted homogeneous with respect to w.

Output: $f \in I$ such that $\mathrm{in}_w(f) = h$

1: Use Algorithm 2.1.16 to compute a homogeneous determinate division with remainder with respect to $>$,

$$(\{q_1, \dots, q_k\}, r) = \mathrm{HDDwR}(h, \{h_1, \dots, h_k\}, >),$$

so that $h = q_1 \cdot h_1 + \ldots + q_k \cdot h_k$ and $r = 0$.

2: **return** $f := q_1 \cdot g_1 + \ldots + q_k \cdot g_k$

Proof. By Proposition 3.1.13, H is a standard basis of $\mathrm{in}_w(I)$, therefore the division of h will always yield remainder 0.

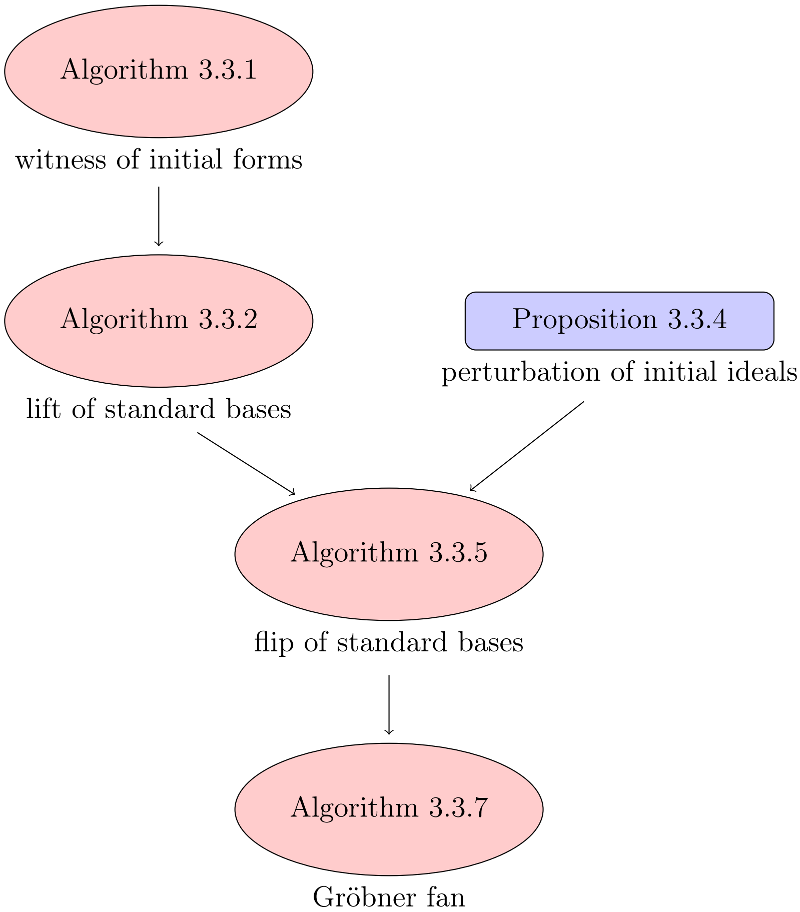


FIGURE 9. outline of Section 3.3

By Remark 2.1.18, since $h, h_1, \ldots, h_k$ are weighted homogeneous with respect to w, so are $q_1, \ldots, q_k$. Hence

$$\mathrm{in}_w(f) = \underbrace{\mathrm{in}_w(q_1) \cdot \mathrm{in}_w(g_1)}_{=q_1 \cdot h_1} + \ldots + \underbrace{\mathrm{in}_w(q_k) \cdot \mathrm{in}_w(g_k)}_{=q_k \cdot h_k} = h.$$

Also note that the division with remainder will always terminate, as the weighted degree cannot become arbitrarily small since the ideal is homogeneous in x and weighted homogeneous overall. $\square$

As announced, we immediately obtain an algorithm which allows us to lift a standard basis of an initial ideal to an initially reduced standard basis of I, assuming we have a standard basis of I with respect to an adjacent ordering at our disposal.

Algorithm 3.3.2 Lift

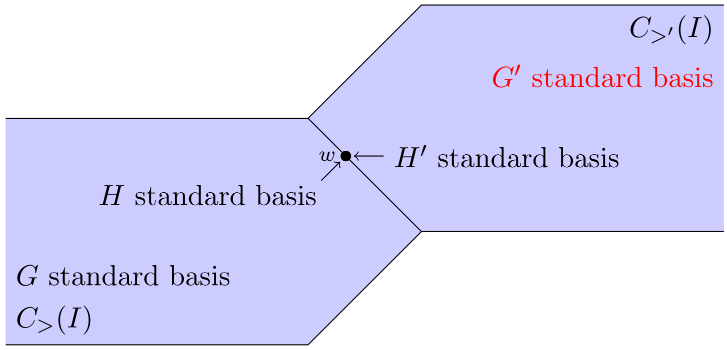


FIGURE 10. lift of standard bases

Input: $(H', >', H, G, >)$, where

- $>$ a weighted t-local monomial ordering on $\mathrm{Mon}(t, x)$ with weight vector in $\mathbb{R}_{<0} \times \mathbb{R}^n$,
- $G = \{g_1, \ldots, g_k\} \subseteq I$ an initially reduced standard basis of I with respect to $>$,
- $H = \{h_1, \ldots, h_k\} \subseteq \mathrm{in}_w(I)$ with $h_i = \mathrm{in}_w(g_i)$ for some $w \in C_>(I)$,
- $>'$ a t-local monomial ordering such that $w \in C_>(I) \cap C_{>'}(I)$,
- $H' \subseteq \mathrm{in}_w(I)$ a weighted homogeneous standard basis with respect to $>'$.

Output: $G' \subseteq I$, an initially reduced standard basis of I with respect to $>'$ such that

$$H' = \{\mathrm{in}_w(g) \mid g \in G'\}.$$

Set $G' := \{\mathrm{Witness}(h, H, G, >) \mid h \in H'\}$.
Reduce G' initially with respect to $>'$.
return G'.

PROOF. Consider a witness $g := \mathrm{Witness}(h, w, G, >)$ for some $h \in H'$. Then, by Lemma 3.1.11, we have $\mathrm{LT}_{>'}(g) = \mathrm{LT}_{>'}(\mathrm{in}_w(g)) = \mathrm{LT}_{>'}(h)$, and thus

$$\langle \mathrm{LT}_{>'}(g) \mid g \in G' \rangle = \langle \mathrm{LT}_{>'}(h) \mid h \in H' \rangle = \mathrm{LT}_{>'}(\mathrm{in}_w(I)) \overset{\text{Lem.}}{\underset{3.1.12}{=}} \mathrm{LT}_{>'}(I). \qquad \square$$

Example 3.3.3 Consider again the ideal

$$I := \langle \underbrace{x - t^3x + t^3z - t^4z}_{=:g_1}, \underbrace{y - t^3y + t^2z - t^4z}_{=:g_2} \rangle \trianglelefteq \mathbb{Z}[\![t]\!][x, y, z]$$

and the weighted monomial ordering $> = >_v$ on $\mathrm{Mon}(t, x, y, z)$ with weight vector $v = (-1, 3, 3, 3) \in \mathbb{R}_{<0} \times \mathbb{R}^3$ and the t-local lexicographical ordering such that $x > y > 1 > t$ as tiebreaker.

In Example 3.1.10, it is shown that its Gröbner cone is of the form

$$C_>(I) = \overline{\{w \in \mathbb{R}_{<0} \times \mathbb{R}^n \mid w_1 \geq 3w_0 + w_3 \text{ and } w_2 \geq 2w_0 + w_3\}}.$$

Picking $w := (-1, 2, -1, 1) \in C_>(I)$, we know by Proposition 3.1.13 that

$$\mathrm{in}_w(I) = \langle \mathrm{in}_w(g_1), \mathrm{in}_w(g_2)\rangle = \langle x, y + t^2 z\rangle.$$

It is easy to see that $\{x, y + t^2 z\}$ is a standard basis of $\mathrm{in}_w(I)$ regardless which monomial ordering is chosen. Since using Algorithm 3.3.1 on $\mathrm{in}_w(g_1)$ and $\mathrm{in}_w(g_2)$ yields g_1 and g_2 respectively, Algorithm 3.3.2 therefore implies that $\{g_1, g_2\}$ is also a standard basis for the adjacent monomial ordering $>'$ on the other side of the facet containing w.

Moreover, since $>'$ has to induce a different leading ideal by definition, and the leading terms of g_1 and g_2 with respect to $>'$ have to occur in their initial forms by Lemma 3.1.11, we see that the adjacent leading ideal is $\langle x, t^2 z\rangle$.

An easy way to construct orderings adjacent to $>$ is by connecting two weight vectors in series, the first a weight vector lying on a facet and the second an outer normal vector of the facet.

Proposition 3.3.4 *Let $>$ be a t-local monomial ordering, $w \in C_>(I)$ and $v \in \mathbb{R}^{n+1}$. Let $>_{(w,v)}$ denote the t-local monomial ordering given by*

$$\begin{aligned} t^\beta \cdot x^\alpha >_{(w,v)} t^{\beta'} \cdot x^{\alpha'} \quad &:\Longleftrightarrow \\ &(\beta,\alpha)\cdot w > (\beta',\alpha')\cdot w, \\ &\text{or } (\beta,\alpha)\cdot w = (\beta',\alpha')\cdot w \text{ and } (\beta,\alpha)\cdot v > (\beta',\alpha')\cdot v, \\ &\text{or } (\beta,\alpha)\cdot w = (\beta',\alpha')\cdot w \text{ and } (\beta,\alpha)\cdot v = (\beta',\alpha')\cdot v \\ &\ \text{and } t^\beta \cdot x^\alpha > t^{\beta'} \cdot x^{\alpha'}. \end{aligned}$$

Then $w = C_>(I) \cap C_{>_{(w,v)}}(I)$ and

$$w + \varepsilon \cdot v \in C_{>_{(w,v)}}(I) \text{ for } \varepsilon > 0 \text{ sufficiently small.}$$

In particular

$$\mathrm{in}_{w+\varepsilon v}(I) = \mathrm{in}_v(\mathrm{in}_w(I)) \text{ for } \varepsilon > 0 \text{ sufficiently small.}$$

Proof. By definition we have $\mathrm{LT}_{>_{(w,v)}}(g) = \mathrm{LT}_{>_{(w,v)}}(\mathrm{in}_w(g))$ for any $g \in R[\![t]\!][x]$, which implies $w \in C_{>_{(w,v)}}(I)$ by Lemma 3.1.11.

Next, let G be an initially reduced standard basis of I with respect to that ordering. Observe that every $g \in G$,

$$g = \underbrace{\ldots\ldots\ldots\ldots\ldots}_{\mathrm{in}_w(g)} + \underbrace{\ldots\ldots\ldots\ldots\ldots}_{\text{rest}},$$

has a distinct degree gap between the terms of highest weighted degree and the rest. As the weighted degree varies continuously under the weight vector, choosing $\varepsilon > 0$ sufficiently small ensures that the $(w + \varepsilon \cdot v)$-weighted degrees of the terms in $\mathrm{in}_w(g)$ remain higher than those of the rest. Thus $\mathrm{in}_{w+\varepsilon\cdot v}(g)$ are the terms of $\mathrm{in}_w(g)$ that have maximal v-weighted degree, i.e. $\mathrm{in}_{w+\varepsilon\cdot v}(g) = \mathrm{in}_v(\mathrm{in}_w(g))$. In particular, we have

$$\mathrm{LT}_{>_{(w,v)}}(\mathrm{in}_{w+\varepsilon\cdot v}(g)) = \mathrm{LT}_{>_{(w,v)}}(g),$$

and hence $w + \varepsilon \cdot v \in C_{>_{(w,v)}}(I)$ by Lemma 3.1.11 again.

The final claim now follows from Proposition 3.1.13:

$$\mathrm{in}_{w+\varepsilon\cdot v}(I) \overset{\text{Prop.}}{\underset{3.1.13}{=}} \langle \mathrm{in}_{w+\varepsilon\cdot v}(g) \mid g \in G\rangle = \langle \mathrm{in}_v(\mathrm{in}_w(g)) \mid g \in G\rangle \overset{\text{Prop.}}{\underset{3.1.13}{=}} \mathrm{in}_v(\mathrm{in}_w(I)).$$

□

With this easy method of constructing adjacent orderings, we are now able to write an algorithm for flipping initially reduced standard bases.

Algorithm 3.3.5 Flip

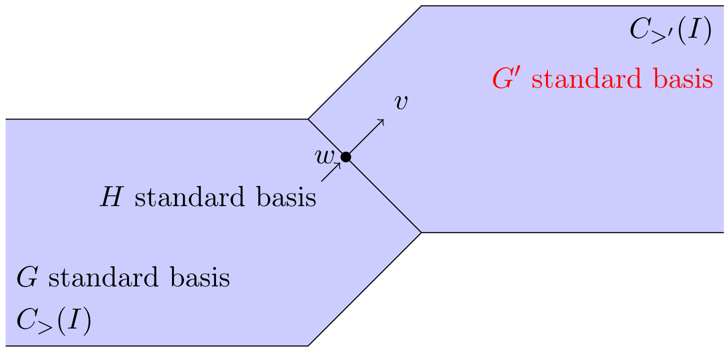


FIGURE 11. flip of standard bases

Input: $(G, H, v, >)$, where

- $>$ a weighted t-local monomial ordering on $\mathrm{Mon}(t, x)$ with weight vector in $\mathbb{R}_{<0} \times \mathbb{R}^n$,
- $G = \{g_1, \ldots, g_k\} \subseteq I$ an initially reduced standard basis of I with respect to $>$,
- $H = \{h_1, \ldots, h_k\} \subseteq \mathrm{in}_w(I)$ with $h_i = \mathrm{in}_w(g_i)$ for some $w \in C_>(I) \cap \mathbb{R}_{<0} \times \mathbb{R}^n$ relative interior point on a lower facet $\tau \leq C_>(I)$, $\tau \nsubseteq \{0\} \times \mathbb{R}^n$.
- $v \in \mathbb{R} \times \mathbb{R}^n$ an outer normal vector of the facet τ.

Output: $(G', >')$, where $>'$ is an adjacent t-local monomial ordering with

$$\tau = C_>(I) \cap C_{>'}(I) \quad \text{and} \quad C_>(I) \neq C_{>'}(I),$$

and $G' \subseteq I$ is an initially reduced standard basis with respect to $>'$.

1: Compute a standard basis H' of $\mathrm{in}_w(I)$ with respect to $>_{(w,v)}$.
2: Set $G' := \mathrm{Lift}(H', >_{(w,v)}, H, G, >)$.
3: **return** $(G', >_{(w,v)})$

PROOF. By our Lifting Algorithm 3.3.2, G' is an initially reduced standard basis of I with respect to $>_{(w,v)}$ and the remaining conditions follow from Proposition 3.3.4. □

Example 3.3.6 Consider the ideal

$$I := \langle 2-t, xy^2 - t^2y^3, x^2 - t^3y^2 \rangle \trianglelefteq \mathbb{Z}[\![t]\!][x,y]$$

and the weighted monomial ordering $>=>_u$ on $\mathrm{Mon}(t,x)$ with weight vector $u := (-1,1,1) \in \mathbb{R}_{<0} \times \mathbb{R}^2$ and t-local lexicographical ordering such that $x > y > 1 > t$ as tiebreaker. An initially reduced standard basis of I is then given by

$$G := \{2-t, xy^2 - t^2y^3, x^2 - t^3y^2, t^3y^4\}.$$

The maximal Gröbner cone $C_>(I) \subseteq \mathbb{R}_{\leq 0} \times \mathbb{R}^2$ is determined by the inequalities

$$\begin{aligned} (w_t, w_x, w_y) \in C_>(I) \quad &\Longleftrightarrow \quad \begin{cases} w_x + 2w_y \geq 2w_t + 3w_y \\ 2w_x \geq 3w_t + 2w_y \end{cases} \\ &\Longleftrightarrow \quad \begin{cases} w_x \geq 2w_t + w_y \\ 2w_x \geq 3w_t + 2w_y \end{cases} \end{aligned}$$

It is easy to see how $w := (-4,1,7)$ is contained in $C_>(I)$. In fact, it lies on its boundary since $2w_x = 3w_t + 2w_y = 2$. Then $v := (3,5,1) \in \mathbb{R}^3$ is an outer normal vector, as even for small $\varepsilon > 0$

$$\underbrace{2(w_x + \varepsilon \cdot v_x)}_{2+10\varepsilon} \not\geq \underbrace{3(w_t + \varepsilon \cdot v_t)}_{-12+9\varepsilon} + \underbrace{2(w_y + \varepsilon \cdot v_y)}_{14+2\varepsilon}.$$

An initially reduced standard basis of $\mathrm{in}_w(I)$ is then given by

$$H := \{\mathrm{in}_w(g) \mid g \in G\} = \{2, xy^2, x^2 - t^3y^2, t^3y^4\},$$

and computing standard basis of $\mathrm{in}_w(I)$ with respect to the doubly weighted ordering $>_{(w,v)}$ yields

$$H' := \{2, xy^2, t^3y^2 - x^2, x^3\},$$

which can then be lifted to a standard basis of I with respect to the same ordering $>_{(w,v)}$ that is adjacent to $>$

$$G' = \{2-t, xy^2 - t^2y^3, t^3y^2 - x^2, x^3 - t^5y^3\}.$$

The Gröbner fan algorithm is a so-called fan traversal algorithm. We start with computing a starting cone and repeatedly use Algorithm 3.3.5 to compute adjacent cones until we obtain the whole fan. The whole process is commonly illustrated on a bipartite graph as shown in Figure 12. This bipartite graph also satisfies the so-called reverse search property, which can be used for further optimization. See Chapter 3.2 in **[Jen07]** for more information about the reverse search property of Gröbner fans.

Note that since the Gröbner fan spans the whole weight space $\mathbb{R}_{\leq 0} \times \mathbb{R}^n$, each facet is contained in exactly two maximal cones. That means, traversing a facet $\tau \leq C_>(I)$ of a maximal Gröbner cone $C_>(I)$ can be omitted if τ is contained in any other of already computed maximal Gröbner cones.

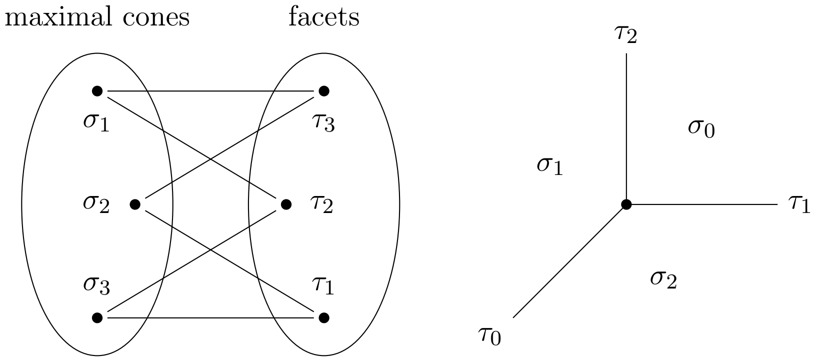


FIGURE 12. The bipartite graph of a Gröbner fan $\Sigma(\langle x+y+z\rangle)$

Algorithm 3.3.7 Gröbner fan

Input: $F \subseteq I \trianglelefteq R[\![t]\!][x]$ an x-homogeneous generating set.
Output: Δ the maximal cones Gröbner fan $\Sigma(I)$ of I.

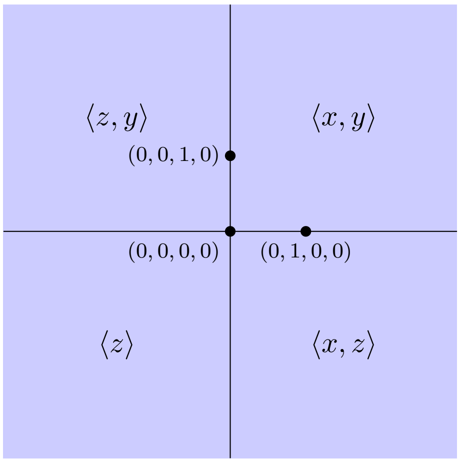
Pick a random weight $u \in \mathbb{R}_{<0} \times \mathbb{R}^n$ and a t-local monomial ordering $>$.
Compute an initially reduced standard basis G of I with respect to $>_u$ using Algorithm 3.2.9.
Construct the maximal Gröbner cone $C_{>_u}(I) = C(G, >_u)$.
Initialize the Gröbner fan $\Sigma := \{C_{>_u}(I)\}$.
Initialize a working list $L := \{(G, >_u, C_{>_u}(I))\}$.
while $L \neq \emptyset$ **do**
  Pick $(G, >_u, C_{>_u}(I)) \in L$.
  for all facets $\tau \leq C_{>_u}(I)$, $\tau \not\subseteq \{0\} \times \mathbb{R}^n$ **do**
    Compute a relative interior point $w \in \tau$.
    if $w \notin C_{>'}(I)$ for all $C_{>'}(I) \in \Sigma \setminus \{C_{>_u}(I)\}$ **then**
      Compute an outer normal vector v of τ.
      Set $H := \{\mathrm{in}_w(g) \mid g \in G\}$.
      Compute $(G', >') := \mathrm{Flip}(G, H, v, >_u)$ using Algorithm 3.3.5.
      Construct the adjacent Gröbner cone $C_{>'}(I) = C(G', >')$.
      Compute a relative interior point $u' \in C_{>'}(I)$, so that G' is a standard basis with respect to $>_{u'}$ and $C_{>'}(I) = C_{>_{u'}}(I)$.
      Set $\Sigma := \Sigma \cup \{C_{>'_u}(I)\}$.
      Set $L := L \cup \{(G', >_{u'}, C_{>_{u'}}(I)\}$.
  Set $L := L \setminus \{(G, >_u, C_{>_u}(I))\}$.
return Δ

Example 3.3.8 For an easy but clear example, consider the ideal

$$I := \langle x+z, y+z\rangle \trianglelefteq \mathbb{Z}[\![t]\!][x,y,z].$$

Because it is weighted homogeneous with respect to $(-1,0,0,0) \in \mathbb{R}_{<0} \times \mathbb{R}^3$ and $(0,1,1,1) \in \{0\} \times \mathbb{R}^3$, its Gröbner fan is closed under translation by $(-1,0,0,0)$ and invariant under translation by $(0,1,1,1)$. We therefore, concentrate on weight vectors on the hyperplane $\{0\} \times \mathbb{R}^2 \times \{0\}$, since any other weight vector in the closed lower halfspace can be generated out of them via the translations.

Looking only at potential leading terms of the generators, one might be led to believe that the Gröbner fan $\Sigma(I)$ restricted to $\{0\} \times \mathbb{R}^2 \times \{0\}$ is of the form

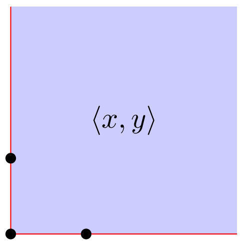


Let us use our algorithm to see why this is not the case:

We start with a random weight vector u, say $u = (0,1,1,0)$, and a random t-local monomial ordering $>$ to be used as tiebreaker. Then $\{\underline{x} + z, \underline{y} + z\}$ already is an initially reduced standard basis with respect to $>_u$, leading terms underlined, so that by Lemma 3.1.11

$$w' \in C_u(I) \quad \Longleftrightarrow \quad \begin{cases} \deg_{w'}(x) \geq \deg_{w'}(z) = 0, \\ \deg_{w'}(y) \geq \deg_{w'}(z) = 0. \end{cases}$$

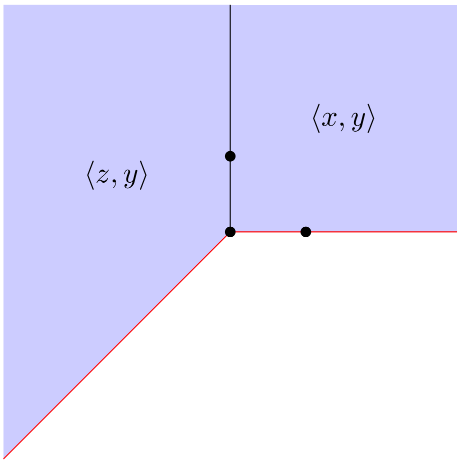
Hence, $C_u(I)$ is the upper left quadrant of the image above, with two facets available for the traversal.



Picking τ to be the upper ray of $C_u(I)$, $w = (0,0,1,0)$ a relative interior point inside of it and $v = (0,-1,0,0)$ an outer normal vector on it, we see that $\mathrm{in}_w(x+z) = \underline{z} + x$ and $\mathrm{in}_w(y+z) = \underline{y}$ already form an initially reduced standard basis of $\mathrm{in}_w(I)$ with respect to $>_{(w,v)}$. Therefore, this standard basis of $\mathrm{in}_w(I)$ lifts again to the very same standard basis $\{\underline{z} + x, \underline{y} + z\}$ of I for the adjacent ordering.

However that standard basis is not initially reduced anymore, and a quick calculation yields the initially reduced standard basis $\{\underline{z} + x, \underline{y} - x\}$, and we obtain

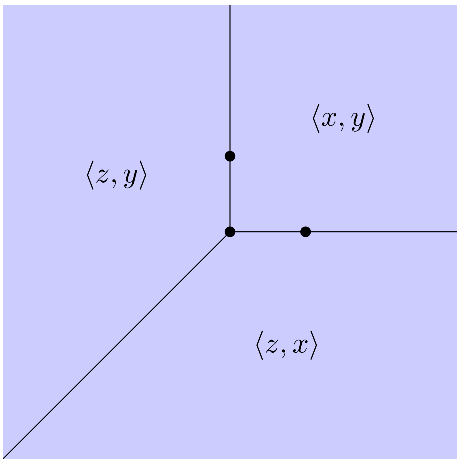
$$w' \in C_{>_{(w,v)}}(I) \quad \Longleftrightarrow \quad \begin{cases} 0 = \deg_{w'}(z) \geq \deg_{w'}(x), \\ \deg_{w'}(y) \geq \deg_{w'}(x). \end{cases}$$



Now let τ be the lower ray of our newly computed Gröbner cone, $w = (0,-1,-1,0)$ a relative interior point and $v = (0,1,-1,0)$ an outer normal vector. We see that $\mathrm{in}(z+x) = \underline{z}$ and $\mathrm{in}_w(y-x) = -\underline{x}+y$ already form an initially reduced standard basis of $\mathrm{in}_w(I)$ with respect to $>_{(w,v)}$, which is why it will lift again to the same standard basis $\{\underline{z}+x, -\underline{x}+y\}$ of I for the adjacent ordering.

As before, this standard basis is not initially reduced anymore, and a quick calculation yields the initially reduced standard basis $\{\underline{z}+y, -\underline{x}+y\}$, which means

$$w' \in C_{>_{(w,v)}}(I) \quad \Longleftrightarrow \quad \begin{cases} 0 = \deg_{w'}(z) \geq \deg_{w'}(y), \\ \deg_{w'}(x) \geq \deg_{w'}(y). \end{cases}$$



And this is how the Gröbner fan $\Sigma(I)$ actually looks like. The misconception at the beginning of the example was due to the oversight that $\mathrm{in}_w(x+z) =$

$\mathrm{in}_w(y+z) = z$ do not generate $\mathrm{in}_w(I)$, because $\{x + \underline{z}, y + \underline{z}\}$ is no initially reduced standard basis for $>_w$.

CHAPTER 4

Computing tropical varieties

In this chapter, we present algorithms to compute the tropical varieties of x-homogeneous ideals in $R[\![t]\!][x]$, applying techniques developed in [**BJS$^+$07**]. Moreover, we discuss specific optimizations for ideals which are obtained through Theorem 1.2.13. For necessary conditions on our ground ring, please note Convention 4.1.1.

This chapter also concludes our proof, that tropical varieties over valued fields may be computed using normal standard bases techniques. In Theorem 1.2.13 of Chapter 1, we reduced the problem to computing tropical varieties over rings, for which we extended the classical standard basis theory in Chapter 2, allowing us to introduce Gröbner fans in Chapter 3.

