Linear diffusions conditioned on long-term survival

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Introduction

Nowadays, diffusion models are used in many different areas such as engineering, physics, medicine and a lot more. References pointing to the various fields of applications can be found in [Bis08]. In this thesis, we investigate diffusions under the condition of 'long-term survival'. Survival means there must be killing. The first thought most people may have in mind is the death of some single individual. No doubt there are certain species on earth whose members have a realistic chance to grow extremely old in age compared to the life span of a human. If this happens one may say they survived very long. But if we stop to think about single individuals and rather look at whole populations 'long-term survival' becomes even more likely.

In this context diffusions are well-suited to model large populations or the competition between two or more species. In [Lip77] and [JM86] it is shown that certain diffusions can arise as scaling limits forcing the initial population to grow towards infinity. This large initial size is what we mean by 'large population'. The usual case is that a population eventually becomes extinct, i.e. killed at a random time τ which we will call 'killing time'. The simplest possibility this can happen is if the population size reaches zero due to denatality and no offspring can be produced anymore. In the context of one-dimensional diffusions $Y = (Y_t)_{t\geq 0}$ this means that Y takes its values in the 'state space' $[0, \infty)$ and zero is a 'killing boundary', i.e. if $Y_s = 0$ then $Y_t = 0$ for all $t \geq s$.

In contrast, the competition of two species is a situation where two killing boundaries naturally come up. Suppose the first species is of size $N_1(t)$ and the second one of size $N_2(t)$ but only the proportion $\frac{N_1(t)}{N_1(t)+N_2(t)}$ of the first species is measured. If we use a diffusion Y to model this situation, it is clear that Y takes its values in $[0, 1]$ and there are two killing boundaries: If Y reached zero before one, this means that only the second species has survived and vice versa. (If Y reached one before zero the second species became extinct.) Thus, the killing time τ is the first time t such that Y_t hits zero or one. Similarly, we can look at the evolution of two interacting genes in a population. For an application of diffusions in genetics we refer to Chapter 15 in [Lan03], a more recent publication is [Les09].

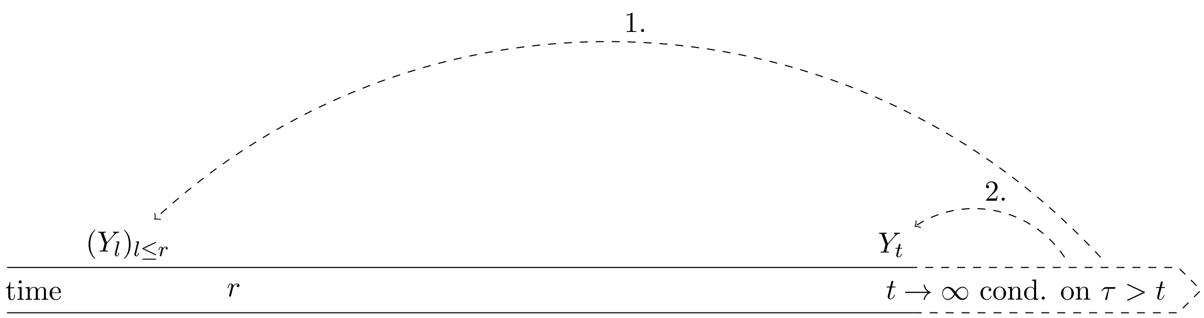
There are also models where killing may occur in the 'interior'. One example is a patient who suffers from a tumor. If the tumor reaches a critical mass or even spreads out, the chance that the patient will die in a short time period is significantly larger than zero. In the context of diffusions this is modelled by a 'killing rate' $x \mapsto \kappa(x)$ defined on the state space. The killing rate is high at hazardous states, i.e. at values for Y where the chance of being killed is high. For an eloborated model of a diffusion under the influence of internal killing we refer to [KT83].

We dedicate ourselfes to the long-term behaviour of diffusion under killing which follow a stochastic differential equation $dY_t = dX_t + \boldsymbol{a}(Y_t)dt$ driven by a Brownian motion X. The diffusions we are working with in this thesis will almost surely be killed. Of course, the analysis of the long-term behaviour is only interesting for the 'long-term survivors', i.e. only for those paths $[0, t] \ni s \mapsto Y_s(w)$ which are *not* killed up to a long time t. Hence, investigating $[0, t] \ni s \mapsto Y_s$ conditioned on survival up to time t seems like a good thing to do. Here we let t tend to infinity looking for a limiting behaviour. If a limit exists, it could serve as an approximation of the diffusion conditioned on long-term survival. In literature we find essentially two kinds of limit results.

1. The first is the one which investigates the so-called 'Q-process' of Y under τ. We look at the whole process under 'infinite survival' as explained above. In chapter 2 we will identify the Q-process for certain diffusions Y and killing times τ. We recommend [CCL$^+$09] and [CMSM13a]. Further, we will see that a Q-process is a special case of a 'penalisation limit'. The corresponding theory was developed several years ago by Yor et al. See the series of papers [RVY06b, RVY06a, RVY05, RVY07] or the book [RY09].

2. The other class of limit results focuses on the distribution of Y_t conditioned on $\tau > t$ if t tends to infinity. If this limit exists, it is referred to as 'quasi-limiting distribution'. A special case is the so-called 'Yaglom limit' $\lim_{t\to\infty} \mathbb{P}_x(Y_t \in \bullet \mid \tau > t)$. We will look into this phenomenon of long-term survival in chapter 1. We also refer to [MV12], [KS12] and [CMSM13a].

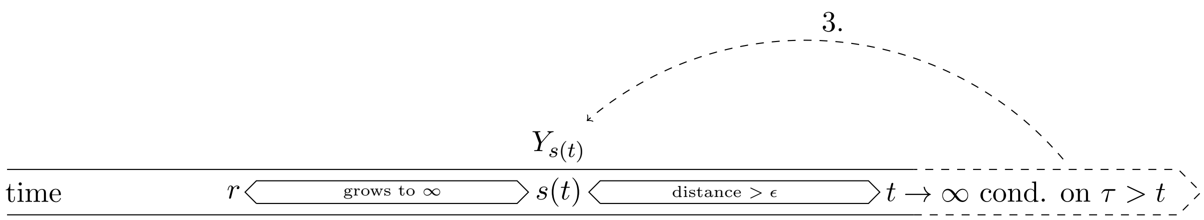
These 2 possibilities are illustrated in the following figure.



Remarkably, even if the Q-process for Y under killing at τ has a unique stationary distribution, in general this is *not* the quasi-limiting distribution of Y under τ. Nevertheless, Q-process and quasi-limiting distribution are closely connected, at least formally. We will illustrate this in chapters 1 and 2.

We also investigate a third possibility to condition the process Y on infinite survival:

3. In chapter 3 we go one step further and 'look between' the Q-process and the quasi-limiting distribution. As the following figure shows, we are interested in the long-term behaviour of $Y_{s(t)}$ under $\tau > t$ for some sequence $s(t) \to \infty$ with $t - s(t) > \epsilon > 0$.



As indicated above, chapter 1 deals with the quasi-limiting behaviour of certain diffusions. In particular, we give criteria for the existence of a Yaglom limit. An inspiration for us was the article [KS12] of Kolb and Steinsaltz. Their Corollary 4.8 sharply separates the case of existence of a Yaglom limit from the case that all the mass goes to infinity in the limiting process described above. Though all their other results also allow a non-zero killing rate, in Corollary 4.8 it is assumed that κ is zero. So we tried to fill this gap. By theorem 1.12 and its corollary 1.13 we see that this dichotomy also holds for a non-zero κ such that $\lim_{x\to\infty}\kappa(x)$ exists:

- $\underline{\lambda} > \lim_{x\to\infty}\kappa(x) \Rightarrow \lim_{t\to\infty}\mathbb{P}_x(Y_t \in \bullet \mid \tau > t)$ is a probability distribution on $[0,\infty)$,
- $\underline{\lambda} = \lim_{x\to\infty}\kappa(x) \Rightarrow \lim_{t\to\infty}\mathbb{P}_x(Y_t \in A \mid \tau > t) = 0$ for all bounded and measurable A

where $-\underline{\lambda}$ is the end of the spectrum of the self-adjoint L^2-generator.

When it comes to the investigation of long-term survival the asymptotic behaviour of $t \mapsto \mathbb{P}(\tau > t)$ is of particular importance as can be see in [MSM01] and [KS12]. In [MSM01] it is shown that the exponential rate of decay for $t \mapsto \mathbb{P}(\tau > t)$ equals $\underline{\lambda}$ under killing at zero and if the drift coefficient $\boldsymbol{a}$ is C^1. In their Corollary 1 the authors use this fact to show that a larger drift 'towards killing' results in a higher $\underline{\lambda}$. In combination with the preceding dichotomy we get a comparison result of the form:

If we have a Yaglom limit for a diffusion $\tilde{Y}$ under τ and Y is another diffusion with a stronger drift 'towards killing', we also get the existence of a Yaglom limit for Y under τ.

We prove such comparison results in theorem 1.17 and, under assumptions similar to the ones made in [KS12], in theorem 1.22. Nevertheless, these theorems are not very much suitable for a quick check for a Yaglom limit of Y. A priori these results do not provide us with diffusions $\tilde{Y}$ to compare with. Hence, to increase applicability we formulate these theorems for the special cases of $\tilde{Y}$ being an Ornstein-Uhlenbeck process and $\tilde{Y}$ being a Brownian motion with constant drift. Furthermore, we show that in many cases the existence of a Yaglom limit for Y with drift coefficient $\boldsymbol{a}$ and for $\tilde{Y}$ with drift coefficient $\tilde{\boldsymbol{a}}$ implies that the diffusions corresponding to the drift coefficients $\min(\boldsymbol{a}, \tilde{\boldsymbol{a}})$ and $\max(\boldsymbol{a}, \tilde{\boldsymbol{a}})$ also have a Yaglom limit.

At the end of chapter 1 we mention a result obtained by Steinsaltz and Evans in [SE07] giving the Yaglom limit if the state space of Y is a finite interval and the drift coefficient is from C^1. We included it since it is some sort of counterpart to proposition 2.3 which is a corresponding penalisation result, i.e. it gives the Q-process of Y living on a finite interval.

As already mentioned, chapter 2 is devoted to the investigation of penalisation limits with special emphasis on the Q-process for diffusions Y on $[0,\infty)$. We pay special attention to the observation that whether a Yaglom limit exists or not has, if at all, only a minor effect on the Q-process. We prove appropriate results in theorems 2.11, 2.13 and 2.14. These results even show that the general form of the Q-process is the same in both cases. Furthermore, in section 2.3 we identify the penalisation limit as the measure of a diffusion $(Z_t)_t$ using certain results about martingale problems. This Z solves a stochastic differential equation similar to the one for Y but with an additional drift term. This drift term as well as the penalisation limit are fully determined by the eigenfunction ψ of the generator corresponding to the asymptotic exponential rate of $t \mapsto \mathbb{P}_x(\tau > t)$. We will also see that the existence of a stationary distribution of Z goes by the square integrability of ψ. To make a connection with quasi-limiting behaviour, there are equivalent statements for the Yaglom limit:

The existence of the Yaglom limit goes by the integrability of ψ.

This can be seen directly from Theorem 3.3 in [SE07] or from Theorem 2.6 in [KS12] as well as from [Miu14], respectively. Anyway, in the results of this thesis the density of the Yaglom limit turns out to be the normalized eigenfunction ψ. Hence, if the Yaglom limit exists then ψ has to be integrable.

In the last section of chapter 2 we are dealing with so-called 'universal measures'. As we can see from [NRY09] they can be related to certain penalisations of a diffusion Y and bear many interesting properties. However, our focus is on the the fact that these measures can provide the opportunity to express the penalisation limit in terms of these universal measures and, what is more important here, without a limiting procedure.

By the second figure from above the basic objective in chapter 3 is the investigation of the long-term behaviour of $t \mapsto \mathbb{P}_x(Y_{s(t)} \in \bullet \mid \tau > t)$. At first we let $s(t)$ follow t in a short distance, i.e. $s(t) = t - u$ for some fixed $u > 0$. One may conjecture that the qualitative long-term behaviour is the same as for $t \mapsto \mathbb{P}_x(Y_t \in \bullet \mid \tau > t)$. Indeed, this turns out to be true. We further establish ties between the Yaglom limit and the Q-process by taking a second limit in u and obtain that

1. for $u \to 0$ the result is the Yaglom limit if existent.
2. if u tends to ∞ and under a non-decreasing κ we get that

$$\lim_{u\to\infty} \left(\lim_{t\to\infty} P_x(Y_{t-u} \in \bullet \mid \tau > t) \right)$$

is the unique stationary distribution of the penalisation limit.

Intuitively, the asymptotic behaviour of $P_x(Y_{t-u} \in \bullet \mid \tau > t)$ should be the same as the long-term behaviour of $P_x(Y_{t-u(t)} \in \bullet \mid \tau > t)$ for a sequence $(u(t))_t$ with $\lim_{t\to\infty} u(t) = u$. We show this under the the existence of a Yaglom limit as well as under the assumption that the mass goes to ∞, i.e. under the second case in the above dichotomy. As a matter of fact, in the latter case we observe again that the mass escapes from every bounded set. This can be found in proposition 3.2. We see by theorem 3.4 that, in the first case, we have the same asymptotic behaviour as $P_x(Y_{t-u} \in \bullet \mid \tau > t)$. To prove this we show uniform convergence on compacts of the semigroup of Y under killing.

But we also look 'really between' the Yaglom limit and the Q-process, i.e. if $t - s(t)$ also tends to infinity as $t \to \infty$. In principle, we observe two different cases. We get the first case if the order of $t \mapsto \mathbb{P}_x(\tau > t)$ is not exactly exponential but has an additional polynomial order. One example we will run into several times in this thesis is a Brownian motion with constant drift coefficient under killing at zero. Under the assumption of a non-increasing κ we show that $\mathbb{P}_x(Y_{s(t)} \in A \mid \tau > t) \to 0$ for every bounded A. This is the content of proposition 3.5.

In contrast we look at a case where $t \mapsto \mathbb{P}_x(\tau > t)$ has an exact exponential order and the Q-process $(Z_t)_t$ has a unique stationary distribution. With the help of a Girsanov transformation we get rid of the drift coefficient and obtain a Brownian motion under killing. This enables us in theorem 3.6 to use previous results to show that $\mathbb{P}_x(Y_{s(t)} \in \bullet \mid \tau > t)$ converges to the stationary distribution of Z. Note that we qualitatively obtain the same result as in point 2. from above.

0. One-dimensional diffusions

In the next sections we are going to introduce *one-dimensional diffusions* which are also called *linear diffusions*. We will focus on them for most parts of this thesis. Essentially a linear or one-dimensional diffusion is a strong Markov process $(Y_t)_{t\geq 0}$ with values in $\mathbb{R}$ and such that $t \mapsto Y_t$ is almost surely continuous; except for one possible jump which occurs when the process is 'killed' at some stopping time τ. The easiest killing one can think of is at a *first hitting time*

$$T_A := \inf\{t \geq 0\,;\ Y_t \in A\},$$

i.e. when the process hits a certain measurable subset A of the *state space* E for the first time. In general E is some interval I together with a so-called *cemetary point* '$\dagger$' which is an absorbing state. Y gets stuck in $\dagger$ after killing. Most of the time we will merely write 'Y' instead of '$(Y_t)_t$'.

As a probabilistic model of an *unkilled* diffusion we have a set

$$(\Omega, \mathcal{F}_\infty, (\mathcal{F}_t)_t, (\mathbb{P}_x)_{x\in E}, (Y_t)_t) \tag{0.1}$$

where

- $(\Omega, \mathcal{F}_\infty, (\mathcal{F}_t)_t, \mathbb{P}_x)$ are filtered probability spaces.
- $(Y_t)_t$ is $(\mathcal{F}_t)_t$-adapted.
- $\mathbb{P}_x(Y_t = 0) = 1$ for each $x \in E$.
- $(Y_t)_t$ is a strong Markov process w.r.t. $(\Omega, \mathcal{F}_\infty, (\mathcal{F}_t)_t, (\mathbb{P}_x)_{x\in E})$.
- $(Y_t)_t$ has almost sure continuous paths under each $\mathbb{P}_x$.

If we include killing at τ the tuple, (0.1) becomes

$$(\Omega, \mathcal{F}_\infty, (\mathcal{F}_t)_t, (\mathbb{P}_x)_{x\in E}, (Y_t)_t, \tau).$$

and the last point from above turns into:

- τ is a stopping time w.r.t. $(\mathcal{F}_t)_{t\geq 0}$ such that
  - $[0, \tau) \ni t \mapsto Y_t$ is almost surely continuous under each $\mathbb{P}_x$.
  - $Y_t = \dagger$ for all $t > \tau$.

0.1. (Constructing) diffusions as solutions to SDEs in $\mathbb{R}$

In a lot of cases we can construct a diffusion Y on $E = \mathbb{R}$ as a solution to a certain *stochastic differential equation (SDE)* and introduce killing in a second step. We can use objects like *standard Brownian motion (BM)* $(X_t)_t$ with which we can built our processes $(Y_t)_t$. The goal is to construct Y with given *infinitesimal diffusion rate* $\sigma(x,t)$ and *drift coefficient* $\boldsymbol{a}(x,t)$; i.e. such that

$$\begin{aligned} \mathbb{E}_x(Y_{t+h} - Y_t \mid Y_t = y) &= h\,\boldsymbol{a}(y,t) + o(h) \\ \mathbb{E}_x((Y_{t+h} - Y_t)^2 \mid Y_t = y) &= h\sigma^2(y,t) + o(h). \end{aligned} \tag{0.2}$$

For us '$o(g)$' is the *Landau notation* for an f such that $\frac{|f|}{g} \to 0$. Here this simply means $\lim_{h\downarrow 0} \frac{|o(h)|}{h} = 0$.

remark 0.1.
- Including killing later on will force the probabilistic model (0.1) to be rich enough or to be suitably enlarged to carry additional objects like the mentioned cemetary point or some exponentially distributed random variable ξ independent from Y. See subsection 0.3 for a typical situation.
- If $E \neq \mathbb{R}$, the assumptions for Y in this thesis will be such that we can always identify Y as a solution to an SDE up to the first hitting $T_{\partial E}$ of ∂E.
- For Y being a BM, i.e. $Y = X$, we have $\boldsymbol{a} \equiv 0$ and $\sigma \equiv 1$ and (0.2) takes a particularly nice form.

∎

Actually there are a few different ways to get the existence of such an Y depending on the model one chooses:

1. If $\sigma = \sigma(x) > 0$ is measurable, bounded and has a positive distance to zero, one can start with Brownian motion and first do a certain time change to incorporate $\sigma(x)$. Essentially the mentioned time change is the inverse τ_t of

   $$F(\tau) = \int_0^\tau \frac{dr}{\sigma(X_r)},$$

   i.e. $F(\tau_t) = t$. Now on the one hand $(X_{\tau_t})_t$ has a generator of the form

   $$\mathbf{L} = \frac{1}{2}\sigma^2(x)\frac{d^2}{dx^2}$$

   and if the diffusion Y_t corresponding to this $\mathbf{L}$ is a solution to an SDE it will be

   $$dY_t = \sigma(Y_t)dX_t. \tag{0.3}$$

   On the other hand existence and uniqueness of a solution to (0.3) can be accomplished by assuming that σ is locally lipschitz continuous.

   Next to incorporate an additional drift we can use Girsanov's formula. Therefore, let $\boldsymbol{a}$ be locally lipschitz continuous and suppose there exists $M > 0$ such that $|\boldsymbol{a}(x)| \leq M \cdot \sigma(x)$ for all $x \in \mathbb{R}$. Then we are allowed to use 'Girsanov' to transform the measure under which Y is the unique strong solution of (0.3) to a measure such that Y now is the unique strong solution of

   $$dY_t = \sigma(Y_t)dX_t + \boldsymbol{a}(Y_t)dt. \tag{0.4}$$

Note that the 'Radon-Nykodim martingale' for this change of measure can be given explicitly. (See chapter 6 in [Var07].)

2. Suppose we are still in the case that σ and $\boldsymbol{a}$ not depend on time t. But this time let $\boldsymbol{a}, \sigma \in C(\mathbb{R})$ with $\sigma > 0$. Now we want to show a way to construct a weak solution to

$$dY_t = \sigma(Y_t)dX_t + \boldsymbol{a}(Y_t)dt \tag{0.5}$$

by a method presented in [Dur96].

Take a Brownian motion $\tilde{X}$, fix any $d \in \mathbb{R}$ and set

$$s(x) := \int_d^x \exp\left(-2\int_d^y \frac{\boldsymbol{a}(z)}{\sigma^2(z)}dz\right) dy. \tag{0.6}$$

(Note that we write $\tilde{X}$ instead of X since this will not be the 'driving' Brownian motion of the SDE (0.5).) Next we define

$$h(x) := \left[s'(s^{-1}(x))\right]^2 \cdot \left[\sigma^2(s^{-1}(x))\right] \quad \text{and} \quad g(x) := \frac{1}{h(x)}.$$

To the end let $t \mapsto \tau_t$ be the inverse of $\tau \mapsto \int_0^\tau g(\tilde{X}_r)dr$. In [Dur96] Chapter 6 it is shown that $Z_t := \tilde{X}_{\tau_t}$ solves the martingale problem for the drift 0 and the diffusion coefficient h. We do not want to talk about martingale problems here. Rather we advice the reader to read into [Dur96] or [Pin95] on this topic. Alternatively one can dip into section 2.3 where some results about martingale problems are used and written down explicitely.

The important fact here is, solving the mentioned martingale problem implies that Z_t 'is a weak solution' to

$$dZ_t = \sqrt{h(Z_t)}dX_t. \tag{0.7}$$

X is a particular Brownian motion which can always be constructed after suitably enlarging the underlying probability space. (This is the reason why we talk of a 'weak' solution! See the proof of Theorem (4.5) in Chapter 5 of [Dur96].)

Up to now it seems like we have not made any improvement. But if we use Itô's formula (and some other calculations exploiting the special form of s) we derive

$$s^{-1}(Z_t) - s^{-1}(Z_0) = \int_0^t (s^{-1})'(Z_r)dZ_r + \frac{1}{2}\int_0^t (s^{-1})''(Z_r)h(Z_r)dr \tag{0.8}$$

and

$$\begin{aligned} \frac{1}{2}(s^{-1})''(z)h(z) &= \boldsymbol{a} \circ s^{-1}(z) \\ (s^{-1})(z)\sqrt{h(z)} &= \sigma \circ s^{-1}(z). \end{aligned} \tag{0.9}$$

And indeed, plugging (0.7) and (0.9) into (0.8) we see that

$$Y_t := s^{-1}(Z_t)$$

solves

$$\begin{aligned} dY_t &= \boldsymbol{a}(Y_t)dt + (s^{-1})'(Z_t)dZ_t \\ &= \boldsymbol{a}(Y_t)dt + (s^{-1})'(Z_t)\sqrt{h(Z_t)}dX_t \\ &= \boldsymbol{a}(Y_t)dt + \sigma(Y_t)dX_t. \end{aligned}$$

Note that we also have uniqueness in law of the solution in a bounded interval. (I.e. we have uniqueness until $T_{[b,c]}$ if we start at $x \in (b,c)$ with $-\infty < b < c < \infty$. See Theorem (1.7) of chapter 6 in [Dur96].)

By the way: The function s from (0.6) is a 'scale function' for Y. More on scale functions will follow in section 0.2.

3. If one allows dependence on t, one usually assumes that σ and $\boldsymbol{a}$ satisfy some Lipschitz condition. Then one can show existence and uniqueness of solutions of

$$dY_t = \sigma(Y_t, t)dX_t + \boldsymbol{a}(Y_t, t)dt$$

using stochastic calculus. In this thesis we do not deal with diffusions where σ and $\boldsymbol{a}$ explicitely depend on time. Thus, we merely recommend the short but readable part in Chapter 6 of [Var07] on this topic.

0.2. Appearance and meaning of scale function and speed measure

The *scale function* s is a characteristic for one-dimensional diffusions which can be used as a space transformation to get rid of the drift coefficient. Its existence is assured for any *regular diffusion*. By *regular diffusion* we mean a diffusion such that

$$\mathbb{P}_x(T_y < \infty) > 0$$

for all $x, y \in I$. See [Bre92] or [BS02]. (Note that we also write T_y instead of $T_{\{y\}}$ for the first hitting time of $A = \{y\}$.) The defining property of s, which now should not be surprising, involves probabilities of leaving intervals at the left boundary earlier then at the right:

For any $x < y < z$ from I it holds:

$$\mathbb{P}_y(T_x < T_z) = \frac{s(z) - s(y)}{s(z) - s(x)}$$

In particular, s is continuous and strictly increasing.

After we have done a space transformation with the scale function we say that the process is on its *natural scale*. If we have the process on its natural scale, we still have a diffusion rate which essentially is responsible for how fast the process leaves certain areas. Here comes the second characteristic; the *speed measure* $m(dx)$. With respect to the speed measure we can integrate to compare expected escape times from subintervals of I.

Mainly when we talk about diffusions (without killing in the interior) we will only look at processes whose generator is of the form

$$\mathbf{L} = \frac{1}{2}\frac{d^2}{dx^2} + \boldsymbol{a}(x)\frac{d}{dx},$$

i.e. a Brownian motion with some additional drift (function) '$\boldsymbol{a}$', which is taken to be continuous in the interior $\mathring{I}$ of I. (More general the *interior* can be defined for any subset U of a metric space V as the set of points which can not be reached by convergent subsequences from $V \setminus U$. For an interval I this is just I without endpoints.) Then we have

$$s(x) = \int_d^x s'(y)\,dy = \int_d^x \exp\left(-2\int_d^y \boldsymbol{a}(z)\,dz\right)dy. \tag{0.10}$$

and

$$m(dy) = m(y)\,dy = 2\exp\left(2\int_d^y \boldsymbol{a}(z)\,dz\right)dy = \frac{2}{s'(y)}\,dy. \tag{0.11}$$

(with some d) in the interior $\mathring{I}$ of the state space I.

remark 0.2. Since s is only unique up to linear transformations with positive slope and m depends on $s(y)$, there is a parameter dependence on 'd'. Hence both, m and s, are only unique up to positive multiples. See for instance [Bre92]. ∎

There is more to the speed measure than it may seem at first sight. To investigate this let us start with

$$\mathbf{L}_c := \frac{1}{2}\frac{d^2}{dx^2} + \boldsymbol{a}(x)\frac{d}{dx}$$

acting on $C_c^2(\mathring{I})$, i.e. on functions which are twice continuously differentiable with compact support in the interior of I.

remark 0.3. $\mathbf{L}_c$ may be defined in the space $C_0(I)$ of continuous functions $f : I \to \mathbb{R}$ with $f(x) \to 0$ if $|x| \to \infty$, the space $C_b(I)$ of bounded continuous functions or in some weighted L^2-space. In section A.2.1 we show how one can get the 'whole' generator as a self-adjoint extension of $\mathbf{L}_c$. ∎

We call an operator $\mathbf{L}_c^*$ *formal adjoint* to $\mathbf{L}_c$ w.r.t. the Lebesgue measure if

$$\langle \mathbf{L}_c f, g\rangle_{L^2(dx)} = \int_{\mathbb{R}} (\mathbf{L}_c f)(x)g(x)dx = \int_{\mathbb{R}} f(x)(\mathbf{L}_c^* g)(x)dx = \langle f, \mathbf{L}_c^* g\rangle_{L^2(dx)}$$

for all $f, g \in C_c^2(\mathring{I})$. (This definition is also used in [CMSM13b], Chapter 6.) For instance if $\boldsymbol{a}$ is piecewise differentiable then it is easily verified that

$$\mathbf{L}_c^* = m\mathbf{L}_c\frac{1}{m}$$

on $C_c^2(\mathring{I})$. To see this we use $(\frac{1}{m})' = -\frac{2\boldsymbol{a}}{m}$ (and we omit to write '(x)' for readability):

$$
\begin{aligned}
&\int f\cdot m\mathbf{L}_c(\frac{g}{m})dx \\
&= \int f\cdot m\left[\frac{1}{2}\left(g'\frac{1}{m} - g\frac{2\boldsymbol{a}}{m}\right)' + \boldsymbol{a}\left(g'\frac{1}{m} - g\frac{2\boldsymbol{a}}{m}\right)\right]dx \\
&= \int f\cdot m\left[\frac{1}{2}g''\frac{1}{m} - \frac{1}{2}g'\frac{2\boldsymbol{a}}{m} - \frac{1}{2}2(g\,\boldsymbol{a})'\frac{1}{m} + \frac{1}{2}2g\,\boldsymbol{a}\frac{2\boldsymbol{a}}{m} + \boldsymbol{a}\,g'\frac{1}{m} - \boldsymbol{a}\,g\frac{2\boldsymbol{a}}{m}\right]dx \\
&= \int f\cdot m\left[\frac{1}{2}g''\frac{1}{m} - \frac{1}{2}2(g\,\boldsymbol{a})'\frac{1}{m}\right]dx \\
&= \int f\cdot\left[\frac{1}{2}g'' - (g\,\boldsymbol{a})'\right]dx \\
&= \int\left(\frac{1}{2}f''\cdot g + \boldsymbol{a}\,f'\cdot g\right)dx.
\end{aligned}
\tag{0.12}
$$

For the last line we used partial integration.
This also implies that $\mathbf{L}_c$ is symmetric in $L^2(dm)$, since

$$
\begin{aligned}
\langle \mathbf{L}_c f, g\rangle_{L^2(dm)} &= \langle \frac{1}{m}\mathbf{L}_c^*(mf), g\rangle_{L^2(dm)} \\
&= \int \frac{1}{m}\mathbf{L}_c^*(mf)gm\,dx \\
&= \int mf\mathbf{L}_c g\,dx \\
&= \langle f, \mathbf{L}_c g\rangle_{L^2(dm)}.
\end{aligned}
$$

Thus, the space $L^2(dm)$ will be the right choice to get a self-adjoint extension. See section A.2.1.

remark 0.4. Up to this point everything stays true if we incorporate some 'potential' $0 \leq \kappa \in C(I)$, i.e.

$$
\mathbf{L} \rightsquigarrow \mathbf{L}^\kappa = \frac{1}{2}\frac{d^2}{dx^2} + \boldsymbol{a}(x)\frac{d}{dx} - \kappa(x).
$$

∎

Using the form $\mathbf{L}_c^* f = \frac{1}{2}f'' - (f\,\boldsymbol{a})'$ from the calculations in (0.12) we can define

$$
\mathbf{L}^* f := \frac{1}{2}f'' - (f\,\boldsymbol{a})'
$$

on functions $f \in C^1$ such that f' is piecewise differentiable as an extension to $\mathbf{L}_c^*$. Now $\mathbf{L}^* m$ makes sense and we derive

$$
\mathbf{L}^* m = 0. \tag{0.13}
$$

In fact, (0.13) is necessary for $m(dx) = m(x)dx$ to be an 'invariant measure' for Y. (See [Var07] section 7.4.) An *invariant measure* for Y is a measure ν on $E = I$ such that

$$
\mathbb{P}_\nu(Y_t \in A) := \int_I \mathbb{P}_x(Y_t \in A)\nu(dx) = \int_A \nu(dx)
$$

for all measurable A and all $t \geq 0$. If this ν is a probability measure, we also call it *invariant distribution.* It even holds that Y is positively recurrent iff m is finite. In that case $\frac{m(dx)}{m(I)}$ is the unique invariant distribution for Y. (See section I.II.6 of [BS02])

0.3. How to incorporate killing on a probabilistic level

Besides killing at certain first hitting times there may also be 'slow killing'. Slow killing can be accomplished by switching from the semigroup $\mathbf{T}_t f(x) = \mathbb{P}_x\left(f(Y_t)\right)$ to a so-called *Feynman-Kac semigroup*

$$\mathbf{T}_t^\kappa f(x) = \mathbb{P}_x\left(f(Y_t)\,\mathrm{e}^{-\int_0^t \kappa(Y_s)ds}\right) \tag{0.14}$$

for some 'potential' $\kappa \geq 0$. In contrast to the generator of $(\mathbf{T}_t)_t$ we now have a zero order term $-\kappa$ in the generator $\mathbf{L}$ of Y:

$$\mathbf{L}^\kappa = \frac{1}{2}\frac{d^2}{dx^2} + \boldsymbol{a}(x)\frac{d}{dx} - \kappa(x).$$

On a probabilistic level, to $(\mathbf{T}_t^\kappa)_t$ corresponds the process $(Y_t)_t$ killed at the time

$$\tau := \inf\{t > 0\,;\ \int_0^t \kappa(Y_s)ds > \xi\},$$

where ξ is a standard exponentially distributed random variable independent of $(Y_t)_t$. This can be seen by the following calculation.

$$\begin{aligned}
\mathbb{P}_x(f(Y_t),\, \tau > t) &= \mathbb{P}_x\Big(f(Y_t)\cdot \mathbb{1}_{\{\int_0^t \kappa(Y_s)ds \leq \xi\}}\Big) \\
&= \mathbb{P}_x\Big(f(Y_t)\cdot \mathbb{E}_x\big(\mathbb{1}_{\{\int_0^t \kappa(Y_s)ds \leq \xi\}} \,\big|\, \sigma(Y_s, s \leq t)\big)\Big) \\
&= \mathbb{P}_x\Big(f(Y_t)\cdot \left[\mathbb{P}_x(r \leq \xi)\right]_{r = \int_0^t \kappa(Y_s)ds}\Big) \\
&= \mathbb{P}_x\left(f(Y_t)\cdot \mathrm{e}^{-\int_0^t \kappa(Y_s)ds}\right)
\end{aligned}$$

We write '$\mathbb{1}_F$' for the *characteristic function of the set F*. And as the reader may have already noticed we will also use the following notation.

convention 0.1. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $X : (\Omega, \mathcal{F}) \to (\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ a random variable with $\mathbb{E}\,|X| < \infty$. Then we will also write

$$\mathbb{P}(X) := \int_\Omega X(\omega)\mathbb{P}(d\omega)$$

instead of '$\mathbb{E}(X)$'.

Let us take a short look at the case of $\kappa \equiv k > 0$ to understand why κ is also called *killing potential* or *killing rate*:

$$\mathbb{P}_x(\tau \leq t) = \mathbb{P}_x\Big(\int_0^t \kappa(Y_s)ds > \xi\Big) = \mathbb{P}_x(kt > \xi) = 1 - e^{-kt}.$$

This implies

$$\mathbb{P}_x(\tau \leq t) = kt + o(t)$$

for small t. Now one may suspect that also, more general,

$$\mathbb{P}_x(\tau \leq t) \approx \kappa(x)t. \tag{0.15}$$

In particular, a higher rate (at the starting point x) gives a higher probability of the process to be killed in the next short time period.

More general killing for diffusions can be declared through a so-called *killing measure* $\kappa(dx)$ on I. 'More general' in the following sense.

Killing under $\kappa(dx)$ means essentially that we exchange '$\int_0^t \kappa(Y_s)ds$' in (0.14) by the additive functional(AF)

$$A_t = \int_I L_t^x(Y)\kappa(dx).$$

(See [BS02] part I chapter II section 4.) Here, $L_t^x(Y)$ is a special AF (w.r.t. Y) called *local time* of Y at x w.r.t. m. The name 'local time' comes from the fact that $\int_b^c L_t^x(Y)m(x)dx$ is the time spend by Y in (b, c) up to t.

If we now choose again a killing potential, i.e. $\kappa(dx) = \kappa(x)m(x)dx$, then we arrive in our old situation of

$$A_t = \int_I L_t^x(Y)\kappa(x)m(x)dx = \int_0^t \kappa(Y_s)ds.$$

But even more is true. Basically, every AF has the above form:

Suppose we exchange the AF from (0.14) by an arbitrary AF $(A_t)_t$ w.r.t. Y in the sense of [BS02]. Then

$$A_t = \int_I L_t^x(Y)\kappa(dx),$$

where $\kappa(dx)$ is exactly the corresponding killing measure. Finally there is a formula which directly connects the distribution of killing time and the place of killing to κ. (See page 13 of [BS02].) In particular it holds

$$\mathbb{P}_x(\tau < t) = \int_0^t \left(\int_I p_s(x, y)\kappa(dy) \right) ds,$$

where κ is a corresponding killing measure and $p_s(x, y)$ is the transition kernel of $(Y_t)_t$ under killing by κ w.r.t. m. From this it is can be seen that (0.15) holds more general. Thus the name killing rate for $\kappa(x)$ is justified. (Calculate $\frac{d}{dt}\mathbb{P}_x(\tau < t)$ at $t = 0$ with the above formula!)

example 0.1. One particular example of killing is *slow killing* at a certain point, e.g. at 0. This is accomplished by taking $A_t = 2\alpha L_t^0$ for some $\alpha > 0$ or by choosing $2\alpha\delta_0(dx)$ as killing measure. For $\alpha = 0$ there will be no killing and the killing occurs 'faster' for larger α. The extreme case of $\alpha \to \infty$ corresponds to *instant killing* which means that $\tau = T_0$. ∎

If the reader is interested in more details, we advise to take a look at [BG68] or [Wil79].

0.4. One-dimensional diffusions at a boundary

As explained in section 0.1 we can get a diffusions on $I = \mathbb{R}$ as solutions to SDEs. But what if we restrict ourselfes to some smaller state space, e.g. $I = (c, d)$ with $c, d \in \mathbb{R}$? This may result in possible *explosion*, meaning that Y could actually reach a boundary point in finite time with positive probability. Criteria for this have been given the name *Feller tests.* (For instance see [Var07] or [Dur96].) Once we have a boundary that could be reached we have some freedom to choose what may happen there (reflection, absorption for an exponential time, killing...). We also call such boundary points *accessible* or *exit boundary.* And of course one may ask:

If there is the possibility to go to a boundary in finite time, is there also a possibility to 'come in' from this boundary respectively start the diffusion in this boundary point. This topic can be found in literature under the name *classification of boundary points.* A good overview is given in [BS02]. (For details see [Dur96] or [Bre92].)

For example boundary points of regular diffusions are also *regular*, i.e. Y can go there from $\mathring{I}$ in finite time and can also start at this point. Since we will mainly deal with regular diffusions on $[0, \infty)$ there will be one boundary point $c = 0$. Note that for a regular boundary point c we have that $m([c, c + \epsilon)) < \infty$ and instead of choosing $d \in \mathring{I}$ in (0.11) we could as well choose c instead. We set thing straight for the rest of the thesis by the following.

convention 0.2. Whenever the lower bound c of the state space I is regular for the diffusion Y with drift coefficient $\boldsymbol{a}$ we take

$$m(x) = 2 \exp \left(2 \int_c^x \boldsymbol{a}(z)\, dz \right)$$

as the density of the speed measure.

There may also be slow killing, 'part-time absorption' and/or reflection at a boundary.

Part-time absorption is often called *stickiness* in literature. (See [BS02] or [SE07].) This simply means that the diffusion gets stuck for an exponentially distributed time at the boundary before it is released again. (Of course, only if it is not killed in the meantime.) Stickiness at c is equivalent to $m(\{c\}) > 0$, i.e. the speed measure puts mass on the boundary c.

We are not interested in diffusions with sticky boundaries. Therefore we make the following assumption for the rest of this thesis:

A. Boundary points of the diffusion Y are *not* sticky whenever accessible.

In fact, we will concentrate on diffusions with boundary points where both, reflection *and* killing, could occur. For this we have to impose boundary conditions on functions in the domain of the generator $\mathbf{L}^\kappa$. These conditions may be parametrized by some $\alpha \in [0, \infty]$ and are usually of the form

$$\begin{cases} f'(c) = 2\alpha f(c) & \text{if } \alpha < \infty \\ f(c) = 0 & \text{if } \alpha = \infty. \end{cases} \tag{0.16}$$

remark 0.5.
- For the latter we find the following memory hook: We kill at $\tau = T_c$. Hence, it holds $(\mathbf{T}_t f)(c) = \mathbb{E}_c(f(Y_t); \tau > t) = 0$. But the image of each $\mathbf{T}_t$, $t > 0$, is a subset in $D(\mathbf{L})$, which is our notation for *domain* of $\mathbf{L}$.

- If $\alpha < \infty$, then it is the killing rate at zero.

∎

example 0.2. We take a Brownian motion with constant negative drift $\boldsymbol{a} \equiv -b$, $b > 0$, on $\mathbb{R}_+$ with reflection at zero. Thus, we get

$$m(x) = 2\exp\left(2\int_0^x \boldsymbol{a}(y)\,dy\right) = 2\,\mathrm{e}^{-2bx}$$

for the density of a speed measure. Obviously $m(I) = m(\mathbb{R}_+) < \infty$. According to section 0.2 we have that $\nu(dx) = 2b\,\mathrm{e}^{-2bx}\,dx$ is the unique invariant distribution and Y is positive recurrent. For a better understanding of reflected Brownian motion with constant drift we refer to [GS00]. ∎

In the next chapters we will use $\mathbf{L}^{\kappa,\alpha}$, respectively $\mathbf{L}^{\kappa,\alpha}(\boldsymbol{a})$, for the $L^2(m)$-generator of $(Y_t)_t$ under the killing rate κ and under (slow) killing with parameter α. Though, in some cases we will have two boundaries. For the second boundary we also need a second killing parameter β and we will write $\mathbf{L}^{\kappa,\alpha,\beta}$ for the generator. We want to use this notation also for other objects when we need to emphasize the dependence on the killing parameters. Thus, we set things straight with the following convention.

convention 0.3.

- Whenever we have an underlying diffusion on $[0,\infty)$ we may use the notation '$(\bullet)^{\kappa,\alpha}$' for killing under the killing rate κ and killing at zero with rate α.
- If the underlying diffusion lives on $[c,d]$, we may use '$(\bullet)^{\kappa,\alpha,\beta}$' for killing under κ, killing at c with rate α and killing at d with rate β.

1. Some results on quasistationarity

From here on we will mostly investigate linear diffusion under killing at some stopping time τ. Amongst other reasons but clearly to avoid trivialities, we generally impose that there are $x, y \in E$ such that

$$\mathbb{P}_x(\tau < \infty) > 0 \quad \text{and} \quad \mathbb{P}_y(\tau > t) > 0 \quad \forall\, t \geq 0. \tag{1.1}$$

Imposing this it is *impossible* to get a stationary distribution ν under τ, i.e. that

$$\mathbb{P}_\nu(Y_t \in A;\, \tau > t) = \nu(A)$$

for any measurable A and $t \geq 0$:

The second assumption from (1.1) dictates that $\nu(E \setminus \{\dagger\}) > 0$. ($Y$ jumps to the absorbing state '$\dagger$' at time τ!) Since we deal with regular diffusions Y we can deduce

$$\begin{aligned} &\int_{E\setminus\{\dagger\}} \mathbb{P}_x(\tau < \infty)\nu(dx) > 0 \\ &\int_{E\setminus\{\dagger\}} \mathbb{P}_x(\tau > t)\nu(dx) > 0 \quad \forall\, t \geq 0. \end{aligned} \tag{1.2}$$

from (1.1). Therefore, we get

$$\int \mathbb{P}_x(Y_t \in E \setminus \{\dagger\};\, \tau > t)\nu(dx) < \nu(E \setminus \{\dagger\})$$

which contradicts the stationarity of ν.

Nevertheless, in many cases we can still observe some kind of stationary behaviour called 'quasistationarity'. In section 1.1 we will introduce the reader to this concept. We will work through a few easy examples and present some basic properties. In particular we will characterize 'quasistationary distributions' ϕ to be the only distributions that fulfil

$$\mathbb{P}_\phi(Y_t \in \bullet;\, \tau > t) = C\phi(\bullet)$$

for any time $t > 0$ with a corresponding $C = C(t) \in (0, 1)$.

From section 1.2 on we will focus on the 'Yaglom limit' which is a special quasistationary distribution. If existent, the Yaglom limit is $\lim_{t\to\infty} \mathbb{P}_x(Y_t \in \bullet \mid \tau > t)$. We hope that the reader will gain some insight into Yaglom limits while we work through the example in 1.2.

In section 1.3 we introduce assumptions under which we will work throughout the rest of the chapter. This section will be completed by a short presentation of diffusions which can not be handled with the results of this thesis.

Section 1.4 will provide us with a theorem giving a sharp distinction (dichotomy) between the existence and non-existence of a Yaglom limit on $E = [0, \infty)$. This will be a generalisation of a former result due to [KS12].

In the following section 1.5 we show further results on the existence of Yaglom limits. They will be of the following kind: If we have existence for (the diffusion with) drift $\boldsymbol{b}$ and $\boldsymbol{b} \geq \boldsymbol{a}$, then we also have existence for drift $\boldsymbol{a}$.

We will close this chapter with section 1.6 which gives an answer to the following question: What can we say about the existence of a Yaglom limit if the state space E is a bounded interval?

Several results, in particular from section 1.4, will be used in chapter 2 to prove penalising theorems built upon the existence of a Yaglom limit.

1.1. Introduction to QSDs

The main ingredients for this section come from [CMSM13c]. Just like the authors of [CMSM13c] we will introduce quasistationary distributions and 'quasi-limiting distributions' for a larger class of processes than just for linear diffusions:

We take some strong Markov process $(Y_t)_{t\in I}$, with a 'time-like' index set; $I = \mathbb{N}$ or $I = \mathbb{R}_+$ for example (equipped with some σ-algebra). But most of the time we will work with $I = \mathbb{R}_+$. Let the process be strong Markov w.r.t. the filtration $(\mathcal{F}_t)_{t\in I}$ and the family of measures $(\mathbb{P}_x)_{x\in E}$, where $(E, \mathcal{G})$ is the state space of Y_t. Now take some random time τ, i.e. a measurable mapping from Ω to I. Note that there is no need to regard τ as a killing time. Moreover, in this section Y will not have any 'built-in' killing. We even impose that the process Y is irreducible on all of E. (For us *irreducibility* means the same as regularity means for linear diffusions.) We further allow only random times τ which fulfil (1.1). And just as we derived (1.2) we also get

$$\mathbb{P}_\nu(\tau > t) = \int \mathbb{P}_x(\tau > t)\nu(dx) > 0$$

for any probability measure ν on E and any $t \geq 0$. Thus, we can write an expression of the form

$$\mathbb{P}_\nu(Y_t \in A \mid \tau > t).$$

A probability measure ϕ (on $(E, \mathcal{G})$) is called a *quasi-limiting distribution (QLD)* for the initial probability ν, if

$$\lim_{t\to\infty} \mathbb{P}_\nu(Y_t \in A \mid \tau > t) = \phi(A) \tag{1.3}$$

for all $A \in \mathcal{G}$. The reader may see τ as a time were the process will be killed. Then we have in particular: The measure of Y_t conditioned on survival up to t converges weakly to ϕ.

From now on, if not said otherwise, we assume τ to be an arbitrary stopping time w.r.t. $(\mathcal{F}_t)_t$. Then we have a corresponding *semigroup of* Y on, respectively under, τ, i.e.

$$\mathbf{T}_t f(x) := \mathbb{P}_x(f(Y_t); \tau > t) = \mathbb{E}_x(f(Y_t)\mathbb{1}_{\{\tau>t\}})$$

for all $f \in bm(E)$. If $(G, \mathcal{G})$ is a measurable space, then $bm(G) = bm(G, \mathcal{G})$ is defined as the space of all *bounded functions* $f : G \to \mathbb{R}$ *measurable w.r.t.* $\mathcal{G}$. (Here $\mathbb{R}$ is equipped with its usual *Borel* σ*-field.*)

Now suppose ν and ϕ fulfil (1.3). Then we have for all $A \in \mathcal{G}$ and $s \geq 0$:

$$\begin{aligned}
\phi(A) &= \lim_{t\to\infty} \mathbb{P}_\nu(Y_{t+s} \in A \mid \tau > t+s) \\
&= \lim_{t\to\infty} \frac{\int \mathbf{T}_{t+s}\mathbb{1}_A(x)\nu(dx)}{\int \mathbf{T}_{t+s}\mathbb{1}_E(x)\nu(dx)} \\
&= \lim_{t\to\infty} \frac{\int \mathbf{T}_t\mathbf{T}_s\mathbb{1}_A(x)\nu(dx)}{\int \mathbf{T}_t\mathbf{T}_s\mathbb{1}_E(x)\nu(dx)} \\
&= \lim_{t\to\infty} \frac{\int \mathbf{T}_t\mathbf{T}_s\mathbb{1}_A(x)\nu(dx)}{\int \mathbf{T}_t\mathbb{1}_E(x)\nu(dx)} \cdot \frac{\int \mathbf{T}_t\mathbb{1}_E(x)\nu(dx)}{\int \mathbf{T}_t\mathbf{T}_s\mathbb{1}_E(x)\nu(dx)} \\
&= \lim_{t\to\infty} \frac{\int \mathbf{T}_t f_s(x)\nu(dx)}{\int \mathbf{T}_t\mathbb{1}_E(x)\nu(dx)} \cdot \frac{\int \mathbf{T}_t\mathbb{1}_E(x)\nu(dx)}{\int \mathbf{T}_t g_s(x)\nu(dx)} \\
&= \lim_{t\to\infty} \mathbb{P}_\nu(f_s(Y_t) \mid \tau > t) \cdot (\mathbb{P}_\nu(g_s(Y_t) \mid \tau > t))^{-1} \\
&= \int f_s(x)\phi(dx) \cdot \left(\int g_s(x)\phi(dx)\right)^{-1},
\end{aligned}$$

where $f_s(x) := \mathbb{P}_x(Y_s \in A, \tau > s)$ and $g_s(x) := \mathbb{P}_x(\tau > s)$. The above calculation gives

$$\phi(A) = \frac{\mathbb{P}_\phi(Y_s \in A, \tau > s)}{\mathbb{P}_\phi(\tau > s)} = \mathbb{P}_\phi(Y_s \in A \mid \tau > s). \tag{1.4}$$

Any probability measure ϕ fulfilling (1.4), is called a *quasistationary distribution* or *QSD* for short.

In the above language: each QLD is also a QSD and each QSD is a QLD, e.g. with itself as initial distribution.

remark 1.1. The last sentence may raise the following question:

Take any QSD ϕ. For which initial ν, besides ϕ, is ϕ a QLD? This means that one is interested in all ν fulfilling (1.3). This set of distributions is called *domain of attraction of* ϕ. Cases in which the domain of attraction is fully known are very rare. Nevertheless, there exist results in [LSM00] for the case of an Ornstein-Uhlenbeck process and in [MPSM98] for the case of Brownian motion with constant drift under killing at zero. ∎

There exist a few quite general results on (the existence of) QSDs, which can be found in [CMSM13c]. Though the results are only stated for τ being the killing time at a 'trap' $E^{tr} \in \mathcal{G}$, i.e. $\tau = T_{E^{tr}} = \inf\{t\,;\; Y_t \in E^{tr}\}$, some of them can be easily seen to hold in the case of an arbitrary stopping time fulfilling the above conditions. This allows us, later on, to make use of these results. E.g. if we do not kill the process instantaneously at a trap but allow Y to cross over up to a time where the trap really snaps.

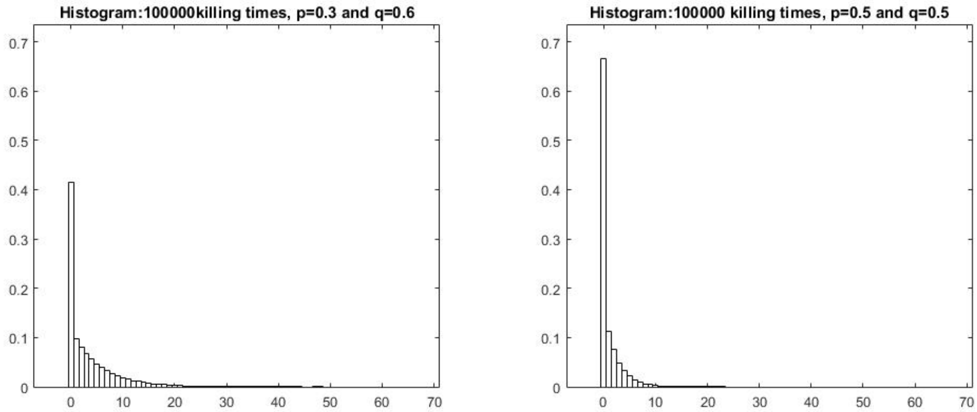
example 1.1. As a simple example we take a mouse which can get its cheese from a trap. The trap is not working very well. So the mouse is caught by the trap only with probability $p < 1$ whenever it goes for the cheese. But when the cheese is taken, the next day the landlord tries again to get rid of the mouse using his trap. It should be clear, that the trap snaps, when the number of cheese-thefts has reached an independent geometric distributed random variable with parameter p. On the other hand: Sometimes the mouse is getting its daily food elsewhere. Let the probabability for this be q. (In the context of diffusions one can compare with slow killing or killing under some potential; see section 0.) The whole process can be modelled by a three-state markov chain (e.g. state 1 is feeding from the trap and survive, state 2 is feeding from the secure alternative and state 3 is feeding from the trap and getting killed). But suppose we are only interested in the distribution of the killing time. One can see that this killing time has the same distribution as

$$\tau = \sum_{i=1}^{\tilde{G}} G_i,$$

where $\tilde{G}$ has a geometric distribution with parameter p and the G_i are independent from $\tilde{G}$ and from each other and also have a geometric distribution, but with parameter $1-q$. By partitioning on $\{\tilde{G}=N\}$, $N=1,2,\ldots$, we derive

$$\mathbb{P}(\tau=j) = \sum_{N=1}^{\infty} \mathbb{P}(\tau=j \mid \tilde{G}=N)\cdot\mathbb{P}(\tilde{G}=N) = \sum_{N=1}^{j} \binom{j-1}{N-1}(1-q)^N q^{j-N}\cdot p(1-p)^{N-1}.$$

Here we used, that the sum of independent identically geometrically distributed random variables has a negative binomial distribution. Below you see a figure of distributions of τ for two different sets of p and q.



∎

Note, that, in general, we need to impose, that $\tau = \inf\{t\,;\ Y_t \in E^{tr}\}$ are stopping times. If $I=\mathbb{R}_+$, as in most parts of this thesis, we overcome this with the following.

convention 1.1. If $I=\mathbb{R}_+$, then Y is *standard* in the sense of [BG68].

Though most of the time Y will be a regular diffusion which already implies that Y is standard. (The standard property even implies that T_A is a stopping time for every *analytic* A. See section 1.10 of [BG68].)

The next two results (propositions 1.1 and 1.3) are taken from [CMSM13c].

proposition 1.1. *(a) Let ϕ be a QSD. Then*

- *τ is exponentially distributed under ϕ, i.e. there is some $\lambda=\lambda(\phi)>0$ such that*

$$\mathbb{P}_\phi(\tau>t) = e^{-\lambda t}.$$

- *there is an $x \in E$ such that for all $\tilde{\lambda} < \lambda$*

$$\mathbb{E}_x e^{\tilde{\lambda}\tau} < \infty.$$

In particular, an exponential moment of the stopping time is necessary for the existence of a QSD or QLD, respectively.

(b) If the initial distribution is $\nu = \delta_x$ for some $x \in E$, then

$$\liminf_{t\to\infty} -\frac{1}{t}\log \mathbb{P}_x(\tau > t) = \sup\{\lambda\,;\ \mathbb{E}_x e^{\lambda\tau} < \infty\}.$$

Next we define

$$(\mathbf{T}_t^d \nu)(f) := \int \mathbf{T}_t f(x)\nu(dx)$$

for each $f \in bm(E)$. $(\mathbf{T}_t^d)_t$ is the dual semigroup of $(\mathbf{T}_t)_t$ acting on the space of all finite signed measures and in particular on probability measures. If ϕ is a QSD, we get

$$\begin{aligned}(T_t^d\phi)(f) &= \mathbb{P}_\phi(f(Y_t)\mid \tau > t)\mathbb{P}_\phi(\tau > t)\\ &= \int f(x)\phi(dx)e^{-\lambda(\phi)t}\end{aligned}$$

by the above proposition; in short

$$\mathbf{T}_t^d\phi = e^{-\lambda t}\phi \tag{1.5}$$

for all $t \geq 0$. In particular we get a small corollary on an equivalent formulation of quasistationarity.

corollary 1.2. *Let ϕ be a probability distribution on $(E, \mathcal{G})$. Then, ϕ is a QSD for Y under τ iff* (1.5) *is fulfilled for all $t > 0$ and some $\lambda > 0$.*

This may also serve to remember the following sufficient (, hence, equivalent) condition for the existence of a QSD.

proposition 1.3. *If there are some $r, C > 0$ and a probability measure μ on $(E, \mathcal{G})$ with*

$$\mathbf{T}_r^d\mu = C\mu,$$

then, there is a QSD ϕ with corresponding $\lambda(\phi)$ as in proposition 1.1 such that

$$C = e^{-\lambda(\phi)r}.$$

remark 1.2. Besides the above proposition, two other (sets of) sufficient conditions are given for the existence of a QSD in [CMSM13c]. For this, let the σ-algebra $\mathcal{G}$ on E be a Borel-σ-algebra such that we can talk about continuity of functions $f : E \to \mathbb{R}$. Thus, we can also declare the continuity on $E^a := E \setminus E^{tr}$ w.r.t. the trace topology. Then one condition is that bounded continuous functions are preserved by the semigroup, i.e.

$$\mathbf{T}_r(C_b(E^{(a)})) \subset C_b(E^{(a)}) \tag{1.6}$$

for some $r > 0$. In fact, if E or E^a is compact and $\tau = T_{E^{tr}}$, originally used in [CMSM13c], (1.6) is the only condition needed to get a QSD. Now, we get a bunch of examples, where a QSD exists:

This holds for any regular linear diffusion Y on some $(-\infty, c]$ with instant killing boundary $b < c$. Let c be reflecting. Thus, we have $\tau = T_b = \inf\{t\,;\ Y_t < b\}$, $E = (-\infty, c]$ and $E^a = [b, c]$. That $\mathbf{T}_r(C_b) \subseteq C_b$ can be seen from section 1.II.1 in [BS02]. ∎

There is some doubt that QSDs are the same when killing at E^{tr} and when killing at ∂E^a. We emphasize this by the following example. (Note, that in remark 1.2 we implicitly claimed that for one-dimensional regular diffusions it is the same. Indeed, this is true as one can check by using points 1.II.1.2 and 1.II.1.3 of [BS02].)

example 1.2. Take three one-dimensional independent diffusions Y^1, Y^2 and Y^3 on the same probability space:

- Y^2 is recurrent on $(-\infty, \infty)$.
- Y^1 follows the same stochastics on $(0, \infty)$ as Y^2 but 0 is reflecting.
- Take Y^3 on $(-\infty, 0)$ to be regular on any closed subinterval. Let 0 be a non-entry boundary.

In (1.7) and (1.8) we do two concatenations to properly introduce the process Y. Essentially a *concatentation* is a process which consists of two or more parts of paths, usually from different processes, 'glued' together. (For a definition see [Pro12].)

1. Set $\tilde{\tau} := \inf\{t\,;\ L_t^0(Y^1) \geq \xi\}$.

   We remind the reader that $t \mapsto L_t^0(Y^1)$ is the local time of Y^1 at 0 and ξ is a standard exponentially distributed random variable independent of the rest. Now we set

$$\tilde{Y}_t := \begin{cases} Y_t^1 & , t < \tilde{\tau} \\ Y_t^2 + (Y_{\tilde{\tau}}^1 - Y_{\tilde{\tau}}^2) & , t \geq \tilde{\tau}. \end{cases} \tag{1.7}$$

2. Next define $\tau := \inf\{t\,;\ \tilde{Y}_t < 0\}$ and observe that $\tau \geq \tilde{\tau}$. Finally, define Y by

$$Y_t := \begin{cases} \tilde{Y}_t & , t < \tau \\ Y_t^3 + (\tilde{Y}_\tau - Y_\tau^3) & , t \geq \tau. \end{cases} \tag{1.8}$$

Now we have

$$\tau = \inf\{t\,;\ \tilde{Y}_t < 0\} = \inf\{t\,;\ Y_t < 0\}.$$

We use τ as killing time. This implies $E^{tr} = (-\infty, 0)$ and $\partial E^a = \{0\}$.

