The Inductive Blockwise Alperin Weight Condition for the Finite Groups $\mathrm{SL}_3(q)$ $(3 \nmid (q-1))$, $G_2(q)$ and ${}^3D_4(q)$

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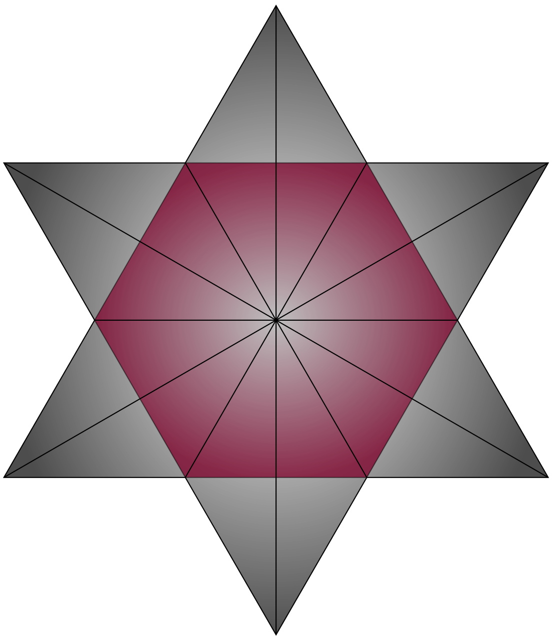

Vom Fachbereich Mathematik der Technischen Universität Kaiserslautern
zur Verleihung des akademischen Grades Doktor der Naturwissenschaften
(Doctor rerum naturalium, Dr. rer. nat.) genehmigte Dissertation

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Datum der Disputation: 16. Oktober 2015

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Preface

The field of research giving rise to the problems considered in this thesis is the *representation theory of finite groups.* As the name suggests, this theory is concerned with the examination of *representations* of finite groups, that is to say, group homomorphisms

$$\mathfrak{X}\colon G \longrightarrow \mathrm{GL}_n(K)$$

from a finite group G to a general linear group $\mathrm{GL}_n(K)$ over some field K. Their smallest constituents, the *irreducible* representations, play a special role inasmuch as they may be regarded as the basic building blocks of all representations. It is well-known that up to isomorphism for any field K and any finite group G there exist only finitely many distinct irreducible K-representations of G. Moreover, there is a one-to-one correspondence between the isomorphism classes of irreducible complex representations of G and the set of its irreducible *complex characters.* Similarly, for fields K of positive characteristic the isomorphism classes of irreducible K-representations of G are in bijection with the set of *irreducible Brauer characters* of G.

The irreducible complex characters of finite groups as well as their irreducible Brauer characters are of considerable interest since, even though information may be lost by passing from representations to characters, many properties of the group in question are encoded in the set of its irreducible characters. Moreover, knowledge of the irreducible complex and Brauer characters of a finite group may provide information about its subgroups, and vice versa, from information about irreducible characters of its subgroups it might be possible to derive information about the group itself, which leads us to the topic of this thesis.

Alperin's weight conjecture is classified as a so-called *global-local* conjecture. This name originates from the fact that for a prime ℓ the conjecture relates information about a finite group G (*global*) to properties of ℓ-local subgroups of G, that is, normalizers of ℓ-subgroups of G (*local*).

An (ℓ-)*weight* of G is defined as a pair (R, ψ), where

* R is an ℓ-subgroup of G, and
* ψ is an irreducible complex character of the normalizer $\mathrm{N}_G(R)$ containing R in its kernel (that is, $\psi(r) = \psi(1)$ for all $r \in R$) and satisfying

$$\psi(1)_\ell = \frac{|\,\mathrm{N}_G(R)\,|_\ell}{|R|}$$

(that is, the ℓ-part of the degree of ψ coincides with the ℓ-part of the order of the factor group $\mathrm{N}_G(R)/R$).

It is well-known that in this situation ψ may be considered as an irreducible character of the quotient $\mathrm{N}_G(R)/R$, so the above condition requests ψ to be of (ℓ-)*defect zero* when regarded as a character of $\mathrm{N}_G(R)/R$. The group G acts on the set of its ℓ-weights by conjugation, whence it is reasonable to speak about G-conjugacy classes of ℓ-weights of G.

Now the conjecture proposed by J. L. Alperin in 1986 on the Arcata Conference on Representations of Finite Groups [Alp87] asserts that ...

... the number of G-conjugacy classes of ℓ-weights of a finite group G equals the number of its irreducible Brauer characters defined over characteristic ℓ.

In addition to the original version of Alperin's conjecture one can refine the assertion as follows: The set consisting of both the irreducible complex characters of G and its irreducible Brauer characters with respect to the prime ℓ may be partitioned into subsets called the *ℓ-blocks* of G. Moreover, each ℓ-weight of G may be assigned to a unique ℓ-block of G, where the assignment is induced by Brauer block induction (cf. Sections 1.2 and 2.2). If an ℓ-weight (R, ψ) is associated to the ℓ-block B, then (R, ψ) is called a *B-weight* of G. As above, the group G acts by conjugation on the set of its B-weights. The *blockwise* Alperin weight conjecture then suggests that ...

... for every ℓ-block B of a finite group G the number of irreducible Brauer characters in B coincides with the number of G-conjugacy classes of B-weights of G.

Since its proposal in 1986 Alperin's weight conjecture has attracted a lot of interest among group and representation theorists, and its validity has been confirmed for a wide range of finite groups including those groups studied in this thesis. However, it has not been possible so far to find a general proof of this conjecture for arbitrary finite groups. Nevertheless, in recent years there has been considerable progress towards a solution for this question, the central focus of this development being a reduction of the original problem to a question on finite (quasi-) simple groups. The reduction theorem for the blockfree version of Alperin's conjecture was obtained by G. Navarro and P. H. Tiep in 2011 [NT11]. They give an inductive proof showing that if all finite non-abelian simple groups satisfy a certain system of conditions, the so-called *inductive Alperin weight condition*, then Alperin's weight conjecture holds for any finite group. Two years later, in 2013, B. Späth [Spä13] refined this result to achieve a reduction theorem for the blockwise version of the weight conjecture along with a corresponding system of inductive conditions, the *inductive blockwise Alperin weight condition*. As for the blockfree case, a verification of these conditions for every finite non-abelian simple group would prove the blockwise Alperin weight conjecture. Since the blockwise version of Alperin's conjecture implies the blockfree one, particular interest lies in the establishment of the inductive blockwise Alperin weight condition for every finite non-abelian simple group. For such a finite non-abelian simple group G, a prime ℓ dividing $|G|$, the universal ℓ'-covering group X of G and an ℓ-block B of X this involves

- the construction of a certain partition of the set of irreducible Brauer characters associated to B respecting the action of automorphisms of X,
- the construction of bijections between certain sets of irreducible Brauer characters in B (coming from the above partition) and sets of B-weights of X equivariant under the action of automorphisms of X,
- the verification of certain conditions regarding the extendibility of the irreducible (Brauer) characters involved in the bijections above.

The blockwise Alperin weight condition for a prime ℓ has already been proven to hold for many of the 26 sporadic groups by work of various authors. Moreover, G. Malle gave its verification in [Mal14] for the simple alternating groups as well as for the Suzuki and Ree groups of types ${}^2B_2$, ${}^2G_2$ and ${}^2F_4$. B. Späth proved it in [Spä13] for finite simple groups of Lie type defined over characteristic ℓ, and together with S. Koshitani in [KS14, KS15] for ℓ-blocks of cyclic defect. Apart from some special cases, the problem of establishing the blockwise Alperin weight condition for finite simple groups of Lie type for all primes dividing their order is still open for types other than ${}^2B_2$, ${}^2G_2$ and ${}^2F_4$.

By means of this thesis we wish to contribute to a verification of Alperin's weight conjecture by proving the inductive condition established by B. Späth concerning the blockwise version of the conjecture for three infinite series of finite simple groups of Lie type:

* the special linear groups $\mathrm{SL}_3(q)$ for prime powers $q > 2$ with $q \not\equiv 1 \bmod 3$,
* the Chevalley groups $G_2(q)$ for prime powers $q \geqslant 5$, and
* the Steinberg triality groups ${}^3D_4(q)$ for arbitrary prime powers q.

The structure of this thesis is as follows: We divided our work into four main parts and a fifth part consisting of appendices that provide information on irreducible characters and decomposition numbers of the groups $\mathrm{SL}_3(q)$, $G_2(q)$ and ${}^3D_4(q)$.

In Part I we create a foundation for the later examination of the three infinite series of finite simple groups specified above in view of the inductive blockwise Alperin weight condition. Chapter 1 begins with an introduction of the notation applied throughout this thesis and a recollection of notions related to group or character theory that play a role for our purposes. Moreover, we give an overview over important results concerning character and block theory.

A more detailed introduction to Alperin's weight conjecture is given in Chapter 2, and the above mentioned inductive condition by B. Späth for the blockwise version of the conjecture is presented in detail in Chapter 3, along with a summary of certain cases for which the inductive blockwise Alperin weight condition has already been verified.

The simple groups in the infinite series we consider here belong to the class of finite groups of Lie type. Regarding them as such allows us to benefit from the large machinery of tools and knowledge that has been established for this class of groups over time. Hence, we give an introduction to such groups in Chapter 4.

In Part II we establish the blockwise Alperin weight condition for the special linear groups $\mathrm{SL}_3(q)$ for prime powers $q > 2$ with $q \not\equiv 1 \bmod 3$. We start by giving an introduction to these groups in Chapter 5 including a description of their automorphism group and further structural properties.

Even though these are already known in principle, a need for more detailed information motivates a description of the ℓ-blocks and ℓ-decomposition numbers of the groups $\mathrm{SL}_3(q)$ for certain primes ℓ in Chapter 6. This provides us with a convenient description of the irreducible Brauer characters of $\mathrm{SL}_3(q)$ with respect to those primes.

In Chapter 7 we then examine the action of automorphisms of $\mathrm{SL}_3(q)$ on its Brauer characters. Furthermore, in this chapter we determine the ℓ-weights of $\mathrm{SL}_3(q)$ in relevant situations and study also their behaviour under the action of automorphisms of $\mathrm{SL}_3(q)$.

We recall that a good understanding of the action of automorphisms of $\mathrm{SL}_3(q)$ on its Brauer characters and weights is necessary to establish the partitions and equivariant bijections demanded by the inductive blockwise Alperin weight condition. These are constructed in Chapter 8.

Subsequently, in Chapter 9 we prove that our partitions and bijections defined for $\mathrm{SL}_3(q)$ satisfy the extendibility requirements that are part of the inductive condition.

Then finally, in Chapter 10, Theorem 10.1, we obtain a proof of the inductive blockwise Alperin weight condition for the groups $\mathrm{SL}_3(q)$, $q > 2$ with $q \not\equiv 1 \bmod 3$, for any prime dividing their order as a summary of the results of the previous chapters.

Part III is dedicated to the study of the Chevalley groups $G_2(q)$ for $q \geqslant 5$. As for the special linear groups, we begin our investigation by providing an introduction to these groups in Chapter 11. Moreover, we describe their automorphism group and certain properties inherent to the groups $G_2(q)$ being finite groups of Lie type.

Subsequently, in Chapter 12 we study the action of automorphisms of $G_2(q)$ on its irreducible Brauer characters and its weights. Both the decomposition numbers and the weights for $G_2(q)$ are already well-known and published in detail in a number of papers, which facilitates our work here.

After having examined the behaviour of Brauer characters and weights of $G_2(q)$ under automorphisms, we establish the partitions and equivariant bijections for the inductive blockwise Alperin weight condition in Chapter 13.

As it turns out, for the groups $G_2(q)$ one may omit the verification of the extendibility conditions for the characters involved in the constructed bijections. Thus, in Chapter 14 we can already summarize the results of the preceding chapters to obtain a proof of the inductive blockwise Alperin weight condition for the groups $G_2(q)$ with $q \geqslant 5$ and all primes dividing their order. This is given in Theorem 14.1.

Finally, in Part IV we examine Steinberg's triality groups ${}^3D_4(q)$ in view of the inductive blockwise Alperin weight condition. Chapter 15 begins with an introduction of the groups ${}^3D_4(q)$ along with a description of their automorphism group and properties characteristic to them as finite groups of Lie type. Moreover, we take a look at certain subgroups of ${}^3D_4(q)$ that are of importance for our work. In particular, we note that $G_2(q)$ occurs as a maximal subgroup of ${}^3D_4(q)$. Having studied the groups $G_2(q)$ already, certain situations permit us to obtain results for ${}^3D_4(q)$ by reducing to questions concerning problems in $G_2(q)$.

In Chapter 16 we examine the behaviour of the irreducible Brauer characters and weights of the groups ${}^3D_4(q)$ under the action of their automorphisms. Similarly as for the groups $G_2(q)$, the decomposition numbers and weights of ${}^3D_4(q)$ are well-known. As indicated above, a number of results concerning the action of automorphisms on weights of ${}^3D_4(q)$ are obtained by reduction to $G_2(q)$.

Knowledge of the action of automorphisms of ${}^3D_4(q)$ on Brauer characters and weights then enables us to construct the desired partitions and equivariant bijections for the inductive blockwise Alperin weight condition in Chapter 17.

As for the groups $G_2(q)$, the extendibility conditions may be omitted, such that at this point sufficient results have been obtained to prove Theorem 18.1 in Chapter 18, the inductive blockwise Alperin weight condition for Steinberg's triality groups ${}^3D_4(q)$ and all primes dividing their order.

Acknowledgement

I would like to take the opportunity to thank my supervisor Prof. Dr. Gunter Malle for giving me the chance to work on this topic of current research interest and for his support throughout my work on this thesis by giving helpful advice and answers to my many questions.

Furthermore, I am grateful to Prof. Dr. Gerhard Hiß for his willingness to serve as the second referee for this thesis.

I thank Britta Späth for reading through this thesis and for plenty of useful hints and ideas that were of great help particularly concerning the extendibility of characters in Part II of this thesis and the inductive condition in general.

Let me also thank friends and family, in particular my parents, who offered moral assistance throughout my three years of writing this thesis. Special thanks go to Simon Schmider for understanding and encouraging me and for enduring my moods and my ups and downs especially during the final stages of completing this thesis.

Last but not least, I gratefully acknowledge financial support by ERC Advanced Grant 291512 during the past three years.

Part I

Introduction and Preparation

Chapter 1

Characters and Blocks

Character and block theory of finite groups as well as finite group actions play prominent roles throughout our study of the blockwise version of Alperin's weight conjecture. For the sake of consistency and transparency we recall certain notions and introduce notation related to these topics here, which shall hence be fixed for the rest of this thesis.

Afterwards we summarize more or less well-known statements involving ordinary and Brauer characters of groups that will be of importance in the course of this thesis. Since a character's affiliation to a block of the corresponding group and questions on its extendibility to larger groups play a major role in the inductive condition for the blockwise Alperin weight conjecture (see Definition 3.2), our main focus will lie on these issues.

1.1 Notation

General Notation. Let us begin with some general notation. As is customary, for the course of this thesis we use the following symbols:

- $\mathbb{C}$: the field of complex numbers with
  - imaginary unit $\mathfrak{i} \in \mathbb{C}$ satisfying $\mathfrak{i}^2 = -1$ and $\exp(2\pi\mathfrak{i}) = 1$, and
  - $\overline{z}$ the complex conjugate of $z \in \mathbb{C}$;
- $\mathbb{R}$: the field of real numbers;
- $\mathbb{Z}$: the ring of integers;
- $\mathbb{N}$: the set of natural numbers, where we always assume that $0 \in \mathbb{N}$.

If $x, y \in \mathbb{Z} \setminus \{0\}$ are non-zero integers, then we denote by

- $\gcd(x, y)$ the (positive) greatest common divisor of x and y.

For a matrix A we write

- A^{tr} to denote the transpose of A, and
- $A = \mathrm{diag}(a_1, \ldots, a_n)$ if A is a square diagonal matrix of size $n \geqslant 1$ with diagonal entries $a_1, \ldots, a_n$.

For a positive integer $m \in \mathbb{Z}_{>0}$ (and a prime number ℓ) we denote by

- $m_\ell, m_{\ell'} \in \mathbb{Z}_{>0}$ the uniquely determined positive integers satisfying $m = m_\ell m_{\ell'}$ such that m_ℓ is a power of ℓ and $m_{\ell'}$ is prime to ℓ;

- $\Phi_m(X) \in \mathbb{Z}[X]$ the m-th cyclotomic polynomial. These polynomials will be important to us in the cases $m \in \{1, 2, 3, 6, 12\}$, where, as is certainly known, we have
  - $\Phi_1(X) = X - 1$,
  - $\Phi_2(X) = X + 1$,
  - $\Phi_3(X) = X^2 + X + 1$,
  - $\Phi_6(X) = X^2 - X + 1$,
  - $\Phi_{12}(X) = X^4 - X^2 + 1$.

Certain well-known finite groups occur frequently throughout our work, whence we shall decide on a fixed notation for these. Let $n \in \mathbb{N}_{>0}$. Then we denote by

- C_n the cyclic group of order n,
- $\mathfrak{S}_n$ the symmetric group on n letters,
- D_{2n} the dihedral group of order $2n$,
- $\mathrm{GL}_n(K)$ or $\mathrm{GL}_n(q)$ the general linear group of degree n over some field K or over a finite field consisting of q elements, respectively,
- $\mathrm{Sp}_n(q)$ the symplectic group of degree n over a finite field consisting of q elements.

Furthermore, we use the following symbols for certain products of groups:

- $G \times H$: the direct product of G and H;
- $G \rtimes H$: the semidirect product of G and H with H acting on G;
- $G \circ H$: the central product of G with H;
- $G \wr H \cong G^n \rtimes H$: the wreath product of G and $H \leqslant \mathfrak{S}_n$, $n \in \mathbb{N}_{>0}$.

Let us now turn towards notation related to more specific aspects of group and character theory. Henceforth, until the end of this section we suppose that G is a finite group.

Groups and Their Elements. For group elements $g, h \in G$ and a subgroup $H \leqslant G$ we adhere to the following notation:

- $|G|$: the cardinality of G;
- $o(g)$: the order of g;
- $[g, h] := g^{-1}h^{-1}gh$: the commutator of g and h;
- $[G, G] := \langle [x, y] \mid x, y \in G \rangle$: the commutator subgroup or derived subgroup of G;
- $\mathrm{Z}(G)$: the center of G;
- $\mathrm{C}_G(h) = \{x \in G \mid xhx^{-1} = h\}$: the centralizer of h in G;
- $\mathrm{N}_G(H) = \{x \in G \mid xHx^{-1} = H\}$: the normalizer of H in G;
- $|G \colon H| = |G|/|H|$: the index of H in G;

- $\mathrm{Aut}(G)$, $\mathrm{Inn}(G)$, $\mathrm{Out}(G) = \mathrm{Aut}(G)/\mathrm{Inn}(G)$: the groups of automorphisms, inner automorphisms and outer automorphisms of G, respectively.

Moreover, for a prime ℓ we denote by

- $g_\ell, g_{\ell'} \in G$ the uniquely determined elements satisfying $g = g_\ell g_{\ell'} = g_{\ell'} g_\ell$ such that g_ℓ is of ℓ-power order and $g_{\ell'}$ has order not divisible by ℓ;

- $\mathrm{Syl}_\ell(G)$ the set of Sylow ℓ-subgroups of G.

Apart from the cardinality, possibly the order of a group element, the index and the set of Sylow subgroups, these notations are used in the same way for infinite groups.

Complex Characters. Let us recall that a *complex* (or *ordinary*) *character* of G is a trace function afforded by a complex matrix representation of G, and that two characters of G agree if and only if the associated complex representations are similar. We denote by

- $\mathrm{Irr}(G)$ the set of all irreducible complex characters of G, that is, the set of all trace functions associated to the irreducible complex representations of G;

- $1_G \in \mathrm{Irr}(G)$ the trivial character of G;

- $\ker(\chi) = \{g \in G \mid \chi(g) = \chi(1)\}$ the kernel of a character χ of G.

For a subgroup $H \leqslant G$, a character χ of G and $\psi \in \mathrm{Irr}(H)$ we denote by

- $\mathrm{Res}^G_H(\chi)$, or simply $\chi_{|H}$, the character of H obtained by restriction of χ to H;

- $\mathrm{Irr}(G \mid \psi)$ the set of irreducible characters of G *lying over* $\psi \in \mathrm{Irr}(H)$, that is, the set of all $\chi \in \mathrm{Irr}(G)$ such that $\chi_{|H}$ has ψ as an irreducible constituent;

- $\mathrm{Ind}^G_H(\psi)$, or simply ψ^G, the induced character of G given by the formula

$$\mathrm{Ind}^G_H(\psi)(g) := \frac{1}{|H|} \sum_{x \in G} \dot{\psi}(x^{-1} g x)$$

  for $g \in G$, where $\dot{\psi}(y) := \psi(y)$ if $y \in H$ and $\dot{\psi}(y) := 0$ if $y \in G \setminus H$. (Note that, in more generality, one may define $\mathrm{Ind}^G_H(\psi)$ in the same way if ψ is not a character but only a $\mathbb{C}$-valued class function on H.)

Brauer Characters. Let us now recall the definition of a Brauer character of G. We adhere to the construction given in [Nav98]. Let ℓ be a prime and let $\mathcal{O}$ denote the ring of algebraic integers in $\mathbb{C}$. Following [Nav98, Ch. 2] we fix a maximal ideal M of $\mathcal{O}$ containing the ideal $\ell\mathcal{O}$. Then by [Nav98, Lemma 2.1] the field $\mathbb{F} := \mathcal{O}/M$ is an algebraic closure of its prime field $\mathbb{F}_\ell$ of characteristic ℓ, and restriction to the group U of ℓ'-roots of unity in $\mathbb{C}^\times$ of the natural epimorphism

$$^*\colon \mathcal{O} \longrightarrow \mathbb{F}$$

induces an isomorphism between U and $\mathbb{F}^\times$.
Henceforth, we shall denote by

- G^0 the set of all ℓ-regular elements of G, that is to say, the set of all elements of G whose order is coprime to ℓ.

We recall that the *Brauer character* φ afforded by an $\mathbb{F}$-representation

$$\mathfrak{X}\colon G \longrightarrow \mathrm{GL}_n(\mathbb{F})$$

is a complex class function on G^0 defined by

$$\varphi(g) := \xi_1 + \cdots + \xi_n$$

for $g \in G^0$, where $\xi_1, \ldots, \xi_n \in U$ are the preimages of the eigenvalues of $\mathfrak{X}(g)$ under the isomorphism ${}^*{}_{|U}\colon U \longrightarrow \mathbb{F}^\times$ (see [Nav98, p. 17]). Moreover, we recall that two irreducible $\mathbb{F}$-representations yield the same Brauer character if and only if they are similar. Note that in general this does not extend to reducible $\mathbb{F}$-representations of G if ℓ divides $|G|$ since in this case the group algebra $\mathbb{F}G$ is not semisimple.
From now on we denote by

- $\mathrm{IBr}_\ell(G)$ the set of irreducible Brauer characters of G with respect to ℓ, that is, the set of Brauer characters afforded by the irreducible $\mathbb{F}$-representations of G.

The kernel of a Brauer character φ of G afforded by an $\mathbb{F}$-representation $\mathfrak{X}$ is defined as

- $\ker(\varphi) = \{g \in G \mid \mathfrak{X}(g) = \mathfrak{X}(1)\}$.

Analogously to the case of complex characters, for a subgroup $H \leqslant G$, a Brauer character φ of G and $\vartheta \in \mathrm{IBr}_\ell(H)$ we set:

- $\mathrm{Res}^G_H(\varphi)$ or $\varphi_{|H}$: the restriction of φ to H;

- $\mathrm{IBr}_\ell(G \mid \vartheta)$: the set of $\varphi \in \mathrm{IBr}_\ell(G)$ such that $\varphi_{|H}$ has ϑ as an irreducible constituent, that is, the set of all irreducible Brauer characters of G *lying over* ϑ;

- $\mathrm{Ind}^G_H(\vartheta)$, or simply ϑ^G, the induced Brauer character of G given by the formula

  $$\mathrm{Ind}^G_H(\vartheta)(g) := \frac{1}{|H|}\sum_{x \in G} \dot{\vartheta}(x^{-1}gx)$$

  for $g \in G^0$, where $\dot{\vartheta}(y) := \vartheta(y)$ if $y \in H^0$ and $\dot{\vartheta}(y) := 0$ if $y \in G^0 \setminus H^0$. (As mentioned above for complex characters, this definition may as well be applied in the case that ϑ is a $\mathbb{C}$-valued class function on H^0.)

Decomposition Numbers and Decomposition Matrix. Let ℓ be a prime as before. If χ is a complex character of G, then we denote by

- χ^0 the restriction of χ to the ℓ-regular elements G^0,

which yields a Brauer character of G (cf. [Nav98, Cor. 2.9]). Moreover, if $\chi \in \mathrm{Irr}(G)$, then we may write

- $\chi^0 = \sum_{\varphi \in \mathrm{IBr}_\ell(G)} d_{\chi\varphi}\varphi$ for uniquely determined integers $d_{\chi\varphi} \in \mathbb{Z}_{\geqslant 0}$ (cf. [Nav98, p. 23]),

which are called (ℓ-)*decomposition numbers*. The associated matrix $(d_{\chi\varphi})_{\chi\in\mathrm{Irr}(G),\varphi\in\mathrm{IBr}_\ell(G)}$ is the (ℓ-)*decomposition matrix* of G.

Conjugacy Classes and Class Sums. Regarding the action of G on itself by conjugation we stick to the following notation:

- $x^g := g^{-1}xg$, ${}^g x := gxg^{-1}$ for $x, g \in G$;
- $[x]_G := \{g^{-1}xg \mid g \in G\}$: the G-conjugacy class of an element $x \in G$;
- $\mathrm{Cl}(G)$: the set of G-conjugacy classes of G;
- $H \sim_G K$: the subgroup $H \leqslant G$ is G-conjugate to the subgroup $K \leqslant G$;
- $\widehat{C} := \sum_{x \in C} x \in \mathbb{Z}G$: the *class sum* of a G-conjugacy class $C \in \mathrm{Cl}(G)$.

The set $\{\widehat{C} \mid C \in \mathrm{Cl}(G)\}$ forms a $\mathbb{C}$-basis of $\mathrm{Z}(\mathbb{C}G)$, the center of the group algebra $\mathbb{C}G$, and an $\mathbb{F}$-basis of $\mathrm{Z}(\mathbb{F}G)$ (see, e.g., [Isa94, Thm. 2.4 and p. 271]), and if $C_1, C_2 \in \mathrm{Cl}(G)$ and

$$\widehat{C_1}\widehat{C_2} = \sum_{C \in \mathrm{Cl}(G)} a_{C_1C_2C}\widehat{C},$$

then clearly the structure constants $a_{C_1C_2C}$ are non-negative integers.

Central Characters. Let $\chi \in \mathrm{Irr}(G)$ be afforded by a complex representation $\mathfrak{X}$. Then the *central character* associated to χ is the algebra homomorphism

- $\omega_\chi\colon \mathrm{Z}(\mathbb{C}G) \longrightarrow \mathbb{C}, \quad \widehat{C} \longmapsto \omega_\chi(\widehat{C}) := \dfrac{|C|\chi(x)}{\chi(1)}, \quad$ for $C \in \mathrm{Cl}(G)$ and $x \in C$.

Recall that the notion "central character" is derived from the fact that for $C \in \mathrm{Cl}(G)$ the class sum $\widehat{C} \in \mathrm{Z}(\mathbb{C}G)$ is represented under $\mathfrak{X}$ by the scalar matrix $\mathfrak{X}(\widehat{C}) = \omega_\chi(\widehat{C})\mathbb{1}_{\chi(1)}$, where $\mathbb{1}_{\chi(1)}$ denotes the unity matrix of size $\chi(1)$ (compare [Isa94, pp. 35/36]).

Let ℓ be a prime and $\mathbb{F}$ be defined as before. By [Isa94, Thm. 3.7] the value $\omega_\chi(\widehat{C}) \in \mathbb{C}$ is an algebraic integer for every $C \in \mathrm{Cl}(G)$, so it is reasonable to consider its image under the epimorphism ${}^*\colon \mathcal{O} \longrightarrow \mathbb{F}$ defined above. This yields an algebra homomorphism

- $\lambda_\chi\colon \mathrm{Z}(\mathbb{F}G) \longrightarrow \mathbb{F}, \quad \widehat{C} \longmapsto \lambda_\chi(\widehat{C}) := \omega_\chi(\widehat{C})^*, \quad$ for $C \in \mathrm{Cl}(G)$

(cf. [Nav98, p. 48]).

For $\varphi \in \mathrm{IBr}_\ell(G)$ we may as well define an algebra homomorphism $\lambda_\varphi\colon \mathrm{Z}(\mathbb{F}G) \longrightarrow \mathbb{F}$: If $\mathfrak{X}$ is an $\mathbb{F}$-representation yielding the Brauer character φ, then we obtain an algebra homomorphism

- $\lambda_\varphi\colon \mathrm{Z}(\mathbb{F}G) \longrightarrow \mathbb{F}$ induced by $\mathfrak{X}(\widehat{C}) = \lambda_\varphi(\widehat{C})\mathbb{1}_{\varphi(1)}$ for some scalar $\lambda_\varphi(\widehat{C}) \in \mathbb{F}$

(see [Nav98, pp. 48/49]).

Blocks and Defect Groups. Let ℓ be a prime. For the equivalence relation $\sim$ on the set $\mathrm{Irr}(G) \cup \mathrm{IBr}_\ell(G)$ defined by

$$\chi \sim \varphi \qquad \text{if and only if} \qquad \lambda_\chi = \lambda_\varphi$$

for $\chi, \varphi \in \mathrm{Irr}(G) \cup \mathrm{IBr}_\ell(G)$ we recall that the *ℓ-blocks* of G are exactly the equivalence classes of $\mathrm{Irr}(G) \cup \mathrm{IBr}_\ell(G)$ under this relation (compare [Nav98, Def. 3.1]). For the decomposition numbers introduced above it holds that $d_{\chi\varphi} \neq 0$ implies $\lambda_\chi = \lambda_\varphi$, so χ and φ belong to the same ℓ-block in this case (see [Nav98, Thm. 3.3]). We denote by

- $\mathrm{Bl}_\ell(G)$ the set of ℓ-blocks of G.

Moreover, for $B \in \mathrm{Bl}_\ell(G)$ we use the following notation:

- λ_B: the algebra homomorphism associated to B, that is, $\lambda_B = \lambda_\psi$ for any $\psi \in B$;
- $\mathrm{Irr}(B) := \mathrm{Irr}(G) \cap B$;
- $\mathrm{IBr}(B) := \mathrm{IBr}_\ell(G) \cap B$;
- $k(B) := |\,\mathrm{Irr}(B)|$;
- $l(B) := |\,\mathrm{IBr}(B)|$.

For a (Brauer) character $\chi \in \mathrm{Irr}(G) \cup \mathrm{IBr}_\ell(G)$ we denote by

- $\mathrm{bl}(\chi)$ the ℓ-block of G containing χ.

The ℓ-block $\mathrm{bl}(1_G)$ containing the trivial character of G is called the *principal ℓ-block* of G and will often be denoted by B_0.

Furthermore, we recall that for each ℓ-block B of G there is a unique G-conjugacy class of ℓ-subgroups of G associated to it, the *defect groups* of B (see, for instance, [Nav98, pp. 81/82]). If $D \leqslant G$ is an ℓ-subgroup of G, then we denote by

- $\mathrm{Bl}_\ell(G \mid D)$ the set of all ℓ-blocks of G having D as a defect group.

The (ℓ-)*defect* of an ℓ-block B of G is defined as

- $d(B) \in \mathbb{Z}_{\geqslant 0}$ satisfying $\dfrac{|G|_\ell}{\ell^{d(B)}} = \min\{\chi(1)_\ell \mid \chi \in \mathrm{Irr}(B)\}$,

and a character $\chi \in \mathrm{Irr}(B)$ with $\chi(1)_\ell = |G|_\ell/\ell^{d(B)}$ is called a *height zero* character of B. For any defect group D of B its order is given by $|D| = \ell^{d(B)}$ (see, e.g., [Nav98, Thm. 4.6]).

Group Actions. Suppose that X is a set with a finite group A acting on it. For $x \in X$, $a \in A$ and $X' \subseteq X$ we denote by

- ${}^a x = a.\,x$, $x^a = a^{-1}.\,x$ the images of a applied to x from left and right, respectively;
- $A_x := \{a \in A \mid a.x = x\}$ the stabilizer of x in A;
- ${}^a X' := \{a.x \mid x \in X'\}$ the image of X' under a;
- $X'^a := \{a^{-1}.x \mid x \in X'\}$ the image of X' under a^{-1};
- $A_{X'} := \{a \in A \mid X'^a = X'\}$ the stabilizer of X' in A.

If, moreover, Y is an $A_{X'}$-stable set, then we denote by

- $A_{X',y}$ the stabilizer of $y \in Y$ in $A_{X'}$.

If A acts on the finite group G by automorphisms, then there is an induced action of A on the set $\mathrm{Irr}(G)$ of irreducible characters of G via

- $\chi^a(g) = \chi(a.g)$ for $\chi \in \mathrm{Irr}(G)$ and $a \in A$.

As an example consider the case where G is a normal subgroup of A and A acts on G by conjugation. Then for $a \in A$ and $g \in G$ we have $a.g = aga^{-1}$, and hence $\chi^a(g) = \chi(aga^{-1})$. Analogously, one obtains an action of A on $\mathrm{IBr}_\ell(G)$ for every prime ℓ.
For $g \in G$ we denote by

- $c_g\colon G \to G$, $x \mapsto gxg^{-1}$, the automorphism of G induced by conjugation action of g,

where for $g, h \in G$ the composition of c_g and c_h is given by $c_g \circ c_h = c_{gh}$.

1.2 Blocks of Finite Groups

Let G be a finite group and suppose that ℓ is a prime. In this section we provide a brief selection of some well-known facts regarding block theory for G that will occur frequently throughout this thesis.

We first recall that for a subgroup H of G and an algebraic closure $\mathbb{F}$ of the finite field $\mathbb{F}_\ell$ consisting of ℓ elements one may extend the algebra homomorphism $\lambda_b\colon \mathrm{Z}(\mathbb{F}H) \longrightarrow \mathbb{F}$ associated to an ℓ-block $b \in \mathrm{Bl}_\ell(H)$ to a linear map

$$\lambda_b^G\colon \mathrm{Z}(\mathbb{F}G) \longrightarrow \mathbb{F}, \quad \widehat{C} \longmapsto \lambda_b\left(\sum_{x\in C\cap H} x\right) \text{ for } C \in \mathrm{Cl}(G).$$

If this map is an algebra homomorphism, then there exists a unique ℓ-block $B \in \mathrm{Bl}_\ell(G)$ such that $\lambda_b^G = \lambda_B$ (see [Nav98, p. 87]), and we say that $b^G := B$ *is defined* and call it the *induced ℓ-block*. In this case the following holds by [Nav98, Lemma 4.13]:

Lemma 1.1. *Let H be a subgroup of G and let $b \in \mathrm{Bl}_\ell(H)$ such that b^G is defined. Then any defect group of b is contained in some defect group of b^G.*

Furthermore, the following criterion is an important result concerning the question as to when induced blocks are defined (see, for instance, [Nav98, Thm. 4.14]):

Proposition 1.2. *Let H be a subgroup of G and suppose that P is an ℓ-subgroup of G such that $P\,\mathrm{C}_G(P) \subseteq H \subseteq \mathrm{N}_G(P)$. Then b^G is defined for any ℓ-block $b \in \mathrm{Bl}_\ell(H)$.*

Moreover, the following statement can be found as an exercise in [Nav98, Ex. 4.4]:

Proposition 1.3. *Let H be a subgroup of G with $b \in \mathrm{Bl}_\ell(H)$ such that b^G is defined. If $a \in \mathrm{Aut}(G)$, then $b^a = \{\chi^a \mid \chi \in b\}$, where*

$$\chi^a(x) = \chi(a.x) \quad \text{for } x \in H^a,$$

is an ℓ-block of H^a such that $(b^a)^G$ is defined and equals $(b^G)^a$.

So-called *canonical characters* occur frequently throughout the course of this thesis. They are defined as follows:

Definition 1.4 (*Canonical character*)**.** Let P be an ℓ-subgroup of G and $b \in \mathrm{Bl}_\ell(P\,\mathrm{C}_G(P))$ with defect group P. Then there exists a unique character $\theta \in \mathrm{Irr}(b)$ such that $P \subseteq \ker(\theta)$ (cf. [NT89, Lemma 5.8.12(ii)]). This character is called the *canonical character* of b.

Let us now turn to the main theorems of Brauer, which play a major role in the study of blocks of finite groups. The first of these theorems asserts a relation between certain ℓ-blocks of a finite group and those of an ℓ-local subgroup (e.g. [Nav98, Thm. 4.17]):

Theorem 1.5 (Brauer's first main theorem)**.** *Let P be an ℓ-subgroup of G. Then the map $\mathrm{Bl}_\ell(\mathrm{N}_G(P) \mid P) \longrightarrow \mathrm{Bl}_\ell(G \mid P)$, $b \longmapsto b^G$, is a bijection.*

Definition 1.6 (*Brauer correspondent*)**.** Let P be an ℓ-subgroup of G and $B \in \mathrm{Bl}_\ell(G \mid P)$. The (according to Theorem 1.5) unique ℓ-block $b \in \mathrm{Bl}_\ell(\mathrm{N}_G(P) \mid P)$ with $b^G = B$ is called the *Brauer (first main) correspondent* of B.

The first main theorem of Brauer may be extended to the following statement (see, for instance, [Nav98, Thm. 9.7]):

Theorem 1.7 (Extended first main theorem of Brauer)**.** *Let P be an ℓ-subgroup of G and $B \in \mathrm{Bl}_\ell(G \mid P)$. There exists a unique $\mathrm{N}_G(P)$-conjugacy class of ℓ-blocks $b \in \mathrm{Bl}_\ell(P\,\mathrm{C}_G(P))$ such that $b^G = B$. Moreover, all these ℓ-blocks b have defect group P, and the Brauer first main correspondent of B is given by $b^{\mathrm{N}_G(P)}$ for any such b.*

Definition 1.8 (*Root*)**.** Let P be an ℓ-subgroup of G and $B \in \mathrm{Bl}_\ell(G \mid P)$. An ℓ-block $b \in \mathrm{Bl}_\ell(P\,\mathrm{C}_G(P) \mid P)$ as in Theorem 1.7 is called a *root* of B.

We refrain from stating Brauer's second main theorem here since it does not play a role for our purposes. Instead, we continue with the third main theorem (see [Nav98, Thm. 6.7]).

Theorem 1.9 (Brauer's third main theorem)**.** *Let H be a subgroup of G and denote by b_0 and B_0 the principal ℓ-blocks of H and G, respectively. If $b \in \mathrm{Bl}_\ell(H)$ is such that b^G is defined, then $b^G = B_0$ if and only if $b = b_0$.*

For a proof of the following theorem we refer to [Nav98, Thm. 9.22]:

Theorem 1.10. *Let P be an ℓ-subgroup of G and let b be an ℓ-block of $P\,\mathrm{C}_G(P)$ with defect group P. Denote by $T(b)$ the stabilizer of b in $\mathrm{N}_G(P)$. Then the induced ℓ-block b^G has defect group P if and only the index $|T(b) \colon P\,\mathrm{C}_G(P)|$ is not divisible by ℓ.*

In certain situations we will need to decide whether two given ℓ-blocks of G agree. The following statement, which is given as an exercise in [Nav98, Ex. 4.5], turns out to be a useful criterion. We recall that the defect groups of a conjugacy class $C \in \mathrm{Cl}(G)$ are exactly the Sylow ℓ-subgroups of the centralizers $\mathrm{C}_G(x)$ for $x \in C$ (cf. [Nav98, p. 80]).

Proposition 1.11. *Let $B_1, B_2 \in \mathrm{Bl}_\ell(G \mid P)$ for some ℓ-subgroup $P \leqslant G$. Then $B_1 = B_2$ if and only if $\lambda_{B_1}(\widehat{C}) = \lambda_{B_2}(\widehat{C})$ for every $C \in \mathrm{Cl}(G)$ with defect group P such that the elements in C are of order prime to ℓ.*

Further important information is provided by Lemma 1.13 below, which can, for instance, be found as Theorem 4.8 in [Nav98], and for which we need the following definition:

Definition 1.12 (*ℓ-core*)**.** We denote by $\mathcal{O}_\ell(G)$ the largest normal ℓ-subgroup of G and call it the *ℓ-core* of G. Note that $\mathcal{O}_\ell(G)$ is uniquely determined in G (e.g. [Gor80, p. 226]).

Lemma 1.13. *Any defect group of any ℓ-block of G contains $\mathcal{O}_\ell(G)$.*

Finally, we recall the following notion concerning blocks of normal subgroups:

Definition 1.14. Let N be a normal subgroup of G and suppose that B is an ℓ-block of G and b is an ℓ-block of N. Then by [Nav98, Thm. 9.2] the following two statements are equivalent:

- If $\chi \in B$, then every irreducible constituent of $\chi_{|N}$ belongs to a G-conjugate of b.
- There exists $\chi \in B$ such that $\chi_{|N}$ has an irreducible constituent lying in b.

In this situation we say that the ℓ-block B *covers* b. (See also [Nav98, p. 193] for a different (but equivalent) definition.)

1.3 Induction and Restriction of Characters

In the following we consider questions concerning the induction and restriction of characters of finite groups. In particular, we state some important results of Clifford theory that will turn out as valuable tools for our work. Clifford theory establishes a relation between the irreducible characters of a finite group and those of a normal subgroup. The first result is the following statement on the behaviour of an irreducible character upon restriction to a normal subgroup:

Theorem 1.15 (Clifford)**.** *Let $N \trianglelefteq G$ be finite groups and ℓ be a prime.*

(i) Let $\chi \in \mathrm{Irr}(G)$ and $\vartheta \in \mathrm{Irr}(N)$ be an irreducible constituent of $\mathrm{Res}^G_N(\chi)$. Then for $t = |G\colon G_\vartheta|$ we have

$$\mathrm{Res}^G_N(\chi) = e \cdot \sum_{i=1}^{t} \vartheta_i,$$

where $e \in \mathbb{N}_{>0}$ and $\vartheta_1 = \vartheta,\ \vartheta_2, \ldots, \vartheta_t$ denote the distinct G-conjugates of ϑ.

(ii) Let $\varphi \in \mathrm{IBr}_\ell(G)$ and $\theta \in \mathrm{IBr}_\ell(N)$ be an irreducible constituent of $\mathrm{Res}^G_N(\varphi)$. Then for $t = |G\colon G_\theta|$ we have

$$\mathrm{Res}^G_N(\varphi) = e \cdot \sum_{i=1}^{t} \theta_i,$$

where $e \in \mathbb{N}_{>0}$ and $\theta_1 = \theta,\ \theta_2, \ldots, \theta_t$ denote the distinct G-conjugates of θ.

Proof. See, for example, [Isa94, Thm. 6.5] for (i) and [Nav98, Cor. 8.7] for (ii). □

A further major tool is the so-called Clifford correspondence. In general, induction of an irreducible character of a finite group to some larger group does not again yield an irreducible character. However, it turns out that in a particular situation irreducibility is preserved by induction.

Theorem 1.16 (Clifford correspondence)**.** *Let $N \trianglelefteq G$ be finite groups and ℓ be a prime.*

(i) Let $\vartheta \in \mathrm{Irr}(N)$. The induction map $\mathrm{Ind}^G_{G_\vartheta}$ induces a bijection

$$\mathrm{Irr}(G_\vartheta \mid \vartheta) \longrightarrow \mathrm{Irr}(G \mid \vartheta), \quad \psi \longmapsto \mathrm{Ind}^G_{G_\vartheta}(\psi).$$

In particular, the character $\mathrm{Ind}^G_{G_\vartheta}(\psi)$ is irreducible for every $\psi \in \mathrm{Irr}(G_\vartheta \mid \vartheta)$.

(ii) Let $\theta \in \mathrm{IBr}_\ell(N)$. The induction map $\mathrm{Ind}^G_{G_\theta}$ induces a bijection

$$\mathrm{IBr}_\ell(G_\theta \mid \theta) \longrightarrow \mathrm{IBr}_\ell(G \mid \theta), \quad \varphi \longmapsto \mathrm{Ind}^G_{G_\theta}(\varphi).$$

In particular, the character $\mathrm{Ind}^G_{G_\theta}(\varphi)$ is irreducible for every $\varphi \in \mathrm{IBr}_\ell(G_\theta \mid \theta)$.

Proof. See, for example, [Isa94, Thm. 6.11] for (i) and [Nav98, Thm. 8.9] for (ii). □

Let us now take a look at the decomposition of characters obtained by applying a composition of the induction and the restriction map. We refer to [NT89, Thm. 3.1.9] for the following statement:

Theorem 1.17 (Mackey formula). *Let H, K be subgroups of a finite group G. For an irreducible character $\theta \in \mathrm{Irr}(H)$ it holds that*

$$\mathrm{Res}_K^G \mathrm{Ind}_H^G(\theta) = \sum_{s \in S} \mathrm{Ind}_{s^{-1}Hs \cap K}^K \mathrm{Res}_{s^{-1}Hs \cap K}^{s^{-1}Hs}(\theta^s),$$

where S denotes a set of (H,K)-double coset representatives, that is, G is the disjoint union $\dot{\bigcup}_{s \in S} HsK$.

As is well-known, for a finite group G the set $\mathrm{Irr}(G)$ of irreducible characters constitutes an orthonormal basis of the $\mathbb{C}$-vector space $\mathfrak{cf}_{\mathbb{C}}(G)$ of $\mathbb{C}$-valued class functions of G with respect to the scalar product

$$\langle -, - \rangle_G \colon \mathfrak{cf}_{\mathbb{C}}(G) \times \mathfrak{cf}_{\mathbb{C}}(G) \longrightarrow \mathbb{C}, \quad (\vartheta_1, \vartheta_2) \longmapsto \frac{1}{|G|} \sum_{g \in G} \vartheta_1(g) \overline{\vartheta_2(g)}$$

(see, for instance, [Isa94, p. 21]). In particular, we may write a complex character ψ of G as

$$\psi = \sum_{\chi \in \mathrm{Irr}(G)} \langle \chi, \psi \rangle_G \chi,$$

so the multiplicity of $\chi \in \mathrm{Irr}(G)$ as an irreducible summand of ψ may be determined by means of the scalar product $\langle -, - \rangle_G$. An important observation is the following statement, which expresses the scalar product in G involving the induction of a class function of a subgroup $H \leqslant G$ as a scalar product in H (see, e.g., [Isa94, Lemma 5.2]):

Proposition 1.18 (Frobenius reciprocity). *Let $H \leqslant G$ be finite groups and suppose that χ and ϑ are $\mathbb{C}$-valued class functions on G and H, respectively. Then*

$$\langle \mathrm{Ind}_H^G(\vartheta), \chi \rangle_G = \langle \vartheta, \mathrm{Res}_H^G(\chi) \rangle_H.$$

In particular, χ is an irreducible constituent of $\mathrm{Ind}_H^G(\vartheta)$ if and only if χ lies over ϑ.

1.4 Extendibility of Characters

Throughout this section we let G denote a finite group and suppose that ℓ is a prime. Let us first summarize a few results concerning the question on when a (Brauer) character of a normal subgroup of G is extendible.

Proposition 1.19. *Let N be a normal subgroup of G.*

(i) Let $\vartheta \in \mathrm{Irr}(N)$ be G-invariant. If for every prime r dividing the index $|G \colon N|$ and every $N \leqslant P \leqslant G$ such that P/N is a Sylow r-subgroup of G/N there exists an extension of ϑ to P, then ϑ extends to G.

(ii) Let $\varphi \in \mathrm{IBr}_\ell(N)$ be G-invariant. If for every prime r dividing the index $|G \colon N|$ and every $N \leqslant P \leqslant G$ such that P/N is a Sylow r-subgroup of G/N there exists an extension of φ to P, then φ extends to G.

Proof. Part (i) is Corollary 11.31 of [Isa94], part (ii) is [Nav98, Thm. 8.29]. □

Proposition 1.20. *Let N be a normal subgroup of G.*

(i) Let $\vartheta \in \mathrm{Irr}(N)$ be G-invariant. If G/N is cyclic, then ϑ extends to G.

(ii) Let $\varphi \in \mathrm{IBr}_\ell(N)$ be G-invariant. If G/N is cyclic, then φ extends to G.

Proof. This is [Isa94, Problem 6.17] for (i). Part (ii) is [Nav98, Thm. 8.12]. □

If an extension of an irreducible (Brauer) character of a normal subgroup N of G is known, then all extensions may be obtained by application of the following theorem (where we refer to pp. 17/18 for an interpretation of the irreducible (Brauer) characters of the quotient G/N as irreducible (Brauer) characters of G):

Theorem 1.21 (Gallagher). *Let N be a normal subgroup of G.*

(i) If $\chi \in \mathrm{Irr}(G)$ satisfies $\vartheta := \chi_{|N} \in \mathrm{Irr}(N)$, then the characters $\eta \cdot \chi$ for $\eta \in \mathrm{Irr}(G/N)$ are irreducible, distinct for distinct characters η, and are exactly the irreducible constituents of ϑ^G.

(ii) If $\psi \in \mathrm{IBr}_\ell(G)$ satisfies $\varphi := \psi_{|N} \in \mathrm{IBr}_\ell(N)$, then the Brauer characters $\beta \cdot \psi$ for $\beta \in \mathrm{IBr}_\ell(G/N)$ are irreducible, distinct for distinct β, and are exactly the irreducible constituents of φ^G.

Proof. Part (i) is [Isa94, Cor. 6.17], part (ii) can be found in [Nav98, Cor. 8.20]. □

As a corollary of Gallagher's theorem, B. Späth states the following result in [Spä09, Rmk. 9.3(i)], which gives a criterion for when two irreducible characters of G extending the same character of a normal subgroups coincide:

Corollary 1.22. *Let N be a normal subgroup of G and $\vartheta \in \mathrm{Irr}(N)$, $\chi_1, \chi_2 \in \mathrm{Irr}(G)$ such that $\chi_{1|N} = \chi_{2|N} = \vartheta$. If $T \subseteq G$ is a subset of G satisfying $\langle N, T\rangle = G$ and $\chi_1(u) = \chi_2(u) \neq 0$ for every $u \in T$, then $\chi_1 = \chi_2$.*

Further extendibility results in the case that G is the semidirect product of two subgroups are given below. The first statement may also be found as Exercise 6.18 in [Isa94].

Lemma 1.23. *Let $G = N \rtimes S$ be a semidirect product of two finite groups N and S, and let $\lambda \in \mathrm{Irr}(N)$ be a G-invariant linear character of N. Then λ extends to G.*

Proof. For $n \in N$ and $s \in S$ we set $\widehat{\lambda}(ns) := \lambda(n)$. We prove that $\widehat{\lambda}$ is a group homomorphism, in which case it constitutes a linear character of G extending λ. Let hence $m, n \in N$ and $s, t \in S$. We have

$$\widehat{\lambda}(msnt) = \widehat{\lambda}(m(sns^{-1})st) = \lambda(m(sns^{-1})) = \lambda(m)\lambda^s(n) = \lambda(m)\lambda(n) = \widehat{\lambda}(ms)\widehat{\lambda}(nt),$$

so $\widehat{\lambda}$ is a group homomorphism, and hence an extension of λ to G as claimed. □

Lemma 1.24. *Suppose that $G = N \rtimes S$ is a semidirect product of two finite groups N and S. Moreover, let A be a group containing both N and G as normal subgroups. Then for $\theta \in \mathrm{Irr}(N)$ the following statements hold:*

(i) The stabilizer of the induced character θ^G in A is given by $A_{\theta^G} = A_\theta S$.

(ii) If $G_\theta = N$ and there exists an extension $\widetilde{\theta}$ of θ to A_θ, then $\widetilde{\theta}^{A_{\theta^G}}$ is an extension of θ^G to A_{θ^G}.

Proof. For part (i) we first observe that, since $N \trianglelefteq G$, the induced character θ^G satisfies

$$\theta^G{}_{|N} = \sum_{s \in S} \theta^s.$$

Now let $x \in A_{\theta^G}$. Then x acts on N, so $\theta^x \in \mathrm{Irr}(N)$, and we have $(\theta^x)^G = (\theta^G)^x = \theta^G$, so by the previous observation there exists $s \in S$ such that $\theta^x = \theta^s$. Hence, $\theta^{xs^{-1}} = \theta$ and $x \in A_\theta S$. Suppose now, conversely, that $x = ys$ with $y \in A_\theta$ and $s \in S$. Then

$$(\theta^G)^x = (\theta^G)^{ys} = (\theta^{ys})^G = (\theta^s)^G = \theta^G,$$

and we conclude that $x \in A_{\theta^G}$. Since for $y' \in A_\theta$, $s' \in S$ we have $s'y' = y'y'^{-1}s'y'$ with $y'^{-1}s'y' \in G = N \rtimes S$ and $N \leqslant A_\theta$, it follows that any element of $A_\theta S$ may be written in the form ys with $y \in A_\theta$ and $s \in S$, so the claim follows.

(ii) According to Clifford correspondence, Theorem 1.16, the character θ^G is irreducible in the present situation. Following part (i) we have $A_{\theta^G} = A_\theta S$ and by assumption $G_\theta = N$, so $A_\theta \cap S = G_\theta \cap S = N \cap S = \{1\}$, and hence $|A_{\theta^G} : A_\theta| = |S|$. Therefore, if we induce $\widetilde{\theta}$ to A_{θ^G}, then we obtain a character of degree

$$\widetilde{\theta}^{\,A_{\theta^G}}(1) = |A_{\theta^G} : A_\theta|\,\theta(1) = |S|\,\theta(1) = |G\colon N|\,\theta(1) = \theta^G(1)$$

with

$$\langle (\widetilde{\theta}^{\,A_{\theta^G}})_{|G}, \theta^G \rangle_G = \langle (\widetilde{\theta}^{\,A_{\theta^G}})_{|N}, \theta \rangle_N \neq 0$$

by Frobenius reciprocity, Proposition 1.18, and the fact that $\widetilde{\theta}$ is an irreducible constituent of $(\widetilde{\theta}^{\,A_{\theta^G}})_{|A_\theta}$, again by Frobenius reciprocity. Due to the equality of the degrees of $\widetilde{\theta}^{\,A_{\theta^G}}$ and θ^G and the irreducibility of θ^G, the character $\widetilde{\theta}^{\,A_{\theta^G}}$ must be an extension of θ^G to A_{θ^G}. □

Lemma 1.25. *Let N be a normal subgroup of G and let $\psi \in \mathrm{Irr}(N)$. If a is an automorphism of G stabilizing N such that $\psi^a = \psi$, then the stabilizer G_ψ of ψ in G is left invariant by a.*

Proof. Let $h \in G_\psi$ and $n \in N$. Then $a(h) \in G$ and

$$\begin{aligned}
\psi^{a(h)}(n) &= \psi(a(h)na(h)^{-1}) \\
&= \psi(a(ha^{-1}(n)h^{-1})) \\
&= \psi^a(ha^{-1}(n)h^{-1}) \\
&= \psi(ha^{-1}(n)h^{-1}) \\
&= \psi^h(a^{-1}(n)) \\
&= \psi(a^{-1}(n)) \\
&= \psi^{a^{-1}}(n) \\
&= \psi(n)
\end{aligned}$$

since for $\psi^a = \psi$ also $\psi^{a^{-1}} = \psi$. Thus, $a(h) \in G_\psi$ as claimed. □

Later on we will be interested in the action of $\mathrm{Aut}(G)_B$ on the set $\mathrm{IBr}(B)$ of irreducible Brauer characters of G belonging to an ℓ-block B of G, and moreover, in some cases the question of extendibility of a Brauer character in B to its stabilizer in $\mathrm{Aut}(G)_B$ arises. In certain situations a particular shape of the ℓ-decomposition matrix may be of assistance in answering this question as the following two results show, for which we need to introduce the notion of a *basic set* given below:

Definition 1.26 (*Basic set*)**.** Let B be an ℓ-block of the finite group G. We call a subset $\mathcal{B} \subseteq \mathrm{Irr}(B)$ a *basic set* for B if $\{\chi^0 \mid \chi \in \mathcal{B}\}$ forms a basis of the abelian group $\sum_{\chi \in \mathrm{Irr}(B)} \mathbb{Z}\chi^0$.

Lemma 1.27. *Let B be an ℓ-block of G and suppose that $\mathcal{B} \subseteq \mathrm{Irr}(B)$ is a basic set for B. Assume further that the ℓ-decomposition matrix for $\mathcal{B}$ is lower unitriangular with respect to a suitable ordering of the characters in $\mathcal{B}$ and $\mathrm{IBr}(B)$, and that every character in $\mathcal{B}$ is left invariant by $\mathrm{Aut}(G)_B$. Then every element in $\mathrm{IBr}(B)$ is left invariant by $\mathrm{Aut}(G)_B$.*

Proof. Let $l := l(B)$ denote the number of irreducible Brauer characters contained in B. Then $|\mathcal{B}| = l$ and we may fix orderings $\mathcal{B} = \{\chi_1, \ldots, \chi_l\}$ and $\mathrm{IBr}(B) = \{\phi_1, \ldots, \phi_l\}$ of $\mathcal{B}$ and $\mathrm{IBr}(B)$, respectively, with respect to which the decomposition matrix D for $\mathcal{B}$ is lower unitriangular, so $d_{\chi_i\phi_i} = 1$ for $1 \leqslant i \leqslant l$ and $d_{\chi_i\phi_j} = 0$ for $1 \leqslant i < j \leqslant l$ if $D = (d_{\chi_i\phi_j})_{1 \leqslant i,j \leqslant l}$. We prove by induction on k that every Brauer character ϕ_k is left invariant by $\mathrm{Aut}(G)_B$.

Let $k = 1$. Then $\phi_1 = \chi_1^0$, and since by assumption χ_1 is invariant under $\mathrm{Aut}(G)_B$, so is ϕ_1. Hence, let $k > 1$ and assume that ϕ_i is invariant under $\mathrm{Aut}(G)_B$ for $1 \leqslant i < k$. We have

$$\phi_k = \chi_k^0 - \sum_{i=1}^{k-1} d_{\chi_k\phi_i}\phi_i,$$

so for $a \in \mathrm{Aut}(G)_B$ we obtain

$$\phi_k^a = (\chi_k^0)^a - \sum_{i=1}^{k-1} d_{\chi_k\phi_i}\phi_i^a = (\chi_k^0) - \sum_{i=1}^{k-1} d_{\chi_k\phi_i}\phi_i = \phi_k$$

as claimed, since $(\chi_k^0)^a = (\chi_k^a)^0 = \chi_k^0$ by assumption and $\phi_i^a = \phi_i$ for $1 \leqslant i < k$ by the induction hypothesis. □

Remark 1.28. From the proof above it becomes clear that the statement of Lemma 1.27 remains valid if one does not assume that every character in $\mathcal{B}$ is left invariant by $\mathrm{Aut}(G)_B$ but only that every χ^0, $\chi \in \mathcal{B}$, is left invariant by $\mathrm{Aut}(G)_B$.

Proposition 1.29. *Suppose the assumptions of Lemma 1.27 and assume, moreover, that G is simple. If every character in $\mathcal{B}$ extends to $\mathrm{Aut}(G)_B$, then every Brauer character in $\mathrm{IBr}(B)$ extends to $\mathrm{Aut}(G)_B$.*

Proof. Let the notation be as in the proof of Lemma 1.27 such that for $1 \leqslant k \leqslant l$ we have

$$\chi_k^0 = \phi_k + \sum_{i=1}^{k-1} d_{\chi_k\phi_i}\phi_i.$$

Denote by $\widetilde{\chi}_k$ an extension of χ_k to $\mathrm{Aut}(G)_B$, which exists by assumption, and let ψ be an irreducible ℓ-modular constituent of $\widetilde{\chi}_k^0$. Since G is assumed to be simple, we may embed it as a normal subgroup of $\mathrm{Aut}(G)$ such that in particular $G \trianglelefteq \mathrm{Aut}(G)_B$. Hence, we may use Clifford theory, Theorem 1.15, to write

$$\psi_{|G} = e\sum_{j=1}^{t}\theta_j$$

for some irreducible constituent $\theta = \theta_1$ of $\psi_{|G}$, where $e \in \mathbb{N}_{>0}$, $t = |\,\mathrm{Aut}(G)_B \colon \mathrm{Aut}(G)_{B,\theta}|$ and $\theta_1, \ldots, \theta_t$ are exactly the distinct $\mathrm{Aut}(G)_B$-conjugates of θ. But since $(\widetilde{\chi}_k^0)_{|G} = \chi_k^0$, we conclude that θ is an irreducible constituent of χ_k^0, so $\theta = \phi_i$ for some $1 \leqslant i \leqslant k$. In particular, θ is invariant under $\mathrm{Aut}(G)_B$ by Lemma 1.27, so $t = 1$ and $\psi_{|G} = e\theta$. Since $(\widetilde{\chi}_k^0)_{|G} = \chi_k^0$, we may choose the irreducible constituent ψ of $\widetilde{\chi}_k^0$ such that ϕ_k is a constituent of $\psi_{|G}$. But then by the considerations above we have $\psi_{|G} = e\phi_k$ for some $e \in \mathbb{N}_{>0}$. Now $d_{\chi_k \phi_k} = 1$, so we must have $e = 1$, whence ψ is an extension of ϕ_k to $\mathrm{Aut}(G)_B$. $\square$

Chapter 2

The Conjecture

We are now ready to turn towards the central topic of this thesis. This chapter is aimed at providing an introduction to Alperin's weight conjecture. We give a definition of the objects and notions involved and present Alperin's hypothesis. Afterwards, we shall give particular attention to the blockwise version of this conjecture, which will be the main focus of the upcoming discussion.

2.1 Alperin's Weight Conjecture

Alperin's weight conjecture asserts that the number of irreducible Brauer characters of a finite group may be determined by counting the number of conjugacy classes of so-called weights for the finite group. Before defining these let us first recall the notion of a defect zero character of a finite group:

Definition 2.1 (*Defect zero character*)**.** Let G be a finite group and ℓ be a prime. If a character $\chi \in \operatorname{Irr}(G)$ satisfies $\chi(1)_\ell = |G|_\ell$, then χ is said to be *of (ℓ-)defect zero* or an *(ℓ-)defect zero character* of G.

The set of ℓ-defect zero characters of G will be denoted by $\operatorname{dz}_\ell(G)$ in the following. One should note that this set might be empty.

Remark 2.2. Let G be a finite group and ℓ be a prime. Following the definition of the ℓ-defect of an ℓ-block of G, a character $\chi \in \operatorname{Irr}(G)$ of ℓ-defect zero must lie in an ℓ-block of G that has ℓ-defect zero. Even more, it is the unique irreducible character of its ℓ-block since for an ℓ-block B of G the following are equivalent (cf. [Nav98, Thm. 3.18]):

- B is of ℓ-defect zero,
- there exists a character $\chi \in \operatorname{Irr}(B)$ with $\chi(1)_\ell = |G|_\ell$,
- $|\operatorname{Irr}(B)| = 1$,
- $|\operatorname{Irr}(B)| = |\operatorname{IBr}(B)|$.

The same reference tells us that the restriction χ^0 of χ to the ℓ-regular elements of G is irreducible in this situation, such that we have $\operatorname{IBr}(B) = \{\chi^0\}$.

Let us now recall that for a finite group G with a normal subgroup N there exists a natural one-to-one correspondence between the set of irreducible characters of the quotient G/N and the set of those irreducible characters of the group G that are trivial on N, which is given as follows:

If $\chi \in \operatorname{Irr}(G)$ is such that $N \subseteq \ker(\chi)$, then $\widehat{\chi}$ defined by $\widehat{\chi}(gN) := \chi(g)$ for $g \in G$ is an irreducible character of G/N.

Vice versa, if $\widehat{\chi} \in \operatorname{Irr}(G/N)$, then we obtain an irreducible character of G containing N in its kernel by setting $\chi(g) := \widehat{\chi}(gN)$. This is also called the *inflation* of $\widehat{\chi}$ to G.

An analogous one-to-one correspondence also exists for the irreducible Brauer characters of G with N in their kernels and the irreducible Brauer characters of G/N.

Bearing this in mind we may now introduce the concept of weights of finite groups:

Definition 2.3 (*ℓ-weights*)**.** Let G be a finite group, ℓ a prime and R an ℓ-subgroup of G. If $\varphi \in \operatorname{Irr}(\mathrm{N}_G(R))$ with $R \subseteq \ker(\varphi)$ is of ℓ-defect zero when regarded as an irreducible character of $\mathrm{N}_G(R)/R$, then the pair (R, φ) is called an (ℓ-)*weight* of G. The character ψ is called *weight character*.

Remark 2.4. Let G be a finite group and ℓ be a prime. If (R, φ) is an ℓ-weight of G, then also

$$(R, \varphi)^a := (R^a, \varphi^a)$$

is an ℓ-weight of G for every automorphism $a \in \operatorname{Aut}(G)$. In particular, we observe that G acts on the set of its ℓ-weights by conjugation. For the G-conjugacy class of an ℓ-weight (R, φ) we write $[(R, \varphi)]_G$. Moreover, the set of G-conjugacy classes of ℓ-weights of G will in the following be denoted by $\mathcal{W}_\ell(G)$.

Alperin's weight conjecture [Alp87, p. 369] now reads as follows:

Conjecture 1 (Alperin, 1986)**.** *Let G be a finite group and ℓ be a prime. Then*

$$|\mathcal{W}_\ell(G)| = |\operatorname{IBr}_\ell(G)|.$$

Note that Definition 2.3 only requires the subgroup R of the finite group G giving rise to an ℓ-weight of G to be an ℓ-group. However, it turns out that not every ℓ-subgroup of G qualifies to constitute the first component of an ℓ-weight of G as we are about to observe in Lemma 2.7 below.

Definition 2.5 (*Radical subgroup*)**.** For a prime ℓ and a finite group G an ℓ-subgroup R of G is called a *radical ℓ-subgroup* of G if $R = \mathcal{O}_\ell(\mathrm{N}_G(R))$. We also say that R *is (ℓ-) radical in G*. The set of radical ℓ-subgroups of G will be denoted by $\operatorname{Rad}_\ell(G)$.

Example 2.6. If G is a finite group and ℓ is a prime, then examples for radical ℓ-subgroups of G are the following:

(i) $R \in \operatorname{Syl}_\ell(G)$ is trivially ℓ-radical in G;

(ii) if B is an ℓ-block of G with defect group D, then D is ℓ-radical in G (see, for instance, [Nav98, Cor. 4.18]);

(iii) if G is simple, or more generally, if $\mathcal{O}_\ell(G) = \{1\}$, then $\{1\}$ is ℓ-radical in G.

Lemma 2.7. *Let G be a finite group and ℓ be a prime. If (R, φ) is an ℓ-weight of G, then R is a radical ℓ-subgroup of G.*

Proof. Since φ is of ℓ-defect zero as an irreducible character of the quotient $\mathrm{N}_G(R)/R$, the ℓ-block of $\mathrm{N}_G(R)/R$ containing φ must have defect group $\{1\}$. But by Lemma 1.13 the ℓ-core $\mathcal{O}_\ell(\mathrm{N}_G(R)/R)$ is contained in any defect group of any block of $\mathrm{N}_G(R)/R$. Hence,

$$\mathcal{O}_\ell(\mathrm{N}_G(R)/R) = \{1\},$$

or equivalently, $R = \mathcal{O}_\ell(\mathrm{N}_G(R))$ as claimed. □

2.2 The Blockwise Version

In this section we concern ourselves with the blockwise version of Alperin's conjecture. Each ℓ-weight of a finite group G may be uniquely assigned to an ℓ-block of G as we will see below. The weight conjecture may then be refined to a question on the number of irreducible Brauer characters and ℓ-weights belonging to the ℓ-blocks of G.

Definition 2.8 (B-*weights*)**.** Let G be a finite group and ℓ be a prime. An ℓ-weight (R, φ) of G belongs to a unique ℓ-block of G in the following way: Let b be the ℓ-block of $\mathrm{N}_G(R)$ containing the weight character φ. Then (R, φ) is called a B-*weight* of G, where $B := b^G$.

Remark 2.9. Let G be a finite group, ℓ a prime and B an ℓ-block of G. In addition, we suppose that (R, φ) is a B-weight. Then for the ℓ-weight (R^a, φ^a), $a \in \mathrm{Aut}(G)_B$, we have

$$\mathrm{bl}(\varphi^a)^G = (\mathrm{bl}(\varphi)^a)^G = (\mathrm{bl}(\varphi)^G)^a = B^a = B$$

by Proposition 1.3. Thus, $\mathrm{Aut}(G)_B$ acts on the set of B-weights of G. In particular, it follows that the group G acts on the set of its B-weights by conjugation. We denote the set of G-conjugacy classes of B-weights of G by $\mathcal{W}(B)$. Note that we drop the prime ℓ in this notation since it is already inherent in the ℓ-block B.

The blockwise version of Alperin's conjecture [Alp87, p. 371] then asserts the following:

Conjecture 2 (Alperin, 1986)**.** *Let G be a finite group, ℓ a prime and B an ℓ-block of G. Then*

$$|\mathcal{W}(B)| = |\,\mathrm{IBr}(B)|.$$

Due to the fact that each ℓ-weight of a finite group G is associated to a unique ℓ-block of G, the sets $\mathcal{W}(B)$, $B \in \mathrm{Bl}_\ell(G)$, form a partition of $\mathcal{W}_\ell(G)$. Similarly, the sets $\mathrm{IBr}(B)$, $B \in \mathrm{Bl}_\ell(G)$, form a partition of $\mathrm{IBr}_\ell(G)$. This leads to the observation that the blockwise variant of Alperin's conjecture implies the blockfree version.

We are hence particularly interested in counting the number of weights belonging to a block B of a finite group G. By [AF90, p. 3] the B-weights of G may be constructed in the following manner:

Construction 2.10. Let ℓ be a prime and B be an ℓ-block of a finite group G. For a radical ℓ-subgroup R of G and an ℓ-block $b \in \mathrm{Bl}_\ell(R\,\mathrm{C}_G(R) \mid R)$ with $b^G = B$ we denote by θ the canonical character of b. Then for every $\psi \in \mathrm{Irr}(\mathrm{N}_G(R)_\theta \mid \theta)$ with

$$\frac{\psi(1)_\ell}{\theta(1)_\ell} = |\,\mathrm{N}_G(R)_\theta \colon R\,\mathrm{C}_G(R)|_\ell$$

the pair

$$(R,\, \mathrm{Ind}_{\mathrm{N}_G(R)_\theta}^{\mathrm{N}_G(R)}(\psi))$$

constitutes a B-weight of G. As a result of Clifford correspondence, Theorem 1.16, distinct characters ψ yield distinct B-weights.

Letting R run over a complete set of representatives for the G-conjugacy classes of radical ℓ-subgroups of G, and for each such R letting b run over a complete set of representatives for the $\mathrm{N}_G(R)$-conjugacy classes of ℓ-blocks $b \in \mathrm{Bl}_\ell(R\,\mathrm{C}_G(R) \mid R)$ with $b^G = B$ provides all B-weights of G.

In general there can be various types of radical ℓ-subgroups giving rise to ℓ-weights belonging to an ℓ-block B. If, however, B is an ℓ-block of abelian defect, then the situation is more restrictive. Compare [An94a, pp. 24/25] for the following:

Lemma 2.11. *Let ℓ be a prime and B be an ℓ-block of a finite group G. Suppose, moreover, that (R, φ) is a B-weight of G. If B is an ℓ-block of abelian defect, then R is a defect group of B.*

Proof. Denote by D a defect group of B. Following Construction 2.10 there exists an ℓ-block $b \in \mathrm{Bl}_\ell(R\,\mathrm{C}_G(R) \mid R)$ such that $B = b^G$, so by [NT89, Thm. 5.5.21] we may assume that

$$\mathrm{Z}(D) \leqslant \mathrm{Z}(R) \leqslant R \leqslant D.$$

Thus, if D is abelian, then $R = D$ is a defect group of B. □

Chapter 3

Reduction Theorems

Even though Alperin's (blockwise) weight conjecture has been verified for a broad range of finite groups, it has not been possible so far to prove this conjecture in general for arbitrary finite groups. As a step towards a solution of this problem, G. Navarro and P. H. Tiep [NT11] reduced the conjecture to a question on finite (quasi-) simple groups, resulting in the *inductive Alperin weight condition*. Following this lead, B. Späth [Spä13] presented a system of inductive conditions, the *inductive blockwise Alperin weight condition*, and proved that a verification of these for all non-abelian finite simple groups implies the blockwise Alperin weight conjecture.

In the following we are interested in the inductive conditions for the blockwise version of Alperin's weight conjecture established by Späth. This is explained by the fact that these conditions constitute a refinement of the conditions given by Navarro and Tiep, which is to say, their verification implies a proof of the conditions required in the blockfree reduction.

3.1 The Inductive Blockwise Alperin Weight Condition

There are several versions of the inductive blockwise Alperin weight condition (iBAW). Apart from the original version given by Späth in [Spä13, Def. 4.1] there is also a version treating only blocks with defect groups involved in certain sets of ℓ-groups [Spä13, Def. 5.17], or a version handling single blocks [KS14, Def. 3.2].

We state here the inductive condition for a single block. For a non-abelian finite simple group this condition may then be verified block by block, this way proving the original inductive condition for the whole group.

Notation 3.1. Let G be a finite group and ℓ be a prime.

(i) If Q is a radical ℓ-subgroup of G and B an ℓ-block of G, then we define the set

$$\mathrm{dz}(\mathrm{N}_G(Q)/Q,\, B) := \{\chi \in \mathrm{dz}_\ell(\mathrm{N}_G(Q)/Q) \mid \mathrm{bl}(\chi)^G = B\},$$

where we regard χ as an irreducible character of $\mathrm{N}_G(Q)$ containing Q in its kernel when considering the induced ℓ-block $\mathrm{bl}(\chi)^G$. One should observe that this set is in one-to-one-correspondence with the set of all B-weights of G having Q as their first component.

(ii) By $\mathrm{Rad}_\ell(G)/\!\sim_G$ we denote a complete system of representatives for the G-conjugacy classes of radical ℓ-subgroups of G.

Definition 3.2 (*(iBAW) condition for an ℓ-block* [KS14, Def. 3.2])**.** Let ℓ be a prime, S a finite non-abelian simple group and X the universal ℓ'-covering group of S. Let B be an ℓ-block of X. We say that the *inductive blockwise Alperin weight (iBAW) condition holds for B* if the following statements are satisfied:

(i) (**Partitions**) There exist subsets $\mathrm{IBr}(B \mid Q) \subseteq \mathrm{IBr}(B)$ for $Q \in \mathrm{Rad}_\ell(X)$ with the following properties:

(1) $\mathrm{IBr}(B \mid Q)^a = \mathrm{IBr}(B \mid Q^a)$ for every $Q \in \mathrm{Rad}_\ell(X)$, $a \in \mathrm{Aut}(X)_B$,

(2) $\mathrm{IBr}(B) = \dot{\bigcup}_{Q \in \mathrm{Rad}_\ell(X)/\sim_X} \mathrm{IBr}(B \mid Q)$.

(ii) (**Bijections**) For every $Q \in \mathrm{Rad}_\ell(X)$ there exists a bijection

$$\Omega_Q^X \colon\ \mathrm{IBr}(B \mid Q) \longrightarrow \mathrm{dz}(\mathrm{N}_X(Q)/Q,\, B)$$

such that $\Omega_Q^X(\phi)^a = \Omega_{Q^a}^X(\phi^a)$ for every $\phi \in \mathrm{IBr}(B \mid Q)$ and $a \in \mathrm{Aut}(X)_B$.

(iii) (**Normally Embedded Conditions**) For every $Q \in \mathrm{Rad}_\ell(X)$ and every Brauer character $\phi \in \mathrm{IBr}(B \mid Q)$ there exists a finite group $A(\phi, Q)$ and Brauer characters $\widetilde{\phi} \in \mathrm{IBr}_\ell(A(\phi, Q))$ and $\widetilde{\phi}' \in \mathrm{IBr}_\ell(N_{A(\phi,Q)}(\overline{Q}))$, where we use the notation $\overline{Q} := QZ/Z$ and $Z := \mathrm{Z}(X) \cap \ker \phi$, with the following properties:

(1) for $\overline{X} := X/Z$ the group $A := A(\phi, Q)$ satisfies $\overline{X} \trianglelefteq A$, $A/\,\mathrm{C}_A(\overline{X}) \cong \mathrm{Aut}(X)_\phi$, $\mathrm{C}_A(\overline{X}) = \mathrm{Z}(A)$ and $\ell \nmid |\mathrm{Z}(A)|$,

(2) $\widetilde{\phi} \in \mathrm{IBr}_\ell(A)$ is an extension of the Brauer character of $\overline{X}$ associated with ϕ,

(3) $\widetilde{\phi}' \in \mathrm{IBr}_\ell(\mathrm{N}_A(\overline{Q}))$ is an extension of the Brauer character of $\mathrm{N}_{\overline{X}}(\overline{Q})$ associated with the inflation of $\Omega_Q^X(\phi)^0 \in \mathrm{IBr}_\ell(\mathrm{N}_X(Q)/Q)$ to $\mathrm{N}_X(Q)$,

(4) $\mathrm{bl}(\widetilde{\phi}_{|J}) = \mathrm{bl}((\widetilde{\phi}')_{|\mathrm{N}_J(\overline{Q})})^J$ for every subgroup J satisfying $\overline{X} \leqslant J \leqslant A$.

(iv) If B is of ℓ-defect zero, then $\Omega_{\{1\}}^X(\psi^0) = \psi$ for every $\psi \in \mathrm{Irr}(B)$, and $\widetilde{\phi} = \widetilde{\phi}'$ for every $\phi \in \mathrm{IBr}(B \mid \{1\})$.

Definition 3.3 (*(iBAW) condition for S and ℓ*)**.** Let ℓ be a prime, S a finite non-abelian simple group and X the universal ℓ'-covering group of S. We say that the *inductive blockwise Alperin weight (iBAW) condition holds for S and ℓ* if the (iBAW) condition holds for every ℓ-block of X.

Remark 3.4. For S, ℓ and X as above note that Späth gives a slightly different definition of the (iBAW) condition for S and ℓ in [Spä13, Def. 4.1]. However, the conditions demanded there immediately imply those stated in Definition 3.3, and vice versa, by [KS14, Lemma 3.3] the (iBAW) condition holds for S and ℓ in the sense of Späth if it holds for some $\mathrm{Aut}(X)$-transversal in $\mathrm{Bl}_\ell(X)$ in the sense of Definition 3.2. Thus, as a matter of fact both definitions coincide.

Definition 3.5. We call a group K *involved* in a finite group H if there exist subgroups $H_1 \trianglelefteq H_2 \leq H$ such that $K \cong H_2/H_1$.

Due to the following theorem by Späth [Spä13, Th. A] there is considerable interest in verifying the (iBAW) condition for all finite non-abelian simple groups:

Theorem 3.6 (Späth)**.** *Let G be a finite group and ℓ be a prime. Assume that every non-abelian simple group S involved in G satisfies the (iBAW) condition for ℓ. Then Conjecture 2 holds for every ℓ-block of G.*

Let us now state some observations which will later on allow us to construct partitions and bijections as in parts (i) and (ii) of Definition 3.2 from an $\mathrm{Aut}(X)_B$-*equivariant* bijection between $\mathrm{IBr}(B)$ and $\mathcal{W}(B)$, that is, from a bijection

$$\Omega_B\colon \mathrm{IBr}(B) \longrightarrow \mathcal{W}(B)$$

with $\Omega_B(\phi)^a = \Omega_B(\phi^a)$ for every $a \in \mathrm{Aut}(X)_B$ and $\phi \in \mathrm{IBr}(B)$.

Lemma 3.7. *Let X be a finite group and ℓ be a prime. Moreover, let $K \in \mathcal{W}(B)$ be an X-conjugacy class of B-weights of X for some ℓ-block B of X. Then for each $Q \in \mathrm{Rad}_\ell(X)$ there exists at most one $\overline{\psi} \in \mathrm{dz}(\mathrm{N}_X(Q)/Q,\, B)$ such that for its inflation ψ to $\mathrm{N}_X(Q)$ we have $K = [(Q,\psi)]_X$.*

In particular, if K has a representative with first component given by a fixed radical ℓ-subgroup $Q \in \mathrm{Rad}_\ell(X)$, then this representative is uniquely determined by Q.

Proof. This is clear since for B-weights (Q,ψ_1), (Q,ψ_2) of X with $(Q,\psi_1) = (Q,\psi_2)^g$ for some $g \in X$, we have $g \in \mathrm{N}_X(Q)$ and hence $\psi_2 = \psi_2^g = \psi_1$. $\square$

For an ℓ-block B of a finite group X and a radical ℓ-subgroup $Q \in \mathrm{Rad}_\ell(X)$ we use the following notation in the lemma below:

$$\mathrm{Irr}^0(\mathrm{N}_X(Q),\, B) := \{\psi \in \mathrm{Irr}^0(\mathrm{N}_X(Q)) \mid \mathrm{bl}(\psi)^X = B\},$$

where

$$\mathrm{Irr}^0(\mathrm{N}_X(Q)) := \{\psi \in \mathrm{Irr}(\mathrm{N}_X(Q)) \mid Q \subseteq \ker(\psi),\ \psi(1)_\ell = |\,\mathrm{N}_X(Q)/Q\,|_\ell\},$$

i.e., $\mathrm{Irr}^0(\mathrm{N}_X(Q),\, B)$ consists of exactly those irreducible characters of $\mathrm{N}_X(Q)$ whose kernel contains Q, which have ℓ-defect zero as characters of $\mathrm{N}_X(Q)/Q$ and which lie in ℓ-blocks of $\mathrm{N}_X(Q)$ that induce to B. These are exactly the inflations to $\mathrm{N}_X(Q)$ of the characters in $\mathrm{dz}(\mathrm{N}_X(Q)/Q,\, B)$.

Lemma 3.8. *Let X be a finite group, ℓ a prime and suppose that for an ℓ-block B of X we have a bijection*

$$\Omega_B\colon \mathrm{IBr}(B) \longrightarrow \mathcal{W}(B)$$

satisfying $\Omega_B(\chi)^a = \Omega_B(\chi^a)$ for all $\chi \in \mathrm{IBr}(B)$ and $a \in \mathrm{Aut}(X)_B$. For every $Q \in \mathrm{Rad}_\ell(X)$ we set

$$\mathrm{IBr}(B \mid Q) := \bigcup_{\psi \in \mathrm{Irr}^0(\mathrm{N}_X(Q),\, B)} \{\Omega_B^{-1}([(Q,\psi)]_X)\}$$

and define a map

$$\begin{aligned} \Omega_Q^X\colon \mathrm{IBr}(B \mid Q) &\longrightarrow \mathrm{dz}(\mathrm{N}_X(Q)/Q,\, B), \\ \chi &\longmapsto \widetilde{\Omega_B(\chi)}, \end{aligned}$$

where $\widetilde{\Omega_B(\chi)}$ *denotes the unique (compare Lemma 3.7) element in* $\mathrm{dz}(\mathrm{N}_X(Q)/Q,\, B)$ *whose inflation* ψ *to* $\mathrm{N}_X(Q)$ *satisfies* $\Omega_B(\chi) = [(Q,\psi)]_X$*. Then for every* $Q \in \mathrm{Rad}_\ell(X)$ *and every automorphism* $a \in \mathrm{Aut}(X)_B$ *it holds that*

$$\mathrm{IBr}(B \mid Q)^a = \mathrm{IBr}(B \mid Q^a)$$

and we have a disjoint union

$$\mathrm{IBr}(B) = \bigcup_{Q \in \mathrm{Rad}_\ell(X)/\sim_X} \mathrm{IBr}(B \mid Q),$$

so part (i) of Definition 3.2 is fulfilled for the ℓ*-block* B*. Moreover, the map* Ω_Q^X *is well-defined, bijective, and satisfies part (ii) of Definition 3.2.*

Proof. Let $a \in \mathrm{Aut}(X)_B$, $Q \in \mathrm{Rad}_\ell(X)$ and $\psi \in \mathrm{Irr}^0(\mathrm{N}_X(Q),\, B)$. By assumption we have

$$\Omega_B(\Omega_B^{-1}([(Q,\psi)]_X)^a) = \Omega_B(\Omega_B^{-1}([(Q,\psi)]_X))^a = [(Q,\psi)]_X^a,$$

so

$$\Omega_B^{-1}([(Q,\psi)]_X)^a = \Omega_B^{-1}([(Q^a,\psi^a)]_X),$$

and we obtain

$$\begin{aligned}
\mathrm{IBr}(B \mid Q)^a &= \bigcup_{\psi \in \mathrm{Irr}^0(\mathrm{N}_X(Q),\, B)} \{\Omega_B^{-1}([(Q,\psi)]_X)^a\} \\
&= \bigcup_{\psi \in \mathrm{Irr}^0(\mathrm{N}_X(Q),\, B)} \{\Omega_B^{-1}([(Q^a,\psi^a)]_X)\} \\
&= \bigcup_{\psi \in \mathrm{Irr}^0(\mathrm{N}_X(Q^a),\, B)} \{\Omega_B^{-1}([(Q^a,\psi)]_X)\} \\
&= \mathrm{IBr}(B \mid Q^a).
\end{aligned}$$

Furthermore, by construction we have $\mathrm{IBr}(B \mid Q) \subseteq \mathrm{IBr}(B)$ for every $Q \in \mathrm{Rad}_\ell(X)$, and if $Q_1, Q_2 \in \mathrm{Rad}_\ell(X)$ are such that there exists $\varphi \in \mathrm{IBr}(B \mid Q_1) \cap \mathrm{IBr}(B \mid Q_2)$, then there exist $\psi_1 \in \mathrm{Irr}^0(\mathrm{N}_X(Q_1),\, B)$ and $\psi_2 \in \mathrm{Irr}^0(\mathrm{N}_X(Q_2),\, B)$ with

$$[(Q_1,\psi_1)]_X = \Omega_B(\varphi) = [(Q_2,\psi_2)]_X,$$

so Q_1 and Q_2 are X-conjugate. Hence, the union $\bigcup_{Q \in \mathrm{Rad}_\ell(X)/\sim_X} \mathrm{IBr}(B \mid Q)$ is disjoint. Since Ω_B is a bijection, for every $\varphi \in \mathrm{IBr}(B)$ there exists a B-weight (Q,ψ) such that $\Omega_B(\varphi) = [(Q,\psi)]_X$, so $\varphi \in \mathrm{IBr}(B \mid Q)$. Thus, indeed

$$\mathrm{IBr}(B) = \dot{\bigcup}_{Q \in \mathrm{Rad}_\ell(X)/\sim_X} \mathrm{IBr}(B \mid Q),$$

which completes the first part of the proof.

By construction and Lemma 3.7 it is clear that Ω_Q^X is well-defined and bijective for any $Q \in \mathrm{Rad}_\ell(X)$. Moreover, for $Q \in \mathrm{Rad}_\ell(X)$ and $\chi \in \mathrm{IBr}(B \mid Q)$ we have $\chi^a \in \mathrm{IBr}(B \mid Q^a)$, and hence

$$\Omega_Q^X(\chi)^a = \widetilde{\Omega_B(\chi)}^a = \widetilde{\Omega_B(\chi^a)} = \Omega_{Q^a}^X(\chi^a)$$

for any $a \in \mathrm{Aut}(X)_B$ as claimed. This finishes the proof. □

3.2 Special Cases

For certain cases some important results have already been established. One example is the statement below for blocks of cyclic defect, which S. Koshitani and B. Späth proved in [KS14] and [KS15]. We benefit greatly from their result since it allows us to omit blocks with cyclic defect groups in the course of our investigations.

Proposition 3.9. *Let ℓ be a prime, S a finite non-abelian simple group and X its universal ℓ'-covering group. Then the (iBAW) condition holds for every ℓ-block of X that has cyclic defect groups.*

Proof. For odd ℓ this is stated in [KS14, Th. 1.1]. For the case $\ell = 2$ the statement follows from the fact that according to [KS15, Lemma 2.3] any 2-block of X with cyclic defect groups is nilpotent, and by [KS14, Thm. 1.3] the (iBAW) condition holds for nilpotent ℓ-blocks of X. □

As a direct consequence of the above statement we obtain the following result for finite simple groups whose Sylow subgroups with respect to a given prime are cyclic:

Corollary 3.10. *Let S be a finite non-abelian simple group with cyclic Sylow ℓ-subgroups. Then the (iBAW) condition holds for S and ℓ.*

Even though the notion of a finite group of Lie type will only be introduced in more detail in the next chapter, we already give an important result for such groups at this point, which has been proven by B. Späth in [Spä13, Th. C]:

Proposition 3.11. *Suppose that S is a finite simple group of Lie type defined over a field of characteristic p. Then the (iBAW) condition holds for S and p.*

Let us now consider the case of simple groups with cyclic outer automorphism groups. As proven by B. Späth, in this situation the normally embedded conditions, that is, point (iii) in the (iBAW) condition, will automatically be satisfied for all blocks once these fulfil the remaining conditions of Definition 3.2:

Proposition 3.12. *Let ℓ be a prime and S be a non-abelian finite simple group such that the quotient $\mathrm{Aut}(S)/S$ is cyclic. Moreover, let X denote the universal ℓ'-covering group of S and suppose that parts (i) and (ii) of the (iBAW) condition hold for every ℓ-block of X. Then the (iBAW) condition holds for S and ℓ.*

Proof. From the proof of [Spä13, Lemma 6.1] it becomes apparent that in the given situation the (iBAW) condition holds for S and ℓ if the following conditions are satisfied:

(i') There exist subsets $\mathrm{IBr}_\ell(X \mid Q) \subseteq \mathrm{IBr}_\ell(X)$ for $Q \in \mathrm{Rad}_\ell(X)$ with the following properties:

(1) $\mathrm{IBr}_\ell(X \mid Q)^a = \mathrm{IBr}_\ell(X \mid Q^a)$ for every $Q \in \mathrm{Rad}_\ell(X)$, $a \in \mathrm{Aut}(X)$,

(2) $\mathrm{IBr}_\ell(X) = \dot{\bigcup}_{Q \in \mathrm{Rad}_\ell(X)/\sim_X} \mathrm{IBr}_\ell(X \mid Q)$.

(ii') For every $Q \in \mathrm{Rad}_\ell(X)$ there exists a bijection

$$\Omega_Q^X \colon \mathrm{IBr}_\ell(X \mid Q) \longrightarrow \mathrm{dz}_\ell(\mathrm{N}_X(Q)/Q)$$

such that

(1) $\Omega_Q^X(\phi)^a = \Omega_{Q^a}^X(\phi^a)$ for every $\phi \in \mathrm{IBr}(X \mid Q)$ and $a \in \mathrm{Aut}(X)$,

(2) $\mathrm{bl}(\phi) = \mathrm{bl}(\phi')^X$, where $\phi' \in \mathrm{Irr}(\mathrm{N}_X(Q))$ denotes the inflation of $\Omega_Q^X(\phi)$.

By assumption we have subsets $\mathrm{IBr}(B \mid Q) \subseteq \mathrm{IBr}(B)$ and bijections $\Omega_{B,Q}^X$ between the sets $\mathrm{IBr}(B \mid Q)$ and $\mathrm{dz}(\mathrm{N}_X(Q)/Q, B)$ for all ℓ-blocks B of X and radical ℓ-subgroups $Q \in \mathrm{Rad}_\ell(X)$ satisfying parts (i) and (ii) of Definition 3.2. From these we may easily obtain subsets $\mathrm{IBr}_\ell(X \mid Q) \subseteq \mathrm{IBr}_\ell(X)$ and bijections Ω_Q^G fulfilling conditions (i') and (ii') above by setting

$$\mathrm{IBr}_\ell(X \mid Q) := \bigcup_{B \in \mathrm{Bl}_\ell(X)} \mathrm{IBr}(B \mid Q)$$

and

$$\Omega_Q^X \colon \mathrm{IBr}_\ell(X \mid Q) \longrightarrow \mathrm{dz}_\ell(\mathrm{N}_X(Q)/Q), \quad \phi \longmapsto \Omega_{\mathrm{bl}(\phi),Q}^X(\phi),$$

for $Q \in \mathrm{Rad}_\ell(X)$. This proves the claim. $\square$

Chapter 4

Chevalley Groups and Finite Groups of Lie Type

The finite simple groups we investigate in this thesis with regard to the inductive blockwise Alperin weight condition are constructed from so-called *Chevalley groups.* This chapter is intended to give a short introduction to such groups and provide information on their properties relevant for our purposes. We mainly follow the construction given in [Car89].

4.1 Root Systems and Simple Lie Algebras over $\mathbb{C}$

In this section we provide a brief overview of certain important properties of simple finite-dimensional complex Lie algebras. From these we will later construct Lie algebras over arbitrary fields K. Certain automorphisms of the complex Lie algebra may then be transferred to the Lie algebra over K, and the corresponding Chevalley group over K will be defined as the group generated by these automorphisms.

Let us first recall the notion of a *root system.* Note that there exist several definitions of root systems. Here we work with the definition given in [Car89, Def. 2.1.1].

Definition 4.1 (*Root system, root, base, Weyl group*)**.** Let V be a Euclidean vector space over the field $\mathbb{R}$ of real numbers with scalar product $(\_,\_)\colon V \times V \longrightarrow \mathbb{R}$. For a vector $0 \neq r \in V$ we denote by ω_r the reflection along the hyperplane orthogonal to r, i.e., the linear map

$$\begin{aligned}\omega_r\colon V &\longrightarrow V,\\ v &\longmapsto \omega_r(v) := v - 2\frac{(r,v)}{(r,r)}r.\end{aligned}$$

We set $\langle r, v\rangle := 2\frac{(r,v)}{(r,r)}$ for short. Note that this defines a map $\langle \_,\_\rangle\colon V \setminus \{0\} \times V \longrightarrow \mathbb{R}$ which is linear only in the second component and for which $\langle v, v\rangle = 2$ for all $0 \neq v \in V$.

We call a subset $\Sigma \subseteq V$ a *(crystallographic) root system* in V if it fulfils the following set of conditions:

(R1) $0 \notin \Sigma$ and Σ is a finite set;

(R2) Σ spans V;

(R3) $\omega_r(s) \in \Sigma$ for all $r, s \in \Sigma$;

(R4) $\langle r, s\rangle \in \mathbb{Z}$ for all $r, s \in \Sigma$;

(R5) $r, \lambda r \in \Sigma$ for $\lambda \in \mathbb{R}$ implies $\lambda \in \{\pm 1\}$.

The elements in Σ are called *roots*.

A linearly independent subset $\Pi \subseteq \Sigma$ such that each root $r \in \Sigma$ is a linear combination of elements in Π with either all coefficients non-negative or all coefficients non-positive is called a *fundamental system* or *base* in Σ. It is a fact that such a base Π always exists in Σ (see, for instance, [Car89, Prop. 2.1.2] or [MT11, Prop. A.7]).

Let $W(\Sigma) := \langle \omega_r \mid r \in \Sigma \rangle$ be the group generated by all reflections ω_r, $r \in \Sigma$. We call $W(\Sigma)$ the *Weyl group* of Σ. In fact, $W(\Sigma) = \langle \omega_r \mid r \in \Pi \rangle$ if Π is a base in Σ (e.g. by [MT11, Prop. A.11]).

Definition 4.2. Let Σ be a root system. For a fixed base Π in Σ there exists a unique element $w_0 \in W(\Sigma)$ such that $w_0(\Pi) = -\Pi$ (see, for instance, [MT11, Cor. A.23]). This element is called the *longest element of* $W(\Sigma)$ *(with respect to* Π*)*.

Definition 4.3. Let Σ be a root system in some Euclidean vector space over $\mathbb{R}$ with scalar product $(\_,\_)$. If there exist non-empty subsets $\Sigma_1, \Sigma_2 \subset \Sigma$ such that $\Sigma = \Sigma_1 \dot{\cup} \Sigma_2$ and $(r_1, r_2) = 0$ for all $r_1 \in \Sigma_1$, $r_2 \in \Sigma_2$, then Σ is said to be *decomposable*. Otherwise we call Σ *indecomposable*.

Definition 4.4. Let Σ denote an indecomposable root system with base Π. Then (e.g. by [GLS98, Thm. 1.8.5(d)] or [MT11, Prop. B.5]) the map

$$\begin{aligned} \mathrm{ht}\colon \Sigma &\longrightarrow \mathbb{Z}, \\ \alpha = \sum_{r \in \Pi} a_r r &\longmapsto \mathrm{ht}(\alpha) := \sum_{r \in \Pi} a_r \end{aligned}$$

takes maximum and minimum values on unique roots. These roots are called *highest* and *lowest roots (with respect to* Π*)*, respectively.

As a first step in the construction of Chevalley groups we will consider certain Lie algebras. The Chevalley groups will then arise as groups of automorphisms for these Lie algebras. We start with the following well-known result (compare, e.g., [Car05, Ch. 4–6]):

Proposition 4.5 (Cartan decomposition)**.** *Let $\mathfrak{L}$ be a finite dimensional simple Lie algebra over $\mathbb{C}$. Then there is an indecomposable root system Σ associated to $\mathfrak{L}$, and $\mathfrak{L}$ may be decomposed as*

$$\mathfrak{L} = \mathfrak{H} \oplus \bigoplus_{r \in \Sigma} \mathfrak{L}_r,$$

where $\mathfrak{H}$ is a Cartan subalgebra of $\mathfrak{L}$, and $\mathfrak{L}_r$ is the 1-dimensional root space associated to $r \in \Sigma$ invariant under the Lie bracket with $\mathfrak{H}$. More precisely, the root system Σ may be regarded as a subset of the dual space $\mathfrak{H}^* = \mathrm{Hom}_{\mathbb{C}}(\mathfrak{H}, \mathbb{C})$ of $\mathfrak{H}$ such that it holds that $\mathfrak{L}_r = \{x \in \mathfrak{L} \mid [h, x] = r(h)x \text{ for all } h \in \mathfrak{H}\}$ for $r \in \Sigma$.

Remark 4.6 (Scalar product on $\mathfrak{H}^*$). The vector space $\mathfrak{H}^*$ in Proposition 4.5 is equipped with a scalar product constructed from the non-degenerate Killing form κ of the Lie algebra $\mathfrak{L}$ in the following way:

By [Car05, Lemma 4.16] there is an isomorphism ${}^*\colon \mathfrak{H} \longrightarrow \mathfrak{H}^*$ defined by

$$h^*\colon \mathfrak{H} \longrightarrow \mathbb{C}, \quad x \longmapsto \kappa(h, x).$$

Hence, for each $\alpha \in \mathfrak{H}^*$ there exists a unique element $t_\alpha \in \mathfrak{H}$ such that $\alpha = t_\alpha^*$. Then the scalar product of $\alpha, \beta \in \mathfrak{H}^*$ is given by $(\alpha, \beta) := \kappa(t_\alpha, t_\beta)$.

Let Σ_1 and Σ_2 be two root systems. Then we call Σ_1 and Σ_2 *equivalent* if there exists a bijection $\alpha\colon \Sigma_1 \longrightarrow \Sigma_2$ and some scalar $\lambda \in \mathbb{R}_{>0}$ such that

$$(\alpha(r), \alpha(s)) = \lambda(r, s)$$

for all $r, s \in \Sigma_1$. The following two important results, which can be found as Theorems 3.5.1 and 3.5.2 in [Car89], show that up to equivalence of root systems and isomorphism of Lie algebras there is a one-to-one-correspondence between indecomposable root systems and simple Lie algebras over $\mathbb{C}$:

Theorem 4.7. *Let Σ be an indecomposable root system. Then there exists a simple Lie algebra over $\mathbb{C}$ such that the root system associated to it is equivalent to Σ.*

Theorem 4.8. *Let $\mathfrak{L}_1$ and $\mathfrak{L}_2$ be two simple Lie algebras over $\mathbb{C}$ with associated root systems Σ_1 and Σ_2, respectively. If Σ_1 and Σ_2 are equivalent, then $\mathfrak{L}_1$ and $\mathfrak{L}_2$ are isomorphic.*

Definition 4.9 (*Co-roots*)**.** Let $\mathfrak{L}$ be a simple Lie algebra over $\mathbb{C}$ with Cartan subalgebra $\mathfrak{H}$ and associated root system $\Sigma \subseteq \mathfrak{H}^*$. For each $r \in \Sigma$ define the *co-root associated to* r as

$$h_r := \frac{2t_r}{\kappa(t_r, t_r)} = \frac{2t_r}{(r, r)} \in \mathfrak{H},$$

where t_r is as in Remark 4.6.

It has been shown that for a simple Lie algebra $\mathfrak{L}$ over $\mathbb{C}$ the set of co-roots associated to its root system can be extended to a basis of $\mathfrak{L}$ with particularly nice multiplication properties, the so-called *Chevalley basis*. See, for instance, [Car89, Thm. 4.2.1] for the following:

Theorem 4.10 (Chevalley basis)**.** *Let $\mathfrak{L}$ be a finite dimensional simple Lie algebra over $\mathbb{C}$ with root system Σ and Cartan decomposition*

$$\mathfrak{L} = \mathfrak{H} \oplus \bigoplus_{r \in \Sigma} \mathfrak{L}_r,$$

and let the co-roots h_r for $r \in \Sigma$ be as in Definition 4.9. Moreover, for roots $r, s \in \Sigma$ with $s \neq \pm r$ denote by ρ_{rs} the largest integer such that $s - \rho_{rs} r \in \Sigma$. Then for each root $r \in \Sigma$ there exists an element $e_r \in \mathfrak{L}_r$ such that

$$\begin{aligned} [e_r e_{-r}] &= h_r, \\ [e_r e_s] &\in \{\pm(\rho_{rs} + 1) e_{r+s}\} \text{ for } s \neq \pm r, \end{aligned}$$

and the elements $\{h_r \mid r \in \Pi\} \cup \{e_r \mid r \in \Sigma\}$, where Π denotes a base in Σ, form a basis for $\mathfrak{L}$. This basis is called a Chevalley basis for $\mathfrak{L}$. The multiplication of basis elements is as follows:

$$\begin{aligned} [h_r h_s] &= 0, \\ [h_r e_s] &= 2\frac{(r, s)}{(r, r)} e_s = \langle r, s \rangle e_s, \\ [e_r e_s] &= \begin{cases} h_r & \text{if } s = -r, \\ 0 & \text{if } s \neq -r \text{ and } r + s \notin \Sigma, \\ N_{rs} e_{r+s} & \text{if } r + s \in \Sigma, \end{cases} \end{aligned}$$

where $N_{rs} \in \{\pm(\rho_{rs} + 1)\}$.

Moreover, the multiplication constants of the Lie algebra $\mathfrak{L}$ with respect to the Chevalley basis are all integers.

Remark 4.11. It should be noted that a Chevalley basis for a simple Lie algebra is not uniquely determined. Rather, there exist many different Chevalley bases depending on the chosen Cartan decomposition and the base of the corresponding root system. Even after fixing a Cartan decomposition and a base inside the associated root system only the h_r's are uniquely determined. There are still choices to be made for the basis vectors e_r. However, once the e_r's have been chosen for the fixed base of the root system, the remaining e_r's are determined up to a sign. (See [Car89, Remark after Thm. 4.2.1].)

We are now able to define certain automorphisms of our Lie algebra that will later play a crucial role in the construction of Chevalley groups.

Definition 4.12. Recall that for a Lie algebra $\mathfrak{L}$ and any element $x \in \mathfrak{L}$ the map

$$\operatorname{ad} x \colon \mathfrak{L} \longrightarrow \mathfrak{L},\ y \longmapsto [xy],$$

is a derivation of $\mathfrak{L}$ (see, e.g., [Car89, p. 34]). Now suppose that $\mathfrak{L}$ is a simple Lie algebra over $\mathbb{C}$ with associated root system Σ and Chevalley basis $\{h_r \mid r \in \Pi\} \cup \{e_r \mid r \in \Sigma\}$. Then by [Car89, p. 61] the map $\operatorname{ad} e_r$ is nilpotent for each $r \in \Sigma$, so the map

$$\begin{aligned} \exp(\operatorname{ad} e_r) : &= \sum_{k=0}^{\infty} \frac{1}{k!} (\operatorname{ad} e_r)^k \\ &= \mathrm{id} + \operatorname{ad} e_r + \frac{1}{2!}(\operatorname{ad} e_r)^2 + \cdots + \frac{1}{(n-1)!}(\operatorname{ad} e_r)^{n-1}, \end{aligned}$$

where $n \in \mathbb{N}$ is such that $(\operatorname{ad} e_r)^n = 0$, is well-defined. By [Car89, Lemma 4.3.1] the map $\exp(\operatorname{ad} e_r)$ is an automorphism of $\mathfrak{L}$ for every $r \in \Sigma$ since $\operatorname{ad} e_r$ is a nilpotent derivation.