4.1. Extension of known techniques

In this section, we present an algorithm to compute the tropical variety of an x-homogeneous ideal in $R[\![t]\!][x]$ (see Definition 1.2.11). Similar to the techniques developed in Bogart, Jensen, Speyer, Sturmfels and Thomas' work on tropical varieties of polynomial ideals over ground fields [**BJS$^+$07**], this is done in three stages.

First, we begin with an easy algorithm to determine tropical varieties of principal ideals. From it, we derive an algorithm to compute tropical varieties with one-codimensional homogeneity space. The simplicity of the convex geometry makes these kind of tropical varieties significantly easier to handle with the help of generic weights and tropical witnesses. Finally we come to the general case, and show how we can apply the previous algorithm to traverse the tropical variety, scouring for all Gröbner cones contained in it. See Figure 1 for a rough outline with numbering.

Convention 4.1.1 In this section, we again adopt Convention 1.2.2, in which we have: K a field with discrete non-trivial valuation, $\mathfrak{K}$ its residue field, R_ν its valuation ring, $p \in R_\nu$ a uniformizing parameter and $R \subseteq R_\nu$ is a dense, noetherian subring with $p \in R$. Both K and R_ν are assumed to be complete, so that we have exact sequences

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \langle p-t\rangle \cdot R[\![t]\!]_{\langle p-t\rangle}[x] & \longrightarrow & R[\![t]\!]_{\langle p-t\rangle}[x] & \longrightarrow & K[x] & \longrightarrow & 0, \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \langle p-t\rangle \cdot R[\![t]\!][x] & \longrightarrow & R[\![t]\!][x] & \xrightarrow[t\,\longmapsto\, p]{} & R_\nu[x] & \longrightarrow & 0.
\end{array}$$

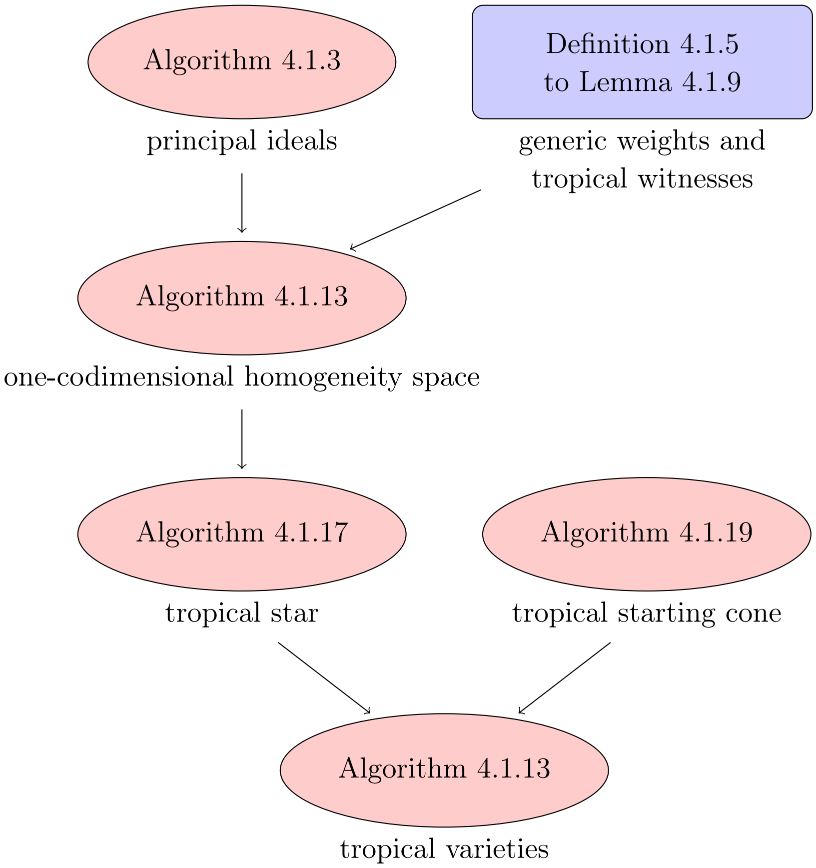


FIGURE 1. outline of Section 4.1

Moreover, we continue to require that R satisfies Convention 2.1.1 (which coincides with Convention 3.1.1), so that standard bases and the Gröbner fans both exist and are computable.

Fix a preimage $I \trianglelefteq R[\![t]\!][x]$ of a homogeneous prime ideal in $K[x]$. Recall that, by Corollary 1.2.14, $\mathrm{Trop}(I)$ is the support of a pure polyhedral complex of a fixed dimension.

Lemma 4.1.2 *Let $\Sigma(I)$ be the Gröbner fan of the x-homogeneous $I \trianglelefteq R[\![t]\!][x]$. Then*

$$\mathrm{Trop}(I) = \bigcup_{w \in \mathrm{Trop}(I)} C_w(I).$$

In particular, $\{C_w(I) \in \Sigma(I) \mid w \in \mathrm{Trop}(I)\}$ is a pure subfan of $\Sigma(I)$ with support $\mathrm{Trop}(I)$.

PROOF. Since, by Definition 3.1.5, all weight vectors in the relative interior of any Gröbner cone $C_w(I)$ yield the same initial ideal, it is clear that either $\mathrm{relint}(C_w(I)) \subseteq \mathrm{Trop}(I)$ or $\mathrm{relint}(C_w(I)) \cap \mathrm{Trop}(I) = \emptyset$.

And, since $\mathrm{Trop}(I)$ is closed in the euclidean topology by Definition 1.2.11, and $C_w(I)$ is a closed polyhedral cone by Proposition 3.1.19, $\mathrm{relint}(C_w(I)) \subseteq$

$\operatorname{Trop}(I)$ implies $C_w(I) \subseteq \operatorname{Trop}(I)$. Therefore,

$$\bigcup_{w \in \operatorname{Trop}(I)} C_w(I) \subseteq \operatorname{Trop}(I).$$

The opposite inclusion follows from $w \in C_w(I)$ for all weight vectors w. Also, the subfan is pure by Corollary 1.2.14. $\square$

We now start with the actual algorithms, as described in the beginning of the section.

Algorithm 4.1.3 $\operatorname{Trop}^{(1)}$, tropical varieties of elements in $R[\![t]\!][x]$
Input: $g \in R[\![t]\!][x]$.
Output: $\Delta = \operatorname{Trop}^{(1)}(g)$, a collection of maximal-dimensional polyhedral cones in $\mathbb{R}_{\leq 0} \times \mathbb{R}^n$ covering $\operatorname{Trop}(g)$, i.e.

$$\operatorname{Trop}(g) := \operatorname{Trop}(\langle g \rangle) = \textstyle\bigcup_{\sigma \in \Delta} \sigma.$$

Suppose $g = \sum_{\beta,\alpha} c_{\alpha,\beta} \cdot t^\beta x^\alpha$.
Construct the finite set of exponent vectors with minimal entry in t,

$$\Lambda := \left\{ (\beta, \alpha) \in \mathbb{N} \times \mathbb{N}^n \;\middle|\; \begin{array}{c} \alpha \in \mathbb{N}^n \text{ with } c_{\alpha,\beta'} \neq 0 \text{ for some } \beta' \in \mathbb{N} \\ \beta = \min\{\beta' \in \mathbb{N} \mid c_{\alpha,\beta'} \neq 0\} \end{array} \right\}.$$

Initialize $\Delta := \emptyset$.
for any two exponent vectors $a, b \in \Lambda$, $a \neq b$ **do**
  Construct

$$\sigma := \left\{ w \in \mathbb{R}_{\leq 0} \times \mathbb{R}^n \;\middle|\; w \cdot a = w \cdot b \geq w \cdot a' \text{ for all other } a' \in \Lambda \right\}.$$

  if $\dim(\sigma) = n$ **then**
    Set $\Delta := \Delta \cup \{\sigma\}$.
return Δ.

Proof. Follows from

$$w \in \operatorname{Trop}(g) \quad \Longleftrightarrow \quad \operatorname{in}_w(g) \text{ is no monomial.}$$ $\square$

Example 4.1.4 Given the element

$$\begin{aligned} g :=& (x + y + 1)(t^5 x + y + t^2) \\ =& \underline{t^5 x^2} + (t^5 + \underline{1) xy} + \underline{y^2} + (t^5 + \underline{t^2) x} + (t^2 + \underline{1) y} + \underline{t^2} \in \mathbb{Z}[\![t]\!][x, y]. \end{aligned}$$

its tropical variety is shown as in Figure 2. The polynomials next to the edges signify the initial form of g with respect to weights on them.

Note that unlike the easy example above suggests, not all initial forms with respect to weights in the tropical varieties need to be binomials. For example, for $g := x^2 + xy + y^2$ it is easy to see that if $\operatorname{in}_w(g)$ were to contain two of the terms, then it would also contain the third.

Next, we introduce generic weights and show how initial ideals with respect to generic weights can be computed without knowing what the generic

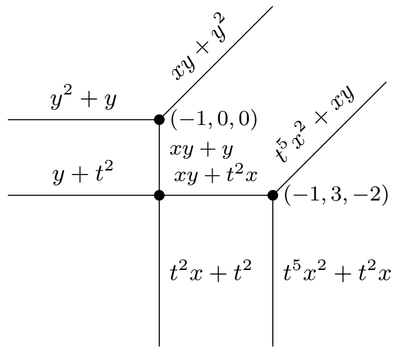


FIGURE 2. $\mathrm{Trop}((x + y + 1)(t^5x + y + t^2)) \cap \{-1\} \times \mathbb{R}^2$

weight is. This is similar to Proposition 3.3.4 and will be necessary in Algorithm 4.1.13 to decide whether an arbitrary polyhedral cone, generally not a Gröbner cone, is contained in the tropical variety.

Definition 4.1.5 (genericity of weight vectors) Let $I \trianglelefteq R[\![t]\!][x]$ be an ideal, and let $\sigma \subseteq \mathbb{R}_{\leq 0} \times \mathbb{R}^n$, $\sigma \nsubseteq \{0\} \times \mathbb{R}^n$, be a polyhedral cone in the weight space, not necessarily a Gröbner cone of I. Suppose $d = \dim(\sigma)$. Then we call a weight $w \in \mathrm{relint}(\sigma)$ *generic* with respect to I, if w does not lie in a Gröbner cone of I of dimension less than d.

Definition 4.1.6 (multiweights) Given weight vectors $w \in \mathbb{R}_{<0} \times \mathbb{R}^n$ and $v_1, \ldots, v_d \in \mathbb{R} \times \mathbb{R}^n$, we define the *initial form* of an element $g \in R[\![t]\!][x]$ with respect to the weight vectors $(w, v_1, \ldots, v_d)$ to be

$$\mathrm{in}_{(w,v_1,\ldots,v_d)}(g) = \mathrm{in}_{v_d} \ldots \mathrm{in}_{v_1} \mathrm{in}_w(g),$$

and we define the *initial form* of an ideal $I \trianglelefteq R[\![t]\!][x]$ with respect to the multidegree $(w, v_1, \ldots, v_d)$ to be

$$\mathrm{in}_{(w,v_1,\ldots,v_d)}(I) = \mathrm{in}_{v_d} \ldots \mathrm{in}_{v_1} \mathrm{in}_w(I) = \langle \mathrm{in}_{(w,v_1,\ldots,v_d)}(g) \mid g \in I \rangle.$$

Also given a t-local monomial ordering on $\mathrm{Mon}(t, x)$ we define the multiweighted ordering $>_{(w,v_1,\ldots,v_k)}$ to be

$$t^\beta \cdot x^\alpha >_{(w,v_1,\ldots,v_k)} t^\delta \cdot x^\gamma \quad \Longleftrightarrow \quad \text{either :}$$

- $w \cdot (\beta, \alpha) > w \cdot (\delta, \gamma)$ or
- $w \cdot (\beta, \alpha) = w \cdot (\delta, \gamma)$ and there exists an $1 \leq l \leq d$ with $v_i \cdot (\beta, \alpha) = v_i \cdot (\delta, \gamma)$ for all $1 \leq i < l$ and $v_l \cdot (\beta, \alpha) > v_l \cdot (\delta, \gamma)$ or
- $w \cdot (\beta, \alpha) = w \cdot (\delta, \gamma)$ and $v_i \cdot (\beta, \alpha) = v_i \cdot (\delta, \gamma)$ for all $1 \leq i \leq d$ and $t^\beta \cdot x^\alpha > t^\delta \cdot x^\gamma$.

Algorithm 4.1.7 initial ideal with respect to a generic weight
Input: (σ, w, G), where
(1) $\sigma \leq \mathbb{R}_{\leq 0} \times \mathbb{R}^n$, $\sigma \nsubseteq \{0\} \times \mathbb{R}^n$, a polyhedral cone,

(2) $w \in \mathrm{relint}(\sigma)$ a relative interior point,
(3) $G \subseteq I$ a generating set of an ideal $I \trianglelefteq R[\![t]\!][x]$.

Output: $(H', G', >') = \mathrm{in}_{(\sigma,w)}(G)$, where
(1) $>'$ a t-local monomial ordering on $\mathrm{Mon}(t,x)$,
(2) G' an initially reduced standard basis of I with respect to $>'$,
(3) $H' = \{\mathrm{in}_u(g) \mid g \in G\}$ for a weight vector $u \in \tau$ that is generic with respect to I.

Moreover, u can be chosen to lie arbitrary close to w.

1: Choose a basis $v_1, \ldots, v_d$, $d = \dim(\sigma)$, of the linear span of σ.
2: Pick a t-local monomial ordering $>$ on $\mathrm{Mon}(t,x)$ and consider the monomial ordering $>_{(w,v_1,\ldots,v_d)}$
3: Compute an initially reduced standard basis G' of $I = \langle g \mid g \in G\rangle$ with respect to $>_{(w,v_1,\ldots,v_d)}$.
4: Set $H' := \{\mathrm{in}_{(w,v_1,\ldots,v_d)}(g) \mid g \in G'\}$
5: **return** $(H', G', >_{(w,v_1,\ldots,v_d)})$.

PROOF. For sake of simplicity, let $>'$ denote the ordering $>_{(w,v_1,\ldots,v_k)}$. By definition of $>'$, it is easy to see that $\mathrm{LT}_{>'}(g_i') = \mathrm{LT}_{>'}(\mathrm{in}_w(g_i'))$ for all i. Lemma 3.1.11 then implies that $w \in C_{>'}(I)$, and hence Proposition 3.1.13 implies that $\{\mathrm{in}_w(g_1'), \ldots, \mathrm{in}_w(g_k')\}$ is an initially reduced standard basis of $\mathrm{in}_w(I)$.
Similarly, it is easy to see that $\mathrm{LT}_{>'}(\mathrm{in}_w(g_i')) = \mathrm{LT}_{>'}(\mathrm{in}_{v_1}\mathrm{in}_w(g_i'))$ for all i, leading to $\{\mathrm{in}_{(w,v_1)}(g_1'), \ldots, \mathrm{in}_{(w,v_1)}(g_k')\}$ being an initially reduced standard basis of $\mathrm{in}_{(w,v_1)}(I)$.
Continuing this train of thought, we obtain that $\{h_i', \ldots, h_k'\}$ is an initially reduced standard basis of $\mathrm{in}_{(w,v_1,\ldots,v_d)}(I)$ with respect to $>'$

Applying Proposition 3.3.4 repeatedly yields

$$\mathrm{in}_{w,v_1,\ldots,v_d}(I) = \mathrm{in}_{v_d} \ldots \mathrm{in}_{v_1} \mathrm{in}_w(I) \overset{\text{Prop.}}{\underset{3.3.4}{=}} \mathrm{in}_u(I)$$

for all $u = w + \varepsilon_1 \cdot v_1 + \ldots + \varepsilon_d \cdot v_d \in \mathbb{R}_{<0} \times \mathbb{R}^n$, provided all $\varepsilon_i > 0$ sufficiently small. In particular, these weight vectors u cannot lie in a Gröbner cone of dimension less than d, making these weights generic with respect to I. Moreover, $w \in \mathrm{relint}(\sigma)$ implies $u \in \mathrm{relint}(\sigma)$. □

Next, we introduce tropical witnesses for monomials in initial ideals and show that Algorithm 3.3.1 is sufficient for computing them. This is a necessary tool for Algorithm 4.1.13 for eliminating Gröbner cones from our consideration.

Definition 4.1.8 Suppose $C_w(I) \nsubseteq \mathrm{Trop}(I)$. We call an element $f \in I$ a *tropical witness* of $C_w(I)$ if

$$\mathrm{in}_v(f) \text{ is a monomial for all } v \in \mathrm{Relint}(C_w(I)).$$

Lemma 4.1.9 *Suppose* $C_w(I) \nsubseteq \mathrm{Trop}(I)$*, so that there exists a monomial* $m \in \mathrm{in}_w(I)$*. Let* G *be an initially reduced standard basis of* I *with respect* *to a t-local monomial ordering $>$ with $w \in C_>(I)$, and set $H := \{\mathrm{in}_w(g) \mid g \in G\}$. Then Algorithm 3.3.1 computes a tropical witness given the input $(m, H, G, >)$.*

PROOF. We know that the output $f \in I$ satisfies $\mathrm{in}_w(f) = m$. Now suppose $v \in \mathrm{Relint}(C_w(I))$. Then $\mathrm{in}_v(I) = \mathrm{in}_w(I)$, which by Proposition 3.1.17 implies $\mathrm{in}_v(g) = \mathrm{in}_w(g)$ for all $g \in G$. This means that the input for our homogeneous division with remainder, $(m, H, >)$, is homogeneous with respect to v, the monomial m being trivially homogeneous with respect to any weight. By Remark 2.1.18, our output, $(\{q_1, \ldots, q_k\}, 0)$, is then also homogeneous with respect to v, which allows us to show that $\mathrm{in}_v(f) = m$ using the very same arguments as in the proof of Algorithm 3.3.1. □

We now come to the algorithm for computing tropical varieties with one-codimensional homogeneity space. In the final part of its proof it becomes apparent how the homogeneity space is used to simplify its computation.

Definition 4.1.10 (homogeneity space) Given an ideal $I \trianglelefteq R[\![t]\!][x]$, we define the *homogeneity space* of I or of $\mathrm{Trop}(I)$ to be the intersection of all its interior Gröbner cones

$$C_0(I) := \bigcap_{w \in \mathbb{R}_{<0} \times \mathbb{R}^n} C_w(I).$$

Example 4.1.11 In this example, we illustrate that $C_0(I)$ in general is no linear subspace and is not the set of all vectors with respect to whom I is weighted homogeneous.

Consider the principal ideal

$$I = \langle x + y + t \cdot (x + y) \rangle \trianglelefteq \mathbb{Z}[\![t]\!][x, y],$$

whose Gröbner Fan splits the weight space $\mathbb{R}_{\leq 0} \times \mathbb{R}^2$ into two maximal cones, cf. Figure 3, and whose homogeneity space is given by

$$C_0(I) = \{(w_t, w_x, w_y) \in \mathbb{R}_{\leq 0} \times \mathbb{R}^n \mid w_x = w_y\}.$$

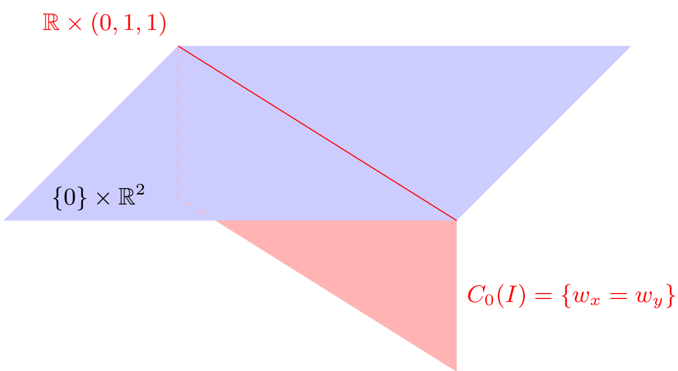


FIGURE 3. $C_0(I) \in \Sigma(\langle x + y + t \cdot (x + y) \rangle)$

In particular, note that $\mathbb{R}\cdot(0,1,1) \subseteq C_0(I)$, which is not surprising since I is (x,y)-homogeneous, making all Gröbner cones $C_w(I)$ invariant under translation by $(0,1,1)$.

We also see that every interior Gröbner cone $C_w(I)$ is closed under translation by weight vectors in $C_0(I)$,

$$u+v \in C_w(I) \text{ for all } u \in C_0(I) \text{ and all } v \in C_w(I),$$

which is again not surprising because $C_0(I)$ is a face of every interior Gröbner cone.

Observe, however, that $(-1,0,0) \in C_0(I)$ despite $x+y+t\cdot(x+y)$ not being weighted homogeneous with respect to it. This deviates from the classical of Gröbner fans of homogeneous ideals, in which $C_0(I)$ consists of all weight vectors under which I is weighted homogeneous.

We have seen that the homogeneity space is not the closure of the set of all weight vectors in $\mathbb{R}_{<0}\times\mathbb{R}$ under which the ideal is weighted homogeneous. However, this can only happen, if there are none of these vectors. Because if there are, we do have:

Lemma 4.1.12 *Suppose there exists a weight vector $w \in \mathbb{R}_{<0}\times\mathbb{R}^n$ with $\mathrm{in}_w(I)\cdot R[\![t]\!][x] = I$. Then*

$$C_0(I) = C_w(I) = \overline{\{w' \in \mathbb{R}_{<0}\times\mathbb{R}^n \mid \mathrm{in}_{w'}(I)\cdot R[\![t]\!][x] = I\}}.$$

Proof. The inclusion $C_0(I) \subseteq C_w(I)$ is clear, as $C_0(I)$ is a subset of every interior Gröbner cone by definition.

For the other inclusion, let v be any vector in $\mathbb{R}_{<0}\times\mathbb{R}^n$. Then $\mathrm{in}_{w+\varepsilon\cdot v}(I) = \mathrm{in}_v(\mathrm{in}_w(I)) = \mathrm{in}_v(I)$ for $\varepsilon > 0$ sufficiently small. Thus $w+\varepsilon\cdot v$ is in $C_v(I)$ for all small $\varepsilon > 0$, which implies that w is in $C_v(I)$, and hence $C_w(I) \subseteq C_v(I)$. $\square$

Algorithm 4.1.13 $\mathrm{Trop}^{(2)}$, tropical varieties in $R[\![t]\!][x]$ with one-codimensional homogeneity space

Input: $(G,>)$, where $>$ a t-local monomial ordering on $\mathrm{Mon}(t,x)$ and G an x-homogeneous standard basis of an ideal $I \trianglelefteq R[\![t]\!][x]$ with respect to $>$. such that $\dim(\mathrm{Trop}(I)) = \dim(C_0(I)) + 1$.

Output: $\Delta = \mathrm{Trop}^{(2)}(I)$, a collection Δ of maximal-dimensional polyhedral cones in $\mathbb{R}_{\leq 0}\times\mathbb{R}^n$ covering $\mathrm{Trop}(I)$, i.e.

$$\mathrm{Trop}(I) = \textstyle\bigcup_{\sigma\in\Delta}\sigma.$$

1: Use Algorithm 4.1.3 and compute for each $g \in G$ a collection $\Delta(g) := \mathrm{Trop}^{(1)}(g)$ of polyhedral cones such that

$$\mathrm{Trop}(g) = \textstyle\bigcup_{\sigma\in\Delta(g)}\sigma.$$

2: Compute the common refinement of all these collections,

$$\Delta := \textstyle\bigwedge_{g\in G}\Delta(g).$$

3: Set $L := \Delta$.

while $L \neq \emptyset$ **do**
  Pick $\sigma \in L$ maximal and $w \in \operatorname{relint}(\sigma)$.
  Suppose $w = (w_t, w_x) \in \mathbb{R}_{<0} \times \mathbb{R}^n$, set $w_{\text{neg}} := (w_t, -w_x) \in \mathbb{R}_{<0} \times \mathbb{R}^n$.
  Use Algorithm 4.1.7 to compute, amongst others, generators H' of an initial ideal $\operatorname{in}_u(I)$ with respect to a generic weight $u \in \operatorname{relint}(\sigma)$ around w:
$$(H', G', >') = \operatorname{in}_{(\sigma,w)}(G).$$
  if $\operatorname{in}_u(I)$ contains a monomial $s \neq 0$ **then**
    Compute $g := \operatorname{Witness}(s, H', G', >')$.
    Set
$$\begin{aligned} G &:= G \cup \{g\}, \\ \Delta &:= \Delta \wedge \operatorname{Trop}(g), \\ L &:= L \wedge \operatorname{Trop}(g). \end{aligned}$$
  else if $w_{\text{neg}} \in \sigma$ **then**
    Use Algorithm 4.1.7 to compute, amongst others, generators H' an initial ideal $\operatorname{in}_u(I)$ with respect to a generic weight $u \in \operatorname{relint}(\sigma)$ around w_{neg}:
$$(H', G', >') = \operatorname{in}_{(\sigma,w_{\text{neg}})}(G).$$
    if $\operatorname{in}_u(I)$ contains a monomial $s \neq 0$ **then**
      Compute $g := \operatorname{Witness}(s, H', G', >)$.
      Set
$$\begin{aligned} G &:= G \cup \{g\}, \\ \Delta &:= \Delta \wedge \operatorname{Trop}(g), \\ L &:= L \wedge \operatorname{Trop}(g). \end{aligned}$$
  else
    Set $L := L \setminus \{\sigma\}$.
return Δ

Proof. To show the termination, observe that the number of Gröbner cones whose interiors intersect Δ non-trivially is overall decreasing due to Lemma 4.1.9. In fact, in each iteration, either the finite working list L or the number of these Gröbner cones decreases, hence the termination.

To show correctness, first note that $\operatorname{Trop}(I) \subseteq \bigcap_{g \in G} \operatorname{Trop}(g) = \bigcup_{\sigma \in \Delta} \sigma$, because $G \subseteq I$.

For the opposite inclusion, observe that for all $\sigma \in \Delta$ maximal the first inclusion implies $\dim(\sigma) \geq \dim(\operatorname{Trop}(I))$. And if $\dim(\sigma) > \dim(\operatorname{Trop}(I))$, any initial ideal with respect to a generic weight would contain a monomial, which contradicts σ successfully passing the tests for monomials. Hence $\dim(\sigma) = \dim(\operatorname{Trop}(I))$. Moreover, $C_0(I) \subseteq \operatorname{Trop}(I) \subseteq \operatorname{Trop}(g)$ implies that $C_0(I) \subseteq \sigma$, so that all $\sigma \in \Delta$ maximal are cones of dimension $\dim(\operatorname{Trop}(I))$ with a one-codimensional subset $C_0(I) \subseteq \sigma$.

Now consider a weight vector $u \in \sigma$ for some $\sigma \in \Delta$ maximal. Let $w \in \sigma$ be the relative interior point chosen in Step 5 and w_{neg} the weight vector constructed in Step 6. Because u and w are linearly dependent modulo the span of $C_0(I)$ for dimensional reasons, there exists a $v \in C_0(J)$ and a $\lambda > 0$ such that one of the following cases holds:

(1) $u = \lambda \cdot w + v$,
(2) $u + v = \lambda \cdot w$,
(3) $u = \lambda \cdot w_{\text{neg}} + v$,
(4) $u + v = \lambda \cdot w_{\text{neg}}$.

Because all Gröbner cones are closed under translation by v, the first case yields

$$\text{in}_u(I) = \text{in}_{\lambda\cdot w+v}(I) = \text{in}_{\lambda\cdot w}(I) = \text{in}_w(I).$$

Similarly, in the second case we obtain

$$\text{in}_u(I) = \text{in}_{u+v}(I) = \text{in}_{\lambda\cdot w}(I) = \text{in}_w(I).$$

Cases (3) and (4) are analogous, and in either case $\text{in}_u(I)$ is monomial free, implying that $u \in \text{Trop}(I)$. $\square$

Example 4.1.14 Consider the ideal $I \trianglelefteq \mathbb{Z}[\![t]\!][x_1, \ldots, x_4]$ generated by

$$g_0 := 3, \qquad g_1 := tx_3x_4 - tx_1x_2 + x_1^2,$$
$$g_2 := tx_1x_2^2 - x_1^2x_2 - t^3x_1x_2x_3 + t^2x_1^2x_3,$$

which, one can show, is a 3-dimensional ideal with homogeneity space

$$C_0(I) = \text{Cone}((-2, -1, 1, 5, -5)) + \mathbb{R} \cdot (0, 1, 1, 1, 1).$$

Figure 4 shows the combinatorial structure of $\bigcap_{i=0}^2 \text{Trop}(g_i)$.