Now $(Y_t)_t$ is a strong Markov process with a.s. continuous paths (up to τ) such that

$$T_{\partial E^a} = \inf\{t\,;\ Y_t = 0\} \neq \inf\{t\,;\ Y_t < 0\} = \tau$$

with probability 1. (Once Y 'tears down' the reflecting barrier at time τ it immediately hits $E^{tr} = (-\infty, 0)$ which ensures the Markov property.)

Actually, $T_{\partial E^a} = 0$ holds $\mathbb{P}_0$-a.s. by section I.II.1 of [BS02]. This stays in sharp contrast to $\mathbb{P}_0(\tau > 0) = 1$.

∎

Next we want to introduce a very special QLD:
A QLD/QSD ϕ is called *Yaglom limit*, if for all $x \in E^{(a)}$ and $A \in \mathcal{G}$:

$$\mathbb{P}_x(Y_t \in A \mid \tau > t) \to \phi(A)$$

as $t \to \infty$.

remark 1.3.
- This name results from the pioneering article [Yag47]. Here the author looks at discrete time branching processes. More precise, he investigates the long-term behaviour in the subcritical and critical case of Galton-Watson processes, i.e. when the expected number of offspring is less or equal then 1. From our knowledge so far, the critical case can not produce a QSD, i.e. can not have a QLD. This is due to the fact that here $\tau = T_0$ and $\mathbb{E}_x(T_0) = \infty$ resulting in $\mathbb{E}_x(e^{\epsilon T_0}) = \infty$ for all $\epsilon > 0$. And this is a violation of the necessary condition for a QSD given in proposition 1.1.
- To ensure that the above definition is not meaningless, in the sense that for different initial points x there are different (Yaglom) limits, one may assume at least that each two states $x, y \in E^a$ have a positive chance to 'communicate through Y before τ', i.e. $\mathbb{P}_x(T_y < \tau) > 0$. To clarify the problem, one can think of a diffusion in $\mathbb{R}$ killed at 0, such that $E^a = \mathbb{R}\setminus\{0\}$. If the process starts on the positive reals, it is not going to reach the negative reals in the (Yaglom) limit and vice versa. But, since we will mostly deal with regular diffusions on an interval where instant killing only appears at the endpoints such restrictions are intrinsic to our situation. (Compare section 1.2 ff.)

∎

The reader may consult section 1.2 for a first example of a Yaglom limit.
The final concept which can be introduced in this general context, is the concept of an 'asymptotic mortality rate'. We start with the simple observation that

$$\mathbb{P}_\phi(\tau > t + s \mid \tau > t) = e^{-\lambda s}$$

whenever ϕ is a QSD with rate λ. (See proposition 1.1.) Now take some initial distribution ν. If we also have

$$\lim_{t\to\infty} \mathbb{P}_\nu(\tau > t + s \mid \tau > t) = e^{-\eta s} \tag{1.9}$$

for some $\eta = \eta(\nu) \geq 0$, then we say that η is the *asymptotic mortality* or *asymptotic killing rate* for the initial distribution ν.

remark 1.4. Actually, we have (1.9) as soon as we have any limiting behaviour towards 'something positive', since

$$\begin{aligned}
&\lim_{t\to\infty} \mathbb{P}_\nu(\tau > t+s+r \mid \tau > t) \\
&= \lim_{t\to\infty} \frac{\mathbb{P}_\nu(\tau > t+s+r)}{\mathbb{P}_\nu(\tau > t+s)} \cdot \frac{\mathbb{P}_\nu(\tau > t+s)}{\mathbb{P}_\nu(\tau > t)} \\
&= \lim_{t\to\infty} \frac{\mathbb{P}_\nu(\tau > t+s+r)}{\mathbb{P}_\nu(\tau > t+s)} \cdot \lim_{t\to\infty} \frac{\mathbb{P}_\nu(\tau > t+s)}{\mathbb{P}_\nu(\tau > t)} \\
&= \lim_{t\to\infty} \mathbb{P}_\nu(\tau > t+r \mid \tau > t) \cdot \lim_{t\to\infty} \mathbb{P}_\nu(\tau > t+s \mid \tau > t),
\end{aligned}$$

which is a defining equation of the exponential. In contrast, if we would have $\lim_{t\to\infty} \mathbb{P}_\nu(\tau > t+s \mid \tau > t) = 0$ for some $s > 0$, then the above calculation would tell us that this must hold for every $s > 0$. ▮

To justify the name 'killing rate', think again of τ as a killing time and observe that

$$\mathbb{P}_\nu(\tau \leq t+s \mid \tau > t) = \eta s + o(s)$$

in the limit as $t \to \infty$. There is also some short lemma which shows that η is actually the decay rate of the probability to survive:

lemma 1.4. *If* (1.9) *holds, then*

$$-\frac{1}{t} \log \mathbb{P}_\nu(\tau > t) \to \eta.$$

proof. Just use

$$\frac{1}{\lfloor t \rfloor} \sum_{n=0}^{\lfloor t \rfloor} \log \left(\frac{\mathbb{P}_\nu(\tau > n+1)}{\mathbb{P}_\nu(\tau > n)} \right) \leq \frac{1}{t} \log \mathbb{P}_\nu(\tau > t) \leq \frac{1}{\lfloor t \rfloor + 1} \sum_{n=0}^{\lfloor t \rfloor - 1} \log \left(\frac{\mathbb{P}_\nu(\tau > n+1)}{\mathbb{P}_\nu(\tau > n)} \right)$$

and take a $\delta > 0$ and a corresponding $m \in \mathbb{N}$ such that

$$e^{-\eta}(1-\delta) \leq \frac{\mathbb{P}_\nu(\tau > n+1)}{\mathbb{P}_\nu(\tau > n)} \leq e^{-\eta}(1+\delta)$$

for all $n > m$. Now, for $\epsilon > 0$, we can always take $\delta > 0$ small enough and t large enough such that

$$-\epsilon - \eta(1+\epsilon) \leq \frac{1}{t} \log \mathbb{P}_\nu(\tau > t) \leq \epsilon - \eta(1-\epsilon).$$

■

As we observed, the Yaglom limit ϕ (or any QLD in general) is also a QSD. Intuitively the mortality rate should carry over to ϕ. This results in the following observation.

lemma 1.5. *Let ϕ be a QLD for the initial distribution ν and suppose we have an asymptotic mortality rate $\eta(\nu)$. Further let $\lambda(\phi)$ be the parameter of the exponential distribution corresponding to ϕ (seen as a QSD) due to proposition 1.1.*

- *Then we have that $\lambda(\phi) = \eta(\phi) = \eta(\nu) > 0$.*

- *In particular, a necessary condition for the existence of a Yaglom limit ϕ is that the asymptotic mortality rates η are independent of the starting point and $\eta > 0$.*

Before we prove this lemma we give the following remark.

remark 1.5.

- Lemma 1.5 combined with lemma 1.4 gives that

$$\liminf_{t\to\infty} -\frac{1}{t}\log \mathbb{P}_\phi(\tau > t) > 0 \tag{1.10}$$

is necessary for the existence of a QLD.

- Observe that (1.10) can only be true if $\mathbb{P}_x(\tau < \infty) > 0$ for some $x \in E$. This is one of the two natural assumptions from (1.1) we made at the beginning of the chapter.

▮

proof of lemma 1.5. Suppose we have

$$e^{-s\eta} = \lim_{t\to\infty} \frac{\mathbb{P}_\nu(\tau > t+s)}{\mathbb{P}_\nu(\tau > t)}$$

and ϕ is a QLD for the initial ν. Then

$$\begin{aligned}
e^{-s\eta} &= \lim_{r\to\infty} \frac{\mathbb{P}_\nu(\tau > r+s)}{\mathbb{P}_\nu(\tau > r)} \cdot \frac{\langle\!\langle \nu, \mathbf{T}_{r+s+t}\mathbb{1}\rangle\!\rangle}{\langle\!\langle \nu, \mathbf{T}_{r+s+t}\mathbb{1}\rangle\!\rangle} \\
&= \lim_{r\to\infty} \frac{\mathbb{P}_\nu(\tau > r+s)}{\mathbb{P}_\nu(\tau > r)} \cdot \frac{\langle\!\langle \nu, \mathbf{T}_{r}\mathbf{T}_{s+t}\mathbb{1}\rangle\!\rangle}{\langle\!\langle \nu, \mathbf{T}_{r+s}\mathbf{T}_{t}\mathbb{1}\rangle\!\rangle} \\
&= \lim_{r\to\infty} \frac{\langle\!\langle \frac{\mathbf{T}^d_r\nu}{\mathbb{P}_\nu(\tau>r)}, \mathbf{T}_{s+t}\mathbb{1}\rangle\!\rangle}{\langle\!\langle \frac{\mathbf{T}^d_{r+s}\nu}{\mathbb{P}_\nu(\tau>r+s)}, \mathbf{T}_{t}\mathbb{1}\rangle\!\rangle} \\
&= \frac{\lim_{r\to\infty} \langle\!\langle \frac{\mathbb{P}_\nu(Y_r\in dy\,;\,\tau>r)}{\mathbb{P}_\nu(\tau>r)}, \mathbf{T}_{t+s}\mathbb{1}\rangle\!\rangle}{\lim_{r\to\infty} \langle\!\langle \frac{\mathbb{P}_\nu(Y_{r+s}\in dy\,;\,\tau>r+s)}{\mathbb{P}_\nu(\tau>r+s)}, \mathbf{T}_{t}\mathbb{1}\rangle\!\rangle} \\
&= \frac{\langle\!\langle \phi, \mathbf{T}_{t+s}\mathbb{1}\rangle\!\rangle}{\langle\!\langle \phi, \mathbf{T}_{t}\mathbb{1}\rangle\!\rangle} = \frac{\langle\!\langle \mathbf{T}^d_{t+s}\phi, \mathbb{1}\rangle\!\rangle}{\langle\!\langle \mathbf{T}^d_{t}\phi, \mathbb{1}\rangle\!\rangle} \\
&= \frac{\langle\!\langle e^{-(s+t)\lambda(\phi)}\phi, \mathbb{1}\rangle\!\rangle}{\langle\!\langle e^{t\lambda(\phi)}\phi, \mathbb{1}\rangle\!\rangle} = e^{-s\lambda}.
\end{aligned}$$

(Here $\langle\!\langle \nu, f\rangle\!\rangle$ means the *dual pairing of finite signed measures ν with bounded measurable functions f*. Hence, in our case it is simply the expectation of f under ν.) Finally, $\eta = \lambda > 0$ by proposition 1.1. ■

At the end of this section we want to give an example which shows that there can be a whole 'continuum' of QSDs and of corresponding mortality rates.

example 1.3. In [MSM94] the authors investigate Brownian motion on $\mathbb{R}_+$ with constant drift $-a$, $a > 0$, and $\tau = T_0$. By the preceding observations, we know that a probability measure ϕ is a QSD iff (1.5) is true for all $t > 0$ and some $\lambda > 0$. The authors use this to identify

$$\varphi_\lambda(x) = \frac{e^{-ax}\sinh(wx)}{\int e^{-ax}\sinh(wx)dx}$$

as densities of QSDs ϕ_λ for each $0 < \lambda \leq \frac{a^2}{2}$, with $w = \sqrt{a^2 - 2\lambda}$. This can be checked by using the kernel (w.r.t. dx) of Y_t under τ

$$p_t(x,y) = e^{\left(-a(y-x)-\frac{a^2}{2}t\right)} \cdot (p(t,x,y) - p(t,x,-y)),$$

where $p(t,x,y)$ is the kernel of the standard Brownian motion. The corresponding mortality rates are $\eta(\varphi_\lambda) = \lambda$, since

$$\begin{aligned}\mathbb{P}_{\phi_\lambda}(\tau > t+s \mid \tau > t) &= \frac{\mathbb{P}_{\phi_\lambda}(\tau > t+s)}{\mathbb{P}_{\phi_\lambda}(\tau > t)} \\ &= \frac{\langle\!\langle \mathbf{T}^d_{t+s}\phi_\lambda, \mathbb{1} \rangle\!\rangle}{\langle\!\langle \mathbf{T}^d_t \phi_\lambda, \mathbb{1} \rangle\!\rangle} \\ &= \frac{e^{-\lambda(t+s)} \langle\!\langle \phi_\lambda, \mathbb{1} \rangle\!\rangle}{e^{-\lambda t} \langle\!\langle \phi_\lambda, \mathbb{1} \rangle\!\rangle} = e^{-\lambda s}.\end{aligned}$$

∎

1.2. Yaglom limits: Approach by an example

We take $E = [0,\infty)$ as the statespace and on it a Brownian motion with constant drift a under $\tau = T_0 \wedge T_\pi$. In terms of the last section we identify $E^a = (0,\pi)$. The generator of Y under τ acts on C^2-functions with boundary conditions

$$f(0) = f(\pi) = 0.$$

It looks like

$$\mathbf{L} = \frac{1}{2}\frac{d^2}{dx^2} + a\frac{d}{dx}$$

and is symmetric in $L^2(m)$ with

$$m(dx) = m(x)dx = e^{2\int_0^x a\,dy}\,dx = e^{2ax}\,dx.$$

Furthermore, $\mathbf{L}$ has some self-adjoint extension in $L^2(m)$ which can be taken to be the minimal extension or Friedrich's extension. (In fact, the domain of $\mathbf{L}$ consists of all elements $f \in C^1(0,\pi)$ such that f' is *absolutely continuous (a.c.)* w.r.t. the Lebesgue measure, $f(0) = f(\pi) = 0$ and $\int_0^\pi (\mathbf{L}f)^2(x)dx < \infty$. See the appendix and references given therein.) But the crucial and very nice part now is that $\mathbf{L}$ has a discrete *spectrum*

$$\Sigma(\mathbf{L}) = \{-\lambda_1, -\lambda_2, -\lambda_3, \ldots\}$$

with $\lambda_n = \frac{n^2+a^2}{2}$. The corresponding eigenfunctions are $\psi_{\lambda_n}(x) = e^{-ax}\sin(nx)$. Moreover, $(\psi_{\lambda_n})_{n\geq 1}$ is a complete orthogonal system for the domain of $\mathbf{L}$. Therefore, we can write every $f \in D(\mathbf{L})$ as

$$\sum_{n\geq 1} \langle f, \psi_n \rangle_{L^2(m)} \psi_n(x)$$

with $\psi_n := \frac{\psi_{\lambda_n}}{\|\psi_{\lambda_n}\|_{L^2(m)}}$. Here '$\langle \cdot, \cdot \rangle_{L^2}$' is the usual *scalar product* in L^2. If we now take $f, g \in L^2(m)$, we may write

$$\begin{aligned}
\mathbb{P}_{gm}(f(Y_t); \tau > t) &= \langle g, \mathbf{T}_t f \rangle_{L^2(m)} \\
&= \langle \mathbf{T}_t g, f \rangle_{L^2(m)} \\
&= \sum_{n \geq 1} \mathrm{e}^{-\lambda_n t} \langle g, \psi_n \rangle \langle \psi_n, f \rangle
\end{aligned}$$

and this behaves like

$$\mathrm{e}^{-\lambda_1 t} \langle g, \psi_1 \rangle \langle \psi_1, f \rangle.$$

Here we used that $\mathbf{L}$ as well as $\mathbf{T}_t$ are self-adjoint in $L^2(m)$. Next we can take g non-negative with $\int g(x)m(x)dx = 1$, such that gm can be regarded as an initial distribution and we also have that

$$\langle g, \psi_1 \rangle > 0.$$

By the same reason, we have also

$$\langle \psi_1, f \rangle > 0,$$

if $f = \mathbb{1}_A$ with $\int_A > 0$. (This is because of $\int fm \geq e^{-|a|\pi} \int_A > 0$.) In particular we get, for each $z \in (0, \pi)$:

$$\begin{aligned}
\mathbb{P}_{gm}(Y_t \leq z \,|\, \tau > t) &= \frac{\mathbb{P}_{gm}(\mathbb{1}_{(0,z]}(Y_t); \tau > t)}{\mathbb{P}_{gm}(\mathbb{1}_{(0,\pi)}(Y_t); \tau > t)} \\
&\propto \frac{e^{-\lambda_1 t} \langle g, \psi_1 \rangle \langle \psi_1, \mathbb{1}_{(0,z]} \rangle}{e^{-\lambda_1 t} \langle g, \psi_1 \rangle \langle \psi_1, \mathbb{1}_{(0,\pi)} \rangle}.
\end{aligned} \tag{1.11}$$

In this example and for the rest of the thesis we will use the notation '$g(t) \propto h(t)$' . This means $\lim_t \frac{g(t)}{h(t)} = 1$. From the context it should always be clear wether we take the limit to ∞ or to some particular element from $\mathbb{R}_+$.

By (1.11) for each starting distribution gm with $g \in L^2(m)$ we get the same QLD/QSD:

$$\mathbb{P}_{gm}(Y_t \in A \,|\, \tau > t) \to \frac{\int_A \psi_1(x)m(x)dx}{\int_{(0,\pi)} \psi_1(x)m(x)dx}.$$

Now observe the following:

$$\varphi_1 := \psi_1 \cdot m$$

is (up to a normalising factor) the density of the QLD and one sees immediately that

$$-\mathbf{L}^* \varphi_1 = -m\mathbf{L}\frac{1}{m}\varphi_1 = \lambda_1 \varphi_1. \tag{1.12}$$

By $\mathbf{L}^*$ we mean the adjoint w.r.t. the Lebesgue measure, since $\psi_1 \in D(\mathbf{L})$ and

$$\langle \mathbf{L}f, \varphi_1 \rangle_{L^2([0,\pi],dx)} = \langle \mathbf{L}f, \frac{\varphi_1}{m} \rangle_{L^2([0,\pi],dm)} = \langle f, \mathbf{L}\frac{\varphi_1}{m} \rangle_{L^2([0,\pi],dm)} = \langle f, m\mathbf{L}\frac{\varphi_1}{m} \rangle_{L^2([0,\pi],dx)}$$

for every $f \in D(\mathbf{L})$. (Note that we have already seen formula (1.12) for the 'formal adjoint' in section 0.2.)

At this stage one may ask about the set of φ's which are:

- positive & integrable
- fulfil (1.12) for some λ
- have the appropriate boundary conditions 'translated' from a corresponding $\psi \in D(\mathbf{L})$.

In the present example the only φ meeting these terms is φ_1. In more general situations, we can at least restrict the range of possible λ's: Because positivity is preserved by '$\varphi \leftrightarrow \psi$' and we have to take care of the boundary conditions, the candidates for such QSD's are only φ's where

$$\lambda \leq \underline{\lambda}$$

with $\underline{\lambda}$ being the bottom of $\Sigma(-\mathbf{L})$. On the other hand it must hold

$$\lambda > 0.$$

(This is to avoid explosion, e.g. of $\mathbb{P}_\phi(\tau > t) = e^{-\lambda t}$, $\phi(dx) = \varphi(x)dx$, and is due to the fact that Y dies eventually. See proposition 1.1.)

Finally we want to see that

$$\underline{\varphi} := m\underline{\psi} := m\psi_1 = \varphi_1 \tag{1.13}$$

is nothing else but the Yaglom limit. We only need some small $\epsilon > 0$ to write

$$\frac{\mathbb{P}_x(Y_t \in A, \tau > t)}{\mathbb{P}_x(\tau > t)} = \frac{\mathbb{P}_x(\mathbb{P}_{Y_\epsilon}(Y_{t-\epsilon} \in A, \tau > t-\epsilon); \tau > \epsilon)}{\mathbb{P}_x(\mathbb{P}_{Y_\epsilon}(\tau > t-\epsilon); \tau > \epsilon)}$$

and use that Y_t under τ has a nice transition kernel $p_t(x, y)$. (Confer [MSM94], where in particular the Yaglom limit for Y_t under $\tau = T_0$ is calculated.) Since $m(x)dx$ and the Lebesgue measure on $[0, \pi]$ are equivalent, we may take $p_t(x, y)$ to be the kernel w.r.t. $m(dx)$. There is also a positive chance of survival up to time $\epsilon > 0$, i.e.

$$\int p_\epsilon(x, y)m(y)dy > 0.$$

Therefore, we may take $g(y) := \frac{p_\epsilon(x,y)}{\int p_\epsilon(x,y)m(y)dy}$ to get

$$\begin{aligned}\mathbb{P}_x(Y_t \in A \,|\, \tau > t) &= \frac{\mathbb{P}_x(\mathbb{P}_{Y_\epsilon}(Y_{t-\epsilon} \in A, \tau > t-\epsilon); \tau > \epsilon)}{\mathbb{P}_x(\mathbb{P}_{Y_\epsilon}(\tau > t-\epsilon); \tau > \epsilon)} \\ &= \frac{\mathbb{P}_{gm}(Y_{t-\epsilon} \in A, \tau > t-\epsilon)}{\mathbb{P}_{gm}(\tau > t-\epsilon)} \\ &\to \frac{\int_A \underline{\varphi}(y)dy}{\int_0^\pi \underline{\varphi}(y)dy}.\end{aligned}$$

For the first equation we used the *Markov property (MP)* of Y.

There are essentially three points we should remember from the above example:

1. A good candidate for the Yaglom limit is $\underline{\varphi} := \underline{\psi} m$ with $\underline{\psi}$ being the 'principal eigenfunction' of $-\mathbf{L}\psi = \lambda\psi$.

   (With *principal* we mean here a positive eigenfunction, not necessarily in the L^2-sense, corresponding to the lowest eigenvalue of $-\mathbf{L}$. (Point 2 below indicates that positivity of this principal eigenfunction holds under quite general conditions.)

2. The bottom $\underline{\lambda}$ of the spectrum $\Sigma(-\mathbf{L})$ is

$$\underline{\lambda} = \max\{\lambda\,;\ -\mathbf{L}\psi = \lambda\psi,\ \psi \text{ is positive}\}. \tag{1.14}$$

   In our example this is obvious since we can calculate everything exactly. But (1.14) holds even in the context of diffusions on $\mathbb{R}_+$. (See theorem A.2.6 in the appendix.) Then, the elements of $D(\mathbf{L})$ have to fulfil only *one* boundary condition of the form (0.16) at zero. In contrast to the given example, this will always allow solutions of $-\mathbf{L}\psi = \lambda\psi$ (under this boundary condition) with $\lambda < \underline{\lambda}$.

3. If 1 & 2 are true, it is reasonable to use the expression *minimal QSD* for the Yaglom limit: Under all candidates $\phi(dx) = \varphi(x)dx$ for a QSD/QLD fulfilling

$$-\mathbf{L}^*\varphi = \lambda\varphi,$$

   $\underline{\varphi}(x)dx$ is the one such that the probability of survival is minimal in the sense of proposition 1.1.

convention 1.2. If not said otherwise, we will also use *(minimal) QSD* or *QLD* when talking about the Yaglom limit.

1.3. Limiting the scope

Here we are going to present some conditions under which the results of the next section 1.4 could be achieved.

For the rest of chapter 2 we are mainly concerned with diffusions which are solutions to

$$dY_t = dX_t + \boldsymbol{a}(Y_t)dt \tag{1.15}$$

on $[0,\infty)$, respectively on $[0,d]$, $d > 0$, under some *killing potential* $\kappa \geq 0$ and reflection at 0, respectively at d. (See subsection 0.3.) If not said otherwise, X will always be a standard Brownian Motion. Thus, formally the generator of Y under killing is

$$\mathbf{L}^\kappa = \frac{1}{2}\frac{d^2}{dx} + \boldsymbol{a}(x)\frac{d}{dx} - \kappa(x).$$

Killing may also happen at the boundaries. We denote the corresponding 'killing rates' by $\alpha \geq 0$, respectively by $\beta \geq 0$. (For more information confer section 0.4.)

Furthermore, we will write $\mathbf{L}^{\kappa,\alpha}$, respectively $\mathbf{L}^{\kappa,\alpha,\beta}$, instead of $\mathbf{L}^\kappa$. If we still write '$\mathbf{L}^\kappa$', this should be understood as $\alpha = \beta = 0$.

remark 1.6. Of course we need to know how to 'interpret' (1.15). The reader may ask: Is there a solution to (1.15) at all?

Our answer is: There is a unique weak solution on any compact interval $I \subset (0,\infty)$ if $\boldsymbol{a} \in C((0,\infty),\mathbb{R})$ holds. For more information see section 0.1. ∎

The following assumption is adopted from [KS12]. This should be no surprise since we will use parts of [KS12] several times in this thesis.

> **A 1.** We have that
>
> (i) the killing rate κ should be in $C(\bar{E})$.
>
> (ii) the drift $\boldsymbol{a} \in C(\mathring{E})$ is locally integrable at ∂E.

With '$\bar{E}$' we mean the *topological closure of* E, i.e. $\bar{E} := E \cup \partial E$.

We make another assumption concerning the behaviour of $\boldsymbol{a}$ at infinity. This is of course irrelevant if the state space E is a bounded interval.

> **A 2.** Infinity is a 'natural boundary'.

Infinity being a *natural boundary* is equivalent to some integrability conditions on the drift $\boldsymbol{a}$, namely that we have

$$\begin{aligned}\int_c^\infty \left(\int_c^x m(y)dy\right) s'(x)dx &= \int_c^\infty \left(\int_c^x m(y)dy\right) m(x)^{-1}dx &&= \infty \\ \int_c^\infty \left(\int_c^x s'(y)dy\right) m(x)dx &= \int_c^\infty (s(x)-s(c))m(x)dx &&= \infty\end{aligned} \tag{1.16}$$

for some $c \in \mathring{E}$, i.e. in the interior of the state space. (See e.g. [BS02] or [MSM01].)

Here s and m are a scale function and (a density of) the corresponding speed measure of Y. We already collected a few properties of these functions in subsection 0.2.

By assumption A 1 (ii) we would have that

$$\begin{aligned}\int_b^c m(x)dx &= \int_b^c e^{2\int_b^x \boldsymbol{a}(y)dy}dx < \infty \\ \int_b^c \frac{1}{m(x)}dx &= \int_b^c e^{-2\int_b^x \boldsymbol{a}(y)dy}dx < \infty\end{aligned} \tag{1.17}$$

for $b \in \partial E$ and any $c \in \mathring{E}$. The first one may be interpreted as follows: the drift is 'small enough' around b, such that the diffusion has a chance to reach this boundary point. If there is such a chance, then the boundary point is called *accessible* or an *exit boundary*. Now the reader could guess some interpretation of finiteness of the second integral: the drift is large enough that the process has a chance to enter $\mathring{E}$ from the boundary. If there is such a chance, the boundary point is called *entrance boundary*. There is a little bit more to it and the usual definition of a *regular boundary* (exit & entrance) looks more complicated:

The conditions are finiteness of the integrals in (1.16) with the boundary ∞ replaced by 0; see [BS02] for example. But, as pointed out in [KS12], it is rather easy to see that a boundary point is indeed regular under (1.17).

Hopefully, the probabilistic interpretation of a 'natural' boundary point now becomes more clear to the reader. We can put it this way: The process takes an 'infinite amount' of time to go to and to come from this point. Actually in the language of boundary point classification the first integral condition means that ∞ is *inaccessible* or a *non-exit boundary.* The second condition is the same as to say that ∞ is a *non-entrance boundary.* To emphasize this, note that (1.16) also implies

$$\lim_{x\to\infty} \mathbb{P}_y(T_x \leq s) = 0$$

and

$$\lim_{x\to\infty} \mathbb{P}_x(T_y \leq s) = 0$$

for all $s > 0$ and any $y > 0$. (This is for example mentioned in [MSM01]. The fact that the first limit is zero, meaning that there is no explosion at ∞, can be found in [Aze74].)

For example one can easily see that ∞ is natural under a constant drift. One can also show that a linear drift with negative slope gives that ∞ is natural.

lemma 1.6. *Let $\boldsymbol{a}$ be measurable and suppose $\boldsymbol{a}$ converges to some $\boldsymbol{a}(\infty) \in \mathbb{R}$ with rate $o(\frac{1}{x})$. Then $\boldsymbol{a}$ respectively a corresponding diffusion Y satisfies A 2.*

proof. The proof is only calculation and essentially consists in taking c in the above integral conditions large enough. In fact, the only condition needed is

$$\boldsymbol{a}(\infty) - \frac{1}{4x} \leq \boldsymbol{a}(x) \leq \boldsymbol{a}(\infty) + \frac{1}{4x}$$

for all $x \geq c$. Now we are going to show that

$$\int_c^\infty \left(\mathrm{e}^{\pm 2\int_c^x \boldsymbol{a}(y)dy} \cdot \int_c^x \mathrm{e}^{\mp 2\int_c^y \boldsymbol{a}(z)dz}\, dy \right) dx = \infty.$$

Using the above inequalities (when c is large enough), we see that the last term is larger then

$$\int_c^\infty \left(\mathrm{e}^{\pm 2\boldsymbol{a}(\infty)(x-c) - \int_c^x \frac{1}{2y}dy} \cdot \int_c^x \mathrm{e}^{\mp 2\boldsymbol{a}(\infty)(y-c) - \int_c^y \frac{1}{2z}dz}\, dy \right) dx.$$

But this is infinite iff

$$\int_c^\infty \left(\frac{\mathrm{e}^{\pm 2\boldsymbol{a}(\infty)x}}{\sqrt{x}} \cdot \int_c^x \frac{\mathrm{e}^{\mp 2\boldsymbol{a}(\infty)y}}{\sqrt{y}} dy \right) dx = \infty.$$

Suppose now that $\boldsymbol{a}(\infty) > 0$. (There is no loss of generality since the case '$\boldsymbol{a}(\infty)$=0' is obvious.) Now the '$+-$'-case is easily seen to be true. For the '$-+$'-case observe that the last equation is true iff

$$\int_{c+1}^\infty \left(\frac{\mathrm{e}^{-2\boldsymbol{a}(\infty)x}}{\sqrt{x}} \cdot \int_c^x \frac{\mathrm{e}^{2\boldsymbol{a}(\infty)y}}{\sqrt{y}} dy \right) dx = \infty.$$

Of course, we can take c even large enough such that $y \mapsto \frac{\mathrm{e}^{2\boldsymbol{a}(\infty)y}}{\sqrt{y}}$ is increasing for all $y \geq c$.

Thus,

$$\begin{aligned}
&\int_{c+1}^{\infty}\left(\frac{\mathrm{e}^{-2\boldsymbol{a}(\infty)x}}{\sqrt{x}}\cdot\int_{c}^{x}\frac{\mathrm{e}^{2\boldsymbol{a}(\infty)y}}{\sqrt{y}}dy\right)dx\\
&\geq\int_{c+1}^{\infty}\left(\frac{\mathrm{e}^{-2\boldsymbol{a}(\infty)x}}{\sqrt{x}}\cdot\int_{x-1}^{x}\frac{\mathrm{e}^{2\boldsymbol{a}(\infty)y}}{\sqrt{y}}dy\right)dx\\
&\geq\int_{c+1}^{\infty}\left(\frac{\mathrm{e}^{-2\boldsymbol{a}(\infty)x}}{\sqrt{x}}\cdot\frac{\mathrm{e}^{2\boldsymbol{a}(\infty)(x-1)}}{\sqrt{x-1}}\right)dx=\infty.
\end{aligned}$$

■

Let us stress the fact that $\boldsymbol{a}$ is not restricted to any continuity condition in the above result. But if we assume that $\boldsymbol{a}$ is Lipschitz continuous (for x large) then we can allow $\boldsymbol{a}$ to be (eventually) bounded:

Take x_0 be large enough such that $\boldsymbol{a}$ has the mentioned properties for $x \geq x_0 - 1$. Set $\boldsymbol{a}_0(x) := \boldsymbol{a}(x)\mathbb{1}_{\{x\geq x_0\}} + \boldsymbol{a}(x_0)\mathbb{1}_{\{x<x_0\}}$. Take $\boldsymbol{a}_0$ as the drift coefficient of a diffusion driven by Brownian motion and call this diffusion $Y(\boldsymbol{a}_0)$. By a theorem of [Lin92] the solutions $Y(m)$ and $Y(M)$ corresponding to the constant drift coefficients m and M satisfying $m \leq \boldsymbol{a}_0 \leq M$ are 'dominating' $Y(\boldsymbol{a}_0)$, i.e. $\mathbb{P}_x(\forall t \geq 0 : Y(m)_t \leq Y(\boldsymbol{a}_0)_t \leq Y(M)_t) = 1$. Since $Y(m)$ and $Y(M)$ behave naturally at ∞ the same is true for $Y(\boldsymbol{a}_0)$ and, therefore, also for Y respectively $\boldsymbol{a}$.

We will need a few more assumptions on our diffusion processes:

A 3. The purely reflected process, this is Y without any killing, is recurrent, i.e. $s(\infty) = \infty$ for some scale function of Y.

As can be seen in [CMSM13b] this assumption is equivalent to

$$\mathbb{P}_x(T_0 < \infty) = 1$$

for all $x \geq 0$, provided we assume A 1 and A 2.

Finally, following [KS12], we also give a stronger assumption which is used in the lemma below to identify the mortality rates $\eta(\delta_x) \equiv \underline{\lambda}$, namely

A 4. The purely reflected process is positive recurrent, i.e. $m(E) < \infty$ for the speed measure m of Y.

To see that $\eta = \underline{\lambda}$ is not always the case, one should assume in some sense the opposite of A4: transience. Take for example $dY_t = dX_t + a\,dt$ with $a > 0$ on $E = [0, \infty)$ and under $\tau = T_0$. Even with $a = 0$ we already have $\eta = 0$. On the other hand we have $\underline{\lambda} = \frac{a^2}{2}$. (Confer [MSM94] or example 6.17 in [CMSM13b].)

lemma 1.7. *Let A 1, A 2 and A 4 be fulfilled. Then*

$$-\frac{\log \mathbb{P}_x(\tau > t)}{t} \to \underline{\lambda}.$$

Actually A 2 is not entirely necessary for the lemma to be true. (see [KS12])

1.3.1. Particular cases which are not covered

In the spirit of lemma 1.6 let us have a look at $\boldsymbol{a}(x) = \frac{1}{2}(bx - \frac{1}{x})$ for some $b \in \mathbb{R}$. For $b < 0$ we have an Ornstein-Uhlenbeck process with additional drift $-\frac{1}{2x}$. We can show that in either case $b = 0, > 0, < 0$ we have that ∞ is natural. (At some point we need to change integrals to see what is going on. This is allowed since we are integrating only exponentials which are non-negative functions.) But zero is a non-entrance boundary; therefore, falls *not* in the framework of section 1.4. (Just calculate the second line in (1.16), with ∞ replaced by zero.) Now the interesting thing is, this process on $(0, \infty)$ is distributed like $F(Z_t)$, where $F(x) = \int_0^x \frac{dy}{\sigma y^{\frac{1}{2}}}$ and

$$dZ_t = \sigma \sqrt{Z_t} dX_t + bZ_t dt \tag{1.18}$$

is a so-called *branching diffusion*. (This can be shown by the use of Ito's formula.) More generally, a space transform like F could be used to get rid of a diffusion coefficient; in the present case $\sigma\sqrt{x}$ from (1.18).

By the way, if we start out with a 'pure' Ornstein-Uhlenbeck process $dY_t = dX_t + \frac{1}{2}bY_t dt$ and search for Z such that $F(Z)$ is distributed like Y we find

$$dZ_t = \sigma \sqrt{Z_t} dX_t + (bZ_t + \frac{\sigma^2}{4}) dt. \tag{1.19}$$

Since F is such that $F(x) \uparrow \infty$ for $x \to \infty$, it should preserve $\lim_{x\to\infty} \mathbb{P}_y(T_x \leq s) = 0$ and $\lim_{x\to\infty} \mathbb{P}_x(T_y \leq s) = 0$. Indeed ∞ is also natural for Z. This may be found in [Dur96]. There it is also explained where the name 'branching diffusion' comes from:

Basically Z_t is the weak limit of a sequence of rescaled *branching processes* $Z_t^n := \frac{B^n_{\lfloor nt \rfloor}}{n}$. For every n we have a seperate branching process $(B_k^n)_k$, where b and σ are directly related to the mean and variance of the offspring distributions. For instance if $b = -1$ and $\sigma^2 = 1$ we can take $\mathrm{Poi}(1 - \frac{1}{n})$ as offspring distribution for B^n to meet the conditions of the convergence theorem (8.3) of [Dur96] chapter 8. In particular

$$\mathbb{P}(\frac{B^n_{nk}}{n} \in \bullet \mid B_0^n = n) = \mathbb{P}(Z_k^n \in \bullet \mid Z_0^n = 1) \to \mathbb{P}_1(Z_k \in \bullet) \tag{1.20}$$

weakly for any $k \in \mathbb{N}$.

There is another possibility to get a solution to (1.18) as a scaling limit: By chapter 8 of [CCL+09] we can as well use certain sequences of rescaled *birth-and-death processes* $(Z_t^n)_n$ which already evolve in continuous time. Thus, we have

$$\mathbb{P}(Z_t^n \in \bullet \mid Z_0^n = x) \to \mathbb{P}_x(Z_t \in \bullet) \tag{1.21}$$

instead of (1.20). In what follows we will use this method for approximation, respectively for simulation.

If we take a closer look, we can ask the following:

1. Does Z_t, under killing at zero, has a Yaglom limit ϕ, if $b < 0$?

2. If so, can we 'merge' this quasistationarity and the convergence in (1.20) and take for example $t = n$ to obtain

$$\mathbb{P}(Z_n^n \in \bullet \,|\, Z_0^n = x,\, Z_n^n > 0) \to \phi(\bullet)$$

weakly?

3. Do the Yaglom limits for Z^n converge (under the correct scaling) to the Yaglom limit of Z?

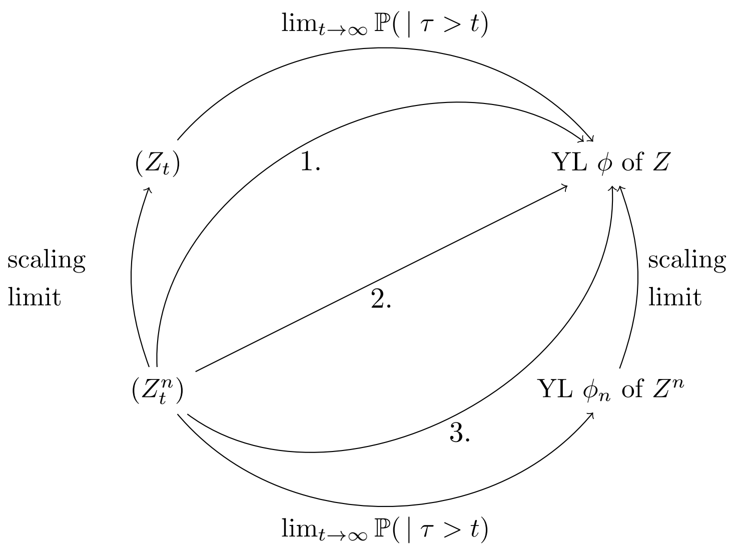
The first question was tackled and answered in [CCL+09]. Besides branching diffusions, the authors of [CCL+09] also pay special attention to the case of *logistic Feller diffusions*

$$dZ_t = \sigma\sqrt{Z_t}\,dX_t + (bZ_t - cZ_t^2)\,dt \tag{1.22}$$

the linear drift term is complemented by a concave ($c > 0$) quadratic term. They also make an additional assumption assuring the uniqueness of a QLD, thus also the uniqueness of a QSD. Though for branching diffusions the mentioned condition is not fulfilled and indeed, it is shown in [Lam07] that there is an infinite number of QSDs for Branching diffusions if $b < 0$.

In [MV12] the third question is raised, again in the context of QSDs for logistic diffusions under $\tau = T_0$. They formulate this as an open problem.

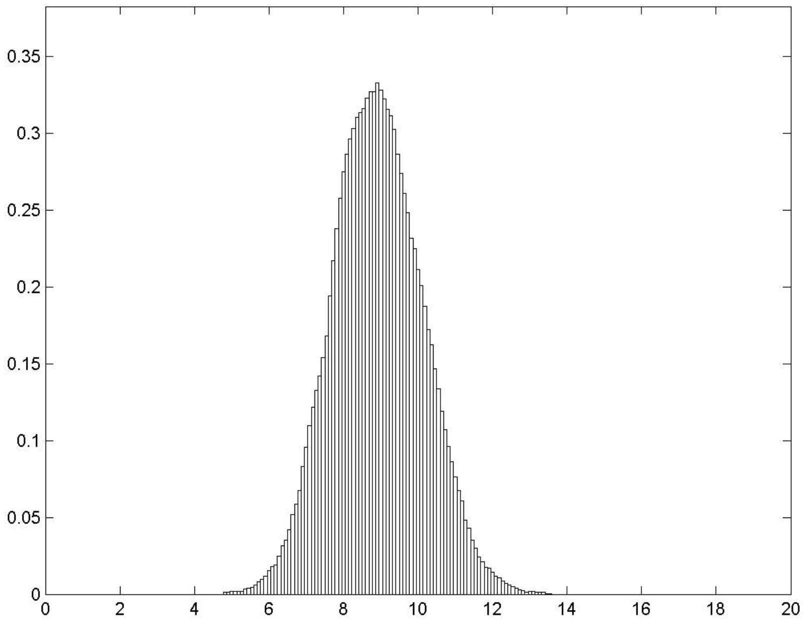
The second question suggests somhow a middle way. The following diagram shows the 3 mentioned possibilities:



Though if we want to use the scaling limit approximation for simulation of a QLD we are forced to use a 'middle way' as above. The obvious reason is that we can not 'compute' one limit before initializing a second limit procedure.

We simulated Yaglom limits for certain diffusions solving (1.18) or (1.22) by using the 'middle way'. Thus, we combined the scaling limit approximation with conditioning on not hitting zero. The reader should be aware that this was already done in [MV12] for the logistic case (1.22). One reason that we decided to do this by ourselfes was that it is unclear whether the authors of [MV12] used the scaling limit method or an Euler method to approximate Z. Another reason is of course that we can easily compare our simulation results with those from [MV12]. For the simulation of $\mathbb{P}_x(Z_t^n \in \bullet \mid T_0 > t)$ we use a Fleming-Viot-type particle system. The heuristics behind this is that we start with a large number $(Z^{n,l})_{l\leq L}$ of independent copies of Z^n under $\mathbb{P}_x$. But they interact in the following way: Once a $Z^{n,l}$ hits zero or gets killed at time s it jumps immidiately to a point uniformly choosen from $\{Z_s^{n,1}, \dots, Z_s^{n,L}\} \setminus \{Z_s^{n,l}\}$. Then the particle system moves again like L independent copies of Z^n (until the next kill). As explained in [MV12] the empirical distribution of $\{Z_t^{n,1}, \dots, Z_t^{n,L}\}$ converges to $\mathbb{P}_x(Z_t^n \in \bullet \mid T_0 > t)$ for $L \to \infty$. We stress the fact that this method already involves three limit procedures: At first $n \to \infty$ followed by $L \to \infty$ and finally $t \to \infty$. Clearly, a simulation can only be performed by taking n, L and t 'large'.

The following picture is a simulation of a QSD for a logistic diffusion with parameters $\sigma = 1$, $b = 9$ and $c = 1$. We chose the parameters (n, L, t) as $(11, 10000, 75)$ and smoothed the approximated density by a simple moving average over five values: For each bar, we averaged over the height of the original bar and the heights of the two bars to the left and the two to the right.



We used appropriately rescaled birth-and-death processes on $\frac{1}{n}\mathbb{N}$ and birth rate $= \frac{1}{2}n + 10$ as well as death rate $= \frac{1}{2}n + 1 + c\frac{i-1}{n}$ for the approximation of the diffusion. Here '$\frac{i}{n}$' is the current state, i.e. the state right before the next jump. Furthermore, the probability that an individual dies before another individual is born is

$$\frac{\left(\frac{1}{2}n + 1 + c\frac{i-1}{n}\right)}{\left(\frac{1}{2}n + 10 + \frac{1}{2}n + 1 + c\frac{i-1}{n}\right)} = \frac{\left(\frac{1}{2}n + 1 + c\frac{i-1}{n}\right)}{\left(n + 11 + c\frac{i-1}{n}\right)}.$$

And if this is the case then the time of death, i.e. when the value of the process changes from

$\frac{i}{n}$ to $\frac{i-1}{n}$, is exponentially distributed with parameter

$$(\frac{1}{2}n + 1 + c\frac{i-1}{n})\cdot\frac{i}{n} = \frac{1}{2}i + \frac{i}{n} + c\frac{i(i-1)}{n^2}.$$

Further we used $L = 10000$ particles in the Fleming-Viot-type system which ran up to time $t = 75$.

Note that the QSD for the logistic diffusion has its maximum at 9. The 9 is the so-called *charge capacity* $\frac{b}{c}$ which is the point where the drift coefficient $bx - cx^2$ switches from > 0 to < 0.

We want to close this subsection with an observation concerning the transformation of Yaglom limits under a differentiable one-on-one map F. Suppose one of the Yaglom limits φ_Y, respectively φ_Z, under some killing time τ for Z, respectively $Y_t = F(Z_t)$, exists. Then the other exists as well and they are connected via

$$\varphi_Z(x) = F'(x)\cdot(\varphi_Y(F(x)). \tag{1.23}$$

This is not hard to verify once we recognize that the killing time for Y is also τ. To prove this let E^a_Y, respectively E^a_Z, be the 'allowed' regions for Y, respectively Z, under τ (as in section 1.1). Thus, F is differentiable and one-on-one from E^a_Z to E^a_Y and we calculate

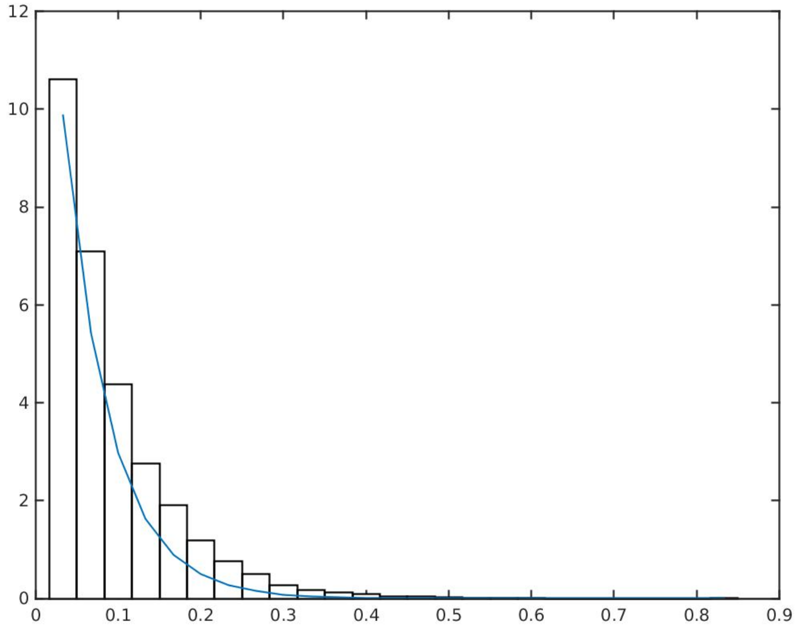
$$\begin{aligned}
&\frac{\int_A \varphi_Z(v)dv}{\int_{E^a_Z} \varphi_Z(v)dv} \\
&= \lim_{t\to\infty} \mathbb{P}_y(Z \in A \mid \tau > t) \\
&= \lim_{t\to\infty} \mathbb{P}_{F(y)}(Y \in F(A) \mid \tau\circ F^{-1} > t) \\
&= \frac{\int_{F(A)} \varphi_Y(w)dw}{\int_{F(E^a_Z)} \varphi_Y(w)dw} \\
&= \frac{\int_A F'(v)\varphi_Y(F(v))dv}{\int_{E^a_Z} F'(v)\varphi_Y(F(v))dv}.
\end{aligned}$$

We want to use this to calculate the QSD for (1.19).

example 1.4. For simplicity choose $\sigma = 1$ and $b = -9$. The QSD φ_Y for $dY_t = dX_t - \frac{9}{2}Y_t dt$ is a multiple of $x\,\mathrm{e}^{-4.5x^2}$. If we now use the transformation (1.23) with $F(x) = \int_0^x \frac{1}{\sigma\sqrt{y}}dy = 2\sqrt{x}$, we deduce that $\varphi_Z(x)$ is a multiple of e^{-18x}. We observe a qualitative difference to the logistic case. In the logistic case we had a maximum of the QSD at the 'charge capacity'. Though for Z from (1.19) we also have a point where the drift coefficient switches from > 0 to < 0 but there is *no* maximum of the QSD.

If we compare the transformed Ornstein-Uhlenbeck-process from (1.19) with the branching diffusion from (1.18) with $\sigma = 1$ and $b = -9$, we see that the difference in the drift coefficients has only a minor influence on their stochastic behaviour if the processes are far away from zero. Interestingly, this is not so much reflected by the corresponding quasistationary distributions. The reason is that, in contrast to the logistic case, the majority of the mass 'returns' and is located in immediate vicinity of zero. But near zero the effect of the different drift coefficients is largest. The next figure shows the calculated QSD $x \mapsto \mathrm{e}^{-18x}$ of (1.19) together with a QSD for

(1.18) simulated by the method we described above and already have used in the logistic case.



For the simulation we used appropriately rescaled birth-and-death processes on $\frac{1}{n}\mathbb{N}$ with $n = 30$ and birth rate $= \frac{1}{2}n + 1$ as well as death rate $= \frac{1}{2}n + 10$ for the approximation of the diffusion. This means the probability that an individual is born before another individual dies is $\left(\frac{1}{2}n+1\right)/\left(\frac{1}{2}n+1+\frac{1}{2}n+10\right) = \left(\frac{1}{2}n+1\right)/(n+11)$. And if this is the case then the time birth is exponentially distributed with parameter $(\frac{1}{2}n + 1)\cdot\frac{i}{n} = \frac{1}{2}i + \frac{i}{n}$. Further we used $L = 20000$ particles in the Fleming-Viot-type system which ran up to time 200. ∎

1.4. Some existence results for QLDs

In this section we prove several results on the existence of the Yaglom limit. We will see that there is a sharp distinction (dichotomy) between the cases '$\underline{\lambda} = 0$' and '$\underline{\lambda} > 0$'. In particular, by theorem 1.12 we find a generalisation of a prior result from [KS12].

Throughout this section we will assume that κ has a limit:

A 5. The limit $K := \lim_{x\to\infty} \kappa(x)$ exists.

In particular, an expression of the form 'K fulfils ...' implicitly means, that A 5 is satisfied and $K = \lim_{x\to\infty} \kappa(x)$.

proposition 1.8. *Suppose we have A 1, A 2, A 3 and $K < \underline{\lambda}$. Then*

(a) we have existence of the QLD which is the normalized version of $\underline{\varphi}$.

(b) the asymptotic mortality rate is $\underline{\lambda}$, independent of any compactly supported initial distribution.

convention 1.3. Unless otherwise specified

- when talking about a QLD we mean the Yaglom limit under the killing time τ.
- $E = [0, \infty)$ and $\tau = \tau^{\kappa,\alpha}$ results from (slow) killing at zero parametrized by α and killing under the killing rate κ.

The next result is, as the above, taken from [KS12] and as the authors have done, we stick at formulating it as a corollary. (It is Corollary 4.8 from [KS12] to be precise.) We hereby point out that it is some sort of generalization to a theorem from [CMSM95] to *slow* killing at the boundary.

corollary 1.9. *Suppose $\kappa \equiv 0$ and we have killing at the boundary, i.e. $\alpha > 0$. Again suppose A 1, A 2 and A 3. Then we have the following dichotomy.*

- *If $\underline{\lambda} > 0$, then we have the above existence of the QLD.*
- *But if $\underline{\lambda} = 0$, then we have escape to ∞ under $\tau^{\kappa,\alpha}$.*

Here *escape to infinity under τ* means that the mass goes to infinity. We formalize this in the following way:

For all $x, y > 0$

$$\mathbb{P}_x(Y_t \leq y \mid \tau > t) \to 0.$$

remark 1.7. The above dichotomy, even the existence of a QLD, depends heavily on the initial distribution. Take for example Y as a Brownian motion with constant negative drift and $\tau = T_0$. In Theorem 1.4 of [MPSM98], it is shown that there exists an initial ν and a sequence $t_n \uparrow \infty$, such that the limits of $\mathbb{P}_\nu(Y_{t_{2n}} \in \bullet \mid \tau > t_{2n})$ and $\mathbb{P}_\nu(Y_{t_{2n+1}} \in \bullet \mid \tau > t_{2n+1})$ exist, but do not agree. (Note that $\nu = \delta_x$ in 1.9!) ∎

What if we only assume A 5?

Before we answer this question, we first present the easily proved result for '$\kappa = 0$ eventually'.

proposition 1.10. *Suppose $\kappa = 0$ eventually and we have killing at the boundary, i.e. $\alpha > 0$. Further suppose A 1, A 2, A 3. Then we have the following dichotomy.*

- *If $\underline{\lambda} > 0$, then we have existence of a QLD with density $\frac{\underline{\varphi}(x)}{\int_0^\infty \underline{\varphi}(y)dy}$.*
- *If $\underline{\lambda} = 0$, then we have escape to ∞.*

proof. Looking at corollary 1.9 (written in [KS12]) we see that the only thing which needs proof is the second part. And this is done by showing that $\underline{\psi}$ is not intergrable w.r.t. $m(dy)$, whenever $\underline{\lambda} = 0$: For this purpose take $x > 0$ such that $\kappa(y) = 0$ for all $y \geq x$ and such that there is a $\beta > 0$ with $\underline{\psi}'(x) = 2\beta\underline{\psi}(x)$. Note that such an x and $\beta = \beta(x)$ must exist by theorem A.2.6. The reason is as follows:

The only situation where it would not exist will be if $\underline{\psi}'(x) = 0$ whenever $\kappa(y) = 0$ for all $y \geq x$. Let x_0 have this property. Then $f(x) := \underline{\psi}(x_0 - x)$ is the unique solution for the 'time-reversed' differential equation

$$\frac{1}{2}f''(x) - \boldsymbol{a}(-x)f'(x) + (\underline{\lambda} - \kappa(-x))f(x) = 0$$

with boundary conditions $f(0) = y_0 > 0$ and $f'(0) = f''(0) = 0$ on any interval $[0, x_0 - \epsilon]$. Thus, it must hold $f(x) \equiv y_0$ and, therefore, $\underline{\psi}(x) \equiv y_0$ and $\underline{\psi}'(x) \equiv 0$ for all $x > 0$. And this contradicts the boundary conditions for $\underline{\psi}$ at 0.

The next step is to take $\underline{\psi}'(x_0) = 2\beta\underline{\psi}(x_0)$ as a boundary condition of $-\mathbf{L}^\kappa\psi = \lambda\psi$ on $[x_0, \infty)$. Then, due to theorem A.2.6 including the positivity of $\underline{\psi}$, $\tilde{\underline{\psi}} := \underline{\psi}|_{[x_0,\infty)}$ is 'the corresponding $\underline{\psi}$ for the problem on $[x_0, \infty)$'. Now we can see (as in the proof of 1.9) that $\tilde{\underline{\psi}}$ is not integrable w.r.t. $m(dy)$ (on $[x_0, \infty)$). Thus, $\underline{\psi}$ is also not integrable w.r.t. to $m(dy)$.

Finally we want to establish the desired result: Theorem 3.15 of [KS12] tells us that Y_t (under $\tau^{\kappa,\alpha}$) 'converges to $\underline{\psi}$ on compacta'. Combining this, the non-integrability of $\underline{\psi}$ and proposition 2.3 of [SE07] we see that Y escapes to ∞. Note that proposition 2.3 of [SE07] does *not* depend on $\boldsymbol{a} \in C^1$ which is actually a general condition of [SE07]. ■

For a proof of the more general result 1.12, we first show the following lemma:

lemma 1.11. *If we choose ψ as a solution to $-\mathbf{L}^\kappa\psi = \lambda\psi$, under $\psi'(0) = 2\alpha\psi(0)$, such that $\psi(0) = 1$, then*

$$\psi(x) = \int_0^x \frac{2}{m(y)} \left(2\alpha - \int_0^y (\lambda - \kappa(z))m(z)\psi(z)dz \right) dy + 1. \tag{1.24}$$

proof. We are done if we show that

$$\psi'(x)\frac{m(x)}{2} = 2\alpha - \int_0^x (\lambda - \kappa(y))m(y)\psi(y)\, dy.$$

(We only need to integrate this equation.) To see this, first observe that the left and right hand side are equally 2α at zero. If we take the derivative of the right hand, we get

$$-(\lambda - \kappa(x))m(x)\psi(x).$$

Due to $m(x) = 2\,e^{2\int_0^x \boldsymbol{a}(y)dy}$, this is the same as the derivative of the left hand side:

$$\begin{aligned}(\psi'\frac{m}{2})' &= \psi''\frac{m}{2} + \boldsymbol{a}m\psi' \\ &= m(\mathbf{L}^\kappa + \kappa)\psi \\ &= m(-\lambda\psi + \kappa\psi).\end{aligned}$$

■

remark 1.8. What if we have instant killing at zero, which shows as $\alpha = \infty$, i.e. $\psi(0) = 0$? For a similar result as above, we have to skip '1' at the end of (1.24) and write $\psi'(0)$ rather than 2α. The proof remains the same. ▮

Now we come to the more general case where the killing rate converges to zero:

theorem 1.12. *Suppose that we have killing at the boundary, i.e. $\alpha > 0$, and that A 1, A 2 and A 3 are fulfilled.*

- *If $\underline{\lambda} = 0$, then we have escape to ∞.*

Now we additionally assume A 5 with $K = 0$.

- *If $\underline{\lambda} > 0$, then we have the existence of a QLD.*

proof. As in the previous proof for $\kappa = 0$ eventually, we deduce the second part from Th 4.7 of [KS12]. For the first part we are going to show that

$$\int_0^\infty \underline{\psi}(x)m(x)dx = \int_0^\infty \underline{\varphi}(x)dx = \infty.$$

By the use of Theorem 3.15 of [KS12] and proposition 2.3 of [SE07], this would imply escape to infinity.

Hence, suppose $\underline{\lambda} = 0$. We only deal with the case $\alpha < \infty$. (The proof for '$\alpha = \infty$' is the same due to remark 1.8.) Because of (1.24), we get

$$\underline{\psi}(x) = \int_0^x \frac{2}{m(y)} \left(2\alpha + \int_0^y \kappa(z)m(z)\underline{\psi}(z)dz \right) dy + 1. \tag{1.25}$$

(A.2.6) and (1.25) give us

$$\underline{\psi}(x) \geq \alpha \int_0^x \frac{1}{m(y)} dy.$$

This would imply

$$\int_0^\infty \underline{\psi}(x)m(x)dx \geq \alpha \int_0^\infty m(x) \int_0^x \frac{1}{m(y)}\, dy\, dx = \infty.$$

The last equality is due to condition A 2 (or simply due to the fact that ∞ is no entrance boundary). ■

We can do exactly the same argumentation (as in the last proof) but under $K > 0$. We only need to 'rewrite' everything in terms of $\tilde{\kappa} := \kappa - K$:

corollary 1.13. *Suppose A 5, $\kappa \geq K$ eventually and that we have killing at the boundary, i.e. $\alpha > 0$. Further suppose A 1, A 2 and A 3. Then we have the following dichotomy.*

- *If $\underline{\lambda} > K$, then we have the existence of a QLD with density $\frac{\underline{\varphi}(x)}{\int_0^\infty \underline{\varphi}(y)dy}$.*
- *And if $\underline{\lambda} = K$, then we have escape to ∞.*

1.5. Comparison theorems for the existence of QLDs

Next we give some comparison results for the bottom of $\Sigma(-\mathbf{L})$ in terms of the drift corresponding to the generator $\mathbf{L}$. Propositions 1.15 and 1.20 are similar to Corollary 1 from [MSM01]; but also allow slow killing at the boundary. The main results are theorems 1.17 and 1.22 which are similar to 1.15 and 1.20 but we can have $\kappa \neq 0$. They will lead to simple conditions for the existence of a QLD. (Confer corollaries 1.24 and 1.25)

We use '$\boldsymbol{a}$' in brackets to indicate that we are working with drift coefficient $\boldsymbol{a}$. For example, $T_0(\boldsymbol{a})$ denotes the first hitting time of a diffusion corresponding to the drift $\boldsymbol{a}$. This diffusion under T_0 has $\mathbf{L}(\boldsymbol{a}) = \mathbf{L}^{0,\infty}(\boldsymbol{a})$ of appendix A.2.1 as $L^2(m)$-generator. (Note that m also depends on $\boldsymbol{a}$.) Consequently, we write $\underline{\lambda}(\boldsymbol{a}) = \underline{\lambda}^{0,\infty}(\boldsymbol{a})$ for the bottom of $\Sigma(-\mathbf{L}(\boldsymbol{a}))$.