Now let $\zeta \in \mathbb{C}$. Then also $\operatorname{ad}(\zeta e_r) = \zeta \operatorname{ad} e_r$ is a nilpotent derivation, so $\exp(\zeta \operatorname{ad} e_r)$ is an automorphism of $\mathfrak{L}$ as well according to [Car89, Lemma 4.3.1]. We denote these automorphisms by

$$x_r(\zeta) := \exp(\zeta \operatorname{ad} e_r)$$

for all $r \in \Sigma$ and $\zeta \in \mathbb{C}$.

With respect to a Chevalley basis the automorphisms $x_r(\zeta)$ may be represented in a handsome manner (see [Car89, pp. 61–63]):

Proposition 4.13. *Let $\mathfrak{L}$ be a simple Lie algebra over $\mathbb{C}$ with associated root system Σ. For $r \in \Sigma$ and $\zeta \in \mathbb{C}$ let $x_r(\zeta)$ be defined as in Definition 4.12 for some fixed Chevalley basis $\mathfrak{B}$ of $\mathfrak{L}$. Denote by $A_r(\zeta)$ the matrix representation of $x_r(\zeta)$ with respect to $\mathfrak{B}$. Then the coefficients of $A_r(\zeta)$ are all given as the product of an integer and some non-negative integral power of ζ, i.e., for each coefficient a of $A_r(\zeta)$ there exists $i \in \mathbb{N}$ and $z \in \mathbb{Z}$ such that $a = z\zeta^i$.*

The automorphisms defined above will play a key role in the construction of Chevalley groups in the subsequent section.

4.2 Chevalley Groups

In the previous section we gathered information on simple Lie algebras over $\mathbb{C}$. Here we apply those results to define certain Lie algebras over arbitrary fields and construct the Chevalley groups from these. The whole section is a summary of [Car89, Sec. 4.4].

Let Σ be an indecomposable root system and K an arbitrary field. Our aim is to construct a group defined over K associated to Σ, the Chevalley group, which will then be denoted $\Sigma(K)$. Let $\mathfrak{L}$ be a simple Lie algebra over $\mathbb{C}$ with associated root system Σ. Such a Lie algebra exists by Theorem 4.7. Moreover, let $\mathfrak{B}$ be a Chevalley basis of $\mathfrak{L}$. By Theorem 4.10 the multiplication constants with respect to $\mathfrak{B}$ are integers, so the $\mathbb{Z}$-span $\mathfrak{L}_\mathbb{Z}$ of $\mathfrak{B}$ is a Lie algebra over $\mathbb{Z}$. In particular, both K and $\mathfrak{L}_\mathbb{Z}$ are abelian groups, so one can define

$$\mathfrak{L}_K := K \otimes \mathfrak{L}_\mathbb{Z}$$

as the tensor product of two abelian groups. Then $\mathfrak{L}_K$ is an (additive) abelian group itself, and if 1_K denotes the unit element in K, then each element $x \in \mathfrak{L}_K$ may be written as

$$x = \sum_{\mathfrak{b} \in \mathfrak{B}} \lambda_\mathfrak{b} (1_K \otimes \mathfrak{b})$$

for suitable scalars $\lambda_\mathfrak{b} \in K$ depending on x. If one sets

$$\overline{\mathfrak{b}} := 1_K \otimes \mathfrak{b}$$

for $\mathfrak{b} \in \mathfrak{B}$, then $\overline{\mathfrak{B}} := \{\overline{\mathfrak{b}} \mid \mathfrak{b} \in \mathfrak{B}\}$ is a basis for the K-vector space $\mathfrak{L}_K$.

In our next step we define a multiplication of elements in $\mathfrak{L}_K$ that induces a Lie algebra structure on $\mathfrak{L}_K$:

Proposition 4.14. *For two elements $\mathfrak{b}_1, \mathfrak{b}_2 \in \mathfrak{B}$ define the product*

$$\left[\, \overline{\mathfrak{b}_1}\ \overline{\mathfrak{b}_2} \,\right] := 1_K \otimes [\mathfrak{b}_1\, \mathfrak{b}_2].$$

By linear extension this multiplication induces a Lie algebra structure on $\mathfrak{L}_K$.

The multiplication constants of $\mathfrak{L}_K$ with respect to the basis $\overline{\mathfrak{B}}$ are given by the multiplication constants of $\mathfrak{L}$ with respect to the basis $\mathfrak{B}$ interpreted as elements of the prime subfield of K.

Proof. This is [Car89, Prop. 4.4.1] and follows from the fact that the multiplication constants of $\mathfrak{L}$ with respect to $\mathfrak{B}$ are integers (see Theorem 4.10). □

After transferring the Lie algebra structure of $\mathfrak{L}$ to $\mathfrak{L}_K$, the next step is to construct automorphisms of $\mathfrak{L}_K$ from the automorphisms $x_r(\zeta)$, $r \in \Sigma$, $\zeta \in \mathbb{C}$, defined in the previous section (cf. [Car89, p. 63]).

Definition 4.15. Let Σ, K, $\mathfrak{L}$, $\mathfrak{B}$, $\mathfrak{L}_K$, $\overline{\mathfrak{B}}$ be defined as before. For $r \in \Sigma$ and $\zeta \in \mathbb{C}$ let the automorphism $x_r(\zeta)$ be represented by the matrix $A_r(\zeta)$ with respect to $\mathfrak{B}$. By Proposition 4.13 each coefficient of $A_r(\zeta)$ is the product of an integer and some non-negative integral power of ζ.

Now let $t \in K$. For $z \in \mathbb{Z}$ we let $\overline{z}$ be the corresponding element of the prime field of K. Then we denote by $\bar{A}_r(t)$ the matrix obtained from $A_r(\zeta)$ by replacing all entries $z\zeta^i$ by the elements $\overline{z}t^i$ of K. Then $\bar{A}_r(t)$ represents an endomorphism of $\mathfrak{L}_K$ with respect to the basis $\overline{\mathfrak{B}}$. We denote this endomorphism by $x_r(t)$. Note that for $K = \mathbb{C}$ this yields exactly the endomorphism of $\mathfrak{L}$ started with, whence this choice of notation should not produce any confusion.

Proposition 4.16. *In the notation of Definition 4.15 the endomorphisms $x_r(t)$ are in fact automorphisms of $\mathfrak{L}_K$ for every $r \in \Sigma$ and $t \in K$.*

Proof. This is [Car89, Prop. 4.4.2]. □

We are now ready to define the so-called adjoint Chevalley groups:

Definition 4.17 (*Adjoint Chevalley groups*)**.** Let $\mathfrak{L}$ be a simple Lie algebra over $\mathbb{C}$ with root system Σ and let K be any field. Fix a Chevalley basis for $\mathfrak{L}$ and let $x_r(t)$, $r \in \Sigma$, $t \in K$, be the corresponding automorphisms of $\mathfrak{L}_K$ as above. Then we denote by

$$\mathfrak{L}(K)_{\mathrm{ad}} := \langle x_r(t) \mid r \in \Sigma,\, t \in K \rangle$$

the group of automorphisms of $\mathfrak{L}_K$ generated by the automorphisms $x_r(t)$, $r \in \Sigma$, $t \in K$, the *Steinberg generators* of $\mathfrak{L}(K)_{\mathrm{ad}}$. This group is called the *adjoint Chevalley group of type* $\mathfrak{L}$ *over* K.

The group $\mathfrak{L}(K)_{\mathrm{ad}}$ is well-defined and does not depend on the chosen Chevalley basis of the Lie algebra $\mathfrak{L}$ as shown in [Car89, Prop. 4.4.3]:

Proposition 4.18. *Up to isomorphism the group* $\mathfrak{L}(K)_{\mathrm{ad}}$ *is uniquely determined by the simple Lie algebra* $\mathfrak{L}$ *and the field* K*.*

Remark 4.19. Clearly, if $\mathfrak{L}_1$ and $\mathfrak{L}_2$ are two isomorphic simple Lie algebras over $\mathbb{C}$, then $\mathfrak{L}_1(K)_{\mathrm{ad}}$ and $\mathfrak{L}_2(K)_{\mathrm{ad}}$ are isomorphic for any field K. We have seen that up to equivalence of root systems and isomorphism of Lie algebras there is a one-to-one-correspondence between indecomposable root systems and simple Lie algebras over $\mathbb{C}$. Hence, we will in the following write $\Sigma(K)_{\mathrm{ad}}$ instead of $\mathfrak{L}(K)_{\mathrm{ad}}$, where Σ is the root system type corresponding to the simple Lie algebra $\mathfrak{L}$ over $\mathbb{C}$, and we will call $\Sigma(K)_{\mathrm{ad}}$ the adjoint Chevalley group *of type* Σ over K.

Theorem 4.20 (Steinberg relations for adjoint Chevalley groups)**.** *Let* $\Sigma(K)_{\mathrm{ad}}$ *be an adjoint Chevalley group over some field* K*. Define*

$$n_r(t) := x_r(t)x_{-r}(-t^{-1})x_r(t) \text{ and } h_r(t) := n_r(t)n_r(-1),$$

where $r \in \Sigma$ *and* $t \in K^\times$*. Then the following relations hold for all* $r, s \in \Sigma$ *and* $t, u \in K$ *(with* $t, u \neq 0$ *in part (iii)):*

(i) $x_r(t)x_r(u) = x_r(t+u)$*;*

(ii) if r *and* s *are linearly independent, then*

$$[x_r(t), x_s(u)] = \prod_{i,j} x_{ir+js}(c_{ijrs}(-t)^i u^j),$$

for certain scalars $c_{ijrs} \in \{\pm 1, \pm 2, \pm 3\}$ *independent of the field* K*, where the product ranges over all pairs* i*,* j *of positive integers such that* $ir + js \in \Sigma$*, and the terms occur in a fixed order independent of* t *and* u *with* $i + j$ *non-decreasing; if there are no positive integers* i*,* j *such that* $ir + js \in \Sigma$*, then* $[x_r(t), x_s(u)] = 1$*;*

(iii) $h_r(t)h_r(u) = h_r(tu)$*.*

Proof. Part (i) is given by [Car89, p. 68], (ii) is [Car89, Thm. 5.2.2], and part (iii) follows from [Car89, p. 92 and Lemma 6.4.4]. □

Proposition 4.21. *For a prime* p *denote by* $\mathbb{F}$ *an algebraic closure of the finite field* $\mathbb{F}_p$ *consisting of* p *elements. Moreover, let* $\mathbf{G} := \Sigma(\mathbb{F})_{\mathrm{ad}}$ *be an adjoint Chevalley group over* $\mathbb{F}$ *and let* $\mathbf{T} := \langle h_r(t) \mid r \in \Sigma,\, t \in \mathbb{F}^\times \rangle$*. Then*

$$\mathrm{N}_{\mathbf{G}}(\mathbf{T}) = \langle \mathbf{T},\, n_r(1) \mid r \in \Sigma \rangle,$$

and there is an isomorphism between the groups $\mathbf{W} := \mathrm{N}_{\mathbf{G}}(\mathbf{T})/\,\mathbf{T}$ *and* $W(\Sigma)$ *induced by the map* $n_r(1) \mapsto \omega_r$*.*

Proof. This follows from [Car89, Thm. 7.2.2] and [GLS98, Thm. 1.9.5(f)]. $\square$

We will now use the relations in Theorem 4.20 to define the so-called universal Chevalley groups as abstract groups generated by certain elements behaving according to relations analogous to those stated in Theorem 4.20. See [Car89, Thm. 12.1.1] for the following:

Theorem 4.22. *Let Σ be an indecomposable root system not of type A_1 and let K be any field. For every $r \in \Sigma$ and $t \in K$ let a symbol $\bar{x}_r(t)$ be given. Define $\Sigma(K)_{\text{uni}}$ to be the abstract group generated by all $\bar{x}_r(t)$, $r \in \Sigma$, $t \in K$, subject to the relations*

$$\begin{aligned}
&\bar{x}_r(t_1)\bar{x}_r(t_2) = \bar{x}_r(t_1 + t_2) \text{ for } t_1, t_2 \in K,\ r \in \Sigma,\\
&[\bar{x}_r(t), \bar{x}_s(u)] = \prod_{i,j} \bar{x}_{ir+js}(c_{ijrs}(-t)^i u^j) \text{ for linearly independent } r, s \in \Sigma,\ t, u \in K,\\
&\bar{h}_r(t_1)\bar{h}_r(t_2) = \bar{h}_r(t_1 t_2) \text{ for } t_1, t_2 \in K^\times,\ r \in \Sigma,
\end{aligned}$$

where $\bar{h}_r(t) := \bar{n}_r(t)\bar{n}_r(-1)$ for $\bar{n}_r(t) := \bar{x}_r(t)\bar{x}_{-r}(-t^{-1})\bar{x}_r(t)$, and the product runs over the same i, j as in Theorem 4.20 with the same scalars c_{ijrs}.
Then the center $\mathrm{Z}(\Sigma(K)_{\text{uni}})$ of $\Sigma(K)_{\text{uni}}$ is contained in $\overline{\mathbf{T}} := \langle \bar{h}_r(t) \mid r \in \Sigma,\ t \in K^\times \rangle$, and the quotient

$$\Sigma(K)_{\text{uni}} / \mathrm{Z}(\Sigma(K)_{\text{uni}})$$

is isomorphic to the adjoint Chevalley group $\Sigma(K)_{\text{ad}}$ such that via this isomorphism the generators $x_r(t)$ of $\Sigma(K)_{\text{ad}}$ correspond to the cosets of $\bar{x}_r(t)$ for $r \in \Sigma$, $t \in K$.

Remark 4.23. If Σ is a root system of type A_1, then one obtains a statement analogous to Theorem 4.22 by replacing the commutator relations by the relations

$$\bar{n}_r(t)\bar{x}_r(u)\bar{n}_r(t)^{-1} = \bar{x}_{-r}(-t^{-2}u)$$

for all $u \in K$, $t \in K^\times$ and $r \in \Sigma$ (cf. [Car89, p. 198]).

Definition 4.24 (*Universal Chevalley groups*)**.** The group $\Sigma(K)_{\text{uni}}$ in Theorem 4.22 is called the *universal Chevalley group of type Σ over K* or the *simply connected Chevalley group of type Σ over K*. Its generators $\bar{x}_r(t)$, $r \in \Sigma$, $t \in K$, are the *Steinberg generators* of $\mathfrak{L}(K)_{\text{uni}}$.

From the proof of Theorem 4.22 given in [Car89] one may derive some of the following relations for universal Chevalley groups:

Theorem 4.25 (Steinberg relations for universal Chevalley groups)**.** *Let p be a prime and denote by $\mathbb{F}$ an algebraic closure of the field $\mathbb{F}_p$ consisting of p elements. Moreover, let $\Sigma(\mathbb{F})_{\text{uni}}$ be a universal Chevalley group over $\mathbb{F}$ with $\Sigma \neq A_1$ and with generators $\bar{x}_r(t)$, $r \in \Sigma$, $t \in \mathbb{F}$, and $\bar{n}_r(t) = \bar{x}_r(t)\bar{x}_{-r}(-t^{-1})\bar{x}_r(t)$, $\bar{h}_r(t) = \bar{n}_r(t)\bar{n}_r(-1)$ for $r \in \Sigma$ and $t \in \mathbb{F}^\times$.*

Then the following relations hold for all $r, s \in \Sigma$ and $t, u \in \mathbb{F}$ (with $t \neq 0$ or $u \neq 0$ whenever appropriate):

(i) $\bar{x}_r(t)\bar{x}_r(u) = \bar{x}_r(t + u)$;

(ii) if r and s are linearly independent, then

$$[\bar{x}_r(t), \bar{x}_s(u)] = \prod_{i,j} \bar{x}_{ir+js}(c_{ijrs}(-t)^i u^j),$$

for c_{ijrs} as in Theorem 4.20, where the product ranges over all pairs i, j of positive integers such that $ir + js \in \Sigma$, and the terms occur in a fixed order independent of t and u with $i + j$ non-decreasing; if there are no positive integers i, j such that $ir + js \in \Sigma$, then $[\bar{x}_r(t), \bar{x}_s(u)] = 1$;

(iii) $[\bar{h}_r(t), \bar{h}_s(u)] = 1$;

(iv) $\bar{h}_r(t)\bar{h}_r(u) = \bar{h}_r(tu)$;

(v) *if* $\{r_1, \ldots, r_m\}$ *is a base in* Σ *and we set* $\check{r} := \dfrac{2r}{(r,r)}$ *for all* $r \in \Sigma$, *then*

$$\bar{h}_r(t) = \prod_{i=1}^{m} \bar{h}_{r_i}(t^{c_i}),$$

where $c_i \in \mathbb{Z}$ *are such that* $\check{r} = \sum_{i=1}^{m} c_i \check{r}_i$;

(vi) *it holds*

$$\prod_{i=1}^{m} \bar{h}_{r_i}(t_i) = 1 \quad \text{if and only if} \quad t_i = 1 \text{ for all } 1 \leqslant i \leqslant m,$$

where $\{r_1, \ldots, r_m\}$ *is a base in* Σ *as above;*

(vii) $\bar{h}_r(t)\bar{x}_s(u)\bar{h}_r(t)^{-1} = \bar{x}_s(t^{\langle r,s\rangle}u)$;

(viii) $\bar{n}_r(t)\bar{x}_s(u)\bar{n}_r(t)^{-1} = \bar{x}_{\omega_r(s)}(\eta_{r,s}t^{-\langle r,s\rangle}u)$ *for some sign* $\eta_{r,s} \in \{\pm 1\}$;

(ix) $\bar{n}_r(t)\bar{n}_s(u)\bar{n}_r(t)^{-1} = \bar{n}_{\omega_r(s)}(\eta_{r,s}t^{-\langle r,s\rangle}u)$ *with* $\eta_{r,s}$ *as in (viii);*

(x) $\bar{n}_r(t)\bar{h}_s(u)\bar{n}_r(t)^{-1} = \bar{h}_{\omega_r(s)}(u)$;

(xi) $\bar{n}_r(1)^2 = \bar{h}_r(-1)$.

Proof. Relations (i), (ii) and (iv) are the defining relations for $\Sigma(\mathbb{F})_{\mathrm{uni}}$. The statements (iii) and (vii) to (xi) can be found in the proof of [Car89, Thm. 12.1.1]. The remaining statements are part of [GLS98, Thm. 1.12.1]. □

Remark 4.26. (i) Suppose that $\Sigma(\mathbb{F})_{\mathrm{ad}}$ is an adjoint Chevalley group over the field $\mathbb{F}$ defined in Theorem 4.25. Except for part (vi) the relations in Theorem 4.25 hold true for the generators $x_r(t)$, $n_r(t)$ and $h_r(t)$ of $\Sigma(\mathbb{F})_{\mathrm{ad}}$ by [GLS98, Thm. 1.12.1] and the fact that $\Sigma(\mathbb{F})_{\mathrm{ad}}$ is the quotient of $\Sigma(\mathbb{F})_{\mathrm{uni}}$ by its center. We should note, however, that the notation in [GLS98] differs slightly from our notation and that [GLS98, Thm. 1.12.1(i)] wrongly claims that $n_r(1)^{-1}h_s(u)n_r(1) = h_{\omega_r(s)}(c_{s,r}u)$ for the sign $c_{s,r} \in \{\pm 1\}$ satisfying $n_r(1)^{-1}x_s(u)n_r(1) = x_{\omega_r(s)}(c_{s,r}u)$. Moreover, part (vi) may be replaced by the statement

(vi)' It holds

$$\prod_{i=1}^{m} h_{r_i}(t_i) = 1 \quad \text{if and only if} \quad \prod_{i=1}^{m} t_i^{\langle r_i, r_j\rangle} = 1 \text{ for all } 1 \leqslant j \leqslant m,$$

where $\{r_1, \ldots, r_m\}$ *is a base in* Σ.

In [GLS98, Thm. 1.12.1] there can also be found more information on the constants c_{ijrs} occurring for both the adjoint and universal type of Chevalley groups.

(ii) The signs $\eta_{r,s}$ in Theorem 4.25 are not uniquely determined by the root system Σ but rather they also depend on the chosen Chevalley basis of the underlying Lie algebra. However, they are independent of the prime p (cf., for instance, the proof of [Car89, Prop. 6.4.3]).

As in the case of adjoint Chevalley groups the following holds:

Proposition 4.27. *Let p be a prime and denote by $\mathbb{F}$ an algebraic closure of the finite field $\mathbb{F}_p$ consisting of p elements. Moreover, let $\overline{\mathbf{G}} := \Sigma(\mathbb{F})_{\text{uni}}$ be a universal Chevalley group over $\mathbb{F}$ and define $\overline{\mathbf{T}} := \langle \bar{h}_r(t) \mid r \in \Sigma,\, t \in \mathbb{F}^{\times} \rangle$. Then*

$$\mathrm{N}_{\overline{\mathbf{G}}}(\overline{\mathbf{T}}) = \langle \overline{\mathbf{T}},\, \bar{n}_r(1) \mid r \in \Sigma \rangle,$$

and there is an isomorphism between the groups $\overline{\mathbf{W}} := \mathrm{N}_{\overline{\mathbf{G}}}(\overline{\mathbf{T}})/\overline{\mathbf{T}}$ and $W(\Sigma)$ induced by the map $\bar{n}_r(1) \mapsto \omega_r$.

Proof. Let $\overline{Z}$ denote the center of $\overline{\mathbf{G}}$. Following Theorem 4.22 it holds that $\overline{Z} \leqslant \overline{\mathbf{T}}$ and $\overline{\mathbf{G}}/\overline{Z} \cong \Sigma(K)_{\text{ad}} =: \mathbf{G}$ via $x_r(t) \mapsto \bar{x}_r(t)\overline{Z}$. Moreover, by Theorem 4.25(x) the group $\overline{\mathbf{T}}$ is normal in $\langle \overline{\mathbf{T}},\, \bar{n}_r(1) \mid r \in \Sigma \rangle$, and according to [GLS98, Thm. 1.9.5(f)] the quotient $\mathrm{N}_{\overline{\mathbf{G}}}(\overline{\mathbf{T}})/\overline{\mathbf{T}}$ is isomorphic to $W(\Sigma)$. For the subgroup $\mathbf{T}$ of $\mathbf{G}$ as in Proposition 4.21 it holds that $\mathbf{T} \cong \overline{\mathbf{T}}/\overline{Z}$. Hence, the statement follows from Proposition 4.21 and the fact that we have $\mathrm{N}_{\overline{\mathbf{G}}}(\overline{\mathbf{T}})/\overline{\mathbf{T}} \cong (\mathrm{N}_{\overline{\mathbf{G}}}(\overline{\mathbf{T}})/\overline{Z})/(\overline{\mathbf{T}}/\overline{Z}) \cong \mathrm{N}_{\mathbf{G}}(\mathbf{T})/\,\mathbf{T}$. □

Definition 4.28 (*Chevalley groups*)**.** Let $\Sigma(K)_{\text{uni}}$ be a universal Chevalley group of type Σ over some field K. Then for any subgroup $Z \leqslant \mathrm{Z}(\Sigma(K)_{\text{uni}})$ of the center of $\Sigma(K)_{\text{uni}}$ the quotient $\mathbf{G} := \Sigma(K)_{\text{uni}}/Z$ will be called a *Chevalley group* of type Σ over K.

The corresponding cosets of the generators $\bar{x}_r(t)$, $\bar{n}_r(t)$ and $\bar{h}_r(t)$ of $\Sigma(K)_{\text{uni}}$ will simply be denoted $x_r(t)$, $n_r(t)$ and $h_r(t)$, and analogously we write $\mathbf{T}$ and $\mathbf{W}$ to denote the images of $\overline{\mathbf{T}}$ and $\overline{\mathbf{W}}$.

Proposition 4.29. *Let $\mathbf{G}$ be a Chevalley group over an algebraically closed field. Then $\mathbf{G}$ is a simple linear algebraic group.*

Proof. It was proven in [Ste68, Thm. 6(a)] that $\mathbf{G}$ is a semisimple algebraic group. Note, however, that [Ste68, Thm. 6(a)] is a statement on a larger class of groups than we are considering here, that is to say, [Ste68, Thm. 6(a)] includes all groups constructed analogously to our construction of Chevalley groups but under the slightly weaker condition that the underlying Lie algebra is semisimple (and not simple as we require here), or in other words, the underlying root system is not required to be indecomposable. Starting with an indecomposable root system, one can show that $\mathbf{G}$ is indeed simple, compare, for instance, with the proof of [MT11, Thm. 8.21]. Since $\mathbf{G}$ is linear by construction, the claim follows. □

4.3 Finite Groups of Lie Type

The Chevalley groups $\mathrm{SL}_3(q)$ and $G_2(q)$, and Steinberg's triality groups ${}^3D_4(q)$, on which the main focus of this thesis lies, belong to the class of finite groups of Lie type. In this section we describe briefly how such groups can be constructed from infinite Chevalley groups as subgroups of fixed points under certain endomorphisms. Essentially, we follow the notation and construction given in [MT11, Part III].

Until the end of this chapter we let p be a prime number and denote by $\mathbb{F}$ an algebraic closure of a finite field $\mathbb{F}_p$ consisting of p elements. Moreover, for a power q of p we let $\mathbb{F}_q$ be the unique subfield of $\mathbb{F}$ consisting of q elements.

Definition 4.30 (*Frobenius endomorphism*)**.** Let $\mathbf{G}$ be a Chevalley group of type Σ over $\mathbb{F}$ with generators $x_r(t)$, $r \in \Sigma$, $t \in \mathbb{F}$. The endomorphism F_q defined by

$$F_q\colon\ \mathbf{G} \longrightarrow \mathbf{G},\ x_r(t) \longmapsto x_r(t^q),\ r \in \Sigma,\ t \in \mathbb{F},$$

is called the *Frobenius endomorphism* of $\mathbf{G}$ with respect to $\mathbb{F}_q$.

Definition 4.31 (*Steinberg endomorphism*)**.** Let $\mathbf{G}$ be a Chevalley group over $\mathbb{F}$. An endomorphism F of $\mathbf{G}$ is called a *Steinberg endomorphism* of $\mathbf{G}$ if there is some $m \in \mathbb{N}_{>0}$ such that F^m is the Frobenius endomorphism of $\mathbf{G}$ with respect to some power q of p.

Remark 4.32. The designation of the endomorphisms introduced in the previous two definitions follows the notation of [MT11]. Note that this is not standard in the literature. For instance in [GLS98] endomorphisms as in Definition 4.31 are called Frobenius endomorphisms, while an endomorphism of $\mathbf{G}$ is called a Steinberg endomorphism if it is surjective and fixes only finitely many elements of $\mathbf{G}$.

Definition 4.33 (*Finite group of Lie type*)**.** Let $\mathbf{G}$ be a Chevalley group over $\mathbb{F}$ and let F be a Steinberg endomorphism of $\mathbf{G}$. Then we denote the subgroup of fixed points of F on $\mathbf{G}$ by $\mathbf{G}^F$, i.e.,

$$\mathbf{G}^F := \{g \in \mathbf{G} \mid F(g) = g\}.$$

The group $\mathbf{G}^F$ is finite (see, for instance, [MT11, Thm. 21.5]), and it will henceforth be called a *finite group of Lie type*.

As observed in Proposition 4.29, Chevalley groups are simple linear algebraic groups, so in particular by [MT11, Thm. 21.7] the following famous result, known as the theorem of Lang–Steinberg, holds for these groups:

Theorem 4.34 (Lang–Steinberg)**.** *Let $\mathbf{G}$ be a connected linear algebraic group over $\mathbb{F}$ and let F be a Steinberg endomorphism of $\mathbf{G}$. Then*

$$L\colon\ \mathbf{G} \longrightarrow \mathbf{G},\ g \longmapsto F(g)g^{-1},$$

defines a surjective morphism.

The theorem of Lang–Steinberg is of great importance for the theory of linear algebraic groups and finite groups of Lie type. One well-known consequence that derives from this result is the following statement:

Corollary 4.35. *Let $\mathbf{G}$ be a Chevalley group over $\mathbb{F}$ with Steinberg endomorphism F. Then for any element $g \in \mathbf{G}$ the subgroups of fixed points of $\mathbf{G}$ under F and gF, respectively, are $\mathbf{G}$-conjugate.*

Proof. Let $g \in \mathbf{G}$. By Lang–Steinberg there exists some $x \in \mathbf{G}$ such that $g^{-1} = F(x)x^{-1}$. We define $\iota(h) := xhx^{-1}$ for $h \in \mathbf{G}^F$. Then

$$(gF)(\iota(h)) = gF(xhx^{-1})g^{-1} = xF(h)x^{-1} = \iota(F(h)) = \iota(h),$$

so $\iota(h) \in \mathbf{G}^{gF}$ for all $h \in \mathbf{G}^F$. On the other hand, for every $y \in \mathbf{G}^{gF}$ we have

$$F(x^{-1}yx) = x^{-1}gF(y)g^{-1}x = x^{-1}yx,$$

so $x^{-1}yx \in \mathbf{G}^F$ is a preimage in $\mathbf{G}^F$ of y under ι. Clearly, ι is injective as it is given by conjugation. Hence, we conclude that ι defines an isomorphism between $\mathbf{G}^F$ and $\mathbf{G}^{gF}$ as claimed. $\square$

In the situation of Corollary 4.35 above we call the finite group $\mathbf{G}^{gF}$ *twisted* since it is obtained by "twisting" the group $\mathbf{G}^F$, that is by conjugating $\mathbf{G}^F$ with a certain element of $\mathbf{G}$. At later stages, when we work with concrete examples for groups of Lie type it will sometimes be convenient to consider twisted versions of the groups in question as they may provide particularly nice descriptions for maximal tori, which are to be regarded next.

4.4 Maximal Tori

Chevalley groups over $\mathbb{F}$ being algebraic groups it is reasonable to speak about their tori. In this section we gather some information on this important kind of subgroups. Let us first note that in Section 4.2 we have already encountered an important example for a (maximal) torus of a Chevalley group:

Proposition 4.36. *Let $\mathbf{G}$ be a Chevalley group of type Σ over $\mathbb{F}$. The subgroup $\mathbf{T}$ of $\mathbf{G}$ defined in Section 4.2 as*

$$\mathbf{T} := \langle h_r(t) \mid r \in \Sigma,\, t \in \mathbb{F}^{\times} \rangle$$

is a maximal torus of $\mathbf{G}$.

Proof. This is [Ste68, Thm. 6(c)] □

An important fact about the maximal tori of Chevalley groups over $\mathbb{F}$ or, more generally, of linear algebraic groups is the following (see, e.g., [MT11, Cor. 6.5]):

Proposition 4.37. *All maximal tori of a linear algebraic group $\mathbf{G}$ defined over the field $\mathbb{F}$ are conjugate in $\mathbf{G}$.*

Definition 4.38 (*Weyl group*)**.** Let $\mathbf{G}$ be a Chevalley group over $\mathbb{F}$ and suppose that $\mathbf{H}$ is a torus of $\mathbf{G}$. Then the group $\mathbf{W}(\mathbf{H}) := \mathrm{N}_{\mathbf{G}}(\mathbf{H})/\mathbf{H}$ is called the *Weyl group of* $\mathbf{H}$.

If $\mathbf{G}$ is of type Σ and $\mathbf{T}$ is a maximal torus of $\mathbf{G}$, then by Propositions 4.21 and 4.27 we have $\mathbf{W} := \mathbf{W}(\mathbf{T}) \cong W(\Sigma)$, and we call $\mathbf{W}$ the *Weyl group* of $\mathbf{G}$.

For a given Chevalley group $\mathbf{G}$ over $\mathbb{F}$ with Steinberg endomorphism F and an F-stable maximal torus $\mathbf{T}$ let us consider a second F-stable maximal torus $\mathbf{T}'$. By Proposition 4.37 there exists an element $g \in \mathbf{G}$ such that $\mathbf{T}' = g\,\mathbf{T}\,g^{-1}$, and since $\mathbf{T}$ and $\mathbf{T}'$ are F-stable, $g^{-1}F(g)$ normalizes $\mathbf{T}$. We set $w := g^{-1}F(g)\,\mathbf{T} \in \mathrm{N}_{\mathbf{G}}(\mathbf{T})/\,\mathbf{T} = \mathbf{W}$ and introduce the notation $\mathbf{T}_w := \mathbf{T}'$.

Definition 4.39 (*F-conjugacy*)**.** Suppose that $\mathbf{G}$ is a Chevalley group over $\mathbb{F}$ with a Steinberg endomorphism F and let $\mathbf{T}$ be an F-stable maximal torus of $\mathbf{G}$ with corresponding Weyl group $\mathbf{W}$. Two elements $w_1, w_2 \in \mathbf{W}$ are called *F-conjugate* if there exists some $w \in \mathbf{W}$ such that $w_2 = F(w)w_1w^{-1}$. Clearly, this defines an equivalence relation, and the corresponding equivalence classes are called *F-conjugacy classes* (cf., for instance, [MT11, Def. 21.9]).

By [MT11] the following result holds for connected reductive groups, so in particular for Chevalley groups over $\mathbb{F}$:

Proposition 4.40. *Let $\mathbf{G}$ be a Chevalley group over $\mathbb{F}$ with a Steinberg endomorphism F and let $\mathbf{T}$ be an F-stable maximal torus of $\mathbf{G}$ with corresponding Weyl group $\mathbf{W}$.*

(i) The map

$$\begin{array}{ccc} \left\{ \begin{array}{c} \mathbf{G}^F\text{-conjugacy classes of } F\text{-stable} \\ \text{maximal tori of } \mathbf{G} \end{array} \right\} & \longrightarrow & \left\{ F\text{-conjugacy classes in } \mathbf{W} \right\} \\ [\mathbf{T}_w]_{\mathbf{G}^F} & \longmapsto & [w]_F, \end{array}$$

is a bijection, where $[\mathbf{T}_w]_{\mathbf{G}^F}$ denotes the $\mathbf{G}^F$-conjugacy class of $\mathbf{T}_w$ in $\mathbf{G}$, and $[w]_F$ denotes the F-conjugacy class of w in $\mathbf{W}$.

(ii) If $\mathbf{T}_w$ is an F-stable $\mathbf{G}$-conjugate of $\mathbf{T}$ for some $w \in \mathbf{W}$, then

$$\mathrm{N}_{\mathbf{G}^F}(\mathbf{T}_w)/\,\mathbf{T}_w^F \cong (\mathrm{N}_{\mathbf{G}}(\mathbf{T}_w)/\,\mathbf{T}_w)^F \cong \mathbf{W}^{wF},$$

where $\mathbf{W}^{wF} = \{v \in \mathbf{W} \mid wF(v)w^{-1} = v\}$ is the group of fixed points of wF on $\mathbf{W}$.

Proof. Part (i) is proven in [MT11, Prop. 25.1], part (ii) is [MT11, Prop. 23.2] for the first isomorphism and [MT11, Prop. 25.3(a)] for the second. Note that the cited statements in [MT11] involve an automorphism ϕ acting on $\mathbf{W}$, and instead of the F-conjugacy classes in $\mathbf{W}$ in part (i) and the group $\mathbf{W}^{wF}$ in part (ii) one considers the ϕ-conjugacy classes in $\mathbf{W}$ and the group $\mathbf{W}^{w\phi} = \{v \in \mathbf{W} \mid w\phi(v)w^{-1} = v\}$, respectively. However, as becomes apparent in the proofs of both statements, the F-conjugacy classes in $\mathbf{W}$ agree with the ϕ-conjugacy classes, and we have $\mathbf{W}^{wF} = \mathbf{W}^{w\phi}$. □

In the notation of Proposition 4.40 the group $\mathbf{T}^F$ of fixed points of the F-stable maximal torus $\mathbf{T}$ of $\mathbf{G}$ under the Steinberg endomorphism F is called a *maximal torus* of the finite group $\mathbf{G}^F$. Part (i) of this proposition tells us that there is a one-to-one correspondence between $\mathbf{G}^F$-conjugacy classes of maximal tori of $\mathbf{G}^F$ and F-conjugacy classes in the Weyl group $\mathbf{W}$.

Remark 4.41. Following Proposition 4.40 for every F-stable maximal torus $\mathbf{T}$ of a Chevalley group $\mathbf{G}$ with Steinberg endomorphism F we have $\mathrm{N}_{\mathbf{G}^F}(\mathbf{T})/\,\mathbf{T}^F \cong \mathbf{W}^F$, where $\mathbf{W}$ denotes the Weyl group corresponding to $\mathbf{T}$. It is remarked in [MT11, p. 198], however, that in general it is not true that $\mathrm{N}_{\mathbf{G}^F}(\mathbf{T}^F) = \mathrm{N}_{\mathbf{G}^F}(\mathbf{T})$, for instance in the case where $\mathbf{T} \neq 1$ but the group of fixed points of F on $\mathbf{T}$ is trivial.

Part II

The Special Linear Groups $\mathrm{SL}_3(q)$ for $q > 2$ and $3 \nmid (q-1)$

Chapter 5

Properties of $\mathrm{SL}_3(q)$

The special linear groups $\mathrm{SL}_3(q)$ for prime powers $q > 2$ such that $3 \nmid (q-1)$ constitute the first series of finite groups of Lie type for which we establish the inductive blockwise Alperin weight condition in this thesis. This chapter gives a brief introduction to these groups and provides a summary of certain of their properties that will be of importance in due course. Our main references are Gorenstein–Lyons–Solomon [GLS98], Malle–Testerman [MT11] and Steinberg [Ste68].

5.1 Construction of $\mathrm{SL}_3(q)$

For a fixed prime power q we define the group $\mathrm{SL}_3(q)$ as the group of fixed points of a universal Chevalley group under a certain Frobenius endomorphism. Let p be a prime and $f \in \mathbb{N}_{>0}$ such that $q = p^f$. Moreover, let $\mathbb{F}$ denote an algebraic closure of the field $\mathbb{F}_p$ consisting of p elements and let $\mathbb{F}_q$ be the unique subfield of $\mathbb{F}$ containing exactly q elements. We denote by

$$\mathbf{G} := \mathrm{SL}_3(\mathbb{F}) := \{A \in \mathrm{GL}_3(\mathbb{F}) \mid \det(A) = 1\}$$

the subgroup of the general linear group $\mathrm{GL}_3(\mathbb{F})$ consisting of all 3×3-matrices defined over $\mathbb{F}$ with determinant 1. According to [Ste68, p. 45] the group $\mathbf{G}$ is a universal Chevalley group of type A_2. Now the field automorphism $\mathbb{F} \longrightarrow \mathbb{F}$, $a \longmapsto a^q$, induces an endomorphism F acting on $\mathbf{G}$ by raising each entry of a matrix in $\mathbf{G}$ to its q-th power:

$$F\colon\ \mathbf{G} \longrightarrow \mathbf{G}, \quad (a_{ij})_{1\leqslant i,j\leqslant 3} \longmapsto (a_{ij}^q)_{1\leqslant i,j\leqslant 3}.$$

This endomorphism is a Frobenius endomorphism in the sense of Definition 4.30, hence a Steinberg endomorphism. The finite group $\mathrm{SL}_3(q)$ of Lie type is now defined as the group

$$G := \mathrm{SL}_3(q) := \mathrm{SL}_3(\mathbb{F})^F := \{A \in \mathrm{SL}_3(\mathbb{F}) \mid F(A) = A\}$$

of fixed points of $\mathrm{SL}_3(\mathbb{F})$ under the Steinberg endomorphism F. Clearly, it consists of all 3×3-matrices with determinant 1 whose entries lie in $\mathbb{F}_q$. Henceforth, until the examination of the special linear groups is completed, the notation G will always refer to the special linear group $\mathrm{SL}_3(q)$. According to [MT11, Table 24.1] the order of this group is as follows:

Lemma 5.1. *The order of the finite group* $\mathrm{SL}_3(q)$ *is given by*

$$\begin{aligned} |\,\mathrm{SL}_3(q)| &= q^3(q-1)^2(q+1)(q^2+q+1) \\ &= q^3\Phi_1(q)^2\Phi_2(q)\Phi_3(q). \end{aligned}$$

For later use we make the following observation on the common divisors of the factors of the group order $|G| = q^3\Phi_1(q)^2\Phi_2(q)\Phi_3(q)$:

Lemma 5.2. *For $q \in \mathbb{N}_{>1}$ a power of a prime p the following statements hold:*

(i) $\gcd(\Phi_2(q), \Phi_3(q)) = 1$*;*

(ii) $\gcd(\Phi_1(q), \Phi_2(q)) = \begin{cases} 2 & \text{if } q \text{ is odd,} \\ 1 & \text{if } q \text{ is even;} \end{cases}$

(iii) $\gcd(\Phi_1(q), \Phi_3(q)) = \begin{cases} 3 & \text{if } q \equiv 1 \bmod 3, \\ 1 & \text{otherwise.} \end{cases}$

In particular, if $q \not\equiv 1 \bmod 3$ and ℓ is a prime dividing the order of $\mathrm{SL}_3(q)$, then either $\ell = 2$, $\ell = p$, or there exists a unique $d \in \{1, 2, 3\}$ such that $\ell \mid \Phi_d(q)$.

Proof. This follows easily from Lemma 5.2 in [Mal07], which states that for a prime r not dividing q, and e the multiplicative order of q modulo r one has

- $r \mid \Phi_f(q)$ if and only if $f = er^i$ for some $i \in \mathbb{N}$, and
- $f = e_r(q)$ if $r^2 \mid \Phi_f(q)$,

where $e_r(q) = e$ if $r \neq 2$ and $e_2(q) = 1$ or 2 for $q \equiv 1 \bmod 4$ or $q \equiv -1 \bmod 4$, respectively. In particular, if a prime r divides $\Phi_{f_1}(q)$ and $\Phi_{f_2}(q)$ for $f_1 < f_2$, then $f_2/f_1 > 1$ is a power of r, and r^2 divides at most one of $\Phi_{f_1}(q)$ and $\Phi_{f_2}(q)$. In our case, this is only possible if $r = 2$ or $r = 3$ with $e = 1$, $f_1 = 1$ and $f_2 = 2$ or $f_2 = 3$, respectively, and then q must be odd for $r = 2$, while for $r = 3$ it follows that $q = q^e \equiv 1 \bmod 3$. □

Let us now turn to the universal covering group of G. This is relevant to us since despite the fact that we say that the (iBAW) condition holds "for a finite simple group S" and a prime ℓ, the conditions that need to be verified are in fact questions concerning the universal ℓ'-covering group of S and not S itself (cf. Definitions 3.2 and 3.3). In our case of $G = \mathrm{SL}_3(q)$ the situation is as follows:

Proposition 5.3. *Suppose that $q \geqslant 2$. Then the finite group $G = \mathrm{SL}_3(q)$ is perfect and its quotient $\mathrm{PSL}_3(q) = G/\mathrm{Z}(G)$ is simple. Moreover, if $q \neq 2, 4$, then $\mathrm{SL}_3(q)$ is the universal covering group of $\mathrm{PSL}_3(q)$.*

Proof. See, for instance, [MT11, Thm. 24.17 and Rmk. 24.19]. □

Remark 5.4. The center of $\mathrm{SL}_3(q)$ consists of all scalar matrices in $\mathrm{SL}_3(q)$. By reason of all matrices in $\mathrm{SL}_3(q)$ having determinant 1 it follows that $\mathrm{SL}_3(q)$ has a non-trivial center if and only if $3 \mid (q-1)$. Now the groups $\mathrm{PSL}_3(q) = \mathrm{SL}_3(q)/\mathrm{Z}(\mathrm{SL}_3(q))$ are simple for all $q \geqslant 2$ according to the above proposition, so if $3 \nmid (q-1)$ and $q \geqslant 2$, then the group $\mathrm{SL}_3(q) = \mathrm{PSL}_3(q)$ is simple. If, in addition, $q > 2$, then again by Proposition 5.3 it even is its own universal covering group. In this thesis we will focus on this situation.

During the subsequent examination of $\mathrm{SL}_3(q)$ in view of the inductive condition for the blockwise Alperin weight conjecture we work under the following assumption:

From now on we assume that $q > 2$ and $q \not\equiv 1 \bmod 3$.

5.2 Weyl Group and Maximal Tori of $\mathrm{SL}_3(q)$

In this section we briefly describe the maximal tori of G. Let us denote by $\mathbf{T}$ the F-stable maximal torus of $\mathbf{G}$ consisting of the diagonal matrices in $\mathbf{G}$. By Proposition 4.40 there is a bijection between the G-conjugacy classes of F-stable maximal tori of $\mathbf{G}$ and the F-conjugacy classes of $\mathbf{W} := \mathrm{N}_{\mathbf{G}}(\mathbf{T})/\,\mathbf{T}$. As is well known, $\mathbf{W} \cong \mathfrak{S}_3$ and $\mathrm{N}_{\mathbf{G}}(\mathbf{T})$ consists of the monomial matrices in $\mathbf{G}$. In particular, the elements of $\mathbf{W}$ may be represented by monomial matrices in $\mathbf{G}$ whose non-zero entries are contained in $\{\pm 1\}$. Since F acts trivially on such matrices, $\mathbf{W}$ is stabilized pointwise by F, so the F-conjugacy classes of $\mathbf{W}$ coincide with its $\mathbf{W}$-conjugacy classes. Now in $\mathbf{W} \cong \mathfrak{S}_3$ there exist exactly three $\mathbf{W}$-conjugacy classes, represented by the permutations $\mathrm{id} = (\,)$, (12) and (123). If we set

$$v_2 := \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \in \mathbf{G} \quad \text{and} \quad v_3 := \begin{bmatrix} 0 & 0 & -1 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \in \mathbf{G},$$

then the left cosets $w_2 := v_2\,\mathbf{T}$ and $w_3 := v_3\,\mathbf{T}$ correspond to the permutations (12) and (123), respectively, and one easily verifies that $\langle v_2, v_3\rangle \cong \mathfrak{S}_3$ (e.g. using GAP [GAP08]). We obtain three representatives in $\mathbf{G}$ for the G-conjugacy classes of maximal tori in G. These are given in Table 5.1 below.

$w \in \mathbf{W}$	$\mathbf{T}^{wF}$	$\mathbf{W}^{wF}$
1	$T_1 = \{\mathrm{diag}(z_1, z_2, z_1^{-1}z_2^{-1}) \mid z_1^{q-1} = z_2^{q-1} = 1\}$ $\cong C_{q-1} \times C_{q-1}$	$\mathfrak{S}_3$
w_2	$T_2 = \{\mathrm{diag}(z, z^q, z^{-(q+1)}) \mid z^{q^2-1} = 1\}$ $\cong C_{q^2-1}$	C_2
w_3	$T_3 = \{\mathrm{diag}(z, z^q, z^{q^2}) \mid z^{q^2+q+1} = 1\}$ $\cong C_{q^2+q+1}$	C_3

Table 5.1: Maximal tori of $\mathrm{SL}_3(q)$

Definition 5.5. We say that a maximal torus T of G is *of type* T_i for $i \in \{1, 2, 3\}$ if T is $\mathbf{G}$-conjugate to the torus T_i defined in Table 5.1.

Remark 5.6. The maximal torus $\mathbf{T}$ of $\mathbf{G}$ consisting of the diagonal matrices in $\mathbf{G}$ is stable under F and lies in the F-stable Borel subgroup of $\mathbf{G}$ given by all upper (respectively lower) triangular matrices in $\mathbf{G}$. Hence, $\mathbf{T}$ is maximally split with respect to F and we will sometimes refer to the torus $T_1 = \mathbf{T}^F$ as the *maximally split torus* of G.

Proposition 5.7. *For the normalizers of the maximal tori of the special linear group the following hold (under the assumption that $q > 2$):*

(i) $\mathrm{N}_{\mathbf{G}^F}(\mathbf{T}^F) = \mathbf{T}^F \rtimes \langle v_2, v_3\rangle \cong \mathbf{T}^F \rtimes \mathfrak{S}_3$,

(ii) $\mathrm{N}_{\mathbf{G}^{w_2F}}(\mathbf{T}^{w_2F}) = \mathbf{T}^{w_2F} \rtimes \langle v_2\rangle \cong \mathbf{T}^{w_2F} \rtimes C_2$,

(iii) $\mathrm{N}_{\mathbf{G}^{w_3F}}(\mathbf{T}^{w_3F}) = \mathbf{T}^{w_3F} \rtimes \langle v_3\rangle \cong \mathbf{T}^{w_3F} \rtimes C_3$.

Proof. These statements are well-known but since they will be used frequently, we give a short idea of the proof here. Let T be one of the maximal tori $\mathbf{T}^F$, $\mathbf{T}^{w_2F}$ or $\mathbf{T}^{w_3F}$. Since we assume that $q > 2$, the eigenvectors in $\mathbb{F}^3$ common to all diagonal matrices in T are exactly the scalar multiples of the standard basis vectors $(1,0,0)^{\mathrm{tr}}$, $(0,1,0)^{\mathrm{tr}}$ and $(0,0,1)^{\mathrm{tr}}$ of $\mathbb{F}^3$. Suppose now that $N \in \mathrm{N}_{\mathbf{G}}(T)$ and let $v \in \mathbb{F}^3$ be a common eigenvector for all elements of T. Then also Nv is an eigenvector for all elements of T, hence, a scalar multiple of a standard basis vector. Letting v run over the three standard basis vectors, we may thus conclude that N is a monomial matrix, that is, each column and each row of N contains exactly one non-zero entry. It is now straightforward to check which monomial matrices of $\mathbf{G}$ are fixed points under F, w_2F and w_3F, respectively. □

Proposition 5.8. *For a prime $\ell \neq p$ dividing $|G|$ (and $q \not\equiv 1 \bmod 3$) we denote by d an integer in $\{1,2,3\}$ such that $\ell \mid \Phi_d(q)$ (note that by Lemma 5.2 this is uniquely determined if $\ell > 2$). Then*

(i) $\mathrm{N}_G(\mathcal{O}_\ell(T)) = \mathrm{N}_G(T)$ *and*

(ii) $\mathrm{C}_G(\mathcal{O}_\ell(T)) = T$

for every maximal torus T of G of type T_d.

Proof. (i) If $d \in \{2,3\}$, then by Corollary 4.35 we may work in the group $\mathbf{G}^{w_dF}$ instead of G and assume that $T = \mathbf{T}^{w_dF}$. Else, we assume that $T = \mathbf{T}^F$. The claim then follows by the same arguments as in the proof of Proposition 5.7 if we can ensure that the set of common eigenvectors in $\mathbb{F}^3$ of $\mathcal{O}_\ell(T)$ is the set of $\mathbb{F}$-multiples of the three standard basis vectors of $\mathbb{F}^3$. Since T consists of diagonal matrices, this is the case if and only if for every $i, j \in \{1,2,3\}$ with $i \neq j$ there exists a diagonal matrix $\mathrm{diag}(t_1,t_2,t_3) \in \mathcal{O}_\ell(T)$ such that $t_i \neq t_j$. This is clearly satisfied if $d = 1$, that is, if $\ell \mid (q-1)$.

For $d = 2$ we have $T = \{\mathrm{diag}(t, t^q, t^{-(q+1)}) \mid t^{q^2-1} = 1\}$, and for $s \in \mathbb{F}^\times$ of order $(q^2-1)_\ell$ we hence obtain $\mathcal{O}_\ell(T) = \langle \mathrm{diag}(s, s^q, s^{-(q+1)})\rangle$. Since $d = 2$, we have $\ell \mid (q+1)$, so $s \notin \mathbb{F}_q^\times$, that is, $s^q \neq s$, and $s^q \notin \mathbb{F}_q^\times$, while $s^{-(q+1)} \in \mathbb{F}_q^\times$. Thus, the diagonal entries of $\mathrm{diag}(s, s^q, s^{-(q+1)})$ are pairwise distinct, so in particular the set of common eigenvectors in $\mathbb{F}^3$ of $\mathcal{O}_\ell(T)$ is as desired.

For $d = 3$ the maximal torus T is given by $\{\mathrm{diag}(t, t^q, t^{q^2}) \mid t^{q^2+q+1} = 1\}$. Since in this situation $\ell \mid (q^2+q+1)$, for $s \in \mathbb{F}^\times$ of order $(q^2+q+1)_\ell$ the matrix $\mathrm{diag}(s, s^q, s^{q^2})$ generates $\mathcal{O}_\ell(T)$ and has pairwise distinct diagonal entries. Accordingly, also in this case we can find an element of $\mathcal{O}_\ell(T)$ whose eigenvectors are exactly the $\mathbb{F}$-multiples of the standard basis vectors of $\mathbb{F}^3$.

Proceeding as outlined in the proof of Proposition 5.7 hence proves (i).

For (ii) one easily verifies that the only elements of $\mathrm{N}_G(T) \supseteq \mathrm{C}_G(\mathcal{O}_\ell(T))$ centralizing all elements of $\mathcal{O}_\ell(T)$ are those contained in T. As in the proof of part (i) this is due to the fact that for any $i, j \in \{1,2,3\}$ with $i \neq j$ one can always find a diagonal matrix $\mathrm{diag}(t_1,t_2,t_3) \in \mathcal{O}_\ell(T)$ with $t_i \neq t_j$. □

For the Sylow ℓ-subgroups of G, where $\ell \neq p$, we observe the following:

Proposition 5.9. *Let $\ell \neq p$ be a prime dividing $|G|$ (and recall that $q \not\equiv 1 \bmod 3$). For a Sylow ℓ-subgroup $P \in \mathrm{Syl}_\ell(G)$ one of the following holds:*

(i) $\ell > 2$ and $P = \mathcal{O}_\ell(T)$, where T is a maximal torus of G of type T_d with d the unique integer in $\{1,2,3\}$ such that $\ell \mid \Phi_d(q)$. Moreover, $\mathrm{C}_G(P) = T$ and $\mathrm{N}_G(P) = \mathrm{N}_G(T)$.

In particular, P is abelian, and P is cyclic if $d \in \{2,3\}$.

(ii) $\ell = 2$ *and* $P \in \mathrm{Syl}_2(\mathrm{N}_G(T))$*, where* T *is a maximal torus of* G *of type* T_d *such that* $d \in \{1,2\}$ *with* $4 \mid \Phi_d(q)$*. Moreover,* $\mathrm{N}_G(P) = P\,\mathrm{C}_G(P)$ *and* $P\,\mathrm{C}_G(P)/P \cong C_{(q-1)_{2'}}$*.*

Proof. The structure of P follows as an easy consequence of Lemma 5.2, Proposition 5.7 and the description of the maximal tori in Table 5.1. The remaining statement in part (i) follows from Proposition 5.8.

For (ii) suppose hence that $\ell = 2$ and let $d \in \{1,2\}$ be such that 4 divides $\Phi_d(q)$. We consider the Sylow 2-subgroup $P_d := \mathcal{O}_2(\mathbf{T}^{w_dF}) \rtimes \langle v_2 \rangle$ of $\mathbf{G}^{w_dF}$, where for $d=1$ we use the notation $w_1 := 1$. Since v_2 acts on $\mathbf{T}^{w_dF}$ by interchanging the first two diagonal entries of the matrices in $\mathbf{T}^{w_dF}$, one easily verifies that

$$[P_d, P_d] = \begin{cases} \langle \mathrm{diag}(\xi^{(q+1)(q-1)_{2'}}, \xi^{-(q+1)(q-1)_{2'}}, 1) \rangle & \text{if } d = 1, \\ \langle \mathrm{diag}(\xi^{(q-1)(q+1)_{2'}}, \xi^{-(q-1)(q+1)_{2'}}, 1) \rangle & \text{if } d = 2, \end{cases}$$

where $\xi \in \mathbb{F}^\times$ is of order q^2-1. In both cases the group $[P_d, P_d]$ contains matrices with pairwise distinct diagonal entries, so as before we may apply the argument on common eigenvectors used in the proof of Proposition 5.7 to find that $\mathrm{N}_{\mathbf{G}^{w_dF}}([P_d, P_d]) = \mathbf{T}^{w_dF} \rtimes \langle v_2 \rangle$. Now the commutator subgroup $[P_d, P_d]$ is characteristic in P_d, so in particular we obtain

$$\mathrm{N}_{\mathbf{G}^{w_dF}}(P_d) \subseteq \mathrm{N}_{\mathbf{G}^{w_dF}}([P_d, P_d]) = \mathbf{T}^{w_dF} \rtimes \langle v_2 \rangle.$$

Since clearly v_2 normalizes P_d, we need to check which elements of $\mathbf{T}^{w_dF}$ normalize P_d, or, in other words, which elements of $\mathbf{T}^{w_dF}$ commute with v_2 modulo $\mathcal{O}_2(\mathbf{T}^{w_dF})$. Straightforward calculations prove that these are exactly the elements in $\langle \mathcal{O}_2(\mathbf{T}^{w_dF}), \mathrm{diag}(\zeta, \zeta, \zeta^{-2}) \rangle$, where $\zeta := \xi^{q+1}$. Moreover, we have

$$\mathrm{C}_{\mathbf{G}^{w_dF}}(P_d) = \mathrm{C}_{\mathrm{N}_{\mathbf{G}^{w_dF}}(P_d)}(P_d) = \mathrm{C}_{\mathbf{T}^{w_dF}}(v_2) = \langle \mathrm{diag}(\zeta, \zeta, \zeta^{-2}) \rangle,$$

so in summary it follows that

$$\mathrm{N}_{\mathbf{G}^{w_dF}}(P_d) = \langle \mathcal{O}_\ell(\mathbf{T}^{w_dF}), \mathrm{diag}(\zeta, \zeta, \zeta^{-2}) \rangle \rtimes \langle v_2 \rangle = P_d\,\mathrm{C}_{\mathbf{G}^{w_dF}}(P_d),$$

and

$$P_d\,\mathrm{C}_{\mathbf{G}^{w_dF}}(P_d)/P_d \cong \mathcal{O}_{2'}(\langle \mathrm{diag}(\zeta, \zeta, \zeta^{-2}) \rangle) \cong C_{(q-1)_{2'}}.$$

By application of Corollary 4.35 the claim follows. ☐

5.3 Automorphisms of $\mathrm{SL}_3(q)$

In this section we describe the automorphism group of G. Let us begin with the introduction of certain automorphisms of G.

Definition 5.10. We denote by F_p the automorphism induced on G via the field automorphism $\mathbb{F}_q \longrightarrow \mathbb{F}_q$, $a \longmapsto a^p$, that is,

$$F_p \colon G \longrightarrow G, \quad (a_{ij})_{1 \leqslant i,j \leqslant 3} \longmapsto (a_{ij}^p)_{1 \leqslant i,j \leqslant 3}.$$

We call the automorphism F_p a *field automorphism* of G. Note that its order in $\mathrm{Aut}(G)$ is given by f, where we recall that $f \in \mathbb{N}_{>0}$ is such that $q = p^f$.

Definition 5.11. Recall that for a matrix X we denote its transpose by X^{tr}. The automorphism Γ of G defined by

$$\Gamma\colon G \longrightarrow G, \quad A \longmapsto (A^{-1})^{\mathrm{tr}},$$

is called a *graph automorphism* of G. This is motivated by the fact that Γ is induced by the non-trivial symmetry of the Dynkin diagram associated to the root system of type A_2 of the Chevalley group $\mathbf{G}$.

The two automorphisms of G defined above play a crucial role throughout the investigation of G in view of the (iBAW) condition. This is due to the fact they generate the whole outer automorphism group of G:

Proposition 5.12. *For $q \not\equiv 1 \bmod 3$ the automorphism group of $G = \mathrm{SL}_3(q)$ is given by*

$$\mathrm{Aut}(G) = G \rtimes \langle \Gamma, F_p \rangle,$$

where $\langle \Gamma, F_p \rangle = \langle \Gamma \rangle \times \langle F_p \rangle \cong C_2 \times C_f$ for $f \in \mathbb{N}_{>0}$ with $q = p^f$.

Proof. Following [GLS98, Thm. 2.5.12 and p. 68] we have $\mathrm{Aut}(G) = \mathrm{PGL}_3(q) \rtimes \langle F_p, \Gamma' \rangle$ with $\langle F_p, \Gamma' \rangle = \langle \Gamma' \rangle \times \langle F_p \rangle$, where Γ' is defined as

$$\Gamma'\colon G \longrightarrow G, \quad A \longmapsto J\Gamma(A)J^{-1}$$

with $J := \left(\begin{smallmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{smallmatrix}\right)$. Now, since $q \not\equiv 1 \bmod 3$, we have

$$\mathrm{GL}_3(q) = \mathrm{SL}_3(q) \times \mathrm{Z}(\mathrm{GL}_3(q)),$$

so $\mathrm{PGL}_3(q)$ and G may be identified here. Moreover, the matrix J has determinant 1 and is, thus, contained in G. Consequently, conjugation by J defines an inner automorphism of G, and we may replace Γ' by Γ in $\mathrm{Aut}(G) = G \rtimes \langle F_p, \Gamma' \rangle$, which yields the proposed statement, where we clearly have $\langle \Gamma, F_p \rangle = \langle \Gamma \rangle \times \langle F_p \rangle \cong C_2 \times C_f$ as claimed. $\square$

Chapter 6

Blocks and Decomposition Numbers of $\mathrm{SL}_3(q)$

In this chapter we describe the ℓ-blocks of G and the associated ℓ-decomposition numbers for all primes $\ell \neq p$ dividing the order of G such that the Sylow ℓ-subgroups of G are non-cyclic, that is to say, for all primes ℓ such that $\ell \mid (q-1)$ following Proposition 5.9. We start by describing the distribution of the ordinary irreducible characters of G into ℓ-blocks. Afterwards we examine their reductions modulo ℓ, which will yield the ℓ-decomposition numbers and hence a description of the Brauer characters of G. Most of the results presented in this chapter are already known in principle. However, we often rely on rather explicit information, whence this chapter is aimed at providing details concerning ℓ-blocks and Brauer characters of G in a manner appropriate for our purposes.

6.1 Conjugacy Classes and the Complex Character Table

In order to describe the ℓ-blocks and ℓ-modular characters of G we employ information on the G-conjugacy classes and the complex irreducible characters of G.

6.1.1 Conjugacy Classes

Let us first describe the conjugacy classes of G. We use the following notation for certain roots of unity in $\mathbb{F}^\times$, which shall be fixed from now until the examination of the special linear groups with regard to the (iBAW) condition is completed:

Notation 6.1. Henceforth, let ζ, ξ and τ denote roots of unity in $\mathbb{F}^\times$ subject to the following conditions:

$$\begin{aligned} \zeta \in \mathbb{F}^\times &\text{ is of order } \Phi_1(q) = q-1, \\ \xi \in \mathbb{F}^\times &\text{ is of order } \Phi_1(q)\Phi_2(q) = q^2-1 \text{ such that } \xi^{q+1} = \zeta, \\ \text{and } \tau \in \mathbb{F}^\times &\text{ is of order } \Phi_3(q) = q^2+q+1. \end{aligned}$$

In particular, we have $\mathbb{F}_q^\times = \langle \zeta \rangle$ and $\mathbb{F}_{q^2}^\times = \langle \xi \rangle$, and τ generates the unique subgroup of $\mathbb{F}_{q^3}^\times$ of order q^2+q+1.

In Table 6.1 below, which is taken from [FS73, Table 1a], we provide a description of all conjugacy classes of G. There are eight distinct *types* of G-conjugacy classes, namely types $C_1, \dots, C_8$, which will in the following be described by representatives in Jordan normal form (possibly not lying in G but only in the infinite algebraic group $\mathbf{G}$). For a class type C indexed by parameters the notation

$$C^{(a)} = C^{(a')} \bmod x$$

for $x \in \mathbb{Z}_{>1}$, $0 \leqslant a < x$, and $a' \in \mathbb{Z}$ indicates that $C^{(a)} = C^{(a'')}$, where $0 \leqslant a'' < x$ is such that $a' \equiv a'' \bmod x$.

Class	Representative	Parameters	Centralizer order
C_1	$\begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix}$	-	$\lvert G\rvert$
C_2	$\begin{bmatrix} 1 & & \\ 1 & 1 & \\ & & 1 \end{bmatrix}$	-	$q^3\Phi_1(q)$
C_3	$\begin{bmatrix} 1 & & \\ 1 & 1 & \\ & 1 & 1 \end{bmatrix}$	-	q^2
$C_4^{(a)}$	$\begin{bmatrix} \zeta^a & & \\ & \zeta^a & \\ & & \zeta^{-2a} \end{bmatrix}$	$1 \leqslant a < \Phi_1(q)$	$q\Phi_1(q)^2\Phi_2(q)$
$C_5^{(a)}$	$\begin{bmatrix} \zeta^a & & \\ 1 & \zeta^a & \\ & & \zeta^{-2a} \end{bmatrix}$	$1 \leqslant a < \Phi_1(q)$	$q\Phi_1(q)$
$C_6^{(a,b)}$	$\begin{bmatrix} \zeta^a & & \\ & \zeta^b & \\ & & \zeta^{-a-b} \end{bmatrix}$	$1 \leqslant a < b < \Phi_1(q)$: $\exists\, b < c \leqslant \Phi_1(q)$ with $a+b+c \equiv 0 \bmod \Phi_1(q)$	$\Phi_1(q)^2$
$C_7^{(a)}$	$\begin{bmatrix} \zeta^a & & \\ & \xi^{-a} & \\ & & \xi^{-qa} \end{bmatrix}$	$1 \leqslant a < \Phi_1(q)\Phi_2(q)$: $a \not\equiv 0 \bmod \Phi_2(q)$, $C^{(a)} = C^{(aq)} \bmod \Phi_1(q)\Phi_2(q)$	$\Phi_1(q)\Phi_2(q)$
$C_8^{(a)}$	$\begin{bmatrix} \tau^a & & \\ & \tau^{aq} & \\ & & \tau^{aq^2} \end{bmatrix}$	$1 \leqslant a < \Phi_3(q)$, $C^{(a)} = C^{(aq)} = C^{(aq^2)} \bmod \Phi_3(q)$	$\Phi_3(q)$

Table 6.1: Conjugacy classes of $\mathrm{SL}_3(q)$, $q \not\equiv 1 \bmod 3$

Remark 6.2. The elements of G belonging to conjugacy classes of type C_6 are exactly those matrices that are G-conjugate to a diagonal matrix with pairwise distinct diagonal entries in $\mathbb{F}_q^\times$. In particular, these classes may also be parametrized by tuples (a, b), where $a, b \in \{1, \ldots, q-1\}$ such that $a \neq b$ and $a, b \not\equiv -(a+b) \bmod q-1$. Two conjugacy classes of type C_6 labelled by (a, b) and (a', b') then coincide if and only if (a', b') is one of (a, b), (b, a), $(a, -(a+b))$, $(-(a+b), a)$, $(b, -(a+b))$, $(-(a+b), b)$, where the components are regarded modulo $q-1$.

For later use we determine the defect groups of the conjugacy classes of G for all primes ℓ dividing $q-1$. Recall that an ℓ-group D is called a *defect group* of a conjugacy class C of G if $D \in \mathrm{Syl}_\ell(\mathrm{C}_G(x))$ for some $x \in C$.

Proposition 6.3. *Let ℓ be a prime such that $\ell \mid (q-1)$. Moreover, let C be a G-conjugacy class of G and D_C a defect group of C with respect to ℓ. Then up to G-conjugation the following statements hold:*

(i) If $C = C_1$, then $D_C \in \mathrm{Syl}_\ell(G)$;

(ii) if $C = C_2$, then $D_C = \mathcal{O}_\ell(\langle \mathrm{diag}(\zeta, \zeta, \zeta^{-2})\rangle)$;

(iii) if $C = C_3$, then $D_C = \{1\}$;

(iv) if C is of type C_4, then $D_C = \left\langle \left(\begin{smallmatrix} A & \\ & d \end{smallmatrix}\right) \;\middle|\; A \in D,\ d = (\det A)^{-1} \right\rangle$, $D \in \mathrm{Syl}_\ell(\mathrm{GL}_2(q))$;

(v) if C is of type C_5, then $D_C = \mathcal{O}_\ell(\langle \mathrm{diag}(\zeta, \zeta, \zeta^{-2})\rangle)$;

(vi) if C is of type C_6, then $D_C = \mathcal{O}_\ell(T)$ for a maximal torus T of G of type T_1;

(vii) if C is of type C_7, then $D_C = \mathcal{O}_\ell(T')$ for a maximal torus T' of G of type T_2;

(viii) if C is of type C_8, then $D_C = \{1\}$.

Proof. Statement (i) is clear. For (ii) observe that by Table 6.1 the centralizer order of an element in C_2 is given by $q^3(q-1)$, and the group $\langle \mathrm{diag}(\zeta, \zeta, \zeta^{-2})\rangle$ of order $q-1$ centralizes the representative of C_2 given in Table 6.1. Since $(q^3(q-1))_\ell = (q-1)_\ell$, (ii) follows.

The centralizer order of elements in C_3 is given by q^2, so the corresponding Sylow ℓ-subgroups are trivial for $\ell \neq p$. Hence, statement (iii) follows.

For (iv) we observe that the group $\left\langle \left(\begin{smallmatrix} A & \\ & d \end{smallmatrix}\right) \;\middle|\; A \in \mathrm{GL}_2(q),\ d = (\det A)^{-1} \right\rangle$, which is of order $|\mathrm{GL}_2(q)| = q(q-1)^2(q+1)$, centralizes the representatives of the conjugacy classes of type C_4 given in Table 6.1. Since the centralizer order of elements in classes of type C_4 is exactly $q(q-1)^2(q+1)$, the claim follows.

The group $\langle \mathrm{diag}(\zeta, \zeta, \zeta^{-2})\rangle$ of order $q-1$ centralizes the representatives of conjugacy classes of type C_5 given in Table 6.1. As the centralizer of elements in classes of type C_5 has order $q(q-1)$ by Table 6.1, part (v) follows.

For (vi), (vii) and (viii) we observe that the maximally split torus $\mathbf{T}$ of the algebraic group $\mathbf{G}$ is the centralizer in $\mathbf{G}$ of the representatives of the classes of types C_6, C_7 and C_8 given in Table 6.1, which lie in $G = \mathbf{G}^F$, $\mathbf{G}^{w_2F}$ and $\mathbf{G}^{w_3F}$, respectively. Taking fixed points under F, w_2F and w_3F, respectively, in conjunction with Corollary 4.35 yields the claim. □

6.1.2 Character Table

The complex character table of $\mathrm{SL}_3(q)$ is provided by Frame–Simpson in [FS73, Table 1b]. We should note, however, that this table contains some misprints. These can, for instance, be found and emended on comparison with the generic character table of $\mathrm{SL}_3(q)$ provided by Chevie ([GHL+96]). Furthermore, let us mention that one inconsistency in Chevie was found and corrected by our calculations (cf. Remark A.1). We present the complete complex character table of G in Appendix A, Table A.1.

6.2 Blocks and Decomposition Matrices

In this section we determine the ℓ-blocks and corresponding ℓ-decomposition matrices of G for primes ℓ dividing $q-1$. In principle, these are already known by work of Fong–Srinivasan [FS82], Dipper–James [DJ86, DJ89], and James [Jam90], but since we rely on detailed information on the ℓ-blocks and the ℓ-modular characters of G in the further course of this thesis, we present some explicit results here.

In a first step we determine how the irreducible ordinary characters of G divide into ℓ-blocks. This will be accomplished by comparison of the central characters attached to the irreducible characters of G. As we recalled in Section 1.1, two such characters belong to the same ℓ-block if and only if the reductions modulo ℓ of their corresponding central characters agree. It will hence be our aim to decide when two central characters of G agree modulo ℓ. Below we state some tools that will prove useful for this purpose. From [Nav98, Thm. 9.12] we obtain the following statement:

Proposition 6.4. *Let H be a finite group, ℓ a prime and $P \leqslant H$ an ℓ-subgroup such that $H = P\,\mathrm{C}_H(P)$. If $B \in \mathrm{Bl}_\ell(H \mid P)$, then there exists a unique $\vartheta \in \mathrm{Irr}(B)$ such that $P \subseteq \ker(\vartheta)$. Furthermore, it holds that $\mathrm{IBr}(B) = \{\vartheta^0\}$. For $\lambda \in \mathrm{Irr}(P)$ we define*

$$\vartheta_\lambda(h) := \begin{cases} \lambda(h_\ell)\vartheta(h_{\ell'}) & \text{for } h \in H \text{ with } h_\ell \in P, \\ 0 & \text{for } h \in H \text{ with } h_\ell \notin P. \end{cases}$$

Then the map

$$\mathrm{Irr}(P) \longrightarrow \mathrm{Irr}(B), \quad \lambda \longmapsto \vartheta_\lambda,$$

is a bijection.

Lemma 6.5. *Let H be a finite group, ℓ a prime and $P \leqslant H$ an ℓ-subgroup. Suppose, moreover, that $N \trianglelefteq H$ is a normal subgroup of H such that $\mathrm{C}_H(P) \subseteq N$. If $\chi \in \mathrm{Irr}(H)$ is such that $\mathrm{bl}(\chi)$ has defect group P and $\psi \in \mathrm{Irr}(N)$ is an irreducible constituent of $\chi_{|N}$, then $\mathrm{bl}(\psi)^H$ is defined and equals $\mathrm{bl}(\chi)$.*

Proof. Let us set $B := \mathrm{bl}(\chi)$. Since B has defect group P with $\mathrm{C}_H(P) \subseteq N \trianglelefteq H$, from [Nav98, Lemma 9.20] it follows that this ℓ-block has the property of being *regular with respect to N*. Moreover, as $\psi \in \mathrm{Irr}(N)$ is an irreducible constituent of $\chi_{|N}$ with $\chi \in \mathrm{Irr}(B)$, the ℓ-block B covers $\mathrm{bl}(\psi)$, whence by [Nav98, Thm. 9.19] the regularity of B with respect to N implies that $\mathrm{bl}(\psi)^H$ is defined and satisfies $\mathrm{bl}(\psi)^H = B = \mathrm{bl}(\chi)$. $\square$

The above two statements allow us to prove the following lemma regarding the question of when two blocks of the normalizer of the maximally split torus of $G = \mathrm{SL}_3(q)$ agree:

Lemma 6.6. *Suppose that $\ell \mid (q-1)$ and let T be the maximally split torus of G consisting of the diagonal matrices in G. Moreover, let $\theta, \theta' \in \mathrm{Irr}(T)$ and $\eta, \eta' \in \mathrm{Irr}(\mathrm{N}_G(T))$ be such that η and η' lie above θ and θ', respectively. Then it holds that*

$$\mathrm{bl}(\eta) = \mathrm{bl}(\eta') \quad \text{if and only if} \quad \theta_{|\mathcal{O}_{\ell'}(T)} \text{ and } \theta'_{|\mathcal{O}_{\ell'}(T)} \text{ are } \mathrm{N}_G(T)\text{-conjugate},$$

provided that both $\mathrm{bl}(\eta)$ and $\mathrm{bl}(\eta')$ have defect group $\mathcal{O}_\ell(T)$.

Proof. Let us suppose that both $\mathrm{bl}(\eta)$ and $\mathrm{bl}(\eta')$ have defect group $\mathcal{O}_\ell(T)$. By assumption η lies above θ, and moreover $\mathcal{O}_\ell(T)$ is an ℓ-subgroup of $\mathrm{N}_G(T)$ with

$$\mathrm{C}_{\mathrm{N}_G(T)}(\mathcal{O}_\ell(T)) = T$$

as seen in Proposition 5.8. Hence, we may apply Lemma 6.5 for $H := \mathrm{N}_G(T)$, $N := T$, $P := \mathcal{O}_\ell(T)$, $\chi := \eta$ and $\psi := \theta$ to find that $\mathrm{bl}(\theta)^{\mathrm{N}_G(T)} = \mathrm{bl}(\eta)$. Analogously, we further obtain that $\mathrm{bl}(\theta')$ induces to $\mathrm{bl}(\eta')$.

Let us now suppose that $\mathrm{bl}(\eta) = \mathrm{bl}(\eta')$. Then we have $\mathrm{bl}(\theta)^{\mathrm{N}_G(T)} = \mathrm{bl}(\theta')^{\mathrm{N}_G(T)}$, whence by the extended first main theorem of Brauer, Theorem 1.7, there exists an element $n \in \mathrm{N}_G(T)$ such that $\mathrm{bl}(\theta') = \mathrm{bl}(\theta)^n$. In particular, there is a character $\psi \in \mathrm{Irr}(\mathrm{bl}(\theta))$ such that $\theta' = \psi^n$. As T is abelian, all complex irreducible characters of T are linear and hence we have $\nu^0 = \theta^0$ for all $\nu \in \mathrm{Irr}(\mathrm{bl}(\theta))$ and $\nu'^0 = \theta'^0$ for all $\nu' \in \mathrm{Irr}(\mathrm{bl}(\theta'))$. We conclude that

$$\theta'_{|\mathcal{O}_{\ell'}(T)} = \theta'^0 = (\psi^n)^0 = (\psi^0)^n = (\theta^0)^n = (\theta_{|\mathcal{O}_{\ell'}(T)})^n,$$

so $\theta'_{|\mathcal{O}_{\ell'}(T)}$ and $\theta_{|\mathcal{O}_{\ell'}(T)}$ are $\mathrm{N}_G(T)$-conjugate.

In order to prove the converse implication let us now assume the existence of an element $n \in \mathrm{N}_G(T)$ satisfying

$$\theta'_{|\mathcal{O}_{\ell'}(T)} = (\theta_{|\mathcal{O}_{\ell'}(T)})^n.$$

Let the characters $\vartheta \in \mathrm{Irr}(\mathrm{bl}(\theta))$ and $\vartheta' \in \mathrm{Irr}(\mathrm{bl}(\theta'))$ be defined as in Proposition 6.4 for $H := T$ and $P := \mathcal{O}_\ell(T)$, that is, $\mathcal{O}_\ell(T)$ lies in the kernels of both ϑ and ϑ', and let $\lambda, \lambda' \in \mathrm{Irr}(\mathcal{O}_\ell(T))$ be such that $\theta = \vartheta_\lambda$ and $\theta' = \vartheta'_{\lambda'}$ in the notation of Proposition 6.4. By construction it follows that

$$\vartheta'_{|\mathcal{O}_{\ell'}(T)} = (\vartheta'_{\lambda'})_{|\mathcal{O}_{\ell'}(T)} = \theta'_{|\mathcal{O}_{\ell'}(T)} = (\theta_{|\mathcal{O}_{\ell'}(T)})^n = ((\vartheta_\lambda)_{|\mathcal{O}_{\ell'}(T)})^n = (\vartheta_{|\mathcal{O}_{\ell'}(T)})^n.$$

But $\vartheta^n(x) = \vartheta(nxn^{-1}) = 1$ for all $x \in \mathcal{O}_\ell(T)$ since $\mathrm{N}_G(T)$ normalizes $\mathcal{O}_\ell(T)$, which lies in the kernel of ϑ. Hence, we conclude that

$$\vartheta^n(g) = \vartheta^n(g_\ell \cdot g_{\ell'}) = \vartheta^n(g_{\ell'}) = \vartheta'(g_{\ell'})$$

for all $g \in T$ and thus

$$\vartheta^n = \vartheta'_{1_{\mathcal{O}_\ell(T)}} \in \mathrm{Irr}(\mathrm{bl}(\theta'))$$

again in the notation of Proposition 6.4. By uniqueness of ϑ' with respect to the property that $\vartheta' \in \mathrm{bl}(\theta')$ and $\mathcal{O}_\ell(T) \subseteq \ker(\vartheta')$ we conclude that $\vartheta' = \vartheta^n$ since $\vartheta^n \in \mathrm{Irr}(\mathrm{bl}(\theta'))$ and $\mathcal{O}_\ell(T) \subseteq \ker(\vartheta^n)$. Thus,

$$\mathrm{bl}(\theta') = \mathrm{bl}(\vartheta') = \mathrm{bl}(\vartheta^n) = \mathrm{bl}(\vartheta)^n = \mathrm{bl}(\theta)^n.$$

Consequently, we conclude that $\mathrm{bl}(\theta')^{\mathrm{N}_G(T)} = (\mathrm{bl}(\theta)^n)^{\mathrm{N}_G(T)} = \mathrm{bl}(\theta)^{\mathrm{N}_G(T)}$, and it follows that $\mathrm{bl}(\eta) = \mathrm{bl}(\eta')$ as claimed. □

In the following we treat the cases of odd and even ℓ separately. In both situations we shall adhere to the notation specified below:

Notation 6.7. (i) For integral tuples $(u, v), (u', v') \in \mathbb{Z} \times \mathbb{Z}$ and $x \in \mathbb{Z}_{>0}$ we write $(u, v) \equiv (u', v') \bmod x$ if it holds that $u \equiv u' \bmod x$ and $v \equiv v' \bmod x$.

(ii) For a subset $M \subseteq \mathbb{Z} \times \mathbb{Z}$ and $x \in \mathbb{Z}_{>0}$ we use the notation $(u, v) \in_{\bmod x} M$ to indicate that $(u, v) \equiv (u', v') \bmod x$ for some $(u', v') \in M$.

(iii) For $u, v \in \mathbb{Z}$ we set

$$\mathcal{S}(u, v) := \{(u, v), (v, u), (-u, v - u), (v - u, -u), (u - v, -v), (-v, u - v)\}.$$

(iv) We define the set

$$\mathcal{M}(\Phi_2\Phi_3) := \{(u,v) \mid u,v \in \mathbb{Z},\ u,v,u-v,2u+v,2v+u \not\equiv 0 \bmod q-1\}.$$

As stated in Remark A.2 the irreducible characters of G of type $\chi_{\Phi_2\Phi_3}$ may then be parametrized by the elements in $\mathcal{M}(\Phi_2\Phi_3)$ such that for $(u,v),(u',v') \in \mathcal{M}(\Phi_2\Phi_3)$ we have

$$\chi_{\Phi_2\Phi_3}^{(u,v)} = \chi_{\Phi_2\Phi_3}^{(u',v')}$$

if and only if $(u',v') \in_{\bmod (q-1)} \mathcal{S}(u,v)$.

6.2.1 The Case $2 \neq \ell \mid (q-1)$

The central characters associated to the irreducible characters of G can be retrieved from Chevie [GHL+96] via the 'Omega'-command (cf. Table A.2 in Appendix A). We are only interested in ℓ-blocks of non-cyclic defect, so in particular, we will not care about ℓ-blocks of ℓ-defect zero in the following. In other words, in this case we do not consider ℓ-blocks containing characters of type $\chi_{\Phi_1^2\Phi_2}$. For all other irreducible characters χ of G the ℓ-modular reduction λ_χ of the corresponding central character ω_χ is given in Table 6.2 below, where the entries are to be regarded modulo ℓ and where we use the short notation

$$\begin{aligned} A_{(v,u),a} &:= \varepsilon^{ua-2va} + \varepsilon^{va-2ua} + \varepsilon^{ua+va}, \\ B_{(v,u),(a,b)} &:= \varepsilon^{va+ub} + \varepsilon^{vb+ua} + \varepsilon^{-v(a+b)+ua} + \varepsilon^{va-u(a+b)} + \varepsilon^{-v(a+b)+ub} + \varepsilon^{vb-u(a+b)}, \end{aligned}$$

with $\varepsilon := \exp((2\pi\mathfrak{i})/(q-1))$ and $\eta := \exp((2\pi\mathfrak{i})/(q^2-1))$.

	C_1	C_2	C_3	$C_4^{(a)}$	$C_5^{(a)}$	$C_6^{(a,b)}$	$C_7^{(a)}$	$C_8^{(a)}$
λ_{χ_1}	1	·	·	3	·	6	·	·
$\lambda_{\chi_{q\Phi_2}}$	1	·	·	3	·	6	·	·
$\lambda_{\chi_{q^3}}$	1	·	·	3	·	6	·	·
$\lambda_{\chi_{\Phi_3}^{(u)}}$	1	·	·	$2\varepsilon^{ua}+\varepsilon^{-2ua}$	·	$2(\varepsilon^{ua}+\varepsilon^{ub}+\varepsilon^{-u(a+b)})$	·	·
$\lambda_{\chi_{q\Phi_3}^{(u)}}$	1	·	·	$2\varepsilon^{ua}+\varepsilon^{-2ua}$	·	$2(\varepsilon^{ua}+\varepsilon^{ub}+\varepsilon^{-u(a+b)})$	·	·
$\lambda_{\chi_{\Phi_2\Phi_3}^{(u,v)}}$	1	·	·	$A_{(u,v),a}$	·	$B_{(u,v),(a,b)}$	·	·
$\lambda_{\chi_{\Phi_1\Phi_3}^{(u)}}$	1	-2	·	ε^{ua}	$-2\varepsilon^{ua}$	·	$-(\eta^{-ua}+\eta^{-uaq})$	·

Table 6.2: Central characters of $\mathrm{SL}_3(q)$, $q \not\equiv 1 \bmod 3$, modulo $\ell \neq 2$, $\ell \mid (q-1)$

Proposition 6.8. *Let $2 \neq \ell \mid (q-1)$. The irreducible characters of* $\mathrm{SL}_3(q)$*, $q \not\equiv 1 \bmod 3$, divide into five distinct types of ℓ-blocks:*

- *the principal ℓ-block B_0:*

$$\begin{aligned}\mathrm{Irr}(B_0) = \;&\{\chi_1,\ \chi_{q\Phi_2},\ \chi_{q^3}\}\\ &\cup\{\chi_{\Phi_3}^{(u)} \mid u \equiv 0 \bmod (q-1)_{\ell'}\}\\ &\cup\{\chi_{q\Phi_3}^{(u)} \mid u \equiv 0 \bmod (q-1)_{\ell'}\}\\ &\cup\{\chi_{\Phi_2\Phi_3}^{(u,v)} \mid u,v \equiv 0 \bmod (q-1)_{\ell'}\};\end{aligned}$$

- *the ℓ-blocks of type $B_{(\text{-})}$:*

$$\begin{aligned}\mathrm{Irr}(B_{(k)}) = \;&\{\chi_{\Phi_2\Phi_3}^{(u,v)} \mid (u,v) \in_{\bmod (q-1)_{\ell'}} \{(k,0),(0,k),(-k,-k)\}\}\\ &\cup\{\chi_{\Phi_3}^{(u)} \mid u \equiv k \bmod (q-1)_{\ell'}\}\\ &\cup\{\chi_{q\Phi_3}^{(u)} \mid u \equiv k \bmod (q-1)_{\ell'}\}\end{aligned}$$

  for $k \in \{1,\dots,(q-1)_{\ell'}-1\}$;

- *the ℓ-blocks of type $B_{(\text{-},\text{-})}$:*

$$\mathrm{Irr}(B_{(u,v)}) = \{\chi_{\Phi_2\Phi_3}^{(u',v')} \mid (u',v') \in_{\bmod (q-1)_{\ell'}} \mathcal{S}(u,v)\}$$

  for $u,v \in \{1,\dots,(q-1)_{\ell'}-1\}$ such that $(u,v) \in \mathcal{M}(\Phi_2\Phi_3)$, and for further parameters $w,z \in \{1,\dots,(q-1)_{\ell'}-1\}$ such that $(w,z) \in \mathcal{M}(\Phi_2\Phi_3)$ it holds that $B_{(u,v)} = B_{(w,z)}$ if and only if $(w,z) \in_{\bmod (q-1)_{\ell'}} \mathcal{S}(u,v)$.

- *the ℓ-blocks of type $B'_{(\text{-})}$:*

$$\mathrm{Irr}(B'_{(k)}) = \{\chi_{\Phi_1\Phi_3}^{(u)} \mid (u \equiv k \bmod (q^2-1)_{\ell'}) \text{ or } (uq \equiv k \bmod (q^2-1)_{\ell'})\}$$

  for $k \in \{1,\dots,(q^2-1)_{\ell'}\}$, $k \not\equiv 0 \bmod q+1$, and for $k' \in \{1,\dots,(q^2-1)_{\ell'}\}$ with $k' \not\equiv 0 \bmod q+1$ it holds that $B'_{(k)} = B'_{(k')}$ if and only if $k \equiv k' \bmod (q^2-1)_{\ell'}$ or $kq \equiv k' \bmod (q^2-1)_{\ell'}$;

- *the ℓ-blocks of ℓ-defect zero.*

Proof. Clearly, the characters χ_1, $\chi_{q\Phi_2}$ and χ_{q^3} belong to the same ℓ-block of G since their central characters agree modulo ℓ according to Table 6.2.

Let us now suppose that two characters $\chi_{\Phi_3}^{(u)}$ and $\chi_{q\Phi_3}^{(v)}$ lie in the same ℓ-block (note that such characters only exist for $q > 2$, which is clearly satisfied here since $\ell \mid (q-1)$ and $\ell > 3$, so in fact $q > 7$). By comparison with Table 6.2 this is equivalent to the two conditions

$$2\varepsilon^{ua} + \varepsilon^{-2ua} \equiv 2\varepsilon^{va} + \varepsilon^{-2va} \bmod \ell \qquad \text{for all } a \in \{1,\dots,q-2\}, \tag{6.1}$$

$$\varepsilon^{ua} + \varepsilon^{ub} + \varepsilon^{-u(a+b)} \equiv \varepsilon^{va} + \varepsilon^{vb} + \varepsilon^{-v(a+b)} \bmod \ell \qquad \begin{array}{l}\text{for all } a,b \in \{1,\dots,q-1\}\\ \text{with } a \neq b \text{ and}\\ a,b \not\equiv -(a+b) \bmod q-1.\end{array} \tag{6.2}$$

We choose $b = q-1$, so $\varepsilon^b = 1$. Furthermore, we fix a class parameter $a \in \{1,\dots,q-1\}$ with $a \neq b$, $a,b \not\equiv -(a+b) \bmod q-1$, and set $\vartheta := \varepsilon^a$. Then from (6.2) it follows that

$(\vartheta^u - \vartheta^v)\,(\vartheta^v - \vartheta^{-u}) \equiv 0 \bmod \ell$:

$$\begin{aligned}
& & \vartheta^u + \vartheta^{-u} &\equiv \vartheta^v + \vartheta^{-v} && \bmod \ell\\
&\overset{\cdot\vartheta^v}{\Longrightarrow} & \vartheta^{u+v} + \vartheta^{v-u} &\equiv \vartheta^{2v} + 1 && \bmod \ell\\
&\Longrightarrow & \vartheta^{u+v} + \vartheta^{v-u} - \vartheta^{2v} - 1 &\equiv 0 && \bmod \ell\\
&\Longrightarrow & (\vartheta^u - \vartheta^v)\left(\vartheta^v - \vartheta^{-u}\right) &\equiv 0 && \bmod \ell.
\end{aligned}$$

If we suppose that $\vartheta^u - \vartheta^v \not\equiv 0 \bmod \ell$, then necessarily $\vartheta^v - \vartheta^{-u} \equiv 0 \bmod \ell$. Inserting this in (6.1) implies

$$\begin{aligned}
& & 2\vartheta^u + \vartheta^{-2u} &\equiv 2\vartheta^{-u} + \vartheta^{2u} && \bmod \ell\\
&\Longrightarrow & \vartheta^{2u} - 2\vartheta^u + 2\vartheta^{-u} - \vartheta^{-2u} &\equiv 0 && \bmod \ell\\
&\overset{\cdot\vartheta^{2u}}{\Longrightarrow} & \vartheta^{4u} - 2\vartheta^{3u} + 2\vartheta^u - 1 &\equiv 0 && \bmod \ell\\
&\Longrightarrow & (\vartheta^u - 1)^3\,(\vartheta^u + 1) &\equiv 0 && \bmod \ell.
\end{aligned}$$

If $\vartheta^u \equiv 1 \bmod \ell$, then also $\vartheta^v \equiv \vartheta^{-u} \equiv 1 \bmod \ell$ in contradiction to the assumption that $\vartheta^u - \vartheta^v \not\equiv 0 \bmod \ell$. But then $\vartheta^u \equiv -1 \bmod \ell$, which gives $\vartheta^u \equiv \vartheta^{-u} \equiv \vartheta^v \bmod \ell$ and again contradicts the assumption. Thus, we have $\vartheta^u \equiv \vartheta^v \bmod \ell$, so $ua \equiv va \bmod (q-1)_{\ell'}$. This needs to hold for all $a \in \{1, \dots, q-1\}$ such that $a \neq b$ and $a, b \not\equiv -(a+b) \bmod q-1$, so in particular (since $q > 7$) $a = 1$ yields $u \equiv v \bmod (q-1)_{\ell'}$. Conversely, it is straightforward to verify that $\chi_{\Phi_3}^{(u)}$ and $\chi_{q\Phi_3}^{(v)}$ lie in the same ℓ-block if $u \equiv v \bmod (q-1)_{\ell'}$. Furthermore, if $u \equiv 0 \bmod (q-1)_{\ell'}$, then $2\varepsilon^{ua} + \varepsilon^{-2ua} \equiv 3 \bmod \ell$ and $2(\varepsilon^{ua} + \varepsilon^{ub} + \varepsilon^{-u(a+b)}) \equiv 6 \bmod \ell$ for all a, b, so in this case $\chi_{\Phi_3}^{(u)}$ and $\chi_{q\Phi_3}^{(u)}$ belong to the principal ℓ-block according to Table 6.2.

Our next step is to consider the characters of type $\chi_{\Phi_1\Phi_3}$. By Table 6.2 (for instance on comparison of the values of the central characters on class C_2) these do not share ℓ-blocks with characters of different types. Hence, let us suppose that u and v are parameters such that $\chi_{\Phi_1\Phi_3}^{(u)}$ and $\chi_{\Phi_1\Phi_3}^{(v)}$ belong to the same ℓ-block of G. Then

$$\varepsilon^{ua} \equiv \varepsilon^{va} \bmod \ell \qquad \text{for all } a \in \{1, \dots, q-2\}, \tag{6.3}$$

$$\eta^{-ua} + \eta^{-uaq} \equiv \eta^{-va} + \eta^{-vaq} \bmod \ell \qquad \text{for all } a \in \{1, \dots, q^2-2\} \text{ with } a \not\equiv 0 \bmod q+1, \tag{6.4}$$

which yields $(\eta^{-va} - \eta^{-ua})\,(\eta^{-uaq} - \eta^{-va}) \equiv 0 \bmod \ell$ as follows: by multiplication of condition (6.4) with the term η^{-va} we obtain

$$\begin{aligned}
& & \eta^{-ua-va} + \eta^{-uaq-va} - \eta^{-2va} - \eta^{-va(q+1)} &\equiv 0 && \bmod \ell\\
&\overset{(6.3)}{\Longrightarrow} & \eta^{-ua-va} + \eta^{-uaq-va} - \eta^{-2va} - \eta^{-ua(q+1)} &\equiv 0 && \bmod \ell\\
&\Longrightarrow & \left(\eta^{-va} - \eta^{-ua}\right)\left(\eta^{-uaq} - \eta^{-va}\right) &\equiv 0 && \bmod \ell.
\end{aligned}$$

Hence, similarly as above we conclude that $\chi_{\Phi_1\Phi_3}^{(u)}$ and $\chi_{\Phi_1\Phi_3}^{(v)}$ belong to the same ℓ-block of G if and only if ($u \equiv v \bmod (q^2-1)_{\ell'}$) or ($uq \equiv v \bmod (q^2-1)_{\ell'}$).

It remains to examine the characters of type $\chi_{\Phi_2\Phi_3}$. To handle these we denote by T the maximally split torus of G consisting of all diagonal matrices in G. The characters of type $\chi_{\Phi_2\Phi_3}$ are of degree $\Phi_2(q)\Phi_3(q)$, which is prime to ℓ. The only irreducible characters of G with degree divisible by ℓ are the ℓ-defect zero characters and those of type $\chi_{\Phi_1\Phi_3}$.

As observed before these do not share blocks with characters of different types. Hence, the characters of type $\chi_{\Phi_2\Phi_3}$ belong to ℓ-blocks of maximal ℓ-defect, that is to say, ℓ-blocks with defect group given by the Sylow ℓ-subgroup $\mathcal{O}_\ell(T)$ of G. For suitable u, v we let $B_{u,v}$ denote the ℓ-block of a character $\chi_{\Phi_2\Phi_3}^{(u,v)}$ and $b_{u,v}$ its Brauer correspondent in $\mathrm{N}_G(\mathcal{O}_\ell(T)) = \mathrm{N}_G(T)$. Then by definition we have

$$\lambda_{B_{u,v}}(\widehat{K}) = \lambda_{b_{u,v}}(\widehat{K \cap \mathrm{N}_G(T)}) \quad \text{for all } K \in \mathrm{Cl}(G).$$

However, from the proof of [Nav98, Thm. 4.14] it follows that in fact it holds

$$\lambda_{B_{u,v}}(\widehat{K}) = \lambda_{b_{u,v}}(\widehat{K \cap \mathrm{N}_G(T)}) = \lambda_{b_{u,v}}(\widehat{K \cap T}) \quad \text{for all } K \in \mathrm{Cl}(G)$$

since T is the centralizer of $\mathcal{O}_\ell(T)$ in G according to Proposition 5.8. Let us now suppose that $\theta, \theta' \in \mathrm{Irr}(T)$ satisfy

$$\lambda_{B_{u,v}}(\widehat{K}) = \lambda_{b_{u,v}}(\widehat{K \cap T}) = \lambda_\theta(\widehat{K \cap T}) \qquad \text{for all } K \in \mathrm{Cl}(G), \tag{6.5}$$

$$\lambda_{B_{u',v'}}(\widehat{K}) = \lambda_{b_{u',v'}}(\widehat{K \cap T}) = \lambda_{\theta'}(\widehat{K \cap T}) \qquad \text{for all } K \in \mathrm{Cl}(G), \tag{6.6}$$

respectively. Then the ℓ-blocks $\mathrm{bl}(\theta)^{\mathrm{N}_G(T)}$ and $\mathrm{bl}(\theta)^G$ are defined according to Proposition 1.2, whence $(\mathrm{bl}(\theta)^{\mathrm{N}_G(T)})^G = \mathrm{bl}(\theta)^G$. Moreover, it holds that $\mathrm{bl}(\theta)^G = B_{u,v}$, so we conclude that

$$(\mathrm{bl}(\theta)^{\mathrm{N}_G(T)})^G = B_{u,v}.$$

But $\mathrm{bl}(\theta)^{\mathrm{N}_G(T)}$ is an ℓ-block of $\mathrm{N}_G(T)$ with defect group $\mathcal{O}_\ell(T)$ following Lemma 1.1 and Lemma 1.13, so it is in fact the Brauer correspondent of $B_{u,v}$ in $\mathrm{N}_G(T)$, that is to say,

$$\mathrm{bl}(\theta)^{\mathrm{N}_G(T)} = b_{u,v}.$$

Hence, by [Nav98, Thm. 9.4 and Lemma 9.8] there is some irreducible character in $b_{u,v}$ lying over θ. Moreover, analogous statements hold for $b_{u',v'}$ and θ'. Then by Lemma 6.6 we have $b_{u,v} = b_{u',v'}$ if and only if $\theta_{|\mathcal{O}_{\ell'}(T)}$ and $\theta'_{|\mathcal{O}_{\ell'}(T)}$ are $\mathrm{N}_G(T)$-conjugate. Hence, by Brauer's first main theorem, Theorem 1.5, we have $B_{u,v} = B_{u',v'}$ if and only if the characters $\theta_{|\mathcal{O}_{\ell'}(T)}$ and $\theta'_{|\mathcal{O}_{\ell'}(T)}$ are $\mathrm{N}_G(T)$-conjugate.

Let ν be the faithful irreducible character of $\mathbb{F}_q^\times = \langle \zeta \rangle$ mapping ζ to ε. For given u, v we define

$$\theta_{u,v}\colon\ T \longrightarrow \mathbb{C}^\times, \quad \mathrm{diag}(t_1, t_2, (t_1t_2)^{-1}) \longmapsto \nu^u(t_1)\nu^v(t_2),$$

and claim that $\theta_{u,v}$ and $\theta_{u',v'}$ satisfy (6.5) and (6.6), respectively. Since for G-conjugacy classes K not of type C_1, C_4 or C_6 we have $K \cap T = \emptyset$ and

$$\lambda_{B_{(u,v)}}(\widehat{K}) = \lambda_{B_{(u',v')}}(\widehat{K}) = 0$$

according to Table 6.2, it remains to consider the G-conjugacy classes $C_4^{(a)}$ and $C_6^{(a,b)}$. The class $C_4^{(a)}$ consists of all matrices in G with Jordan normal form given by $\mathrm{diag}(\zeta^a, \zeta^a, \zeta^{-2a})$, so

$$C_4^{(a)} \cap T = \{\mathrm{diag}(\zeta^a, \zeta^a, \zeta^{-2a}),\ \mathrm{diag}(\zeta^{-2a}, \zeta^a, \zeta^a),\ \mathrm{diag}(\zeta^a, \zeta^{-2a}, \zeta^a)\},$$

and these three elements each constitute a single conjugacy class in T. Thus, it follows that

$$\begin{aligned}\lambda_{\theta_{u,v}}(\widehat{C_4^{(a)}\cap T}) &= \lambda_{\theta_{u,v}}(\mathrm{diag}(\zeta^a,\zeta^a,\zeta^{-2a})) + \lambda_{\theta_{u,v}}(\mathrm{diag}(\zeta^{-2a},\zeta^a,\zeta^a)) \\ &\quad + \lambda_{\theta_{u,v}}(\mathrm{diag}(\zeta^a,\zeta^{-2a},\zeta^a)) \\ &= \big(\theta_{u,v}(\mathrm{diag}(\zeta^a,\zeta^a,\zeta^{-2a})) + \theta_{u,v}(\mathrm{diag}(\zeta^{-2a},\zeta^a,\zeta^a)) \\ &\quad +\theta_{u,v}(\mathrm{diag}(\zeta^a,\zeta^{-2a},\zeta^a))\big)^* \\ &= \big(\varepsilon^{ua+va} + \varepsilon^{-2ua+va} + \varepsilon^{ua-2va}\big)^* \\ &= \lambda_{\chi^{(u,v)}_{\Phi_2\Phi_3}}(\widehat{C_4^{(a)}}).\end{aligned}$$

Similarly, one shows that

$$\lambda_{\theta_{u,v}}(\widehat{C_6^{(a,b)}\cap T}) = \lambda_{\chi^{(u,v)}_{\Phi_2\Phi_3}}(\widehat{C_6^{(a,b)}}),$$

so $\theta_{u,v}$ and $\theta_{u',v'}$ are as claimed. The question now is as to when $\theta_{u,v|\mathcal{O}_{\ell'}(T)}$ and $\theta_{u',v'|\mathcal{O}_{\ell'}(T)}$ are $\mathrm{N}_G(T)$-conjugate. We may write

$$\mathcal{O}_{\ell'}(T) = \{\mathrm{diag}(\zeta^{a(q-1)_\ell},\zeta^{b(q-1)_\ell},\zeta^{-(a+b)(q-1)_\ell}) \mid 1 \leqslant a,b \leqslant (q-1)_{\ell'}\},$$

and we recall that $\mathrm{N}_G(T) \cong T \rtimes \mathfrak{S}_3$ by Proposition 5.7, where the action of $\mathfrak{S}_3$ on T is given by permutation of the diagonal entries of the matrices in T. Clearly, it is true that $\theta_{u,v|\mathcal{O}_{\ell'}(T)}$ and $\theta_{u',v'|\mathcal{O}_{\ell'}(T)}$ are $\mathrm{N}_G(T)$-conjugate if and only if there exists some $n \in \mathrm{N}_G(T)$ such that

$$\theta_{u,v}(\mathrm{diag}(\zeta^{a(q-1)_\ell},\zeta^{b(q-1)_\ell},\zeta^{(-a-b)(q-1)_\ell})^n) = \theta_{u',v'}(\mathrm{diag}(\zeta^{a(q-1)_\ell},\zeta^{b(q-1)_\ell},\zeta^{(-a-b)(q-1)_\ell}))$$

for all $1 \leqslant a,b \leqslant (q-1)_{\ell'}$, which in turn is the case if and only if

$$(u',v') \in_{\mathrm{mod}\ (q-1)_{\ell'}} \{(u,v),(v,u),(-u,v-u),(v-u,-u),(u-v,-v),(-v,u-v)\}.$$

Straightforward calculations show that $\chi^{(u,v)}_{\Phi_2\Phi_3}$ belongs to the principal ℓ-block if and only if $u,v \equiv 0 \bmod (q-1)_{\ell'}$. Furthermore, let us fix parameters $u,v \in \{0,\dots,(q-1)_{\ell'}-1\}$ such that $v \equiv 0 \bmod (q-1)_{\ell'}$. Then it is easy to verify that $\chi^{(u,v)}_{\Phi_2\Phi_3}$ belongs to the same ℓ-block as $\chi^{(u)}_{\Phi_3}$ and $\chi^{(u)}_{q\Phi_3}$, and a further character $\chi^{(u',v')}_{\Phi_2\Phi_3}$ belongs to this ℓ-block if and only if

$$\begin{aligned}(u',v') \in_{\mathrm{mod}\ (q-1)_{\ell'}} &\{(u,v),(v,u),(-u,v-u),(v-u,-u),(u-v,-v),(-v,u-v)\} \\ =&\{(u,0),(0,u),(-u,-u)\}.\end{aligned}$$

Finally, if $u,v \not\equiv 0 \bmod (q-1)_{\ell'}$, then the ℓ-block of $\chi^{(u,v)}_{\Phi_2\Phi_3}$ does not contain any characters of other types, i.e., all characters in this ℓ-block are of type $\chi_{\Phi_2\Phi_3}$. □

Proposition 6.9. *Let $2 \neq \ell \mid (q-1)$. The decomposition matrices for the ℓ-blocks of G not of ℓ-defect zero (cf. Proposition 6.8) are as follows:*

- *for the principal ℓ-block B_0:*

B_0		φ_1	φ_2	φ_3
1_G		1	·	·
$\chi_{q\Phi_2}$		·	1	·
χ_{q^3}		·	·	1
$\chi^{(u)}_{\Phi_3}$,	$u \equiv 0 \bmod (q-1)_{\ell'}$	1	1	·
$\chi^{(u)}_{q\Phi_3}$,	$u \equiv 0 \bmod (q-1)_{\ell'}$	·	1	1
$\chi^{(u,v)}_{\Phi_2\Phi_3}$,	$u, v \equiv 0 \bmod (q-1)_{\ell'}$	1	2	1

- *for the ℓ-blocks of type $B_{(\_)}$:*

$B_{(k)}$		$\varphi^{(k;1)}$	$\varphi^{(k;2)}$
$\chi^{(u)}_{\Phi_3}$,	$u \equiv k \bmod (q-1)_{\ell'}$	1	·
$\chi^{(u)}_{q\Phi_3}$,	$u \equiv k \bmod (q-1)_{\ell'}$	·	1
$\chi^{(u,v)}_{\Phi_2\Phi_3}$,	$(u,v) \in_{\bmod (q-1)_{\ell'}} \{(k,0),(0,k),(-k,-k)\}$	1	1

for $k \in \{1, \dots, (q-1)_{\ell'} - 1\}$;

- *for the ℓ-blocks of type $B_{(\_,\_)}$:*

$B_{(u,v)}$	$\varphi^{(u,v)}$
$\chi^{(u',v')}_{\Phi_2\Phi_3}$, $(u',v') \in_{\bmod (q-1)_{\ell'}} \mathcal{S}(u,v)$	1

for $u, v \in \{1, \dots, (q-1)_{\ell'} - 1\}$ with $(u,v) \in \mathcal{M}(\Phi_2\Phi_3)$;

- *for the ℓ-blocks of type $B'_{(\_)}$:*

$B'_{(k)}$		$\varphi^{(k)'}$
$\chi^{(u)}_{\Phi_1\Phi_3}$,	$u \equiv k \bmod (q^2-1)_{\ell'}$ *or* $uq \equiv k \bmod (q^2-1)_{\ell'}$	1

for $k \in \{1, \dots, (q^2-1)_{\ell'}\}$ with $k \not\equiv 0 \bmod q+1$.