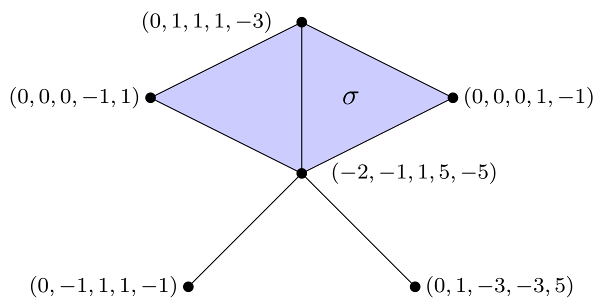


FIGURE 4. combinatorial structure of $\bigcap_{i=0}^2 \text{Trop}(g_i)$

For the sake of simplicity, we will omit the computation of $\bigcap_{i=0}^2 \text{Trop}(g_i)$. Nonetheless, one cone σ that can be easily seen to be contained in it is

$$\sigma := \text{Cone}(\underbrace{(0,0,0,-1,1)}_{=:w_1}, \underbrace{(-2,-1,1,5,-5)}_{=:w_2}, \underbrace{(0,1,1,1,-3)}_{=:w_3}) + \mathbb{R} \cdot (0,1,1,1,1).$$

Ignoring g_0 since $\mathrm{Trop}(g_0) = \mathbb{R}_{\leq 0} \times \mathbb{R}^4$ doesn't contribute anything to the intersection, we have

$$g_1 = \underbrace{tx_3x_4 \overbrace{-tx_1x_2 + x_1^2}^{\mathrm{in}_{w_3}(g_1)}}_{\mathrm{in}_{w_1}(g_1),\, \mathrm{in}_{w_2}(g_1)}, \quad g_2 = \underbrace{\overbrace{tx_1x_2^2 - x_1^2x_2}^{\mathrm{in}_{w_1}(g_2)} - t^3x_1x_2x_3 + t^2x_1^2x_3}_{\mathrm{in}_{w_2}(g_2),\, \mathrm{in}_{w_3}(g_2)}.$$

Therefore, for any weight $w \in \sigma$, $\mathrm{in}_w(g_1)$ contains at least the binomial $-tx_1x_2 + x_1^2$, while $\mathrm{in}_w(g_2)$ contains at least the binomial $tx_1x_2 - x_1^2x_2$. In particular, neither are monomials. As a side note, it is not surprising that w_2 does not cut out a single proper initial form while w_1 and w_3 do considering that it is contained in the homogeneity space.

However, it can be shown that, for

$$g_3 := tx_2x_3x_4 + t^2x_1^2x_3 - t^3x_1x_2x_3 \in I \quad \text{and} \quad w := (-1, 1, 2, 2, 0) \in \sigma,$$

$\mathrm{in}_w(g_3) = tx_2x_3x_4$ is in fact a monomial, which implies that $\sigma \not\subseteq \mathrm{Trop}(I)$. Not that we would have expected otherwise considering the dimensions, $\dim(\sigma) = 4 > 3 = \dim(\mathrm{Trop}(I))$. Figure 5 illustrates the combinatorial structure of $\bigcap_{i=0}^{3} \mathrm{Trop}(g_i)$, red highlighting what has been eliminated when intersecting our previous structure with $\mathrm{Trop}(g_3)$.

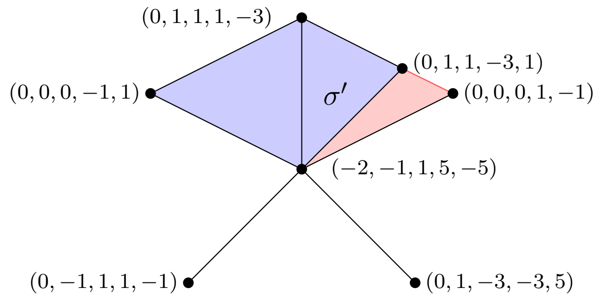


FIGURE 5. combinatorial structure of $\bigcap_{i=0}^{3} \mathrm{Trop}(g_i)$

Continuing, the following cone can easily seen to be contained in the intersection of the tropical varieties of $g_0, \ldots, g_3$,

$$\sigma' := \mathrm{Cone}(\underbrace{(0,1,1,-3,1)}_{w_1}, \underbrace{(-2,-1,1,5,-5)}_{w_2}, \underbrace{(0,1,1,1,-3)}_{w_3}) + \mathbb{R}\cdot(0,1,1,1,1),$$

since

$$g_1 = \underbrace{tx_3x_4 \overbrace{-tx_1x_2 + x_1^2}^{\mathrm{in}_{w_1}(g_1),\, \mathrm{in}_{w_3}(g_1)}}_{\mathrm{in}_{w_2}(g_1)}, \quad g_2 = \underbrace{\overbrace{tx_1x_2^2 - x_1^2x_2}^{\mathrm{in}_{w_1}(g_2)} - t^3x_1x_2x_3 + t^2x_1^2x_3}_{\mathrm{in}_{w_2}(g_2),\, \mathrm{in}_{w_3}(g_2)},$$

$$g_3 = \underbrace{tx_2x_3x_4 \overbrace{+t^2x_1^2x_3 - t^3x_1x_2x_3}^{\mathrm{in}_{w_3}(g_3)}}_{\mathrm{in}_{w_1}(g_3),\, \mathrm{in}_{w_2}(g_3)}.$$

However, setting

$$g_4 := tx_2x_3x_4 - t^3x_3^2x_4 \in I \quad \text{and} \quad w' := (-1,3,4,5,0) \in \sigma',$$

$\mathrm{in}_{w'}(g_4) = tx_2x_3x_4$ is a monomial, implying that $\sigma' \nsubseteq \mathrm{Trop}(I)$, which is again not surprising because of the same dimensional reasons. Figure 6 illustrates the combinatorial structure of $\bigcap_{i=0}^4 \mathrm{Trop}(g_i)$, red highlighting what has been eliminated when intersecting our previous structure with $\mathrm{Trop}(g_4)$.

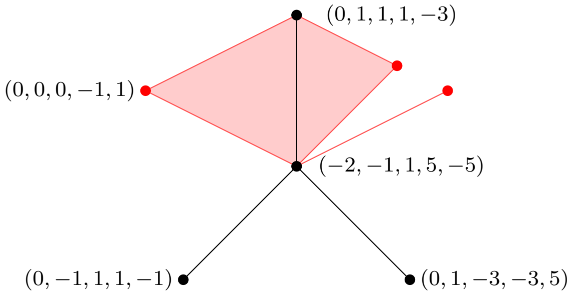


FIGURE 6. combinatorial structure of $\bigcap_{i=0}^4 \mathrm{Trop}(g_i)$

Further calculations will yield that $\mathrm{Trop}(I) = \bigcap_{i=0}^4 \mathrm{Trop}(g_i)$.

Remark 4.1.15 (searching for monomials via saturation) In Algorithm 4.1.13, we are required to decide whether $L := \mathrm{in}_u(I)$ is monomial free, and, if yes, find a monomial inside of it. This is a highly non-trivial task, as the containment of a monomial is generally not a property which can be easily read off from a Gröbner basis or a standard basis.

Consider for example, the ideal

$$L = \langle x^2 \cdot (x-z), y^2 \cdot (y-z), (x+y)\cdot(x+y+z)\rangle \trianglelefteq \mathbb{Q}[x,y,z],$$

whose unique reduced Gröbner basis with respect to the degree lexicographical ordering is

$$\{x^2 + 2xy + y^2 + xz + yz,\ y^3 - y^2z,\ 3xy^2 + 5xyz + 6y^2z + 2xz^2 + 2yz^2,$$
$$10y^2z^2 + xz^3 + yz^3,\ 5xyz^2 + 2xz^3 + 2yz^3,\ xz^4 + yz^4\},$$

yet who, as the intersection of three lines on the coordinate hyperplanes, definitely contains a monomial, for example $(xyz)^2$.

One possible way of seeing it, is to compute the saturation of L with respect to the product of all variables through repeated quotient constructions until it stabilizes, and keep track how many quotients were necessary. In our example, we would see that the quotients immediately stabilize after the second iteration,

$$L:(xyz)^1 \subsetneq L:(xyz)^2 = L:(xyz)^2 = \ldots = L:(xyz)^\infty = \langle 1\rangle.$$

Because the saturation yields the whole ring, L contains a monomial, and, since two quotients are necessary, L contains the monomial $(xyz)^2$. If $L:(xyz)^\infty \neq \langle 1\rangle$, then L would contain no monomial.

As a caveat, this method will generally not yield the monomial of lowest degree, which might be desirable for keeping the tropical witness as simple as possible. For example, it is possible to show that $x^5 \in L$ in our given ideal.

There are methods which are guaranteed to find a monomial whose exponent vector is minimal lexicographically, see Chapter 5 in [**Stu96**], which yields strictly smaller monomials than the method above, but those results are generally not of lowest degree either. Instead of saturating with respect to the product of all variables, the method saturates with respect to each variable individually.

Remark 4.1.16 (searching for monomials via Rabinowitsch) Another method for finding monomials in an ideal is the so-called *Rabinowitsch trick*. Originally introduced in order to prove the weak Nullstellensatz, it also works over general domains.

Let R be a domain, and let $I \trianglelefteq R$ be an ideal. Consider a univariate polynomial ring $R[u]$ over R. Then

$$f \in \sqrt{I} \quad \Longleftrightarrow \quad \langle 1 - u \cdot f \rangle + I = \langle 1 \rangle = R[u].$$

One direction is straight-forward: Suppose $f^l \in I$ for $l \in \mathbb{N}$ sufficiently high. Because

$$1 - u^l \cdot f^l = (1 - u \cdot f) \cdot (1 + uf + u^2 f + \ldots + u^{l-1} f^{l-1}) \in \langle 1 - u \cdot f \rangle,$$

we then have

$$1 = 1 - u^l \cdot f^n + u^l \cdot f^l \in \langle 1 - u \cdot f \rangle + I.$$

For the other direction, suppose $I = \langle b_1, \ldots, b_k \rangle \trianglelefteq R$ and let $a_0, a_1, \ldots, a_k \in R[u]$ so that

$$1 = a_0 \cdot (1 - u \cdot f) + a_1 \cdot b_1 + \ldots + a_k \cdot b_k.$$

Substituting u with $\frac{1}{f}$, this implies

$$\begin{aligned} 1 &= a_0 \left(\frac{1}{f}\right) \cdot \left(1 - \frac{1}{f} \cdot f\right) + a_1 \left(\frac{1}{f}\right) \cdot b_1 + \ldots + a_k \left(\frac{1}{f}\right) \cdot b_k \\ &= a_1 \left(\frac{1}{f}\right) \cdot b_1 + \ldots + a_k \left(\frac{1}{f}\right) \cdot b_k \in \mathrm{Quot}(R). \end{aligned}$$

Multiplying with a sufficiently high power f^l, $l \in \mathbb{N}$, we thus obtain $f^l \in I$.

Now if our ring is $R[\![t]\!][x]$ and $f = t \cdot x_1 \cdot \ldots \cdot x_n$ is the product of all variables, then

$$I \text{ contains a monomial} \quad \Longleftrightarrow \quad \langle 1 - u \cdot f \rangle + I = \langle 1 \rangle = R[u].$$

Using syzygies, we are able decide this question and compute a representation

$$1 = a_0 \cdot (1 - u \cdot f) + a_1 \cdot b_1 + \ldots + a_k \cdot b_k.$$

Keeping track of the powers of u in all the a_i, we can find a monomial in I as a linear combination of $b_1, \ldots, b_k$. But as we see, it has the very same caveat as the method using saturation in 4.1.15.

Given a facet of a maximal Gröbner cone in the tropical variety $\mathrm{Trop}(I)$, Algorithm 4.1.13 can now be used to determine directions in which adjacent maximal Gröbner cones the tropical variety lie.

Algorithm 4.1.17 tropical star

Input: $(H, G, >)$, where for a maximal Gröbner cone $C_w(I) \subseteq \mathrm{Trop}(I)$ in our tropical variety and $u \in C_w(I) \cap \mathbb{R}_{<0} \times \mathbb{R}^n$ a relative interior point on one of its facets,

(1) $>$ a t-local monomial ordering with $C_w(I) \leq C_>(I)$,
(2) $G \subseteq I$ an initially reduced x-homogeneous standard basis with respect to $>$,
(3) $H = \{\mathrm{in}_u(g) \mid g \in G\} \subseteq \mathrm{in}_u(I)$.

Output: $N \subseteq \mathbb{R} \times \mathbb{R}^n$ a minimal finite subset such that for any maximal Gröbner cone $C_{w'}(I) \subseteq \mathrm{Trop}(I)$ with $u \in C_{w'}(I)$ there exists a $v \in N$ such that $u + \varepsilon \cdot v \in C_{w'}(I)$ for $\varepsilon > 0$ sufficiently small.

Use Algorithm 4.1.13 to compute

$$\Delta := \mathrm{Trop}^{(2)}(H, >),$$

so that $\mathrm{Trop}(\mathrm{in}_u(I)) = \bigcup_{\sigma \in \Delta} \sigma$.

for $\sigma \in \Delta$ maximal **do**

  Compute a relative interior point $v_\sigma \in \sigma$.

return $N := \{v_\sigma \mid \sigma \in \Delta \text{ maximal}\}$.

Proof. Let $I := \mathrm{in}_u(I)$. By Proposition 3.3.4, we have $\mathrm{in}_v(I) = \mathrm{in}_{u+\varepsilon\cdot v}(I)$ for $\varepsilon > 0$ sufficiently small, which implies that $u + \varepsilon \cdot v \in \mathrm{Trop}(I)$ for $\varepsilon > 0$ sufficiently small if and only if $v \in \mathrm{Trop}(I)$. This implies the correctness of our output, provided we may use Algorithm 4.1.13 in the first place.

Because all maximal cones are of a fixed dimension by Lemma 4.1.2, $C_u(I) \subseteq \mathrm{Trop}(I)$ is one-codimensional, say $\dim(\mathrm{Trop}(I)) = k + 1$ and $\dim(C_u(I)) = k$. On the one hand, our previous considerations imply $\dim(\mathrm{Trop}(I)) = \dim(\mathrm{Trop}(I)) = k + 1$. On the other hand, by Definition 4.1.10 of the homogeneity space, we have $C_u(I) \subseteq C_0(I)$.

Therefore, either $\dim(C_0(I)) = k = \dim(\mathrm{Trop}(I)) - 1$ or $\dim(C_0(I)) = k+1 = \dim(\mathrm{Trop}(I))$. The latter is not possible since any inner normal vector of $C_u(I) \leq C_w(I)$ lies in $\mathrm{Trop}(I)$ but not in $C_0(I)$. Hence all prerequisites of Algorithm 4.1.13 are fulfilled. $\square$

Example 4.1.18 Consider the ideal $I \trianglelefteq \mathbb{Z}[\![t]\!][x_1, \ldots, x_4]$ generated by

$$\{3 - t, \quad 8tx_3x_4 + tx_1x_2 + 2x_1^2, \\ tx_1x_2^2 + 2x_1^2x_2 + 2t^3x_1x_2x_3 + 4t^2x_1^2x_3 - 64tx_1^3\}.$$

Its tropical variety has a homogeneity space generated by $(0, 1, 1, 1, 1)$, which is not surprising since the ideal is x-homogeneous, and it is combinatorially of the form as displayed in Figure 7. Note that the upper three vertices and the lower vertex have first component 0, thus they yield points at infinite if intersected with the affine hyperplane $\{-1\} \times \mathbb{R}^4$.

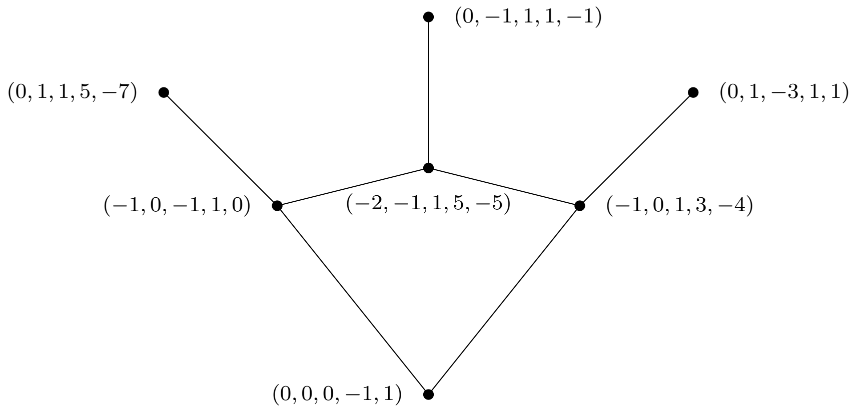


FIGURE 7. combinatorial structure of Trop(I)

In Example 4.1.14 we have computed the tropical variety of its initial ideal I with respect to $w = (-2,-1,1,5,-5) \in \mathbb{R}_{<0} \times \mathbb{R}^n$. Comparing Figure 6 and Figure 7, we see how Trop(I) determines the structure of Trop(I) around the weight vector w, see Figure 8.

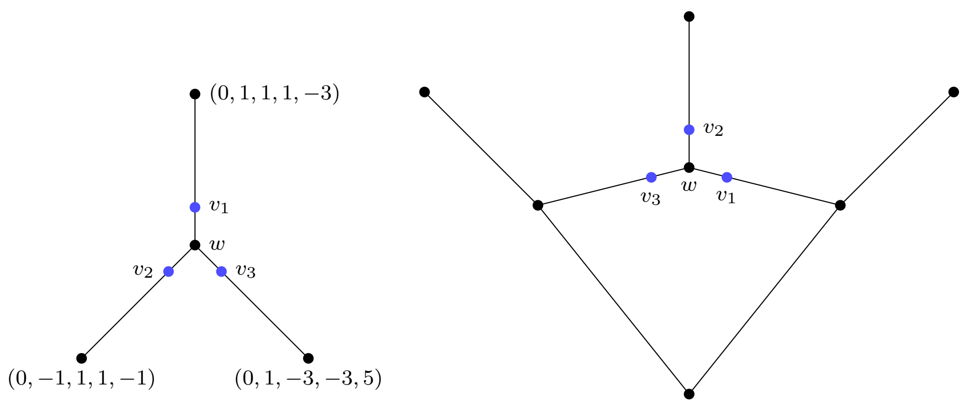


FIGURE 8. Trop($\mathrm{in}_w(I)$) and Trop(I)

Points v_1 that lie in Trop(I) on the edge to $(0,1,1,1,-3)$ and sufficiently close to w lie in Trop(I) on the edge to $(-1,0,1,3,-4)$, as both edge directions are linearly dependent,

$$(0,1,1,1,-3) - (-2,-1,1,5,-5) = (2,2,0,-4,2),$$
$$(-1,0,1,3,-4) - (-2,-1,1,5,-5) = (1,1,0,-2,1).$$

And obviously, points v_2 that lie in Trop(I) on the edge to $(0,-1,1,1,-1)$ also lie in Trop(I) on the edge to $(0,-1,1,1,-1)$.

Finally, points v_3 that lie in Trop(I) on the edge to $(0,1,-3,-3,5)$ lie in Trop(I) on the edge to $(-1,0,-1,1,0)$, as both edge directions are again linearly dependent,

$$(0,1,-3,-3,5)-(-2,-1,1,5,-5)=(2,2,-4,-8,10),$$
$$(-1,0,-1,1,0)-(-2,-1,1,5,-5)=(1,1,-2,-4,5).$$

Next we describe an algorithm to compute a starting cone in our tropical variety.

Algorithm 4.1.19 starting cone

Input: $(G,>)$, where G is an initially reduced standard basis of I with respect to the t-local monomial ordering $>$.

Output: $(H',G',>')$, where

(1) $>'$ a t-local monomial ordering with $C_{w'}(I)\leq C_{>'}(I)$ for a maximal Gröbner cone $C_{w'}(I)\subseteq \mathrm{Trop}(I)$,

(2) $G'\subseteq I$ an initially reduced standard basis with respect to $>'$,

(3) $H'=\{\mathrm{in}_{w'}(g)\mid g\in G'\}\subseteq \mathrm{in}_{w'}(I)$.

1: **if** $\dim(I)=\dim(C_0(I))$ **then**

2: **return** $(G,G,>)$

3: Find a weight vector $w\in(\mathrm{Trop}(I)\setminus C_0(I))\cap(\mathbb{R}_{<0}\times\mathbb{R}^n)$.

4: Let $>_w$ denote a weighted ordering with weight vector w and an arbitrary tiebreaker.

5: Compute an initially reduced standard basis G'' of I with respect to $>_w$.

6: Set $H'':=\{\mathrm{in}_w(g)\mid g\in G''\}$.

7: Rerun
$$(H',G'_0,>'_0)=\mathrm{startingCone}(H'',>_w).$$

8: Let $>'$ be the weighted monomial ordering with weight vector w and tiebreaker $>'_0$.

9: Lift G'_0 to an initially reduced standard basis G' of I:
$$G'=\mathrm{Lift}(G'_0,>',H'',G'',>_w).$$

10: **return** $(H',G',>')$

Proof. Labelling all the objects appearing in the ν-th recursion step by a subscript ν we have

$$\dim C_0(I_0)<\dim C_0(I_1)<\dim C_0(I_2)<\ldots,$$

as $w_\nu\notin C_0(I_\nu)$ yet $w_\nu\in C_0(I_{\nu+1})$ for all ν. And since all $\dim C_0(I_\nu)$ are strictly bounded from above by the dimension of $\mathbb{R}_{\leq 0}\times\mathbb{R}^n$, the recursions stop eventually and our algorithm terminates.

For the correctness of the output make an induction on the number of recursions. If there are no recursions, i.e. $\dim(I)=\dim(C_0(I))$, then $(G,G,>)$ obviously satisfies the conditions (1), (2) and (3).

If recursions do happen, then, by induction hypothesis, the output $(H',G'_0,>'_0)$ in Step 7 is correct. Setting $J:=\mathrm{in}_w(I)$, this means:

(i) $>'_0$ is a t-local monomial ordering with $C_{w'_0}(J)\leq C_{>'_0}(J)$ for a maximal Gröbner cone $C_{w'_0}(J)\subseteq\mathrm{Trop}(J)$,

(ii) $G_0' \subseteq J$ is an initially reduced standard basis with respect to $>_0'$,

(iii) $H' = \{\mathrm{in}_{w_0'}(g) \mid g \in G_0'\} \subseteq \mathrm{in}_{w_0'}(J)$.

Recall that by Proposition 3.3.4, we have $\mathrm{in}_v(J) = \mathrm{in}_{w+\varepsilon\cdot v}(I)$ for $\varepsilon > 0$ sufficiently small. Setting $w' := w + \varepsilon \cdot w_0'$ for $\varepsilon > 0$ sufficiently small, this implies that $C_{w'}(I) \subseteq \mathrm{Trop}(I)$ and

$$\dim C_{w'}(I) = \dim C_{w_0'}(J) \overset{\text{(i)}}{=} \dim \mathrm{Trop}(J) = \dim \mathrm{Trop}(I).$$

Moreover, because we have set $>'$ to be a weighted ordering with weight vector w', we have $C_{w'}(I) \leq C_{>'}(I)$ and condition (1) on the t-local ordering $>'$ is fulfilled.

The remaining conditions (2) and (3) now follow directly from the correctness of our lifting algorithm, which we may use by construction of G'' and H'' and because of $w \in C_{>'}(I) \cap C_{>_w}(I)$:

(a) G' is an initially reduced standard basis with respect to $>'$,

(b) $G_0' = \{\mathrm{in}_w(g) \mid g \in G'\}$.

Condition (2) is equivalent to condition (a), and condition (3) follows from

$$H' \overset{\text{(iii)}}{=} \{\mathrm{in}_{w_0'}(g) \mid g \in G_0'\} \overset{\text{(b)}}{=} \{\mathrm{in}_{w_0'} \mathrm{in}_w(g) \mid g \in G'\}$$

and $w' := w + \varepsilon \cdot w_0'$. $\square$

Remark 4.1.20 In Step 3 of the previous algorithm, it is necessary to find a non-trivial point w in the tropical variety. This can be achieved by traversing the Gröbner fan as in Algorithm 3.3.7 and checking all Gröbner cones whether they have a ray that is contained inside the tropical variety.

Due to the repeated transition to initial ideals, see Steps 6 and 7, the ideal and hence its Gröbner fan become simpler in each iteration. For example, the dimension of the homogeneity space is strictly increasing, as $w \notin C_0(I)$ but $w \in C_0(\mathrm{in}_w(I))$.

Example 4.1.21 Consider the ideal $I \trianglelefteq \mathbb{Z}[\![t]\!][x_1, \ldots, x_4]$ from the previous Example 4.1.18 generated by

$$\{3 - t, \quad 24x_3x_4 + 3x_1x_2 + 2x_1^2, \quad x_2x_3x_4 + 18x_3^2x_4 + 8x_1^3\}.$$

A short calculation reveals that $\dim(\mathrm{Trop}(I)) = \dim(I) = 3 > 1 = \dim(C_0(I))$ with

$$C_0(I) = \mathbb{R} \cdot (0, 1, 1, 1, 1).$$

Picking a weighted monomial ordering $>$ with weight vector $(-1, 1, 11, 3, 19)$, one can show that its maximal Gröbner cone $C_>(I)$ has a ray generated by $w := (-2, -1, 1, 5, -5)$, which is contained in the tropical variety. The initial ideal $\mathrm{in}_w(I)$ is then generated by

$$\{3, \quad tx_3x_4 - tx_1x_2 + x_1^2, \quad tx_1x_2^2 - x_1^2x_2 - t^3x_1x_2x_3 + t^2x_1^2x_3\}.$$

Another short calculation shows that $\dim(\mathrm{in}_w(I)) = 3 > 2 = \dim(C_0(\mathrm{in}_w(I)))$ with

$$C_0(\mathrm{in}_w(I)) = \mathbb{R}_{\geq 0} \cdot w + \mathbb{R} \cdot (0, 1, 1, 1, 1).$$

Sticking to the same weighted monomial ordering $>$, we see that its maximal Gröbner cone $C_>(\mathrm{in}_w(I))$ has a ray generated by $v := (6, -11, 11, -1, 1)$, which is again contained in the tropical variety $\mathrm{Trop}(\mathrm{in}_w(I))$. The initial ideal $\mathrm{in}_v\,\mathrm{in}_w(I)$ is then generated by

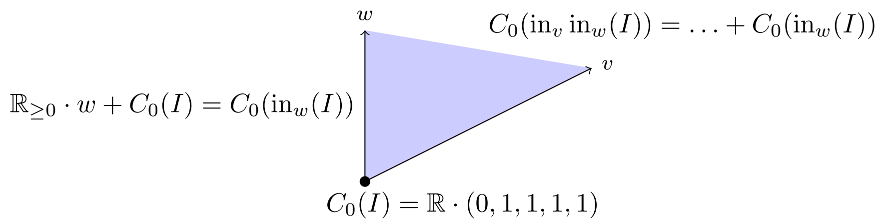
$$\{3, \quad tx_3x_4 - tx_1x_2, \quad x_2x_3x_4 - t^2x_1x_2x_3\}.$$

It follows that $\dim(\mathrm{in}_v\,\mathrm{in}_w(I)) = 3 = \dim(C_0(\mathrm{in}_v\,\mathrm{in}_w(I)))$ with

$$C_0(\mathrm{in}_v\,\mathrm{in}_w(I)) = \mathbb{R}_{\geq 0} \cdot (-1, 1, -1, 1, -1) + \mathbb{R} \cdot (0, 1, 0, 0, 1) + \mathbb{R} \cdot (0, 0, 1, 1, 0),$$

so that the recursions end.