1.5.1. Comparison under C^1-drift coefficients

For the proof of the next proposition we are using a Theorem from [SE07]. Therefore, we have to impose a stronger smoothness condition than A 1; namely:

A 6. We have that

(i) the killing rate κ is non-negative and $C[0,\infty)$.

(ii) the drift coefficient $\boldsymbol{a}$ is in $C^1(0,\infty)$ with 0 being regular.

remark 1.9. As in A 1, we may replace 'zero regular' by the more handy condition $\boldsymbol{a}\cdot\mathbb{1}_{[0,\epsilon)} \in L^1$. This already implies regularity. ∎

lemma 1.14. *Let $\boldsymbol{a}$ and $\boldsymbol{b}$ be drift coefficients which are locally Lipschitz continuous. Let the boundary 0 be regular for $\boldsymbol{a}$ and $\boldsymbol{b}$. Furthermore, let $\boldsymbol{a} \le \boldsymbol{b}$ on $(0,\infty)$ and let ∞ be a natural boundary for $\boldsymbol{a}$ and $\boldsymbol{b}$. Then*

$$\mathbb{P}_x(T_0(\boldsymbol{a}) \ge s) \le \mathbb{P}_x(T_0(\boldsymbol{b}) \ge s)$$

for all $x, s \ge 0$.

proof. It holds

$$\mathbb{P}_x(T_\epsilon(\boldsymbol{a}) \ge s) - \mathbb{P}_x(T_\epsilon(\boldsymbol{a}) \ge s,\, T_N(\boldsymbol{a}) \ge s) \le \mathbb{P}_x(T_N(\boldsymbol{a}) < s) \to 0$$

as $N \to \infty$. The same is true for $\boldsymbol{b}$ instead of $\boldsymbol{a}$. Thus, we have that

$$\begin{aligned} &\mathbb{P}_x(T_\epsilon(\boldsymbol{b}) \ge s) - \mathbb{P}_x(T_\epsilon(\boldsymbol{a}) \ge s) \\ &\ge \mathbb{P}_x(T_\epsilon(\boldsymbol{b}) \wedge T_N(\boldsymbol{b}) \ge s) - \mathbb{P}_x(T_\epsilon(\boldsymbol{a}) \wedge T_N(\boldsymbol{a}) \ge s) - o_N(1). \end{aligned} \tag{1.26}$$

Let the superscript 'N', e.g. in $\boldsymbol{a}^N$ inidcate that we are working with the diffusion which is the strong solution to

$$dY_t = \boldsymbol{a}^N(Y_t)dt + X_t.$$

Here $\boldsymbol{a}^N$ is some Lipschitz continuous function on $\mathbb{R}$ such that $\boldsymbol{a}^N = \boldsymbol{a}$ on $[\frac{1}{N}, N]$. Now for large N such that $\frac{1}{N} \le \epsilon$ we have that

$$\mathbb{P}_x(T_\epsilon(\boldsymbol{a}) \wedge T_N(\boldsymbol{a}) \ge s) = \mathbb{P}_x(T_\epsilon(\boldsymbol{a}^N) \wedge T_N(\boldsymbol{a}^N) \ge s). \tag{1.27}$$

We also have

$$\mathbb{P}_x(T_\epsilon(\boldsymbol{a}^N) \ge s) - \mathbb{P}_x(T_\epsilon(\boldsymbol{a}^N) \ge s,\, T_N(\boldsymbol{a}^N) \ge s) \le \mathbb{P}_x(T_N(\boldsymbol{a}^N) < s).$$

Though the state space of the diffusion with drift coefficient $\boldsymbol{a}^N$ is $\mathbb{R}$ can still take e.g. $\frac{x}{2}$ as a reflecting boundary and use a coupling argument to obtain

$$\mathbb{P}_x(T_N(\boldsymbol{a}^N) < s) \le \mathbb{P}_x(T_N(\boldsymbol{a}^N_{\frac{x}{2}}) < s).$$

Here we indicated the reflection at $\frac{x}{2}$ by the same subscript: $\boldsymbol{a}^N_{\frac{x}{2}}$. The advantage now is that we can replace $\boldsymbol{a}^N_{\frac{x}{2}}$ by $\boldsymbol{a}_{\frac{x}{2}}$ for N large enough such that $\frac{x}{2} > \frac{1}{N}$. Hence,

$$\mathbb{P}_x(T_\epsilon(\boldsymbol{a}^N) \geq s) - \mathbb{P}_x(T_\epsilon(\boldsymbol{a}^N) \geq s,\, T_N(\boldsymbol{a}^N) \geq s) \leq \mathbb{P}_x(T_N(\boldsymbol{a}_{\frac{x}{2}}) < s). \tag{1.28}$$

Note that the diffusion with drift coeficient $\boldsymbol{a}$ reflected at $\frac{x}{2}$ still has the property that ∞ is natural. Therefore,

$$\mathbb{P}_x(T_N(\boldsymbol{a}_{\frac{x}{2}}) < s) \to 0$$

as $N \to \infty$. Together with (1.26), (1.27) and (1.28) we arrive at

$$\mathbb{P}_x(T_\epsilon(\boldsymbol{b}) \geq s) - \mathbb{P}_x(T_\epsilon(\boldsymbol{a}) \geq s) \geq \mathbb{P}_x(T_\epsilon(\boldsymbol{b}^N) \geq s) - \mathbb{P}_x(T_\epsilon(\boldsymbol{a}^N) \geq s) - o_N(1).$$

Note that we have used the fact that (1.27) and (1.28) also hold for $\boldsymbol{b}^{(N)}$. Of course, we can choose $\boldsymbol{a}^N$ and $\boldsymbol{b}^N$ such that $\boldsymbol{a}^N \leq \boldsymbol{b}^N$. Hence, by [Lin92] section VI.5 we find that

$$Y(\boldsymbol{a}^N)_t \leq Y(\boldsymbol{b}^N)_t \text{ for all } t$$

$\mathbb{P}_x$-almost surely. ($Y(\boldsymbol{a}^N)$ and $Y(\boldsymbol{b}^N)$ are the diffusions corresponding to the drift coefficients $\boldsymbol{a}^N$ and $\boldsymbol{b}^N$.) This implies

$$\mathbb{P}_x(T_\epsilon(\boldsymbol{b}) \geq s) - \mathbb{P}_x(T_\epsilon(\boldsymbol{b}) \geq s) \geq -o_N(1)$$

and leads to

$$\mathbb{P}_x(T_\epsilon(\boldsymbol{b}) \geq s) \geq \mathbb{P}_x(T_\epsilon(\boldsymbol{b}) \geq s).$$

Because of $\{T_\epsilon \geq s\} \uparrow$ as $\epsilon \downarrow 0$ we deduce

$$\mathbb{P}_x(T_0(\boldsymbol{b}) \geq s) \geq \mathbb{P}_x(T_0(\boldsymbol{a}) \geq s).$$

We show $T_\epsilon \uparrow T_0$ as $\epsilon \downarrow 0$ by contradiction:

Assume $T_\epsilon \uparrow T_0$ is not the case. Now we must have some $\tilde{\Omega} \subseteq \Omega$ with $\mathbb{P}_x(\tilde{\Omega}) > 0$ such that for all $\omega \in \Omega$ there is a $\delta(\omega) > 0$ and a sequence $t_n(\omega)$ with $Y_{t_n} \leq \frac{1}{n}$ and for all zeros s of $Y(\omega)$ we have $s - t_n > \delta$. But t_n has a convergent subsequence $t_{n_k} \to t_0$. By continuity of the sample paths we deduce

$$Y_{t_0(\omega)}(\omega) = Y_{\lim_{k\to\infty} t_{n_k}(\omega)}(\omega) = \lim_{k\to\infty} Y_{t_{n_k}(\omega)}(\omega) = 0$$

giving the contradiction

$$\inf\{t\,;\ Y_t = 0\} \leq t_0 \leq T_0 - \delta < T_0$$

on $\tilde{\Omega}$. ∎

proposition 1.15. *Let $\kappa = 0$ and let $\boldsymbol{a}, \boldsymbol{b}$ both satisfy A 2 and A 6. If $\boldsymbol{b}$ additionally fulfils A 3, we have that*

$$\boldsymbol{a} \leq \boldsymbol{b}$$

implies

$$\underline{\lambda}(\boldsymbol{b}) > 0 \Rightarrow \underline{\lambda}(\boldsymbol{a}) \geq \underline{\lambda}(\boldsymbol{b}).$$

In particular we have

$$\exists\,\text{QLD for } \boldsymbol{b} \Rightarrow \exists\,\text{QLD for } \boldsymbol{a}\,.$$

Since we are sometimes dealing with different 'α', as in the proof below, we will occasionally indicate this with a superscript like '$\lambda^\alpha(\boldsymbol{a})$'.

proof. By theorem 1.12 we have: $\exists$ QLD for $\boldsymbol{b} \Rightarrow \underline{\lambda}^\alpha(\boldsymbol{b}) > 0$. Suppose $\underline{\lambda}^\infty(\boldsymbol{b}) = 0$. Since zero is no isolated eigenvalue we have that 0 is the supremum of the essential spectrum of $\mathbf{L}^\infty(\boldsymbol{b})$. Because the essential spectra of $\mathbf{L}^\infty(\boldsymbol{b})$ and $\mathbf{L}^\alpha(\boldsymbol{b})$ coincide it must hold $\underline{\lambda}^\alpha(\boldsymbol{b}) = 0$, which gives a contradiction. Using proposition 1.8 gives $\eta^\infty(\boldsymbol{b}) > 0$. Now Lemma 1.14 gives us

$$0 < \eta^\infty(\boldsymbol{b}) = -\lim \frac{\log \mathbb{P}_x(\tau^\infty(Y(\boldsymbol{b})) > t)}{t} \leq -\lim \frac{\log \mathbb{P}_x(\tau^\infty(Y(\boldsymbol{a})) > t)}{t} = \eta^\infty(\boldsymbol{a}).$$

Note that every C^1-function is also locally Lipschitz continuous. We use Theorem 3.4 of [SE07] to obtain

$$\underline{\lambda}^\infty(\boldsymbol{a}) = \eta^\infty(\boldsymbol{a}) > 0.$$

(Note that due to 1.16 v we have A 7 satisfied for κ and $\boldsymbol{a}$. That we are in the limit point case is due to [KS12]; see remark 1.11.)

As above we can assume $\underline{\lambda}^\alpha(\boldsymbol{a}) = 0$ giving a contradiction. Finally we use again theorem 1.12 or Theorem 3.4 of [SE07] (together with $\underline{\lambda}^\alpha(\boldsymbol{a}) > 0$) to get the existence of a QLD. ■

remark 1.10*.* The only point in the proof were we need that $\boldsymbol{a}$ is C^1 is to get $\underline{\lambda}^\infty(\boldsymbol{a}) > 0$ if $\eta^\infty(\boldsymbol{a}) > 0$. ▮

In fact, the same argumentation can be used to get the result if $\kappa \to 0$. (See theorem 1.17.) To do so we define $w_t(x) := \frac{\mathbb{P}_x(\tau>t)}{\mathbb{P}_1(\tau>t)}$ and introduce the following assumption.

A 7. For each $s \geq 0$ we have $\mathbb{P}_x\big(\sup_{t\geq 0} w_t(Y_s)\cdot\mathbb{1}_{\{\tau>s\}}\big) < \infty$.

This assumption will also play an important role in chapter 2. It will be necessary to use *Lebesgue's dominated convergence theorem (DOM)* . At this point we also provide some sufficient conditions for this assumption.

lemma 1.16. *Let any of the following conditions on* $\boldsymbol{a}$ *and* κ *be fulfilled.*

(i) $\boldsymbol{a} = O(x)$ *and* $\kappa = O(x^2)$*.*

(ii) Pure reflection at the boundary, i.e. $\alpha = 0$*, but* κ *increasing.*

(iii) $\boldsymbol{a}^+$ *has at most linear growth.* $\boldsymbol{a}^-$ *has polynomial growth and* $\boldsymbol{a}' = O(x^2)$ *as well as* $\kappa = O(x^2)$*.*

(iv) $\kappa = 0$

(v) Let $\kappa \downarrow 0$ *and let* $\boldsymbol{a} \leq \boldsymbol{b}$ *be such that* $\boldsymbol{b}$ *and* κ *fulfil A 7.*

Then A 7 is satisfied.

For us '$O(g)$' is the *Landau notation* for an 'f' such that $\frac{|f|}{g}$ is bounded eventually. Further we use f^-, resp. f^+, for the negative part $-(f \wedge 0)$, respectively the positive part $f \vee 0$, of f.

proof of lemma 1.16. The first three points are more or less proved in and before Lemma 2.5 of [SE07]. We only added that κ can be taken to be even $O(x^2)$ (instead of $O(x)$). But this can be seen immidiately from the proof in [SE07]. For the forth point we arrive, as right before Lemma 2.5, at

$$w_t(x) \leq 1 + \frac{1}{\mathbb{P}_1(\tau > T_x)} \leq 1 + \frac{1}{\mathbb{P}_1(T_0 > T_x)} = 1 + \frac{s(x) - s(0)}{s(1) - s(0)}.$$

Now we use some trick due to [CMSM95]. At first we see that

$$\mathbb{P}_x(s(Y_{\tau \wedge T_m \wedge l}) - s(x)) = \mathbb{P}_x \int_0^{\tau \wedge T_M \wedge l} ds(Y_r).$$

Using Ito's formula on $ds(Y_r)$ and using that

$$\mathbf{L}s = (\frac{1}{2}\frac{d^2}{dx^2} + \boldsymbol{a}\frac{d}{dx})s = 0,$$

we obtain

$$\mathbb{P}_x(s(Y_{\tau \wedge T_m \wedge l}) - s(x)) = 0.$$

Therefore, we have

$$\mathbb{P}_x(s(Y_r)\mathbb{1}_{\{\tau \wedge T_M \geq r\}}) \leq \mathbb{P}_x(s(Y_{\tau \wedge T_M \wedge r})) = s(x).$$

Letting M tend to ∞ now gives a dominant for $(w_t(Y_l)\mathbb{1}_{\{\tau > l\}})_t$.

Point five should be rather obvious, since the contribution to the term under consideration is larger, where Y_s is large. And due to a smaller diffusion coefficient $\boldsymbol{a}$ it is more likely that Y is 'small'. ■

We also see the need to make the following

remark 1.11. [SE07] persistently assume that

A. We are in the *limit point case.*

Some reader may wonder that we forgot to post this condition when using their results. But we are always in the limit point case.

The proof is a simple use of Lemma 3.1 of [KS12] together with the observations that we have always $\kappa \in L^2_{loc}[0, \infty)$ and the density m of the speed measure is locally Lipschitz on $(0, \infty)$ as soon as we have A 6 respectively A 1. ▮

theorem 1.17. *Let $\kappa \to 0$ and let $\boldsymbol{a}, \boldsymbol{b}$ and κ satisfy A 2 and A 6. Furthermore, let $\boldsymbol{b}$ (and κ) additionally satisfy A 3 and A 7. (E.g. through some suffifient condition from 1.16.) Then we have that*

$$\boldsymbol{a} \leq \boldsymbol{b}$$

implies

$$\underline{\lambda}(\boldsymbol{b}) > 0 \Rightarrow \underline{\lambda}(\boldsymbol{a}) \geq \underline{\lambda}(\boldsymbol{b}).$$

In particular one has

$$\exists\,\text{QLD for } \boldsymbol{b} \Rightarrow \exists\,\text{QLD for } \boldsymbol{a}\,.$$

proof. We use again theorem 1.12 to get: $\exists\,$QLD for $\boldsymbol{b} \Rightarrow \underline{\lambda}^{\alpha,\kappa}(\boldsymbol{b}) > 0$. Now we suppose $\underline{\lambda}^{\infty,0}(\boldsymbol{b}) = 0$. Since 0 is no isolated eigenvalue it is the supremum of the essential spectrum of $\mathbf{L}^{0,\infty}(\boldsymbol{b})$. But the essential spectra of $\mathbf{L}^{0,\infty}(\boldsymbol{b})$ and $\mathbf{L}^{0,\alpha}(\boldsymbol{b})$ coincide. Taking into account that the operators are non-positive, we deduce $\underline{\lambda}^{\alpha,0}(\boldsymbol{b}) = 0$. By Lemma 3.3 (i) of [KS12] the essential spectra of $\mathbf{L}^{0,\alpha}(\boldsymbol{b})$ and $\mathbf{L}^{\kappa,\alpha}(\boldsymbol{b})$ are also the same. This implies $\underline{\lambda}^{\alpha,\kappa}(\boldsymbol{b}) = 0$ in contradiction to $\underline{\lambda}^{\alpha,\kappa}(\boldsymbol{b}) > 0$. Thus, $\underline{\lambda}^{\infty,0}(\boldsymbol{b}) > 0$. (As in the proof of 1.15) we conclude

$$\underline{\lambda}^{\infty,0}(\boldsymbol{a}) = \eta^{\infty,0}(\boldsymbol{a}) > 0$$

(A 7 due to 1.16 v and the limit point case due to remark 1.11!)

As usual we may assume $\underline{\lambda}^{\alpha,\kappa}(\boldsymbol{a}) = 0$ and get a contradiction. Hence, $\underline{\lambda}^{\alpha,\kappa}(\boldsymbol{a}) > 0$ is true and finally use again Th 3.4 of [SE07] to conclude the existence of a QLD. ■

1.5.2. Comparison under continuous drift coefficients

Next we will prove more general version of proposition 1.15 and theorem 1.17 by only assuming A 1 rather than A 6 (or even A 7). The drift coefficient $\boldsymbol{a}$ no longer need to be differentiable. But this comes for a price: we have to impose

A 8. We have that $\boldsymbol{a}^2$ is locally integrable at boundary points.

Here we work on $E = [0,\infty)$. Thus, A 8 simply means that $\boldsymbol{a}\cdot\mathbb{1}_{[0,\epsilon)} \in L^2$ for some $\epsilon > 0$.

lemma 1.18. *Let $\boldsymbol{a}$ be in $C(0,\infty)$ and satisfy A 8. Take any sequence $\boldsymbol{a}_n$ such that*

- *each $\boldsymbol{a}_n$ is locally lipschitz on $[0,\infty)$,*
- *$\boldsymbol{a}_n(x) \to \boldsymbol{a}(x)$ for all $x > 0$,*
- *$(\boldsymbol{a}_n)_n$ is uniformly bounded on compact subsets of $(0,\infty)$.*

Then we have

$$\mathbb{P}_x(T_0(\boldsymbol{a}_n) \geq s) \to \mathbb{P}_x(T_0(\boldsymbol{a}) \geq s)$$

for all $s, x > 0$.

proof. Since the $\boldsymbol{a}_n$ are locally lipschitz, we know that corresponding diffusions Y^n can be realized as solutions to

$$dY_t = dX_t + \boldsymbol{a}_n(Y_t)dt$$

where X is a Brownian motion starting at x w.r.t. $\mathbb{P}_x$ (,at least up to $T_0(\boldsymbol{a}_n)$). By the Cameron-Martin-Girsanov transformation, we know that the corresponding 'Girsanov density' is

$$f_t^n := \exp(\int_0^t \boldsymbol{a}_n(X_r)dX_r - \frac{1}{2}\int_0^t \boldsymbol{a}_n^2(X_r)dr).$$

(By a suitable continuation of $\boldsymbol{a}_n$ to the negative reals we find that) the measure corresponding to Y^n on $\mathcal{F}_t$ is

$$\mathbb{Q}_t^n := f_t^n \circ \mathbb{P}_x|_{\mathcal{F}_t}.$$

(This is a notation used also in the context of penalisation; compare section 2.1. In fact, what we are going to do to prove the lemma may be seen as some sort of penalisation result for $n \to \infty$ with the penaliser $(f_t^n)_{n \geq 1}$.) The reader may have already guessed that we want to show convergence to

$$\mathbb{Q}_t := f_t \circ \mathbb{P}_x|_{\mathcal{F}_t}$$

with $f_t = \exp(\int_0^t \boldsymbol{a}(X_r)dX_r - \frac{1}{2}\int_0^t \boldsymbol{a}^2(X_r)dr)$.

(Why this definition makes sense and the $\mathbb{Q}_t$'s are indeed the measures corresponding to $\mathbf{L}^{\infty,0}(\boldsymbol{a})$, when killed at zero, compare remark 1.12. Note that this is the reason for assuming square-integrability of $\boldsymbol{a}$ at zero.)

To this purpose we define

$$\mathbb{Q}_t^{n,K} := f_{t\wedge\tau_K}^n \circ \mathbb{P}_x|_{\mathcal{F}_t}$$

with $\tau_K := T_K \wedge T_{1/K}$. Notice that $\mathbb{Q}^{n,K} = \mathbb{Q}^n$ on $\mathcal{F}_{t\wedge\tau_K}$ by optional sampling, since $(f_t^n)_t$ is a martingale. For each K we have $(\boldsymbol{a}_n)_n$ uniformly bounded on $[1/K, K]$. Now we can use a 'stochastic-integral version' of the dominated convergence theorem to obtain

$$\int_0^{t\wedge\tau_K} \boldsymbol{a}_n(X_r)dX_r = \int_0^t \boldsymbol{a}_n(X_r)dX_r^K \to \int_0^{t\wedge\tau_K} \boldsymbol{a}(X_r)dX_r$$

in probability with $X_t^K := X_{t\wedge\tau_K}$. (For instance see Theorem 32 in Chapter IV of [Pro05].) We also have

$$\int_0^{t\wedge\tau_K} \boldsymbol{a}_n^2(X_r)dr \to \int_0^{t\wedge\tau_K} \boldsymbol{a}^2(X_r)dr$$

a.s. for $n \to \infty$ by the ordinary dominated convergence theorem. Thus, we find a subsequence $(n(m,K,t))_m$ with

$$f_{t\wedge\tau_K}^{n(m,K,t)} \to f_{t\wedge\tau_K}$$

a.s. for $m \to \infty$.

By the Lemma of Scheffé we get

$$f_{t\wedge\tau_K}^{n(m,K,t)} \to f_{t\wedge\tau_K} \tag{1.29}$$

in $L^1(\mathbb{P}_x)$ as $m \to \infty$. The rest will follow essentially by the proof of Lemma 11.1 of [SV06], but let us be a little bit more detailed:

At first we take some bounded and $\mathcal{F}_{t\wedge\tau_K}$-mb Φ. Due to (1.29) we conclude

$$\mathbb{Q}^{n,K}(\Phi) = \mathbb{P}_x(f_{t\wedge\tau_K}^n \cdot \Phi) \to \mathbb{P}_x(f_{t\wedge\tau_K} \cdot \Phi).$$

(For the rest of the proof we write 'n' instead of '$n(m,K,t)$' to increase readability. For example '$n \to \infty$' implicitely means '$m \to \infty$ in terms depending on $n(m,K,t)$'.) Using optional sampling on $(f_t)_t$ we see that

$$\mathbb{P}_x(f_{t\wedge\tau_K} \cdot \Phi) = \mathbb{P}_x(\mathbb{E}(f_{t\wedge\tau_K} \mid \mathcal{F}_{t\wedge\tau_K}) \cdot \Phi) = \mathbb{P}_x(f_t\Phi).$$

This gives us

$$\mathbb{Q}_t^{n,K} \twoheadrightarrow \mathbb{Q}_t \tag{1.30}$$

on $\mathcal{F}_{t\wedge\tau_K}$ for every $t, K > 0$. (Here we adapt again the notation from chapter 2.) Now take some $s < t$ and a bounded and $\mathcal{F}_{s\wedge T_0}$-measurable Φ. Then $\Phi\cdot\mathbb{1}_{\{t\wedge\tau_K > s\wedge T_0\}}$ is $\mathcal{F}_{t\wedge\tau_K}$-measurable. Note that, although $\{t\wedge\tau_K > s\wedge T_0\} \in \mathcal{F}_{t\wedge\tau_K}$ and, therefore, $\{t\wedge\tau_K \leq s\wedge T_0\} \in \mathcal{F}_{t\wedge\tau_K}$, in general $\Phi\cdot\mathbb{1}_{\{t\wedge\tau_K \leq s\wedge T_0\}}$ is *not* $\mathcal{F}_{t\wedge\tau_K}$-measurable. But this suffices to get

$$|\mathbb{Q}_t^n(\Phi) - \mathbb{Q}_t(\Phi)| \leq \left|\mathbb{Q}_t^{n,K}(\Phi) - \mathbb{Q}_t(\Phi)\right| + \|\Phi\|_\infty \cdot 2\mathbb{Q}_t^n(t\wedge\tau_K \leq s\wedge T_0) \tag{1.31}$$

on the one hand and

$$\lim_{n\to\infty} \mathbb{Q}_t^n(t\wedge\tau_K \leq s\wedge T_0) \to \mathbb{Q}_t(t\wedge\tau_K \leq s\wedge T_0) \tag{1.32}$$

on the other, by (1.30). Furthermore, we have

$$\begin{aligned}
&\limsup_n \left|\mathbb{Q}^{n,K}(\Phi) - \mathbb{Q}(\Phi)\right| \\
\leq &\limsup_n \left|\mathbb{Q}^{n,K}(\Phi\mathbb{1}_{\{t\wedge\tau_K > s\wedge T_0\}}) - \mathbb{Q}(\Phi\mathbb{1}_{\{t\wedge\tau_K > s\wedge T_0\}})\right| \\
+ &\limsup_n \left|\mathbb{Q}^{n,K}(\Phi\mathbb{1}_{\{t\wedge\tau_K \leq s\wedge T_0\}}) - \mathbb{Q}(\Phi\mathbb{1}_{\{t\wedge\tau_K \leq s\wedge T_0\}})\right| \\
\leq &\limsup_n \|\Phi\|_\infty \left[\mathbb{Q}^{n,K}(t\wedge\tau_K \leq s\wedge T_0) + \mathbb{Q}(t\wedge\tau_K \leq s\wedge T_0)\right]
\end{aligned}$$

by (1.30). Now we use (1.31) and (1.32) to conclude

$$\limsup_n |\mathbb{Q}^n(\Phi) - \mathbb{Q}(\Phi)| \leq 4\,\|\Phi\|_\infty\, \mathbb{Q}(t\wedge\tau_K \leq s\wedge T_0).$$

Since $s < t$ and $\tau_K \nearrow T_0$, we deduce

$$\mathbb{Q}_t^n(\Phi) \to \mathbb{Q}_t(\Phi).$$

Hence, we arrive at

$$\mathbb{Q}_t^n|_{\mathcal{F}_{s\wedge\ T_0}} \twoheadrightarrow \mathbb{Q}_t|_{\mathcal{F}_{s\wedge\ T_0}}$$

for any $s < t$. Observe that

$$\{T_0 \geq s\} = \{T_0 \wedge s = s\} = \Omega \setminus \bigcup_{q\text{ rat.},\, q\leq s} \{T_0\wedge s < q\} \in \mathcal{F}_{s\wedge T_0}.$$

Therefore, we may choose $\Phi = \mathbb{1}_{\{T_0\geq s\}}$ and get

$$\mathbb{P}_x(T_0(\boldsymbol{a}_n) \geq s) = \mathbb{Q}_t^n(\Phi) \to \mathbb{Q}_t(\Phi) = \mathbb{P}_x(T_0(\boldsymbol{a}) \geq s).$$

■

remark 1.12. Some words on the linkage between the $L^2(m)$-generator $\mathbf{L}^{0,\infty}(\boldsymbol{a})$ of appendix A.2.1 and the measures $\mathbb{Q}_t$ from the above proof:

First of all, compare [Kuo06] section 8.7 and [MU12] page 4 to get that f_t is a local martingale.

Once we have this, we may use Theorem 2.1 of [MU12] to get that f_t is a real martingale. Thus, the above 'Girsanov change-of-measure' by f_t is valid. ([MU12] also give sufficient conditions for a family of 'Girsanov densities' to be uniformly integrable.) Furthermore, there is a corresponding unique weak solution of $dY_t = dX_t + \boldsymbol{a}(Y_t)dt$. In fact, the *Girsanov theorem* itself gives a 'construction' of such a solution: Plainly, $\mathbb{Q}_t$ is the measure of $(Y_s)_{s\leq t}$. See [Kuo06] section 8.9. Any such unique weak solution gives also a unique solution to the corresponding martingale problem and vice versa. See for example section 5.4 in [KS88]. And the martingale problem is the essential link between the process, respectively weak solution up to killing at zero, and the $L^2(m)$-generator $\mathbf{L}^{0,\infty}(\boldsymbol{a})$ of appendix A.2.1. Compare also section 2.2 of [KS12]. ∎

With the above lemma we want to compare survival probabilities via comparison of drifts.

lemma 1.19. *If $\boldsymbol{a}$ and $\boldsymbol{b}$ are drift coefficients from $C(0,\infty)$ satisfying A 8 and we have $\boldsymbol{a} \leq \boldsymbol{b}$ on $(0,\infty)$, then*

$$\mathbb{P}_x(T_0(\boldsymbol{a}) \geq s) \leq \mathbb{P}_x(T_0(\boldsymbol{b}) \geq s)$$

for all $x, s > 0$.

For the proof we will use a *coupling* which will be indicated by the superscript $(\cdot)'$. (A good reference is [Lin92].)

proof of lemma 1.19. We first prove this for the special case of $\boldsymbol{a}$ and $\boldsymbol{b}$ locally Lipschitz continuous:

By a coupling $(Y'(\boldsymbol{a}), Y'(\boldsymbol{b}))$ of $Y(\boldsymbol{a})$ and $Y(\boldsymbol{b})$ such that $Y'_t(\boldsymbol{a}) \leq Y'_t(\boldsymbol{b})$ for all $t \geq 0$, we conclude

$$\mathbb{P}_x(\tau^\infty(Y'(\boldsymbol{a})) \geq t) \leq \mathbb{P}_x(\tau^\infty(Y'(\boldsymbol{b})) \geq t).$$

(Of course, here $\tau^\infty = T_0$. i.e. the first hitting time of zero.)

In [Lin92] coupling is formulated for $\boldsymbol{a}$ and $\boldsymbol{b}$ Lipschitz continuous. But we can use it here, if we only assume that $\boldsymbol{a}$ and $\boldsymbol{b}$ are locally Lipschitz continuous:

We first look at processes $Y(\boldsymbol{a}_n)$ and $Y(\boldsymbol{b}_n)$ were $\boldsymbol{b}_n = \boldsymbol{b}$ and $\boldsymbol{a}_n = \boldsymbol{a}$ on $[0,n]$ but are globally lipschitz and $\boldsymbol{a}_n \leq \boldsymbol{b}_n$ on whole $\mathbb{R}$. Thus, we have a coupling $(Y'(\boldsymbol{a}_n), Y'(\boldsymbol{b}_n))$ with $Y'_t(\boldsymbol{a}_n) \leq Y'_t(\boldsymbol{b}_n)$. This is in particular true if we stop at $\sigma_n := T_0(Y'(\boldsymbol{a}_n)) \wedge T_n(Y'(\boldsymbol{b}_n))$, giving

$$\mathbb{P}_x(Y'_{t\wedge\sigma_n}(\boldsymbol{a}_n) \leq Y'_{t\wedge\sigma_n}(\boldsymbol{b}_n) \text{ for all } t) = 1$$

for all n. But up to the stopping time σ_n we have that $Y'(\boldsymbol{a}_n)$ and $Y'(\boldsymbol{b}_n)$ are the unique solutions to $dY_t = dX_t + \boldsymbol{a}(Y_t)dt$ and $dY_t = dX_t + \boldsymbol{b}(Y_t)dt$ under $Y_0 = x$. Hence, up to σ_n they are $Y(\boldsymbol{a})$ and $Y(\boldsymbol{b})$. This gives

$$\mathbb{P}_x(Y_{t\wedge\sigma_n}(\boldsymbol{a}) \leq Y_{t\wedge\sigma_n}(\boldsymbol{b}) \text{ for all } t, n) = 1.$$

Of course, the stopping times involved are also the stopping times corresponding to $Y(\boldsymbol{a})$ and $Y(\boldsymbol{b})$, i.e.

$$\sigma_n = T_0(Y(\boldsymbol{a})) \wedge T_n(Y(\boldsymbol{b})).$$

Since ∞ is natural for $\boldsymbol{b}$ (and $\boldsymbol{a}$), we know that $T_n(Y(\boldsymbol{b})) \to \infty$ a.s.. But then we have

$$\mathbb{P}_x(Y_{t\wedge T_0(Y(\boldsymbol{a}))}(\boldsymbol{a}) \leq Y_{t\wedge T_0(Y(\boldsymbol{a}))}(\boldsymbol{b}) \text{ for all } t) = 1.$$

This implies $\mathbb{P}_x(T_0(Y(\boldsymbol{a})) \leq T_0(Y(\boldsymbol{b}))) = 1$, which finally gives $\mathbb{P}_x(\tau^\infty(\boldsymbol{a}) \geq t) \leq \mathbb{P}_x(\tau^\infty(\boldsymbol{b}) \geq t)$.

Now let $\boldsymbol{a} \leq \boldsymbol{b}$ be only in $C(0,\infty)$ with zero accessible: Then we can find (and take) $\boldsymbol{a}_n$ and $\boldsymbol{b}_n$ locally lipschitz with $(\boldsymbol{a}_n)_n$ and $(\boldsymbol{b}_n)_n$ uniformly bounded on compacts such that $\boldsymbol{a}_n \to \boldsymbol{a}$ and $\boldsymbol{b}_n \to \boldsymbol{b}$ pointwise on $(0,\infty)$. But then $\boldsymbol{a}_n \wedge \boldsymbol{b}_n \to \boldsymbol{a}$ and $\boldsymbol{a}_n \wedge \boldsymbol{b}_n$ also fulfils the assumptions from lemma 1.18. Hence, there is no loss of generality in assuming $\boldsymbol{a}_n \leq \boldsymbol{b}_n$ for each n. In the end we only have to use lemma 1.18, to get

$$\mathbb{P}_x(T_0(\boldsymbol{a}) \geq s) = \lim_n \mathbb{P}_x(T_0(\boldsymbol{a}_n) \geq s) \leq \lim_n \mathbb{P}_x(T_0(\boldsymbol{b}_n) \geq s) = \mathbb{P}_x(T_0(\boldsymbol{b}) \geq s).$$

■

proposition 1.20. *Suppose the drift* $\boldsymbol{b}$ *satisfies A 1, A 2, A 3 and A 8. Let* $\kappa = 0$ *and let* $\boldsymbol{a}$ *be another drift coefficient which satisfies A 1, A 2 and A 8. Further suppose that* $\alpha > 0$*. Then*

$$\boldsymbol{a} \leq \boldsymbol{b}$$

implies

$$\exists\,\text{QLD for } \boldsymbol{b} \Rightarrow \exists\,\text{QLD for } \boldsymbol{a}\,.$$

Before proving proposition 1.20 let us make the following observation:

remark 1.13. Suppose we are only interested in obtaining the above result in the case $\alpha = \infty$, i.e. $\tau = T_0$. In this case we may use directly lemma 1.19 to see that

$$\mathbb{E}_x\,\mathrm{e}^{\epsilon T_0(\boldsymbol{a})} = \int_1^\infty \mathbb{P}_x(\mathrm{e}^{\epsilon T_0(\boldsymbol{a})} \geq s)ds \leq \int_1^\infty \mathbb{P}_x(\mathrm{e}^{\epsilon T_0(\boldsymbol{b})} \geq s)ds = \mathbb{E}_x\,\mathrm{e}^{\epsilon T_0(\boldsymbol{b})}\,.$$

Now we only have to use the well fitting criterion for the existence of a QSD shown in Theorem 4.14 of [KS12]. ▮

proof of proposition 1.20. The main argument will be Corollary 1.9. Obviously we have that $\underline{\lambda}^\alpha(\boldsymbol{b}) > 0$. By 1.8 we know that

$$\underline{\lambda}^\alpha(\boldsymbol{b}) = \eta^\alpha(\boldsymbol{b}) = -\lim \frac{\log \mathbb{P}_x(\tau(\boldsymbol{b}) > t)}{t} \leq -\lim \frac{\log \mathbb{P}_x(\tau^\infty(Y(\boldsymbol{b})) > t)}{t} = \eta^\infty(\boldsymbol{b})$$

for any $\alpha \in (0,\infty]$.

Now we use a comparison technique as in the proof of 1.15 to get

$$0 < \eta^\infty(\boldsymbol{b}) \leq \eta^\infty(\boldsymbol{a}).$$

Though here we have to use lemma 1.19 instead of 1.14. Next assume we would have $\underline{\lambda}^\alpha(\boldsymbol{a}) = 0$. As in the proof of Lemma 3.3 (iii) of [KS12] we see that the essential spectra of $\mathbf{L}^\alpha(\boldsymbol{a})$ and $\mathbf{L}^\infty(\boldsymbol{a})$ coincide. (This can be seen through Th 10.17 of [Wei00] which states that any 'resolvent-difference' of two self-adjoint extensions is compact if the defect indices of the 'generating' symmetric operator coincide.) Since, by Lemma 3.3 (vi), 0 is no isolated eigenvalue, the only possibility left is that $\underline{\lambda}^\infty(\boldsymbol{a}) = 0$, too. Finally we use lemma 1.21 to conclude that $\eta^\infty(\boldsymbol{a}) = 0$ giving a contradiction.

Now that we have $\underline{\lambda}^\alpha(\boldsymbol{a}) > 0$ we again use corollary 1.9 to obtain the existence of a quasi-limiting distribution for $\boldsymbol{a}$. ■

The following lemma is a generalization of Corollary 6.25 of [CMSM13b] and uses the same idea of the proof. (Note that the authors of [CMSM13b] assume $\boldsymbol{a} \in C^1$. In that case we could as well use Theorem 1 of [MSM01] in the proof above.) Though, in order to prove this needed lemma 1.21 we make one additional assumption. For this let $p_t(x,y)$ be the kernel of Y under τ w.r.t. the speed measure $m(dx)$.

A 9. We have that $y \mapsto p_t(x,y)$ is square-integrable w.r.t. m for any $x \in E$ and $t \geq 0$.

remark 1.14. One sees immediately, that Brownian motion with constant drift and the Ornstein-Uhlenbeck process (both under instant killing at zero) carry this property. More general, it holds under the C^1-assumption of the drift coefficient. For a proof see [MV12] and [CCL+09]. ∎

lemma 1.21. *Suppose the drift* $\boldsymbol{a}$ *satisfies A 1, A 2, A 3 and A 9. Then*

$$\limsup_{t\to\infty} -\frac{\log \mathbb{P}_x(T_0 > t)}{t} \leq \underline{\lambda}^{0,\infty}.$$

To show this we are going to use *Weyl's spectral theorem* in the version of theorem A.2.2.

proof. First of all, take any measurable and bounded $B \subset \mathbb{R}_+$ and some $\delta > 0$. Then we have (by the Markov property)

$$\mathbb{P}_x(Y_{t+\delta} \in B,\, \tau > t+\delta) = \mathbb{P}_x(\tau > \delta) \cdot \mathbb{P}_\nu(Y_t \in B,\, \tau > t) = \mathbb{P}_x(\tau > \delta) \int e^{t\mathbf{L}} \mathbb{1}_B(y)\nu(dy),$$

where $\nu(dy) = \frac{p_\delta(x,y)m(dy)}{\int p_\delta(x,z)m(dz)}$. $p_t(x,y)$ is the transition kernel of Y under τ (w.r.t. m).

Next we use Weyl's spectral theorem with $F(\lambda) = e^{\lambda t}$. Thus, we arrive at

$$\mathbb{P}_x(Y_{t+\delta} \in B,\, \tau > t+\delta) = \mathbb{P}_x(\tau > \delta) \cdot \int_0^\infty \int_{-\infty}^{-\underline{\lambda}} e^{\lambda t} \int_0^\infty \mathbb{1}_B(z)\psi_{-\lambda}(z)m(dz)\psi_{-\lambda}(y)\varrho(d\lambda)\nu(dy).$$

Now take $\epsilon > 0$ and split the above expression at the 'ϱ-integral' into the parts

$$I_1 := \mathbb{P}_x(\tau > \delta) \cdot \int_0^\infty \int_{(-(\underline{\lambda}+\epsilon),-\underline{\lambda}]} e^{\lambda t} \int_0^\infty \mathbb{1}_B(z)\psi_{-\lambda}(z)m(dz)\psi_{-\lambda}(y)\varrho(d\lambda)\nu(dy)$$

and

$$I_2 := \mathbb{P}_x(\tau > \delta) \cdot \int_0^\infty \int_{(-\infty,-(\underline{\lambda}+\epsilon)]} e^{\lambda t} \int_0^\infty \mathbb{1}_B(z)\psi_{-\lambda}(z)m(dz)\psi_{-\lambda}(y)\varrho(d\lambda)\nu(dy).$$

By Weyl's spectral theorem A.2.2 and remark A.2.4, we know that $\psi_{\underline{\lambda}}$ is a multiple of $\underline{\psi}$. Hence, both functions are positive on $(0,\infty)$. By the continuity of $(\lambda, x) \mapsto \psi_\lambda(x)$ (see Theorem 1.7.4 of [CL55]) we can take ϵ and '$B > 0$' so small that we can find some $c > 0$, such that:

(i) $\int \mathbb{1}_B(z)\psi_\lambda(z)m(dz) > c$ for all $\lambda \in [\underline{\lambda}, \underline{\lambda}+\epsilon]$.

Remember some natural main assumption from section 1.1:

$$\mathbb{P}_x(T_0 > t) > 0$$

for all $t \geq 0$ and some (, hence, each) $x > 0$ (in order to give meaning to the condition of asymptotic survival.) With this in mind, we see that at least for $\delta > 0$ small enough, there exists an interval $[u, v]$, $u > 0$, such that

$$\mathbb{P}_x(Y_\delta \in [u, v],\, \tau > \delta) > 0.$$

(Observe that $\mathbb{P}_x(Y_\delta > \frac{x}{2},\, \tau > \delta) \geq \mathbb{P}_x(\inf_{s\in[0,\delta]} Y_s > \frac{x}{2},\, \tau > \delta) \geq \mathbb{P}_x(T_{\frac{x}{2}} > \delta) > 0$, for $\delta > 0$ small enough, since $\mathbb{P}_x(T_{\frac{x}{2}} > \delta) \to \mathbb{P}_x(T_{\frac{x}{2}} > 0) = 1$. Because of $\lim_{v\to\infty} \mathbb{P}_x(Y_\delta \in [\frac{x}{2}, v],\, \tau > \delta) = \mathbb{P}_x(Y_\delta > \frac{x}{2},\, \tau > \delta)$ there must be some v such that $\mathbb{P}_x(Y_\delta \in [\frac{x}{2}, v],\, \tau > \delta) > 0$.)

Thus, $p_\delta(x, y)$ must be positive at some $y_0 \in [u, v]$. Now use (3.6) of [KS12] with $K = [\frac{x}{2}, \frac{3x}{2}] \times [u, v]$ and $\delta = T$ to get the existence of a $\gamma > 0$ such that

$$\gamma p_\delta(x, y_0) \leq p_\delta(x, y)$$

for all $y \in [u, v]$. In fact, we may choose $c > 0$ from (i) such that

(ii) $\int \psi_\lambda(y)\nu(dy) \geq \int_u^v \psi_\lambda(y)\nu(dy) > c$ for all $\lambda \in [\underline{\lambda}, \underline{\lambda} + \epsilon]$.

Thus, by (i) and (ii),

$$\begin{aligned} I_1 &\geq c^2 \mathbb{P}_x(\tau > \delta) \int_{(-(\underline{\lambda}+\epsilon), \underline{\lambda}]} e^{\lambda t}\Gamma(d\lambda) \\ &\geq c^2 \mathbb{P}_x(\tau > \delta) \int_{(-(\underline{\lambda}+\frac{\epsilon}{2}), \underline{\lambda}]} e^{\lambda t}\Gamma(d\lambda) \\ &\geq c^2 \mathbb{P}_x(\tau > \delta) e^{-(\underline{\lambda}+\frac{\epsilon}{2})t}\Gamma((-(\underline{\lambda} + \frac{\epsilon}{2}), \underline{\lambda}]) \end{aligned}$$

Since $\underline{\lambda}$ is the largest point of increase for ϱ we know that $\varrho(-(\underline{\lambda} + \frac{\epsilon}{2}), -\underline{\lambda}]) > 0$ for all $\epsilon > 0$. Therefore,

$$\limsup_{t\to\infty} -\frac{\log(I_1)}{t} \leq \underline{\lambda} + \frac{\epsilon}{2}.$$

Next we show that I_2 has a faster exponential decay, then I_1. This would imply that

$$\mathbb{P}_x(Y_{t+\delta} \in B,\, \tau > t + \delta) \propto I_1(t)$$

and finally give us

$$\begin{aligned} \eta &= \lim_{t\to\infty} -\frac{\log \mathbb{P}_x(\tau > t)}{t} \\ &\leq \limsup_{t\to\infty} -\frac{\log \mathbb{P}_x(Y_{t+\delta} \in B,\, \tau > t + \delta)}{t} \\ &\leq \limsup_{t\to\infty} -\frac{\log(I_1)}{t} \\ &\leq \underline{\lambda} + \frac{\epsilon}{2} \end{aligned}$$

for each $\epsilon > 0$ small enough. This would prove $\eta \leq \underline{\lambda}$.

To prove the faster decay of I_2, we cancel out the normalising constant $\mathbb{P}_x(\tau > \delta)$. Next we introduce the measures $\tilde{\varrho}(t)$ by $\frac{d\tilde{\varrho}}{d\varrho}(\lambda) = e^{\lambda t}\mathbb{1}_{(-\infty,-(\underline{\lambda}+\epsilon)]}(\lambda)$. Now I_2 is nothing else then

$$\Big\langle \int \psi_{-\lambda}(y)p_\delta(x,y)m(dy), \int \mathbb{1}_B(y)\psi_{-\lambda}(y)m(dy)\Big\rangle_{L^2(\tilde{\varrho})}.$$

Hence, by Cauchy-Schwartz,

$$\begin{aligned} I_2^2 &\le \left\| \int p_\delta(x,y)\psi_{-\lambda}(y)m(dy)\right\|^2_{L^2(\tilde{\varrho})} \cdot \left\| \int \mathbb{1}_B(y)\psi_{-\lambda}(y)m(dy)\right\|^2_{L^2(\tilde{\varrho})} \\ &\le e^{-(\underline{\lambda}+\epsilon)t} \left\|\mathbf{U}(p_\delta(x,\cdot))\right\|^2_{L^2(\varrho)} \cdot e^{-(\underline{\lambda}+\epsilon)t} \left\|\mathbf{U}(\mathbb{1}_B)\right\|^2_{L^2(\varrho)} \\ &= e^{-2(\underline{\lambda}+\epsilon)t} \left\|p_\delta(x,\cdot)\right\|^2_{L^2(m)} \cdot \left\|\mathbb{1}_B\right\|^2_{L^2(m)}. \end{aligned}$$

■

We may use similar arguments as in the proof of proposition 1.20 to derive a corresponding result for $\kappa \to 0$:

theorem 1.22. *Suppose the drift* $\boldsymbol{b}$ *satisfies A 1, A 2, A 3 and A 8. Let* $\kappa \to 0$ *and let* $\boldsymbol{a}$ *be another drift coefficient, which satisfies A 1, A 2, A 8 and A 9. Then*

$$\boldsymbol{a} \le \boldsymbol{b}$$

implies

$$\exists\,\text{QLD for } \boldsymbol{b} \Rightarrow \exists\,\text{QLD for } \boldsymbol{a}.$$

To prove this we use the following lemma.

lemma 1.23. *Suppose* $\alpha > 0$ *and the drift* $\boldsymbol{b}$ *and the killing rate* κ *satisfy A 1, A 2 and A 3. Then* 0 *is no isolated eigenvalue of* $\mathbf{L}^{\kappa,\alpha}(\boldsymbol{b})$.

Let us stress that we do not need the L^2-conditions on the kernel and on the drift for the proof of the lemma.

proof of lemma 1.23. The proof is essentially the same as the one from Lemma 3.3 point (vi) of [KS12]. (Here the authors assumed $\kappa \equiv 0$.) The authors did some calculations based on a spectral resolution for $\mathbf{L}^{\kappa,\alpha}(\boldsymbol{a})$, which are also valid here. Next they used that the principal eigenfunction can be chosen to be everywhere positive under the assumption of a spectral gap. But if one assumes that 0 is isolated, we see that the 'supremum' in (A.2.6) becomes a maximum. So, this is also true in our case. Their proof relies upon their Lemma 2.1 which they showed to hold under our conditions. The last ingredient is that the image of $e^{-t\mathbf{L}}$ is included in the domain of $\mathbf{L}$. But this is also true, as pointed out on page 168 of [KS12]. ■

proof of theorem 1.22. We use theorem 1.12 to conclude that $\underline{\lambda}^{\kappa,\alpha}(\boldsymbol{b}) > 0$. Now suppose we would have $\underline{\lambda}^{0,\infty}(\boldsymbol{b}) = 0$. By lemma 1.23 we conclude that 0 is the supremum of the essential spectrum. See the proof of Lemma 3.3 (iii) of [KS12] observe that the essential spectra of $\mathbf{L}^{0,\alpha}(\boldsymbol{b})$ and $\mathbf{L}^{0,\infty}(\boldsymbol{b})$ coincide. Thus, $\underline{\lambda}^{0,\alpha}(\boldsymbol{b}) = 0$. Since the essential spectra of $\mathbf{L}^{0,\alpha}(\boldsymbol{b})$ and $\mathbf{L}^{\kappa,\alpha}(\boldsymbol{b})$ coincide (Lemma 3.3 (i) of [KS12]) we conclude that $\underline{\lambda}^{\kappa,\alpha}(\boldsymbol{b}) = 0$, giving a contradiction.

Hence, $\underline{\lambda}^{0,\infty}(\boldsymbol{b}) > 0$. Now proposition 1.8 implies that $0 < \eta^{0,\infty}(\boldsymbol{b})$. Next we use lemma 1.19 to get

$$0 < \eta^{0,\infty}(\boldsymbol{b}) \leq \eta^{0,\infty}(\boldsymbol{a}). \tag{1.33}$$

By lemma 1.21 we see that $\underline{\lambda}^{0,\infty}(\boldsymbol{a}) > 0$. Similar as above, we assume $\underline{\lambda}^{\kappa,\alpha}(\boldsymbol{a}) = 0$ giving a contradiction. Hence, $\underline{\lambda}^{\kappa,\alpha}(\boldsymbol{a}) > 0$ and we see that there must be a QSD for $\boldsymbol{a}$, due to theorem 1.12. ■

remark 1.15. Compared with theorem 1.17 we imposed

1. local L^2-integrability of the drift $\boldsymbol{a}$
2. L^2-integrability of the kernel of Y under $\tau = T_0$

to obtain theorem 1.22.

- 1. is due to the fact that we needed to make sure that $(f_t)_t$ from the proof of lemma 1.18 is a proper martingale. Thus, we have used results from [MU12]. (See also remark 1.12.)
- The reason for 2. is the usage of a Cauchy-Schwartz inequality after applying Weyl's spectral theorem A.2.2. (See the proof of lemma 1.21.)

It would be desirable to find easy to verify sufficient conditions for the assumptions 1. or 2. or even weaken them considerably. ▮

The next corollary gives sufficient conditions for the existence of a QLD by comparing with a negative constant drift and then with an Ornstein-Uhlenbeck drift.

corollary 1.24. *Let A 2 and at least one of the following condition be satisfied for* $\boldsymbol{a}$ *and* κ*:*

(i) A 6 is true.

(ii) A 1, A 8 and A 9.

Furthermore, let $\kappa \to 0$ *and let* $\boldsymbol{a}$ *have any of the following properties:*

(iii) $\boldsymbol{a}(x) < -\epsilon$ *for some* $\epsilon > 0$ *all* x*.*

(iv) $\boldsymbol{a}(x) < -bx$ *for some* $b > 0$ *and all* x*.*

Then we have existence of a QLD respectively $\underline{\lambda} > 0$*.*

proof. First of all, we assume that $\kappa \equiv 0$:

One part of the proof consists in seeing that for $\boldsymbol{b}(x) = -\epsilon$ one has $\underline{\lambda}^{\infty,0}(\boldsymbol{b}) = \frac{\epsilon^2}{2} > 0$; e.g. by taking a look in [MSM94]. The second part is using the same trick as in the proof of 1.20: The essential spectra of $\mathbf{L}^{0,\infty}(\boldsymbol{b})$ and $\mathbf{L}^{0,\alpha}(\boldsymbol{b})$ are the same and since 0 is not an isolated eigenvalue, we must have $\underline{\lambda}^{0,\alpha}(\boldsymbol{b}) > 0$. (By (A.2.6) one can also calculate directly $\underline{\lambda}^{0,\alpha}(\boldsymbol{b})$.)

The next part is a simple application of proposition 1.15 in case (i) and proposition 1.20 in case (ii).

As for the Ornstein-Uhlenbeck case:

One can see, respectively calculate, that an Ornstein-Uhlenbeck process satisfies all the assumptions A 1, 2, 3 and 7. Furthermore, the spectrum of $\mathbf{L}^{0,\infty}(-bx)$ is discrete with $\underline{\lambda}^{0,\infty} =$

$b > 0$, respectively we have existence of a QLD; the eigenfunctions are essentially the *Hermite polynomials* of odd order, i.e. the $f_n(x) = H_n(\sqrt{2b}x)$ with $H_n(x) := (-1)^n \mathrm{e}^{\frac{x^2}{2}} \frac{d^n}{dx^n} \mathrm{e}^{-\frac{x^2}{2}}$ and $n \in \mathbb{N} \setminus 2\mathbb{N}$. At this stage we do exactly the same as in the case of a constant drift .

Now let $\kappa \to 0$:

Above we have seen that $\underline{\lambda}^{0,\alpha}(\boldsymbol{b}) > 0$. By Lemma 1.23 we get that 0 is no isolated eigenvalue of $\mathbf{L}^{\kappa,\alpha}(\boldsymbol{b})$. Together with (i) from Lemma 3.3 of [KS12] we arrive at $\underline{\lambda}^{\kappa,\alpha}(\boldsymbol{b}) > 0$.

Now we can apply theorem 1.17 in case (i) and theorem 1.22 in case (ii). ■

Of course we could compare with other $\boldsymbol{b}$ then constant drift and Ornstein-Uhlenbeck drift. One can also raise the following question: If we have a QLD for $\boldsymbol{a}(x)$ and $\boldsymbol{b}(x)$, what can we say about $(\boldsymbol{a}\vee\boldsymbol{b})(x)$? For instance, the next result tells us that a QLD exists for the drift coefficient $-\epsilon \vee (-bx)$ with $\epsilon, b > 0$.

corollary 1.25. *Let the drift coefficients* $\boldsymbol{a}$ *and* $\boldsymbol{b}$ *and the killing rate* κ *fulfil A 1, A 2, A 3 and* $\kappa \to 0$. *Let* $\alpha > 0$. *Furthermore, let* $\boldsymbol{a}$ *and* $\boldsymbol{b}$ *be integrable at zero and let one of them eventually dominate the other, e.g.* $\boldsymbol{a}(x) \leq \boldsymbol{b}(x)$ *for* x *large enough. Then*

$$\exists\,\text{QLD for } \boldsymbol{a} \text{ and } \boldsymbol{b} \Rightarrow \exists\,\text{QLD for } \boldsymbol{a}\wedge\boldsymbol{b} \text{ and } \boldsymbol{a}\vee\boldsymbol{b}.$$

proof. By theorem 1.12 we have that $\underline{\lambda}^{\alpha,\kappa}(\boldsymbol{a}) \wedge \underline{\lambda}^{\alpha,\kappa}(\boldsymbol{b}) > 0$. Thus, the supremum of the essential spectra of $\mathbf{L}^{\kappa,\alpha}(\boldsymbol{a})$ and $\mathbf{L}^{\kappa,\alpha}(\boldsymbol{b})$ is less then zero. But the essential spectra of these operators are identical to the essential spectra of $\mathbf{L}^{0,\infty}(\boldsymbol{a})$ and $\mathbf{L}^{0,\infty}(\boldsymbol{b})$. By theorem A.2.7 (using the notation therein) these essential spectra are the same as the ones from $\mathbf{L}_M(\boldsymbol{a})$ and $\mathbf{L}_M(\boldsymbol{b})$ for any $M > 0$. W.l.o.g. let $\boldsymbol{a}(x) \leq \boldsymbol{b}(x)$ for $x \geq M$ be true. Thus, $\mathbf{L}_M(\boldsymbol{a}) = \mathbf{L}_M(\boldsymbol{a}\wedge\boldsymbol{b})$ and $\mathbf{L}_M(\boldsymbol{b}) = \mathbf{L}_M(\boldsymbol{a}\vee\boldsymbol{b})$. Vice versa the essential spectra of $\mathbf{L}^{0,\infty}(\boldsymbol{a}\wedge\boldsymbol{b})$ and of $\mathbf{L}^{0,\infty}(\boldsymbol{a}\vee\boldsymbol{b})$ and, therefore, also the ones from $\mathbf{L}^{\kappa,\alpha}(\boldsymbol{a}\wedge\boldsymbol{b})$ and $\mathbf{L}^{\kappa,\alpha}(\boldsymbol{a}\vee\boldsymbol{b})$ have a supremum smaller than zero. Since 0 is no isolated eigenvalue by lemma 1.23 we get $\underline{\lambda}^{\alpha,\kappa}(\boldsymbol{a}\wedge\boldsymbol{b}) \wedge \underline{\lambda}^{\alpha,\kappa}(\boldsymbol{a}\vee\boldsymbol{b}) > 0$. Theorem 1.12 gives us the existence of the QLDs for $\boldsymbol{a}\wedge\boldsymbol{b}$ and $\boldsymbol{a}\vee\boldsymbol{b}$.