Proof. In proving our assertion we make use of the fact that under the assumption that $q \not\equiv 1 \bmod 3$ it holds $\mathrm{GL}_3(q) = \mathrm{SL}_3(q) \times \mathrm{Z}(\mathrm{GL}_3(q))$. Let us assume for a moment that $\ell \neq p$ is an arbitrary prime dividing $|\,\mathrm{SL}_3(q)|$. By [DJ86, Thm. 6.5 and Thm. 6.6] and [DJ89, Thm. 6.1] up to permutation of rows and columns the ℓ-decomposition matrix of $\mathrm{GL}_3(q)$ contains a lower unitriangular $|\,\mathrm{IBr}_\ell(\mathrm{GL}_3(q))| \times |\,\mathrm{IBr}_\ell(\mathrm{GL}_3(q))|$-submatrix D of the form

$$D = \begin{bmatrix} \Delta(x_1) & & 0 \\ & \ddots & \\ 0 & & \Delta(x_m) \end{bmatrix},$$

where for each $i \in \{1, \dots, m\}$ the term x_i indexes a certain set of irreducible ordinary and modular characters of $\mathrm{GL}_3(q)$. If x is one of these indices, then to x one can associate an array

$$(d_1, \dots, d_N; k_1, \dots, k_N)$$

with $N \in \mathbb{N}_{>0}$ and $d_j, k_j \in \mathbb{N}_{>0}$ for all $1 \leqslant j \leqslant N$ such that $d_1k_1 + \cdots + d_Nk_N = 3$. Moreover, one has that $\Delta(x)$ is the Kronecker product

$$\Delta(x) = \Delta(d_1, k_1) \otimes \cdots \otimes \Delta(d_N, k_N)$$

of certain $p(k_i) \times p(k_i)$-matrices $\Delta(d_i, k_i)$, where $p(k_i)$ denotes the number of partitions of k_i. By [DJ89, Thm. 6.2] such a matrix $\Delta(d_i, k_i)$ coincides with the ℓ-decomposition matrix Δ_{k_i} for the unipotent characters of $\mathrm{GL}_{k_i}(q^{d_i})$. These matrices Δ_k for $k \leqslant 10$ and $d \in \mathbb{N}_{>0}$ are given in [Jam90, Appendix 1] and depend not only on k but also on ℓ and q^d, or rather, they depend on k, ℓ and $e_q(d)$, where $e_q(d)$ is the least natural number r such that ℓ divides $1 + q^d + \ldots + q^{(r-1)d}$. As noted in [DJ86, (2.1)] we have $e_q(d) = \ell$ if and only if $\ell \mid (q^d - 1)$. Now in our case it holds that $2 \neq \ell \mid (q-1)$, so in particular for any $d \in \mathbb{N}_{>0}$ we have $e_q(d) = \ell$. For given k and d by [Jam90, Thm. 6.4] the matrix Δ_k is the identity matrix of size $p(k)$ if $e_q(d) > k$. Since $2 \neq \ell \mid (q-1)$ and $q \not\equiv 1 \bmod 3$, we have $e_q(d) = \ell \geqslant 5$. On the other hand every array $(d_1, \dots, d_N; k_1, \dots, k_N)$ associated to one of the indices x_i, $1 \leqslant i \leqslant m$, is subject to the condition $d_1k_1 + \cdots + d_Nk_N = 3$. In particular, $k_j \leqslant 3 < e_q(d)$ for $1 \leqslant j \leqslant N$. Hence, any matrix $\Delta(d_j, k_j)$ occurring in this situation is the identity matrix of size $p(k_i)$, so D is the identity matrix of size $|\,\mathrm{IBr}_\ell(\mathrm{GL}_3(q))|$.

It is well-known that for a direct product $H = H_1 \times H_2$ of two finite groups H_1 and H_2 one has

$$\begin{aligned} \mathrm{Irr}(H) &= \{\chi_1 \times \chi_2 \mid \chi_i \in \mathrm{Irr}(H_i)\ i = 1, 2\}, \\ \mathrm{IBr}_\ell(H) &= \{\varphi_1 \times \varphi_2 \mid \varphi_i \in \mathrm{IBr}_\ell(H_i),\ i = 1, 2\}, \end{aligned}$$

and if B is an ℓ-block of H, then there exist unique ℓ-blocks B_1 and B_2 of H_1 and H_2, respectively, such that

$$\begin{aligned} \mathrm{Irr}(B) &= \{\chi_1 \times \chi_2 \mid \chi_i \in \mathrm{Irr}(B_i),\ i = 1, 2\}, \\ \mathrm{IBr}(B) &= \{\varphi_1 \times \varphi_2 \mid \varphi_i \in \mathrm{IBr}(B_i),\ i = 1, 2\}. \end{aligned}$$

Moreover, if D_B, D_{B_1}, and D_{B_2} denote the ℓ-decomposition matrices of B, B_1, and B_2, respectively, then $D_B = D_{B_1} \otimes D_{B_2}$.

Let now b_0 denote the principal ℓ-block of $\mathrm{Z}(\mathrm{GL}_3(q))$ and let B be any ℓ-block of $\mathrm{SL}_3(q)$. Then the decomposition matrix of b_0 is a $|\,\mathrm{Irr}(b_0)| \times 1$-matrix with all entries equal to 1 since $\mathrm{Z}(\mathrm{GL}_3(q))$ is cyclic, so $\mathrm{GL}_3(q)$ has an ℓ-block with decomposition matrix

$$D_B \otimes D_{b_0} = \left.\begin{bmatrix} D_B \\ \vdots \\ D_B \end{bmatrix}\right\} |\,\mathrm{Irr}(b_0)| \text{ times.}$$

From the observation that the matrix D above is the identity matrix we conclude that also $D_B \otimes D_{b_0}$, and hence D_B, contains an identity matrix of size $|\,\mathrm{IBr}(B)|$ as a submatrix. This yields the proposed decomposition matrices for ℓ-blocks of types $B_{(\text{-},\text{-})}$ or $B'_{(\text{-})}$.

For an ℓ-block B of type $B_{(\text{-})}$ one easily verifies that characters in B of the same type reduce modulo ℓ to the same Brauer character. The characters in B are of types χ_{Φ_3}, $\chi_{q\Phi_3}$ and $\chi_{\Phi_2\Phi_3}$. Since D_B contains a 2×2-identity matrix, the ℓ-modular reductions of the irreducible characters in B of two of these types stay irreducible. The characters in B of the third type must have both of these irreducible reductions as constituents because B is a single ℓ-block. Hence, by considering the degrees of the characters in $\mathrm{Irr}(B)$ one may conclude that the characters of types χ_{Φ_3} and $\chi_{q\Phi_3}$ stay irreducible under ℓ-modular reduction, while the reductions of characters of type $\chi_{\Phi_2\Phi_3}$ split into two distinct irreducible ℓ-modular characters as claimed.

Finally, let us consider the principal ℓ-block B_0. This contains the three unipotent characters χ_1, $\chi_{q\Phi_2}$ and χ_{q^3} of $\mathrm{SL}_3(q)$, and the characters $\chi_1 \times 1_{Z(\mathrm{GL}_3(q))}$, $\chi_{q\Phi_2} \times 1_{Z(\mathrm{GL}_3(q))}$ and $\chi_{q^3} \times 1_{Z(\mathrm{GL}_3(q))}$ are exactly the unipotent characters of $\mathrm{GL}_3(q)$. By [Jam90, p. 225] these have ℓ-decomposition matrix Δ_3 for $e_q(1) = \ell$, which is the identity matrix of size 3 as we have observed before. Hence, the head of the decomposition matrix D_{B_0} is known, and the remaining proposed decomposition numbers for B_0 may easily be verified. $\square$

Lemma 6.10. *Suppose that $2 \neq \ell \mid (q-1)$. The ℓ-blocks of G of type $B'_{(\text{-})}$ have cyclic defect group $\mathcal{O}_\ell(\langle \mathrm{diag}(\zeta, \zeta, \zeta^{-2})\rangle)$. Any other ℓ-block of G which is not of ℓ-defect zero has maximal ℓ-defect.*

Proof. By looking at the character degrees one may easily identify the ℓ-blocks of maximal defect. Let us hence consider an ℓ-block B of type $B'_{(\text{-})}$. Then λ_B does not take value 0 on $\widehat{C_2}$ according to Table 6.2. Consequently, by the Min-Max-Theorem [Nav98, Thm. 4.4] a defect group of B is contained in a defect group of the conjugacy class C_2, which is G-conjugate to $\mathcal{O}_\ell(\langle \mathrm{diag}(\zeta, \zeta, \zeta^{-2})\rangle)$ by Proposition 6.3. Moreover, this group has order $|\mathcal{O}_\ell(\langle \mathrm{diag}(\zeta, \zeta, \zeta^{-2})\rangle)| = (q-1)_\ell$ and one easily verifies that B is of ℓ-defect $\log_\ell (q-1)_\ell$. Hence, $\mathcal{O}_\ell(\langle \mathrm{diag}(\zeta, \zeta, \zeta^{-2})\rangle)$ is a defect group of B. $\square$

6.2.2 The Case $\ell = 2$

The reductions modulo 2 of the central characters of $\mathrm{SL}_3(q)$, which can be found in Table A.2 in Appendix A, are given below, where as for the case $2 \neq \ell \mid (q-1)$ only the characters not of 2-defect zero are listed. Moreover, $A_{(u,v),a}$ and $B_{(u,v),(a,b)}$ are defined as in Section 6.2.1.

	C_1	C_2	C_3	$C_4^{(a)}$	$C_5^{(a)}$	$C_6^{(a,b)}$	$C_7^{(a)}$	$C_8^{(a)}$
λ_{χ_1}	1	·	·	1	·	·	·	·
$\lambda_{\chi_{q\Phi_2}}$	1	·	·	1	·	·	·	·
$\lambda_{\chi_{q^3}}$	1	·	·	1	·	·	·	·
$\lambda_{\chi^{(u)}_{\Phi_3}}$	1	·	·	ε^{-2ua}	·	·	·	·
$\lambda_{\chi^{(u)}_{q\Phi_3}}$	1	·	·	ε^{-2ua}	·	·	·	·
$\lambda_{\chi^{(u,v)}_{\Phi_2\Phi_3}}$	1	·	·	$A_{(u,v),a}$	·	$B_{(u,v),(a,b)}$	·	·
$\lambda_{\chi^{(u)}_{\Phi_1\Phi_3}}$	1	·	·	ε^{ua}	·	·	$\eta^{-ua} + \eta^{-uaq}$	·

Table 6.3: Central characters of $\mathrm{SL}_3(q)$, $q \not\equiv 1 \bmod 3$, modulo 2

Proposition 6.11. *The irreducible characters of* $\mathrm{SL}_3(q)$, $q \not\equiv 1 \bmod 3$, *divide into five distinct types of* 2*-blocks:*

- *the principal* 2*-block* B_0*:*

$$\begin{aligned}\mathrm{Irr}(B_0) = \{&\chi_1,\ \chi_{q\Phi_2},\ \chi_{q^3}\}\\ &\cup\{\chi_{\Phi_3}^{(u)} \mid u \equiv 0 \bmod (q-1)_{2'}\}\\ &\cup\{\chi_{q\Phi_3}^{(u)} \mid u \equiv 0 \bmod (q-1)_{2'}\}\\ &\cup\{\chi_{\Phi_1\Phi_3}^{(u)} \mid u \equiv 0 \bmod (q^2-1)_{2'}\}\\ &\cup\{\chi_{\Phi_2\Phi_3}^{(u,v)} \mid u,v \equiv 0 \bmod (q-1)_{2'}\};\end{aligned}$$

- *the* 2*-blocks of type* $B_{(\_)}$*:*

$$\begin{aligned}\mathrm{Irr}(B_{(k)}) = \{&\chi_{\Phi_3}^{(u)} \mid u \equiv k \bmod (q-1)_{2'}\}\\ &\cup\{\chi_{q\Phi_3}^{(u)} \mid u \equiv k \bmod (q-1)_{2'}\}\\ &\cup\{\chi_{\Phi_2\Phi_3}^{(u,v)} \mid (u,v) \in_{\bmod (q-1)_{2'}} \{(k,0),(0,k),(-k,-k)\}\}\\ &\cup\{\chi_{\Phi_1\Phi_3}^{(u)} \mid u \equiv -(q+1)k \bmod (q^2-1)_{2'}\};\end{aligned}$$

  for $k \in \{1,\dots,(q-1)_{2'}-1\}$*;*

- *the* 2*-blocks of type* $B_{(\_,\_)}$*:*

$$\mathrm{Irr}(B_{(u,v)}) = \{\chi_{\Phi_2\Phi_3}^{(u',v')} \mid (u',v') \in_{\bmod (q-1)_{2'}} \mathcal{S}(u,v)\}$$

  for $u,v \in \{1,\dots,(q-1)_{2'}-1\}$ *such that* $(u,v) \in \mathcal{M}(\Phi_2\Phi_3)$*, and for further parameters* $w,z \in \{1,\dots,(q-1)_{2'}-1\}$ *such that* $(w,z) \in \mathcal{M}(\Phi_2\Phi_3)$ *it holds that* $B_{(u,v)} = B_{(w,z)}$ *if and only if* $(w,z) \in_{\bmod (q-1)_{2'}} \mathcal{S}(u,v)$*.*

- *the* 2*-blocks of type* $B'_{(\_)}$*:*

$$\mathrm{Irr}(B'_{(k)}) = \{\chi_{\Phi_1\Phi_3}^{(u)} \mid (u \equiv k \bmod (q^2-1)_{2'}) \text{ or } (uq \equiv k \bmod (q^2-1)_{2'})\}$$

  for $k \in \{1,\dots,(q^2-1)_{2'}-1\}$, $k \not\equiv 0 \bmod (q+1)_{2'}$*, and for* $k' \in \{1,\dots,(q^2-1)_{2'}-1\}$ *with* $k' \not\equiv 0 \bmod (q+1)_{2'}$ *it holds that* $B'_{(k)} = B'_{(k')}$ *if and only if* $k \equiv k' \bmod (q^2-1)_{2'}$ *or* $kq \equiv k' \bmod (q^2-1)_{2'}$*;*

- *the* 2*-blocks of* 2*-defect zero.*

Proof. From Table 6.3 we can directly read off that χ_1, $\chi_{q\Phi_2}$ and χ_{q^3} belong to the principal 2-block B_0. Moreover, we observe that for any fixed parameter u the characters $\chi_{\Phi_3}^{(u)}$ and $\chi_{q\Phi_3}^{(u)}$ belong to the same 2-block. Suppose now that two parameters u and v are given. Then $\chi_{\Phi_3}^{(u)}$ and $\chi_{q\Phi_3}^{(v)}$ lie in the same 2-block if and only if $\varepsilon^{-2ua} \equiv \varepsilon^{-2va} \bmod 2$ for all $a \in \{1,\dots,q-2\}$, or, equivalently, if and only if $-2ua \equiv -2va \bmod (q-1)_{2'}$ for all those a. Thus, since $(q-1)_{2'}$ is odd, we conclude that the characters in question belong to the same 2-block exactly if $u \equiv v \bmod (q-1)_{2'}$. For $u \equiv 0 \bmod (q-1)_{2'}$ the corresponding characters of types χ_{Φ_3} and $\chi_{q\Phi_3}$ belong to the principal 2-block B_0.

Let us now examine the characters of type $\chi_{\Phi_1\Phi_3}$. According to Table 6.3 it holds that $\chi_{\Phi_1\Phi_3}^{(u)}$ and $\chi_{\Phi_1\Phi_3}^{(v)}$ belong to the same 2-block if and only if the two conditions

$$\begin{aligned}
&(6.7) && \varepsilon^{ua} \equiv \varepsilon^{va} \bmod 2 && \text{for all } a \in \{1, \dots, q-2\},\\
&(6.8) && \eta^{-ua} + \eta^{-uaq} \equiv \eta^{-va} + \eta^{-vaq} \bmod 2 && \text{for all } a \in \{1, \dots, q^2-2\},\ a \not\equiv 0 \bmod q+1,
\end{aligned}$$

are fulfilled. Suppose, hence, that conditions (6.7) and (6.8) are satisfied and set $\vartheta := \eta^a$. Then ($\vartheta^{-u} \equiv \vartheta^{-vq} \bmod 2$) or ($\vartheta^{-u} \equiv \vartheta^{-v} \bmod 2$) since

$$\begin{aligned}
0 &\overset{(6.8)}{\equiv} \vartheta^{-u}\left(\vartheta^{-u} + \vartheta^{-uq} - \vartheta^{-v} - \vartheta^{-vq}\right)\\
&\equiv \vartheta^{-2u} - \vartheta^{-u-v} - \vartheta^{-vq-u} + \vartheta^{-u(q+1)}\\
&\overset{(6.7)}{\equiv} \vartheta^{-2u} - \vartheta^{-u-v} - \vartheta^{-vq-u} + \vartheta^{-v(q+1)}\\
&\equiv \left(\vartheta^{-u} - \vartheta^{-vq}\right)\left(\vartheta^{-u} - \vartheta^{-v}\right) \bmod 2.
\end{aligned}$$

We conclude that in this case ($u \equiv v \bmod (q^2-1)_{2'}$) or ($u \equiv vq \bmod (q^2-1)_{2'}$) because the above conditions need to hold true for all possible choices of a in $\vartheta = \eta^a$. Conversely, one easily verifies that $\chi_{\Phi_1\Phi_3}^{(u)}$ and $\chi_{\Phi_1\Phi_3}^{(v)}$ belong to the same block if ($u \equiv v \bmod (q^2-1)_{2'}$) or ($u \equiv vq \bmod (q^2-1)_{2'}$).

From Table 6.3 we deduce that a character $\chi_{\Phi_1\Phi_3}^{(u)}$ lies in the principal 2-block if and only if its parameter u satisfies

$$\begin{aligned}
&(6.9) && \varepsilon^{ua} \equiv 1 \bmod 2 && \text{for all } a \in \{1, \dots, q-2\},\\
&(6.10) && \eta^{-ua} + \eta^{-uaq} \equiv 0 \bmod 2 && \text{for all } a \in \{1, \dots, q^2-2\},\ a \not\equiv 0 \bmod q+1.
\end{aligned}$$

It is easy to see that (6.9) is satisfied exactly if $u \equiv 0 \bmod (q-1)_{2'}$. Now for (6.10) we observe that $\eta^{-ua} + \eta^{-uaq} \equiv 0 \bmod 2$ if and only if $ua(q-1) \equiv 0 \bmod (q^2-1)_{2'}$, so $ua \equiv 0 \bmod (q+1)_{2'}$, and since $\gcd((q-1)_{2'}, (q+1)_{2'}) = 1$ we conclude that (6.9) and (6.10) are satisfied if and only if $u \equiv 0 \bmod (q^2-1)_{2'}$.

Now suppose that two characters $\chi_{\Phi_1\Phi_3}^{(u)}$ and $\chi_{\Phi_3}^{(v)}$ (or $\chi_{q\Phi_3}^{(v)}$) lie in the same 2-block. Then by Table 6.3 we have

$$\eta^{-ua} + \eta^{-uaq} \equiv 0 \bmod 2 \text{ for all } a \in \{1, \dots, q^2-2\},\ a \not\equiv 0 \bmod q+1,$$

which by the previous observations means nothing but $u \equiv 0 \bmod (q+1)_{2'}$. Furthermore, we have $\varepsilon^{ua} \equiv \varepsilon^{-2va}$ for all $a \in \{1, \dots, q-2\}$, or, equivalently, $ua + 2va \equiv 0 \bmod (q-1)_{2'}$ for all such a, which gives $u \equiv -2v \equiv -(q+1)v \bmod (q-1)_{2'}$. But due to the facts that $-(q+1)v \equiv 0 \equiv u \bmod (q+1)_{2'}$ and $\gcd((q-1)_{2'}, (q+1)_{2'}) = 1$, we may conclude that $u \equiv -(q+1)v \bmod (q^2-1)_{2'}$.

We are left to determine the distribution of characters of type $\chi_{\Phi_2\Phi_3}$ into 2-blocks. Let a character of type $\chi_{\Phi_2\Phi_3}$ be given. If its parameters u, v satisfy $u \equiv v \equiv 0 \bmod (q-1)_{2'}$, then one easily verifies that this character belongs to the principal 2-block. Now suppose that $u \equiv 0 \bmod (q-1)_{2'}$ but $v \not\equiv 0 \bmod (q-1)_{2'}$. Then

$$A_{(u,v),a} = \varepsilon^{ua-2va} + \varepsilon^{va-2ua} + \varepsilon^{ua+va} \equiv \varepsilon^{-2va} + \varepsilon^{va} + \varepsilon^{va} \equiv \varepsilon^{-2va} \bmod 2$$

for all $a \in \{1, \dots, q-2\}$ and

$$\begin{aligned} B_{(u,v),(a,b)} &= \varepsilon^{ua+vb} + \varepsilon^{ub+va} + \varepsilon^{-ua-ub+va} + \varepsilon^{ua-va-vb} + \varepsilon^{-ua-ub+vb} + \varepsilon^{ub-va-vb} \\ &\equiv \varepsilon^{vb} + \varepsilon^{va} + \varepsilon^{va} + \varepsilon^{-va-vb} + \varepsilon^{vb} + \varepsilon^{-va-vb} \\ &\equiv 0 \bmod 2 \end{aligned}$$

for all $a, b \in \{1, \ldots, q-1\}$ with $a \neq b$ and $a, b \not\equiv -(a+b) \bmod q-1$. Note that the second condition is only relevant if $q > 3$. From Table 6.3 we deduce that the character $\chi_{\Phi_2\Phi_3}^{(u,v)}$ is contained in the same block as the characters $\chi_{\Phi_3}^{(w)}$ and $\chi_{q\Phi_3}^{(w)}$, where the parameter w is such that $w \equiv v \bmod (q-1)_{2'}$. Moreover, the same holds true if either $u \equiv w \not\equiv 0 \bmod (q-1)_{2'}$ and $v \equiv 0 \bmod (q-1)_{2'}$ or if $u \equiv v \equiv -w \bmod (q-1)_{2'}$ since

$$\chi_{\Phi_2\Phi_3}^{(u,v)} = \chi_{\Phi_2\Phi_3}^{(v,u)} = \chi_{\Phi_2\Phi_3}^{(v-u,-u)}.$$

Let us now suppose that we neither have $u \equiv 0 \bmod (q-1)_{2'}$ nor $v \equiv 0 \bmod (q-1)_{2'}$ nor $u \equiv v \bmod (q-1)_{2'}$. In our next step we determine the defect group D of the 2-block containing $\chi_{\Phi_2\Phi_3}^{(u,v)}$. For this we fix the class parameters $a = q-2$ and $b = 1$ for a G-conjugacy class of type C_6. It follows that

$$\begin{aligned} \omega_{\chi_{\Phi_2\Phi_3}^{(u,v)}}\left(\widehat{C_6^{(a,b)}}\right) &= \varepsilon^{ua+vb} + \varepsilon^{ub+va} + \varepsilon^{-ua-ub+va} + \varepsilon^{ua-va-vb} + \varepsilon^{-ua-ub+vb} + \varepsilon^{ub-va-vb} \\ &\equiv \left(\varepsilon^{v} - 1\right)\left(\varepsilon^{-u} - 1\right)\left(\varepsilon^{u-v} - 1\right) \bmod 2. \end{aligned}$$

By choice of the parameters u, v to satisfy $u, v, u-v \not\equiv 0 \bmod (q-1)_{2'}$, we may conclude that

$$\lambda_{\chi_{\Phi_2\Phi_3}^{(u,v)}}\left(\widehat{C_6^{(a,b)}}\right) \neq 0.$$

Following [Nav98, Thm. 4.4] this implies that up to G-conjugation the group D is contained in a defect group D_C of the class $C = C_6^{(a,b)}$. But $C_6^{(a,b)} = \left[\mathrm{diag}(\zeta^a, \zeta^b, \zeta^{-a-b})\right]_G$ with mutually distinct diagonal entries, so $\mathrm{C}_G(\mathrm{diag}(\zeta^a, \zeta^b, \zeta^{-a-b})) = T$, where we denote by T the maximally split torus of G consisting of the diagonal matrices in G. Thus, the Sylow 2-subgroup $\mathcal{O}_2(T)$ of T is a defect group of C and we obtain that $\mathcal{O}_2(T)$ contains a G-conjugate of D. On the other hand, we have

$$|D| \cdot \chi_{\Phi_2\Phi_3}^{(u,v)}(1)_2 \geqslant |G|_2$$

since D is a defect group of the 2-block containing $\chi_{\Phi_2\Phi_3}^{(u,v)}$, that is, to state it differently, we have $|D| \cdot (q+1)_2 \geqslant (q-1)_2^2 (q+1)_2$, hence $|D| \geqslant (q-1)_2^2 = |\mathcal{O}_2(T)|$. Thus, we finally conclude that $D = \mathcal{O}_2(T)$ up to G-conjugation.

We may now apply Lemma 6.6 to compute the distribution of characters $\chi_{\Phi_2\Phi_3}^{(u,v)}$ with $u, v, u-v \not\equiv 0 \bmod (q-1)_{2'}$ into 2-blocks. We have shown that there exists a G-conjugacy class of type C_6 on which the reduction modulo 2 of the central character associated to $\chi_{\Phi_2\Phi_3}^{(u,v)}$ does not take value zero, so comparison with Table 6.3 shows that the 2-block containing $\chi_{\Phi_2\Phi_3}^{(u,v)}$ only contains characters of type $\chi_{\Phi_2\Phi_3}$. Now by the same calculation as in the case that $2 \neq \ell \mid (q-1)$ one can prove that two characters $\chi_{\Phi_2\Phi_3}^{(u,v)}$ and $\chi_{\Phi_2\Phi_3}^{(u',v')}$ belong to the same 2-block if and only if

$$(u', v') \in_{\bmod (q-1)_{2'}} \{(u,v), (v,u), (-u, v-u), (v-u, -u), (u-v, -v), (-v, u-v)\}$$

as claimed. $\square$

Proposition 6.12. *The decomposition matrices for the 2-blocks of G not of 2-defect zero (cf. Proposition 6.11) are as follows:*

- *for the principal 2-block B_0:*

B_0	φ_1	φ_2	φ_3
1_G	1	·	·
$\chi_{q\Phi_2}$	·	1	·
χ_{q^3}	1	·	1
$\chi^{(u,v)}_{\Phi_2\Phi_3}$, $u, v \equiv 0 \bmod (q-1)_{2'}$	2	2	1
$\chi^{(u)}_{\Phi_3}$, $u \equiv 0 \bmod (q-1)_{2'}$	1	1	·
$\chi^{(u)}_{q\Phi_3}$, $u \equiv 0 \bmod (q-1)_{2'}$	1	1	1
$\chi^{(u)}_{\Phi_1\Phi_3}$, $u \equiv 0 \bmod (q^2-1)_{2'}$	·	·	1

- *for the 2-blocks of type $B_{(\text{-})}$:*

$B_{(k)}$	$\varphi^{(k;1)}$	$\varphi^{(k;2)}$
$\chi^{(u)}_{\Phi_3}$, $u \equiv k \bmod (q-1)_{2'}$	1	·
$\chi^{(u)}_{q\Phi_3}$, $u \equiv k \bmod (q-1)_{2'}$	$\alpha+1$	1
$\chi^{(u)}_{\Phi_1\Phi_3}$, $u \equiv -(q+1)k \bmod (q^2-1)_{2'}$	α	1
$\chi^{(u,v)}_{\Phi_2\Phi_3}$, $(u,v) \in_{\bmod (q-1)_{2'}} \{(k,0),(0,k),(-k,-k)\}$	$\alpha+2$	1

for $k \in \{1, \dots, (q-1)_{2'}-1\}$ and $\alpha \in \{0,1\}$;

- *for the 2-blocks of type $B_{(\text{-},\text{-})}$:*

$B_{(u,v)}$	$\varphi^{(u,v)}$
$\chi^{(u',v')}_{\Phi_2\Phi_3}$, $(u',v') \in_{\bmod (q-1)_{2'}} \mathcal{S}(u,v)$	1

for $u, v \in \{1, \dots, (q-1)_{2'}-1\}$ with $(u,v) \in \mathcal{M}(\Phi_2\Phi_3)$;

- *for the 2-blocks of type $B'_{(\text{-})}$:*

$B'_{(k)}$	$\varphi^{(k)'}$
$\chi^{(u)}_{\Phi_1\Phi_3}$, $u \equiv k \bmod (q^2-1)_{2'}$ or $uq \equiv k \bmod (q^2-1)_{2'}$	1

for $k \in \{1, \dots, (q^2-1)_{2'}-1\}$ with $k \not\equiv 0 \bmod (q+1)_{2'}$.

Proof. The proof works similarly as for the case $2 \neq \ell \mid (q-1)$. Since 2 divides $q-1$, for any choice of $d \in \mathbb{N}_{>0}$ we have $e_q(d) = 2$, where $e_q(d)$ is defined as in the proof of

Proposition 6.9, and for $k \in \{1,2,3\}$ the 2-modular decomposition matrices Δ_k for the unipotent characters of $\mathrm{GL}_k(q^d)$ are given by

$$\Delta_1 = \begin{bmatrix} 1 \end{bmatrix}, \quad \Delta_2 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \quad \Delta_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

according to [Jam90, Appendix 1]. By the same argument as in the proof of Proposition 6.9 the decomposition matrix for the three unipotent characters χ_1, $\chi_{q\Phi_2}$ and χ_{q^3} in the principal 2-block B_0 is given by Δ_3. The decomposition numbers for the remaining characters in B_0 follow by easy calculations.

For the 2-blocks B of type $B_{(\text{-})}$ it follows that their decomposition matrix contains either Δ_2 or an identity matrix of size 2 (or both) as a submatrix. Hence, at least one irreducible character in B stays irreducible under 2-modular reduction. One easily verifies that if $B = B_{(k)}$, $k \in \{1, \ldots, (q-1)_{2'} - 1\}$, then the relations

$$\begin{aligned} \chi_{q\Phi_3}^{(k)0} &= \chi_{\Phi_3}^{(k)0} + \chi_{\Phi_1\Phi_3}^{(u)0}, \\ \chi_{\Phi_2\Phi_3}^{(k,0)0} &= 2\chi_{\Phi_3}^{(k)0} + \chi_{\Phi_1\Phi_3}^{(u)0} \end{aligned}$$

hold for $u \equiv -(q+1)k \bmod (q^2-1)_{2'}$, and characters of the same type have the same 2-modular reduction. If both $\chi_{\Phi_3}^{(k)0}$ and $\chi_{\Phi_1\Phi_3}^{(u)0}$ are irreducible, then the claim follows for B with $\alpha = 0$. Hence, suppose that this is not the case. Then by the previous observations the decomposition matrix associated to $\chi_{\Phi_3}^{(k)}$ and $\chi_{\Phi_1\Phi_3}^{(u)}$ has the form Δ_2 with $\chi_{\Phi_3}^{(k)0}$ irreducible and

$$\chi_{\Phi_1\Phi_3}^{(u)0} = \chi_{\Phi_3}^{(k)0} + \varphi$$

for some $\varphi \in \mathrm{IBr}(B)$ with $\varphi \neq \chi_{\Phi_3}^{(k)0}$. This yields the proposed 2-decomposition matrix for B with $\alpha = 1$. $\square$

Lemma 6.13. *The 2-blocks of G of type $B_{(\text{-},\text{-})}$ have defect group $\mathcal{O}_2(T)$, where T denotes a maximal torus of G of type T_1, while the 2-blocks of type $B'_{(\text{-})}$ have defect group $\mathcal{O}_2(T')$ for a maximal torus T' of G of type T_2. The 2-blocks of G of type $B_{(\text{-})}$ are of maximal 2-defect.*

Proof. As we have already seen in the proof of Proposition 6.11 the 2-blocks of type $B_{(\text{-},\text{-})}$ have defect group $\mathcal{O}_2(T)$.

Now suppose that B is a 2-block of type $B'_{(\text{-})}$. Then $\lambda_B(\widehat{C_7^{(1)}}) \neq 0$ as can easily be verified (cf. Table 6.3). Hence, by the Min-Max-Theorem [Nav98, Thm. 4.4] a defect group of B is contained in a defect group of $C_7^{(1)}$. By Proposition 6.3 the defect groups of conjugacy classes of type C_7 are conjugate to $\mathcal{O}_2(T')$, which is of order $(q^2-1)_2$. Now a study of the character degrees for B yields that B has defect $\log_2(q^2-1)_2$, whence $\mathcal{O}_2(T')$ must be a defect group of B.

For the 2-blocks of type $B_{(\text{-})}$ the examination of the corresponding character degrees shows that these have maximal 2-defect. $\square$

Chapter 7

Action of Automorphisms

The upcoming chapter is aimed at providing a good understanding of the action of automorphisms of the group $G = \mathrm{SL}_3(q)$, $q \not\equiv 1 \bmod 3$, on its weights and Brauer characters. This is crucial for the successful establishment of the inductive blockwise Alperin weight condition for G. Let us recall from Section 5.3 that

$$\mathrm{Aut}(G) = G \rtimes \langle \Gamma, F_p \rangle,$$

where Γ denotes the transpose-inverse automorphism of G and F_p acts on G by raising each matrix entry to its p-th power.

7.1 Action on the (Brauer) Characters of $\mathrm{SL}_3(q)$

Before we turn to the characters of G let us examine the behaviour of the conjugacy classes of G under the action of $\mathrm{Aut}(G)$. In order to understand the automorphism action on the semisimple classes of G the following observation will be a helpful tool:

Lemma 7.1. *Let $s, s' \in G$ be semisimple elements. If s and s' are conjugate in the infinite algebraic group $\mathbf{G} = \mathrm{SL}_3(\mathbb{F})$, then they are conjugate in G.*

Proof. Recall that for the Steinberg endomorphism F of $\mathbf{G}$ defined in Section 5.1 we have $G = \mathbf{G}^F$. Suppose that $s \in G$ is semisimple and define $V := [s]_{\mathbf{G}}$, the $\mathbf{G}$-conjugacy class of s in $\mathbf{G}$. Then $\mathrm{C}_{\mathbf{G}}(s)$ is closed and connected by [MT11, Thm. 14.16]. Moreover, $\mathbf{G}$ acts transitively on V, for $h \in \mathbf{G}$ it holds that $F(hsh^{-1}) = F(h)sF(h)^{-1} \in V$, so $F(V) \subseteq V$, and we have

$$F(g.v) = F(gvg^{-1}) = F(g)F(v)F(g^{-1}) = F(g)F(v)F(g)^{-1} = F(g).F(v)$$

for all $g \in G$ and $v \in V$. Since $\mathbf{G}$ is connected reductive, according to [MT11, Thm. 21.11] this implies the existence of a one-to-one correspondence

$$\{G\text{-orbits on } V^F\} \xleftrightarrow{1:1} \{F\text{-classes in } \mathrm{C}_{\mathbf{G}}(s)/\,\mathrm{C}_{\mathbf{G}}(s)^\circ\},$$

where $\mathrm{C}_{\mathbf{G}}(s)^\circ$ denotes the connected component of $\mathrm{C}_{\mathbf{G}}(s)$. But since $\mathrm{C}_{\mathbf{G}}(s)$ is connected, we have $\mathrm{C}_{\mathbf{G}}(s)/\,\mathrm{C}_{\mathbf{G}}(s)^\circ = \{1\}$, and hence there is only one G-orbit on V^F. In particular, any semisimple $s' \in G$ that is $\mathbf{G}$-conjugate to s must even be conjugate to s under G. □

Remark 7.2. Note that an analogous statement to Lemma 7.1 holds true for the general linear group $\mathrm{GL}_n(\mathbb{F})$ for any $n \in \mathbb{N}_{>0}$ and its group of fixed points $\mathrm{GL}_n(q)$ under the Steinberg endomorphism which raises each entry of a matrix in $\mathrm{GL}_n(\mathbb{F})$ to its q-th power. Even more generally, it holds for any connected reductive algebraic group $\mathbf{H}$ such that $[\mathbf{H},\mathbf{H}]$ is simply connected and its finite group $\mathbf{H}^\sigma$ of fixed points under a Steinberg endomorphism σ. This can be proven by the same arguments employed to verify Lemma 7.1.

Proposition 7.3. *Let $s \in G$ be semisimple. Then*

(i) $\Gamma(s)$ is conjugate to s^{-1} in G,

(ii) $F_p(s)$ is conjugate to s^p in G.

Proof. Since s is semisimple, there exists a diagonal matrix $d \in \mathbf{G}$ conjugate to s in $\mathbf{G}$, i.e., $s = x^{-1}dx$ for some $x \in \mathbf{G}$. We obtain

$$\Gamma(s) = s^{-\operatorname{tr}} = x^{\operatorname{tr}} d^{-\operatorname{tr}} x^{-\operatorname{tr}} = x^{\operatorname{tr}} d^{-1} x^{-\operatorname{tr}},$$

and hence $\Gamma(s)$ is $\mathbf{G}$-conjugate to d^{-1}, which in turn is $\mathbf{G}$-conjugate to s^{-1}. By transitivity, $\Gamma(s)$ and s^{-1} are conjugate in $\mathbf{G}$, so following Proposition 7.1 they must already be conjugate in G. This gives part (i). Statement (ii) is proven similarly. □

Let us now turn to the action of the automorphisms Γ and F_p on the non-semisimple conjugacy classes of G, that is, the classes C_1, C_2, C_3 and the classes of type C_5.

Proposition 7.4. *Let C be a non-semisimple conjugacy class of G.*

(i) If $C \in \{C_1, C_2, C_3\}$, then C is left invariant by $\operatorname{Aut}(G)$.

(ii) If $C = C_5^{(a)}$ for some $1 \leqslant a < q-1$, then $\Gamma(C) = C_5^{((q-1)-a)}$ and $F_p(C) = C_5^{(ap)}$.

Proof. This follows by direct calculation and the fact that $\operatorname{Aut}(G) = G \rtimes \langle \Gamma, F_p\rangle$. □

We can now describe the action of the automorphisms of G on $\operatorname{Irr}(G)$ as follows:

Corollary 7.5. *For $\chi \in \operatorname{Irr}(G)$ the following statements hold:*

(i) If $\chi \in \{1_G, \chi_{q\Phi_2}, \chi_{q^3}\}$, then χ is left invariant by $\operatorname{Aut}(G)$.

(ii) If $\chi = \chi_\gamma^{(u)}$ for $\gamma \in \{\Phi_3,\ q\Phi_3,\ \Phi_1\Phi_3,\ \Phi_1^2\Phi_2\}$ and a corresponding parameter $u \in \mathbb{Z}$, then $\chi^{F_p} = \chi_\gamma^{(up)}$ and $\chi^\Gamma = \chi_\gamma^{(-u)}$.

(iii) If $\chi = \chi_{\Phi_2\Phi_3}^{(u,v)}$ for parameters $u, v \in \mathbb{Z}$, then $\chi^{F_p} = \chi_{\Phi_2\Phi_3}^{(up,vp)}$ and $\chi^\Gamma = \chi_{\Phi_2\Phi_3}^{(-u,-v)}$.

Proof. (i) If $\chi \in \{1_G, \chi_{q\Phi_2}, \chi_{q^3}\}$, then it is the unique irreducible character of G of degree $\chi(1)$, hence left invariant by $\operatorname{Aut}(G)$.

For (ii) and (iii) suppose that χ is of type χ_γ for $\gamma \in \{\Phi_3,\ q\Phi_3,\ \Phi_1\Phi_3,\ \Phi_1^2\Phi_2,\ \Phi_2\Phi_3\}$. Under our assumption that $q \geqslant 3$ and $q \not\equiv 1 \bmod 3$ it follows that irreducible characters of G of distinct types have distinct degrees (cf. Table A.1). In particular, the type of χ cannot be changed by $\operatorname{Aut}(G)$, that is, both χ^{F_p} and χ^Γ are of type χ_γ. Now a direct comparison of the character values given in Table A.1 in conjunction with Propositions 7.3 and 7.4 yields the claim. □

Concerning the action of automorphisms of G on its irreducible Brauer characters we recall that we only need to concentrate on the case of characteristic ℓ dividing $q-1$, and moreover, we note that for ℓ-blocks B of G our interests are in fact restricted to the behaviour of $\operatorname{IBr}(B)$ under $\operatorname{Aut}(G)_B$. This question, however, is easily answered below, even without the use of the results on the action of $\operatorname{Aut}(G)$ on $\operatorname{Irr}(G)$. Nevertheless, this information has not been obtained in vain but will be of importance in the investigation of the extendibility of (Brauer) characters of G in Chapter 9. In particular, in that context it will once more confirm the following statement:

Proposition 7.6. *Suppose that $\ell \mid (q-1)$ and let B be an ℓ-block of G. Then* $\mathrm{Aut}(G)_B$ *acts trivially on* $\mathrm{IBr}(B)$.

Proof. We go through the distinct types of ℓ-blocks of G. Clearly, if B is an ℓ-block of ℓ-defect zero, then $l(B) = 1$, i.e., B contains only one irreducible Brauer character, which is hence left invariant by $\mathrm{Aut}(G)_B$. The same holds true if B is of type $B_{(\text{-},\text{-})}$ or $B'_{(\text{-})}$ by Proposition 6.9 for $\ell \neq 2$ and Proposition 6.12 for $\ell = 2$.

Suppose now that $B = B_0$ is the principal ℓ-block. Then by Propositions 6.9 and 6.12 we have $l(B) = 3$, and the degrees of the irreducible Brauer characters in B are given by 1, $q(q+1)$ and q^3 if $\ell \neq 2$ and by 1, $q(q+1)$ and $q^3 - 1$ if $\ell = 2$. Since these three numbers are distinct for all choices of $q > 2$, we conclude that $\mathrm{Aut}(G)_B = \mathrm{Aut}(G)$ leaves each element of $\mathrm{IBr}(B)$ invariant.

Let us finally suppose that B is of type $B_{(\text{-})}$. Then by Propositions 6.9 and 6.12 there are two irreducible Brauer characters in B, whose degrees are given by $q^2 + q + 1$ and $q(q^2 + q + 1)$ if $\ell \neq 2$ and by $q^2 + q + 1$ and $(q - (\alpha + 1))(q^2 + q + 1)$ for some $\alpha \in \{0, 1\}$ if $\ell = 2$. Now by Proposition 6.11 there can only exist 2-blocks of type $B_{(\text{-})}$ if $(q-1)_{2'} > 1$. Thus, we may assume that $q \geqslant 7$ (even $q \geqslant 11$ since we consider the case that $q \not\equiv 1 \bmod 3$), so in particular $q - (\alpha + 1) > 1$. Accordingly, the two irreducible Brauer characters in B are of distinct degrees and cannot be interchanged by any element of $\mathrm{Aut}(G)_B$ as claimed. □

7.2 Action on the Weights of $\mathrm{SL}_3(q)$

In this section we construct the B-weights of G for various blocks B of G of non-cyclic defect and study their behaviour under $\mathrm{Aut}(G)_B$. We start by classifying the radical ℓ-subgroups of G for primes $\ell \neq p$ such that G has non-cyclic Sylow ℓ-subgroups, that is, for all ℓ dividing $q - 1$. Here, we always have $\ell \neq 3$ by the assumption that $q \not\equiv 1 \bmod 3$. Afterwards we apply Construction 2.10 to find the B-weights of G for any relevant ℓ-block B and examine how these behave under automorphisms of G.

7.2.1 Radical Subgroups

We classify the radical ℓ-subgroups of G individually for the two cases $2 \neq \ell \mid (q-1)$ and $\ell = 2$. As it turns out, this job is easily accomplished for odd ℓ, while for $\ell = 2$ more work needs to be done. Let us state here the following lemma, which will be applied in both cases:

Lemma 7.7. *Let $\zeta \in \mathbb{F}^\times$ be of order $q - 1$ and set $Q := \langle \mathrm{diag}(\zeta, \zeta, \zeta^{-2}) \rangle \leqslant G$. Then*

$$\mathrm{C}_G(Q) = \mathrm{N}_G(Q) = \left\{ \begin{pmatrix} A & 0 \\ 0 & (\det A)^{-1} \end{pmatrix} \;\middle|\; A \in \mathrm{GL}_2(q) \right\} \cong \mathrm{GL}_2(q).$$

Moreover, we have $\mathrm{C}_G(\mathcal{O}_\ell(Q)) = \mathrm{N}_G(\mathcal{O}_\ell(Q)) = \mathrm{N}_G(Q)$ for any prime ℓ dividing $q - 1$.

Proof. Similarly as in the proof of Proposition 5.7 we consider common $\mathbb{F}_q$-eigenspaces of the matrices in $Q := \langle \mathrm{diag}(\zeta, \zeta, \zeta^{-2}) \rangle$. In the notation used there, that is, $e_1 = (1,0,0)^{\mathrm{tr}}$, $e_2 = (0,1,0)^{\mathrm{tr}}$ and $e_3 = (0,0,1)^{\mathrm{tr}} \in \mathbb{F}_q^3$, the $\mathbb{F}_q$-eigenspaces common to all elements in Q are $V_1 := \langle e_1, e_2 \rangle_{\mathbb{F}_q}$ and $V_2 := \langle e_3 \rangle_{\mathbb{F}_q}$. Note that this requires $\zeta^{-2} \neq \zeta$, which is guaranteed by our assumption that $q \not\equiv 1 \bmod 3$. Since the common eigenspaces of Q are permuted by $\mathrm{N}_G(Q)$, and $\dim V_1 \neq \dim V_2$, we conclude that V_1 and V_2 are stabilized by any element of $\mathrm{N}_G(Q)$, which proves that

$$\mathrm{N}_G(Q) \subseteq \left\{ \begin{pmatrix} A & 0 \\ 0 & (\det A)^{-1} \end{pmatrix} \;\middle|\; A \in \mathrm{GL}_2(q) \right\}.$$

The converse inclusion is obviously satisfied, so indeed $\mathrm{N}_G(Q)$ is as claimed. Since we have $\mathrm{C}_G(Q) \subseteq \mathrm{N}_G(Q)$ and $\mathrm{N}_G(Q)$ centralizes Q, this completes the proof of the first claim.

Suppose now that ℓ is a prime dividing $q-1$. Then the same argumentation applies since the common eigenspaces of $\mathcal{O}_\ell(\langle \mathrm{diag}(\zeta, \zeta, \zeta^{-2})\rangle)$ are still given by V_1 and V_2. □

7.2.1.1 The Case $2 \neq \ell \mid (q-1)$

Let $\ell \neq 2$ be a prime dividing $q-1$. As we will see, G possesses three distinct G-conjugacy classes of radical ℓ-subgroups. Before we prove this, let us state the following lemma.

Lemma 7.8. *Let H be a finite almost simple group. Then H does not possess any non-trivial solvable normal subgroups.*

Proof. Let S be a non-abelian simple group such that $S \leqslant H \leqslant \mathrm{Aut}(S)$. Moreover, suppose that $N \trianglelefteq H$ is a solvable normal subgroup of H. Then we have $[N, S] \trianglelefteq N \cap S$, and as a subgroup of N, this is also solvable. But then we have $[N, S] = \{1\}$ since the only solvable normal subgroup of S is the trivial one. Hence, N commutes with S, and as $N \leqslant \mathrm{Aut}(S)$, we obtain $N = \{1\}$. □

Proposition 7.9. *Let $2 \neq \ell \mid (q-1)$ and suppose that R is a radical ℓ-subgroup of G. Then up to G-conjugation one of the following holds:*

(i) $R = \{1\}$;

(ii) $R = \mathcal{O}_\ell(\langle \mathrm{diag}(\zeta, \zeta, \zeta^{-2})\rangle)$, *where $\zeta \in \mathbb{F}^\times$ is of order $q-1$;*

(iii) $R = \mathcal{O}_\ell(T) \in \mathrm{Syl}_\ell(G)$, *where T denotes the maximally split torus of G consisting of diagonal matrices.*

Proof. Clearly, the groups in (i) and (iii) are ℓ-radical in G. Hence, let us consider the group $Q := \mathcal{O}_\ell(\langle \mathrm{diag}(\zeta, \zeta, \zeta^{-2})\rangle)$ in (ii). We first show that Q is indeed ℓ-radical in G. To prove this we use that

$$Q \cong \mathcal{O}_\ell(\langle \mathrm{diag}(\zeta, \zeta)\rangle) = \mathcal{O}_\ell(\mathrm{Z}(\mathrm{GL}_2(q)))$$

and $\mathrm{N}_G(Q) \cong \mathrm{GL}_2(q)$ according to Lemma 7.7. As $\mathrm{PGL}_2(q) = \mathrm{GL}_2(q)/\mathrm{Z}(\mathrm{GL}_2(q))$ is almost simple for $q \geqslant 4$ (since $\mathrm{PSL}_2(q) \leqslant \mathrm{PGL}_2(q) \leqslant \mathrm{Aut}(\mathrm{PSL}_2(q))$ and $\mathrm{PSL}_2(q)$ is simple for $q \geqslant 4$ by [Wil09, p. 46 and pp. 48–50]), we apply Lemma 7.8 to deduce that this group does not contain any non-trivial solvable normal subgroups. We set $Z := \mathrm{Z}(\mathrm{GL}_2(q))$. Now suppose that $P/\mathcal{O}_\ell(Z)$ is a normal ℓ-subgroup of $\mathrm{GL}_2(q)/\mathcal{O}_\ell(Z)$ for some group P with $\mathcal{O}_\ell(Z) \subseteq P \trianglelefteq \mathrm{GL}_2(q)$. Then P must be an ℓ-group and PZ/Z is a normal ℓ-subgroup of $\mathrm{PGL}_2(q)$, hence solvable, whence by Lemma 7.8 we have $PZ/Z = \{1\}$. But then $P \subseteq Z$ and thus $P = \mathcal{O}_\ell(Z)$. We conclude that

$$\mathcal{O}_\ell(\mathrm{N}_G(Q)/Q) \cong \mathcal{O}_\ell(\mathrm{GL}_2(q)/\mathcal{O}_\ell(Z)) = \{1\}.$$

Now suppose that a non-trivial ℓ-subgroup $H \leqslant G$ is not G-conjugate to any of the subgroups in the claim. If $\{1\} \neq H \leqslant Q$, then $\mathrm{N}_G(H) = \mathrm{N}_G(Q) \cong \mathrm{GL}_2(q)$, so H cannot be ℓ-radical. Thus, assume that H is not G-conjugate to a subgroup of Q and suppose that $H \subsetneq \mathcal{O}_\ell(T) \in \mathrm{Syl}_\ell(G)$. Since $\mathcal{O}_\ell(T)$ consists of diagonal matrices, the group H must contain an element $\mathrm{diag}(x, y, z)$ with x, y, $z \in \mathbb{F}_q^\times$ pairwise distinct. As in the proof of Proposition 5.8 it follows that

$$\mathrm{N}_G(H) \leqslant \{\text{monomial matrices in } G\} = \mathrm{N}_G(\mathcal{O}_\ell(T)).$$

Of course we have $\mathcal{O}_\ell(T) \leqslant \mathrm{N}_G(H) \leqslant \mathrm{N}_G(\mathcal{O}_\ell(T))$, so $H \subsetneq \mathcal{O}_\ell(T) \trianglelefteq \mathrm{N}_G(H)$, whence H cannot be ℓ-radical in G. □

7.2.1.2 The Case $\ell = 2$

The aim of this section is to determine the radical 2-subgroups of $G = \mathrm{SL}_3(q)$, where we always assume that q is odd. This task turns out to be somewhat more elaborate than the classification of the radical subgroups of G for odd primes dividing $q - 1$ in the previous section. We approach this problem by regarding G as a subgroup of the general linear group $\mathrm{GL}_3(q)$. Even more, since by assumption $q \not\equiv 1 \bmod 3$, we have

$$\mathrm{GL}_3(q) = \mathrm{SL}_3(q) \times \mathrm{Z}(\mathrm{GL}_3(q)),$$

that is, G is a direct factor of $\mathrm{GL}_3(q)$. By [NT11, Lemma 2.3(b)] there exists a bijection

$$\begin{aligned} \mathrm{Rad}_\ell(\mathrm{SL}_3(q)) \times \mathrm{Rad}_\ell(\mathrm{Z}(\mathrm{GL}_3(q))) &\longrightarrow \mathrm{Rad}_\ell(\mathrm{GL}_3(q)), \\ (R_1, R_2) &\longmapsto R_1 \times R_2, \end{aligned}$$

where we recall that $\mathrm{Rad}_\ell(H)$ denotes the set of all radical ℓ-subgroups of a finite group H. Since $\mathrm{Z}(\mathrm{GL}_3(q))$ is cyclic, its only radical ℓ-subgroup is $\mathcal{O}_\ell(\mathrm{Z}(\mathrm{GL}_3(q)))$, so in fact we have a bijection

$$\begin{aligned} \mathrm{Rad}_\ell(\mathrm{SL}_3(q)) &\longrightarrow \mathrm{Rad}_\ell(\mathrm{GL}_3(q)) \\ R &\longmapsto R \times \mathcal{O}_\ell(\mathrm{Z}(\mathrm{GL}_3(q))), \end{aligned}$$

with inverse

$$\begin{aligned} \mathrm{Rad}_\ell(\mathrm{GL}_3(q)) &\longrightarrow \mathrm{Rad}_\ell(\mathrm{SL}_3(q)) \\ R &\longmapsto R \cap \mathrm{SL}_3(q). \end{aligned}$$

Hence, in order to find all radical ℓ-subgroups of $\mathrm{SL}_3(q)$ it suffices to know those of $\mathrm{GL}_3(q)$. Here we are in the fortunate position that the radical subgroups of the general linear groups, however abstract their description may be, are known. The structure of radical 2-subgroups of $\mathrm{GL}_n(q)$ is described in [An92]. In order to present the result proven there we first need to introduce the notion of basic subgroups of general linear groups, which turn out to be the building blocks of radical subgroups for these groups.

Construction/Definition 7.10 (*Basic subgroups*)**.** In the subsequent construction we follow [An92, pp. 509–513]. For q odd we let $a \in \mathbb{N}$ be such that $2^{a+1} = (q^2 - 1)_2$.

For $\gamma \geqslant 0$ let E_γ be an extraspecial group of order $2^{1+2\gamma}$. Moreover, for $\alpha \geqslant 0$ let Z_α be a cyclic group of order $2^{a+\alpha}$ if $q \equiv 1 \bmod 4$ or $\alpha \geqslant 1$. Then by [An92, pp. 509] the central product $E_\gamma Z_\alpha$ over $\mathrm{Z}(E_\gamma) = \Omega_1(Z_\alpha)$, where $\Omega_1(Z_\alpha) = \langle z \in Z_\alpha \mid z^2 = 1 \rangle$, may be embedded into the general linear group $\mathrm{GL}_{2^\gamma}(q^{2^\alpha})$ in such a way that Z_α is identified with $\mathcal{O}_2(\mathrm{Z}(\mathrm{GL}_{2^\gamma}(q^{2^\alpha})))$. For $q \equiv -1 \bmod 4$ we set $Z_0 = \mathrm{Z}(E_\gamma)$ (where $\gamma \geqslant 0$ as above), such that $E_\gamma Z_0 = E_\gamma$, which may be embedded into $\mathrm{GL}_{2^\gamma}(q)$ by [An92, (1A)].

There exists an embedding

$$\mathrm{GL}_{2^\gamma}(q^{2^\alpha}) \hookrightarrow \mathrm{GL}_{2^{\alpha+\gamma}}(q)$$

induced by the Galois field extension $\mathbb{F}_{q^{2^\alpha}} / \mathbb{F}_q$, i.e., an embedding obtained by regarding a matrix $A \in \mathrm{GL}_{2^\gamma}(q^{2^\alpha})$, which represents an invertible $\mathbb{F}_{q^{2^\alpha}}$-linear map of a 2^γ-dimensional $\mathbb{F}_{q^{2^\alpha}}$-vector space V, as an invertible $\mathbb{F}_q$-linear map of the space V viewed as an $\mathbb{F}_q$-vector space of $\mathbb{F}_q$-dimension $2^\alpha \cdot 2^\gamma = 2^{\alpha+\gamma}$. In [An92, p. 509] the author denotes by $R_{\alpha,\gamma}$ the image of $E_\gamma Z_\alpha$ in $\mathrm{GL}_{2^{\alpha+\gamma}}(q)$ under the arising embedding

$$E_\gamma Z_\alpha \hookrightarrow \mathrm{GL}_{2^\gamma}(q^{2^\alpha}) \hookrightarrow \mathrm{GL}_{2^{\alpha+\gamma}}(q)$$

and claims that this image is "determined up to conjugacy" in $\mathrm{GL}_{2^{\alpha+\gamma}}(q)$. We should mention that we are not entirely sure about how the notion *determined up to conjugacy* is to be interpreted, that is to say, whether there exists only one conjugacy class of subgroups of $\mathrm{GL}_{2^{\alpha+\gamma}}(q)$ isomorphic to $E_\gamma Z_\alpha$, or whether this merely means that all images of $E_\gamma Z_\alpha$ in $\mathrm{GL}_{2^{\alpha+\gamma}}(q)$ under embeddings of the kind described above are conjugate in $\mathrm{GL}_{2^{\alpha+\gamma}}(q)$. Due to this uncertainty, we will not assume any knowledge on uniqueness in the following, but we give a proof of uniqueness in the required manner whenever necessary and possible.

Now for $m \geqslant 1$ we further denote by $R_{m,\alpha,\gamma}$ the image of $R_{\alpha,\gamma}$ in $\mathrm{GL}_{m2^{\alpha+\gamma}}(q)$ under the m-fold diagonal mapping

$$\mathrm{GL}_{2^{\alpha+\gamma}}(q) \hookrightarrow \mathrm{GL}_{m2^{\alpha+\gamma}}(q), \quad g \longmapsto \left.\begin{pmatrix} g & & & \\ & g & & \\ & & \ddots & \\ & & & g \end{pmatrix}\right\} m \text{ times.}$$

Then also the group $R_{m,\alpha,\gamma}$ is "determined up to conjugacy" in $\mathrm{GL}_{m2^{\alpha+\gamma}}(q)$ according to [An92, p. 510].

Assume now that $q \equiv -1 \bmod 4$. We denote by P a semidihedral group of order 2^{a+2} and let $E_\gamma P$ be the central product of the extraspecial group E_γ of order $2^{1+2\gamma}$ with P. Then by [An92, p. 511] the group $E_\gamma P$ may be embedded into $\mathrm{GL}_{2^{\gamma+1}}(q)$ via a faithful absolutely irreducible representation, and the image $S_{1,\gamma}$ of $E_\gamma P$ under this embedding is "uniquely determined up to conjugacy". Moreover, $S_{1,\gamma}$ is independent of the type of E_γ. For $m \geqslant 1$ we let $S_{m,1,\gamma}$ denote the image of $S_{1,\gamma}$ in $\mathrm{GL}_{m2^{\gamma+1}}(q)$ under the m-fold diagonal mapping similar as above. Then $S_{m,1,\gamma}$ is "determined up to conjugacy" in $\mathrm{GL}_{m2^{\gamma+1}}(q)$ by [An92, p. 512].

Now for given integers $\alpha \geqslant 0$, $\gamma \geqslant 0$ and $m \geqslant 1$ we define two subgroups of $\mathrm{GL}_{m2^{\alpha+\gamma}}(q)$ by setting

$$R^1_{m,\alpha,\gamma} := R_{m,\alpha,\gamma}$$

and

$$R^2_{m,\alpha,\gamma} := \begin{cases} S_{m,1,\gamma-1} & \text{if } q \equiv -1 \bmod 4,\ \alpha = 0 \text{ and } \gamma \geqslant 1, \\ R_{m,\alpha,\gamma} & \text{else.} \end{cases}$$

For $c \geqslant 0$ we denote by A_c the elementary abelian 2-group of order 2^c represented by its regular permutation representation. Then for any sequence $\mathbf{c} = (c_1, \ldots, c_t)$ with $c_i \in \mathbb{Z}_{\geqslant 0}$ we set $A_{\mathbf{c}} = A_{c_1} \wr A_{c_2} \wr \cdots \wr A_{c_t}$ and define the wreath product

$$R^i_{m,\alpha,\gamma,\mathbf{c}} := R^i_{m,\alpha,\gamma} \wr A_{\mathbf{c}} \leqslant \mathrm{GL}_d(q)$$

for $i = 1, 2$, where $d = m2^{\alpha+\gamma+c_1+\cdots+c_t}$. By [An92, p. 513] the group $R^i_{m,\alpha,\gamma,\mathbf{c}}$ "is determined up to conjugacy" in $\mathrm{GL}_d(q)$. Unless $q \equiv -1 \bmod 4$, $\alpha = 0$, $\gamma = 0$ and $c_1 = 1$, the groups $R^i_{m,\alpha,\gamma,\mathbf{c}}$ are called *basic subgroups* of $\mathrm{GL}_d(q)$.

We may now move on to the statement about the structure of radical 2-subgroups of general linear groups which we have already referred to at the beginning of this section. By [An92, (2B)] the following holds:

Theorem 7.11. *Let V be an $\mathbb{F}_q$-vector space and suppose that R is a radical 2-subgroup of the group $\mathrm{GL}(V)$ of its $\mathbb{F}_q$-linear automorphisms. Then there exists a corresponding decomposition*

$$\begin{aligned} V &= V_1 \oplus \cdots \oplus V_s \oplus V_{s+1} \oplus \cdots \oplus V_t \\ R &= R_1 \times \cdots \times R_s \times R_{s+1} \times \cdots \times R_t \end{aligned}$$

such that $R_i = \{\pm 1_{V_i}\}$ for $1 \leqslant i \leqslant s$ and R_i are basic subgroups of $\mathrm{GL}(V_i)$ for $i \geqslant s+1$. Moreover, if $q \equiv 1 \bmod 4$, then $s = 0$.

This result will be of great use for the determination of the radical 2-subgroups of G. We now introduce certain 2-subgroups of $\mathrm{GL}_2(q)$, $\mathrm{SL}_3(q)$ and $\mathrm{GL}_3(q)$, respectively, that will be of interest to us in due course. To this end we recall the following widely known fact about finite fields:

Lemma 7.12. *In any finite field F there exist elements $a, b \in F$ such that $a^2 + b^2 = -1$.*

Proof. This is a well-known result. It trivially holds for finite fields of characteristic 2, and for odd characteristic see, for instance, [Kap72, p. 15]. □

Keeping this result in mind we may consider the following subgroup of $\mathrm{GL}_2(q)$:

Proposition 7.13. *Let $a, b \in \mathbb{F}_p$ such that $a^2 + b^2 = -1$ (these exist by Lemma 7.12). Then the subgroup*

$$Q_8' := \left\{ \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \pm \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \pm \begin{pmatrix} a & b \\ b & -a \end{pmatrix}, \pm \begin{pmatrix} -b & a \\ a & b \end{pmatrix} \right\}$$

of $\mathrm{GL}_2(p) \leqslant \mathrm{GL}_2(q)$ is isomorphic to the quaternion group Q_8.

Proof. This is well-known and may easily be verified by straightforward calculations. □

Let us observe that we may choose the elements $a, b \in \mathbb{F}_p$ in the above lemma such that $a^2 = -1$ and $b = 0$ if $q \equiv 1 \bmod 4$, so in this case one obtains a particularly nice embedding of the quaternion group into $\mathrm{GL}_2(q)$.

We wish to prove now that up to $\mathrm{GL}_2(q)$-conjugation the group Q_8' as defined above is the only subgroup of $\mathrm{GL}_2(q)$ isomorphic to the quaternion group Q_8. This information will be needed later on to determine the basic subgroups of $\mathrm{GL}_2(q)$. In order to accomplish this task we use the following statement on the realizability of certain group representations in a specific situation:

Proposition 7.14. *Let H be a finite group and $\chi \in \mathrm{Irr}(H)$ such that $\chi(h) \in \mathbb{Z}$ for all $h \in H$. If r is a prime such that the Brauer character χ^0 with respect to r is irreducible, then for every $s \in \mathbb{N}_{>0}$ there exists an absolutely irreducible $\mathbb{F}_{r^s}$-representation affording the Brauer character χ^0.*

In particular, if moreover H is an r'-group and χ is the unique faithful character of H of degree $\chi(1)$, then for every $s \in \mathbb{N}_{>0}$ there exists a unique $\mathrm{GL}_{\chi(1)}(r^s)$-conjugacy class of subgroups of $\mathrm{GL}_{\chi(1)}(r^s)$ isomorphic to H.

Proof. Denote by F an algebraic closure of the finite field $\mathbb{F}_r$ consisting of r elements and let $\mathfrak{X}$ be an irreducible F-representation with associated Brauer character χ^0. Moreover, let ψ denote the character of $\mathfrak{X}$. Then for all $h \in H$ the character value $\psi(h)$ is given by the reduction modulo r of $\chi(h_{r'}) \in \mathbb{Z}$ (see, e.g., [Nav98, Lemma 2.4]). Thus, $\psi(h) \in \mathbb{F}_r \leqslant \mathbb{F}_{r^s}$ for all $h \in H$ and $s \in \mathbb{N}_{>0}$.

Now, constituting an irreducible F-representation, $\mathfrak{X}$ is absolutely irreducible, and hence by [Isa94, Cor. 9.20] for every $s \in \mathbb{N}_{>0}$ there exists an irreducible $\mathbb{F}_{r^s}$-representation $\mathfrak{Y}_s$ affording the character $m_s\psi$ for some positive integer m_s. Even more, according to [Isa94, Thm. 9.21(b)] we have $m_s = 1$ since F has positive characteristic, so we conclude that $\mathfrak{Y}_s$ is an absolutely irreducible $\mathbb{F}_{r^s}$-representation with character ψ affording the Brauer character χ^0.

Let us assume now that H is an r'-group and χ is the unique faithful character of H of degree $\chi(1)$. Then $\mathrm{IBr}_r(H) = \mathrm{Irr}(H)$ and, in particular, up to $\mathrm{GL}_{\chi(1)}(r^s)$-conjugation $\mathfrak{Y}_s$ is the unique faithful $\mathbb{F}_{r^s}$-representation of H of degree $\chi(1)$. Since non-$\mathrm{GL}_{\chi(1)}(r^s)$-conjugate subgroups of $\mathrm{GL}_{\chi(1)}(r^s)$ isomorphic to H would yield non-$\mathrm{GL}_{\chi(1)}(r^s)$-conjugate faithful $\mathbb{F}_{r^s}$-representations of H of degree $\chi(1)$, which would hence afford distinct faithful characters of H of degree $\chi(1)$, we may conclude that there exists a unique $\mathrm{GL}_{\chi(1)}(r^s)$-conjugacy class of subgroups of $\mathrm{GL}_{\chi(1)}(r^s)$ isomorphic to H as claimed. □

Corollary 7.15. *Let $r \neq 2$ be a prime and $s \in \mathbb{N}_{>0}$. Then in $\mathrm{GL}_2(r^s)$ there exists exactly one $\mathrm{GL}_2(r^s)$-conjugacy class of subgroups isomorphic to the quaternion group Q_8.*

Proof. It is well-known that the character table of Q_8 is given by

	C_1	C_2	C_3	C_4	C_5
χ_1	1	1	1	1	1
χ_2	1	−1	1	1	−1
χ_3	1	1	−1	1	−1
χ_4	1	−1	−1	1	1
χ_5	2	·	·	−2	·

Table 7.1: Character table of Q_8

for $\mathrm{Irr}(Q_8) = \{\chi_1, \ldots, \chi_5\}$ and $\mathrm{Cl}(Q_8) = \{C_1, \ldots, C_5\}$. Since $\chi_i + \chi_j$ contains C_4 in its kernel for every choice of $i, j \in \{1, \ldots, 4\}$, we conclude that χ_5 is the unique faithful character of Q_8 of degree 2. Moreover, $\chi_5(x) \in \mathbb{Z}$ for all $x \in Q_8$, and since Q_8 is a 2-group we have $\chi_5^0 \in \mathrm{IBr}_r(Q_8)$, so by Proposition 7.14 the claim follows. □

Corollary 7.16. *Let $r \neq 2$ be a prime and $s \in \mathbb{N}_{>0}$. Then in $\mathrm{GL}_2(r^s)$ there exists exactly one $\mathrm{GL}_2(r^s)$-conjugacy class of subgroups isomorphic to the dihedral group D_8 of order 8.*

Proof. The proof is the same as for Corollary 7.15 since the character tables of Q_8 and D_8 agree (in fact, it is commonly known that these two groups provide the smallest example of non-isomorphic groups having the same complex character table), and the arguments used for Q_8 in Corollary 7.15 apply to D_8 as well. □

Let us now define the following groups:

Definition 7.17. We consider the group Q_8' introduced in Proposition 7.13 for elements $a, b \in \mathbb{F}_p$ with $a^2 + b^2 = -1$ as a subgroup of $\mathrm{GL}_2(q)$. Then we define the subgroup

$$\widetilde{Q_8'} := \langle Q_8', \mathcal{O}_2(Z(\mathrm{GL}_2(q)))\rangle$$

of the general linear group $\mathrm{GL}_2(q)$. This group may as well be regarded as a subgroup of the special linear group $G = \mathrm{SL}_3(q)$ via the embedding

$$\mathrm{GL}_2(q) \hookrightarrow \mathrm{SL}_3(q),\ A \longmapsto \begin{pmatrix} A & \\ & (\det A)^{-1} \end{pmatrix}.$$

From now we denote the image of $\widetilde{Q_8'}$ under this embedding by $\widetilde{Q_8}$. Clearly, $\widetilde{Q_8}$ forms a 2-subgroup of G.

Proposition 7.18. *Let $R = \widetilde{Q_8} \leqslant G$ be as in Definition 7.17 and $\zeta \in \mathbb{F}^\times$ of order $q-1$. Then we have $\mathrm{C}_G(R) = \langle \mathrm{diag}(\zeta, \zeta, \zeta^{-2})\rangle$ and $\mathrm{N}_G(R)/R\,\mathrm{C}_G(R) \cong \mathfrak{S}_3$.*

Proof. Let us set $Q := \langle \mathrm{diag}(\zeta, \zeta, \zeta^{-2})\rangle$. Due to the fact that the group $R = \widetilde{Q_8}$ is defined as the image of the embedding of $\widetilde{Q_8'} = \langle Q_8', \mathcal{O}_2(\mathrm{Z}(\mathrm{GL}_2(q)))\rangle$ into G via the map

$$\iota\colon\ \mathrm{GL}_2(q) \hookrightarrow \mathrm{SL}_3(q),\ A \longmapsto \iota(A) := \begin{pmatrix} A & \\ & (\det A)^{-1} \end{pmatrix}$$

with Q_8' as in Proposition 7.13, it holds that $\mathrm{C}_G(R) \supseteq \iota(\mathrm{C}_{\mathrm{GL}_2(q)}(\widetilde{Q_8'}))$. We claim that in fact we have equality here. Let us observe that by Schur's lemma and the fact that the representation yielding Q_8' is absolutely irreducible, we have

$$\mathrm{C}_{\mathrm{GL}_2(q)}(\widetilde{Q_8'}) = \mathrm{C}_{\mathrm{GL}_2(q)}(Q_8') = \mathrm{Z}(\mathrm{GL}_2(q)),$$

whence $\iota(\mathrm{C}_{\mathrm{GL}_2(q)}(\widetilde{Q_8'})) = Q$. Now $\mathcal{O}_2(Q) \subseteq R$, so in particular we have

$$\mathrm{C}_G(R) \subseteq \mathrm{C}_G(\mathcal{O}_2(Q)) = \iota(\mathrm{GL}_2(q))$$

following Lemma 7.7. We conclude that $\mathrm{C}_G(R) = \iota(\mathrm{C}_{\mathrm{GL}_2(q)}(\widetilde{Q_8'})) = Q$ as claimed.

Let us now concentrate on the normalizer of R in G. As above we have the inclusion $\iota(\mathrm{N}_{\mathrm{GL}_2(q)}(\widetilde{Q_8'})) \subseteq \mathrm{N}_G(R)$, and indeed it turns out that one has equality here as well. Since $\mathrm{Z}(R)$, which is a characteristic subgroup of R, is given by $\mathcal{O}_2(Q)$, we have

$$\mathrm{N}_G(R) \subseteq \mathrm{N}_G(\mathcal{O}_2(Q)) = \iota(\mathrm{GL}_2(q))$$

according to Lemma 7.7, so as above we conclude that $\mathrm{N}_G(R) = \iota(\mathrm{N}_{\mathrm{GL}_2(q)}(\widetilde{Q_8'}))$. This implies that we have

$$\mathrm{N}_G(R)/R\,\mathrm{C}_G(R) \cong \mathrm{N}_{\mathrm{GL}_2(q)}(\widetilde{Q_8'})/\widetilde{Q_8'}\,\mathrm{C}_{\mathrm{GL}_2(q)}(\widetilde{Q_8'}).$$

Suppose that $q \equiv 1 \bmod 4$. Then by [An92, (1B)] we have

$$\mathrm{N}_{\mathrm{GL}_2(q)}(\widetilde{Q_8'})/\widetilde{Q_8'}\,\mathrm{C}_{\mathrm{GL}_2(q)}(\widetilde{Q_8'}) \cong \mathfrak{S}_3$$

as claimed. Hence, we now let $q \equiv -1 \bmod 4$. In this case the group $\mathcal{O}_2(\mathrm{Z}(\mathrm{GL}_2(q)))$ is contained in Q_8', so $\widetilde{Q_8'} = Q_8'$, and by [An92, (1C)] there exists a subgroup $H \leqslant \mathrm{GL}_2(q)$ such that $Q_8' \trianglelefteq H$, $\mathrm{C}_H(Q_8') = \mathrm{Z}(Q_8')$ and $H/Q_8' \cong \mathfrak{S}_3$. Since $\mathrm{Z}(Q_8') = \mathcal{O}_2(\mathrm{Z}(\mathrm{GL}_2(q)))$ in this case, we conclude that $H \cap \mathrm{Z}(\mathrm{GL}_2(q)) = \mathcal{O}_2(\mathrm{Z}(\mathrm{GL}_2(q)))$. Furthermore, it holds that $Q_8' \cap \mathrm{Z}(\mathrm{GL}_2(q)) = \mathcal{O}_2(\mathrm{Z}(\mathrm{GL}_2(q)))$, so we obtain

$$H\,\mathrm{C}_{\mathrm{GL}_2(q)}(Q_8')/Q_8'\,\mathrm{C}_{\mathrm{GL}_2(q)}(Q_8') = H\,\mathrm{Z}(\mathrm{GL}_2(q))/Q_8'\,\mathrm{Z}(\mathrm{GL}_2(q)) \cong \mathfrak{S}_3$$

with $H\,\mathrm{C}_{\mathrm{GL}_2(q)}(Q_8') \subseteq \mathrm{N}_{\mathrm{GL}_2(q)}(Q_8')$. We claim that equality holds here. To prove this we first observe that $\mathrm{Aut}(Q_8') = \mathfrak{S}_4$, which is well-known and easy to verify, and that the quotient $\mathrm{N}_{\mathrm{GL}_2(q)}(Q_8')/\,\mathrm{C}_{\mathrm{GL}_2(q)}(Q_8')$ may be embedded into $\mathrm{Aut}(Q_8')$. We have

$$|H\,\mathrm{C}_{\mathrm{GL}_2(q)}(Q_8')| = |\mathfrak{S}_3| \cdot |Q_8'\,\mathrm{C}_{\mathrm{GL}_2(q)}(Q_8')| = 6 \cdot \frac{|Q_8'| \cdot |\,\mathrm{Z}(\mathrm{GL}_2(q))|}{|Q_8' \cap \mathrm{Z}(\mathrm{GL}_2(q))|} = 24(q-1),$$

which yields

$$\big|H\,\mathrm{C}_{\mathrm{GL}_2(q)}(Q_8')/\,\mathrm{C}_{\mathrm{GL}_2(q)}(Q_8')\big| = 24 = |\mathfrak{S}_4|.$$

Now recall that $\mathrm{N}_{\mathrm{GL}_2(q)}(Q_8')/\,\mathrm{C}_{\mathrm{GL}_2(q)}(Q_8')$ embeds into $\mathrm{Aut}(Q_8') = \mathfrak{S}_4$, so we must already have

$$\mathrm{N}_{\mathrm{GL}_2(q)}(Q_8') = H\,\mathrm{C}_{\mathrm{GL}_2(q)}(Q_8'),$$

and consequently

$$\mathrm{N}_G(R)/R\,\mathrm{C}_G(R) \cong \mathrm{N}_{\mathrm{GL}_2(q)}(Q_8')/Q_8'\,\mathrm{C}_{\mathrm{GL}_2(q)}(Q_8') \cong \mathfrak{S}_3$$

also for $q \equiv -1 \bmod 4$ as claimed. This completes the proof. $\square$

Definition 7.19. We define the group D_8' to be the subgroup of $\mathrm{GL}_2(q)$ given by

$$D_8' := \left\{ \pm\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\ \pm\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},\ \pm\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},\ \pm\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\}.$$

Clearly, this group is dihedral of order 8, and by Corollary 7.16 up to $\mathrm{GL}_2(q)$-conjugation it is the unique subgroup of $\mathrm{GL}_2(q)$ of this isomorphism type. We may embed D_8' into G via the map

$$\mathrm{GL}_2(q) \hookrightarrow G,\ A \longmapsto \begin{pmatrix} A & \\ & (\det A)^{-1} \end{pmatrix}.$$

The image of D_8' under this embedding will in the following be denoted by $\widetilde{D_8}$. It should be noted that in contrast to the subgroup $\widetilde{Q_8}$ of G given in Definition 7.17 the group $\widetilde{D_8}$ does not contain the group $\mathcal{O}_2(\langle \mathrm{diag}(\zeta,\zeta,\zeta^{-2})\rangle)$ if $q \equiv 1 \bmod 4$. However, in order to justify the analogous notation we point out that the group $\widetilde{D_8}$ will only play a role here if $q \equiv -1 \bmod 4$, in which case we have $\mathcal{O}_2(\langle \mathrm{diag}(\zeta,\zeta,\zeta^{-2})\rangle) = \mathrm{Z}(\widetilde{D_8})$.

Proposition 7.20. *Suppose that $q \equiv -1 \bmod 4$ and let $R = \widetilde{D_8} \leqslant G$ be as in Definition 7.19. Then we have $\mathrm{C}_G(R) = \langle \mathrm{diag}(\zeta,\zeta,\zeta^{-2})\rangle$, where $\zeta \in \mathbb{F}^\times$ is of order $q-1$, and $\mathrm{N}_G(R)/R\,\mathrm{C}_G(R) \cong C_2$.*

Proof. Let us first consider the subgroup D_8' of $\mathrm{GL}_2(q)$ as in Definition 7.19. According to [An92, (1C)] there is a subgroup $H \leqslant \mathrm{GL}_2(q)$ such that $D_8' \trianglelefteq H$, $\mathrm{C}_H(D_8') = \mathrm{Z}(D_8')$ and $H/D_8' \cong C_2$. We claim that

$$\mathrm{N}_{\mathrm{GL}_2(q)}(D_8') = H\,\mathrm{C}_{\mathrm{GL}_2(q)}(D_8').$$

Analogously to the procedure in the proof of Proposition 7.18 we consider the quotient group $H\,\mathrm{C}_{\mathrm{GL}_2(q)}(D_8')/D_8'\,\mathrm{C}_{\mathrm{GL}_2(q)}(D_8')$. Since the representation of the dihedral group of order 8 yielding the group D_8' as a subgroup of $\mathrm{GL}_2(q)$ is absolutely irreducible, we have

$$\mathrm{C}_{\mathrm{GL}_2(q)}(D_8') = \mathrm{Z}(\mathrm{GL}_2(q))$$

according to Schur's lemma. Moreover, due to the assumption that $q \equiv -1 \bmod 4$ we have $\mathrm{Z}(D_8') = \mathcal{O}_2(\mathrm{Z}(\mathrm{GL}_2(q)))$, and hence $H \cap \mathrm{Z}(\mathrm{GL}_2(q)) = \mathcal{O}_2(Z(\mathrm{GL}_2(q)))$. Thus, we deduce that

$$H\,\mathrm{C}_{\mathrm{GL}_2(q)}(D_8')/D_8'\,\mathrm{C}_{\mathrm{GL}_2(q)}(D_8') = HZ(\mathrm{GL}_2(q))/D_8'Z(\mathrm{GL}_2(q)) \cong H/D_8' \cong C_2,$$

and hence we obtain

$$|H\,\mathrm{C}_{\mathrm{GL}_2(q)}(D_8')| = |C_2| \cdot |D_8'\,\mathrm{C}_{\mathrm{GL}_2(q)}(D_8')| = 2 \cdot \frac{|D_8'| \cdot |\,\mathrm{Z}(\mathrm{GL}_2(q))|}{|D_8' \cap \mathrm{Z}(\mathrm{GL}_2(q))|} = 8(q-1).$$

We conclude that

$$|H\,\mathrm{C}_{\mathrm{GL}_2(q)}(D_8')/\,\mathrm{C}_{\mathrm{GL}_2(q)}(D_8')| = 8 = |D_8'|.$$

Now, as commonly known, the automorphism group of D_8' is given by $\mathrm{Aut}(D_8') \cong D_8'$, and since $H\,\mathrm{C}_{\mathrm{GL}_2(q)}(D_8') \subseteq \mathrm{N}_{\mathrm{GL}_2(q)}(D_8')$ and the quotient $\mathrm{N}_{\mathrm{GL}_2(q)}(D_8')/\,\mathrm{C}_{\mathrm{GL}_2(q)}(D_8')$ embeds into $\mathrm{Aut}(D_8')$, we consequently have $H\,\mathrm{C}_{\mathrm{GL}_2(q)}(D_8') = \mathrm{N}_{\mathrm{GL}_2(q)}(D_8')$.

Let us now consider the centralizer and normalizer of R in G. Similarly as in the proof of Proposition 7.18 one can show that $\mathrm{C}_G(R)$ and $\mathrm{N}_G(R)$ are given by the images of $\mathrm{C}_{\mathrm{GL}_2(q)}(D_8')$ and $\mathrm{N}_{\mathrm{GL}_2(q)}(D_8')$, respectively, under the embedding

$$\mathrm{GL}_2(q) \hookrightarrow G,\ A \longmapsto \begin{pmatrix} A & \\ & (\det A)^{-1} \end{pmatrix},$$

so in particular the centralizer of R is given by $\mathrm{C}_G(R) = \langle \mathrm{diag}(\zeta, \zeta, \zeta^{-2}) \rangle$, and we have an isomorphism

$$\mathrm{N}_G(R)/R\,\mathrm{C}_G(R) \cong \mathrm{N}_{\mathrm{GL}_2(q)}(D_8')/D_8'\,\mathrm{C}_{\mathrm{GL}_2(q)}(D_8') \cong C_2$$

as claimed. □

Corollary 7.21. *In the situation of Proposition 7.20 the group* $\mathrm{N}_G(R)/R$ *centralizes its subgroup* $R\,\mathrm{C}_G(R)/R$.

Proof. This follows immediately from the fact that $\mathrm{C}_G(R) = \langle \mathrm{diag}(\zeta, \zeta, \zeta^{-2}) \rangle$ and $\mathrm{N}_G(R)$ is contained in the image of $\mathrm{GL}_2(q)$ in G under the embedding

$$\mathrm{GL}_2(q) \hookrightarrow G,\ A \longmapsto \begin{pmatrix} A & \\ & (\det A)^{-1} \end{pmatrix},$$

as observed in the proof of Proposition 7.20. □

We next determine the basic subgroups of the general linear groups $\mathrm{GL}_1(q)$, $\mathrm{GL}_2(q)$ and $\mathrm{GL}_3(q)$. Knowing these we may apply Theorem 7.11 to determine all radical 2-subgroups of $\mathrm{GL}_3(q)$, and hence of $\mathrm{SL}_3(q)$ by the considerations at the beginning of this section. The lemma below will help to understand the structure of some of the arising basic subgroups.

Lemma 7.22. *The central product of C_2 with C_{2^k} is isomorphic to C_{2^k} for any natural number $k > 0$.*

Proof. The central product of C_2 with C_{2^k} is isomorphic to $(C_2 \times C_{2^k})/\langle(-1,-1)\rangle$. We denote by $\overline{}\colon C_2 \times C_{2^k} \longrightarrow (C_2 \times C_{2^k})/\langle(-1,-1)\rangle$ the natural epimorphism. Let x be a generator of C_{2^k}. Then the group $C_2 \times C_{2^k}$ is generated by $(-1,1)$ and $(1,x)$, and we have

$$\overline{(-1,1)} = \overline{(1,-1)} = \overline{(1,x)}^{2^{k-1}}.$$

Hence, the quotient $(C_2 \times C_{2^k})/\langle(-1,-1)\rangle$ is generated by $\overline{(1,x)}$, which is of order 2^k. □

Proposition 7.23. *The basic subgroups of the general linear groups $\mathrm{GL}_n(q)$, $n \in \{1,2,3\}$, are given as follows, where $\zeta, \xi \in \mathbb{F}^\times$ are of orders $q-1$ and q^2-1, respectively:*

(i) For $n = 1$ the only basic subgroup of $\mathrm{GL}_1(q)$ is $\mathcal{O}_2(\mathrm{Z}(\mathrm{GL}_1(q))) = \mathcal{O}_2(\langle(\zeta)\rangle)$.

(ii) For $n = 2$ and a basic subgroup R of $\mathrm{GL}_2(q)$ up to $\mathrm{GL}_2(q)$-conjugation one of the following holds:

(a) $R = \mathcal{O}_2(\mathrm{Z}(\mathrm{GL}_2(q))) = \mathcal{O}_2(\langle\mathrm{diag}(\zeta,\zeta)\rangle)$,

(b) $R \sim_{\mathrm{GL}_2(\mathbb{F})} \mathcal{O}_2(\langle\mathrm{diag}(\xi,\xi^q)\rangle)$,

(c) $R \in \mathrm{Syl}_2(\mathrm{GL}_2(q))$, *i.e.*,

$$\begin{array}{lll} & R = \mathcal{O}_2(\langle\mathrm{diag}(\zeta,1)\rangle \times \langle\mathrm{diag}(1,\zeta)\rangle) \rtimes \langle\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)\rangle & \textit{if } q \equiv +1 \bmod 4, \\ \textit{or} & R \sim_{\mathrm{GL}_2(\mathbb{F})} \mathcal{O}_2(\langle\mathrm{diag}(\xi,\xi^q)\rangle) \rtimes \langle\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)\rangle & \textit{if } q \equiv -1 \bmod 4, \end{array}$$

(d) $q \equiv +1 \bmod 4$ *and* $R = \widetilde{Q_8'}$,

(e) $q \equiv -1 \bmod 4$ *and* $R \in \{Q_8', D_8'\}$.

(iii) For $n = 3$ the only basic subgroup of $\mathrm{GL}_3(q)$ is $\mathcal{O}_2(\mathrm{Z}(\mathrm{GL}_3(q))) = \mathcal{O}_2(\langle\mathrm{diag}(\zeta,\zeta,\zeta)\rangle)$.

Proof. We use the notation introduced in Definition 7.10 for the basic subgroups, that is, if R is a basic subgroup of $\mathrm{GL}_n(q)$, then $R = R^i_{m,\alpha,\gamma,\mathbf{c}}$ for certain integers $i \in \{1,2\}$, $m \geqslant 1$, $\alpha, \gamma \geqslant 0$ and a sequence $\mathbf{c} = (c_1,\ldots,c_t)$ of non-negative integers $c_j \in \mathbb{Z}_{\geqslant 0}$ such that we have $n = m2^{\alpha+\gamma+c_1+\cdots+c_t}$. Moreover, $a \in \mathbb{N}$ is such that $(q^2-1)_2 = 2^{a+1}$.

(i) Let us first determine the basic subgroups of the group $\mathrm{GL}_1(q)$. In this case we have $n = 1 = 1 \cdot 2^0$, so $m = 1$, $\alpha = \gamma = c_i = 0$. We deduce that $A_{\mathbf{c}} = 1$, whence it follows that $R^i_{1,0,0,\mathbf{c}} = R_{1,0,0} = R_{0,0}$. Hence, there exists only one basic subgroup of $\mathrm{GL}_1(q)$. For $q \equiv 1 \bmod 4$ this is an embedding of the group E_0Z_0 into $\mathrm{GL}_1(q)$, which is the central product of the cyclic groups C_2 and C_{2^a}, i.e., an embedding of C_{2^a} into $\mathrm{GL}_1(q)$ by Lemma 7.22. Since $\mathrm{GL}_1(q)$ is cyclic, this is uniquely determined, given by $\mathcal{O}_2(\mathrm{Z}(\mathrm{GL}_1(q))) = \mathcal{O}_2(\langle(\zeta)\rangle)$ as claimed. For $q \equiv -1 \bmod 4$ the group Z_0 was defined to be the center of E_γ, so in this case $R_{0,0}$ is the embedding of $E_0Z_0 = C_2$ into $\mathrm{GL}_1(q)$. As before, this is uniquely determined and given by $\mathcal{O}_2(\mathrm{Z}(\mathrm{GL}_1(q))) = \mathcal{O}_2(\langle(\zeta)\rangle)$.

(ii) For the basic subgroups of $\mathrm{GL}_2(q)$ let us first observe that $n = 2 = 1 \cdot 2^1 = 2 \cdot 2^0$, so we have to consider several cases:

(1) $m = 1$, $\alpha = 1$, $\gamma = 0$, $c_j = 0$ for all $j \geqslant 1$,

(2) $m = 1$, $\alpha = 0$, $\gamma = 1$, $c_j = 0$ for all $j \geqslant 1$,

(3) $m = 1$, $\alpha = 0$, $\gamma = 0$, $c_j = 0$ for all $j \geqslant 2$, $c_1 = 1$,

(4) $m = 2$, $\alpha = 0$, $\gamma = 0$, $c_j = 0$ for all $j \geqslant 1$.

In Case (1) we have $R^i_{m,\alpha,\gamma,\mathbf{c}} = R_{1,0}$. This is an embedding of the central product $C_2 C_{2^{a+1}}$, which is isomorphic to $C_{2^{a+1}}$ by Lemma 7.22, into the general linear group $\mathrm{GL}_2(q)$. We prove that up to $\mathrm{GL}_2(q)$-conjugation there exists only one such subgroup inside $\mathrm{GL}_2(q)$. Let $A \in \mathrm{GL}_2(q)$ be of order $2^{a+1} = (q^2-1)_2$. Then there exists an eigenvalue of A of order $(q^2-1)_2$, say $\beta \in \mathbb{F}^\times_{q^2} \setminus \mathbb{F}^\times_q$. Since $\beta \notin \mathbb{F}^\times_q$, by [Ste51, p. 226] the second eigenvalue of A is $\beta^q \neq \beta$. Hence, A is a semisimple element of $\mathrm{GL}_2(q)$, which is conjugate to $\mathrm{diag}(\beta, \beta^q)$ in $\mathrm{GL}_2(\mathbb{F})$. Now let $A' \in \mathrm{GL}_2(q)$ be another element of order $(q^2-1)_2$ with eigenvalue $\beta' \in \mathbb{F}^\times_{q^2} \setminus \mathbb{F}^\times_q$ of order $(q^2-1)_2$. As before, the second eigenvalue of A' is $\beta'^q \neq \beta'$. Being cyclic, the group $\mathbb{F}^\times_{q^2}$ contains only one subgroup of order $(q^2-1)_2$, so in particular there exists an odd element $x \in \mathbb{N}$ such that $\beta'^x = \beta$. The matrix A'^x is hence still of order $(q^2-1)_2$ and has eigenvalues β and β^q. Thus, in $\mathrm{GL}_2(\mathbb{F})$ it is conjugate to $\mathrm{diag}(\beta, \beta^q)$, and hence to A. By Remark 7.2 the matrices A and A'^x are already conjugate in $\mathrm{GL}_2(q)$. In particular, the subgroups of $\mathrm{GL}_2(q)$ generated by A respectively A'^x lie in the same $\mathrm{GL}_2(q)$-conjugacy class. Since A' and A'^x generate the same subgroup of $\mathrm{GL}_2(q)$, we may now conclude that any two cyclic subgroups of $\mathrm{GL}_2(q)$ of order $(q^2-1)_2$ are conjugate in $\mathrm{GL}_2(q)$. In particular, we have

$$R^i_{m,\alpha,\gamma,\mathbf{c}} \sim_{\mathrm{GL}_2(\mathbb{F})} \mathcal{O}_2(\langle \mathrm{diag}(\xi, \xi^q)\rangle),$$

which gives (ii)(b).

Case (2) gives $R^i_{m,\alpha,\gamma,\mathbf{c}} = R_{0,1}$ if $q \equiv 1 \bmod 4$. In this situation $R_{0,1}$ is an embedding of the central product $E_1 C_{2^a}$ into $\mathrm{GL}_2(q)$, where E_1 is an extraspecial group of order 2^{1+2}, i.e., E_1 is either a quaternion group Q_8 or a dihedral group D_8 of order 8, depending on the type of E_1. By [An92, p. 501] the product $E_1 C_{2^a}$ does not depend on the type of E_1, so we may assume that $E_1 = Q_8$. As described in Definition 7.10 the group $R_{0,1}$ is obtained by embedding $E_1 C_{2^a}$ into $\mathrm{GL}_2(q)$ in such a way that C_{2^a} is identified with $\mathcal{O}_2(\mathrm{Z}(\mathrm{GL}_2(q)))$. Moreover, by Corollary 7.15 there exists exactly one conjugacy class of subgroups in $\mathrm{GL}_2(q)$ isomorphic to Q_8. Since $\mathrm{GL}_2(q)$-conjugation stabilizes the central subgroup $\mathcal{O}_2(\mathrm{Z}(\mathrm{GL}_2(q)))$ of $\mathrm{GL}_2(q)$, we may hence assume that $R^i_{m,\alpha,\gamma,\mathbf{c}} = \langle Q'_8, \mathrm{diag}(\zeta, \zeta)_2\rangle = \widetilde{Q'_8} \leqslant \mathrm{GL}_2(q)$, so we obtain (ii)(d).

Assume now that $q \equiv -1 \bmod 4$. Then $R^1_{m,\alpha,\gamma,\mathbf{c}} = R_{0,1}$ and $R^2_{m,\alpha,\gamma,\mathbf{c}} = S_{1,1,0} = S_{1,0}$. Consider first $R^1_{m,\alpha,\gamma,\mathbf{c}}$. We have $R_{0,1} = E_1 Z_0 = E_1$, and as for $q \equiv 1 \bmod 4$, the group E_1 is either quaternion or dihedral of order 8. If E_1 is quaternion, then by Corollary 7.15 we may assume that $R^1_{m,\alpha,\gamma,\mathbf{c}} = Q'_8$, and if E_1 is dihedral, then by Corollary 7.16 we have $R = D'_8$ up to $\mathrm{GL}_2(q)$-conjugation. This yields (ii)(e). Let us now examine $R^2_{m,\alpha,\gamma,\mathbf{c}}$. By definition $S_{1,0}$ is the image of the central product $E_0 P$ embedded into $\mathrm{GL}_2(q)$, where P is a semidihedral group of order 2^{a+2} and $E_0 = C_2$. As $|\mathrm{GL}_2(q)|_2 = (q-1)^2_2(q+1)_2 = 2^{a+2}$, we have $S_{1,0} \in \mathrm{Syl}_2(\mathrm{GL}_2(q))$ and hence

$$R^2_{m,\alpha,\gamma,\mathbf{c}} \sim_{\mathrm{GL}_2(\mathbb{F})} \mathcal{O}_2(\langle \mathrm{diag}(\xi, \xi^q)\rangle) \rtimes \langle \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)\rangle\,,$$

which gives (ii)(c) for $q \equiv -1 \bmod 4$.

In case (3) we only need to determine the groups $R^i_{m,\alpha,\gamma,\mathbf{c}}$ for $q \equiv 1 \bmod 4$ since by definition they do not constitute basic subgroups if $q \equiv -1 \bmod 4$. We have

$$R^i_{m,\alpha,\gamma,\mathbf{c}} = R^i_{m,\alpha,\gamma} \wr A_{\mathbf{c}} = R_{0,0} \wr \mathfrak{S}_2 \leqslant \mathrm{GL}_2(q).$$

As we computed for $\mathrm{GL}_1(q)$, the group $R_{0,0}$ is the uniquely determined image of the embedding of C_{2^a} into $\mathrm{GL}_1(q)$. In particular, it follows that the group $R^i_{m,\alpha,\gamma,\mathbf{c}}$ is of order given by $2 \cdot (2^a)^2 = |\operatorname{GL}_2(q)|_2$. Hence, it forms a Sylow 2-subgroup of $\operatorname{GL}_2(q)$, and we have

$$R^i_{m,\alpha,\gamma,\mathbf{c}} = \mathcal{O}_2(\langle \operatorname{diag}(\zeta,1)\rangle \times \langle \operatorname{diag}(1,\zeta)\rangle) \rtimes \langle (\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix})\rangle$$

up to $\operatorname{GL}_2(q)$-conjugation. This gives (ii)(c) for $q \equiv 1 \bmod 4$.

Let us now consider case (4). Then $R^i_{m,\alpha,\gamma,\mathbf{c}} = R_{m,\alpha,\gamma}$ is the image of $R_{0,0}$ under the two-fold diagonal embedding, that is, we have $R^i_{m,\alpha,\gamma,\mathbf{c}} = \mathcal{O}_2(\langle \operatorname{diag}(\zeta,\zeta)\rangle) \leqslant \operatorname{GL}_2(q)$. This gives the basic subgroup in (ii)(a).

(iii) It remains to find the basic subgroups of $\operatorname{GL}_3(q)$. We have $n = 3 = 3 \cdot 2^0$, so $m = 3$ and $\alpha = \gamma = c_i = 0$. Then $R^i_{m,\alpha,\gamma,\mathbf{c}} = R^i_{m,\alpha,\gamma} = R_{m,\alpha,\gamma}$ is the image of $R_{0,0}$ in $\operatorname{GL}_3(q)$ under the 3-fold diagonal mapping, i.e., we have $R^i_{m,\alpha,\gamma,\mathbf{c}} = \mathcal{O}_2(\langle \operatorname{diag}(\zeta,\zeta,\zeta)\rangle)$ as claimed. $\square$

We have now obtained all necessary information to construct the radical 2-subgroups of the special linear group G.

Proposition 7.24. *Let $R \leqslant G$ be a radical 2-subgroup of G. Then up to G-conjugation one of the following holds:*

(i) $R = \{1\}$,

(ii) $R = \mathcal{O}_2(\langle \operatorname{diag}(\zeta,\zeta,\zeta^{-2})\rangle)$, *where $\zeta \in \mathbb{F}^\times$ is of order $q-1$,*

(iii) $(q+1)_{2'} \neq 1$ *and $R = \mathcal{O}_2(T')$, where T' denotes a maximal torus of G of type T_2,*

(iv) $R = \widetilde{Q_8}$,

(v) $R = \mathcal{O}_2(T)$, *where T denotes a maximal torus of G of type T_1,*

(vi) $R \in \operatorname{Syl}_2(G)$.

Proof. Let us first verify that the above specified subgroups of G are indeed 2-radical in G. This is clearly true for the Sylow 2-subgroups of G and for the trivial subgroup $\{1\}$ as G is simple.

For $R = \mathcal{O}_2(\langle \operatorname{diag}(\zeta,\zeta,\zeta^{-2})\rangle)$ as in (ii) we observe that $\operatorname{N}_G(R) \cong \operatorname{GL}_2(q)$ by Lemma 7.7. Analogously to the proof of Proposition 7.9 one can show that R is 2-radical in G.

For $R = \mathcal{O}_2(T')$ as in (iii) we recall that by Proposition 5.7 we have $\operatorname{N}_G(T') \cong T' \rtimes C_2$, where the generating element c of C_2 acts trivially only on elements of T' whose order divides $q-1$. Moreover, $\operatorname{N}_G(R) = \operatorname{N}_G(T')$ by Proposition 5.8, so

$$\operatorname{N}_G(R)/R \cong C_{(q^2-1)_{2'}} \rtimes C_2$$

such that c commutes only with elements of $C_{(q^2-1)_{2'}}$ of order dividing $q-1$. Now for $(q+1)_{2'} \neq 1$ there exists an element s in $C_{(q^2-1)_{2'}}$ whose order does not divide $q-1$, for instance the generator of this cyclic group. Hence, s and c do not commute. Now suppose that $t \in C_{(q^2-1)_{2'}}$ is such that the element tc has order 2 in $C_{(q^2-1)_{2'}} \rtimes C_2$. Then $s^{-1}tcs = ts^{-1}cs \neq tc$, i.e., s does not normalize the subgroup of $C_{(q^2-1)_{2'}} \rtimes C_2$ generated by tc. In particular, $\operatorname{N}_G(R)/R$ with $|\operatorname{N}_G(R)/R|_2 = 2$ does not contain a non-trivial normal 2-subgroup, so R is a radical 2-subgroup of G in this case.

Let us now consider the case $R = \widetilde{Q_8}$ as in (iv). From Proposition 7.18 it follows that $\operatorname{C}_G(R) = \langle \operatorname{diag}(\zeta,\zeta,\zeta^{-2})\rangle$ and $\operatorname{N}_G(R)/R\operatorname{C}_G(R) \cong \mathfrak{S}_3$. Hence,

$$R\operatorname{C}_G(R)/R \cong \mathcal{O}_{2'}(\langle \operatorname{diag}(\zeta,\zeta,\zeta^{-2})\rangle)$$

since $R \cap \langle \mathrm{diag}(\zeta,\zeta,\zeta^{-2})\rangle = \mathcal{O}_2(\langle \mathrm{diag}(\zeta,\zeta,\zeta^{-2})\rangle)$, so $R\,\mathrm{C}_G(R)/R$ is a $2'$-group. Let us now suppose that P is a normal 2-subgroup of $\mathrm{N}_G(R)/R$. Then

$$(P(R\,\mathrm{C}_G(R)/R))/(R\,\mathrm{C}_G(R)/R)$$

is a normal 2-subgroup of $(\mathrm{N}_G(R)/R)/(R\,\mathrm{C}_G(R)/R) \cong \mathrm{N}_G(R)/R\,\mathrm{C}_G(R) \cong \mathfrak{S}_3$. But the symmetric group $\mathfrak{S}_3$ does not contain any non-trivial normal 2-subgroups, so P is contained in $R\,\mathrm{C}_G(R)/R$. Since this is a $2'$-group, we conclude that $P = \{1\}$, that is, $\mathcal{O}_2(\mathrm{N}_G(R)/R) = \{1\}$ and R is a radical 2-subgroup of G as claimed.

Suppose now that $R = \mathcal{O}_2(T)$ as in (v), where T denotes a maximal torus of G of type T_1. By Proposition 5.8 the normalizer of R is given by $\mathrm{N}_G(R) = \mathrm{N}_G(T) \cong T \rtimes \mathfrak{S}_3$, so $\mathrm{N}_G(R)/R \cong (C_{(q-1)_{2'}} \times C_{(q-1)_{2'}}) \rtimes \mathfrak{S}_3$, which does not have any non-trivial normal 2-subgroups since $\mathfrak{S}_3$ does not possess any. Thus, R is 2-radical in G.

