Siegfried Wendt
University of Kaiserslautern

The Concept of "State" in System Theory

In system theory, state is a key concept. Here, the word "state" refers to condition, as in the example "Since he went into the hospital, his state of health worsened daily." This colloquial meaning was the starting point for defining the concept of state in system theory.

System theory describes the relationship between input X and output Y, that is, between influence and reaction. In system theory, a system is something that shows an observable behavior that may be influenced. Therefore, apart from the system, there must be something else influencing and observing the reaction of the system. This is called the environment of the system.

System theory distinguishes between continuous systems and discrete systems. A discrete system accepts influences X and produces reactions Y that are elements of discrete sets. Placing a coin into a subway ticket vending machine and the subsequent dispensing of a ticket from the machine provide an example of an individual influence X and reaction Y in a discrete system. An example of a continuous system would be equipment for acoustic amplification. In this system, the influence X is generated as a variation of the air pressure in front of the microphone. The system output Y is produced as a variation of the air pressure in front of the speaker.

Concerning discrete systems, there is usually no conflict in determining which occurrence is an individual influence or which is an individual reaction. In the example of the ticket vending machine, there is no other choice than to regard the insertion of a coin or the pushing of a button as input individuals.

Regarding continuous systems however, examples do not provide a definite answer to the question what should be considered to be a single influence or a single reaction. Using the example of the amplifier system again, there is continuous air pressure "X(t)" [1)] in front of the microphone from the time t_{ON} (when the system is taken into operation) until the time t_{OFF} (when the system is shut off). During the same time interval, there is air pressure "Y(t)" in front of the speaker. One could consider the course of the air pressure "X(t)" in the interval between t_{ON} and t_{OFF} as a single influence on the system to which the course "Y(t)" within the same interval occurs as the reaction of the system.

However, instead of considering the "X(t)" as a single influence, it may also be seen as a continuous sequence of influences X(t). When looking at it this way, one should not be disturbed by the fact that an air pressure value X(t) at only one single point in time cannot really have an influence on the system. Realistically, to cause a reaction, it is necessary to move matter or energy into the system, and this requires a time interval.

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1) To distinguish a functional value at a certain point in time from the function itself, i.e. from the functional course over the time axis, the following symbols are used:

are used:	X(t)	is a symbol for the value of the input X at the time t.
	"X(t)"	is a symbol for the course of the input over the time axis.

Nevertheless, it is convenient in system theory to use the measuring value X(t) as an input into the system when dealing with continuous systems. Therefore, in the following discussion, an influence or input X is meant to be
– a value X(t) measured at a certain point in time for continuous systems
– a discrete input individual in the case of discrete systems.

The same distinction applies to the reactions or outputs Y.

There are simple cases where the relationship between input and output can be represented as a function, in which the input X is the argument and the output Y is the result:

$$Y = f(X)$$

Each system with such functional behavior is called a *mapper.* An example of a mapper is a system that relates the position of a light switch to the burning of a light bulb.

Not all systems are mappers. This means that the reaction Y cannot always be predicted by knowing the influence X. Using the telephone system as an example, an influence occurs by dialing a certain phone number. The reaction to the number dialed is not always definite. If the number dialed belongs to a married couple, then either the wife or the husband may answer the call. However, it could also be that no one is at home and the call goes unanswered.

As a consequence, the system theorist is faced with the problem of describing systems where the relationship between an influence X and a reaction Y cannot be a function Y=f(X). In cases like this, it is obvious to look for another functional relationship g instead of the non–existing functional relationship f. The formula is

$$Y = g(X, \textit{context})$$

where *context* includes everything that must be known in addition to the influence X, in order to conclude a reaction Y. The context can include three different types of information:

(1) information which, principally, cannot be found at all;
(2) information which can be found by looking at the environment;
(3) information which can be found by looking into the system.

These three types of context information will now be illustrated with examples.

As an example for the first type, take the game of roulette at a casino. The roulette wheel with its numbered compartments and the ball may move towards each other relatively. The croupier's influence exists in setting the wheel in motion and throwing the ball in the opposite direction of the spinning wheel. It cannot be predicted which numbered compartment the ball will eventually fall into; there is no context knowledge that would allow such a prediction.