Of course, we need that 0 is regular and ∞ is natural to apply 1.12 and 1.23:

Infinity is obviously natural (for $\boldsymbol{a}\wedge\boldsymbol{b}$ and $\boldsymbol{a}\vee\boldsymbol{b}$) because it is assumed for $\boldsymbol{a}$ and $\boldsymbol{b}$. Zero is regular since

$$\begin{aligned}\left|\int_x^c (\boldsymbol{a}\vee\boldsymbol{b}(z) - \boldsymbol{a}(z))dz\right| &\leq \left|\int_x^c (\boldsymbol{a}\vee\boldsymbol{b}(z) - \boldsymbol{a}\wedge\boldsymbol{b}(z))dz\right| \\ &\leq \int_x^c |\boldsymbol{a}(z) - \boldsymbol{b}(z)|\,dz \\ &\leq \int_0^c |\boldsymbol{a}(z)|\,dz + \int_0^c |\boldsymbol{b}(z)|\,dz < \infty\end{aligned}$$

for any $c > 0$ and $x \in [0, c]$. This leads to

$$\begin{aligned}&\int_0^c \left(\int_x^c \exp^{-2\int_y^c \boldsymbol{a}\vee\boldsymbol{b}(z)dz}\,dy\right) \exp^{2\int_x^c \boldsymbol{a}\vee\boldsymbol{b}(z)dz}\,dx \\ \leq &\int_0^c \left(\int_x^c \exp^{-2\int_y^c \boldsymbol{a}(z)dz}\,dy\right) \exp^{2\int_x^c \boldsymbol{a}(z)dz} \left[\exp^{2\int_x^c (\boldsymbol{a}\vee\boldsymbol{b}(z) - \boldsymbol{a}(z))dz}\right] dx \\ <&\infty\end{aligned}$$

and the same trick applies to show that

$$\int_0^c \left(\int_x^c \exp^{2\int_y^c \boldsymbol{a}\vee \boldsymbol{b}(z)dz} dy \right) \exp^{-2\int_x^c \boldsymbol{a}\vee \boldsymbol{b}(z)dz} dx < \infty$$

and

$$\int_0^c \left(\int_x^c \exp^{\pm 2\int_y^c \boldsymbol{a}\wedge \boldsymbol{b}(z)dz} dy \right) \exp^{\mp 2\int_x^c \boldsymbol{a}\wedge \boldsymbol{b}(z)dz} dx < \infty.$$

■

Actually the proof above is more easy for $\boldsymbol{a}\wedge\boldsymbol{b}$ (once we know that 0 is regular and ∞ is natural). We only have to use theorem 1.22.

1.6. If the state space is a finite interval

Finally we want to state some existence result for a Yaglom limit in case that Y lives in a bounded region, i.e. in $[0, c]$. This result can be found in [SE07]. Essentially c is just another boundary point of the kind like zero. This means we impose similar boundary conditions, such that, besides reflection at c, there may be (slow) killing, and its 'rate' is parametrized by some $\beta > 0$. (See section 0.4.)

proposition 1.26. *Let the drift* $\boldsymbol{a} \in C^1(0, c)$*. Let* 0 *and* c *be regular. Furthermore, let the killing rate* $\kappa \in C[0, c]$*. Then we have*

$$\mathrm{e}^{\underline{\lambda} t}\, \mathbb{P}_\nu(Y_t \in A, \tau > t) \longrightarrow \frac{\int_0^c \underline{\varphi}(y)/m(y)\nu(dy)}{\int_0^c \underline{\varphi}(y)^2/m(y)dy} \int_A \underline{\varphi}(y)dy$$

for all measurable $A \subseteq [0, c]$ *and distributions* ν *on* $[0, c]$*.*

Of course we get (the normalized version of) $\underline{\varphi}$ as Yaglom limit or unique QLD, respectively:

$$\mathbb{P}_\nu(Y_t \in A \mid \tau > t) = \frac{\mathrm{e}^{\underline{\lambda} t}\, \mathbb{P}_\nu(Y_t \in A, \tau > t)}{\mathrm{e}^{\underline{\lambda} t}\, \mathbb{P}_\nu(Y_t \in [0, c], \tau > t)} \to \frac{\int_A \underline{\varphi}(y)dy}{\int_0^c \underline{\varphi}(y)dy}.$$

remark 1.16. If we would only be interested in the asymptotic behaviour of $\mathbb{P}_\nu(Y_t \in A, \tau > t)$ if $\tau = T_0 \wedge T_c$, then we can use a representation of the transition kernel (w.r.t. $m(dy)$) due to [CMSM13b] proposition 6.2:

$$\sum_{n=1}^N \mathrm{e}^{-\lambda_n t}\, \psi_n(x)\psi_n(y) \to p_t(x, y), \tag{1.34}$$

where the convergence is absolute and uniform on $[t_0, \infty) \times [0, c]^2$ for every $t_0 > 0$. Intuitively (1.34) gives us some sort of spectral representation of the corresponding semigroup $(\mathbf{T}_t)_t$:

$$\mathbf{T}_t h(x) = \int \mathrm{e}^{-\lambda t}\, \psi_\lambda(x)\psi_\lambda(y)\varrho(d\lambda) \int h(y)m(dy) \tag{1.35}$$

with ϱ being the counting measure on $\{\lambda_1, \lambda_2, \ldots\}$. Of course, we should mention:

- $0 < \lambda_1 < \lambda_2 < \dots$ constitute the $L^2(m)$-spectrum of $-\mathbf{L}$, which is simple and discrete.
- The ψ_n are the normalized $L^2(m)$-eigenfunctions for λ_n and we use the notation $\underline{\lambda} = \lambda_1$ as well as $\underline{\psi} = \psi_1$ for the principal eigenvalue and eigenfunction. (This notation was already introduced at the end of the example of 1.2.)

Note that in the proof of [CMSM13b] it is presumed that $\boldsymbol{a} \in C^1[0, c]$. (Mainly due to the extensive use of results from the theory of boundary value problems.) We also point out that ϱ is some sort of analogue to the measure in Weyl's spectral theorem A.2.2. Though, we may not be able to "normalise" the ψ_λ (and, therefore, ϱ) in such a way that $\psi'_\lambda(0) = \psi'_\lambda(c) = 1$. (Only, if $\boldsymbol{a}$ is antisymmetric around $\frac{c}{2}$!)

Now we want to see point (i) of Th6.4 from [CMSM13b], namely:

$$\mathrm{e}^{\underline{\lambda} t}\, p(t, x, y) \to \underline{\psi}(x)\underline{\psi}(y) \tag{1.36}$$

uniformly on $[0, c]^2$.

proof of (1.36).

$$\left|\mathrm{e}^{\underline{\lambda} t}\, p(t, x, y) - \underline{\psi}(x)\underline{\psi}(y)\right| = \left|\sum_{n\geq 2} \mathrm{e}^{(\underline{\lambda}-\lambda_n)t}\, \psi_n(x)\psi_n(y)\right|.$$

Now we know by (1.34) that $\sum_{n=2}^{N} \mathrm{e}^{\lambda_1-\lambda_n} |\psi_n(x)|\,|\psi_n(y)|$ converges uniformly on $[0, c]^2$. Therefore, we have

$$\forall x, y : \quad \sum_{n\geq 2} \mathrm{e}^{\lambda_1-\lambda_n} |\psi_n(x)|\,|\psi_n(y)| \leq M < \infty.$$

(Be aware that the above series is the uniform limit of continuous functions on a compact set.) This gives

$$\begin{aligned}\left|\mathrm{e}^{\underline{\lambda} t}\, p(t, x, y) - \underline{\psi}(x)\underline{\psi}(y)\right| &\leq \sum_{n\geq 2} \mathrm{e}^{(\lambda_1-\lambda_n)t} |\psi_n(x)|\,|\psi_n(y)| \\ &\leq \mathrm{e}^{(\lambda_1-\lambda_2)(t-1)} \sum_{n\geq 2} \mathrm{e}^{\lambda_1-\lambda_n} |\psi_n(x)|\,|\psi_n(y)| \\ &\leq \mathrm{e}^{(\lambda_1-\lambda_2)(t-1)} \cdot M \to 0 \end{aligned} \tag{1.37}$$

■

Let us see now that

$$\mathrm{e}^{\underline{\lambda} t}\, \mathbb{P}_\nu(h(Y_t); \tau > t) \to \int\int h(y)\underline{\psi}(x)\underline{\psi}(y) m(dy)\nu(dx) \tag{1.38}$$

for all $h \in L^2(m)$ and each initial ν.

proof of (1.38). This is mainly due to the estimate in (1.37) (, i.e. the uniform convergence in

(1.36),) since:

$$
\begin{aligned}
&\left| e^{\underline{\lambda} t}\, \mathbb{P}_\nu(h(Y_t); \tau > t) - \int\int h(y)\underline{\psi}(x)\underline{\psi}(y)m(dy)\nu(dx) \right| \\
=&\left| \int\int h(y) \sum_{n\geq 2} e^{(\lambda_1-\lambda_n)t}\, \psi_n(x)\psi_n(y)m(dy)\nu(dx) \right| \\
\leq &\int\int |h(y)| \sum_{n\geq 2} e^{(\lambda_1-\lambda_n)t}\, |\psi_n(x)|\, |\psi_n(y)|\, m(dy)\nu(dx) \\
\leq &\, e^{(\lambda_1-\lambda_2)(t-1)}\, M \int\int |h(y)|\, m(dy)\nu(dx) \\
=&M\, e^{(\lambda_1-\lambda_2)(t-1)}\, \|h\|_{L^1(m)} \to 0.
\end{aligned}
$$

The limit in the last line holds because of $\boldsymbol{a} \in C^1[0,c]$, which leads to m being finite and $\|h\|_{L^1(m)} < \infty$. ■

Finally (1.38) gives the same as proposition 1.26 under $\tau = T_0 \wedge T_c$:

$$
e^{\underline{\lambda} t}\, \mathbb{P}_\nu(Y_t \in A, \tau > t) \to \int_0^c \underline{\psi}(x)\nu(dx) \cdot \int_A \underline{\psi}(y)m(dy).
$$

Reminder: $\underline{\varphi} = \underline{\psi} m$ is the unique QLD, up to a constant multiple.

In the setting of [CMSM13b], $\underline{\psi} = \underline{\varphi}/m$ is normalized. Thus, $\|\underline{\psi}\|_{L^2(m)} = \int \underline{\varphi}^2(y)/m(y)dy = 1!$ ▮

2. Penalisation results

In the last chapter we gave some results concerning the existence and appearance of $\lim_{t\to\infty} \mathbb{P}_x(Y_t \in \bullet \mid \tau > t)$ for certain diffusions Y and killing times τ. In this chapter we will continue the investigation of Y under 'infinite survival'. But this time we will not condition Y_t on $\{\tau > t\}$ and let t tend to infinity. Instead, we condition $(Y_r)_{r\le s}$ for any fixed s on $\{\tau > t\}$ an let $t \to \infty$.

This chapter is organized as follows.

In section 2.1 we will put this in a more general framework due to Roynette, Vallois and Yor. (See [RY09].) The idea is to give paths a lower weight which are responsible for an undesired event, i.e. we 'penalise' these paths. For instance we want to avoid that Y hits zero. For this we translate 'not hitting zero up to time t' into weighing/penalising with

$$\Gamma_t = \begin{cases} 1 & , T_0 > t \\ 0 & , T_0 \le t\,. \end{cases} \tag{2.1}$$

And as a second step we let t tend to ∞ to obtain a so-called 'penalisation limit'.

In sections 2.2 and 2.3 we mainly deal with diffusions Y as in chapter 1 which are solutions to an SDE

$$dY_t = dX_t + \boldsymbol{a}(Y_t)dt$$

They will be penalised with processes $(\Gamma_t)_t$ similar to (2.1). But T_0 can be replaced by a more general killing time $\tau = \tau^{\kappa,\alpha(,\beta)}$. In section 2.2 we will prove the existence of certain penalisation limits. It will turn out in section 2.3 that all these penalisation limits are again distributed like diffusions Z which

(a) have a generator of the form

$$\frac{1}{\psi}\mathbf{L}\psi.$$

Here the function ψ is a positive eigenfunction for $\mathbf{L} = \mathbf{L}^{\kappa,\alpha(,\beta)}$, the generator of Y under τ.

(b) solve

$$dZ_t = dX_t + \left[\boldsymbol{a}(Z_t) + \frac{\psi'(Z_t)}{\psi(Z_t)}\right] dt$$

on $\mathring{E}$.

The last section 2.4 discusses two topics:

1. We are concerned with so-called 'universal measures' $\mathbb{U}$. A lot of interesting properties of these universal measures are known in the case of Brownian motion. (See [NRY09] and references therein.) Our interest lies in the fact that one can penalise $\mathbb{U}$ directly with Γ_∞ *without any limiting procedure* and the result is the penalisation limit. (See (2.36).)

2. Other mathematicians like Profeta also proved penalisation results for certain diffusions. (See [Pro10, Pro12].) We give a short overview and compare our results with previously obtained results.

2.1. Introduction to penalisation

As the headline suggests, in this section we will lay the foundation for the rest of the chapter. It is meant to help the reader in understanding as much as to present the main objects and some basic results. The notion and most of the results are adopted from [RY09]. Although some of the following results could also be put into a more general framework we decided to restrict ourselfes to the following set-up. This will allow an immediate transfer to the case of linear diffusions.

Our measure space will be $\Omega = C[0,\infty)$ equipped with $\mathcal{F} = \sigma_{min}(C[0,\infty))$ and $\mathcal{F}_s = \sigma^s_{min}(C[0,\infty))$. By $\sigma_{min}(\Omega)$ we mean *the smallest σ-algebra on Ω such that all components $\Omega \ni \omega \mapsto \omega(t) =: X_t(\omega)$ are measurable.* Likewise $\sigma^s_{min}(\Omega)$ is *the smallest σ-algebra such that all components $(X_r)_{r\leq s}$ are measurable.* (Later on we may need to enlarge our model to supply additional random variables, e.g. exponentially distributed ξ's independent from X.)

The following definition introduces the so-called 'penalising principle'.

definition 2.1. Let $\mathbb{P}$ be any probability on $(\Omega, \mathcal{F})$. For all $t \geq 0$ we take a $\Gamma_t : C[0,\infty) \to (\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+))$ measurable with the additional property that $0 < \mathbb{P}(\Gamma_t) < \infty$. We say that a *penalising principle* holds for $(\Omega, \mathcal{F}, \mathbb{P})$ and Γ whenever

$$\mathbb{S}^t_s(f) := \frac{\mathbb{P}(f \cdot \Gamma_t)}{\mathbb{P}(\Gamma_t)} \to \mathbb{S}_s(f)$$

for all $s \geq 0$ and $f \in bm(\Omega, \mathcal{F}_s)$.

In this context we occasionally talk about processes Γ as *penalisers*.

For convenience think of $\mathbb{P}$ as the image measure of $\mathbb{P}_x$ under Y. Here Y is some linear diffusion *without killing* under $(\mathbb{P}_x)_{x\in E}$ (as in the beginning of chapter 0).

convention 2.2. If $\mathbb{P}$ is a probability distribution on some measure space $(\Omega, \mathcal{F})$ and $Z \geq 0$ a real random variable on $(\Omega, \mathcal{F})$ with $0 < \mathbb{P}(Z) < \infty$, then

$$\mathbb{Q}(F) := \frac{\mathbb{P}(Z \cdot \mathbb{1}_F)}{\mathbb{P}(Z)}$$

defines a probability on $(\Omega, \mathcal{F})$ and we will often use the notation

$$Z \circ \mathbb{P} := \mathbb{Q}.$$

The penalising principle was introduced by Roynette, Vallois and Yor. For $\mathbb{P}$ being the Wiener measure they proved that it holds for various kinds of penalisers. (See [RY09] and references therein.)

remark 2.1. If the penalising principle is fulfilled, we obtain, as expected, a probability measure $\mathbb{S}_s$ on $(C[0,\infty), \mathcal{F}_s)$. And we get

$$\mathbb{S}_s^t \twoheadrightarrow \mathbb{S}_s$$

which is a sort of convergence we introduce in definition A.1.1. (Compare e.g. [Nev65], SecIV.2.) ∎

proposition 2.1. *Let the penalising principle be fulfilled for* $(\Omega, \mathcal{F}, \mathbb{P})$ *and* Γ. *Then*

(a) we get that

$$\mathbb{S}_s = M_s \circ \mathbb{P}\big|_{\mathcal{F}_s}$$

for each $s \geq 0$. *The process* $(M_s)_s$ *is a positive* $(\mathcal{F}_s)_s$*-martingale with* $\mathbb{P}(M_s) \equiv 1$.

(b) there is a probability measure $\mathbb{S}$ *on* $(\Omega, \mathcal{F})$ *such that* $\mathbb{S}_s = \mathbb{S}\big|_{\mathcal{F}_s}$, *for any* $s \geq 0$. $(\mathbb{S}_s)_s$ *is characterised by the marginal distributions.*

proof. The $\mathbb{S}_s$ are projective, since

$$\mathbb{S}_s(F) = \lim_{t\to\infty} \frac{\mathbb{P}(\mathbb{1}_F \cdot \Gamma_t)}{\mathbb{P}(\Gamma_t)} = \lim_{t\to\infty} \mathbb{S}_r^t(F) = \mathbb{S}_r(F) \tag{2.2}$$

for $r < s$ and each $F \in \mathcal{F}_r$.

(a) Furthermore, we have $\mathbb{S}_s^t \ll \mathbb{P}_s = \mathbb{P}|_{\mathcal{F}_s}$, for all t and s. We observe that

$$\frac{d\mathbb{S}_s^t}{d\mathbb{P}_s} = \mathbb{E}\Big(\frac{\Gamma_t}{\mathbb{E}(\Gamma_t)}\Big| \mathcal{F}_s\Big). \tag{2.3}$$

Hence, we have

$$\mathbb{S}_s(F) = \lim_{t\to\infty} \mathbb{S}_s^t(F) = 0$$

for all $F \in \mathcal{F}_s$ with $\mathbb{P}(F) = 0$. But this means that $\mathbb{S}_s \ll \mathbb{P}_s$ and $\mathbb{S}_s = M_s \circ \mathbb{P}$ with $M_s = \frac{d\mathbb{S}_s}{d\mathbb{P}_s}$. Because $\mathbb{S}_s$ is a probability measure, we also get $\mathbb{E}M_s \equiv 1$ and $M_s \geq 0$. Finally, for $F \in \mathcal{F}_r$, we have

$$\mathbb{E}(M_s \mathbb{1}_F) = \mathbb{S}_s(F) \overset{(2.2)}{=} \mathbb{S}_r(F) = \mathbb{E}(M_r \mathbb{1}_F),$$

which proves part (a).

(b) See appendix A.3.

∎

This is the reason why we will call $\mathbb{S}$ the *penalisation measure* or the *penalisation limit* (for $\mathbb{P}$ and Γ). In the remaining part of this introduction we are merely concerned about general results on the existence of such a penalisation measure.

Since we will penalize measures of processes which also depend on an initial distribution ν we will also write '$\mathbb{S}_\nu$' for the penailsation limit or '$\mathbb{S}_x$' in the special case of $\nu = \delta_x$, if we want to emphasize this dependence.

remark 2.2.

- We got proposition 2.1 because of the projectivity of $\mathbb{S}_s$. In general the family $(\mathbb{S}_s^t)_s$ is not projective. Nevertheless, it can be seen that this family is projective if and only if $\big(\frac{\Gamma_t}{\mathbb{P}(\Gamma_t)}\big)_t$ is a $(\mathcal{F}_t)_t$-martingale. But this would be a rare situation and nothing new happens in the limit procedure: We only take the limit of eventually constant terms.

- Though we shifted the proof of point (b) to the appendix, we hereby want to attract the reader's attention to the following: Proposition 2.1 holds in greater generality. In fact, we only need to make assumptions such that *Kolmogorov's extension theorem* is applicable.

∎

In the following we present a criterion for penalising which can be found in Chapter 1 of [RY09]. It will help us to gain more insight into the machinery of penalising.

proposition 2.2. *Suppose the following two points are fulfilled.*

(i) $\frac{d\mathbb{S}_s^t}{d\mathbb{P}_s} \to M_s$ *$\mathbb{P}_s$-almost surely for all* $s \geq 0$.

(ii) $\mathbb{E}(M_s) = 1$ *for all* $s \geq 0$.

Then we have that

(a) $\mathbb{S}_s^t \twoheadrightarrow M_s \circ \mathbb{P}$ *for all* $s \geq 0$.

(b) $(M_s)_s$ *is a* $(\mathcal{F}_s)_s$*-martingale.*

proof.

(a) We can use *Scheffé's Lemma*, since the assumptions imply that the 'densities' $\frac{d\mathbb{S}_s^t}{d\mathbb{P}_s}$ converge to M_s. Thus, we get L_1-convergence. With (2.3) we see that

$$\mathbb{S}_s^t(f) = \mathbb{P}\Big(f \cdot \mathbb{E}\big(\frac{\Gamma_t}{\mathbb{E}(\Gamma_t)}\big|\,\mathcal{F}_s\big)\Big) \to \mathbb{P}(f \cdot M_s),$$

for arbitrary $f \in bm(\mathcal{F}_s)$.

(b) This follows from (a) because of proposition 2.1.

∎

By point (i) in proposition 2.2 the central objects are the Radon-Nykodim derivatives $\frac{d\mathbb{S}_s^t}{d\mathbb{P}_s}$. To investigate them further we need more assumptions on $\mathbb{P}$ and on $(\Gamma_t)_t$. For this let $(Y_t)_t$ be a linear diffusion *without killing* on $E \subset \mathbb{R}$ and for each $x \in E$ let $\mathbb{P}_x$ be the measure induced by Y, $Y_0 = x$, on $\Omega = C[0,\infty)$. Let $(\mathcal{F}_t)_t$ be the corresponding filtration. Next we choose $\mathbb{P} = \mathbb{P}_x$ for a fixed x. Assume further that $(\Gamma_t)_t$ is a *multiplicative functional*, or *MF* for short. We take the definition of a 'multiplicative functional' due to [BG68]. In particular, we will use that

- $\Gamma_t = \Gamma_s \cdot \Gamma_{t-s}\Theta_s$ for all $0 \leq s \leq t$.
- Γ_t is $\mathcal{F}_t$-measurable for each $t \geq 0$.

The Θ_s are *shift operators* which act on functions $[0,\infty) \ni t \mapsto w_t$ by $\Theta_s(w) := (t \mapsto w_{t+s})$.

A particular class of MF's are those of the 'Feynman-Kac type'. Essentially one takes $\Gamma_t = \exp(-\int_0^t \kappa(X_s)ds)$ for some appropriate function κ. They are investigated in section 2.2.

Furthermore, we impose that $0 < \mathbb{P}_y(\Gamma_t) < \infty$ for all $y \in E$ and $t \geq 0$.

Because of the above assumptions we can use the Markov property to get

$$\begin{aligned}\mathbb{S}_s^t(f) &= \mathbb{P}_x\Big(f\cdot\mathbb{E}\big(\frac{\Gamma_t}{\mathbb{P}_x(\Gamma_t)}\big|\,\mathcal{F}_s\big)\Big)\\ &= \mathbb{P}_x\Big(f\cdot\Gamma_s\frac{\mathbb{P}_{X_s}(\Gamma_{t-s})}{\mathbb{P}_x(\Gamma_t)}\Big),\end{aligned}$$

for any $f \in bm(\Omega, \mathcal{F}_s)$.

By proposition 2.2 one is eager to find a limit of

$$\frac{d\mathbb{S}_s^t}{d\mathbb{P}_s} = \Gamma_s\cdot\frac{\mathbb{P}_{X_s}(\Gamma_{t-s})}{\mathbb{P}_x(\Gamma_t)}$$

as t goes to infinity. Hence, the investigation of $\frac{\mathbb{P}_y(\Gamma_{t-s})}{\mathbb{P}_x(\Gamma_t)}$ for $t \to \infty$ is the key.

We feel that now will be a good time for an example. We give a rather simple one which can also be found in [Var07] chapter 6 and, more general, also in [MSM94].

example 2.1. Let $\mathbb{P} = \mathbb{P}_0$ be the Wiener measure and let $\mathbb{P}_x$ equal $\mathbb{P}_0$ 'shifted by x'. Take $\Gamma_t = \mathbb{1}_{\{T_0 > t\}}$. Due to the previous observations we try to find the limit of

$$\frac{\mathbb{P}_y(T_0 > t - s)}{\mathbb{P}_x(T_0 > t)}.$$

But

$$\mathbb{P}_x(T_0 > t) = \int_t^\infty \frac{|x|}{\sqrt{2\pi r^3}} e^{-\frac{x^2}{2r}}\,dr \tag{2.4}$$

(e.g. by [KS88], section 2.8). This gives

$$\frac{\mathbb{P}_y(T_0 > t - s)}{\mathbb{P}_x(T_0 > t)} \to \frac{y}{x}.$$

(Note that we do not need to take the absolute values since x and y will have the same sign.) Therefore, we have the existence of M_s, needed in proposition 2.2, and we identify it as

$$M_s = \mathbb{1}_{\{T_0 > s\}}\frac{X_s}{x}.$$

It remains to check the second point, namely that

$$\mathbb{P}_x(M_s) = \frac{\mathbb{P}_x(X_s; T_0 > s)}{x} = 1$$

for all $s \geq 0$. From here on we choose $x > 0$ without loss of generality. Then

$$\mathbb{P}_x(X_s; T_0 > s) = \int_0^\infty y\cdot p_s(x, y)dy$$

with the kernel

$$p_s(x, y) = \frac{1}{\sqrt{2\pi s}}\left(\mathrm{e}^{-\frac{(x-y)^2}{2s}} - \mathrm{e}^{-\frac{(x+y)^2}{2s}}\right)$$

on $\mathbb{R}_+$. (See e.g. [MSM94].) This equals x as you can see by the following calculation:

$$\begin{aligned}
&\int_0^\infty y{\cdot}p_s(x,y)dy\\
=&\int_0^\infty y{\cdot}\frac{1}{\sqrt{2\pi s}}\left(\mathrm{e}^{-\frac{(x-y)^2}{2s}}-\mathrm{e}^{-\frac{(x+y)^2}{2s}}\right)dy\\
=&\int_{-x}^\infty (\tilde{y}+x){\cdot}\frac{1}{\sqrt{2\pi s}}\,\mathrm{e}^{-\frac{\tilde{y}^2}{2s}}\,d\tilde{y}-\int_x^\infty (\tilde{y}-x){\cdot}\frac{1}{\sqrt{2\pi s}}\,\mathrm{e}^{-\frac{\tilde{y}^2}{2s}}\,d\tilde{y}\\
=&2x{\cdot}\int_0^\infty \frac{1}{\sqrt{2\pi s}}\,\mathrm{e}^{-\frac{\tilde{y}^2}{2s}}\,d\tilde{y}+\int_{-x}^0 (\tilde{y}+x){\cdot}\frac{1}{\sqrt{2\pi s}}\,\mathrm{e}^{-\frac{\tilde{y}^2}{2s}}\,d\tilde{y}+\int_0^x (\tilde{y}-x){\cdot}\frac{1}{\sqrt{2\pi s}}\,\mathrm{e}^{-\frac{\tilde{y}^2}{2s}}\,d\tilde{y}\\
=&2x{\cdot}\int_0^\infty \frac{1}{\sqrt{2\pi s}}\,\mathrm{e}^{-\frac{\tilde{y}^2}{2s}}\,d\tilde{y}\\
=&x.
\end{aligned}$$

So, we get a penalisation limit and identify it as the measure of a three-dimensional Bessel process starting at x. (See [Bov12] section 4.6.) ∎

By looking at (2.4), we can see that

$$\mathbb{P}_x(\Gamma_t)\sim t^{-1/2}$$

in the above example. But if the penaliser Γ is indeed some MF, such an 'exact order and rate of decay' is all we need to get the existence of M_s. To understand this assume that

$$\mathbb{P}_x(\Gamma_t)\sim t^{-k}e^{-\eta t}, \tag{2.5}$$

for some k and $\eta\geq 0$, in the sense that

$$t^k e^{\eta t}{\cdot}\mathbb{P}_x(\Gamma_t)\to\psi(x)>0.$$

(In the context of chapter 1 we recognize η as the *mortality rate*; see lemma 1.4.)

This implies

$$\frac{\mathbb{P}_y(\Gamma_{t-s})}{\mathbb{P}_x(\Gamma_t)}\to e^{\eta s}\frac{\psi(y)}{\psi(x)}$$

and, therefore,

$$M_s=\Gamma_s e^{\eta s}\frac{\psi(X_s)}{\psi(x)}.$$

As we will see, this form is more or less universal. (See propositions 2.3 and 2.6 or theorems 2.13 and 2.14.) We can go even one step further and define a natural semigroup $(\mathbf{T}_t^\Gamma)_t$ given by Γ on the space of non-negative measurable functions $f: E\to\mathbb{R}_+$ via

$$\mathbf{T}_t^\Gamma f(x):=\mathbb{P}_x(f(X_t){\cdot}\Gamma_t).$$

With these notations, if

$$\mathbf{T}_t^\Gamma \psi = e^{-\eta t}\psi \tag{2.6}$$

we see that

$$\mathbb{E}_x(M_s) = 1$$

is valid. We will see that (2.6) can result from the fact that $-\eta$ is the principal eigenvalue and ψ is the principal eigenfunction of the generator $\mathbf{L}$ under killing. For instance if $\mathbf{L}$ has a spectral gap. One example is an Ornstein-Uhlenbeck process on $[0,\infty)$ with $\Gamma_t = \mathbb{1}_{\{T_0>t\}}$.

From this point of view, about example 2.1 so much could be said: $\mathbf{L} = \frac{1}{2}\frac{d}{dx}$ acts on functions f with $f(0) = 0$. (See the appendix and subsection 0.4.) On the other hand $\psi(x) = x$ fulfils $\frac{1}{2}\frac{d}{dx}\psi = 0 = 0\cdot\psi$ and $\psi(0) = 0$ (though there is no spectral gap, $\psi(x) = x$ is not in $D(\mathbf{L})$ and (2.6) is not true). But the penalisation limit has density

$$M_s = \Gamma_s \, \mathrm{e}^{0s}\, \frac{\psi(Y_s)}{\psi(Y_0)} = \mathbb{1}_{\{T_0>s\}}\frac{X_s}{x}$$

on $\mathcal{F}_s$ under $\mathbb{P}_x$.

We have choosen another example, taken from [RVY06a], in order to show that there are also other cases than penalising with MF's.

example 2.2 (Weighing BM by its running maximum). Let $\mathbb{P} = \mathbb{P}_x$ be the shifted Wiener measure and $S_t := \sup_{s\le t} X_s$ the corresponding running maximum. We suppose here that $x < 0$. (Otherwise we should formulate everything with $S_t = \inf_{s\le t} X_s$ in what follows.) Choose a map $h : \mathbb{R} \to \mathbb{R}_+$. To get a penaliser by

$$\Gamma_t := h(S_t)$$

we should assume $0 < \mathbb{E}_x h(S_t) < \infty$ for all $t \ge 0$. This could be accomplished by assuming

$$\int h(y)dy < \infty$$

and

$$\int_x^\infty h(y)dy > 0.$$

The reason is that we can calculate

$$\mathbb{P}_x(h(S_t)) = \sqrt{\frac{2}{\pi t}}\int_0^\infty h(y+x)e^{-\frac{y^2}{2t}}dy \overset{DOM}{\propto} \sqrt{\frac{2}{\pi t}}\int_x^\infty h(y)dy.$$

To identify M_s notice that

$$
\begin{aligned}
\mathbb{E}_x(h(S_t) \mid \mathcal{F}_s) &= \mathbb{E}_x\big(h\big(S_s \vee \big(X_s + \sup_{s\le r\le t}(X_r - X_s)\big)\big) \mid \mathcal{F}_s\big) \\
&= \mathbb{P}_0\left(h(a \vee (b + S_{t-s}))\right) \mid_{(a,b)=(S_s,X_s)} \\
&= \sqrt{\frac{2}{\pi(t-s)}} \int_0^\infty h(a \vee (b+y)) e^{-\frac{y^2}{2(t-s)}} dy \mid_{(a,b)=(S_s,X_s)} \\
&= \sqrt{\frac{2}{\pi(t-s)}} \left[\int_0^{a-b} h(a) e^{-\frac{y^2}{2(t-s)}} dy + \int_{a-b}^\infty h(b+y) e^{-\frac{y^2}{2(t-s)}} dy\right]_{(a,b)=(S_s,X_s)} \\
&\overset{DOM}{\propto} \sqrt{\frac{2}{\pi(t-s)}} \left[h(a)(a-b) + \int_a^\infty h(y) dy\right]_{(a,b)=(S_s,X_s)}.
\end{aligned}
$$

This implies

$$
M_s = \lim_{t\to\infty} \mathbb{E}\Big(\frac{h(S_t)}{\mathbb{E}_x h(S_t)}\Big| \mathcal{F}_s\Big) = \frac{h(S_s)(S_s - X_s) + \int_{S_s}^\infty h(y)dy}{\int_x^\infty h(y)dy}. \tag{2.7}
$$

$(M_s)_s$ is known as a local martingale (at least for h bounded) from [AY79]. Next we verify the conditions of proposition 2.2. To do so we first show the existence of an upper bound in order to apply the Lebesgue's dominated convergence theorem. From the above calculations we see that

$$
\mathbb{E}(\Gamma_t \mid \mathcal{F}_s) \le \sqrt{\frac{2}{\pi(t-s)}} \left[h(S_s)(S_s - X_s) + \int_{S_s}^\infty h(y)dy\right]
$$

whenever $t > s$. We also get

$$
\mathbb{E}_x(\Gamma_t) \ge \frac{1}{\sqrt{2\pi t}} \int_x^\infty h(y)dy
$$

for t large enough. Thus, if we choose t large enough we arrive at

$$
\mathbb{E}\Big(\frac{h(S_t)}{\mathbb{E}_x(h(S_t))}\Big| \mathcal{F}_s\Big) \le 3M_s.
$$

Especially if we take $h_k := h \wedge k$ instead of h. Observe that, for large k, the corresponding

$$
M_s^k = \frac{h_k(S_s)(S_s - X_s) + \int_{S_s}^\infty h_k(y)dy}{\int_x^\infty h_k(y)dy}
$$

are in L^1. By the dominated convergence theorem we get

$$
\mathbb{P}_x\left(f \cdot \frac{h_k(S_t)}{\mathbb{P}_x(h_k(S_t))}\right) \to \mathbb{P}_x\big(M_s^k \cdot f\big)
$$

for each $f \in bm(\Omega, \mathcal{F}_s)$. Thus, the penalising principle is fulfilled and by proposition 2.1 we have that

- $\mathbb{E}_x(M_s^k) \equiv 1$ for all s, k.
- $(M_s^k)_s$ is a martingale for each k.

Therefore,

$$N_s^k := h_k(S_s)(S_s - X_s) + \int_{S_s}^{\infty} h_k(y)dy$$

is also a martingale since it is just some multiple of M_s^k. Ergo

$$\mathbb{E}_x(N_s^k) = \mathbb{E}_x(N_0^k) = \int_x^{\infty} h_k(y)dy.$$

If we now use monotone convergence twice, we see that

$$\mathbb{E}_x(N_s^k) = \mathbb{E}_x(N_0^k) \to \int_x^{\infty} h(y)dy < \infty$$

and on the other hand

$$\mathbb{E}_x(N_s^k) \to \mathbb{E}_x\left(h(S_s)(S_s - X_s) + \int_{S_s}^{\infty} h(y)dy\right).$$

This implies

$$\mathbb{E}_x(M_s) = 1.$$

Finally we use proposition 2.2 to derive that

$$\mathbb{P}_x\left(f \cdot \frac{h(S_t)}{\mathbb{P}_x(h(S_t))}\right) \to \mathbb{P}_x(f \cdot M_s) = \mathbb{P}_x\left(f \cdot \frac{h(S_s)(S_s - X_s) + \int_{S_s}^{\infty} h(y)dy}{\int_x^{\infty} h(y)dy}\right)$$

for all $s \geq 0$ and $f \in bm(\Omega, \mathcal{F}_s)$.

Furthermore, we have shown that the local martingale from in (2.7) is a proper martingale. ∎

remark 2.3.

- Example 2.2 is a generalisation of example 2.1. To see this define $h(y) := \mathbb{1}_{[x,0]}(y)$ and observe that penalising $\mathbb{P}_x$ with $\Gamma_t = h(S_t)$ is the same as penalising with $\Gamma_t = \mathbb{1}_{\{T_0 > t\}}$.
- The big improvement in comparison to example 2.1 is the following: Once you choose h such that $h(y) > 0$ for all y greater than the initial value x, there is *no* exclusion of paths under $\mathbb{S}_s^t$. I.e. paths are not 'cut out' from the path space. They rather get a lower weight instead of weight zero.

∎

There are also works where Brownian motion is penalised by $\Gamma_t = f(\inf_{s \leq t} X_s, \sup_{s \leq t} X_t)$. See for instance [RVY06a] or [RVY08a]. The latter deals with the case of compactly supported f. In particular the authors take $f(y,z) = \mathbb{1}_{\{[b,c]\times[b,c]\}}(y,z)$. This is the same as conditioning on survival under killing at the boundaries b and c, i.e. $\tau = T_b \wedge T_c$. Surprisingly they get the same result for a whole class of f of the form $f(y,z) = g(y,z) \cdot \mathbb{1}_{\{[b,c]\times[b,c]\}}(y,z)$. Just compare Theorem 5.1 from [RVY08a] with proposition 2.3 below (for the case $Y = X$ and $\tau = T_0 \wedge T_c$).

2.2. Feynman-Kac penalisation

In this section will specialize to a certain kind of penalisation known in the literature as 'Feynman-Kac penalisation'. (See e.g. [RY09].) In particular, we will condition regular diffusions Y as in the preceding chapters *not* to be killed under some killing time τ. The resulting process is also called *Q-process*. As far as we know this name was first used in [AN72] for branching processes conditioned not to be killed.

In subsection 2.2.1 we will consider the case of a bounded state space E. (See proposition 2.3.) We also illustrate the connection between the penalisation limit and the minimal QSD.

Subsection 2.2.2 essentially consists of known results when Y is a Brownian motion on $\mathbb{R}$.

This section will be complemented by penalisation results for more general diffusions Y. We will investigate Y conditioned on survival under the assumption that a QLD exists as well as under the contrary assumption that there is no Yaglom limit. (Compare theorems 2.13 and 2.14.)

Let us start by formulating the problem. Take $(\Omega, \mathcal{F}_\infty, (\mathcal{F}_t)_t, (\mathbb{P}_x)_{x\in E}, (Y_t)_t)$ such that

- $(Y_t)_t$ is a diffusion on E with $\partial E = \{b, c\}$ and $-\infty \leq b < c \leq \infty$.
- Here b and c are supposed to be regular boundaries and reflecting whenever finite.
- Y solves

$$dY_t = dX_t + \boldsymbol{a}(Y_t)dt \tag{2.8}$$

  until $T_{\partial I}$ for every interval I with $\bar{I} \subseteq \mathring{E}$.
- $\mathbb{P}_x(Y_0 = x) = 1$ for every $x \in E$.
- X is a Brownian motion on each $(\Omega, \mathcal{F}_\infty, (\mathcal{F}_t)_t, \mathbb{P}_x)$ with $\mathbb{P}_x(X_0 = x) = 1$.

remark 2.4.
- We can always find such a solution and it is unique on closed and bounded intervals $I \subset (b, c)$ if $\boldsymbol{a} \in C$. (See section 0.1.)
- In a lot of cases, e.g. if $\boldsymbol{a} \in C^1$, we can take $\mathbb{P}_x$ to be the Wiener measure 'shifted to x'. We may even take $\mathbb{P}_x$ as the image measures under Y and set $\mathbb{P} := \mathbb{P}_x$ for some fixed x. This is a way of putting everything into the framework of section 2.1.

∎

Under *Feynman-Kac Penalisation (FKP)* we understand penalising with

$$\Gamma_t := \exp\left(-\int L_t^x(Y)\kappa(dx)\right), \tag{2.9}$$

where $L_t^x(Y)$ is the local time and κ is some non-zero and non-negative Radon measure. (We only deal with regular diffusions. This ensures the existence of $L_t^x(Y)$. See section I.II.2 in [BS02].)

Note that, if κ has a density with respect to the Lebesgue measure, the above formula looks particularly nice. If we abuse notation by writing also κ for the density, we obtain

$$\Gamma_t = \exp\left(-\int_0^t \kappa(Y_s)ds\right)$$

almost surely for arbitrary $t \geq 0$. This formula holds in greater generality for semimartingales; especially for solutions to (2.8). See equation (7.3) in [KS88].

Note also that whenever b or c is a finite endpoint, κ may have some mass there (possibly ∞ mass). This corresponds to (instant) killing at this boundary. Take for example Y on $\mathbb{R}_+$ with $\kappa(dx) = \infty\cdot\delta_0(dx)$. Thus, penalising the process Y with (2.9) is the same as conditioning the process on not hitting zero or not being killed at zero, respectively.

On the connection with killing see section 0.3.

2.2.1. What should be expected

Here we want to give some idea on the connections with objects such as the minimal QLD which we already introduced in chapter 1. First of all we formulate a penalisation result for Y on $[0, c]$ with $c < \infty$. After presenting a proof we turn to the main objective of this section: Convince the reader that we expect the penalisation limit to be Y plus some remarkable additional drift. Next we review the example of section 1.2. Finally we discuss whether the penalisation limit has an invariant distribution.

proposition 2.3. *Assume the same as in proposition 1.26. Then we have*

$$\mathbb{P}_x(F \mid \tau > t) \to \mathrm{e}^{\underline{\lambda} s} \frac{\int_0^c \underline{\varphi}(y)/m(y) \mathbb{P}_x(F, \tau > s, Y_s \in dy)}{\underline{\varphi}(x)/m(x)}$$

for all $F \in \mathcal{F}_s$, $s > 0$ and $x \in (0, c)$.

Before we prove this let us recall that $\underline{\varphi} = \underline{\psi} m$ with $\underline{\psi}$ being a non-trivial solution to

$$-\left[\frac{1}{2}\frac{d^2}{dx^2} + \boldsymbol{a}(x)\frac{d}{dx} - \kappa(x)\right]\underline{\psi} = \underline{\lambda}\cdot\underline{\psi}.$$

The value $\underline{\lambda}$ is the infimum of the $L^2(m)$-spectrum of $-\mathbf{L}^{\kappa,\alpha,\beta} = -\frac{1}{2}\frac{d^2}{dx^2} - \boldsymbol{a}(x)\frac{d}{dx} + \kappa(x)$. In general $\underline{\psi}$ it is not an element of $L^2(m)$. But it fulfils the same boundary conditions as all elements of $D(\mathbf{L}^{\kappa,\alpha,\beta})$, i.e. (0.16) at 0 and (0.16) at c with β instead of α. We have that $\underline{\psi}$ is unique up a positive multiple.

proof of proposition 2.3. We use the Markov property of Y at $s < t$ to obtain

$$\mathbb{P}_x(F \mid \tau > t) = \frac{\mathbb{P}_x(F, \tau > s, \mathbb{P}_{Y_s}(\tau > t - s))}{\mathbb{P}_x(\tau > t)}.$$

By proposition 1.26 we know that

$$\mathrm{e}^{\underline{\lambda} t}\, \mathbb{P}_x(\tau > t) = \mathrm{e}^{\underline{\lambda} t}\, \mathbb{P}_x(Y_t \in [0, c]) \to \underline{\varphi}(x)/m(x)\cdot\frac{\int_0^c \underline{\varphi}(y)dy}{\int_0^c \underline{\varphi}^2(y)/m(y)\, dy}$$

with $\nu = \delta_x$. Ergo

$$\mathbb{P}_x(\tau > t) \propto \mathrm{e}^{-\underline{\lambda} t}\, \underline{\varphi}(x)/m(x)\cdot\frac{\int \underline{\varphi}}{\int \underline{\varphi}^2/m}.$$

Now we also rewrite the numerator of the initial term:

$$\mathbb{P}_x(F \mid \tau > t) \propto \frac{\int \mathbb{P}_y(\tau > t-s)\mathbb{P}_x(F, \tau > s, Y_s \in dy)}{\mathrm{e}^{-\underline{\lambda} t}\, \underline{\varphi}(x)/m(x) \cdot \frac{\int \underline{\varphi}(y)dy}{\int \underline{\varphi}^2(y)/m(y)\, dy}}$$

$$= \frac{\mathbb{P}_{\tilde{\nu}}(\tau > t-s) \cdot \mathbb{P}_x(F, \tau > s)}{\mathrm{e}^{-\underline{\lambda} t}\, \underline{\varphi}(x)/m(x) \cdot \frac{\int \underline{\varphi}(y)dy}{\int \underline{\varphi}^2(y)/m(y)\, dy}},$$

with $\tilde{\nu}(B) := \frac{\mathbb{P}_x(F, \tau > s, Y_s \in B)}{\mathbb{P}_x(F, \tau > s)}$.

Note that if $\mathbb{P}_x(F, \tau > s) = 0$, then both terms in the proposition are zero for $t > s$ and there is nothing to prove.

Next we rewrite the numerator again by 1.26 and get

$$\mathbb{P}_x(F \mid \tau > t) \propto \frac{\mathrm{e}^{-\underline{\lambda}(t-s)} \int \underline{\varphi}(y)/m(y) \tilde{\nu}(dy) \frac{\int \underline{\varphi}(y)dy}{\int \underline{\varphi}^2(y)/m(y)\, dy} \cdot \mathbb{P}_x(F, \tau > s)}{\mathrm{e}^{-\underline{\lambda} t}\, \underline{\varphi}(x)/m(x) \cdot \frac{\int \underline{\varphi}(y)dy}{\int \underline{\varphi}^2(y)/m(y)\, dy}}$$

$$= \mathrm{e}^{\underline{\lambda} s} \frac{\int \underline{\varphi}(y)/m(y) \tilde{\nu}(dy) \mathbb{P}_x(F, \tau > s)}{\underline{\varphi}(x)/m(x)}$$

$$= \mathrm{e}^{\underline{\lambda} s} \frac{\int \underline{\varphi}(y)/m(y) \mathbb{P}_x(F, \tau > s, Y_s \in dy)}{\underline{\varphi}(x)/m(x)}.$$

■

The result looks a little bit nicer if we use

$$\underline{\psi}(x) = \underline{\varphi}(x)/m(x).$$

(See (1.13) in section 1.2.) If $F = \{Y_s \in A\}$ for some measurable A, then

$$\mathbb{P}_x(Y_s \in A \mid \tau > t) \to \mathrm{e}^{\underline{\lambda} s} \frac{\int_A \underline{\psi}(y) \mathbb{P}_x(Y_s \in dy, \tau > s)}{\underline{\psi}(x)}. \tag{2.10}$$

Suppose for the rest of this section that (2.10) holds more generally. E.g. if we also allow $c = \infty$. (We will prove appropriate results in the next sections.) With the help of the semigroup $(\mathbf{T}_t)$ corresponding to $\mathbf{L} = \mathbf{L}^{\kappa,\alpha(,\beta)}$, respectively to Y under $\tau = \tau^{\kappa,\alpha(,\beta)}$, we may also write

$$\mathbb{P}_x(f(Y_s) \mid \tau > t) \to \mathrm{e}^{\underline{\lambda} s} \frac{(\mathbf{T}_s \underline{\psi} f)(x)}{\underline{\psi}(x)}. \tag{2.11}$$

If we speak of '$\mathbf{L}$', we mean here an operator in $L^2(dm)$. Hence, $\mathbf{T}_s$ acts on $L^2(dm)$-functions.

This means we should at least impose $\underline{\psi} f \in L^2(dm)$ in (2.11) and $\underline{\psi} f \in D(\mathbf{L})$ in the following calculation:

$$\begin{aligned}
&\frac{\mathrm{e}^{\underline{\lambda} s}\,\frac{\mathbf{T}_s(\underline{\psi} f)(x)}{\underline{\psi}(x)} - \frac{\mathbf{T}_0(\underline{\psi} f)(x)}{\underline{\psi}(x)}}{s} \\
=&\frac{1}{\underline{\psi}(x)} \cdot \left[\frac{\mathrm{e}^{\underline{\lambda} s}\,\mathbf{T}_s(\underline{\psi} f)(x) - \mathbf{T}_s(\underline{\psi} f)(x)}{s} + \frac{\mathbf{T}_s(\underline{\psi} f)(x) - \mathbf{T}_0(\underline{\psi} f)(x)}{s} \right] \\
\to&\frac{1}{\underline{\psi}(x)} \cdot \left[\underline{\lambda}\underline{\psi} f(x) + \mathbf{L}(\underline{\psi} f)(x) \right] \\
=&\frac{1}{\underline{\psi}(x)} \cdot (\mathbf{L} + \underline{\lambda})(\underline{\psi} f)(x).
\end{aligned}$$

We define the resulting operator

$$\mathbf{L}^{\underline{\psi}} := \frac{1}{\underline{\psi}}(\mathbf{L} + \underline{\lambda})\underline{\psi} \tag{2.12}$$

on $\{f \in L^2(\underline{\psi}^2 m)\,;\ \underline{\psi} f \in D(\mathbf{L}) \text{ and } \frac{1}{\underline{\psi}}\mathbf{L}(\underline{\psi} f) \in L^2(\underline{\psi}^2 m)\}$. Though (2.12) looks neat it does not help us to further investigate the penalisation limit. Thus, we calculate

$$\begin{aligned}
\mathbf{L}^{\underline{\psi}} f &= \frac{1}{\underline{\psi}} \left(\frac{1}{2}(\underline{\psi} f)'' + \boldsymbol{a}(\underline{\psi} f)' - \kappa(\underline{\psi} f) + \underline{\lambda}(\underline{\psi} f) \right) \\
&= \frac{1}{\underline{\psi}} \left(\frac{1}{2}\underline{\psi}'' f + \underline{\psi}' f' + \frac{1}{2}\underline{\psi} f'' + \boldsymbol{a}\underline{\psi}' f + \boldsymbol{a}\underline{\psi} f' - \kappa\underline{\psi} f + \underline{\lambda}\underline{\psi} f \right)
\end{aligned}$$

and if we factor out f in the last line and use that $-\mathbf{L}\underline{\psi} = \underline{\lambda}\underline{\psi}$ we arrive at

$$\mathbf{L}^{\underline{\psi}} = \frac{1}{2}\frac{d^2}{dx^2} + \left(\boldsymbol{a}(x) + \frac{\underline{\psi}'(x)}{\underline{\psi}(x)} \right) \frac{d}{dx}. \tag{2.13}$$

Formally this looks just like the generator of Y with additional drift coefficient $\frac{d}{dx}\log\underline{\psi}(x)$. Indeed, this is true for a large class of penalisation limits. We will rigorously prove this in section 2.3.

example 2.3. Take $a \in \mathbb{R}$ constant, i.e. $dY_t = dX_t + a dt$ on $(0, \pi)$ under $\tau = T_0 \wedge T_\pi$. In the example of section 1.2, we have seen that

$$\underline{\psi} = \mathrm{e}^{-ax}\sin(x)$$

with $\underline{\lambda} = \frac{1+a^2}{2}$. Thus, the additional drift term will be

$$\frac{d}{dx}\log\underline{\psi}(x) = -a + \frac{\cos(x)}{\sin(x)}.$$

Surprisingly, we get a penalised process corresponding to the generator

$$\mathbf{L}^{\underline{\psi}} = \frac{1}{2}\frac{d^2}{dx^2} + \frac{\cos(x)}{\sin(x)}\frac{d}{dx} \tag{2.14}$$

which does not depend on the original drift a. Since $\lim_{x\downarrow 0} \frac{\cos(x)}{\sin(x)} = \infty$ and $\lim_{x\uparrow \pi} \frac{\cos(x)}{\sin(x)} = -\infty$ the penalisation limit has a very strong drift away from the boundary points. So strong that we can show that there are no exit boundaries. (Simply use $\sin(x) \sim x$ and compute the first term of (1.16) with ∞ replaced by 0 and π, respectively.) In particular, no boundary conditions are needed for (2.14) to be the generator of the penalising limit. Intuitively, this should be clear since we condition a BM on not hitting the boundaries 0 and π. ∎

Finally, we ask whether a penalisation limit has an invariant distribution and, if so, can we identify it? Let us put this in a little bit larger framework:

Let ψ_λ be a positive solution to

$$-\mathbf{L}\psi = \lambda\psi$$

with $\mathbf{L} = \mathbf{L}^{\kappa,\alpha(,\beta)}$. Now we can do a so-called *eigenstate-transformation*

$$\mathbf{L}^{\psi_\lambda} := \frac{1}{\psi_\lambda}(\mathbf{L} + \lambda)\psi_\lambda.$$

We see by (2.12) that $\mathbf{L}^{\psi_{\underline{\lambda}}} = \mathbf{L}^{\underline{\psi}}$ and we (again) calculate

$$\mathbf{L}^{\psi_\lambda} = \mathbf{L} + \kappa + \frac{d}{dx}\log\psi_\lambda(x)\frac{d}{dx} = \frac{1}{2}\frac{d}{dx} + \boldsymbol{a}(x)\frac{d}{dx} + \frac{d}{dx}\log\psi_\lambda(x)\frac{d}{dx}.$$

One may ask now, if we can get a process $(Z_t^\lambda)_t$ which has a generator of the form $\mathbf{L}^{\psi_\lambda}$, by a 'penalisation-like' limiting procedure. This is for example done (for diffusions on $\mathbb{R}_+$ under $\tau = T_0$) in [MSM01] or [CMSM13b] section 6.4. In the spirit of the preceding observations we define the diffusion Z^λ on $[0,\infty)$ as follows:

- Z^λ is a (weak) solution to

$$dZ_t = dX_t + \left[\boldsymbol{a}(Z_t) + \frac{\psi'_\lambda(Z_t)}{\psi_\lambda(Z_t)}\right] dt$$

  on $(0,\infty)$ with ψ_λ as above.

- If 0 is accessible, then it is 'purely reflecting'.

Now we can formulate the following proposition:

proposition 2.4. *Let Z^λ be as above. Then*

(a) $\lambda \leq \underline{\lambda}$.

(b) If $\psi_\lambda \in L^2(m)$, then it must hold $\lambda = \underline{\lambda}$.

(c) Now let $\underline{\psi} \in L^2(m)$. Then

$$\frac{\underline{\psi}^2(y)\cdot m(y)dy}{\int \underline{\psi}^2(x)\cdot m(x)dx}.$$

is the unique stationary distribution for $Z^{\underline{\lambda}}$.

proof.

(a) This is clear by the introduction of Z^{λ} through positivity of ψ_{λ} and by theorem A.2.6.

(b) This is again theorem A.2.6.

(c) If this is true, then we must have $\underline{\psi}(x)m(x) \to 0$ for $x \to \infty$ implying

$$\int_{d}^{\infty} (\underline{\psi}^2(x)m(x))^{-1}dx = \infty \tag{2.15}$$

for any $d > 0$. Let $\underline{s}$ denote a scale function of $Z^{\underline{\lambda}}$. By (2.15) and formula (0.10) we see that $\underline{s}(x) \to \infty$ for $x \to \infty$. Hence, $Z^{\underline{\lambda}}$ is recurrent. This implies in particular that a stationary distribution, if existent, is unique. (See section I.II.6. in [BS02].) Further observe that $Z_s^{\underline{\lambda}}$ has the distribution

$$\begin{aligned} &\int_E \frac{\underline{\psi}^2(x)m(x)}{\int_E \underline{\psi}^2(z)m(z)dz} \cdot e^{\underline{\lambda} s} \frac{\mathbb{P}_x(Y_s \in dy,\, \tau > s)}{\underline{\psi}(x)} dx \\ =& \frac{1}{\int_E \underline{\psi}^2(z)m(z)dz} \int_E \underline{\psi}(x)m(x) \cdot e^{\underline{\lambda} s}\, \mathbb{P}_x(Y_s \in dy,\, \tau > s)dx \end{aligned} \tag{2.16}$$

if $Z_0^{\underline{\lambda}}$ has distribution $\frac{\underline{\psi}^2(x)m(x)}{\int_E \underline{\psi}^2(z)m(z)dz}dx$. (We use '$E$' for the state space.) Due to [BS02] section I.II. for each x the measure $\mathbb{P}_x(Y_s \in dy, \tau > s)$ has a density $p_s(x,y)$ w.r.t. $m(y)dy$ and $p_s(x,y) = p_s(y,x)$. Thus, (2.16) becomes

$$\begin{aligned} &\frac{1}{\int_E \underline{\psi}^2(z)m(z)dz} \int_E \left[\underline{\psi}(x)m(x)\, e^{\underline{\lambda} s}\, p_s(x,y)m(y)dy\right] dx \\ =& \frac{1}{\int_E \underline{\psi}^2(z)m(z)dz} m(y)\, e^{\underline{\lambda} s} \left[\int_E p_s(y,x)\underline{\psi}(x)m(x)dx\right] dy \\ =& \frac{1}{\int_E \underline{\psi}^2(z)m(z)dz} m(y)\, e^{\underline{\lambda} s}\, \mathbf{T}_s\underline{\psi}(y)dy. \end{aligned}$$

To show now that $\underline{\psi}^2(x)m(x)dx$ is stationary it suffices prove

$$\frac{d}{ds} e^{\underline{\lambda} s}\, \mathbf{T}_s\underline{\psi}(y) = 0.$$

But

$$\frac{d}{ds}\mathbf{T}_s\underline{\psi}(y) = \lim_{\epsilon \downarrow 0} \frac{\mathbf{T}_{s+\epsilon}\underline{\psi}(y) - \mathbf{T}_s\underline{\psi}(y)}{\epsilon} = \mathbf{T}_s\mathbf{L}\underline{\psi}(y) = -\underline{\lambda}\mathbf{T}_s\underline{\psi}(y)$$

in $L^2(m)$ since $\underline{\psi} \in D(\mathbf{L})$. This implies

$$\frac{d}{ds} e^{\underline{\lambda} s}\, \mathbf{T}_s\underline{\psi}(y) = \underline{\lambda}\, e^{\underline{\lambda} s}\, \mathbf{T}_s\underline{\psi}(y) + e^{\underline{\lambda} s} \frac{d}{ds}\mathbf{T}_s\underline{\psi}(y) = 0.$$

■

In particular, $\underline{\psi}^2 m$ is always a stationary distribution for $Z^{\underline{\lambda}}$, whenever the principal eigenfunction $\underline{\psi}$ is a 'real' eigenstate, i.e. $\underline{\psi} \in L^2(m)$. One sufficient condition for Y on $\mathbb{R}_+$, is

$$\liminf_{x\to\infty} \kappa(x) > \underline{\lambda}. \tag{2.17}$$

(See section 4.2 of [KS12].) Note that Z must be recurrent; even if there is some drift $\boldsymbol{a} > 0$ making Y transient. (Confer the proof of proposition 2.4.) For convenience think of zero as a reflecting boundary and of a strictly increasing κ. Therefore, conditioning on survival 'drives the process in again'; away from the region where κ is high. The remarkable thing is that conditioning on survival up to t does not lose its full power 'at the end':

The Yaglom limit exists under (2.17) and is $\underline{\varphi} = \underline{\psi} m$.

(This is essentially Theorem 4.3 of [KS12].)

2.2.2. The case of Brownian motion

For this subsection we take $Y = X$ on the state space $E = \mathbb{R}$. We formulate a rather obvious penalising criterion as it is done in [RVY06b] under the name *Generic Theorem.*

proposition 2.5. *Suppose we have* Γ *of the form* (2.9) *with* $\kappa(\mathbb{R}) < \infty$*. Further assume there are* $k \geq 0$ *and* $\psi : \mathbb{R} \to (0, \infty)$*, such that*

(i) $t^k \mathbb{P}_x(\Gamma_t) \to \psi(x)$ *for all* x*.*

(ii) $\mathbb{P}_x \left(\sup_t t^k \mathbb{P}_{X_s}(\Gamma_t) \right) < \infty$ *for all* x *and all* $s \geq 0$*.*

Then the penalising principle 2.1 holds with

$$M_s = \frac{\psi(X_s)}{\psi(X_0)} \Gamma_s,$$

i.e. we have

$$\frac{\mathbb{P}_x(\mathbb{1}_F \cdot \Gamma_t)}{\mathbb{P}_x(\Gamma_t)} \to \mathbb{S}_x(F) = \mathbb{P}_x(\mathbb{1}_F \cdot M_s)$$

for all $s \geq 0$ *and* $F \in \mathcal{F}_s$*.*

proof. The proof simply uses the Markov property of Brownian motion at time s, Lebesgue's dominated convergence theorem and that $\frac{t^k}{(t-s)^k} \to 1$ for all $s \geq 0$. ■

Of course, the above result may be formulated in greater generality, e.g. under

- $(Y_t)_t$ a Markov process

and

- $(\Gamma_t)_t$ a multiplicative functional.