It remains to prove that any radical 2-subgroup of G is G-conjugate to one of the subgroups specified in the proposition. By Theorem 7.11 and the preceding considerations, in order to prove that we have found all radical 2-subgroups of $\mathrm{SL}_3(q)$ it suffices to find all subgroups R' of $\mathrm{GL}_3(q)$ that have a decomposition as in Theorem 7.11 and check which of those are 2-radical. We have $\mathrm{GL}_3(q) \cong \mathrm{GL}(V)$ for any 3-dimensional $\mathbb{F}_q$-vector space V, and the possible decompositions of V up to isomorphism are $V = V_1$ with $\dim V_1 = 3$, $V = V_1 \oplus V_2$ with $\dim V_1 = 2$, $\dim V_2 = 1$, and $V = V_1 \oplus V_2 \oplus V_3$ with $\dim V_i = 1$, $i = 1, 2, 3$.

Suppose first that $V = V_1$ with $\dim V_1 = 3$. Then $R' = R_1$ is a basic subgroup of $\mathrm{GL}(V_1) \cong \mathrm{GL}_3(q)$ or possibly $R' = \{\pm 1_{V_1}\}$ if $q \equiv -1 \bmod 4$. According to Proposition 7.23 it holds that the only basic subgroup of $\mathrm{GL}_3(q)$ is given by $\mathcal{O}_2(\langle \mathrm{diag}(\zeta,\zeta,\zeta)\rangle)$. Moreover, if $q \equiv -1 \bmod 4$, then $\{\pm 1_{V_1}\}$ and $\mathcal{O}_2(\langle \mathrm{diag}(\zeta,\zeta,\zeta)\rangle)$ agree, so in any case we obtain that $R' = \mathcal{O}_2(\langle \mathrm{diag}(\zeta,\zeta,\zeta)\rangle)$. Hence, $R' \cap \mathrm{SL}_3(q) = \{1\}$, which is 2-radical. This gives (i).

Let now $V = V_1 \oplus V_2$ with $\dim V_1 = 2$ and $\dim V_2 = 1$. Suppose first that $q \equiv 1 \bmod 4$. Then $R' = R_1 \times R_2$ with R_1, R_2 basic subgroups of $\mathrm{GL}_2(q)$, $\mathrm{GL}_1(q)$, respectively. By Proposition 7.23 up to conjugation (possibly in $\mathrm{GL}_3(\mathbb{F})$) one of the following holds:

(a) $R' = \mathcal{O}_2(\langle \mathrm{diag}(\xi,\xi^q,1)\rangle) \times \mathcal{O}_2(\langle \mathrm{diag}(1,1,\zeta)\rangle)$, where $\xi \in \mathbb{F}^\times$ is of order q^2-1,

(b) $R' = \mathcal{O}_2(\langle Q_8', \mathrm{diag}(\zeta,\zeta,1)\rangle) \times \mathcal{O}_2(\langle \mathrm{diag}(1,1,\zeta)\rangle)$ where we identify Q_8' with its image under the embedding $\mathrm{GL}_2(q) \to \mathrm{GL}_3(q)$, $A \mapsto \begin{pmatrix} A & \\ & 1 \end{pmatrix}$,

(c) $R' = \mathcal{O}_2((\langle \mathrm{diag}(\zeta,1,1)\rangle \times \langle \mathrm{diag}(1,\zeta,1)\rangle \times \langle \mathrm{diag}(1,1,\zeta)\rangle)) \rtimes \left\langle \begin{pmatrix} 0&1&0\\1&0&0\\0&0&1 \end{pmatrix} \right\rangle$,

(d) $R' = \mathcal{O}_2(\langle \mathrm{diag}(\zeta,\zeta,1)\rangle) \times \mathcal{O}_2(\langle \mathrm{diag}(1,1,\zeta)\rangle)$.

These correspond to the G-conjugacy classes of radical 2-subgroups of G represented by $\mathcal{O}_2(T') \sim_{\mathbf{G}} \mathcal{O}_2(\langle \mathrm{diag}(\xi,\xi^q,\xi^{-(q+1)})\rangle)$ in (a), $\widetilde{Q_8}$ in (b), a Sylow 2-subgroup of G in (c) and $\mathcal{O}_2(\langle \mathrm{diag}(\zeta,\zeta,\zeta^{-2})\rangle)$ in (d).

For $q \equiv -1 \bmod 4$ the groups R_i are either basic subgroups or $R_i = \{\pm 1_{V_i}\}$. In any case, $R_2 = \{\pm 1_{V_2}\}$ as we observed above, so the basic subgroups of $\mathrm{GL}_2(q)$ given in Proposition 7.23 yield the following candidates for radical 2-subgroups of $\mathrm{GL}_3(q)$:

(a) $R' = \langle \mathrm{diag}(-1,-1,1)\rangle \times \langle \mathrm{diag}(1,1,-1)\rangle$,

(b) $R' \sim_{\mathrm{GL}_3(\mathbb{F})} \mathcal{O}_2(\langle \mathrm{diag}(\xi,\xi^q,1)\rangle) \times \langle \mathrm{diag}(1,1,-1)\rangle$,

(c) $R' = \mathcal{O}_2(\langle \mathrm{diag}(\xi,\xi^q,1), \mathrm{diag}(1,1,-1)\rangle) \rtimes \left\langle \begin{pmatrix} 0&1&0\\1&0&0\\0&0&1 \end{pmatrix} \right\rangle$,

(d) $R' = Q_8' \times \mathcal{O}_2(\langle \mathrm{diag}(1,1,-1)\rangle)$, where we identify Q_8' with its image under the embedding $\mathrm{GL}_2(q) \to \mathrm{GL}_3(q)$, $A \mapsto \left(\begin{smallmatrix} A & \\ & 1 \end{smallmatrix}\right)$,

(e) $R' = D_8' \times \mathcal{O}_2(\langle \mathrm{diag}(1,1,-1)\rangle)$, where we identify D_8' with its image under the embedding $\mathrm{GL}_2(q) \to \mathrm{GL}_3(q)$, $A \mapsto \left(\begin{smallmatrix} A & \\ & 1 \end{smallmatrix}\right)$.

The subgroups of $\mathrm{GL}_3(q)$ specified in points (a), (b), (c) and (d) above correspond to the conjugacy classes of radical 2-subgroups of G represented by the groups $\mathcal{O}_2(\mathrm{diag}(\zeta,\zeta,\zeta^{-2}))$, $\mathcal{O}_2(T') \sim_{\mathbf{G}} \mathcal{O}_2(\langle \mathrm{diag}(\xi,\xi^q,\xi^{-(q+1)})\rangle)$, a Sylow 2-subgroup of G, and $\widetilde{Q_8}$, respectively.

We prove now that the group in (e) does not admit a radical 2-subgroup of G. Suppose that R' is as in case (e). Then $R' \cap G = \widetilde{D_8}$ and by Proposition 7.20 we have

$$\mathrm{C}_G(\widetilde{D_8}) = \langle \mathrm{diag}(\zeta,\zeta,\zeta^{-2})\rangle$$

and

$$\mathrm{N}_G(\widetilde{D_8})/\widetilde{D_8}\,\mathrm{C}_G(\widetilde{D_8}) \cong C_2.$$

In particular, this yields

$$(\mathrm{N}_G(\widetilde{D_8})/\widetilde{D_8})/(\widetilde{D_8}\,\mathrm{C}_G(\widetilde{D_8})/\widetilde{D_8}) \cong \mathrm{N}_G(\widetilde{D_8})/\widetilde{D_8}\,\mathrm{C}_G(\widetilde{D_8}) \cong C_2.$$

Moreover, by Corollary 7.21 the group $\widetilde{D_8}\,\mathrm{C}_G(\widetilde{D_8})/\widetilde{D_8}$ lies in the center of $\mathrm{N}_G(\widetilde{D_8})/\widetilde{D_8}$, and since its index in $\mathrm{N}_G(\widetilde{D_8})/\widetilde{D_8}$ is 2, we conclude that $\mathrm{N}_G(\widetilde{D_8})/\widetilde{D_8}$ is abelian. Now 2 divides the order of $\mathrm{N}_G(\widetilde{D_8})/\widetilde{D_8}$, so we conclude that $\mathrm{N}_G(\widetilde{D_8})/\widetilde{D_8}$ contains a non-trivial normal 2-subgroup, i.e., $\widetilde{D_8}$ is not 2-radical in G.

Finally we suppose that V decomposes as $V = V_1 \oplus V_2 \oplus V_3$ with $\dim V_i = 1$ for all i. Then we get

$$R' = \mathcal{O}_2(\langle \mathrm{diag}(\zeta,1,1)\rangle) \times \mathcal{O}_2(\langle \mathrm{diag}(1,\zeta,1)\rangle) \times \mathcal{O}_2(\langle \mathrm{diag}(1,1,\zeta)\rangle),$$

which corresponds to the radical 2-subgroup $\mathcal{O}_2(T)$ of G as in (v).

We summarize that any subgroup of $\mathrm{GL}_3(q)$ of the form described in Theorem 7.11 either admits a subgroup of G conjugate to one of the groups specified in the claim of this proposition or a non-radical 2-subgroup of G. This finishes the proof. □

We have now obtained a classification of the radical ℓ-subgroups of G for all primes ℓ dividing $q-1$. Applying Construction 2.10 we will in the following make use of this classification to find the B-weights of G for all ℓ-blocks B of G of non-cyclic defect.

7.2.2 B-Weights and the Action of $\mathrm{Aut}(G)_B$ in the Case $2 \neq \ell \mid (q-1)$

Let us suppose that ℓ is an odd prime dividing $q-1$. We recall from Proposition 6.8 that there exist five distinct types of ℓ-blocks for G, namely the principal ℓ-block B_0, the ℓ-blocks of types $B_{(\text{-})}$, $B_{(\text{-},\text{-})}$ or $B'_{(\text{-})}$, and the ℓ-blocks of ℓ-defect zero. As observed in Lemma 6.10 only the ℓ-blocks of types

$$B_0,\ B_{(\text{-})} \text{ and } B_{(\text{-},\text{-})}$$

have non-cyclic defect groups. More precisely, these are of maximal defect, that is to say, their defect groups are given by the Sylow ℓ-subgroups of G. Since these are abelian according to Proposition 5.9, it follows from Lemma 2.11 that $R \in \mathrm{Syl}_\ell(G)$ for every ℓ-weight (R,φ) belonging to such an ℓ-block. In consequence of Construction 2.10, representatives for the G-conjugacy classes of B-weights for an ℓ-block B of type B_0, $B_{(\text{-})}$ or $B_{(\text{-},\text{-})}$ are hence constructed as follows:

Throughout this section we let T denote the maximally split torus of G consisting of the diagonal matrices in G. Its ℓ-core $\mathcal{O}_\ell(T)$ is a Sylow ℓ-subgroup of G by Proposition 5.9 and a defect group of B as observed above. Furthermore, we have

$$\mathcal{O}_\ell(T)\,\mathrm{C}_G(\mathcal{O}_\ell(T)) = T$$

and

$$\mathrm{N}_G(\mathcal{O}_\ell(T)) = \mathrm{N}_G(T)$$

by Proposition 5.9, and by Theorem 1.7, Brauer's extended first main theorem, there exists only one $\mathrm{N}_G(T)$-conjugacy class of ℓ-blocks $b \in \mathrm{Bl}_\ell(T \mid \mathcal{O}_\ell(T))$ inducing to B. Note that due to T being abelian each of its ℓ-blocks has defect group $\mathcal{O}_\ell(T)$ following Lemma 1.13, so in other words we have $\mathrm{Bl}_\ell(T \mid \mathcal{O}_\ell(T)) = \mathrm{Bl}_\ell(T)$. Hence, according to Construction 2.10 we obtain a complete set of representatives for the G-conjugacy classes of B-weights of G by fixing one ℓ-block $b \in \mathrm{Bl}_\ell(T)$ with $b^G = B$, determining all characters $\psi \in \mathrm{Irr}(\mathrm{N}_G(T)_\theta \mid \theta)$ such that $\psi(1)_\ell = |\,\mathrm{N}_G(T)_\theta \colon T|_\ell$, where $\theta \in \mathrm{Irr}(T)$ denotes the canonical character of b, and finally constructing the ℓ-weights

$$(\mathcal{O}_\ell(T),\ \mathrm{Ind}_{\mathrm{N}_G(T)_\theta}^{\mathrm{N}_G(T)}(\psi))$$

for those ψ. The ℓ-weights obtained this way belong to B, they are distinct for distinct ψ and up to G-conjugation they constitute all ℓ-weights of G associated to B.

Consequently, for each ℓ-block B in question our task is to find a root $b \in \mathrm{Bl}_\ell(T)$ of B with associated canonical character $\theta \in \mathrm{Irr}(T)$, and then to construct all characters $\psi \in \mathrm{Irr}(\mathrm{N}_G(T)_\theta \mid \theta)$ satisfying $\psi(1)_\ell = |\,\mathrm{N}_G(T)_\theta \colon T|_\ell$. Induction of these characters as above then yields the B-weights aimed for.

As has become apparent from the considerations above, the irreducible characters of the maximally split torus T play an important role for the construction of ℓ-weights in the present situation. It will therefore be convenient to fix the following parametrization of $\mathrm{Irr}(T)$:

Notation 7.25 (Parametrization of $\mathrm{Irr}(T)$)**.** As before we choose $\zeta \in \mathbb{F}^\times$ to generate $\mathbb{F}_q^\times$. Using this notation we consider the linear character θ_0 of the cyclic group $\mathbb{F}_q^\times$ defined by

$$\theta_0 \colon \mathbb{F}_q^\times \longrightarrow \mathbb{C}^\times, \quad \zeta \longmapsto \varepsilon := \exp\left(\frac{2\pi\mathrm{i}}{q-1}\right).$$

Then θ_0 generates the character group of $\mathbb{F}_q^\times$, that is, the set of irreducible characters of $\mathbb{F}_q^\times$ may be written as

$$\mathrm{Irr}(\mathbb{F}_q^\times) = \{\theta_0^i \mid 0 \leqslant i < q-1\}.$$

Now the maximal torus T is isomorphic to the direct product $\mathbb{F}_q^\times \times \mathbb{F}_q^\times$, whence the above parametrization of $\mathrm{Irr}(\mathbb{F}_q^\times)$ yields a parametrization of $\mathrm{Irr}(T)$ given by

$$\mathrm{Irr}(T) = \{\theta_0^i \times \theta_0^j \times 1 \mid 0 \leqslant i, j < q-1\},$$

where we have

$$(\theta_0^i \times \theta_0^j \times 1)(\mathrm{diag}(t_1, t_2, (t_1t_2)^{-1})) = \theta_0^i(t_1)\theta_0^j(t_2)$$

for every diagonal matrix $\mathrm{diag}(t_1, t_2, (t_1t_2)^{-1}) \in T$ with $t_1, t_2 \in \mathbb{F}_q^\times$.

Having decided on a parametrization of $\mathrm{Irr}(T)$ we are now interested in the action of $\mathrm{N}_G(T)$ on this set, which will be of importance when it comes to determining the stabilizers $\mathrm{N}_G(T)_\theta$ for various canonical characters $\theta \in \mathrm{Irr}(T)$.

Lemma 7.26. *Let $\theta = \theta_0^i \times \theta_0^j \times 1 \in \mathrm{Irr}(T)$, and for $n \in \mathrm{N}_G(T)$ denote by $w \in \mathfrak{S}_3$ the image of n in $\mathrm{N}_G(T)/T \cong \mathfrak{S}_3$. Then we have*

(i) $\theta^n = \theta_0^j \times \theta_0^i \times 1$ *if* $w = (12)$,

(ii) $\theta^n = \theta_0^{i-j} \times \theta_0^{-j} \times 1$ *if* $w = (23)$,

(iii) $\theta^n = \theta_0^{-i} \times \theta_0^{j-i} \times 1$ *if* $w = (13)$,

(iv) $\theta^n = \theta_0^{j-i} \times \theta_0^{-i} \times 1$ *if* $w = (123)$,

(v) $\theta^n = \theta_0^{-j} \times \theta_0^{i-j} \times 1$ *if* $w = (132)$.

Proof. These are straightforward calculations in view of $\mathrm{N}_G(T)/T \cong \mathfrak{S}_3$ acting on T by permutation of the diagonal entries of the matrices in T. □

We are now fully prepared to construct the B-weights for all ℓ-blocks B of G of non-cyclic defect, i.e., for the ℓ-blocks of types B_0, $B_{(\text{-})}$ and $B_{(\text{-},\text{-})}$. For this purpose we treat each of these three cases individually.

7.2.2.1 The Principal Block B_0

Let $B = B_0$ be the principal ℓ-block of G. In consequence of the third main theorem of Brauer, Theorem 1.9, the principal ℓ-block b_0 of T is the unique ℓ-block of T that induces to B, and the associated canonical character θ is the trivial character of T, whence $\mathrm{N}_G(T)_\theta = \mathrm{N}_G(T)$. In particular, the trivial character $1_{\mathrm{N}_G(T)}$ of $\mathrm{N}_G(T)$ is an extension of θ to $\mathrm{N}_G(T)_\theta = \mathrm{N}_G(T)$, and by Gallagher's theorem, Theorem 1.21, we have

$$\mathrm{Irr}(\mathrm{N}_G(T) \mid \theta) = \{1_{\mathrm{N}_G(T)} \cdot \nu \mid \nu \in \mathrm{Irr}(\mathrm{N}_G(T)/T)\}.$$

As a result we obtain the following:

Proposition 7.27. *Suppose that $2 \neq \ell \mid (q-1)$ and let $B = B_0$ be the principal ℓ-block of G. Then $|\mathcal{W}(B)| = 3$. More precisely, if (R, φ) is a B-weight, then up to G-conjugation it holds that $R = \mathcal{O}_\ell(T)$ and $\varphi = 1_{\mathrm{N}_G(T)} \cdot \nu$, where ν denotes one of the three irreducible characters of $\mathrm{N}_G(T)/T \cong \mathfrak{S}_3$.*

Proof. This follows from the previous considerations seeing that the degrees of the irreducible characters of $\mathfrak{S}_3$ are either 1 or 2, whence, in the notation used above, for all $\psi \in \mathrm{Irr}(\mathrm{N}_G(T)_\theta \mid \theta)$ we have $\psi(1)_\ell = 1 = |6|_\ell = |\mathrm{N}_G(T)_\theta \colon T|_\ell$. Accordingly, each of the three irreducible characters of $\mathrm{N}_G(T)/T$ gives rise to a distinct G-conjugacy class of B-weights as claimed. □

Let us now consider the action of the automorphisms of G on the G-conjugacy classes of B-weights. We observe the following:

Proposition 7.28. *Suppose that $2 \neq \ell \mid (q-1)$ and let $B = B_0$ be the principal ℓ-block of G. Then $\mathrm{Aut}(G)_B = \mathrm{Aut}(G)$ acts trivially on $\mathcal{W}(B)$.*

Proof. Let (R, φ) be a B-weight and $a \in \mathrm{Aut}(G)$. By Proposition 7.27 we may assume that $R = \mathcal{O}_\ell(T) \in \mathrm{Syl}_\ell(G)$. In particular, constituting a Sylow ℓ-subgroup of G the image of R under a is a G-conjugate of R, and since our interests only lie in the G-conjugacy classes of B-weights, we may without loss of generality assume that $a \in \mathrm{Aut}(G)_R$. Moreover, following Proposition 7.27 the weight character φ is given by $\varphi = 1_{\mathrm{N}_G(T)} \cdot \nu$ for some irreducible character ν of $\mathrm{N}_G(T)/T \cong \mathfrak{S}_3$. If ν is the trivial character of $\mathrm{N}_G(T)/T$, then φ is clearly fixed by a. Moreover, a also leaves the two non-trivial irreducible characters of $\mathrm{N}_G(T)/T$ invariant since they have distinct degrees and can hence not be interchanged by a. Thus, for any choice of $\nu \in \mathrm{Irr}(\mathrm{N}_G(T)/T)$ the character $\varphi = 1_{\mathrm{N}_G(T)} \cdot \nu$ is left invariant by a, which finishes the proof. □

7.2.2.2 The Blocks of Type $B_{(\text{-})}$

Let us now turn to the ℓ-blocks of G of type $B_{(\text{-})}$. In a first step towards the construction of the ℓ-weights belonging to such blocks we take a look at the canonical characters associated to roots of ℓ-blocks of this type.

Lemma 7.29. *Suppose that $2 \neq \ell \mid (q-1)$ and denote by B an ℓ-block of G of type $B_{(\text{-})}$. Moreover, suppose that $b \in \mathrm{Bl}_\ell(T)$ is a root of B. Then there exists $1 \leqslant k < (q-1)_{\ell'}$ such that up to $\mathrm{N}_G(T)$-conjugation the canonical character of b is given by the linear character $\theta = \theta_0^{k(q-1)_\ell} \times 1 \times 1 \in \mathrm{Irr}(T)$.*

Proof. Suppose that $B = B_{(u)}$ for some parameter $u \in \{1, \dots, (q-1)_{\ell'} - 1\}$. As a result of Proposition 6.8 the ℓ-block B contains every irreducible character of G of type χ_{Φ_3} whose parameter v satisfies $u \equiv v \bmod (q-1)_{\ell'}$. If v is such a parameter, then there exists $i \in \{0, \dots, (q-1)_\ell - 1\}$ such that $v = u + i(q-1)_{\ell'}$, and we have

$$v \equiv 0 \bmod (q-1)_\ell \quad \text{ if and only if } \quad i(q-1)_{\ell'} \equiv -u \bmod (q-1)_\ell.$$

Since $\gcd((q-1)_\ell, (q-1)_{\ell'}) = 1$, the integer $(q-1)_{\ell'}$ is invertible modulo $(q-1)_\ell$, so in particular $i(q-1)_{\ell'} \equiv -u \bmod (q-1)_\ell$ has exactly one solution $i \in \{0, \dots, (q-1)_\ell - 1\}$. Consequently, there exists exactly one $v \in \{1, \dots, q-2\}$ with $v \equiv u \bmod (q-1)_{\ell'}$ and $v \equiv 0 \bmod (q-1)_\ell$, say $v = k(q-1)_\ell$ for some $1 \leqslant k < (q-1)_{\ell'}$. Thus, B contains exactly one character of type χ_{Φ_3} whose parameter is a multiple of $(q-1)_\ell$, namely the character with parameter $v = k(q-1)_\ell$. We show that the character

$$\theta = \theta_0^{k(q-1)_\ell} \times 1 \times 1 \in \mathrm{Irr}(T)$$

satisfies the claim.

As noted before, any ℓ-block b' of T has defect group $\mathcal{O}_\ell(T)$ since T is abelian, so in particular b'^G has defect group $\mathcal{O}_\ell(T) \in \mathrm{Syl}_\ell(G)$ by Lemma 1.1. Following Proposition 1.11 we hence have $b'^G = B$ if and only if

$$\lambda_B(\widehat{C}) = \lambda_{b'^G}(\widehat{C})$$

for every ℓ'-conjugacy class C of G with defect group $\mathcal{O}_\ell(T)$. Moreover, by the extended first main theorem of Brauer, Theorem 1.7, all ℓ-blocks $b' \in \mathrm{Bl}_\ell(T)$ with $b'^G = B$ are conjugate under $\mathrm{N}_G(\mathcal{O}_\ell(T)) = \mathrm{N}_G(T)$. Hence, it suffices to prove that $b' := \mathrm{bl}(\theta)$ satisfies this condition.

As a result of Proposition 6.3 the conjugacy classes of G with defect group $\mathcal{O}_\ell(T)$ are the classes C_1, $C_4^{(a)}$ and $C_6^{(a,b)}$ for appropriate parameters a, b (cf. Table 6.1), where for the trivial class C_1 there is nothing to show. We have

$$C_4^{(a)} \cap T = \{\mathrm{diag}(\zeta^a, \zeta^a, \zeta^{-2a}),\ \mathrm{diag}(\zeta^a, \zeta^{-2a}, \zeta^a),\ \mathrm{diag}(\zeta^{-2a}, \zeta^a, \zeta^a)\}$$

and

$$\begin{aligned}C_6^{(a,b)} \cap T = \{&\operatorname{diag}(\zeta^a, \zeta^b, \zeta^{-(a+b)}), \operatorname{diag}(\zeta^b, \zeta^a, \zeta^{-(a+b)}), \operatorname{diag}(\zeta^a, \zeta^{-(a+b)}, \zeta^b),\\ &\operatorname{diag}(\zeta^{-(a+b)}, \zeta^a, \zeta^b), \operatorname{diag}(\zeta^b, \zeta^{-(a+b)}, \zeta^a), \operatorname{diag}(\zeta^{-(a+b)}, \zeta^b, \zeta^a)\}.\end{aligned}$$

Hence,

$$\begin{aligned}\lambda_{\mathrm{bl}(\theta)^G}(\widehat{C_4^{(a)}}) &= \lambda_{\mathrm{bl}(\theta)}^G(\widehat{C_4^{(a)}}) = \lambda_{\mathrm{bl}(\theta)}(\widehat{C_4^{(a)} \cap T})\\ &= \big(\theta(\operatorname{diag}(\zeta^a, \zeta^a, \zeta^{-2a})) + \theta(\operatorname{diag}(\zeta^a, \zeta^{-2a}, \zeta^a)) + \theta(\operatorname{diag}(\zeta^{-2a}, \zeta^a, \zeta^a))\big)^*\\ &= \left(\theta_0^{k(q-1)_\ell}(\zeta^a) + \theta_0^{k(q-1)_\ell}(\zeta^a) + \theta_0^{k(q-1)_\ell}(\zeta^{-2a})\right)^*\\ &= \left(2\varepsilon^{ak(q-1)_\ell} + \varepsilon^{-2ak(q-1)_\ell}\right)^*.\end{aligned}$$

On the other hand, we also have

$$\lambda_B(\widehat{C_4^{(a)}}) = \lambda_{\chi_{\Phi_3}^{(k(q-1)_\ell)}}(\widehat{C_4^{(a)}}) = \left(2\varepsilon^{ak(q-1)_\ell} + \varepsilon^{-2ak(q-1)_\ell}\right)^*$$

by Table 6.2, so $\lambda_{\mathrm{bl}(\theta)^G}(\widehat{C_4^{(a)}}) = \lambda_B(\widehat{C_4^{(a)}})$. Moreover, for the conjugacy classes of type C_6 we observe that

$$\begin{aligned}\lambda_{\mathrm{bl}(\theta)^G}(\widehat{C_6^{(a,b)}}) &= \lambda_{\mathrm{bl}(\theta)}^G(\widehat{C_6^{(a,b)}}) = \lambda_{\mathrm{bl}(\theta)}(\widehat{C_6^{(a,b)} \cap T})\\ &= \Big(\theta(\operatorname{diag}(\zeta^a, \zeta^b, \zeta^{-(a+b)})) + \theta(\operatorname{diag}(\zeta^b, \zeta^a, \zeta^{-(a+b)}))\\ &\quad + \theta(\operatorname{diag}(\zeta^a, \zeta^{-(a+b)}, \zeta^b)) + \theta(\operatorname{diag}(\zeta^{-(a+b)}, \zeta^a, \zeta^b))\\ &\quad + \theta(\operatorname{diag}(\zeta^b, \zeta^{-(a+b)}, \zeta^a)) + \theta(\operatorname{diag}(\zeta^{-(a+b)}, \zeta^b, \zeta^a))\Big)^*\\ &= \left(2\left(\theta_0^{k(q-1)_\ell}(\zeta^a) + \theta_0^{k(q-1)_\ell}(\zeta^b) + \theta_0^{k(q-1)_\ell}(\zeta^{-(a+b)})\right)\right)^*\\ &= \left(2\left(\varepsilon^{ak(q-1)_\ell} + \varepsilon^{bk(q-1)_\ell} + \varepsilon^{-(a+b)k(q-1)_\ell}\right)\right)^*,\end{aligned}$$

which equals $\lambda_B(\widehat{C_6^{(a,b)}})$ according to Table 6.2. Hence, the claim follows since $\mathcal{O}_\ell(T)$ is clearly contained in the kernel of θ. $\square$

By application of the information provided by the above lemma we may now construct the ℓ-weights of G belonging to ℓ-blocks of type $B_{(\text{-})}$.

Proposition 7.30. *Suppose that $2 \neq \ell \mid (q-1)$ and let B be an ℓ-block of G of type $B_{(\text{-})}$. Then $|\mathcal{W}(B)| = 2$. More precisely, denote by $\theta \in \operatorname{Irr}(T)$ the canonical character of a root of B and assume that (R, φ) is a B-weight. Then up to G-conjugation it holds that $R = \mathcal{O}_\ell(T)$ and*

$$\varphi = \operatorname{Ind}_{\mathrm{N}_G(T)_\theta}^{\mathrm{N}_G(T)}(\widetilde{\theta} \cdot \nu),$$

where $\widetilde{\theta}$ is an extension of θ to $\mathrm{N}_G(T)_\theta$ and $\nu \in \operatorname{Irr}(\mathrm{N}_G(T)_\theta/T)$ with $\mathrm{N}_G(T)_\theta/T \cong C_2$.

Proof. As discussed before we have $R = \mathcal{O}_\ell(T)$ up to G-conjugation, and by Lemma 7.29 we may assume that $\theta = \theta_0^{k(q-1)_\ell} \times 1 \times 1$ for a suitable parameter k. Now $\mathrm{N}_G(T)/T \cong \mathfrak{S}_3$ and $\mathrm{N}_G(T)$ acts on $\operatorname{Irr}(T) = \{\theta_0^i \times \theta_0^j \times 1 \mid 0 \leqslant i, j < q-1\}$ as described in Lemma 7.26, from which we deduce that modulo T there exists only one element in $\mathrm{N}_G(T)$ stabilizing

θ, namely the element corresponding to the transposition $(2\,3) \in \mathfrak{S}_3$, so $\mathrm{N}_G(T)_\theta/T \cong C_2$. By Proposition 1.20(i) the character θ has an extension $\widetilde{\theta}$ to $\mathrm{N}_G(T)_\theta$, whence the claim follows from the considerations at the beginning of Section 7.2.2 since by the theorem of Gallagher, Theorem 1.21, we have

$$\mathrm{Irr}(\mathrm{N}_G(T)_\theta \mid \theta) = \{\widetilde{\theta} \cdot \nu \mid \nu \in \mathrm{Irr}(\mathrm{N}_G(T)_\theta/T)\},$$

and both characters ψ in this set are linear, hence satisfy $\psi(1)_\ell = 1 = |\,\mathrm{N}_G(T)_\theta \colon T|_\ell$. □

Proposition 7.31. *Suppose that $2 \neq \ell \mid (q-1)$ and let B be an ℓ-block of G of type $B_{(\text{-})}$. Then $\mathrm{Aut}(G)_B$ acts trivially on $\mathcal{W}(B)$.*

Proof. Let (R, φ) be a B-weight. In consequence of Proposition 7.30 we may assume that $R = \mathcal{O}_\ell(T)$ and φ is the induction to $\mathrm{N}_G(T)$ of the character $\widetilde{\theta} \cdot \nu$, where $\widetilde{\theta} \in \mathrm{Irr}(\mathrm{N}_G(T)_\theta)$ is an extension of

$$\theta = \theta_0^{k(q-1)_\ell} \times 1 \times 1$$

for some $k \in \mathbb{Z}$ and $\nu \in \mathrm{Irr}(\mathrm{N}_G(T)_\theta/T)$ with $\mathrm{N}_G(T)_\theta/T \cong C_2$. Let $a \in \mathrm{Aut}(G)_B$ and assume without loss of generality that a stabilizes T and hence also R. Then by Brauer's extended first main theorem, Theorem 1.7, the ℓ-block $\mathrm{bl}(\theta)^a = \mathrm{bl}(\theta^a)$, which induces to B by Proposition 1.3, is $\mathrm{N}_G(T)$-conjugate to $\mathrm{bl}(\theta)$, and since θ^a is the canonical character of $\mathrm{bl}(\theta^a)$, we deduce that $\theta^a = \theta^n$ for some $n \in \mathrm{N}_G(T)$. Hence, replacing a by $ac_{n^{-1}}$, where we recall that

$$c_{n^{-1}} \colon G \longrightarrow G, \quad g \longmapsto n^{-1}gn,$$

we may assume that a fixes θ. Due to the fact that $\mathrm{Aut}(G) = G \rtimes \langle \Gamma, F_p\rangle$ and $R = \mathcal{O}_\ell(T)$ is stable under Γ and F_p, we must have $a \in c_x\langle \Gamma, F_p\rangle$ for some $x \in \mathrm{N}_G(R) = \mathrm{N}_G(T)$. One easily verifies that the element

$$v := v_2 v_3 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix} \in \mathrm{N}_G(T) \setminus T,$$

which corresponds to the permutation (23) in $\mathfrak{S}_3$, stabilizes θ, such that $\mathrm{N}_G(T)_\theta = \langle T, v\rangle$ as a result of Proposition 7.30. Clearly, v is left invariant by both Γ and F_p, so according to Lemma 1.25 we have $c_x(v) = a(v) \in \mathrm{N}_G(T)_\theta$. Hence, the elements $c_x(v)$ and v agree modulo T, which leads to the conclusion that

$$xT \in \mathrm{C}_{\mathrm{N}_G(T)/T}(vT) \cong \mathrm{C}_{\mathfrak{S}_3}((2\,3)) = \langle (2\,3)\rangle,$$

or, in other words, $xT \in \langle vT\rangle$, that is, $x \in \langle T, v\rangle = \mathrm{N}_G(T)_\theta$. Thus,

$$\widetilde{\theta}^a(v) = \widetilde{\theta}^{c_x}(v) = \widetilde{\theta}(v),$$

such that by application of Corollary 1.22 it follows that $\widetilde{\theta}^a = \widetilde{\theta}$ because both $\widetilde{\theta}^a$ and $\widetilde{\theta}$ are extensions of θ to its stabilizer $\mathrm{N}_G(T)_\theta = \langle T, v\rangle$ satisfying $\widetilde{\theta}^a(v) = \widetilde{\theta}(v) \neq 0$ since $\widetilde{\theta}$ is linear. Clearly, a stabilizes ν being one of the two irreducible characters of $\mathrm{N}_G(T)_\theta/T \cong C_2$, so in particular a stabilizes $\widetilde{\theta} \cdot \nu$ and hence also φ. This proves the claim. □

7.2.2.3 The Blocks of Type $B_{(\text{-},\text{-})}$

The last type of ℓ-blocks of G we consider for the present choice of prime ℓ are the ℓ-blocks of type $B_{(\text{-},\text{-})}$. As for the ℓ-blocks of type $B_{(\text{-})}$ we begin our examination by the following observation on canonical characters associated to such blocks:

Lemma 7.32. *Suppose that $2 \neq \ell \mid (q-1)$ and let B be an ℓ-block of G of type $B_{(\text{-},\text{-})}$. Moreover, let $b \in \mathrm{Bl}_\ell(T)$ be a root of B. Then there exist parameters $1 \leqslant u, v < (q-1)_{\ell'}$ with $(u(q-1)_\ell,\ v(q-1)_\ell) \in \mathcal{M}(\Phi_2\Phi_3)$ (cf. Notation 6.7) such that up to $\mathrm{N}_G(T)$-conjugation the canonical character of b is given by $\theta = \theta_0^{u(q-1)_\ell} \times \theta_0^{v(q-1)_\ell} \times 1 \in \mathrm{Irr}(T)$.*

Proof. Similarly as in the proof of Lemma 7.29 one can show that there exists a unique irreducible character of type $\chi_{\Phi_2\Phi_3}$ in B both of whose parameters are multiples of $(q-1)_\ell$, say parametrized by the tuple $(u(q-1)_\ell,\ v(q-1)_\ell) \in \mathcal{M}(\Phi_2\Phi_3)$ with $1 \leqslant u, v < (q-1)_{\ell'}$. We prove that

$$\theta = \theta_0^{u(q-1)_\ell} \times \theta_0^{v(q-1)_\ell} \times 1 \in \mathrm{Irr}(T)$$

meets the claim.

As in the proof of Lemma 7.29 we show that $\lambda_B(\widehat{C}) = \lambda_{\mathrm{bl}(\theta)^G}(\widehat{C})$ for every ℓ'-conjugacy class C of G with defect group $\mathcal{O}_\ell(T)$, that is, we need to check this condition for the classes $C_4^{(a)}$ and $C_6^{(a,b)}$ for appropriate parameters a and b. We have

$$\begin{aligned}
\lambda_{\mathrm{bl}(\theta)^G}(\widehat{C_4^{(a)}}) =& \lambda_{\mathrm{bl}(\theta)}^G(\widehat{C_4^{(a)}}) = \lambda_{\mathrm{bl}(\theta)}(\widehat{C_4^{(a)} \cap T}) \\
=& \big(\theta(\mathrm{diag}(\zeta^a, \zeta^a, \zeta^{-2a})) + \theta(\mathrm{diag}(\zeta^a, \zeta^{-2a}, \zeta^a)) + \theta(\mathrm{diag}(\zeta^{-2a}, \zeta^a, \zeta^a))\big)^* \\
=& \Big(\theta_0^{u(q-1)_\ell}(\zeta^a)\theta_0^{v(q-1)_\ell}(\zeta^a) + \theta_0^{u(q-1)_\ell}(\zeta^a)\theta_0^{v(q-1)_\ell}(\zeta^{-2a}) \\
& + \theta_0^{u(q-1)_\ell}(\zeta^{-2a})\theta_0^{v(q-1)_\ell}(\zeta^a)\Big)^* \\
=& \Big(\varepsilon^{a(u+v)(q-1)_\ell} + \varepsilon^{a(u-2v)(q-1)_\ell} + \varepsilon^{a(v-2u)(q-1)_\ell}\Big)^* \\
=& \lambda_{\chi_{\Phi_2\Phi_3}^{(u(q-1)_\ell, v(q-1)_\ell)}}(\widehat{C_4^{(a)}}) \\
=& \lambda_B(\widehat{C_4^{(a)}})
\end{aligned}$$

according to Table 6.2. Furthermore, for the G-conjugacy classes of type C_6 we similarly observe that

$$\begin{aligned}
\lambda_{\mathrm{bl}(\theta)^G}(\widehat{C_6^{(a,b)}}) =& \lambda_{\mathrm{bl}(\theta)}^G(\widehat{C_6^{(a,b)}}) = \lambda_{\mathrm{bl}(\theta)}(\widehat{C_6^{(a,b)} \cap T}) \\
=& \Big(\theta(\mathrm{diag}(\zeta^a, \zeta^b, \zeta^{-(a+b)})) + \theta(\mathrm{diag}(\zeta^b, \zeta^a, \zeta^{-(a+b)})) \\
& + \theta(\mathrm{diag}(\zeta^a, \zeta^{-(a+b)}, \zeta^b)) + \theta(\mathrm{diag}(\zeta^{-(a+b)}, \zeta^a, \zeta^b)) \\
& + \theta(\mathrm{diag}(\zeta^b, \zeta^{-(a+b)}, \zeta^a)) + \theta(\mathrm{diag}(\zeta^{-(a+b)}, \zeta^b, \zeta^a))\Big)^* \\
=& \Big(\theta_0^{u(q-1)_\ell}(\zeta^a)\theta_0^{v(q-1)_\ell}(\zeta^b) + \theta_0^{u(q-1)_\ell}(\zeta^b)\theta_0^{v(q-1)_\ell}(\zeta^a) \\
& + \theta_0^{u(q-1)_\ell}(\zeta^a)\theta_0^{v(q-1)_\ell}(\zeta^{-(a+b)}) + \theta_0^{u(q-1)_\ell}(\zeta^{-(a+b)})\theta_0^{v(q-1)_\ell}(\zeta^a) \\
& + \theta_0^{u(q-1)_\ell}(\zeta^b)\theta_0^{v(q-1)_\ell}(\zeta^{-(a+b)}) + \theta_0^{u(q-1)_\ell}(\zeta^{-(a+b)})\theta_0^{v(q-1)_\ell}(\zeta^b)\Big)^*
\end{aligned}$$

$$= \Big(\varepsilon^{(au+bv)(q-1)_\ell} + \varepsilon^{(bu+av)(q-1)_\ell} + \varepsilon^{(au-(a+b)v)(q-1)_\ell}$$
$$+ \varepsilon^{(-(a+b)u+av)(q-1)_\ell} + \varepsilon^{(bu-(a+b)v)(q-1)_\ell} + \varepsilon^{(-(a+b)u+bv)(q-1)_\ell}\Big)^*$$
$$= \lambda_B(\widehat{C_6^{(a,b)}})$$

on comparison with Table 6.2. We have thus shown that $\mathrm{bl}(\theta)^G = B$, and since clearly $\mathcal{O}_\ell(T) \subseteq \ker(\theta)$, the claim follows. □

Proposition 7.33. *Suppose that $2 \neq \ell \mid (q-1)$ and let B be an ℓ-block of G of type $B_{(\text{-},\text{-})}$. Moreover, let $\theta \in \mathrm{Irr}(T)$ be the canonical character of a root of B. Then $|\mathcal{W}(B)| = 1$, and if (R, φ) is a B-weight, then $R = \mathcal{O}_\ell(T)$ and $\varphi = \mathrm{Ind}_T^{\mathrm{N}_G(T)}(\theta)$ up to G-conjugation.*

Proof. As before, we have $R = \mathcal{O}_\ell(T)$ up to G-conjugation since R is a defect group of B. Following Lemma 7.32 we may assume that

$$\theta = \theta_0^{u(q-1)_\ell} \times \theta_0^{v(q-1)_\ell} \times 1$$

for suitable parameters $1 \leqslant u, v < (q-1)_{\ell'}$ with $(u(q-1)_\ell, v(q-1)_\ell) \in \mathcal{M}(\Phi_2\Phi_3)$, that is, u, v, $u-v$, $2u+v$, $2v+u \not\equiv 0 \bmod (q-1)_{\ell'}$. By application of Lemma 7.26 one easily deduces that $\mathrm{N}_G(T)_\theta = T$. Hence, the considerations at the beginning of Section 7.2.2 imply that there exists only one G-conjugacy class of B-weights, which is represented by the weight $(\mathcal{O}_\ell(T), \mathrm{Ind}_T^{\mathrm{N}_G(T)}(\theta))$ as claimed. □

As an immediate consequence of this result we obtain the following:

Proposition 7.34. *Suppose that $2 \neq \ell \mid (q-1)$ and let B be an ℓ-block of G of type $B_{(\text{-},\text{-})}$. Then $\mathrm{Aut}(G)_B$ acts trivially on $\mathcal{W}(B)$.*

Proof. This follows from the fact that $|\mathcal{W}(B)| = 1$ by Proposition 7.33. □

As an important result of this section we record the following statement:

Theorem 7.35. *Let $2 \neq \ell \mid (q-1)$. Then the blockwise Alperin weight conjecture holds for each ℓ-block of G.*

Proof. This statement holds for ℓ-blocks of cyclic defect by Proposition 3.9. For an ℓ-block B of non-cyclic defect we summarize the following:

- $|\mathcal{W}(B)| = 3 = l(B)$ if B is the principal ℓ-block B_0,
- $|\mathcal{W}(B)| = 2 = l(B)$ if B is of type $B_{(\text{-})}$,
- $|\mathcal{W}(B)| = 1 = l(B)$ if B is of type $B_{(\text{-},\text{-})}$

according to Proposition 6.9 for $l(B)$ and Propositions 7.27, 7.30 and 7.33 for $|\mathcal{W}(B)|$. □

7.2.3 B-Weights and the Action of $\mathrm{Aut}(G)_B$ in the Case $\ell = 2$

Suppose now that $\ell = 2$ and q is odd. As for the previous case, by Proposition 6.11 there exist five distinct types of 2-blocks for G, which are the principal 2-block B_0, the 2-blocks of type $B_{(\text{-})}$, $B_{(\text{-},\text{-})}$ or $B'_{(\text{-})}$, and the 2-blocks of 2-defect zero.

As proposed by Lemma 6.13, the 2-blocks of G of type $B'_{(\text{-})}$ have cyclic defect groups, whence analogously to the previous case we again only consider the principal 2-block and the 2-blocks of types $B_{(\text{-})}$ and $B_{(\text{-},\text{-})}$.

Once more we follow Construction 2.10 to obtain all G-conjugacy classes of 2-weights for these three types of 2-blocks. We note that in contrast to the previous case the Sylow 2-subgroups of G are not abelian, such that in general, if (R, φ) is a 2-weight for some 2-block B of G, the group R does not need to be a defect group of B. In particular, following Construction 2.10 in principle we have to consider all types of radical 2-subgroups of G. However, it turns out that in fact some of the six G-conjugacy classes of radical 2-subgroups of G (cf. Proposition 7.24) do not give rise to any 2-weights belonging to 2-blocks of types B_0, $B_{(\text{-})}$ or $B_{(\text{-},\text{-})}$. These radical 2-subgroups will hence not be taken account of in the construction of 2-weights for these blocks. Our first result is the following observation:

Proposition 7.36. *Let $R = \{1\}$ be the trivial subgroup of G. If (R, φ) is a B-weight of G for some 2-block B of G, then B is of 2-defect zero.*

In particular, $R = \{1\}$ does not give rise to any 2-weights of G belonging to 2-blocks of types B_0, $B_{(\text{-})}$ or $B_{(\text{-},\text{-})}$.

Proof. Clearly, we have $\mathrm{N}_G(R)/R = G$, so φ is an irreducible character of 2-defect zero associated to B. Hence, B must have 2-defect zero. $\square$

Secondly, the following statement disproves the existence of any 2-weights of G whose first component is G-conjugate to the radical 2-subgroup $\mathcal{O}_2(\langle \mathrm{diag}(\zeta, \zeta, \zeta^{-2}) \rangle)$, where we recall that $\zeta \in \mathbb{F}^\times$ denotes a primitive root of unity of order $q-1$.

Proposition 7.37. *Let $R = \mathcal{O}_2(\langle \mathrm{diag}(\zeta, \zeta, \zeta^{-2}) \rangle)$ for $\zeta \in \mathbb{F}^\times$ of order $q-1$. Then $\mathrm{N}_G(R)$ does not have any irreducible character that is of 2-defect zero if regarded a character of the quotient $\mathrm{N}_G(R)/R$. In particular, there do not exist any 2-weights for G whose first component is G-conjugate to R.*

Proof. As already observed in Lemma 7.7, we have

$$\mathrm{N}_G(R)/R \cong \mathrm{GL}_2(q)/\mathcal{O}_2(\mathrm{Z}(\mathrm{GL}_2(q))),$$

which is of order

$$|\,\mathrm{N}_G(R)/R\,| = \frac{q(q-1)^2(q+1)}{(q-1)_2}.$$

Hence, an irreducible character $\varphi \in \mathrm{Irr}(\mathrm{N}_G(R))$ that is of 2-defect zero when regarded as a character of $\mathrm{N}_G(R)/R$ would satisfy $\varphi(1)_2 = (q^2-1)_2$. However, by [Ste51, Table II] the degrees of the irreducible characters of $\mathrm{GL}_2(q)$ are 1, q, $q-1$ and $q+1$. Hence, such a character φ does not exist, whence R does not admit any 2-weights for G. $\square$

As a next step we focus on the question which types of 2-blocks of G hold 2-weights with first component G-conjugate to the radical 2-subgroup $\mathcal{O}_2(T')$, where T' denotes a maximal torus of G of type T_2.

Proposition 7.38. *Suppose that T' is a maximal torus of G of type T_2 and let $R = \mathcal{O}_2(T')$. Then any 2-weight (R, φ) of G belongs to a 2-block of type $B'_{(\text{-})}$.*

Proof. Suppose that (R, φ) is a 2-weight for G and denote by B the 2-block of G to which (R, φ) is associated. By Propositions 5.7 and 5.8 it holds that $\mathrm{N}_G(R) = \mathrm{N}_G(T') \cong T' \rtimes C_2$ and $R\,\mathrm{C}_G(R) = T'$. Hence, by Construction 2.10 we have

$$\varphi = \mathrm{Ind}_{\mathrm{N}_G(T')_\theta}^{\mathrm{N}_G(T')}(\psi)$$

for some $\psi \in \mathrm{Irr}(\mathrm{N}_G(T')_\theta \mid \theta)$ with $\psi(1)_2 = |\,\mathrm{N}_G(T')_\theta \colon T'|_2$, where $\theta \in \mathrm{Irr}(T')$ is the canonical character of a 2-block of T' inducing to B. Since $\mathrm{N}_G(T')/T' \cong C_2$, it follows that $|\,\mathrm{N}_G(T')_\theta \colon T'| \in \{1,2\}$, so in particular $\mathrm{N}_G(T')_\theta/T'$ is cyclic, such that by Proposition 1.20(i) there exists an extension $\widetilde{\theta}$ of θ to $\mathrm{N}_G(T')_\theta$. Then by Gallagher, Theorem 1.21, we have

$$\mathrm{Irr}(\mathrm{N}_G(T')_\theta \mid \theta) = \{\widetilde{\theta} \cdot \beta \mid \beta \in \mathrm{Irr}(\mathrm{N}_G(T')_\theta/T')\}.$$

Clearly, all characters in this set are linear, so ψ as above satisfies $\psi(1)_2 = 1$. We conclude that also $|\,\mathrm{N}_G(T')_\theta \colon T'|_2 = 1$, so $\mathrm{N}_G(T')_\theta = T$. Now a defect group of the 2-block $\mathrm{bl}(\theta)$ is given by $R = \mathcal{O}_2(T') \in \mathrm{Syl}_2(T')$ in consequence of Lemma 1.13, and since $\mathrm{N}_G(T')_\theta = T$, Theorem 1.10 implies that B must have defect group R' as well. But by Lemma 6.13 any 2-block of G with defect group R' is of type $B'_{(\text{-})}$, so the claim follows. $\square$

Remark 7.39. We recall from Proposition 7.24 that the group $\mathcal{O}_2(T')$ for a maximal torus T' of G of type T_2 is only 2-radical in G if $(q+1)_{2'} \neq 1$. Note that this is in accordance with the fact that 2-blocks of type $B'_{(\text{-})}$ only exist for G if this condition is satisfied (cf. Proposition 6.11).

Finally, we observe that only for the principal 2-block B_0 and 2-blocks of type $B_{(\text{-},\text{-})}$ there exist 2-weights with first component $\mathcal{O}_2(T)$ for a maximal torus T of G of type T_1:

Proposition 7.40. *Suppose that T is a maximal torus of G of type T_1 and let $R = \mathcal{O}_2(T)$. If (R,φ) is a B-weight for some 2-block B of G, then one of the following holds:*

(i) $B = B_0$ is the principal 2-block of G and $\varphi = 1_{\mathrm{N}_G(R)} \cdot \chi_2$, where χ_2 denotes the unique irreducible character of degree 2 of $\mathrm{N}_G(R)/R\,\mathrm{C}_G(R) \cong \mathfrak{S}_3$;

(ii) B is of type $B_{(\text{-},\text{-})}$, and (R,φ) is the unique B-weight with first component R.

Proof. By Proposition 5.8 we have $R\,\mathrm{C}_G(R) = \mathrm{C}_G(R) = T$ and $\mathrm{N}_G(R) = \mathrm{N}_G(T) = T \rtimes \mathfrak{S}_3$, with $\mathfrak{S}_3$ acting on T by permutation of the diagonal entries. Let $\theta \in \mathrm{Irr}(T)$ denote an irreducible constituent of $\varphi_{|T}$. Following Construction 2.10 there exists a character $\psi \in \mathrm{Irr}(\mathrm{N}_G(R)_\theta \mid \theta)$ with $\psi(1)_2 = |\,\mathrm{N}_G(R)_\theta \colon T\,|_2$ such that

$$\varphi = \mathrm{Ind}_{\mathrm{N}_G(R)_\theta}^{\mathrm{N}_G(R)}(\psi)$$

and for $b = \mathrm{bl}(\theta)$ we have $b^G = B$.

Let us assume first that b is the principal 2-block of T. Then $B = b^G$ is the principal 2-block of G by Brauer's third main theorem, Theorem 1.9, and $\theta = 1_T$ is the trivial character of T with $\mathrm{N}_G(R)_\theta = \mathrm{N}_G(R)$. By Gallagher, Theorem 1.21, we have

$$\mathrm{Irr}(\mathrm{N}_G(R) \mid \theta) = \{1_{\mathrm{N}_G(R)} \cdot \nu \mid \nu \in \mathrm{Irr}(\mathrm{N}_G(R)/T)\}.$$

From the fact that $\mathrm{N}_G(R)/T \cong \mathfrak{S}_3$ we conclude that this set contains exactly one character ψ satisfying $\psi(1)_2 = |\,\mathrm{N}_G(R) \colon T\,|_2 = 2$, namely the character $\psi = 1_{\mathrm{N}_G(R)} \cdot \chi_2$, where χ_2 denotes the unique irreducible character of degree 2 of $\mathrm{N}_G(R)/T \cong \mathfrak{S}_3$. Hence, we have $\varphi = 1_{\mathrm{N}_G(R)} \cdot \chi_2$ as claimed.

Suppose now that b is non-principal. In the notation of Section 7.2.2 the canonical character θ of b is given by

$$\theta = \theta_0^{u(q-1)_2} \times \theta_0^{v(q-1)_2} \times 1$$

for some $0 \leqslant u,v < (q-1)_{2'}$, and since b is non-principal, we may assume without loss of generality that $u \neq 0$. Suppose that $v = 0$. Then $\mathrm{N}_G(R)_\theta/T \cong C_2$ by Lemma 7.26, and according to Gallagher, Theorem 1.21, we have

$$\mathrm{Irr}(\mathrm{N}_G(R)_\theta \mid \theta) = \{\widetilde{\theta} \cdot \nu \mid \nu \in \mathrm{Irr}(\mathrm{N}_G(R)_\theta/T)\}$$

for some extension $\widetilde{\theta}$ of θ to $\mathrm{N}_G(R)_\theta$, which exists by Proposition 1.20(i) since $\mathrm{N}_G(R)_\theta/T$ is cyclic. Hence, the elements in $\mathrm{Irr}(\mathrm{N}_G(R)_\theta \mid \theta)$ are linear, while $|\,\mathrm{N}_G(R)_\theta \colon T\,|_2 = 2$, in contradiction to the fact that there exists $\psi \in \mathrm{Irr}(\mathrm{N}_G(R)_\theta \mid \theta)$ with $\psi(1)_2 = |\,\mathrm{N}_G(R)_\theta \colon T\,|_2$. Hence, $v \neq 0$. Similarly, one proves that $v \neq u$.

Now analogously to the proof of Lemma 7.32 one can show that $B = b^G$ is a 2-block of type $B_{(\text{-},\text{-})}$. Moreover, since $u, v \neq 0$ and $u \neq v$, we have $\mathrm{N}_G(R)_\theta = T$ following Lemma 7.26. Hence, $\mathrm{Irr}(\mathrm{N}_G(R)_\theta \mid \theta) = \{\theta\}$, and we conclude that φ is the induction of θ to $\mathrm{N}_G(R)$. By Brauer's extended first main theorem, Theorem 1.7, up to $\mathrm{N}_G(R)$-conjugation b is the unique 2-block of T inducing to B. In particular, θ is determined up to $\mathrm{N}_G(R)$-conjugation, so φ is uniquely determined by R. $\square$

7.2.3.1 The Principal Block B_0

Let us now move on to the determination of the 2-weights of G associated to the principal 2-block of G. These may be classified as follows:

Proposition 7.41. *Let $B = B_0$ be the principal 2-block of G. Then $|\mathcal{W}(B)| = 3$, and if (R, φ) is a B-weight, then up to G-conjugation one of the following holds:*

(i) $R = \widetilde{Q_8}$ and $\varphi = 1_{\mathrm{N}_G(R)} \cdot \chi_2$, where χ_2 denotes the unique irreducible character of degree 2 of $\mathrm{N}_G(R)/R\,\mathrm{C}_G(R) \cong \mathfrak{S}_3$;

(ii) $R = \mathcal{O}_2(T)$ for a maximal torus T of G of type T_1, and $\varphi = 1_{\mathrm{N}_G(R)} \cdot \chi_2$, where χ_2 denotes the unique irreducible character of degree 2 of $\mathrm{N}_G(R)/R\,\mathrm{C}_G(R) \cong \mathfrak{S}_3$;

(iii) $R \in \mathrm{Syl}_2(G)$ and $\varphi = 1_{\mathrm{N}_G(R)}$.

Proof. We construct the B-weights following Construction 2.10, that is, we go through the list of radical 2-subgroups of G given in Proposition 7.24. We have already shown in Propositions 7.36, 7.37 and 7.38 that the radical 2-subgroups G-conjugate to $\{1\}$, $\mathcal{O}_2(\langle \mathrm{diag}(\zeta, \zeta, \zeta^{-2})\rangle)$ for $\zeta \in \mathbb{F}^\times$ of order $q-1$, or $\mathcal{O}_2(T')$ for a maximal torus T' of G of type T_2 do not admit any B-weights.

Hence, let us first suppose that we have $R = \widetilde{Q_8}$. Then according to Proposition 7.18 this satisfies

$$R\,\mathrm{C}_G(R) = \langle R, \mathrm{diag}(\zeta, \zeta, \zeta^{-2})\rangle$$

and

$$\mathrm{N}_G(R)/R\,\mathrm{C}_G(R) \cong \mathfrak{S}_3.$$

Moreover, $\mathrm{diag}(\zeta, \zeta, \zeta^{-2})_2 \in R$. Now by the third main theorem of Brauer, Theorem 1.9, the principal 2-block $b_0 \in \mathrm{Bl}_2(R\,\mathrm{C}_G(R))$ is the unique 2-block of $R\,\mathrm{C}_G(R)$ inducing to B. Its canonical character is the trivial character $1_{R\,\mathrm{C}_G(R)}$, so according to Construction 2.10 the B-weights with first component R are given by the pairs (R, ψ), where the irreducible character $\psi \in \mathrm{Irr}(\mathrm{N}_G(R) \mid 1_{R\,\mathrm{C}_G(R)})$ meets the condition $\psi(1)_2 = |\,\mathrm{N}_G(R) \colon R\,\mathrm{C}_G(R)|_2 = 2$. Following Gallagher, Theorem 1.21, we have

$$\mathrm{Irr}(\mathrm{N}_G(R) \mid 1_{R\,\mathrm{C}_G(R)}) = \{1_{\mathrm{N}_G(R)} \cdot \nu \mid \nu \in \mathrm{Irr}(\mathrm{N}_G(R)/R\,\mathrm{C}_G(R))\},$$

which contains exactly one character ψ with $\psi(1)_2 = 2$, namely the character $1_{\mathrm{N}_G(R)} \cdot \chi_2$, where χ_2 denotes the unique irreducible character of degree 2 of $\mathrm{N}_G(R)/R\,\mathrm{C}_G(R) \cong \mathfrak{S}_3$. Hence, up to G-conjugation $(R,\, 1_{\mathrm{N}_G(R)} \cdot \chi_2)$ is the unique 2-weight of G belonging to B.

Suppose now that $R = \mathcal{O}_2(T)$ for a maximal torus T of G of type T_1. By Proposition 5.8 we have $\mathrm{C}_G(R) = T$ and $\mathrm{N}_G(R) \cong \mathrm{C}_G(R) \rtimes \mathfrak{S}_3$. As before, the unique 2-block of $R\,\mathrm{C}_G(R)$ inducing to B is the principal 2-block of $R\,\mathrm{C}_G(R)$, which has canonical character $1_{R\,\mathrm{C}_G(R)}$. Again $\mathrm{N}_G(R)/R\,\mathrm{C}_G(R) \cong \mathfrak{S}_3$, so by the same argumentation as in the previous case we obtain that up to G-conjugation $(R, 1_{\mathrm{N}_G(R)} \cdot \chi_2)$ is the unique 2-weight belonging to B, where once more χ_2 denotes the unique irreducible character of degree 2 of $\mathrm{N}_G(R)/R\,\mathrm{C}_G(R) \cong \mathfrak{S}_3$.

Finally, let $R \in \mathrm{Syl}_2(G)$. By Proposition 5.9 we have $\mathrm{N}_G(R) = R\,\mathrm{C}_G(R)$. Hence, using the same arguments as before we conclude that the unique B-weight with first component R is given by the 2-weight $(R, 1_{\mathrm{N}_G(R)})$. This completes the proof. □

The behaviour of the B-weights under the action of $\mathrm{Aut}(G)$ is obtained easily:

Proposition 7.42. *Let $B = B_0$ be the principal 2-block of G. Then $\mathrm{Aut}(G)_B = \mathrm{Aut}(G)$ acts trivially on $\mathcal{W}(B)$.*

Proof. Clearly, $\mathrm{Aut}(G)_B = \mathrm{Aut}(G)$. Following Proposition 7.41 the G-conjugacy class of a B-weight is uniquely determined by the isomorphism type of the corresponding radical 2-subgroup. Since $\mathrm{Aut}(G)$ preserves isomorphism types, the claim follows immediately. □

7.2.3.2 The Blocks of Type $B_{(\text{-})}$

We next consider the 2-weights associated to 2-blocks of G of type $B_{(\text{-})}$, for which we observe the following:

Proposition 7.43. *Let B be a 2-block of G of type $B_{(\text{-})}$. Then $|\mathcal{W}(B)| = 2$, and if (R, φ) is a B-weight, then up to G-conjugation one of the following holds:*

(i) $R = \widetilde{Q_8}$, *there exists a unique 2-block $b \in \mathrm{Bl}_2(R\,\mathrm{C}_G(R))$ with $b^G = B$, and $\varphi = \widetilde{\theta} \cdot \chi_2$, where the character $\widetilde{\theta}$ is an extension of the canonical character θ of b to $\mathrm{N}_G(R)$ and χ_2 denotes the unique irreducible character of degree 2 of $\mathrm{N}_G(R)/R\,\mathrm{C}_G(R) \cong \mathfrak{S}_3$;*

(ii) $R \in \mathrm{Syl}_2(G)$, $R\,\mathrm{C}_G(R) = \mathrm{N}_G(R)$, *and φ is the canonical character of the Brauer correspondent $b \in \mathrm{Bl}_2(\mathrm{N}_G(R) \mid R)$ of B.*

Proof. As for the principal 2-block B_0, we construct the B-weights (R, φ) by following Construction 2.10 and recall that from Propositions 7.36, 7.37, 7.38 and 7.40 it is known that we only need to consider radical 2-subgroups of G which are either G-conjugate to $\widetilde{Q_8}$ or Sylow 2-subgroups of G.

Hence, let us first suppose that $R = \widetilde{Q_8}$. Then from Proposition 7.18 it follows that $R\,\mathrm{C}_G(R) = \langle R, \mathrm{diag}(\zeta, \zeta, \zeta^{-2})\rangle$, where ζ generates $\mathbb{F}_q^\times$, and $\mathrm{N}_G(R)/R\,\mathrm{C}_G(R) \cong \mathfrak{S}_3$. Since $\mathcal{O}_2(R\,\mathrm{C}_G(R)) = R \in \mathrm{Syl}_2(R\,\mathrm{C}_G(R))$, any 2-block of $R\,\mathrm{C}_G(R)$ has defect group R according to Lemma 1.13. In particular, there exists a bijection between the sets $\mathrm{Bl}_2(R\,\mathrm{C}_G(R))$ and $\mathrm{Irr}(R\,\mathrm{C}_G(R)/R)$ via the associated canonical characters. We have

$$R\,\mathrm{C}_G(R)/R \cong \mathcal{O}_{2'}(\langle \mathrm{diag}(\zeta, \zeta, \zeta^{-2})\rangle),$$

whence the canonical character corresponding to a 2-block $b \in \mathrm{Bl}_2(R\,\mathrm{C}_G(R))$ may be described as follows: for the maximally split torus T of G consisting of the diagonal matrices in G we use the parametrization for $\mathrm{Irr}(T)$ introduced in Notation 7.25 and set $\theta_{(k)} := \theta_0^k \times 1 \times 1 \in \mathrm{Irr}(T)$ for $0 \leqslant k < q-1$. Then the set of irreducible characters of $\mathcal{O}_{2'}(\langle \mathrm{diag}(\zeta, \zeta, \zeta^{-2})\rangle)$ may be parametrized as

$$\mathrm{Irr}(\mathcal{O}_{2'}(\langle \mathrm{diag}(\zeta, \zeta, \zeta^{-2})\rangle)) = \{(\theta_{(k(q-1)_2)})_{|\mathcal{O}_{2'}(\langle \mathrm{diag}(\zeta,\zeta,\zeta^{-2})\rangle)} \mid 0 \leqslant k < (q-1)_{2'}\},$$

and the set of irreducible characters of $R\,\mathrm{C}_G(R)$ trivial on R, that is, the set of canonical characters associated to the 2-blocks of $R\,\mathrm{C}_G(R)$, is given by

$$\{\theta'_{(k(q-1)_2)} \mid 0 \leqslant k < (q-1)_{2'}\} \subseteq \mathrm{Irr}(R\,\mathrm{C}_G(R)),$$

where we set

$$\theta'_{(j)}(xd) := \theta_{(j)}(d)$$

for $x \in R$ and $d \in \langle \mathrm{diag}(\zeta, \zeta, \zeta^{-2})\rangle$. If we now take a 2-block b of $R\,\mathrm{C}_G(R)$ with canonical character $\theta'_{(k(q-1)_2)}$ for some $0 \leqslant k < (q-1)_{2'}$, then the question is to decide in which cases b induces to B. As we observed above, a defect group of b is given by R, so up to G-conjugation the defect groups of b^G contain R following Lemma 1.1. From Lemma 6.13 it is known that the 2-blocks of G have defect groups conjugate to $\{1\}$, $\mathcal{O}_2(T')$ for a maximal torus T' of type T_2, $\mathcal{O}_2(T)$ or the Sylow 2-subgroups of G. Of these the Sylow 2-subgroups of G are the only non-abelian groups, and as R is non-abelian, we conclude that the induced block b^G has maximal defect. Hence, by Proposition 1.11 in order to determine b^G it suffices to compute $\lambda_{b^G}(\widehat{C})$ for every $2'$-conjugacy class C with maximal defect group. The G-conjugacy classes with maximal defect groups are the classes of types C_1 and C_4 by Proposition 6.3. We claim that

$$C_4^{(a)} \cap R\,\mathrm{C}_G(R) = \{\mathrm{diag}(\zeta^a, \zeta^a, \zeta^{-2a})\}$$

if the parameter a is such that $C_4^{(a)}$ is a $2'$-conjugacy class. Suppose $y \in C_4^{(a)} \cap R\,\mathrm{C}_G(R)$. Then y may be written as $y = xd$ for some $x \in R$ and $d \in \mathcal{O}_{2'}(\langle \mathrm{diag}(\zeta, \zeta, \zeta^{-2})\rangle)$. But since x and d commute and are of coprime order, we have $x = 1$ as y has odd order and $x \in R$. Hence, $C_4^{(a)} \cap R\,\mathrm{C}_G(R) \subseteq \mathcal{O}_{2'}(\langle \mathrm{diag}(\zeta, \zeta, \zeta^{-2})\rangle)$. On the other hand, the only element of $\mathcal{O}_{2'}(\langle \mathrm{diag}(\zeta, \zeta, \zeta^{-2})\rangle)$ contained in $C_4^{(a)}$ is $\mathrm{diag}(\zeta^a, \zeta^a, \zeta^{-2a})$, so the claim follows. Thus,

$$\begin{aligned}
\lambda_{b^G}(\widehat{C_4^{(a)}}) &= \lambda_b(\widehat{C_4^{(a)} \cap R\,\mathrm{C}_G(R)}) \\
&= \Big(\theta'_{(k(q-1)_2)}(\mathrm{diag}(\zeta^a, \zeta^a, \zeta^{-2a}))\Big)^* \\
&= \Big(\theta_{(k(q-1)_2)}(\mathrm{diag}(\zeta^a, \zeta^a, \zeta^{-2a}))\Big)^* \\
&= \Big(\theta_0^{k(q-1)_2}(\zeta^a)\Big)^* \\
&= \Big(\varepsilon^{k(q-1)_2 a}\Big)^*,
\end{aligned}$$

where we recall that $\varepsilon = \exp(2\pi \mathrm{i}/(q-1))$. Hence, if $B = B_{(u)}$ for some $u \in \mathbb{Z}$, then by comparison with Table 6.3 we see that $b^G = B$ if and only if $\varepsilon^{k(q-1)_2 a} \equiv \varepsilon^{-2ua} \bmod 2$ for every $a \in \mathbb{Z}$ such that $C_4^{(a)}$ is a $2'$-conjugacy class, that is, if and only if we have $k(q-1)_2 a \equiv -2ua \bmod (q-1)_{2'}$ for every such a, in particular for $a = (q-1)_2$. It follows that $k(q-1)_2 \equiv -2u \bmod (q-1)_{2'}$. Clearly, $(q-1)_2$ is invertible modulo $(q-1)_{2'}$, whence we conclude that k is uniquely determined by the condition $0 \leqslant k < (q-1)_{2'}$. Hence, there exists a unique 2-block $b \in \mathrm{Bl}_2(R\,\mathrm{C}_G(R))$ inducing to B, namely the 2-block with associated canonical character given by $\theta = \theta'_{(k(q-1)_2)}$, where $0 \leqslant k < (q-1)_{2'}$ is the unique element satisfying $k(q-1)_2 \equiv -2u \bmod (q-1)_{2'}$.

As observed in the proof of Proposition 7.18 we have

$$\mathrm{N}_G(R) = \left\{ \begin{pmatrix} X & \\ & (\det X)^{-1} \end{pmatrix} \,\middle|\, X \in \mathrm{N}_{\mathrm{GL}_2(q)}(\widetilde{Q'_8}) \right\},$$

from which we deduce that $\mathrm{N}_G(R)$ acts trivially on $R\,\mathrm{C}_G(R)/R \cong \mathcal{O}_{2'}(\mathrm{diag}(\zeta,\zeta,\zeta^{-2}))$, so in particular $\mathrm{N}_G(R)$ stabilizes the canonical character θ of b. Since $\mathrm{N}_G(R)/R\,\mathrm{C}_G(R) \cong \mathfrak{S}_3$, there exists an extension $\widetilde{\theta}$ of θ to $\mathrm{N}_G(R)_\theta = \mathrm{N}_G(R)$ by Propositions 1.19(i) and 1.20(i). According to Gallagher, Theorem 1.21, we have

$$\mathrm{Irr}(\mathrm{N}_G(R) \mid \theta) = \{\widetilde{\theta} \cdot \nu \mid \nu \in \mathrm{Irr}(\mathrm{N}_G(R)/R\,\mathrm{C}_G(R))\},$$

and the only element ψ in this set with $\psi(1)_2 = |\,\mathrm{N}_G(R)\colon R\,\mathrm{C}_G(R)|_2 = 2$ is the character $\widetilde{\theta} \cdot \chi_2$ with χ_2 the unique irreducible character of degree 2 of $\mathrm{N}_G(R)/R\,\mathrm{C}_G(R) \cong \mathfrak{S}_3$. Hence, R admits exactly one B-weight, namely the 2-weight $(R, \widetilde{\theta} \cdot \chi_2)$.

Let us now assume that $R \in \mathrm{Syl}_2(G)$. Then according to Proposition 5.9 we have

$$\mathrm{N}_G(R) = R\,\mathrm{C}_G(R) = \langle R, \mathrm{diag}(\zeta,\zeta,\zeta^{-2})\rangle.$$

By Brauer's first main theorem, Theorem 1.5, block induction yields a bijection between the sets $\mathrm{Bl}_2(\mathrm{N}_G(R) \mid R)$ and $\mathrm{Bl}_2(G \mid R)$, so in particular there exists a unique 2-block b of $R\,\mathrm{C}_G(R) = \mathrm{N}_G(R)$ with defect group R such that $b^G = B$. Let θ denote its canonical character. Then θ may be regarded as an irreducible character of

$$R\,\mathrm{C}_G(R)/R \cong \mathcal{O}_{2'}(\langle \mathrm{diag}(\zeta,\zeta,\zeta^{-2})\rangle),$$

so θ is linear. Thus, we have $\mathrm{Irr}(\mathrm{N}_G(R) \mid \theta) = \{\theta\}$ and $\theta(1)_2 = 1 = |\,\mathrm{N}_G(R)\colon R\,\mathrm{C}_G(R)|_2$, whence by Construction 2.10 the pair (R, θ) is the unique B-weight of G with first component R. This completes the proof. □

Proposition 7.44. *Suppose that B is a 2-block of G of type $B_{(\text{-})}$. Then $\mathrm{Aut}(G)_B$ acts trivially on $\mathcal{W}(B)$.*

Proof. The proof is the same as for Proposition 7.42 since the G-conjugacy class of a B-weight is uniquely determined by the isomorphism type of its first component. □

7.2.3.3 The Blocks of Type $B_{(\text{-},\text{-})}$

Finally, we consider the 2-blocks of G of type $B_{(\text{-},\text{-})}$ and describe the 2-weights associated to such blocks as follows:

Proposition 7.45. *Let B be a 2-block of G of type $B_{(\text{-},\text{-})}$ and let $\theta \in \mathrm{Irr}(T)$ be the canonical character of a root of B, where T is a maximal torus of G of type T_1. Then $|\mathcal{W}(B)| = 1$, and if (R, φ) is a B-weight, then up to G-conjugation $R = \mathcal{O}_2(T)$ and $\varphi = \mathrm{Ind}_T^{\mathrm{N}_G(T)}(\theta)$.*

Proof. Since the 2-blocks of G of type $B_{(\text{-},\text{-})}$ have abelian defect groups $\mathcal{O}_2(T)$ according to Lemma 6.13, it follows from Lemma 2.11 that any G-conjugacy class of B-weights may be represented by a 2-weight with first component $\mathcal{O}_2(T)$, so up to G-conjugation we have $R = \mathcal{O}_2(T)$. But following Proposition 7.40 there exists exactly one such 2-weight for B, which, as seen in the proof of that proposition, has weight character $\varphi = \mathrm{Ind}_T^{\mathrm{N}_G(T)}(\theta)$ as claimed. □

As an immediate consequence of this result we obtain the following:

Proposition 7.46. *Let B be a 2-block of G of type $B_{(\text{-},\text{-})}$. Then $\mathrm{Aut}(G)_B$ acts trivially on $\mathcal{W}(B)$.*

Proof. This is clear since by Proposition 7.45 we have $|\mathcal{W}(B)| = 1$. □

Analogously to Theorem 7.35 for the case $2 \neq \ell \mid (q-1)$ we have proven the following:

Theorem 7.47. *The blockwise Alperin weight conjecture holds for each* 2*-block of* G*.*

Proof. This is true for 2-blocks of G with cyclic defect groups by Proposition 3.9. If B is a 2-block of G of non-cyclic defect, then we summarize that

- $|\mathcal{W}(B)| = 3 = l(B)$ if B is the principal 2-block B_0,
- $|\mathcal{W}(B)| = 2 = l(B)$ if B is of type $B_{(\text{-})}$,
- $|\mathcal{W}(B)| = 1 = l(B)$ if B is of type $B_{(\text{-},\text{-})}$

following Proposition 6.12 for $l(B)$ and Propositions 7.41, 7.43 and 7.45 for $|\mathcal{W}(B)|$. $\square$

Chapter 8

Partitions and Equivariant Bijections

After the preparatory work of the previous chapters we are now fully equipped to establish parts (i) and (ii) of the inductive blockwise Alperin weight condition given in Definition 3.2 for every ℓ-block B of G of non-cyclic defect, where ℓ is a prime dividing $q-1$. More precisely, since by Proposition 5.3 in our case of $q > 2$ and $q \not\equiv 1 \bmod 3$ the group G is its own universal covering group, we show the following:

(i) There exist subsets $\mathrm{IBr}(B \mid Q) \subseteq \mathrm{IBr}(B)$ for $Q \in \mathrm{Rad}_\ell(G)$ with the following properties:

   (1) $\mathrm{IBr}(B \mid Q)^a = \mathrm{IBr}(B \mid Q^a)$ for every $Q \in \mathrm{Rad}_\ell(G)$, $a \in \mathrm{Aut}(G)_B$,

   (2) $\mathrm{IBr}(B) = \dot{\bigcup}_{Q \in \mathrm{Rad}_\ell(G)/\sim_G} \mathrm{IBr}(B \mid Q)$.

(ii) For every $Q \in \mathrm{Rad}_\ell(G)$ there exists a bijection

$$\Omega_Q^G \colon \mathrm{IBr}(B \mid Q) \longrightarrow \mathrm{dz}(\mathrm{N}_G(Q)/Q,\, B)$$

such that $\Omega_Q^G(\phi)^a = \Omega_{Q^a}^G(\phi^a)$ for every $\phi \in \mathrm{IBr}(B \mid Q)$ and $a \in \mathrm{Aut}(G)_B$.

Under consideration of our results on the action of $\mathrm{Aut}(G)$ on the weights and Brauer characters of G and by application of Lemma 3.8 we may easily define partitions and bijections as demanded by parts (i) and (ii) above:

Proposition 8.1. *Let ℓ be a prime such that $\ell \mid (q-1)$ and let B be an ℓ-block of G of non-cyclic defect. Then conditions (i) and (ii) of Definition 3.2 are satisfied for B.*

Proof. Let $\Omega_B \colon \mathrm{IBr}(B) \longrightarrow \mathcal{W}(B)$ be any bijection between the set of irreducible Brauer characters in B and the set of G-conjugacy classes of B-weights in G (this exists by Theorems 7.35 and 7.47). According to Proposition 7.6 and Propositions 7.28, 7.31, 7.34, 7.42, 7.44 and 7.46 the group $\mathrm{Aut}(G)_B$ acts trivially on both $\mathrm{IBr}(B)$ and $\mathcal{W}(B)$, so in particular the bijection Ω_B is $\mathrm{Aut}(G)_B$-equivariant. Hence, in consequence of Lemma 3.8 it is possible to construct subsets $\mathrm{IBr}(B \mid Q)$ of $\mathrm{IBr}(B)$ with corresponding bijections $\Omega_Q^G \colon \mathrm{IBr}(B \mid Q) \longrightarrow \mathrm{dz}(\mathrm{N}_G(Q)/Q,\, B)$ for every $Q \in \mathrm{Rad}_\ell(G)$ such that conditions (i) and (ii) of Definition 3.2 are satisfied for B. ☐

Chapter 9

Normally Embedded Conditions

In this chapter we will be concerned with the extendibility of the characters of G we have studied so far to their stabilizer in $\mathrm{Aut}(G)$. The aim is to prove part (iii) of the (iBAW) condition for all ℓ-blocks of G of non-cyclic defect, where $\ell \mid (q-1)$, with respect to the equivariant bijections Ω_Q^G, $Q \in \mathrm{Rad}_\ell(G)$, whose existence we proved in Proposition 8.1. In our case of $G = \mathrm{SL}_3(q)$ being its own universal covering group, for an ℓ-block B of G part (iii) of Definition 3.2 reduces to the conditions below:

(iii) For every $Q \in \mathrm{Rad}_\ell(G)$ and every $\phi \in \mathrm{IBr}(B \mid Q)$ there exists a finite group $A(\phi, Q)$ and Brauer characters $\widetilde{\phi} \in \mathrm{IBr}_\ell(A(\phi, Q))$ and $\widetilde{\phi}' \in \mathrm{IBr}_\ell(N_{A(\phi,Q)}(Q))$ with the following properties:

1. the group $A := A(\phi, Q)$ satisfies $G \trianglelefteq A$, $A/\,\mathrm{C}_A(G) \cong \mathrm{Aut}(G)_\phi$, $\mathrm{C}_A(G) = \mathrm{Z}(A)$ and $\ell \nmid |\,\mathrm{Z}(A)|$,
2. $\widetilde{\phi} \in \mathrm{IBr}_\ell(A)$ is an extension of ϕ,
3. $\widetilde{\phi}' \in \mathrm{IBr}_\ell(\mathrm{N}_A(Q))$ is an extension of the inflation of $\Omega_Q^G(\phi)^0 \in \mathrm{IBr}_\ell(\mathrm{N}_G(Q)/Q)$,
4. $\mathrm{bl}(\widetilde{\phi}_{|J}) = \mathrm{bl}((\widetilde{\phi}')_{|\,\mathrm{N}_J(Q)})^J$ for every subgroup J satisfying $G \leqslant J \leqslant A$.

Remark 9.1. Let us fix an ℓ-block B of G and a radical ℓ-subgroup $Q \in \mathrm{Rad}_\ell(G)$ as above. Since G is simple, we observe that for every $\phi \in \mathrm{IBr}(B \mid Q)$ the group $A(\phi, Q) := \mathrm{Aut}(G)_\phi$ satisfies

$$G \trianglelefteq A(\phi, Q), \;\; \mathrm{C}_{A(\phi,Q)}(G) = \{1\} \;\text{ and }\; \mathrm{Z}(A(\phi, Q)) = \{1\},$$

so $A(\phi, Q)$ fulfils condition (1) of Definition 3.2(iii). It will hence be our objective to show that there also exist Brauer characters $\widetilde{\phi} \in \mathrm{IBr}_\ell(\mathrm{Aut}(G)_\phi)$ and $\widetilde{\phi}' \in \mathrm{IBr}_\ell(N_{\mathrm{Aut}(G)_\phi}(Q))$ satisfying conditions (2) to (4).

9.1 Results on Extendibility

In this section we concentrate on conditions (2) and (3) of Definition 3.2(iii). We prove that for $\ell \mid (q-1)$ every $\phi \in \mathrm{IBr}_\ell(G)$ extends to $\mathrm{Aut}(G)_\phi$, and that, moreover, for every ℓ-weight (R, φ) of G the weight character φ extends to $\mathrm{Aut}(G)_{R,\varphi}$.

9.1.1 Extendibility of (Brauer) Characters of $\mathrm{SL}_3(q)$

The aim of this section is to obtain extendibility results concerning the characters of G. For this purpose it will prove useful to consider the extendibility of characters of general linear groups first.

As before, $q = p^f$ is a power of the prime p. For a general linear group $\mathrm{GL}_n(q)$ we denote by F_p the automorphism of $\mathrm{GL}_n(q)$ induced by the field automorphism $\mathbb{F}_q \longrightarrow \mathbb{F}_q$, $a \longmapsto a^p$, and let Γ be the transpose-inverse automorphism of $\mathrm{GL}_n(q)$. Clearly, for $n = 3$ restricting Γ and F_p to $G = \mathrm{SL}_3(q)$ gives exactly the automorphisms Γ and F_p that we have defined for G, which justifies the double use of this notation.

Our first aim is a result on the structure of the subgroups of $\mathrm{Out}(G)$. To this end we state the following lemma:

Lemma 9.2. *Let H be a finite abelian group with presentation*

$$H := \langle x, y \mid x^2 = y^m = 1,\ [x, y] = 1\rangle$$

for some $m \in \mathbb{N}_{>0}$, $x, y \in H$. If K is a subgroup of H such that $x \notin K$, then K is cyclic.

Proof. This is well-known and is easily verified by considering the group homomorphism $\varphi\colon H \longrightarrow \langle y\rangle \cong C_m$ defined by $\varphi(x) = 1$ and $\varphi(y) = y$. This has $\ker(\varphi) = \langle x\rangle$, and if K is a subgroup of H, then the restriction $\varphi_{|K}\colon K \longrightarrow \langle y\rangle$ has kernel $\ker(\varphi_{|K}) = \langle x\rangle \cap K$. Since $K/\ker(\varphi_{|K})$ is isomorphic to a (cyclic) subgroup of $\langle y\rangle$ and $\ker(\varphi_{|K}) = \{1\}$ if $x \notin K$, the claim follows. □

Corollary 9.3. *Let $K \leq \langle \Gamma, F_p\rangle \leqslant \mathrm{Aut}(G)$. Then either $\Gamma \in K$ or K is cyclic. In particular, if K is not cyclic, then we have $K = \langle \Gamma, F_p^i\rangle$ for some $i \in \mathbb{Z}$ with $i \mid f = o(F_p)$.*

Proof. The first part is Lemma 9.2 with $H = \langle \Gamma, F_p\rangle$, $x = \Gamma$ and $y = F_p$. Hence, if K is not cyclic, we have $\Gamma \in K$, so $K = \langle \Gamma, F_p^i\rangle$ for some $i \in \mathbb{Z}$ such that $K \cap \langle F_p\rangle = \langle F_p^i\rangle$. By replacing i by $\gcd(i, f)$, which is possible by Bézout's identity, we may assume that i divides f. □

The following result by C. Bonnafé is a consequence of a theorem of T. Shintani (which in fact is not restricted to the case that $q \not\equiv 1 \bmod 3$):

Proposition 9.4. *If a character $\theta \in \mathrm{Irr}(\mathrm{GL}_n(q))$ is invariant under F_p^i for some $i \mid f$, then there exists an extension $\widetilde{\theta} \in \mathrm{Irr}(\mathrm{GL}_n(q) \rtimes \langle F_p^i\rangle)$ of θ satisfying $\widetilde{\theta}(F_p^i) \in \mathbb{Z}_{>0}$.*

Proof. In [Bon99, Thm. 4.3.1] Bonnafé proves the existence of a certain extension $\widetilde{\theta}$ of θ to $\mathrm{GL}_n(q) \rtimes \langle F_p^i\rangle$, which by [Bon99, Lemma 4.3.2] has the property that $\widetilde{\theta}(F_p^i) \in \mathbb{Z}_{>0}$. □

We will also need the following statement (compare [Spä09, Lemma 9.7]):

Corollary 9.5. *Suppose that $\chi \in \mathrm{Irr}(G)$ with $\chi^\Gamma = \chi^{F_p^i} = \chi$ for some $i \in \mathbb{Z}$. Then χ extends to $G \rtimes \left\langle \Gamma, F_p^i\right\rangle$.*

Proof. Following our assumption that $q \not\equiv 1 \bmod 3$, we have $\mathrm{GL}_3(q) = \mathrm{SL}_3(q) \times \mathrm{Z}(\mathrm{GL}_3(q))$. In particular, the character $\chi' := \chi \times 1_{\mathrm{Z}(\mathrm{GL}_3(q))}$ is an extension of χ to $\mathrm{GL}_3(q)$. Since χ is invariant under F_p^i and Γ, so is χ'. As in the proof of Corollary 9.3 we may assume that $i \mid f$, so by Proposition 9.4 there exists an extension

$$\chi'' \in \mathrm{Irr}(\mathrm{GL}_3(q) \rtimes \langle F_p^i\rangle)$$

of χ' such that $\chi''(F_p^i) \neq 0$. Now Γ normalizes $\mathrm{GL}_3(q) \rtimes \langle F_p^i \rangle$ and stabilizes χ', so ${\chi''}^{\Gamma}$ is an extension of χ' to $\mathrm{GL}_3(q) \rtimes \langle F_p^i \rangle$ as well, and since Γ and F_p commute we have

$$ {\chi''}^{\Gamma}(F_p^i) = \chi''(F_p^i) \neq 0. $$

Now denote by ψ and ψ' the restrictions of χ'' and ${\chi''}^{\Gamma}$ to $\mathrm{SL}_3(q) \rtimes \langle F_p^i \rangle$, respectively. Then both ψ and ψ' are extensions of χ satisfying

$$ \psi(F_p^i) = \psi'(F_p^i) \neq 0. $$

Hence, by Corollary 1.22 we have $\psi = \psi'$, i.e., ψ is a Γ-invariant extension of χ to $\mathrm{SL}_3(q) \rtimes \langle F_p^i \rangle$. Then by Proposition 1.20(i) there exists an extension of ψ to $\mathrm{SL}_3(q) \rtimes \langle F_p^i, \Gamma \rangle$ as claimed. □

Corollary 9.6. *Every irreducible character of G extends to its stabilizer in* $\mathrm{Aut}(G)$*.*

Proof. Let $\chi \in \mathrm{Irr}(G)$. We have $\mathrm{Aut}(G)_\chi = G \rtimes \langle \Gamma, F_p \rangle_\chi$, and if we set $K := \langle \Gamma, F_p \rangle_\chi$, then by Corollary 9.3 either K is cyclic or we may write $K = \langle \Gamma, F_p^i \rangle$ for some $i \mid f$. In the case of K being cyclic χ extends to $\mathrm{Aut}(G)_\chi$ by Proposition 1.20(i). Otherwise, χ extends to $\mathrm{Aut}(G)_\chi$ in consequence of Corollary 9.5. □

We now turn to the question concerning the extendibility of the Brauer characters associated to G. In answering this question we will make use of the lower unitriangular shape of the decomposition matrix of G. More precisely, we use Proposition 1.29 to show that all irreducible Brauer characters of G with respect to primes ℓ dividing $q-1$ extend to their stabilizer in $\mathrm{Aut}(G)$. Note that for a Brauer character $\phi \in \mathrm{IBr}_\ell(G)$ contained in an ℓ-block B, its stabilizer in $\mathrm{Aut}(G)$ necessarily fixes the whole ℓ-block B, so we always have $\mathrm{Aut}(G)_\phi \subseteq \mathrm{Aut}(G)_B$.

Lemma 9.7. *Let $\ell \mid (q-1)$ and suppose that B is an ℓ-block of G. Then there exists a basic set $\mathcal{B} \subseteq \mathrm{Irr}(B)$ for B such that the ℓ-decomposition matrix for $\mathcal{B}$ is lower unitriangular with respect to a suitable ordering of the characters in $\mathcal{B}$ and* $\mathrm{IBr}(B)$*, and every character in $\mathcal{B}$ is left invariant by* $\mathrm{Aut}(G)_B$*.*

Proof. We prove the claim by considering the distinct types of ℓ-blocks of G described in Propositions 6.8 and 6.11. Clearly, for ℓ-blocks of ℓ-defect zero there is nothing to show.

Suppose first that B is the principal ℓ-block. Then by Propositions 6.9 and 6.12 we may choose $\mathcal{B}$ to consist of the three unipotent characters 1_G, $\chi_{q\Phi_2}$ and χ_{q^3}, which are clearly left invariant by $\mathrm{Aut}(G)_B = \mathrm{Aut}(G)$ being the unique irreducible characters of G of their degree (cf. Table A.1).

Let B be of type $B_{(\_)}$. Then similarly as in the proof of Lemma 7.29 one can show that there exist unique characters in B of types χ_{Φ_3} and $\chi_{q\Phi_3}$, respectively, whose parameter is a multiple of $(q-1)_\ell$, say their parameter is given by u (note that this is the same parameter for both characters in consequence of Propositions 6.8 and 6.11). Let $a \in \mathrm{Aut}(G)_B$. Since $\mathrm{Aut}(G) = G \rtimes \langle \Gamma, F_p \rangle$, the automorphism a is the product of an inner automorphism of G and powers of Γ and F_p, so by Corollary 7.5 it fixes character types and the parameters of $\chi_{\Phi_3}^{(u)a}$ and $\chi_{q\Phi_3}^{(u)a}$ are still multiples of $(q-1)_\ell$. Since a fixes B, it thus leaves $\chi_{\Phi_3}^{(u)}$ and $\chi_{q\Phi_3}^{(u)}$ invariant. Hence, we choose

$$ \mathcal{B} = \{\chi_{\Phi_3}^{(u)},\ \chi_{q\Phi_3}^{(u)}\}. $$

Then by Propositions 6.9 and 6.12 this set is as required.

If B is of type $B_{(\text{-},\text{-})}$ or $B'_{(\text{-})}$, then $|\operatorname{IBr}(B)| = 1$ and following Propositions 6.9 and 6.12 any one-element subset of $\operatorname{Irr}(B)$ trivially yields a basic set with corresponding lower unitriangular decomposition matrix. Similarly as before there exists a unique character in $\operatorname{Irr}(B)$ whose parameter is a multiple of $(q^2-1)_\ell$ in the case that B is of type $B'_{(\text{-})}$ and whose parameters are multiples of $(q-1)_\ell$ in the case that B is of type $B_{(\text{-},\text{-})}$, respectively. By the same arguments as for the blocks of type $B_{(\text{-})}$, this character is left invariant by $\operatorname{Aut}(G)_B$. Hence, we may choose $\mathcal{B}$ to consist of exactly this character. $\square$

Corollary 9.8. *Let $\ell \mid (q-1)$. Then every Brauer character $\phi \in \operatorname{IBr}_\ell(G)$ extends to its stabilizer $\operatorname{Aut}(G)_\phi$ in $\operatorname{Aut}(G)$.*

Proof. Let $\phi \in \operatorname{IBr}_\ell(G)$ and $B := \operatorname{bl}(\phi)$. By Lemma 9.7 there exists a basic set $\mathcal{B} \subseteq \operatorname{Irr}(B)$ for B such that the ℓ-decomposition matrix for $\mathcal{B}$ is lower unitriangular with respect to a suitable ordering of the characters in $\mathcal{B}$ and $\operatorname{IBr}(B)$, and every character in $\mathcal{B}$ is left invariant by $\operatorname{Aut}(G)_B$. Hence, for every character $\chi \in \mathcal{B}$ we have $\operatorname{Aut}(G)_\chi = \operatorname{Aut}(G)_B$, and by Lemma 1.27 every element in $\operatorname{IBr}(B)$ is left invariant by $\operatorname{Aut}(G)_B$ as well, so in particular $\operatorname{Aut}(G)_\phi = \operatorname{Aut}(G)_B$. Following Corollary 9.6 every character in $\mathcal{B}$ extends to $\operatorname{Aut}(G)_B$. Thus, by Proposition 1.29 every irreducible Brauer character belonging to B extends to $\operatorname{Aut}(G)_B$, so in particular ϕ does. $\square$

9.1.2 Extendibility of Weight Characters

Here we prove that for $\ell \mid (q-1)$ and (R,φ) an ℓ-weight of G associated to an ℓ-block of non-cyclic defect there always exist extensions of φ and the associated Brauer character of $\mathrm{N}_G(R)$ to $\operatorname{Aut}(G)_{R,\varphi}$.

9.1.2.1 The Case $2 \neq \ell \mid (q-1)$

We assume that ℓ is an odd prime dividing $q-1$ and denote by $T = \mathbf{T}^F$ the maximally split torus of G consisting of the diagonal matrices in G. If B is an ℓ-block of G of non-cyclic defect, then according to Propositions 7.27, 7.30 and 7.33 for every B-weight (R,φ) of G we have

$$R = \mathcal{O}_\ell(T) \quad \text{and} \quad \varphi = \psi^{\mathrm{N}_G(T)}$$

up to G-conjugation, where $\theta \in \operatorname{Irr}(T)$ is an irreducible constituent of $\varphi_{|T}$ and ψ is an irreducible character of $\mathrm{N}_G(T)_\theta$ lying above θ. In this section we prove that φ always extends to $\operatorname{Aut}(G)_{\mathcal{O}_\ell(T),\varphi}$.

Since we assume that $q \not\equiv 1 \bmod 3$, one may easily check that the only irreducible character θ of T such that the quotient $\mathrm{N}_G(T)_\theta/T$ has order divisible by 3 is the trivial character 1_T (cf. Lemma 7.26). Consequently, the following proposition does indeed handle all possibilities that can arise during the examination of irreducible characters of T.

Proposition 9.9. *Let $D := \langle F_p, \Gamma\rangle \leqslant \operatorname{Aut}(G)$ and $\theta \in \operatorname{Irr}(T)$. Then*

(i) θ extends to $(\mathrm{N}_G(T) \rtimes D)_\theta$;

(ii) for $\mathrm{N}_G(T)_\theta = T$ the character $\chi := \theta^{\mathrm{N}_G(T)} \in \operatorname{Irr}(\mathrm{N}_G(T))$ extends to $(\mathrm{N}_G(T) \rtimes D)_\chi$;

(iii) for $\mathrm{N}_G(T)_\theta/T \cong C_2$ the character $\chi := \psi^{\mathrm{N}_G(T)} \in \operatorname{Irr}(\mathrm{N}_G(T))$ has an extension to $(\mathrm{N}_G(T) \rtimes D)_\chi$, where $\psi \in \operatorname{Irr}(\mathrm{N}_G(T)_\theta \mid \theta)$ is an extension of θ to $\mathrm{N}_G(T)_\theta$;

(iv) for $\theta = 1_T$ the three irreducible characters in $\operatorname{Irr}(\mathrm{N}_G(T) \mid \theta)$ extend to $\mathrm{N}_G(T) \rtimes D$.

Proof. (i) Let us denote by V the group of monomial matrices in G with entries ± 1 and set

$$W := \langle V, \Gamma, F_p \rangle = V \times D.$$

Then following Proposition 5.7 we have $\mathrm{N}_G(T) = \langle T, V\rangle$ and $\mathrm{N}_G(T) \rtimes D = \langle T, W\rangle$, and moreover, there exists a subgroup $V' \subseteq V$ with $V' \cong \mathfrak{S}_3$ and $\mathrm{N}_G(T) = T \rtimes V'$, for instance $V' = \langle v_2, v_3 \rangle$. Let now $H := V \cap T$ and $\nu := \theta_{|H} \in \mathrm{Irr}(H)$. Then $W_\nu = V_\nu \times D$ since the elements of H are invariant under Γ and F_p. We have

$$V_\nu = (H \rtimes V')_\nu = H \rtimes (V')_\nu,$$

so ν has an extension ν' to V_ν according to Lemma 1.23. As $W_\nu = V_\nu \times D$, there exists an extension $\widetilde{\nu}$ of ν' to W_ν. Now for $t \in T$ and $w \in W_\theta \leqslant W_\nu$ we set

$$\widetilde{\theta}(tw) := \theta(t)\widetilde{\nu}(w)$$

and claim that $\widetilde{\theta}$ defines an extension of θ to $(\mathrm{N}_G(T) \rtimes D)_\theta = \langle T, W_\theta\rangle$. First of all, we observe that $\widetilde{\theta}$ is well-defined since any element of $\langle T, W_\theta \rangle$ may be written in the form tw as above, and for elements $t, t' \in T$ and $w, w' \in W_\theta$ such that $tw = t'w'$, we have $t^{-1}t' = ww'^{-1} \in T \cap W_\theta = H$, and hence $\widetilde{\nu}(ww'^{-1}) = \theta(ww'^{-1})$, so we obtain

$$\begin{aligned}
\theta(t)\widetilde{\nu}(w) &= \theta(t't'^{-1}t)\widetilde{\nu}(ww'^{-1}w') \\
&= \theta(t')\theta(t'^{-1}t)\widetilde{\nu}(ww'^{-1})\widetilde{\nu}(w') \\
&= \theta(t')\theta(t'^{-1}t)\theta(ww'^{-1})\widetilde{\nu}(w') \\
&= \theta(t')\theta(t'^{-1}tww'^{-1})\widetilde{\nu}(w') \\
&= \theta(t')\theta(1)\widetilde{\nu}(w') \\
&= \theta(t')\widetilde{\nu}(w').
\end{aligned}$$

Next we prove that $\widetilde{\theta}$ is a group homomorphism, whence a linear character of $(\mathrm{N}_G(T) \rtimes D)_\theta$. For $t, t' \in T$ and $w, w' \in W_\theta$ we have

$$\begin{aligned}
\widetilde{\theta}(twt'w') &= \widetilde{\theta}(twt'w^{-1}ww') \\
&= \theta(twt'w^{-1})\widetilde{\nu}(ww') \\
&= \theta(t)\theta^w(t')\widetilde{\nu}(w)\widetilde{\nu}(w') \\
&= \theta(t)\widetilde{\nu}(w)\theta(t')\widetilde{\nu}(w') \\
&= \widetilde{\theta}(tw)\widetilde{\theta}(t'w'),
\end{aligned}$$

so $\widetilde{\theta}$ is a linear character of $(\mathrm{N}_G(T) \rtimes D)_\theta$ satisfying $\widetilde{\theta}_{|T} = \theta$ by construction, hence an extension of θ to $(\mathrm{N}_G(T) \rtimes D)_\theta$ as claimed.

(ii) Let θ' be an extension of θ to $(\mathrm{N}_G(T) \rtimes D)_\theta$, which exists by (i). Then $\theta'^{(\mathrm{N}_G(T) \rtimes D)_\chi}$ is an extension of χ to $(\mathrm{N}_G(T) \rtimes D)_\chi$ by Lemma 1.24(ii) applied to $N = T$, $S \cong \mathfrak{S}_3$ and $A = \mathrm{N}_G(T) \rtimes D$.

(iii) Since $\mathrm{N}_G(T)$ stabilizes χ, we have

$$(\mathrm{N}_G(T) \rtimes D)_\chi = \mathrm{N}_G(T) \rtimes D_\chi.$$

By Corollary 9.3 either D_χ is cyclic or we have $D_\chi = \langle \Gamma, F_p^i \rangle$ for some $i \in \mathbb{Z}$. If D_χ is cyclic, then χ extends to $\mathrm{N}_G(T) \rtimes D_\chi$ by Proposition 1.20(i). Hence, suppose that $D_\chi = \langle \Gamma, F_p^i \rangle$ for some $i \in \mathbb{Z}$. Again by Proposition 1.20(i) there exists an extension $\widetilde{\chi}$ of χ to $\mathrm{N}_G(T) \rtimes \langle \Gamma \rangle$. We have

$$\widetilde{\chi}(1) = \chi(1) = |\,\mathrm{N}_G(T)\colon\ \mathrm{N}_G(T)_\theta| = 3,$$

so $\widetilde{\chi}(\Gamma)$ must be the sum of three $o(\Gamma)$-th roots of unity. Since $o(\Gamma) = 2$, we conclude that $\widetilde{\chi}(\Gamma) \in \{\pm 1, \pm 3\}$, so in particular $\widetilde{\chi}(\Gamma) \neq 0$. Moreover, F_p^i commutes with Γ and hence normalizes $\mathrm{N}_G(T) \rtimes \langle\Gamma\rangle$, so $\widetilde{\chi}^{F_p^i}$ is an irreducible character of $\mathrm{N}_G(T) \rtimes \langle\Gamma\rangle$ with

$$(\widetilde{\chi}^{F_p^i})_{|\,\mathrm{N}_G(T)} = \chi^{F_p^i} = \chi,$$

and we have $\widetilde{\chi}^{F_p^i}(\Gamma) = \widetilde{\chi}(\Gamma) \neq 0$. Thus, from Corollary 1.22 it follows that $\widetilde{\chi}$ is F_p^i-invariant. Since the quotient

$$(\mathrm{N}_G(T) \rtimes D_\chi)/(\mathrm{N}_G(T) \rtimes \langle\Gamma\rangle) = (\mathrm{N}_G(T) \rtimes \langle\Gamma, F_p^i\rangle)/(\mathrm{N}_G(T) \rtimes \langle\Gamma\rangle)$$

is cyclic, once again by Proposition 1.20(i) there exists an extension of $\widetilde{\chi}$ to $\mathrm{N}_G(T) \rtimes D_\chi$, hence an extension of χ to $\mathrm{N}_G(T) \rtimes D_\chi$ as claimed.

(iv) Since $\mathrm{N}_G(T)/T \cong \mathfrak{S}_3$, we consider the irreducible characters of $\mathfrak{S}_3$. We write

$$\mathrm{Irr}(\mathfrak{S}_3) = \{\gamma_1, \gamma_2, \gamma_3\},$$

where $\gamma_1 = 1_{\mathfrak{S}_3}$, $\gamma_2 \neq 1_{\mathfrak{S}_3}$ is linear and γ_3 is irreducible of degree two. Then we fix an isomorphism $\Theta_1\colon\ \mathrm{N}_G(T)/T \longrightarrow \mathfrak{S}_3$ such that we may write

$$\mathrm{Irr}(\mathrm{N}_G(T) \mid 1_T) = \{\psi_1, \psi_2, \psi_3\}$$

with $\psi_i(n) = \gamma_i(\Theta_1(nT))$ for $n \in \mathrm{N}_G(T)$ and $i = 1, 2, 3$. Let us now consider the irreducible characters of $\mathrm{N}_G(T) \rtimes D$ lying above $1_{T \rtimes D}$. The map

$$\begin{aligned} \pi\colon\ \mathrm{N}_G(T)/T &\longrightarrow (\mathrm{N}_G(T) \rtimes D)/(T \rtimes D), \\ nT &\longmapsto n(T \rtimes D), \end{aligned}$$

is an isomorphism as one easily checks, so $\Theta_2 := \Theta_1 \circ \pi^{-1}$ defines an isomorphism between $(\mathrm{N}_G(T) \rtimes D)/(T \rtimes D)$ and $\mathfrak{S}_3$. Then we may write

$$\mathrm{Irr}(\mathrm{N}_G(T) \rtimes D \mid 1_{T \rtimes D}) = \{\widetilde{\psi}_1, \widetilde{\psi}_2, \widetilde{\psi}_3\},$$

where $\widetilde{\psi}_i(x) = \gamma_i(\Theta_2(x(T \rtimes D)))$ for $x \in \mathrm{N}_G(T) \rtimes D$ and $i = 1, 2, 3$. Now for any $i = 1, 2, 3$ and $n \in \mathrm{N}_G(T)$ we have

$$\widetilde{\psi}_i(n) = \gamma_i(\Theta_2(n(T \rtimes D))) = \gamma_i(\Theta_2(\pi(nT))) = \gamma_i(\Theta_1(nT)) = \psi_i(n),$$

so $(\widetilde{\psi}_i)_{|\,\mathrm{N}_G(T)} = \psi_i$, that is, we have found an extension of ψ_i to $\mathrm{N}_G(T) \rtimes D$. $\square$

In order to show that for every ℓ-block B of G with non-cyclic defect groups and every B-weight (R, φ) the character φ extends to $\mathrm{Aut}(G)_{R,\varphi}$, we need the assertion of the lemma below.