The behavior of such systems cannot be described without leaving room for random effects. Systems like these are called *indeterministic systems*. On the other hand, systems are called *deterministic* if, in order to predict a reaction Y to an influence X, there is either no context knowledge necessary or the necessary context knowledge may be found in the environment or in the system.

In the case where context information can be found in the environment, it must be assumed that the interface between the system and its environment is such that – apart from the influence X – there may be other simultaneous influences by the environment on the system.

An example of this would be the flight behavior of a ball that is influenced by certain forces. In this case, the influence X could be an athlete throwing the ball at a certain speed in a certain direction. Another influence could be the wind influencing the flight path. Therefore, if the influence X by the thrower is known as well as the external context, which is the wind, then the flight path may be predicted, but not by knowing only X.

In the case where context information can be found by looking into the system, this context information is called *the state of the system*.

Take, for example, a machine that dispenses bottles filled with mineral water. Assume that this machine only accepts coins of a certain value, and that the price of a bottle is three times the value of a coin. The reaction of the machine to the insertion of a coin will not always be the same. The observed reaction of the machine to the insertion of the first or the second coin will be that – at the interface between the system and its environment – nothing happens. However, the reaction to the insertion of the third coin will be that a bottle of mineral water rumbles into the dispenser. In this example, the insertion of each coin changes the state of the system because the machine must know each time the total amount of money inserted so far.

In all cases where apart from the state, no other context information need to be taken into consideration, it is sufficient to know the present system state Z and the influence X in order to predict the reaction Y. For continuous systems, the formula is

$$Y(t) = \omega \, [\; Z(t), X(t) \;]$$

where ω is the so–called *output function* which provides the system reaction Y, which can be observed at the interface between the system and its environment.

Of course, the influence X may also cause changes inside the system, which cannot be observed at the interface. Changes inside the system can be context changes; that is, the state Z may change which could cause the system to show a different reaction to the same X in the future. The difference between a later state Z(t+Δt) and an earlier state Z(t) must be the result of the influences that the system has been exposed to in the interim. For continuous systems, the formula is

$$Z(t+\Delta t) = \delta \, [\; Z(t), \text{"}X(\tau)\text{"} \text{ in the interval } \; t \le \tau < t+\Delta t \;]$$

where δ is the so–called *state transition function*. The argument of δ is a certain course, and a value is supplied as a result. This can be seen as a kind of integration, because in the normal case of integration, a curve is given as the argument, and the size of the area beneath the curve is supplied as the result. It may be said that in a given state, knowledge about occurrences from the past is concentrated.

For discrete systems, the continuous time variable t is not used, because the inputs and outputs are individuals that are obtained by abstraction of certain occurrences. Thus, for example, the insertion of a coin is a process that takes a certain amount of time. To determine that a coin was inserted, it is unimportant how long the insertion process lasted. The time period would only be relevant if it were possible for several discrete input or output occurrences to overlap simultaneously. This is only possible with systems that have more than one input channel or more than one output channel.

If, for example, a vending machine for certain goods has more than one coin insertion slot, two people could insert coins independent of each other. In this case, it would be relevant for the behavior of the machine to know how the individual coin insertions occurred in relation to time. In cases like this the processes are said to be *concurrent*; this refers to their causal independence.

Concurrency is only defined in the world of discrete systems. The term *concurrency* has no well–defined meaning in continuous systems, where the input X, the output Y and the state Z are observed at each point t of the time continuum.

Formulas describing the behavior of discrete systems, which are structured in the same way as the formulas describing the behavior of continuous systems, can only be determined for *sequential systems* which, by definition, exclude concurrency. These formulas are as follows:

$$Y(n) = \omega [Z(n), X(n)]$$

$$Z(n+\Delta n) = \delta [Z(n), \text{"}X(j)\text{" in the interval } n \le j < n+\Delta n]$$

The formulas for continous and discrete systems differ only in one aspect: The variables t and τ for continuous time are replaced by the variables n and j, which are indices for counting the successive input, output and state individuals.