Some justification, why we have formulated it only in the setting of BM may be that the assumptions of the principle are shown for a rather large class $\mathcal{I}$ of κ's given in the next definition.

definition 2.3. We say that

$$\Gamma_t = \exp\left(-\int L_t^x \kappa(dx)\right)$$

is in the class $\mathcal{I}$, if

1. κ is a positive Radon measure on $\mathbb{R}$
2. $\int(1+|x|)\kappa(dx) < \infty$.

In [RVY06b] the following is shown.

proposition 2.6. *If Γ is in class $\mathcal{I}$, then*

(a) proposition 2.5 is fulfilled with $k = \frac{1}{2}$.

(b) the function ψ may be characterised as the unique solution of

$$\mathbf{L}^\kappa \psi = \frac{1}{2}\psi'' - \kappa\psi = 0$$

under

$$\lim_{x\to\infty} \psi'(x) = -\lim_{x\to-\infty} \psi'(x) = \sqrt{\frac{2}{\pi}}.$$

(c) the penalisation measure $\mathbb{S}_x$ is the measure of Z solving

$$dZ_t = dX_t + \frac{\psi'(Z_t)}{\psi(Z_t)}dt$$

under $Z_0 = x$.

Actually there a several proofs of proposition 2.6. (See [RVY06b] as well as [RY09].)

2.2.3. On more general diffusions

We now concentrate on diffusions Y introduced at the beginning of this section 2.2. The state space will be $E = [0,\infty)$. Our penalisers Γ will be

$$\Gamma_t := \exp\left(-\int L_t^x(Y)\tilde{\kappa}(dx)\right),$$

as in (2.9). We take

$$\tilde{\kappa}(dx) = \kappa(x)dx + 2\alpha\delta_0(dx)$$

with $\kappa \geq 0$ and $\alpha \in [0,\infty) \cup \{\infty\}$. We explained in section 0.3 that this corresponds to killing under the killing rate κ and killing at zero with 'rate' α. As before we denote the killing time by $\tau = \tau^{\kappa,\alpha}$. Thus, we have

$$\mathbb{P}_x(f\cdot\Gamma_t) = \mathbb{P}_x(f;\, \tau > t)$$

for every $f \in L^1(\mathbb{P}_x)$.

Under the existence of a QLD

In order to prove penalisation theorems under the existence of a QLD we use the next result which can be found in [SE07]. It gives an exact ratio-limit of survival probabilities (depending on the initial value).

proposition 2.7. *Suppose we have A 2, A 5, A 6 and A 7. Suppose further that* $K \neq \underline{\lambda}$*. Then*

$$w_t(x) \to \psi_\eta(x)$$

for every x*.*

We remind the reader that η is the mortality rate and, loosely speaking, that ψ_η is the corresponding eigenfunction of the generator $\mathbf{L} = \mathbf{L}^{\kappa,\alpha}$. We already introduced these objects as well as the assumptions from proposition 2.7 in chapter 1.

Now we present a first penalisation result assuming $\kappa \equiv 0$ but under slow killing at the boundary. In this direction it is a generalisation to a result given in [CMSM95].

proposition 2.8. *Suppose A 2, A 3 and A 6 as well as* $\kappa \equiv 0$*. Then, under the existence of a QLD,*

(a) the mortality rate is $\eta = \underline{\lambda} > 0$*.*

(b) for all $s \geq 0$ *and* $F \in \mathcal{F}_s$ *we have*

$$\mathbb{P}_x(F \mid \tau > t) \to \mathrm{e}^{\eta s}\, \mathbb{P}_x\left(\mathbb{1}_F \cdot \mathbb{1}_{\{\tau > s\}} \frac{\psi_\eta(Y_s)}{\psi_\eta(Y_0)}\right).$$

Before we start with the proof observe that proposition 2.8 says that a 'penalising principle' as in definition 2.1 is fulfilled. Indeed, we have

$$\mathbb{P}_x(F \mid \tau > t) = \frac{\mathbb{P}_x(F,\, \tau > t)}{\mathbb{P}_x(\tau > t)} = \frac{\mathbb{P}_x(\mathbb{1}_F \cdot \Gamma_t)}{\mathbb{P}_x(\Gamma_t)}$$

with

$$\Gamma_t = \exp\left(-\int L_t^x 2\alpha\delta_0(dx)\right) = \mathrm{e}^{-2\alpha L_t^0}.$$

proof. By the Markov property at time s we write

$$\mathbb{P}_x(F \mid \tau > t) = \frac{\mathbb{P}_x(\mathbb{1}_F \mathbb{1}_{\{\tau > s\}} \mathbb{P}_{Y_s}(\tau > t - s))}{\mathbb{P}_x(\mathbb{1}_{\{\tau > t\}})}$$

giving

$$\mathbb{P}_x(F \mid \tau > t) = \frac{\mathbb{P}(\mathbb{1}_F \mathbb{1}_{\{\tau > s\}} w_{t-s}(Y_s))}{w_t(x)} \cdot \frac{\mathbb{P}_1(\tau > t - s)}{\mathbb{P}_1(\tau > t)}.$$

Since a QLD exists corollary 1.9 ensures that we are in the case $\underline{\lambda} > 0$. Now we make use of 1.8 and see that $\eta = \underline{\lambda}$ and the second term from above converges to

$$\mathrm{e}^{\eta s},$$

by applying lemma 4.1 of [KS12].

By lemma 1.16 we have that A 7 is automatically satisfied. Finally we use proposition 2.7 and the dominated convergence theorem to obtain the desired result. ■

For sufficient conditions ensuring the existence of a QLD confer section 1.4. For example, we have a QLD if the drift has some positive distance to zero from below.

remark 2.5.

- For the special case of $\tau = T_0$ see also Theorem 6.26 of [CMSM13b].
- For other penalisation results of linear diffusions conditioned on not hitting zero see section 3.1.4 in [NRY09] and references given therein.
- Similar results as in proposition 2.8 are obtained in [Pro10, SV09]. The main difference is that the authors of [Pro10] and [SV09] make an additional assumption on the inverse of $t \mapsto L^0_t$.

▮

Next we are going to generalize the above proposition from '$\kappa \equiv 0$' to 'A 5', i.e. that κ has some limit. First of all we need a small lemma

lemma 2.9. *Suppose that Y is a regular diffusion on $[0,\infty)$ (respectively on $[0,d]$) as in section 1.2. Let κ be a killing rate from $C[0,\infty)$ (respectively from $C[0,d]$). And let there be killing at zero with a killing rate parametrized by $\alpha \geq 0$ (and killing at d parametrized by $\beta \geq 0$) as in section 1.2. Finally, let $\tau = \tau^{\kappa,\alpha(,\beta)}$ be the corresponding killing time and*

$$\eta^{\kappa,\alpha(,\beta)} = -\lim_{t\to\infty} \frac{\log \mathbb{P}_x(\tau^{\kappa,\alpha(,\beta)} > t)}{t}$$

the corresponding mortality rate. Then

$$\mathbb{P}_x\big(f\cdot \mathbb{1}_{\{\tau^{\kappa+p,\alpha(,\beta)}>t\}}\big) = \mathrm{e}^{-pt}\, \mathbb{P}_x\big(f\cdot \mathbb{1}_{\{\tau^{\kappa,\alpha(,\beta)}>t\}}\big)$$

for each $f \in bm(\Omega, \mathcal{F})$ and every real $p \geq 0$. Furthermore, we have

$$\eta^{\kappa+p,\alpha(,\beta)} = \eta^{\kappa,\alpha(,\beta)} + p.$$

proof. We only show the assertion if we have two boundaries 0 and d. (The proof is analog if the state space is $E = [0,\infty)$.) We divide the proof in 3 cases:

1. There is instant killing at the boundaries, i.e. $\alpha = \beta = \infty$.

2. The process is instantly killed only at one boundary. W.l.o.g. we regard $\alpha < \beta = \infty$.

3. We assume that $\alpha, \beta < \infty$.

on 1. Here we have

$$\begin{aligned}
&\mathbb{P}_x(f\cdot \mathbb{1}_{\{\tau^{\kappa+p,\infty,\infty}>t\}}) \\
=&\mathbb{P}_x(f\cdot \mathbb{1}_{\{T_0\wedge T_d>t\}}\mathbb{1}_{\{\tau^{\kappa+p,0,0}>t\}}) \\
=&\mathbb{P}_x(f\cdot \mathbb{1}_{\{T_0\wedge T_d>t\}}\,\mathrm{e}^{-\int_0^t(\kappa+p)(Y_s)ds})
\end{aligned}$$

The last equality is due to $\tau^{\kappa+p} = \inf\{t\,;\ \int_0^t (\kappa+p)(Y_s)ds > \xi\}$ with a standard exponentially distributed ξ independent of $\mathcal{F}^Y = \sigma(Y_s,\, s \geq 0)$.

Hence,

$$\mathbb{P}_x(f\cdot\mathbb{1}_{\{\tau^{\kappa+p,\infty,\infty}>t\}}) = \mathbb{P}_x(f\cdot\mathbb{1}_{\{T_0\wedge T_d>t\}}\cdot \mathrm{e}^{-\int_0^t \kappa(Y_s)ds}\,\mathrm{e}^{-pt}) = \mathrm{e}^{-pt}\,\mathbb{P}_x(f\cdot\mathbb{1}_{\{\tau^{\kappa,\infty,\infty}>t\}}).$$

By taking $f \equiv 1$ we see that

$$\begin{aligned}\eta^{\kappa+p,\infty,\infty} &= -\lim_{t\to\infty} \frac{\log \mathbb{P}_x(\tau^{\kappa+p,\infty,\infty} > t)}{t} \\ &= -\lim_{t\to\infty} \frac{\log\left[\mathrm{e}^{-pt}\,\mathbb{P}_x(f\cdot\mathbb{1}_{\{\tau^{\kappa,\infty,\infty}>t\}})\right]}{t} \\ &= \eta^{\kappa,\infty,\infty} + p.\end{aligned}$$

on 2. Now we have

$$\begin{aligned}&\mathbb{P}_x(f\cdot\mathbb{1}_{\{\tau^{\kappa,\alpha,\infty}>t\}}) \\ =&\mathbb{P}_x(f\cdot\mathbb{1}_{\{T_d>t\}}\mathbb{1}_{\{\tau^{\kappa+p,\alpha,0}>t\}}) \\ =&\mathbb{P}_x(f\cdot\mathbb{1}_{\{T_d>t\}}\,\mathrm{e}^{-2\alpha L_t^0 - \int_0^t (\kappa+p)(Y_s)ds}).\end{aligned}$$

We remind the reader that $(t,y) \mapsto L_t^y = L_t^y(Y)$ is the a.s. continuous local time of Y w.r.t. $m(dx)$. Now we proceed as in case **1.**, e.g.

$$\eta^{\kappa+p,\alpha,\infty} = -\lim_{t\to\infty} \frac{\log\left[\mathbb{P}_x(\mathbb{1}_{\{T_d>t\}}\cdot \mathrm{e}^{-2\alpha L_t^0 - \int_0^t \kappa(Y_s)ds})\,\mathrm{e}^{-pt}\right]}{t} = \eta^{\kappa,\alpha,\infty} + p.$$

on 3. This is analog to the previous case. Simply use

$$\mathbb{P}_x(f\cdot\mathbb{1}_{\{\tau^{\kappa,\alpha,\beta}>t\}}) = \mathbb{P}_x(f\cdot \mathrm{e}^{-2\alpha L_t^0 - 2\beta L_t^d - \int_0^t \kappa(Y_s)ds}).$$

■

proposition 2.10. *Suppose A 2, A 3, A 5, A 6 and A 7 as well as $\kappa(x) \geq K$ for all x. Further suppose we have $\underline{\lambda}^{0,\alpha} > 0$ (e.g. through some sufficient condition from chapter 1). Then*

(a) the mortality rate is $\eta = \underline{\lambda}^{\kappa,\alpha} > K$.

(b) for all $s \geq 0$ and $F \in \mathcal{F}_s$ we have

$$\mathbb{P}_x(F \mid \tau > t) \to \mathrm{e}^{\eta s}\,\mathbb{P}_x\left(\mathbb{1}_F\cdot\mathbb{1}_{\{\tau>s\}}\frac{\psi_\eta(Y_s)}{\psi_\eta(Y_0)}\right).$$

proof. The proof has two steps:

1. We show the assertion for $K = \lim_{x\to\infty} \kappa(x) = 0$.

2. We use this to prove 2.10.

on 1. As soon as we have $\underline{\lambda}^{\kappa,\alpha} > 0$ we are able to show that

$$\lim_{t\to\infty} \frac{\mathbb{P}_x(\mathbb{1}_F \mathbb{1}_{\{\tau^\kappa > s\}} w_{t-s}(Y_s))}{w_t(x)} = \mathbb{P}_x(\mathbb{1}_F \mathbb{1}_{\{\tau^\kappa > s\}} \frac{\psi_\eta(Y_s)}{\psi_\eta(Y_0)})$$

just like we did in the proof of proposition 2.8. So let us try to see $\underline{\lambda}^{\kappa,\alpha} > 0$:
If we have $\underline{\lambda}^{0,\alpha} > 0$, we have existence of a QLD and also that

$$\eta^{0,\alpha} = \underline{\lambda}^{0,\alpha}$$

by proposition 1.8. But the exponential rate of killing becomes only higher if we have killing in the interior, i.e.

$$\eta_x^{\kappa,\alpha} = -\lim_{t\to\infty} \frac{\log \mathbb{P}_x(\tau^\kappa > t)}{t} \geq -\lim_{t\to\infty} \frac{\log \mathbb{P}_x(\tau^0 > t)}{t} = \eta^{0,\alpha} > 0.$$

Now we use Theorem 3.4 (i) of [SE07] to derive

$$\underline{\lambda}^{\kappa,\alpha} = \eta^{\kappa,\alpha} = \eta_x^{\kappa,\alpha} > 0.$$

Finally we have to verify that

$$\frac{\mathbb{P}_x(\tau^\kappa > t-s)}{\mathbb{P}_x(\tau^\kappa > t)} \to e^{\eta s}.$$

For this we apply, as in the proof of the preceding proposition, lemma 4.1 of [KS12].

on 2. Consider $\tilde{\kappa} := \kappa - K$ instead of κ. To surpress unnecessary notation write 'tilde' on top of everything concerning $\tilde{\kappa}$. By point **1** we have

$$\tilde{\eta} = \tilde{\underline{\lambda}} > 0.$$

Using lemma 2.9 we get

$$\eta = \tilde{\eta} + K$$

giving in particular that $\eta > K$. By the 'initial value problem characterization' of $\underline{\lambda}$ (via adding $K\psi$ to the equation) we see that

$$-\underline{\lambda} + K = -\tilde{\underline{\lambda}}.$$

This gives

$$\eta = \tilde{\eta} + K = \tilde{\underline{\lambda}} + K = \underline{\lambda}.$$

We can factor out e^{-Kt} by lemma 2.9 to obtain

$$\mathbb{P}_x(F \mid \tau > t) = \mathbb{P}_x(F \mid \tilde{\tau} > t).$$

This converges to

$$e^{\tilde{\eta}s}\mathbb{P}_x\left(\mathbb{1}_F\mathbb{1}_{\{\tilde{\tau}>s\}}\frac{\tilde{\psi}_{\tilde{\eta}}(Y_s)}{\tilde{\psi}_{\tilde{\eta}}(Y_0)}\right)$$

by point **1**.

Writing carefully the 'defining' initial value problem characterizations for $\tilde{\psi}_{\tilde{\eta}} = \tilde{\psi}_{\eta-K}$ and ψ_η we see that

$$\tilde{\psi}_{\tilde{\eta}} = \psi_\eta.$$

(Because $(\eta - K)\tilde{\psi}_{\tilde{\eta}} = \tilde{\eta}\cdot\tilde{\psi}_{\tilde{\eta}} = -\tilde{\mathbf{L}}\tilde{\psi}_{\tilde{\eta}} = -\mathbf{L}\tilde{\psi}_{\tilde{\eta}} - K\tilde{\psi}_{\tilde{\eta}}$ and $\tilde{\psi}_{\tilde{\eta}}$ is positive on $(0,\infty)$ and fulfils the correct boundary condition, e.g. $\tilde{\psi}'_{\tilde{\eta}}(0) = 2\alpha\tilde{\psi}_{\tilde{\eta}}(0)$ if $\alpha < \infty$.) Finally observe that $\mathbb{P}_x(F \mid \tau > t)$ converges to

$$e^{(\eta-K)s}\mathbb{P}_x\left(\mathbb{1}_F\mathbb{1}_{\{\tau>s\}}\frac{\psi_\eta(Y_s)}{\psi_\eta(Y_0)}\right)e^{Ks} = e^{\eta s}\mathbb{P}_x\left(\mathbb{1}_F\mathbb{1}_{\{\tau>s\}}\frac{\psi_\eta(Y_s)}{\psi_\eta(Y_0)}\right).$$

■

By theorem 1.12 we could as well give a more general version of proposition 2.8. More general in the sense that we assume only $K = \lim_{x\to\infty}\kappa(x) = 0$:

theorem 2.11. *Suppose A 2, A 3, A 5, A 6 and A 7 as well as $\kappa(x) \geq K$ for all x. Then, under the existence of a QLD,*

(a) the mortality rate is $\eta = \underline{\lambda}^{\kappa,\alpha} > K$.

(b) for all $s \geq 0$ and $F \in \mathcal{F}_s$ we have

$$\mathbb{P}_x(F \mid \tau > t) \to \mathrm{e}^{\eta s}\,\mathbb{P}_x\left(\mathbb{1}_F\cdot\mathbb{1}_{\{\tau>s\}}\frac{\psi_\eta(Y_s)}{\psi_\eta(Y_0)}\right).$$

proof. It is essentially the proof of proposition 2.10 above. Because, by theorem 1.12, we know that the existence of a QLD implies $\underline{\lambda} > 0$. ■

Note that the above penalising theorems use proposition 2.7. This is why we assumed A 6 and, therefore, the drift coefficient is from $C^1(0,\infty)$. Let us now formulate and prove a slightly stronger version of proposition 2.7 under the existence of a QLD. This will enable us to show a version of theorem 2.11 under $\boldsymbol{a} \in C(0,\infty)$. (See theorem 2.13.)

lemma 2.12. *Suppose we have a killing rate κ and a drift coefficient $\boldsymbol{a}$ which fulfil A 1, A 2, A 3, A 5 and A 7. Further assume that $K = \lim_{y\to\infty}\kappa(y) \leq \kappa(x)$ eventually and that we have killing at the boundary, i.e. $\alpha > 0$. Then, under the existence of a QLD,*

(a) $w_t \to \psi_{\underline{\lambda}}$ uniformly on compacts.

(b) $\eta = \underline{\lambda}$ is the asymptotic mortality rate.

proof. We use the first part of Lemma 5.1 from [SE07]:

Any sequence $t_n^* \uparrow \infty$ has a subsequence t_n such that w_{t_n} converges (uniformly on compacts) to a continuous limit $g^{(t_n)}$ with $g^{(t_n)}(x) > 0$ for all $x > 0$.

Note that the proof which is given in [SE07] also holds under the assumptions of lemma 2.12. Using this in connection with A 7 gives us

$$\lim_{n\to\infty} \mathbb{P}_x(w_{t_n}(Y_s),\, \tau > s) = \mathbb{P}_x(g^{(t_n)}(Y_s),\, \tau > s). \tag{2.18}$$

(We skip the superscripts for κ and α to increase readability.)

On the other hand

$$\begin{aligned} \lim_{n\to\infty} \mathbb{P}_x(w_{t_n}(Y_s),\, \tau > s) &= \lim_{n\to\infty} \frac{\mathbb{P}_x(\mathbb{P}_{Y_s}(\tau > t_n),\, \tau > s)}{\mathbb{P}_1(\tau > t_n)} \cdot \frac{\mathbb{P}_x(\tau > t_n)}{\mathbb{P}_1(\tau > t_n)} \\ &= \lim_{n\to\infty} \frac{\mathbb{P}_x(\tau > t_n + s)}{\mathbb{P}_x(\tau > t_n)} \cdot g^{(t_n)}(x) \\ &= \mathrm{e}^{\underline{\lambda} s} \cdot g^{(t_n)}(x). \end{aligned} \tag{2.19}$$

Here we have used the existence of a QLD together with corollary 1.13 to get $\underline{\lambda} > K$. Then we used proposition 1.8 (b) to obtain $\eta = \underline{\lambda}$. By (2.18) and (2.19) we see that

$$\mathbf{T}_s g^{(t_n)} = \mathrm{e}^{-\underline{\lambda} s}\, g^{(t_n)}.$$

Furthermore,

$$\mathbf{T}_s g^{(t_n)} \in D(\mathbf{L})$$

as pointed out in [KS12]. This implies

$$g^{(t_n)} \in D(\mathbf{L}). \tag{2.20}$$

Therefore, we get

$$\frac{\mathbf{T}_s g^{(t_n)} - g^{(t_n)}}{s} = \frac{\mathrm{e}^{-\eta s}\, g^{(t_n)} - g^{(t_n)}}{s} \to -\eta g^{(t_n)}$$

in $L^2(m)$ as $s \to 0$. Because of (2.20) we deduce

$$\mathbf{L} g^{(t_n)} = -\eta g^{(t_n)}.$$

Finally $g^{(t_n)} > 0$ gives

$$g^{(t_n)} = \psi_\eta = \underline{\psi}.$$

Since any $t_n^* \uparrow \infty$ has a subsequence (t_n) such that $\lim_{n\to\infty} w_{t_n} = \psi_\eta$ we have

$$\lim_{t\to\infty} w_t = \psi_\eta.$$

■

theorem 2.13. *Suppose we have a killing rate κ and a drift coefficient $\boldsymbol{a}$ which fulfil A 1, A 2, A 3, A 5 and A 7. Further assume that $K = \lim_{y\to\infty} \kappa(y) \leq \kappa(x)$ eventually and that we have killing at the boundary, i.e. $\alpha > 0$. Then, under the existence of a QLD,*

(a) the mortality rate is $\eta = \underline{\lambda} > K$.

(b) for all $s \geq 0$ and $F \in \mathcal{F}_s$ we have

$$\mathbb{P}_x(F \mid \tau > t) \to \mathrm{e}^{\eta s}\, \mathbb{P}_x\left(\mathbb{1}_F \cdot \mathbb{1}_{\{\tau > s\}} \frac{\psi_\eta(Y_s)}{\psi_\eta(Y_0)}\right).$$

proof. By the Markov property at time s we write

$$\mathbb{P}_x(F \mid \tau > t) = \frac{\mathbb{P}_x(\mathbb{1}_F \mathbb{1}_{\{\tau>s\}} \mathbb{P}_{Y_s}(\tau > t - s))}{\mathbb{P}_x(\mathbb{1}_{\{\tau>t\}})}$$

giving

$$\mathbb{P}_x(F \mid \tau > t) = \frac{\mathbb{P}_x(\mathbb{1}_F \mathbb{1}_{\{\tau>s\}} w_{t-s}(Y_s))}{w_t(x)} \cdot \frac{\mathbb{P}_1(\tau > t - s)}{\mathbb{P}_1(\tau > t)}.$$

By the existence of a QLD and by corollary 1.13 we obtain $\underline{\lambda} > K$. Furthermore, the second term from above converges to

$$\mathrm{e}^{\eta s},$$

by applying Lemma 4.1 of [KS12]. $\eta = \underline{\lambda}$ is one part of lemma 2.12.

Finally we can use Lebesgue's dominated convergence because of lemma 2.12 and A 7. ■

Under the abscence of a QLD

As the title of this subsection states, we suppose that there is *no* Yaglom limit.

We deal with the case of 'low killing at infinity' (as named in [KS12]). This means that we assume

A 10. We have that $K = \lim_{x\to\infty} \kappa(x) < \underline{\lambda}$.

A typical situation may be a drift coefficient $\boldsymbol{a} \geq \epsilon > 0$, which makes Y transient, and a decreasing killing rate κ. Although this situation is rather unspectacular since the penalisation limit will inherit the transient behaviour we still want to give the following penalisation result.

theorem 2.14. *Suppose A 2, A 6 and A 10; in particular $K \neq \underline{\lambda}$. Let Y be transient. Suppose further that $\kappa \downarrow$ and that κ and the drift coefficient $\boldsymbol{a}$ fulfil any of the conditions of lemma 1.16. Then we have*

$$\mathbb{P}_x(F \mid \tau > t) \to \mathrm{e}^{K s}\, \mathbb{P}_x\left(\mathbb{1}_F \cdot \mathbb{1}_{\{\tau > s\}} \frac{\psi_K(Y_s)}{\psi_K(x)}\right),$$

for all $x > 0, s > 0$ and $F \in \mathcal{F}_s$.

remark 2.6.

- Under $\kappa \downarrow$, the assumption $K \neq \underline{\lambda}$ is the same as to say that A 10 is valid. This is due to the fact that we only have to look at the case $K = 0$ and because of $\underline{\lambda} \geq 0$.
- In the case of $\kappa \equiv 0$ we see by Corollary 4.8 of [KS12] that we must have transience of Y. To see this we assume that Y is recurrent. Then we have the existence of a *QLD* by $\underline{\lambda} > 0$ which obviously 'conctradicts the headline' of this subsection.
- If we additionally assume that $\boldsymbol{a}$ is locally integrable at zero, we may use corollary 1.13, instead, to see that transience holds. This implies that the assumption 'Let Y be transient.' would be unnecessary.

■

proof of theorem 2.14. Using lemma 1.16 we get the existence of a dominant, i.e.

$$\mathbb{P}_x\big(\sup_t w_t(Y_s)\cdot\mathbb{1}_{\{\tau>s\}}\big) < \infty.$$

By lemma 2.9 this holds also if we use the killing rate $\tilde{\kappa} := \kappa - K$, instead. Thus, we have

$$\mathbb{P}_x\big(\sup_t \tilde{w}_t(Y_s)\cdot\mathbb{1}_{\{\tilde{\tau}>s\}}\big) < \infty.$$

(We will write a tilde above all connected to $\tilde{\kappa}$.) By proposition 2.7 we get

$$\tilde{w}_t(x) \to \frac{\tilde{\psi}_{\tilde{\eta}}(x)}{\tilde{\psi}_{\tilde{\eta}}(1)}. \tag{2.21}$$

Here $\tilde{\psi}_{\tilde{\eta}}$ is a solution to

$$\tilde{\mathbf{L}}f := (\mathbf{L}^\kappa + K)f = -\tilde{\eta}f$$

under $f'(0) = 2\alpha f(0)$. Now we need to see that the mortality rate $\tilde{\eta} = 0$:

For this we make an additional assumption, which will later turn out to hold anyway; namely

$$\mathbb{P}_y(T_x < \tau) > 0 \tag{2.22}$$

for all y, x. (See the end of the proof.) Now we can use the Markov property to obtain

$$\begin{aligned}
\mathbb{P}_y(\tilde{\tau} > t) &\geq \mathbb{P}_y(\tilde{\tau} > t + T_x) \\
&= \mathbb{P}_y(\tilde{\tau} > T_x, \mathbb{P}_x(\tilde{\tau} > t)) \\
&= \mathbb{P}_y(\tilde{\tau} > T_x)\mathbb{P}_x(\tilde{\tau} > t) \\
&\geq \mathbb{P}_y(\tilde{\tau} > T_x)\mathbb{P}_x(T_{\frac{x}{2}} \wedge \tilde{\tau} > t) \\
&\geq \mathbb{P}_y(\tilde{\tau} > T_x)\mathbb{P}_x(T_{\frac{x}{2}} > t, \int_0^t \tilde{\kappa}(Y_s)ds < \xi) \\
&\geq \mathbb{P}_y(\tilde{\tau} > T_x)\mathbb{P}_x(T_{\frac{x}{2}} > t, \int_0^t M ds < \xi) \\
&= \mathbb{P}_y(\tilde{\tau} > T_x)e^{-Mt}\mathbb{P}_x(T_{\frac{x}{2}} > t).
\end{aligned}$$

ξ is standard exponentially distributed and independent of 'the rest' and $M = M(\tilde{\kappa}, \frac{x}{2}) := \max_{z\geq\frac{x}{2}} \tilde{\kappa}(z)$.

Since we are in the transient case, we have that

$$\mathbb{P}_x(T_{\frac{x}{2}} > t) \to \mathbb{P}_x(T_{\frac{x}{2}} = \infty) > 0.$$

Thus, the mortality rate, when starting at some y, satisfies

$$\tilde{\eta}(\delta_y) \leq M.$$

This can be made arbitrarily small because of $\tilde{\kappa} \downarrow 0$. With this in mind, we get the limit in

(2.21) is equal to

$$\frac{\psi_K(x)}{\psi_K(1)}.$$

(Because $(K-K)\tilde{\psi}_{\tilde{\eta}} = \tilde{\eta}\cdot\tilde{\psi}_{\tilde{\eta}} = -\tilde{\mathbf{L}}\tilde{\psi}_{\tilde{\eta}} = -\mathbf{L}^{\kappa}\tilde{\psi}_{\tilde{\eta}} - K\tilde{\psi}_{\tilde{\eta}}$ implies $-\mathbf{L}^{\kappa}\tilde{\psi}_{\tilde{\eta}} = K\tilde{\psi}_{\tilde{\eta}}$. Furthermore, $\tilde{\psi}_{\tilde{\eta}}$ is positive on $(0,\infty)$ and fulfils the correct boundary condition, i.e. $\tilde{\psi}'_{\tilde{\eta}}(0) = 2\alpha\tilde{\psi}_{\tilde{\eta}}(0)$ if $\alpha < \infty$.) Now we use a trick from the proof of Th 6.26 of [CMSM13b]: Since $\tilde{\kappa} \downarrow 0$, we get, for all $z > 0$,

$$\begin{aligned}
\mathbb{P}_x(\tilde{\tau} > t+s) &= \mathbb{P}_x(\tilde{\tau} > t, \mathbb{P}_{Y_t}(\tilde{\tau} > s)) \\
&\geq \mathbb{P}_x(\tilde{\tau} > t, Y_t \geq z, \mathbb{P}_{Y_t}(\tilde{\tau} > s)) \\
&\geq \mathbb{P}_x(\tilde{\tau} > t, Y_t \geq z, \mathbb{P}_z(\tilde{\tau} > s)).
\end{aligned}$$

This implies

$$1 \geq \liminf_t \frac{\mathbb{P}_x(\tilde{\tau} > t+s)}{\mathbb{P}_x(\tilde{\tau} > t)} \geq \liminf_t \mathbb{P}_x(Y_t \geq z \mid \tilde{\tau} > t)\mathbb{P}_z(\tilde{\tau} > s) = \mathbb{P}_z(\tilde{\tau} > s),$$

where the last equality is due to the fact that we have always 'escape to infinity' under low killing at ∞. (Confer Theorem 4.9 of [KS12].) Using that ∞ is natural, we see that $\lim_{z\to\infty}\mathbb{P}_z(\tilde{\tau} > s) = 1$. Hence,

$$\lim_{t\to\infty} \frac{\mathbb{P}_x(\tilde{\tau} > t+s)}{\mathbb{P}_x(\tilde{\tau} > t)} = 1.$$

Putting the parts together, we get, as usual,

$$\begin{aligned}
\mathbb{P}_x(F \mid \tau > t) &= \frac{\mathbb{P}_x(F, \tilde{\tau} > t)}{\mathbb{P}_x(\tilde{\tau} > t)} \\
&= \frac{\mathbb{P}_x(F, \tilde{\tau} > s, \mathbb{P}_{Y_s}(\tilde{\tau} > t-s))}{\mathbb{P}_x(\tilde{\tau} > t)} \\
&= \frac{\mathbb{P}_x(F, \tilde{w}_{t-s}(Y_s), \tilde{\tau} > s)}{\tilde{w}_t(x)} \cdot \frac{\mathbb{P}_1(\tilde{\tau} > t-s)}{\mathbb{P}_1(\tilde{\tau} > t)} \\
&\to \mathbb{P}_x(F, \frac{\psi_K(Y_s)}{\psi_K(x)}, \tilde{\tau} > s) = \mathbb{P}_x(F, \mathrm{e}^{Ks}\frac{\psi_K(Y_s)}{\psi_K(x)}, \tau > s).
\end{aligned}$$

Finally one little fact has to be verified. This is that the additional assumption (2.22) is always true in the present case:

Of course, the diffusion Y is nice enough to have $\mathbb{P}_x(T_y < \infty) > 0$ for all $x, y \geq 0$ (due to regularity). Thus, we find a t_0 such that $\mathbb{P}_x(T_y \leq t_0) > 0$. Hence,

$$\begin{aligned}
\mathbb{P}_x(T_y \leq \tau) &= \mathbb{P}_x(\int_0^{T_y} \kappa(Y_s)ds < \xi) \\
&= \mathbb{P}_x(e^{-\int_0^{T_y}\kappa(Y_s)ds}) \\
&\geq \mathbb{P}_x(e^{-\int_0^{T_y}\kappa(Y_s)ds}, T_y \leq t_0) \\
&\geq \mathbb{P}_x(e^{-\int_0^{t_0} Mds}, T_y \leq t_0) \\
&\geq e^{-Mt_0}\mathbb{P}_x(T_y \leq t_0) > 0.
\end{aligned}$$

Here, ξ is standard exponentially distributed and independent of Y and $M = M(\kappa, y)$ is an upper bound for κ on $[y, \infty)$. (It is the same notation as in the last proof.)

■

remark 2.7. Collet et al. proved this result for the special case of $\tau = T_0$ in Th 6.16 of [CMSM13b]. ▮

2.3. The penalisation limit as Doob's h-transform

In section 2.2 we gave several penalisation results for regular diffusions Y in $[0, \infty)$ (and in $[0, d]$). See theorems 2.13 and 2.14 (and proposition 2.3). Essentially our penalisers were $t \mapsto \mathbb{1}_{\{\tau^{\kappa,\alpha(,\beta)} > t\}}$.

This section has two goals:

- to see that the penalisation limits, in particular from the mentioned propositions, can be regarded as 'h-transforms' were the 'h' stands for 'harmonic'.
- to identify the penalisation limit as the original diffusion but with an additional drift as already suggested by the calculations surrounding equation (2.13).

On the first point so much could be said: All penalisation limits obtained in this thesis for $\Gamma_t = \mathbb{1}_{\{\tau^{\kappa,\alpha(,\beta)} > t\}}$ (of $\mathbb{P}_x$ on $\mathcal{F}_s$) are of the form

$$\mathrm{e}^{\lambda s} \frac{\psi_\lambda(Y_s)}{\psi_\lambda(Y_0)} \circ \mathbb{P}_x \tag{2.23}$$

with

$$\left[\big(\frac{1}{2} \frac{d^2}{dx^2} + \boldsymbol{a}(x) \frac{d}{dx} - \kappa(x) \big) + \lambda \right] \psi_\lambda = 0 \tag{2.24}$$

were ψ_λ satisfies the boundary condition (0.16) at 0 (and a similar boundary condition but with α replaced by β at c). Equation (2.24) is the reason why the penalisation limit is called *harmonic-transformation*. For further information on h-transforms in the context of diffusions we recommend section VIII.3 of [RY99], section 3.1.4 of [NRY09] and section I.II.5 of [BS02].

remark 2.8. If $E = [0, c]$ and the diffusion is indeed regular, we have that $\underline{\psi} = \psi_{\underline{\lambda}} \in D(\mathbf{L}^{\kappa,\alpha,\beta})$. And we can write

$$\left[\mathbf{L}^{\kappa,\alpha,\beta} + \underline{\lambda} \right] \underline{\psi} = 0$$

replacing (2.24). ▮

Next we concentrate on the second point. We first give an outline and then work through the details:

1. We first give some information how a process under killing is modelled such that killing turns into absorbtion in a point added to the state space.
2. We introduce 'Feller-Dynkin processes' and 'Feller-Dynkin semigroups' and show that our diffusions under possible killing have this Feller-Dynkin property.

3. This helps to show that the h-transformed diffusions solve a certain 'martingale problem'.

4. We use this to prove that the h-transformed diffusions are the unique solution to certain 'generalized martingale problem'.

5. Finally we deduce that the h-transformed diffusions solve a SDE in the interior of the state space, which looks similar to the SDE for Y but with an additional drift term. Furthermore, the boundaries for the h-transformed diffusions reflecting whenever accessible. This is proposition 2.18 which is the main result of section 2.3.

on 1.

We need to understand that killing at a certain stopping time is modelled in literature by introducing an additional point '†', often called *cemetary point*, to the state space. Of course the whole probabilistic setting must be enlarged as well. (See e.g. [BG68].) The next table gives an overview and introduces notations we are going to use later on.

the picture on $\mathcal{F}_\tau$	'death' at τ
state space E	state space $\hat{E} = E \cup \{\dagger\}$
the model: $(\Omega, \mathcal{F}_\infty, (\mathcal{F}_t)_t, (\mathbb{P}_x)_{x \in E})$	enlarged model: $(\hat{\Omega}, \hat{\mathcal{F}}_\infty, (\hat{\mathcal{F}}_t)_t, (\hat{\mathbb{P}}_x)_{x \in \hat{E}})$
killing at τ	entry time $\hat{\tau}$ into absorbing state †
$t \mapsto Y_t$ is continuous	$t \mapsto \hat{Y}_t$ is continuous except for a jump at $\hat{\tau}$

Moreover $(\mathcal{F}_\infty, (\mathcal{F}_t)_t, (\mathbb{P}_x)_{x \in E})$ must be large enough to carry additional independent random variables for the killing mechanism as described in chapter 0. Note that in what follows we will not always mention the parameters κ, α, β explicitely which control the killing mechanism. This was already done in the table above.

convention 2.4. We will also talk of the $(\hat{\bullet})$-*picture* if we want to emphasize that our model is enlarged through †. Furthermore, we will use the notation '$(\hat{\bullet})$' for every other object given in the $(\hat{\bullet})$-picture.

on 2.

Y_t under τ is a *Feller-Dynkin process* if the semigroup $\mathbf{T}_t f(x) = \mathbb{P}_x(f(Y_t)\mathbb{1}_{\{\tau > t\}})$ of $(Y_t)_t$ under killing or the semigroup $(\hat{\mathbf{T}}_t)_t$ of $(\hat{Y}_t)_t$ is a *Feller-Dynkin semigroup*, i.e.

1. For each $t \geq 0$ the operator $\mathbf{T}_t$ acts on the space C_0 of continuous functions vanishing for $\|x\| \to \infty$ and at †, respectively. Briefly, $\mathbf{T}_t(C_0) \subseteq C_0$ for all $t \geq 0$.
2. $\mathbf{T}_{s+t} = \mathbf{T}_s \mathbf{T}_t$ for all $s, t \geq 0$.
3. $\|\mathbf{T}_t\|_\infty \leq 1$ for all $t \geq 0$.
4. $\|\mathbf{T}_t f - f\|_\infty \to 0$ for all $f \in C_0$ as $t \to 0$.

5. It is *sub-Markovian* which means that $0 \leq \mathbf{T}_t(f) \leq 1$ whenever $0 \leq f \leq 1$.

Here $\|\cdot\|_\infty$ is the *sup norm.*

remark 2.9. By the one-on-one correspondence

$$C_0(\hat{E}) \ni f \mapsto f|_E \in C_0(E)$$

and

$$\hat{\mathbf{T}}_t f(x) = \begin{cases} 0, & x = \dagger \\ \mathbf{T}_t f|_E(x), & x \in E \end{cases}$$

it is easy to see that $(\mathbf{T}_t)$ is Feller-Dynkin if and only if $(\hat{\mathbf{T}}_t)$ is Feller-Dynkin. ∎

The next lemma shows that Y possesses the Feller-Dynkin property. Though the proof may use a standard argumentation we include it for the sake of completeness.

lemma 2.15. *Suppose that Y is a regular diffusion on $[0,\infty)$ (respectively on $[0,c]$). Let κ be a killing rate from $C_b[0,\infty)$ (respectively from $C[0,c]$). And let there be killing at zero with a killing rate parametrized by $\alpha \geq 0$ (and killing at c parametrized by $\beta \geq 0$). This means $\tau = \tau^{\kappa,\alpha(,\beta)}$. Then $(\mathbf{T}_t)_t$ has the Feller-Dynkin property.*

For the case of $E = [0,c]$ nothing is to prove anymore: Since, by definition, we have $C_0[0,c] = C_b[0,c]$ and our diffusions already have the 'Feller property'. (See [BS02].) Though the following proof for $E = [0,\infty)$ also works on $E = [0,c]$.

proof. We concentrate on proving the lemma for $E = [0,\infty)$. Since we are dealing with a regular diffusion on $[0,\infty)$, we have the existence of a jointly continuous local time $L_t^x(Y)$, or simply L_t^x when there is no danger of confusion. With this in mind we show the Feller-Dynkin property (as in [Wil79] or [BS02]) of the semigroup under (slow) killing. Therefore, set

$$\mathbf{T}_t^\alpha f(x) := \mathbb{E}_x\left[f(Y_t)\,\mathrm{e}^{-2\alpha L_t^0}\right]$$

which is exactly this semigroup under slow killing at zero (; see section 0). Now we use a trick due to [Wil79] p. 156 ff. The C_0-*resolvents* of $(\mathbf{T}_t^\alpha)_{t\geq 0}$ are

$$\mathbf{R}_\lambda^\alpha f(x) := \int_0^\infty \mathrm{e}^{-\lambda s}\,\mathbf{T}_s^\alpha f(x) ds,$$

for any $f \in C_0[0,\infty)$ and $\lambda > 0$. ($f \in C_0$ means here that $f(x) \to 0$ as $x \to \infty$.)

By the strong MP of Y we calculate

$$\begin{aligned}
&\mathbf{R}^\alpha_\lambda f(x)\\
=&\mathbb{E}_x\left[\int_0^\infty \mathrm{e}^{-\lambda s}\,\mathrm{e}^{-2\alpha L^0_s} f(Y_s)ds\right]\\
=&\mathbb{E}_x\left[\int_0^{T_y} \mathrm{e}^{-\lambda s}\,\mathrm{e}^{-2\alpha L^0_s} f(Y_s)ds\right]+\mathbb{E}_x\left[\int_0^\infty \mathrm{e}^{-\lambda(s+T_y)}\,\mathrm{e}^{-2\alpha L^0_{s+T_y}} f(Y_{s+T_y})ds\right]\\
=&\mathbb{E}_x\left[\int_0^{T_y} \mathrm{e}^{-\lambda s}\,\mathrm{e}^{-2\alpha L^0_s} f(Y_s)ds\right]+\int_0^\infty \mathrm{e}^{-\lambda s}\,\mathbb{E}_x\left[\mathrm{e}^{-\lambda T_y}\,\mathrm{e}^{-2\alpha L^0_{T_y}}\right]\mathbb{E}_y\left[\mathrm{e}^{-2\alpha L^0_s} f(Y_s)\right]ds\\
=&\mathbb{E}_x\left[\int_0^{T_y} \mathrm{e}^{-\lambda s}\,\mathrm{e}^{-2\alpha L^0_s} f(Y_s)ds\right]+\mathbb{E}_x\left[\mathrm{e}^{-\lambda T_y}\,\mathrm{e}^{-2\alpha L^0_{T_y}}\right]\mathbf{R}^\alpha_\lambda f(y).
\end{aligned} \tag{2.25}$$

Since $\|\mathbf{R}_\lambda f\|_\infty \leq \frac{1}{\lambda}\|f\|_\infty$, we get

$$|\mathbf{R}^\alpha_\lambda f(x)-\mathbf{R}^\alpha_\lambda f(y)| \leq \frac{\|f\|_\infty}{\lambda}\left(1-\mathbb{E}_x\left[\mathrm{e}^{-\lambda T_y}\right]+1-\mathbb{E}_x\left[\mathrm{e}^{-\lambda T_y-2\alpha L^0_{T_y}}\right]\right). \tag{2.26}$$

Now we want to use that $\mathbb{P}_x(T_y \geq T_0)\vee\mathbb{P}_x(T_y\geq r)\to 0$, for any $r>0$ and $x,y>0$, as $y\to x$. (That $\mathbb{P}_x(T_y\geq T_0)\to 0$ is clear for $x\geq y$. If $x\leq y$, we have $\mathbb{P}_x(T_y\geq T_0)\leq\mathbb{P}_x(T_y\geq T_\epsilon)\to 0$, which is easily seen by the finiteness and continuity of the scale function for the diffusion on, let us say, $(\frac{\epsilon}{2},2y)$. And that $\mathbb{P}_x(T_y\geq r)\to 0$ comes from the fact that any regular diffusion excurses to the left and to the right of its starting point with probability 1 in any time window $[0,r]$, $r>0$. See part 1 chapter II section 1 in [BS02].)

Take r small enough such that $\mathrm{e}^{-\lambda r}>1-\delta$ for an arbitrary $\delta>0$. Next, take $|y-x|$ so small that $\mathbb{P}_x(T_y\geq r\wedge T_0)<\epsilon$, for any $\epsilon>0$. Hence, $\mathbb{E}_x\left[\,\mathrm{e}^{-\lambda T_y}\,\right]$ and $\mathbb{E}_x\left[\,\mathrm{e}^{-\lambda T_y-2\alpha L^0_{T_y}}\,\right]$ are greater or equal to $(1-\epsilon)\,\mathrm{e}^{-\lambda r}\geq(1-\epsilon)(1-\delta)$. This gives $|\mathbf{R}^\alpha_\lambda f(x)-\mathbf{R}^\alpha_\lambda f(y)|\to 0$ due to (2.26), as $y\to x$. Since $\mathbb{P}_x(T_y\leq t)\to 0$ as $x\to\infty$ we can use the third line of (2.25) to show that $\mathbf{R}^\alpha_\lambda f(x)\to 0$ as $x\to\infty$. (Just take y and t large enough to ensure that both $\sup_{z\geq y}|f(z)|$ and $\int_t^\infty \mathrm{e}^{-\lambda s}\,ds$ are small.) This implies

$$\mathbf{R}^\alpha_\lambda(C_0[0,\infty))\subseteq C_0[0,\infty).$$

Of course, we have also $0\leq T^\alpha_t f\leq 1$, if $0\leq f\leq 1$ and $\mathbf{T}^\alpha_{s+t}=\mathbf{T}^\alpha_s\mathbf{T}^\alpha_t$ with $\mathbf{T}_0=Id$. These are points 8.2(ii) and 8.2(iii) in chapter III of [Wil79]. If f is C_0, we also have that $T_tf(x)\to f(x)$ as $t\to 0$, which is due to the boundedness and (right-)continuity of $t\mapsto\mathrm{e}^{-2\alpha L^0_t} f(Y_t)$. This is point (8.2(iv))* in chapter III of [Wil79]. And if you look at the proof following point (8.2(iv))* you will find that for every $f\in C_0[0,\infty)$ we have

$$\|\lambda\mathbf{R}_\lambda f-f\|_\infty\to 0$$

as $\lambda\to\infty$. Thus, by the *Hille-Yosida theorem*, there is a unique semigroup $(\tilde{\mathbf{T}}^\alpha_t)_t$, which is, in particular strongly contiuous at $t=0$ such that

$$\mathbf{R}^\alpha_\lambda f(x):=\int_0^\infty \mathrm{e}^{-\lambda s}\,\tilde{\mathbf{T}}^\alpha_s f(x)ds,$$

for all $\lambda>0$ and $f\in C_0$. By inverse Laplace transform, we see that $\tilde{\mathbf{T}}^\alpha_t=\mathbf{T}^\alpha_t$. (For the suitable version of the Hille-Yosida theorem see Chapter III point (4) of [Wil79]; one can check directly via points (4.5) and (4.6) therein that $\tilde{\mathbf{T}}_t^\alpha$, hence, also $\mathbf{T}_t^\alpha$ preserves the C_0-property.) Finally, if we additionally kill in the interior by some $\kappa \in C_b([0,\infty), \mathbb{R}_+)$, then the corresponding semigroup

$$\mathbf{T}_t^{\kappa,\alpha} f(x) := \mathbb{E}_x \left[\mathrm{e}^{-2\alpha L_t^0 - \int_0^t \kappa(Y_s) ds} f(Y_t) \right]$$

is still a Feller-Dynkin semigroup and in particular strongly continuous at $t = 0$. (Indeed, this holds as you can see from III.39 of [Wil79].) ■

on 3.

Our Feller-Dynkin processes are also strong Markov processes and we have the existence of a generator $\hat{\mathbf{L}}_0$ in $(C_0(\hat{E}, \mathbb{R}), \|\cdot\|_\infty)$. (In fact, this is true in general as we can see by section III.3 in [Wil79].) Our interest in the Feller-Dynkin property comes from the fact that a Feller-Dynkin process $(\hat{\Omega}, \hat{\mathcal{F}}_\infty, (\hat{\mathcal{F}}_t), (\hat{\mathbb{P}}_x)_{x\in\hat{E}})$ solves the following 'martingale problem'.

For every $f \in D(\hat{\mathbf{L}}_0)$ and $x \in \hat{E}$

$$f(\hat{Y}_t) - \int_0^t \hat{\mathbf{L}}_0 f(\hat{Y}_s) ds \tag{2.27}$$

is a $(\hat{\mathcal{F}}_t)_t$-martingale under $\hat{\mathbb{P}}_x$. (See [Bov12].)

In our context we know that $\hat{\mathbf{L}}_0 = \hat{\mathbf{L}}_0^{\kappa,\alpha(,\beta)}$ is the trivial extension to $C_0(\hat{E}, \mathbb{R})$ of $\mathbf{L}_0 = \frac{1}{2}\frac{d^2}{dx^2} + \boldsymbol{a}(x)\frac{d}{dx} - \kappa(x)$ with

$$D(\mathbf{L}_0) = \{f \in C_0^2(E)\,;\ \frac{1}{2}f'' + \boldsymbol{a} f' - \kappa f \in C_0(E),\ f'(0) = 2\alpha f(0)(,\ f'(c) = 2\beta f(c))\}.$$

(See [BS02].) We remind the reader that E is either $[0,\infty)$ or $[0,c]$.) Define the first exit time from a measurable set A as

$$l_A = l_A(Y) := \inf\{t \geq 0\,;\ Y \notin A\}$$

and let $\hat{l}_A = \hat{l}_A(\hat{Y})$ be the corresponding object in the $(\hat{\bullet})$-picture. If we take a bounded closed interval I such that $I \subset (0,\infty)$, respectively $I \subset (0,c)$, then

$$f(\hat{Y}_{t\wedge\hat{l}_I}) - \int_0^{t\wedge\hat{l}_I} \hat{\mathbf{L}}_0 f(\hat{Y}_s) ds \tag{2.28}$$

is also a $(\hat{\mathcal{F}}_t)_t$-martingale under $\hat{\mathbb{P}}_x$. This can be seen by Corollary 3.2.8 of [vWW90]. Nevertheless, we even prove that optional stopping of the martingale from (2.27) holds at $\hat{l}_I$ and, on the way, we recall a few properties of the speed measure and scale function:

Note that $\hat{l}_I$ only grows if we set $\kappa \equiv 0$. Thus, it suffices to show $\hat{\mathbb{E}}_x[\hat{l}_I] = \mathbb{E}_x[l_I] < \infty$ in this case to apply optional stopping. The rest of the proof is essentially Theorem 16.36 of [Bre92]: We first bring the process on its natural scale, i.e. $Y_t^s := s(Y_t)$, and observe that $\mathbb{E}_x[l_I] = \mathbb{E}_x[l_{s(I)}(Y^s)]$. But I is finite and so is $s(I)$. Thus, we have $m(s(I)) < \infty$ since the speed measure is locally finite in the interior of the state space. Now optional stopping could be applied since $\mathbb{E}_x[l_{s(I)}(Y^s)] = \int_{s(I)} g(x) m(dx)$ for some bounded function g and, therefore, $\hat{\mathbb{E}}_x[\hat{l}_I] = \mathbb{E}_x[l_I] < \infty$.

Next we 'define the penalisation limit' in the $(\hat{\bullet})$-picture:

$$\hat{\mathbb{P}}_x^{\lambda,t}(F) := \mathrm{e}^{\lambda t}\,\hat{\mathbb{P}}_x\Big(\mathbb{1}_F\cdot\frac{\psi_\lambda(\hat{Y}_t)}{\psi_\lambda(\hat{Y}_0)}\mathbb{1}_{\{\hat{\tau}>t\}}\Big)$$

on $\hat{\mathcal{F}}_t$ and $x \in \mathring{E}$. We remind the reader that ψ_λ is the harmonic function satisfying (2.24), is positive in $\mathring{E}$ and satisfies the boundary condition (0.16) at 0 with α and at c with β, respectively.

The next remark is to make clear that for certain λ the $\hat{\mathbb{P}}_x^{\lambda,t}$ give rise to a probability distribution $\hat{\mathbb{P}}_x^\lambda$ on $\hat{\mathcal{F}}_\infty$ which is indeed the corresponding penalisation limit in the $(\hat{\bullet})$-picture.

remark 2.10.

- Note that in the penalisation results under consideration, e.g. theorems 2.13 and 2.14 and proposition 2.3, we have either $\lambda = \underline{\lambda}$ or $\lambda = K = \lim_{x\to\infty}\kappa(x)$ if $K > \underline{\lambda}$. In these situations we showed the existence of such a ψ_λ. But whenever $\lambda < \underline{\lambda}$ the existence is not clear. Thus, in such cases we would take the existence as an assumption.

- Furthermore, we need that

$$M_t^\lambda := \mathrm{e}^{\lambda t}\cdot\frac{\psi_\lambda(\hat{Y}_t)}{\psi_\lambda(\hat{Y}_0)}\mathbb{1}_{\{\hat{\tau}>t\}} \tag{2.29}$$

  is an $(\hat{\mathcal{F}}_t)_t$-martingale under $\hat{P}_x$. The purpose is to get the existence of the measure $\mathbb{P}_x^\lambda$ on $(\hat{\Omega}, \hat{\mathcal{F}}_\infty)$ such that $\hat{\mathbb{P}}_x^\lambda|_{\hat{\mathcal{F}}_s} = \mathbb{P}_x^{\lambda,s}$ for all $s \geq 0$. For this we intend to use proposition 2.1. Hence, we need to know that the measures $\hat{P}_x^{\lambda,s}$ arise as penalisation limits and that proposition 2.1 is still holds in the $(\hat{\bullet})$-picture:

  - Results such as theorems 2.13 and 2.14 and proposition 2.3 are still valid in the $(\hat{\bullet})$-picture. To see this we only need to know that we can 'rewrite'

$$\mathbb{P}_x\big(\mathbb{1}_F\cdot\mathbb{P}_{Y_s}(f(Y_r);\,\tau > r);\,\tau > s\big) \tag{2.30}$$

    with $F \in \mathcal{F}_s$, $s, r \geq 0$ and f integrable w.r.t. $\mathbb{P}_y(Y_r \in dz;\, \tau > r)$ for all $y \in E$ (such that the term in (2.30) is $< \infty$). But whenever $\tau > t$ then $Y_t \neq \dagger$ and the term in (2.30) equals

$$\hat{\mathbb{P}}_x\big(\mathbb{1}_F\cdot\hat{\mathbb{P}}_{\hat{Y}_s}(f(\hat{Y}_r);\,\hat{\tau} > r);\,\hat{\tau} > s\big).$$

    Finally it should be mentioned that we can take $F \in \hat{\mathcal{F}}_s$ and the proofs of the penalisation results we presented still work.

  - We get the results from proposition 2.1 in the $(\hat{\bullet})$-picture since we are still in a setting were Kolmogorov's extension theorem can be applied.

  Thus, we can use proposition 2.1 to see that $(M_t^\lambda)_t$ is a martingale.