Summing up, this proves that $w + \varepsilon \cdot v \in \mathrm{Trop}(I)$ for $\varepsilon > 0$ sufficiently small. Together with $C_0(I)$ generated by $(0, 1, 1, 1, 1)$ that is naturally contained in every Gröbner cone, this determines a maximal Gröbner cone in our tropical variety. What remains is to compute an initially reduced standard basis of $\mathrm{in}_{w+\varepsilon\cdot v}(I)$ and I with respect to $>$, which we will return. Using that data, Algorithm 3.1.28 constructs a maximal Gröbner cone in our tropical variety.



$$1 = \dim(C_0(I)) < \dim(C_0(\mathrm{in}_w(I))) < \dim(C_0(\mathrm{in}_v\,\mathrm{in}_w(I))) = \dim \mathrm{Trop}(I) = 3$$

FIGURE 9. computing a tropical starting cone

Algorithm 4.1.22 tropical variety

Input: $(G_{\text{input}}, >_{\text{input}})$, where G_{input} is an initially reduced standard basis of I with respect to the t-local monomial ordering $>_{\text{input}}$.

Output: $\Delta = \mathrm{Trop}(G, >)$, a collection of maximal Gröbner cones $C_w(I) \subseteq \mathrm{Trop}(I)$ covering $\mathrm{Trop}(I)$, i.e.

$$\mathrm{Trop}(I) = \textstyle\bigcup_{C_w(I) \in \Delta} C_w(I).$$

Compute an initially reduced standard bases for a starting cone

$$(H, G, >) = \mathrm{startingCone}(G_{\text{input}}, >_{\text{input}}).$$

Construct the corresponding Gröbner cone

$$C_w(I) = C(H, G, >),$$

so that $w \in \mathrm{Trop}(I)$ and $C_w(I) \subseteq \mathrm{Trop}(I)$ maximal.

Initialize $\Delta := \{C_w(I)\}$.

Initialize a working list $L := \{(G, >, C_w(I))\}$.

while $L \neq \emptyset$ **do**

Pick $(G, >, C_w(I)) \in L$.
for all $\tau \leq C_w(I)$, $\tau \nsubseteq \{0\} \times \mathbb{R}^n$ **do**
Compute a relative interior point $u \in \tau$.
Set $H := \{\mathrm{in}_u(g) \mid g \in G\}$
Compute a set of normal vectors based on its tropical star

$$N = \mathrm{tropicalStar}(H, G, >).$$

for $v \in N$ **do**
if $u + \varepsilon \cdot v \notin C_w(I)$, $\varepsilon > 0$ suff. small, for all $C_w(I) \in \Delta$ **then**
Compute an initially reduced standard basis with respect to the adjacent ordering

$$(G', >') := \mathrm{Flip}(G, H, v, >).$$

Set $H' := \{\mathrm{in}_{(u,v)}(g) \mid g \in G'\}$.
Construct the adjacent Gröbner cone

$$C_{w'}(I) := C(H', G', >').$$

Set $\Delta := \Delta \cup \{C_{w'}(I)\}$.
Set $L := L \cup \{(G', >', C_{w'}(I))\}$.
Set $L := L \setminus \{(G, >, C_w(I))\}$.
return Δ.

PROOF. Termination and correctness follow from Lemma 4.1.2 as well as the correctness of all the algorithms used. $\square$

Example 4.1.23 For a visual example, consider the 3-dimensional ideal

$$\begin{aligned} I &= \langle 12x^2 + 13xy + 16y^2 + 9xz + 8z^2, 2 - t\rangle \\ &= \langle 3t^2x^2 + 13xy + t^4y^2 + 9xz + t^3z^2, 2 - t\rangle \in \mathbb{Z}[\![t]\!][x, y, z]. \end{aligned}$$

It is actually not hard to see that

$$\mathrm{Trop}(I) = \mathrm{Trop}(\underbrace{3t^2x^2 + 13xy + t^4y^2 + 9xz + t^3z^2}_{=:g}),$$

so that we can actually compute $\mathrm{Trop}(I)$ using Algorithm 4.1.3. But if we were to use the general algorithm for its computation, it would look something like this:

For the starting cone, we begin with a random monomial ordering, say the weighted ordering $>_w$, where $w = (-3, -7, 4, 3) \in \mathbb{R}_{<0} \times \mathbb{R}^3$ and $>$ denotes the t-local lexicographical ordering with $x > y > z > 1 > t$.

Then we compute an initially reduced standard basis (terms are ordered by their monomials),

$$\{\underline{t^3z^2} + 13xy + t^4y^2 + 9xz + 3t^2x^2, \quad \underline{2} - t\}.$$

Because

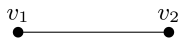
$$\mathrm{in}_w(3t^2x^2 + 13xy + t^4y^2 + 9xz + t^3z^2) = t^3z^2 + 13xy,$$

we see that $w \in \mathrm{Trop}(I)$. In fact, w lies on a maximal Gröbner cone in the tropical variety. Because its initial form is binomial, the only weight vectors w' such that $\mathrm{in}_{w+\varepsilon w'}(g)$ is no monomial are the w' such that $\mathrm{in}_{w+\varepsilon w'}(g) = \mathrm{in}_w(g)$, i.e. $w' \in C_w(I)$.

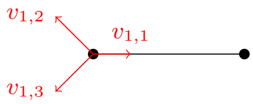
Note that all Gröbner cones are invariant under translation by $(0,1,1,1)$. Hence the 3-dimensional Gröbner cone $C_w(I)$ is spanned by two rays, which are generated by $v_1 = (-2,-7,1,0)$ and $v_2 = (-1,-1,0,0)$ respectively. This can be seen from their respective initial forms, which only gain one additional term compared to $\mathrm{in}_w(g)$,

$$\mathrm{in}_{v_1}(g) = t^3z^2 + 13xy + t^4y^2 \text{ and } \mathrm{in}_{v_2}(g) = t^3z^2 + 13xy + 9xz.$$

We have thus finished computing a starting cone and its two facets, which we need to traverse.

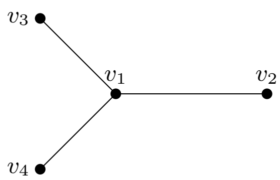


If we pick one of the facets, say the one generated by v_1, we see that its tropical star consists of three rays. One ray points in the direction $v_{1,1} = (0,0,0,-1)$ so that $\mathrm{in}_{v_1+\varepsilon \cdot v_{1,1}}(g) = t^4y^2 + 13xy = \mathrm{in}_w(g)$, which undoubtedly points into our starting cone. Another ray points in the direction $v_{1,2} = (0,0,1,1)$ so that $\mathrm{in}_{v_1+\varepsilon \cdot v_{1,2}}(g) = t^4y^2 + t^3z^2$. The last ray points in the direction $v_{1,3} = (0,0,-2,-1)$ so that $\mathrm{in}_{v_1+\varepsilon \cdot v_{1,3}}(g) = t^3z^2 + 13xy$.

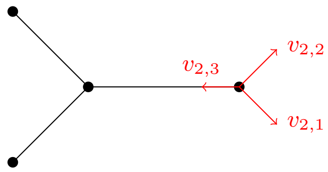


Continuing with direction $v_{1,2} = (0,0,1,1)$, to whose side lies the closure of equivalence class such that $\mathrm{in}_{w'}(g) = t^4y^2 + t^3z^2$, we see that the other ray of the maximal Gröbner cone is generated by $v_3 = (0,0,1,1)$ with $\mathrm{in}_{v_3}(g) = t^4y^2 + t^3z^2$. The ray lies on the boundary of the maximal Gröbner cone because it lies on the boundary of the lower halfspace.

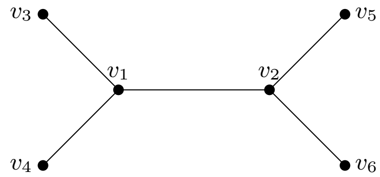
Continuing with the direction $v_{1,3} = (0,0,-2,-1)$, which is the closure of the equivalence class such that $\mathrm{in}_{w'}(g) = t^3z^2 + 13xy$, we get that the other ray of the maximal Gröbner cone is $v_4 = (0,1,1,-2)$ with $\mathrm{in}_{v_4}(g) = 13xy + 3t^2x^2 + t^4y^2$.



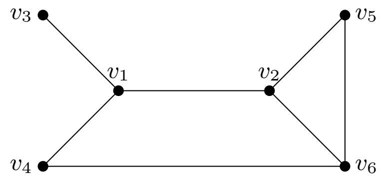
Because both v_3 and v_4 lie on the boundary of the lower halfspace, the only facet left to traverse is the one generated by v_2. The tropical star around v_2 consists of three rays. One ray points in the direction of $v_{2,1} = (0,0,-1,-1)$ so that $\mathrm{in}_{v_1+\varepsilon \cdot v_{2,1}} = 13xy + 9xz$. Another ray points in the direction of $v_{2,2} = (0,0,-1,0)$ so that $\mathrm{in}_{v_1+\varepsilon \cdot v_{2,2}} = 9xz + t^3z^2$. The final ray points in the direction of $v_{2,3} = (0,0,2,1)$ so that $\mathrm{in}_{v_1+\varepsilon\cdot v_{2,3}} = 13xy + t^3z^2$, this is the vector pointing into our starting cone.



Continuing in the direction of $v_{2,1}$, the other ray of the maximal Gröbner cone is generated by $v_5 = (-1,2,0,0)$ as $\mathrm{in}_{v_5}(g) = 13xy + 9xz + 3t^2x^2$. And continuing in the direction of $v_{2,2}$, the other ray is generated by $v_6 := (0,0,-1,0)$ as $\mathrm{in}_{v_6}(g) = 9xz + 3t2x2 + t3z2$.



Because v_5 lies on the boundary of the lower halfspace, v_6 generates the only facet left to traverse. A quick glance at the initial forms imply that it is connected to the facet generated by v_4 and the facet generated by v_5.



We obtain that $\mathrm{Trop}(I)$ is covered by a polyhedral fans with 6 rays, of which the ones generated by v_1, v_2, v_6 lie in the interior of the lower halfspace $\mathbb{R}_{\leq 0} \times \mathbb{R}^n$, while the ones generated by v_3, v_4, v_5 lie on its boundary.

The 6 rays are pairwise connected via 7 cones of one higher dimension. Note that all of these cones intersect the interior of the lower halfspace non-trivially, and none of them lies on the boundary. In fact, the edges connecting (v_1, v_3), (v_2, v_5), (v_6, v_4) and (v_6, v_5) intersect the boundary in codimension one, while the cones connecting (v_1, v_3) and (v_2, v_6) intersect the boundary in codimension 2, which has to be the homogeneity space, similar to the figure in Example 3.1.6.

4.2. Application to fields with valuation

With the algorithms of the last section, we can now describe an algorithm for computing tropical varieties over fields with non-trivial valuation.

Lemma 4.2.1 *Let $I \trianglelefteq K[x]$ be an ideal, and let $G = \{g_1, \ldots, g_k\} \subseteq I \cap R_\nu[x]$ be a generating set over the valuation ring. Because $\pi : R[\![t]\!][x] \to R_\nu[x]$ is surjective, there exist $g_1', \ldots, g_k' \in R[\![t]\!][x]$ such that $\pi(g_i') = g_i \in R[x]$. Then*

$$\pi^{-1} I = \Big(\langle g_1', \ldots, g_k' \rangle + \langle p - t \rangle \Big) : p^\infty \trianglelefteq R[\![t]\!][x].$$

Proof. $\pi^{-1} I \supseteq (\langle g_1, \ldots, g_k \rangle + \langle p - t \rangle) : p^\infty$ is obvious, as $p - t$ is mapped to 0 and p is invertible in K.

For the converse inclusion, let $f \in \pi^{-1} I$. Then there are $q_1, \ldots, q_k \in K[x]$ such that

$$\pi(f) = q_1 \cdot g_1 + \ldots + q_k \cdot g_k \in K[x],$$

which means that for a sufficiently high power $l \in \mathbb{N}$ we have

$$p^l \cdot \pi(f) = \underbrace{p^l q_1}_{\in R_\nu[x]} \cdot g_1 + \ldots + \underbrace{p^l q_k}_{\in R_\nu[x]} \cdot g_k \in R_\nu[x].$$

Because the map $\pi : R[\![t]\!][x] \to R_\nu[x]$ is surjective, there exist $q_1', \ldots, q_k' \in R[\![t]\!][x]$ such that

$$p^l \cdot \pi(f) = \pi(q_1' \cdot g_1' + \ldots + q_k' \cdot g_k'),$$

or rather

$$p^l \cdot f - q_1' \cdot g_1' + \ldots + q_k' \cdot g_k' \in \ker(\pi) = \langle p - t \rangle.$$

Thus $p^l \cdot f \in \langle g_1', \ldots, g_k' \rangle + \langle p - t \rangle$, and hence

$$f \in (\langle g_1', \ldots, g_k' \rangle + \langle p - t \rangle) : p^\infty. \qquad \square$$

Proposition 4.2.2 *Let $I \trianglelefteq K[x]$ be an ideal, and let $G = \{g_1', \ldots, g_k'\} \subseteq \pi^{-1} I$ such that $I = \langle \pi(g_1'), \ldots, \pi(g_k') \rangle$. Then*

$$\operatorname{Trop}(\pi^{-1} I) = \operatorname{Trop}(\langle g_1', \ldots, g_k' \rangle + \langle p - t \rangle).$$

Proof. By Lemma 4.2.1, we have

$$\pi^{-1} I = \Big(\underbrace{\langle g_1', \ldots, g_k' \rangle + \langle p - t \rangle}_{=:I'} \Big) : p^\infty \trianglelefteq R[\![t]\!][x].$$

Consider a weight vector $w \in \mathbb{R}_{<0} \times \mathbb{R}^n$ and suppose $\operatorname{in}_w(I')$ contains a monomial $t^\beta x^\alpha$. By Algorithm 3.3.1, there exists a witness $f \in I'$ with $\operatorname{in}_w(f) = t^\beta x^\alpha$. However since $I' \subseteq \pi^{-1} I$, $\operatorname{in}_w(I)$ then contains the monomial $t^\beta x^\alpha$ as well.

Now suppose $\operatorname{in}_w(\pi^{-1} I)$ contains a monomial $t^\beta x^\alpha$. By Algorithm 3.3.1, there exists a witness $f \in \pi^{-1} I$ with $\operatorname{in}_w(f) = t^\beta x^\alpha$. Let $l \in \mathbb{N}$ be sufficiently high such that $p^l \cdot f \in I'$. Now since $p - t \in I'$, this implies $t^l \cdot f \in I'$ and $\operatorname{in}_w(I')$ then contains the monomial $\operatorname{in}_w(t^l \cdot f) = t^{\beta + l} x^\alpha$. $\square$

Algorithm 4.2.3 tropical variety of an ideal $I \trianglelefteq K[x]$

Input: G, where $G \subseteq I \trianglelefteq K[x]$ a generating set.

Output: $\Delta = \mathrm{Trop}(G)$, a collection of maximal Gröbner cones $C_w(I) \subseteq \mathrm{Trop}(I)$ covering $\mathrm{Trop}(I)$, i.e.

$$\mathrm{Trop}(I) = \textstyle\bigcup_{C_w(I)\in\Delta} C_w(I).$$

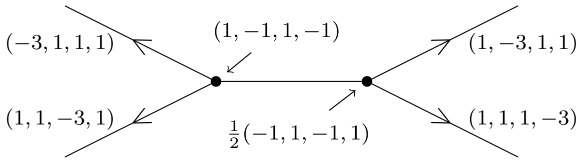
Set $G' := \emptyset$.
for $g \in G$ **do**
    Pick $l \in \mathbb{N}$ sufficiently high such that $p^l \cdot g \in R_\nu[x]$.
    Find a preimage $g' \in R[\![t]\!][x]$ such that $\pi(g') = p^l \cdot g$.
    Set $G' := G' \cup \{g'\}$.
Construct a set of maximal Gröbner cones covering $\mathrm{Trop}(\langle G' \rangle + \langle p - t \rangle)$ using Algorithm 4.1.22.
Intersect each cone of Δ with $\{-1\} \times \mathbb{R}^n$.
return Δ.

Proof. Follows from Theorem 1.2.13 and Proposition 4.2.2. $\square$

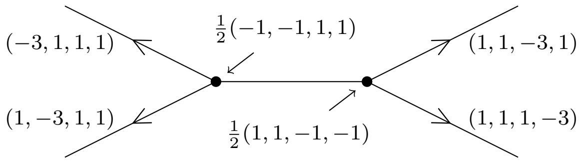
Example 4.2.4 Consider the ideal from Example 1.1.20,

$$I := \langle x_1 + 2x_2 - 3x_3, 3x_2 - 4x_3 + 5x_4 \rangle \trianglelefteq \mathbb{Q}[x].$$

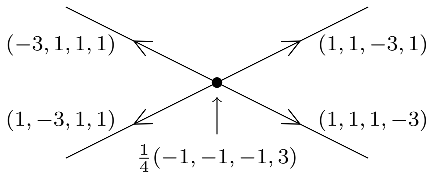
As one might expect, the tropical varieties differ depending on the valuation. Figure 10 shows $\mathrm{Trop}(I)$ for various p-adic valuations on $\mathbb{Q}$. For p sufficiently high, it coincides with the tropical variety for the trivial valuation, which is not surprising because the prime number is simply too high for $p - t$ to play a role in our standard basis calculations.



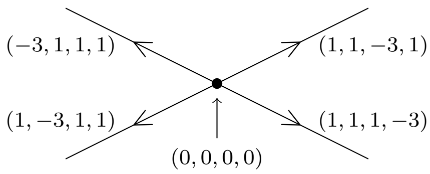
$\mathrm{Trop}_{\nu_2}(I)$



$\mathrm{Trop}_{\nu_3}(I)$



$\mathrm{Trop}_{\nu_5}(I)$



$\mathrm{Trop}_{\nu_p}(I) = \mathrm{Trop}(I)$ for $p = 7, 11, 13, 17$

FIGURE 10. $\mathrm{Trop}_{\nu}(I)$ for various p-adic and the trivial valuations.

Example 4.2.5 Consider the Grassmann-Plücker ideal from Example 6.2.3 of [**Jen07**] which sits in a polynomial ring with 10 variables,

$$\begin{aligned} I &:= \langle bf - ah - ce, bg - ai - de, cg - aj - df, ci - bj - dh, fi - ej - gh\rangle \\ &\trianglelefteq \mathbb{Q}[a, \ldots, j]. \end{aligned}$$

Its tropical variety does not seem dependent on any valuation on $\mathbb{Q}$. Figure 11 shows a shortened output of SINGULAR when computing its tropical variety with respect to the 2-adic valuation. It is describing a polyhedral fan whose intersection with the affine hyperplane $\{-1\} \times \mathbb{R}^n$ yields a polyhedral complex Δ covering $\mathrm{Trop}_{\nu_2}(I)$. Here is a list of the cones missing in the figure, all non-maximal:

```
CONES
{}# Dimension 5
{0}# Dimension 6
{1}
{2}
{3}
{4}
{5}
{6}
{7}
{8}
{9}
{10}
{0 1}# Dimension 7
{0 2}
{0 3}
{0 4}
{0 5}
{0 6}
{0 7}
{0 8}
{0 9}
{0 10}
{1 2}
{1 3}
{1 4}
{2 5}
{2 9}
{3 7}
{4 6}
{3 8}
{4 10}
{5 6}
{5 7}
{6 8}
{7 10}
{8 9}
{9 10}
```

The cone `{}` represents the lineality space of the polyhedral complex, while the cone `{0}` describes the only vertex of Δ sitting at the origin. The cones `{0 1}` to `{0 10}` describe ten polyhedra which sit at the origin and are unbounded in the directions #1 to #10. And the maximal cones `{0 i j}` describe polyhedra which are spanned by `{0 i}` and `{0 j}`. It is not hard to realize, that Δ is in fact a polyhedral fan, with rays `{0 1}` to `{0 10}` and maximal cones `{0 i j}`.

Note that, from a perspective of Δ in $\mathbb{R}^n = \{-1\} \times \mathbb{R}^n$, all data is given in projective coordinates, and the cones `{i j}` with $i \neq 0 \neq j$ lie on the hyperplane at infinite. They distort the f-Vector shown, so that it is not the f-Vector of Δ.

Figure 12 illustrates the combinatorial structure of Δ. Each vertex represents a ray of Δ, while each edge represents a maximal cone of Δ. The graph shown should be thought of as lying on a sphere S^2, on which the colored edges connect with their counterpart on the other side.

```
                     SINGULAR                               /  Development
 A Computer Algebra System for Polynomial Computations   /   version 4.0.1
                                                       0<
 by: W. Decker, G.-M. Greuel, G. Pfister, H. Schoenemann \ Sep 2014
FB Mathematik der Universitaet, D-67653 Kaiserslautern    \
> LIB "gfanlib.so";
// ** loaded /usr/local/bin/../libexec/singular/MOD/gfanlib.so
> printlevel = 1;
> ring r=0,(a,b,c,d,e,f,g,h,i,j),dp;
> ideal I = bf-ah-ce, bg-ai-de, cg-aj-df, ci-bj-dh, fi-ej-gh;
> tropicalVariety(I,number(2));
cones finished: 1 cones in working list: 4
[...] information on the state of the traversal because printlevel=1 was set
cones finished: 14 cones in working list: 1
cones finished: 15 cones in working list: 0
```

```
_application PolyhedralFan
_version 2.2
_type PolyhedralFan

AMBIENT_DIM
11

DIM
8

LINEALITY_DIM
5

RAYS
-1 0 0 0 0 0 0 0 0 0 0 # 0
0 -3 1 1 1 1 1 1 -1 -1 -1# 1
0 -1 -1 1 1 -1 1 1 1 1 -3# 2
0 -1 1 -1 1 1 -1 1 1 -3 1# 3
0 -1 1 1 -1 1 1 -1 -3 1 1# 4
0 1 -3 1 1 1 -1 -1 1 1 -1# 5
0 1 -1 -1 1 1 1 -3 -1 1 1# 6
0 1 -1 1 -1 1 -3 1 1 -1 1# 7
0 1 1 -3 1 -1 1 -1 1 -1 1# 8
0 1 1 -1 -1 -3 1 1 1 1 -1# 9
0 1 1 1 -3 -1 -1 1 -1 1 1# 10
```

```
LINEALITY_SPACE
0 -1 0 0 0 0 0 0 1 1 1   # 0
0 0 -1 0 0 0 1 1 0 0 1   # 1
0 0 0 1 0 0 1 0 1 0 1    # 2
0 0 0 0 1 0 0 1 0 1 1    # 3
0 0 0 0 0 -1 -1 -1 -1 -1 -1# 4

F_VECTOR
1 11 25 15

MAXIMAL_CONES
{0 1 2}# Dimension 8
{0 1 3}
{0 1 4}
{0 2 5}
{0 2 9}
{0 3 7}
{0 4 6}
{0 3 8}
{0 4 10}
{0 5 6}
{0 5 7}
{0 6 8}
{0 7 10}
{0 8 9}
{0 9 10}
```

FIGURE 11. SINGULAR output for the Grassmann-Plücker ideal

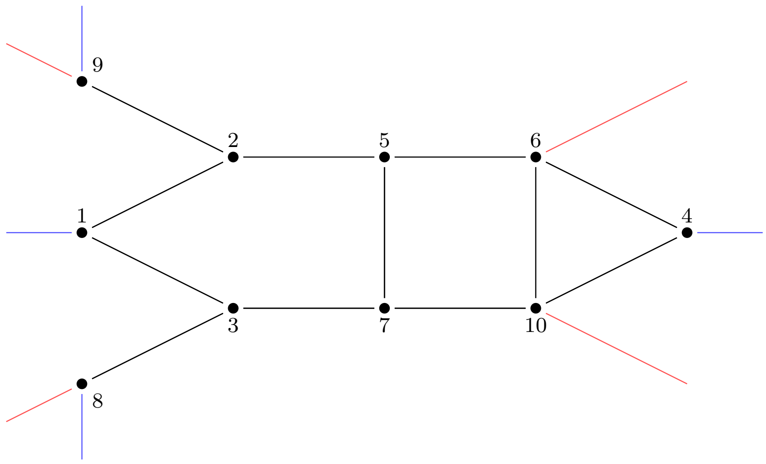


FIGURE 12. tropical variety of the Grassmann-Plücker ideal

4.3. Optimizations and shortcuts

In this section, we consider some mathematically simple optimizations for ideals arising from Theorem 1.2.13, which will speed up the computations considerably for our application. They mainly apply to the ideals J, which are initial ideals $\mathrm{in}_u(I)$ of I with respect to some weight vector in $u \in \mathbb{R}_{<0} \times \mathbb{R}^n$.

Let us recall the previous convention, which we will also require for this section.

Convention 4.3.1 We assume Convention 1.2.2 to hold, in which we have: K a field with discrete valuation, $\mathfrak{K}$ its residue field, R_ν its discrete valuation ring, $p \in R_\nu$ a uniformizing parameter and $R \subset R_\nu$ a dense noetherian subring with $p \in R$. Both K and R_ν are assumed to be complete so that we have

$$\begin{tikzcd} 0 \arrow[r] & \langle p-t\rangle \cdot R[\![t]\!]_{\langle p-t\rangle}[x] \arrow[r] & R[\![t]\!]_{\langle p-t\rangle}[x] \arrow[r] & K[x] \arrow[r] & 0, \\ 0 \arrow[r] & \langle p-t\rangle \cdot R[\![t]\!][x] \arrow[r] \arrow[u] & R[\![t]\!][x] \arrow[r, "t \longmapsto p"'] \arrow[u] & R_\nu[x] \arrow[r] \arrow[u] & 0. \end{tikzcd}$$

and $R/\langle p\rangle = \mathfrak{K}$.

Moreover, we continue to require that R satisfies Convention 2.1.1 (which coincides with Convention 3.1.1), so that standard bases and the Gröbner fans both exist and are computable.

Fix a preimage $I \trianglelefteq R[\![t]\!][x]$ of a homogeneous ideal in $K[x]$, which in particular implies that I is x-homogeneous and $p - t \in I$. Moreover, fix a t-local monomial ordering $>$ on $\mathrm{Mon}(t, x)$. Let $J \trianglelefteq R[t, x]$ be an initial ideal of I with respect to a weight vector in $\mathbb{R}_{<0} \times \mathbb{R}^n$. In particular, $p \in J$.

We will begin with the obvious: Because $p \in J$, many computations involving J may be sped up by doing them over $R/\langle p\rangle = \mathfrak{K}$ instead.

Definition 4.3.2 Given a residue class $\overline{f} \in \mathfrak{K}[t, x]$ we call a representative $f := \sum_{\beta,\alpha} c_{\alpha,\beta} \cdot t^\beta x^\alpha \in R[t, x]$ of $\overline{f}$ a *canonical choice*, if for all $\beta \in \mathbb{N}$ and $\alpha \in \mathbb{N}^n$

$$\begin{aligned} \overline{c}_{\alpha,\beta} = \overline{0} \in \mathfrak{K} \quad &\Longleftrightarrow \quad c_{\alpha,\beta} = 0 \in R, \text{ and} \\ \overline{c}_{\alpha,\beta} = \overline{1} \in \mathfrak{K} \quad &\Longleftrightarrow \quad c_{\alpha,\beta} = 1 \in R. \end{aligned}$$

In particular, the monomials occurring in f are the monomials occurring in $\overline{f}$ and vice versa.

In Algorithm 3.3.5 for flipping standard bases of I from one ordering to an adjacent one, it is required to compute a standard basis of J with respect to that adjacent ordering. Similarly, in Algorithm 4.1.7 for initial ideals with respect to generic weights, which is repeatedly called from Algorithm 4.1.13, it is required to compute a standard basis of J with respect to a multiweighted ordering. This computation can be done entirely over the residue field.

Algorithm 4.3.3 an initially reduced standard basis of J

Input: $(H, >)$, where $H = \{h_0, \ldots, h_k\} \subseteq J$ a generating set, J such that $p \in J$, and $>$ a t-local monomial ordering on $\mathrm{Mon}(t, x)$.