Lemma 9.10. *Let A be a finite group, $N \trianglelefteq A$ a normal subgroup and r a prime dividing the order of N. Moreover, let (R, φ) be an r-weight of N such that φ extends to $A_{R,\varphi}$. If (Q, χ) is an r-weight of N that is N-conjugate to (R, φ), then χ extends to $A_{Q,\chi}$*

Proof. Let $x \in N$ such that $(Q, \chi) = (R, \varphi)^x$ and denote by φ' an extension of φ to $A_{R,\varphi}$. We prove that φ'^x is an extension of χ to $A_{Q,\chi}$. Let us first show that $A_{Q,\chi} = x^{-1}A_{R,\varphi}x$. Suppose that $a \in A_{R,\varphi}$. Then

$$\begin{aligned} (x^{-1}ax)^{-1}Q(x^{-1}ax) &= x^{-1}a^{-1}(xQx^{-1})ax \\ &= x^{-1}a^{-1}Rax \\ &= x^{-1}Rx \\ &= Q, \end{aligned}$$

so $x^{-1}ax \in A_Q$. Moreover, for all elements $y \in \mathrm{N}_N(Q)$ we observe that

$$\begin{aligned}\chi^{x^{-1}ax}(y) &= \chi((x^{-1}ax)y(x^{-1}ax)^{-1})\\ &= (\chi^{x^{-1}})^a(xyx^{-1})\\ &= \varphi^a(xyx^{-1})\\ &= \varphi(xyx^{-1})\\ &= \varphi^x(y)\\ &= \chi(y),\end{aligned}$$

that is, $x^{-1}ax \in A_{Q,\chi}$, so we conclude that $x^{-1}A_{R,\varphi}x \subseteq A_{Q,\chi}$. Analogously one proves that $xA_{Q,\chi}x^{-1} \subseteq A_{R,\varphi}$, and hence $A_{Q,\chi} = x^{-1}A_{R,\varphi}x$. Now φ'^x is an irreducible character of $(A_{R,\varphi})^x = A_{Q,\chi}$, and since

$$\mathrm{N}_N(Q) = \mathrm{N}_N(R^x) = \mathrm{N}_N(R)^x = x^{-1}\,\mathrm{N}_N(R)x,$$

for all $y \in \mathrm{N}_N(Q)$ we have $xyx^{-1} \in \mathrm{N}_N(R)$ and hence

$$\varphi'^x(y) = \varphi'(xyx^{-1}) = \varphi(xyx^{-1}) = \varphi^x(y) = \chi(y),$$

so φ'^x is indeed an extension of χ to $A_{Q,\chi}$, which concludes the proof. $\square$

We can now prove the desired statement on the extendibility of weight characters:

Proposition 9.11. *Suppose that $2 \neq \ell \mid (q-1)$. Let B be an ℓ-block of G with non-cyclic defect groups and suppose that (R,φ) is a B-weight. Then φ extends to $\mathrm{Aut}(G)_{R,\varphi}$.*

Proof. Following Propositions 7.27, 7.30 and 7.33 there exists $g \in G$ such that $R^g = \mathcal{O}_\ell(T)$. We set $\chi := \varphi^g$ and consider the weight

$$(\mathcal{O}_\ell(T),\chi) = (R,\varphi)^g.$$

By the propositions referred to above we have $\chi = \psi^{\mathrm{N}_G(T)}$ for some $\psi \in \mathrm{Irr}(\mathrm{N}_G(T)_\theta \mid \theta)$, where either

- $\theta = 1_T$ and $\chi = \psi \in \mathrm{Irr}(\mathrm{N}_G(T) \mid 1_T)$ if $B = B_0$ is the principal ℓ-block, or
- $\mathrm{N}_G(T)_\theta/T \cong C_2$ if B is of type $B_{(\text{-})}$, or
- $\mathrm{N}_G(T)_\theta = T$ and $\psi = \theta$ in the case of B being of type $B_{(\text{-},\text{-})}$.

Hence, by Proposition 9.9 there exists an extension $\widetilde{\chi}$ of χ to $(\mathrm{N}_G(T) \rtimes \langle \Gamma, F_p\rangle)_\chi$. Now we have $\mathrm{Aut}(G) = G \rtimes \langle \Gamma, F_p\rangle$, where Γ and F_p stabilize $\mathcal{O}_\ell(T)$, so

$$\mathrm{Aut}(G)_{\mathcal{O}_\ell(T)} = \mathrm{N}_G(\mathcal{O}_\ell(T)) \rtimes \langle \Gamma, F_p\rangle = \mathrm{N}_G(T) \rtimes \langle \Gamma, F_p\rangle$$

according to Proposition 5.8. Thus, the character $\widetilde{\chi}$ constitutes an extension of χ to its stabilizer $\mathrm{Aut}(G)_{\mathcal{O}_\ell(T),\chi} = (\mathrm{N}_G(T) \rtimes \langle \Gamma, F_p\rangle)_\chi$. Hence, by Lemma 9.10 also φ extends to $\mathrm{Aut}(G)_{R,\varphi}$ as claimed. $\square$

9.1.2.2 The Case $\ell = 2$

In this section we suppose that $\ell = 2$ and q is odd and prove that for every 2-weight (R, φ) of G the weight character φ extends to its stabilizer $\mathrm{Aut}(G)_{R,\varphi}$ in $\mathrm{Aut}(G)$.

Lemma 9.12. *Let (R, φ) be a 2-weight of G and denote by $\overline{\varphi}$ the 2-defect zero character of $\mathrm{N}_G(R)/R$ associated to φ. If $\overline{\varphi}$ extends to $(\mathrm{Aut}(G)_R/R)_{\overline{\varphi}}$, then φ extends to $\mathrm{Aut}(G)_{R,\varphi}$.*

Proof. This follows directly from the fact that $(\mathrm{Aut}(G)_R/R)_{\overline{\varphi}} = \mathrm{Aut}(G)_{R,\varphi}/R$, so the extension of $\overline{\varphi}$ to $(\mathrm{Aut}(G)_R/R)_{\overline{\varphi}}$ may be regarded as a character of $\mathrm{Aut}(G)_{R,\varphi}$ containing R in its kernel and extending φ. $\square$

Lemma 9.13. *Suppose that (R, φ) is a 2-weight of G and denote by $\mathrm{Aut}_2(G)$ the subgroup $G \rtimes \mathcal{O}_2(\langle \Gamma, F_p \rangle)$ of $\mathrm{Aut}(G)$. If the 2-defect zero character $\overline{\varphi}$ of $\mathrm{N}_G(R)/R$ associated to φ extends to $(\mathrm{Aut}_2(G)_R/R)_{\overline{\varphi}}$, then $\overline{\varphi}$ extends to $(\mathrm{Aut}(G)_R/R)_{\overline{\varphi}}$.*

Proof. Since $\mathrm{N}_G(R)/R \trianglelefteq \mathrm{Aut}(G)_R/R$, by Proposition 1.19(i) the character $\overline{\varphi}$ extends to $(\mathrm{Aut}(G)_R/R)_{\overline{\varphi}}$ if it extends to every group P lying between $\mathrm{N}_G(R)/R$ and $(\mathrm{Aut}(G)_R/R)_{\overline{\varphi}}$ such that $P/(\mathrm{N}_G(R)/R)$ is a Sylow r-subgroup of

$$H := (\mathrm{Aut}(G)_R/R)_{\overline{\varphi}}/(\mathrm{N}_G(R)/R)$$

for some prime r dividing the order of H. Due to the fact that $\langle \Gamma, F_p \rangle \cong C_2 \times C_f$ is abelian, there exist unique Sylow r-subgroups of H for every prime r dividing $|H|$, and moreover these are cyclic whenever r is odd. In particular, for $r \neq 2$ there exists an extension of $\overline{\varphi}$ to P for every Sylow r-subgroup $P/(\mathrm{N}_G(R)/R)$ of H according to Proposition 1.20(i). Consequently, if the character $\overline{\varphi}$ extends to $(\mathrm{Aut}_2(G)_R/R)_{\overline{\varphi}}$, then it extends to $(\mathrm{Aut}(G)_R/R)_{\overline{\varphi}}$ since

$$(\mathrm{Aut}_2(G)_R/R)_{\overline{\varphi}}/(\mathrm{N}_G(R)/R)$$

is the unique Sylow 2-subgroup of the group $H = (\mathrm{Aut}(G)_R/R)_{\overline{\varphi}}/(\mathrm{N}_G(R)/R)$. (This follows since for every $a \in \mathrm{Aut}_2(G)_R$ there exists a 2-power m such that $a^m \in G$, so even $a^m \in \mathrm{N}_G(R)$, and conversely, every element $a \in \mathrm{Aut}(G)_R$ that has 2-power order modulo $\mathrm{N}_G(R)$ must also be a 2-element modulo G, i.e., $a \in \mathrm{Aut}_2(G)_R$.) $\square$

Proposition 9.14. *Let (R, φ) be a 2-weight of G. Then the 2-defect zero character $\overline{\varphi}$ of $\mathrm{N}_G(R)/R$ associated to φ extends to $(\mathrm{Aut}(G)_R/R)_{\overline{\varphi}}$.*

Proof. Let b denote the 2-block of $\mathrm{N}_G(R)/R$ containing $\overline{\varphi}$. Since $\overline{\varphi}$ is of 2-defect zero, we have $\mathrm{Irr}(b) = \{\overline{\varphi}\}$. Further, denote by $\mathrm{Aut}_2(G)$ the subgroup $G \rtimes \mathcal{O}_2(\langle \Gamma, F_p \rangle)$ of $\mathrm{Aut}(G)$ as in Lemma 9.13. We let

$$b' \in \mathrm{Bl}_2((\mathrm{Aut}_2(G)_R/R)_{\overline{\varphi}})$$

be a 2-block covering b and we choose a character $\chi \in \mathrm{Irr}(b')$ of height zero. By Definition 1.14 every irreducible constituent of the restricted character $\chi_{|\,\mathrm{N}_G(R)/R}$ is contained in an $(\mathrm{Aut}_2(G)_R/R)_{\overline{\varphi}}$-conjugate of b. But since b is the 2-block containing $\overline{\varphi}$, it clearly stays invariant under the action of $(\mathrm{Aut}_2(G)_R/R)_{\overline{\varphi}}$, so in fact every irreducible constituent of $\chi_{|\,\mathrm{N}_G(R)/R}$ lies in b. Now $\mathrm{Irr}(b) = \{\overline{\varphi}\}$, whence we conclude that $\chi_{|\,\mathrm{N}_G(R)/R} = e\overline{\varphi}$ for some integer $e \geqslant 1$. The quotient

$$(\mathrm{Aut}_2(G)_R/R)_{\overline{\varphi}}/(\mathrm{N}_G(R)/R)$$

is a 2-group, implying by [Nav98, Cor. 9.6] that the 2-block b' is the unique 2-block of $(\mathrm{Aut}_2(G)_R/R)_{\overline{\varphi}}$ which covers b. Since χ was chosen to be of height 0 in b', we may apply [Nav98, Cor. 9.18] to deduce that the integer

$$e = \langle \chi_{|\,\mathrm{N}_G(R)/R}, \overline{\varphi} \rangle_{\mathrm{N}_G(R)/R}$$

is prime to 2. On the other hand, e divides $|(\mathrm{Aut}_2(G)_R/R)_{\overline{\varphi}} \colon \mathrm{N}_G(R)/R\,|$ according to [Hup98, Thm. 21.3(a)], so we conclude that $e = 1$. Thus, $\chi_{|\,\mathrm{N}_G(R)/R} = \overline{\varphi}$, that is, χ is an extension of $\overline{\varphi}$ to $(\mathrm{Aut}_2(G)_R/R)_{\overline{\varphi}}$. We may finally apply Lemma 9.13 and conclude that there exists an extension of $\overline{\varphi}$ to $(\mathrm{Aut}(G)_R/R)_{\overline{\varphi}}$ as claimed. □

Corollary 9.15. *Let (R, φ) be a 2-weight of G. Then φ extends to $\mathrm{Aut}(G)_{R,\varphi}$.*

Proof. By Proposition 9.14 the 2-defect zero character $\overline{\varphi} \in \mathrm{Irr}(\mathrm{N}_G(R)/R)$ associated to φ extends to $(\mathrm{Aut}(G)_R/R)_{\overline{\varphi}}$. Hence, by Lemma 9.12 there exists an extension of φ to $\mathrm{Aut}(G)_{R,\varphi}$. □

9.1.3 Summary

Let us briefly outline how far we have progressed up to now in establishing the (iBAW) condition for $\mathrm{SL}_3(q)$, $q > 2$, $q \not\equiv 1 \bmod 3$. We need Lemma 9.16 below. (We should note that this lemma does in fact not only hold for $G = \mathrm{SL}_3(q)$ but for an arbitrary finite simple group. However, we only need it in the case $G = \mathrm{SL}_3(q)$.)

Lemma 9.16. *Let ℓ be a prime and B be an ℓ-block of G. Suppose that there exist subsets $\mathrm{IBr}(B \mid Q) \subseteq \mathrm{IBr}(B)$ and bijections*

$$\Omega_Q^G \colon \mathrm{IBr}(B \mid Q) \longrightarrow \mathrm{dz}(\mathrm{N}_G(Q)/Q,\, B)$$

for every $Q \in \mathrm{Rad}_\ell(G)$ satisfying conditions (i) and (ii) of Definition 3.2. Then for all $Q \in \mathrm{Rad}_\ell(G)$ and all $\phi \in \mathrm{IBr}(B \mid Q)$ we have

$$\mathrm{N}_{\mathrm{Aut}(G)_\phi}(Q) = \mathrm{Aut}(G)_{Q,\chi},$$

where $\chi \in \mathrm{Irr}(\mathrm{N}_G(Q))$ denotes the inflation of $\Omega_Q^G(\phi) \in \mathrm{Irr}(\mathrm{N}_G(Q)/Q)$ to $\mathrm{N}_G(Q)$.

Proof. Let $Q \in \mathrm{Rad}_\ell(G)$ and $\phi \in \mathrm{IBr}(B \mid Q)$ and denote by χ the inflation of $\Omega_Q^G(\phi)$ to $\mathrm{N}_G(Q)$. Suppose that $a \in \mathrm{N}_{\mathrm{Aut}(G)_\phi}(Q)$. Since $\phi \in \mathrm{IBr}(B)$, we have $a \in \mathrm{Aut}(G)_B$, so by assumption

$$\Omega_Q^G(\phi)^a = \Omega_{Q^a}^G(\phi^a) = \Omega_Q^G(\phi),$$

that is, $\Omega_Q^G(\phi)$ is left invariant by a. Hence, also its inflation χ is invariant under a, since

$$\chi^a(n) = \chi(a(n)) = \Omega_Q^G(\phi)(a(n)Q) = \Omega_Q^G(\phi)^a(nQ) = \chi(n)$$

for all $n \in \mathrm{N}_G(Q)$. This yields the inclusion $\mathrm{N}_{\mathrm{Aut}(G)_\phi}(Q) \subseteq \mathrm{Aut}(G)_{Q,\chi}$.

Let now $a \in \mathrm{Aut}(G)_{Q,\chi}$. As the ℓ-block b of $\mathrm{N}_G(Q)$ containing χ satisfies $b^G = B$ by assumption and χ is left invariant by a, we have

$$B^a = (b^G)^a = (b^a)^G = b^G = B$$

by Proposition 1.3, so $a \in \mathrm{Aut}(G)_B$. In particular, $\phi^a \in \mathrm{IBr}(B \mid Q)^a = \mathrm{IBr}(B \mid Q)$, and once more, since χ and hence $\Omega_Q^G(\phi)$ is invariant under a, we have

$$\Omega_Q^G(\phi^a) = \Omega_Q^G(\phi)^a = \Omega_Q^G(\phi).$$

Thus, by bijectivity of Ω_Q^G it follows that $\phi^a = \phi$, i.e.,

$$a \in \mathrm{Aut}(G)_\phi \cap \mathrm{N}_{\mathrm{Aut}(G)}(Q) = \mathrm{N}_{\mathrm{Aut}(G)_\phi}(Q).$$

We conclude that $\mathrm{N}_{\mathrm{Aut}(G)_\phi}(Q) = \mathrm{Aut}(G)_{Q,\chi}$. □

At this point we have nearly established all parts of the (iBAW) condition for the ℓ-blocks of G with non-cyclic defect groups, where $\ell \mid (q-1)$:

Theorem 9.17. *Let $\ell \mid (q-1)$ be a prime and B be an ℓ-block of G with non-cyclic defect groups. The subsets $\mathrm{IBr}(B \mid Q) \subseteq \mathrm{IBr}(B)$ and the $\mathrm{Aut}(G)_{B,Q}$-equivariant bijections Ω_Q^G for $Q \in \mathrm{Rad}_\ell(G)$ defined as in the proof of Proposition 8.1 satisfy conditions (1) - (3) of Definition 3.2(iii).*

More precisely, for every $Q \in \mathrm{Rad}_\ell(G)$ and every $\phi \in \mathrm{IBr}(B \mid Q)$ the finite group $\mathrm{Aut}(G)_\phi$ satisfies the following conditions: There exist characters $\widetilde{\phi} \in \mathrm{IBr}_\ell(\mathrm{Aut}(G)_\phi)$ and $\widetilde{\phi}' \in \mathrm{IBr}_\ell(N_{\mathrm{Aut}(G)_\phi}(Q))$ such that:

(1) the group $A := \mathrm{Aut}(G)_\phi$ satisfies $G \trianglelefteq A$, $A/\mathrm{C}_A(G) \cong \mathrm{Aut}(G)_\phi$, $\mathrm{C}_A(G) = \mathrm{Z}(A)$ and $\ell \nmid |\mathrm{Z}(A)|$,

(2) $\widetilde{\phi} \in \mathrm{IBr}_\ell(A)$ is an extension of ϕ,

(3) $\widetilde{\phi}' \in \mathrm{IBr}_\ell(\mathrm{N}_A(Q))$ is an extension of the inflation of $\Omega_Q^G(\phi)^0 \in \mathrm{IBr}_\ell(\mathrm{N}_G(Q)/Q)$.

Proof. We let $Q \in \mathrm{Rad}_\ell(G)$ and $\phi \in \mathrm{IBr}(B \mid Q)$. Then for the group $A := \mathrm{Aut}(G)_\phi$ we clearly have $G \trianglelefteq A$ and

$$\mathrm{C}_A(G) = \{1\} = \mathrm{Z}(A),$$

so (1) is satisfied.

For (2) recall that by Corollary 9.8 every irreducible Brauer character of G extends to its stabilizer in $\mathrm{Aut}(G)$, so in particular there exists $\widetilde{\phi} \in \mathrm{IBr}_\ell(A)$ extending ϕ.

Finally, for (3) we let $\chi \in \mathrm{Irr}(\mathrm{N}_G(Q))$ be the inflation of $\Omega_Q^G(\phi)$ to $\mathrm{N}_G(Q)$. Then χ^0 is the irreducible Brauer character of $\mathrm{N}_G(Q)$ associated with

$$\Omega_Q^G(\phi)^0 \in \mathrm{IBr}_\ell(\mathrm{N}_G(Q)/Q),$$

so condition (3) demands an extension of χ^0 to $\mathrm{N}_A(Q)$. The pair (Q, χ) is a B-weight, and hence by Proposition 9.11 for odd ℓ and Corollary 9.15 for $\ell = 2$ there exists an extension $\widetilde{\chi}$ of χ to $\mathrm{Aut}(G)_{Q,\chi}$. Now the sets $\mathrm{IBr}(B \mid Q)$ and the bijections Ω_Q^G satisfy conditions (i) and (ii) of Definition 3.2 by Proposition 8.1, so following Lemma 9.16 we have $\mathrm{Aut}(G)_{Q,\chi} = \mathrm{N}_{\mathrm{Aut}(G)_\phi}(Q)$. Hence, $\widetilde{\chi}$ is an extension of χ to $\mathrm{N}_{\mathrm{Aut}(G)_\phi}(Q) = \mathrm{N}_A(Q)$. We have

$$(\widetilde{\chi}^0)_{\mid \mathrm{N}_G(Q)} = (\widetilde{\chi}_{\mid \mathrm{N}_G(Q)})^0 = \chi^0,$$

so the Brauer character $\widetilde{\phi}' := \widetilde{\chi}^0$ is an extension of χ^0 to $\mathrm{N}_A(Q)$, and irreducibility of $\widetilde{\phi}'$ is necessarily implied by irreducibility of χ^0. This completes the proof. □

9.2 Block Correspondence

Let $\ell \mid (q-1)$. So far we have established parts (i), (ii) and (iii)(1)-(3) of the inductive conditions in Definition 3.2 for all ℓ-blocks of G of non-cyclic defect. The objective of this section is to show that the extensions in part (iii) can be chosen such that also part (iii)(4) of the inductive condition is satisfied. In order to reach this goal we exploit a number of results achieved by Koshitani–Späth in [KS13].

Lemma 9.18. *Let $K \trianglelefteq A$ be finite groups and $H \leqslant A$ a subgroup of A. Moreover, let $M := K \cap H$, and for a prime r let $b' \in \mathrm{Bl}_r(M)$ and $c' \in \mathrm{Bl}_r(H)$ be an r-block covering b'. If both $(b')^K$ and $(c')^A$ are defined, then $(c')^A$ covers $(b')^K$.*

Proof. This is [KS13, Lemma 2.3]. □

Lemma 9.19. *Let $K \trianglelefteq A$ be finite groups and $H \leqslant A$ a subgroup of A with $A = KH$. Moreover, let r be a prime. For $M := K \cap H$ let $b' \in \mathrm{Bl}_r(M)$ be an r-block that has a defect group D with $C_A(D) \subseteq H$. Suppose that some $\widetilde{\phi} \in \mathrm{IBr}_r(A)$ and $\widetilde{\phi}' \in \mathrm{IBr}_r(H)$ satisfy $\phi := \widetilde{\phi}_{|K} \in \mathrm{IBr}(b)$ and $\phi' := \widetilde{\phi}'_{|M} \in \mathrm{IBr}(b')$, where $b := (b')^K$. If it holds that $r \nmid |A/K|$ and $\mathrm{bl}(\widetilde{\phi}')^A = \mathrm{bl}(\widetilde{\phi})$, then*

$$\mathrm{bl}(\widetilde{\phi}'_{|\langle M, x\rangle})^{\langle K, x\rangle} = \mathrm{bl}(\widetilde{\phi}_{|\langle K, x\rangle})$$

for all $x \in H$.

Proof. This is [KS13, Lemma 2.4]. □

Lemma 9.20. *Let $K \trianglelefteq A$ be finite groups and $H \leqslant A$ a subgroup of A with $A = KH$. Moreover, let r be a prime. For $M := K \cap H$ let $b' \in \mathrm{Bl}_r(M)$ be an r-block that has a defect group D with $C_A(D) \subseteq H$. Suppose that some $\widetilde{\phi} \in \mathrm{IBr}_r(A)$ and $\widetilde{\phi}' \in \mathrm{IBr}_r(H)$ satisfy $\phi := \widetilde{\phi}_{|K} \in \mathrm{IBr}(b)$ and $\phi' := \widetilde{\phi}'_{|M} \in \mathrm{IBr}(b')$, where $b := (b')^K$. If*

$$\mathrm{bl}(\widetilde{\phi}'_{|\langle M, x\rangle})^{\langle K, x\rangle} = \mathrm{bl}(\widetilde{\phi}_{|\langle K, x\rangle})$$

for all $x \in H^0 = \{y \in H \mid r \nmid o(y)\}$, then $\mathrm{bl}(\widetilde{\phi}')^A = \mathrm{bl}(\widetilde{\phi})$.

Proof. This is [KS13, Lemma 2.5(a)]. □

Let us now return to our situation of $G = \mathrm{SL}_3(q)$ with $q > 2$ and $q \not\equiv 1 \bmod 3$. By application of the above tools and use of similar ideas as in the proof of [CS14, Lemma 3.2] we are finally able to verify the last missing piece for the (iBAW) condition for G:

Proposition 9.21. *Suppose that $\ell \mid (q-1)$ and denote by B an ℓ-block of G of non-cyclic defect. Moreover, suppose that $\mathrm{IBr}(B \mid Q) \subseteq \mathrm{IBr}(B)$ and Ω_Q^G are defined as in the proof of Proposition 8.1 for $Q \in \mathrm{Rad}_\ell(G)$.*

Let (R, ψ) be a B-weight of G. Moreover, let $\phi \in \mathrm{IBr}(B \mid R)$ be defined such that $\phi' := \psi^0 \in \mathrm{IBr}_\ell(\mathrm{N}_G(R))$ is the inflation of $\Omega_R^G(\phi)^0 \in \mathrm{IBr}_\ell(\mathrm{N}_G(R)/R)$ to $\mathrm{N}_G(R)$ and denote by $A := \mathrm{Aut}(G)_\phi$ the stabilizer of ϕ in $\mathrm{Aut}(G)$. Let $\widetilde{\phi}' \in \mathrm{IBr}_\ell(\mathrm{N}_A(R))$ be an extension of ϕ' to $\mathrm{N}_A(R)$ (exists by Theorem 9.17). Then there exists an extension $\widetilde{\phi} \in \mathrm{IBr}_\ell(A)$ of ϕ to A satisfying

$$\mathrm{bl}(\widetilde{\phi}'_{|\mathrm{N}_J(R)})^J = \mathrm{bl}(\widetilde{\phi}_{|J})$$

for all groups $G \leqslant J \leqslant A$.

$$\begin{tikzcd}
 & & \widetilde{\phi}_{|J} & A \arrow[dd, dash] & \widetilde{\phi} \\
 & & J \arrow[ur, dash] \arrow[dd, dash] & & \\
\phi & G \arrow[ur, dash] \arrow[dd, dash] & & \mathrm{N}_A(R) & \widetilde{\phi}' \\
 & & \mathrm{N}_J(R) \arrow[ur, dash] & & \\
\phi' & \mathrm{N}_G(R) \arrow[ur, dash] \arrow[d, dash] & \widetilde{\phi}'_{|\mathrm{N}_J(R)} & & \\
 & R & & &
\end{tikzcd}$$

In particular, in the situation of Theorem 9.17 for every $Q \in \mathrm{Rad}_\ell(G)$ *and every* $\phi \in \mathrm{IBr}(B \mid Q)$ *the Brauer characters* $\widetilde{\phi} \in \mathrm{IBr}_\ell(\mathrm{Aut}(G)_\phi)$ *and* $\widetilde{\phi}' \in \mathrm{IBr}_\ell(\mathrm{N}_{\mathrm{Aut}(G)_\phi}(Q))$ *can be chosen such that in addition to parts (1) to (3) of the inductive condition in Definition 3.2(iii) they also satisfy part (4) of Definition 3.2(iii).*

Proof. Let us consider the quotient group A/G. Since $A \leqslant \mathrm{Aut}(G) = G \rtimes \langle \Gamma, F_p \rangle$ with $[\Gamma, F_p] = 1$, the group A/G is abelian. Hence, there exists a unique subgroup $A_1 \leqslant A$ such that A_1/G is a Hall ℓ'-subgroup of A/G, i.e., such that $|A_1/G| = |A/G|_{\ell'}$. The Brauer character $\widehat{\phi}' := \widetilde{\phi}'_{|\mathrm{N}_{A_1}(R)}$ is an extension of ϕ' to $\mathrm{N}_{A_1}(R)$, and since $\mathrm{N}_G(R) \trianglelefteq \mathrm{N}_{A_1}(R)$, the ℓ-block $\mathrm{bl}(\widehat{\phi}')$ covers $\mathrm{bl}(\phi')$ by Definition 1.14. Since $\mathrm{bl}(\widehat{\phi}')$ and $\mathrm{bl}(\phi')$ are ℓ-blocks of $\mathrm{N}_{A_1}(R)$ and $\mathrm{N}_G(R)$, respectively, with R an ℓ-group, the induced blocks $\mathrm{bl}(\widehat{\phi}')^{A_1}$ and $\mathrm{bl}(\phi')^G$ are defined according to Proposition 1.2, and moreover, by assumption we have $\mathrm{bl}(\phi')^G = \mathrm{bl}(\psi)^G = B = \mathrm{bl}(\phi)$. We may apply Lemma 9.18 for $G \trianglelefteq A_1$, $H = \mathrm{N}_{A_1}(R)$ and $M = \mathrm{N}_G(R)$ to deduce that the ℓ-block $\mathrm{bl}(\widehat{\phi}')^{A_1}$ covers $\mathrm{bl}(\phi)$. Hence, by [Nav98, Thm. 9.4] there exists a character in $\mathrm{bl}(\widehat{\phi}')^{A_1}$ lying above ϕ, that is

$$\mathrm{IBr}_\ell(A_1 \mid \phi) \cap \mathrm{IBr}(\mathrm{bl}(\widehat{\phi}')^{A_1}) \neq \emptyset.$$

As we have proven in Theorem 9.17, the Brauer character ϕ extends to A, so let us suppose that $\overline{\phi} \in \mathrm{IBr}_\ell(A)$ is such an extension. Then $\overline{\phi}_{|A_1} \in \mathrm{IBr}_\ell(A_1)$ extends ϕ to A_1, and hence by Gallagher, Theorem 1.21, we have

$$\mathrm{IBr}_\ell(A_1 \mid \phi) = \{\overline{\phi}_{|A_1} \cdot \nu \mid \nu \in \mathrm{IBr}_\ell(A_1/G)\},$$

which consists only of extensions of ϕ to A_1 since A_1/G is abelian. Thus, we conclude that $\mathrm{IBr}(\mathrm{bl}(\widehat{\phi}')^{A_1})$ contains an extension of ϕ, say $\widehat{\phi}$. Again by Gallagher and the fact that A/G is abelian, the set

$$\mathrm{IBr}_\ell(A \mid \phi) = \{\overline{\phi} \cdot \nu \mid \nu \in \mathrm{IBr}_\ell(A/G)\}$$

consists of extensions of ϕ to A. We have $\widehat{\phi} = \overline{\phi}_{|A_1} \cdot \nu$ for some $\nu \in \mathrm{IBr}_\ell(A_1/G)$, and since A/G is abelian, ν extends to A/G, say $\beta \in \mathrm{IBr}_\ell(A/G)$ with $\beta_{|A_1/G} = \nu$. Hence, there exists an extension $\widetilde{\phi} \in \mathrm{IBr}_\ell(A)$ of $\widehat{\phi}$ to A, namely $\widetilde{\phi} = \overline{\phi} \cdot \beta \in \mathrm{IBr}_\ell(A \mid \phi)$.

Note that in our case the representatives of any two distinct G-conjugacy classes of radical ℓ-subgroups are non-isomorphic, so for any $a \in \mathrm{Aut}(G)$ there exists $g \in G$ such that $R^a = R^g$. Hence, for $x \in A$ we conclude that $x \in G\,\mathrm{N}_A(R)$, and since $G\,\mathrm{N}_A(R) \subseteq A$, we have $A = G\,\mathrm{N}_A(R)$. Analogously, it follows that $A_1 = G\,\mathrm{N}_{A_1}(R)$.

Moreover, by Lemma 1.13 the group $\mathcal{O}_\ell(\mathrm{N}_G(R))$ is contained in any defect group of any ℓ-block of $\mathrm{N}_G(R)$. Since R is a radical ℓ-subgroup of G, we have $\mathcal{O}_\ell(\mathrm{N}_G(R)) = R$, so in particular, R is contained in any defect group of $\mathrm{bl}(\phi')$. Hence, for every defect group D of $\mathrm{bl}(\phi')$ we have $\mathrm{C}_{A_1}(D) \subseteq \mathrm{C}_{A_1}(R) \subseteq \mathrm{N}_{A_1}(R)$, and since $\ell \nmid |A_1/G|$, we may apply Lemma 9.19 for $G \trianglelefteq A_1$ with $H = \mathrm{N}_{A_1}(R)$, which gives

$$\mathrm{bl}(\widehat{\phi'}_{|\langle \mathrm{N}_G(R),\, x\rangle})^{\langle G,\, x\rangle} = \mathrm{bl}(\widehat{\phi}_{|\langle G,\, x\rangle})$$

for all $x \in \mathrm{N}_{A_1}(R)$.

Let now $G \leqslant J \leqslant A$. Then $G \trianglelefteq J$ and by similar arguments as above it follows that $J = G\,\mathrm{N}_J(R)$ and $\mathrm{C}_J(D) \subseteq \mathrm{N}_J(R)$ for every defect group D of $\mathrm{bl}(\phi')$. Since A_1/G is the unique Hall ℓ'-subgroup of A/G, the set $\mathrm{N}_J(R)^0$ of ℓ'-elements of $\mathrm{N}_J(R)$ is contained in A_1, so we have

$$\begin{aligned}
\mathrm{bl}((\widetilde{\phi'}_{|\mathrm{N}_J(R)})_{|\langle \mathrm{N}_G(R),\, x\rangle})^{\langle G,\, x\rangle} &= \mathrm{bl}(\widehat{\phi'}_{|\langle \mathrm{N}_G(R),\, x\rangle})^{\langle G,\, x\rangle} \\
&= \mathrm{bl}(\widehat{\phi}_{|\langle G,\, x\rangle}) \\
&= \mathrm{bl}((\widetilde{\phi}_{|J})_{|\langle G,\, x\rangle})
\end{aligned}$$

for all $x \in \mathrm{N}_J(R)^0 \subseteq \mathrm{N}_{A_1}(R)$ by the above considerations. Following Lemma 9.20 for $G \trianglelefteq J$ and $H = \mathrm{N}_J(R)$ we conclude that $\mathrm{bl}(\widetilde{\phi'}_{|\mathrm{N}_J(R)})^J = \mathrm{bl}(\widetilde{\phi}_{|J})$ as claimed. □

Chapter 10

The Main Result for $\mathrm{SL}_3(q)$

Having examined the inductive conditions from Definition 3.2 for all relevant blocks of $\mathrm{SL}_3(q)$, $q > 2$, $q \not\equiv 1 \bmod 3$, we are now able to combine our observations and results obtained in the previous chapters to state and verify the one main assertion we were aiming for in this part of the thesis:

Theorem 10.1. *Let $q > 2$ be a prime power with $q \not\equiv 1 \bmod 3$. Then the inductive blockwise Alperin weight condition (cf. Definition 3.3) holds for the group $\mathrm{SL}_3(q)$ and every prime ℓ dividing its order.*

Proof. Since according to Remark 5.4 the group $G = \mathrm{SL}_3(q)$ is its own universal covering group in the given situation, we need to verify conditions (i) to (iii) of Definition 3.2 for every ℓ-block B of $X = G$ for every prime ℓ dividing $|G|$. If ℓ is such a prime, then either $\ell = p$, $\ell = 2$ or $\ell > 2$ divides exactly one of $\Phi_1(q)$, $\Phi_2(q)$ or $\Phi_3(q)$ following Lemma 5.2.

For $\ell = p$ the claim holds by Proposition 3.11, and for $\ell > 2$ dividing $\Phi_2(q)\Phi_3(q)$ the Sylow ℓ-subgroups of G are cyclic by Proposition 5.9(i), so in consequence of Proposition 3.9 the assertion is true in this case as well. Hence, it remains to consider the case where $\ell \mid \Phi_1(q) = q - 1$ and B is an ℓ-block of G of non-cyclic defect. This case, however, is completely solved by our results in Proposition 8.1, Theorem 9.17 and Proposition 9.21. Hence, the (iBAW) condition holds for $\mathrm{SL}_3(q)$ and every prime ℓ dividing its order as claimed. □

Part III

The Chevalley Groups $G_2(q)$ for $q \geqslant 5$

Chapter 11

Properties of $G_2(q)$

The second infinite series of groups of Lie type we examine in this thesis under the objective of verifying the inductive blockwise Alperin weight condition is the series consisting of the finite Chevalley groups $G_2(q)$. We begin our study by presenting a brief introduction to these groups and giving a summary of those of their properties that are of importance in the subsequent investigation. Our main references are Carter [Car89], Chang [Cha68], Gorenstein–Lyons–Solomon [GLS98], Malle–Testerman [MT11] and Steinberg [Ste68].

11.1 Construction of $G_2(q)$

As a finite group of Lie type the group $G_2(q)$ may be constructed as the subgroup of fixed points with respect to a Frobenius endomorphism of an infinite universal Chevalley group with root system of type G_2.

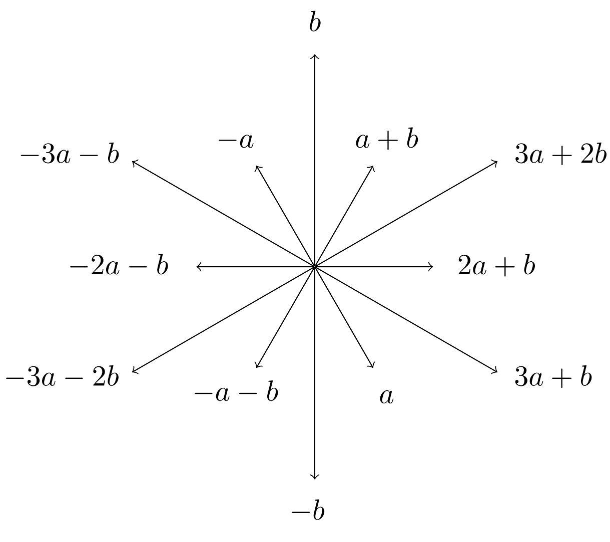

Figure 11.1: Root system of type G_2

To give a more precise definition of $G_2(q)$ we let p be a prime and $f \in \mathbb{N}_{>0}$ be such that $q = p^f$. Moreover, let us consider a root system Σ of type G_2, that is,

$$\Sigma := \{\pm a,\ \pm b,\ \pm(a+b),\ \pm(2a+b),\ \pm(3a+b),\ \pm(3a+2b)\} \subset \mathbb{R}^2,$$

where $\Pi := \{a, b\}$ is a base for Σ (see, for instance, [GLS98, Rmk. 1.8.8]). This may be illustrated as in Figure 11.1 above (cf., e.g., [Car89, p. 46]). Note that the roots in Σ do not all have the same length, but rather there are two root lengths in Σ. The roots $\pm b$, $\pm(3a+2b)$ and $\pm(3a+b)$ are the long roots of Σ, the remaining roots are short.

Let us now come to the definition of $G_2(q)$. We denote by $\mathbb{F}$ an algebraic closure of the finite field $\mathbb{F}_p$ consisting of p elements and let $\mathbb{F}_q$ be its subfield containing exactly q elements. Moreover, we define $\mathbf{G}$ to be a universal Chevalley group over $\mathbb{F}$ with root system Σ and Steinberg generators $x_r(t)$, $h_r(s)$ and $n_r(s)$, $r \in \Sigma$, $t \in \mathbb{F}$, $s \in \mathbb{F}^\times$, as in Theorem 4.22. The linear map given by

$$F\colon \mathbf{G} \longrightarrow \mathbf{G}, \quad x_r(t) \longmapsto x_r(t^q), \quad r \in \Sigma, t \in \mathbb{F},$$

is the Frobenius endomorphism of $\mathbf{G}$ with respect to $\mathbb{F}_q$, hence a Steinberg endomorphism of $\mathbf{G}$, and the finite group $\mathbf{G}^F$ of fixed points of $\mathbf{G}$ under F is exactly the finite group of Lie type which we will henceforth denote by $G := G_2(q) := \mathbf{G}^F$. This group is generated by all $x_r(t)$ for $r \in \Sigma$ and $t \in \mathbb{F}_q$.

In some cases it will turn out to be convenient to use a different notation for the roots in Σ obtained by setting $\xi_1 := a+b$, $\xi_2 := a$ and $\xi_3 := -\xi_1 - \xi_2$, which yields

$$\Sigma = \{\pm\xi_1, \pm\xi_2, \pm\xi_3, \pm(\xi_1-\xi_2), \pm(\xi_2-\xi_3), \pm(\xi_3-\xi_1)\}.$$

This notation allows a uniform description of the action of the Weyl group of $\mathbf{G}$ on the associated maximal torus as will become apparent in the subsequent section.

Lemma 11.1. *The order of the finite group $G_2(q)$ is given by*

$$\begin{aligned}|G_2(q)| &= q^6(q-1)^2(q+1)^2(q^2+q+1)(q^2-q+1)\\ &= q^6\Phi_1(q)^2\Phi_2(q)^2\Phi_3(q)\Phi_6(q).\end{aligned}$$

Proof. This is, for instance, stated in [MT11, Table 24.1]. □

As it turns out, apart from a few exceptions, the groups $G_2(q)$ are simple and constitute their own universal covering groups:

Proposition 11.2. *Suppose that $q \geqslant 3$. Then the finite group $G_2(q)$ is simple. Moreover, if $q \geqslant 5$, then $G_2(q)$ has trivial Schur multiplier, that is, it is its own universal covering group.*

Proof. See, for instance, [MT11, Table 24.2, Thm. 24.17 and Rmk. 24.19]. □

In this thesis we are interested in the case where $G_2(q)$ is simple and has trivial Schur multiplier, whence henceforth, unless stated differently, the following is assumed:

From now on we assume that $q \geqslant 5$.

11.2 Weyl Group and Maximal Tori of $G_2(q)$

Our next objective is to describe the maximal tori of the finite group G. To this end we consider the maximal torus $\mathbf{T}$ of $\mathbf{G}$ generated by all $h_r(t)$, $r \in \Sigma$, $t \in \mathbb{F}^\times$, as in Proposition 4.36 and denote by $\mathbf{W} := \mathrm{N}_{\mathbf{G}}(\mathbf{T})/\,\mathbf{T}$ the corresponding Weyl group. It is a well-known fact that $\mathbf{W}$ is a dihedral group of order 12 (e.g. [Hum72, Table 12.1]).

Notation 11.3. For the purpose of a convenient description of the action of the Weyl group $\mathbf{W}$ on the maximal torus $\mathbf{T}$, let us introduce a different way of writing the elements of $\mathbf{T}$ following the notation of [Cha68]. We recall the root system Σ of $\mathbf{G}$ and consider the additive group $\mathbb{Z}\Sigma$ generated by Σ. By [Car89, p. 98] each group homomorphism $\chi\colon \mathbb{Z}\Sigma \longrightarrow \mathbb{F}^\times$ (called an $\mathbb{F}$-*character* of $\mathbb{Z}\Sigma$) gives rise to an element $h(\chi)$ of the maximal torus $\mathbf{T}$. Even more, according to [Car89, Thm. 7.1.1] we have

$$\mathbf{T} = \{h(\chi) \mid \chi\colon \mathbb{Z}\Sigma \longrightarrow \mathbb{F}^\times \text{ is a group homomorphism}\}.$$

Since $\mathbb{Z}\Sigma$ is a free abelian group, it is evident that for any fixed basis $B \subseteq \mathbb{Z}\Sigma$ a group homomorphism $\chi\colon \mathbb{Z}\Sigma \longrightarrow \mathbb{F}^\times$ (and hence $h(\chi)$) is uniquely determined by the images of the basis elements under χ. Now let us fix the basis $B = \{\xi_1, \xi_2\}$ (this clearly constitutes a basis of $\mathbb{Z}\Sigma$). Then we denote by $h(z_1, z_2, z_3)$ with $z_1, z_2, z_3 \in \mathbb{F}^\times$ the element $h(\chi)$ of $\mathbf{T}$ defined by $\chi(\xi_1) = z_1$, $\chi(\xi_2) = z_2$, and the condition $z_1 z_2 z_3 = 1$. We have

$$h_r(z) = h(z^{\langle r,\, \xi_1\rangle}, z^{\langle r,\, \xi_2\rangle}, z^{\langle r,\, \xi_3\rangle})$$

for $r \in \Sigma$ and $z \in \mathbb{F}^\times$ (see [Car89, p. 98]).

In the above notation the action of $\mathrm{N}_{\mathbf{G}}(\mathbf{T})$, and hence of $\mathbf{W}$, on the maximal torus $\mathbf{T}$ may now be uniformly described as follows (note that this statement is also given in [Cha68, p. 193]):

Lemma 11.4. *Let $z_1, z_2, z_3 \in \mathbb{F}^\times$ with $z_1 z_2 z_3 = 1$ and suppose that $i, j, k \in \{1, 2, 3\}$ are pairwise distinct. Then we have*

$$\begin{aligned} n_{\xi_i - \xi_j}(1)^{-1} h(z_1, z_2, z_3) n_{\xi_i - \xi_j}(1) &= h(z_{\pi(1)}, z_{\pi(2)}, z_{\pi(3)}), \\ n_{\xi_k}(1)^{-1} h(z_1, z_2, z_3) n_{\xi_k}(1) &= h(z_{\pi(1)}^{-1}, z_{\pi(2)}^{-1}, z_{\pi(3)}^{-1}), \end{aligned}$$

where π denotes the transposition $(ij) \in \mathfrak{S}_3$.

Proof. These are easy calculations applying Theorem 4.25(x) and the fact that for any root $r \in \Sigma$ and any field element $z \in \mathbb{F}^\times$ we have $h_r(z) = h(z^{\langle r,\, a+b\rangle}, z^{\langle r,\, a\rangle}, z^{\langle r,\, -(2a+b)\rangle})$. □

Let us now take a look at the maximal tori of G. Up to G-conjugation there exist six of these, denoted T_+, T_-, T_a, T_b, T_3 and T_6 (see, e.g., [Asc87, p. 254]). Representatives in $\mathbf{G}$ of these maximal tori are given in Table 11.1 below, where we write

$$\begin{aligned} v_2 &:= n_b(1) n_{-(2a+b)}(1), \\ v_3 &:= n_{3a+b}(1) n_{-b}(1), \\ v_6 &:= v_2 v_3. \end{aligned}$$

These elements will henceforth occur frequently.

$w \in W$	$\mathbf{T}^{wF}$	$\mathbf{W}^{wF}$
1	$T_+ = \{h(z_1, z_2, z_3) \mid z_i^{q-1} = 1,\ z_1 z_2 z_3 = 1\}$ $\cong C_{q-1} \times C_{q-1}$	D_{12}
$v_2 T$	$T_- = \{h(z_1, z_2, z_3) \mid z_i^{q+1} = 1,\ z_1 z_2 z_3 = 1\}$ $\cong C_{q+1} \times C_{q+1}$	D_{12}
$n_a(1)T$	$T_a = \{h(z, z^{q-1}, z^{-q}) \mid z^{q^2-1} = 1\}$ $\cong C_{q^2-1}$	$C_2 \times C_2$
$n_b(1)T$	$T_b = \{h(z, z^{q}, z^{-(q+1)}) \mid z^{q^2-1} = 1\}$ $\cong C_{q^2-1}$	$C_2 \times C_2$
$v_3 T$	$T_3 = \{h(z, z^{q}, z^{q^2}) \mid z^{q^2+q+1} = 1\}$ $\cong C_{q^2+q+1}$	C_6
$v_6 T$	$T_6 = \{h(z, z^{-q}, z^{q^2}) \mid z^{q^2-q+1} = 1\}$ $\cong C_{q^2-q+1}$	C_6

Table 11.1: Maximal tori of $G_2(q)$

We refer to [Cha68, p. 194] and [Eno70, p. 507] for the statement on the representatives of the maximal tori $\mathbf{T}^{wF}$ and to [Kle88b, Table I] or [Car70, Table 3] for the corresponding fixed points of the Weyl group $\mathbf{W}$.

For future use we set $F_+ := F$ and $F_- := v_2 F$, and, accordingly, $G_\varepsilon := \mathbf{G}^{F_\varepsilon}$ for $\varepsilon \in \{\pm\}$ (or $\varepsilon \in \{\pm 1\}$ slightly abusing notation). Then $G_+ = G$, and in consequence of Corollary 4.35 the finite group G_- is $\mathbf{G}$-conjugate to G.

11.3 Relations in $G_2(q)$

Let us recall from Remark 4.26 that the signs $\eta_{r,s}$ for roots $r, s \in \Sigma$ occurring in the Steinberg relations given in Theorem 4.25 depend on the chosen Chevalley basis underlying $\mathbf{G}$, that is, they depend on the parametrization of the generators $x_r(t)$ of $\mathbf{G}$. Hence, for consistency of our future calculations it seems reasonable to fix such a parametrization for the remainder of this work:

Proposition 11.5. *There exists a Chevalley basis $\{h_r \mid r \in \Pi\} \cup \{e_r \mid r \in \Sigma\}$ as in Theorem 4.10 for the simple Lie algebra of type G_2 underlying $\mathbf{G}$ such that the corresponding multiplication constants N_{rs} for $r, s \in \Sigma$ with $r + s \in \Sigma$ are given by the relation $N_{rs} = -N_{sr}$ and the identities*

$$\begin{aligned}
N_{\xi_i-\xi_j,\xi_j-\xi_k} &= 1, \\
N_{-\xi_i,\xi_i-\xi_j} &= 1, \\
N_{\xi_i-\xi_j,\xi_j} &= 1, \\
N_{\xi_i,\xi_j} &= 2(ijk), \\
N_{-\xi_i,-\xi_j} &= -2(ijk), \\
N_{\xi_i,-\xi_j} &= 3
\end{aligned}$$

for pairwise distinct $i, j, k \in \{1, 2, 3\}$*, where* (ijk) *denotes the sign of the permutation* $\sigma \in \mathfrak{S}_3$ *given by* $\sigma(1) = i$*,* $\sigma(2) = j$ *and* $\sigma(3) = k$*.*

Moreover, with respect to the above Chevalley basis the signs $\eta_{r,s}$*,* $r, s \in \Sigma$*, occurring in Theorem 4.25 are given by Table 11.2 and the relations* $\eta_{r,r} = -1$*,* $\eta_{r,-s} = \eta_{r,s}$ *and* $\eta_{-r,s} = \eta_{r,\omega_r(s)}$*:*

$r \backslash s$	$a+b$	a	$-(2a+b)$	b	$3a+b$	$-(3a+2b)$
$a+b$	-1	-1	1	-1	1	1
a	1	-1	-1	1	-1	1
$-(2a+b)$	-1	1	-1	1	1	-1
b	-1	1	1	-1	1	-1
$3a+b$	1	-1	1	-1	-1	1
$-(3a+2b)$	1	1	-1	1	-1	-1

Table 11.2: Signs $\eta_{r,s}$ for $G_2(q)$

Proof. This is stated in [Ree61, p. 439 and pp. 441/442]. $\square$

We shall henceforth assume that the generators $x_r(t)$ of $\mathbf{G}$ are derived from a Chevalley basis as in Proposition 11.5, so in particular in our computations we use the signs $\eta_{r,s}$ as given in Table 11.2. As a first observation we obtain the following relations for the elements $v_2, v_3, v_6 \in \mathrm{N}_{\mathbf{G}}(\mathbf{T})$ defined in Section 11.2 above:

Lemma 11.6. *We have* $v_2^2 = v_3^3 = v_6^6 = 1$*,* $[v_2, v_3] = 1$*, and moreover*

(i) $v_2^{-1} h(z_1, z_2, z_3) v_2 = h(z_1^{-1}, z_2^{-1}, z_3^{-1})$*,*

(ii) $v_3^{-1} h(z_1, z_2, z_3) v_3 = h(z_3, z_1, z_2)$

for all $z_1, z_2, z_3 \in \mathbb{F}^\times$ *with* $z_1 z_2 z_3 = 1$*.*

Proof. The first part of the claim follows from Theorem 4.25(ix) in combination with Proposition 11.5. Statements (i) and (ii) are immediate consequences of Lemma 11.4. $\square$

Moreover, the following relations hold:

Lemma 11.7. *Let* $t \in \mathbb{F}$ *(with* $t \neq 0$ *in (ii)) and* $r \in \Sigma$*. Then*

(i) $v_2 x_r(t) v_2^{-1} = x_{-r}(-t)$*,*

(ii) $v_2 n_r(t) v_2^{-1} = n_{-r}(-t)$*.*

In particular, v_2 *commutes with all* $n_r(1)$*,* $r \in \Sigma$*.*

Proof. By definition we have $v_2 = n_b(1) n_{-(2a+b)}(1)$, so by application of Theorem 4.25(viii) we obtain that

$$v_2 x_r(t) v_2^{-1} = x_{\omega_b(\omega_{-(2a+b)}(r))}(\eta_{b,\omega_{-(2a+b)}(r)} \eta_{-(2a+b),r} t).$$

Since b and $-(2a+b)$ are perpendicular to each other, $\omega_b(\omega_{-(2a+b)}(r)) = -r$ for any $r \in \Sigma$. Moreover, by Proposition 11.5 we have $\eta_{b,\omega_{-(2a+b)}(r)} \eta_{-(2a+b),r} = -1$ for all $r \in \Sigma$, hence (i) follows. Statement (ii) is a direct consequence of (i) since $n_r(t) = x_r(t) x_{-r}(-t^{-1}) x_r(t)$. Moreover, $n_r(1) = n_{-r}(-1)$ by Theorem 4.25(ix), which yields the final implication. $\square$

Let us now come to a well-known statement which does in fact not just hold for type G_2 but also for arbitrary Chevalley groups. Nevertheless, we only state this result here for Chevalley groups of type G_2 as for our purpose it is only needed in the context of these groups.

Proposition 11.8. *For every root $r \in \Sigma$ there exists an epimorphism*

$$\phi_r \colon \mathrm{SL}_2(\mathbb{F}) \longrightarrow \langle x_r(t), x_{-r}(t) \mid t \in \mathbb{F} \rangle$$

such that

$$\begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \longmapsto x_r(t) \quad \textit{and} \quad \begin{bmatrix} 1 & 0 \\ t & 1 \end{bmatrix} \longmapsto x_{-r}(t)$$

for $t \in \mathbb{F}$. Moreover, under ϕ_r we have

$$\begin{bmatrix} t & 0 \\ 0 & t^{-1} \end{bmatrix} \longmapsto h_r(t) \quad \textit{and} \quad \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \longmapsto n_r(1).$$

Proof. See, for instance, [Car89, Th. 6.3.1] for the existence of the epimorphism ϕ_r. The statements on the preimages for $h_r(t)$ and $n_r(t)$ may easily be derived from the identities $n_r(t) = x_r(t)x_{-r}(-t^{-1})x_r(t)$ and $h_r(t) = n_r(t)n_r(-1)$. □

For later use we also state the values of the Cartan integers $\langle r, s\rangle$ for $r, s \in \Sigma$:

Lemma 11.9. *The Cartan integers $\langle r, s\rangle$, $r, s \in \Sigma$, for the root system Σ of type G_2 can be found in Table 11.3 below, where it should be noted that $\langle -r, s\rangle = \langle r, -s\rangle = -\langle r, s\rangle$ for all $r, s \in \Sigma$.*

r \ s	a	b	$a+b$	$2a+b$	$3a+b$	$3a+2b$
a	2	−3	−1	1	3	0
b	−1	2	1	0	−1	1
$a+b$	−1	3	2	1	0	3
$2a+b$	1	0	1	2	3	3
$3a+b$	1	−1	0	1	2	1
$3a+2b$	0	1	1	1	1	2

Table 11.3: Cartan integers for a root system of type G_2

Proof. Following [Car89, p. 40] the Cartan integers are given by $\langle r, s\rangle = p(r, s) - q(r, s)$ for $r, s \in \Sigma$ with $r \neq \pm s$, where

$$\begin{aligned} p(r, s) &= \max\{i \geqslant 0 \mid -jr + s \in \Sigma \text{ for all } 0 \leqslant j \leqslant i\}, \\ q(r, s) &= \max\{i \geqslant 0 \mid jr + s \in \Sigma \text{ for all } 0 \leqslant j \leqslant i\}. \end{aligned}$$

For $r = s$ we clearly have $\langle r, r\rangle = 2$. Application of these formulae yields the claim. □

11.4 Automorphisms of $G_2(q)$

The aim of this section is a description of the automorphism group of G. Let us recall that $q = p^f$ for a prime p and $f \in \mathbb{N}_{>0}$. Similarly as for the special linear groups we obtain a first automorphism of G by the following definition:

Definition 11.10. By [Ste68, p. 158] the field automorphism $\mathbb{F}_q \longrightarrow \mathbb{F}_q$, $a \longmapsto a^p$, induces an automorphism F_p of the group $G = G_2(q)$ via

$$F_p \colon G \longrightarrow G, \quad x_r(t) \longmapsto x_r(t^p), \quad r \in \Sigma,\ t \in \mathbb{F}_q.$$

The automorphism F_p will be called a *field automorphism* of G. Note that its order in $\mathrm{Aut}(G)$ is given by f.

In the case that $p = 3$ we may define another automorphism of G:

Definition 11.11. Suppose that $p = 3$. By [Ste68, p. 156] there exists a unique angle-preserving and length-changing bijection $\rho \colon \Sigma \longrightarrow \Sigma$ satisfying $\rho(\Delta) = \Delta$. This bijection may be imagined as reflecting along the line bisecting the angle between a and b, and interchanging root lengths (see Figure 11.2). Now ρ induces an automorphism Γ of $G_2(q)$ via

$$\Gamma \colon G \longrightarrow G, \quad x_r(t) \longmapsto \begin{cases} x_{\rho(r)}(\epsilon_r t) & \text{if } r \text{ is long,} \\ x_{\rho(r)}(\epsilon_r t^3) & \text{if } r \text{ is short,} \end{cases}$$

for suitable signs $\epsilon_r \in \{\pm 1\}$ with $\epsilon_r = 1$ if one of $\pm r$ is contained in Δ (compare [Ste68, pp. 156/157]). This automorphism is called a *graph automorphism* of G.

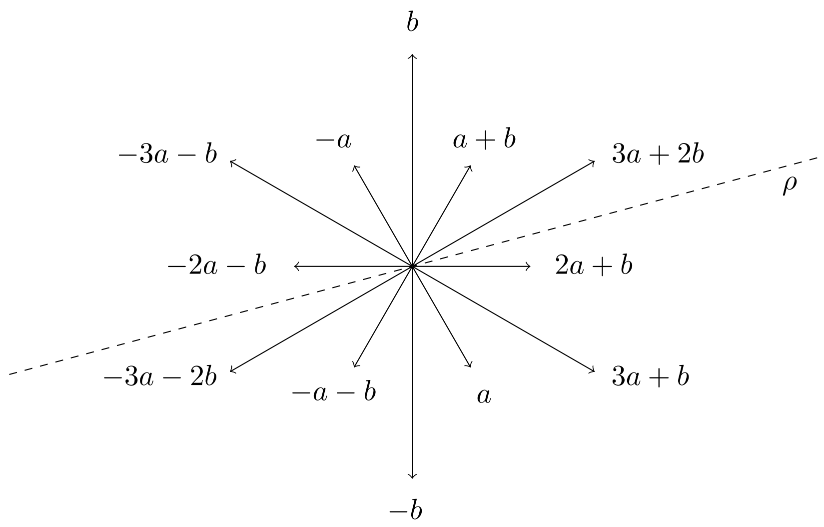

Figure 11.2: Permutation ρ of root system of type G_2

Remark 11.12. Let us assume again that $p = 3$. The signs ϵ_r in the definition of the graph automorphism Γ above depend, like the signs $\eta_{r,s}$, on the chosen Chevalley basis of the Lie algebra underlying $\mathbf{G}$. Since we are working with the fixed Chevalley basis given by Proposition 11.5, the question arises of determining the values of the signs ϵ_r with respect to this particular basis.

Following the proof of [Car89, Prop. 12.4.1] we may assume that the signs ϵ_r equal 1 for all $r \in \Sigma$ if the following condition on the multiplication constants is satisfied:

$$-\frac{1}{2}N_{\xi_2,\xi_1} = \frac{1}{3}N_{\xi_2,-\xi_3} = N_{\xi_1-\xi_2,\xi_2-\xi_3}.$$

In consequence of Proposition 11.5 we have

$$\begin{aligned} N_{\xi_2,\xi_1} &= 2(213) = -2, \\ N_{\xi_2,-\xi_3} &= 3, \\ N_{\xi_1-\xi_2,\xi_2-\xi_3} &= 1, \end{aligned}$$

whence the above condition is clearly fulfilled. Thus, in the following we work with the graph automorphism

$$\Gamma\colon G \longrightarrow G, \quad x_r(t) \longmapsto \begin{cases} x_{\rho(r)}(t) & \text{if } r \text{ is long,} \\ x_{\rho(r)}(t^3) & \text{if } r \text{ is short.} \end{cases}$$

We observe that since ρ^2 is the identity map on Σ and ρ interchanges root lengths, we have $\Gamma^2(x_r(t)) = x_r(t^3)$ for all $r \in \Sigma$, so $\Gamma^2 = F_3 = F_p$. If for $r \in \Sigma$ we define λ_r to be 1 if r is long, and 3 if r is short, then the above definition of Γ reduces to $\Gamma(x_r(t)) = x_{\rho(r)}(t^{\lambda_r})$.

For our future calculations it will be convenient to understand the action of the graph automorphism Γ on the elements $h_r(t)$ and $n_r(t)$:

Lemma 11.13. *Suppose that $p = 3$. Then it holds that*

(i) $\Gamma(n_r(t)) = n_{\rho(r)}(t^{\lambda_r})$,

(ii) $\Gamma(h_r(t)) = h_{\rho(r)}(t^{\lambda_r})$

for every root $r \in \Sigma$ and every $t \in \mathbb{F}_q^\times$.

Proof. We have

$$\begin{aligned} \Gamma(n_r(t)) &= \Gamma(x_r(t))\Gamma(x_{-r}(-t^{-1}))\Gamma(x_r(t)) \\ &= x_{\rho(r)}(t^{\lambda_r})x_{\rho(-r)}((-t^{-1})^{\lambda_{-r}})x_{\rho(r)}(t^{\lambda_r}) \\ &= x_{\rho(r)}(t^{\lambda_r})x_{-\rho(r)}(-(t^{\lambda_r})^{-1})x_{\rho(r)}(t^{\lambda_r}) \\ &= n_{\rho(r)}(t^{\lambda_r}) \end{aligned}$$

since $\rho(-r) = -\rho(r)$, the roots r and $-r$ have the same length, and $(-1)^{\lambda_r} = -1$. From this we deduce that

$$\begin{aligned} \Gamma(h_r(t)) &= \Gamma(n_r(t))\Gamma(n_r(1))^{-1} \\ &= n_{\rho(r)}(t^{\lambda_r})n_{\rho(r)}(1)^{-1} \\ &= h_{\rho(r)}(t^{\lambda_r}) \end{aligned}$$

for all $r \in \Sigma$ and $t \in \mathbb{F}_q^\times$ as claimed. $\square$

By means of the two automorphisms Γ and F_p the automorphism group of G may now be described as follows:

Proposition 11.14. *For $G = G_2(q)$ with $q = p^f$ one has*

$$\mathrm{Aut}(G) = \begin{cases} G \rtimes \langle F_p \rangle & \text{if } p \neq 3, \\ G \rtimes \langle \Gamma \rangle & \text{if } p = 3. \end{cases}$$

In particular, the outer automorphism group of G is cyclic for all primes p.

Proof. This is well-known and follows, for instance, from [GLS98, Th. 2.5.12(a)–(e)] by taking into account that the algebraic group $\mathbf{G}$ is both adjoint and universal (cf. [MT11, Table 9.2]). □

Remark 11.15. (i) As indicated earlier, in certain situations it will be convenient to work inside the group $G_- = \mathbf{G}^{v_2 F}$, which we defined in Section 11.2, instead of the group $G = G_+$. This is due to the fact that inside the group G_- the maximal tori of type T_- have a representative lying in the maximal torus $\mathbf{T}$ of $\mathbf{G}$ (cf. Table 11.1), which allows a nice description of the action of the field automorphism on this torus. For this we should note that as for the group G_+ the endomorphism of $\mathbf{G}$ defined by $x_r(t) \mapsto x_r(t^p)$ induces an automorphism of G_-. Similarly as for G_+ we shall denote this automorphism by F_p, and if $p \neq 3$, then $\mathrm{Aut}(G_-) = \langle G_-, F_p \rangle$. The action of F_p on the maximal torus T_- of G_- lying inside $\mathbf{T}$ is then given by raising each element to its p-th power, which is analogous to the action of F_p on the maximal torus T_+ of G_+ inside $\mathbf{T}$.
However, contrary to the case of the group G_+ it does not hold true any longer that F_p has order f as an automorphism of G_-, where $q = p^f$. Rather, the order of F_p is given by $2f$ in this case since $(v_2 F_q)^2 = F_{q^2} = F_p^{2f}$ acts trivially on G_-, while $F_p^f = F_q$ acts on G_- by conjugation with v_2.

(ii) Following Definition 11.10 for $\delta \in \{\pm\}$ the field automorphism F_p acts trivially on the quotient

$$\mathrm{N}_{G_\delta}(\mathbf{T})/T_\delta = \mathrm{N}_{\mathbf{G}^{F_\delta}}(\mathbf{T})/\,\mathbf{T}^{F_\delta} = \langle T_\delta, n_r(1) \mid r \in \Sigma \rangle / T_\delta \cong D_{12},$$

where we have $\mathrm{N}_{\mathbf{G}^{F_\delta}}(\mathbf{T}) = \langle T_\delta, n_r(1) \mid r \in \Sigma \rangle$ in consequence of Lemma 11.7 (cf. also Propositions 4.27 and 4.40 for the statement on the isomorphism). Furthermore, any element of $\mathrm{N}_{G_\delta}(\mathbf{T})$ may be written in the form $n \cdot t$ for some $t \in T_\delta$ and $n \in \mathrm{N}_{G_\delta}(\mathbf{T})$ with $F_p(n) = n$.

Chapter 12

Action of Automorphisms

In this chapter we determine the action of the automorphisms of the group $G = G_2(q)$ on its irreducible Brauer characters as well as on its ℓ-weights in the cases $\ell = 2$ and $\ell = 3$, where $\ell \nmid q$. The case where $\ell \geqslant 5$ is not considered here since in this situation the (iBAW) condition has already been proven to hold for G either by reason of the ℓ-blocks having cyclic defect groups or by results of Cabanes–Späth [CS13]. More details on this will be provided in the proof of Theorem 14.1.

In the previous chapter we saw that the outer automorphism group of G is cyclic, generated either by the field automorphism F_p if $3 \nmid q$ or by the graph automorphism Γ in the case where q is a power of 3. Hence, in order to understand the action of $\mathrm{Aut}(G)$ on the Brauer characters and G-conjugacy classes of weights of G it suffices to know how these behave under the action of F_p or Γ, accordingly.

12.1 Action on the Brauer Characters of $G_2(q)$

Let us first study the action of the automorphism group $\mathrm{Aut}(G)$ on the Brauer characters of G. A set of representatives for the conjugacy classes of G as well as a brief description of the irreducible characters of G and the ℓ-blocks and ℓ-decomposition numbers for $\ell \in \{2, 3\}$ may be found in Sections B.1, B.2 and B.3 of Appendix B. For the notation used here we also refer to those sections.

12.1.1 The Case $\ell = 2$

Here we assume that $\ell = 2$ and q is odd.

Proposition 12.1. *Let B be a 2-block of G. Then one of the following holds:*

(i) $B \neq B_0$ *and* $Aut(G)_B$ *acts trivially on* $\mathrm{IBr}(B)$*,*

(ii) $B = B_0$*,* $3 \nmid q$*, and* $Aut(G)_B = \mathrm{Aut}(G)$ *acts trivially on* $\mathrm{IBr}(B)$*,*

(iii) $B = B_0$*,* $3 \mid q$*, and* Γ *interchanges* φ_{13} *and* φ_{14}*, and leaves the remaining elements of* $\mathrm{IBr}(B)$ *invariant.*

Proof. We go through the different types of 2-blocks of G (cf. Section B.3.1 for the 2-blocks of G and the corresponding decomposition matrices). Certainly, the claim holds for 2-blocks of defect zero. Suppose now that B is the 2-block B_3, in which case we must have $3 \nmid q$. Then a basic set for $\mathrm{IBr}(B)$ is given by $\mathcal{B} = \{X_{31}, X_{32}, X_{33}\}$, and since according to Table B.2 each of the characters in $\mathcal{B}$ is the unique irreducible character of G of its degree, it follows that $\mathrm{Aut}(G)_B = \mathrm{Aut}(G)$ acts trivially on $\mathcal{B}$. Moreover, the decomposition matrix for B is unitriangular with respect to a suitable ordering of $\mathcal{B}$ and $\mathrm{IBr}(B)$, whence by Lemma 1.27 each element of $\mathrm{IBr}(B)$ is left invariant by $\mathrm{Aut}(G)_B = \mathrm{Aut}(G)$.

Let us next assume that B is of type B_I for some $I \in \{1a, 1b, 2a, 2b\}$. Then we have $|\,\mathrm{IBr}(B)| = 2$ and the irreducible Brauer characters in B are of degrees $\varphi'_I(1)$ and $\varphi_I(1) = (q-1)\varphi'_I(1)$ following Table B.2 and Section B.3.1. Since by assumption $q \geqslant 5$, these degrees are distinct, whence in particular $\mathrm{Aut}(G)_B$ does not interchange the two elements of $\mathrm{IBr}(B)$.

If B is of type B_{X_I} for some $I \in \{1, 2, a, b\}$, then $|\,\mathrm{IBr}(B)| = 1$, so the claim follows trivially. We have thus proven (i).

Let us finally suppose that B is the principal 2-block B_0. Then a basic set for $\mathrm{IBr}(B)$ is given by $\mathcal{B} = \{X_{11}, X_{17}, X_{18}, X_{13}, X_{14}, X_{15}, X_{12}\}$, and, apart from the characters X_{13} and X_{14}, according to Table B.2 the elements in this set are the unique irreducible characters of G of their degree, hence left invariant by $\mathrm{Aut}(G) = \mathrm{Aut}(G)_B$. We take a closer look at X_{13} and X_{14}. These are the only irreducible characters of G of degree $\frac{1}{3}q\Phi_3(q)\Phi_6(q)$, whence any automorphism of G either stabilizes both of these characters or it interchanges them. Let us assume first that $3 \nmid q$ and consider the conjugacy class of G represented by $u_2 = x_{2a+b}(1)$ (cf. Table B.1). Clearly, this class is stabilized by the field automorphism F_p, so $X_{13}^{F_p}(u_2) = X_{13}(u_2)$. Now by Table B.3 we have

$$X_{13}(u_2) = q \neq 0 = X_{14}(u_2),$$

from which we deduce that $X_{13}^{F_p} \neq X_{14}$, that is, $X_{13}^{F_p} = X_{13}$ and $X_{14}^{F_p} = X_{14}$. By Proposition 11.14 we have $\mathrm{Aut}(G) = G \rtimes \langle F_p \rangle$ for $3 \nmid q$, which implies that also X_{13} and X_{14} are in fact stabilized by all of $\mathrm{Aut}(G)$ in this case, yielding (ii).

Let now $3 \mid q$ and consider the conjugacy classes of G represented by $u_1 = x_{3a+2b}(1)$ and $u_2 = x_{2a+b}(1)$. In this case the automorphism group of G is given by $\mathrm{Aut}(G) = G \rtimes \langle \Gamma \rangle$ according to Proposition 11.14, and from Definition 11.11 and Remark 11.12 it follows that

$$\Gamma(u_1) = \Gamma(x_{3a+2b}(1)) = x_{2a+b}(1) = u_2.$$

By Table B.3 we have $X_{13}(u_1) \neq X_{13}(u_2) = X_{13}^{\Gamma}(u_1)$ but $X_{13}^{\Gamma}(u_1) = X_{13}(u_2) = X_{14}(u_1)$, and we conclude that $X_{13}^{\Gamma} = X_{14}$. Let us now take a look at the 2-decomposition numbers of B_0, which are given by the following matrix with respect to the basic set $\mathcal{B}$:

	φ_{11}	φ_{17}	φ_{18}	φ_{13}	φ_{14}	φ_{15}	φ_{12}
X_{11}	1	·	·	·	·	·	·
X_{17}	·	1	·	·	·	·	·
X_{18}	·	·	1	·	·	·	·
X_{13}	1	·	·	1	·	·	·
X_{14}	1	·	·	·	1	·	·
X_{15}	·	1	·	·	·	1	·
X_{12}	1	α	β	1	1	·	1

for suitable $\alpha, \beta \geqslant 0$ (cf. Section B.3.1). Clearly, since X_{11}, X_{17} and X_{18} are invariant under $\mathrm{Aut}(G)$, so are φ_{11}, φ_{17} and φ_{18}. Now $\varphi_{13} = X_{13}^0 - \varphi_{11}$ and $\varphi_{14} = X_{14}^0 - \varphi_{11}$, so by the previous observations Γ interchanges φ_{13} and φ_{14}. Since X_{15} and X_{12} are left invariant by $\mathrm{Aut}(G)$ and the multiplicity of φ_{13} as an irreducible constituent of X_{15}^0 and X_{12}^0, respectively, agrees with that of φ_{14} in each case, we conclude that φ_{15} and φ_{12} are stabilized by Γ, and hence by $\mathrm{Aut}(G) = G \rtimes \langle \Gamma \rangle$ as claimed in (iii). □

12.1.2 The Case $\ell = 3$

Here we assume that $\ell = 3$ and $3 \nmid q$. In particular, we have $\mathrm{Aut}(G) = G \rtimes \langle F_p \rangle$.

Proposition 12.2. *Let B be a 3-block of G. Then $\mathrm{Aut}(G)_B$ acts trivially on $\mathrm{IBr}(B)$.*

Proof. As for the case $\ell = 2$ we go through the different types of 3-blocks of G (cf. Section B.3.2 for the 3-blocks of G and the corresponding decomposition matrices), and we do not need to deal with 3-blocks of defect zero.

Suppose first that B is the 3-block B_2. Then q is odd and $|\mathrm{IBr}(B)| = 4$ with basic set given by $\mathcal{B} = \{X_{21}, X_{22}, X_{23}, X_{24}\}$. Following Table B.2 the characters X_{21} and X_{22} are the unique irreducible characters of G of their degrees, hence invariant under the action of $\mathrm{Aut}(G) = \mathrm{Aut}(G)_B$, while $X_{23}(1) = X_{24}(1)$. However, there are no further irreducible characters of G of this degree, so an element of $\mathrm{Aut}(G)$ not stabilizing X_{23} and X_{24} has to interchange these. But by Table B.4 the character X_{23} takes the value $q(q^2+1)$ on the conjugacy class represented by u_1, while the character X_{24} does not take this value on any conjugacy class, so in consequence X_{23} and X_{24} cannot be interchanged by any automorphism of G. We conclude that $\mathrm{Aut}(G)_B = \mathrm{Aut}(G)$ acts trivially on $\mathcal{B}$, and since the decomposition matrix for B with respect to suitable orderings of $\mathcal{B}$ and $\mathrm{IBr}(B)$ is unitriangular, we may apply Lemma 1.27 to deduce that $\mathrm{IBr}(B)$ is stabilized pointwise by $\mathrm{Aut}(G)_B$.

Similarly as in the case $\ell = 2$, for 3-blocks B of type B_I, where $I \in \{1a, 1b, 2a, 2b\}$, there exist exactly two irreducible Brauer characters in B, which are of distinct degrees since $q \geqslant 5$. Hence, $\mathrm{Aut}(G)_B$ acts trivially on $\mathrm{IBr}(B)$.

If $q \equiv 1 \bmod 3$ and B is a 3-block of type B_{X_I} for $I \in \{1, 3, a, b\}$ or if $q \equiv -1 \bmod 3$ and B is a 3-block of type B_{X_I} for $I \in \{2, 6, a, b\}$, then $\mathrm{IBr}(B)$ contains only one element, which is hence trivially stabilized by $\mathrm{Aut}(G)_B$.

Finally, we suppose that $B = B_0$ is the principal 3-block of G. Then a basic set for $\mathrm{IBr}(B)$ is given by the set $\mathcal{B} = \{X_{11}, X_{18}, X_{19,1}, X_{14}, X_{15}, X_{16}, X_{12}\}$ for $q \equiv 1 \bmod 3$ or by the set $\mathcal{B} = \{X_{11}, X_{18}, X_{19,1}, X_{14}, X_{17}, X_{16}, X_{12}\}$ for $q \equiv -1 \bmod 3$. With the exception of the character $X_{19,1}$ any element of $\mathcal{B}$ is the unique irreducible character of G of its degree (see Table B.2), hence invariant under $\mathrm{Aut}(G)$. For the character $X_{19,1}$ we have

$$X_{19,1}(1) = \frac{1}{3} q \Phi_1(q)^2 \Phi_2(q)^2,$$

and according to Table B.2 it holds that $X_{19,2}$ is the only other irreducible character of G of this degree. Moreover, by Remark B.3 we have $X_{19,1}^0 = X_{19,2}^0$, so we conclude that $\mathrm{Aut}(G)$ leaves $X_{19,1}^0$ invariant (even though some automorphism of G might interchange $X_{19,1}$ and $X_{19,2}$). In summary, we conclude that for any element $\chi \in \mathcal{B}$ the Brauer character χ^0 must be left invariant by $\mathrm{Aut}(G) = \mathrm{Aut}(G)_B$. With respect to a suitable ordering of both $\mathcal{B}$ and $\mathrm{IBr}(B)$ the corresponding decomposition matrix is unitriangular according to Section B.3.2, so again by application of Lemma 1.27 and Remark 1.28 we conclude that the irreducible Brauer characters in B are left invariant by $\mathrm{Aut}(G)_B$ as claimed. □

12.2 Action on the Weights of $G_2(q)$

The B-weights of G for ℓ-blocks B of non-cyclic defect have already been determined by J. An in [An94a] for various primes ℓ dividing the order of G. We give here a summary of his results for the primes 2 and 3. Furthermore, our task is now to examine how these weights behave under the action of automorphisms of G.

For the course of this section we also allow $q < 5$. The reason for this is that in certain cases we need to consider weights of the subgroup $G_2(p)$ of $G_2(q)$ defined over the prime field $\mathbb{F}_p$ of $\mathbb{F}_q$, which makes it necessary to include the groups $G_2(2)$ and $G_2(3)$ in our investigations. However, this shall not cause any problems here since neither any of our proofs nor any of the results we refer to rely on the condition that $q \geqslant 5$.

12.2.1 The Case $\ell = 2$

Throughout this section we assume that $\ell = 2$ and q is odd. Moreover, we let $\varepsilon \in \{\pm 1\}$ be such that $q \equiv \varepsilon \bmod 4$. In the first part of this section we examine certain 2-subgroups and 2-weights of G in particular situations in order to understand their behaviour under the action of $\mathrm{Aut}(G)$. These groups will occur in Proposition 12.16 as radical 2-subgroups of G giving rise to 2-weights belonging to the principal 2-block of G.

To begin with, note that any Sylow 2-subgroup of G is contained in the centralizer of an involution, and thus so is any 2-subgroup of G. Let us hence fix the involution $y := h_a(-1)h_b(-1) \in G$ for the following investigations. By application of Lemma 11.9 and the fact that

$$h_\gamma(z) = h(z^{\langle \gamma,\, \xi_1 \rangle}, z^{\langle \gamma,\, \xi_2 \rangle}, z^{\langle \gamma,\, \xi_3 \rangle})$$

for $\gamma \in \Sigma$ and $z \in \mathbb{F}^\times$ (cf. Section 11.2) we obtain that $y = h(1, -1, -1) \in T_+$. Moreover, by [Ree67, Thm. 2.5] all involutions in G are G-conjugate to y.

Lemma 12.3. *The centralizer in G of the involution $y = h_a(-1)h_b(-1)$ is given by*

$$\mathrm{C}_G(y) = \langle T_+,\ x_{a+b}(t),\ x_{3a+b}(t),\ n_{a+b}(1),\ n_{3a+b}(1) \mid t \in \mathbb{F}_q \rangle.$$

Proof. As observed in Section 11.2, there exists an $\mathbb{F}$-character χ such that $y = h(\chi)$ since $y \in T_+ \subseteq \mathbf{T}$. By [Cha68, Th. 4.1] (for $p \geqslant 5$) and [Eno70, Prop. 3.1 and Sec. 7] (for $p = 3$) we have

$$\mathrm{C}_G(h(\chi)) = \langle T_+,\ x_{\pm r}(t) \mid r \in \Sigma,\ \chi(r) = 1,\ t \in \mathbb{F}_q \rangle.$$

We have already noted above that $y = h(1, -1, -1)$. Hence, it follows that y corresponds to the $\mathbb{F}$-character $\chi\colon \mathbb{Z}\Sigma \longrightarrow \mathbb{F}^\times$ with $\chi(a + b) = 1$ and $\chi(a) = -1$, and since

$$\chi(c_1 a + c_2 b) = \chi(a)^{c_1}\chi(b)^{c_2} = (-1)^{c_1+c_2}$$

for $c_1, c_2 \in \mathbb{Z}$, we deduce that $\chi(r) = 1$ for $r \in \Sigma$ if and only if $r \in \{\pm(a+b), \pm(3a+b)\}$. $\square$

In the following we will consider a number of statements involving certain orthogonal groups. Let us fix some notation:

Notation 12.4. For an even natural number $n \geqslant 2$, a prime power q and a sign $\delta \in \{\pm\}$ we denote by $\mathrm{CO}_n^\delta(q)$, $\mathrm{GO}_n^\delta(q)$ and $\mathrm{SO}_n^\delta(q)$ with

$$\mathrm{CO}_n^\delta(q) \trianglerighteq \mathrm{GO}_n^\delta(q) \trianglerighteq \mathrm{SO}_n^\delta(q) \quad \text{and} \quad \mathrm{CO}_n^\delta(q) \trianglerighteq \mathrm{SO}_n^\delta(q)$$

the *conformal orthogonal group*, the *general orthogonal group* and the *special orthogonal group over $\mathbb{F}_q$ of degree n and δ-type*, respectively. We refer to [KL90, Sec. 2.5] for a detailed description of these groups. One should note that the notation in [KL90] differs from the notation used here in the way that in [KL90] the conformal orthogonal groups are denoted by $\mathrm{GO}_n^\pm(q)$, while the general orthogonal groups are written as $\mathrm{O}_n^\pm(q)$.

Remark 12.5. Since by [Ree67, Thm. 2.5] all involutions of G are conjugate in G, it follows from [An94a, (1A) and (3D)] that $\mathrm{C}_G(y)$ is isomorphic to $\mathrm{SO}_4^+(q)$, the special orthogonal group over $\mathbb{F}_q$ of degree 4 and plus type.

The first type of 2-weights we consider now are those arising from 2-subgroups of G isomorphic to the extraspecial group 2_+^{1+4} of order 2^{1+4} and plus type.

Lemma 12.6. *Suppose that $q \equiv \pm 3 \bmod 8$. Then any two subgroups of $\mathrm{SO}_4^+(q)$ that are isomorphic to 2_+^{1+4} are conjugate in $\mathrm{CO}_4^+(q)$.*

Moreover, any subgroup $R \leqslant \mathrm{SO}_4^+(q)$ isomorphic to 2_+^{1+4} is 2-radical in $\mathrm{SO}_4^+(q)$ with

$$\mathrm{N}_{\mathrm{SO}_4^+(q)}(R)/R \cong (C_3 \times C_3) \rtimes C_2,$$

where the non-trivial element of C_2 acts on $C_3 \times C_3$ by inversion.

Proof. The first statement is part of [An93, (1G)(b)]. For the second statement we note that, in consequence of the first part, for subgroups $R_1, R_2 \leqslant \mathrm{SO}_4^+(q)$ isomorphic to 2_+^{1+4} we have

$$\mathrm{N}_{\mathrm{SO}_4^+(q)}(R_1)/R_1 \cong \mathrm{N}_{\mathrm{SO}_4^+(q)}(R_2)/R_2,$$

so R_1 is a radical 2-subgroup of $\mathrm{SO}_4^+(q)$ if and only if R_2 is. Moreover, according to [An94a, (2B)] the group $\mathrm{SO}_4^+(q)$ contains a radical 2-subgroup $R \cong 2_+^{1+4}$, which satisfies $\mathrm{N}_{\mathrm{SO}_4^+(q)}(R)/R \cong (C_3 \times C_3) \rtimes C_2$ with the non-trivial element of C_2 acting on $C_3 \times C_3$ by inversion. Hence, the claim follows. $\square$

Before we state the next result let us observe that in the case of $3 \mid q$ we always have $q \equiv 1 \bmod 8$ if q is an even power of 3, and $q \equiv 3 \bmod 8$ otherwise. Hence, $q \equiv \pm 3 \bmod 8$ immediately implies that $q \equiv 3 \bmod 8$ in this situation.

Proposition 12.7. *Suppose that $3 \mid q$ and let $q \equiv 3 \bmod 8$. Moreover, define*

$$R := \langle x_{3a+b}(-1)x_{-(3a+b)}(-1),\ x_{a+b}(-1)x_{-(a+b)}(-1),\ n_{3a+b}(1),\ n_{a+b}(1)\rangle \leqslant G_2(3) \leqslant G.$$

Then $R \cong 2_+^{1+4}$ is a radical 2-subgroup of both $G_2(3)$ and $G_2(q)$.

Proof. We first prove that $R \cong 2_+^{1+4}$ by making use of the fact that $2_+^{1+4} \cong D_8 \circ D_8$ is the central product of two dihedral groups of order 8. For $r \in \Sigma$ recall the epimorphisms ϕ_r from Proposition 11.8. As $3 \mid q$, we have

$$x_{a+b}(-1)x_{-(a+b)}(-1) = \phi_{a+b}\left(\left[\begin{smallmatrix} 1 & -1 \\ 0 & 1 \end{smallmatrix}\right]\right)\phi_{a+b}\left(\left[\begin{smallmatrix} 1 & 0 \\ -1 & 1 \end{smallmatrix}\right]\right) = \phi_{a+b}\left(\left[\begin{smallmatrix} -1 & -1 \\ -1 & 1 \end{smallmatrix}\right]\right)$$

with $\left[\begin{smallmatrix} -1 & -1 \\ -1 & 1 \end{smallmatrix}\right]^2 = \left[\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right]$. Since by Proposition 11.8 it holds that

$$\phi_{a+b}\left(\left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right)\right) = h_{a+b}(-1),$$

which does not equal 1 by Theorem 4.25(v),(vi), we conclude that $x_{a+b}(-1)x_{-(a+b)}(-1)$ has order 4. Analogously, one can show that $x_{3a+b}(-1)x_{-(3a+b)}(-1)$ has order 4. Moreover, from Theorem 4.25(ii) it follows that $x_{a+b}(-1)x_{-(a+b)}(-1)$ and $x_{3a+b}(-1)x_{-(3a+b)}(-1)$ commute, and by part (v) of that theorem we have

$$h_{a+b}(-1) = h_{3a+b}(-1) = y.$$

By Theorem 4.25(viii) and Proposition 11.5 the element $n_{a+b}(1)$ acts trivially on the group $\langle x_{3a+b}(-1)x_{-(3a+b)}(-1)\rangle$ and inverts the elements of $\langle x_{a+b}(-1)x_{-(a+b)}(-1)\rangle$. The same holds true if one interchanges the roles of $a+b$ and $3a+b$. Finally, from Theorem 4.25(ix) and Proposition 11.5 we deduce that $[n_{a+b}(1), n_{3a+b}(1)] = 1$. Hence, we conclude that

$$\begin{aligned} R &= \big(\langle x_{a+b}(-1)x_{-(a+b)}(-1)\rangle \rtimes \langle n_{a+b}(1)\rangle\big) \circ \big(\langle x_{3a+b}(-1)x_{-(3a+b)}(-1)\rangle \rtimes \langle n_{3a+b}(1)\rangle\big) \\ &\cong D_8 \circ D_8 \end{aligned}$$

as claimed.

As observed before, we have $y = h_{3a+b}(-1) = n_{3a+b}(1)^2 \in R$, and moreover one easily verifies that $\mathrm{Z}(R) = \langle y\rangle$. Hence, $R \subseteq \mathrm{C}_G(y) \cong \mathrm{SO}_4^+(q)$ (and $R \subseteq \mathrm{C}_{G_2(3)}(y) \cong \mathrm{SO}_4^+(3)$, respectively), and from Lemma 12.6 it thus follows that R is 2-radical in both $\mathrm{SO}_4^+(q)$ and $\mathrm{SO}_4^+(3)$. The center of R is a characteristic subgroup of R, so in particular we obtain that $\mathrm{N}_G(R) \subseteq \mathrm{N}_G(\mathrm{Z}(R)) = \mathrm{C}_G(y)$. Consequently, R is 2-radical in $G_2(q)$, and analogously it also follows that R is 2-radical in $G_2(3)$. $\square$

Proposition 12.8. *Assume that $3 \mid q$ and let $q \equiv 3 \bmod 8$. Moreover, suppose that the group $R \cong 2_+^{1+4}$ is as in Proposition 12.7. Then $\mathrm{N}_G(R) = \mathrm{C}_{G_2(3)}(y)$.*

Proof. In the proof of the Proposition 12.7 we observed that $\mathrm{N}_G(R) \subseteq \mathrm{C}_G(y) \cong \mathrm{SO}_4^+(q)$, whence we may apply Lemma 12.6 to deduce that

$$\mathrm{N}_G(R)/R \cong (C_3 \times C_3) \rtimes C_2,$$

with the action of the non-trivial element of C_2 on $C_3 \times C_3$ given by inversion. E.g. by application of Theorem 4.25 one verifies that the group

$$\mathrm{C}_{G_2(3)}(y) = \langle h_a(-1),\ h_b(-1),\ x_{a+b}(1),\ x_{3a+b}(1),\ n_{a+b}(1),\ n_{3a+b}(1)\rangle \geqslant R$$

(cf. Lemma 12.3) is contained in $\mathrm{N}_G(R)$. Now $h_a(-1)$ and $h_b(-1)$ agree modulo R since $y = h_a(-1)h_b(-1)$ lies in R, and hence according to Theorem 4.25(ii),(vii) and Lemma 11.9 we have

$$\begin{aligned} \mathrm{C}_{G_2(3)}(y)/R &= \langle \overline{h_a(-1)},\ \overline{x_{a+b}(1)},\ \overline{x_{3a+b}(1)}\rangle \\ &= (\langle \overline{x_{a+b}(1)}\rangle \times \langle \overline{x_{3a+b}(1)}\rangle) \rtimes \langle \overline{h_a(-1)}\rangle \end{aligned}$$

with the action of $\overline{h_a(-1)}$ given by inversion and $\overline{}\colon \mathrm{C}_{G_2(3)}(y) \twoheadrightarrow \mathrm{C}_{G_2(3)}(y)/R$ denoting the natural epimorphism. Since $3 \mid q$, it follows that $x_{a+b}(1)^3 = x_{3a+b}(1)^3 = 1$, and hence we may conclude that $\mathrm{N}_G(R) = \mathrm{C}_{G_2(3)}(y)$. $\square$

With R as above, the previous result allows us to examine the action of the graph automorphism Γ on $\mathrm{N}_G(R)/R$ in the situation of $3 \mid q$ and $q \equiv 3 \bmod 8$.

Lemma 12.9. *Assume that $3 \mid q$ and let $q \equiv 3 \bmod 8$. Moreover, let $R \cong 2_+^{1+4}$ be as in Proposition 12.7 and suppose the notation of the proof of Proposition 12.8, that is,*

$$\mathrm{N}_G(R)/R = (\langle \overline{x_{a+b}(1)}\rangle \times \langle \overline{x_{3a+b}(1)}\rangle) \rtimes \langle \overline{h_a(-1)}\rangle.$$

Let us parametrize the elements of $\mathrm{N}_G(R)/R$ by setting

$$((s,t),z) := \overline{x_{a+b}(s)}\ \overline{x_{3a+b}(t)}\, z \in \mathrm{N}_G(R)/R$$

for $s,t \in \mathbb{F}_3$ and $z \in \langle \overline{h_a(-1)}\rangle$. Then $\Gamma(((s,t),z)) = ((t,s),z)$.

Proof. This is obvious since by definition of Γ it holds that $\Gamma(x_{a+b}(t)) = x_{3a+b}(t^3)$ and $\Gamma(x_{3a+b}(t)) = x_{a+b}(t)$, and we have $t^3 = t$ for all $t \in \mathbb{F}_3$. Moreover, from Lemma 11.13 it follows that $\Gamma(h_a(-1)) = h_b(-1)$, which coincides with $h_a(-1)$ modulo R. $\square$

Let us now determine the conjugacy classes of the quotient $\mathrm{N}_G(R)/R$, where R is as in Proposition 12.7. Information on these will allow us to understand the action of Γ on the irreducible characters of $\mathrm{N}_G(R)/R$ in the case that $3 \mid q$.

Lemma 12.10. *Assume that $3 \mid q$ and let $q \equiv 3 \bmod 8$. Moreover, let $R \cong 2_+^{1+4}$ be as in Proposition 12.7 and assume the notation of Lemma 12.9. Then the conjugacy classes of $\mathrm{N}_G(R)/R$ are given by*

$$C_{s,t} := \{((s,t),1),\ ((-s,-t),1)\},\ s,t \in \mathbb{F}_3,\ \textit{and}$$
$$C_\star := \{((u,v),z) \mid u,v \in \mathbb{F}_3,\ z \neq 1\},$$

where $C_{s,t} = C_{s',t'}$ if and only if $(s,t) = (s',t')$ or $(s,t) = (-s',-t')$.

In particular, the quotient $\mathrm{N}_G(R)/R$ has 6 pairwise distinct conjugacy classes, given by $C_{0,0}$, $C_{0,1}$, $C_{1,0}$, $C_{1,1}$, $C_{1,-1}$ and $C_\star$.

Proof. This follows by easy computations. $\square$

Lemma 12.11. *Assume that $3 \mid q$ and let $q \equiv 3 \bmod 8$. Moreover, suppose that $R \cong 2_+^{1+4}$ is as in Proposition 12.7. Then the character table of the quotient group $\mathrm{N}_G(R)/R$ is given by:*

	$C_{0,0}$	$C_\star$	$C_{0,1}$	$C_{1,0}$	$C_{1,1}$	$C_{1,-1}$
$\chi_{0,0}$	1	1	1	1	1	1
$\chi_\star$	1	-1	1	1	1	1
$\chi_{0,1}$	2	·	2	-1	-1	-1
$\chi_{1,0}$	2	·	-1	2	-1	-1
$\chi_{1,1}$	2	·	-1	-1	2	-1
$\chi_{1,-1}$	2	·	-1	-1	-1	2

Proof. According to Lemma 12.6 it holds that $\mathrm{N}_G(R)/R \cong (C_3 \times C_3) \rtimes C_2$, where the non-trivial element of C_2 acts on $C_3 \times C_3$ by inversion. This group has identifier $[18, 4]$ in the SmallGroups Library [Bre12] provided by GAP, whence the GAP-command

```
gap> Display(CharacterTable(SmallGroup([18,4])));
```

provides the character table of $\mathrm{N}_G(R)/R$ as given above. $\square$

Proposition 12.12. *Assume that $3 \mid q$ and let $q \equiv 3 \bmod 8$. Moreover, let $R \cong 2_+^{1+4}$ be as in Proposition 12.7 such we have that $\mathrm{Irr}(\mathrm{N}_G(R)/R) = \{\chi_{0,0}, \chi_*, \chi_{0,1}, \chi_{1,0}, \chi_{1,1}, \chi_{1,-1}\}$ as in Lemma 12.11. Then the automorphism Γ interchanges $\chi_{0,1}$ and $\chi_{1,0}$ but leaves the remaining irreducible characters of $\mathrm{N}_G(R)/R$ invariant.*

Proof. By Lemma 12.9 and Lemma 12.10 we have $\Gamma(C_{0,1}) = C_{1,0}$, while the remaining conjugacy classes are left invariant by Γ. Hence, the claim follows immediately upon comparison with the character table of $\mathrm{N}_G(R)/R$ given in Lemma 12.11. $\square$

For now this completes our study of the groups 2_+^{1+4}. The second type of 2-groups we examine now are those isomorphic to $2_+^{1+2} \circ D_{(q^2-1)_2}$, the central product of an extraspecial group 2_+^{1+2} of order 8 and plus type with a dihedral group $D_{(q^2-1)_2}$ of order $(q^2-1)_2$.

Lemma 12.13. *Suppose that $q \equiv 1 \bmod 8$. Then any two subgroups of $\mathrm{SO}_4^+(q)$ that are isomorphic to $2_+^{1+2} \circ D_{(q^2-1)_2}$ are conjugate in $\mathrm{CO}_4^+(q)$.*

Moreover, any subgroup $R \leqslant \mathrm{SO}_4^+(q)$ isomorphic to $2_+^{1+2} \circ D_{(q^2-1)_2}$ is 2-radical in $\mathrm{SO}_4^+(q)$ with $\mathrm{N}_{\mathrm{SO}_4^+(q)}(R)/R \cong \mathfrak{S}_3$.

Proof. For the first statement we note that both groups 2_+^{1+2} and $D_{(q^2-1)_2}$ may be embedded into $\mathrm{GO}_2^+(q)$ (e.g. by [An93, (1G)(b)] and the fact that $D_{(q^2-1)_2}$ is isomorphic to a Sylow 2-subgroup of $\mathrm{GO}_2^+(q)$ by [KL90, Prop. 2.9.1(iii)]). Now [An93, p. 176] states that the image of $2_+^{1+2} \circ D_{(q^2-1)_2}$ under the embedding

$$2_+^{1+2} \circ D_{(q^2-1)_2} \hookrightarrow \mathrm{GO}_2^+(q) \circ \mathrm{GO}_2^+(q) \hookrightarrow \mathrm{GO}_4^+(q)$$

is "determined uniquely [...] up to conjugacy". We strongly suspect that this is saying that there is only one conjugacy class of such subgroups in $\mathrm{GO}_4^+(q)$. This would imply the first claim but unfortunately it is not entirely clear what exactly is meant here. However, the representation theory of dihedral groups is well-known and one easily verifies that any faithful absolutely irreducible $\mathbb{F}_q$-representation of $2_+^{1+2} \circ D_{(q^2-1)_2}$ is 4-dimensional, and that, moreover, up to $\mathrm{GL}_4(q)$-conjugation the images in $\mathrm{GL}_4(q)$ of all faithful irreducible 4-dimensional $\mathbb{F}_q$-representations of $2_+^{1+2} \circ D_{(q^2-1)_2}$ agree. Thus, any two subgroups of $\mathrm{SO}_4^+(q)$ that are isomorphic to $2_+^{1+2} \circ D_{(q^2-1)_2}$ must be conjugate in $\mathrm{GL}_4(q)$, whence from [KL90, Cor. 2.10.4(iii)] it follows that they must even be conjugate in $\mathrm{CO}_4^+(q)$ as claimed.

Now, if $R_1, R_2 \leqslant \mathrm{SO}_4^+(q)$ with $R_1, R_2 \cong 2_+^{1+2} \circ D_{(q^2-1)_2}$, then

$$\mathrm{N}_{\mathrm{SO}_4^+(q)}(R_1)/R_1 \cong \mathrm{N}_{\mathrm{SO}_4^+(q)}(R_2)/R_2$$

by the first part, so it suffices to know that there exists at least one radical 2-subgroup of $\mathrm{SO}_4^+(q)$ isomorphic to $2_+^{1+2} \circ D_{(q^2-1)_2}$. This, as well as the statement on the normalizer of R in $\mathrm{SO}_4^+(q)$, holds by [An94a, (2B)]. □

Lemma 12.14. *Suppose that $q \equiv 1 \bmod 8$. The groups*

$$R_{a+b} := \langle n_{a+b}(1),\ n_{3a+b}(1),\ h_b(-1),\ h_{a+b}(t)\ \mid t \in \mathbb{F}_q^\times,\ t^{(q-1)_2} = 1\rangle \leqslant G,$$
$$R_{3a+b} := \langle n_{a+b}(1),\ n_{3a+b}(1),\ h_a(-1),\ h_{3a+b}(t) \mid t \in \mathbb{F}_q^\times,\ t^{(q-1)_2} = 1\rangle \leqslant G$$

are isomorphic to $2_+^{1+2} \circ D_{(q^2-1)_2}$ and 2-radical in G. Moreover, R_{a+b} and R_{3a+b} are not conjugate in G.

Proof. Let us first show that R_{a+b} and R_{3a+b} are isomorphic to $2_+^{1+2} \circ D_{(q^2-1)_2}$. We have $y = h_a(-1)h_b(-1) = h_{a+b}(-1) \in R_{a+b}$, so in particular also $h_a(-1) \in R_{a+b}$. Using the relations given in Theorem 4.25 one can show that $h_a(-1)n_{a+b}(1)$ has order 2 and acts on $\langle h_{a+b}(t) \mid t \in \mathbb{F}_q^\times,\ t^{(q-1)_2} = 1\rangle$ by inversion. Moreover, $h_a(-1)n_{3a+b}(1)$ has order 2 and acts on $\langle h_a(-1)h_{a+b}(i)\rangle$ by inversion, where $i \in \mathbb{F}_q^\times$ denotes an element of order 4 (which exists by assumption). Now the subgroups

$$\langle h_{a+b}(t) \mid t \in \mathbb{F}_q^\times,\ t^{(q-1)_2} = 1\rangle \rtimes \langle h_a(-1)n_{a+b}(1)\rangle \cong D_{(q^2-1)_2}$$

and

$$\langle h_a(-1)h_{a+b}(i)\rangle \rtimes \langle h_a(-1)n_{3a+b}(1)\rangle \cong D_8 \cong 2_+^{1+2}$$

of R_{a+b} commute with each other (following Lemma 11.4), both have center $\langle h_{a+b}(-1)\rangle$, and together they generate R_{a+b}. We conclude that $R_{a+b} \cong 2_+^{1+2} \circ D_{(q^2-1)_2}$. Analogously, one can also show that $R_{3a+b} \cong 2_+^{1+2} \circ D_{(q^2-1)_2}$.

Since $y = h_a(-1)h_b(-1) = h_{a+b}(-1) = h_{3a+b}(-1)$ as observed before, it is contained in both groups, and in fact both groups have center generated by y. Hence,

$$R_{a+b},\, R_{3a+b} \subseteq \mathrm{C}_G(y) \cong \mathrm{SO}_4^+(q),$$

and from Lemma 12.13 we deduce that R_{a+b} and R_{3a+b} are 2-radical in $\mathrm{C}_G(y)$. Moreover, since $\mathrm{Z}(R_{a+b}) = \mathrm{Z}(R_{3a+b}) = \langle y\rangle$, also

$$\mathrm{N}_G(R_{a+b}),\, \mathrm{N}_G(R_{3a+b}) \subseteq \mathrm{C}_G(y) \cong \mathrm{SO}_4^+(q).$$

Hence, R_{a+b} and R_{3a+b} are 2-radical in G with $\mathrm{N}_G(R_{a+b})/R_{a+b}$ and $\mathrm{N}_G(R_{3a+b})/R_{3a+b}$ isomorphic to $\mathfrak{S}_3$ following Lemma 12.13.

Now suppose there is $g \in G$ such that $R_{3a+b} = (R_{a+b})^g$. Since commutator subgroups are characteristic, we then have $[R_{3a+b}, R_{3a+b}] = [R_{a+b}, R_{a+b}]^g$ as well. Easy calculations under consideration of Theorem 4.25 yield

$$[R_{a+b}, R_{a+b}] = \langle\, h_{a+b}(t^2) \,\mid t \in \mathbb{F}_q^\times,\ t^{(q-1)_2} = 1\rangle,$$
$$[R_{3a+b}, R_{3a+b}] = \langle h_{3a+b}(t^2) \mid t \in \mathbb{F}_q^\times,\ t^{(q-1)_2} = 1\rangle.$$

Now let $t \in \mathbb{F}_q^\times$ be of order $(q-1)_2$. Then there must exist another element $z \in \mathbb{F}_q^\times$ of order $(q-1)_2$ such that $h_{3a+b}(t^2) = h_{a+b}(z^2)^g$. It is well-known that all elements of $T_+ = \mathbf{T}^F$ which are conjugate in G must be conjugate by an element of the Weyl group $\mathbf{W}$ (see, for instance, [Car85, Prop. 3.7.1]), which equals

$$\mathbf{W} = \langle \mathbf{T},\ n_r(1) \mid r \in \Sigma\rangle / \,\mathbf{T}$$

according to Proposition 4.27. Moreover, by Theorem 4.25(x) we have

$$n_r(1)h_s(z^2)n_r(1)^{-1} = h_{\omega_r(s)}(z^2)$$

for all $r, s \in \Sigma$, and $\omega_r(s)$ is a short root if and only if s is short, so there must exist a short root s such that $h_{3a+b}(t^2) = h_s(z^2)$. From [GLS98, Th. 1.9.5(d)] it follows that

$$t^{2\langle 3a+b,\, r\rangle} = z^{2\langle s,\, r\rangle}$$

for all $r \in \Sigma$ (note that one has to be careful here since the definition of $\langle \_\,,\_\rangle$ in [GLS98] differs from the one used here in the way that the roles of the two arguments are interchanged). For $r = a + b$ we have

$$\langle 3a + b,\, r\rangle = 2\frac{(3a + b,\, a + b)}{(3a + b,\, 3a + b)} = 0$$

since $a+b$ and $3a+b$ are orthogonal roots. Hence, $z^{2\langle s,\, a+b\rangle} = 1$. Now, being a short root, s is contained in $\{\pm a,\ \pm(a + b),\ \pm(2a + b)\}$ and we deduce that $\langle s,\, a + b\rangle \in \{\pm 1,\ \pm 2\}$ following Lemma 11.9. We conclude that $z^4 = 1$, which contradicts the assumption that z has order $(q-1)_2$ and $q \equiv 1 \bmod 8$. □

Proposition 12.15. *Let $q \equiv 1 \bmod 8$. The groups R_{a+b} and R_{3a+b} from Lemma 12.14 are stabilized by the field automorphism F_p. If $3 \mid q$, the graph automorphism Γ interchanges R_{a+b} and R_{3a+b}.*

Proof. The first statement is obvious. The second claim follows from Lemma 11.13 and the fact that $\rho\colon \Sigma \longrightarrow \Sigma$ interchanges $a+b$ and $3a+b$, as well as a and b. □

We are now ready to describe the action of $\mathrm{Aut}(G)$ on the 2-weights of G. This will be accomplished by a case-by-case analysis of the different types of 2-blocks of G of non-cyclic defect. Following [An94a, p. 36] or Section B.3.1 these are the 2-blocks B_0, B_3 (if $3 \nmid q$), and the 2-blocks of types B_{1a}, B_{1b}, B_{2a}, B_{2b}, B_{X_1} and B_{X_2}.

