Hecke algebras of type A: Auslander–Reiten quivers and branching rules

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Preface

The field of mathematical research in which the content of this thesis is embedded is the *representation theory of finite-dimensional selfinjective algebras.* These algebras are amongst the most studied algebras in representation theory since they include many classical algebras like group algebras of finite groups.
For a finite-dimensional algebra A over a field $\mathbb{k}$, the main object of study is the category of *finite-dimensional representations* of A, that is to say, its objects are finite-dimensional $\mathbb{k}$-vector spaces V together with a prescribed $\mathbb{k}$-algebra homomorphism

$$\Phi : A \longrightarrow \mathrm{End}_{\mathbb{k}}(V),$$

where we assume that $\mathrm{End}_{\mathbb{k}}(V)$ acts on V from the left. The morphisms are the $\mathbb{k}$-linear homomorphism that commute with the action of A on V induced by Φ. This category is abelian and constitutes an important invariant of the $\mathbb{k}$-algebra A. Equivalently, one can also consider the category $A-\mathrm{mod}$ of finitely generated *left A-modules.* In this case one speaks of finite-dimensional left A-modules.

The overall objective is to classify the finite-dimensional A-modules up to isomorphism. Fortunately, every such module decomposes as a sum of modules that cannot be decomposed any further. These A-modules are called *indecomposable.* But even then, in most of the cases, classifying all indecomposable A-modules up to isomorphism is an impossible task. Therefore, describing invariants for $A-\mathrm{mod}$ whose computation is achievable and which still give enough information is very desirable.

In the first part of this thesis we are concerned with an invariant that was given by M. Auslander and I. Reiten in a series of papers adapting a completely new viewpoint on the subject: Instead of working in the module category itself, they considered the category $\mathrm{Fun}(A)$, whose objects are the covariant additive functors

$$A-\mathrm{mod} \longrightarrow \mathbb{k}-\mathrm{mod},$$

and whose morphisms are the natural transformations between them. Then, by considering the irreducible objects in $\mathrm{Fun}(A)$, one gains new insight into the category $A-\mathrm{mod}$. Here, it is crucial that the finitely generated projective objects in $\mathrm{Fun}(A)$ are given by the *representable functors* $\mathrm{Hom}_A(M,-)$, where M is in $A-\mathrm{mod}$, and that each irreducible object of $\mathrm{Fun}(A)$ is finitely presented. Assuming that A is not semisimple, a minimal projective resolution of an irreducible object in $\mathrm{Fun}(A)$ then corresponds to a short exact sequence

$$0 \longrightarrow L \xrightarrow{f} M \xrightarrow{g} N \longrightarrow 0$$

in $A-\mathrm{mod}$, known as an *Auslander–Reiten sequence* or *AR-sequence* for short.
These sequences determine a directed graph $\Gamma(A)$, the *Auslander–Reiten quiver* or *AR-quiver* for short, whose vertices are in one-to-one correspondence with the isomorphism classes of indecomposable finite-dimensional A-modules, and the arrows between the vertices correspond to certain maps that are determined by the homomorphisms occurring in AR-sequences.
One objective then is to determine for a given finite-dimensional $\Bbbk$-algebra A the graph structure of its AR-quiver $\Gamma(A)$. In the case where A is selfinjective, instead of computing $\Gamma(A)$, one usually considers the subgraph $\Gamma_s(A)$ of $\Gamma(A)$, the *stable* AR-quiver of A, obtained by deleting the projective vertices and all arrows attached to them. In practice, this graph is easier to describe since, by a general theorem of C. Riedtmann, the graph structures of the components of $\Gamma_s(A)$ are very limited in shape. Namely, to each connected component $\mathcal{C}$ of $\Gamma_s(A)$ one can associate a directed tree T and an admissible group G of automorphisms of the translation quiver $\mathbb{Z}T$ such that $\mathbb{Z}T/G$ is isomorphic to $\mathcal{C}$. The undirected tree obtained from T is then uniquely determined up to isomorphism by $\mathcal{C}$ and is called the *tree class* of $\mathcal{C}$.

In Part I of the thesis we determine the possible tree classes occurring for components of the stable AR-quiver of a Hecke algebra associated with a symmetric group in characteristic zero. These algebras can be thought of as deformations of group algebras and are instances of finite-dimensional selfinjective algebras. In the following we write $H_n^f(q)$ for the Hecke algebra of the symmetric group on n letters with defining parameter $q \neq 1$, a primitive eth root of unity in $\Bbbk$.
The main motivation for this is the landmark paper of K. Erdmann [33], where she showed that for a block of a group algebra of a finite group, the tree class of a stable component is almost always A_∞. The main result there states that if the block under consideration is of wild representation type, then the tree class of every stable component is A_∞.
Since $H_n^f(q)$ is a deformation of a group algebra, it is likely that a similar result also holds in that case.
In fact, the first main result of the thesis is the following: If the ground field $\Bbbk$ is algebraically closed and of characteristic zero, then we show that if B is a block of $H_n^f(q)$ of wild representation type, then all the connected components of $\Gamma_s(B)$ have tree class A_∞.