Output: G, an initially reduced standard basis of J with respect to $>$.

1: Let $\overline{h}_0, \ldots, \overline{h}_k \in \mathfrak{K}[t, x]$ denote the residues of $h_0, \ldots, h_k \in R[t, x]$.
2: Compute a monic, initially reduced standard basis $\{\overline{g}_1, \ldots, \overline{g}_l\}$ of $\overline{J} = \langle \overline{h}_0, \ldots, \overline{h}_k \rangle$ with respect to $>$.
3: Let $g_1, \ldots, g_l \in R[t, x]$ be canonical representatives for $\overline{g}_1, \ldots, \overline{g}_l$.
4: **return** $G := \{p, g_1, \ldots, g_l\}$.

Proof. First, we show that G is indeed a standard basis. Since $p \in J$, it is clear that $\langle G \rangle \subseteq J$ and therefore $\langle \mathrm{LT}_>(g) \mid g \in G \rangle \subseteq \mathrm{LT}_>(J)$.

For the opposite inclusion, consider a term $s = c \cdot t^\beta x^\alpha \in \mathrm{LT}_>(J)$. Now if $p \mid c$, then $s \in \langle \mathrm{LT}_>(g) \mid g \in G \rangle$, since $p \in G$ and $\mathrm{LT}_>(p) = p$. And if $p \nmid c$, we may use $p \in \mathrm{LT}_>(J)$ and the fact that $\langle p, c \rangle = R$ to normalize s, and get $t^\beta x^\alpha \in \mathrm{LT}_>(J)$. Thus $t^\beta x^\alpha \in \mathrm{LT}_>(\overline{J})$, and hence there is a $\overline{g_i}$ such that $\mathrm{LM}_>(\overline{g}_i) \mid t^\beta x^\alpha$. Since all $\overline{g}_i$ were chosen to be monic, this implies $\mathrm{LT}_>(\overline{g}_i) \mid t^\beta x^\alpha$, and because all g_i were chosen to be canonical representatives, this implies $\mathrm{LT}_>(g_i) \mid s$.

Next, it remains to show that G is initially reduced. This is easy to see. For each g_i let g_i' be the sum of all terms of g_i with minimal degree in t, as in Definition 3.1.7, the same for $(\overline{g}_i)'$ and $\overline{g}_i$. Because g_i is a canonical representative of $\overline{g}_i$, so is g_i' for $(\overline{g}_i)'$. Now if G were not initially reduced and one g_i' would have a term contained in $\mathrm{LT}_>(J)$ that is not the leading term, then so would $(\overline{g}_i)'$ because g_i' is a canonical representative of $(\overline{g}_i)'$. ☐

In Algorithm 4.1.13 for computing tropical varieties with one-codimensional homogeneity space, it is repeatedly required to decide whether monomials lie in J or in the initial ideals of J. This is another problem, that can be entirely solved over the residue field.

Algorithm 4.3.4 monomial in J

Input: $(H, >)$, where $H = \{h_0, \ldots, h_k\} \subseteq J$ a generating set and J such that $p \in J$.

Output: if it exists, a monomial $m \in J$, 0 otherwise.

1: Let $\overline{h}_0, \ldots, \overline{h}_k \in \mathfrak{K}[t, x]$ denote the residues of $h_1, \ldots, h_k \in R[t, x]$.
2: **if** there exists a monomial $m \in \langle \overline{h}_0, \ldots, \overline{h}_k \rangle$ **then**
3: **return** m
4: **else**
5: **return** 0

Proof. Clear, since J contains a monomial if and only if $\overline{J}$ does. ☐

In Algorithm 3.3.2 for lifting standard bases of J to standard basis of I, it is necessary to compute witnesses in I for the standard basis elements of J. This is also something, which can be mainly done over the residue field.

Algorithm 4.3.5 Witness in I of an element in J

Input: $(h, H, G, >)$, where

- $>$ a weighted t-local monomial ordering on $\mathrm{Mon}(t, x)$,
- $G = \{g_0, g_1, \ldots, g_k\} \subseteq J$ an initially reduced, polynomial standard basis w.r.t. $>$ such that $g_0 = p$, and all g_i with $i > 0$ canonical representatives of their residues,
- $H = \{h_0, h_1, \ldots, h_k\} \subseteq \mathrm{in}_w(I)$ with $h_i = \mathrm{in}_w(g_i)$ for some $w \in C_>(I)$,
- $h \in \mathrm{in}_w(I)$ weighted homogeneous with respect to w and also a canonical representative of its residue.

Output: $f \in I$ such that $\mathrm{in}_w(f) = h$

1: Let $\overline{h}, \overline{h}_i, \overline{g}_i \in \mathfrak{K}[t, x]$ denote the residues of $h, h_i, g_i \in R[t, x]$.

2: Compute a homogeneous determinate division with remainder w.r.t. $>_u$:

$$\overline{h} = \overline{q}_1 \cdot \overline{h}_1 + \ldots + \overline{q}_k \cdot \overline{h}_k + 0.$$

3: Let $q_1, \ldots, q_k \in R[t, x]$ be canonical representatives of $\overline{q}_1, \ldots, \overline{q}_k$.

4: Set

$$q_0 := \frac{h - (q_1 \cdot h_1 + \ldots + q_k \cdot h_k)}{p} \in R[t, x].$$

5: **return** $f := q_0 \cdot g_0 + q_1 \cdot g_1 + \ldots + q_k \cdot g_k$.

Proof. To show: $\mathrm{in}_w(f) = h$.
Because all g_i are canonical representatives, so are all h_i, and hence the monomials occurring in h, h_i or g_i coincide with the monomials occurring in their residues.

In particular, $\overline{h}$ and all $\overline{h}_i$ are weighted homogeneous, and thus all $\overline{q}_i \cdot \overline{h}_i$ are weighted homogeneous with weighted degree $\deg_w(\overline{h}) = \deg_w(h)$. The same holds true for $q_i \cdot h_i$ and hence also q_0. Hence

$$\begin{aligned}\mathrm{in}_w(f) &= \underbrace{\mathrm{in}_w(q_0) \cdot \mathrm{in}_w(g_0)}_{=q_0 \cdot p} + \underbrace{\mathrm{in}_w(q_1) \cdot \mathrm{in}_w(g_1)}_{=q_1 \cdot h_1} + \ldots + \underbrace{\mathrm{in}_w(q_k) \cdot \mathrm{in}_w(g_k)}_{=q_k \cdot h_k} \\ &= h\end{aligned}$$

□

For the final optimization, we look at ways to exploit the weighted homogeneity of $J = \mathrm{in}_w(I)$.

First note, as an addendum to Algorithm 4.3.3, that there is no distinction between initially reduced and reduced standard bases for J, assuming the standard bases are weighted homogeneous.

Lemma 4.3.6 *Let G be a weighted homogeneous standard basis of J with respect to $>$. Then G is initially reduced with respect to $>$ if and only if it is reduced with respect to $>$.*

Proof. It is clear that any reduced standard basis is initially reduced. For the converse, suppose that $G = \{g_1, \ldots, g_k\}$ with $g_i = \sum_{\alpha\in\mathbb{N}^n} g_{i,\alpha} \cdot x^\alpha$, $g_{i,\alpha} \in R[t]$. Recall that, by definition, G is initially reduced if the following set is reduced:

$$G' := \left\{ \sum_{\alpha\in\mathbb{N}} \mathrm{LT}_>(g_{i,\alpha}) \cdot x^\alpha \;\middle|\; i = 1, \ldots, k \right\}.$$

However, since G is weighted homogeneous, we necessarily have $\mathrm{LT}_>(g_{i,\alpha}) = g_{i,\alpha}$ and hence G' coincides with G. □

Also, recall that working with a monomial ordering $>$ on $\mathrm{Mon}(t, x)$ is generally harder, if $>$ is no well-ordering. However, because $J \trianglelefteq R[t, x]$ is polynomial we can apply a well-known trick for homogeneous ideals in polynomial rings:

Should J be weighted homogeneous with respect to some positive weight vector $w' \in \mathbb{R}_{>0} \times \mathbb{R}^n_{>0}$, we may work with the weighted well-ordering $>_{w'}$ instead of $>$. A reduced Gröbner basis of J with respect to $>_{w'}$ is a reduced standard basis of J with respect to $>$ and vice versa.

The next lemma shows that all initial ideals J that arise in our computations satisfy this property, so that the trick mentioned above is applicable.

Lemma 4.3.7 *Let $I \trianglelefteq R[\![t]\!][x]$ be x-homogeneous and $J := \mathrm{in}_w(I) \trianglelefteq R[t, x]$ for some weight vector $w \in \mathbb{R}_{<0} \times \mathbb{R}^n$. Then there exists a positive weight vector $w' \in \mathbb{R}_{>0} \times \mathbb{R}^n_{>0}$ such that J is homogeneous with respect to it.*

Proof. Because the first entry in w is negative, $-w \in \mathbb{R}_{>0} \times \mathbb{R}^n$. Hence, for $k \in \mathbb{N}$ sufficiently high, $w' := k \cdot (0, 1, \ldots, 1) - w \in \mathbb{R}_{>0} \times \mathbb{R}^n_{>0}$.

And since J is both x-homogeneous, i.e. weighted homogeneous with respect to $(0, 1, \ldots, 1)$, and weighted homogeneous with respect to w, it is also weighted homogeneous with respect to w'. □

4.4. Concluding words

In Section 4.1 we have seen that computing the tropical variety Trop(I) consists of two distinct parts: We first compute a starting cone to begin with and then traverse the entire tropical variety until all Gröbner cones in it are known.

Traversing the tropical variety consists of repeated computations of tropical stars using Algorithm 4.1.17 and repeated computations of flips of standard bases using Algorithm 3.3.5.

Since computing the tropical star boils down to repeated search for monomials in initial ideals with respect to generic weights, which mainly consists of a single standard basis computation of J with respect to a multiweighted ordering, and computation of witnesses thereof, it can be entirely done over the residue field thanks to Algorithm 4.3.4 and Algorithm 4.3.5.

And because computing the flip boils down to computing a standard basis of an initial ideal and lifting it to the original ideal, it can also be done over the residue field thanks to Algorithm 4.3.3 and Algorithm 4.3.5.

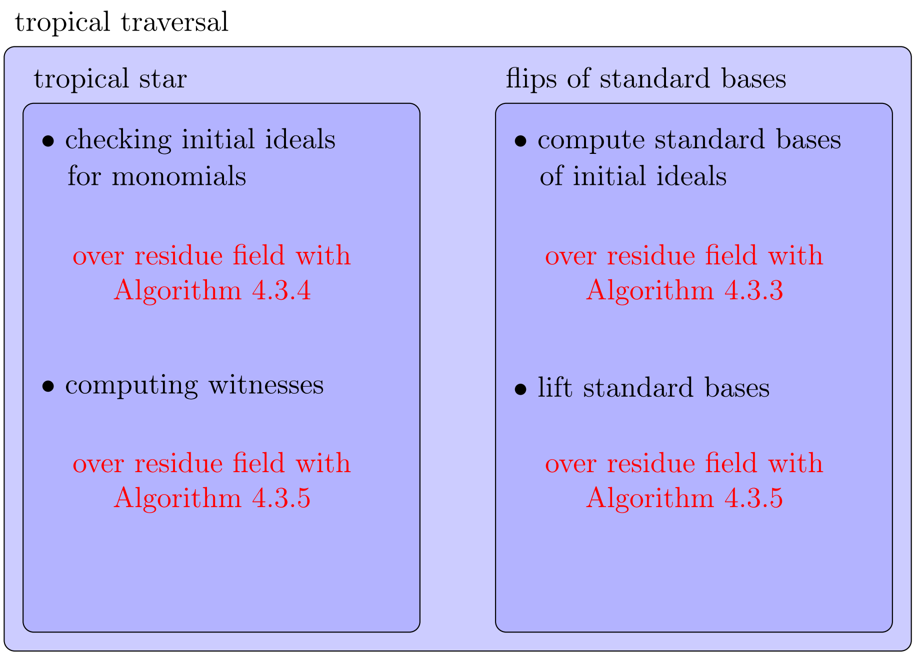


FIGURE 13. algebraic computations during the traversal

Computing the starting cone using Algorithm 4.1.19 consists of recursive searches for points in the tropical variety and lifts of Gröbner bases. The latter is something that can always be done over the residue field by Algorithm 4.3.5.

The first, as explained in Remark 4.1.20, is essentially computing the Gröbner fan until we encounter a point in the tropical variety. Since we are working with initial ideals containing p in the recursions, all computations can be done over the residue field. For the beginning of the recursions however, we do need a standard basis computation of I to start our Gröbner fan traversal.

tropical starting cone computation

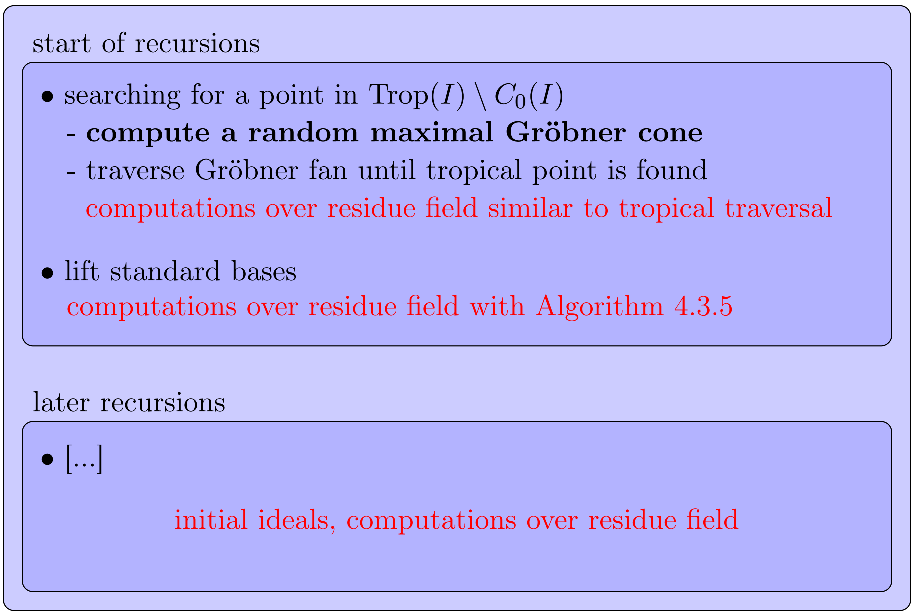


FIGURE 14. algebraic computations during the starting cone computation

We see that, apart from the reduction process which is necessary for determining the inequalities and equations of the Gröbner cones, the only computation in $R[\![t]\!][x]$ required in the entire algorithm is a single standard bases computation at the very beginning. The remaining computations such as divisions with remainder, normal form computations or initially reduced standard basis computations, all happen over the residue field (and with respect to well-orderings thanks to Lemma 4.3.7).

However, standard basis computations over coefficient rings, in SINGULAR as well as in other computer algebra systems, are still in a highly experimental state. In fact, all examples up till now, which have fail to be computed in reasonable time, have been because of it. And examples which managed to pass the initial standard basis computation all successfully finished in reasonable time.

But this is not surprising. As if finding the right balance between the plethora of strategies for standard basis computations over fields is not a hard problem already, the difficulty of having leading coefficients in a ring adds a whole new dimension to it.

Nevertheless, our ideals $\pi^{-1}I$ in $R[\![t]\!][x]$ always contain a generator $p-t$, which has leading term p. And because $R/\langle p\rangle = \mathfrak{K}$ is a field, one would expect that a strategy, which aggressively uses this element to normalize every other leading coefficient during our computation to 1, should make the standard basis computation almost as easy as over a ground field. In the context of this work, this is certainly something that needs to be investigated next.

Finally, I want to mention two other interesting topics that are worth pursuing, though they are not only interesting for this approach of computing tropical varieties over valued fields, but interesting for computing tropical varieties over rings in general.

The first is Andrew Chan's work on computing tropical curves via coordinate projections as in Chapter 4 of [**Cha13**]. If this method can be adjusted to work over rings, it might prove very useful for computing tropical varieties with one-codimensional homogeneity spaces. Tests have shown that it is significantly faster than the technique shown in Algorithm 4.1.13. The method works by computing several projections of the tropical curve onto coordinate hypersurfaces and using them to reconstruct the original tropical curve.

The second is a method by Anders Jensen for finding points in the tropical variety, which is necessary for computing a starting cone, see Algorithm 4.1.19. As of today, there exists no citeable source, but the rough idea is to compute stable intersections with hyperplanes to reduce the dimension of our problem to a certain degree. Any point in the stable intersection naturally lies in our original tropical variety to begin with. Computing stable intersections is something that is made possible by the recent work of Anders Jensen and Josephine Yu [**JY13**].

However it is neither clear whether computing stable intersections over a ring may be done in a similar fashion theoretically, nor whether it is feasible practically. Computing the stable intersections over fields generally requires transcendental extensions and, as of today, none of the big computer algebra systems (`Singular`, `Macaulay2`, `Magma`) supports transcendental extensions over rings.

CHAPTER 5

Software and examples

5.1. Implementation

Before we come to the implementation of computing tropical varieties over valued fields, it should be noted that Singular has no native support for convex geometry. It relies on the following two software packages instead:

(1) Gfanlib is a C++ library by Anders N. Jensen on convex polyhedral cones, polyhedral fans and fundamental operations thereon. It is part of the official Gfan ([**Jen11**]) sources.

(2) Polymake ([**GJ00**]) is a tool to study the combinatorics and the geometry of convex polytopes and polyhedra. It is also capable of dealing with simplicial complexes, matroids, polyhedral fans, graphs, tropical objects, and other objects.

These software packages are connected to Singular via two interfaces:

Gfanlib.so is a binary Singular interface to Gfanlib. It was written by Anders Jensen, Frank Seelisch and myself, and it provides Singular with data structures for polyhedral cones and polyhedral fans, as well as some basic functionality thereon.

Polymake.so is a binary Singular interface to Polymake. It was written by myself, and it gives Singular the ability to call the higher algorithms of Polymake, for example routines for computing matroid fans in a-tint ([**Ham12**]).

Now if we compare our approach to computing tropical varieties over valued fields to existing techniques for computing tropical varieties over fields with trivial valuation, the key difference that springs into mind is that we are working with inhomogeneous ideals $\pi^{-1}I$ in $R[\![t]\!][x]$ while the existing techniques deal with homogeneous ideals I in $K[x]$.

Consequently, many fundamental algorithms in computer algebra, normal form and standard bases computation, reduction algorithm, are completely different, even though they serve the same purpose respectively. Also, some central algorithms in convex geometry, computing tropical varieties of polynomials, computing interior points on facets for flips, differ slightly because we are either restricting ourselves to a closed lower halfspace $\mathbb{R}_{\leq 0} \times \mathbb{R}^n$ or we are working in the whole weight space $\mathbb{R}^n$. Another divergence are the possible optimizations for ideals $\pi^{-1}I \trianglelefteq R[\![t]\!][x]$ resp. initial ideals thereof, cf. Chapter 4.2. Nevertheless, as we have seen in Chapter 3.3 and Chapter 4.1, the overall structure of the algorithms is very similar.

Therefore, a major challenge during the implementation was designing a simple, common framework for these two widely similar, yet fundamentally different settings. And this framework should follow the following rules:

(1) It needs to know which algorithms are eligible and how they are supposed to be used. For example, when computing a standard basis, we would like to pass on information whether the ideal is weighted homogeneous or not, as it can be used for optimizations (see Hilbert driven standard basis computation).
(2) It needs to know which optimizations are possible and how they are supposed to be done. For example, when to adjust weight vectors, and how it can be done, whether using $(0, 1, \ldots, 1)$ because $\pi^{-1}I \trianglelefteq R[\![t]\!][x]$ is x-homogeneous, or using $(1, 1, \ldots, 1)$ because $I \trianglelefteq K[x]$ is homogeneous in general. Note that the latter is always possible, while the first is not.
(3) It needs to know which optimizations are necessary. This is mainly due to the design of Singular, which is aggressively optimized for standard basis computations, not for Gröbner fan computations. Suppose we had several ideals $I, J_1, \ldots, J_k$ in an active ring $R[\![t]\!][x]$ with a fixed weighted ordering $>_w$. Adjusting the weight vector, would then force us to create copies of all ideals and change all exponent vectors, as they are stored in a highly condensed fashion for performance reasons.
(4) The code for the non-trivial valuation case and the trivial valuation case needs to be as overlapping as mathematically and implementationally feasible. For example, we want a common black box for computing tropical varieties with a one-codimensional homogeneity space, which takes as input an ideal as well as recipes for computing generically weighted initial ideals and tropical varieties of polynomials, rather than two separate algorithms for the non-trivial valuation and the trivial valuation case. The reason being, that any improvement for one algorithm, which does not involve generically weighted initial ideals or tropical varieties of polynomials, should be a sensible improvement for the other.

To facilitate all that, the library is structured in multiple layers (note that each **bolded** name represents a file in the source code):

(0) The zeroth layer consists of functions that are used frequently throughout the entire library regardless whether the valuation is trivial or not. **std_wrapper** contains a wrapper for the standard basis method of `Singular`, and **initial** contains various routines for computing initial forms of polynomials and initial forms of generators of an ideal.
(1) The first layer represents the mutable part of the framework, capturing all possible scenarios which might arise in our algorithm. As of now, there are only three fundamentally different scenarios, which are working with an ideal $\pi^{-1}I \trianglelefteq R[\![t]\!][x]$, one of its initial ideals $J = \mathrm{in}_w(I)$ or an ideal $I \trianglelefteq K[x]$. However, for the sake of flexibility, we distinguish between:
   (a) whether the valuation is non-trivial,
   (b) whether the coefficient ring is a field,

(c) whether the ideal is homogeneous,
(d) whether the ideal is principal,
(e) whether the homogeneity space has codimension greater than 1.

Depending on which conditions holds, different algorithms will be called for the same task. For that, **tropicalStrategy** contains various wrappers and containers, in order to call functions which suit our current scenario best. Functions wrapped by **tropicalStrategy** can be found in the following files:

- **witness**, **lift** and **flip** contain implementations of the algorithms of the same name in Chapter 3.2.
  If our valuation is non-trivial, Algorithms 4.3.3 and 4.3.5 show that the core computations may be done over the residue field. Calling these functions through **tropicalStrategy** therefore may lead to a temporary switch to the residue field.
- **containsMonomial** contains functions that search for monomials, see Remarks 4.1.15 and 4.1.16.
  Same as above, calling these functions through **tropicalStrategy** may result in a temporary switch to the residue field, see Algorithm 4.3.4.
- **adjustWeights** consists of routines for adjusting weight vectors.
  In case our ideal is homogeneous, a sufficiently high multiple of the weight vector $(1, \ldots, 1)$ is added to the input until it is strictly positive. If our ideal is x-homogeneous, we use the method described in Lemma 4.3.7.
- **initialReduction** consists of functions for reducing an x-homogeneous ideal initially, which is only needed if our ideal is not homogeneous. If our ideal is homogeneous, we use the native reduction methods of SINGULAR, as explained in Lemma 4.3.6.

Moreover, **tropicalStrategy** computes and stores pieces of information, which are potentially useful to have at hand during any point of our computation, such as the expected dimension of the tropical variety.

(2) The second layer consists of methods which are immediately build upon the first. They do not distinguish between the non-trivial and trivial valuation case, but rather depend on the first layer to do that for them.
- **groebnerCone** holds a class that represents a Gröbner cone in our algorithm. It carries all relevant information such as the inequalities and equations of the Gröbner cone $C_w(I)$ as well as an initially reduced standard basis of I with respect to a monomial ordering $>$ such that $w \in C_>(I)$.
- **tropicalCurves** holds an implementation of Algorithm 4.1.13 for computing tropical varieties with a one-codimensional homogeneity space. It relies on tropicalStrategy to check for monomials in initial ideals to compute tropical varieties of polynomials and to check initial ideals with respect to generic weights for monomials.

(3) In the third layer, **startingCone** contains Algorithm 4.1.19 for computing a maximal cone in the tropical variety, and **tropicalTraversal** contains an algorithm for traversing the remainder of the tropical variety based on repeated usage of Algorithm 4.1.17 for computing its tropical neighbours.
(4) On the final layer, there are **groebnerFan**, **groebnerComplex** and **tropicalVariety**. They contain algorithms for computing their namesakes and, building on the layers below them, their implementation becomes very simple.

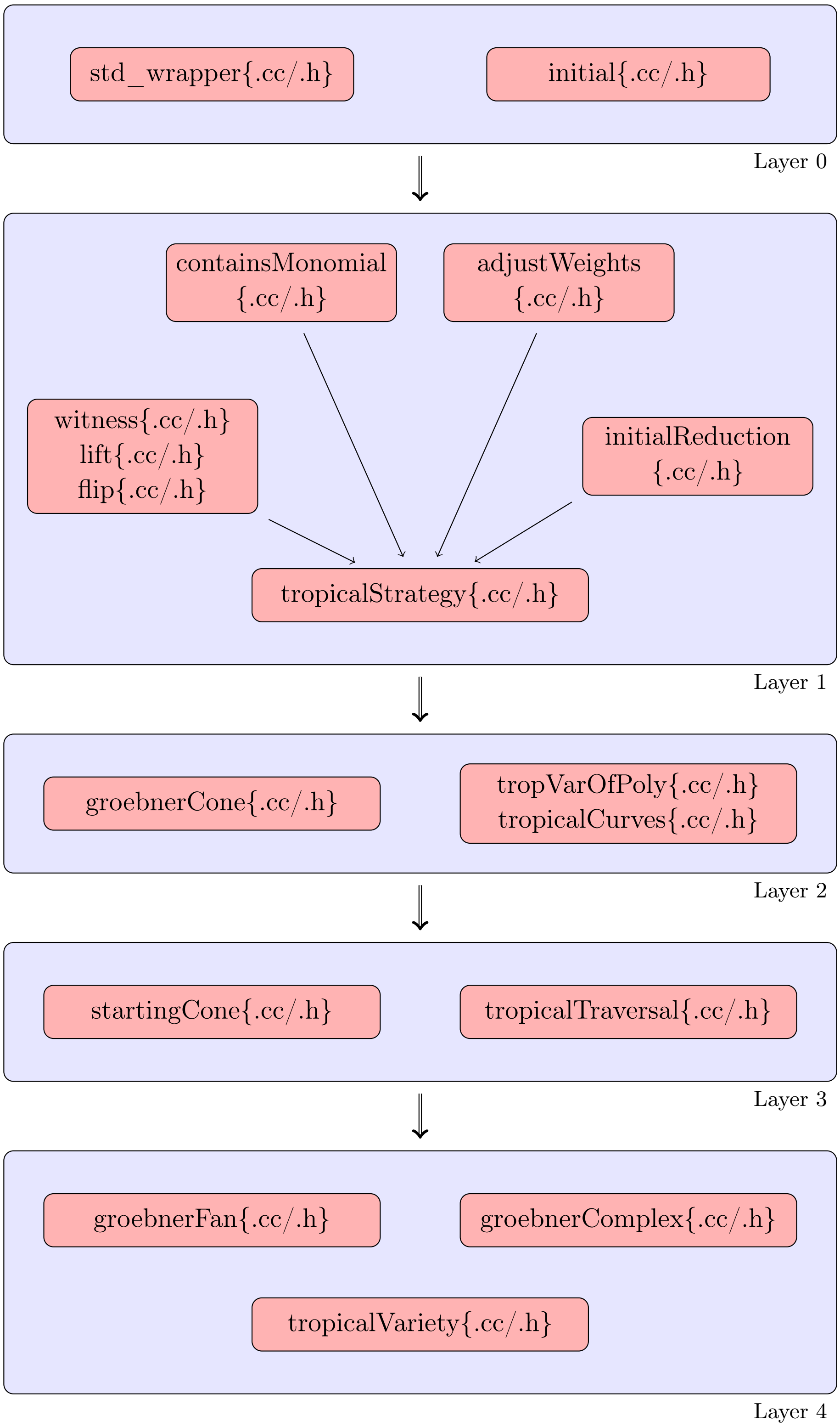


FIGURE 1. structure of the source code

5.2. Examples and timings

Figure 2 compares the timings of GFAN and SINGULAR on some determinantal and Plücker ideals over $\mathbb{Q}$ taken from Chapter 9.2 of [**Jen07**]. Since GFAN does not support the computation of tropical varieties with respect to p-adic orderings, all computations were done with respect to the trivial valuation. We see that the times are relatable, though SINGULAR is undoubtedly trailing behind. At one example SINGULAR takes more than four times longer, while at another examples GFAN terminates prematurely with an exception.