∎

Next we show that $\hat{\mathbb{P}}_x^\lambda$ solves a certain martingale problem. To do so we first define $\hat{\mathbf{L}}_0^\lambda$ as the 'trivial extension' of

$$\mathbf{L}_0^\lambda f(x) := \frac{1}{\psi_\lambda(x)}\left[\frac{1}{2}\frac{d^2}{dx^2} + \boldsymbol{a}(x)\frac{d}{dx} - \kappa(x) + \lambda\right](\psi_\lambda f)(x) \tag{2.31}$$

on

$$D(L_0^\lambda) = \left\{ f \in C_0^2(E)\,;\ \frac{1}{\psi_\lambda}\left[\frac{1}{2}(f\psi_\lambda)'' + \boldsymbol{a}(f\psi_\lambda)' - \kappa(f\psi_\lambda)\right] \in C_0(E),\ f'(0) = 0(, f'(c) = 0) \right\}$$

to $C_0(\hat{E})$.

lemma 2.16. *Suppose that Y is a regular diffusion on $[0,\infty)$ (respectively on $[0,d]$). Let κ be a killing rate from $C_b[0,\infty)$ (respectively from $C[0,d]$). And let there be killing at zero with a killing rate parametrized by $\alpha \geq 0$ (and killing at d parametrized by $\beta \geq 0$). Now choose an interval I and a bounded open interval J such that*

$$\bar{I} \subset J \subset \bar{J} \subset \mathring{E}.$$

Then, in the $(\hat{\bullet})$-picture, we have that

$$f(\hat{Y}_{t\wedge\hat{l}_I}) - \int_0^{t\wedge\hat{l}_I} \hat{\mathbf{L}}_0^\lambda f(\hat{Y}_s)ds \tag{2.32}$$

is a $(\hat{\mathcal{F}}_t)_t$-martingale under $\hat{\mathbb{P}}_x^\lambda$. This holds for every $x \in J$ and every $f \in C^2$ with $\operatorname{supp}(f) \subset J$.

proof. At first it should be clear for $x \in J \setminus \mathring{I}$ since the expression in (2.32) is a.s. $f(x)$ in this case. So let us assume that $x \in \mathring{I}$. Note that $\psi_\lambda f \in D(\mathbf{L}_0)$. (We sometimes write the same for a function on E as well as for its trivial extension to $\hat{E}$.) Using (2.27) one shows that

$$\mathbb{R}_t := \mathrm{e}^{\lambda t}\,\psi_\lambda(\hat{Y}_t)f(\hat{Y}_t) - \int_0^t \mathrm{e}^{\lambda t}\left[\hat{\mathbf{L}}_0 + \lambda\right](\psi_\lambda f)(\hat{Y}_r)dr$$

is also a martingale under $\hat{\mathbb{P}}_x$ for every $x \in \hat{E}$. (See [Bov12].) But this implies

$$\begin{aligned}
&\hat{\mathbb{E}}_x(R_t - R_s)\mathbb{1}_F \\
=&\hat{\mathbb{E}}_x\left[\mathrm{e}^{\lambda t}(\psi_\lambda f)(\hat{Y}_t) - \mathrm{e}^{\lambda s}(\psi_\lambda f)(\hat{Y}_s) - \int_s^t \mathrm{e}^{\lambda r}[\hat{\mathbf{L}}_0 + \lambda](\psi_\lambda f)(\hat{Y}_r)dr\right]\mathbb{1}_F \\
=&0
\end{aligned}$$

for all $F \in \hat{\mathcal{F}}_s$ and $s \leq t$. Because of $\psi_\lambda f \in C_0(\hat{E})$ the equality still holds if we multiply the first two terms in the second row by '$\mathbb{1}_{\{\hat{\tau}>t\}}$', respectively '$\mathbb{1}_{\{\hat{\tau}>s\}}$'. If we also use Fubini's Theorem on the third term, we obtain

$$\hat{\mathbb{E}}_x\left[\mathrm{e}^{\lambda t}\,\psi_\lambda(\hat{Y}_t)\mathbb{1}_{\{\hat{\tau}>t\}}\cdot f(\hat{Y}_t) - \mathrm{e}^{\lambda s}\,\psi_\lambda(\hat{Y}_s)\mathbb{1}_{\{\hat{\tau}>s\}}\cdot f(\hat{Y}_s)\right] - \int_s^t \mathrm{e}^{\lambda r}\,\hat{\mathbb{E}}_x\left[[\hat{\mathbf{L}}_0 + \lambda](\psi_\lambda f)(\hat{Y}_r)\mathbb{1}_F\right]dr = 0.$$

Now we use that $[\hat{\mathbf{L}}_0 + \lambda](\psi_\lambda f) \in C_0(\hat{E})$ to multiply with '$\mathbb{1}_{\{\hat{\tau}>r\}}$' under $\hat{\mathbb{E}}_x$ in the last term. This gives

$$
\begin{aligned}
&\hat{\mathbb{E}}_x\left[M_t^\lambda \cdot f(\hat{Y}_t) - M_s^\lambda \cdot f(\hat{Y}_s)\right]\mathbb{1}_F - \frac{1}{\psi_\lambda(x)}\int_s^t \mathrm{e}^{\lambda r}\,\hat{\mathbb{E}}_x\left[[\hat{\mathbf{L}}_0+\lambda](\psi_\lambda f)(\hat{Y}_r)\mathbb{1}_F\cdot\mathbb{1}_{\{\hat{\tau}>r\}}\right]dr\\
=&\hat{\mathbb{E}}_x\left[M_t^\lambda \cdot f(\hat{Y}_t) - M_s^\lambda \cdot f(\hat{Y}_s)\right]\mathbb{1}_F - \int_s^t \hat{\mathbb{E}}_x\left[M_r^\lambda\cdot\hat{\mathbf{L}}_0^\lambda f(\hat{Y}_r)\mathbb{1}_F\right]dr\\
=&\hat{\mathbb{E}}_x\left[M_t^\lambda \cdot (f(\hat{Y}_t) - f(\hat{Y}_s))\mathbb{1}_F\right] - \int_s^t \hat{\mathbb{E}}_x\left[M_t^\lambda\cdot\hat{\mathbf{L}}_0^\lambda f(\hat{Y}_r)\mathbb{1}_F\right]dr\\
=&0
\end{aligned}
$$

using M_t^λ from (2.29). Next we use again Fubini's Theorem and get

$$
\begin{aligned}
&\hat{\mathbb{E}}_x\left[M_t^\lambda \cdot (f(\hat{Y}_t) - f(\hat{Y}_s))\mathbb{1}_F\right] - \hat{\mathbb{E}}_x\left[M_t^\lambda\cdot\int_s^t \hat{\mathbf{L}}_0^\lambda f(\hat{Y}_r)dr\mathbb{1}_F\right]\\
=&\hat{\mathbb{E}}_x^\lambda\left[f(\hat{Y}_t) - f(\hat{Y}_s) - \int_s^t \hat{\mathbf{L}}_0^\lambda f(\hat{Y}_r)dr\right]\cdot\mathbb{1}_F\\
=&0.
\end{aligned}
$$

Therefore, we have that $t \mapsto f(\hat{Y}_t) - \int_0^t \hat{\mathbf{L}}_0^\lambda f(\hat{Y}_r)dr$ and ,thus, also

$$
t \mapsto f(\hat{Y}_{t\wedge\hat{l}_I}) - \int_0^{t\wedge\hat{l}_I} \hat{\mathbf{L}}_0^\lambda f(\hat{Y}_r)dr
$$

is a $(\hat{\mathcal{F}}_t)_t$-martingale under $\hat{\mathbb{P}}_x^\lambda$. ■

on 4.

For this point we use a result about uniqueness and existence of certain 'generalized martingale problems'; Theorem 13.1 from [Pin95]. We reformulate this theorem and extract the facts we need into the next lemma.

lemma 2.17. *Let $E = (m, M)$ and let $\tilde{\boldsymbol{a}}$ be continuous on $[m, M]$. Define*

$$
\tilde{D} := D_{\tilde{\boldsymbol{a}}} := \frac{1}{2}\frac{d^2}{dx^2} + \tilde{\boldsymbol{a}}(x)\frac{d}{dx}.
$$

and

$$
E_n := (m + \frac{1}{n}, M - \frac{1}{n})
$$

for every $n \geq 1$. Then there is a unique solution $(\hat{\Omega}, \hat{\mathcal{F}}_\infty, (\hat{\mathcal{F}}_t)_t, (\hat{\mathbb{P}}_x)_{x\in\hat{E}}, (\hat{Y}_t)_t)$ to the following 'generalized martingale problem':

- *For every $x \in \hat{E}$ we have $\hat{\mathbb{P}}_x(\hat{Y}_0 = x) = 1$.*
- *For every $f \in C^2((m, M), \mathbb{R})$, $x \in \hat{E}$ and $n \geq 1$ we have that*

$$
f(\hat{Y}_{t\wedge\sigma_n}) - \int_0^{t\wedge\sigma_n} \tilde{D}f(\hat{Y}_s)ds \qquad (2.33)
$$

is a $(\hat{\mathcal{F}}_t)_t$*-martingale under* $\hat{\mathbb{P}}_x$.

Here $\sigma_n := \hat{l}_{E_n}$.

Now we want to show how to use this lemma in our situation:

Take arbitrary numbers $m \le M$ such that $(m, M) \subset \mathring{E}$. For any $f \in C^2(m, M)$ and any $n \in \mathbb{N}$ there is a $\bar{f}_n \in C^2(E)$ with $\text{supp}(\bar{f}_n) \subset (m, M)$ as well as $f = \bar{f}_n$ on E_n. (This can be done by connecting $(m + \frac{1}{n}, f(m + \frac{1}{n}))$ and e.g. $(m + \frac{1}{2n}, 0)$ with a polynomial of degree ≥ 5 such that the C^2-property is preserved. In the same manner we can connect $(M - \frac{1}{n}, f(M - \frac{1}{n}))$ and $(M - \frac{1}{2n}, 0)$.) Further observe that the differential expression $\frac{1}{\psi_\lambda(x)}\left[\frac{1}{2}\frac{d^2}{dx^2} + \boldsymbol{a}(x)\frac{d}{dx} - \kappa(x) + \lambda\right]\psi_\lambda(x)$ for $\mathbf{L}_0^\lambda$ equals

$$\tilde{D} := \frac{1}{2}\frac{d^2}{dx^2} + \tilde{\boldsymbol{a}}(x)\frac{d}{dx} = \frac{1}{2}\frac{d^2}{dx^2} + \left[\boldsymbol{a}(x) + \frac{\psi'_\lambda(x)}{\psi_\lambda(x)}\right]\frac{d}{dx}$$

on $C^2(0, \infty)$, respectively on $C^2(0, c)$. Now take $J = (m, M)$, $I = E_n$ and $f = \bar{f}_n$ in lemma 2.16. We deduce that the expression in (2.33) is an $(\hat{\mathcal{F}}_t)_t$-martingale under $\hat{\mathbb{P}}_x^\lambda$. To get a unique solution to this martingale problem we have to restrict $\hat{Y}$ to J. This is realized by taking the image measure of

$$t \mapsto \begin{cases} \hat{Y}_t & , t < \hat{l}_J \\ \dagger & , t \ge \hat{l}_J \end{cases}.$$

on 5.

In the last point we have seen that the distribution of $(Y_t)_t$ under $\mathbb{P}_x^\lambda$ killed at l_J is the unique solution to the 'generalized martingale problem'. The final step is to identify the penalisation limit as a diffusion on $[0, \infty)$, respectively $[0, d]$.

theorem 2.18. *Suppose that* Y *is a regular diffusion on* $[0, \infty)$ *(respectively on* $[0, d]$*) with drift* $\boldsymbol{a}$*. Let* κ *be a killing rate from* $C_b[0, \infty)$ *(respectively from* $C[0, d]$*). And let there be killing at zero with a killing rate parametrized by* $\alpha \ge 0$ *(and killing at d parametrized by* $\beta \ge 0$*). Then*

(a) the penalisation limit Z_t *is the diffusion which solves*

$$dZ_t = dX_t + \tilde{\boldsymbol{a}}(Z_t)dt$$

on $\mathring{E}$ *with* $\tilde{\boldsymbol{a}}(x) = \boldsymbol{a}(x) + \frac{\psi'_\lambda(x)}{\psi_\lambda(x)}$.

(b) a boundary of Z_t *is purely reflecting whenever it is accessible. This means that elements of the generator fulfil* $f'(0) = 0$ *(and* $f'(d) = 0$*). In addition there is no stickiness at an accessible boundary.*

proof.

(a): We show that the measure of Z_t instantly killed at l_J solves the corresponding generalized martingale problem from lemma 2.17 (for any bounded interval J with $\bar{J} \subset \mathring{E}$). Since this martingale problem has only the penalising limit as solution we are done. To do so take $J = (m, M)$ and any $\epsilon > 0$ such that $(m - \epsilon, M + \epsilon) \subset \mathring{E}$. Now take (a family of) filtered probability spaces $(\tilde{\Omega}, \tilde{F}_\infty, (\tilde{\mathcal{F}}_t), (\tilde{\mathbb{P}}_x)_{x \in \tilde{E}})$ and an $\tilde{\mathcal{F}}_t$-adapted process $\tilde{Z}_t$ such that

- we have $\tilde{E} = (m - \epsilon, M + \epsilon) \cup \{\dagger\}$.

– $\tilde{\mathbb{P}}_x(\tilde{Z}_0 = x) = 1$ for all $x \in \tilde{E}$.

– $\tilde{Z}$ has the same distribution as Z up to $\tilde{l}_{(m-\epsilon,M+\epsilon)}$. ($\tilde{l}_U := \inf\{t \geq 0\,;\ \tilde{Z} \notin U\}$).

– $\tilde{Z}_t = \dagger$ if $t \geq \tilde{l}_{(m-\epsilon,M+\epsilon)}$.

Loosely speaking, $\tilde{Z}$ is Z instantly killed at $\tilde{l}_{(m-\epsilon,M+\epsilon)}$. Since the drift $\tilde{\boldsymbol{a}}$ of Z (respectively $\tilde{Z}$) is continuous on $[m-\epsilon, M+\epsilon]$ it surely is a regular diffusion. Let $\tilde{L}_0$ be its C_0-generator. Thus, we have that $f(\tilde{Z}_t) - \int_0^t \tilde{L}_0 f(\tilde{Z}_s)ds$ is a $(\tilde{\mathcal{F}}_t)_t$-martingale under each $\tilde{\mathbb{P}}_x$ and for every $f \in D(\tilde{L}_0)$. (See [Bov12].) And just as we showed in point **4.** we deduce that

$$f(\tilde{Z}_t \wedge \sigma_n) - \int_0^t \tilde{L}_0 f(\tilde{Z}_s)ds$$

is a $(\tilde{\mathcal{F}}_t)_t$-martingale under each $\tilde{\mathbb{P}}_x$ for every $f \in C^2(m, M)$ and n. (We use the notation from lemma 2.17, i.e. $\sigma_n := \inf\{t \geq 0\,;\ \hat{Y}_t \notin (m + \frac{1}{n}, M - \frac{1}{n})\}$.) But then lemma 2.17 implies that the image measure of

$$\begin{cases} \tilde{Z}_t & , t < \tilde{l}_{(m,M)} \\ \dagger & , t \geq \tilde{l}_{(m,M)} \end{cases}.$$

under $\tilde{Q}_x$ is the same as the image measure of the penalising limit $\hat{\mathbb{P}}_x^\lambda$ instantly killed at $\hat{l}_{(m,M)}$. This proves (a) because it holds for all m and M with $[m, M] \subset \mathring{E}$.

(b): Note that the first assertion follows from the fact that the penalisation limit has the generator $\mathbf{L}_0$ defined in (2.31). For the second assertion assume w.l.o.g. that the left boundary of Y, call it 'b', is accessible for Z. Recall that Z is the *recurrent* Y conditioned on 'infinite survival'. This implies that there could only be *slow* killing at b for Y. But if there is only slow killing at b then we have $\underline{\psi}(b) > 0$. (If not, we would have $\underline{\psi}(b) = \underline{\psi}'(b) = 0$ and, therefore, $\underline{\psi} \equiv 0$.) Next let $t > 0$ be arbitrary. By the general form of the penalising measure which we obtained in the theorems of the last sections, we get

$$\mathbb{P}_b(Z_t = b) = \lim_{\epsilon \downarrow 0} \mathbb{P}_b(Z_t \in [b, b+\epsilon)) = \lim_{\epsilon \downarrow 0} \int_{[b,b+\epsilon)} \mathrm{e}^{Ct} \frac{\underline{\psi}(y)}{\underline{\psi}(b)} \mathbb{P}_b(Y_t \in dy;\, \tau > t) \tag{2.34}$$

with $C = K$ or $C = \underline{\lambda}$. But Y under τ has a kernel $p_t(x, y)$ w.r.t. m. (See [BS02].) Hence, (2.34) equals 0 since

$$\begin{aligned} &\lim_{\epsilon \downarrow 0} \int_{[b,b+\epsilon)} \frac{\underline{\psi}(y)}{\underline{\psi}(b)} p_t(b, y) m(dy) \\ \leq &\sup_{z \in (b, b+\epsilon_0)} \frac{\underline{\psi}(z)}{\underline{\psi}(b)} \cdot \lim_{\epsilon \downarrow 0} \int_b^{b+\epsilon} p_t(b, y) m(y) dy + p_t(b, b) m(\{b\}) \\ =&0. \end{aligned}$$

We chose ϵ_0 such that $b + \epsilon_0 \in \mathring{E}$ and used that there is no stickiness at b which implies

$m(\{b\}) = 0$. But $\mathbb{P}_b(Z_t = b) = 0$ for all $t > 0$ gives already that there is no stickiness at boundaries for Z. (See [Bre92] section 16.7.)

■

example 2.4. One particular example is a Brownian motion on $E = [0, \infty)$ with constant drift $a > 0$ toghether with slow killing at zero. It is obvious that $K = 0 < \underline{\lambda}$. According to theorem 2.14 we calculate

$$\psi_K(x) = \psi_0(x) = a + \alpha(1 - e^{-2ax})$$

(up to a positive multiple). We deduce that the penalisation limit has the additional drift term

$$\nabla \log \psi(x) = \frac{\psi_0'(x)}{\psi_0(x)} = \frac{2\alpha a}{(a+\alpha)e^{2ax} - \alpha}.$$

The result is also plausible in the following ways:

- As a tends becomes large, the additional drift goes faster to zero. The condition of survival has a smaller effect since there are more and more paths where Y already survives for a long time.
- As α tends to ∞ we get the same result as in the case of $\tau = T_0$. (Compare Theorem 6.16 of [CMSM13b].)

■

2.4. From FKP to general penalisation results

In subsection 2.4.1 we concentrate on the case $Y = X$. We are going to see a way to come from penalisation with some $\Gamma \in \mathcal{I}$ to penalisation results for another class of penalisers. So-called 'universal measures' will play an important role. These universal measures provide the opportunity to get penalisation limits *without* a limiting procedure. (Confer equation (2.36).) From this point of view we find it reasonable to ask if such 'universal measures' also exist for diffusions other than BM. Amongst other things, subsection 2.4.2 deals with this question.

2.4.1. A universal measure for the penalisation of BM

One way to introduce the universal measures for Brownian motion is given by the next proposition.

proposition 2.19. *Let Γ be of class $\mathcal{I}$ (as in definition 2.3) and $(M_t^\Gamma)_t$ the weight martingale from the corresponding penalisation, i.e. on $\mathcal{F}_s$ the penalisation measure $\mathbb{S}$ has the form*

$$\mathbb{S}_x^\Gamma = M_s^\Gamma \circ \mathbb{P}_x$$

(as in proposition 2.6). Then

(a) $\frac{1}{\Gamma_\infty}$ is $\mathbb{S}_x^\Gamma$-a.s. finite.

(b) (using the notation of 2.6) we define a proper measure by

$$\mathbb{U} := \frac{\psi(x)}{\Gamma_\infty} \circ \mathbb{S}_x^\Gamma \tag{2.35}$$

on $(\Omega, \mathcal{F})$ with the following properties:

- *$\mathbb{U} = \mathbb{U}_x$ may depend on x, but is independent from the choice of Γ.*
- *$\mathbb{U}_x$ is σ-finite for every x.*

proof. A proof can be found on the very first pages of [NRY09]. ■

From now on we will call the family of measures $(\mathbb{U}_x)_x$ won by the procedure above *universal measures of penalisation.* Next we define another class of penalisers which we are going to call a *target class $\mathcal{T}$.*

definition 2.5. An adapted process $\Gamma \geq 0$ is said to be of class $\mathcal{T}$ if

1. $t \mapsto \Gamma_t$ is non-increasing.
2. Γ_∞ is integrable with respect to $\mathbb{U}_x$.
3. There is some compact $C \subset \mathbb{R}$ such that $t \mapsto \Gamma_t$ is constant after the last visit of X to C.

Maybe the reader noticed that the above definition is not well defined since $\mathcal{T}$ can depend on the initial value x. Instead, we should write something like '$\mathcal{T}(\mathbb{P}_x)$'. We solve this as follows.

convention 2.6. If we do not mention the initial distribution explicitely, we will always use the measure of the diffusion under consideration starting at 'x'.

Note that for a non-trivial $\kappa \geq 0$ with compact support we have $t \mapsto \mathrm{e}^{-\int_0^t \kappa(X_s)ds} \in \mathcal{I} \cap \mathcal{T}$. But we also find examples of processes contained in $\mathcal{T}$ but not $\mathcal{I}$ and vice versa: On the one hand take a κ, due to definition 2.3, with non-compact support. Then we have that $t \mapsto \mathrm{e}^{-\int_0^t \kappa(X_s)ds} \in \mathcal{I} \backslash \mathcal{T}$. On the other hand take $\Gamma_t = f(L_t^y)$ for some non-increasing $f \geq 0$, e.g. $f = \mathbb{1}_{[0,1]}$, and $y \in \mathbb{R}$. Now we get $\Gamma \in \mathcal{T} \setminus \mathcal{I}$. (For C from point 3. of definition 2.5 we can take any closed interval which contains y.)

In the next result we connect penalisation by $\Gamma^0 \in \mathcal{I}$ with penalisation by $\Gamma \in \mathcal{T}$.

proposition 2.20. *Let $\Gamma^0 \in \mathcal{I}$ and $\Gamma \in \mathcal{T}$. Let $\mathbb{S}_x^0$ be the penalisation measure (for $\mathbb{P}_x$) under Γ^0 and M_s^0 the corresponding weight martingale. Then the penalisation measure $\mathbb{S}_x$ under Γ exists and is given on $\mathcal{F}_s$ by*

$$\mathbb{S}_x = N_s \circ \mathbb{S}_x^0$$

where N_s is a $\mathbb{S}_x^0$-martingale closed by (the normalised version of) $\frac{\Gamma_\infty}{\Gamma_\infty^0}$, i.e.

$$N_s = \frac{\mathbb{S}_x^0(\frac{\Gamma_\infty}{\Gamma_\infty^0} \mid \mathcal{F}_s)}{\mathbb{S}_x^0(\frac{\Gamma_\infty}{\Gamma_\infty^0})}.$$

The weight martingale of $\mathbb{S}_x$ is then given by

$$M_s = M_s^0 \cdot N_s.$$

proof. The above statement is a mixture of Theorems 1.2.1 and 1.2.15 from [NRY09]. ■

The procedure from proposition 2.19 to get $\mathbb{U}_x$ gives rise to another question:

If the definition from (2.35) does not depend on the choice of $\Gamma^0 \in \mathcal{I}$, can we find an alternative definition of the universal measures not involving a particular element of $\mathcal{I}$?

Corresponding 'constructions' can be found in [NRY09]. In this spirit of 'bypassing' we have a nice formula for $\mathbb{S}$, corresponding to penalisation by $\Gamma \in \mathcal{T}$, which only contains the universal measures and the penaliser, namely

$$\mathbb{S}_x = \frac{\Gamma_\infty}{\mathbb{U}_x(\Gamma_\infty)} \circ \mathbb{U}_x. \tag{2.36}$$

example 2.5. Take $a < x < b$ and some additive functional $\Gamma_t^0 = \exp(\int_0^t \kappa(X_s)ds)$ of Feynman-Kac type such that $\kappa|_{[a,b]} \in \mathcal{I}$. Next we multiply it with $\mathbb{1}_{\{T_a > t\}}$ and $\mathbb{1}_{\{T_b > t\}}$. We obtain

$$\Gamma_t = \mathbb{1}_{\{T_a \wedge T_b > t\}} \cdot \mathrm{e}^{\int_0^t \kappa(X_s)ds}$$

which is *not* contained in $\mathcal{I}$ but in $\mathcal{T}$. Note also that we can not use proposition 2.3 to get a penalisation result since κ is not continuous. As a concrete example choose $b = -a = \frac{1}{\sqrt{2}}$ and $\kappa(dx) = \mathbb{1}_{[0,\frac{1}{\sqrt{2}}]}(x)dx$. Next we apply proposition 2.6 to Γ^0: Using basic calculus we can show that $\psi(x)$ from proposition 2.6 is proportional to

$$\begin{cases} -\sqrt{2}(\mathrm{e}-1)x + \mathrm{e}+, & ,x \le 0 \\ \mathrm{e}^{\sqrt{2}x} + \mathrm{e} \cdot \mathrm{e}^{-\sqrt{2}x} & ,0 \le x \le \frac{1}{\sqrt{2}} \\ \sqrt{2}(\mathrm{e}-1)x + 2 & ,x \ge \frac{1}{\sqrt{2}} \end{cases}.$$

Now proposition 2.5 tells us that

$$\mathbb{S}_x^0 = \frac{\psi(X_s)}{\psi(x)} \Gamma_s^0 \circ \mathbb{P}_x.$$

Finally, by proposition 2.20 we can express $\mathbb{S}_x$ in terms of Γ and $\mathbb{S}_x^0$. Although a deeper study of the measures $\mathbb{S}_x^0$ is needed to go any further. ▮

Incorporating instant killing at some boundary point may also be accomplished via multiplication of $\mathbb{1}_{\{L_t^b = 0\}}$ and/or $\mathbb{1}_{\{L_t^a = 0\}}$. Penalisations under $\Gamma_t = \mathbb{1}_{\{L_t^a = L_t^b = 0\}}$ already exist in literature. For instance it is included in the results of [NN13].

2.4.2. Previous penalisation results for diffusions and more universal measures

The purpose of this section is to present conditions under which previous penalisation results for diffusions Y other than BM were achieved. Along the way we will see measures $\mathbb{U}_x^Y$ whose construction is the same as the universal measures $\mathbb{U}_x^X = \mathbb{U}_x$ from the preceding subsection. The articles about penalisation which we will mainly refer to are [SV09] and [Pro12].

Let us first say which kind of diffusions are penalised in [SV09]. As in this thesis the authors take regular diffusions Y on $E = [0, \infty)$. They also assume that ∞ is natural and that 0 is purely reflecting without stickiness, i.e. $m(\{0\}) = 0$. But the authors also make two additional assumptions which we do not or not entirely asssume:

(i) The tail distributions of the Lévy measure for τ_l^0 are subexponential. Here $l \mapsto \tau_l^0$ is the inverse of $t \mapsto L_t^0(Y)$. (The process $l \mapsto \tau_l^0$ is a subordinator. Thus, it is a special kind of Lévy process.)

(ii) Y is recurrent, i.e. $s(\infty) = \infty$.

The penalisers in [SV09] have the form $\Gamma_t = h(L_t^0)$ where

(i) $h \geq 0$.

(ii) h is non-increasing.

(iii) h has compact support.

The corresponding penalising result is Theorem 5.2 from [SV09].

remark 2.11. For example the Bessel process with dimension $d \in (0, 2)$, i.e. a diffusion Y which is a solution to

$$dY_t = dX_t + \frac{d-1}{2Y_t} dt$$

on $(0, \infty)$ and reflected at 0, fulfils all the above assumptions. In fact, Theorem 5.2 of [SV09] is a sort of generalisation of Theorem 1.1 from [RVY08b]. The latter article exclusively deals with Bessel processes of dimension $d \in (0, 2)$. ∎

In contrast to [SV09] the author of [Pro12] penalises with functionals Γ_t which depend on the last passage time g_c^t of some $c \in E$ up to time t. He assumes also that Y is null-recurrent. His argument is that there is a qualitative difference to the case of positive recurrence: In [Pro10] it is shown that penalising a positive recurrent diffusion by $\Gamma_t = \mathrm{e}^{\alpha L_t^0}$ yields again a diffusion. But in the null-recurrent case this is not true anymore if we choose $\Gamma_t = \exp(\alpha L_{g_0^t}^0) = \mathrm{e}^{\alpha L_t^0}$. In fact, the resulting penalisation martingale $(M_t^\Gamma)_t$ looks more like the 'Azéma-Yor martingale' in example 2.2 than a 'Girsanov martingale'. (See remark 1.7 and Theorem 1.5 in [Pro12].)

Let us come to the prementioned measures $(\mathbb{U}_x^Y)_{x \in E}$. Such measures can be found in [SV09] as well as [Pro12]. And indeed, they have exactly the same structure as $\mathbb{U}_x^X = \mathbb{U}_x$. For the 'construction' of $\mathbb{U}_x^Y$ we refer to [NRY09]; especially to (3.2.22) therein.

Though we are careful to use the word 'universal' here for several reasons:

- So far we have seen no attempt to come from penalisation by an initial class '$\mathcal{I}(Y)$' to penalisations by a target class '$\mathcal{T}(Y)$' in the spirit of proposition 2.20.
- We do not see the possibility to derive a formula equivalent to (2.36) from the calculations or the results in [Pro12].

At least from the penalising result of [SV09] and chapter 3 in [NRY09] we are able to derive a very similar formula:

proposition 2.21. *Let Y be a diffusion fulfilling the assumptions in [SV09]. (These assumptions are already given above.) Further let $\mathbb{S}_0^l$ be the penalisation limit of $\mathbb{P}_0$ by $\Gamma_t^l := \mathbb{1}_{\{L_t^0 < l\}}$ for some $l > 0$. Then we have*

$$\mathbb{S}_0^l = \frac{\Gamma_\infty^l}{\mathbb{U}_0^Y(\Gamma_\infty^l)} \circ \mathbb{U}_0^Y.$$

proof. We combine equations (3.2.28), (3.2.38) and (3.2.40) from [NRY09]. ■

But in our opinion proposition 2.21 is not sufficient to call $\mathbb{U}_0^Y$ a 'universal measure'. Finally, we have to mention two papers [NN13, NN12] by Najnudel and Nikeghbali. These are also concerned with measures $\mathbb{U}_x^Y$. They do not give a construction of $\mathbb{U}_x^Y$ similar to the one from [NRY09]. But they prove an equivalent result to (2.36). Though their penalisation results for diffusions are more restrictive: For instance they assume that Y is already on a natural scale, i.e. they penalise $\tilde{Y} = s(Y)$ instead of Y.

3. On the way from penalisation to quasistationarity

In chapter 1 we have seen several conditions which ensure the existence of a Yaglom limit for a diffusion Y under certain stopping times τ, i.e.

$$\lim_{t\to\infty} \mathbb{P}_x(Y_t \in \bullet \mid \tau > t)$$

exists and its mass is concentrated in the state space E of the diffusion Y. On the other hand in chapter 2 we proved penalisation results for these diffusions Y under the same stopping times τ. We have seen that the existence and form of the penalisation limit does not depend on the existence of a Yaglom limit. In particular we derived the limit of

$$\mu_x^{s|t}(\bullet) := \mathbb{P}_x(Y_s \in \bullet \mid \tau > t)$$

for any fixed $s > 0$ as $t \to \infty$. In what follows we want to look 'between' the penalisation limit and the Yaglom limit. The main question in this chapter is:

What happens with

$$\mathbb{P}_x(Y_s \in \bullet \mid \tau > t)$$

for $t \to \infty$ if we allow s to depend on t such that

- $s(t) < t$
- $\lim_{t\to\infty} s(t) = \infty$?

Note that the penalisation limit already bares the answer if $\lim_{t\to\infty} s(t) < \infty$.

This chapter is organized as follows.

At first we handle the case of $s(t) = t - u$ for some u independent from t. This will be complemented by the investigations in section 3.1. In particular theorem 3.4 shows that we get the same result if we allow $u = u(t)$ to depend on t such that $u(t) \to u$.

In section 3.2 we discuss the case of $s(t) \to \infty$. Comparing the results of proposition 3.5 and theorem 3.6 we observe a qualitative difference between 'high killing at infinity' ($\liminf_{x\to\infty} \kappa(x) > \underline{\lambda}$) and a situation typical for 'low killing at infinity'.

remark 3.1. The notion of 'high killing at ∞' and 'low killing at ∞' was introduced in [KS12]. Therein the reader can also find results about the influence of high or low killing at ∞ on the quasistationary behaviour. ∎

Let us start now with the case $s(t) = t - u$ which can be dealt with quickly:

Suppose we are in the setting of section 1.1. Thus, we have a process Y whose behaviour is influencing a killing time τ with $\mathbb{P}_x(\tau < \infty) = 1$ for all $x \in E$. (We recall that $(E, \mathcal{G})$ is the state space of Y, i.e. $Y_t \in E$ for all t.) Suppose further that Y has a Yaglom limit $\phi(dx)$ w.r.t. τ.

What happens with Y at $t-u$ in the long run if we condition on $\tau > t$? Hence, we want to know if

$$\lim_{t\to\infty} \mu_x^{t-u|t}(A) = \lim_{t\to\infty} \mathbb{P}_x(Y_{t-u} \in A \mid \tau > t)$$

exists and if so we want to identify the limit. But this is rather easy to see since

$$\begin{aligned} \mu_x^{t-u|t}(A) &= \frac{\mathbb{P}_x(Y_{t-u} \in A, \mathbb{P}_{Y_{t-u}}(\tau > u), \tau > t-u)}{\mathbb{P}_x(\mathbb{P}_{Y_{t-u}}(\tau > u), \tau > t-u)} \\ &= \frac{\mathbb{P}_x(Y_{t-u} \in A, \mathbb{P}_{Y_{t-u}}(\tau > u) \mid \tau > t-u)}{\mathbb{P}_x(\mathbb{P}_{Y_{t-u}}(\tau > u) \mid \tau > t-u)} \\ &\to \frac{\int_A \mathbb{P}_r(\tau > u)\phi(dr)}{\int_E \mathbb{P}_r(\tau > u)\phi(dr)} \end{aligned} \tag{3.1}$$

as t tends to ∞. This is plausible in two ways:

1. **$\mathbf{u} \approx \mathbf{0}$**
   Here assume that $\mathbb{P}_x(\tau > 0) = 1$ for $\phi(dx)$-a.e. x. (For example this is true for the diffusions Y with a Yaglom limit under τ considered in section 1.4.) By the monotone convergence theorem we see that the limit in (3.1) tends towards

   $$\frac{\int_A \phi(dr)}{\int \phi(dr)} = \int_A \phi(dr)$$

   as $u \downarrow 0$ which is the exact solution(Yaglom limit) if we would take $u = 0$.

2. **$\mathbf{u} \to \infty$**
   Now we want to go 'back in time'. Thus, our suggestion is that we get a result which has something to do with the corresponding penalisation limit. (See chapter 2.) Indeed, we are going to see now that we can get an invariant distribution of the penalisation limit as $u \to \infty$.

proposition 3.1. *Suppose Y is a linear diffusion on $E = [0, \infty)$ with drift coefficient $\boldsymbol{a}$ under killing at $\tau = \tau^{\kappa,0}$. Let $\boldsymbol{a}$ and κ fulfil A 1 and A 2. Further suppose that κ is non-decreasing such that* $\lim_{x\to\infty} \kappa(x) > \underline{\lambda}$. *Then we have that the limit from* (3.1) *tends to*

$$\nu(A) := \frac{\int_A \underline{\psi}^2(r)m(r)dr}{\int_E \underline{\psi}^2(r)m(r)dr} \tag{3.2}$$

as $u \to \infty$. Furthermore, ν is the unique stationary distribution of the penalisation limit of Y under τ.

As usual, $\underline{\lambda}$ is the bottom of the spectrum of $-\mathbf{L}^{\kappa,0}$ and $\mathbf{L}^{\kappa,0}$ is the generator of Y under $\tau = \tau^{\kappa,0}$. We also recall that m is the density of the speed measure of Y.

proof of proposition 3.1. By Theorem 4.3 of [KS12] we have that $e^{\eta u} \cdot \mathbb{P}_r(\tau > u) \to \underline{\psi}(r) \int_0^\infty \underline{\psi}(s)m(ds)$ as $u \to \infty$ and the mortality rate is $\eta = \underline{\lambda}$ independent of r. Moreover, the Yaglom limit exists and has the density $\underline{\varphi} = \underline{\psi} \cdot m$. Hence, the limit in (3.1) equals

$$\frac{\int_A \mathbb{P}_r(\tau > u)\underline{\psi}(r)m(dr)}{\int_E \mathbb{P}_r(\tau > u)\underline{\psi}(r)m(dr)} \tag{3.3}$$

and tends to (3.2) as $u \to \infty$. This is reasonable and can be made precise by $\kappa \uparrow$:

By a coupling-argument we see that $r \mapsto \mathbb{P}_r(\tau > u) \downarrow$ is decreasing. For this take two filtered probability spaces $(\Omega^i, \mathcal{F}^i, (\mathcal{F}^i_t), \mathbb{P}^i)$, $i = 1, 2$, with adapted processes $(\tilde{Y}^1_t)_t$ and $(\tilde{Y}^2_t)_t$ on them such that

- $\tilde{Y}^1$ and $\tilde{Y}^2$ satisfy $dY_t = dX_t + \boldsymbol{a}(Y_t)dt$ on $(0, \infty)$.
- $\tilde{Y}^1$ and $\tilde{Y}^2$ are purely reflected at 0.
- $\tilde{Y}^1_0 = x$ $\mathbb{P}^1$-a.s. and $\tilde{Y}^2_0 = y > x$ $\mathbb{P}^2$-almost surely.

Now let $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ be the direct product of $(\Omega^1, \mathcal{F}^1, (\mathcal{F}^1_t), \mathbb{P}^1)$ and $(\Omega^2, \mathcal{F}^2, (\mathcal{F}^2_t), \mathbb{P}^2)$. In particular $Y^1(w) := \tilde{Y}^1(w_1)$ and $Y^2(w) := \tilde{Y}^2(w_2)$ are independent. The next step is to couple at

$$\sigma = \inf\{t > 0\,;\ Y^1_t = Y^2_t\},$$

i.e. define

$$Y^\sigma_t := \begin{cases} Y^2_t & , t < \sigma \\ Y^1_t & , t \geq \sigma. \end{cases}$$

Note that the distributions of Y^2 and Y^σ are the same. Thus, we deduce

$$\begin{aligned} \mathbb{P}_x(\tau > u) &= \mathbb{P}_x\big(\mathrm{e}^{-\int_0^u \kappa(Y_s)ds}\big) \\ &= \mathbb{P}^1\big(\mathrm{e}^{-\int_0^u \kappa(\tilde{Y}^1_s)ds}\big) \\ &= \mathbb{P}^1 \otimes \mathbb{P}^2\left(\mathrm{e}^{-\int_0^u \kappa(Y^1_s)ds}\right) \\ &\geq \mathbb{P}^1 \otimes \mathbb{P}^2\left(\mathrm{e}^{-\int_0^u \kappa(Y^\sigma_s)ds}\right) \\ &= \mathbb{P}^1 \otimes \mathbb{P}^2\left(\mathrm{e}^{-\int_0^u \kappa(Y^2_s)ds}\right) \\ &= \mathbb{P}^2\big(\mathrm{e}^{-\int_0^u \kappa(\tilde{Y}^2_s)ds}\big) \\ &= \mathbb{P}_y(\tau > u) \end{aligned} \tag{3.4}$$

which shows that $\mathbb{P}_r(\tau > u) \downarrow$ as $r \uparrow$.

If we now take a look at the proof of Lemma 4.4 from [KS12], we see that $\mathrm{e}^{\eta u} \cdot \mathbb{P}_r(\tau > u)$ is dominated by a constant independent of u and r. Thus, we can use Lebesgue's dominated convergence theorem in the numerator and also in the denominator of (3.3) to get (3.2).

Since the penalisation limit has a generator of the form

$$\mathbf{L}^{\underline{\psi}} = \frac{1}{\underline{\psi}}(\mathbf{L}^{\kappa,0} + \underline{\lambda})\underline{\psi}$$

it is an immediate consequence that $(\mathbf{L}^{\underline{\psi}})^* = \underline{\psi}((\mathbf{L}^{\kappa,0})^* + \underline{\lambda})\frac{1}{\underline{\psi}}$. (We remind the reader that $\mathbf{L}^*$ is the formal adjoint of $\mathbf{L}$ w.r.t. the Lebesgue measure introduced in section 0.2.)

Thus, we get

$$(\mathbf{L}^{\underline{\psi}})^*(\underline{\psi}^2 \cdot m) = \underline{\psi}((\mathbf{L}^{\kappa,0})^* + \underline{\lambda})\underline{\psi} m = \underline{\psi} \cdot 0 = 0.$$

Therefore, the limit (3.2) of (3.3) as $u \to \infty$ is nothing else but an invariant distribution for the penalisation limit. Finally, proposition 2.4 tells us that the distribution given by (3.2) is the unique stationary distribution of the penalisation limit. ■

remark 3.2.

- For an argumentation which does not involve the formal adjoint see the proof of proposition 2.4.
- Note that we do not have to impose $\underline{\psi} \in L^2(m)$ since it is true anyway. This is shown in Theorem 4.3 of [KS12].

∎

Point **2** from above raises the question if we can find a process which starts with the (existing) quasistationary distribution and behaves like Y coming back from ∞ under the condition of infinite survival. First 'going to ∞' under the condition of survival and 'then coming back from ∞' sounds odd. Indeed, the following calculations will show that such a process can not exist in general.

For this let Y be any regular linear diffusion which has a Yaglom limit ϕ under killing at τ. Let f be some measurable bounded functional from $C[0,u]$ to $\mathbb{R}$. Thus,

$$\begin{aligned}\mathbb{Q}_t^u(f) := \mathbb{P}_x(f(Y_{[t-u,t]}) \mid \tau > t) &= \frac{\mathbb{P}_x(\mathbb{P}_{Y_{t-u}}(f(Y_{[0,u]}),\, \tau > u),\, \tau > t-u)}{\mathbb{P}_x(\tau > t)} \\ &= \frac{\mathbb{P}_x(g(Y_{t-u}) \mid \tau > t-u)}{\mathbb{P}_x(h(Y_{t-u}) \mid \tau > t-u)},\end{aligned}$$

with

$$\begin{aligned} g(y) &:= \mathbb{P}_y(f(Y_{[0,u]}),\, \tau > u) \\ h(y) &:= \mathbb{P}_y(\tau > u).\end{aligned}$$

(We use the notation $Y_{[r,t]}$ for $(Y_s)_{s\in[r,t]}$.) We deduce that

$$\mathbb{Q}_\infty^u(f) := \lim_{t\to\infty} \mathbb{Q}_t^u(f) = \frac{\int g(y)\phi(dy)}{\int h(y)\phi(dy)}.$$

But $\int g(y)\phi(dy) = \mathbb{P}_\phi(f(Y_{[0,u]}),\, \tau > u)$ and, therefore,

$$\mathbb{Q}_\infty^u(f) = \mathbb{P}_\phi(f(Y_{[0,u]}) \mid \tau > u). \tag{3.5}$$

Now we have 'gone to ∞ under survival'. If we want to have a process which is 'coming back from ∞', i.e. $u \uparrow$ in (3.5), we need that $(\mathbb{Q}_\infty^u)_u$ consistent in u, i.e.

$$\mathbb{Q}_\infty^w|_{\mathcal{F}_v} = \mathbb{Q}_\infty^v$$

for any $v < w$. We will see that this is not true in general. For this take the special case of $\mathbb{1}_{\{Y_u\in A\}}$ for some measurable A and $u < v < w$.

As this functional is $\mathcal{F}_v$-measurable, we must have

$$\begin{aligned}\mathbb{Q}^w_\infty(Y_u \in A) &= \frac{\mathbb{P}_\phi(Y_u \in A,\, \tau > w)}{\mathbb{P}_\phi(\tau > w)} \\ &= \frac{\mathbb{P}_\phi(Y_u \in A,\, \mathbb{P}_{Y_u}(\tau > w-u),\, \tau > u)}{\mathbb{P}_\phi(\tau > w)} \\ &= \frac{\int\int_A p_u(y,z)\,\mathbb{P}_z(\tau > w-u)\,m(dz)\,\phi(dy)}{\mathbb{P}_\phi(\tau > w)}\end{aligned}$$

equal to

$$\mathbb{Q}^v_\infty(Y_u \in A) = \frac{\int\int_A p_u(y,z)\,\mathbb{P}_z(\tau > v-u)\,m(dz)\,\phi(dy)}{\mathbb{P}_\phi(\tau > v)}$$

where $p_t(y,z)$ is a kernel for Y under τ w.r.t. the speed measure m. Since this holds for any A, we can use Fubini's theorem and proposition 1.1 to obtain

$$\int p_u(y,z)\mathbb{P}_z(\tau > v-u)\phi(dy) = \mathrm{e}^{\lambda(w-v)} \int p_u(y,z)\mathbb{P}_z(\tau > w-u)\phi(dy)$$

for some $\lambda = \lambda(\phi) > 0$ and almost every $z > 0$. But here the y-integrals cancel and we get the necessary condition

$$\frac{\mathbb{P}_z(\tau > v-u)}{\mathbb{P}_z(\tau > w-u)} = \mathrm{e}^{\lambda(w-v)}\,. \tag{3.6}$$

We can even take $u = 0$. In particular, we see that (3.6) must be true for every $z > 0$. But this is *not* true in general. Just take the standard example of BM with constant negative drift under $\tau = T_0$. (For the kernel confer example 3.1 below.)

3.1. Near quasistationarity

In this section we allow a dependence $u(t)$ of t. But in direct continuation of the preceding discussion we suppose that

$$u(t) \to u.$$

As a warm-up we will go through the 'standard examples' of Brownian motion with constant drift and the Ornstein-Uhlenbeck process.

The obvious way to deal with this situation is to take the difference

$$\mu_x^{t-u(t)|t}(A) - \mu_x^{t-u|t}(A) = \mathbb{P}_x(Y_{t-u(t)} \in A \mid \tau > t) - \mathbb{P}_x(Y_{t-u} \in A \mid \tau > t)$$

and show that this converges to zero to get the same limit for

$$\mu_x^{t-u(t)|t}(\bullet) = \mathbb{P}_x(Y_{t-u(t)} \in \bullet \mid \tau > t)$$

as in the case of $u(t) \equiv u$.

It should be clear that we can assume that A is bounded which is no restriction after all: Suppose we have $\lim_{t\to\infty}\left(\mu_x^{t-u(t)|t}(A) - \mu_x^{t-u|t}(A)\right) = 0$ for all bounded A. Then we have in particular that

$$
\begin{aligned}
&\lim_{t\to\infty} \mathbb{P}_x(Y_{t-u(t)} \le r \mid \tau > t) \\
&= \lim_{t\to\infty} \mu_x^{t-u(t)|t}([0,r]) \\
&= \lim_{t\to\infty} \mu_x^{t-u|t}([0,r]) \\
&= \lim_{t\to\infty} \mathbb{P}_x(Y_{t-u} \le r \mid \tau > t)
\end{aligned}
$$

for all $r \in \mathbb{R}$. Thus, the distribution functions of $\mu_x^{t-u(t)|t}$ converges pointwise to the distribution function of $\frac{\mathbb{P}_r(\tau>u)\phi(dr)}{\int \mathbb{P}_r(\tau>u)\phi(dr)}$. Though we may not get a convergence in the sense of A.1 we still have

$$
\mu_x^{t-u(t)|t}(dr) \to \frac{\mathbb{P}_r(\tau > u)\phi(dr)}{\int \mathbb{P}_r(\tau > u)\phi(dr)}
$$

in distribution.

This will be shown for Brownian motion with constant drift and the Ornstein-Uhlenbeck process in examples 3.1 and 3.2. We are going to use the existence of the kernel $p_t(x,y)$ of Y under τ w.r.t. the Lebesgue measure, i.e.

$$
\mathbb{P}_x(Y_t \in A, \tau > t) = \int_A p_t(x,y)dy
$$

for any $x \in E$, any measurable A and any $t > 0$.

By using this and the Markov property at $t - u(t)$, respectively at $t - u$, we get

$$
\mathbb{P}_x(Y_{t-u(t)} \in A \mid \tau > t) - \mathbb{P}_x(Y_{t-u} \in A \mid \tau > t) = I_1 + I_2
$$

with

$$
\begin{aligned}
I_1 &:= \frac{\int_A \left[\mathbb{P}_y(\tau > u(t)) - \mathbb{P}_y(\tau > u)\right] p_{t-u(t)}(x,y)dy}{\mathbb{P}_x(\tau > t)} \\
&\text{and} \\
I_2 &:= \frac{\int_A \mathbb{P}_y(\tau > u) \left[p_{t-u(t)}(x,y) - p_{t-u}(x,y)\right] dy}{\mathbb{P}_x(\tau > t)}.
\end{aligned}
\tag{3.7}
$$

example 3.1. Take Y to be a Brownian motion with constant drift $a \neq 0$ and $\tau = T_0$. As announced above we want to show

$$
\mu_x^{t-u(t)|t}(A) - \mu_x^{t-u|t}(A) = \mathbb{P}_x(Y_{t-u(t)} \in A \mid \tau > t) - \mathbb{P}_x(Y_{t-u} \in A \mid \tau > t) \to 0
$$

for each bounded A which already implies

$$
\lim_{t\to\infty} \mu_x^{t-u(t)|t} = \lim_{t\to\infty} \mu_x^{t-u|t}
$$

in distribution. The kernel w.r.t. to the Lebesgue measure is

$$
p_t(x,y) = \exp\left(a(y-x) - \frac{a^2}{2}t\right) \cdot \frac{1}{\sqrt{2\pi t}} \left[\exp\left(-\frac{(x-y)^2}{2t}\right) - \exp\left(-\frac{(x+y)^2}{2t}\right)\right].
$$

(See [MSM94].)

Note that for this example we will assume

$$u(t) - u = o\left(\frac{1}{t}\right)$$

which is due to the derivation of (3.9) below.

We now show that I_1 and I_2 tend to zero as $t \to \infty$.

$\underline{I_1}$: Note that

$$\frac{\int_A p_{t-u(t)}(x,y)dy}{\mathbb{P}_x(\tau > t)} \leq \frac{\mathbb{P}_x(\tau > t - u(t))}{\mathbb{P}_x(\tau > t)}.$$

Further, we are in a situation where $\frac{\mathbb{P}_x(\tau>t-u)}{\mathbb{P}_x(\tau>t)} \to \mathrm{e}^{\eta u}$. Therefore, it is rather easy to see that $\frac{\mathbb{P}_x(\tau>t-u)}{\mathbb{P}_x(\tau>t)} - \frac{\mathbb{P}_x(\tau>t-u(t))}{\mathbb{P}_x(\tau>t)} \to 0$. This implies

$$\limsup_{t\to\infty} \frac{\int_A p_{t-u(t)}(x,y)dy}{\mathbb{P}_x(\tau > t)} < \infty.$$

Hence, it is enough to show that for every $\epsilon > 0$ we can find a $\delta > 0$ such that

$$\sup_{y\in A}\{|\mathbb{P}_y(u-\delta \leq \tau \leq u+\delta)|\} = \sup_{y\in A}\{\mathbb{P}_y(\tau > u-\delta) - \mathbb{P}_y(\tau > u+\delta)\} < \epsilon. \tag{3.8}$$

Unravelling (3.8) with the help of $p_t(x,y)$, we see that (3.8) is true if

$$\int_0^\infty \exp(az)\cdot\left[\exp\left(-\frac{(y-z)^2}{2(u-\delta)}\right) - \exp\left(-\frac{(y-z)^2}{2(u+\delta)}\right)\right]dz +$$
$$\int_0^\infty \exp(az)\cdot\left[\exp\left(-\frac{(y+z)^2}{2(u-\delta)}\right) - \exp\left(-\frac{(y+z)^2}{2(u+\delta)}\right)\right]dz$$

becomes arbitrarily small (uniformly in $y \in A$) if $\delta > 0$ is small enough. Now let $M > 0$ be such that $A \subseteq B_M(0)$. Then we get

$$2\int_{-M}^\infty \exp(az)\cdot\left[\exp\left(-\frac{z^2}{2(u-\delta)}\right) - \exp\left(-\frac{z^2}{2(u+\delta)}\right)\right]dz$$

as upper bound of the former expression. If $a < 0$, then we can use Lebesgue's dominated convergence theorem directly to obtain zero in the limit as $\delta \to 0$. For $a \geq 0$, we first rewrite $\exp(az) = \exp(bz)\exp((a-b)z)$ with some $b < 0$. Next we complete the squares between $\exp((a-b)z)$ and the terms in '[...]'.

$\underline{I_2}$: We again rewrite this expression by using $p_t(x,y)$. Thus, after multiplication with $\frac{\sqrt{t}}{\sqrt{t}}$ and rearranging, we have

$$\mathrm{e}^{-ax-\frac{a^2}{2}t}\frac{1}{\sqrt{2\pi t}}\cdot[J_- - J_+]$$

with

$$J_\pm = \int_A \mathbb{P}_y(\tau > u) \cdot \mathrm{e}^{ay} \left(\mathrm{e}^{\frac{a^2}{2}u(t)} \frac{\sqrt{t}}{\sqrt{t-u(t)}} \mathrm{e}^{-\frac{(x\pm y)^2}{2(t-u(t))}} - \mathrm{e}^{\frac{a^2}{2}u} \frac{\sqrt{t}}{\sqrt{t-u}} \mathrm{e}^{-\frac{(x\pm y)^2}{2(t-u)}} \right) dy$$

in the numerator and

$$\mathrm{e}^{-ax-\frac{a^2}{2}t} \frac{1}{\sqrt{2\pi t}} \cdot \int_0^\infty \mathrm{e}^{ay} \left(\mathrm{e}^{-\frac{(x-y)^2}{2t}} - \mathrm{e}^{-\frac{(x+y)^2}{2t}} \right) dy$$

in the denominator. Hence, a lot of terms cancel out and, because one can show that

$$t \cdot \int_0^K \mathrm{e}^{ay} \left(\mathrm{e}^{-\frac{(x-y)^2}{2t}} - \mathrm{e}^{-\frac{(x+y)^2}{2t}} \right) \to 2x \left(\frac{1}{a^2} - \frac{1}{a^2} \mathrm{e}^{-aK} + \frac{K}{a} \mathrm{e}^{aK} \right) > 0$$

(at least for K large enough), it suffices to show that the '[...]-term' in the numerator is $o\left(\frac{1}{t}\right)$. We can further reduce this condition to

$$\int_A \mathbb{P}_y(\tau > u) \cdot \mathrm{e}^{ay} \left(\mathrm{e}^{-\frac{(x\pm y)^2}{2(t-u(t))}} - \mathrm{e}^{-\frac{(x\pm y)^2}{2(t-u)}} \right) dy = o\left(\frac{1}{t}\right) \tag{3.9}$$

using

$$u(t) - u = o\left(\frac{1}{t}\right) \Rightarrow t \cdot \left(\mathrm{e}^{\frac{a^2}{2}u(t)} - \mathrm{e}^{\frac{a^2}{2}u} \right) \to 0$$

and

$$t \cdot \left(\frac{\sqrt{t}}{\sqrt{t-u(t)}} - \frac{\sqrt{t}}{\sqrt{t-u}} \right) \to 0.$$

Since A is assumed to be bounded, we can always find an integrable upper bound for the (bounded!) integrand of (3.9). Thus, (3.9) would be true if we have $t \cdot \left(\mathrm{e}^{-\frac{b}{t-u(t)}} - \mathrm{e}^{-\frac{b}{t-u}} \right) \to 0$. But this holds since $(u(t))_t$ is bounded. (Just use Taylor formula $\mathrm{e}^{-l} = 1 - l + o(|l|)$ if $l \to 0$.)

∎

There are some basic points which can be extracted from the above example:

First of all, under certain assumptions on the rate of $u(t) - u$, the 'quasistationary behaviour' of $Y_{t-u(t)}$ (conditioned on $\{\tau > t\}$) may be the same as for Y_{t-u}. In particular in the case of linear diffusions on $\mathbb{R}_+$ from chapter 1 we would expect that

- $\mathbb{P}_x(Y_{t-u(t)} \in A \,|\, \tau > t) \to \frac{\int_A \mathbb{P}_r(\tau > u)\underline{\varphi}(r)dr}{\int_E \mathbb{P}_r(\tau > u)\underline{\varphi}(r)dr}$ if there exists a Yaglom limit $\underline{\varphi}$.
- $\mathbb{P}_x(Y_{t-u(t)} \leq M \,|\, \tau > t) \to 0$ for all $M > 0$, provided we have escape to ∞ and the existence of $\lim_{t\to\infty} \frac{\mathbb{P}_x(\tau > t)}{\mathbb{P}_x(\tau > t-u)} > 0$.

Of course, we should say some words on the second point:

A sufficient condition for the existence of $\lim_{t\to\infty} \frac{\mathbb{P}_x(\tau > t)}{\mathbb{P}_x(\tau > t-u)}$ may be found in Lemma 4.1 of [KS12]. Example 3.1 fulfils this conditions if $a \neq 0$. (See also remark 1.4.) Once we have this, we can calculate

$$\mathbb{P}_x(Y_{t-u} \le M \mid \tau > t) = \frac{\mathbb{P}_x(Y_{t-u} \le M, \mathbb{P}_{Y_{t-u}}(\tau > u) \mid \tau > t-u)}{\mathbb{P}_x(\tau>t)/\mathbb{P}_x(\tau>t-u)} \to 0. \tag{3.10}$$

By the way, Y from the above example escapes to ∞ if $a > 0$. This is as expected and can be shown(calculated) directly.

The second point can be shown to hold in a quite general setting:

proposition 3.2. *Suppose $(Y_t)_{t\ge 0}$ escapes to infinity under some stopping time τ and assume $\limsup_{t\to\infty} \frac{\mathbb{P}_x(\tau>t-u)}{\mathbb{P}_x(\tau>t)} < \infty$ for some $u > 0$. Then we have*

$$\mathbb{P}_x(Y_{t-u(t)} \in A \mid \tau > t) \to 0$$

for each bounded measurable A and bounded $u(t)$.

In this proposition we can take any Markov process with values in $\mathbb{R}^n$ (not only linear diffusions).

proof. We must have $\limsup_{t\to\infty} \frac{\mathbb{P}_x(\tau>t-v)}{\mathbb{P}_x(\tau>t)} < \infty$ for any $v \ge 0$:

If not, we would have some $v > u$ such that $\limsup_{t\to\infty} \frac{\mathbb{P}_x(\tau>t-v)}{\mathbb{P}_x(\tau>t)} = \infty$. Now take $l \in \mathbb{N}$ and $w \le u$ with $v = l{\cdot}u + w$. Then we can calculate

$$\limsup_{t\to\infty} \frac{\mathbb{P}_x(\tau > t-v)}{\mathbb{P}_x(\tau > t)} \le \left(\limsup_{t\to\infty} \frac{\mathbb{P}_x(\tau > t-u)}{\mathbb{P}_x(\tau > t)}\right)^l \cdot \limsup_{t\to\infty} \frac{\mathbb{P}_x(\tau > t-w)}{\mathbb{P}_x(\tau > t)}$$

using the same trick as in remark 1.4. Thus, we must have $\limsup_{t\to\infty} \frac{\mathbb{P}_x(\tau>t-w)}{\mathbb{P}_x(\tau>t)} = \infty$ which implies that $\limsup_{t\to\infty} \frac{\mathbb{P}_x(\tau>t-u)}{\mathbb{P}_x(\tau>t)} = \infty$ giving a contradiction.

Next we use the same trick as in (3.10) to get:

$$\begin{aligned}
&\mathbb{P}_x(Y_{t-u(t)} \in A \mid \tau > t)\\
=&\frac{\mathbb{P}_x(\tau > t-u(t))}{\mathbb{P}_x(\tau > t)} \cdot \mathbb{P}_x(Y_{t-u(t)} \in A, \mathbb{P}_{Y_{t-u}}(\tau > u) \mid \tau > t-u(t))\\
\le&\frac{\mathbb{P}_x(\tau > t-v)}{\mathbb{P}_x(\tau > t)} \cdot \mathbb{P}_x(Y_{t-u(t)} \in A \mid \tau > t-u(t)),
\end{aligned}$$

for every $u(t) \le v \le t$. By the assumptions we can choose v as a global bound for $u(t)$. Therefore,

$$\begin{aligned}
&\limsup_{t\to\infty} \mathbb{P}_x(Y_{t-u(t)} \in A \mid \tau > t)\\
\le&\limsup_{t\to\infty} \frac{\mathbb{P}_x(\tau > t-v)}{\mathbb{P}_x(\tau > t)} \cdot \limsup_{t\to\infty} \mathbb{P}_x(Y_{t-u(t)} \in A \mid \tau > t-u(t)) = 0
\end{aligned}$$

since we supposed that Y escapes to ∞. ■

Let us do a second example. (At least to get familiar with the kind of calculations one can do if we have a nice transition kernel.)

example 3.2. For Y we take the Ornstein-Uhlenbeck process which is a solution to $dY_t = dX_t - aY_t dt$ with $a > 0$ on $\mathbb{R}_+$ under killing at $\tau = T_0$. (Note that in the case $a < 0$ we have escape to infinity. This is plausible, since the 'purely reflecting version' is transient and we even condition on not hitting the reflecting barrier 0. Hence, this case is covered by proposition 3.2.)

As in the preceding example 3.1 we want to show

$$\lim_{t\to\infty} \mu_x^{t-u(t)|t} = \lim_{t\to\infty} \mu_x^{t-u|t}$$

in distribution by proving

$$\mu_x^{t-u(t)|t}(A) - \mu_x^{t-u|t}(A) = \mathbb{P}_x(Y_{t-u(t)} \in A \mid \tau > t) - \mathbb{P}_x(Y_{t-u} \in A \mid \tau > t) \to 0$$

for each bounded A.