The roadmap for this is as follows: In Chapter 1 we introduce the notation that will be valid throughout the first part of this thesis. In Chapter 2 we will collect the basic definitions and results of the Auslander–Reiten theory of a finite-dimensional algebra that are needed in the course of this thesis. Moreover, we state important results concerning the representation theory of a finite-dimensional selfinjective algebra.

In Chapter 3 we will be concerned with the representation theory of skew group algebras, an important class of algebras that will play a decisive role throughout the first part. We give a brief introduction into this subject by stating the crucial results relating the representation theory of the skew group algebra $A \star G$ with that of the underlying algebra A. Afterwards we will prove a theorem stating that under some mild conditions on the algebra A and the finite group G all stable components of $\Gamma_s(A \star G)$ have tree class A_∞ if this is true for $\Gamma_s(A)$.

In Chapter 4 we proceed by studying the shape of the stable AR-quiver of a *quantum complete intersection.* These selfinjective algebras are deformations of truncated polynomial rings, which form an important class of algebras. The reason for considering such algebras is that they occur as vertices of the Hecke algebra $H_n^f(-1)$ in characteristic zero. Note that there is a vertex theory for Hecke algebras of the symmetric group in characteristic zero similar to that for group algebras in positive characteristic. The definition of a vertex is given in Chapter 1 of Part I.

In [11] it is shown that for a homogeneous quantum complete intersection of wild representation type, a connected component of the stable AR-quiver has tree class A_∞. The main theorem of the chapter will be a generalization of this to quantum complete intersections whose commutation matrix is given by arbitrary roots of unity. The crucial observation here is that such algebras arise from skew group algebras over homogeneous quantum complete intersections.

Having determined the shape of the stable AR-quiver of vertices of Hecke algebras at a primitive second root of unity in characteristic zero, we proceed in Chapter 5 by proving a similar result for Hecke algebras at a primitive eth root of unity, where $e > 2$.

In this case a vertex of an indecomposable $H_n^f(q)$-module is isomorphic to the outer tensor product $H_e^f(q)^{\otimes k}$, $k \geq 0$. The algebra $H_e^f(q)$ has a unique non-semisimple block B, which is a Brauer tree algebra whose tree is a line with no exceptional vertex. Thus, every block of $H_e^f(q)^{\otimes k}$ is Morita equivalent to an outer tensor product of some copies of B.

The main theorem of Chapter 5 states that if $k > 1$, then a connected component of $\Gamma_s(B^{\otimes k})$ has tree class A_∞. Amazingly, a skew group algebra construction over a truncated polynomial ring will be the main ingredient of the latter proof.

Chapter 6 preludes the main result of Part I. It is intended to explain a theorem for blocks of Hecke algebras in characteristic zero giving a Morita equivalence between a Rouquier block of weight w and a wreath product of an outer product of the principal block B of $H_e^f(q)$ and a symmetric group on w letters. This was first proven by J. Chuang and R. Kessar in the case of group algebras of symmetric groups, which then was extended to general linear groups in non-defining characteristic independently by W. Turner and H. Miyachi. The Hecke algebra case was then established by J. Chuang and H. Miyachi.

For the convenience of the reader we briefly review the necessary definitions and results of the representation theory of finite general linear groups in non-defining characteristic that are needed to state the theorem. We also describe the construction of the bimodule inducing the Morita equivalence in detail.
In Chapter 7 we finally obtain our main result on the tree classes of stable components of blocks of $H_n^f(q)$. To this end we require the famous theorem of J. Chuang and R. Rouquier on derived equivalences between blocks of Hecke algebras of the same weight.
The main theorem then classifies the possible tree classes which may occur for blocks of $H_n^f(q)$ of weight w. As expected, this classification only depends on the parameter $q \in \Bbbk$ and the weight w of the block.