Once again, it should be stressed that SINGULAR depends on GFANLIB for all computations in convex geometry and would not be able to make the following computations on its own.

Example	n	h	d	f-vector	GFAN	SINGULAR
$\mathrm{Det}_{3,3,4}$	12	6	10	(1,12,66,132,90)	2	2
$\mathrm{Det}_{3,3,5}$	15	7	12	(1,45,315,930,1260,630)	30	39
$\mathrm{Det}_{3,4,4}$	16	7	12	(1,50,360,1128,1680,936)	89	377
$\mathrm{Detsym}_{3,4}$	10	4	7	(1,20,75,75)	4	4
$\mathrm{Detsym}_{3,5}$	15	5	9	(1,75,495,1155,855)	$*$	6371
$\mathrm{Grass}_{2,5}$	10	5	7	(1,10,15)	0	0
$\mathrm{Grass}_{3,6}$	20	6	10	(1,65,550,1395,1035)	707	914

n : number of variables in the polynomial ring
h : dimension of the homogeneity space in the tropical variety
d : dimension of the tropical variety
GFAN: seconds used by GFAN
SINGULAR: seconds used by SINGULAR

FIGURE 2. comparison: determinantal ideals and Grassmannians

Figure 3 shows the computation of random tropical linear spaces in $\mathbb{R}^{10}$. For this class of examples, SINGULAR seems to be faster than GFAN.

Example	n	h	d	f-vector	GFAN	SINGULAR
L_1	10	1	6	(1,15,88,263,416,296)	28	10
L_2	10	2	6	(1,11,47,108,125)	33	6
L_3	10	2	6	(1,12,59,138,149)	32	7

FIGURE 3. comparison: tropical linear spaces

Here are all ideals in the two figures above:

Example 5.2.1 (determinantal ideals) We set $\mathrm{Det}_{t,m,n}$ to be the ideal in the polynomial ring $\mathbb{Q}[x_{11}, x_{12}, \dots, x_{mn}]$ generated by the $t \times t$ minors of the matrix of variables

$$\begin{pmatrix} x_{11} & x_{12} & \dots & x_{1n} \\ x_{21} & x_{22} & \dots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{m1} & x_{m2} & \dots & x_{mn} \end{pmatrix}$$

Renaming the variables $x_{11}, x_{12}, \dots, x_{mn}$ to $x_1, x_2, \dots, x_{m \cdot n}$, they can easily be constructed in SINGULAR as follows:

```
int t = 3; int n = 3; int m = 5;
ring r = 0,x(1..n*m),dp;
matrix M[n][m] = x(1..n*m);
ideal Det = minor(M, t);
```

Their tropical variety computation can be timed with:

```
LIB "gfanlib.lib";
int t = timer;
fan f = tropicalVariety(I);
t = timer-t; t;
```

Set `printlevel = 1;` for a log during the tropical traversal.

Example 5.2.2 (symmetric determinantal ideals) We set $\mathrm{Detsym}_{t,n}$ to be the ideal in the polynomial ring in $\frac{n(n+1)}{2}$ variables generated by the $t \times t$ minors of the symmetric matrix of variables.

Then $\mathrm{Detsym}_{3,4}$ is constructed as follows

```
ring r = 0,x(1..10),dp;
matrix M[4][4] =
  x(1), x(2), x(3), x(4),
  x(2), x(5), x(6), x(7),
  x(3), x(6), x(8), x(9),
  x(4), x(7), x(9), x(10);
ideal Detsym = minor(M,3);
```

and $\mathrm{Detsym}_{3,5}$ can be constructed by

```
ring r = 0,x(1..15),dp;
matrix M[5][5] =
  x(1), x(2), x(3), x(4), x(5),
  x(2), x(6), x(7), x(8), x(9),
  x(3), x(7), x(10),x(11),x(12),
  x(4), x(8), x(11),x(13),x(14),
  x(5), x(9), x(12),x(14),x(15);
ideal Detsym = minor(M,3);
```

Example 5.2.3 (Plücker ideals) Let $\mathrm{Grass}_{d,n}$ be the ideal in the polynomial ring in $\binom{n}{d}$ variables generated by the relations on the $d \times d$ minors of a $d \times n$ matrix.

For $\mathrm{Grass}_{2,5}$, this ideal is given by

```
ring r=0,x(1..10),dp;
ideal G25 =
  -x(3)*x(5)+x(2)*x(6)-x(1)*x(8),
  -x(4)*x(5)+x(2)*x(7)-x(1)*x(9),
  -x(4)*x(6)+x(3)*x(7)-x(1)*x(10),
  -x(4)*x(8)+x(3)*x(9)-x(2)*x(10),
  -x(7)*x(8)+x(6)*x(9)-x(5)*x(10);
```

and for $\mathrm{Grass}_{3,6}$ it is given by (with courtesy of Sebastian Muskalla)

```
ring r = 0,(x_1_2_3,x_1_2_4,x_1_2_5,x_1_2_6,
            x_1_3_4,x_1_3_5,x_1_3_6,x_1_4_5,x_1_4_6,x_1_5_6,
            x_2_3_4,x_2_3_5,x_2_3_6,x_2_4_5,x_2_4_6,x_2_5_6,
            x_3_4_5,x_3_4_6,x_3_5_6,x_4_5_6),dp;
ideal G36 =
  x_2_5_6*x_3_4_6-x_2_4_6*x_3_5_6+x_2_3_6*x_4_5_6,
  x_1_5_6*x_3_4_6-x_1_4_6*x_3_5_6+x_1_3_6*x_4_5_6,
  x_2_5_6*x_3_4_5-x_2_4_5*x_3_5_6+x_2_3_5*x_4_5_6,
  x_2_4_6*x_3_4_5-x_2_4_5*x_3_4_6+x_2_3_4*x_4_5_6,
  x_2_3_6*x_3_4_5-x_2_3_5*x_3_4_6+x_2_3_4*x_3_5_6,
  x_1_5_6*x_3_4_5-x_1_4_5*x_3_5_6+x_1_3_5*x_4_5_6,
  x_1_4_6*x_3_4_5-x_1_4_5*x_3_4_6+x_1_3_4*x_4_5_6,
  x_1_3_6*x_3_4_5-x_1_3_5*x_3_4_6+x_1_3_4*x_3_5_6,
  x_1_2_6*x_3_4_5-x_1_2_5*x_3_4_6+x_1_2_4*x_3_5_6
    -x_1_2_3*x_4_5_6,
  x_1_5_6*x_2_4_6-x_1_4_6*x_2_5_6+x_1_2_6*x_4_5_6,
  x_2_3_6*x_2_4_5-x_2_3_5*x_2_4_6+x_2_3_4*x_2_5_6,
  x_1_5_6*x_2_4_5-x_1_4_5*x_2_5_6+x_1_2_5*x_4_5_6,
  x_1_4_6*x_2_4_5-x_1_4_5*x_2_4_6+x_1_2_4*x_4_5_6,
  x_1_3_6*x_2_4_5-x_1_3_5*x_2_4_6+x_1_3_4*x_2_5_6
    +x_1_2_3*x_4_5_6,
  x_1_2_6*x_2_4_5-x_1_2_5*x_2_4_6+x_1_2_4*x_2_5_6,
  x_1_5_6*x_2_3_6-x_1_3_6*x_2_5_6+x_1_2_6*x_3_5_6,
  x_1_4_6*x_2_3_6-x_1_3_6*x_2_4_6+x_1_2_6*x_3_4_6,
  x_1_4_5*x_2_3_6-x_1_3_5*x_2_4_6+x_1_3_4*x_2_5_6
    +x_1_2_5*x_3_4_6-x_1_2_4*x_3_5_6+x_1_2_3*x_4_5_6,
  x_1_5_6*x_2_3_5-x_1_3_5*x_2_5_6+x_1_2_5*x_3_5_6,
  x_1_4_6*x_2_3_5-x_1_4_5*x_2_3_6-x_1_3_4*x_2_5_6
    +x_1_2_4*x_3_5_6,
  x_1_4_5*x_2_3_5-x_1_3_5*x_2_4_5+x_1_2_5*x_3_4_5,
  x_1_3_6*x_2_3_5-x_1_3_5*x_2_3_6+x_1_2_3*x_3_5_6,
  x_1_2_6*x_2_3_5-x_1_2_5*x_2_3_6+x_1_2_3*x_2_5_6,
  x_1_5_6*x_2_3_4-x_1_3_4*x_2_5_6+x_1_2_4*x_3_5_6
    -x_1_2_3*x_4_5_6,
  x_1_4_6*x_2_3_4-x_1_3_4*x_2_4_6+x_1_2_4*x_3_4_6,
  x_1_4_5*x_2_3_4-x_1_3_4*x_2_4_5+x_1_2_4*x_3_4_5,
```

```
x_1_3_6*x_2_3_4-x_1_3_4*x_2_3_6+x_1_2_3*x_3_4_6,
x_1_3_5*x_2_3_4-x_1_3_4*x_2_3_5+x_1_2_3*x_3_4_5,
x_1_2_6*x_2_3_4-x_1_2_4*x_2_3_6+x_1_2_3*x_2_4_6,
x_1_2_5*x_2_3_4-x_1_2_4*x_2_3_5+x_1_2_3*x_2_4_5,
x_1_3_6*x_1_4_5-x_1_3_5*x_1_4_6+x_1_3_4*x_1_5_6,
x_1_2_6*x_1_4_5-x_1_2_5*x_1_4_6+x_1_2_4*x_1_5_6,
x_1_2_6*x_1_3_5-x_1_2_5*x_1_3_6+x_1_2_3*x_1_5_6,
x_1_2_6*x_1_3_4-x_1_2_4*x_1_3_6+x_1_2_3*x_1_4_6,
x_1_2_5*x_1_3_4-x_1_2_4*x_1_3_5+x_1_2_3*x_1_4_5;
```

Example 5.2.4 (random tropical linear spaces) The tropical linear spaces in Figure 3 are given by

```
ring s = 0,x(1..10),dp;
ideal L1 =
  4*x(1)+7*x(2)+4*x(7),
  7*x(1)+2*x(4)+6*x(5)+7*x(9)+4*x(10),
  4*x(1)+4*x(3)+8*x(4)+5*x(5)+9*x(6)+4*x(7)+5*x(8)+9*x(10),
  x(5)+5*x(7)+8*x(8)+7*x(9);
ideal L2 =
  3*x(3)+9*x(4)+8*x(6)+6*x(7)+4*x(8)+x(9),
  8*x(2)+x(4)+x(5),
  5*x(1)+4*x(2)+9*x(4)+9*x(6)+9*x(7)+9*x(9),
  2*x(1)+x(2)+4*x(3)+9*x(6)+7*x(9);
ideal L3 =
  8*x(2)+9*x(3)+3*x(4)+8*x(9),
  3*x(4)+5*x(5)+2*x(6)+x(8),
  9*x(1)+5*x(2)+2*x(4)+7*x(6)+5*x(8)+9*x(9)+4*x(10),
  x(1)+8*x(3)+7*x(6)+x(9)+8*x(10);
```

They were originally created through

```
LIB "random.lib";
ring s = 0,x(1..10),dp;
ideal L = sparseid(4,1,1,50,9);
```

5.3. List of algorithms

Here is a complete list of algorithms found in this thesis.

Chapter 2. Standard bases in $R[\![t]\!][x]^s = R[t_1, \ldots, t_m][x_1, \ldots, x_n]^s$

(1) division in the ground ring, Algorithm 2.1.11,
(2) homogeneous determiate division with remainder, Algorithm 2.1.16,
(3) weak division with remainder, Algorithm 2.1.21,
(4) normal form, Algorithm 2.2.3,
(5) standard basis, Algorithm 2.2.17,
(6) standard basis over factorial rings, Algorithm 2.2.19,
(7) reduction, Algorithm 2.2.21.

Chapter 3. Gröbner fans in $R[\![t]\!][x] = R[t][x_1, \ldots, x_n]$

(1) Gröbner cone, input (G, w), Algorithm 3.1.27,
(2) Gröbner cone, input (G, H), Algorithm 3.1.28,
(3) maximal Gröbner cone, Algorithm 3.1.29
(4) initial reduction with with respect to $(p - t)$, Algorithm 3.2.3,
(5) initial reduction same x-degree, Algorithm 3.2.4,
(6) initial reduction all at once, Algorithm 3.2.6,
(7) initial reduction step by step, Algorithm 3.2.7,
(8) initially reduced standard basis, Algorithm 3.2.9,
(9) witness of elements in initial ideals, Algorithm 3.3.1,
(10) lift of standard bases, Algorithm 3.3.2,
(11) flip of standard bases, Algorithm 3.3.5,
(12) Gröbner fan, Algorithm 3.3.7.

Chapter 4. Tropical varieties in $R[\![t]\!][x] = R[t][x_1, \ldots, x_n]$

(1) tropical variety of an element, Algorithm 4.1.3,
(2) initial ideal with respect to a generic weight, Algorithm 4.1.7,
(3) tropical variety with one-codimensional homogeneity space, Alg. 4.1.13,
(4) tropical star, Algorithm 4.1.17,
(5) tropical starting cone, Algorithm 4.1.19,
(6) tropical variety, Algorithm 4.1.22,
(7) tropical variety over valued fiels, Algorithm 4.2.3,
(8) optimization: initially reduced standard basis of J, Algorithm 4.3.3,
(9) optimization: monomials in J, Algorithm 4.3.4,
(10) optimization: witnesses of elements in J, Algorithm 4.3.5.

Part 2

Strongly symmetric smooth toric varieties

CHAPTER 6

Strongly symmetric smooth toric varieties

DISCLAIMER: The content of this chapter is published in the Kyoto Journal of Mathematics, volume 52 in the year 2012, see [**CRT12**].

6.0. Introduction

In this part, we investigate toric varieties which are defined by fans of arrangements of hyperplanes, thereby generalizing the definition and construction of toric varieties which are associated to classical root systems. Toric varieties arising from root systems had previously been considered, investigated and used by De Concini-Procesi [**DCP83**], Voskresenskij-Klyachko [**KV85**], Procesi [**Pro90**], Dolgachev-Lunts [**DL94**], Stembridge [**Ste94**], Klyachko [**Kly95**], and Brion-Joshua [**BJ08**]. Recently Batyrev-Blume, [**BB11a**], [**BB11b**], found generalizations of the Losev-Manin moduli spaces by investigating the functor of toric varieties associated with Weyl chambers. The so called crystallographic arrangements are generalizations of the classical root systems and their Weyl chamber structure. In this paper we establish a one to one correspondence between crystallographic arrangements and toric varieties which are smooth and projective, and which have the property of being strongly symmetric, see Def. 6.2.1, a property which has not been used in the previous papers mentioned above.

Crystallographic arrangements have originally been used in the theory of pointed Hopf algebras: Classical Lie theory leads to the notion of Weyl groups which are special reflection groups characterized by a certain integrality and which are therefore also called *crystallographic* reflection groups. A certain generalization of the universal enveloping algebras of Lie algebras yields Hopf algebras to which one can associate *root systems* and *Weyl groupoids* (see [**Hec06**], [**HS10**], [**AHS10**]). The case of *finite Weyl groupoids* has recently been treated including a complete classification in a series of papers [**CH09a**], [**CH09b**], [**CH11**], [**CH12**], [**CH10**].

The theorems needed for the classification reveal an astonishing connection: It turns out that finite Weyl groupoids correspond to certain simplicial arrangements called *crystallographic* [**Cun11**]: Let $\mathcal{A}$ be a simplicial arrangement of finitely many real hyperplanes in a Euclidean space V and let R be a set of nonzero covectors such that $\mathcal{A} = \{\alpha^\perp \mid \alpha \in R\}$. Assume that $\mathbb{R}\alpha \cap R = \{\pm\alpha\}$ for all $\alpha \in R$. The pair $(\mathcal{A}, R)$ is called crystallographic, see [**Cun11**, Def. 2.3] or Def. 6.1.1, if for any chamber K the elements of R are integer linear combinations of the covectors defining the walls of K.

For example, crystallographic Coxeter groups give rise to crystallographic arrangements in this sense, but there are many other.

Thus the main feature of crystallographic arrangements is the integrality. But integrality is also the fundamental property of a fan in toric geometry. Indeed, the set of closed chambers of a rational simplicial arrangement is a fan which is strongly symmetric. A closer look reveals that the property *crystallographic* corresponds to the smoothness of the variety. We obtain (see Thm. 6.3.3):

Theorem 6.0.1 *There is a one to one correspondence between crystallographic arrangements and strongly symmetric smooth fans.*

Thus the classification of finite Weyl groupoids [**CH10**] gives:

Corollary 6.0.2 *Any strongly symmetric smooth complete toric variety is isomorphic to a product of*

(1) varieties of dimension two corresponding to triangulations of a convex n-gon by non-intersecting diagonals (see Section 6.5),
(2) varieties of dimension $r > 2$ corresponding to the reflection arrangements of type A_r, B_r, C_r and D_r, or out of a series of $r-1$ further varieties,
(3) 74 further "sporadic" varieties.

To each crystallographic arrangement $\mathcal{A}$, we construct a polytope P such that the toric variety of P is isomorphic to the toric variety corresponding to $\mathcal{A}$. Thus we obtain that the variety is projective, see Section 6.4. Further, the strong symmetry of the fan Σ associated to $\mathcal{A}$ gives rise to a system $\{Y^E\}_{E\in L(\mathcal{A})}$ of smooth strongly symmetric toric varieties $Y^E \subseteq X_\Sigma$ (here $L(\mathcal{A})$ is the poset of intersections of hyperplanes of $\mathcal{A}$). This system mirrors the arrangement $\mathcal{A}$ in X_Σ in a remarkable way, see Section 6.6.2, and will be called the associated toric arrangement. The intersections $Y^H \cap T$ with the torus T of X_Σ for $H \in \mathcal{A}$ are subtori of T and form a toric arrangement.

This note is organized as follows. After recalling the notions of fans and arrangements of hyperplanes in Section 6.1, we collect some results on strongly symmetric fans in Section 6.2. We then prove the main theorem (the correspondence) in Section 6.3. In Section 6.4 we construct a polytope for each crystallographic arrangement. In Section 6.5 we compare the well-known classifications of smooth complete surfaces (specified for the centrally symmetric case) and the corresponding arrangements of rank two. In the following section we discuss the toric arrangements associated to the crystallographic arrangements. The last section consists of further remarks on irreducibility, blowups, and automorphisms.

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6.1. Preliminaries

Let us first recall the notions of hyperplane arrangements and of fans for normal toric varieties.

For subsets A in a real vector space V of dimension r and a subset B of its dual V^* we set

$$\begin{aligned} A^\perp &= \{b \in V^* \mid b(a) = 0\ \forall\, a \in A\}, \\ B^\vee &= \{a \in V \mid b(a) \geq 0\ \forall\, b \in B\}, \\ B^\perp &= \{a \in V \mid b(a) = 0\ \forall\, b \in B\}. \end{aligned}$$

An *open* or *closed simplicial cone* σ is a subset $\sigma \subseteq V$ such that there exist linearly independent $n_1, \ldots, n_d$, $d \in \mathbb{N}$ with

$$\begin{aligned} &\sigma = \langle n_1, \ldots, n_d \rangle_{\mathbb{R}_{>0}} := \mathbb{R}_{>0} n_1 + \ldots + \mathbb{R}_{>0} n_d \\ \text{or}\qquad &\sigma = \langle n_1, \ldots, n_d \rangle_{\mathbb{R}_{\geq 0}} := \mathbb{R}_{\geq 0} n_1 + \ldots + \mathbb{R}_{\geq 0} n_d \end{aligned}$$

respectively.

6.1.1. Fans and toric varieties. Given a lattice N in V of rank r, its dual lattice $M = \mathrm{Hom}(N, \mathbb{Z})$ is viewed as lattice in V^*. A subset $\sigma \subseteq V$ is called a (closed) *strongly convex rational polyhedral cone* if there exist $n_1, \ldots, n_d \in N$ such that

$$\sigma = \langle n_1, \ldots, n_d \rangle_{\mathbb{R}_{\geq 0}} \quad \text{and} \quad \sigma \cap -\sigma = \{0\}.$$

We say that $n_1, \ldots, n_d$ are *generators* of σ. By abuse of notation we will call such a cone simply an "N-cone".
We call σ *simplicial*, if σ is a closed simplicial cone. If σ is generated by a subset of a $\mathbb{Z}$-basis of N, then we say that σ is *smooth*.
Let σ be an N-cone. We write $\langle \sigma \rangle_{\mathbb{R}} := \sigma + (-\sigma)$ for the subspace spanned by σ. The *dimension* $\dim(\sigma)$ of σ is the dimension of $\langle \sigma \rangle_{\mathbb{R}}$.

Identifying $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$ with V, we consider fans Σ in $N_{\mathbb{R}}$ of strongly convex rational polyhedral cones as defined in the standard theory of toric varieties, see [**Oda88**], [**CLS11**]:

A *face* of σ is the intersection of σ with a supporting hyperplane, $\sigma \cap m^\perp$, $m \in V^*$, $m(a) \geq 0$ for all $a \in \sigma$. Faces of codimension 1 are called *facets.*
A *fan* in N is a nonempty collection of N-cones Σ such that

(1) any face τ of a cone $\sigma \in \Sigma$ is contained in Σ,
(2) any intersection $\sigma_1 \cap \sigma_2$ of two cones $\sigma_1, \sigma_2 \in \Sigma$ is a face of σ_1 and σ_2.

For $k \in \mathbb{N}$ we write $\Sigma(k) = \{\sigma \in \Sigma \mid \dim(\sigma) = k\}$. For $S \subseteq \Sigma$ we write $\mathrm{Supp}\, S = \bigcup_{\sigma \in S} \sigma$ for the *support* of S.

The fan Σ and its associated toric variety X_Σ (over the ground field $\mathbb{C}$) is called *simplicial* if any cone of Σ is simplicial. It is well-known that X_Σ for finite Σ is nonsingular (smooth) if and only if each cone σ of Σ is smooth. Moreover, X_Σ is complete (compact) if and only if Σ is finite and $\mathrm{Supp}\, \Sigma = N_{\mathbb{R}}$.

6.1.2. Crystallographic arrangements. Let $\mathcal{A}$ be a simplicial arrangement in $V = \mathbb{R}^r$, i.e. $\mathcal{A} = \{H_1, \dots, H_n\}$ where $H_1, \dots, H_n$ are distinct linear hyperplanes in V and every component of $V \smallsetminus \bigcup_{H \in \mathcal{A}} H$ is an open simplicial cone. Let $\mathcal{K}(\mathcal{A})$ be the set of connected components of $V \smallsetminus \bigcup_{H \in \mathcal{A}} H$; they are called the *chambers* of $\mathcal{A}$.

For each H_i, $i = 1, \dots, n$ we choose an element $x_i \in V^*$ such that $H_i = x_i^\perp$. Let

$$R = \{\pm x_1, \dots, \pm x_n\} \subseteq V^*.$$

For each chamber $K \in \mathcal{K}(\mathcal{A})$ set

$$\begin{aligned} W^K &= \{H \in \mathcal{A} \mid \dim(H \cap \overline{K}) = r-1\}, \\ B^K &= \{\alpha \in R \mid \alpha^\perp \in W^K, \quad \{\alpha\}^\vee \cap K = K\} \subseteq R. \end{aligned}$$

Here, $\overline{K}$ denotes the closure of K. The elements of W^K are the *walls* of K and B^K "is" the set of normal vectors of the walls of K pointing to the inside. Note that

$$\overline{K} = \bigcap_{\alpha \in B^K} \{\alpha\}^\vee,$$

and that B^K is a basis of V^* because $\mathcal{A}$ is simplicial. Moreover, if $\alpha_1^\vee, \dots, \alpha_r^\vee$ is the dual basis to $B^K = \{\alpha_1, \dots, \alpha_r\}$, then

$$K = \Big\{ \sum_{i=1}^{r} a_i \alpha_i^\vee \mid a_i > 0 \quad \text{for all} \quad i = 1, \dots, r \Big\}. \tag{15}$$

Definition 6.1.1 Let $\mathcal{A}$ be a simplicial arrangement and $R \subseteq V^*$ a finite set such that $\mathcal{A} = \{\alpha^\perp \mid \alpha \in R\}$ and $\mathbb{R}\alpha \cap R = \{\pm \alpha\}$ for all $\alpha \in R$. For $K \in \mathcal{K}(\mathcal{A})$ set

$$R_+^K = R \cap \sum_{\alpha \in B^K} \mathbb{R}_{\geq 0} \alpha.$$

We call $(\mathcal{A}, R)$ a *crystallographic arrangement* if for all $K \in \mathcal{K}(\mathcal{A})$:

(I) $\quad R \subseteq \sum_{\alpha \in B^K} \mathbb{Z}\alpha$.

Remark 6.1.2 Notice that one can prove that in fact if $(\mathcal{A}, R)$ is crystallographic, then $R \subseteq \pm \sum_{\alpha \in B^K} \mathbb{N}_0 \alpha$ (see **[Cun11]**).

6.2. Strong symmetry of fans

Definition 6.2.1 We call a fan Σ in V *strongly symmetric* if it is complete and if there exist hyperplanes $H_1, \dots, H_n$ in V such that

$$\operatorname{Supp} \Sigma(r-1) = H_1 \cup \ldots \cup H_n.$$

We write $\mathcal{A}(\Sigma) := \{H_1, \dots, H_n\}$. We call a toric variety X_Σ *strongly symmetric* if Σ is strongly symmetric.

We call a fan Σ *centrally symmetric* if $\Sigma = -\Sigma$. We call a toric variety X_Σ *centrally symmetric* if Σ is centrally symmetric.

Remark 6.2.2 One could also call a strongly symmetric fan *strongly complete* because for any $\tau \in \Sigma$ the collection of $\sigma \cap \langle\tau\rangle_{\mathbb{R}}$, $\sigma \in \Sigma$, is a complete fan in $\langle\tau\rangle_{\mathbb{R}}$ as a subfan of Σ.

Lemma 6.2.3 *Let τ be an $(r-1)$-dimensional cone in $\mathbb{R}^r$ and $H_1, \ldots, H_n$ be hyperplanes in $\mathbb{R}^r$. If $\tau \subseteq H_1 \cup \ldots \cup H_n$, then $\tau \subseteq H_i$ for some $1 \leq i \leq n$.*

PROOF. We construct inductively sets $T_i \subseteq \tau$ with $i + r - 1$ elements such that each subset B, $|B| = r - 1$, is linearly independent: Let $T_0 := \{n_1, \ldots, n_{r-1}\}$, where $n_1, ..., n_{r-1} \in \tau$ are linearly independent and span $\langle\tau\rangle_{\mathbb{R}}$. Given T_i, let

$$\Xi_i := \{\langle v_1, \ldots, v_{r-2}\rangle \mid v_1, \ldots, v_{r-2} \in T_i\}$$

be the set of subspaces generated by $r-2$ elements of T_i. Since τ has dimension $r-1$, $\bigcup_{U \in \Xi_i} U \neq \langle\tau\rangle_{\mathbb{R}}$. For any $w \in \tau \setminus \bigcup_{U \in \Xi_i} U$, $T_{i+1} := T_i \cup \{w\}$ has the required property.