12.2.1.1 The Principal Block B_0

The 2-weights of G belonging to the principal 2-block B_0 have been described by J. An in [An94a] as follows:

Proposition 12.16. *Suppose that $B = B_0$ is the principal 2-block of G. Then $|\mathcal{W}(B)| = 7$. Moreover, if (R, φ) is a B-weight of G, then up to G-conjugation one of the following holds:*

- *(i)* $R \cong (C_2)^3$, *an elementary abelian group of order 8,* $\mathrm{N}_G(R)/R \cong \mathrm{GL}_3(2)$, *and φ is the inflation of the Steinberg character of* $\mathrm{N}_G(R)/R \cong \mathrm{GL}_3(2)$. *There exists exactly one G-conjugacy class of such B-weights in G.*
- *(ii)* $R \sim_{\mathbf{G}} \langle \mathcal{O}_2(T_\varepsilon), v_2\rangle$, $\mathrm{N}_G(R)/R \cong \mathfrak{S}_3$, *and φ is the inflation of the unique irreducible character of* $\mathrm{N}_G(R)/R \cong \mathfrak{S}_3$ *of degree* 2. *There exists exactly one G-conjugacy class of such B-weights in G.*
- *(iii)* $R \in \mathrm{Syl}_2(G)$ *is a Sylow 2-subgroup of G,* $\mathrm{N}_G(R) = R$, *and φ is the trivial character of* $\mathrm{N}_G(R)$. *There exists exactly one G-conjugacy class of such B-weights in G.*
- *(iv)* $q \equiv \pm 1 \bmod 8$, $R \cong 2_+^{1+2} \circ D_{(q^2-1)_2}$, *the central product of an extraspecial 2-group of order* 8 *and plus type with a dihedral group of order* $(q^2-1)_2$, $\mathrm{N}_G(R)/R \cong \mathfrak{S}_3$, *and φ is the inflation of the unique irreducible character of* $\mathrm{N}_G(R)/R \cong \mathfrak{S}_3$ *of degree* 2. *There exist exactly two G-conjugacy classes of such B-weights in G.*
- *(v)* $q \equiv \pm 1 \bmod 8$, $R \cong 2_+^{1+4}$, *an extraspecial 2-group of order* 2^{1+4} *and plus type,* $\mathrm{N}_G(R)/R \cong \mathfrak{S}_3 \times \mathfrak{S}_3$, *and φ is the inflation of the unique irreducible character of* $\mathrm{N}_G(R)/R \cong \mathfrak{S}_3 \times \mathfrak{S}_3$ *of degree 4. There exist exactly two G-conjugacy classes of such B-weights in G.*
- *(vi)* $q \equiv \pm 3 \bmod 8$, $R \cong 2_+^{1+4}$, $\mathrm{N}_G(R)/R \cong (C_3 \times C_3) \rtimes C_2$ *with the action of the non-trivial element of C_2 on $C_3 \times C_3$ given by inversion, and φ is the inflation of one of the four irreducible characters of* $\mathrm{N}_G(R)/R$ *of degree 2. There exists exactly one G-conjugacy class of such R in G.*

Proof. This follows from [An94a, (3C)] and the proof of [An94a, (3H)]. □

The action of $\mathrm{Aut}(G)$ on the B_0-weights of G is given as follows:

Proposition 12.17. *Let $B = B_0$ be the principal* 2*-block of G and suppose that (R, φ) is a B-weight of G. The following statements hold:*

- *(i) If R is as in (i), (ii) or (iii) of Proposition 12.16, then up to G-conjugation (R, φ) is invariant under* $\mathrm{Aut}(G)$.

(ii) *If $q \equiv \pm 1 \bmod 8$ and $R \cong 2_+^{1+2} \circ D_{(q^2-1)_2}$, then F_p stabilizes the G-conjugacy class of (R, φ), while Γ (if existent, i.e., if $3 \mid q$) interchanges the two G-conjugacy classes of type (R, φ).*

(iii) *If $q \equiv \pm 1 \bmod 8$ and $R \cong 2_+^{1+4}$, then up to G-conjugation (R, φ) is invariant under $\mathrm{Aut}(G)$.*

(iv) *If $q \equiv \pm 3 \bmod 8$ and $R \cong 2_+^{1+4}$, then F_p stabilizes the G-conjugacy class of (R, φ). By Proposition 12.16(vi) the weight character φ is the inflation of one of the four irreducible characters of $\mathrm{N}_G(R)/R$ of degree 2. If $3 \mid q$, then two of these characters are stabilized by Γ, the remaining two are interchanged.*

Proof. (i) This is clear since in cases (i), (ii) or (iii) of Proposition 12.16 the character φ is uniquely determined by R and there exists a unique G-conjugacy class of B-weights with first component isomorphic to R.

For (ii) we observe that in consequence of Lemma 12.14 we may assume that up to G-conjugation R is one of R_{a+b} or R_{3a+b}. Since R_{a+b} and R_{3a+b} are not G-conjugate by Lemma 12.14, it follows from Proposition 12.15 that the claim holds if $q \equiv 1 \bmod 8$. Now suppose that $q \equiv -1 \bmod 8$. Then we have $q = p^f$ for odd f and $p \neq 3$. Hence, the automorphism Γ does not exist, and F_p has odd order f on G. Consequently, F_p must stabilize both G-conjugacy classes of type (R, φ).

(iii) According to Proposition 12.16(v) there exist exactly two G-conjugacy classes of B-weights of type (R, φ) with φ uniquely determined by R, and at least one of these has a representative with first component lying in $G_2(p)$, so both classes are stabilized by F_p. If $p = 3$, then the automorphism Γ also acts on $G_2(p)$, which contains exactly one conjugacy class of such R by Proposition 12.16(vi). Hence, Γ stabilizes both G-conjugacy classes of R that exist in $G = G_2(q)$, so in particular, the G-conjugacy class of (R, φ) is stabilized.

For (iv) we observe that by Proposition 12.16(vi) there exists a unique G-conjugacy class of B-weights of type (R, φ) in G, and this has a representative with first component contained in $G_2(p)$, so assume that $R \leqslant G_2(p)$. Moreover, $q \equiv \pm 3 \bmod 8$ implies that also $p \equiv \pm 3 \bmod 8$, so by the same proposition

$$\mathrm{N}_G(R)/R \cong \mathrm{N}_{G_2(p)}(R)/R,$$

and hence $\mathrm{N}_G(R) = \mathrm{N}_{G_2(p)}(R)$. We deduce that F_p stabilizes all characters of $\mathrm{N}_G(R)/R$, so in particular (R, φ) is stabilized. For $p = 3$ we may assume that R is as in Proposition 12.7, whence the action of Γ on the irreducible characters of $\mathrm{N}_G(R)/R$ of degree two is described in Proposition 12.12. □

12.2.1.2 The Block B_3

Note that this 2-block does only exist if $3 \nmid q$ (cf. Section B.3.1), and that the outer automorphism group of $G_2(q)$ is generated by the field automorphism F_p in this case.

Proposition 12.18. *Let $B = B_3$. Then $|\mathcal{W}(B)| = 3$. Moreover, if (R_1, φ_1) and (R_2, φ_2) are B-weights of G that are not G-conjugate, then R_1 and R_2 are not isomorphic.*

Proof. This follows immediately from the proof of [An94a, (3I)]. □

In consequence, we may easily derive the following observation about the action of the automorphism group $\mathrm{Aut}(G)$ on the B_3-weights of G:

Corollary 12.19. *Let $B = B_3$. Then $\mathrm{Aut}(G)_B = \mathrm{Aut}(G)$ acts trivially on $\mathcal{W}(B)$.*

Proof. This follows from Proposition 12.18 since $\mathrm{Aut}(G)_B = \mathrm{Aut}(G)$ and any automorphism of G clearly preserves the isomorphism type of R for a B-weight (R, φ) of G. □

12.2.1.3 The Blocks of Types B_{1a}, B_{1b}, B_{2a} and B_{2b}

Let B be a 2-block of G. If B is of type B_c for some $c \in \{1a, 1b, 2a, 2b\}$, then we write $B \in \{B_{1a}, B_{1b}, B_{2a}, B_{2b}\}$. See Section B.3.1 for further information on such 2-blocks. Let us recall for the following statement that $\varepsilon \in \{\pm 1\}$ was chosen to satisfy $q \equiv \varepsilon \bmod 4$.

Proposition 12.20. *Let $B \in \{B_{1a}, B_{1b}, B_{2a}, B_{2b}\}$ be a 2-block of G. Then $|\mathcal{W}(B)| = 2$. Moreover, for a B-weight (R, φ) the following statements hold:*

(i) If $B \in \{B_{1a}, B_{1b}\}$ and $\varepsilon = 1$ or if $B \in \{B_{2a}, B_{2b}\}$ and $\varepsilon = -1$, then we either have $R \cong 2_+^{1+2} \circ C_{(q-\varepsilon)_2}$, the central product of 2_+^{1+2} with $C_{(q-\varepsilon)_2}$, or $R \cong C_{(q-\varepsilon)_2} \wr C_2$, the wreath product of $C_{(q-\varepsilon)_2}$ by C_2, and φ is uniquely determined by R and B. Moreover, for each such R there exists exactly one G-conjugacy class of B-weights of type (R, φ) in G.

(ii) If $B \in \{B_{1a}, B_{1b}\}$ and $\varepsilon = -1$ or if $B \in \{B_{2a}, B_{2b}\}$ and $\varepsilon = 1$, then we either have $R \cong 2_-^{1+2}$ or $R \cong S_{2(q^2-1)_2}$, a semidihedral group of order $2(q^2-1)_2$, and φ is uniquely determined by R and B. Moreover, for each such R there exists exactly one G-conjugacy class of B-weights of type (R, φ) in G.

Proof. This follows from the proof of [An94a, (3I)]. □

The above description implies the following action of automorphisms of G on the 2-weights associated to 2-blocks of G of types B_{1a}, B_{1b}, B_{2a} or B_{2b}:

Corollary 12.21. *Suppose that $B \in \{B_{1a}, B_{1b}, B_{2a}, B_{2b}\}$ is a 2-block of G. Then* $\mathrm{Aut}(G)_B$ *acts trivially on $\mathcal{W}(B)$.*

Proof. According to Proposition 12.20 the G-conjugacy class of any B-weight (R, φ) is uniquely determined by B and the isomorphism type of R. Hence, since for any automorphism $a \in \mathrm{Aut}(G)_B$ the 2-weight $(R, \varphi)^a = (R^a, \varphi^a)$ belongs to B and $R^a \cong R$, it follows that $\mathrm{Aut}(G)_B$ stabilizes the G-conjugacy class of (R, φ). □

Remark 12.22. In the situation of Proposition 12.20(i) a defect group of the 2-block B is given by $C_{(q-\varepsilon)_2} \wr C_2$, while in (ii) B has $S_{2(q^2-1)_2}$ as a defect group (cf. [An94a, (3I)]).

12.2.1.4 The Blocks of Types B_{X_1} and B_{X_2}

In this section we examine the 2-blocks B of G that are of type B_{X_1} or B_{X_2} (compare Section B.3.1 in Appendix B). We use the notation $B \in \{B_{X_1}, B_{X_2}\}$ in this situation.

Proposition 12.23. *Suppose that $B \in \{B_{X_1}, B_{X_2}\}$ is a 2-block of G. Then $|\mathcal{W}(B)| = 1$.*

Proof. This is [An94a, (3I)(a)]. □

Corollary 12.24. *Suppose that $B \in \{B_{X_1}, B_{X_2}\}$ is a 2-block of G. Then* $\mathrm{Aut}(G)_B$ *acts trivially on $\mathcal{W}(B)$.*

Proof. This result follows directly from Proposition 12.23 since G possesses exactly one G-conjugacy class of B-weights. □

12.2.2 The Case $\ell = 3$

Throughout this section we let $\ell = 3$ and suppose that $3 \nmid q$. Recall that we also allow $q < 5$ here. Let us define $\varepsilon \in \{\pm 1\}$ to be such that $q \equiv \varepsilon \bmod 3$. Now we recall from Section 11.2 the notation $F_+ = F$ and $F_- = v_2 F$, the definition $G_\varepsilon = \mathbf{G}^{F_\varepsilon}$ and the fact that G_+ and G_- are conjugate in $\mathbf{G}$ by Corollary 4.35. For convenience we will mainly work with the group G_ε throughout this section since it provides a particularly nice description for the maximal torus $T_\varepsilon = \mathbf{T}^{F_\varepsilon}$ (see Table 11.1). According to [An94a, (1E)] we have

$$\mathrm{N}_{G_\varepsilon}(T_\varepsilon)/T_\varepsilon \cong D_{12}.$$

Thus, since we also have $\mathrm{N}_{G_\varepsilon}(\mathbf{T})/\, \mathbf{T}^{F_\varepsilon} \cong \mathbf{W}^{F_\varepsilon} \cong D_{12}$ by Proposition 4.40 and Table 11.1 with $\mathrm{N}_{G_\varepsilon}(\mathbf{T}) \subseteq \mathrm{N}_{G_\varepsilon}(T_\varepsilon)$, it follows that

$$\mathrm{N}_{G_\varepsilon}(T_\varepsilon) = \mathrm{N}_{G_\varepsilon}(\mathbf{T}) = \langle T_\varepsilon, n_r(1) \mid r \in \Sigma \rangle$$

in this case (cf. also Proposition 4.27 and Lemma 11.7).

Let us now consider the group

$$\mathbf{L} := \langle x_b(t),\ x_{-b}(t),\ x_{3a+b}(t),\ x_{-(3a+b)}(t) \mid t \in \mathbb{F} \rangle \leqslant \mathbf{G} = G_2(\mathbb{F}).$$

This group satisfies $\mathbf{L} \cong \mathrm{SL}_3(\mathbb{F})$ via

$$x_b(t) \longmapsto \begin{bmatrix} 1 & t & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad x_{3a+b}(t) \longmapsto \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix},$$
$$x_{-b}(t) \longmapsto \begin{bmatrix} 1 & 0 & 0 \\ t & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad x_{-(3a+b)}(t) \longmapsto \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & t & 1 \end{bmatrix}.$$

Moreover, the group $\mathbf{L}$ is stable under both F_+ and F_- (cf. Lemma 11.7), and we set

$$L_+ := \mathbf{L}^{F_+} \cong \mathrm{SL}_3(q),$$
$$L_- := \mathbf{L}^{F_-} \cong \mathrm{SU}_3(q) =: \mathrm{SL}_3(-q),$$

where $\mathrm{SU}_3(q)$ denotes the special unitary group of degree 3 over $\mathbb{F}_q$. In addition, we set $K_\varepsilon := \langle L_\varepsilon, v_2 \rangle \leqslant G_\varepsilon$. This yields the semidirect product $K_\varepsilon = L_\varepsilon \rtimes \langle v_2 \rangle$ with v_2 acting on L_ε as the transpose-inverse automorphism does on $\mathrm{SL}_3(\varepsilon q)$ (compare Lemma 11.7). Clearly, the field automorphism F_p of G_ε acts on both L_ε and K_ε, and $F_p(v_2) = v_2$.

Remark 12.25. Note that in [An94a] the group K_ε is defined in a different way, with L_ε defined as a subgroup of K_ε of index 2, such that $L_\varepsilon \cong \mathrm{SL}_3(\varepsilon q)$, and K_ε is obtained from L_ε by extending by an involutory outer automorphism (compare [An94a, p. 24]). However, up to G_ε-conjugation this definition of K_ε agrees with the one given here:

The group $\mathrm{SL}_3(\varepsilon q)$ has a non-trivial center of order 3, i.e., it centralizes an element of order 3. In G_ε there exists only one conjugacy class of non-trivial elements that have centralizer order divisible by $|\,\mathrm{SL}_3(\varepsilon q)|$. This conjugacy class is represented by the element $k_3 = h(\omega, \omega, \omega)$ with $\omega \in \mathbb{F}^\times$ of order 3, which has centralizer order exactly $|\,\mathrm{SL}_3(\varepsilon q)|$ (compare Table B.1). Note that k_3 is contained in L_ε as in our definition, so in fact $L_\varepsilon = \mathrm{C}_{G_\varepsilon}(k_3)$, and we conclude that the groups arising from the two definitions for L_ε are conjugate in G_ε. Moreover, the element v_2 acts on L_ε as an involutory outer automorphism. If $\iota_1, \iota_2 \in G_\varepsilon \setminus L_\varepsilon$ both act on L_ε as involutory outer automorphisms, then $k_3^{\iota_1} = k_3^{-1} = k_3^{\iota_2}$, such that $\iota_1 \iota_2^{-1} \in \mathrm{C}_{G_\varepsilon}(k_3) = L_\varepsilon$, and hence $\langle L_\varepsilon, \iota_1 \rangle = \langle L_\varepsilon, \iota_2 \rangle$. We conclude that the group K_ε is unique up to G_ε-conjugation.

Considerable information on the defect groups of 3-blocks of G_ε is given by the subsequent result, part of which is due to G. Hiß and J. Shamash, who determined the 3-blocks and corresponding Brauer characters of $G_2(q)$ for q not divisible by 3 in [HS90]:

Proposition 12.26. *Let B be a 3-block of G_ε. Then*

(i) B has maximal defect if and only if B is the principal 3-block of G_ε,

(ii) B has abelian defect groups if and only if B is non-principal, and

(iii) if B has non-cyclic abelian defect groups, then $\mathcal{O}_3(T_\varepsilon)$ is a defect group of B. In this case we have

$$\begin{aligned} &\mathrm{C}_{G_\varepsilon}(\mathcal{O}_3(T_\varepsilon)) = T_\varepsilon \\ \textit{and}\quad &\mathrm{N}_{G_\varepsilon}(\mathcal{O}_3(T_\varepsilon)) = \mathrm{N}_{G_\varepsilon}(T_\varepsilon), \end{aligned}$$

with $\mathrm{N}_{G_\varepsilon}(T_\varepsilon)/T_\varepsilon \cong D_{12}$.

Proof. The statements on the defect groups of B hold by [HS90, Sec. 2.2, 2.3]. For the normalizer and centralizer in (iii) see, for instance, [An94a, (1D)]. □

Hence, if B is a non-principal 3-block of G_ε that has non-cyclic defect groups, then up to G_ε-conjugation all B-weights of G_ε derive from the radical 3-subgroup $R = \mathcal{O}_3(T_\varepsilon)$ of G_ε. According to Construction 2.10 the irreducible characters of $R\,\mathrm{C}_{G_\varepsilon}(R) = T_\varepsilon$ play a major role in this case. Let us hence fix a parametrization for $\mathrm{Irr}(T_\varepsilon)$:

Notation 12.27 (Parametrization of $\mathrm{Irr}(T_\varepsilon)$)**.** Let us first recall from Table 11.1 that we have a parametrization of the maximal torus $T_\varepsilon = \mathbf{T}^{F_\varepsilon}$ given by

$$\mathbf{T}^{F_\varepsilon} = \{h(z_1, z_2, z_3) \mid z_i \in \mathbb{F}^\times,\ z_i^{q-\varepsilon} = 1,\ z_1 z_2 z_3 = 1\}.$$

Now let $z \in \mathbb{F}^\times$ be of order $q - \varepsilon$ and denote by θ_0 the irreducible character of $\langle z \rangle$ given by $\theta_0(z) = \exp((2\pi\mathrm{i})/(q-\varepsilon))$. Then the irreducible characters of T_ε may be parametrized as

$$\mathrm{Irr}(T_\varepsilon) = \{\theta_0^i \times \theta_0^j \times 1 \mid 0 \leqslant i, j < q - \varepsilon\},$$

where $(\theta_0^i \times \theta_0^j \times 1)(h(z_1, z_2, z_3)) = \theta_0^i(z_1)\theta_0^j(z_2)$. We shall fix this parametrization until the end of this section.

Let us now examine how the normalizer of T_ε in G_ε acts on $\mathrm{Irr}(T_\varepsilon)$. At the beginning of the present section we observed that $\mathrm{N}_{G_\varepsilon}(T_\varepsilon) = \langle T_\varepsilon, n_r(1) \mid r \in \Sigma\rangle$, whence it suffices to understand the action of $n_r(1)$, $r \in \Sigma$, on $\mathrm{Irr}(T_\varepsilon)$.

Lemma 12.28. *Let $\theta \in \mathrm{Irr}(T_\varepsilon)$ and $0 \leqslant i, j < q - \varepsilon$ be such that $\theta = \theta_0^i \times \theta_0^j \times 1$. Then the following statements hold:*

$$\begin{array}{rlclclcl}
\textit{(i)} & \theta^{n_a(1)} & = & \theta_0^i & \times & \theta_0^{i-j} & \times & 1, \\
\textit{(ii)} & \theta^{n_b(1)} & = & \theta_0^j & \times & \theta_0^i & \times & 1, \\
\textit{(iii)} & \theta^{n_{a+b}(1)} & = & \theta_0^{j-i} & \times & \theta_0^j & \times & 1, \\
\textit{(iv)} & \theta^{n_{2a+b}(1)} & = & \theta_0^{-j} & \times & \theta_0^{-i} & \times & 1, \\
\textit{(v)} & \theta^{n_{3a+b}(1)} & = & \theta_0^{i-j} & \times & \theta_0^{-j} & \times & 1, \\
\textit{(vi)} & \theta^{n_{3a+2b}(1)} & = & \theta_0^{-i} & \times & \theta_0^{j-i} & \times & 1, \\
\textit{(vii)} & \theta^{v_2} & = & \theta_0^{-i} & \times & \theta_0^{-j} & \times & 1.
\end{array}$$

Proof. This is a direct consequence of Lemma 11.4 and Lemma 11.6. □

Corollary 12.29. *Let $\theta \in \operatorname{Irr}(T_\varepsilon)$ be such that $\mathrm{N}_{G_\varepsilon}(T_\varepsilon)_\theta/T_\varepsilon \cong C_2$. Then up to $\mathrm{N}_{G_\varepsilon}(T_\varepsilon)$-conjugation it holds that*

$$\theta = \theta_0^i \times \theta_0^i \times 1 \quad \textit{or} \quad \theta = \theta_0^i \times \theta_0^{-i} \times 1$$

for a suitable $0 < i < q - \varepsilon$.

Proof. As observed above, the group $\mathrm{N}_{G_\varepsilon}(T_\varepsilon)/T_\varepsilon = \langle T_\varepsilon, n_r(1) \mid r \in \Sigma\rangle/T_\varepsilon$ is dihedral of order 12, and as such it contains exactly seven involutions (check, e.g., with GAP). These must given by the elements $n_r(1)T_\varepsilon$, $r \in \Sigma$ a positive root, and v_2T_ε since these seven involutions act pairwise distinctly on T_ε according to Lemma 11.4 and Lemma 11.6, whence they must be pairwise distinct. They correspond to the six reflections along the hyperplanes orthogonal to r, $r \in \Sigma$, and the involution mapping $r \in \Sigma$ to $-r$, which may be imagined as the reflection through the intersection point in Figure 11.1. Now suppose that $\theta = \theta_0^i \times \theta_0^j \times 1$ for some $0 \leqslant i, j < q - \varepsilon$. Then one of the following occurs in consequence of Lemma 12.28:

$$\begin{aligned} i &\equiv 2j \mod q - \varepsilon, \\ i &= j, \\ j &\equiv 2i \mod q - \varepsilon, \\ i &\equiv -j \mod q - \varepsilon, \\ j &= 0, \\ i &= 0. \end{aligned}$$

If $i \equiv 2j \bmod q - \varepsilon$ or $j \equiv 2i \bmod q - \varepsilon$, then one verifies by help of Lemma 12.28 that θ is conjugate to $\theta_0^j \times \theta_0^{-j} \times 1$ or $\theta_0^i \times \theta_0^{-i} \times 1$ via $v_6 = v_2v_3$ or v_3^2, respectively. It then holds that $j > 0$ or $i > 0$, respectively, since θ is non-trivial.

Similarly, for $j = 0$ or $i = 0$ one shows that θ is conjugate to $\theta_0^i \times \theta_0^i \times 1$ or $\theta_0^j \times \theta_0^j \times 1$ via $n_a(1)$ or $n_{a+b}(1)$, respectively, with $i > 0$ or $j > 0$, accordingly. □

Let us now come to the examination of the B-weights for 3-blocks B of G_ε of non-cyclic defect. Following [An94a, p. 36] or Section B.3.2 these are the 3-blocks B_1, B_2 (if q is odd) and the 3-blocks of types $B_{\delta a}$, $B_{\delta b}$ and B_{X_δ}, where $\delta = 1$ if $\varepsilon = 1$ and $\delta = 2$ if $\varepsilon = -1$.

12.2.2.1 The Principal Block B_0

Before we turn to the 3-weights of the principal 3-block of G_ε, let us consider the following proposition, which provides information on the local properties of extraspecial 3-subgroups of G_ε of order 3^{1+2} and exponent 3. As we will observe below, along with the Sylow 3-subgroups of G_ε these groups give rise to the 3-weights for the principal 3-block B_0.

Proposition 12.30. *For $R \leqslant G_\varepsilon$ an extraspecial group of order 3^{1+2} and exponent 3 the following statements hold:*

(i) Up to G_ε-conjugation we have $R \leqslant L_\varepsilon$ and $\mathrm{N}_{G_\varepsilon}(R) \leqslant K_\varepsilon$.

(ii) If R is contained in L_ε, then we have

$$\mathrm{N}_{L_\varepsilon}(R)/R \cong \begin{cases} Q_8 & \textit{if } (q^2-1)_3 = 3, \\ \mathrm{Sp}_2(3) & \textit{if } (q^2-1)_3 > 3. \end{cases}$$

Moreover, L_ε contains exactly one L_ε-conjugacy class of subgroups isomorphic to R if $(q^2-1)_3 = 3$, and three such L_ε-conjugacy classes if $(q^2-1)_3 > 3$.

(iii) *(1)* *If* $(q^2-1)_3 = 3$*, then* $R \in \mathrm{Syl}_3(G)$*, so* G_ε *contains exactly one* G_ε*-conjugacy class of subgroups isomorphic to* R*, and we have* $\mathrm{N}_{G_\varepsilon}(R) = \langle \mathrm{N}_{L_\varepsilon}(R), \rho\rangle$ *for some* $\rho \in K_\varepsilon \setminus L_\varepsilon$*.*

(2) *If* $(q^2-1)_3 > 3$*, then* G_ε *contains two* G_ε*-conjugacy classes of subgroups isomorphic to* R*. One of these has* $\mathrm{N}_{G_\varepsilon}(R) = \mathrm{N}_{L_\varepsilon}(R)$*, the other one satisfies* $\mathrm{N}_{G_\varepsilon}(R) = \langle \mathrm{N}_{L_\varepsilon}(R), \rho\rangle$ *for some* $\rho \in K_\varepsilon \setminus L_\varepsilon$*.*

Proof. This is proven in [An94a, (1E)] and [An94a, (1G)]. □

Proposition 12.31. *For* $B = B_0$ *the principal* 3*-block of* G_ε *we have* $|\mathcal{W}(B)| = 7$*. Moreover, if* (R,φ) *is a* B*-weight of* G_ε*, then up to* G_ε*-conjugation one of the following holds:*

(i) $(q^2-1)_3 = 3$*,* $R \in \mathrm{Syl}_3(G_\varepsilon)$ *is an extraspecial group of order* 3^{1+2} *and exponent* 3*, and* φ *is the inflation of one of the seven irreducible characters of* $\mathrm{N}_{G_\varepsilon}(R)/R$*.*

(ii) $(q^2-1)_3 > 3$*,* $R \in \mathrm{Syl}_3(G_\varepsilon)$ *is a Sylow* 3*-subgroup of* G_ε*,* $\mathrm{N}_{G_\varepsilon}(R)/R \cong C_2 \times C_2$*, and* φ *is the inflation of one of the four linear characters of* $\mathrm{N}_{G_\varepsilon}(R)/R$*.*

(iii) $(q^2-1)_3 > 3$*,* $R \leqslant L_\varepsilon$ *is an extraspecial group of order* 3^{1+2} *and exponent* 3 *such that* $\mathrm{N}_{G_\varepsilon}(R) = \mathrm{N}_{K_\varepsilon}(R)$*,* $|\,\mathrm{N}_{K_\varepsilon}(R)\colon \mathrm{N}_{L_\varepsilon}(R)| = 2$*, and* φ *is the inflation of one of the two extensions of the Steinberg character of* $\mathrm{N}_{L_\varepsilon}(R)/R \cong \mathrm{Sp}_2(3)$ *to* $\mathrm{N}_{G_\varepsilon}(R)/R$*. There exists exactly one* G_ε*-conjugacy class of such* R *in* G_ε*.*

(iv) $(q^2-1)_3 > 3$*,* $R \leqslant L_\varepsilon$ *is an extraspecial group of order* 3^{1+2} *and exponent* 3 *where it holds that* $\mathrm{N}_{G_\varepsilon}(R) = \mathrm{N}_{L_\varepsilon}(R)$*, and* φ *is the inflation of the Steinberg character of* $\mathrm{N}_{G_\varepsilon}(R)/R \cong \mathrm{Sp}_2(3)$*. There exists exactly one* G_ε*-conjugacy class of such* R *in* G_ε*.*

Proof. This follows from the proof of [An94a, (3A)] and Proposition 12.30. □

Proposition 12.32. *Suppose that* $B = B_0$ *is the principal* 3*-block of* G_ε*. Then the action of* $\mathrm{Aut}(G_\varepsilon)_B = \mathrm{Aut}(G_\varepsilon)$ *on* $\mathcal{W}(B)$ *is trivial.*

Proof. Since $3 \nmid q$, it is known from Proposition 11.14 that $\mathrm{Out}(G_\varepsilon)$ is generated by the field automorphism F_p, so it suffices to prove that this automorphism stabilizes any conjugacy class of B-weights in G_ε. We go through the cases listed in Proposition 12.31.

Let (R,φ) be as Proposition 12.31(i). Then $(q^2-1)_3 = 3$, R is a Sylow 3-subgroup of $G_2(q)$, and by Proposition 12.30 we have $|\,\mathrm{N}_{G_2(q)}(R)/R\,| = 2|Q_8| = 16$. Now we consider the group $G_2(p)$. Since $p \neq 3$, this has

$$|G_2(p)|_3 = 3(p^2-1)_3^2 \geqslant 3^3 = |R| = |G_2(q)|_3$$

in consequence of Lemma 11.1. But $G_2(p)$ is a subgroup of $G_2(q)$, so we conclude that $|G_2(p)|_3 = |R|$, and by changing to a $G_2(q)$-conjugate we may thus assume that $R \leqslant G_2(p)$. Again by Proposition 12.30 it follows that also $|\,\mathrm{N}_{G_2(p)}(R)/R\,| = 2|Q_8| = 16$, whence we must have

$$\mathrm{N}_{G_2(q)}(R) = \mathrm{N}_{G_2(p)}(R).$$

In particular, the field automorphism F_p acts trivially on R and $\mathrm{N}_{G_2(q)}(R)$, and thus leaves (R,φ) invariant.

Now suppose that we are in the situation of Proposition 12.31(ii), such that we have $(q^2-1)_3 > 3$ and $R \in \mathrm{Syl}_3(G_\varepsilon)$. Following the proof of [An94a, (1E)] we may assume that

$$R = \langle \mathcal{O}_3(T_\varepsilon), v_3\rangle$$

such that $\mathrm{N}_{G_\varepsilon}(R)T_\varepsilon = \mathrm{N}_{G_\varepsilon}(T_\varepsilon)$, $\mathrm{N}_{G_\varepsilon}(R) \cap T_\varepsilon = \mathcal{O}_3(T_\varepsilon)$ and $\mathrm{N}_{G_\varepsilon}(R)/R \cong C_2 \times C_2$. Moreover, one easily verifies that $\langle T_\varepsilon, v_2, v_3, n_b(1)\rangle/T_\varepsilon \cong D_{12}$, for instance by application of Lemma 11.4, so it follows that

$$\mathrm{N}_{G_\varepsilon}(T_\varepsilon) = \langle T_\varepsilon, v_2, v_3, n_b(1)\rangle.$$

Hence, we have

$$\mathrm{N}_{G_\varepsilon}(R) = \langle \mathcal{O}_3(T_\varepsilon), v_3, sv_2, tn_b(1)\rangle$$

for suitable torus elements $s, t \in T_\varepsilon$ and $\mathrm{N}_{G_\varepsilon}(R)/R = \langle \overline{sv_2}\rangle \times \langle \overline{tn_b(1)}\rangle$, where we denote by $\overline{}\colon \mathrm{N}_{G_\varepsilon}(R) \twoheadrightarrow \mathrm{N}_{G_\varepsilon}(R)/R$ the natural epimorphism. We prove that F_p acts trivially on $\mathrm{N}_{G_\varepsilon}(R)/R$.

Suppose that $F_p(\overline{sv_2}) = \overline{tn_b(1)}$. Since v_2 and $n_b(1)$ do not coincide modulo T_ε and v_3 has order 3 by Lemma 11.6, we must have $F_p(sv_2) = s^p v_2 = tn_b(1)xv_3^i$ for some $x \in \mathcal{O}_3(T_\varepsilon)$ and $i \in \{1,2\}$. Hence, modulo T_ε the elements v_2 and $n_b(1)v_3^i$ coincide. Now following Lemma 11.4 and Lemma 11.6 we have

$$h(z_1, z_2, z_3)^{v_2} = h(z_1^{-1}, z_2^{-1}, z_3^{-1})$$

but

$$\begin{aligned} h(z_1, z_2, z_3)^{n_b(1)v_3} &= h(z_3, z_2, z_1),\\ h(z_1, z_2, z_3)^{n_b(1)v_3^2} &= h(z_1, z_3, z_2) \end{aligned}$$

for any $z_1, z_2, z_3 \in \mathbb{F}^\times$ with $z_1z_2z_3 = 1$. In particular, since $3 \mid (q-\varepsilon)$, we may always find $z_1, z_2, z_3 \in \mathbb{F}^\times$ such that $h(z_1, z_2, z_3) \in T_\varepsilon$ and

$$h(z_1, z_2, z_3)^{v_2} \notin \{h(z_1, z_2, z_3)^{n_b(1)v_3},\ h(z_1, z_2, z_3)^{n_b(1)v_3^2}\},$$

so $F_p(\overline{sv_2}) = \overline{tn_b(1)}$ is impossible.

Similarly, if we assume that $F_p(\overline{sv_2}) = \overline{sv_2} \cdot \overline{tn_b(1)}$, then $F_p(sv_2) = s^p v_2 = sv_2tn_b(1)xv_3^i$ for some $x \in \mathcal{O}_3(T_\varepsilon)$ and $i \in \{1,2\}$. We deduce that modulo T_ε the elements $n_b(1)v_3^i$ and 1 coincide, which contradicts the above observation on the action of $n_b(1)v_3^i$ on T_ε. Hence, $\overline{sv_2}$ must be left invariant by F_p.

Finally, suppose $F_p(\overline{tn_b(1)}) = \overline{sv_2} \cdot \overline{tn_b(1)}$. Then $F_p(tn_b(1)) = t^p n_b(1) = xv_3^i sv_2 tn_b(1)$ for some $x \in \mathcal{O}_3(T_\varepsilon)$ and $i \in \{1,2\}$. Accordingly, modulo T_ε the elements $v_3^i v_2$ and 1 agree, in contradiction to the fact that v_2 has order 2 modulo T_ε while v_3 has order 3. Hence, also $\overline{tn_b(1)}$ is stabilized by F_p. We conclude that F_p stabilizes R and acts trivially on $\mathrm{N}_{G_\varepsilon}(R)/R$, and thus on any B-weight (R, φ).

Let now $(q^2-1)_3 > 3$ and R be an extraspecial group of order 3^{1+2} and exponent 3 as in (iii) or (iv) of Proposition 12.31. By Proposition 12.30 we may assume that $R \leqslant L_\varepsilon$ and $\mathrm{N}_{G_\varepsilon}(R) \leqslant K_\varepsilon$, with $\mathrm{N}_{L_\varepsilon}(R)/R \cong \mathrm{Sp}_2(3)$. Let $\omega \in \mathbb{F}^\times$ be of order 3. We consider the group

$$R' := \left\langle \begin{bmatrix} \omega & 0 & 0\\ 0 & \omega^{-1} & 0\\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1\\ 1 & 0 & 0\\ 0 & 1 & 0 \end{bmatrix} \right\rangle \leqslant \mathrm{SL}_3(\varepsilon q),$$

which is an extraspecial group of order 3^{1+2} and exponent 3 (cf. the proof of [An94a, (1G)]). Recall that by Proposition 12.30 there exist exactly three L_ε-conjugacy classes of subgroups isomorphic to R in $L_\varepsilon \cong \mathrm{SL}_3(\varepsilon q)$, two of which are conjugate under G_ε, and the other one having a representative that is stabilized by v_2 respectively by the transpose-inverse automorphism (cf. also the proof of [An94a, (1G)]). It is easy to check that the transpose-inverse automorphism fixes R', so if R is as in (iii) then we may suppose that R corresponds to the group R' in $\mathrm{SL}_3(\varepsilon q) \cong L_\varepsilon$ and $\mathrm{N}_{G_\varepsilon}(R) = \langle \mathrm{N}_{L_\varepsilon}(R), v_2 \rangle$. In particular, R is stabilized by F_p. Let $\mathrm{St}^{\pm} \in \mathrm{Irr}(\mathrm{N}_{G_\varepsilon}(R)/R)$ denote the two extensions of the Steinberg character St of

$$\mathrm{N}_{L_\varepsilon}(R)/R \cong \mathrm{Sp}_2(3)$$

to $\langle \mathrm{N}_{L_\varepsilon}(R), v_2 \rangle / R$. Then by Gallagher's theorem, Theorem 1.21, we have

$$\mathrm{St}^-(v_2) = -\,\mathrm{St}^+(v_2).$$

By Lemma 11.6 it holds that $v_2^2 = 1$. Moreover, $\mathrm{St}^+(1) = 3$ since St has degree 3 being the Steinberg character of $\mathrm{Sp}_2(3)$. Hence, the character value $\mathrm{St}^+(v_2)$ is the sum of three square roots of unity, so $\mathrm{St}^+(v_2) \in \{\pm 1, \pm 3\}$. In particular, we have $\mathrm{St}^+(v_2), \mathrm{St}^-(v_2) \neq 0$. Now, since F_p fixes v_2, we have

$$(\mathrm{St}^+)^{F_p}(v_2) = \mathrm{St}^+(v_2) \neq 0,$$
$$(\mathrm{St}^-)^{F_p}(v_2) = \mathrm{St}^-(v_2) \neq 0.$$

Moreover, being the unique irreducible character of $\mathrm{N}_{G_\varepsilon}(R)/R$ of degree 3, the Steinberg character St is left invariant by F_p, so

$$((\mathrm{St}^{\pm})^{F_p})_{|\,\mathrm{N}_{L_\varepsilon}(R)/R} = \mathrm{St}^{F_p} = \mathrm{St},$$

and we conclude by application of Corollary 1.22 that F_p leaves both St^+ and St^- invariant. Hence, it follows that F_p fixes the G_ε-conjugacy class of any B-weight (R, φ) as φ is the inflation of one of St^+ or St^-.

Finally, suppose that (R, φ) is as in (iv). Then up to G_ε-conjugation R is uniquely determined in G_ε by its normalizer $\mathrm{N}_{G_\varepsilon}(R)$, and φ is uniquely determined by R, so the G_ε-conjugacy class of (R, φ) is stabilized by $\mathrm{Aut}(G_\varepsilon)$. This completes the proof. □

12.2.2.2 The Block B_2

In this section we examine the 3-weights of G_ε associated to the 3-block B_2. Note that this 3-block only exists if q is odd (cf. Section B.3.2 of Appendix B).

Proposition 12.33. *Let $B = B_2$. Then $|\mathcal{W}(B)| = 4$. Moreover, if (R, φ) is a B-weight of G_ε, then up to G_ε-conjugation it holds that $R = \mathcal{O}_3(T_\varepsilon)$ with $\mathrm{N}_{G_\varepsilon}(R) = \mathrm{N}_{G_\varepsilon}(T_\varepsilon)$ and $\mathrm{C}_{G_\varepsilon}(R) = T_\varepsilon$, and if a linear character $\theta \in \mathrm{Irr}(T_\varepsilon)$ is an irreducible constituent of $\varphi_{|T_\varepsilon}$, then $\theta^2 = 1_{T_\varepsilon} \neq \theta$ and*

$$\mathrm{N}_{G_\varepsilon}(R)_\theta / T_\varepsilon \cong C_2 \times C_2.$$

Furthermore, the set $\mathrm{Irr}(\mathrm{N}_{G_\varepsilon}(R)_\theta \mid \theta)$ consists of four distinct extensions of θ and it holds that $\varphi = \psi^{\mathrm{N}_{G_\varepsilon}(T_\varepsilon)}$ for some $\psi \in \mathrm{Irr}(\mathrm{N}_{G_\varepsilon}(R)_\theta \mid \theta)$.

Proof. This statement follows from Proposition 12.26 and the proof of [An94a, (3B)]. □

Proposition 12.34. *Let $B = B_2$. Then $\mathrm{Aut}(G_\varepsilon)_B = \mathrm{Aut}(G_\varepsilon)$ acts trivially on $\mathcal{W}(B)$.*

Proof. As before, it suffices to check invariance under the action of F_p. We let (R, φ) be a B-weight of G_ε. Following Proposition 12.33 we may assume that $R = \mathcal{O}_3(T_\varepsilon)$ with $R\,\mathrm{C}_{G_\varepsilon}(R) = T_\varepsilon$ and $\mathrm{N}_{G_\varepsilon}(R) = \mathrm{N}_{G_\varepsilon}(T_\varepsilon)$. Let θ be an irreducible constituent of $\varphi_{|T_\varepsilon}$. Then according to Proposition 12.33 the linear character θ has order 2, so in the parametrization of Notation 12.27 it follows that θ is one of

$$\theta_0^{\frac{q-\varepsilon}{2}} \times \theta_0^{\frac{q-\varepsilon}{2}} \times 1, \quad \theta_0^{\frac{q-\varepsilon}{2}} \times 1 \times 1, \quad 1 \times \theta_0^{\frac{q-\varepsilon}{2}} \times 1.$$

The latter two characters are conjugate to the first via $n_a(1)$ and $n_{a+b}(1)$, respectively (cf. Lemma 12.28), so since by Clifford theory, Theorem 1.15, all irreducible constituents of $\varphi_{|T_\varepsilon}$ are $\mathrm{N}_{G_\varepsilon}(T_\varepsilon)$-conjugate, we may assume that

$$\theta = \theta_0^{\frac{q-\varepsilon}{2}} \times \theta_0^{\frac{q-\varepsilon}{2}} \times 1,$$

which is left invariant by $n_b(1)$, $n_{2a+b}(1)$ and $v_2 = n_b(1)n_{2a+b}(1)^{-1}$ following Lemma 12.28. By Proposition 12.33 we have $\mathrm{N}_{G_\varepsilon}(R)_\theta/T_\varepsilon \cong C_2 \times C_2$, so in fact $\mathrm{N}_{G_\varepsilon}(R)_\theta = \langle T_\varepsilon, n_b(1), v_2\rangle$. Since F_p acts trivially on both $n_b(1)$ and v_2, it follows that F_p stabilizes $\mathrm{N}_{G_\varepsilon}(R)_\theta$ and we have

$$(\psi^{F_p})^{\mathrm{N}_{G_\varepsilon}(R)} = (\psi^{\mathrm{N}_{G_\varepsilon}(R)})^{F_p}$$

for all $\psi \in \mathrm{N}_{G_\varepsilon}(R)_\theta$. Thus, we only need to check that any extension of θ to $\mathrm{N}_{G_\varepsilon}(R)_\theta$ stays invariant under F_p. But this follows from Corollary 1.22 since θ is linear,

$$(\psi^{F_p})_{|T_\varepsilon} = \theta^{F_p} = \theta^p = \theta = \psi_{|T_\varepsilon},$$

$\psi^{F_p}(n_b(1)) = \psi(n_b(1)) \neq 0$ and $\psi^{F_p}(v_2) = \psi(v_2) \neq 0$ for any $\psi \in \mathrm{N}_{G_\varepsilon}(R)_\theta$. □

Remark 12.35. By Proposition 12.33 for the 3-block $B = B_2$, a B-weight $(\mathcal{O}_3(T_\varepsilon), \varphi)$ and an irreducible constituent $\theta \in \mathrm{Irr}(T_\varepsilon)$ of $\varphi_{|T_\varepsilon}$ we have $\theta^2 = 1_{T_\varepsilon}$, $\theta \neq 1_{T_\varepsilon}$, and θ extends to its stabilizer $\mathrm{N}_{G_\varepsilon}(T_\varepsilon)_\theta$ in $\mathrm{N}_{G_\varepsilon}(T_\varepsilon)$. This is stated in [An94a, (3B)] but the proof of the extendibility of θ given there is not very precise. For later use it will be convenient to reprove it here in more detail. As in the proof of Proposition 12.34 we may assume that

$$\theta = \theta_0^{\frac{q-\varepsilon}{2}} \times \theta_0^{\frac{q-\varepsilon}{2}} \times 1$$

and $\mathrm{N}_{G_\varepsilon}(T_\varepsilon)_\theta = \langle T_\varepsilon, n_b(1), v_2\rangle$. By definition we have $v_2 = n_b(1)n_{-(2a+b)}(1)$. We claim that $n_b(1)$ and v_2 commute. Then by Proposition 1.20(i) there exists an extension η of θ to $\langle T_\varepsilon, n_b(1)\rangle$, and since $[n_b(1), v_2] = 1$, $n_b(1)^2 \in T_\varepsilon$ and v_2 normalizes $\langle T_\varepsilon, n_b(1)\rangle$ and stabilizes θ, we have

$$\begin{aligned}
\eta^{v_2}(tn_b(1)) &= \eta(v_2tv_2^{-1}n_b(1)) = \eta(v_2tv_2^{-1})\eta(n_b(1)) \\
&= \theta(v_2tv_2^{-1})\eta(n_b(1)) = \theta(t)\eta(n_b(1)) \\
&= \eta(t)\eta(n_b(1)) = \eta(tn_b(1))
\end{aligned}$$

for all $t \in T_\varepsilon$, so v_2 leaves η invariant. Again it follows by Proposition 1.20(i) that η has an extension η' to $\mathrm{N}_{G_\varepsilon}(T_\varepsilon)_\theta = \langle T_\varepsilon, n_b(1), v_2\rangle$ since $\langle T_\varepsilon, n_b(1), v_2\rangle/\langle T_\varepsilon, n_b(1)\rangle \cong C_2$ is cyclic. Then η' is an extension of θ to $\mathrm{N}_{G_\varepsilon}(T_\varepsilon)_\theta$ as claimed.

Let us hence show that $[n_b(1), v_2] = 1$. For this it suffices to prove that $n_b(1)$ and $n_{-(2a+b)}(1)$ commute. Following Theorem 4.25(ix) we have

$$n_b(1)n_{-(2a+b)}(1)n_b(1)^{-1} = n_{\omega_b(-(2a+b))}(\eta_{b,-(2a+b)}).$$

Now b and $-(2a+b)$ are orthogonal roots, so $\omega_b(-(2a+b)) = -(2a+b)$, and by Proposition 11.5 we may suppose that $\eta_{b,-(2a+b)} = 1$. This proves the claim.

12.2.2.3 The Blocks of Types B_{1a}, B_{1b}, B_{2a} and B_{2b}

As usual we are only interested in 3-blocks of non-cyclic defect. The 3-blocks of types B_{1a}, B_{1b}, B_{2a} and B_{2b} are described in Section B.3.2, and we have

- B_{1a}, B_{1b} have non-cyclic defect if and only if $\varepsilon = +1$, and
- B_{2a}, B_{2b} have non-cyclic defect if and only if $\varepsilon = -1$.

In the following we consider only these cases and set

$$B_a := \begin{cases} B_{1a} & \text{if } \varepsilon = +1, \\ B_{2a} & \text{if } \varepsilon = -1, \end{cases} \qquad B_b := \begin{cases} B_{1b} & \text{if } \varepsilon = +1, \\ B_{2b} & \text{if } \varepsilon = -1, \end{cases}$$

so if a 3-block B is of type B_a or B_b, then it possesses non-cyclic defect groups. We shall write $B \in \{B_a, B_b\}$ in this case. According to Proposition 12.26 the defect groups of B are abelian and G_ε-conjugate to $\mathcal{O}_3(T_\varepsilon)$. Hence, from Lemma 2.11 we deduce that R is G_ε-conjugate to $\mathcal{O}_3(T_\varepsilon)$ whenever (R, φ) is a B-weight for some φ, and there is no loss in generality in assuming that $R = \mathcal{O}_3(T_\varepsilon)$ in this case.

Proposition 12.36. *Suppose that $B \in \{B_a, B_b\}$ is a 3-block of G_ε. Then $|\mathcal{W}(B)| = 2$. Moreover, if (R, φ) is a B-weight of G_ε, then up to G_ε-conjugation it holds that $R = \mathcal{O}_3(T_\varepsilon)$ with $\mathrm{N}_{G_\varepsilon}(R) = \mathrm{N}_{G_\varepsilon}(T_\varepsilon)$ and $\mathrm{C}_{G_\varepsilon}(R) = T_\varepsilon$, and for any irreducible constituent $\theta \in \mathrm{Irr}(T_\varepsilon)$ of $\varphi_{|T_\varepsilon}$ we have*

$$\mathrm{N}_{G_\varepsilon}(R)_\theta / T_\varepsilon \cong C_2$$

and $\varphi = \psi^{\mathrm{N}_{G_\varepsilon}(R)}$ for one of the two extensions $\psi \in \mathrm{Irr}(\mathrm{N}_{G_\varepsilon}(T_\varepsilon)_\theta \mid \theta)$ of θ.

Proof. This follows from Proposition 12.26 and the proof of [An94a, (3B)]. □

Let us now examine the action of automorphisms of G_ε on 3-weights of G_ε associated to 3-blocks of type B_a or B_b.

Proposition 12.37. *Suppose that $B \in \{B_a, B_b\}$ is a 3-block of G_ε. Then $\mathrm{Aut}(G_\varepsilon)_B$ acts trivially on $\mathcal{W}(B)$.*

Proof. Let (R, φ) be a B-weight of G_ε. Following Proposition 12.36 we may assume that $R = \mathcal{O}_3(T_\varepsilon)$ with $\mathrm{N}_{G_\varepsilon}(R) = \mathrm{N}_{G_\varepsilon}(T_\varepsilon)$, and that, moreover, $\mathrm{N}_{G_\varepsilon}(R)_\theta / T_\varepsilon \cong C_2$ if $\theta \in \mathrm{Irr}(T_\varepsilon)$ is an irreducible constituent of $\varphi_{|T_\varepsilon}$. Since by Clifford theory, Theorem 1.15, all irreducible constituents of $\varphi_{|T_\varepsilon}$ are $\mathrm{N}_{G_\varepsilon}(T_\varepsilon)$-conjugate, in consequence of Corollary 12.29 we may assume that

$$\theta = \theta_0^i \times \theta_0^i \times 1 \quad \text{or} \quad \theta = \theta_0^i \times \theta_0^{-i} \times 1$$

for a suitable $0 < i < q - \varepsilon$, whence by Lemma 12.28 we have

$$\mathrm{N}_{G_\varepsilon}(R)_\theta = \begin{cases} \langle T_\varepsilon, n_b(1) \rangle & \text{if } \theta = \theta_0^i \times \theta_0^i \times 1, \\ \langle T_\varepsilon, n_{2a+b}(1) \rangle & \text{if } \theta = \theta_0^i \times \theta_0^{-i} \times 1. \end{cases}$$

For this one should note that $v_2 \notin \mathrm{N}_{G_\varepsilon}(R)_\theta$ as otherwise θ would be as in the proof of Proposition 12.34 with $\mathrm{N}_{G_\varepsilon}(R)_\theta \cong C_2 \times C_2$. In particular, $i \neq (q - \varepsilon)/2$.

Let us now suppose that $a \in \mathrm{Aut}(G_\varepsilon)_B$. Since any inner automorphism of G_ε stabilizes B, we may assume that $a = F_p^k$ for some $k \in \mathbb{N}$ (cf. Proposition 11.14), so that

$$(R, \varphi)^a = (R, \varphi)^{F_p^k} = (R, \varphi^{F_p^k})$$

and we need to prove that φ is left invariant by F_p^k. Due to the fact that F_p, and hence also $a = F_p^k$, acts on T_ε, we have that θ^a is an irreducible character of T_ε. Since (R, φ) is a B-weight, it follows from Construction 2.10 that $\mathrm{bl}(\theta)^{G_\varepsilon} = B$, and from the fact that a stabilizes B in conjunction with Proposition 1.3 we may deduce that

$$\mathrm{bl}(\theta^a)^{G_\varepsilon} = (\mathrm{bl}(\theta)^a)^{G_\varepsilon} = (\mathrm{bl}(\theta)^{G_\varepsilon})^a = B^a = B,$$

that is, $\mathrm{bl}(\theta^a)$ induces to B. But B has defect group $\mathcal{O}_3(T_\varepsilon)$ by Proposition 12.26, just like any 3-block of T_ε in consequence of Lemma 1.13, so the extended first main theorem of Brauer, Theorem 1.7, implies that θ^a and θ, the canonical characters of $\mathrm{bl}(\theta^a)$ and $\mathrm{bl}(\theta)$, respectively, are conjugate under $\mathrm{N}_{G_\varepsilon}(T_\varepsilon)$. Now Lemma 12.28 yields the following $\mathrm{N}_{G_\varepsilon}(T_\varepsilon)$-conjugates of θ in the case $\theta = \theta_0^i \times \theta_0^i \times 1$:

$$\theta_0^i \times \theta_0^i \times 1, \quad \theta_0^i \times 1 \times 1, \quad 1 \times \theta_0^i \times 1, \quad \theta_0^{-i} \times \theta_0^{-i} \times 1, \quad 1 \times \theta_0^{-i} \times 1, \quad \theta_0^{-i} \times 1 \times 1.$$

These are pairwise distinct as $i \neq (q-\varepsilon)/2$, so since $|\,\mathrm{N}_{G_\varepsilon}(T_\varepsilon) \colon \mathrm{N}_{G_\varepsilon}(T_\varepsilon)_\theta| = 6$, these must indeed be all $\mathrm{N}_{G_\varepsilon}(T_\varepsilon)$-conjugates of θ in this case. Similarly, for $\theta = \theta_0^i \times \theta_0^{-i} \times 1$ the $\mathrm{N}_{G_\varepsilon}(T_\varepsilon)$-conjugates of θ are given by

$$\theta_0^i \times \theta_0^{-i} \times 1, \ \theta_0^i \times \theta_0^{2i} \times 1, \ \theta_0^{-i} \times \theta_0^i \times 1, \ \theta_0^{-2i} \times \theta_0^{-i} \times 1, \ \theta_0^{2i} \times \theta_0^i \times 1, \ \theta_0^{-i} \times \theta_0^{-2i} \times 1.$$

Note that these characters are pairwise distinct as $i \neq (q-\varepsilon)/2$ and $2i \not\equiv -i \bmod (q-\varepsilon)$, where the latter holds since otherwise θ would be of order 3, which is not possible since $\mathcal{O}_3(T_\varepsilon) \subseteq \ker(\theta)$. Now $a = F_p^k$ acts on $\mathrm{Irr}(T_\varepsilon)$ by raising the linear characters of T_ε to their p^k-th power, that is,

$$\theta^a = \begin{cases} \theta_0^{ip^k} \times \theta_0^{ip^k} \times 1 & \text{if } \theta = \theta_0^i \times \theta_0^i \times 1, \\ \theta_0^{ip^k} \times \theta_0^{-ip^k} \times 1 & \text{if } \theta = \theta_0^i \times \theta_0^{-i} \times 1. \end{cases}$$

Hence, since θ^a and θ are $\mathrm{N}_{G_\varepsilon}(T_\varepsilon)$-conjugate, the above observations on the shapes of the $\mathrm{N}_{G_\varepsilon}(T_\varepsilon)$-conjugates of θ imply that $\theta^a \in \{\theta, \theta^{-1}\}$. According to Lemma 12.28 we have

$$\theta^{-1} = \begin{cases} \theta^{n_{2a+b}(1)} & \text{if } \theta = \theta_0^i \times \theta_0^i \times 1, \\ \theta^{n_b(1)} & \text{if } \theta = \theta_0^i \times \theta_0^{-i} \times 1. \end{cases}$$

Let us now recall the notation c_x, $x \in G_\varepsilon$, for the automorphism of the group G_ε defined by $c_x(g) = xgx^{-1}$ for all $g \in G_\varepsilon$, and set

$$a' := \begin{cases} a = F_p^k & \text{if } \theta^a = \theta, \\ ac_{n_{2a+b}(1)} = F_p^k c_{n_{2a+b}(1)} & \text{if } \theta^a = \theta^{-1} \text{ and } \theta = \theta_0^i \times \theta_0^i \times 1, \\ ac_{n_b(1)} = F_p^k c_{n_b(1)} & \text{if } \theta^a = \theta^{-1} \text{ and } \theta = \theta_0^i \times \theta_0^{-i} \times 1. \end{cases}$$

Then $\theta^{a'} = \theta$ and φ is left invariant by $a = F_p^k$ if and only if it is stabilized by a' since $n_{2a+b}(1), n_b(1) \in \mathrm{N}_{G_\varepsilon}(T_\varepsilon) = \mathrm{N}_{G_\varepsilon}(R)$. Moreover, in consequence of Theorem 4.25(ix) and Proposition 11.5 we have $[n_{2a+b}(1), n_b(1)] = 1$, whence it follows that a' stabilizes both $n_b(1)$ and $n_{2a+b}(1)$. Now following Proposition 12.36 there is $\psi \in \mathrm{Irr}(\mathrm{N}_{G_\varepsilon}(T_\varepsilon)_\theta \mid \theta)$ such that $\varphi = \psi^{\mathrm{N}_{G_\varepsilon}(T_\varepsilon)}$ and hence

$$\varphi^{a'} = (\psi^{\mathrm{N}_{G_\varepsilon}(T_\varepsilon)})^{a'} = (\psi^{a'})^{\mathrm{N}_{G_\varepsilon}(T_\varepsilon)},$$

where $\psi^{a'} \in \mathrm{Irr}(\mathrm{N}_{G_\varepsilon}(T_\varepsilon)_\theta \mid \theta)$ since a' stabilizes T_ε, $n_b(1)$, $n_{2a+b}(1)$ and θ. We prove that $\psi^{a'} = \psi$. It holds that $(\psi^{a'})_{|T_\varepsilon} = \theta = \psi_{|T_\varepsilon}$. Moreover, we have $\mathrm{N}_{G_\varepsilon}(T_\varepsilon)_\theta = \langle T_\varepsilon, n \rangle$, where n equals $n_b(1)$ or $n_{2a+b}(1)$ depending on θ, and n is left invariant by a'. Hence, since ψ is linear, we conclude that

$$\psi^{a'}(n) = \psi(n) \neq 0,$$

implying in accordance with Corollary 1.22 that $\psi^{a'} = \psi$. Thus, we have $\varphi^a = \varphi^{a'} = \varphi$ as claimed, which concludes the proof. $\square$

12.2.2.4 The Blocks of Types B_{X_1} and B_{X_2}

Finally, we consider the 3-blocks of G_ε of types B_{X_1} and B_{X_2}. According to Section B.3.2 of Appendix B a 3-block of type B_{X_1} has non-cyclic defect groups if and only if $q \equiv 1 \bmod 3$, while a 3-block of type B_{X_2} is of non-cyclic defect if and only if $q \equiv -1 \bmod 3$.

Proposition 12.38. *Let $B \in \{B_{X_1}, B_{X_2}\}$ be a 3-block of G_ε with non-cyclic defect groups. Then $|\mathcal{W}(B)| = 1$.*

Proof. This is [An94a, (3B)]. □

We may immediately conclude the following:

Corollary 12.39. *Let $B \in \{B_{X_1}, B_{X_2}\}$ be a 3-block of G_ε with non-cyclic defect groups. Then $\mathrm{Aut}(G_\varepsilon)_B$ acts trivially on $\mathcal{W}(B)$.*

Proof. This follows directly from Proposition 12.38 as there exists only one G_ε-conjugacy class of B-weights in G_ε. □

Chapter 13

Partitions and Equivariant Bijections

This chapter is aimed at establishing parts (i) and (ii) of the inductive blockwise Alperin weight condition in Definition 3.2 for every ℓ-block B of $G = G_2(q)$, $q \geqslant 5$, of non-cyclic defect, where $\ell \in \{2, 3\}$. Following Proposition 11.2 the group G is its own universal covering group, whence as for the special linear groups $\mathrm{SL}_3(q)$ in the previous part of this thesis, our objective is to prove the following statements for every $\ell \in \{2, 3\}$ and every ℓ-block B of G of non-cyclic defect:

(i) There exist subsets $\mathrm{IBr}(B \mid Q) \subseteq \mathrm{IBr}(B)$ for $Q \in \mathrm{Rad}_\ell(G)$ with the following properties:

 (1) $\mathrm{IBr}(B \mid Q)^a = \mathrm{IBr}(B \mid Q^a)$ for every $Q \in \mathrm{Rad}_\ell(G)$, $a \in \mathrm{Aut}(G)_B$,

 (2) $\mathrm{IBr}(B) = \dot{\bigcup}_{Q \in \mathrm{Rad}_\ell(G)/\sim_G} \mathrm{IBr}(B \mid Q)$.

(ii) For every $Q \in \mathrm{Rad}_\ell(G)$ there exists a bijection

$$\Omega_Q^G\colon\ \mathrm{IBr}(B \mid Q) \longrightarrow \mathrm{dz}(\mathrm{N}_G(Q)/Q,\, B)$$

such that $\Omega_Q^G(\phi)^a = \Omega_{Q^a}^G(\phi^a)$ for every $\phi \in \mathrm{IBr}(B \mid Q)$ and $a \in \mathrm{Aut}(G)_B$.

This will be proven similarly as for the special linear groups in Proposition 8.1. During our previous observations we discovered that, unless B is the principal 2-block of G, the action of $\mathrm{Aut}(G)_B$ on both the G-conjugacy classes of B-weights and the set of irreducible Brauer characters in B is trivial, which allows us to easily derive the following result:

Proposition 13.1. *Let $\ell \in \{2, 3\}$ and let B be an ℓ-block of G of non-cyclic defect. Then conditions (i) and (ii) of Definition 3.2 are satisfied for B.*

Proof. Following [An94a, p. 36] or Sections B.3.1 and B.3.2 the 2-blocks of G of non-cyclic defect are the 2-blocks B_0, B_3 (if $3 \nmid q$), and the 2-blocks of types B_{1a}, B_{1b}, B_{2a}, B_{2b}, B_{X_1} and B_{X_2}, while the 3-blocks of G of non-cyclic defect are given by B_0, B_2 (if $2 \nmid q$), and the 3-blocks of types $B_{\delta a}$, $B_{\delta b}$ and B_{X_δ}, where $\delta = 1$ if $q \equiv 1 \bmod 3$ and $\delta = 2$ if $q \equiv -1 \bmod 3$.

We assume first that B is not the principal 2-block of G. Following the results of [An94a], which we summarized in Propositions 12.18, 12.20, 12.23, 12.31, 12.33, 12.36 and 12.38, the blockwise Alperin weight conjecture holds for B, that is, $|\operatorname{IBr}(B)| = |\mathcal{W}(B)|$. We may hence choose a bijection $\Omega_B\colon \operatorname{IBr}(B) \longrightarrow \mathcal{W}(B)$. In consequence of Propositions 12.1 and 12.2 the action of $\operatorname{Aut}(G)_B$ on $\operatorname{IBr}(B)$ is trivial, and moreover, by the results of Sections 12.2.1 and 12.2.2, also $\mathcal{W}(B)$ is stabilized pointwise by $\operatorname{Aut}(G)_B$. In particular, Ω_B is trivially $\operatorname{Aut}(G)_B$-equivariant, whence according to Lemma 3.8 it is possible to define subsets $\operatorname{IBr}(B \mid Q) \subseteq \operatorname{IBr}(B)$ and bijections $\Omega_Q^G\colon \operatorname{IBr}(B \mid Q) \longrightarrow \operatorname{dz}(\mathrm{N}_G(Q)/Q,\, B)$ for every radical ℓ-subgroup $Q \in \operatorname{Rad}_\ell(G)$ such that conditions (i) and (ii) of Definition 3.2 are satisfied for B.

Let us now assume that B is the principal 2-block of G. By [An94a] (cf. Proposition 12.16 and Section B.3.1) we have $|\operatorname{IBr}(B)| = |\mathcal{W}(B)| = 7$. If we can find an $\operatorname{Aut}(G)_B$-equivariant bijection $\Omega_B\colon \operatorname{IBr}(B) \longrightarrow \mathcal{W}(B)$, then by the same arguments as above the claim follows. According to Propositions 12.1 and 12.17 the action of $\operatorname{Aut}(G)_B = \operatorname{Aut}(G)$ is trivial on both $\operatorname{IBr}(B)$ and $\mathcal{W}(B)$ if $3 \nmid q$ as in this case $\operatorname{Aut}(G) = \langle G, F_p\rangle$ by Proposition 11.14, so there is nothing to prove in this situation. Assume hence that $3 \mid q$, in which case Proposition 11.14 gives $\operatorname{Aut}(G) = \langle G, \Gamma\rangle$. We proved in Proposition 12.1 that Γ interchanges exactly two of the seven irreducible Brauer characters in B and leaves the remaining ones invariant. Moreover, as a result of Proposition 12.17 we know that Γ interchanges two of the seven G-conjugacy classes of B-weights and acts trivially on the remaining five classes. Thus, since inner automorphisms of G leave the elements of $\operatorname{IBr}(B)$ and $\mathcal{W}(B)$ invariant, an $\operatorname{Aut}(G)$-equivariant bijection $\Omega_B\colon \operatorname{IBr}(B) \longrightarrow \mathcal{W}(B)$ exists, so as above the claim follows as a consequence of Lemma 3.8. $\square$

Chapter 14

The Main Result for $G_2(q)$

Our objective was to establish the inductive blockwise Alperin weight condition for the Chevalley groups $G_2(q)$ with $q \geqslant 5$. Thanks to the fact that these groups possess cyclic outer automorphism groups, we do not need to concern ourselves with the normally embedded conditions in part (iii) of Definition 3.2 (cf. Proposition 3.12). In fact, the results we have obtained so far suffice to prove the desired statement:

Theorem 14.1. *Let $q \geqslant 5$ be a prime power. Then the inductive blockwise Alperin weight condition (cf. Definition 3.3) holds for the group $G_2(q)$ and every prime ℓ dividing its order.*

Proof. According to Proposition 11.2 the simple group $G = G_2(q)$ is its own universal covering group if $q \geqslant 5$. Moreover, as seen in Proposition 11.14 the outer automorphism group of G is cyclic. Thus, following Proposition 3.12 we need to verify conditions (i) and (ii) of Definition 3.2 for every ℓ-block B of $X = G$ for every prime ℓ dividing $|G|$.

If $\ell = p$, then the claim holds by Proposition 3.11, so let us assume that $\ell \neq p$. By Lemma 11.1 the order of G is given by

$$|G_2(q)| = q^6 \Phi_1(q)^2 \Phi_2(q)^2 \Phi_3(q) \Phi_6(q),$$

so ℓ divides at least one of the factors $\Phi_1(q)$, $\Phi_2(q)$, $\Phi_3(q)$ and $\Phi_6(q)$. Similarly as in the proof of Lemma 5.2 one can show that

$$\gcd(\Phi_1(q), \Phi_6(q)) = \gcd(\Phi_2(q), \Phi_3(q)) = \gcd(\Phi_3(q), \Phi_6(q)) = 1,$$

$$\gcd(\Phi_1(q), \Phi_3(q)) = \begin{cases} 3 & \text{if } q \equiv 1 \bmod 3, \\ 1 & \text{else,} \end{cases}$$

$$\gcd(\Phi_2(q), \Phi_6(q)) = \begin{cases} 3 & \text{if } q \equiv -1 \bmod 3, \\ 1 & \text{else.} \end{cases}$$

Hence, if $\ell \geqslant 5$, then it divides exactly one of $\Phi_1(q)$, $\Phi_2(q)$, $\Phi_3(q)$ or $\Phi_6(q)$. Suppose that $5 \leqslant \ell \mid \Phi_3(q)\Phi_6(q)$. Then a Sylow ℓ-subgroup of G is contained in a maximal torus of G of type T_3 or T_6 depending on whether ℓ divides $\Phi_3(q)$ or $\Phi_6(q)$, so in particular the Sylow ℓ-subgroups of G are cyclic in this case (cf. Table 11.1), whence the (iBAW) conditions holds for G and ℓ by Proposition 3.9.

Let us now assume that $5 \leqslant \ell \mid \Phi_1(q)\Phi_2(q)$. Then the ℓ-blocks of G have either cyclic or maximal defect (see [Sha89a, Prop. 3.1, 4.1] and [Sha92, p. 1379]). For the ℓ-blocks of cyclic defect the (iBAW) condition holds again by Proposition 3.9, while it has been proven to hold for 3-blocks of maximal defect by Cabanes–Späth in [CS13, Cor. 7.6]. Hence, the inductive blockwise Alperin weight condition holds for G and $\ell \geqslant 5$ dividing $|G|$.

It remains to consider the non-cyclic ℓ-blocks of G in the case $\ell \in \{2, 3\}$. This has been done throughout the previous chapters, and as a result we obtained in Proposition 13.1 that every ℓ-block of G of non-cyclic defect satisfies conditions (i) and (ii) of Definition 3.2. In summary, the (iBAW) condition holds for $G = G_2(q)$ and every prime ℓ dividing its order. This concludes the proof. $\square$

Part IV

The Steinberg Triality Groups $^3D_4(q)$

Chapter 15

Properties of ${}^3D_4(q)$

This chapter provides an introduction to the groups ${}^3D_4(q)$, *Steinberg's triality groups* as they are commonly known. These are finite groups of Lie type and may be constructed as fixed point groups of universal Chevalley groups of type D_4 under a certain endomorphism deriving from an exceptional symmetry of order 3 of the associated Dynkin diagram of type D_4.

The results described here do not present any new findings, but rather they are meant to provide an overview over important properties of the groups ${}^3D_4(q)$ that have been well-known for a long time. Our main references are Carter [Car89], Deriziotis–Michler [DM87], Gorenstein–Lyons–Solomon [GLS98] and Steinberg [Ste68].

15.1 Construction of ${}^3D_4(q)$

We start the construction of the groups ${}^3D_4(q)$ by considering a root system Σ of type D_4 over the field $\mathbb{R}$ of real numbers, i.e.,

$$\Sigma = \{\pm e_i \pm e_j \mid 1 \leqslant i < j \leqslant 4\},$$

where e_1, e_2, e_3, e_4 form an orthonormal basis of $\mathbb{R}^4$. Inside Σ we fix a base for this root system given by

$$\Pi = \{e_1 - e_2,\ e_2 - e_3,\ e_3 - e_4,\ e_3 + e_4\},$$

whose elements we denote by $r_1 := e_1 - e_2$, $r_2 := e_2 - e_3$, $r_3 := e_3 - e_4$ and $r_4 := e_3 + e_4$. The Dynkin diagram associated to Σ is given in Figure 15.1 below:

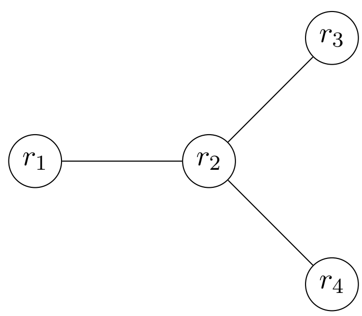

Figure 15.1: Dynkin diagram of type D_4

Now we fix a prime number p and denote by $\mathbb{F}$ an algebraic closure of the finite field $\mathbb{F}_p$ consisting of p elements. Then let $\mathbf{G} = D_4(\mathbb{F})$ be a universal Chevalley group of type D_4 over $\mathbb{F}$. For $q = p^f$, $f \in \mathbb{N}_{>0}$, let $\mathbb{F}_q$ be the unique subfield of $\mathbb{F}$ consisting of q elements. The field automorphism $\mathbb{F} \longrightarrow \mathbb{F}$, $a \longmapsto a^q$, induces a Frobenius endomorphism F_q of $\mathbf{G}$ via its action on the Steinberg generators $x_r(t)$, $r \in \Sigma$, $t \in \mathbb{F}$, of $\mathbf{G}$, that is,

$$F_q\colon \mathbf{G} \longrightarrow \mathbf{G}, \quad x_r(t) \longmapsto x_r(t^q), \quad r \in \Sigma,\ t \in \mathbb{F}.$$

Another endomorphism of $\mathbf{G}$ is obtained as follows. We consider the symmetry ρ of the Dynkin diagram of type D_4 given by

$$\rho : \Pi \longrightarrow \Pi, \quad r_1 \longmapsto r_3 \longmapsto r_4 \longmapsto r_1, \quad \rho(r_2) = r_2.$$

Then ρ has order 3, whence it is called a *triality*. Its action on the Dynkin diagram is illustrated in Figure 15.2 below:

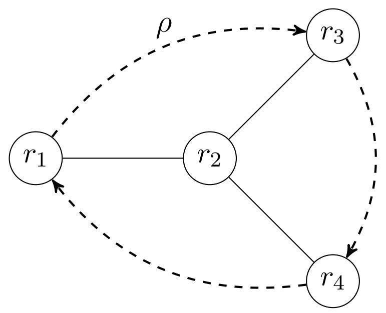

Figure 15.2: Symmetry ρ of the Dynkin diagram of type D_4

There exists a unique isometry of $\mathbb{R}^4 = \mathbb{R}\Sigma$ which maps each $r \in \Pi$ to its image $\rho(r)$ under ρ (cf. [Car89, p. 217]). We denote this isometry by ρ as well. According to [Car89, Prop. 12.2.2] it holds that ρ is a linear transformation of $\mathbb{R}^4$ satisfying $\rho(\Sigma) = \Sigma$, so ρ may be regarded as a permutation of the root system Σ of $\mathbf{G}$. Following [Car89, Prop. 12.2.3 and Lemma 13.6.2] by choosing a suitable Chevalley basis for the simple Lie algebra of type D_4 underlying $\mathbf{G}$ we may assume that ρ induces an automorphism τ of $\mathbf{G}$ given by

$$\tau\colon \mathbf{G} \longrightarrow \mathbf{G}, \quad x_r(t) \longmapsto x_{\rho(r)}(t), \quad r \in \Sigma,\ t \in \mathbb{F}.$$

This satisfies $\tau(n_r(t)) = n_{\rho(r)}(t)$ and $\tau(h_r(t)) = h_{\rho(r)}(t)$ for $r \in \Sigma$, $t \in \mathbb{F}^\times$, since for $r \in \Sigma$ it holds that $\rho(-r) = -\rho(r)$. Arising from a symmetry of the Dynkin diagram of type D_4, the automorphism τ is classified as a *graph automorphism*. Note that τ commutes with F_q and has order 3 as an automorphism of $\mathbf{G}$.

We let the endomorphism F of $\mathbf{G}$ be the product $\tau F_q = F_q \tau$. Then $F^3 = \tau^3 F_q^3 = F_q^3$, so F is a Steinberg endomorphism of $\mathbf{G}$. The group ${}^3D_4(q)$ is defined as the group

$$G := {}^3D_4(q) := \mathbf{G}^F = \{g \in \mathbf{G} \mid F(g) = g\}$$

of fixed points of $\mathbf{G}$ under F. Since $F^3 = F_q^3$, it follows that the finite group G is contained in the group $D_4(q^3) := D_4(\mathbb{F}_{q^3}) = \mathbf{G}^{F_q^3}$.

Lemma 15.1. *The order of the finite group ${}^3D_4(q)$ is given by*

$$\begin{aligned}\left|{}^3D_4(q)\right| &= q^{12}(q-1)^2(q+1)^2(q^2+q+1)^2(q^2-q+1)^2(q^4-q^2+1)\\ &= q^{12}\Phi_1(q)^2\Phi_2(q)^2\Phi_3(q)^2\Phi_6(q)^2\Phi_{12}(q).\end{aligned}$$

Proof. See, for instance, [Car89, Thm. 14.3.2]. □

Let us once more consider the isometry ρ of $\mathbb{R}^4 = \mathbb{R}\Sigma$ defined above. For a root $r \in \Sigma$ we let

$$\widetilde{r} := \frac{1}{3}(r + \rho(r) + \rho^2(r))$$

denote the orthogonal projection of r onto the subspace of $\mathbb{R}\Sigma$ invariant under ρ. If we set $\widetilde{\Sigma} := \{\widetilde{r} \mid r \in \Sigma\}$, then $\widetilde{\Sigma}$ forms a root system of type G_2 by [Ste68, Thm. 32 and example afterwards]. For $S \in \widetilde{\Sigma}$ we define $\Sigma(S) := \{r \in \Sigma \mid \widetilde{r} = S\}$ to be the preimage of S in Σ under the orthogonal projection induced by ρ. Hence,

$$\Sigma(S) = \begin{cases} \{r\} & \text{if } S = \widetilde{r} \text{ for some } r \in \Sigma \text{ with } \rho(r) = r,\\ \{r, \rho(r), \rho^2(r)\} & \text{if } S = \widetilde{r} \text{ for some } r \in \Sigma \text{ with } \rho(r) \neq r, \end{cases}$$

for $S \in \widetilde{\Sigma}$. Then by [Ste68, Corollary of Thm. 32] for each $S \in \widetilde{\Sigma}$ the set $\Sigma(S)$ is the positive system of roots of a root system of type A_1 or A_1^3.

Analogously to the root subgroups $\mathbf{X}_r := \{x_r(t) \mid t \in \mathbb{F}\}$, $r \in \Sigma$, which generate the group $\mathbf{G}$, we define for $S \in \widetilde{\Sigma}$ the group

$$\mathbf{X}_S := \langle x_r(t) \mid r \in \Sigma(S),\ t \in \mathbb{F}\rangle \leqslant \mathbf{G}\,.$$

Moreover, for $S \in \widetilde{\Sigma}$, $r \in \Sigma$ with $\widetilde{r} = S$, and $t \in \mathbb{F}_{q^3}$ we set

$$x_S(t) = \begin{cases} x_r(t) & \text{if } \Sigma(S) \text{ has type } A_1,\\ x_r(t)x_{\rho(r)}(t^q)x_{\rho^2(r)}(t^{q^2}) & \text{if } \Sigma(S) \text{ has type } A_1^3. \end{cases}$$

Then by [Ste68, Lemma 63] for $S \in \widetilde{\Sigma}$ we have

$$\mathbf{X}_S^F = \begin{cases} \{x_S(t) \mid t \in \mathbb{F}_q\} & \text{if } \Sigma(S) \text{ has type } A_1,\\ \{x_S(t) \mid t \in \mathbb{F}_{q^3}\} & \text{if } \Sigma(S) \text{ has type } A_1^3, \end{cases}$$

and since $\mathbf{G}$ is universal, according to [Ste68, Lemma 64] our group $G = \mathbf{G}^F$ is generated by the fixed point groups $\mathbf{X}_S^F$ with $S \in \widetilde{\Sigma}$.

Concerning simplicity and universal coverings of Steinberg's triality groups ${}^3D_4(q)$ the following statement is well-known:

Proposition 15.2. *For any prime power q the finite group ${}^3D_4(q)$ is simple and constitutes its own universal covering group.*

Proof. See, for instance, [MT11, Table 24.2, Thm. 24.17 and Rmk. 24.19]. □

15.2 Weyl Group and Maximal Tori of ${}^3D_4(q)$

In this section we give a brief overview over the maximal tori existing in G. Moreover, we examine the Weyl group action on the maximal tori of $\mathbf{G}$. We denote by $\mathbf{T}$ the F-stable maximal torus of $\mathbf{G}$ generated by all elements $h_r(t)$, $r \in \Sigma$, $t \in \mathbb{F}^\times$, as in Proposition 4.36 and let $\mathbf{W} = \mathrm{N}_{\mathbf{G}}(\mathbf{T})/\,\mathbf{T}$ be the corresponding Weyl group. Then F acts on $\mathbf{W}$, and by Proposition 4.40 there is a one-to-one correspondence between the F-conjugacy classes of $\mathbf{W}$ and the G-conjugacy classes of F-stable maximal tori of $\mathbf{G}$, such that, up to F-conjugacy, for each F-stable maximal torus $\mathbf{T}'$ of $\mathbf{G}$ there exists a unique $w \in \mathbf{W}$ such that the maximal torus $\mathbf{T}'^F$ of G is $\mathbf{G}$-conjugate to $\mathbf{T}^{wF}$. By Proposition 4.27 the Weyl group $\mathbf{W}$ is isomorphic to $\langle \omega_r \mid r \in \Sigma \rangle$ via $n_r(1)\,\mathbf{T} \mapsto \omega_r$, so we may identify those two groups in the following.

Now we denote by ω_0 the longest element of $\mathbf{W}$ and by r_* the highest root of Σ. Then $\omega_0 = -1$ by [GLS98, Rmk. 1.8.9] and one easily verifies that the root $e_1+e_2 \in \Sigma$ has height 5 with respect to the chosen base Π, which makes it the highest root of Σ. Moreover, we use the short notation

$$h(z_1, z_2, z_3, z_4) := h_{r_1}(z_1)h_{r_2}(z_2)h_{r_3}(z_3)h_{r_4}(z_4)$$

for $z_1, z_2, z_3, z_4 \in \mathbb{F}^\times$. There exist seven G-conjugacy classes of maximal tori in G, whose representatives in $\mathbf{G}$ are given in Table 15.1 below (see, for instance, [DM87, p. 42] and [Kle88a, Table I]).

$w \in W$	$\mathbf{T}^{wF}$	$\mathbf{W}^{wF}$
1	$T_+ = \{h(z_1, z_2, z_1^q, z_1^{q^2}) \mid z_1^{q^3-1} = z_2^{q-1} = 1\}$ $\cong C_{q^3-1} \times C_{q-1}$	D_{12}
$\omega_{1,+} := \omega_{r_*}$	$T_{1,+} = \{h(z, z^{1-q^3}, z^{q^4}, z^{q^2}) \mid z^{(q^3-1)(q+1)} = 1\}$ $\cong C_{(q^3-1)(q+1)}$	$C_2 \times C_2$
$\omega_{1,-} := \omega_0\omega_{1,+}$	$T_{1,-} = \{h(z, z^{1+q^3}, z^{q^4}, z^{q^2}) \mid z^{(q^3+1)(q-1)} = 1\}$ $\cong C_{(q^3+1)(q-1)}$	$C_2 \times C_2$
$\omega_{2,+} := \omega_{r_*}\omega_{r_2}$	$T_{2,+} = \{h(z_1, z_2, z_1^q z_2, (z_1^{-1}z_2)^{q+1}) \mid z_1^{q^2+q+1} = z_2^{q^2+q+1} = 1\}$ $\cong C_{q^2+q+1} \times C_{q^2+q+1}$	$\mathrm{SL}_2(3)$
$\omega_{2,-} := \omega_0\omega_{2,+}$	$T_{2,-} = \{h(z_1, z_2, z_1^{-q} z_2, (z_1 z_2^{-1})^{q-1}) \mid z_1^{q^2-q+1} = z_2^{q^2-q+1} = 1\}$ $\cong C_{q^2-q+1} \times C_{q^2-q+1}$	$\mathrm{SL}_2(3)$
$\omega_3 := \omega_{r_1}\omega_{r_2}$	$T_3 = \{h(z, z^{1+q^3}, z^q, z^{q^2}) \mid z^{q^4-q^2+1} = 1\}$ $\cong C_{q^4-q^2+1}$	C_4
ω_0	$T_- = \{h(z_1, z_2, z_1^{-q}, z_1^{q^2}) \mid z_1^{q^3+1} = z_2^{q+1} = 1\}$ $\cong C_{q^3+1} \times C_{q+1}$	D_{12}

Table 15.1: Maximal tori of ${}^3D_4(q)$

As for the past two series of groups we have studied, for the further course of this last part of the thesis it will be important to understand the action of the Weyl group $\mathbf{W}$ on the maximal torus $\mathbf{T}$. Since for the Weyl group we have

$$\mathbf{W} \cong \langle \omega_r \mid r \in \Sigma \rangle = \langle \omega_r \mid r \in \Pi \rangle,$$

it suffices to consider the action of its generators ω_r, $r \in \Pi$, on $\mathbf{T}$.

Lemma 15.3. *The generators ω_r, $r \in \Pi$, of the Weyl group $\mathbf{W}$ of $\mathbf{G}$ act on the maximal torus $\mathbf{T}$ as follows:*

$$\begin{aligned}
\omega_{r_1} h(t_1, t_2, t_3, t_4)\omega_{r_1}^{-1} &= h(t_1^{-1}t_2, t_2, t_3, t_4),\\
\omega_{r_2} h(t_1, t_2, t_3, t_4)\omega_{r_2}^{-1} &= h(t_1, t_1t_2^{-1}t_3t_4, t_3, t_4),\\
\omega_{r_3} h(t_1, t_2, t_3, t_4)\omega_{r_3}^{-1} &= h(t_1, t_2, t_2t_3^{-1}, t_4),\\
\omega_{r_4} h(t_1, t_2, t_3, t_4)\omega_{r_4}^{-1} &= h(t_1, t_2, t_3, t_2t_4^{-1})
\end{aligned}$$

for all choices of $t_1, t_2, t_3, t_4 \in \mathbb{F}^\times$.

Proof. The element ω_r, $r \in \Pi$, is identified with $n_r(1)\,\mathbf{T} \in \mathbf{W}$, so by Theorem 4.25(x) we have

$$\omega_r h_s(t)\omega_r^{-1} = h_{\omega_r(s)}(t)$$

for $t \in \mathbb{F}^\times$, $s \in \Sigma$. Hence, for $r = r_1$ it holds that ω_{r_1} leaves $h_{r_3}(t)$ and $h_{r_4}(t)$ invariant for all $t \in \mathbb{F}^\times$ as r_1 is orthogonal to both r_3 and r_4. Moreover, it follows from Theorem 4.25(v) that $\omega_{r_1} h_{r_1}(t)\omega_{r_1}^{-1} = h_{-r_1}(t) = h_{r_1}(t^{-1})$ for all $t \in \mathbb{F}^\times$. Now

$$\begin{aligned}
\omega_{r_1}(r_2) &= r_2 - 2\frac{(r_1, r_2)}{(r_1, r_1)} r_1\\
&= e_2 - e_3 - (e_1 - e_2,\, e_2 - e_3)(e_1 - e_2)\\
&= e_2 - e_3 - (-1)(e_1 - e_2)\\
&= r_1 + r_2.
\end{aligned}$$

Thus, again by Theorem 4.25(v) we conclude that $\omega_{r_1} h_{r_2}(t)\omega_{r_1}^{-1} = h_{r_1+r_2}(t) = h_{r_1}(t)h_{r_2}(t)$ for all $t \in \mathbb{F}^\times$. Combining these results we obtain

$$\begin{aligned}
\omega_{r_1} h(t_1, t_2, t_3, t_4)\omega_{r_1}^{-1} &= h_{r_1}(t_1^{-1})h_{r_1}(t_2)h_{r_2}(t_2)h_{r_3}(t_3)h_{r_4}(t_4)\\
&= h_{r_1}(t_1^{-1}t_2)h_{r_2}(t_2)h_{r_3}(t_3)h_{r_4}(t_4)\\
&= h(t_1^{-1}t_2, t_2, t_3, t_4)
\end{aligned}$$

as claimed. The remaining three statements are proven similarly. □

Let us now take a closer look at the Weyl groups associated to the maximal tori of types T_+ and T_-. These maximal tori will occur most frequently in the following and a good understanding of their Weyl groups will prove useful.

Lemma 15.4. *Let $w \in \{1, \omega_0\}$. Then the elements ω_{r_2} and $\omega_{r_1}\omega_{r_3}\omega_{r_4}$ are fixed points of $\mathbf{W}$ under wF.*

Proof. Since $w \in \mathrm{Z}(\mathbf{W})$, it suffices to show that ω_{r_2} and $\omega_{r_1}\omega_{r_3}\omega_{r_4}$ are fixed under the action of F. The element ω_{r_2} is identified with $n_{r_2}(1)\,\mathbf{T}$ and we have

$$F(n_{r_2}(1)) = n_{\rho(r_2)}(1^q) = n_{r_2}(1)$$

by definition of F, so clearly $\omega_{r_2} \in \mathbf{W}^{wF}$. Again by definition of F we moreover have

$$\begin{aligned} F(n_{r_1}(1)n_{r_3}(1)n_{r_4}(1)) &= n_{\rho(r_1)}(1)n_{\rho(r_3)}(1)n_{\rho(r_4)}(1) \\ &= n_{r_3}(1)n_{r_4}(1)n_{r_1}(1). \end{aligned}$$

Now the roots r_1, r_3 and r_4 are pairwise perpendicular, whence the corresponding reflections commute, i.e., $n_{r_3}(1)n_{r_4}(1)n_{r_1}(1)$ and $n_{r_1}(1)n_{r_3}(1)n_{r_4}(1)$ agree modulo $\mathbf{T}$. Consequently, $\omega_{r_1}\omega_{r_3}\omega_{r_4}$ is fixed by F, and hence by wF, as claimed. □

Lemma 15.5. *The element* $\omega_{r_1}\omega_{r_3}\omega_{r_4}\omega_{r_2}$ *has order* 6 *in* $\mathbf{W}$.

Proof. These are direct calculations, e.g., using the relations in Lemma 15.3 or GAP. □

Corollary 15.6. *Let* $w \in \{1, \omega_0\}$. *Then* $\mathbf{W}^{wF} = \langle \omega_{r_2}, \omega_{r_1}\omega_{r_3}\omega_{r_4} \rangle$.

Proof. Following Lemma 15.4 we have $\omega_{r_2}, \omega_{r_1}\omega_{r_3}\omega_{r_4} \in \mathbf{W}^{wF}$. Moreover, by Lemma 15.5 the element $\bar{\omega} := \omega_{r_1}\omega_{r_3}\omega_{r_4}\omega_{r_2} \in \mathbf{W}^{wF}$ has order 6, and, naturally, the order of the reflection ω_{r_2} in $\mathbf{W}^{wF}$ is 2. Now direct calculations show that ω_{r_2} acts on $\langle \bar{w} \rangle$ by inversion. Hence, the claim follows since by Table 15.1 the group $\mathbf{W}^{wF}$ is dihedral of order 12. □

Remark 15.7. In the situation of Lemma 15.4 suppose that q is even. For $r, s \in \Sigma$ it follows from Theorem 4.25(ix) that

$$n_r(1)n_s(1)n_r(1)^{-1} = n_{\omega_r(s)}(1)$$

since the sign $\eta_{r,s}$ has to equal 1 in this case. In particular, the elements $n_{r_1}(1)n_{r_3}(1)n_{r_4}(1)$ and $n_{r_3}(1)n_{r_4}(1)n_{r_1}(1)$ agree (not only modulo $\mathbf{T}$ as in the proof of Lemma 15.4), and for $r = s$ it follows that $n_r(1) = n_{-r}(1)$. Moreover, by Theorem 4.25(xi) we have $n_r(1)^2 = h_r(-1) = h_r(1) = 1$. Following Proposition 4.27 we may choose a preimage $n_0 := n_{r_{i_1}}(1) \cdots n_{r_{i_k}}(1) \in \mathrm{N}_{\mathbf{G}}(\mathbf{T})$ of ω_0 with $r_{i_1}, \ldots, r_{i_k} \in \Sigma$, $k \in \mathbb{N}$. Since ω_0 acts on Σ by sending a root $r \in \Sigma$ to $-r$, it follows that

$$n_0 n_r(1) n_0^{-1} = n_{-r}(1) = n_r(1),$$

so n_0 commutes with all $n_r(1)$, $r \in \Sigma$. Under consideration of the proof of Lemma 15.4 we conclude that $n_{r_2}(1)$ and $n_{r_1}(1)n_{r_3}(1)n_{r_4}(1)$ are fixed points of $\mathbf{G}$ under both F and n_0F if q is even. In particular, we have

$$\begin{aligned} \mathrm{N}_{\mathbf{G}^F}(\mathbf{T}) &= \langle \mathbf{T}^F, n_{r_2}(1), n_{r_1}(1)n_{r_3}(1)n_{r_4}(1) \rangle, \\ \mathrm{N}_{\mathbf{G}^{n_0F}}(\mathbf{T}) &= \langle \mathbf{T}^{n_0F}, n_{r_2}(1), n_{r_1}(1)n_{r_3}(1)n_{r_4}(1) \rangle = \langle \mathbf{T}^{\omega_0F}, n_{r_2}(1), n_{r_1}(1)n_{r_3}(1)n_{r_4}(1) \rangle \end{aligned}$$

in consequence of Corollary 15.6 (cf. also Propositions 4.27 and 4.40).

15.3 Automorphisms of ${}^3D_4(q)$

The automorphism group of the triality group $G = {}^3D_4(q)$ is particularly easy to describe. As usual we define the field automorphism F_p of G as follows:

Definition 15.8. By [Ste68, p. 158] the field automorphism $\mathbb{F} \longrightarrow \mathbb{F}$, $a \longmapsto a^p$, induces an automorphism F_p of the group $\mathbf{G} = D_4(\mathbb{F})$ via

$$F_p\colon\ \mathbf{G} \longrightarrow \mathbf{G}, \quad x_r(t) \longmapsto x_r(t^p), \quad r \in \Sigma,\ t \in \mathbb{F}.$$

Since F_p clearly commutes with F, it also induces an automorphism of $G = \mathbf{G}^F$, which will be called the *field automorphism* F_p of G. Note that its order in $\mathrm{Aut}(G)$ is given by $3f$, where $q = p^f$, since $G \leqslant D_4(q^3)$ as observed before.

Having defined the field automorphism of G we may already give the following statement on the shape of the automorphism group of G:

Proposition 15.9. *For $G = {}^3D_4(q)$ with $q = p^f$, $f \in \mathbb{N}_{>0}$, we have*

$$\mathrm{Aut}(G) = G \rtimes \langle F_p \rangle.$$

In particular, the outer automorphism group of G is cyclic.

Proof. This is a well-known statement, which may, for instance, be derived from [MT11, Table 22.1] together with [GLS98, Th. 2.5.12(a),(d),(f)]. □

Remark 15.10. Let q be even and suppose the notation of Remark 15.7. Since the field automorphism F_p commutes with n_0F, it is an automorphism of $\mathbf{G}^{n_0F}$ and we have

$$\mathrm{Aut}(\mathbf{G}^{n_0F}) = \langle \mathbf{G}^{n_0F}, F_p \rangle.$$

Clearly, F_p also acts on $\mathbf{T}^F$ and $\mathbf{T}^{n_0F} = \mathbf{T}^{\omega_0 F}$, and since by Remark 15.7 we have

$$\mathrm{N}_{\mathbf{G}^F}(\mathbf{T}) = \langle \mathbf{T}^F, n_{r_2}(1), n_{r_1}(1)n_{r_3}(1)n_{r_4}(1) \rangle,$$
$$\mathrm{N}_{\mathbf{G}^{n_0F}}(\mathbf{T}) = \langle \mathbf{T}^{\omega_0F}, n_{r_2}(1), n_{r_1}(1)n_{r_3}(1)n_{r_4}(1) \rangle,$$

we conclude that F_p acts trivially on the quotients $\mathrm{N}_{\mathbf{G}^F}(\mathbf{T})/\,\mathbf{T}^F$ and $\mathrm{N}_{\mathbf{G}^{n_0F}}(\mathbf{T})/\,\mathbf{T}^{\omega_0F}$.

15.4 Special Subgroups of ${}^3D_4(q)$

Let us now introduce certain subgroups of ${}^3D_4(q)$ which will play a role in the description and examination of the weights of ${}^3D_4(q)$.

15.4.1 $G_2(q)$ as a Maximal Subgroup of ${}^3D_4(q)$

As shown in [Kle88a] the group $G = {}^3D_4(q)$ contains maximal subgroups isomorphic to the Chevalley group $G_2(q)$, all of which are conjugate under G.

Proposition 15.11. *For $G = {}^3D_4(q)$ we denote by $\widetilde{G} := G^\tau$ $(= G^{F_q})$ the subgroup of fixed points of G under τ (or equivalently under F_q). Then $\widetilde{G} \cong G_2(q)$.*

Proof. This is well-known and in the notation of Section 15.1 it follows from the fact that G is generated by the groups $\mathbf{X}_S^F$, $S \in \widetilde{\Sigma}$, with

$$(X_S^F)^{F_q} = \{x_S(t) \mid t \in \mathbb{F}_q\},$$

and the generators $x_S(t)$, $S \in \widetilde{\Sigma}$, $t \in \mathbb{F}_q$, satisfy the same relations as for type G_2 (cf., e.g., [GLS98, Thm. 1.12.1, Table 2.4, Thm. 2.4.5 and Thm. 2.4.7]). □

Corollary 15.12. *Let $\widetilde{G} \cong G_2(q)$ be as in Proposition 15.11. Then the field automorphism F_p of G acts on $\widetilde{G}$ as the field automorphism of $G_2(q)$ in Definition 11.10.*

Proof. This follows immediately from the proof of the previous proposition since for $S \in \widetilde{\Sigma}$, $t \in \mathbb{F}_q$, we have $F_p(x_S(t)) = x_S(t^p)$, i.e., the operation of F_p on the generators $x_S(t)$ is as for type G_2 (cf. Definition 11.10). □

15.4.2 Other Subgroups of ${}^3D_4(q)$

In the previous section we observed that G contains maximal subgroups of isomorphism type $G_2(q)$. For further use we now define a few more subgroups, some of which we have already encountered as subgroups of $G_2(q)$ in Part III of this thesis. We will stick to the notation used in [An95].

The first subgroups of G that we introduce here are the groups denoted K_δ and L_δ. These groups already appeared in Section 12.2.2 as subgroups of $G_2(q)$.

Definition 15.13. We fix a maximal subgroup $\widetilde{G} \cong G_2(q)$ of G. Then following [An95, p. 275] for $\delta \in \{\pm 1\}$ there exist maximal subgroups K_δ of $\widetilde{G}$ such that these contain subgroups

$$L_\delta \cong \mathrm{SL}_3(\delta q) = \begin{cases} \mathrm{SL}_3(q) & \text{if } \delta = +1, \\ \mathrm{SU}_3(q) & \text{if } \delta = -1, \end{cases}$$

and K_δ is the extension of L_δ by an involutory outer automorphism. Compare also Section 12.2.2 for the construction of K_δ, L_δ as subgroups of $G_2(q)$.

Later on two further subgroups will play a role in the examination of 3-weights of G. For the definition of these we need the classification of semisimple elements of G given in Section C.1 of Appendix C, where we define a semisimple element $s \in G$ to be *of type* s_i, $i \in \{1, \ldots, 15\}$, if s belongs to the equivalence class $[s_i]$ of semisimple elements of G (see Table C.1 for the distinct equivalence classes).

Definition 15.14. Let $\delta \in \{\pm 1\}$ be such that $q \equiv \delta \bmod 3$. Moreover, let $s \in G$ be a semisimple element of type s_4 if $\delta = +1$ or of type s_9 if $\delta = -1$. Then as in [An95, pp. 274/275] we set $M_\delta := \mathrm{N}_G(\langle s \rangle)$ and $H_\delta := \mathrm{C}_G(s)$.

Remark 15.15. By [An95, p. 275] it holds that $M_\delta / H_\delta \cong C_2$. Moreover, as done in [An95] we may assume that the semisimple element s in the previous definition is chosen such that $K_\delta \leqslant M_\delta$ and $L_\delta \leqslant H_\delta$. This is possible by choosing $1 \neq s \in \mathrm{Z}(L_\delta)$, which then necessarily is of the desired type (cf. [DM87, Tables 2.2a and 2.2b]).

Chapter 16

Action of Automorphisms

In this chapter we examine the action of the automorphisms of the group G on its irreducible Brauer characters as well as on its ℓ-weights for the primes $\ell = 2$ and $\ell = 3$ in the case where $\ell \nmid q$. By similar reasons as for the Chevalley groups $G_2(q)$ we do not need to consider primes $\ell \geqslant 5$ here as will become apparent in the proof of Theorem 18.1.

According to Proposition 15.9 the outer automorphism group of G is cyclic, generated by the field automorphism F_p, whence in order to understand the action of $\mathrm{Aut}(G)$ on the Brauer characters and G-conjugacy classes of weights of G it is enough to study their behaviour under the action of F_p.

16.1 Action on the Brauer Characters of ${}^3D_4(q)$

For an ℓ-block B of G we wish to understand the action of the automorphisms in $\mathrm{Aut}(G)_B$ on the irreducible Brauer characters belonging to B. In order to meet this purpose we use the information on the ℓ-decomposition numbers of G provided by Geck, Himstedt–Huang and Himstedt in [Gec91], [HH13] and [Him07], respectively. We refer to Section C.3 of Appendix C for a short summary of their results. The basic setting of their work is as follows:

For $(\mathbf{G}^*, F^*)$ dual to $(\mathbf{G}, F)$ (cf., e.g., [CE04, Sec. 8.2]) there exists an isomorphism between $G^* := \mathbf{G}^{*F^*}$ and $G = \mathbf{G}^F$. Hence, the classification of semisimple elements of G described in Appendix C.1 may be applied to the semisimple elements of G^*, so any semisimple element in G^* may be uniquely assigned to one of the types $s_1, \dots, s_{15}$.

For semisimple elements $s \in G^*$ there are Lusztig series $\mathcal{E}(G, s) \subset \mathrm{Irr}(G)$ of irreducible characters of G corresponding to s (cf. [CE04, Def. 8.23]) such that by [CE04, Thm. 8.24] we have a disjoint union

$$\mathrm{Irr}(G) = \bigcup_s \mathcal{E}(G, s),$$

where the union is indexed by a complete system of representatives for the G^*-conjugacy classes of semisimple elements in G^*. Moreover, if $s \in G^*$ is a semisimple ℓ'-element, then the set

$$\mathcal{E}_\ell(G, s) := \bigcup_t \mathcal{E}(G, st),$$

where the union runs over all elements $t \in \mathrm{C}_{G^*}(s)$ of ℓ-power order, is a union of ℓ-blocks of G (see, e.g., [CE04, Thm. 9.12]). Furthermore, if $s \in G^*$ is a semisimple ℓ'-element with $s \neq 1$ if $\ell = 2$, then the set $\mathcal{E}(G, s)$ forms a basic set for $\mathcal{E}_\ell(G, s)$ (see [Gec91, p. 3258, p. 3267] and [HH13, p. 2]). For $s = 1$ and $\ell = 2$ the set

$$\mathcal{E}(G, 1) = \{1_G,\ [\varepsilon_1],\ [\varepsilon_2],\ \mathrm{St},\ \rho_1,\ \rho_2,\ {}^3D_4[-1],\ {}^3D_4[1]\}$$

of unipotent characters of G (cf. Section C.2) is not a basic set for $\mathcal{E}_2(G,1)$. However, the set $\mathcal{E}(G,1)\setminus\{{}^3D_4[-1]\}$ forms a basic set for $\mathcal{E}_2(G,1)$ according to [Gec91, p. 3267]. Hence, to understand the ℓ-decomposition numbers for a union $\mathcal{E}_\ell(G,s)$ of ℓ-blocks of G it suffices to describe the ℓ-decomposition numbers for the characters in the Lusztig series $\mathcal{E}(G,s)$ (or in $\mathcal{E}(G,1)\setminus\{{}^3D_4[-1]\}$ if $\ell=2$ and $s=1$).

In the publications specified above, Geck, Himstedt–Huang and Himstedt provide decomposition numbers for the basic sets $\mathcal{E}(G,s)$ (or $\mathcal{E}(G,1)\setminus\{{}^3D_4[-1]\}$ if $\ell=2$ and $s=1$) of $\mathcal{E}_\ell(G,s)$ for semisimple elements s of all possible types (cf. Section C.3 in Appendix C).

16.1.1 The Case $\ell = 2$

Let us first suppose that $\ell=2$. Since we are only interested in the case of non-defining characteristic, we assume moreover that q is odd.

Proposition 16.1. *Let B be a 2-block of G. Then any Brauer character in* $\mathrm{IBr}(B)$ *is left invariant by* $\mathrm{Aut}(G)_B$.

Proof. As described above, there exists a semisimple element $s\in G^*$ of odd order such that $\mathrm{Irr}(B)$ is contained in the union $\mathcal{E}_2(G,s)$ of 2-blocks.