In Part II of the thesis we will be concerned with another important invariant of a finite-dimensional algebra A. Instead of looking at all isomorphism classes of indecomposable A-modules, one usually imposes additional structure on the latter. An important class is given by the isomorphism classes of *irreducible* A-modules, which are not only indecomposable but also have no non-trivial submodules. As it turns out, in many important cases, labelling the irreducible modules is an achievable task. Moreover, since any finite-dimensional A-module has a filtration by irreducibles, one may also get new information on the indecomposables themselves and, hence, on $A - \text{mod}$.
In the second part we again focus on the representation theory of Hecke algebras of type A at a root of unity. We will give a detailed overview of the theory initiated by A. Lascoux, B. Leclerc, J.-Y. Thibon in [62], S. Ariki in [2] and I. Grojnowski in [44] concerning the representation theory of cyclotomic Hecke algebras, where we follow the approach of I. Grojnowski.
In particular, we give a thorough description of the various functors defined on the module categories of these algebras, based on the ideas given in [44].
We will study the divided power functors in great detail, and this enables us to deduce results about the vertices of Hecke algebras of the symmetric group.
The vertex theory for Hecke algebras of the symmetric group can be seen as an analogue of the vertex theory for group algebras. In the context of Hecke algebras, a vertex of an indecomposable $H_n^f(q)$-module M is a standard parabolic subgroup W of the symmetric group on n letters which is minimal with respect to the property that M is projective relative to the subalgebra of $H_n^f(q)$ determined by W. Since the standard parabolic subgroups are indexed by compositions of n, it is natural to ask what the compositions indexing the vertices may look like.
In characteristic zero the answer to this question was given by J. Du in [32], where he showed that the vertices of indecomposable $H_n^f(q)$-modules are l-parabolic subgroups, with l the order of q in the underlying field.
In [27] a conjecture was given by R. Dipper and J. Du, describing the structure of the compositions indexing the vertices in general. Namely, the conjecture says that the vertices are $l - p$-parabolic subgroups, where p relates to the characteristic of the underlying field.

Using properties of the introduced functors, we are able to give an application of Grojnowski's approach to the Dipper–Du Conjecture: We show that the vertex of an indecomposable $H_n^f(q)$-module lying in a block of finite representation type is $l-p$-parabolic.
We make use of various structures that the sum over the Grothendieck groups of $H_n^f(q)$-modules

$$\mathcal{G} := \bigoplus_{n\geq 0} \mathcal{K}(H_n^f(q) - \mathrm{mod})$$

carries. In [2] S. Ariki made the crucial observation that in the case where the ground field is the field of complex numbers, the abelian group $\mathcal{G}$ can be given the structure of an irreducible highest weight module over the infinite-dimensional Lie algebra $\widehat{\mathfrak{sl}}_l$.
I. Grojnowski [44] gave similar results in the case where the ground field has arbitrary characteristic. There, he uses different techniques originating in the theory of affine Hecke algebras, giving a new functorial approach to the setting. Namely, he gives an alternative description of the functors that are used in Ariki's work, which enables one to define the divided power functors on the various categories $H_n^f(q) - \mathrm{mod}$, $n \geq 0$. As the notion suggests, they restrict to operators on the Grothendieck groups, which then coincide with the divided powers coming from the action of $\widehat{\mathfrak{sl}}_l$.
Furthermore, we make use of the theory of crystal bases developed by M. Kashiwara. He showed that to an irreducible highest weight module of the quantized universal enveloping algebra over an affine Kac–Moody algebra one can associate a certain graph, the *crystal graph*. This graph encodes combinatorial data of the highest weight module under consideration.
In particular, the $\widehat{\mathfrak{sl}}_l$-module $\mathcal{G}$ gives rise to a crystal graph whose structure was determined by K. C. Misra and T. Miwa in [70].
From this one obtains a labelling of the isomorphism classes of irreducible $H_n^f(q)$-modules, where $n \geq 0$ varies.
A. Kleshchev established in [58] another labelling of the irreducible modules for the group algebras of the symmetric group by investigating the socles of the restrictions of the latter. This was later generalized to Hecke algebras of the symmetric group by J. Brundan [16].
Amazingly, these two labellings coincide, which enables us to reformulate the famous result obtained by J. Scopes in [80] in the symmetric group case and T. Jost in [52] in the Hecke algebra case in terms of the functors given by I. Grojnowski. The theorems of J. Scopes and T. Jost establish Morita equivalences of blocks of group algebras of symmetric groups of the same weight and the associated Hecke algebras, respectively.
From this and the properties of the functors inducing the equivalence, we deduce, for blocks B and B' of Hecke algebras of the symmetric group forming a Scopes pair, that two indecomposable modules corresponding under the equivalence have a common vertex.

As an application, this gives a proof of the Dipper–Du Conjecture in the case where the blocks are of finite representation type.

The roadmap for the second part will be: In Chapter 8 we introduce the notation used in what follows.
In Chapter 9 we recall the notion of an affine Hecke algebra of type A, denoted by $H_n(q)$, $n \geq 0$. We begin with the basic definitions and then turn our attention to the subalgebra of Laurent polynomials of $H_n(q)$, which will become important in the further investigations. By computing generalized eigenspaces, we explain why this subalgebra is important in the representation theory of $H_n(q)$.
Furthermore, we define refinements of the restriction functor

$$\mathrm{Res} : H_n(q) - \mathrm{mod} \longrightarrow H_{n-1}(q) - \mathrm{mod},$$

as well as their iterations. Finally, for each $r \geq 0$, we define divided power functors