Now consider the $(r-1)n$ elements of $T_{(r-1)(n-1)}$. Let ℓ be the maximal number of elements in any H_i. Then $\ell \geq r-1$. Then there is an $1 \leq i \leq n$ such that $r-1$ of these elements lie in H_i. These are linearly independent and belong to τ, so $\tau \subseteq \langle\tau\rangle_{\mathbb{R}} \subseteq H_i$. □

Lemma 6.2.4 *Let Σ be an r-dimensional fan. Then the following are equivalent:*

(1) Σ is complete, and for all $\tau \in \Sigma(r-1)$, $\sigma \in \Sigma$,

$$\sigma \cap \langle\tau\rangle_{\mathbb{R}} \in \Sigma,$$

(2) the fan Σ is strongly symmetric.

PROOF. Assume (1). Let $\tau \in \Sigma(r-1)$. Since Σ is complete, $\langle\tau\rangle_{\mathbb{R}} \subseteq \operatorname{Supp}\Sigma$. Thus $\langle\tau\rangle_{\mathbb{R}} = \bigcup_{\sigma\in\Sigma}\langle\tau\rangle_{\mathbb{R}} \cap \sigma$. By (1), $\Sigma' := \{\langle\tau\rangle_{\mathbb{R}} \cap \sigma \mid \sigma \in \Sigma\}$ is a subfan of Σ. Further, $\operatorname{Supp}\Sigma'(r-1) = \operatorname{Supp}\Sigma' = \langle\tau\rangle_{\mathbb{R}}$ because Σ' is complete in $N \cap \langle\tau\rangle_{\mathbb{R}}$ and the maximal cones in Σ' have dimension $r-1$. Hence for each τ of codimension 1, $\langle\tau\rangle_{\mathbb{R}}$ is a union of elements of $\Sigma(r-1)$. This implies $\operatorname{Supp}\Sigma(r-1) = \bigcup_{\tau\in\Sigma(r-1)}\langle\tau\rangle_{\mathbb{R}}$ (finite union by definition of complete).

Now assume $\operatorname{Supp}\Sigma(r-1) = H_1 \cup \ldots \cup H_n$ for some hyperplanes $H_1, \ldots, H_n$. Let $\tau \in \Sigma(r-1)$ and $\sigma \in \Sigma$. Then by Lemma 6.2.3, $\langle\tau\rangle_{\mathbb{R}} = H_i$ for some $1 \leq i \leq n$, and there exist $\eta_1, \ldots, \eta_k \in \Sigma(r-1)$ with $H_i = \eta_1 \cup \ldots \cup \eta_k$. But $\sigma \cap H_i = \bigcup_{j=1}^k \sigma \cap \eta_j$, so $\overset{\circ}{\sigma} \cap H_i = \emptyset$, i.e. H_i is a supporting hyperplane and $\sigma \cap H_i$ is a face of σ and thus an element of Σ. □

Lemma 6.2.5 *Let Σ be an r-dimensional strongly symmetric fan. Then the set of all intersections of closed chambers of $\mathcal{A}(\Sigma)$ is Σ. In particular, Σ is centrally symmetric.*

PROOF. Let $\sigma \in \Sigma(r)$. Then the facets of σ are in $\operatorname{Supp}\Sigma(r-1) = H_1 \cup \ldots \cup H_n$ and $\overset{\circ}{\sigma} \subseteq \mathbb{R}^r \setminus \operatorname{Supp}\Sigma(r-1)$. Since Σ is complete, $\Sigma(r)$ is the set of closed chambers of $\mathcal{A}$. □

Definition 6.2.6 Let Σ be a fan in N, $\delta \in \Sigma$, and write $\kappa : V \to V/\langle\delta\rangle_{\mathbb{R}}$ for the canonical projection. Then

$$\operatorname{Star}(\delta) = \{\overline{\sigma} = \kappa(\sigma) \subseteq V/\langle\delta\rangle_{\mathbb{R}} \mid \delta \subseteq \sigma \in \Sigma\}$$

is a fan in $N(\delta) := \kappa(N)$ (compare [**CLS11**, Ex. 3.2.7]). Its toric variety is isomorphic to the orbit closure $V(\delta)$ in X_Σ.

Lemma 6.2.7 *Let Σ be an r-dimensional fan. Then the following are equivalent:*

(1) *The fan Σ is strongly symmetric,*
(2) *the fan $\operatorname{Star}(\sigma)$ is strongly symmetric for all $\sigma \in \Sigma$.*

PROOF. We use Lemma 6.2.4. Assume (1). Let $\sigma \in \Sigma$ and consider a cone $\overline{\tau} \in \operatorname{Star}(\sigma)$ of codimension one. Then $\langle\overline{\tau}\rangle_{\mathbb{R}} = \overline{\langle\tau\rangle_{\mathbb{R}}} \subseteq V/\langle\sigma\rangle_{\mathbb{R}}$ and hence for any cone $\overline{\pi} \in \operatorname{Star}(\sigma)$ we have $\overline{\pi} \cap \langle\overline{\tau}\rangle_{\mathbb{R}} = \overline{\pi \cap \langle\tau\rangle_{\mathbb{R}}} \in \operatorname{Star}(\sigma)$, because $\pi \cap \langle\tau\rangle_{\mathbb{R}}$ is a cone in Σ containing σ; thus $\operatorname{Star}(\sigma)$ is strongly symmetric.

Since $\Sigma = \operatorname{Star}(\{0\})$, (1) follows from (2). □

Proposition 6.2.8 *Let Σ be an r-dimensional complete fan. Then the following are equivalent:*

(1) *The fan Σ is strongly symmetric,*
(2) *the fan $\operatorname{Star}(\sigma)$ is centrally symmetric for all $\sigma \in \Sigma$,*
(3) *the fan $\operatorname{Star}(\delta)$ is centrally symmetric for all $\delta \in \Sigma(r-2)$.*

PROOF. The implication (1) ⇒ (2) follows from Lemma 6.2.5 and Lemma 6.2.7; (2) ⇒ (3) is obvious.

Suppose that $\operatorname{Star}(\delta)$ is centrally symmetric for any $\delta \in \Sigma(r-2)$. We have to show that for any $\tau_0 \in \Sigma(r-1)$, $H := \langle\tau_0\rangle_{\mathbb{R}} \subseteq S := \operatorname{Supp}\Sigma(r-1)$. Suppose $H \nsubseteq S$. Let $\{\tau_0, \ldots, \tau_k\} = \{\tau \in \Sigma(r-1) \mid \tau \subseteq H\}$. Then

$$\tau_0 \cup \ldots \cup \tau_k \subsetneqq H.$$

Let p be a point of the relative border $\partial(\tau_0 \cup \ldots \cup \tau_k)$ in H. Then there is an i with $p \in \partial\tau_i$ and a $\delta \in \Sigma(r-2)$, $\delta \subseteq \tau_i$ such that $p \in \delta \subseteq \tau_i \subseteq \langle\tau_i\rangle_{\mathbb{R}} = H$. We have $\overline{\tau_i} \in \operatorname{Star}(\delta)$, $\overline{\tau_i} \subseteq \overline{H}$, and $\dim \overline{H} = 1$. Because $\operatorname{Star}(\delta)$ is centrally symmetric, $-\overline{\tau_i} \in \operatorname{Star}(\delta)$. Then $-\overline{\tau_i} = \overline{\tau'}$ for some $\delta \subseteq \tau' \in \Sigma(r-1)$ with $\overline{\tau'} \subseteq \overline{H}$. Then $\tau' \subseteq H$, $\delta \subseteq \tau_i \cap \tau'$ and $\tau' \neq \tau_i$. Hence $\delta = \tau_i \cap \tau'$ because $\dim(\delta) = r-2$. But then $p \notin \partial(\tau_0 \cup \ldots \cup \tau_k)$, contradicting the assumption. □

Example 6.2.9 There are of course fans which are centrally symmetric but not strongly symmetric. Here is such an example which is smooth: Let R be the standard basis of $\mathbb{R}^3$ and Σ_R be the fan as defined in Lemma

6.3.1. Blowing up along two opposite cones $\sigma, -\sigma \in \Sigma_R$ preserves the central symmetry, but the resulting fan is not strongly symmetric.

In the case of smooth strongly symmetric fans, we obtain

Lemma 6.2.10 *Let Σ be a smooth strongly symmetric fan in N, $\sigma \in \Sigma$ and $E := \langle\sigma\rangle_{\mathbb{R}}$. Then $N \cap E$ is a lattice of rank $\dim(\sigma)$ and $\Sigma^E := \{\eta \cap E \mid \eta \in \Sigma\} \subseteq \Sigma$ is a smooth strongly symmetric fan in $N \cap E$.*

Proof. Using a $\mathbb{Z}$-basis of σ one finds that $N \cap E$ is a sublattice of N of rank $\dim(\sigma)$ and that the inclusion $N \cap E \hookrightarrow N$ is split. Consider first a $\sigma \in \Sigma(r-1)$ and let $E := \langle\sigma\rangle_{\mathbb{R}}$. By Lemma 6.2.4, $\eta \cap E \in \Sigma$ for all $\eta \in \Sigma$. Thus Σ^E is a subfan of Σ and it is complete since $\operatorname{Supp}\Sigma = V$. Write $\operatorname{Supp}\Sigma(r-1) = E \cup H_2 \cup \ldots \cup H_n$ for hyperplanes $H_2, \ldots, H_n$. Then

$$\operatorname{Supp}\Sigma^E(r-2) = (H_2 \cup \ldots \cup H_n) \cap E = (H_2 \cap E) \cup \ldots \cup (H_n \cap E),$$

i.e. Σ^E is strongly symmetric. The claim is true for arbitrary $\sigma \in \Sigma$ by induction on $\dim(\sigma)$. $\square$

6.3. The correspondence

Lemma 6.3.1 *Let $(\mathcal{A}, R)$ be a crystallographic arrangement in V. Set*

$$M_R := \sum_{\alpha \in R} \mathbb{Z}\alpha \cong \mathbb{Z}^r$$

and let N_R be the dual lattice to M_R. Then the set Σ_R of all intersections of closed chambers of $\mathcal{A}$ is a strongly symmetric smooth fan in N_R.

Proof. It is clear that Σ_R is a strongly symmetric fan. Let $\sigma \in \Sigma_R$ be of maximal dimension, i.e. $\sigma = \overline{K}$ for a chamber $K \in \mathcal{K}(\mathcal{A})$. By Equation 15, σ is generated by the basis of N_R dual to B^K, hence σ is smooth. $\square$

Lemma 6.3.2 *Let Σ be a strongly symmetric smooth fan in $N \subseteq V = \mathbb{R}^r$. Then there exists a set $R \subseteq V^*$ such that $(\mathcal{A}, R)$ is a crystallographic arrangement, where*

$$\mathcal{A} = \mathcal{A}(\Sigma) = \{\langle\tau\rangle_{\mathbb{R}} \mid \tau \in \Sigma(r-1)\}.$$

Proof. Since Σ is strongly symmetric, $\mathcal{A}$ is a finite set of hyperplanes, and by Lemma 6.2.5, the set of all intersections of closed chambers of $\mathcal{A}$ is Σ. Further,

$$\bigcup_{\sigma \in \Sigma(r)} \mathring{\sigma} = V \setminus \bigcup_{H \in \mathcal{A}} H$$

since each facet of a $\sigma \in \Sigma(r)$ is contained in a hyperplane of $\mathcal{A}$ and since Σ is complete. The cones $\mathring{\sigma}$ in the above union are open simplicial cones, because σ is smooth, hence $\mathcal{A}$ is a simplicial arrangement.

Let $\sigma \in \Sigma$ be a cone of maximal dimension. Since σ is smooth, there exists a unique $\mathbb{Z}$-basis of N generating σ. We will prematurely denote B^{K_σ} its dual basis, where K_σ is the chamber with $\overline{K_\sigma} = \sigma$.

Now set R to be the union of all the B^{K_σ} for $\sigma \in \Sigma(r)$. Clearly,

$$R \subseteq \sum_{\alpha \in B^{K_\sigma}} \mathbb{Z}\alpha,$$

since each B^{K_σ} is a $\mathbb{Z}$-basis of $M = \mathrm{Hom}(N, \mathbb{Z})$ and $R \subseteq M$.

It remains to show that for each hyperplane $H = \langle \tau \rangle_{\mathbb{R}} \in \mathcal{A}$, $\tau \in \Sigma(r-1)$, there is a vector $x \in R$ such that $R \cap H^\perp = \{\pm x\}$.

Let $\sigma \in \Sigma(r)$ containing τ, and x be the element with $\{x\} = B^{K_\sigma} \cap H^\perp$. In particular x is primitive. Assume $\lambda x \in R$ for a $\lambda \in \mathbb{Z}$. Then there exists a $\sigma' \in \Sigma$ with $\lambda x \in B^{K_{\sigma'}}$. Thus $\lambda = \pm 1$ since $B^{K_{\sigma'}}$ is a $\mathbb{Z}$-basis of M. □

Theorem 6.3.3 *The map $(\mathcal{A}, R) \mapsto \Sigma_R$ from the set of crystallographic arrangements to the set of strongly symmetric smooth fans is a bijection.*

Proof. This is Lemma 6.3.1 and Lemma 6.3.2. □

Corollary 6.3.4 *A complete classification of strongly symmetric smooth toric varieties is now known.*

Proof. This is [**CH10**, Thm. 1.1]. □

Definition 6.3.5 We denote the toric variety of the fan Σ_R by $X(\mathcal{A}, R)$ or $X(\mathcal{A})$ and call it the toric variety of the arrangement $(\mathcal{A}, R)$.

Remark 6.3.6 For a fixed crystallographic arrangement $(\mathcal{A}, R)$, choosing another lattice than M_R may result in a strongly symmetric fan which is not smooth. Further, the correspondence $(\mathcal{A}, R) \mapsto \Sigma_R$ extends by its definition to a correspondence between rational simplicial arrangements and simplicial strongly symmetric fans. However, there exist rational simplicial non-crystallographic arrangements, i.e., there is a basis with respect to which all covectors of the hyperplanes have rational coordinates, although there is no lattice M for which the corresponding fan is smooth. The smallest example in dimension three has 12 hyperplanes and is denoted $\mathcal{A}(12, 1)$ in [**Grü09**] (compare the catalogue [**Grü09**] with the list in [**CH12**]).

Remark 6.3.7 Any smooth complete fan in N can be visualized by a triangulation of the sphere $S = V \smallsetminus \{0\}/\mathbb{R}_{>0}$, see [**Oda88**, Sect. 1.7]. Such a fan is centrally symmetric if and only if its triangulation is invariant under the reflection $p \leftrightarrow -p$ of S, and the strong symmetry of the fan Σ_R of a crystallographic arrangement $(\mathcal{A}, R)$ means that its triangulation is induced by the hyperplane sections $H \cap S$, $H \in \mathcal{A}$.

In particular in dimension 3 Tsuchihashi's characterization by admissible N-weights (see [**Oda88**, Cor. 1.32]) for strongly symmetric fans agrees with the classification in [**CH12**]. For higher dimension the correspondence to Weyl groupoids produces similar conditions if one considers certain products of reflections.

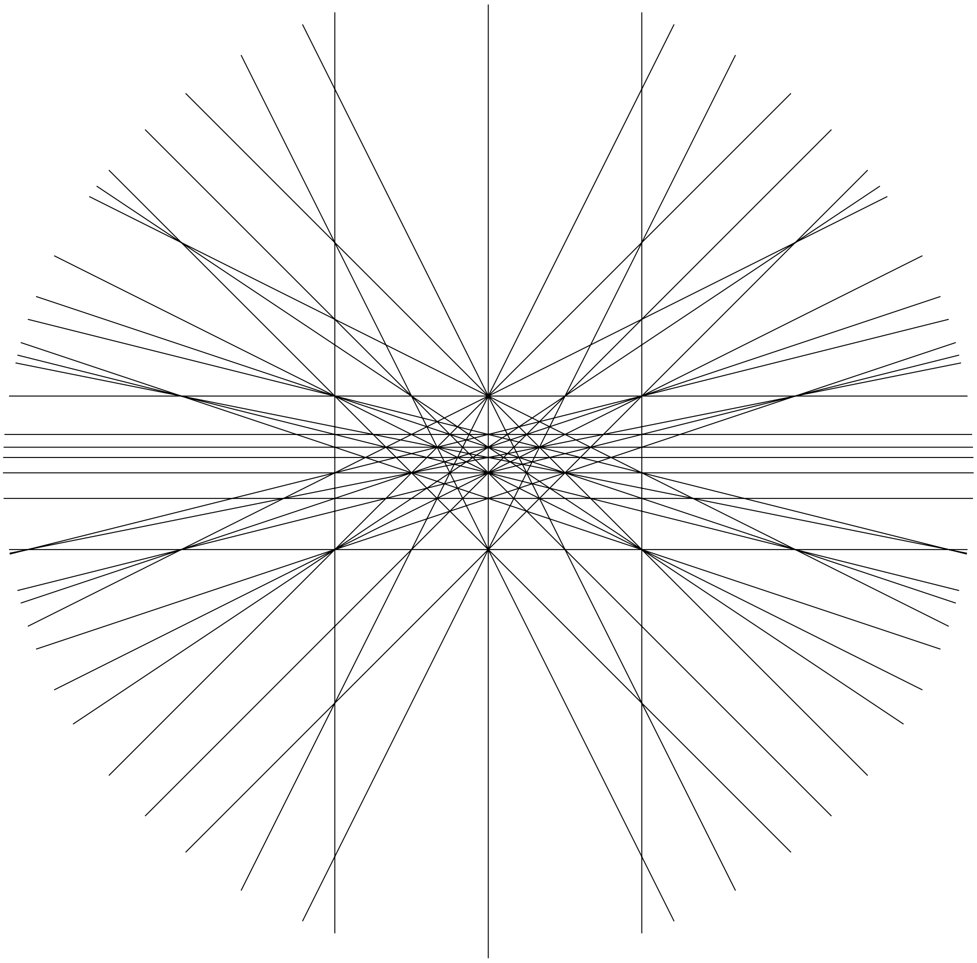

FIGURE 1. The largest crystallographic arrangement in dimension three (see Example 6.3.8)

For a geometric interpretation of the strong symmetry of $X(\mathcal{A})$ see Rem. 6.6.9.

Example 6.3.8 The crystallographic arrangement with the largest number of hyperplanes in dimension three has 37 hyperplanes. Fig. 1 is a projective image of this *sporadic* arrangement: The triangles correspond to the maximal cones; one hyperplane is the line at infinity.

We further obtain a new proof of [**BC12**, Prop. 5.3]:

Corollary 6.3.9 *Let $\mathcal{A}$ be a crystallographic arrangement and E be an intersection of hyperplanes of $\mathcal{A}$. Then the restriction $\mathcal{A}^E$ of $\mathcal{A}$ to E,*

$$\mathcal{A}^E := \{E \cap H \mid H \in \mathcal{A},\ E \not\subseteq H\}$$

is a crystallographic arrangement.

PROOF. This follows from Thm. 6.3.3, the fact that subfans of smooth fans are smooth, and Lemma 6.2.10. $\square$

6.4. Projectivity

Let $(\mathcal{A}, R)$ be a crystallographic arrangement and N, M, V, V^* be as in Section 6.3, $\Sigma := \Sigma_R$. We first prove that $X(\mathcal{A}) = X_\Sigma$ is projective by constructing a polytope P such that $X_P \cong X_\Sigma$.

Proposition 6.4.1 *Let $\mathcal{A}$ be a crystallographic arrangement. For a chamber K let*

$$\rho^K := \frac{1}{2} \sum_{\alpha \in R_+^K} \alpha.$$

Then the set $\{\rho^K \mid K \in \mathcal{K}(\mathcal{A})\}$ is the set of vertices of an integral convex polytope P in $\frac{1}{2}M$.

PROOF. For each chamber K define a simplicial cone by

$$S^K := \rho^K - \langle \alpha \mid \alpha \in B^K \rangle_{\mathbb{R}_{\geq 0}}.$$

Let P be the polytope

$$P := \bigcap_{K \in \mathcal{K}(\mathcal{A})} S^K.$$

Let K be a chamber. We prove that ρ^K is a vertex of P by showing $\rho^K \in P$: Let K' be a chamber. Notice first that for $\alpha \in R$ we have

$$\alpha \in R_+^K \quad \Longleftrightarrow \quad -\alpha \in R \setminus R_+^K,$$

which implies $R_+^{K'} \setminus R_+^K = -R_+^K \setminus R_+^{K'}$. Thus

$$\rho^K = \rho^{K'} - \frac{1}{2} \sum_{\alpha \in R_+^{K'} \setminus R_+^K} \alpha + \frac{1}{2} \sum_{\alpha \in R_+^K \setminus R_+^{K'}} \alpha = \rho^{K'} - \sum_{\alpha \in R_+^{K'} \setminus R_+^K} \alpha \in S^{K'}.$$

$\square$

Remark 6.4.2 The set $\{\rho^K \mid K \in \mathcal{K}(\mathcal{A})\}$ of the last proposition is the orbit of one fixed ρ^K under the action of the Weyl groupoid $\mathcal{W}(\mathcal{A})$ since for a simple root $\alpha \in B^K$ we have $\sigma_\alpha(\rho^K) = \rho^K - \alpha$ (see [**CH09a**]).

Corollary 6.4.3 *Let $\mathcal{A}$ be a crystallographic arrangement. Then X_Σ is a projective variety isomorphic to X_P, where P is the polytope of Prop. 6.4.1.*

PROOF. This is Prop. 6.4.1 and [**Oda88**, Thm. 2.22]. $\square$

We now describe an explicit immersion of X_Σ into $\mathbb{P}_1^R \cong \mathbb{P}_1^{2n}$.

Definition 6.4.4 For any $\sigma \in \Sigma$, $\alpha \in R$ let

$$s_\alpha(\sigma) = \begin{cases} +1 & \text{if } \alpha(\sigma) = \mathbb{R}_{\geq 0} \\ 0 & \text{if } \alpha(\sigma) = \{0\} \\ -1 & \text{if } \alpha(\sigma) = \mathbb{R}_{\leq 0} \end{cases}$$

and let $s(\sigma) = (s_\alpha(\sigma))_{\alpha\in R}$.

Definition 6.4.5 Let $2n = |R|$, let V' be a $2n$-dimensional vector space over $\mathbb{R}$ and $(e_\alpha)_{\alpha\in R}$ be a basis of V'^*. Further, let $M' := \mathbb{Z}\{e_\alpha \mid \alpha \in R\} \subseteq V'^*$ be the lattice generated by this basis and let N' be the dual lattice. Then $\mathcal{A}' := \{e_\alpha^\perp \mid \alpha \in R\}$ is a Boolean arrangement and we call the corresponding fan $\Sigma' := \Sigma(\mathcal{A}')$ a *Boolean fan.* Notice that

$$X_{\Sigma'} \cong \mathbb{P}_1^{2n}.$$

Consider the homomorphism $M' \to M$, $e_\alpha \mapsto \alpha$ for $\alpha \in R$ and its dual

$$\varphi : N \to N', \quad n \mapsto (\alpha(n))_{\alpha\in R}.$$

Lemma 6.4.6 *Choose a chamber K. Then with respect to the basis B^{K^*} of N the map φ is represented by a matrix of the form*

$$\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \\ * & \cdots & * \\ \vdots & & \vdots \end{pmatrix}.$$

It follows that φ is a split monomorphism and in particular $N'/\varphi(N)$ is torsion free.

Lemma 6.4.7

(1) *The map φ is a map of fans $(N, \Sigma) \to (N', \Sigma')$.*
(2) *For any $\sigma' \in \Sigma'$, $\varphi(V) \cap \sigma' \in \Sigma$.*

Proof. (1) Let $\sigma \in \Sigma$ and let $\sigma' \in \Sigma'$ be the cone with $s(\sigma') = s(\sigma)$. Then $\varphi(\sigma) \subseteq \sigma'$.
(2) If $\sigma' \in \Sigma'$ is maximal, let $s(\sigma') = (\varepsilon_1, \ldots, \varepsilon_{2n})$ with $\varepsilon_\nu \in \{\pm 1\}$, and let

$$\tau = \bigcap_\nu \{x \in V \mid \varepsilon_\nu \alpha_\nu(x) \geq 0\}.$$

Then $\tau \in \Sigma$ and $\tau = \varphi^{-1}(\sigma')$. If σ' is arbitrary, then $\sigma' = \sigma_1' \cap \ldots \cap \sigma_k'$ for maximal σ_i' and then $\varphi^{-1}(\sigma') = \bigcap \varphi^{-1}(\sigma_i') \in \Sigma$. □

Corollary 6.4.8 *The induced toric morphism $f = \varphi_* : X_\Sigma \to X_{\Sigma'}$ is proper and $X_\Sigma \twoheadrightarrow f(X_\Sigma)$ is the normalization of the closed (reduced) image.*

Proof. See [**Oda88**, Prop. 1.14]. □

Proposition 6.4.9 *The map $X_\Sigma \to X_{\Sigma'}$ is a closed embedding of nonsingular toric varieties.*

Proof. Let σ be a maximal cone, K the corresponding chamber and $B^K \subseteq R$ the basis of M. If $\sigma' \in \Sigma'$ is the cone with $s(\sigma) = s(\sigma')$ ($\sigma =$

$\varphi(V) \cap \sigma')$, then the dual cone to σ' is

$$\sigma'^\vee = \langle e_\alpha \in R \mid s_\alpha(\sigma') = 1\rangle_{\mathbb{R}_{\geq 0}}.$$

The map $\sigma'^\vee \cap M' \to \langle B^K\rangle_{\mathbb{Z}_{\geq 0}}$ is surjective, so $\mathbb{C}[\sigma'^\vee \cap M'] \to \mathbb{C}[\langle B^K\rangle_{\mathbb{Z}_{\geq 0}}]$ is a surjective homomorphism of $\mathbb{C}$-algebras giving rise to the closed embedding

$$f|_{U_\sigma} : U_\sigma \to U'_{\sigma'},$$

where $f = \varphi_*$ as in Cor. 6.4.8. Because U_σ is dense in X_Σ, the closure of $f(U_\sigma)$ equals $f(X_\Sigma)$, hence $f(U_\sigma) = f(X_\Sigma) \cap U'_{\sigma'}$. It follows that $f(X_\Sigma)$ is smooth and that $X_\Sigma \to f(X_\Sigma)$ is an isomorphism. The injectivity of f follows from that of $f|_{U_\sigma}$ because then $f|_{\mathrm{orb}(\sigma)}$ is an injective map $\mathrm{orb}(\sigma) \to \mathrm{orb}(\sigma')$ for each cone σ of the orbit decomposition of X_Σ. □

6.5. Remarks on surfaces

For 2-dimensional fans of complete toric surfaces obviously strongly symmetric is the same as centrally symmetric. The classification of smooth complete toric surfaces, see [**Oda88**, Cor. 1.29] can be specialized as follows. It turns out that this classification coincides with the classification of crystallographic arrangements of rank two [**CH09b, CH11**].

Let Σ be the fan of a smooth complete toric surface with rays $\rho_1, \dots, \rho_s$ ordered counterclockwise with primitive generators $n_1, \dots, n_s$. There are integers $a_1, \dots, a_s$ such that

$$n_{j-1} + n_{j+1} + a_j n_j = 0$$

for $1 \leq j \leq s$ where $n_{s+1} := n_1$, $n_0 := n_s$. The integers a_j are the self-intersection numbers of the divisors D_j associated to the rays ρ_j. The circular weighted graph $\Gamma(\Sigma)$ has as its vertices on S^1 the rays ρ_j with weights a_j. These weights satisfy the identity

$$\begin{pmatrix} 0 & -1 \\ 1 & -a_s \end{pmatrix} \cdots \begin{pmatrix} 0 & -1 \\ 1 & -a_1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Conversely, to any circular weighted graph with this identity there is a smooth complete toric surface with this graph, unique up to toric isomorphisms.

All these surfaces are obtained from the basic surfaces $\mathbb{P}_2$, $\mathbb{P}_1 \times \mathbb{P}_1$, and the Hirzebruch surfaces $\mathbb{F}_a$, $a \geq 2$, by a finite succession of blowing-ups. If the surface X_Σ is centrally symmetric, then the number s of rays is even, $s = 2t$, and $a_{t+j} = a_j$ for $1 \leq j \leq t$. In this case

$$\begin{pmatrix} 0 & -1 \\ 1 & -a_t \end{pmatrix} \cdots \begin{pmatrix} 0 & -1 \\ 1 & -a_1 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix},$$

which is "dual" to the formula of the classification of crystallographic arrangements of rank two (see [**CH09b**]).