The discrete course "X(j)" in the interval $n \le j < n+\Delta n$ is a finite sequence of input individuals and may be represented explicitly:

$$Z(n+\Delta n) = \delta [Z(n), (X(n), X(n+1), X(n+2), ... X(n+\Delta n-2), X(n+\Delta n-1))]$$

While in the continuous case, the length Δt of the time interval must be left open, the length Δn in the discrete case may be limited to one. This results in the following formulas:

$$Y(n) = \omega [Z(n), X(n)]$$

$$Z(n+1) = \delta [Z(n), (X(n))]$$

In these formulas, the two appearances of X(n) differ formally: In the output function ω, X(n) appears as one single value on principle, while in the state transition function δ, X(n) is an element of a sequence that arbitrarily has been restricted to length one. The formulas are usually written so that this difference is no longer apparent:

$$Y(n) = \omega [Z(n), X(n)]$$

$$Z(n+1) = \delta [Z(n), X(n)]$$

These two formulas make up the so–called *state machine model* by Mealy, which has been extensively detailed in the author's essay "The models by Moore and Mealy – Clarification of a confusion in terms."

Though these formulas look simpler than those of the continuous case, one should not draw the erroneous conclusion that the formulas for sequential systems may be interpreted as easily or even easier than the formulas for continuous systems.

For continuous systems, all three variables X, Y and Z fall into the same category, which is the category of measured values. That is why the courses "X(t)", "Y(t)" and "Z(t)" can be pictured in a uniform way as curves over the time axis. In contrast to this, the variables X, Y and Z of the discrete system do not fall into the same category. In the most simple interpretation, X and Y are to be interpreted as variables for events, and Z is to be interpreted as a variable for a static situation. In the state Z(n), the system is static until it is disturbed by an influence in the form of an occurrence X(n). The system then produces the result Y(n) and is then again at rest, which we describe as the new system state Z(n+1).

In the aforementioned essay ”The models by Moore and Mealy – Clarification of a confusion in terms”, the various possibilities of interpreting the formulas of the sequential system are described in detail. In particular, if these formulas are used to describe sequential circuits, the relationships between the formal variables X, Y and Z, and the appearances observed during the operation of a sequential circuit are not at all trivial.

Based on the uniform category of the three variables X, Y and Z, there is the possibility that all three variables may be vector variables in a continuous system. Input, output or state do not have to be limited to single measured values. Rather, a definite number of measured values may be grouped together:

$$X(t) = (x_1(t), x_2(t), x_3(t), \ldots \; x_k(t))$$

$$Y(t) = (y_1(t), y_2(t), y_3(t), \ldots \; y_l(t))$$

$$Z(t) = (z_1(t), z_2(t), z_3(t), \ldots \; z_m(t))$$

In discrete systems, the possibility of a vectorization does not exist for all three variables X, Y and Z in the same way. The vectorization of Y and Z creates no problems, whereas the vectorization of X is difficult.

Since the state Z is static, it may be composed of a definite number of smaller circumstances. Furthermore, an output individual Y may be composed of smaller individuals, each having its own output channel.

A separation of an input individual X into smaller individuals that have their individual channels, may only be admitted on the condition that there will not be a problem of concurrency. Otherwise, the system would no longer be a sequential system and the given formulas would no longer be appropriate. If there are several input channels, then there are only two possibilities for excluding concurrency:

(1) One possibility would be that the environment is obliged to actually use only one of the input channels at a time, through which an input individual is sent. At the same time, the remaining input channels are not to be used. It would not be reasonable to consider this mode of operation as a vectorization of X.

(2) The other possibility would be that one of the input channels is the *trigger channel* and the remaining channels would only be used as *scannable channels*. A reaction of the system may only be triggered by an event on the trigger channel. An example for such a triggering event would be the insertion of a coin or the pressing of a button. Through the scannable channels, the environment offers information to the system, which the system may or may not use. When a triggering event occurs on the trigger channel, the reaction of the system may depend on the information offered on the scannable channels. An example of this would be a person’s reaction to being kicked on the shinbone. The reaction may be influenced by what the person sees while looking around at the time of the kick.

The so–called clocked sequential circuits are typical examples for such systems that have several input channels, but only one of them, the channel for the clock–pulse, being the trigger channel.