$$H_n(q) - \mathrm{mod} \longrightarrow H_{n-r}(q) - \mathrm{mod},$$

and show how these are related to the refinements of the restriction functor.
For the convenience of the reader, we will recall in Chapter 10 the basic notions of the representation theory of Kac–Moody algebras as they will become important in later chapters. In particular, we will explain how the representation theory of the latter and the representation theory of its derived algebra are related since in this thesis we will mainly work over the derived algebra.
Chapter 11 recalls the definition of various factor algebras of $H_n(q)$, the *cyclotomic Hecke algebras*, which will be denoted by $H_n^\Lambda(q)$. In particular, choosing $\Lambda = \Lambda_0$, we obtain the finite-dimensional Hecke algebra $H_n^f(q)$. The main goal of this chapter is to define the cyclotomic analogues of the functors defined in Chapter 9 as well as their adjoints.
In Chapter 12 we review the multiplicity-one results given by I. Grojnowski and M. Vazirani in [45] for the affine Hecke algebras as well as their cyclotomic quotients. We will derive some combinatorial results of the action of the operators on the Grothendieck groups induced by the functors defined in Chapter 11.
Chapter 13 explains how the sum of the Grothendieck groups

$$\mathcal{G}(\Lambda) := \bigoplus_{n \geq 0} \mathcal{K}(H_n^\Lambda(q) - \mathrm{mod})$$

can be given a module structure over the derived algebra of $\widehat{\mathfrak{sl}}_l$. Moreover, we state the result that identifies the latter as the irreducible highest weight module of highest weight Λ of this algebra.
In Chapter 14 we introduce the theory of M. Kashiwara concerning crystal bases of highest weight modules of quantized enveloping algebras of Kac–Moody algebras and their associated crystal graphs. The main aim is to explain how a crystal graph is constructed from $\mathcal{G}(\Lambda)$ and that it is isomorphic to the crystal graph of the irreducible highest weight module of highest weight Λ of the quantized enveloping algebra of $\widehat{\mathfrak{sl}}_l$.

Chapter 15 is a review of the famous theorem by K. C. Misra and T. Miwa giving the structure of the crystal graph associated to $\mathcal{G}(\Lambda_0)$. From this we obtain a labelling of the irreducible $H_n^f(q)$-modules, $n \geq 0$.
In Chapter 16 we explain another labelling of the irreducible $H_n^f(q)$-modules, $n \geq 0$, and state the branching rules due to A. Kleshchev and J. Brundan. Furthermore, we show that this labelling coincides with the labelling given in Chapter 15.
Chapter 17 contains one of the milestones in the proof of the main result. There we reformulate the Morita equivalence for blocks of Hecke algebras given originally by J. Scopes in terms of our functors defined in Chapter 11.
Finally, in Chapter 18 we prove that the Dipper–Du Conjecture is true for $H_n^f(q)$-modules lying in blocks of finite representation type.
We will also discuss the supposed counterexample to the conjecture given in [48]. As is well known, this does not disprove the conjecture.

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Part I

Stable Auslander–Reiten components of Hecke algebras of the symmetric group

Chapter 1

Preliminaries

1.1 Goal

In the first part of this thesis, we will prove that if B is a wild block of a Hecke algebra of the symmetric group over an algebraically closed field of characteristic zero, then every connected component of the stable Auslander–Reiten quiver $\Gamma_s(B)$ has tree class A_∞.

1.2 Notation

Throughout the first part of this thesis, the symbol $\Bbbk$ will always denote a field.

Categories and functors. All categories under consideration will be additive $\Bbbk$-categories, that is to say, preadditive categories that have finite direct sums such that each morphism set carries the structure of a $\Bbbk$-vector space, and where the composition morphism is $\Bbbk$-bilinear. Throughout, if $\mathcal{C}$ is any category, we denote by $1_{\mathcal{C}}$ the identity functor of $\mathcal{C}$.
Furthermore, if $\mathcal{C}$ and $\mathcal{D}$ are $\Bbbk$-categories, then, if not stated otherwise, by a functor $\mathcal{F} : \mathcal{C} \to \mathcal{D}$ we shall always mean an additive functor such that the induced homomorphisms $\mathrm{Hom}_{\mathcal{C}}(A, B) \to \mathrm{Hom}_{\mathcal{D}}(\mathcal{F}(A), \mathcal{F}(B))$ of abelian groups are $\Bbbk$-vector space homomorphisms.
For a finite-dimensional algebra A defined over $\Bbbk$, we denote by $A - \mathrm{mod}$ (resp. $\mathrm{mod} - A$) the abelian $\Bbbk$-category of finite-dimensional *left* (resp. *right*) A-modules. Moreover, we identify the category $\mathrm{mod} - A$ with the category $A^{\mathrm{op}} - \mathrm{mod}$ of finite-dimensional left A^{op}-modules, where A^{op} denotes the opposite ring of A.
By $A - \mathrm{proj}$ (resp. $\mathrm{proj} - A$) we denote the full subcategory of projective left A-modules (resp. right A-modules). Again, the category $\mathrm{proj} - A$ can also be identified with the subcategory $A^{\mathrm{op}} - \mathrm{proj}$ of $A^{\mathrm{op}} - \mathrm{mod}$ of projective left A^{op}-modules.
By $A - \underline{\mathrm{mod}}$, we denote the stable module category of $A - \mathrm{mod}$, i.e., the category whose objects are that of $A - \mathrm{mod}$, and the morphisms are given by equivalence classes