Note further that sequences $a_1, \dots, a_t$ satisfying this formula are in bijection with triangulations of a convex t-gon by non-intersecting diagonals. The numbers in Fig. 2 are $-a_1, \dots, -a_t$; these are certain entries of the Cartan

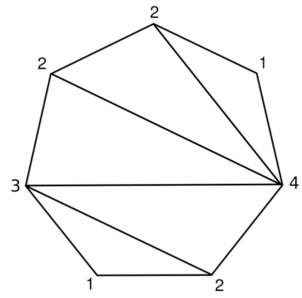

FIGURE 2. Triangulation of a t-gon

matrices of the corresponding Weyl groupoid (see [**CH11**] for more details). Attaching a triangle to the t-gon corresponds to a double blowing-up on the variety.

One can subdivide a smooth complete 2-dimensional fan Σ by filling in the opposite $-\rho$ of each ray ρ in order to get a complete centrally symmetric fan Σ_C. However, Σ_C need not be smooth as in Example 6.5.1. But by inserting further pairs $\rho, -\rho$ of rays one can desingularize the surface X_{Σ_C} in an even succession of blowing-ups to obtain a smooth complete centrally symmetric surface $X_{\tilde{\Sigma}}$ with a surjective toric morphism $X_{\tilde{\Sigma}} \to X_\Sigma$.

Example 6.5.1 Let Σ be the fan of the Hirzebruch surface $\mathbb{F}_a$, $a \geq 2$, with the primitive generators

$$n_1 = (1,0), \quad n_2 = (0,1), \quad n_3 = (-1,a), \quad n_4 = (0,-1).$$

The fan Σ_C is then obtained by adding the rays spanned by $(-1,0)$ and $(1,-a)$. This fan is no longer smooth. After filling in the rays spanned by $(1,-\nu)$ for $1 \leq \nu < a$, we obtain a smooth complete centrally symmetric fan $\tilde{\Sigma}$ with $2a$ rays. In case $a = 2$ its circular graph has the weights $(-1,-2,-1,-2;-1,-2,-1,-2)$ (this corresponds to the reflection arrangement of type B and C).

Example 6.5.2 In good cases the centrally symmetric fan Σ_C may already be smooth. As an example let Σ be the fan of $\mathbb{P}_2$ spanned by $(1,0)$, $(0,1)$ and $(-1,-1)$. Then the fan Σ_C is spanned, in counterclockwise order, by

$$(1,0),(1,1),(0,1),(-1,0),(-1,-1),(0,-1).$$

This is the fan of the blow up $\tilde{\mathbb{P}}_2$ of $\mathbb{P}_2$ at the three fixed points of the torus action. The corresponding arrangement is the reflection arrangement of type A_2. Its circular graph has the weights

$$(-1,-1,-1;-1,-1,-1).$$

The same surface can be obtained by blowing up $\mathbb{P}_1 \times \mathbb{P}_1$ in two points corresponding to the enlargement of the weighted graph $(0,0,0,0)$ by inserting -1 after the first and third place, see [**Oda88**, Cor. 1.29].

Notice that $\mathbb{P}_1 \times \mathbb{P}_1$ corresponds to the reducible reflection arrangement of type $A_1 \times A_1$. One should also note here that $\tilde{\mathbb{P}}_2$ and $\mathbb{P}_1 \times \mathbb{P}_1$ are the only toric Del Pezzo surfaces which are centrally symmetric.

6.6. Parabolic subgroupoids and toric arrangements

If $(\mathcal{A}, R)$ is a crystallographic arrangement in V and E is an intersection of hyperplanes of $\mathcal{A}$, then by Cor. 6.3.9 the restriction $\mathcal{A}^E$ is again crystallographic. The dual statement is that $\mathrm{Star}(\delta)$ for $\delta \in \Sigma_R$ is the fan of a crystallographic arrangement which corresponds to a parabolic subgroupoid, see below. Both constructions may be translated to the corresponding toric varieties in a compatible way. This gives rise to posets of toric varieties which we call *toric arrangements* (see Section 6.6.2).

6.6.1. Star fans and parabolic subgroupoids. Let $(\mathcal{A}, R)$ be a crystallographic arrangement, Σ_R be the corresponding smooth strongly symmetric fan in $\mathbb{R}^r$, $\delta \in \Sigma$, $E := \langle \delta \rangle_{\mathbb{R}}$ and $d := \dim(E)$. Let $R_E := R \cap E^\perp$ and

$$\mathcal{A}_E := \{\overline{\alpha^\perp} \subseteq V/E \mid \alpha \in R_E\},$$

and notice that $\overline{\alpha^\perp}$ are hyperplanes in V/E because $\alpha \in E^\perp$. Remark also that $\mathcal{A}_E$ only depends on E. By [**CH12**, Cor. 2.5], R_E is a set of real roots of a parabolic subgroupoid of $\mathcal{W}(\mathcal{A}(\Sigma))$ (see [**HW11**, Def. 2.3] for the precise definition of a parabolic subgroupoid). Here, $\mathcal{W}(\mathcal{A}(\Sigma))$ is the Weyl groupoid of the Cartan scheme given by the crystallographic arrangement $\mathcal{A}(\Sigma)$ as described in [**Cun11**, Prop. 4.5]. Thus $(\mathcal{A}_E, R_E)$ is a crystallographic arrangement. It corresponds to the fan $\mathrm{Star}(\delta)$:

Proposition 6.6.1 *Let $(\mathcal{A}, R)$ be a crystallographic arrangement and let δ be a d-dimensional cone of the fan Σ_R. Then the orbit closure $V(\delta) \subseteq X(\mathcal{A})$ of* $\mathrm{orb}(\delta)$ *corresponds to the crystallographic arrangement*

$$\mathcal{A}_E = \{\overline{H} \subseteq V/E \mid H \in \mathcal{A}\} = \{\langle \overline{\tau} \rangle_{\mathbb{R}} \mid \overline{\tau} \in \mathrm{Star}(\delta)(r-d-1)\},$$

where $E = \langle \delta \rangle_{\mathbb{R}}$ as above.

Proof. Let $\overline{H}$ be in the left set. Then $\delta \subseteq E \subseteq H$, thus there exists a $\tau \in \Sigma(r-1)$ with $\delta \subseteq \tau \subseteq H$. Hence $\langle \overline{\tau} \rangle_{\mathbb{R}}$ is in the right hand set.

Now let $\langle \overline{\tau} \rangle_{\mathbb{R}}$ be in the right hand set. Then $E \subseteq \langle \tau \rangle_{\mathbb{R}} \subseteq H$ for an $H \in \mathcal{A}$ and so $\langle \overline{\tau} \rangle_{\mathbb{R}} \subseteq \overline{H}$. But since these have the same dimension, they are equal. □

Corollary 6.6.2 *Let Σ be a strongly symmetric fan in $\mathbb{R}^r$ and $\delta, \delta' \in \Sigma$ with $\langle \delta \rangle_{\mathbb{R}} = \langle \delta' \rangle_{\mathbb{R}}$. Then* $\mathrm{Star}(\delta) = \mathrm{Star}(\delta')$ *and $V(\delta) \cong V(\delta')$ even so $V(\delta) \neq V(\delta')$.*

Proof. As in Prop. 6.6.1, $\mathrm{Star}(\delta)$ only depends on $\langle \delta \rangle_{\mathbb{R}}$ because $\mathrm{Star}(\delta)$ is strongly symmetric. Note that here smoothness is not used. □

Corollary 6.6.3 *Let Σ be a smooth strongly symmetric fan in $\mathbb{R}^r$, $\mathcal{W}(\mathcal{A}(\Sigma))$ the corresponding Weyl groupoid, and $\delta \in \Sigma$. Then the Weyl groupoid $\mathcal{W}(\mathcal{A}(\mathrm{Star}(\delta)))$ is equivalent to a connected component of a parabolic subgroupoid of $\mathcal{W}(\mathcal{A}(\Sigma))$.*

6.6.2. Associated toric arrangements. Let as before Σ be the fan of a crystallographic arrangement $(\mathcal{A}, R)$ and as in [**OT92**, Def. 2.1] let $L(\mathcal{A})$ be the poset of nonempty intersections of elements of $\mathcal{A}$. By Lemma 6.2.10, for any $E \in L(\mathcal{A})$ we are given the strongly symmetric smooth subfan

$$\Sigma^E = \{\sigma \cap E \mid \sigma \in \Sigma\} = \{\sigma \in \Sigma \mid \sigma \subseteq E\}$$

of Σ. Let X^E denote its toric variety. The inclusion $\iota : N^E = N \cap E \hookrightarrow N$ is then a sublattice and compatible with the fans Σ^E and Σ and induces a toric morphism

$$f^E : X^E \to X(\mathcal{A}) = X_\Sigma.$$

Lemma 6.6.4 *The map f^E is a closed immersion with image $Y^E \subseteq X(\mathcal{A})$ of dimension* $\dim E$.

Proof. The subspace E is spanned by any cone $\tau \in \Sigma^E$ of maximal dimension $s := \dim E$. Using a $\mathbb{Z}$-basis of τ as in the proof of Lemma 6.2.10 one finds that $N^E = N \cap E$ is a sublattice of N of rank s and that the inclusion $\iota : N^E \hookrightarrow N$ is split. The induced map $\iota_\mathbb{R}$ sends a cone σ to itself and thus gives rise to a proper toric morphism f^E. Let M^E be the dual lattice of N^E and $\sigma \in \Sigma^E$. Using the duals of bases of N^E and N, one finds that the induced dual map $\iota^* : M \cap \sigma^\vee \to M^E \cap \sigma^\vee$ is surjective. Then

$$f^E|_{U_\sigma^E} : U_\sigma^E \to U_\sigma$$

is a closed immersion, where $U_\sigma^E \subseteq X^E$ and $U_\sigma \subseteq X_\Sigma$ denote the open affine spectra defined by $M^E \cap \sigma^\vee$ resp. $M \cap \sigma^\vee$. As in the proof of Prop. 6.4.9 we conclude that f^E is globally a closed immersion. □

Remark 6.6.5 Note that Y^E is not invariant under the torus action on X_Σ but is a strongly symmetric smooth toric variety on its own with torus $T^E = N^E \otimes \mathbb{C}^* \subseteq T$.

Proposition 6.6.6 *With the above notation the subvarieties $Y^E \subseteq X_\Sigma$ have the following properties.*

(1) *Each Y^E, $E \in L(\mathcal{A})$, is invariant under the involution of X_Σ defined by the central symmetry of Σ.*
(2) *For each cone $\sigma \in \Sigma$,*

$$Y^E \cap \mathrm{orb}(\sigma) = \begin{cases} \mathrm{orb}^E(\sigma) & \text{if } \sigma \subseteq E, \\ \emptyset & \text{if } \sigma \not\subseteq E, \end{cases}$$

and

$$Y^E \cap V(\sigma) = \begin{cases} V^E(\sigma) & \text{if } \sigma \subseteq E, \\ \emptyset & \text{if } \sigma \not\subseteq E, \end{cases}$$

where $\mathrm{orb}^E(\sigma)$ *resp.* $V^E(\sigma)$ *denote the images of the orbit of* σ *resp. its closure in* X^E.

(3) *When* $F, E \in L(\mathcal{A})$ *with* $F \subseteq E$, *then the composition* $X^F \hookrightarrow X^E \hookrightarrow X_\Sigma$ *is the inclusion* $X^F \hookrightarrow X_\Sigma$.

(4) *For any* $E, F \in L(\mathcal{A})$, $Y^{E\cap F} = Y^E \cap Y^F$.

(5) *The intersections* $Y^E \cap T$ *of* Y^E *with the torus* T *of* X_Σ *are the subtori* $T^E = N^E \otimes \mathbb{C}^*$ *of* T *of dimension* $\dim(E)$ *and constitute a toric arrangement.*

Definition 6.6.7 We call the system $\{Y^E\}_{E\in L(\mathcal{A})}$ the associated *toric arrangement* of the strongly symmetric smooth toric variety $X(\mathcal{A})$.

Remark 6.6.8 Prop. 6.6.6 (4) shows that the assignment $E \mapsto Y^E$ is an isomorphism of posets.

Remark 6.6.9 Prop. 6.6.6 yields a geometric interpretation of the strong symmetry of $X(\mathcal{A})$ by its toric arrangement: For any hyperplane $H \in \mathcal{A}$ the union of the curves $V(\tau)$, $\tau \subseteq H$, $\dim(\tau) = r-1$, is the set of fixed points of $X(\mathcal{A})$ under the action of the subtorus $T^H = N^H \otimes \mathbb{C}^* = Y^H \cap T$ of T. This union meets the hypersurface Y^H exactly in the set of its fixed points under the action of its torus T^H, and does not meet any other $Y^{H'}$.

The same holds for any $E \in L(\mathcal{A})$ for Y^E and the varieties $V(\tau)$, $\tau \subseteq E$, $\dim(\tau) = \dim E$, inside any other Y^F, $E \subseteq F \in L(\mathcal{A})$.

Proof. (i) follows from the fact that f^E is induced by the map ι between strongly symmetric fans.

(ii) follows from the orbit decompositions of Y^E and $V(\sigma)$ and the fact that f^E maps $\mathrm{orb}^E(\sigma)$ into $\mathrm{orb}(\sigma)$, because $\iota_{\mathbb{R}}(\sigma) = \sigma$ for $\sigma \in \Sigma^E$. If $\sigma \not\subseteq E$, no $\mathrm{orb}^E(\tau)$, $\tau \subseteq E$, can meet $\mathrm{orb}(\sigma)$. If $\sigma \subseteq E$, $\mathrm{orb}^E(\sigma) = \mathrm{orb}(\sigma) \cap Y^E$.

(iii) follows directly from the definition of the morphisms f^E.

(iv) It is sufficient to assume that F is a hyperplane $H \in \mathcal{A}$ with $E \not\subseteq H$. Let $s = \dim E$. Then $\dim Y^{E\cap H} = s-1$ and $Y^{E\cap H} \subseteq Y^E \cap Y^H$. Suppose that there is a point $x \in Y^E \cap Y^H$ and $x \notin Y^{E\cap H}$. Let then $\sigma \in \Sigma$ be a maximal cone with $x \in \mathrm{orb}(\sigma)$. Then $Y^{E\cap H} \cap \mathrm{orb}(\sigma) \subsetneq Y^E \cap Y^H \cap \mathrm{orb}(\sigma)$. By property (ii), $\sigma \subseteq E \cap H$ and

$$Y^{E\cap H} \cap \mathrm{orb}(\sigma) = \mathrm{orb}^{E\cap H}(\sigma), \quad Y^E \cap \mathrm{orb}(\sigma) = \mathrm{orb}^E(\sigma),$$

$$Y^H \cap \mathrm{orb}(\sigma) = \mathrm{orb}^H(\sigma)$$

are subtori of $\mathrm{orb}(\sigma)$ of dimensions $s-1-\dim(\sigma)$, $s-\dim(\sigma)$, $r-1-\dim(\sigma)$ and $\dim(\mathrm{orb}(\sigma)) = r-\dim(\sigma)$. It follows that $Y^E \cap Y^H \cap \mathrm{orb}(\sigma)$ is a subtorus of dimension $s-1-\dim(\sigma)$, too. Hence $Y^{E\cap H} \cap \mathrm{orb}(\sigma) = Y^E \cap Y^H \cap \mathrm{orb}(\sigma)$, contradiction.

(v) follows from (ii) for the special case $T = \mathrm{orb}(\{0\})$. Then the definition of a toric arrangement as in [**DCP05**] is satisfied. □

Property (ii) of Prop. 6.6.6 also includes that the intersections $Y^E \cap V(\sigma)$ are smooth, irreducible and proper of dimension $\dim E - \dim \sigma$. Moreover, we have the

Proposition 6.6.10 *With the above notation:*

(1) *For any fixed orbit closure $V(\tau) \subseteq X(\mathcal{A})$ the intersections $Y^E \cap V(\tau)$, $\tau \subseteq E$ constitute the toric arrangement $\{Y^{E/\langle\tau\rangle_{\mathbb{R}}}\}$ of the variety $V(\tau)$ corresponding to the crystallographic arrangement $\mathcal{A}_D$, $D = \langle\tau\rangle_{\mathbb{R}}$ with fan* $\mathrm{Star}(\tau)$ *as in Prop. 6.6.1.*
(2) *The intersections $Y^E \cap \mathrm{orb}(\tau)$, $\tau \subseteq E$, form a toric arrangement of subtori in each orbit* $\mathrm{orb}(\tau)$ *of $X(\mathcal{A})$.*

Proof. Let $D = \langle\tau\rangle_{\mathbb{R}} \subseteq E$ and $\overline{E} = E/D$. Under the isomorphism $X_{\mathrm{Star}(\tau)} \cong V(\tau)$ an orbit $\mathrm{orb}(\sigma) \subseteq V(\tau)$, $\tau \subseteq \sigma$, is identified with the orbit $\mathrm{orb}(\overline{\sigma})$ with $\overline{\sigma} \subseteq V/D$ the image of σ. Likewise, an orbit $\mathrm{orb}^E(\sigma)$ in X^E with $\tau \subseteq \sigma \subseteq E$ can be identified with the orbit $\mathrm{orb}^{\overline{E}}(\overline{\sigma})$ in the variety $X_{\mathrm{Star}(\tau)^{\overline{E}}} \cong V^E(\tau)$ in X^E. It follows that the embeddings $X^E \hookrightarrow X(\mathcal{A})$ and $X_{\mathrm{Star}(\tau)^{\overline{E}}} \hookrightarrow X_{\mathrm{Star}(\tau)} = V(\tau)$ are compatible and thus that $Y^E \cap V(\tau)$ is the image of the latter.

(2) follows from (v) of Prop. 6.6.6 since $\mathrm{orb}(\tau)$ is the torus of $V(\tau)$. □

Example 6.6.11 The system $\{Y^E\}_{E \in L(\mathcal{A})}$ for strongly symmetric toric surfaces has the following special features (see Fig. 3). Here each E is a line of $\mathcal{A}$.

(1) For $\rho \subseteq E$, $Y^E \cap D_\rho = \mathrm{orb}^E(\rho)$ is a point $p_\rho \in \mathrm{orb}(\rho)$.
(2) $Y^E \setminus (D_\rho \cup D_{-\rho})$ is the torus $T^E \cong k^*$ of Y^E.
(3) $Y^E \cap D_{\rho'} = \emptyset$ for $\rho' \nsubseteq E$.
(4) $Y^E \cap Y^F = \{1\} \subseteq T$ for any $E, F \in L(\mathcal{A})$.

Notice here that all the divisors D_ρ and Y^E are isomorphic to $\mathbb{P}_1$ and that the intersections are transversal.

There is an interesting formula for the divisor classes of the curves Y^E in terms of the toric divisors D_ρ as follows. Keeping the notation of Section 6.5, let $a_1, \ldots, a_{2t}$ be a chosen order of the weights of the circular graph of the surface $X(\mathcal{A})$ with corresponding divisors $D_1, \ldots, D_{2t}$, and let $Y_1 = Y^E$ in case $E := \langle n_1\rangle_{\mathbb{R}}$.

Then the standard sequence $0 \to M \to \mathbb{Z}^{\Sigma(1)} \to \mathrm{Pic}\, X(\mathcal{A}) \to 0$ can be represented by the exact sequence

$$0 \longrightarrow \mathbb{Z}^2 \xrightarrow{(Q,-Q)} \mathbb{Z}^t \oplus \mathbb{Z}^t \xrightarrow{\begin{pmatrix} A & I \\ 0 & I \end{pmatrix}} \mathbb{Z}^{t-2} \oplus \mathbb{Z}^t \longrightarrow 0$$

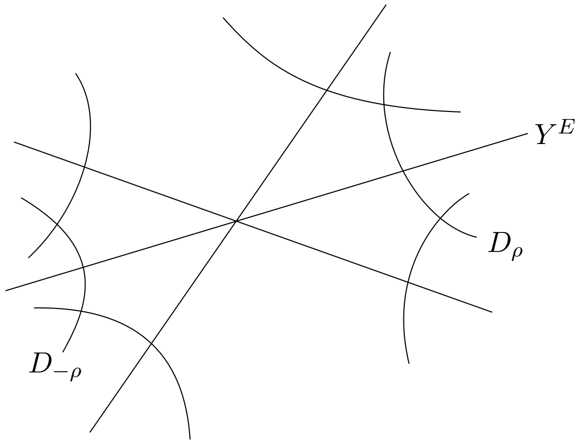

FIGURE 3. Example 6.6.11

where $Q^\vee = (n_1, \ldots, n_t)$ is the matrix of the first t primitive elements and $A^\vee$ is the matrix

$$A^\vee = \begin{pmatrix} a_1 & 1 & & & -1 \\ 1 & a_2 & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & 1 \\ -1 & & & 1 & a_t \end{pmatrix}$$

of rank $t-2$ expressing the relations $n_{j-1} + a_j n_j + n_{j+1} = 0$. To deduce the formula for Y_1 we choose n_1, n_t as the basis of the lattice N. Then

$$Q^\vee = \begin{pmatrix} 1 & x_2 & \cdots & x_{t-1} & 0 \\ 0 & y_2 & \cdots & y_{t-1} & 1 \end{pmatrix}$$

and $y_2 = 1$ since $A \cdot Q = 0$.

Proposition 6.6.12 *With the above notation,*

$$Y_1 \sim D_2 + \sum_{\nu=3}^{t-1} y_\nu D_\nu + D_t \sim D_{t+2} + \sum_{\nu=3}^{t-1} y_\nu D_{\nu+2} + D_{2t} \tag{16}$$

up to rational equivalence and Y_1 *has selfintersection* $Y_1^2 = 0$.

Remark 6.6.13 Choosing n_1, n_t as a basis, the columns of $Q^\vee$ become the positive roots of the associated Weyl groupoid at the object corresponding to Y_1.

The formula for the other $Y_\nu = Y^E$, $n_\nu \in E$, follows by cyclic permutation of the indices. Note that the classes of $D_2, \ldots, D_t$ are part of a basis of $\operatorname{Pic} X(\mathcal{A})$. The formula can be derived as follows. If Y_1 is equivalent to $\sum c_\nu D_\nu$, the intersection numbers $D_\nu^2 = a_\nu$, $D_\mu D_\nu \in \{0, 1\}$ for $\mu \neq \nu$ and

$$Y_1 D_\nu = \begin{cases} 1 & \nu \in \{1, t+1\} \\ 0 & \text{else} \end{cases}$$

yield a system of equations for the coefficients $c_2, \ldots, c_{2t}$. This system has a unique solution modulo $(Q, -Q)$ such that $c_1 = 0$, $c_2 = 1$, which is

$$(c_2, \ldots, c_{2t}) = (y_2, \ldots, y_{t-1}, 1, 0, \ldots, 0) \bmod (Q, -Q).$$

For that one has to use the relations between the weights $a_1, \ldots, a_t$, see Section 6.5. The proof for $Y_1^2 = 0$ follows from the second equivalence of Equation 16.

Remark 6.6.14 The relations between the weights $a_1, \ldots, a_{2t}$ naturally lead to the Grassmanian and to cluster algebras of type A, see [**CH11**] for more details.

6.7. Further remarks

6.7.1. Reducibility. An arrangement $(\mathcal{A}, V)$ is called *reducible* if there exist arrangements $(\mathcal{A}_1, V_1)$ and $(\mathcal{A}_2, V_2)$ such that $V = V_1 \oplus V_2$ and

$$\mathcal{A} = \mathcal{A}_1 \times \mathcal{A}_2 := \{H \oplus V_2 \mid H \in \mathcal{A}_1\} \cup \{V_1 \oplus H \mid H \in \mathcal{A}_2\},$$

compare [**OT92**, Def. 2.15]. It is easy to see that a crystallographic arrangement $(\mathcal{A}, V)$ is reducible if and only if the corresponding Cartan scheme is reducible in the sense of [**CH09a**, Def. 4.3], i.e. the generalized Cartan matrices are decomposable. For the fan Σ corresponding to $\mathcal{A}$, reducibility translates to the fact that there are fans Σ_1 and Σ_2 such that

$$\Sigma = \Sigma_1 \times \Sigma_2 = \{\sigma \times \tau \mid \sigma \in \Sigma_1, \tau \in \Sigma_2\}.$$

Notice that by Lemma 6.2.10 the fans Σ_1 and Σ_2 are strongly symmetric and smooth as well.

6.7.2. Inserting one hyperplane and blowups. In higher dimension, the situation is much more complicated. There are only finitely many crystallographic arrangements for each rank $r > 2$. Whether the insertion of new hyperplanes corresponds to a series of blowing-ups is unclear. The case of a single new hyperplane may be explained in the following way:

Proposition 6.7.1 *Let $(\mathcal{A}, R)$ and $(\mathcal{A}', R')$ be crystallographic arrangements of rank r with $\mathcal{A}' = \mathcal{A} \dot\cup \{H\}$. Then the toric morphism $X_{\Sigma'} \to X_\Sigma$ induced by the subdivision is a blowup along two-dimensional torus invariant subvarieties of X_Σ.*

Proof. Let $\sigma \in \Sigma := \Sigma_R$ be a maximal cone with $H \cap \overset{\circ}{\sigma} \neq \emptyset$. We prove that H star subdivides σ. The hyperplane H divides σ into two parts σ_1' and σ_2' which intersect in a codimension one cone τ'. Note that $|\sigma(1)| = r$, $|\sigma_1'(1) \cup \sigma_2'(1)| = r + 1$, thus there is exactly one ray ρ' involved which is not in Σ. Let $\rho_1 \subseteq \sigma_1', \rho_2 \subseteq \sigma_2'$ be the rays which are not subsets of τ', and $\tau \subseteq \sigma$ be the cone generated by ρ_1, ρ_2. Then $H \cap \tau = \rho'$. But by Cor. 6.3.9, $\mathcal{A}'^{\langle\tau\rangle_\mathbb{R}}$ is a crystallographic arrangement in which $\langle\rho_1\rangle_\mathbb{R}$, $\langle\rho'\rangle_\mathbb{R}$, $\langle\rho_2\rangle_\mathbb{R}$

are subsequent hyperplanes. By 6.5 we obtain that ρ is generated by the sum of the generators of ρ'_1, ρ'_2. $\square$

6.7.3. Automorphisms. Let Σ be a strongly symmetric smooth fan, $(\mathcal{A}, R)$ the corresponding crystallographic arrangement.

Definition 6.7.2 If $\mathcal{A}$ comes from the connected simply connected Cartan scheme $\mathcal{C} = \mathcal{C}(I, A, (\rho_i)_{i\in I}, (C^a)_{a\in A})$, and $a \in A$, then we call

$$\mathrm{Aut}(\mathcal{C}, a) := \{w \in \mathrm{Hom}(a, b) \mid b \in A,\ \ R^a = R^b\}$$

the *automorphism group of* $\mathcal{C}$ *at* a. This is a finite subgroup of $\mathrm{Aut}(\mathbb{Z}^r) \cong \mathrm{Aut}(M)$ because the number of all morphisms is finite.

Since $\mathcal{C}$ is connected, $\mathrm{Aut}(\mathcal{C}, a) \cong \mathrm{Aut}(\mathcal{C}, b)$ for all $a, b \in A$. The choice of $a \in A$ corresponds to the choice of a chamber and thus of an isomorphism $\mathbb{Z}^r \cong M$. Every element of $\mathrm{Aut}(\mathcal{C}, a)$ clearly induces a toric automorphism of Σ. The groups $\mathrm{Aut}(\mathcal{C}, a)$ have been determined in [**CH10**], see [**CH10**, Thm. 3.18] and [**CH10**, A.3]. However, sometimes there are elements of $\mathrm{Aut}(\Sigma)$ which are not induced by an element of $\mathrm{Aut}(\mathcal{C}, a)$. For example, we always have the toric automorphism

$$N \to N, \quad v \mapsto -v,$$

but there is a sporadic Cartan scheme of rank three with trivial automorphism group.

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