We will use that Y under τ has a nice transition kernel

$$\begin{aligned} p_t(x,y) &= \sqrt{\frac{2}{\pi h(t)}} \exp\left(-\frac{\mathrm{e}^{-2at}\,x}{2h(t)} - \frac{y^2}{2h(t)}\right) \sinh\left(\frac{\mathrm{e}^{-at}\,xy}{h(t)}\right) \\ &= \frac{1}{\sqrt{2\pi h(t)}} \left[\exp\left(-\frac{(\mathrm{e}^{-at}\,x - y)^2}{2h(t)}\right) - \exp\left(-\frac{(\mathrm{e}^{-at}\,x + y)^2}{2h(t)}\right)\right]. \end{aligned}$$

w.r.t. the Lebesgue measure. Here $h(t) := \frac{1-\mathrm{e}^{-2at}}{2a}$. As in example 3.1 we start with the partition into I_1 and I_2 from (3.7) and show that both converge to zero as $t \to \infty$.

on $\mathbf{I_1}$: As in the last example it suffices to show that

$$\begin{aligned} &\mathbb{P}_y(\tau > u - \delta) - \mathbb{P}_y(\tau > u + \delta) \\ &= \int_0^\infty \frac{1}{\sqrt{2\pi h(u-\delta)}} \left[\exp\left(-\frac{(\mathrm{e}^{-a(u-\delta)}\,y - z)^2}{2h(u-\delta)}\right) - \exp\left(-\frac{(\mathrm{e}^{-a(u-\delta)}\,y + z)^2}{2h(u-\delta)}\right)\right] dz \\ &- \int_0^\infty \frac{1}{\sqrt{2\pi h(u+\delta)}} \left[\exp\left(-\frac{(\mathrm{e}^{-a(u+\delta)}\,y - z)^2}{2h(u+\delta)}\right) - \exp\left(-\frac{(\mathrm{e}^{-a(u+\delta)}\,y + z)^2}{2h(u+\delta)}\right)\right] dz \\ &< \epsilon \end{aligned}$$

for $\delta > 0$ small enough and all $y \in A$. Since expressions of the form $\int \exp\big(-\frac{(z+b)^2}{2c}\big)dz$ are globally bounded for all $c \geq m > 0$ and $\frac{1}{h(u+\delta)} - \frac{1}{h(u-\delta)} \to 0$ as $\delta \to 0$ we see that the above is true iff the absolute value of

$$\begin{aligned} &\int_0^\infty \left[\exp\left(-\frac{(\mathrm{e}^{-a(u-\delta)}\,y - z)^2}{2h(u-\delta)}\right) - \exp\left(-\frac{(\mathrm{e}^{-a(u+\delta)}\,y - z)^2}{2h(u+\delta)}\right)\right] dz \\ -&\int_0^\infty \left[\exp\left(-\frac{(\mathrm{e}^{-a(u-\delta)}\,y + z)^2}{2h(u-\delta)}\right) - \exp\left(-\frac{(\mathrm{e}^{-a(u+\delta)}\,y + z)^2}{2h(u+\delta)}\right)\right] dz \end{aligned}$$

is less then $\epsilon > 0$ for δ small enough. And of course, this would be true if

$$\left|\int_0^\infty \left[\exp\left(-\frac{(\mathrm{e}^{-a(u-\delta)}\,y \pm z)^2}{2h(u-\delta)}\right) - \exp\left(-\frac{(\mathrm{e}^{-a(u+\delta)}\,y \pm z)^2}{2h(u+\delta)}\right)\right] dz\right| < \frac{\epsilon}{2}. \tag{3.11}$$

We observe that (3.11) is valid once we check the $\epsilon - \delta$ criterion for the expressions

$$\int_0^\infty \left[\exp\left(-\frac{(\mathrm{e}^{-a(u-\delta)}\, y \pm z)^2}{2h(u-\delta)}\right) - \exp\left(-\frac{(\mathrm{e}^{-a(u+\delta)}\, y \pm z)^2}{2h(u-\delta)}\right)\right] dz$$

and

$$\int_0^\infty \left[\exp\left(-\frac{(\mathrm{e}^{-a(u+\delta)}\, y \pm z)^2}{2h(u-\delta)}\right) - \exp\left(-\frac{(\mathrm{e}^{-a(u+\delta)}\, y \pm z)^2}{2h(u+\delta)}\right)\right] dz.$$

<u>first term:</u> To clarify what is really important here, we use a small change of variables $\frac{z}{\sqrt{2h(u-\delta)}} \rightsquigarrow w$ to see that this expression is

$$\sqrt{2h(u-\delta)}\left[\int_0^\infty \exp\left(-(w \pm h_1(u,\delta,y))^2\right) dw - \int_0^\infty \exp\left(-(w \pm h_2(u,\delta,y))^2\right) dw\right].$$

h_1 and h_2 converge uniformly in $y \in A$ to $h_0(u,y) = \frac{\mathrm{e}^{-au}\, y}{\sqrt{2h(u)}}$ which is bounded in $y \in A$. Thus, it suffices to show the following:

$$\sup_{y\in A}\left|\int_0^\infty \left[\exp\left(-(w+g_\delta(y))^2\right) - \exp\left(-(w+g_0(y))^2\right)\right] dw\right| \to 0 \qquad (3.12)$$

as $\delta \to 0$, whenever $\sup_{y\in A}|g_\delta(y) - g_0(y)| \to 0$ as $\delta \to 0$ and $\sup_{y\in A}|g_\delta(y)| \le M < \infty$ for small enough δ.

proof of (3.12)*:* The integral can be written as

$$\int_0^\infty \exp(-w^2)\left[\exp(-2wg_\delta - g_\delta^2) - \exp(-2wg_0 - g_0^2)\right].$$

This equals

$$\int_0^\infty \exp(-w^2 + 4Mw)\cdot\left[\exp(-2w(g_\delta + 2M) - g_\delta^2) - \exp(-2w(g_0 + 2M) - g_0^2)\right] dw.$$

Now we use that $g_\delta + 2M \ge 0$ for small δ and that $\left|\mathrm{e}^{-c} - \mathrm{e}^{-d}\right| \le |c-d|$ for all $c, d \ge 0$ to get the following upper bound:

$$\begin{aligned}&\int_0^\infty \exp(-w^2 + 4Mw)\left[2w\,|g_0 - g_\delta| + \left|g_0^2 - g_\delta^2\right|\right] dw\\ \le &\int_0^\infty \exp(-w^2 + 4Mw)\left[2w\,|g_0 - g_\delta| + 3M\,|g_0 - g_\delta|\right] dw\\ \le &\tilde{\epsilon}\cdot\int_0^\infty \exp(-w^2 + 4Mw)(2w + 3M)dw\end{aligned}$$

for every $\tilde{\epsilon} > 0$ as δ is small enough. ■

<u>second term:</u> By the same trick as in example 3.1 for I_1 we show that this gets arbitrarily small for δ small enough.

Thus, we proved that $I_1 \to 0$ as t tends to infinity.

on $\mathbf{I_2}$: For the decay of the denominator of I_2 the following holds:

$$\liminf_{t\to\infty} \mathrm{e}^{at}\, \mathbb{P}_x(\tau > t) > 0. \tag{3.13}$$

proof of (3.13). We have

$$\begin{aligned} p_t(x,y) &= \frac{1}{\sqrt{2\pi h(t)}} \exp\left(-\frac{\mathrm{e}^{-2at}\, x}{2h(t)} - \frac{y^2}{2h(t)}\right) \cdot 2\sinh\left(\frac{\mathrm{e}^{-at}\, xy}{h(t)}\right) \\ &= \frac{1}{\sqrt{2\pi h(t)}} \exp\left(-\frac{\mathrm{e}^{-2at}\, x}{2h(t)} - \frac{y^2}{2h(t)}\right) \cdot 2\left[\frac{\mathrm{e}^{-at}\, xy}{h(t)} + o\left(\left|\frac{\mathrm{e}^{-at}\, xy}{h(t)}\right|\right)\right], \end{aligned}$$

where we used Taylor formula for sinh. Since $h(t) \to \frac{1}{2a} > 0$, we see that the term right before '[...]' converges to the density of $\mathcal{N}(0, \frac{1}{a})$ and is dominated by a constant (e.g. by $\sqrt{a}$) for large t. We also observe that $\mathrm{e}^{at} \cdot [\ldots] \to 2axy$ and

$$\mathrm{e}^{at} \cdot \sinh\left(\frac{\mathrm{e}^{-at}\, xy}{h(t)}\right) \le \mathrm{e}^{at} \cdot \left[\frac{\mathrm{e}^{-at}\, x{\cdot}K}{h(t)} + \left|\frac{\mathrm{e}^{-at}\, x{\cdot}K}{h(t)}\right|\right] \le 3{\cdot}2axK < \infty$$

for all $y \le K$ and t large enough. Now we can use the dominated convergence theorem to obtain

$$\begin{aligned} &\liminf_{t\to\infty} \mathrm{e}^{at}\, \mathbb{P}_x(\tau > t) \\ =&\liminf_{t\to\infty} \mathrm{e}^{at} \cdot \int_0^\infty \frac{1}{\sqrt{2\pi h(t)}} \exp\left(-\frac{\mathrm{e}^{-2at}\, x}{2h(t)} - \frac{y^2}{2h(t)}\right) \cdot 2\sinh\left(\frac{\mathrm{e}^{-at}\, xy}{h(t)}\right) dy \\ \ge& \lim_{t\to\infty} \int_0^K \frac{1}{\sqrt{2\pi h(t)}} \exp\left(-\frac{\mathrm{e}^{-2at}\, x}{2h(t)} - \frac{y^2}{2h(t)}\right) \cdot 2\,\mathrm{e}^{at} \sinh\left(\frac{\mathrm{e}^{-at}\, xy}{h(t)}\right) dy > 0 \end{aligned}$$

■

Of course we should prove now that the numerator of I_2 is $o(\mathrm{e}^{-at})$. To this purpose we separate the numerator of I_2 as we have done in example 3.1:

We have

$$\begin{aligned} &\int_A \mathbb{P}_y(\tau > u) \frac{1}{\sqrt{2\pi h(t-u(t))}} \left[\mathrm{e}^{-\frac{(\mathrm{e}^{-a(t-u(t))}\, x - y)^2}{2h(t-u(t))}} - \mathrm{e}^{-\frac{(\mathrm{e}^{-a(t-u(t))}\, x + y)^2}{2h(t-u(t))}}\right] dy \\ -&\int_A \mathbb{P}_y(\tau > u) \frac{1}{\sqrt{2\pi h(t-u)}} \left[\mathrm{e}^{-\frac{(\mathrm{e}^{-a(t-u)}\, x - y)^2}{2h(t-u)}} - \mathrm{e}^{-\frac{(\mathrm{e}^{-a(t-u)}\, x + y)^2}{2h(t-u)}}\right] dy \\ =&\frac{1}{\sqrt{2\pi h(t)}} \left[J_- - J_+\right], \end{aligned}$$

where

$$J_\pm = \int_A \mathbb{P}_y(\tau > u)\sqrt{\frac{h(t)}{h(t-u(t))}} \exp\left(-\frac{(\mathrm{e}^{-a(t-u(t))}\,x \pm y)^2}{2h(t-u(t))}\right) dy$$
$$- \int_A \mathbb{P}_y(\tau > u)\sqrt{\frac{h(t)}{h(t-u)}} \exp\left(-\frac{(\mathrm{e}^{-a(t-u)}\,x \pm y)^2}{2h(t-u)}\right) dy.$$

By adding and substracting

$$\sqrt{\frac{h(t)}{h(t-u(t))}} \exp\left(-\frac{(\mathrm{e}^{-a(t-u)}\,x \pm y)^2}{2h(t-u)}\right)$$

inside the integral of $J_\pm$ we see that we have to check the rates of decay of

$$\int_A \mathbb{P}_y(\tau > u)\sqrt{\frac{h(t)}{h(t-u(t))}} \left[\mathrm{e}^{-\frac{(\mathrm{e}^{-a(t-u(t))}\,x\pm y)^2}{2h(t-u(t))}} - \mathrm{e}^{-\frac{(\mathrm{e}^{-a(t-u)}\,x\pm y)^2}{2h(t-u)}}\right] dy$$

and (3.14)

$$\int_A \mathbb{P}_y(\tau > u)\left[\sqrt{\frac{h(t)}{h(t-u(t))}} - \sqrt{\frac{h(t)}{h(t-u)}}\right] \mathrm{e}^{-\frac{(\mathrm{e}^{-a(t-u)}\,x\pm y)^2}{2h(t-u)}}\, dy.$$

first term: We concentrate on the '[...]'-term

$$\left[\exp\left(-\frac{(\mathrm{e}^{-a(t-u(t))}\,x \pm y)^2}{2h(t-u(t))}\right) - \exp\left(-\frac{(\mathrm{e}^{-a(t-u)}\,x \pm y)^2}{2h(t-u)}\right)\right] \tag{3.15}$$

since we intend to use the dominated convergence theorem and $h(t)/h(t-u(t)) \to 1$. We extend [...] by $\exp\big(-\frac{(\mathrm{e}^{-a(t-u)}\,x\pm y)^2}{2h(t-u)}\big)$ to get that [...] is equal to

$$\exp\left(-\frac{(\mathrm{e}^{-a(t-u)}\,x \pm y)^2}{2h(t-u)}\right) \cdot \left[\mathrm{e}^{\,g(t,y)} - 1\right]$$
$$= \exp\left(-\frac{(\mathrm{e}^{-a(t-u)}\,x \pm y)^2}{2h(t-u)}\right) \cdot \left[g(t,y) + o_t(|g(t,y)|)\right]$$

with

$$g(t,y) := -\frac{(\mathrm{e}^{-a(t-u(t))}\,x \pm y)^2}{2h(t-u(t))} + \frac{(\mathrm{e}^{-a(t-u)}\,x \pm y)^2}{2h(t-u)}.$$

Here we have used a Taylor formula for the exponential function. Next we want to show that $\mathrm{e}^{at} \cdot g(t,y)$ converges to zero but this is rather easy by splitting $g(t,y)$ into

$$\frac{(\mathrm{e}^{-a(t-u)}\,x \pm y)^2}{2h(t-u)} - \frac{(\mathrm{e}^{-a(t-u)}\,x \pm y)^2}{2h(t-u(t))}$$

and

$$\frac{(\mathrm{e}^{-a(t-u)}\,x \pm y)^2}{2h(t-u(t))} - \frac{(\mathrm{e}^{-a(t-u(t))}\,x \pm y)^2}{2h(t-u(t))}.$$

Thus, we have the convergence of $\mathrm{e}^{at}\cdot[\ldots]$ to zero (pointwise in y); $[\ldots]$ being the term from (3.15). Now we want to identify some integrable upper bound of $\left|\mathrm{e}^{at}\cdot[\ldots]\right|$ to apply Lebesgue's dominated convergence theorem:

For this purpose we use again that $\left|\mathrm{e}^{-c}-\mathrm{e}^{-d}\right| \le |c-d|$ for all $c, d \ge 0$ to obtain

$$\mathrm{e}^{at}\cdot[\ldots] \le \mathrm{e}^{at}\cdot\left|\frac{(\mathrm{e}^{-a(t-u(t))}\,x \pm y)^2}{2h(t-u(t))} - \frac{(\mathrm{e}^{-a(t-u)}\,x \pm y)^2}{2h(t-u)}\right|.$$

By solving the squares and using the triangle inequality twice, we see that the only term in question (i.e. may not be dominated by something integrable) is the last term

$$\mathrm{e}^{at}\cdot\left|\frac{y^2}{2h(t-u(t))} - \frac{y^2}{2h(t-u)}\right|.$$

But this also has an integrable upper bound which can be seen by

$$\begin{aligned}\mathrm{e}^{at}\cdot\left|\frac{1}{h(t-u(t))} - \frac{1}{h(t-u)}\right| &= \mathrm{e}^{at}\cdot\left|\frac{2a\big(\mathrm{e}^{-2a(t-u(t))}-\mathrm{e}^{-2a(t-u)}\big)}{\big(1-\mathrm{e}^{-2a(t-u(t))}\big)\big(1-\mathrm{e}^{-2a(t-u)}\big)}\right| \\ &\propto 2a\,\mathrm{e}^{-at}\left|\mathrm{e}^{2au}-\mathrm{e}^{2au(t)}\right| \to 0.\end{aligned}$$

second term: It is rather easy to see that

$$\begin{aligned}&\mathrm{e}^{at}\cdot\left(\sqrt{\frac{h(t)}{h(t-u(t))}} - \sqrt{\frac{h(t)}{h(t-u)}}\right) \\ =\,&\mathrm{e}^{at}\cdot\left(\frac{\sqrt{1-\mathrm{e}^{-2at}}}{\sqrt{1-\mathrm{e}^{-2a(t-u(t))}}} - \frac{\sqrt{1-\mathrm{e}^{-2at}}}{\sqrt{1-\mathrm{e}^{-2a(t-u)}}}\right) \\ \propto\,&\mathrm{e}^{at}\cdot\left(\frac{1}{\sqrt{1-\mathrm{e}^{-2a(t-u(t))}}} - \frac{1}{\sqrt{1-\mathrm{e}^{-2a(t-u)}}}\right) \\ \propto\,&\mathrm{e}^{at}\cdot\left(\sqrt{1-\mathrm{e}^{-2a(t-u)}} - \sqrt{1-\mathrm{e}^{-2a(t-u(t))}}\right) \\ \propto\,&\mathrm{e}^{at}\cdot\left(1-\frac{\mathrm{e}^{-2a(t-u)}}{2} - \left(1-\frac{\mathrm{e}^{-2a(t-u(t))}}{2}\right)\right) \\ =\,&\mathrm{e}^{-at}\cdot\frac{\mathrm{e}^{2a\,u(t)}-\mathrm{e}^{2a\,u}}{2} \to 0.\end{aligned}$$

For the last '$\propto$-relation' we used a Taylor approximation of $x \mapsto \sqrt{1-x}$ at zero. Now the integrand of the second term from (3.14) is nicely bounded and we apply the dominated convergence theorem.

∎

In proposition 3.4 we are going to see that the existence of a Yaglom limit, i.e. the existence of $\lim_{t\to\infty} \mu_x^{t|t}$, implies the existence of $\lim_{t\to\infty} \mu_x^{t-u(t)|t}$ whenever $u(t) \to u \geq 0$. But at first we want to prove a lemma which may have a right on its own. In particular, one can deduce from it that the Feller semigroup corresponding to Y under τ is uniformly continuous on compacts.

lemma 3.3. *Suppose that Y is a regular diffusion on $[0,\infty)$ as in section 1.2. Let κ be a killing rate continuous on $[0,\infty)$. And let there be killing at zero with a killing rate parametrized by $\alpha > 0$. Finally, let $\tau = \tau^{\kappa,\alpha}$ be the corresponding killing time and $u(t) \to u > 0$ when $t \to \infty$. Then*

$$\sup_{y\in A} |\mathbb{P}_y(\tau > u(t)) - \mathbb{P}_y(\tau > u)| \to 0, \tag{3.16}$$

for all bounded $A \subset \mathbb{R}_+$ as $t \to \infty$.

proof. Obviously we only have to check (3.16) for $A = [0, N]$, $N \in \mathbb{N}$. We have

$$\begin{aligned} &|\mathbb{P}_y(\tau > u(t)) - \mathbb{P}_y(\tau > u)| \\ \leq &\left|\mathbb{E}_y(f(Y_{u(t)}), \tau > u(t)) - \mathbb{E}_y(f(Y_u), \tau > u)\right| \\ + &\left|\mathbb{E}_y(1 - f(Y_{u(t)}), \tau > u(t)) - \mathbb{E}_y(1 - f(Y_u), \tau > u)\right| . \end{aligned} \tag{3.17}$$

Here we choose $f \in C$ such that $0 \leq f \leq 1$, $f \equiv 1$ on $[0, M]$ and $f \equiv 0$ on $[M+1, \infty)$. The second term of (3.17) is smaller than

$$\begin{aligned} &\sup_{y\leq N} \left(\left|\mathbb{P}_y(Y_{u(t)} \geq M)\right| + |\mathbb{P}_y(Y_u \geq M)|\right) \\ \leq &\sup_{y\leq N} \left(\mathbb{P}_y(T_M(Y) \leq u + \delta) + \mathbb{P}_y(T_M(Y) \leq u)\right) \\ \leq &\mathbb{P}_N(T_M(Y) \leq u + \delta) + \mathbb{P}_N(T_M(Y) \leq u) \end{aligned} \tag{3.18}$$

for $M \geq N$ and t large enough. For the last inequality we used that $\mathbb{P}_x(T_z \leq t) \leq \mathbb{P}_y(T_z \leq t)$ for all $x < y < z$ in the state space E and for all $t > 0$. (This can be verified by a coupling argument similar to the one following (3.2).)

By our assumptions we may choose M so large that (3.18) becomes $\leq \frac{\epsilon}{2}$. Let us get a grip on the first term of (3.17).

If $\kappa \in C_b([0,\infty), \mathbb{R}_+)$, the corresponding semigroup $(\mathbf{T}_t^\tau)_t$ is a Feller-Dynkin semigroup and in particular strongly continuous at $t = 0$ by lemma 2.15. Of course, our κ is not bounded. But we take a fixed t_0 and M large enough such that $\mathbb{P}_N(T_M \leq t_0) < \delta$. By a coupling argument we see that $\mathbb{P}_y(T_M \leq t) < \delta$ also holds for all $y \leq N$ and $t \leq t_0$. Now define

$$\kappa_M(x) := \begin{cases} \kappa(x) & x \leq M \\ \kappa(M) & x > M \end{cases} .$$

Then

$$
\begin{aligned}
&\sup_{y\le N} |\mathbf{T}_t^{\alpha,\kappa} f(y) - \mathbf{T}_t^{\alpha,\kappa_M} f(y)| \\
\le &\|f\|_\infty \cdot \sup_{y\le N} \mathbb{E}_y \left| \mathrm{e}^{-\int_0^t \kappa(Y_s)ds} - \mathrm{e}^{-\int_0^t \kappa_M(Y_s)ds} \right| \\
= &\|f\|_\infty \cdot \sup_{y\le N} \mathbb{E}_y \left| \mathrm{e}^{-\int_0^t \kappa(Y_s)ds} - \mathrm{e}^{-\int_0^t \kappa_M(Y_s)ds} \right| \mathbb{1}_{\{T_M \le t\}} \\
\le &\|f\|_\infty \cdot 2\delta =: \tilde{\delta}
\end{aligned}
$$

uniformly for all $t \le t_0$. By the strong continuity of $(T_t^{\alpha,\kappa_M})_t$, this implies

$$
\begin{aligned}
&\sup_{y\le N} \left|\mathbb{E}_y(f(Y_{u(t)}), \tau > u(t)) - \mathbb{E}_y(f(Y_u), \tau > u)\right| \\
= &\sup_{y\le N} \left| T_{u(t)}^{\alpha,\kappa} f(y) - T_u^{\alpha,\kappa} f(y) \right| \\
\le &\sup_{y\le N} \left| T_{u(t)}^{\alpha,\kappa_M} f(y) - T_u^{\alpha,\kappa_M} f(y) \right| + 2\tilde{\delta} \\
\le &\left\| T_{u(t)}^{\alpha,\kappa_M} f - T_u^{\alpha,\kappa_M} f \right\|_\infty + 2\tilde{\delta} \\
\le &\left\| T_{u(t)\wedge u}^{\alpha,\kappa_M} \right\|_\infty \cdot \left\| T_{|u(t)-u|}^{\alpha,\kappa_M} f - f \right\|_\infty + 2\tilde{\delta} \\
\le &3\tilde{\delta}.
\end{aligned}
$$

In the above estimate t_0 can be choosen to be $u+1$ and $\tilde{\delta} = 2\delta \|f\|_\infty$ to be arbitrarily small. ■

Now we are going to show that $\mu_x^{t-u(t)|t}(\bullet) = \mathbb{P}_x(Y_{t-u(t)} \in \bullet \mid \tau > t)$ indeed has the same asymptotic behaviour as $\mu_x^{t-u|t}$.

theorem 3.4. *Suppose we have a regular diffusion Y on $[0,\infty)$ with the drift coefficient $\boldsymbol{a}$ and some killing rate κ such that A 1 and A 2 are fulfilled. There may be killing at zero parametrized by $\alpha > 0$. Further suppose that Y has a Yaglom limit $\underline{\varphi}(r)dr$ under $\tau = \tau^{\kappa,\alpha}$ and that $\underline{\lambda}^{\kappa,\alpha} \ne K := \lim_{x\to\infty} \kappa(x)$. Then*

$$
\mathbb{P}_x(Y_{t-u(t)} \in A \mid \tau > t) \to \frac{\int_A \mathbb{P}_r(\tau > u)\underline{\varphi}(r)dr}{\int_0^\infty \mathbb{P}_r(\tau > u)\underline{\varphi}(r)dr}
$$

for each bounded measurable A and whenever $u(t) \to u$.

Note that on one hand there is no additional assumption on the rate of convergence of $u(t) - u \to 0$ and on the other hand the previous 2 examples fit into the framework of the above proposition.

proof of theorem 3.4. We again take the partition of

$$
\mathbb{P}_x(Y_{t-u(t)} \in A \mid \tau > t) - \mathbb{P}_x(Y_{t-u} \in A \mid \tau > t)
$$

into I_1 and I_2 from (3.7) right before example 3.1 and show that both terms tend to zero as

$t \to \infty$. Thus,

$$I_1 = \frac{\int_A [\mathbb{P}_y(\tau > u(t)) - \mathbb{P}_y(\tau > u)]\, p_{t-u(t)}(x, dy)}{\mathbb{P}_x(\tau > t)}$$

and if we use lemma 3.3 we can always take $t \geq t_0(\epsilon)$ large enough such that

$$\begin{aligned} |I_1| &\leq \sup_{y \in A} |P_y(\tau > u(t)) - \mathbb{P}_y(\tau > u)| \cdot \frac{\mathbb{P}_x(\tau > t - u(t))}{\mathbb{P}_x(\tau > t)} \\ &\leq \epsilon \cdot \frac{\mathbb{P}_x(\tau > t - u(t))}{\mathbb{P}_x(\tau > t)} \end{aligned}$$

for any $\epsilon > 0$. The only thing we still need to know for $I_1 = I_1(t) \to 0$ is that

$$\limsup_{t \to \infty} \frac{\mathbb{P}_x(\tau > t - u(t))}{\mathbb{P}_x(\tau > t)} < \infty.$$

But this is true since $\lim_{t\to\infty} \frac{\mathbb{P}_x(\tau > t - v)}{\mathbb{P}_x(\tau > t)}$ exists for any $v > 0$ by Lemma 4.1 of [KS12]. Next let us show that

$$I_2(t) = \frac{\int_A \mathbb{P}_y(\tau > u) \left[p_{t-u(t)}(x, dy) - p_{t-u}(x, dy)\right]}{P_x(\tau > t)}$$

goes to 0 as $t \to \infty$. W.l.o.g. we suppose that either $u(t) \geq u$ or $u(t) \leq u$ for all t large enough. (If not, we can always split a sequence $t_k \to \infty$ into a subsequences (r_l) and s_l such that $u(r_l)$ is larger and $u(s_l)$ is smaller than u. And if we show that both $I(r_l)$ and $I(s_l)$ converge to 0 the same is true for $I(t_k)$.) Further we concentrate on the case '$u(t) \geq u$'. (The argumentation for '$u(t) \leq u$' is similar.) We use the MP at $t - u(t)$ to obtain

$$\begin{aligned} &\frac{\mathbb{P}_x(\mathbb{P}_{Y_{t-u(t)}}(\tau > u),\, Y_{t-u(t)} \in A,\, \tau > t - u(t)) - \mathbb{P}_x(\mathbb{P}_{Y_{t-u}}(\tau > u),\, Y_{t-u} \in A,\, \tau > t - u)}{\mathbb{P}_x(\tau > t)} \\ =&\frac{\mathbb{P}_x(Y_{t-u(t)} \in A,\, \tau > t - (u(t) - u)) - \mathbb{P}_x(Y_{t-u} \in A,\, \tau > t)}{\mathbb{P}_x(\tau > t)} \\ =&I_{2,1} + \frac{\mathbb{P}_x(\tau > t - u(t))}{\mathbb{P}_x(\tau > t)} \cdot I_{2,2} \end{aligned}$$

with

$$I_{2,1} = \frac{\mathbb{P}_x(Y_{t-u(t)} \in A,\, \tau > t - (u(t) - u)) - \mathbb{P}_x(\tau > t,\, Y_{t-u(t)} \in A)}{\mathbb{P}_x(\tau > t)}$$

and

$$I_{2,2} = \frac{\mathbb{P}_x(\tau > t,\, Y_{t-u(t)} \in A) - \mathbb{P}_x(\tau > t,\, Y_{t-u} \in A)}{\mathbb{P}_x(\tau > t - u(t))}.$$

Since

$$\begin{aligned} |I_{2,1}| &= \frac{\mathbb{P}_x(\{\tau > t\} \setminus \{\tau > t - (u(t) - u)\},\, Y_{t-u(t)} \in A)}{\mathbb{P}_x(\tau > t)} \\ &\leq \frac{\mathbb{P}_x(\{\tau > t\} \setminus \{\tau > t - (u(t) - u)\})}{\mathbb{P}_x(\tau > t)} \\ &= \frac{P_x(\tau > t) - \mathbb{P}_x(\tau > t - (u(t) - u))}{\mathbb{P}_x(\tau > t)} \end{aligned}$$

we have

$$\limsup_{t\to\infty} |I_{2,1}(t)| \leq 1 - \mathrm{e}^{-\gamma\cdot v}$$

for any $v > 0$ (and t large enough such that $u(t) - u \leq v$). By the MP

$$\begin{aligned} I_{2,2} &= \mathbb{P}_x(P_{Y_{t-u(t)}}(\tau > u(t)),\, Y_{t-u(t)} \in A \,|\, \tau > t - u(t)) \\ &\quad - \frac{\mathbb{P}_x(\tau > t-u)}{\mathbb{P}_x(\tau > t-u(t))}\mathbb{P}_x(\mathbb{P}_{Y_{t-u}}(\tau > u),\, Y_{t-u} \in A \,|\, \tau > t-u). \end{aligned}$$

The second term is asymptotically equivalent to

$$\mathbb{P}_x(\mathbb{P}_{Y_{t-u(t)}}(\tau > u),\, Y_{t-u(t)} \in A \,|\, \tau > t-u(t)).$$

Therefore, we get

$$I_{2,2}(t) \propto \mathbb{P}_x(\Big[\mathbb{P}_{Y_{t-u(t)}}(\tau > u) - \mathbb{P}_{Y_{t-u(t)}}(\tau > u(t))\Big]\,,\, Y_{t-u(t)} \in A \,|\, \tau > t-u(t)).$$

Finally this is

$$\leq \sup_{y\in A} |\mathbb{P}_y(\tau > u) - \mathbb{P}_y(\tau > u(t))|$$

which converges to zero by lemma 3.3. ■

3.2. Somewhere in between

In this section we are looking for answers on the following question:

What happens with $\mu_x^{s(t)|t}$, i.e. with

$$\mathbb{P}_x\left(Y_{s(t)} \in \bullet \,|\, \tau > t\right),$$

if

$$s(t) \to \infty \quad \text{and} \quad t - s(t) \to \infty \tag{3.19}$$

as t tends to infinity?

For this purpose we investigate two examples. We will use the notation '$q^t(s(t), x, y)$' for *the density of* $\mu_x^{s(t)|t}$.

The first example is a good old friend: Brownian motion with constant drift under $\tau = T_0$.

example 3.3. At first we want to deduce the density of $q^t(s(t), x, \cdot)$ w.r.t. the Lebesgue measure. Therefore, let A be any Borel-measurable set in $\mathbb{R}_+$. Then

$$\begin{aligned}\mathbb{P}_x(Y_{s(t)} \in A \mid \tau > t) &= \frac{\mathbb{P}_x(\mathbb{P}_{Y_{s(t)}}(\tau > t - s(t)),\, Y_{s(t)} \in A,\, \tau > s(t))}{\mathbb{P}_x(\tau > t)} \\ &= \frac{\int_A \mathbb{P}_y(\tau > t - s(t)) \cdot p_{s(t)}(x,y)dy}{\int_0^\infty p_t(x,y)dy} \\ &= \frac{\int_A \int_0^\infty p_{t-s(t)}(y,z)dz \cdot p_{s(t)}(x,y)dy}{\int_0^\infty p_t(x,y)dy}.\end{aligned}$$

Here

$$\begin{aligned}p_t(x,y) &= \mathrm{e}^{-a(y-x)-\frac{a^2}{2}t} \cdot p_-(t,x,y) \\ &= \mathrm{e}^{-a(y-x)-\frac{a^2}{2}t} \cdot \frac{1}{\sqrt{2\pi t}}\left(\mathrm{e}^{-\frac{(x-y)^2}{2t}} - \mathrm{e}^{-\frac{(x+y)^2}{2t}}\right)\end{aligned} \tag{3.20}$$

is the kernel of (the semigroup of) Y under $\tau = T_0$ w.r.t. dy. This leads to

$$\mathbb{P}_x(Y_{s(t)} \in A \mid \tau > t) = \frac{\int_A \int_0^\infty \mathrm{e}^{-az}\, p_-(t - s(t), y, z)dz \cdot p_-(s(t), x, y)dy}{\int_0^\infty \mathrm{e}^{-az}\, p_-(t,x,z)dz}.$$

Thus,

$$q^t(s(t),x,y) = \frac{\int_0^\infty \mathrm{e}^{-az}\, p_-(t-s(t),y,z)dz \cdot p_-(s(t),x,y)}{\int_0^\infty \mathrm{e}^{-az}\, p_-(t,x,z)dz} \tag{3.21}$$

is the density of $Y_{s(t)}$ conditioned on $\{\tau > t\}$.

Now we want to investigate the asymptotic behaviour of $q^t(s(t),x,y)$ for $t \to \infty$. We have that

$$p_-(s(t),x,y) \cdot s(t)^{\frac{3}{2}} \to \sqrt{\frac{2}{\pi}}xy$$

when $s(t) \to \infty$ since

$$p_-(t,x,y) = \frac{1}{\sqrt{2\pi t}}\left(\mathrm{e}^{-\frac{(x-y)^2}{2t}} - \mathrm{e}^{-\frac{(x+y)^2}{2t}}\right) \propto \frac{1}{\sqrt{2\pi}}\frac{(x+y)^2 - (x-y)^2}{2t^{\frac{3}{2}}} \tag{3.22}$$

which is the first step towards the asymptotics of (3.21). Next we write the integral in the denominator of (3.21) as

$$\frac{1}{t^{\frac{3}{2}}}\int_0^\infty \mathrm{e}^{-az} \cdot \frac{t}{\sqrt{2\pi}}\left(\mathrm{e}^{-\frac{(x-z)^2}{2t}} - \mathrm{e}^{-\frac{(x+z)^2}{2t}}\right)dz. \tag{3.23}$$

Using a Taylor approximation on $y \mapsto \mathrm{e}^{-y}$ we see that there is a $C = C(t_0) > 0$ such that

$$\left|\frac{t}{\sqrt{2\pi}}\left(\mathrm{e}^{-\frac{(x-z)^2}{2t}} - \mathrm{e}^{-\frac{(x+z)^2}{2t}}\right)\right| \leq 2xz + \frac{C}{t_0}g(x,z) \tag{3.24}$$

for any $t \geq t_0 > 0$. The bound $z \mapsto g(x,z)$ can be chosen to be of order $O(z^4)$. By (3.22) and

(3.24) we can use the dominated convergence theorem in (3.23) to obtain

$$\begin{aligned}\int_0^\infty \mathrm{e}^{-az}\, p_-(t,x,z)dz &= \frac{1}{t^{\frac{3}{2}}}\int_0^\infty \mathrm{e}^{-az}\,\frac{t}{\sqrt{2\pi}}\left(\mathrm{e}^{-\frac{(x-z)^2}{2t}} - \mathrm{e}^{-\frac{(x+z)^2}{2t}}\right)dz \\ &\propto \frac{1}{t^{\frac{3}{2}}}\int_0^\infty \mathrm{e}^{-az}\,\frac{1}{\sqrt{2\pi}}\frac{(x+z)^2-(x-z)^2}{2}dz.\end{aligned}$$

Hence,

$$\int_0^\infty \mathrm{e}^{-az}\, p_-(t,x,z)dz \propto \frac{1}{t^{\frac{3}{2}}}\frac{2x}{\sqrt{2\pi}a} \tag{3.25}$$

as well as

$$\int_0^\infty \mathrm{e}^{-az}\, p_-(t-s(t),y,z)dz \propto \frac{1}{(t-s(t))^{\frac{3}{2}}}\frac{2y}{\sqrt{2\pi}a}$$

if $t-s(t)\to\infty$. Plugging everything into (3.21) we get

$$q^t(s(t),x,y) \propto \frac{t^{\frac{3}{2}}}{(t-s(t))^{\frac{3}{2}}s(t)^{\frac{3}{2}}}\sqrt{\frac{2}{\pi}}y^2. \tag{3.26}$$

Therefore, the densities of $Y_{s(t)}$ conditioned on $\{\tau>t\}$ converge to zero on $\mathbb{R}_+$ as t tends to ∞. This looks a bit like the 'escape to infinity'-behaviour from chapter 1. By taking a closer look on the calculations above we could make this rigorous. But this can be seen more quickly: Take any $M>0$. Then

$$\begin{aligned}\mathbb{P}_x(Y_{s(t)}\le M \mid \tau>t) &= \frac{\mathbb{P}_x(\mathbb{P}_{Y_{s(t)}}(\tau>t-s(t)),\, Y_{s(t)}\le M,\, \tau>s(t))}{\mathbb{P}_x(\tau>t)} \\ &\le \frac{\mathbb{P}_x(\mathbb{P}_M(\tau>t-s(t)),\, Y_{s(t)}\le M,\, \tau>s(t))}{\mathbb{P}_x(\tau>t)} \\ &\le \frac{\mathbb{P}_M(\tau>t-s(t))\cdot\mathbb{P}_x(\tau>s(t))}{\mathbb{P}_x(\tau>t)} \\ &= \frac{\int_0^\infty p_{t-s(t)}(M,z)dz\cdot\int_0^\infty p_{s(t)}(x,z)dz}{\int_0^\infty p_t(x,z)dz} \\ &= \frac{\mathrm{e}^{aM}\int_0^\infty \mathrm{e}^{-az}\,p_-(t-s(t),M,z)dz\cdot\int_0^\infty \mathrm{e}^{-az}\,p_-(s(t),x,z)dz}{\int_0^\infty \mathrm{e}^{-az}\,p_-(t,x,z)dz} \\ &\propto \mathrm{e}^{aM}\frac{2M}{\sqrt{2\pi}a}\cdot\frac{t^{\frac{3}{2}}}{(t-s(t))^{\frac{3}{2}}s(t)^{\frac{3}{2}}}\to 0\end{aligned} \tag{3.27}$$

due to (3.25). Here $\frac{t^{\frac{3}{2}}}{(t-s(t))^{\frac{3}{2}}s(t)^{\frac{3}{2}}}\to 0$ can be seen by

$$\left(\frac{t^{\frac{3}{2}}}{(t-s(t))^{\frac{3}{2}}s(t)^{\frac{3}{2}}}\right)^{\frac{2}{3}} = \frac{t-s(t)}{(t-s(t))s(t)} + \frac{s(t)}{(t-s(t))s(t)}\to 0.$$

■

We can even extract a small proposition from the example:

proposition 3.5. *Let Y be an irreducible diffusion on $E = [0, \infty)$. Further let $\tau = \tau^{\kappa,\alpha}$ be the killing time corresponding to killing under the killing rate $\kappa \geq 0$ and killing at zero with rate $\alpha \geq 0$. Let κ be decreasing and assume* (1.1)*. Finally suppose* (2.5)*, i.e. that*

$$\mathbb{P}_x(\tau > t) \sim t^{-k}\, \mathrm{e}^{-\eta t} \tag{3.28}$$

for some $k > 0$ and $\eta \geq 0$. Then we have

$$\lim_{t\to\infty} \mu_x^{s(t)|t}(A) = 0$$

for all bounded and measurable A (if $s(t) \to \infty$ and $t - s(t) \to \infty$).

proof. By taking a closer look at the last example we observe that we only have to retrace the steps in (3.27). Hence,

$$\begin{aligned} \mathbb{P}_x(Y_{s(t)} \in A \mid \tau > t) &\leq \mathbb{P}_x(Y_{s(t)} \leq M \mid \tau > t) \\ &= \frac{\mathbb{P}_x(\mathbb{P}_{Y_{s(t)}}(\tau > t - s(t)),\, Y_{s(t)} \leq M,\, \tau > s(t))}{\mathbb{P}_x(\tau > t)} \\ &\leq \frac{\mathbb{P}_x(\mathbb{P}_M(\tau > t - s(t)),\, Y_{s(t)} \leq M,\, \tau > s(t))}{\mathbb{P}_x(\tau > t)} \end{aligned} \tag{3.29}$$

with some $M > 0$ such that $A \subseteq [0, M]$ and by using a coupling argument as in (3.4). But (3.29) gives us

$$\mathbb{P}_x(Y_{s(t)} \in A \mid \tau > t) \leq \frac{\mathbb{P}_M(\tau > t - s(t)) \cdot \mathbb{P}_x(\tau > s(t))}{\mathbb{P}_x(\tau > t)}.$$

At the end we apply (3.28) to obtain

$$\mathbb{P}_x(Y_{s(t)} \in A \mid \tau > t) = O\left(\frac{t^k}{(t - (s(t))^k s(t)^k)}\right).$$

This delivers the desired result since $\frac{t^k}{(t-(s(t))^k s(t)^k)} \to 0$ as $t \to \infty$. ■

Next we present another example which will yield, as we will see at the end, a qualitatively different result. But at first we recall the following notations already used in the preceding chapters:

- $\tau^{\kappa,\infty}(Y)$ is the killing time that we get when combining instant killing at zero with killing of Y under the rate κ.
- $\mathbf{L}^{\kappa,\infty}(\boldsymbol{a}) = \frac{1}{2}\frac{d^2}{dx^2} + \boldsymbol{a}(x)\frac{d}{dx} - \kappa(x)$ is the $L^2(m)$-generator of (the semigroup of) Y under $\tau^{\kappa,\infty}$.
- $\underline{\lambda}^{\kappa,\infty}(\boldsymbol{a})$ is the infimum of the spectrum of $-\mathbf{L}^{\kappa,\infty}(\boldsymbol{a})$.

- $\underline{\psi}^{\kappa,\infty}(\boldsymbol{a})$ is a corresponding eigenfunction in the sense of Sturm-Liouville, i.e. the unique (up to positive multiples) positive solution on $\mathbb{R}_+$ to

$$\left[\frac{1}{2}\frac{d^2}{dx^2} + \boldsymbol{a}(x)\frac{d}{dx} - \kappa(x)\right]\psi = -\underline{\lambda}^{\kappa,\infty}(\boldsymbol{a})\psi$$

with $\psi(0) = 0$.

example 3.4. Let $(Y_t)_t$ be the Ornstein-Uhlenbeck process which is a solution to

$$dY_t = dX_t + \boldsymbol{a}(Y_t)dt \tag{3.30}$$

with $\boldsymbol{a}(y) = -ay$, $a > 0$, and $Y_0 = x$. Let there be instant killing at zero, i.e. take $\tau = T_0 = T_0(Y)$. Fortunately Y has a kernel τ w.r.t. the Lebesgue measure:

$$p_t(x,y) = \sqrt{\frac{2}{\pi h(t)}}\exp\left(-\frac{\mathrm{e}^{-2at}}{2h(t)} - \frac{y^2}{2h(t)}\right)\sinh\left(\frac{\mathrm{e}^{-at}\,xy}{h(t)}\right)$$

with $h(t) := \frac{1-\mathrm{e}^{-2at}}{2a}$. We also observe that $\underline{\lambda} = \underline{\lambda}^{0,\infty}(\boldsymbol{a}) = a$. To shorten writing we also define

$$g_t(x,y) := \mathrm{e}^{\underline{\lambda}t}\,p_t(x,y) = \mathrm{e}^{at}\,p_t(x,y).$$

We immediately get

$$\begin{aligned}
g_t(x,y) &\propto \sqrt{\frac{2}{\pi\frac{1}{2a}}}\exp\left(-\frac{y^2}{2\frac{1}{2a}}\right)\cdot \mathrm{e}^{at}\sinh\left(\frac{\mathrm{e}^{-at}\,xy}{h(t)}\right)\\
&\propto \sqrt{\frac{4a}{\pi}}\exp(-ay^2)\cdot\frac{xy}{h(t)}\frac{\exp\left(\frac{\mathrm{e}^{-at}\,xy}{h(t)}\right)-\exp\left(-\frac{\mathrm{e}^{-at}\,xy}{h(t)}\right)}{2\frac{\mathrm{e}^{-at}\,xy}{h(t)}}\\
&\to \sqrt{\frac{4a}{\pi}}\exp(-ay^2)\cdot 2axy = \frac{4a^{\frac{3}{2}}}{\sqrt{\pi}}xy\,\mathrm{e}^{-ay^2}.
\end{aligned}$$

Using this we deduce

$$\begin{aligned}
\mathbb{P}_x(Y_{s(t)}\in dy\,|\,\tau>t) &= \frac{p_{s(t)}(x,y)dy\int_0^\infty p_{t-s(t)}(y,z)dz}{\int_0^\infty p_t(x,z)dz}\\
&= g_{s(t)}(x,y)dy\cdot\frac{\int_0^\infty g_{t-s(t)}(y,z)dz}{\int_0^\infty g_t(x,z)dz}\\
&\propto \frac{4a^{\frac{3}{2}}}{\sqrt{\pi}}xy\,\mathrm{e}^{-ay^2}\,dy\cdot\frac{\int_0^\infty g_{t-s(t)}(y,z)dz}{\int_0^\infty g_t(x,z)dz}.
\end{aligned}\tag{3.31}$$

To investigate the limit behaviour of $\mathbb{P}_x(Y_{s(t)}\in dy\,|\,\tau>t)$ it remains to calculate

$$\lim_{t\to\infty}\int_0^\infty g_t(x,y)dy.$$

For this we emphasize the possibility to choose $\mathbb{P}_x$ as the shifted Wiener measure on $(\Omega,\mathcal{F})$,

$\Omega = C[0,\infty)$. Thus, $X_t := w(t)$ is a standard Brownian motion starting at x under $\mathbb{P}_x$. There is a strong solution to (3.30). Let $\mathbb{P}_x^Y$ be the measure of Y on the same space as $\mathbb{P}_x$. By using a 'Cameron-Martin-Girsanov transformation' it is shown in Proposition 21 of [MV12] that

$$\mathbb{P}_x^Y(\Phi, \tau > t) = \mathbb{P}_x(\Phi \cdot \mathbb{1}_{\{\tau > t\}} \frac{\sqrt{m}(X_t)}{\sqrt{m}(X_0)} \mathcal{E}_t^\kappa)$$

for any $t > 0$ and $\Phi \in mb(\mathcal{F})$. Here '$\sqrt{m}$' is the square-root of (the density of) a speed measure for Y,

$$\mathcal{E}_t^\kappa := \mathrm{e}^{-\int_0^t \kappa(X_r)dr}$$

and

$$\kappa(x) := \frac{\boldsymbol{a}^2(x) + \boldsymbol{a}'(x)}{2}.$$

In the present example we have $\sqrt{m}(x) = \mathrm{e}^{-\frac{a}{2}x^2}$ and $\mathcal{E}_t^\kappa = \mathrm{e}^{-\frac{a^2}{2}\int_0^t X_r^2 dr} \cdot \mathrm{e}^{\frac{a}{2}t}$. So we get

$$\begin{aligned} \int_0^\infty g_t(x,y)dy &= \mathrm{e}^{at}\, \mathbb{P}_x(T_0(Y) > t) \\ &= \mathrm{e}^{at}\, \mathbb{P}_x^Y(T_0(X) > t) \\ &= \mathrm{e}^{at}\, \mathbb{P}_x\left(T_0(X) > t, \frac{\exp(-\frac{a}{2}X_t^2)}{\exp(-\frac{a}{2}x^2)}\mathcal{E}_t^\kappa\right). \end{aligned} \tag{3.32}$$

Now we want to interprete κ as a killing rate. For this we have to lift it such that it is positive everywhere, e.g. via

$$\kappa \rightsquigarrow \tilde{\kappa} := \kappa + a.$$

From (3.32) we derive

$$\int_0^\infty g_t(x,y)dy = \mathrm{e}^{\frac{a}{2}x^2} \cdot \mathrm{e}^{2at}\, \mathbb{P}_x\big(\exp\big(-\frac{a}{2}X_t^2\big), \tau^{\tilde{\kappa},\infty} > t\big). \tag{3.33}$$

Luckily we are in a situation were Theorem 4.3 of [KS12] can be applied since infinity is an inaccessible boundary for X and obviously $\infty = \liminf_{x\to\infty} \tilde{\kappa}(x) > \underline{\lambda}^{\tilde{\kappa},\infty}(0)$. Thus, we have

$$\mathrm{e}^{\underline{\lambda}^{\tilde{\kappa},\infty}t}\, \mathbb{P}_x(X_t \in A, \tau^{\tilde{\kappa},\infty} > t) \to \underline{\psi}^{\tilde{\kappa},\infty}(x) \int_A \underline{\psi}^{\tilde{\kappa},\infty}(y)dy$$

for all measurable A. We use (3.33) to obtain

$$\int_0^\infty g_t(x,y)dy \propto \frac{\mathrm{e}^{2at}}{\mathrm{e}^{\underline{\lambda}^{\tilde{\kappa},\infty}t}} \cdot \mathrm{e}^{\frac{a}{2}x^2}\, \underline{\psi}^{\tilde{\kappa},\infty}(x) \int_0^\infty \mathrm{e}^{-\frac{a}{2}y^2}\, \underline{\psi}^{\tilde{\kappa},\infty}(y)dy. \tag{3.34}$$

The next step is to show that

$$\underline{\lambda}^{\tilde{\kappa},\infty} = 2a$$

and to further identify $\underline{\psi}^{\tilde{\kappa},\infty}$. For this observe that

$$\tilde{\psi} \text{ solves } \left[\frac{1}{2}\frac{d^2}{dx^2} - \tilde{\kappa}\right]\tilde{\psi} = -\lambda\tilde{\psi} \text{ under } \tilde{\psi}(0) = 0$$

if and only if

$$\psi = \frac{\tilde{\psi}}{\sqrt{m}} \text{ solves } \left[\frac{1}{2}\frac{d^2}{dx^2} + \boldsymbol{a}\frac{d}{dx}\right]\psi = -(\lambda - a)\psi \text{ under } \psi(0) = 0.$$

Also note that the property that a solution is positive is preserved by the transformation '$\tilde{\psi} \leftrightarrow \frac{\tilde{\psi}}{\sqrt{m}}$'. Thus, we can apply theorem A.2.6 to see that

$$\underline{\lambda}^{0,\infty}(\boldsymbol{a}) = \underline{\lambda}^{\tilde{\kappa},\infty}(0) - a.$$

Because of $\underline{\lambda}^{0,\infty}(\boldsymbol{a}) = a$ we get

$$\underline{\lambda}^{\tilde{\kappa},\infty}(0) = 2a. \tag{3.35}$$

We also deduce that $\underline{\psi}^{0,\infty}(\boldsymbol{a}) = \frac{\underline{\psi}^{\tilde{\kappa},\infty}(0)}{\sqrt{m}}$. Since $\underline{\psi}^{0,\infty}(\boldsymbol{a})$ is proportional to the first Hermite polynomial of odd order and $\sqrt{m}(x) = \mathrm{e}^{-\frac{a}{2}x^2}$ we get

$$\underline{\psi}^{\tilde{\kappa},\infty}(x) = Cx\,\mathrm{e}^{-\frac{a}{2}x^2} \tag{3.36}$$

with a constant C independent from x. Now we use (3.35) and (3.36) to transform (3.34) into

$$\int_0^\infty g_t(x,y)dy \to C^2 x \int_0^\infty y\,\mathrm{e}^{-ay^2}\,dy.$$

We combine this with (3.31) to see that the density $q^t(s(t),x,y)$ of $\mathbb{P}_x(Y_{s(t)} \in \bullet \mid \tau > t)$ converges pointwise to

$$y \mapsto \frac{4a^{\frac{3}{2}}}{\sqrt{\pi}} y^2\,\mathrm{e}^{-ay^2}$$

as $t \to \infty$. The reader may verify that this is indeed a density on $\mathbb{R}_+$. We are almost done. Now we use Scheffé's Lemma to obtain

$$\mathbb{P}_x(Y_{s(t)} \in A \mid \tau > t) = \int_A q^t(s(t),x,y)dy \to \int_A \frac{4a^{\frac{3}{2}}}{\sqrt{\pi}} y^2\,\mathrm{e}^{-ay^2}\,dy$$

for all measurable A. If we look closely, we even see that

$$\mathbb{P}_x(Y_{s(t)} \in A \mid \tau > t) \to \frac{\int_A \underline{\psi}^2(y)m(y)dy}{\int_0^\infty \underline{\psi}^2(y)m(y)dy}$$

with $\underline{\psi} = \underline{\psi}^{0,\infty}(\boldsymbol{a})$. Using proposition 2.4 we deduce that this limit is nothing else but the unique invariant distribution of the penalisation limit $Z = Z^{\underline{\lambda}}$ of Y under $\tau = T_0$. ∎

When looking at the key steps of the last example we see that only a few alterations are needed to get the same result in a general setting. (See theorem 3.6 below.) We will work under the following two assumptions which have not been properly introduced yet:

A 11. We have that $\boldsymbol{a}^2(x) + \boldsymbol{a}'(x) \in C[0,\infty)$ and there is an $l \geq 0$ (which we fix) such that

$$\tilde{\kappa}(x) := \kappa(x) + \frac{\boldsymbol{a}^2(x) + \boldsymbol{a}'(x)}{2} + l \geq 0.$$

A 12. We have that $\liminf_{x\to\infty} \tilde{\kappa}(x) > \underline{\lambda}^{\tilde{\kappa},\infty}(0)$.

We will essentially use notations as in the last example. We also define $\tau := \tau^{\kappa,\infty}(Y)$, $\underline{\lambda} := \underline{\lambda}^{\kappa,\infty}(\boldsymbol{a})$ and $\underline{\psi} := \underline{\psi}^{\kappa,\infty}(\boldsymbol{a})$. Now we are able to formulate the announced generalisation of example 3.4.

theorem 3.6. *Let Y be a regular diffusion on $E = [0,\infty)$ and a solution to $dY_t = dX_t + \boldsymbol{a}(Y_t)dt$ on $(0,\infty)$ as introduced in chapter 0. Suppose that $\boldsymbol{a}$ and κ satisfy A 2, A 3, A 5 with $\kappa \geq K := \lim_{x\to\infty}\kappa(x)$, A 6, A 7, A 11 and A 12. Further suppose that $x \mapsto m(x)$ is locally bounded. Then*

(a) the penalisation limit $(Z_t)_t$ of Y under τ, i.e. with the penaliser $\Gamma_t = \mathbb{1}_{\{\tau>t\}}$, exists.

(b) Z is the h-transform $Z^{\underline{\lambda}}$ of Y by $\underline{\psi}$ and has the unique stationary distribution

$$\mathcal{B}(\mathbb{R}_+) \ni A \mapsto \int_A \underline{\psi}^2(x)m(x)dx.$$

(c) we have that

$$\mathbb{P}_x(Y_{s(t)} \leq M \mid \tau > t) \to \int_0^M \underline{\psi}^2(x)m(x)dx$$

for all $M \geq 0$ under (3.19). Thus, we have convergence in distribution.

remark 3.3.
- For example m is locally bounded if we impose that $\boldsymbol{a}_+$ is integrable at 0. Though this assumption seems somewhat unnecessary because of $\boldsymbol{a}^2(x) + \boldsymbol{a}'(x) \in C[0,\infty)$ from A 11.
- We also stress the fact that A 7 is fulfilled whenever $\kappa \equiv 0$ as in the preceding example. (See lemma 1.16.)

∎

proof of theorem 3.6. The assumptions we made are strong enough to get the existence of a kernel $p_t(x,y)$ of Y under τ w.r.t. the Lebesgue measure. (See also the proof of Theorem 22 in [MV12].) Thus, we get

$$\begin{aligned}\mathbb{P}_x(Y_{s(t)} \in A \mid \tau > t) &= \int_A p_{s(t)}(x,y) \cdot \frac{\int p_{t-s(t)}(y,z)dz}{\int p_t(x,z)dz}dy \\ &= \int_A g_{s(t)}(x,y) \cdot \frac{\int g_{t-s(t)}(y,z)dz}{\int g_t(x,z)dz}dy\end{aligned} \tag{3.37}$$

with $g_t(x,y) := \mathrm{e}^{\underline{\lambda} t}\, p_t(x,y)$ as in example 3.4. We also use the 'Cameron-Martin-Girsanov transformation' from Proposition 21 of [MV12] to see that

$$\begin{aligned}\int_A g_t(x,y)dy &= \mathrm{e}^{\underline{\lambda} t}\, \mathbb{P}_x(Y_t \in A,\, \tau > t)\\ &= \mathrm{e}^{\underline{\lambda}^{\tilde{\kappa},\infty}(0)\cdot t}\, \frac{1}{\sqrt{m}(x)} \mathbb{P}_x(\mathbb{1}_{\{X_t\in A\}}\cdot\sqrt{m}(X_t), \tau^{\tilde{\kappa},\infty}(X) > t)\\ &\to \frac{\underline{\psi}^{\tilde{\kappa},\infty}(x)}{\sqrt{m}(x)} \cdot \int_A \sqrt{m}(y)\underline{\psi}^{\tilde{\kappa},\infty}(y)dy\end{aligned} \tag{3.38}$$

as $t \to \infty$. For the convergence result we used Theorem 4.3 of [KS12]. Note that

$$\underline{\psi}^{\tilde{\kappa},\infty} = \underline{\psi}^{\tilde{\kappa},\infty}(0)$$

is the unique (up to positive multiples) positive solution to

$$\frac{1}{2}\frac{d^2}{dx^2}\tilde{\psi} - \tilde{\kappa}\tilde{\psi} = -\underline{\lambda}^{\tilde{\kappa},\infty}(0)\cdot\tilde{\psi} \text{ under } \tilde{\psi}(0) = 0.$$

As in example 3.4 we use the transformation '$\tilde{\psi} \leftrightarrow \frac{\tilde{\psi}}{\sqrt{m}}$' and (3.38) to derive

$$\int_A g_t(x,y)dy \to \underline{\psi}(x) \cdot \int_A \underline{\psi}(y)m(y)dy$$

for all measurable A. Here $\underline{\psi} = \underline{\psi}^{\kappa,\infty}(\boldsymbol{a})$ is the unique (up to positive multiple) positive solution to

$$\frac{1}{2}\frac{d^2}{dx^2}\psi + \boldsymbol{a}(x)\frac{d}{dx}\psi = -\underline{\lambda}\cdot\psi \text{ under } \psi(0) = 0.$$

In particular we have

$$\frac{\int_0^\infty g_{t-s(t)}(y,z)dz}{\int_0^\infty g_t(x,z)dz} \to \frac{\underline{\psi}(y)}{\underline{\psi}(x)} \tag{3.39}$$

pointwise in y. Up to now we were able to proceed as in example 3.4. Unfortunately we can not identify an almost sure limit of $y \mapsto g_t(x,y)$ as $t \to \infty$. Thus, we can not use Scheffé's Lemma. To overcome this difficulty we first restrict to measurable $A \subseteq [0,M]$ for some $M > 0$ which we fix for the moment. The Lebesgue measure on $[0,M]$ is essentially a probability on $[0,M]$ (suitably normalized by $\frac{1}{M}$). Hence, we can interprete the sequences $t \mapsto (y \mapsto g_t(x,y))$ and $y \mapsto \underline{\psi}(x) \cdot \underline{\psi}(y)m(y)$ as random variables. Since

$$\int_A g_t(x,y)dy \to \int_A \left[\underline{\psi}(x)\underline{\psi}(y)m(y)\right] dy$$

for all measurable $A \subseteq [0,M]$ we get

$$\int_0^M g_t(x,y)\cdot f(y)dy \to \int_0^M \left[\underline{\psi}(x)\underline{\psi}(y)m(y)\right]\cdot f(y)dy \tag{3.40}$$

for all $f \in mb(\mathcal{B}[0,M])$. (See Proposition IV.2.2 in [Nev65].)