$$\underline{\mathrm{Hom}}_A(A, B) = \mathrm{Hom}_A(A, B)/\mathcal{P}(A, B),$$

for objects A and B in $A-\underline{\mathrm{mod}}$. Here, $\mathcal{P}(A,B)$ denotes the subspace of morphisms f of $\mathrm{Hom}_A(A,B)$ such that there is a projective A-module P and morphisms $s: A \to P$, $t: P \to B$ with $f = ts$.
Dually, we define $A-\overline{\mathrm{mod}}$ to be the category whose objects are that of $A-\mathrm{mod}$, and the morphisms are given by equivalence classes of morphisms in $A-\mathrm{mod}$, where two morphisms are considered to be the same if their difference factors through an injective A-module.
Moreover, we denote by $A-\mathrm{mod}_{\mathcal{P}}$ the full subcategory of $A-\mathrm{mod}$ consisting of A-modules having no non-zero direct summands isomorphic to some projective A-module.
If not stated otherwise, in the following, by an A-module we will always mean a left A-module.

Hecke algebras. We use the definition of [29]. Let n be a non-negative integer and denote by $W(n)$ the symmetric group on n letters. If $n \geq 2$, then $W(n)$ is generated by the set $\mathfrak{B} = \{s_1 = (1,2), \ldots, s_{n-1} = (n-1,n)\}$ of basic transpositions.
Let $q \in \Bbbk$ be invertible. The *Iwahori–Hecke algebra* $H_n^f(q)$ of the symmetric group $W(n)$ with parameter q is the $\Bbbk$-algebra generated by the symbols $T_i = T_{s_i}$, $1 \leq i \leq n-1$, subject to the following relations:

(a) $(T_i - q)(T_i + 1) = 0$.

(b) $T_iT_j = T_jT_i$, for $1 \leq i < j-1 \leq n-2$.

(c) $T_{i+1}T_iT_{i+1} = T_iT_{i+1}T_i$, for $1 \leq i \leq n-2$.

Denote by e the least positive integer $i \geq 2$ such that

$$1 + q + \ldots + q^{i-1} = 0.$$

If no such i exists, one sets $e = \infty$. In the first part of this thesis, we will always assume that $q \neq 1$ and e is finite. Then e is the multiplicative order of q in $\Bbbk$.

Vertices. Let $\mu = (\mu_1, \ldots, \mu_k)$ be a composition of the non-negative integer n. The *standard parabolic* W_μ of $W(n)$ is defined as the row stabilizer of the row standard tableau associated to μ. Note that W_μ is generated by $W_\mu \cap \mathfrak{B}$.
For a composition μ of n, we denote by H_μ the subalgebra of $H_n^f(q)$ associated to W_μ generated by the elements T_i, where $s_i \in W_\mu$. The algebra H_μ is called a *standard parabolic* subalgebra of $H_n^f(q)$.

Let M be an indecomposable $H_n^f(q)$-module. We say that M is *projective relative to* H_μ if M is isomorphic to a direct summand of $H_n^f(q) \otimes_{H_\mu} M$. In this case one also says that M is *H_μ-projective.*
A *vertex* of M is a standard parabolic subgroup W_μ of $W(n)$ such that M is projective relative to H_μ and there are no standard parabolic subgroups of $W(n)$ properly contained in W_μ with that property.

Chapter 2

Auslander–Reiten theory

The theory of Auslander–Reiten plays a major role in the representation theory of non-semisimple finite-dimensional algebras Λ defined over a field $\mathbb{k}$. The philosophy behind this theory is to describe the structure of the finite-dimensional Λ-modules via the structure preserving maps between them. This viewpoint leads to a description of important classes of maps, which determine the modules up to isomorphism. One of the milestones is the definition of *almost split sequences*, which gives new insight in the homological structure of the category of Λ-modules.
As a consequence one may define the *Auslander–Reiten quiver* of Λ, which is a locally finite graph constructed from the isomorphism classes of the finite-dimensional indecomposable Λ-modules and certain maps between them. This gives a homological invariant of the category of finite-dimensional Λ-modules.
In this section we give a brief introduction to the Auslander–Reiten theory of a finite-dimensional $\mathbb{k}$-algebra, stating the most important results, which are used throughout this exposition. We will first give the basic definitions and results, most of which are true in the more general context of Artin algebras.
Afterwards, we will focus on the representation theory of finite-dimensional selfinjective $\mathbb{k}$-algebras as these will be the most important algebras in this thesis.
In the following, Λ will always denote a finite-dimensional algebra over a field $\mathbb{k}$.