If s is of type s_i for some $i\in\{6,8,11,12,13,14,15\}$, then by Section C.3.1 the set $\mathcal{E}_2(G,s)$ has only one irreducible Brauer character associated to it, so in particular we have $\mathrm{Irr}(B)=\mathcal{E}_2(G,s)$, and $\mathrm{IBr}(B)$ trivially stays invariant under the action of $\mathrm{Aut}(G)_B$.

Suppose that $s=1$. Then by the proof of [An95, (3G)] we have $\mathrm{Irr}(B)=\mathcal{E}_2(G,s)$ and B is the principal 2-block of G. Moreover, the unipotent characters 1_G, $[\varepsilon_1]$, ${}^3D_4[1]$, ρ_1, ρ_2, $[\varepsilon_2]$ and St of G yield a basic set for B, see Section C.3.1. Since $\mathrm{Aut}(G)$ acts on the set of unipotent characters of G and the degrees of these characters are pairwise distinct by Table C.2, we conclude that $\mathrm{Aut}(G)_B$ fixes each unipotent character in B. From this observation and the fact that the decomposition matrix with respect to the basic set given above can be arranged to be unitriangular following Section C.3.1, it follows from Lemma 1.27 that $\mathrm{IBr}(B)$ is stabilized pointwise by $\mathrm{Aut}(G)_B$.

If s is of type s_i for some $i\in\{3,5,7,10\}$, then from the decomposition numbers for $\mathcal{E}_2(G,s)$ in Section C.3.1 we deduce that $\mathcal{E}_2(G,s)$ is a single block, so $\mathrm{Irr}(B)=\mathcal{E}_2(G,s)$, and that, moreover, B contains exactly two irreducible Brauer characters, which have distinct degrees since q is odd, thus greater than 2. Hence, these are left invariant by $\mathrm{Aut}(G)_B$.

If s is of type s_4, then from the decomposition numbers for $\mathcal{E}_2(G,s)$ in Section C.3.1 and the description of the irreducible characters of G in Table C.2 it follows that there are three irreducible Brauer characters associated to $\mathcal{E}_2(G,s)$, and these have degrees

$$\begin{aligned} A&,\\ q(q+1)A&,\\ (q^3-1)A&, \end{aligned}$$

where A is some positive integer depending on q. Clearly, these are pairwise distinct for any prime power q, whence we conclude that $\mathrm{Aut}(G)_B$ acts trivially on $\mathrm{IBr}(B)$.

If s is of type s_9, then there are exactly three irreducible Brauer characters associated to $\mathcal{E}_2(G,s)$, which have degrees

$$\begin{aligned} A&,\\ q(q-1)A&,\\ (q^3-cq(q-1)-1)A&, \end{aligned}$$

for $c \in \{1, 2\}$ and some $A \in \mathbb{Z}_{>0}$ depending on q by Table C.2 and Section C.3.1. Easy calculations show that these are pairwise distinct for any prime power q. Hence, it follows again that the action of $\mathrm{Aut}(G)_B$ on $\mathrm{IBr}(B)$ is trivial.

Since s has to have odd order, it cannot be of type s_2. Thus, we have regarded all possible situations, which completes the proof. $\square$

16.1.2 The Case $\ell = 3$

Assume now that $\ell = 3$ and q is not a power of 3.

Proposition 16.2. *Let B be a 3-block of G. Then any Brauer character in* $\mathrm{IBr}(B)$ *is left invariant by* $\mathrm{Aut}(G)_B$.

Proof. Similarly as in the proof of Proposition 16.1 above, we let $s \in G^*$ be a semisimple $3'$-element such that $\mathrm{Irr}(B)$ is contained in $\mathcal{E}_3(G, s)$.

If s is of type s_2, then q must be odd, and following Table C.2 and Section C.3.2 there exists some $A \in \mathbb{Z}_{>0}$ depending on q such that the character degrees of the irreducible Brauer characters in $\mathcal{E}_3(G, s)$ are given by

$$\begin{array}{r} A, \\ q^3A, \\ qA, \\ q^4A, \end{array} \quad \text{or} \quad \begin{array}{r} A, \\ (q^3-1)A, \\ (q-1)A, \\ (q-1)(q^3-1)A, \end{array}$$

for $q \equiv 1 \bmod 3$ or $q \equiv -1 \bmod 3$, respectively. Since q is odd, so in particular $q > 2$, these are pairwise distinct, and hence left invariant by $\mathrm{Aut}(G)_B$.

Suppose that s is of type s_i with $i \in \{6, 8, 11, 12, 13, 14, 15\}$. Then from Section C.3.2 it follows that $|\,\mathrm{IBr}(B)| = 1$, so $\mathrm{Aut}(G)_B$ clearly acts trivially on $\mathrm{IBr}(B)$.

Assume now that s is of type s_i with $i \in \{3, 5, 7, 10\}$. Then $|\mathcal{E}(G, s)| = 2$. By Table C.2 and Section C.3.2 the two irreducible Brauer characters belonging to 3-blocks contained in $\mathcal{E}_3(G, s)$ have pairwise distinct degrees if $q \equiv 1 \bmod 3$ and are hence left invariant by $\mathrm{Aut}(G)_B$. If $q \equiv -1 \bmod 3$, then there exists $A \in \mathbb{Z}_{>0}$ depending on q such that the irreducible Brauer characters associated to $\mathcal{E}_3(G, s)$ have degrees

$$\begin{array}{r} A, \\ (q^3-1)A, \end{array} \quad \text{or} \quad \begin{array}{r} A, \\ (q-1)A, \end{array}$$

for $i \in \{3, 7\}$ or $i \in \{5, 10\}$, respectively. Hence, for $i \in \{3, 7\}$ their degrees are distinct, so the action of $\mathrm{Aut}(G)_B$ on $\mathrm{IBr}(B)$ is trivial also for $q \equiv -1 \bmod 3$. For $i \in \{5, 10\}$ the degrees of the irreducible Brauer characters in $\mathcal{E}_3(G, s)$ are distinct if and only if $q > 2$. However, following Table C.1 for $q = 2$ there do not exist any semisimple elements of type s_5, and the semisimple elements of type s_{10} are of order 9 in this case, hence no $3'$-elements. Consequently, this situation cannot occur. We conclude that also in the case $q \equiv -1 \bmod 3$ the action of $\mathrm{Aut}(G)_B$ on $\mathrm{IBr}(B)$ is trivial due to pairwise distinct irreducible character degrees.

For s of type s_4 the character values provided by Table C.2 and the 3-decomposition numbers for $\mathcal{E}_3(G, s)$ given in Section C.3.2 show that there are exactly three irreducible Brauer characters associated to $\mathcal{E}_3(G, s)$, which have degrees

$$\begin{array}{r} A, \\ (q(q+1)-1)A, \\ (q^3-q(q+1)+1)A, \end{array} \quad \text{or} \quad \begin{array}{r} A, \\ q(q+1)A, \\ (q^3-1)A, \end{array}$$

for $q \equiv 1 \bmod 3$ and $q \equiv -1 \bmod 3$, respectively, where $A \in \mathbb{Z}_{>0}$ depends on q. These are pairwise distinct for any prime power q, as one easily verifies. Hence, the action of $\mathrm{Aut}(G)_B$ on $\mathrm{IBr}(B)$ is trivial.

Similarly, for s of type s_9 one has three irreducible Brauer characters associated to $\mathcal{E}_3(G,s)$, and these have pairwise distinct degrees for any prime power $q \equiv 1 \bmod 3$. For $q \equiv -1 \bmod 3$ there exists $A \in \mathbb{Z}_{>0}$ depending on q such that the degrees of the irreducible Brauer characters belonging to $\mathcal{E}_3(G,s)$ have degrees

$$\begin{aligned} A,\\ q(q-1)A,\\ (q^3-eq(q-1)-1)A, \end{aligned}$$

for some integer $e \geqslant 0$. One easily shows that these are pairwise distinct whenever $q > 2$. For $q = 2$ Table C.1 shows that any semisimple element of type s_9 is of order 3, so this situation cannot occur. Thus, $\mathrm{Aut}(G)_B$ acts trivially on $\mathrm{IBr}(B)$ also in this case.

Let now $s = 1$. Following Section C.3.2 the unipotent characters of G form a basic set for $\mathcal{E}_3(G,1)$, and the decomposition matrix with respect to these characters is unitriangular. Since the unipotent characters of G have pairwise distinct degrees by Table C.2, they are left invariant by $\mathrm{Aut}(G)$ as in the proof of Proposition 16.1. Then by unitriangularity of the decomposition matrix corresponding to the unipotent characters of G it follows from Lemma 1.27 that $\mathrm{Aut}(G)_B$ stabilizes $\mathrm{IBr}(B)$ pointwise. □

16.2 Action on the Weights of ${}^3D_4(q)$

In this section we study the action of the automorphisms of G on the G-conjugacy classes of its ℓ-weights for $\ell \in \{2,3\}$. This examination is conducted blockwise, the main distinction being made between ℓ-blocks of abelian defect and ℓ-blocks of non-abelian defect.

16.2.1 The Case $\ell = 2$

Here we assume once more that $\ell = 2$ and q is odd. Moreover, we let $\varepsilon \in \{\pm 1\}$ be defined such that $q \equiv \varepsilon \bmod 4$.

16.2.1.1 Blocks of Non-Abelian Defect

The 2-weights of G for the principal 2-block B_0 have been described by J. An in [An95] as in Proposition 16.3 below. Note that there are significant parallels to the situation of the principal 2-block of $G_2(q)$ (cf. Proposition 12.16).

Proposition 16.3. *Suppose that $B = B_0$ is the principal 2-block of G. Then $|\mathcal{W}(B)| = 7$. Moreover, if (R,φ) is a B-weight of G, then up to G-conjugation one of the following holds:*

(i) $R \cong (C_2)^3$*, an elementary abelian group of order* 8*,* $\mathrm{N}_G(R)/R \cong \mathrm{GL}_3(2)$*, and* φ *is the inflation of the Steinberg character of* $\mathrm{N}_G(R)/R \cong \mathrm{GL}_3(2)$*. There exists exactly one G-conjugacy class of such B-weights in G.*

(ii) $R = \langle \mathcal{O}_2(T_\varepsilon), \rho\rangle$ *for some involution* $\rho \in \mathrm{N}_G(T_\varepsilon)$*,* $\mathrm{N}_G(R)/R \cong \mathfrak{S}_3$*, and* φ *is the inflation of the unique irreducible character of* $\mathrm{N}_G(R)/R \cong \mathfrak{S}_3$ *of degree* 2*. There exists exactly one G-conjugacy class of such B-weights in G.*

(iii) $R \in \mathrm{Syl}_2(G)$ *is a Sylow* 2*-subgroup of* G*,* $\mathrm{N}_G(R) = R$*, and* φ *is the trivial character of* $\mathrm{N}_G(R)$*. There exists exactly one G-conjugacy class of such B-weights in G.*

(iv) $q \equiv \pm 1 \bmod 8$, $R \cong 2_+^{1+2} \circ D_{(q^2-1)_2}$, *the central product of an extraspecial* 2*-group of order* 8 *and plus type with a dihedral group of order* $(q^2-1)_2$, $\mathrm{N}_G(R)/R \cong \mathfrak{S}_3$, *and* φ *is the inflation of the unique irreducible character of* $\mathrm{N}_G(R)/R \cong \mathfrak{S}_3$ *of degree* 2*. There exist exactly two* G*-conjugacy classes of such* B*-weights in* G*.*

(v) $q \equiv \pm 1 \bmod 8$, $R \cong 2_+^{1+4}$, *an extraspecial* 2*-group of order* 2^{1+4} *and plus type,* $\mathrm{N}_G(R)/R \cong \mathfrak{S}_3 \times \mathfrak{S}_3$, *and* φ *is the inflation of the unique irreducible character of* $\mathrm{N}_G(R)/R \cong \mathfrak{S}_3 \times \mathfrak{S}_3$ *of degree* 4*. There exist exactly two* G*-conjugacy classes of such* B*-weights in* G*.*

(vi) $q \equiv \pm 3 \bmod 8$, $R \cong 2_+^{1+4}$, $\mathrm{N}_G(R)/R \cong (C_3 \times C_3) \rtimes C_2$ *with the action of the non-trivial element of* C_2 *on* $C_3 \times C_3$ *given by inversion, and* φ *is the inflation of one of the four irreducible characters of* $\mathrm{N}_G(R)/R$ *of degree 2. There exists exactly one* G*-conjugacy class of such* R *in* G*.*

Proof. This is a consequence of [An95, (2B)–(2D), (3C)] and the proofs of [An95, (3G)] and [An94a, (3H)]. □

Remark 16.4. If (R,φ) is a B_0-weight for G with R as in (iv), (v) or (vi) of Proposition 16.3, then as in the proof of [An95, (3C)] we may assume that R is contained in a maximal subgroup $\widetilde{G}$ of G isomorphic to $G_2(q)$ such that

$$\mathrm{N}_{\widetilde{G}}(R) \leqslant \mathrm{N}_{\widetilde{G}}(\mathrm{Z}(R)) = \mathrm{C}_{\widetilde{G}}(\mathrm{Z}(R)) \cong \mathrm{SO}_4^+(q)$$

and R is 2-radical in $\mathrm{C}_{\widetilde{G}}(\mathrm{Z}(R)) \cong \mathrm{SO}_4^+(q)$ with $\mathrm{N}_{\widetilde{G}}(R)/R \cong \mathrm{N}_G(R)/R$ by [An94a, (2B)]. Thus, we observe that the normalizer $\mathrm{N}_G(R)$ is already contained in $\widetilde{G}$.

Proposition 16.5. *Let* $B = B_0$ *be the principal* 2*-block of* G*. Then* $\mathrm{Aut}(G)_B = \mathrm{Aut}(G)$ *acts trivially on* $\mathcal{W}(B)$*.*

Proof. Let (R,φ) be a B-weight. If R is as in Proposition 16.3(i), (ii) or (iii), then the G-conjugacy class of (R,φ) is uniquely determined by the isomorphism type of R. Hence, it stays invariant under $\mathrm{Aut}(G)$.

Suppose now that R is as in (iv), (v) or (vi) of Proposition 16.3. Following Remark 16.4 we may assume that $\mathrm{N}_G(R)$ is contained in the subgroup $\widetilde{G} \cong G_2(q)$, where $\widetilde{G}$ is as in Proposition 15.11. According to Proposition 15.9 we have $\mathrm{Aut}(G) = \langle G, F_p\rangle$ with F_p stabilizing $\widetilde{G}$ and acting on it as a field automorphism of $G_2(q)$ by Corollary 15.12. Now let a be an element of $\mathrm{Aut}(G)$. Since we are only interested in the action of automorphisms of G on the G-conjugacy classes of B-weights, inner automorphisms of G do not play a role and we may assume that $a = F_p^i$ for some $i \in \mathbb{N}$. Now

$$R \leqslant \mathrm{N}_G(R) \leqslant \widetilde{G},$$

so we may regard (R,φ) as a 2-weight for $\widetilde{G}$ and $a = F_p^i$ as a power of the field automorphism of $\widetilde{G}$. Hence, in consequence of Proposition 12.17 the 2-weight (R,φ) is stabilized by a up to $\widetilde{G}$-conjugation, so in particular the G-conjugacy class of (R,φ) is left invariant by a, and thus by all of $\mathrm{Aut}(G)$ as claimed. □

For the non-principal 2-blocks of G of non-abelian defect J. An showed the following:

Proposition 16.6. *Let B be a non-principal 2-block of G with non-abelian defect groups. Then one of the following situations occurs:*

(i) It holds that $|\mathcal{W}(B)| = 3$, and moreover, if (R_1, φ_1) and (R_2, φ_2) are non-conjugate B-weights, then one has $R_1 \ncong R_2$.

(ii) It holds that $|\mathcal{W}(B)| = 2$, and moreover, if (R_1, φ_1) and (R_2, φ_2) are non-conjugate B-weights, then one has $R_1 \ncong R_2$.

Proof. This follows from the proof of [An95, (3G)]. $\square$

Proposition 16.7. *Let B be a non-principal 2-block of G with non-abelian defect groups. Then $\mathrm{Aut}(G)_B$ acts trivially on $\mathcal{W}(B)$.*

Proof. Let (R, φ) be a B-weight of G and $a \in \mathrm{Aut}(G)_B$. Then $(R, \varphi)^a = (R^a, \varphi^a)$ is a B-weight with $R^a \cong R$. Hence, Proposition 16.6 implies that the B-weights (R, φ) and $(R, \varphi)^a$ are G-conjugate. $\square$

16.2.1.2 Blocks of Abelian Defect

In this section we study the action of automorphisms of G on the 2-weights associated to 2-blocks with non-cyclic abelian defect groups. Recall that the case of cyclic defect is covered by Koshitani–Späth [KS14, KS15].

Proposition 16.8. *Suppose that B is a 2-block of G which has a non-cyclic abelian defect group D. Then the following statements hold:*

(i) The centralizer $\mathrm{C}_G(D) =: T$ is a maximal torus of G of type T_ε, $D = \mathcal{O}_2(T)$ and $\mathrm{N}_G(D) = \mathrm{N}_G(T)$ with $\mathrm{N}_G(T)/T \cong D_{12}$. In particular, in G there exists only one G-conjugacy class of subgroups isomorphic to D.

(ii) Consider $\mathrm{Irr}(T)$ as an abelian group and fix an isomorphism $\hat{}: T \longrightarrow \mathrm{Irr}(T)$. Then up to G-conjugation there exists a unique $2'$-element $s \in T$ and a 2-block $b \in \mathrm{Bl}_2(T)$ of $T = \mathrm{C}_G(D)$ with $b^G = B$ such that the linear character $\theta := \hat{s} \in \mathrm{Irr}(T)$ is the canonical character of b.

(iii) For θ and s as in (ii) we have $\mathrm{N}_G(T)_\theta/T \cong \mathrm{C}_{W(T)}(s)$, where $W(T) := \mathrm{N}_G(T)/T$.

Proof. For (i) we observe that according to [DM87, Prop. 5.8] the centralizer $\mathrm{C}_G(D) =: T$ is a maximal torus of G such that $D = \mathcal{O}_2(T)$. Since maximal tori of G of types $T_{1,\pm}$ and T_3 are cyclic and maximal tori of types $T_{2,\pm}$ have odd order, it follows that T must be of type T_δ for some $\delta \in \{\pm 1\}$. But by [An95, (2D)(a)] the centralizer of $\mathcal{O}_2(T_\delta)$ in G is a maximal torus if and only if $\delta = \varepsilon$, so T must be of type T_ε. Moreover, by the same reference it follows that $\mathrm{N}_G(D) = \mathrm{N}_G(T)$ with $\mathrm{N}_G(T)/T \cong D_{12}$. Since D is the unique Sylow 2-subgroup of the maximal torus T, there exists only one G-conjugacy class of subgroups of G isomorphic to D.

Finally, the statements given in (ii) and (iii) can be found as parts (b) and (c) of Proposition 5.8 in [DM87]. $\square$

Proposition 16.9. *Let B be a 2-block of G of non-cyclic abelian defect. Then $\mathrm{Aut}(G)_B$ acts trivially on $\mathcal{W}(B)$.*

Proof. Suppose that (R, φ) is a B-weight. Since B has abelian defect, it follows from Lemma 2.11 that R is a defect group of B. Hence, due to Proposition 16.8(i) we have $R = \mathcal{O}_2(T)$ for some maximal torus T of G of type T_ε and $\mathrm{N}_G(R) = \mathrm{N}_G(T)$. Let $s \in T$, $\theta = \hat{s}$ and $b \in \mathrm{Bl}_2(T)$ be as in Proposition 16.8(ii) such that $\mathrm{N}_G(T)_\theta/T \cong \mathrm{C}_{W(T)}(s)$ for $W(T) = \mathrm{N}_G(T)/T$.

Since B is non-principal and s is of odd order, it follows that $s^2 \neq 1$. Hence, according to [DM87, Table 3.4] up to isomorphism we have $\mathrm{C}_{W(T)}(s) \in \{\{1\}, C_2, \mathfrak{S}_3\}$. However, b has defect group R, and since $B = b^G$ has defect group R as well, Theorem 1.10 implies that the index $|\,\mathrm{N}_G(T)_\theta\colon T|$ is prime to 2. We conclude that $\mathrm{N}_G(T)_\theta = T$.

Following Construction 2.10 we hence have $\varphi = \theta^{\mathrm{N}_G(T)}$, so φ is uniquely determined by R and B. If $a \in \mathrm{Aut}(G)_B$, then as above the first component R^a of the B-weight $(R, \varphi)^a$ must be a defect group of B, so in particular it is G-conjugate to R. In summary, we conclude that the G-conjugacy class of (R, φ) is left invariant by $\mathrm{Aut}(G)_B$. □

16.2.2 The Case $\ell = 3$

Here we study the action of $\mathrm{Aut}(G)$ on the 3-weights of G. Throughout this section we assume that $3 \nmid q$ and denote by ε the unique element in $\{\pm 1\}$ with $q \equiv \varepsilon \bmod 3$.

If T is a maximal torus of G of type T_ε, then in consequence of [An95, (1A)] it holds that $\mathrm{N}_G(T)/T \cong D_{12}$. We let $\widetilde{G} \cong G_2(q)$ be as in Proposition 15.11 and assume that T is such that $\widetilde{T} := T \cap \widetilde{G}$ is a maximal torus of $\widetilde{G}$ isomorphic to $C_{q-\varepsilon} \times C_{q-\varepsilon}$. Then as in Section 12.2.2 it also holds that $\mathrm{N}_{\widetilde{G}}(\widetilde{T})/\widetilde{T} \cong D_{12}$. This allows us to prove the following two statements:

Lemma 16.10. *Let $q \geqslant 5$. Suppose that $\widetilde{G} \cong G_2(q)$ is as in Proposition 15.11 and let T be a maximal torus of G of type T_ε such that $\widetilde{T} := T \cap \widetilde{G}$ is a maximal torus of $\widetilde{G}$ with $\widetilde{T} \cong C_{q-\varepsilon} \times C_{q-\varepsilon}$. Then it holds that $\mathrm{C}_G(\widetilde{T}) = T$.*

Proof. Clearly, $T \subseteq \mathrm{C}_G(\widetilde{T})$. Moreover, $\mathrm{C}_G(\widetilde{T}) \subseteq \mathrm{C}_G(x)$ for every $x \in \widetilde{T}$. If $\widetilde{T}$ contains a regular element g of T, i.e., an element g such that $\mathrm{C}_G(g) = T$, then $\mathrm{C}_G(\widetilde{T}) \subseteq \mathrm{C}_G(g) = T$, so the claim follows. Since $q \equiv \varepsilon \bmod 3$, it follows that $q - \varepsilon \geqslant 6$, whence according to Table 15.1 the torus T contains elements represented by $h(z, z^{-1}, z, z)$ with $z \in \mathbb{F}^\times$ of order $q - \varepsilon$ (these are of type s_6 or s_{15} for $\varepsilon = +1$ and $\varepsilon = -1$, respectively), and by [Kle88a, Table II] these elements are regular. Since for an element $x \in T$ we have $x \in \widetilde{T}$ if and only if $x^{q-\varepsilon} = 1$, elements of T represented by such $h(z, z^{-1}, z, z)$ already lie in $\widetilde{T}$. Hence, $\widetilde{T}$ contains regular elements of T, which proves the claim. □

Proposition 16.11. *In the situation of Lemma 16.10 we have $\mathrm{N}_G(T) = \mathrm{N}_{\widetilde{G}}(\widetilde{T})T$.*

Proof. We prove that $\mathrm{N}_{\widetilde{G}}(\widetilde{T})$ normalizes T. Then $\mathrm{N}_{\widetilde{G}}(\widetilde{T})T \subseteq \mathrm{N}_G(T)$, and since $\mathrm{N}_G(T)/T$ and $\mathrm{N}_{\widetilde{G}}(\widetilde{T})/\widetilde{T}$ are isomorphic to D_{12} and

$$\mathrm{N}_{\widetilde{G}}(\widetilde{T})T/T \cong \mathrm{N}_{\widetilde{G}}(\widetilde{T})/(\mathrm{N}_{\widetilde{G}}(\widetilde{T}) \cap T) = \mathrm{N}_{\widetilde{G}}(\widetilde{T})/\widetilde{T} \cong D_{12},$$

we obtain equality. By Lemma 16.10 we have $\mathrm{C}_G(\widetilde{T}) = T$, so every element of G which normalizes $\widetilde{T}$ stabilizes $\mathrm{C}_G(\widetilde{T}) = T$. In particular, $\mathrm{N}_{\widetilde{G}}(\widetilde{T}) \subseteq \mathrm{N}_G(T)$ as claimed. □

16.2.2.1 Blocks of Non-Abelian Defect

As for the case $\ell = 2$, we first consider the 3-blocks of G with non-abelian defect groups. For the principal 3-block B_0 the following has been shown to hold by J. An:

Proposition 16.12. *Suppose that $B = B_0$ is the principal 3-block of G. Then $|\mathcal{W}(B)| = 7$. Moreover, if (R, φ) is a B-weight of G, then up to G-conjugation one of the following holds:*

(i) R is an extraspecial group of order 3^{1+2} and exponent 3 such that $R \leqslant L_\varepsilon \leqslant H_\varepsilon$, $\mathrm{C}_G(R) = \mathrm{Z}(H_\varepsilon) \cong C_{q^2+\varepsilon q+1}$, $\mathrm{N}_G(R) = \mathrm{N}_{M_\varepsilon}(R)$, $\mathrm{N}_{H_\varepsilon}(R)/R\,\mathrm{C}_G(R) \cong \mathrm{Sp}_2(3)$, and $|\,\mathrm{N}_G(R) \colon \mathrm{N}_{H_\varepsilon}(R)| = 2$.
The character φ is the inflation of one of the two extensions of the Steinberg character of $\mathrm{N}_{H_\varepsilon}(R)/R\,\mathrm{C}_G(R) \cong \mathrm{Sp}_2(3)$ to $\mathrm{N}_G(R)/R\,\mathrm{C}_G(R)$.
There exists exactly one G-conjugacy class of such R in G.

(ii) $R \in \mathrm{Syl}_3(G)$, $\mathrm{C}_G(R) \cong C_{q^2+\varepsilon q+1}$, $\mathrm{N}_G(R)/R\,\mathrm{C}_G(R) \cong C_2 \times C_2$, and φ is the inflation of one of the four irreducible characters of $\mathrm{N}_G(R)/R\,\mathrm{C}_G(R)$.

(iii) $R = \mathcal{O}_3(T)$ for a maximal torus T of G of type $T_{2,\varepsilon}$, $\mathrm{N}_G(R)/R\,\mathrm{C}_G(R) \cong \mathrm{Sp}_2(3)$ with $\mathrm{C}_G(R) = T$, and φ is the inflation of the Steinberg character of $\mathrm{N}_G(R)/R\,\mathrm{C}_G(R)$.
There exists exactly one G-conjugacy class of such B-weights in G.

Proof. This follows from [An95, (1A)] and the proof of [An95, (3B)]. □

In order to understand the action of $\mathrm{Aut}(G)$ on the 3-weights associated to the principal 3-block of G we need the following two observations:

Lemma 16.13. *Let $R \leqslant G$ be an extraspecial group of order 3^{1+2} and exponent 3. Then there exists exactly one G-conjugacy class of subgroups of G isomorphic to R. Moreover, if $R \leqslant L_\varepsilon \leqslant K_\varepsilon \leqslant M_\varepsilon$, then*

$$\mathrm{N}_G(R)/R\,\mathrm{C}_G(R) \cong \mathrm{Sp}_2(3) \rtimes \langle \rho \rangle$$

for some involution $\rho \in K_\varepsilon \setminus L_\varepsilon$ such that $M_\varepsilon = \langle H_\varepsilon, \rho \rangle$.

Proof. This is [An95, (1C)]. □

For the proof of Lemma 16.14 below we recall the following notation: If $x \in G$, then we denote by $c_x \in \mathrm{Aut}(G)$ the inner automorphism of G given by conjugation with x, i.e., $c_x(g) = xgx^{-1}$ for all $g \in G$.

Lemma 16.14. *Let T be a maximal torus of G of type T_ε. Then there exists an automorphism $a \in \mathrm{Aut}(G)_T$ with $\mathrm{Out}(G) = \langle a\,\mathrm{Inn}(G) \rangle$ which acts trivially on $\mathrm{N}_G(T)/T$.*

Proof. First of all we note that if the claim holds for one fixed T, then it also holds for all its G-conjugates, and hence for any maximal torus of G of type T_ε.

Suppose first that $q < 5$, that is, $q \in \{2, 4\}$ since $q \equiv \varepsilon \bmod 3$. (The following proof does in fact work for any even q as we will see. However, we only need it for $q \in \{2, 4\}$ as we have a different proof for arbitrary $q \geqslant 5$ relying on Lemma 16.10.) Let $F_+ := F$ if $\varepsilon = 1$ and $F_- := n_0 F$ if $\varepsilon = -1$, where n_0 is as in Remark 15.7. As observed above we have $\mathrm{N}_G(T)/T \cong D_{12}$, and since

$$\mathrm{N}_G(T) \cong \mathrm{N}_{\mathbf{G}^{F_\varepsilon}}(\mathbf{T}^{F_\varepsilon}) \geqslant \mathrm{N}_{\mathbf{G}^{F_\varepsilon}}(\mathbf{T})$$

with $\mathrm{N}_{\mathbf{G}^{F_\varepsilon}}(\mathbf{T})/\,\mathbf{T}^{F_\varepsilon} \cong D_{12}$ by Table 15.1, it follows that $\mathrm{N}_{\mathbf{G}^{F_\varepsilon}}(\mathbf{T}) = \mathrm{N}_{\mathbf{G}^{F_\varepsilon}}(\mathbf{T}^{F_\varepsilon})$ here. In particular, we have

$$\mathrm{N}_{\mathbf{G}^{F_\varepsilon}}(\mathbf{T}^{F_\varepsilon}) = \langle \mathbf{T}^{F_\varepsilon}, n_{r_2}(1), n_{r_1}(1)n_{r_3}(1)n_{r_4}(1)\rangle$$

by Remark 15.7 with the field automorphism F_p stabilizing $\mathbf{T}^{F_\varepsilon}$ and acting trivially on the quotient $\mathrm{N}_{\mathbf{G}^{F_\varepsilon}}(\mathbf{T}^{F_\varepsilon})/\,\mathbf{T}^{F_\varepsilon}$ following Remark 15.10. Since F_p generates the outer automorphism group of the group $\mathbf{G}^{F_\varepsilon}$, which is $\mathbf{G}$-conjugate to G by Corollary 4.35, the claim follows in this case.

Now suppose that $q \geqslant 5$ and let $\widetilde{G} \cong G_2(q)$ be as in Proposition 15.11. If necessary, after replacing T by a G-conjugate we may assume that $\widetilde{T} := T \cap \widetilde{G}$ is a maximal torus of $\widetilde{G}$ isomorphic to $C_{q-\varepsilon} \times C_{q-\varepsilon}$. In consequence of Remark 11.15 in conjunction with the observations at the beginning of Section 12.2.2, and possibly after $\widetilde{G}$-conjugation, there exists an automorphism $\tilde{a}$ of $\widetilde{G}$ (e.g. the field automorphism of $\widetilde{G}$) such that

- $\mathrm{Out}(\widetilde{G}) = \langle \tilde{a}\,\mathrm{Inn}(\widetilde{G})\rangle$,
- $\tilde{a}(\widetilde{T}) = \widetilde{T}$, and
- every element of $\mathrm{N}_{\widetilde{G}}(\widetilde{T})$ can be written in the form nt with $t \in \widetilde{T}$ and $n \in \mathrm{N}_{\widetilde{G}}(\widetilde{T})$ such that $\tilde{a}(n) = n$.

Now the field automorphism F_p acts on $\widetilde{G}$ and hence, since $\mathrm{Out}(\widetilde{G}) = \langle \tilde{a}\,\mathrm{Inn}(\widetilde{G})\rangle$, there is some $k \in \mathbb{N}$ and $g \in \widetilde{G}$ such that F_p acts as $\tilde{a}^k c_g$ on $\widetilde{G}$. Then $a := F_p c_{g^{-1}}$ fixes $\widetilde{G}$ and $\widetilde{T}$, and hence T since $T = \mathrm{C}_G(\widetilde{T})$ by Lemma 16.10. Moreover, $\mathrm{Out}(G) = \langle a\,\mathrm{Inn}(G)\rangle$.

Now let $x \in \mathrm{N}_G(T)$. Following Proposition 16.11 it holds that $\mathrm{N}_G(T) = \mathrm{N}_{\widetilde{G}}(\widetilde{T})T$, and hence $x = ms$ for some $m \in \mathrm{N}_{\widetilde{G}}(\widetilde{T})$ and $s \in T$. Now we have $m = nt$ for suitable elements $n \in \mathrm{N}_{\widetilde{G}}(\widetilde{T})$ such that $\tilde{a}(n) = n$ and $t \in \widetilde{T}$. But then $x = nts \in nT$ and

$$a(x) = a(nts) = \tilde{a}^k(n)a(ts) = na(ts) \in nT,$$

so $a(x)$ and x coincide modulo T, i.e., a acts trivially on $\mathrm{N}_G(T)/T$. □

Proposition 16.15. *Let $B = B_0$ be the principal 3-block of G. Then $\mathrm{Aut}(G)_B = \mathrm{Aut}(G)$ acts trivially on $\mathcal{W}(B)$.*

Proof. We prove this claim by going through the distinct cases in Proposition 16.12. Let (R, φ) be a B-weight of G. If $R = \mathcal{O}_3(T)$ for a maximal torus T of G of type $T_{2,\varepsilon}$, then by Proposition 16.12(iii) the G-conjugacy class of (R, φ) is uniquely determined by B and the isomorphism type of R. Hence, it stays invariant under $\mathrm{Aut}(G)$.

Suppose now that R is an extraspecial group of order 3^{1+2} and exponent 3 as in Proposition 16.12(i). Then by Lemma 16.13 the group R is uniquely determined up to G-conjugation. Hence, we may assume that R is contained in the maximal subgroup of G isomorphic to $G_2(q)$ as in Proposition 15.11 and has the same form as in the proof of Proposition 12.32 for case (iii). Then as in that proof, the element ρ in Lemma 16.13 can be chosen such that $\rho \in \mathrm{N}_{K_\varepsilon}(R)$ and F_p acts trivially on ρ. Moreover, R is stabilized by F_p. According to Proposition 16.12(i) the character φ corresponds to one of the two extensions of the Steinberg character of $\mathrm{N}_{H_\varepsilon}/R\,\mathrm{C}_G(R) \cong \mathrm{Sp}_2(3)$ to $\mathrm{N}_G(R)/R\,\mathrm{C}_G(R)$. Let us denote these two extensions by St^+ and St^-. By Proposition 16.12(i) and Lemma 16.13 we have $\mathrm{N}_G(R) = \langle \mathrm{N}_{H_\varepsilon}(R), \rho\rangle$, so

$$\mathrm{St}^-(\rho R\,\mathrm{C}_G(R)) = -\,\mathrm{St}^+(\rho R\,\mathrm{C}_G(R))$$

by Gallagher's theorem, Theorem 1.21. Now $(R, \varphi)^{F_p} = (R, \varphi^{F_p})$ is again a B-weight, so from Proposition 16.12 it follows that φ^{F_p} corresponds to one of St^+ and St^-. It holds that $\mathrm{St}^{\pm}(\rho R\,\mathrm{C}_G(R)) \neq 0$ since $\mathrm{St}^{\pm}$ is of degree three and ρ has order two, so $\mathrm{St}^{\pm}(\rho R\,\mathrm{C}_G(R))$ is the sum of three square roots of unity, i.e., $\mathrm{St}^{\pm}(\rho R\,\mathrm{C}_G(R)) \in \{\pm 1, \pm 3\}$. But then

$$(\mathrm{St}^+)^{F_p}(\rho R\,\mathrm{C}_G(R)) = \mathrm{St}^+(\rho R\,\mathrm{C}_G(R)) \neq -\,\mathrm{St}^+(\rho R\,\mathrm{C}_G(R)) = \mathrm{St}^-(\rho R\,\mathrm{C}_G(R)),$$

so we conclude that $\varphi^{F_p} = \varphi$, and hence (R, φ) stays invariant under the action of F_p. In particular, the G-conjugacy class of (R, φ) is fixed by $\mathrm{Aut}(G)$.

Finally, we suppose that $R \in \mathrm{Syl}_3(G)$. Then from Proposition 16.12 it is known that

$$\mathrm{N}_G(R)/R\,\mathrm{C}_G(R) \cong C_2 \times C_2.$$

We prove that there exists an automorphism $a \in \mathrm{Aut}(G)_R$ which acts trivially on the quotient group $\mathrm{N}_G(R)/R\,\mathrm{C}_G(R)$ and which generates the outer automorphisms of G, i.e., $\mathrm{Out}(G) = \langle a\,\mathrm{Inn}(G)\rangle$. If this is the case, then the claim follows: Let θ be an irreducible constituent of $\varphi_{|R\,\mathrm{C}_G(R)}$. Then by Construction 2.10 we have $\mathrm{bl}(\theta)^G = B_0$, so by Theorem 1.9 it follows that θ must be the trivial character of $R\,\mathrm{C}_G(R)$. Moreover, by Construction 2.10 and Gallagher's theorem it holds that φ is one of the four extensions of θ to $\mathrm{N}_G(R)$, i.e.,

$$\varphi \in \mathrm{Irr}(\mathrm{N}_G(R) \mid \theta) = \{1_{\mathrm{N}_G(R)} \cdot \beta \mid \beta \in \mathrm{Irr}(\mathrm{N}_G(R)/R\,\mathrm{C}_G(R))\}.$$

Hence, since the character $1_{\mathrm{N}_G(R)}$ is clearly left invariant by $\mathrm{Aut}(G)_R$, the claim follows if an automorphism $a \in \mathrm{Aut}(G)_R$ as above exists.

Following the proof of [An95, (1A)] we may assume that $\mathrm{N}_G(R)$ is contained in $\mathrm{N}_G(T)$ for some maximal torus T of type T_ε, and where in addition $\mathrm{N}_G(T) = \langle T, \rho, \tau, \sigma\rangle$ for suitable $\rho, \tau, \sigma \in G$ such that $R = \langle \mathcal{O}_3(T), \sigma\rangle$, σ has order 3 modulo T, ρT generates the center of $\mathrm{N}_G(T)/T \cong D_{12}$, and $\tau^{-1}\sigma\tau$ coincides with σ^{-1} modulo T. Moreover, by the same reference $\mathrm{C}_G(R) \subseteq T$, $\mathrm{N}_G(R) = \langle R\,\mathrm{C}_G(R), \rho, \tau\rangle$, and we have

$$\mathrm{N}_G(R)/R\,\mathrm{C}_G(R) = \langle \rho R\,\mathrm{C}_G(R)\rangle \times \langle \tau R\,\mathrm{C}_G(R)\rangle.$$

By Lemma 16.14 there is an automorphism $\tilde{a} \in \mathrm{Aut}(G)_T$ such that $\mathrm{Out}(G) = \langle \tilde{a}\,\mathrm{Inn}(G)\rangle$ and $\tilde{a}$ acts trivially on $\mathrm{N}_G(T)/T$. Since $R^{\tilde{a}} \leqslant \mathrm{N}_G(T)$ and R is a Sylow 3-subgroup of G and hence of $\mathrm{N}_G(T)$, there exists $y \in \mathrm{N}_G(T)$ such that $R^{\tilde{a}y} = R$, i.e., the automorphism $a := \tilde{a}c_y \in \mathrm{Aut}(G)$ stabilizes R, so in particular it acts on $\mathrm{N}_G(R)/R\,\mathrm{C}_G(R)$. Moreover, it holds that

$$\langle a\,\mathrm{Inn}(G)\rangle = \langle \tilde{a}\,\mathrm{Inn}(G)\rangle = \mathrm{Out}(G).$$

Since $\mathrm{N}_G(T) = \langle T, \rho, \tau, \sigma\rangle$ and $\rho, \tau, \sigma \in \mathrm{N}_G(R)$, we may assume that $y \in T$. We prove that a acts trivially on $\mathrm{N}_G(R)/R\,\mathrm{C}_G(R)$. Since $\rho, \tau \in \mathrm{N}_G(T)$, we have

$$c_y(\rho) = y\rho y^{-1} = \rho\rho^{-1}y\rho y^{-1} \in \rho T,$$

and analogously $c_y(\tau) \in \tau T$. Hence, it follows that $a(\rho) \in \rho T$ and $a(\tau) \in \tau T$ since $\tilde{a}$ acts trivially on $\mathrm{N}_G(T)/T$.

Let us suppose that $a(\rho R\,\mathrm{C}_G(R)) = \tau R\,\mathrm{C}_G(R)$. Then $a(\rho) = \tau x\sigma^i$ for some $x \in T$ and $i \in \{0, 1, 2\}$ since $R = \langle \mathcal{O}_3(T), \sigma\rangle$ and $\mathrm{C}_G(R) \subseteq T$. It follows that $\rho \equiv a(\rho) \equiv \tau\sigma^i \bmod T$, i.e., $\tau \equiv \rho\sigma^{-i} \bmod T$. Since $\rho T \in \mathrm{Z}(\mathrm{N}_G(T)/T)$, we conclude that

$$\sigma^{-1} \equiv \tau^{-1}\sigma\tau \equiv \sigma^i\rho^{-1}\sigma\rho\sigma^{-i} \equiv \sigma \bmod T$$

in contradiction to the fact that σ has order 3 modulo T.

Assume now that $a(\rho R\,\mathrm{C}_G(R)) = \rho\tau R\,\mathrm{C}_G(R)$. Then $a(\rho) = \rho\tau x\sigma^i$ for some $x \in T$ and $i \in \{0,1,2\}$, so $\rho \equiv a(\rho) \equiv \rho\tau\sigma^i \bmod T$, hence $\tau \equiv \sigma^{-i} \bmod T$. But then

$$\sigma^{-1} \equiv \tau^{-1}\sigma\tau \equiv \sigma \bmod T,$$

a contradiction. Thus, we conclude that $\rho R\,\mathrm{C}_G(R)$ is left invariant by a.

Similarly, one checks that $\tau R\,\mathrm{C}_G(R)$ and $\rho\tau R\,\mathrm{C}_G(R)$ are invariant under a, so a acts trivially on $\mathrm{N}_G(R)/R\,\mathrm{C}_G(R)$, and hence on the weight (R,φ). Since $\langle a\,\mathrm{Inn}(G)\rangle = \mathrm{Out}(G)$, we conclude that $\mathrm{Aut}(G)$ stabilizes the G-conjugacy class of (R,φ). □

Let us now turn towards the case of non-principal 3-blocks of G with non-abelian defect groups. For these J. An showed the following:

Proposition 16.16. *Let B be a non-principal 3-block of G with non-abelian defect groups. Then $|\mathcal{W}(B)| = 3$. Moreover, if (R,φ) is a B-weight of G and θ is an irreducible constituent of $\varphi_{|R\,\mathrm{C}_G(R)}$, then θ is a linear character and up to G-conjugation one of the following holds:*

(i) *R is an extraspecial group of order 3^{1+2} and exponent 3 such that $R \leqslant L_\varepsilon \leqslant H_\varepsilon$, $\mathrm{C}_G(R) = \mathrm{Z}(H_\varepsilon) \cong C_{q^2+\varepsilon q+1}$, $\mathrm{N}_G(R) = \mathrm{N}_{M_\varepsilon}(R)$, $\mathrm{N}_{H_\varepsilon}(R)/R\,\mathrm{C}_G(R) \cong \mathrm{Sp}_2(3)$, and $|\,\mathrm{N}_G(R)\colon \mathrm{N}_{H_\varepsilon}(R)| = 2$. Moreover, $\mathrm{N}_G(R)_\theta = \mathrm{N}_{H_\varepsilon}(R)$ and*

$$\varphi = (\widetilde{\theta}\cdot \mathrm{St})^{\mathrm{N}_G(R)},$$

where St denotes the inflation of the Steinberg character of $\mathrm{N}_{H_\varepsilon}(R)/R\,\mathrm{C}_G(R)$ and $\widetilde{\theta}$ is an extension of θ to $\mathrm{N}_G(R)_\theta$. The character φ is uniquely determined by R and B, and there exists exactly one G-conjugacy class of such B-weights in G.

(ii) *$R \in \mathrm{Syl}_3(G)$ with $R \leqslant H_\varepsilon$, $\mathrm{C}_G(R) = \mathrm{Z}(H_\varepsilon) \cong C_{q^2+\varepsilon q+1}$, $\mathrm{N}_{H_\varepsilon}(R)/R\,\mathrm{C}_G(R) \cong C_2$, $\mathrm{N}_G(R)/R\,\mathrm{C}_G(R) \cong C_2 \times C_2$, $\mathrm{N}_G(R)_\theta = \mathrm{N}_{H_\varepsilon}(R)$, and*

$$\varphi = (\widetilde{\theta}\cdot\xi)^{\mathrm{N}_G(R)},$$

where $\xi \in \mathrm{Irr}(\mathrm{N}_G(R)_\theta/R\,\mathrm{C}_G(R))$ and $\widetilde{\theta}$ is an extension of θ to $\mathrm{N}_G(R)_\theta$.

Proof. This follows from [An95, (1A)] and the proof of [An95, (3B)]. □

In order to understand the action of $\mathrm{Aut}(G)$ on the 3-weights of G described in Proposition 16.16 above we need the following lemma:

Lemma 16.17. *Let R be as in Proposition 16.16(ii). Then $H_\varepsilon = U \times \mathrm{Z}(H_\varepsilon)_{3'}$ for some subgroup $U \leqslant H_\varepsilon$ such that $\mathrm{N}_{H_\varepsilon}(R) = \mathrm{N}_U(R) \times \mathrm{Z}(H_\varepsilon)_{3'}$. More precisely,*

$$\mathrm{N}_{H_\varepsilon}(R) = \langle R, x\rangle \times \mathrm{Z}(H_\varepsilon)_{3'}$$

for some $x \in U$ with $x^2 \in R$.

Proof. By the proof of [An95, (3B)] there exists a subgroup $U \leqslant H_\varepsilon$ such that we have $H_\varepsilon = U \times \mathrm{Z}(H_\varepsilon)_{3'}$ and $\mathrm{N}_{H_\varepsilon}(R) = \mathrm{N}_U(R) \times \mathrm{Z}(H_\varepsilon)_{3'}$. Now by Proposition 16.16(ii) it holds that $\mathrm{C}_{H_\varepsilon}(R) = \mathrm{Z}(H_\varepsilon)$ and $\mathrm{N}_{H_\varepsilon}(R)/R\,\mathrm{C}_{H_\varepsilon}(R) \cong C_2$. Since R is a Sylow 3-subgroup of G and hence of H_ε, we have $\mathrm{Z}(H_\varepsilon)_3 \leqslant R$. Thus, $R\,\mathrm{C}_{H_\varepsilon}(R) = R\,\mathrm{Z}(H_\varepsilon) = R \times \mathrm{Z}(H_\varepsilon)_{3'}$, so

$$\mathrm{N}_U(R)/R \cong \mathrm{N}_{H_\varepsilon}(R)/R\,\mathrm{C}_{H_\varepsilon}(R) \cong C_2,$$

which proves the claim. □

Proposition 16.18. *Suppose that B is a non-principal 3-block of G of non-abelian defect. Then $\mathrm{Aut}(G)_B$ acts trivially on $\mathcal{W}(B)$.*

Proof. Let (R,φ) be a B-weight of G. We go through the cases in Proposition 16.16. If R is an extraspecial group of order 3^{1+2} and exponent 3, then by Proposition 16.16(i) the character φ is uniquely determined by R and B, and there exists only one G-conjugacy class of B-weights with first component isomorphic to R. Hence, $\mathrm{Aut}(G)_B$ stabilizes the G-conjugacy class of (R,φ).

Suppose now that $R \in \mathrm{Syl}_3(G)$ such that $R \leqslant H_\varepsilon$ and let θ be an irreducible constituent of $\varphi_{|R\,\mathrm{C}_G(R)}$. Then by Proposition 16.16(ii) the character φ is of the form $\eta^{\mathrm{N}_G(R)}$ for some extension η of θ to N_θ. Let $a \in \mathrm{Aut}(G)$ such that $R^a = R$ and $B^a = B$. Moreover, let b be the 3-block of $R\,\mathrm{C}_G(R)$ containing θ. Then b has defect group $R \in \mathrm{Syl}_3(G)$, and hence by Lemma 1.1 also $B \in \mathrm{Bl}_3(G \mid R)$. Since a fixes B, it follows from Proposition 1.3 that b^a induces to B, so the 3-blocks b and b^a of $R\,\mathrm{C}_G(R)$ must be conjugate under $\mathrm{N}_G(R)$ by Brauer's extended first main theorem, Theorem 1.7, and hence also their canonical characters θ and θ^a are $\mathrm{N}_G(R)$-conjugate.

Let $x \in H_\varepsilon$ be as in Lemma 16.17, i.e., such that $\mathrm{N}_{H_\varepsilon}(R) = \langle R, x\rangle \times \mathrm{Z}(H_\varepsilon)_{3'}$ and $x^2 \in R$. According to Proposition 16.16(ii) we have

$$\mathrm{N}_G(R)/R\,\mathrm{C}_G(R) \cong C_2 \times C_2.$$

Hence, there exist $y \in G$ such that $\mathrm{N}_G(R) = \langle R\,\mathrm{C}_G(R), x, y\rangle$ with $[x,y] \in R\,\mathrm{C}_G(R)$. Now the orbit of θ under the conjugation action of $\mathrm{N}_G(R)$ is given by the set $\{\theta, \theta^x, \theta^y, \theta^{xy}\}$. From Proposition 16.16(ii) it follows that $\mathrm{N}_G(R)_\theta = \mathrm{N}_{H_\varepsilon}(R)$, so in fact $\theta^x = \theta$, and we conclude that $\theta^a \in \{\theta, \theta^y\}$. Since $\mathrm{N}_G(R)_\theta$ has index two in $\mathrm{N}_G(R)$, it is a normal subgroup of $\mathrm{N}_G(R)$, and hence $\mathrm{N}_G(R)_{\theta^y} = (\mathrm{N}_G(R)_\theta)^y = \mathrm{N}_G(R)_\theta$. Thus,

$$\mathrm{N}_G(R)_{\theta^a} = \mathrm{N}_G(R)_\theta = \langle R, x\rangle \times \mathrm{Z}(H_\varepsilon)_{3'}.$$

Clearly, $a^{-1}(x) = x^a \in (\mathrm{N}_G(R)_\theta)^a = \mathrm{N}_G(R)_{\theta^a} = \langle R, x\rangle \times \mathrm{Z}(H_\varepsilon)_{3'}$. Suppose $x^a = v \cdot z$ for some $v \in \langle R, x\rangle$ and $z \in \mathrm{Z}(H_\varepsilon)_{3'}$. Then

$$v^2 \cdot z^2 = (x^a)^2 = (x^2)^a \in R^a = R.$$

Since $v^2 \in R$, we also have $z^2 \in R$. But as $\mathrm{Z}(H_\varepsilon) \cong C_{q^2+\varepsilon q+1}$ (cf. Proposition 16.16(ii)), it follows that $\mathrm{Z}(H_\varepsilon)_{3'}$ has odd order. Hence, we conclude that z^2 can only be contained in the 3-group R if $z = 1$. Thus, $x^a \in \langle R, x\rangle$, so a^{-1} and a act on $\langle R, x\rangle$, and we may write $a(x) = rx$ for some $r \in R$.

Now there exist exactly two extensions of θ to $\mathrm{N}_G(R)_\theta$, say θ_+ and θ_- such that $\theta_+(x) = 1$ and $\theta_-(x) = -1$ (for this recall that $x^2 \in R$ and θ is a linear character of $R\,\mathrm{C}_G(R)$ with R in its kernel, so $\theta_\pm(x)^2 = \theta_\pm(x^2) = 1$). We write $\theta_\pm(x) = \pm 1$. Analogously, we denote by $(\theta^y)_\pm$ the two extensions of θ^y to $\mathrm{N}_G(R)_{\theta^y} = \mathrm{N}_G(R)_\theta$ with $(\theta^y)_\pm(x) = \pm 1$. It holds that

$$(\theta_\pm)^a(c) = \theta_\pm(a(c)) = \theta(a(c)) = \theta^a(c)$$

for all $c \in R\,\mathrm{C}_G(R)$ since a acts on $R\,\mathrm{C}_G(R)$, so $(\theta_\pm)^a$ are extensions of θ^a to $\mathrm{N}_G(R)_\theta$. Moreover,

$$(\theta_\pm)^a(x) = \theta_\pm(a(x)) = \theta_\pm(rx) = \theta_\pm(r)\theta_\pm(x) = \theta_\pm(x) = \pm 1,$$

so we have $(\theta_\pm)^a = (\theta^a)_\pm$. Denote by ψ_+ and ψ_- the induced characters $(\theta_+)^{\mathrm{N}_G(R)}$ and $(\theta_-)^{\mathrm{N}_G(R)}$, respectively. Then $\varphi \in \{\psi_+, \psi_-\}$. We recall that $\mathrm{N}_G(R)_\theta$ is normal in $\mathrm{N}_G(R) = \langle \mathrm{N}_G(R)_\theta, y\rangle$, so by Clifford theory, Theorem 1.15, we have

$$\psi_{\pm|\,\mathrm{N}_G(R)_\theta} = \theta_\pm + (\theta_\pm)^y.$$

Since $\mathrm{N}_G(R)_\theta = \mathrm{N}_{H_\varepsilon}(R) \trianglelefteq \mathrm{N}_G(R)$, we have $yxy^{-1} \in \mathrm{N}_{H_\varepsilon}(R) = \langle R, x\rangle \times \mathrm{Z}(H_\varepsilon)_{3'}$, so we write $yxy^{-1} = v \cdot z$ for some $v \in \langle R, x\rangle$ and $z \in \mathrm{Z}(H_\varepsilon)_{3'}$. Then

$$v^2 \cdot z^2 = (yxy^{-1})^2 = yx^2y^{-1} \in R$$

since $x^2 \in R$ and $y \in \mathrm{N}_G(R)$. As above we conclude that $z^2 \in R$ and hence $z = 1$, so we may write $yxy^{-1} = sx$ for some $s \in R$. Then

$$(\theta_\pm)^y(x) = \theta_\pm(yxy^{-1}) = \theta_\pm(sx) = \theta_\pm(s)\theta_\pm(x) = \theta_\pm(x) = \pm 1.$$

Since $(\theta_\pm)^y$ are extensions of θ^y, we conclude that $(\theta_\pm)^y = (\theta^y)_\pm$. Then

$$\psi_{\pm|\,\mathrm{N}_G(R)_\theta} = \theta_\pm + (\theta^y)_\pm,$$

and by Frobenius reciprocity, Proposition 1.18, and the fact that $(\psi_\pm)^a = ((\theta_\pm)^a)^{\mathrm{N}_G(R)}$ it follows that

$$\langle(\psi_\pm)^a, \psi_\pm\rangle_{\mathrm{N}_G(R)} = \langle(\theta_\pm)^a, \psi_{\pm|\,\mathrm{N}_G(R)_\theta}\rangle_{\mathrm{N}_G(R)_\theta} = \langle(\theta^a)_\pm, \theta_\pm + (\theta^y)_\pm\rangle_{\mathrm{N}_G(R)_\theta} = 1$$

since $\theta_\pm$, $(\theta^y)_\pm$ are distinct, irreducible, and $\theta^a \in \{\theta, \theta^y\}$. Thus, $(\psi_\pm)^a = \psi_\pm$, and hence we conclude that $(R,\varphi)^a = (R,\varphi)$. Since for any automorphism $\sigma \in \mathrm{Aut}(G)_B$ we have $R^\sigma \in \mathrm{Syl}_3(G)$, there always exists $g \in G$ such that $R^{\sigma g} = R$. Clearly, we also have $\sigma c_g \in \mathrm{Aut}(G)_B$. By the previous considerations we know that $(R,\varphi)^{\sigma g} = (R,\varphi)$, so $(R,\varphi)^\sigma = (R,\varphi)^{g^{-1}}$ is G-conjugate to (R,φ) as claimed. □

16.2.2.2 Blocks of Abelian Defect

In this section we study the behaviour of B-weights under automorphisms of G for 3-blocks B of non-cyclic abelian defect. For such a 3-block the following statements hold by results of D. I. Deriziotis and G. O. Michler [DM87] and J. An [An95]:

Proposition 16.19. *Let B be a 3-block of G with a non-cyclic abelian defect group D. Then the following statements hold:*

(i) The centralizer $\mathrm{C}_G(D) =: T$ is a maximal torus of G of type T_ε or $T_{2,\varepsilon}$, $D = \mathcal{O}_3(T)$ and $\mathrm{N}_G(D) = \mathrm{N}_G(T)$. In particular, in G there exists only one G-conjugacy class of subgroups isomorphic to D.

(ii) Consider $\mathrm{Irr}(T)$ as an abelian group and fix an isomorphism $\hat{}\colon T \longrightarrow \mathrm{Irr}(T)$. Then up to G-conjugation there exists a unique $3'$-element $s \in T$ and a 3-block $b \in \mathrm{Bl}_3(T)$ of $T = \mathrm{C}_G(D)$ with $b^G = B$ such that the linear character $\theta := \hat{s} \in \mathrm{Irr}(T)$ is the canonical character of b.

(iii) For θ and s as in (ii) we have $\mathrm{N}_G(T)_\theta/T \cong \mathrm{C}_{W(T)}(s)$, where $W(T) := \mathrm{N}_G(T)/T$.

Proof. Part (i) follows from [An95, (1A)] since being a defect group of B the 3-group D is radical in G (cf. Example 2.6). Statements (ii) and (iii) are part of [DM87, Prop. 5.8]. □

Blocks with Defect Groups $\mathcal{O}_3(T_{2,\varepsilon})$

Let us first treat the case of 3-blocks of G with defect group $\mathcal{O}_3(T)$ for a maximal torus T of G of type $T_{2,\varepsilon}$. Similarly as for the case of 2-blocks of G with non-cyclic abelian defect groups we can prove the following:

Proposition 16.20. *Let B be a 3-block of G with defect group $\mathcal{O}_3(T)$ for a maximal torus T of G of type $\mathrm{T}_{2,\varepsilon}$. Then $\mathrm{Aut}(G)_B$ acts trivially on $\mathcal{W}(B)$.*

Proof. Suppose that (R, φ) is a B-weight. In consequence of Lemma 2.11 we may assume that $R = \mathcal{O}_3(T)$ with $\mathrm{N}_G(R) = \mathrm{N}_G(T)$ by Proposition 16.19(i). Let $s \in T$, $\theta = \hat{s}$ and $b \in \mathrm{Bl}_3(T)$ be as in Proposition 16.19(ii), so $\mathrm{N}_G(T)_\theta/T \cong \mathrm{C}_{W(T)}(s)$ for $W(T) = \mathrm{N}_G(T)/T$. Since B is non-principal, we have $s \neq 1$. Then by [DM87, Lemma 3.4] up to isomorphism it holds that $\mathrm{C}_{W(T)}(s) \in \{\{1\}, C_3\}$.

Now both 3-blocks b and $B = b^G$ have defect group R, so Theorem 1.10 implies that the index $|\mathrm{N}_G(T)_\theta \colon T|$ is not divisible by 3. It follows that $\mathrm{N}_G(T)_\theta = T$, and hence by Construction 2.10 we have $\varphi = \theta^{\mathrm{N}_G(T)}$. Consequently, φ is uniquely determined by R and B, and since R^a is G-conjugate to R for any $a \in \mathrm{Aut}(G)_B$, we conclude that the G-conjugacy class of (R, φ) is left invariant by $\mathrm{Aut}(G)_B$. □

Blocks with Defect Groups $\mathcal{O}_3(T_\varepsilon)$

In the following we concentrate on the case of 3-blocks of G that have a defect group $\mathcal{O}_3(T)$ for a maximal torus T of G of type T_ε. For q large enough we wish to reduce this case to a question on 3-weights of $G_2(q)$. However, this requires the application of Lemma 16.10, whence we have to treat the case of $q < 5$ separately:

Proposition 16.21. *Suppose that $q \in \{2, 4\}$ (or, more generally, that q is even). Let B be a 3-block of G with defect group $\mathcal{O}_3(T)$ for a maximal torus T of G of type T_ε. Then $\mathrm{Aut}(G)_B$ acts trivially on $\mathcal{W}(B)$.*

Proof. Let (R, φ) be a B-weight. As in the proof of Proposition 16.20 we may assume that $R = \mathcal{O}_3(T)$ such that $\mathrm{N}_G(R) = \mathrm{N}_G(T)$ by Proposition 16.19, and we let $s \in T$, $\theta = \hat{s}$ and $b \in \mathrm{Bl}_3(T)$ be as in Proposition 16.19(ii), so $\mathrm{N}_G(T)_\theta/T \cong \mathrm{C}_{W(T)}(s)$, where $W(T) = \mathrm{N}_G(T)/T$. Since B is non-principal, we have $s \neq 1$, and since q is even, it follows that s cannot be of order 2. Hence, from [DM87, Lemma 3.4] we obtain that

$$\mathrm{C}_{W(T)}(s) \in \{\{1\}, C_2, \mathfrak{S}_3, \mathrm{SL}_3(2)\}$$

up to isomorphism. As in the proof of Proposition 16.20 the index $|\mathrm{N}_G(T)_\theta \colon T|$ must be coprime to 3, whence in fact $\mathrm{C}_{W(T)}(s) \in \{\{1\}, C_2\}$ up to isomorphism.

If $\mathrm{N}_G(T)_\theta = T$, then Construction 2.10 yields $\varphi = \theta^{\mathrm{N}_G(T)}$, such that (R, φ) is uniquely determined by R and B, hence left invariant by $\mathrm{Aut}(G)_B$ up to G-conjugation in consequence of Proposition 16.19(i).

Suppose now that $\mathrm{N}_G(T)_\theta/T \cong C_2$. Then from Construction 2.10 it follows that

$$\varphi = \widetilde{\theta}^{\mathrm{N}_G(T)},$$

where $\widetilde{\theta}$ denotes one of the two extensions of θ to $\mathrm{N}_G(T)_\theta$, which exist by Proposition 1.20. If $a \in \mathrm{Aut}(G)_B$, then up to G-conjugation a stabilizes T, so we may without loss of generality assume that $T^a = T$ and hence $R^a = R$. Moreover, as in the proof of Proposition 12.37 it follows that θ^a and θ are $\mathrm{N}_G(T)$-conjugate, so by multiplication of a with a suitable inner automorphism we may assume that a leaves θ invariant. In particular, a stabilizes $\mathrm{N}_G(T)_\theta$ by Lemma 1.25. We prove that $\widetilde{\theta}^a = \widetilde{\theta}$. Then

$$\varphi^a = (\widetilde{\theta}^{\mathrm{N}_G(T)})^a = (\widetilde{\theta}^a)^{\mathrm{N}_G(T)} = \widetilde{\theta}^{\mathrm{N}_G(T)} = \varphi,$$

and the claim follows. In consequence of Corollary 1.22 it suffices to show that

$$\widetilde{\theta}^a(n) = \widetilde{\theta}(n) \neq 0,$$

where $n \in \mathrm{N}_G(T)$ with $n^2 \in T$ is such that $\mathrm{N}_G(T)_\theta = \langle T, n\rangle$. Clearly, both values are non-zero since the character $\widetilde{\theta}$ is linear. Now we have $a(n) \in \mathrm{N}_G(T)_\theta \setminus T$, so there exists $t \in T$ such that $a(n) = tn$, and we have

$$\widetilde{\theta}^a(n) = \widetilde{\theta}(tn) = \widetilde{\theta}(t)\widetilde{\theta}(n).$$

Since $n^2 \in T$ and $\widetilde{\theta}^a(s) = \widetilde{\theta}(s) = \theta(s)$ for all $s \in T$, it follows that

$$\widetilde{\theta}(n)^2 = \widetilde{\theta}(n^2) = \widetilde{\theta}^a(n^2) = \widetilde{\theta}^a(n)^2 = \widetilde{\theta}(t)^2\widetilde{\theta}(n)^2,$$

and hence $\widetilde{\theta}(t)^2 = 1$. Now q is even, so in particular T does not contain elements of even order. It follows that $\widetilde{\theta}(t) = 1$, and hence we have $\widetilde{\theta}^a(n) = \widetilde{\theta}(n)$ as claimed. □

Let us now consider the situation that $q \geqslant 5$. We reduce this case to a question on 3-weights of $G_2(q)$. For this purpose we fix some notation:

Recall that $\varepsilon \in \{\pm 1\}$ was chosen such that $q \equiv \varepsilon \bmod 3$. We let $\widetilde{G} = G^{F_q} \cong G_2(q)$ be a maximal subgroup of G as in Proposition 15.11. In addition, we fix a maximal torus T of G of type T_ε such that $\widetilde{T} := T \cap \widetilde{G}$ is a maximal torus of $\widetilde{G}$ isomorphic to $C_{q-\varepsilon} \times C_{q-\varepsilon}$. As observed at the beginning of Section 16.2.2, we have

$$\mathrm{N}_G(T)/T \cong \mathrm{N}_{\widetilde{G}}(\widetilde{T})/\widetilde{T} \cong D_{12},$$

such that by Proposition 16.11 it holds that $\mathrm{N}_G(T) = \mathrm{N}_{\widetilde{G}}(\widetilde{T})T$ for $q \geqslant 5$. This equality will be used frequently throughout the rest of this section.

Furthermore, we define $R := \mathcal{O}_3(T)$ and $\widetilde{R} := \mathcal{O}_3(\widetilde{T})$ to be the Sylow 3-subgroups of T and $\widetilde{T}$, respectively. Now let B be a 3-block of G with defect group R. Since B has abelian defect, it follows from Lemma 2.11 that a 3-subgroup $Q \leqslant G$ is G-conjugate to R whenever there exists a B-weight with first component Q.

If (R, φ) is a B-weight of G and $\theta \in \mathrm{Irr}(T)$ is an irreducible constituent of the restriction $\varphi_{|T}$ of φ to $R\,\mathrm{C}_G(R) = T$, then $B = \mathrm{bl}(\theta)^G$ following Construction 2.10. We may draw the following diagram of inclusions:

$$\begin{tikzcd}[column sep=small]
 & & G \arrow[d, dash] & \\
 & \widetilde{G} \arrow[ur, dash] \arrow[dd, dash] & & \\
 & & \mathrm{N}_G(T) = \mathrm{N}_{\widetilde{G}}(\widetilde{T})T \arrow[dd, dash] & \varphi \\
 \varphi_{|\mathrm{N}_{\widetilde{G}}(\widetilde{T})} & \mathrm{N}_{\widetilde{G}}(\widetilde{T}) \arrow[ur, dash] \arrow[dd, dash] & & \\
 & & T \arrow[dd, dash] & \theta \\
 \theta_{|\widetilde{T}} & \widetilde{T} \arrow[ur, dash] \arrow[dd, dash] & & \\
 & & R & \\
 & \widetilde{R} \arrow[ur, dash] & &
\end{tikzcd}$$

Let us set $\widetilde{B} := \mathrm{bl}(\theta_{|\widetilde{T}})^{\widetilde{G}}$. Our objective is to relate the B-weight (R, φ) of G to a (set of) $\widetilde{B}$-weight(s) of $\widetilde{G}$. To this end we examine the restriction of the weight character φ to $\mathrm{N}_{\widetilde{G}}(\widetilde{T})$. In this connection the irreducible characters of T will play a key role, whence we fix the following parametrization for $\mathrm{Irr}(T)$ to allow a convenient handling:

Notation 16.22 (Parametrization of $\mathrm{Irr}(T)$)**.** Being of type T_ε the maximal torus T of G satisfies $T \cong C_{q^3-\varepsilon} \times C_{q-\varepsilon}$ according to Table 15.1. We identify $C_{q^3-\varepsilon}$ and $C_{q-\varepsilon}$ with the unique subgroups of the cyclic group $\mathbb{F}_{q^6}^\times$ of orders $q^3-\varepsilon$ and $q-\varepsilon$, respectively. Moreover, we fix a generator of $C_{q^3-\varepsilon}$, that is, an element $z \in \mathbb{F}_{q^6}^\times$ of order $q^3-\varepsilon$. Then the linear map defined by

$$\theta_0 \colon C_{q^3-\varepsilon} \longrightarrow \mathbb{C}^\times, \quad z \longmapsto \exp\left(\frac{2\pi\mathrm{i}}{q^3-\varepsilon}\right),$$

generates the character group of $C_{q^3-\varepsilon}$, i.e., we have

$$\mathrm{Irr}(C_{q^3-\varepsilon}) = \{\theta_0^i \mid 0 \leqslant i < q^3-\varepsilon\}.$$

Now $C_{q-\varepsilon}$ is generated by $z^{q^2+\varepsilon q+1}$. Every irreducible character of $C_{q-\varepsilon}$ may be extended to $C_{q^3-\varepsilon}$ by Proposition 1.20(i), and conversely, every restriction of a character in $\mathrm{Irr}(C_{q^3-\varepsilon})$ to $C_{q-\varepsilon}$ yields an irreducible character of $C_{q-\varepsilon}$. Moreover, for $0 \leqslant i,j < q^3-\varepsilon$ we have $\theta^i_{0|C_{q-\varepsilon}} = \theta^j_{0|C_{q-\varepsilon}}$ if and only if $i \equiv j \bmod q-\varepsilon$. Hence, by abuse of notation (that is, writing θ_0 instead of $\theta_{0|C_{q-\varepsilon}}$) we obtain

$$\mathrm{Irr}(C_{q-\varepsilon}) = \{\theta_0^i \mid 0 \leqslant i < q-\varepsilon\}$$

with $\theta_0^i \neq \theta_0^j$ for $0 \leqslant i,j < q-\varepsilon$ if $i \neq j$. Combination of these two parametrizations for the characters of $C_{q^3-\varepsilon}$ and $C_{q-\varepsilon}$ yields a parametrization for $\mathrm{Irr}(T)$ given by

$$\mathrm{Irr}(T) = \{\theta_0^i \times \theta_0^j \mid 0 \leqslant i < q^3-\varepsilon,\ 0 \leqslant j < q-\varepsilon\},$$

where we have

$$(\theta_0^i \times \theta_0^j)(h(t_1, t_2, t_1^{\varepsilon q}, t_1^{q^2})) = \theta_0^i(t_1)\theta_0^j(t_2)$$

for an element of T represented by $h(t_1, t_2, t_1^{\varepsilon q}, t_1^{q^2})$ with $t_1, t_2 \in \mathbb{F}^\times$ such that $t_1^{q^3-\varepsilon} = 1$ and $t_2^{q-\varepsilon} = 1$.

We now prove a series of lemmata which will eventually allow us to handle all situations which may occur when restricting a weight character φ coming from a B-weight (R,φ) of G to $\mathrm{N}_{\widetilde{G}}(\widetilde{T})$.

Lemma 16.23. *Suppose that $\theta \in \mathrm{Irr}(T)$. Then $\widetilde{T}$ is contained in the kernel of θ if and only if the index $|\,\mathrm{N}_G(T)_\theta \colon T\,|$ is divisible by* 3.

Proof. We consider the element $\bar{\omega}$ in the Weyl group $W(T) = \mathrm{N}_G(T)/T$ of T corresponding to the reflection $\omega_{r_1}\omega_{r_3}\omega_{r_4}\omega_{r_2}$. Note that such an element exists in $W(T)$ as a result of Lemma 15.4, and $\bar{\omega}$ has order 6 in $W(T)$ by Lemma 15.5. We set $\omega := \bar{\omega}^2$, which is hence of order 3. Then for an element of T represented by $h(t_1, t_2, t_1^{\varepsilon q}, t_1^{q^2})$ with $t_1, t_2 \in \mathbb{F}^\times$ such that $t_1^{q^3-\varepsilon} = t_2^{q-\varepsilon} = 1$ we have

$$\omega h(t_1,\, t_2,\, t_1^{\varepsilon q},\, t_1^{q^2})\omega^{-1} = h(t_1 t_2^{-1},\, t_1^{q^2+\varepsilon q+1} t_2^{-2},\, t_1^{\varepsilon q} t_2^{-1},\, t_1^{q^2} t_2^{-1})$$

by application of Lemma 15.3. Suppose that $\widetilde{T} \subseteq \ker(\theta)$. Then we may write $\theta = \theta_0^{i(q-\varepsilon)} \times 1$

for some $0 \leqslant i < q^2 + \varepsilon q + 1$. Hence,

$$\begin{aligned}
\theta^\omega(h(t_1, t_2, t_1^{\varepsilon q}, t_1^{q^2})) &= \theta(\omega h(t_1,\, t_2,\, t_1^{\varepsilon q},\, t_1^{q^2})\omega^{-1}) \\
&= \theta(h(t_1 t_2^{-1},\, t_1^{q^2+\varepsilon q+1} t_2^{-2},\, t_1^{\varepsilon q} t_2^{-1},\, t_1^{q^2} t_2^{-1})) \\
&= \theta_0^{i(q-\varepsilon)}(t_1 t_2^{-1}) \\
&= \theta_0(t_1^{i(q-\varepsilon)} t_2^{-i(q-\varepsilon)}) \\
&= \theta_0(t_1^{i(q-\varepsilon)}) \\
&= \theta_0^{i(q-\varepsilon)}(t_1) \\
&= \theta(h(t_1, t_2, t_1^{\varepsilon q}, t_1^{q^2}))
\end{aligned}$$

for any $t_1, t_2 \in \mathbb{F}^\times$ such that $t_1^{q^3-\varepsilon} = t_2^{q-\varepsilon} = 1$, so $\omega \in W(T)_\theta = \mathrm{N}_G(T)_\theta/T$. This shows that 3 divides $|\,\mathrm{N}_G(T)_\theta \colon T\,|$.

Let us now suppose that 3 divides the index $|\,\mathrm{N}_G(T)_\theta \colon T\,|$. Since $\mathrm{N}_G(T)/T \cong D_{12}$ contains exactly two elements of order 3, it follows that ω and ω^{-1} must stabilize θ. Let $t_1, t_2 \in \mathbb{F}^\times$ be such that $t_1^{q^3-\varepsilon} = t_2^{q-\varepsilon} = 1$. We write $\theta = \theta_0^i \times \theta_0^j$ for suitable $0 \leq i < q^3 - \varepsilon$ and $0 \leqslant j < q - \varepsilon$. Then similarly as above we obtain

$$\begin{aligned}
\theta^\omega(h(t_1, t_2, t_1^{\varepsilon q}, t_1^{q^2})) &= \theta(h(t_1 t_2^{-1},\, t_1^{q^2+\varepsilon q+1} t_2^{-2},\, t_1^{\varepsilon q} t_2^{-1},\, t_1^{q^2} t_2^{-1})) \\
&= \theta_0^i(t_1 t_2^{-1}) \theta_0^j(t_1^{q^2+\varepsilon q+1} t_2^{-2}) \\
&= \theta_0^{i+j(q^2+\varepsilon q+1)}(t_1) \theta_0^{-i-2j}(t_2) \\
&= (\theta_0^{i+j(q^2+\varepsilon q+1)} \times \theta_0^{-i-2j})(h(t_1, t_2, t_1^{\varepsilon q}, t_1^{q^2})).
\end{aligned}$$

Since $\theta^\omega = \theta$, it follows that

$$i \equiv i + j(q^2 + \varepsilon q + 1) \mod q^3 - \varepsilon, \tag{16.1}$$

$$j \equiv -i - 2j \mod q - \varepsilon. \tag{16.2}$$

Condition (16.1) is fulfilled exactly if $j(q^2 + \varepsilon q + 1) \equiv 0 \bmod q^3 - \varepsilon$, or, equivalently, if we have $j \equiv 0 \bmod q - \varepsilon$, whence condition (16.2) reduces to

$$i \equiv 0 \mod q - \varepsilon. \tag{16.2'}$$

Hence, $\widetilde{T} \subseteq \ker(\theta)$. This proves the claim. □

Lemma 16.24. *Let B be a 3-block of G with abelian defect group $\mathcal{O}_3(T)$ and suppose that $\theta \in \mathrm{Irr}(T)$ is the canonical character of a 3-block $b \in \mathrm{Bl}_3(T)$ such that $b^G = B$. Moreover, set $\widetilde{\theta} := \theta_{|\widetilde{T}} \in \mathrm{Irr}(\widetilde{T})$. Then (at least) one of the following situations occurs:*

(i) $\mathrm{N}_G(T)_\theta = T$,

(ii) $\mathrm{N}_G(T)_\theta = \mathrm{N}_G(T)_{\widetilde{\theta}}$,

(iii) $\mathrm{N}_G(T)_\theta/T \cong C_2$ *and* $|\,\mathrm{N}_G(T)_{\widetilde{\theta}} \colon \mathrm{N}_G(T)_\theta\,| = 2$.

Proof. Suppose that neither (i) nor (ii) hold. We show that (iii) holds in this case. The 3-block b has defect group $\mathcal{O}_3(T)$, whence it follows from Theorem 1.10 that 3 does not divide $|\,\mathrm{N}_G(T)_\theta \colon T\,|$. In particular, up to isomorphism we have

$$\mathrm{N}_G(T)_\theta/T \in \{C_2, C_2 \times C_2\}$$

since $\mathrm{N}_G(T)/T \cong D_{12}$.

The group $\mathrm{N}_G(T)/T \cong D_{12}$ contains exactly two elements of order 3, which are given by ω and ω^{-1}, where ω is as in the proof of Lemma 16.23. We prove first that $\omega \notin \mathrm{N}_G(T)_{\widetilde{\theta}}$. To this end we assume the contrary, i.e., $\omega \in \mathrm{N}_G(T)_{\widetilde{\theta}}$. Suppose that $\theta = \theta_0^i \times \theta_0^j$ for suitable parameters i and j. Then similarly as in the proof of Lemma 16.23 we have

$$(\theta_0^i \times \theta_0^j)_{|\widetilde{T}} = (\theta_0^{i+j(q^2+\varepsilon q+1)} \times \theta_0^{-i-2j})_{|\widetilde{T}},$$

which is equivalent to the conditions

$$i \equiv i + j(q^2 + \varepsilon q + 1) \mod q - \varepsilon, \tag{16.3}$$

$$j \equiv -i - 2j \mod q - \varepsilon. \tag{16.4}$$

These are fulfilled if and only if $3j \equiv 0 \bmod q - \varepsilon$ and $i \equiv 0 \bmod q - \varepsilon$. But θ is the canonical character of the 3-block b, whence in particular we have $(q-\varepsilon)_3 \mid j$, so in fact it also holds that $j \equiv 0 \bmod q - \varepsilon$. Hence, $\widetilde{T} \subseteq \ker(\theta)$ and Lemma 16.23 implies that 3 divides the index $|\,\mathrm{N}_G(T)_\theta \colon T\,|$, a contradiction. Thus, $|\,\mathrm{N}_G(T)_{\widetilde{\theta}} \colon T\,|$ divides 4.

In particular, if we had $\mathrm{N}_G(T)_\theta/T \cong C_2 \times C_2$, then this would imply that we also had $\mathrm{N}_G(T)_{\widetilde{\theta}}/T \cong C_2 \times C_2$ since $\mathrm{N}_G(T)_\theta \subseteq \mathrm{N}_G(T)_{\widetilde{\theta}}$, so $\mathrm{N}_G(T)_{\widetilde{\theta}} = \mathrm{N}_G(T)_\theta$ in this case. Hence, we must have $\mathrm{N}_G(T)_\theta/T \cong C_2$ and $\mathrm{N}_G(T)_{\widetilde{\theta}}/T \cong C_2 \times C_2$ as claimed. □

Let us now go through the three cases specified in Lemma 16.24. Case (i) may be treated very quickly:

Proposition 16.25. *Let B be a 3-block of G with abelian defect group $\mathcal{O}_3(T)$ and suppose that $\theta \in \mathrm{Irr}(T)$ is the canonical character of a 3-block $b \in \mathrm{Bl}_3(T)$ such that $b^G = B$ and $\mathrm{N}_G(T)_\theta = T$. Then $|\mathcal{W}(B)| = 1$.*

Proof. This follows immediately from Construction 2.10 and Lemma 2.11, which imply that up to G-conjugation for any B-weight (Q, φ) we have $Q = \mathcal{O}_3(T)$ and $\varphi = \theta^{\mathrm{N}_G(T)}$. □

Corollary 16.26. *Let B be a 3-block of G with abelian defect group $\mathcal{O}_3(T)$ and suppose that $\theta \in \mathrm{Irr}(T)$ is the canonical character of a 3-block $b \in \mathrm{Bl}_3(T)$ such that $b^G = B$ and $\mathrm{N}_G(T)_\theta = T$. Then $\mathrm{Aut}(G)_B$ acts trivially on $\mathcal{W}(B)$.*

Proof. This is clear since by Proposition 16.25 we have $|\mathcal{W}(B)| = 1$. □

We now turn towards cases (ii) and (iii) of Lemma 16.24.

Lemma 16.27. *Suppose that $q \geqslant 5$. Let B be a 3-block of G with abelian defect group $\mathcal{O}_3(T)$ and suppose that $\theta \in \mathrm{Irr}(T)$ is the canonical character of a 3-block $b \in \mathrm{Bl}_3(T)$ such that $b^G = B$. Set $\widetilde{\theta} := \theta_{|\widetilde{T}} \in \mathrm{Irr}(\widetilde{T})$.*

(i) If $\mathrm{N}_G(T)_\theta = \mathrm{N}_G(T)_{\widetilde{\theta}}$, then we also have $\mathrm{N}_{\widetilde{G}}(\widetilde{T})_\theta = \mathrm{N}_{\widetilde{G}}(\widetilde{T})_{\widetilde{\theta}}$.

(ii) If $\mathrm{N}_G(T)_\theta/T \cong C_2$ and $|\,\mathrm{N}_G(T)_{\widetilde{\theta}} : \mathrm{N}_G(T)_\theta| = 2$, then we also have $\mathrm{N}_{\widetilde{G}}(\widetilde{T})_\theta/\widetilde{T} \cong C_2$ and $|\,\mathrm{N}_{\widetilde{G}}(\widetilde{T})_{\widetilde{\theta}} : \mathrm{N}_{\widetilde{G}}(\widetilde{T})_\theta| = 2$.

Proof. This follows easily from the fact that $\mathrm{N}_G(T) = T\,\mathrm{N}_{\widetilde{G}}(\widetilde{T})$ by Proposition 16.11 and T stabilizes both θ and $\widetilde{\theta}$, whence $\mathrm{N}_G(T)_\theta = T\,\mathrm{N}_{\widetilde{G}}(\widetilde{T})_\theta$ and $\mathrm{N}_G(T)_{\widetilde{\theta}} = T\,\mathrm{N}_{\widetilde{G}}(\widetilde{T})_{\widetilde{\theta}}$. □

Lemma 16.28. *Suppose that $q \geqslant 5$. Let B be a 3-block of G with defect group $\mathcal{O}_3(T)$ and let $\theta \in \mathrm{Irr}(T)$ be the canonical character of a 3-block $b \in \mathrm{Bl}_3(T)$ with $b^G = B$. Then θ extends to $\mathrm{N}_G(T)_\theta$. In particular, the set $\mathrm{Irr}(\mathrm{N}_G(T)_\theta \mid \theta)$ only consists of extensions of θ.*

Proof. Clearly, the stabilizer of b in $\mathrm{N}_G(T)$ is given by $\mathrm{N}_G(T)_\theta$. The 3-block b has defect group $\mathcal{O}_3(T) \in \mathrm{Syl}_3(T)$, and since by assumption $b^G = B$ has defect group $\mathcal{O}_3(T)$ as well, for the stabilizer $\mathrm{N}_G(T)_\theta$ of b in $\mathrm{N}_G(T)$ we have $|\,\mathrm{N}_G(T)_\theta\colon T\,|_3 = 1$ in consequence of Theorem 1.10. Hence, $\mathrm{N}_G(T)_\theta/T$ is isomorphic to one of the $3'$-subgroups

$$\{1\},\ C_2 \text{ and } C_2 \times C_2$$

of D_{12}. If $\mathrm{N}_G(T)_\theta/T = \{1\}$ or $\mathrm{N}_G(T)_\theta/T \cong C_2$, then θ extends to $\mathrm{N}_G(T)_\theta$ by Proposition 1.20(i). Thus, we suppose that $\mathrm{N}_G(T)_\theta/T$ is a Klein four-group $C_2 \times C_2$:

By Proposition 16.19 the canonical character θ of b corresponds via an isomorphism between $\mathrm{Irr}(T)$ and T to a semisimple $3'$-element $s \in T$ and we have $\mathrm{N}_G(T)_\theta/T \cong \mathrm{C}_{W(T)}(s)$. By [DM87, Table 3.4] from $\mathrm{C}_{W(T)}(s) \cong C_2 \times C_2$ it follows that s, and hence θ, has order 2, so $\theta = \theta^{-1}$. (Note that this follows from the fact that θ is stabilized by the non-trivial element of the center of $\mathrm{N}_G(T)/T$, which acts on T and $\mathrm{Irr}(T)$ by inversion.) According to Lemma 16.24 we have

$$\mathrm{N}_G(T)_\theta = \mathrm{N}_G(T)_{\widetilde{\theta}},$$

where $\widetilde{\theta} = \theta_{|\widetilde{T}}$, and from Lemma 16.23 we know that $\widetilde{\theta}$ is not the trivial character of $\widetilde{T}$, so $\widetilde{\theta}$ is of order 2 as well. Then by Remark 12.35 there exist $n_1, n_2 \in \mathrm{N}_{\widetilde{G}}(\widetilde{T})$ with $[n_1, n_2] = 1$ and $n_1^2, n_2^2 \in \widetilde{T}$ such that $\mathrm{N}_{\widetilde{G}}(\widetilde{T})_{\widetilde{\theta}} = \langle \widetilde{T}, n_1, n_2\rangle$. But then

$$\mathrm{N}_G(T)_\theta = \mathrm{N}_G(T)_{\widetilde{\theta}} = T\,\mathrm{N}_{\widetilde{G}}(\widetilde{T})_{\widetilde{\theta}} = \langle T, n_1, n_2\rangle$$

in consequence of Proposition 16.11. By Proposition 1.20(i) there exists an extension η of θ to $\langle T, n_1\rangle$, and since $[n_1, n_2] = 1$ and n_2 normalizes T and $\langle T, n_1\rangle$, we have

$$\eta^{n_2}(tn_1) = \eta(n_2tn_2^{-1}n_1) = \eta(n_2tn_2^{-1})\eta(n_1) = \theta^{n_2}(t)\eta(n_1) = \theta(t)\eta(n_1) = \eta(tn_1)$$

for all $t \in T$. Hence, η is invariant under n_2, so by Proposition 1.20(i) we may extend it to $\mathrm{N}_G(T)_\theta = \langle T, n_1, n_2\rangle$ since $\langle T, n_1, n_2\rangle/\langle T, n_1\rangle$ is cyclic of order 2. This extension is then an extension of θ to $\mathrm{N}_G(T)_\theta$ as claimed.

Now since $\mathrm{N}_G(T)_\theta/T$ is abelian in any case, it follows by the theorem of Gallagher, Theorem 1.21, that $\mathrm{Irr}(\mathrm{N}_G(T)_\theta \mid \theta)$ only consists of extensions of θ, which concludes the proof. □

Lemma 16.29. *Suppose that $q \geqslant 5$. Let B be a 3-block of G with abelian defect group $\mathcal{O}_3(T)$ and suppose that $\theta \in \mathrm{Irr}(T)$ is the canonical character of a 3-block $b \in \mathrm{Bl}_3(T)$ satisfying $b^G = B$. Moreover, set $\widetilde{\theta} := \theta_{|\widetilde{T}} \in \mathrm{Irr}(\widetilde{T})$ and assume that $\mathrm{N}_G(T)_\theta/T \cong C_2$ and $|\,\mathrm{N}_G(T)_{\widetilde{\theta}} : \mathrm{N}_G(T)_\theta| = 2$. Then*

(i) $\mathrm{Irr}(\mathrm{N}_{\widetilde{G}}(\widetilde{T})_{\widetilde{\theta}} \mid \widetilde{\theta})$ only consists of extensions of $\widetilde{\theta}$, and

(ii) any element of $\mathrm{Irr}(\mathrm{N}_{\widetilde{G}}(\widetilde{T})_\theta \mid \widetilde{\theta})$ can be obtained as the restriction of a character in $\mathrm{Irr}(\mathrm{N}_{\widetilde{G}}(\widetilde{T})_{\widetilde{\theta}} \mid \widetilde{\theta})$ to $\mathrm{N}_{\widetilde{G}}(\widetilde{T})_\theta$.

In particular, the characters in $\mathrm{Irr}(\mathrm{N}_{\widetilde{G}}(\widetilde{T})_\theta \mid \widetilde{\theta})$ are invariant under $\mathrm{N}_{\widetilde{G}}(\widetilde{T})_{\widetilde{\theta}}$-conjugation.

Proof. (i) It holds that $\mathrm{N}_G(T)_{\widetilde{\theta}}/T \cong C_2 \times C_2$, and hence also $\mathrm{N}_{\widetilde{G}}(\widetilde{T})_{\widetilde{\theta}}/\widetilde{T} \cong C_2 \times C_2$. In particular, $\widetilde{\theta}$ is stabilized by the generator of $\mathrm{Z}(\mathrm{N}_{\widetilde{G}}(\widetilde{T})/\widetilde{T})$, which acts on $\mathrm{Irr}(\widetilde{T})$ by inversion, so it follows that $\widetilde{\theta}^{-1} = \widetilde{\theta} \neq 1_{\widetilde{T}}$. Thus, in Remark 12.35 we have already proven that $\widetilde{\theta}$ extends to $\mathrm{N}_{\widetilde{G}}(\widetilde{T})_{\widetilde{\theta}}$, whence the claim now follows from Gallagher's theorem, Theorem 1.21, and the fact that $\mathrm{N}_{\widetilde{G}}(\widetilde{T})_{\widetilde{\theta}}/\widetilde{T} \cong C_2 \times C_2$ is abelian.

(ii) Let $\chi \in \operatorname{Irr}(\mathrm{N}_{\widetilde{G}}(\widetilde{T})_{\widetilde{\theta}} \mid \widetilde{\theta})$ be an extension of $\widetilde{\theta}$. By Theorem 1.21 we have

$$\operatorname{Irr}(\mathrm{N}_{\widetilde{G}}(\widetilde{T})_{\widetilde{\theta}} \mid \widetilde{\theta}) = \{\chi \cdot \beta \mid \beta \in \operatorname{Irr}(\mathrm{N}_{\widetilde{G}}(\widetilde{T})_{\widetilde{\theta}}/\widetilde{T})\}.$$

Now $\chi_{|\mathrm{N}_{\widetilde{G}}(\widetilde{T})_\theta}$ is an extension of $\widetilde{\theta}$ to $\mathrm{N}_{\widetilde{G}}(\widetilde{T})_\theta$, so again by Theorem 1.21 it holds that

$$\operatorname{Irr}(\mathrm{N}_{\widetilde{G}}(\widetilde{T})_\theta \mid \widetilde{\theta}) = \{\chi_{|\mathrm{N}_{\widetilde{G}}(\widetilde{T})_\theta} \cdot \eta \mid \eta \in \operatorname{Irr}(\mathrm{N}_{\widetilde{G}}(\widetilde{T})_\theta/\widetilde{T})\}.$$

Since $\mathrm{N}_{\widetilde{G}}(\widetilde{T})_\theta/\widetilde{T} \cong C_2$ is normal in $\mathrm{N}_{\widetilde{G}}(\widetilde{T})_{\widetilde{\theta}}/\widetilde{T}$ due to the fact that $|\mathrm{N}_{\widetilde{G}}(\widetilde{T})_{\widetilde{\theta}} : \mathrm{N}_{\widetilde{G}}(\widetilde{T})_\theta| = 2$ by Lemma 16.27(ii), it follows that $\mathrm{N}_{\widetilde{G}}(\widetilde{T})_{\widetilde{\theta}}/\widetilde{T}$ acts on $\operatorname{Irr}(\mathrm{N}_{\widetilde{G}}(\widetilde{T})_\theta/\widetilde{T})$. This set consists of exactly two irreducible characters since $\mathrm{N}_{\widetilde{G}}(\widetilde{T})_\theta/\widetilde{T} \cong C_2$, so these are clearly stabilized by $\mathrm{N}_{\widetilde{G}}(\widetilde{T})_{\widetilde{\theta}}/\widetilde{T}$. Hence, for every $\eta \in \operatorname{Irr}(\mathrm{N}_{\widetilde{G}}(\widetilde{T})_\theta/\widetilde{T})$ there exists $\beta \in \operatorname{Irr}(\mathrm{N}_{\widetilde{G}}(\widetilde{T})_{\widetilde{\theta}}/\widetilde{T})$ which extends η, so the claim follows. □

Lemma 16.30. *Suppose that $q \geqslant 5$. Let $\varphi \in \operatorname{Irr}(\mathrm{N}_G(T))$ and $\theta \in \operatorname{Irr}(T)$ be an irreducible constituent of $\varphi_{|T}$. If $\psi \in \operatorname{Irr}(\mathrm{N}_{\widetilde{G}}(\widetilde{T}))$ is an irreducible constituent of $\varphi_{|\mathrm{N}_{\widetilde{G}}(\widetilde{T})}$, then ψ lies above $\widetilde{\theta} := \theta_{|\widetilde{T}} \in \operatorname{Irr}(\widetilde{T})$.*

In particular, if $\mathcal{O}_3(T) \subseteq \ker(\theta)$, then $\mathcal{O}_3(\widetilde{T})$ is contained in the kernel of ψ.

Proof. Let $r \in \mathbb{N}_{>0}$, $e_i \in \mathbb{Z}_{>0}$ and $\psi_i \in \operatorname{Irr}(\mathrm{N}_{\widetilde{G}}(\widetilde{T}))$ for $i \in \{1, \dots, r\}$ with $\psi_i \neq \psi_j$ if $i \neq j$ be such that

$$\varphi_{|\mathrm{N}_{\widetilde{G}}(\widetilde{T})} = \sum_{i=1}^{r} e_i \psi_i.$$

By Clifford theory, Theorem 1.15, we may write

$$\varphi_{|T} = f \cdot \sum_{j=1}^{t} \theta_j$$

for some $f \in \mathbb{N}_{>0}$, $t = |\mathrm{N}_G(T) : \mathrm{N}_G(T)_\theta|$ and $\theta_1 = \theta, \theta_2, \dots, \theta_t$ the $\mathrm{N}_G(T)$-conjugates of θ. Now let $i \in \{1, \dots, r\}$ be such that $\psi = \psi_i$. The restricted character $\psi_{|\widetilde{T}}$ is a summand of $\varphi_{|\widetilde{T}} = (\varphi_{|T})_{|\widetilde{T}}$, so $\theta_{j|\widetilde{T}}$ is a constituent of $\psi_{|\widetilde{T}}$ for some j. Since

$$\mathrm{N}_G(T) = \mathrm{N}_{\widetilde{G}}(\widetilde{T})T$$

by Proposition 16.11, the character θ_j is in fact conjugate to θ via an element of $\mathrm{N}_{\widetilde{G}}(\widetilde{T})$, so in particular it follows that $(\theta_j)_{|\widetilde{T}}$ and $\widetilde{\theta} = \theta_{|\widetilde{T}}$ are $\mathrm{N}_{\widetilde{G}}(\widetilde{T})$-conjugate. Hence, by Clifford theory the character $\widetilde{\theta}$ is a constituent of $\psi_{|\widetilde{T}}$ as claimed.

If now $\mathcal{O}_3(T) \subseteq \ker(\theta)$, then also $\mathcal{O}_3(\widetilde{T}) \subseteq \ker(\widetilde{\theta})$, and since $\mathcal{O}_3(\widetilde{T})$ is normal in $\mathrm{N}_{\widetilde{G}}(\widetilde{T})$, it lies in the kernel of each $\mathrm{N}_{\widetilde{G}}(\widetilde{T})$-conjugate of $\widetilde{\theta}$, which proves the claim. □

Proposition 16.31. *Suppose that $q \geqslant 5$. Let B be a 3-block of G with abelian defect group $R = \mathcal{O}_3(T)$ and suppose that (R, φ) is a B-weight. Moreover, denote by $\theta \in \operatorname{Irr}(T)$ an irreducible constituent of $\varphi_{|T}$ and define $\widetilde{\theta} := \theta_{|\widetilde{T}} \in \operatorname{Irr}(\widetilde{T})$ and $\widetilde{\varphi} := \varphi_{|\mathrm{N}_{\widetilde{G}}(\widetilde{T})}$.*

(i) If $\mathrm{N}_G(T)_\theta = \mathrm{N}_G(T)_{\widetilde{\theta}}$, then $\widetilde{\varphi}$ is irreducible.

(ii) If $\mathrm{N}_G(T)_\theta/T \cong C_2$ and $|\mathrm{N}_G(T)_{\widetilde{\theta}} : \mathrm{N}_G(T)_\theta| = 2$, then $\widetilde{\varphi}$ is the sum of two distinct irreducible characters of $\mathrm{N}_{\widetilde{G}}(\widetilde{T})$ of the same degree.

In particular, if χ *is an irreducible constituent of* $\widetilde{\varphi}$*, then* $(\widetilde{R}, \chi)$ *is a* $\widetilde{B}$*-weight for* $\widetilde{G}$*, where* $\widetilde{B} = \mathrm{bl}(\widetilde{\theta})^{\widetilde{G}}$ *with defect group* $\widetilde{R} = \mathcal{O}_3(\widetilde{T})$*.*

Proof. Let $r \in \mathbb{N}_{>0}$, $e_i \in \mathbb{Z}_{>0}$ and $\psi_i \in \mathrm{Irr}(\mathrm{N}_{\widetilde{G}}(\widetilde{T}))$ for $i \in \{1, \dots, r\}$ with $\psi_i \neq \psi_j$ if $i \neq j$ be such that

$$\widetilde{\varphi} = \sum_{i=1}^{r} e_i \psi_i.$$

By Construction 2.10 it holds that $\varphi = \psi^{\mathrm{N}_G(T)}$ for some $\psi \in \mathrm{Irr}(\mathrm{N}_G(T)_\theta \mid \theta)$ satisfying $\psi(1)_3 = |\,\mathrm{N}_G(T)_\theta \colon T\,|_3$, and the 3-block $\mathrm{bl}(\theta)$ with $\mathrm{bl}(\theta)^G = B$ has canonical character θ. By Lemma 16.28 the set $\mathrm{Irr}(\mathrm{N}_G(T)_\theta \mid \theta)$ only consists of extensions of θ. In particular, $\psi(1) = 1$, and thus

$$\varphi(1) = |\,\mathrm{N}_G(T) \colon \mathrm{N}_G(T)_\theta\,|.$$

On the other hand, consider the restriction $\psi_{i|\widetilde{T}}$ for $i \in \{1, \dots, r\}$. By Lemma 16.30 all characters ψ_i lie above $\widetilde{\theta}$, so by Clifford theory, Theorem 1.15, there exist $f_i \in \mathbb{Z}_{>0}$ such that

$$\psi_{i|\widetilde{T}} = f_i \sum_{j=1}^{t} \widetilde{\theta}_j,$$

where $t = |\,\mathrm{N}_{\widetilde{G}}(\widetilde{T}) \colon \mathrm{N}_{\widetilde{G}}(\widetilde{T})_{\widetilde{\theta}}\,|$ and $\widetilde{\theta}_1 = \widetilde{\theta}, \dots, \widetilde{\theta}_t$ are the $\mathrm{N}_{\widetilde{G}}(\widetilde{T})$-conjugates of $\widetilde{\theta}$. Using this decomposition the degree of φ is given by

$$\varphi(1) = \widetilde{\varphi}(1) = \sum_{i=1}^{r} e_i \psi_i(1) = \sum_{i=1}^{r} e_i f_i \sum_{j=1}^{t} \widetilde{\theta}_j(1) = |\,\mathrm{N}_{\widetilde{G}}(\widetilde{T}) \colon \mathrm{N}_{\widetilde{G}}(\widetilde{T})_{\widetilde{\theta}}\,| \sum_{i=1}^{r} e_i f_i.$$

If $\mathrm{N}_G(T)_\theta = \mathrm{N}_G(T)_{\widetilde{\theta}}$, then it follows easily that $|\,\mathrm{N}_G(T) \colon \mathrm{N}_G(T)_\theta\,|$ and $|\,\mathrm{N}_{\widetilde{G}}(\widetilde{T}) \colon \mathrm{N}_{\widetilde{G}}(\widetilde{T})_{\widetilde{\theta}}\,|$ coincide, so comparison of the two formulae for the degree of φ yields

$$\sum_{i=1}^{r} e_i f_i = 1,$$

that is, $r = 1$ and $e_1 = f_1 = 1$. Hence, $\widetilde{\varphi} = \psi_1$ is irreducible as claimed in (i).

Suppose hence that $\mathrm{N}_G(T)_\theta / T \cong C_2$ and $|\,\mathrm{N}_G(T)_{\widetilde{\theta}} \colon \mathrm{N}_G(T)_\theta\,| = 2$ as in (ii). It follows that

$$|\,\mathrm{N}_G(T) \colon \mathrm{N}_G(T)_\theta\,| = 2\,|\,\mathrm{N}_{\widetilde{G}}(\widetilde{T}) \colon \mathrm{N}_{\widetilde{G}}(\widetilde{T})_{\widetilde{\theta}}\,|,$$

so in this case we have $\sum_{i=1}^{r} e_i f_i = 2$, whence $\widetilde{\varphi}$ must be the sum of at most two irreducible constituents.