Next we use (3.37) to obtain the following decomposition:

$$\mathbb{P}_x(Y_{s(t)} \leq M \mid \tau > t) = I_1(t) + I_2(t)$$

with

$$I_1(t) := \int_0^M g_{s(t)}(x,y) \cdot \left[\frac{\int_0^\infty g_{t-s(t)}(y,z)dz}{\int_0^\infty g_t(x,z)dz} - \frac{\underline{\psi}(y)}{\underline{\psi}(x)} \right] dy$$

and

$$I_2(t) := \int_0^M g_{s(t)}(x,y) \cdot \frac{\underline{\psi}(y)}{\underline{\psi}(x)} dy.$$

By (3.39) the sequence from $I_1(t)$ in '[...]'-brackets converges pointwise to zero and is continuous on $[0, M]$. Though we should say a few words on the continuity. Out of question is that $y \mapsto \underline{\psi}(y)$ is continuous. But why should $y \mapsto \int_0^\infty g_t(y,z)dz$ be continuous? This will be answered by the following points:

1. $g_t(y,z) = \mathrm{e}^{\lambda t} p_t(x,y)$ is jointly continuous in all variables. (See [BS02].)

2. Point 1. implies that $y \mapsto \int_0^N g_t(y,z)dz$ is continuous for any $N > 0$.

3. We have

$$\begin{aligned}\int_N^\infty g_t(y,z)dz &= \mathrm{e}^{\lambda t} \int_N^\infty p_t(y,z)dz \\ &= \mathrm{e}^{\lambda t}\, \mathbb{P}_y(Y_t \geq N,\, \tau > t) \\ &\leq \mathrm{e}^{\lambda t}\, \mathbb{P}_y(Y_t \geq N) \\ &\leq \mathrm{e}^{\lambda t}\, \mathbb{P}_M(Y_t \geq N)\end{aligned}$$

   for all $y \leq M < N$. The last inequality can be seen by a 'coupling-argument' similar to the one we used in (3.4). This gets arbitrarily small for N large enough.

4. Take $\epsilon > 0$ and $(y_n)_n \subset [0, M]$ with $y_n \to y \leq M$. By 3. we can take N large such that

$$\sup_{y \leq M} \int_N^\infty g_t(y,z)dz < \frac{\epsilon}{3}.$$

   By 2. we have

$$\left| \int_0^N g_t(y_n,z)dz - \int_0^N g_t(y,z)dz \right| < \frac{\epsilon}{3}$$

   for n large enough. Thus, we get

$$
\begin{aligned}
&\left|\int_0^\infty g_t(y_n,z)dz - \int_0^\infty g_t(y,z)dz\right| \\
\leq &\int_N^\infty g_t(y_n,z)dz + \left|\int_0^N g_t(y_n,z)dz - \int_0^N g_t(y,z)dz\right| + \int_N^\infty g_t(y,z)dz \\
<&\epsilon.
\end{aligned}
$$

This shows the contiuity of $y \mapsto \int_0^\infty g_t(y,z)dz$.

Thus, $\frac{\int_0^\infty g_{t-s(t)}(y,z)dz}{\int_0^\infty g_t(x,z)dz} - \frac{\underline{\psi}(y)}{\underline{\psi}(x)}$ also converges uniformly to 0. Therefore,

$$
\begin{aligned}
|I_1(t)| &\leq \int_0^M g_{s(t)}(x,y)\cdot \left|\frac{\int_0^\infty g_{t-s(t)}(y,z)dz}{\int_0^\infty g_t(x,z)dz} - \frac{\underline{\psi}(y)}{\underline{\psi}(x)}\right| dy \\
&\leq \int_0^M g_{s(t)}(x,y)\cdot\epsilon\, dy \\
&\leq \epsilon\cdot\left(\underline{\psi}(x)\int_0^M \underline{\psi}(y)m(y)dy + 1\right)
\end{aligned}
$$

for any $\epsilon > 0$ and $t \geq t_0(\epsilon)$ large enough. This implies

$$
I_1(t) \to 0.
$$

By (3.40) we get that

$$
I_2(t) \to \int_0^M \left[\underline{\psi}(x)\underline{\psi}(y)m(y)\right]\cdot\frac{\underline{\psi}(y)}{\underline{\psi}(x)}dy = \int_0^M \underline{\psi}^2(y)m(y)dy.
$$

This proves point (c) since we can take any $M \geq 0$ in the above argumentation. Obviously $\underline{\psi} \in L^2(dm)$ which implies point (b) if we use proposition 2.4. Finally (a) follows from theorem 2.11. ■

One can see quickly that the above proposition holds also for other 'Ornstein-Uhlenbeck-like' processes, i.e. we can treat any diffusion with an $\boldsymbol{a}$ such that

- $\boldsymbol{a} \in C^1[0,\infty)$.
- $\boldsymbol{a}(x) \to -\infty$ as $x \to \infty$.
- $\boldsymbol{a}'$ is bounded.

But there are also other kinds of diffusions which can be handled as we can see by the following example:

example 3.5. For every $\epsilon > 0$ take $f_\epsilon \in C^1[0,1]$ such that

- $0 \leq f_\epsilon \leq \epsilon$.
- $f_\epsilon(1) = 0$.

- $f_\epsilon'(1) = -1$.

Now define the drift coefficients

$$\boldsymbol{a}_\epsilon(x) := \begin{cases} f_\epsilon(x) & x \le 1 \\ \frac{1}{x} - 1 & x > 1\,. \end{cases}$$

Thus, we have $\boldsymbol{a}_\epsilon \in C^1[0,\infty)$ and we see that

$$\tilde{\kappa}(x) = \kappa_\epsilon(x) := \frac{\boldsymbol{a}_\epsilon^2(x) + \boldsymbol{a}_\epsilon'(x)}{2} = \frac{\left(\frac{1}{x} - 1\right)^2 - \frac{1}{x^2}}{2} = \frac{1}{2} - \frac{1}{x}$$

for $x \ge 1$. For simplicity we took $\kappa \equiv 0$. The density of the speed measure corresponding to the $\boldsymbol{a}_\epsilon$ is

$$m_\epsilon(x) := \exp\left(2\int_0^x \boldsymbol{a}_\epsilon(y)dy\right) = \begin{cases} \exp\left(2\int_0^x f_\epsilon(y)dy\right) & , x \le 1 \\ \exp\left(2\int_0^1 f_\epsilon(y)dy\right) \cdot x^2\, \mathrm{e}^{2(1-x)} & , x > 1\,. \end{cases}$$

This implies

$$\int_0^\infty \frac{1}{m_\epsilon(x)} dx = \infty \tag{3.41}$$

which is equivalent to positive recurrence of the corresponding diffusion Y^ϵ on $E = [0,\infty)$. (The reader may have guessed this already since $\boldsymbol{a}_\epsilon(x) < -\frac{1}{2} < 0$ eventually.) Now every (other) assumption of theorem 3.6 is obvious except for A 12. So we have to check if there is some $\epsilon > 0$ such that

$$\underline{\lambda}^{\kappa_\epsilon,\infty}(0) < \lim_{x\to\infty} \kappa_\epsilon(x) = \frac{1}{2}. \tag{3.42}$$

To see this we use Theorem 1 of [Pin09]; in particular that

$$\underline{\lambda}^{\kappa_\epsilon,\infty}(0) = \underline{\lambda}^{0,\infty}(\boldsymbol{a}_\epsilon) \le \frac{1}{2\Omega_\epsilon^+}$$

with

$$\Omega_\epsilon^+ := \sup_{x\ge 0}\left(\int_0^x \frac{1}{m_\epsilon(y)} dy \cdot \int_x^\infty m_\epsilon(y)dy\right).$$

Hence, to get (3.42) we have to check if

$$\Omega_\epsilon^+ > 1 \tag{3.43}$$

for some $\epsilon > 0$. Observe that, on the one hand, we have

$$\int_0^1 \frac{1}{m_\epsilon(y)} dy \ge \int_0^1 \mathrm{e}^{-2\epsilon y}\, dy = \frac{1}{2\epsilon}(1 - \mathrm{e}^{-2\epsilon})$$

which converges to 1 as $\epsilon \downarrow 0$.

On the other hand we have

$$\begin{aligned}\int_1^\infty m_\epsilon(y)dy &\geq \int_1^\infty y^2\,\mathrm{e}^{2(1-y)}\,dy \\ &= \left[-\,\mathrm{e}^{2(1-y)}\left(\frac{y^2}{2}+\frac{y}{2}+\frac{1}{4}\right)\right]_{y=1}^{y=\infty} \\ &= \frac{1}{2}+\frac{1}{2}+\frac{1}{4} > 1.\end{aligned}$$

Thus, we get

$$\int_0^1 \frac{1}{m_\epsilon(y)}dy \cdot \int_1^\infty m_\epsilon(y)dy > 1$$

if $\epsilon > 0$ is small enough which implies (3.43). ∎

A. Appendix

The following results have no demand for being new. Though the majority of the results may be well known to the experienced reader we included some proofs or at least sketches of proofs; mainly when we could not find an adequate reference or when we show a more general version of a known result.

A.1. The type of convergence we are dealing with

All of the main convergence results in this thesis imply also weak convergence. But most of the time we will prove a stronger convergence. This 'strong weak convergence' is defined as follows.

definition A.1.1. Let $\mathbb{P}^t$, $t \geq 0$, and $\mathbb{P}$ be probability measures on $(\Omega, \mathcal{F})$. Define

$$\mathbb{P}^t \twoheadrightarrow \mathbb{P} :\leftrightarrow \forall f \in bm(\mathcal{F}) : \mathbb{P}^t(f) \to \mathbb{P}(f).$$

A.2. Generators of one-dimensional diffusions and their spectra

Whenever the task is to investigate the asymptotic behaviour of a stochastic process Y, e.g. of a linear diffusion, the attention should turn to the semigroup $(\mathbf{T}_t)_t$ of Y in spectral representation. To do so we have to put 'things' into an L^2-setting. In particular, we identify Y with its self-adjoint L^2-generator. To make use of the spectral respresentation of $(\mathbf{T}_t)_t$ we also take a closer look at some spectral properties of this generator.

For this chapter we suppose that Y is a linear diffusion on $E = [0, \infty)$ which is a solution to

$$dY_t = dX_t + \boldsymbol{a}(Y_t)dt$$

on $(0, \infty)$ in the way described in chapter 0. As before we will write $\mathbf{L} = \mathbf{L}^{\kappa,\alpha}$ for the generator of Y under killing at $\tau = \tau^{\kappa,\alpha}$.

Section A.2.1 is devoted to the introduction of the L^2-generator.

In section A.2.2 we present 'Weyl's spectral theorem'.

Section A.2.3 is all about an equivalent formulation of $\inf \Sigma(-\mathbf{L})$. See theorem A.2.6 which is used several times in this thesis.

We also apply results (from [KS12]) which involve the *essential spectrum* $\Sigma_{ess}(-\mathbf{L})$ of $-\mathbf{L} = -\mathbf{L}^{\kappa,\infty}$. We will see in section A.2.4 that $\inf \Sigma_{ess}(-\mathbf{L})$ is *not* affected by a shift of the instant killing boundary.

A.2.1. A generator in L^2-space

We have one boundary point $c = 0$ respectively the boundary condition (0.16) with $c = 0$. Our goal is to get a self-adjoint densely defined operator on some L^2-space which incoorporates this boundary condition. Since we know by section 0.2 that $\mathbf{L}_c = \frac{1}{2}\frac{d^2}{dx^2} + \boldsymbol{a}(x)\frac{d}{dx}$ defined on C_c^2 is symmetric in $L^2(dm)$, we take this as the mentioned L^2-space. We remind the reader that dm has the density m from (0.11) depending on $\boldsymbol{a}$.

The generator as self-adjoint extension of minimal Sturm-Liouville operator

At first we essentially follow the construction described in [KS12]. Thus, we define a closable quadratic form

$$\tilde{q}^{\kappa,\alpha}(f) := \begin{cases} \alpha\,|f|^2(0) + \frac{1}{2}\int_0^\infty |f'(x)|^2\, m(x)dx + \int_0^\infty \kappa(x)\,|f(x)|^2\, m(x)dx & \text{if } \alpha < \infty \\ \\ \frac{1}{2}\int_0^\infty |f'(x)|^2\, m(x)dx + \int_0^\infty \kappa(x)\,|f(x)|^2\, m(x)dx & \text{if } \alpha = \infty\,. \end{cases}$$

The domain of this form is

$$D_{\kappa,\alpha} := \begin{cases} \{f \in L^2(dm) \cap C([0,\infty),\mathbb{C})\,;\; f, f' \text{ abs. cont. and } \tilde{q}(f) < \infty\} & \text{if } \alpha < \infty \\ \{f \in L^2(dm) \cap C([0,\infty),\mathbb{C})\,;\; f, f' \text{ abs. cont.}, f(0) = 0 \text{ and } \tilde{q}(f) < \infty\} & \text{if } \alpha = \infty\,. \end{cases}$$

remark A.2.1. With 'closable' we mean closable w.r.t.

$$\|f\|_{\tilde{q}} = \tilde{q}(f,f) + (1-\lambda)\,\|f\|_{L^2(dm)}$$

where λ is some lower bound of $\tilde{q} = \tilde{q}^{\kappa,\alpha}$. ∎

Since we need densely defined forms here are some easy to verify conditions which ensure that $\tilde{q}$ is indeed densely defined:

lemma A.2.1. *Let $\boldsymbol{a}$ and m fulfil:*

i) dm has density $m \in C(0,\infty)$ w.r.t. to the Lebesgue measure on $\mathbb{R}_+$.

ii) $m(x) > 0$ for $x > 0$.

iii) $\kappa \in L^2_{loc}(m)$.

Then

$$C_c^\infty := \{f \in C^\infty[0,\infty)\,;\; \operatorname{supp}(f) \subseteq (0,\infty)\, \text{is compact}\} \subseteq D$$

and C_c^∞ is dense in $L^2(m)$.

We intentionally skipped the subscripts on D since the lemma also holds for any 'D' which we will introduce later on.

proof of A.2.1. By the assumptions one easily verifies that C_c^∞ is contained in D. (Use point iii) together with the Cauchy-Schwartz inequality.) It remains to show that C_c^∞ is dense in $L^2(m)$.

Let f be in $L^2(m)$. By the integrability of f^2 we can choose $\delta, \Delta > 0$ with

$$\int_0^\delta f^2 m < \frac{\epsilon}{3} \quad \text{and} \quad \int_\Delta^\infty f^2 m < \frac{\epsilon}{3}.$$

But by the continuity of m and by the local integrability of m we find $c > 0$ and $C > 0$ such that

$$c \int_\delta^\Delta f^2 dx \leq \int_\delta^\Delta f^2 m dx \leq C \int_\delta^\Delta f^2 dx.$$

In particular $f \cdot \mathbb{1}_{[\delta,\Delta]}$ is an element from $L^2(dx)$ and we can find a function $g \in C_c^\infty$ with $\text{supp}(g) \subseteq [\delta, \Delta]$ and $\int_\delta^\Delta (f-g)^2 dx < \frac{\epsilon}{3C}$. Putting things together we arrive at $\int_0^\infty (f-g)^2 dm < \epsilon$. ■

Now that we have a densely defined form $\tilde{q}$ there is a positive self-adjoint operator corresponding to the closure $q^{\kappa,\alpha} := \overline{\tilde{q}^{\kappa,\alpha}}$. It will be denoted (as in [KS12]) by $-\mathbf{L}^{\kappa,\alpha}$. By 'corresponding' we mean the following:

If we have a closed symmetric and densely defined form q, then we can look for all $g \in D(q)$ such that there is some $h \in L^2(dm)$ with

$$q(f,g) = \langle f, h \rangle$$

for every $f \in D(q)$. Setting $\mathbf{L}g := h$ gives a positive self-adjoint operator on this set of g's. Furthermore, this gives a bijective mapping between symmetric and densely defined closed forms and positive self-adjoint operators. (See proposition 1.2.2 of [BH91].)

We also claim that $\mathbf{L}^{\kappa,\alpha}$ is nothing else but a self-adjoint extension of the *minimal Sturm-Liouville operator*. His action is defined as

$$\mathbf{S}f(x) := -\frac{1}{2}f''(x) - \boldsymbol{a}(x)f'(x) + \kappa(x)f(x)$$

and his domain is

$$D_{min} := \{f \in L^2(dm)\,;\ \mathbf{S}f \in L^2(dm), f \text{ and } f' \text{ are a.c. and } f \text{ has compact support}\}.$$

We further restrict ourselves to the case of a regular boundary point at $c = 0$ where *regular* should be understood in the context of Sturm-Liouville theory. In our case this simply means that κ and m are locally integrable and in particular around zero, of course. (See Chapter 10 in [Zet05] or Chapter 13 in [Wei03].) To see that $\mathbf{L}^{\kappa,\alpha}$ extends $\mathbf{S}$ one can easily verify the following points:

(i) $D_{min} \subseteq D_{\kappa,\alpha}$

(ii) $\tilde{q}(f,g) = \langle f, \mathbf{S}g \rangle$ for each $f \in D(\tilde{q})$ and $g \in D_{min}$

(iii) $q(f,g) = \langle f, \mathbf{S}g \rangle$ for each $f \in D(q)$ and $g \in D_{min}$.

Thus, $\mathbf{S} \subseteq \mathbf{L}^{\kappa,\alpha}$ and $\mathbf{L}^{\kappa,\alpha}$ is a self-adjoint extension of $\mathbf{S}$. Now we restrict ourselves to the cases which are investigated in this thesis. We know that we are in the limit point case at ∞. (Confer remark 1.11.) Hence, one can see that

$$D(\mathbf{L}^{\kappa,\alpha}) = \{f \in L^2(dm)\,;\ \mathbf{S}f \in L^2(dm), f, f' \text{ are a.c. and } f \text{ fulfils (0.16)}\}$$

(via integration by parts and comparison of the corresponding forms).

The generator as Friedrich's extension

Starting this time with a symmetric operator $\tilde{\mathbf{L}}^{\kappa,\alpha} = \mathbf{S}$ on $D_{\kappa,\alpha}$ we can define

$$\tilde{q}(f,g) := \langle f, -\tilde{\mathbf{L}}^{\kappa,\alpha} g\rangle$$

on $D_{\kappa,\alpha}$. Once we do this we have a closeable densely defined symmetric form. Its closure q corresponds (as we have seen above) to $\mathbf{L}^{\kappa,\alpha}$. If we proceed in this way, then $\mathbf{L}^{\kappa,\alpha}$ is the so-called *Friedrich's extension* of $\tilde{\mathbf{L}}^{\kappa,\alpha}$. Furthermore, Theorem 4.15 from [Wei03] tells us that $\tilde{q}$ and q, respectively $\mathbf{L}^{\kappa,\alpha}$, have the same lower bounds.

remark A.2.2. Until now we have a densely defined self-adjoint operator $\mathbf{L}$ corresponding to a Dirichlet form q. The next step would be to get an associated Markov process Y by results about solutions of certain martingale problems. For details see sections 2.1 and 2.2 of [KS12] or section 2.1 of [Kol09] and references therein. ∎

A.2.2. Weyl's spectral theorem

In the setting under consideration, we have a regular boundary point at zero and infinity is in the limit point case. Therefore, we can formulate a special version of the spectral theorem in 'multiplication-form'. This theorem is sometimes called *Weyl's spectral theorem*:

theorem A.2.2. *Let $\mathbf{a}$ and κ be such that A 1 and A 2 are fulfilled. Let $-\mathbf{L} = -\mathbf{L}^{\kappa,\alpha}$ be declared as in section A.2.1. Let ψ_λ be the solution to $-\mathbf{L}\psi = \lambda\psi$ under $\psi(0) = 0$ and $\psi'(0) = 2$ if $\alpha = \infty$ or under $\psi(0) = \frac{1}{1+\alpha}$ and $\psi'(0) = \frac{2\alpha}{1+\alpha}$ if $\alpha < \infty$. Then there is a σ-finite measure ϱ on $\mathbb{R}$ such that*

(a) $\mathbf{U} : L^2(m) \to L^2(\varrho)$, *defined by*

$$\mathbf{U}(f)(\lambda) := \int_0^\infty f(x)\psi_\lambda(x)m(dx)$$

is unitary.

(b) we also have

$$F(-\mathbf{L})f = \mathbf{U}^{-1}(F \cdot \mathbf{U}(f))$$

for each continuous F bounded on $\Sigma(-\mathbf{L})$, i.e. $\mathbf{U}$ translates $F(-\mathbf{L})$ into a multiplication operator in $L^2(\varrho)$.

(c) $\mathrm{supp}(\varrho) = \Sigma(-\mathbf{L})$.

(d) $(\mathbf{U}^{-1}g)(x) = \int_{\Sigma(-\mathbf{L})} g(\lambda)\psi_\lambda(x)\varrho(d\lambda)$.

proof. See Theorem 14.1 b) of [Wei03] as well as Proposition VIII.3.1 and Theorem VIII.4 of [RS80]. ■

remark A.2.3. Note that the Sturm-Liouville theory for theorem A.2.2 presented in [Wei03] only assumes that m, m^{-1} and κ are locally integrable on $(0,\infty)$ together with 0 being regular instead of A 1. ▮

A derivation in the C^1-case

A possible derivation of theorem A.2.2 where the connection to Sturm-Liouville theory can be seen immediately is by assuming $\boldsymbol{a} \in C^1$ and connecting diffusion-type operators (no potential term) to Schrödinger-type operators (no drift term):

For simplicity we suppose $\kappa \equiv 0$ and write $\mathbf{L}^d$ for $\mathbf{L}^{0,\alpha}$. ('d' for 'diffusion') Furthermore, we introduce the unitary mapping

$$V : L^2(m) \to L^2(dx)\,, \quad Vf(x) := f(x)\cdot\sqrt{m(x)} = f(x)\cdot \mathrm{e}^{\int_0^x \boldsymbol{a}(y)dy}$$

and define $\mathbf{L}^S := V\mathbf{L}^d V^{-1}$. This is an operator which acts on

$$D(V) = \{f \in L^2(dx) \cap C^2(0,\infty)\,;\ \mathbf{L}^{\kappa,\alpha} f \in L^2(dx) \text{ and } f \text{ fulfills } f'(0) = (\boldsymbol{a}(0) + 2\alpha)f(0)\}$$

if we additionally impose that $\boldsymbol{a} \in C^1$. (Note that we can generalize this a little bit, by switching from $\boldsymbol{a} \in C^1$ to '$\boldsymbol{a}$ a.c. w.r.t. dx' and from $f \in C^2$ to 'f' a.c.'. Nevertheless, we need a more strict condition than just $\boldsymbol{a} \in C$.) For every $f \in D(V)$ we have

$$\mathbf{L}^S f = \left[\frac{1}{2}\frac{d^2}{dx^2} - \frac{\boldsymbol{a}^2 + \boldsymbol{a}'}{2}\right] f.$$

As explained in the introductory part of [Pin09], if $\frac{\boldsymbol{a}^2+\boldsymbol{a}'}{2}$ is bounded from below we may realize $-\mathbf{L}^S$ as a Friedrichs extension on $D(V)$ in $L^2(dx)$.

Next we briefly say something about $\varrho = \varrho^S$ for $\mathbf{L} = \mathbf{L}^S$ originating from the Sturm-Liouville theory:

Denote by ψ_λ^S the solution of $-\mathbf{L}^S\psi = \lambda\psi$ under the (new) boundary conditions $\psi(0) = \frac{1}{1+\alpha}$ and $\psi'(0) = \frac{\boldsymbol{a}(0)+2\alpha}{1+\alpha}$. We can uniquely define the so-called *Weyl-Titchmarsh solutions* for every $\lambda \in \mathbb{C} \setminus \Sigma(-\mathbf{L}^S)$. There always exists a corresponding *Weyl-Titchmarsh function* $m : \mathbb{C} \setminus \Sigma(-\mathbf{L}^S) \to \mathbb{C}$ which is a Herglotz function. Thus, it has a representation in terms of a unique measure ϱ^S on $\mathbb{R}$ with $\int \frac{1}{1+x^2}\varrho^S(dx) < \infty$. All of this can be read in [GZ06]. Essentially, their Theorems 2.6 and 2.9 give us Weyl's spectral theorem with $\mathbf{L} = \mathbf{L}^S$ and $\varrho = \varrho^S$.

Now as we have Weyl's theorem for the Schrödinger case, we can see that it also holds for $\mathbf{L}^d$:

We write $\mathbf{U}^S$ for 'Weyl's unitary operator' from $L^2(dx)$ to $L^2(\varrho)$ corresponding to $\mathbf{L}^S$. Then we see that $\mathbf{U} := \mathbf{U}^S V$ is the corresponding unitary operator from $L^2(m)$ to $L^2(\varrho)$. Note that $\psi_\lambda = V^{-1}\psi_\lambda^S$, because it fulfils the old boundary conditions and

$$\mathbf{L}^d(V^{-1}\psi_\lambda^S) = V^{-1}\mathbf{L}^S\psi_\lambda^S = V^{-1}\lambda\psi_\lambda^S = \lambda V^{-1}\psi_\lambda^S.$$

Hence,

$$\mathbf{U}f(\lambda) = \mathbf{U}^S Vf(\lambda) = \int Vf(x)\psi_\lambda^S(x)dx = \int Vf(x)\psi_\lambda^S(x)(V^{-1})^2 m(x)dx$$
$$= \int f(x)V^{-1}\psi_\lambda^S(x)m(dx) = \int f(x)\psi_\lambda(x)m(dx)$$

and

$$\mathbf{U}F(\mathbf{L}^d)\mathbf{U}^{-1} = \mathbf{U}^S VF(\mathbf{L}^d)V^{-1}(\mathbf{U}^S)^{-1}.$$

To show that $F(\mathbf{L}^d)f = \mathbf{U}^{-1}(F \cdot \mathbf{U}(f))$, we use the *spectral resolutions* $E_{\mathbf{L}^{(\cdot)}}(\lambda)$ of $\mathbf{L}^d$ and $\mathbf{L}^S$. By Theorems 8.8 and 8.14 of [Wei00] we know that

$$E_{\mathbf{L}^d}(\lambda) = E_{V^{-1}\mathbf{L}^S V}(\lambda) = V^{-1}E_{\mathbf{L}^S}(\lambda)V$$

for each λ since $\mathbf{L}^d$ and $\mathbf{L}^S$ are unitary equivalent in the same manner. Thus,

$$\begin{aligned}
\langle F(\mathbf{L}^d)f, g\rangle_{L^2(m)} &= \int F(\lambda)d_\lambda\langle E_{V^{-1}\mathbf{L}^S V}(\lambda)f, g\rangle_{L^2(m)} \\
&= \int F(\lambda)d_\lambda\langle V^{-1}E_{\mathbf{L}^S}(\lambda)Vf, g\rangle_{L^2(m)} \\
&= \int F(\lambda)d_\lambda\langle E_{\mathbf{L}^S}(\lambda)Vf, Vg\rangle_{L^2(dx)} \\
&= \langle F(\mathbf{L}^S)Vf, Vg\rangle_{L^2(dx)} \\
&= \langle V^{-1}F(\mathbf{L}^S)Vf, g\rangle_{L^2(m)}
\end{aligned}$$

for each $g \in L^2(m)$ and $f \in D(F(\mathbf{L}^d))$. Hence, $F(\mathbf{L}^d) = V^{-1}F(\mathbf{L}^S)V$ and we see that

$$(\mathbf{U}F(\mathbf{L}^d)\mathbf{U}^{-1}f)(\lambda) = (\mathbf{U}^S VF(\mathbf{L}^d)V^{-1}(\mathbf{U}^S)^{-1}f)(\lambda) = (\mathbf{U}^S F(\mathbf{L}^S)(\mathbf{U}^S)^{-1}f)(\lambda) = F(\lambda)\cdot f(\lambda)$$

using the results for $\mathbf{L}^S$.

A.2.3. On the infimum of the spectrum

The main in this section will be theorem A.2.6 which gives an alternative describtion of $\underline{\lambda} = \inf \Sigma(-\mathbf{L}^{\kappa,\alpha})$. But at first we need to introduce a few more objects. Along the way we will see yet another way of introducing the 'spectral measure' ϱ from theorem A.2.2.

We mainly follow chapter 9 of [CL55]. Accordingly let us have a look at the following problem

$$\begin{aligned}
-\mathbf{L}^{\kappa,\alpha}\psi &= \lambda\psi \\
\psi'(0) &= 2\alpha\psi(0) \\
\psi(b) &= 0
\end{aligned} \tag{A.2.1}$$

with $b > 0$. Under assumption A 1 this problem submits eigenvalues λ_n^b and a corresponding *complete* set of eigenvectors $\psi_{\lambda_n^b}$ in $L^2([0,b), m)$. (Note that the symmetrizing measure for $\mathbf{L}^{\kappa,\alpha}$ under (A.2.1) is nothing else but $m|_{[0,b)}$. For the sake of readability we again write m as well as $\mathbf{L}^{\kappa,\alpha}$ without giving special emphasis to the second boundary condition at b.) In the case of (A.2.1) there is σ-finite measure ρ_b analogous to ϱ from theorem A.2.2. ρ_b is a point measure with some weights r_n^b at λ_n^b. If we set

$$\mathbf{U}^b f(\lambda) := \langle f, \psi_\lambda \rangle_{L^2([0,b),m)},$$

we have the relation

$$\|f\|^2_{L^2([0,b),m)} = \sum_{n=0}^{\infty} \left| r_n^b \right|^2 \left| \int_0^b f(x)\psi_{\lambda_n^b}(x)m(dx) \right|^2 = \left\| \mathbf{U}^b f \right\|^2_{L^2(\rho_b)}.$$

Clearly, $\mathbf{U}^b : L^2([0,b),m) \to L^2(\rho_b)$ is unitary. (It is the analogue to $\mathbf{U}$ from theorem A.2.2.) Now the 'distribution functions' of ρ_b tend to some non-decreasing F as $b \to \infty$. If we denote the corresponding measure by 'ρ', we can express the following lemma

lemma A.2.3. *Suppose $\boldsymbol{a}$ and κ fulfil A 1 and A 2. Let ϱ be the measure given by theorem A.2.2. Then we have*

$$\rho = \varrho.$$

proof. Theorem 9.3.1 and Theorem 9.3.2 of [CL55] extend to the present case. In particular the unitary mapping $\mathbf{U}$ from Weyl's spectral theorem is also a unitary mapping from $L^2(dm)$ to $L^2(\rho)$. Since the same mapping rule $\mathbf{U}$ is unitary from $L^2(dm)$ to $L^2(\varrho)$ as well as to $L^2(\rho)$, we conclude that $\varrho = \rho$:

Since $\mathbf{U}$ is norm-preserving we have

$$\int_A d\rho = \int_A d\varrho$$

for all A such that A is integrable. In particular, this is true for $A = \emptyset$ or if A is any closed subinterval in $(0,\infty)$. Thus, we can use Theorem 10.3 from [Bil79] to get $\varrho = \rho$. ∎

lemma A.2.4. *Suppose $\boldsymbol{a} \in C(0,\infty)$ is locally integrable at zero, $\kappa \in L^1([0,\infty))$ and $\alpha \geq 0$. Then*

$$\begin{aligned} -\frac{1}{2}y'' - \boldsymbol{a}y' + \kappa y &= \lambda y \\ y'(0) &= 2\alpha y(0). \end{aligned} \tag{A.2.2}$$

has a non-trivial solution $y(x) = y(x,\lambda)$ on $[0,\infty)$ which is unique up to a multiplicative constant. Furthermore, $(x,\lambda) \mapsto y(x,\lambda)$ is uniformly continuous on $[0,M] \times K$ for any compact K and any $M > 0$.

It should be mentioned that the lemma and the following results also hold under '$y(0) = 0$' instead of '$y'(0) = 2\alpha y(0)$'.

proof of lemma A.2.4. We use 2.2.1, 2.3.1 and 2.4.1 of [Zet05]. We put this into the following form:

Take $M > 0$ and let p and q be functions on a $(0,\infty)$ such that $\frac{1}{p}$ and q are integrable on $[0,M]$. Let p be differentiable on $(0,\infty)$. Then

$$-(py')' + qy = 0$$
$$y'(0) = 2\alpha y(0) \tag{A.2.3}$$

has a non-trivial solution $y(x) = y(x,q)$ on $(0,\infty)$ which is unique up to a multiplicative constant. This solution has a continuous extension to 0. Furthermore, for each ϵ there is a δ such that

$$\int_0^M |q_1(z) - q_2(z)|\, dz < \delta \Rightarrow |y(x,q_1) - y(x,q_2)| < \epsilon \tag{A.2.4}$$

for all $x \in [0,M]$. (The q_i have to fulfil the above assumptions.)

For the proof of the lemma we choose $p(x) = \mathrm{e}^{2\int_0^x \boldsymbol{a}(y)dy}$ and $q(x) = -2(\lambda - \kappa(x))p(x)$. Dividing the differential equation in (A.2.3) by $2{\cdot}p(x)$, we see that y is also the unique solution to

$$-\frac{1}{2}y'' - \boldsymbol{a}y' + \kappa y = \lambda y$$
$$y'(0) = 2\alpha y(0).$$

Now y depends on x and λ. Since p is continuous on $[0,M]$ we have $\max_{z\le M} p(z) < \infty$. Therefore,

$$|\lambda_1 - \lambda_2| < \frac{\delta}{2M{\cdot}\max_{z\le M} p(z)} \Rightarrow \int_0^M |q_1(z) - q_2(z)|\, dz = \int_0^M |2\lambda_2 p(z) - 2\lambda_1 p(z)|\, dz < \delta.$$

Together with (A.2.4) and the continuity of $x \mapsto y(x,\lambda)$ we conclude:

$(x,\lambda) \mapsto y(x,\lambda)$ is uniformly continuous on $[0,M]\times K$ for any compact K and any $M > 0$. ■

lemma A.2.5. *Suppose* $\boldsymbol{a} \in C(0,\infty)$ *is locally integrable at zero and* $\kappa \in L^1((0,b))$ $\alpha \ge 0$. *Let* $b > 0$ *and take the Sturm-Liouville problem*

$$-\frac{1}{2}y'' - \boldsymbol{a}y' + \kappa(x)y = \lambda y$$
$$y'(0) = 2\alpha y(0) \tag{A.2.5}$$
$$y(b) = 0$$

which is similar to (A.2.2) *but we 'truncate at b'.*

Then we have the following:

(a) There are infinitely but countably many eigenvalues λ *of* (A.2.5).

(b) All eigenvalues are real and simple.

(c) The set of eigenvalues is bounded below and if we order it such that $\lambda_0 < \lambda_1 < \lambda_2 < \ldots$ *then*

$$\lim_{n\to\infty} \lambda_n = \infty.$$

(d) If ψ_n *is the eigenfunction to* λ_n*, then* ψ_n *has exactly* n *zeros in* $(0,b)$

proof. Essentially, this is Theorem 4.6.2 from [Zet05]. Although in [Zet05] it is formulated in Sturm-Liouville form as in (A.2.3). So we have to take $p(x) = \mathrm{e}^{2\int_0^x \boldsymbol{a}(y)dy}$ and $q(x) = -2(\lambda - \kappa(x))p(x)$ and divide by $2{\cdot}p(x)$ to arrive at the form (A.2.5) (just as we did in the proof of lemma A.2.4). ■

The following result is a slight generalization of Lemma 2.2 from [SE07]. (In the case $\boldsymbol{a} \in C^1$ it was proven first in [Man61].)

theorem A.2.6. *Let A 1 be fulfilled. Then the set of λ such that ψ_λ does not change sign is $(-\infty, \underline{\lambda}]$. In particular, we have*

$$\underline{\lambda} = \max\{\lambda\,;\ \exists\,\psi \neq 0 \text{ not changing sign s.t. } -\frac{1}{2}\psi'' - \boldsymbol{a}\psi' + \kappa\psi = \lambda\psi \text{ under (0.16)}\}. \quad \text{(A.2.6)}$$

proof. We can mimic the proof of Lemma 2.2 in [SE07]. To do so we make a few remarks giving the necessary ingredients:

- Instead of Theorem 8.2.1 of [CL55] use lemma A.2.5.
- Instead of Theorem 1.7.5 of [CL55] use lemma A.2.4.
- λ_ρ is the smallest point of increase of ρ.
- [CL55] directly use some part of the proof by Mandl depending only on the fact that we are in the limit point case at infinity. This is used to get that $\underline{\lambda}$ is a maximum and not only a supremum in (A.2.6). Note that we are indeed in the limit point case at ∞ due to a result from [KS12]. But this part of the proof can also be seen by a different method as explained in remark A.2.4.
- Lemma A.2.3 shows that the 'spectral measure' ρ is nothing else but the measure ϱ from theorem A.2.2.
- $\lambda_\rho = \underline{\lambda}$ by Weyl's spectral theorem.

■

remark A.2.4. That we always have '$\underline{\lambda} = \max\ldots$' in (A.2.6) can be seen as follows:
Let

$$\underline{\lambda} = \sup\{\lambda\,;\ \exists\,\psi > 0 \text{ solving } -\frac{1}{2}\psi'' - \boldsymbol{a}\psi' + \kappa\psi = \lambda\psi \text{ under } \psi(0) = \frac{1}{1+\alpha} \text{ and } \psi'(0) = \frac{2\alpha}{1+\alpha}\}$$

be true. Due to Theorem 2.4.1. of [Zet05] we have that $(x, \lambda) \mapsto \psi_\lambda(x)$ is uniformly continuous on compacta. This implies $\underline{\psi} \geq 0$. Together with $\underline{\psi}(0) = \frac{1}{1+\alpha} > 0$ if $\alpha < \infty$ and $\underline{\psi}'(0) > 0$ otherwise. In either case there must be some $\epsilon > 0$ with $\underline{\psi}(x) \neq 0$ for all $x \in (0, \epsilon)$. Finally, we use an elliptic Harnack inequality to obtain

$$\sup_{x\in[\frac{1}{n},n]} \underline{\psi}(x) \leq C \inf_{x\in[\frac{1}{n},n]} \underline{\psi}(x)$$

for arbitrarily large n. (See Corollary 8.21 in [GT01]. Actually, Corollary 8.21 only needs $\boldsymbol{a}$ and κ to be measurable and locally bounded.)

Finally we see that $\underline{\psi}(x) \neq 0$ for all $x > 0$. ■

A.2.4. On the essential spectrum of the generators

The next result states that the least upper bound of the essential spectrum is independent from the location of the boundary point. For this take $x \geq 0$ and $\boldsymbol{a} : \mathbb{R}_+ \to \mathbb{R}$. We will write '$\mathbf{L}_x(\boldsymbol{a})$' for the $L^2(dm)$-generator of Y on $E = [x, \infty)$ with drift $\boldsymbol{a}$ under *instant* killing at x.

theorem A.2.7. *Let $\boldsymbol{a} \in L^1_{loc}[0, \infty)$. Then*

$$\sup \Sigma_{ess}(\mathbf{L}_x(\boldsymbol{a})) = \sup \Sigma_{ess}(\mathbf{L}_y(\boldsymbol{a}))$$

for all $x, y \geq 0$.

To prove this, we will need a few lemmata. But at first we introduce the following number:

$$l(\mathbf{L}) := \lim_{y\to\infty} \sup_{\substack{\varphi \in C_c^2 \\ \operatorname{supp}(\varphi)\subset[y,\infty)}} \frac{\langle \mathbf{L}\varphi, \varphi\rangle}{\langle \varphi, \varphi\rangle}.$$

It will turn out that $l(\mathbf{L}_x(\boldsymbol{a}))$ is an equivalent formulation of $\sup \Sigma_{ess}(\mathbf{L}_x(\boldsymbol{a}))$. Note that $l(\mathbf{L}_x(\boldsymbol{a}))$ is independent from the choice of x.

lemma A.2.8. *We have that $l(\mathbf{L}_x(\boldsymbol{a})) \geq \sup \Sigma_{ess}(\mathbf{L}_x(\boldsymbol{a}))$.*

proof. The proof is analogous to the one of Lemma 2.1 in [Per60]. Of course, now the quadratic form corresponding to $\mathbf{L}_x(\boldsymbol{a})$ is

$$Q(f, g) := \int f'g'dm$$

(and not $\int f'g'dx + \int V fgdx$ as in the case of a Schrödinger operator). ■

lemma A.2.9. *Let $\lambda \in \Sigma_{ess}(\mathbf{L}_x(\boldsymbol{a}))$. Then there is an orthonormal sequence $(u_k)_k$ such that*

(a) $\langle \mathbf{L}_x(\boldsymbol{a})u_k, u_k\rangle \to \lambda$

(b) $\|u_k\|_{L^2(B,dm)} \to 0$ *for every bounded $B \subset [x, \infty)$.*

proof. We use the first part of the proof of Lemma 2.2. in [Per60]. For readability we simply write '$\mathbf{L}$' instead of $\mathbf{L}_x(\boldsymbol{a})$.

There is an orthonormal sequence $(u_k)_k \subset D(\mathbf{L})$ such that

- $u_k \to 0$ weakly.
- $\|\mathbf{L}u_k - \lambda u_k\|_{L^2(dm)} \to 0$.

(See Nr.133 in [RS73].)

Thus, for abritrary $\epsilon > 0$ we have

$$\epsilon \geq \|\mathbf{L}u_k - \lambda u_k\| \geq |\langle \mathbf{L}u_k, u_k\rangle - \lambda\langle u_k, u_k\rangle| \geq \langle \mathbf{L}u_k, u_k\rangle - \lambda \geq \left\|u_k'\right\|^2 - \lambda$$

provided k is large enough. This already proves point (a). Further, we deduce that there is a constant $M > 0$ such that

$$\left\|u_k'\right\|_{L^2(B,dm)} + \|u_k\|_{L^2(B,dm)} \leq M$$

for all k. Note that $\boldsymbol{a}$ has a nice integrability property such that

$$\|\cdot\|_{L^2(B,dm)} \sim \|\cdot\|_{L^2(B,dx)}\,. \tag{A.2.7}$$

Hence, $(u_k)_k$ is bounded in the Sobolev space $W^{1,2}(B)$. W.l.o.g. we can assume that B is open. Now we use *Sobolev's embedding theorem* from [Alt06] to see that there is a convergent subsequence in $W^{0,2}(B) = L^2(B)$. By (A.2.7) this subsequence also converges in $L^2(B, dm)$ and the limit is zero since $u_k \to 0$ weakly also in $L^2(B, dm)$. Assume that $(u_k)_k$ does *not* converge to zero in $L^2(B, dm)$. Then there is an $\epsilon > 0$ and a subsequence $(v_n)_n$ such that

$$\|v_n\|_{L^2(B,dm)} \geq \epsilon \text{ for all } n. \tag{A.2.8}$$

But the same argumentation we did before also applies to $(v_n)_n$. Hence, there must be a subsequence $(v_{n_j})_j$ such that

$$\lim_{j\to\infty} \left\|v_{n_j}\right\|_{L^2(B,dm)} = 0$$

which obviously contradicts (A.2.8). This proves point (b). ■

lemma A.2.10. *We have* $l(\mathbf{L}_x(\boldsymbol{a})) \leq \sup \Sigma_{ess}(\mathbf{L}_x(\boldsymbol{a}))$.

proof. This is analogous to the proof of Theorem 1 of [Går83]. Everything also works in the case of $Q(f) = \int (f')^2 dm$. Nevertheless, we feel the need to make two remarks:

- One has to use lemma A.2.9 two times.
- To clarify that a certain formula on functions from C_c^2 also holds for elements of $D(\mathbf{L}_x(\boldsymbol{a}))$, you may use Lemma 1.1 of Persson (which turns out to be true also in the case of $Q(f, g) = \int f'g'dm$).

■

proof of theorem A.2.7. First observe that, for $\operatorname{supp}(\varphi) \cap [0, x] = \emptyset$, we have

$$\begin{aligned}
\frac{\langle \mathbf{L}_0(\boldsymbol{a})\varphi, \varphi\rangle}{\langle \varphi, \varphi\rangle} &= \frac{\int_0^\infty \varphi'(y)^2\, \mathrm{e}^{2\int_0^y \boldsymbol{a}(z)dz}\, dy}{\int_0^\infty \varphi(y)^2\, \mathrm{e}^{2\int_0^y \boldsymbol{a}(z)dz}\, dy} \\
&= \frac{\int_x^\infty \varphi'(y)^2\, \mathrm{e}^{2\int_0^y \boldsymbol{a}(z)dz}\, dy}{\int_x^\infty \varphi(y)^2\, \mathrm{e}^{2\int_0^y \boldsymbol{a}(z)dz}\, dy} \\
&= \frac{\int_x^\infty \varphi'(y)^2\, \mathrm{e}^{2\int_x^y \boldsymbol{a}(z)dz}\, dy}{\int_x^\infty \varphi(y)^2\, \mathrm{e}^{2\int_x^y \boldsymbol{a}(z)dz}\, dy} \\
&= \frac{\langle \mathbf{L}_x(\boldsymbol{a})\varphi, \varphi\rangle_{L^2([x,\infty),dm)}}{\langle \varphi, \varphi\rangle_{L^2([x,\infty),dm)}}.
\end{aligned}$$

Thus, by lemmata A.2.8 and A.2.10, we get

$$\begin{aligned}
\sup \Sigma_{ess}(\mathbf{L}_0) &= \lim_{y\to\infty} \sup_{\operatorname{supp}\varphi\subset[y,\infty)} \frac{\langle \mathbf{L}_0\varphi, \varphi\rangle_{L^2([0,\infty),dm)}}{\langle \varphi, \varphi\rangle_{L^2([0,\infty),dm)}} \\
&= \lim_{y\to\infty} \sup_{\operatorname{supp}\varphi\subset[y,\infty)} \frac{\langle \mathbf{L}_x\varphi, \varphi\rangle_{L^2([x,\infty),dm)}}{\langle \varphi, \varphi\rangle_{L^2([x,\infty),dm)}} \\
&= \sup \Sigma_{ess}(\mathbf{L}_x).
\end{aligned}$$

■

remark A.2.5. We take a look on the special case of $\boldsymbol{a} \in C^1[0,\infty)$. Of course, we have the result stated in theorem A.2.7. But now we can obtain it with less effort:

Following Theorem 1 of [Går83] we want the existence of a constant k and some $c \in C[0,\infty)$ such that

$$\|f'\|^2_{L^2(dx)} + \langle \frac{\boldsymbol{a}^2 + \boldsymbol{a}'}{2} f, f \rangle_{L^2(dx)} \geq \int c(x) \left|f'(x)\right|^2 dx - k \|f\|^2 ,$$

loosely speaking, for any $f \in L^2([0,\infty), dx)$ with $f(0) = 0$ and where this expression makes sense. For instance we could suppose that $\boldsymbol{a}$ is bounded from below. Now Theorem 1 in [Går83] tells us that the self-adjoint extension of

$$\mathbf{L}^S := \frac{1}{2}\frac{d^2}{dx^2} - \frac{\boldsymbol{a}^2 + \boldsymbol{a}'}{2}$$

from C_0^1 in $L^2(dx)$ has the property that $l(\mathbf{L}^S) = \sup \Sigma_{ess}(\mathbf{L}^S)$. On the other hand $\boldsymbol{a} \in C^1$ implies that $\mathbf{L}$ is unitary equivalent to the Schrödinger operator

$$\mathbf{L}^S = \sqrt{m}\mathbf{L}\frac{1}{\sqrt{m}}.$$

Hence, the result of theorem A.2.7 follows immediately. (On the unitary correspondence see [Pin09].) ▮

A.3. On the existence of the penalisation measure

Here we give a proof of proposition 2.1 (b).

At first take the family of measures

$$\pi_{t_1,\dots,t_k}(B) := \mathbb{S}_s(\mathrm{pr}^{-1}_{t_1,\dots,t_k}(B)),$$

with $t_1, \dots, t_k \leq s$,$k \in \mathbb{N}$ and $B \subseteq \mathbb{R}^k$ measurable. By $\mathrm{pr}_{t_1,\dots,t_k}$ we mean a *projection operator* $\mathbb{R}^I \ni f \mapsto (f(t_1), \dots, f(t_k))$ where $\{t_1, \dots, t_k\}$ should be a subset of I, of course.

Using equation (2.2) we can show that this family is

- well defined; i.e. independent of s.
- consistent.

Now we use *Kolmogorov's extension theorem* to obtain the existence of $\tilde{\mathbb{S}}$ on $(\mathbb{R}^{[0,\infty)}, \sigma_{min}(\mathbb{R}^{[0,\infty)}))$ such that the marginal distributions of each $\mathbb{S}_s$ coincide with those of $\tilde{\mathbb{S}}|_{\sigma^s_{min}(\mathbb{R}^{[0,\infty)})}$. But we want to have $\tilde{\mathbb{S}}$ on $C[0,\infty)$. For this purpose define:

$$\Omega_0 := \{f \in \mathbb{R}^{[0,\infty)}\,;\; f\big|_{\mathbb{Q}_+} \text{ is uniformly continuous on each compact set}\}.$$

1. First we show that $\tilde{\mathbb{S}}(\Omega_0) = 1$:

   Observe that

$$\Omega_n := \{f \in \mathbb{R}^{[0,\infty)}\,;\; \forall m\ \exists k\ \forall r,s \in \mathbb{Q}\cap[0,n] :\ |r-s| < \frac{1}{k} \to |f(r) - f(s)| < \frac{1}{m}\}$$

is a decreasing sequence of measurable sets. Therefore, we have in particular

$$\tilde{\mathbb{S}}(\Omega_0) = \lim_{n\to\infty} \tilde{\mathbb{S}}(\Omega_n). \tag{A.3.1}$$

We define

$$\Omega_n^{\mathbb{Q}} := \{f \in \mathbb{R}^{\mathbb{Q}\cap[0,n]}\,;\ f \text{ is uniformly continuous}\}.$$

on $\mathbb{Q} \cap [0, n]$ as well as

$$\mathrm{pr}_{\mathbb{Q},n}(f) := \big(\mathbb{Q} \cap [0, n] \ni s \mapsto f(s)\big).$$

Using these definitions we deduce

$$\mathrm{pr}_{\mathbb{Q},n}(\Omega_n) = \Omega_n^{\mathbb{Q}}.$$

We plug this into (A.3.1) to see that

$$\tilde{\mathbb{S}}(\Omega_0) = \lim_{n\to\infty} \tilde{\mathbb{S}}\big(\,\mathrm{pr}_{\mathbb{Q},n}^{-1}\,(\Omega_n^{\mathbb{Q}})\big). \tag{A.3.2}$$

Next we choose subsets $A_k = \{a_1, \dots, a_k\}$ of $\mathbb{Q} \cap [0, n]$ such that

$$A_k \uparrow \mathbb{Q} \cap [0, n].$$

Let p_k be the corresponding projections; i.e.

$$p_k(f) := (f(a_1), \dots, f(a_k))$$

for every $f \in \mathbb{R}^{\mathbb{Q}\cap[0,n]}$. Hence,

$$p_k^{-1} p_k(F) \downarrow F$$

for all $F \subseteq \mathbb{R}^{\mathbb{Q}\cap[0,n]}$ and in particular for $F = \Omega_n^{\mathbb{Q}}$. Thus, we obtain

$$\begin{aligned}
\tilde{\mathbb{S}}\big(\,\mathrm{pr}_{\mathbb{Q},n}^{-1}\,(\Omega_n^{\mathbb{Q}})\big) &= \lim_{k\to\infty} \tilde{\mathbb{S}}\big(\,\mathrm{pr}_{\mathbb{Q},n}^{-1}\,(p_k^{-1} p_k(\Omega_n^{\mathbb{Q}}))\big) \\
&= \lim_{k\to\infty} \tilde{\mathbb{S}}\big(\,\mathrm{pr}_{a_1,\dots,a_k}^{-1}\,(p_k(\Omega_n^{\mathbb{Q}}))\big) \\
&= \lim_{k\to\infty} \mathbb{S}_n\big(\,\mathrm{pr}_{a_1,\dots,a_k}^{-1}\,(p_k(\Omega_n^{\mathbb{Q}}))\big),
\end{aligned}$$

where we have used $p_k p_{\mathbb{Q},n} = \mathrm{pr}_{a_1,\dots,a_k}$. Further, the same argument gives

$$\begin{aligned}
\tilde{\mathbb{S}}\big(\,\mathrm{pr}_{\mathbb{Q},n}^{-1}\,(\Omega_n^{\mathbb{Q}})\big) &= \lim_{k\to\infty} \mathbb{S}_n\big(\,\mathrm{pr}_{\mathbb{Q},n}^{-1}\,(p_k^{-1} p_k(\Omega_n^{\mathbb{Q}}))\big) \\
&= \mathbb{S}_n\big(\,\mathrm{pr}_{\mathbb{Q},n}^{-1}\,(\Omega_n^{\mathbb{Q}})\big) \\
&= \mathbb{S}_n(\Omega_n \cap C[0, \infty)) \\
&= \mathbb{S}_n(C[0, \infty)) \\
&= 1.
\end{aligned}$$

And indeed, by (A.3.2) we arrive at $\tilde{\mathbb{S}}(\Omega_0) = 1$.

2. By the preceding result we could restrict $\tilde{\mathbb{S}}$ to Ω_0. But for $f \in \Omega_0$ there is a unique continuous extension, call it $\tilde{X}(f)$. (Set $\tilde{X}(f) = 0$ or anything else continuous if $f \notin \Omega_0$.) Similar calculations as before give that $\tilde{X}(f)$ is a continuous modification of $X_t(\omega) = \omega(t)$ under $\tilde{\mathbb{S}}$. Hence, the image measure $\tilde{\mathbb{S}}_{\tilde{X}}$ is a probability measure on $C[0, \infty)$. If we finally set $\mathbb{S} := \tilde{\mathbb{S}}_{\tilde{X}}$, then the marginals of $\mathbb{S}\big|_{\mathcal{F}_s}$ (on $\mathbb{Q} \cap [0, s]$ dense in $[0, s]$) are just the same as the marginals of $\mathbb{S}_s$. But we have continuous paths. This immidiately gives $\mathbb{S}\big|_{\mathcal{F}_s} = \mathbb{S}_s$ for all $s \geq 0$.

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Kurzdarstellung

Es wird das Langzeitverhalten von Diffusionen $(Y_t)_{t\geq 0}$ auf $[0,\infty)$ unter 'killing' (Sterben) zu einer zufälligen Zeit τ untersucht. Dieses killing kann sowohl am Randpunkt als auch im Inneren $(0,\infty)$ geschehen. Y ist eine Lösung einer stochatischen Differentialgleichung $dY_t = dX_t + \boldsymbol{a}(Y_t)dt$ getrieben von einer Brownschen Bewegung X. Wir setzen fast immer $\boldsymbol{a} \in C$ voraus und auch, dass 0 regulär ist und ∞ natürlich. Schließlich bedingen wir Y auf Überleben bis zum Zeitpunkt t und lassen t gegen Unendlich streben. Wir untersuchen das daraus resultierende asymptotische Verhalten auf verschiedene Art und Weise.

In Kapitel 1 interessiert uns das Grenzwertverhalten von $\mathbb{P}(Y_t \in \bullet \mid \tau > t)$. Wenn der Limes existiert und ein Wahrscheinlichkeitsmaß ist, so nennt man diesen auch 'quasi-limiting distribution' (Quasi-Grenzwertverteilung). Wir erweitern ein Resultat aus [KS12] auf den Fall, dass auch im Inneren gestorben werden kann, welches besagt, dass entweder eine quasi-limiting distribution existiert oder der Grenzwert dem Zustandsraum $[0,\infty)$ das Maß Null zuteilt. Es werden auch Existenzresultate gezeigt, in denen man $\boldsymbol{a}$ mit einem zu einer anderen Diffusion gehörigen Driftkoeffizienten $\boldsymbol{b}$ vergleicht von der die Existenz einer quasi-limiting distribution bekannt ist.

In Kapitel 2 konzentrieren wir uns auf $\lim_{t\to\infty} \mathbb{P}(F_s \mid \tau > t)$, wobei F_s messbar ist bzgl. der σ-Algebra erzeugt von $(Y_r)_{r\leq s}$ für $s > 0$. Das Ergebnis ist das Maß von $(Y_r)_{r\leq s}$ bedingt auf Überleben; in der Literatur auch als 'Q-process' bezeichnet. Desweiteren fassen wir den Q-process als Spezialfall sogenannter 'penalisation limits' auf. Ob nun eine quasi-limiting distribution existiert oder nicht, hat keinen allzu großen Effekt auf den Q-process. Dies ist an den Theoremen 2.11, 2.13 and 2.14 ersichtlich, an denen man auch erkennt, dass die allgemeine Form immer dieselbe ist.

In Kapitel 3 untersuchen wir das asymptotische Verhalten von $\mathbb{P}(Y_{s(t)} \in \bullet \mid \tau > t)$, wobei $t \mapsto s(t)$ eine Funktion ist mit $t - s(t) > \epsilon$ für ein $\epsilon > 0$ und $s(t) \to \infty$. Wir zeigen, dass das Verhalten qualitativ dasselbe ist wie das asymptotische Verhalten von $\mathbb{P}(Y_t \in \bullet \mid \tau > t)$ im Fall, dass $t - s(t)$ beschränkt ist. Weiterhin betrachten wir für $t - s(t) \to \infty$ zwei Fälle. Einerseits beobachten wir, dass $\lim_{t\to\infty} \mathbb{P}(Y_{s(t)} \in \bullet \mid \tau > t)$ die stationäre Verteilung des Q-process ist, andererseits zeigen wir, dass dieser Grenzwert unter gewissen Vorraussetzungen wieder Null wird. Die Ergebnisse lassen vermuten, dass diese Fälle auch allgemeiner durch die Existenz bzw. Nichtexistenz jener stationären Verteilung entstehen.

Abstract

We investigate the long-term behaviour of diffusions $(Y_t)_{t\geq 0}$ on $[0,\infty)$ under killing at some random time τ. Killing can occur at 0 as well as in the interior $(0,\infty)$ of the state space. Y follows a stochastic differential equation $dY_t = dX_t + \boldsymbol{a}(Y_t)dt$ driven by a Brownian motion X. The diffusions we are working with will almost surely be killed. In large parts of this thesis we only assume that $\boldsymbol{a} \in C$. Further, we suppose that 0 is regular and that ∞ is natural. We condition Y on survival up to time t and let t tend to infinity looking for a limiting behaviour. We investigate the asymptotic behaviour in the following ways.

In chapter 1 we look for the limit of $\mathbb{P}(Y_t \in \bullet \mid \tau > t)$. If this limit is a probability measure on $[0,\infty)$ it is called quasi-limiting distribution. We extend a result from [KS12] to the case of non-trivial internal killing which says that either we have a quasi-limiting distribution or all the mass escapes to infinity. We also show that the existence of a quasi-limiting distribution may follow by comparing the drift coefficient $\boldsymbol{a}$ with a drift coefficient $\boldsymbol{b}$ corresponding to a diffusion which already admits a quasi-limiting distribution.

In chapter 2 the focus is on $\lim_{t\to\infty} \mathbb{P}(F_s \mid \tau > t)$. Here F_s is measurable with respect to the σ-field generated by $(Y_r)_{r\leq s}$ for some fixed $s > 0$. The result is the measure of the process $(Y_r)_{r\leq s}$ conditioned on survival, which is sometimes referred to as Q-process. It is also a special case of a so-called penalisation limit. Whether a quasi-limiting distribution exists or not has only a minor effect on the Q-process. We prove appropriate results in theorems 2.11, 2.13 and 2.14 and see that the general form of the Q-process is the same in both cases.

In chapter 3 we investigate the limiting behaviour of $\mathbb{P}(Y_{s(t)} \in \bullet \mid \tau > t)$. Here $t \mapsto s(t)$ is a function with $t - s(t) > \epsilon$ for some $\epsilon > 0$ and $s(t) \to \infty$. We prove that the qualitative behaviour is the same as the quasi-limiting behaviour if $s(t)$ is 'near' t. Furthermore, we demonstrate that $\lim_{t\to\infty} \mathbb{P}(Y_{s(t)} \in \bullet \mid \tau > t)$ is either the stationary distribution of the penalisation limit or zero if $t - s(t) \to \infty$. Apparently, this is caused by the existence respectively the absence of this stationary distribution.