2.1 Auslander–Reiten sequences and irreducible maps

We will first give the basic definitions of *almost split maps* and *Auslander–Reiten sequences*. After that we state the concepts that are needed to construct Auslander–Reiten sequences. Moreover, we recall the definition of an *irreducible map* and state the relationship to Auslander–Reiten sequences. All the results stated here can be found in [7].

Definition 2.1.1. (a) A homomorphism $f : B \to C$ between finite-dimensional Λ-modules is *right almost split* if the following conditions hold:

(i) f is not a split epimorphism.

(ii) For every homomorphism $h : X \to C$ that is not a split epimorphism, there exists a homomorphism $s : X \to B$ such that $h = fs$.

(b) A homomorphism $g : A \to B$ between finite-dimensional Λ-modules is *left almost split* if the following conditions hold:

(i) g is not a split monomorphism.

(ii) For every homomorphism $h : A \to X$ that is not a split monomorphism, there exists a homomorphism $s : B \to X$ such that $h = sg$.

(c) A short exact sequence

$$0 \longrightarrow A \xrightarrow{g} B \xrightarrow{f} C \longrightarrow 0 \tag{2.1}$$

of finite-dimensional Λ-modules is called an *Auslander–Reiten sequence* (resp. AR–sequence for short) or *almost split sequence* if f is right almost split and g is left almost split.

In the following, recall from [7, I, §2] the definition of a left (right) minimal homomorphism. By [7, V, Proposition 1.14], we have that the modules A and C are indecomposable, whereas the homomorphism g is left minimal and left almost split and the homomorphism f is right minimal and right almost split. In this situation one says that g is a minimal left almost split homomorphism and f is a minimal right almost split homomorphism.
Auslander–Reiten sequences have remarkable properties, for example, up to isomorphism, they are determined by their terms at the beginning and the end:

Theorem 2.1.2. *The following are equivalent for two AR-sequences* $0 \to A \to B \to C \to 0$ *and* $0 \to A' \to B' \to C' \to 0$*:*

(a) $A \cong A'$.

(b) $C \cong C'$.

(c) *The sequences are isomorphic, i.e., there is a commutative diagram*

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & 0
\end{tikzcd}$$

with the vertical homomorphisms being isomorphisms.

Proof. See [7, V, Theorem 1.16]. □

For an arbitrary ring it is far from being clear that for a given Λ-module M there is an AR-sequence starting (resp. ending) in M. For Artin algebras, this is the famous theorem of Auslander and Reiten, see [7, V, Theorem 1.15].

Theorem 2.1.3. (a) *If C is an indecomposable non-projective Λ-module, then there is an AR-sequence $0 \to A \to B \to C \to 0$ in $\Lambda - \mathrm{mod}$.*

(b) *If A is an indecomposable non-injective Λ-module, then there exists an AR-sequence $0 \to A \to B \to C \to 0$ in $\Lambda - \mathrm{mod}$.*

As a direct consequence of this theorem, if Λ is not semisimple, then there exist AR-sequences in $\Lambda - \mathrm{mod}$.
We now recall the important constructions from [7, §IV], from which one may construct the AR-sequence ending in a given module. Denote by $(-)^*$ the contravariant functor

$$\mathrm{Hom}_\Lambda(-,\Lambda) : \Lambda - \mathrm{mod} \longrightarrow \Lambda^{\mathrm{op}} - \mathrm{mod}. \tag{2.2}$$

By [7, §II, Proposition 4.3], on restriction to $\Lambda - \mathrm{proj}$, the functor $(-)^*$ induces a duality between $\Lambda - \mathrm{proj}$ and $\Lambda^{\mathrm{op}} - \mathrm{proj}$.
Moreover, we denote by $D : \Lambda - \mathrm{mod} \to \Lambda^{\mathrm{op}} - \mathrm{mod}$ the duality induced by the usual duality

$$\mathrm{Hom}_{\mathbb{k}}(-,\mathbb{k}) : \mathbb{k} - \mathrm{mod} \longrightarrow \mathbb{k} - \mathrm{mod},$$

see [7, II.3]. By interchanging the roles of Λ and Λ^{op} we also get a duality $D' : \Lambda^{\mathrm{op}} - \mathrm{mod} \to \Lambda - \mathrm{mod}$. Then, we have that

$$D'D \cong 1_{\Lambda-\mathrm{mod}} \text{ and } DD' \cong 1_{\Lambda^{\mathrm{op}}-\mathrm{mod}}$$

as functors. If there is no danger of confusion, the duality D' will also be denoted by D.
Let M be in $\Lambda - \mathrm{mod}$. If we take a minimal projective resolution

$$P_1 \xrightarrow{f} P_0 \longrightarrow M \longrightarrow 0$$

of M in $\Lambda - \mathrm{mod}$, then by applying the duality $(-)^*$ we obtain a minimal projective resolution