Let us go into more detail. As observed above, it holds that $\varphi = \psi^{\mathrm{N}_G(T)}$ for some linear extension $\psi \in \mathrm{Irr}(\mathrm{N}_G(T)_\theta \mid \theta)$ of θ. By Proposition 16.11 we have $\mathrm{N}_G(T) = \mathrm{N}_G(T)_\theta\, \mathrm{N}_{\widetilde{G}}(\widetilde{T})$, so Mackey's theorem, Theorem 1.17, implies that

$$\widetilde{\varphi} = \mathrm{Res}^{\mathrm{N}_G(T)}_{\mathrm{N}_{\widetilde{G}}(\widetilde{T})}(\mathrm{Ind}^{\mathrm{N}_G(T)}_{\mathrm{N}_G(T)_\theta}(\psi)) = \mathrm{Ind}^{\mathrm{N}_{\widetilde{G}}(\widetilde{T})}_{\mathrm{N}_{\widetilde{G}}(\widetilde{T})_\theta}(\mathrm{Res}^{\mathrm{N}_G(T)_\theta}_{\mathrm{N}_{\widetilde{G}}(\widetilde{T})_\theta}(\psi))$$

since $\mathrm{N}_G(T)_\theta \cap \mathrm{N}_{\widetilde{G}}(\widetilde{T}) = \mathrm{N}_{\widetilde{G}}(\widetilde{T})_\theta$. Moreover, since $\mathrm{N}_G(T)_\theta \subsetneqq \mathrm{N}_G(T)_{\widetilde{\theta}} \subsetneqq \mathrm{N}_G(T)$, we have

$$\widetilde{\varphi} = \mathrm{Ind}^{\mathrm{N}_{\widetilde{G}}(\widetilde{T})}_{\mathrm{N}_{\widetilde{G}}(\widetilde{T})_{\widetilde{\theta}}}(\mathrm{Ind}^{\mathrm{N}_{\widetilde{G}}(\widetilde{T})_{\widetilde{\theta}}}_{\mathrm{N}_{\widetilde{G}}(\widetilde{T})_\theta}(\mathrm{Res}^{\mathrm{N}_G(T)_\theta}_{\mathrm{N}_{\widetilde{G}}(\widetilde{T})_\theta}(\psi))).$$

We examine the character $\Upsilon := \operatorname{Ind}_{\mathrm{N}_{\widetilde{G}}(\widetilde{T})_\theta}^{\mathrm{N}_{\widetilde{G}}(\widetilde{T})_{\widetilde{\theta}}}(\operatorname{Res}_{\mathrm{N}_{\widetilde{G}}(\widetilde{T})_\theta}^{\mathrm{N}_G(T)_\theta}(\psi))$. By definition it holds that

$$\Upsilon(x) = \frac{1}{|\,\mathrm{N}_{\widetilde{G}}(\widetilde{T})_\theta\,|} \sum_{n \in \mathrm{N}_{\widetilde{G}}(\widetilde{T})_{\widetilde{\theta}}} \dot{\psi}(nxn^{-1})$$

for $x \in \mathrm{N}_{\widetilde{G}}(\widetilde{T})_{\widetilde{\theta}}$, where we have $\dot{\psi}(h) = \psi(h)$ if $h \in \mathrm{N}_{\widetilde{G}}(\widetilde{T})_\theta$ and $\dot{\psi}(h) = 0$ else. Now for $x \in \widetilde{T}$ it follows that

$$\dot{\psi}(nxn^{-1}) = \widetilde{\theta}(nxn^{-1}) = \widetilde{\theta}^n(x) = \widetilde{\theta}(x)$$

for all $n \in \mathrm{N}_{\widetilde{G}}(\widetilde{T})_{\widetilde{\theta}}$, so

$$\Upsilon_{|\widetilde{T}} = \frac{|\,\mathrm{N}_{\widetilde{G}}(\widetilde{T})_{\widetilde{\theta}}\,|}{|\,\mathrm{N}_{\widetilde{G}}(\widetilde{T})_\theta|}\,\widetilde{\theta} = 2\widetilde{\theta}$$

by Lemma 16.27(ii). From this we deduce that any irreducible constituent of Υ lies above $\widetilde{\theta}$, so in particular any such irreducible constituent is linear in consequence of Lemma 16.29(i). Since Υ has degree 2, it follows that it is the sum of exactly two linear constituents lying above $\widetilde{\theta}$, say

$$\Upsilon = \chi_1 + \chi_2$$

for $\chi_1, \chi_2 \in \operatorname{Irr}(\mathrm{N}_{\widetilde{G}}(\widetilde{T})_{\widetilde{\theta}} \mid \widetilde{\theta})$. Since $\mathrm{N}_{\widetilde{G}}(\widetilde{T})_\theta$ has index 2 in $\mathrm{N}_{\widetilde{G}}(\widetilde{T})_{\widetilde{\theta}}$, it is normal in $\mathrm{N}_{\widetilde{G}}(\widetilde{T})_{\widetilde{\theta}}$. In particular, for $x, n \in \mathrm{N}_{\widetilde{G}}(\widetilde{T})_{\widetilde{\theta}}$ we have

$$nxn^{-1} \in \mathrm{N}_{\widetilde{G}}(\widetilde{T})_\theta \quad \text{if and only if} \quad x \in \mathrm{N}_{\widetilde{G}}(\widetilde{T})_\theta.$$

Thus, $\Upsilon(x) = 0$ whenever $x \in \mathrm{N}_{\widetilde{G}}(\widetilde{T})_{\widetilde{\theta}} \setminus \mathrm{N}_{\widetilde{G}}(\widetilde{T})_\theta$. But Υ is the sum of the two linear characters χ_1 and χ_2, so $\Upsilon(x) = 0$ is only possible if $\chi_1 \neq \chi_2$. We conclude that

$$\widetilde{\varphi} = \Upsilon^{\mathrm{N}_{\widetilde{G}}(\widetilde{T})} = (\chi_1 + \chi_2)^{\mathrm{N}_{\widetilde{G}}(\widetilde{T})} = (\chi_1)^{\mathrm{N}_{\widetilde{G}}(\widetilde{T})} + (\chi_2)^{\mathrm{N}_{\widetilde{G}}(\widetilde{T})}.$$

Now $\chi_1, \chi_2 \in \operatorname{Irr}(\mathrm{N}_{\widetilde{G}}(\widetilde{T})_{\widetilde{\theta}} \mid \widetilde{\theta})$ are distinct, and following Clifford correspondence, Theorem 1.16, the map

$$\operatorname{Irr}(\mathrm{N}_{\widetilde{G}}(\widetilde{T})_{\widetilde{\theta}} \mid \widetilde{\theta}) \longrightarrow \operatorname{Irr}(\mathrm{N}_{\widetilde{G}}(\widetilde{T}) \mid \widetilde{\theta}), \quad \vartheta \longmapsto \operatorname{Ind}_{\mathrm{N}_{\widetilde{G}}(\widetilde{T})_{\widetilde{\theta}}}^{\mathrm{N}_{\widetilde{G}}(\widetilde{T})}(\vartheta),$$

is a bijection. Thus, also $(\chi_1)^{\mathrm{N}_{\widetilde{G}}(\widetilde{T})}$ and $(\chi_2)^{\mathrm{N}_{\widetilde{G}}(\widetilde{T})}$ must be distinct, irreducible, and clearly both of the same degree, as claimed.

For the last statement, suppose that χ is an irreducible constituent of $\widetilde{\varphi}$. We have shown that

$$\chi(1)_3 = \varphi(1)_3 = |\,\mathrm{N}_G(R)/R\,|_3,$$

and since

$$|\,\mathrm{N}_G(R)/R\,|_3 = 3 = |\,\mathrm{N}_{\widetilde{G}}(\widetilde{R})/\widetilde{R}\,|_3$$

and $\widetilde{R} \subseteq \ker(\chi)$ by Lemma 16.30, it follows that $(\widetilde{R}, \chi)$ is a 3-weight for $\widetilde{G}$. Moreover, since by Lemma 16.30 the character χ lies above $\widetilde{\theta}$, we conclude that the 3-weight $(\widetilde{R}, \widetilde{\varphi})$ belongs to $\widetilde{B} = \operatorname{bl}(\widetilde{\theta})^{\widetilde{G}}$ in consequence of Construction 2.10. We observed in the proof of Lemma 16.24 that $|\,\mathrm{N}_G(T)_{\widetilde{\theta}} \colon T\,|$, and hence also $|\,\mathrm{N}_{\widetilde{G}}(\widetilde{T})_{\widetilde{\theta}} \colon \widetilde{T}\,|$, is a divisor of 4. Since $\operatorname{bl}(\widetilde{\theta})$ has defect group $\widetilde{R} = \mathcal{O}_3(T)$, Theorem 1.10 implies that also $\widetilde{B} = \operatorname{bl}(\widetilde{\theta})^{\widetilde{G}}$ must have defect group $\widetilde{R}$. This finishes the proof. $\square$

Proposition 16.32. *Suppose that $q \geqslant 5$. Let B be a 3-block of G with abelian defect group $R = \mathcal{O}_3(T)$ and suppose that (R, φ_1) and (R, φ_2) are B-weights. Then the characters ${\varphi_1}_{|\mathrm{N}_{\widetilde{G}}(\widetilde{T})}$ and ${\varphi_2}_{|\mathrm{N}_{\widetilde{G}}(\widetilde{T})}$ have a common irreducible constituent if and only if $\varphi_1 = \varphi_2$.*

Proof. Let $\theta_1, \theta_2 \in \mathrm{Irr}(T)$ be irreducible constituents of ${\varphi_1}_{|T}$ and ${\varphi_2}_{|T}$, respectively. Following Construction 2.10 we have $\mathrm{bl}(\theta_1)^G = \mathrm{bl}(\theta_2)^G = B$, whence by the extended first main theorem of Brauer, Theorem 1.7, the 3-blocks $\mathrm{bl}(\theta_1)$ and $\mathrm{bl}(\theta_2)$ are conjugate under $\mathrm{N}_G(R) = \mathrm{N}_G(T)$, so in particular their canonical characters θ_1 and θ_2 are conjugate under $\mathrm{N}_G(T)$. According to Clifford, Theorem 1.15, all $\mathrm{N}_G(T)$-conjugates of θ_1 occur with the same multiplicity as irreducible constituents of ${\varphi_1}_{|T}$, so without loss of generality we may assume that $\theta := \theta_1 = \theta_2$.

In consequence of Proposition 16.25 the claim is trivially true if $\mathrm{N}_G(T)_\theta = T$, so let us assume from now that $\mathrm{N}_G(T)_\theta \gneqq T$. In particular, from Lemma 16.24 we deduce that for $\widetilde{\theta} := \theta_{|\widetilde{T}} \in \mathrm{Irr}(\widetilde{T})$ one of the following holds:

(i) $\mathrm{N}_G(T)_\theta = \mathrm{N}_G(T)_{\widetilde{\theta}}$,

(ii) $\mathrm{N}_G(T)_\theta / T \cong C_2$ and $|\,\mathrm{N}_G(T)_{\widetilde{\theta}} : \mathrm{N}_G(T)_\theta| = 2$.

Following Construction 2.10 there exist $\psi_1, \psi_2 \in \mathrm{Irr}(\mathrm{N}_G(T)_\theta \mid \theta)$ such that

$$\varphi_1 = \mathrm{Ind}_{\mathrm{N}_G(T)_\theta}^{\mathrm{N}_G(T)}(\psi_1) \quad \text{and} \quad \varphi_2 = \mathrm{Ind}_{\mathrm{N}_G(T)_\theta}^{\mathrm{N}_G(T)}(\psi_2).$$

Hence, if we set $\widetilde{\varphi_1} := {\varphi_1}_{|\mathrm{N}_{\widetilde{G}}(\widetilde{T})}$ and $\widetilde{\varphi_2} := {\varphi_2}_{|\mathrm{N}_{\widetilde{G}}(\widetilde{T})}$, then we have

$$\widetilde{\varphi_i} = \mathrm{Res}_{\mathrm{N}_{\widetilde{G}}(\widetilde{T})}^{\mathrm{N}_G(T)} \mathrm{Ind}_{\mathrm{N}_G(T)_\theta}^{\mathrm{N}_G(T)}(\psi_i)$$

for $i = 1, 2$. Since $\mathrm{N}_G(T) = \mathrm{N}_{\widetilde{G}}(\widetilde{T})T$ by Proposition 16.11 and $T \subseteq \mathrm{N}_G(T)_\theta$, we have $\mathrm{N}_G(T) = \mathrm{N}_G(T)_\theta \, \mathrm{N}_{\widetilde{G}}(\widetilde{T})$. Thus, as in the proof of Proposition 16.31, by Mackey's formula, Theorem 1.17, it follows that

$$\widetilde{\varphi_i} = \mathrm{Ind}_{\mathrm{N}_{\widetilde{G}}(\widetilde{T})_\theta}^{\mathrm{N}_{\widetilde{G}}(\widetilde{T})} \mathrm{Res}_{\mathrm{N}_{\widetilde{G}}(\widetilde{T})_\theta}^{\mathrm{N}_G(T)_\theta}(\psi_i)$$

for $i = 1, 2$. Let us suppose first that $\mathrm{N}_G(T)_\theta = \mathrm{N}_G(T)_{\widetilde{\theta}}$ as in (i) above. In particular, $\widetilde{\varphi_1}$ and $\widetilde{\varphi_2}$ are irreducible as shown in Proposition 16.31. As in the proof of Proposition 16.31 we consider the bijection

$$\mathrm{Irr}(\mathrm{N}_{\widetilde{G}}(\widetilde{T})_{\widetilde{\theta}} \mid \widetilde{\theta}) \longrightarrow \mathrm{Irr}(\mathrm{N}_{\widetilde{G}}(\widetilde{T}) \mid \widetilde{\theta}), \quad \vartheta \longmapsto \mathrm{Ind}_{\mathrm{N}_{\widetilde{G}}(\widetilde{T})_{\widetilde{\theta}}}^{\mathrm{N}_{\widetilde{G}}(\widetilde{T})}(\vartheta).$$

Since $\mathrm{N}_G(T)_\theta = \mathrm{N}_G(T)_{\widetilde{\theta}}$, we also have $\mathrm{N}_{\widetilde{G}}(\widetilde{T})_\theta = \mathrm{N}_{\widetilde{G}}(\widetilde{T})_{\widetilde{\theta}}$ by Lemma 16.27(i), whence the above bijection is the same as

$$\mathrm{Irr}(\mathrm{N}_{\widetilde{G}}(\widetilde{T})_\theta \mid \widetilde{\theta}) \longrightarrow \mathrm{Irr}(\mathrm{N}_{\widetilde{G}}(\widetilde{T}) \mid \widetilde{\theta}), \quad \vartheta \longmapsto \mathrm{Ind}_{\mathrm{N}_{\widetilde{G}}(\widetilde{T})_\theta}^{\mathrm{N}_{\widetilde{G}}(\widetilde{T})}(\vartheta).$$

By Lemma 16.28 the characters ψ_1 and ψ_2 are extensions of θ. In particular, $\mathrm{Res}_{\mathrm{N}_{\widetilde{G}}(\widetilde{T})_\theta}^{\mathrm{N}_G(T)_\theta}(\psi_i)$ is irreducible for $i = 1, 2$ and lies above $\widetilde{\theta}$. Hence, it follows that

$$\widetilde{\varphi_1} = \widetilde{\varphi_2} \quad \text{if and only if} \quad \mathrm{Res}_{\mathrm{N}_{\widetilde{G}}(\widetilde{T})_\theta}^{\mathrm{N}_G(T)_\theta}(\psi_1) = \mathrm{Res}_{\mathrm{N}_{\widetilde{G}}(\widetilde{T})_\theta}^{\mathrm{N}_G(T)_\theta}(\psi_2).$$

Both the characters ψ_1 and ψ_2 are extensions of θ to $\mathrm{N}_G(T)_\theta$, so according to Gallagher, Theorem 1.21, there exists $\eta \in \mathrm{Irr}(\mathrm{N}_G(T)_\theta/T)$ such that $\psi_2 = \psi_1 \cdot \eta$. Now suppose that

$$\mathrm{Res}^{\mathrm{N}_G(T)_\theta}_{\mathrm{N}_{\widetilde{G}}(\widetilde{T})_\theta}(\psi_1) = \mathrm{Res}^{\mathrm{N}_G(T)_\theta}_{\mathrm{N}_{\widetilde{G}}(\widetilde{T})_\theta}(\psi_2)$$

and let $n \in \mathrm{N}_G(T)_\theta$. Since $\mathrm{N}_G(T)_\theta = T\,\mathrm{N}_{\widetilde{G}}(\widetilde{T})_\theta$, there exist $t \in T$ and $m \in \mathrm{N}_{\widetilde{G}}(\widetilde{T})_\theta$ such that $n = tm$. It follows that

$$\theta(t)\psi_1(m)\eta(nT) = \psi_1(n)\eta(nT) = \psi_2(n) = \theta(t)\psi_2(m) = \theta(t)\psi_1(m),$$

and since ψ_1 and θ are linear, we have $\psi_1(m), \theta(t) \neq 0$, so $\eta(nT) = 1$, that is, $\eta = 1_{\mathrm{N}_G(T)_\theta/T}$ in this case. Hence, ψ_1 and ψ_2 agree, and thus $\varphi_1 = \varphi_2$ as claimed.

Let us finally suppose that $\mathrm{N}_G(T)_\theta/T \cong C_2$ and $|\,\mathrm{N}_G(T)_{\widetilde{\theta}} : \mathrm{N}_G(T)_\theta\,| = 2$ as in case (ii) above. In the proof of Proposition 16.31 we observed that there exist linear characters $\chi_{11}, \chi_{12}, \chi_{21}, \chi_{22} \in \mathrm{Irr}(\mathrm{N}_{\widetilde{G}}(\widetilde{T})_{\widetilde{\theta}} \mid \widetilde{\theta})$ with $\chi_{11} \neq \chi_{12}$ and $\chi_{21} \neq \chi_{22}$ such that

$$\widetilde{\varphi_i} = (\chi_{i1})^{\mathrm{N}_{\widetilde{G}}(\widetilde{T})} + (\chi_{i2})^{\mathrm{N}_{\widetilde{G}}(\widetilde{T})}$$

with $(\chi_{ij})^{\mathrm{N}_{\widetilde{G}}(\widetilde{T})} = (\chi_{i'j'})^{\mathrm{N}_{\widetilde{G}}(\widetilde{T})}$ if and only if $\chi_{ij} = \chi_{i'j'}$ for $i, i', j, j' \in \{1, 2\}$, and moreover

$$\Upsilon_i := \mathrm{Ind}^{\mathrm{N}_{\widetilde{G}}(\widetilde{T})_{\widetilde{\theta}}}_{\mathrm{N}_{\widetilde{G}}(\widetilde{T})_\theta}(\mathrm{Res}^{\mathrm{N}_G(T)_\theta}_{\mathrm{N}_{\widetilde{G}}(\widetilde{T})_\theta}(\psi_i)) = \chi_{i1} + \chi_{i2}$$

for $i = 1, 2$. Now the restriction of ψ_i to $\mathrm{N}_{\widetilde{G}}(\widetilde{T})_\theta$ is an extension of $\widetilde{\theta}$ to $\mathrm{N}_{\widetilde{G}}(\widetilde{T})_\theta$ for $i = 1, 2$. Hence, by Lemma 16.29 it is invariant under the conjugation action of $\mathrm{N}_{\widetilde{G}}(\widetilde{T})_{\widetilde{\theta}}$. Thus, for $x \in \mathrm{N}_{\widetilde{G}}(\widetilde{T})_\theta$ we have

$$\Upsilon_i(x) = \frac{1}{|\,\mathrm{N}_{\widetilde{G}}(\widetilde{T})_\theta\,|} \sum_{n \in \mathrm{N}_{\widetilde{G}}(\widetilde{T})_{\widetilde{\theta}}} \dot{\psi}_i(nxn^{-1}) = \frac{1}{|\,\mathrm{N}_{\widetilde{G}}(\widetilde{T})_\theta\,|} \sum_{n \in \mathrm{N}_{\widetilde{G}}(\widetilde{T})_{\widetilde{\theta}}} \psi_i(x) = 2\psi_i(x),$$

that is,

$$\Upsilon_{i|\,\mathrm{N}_{\widetilde{G}}(\widetilde{T})_\theta} = 2\psi_{i|\,\mathrm{N}_{\widetilde{G}}(\widetilde{T})_\theta},$$

and hence

$$\chi_{i1|\,\mathrm{N}_{\widetilde{G}}(\widetilde{T})_\theta} = \chi_{i2|\,\mathrm{N}_{\widetilde{G}}(\widetilde{T})_\theta} = \psi_{i|\,\mathrm{N}_{\widetilde{G}}(\widetilde{T})_\theta}$$

for $i = 1, 2$. Now suppose that $\widetilde{\varphi_1}$ and $\widetilde{\varphi_2}$ have a common irreducible constituent. Then there exist $j, j' \in \{1, 2\}$ such that $(\chi_{1j})^{\mathrm{N}_{\widetilde{G}}(\widetilde{T})} = (\chi_{2j'})^{\mathrm{N}_{\widetilde{G}}(\widetilde{T})}$, and thus $\chi_{1j} = \chi_{2j'}$. By the observation above it follows that

$$\psi_{1|\,\mathrm{N}_{\widetilde{G}}(\widetilde{T})_\theta} = \chi_{1j|\,\mathrm{N}_{\widetilde{G}}(\widetilde{T})_\theta} = \chi_{2j'|\,\mathrm{N}_{\widetilde{G}}(\widetilde{T})_\theta} = \psi_{2|\,\mathrm{N}_{\widetilde{G}}(\widetilde{T})_\theta}.$$

Now the same argumentation as for case (i) yields that $\psi_1 = \psi_2$, whence also $\varphi_1 = \varphi_2$. This completes the proof. □

Let us pause for a moment and summarize what we have proven so far: If $q \geqslant 5$ and B is a 3-block of G with abelian defect group $R = \mathcal{O}_3(T)$, then either $|\mathcal{W}(B)| = 1$, in which case $\mathrm{Aut}(G)_B$ clearly acts trivially on $\mathcal{W}(B)$, or $|\mathcal{W}(B)| > 1$ and for a B-weight (R, φ) the set

$$\{(\widetilde{R}, \psi) \mid \psi \text{ irreducible constituent of } \varphi_{|\,\mathrm{N}_{\widetilde{G}}(\widetilde{T})}\},$$

where $\widetilde{R} = \mathcal{O}_3(\widetilde{T})$, consists of pairwise distinct $\widetilde{B}$-weights of $\widetilde{G}$, where $\widetilde{B} = \mathrm{bl}(\widetilde{\theta})^{\widetilde{G}}$ with $\widetilde{\theta} := \theta_{|\widetilde{T}} \in \mathrm{Irr}(\widetilde{T})$ for some irreducible constituent $\theta \in \mathrm{Irr}(T)$ of $\varphi_{|T}$. Moreover, two such sets corresponding to distinct B-weights (R, φ_1) and (R, φ_2) are disjoint. Based on this information we may now reduce the question of the behaviour of $\mathcal{W}(B)$ under $\mathrm{Aut}(G)_B$ to a question of the behaviour of $\widetilde{B}$-weights of $\widetilde{G}$ under $\mathrm{Aut}(G)_{B,\widetilde{G}}$, which has already been answered in Part III of this thesis. To this end we introduce the following notation:

Notation 16.33. Suppose that $q \geqslant 5$. We keep the notation $R = \mathcal{O}_3(T)$ and $\widetilde{R} = \mathcal{O}_3(\widetilde{T})$. Moreover we recall that for a 3-weight (R, φ) of G we denote by $[(R, \varphi)]_G$ its G-conjugacy class, and similarly, if $(\widetilde{R}, \psi)$ is a 3-weight of $\widetilde{G}$, then we denote its $\widetilde{G}$-conjugacy class by $[(\widetilde{R}, \psi)]_{\widetilde{G}}$.

Now let B be a 3-block of G with abelian defect group R such that $\mathrm{N}_G(T)_\theta \gneqq T$ if $\theta \in \mathrm{Irr}(T)$ is the canonical character of a root of B. Note that every G-conjugacy class of B-weights of G has exactly one representative whose first component is given by R. We set

$$\widetilde{\mathcal{W}}(B) := \left\{ \{[(\widetilde{R}, \psi)]_{\widetilde{G}} \mid \psi \text{ irreducible constituent of } \varphi_{|\mathrm{N}_{\widetilde{G}}(\widetilde{T})}\} \;\middle|\; [(R, \varphi)]_G \in \mathcal{W}(B) \right\}.$$

Remark 16.34. In the notation above, if (R, φ) is a B-weight of G, then φ lies above θ by Construction 2.10. Hence, following Lemma 16.24 and Proposition 16.31 every element of $\widetilde{\mathcal{W}}(B)$ is a set consisting of either one or two $\widetilde{G}$-conjugacy classes of $\widetilde{B}$-weights, where $\widetilde{B} = \mathrm{bl}(\widetilde{\theta})^{\widetilde{G}}$ for $\widetilde{\theta} := \theta_{|\widetilde{T}} \in \mathrm{Irr}(\widetilde{T})$. Moreover, any two such sets in $\widetilde{\mathcal{W}}(B)$ are disjoint by Proposition 16.32.

Proposition 16.35. *Suppose that $q \geqslant 5$. Let B be a 3-block of G with abelian defect group $R = \mathcal{O}_3(T)$ such that $\mathrm{N}_G(T)_\theta \gneqq T$ if $\theta \in \mathrm{Irr}(T)$ is the canonical character of a root of B. The map*

$$\begin{aligned} \Lambda\colon \quad \mathcal{W}(B) &\longrightarrow \widetilde{\mathcal{W}}(B), \\ [(R, \varphi)]_G &\longmapsto \{[(\widetilde{R}, \psi)]_{\widetilde{G}} \mid \psi \textit{ irreducible constituent of } \varphi_{|\mathrm{N}_{\widetilde{G}}(\widetilde{T})}\}, \end{aligned}$$

is a bijection.

Proof. Surjectivity is obvious by construction of $\widetilde{\mathcal{W}}(B)$, while injectivity follows as a direct consequence of Proposition 16.32. ☐

Our next aim is to prove that Λ is equivariant under the action of $\mathrm{Aut}(G)_{B,\widetilde{G}}$. For this we first verify that $\mathrm{Aut}(G)_{B,\widetilde{G}}$ does indeed act on $\widetilde{\mathcal{W}}(B)$:

Lemma 16.36. *Suppose that $q \geqslant 5$ and let $a \in \mathrm{Aut}(\widetilde{G})$. If $\widetilde{R}^a = \widetilde{R}$, then $R^a = R$.*

Proof. By Lemma 16.10 we have $\mathrm{C}_G(\widetilde{T}) = T$. Moreover, $\widetilde{T} = \mathrm{C}_{\widetilde{G}}(\widetilde{R})$ by Proposition 12.26, so $T = \mathrm{C}_G(\mathrm{C}_{\widetilde{G}}(\widetilde{R}))$. Hence, a stabilizes T. But R is the unique Sylow 3-subgroup of T, so also $R^a = R$. ☐

Proposition 16.37. *Suppose that $q \geqslant 5$. Let B be a 3-block of G with abelian defect group $R = \mathcal{O}_3(T)$ such that $\mathrm{N}_G(T)_\theta \gneqq T$ if $\theta \in \mathrm{Irr}(T)$ is the canonical character of a root of B. Then the group $\mathrm{Aut}(G)_{B,\widetilde{G}}$ acts on $\widetilde{\mathcal{W}}(B)$.*

Proof. Let $[(R,\varphi)]_G \in \mathcal{W}(B)$ and denote by $\widetilde{\varphi}$ the restriction of φ to $\mathrm{N}_{\widetilde{G}}(\widetilde{T})$. Moreover, let $a \in \mathrm{Aut}(G)_{B,\widetilde{G}}$. Since $\widetilde{R}$ is the unique Sylow 3-subgroup of the maximal torus $\widetilde{T}$ of $\widetilde{G}$, there exists an element $g \in \widetilde{G}$ such that

$$\widetilde{R}^a = \widetilde{R}^g,$$

so for every irreducible constituent ψ of $\widetilde{\varphi}$ we have

$$[(\widetilde{R},\psi)]_{\widetilde{G}}^a = [(\widetilde{R}^a,\psi^a)]_{\widetilde{G}} = [(\widetilde{R},\psi^{ag^{-1}})]_{\widetilde{G}}.$$

By Lemma 16.36 we have $R^{ag^{-1}} = R$, whence in particular $(R,\varphi^{ag^{-1}})$ is a B-weight for G with irreducible constituents of ${\varphi^{ag^{-1}}}_{|\mathrm{N}_{\widetilde{G}}(\widetilde{T})}$ given by the set

$$\{\psi^{ag^{-1}} \mid \psi \text{ irreducible constituent of } \widetilde{\varphi}\}.$$

Hence,

$$\begin{aligned}\{[(\widetilde{R},\psi)]_{\widetilde{G}} \mid \psi &\text{ irreducible constituent of } \widetilde{\varphi}\}^a \\ &= \{[(\widetilde{R},\psi^{ag^{-1}})]_{\widetilde{G}} \mid \psi \text{ irreducible constituent of } \widetilde{\varphi}\} \\ &= \{[(\widetilde{R},\chi)]_{\widetilde{G}} \mid \chi \text{ irreducible constituent of } {\varphi^{ag^{-1}}}_{|\mathrm{N}_{\widetilde{G}}(\widetilde{T})}\} \in \widetilde{\mathcal{W}}(B),\end{aligned}$$

so $\mathrm{Aut}(G)_{B,\widetilde{G}}$ acts on $\widetilde{\mathcal{W}}(B)$ as claimed. □

Theorem 16.38. *The bijection Λ in Proposition 16.35 is $\mathrm{Aut}(G)_{B,\widetilde{G}}$-equivariant.*

Proof. Let $a \in \mathrm{Aut}(G)_{B,\widetilde{G}}$ and $[(R,\varphi)]_G \in \mathcal{W}(B)$. Moreover, let $g \in \widetilde{G}$ be such that $\widetilde{R}^a = \widetilde{R}^g$. Then by Lemma 16.36 also $R^a = R^g$, and as in the proof of Proposition 16.37 we have

$$\begin{aligned}\Lambda([(R,\varphi)]_G)^a &= \{[(\widetilde{R},\psi)]_{\widetilde{G}} \mid \psi \text{ irreducible constituent of } \varphi_{|\mathrm{N}_{\widetilde{G}}(\widetilde{T})}\}^a \\ &= \{[(\widetilde{R},\psi)]_{\widetilde{G}}^a \mid \psi \text{ irreducible constituent of } \varphi_{|\mathrm{N}_{\widetilde{G}}(\widetilde{T})}\} \\ &= \{[(\widetilde{R},\chi)]_{\widetilde{G}} \mid \chi \text{ irreducible constituent of } {\varphi^{ag^{-1}}}_{|\mathrm{N}_{\widetilde{G}}(\widetilde{T})}\} \\ &= \Lambda([(R,\varphi^{ag^{-1}})]_G) \\ &= \Lambda([(R^a,\varphi^a)]_G) \\ &= \Lambda([(R,\varphi)]_G^a).\end{aligned}$$

Hence, Λ is equivariant under the action of $\mathrm{Aut}(G)_{B,\widetilde{G}}$ as claimed. □

We are finally able to prove the following result on the behaviour of B-weights under the action of $\mathrm{Aut}(G)_B$:

Corollary 16.39. *Suppose that $q \geqslant 5$. Let B be a 3-block of G with abelian defect group $R = \mathcal{O}_3(T)$ such that $\mathrm{N}_G(T)_\theta \gneqq T$ if $\theta \in \mathrm{Irr}(T)$ is the canonical character of a root of B. Then $\mathrm{Aut}(G)_B$ acts trivially on $\mathcal{W}(B)$.*

Proof. Let $a \in \mathrm{Aut}(G)_B$. Since all maximal subgroups of G isomorphic to $G_2(q)$ are G-conjugate to each other and inner automorphisms of G do not play a role, we may without loss of generality assume that a stabilizes $\widetilde{G}$, i.e., $a \in \mathrm{Aut}(G)_{B,\widetilde{G}}$. Let $[(R,\varphi)]_G \in \mathcal{W}(B)$. By Theorem 16.38 we have

$$[(R,\varphi)]_G^a = \Lambda^{-1}(\Lambda([(R,\varphi)]_G)^a),$$

where Λ denotes the bijection of Proposition 16.35. Since $3 \nmid q$, in consequence of the results obtained in Chapter 12 every $\widetilde{G}$-conjugacy class of 3-weights in $\widetilde{G}$ belonging to a 3-block of $\widetilde{G}$ of non-cyclic defect is left invariant by any automorphism of $\widetilde{G}$. In particular, it follows that

$$\Lambda([(R,\varphi)]_G)^a = \Lambda([(R,\varphi)]_G)$$

since for any irreducible constituent χ of the restriction of φ to $\mathrm{N}_{\widetilde{G}}(\widetilde{T})$ the 3-weight $(\widetilde{R},\chi)$ belongs to a 3-block of $\widetilde{G}$ with non-cyclic abelian defect group given by $\widetilde{R}$ according to Proposition 16.31. Thus,

$$[(R,\varphi)]_G^a = \Lambda^{-1}(\Lambda([(R,\varphi)]_G)) = [(R,\varphi)]_G,$$

which completes the proof. □

Chapter 17

Partitions and Equivariant Bijections

In this chapter we establish parts (i) and (ii) of the inductive blockwise Alperin weight condition in Definition 3.2 for every ℓ-block B of $G = {}^3D_4(q)$ of non-cyclic defect, where $\ell \in \{2, 3\}$. As it was the case for the previous two series of finite groups we studied here, the group G is its own universal covering group following Proposition 15.2. In particular, once more we aim to prove the following statements for every $\ell \in \{2, 3\}$ and every ℓ-block B of G of non-cyclic defect:

(i) There exist subsets $\mathrm{IBr}(B \mid Q) \subseteq \mathrm{IBr}(B)$ for $Q \in \mathrm{Rad}_\ell(G)$ with the following properties:

  (1) $\mathrm{IBr}(B \mid Q)^a = \mathrm{IBr}(B \mid Q^a)$ for every $Q \in \mathrm{Rad}_\ell(G)$, $a \in \mathrm{Aut}(G)_B$,

  (2) $\mathrm{IBr}(B) = \dot{\bigcup}_{Q \in \mathrm{Rad}_\ell(G)/\sim_G} \mathrm{IBr}(B \mid Q)$.

(ii) For every $Q \in \mathrm{Rad}_\ell(G)$ there exists a bijection

$$\Omega_Q^G\colon\ \mathrm{IBr}(B \mid Q) \longrightarrow \mathrm{dz}(\mathrm{N}_G(Q)/Q,\, B)$$

such that $\Omega_Q^G(\phi)^a = \Omega_{Q^a}^G(\phi^a)$ for every $\phi \in \mathrm{IBr}(B \mid Q)$ and $a \in \mathrm{Aut}(G)_B$.

This will be proven in the same way as for the special linear groups, exploiting the fact that $\mathrm{Aut}(G)_B$ acts trivially on both $\mathrm{IBr}(B)$ and the set $\mathcal{W}(B)$ of G-conjugacy classes of B-weights in G for every such ℓ-block B.

Proposition 17.1. *Let $\ell \in \{2, 3\}$ and let B be an ℓ-block of G of non-cyclic defect. Then conditions (i) and (ii) of Definition 3.2 are satisfied for B.*

Proof. In [An95, Rmk. 1, (3B), (3G)] J. An showed that the blockwise Alperin weight conjecture is true for B. Consequently, we may choose a bijection $\Omega_B\colon\ \mathrm{IBr}(B) \longrightarrow \mathcal{W}(B)$. In Propositions 16.1, 16.2, 16.5, 16.7, 16.9, 16.15, 16.18, 16.20 and 16.21 and Corollaries 16.26 and 16.39 we proved that the group $\mathrm{Aut}(G)_B$ acts trivially on both $\mathrm{IBr}(B)$ and $\mathcal{W}(B)$. Accordingly, Ω_B is $\mathrm{Aut}(G)_B$-equivariant. Hence, following Lemma 3.8 it is possible to construct subsets $\mathrm{IBr}(B \mid Q) \subseteq \mathrm{IBr}(B)$ and bijections $\Omega_Q^G\colon\ \mathrm{IBr}(B \mid Q) \longrightarrow \mathrm{dz}(\mathrm{N}_G(Q)/Q,\, B)$ for every $Q \in \mathrm{Rad}_\ell(G)$ such that conditions (i) and (ii) of Definition 3.2 are satisfied for the ℓ-block B. This concludes the proof. ☐

Chapter 18

The Main Result for ${}^3D_4(q)$

As for the Chevalley groups $G_2(q)$, our aim of establishing the inductive blockwise Alperin weight condition for Steinberg's triality groups ${}^3D_4(q)$ is simplified by the fact that these groups possess cyclic outer automorphism groups, whence we do not need to deal with the normally embedded conditions in part (iii) of Definition 3.2 (cf. Proposition 3.12). Applying the results we have obtained so far we may already prove the following:

Theorem 18.1. *Let q be a prime power. Then the inductive blockwise Alperin weight condition (cf. Definition 3.3) holds for the group ${}^3D_4(q)$ and every prime ℓ dividing its order.*

Proof. We saw in Proposition 15.2 that the simple group $G = {}^3D_4(q)$ is its own universal covering group, and moreover the outer automorphism group of G is cyclic by Proposition 15.9. Consequently, following Proposition 3.12 to prove the claim it suffices to verify conditions (i) and (ii) of Definition 3.2 for every ℓ-block B of $X = G$ for every prime ℓ dividing $|G|$.

As usual, if $\ell = p$, then the inductive blockwise Alperin weight condition holds by Proposition 3.11, so we assume that $\ell \neq p$. Following Lemma 15.1 the order of G is given by

$$\left|{}^3D_4(q)\right| = q^{12}\Phi_1(q)^2\Phi_2(q)^2\Phi_3(q)^2\Phi_6(q)^2\Phi_{12}(q),$$

so ℓ divides at least one of the factors $\Phi_1(q)$, $\Phi_2(q)$, $\Phi_3(q)$, $\Phi_6(q)$ and $\Phi_{12}(q)$. As stated in the proof of Theorem 14.1, one can show that

$$\begin{aligned}
&\gcd(\Phi_1(q),\Phi_6(q)) = \gcd(\Phi_2(q),\Phi_3(q)) = \gcd(\Phi_3(q),\Phi_6(q)) = 1,\\
&\gcd(\Phi_1(q),\Phi_3(q)) = \begin{cases} 3 & \text{if } q \equiv 1 \bmod 3,\\ 1 & \text{else,}\end{cases}\\
&\gcd(\Phi_2(q),\Phi_6(q)) = \begin{cases} 3 & \text{if } q \equiv -1 \bmod 3,\\ 1 & \text{else,}\end{cases}
\end{aligned}$$

and moreover, as one easily verifies by application of the arguments employed in the proof of Lemma 5.2, it also holds that

$$\gcd(\Phi_n(q),\Phi_{12}(q)) = 1$$

for all $n \in \{1, 2, 3, 6\}$. Thus, if $\ell \geqslant 5$, then it divides exactly one of $\Phi_1(q)$, $\Phi_2(q)$, $\Phi_3(q)$, $\Phi_6(q)$ or $\Phi_{12}(q)$. Suppose that $5 \leqslant \ell \mid \Phi_{12}(q)$. Then a Sylow ℓ-subgroup of G is contained in a maximal torus of G of type T_3, so it follows that the Sylow ℓ-subgroups of G are cyclic in this case (cf. Table 15.1). In particular, the (iBAW) conditions holds for G and ℓ by Proposition 3.9.

Let now $5 \leqslant \ell \mid \Phi_1(q)\Phi_2(q)\Phi_3(q)\Phi_6(q)$. Then according to [DM87, Lemma 5.2] the ℓ-blocks of G have either cyclic or maximal defect. In the case of cyclic ℓ-blocks the (iBAW) condition holds again by Proposition 3.9, while it has been proven to hold for 3-blocks of maximal defect by Cabanes–Späth in [CS13, Cor. 7.6]. We conclude that the inductive blockwise Alperin weight condition holds for G and $\ell \geqslant 5$.

We are only left with the case of non-cyclic ℓ-blocks of G for $\ell \in \{2, 3\}$. This case has been considered in the course of the previous chapters. In Proposition 17.1 we proved that every such ℓ-block of G of non-cyclic defect satisfies conditions (i) and (ii) of Definition 3.2. In conclusion, the (iBAW) condition holds for $G = {}^3D_4(q)$ and every prime ℓ dividing its order, which finishes the proof. $\square$

Part V

Appendices

Appendix A

Appendix for $\mathrm{SL}_3(q)$

A.1 Characters of $\mathrm{SL}_3(q)$, $3 \nmid (q-1)$

The character table of the special linear group $\mathrm{SL}_3(q)$ was determined by W. A. Simpson and J. S. Frame in [FS73]. We should note that the character table they present contains some mistakes. However, these can be detected and corrected by comparison with the generic character table of $\mathrm{SL}_3(q)$ provided by Chevie [GHL+96]. For the case $q \not\equiv 1 \bmod 3$ the Chevie commands

```
> GenCharTab('SL3.n1');
> PrintVal('SL3.n1');
```

together with [FS73, Table 2] yield the following character table for $\mathrm{SL}_3(q)$, in which for an irreducible character χ the notation $(u) = (u') \bmod x$ for $x \in \mathbb{Z}_{>1}$, $1 \leqslant u < x$ and $u' \in \mathbb{Z}$ indicates that $\chi^{(u)} = \chi^{(u'')}$, where $0 \leqslant u'' < x$ is such that $u' \equiv u'' \bmod x$.

Characters	χ_1	$\chi_{q\Phi_2}$	χ_{q^3}	$\chi_{\Phi_3}^{(u)}$	$\chi_{q\Phi_3}^{(u)}$
Parameters	–	–	–	$1 \leqslant u < q-1$	$1 \leqslant u < q-1$
C_1	1	$q(q+1)$	q^3	q^2+q+1	$q(q^2+q+1)$
C_2	1	q	$\cdot$	$q+1$	q
C_3	1	$\cdot$	$\cdot$	1	$\cdot$
$C_4^{(a)}$	1	$q+1$	q	$(q+1)\varepsilon^{ua}+\varepsilon^{-2ua}$	$(q+1)\varepsilon^{ua}+q\varepsilon^{-2ua}$
$C_5^{(a)}$	1	1	$\cdot$	$\varepsilon^{ua}+\varepsilon^{-2ua}$	ε^{ua}
$C_6^{(a,b)}$	1	2	1	$\varepsilon^{ua}+\varepsilon^{ub}+\varepsilon^{-u(a+b)}$	$\varepsilon^{ua}+\varepsilon^{ub}+\varepsilon^{-u(a+b)}$
$C_7^{(a)}$	1	$\cdot$	-1	ε^{ua}	$-\varepsilon^{ua}$
$C_8^{(a)}$	1	-1	1	$\cdot$	$\cdot$

Table A.1: Character table of $\mathrm{SL}_3(q)$, $q \not\equiv 1 \bmod 3$

Characters	$\chi^{(u,v)}_{\Phi_2\Phi_3}$	$\chi^{(u)}_{\Phi_1\Phi_3}$	$\chi^{(u)}_{\Phi_1^2\Phi_2}$
Parameters	$1 \leqslant u < v < q-1$, $\exists\, v < w \leqslant q-1$ with $u+v+w \equiv 0 \bmod q-1$	$1 \leqslant u < q^2-1$, $u \not\equiv 0 \bmod q+1$, $(u) = (uq) \bmod q^2-1$	$1 \leqslant u < q^2+q+1$, $u \not\equiv 0 \bmod q^2+q+1$, $(u) = (uq) = (uq^2)$ $\bmod q^2-1$
C_1	$(q+1)(q^2+q+1)$	$(q-1)(q^2+q+1)$	$(q-1)^2(q+1)$
C_2	$2q+1$	-1	$-(q-1)$
C_3	1	-1	1
$C_4^{(a)}$	$(q+1)A_{(u,v),a}$	$(q-1)\varepsilon^{ua}$	$\cdot$
$C_5^{(a)}$	$A_{(u,v),a}$	$-\varepsilon^{ua}$	$\cdot$
$C_6^{(a,b)}$	$B_{(u,v),(a,b)}$	$\cdot$	$\cdot$
$C_7^{(a)}$	$\cdot$	$-\eta^{-ua}-\eta^{-uaq}$	$\cdot$
$C_8^{(a)}$	$\cdot$	$\cdot$	$\gamma^{ua}+\gamma^{uaq}+\gamma^{uaq^2}$

Table A.1: Character table of $\mathrm{SL}_3(q)$, $q \not\equiv 1 \bmod 3$ (continued)

For a description of the conjugacy classes of $\mathrm{SL}_3(q)$, $q \not\equiv 1 \bmod 3$, we refer to Section 6.1.1. Moreover, as in Section 6.2 we adhere to the notation

$$\begin{aligned}
\varepsilon &:= \exp((2\pi\mathrm{i})/(q-1)),\\
\eta &:= \exp((2\pi\mathrm{i})/(q^2-1)),\\
\gamma &:= \exp((2\pi\mathrm{i})/(q^2+q+1)),\\
A_{(u,v),a} &:= \varepsilon^{ua-2va}+\varepsilon^{va-2ua}+\varepsilon^{ua+va},\\
B_{(u,v),(a,b)} &:= \varepsilon^{va+ub}+\varepsilon^{vb+ua}+\varepsilon^{-v(a+b)+ua}+\varepsilon^{va-u(a+b)}+\varepsilon^{-v(a+b)+ub}+\varepsilon^{vb-u(a+b)}.
\end{aligned}$$

Remark A.1. In Chevie the value of a character $\chi^{(u)}_{\Phi_1\Phi_3}$ on elements of the conjugacy class $C_7^{(a)}$ was stated to be $-\eta^{ua}-\eta^{uaq}$. We corrected it to $-\eta^{-ua}-\eta^{-uaq}$. This inconsistency became apparent while determining the distribution of the irreducible characters of $\mathrm{SL}_3(q)$ into ℓ-blocks in the case that $2 \neq \ell \mid (q+1)$. More precisely, we detected it during the attempt of deciding in which case characters of types $\chi_{\Phi_1\Phi_3}$ and χ_{Φ_3} (cf. Remark A.2 below) belong to the same ℓ-block. Note that our results in this matter are not part of this thesis since the consideration of the case $2 \neq \ell \mid (q+1)$ has become obsolete after Koshitani–Späth's progress in [KS14] for blocks with cyclic defect groups.

Remark A.2. If χ is an irreducible character of $\mathrm{SL}_3(q)$, $q \not\equiv 1 \bmod 3$, such that $\chi = \chi_\gamma^{(u)}$ or $\chi = \chi_\gamma^{(u,v)}$ for suitable parameters $u, v \in \mathbb{Z}$ and $\gamma \in \{\Phi_3, q\Phi_3, \Phi_2\Phi_3, \Phi_1\Phi_3, \Phi_1^2\Phi_2\}$, then we say that χ is *of type* χ_γ.

Moreover, extending the parametrization given in Table A.1 the characters of type χ_γ may be parametrized by (tuples of) integers subject to the following conditions:

$\gamma \in \{\Phi_3, q\Phi_3\}$**:** Parameters $u \in \mathbb{Z}$ with $u \not\equiv 0 \bmod q-1$, and $\chi_\gamma^{(u)} = \chi_\gamma^{(u')}$ if and only if $u \equiv u' \bmod q-1$.

$\gamma = \Phi_2\Phi_3$**:** Parameters $u, v \in \mathbb{Z}$ with $u, v, u-v, 2u+v, 2v+u \not\equiv 0 \bmod q-1$, and it holds that $\chi_\gamma^{(u,v)} = \chi_\gamma^{(u',v')}$ if and only if in the notation of Notation 6.7 we have $(u', v') \in_{\bmod (q-1)} \{(u,v), (v,u), (-u, v-u), (v-u, -u), (u-v, -v), (-v, u-v)\}$.

$\gamma = \Phi_1\Phi_3$**:** Parameters $u \in \mathbb{Z}$ satisfying $u \not\equiv 0 \bmod q+1$, and $\chi_\gamma^{(u)} = \chi_\gamma^{(u')}$ if and only if $u \equiv u' \bmod q+1$ or $u \equiv u'q \bmod q+1$.

$\gamma = \Phi_1^2\Phi_2$**:** Parameters $u \in \mathbb{Z}$ satisfying $u \not\equiv 0 \bmod q^2+q+1$, and $\chi_\gamma^{(u)} = \chi_\gamma^{(u')}$ if and only if $u \equiv u' \bmod q^2+q+1$ or $u \equiv u'q \bmod q^2+q+1$ or $u \equiv u'q^2 \bmod q^2+q+1$.

A.2 Central Characters of $\mathrm{SL}_3(q)$, $3 \nmid (q-1)$

For the purpose of determining the ℓ-blocks of $\mathrm{SL}_3(q)$ for certain primes ℓ in Chapter 6 we use information on the central characters ω_χ corresponding to the irreducible characters χ of $\mathrm{SL}_3(q)$. We obtain their values by means of the `Omega`-command provided by Chevie. In the notation of the previous section the sequence of Chevie-commands

```
> GenCharTab(‘SL3.n1‘);
> Copy(‘SL3.n1‘,cen,[],[]);
> Omega(‘SL3.n1‘,cen);
> PrintVal(cen);
```

yields the following central characters for $\mathrm{SL}_3(q)$ in the case that $q \not\equiv 1 \bmod 3$:

	C_1	C_2	C_3	$C_4^{(a)}$
ω_{χ_1}	1	$(q^3-1)(q+1)$	$q(q^2-1)(q^3-1)$	$q^2(q^2+q+1)$
$\omega_{\chi_{q\Phi_2}}$	1	q^3-1	$\cdot$	$q(q^2+q+1)$
$\omega_{\chi_{q^3}}$	1	$\cdot$	$\cdot$	q^2+q+1
$\omega_{\chi_{\Phi_3}^{(u)}}$	1	$(q+1)^2(q-1)$	$q(q-1)^2(q+1)$	$q^2((q+1)\varepsilon^{ua}+\varepsilon^{-2ua})$
$\omega_{\chi_{q\Phi_3}^{(u)}}$	1	q^2-1	$\cdot$	$q((q+1)\varepsilon^{ua}+q\varepsilon^{-2ua})$
$\omega_{\chi_{\Phi_2\Phi_3}^{(u,v)}}$	1	$(2q+1)(q-1)$	$q(q-1)^2$	$q^2A_{(u,v),a}$
$\omega_{\chi_{\Phi_1\Phi_3}^{(u)}}$	1	$-(q+1)$	$-q(q^2-1)$	$q^2\varepsilon^{ua}$
$\omega_{\chi_{\Phi_1^2\Phi_3}^{(u)}}$	1	$-(q^2+q+1)$	$q(q^2+q+1)$	$\cdot$

Table A.2: Central characters of $\mathrm{SL}_3(q)$, $q \not\equiv 1 \bmod 3$

	$C_5^{(a)}$	$C_6^{(a,b)}$
ω_{χ_1}	$q^2(q+1)(q^3-1)$	$q^3(q+1)(q^2+q+1)$
$\omega_{\chi_{q\Phi_2}}$	$q(q^3-1)$	$2q^2(q^2+q+1)$
$\omega_{\chi_{q^3}}$	$\cdot$	$(q+1)(q^2+q+1)$
$\omega_{\chi_{\Phi_3}^{(u)}}$	$q^2(q^2-1)(\varepsilon^{ua}+\varepsilon^{-2ua})$	$q^3(q+1)(\varepsilon^{ua}+\varepsilon^{ub}+\varepsilon^{-u(a+b)})$
$\omega_{\chi_{q\Phi_3}^{(u)}}$	$q(q^2-1)\varepsilon^{ua}$	$q^2(q+1)(\varepsilon^{ua}+\varepsilon^{ub}+\varepsilon^{-u(a+b)})$
$\omega_{\chi_{\Phi_2\Phi_3}^{(u,v)}}$	$q^2(q-1)A_{(u,v),a}$	$q^3B_{(u,v),(a,b)}$
$\omega_{\chi_{\Phi_1\Phi_3}^{(u)}}$	$-q^2(q+1)\varepsilon^{ua}$	$\cdot$
$\omega_{\chi_{\Phi_1^2\Phi_3}^{(u)}}$	$\cdot$	$\cdot$

Table A.2: Central characters of $\mathrm{SL}_3(q)$, $q \not\equiv 1 \bmod 3$ (continued)

	$C_7^{(a)}$	$C_8^{(a)}$
ω_{χ_1}	$q^3(q^3-1)$	$q^3(q-1)^2(q+1)$
$\omega_{\chi_{q\Phi_2}}$	$\cdot$	$-q^2(q-1)^2$
$\omega_{\chi_{q^3}}$	$-(q^3-1)$	$(q-1)^2(q+1)$
$\omega_{\chi_{\Phi_3}^{(u)}}$	$q^3(q-1)\varepsilon^{ua}$	$\cdot$
$\omega_{\chi_{q\Phi_3}^{(u)}}$	$-q^2(q-1)\varepsilon^{ua}$	$\cdot$
$\omega_{\chi_{\Phi_2\Phi_3}^{(u,v)}}$	$\cdot$	$\cdot$
$\omega_{\chi_{\Phi_1\Phi_3}^{(u)}}$	$-q^3(\eta^{-ua}-\eta^{-uaq})$	$\cdot$
$\omega_{\chi_{\Phi_1^2\Phi_3}^{(u)}}$	$\cdot$	$q^3(\gamma^{ua}+\gamma^{uaq}+\gamma^{uaq^2})$

Table A.2: Central characters of $\mathrm{SL}_3(q)$, $q \not\equiv 1 \bmod 3$ (continued)

Appendix B

Appendix for $G_2(q)$

B.1 Conjugacy Classes of $G_2(q)$

Let $q = p^f$ for a prime p and a natural number $f \in \mathbb{N}_{>0}$. For the purposes of our study we do not require detailed information on the conjugacy classes of $G_2(q)$ in most instances. Hence, we just give a brief overview of the conjugacy classes of $G_2(q)$ here, and we present explicit information only where necessary.

A complete set of representatives for the conjugacy classes of $G = G_2(q)$ is given in Table B.1 below, where for each conjugacy class we provide notation for a representative of this class and the corresponding centralizer order. In addition, we give explicit representatives for certain of the conjugacy classes. For more detailed information we refer the reader to [Eno70, Tables 1, 2] (for $p = 2, 3$) and [Cha68] (for $p \geqslant 5$). The notation we use here is adopted from [CR74] and [Hiß90] (except that we interchanged the roles of a and b in $h_{1a}(i)$, $h_{1b}(i)$, $h_{1a,1}(i)$, $h_{1b,1}(i)$). In the case that $3 \nmid q$ we denote by ε the unique element in $\{\pm 1\}$ satisfying $q \equiv \varepsilon \bmod 3$.

g	Representative	Remark	$\lvert \mathrm{C}_G(g) \rvert$
1	identity	–	$\lvert G \rvert$
u_1	$x_{3a+2b}(1)$	–	$q^6\Phi_1(q)\Phi_2(q)$
u_2	$x_{2a+b}(1)$	–	$\left\{\begin{array}{l} q^6\Phi_1(q)\Phi_2(q) \text{ if } p = 2,3, \\ q^4\Phi_1(q)\Phi_2(q) \text{ if } p \geqslant 5, \end{array}\right\}$
u_3	*omitted*	–	$\left\{\begin{array}{r} 6q^4 \text{ if } p \neq 3 \\ q^6 \text{ if } p = 3, \end{array}\right\}$
u_4	*omitted*	–	$2q^4$
u_5	*omitted*	–	$\left\{\begin{array}{l} 3q^4 \text{ if } p \neq 3 \\ 2q^4 \text{ if } p = 3, \end{array}\right\}$
u_6	*omitted*	–	$\left\{\begin{array}{l} pq^2 \text{ if } p = 2,3, \\ 3q^4 \text{ if } p \geqslant 5, \end{array}\right\}$

Table B.1: Conjugacy classes of $G_2(q)$

g	Representative	Remark	$\lvert C_G(g)\rvert$
u_7	*omitted*	only for $p = 2, 3$	pq^2
u_8	*omitted*	only for $p = 3$	pq^2
k_2	$h(-1,-1,1)$	only for $p \neq 2$	$q^2\Phi_1(q)^2\Phi_2(q)^2$
$k_{2,1}$	*omitted*	only for $p \neq 2$	$q^2\Phi_1(q)\Phi_2(q)$
$k_{2,2}$	*omitted*	only for $p \neq 2$	$q^2\Phi_1(q)\Phi_2(q)$
$k_{2,3}$	*omitted*	only for $p \neq 2$	$2q^2$
$k_{2,4}$	*omitted*	only for $p \neq 2$	$2q^2$
k_3	$h(\omega,\omega,\omega)$	only for $p \neq 3$, $1 \neq \omega \in \mathbb{F}^\times$, $\omega^3 = 1$	$q^3\Phi_1(q)\Phi_2(q)(q^3-\varepsilon)$
$k_{3,1}$	*omitted*	only for $p \neq 3$	$q^3(q-\varepsilon)$
$k_{3,2}$	*omitted*	only for $p \neq 3$	$3q^2$
$k_{3,3,1}$	*omitted*	only for $p \neq 3$	$3q^2$
$k_{3,3,2}$	*omitted*	only for $p \neq 3$	$3q^2$
$h_{1a}(i)$	$h(z^i, z^{-2i}, z^i)$	$\left\{\begin{matrix} o(z) = \Phi_1(q), \\ z^{2i}, z^{3i} \neq 1 \end{matrix}\right\}$	$q\Phi_1(q)^2\Phi_2(q)$
$h_{1a,1}(i)$	*omitted*	–	$q\Phi_1(q)$
$h_{1b}(i)$	$h(z^i, z^{-i}, 1)$	$\left\{\begin{matrix} o(z) = \Phi_1(q), \\ z^{2i} \neq 1 \end{matrix}\right\}$	$q\Phi_1(q)^2\Phi_2(q)$
$h_{1b,1}(i)$	*omitted*	–	$q\Phi_1(q)$
$h_1(i,j)$	$h(z^i, z^j, z^{-(i+j)})$	$\left\{\begin{matrix} o(z) = \Phi_1(q), \\ z^i z^{\pm j} \neq 1, \\ z^i z^{\pm(i+j)} \neq 1, \\ z^j z^{\pm(i+j)} \neq 1 \end{matrix}\right\}$	$\Phi_1(q)^2$
$h_{2a}(i)$	$h(z^i, z^{-2i}, z^i)$	$\left\{\begin{matrix} o(z) = \Phi_2(q), \\ z^{2i}, z^{3i} \neq 1 \end{matrix}\right\}$	$q\Phi_1(q)\Phi_2(q)^2$
$h_{2a,1}(i)$	*omitted*	–	$q\Phi_2(q)$
$h_{2b}(i)$	$h(z^i, z^{-i}, 1)$	$\left\{\begin{matrix} o(z) = \Phi_2(q), \\ z^{2i} \neq 1 \end{matrix}\right\}$	$q\Phi_1(q)\Phi_2(q)^2$
$h_{2b,1}(i)$	*omitted*	–	$q\Phi_2(q)$
$h_2(i,j)$	$h(z^i, z^j, z^{-(i+j)})$	$\left\{\begin{matrix} o(z) = \Phi_2(q), \\ z^i z^{\pm j} \neq 1, \\ z^i z^{\pm(i+j)} \neq 1, \\ z^j z^{\pm(i+j)} \neq 1 \end{matrix}\right\}$	$\Phi_2(q)^2$

Table B.1: Conjugacy classes of $G_2(q)$ (continued)

g	Representative	Remark	$\lvert C_G(g)\rvert$
$h_a(i)$	$h(z^{iq}, z^{-i(q-1)}, z^{-i})$	$\begin{Bmatrix} o(z) = q^2-1, \\ z^{i(q\pm1)} \neq 1 \end{Bmatrix}$	$\Phi_1(q)\Phi_2(q)$
$h_b(i)$	$h(z^{i}, z^{iq}, z^{-i(q+1)})$	$\begin{Bmatrix} o(z) = q^2-1, \\ z^{i(q\pm1)} \neq 1 \end{Bmatrix}$	$\Phi_1(q)\Phi_2(q)$
$h_3(i)$	$h(z^{i}, z^{iq}, z^{iq^2})$	$\begin{Bmatrix} o(z) = \Phi_3(q), \\ z^{3i} \neq 1 \end{Bmatrix}$	$\Phi_3(q)$
$h_6(i)$	$h(z^{i}, z^{-iq}, z^{iq^2})$	$\begin{Bmatrix} o(z) = \Phi_6(q), \\ z^{3i} \neq 1 \end{Bmatrix}$	$\Phi_6(q)$

Table B.1 Conjugacy classes of $G_2(q)$ (continued)

Remark B.1. (i) Note that in the case that $p \geqslant 5$ the representatives u_1 and u_2 provided by [CR74] are given by $x_b(1)$ and $x_a(1)$, respectively. However, it is clear that $x_b(1)$ is conjugate in $G_2(q)$ to $x_{3a+2b}(1)$, e.g., by conjugation with $n_{3a+b}(1)$ (cf. Theorem 4.25(viii) and Proposition 11.5), and similarly $x_a(1)$ is conjugate in $G_2(q)$ to $x_{2a+b}(1)$ by conjugation with $n_{a+b}(1)$.

(ii) For an explanation concerning the parametrization of the conjugacy classes labelled by parameters (i) or (i,j) in Table B.1 we refer to [Hiß90], [EY86] and [Eno76]. We restrain from going into details here since our computations do not involve any of these classes.

B.2 Characters of $G_2(q)$

In this section we give a brief overview of the irreducible characters of $G_2(q)$. To be more precise, we provide the degrees of all irreducible characters of $G_2(q)$, and for certain characters we also present their character values (this is done in Tables B.3 and B.4 for the characters X_{13} and X_{14}, and for the characters X_{23} and X_{24}, respectively).

The characters of $G_2(q)$ have been determined by B. Chang and R. Ree [CR74] in 1974 for $p \geqslant 5$, by H. Enomoto [Eno76] in 1976 for $3 \mid q$, and by H. Enomoto and H. Yamada [EY86] in 1986 for $2 \mid q$. An explicit description of the character table of $G_2(q)$ for $p \geqslant 5$ has been provided by G. Hiß [Hiß90] in 1990. The notations for the characters employed in each of these papers differ considerably from each other. In order to have a uniform description of the irreducible characters of $G_2(q)$ we adopted the notation of Chang–Ree for all cases. As above, in the case that $3 \nmid q$ we let $\varepsilon \in \{\pm 1\}$ be such that $q \equiv \varepsilon \bmod 3$. Moreover, for odd q we let $\delta \in \{\pm 1\}$ satisfy $q \equiv \delta \bmod 4$. For any q the degrees of the irreducible characters of $G_2(q)$ are given by Table B.2 (compare [CR74], [Eno76] and [EY86]).

Remark B.2. As indicated in Table B.2, some of the irreducible characters of $G_2(q)$ depend on certain parameters (k) or (k,l). These, however, do not play a role throughout this thesis, whence we omit an explicit explanation here but rather refer to [Hiß90], [EY86] and [Eno76] for more details.

Character	Degree	Remark	Notation in [EY86], [Eno76]
X_{11}	1	-	θ_0, θ_0
X_{12}	q^6	-	θ_5, θ_5
X_{13}	$\frac{1}{3}q\Phi_3(q)\Phi_6(q)$	-	θ_3, θ_3
X_{14}	$\frac{1}{3}q\Phi_3(q)\Phi_6(q)$	-	θ_4, θ_4
X_{15}	$\frac{1}{2}q\Phi_2(q)^2\Phi_6(q)$	-	θ_2, θ_2
X_{16}	$\frac{1}{6}q\Phi_2(q)^2\Phi_3(q)$	-	θ_1, θ_1
X_{17}	$\frac{1}{2}q\Phi_1(q)^2\Phi_3(q)$	-	θ_2', θ_{11}
X_{18}	$\frac{1}{6}q\Phi_1(q)^2\Phi_6(q)$	-	θ_1', θ_{10}
$X_{19,1}$	$\frac{1}{3}q\Phi_1(q)^2\Phi_2(q)^2$	-	$\theta_9(1)$, $\theta_{12}(1)$
$X_{19,2}$	$\frac{1}{3}q\Phi_1(q)^2\Phi_2(q)^2$	-	$\theta_9(2)$, $\theta_{12}(-1)$
X_{31}	$q^3(q+\varepsilon)\Phi_6(\varepsilon q)$	only for $3 \nmid q$	θ_7, -
X_{32}	$(q+\varepsilon)\Phi_6(\varepsilon q)$	only for $3 \nmid q$	θ_6, -
X_{33}	$q(q+\varepsilon)^2\Phi_6(\varepsilon q)$	only for $3 \nmid q$	θ_8, -
X_{21}	$q^2\Phi_3(q)\Phi_6(q)$	only for $2 \nmid q$	- , θ_7
X_{22}	$\Phi_3(q)\Phi_6(q)$	only for $2 \nmid q$	- , θ_6
X_{23}	$q\Phi_3(q)\Phi_6(q)$	only for $2 \nmid q$	- , θ_8
X_{24}	$q\Phi_3(q)\Phi_6(q)$	only for $2 \nmid q$	- , θ_9
$X_{1a}(k)$	$q\Phi_2(q)\Phi_3(q)\Phi_6(q)$	-	χ_2, χ_2
$X_{1a}'(k)$	$\Phi_2(q)\Phi_3(q)\Phi_6(q)$	-	χ_1, χ_1
$X_{1b}(k)$	$q\Phi_2(q)\Phi_3(q)\Phi_6(q)$	-	χ_4, χ_4
$X_{1b}'(k)$	$\Phi_2(q)\Phi_3(q)\Phi_6(q)$	-	χ_3, χ_3
$X_{2a}(k)$	$q\Phi_1(q)\Phi_3(q)\Phi_6(q)$	-	χ_4', χ_8
$X_{2a}'(k)$	$\Phi_1(q)\Phi_3(q)\Phi_6(q)$	-	χ_3', χ_7
$X_{2b}(k)$	$q\Phi_1(q)\Phi_3(q)\Phi_6(q)$	-	χ_2', χ_6
$X_{2b}'(k)$	$\Phi_1(q)\Phi_3(q)\Phi_6(q)$	-	χ_1', χ_5
$X_1(k,l)$	$\Phi_2(q)^2\Phi_3(q)\Phi_6(q)$	-	χ_5, χ_9
$X_2(k,l)$	$\Phi_1(q)^2\Phi_3(q)\Phi_6(q)$	-	χ_5', χ_{12}
$X_a(k)$	q^6-1	-	χ_6, χ_{10}
$X_b(k)$	q^6-1	-	χ_6', χ_{11}
$X_3(k)$	$\Phi_1(q)^2\Phi_2(q)^2\Phi_6(q)$	-	χ_7, χ_{13}
$X_6(k)$	$\Phi_1(q)^2\Phi_2(q)^2\Phi_3(q)$	-	χ_7', χ_{14}

Table B.2: Irreducible characters of $G_2(q)$

	X_{13} (for $3 \nmid q$)	X_{14} (for $3 \nmid q$)	X_{13} (for $3 \mid q$)	X_{14} (for $3 \mid q$)
1	$\frac{1}{3}q\Phi_3(q)\Phi_6(q)$	$\frac{1}{3}q\Phi_3(q)\Phi_6(q)$	$\frac{1}{3}q\Phi_3(q)\Phi_6(q)$	$\frac{1}{3}q\Phi_3(q)\Phi_6(q)$
u_1	$-\frac{1}{3}q(q^2-1)$	$\frac{1}{3}q(2q^2+1)$	$-\frac{1}{3}q(q^2-1)$	$\frac{1}{3}q(2q^2+1)$
u_2	q	$\cdot$	$\frac{1}{3}q(2q^2+1)$	$-\frac{1}{3}q(q^2-1)$
u_3	$\frac{1}{3}q(\varepsilon q+5)$	$\frac{1}{3}q(\varepsilon q-1)$	$\frac{1}{3}q$	$\frac{1}{3}q$
u_4	$-\frac{1}{3}q(\varepsilon q-1)$	$-\frac{1}{3}q(\varepsilon q-1)$	$\frac{1}{3}q(q+1)$	$\frac{1}{3}q(q+1)$
u_5	$\frac{1}{3}q(\varepsilon q-1)$	$\frac{1}{3}q(\varepsilon q+2)$	$-\frac{1}{3}q(q-1)$	$-\frac{1}{3}q(q-1)$
u_6	$\cdot$	$\cdot$	$-\frac{2}{3}q$	$-\frac{2}{3}q$
u_7 $(p<5)$	$\cdot$	$\cdot$	$\frac{1}{3}q$	$\frac{1}{3}q$
u_8 $(p=3)$	$\cdot$	$\cdot$	$\frac{1}{3}q$	$\frac{1}{3}q$
k_2 $(p \neq 2)$	q	q	q	q
$k_{2,1}$ $(p \neq 2)$	$\cdot$	q	$\cdot$	q
$k_{2,2}$ $(p \neq 2)$	q	$\cdot$	q	$\cdot$
$k_{2,3}$ $(p \neq 2)$	$\cdot$	$\cdot$	$\cdot$	$\cdot$
$k_{2,4}$ $(p \neq 2)$	$\cdot$	$\cdot$	$\cdot$	$\cdot$
k_3 $(p \neq 3)$	$\frac{1}{3}(2q-\varepsilon)\Phi_3(\varepsilon q)$	$-\frac{1}{3}(q-2\varepsilon)\Phi_3(\varepsilon q)$	$-$	$-$
$k_{3,1}$ $(p \neq 3)$	$\frac{1}{3}(q-\varepsilon)$	$\frac{1}{3}(q+2\varepsilon)$	$-$	$-$
$k_{3,2}$ $(p \neq 3)$	$-\frac{1}{3}(2q+\varepsilon)$	$-\frac{2}{3}(q-\varepsilon)$	$-$	$-$
$k_{3,3,1}$ $(p \neq 3)$	$\frac{1}{3}(q-\varepsilon)$	$\frac{1}{3}(q+2\varepsilon)$	$-$	$-$
$k_{3,3,2}$ $(p \neq 3)$	$\frac{1}{3}(q-\varepsilon)$	$\frac{1}{3}(q+2\varepsilon)$	$-$	$-$
$h_{1a}(i)$	q	1	q	1
$h_{1a,1}(i)$	$\cdot$	1	$\cdot$	1
$h_{1b}(i)$	1	q	1	q
$h_{1b,1}(i)$	1	$\cdot$	1	$\cdot$
$h_{2a}(i)$	q	-1	q	-1
$h_{2a,1}(i)$	$\cdot$	-1	$\cdot$	-1
$h_{2b}(i)$	-1	q	-1	q
$h_{2b,1}(i)$	-1	$\cdot$	-1	$\cdot$
$h_1(i,j)$	1	1	1	1
$h_2(i,j)$	-1	-1	-1	-1
$h_a(i)$	1	-1	1	-1
$h_b(i)$	-1	1	-1	1
$h_3(i)$	$\cdot$	$\cdot$	$\cdot$	$\cdot$
$h_6(i)$	$\cdot$	$\cdot$	$\cdot$	$\cdot$

Table B.3: Characters X_{13} and X_{14} of $G_2(q)$

	X_{23} (for $2, 3 \nmid q$)	X_{24} (for $2, 3 \nmid q$)
1	$q\Phi_3(q)\Phi_6(q)$	$q\Phi_3(q)\Phi_6(q)$
u_1	$q(q^2+1)$	q
u_2	q	$2q$
u_3	q	$3q$
u_4	q	q
u_5	q	$\cdot$
u_6	$\cdot$	$\cdot$
k_2	$q^2\delta + q + \delta$	$q^2\delta + q + \delta$
$k_{2,1}$	$q + \delta$	δ
$k_{2,2}$	δ	$q + \delta$
$k_{2,3}$	δ	δ
$k_{2,4}$	δ	δ
k_3	$\varepsilon\Phi_3(\varepsilon q)$	$q\Phi_3(\varepsilon q)$
$k_{3,1}$	$q + \varepsilon$	q
$k_{3,2}$	ε	$\cdot$
$k_{3,3,1}$	ε	$\cdot$
$k_{3,3,2}$	ε	$\cdot$
$h_{1a}(i)$	$(-1)^i(q+1)+1$	$q+(-1)^i(q+1)$
$h_{1a,1}(i)$	$(-1)^i+1$	$(-1)^i$
$h_{1b}(i)$	$q+(-1)^i(q+1)$	$(-1)^i(q+1)+1$
$h_{1b,1}(i)$	$(-1)^i$	$(-1)^i+1$
$h_{2a}(i)$	$(-1)^i(q-1)-1$	$q+(-1)^i(q-1)$
$h_{2a,1}(i)$	$-1-(-1)^i$	$-(-1)^i$
$h_{2b}(i)$	$q+(-1)^i(q-1)$	$(-1)^i(q-1)-1$
$h_{2b,1}(i)$	$-(-1)^i$	$-1-(-1)^i$
$h_1(i)$	$(-1)^{i+j}+(-1)^i+(-1)^j$	$(-1)^{i+j}+(-1)^i+(-1)^j$
$h_2(i)$	$-(-1)^{i+j}-(-1)^i-(-1)^j$	$-(-1)^{i+j}-(-1)^i-(-1)^j$
$h_a(i)$	$-(-1)^i$	$(-1)^i$
$h_b(i)$	$(-1)^i$	$-(-1)^i$
$h_3(i)$	$\cdot$	$\cdot$
$h_6(i)$	$\cdot$	$\cdot$

Table B.4: Characters X_{23} and X_{24} of $G_2(q)$ for $2, 3 \nmid q$

B.3 Blocks and Decomposition Numbers of $G_2(q)$

The ℓ-blocks and ℓ-decomposition numbers of $G_2(q)$ have been described by G. Hiß and J. Shamash in a series of papers for various primes ℓ (see [Sha89a], [Sha89b], [Hiß89], [Hiß90], [HS90], [Sha92], [HS92]). Below, we give a brief summary of their results obtained for the cases $\ell = 2$ and $\ell = 3$.

B.3.1 The Case $\ell = 2$

In this section we briefly describe the decomposition numbers for 2-blocks of $G_2(q)$, where we assume that q is odd. For a more detailed description we refer the reader to [HS92]. The 2-blocks of $G_2(q)$ may be divided into the following classes of 2-blocks:

- the principal 2-block B_0,
- the 2-block B_3 (only for $3 \nmid q$),
- the 2-blocks of types B_{1a}, B_{1b}, B_{2a}, B_{2b},
- the 2-blocks of types B_{X_1}, B_{X_2}, B_{X_a}, B_{X_b},
- the 2-blocks of 2-defect zero.

By the results of [HS92] the 2-decomposition matrix of $G_2(q)$ is lower unitriangular with respect to a suitable ordering of the irreducible ordinary and Brauer characters. Below we present for each type of 2-block B of $G_2(q)$ not of 2-defect zero a unitriangular square submatrix of size $l(B) = |\operatorname{IBr}(B)|$ of the decomposition matrix associated to B, which we obtain from [HS92, Sec. 2]. The irreducible Brauer characters of B may then be derived from the 2-decomposition numbers given in this matrix.

The Principal 2-Block B_0: The principal 2-block B_0 of $G_2(q)$ contains exactly seven irreducible Brauer characters. The corresponding 2-decomposition numbers may be derived from the following matrix, where $0 \leqslant \alpha \leqslant 2q$ with $\alpha \leqslant q-1$ if $3 \nmid q$, and $0 \leqslant \beta \leqslant \frac{1}{3}(q+2)$:

	φ_{11}	φ_{17}	φ_{18}	φ_{13}	φ_{14}	φ_{15}	φ_{12}
X_{11}	1	·	·	·	·	·	·
X_{17}	·	1	·	·	·	·	·
X_{18}	·	·	1	·	·	·	·
X_{13}	1	·	·	1	·	·	·
X_{14}	1	·	·	·	1	·	·
X_{15}	·	1	·	·	·	1	·
X_{12}	1	α	β	1	1	·	1

The 2-Block B_3: The 2-block B_3 does only exist if $3 \nmid q$. It contains exactly three irreducible Brauer characters and the 2-decomposition matrix for B_3 has a unitriangular submatrix of the following form, where $1 \leqslant \gamma \leqslant \frac{1}{3}(q+1)$ with $\gamma = 1$ if $q \equiv 1 \bmod 4$:

$q \equiv 1 \bmod 3$	φ_{32}	φ_{33}	φ_{31}
X_{32}	1	·	·
X_{33}	·	1	·
X_{31}	1	·	1

$q \equiv -1 \bmod 3$	φ_{32}	φ_{33}	φ_{31}
X_{32}	1	·	·
X_{33}	·	1	·
X_{31}	1	γ	1

Following [HS92, Sec. 2.2, 2.3] and the proof of [An94a, (3I)] the 2-block B_3 has non-cyclic defect groups.

The 2-Blocks of Types B_{1a}, B_{1b}, B_{2a}, B_{2b}: Let B be a 2-block of $G_2(q)$ of type B_I for some $I \in \{1a, 1b, 2a, 2b\}$. Then $|\operatorname{IBr}(B)| = 2$, and the 2-decomposition numbers for B are given by:

	φ_I'	φ_I
X_I'	1	·
X_I	1	1

Following [HS92, Sec. 2.2, 2.3] and the proof of [An94a, (3I)] such 2-blocks of G have non-cyclic defect groups.

The 2-Blocks of Types B_{X_1}, B_{X_2}, B_{X_a}, B_{X_b}: If B is a 2-block of $G_2(q)$ of type B_{X_I} for some $I \in \{1, 2, a, b\}$, then $|\operatorname{IBr}(B)| = 1$ and the 2-decomposition matrix for B contains a submatrix of the following form:

	φ_I
X_I	1

Note that by [An94a, (3.3) and Rmk. after (3D)] a 2-block of G of type B_{X_I} has non-cyclic defect groups if and only if $I \in \{1, 2\}$.

B.3.2 The Case $\ell = 3$

Let us now describe the decomposition numbers for 3-blocks of $G_2(q)$ in the case that $3 \nmid q$. For a more detailed description we refer the reader to [HS90]. The 3-blocks of $G_2(q)$ may be divided into the following classes of 3-blocks:

- the principal 3-block B_0,
- the 3-block B_2 (only for $2 \nmid q$),
- the 3-blocks of types B_{1a}, B_{1b}, B_{2a}, B_{2b},
- the 3-blocks of types B_{X_1}, B_{X_3}, B_{X_a}, B_{X_b} (for $q \equiv 1 \bmod 3$),
- the 3-blocks of types B_{X_2}, B_{X_6}, B_{X_a}, B_{X_b} (for $q \equiv -1 \bmod 3$),
- the 3-blocks of 3-defect zero.

As for the case $\ell = 2$, by the results of Hiß–Shamash [HS90] the 3-decomposition matrix of $G_2(q)$ is lower unitriangular with respect to a suitable ordering of the irreducible ordinary and Brauer characters. Again we present for each type of 3-block B of $G_2(q)$ not of 3-defect zero a unitriangular square submatrix of size $l(B) = |\operatorname{IBr}(B)|$ of the decomposition matrix associated to B, from which the irreducible Brauer characters of B may then be derived. The decomposition numbers we present here may be found in [HS90, Sec. 2, Sec. 3] for the principal 3-block and [Hiß89, Sec. 2] for the remaining 3-blocks (note that by [HS90, pp. 371/372] the non-principal 3-blocks of $G_2(q)$ have abelian defect and their 3-decomposition matrices coincide with those of the corresponding ℓ-blocks of $G_2(q)$ in the case $\ell > 3$, which are given in [Hiß89]). In the case of a 3-block having cyclic defect groups, [HS90] presents the associated Brauer tree, from which we read of the corresponding 3-decomposition numbers.

The Principal 3-Block B_0: The principal 3-block B_0 of $G_2(q)$ contains exactly seven irreducible Brauer characters. In the case of $q \equiv 1 \bmod 3$ the corresponding decomposition numbers may be derived from the following matrix, where $0 \leqslant \alpha \leqslant 1$, $0 \leqslant \beta \leqslant q-2$, and $1 \leqslant \gamma \leqslant q+1$, with $\alpha = \beta = 0$ if $2 \mid q$:

$q \equiv 1 \bmod 3$	φ_{11}	φ_{18}	φ_{19}	φ_{14}	φ_{15}	φ_{16}	φ_{12}
X_{11}	1	·	·	·	·	·	·
X_{18}	·	1	·	·	·	·	·
$X_{19,1}$	·	·	1	·	·	·	·
X_{14}	·	$\alpha+1$	·	1	·	·	·
X_{15}	1	·	·	·	1	·	·
X_{16}	·	α	·	1	·	1	·
X_{12}	·	β	γ	·	1	·	1

In the case that $q \equiv -1 \bmod 3$ the 3-decomposition matrix of B_0 has a submatrix of the following form, where $1 \leqslant \alpha \leqslant q+1$, $1 \leqslant \beta \leqslant q-1$, and $1 \leqslant \gamma \leqslant \frac{1}{2}q$:

$q \equiv -1 \bmod 3$	φ_{11}	φ_{18}	φ_{19}	φ_{14}	φ_{17}	φ_{16}	φ_{12}
X_{11}	1	·	·	·	·	·	·
X_{18}	·	1	·	·	·	·	·
$X_{19,1}$	·	·	1	·	·	·	·
X_{14}	1	1	·	1	·	·	·
X_{17}	·	·	·	·	1	·	·
X_{16}	2	·	·	1	·	1	·
X_{12}	1	α	β	1	γ	1	1

Remark B.3. In both cases the character $X_{19,1}$ may be replaced by $X_{19,2}$ without changing the decomposition matrix. In particular, we have $\varphi_{19} = X^0_{19,1} = X^0_{19,2}$.

The 3-Block B_2: The 3-block B_2 does only exist in the case that $2 \nmid q$. It contains exactly four irreducible Brauer characters, and the 3-decomposition matrix for B_2 has a unitriangular submatrix of the following form:

$q \equiv 1 \bmod 3$	φ_{21}	φ_{22}	φ_{23}	φ_{24}
X_{21}	1	·	·	·
X_{22}	·	1	·	·
X_{23}	·	·	1	·
X_{24}	·	·	·	1

$q \equiv -1 \bmod 3$	φ_{22}	φ_{23}	φ_{24}	φ_{21}
X_{22}	1	·	·	·
X_{23}	1	1	·	·
X_{24}	1	·	1	·
X_{21}	1	1	1	1

According to [HS90] this 3-block has defect group $\mathcal{O}_3(T)$ for a maximal torus T of $G_2(q)$ of type T_ε, where $\varepsilon \in \{\pm 1\}$ is such that $q \equiv \varepsilon \bmod 3$.

The 3-Blocks of Types B_{1a}, B_{1b}, B_{2a}, B_{2b}: Let B be a 3-block of $G_2(q)$ of type B_I for $I \in \{1a, 1b, 2a, 2b\}$. Then $|\operatorname{IBr}(B)| = 2$, and the 3-decomposition numbers for B are given by the matrix below:

$q \equiv 1 \bmod 3$	φ_I'	φ_I
X_I'	1	·
X_I	·	1

$q \equiv -1 \bmod 3$	φ_I'	φ_I
X_I'	1	·
X_I	1	1

Note that by [HS90, Sec. 2] the 3-blocks of types B_{1a} or B_{1b} are of cyclic defect if and only if $q \equiv -1 \bmod 3$, while the 3-blocks of types B_{2a} or B_{2b} are of cyclic defect if and only if $q \equiv 1 \bmod 3$.

The 3-Blocks of Types B_{X_1}, B_{X_3}, B_{X_a}, B_{X_b} for $q \equiv 1 \bmod 3$: Suppose $q \equiv 1 \bmod 3$. If B is a 3-block of $G_2(q)$ of type B_{X_I} for some $I \in \{1, 3, a, b\}$, then $|\operatorname{IBr}(B)| = 1$ and the 3-decomposition matrix for B contains a submatrix of the following form:

	φ_I
X_I	1

Following [HS90, Sec. 2] the 3-block B has cyclic defect groups if and only if $I \neq 1$.

The 3-Blocks of Types B_{X_2}, B_{X_6}, B_{X_a}, B_{X_b} for $q \equiv -1 \bmod 3$: For $q \equiv -1 \bmod 3$ let B be a 3-block of $G_2(q)$ of type B_{X_I} for some $I \in \{2, 6, a, b\}$. Then $|\operatorname{IBr}(B)| = 1$ and the 3-decomposition matrix for B contains a submatrix of the following form:

	φ_I
X_I	1

Following [HS90, Sec. 2] the 3-block B has cyclic defect groups if and only if $I \neq 2$.

Appendix C

Appendix for $^3D_4(q)$

C.1 Semisimple Classes of $^3D_4(q)$

The semisimple elements of the group $G = {}^3D_4(q)$ play an important role, in particular for the description of the irreducible characters and blocks of G. On the set of all G-conjugacy classes of semisimple elements of G one may define the following equivalence relation:

Let $s_1, s_2 \in G$ be semisimple. The corresponding G-conjugacy classes $[s_1]_G$ and $[s_2]_G$ are defined to be equivalent if and only if the centralizers $C_G(s_1)$ and $C_G(s_1)$ are G-conjugate. We simply write $[s]$ for the equivalence classes of the G-conjugacy class of a semisimple element $s \in G$. There are 14 equivalence classes of semisimple elements in G if q is even, denoted $[s_1], [s_3], \dots [s_{15}]$, while for odd q there exist 15 equivalence classes, denoted $[s_1], \dots [s_{15}]$. If a semisimple element $s \in G$ belongs to the equivalence class $[s_i]$, $i \in \{1, \dots, 15\}$, then we call s *of type* s_i. For the representatives of these equivalence classes presented in Table C.1 below we refer to [DM87, Table 2.1]. However, we remark that in the table given in [DM87] some constraints on the parameters t_1 and t_2 are missing for the semisimple classes of types s_6, s_{12}, s_{13} and s_{15} (for instance the condition that $t_1 \neq 1$ for types s_6 and s_{15}). Easy calculations, which we omit here, show that the parameters have to satisfy the conditions we present in Table C.1.

Class	Representative, q even	Representative, q odd
$[s_1]$	$h(1,1,1,1)$	$h(1,1,1,1)$
$[s_2]$	$-$	$h(-1,1,-1,-1)$
$[s_3]$	$h(t,t^2,t,t)$, $t^{q-1}=1;\ t\neq 1$	$h(t,t^2,t,t)$, $t^{q-1}=1;\ t^2\neq 1$
$[s_4]$	$h(t,1,t^q,t^{q^2})$, $t^{q^2+q+1}=1;\ t\neq 1$	$h(t,1,t^q,t^{q^2})$, $t^{q^2+q+1}=1;\ t\neq 1$
$[s_5]$	$h(t,1,t^q,t^{q^2})$, $t^{q^3-1}=1;\ t^{q^2+q+1}\neq 1$	$h(t,1,t^q,t^{q^2})$, $t^{q^3-1}=1;\ t^{q^2+q+1}\neq 1;\ t^2\neq 1$

Table C.1: Semisimple classes of $^3D_4(q)$

Class	Representative, q even	Representative, q odd
$[s_6]$	$h(t_1, t_2, t_1^q, t_1^{q^2})$, $t_1^{q^3-1} = t_2^{q-1} = 1;\ t_1, t_2 \neq 1;$ $t_2 \neq t_1, t_1^2;\ t_1^{q^2+q+1} \neq t_2, t_2^2$	$h(t_1, t_2, t_1^q, t_1^{q^2})$, $t_1^{q^3-1} = t_2^{q-1} = 1;\ t_1, t_2 \neq 1;$ $t_2 \neq t_1, t_1^2;\ t_1^{q^2+q+1} \neq t_2, t_2^2$
$[s_7]$	$h(t, t^2, t, t)$, $t^{q+1} = 1;\ t \neq 1$	$h(t, t^2, t, t)$, $t^{q+1} = 1;\ t^2 \neq 1$
$[s_8]$	$h(t, t^{-q^3+1}, t^{q^4}, t^{q^2})$, $t^{(q^3-1)(q+1)} = 1;\ t^{q^3-1} \neq 1 \neq t^{q+1}$	$h(t, t^{-q^3+1}, t^{q^4}, t^{q^2})$, $t^{(q^3-1)(q+1)} = 1;\ t^{q^3-1} \neq 1 \neq t^{q+1}$
$[s_9]$	$h(t, 1, t^{-q}, t^{q-1})$, $t^{q^2-q+1} = 1;\ t \neq 1$	$h(t, 1, t^{-q}, t^{q-1})$, $t^{q^2-q+1} = 1;\ t \neq 1$
$[s_{10}]$	$h(t, 1, t^{-q}, t^{q^2})$, $t^{q^3+1} = 1;\ t^{q^2-q+1} \neq 1$	$h(t, 1, t^{-q}, t^{q^2})$, $t^{q^3+1} = 1;\ t^{q^2-q+1} \neq 1;\ t^2 \neq 1$
$[s_{11}]$	$h(t, t^{q^3+1}, t^{q^4}, t^{q^2})$, $t^{(q^3+1)(q-1)} = 1;\ t^{q^3+1} \neq 1 \neq t^{q-1}$	$h(t, t^{q^3+1}, t^{q^4}, t^{q^2})$, $t^{(q^3+1)(q-1)} = 1;\ t^{q^3+1} \neq 1 \neq t^{q-1}$
$[s_{12}]$	$h(t_1, t_2, t_1^q t_2, t_1^{-q-1} t_2^{q+1})$, $t_1^{q^2+q+1} = t_2^{q^2+q+1} = 1;\ t_2^{1-q^2} \neq t_1^2;$ $t_2 \neq 1, t_1^2, t_1^{-2q}$	$h(t_1, t_2, t_1^q t_2, t_1^{-q-1} t_2^{q+1})$, $t_1^{q^2+q+1} = t_2^{q^2+q+1} = 1;\ t_2^{1-q^2} \neq t_1^2;$ $t_2 \neq 1, t_1^2, t_1^{-2q}$
$[s_{13}]$	$h(t_1, t_2, t_1^{-q} t_2, t_1^{q-1} t_2^{-q+1})$, $t_1^{q^2-q+1} = t_2^{q^2-q+1} = 1;\ t_2^{1-q^2} \neq t_1^2;$ $t_2 \neq 1, t_1^2, t_1^{2q}$	$h(t_1, t_2, t_1^{-q} t_2, t_1^{q-1} t_2^{-q+1})$, $t_1^{q^2-q+1} = t_2^{q^2-q+1} = 1;\ t_2^{1-q^2} \neq t_1^2;$ $t_2 \neq 1, t_1^2, t_1^{2q}$
$[s_{14}]$	$h(t, t^{q^3+1}, t^q, t^{q^2})$, $t^{q^4-q^2+1} = 1;\ t \neq 1$	$h(t, t^{q^3+1}, t^q, t^{q^2})$, $t^{q^4-q^2+1} = 1;\ t \neq 1$
$[s_{15}]$	$h(t_1, t_2, t_1^{-q}, t_1^{q^2})$, $t_1^{q^3+1} = t_2^{q+1} = 1;\ t_1, t_2 \neq 1;$ $t_2 \neq t_1, t_1^2;\ t_1^{q^2-q+1} \neq t_2, t_2^2$	$h(t_1, t_2, t_1^{-q}, t_1^{q^2})$, $t_1^{q^3+1} = t_2^{q+1} = 1;\ t_1, t_2 \neq 1;$ $t_2 \neq t_1, t_1^2;\ t_1^{q^2-q+1} \neq t_2, t_2^2$

Table C.1: Semisimple classes of $^3D_4(q)$ (continued)

C.2 Characters of $^3D_4(q)$

The irreducible ordinary characters of Steinberg's triality groups $^3D_4(q)$ have been determined by N. Spaltenstein, who gives the values of the unipotent characters of $^3D_4(q)$ in [Spa82], and by D. I. Deriziotis and G. O. Michler, who provide the remaining irreducible characters in [DM87]. For our purposes it suffices to know the degrees of the irreducible characters of $^3D_4(q)$, which we present in Table C.2 below. All information in the table is taken from [DM87, Table 4.4], and we adopt the notation for the irreducible characters used there. The first eight characters

$$1_G,\ [\varepsilon_1],\ [\varepsilon_2],\ \mathrm{St},\ \rho_1,\ \rho_2,\ {}^3D_4[-1] \text{ and } {}^3D_4[1]$$

are exactly the unipotent characters of $^3D_4(q)$, and there exists exactly one irreducible character of each of these types.

The notation for the non-unipotent irreducible characters of ${}^3D_4(q)$ derives from the Jordan decomposition of characters of reductive groups over finite fields introduced by Lusztig in [Lus84]. According to [DM87, pp. 49/50] in our case this yields a bijection between the irreducible characters of $G = {}^3D_4(q)$ and pairs $([s]_G, \psi)$, where $[s]_G$ denotes the conjugacy class of a semisimple element $s \in G$ and ψ is a unipotent character of the centralizer of s in G. We write $([s]_G, \emptyset)$ if s is a regular element. Then the notation for a non-unipotent irreducible character χ is as follows:

If χ corresponds to a pair $([s]_G, \emptyset)$, where s is of type s_i, $2 \leqslant i \leqslant 15$, such that s is regular, then we write $\chi = \chi_i$. Else, if χ corresponds to $([s]_G, \psi)$, where s is of type s_i, $2 \leqslant i \leqslant 15$, with s non-regular, then χ is denoted by χ_{i,a_ψ}, where a_ψ is an expression representing the unipotent character ψ. (Note that, in general, there exist several characters of the same type χ_i or χ_{i,a_ψ} if $i > 2$, while for $i = 2$ there exists only one (or none if $2 \mid q$) G-conjugacy class of semisimple elements of type s_2.)

Character	Degree	Remark
1_G	1	-
$[\varepsilon_1]$	$q(q^4 - q^2 + 1)$	-
$[\varepsilon_2]$	$q^7(q^4 - q^2 + 1)$	-
St	q^{12}	-
ρ_1	$\frac{1}{2}q^3(q^3+1)^2$	-
ρ_2	$\frac{1}{2}q^3(q+1)^2(q^4 - q^2 + 1)$	-
${}^3D_4[-1]$	$\frac{1}{2}q^3(q^3-1)^2$	-
${}^3D_4[1]$	$\frac{1}{2}q^3(q-1)^2(q^4 - q^2 + 1)$	-
$\chi_{2,1}$	$(q^8 + q^4 + 1)$	only for $2 \nmid q$
$\chi_{2,\mathrm{St}}$	$q^3(q^8 + q^4 + 1)$	only for $2 \nmid q$
$\chi_{2,\mathrm{St}'}$	$q(q^8 + q^4 + 1)$	only for $2 \nmid q$
$\chi_{2,\mathrm{St}\,\mathrm{St}'}$	$q^4(q^8 + q^4 + 1)$	only for $2 \nmid q$
$\chi_{3,1}$	$(q+1)(q^8 + q^4 + 1)$	only for $q > 3$
$\chi_{3,\mathrm{St}}$	$q^3(q+1)(q^8 + q^4 + 1)$	only for $q > 3$
$\chi_{4,1}$	$(q+1)(q^2 - q + 1)^2(q^4 - q^2 + 1)$	-
$\chi_{4,\mathrm{St}}$	$q^3(q+1)(q^2 - q + 1)^2(q^4 - q^2 + 1)$	-
$\chi_{4,qs}$	$q(q+1)^2(q^2 - q + 1)^2(q^4 - q^2 + 1)$	-
$\chi_{5,1}$	$(q^3+1)(q^8 + q^4 + 1)$	-
$\chi_{5,\mathrm{St}}$	$q(q^3+1)(q^8 + q^4 + 1)$	-
χ_6	$(q+1)(q^3+1)(q^8 + q^4 + 1)$	-
$\chi_{7,1}$	$(q-1)(q^8 + q^4 + 1)$	-

Table C.2: Irreducible characters of ${}^3D_4(q)$

Character	Degree	Remark
$\chi_{7,\mathrm{St}}$	$q^3(q-1)(q^8+q^4+1)$	-
χ_8	$(q-1)(q^3+1)(q^8+q^4+1)$	-
$\chi_{9,1}$	$(q-1)(q^2+q+1)^2(q^4-q^2+1)$	-
$\chi_{9,\mathrm{St}}$	$q^3(q-1)(q^2+q+1)^2(q^4-q^2+1)$	-
$\chi_{9,qs'}$	$q(q-1)^2(q^2+q+1)^2(q^4-q^2+1)$	-
$\chi_{10,1}$	$(q^3-1)(q^8+q^4+1)$	-
$\chi_{10,\mathrm{St}}$	$q(q^3-1)(q^8+q^4+1)$	-
χ_{11}	$(q+1)(q^3-1)(q^8+q^4+1)$	-
χ_{12}	$(q-1)^2(q^3+1)^2(q^4-q^2+1)$	-
χ_{13}	$(q+1)^2(q^3-1)^2(q^4-q^2+1)$	-
χ_{14}	$(q^6-1)^2$	-
χ_{15}	$(q-1)(q^3-1)(q^8+q^4+1)$	-

Table C.2: Irreducible characters of ${}^3D_4(q)$ (continued)

C.3 Blocks and Decomposition Numbers of ${}^3D_4(q)$

The ℓ-decomposition numbers for the groups ${}^3D_4(q)$, where $\ell \nmid q$, have been determined by Geck in [Gec91], Himstedt–Huang in [HH13] and Himstedt in [Him07]. We present here a summary of their results for the primes $\ell = 2$ and $\ell = 3$, where we use the notation introduced at the beginning of Section 16.1.

C.3.1 The Case $\ell = 2$

The results obtained by Himstedt in [Him07, Thm. 3.1] for the 2-decomposition numbers of $G = {}^3D_4(q)$, $2 \nmid q$, are summarized below.

$\mathcal{E}_2(G,1)$: For the set $\mathcal{E}_2(G,1)$ there exist seven irreducible Brauer characters $\varphi_1, \dots, \varphi_7$, with the following corresponding 2-decomposition matrix for suitable $0 \leqslant a, b \leqslant q$:

	φ_1	φ_2	φ_3	φ_4	φ_5	φ_6	φ_7
1_G	1	·	·	·	·	·	·
$[\varepsilon_1]$	1	1	·	·	·	·	·
${}^3D_4[1]$	·	·	1	·	·	·	·
ρ_1	·	·	1	1	·	·	·
ρ_2	·	·	·	1	1	·	·
$[\varepsilon_2]$	1	·	·	·	·	1	·
St	1	1	a	·	b	1	1

$\mathcal{E}_2(G, s_i)$, $i \in \{3,5,7,10\}$: Given a $2'$-element $s \in G^*$ of type s_i, $i \in \{3,5,7,10\}$, such that one has $\mathcal{E}(G,s) = \{\chi_{i,1}, \chi_{i,\mathrm{St}}\}$, there are exactly two Brauer characters $\varphi_{i,1}$, $\varphi_{i,2}$ associated to $\mathcal{E}_2(G,s)$, and the 2-decomposition numbers for $\mathcal{E}(G,s)$ are as follows:

	$\varphi_{i,1}$	$\varphi_{i,2}$
$\chi_{i,1}$	1	·
$\chi_{i,\mathrm{St}}$	1	1

$\mathcal{E}_2(G, s_4)$ **:** For a $2'$-element $s \in G^*$ of type s_4 such that $\mathcal{E}(G,s) = \{\chi_{4,1}, \chi_{4,qs}, \chi_{4,\mathrm{St}}\}$, there are exactly three irreducible Brauer characters $\varphi_{4,1}$, $\varphi_{4,2}$, $\varphi_{4,3}$ associated to the set $\mathcal{E}_2(G,s)$, and we have the following 2-decomposition matrix:

	$\varphi_{4,1}$	$\varphi_{4,2}$	$\varphi_{4,3}$
$\chi_{4,1}$	1	·	·
$\chi_{4,qs}$	·	1	·
$\chi_{4,\mathrm{St}}$	1	·	1

$\mathcal{E}_2(G, s_9)$ **:** For a $2'$-element $s \in G^*$ of type s_9 with $\mathcal{E}(G,s) = \{\chi_{9,1}, \chi_{9,qs'}, \chi_{9,\mathrm{St}}\}$, there are exactly three irreducible Brauer characters $\varphi_{9,1}$, $\varphi_{9,2}$, $\varphi_{9,3}$ associated to the set $\mathcal{E}_2(G,s)$, and the 2-decomposition numbers are as follows, where $c = 1$ if $q \equiv 1 \bmod 4$ and $c = 2$ if $q \equiv -1 \bmod 4$:

	$\varphi_{9,1}$	$\varphi_{9,2}$	$\varphi_{9,3}$
$\chi_{9,1}$	1	·	·
$\chi_{9,qs'}$	·	1	·
$\chi_{9,\mathrm{St}}$	1	c	1

$\mathcal{E}_2(G, s_i)$, $i \in \{6,8,11,12,13,14,15\}$: If $s \in G^*$ is a $2'$-element of type s_i for a parameter $i \in \{6,8,11,12,13,14,15\}$ such that $\mathcal{E}(G,s) = \{\chi_i\}$, then $\mathcal{E}_2(G,s)$ has exactly one irreducible Brauer character $\varphi_{i,1}$ associated to it, and the 2-decomposition matrix for $\mathcal{E}(G,s)$ is as follows:

	$\varphi_{i,1}$
χ_i	1

C.3.2 The Case $\ell = 3$

The 3-modular decomposition numbers for ${}^3D_4(q)$, $3 \nmid q$, have been determined by Geck in [Gec91] for the case that q is odd, and by Himstedt–Huang in [HH13] for even q. We give here a summary of their results. Note that in most situations one has to distinguish between the cases $q \equiv 1 \bmod 3$ and $q \equiv -1 \bmod 3$.

$\mathcal{E}_3(G,1)$: For the set $\mathcal{E}_3(G,1)$ there exist eight irreducible Brauer characters $\varphi_1, \ldots, \varphi_8$. If $q \equiv 1 \bmod 3$, then the corresponding 3-decomposition matrix is as follows for suitable integers $a, c \geqslant 1$, $b \geqslant 0$:

$q \equiv 1 \bmod 3$	φ_1	φ_2	φ_3	φ_4	φ_5	φ_6	φ_7	φ_8
1_G	1	·	·	·	·	·	·	·
$[\varepsilon_1]$	·	1	·	·	·	·	·	·
ρ_1	·	1	1	·	·	·	·	·
ρ_2	1	·	·	1	·	·	·	·
${}^3D_4[-1]$	·	·	·	·	1	·	·	·
${}^3D_4[1]$	·	·	·	·	·	1	·	·
$[\varepsilon_2]$	·	·	1	·	·	a	1	·
St	·	·	·	1	·	b	c	1

In this case, the character ${}^3D_4[-1]$ has 3-defect zero.

If $q \equiv -1 \bmod 3$, then there are integers $a \geqslant 0$, $b \geqslant a+1$, $c \geqslant 1$, $d \geqslant 1$ such that the 3-decomposition matrix for $\mathcal{E}(G, 1)$ is the following:

$q \equiv -1 \bmod 3$	φ_1	φ_2	φ_3	φ_4	φ_5	φ_6	φ_7	φ_8
1_G	1	·	·	·	·	·	·	·
$[\varepsilon_1]$	1	1	·	·	·	·	·	·
ρ_1	·	·	1	·	·	·	·	·
ρ_2	2	1	·	1	·	·	·	·
${}^3D_4[-1]$	·	·	·	·	1	·	·	·
${}^3D_4[1]$	·	·	·	·	·	1	·	·
$[\varepsilon_2]$	1	·	·	1	a	·	1	·
St	1	1	·	1	b	c	d	1

In this case, the character ρ_1 has 3-defect zero.

$\mathcal{E}_3(G, s_2)$**:** If $s \in G^*$ is of type s_2 with $\mathcal{E}(G, s) = \{\chi_{2,1}, \chi_{2,\mathrm{St}}, \chi_{2,\mathrm{St}'}, \chi_{2,\mathrm{St}\,\mathrm{St}'}\}$, then there are exactly four Brauer characters $\varphi_{2,1}, \dots, \varphi_{2,4}$ associated to $\mathcal{E}_3(G, s)$, and the 3-decomposition numbers for $\mathcal{E}(G, s)$ are as follows:

$q \equiv 1 \bmod 3$	$\varphi_{2,1}$	$\varphi_{2,2}$	$\varphi_{2,3}$	$\varphi_{2,4}$
$\chi_{2,1}$	1	·	·	·
$\chi_{2,\mathrm{St}}$	·	1	·	·
$\chi_{2,\mathrm{St}'}$	·	·	1	·
$\chi_{2,\mathrm{St}\,\mathrm{St}'}$	·	·	·	1

$q \equiv -1 \bmod 3$	$\varphi_{2,1}$	$\varphi_{2,2}$	$\varphi_{2,3}$	$\varphi_{2,4}$
$\chi_{2,1}$	1	·	·	·
$\chi_{2,\mathrm{St}}$	1	1	·	·
$\chi_{2,\mathrm{St}'}$	1	·	1	·
$\chi_{2,\mathrm{St}\,\mathrm{St}'}$	1	1	1	1

$\mathcal{E}_3(G, s_i)$, $i \in \{3, 5, 7, 10\}$: Given a $3'$-element $s \in G^*$ of type s_i, $i \in \{3, 5, 7, 10\}$, such that one has $\mathcal{E}(G, s) = \{\chi_{i,1}, \chi_{i,\mathrm{St}}\}$, there are exactly two Brauer characters $\varphi_{i,1}$, $\varphi_{i,2}$ associated to $\mathcal{E}_3(G, s)$, and the 3-decomposition numbers for $\mathcal{E}(G, s)$ are as follows:

$q \equiv 1 \bmod 3$	$\varphi_{i,1}$	$\varphi_{i,2}$
$\chi_{i,1}$	1	·
$\chi_{i,\mathrm{St}}$	·	1

$q \equiv -1 \bmod 3$	$\varphi_{i,1}$	$\varphi_{i,2}$
$\chi_{i,1}$	1	·
$\chi_{i,\mathrm{St}}$	1	1

$\mathcal{E}_3(G, s_4)$: For a $3'$-element $s \in G^*$ of type s_4 such that $\mathcal{E}(G, s) = \{\chi_{4,1}, \chi_{4,qs}, \chi_{4,\mathrm{St}}\}$, there are exactly three irreducible Brauer characters $\varphi_{4,1}$, $\varphi_{4,2}$, $\varphi_{4,3}$ associated to the set $\mathcal{E}_3(G, s)$, and we have the following 3-decomposition matrix:

$q \equiv 1 \bmod 3$	$\varphi_{4,1}$	$\varphi_{4,2}$	$\varphi_{4,3}$
$\chi_{4,1}$	1	·	·
$\chi_{4,qs}$	1	1	·
$\chi_{4,\mathrm{St}}$	·	1	1

$q \equiv -1 \bmod 3$	$\varphi_{4,1}$	$\varphi_{4,2}$	$\varphi_{4,3}$
$\chi_{4,1}$	1	·	·
$\chi_{4,qs}$	·	1	·
$\chi_{4,\mathrm{St}}$	1	·	1

$\mathcal{E}_3(G, s_9)$: For a $3'$-element $s \in G^*$ of type s_9 with $\mathcal{E}(G, s) = \{\chi_{9,1}, \chi_{9,qs'}, \chi_{9,\mathrm{St}}\}$, there are exactly three irreducible Brauer characters $\varphi_{9,1}$, $\varphi_{9,2}$, $\varphi_{9,3}$ associated to the set $\mathcal{E}_3(G, s)$, and the 3-decomposition numbers are as follows for some integer $e \geqslant 0$:

$q \equiv 1 \bmod 3$	$\varphi_{9,1}$	$\varphi_{9,2}$	$\varphi_{9,3}$
$\chi_{9,1}$	1	·	·
$\chi_{9,qs'}$	·	1	·
$\chi_{9,\mathrm{St}}$	·	·	1

$q \equiv -1 \bmod 3$	$\varphi_{9,1}$	$\varphi_{9,2}$	$\varphi_{9,3}$
$\chi_{9,1}$	1	·	·
$\chi_{9,qs'}$	·	1	·
$\chi_{9,\mathrm{St}}$	1	e	1

$\mathcal{E}_3(G, s_i)$, $i \in \{6, 8, 11, 12, 13, 14, 15\}$: If $s \in G^*$ is a $3'$-element of type s_i for a parameter $i \in \{6, 8, 11, 12, 13, 14, 15\}$ such that $\mathcal{E}(G, s) = \{\chi_i\}$, then $\mathcal{E}_3(G, s)$ has exactly one irreducible Brauer character $\varphi_{i,1}$ associated to it, and the 3-decomposition matrix for $\mathcal{E}(G, s)$ is as follows:

	$\varphi_{i,1}$
χ_i	1

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