$$P_1^* \xrightarrow{f^*} P_0^* \longrightarrow \mathrm{Coker}(f^*) \longrightarrow 0$$

in $\Lambda^{\mathrm{op}} - \mathrm{mod}$. One defines $\mathrm{Tr}(M) = \mathrm{Coker}(f^*)$, which is the *transpose* of M. Then, by [7, IV, Proposition 1.6], the map Tr induces a duality

$$\Lambda - \underline{\mathrm{mod}} \longrightarrow \Lambda^{\mathrm{op}} - \underline{\mathrm{mod}},$$

which is also denoted by Tr. From the duality D we obtain a duality

$$\Lambda^{\mathrm{op}} - \underline{\mathrm{mod}} \longrightarrow \Lambda - \overline{\mathrm{mod}}.$$

If there is no danger of confusion we will denote this duality by the same symbol D. Then, the composition

$$D\mathrm{Tr} : \Lambda - \underline{\mathrm{mod}} \longrightarrow \Lambda - \overline{\mathrm{mod}}$$

is an equivalence of $\Bbbk$-categories with inverse equivalence the composition

$$\mathrm{Tr}D : \Lambda - \overline{\mathrm{mod}} \longrightarrow \Lambda - \underline{\mathrm{mod}}.$$

Next, we look at what happens to the objects of $\Lambda - \mathrm{mod}$ under the map $D\mathrm{Tr}$: $\Lambda - \mathrm{mod} \to \Lambda - \mathrm{mod}$. Similarly, we have a map $\mathrm{Tr}D : \Lambda^{\mathrm{op}} - \mathrm{mod} \to \Lambda^{\mathrm{op}} - \mathrm{mod}$. Then, by [7, IV, Proposition 1.10], these maps afford mutually inverse bijections between the isomorphism classes of indecomposable non-projective Λ-modules and the isomorphism classes of indecomposable non-injective Λ-modules.
The map $D\mathrm{Tr}$ is called the *Auslander–Reiten translation* (AR-translation for short).

Notation 2.1.4. In the following, the map DTr will be also denoted by τ.

The relation with the AR-sequences is the following.

Proposition 2.1.5. *Suppose that* $0 \to A \to B \to C \to 0$ *is an AR-sequence in* $\Lambda - \mathrm{mod}$. *Then* $A \cong D\mathrm{Tr}(C)$ *and* $C \cong \mathrm{Tr}D(A)$.

Proof. This is [7, V, Proposition 1.14]. □

Closely related to the concept of an AR-sequence is the notion of an *irreducible* morphism, whose definition we give next.

Definition 2.1.6. A homomorphism $g : B \to C$ between finite-dimensional Λ-modules is called *irreducible* if it is neither a split monomorphism nor a split epimorphism and whenever the homomorphisms $s : B \to X$ and $t : X \to C$ are such that $f = ts$, then s is a split monomorphism or t is a split epimorphism.

We have the following crucial fact about irreducible homomorphisms.

Proposition 2.1.7. *Let* $g : B \to C$ *be irreducible in* $\Lambda - \mathrm{mod}$. *Then:*

(a) *g is a monomorphism or an epimorphism, but not an isomorphism.*

(b) *If g is a monomorphism, then B is isomorphic to a direct summand of each proper submodule of C containing B.*

(c) *If g is an epimorphism, then C is isomorphic to a direct summand of each factor module of B/U with* $0 \neq U \subseteq \mathrm{Ker}(g)$.

Proof. See [7, V, Lemma 5.1]. □

The next well-known fact is a direct consequence of the last proposition. It characterizes the irreducible maps with target (resp. domain) a projective (resp. injective) indecomposable module. Recall that if M is in $\Lambda - \mathrm{mod}$, then the radical $\mathrm{rad}(M)$ is defined to be the intersection of all maximal submodules of M, whereas the socle $\mathrm{soc}(M)$ is defined to be the sum of all irreducible submodules of M.

Corollary 2.1.8. (a) *Suppose that P is an indecomposable projective Λ-module and $g : X \to P$ an irreducible homomorphism. Then X is isomorphic to a direct summand of* $\mathrm{rad}(P)$.

(b) *Suppose that I is an indecomposable injective Λ-module and let $g : I \to Y$ be an irreducible homomorphism. Then Y is isomorphic to a direct summand of $I/\mathrm{soc}(I)$.*

Proof. By [7, I, Proposition 4.7], $\mathrm{rad}(P)$ is the unique maximal submodule of P. If $g : X \to P$ is irreducible, then X is either a monomorphism or an epimorphism by Proposition 2.1.7. Since P is projective and g is not a split epimorphism, g must be a monomorphism. Hence, by part (b) of Proposition 2.1.7, we infer that X is isomorphic to a direct summand of $\mathrm{rad}(P)$ and part (a) follows.
Since I is injective and indecomposable, the socle $\mathrm{soc}(I)$ of I is irreducible by [7, II, Proposition 4.1]. If $g : I \to Y$ is irreducible, then g must be an epimorphism. It follows that $\mathrm{Ker}(g) \neq 0$, and thus, $\mathrm{soc}(I) \subseteq \mathrm{Ker}(g)$. The claim now follows from part (c) of Proposition 2.1.7. □

The next theorem connects irreducible homomorphisms with AR-sequences. It is the content of [7, V, Theorem 5.3].

Theorem 2.1.9. (a) *Let C in $\Lambda - \mathrm{mod}$ be indecomposable. Then a homomorphism $g : B \to C$ is irreducible if and only if there is a homomorphism $g' : B' \to C$ such that the induced homomorphism*

$$(g, g') : B \oplus B' \to C$$

is minimal right almost split.

(b) *Let A in $\Lambda - \mathrm{mod}$ be indecomposable. Then a homomorphism $f : A \to B$ is irreducible if and only if there is a homomorphism $f' : A \to B'$ such that the induced homomorphism*

$$\begin{pmatrix} f \\ f' \end{pmatrix} : A \to B \oplus B'$$

is minimal left almost split.

In our further investigations it is important to know in which AR-sequences indecomposable projective injective modules occur. Therefore, we state the following:

Proposition 2.1.10. (a) *Let $\delta : 0 \to A \to B \to C \to 0$ be an AR-sequence. If B has an indecomposable projective injective summand P, then the composition series of P has length $l(P) \geq 2$ and δ is isomorphic to the sequence*

$$\delta : 0 \longrightarrow \mathrm{rad}(P) \xrightarrow{\begin{pmatrix} -i \\ p \end{pmatrix}} P \oplus \mathrm{rad}(P)/\mathrm{soc}(P) \xrightarrow{(q,j)} P/\mathrm{soc}(\mathrm{P}) \longrightarrow 0, \tag{2.3}$$

where $i : \mathrm{rad}(P) \to P$ and $j : \mathrm{rad}(P)/\mathrm{soc}(P) \to P/\mathrm{soc}(P)$ are the canonical inclusion homomorphisms and $p : \mathrm{rad}(P) \to \mathrm{rad}(P)/\mathrm{soc}(P)$ and $q : P \to P/\mathrm{soc}(P)$ are the canonical quotient homomorphisms.

(b) *If P is indecomposable projective injective with $l(P) \geq 2$, then there exists some AR-sequence $\delta : 0 \to A \to B \to C \to 0$ such that P is a summand of B.*

Proof. This is [7, V, Proposition 5.5]. □

In other words, if P is a projective injective Λ-module and not irreducible, the only sequence in which P occurs is the one given in (2.3). This sequence is also called a *standard sequence*. We also note the following:

Proposition 2.1.11. *Suppose that*

$$\delta : 0 \longrightarrow A \xrightarrow{g} B \xrightarrow{f} C \longrightarrow 0$$

is an exact sequence. Then δ is an AR-sequence if and only if f and g are both irreducible.

Proof. See [7, V, Proposition 5.9]. □

Next we give the definition of the radical of the module category $\Lambda-$mod. By $\Lambda-$ind we denote a full subcategory of $\Lambda -$ mod whose objects are chosen representatives from isomorphism classes of indecomposable Λ-modules.

Definition 2.1.12. Let A and B be in $\Lambda -$ mod. The set $\mathrm{rad}_\Lambda(A, B) := \{f \in \mathrm{Hom}_\Lambda(A, B) \mid hfg$ is not an isomorphism for any $g : X \to A$ and $h : B \to X$ with X in $\Lambda - \mathrm{ind}\}$ is called the *radical* of $\mathrm{Hom}_\Lambda(A, B)$.

That $\mathrm{rad}_\Lambda(A, B)$ is actually a subspace of $\mathrm{Hom}_\Lambda(A, B)$ is part of the proof of [7, V, Proposition 7.1]. Inductively, one defines the powers $\mathrm{rad}^n_\Lambda(A, B)$ for each positive integer $n \geq 2$ as follows: $\mathrm{rad}^n_\Lambda(A, B) = \{f \in \mathrm{Hom}_\Lambda(A, B) \mid$ there exists X in $\Lambda -$ mod, $g \in \mathrm{rad}_\Lambda(A, X)$ and $h \in \mathrm{rad}^{n-1}_\Lambda(X, B)$ such that $f = hg\}$. Connecting this with the notion of irreducible morphisms, one gets:

Proposition 2.1.13. *Let $f : A \to B$ be a homomorphism between indecomposable modules A and B in $\Lambda - \mathrm{mod}$. Then f is irreducible if and only if $f \in \mathrm{rad}_\Lambda(A, B) \setminus \mathrm{rad}^2_\Lambda(A, B)$.*

Proof. See [7, V, Proposition 7.3]. □

2.2 The Auslander–Reiten quiver

A quiver $\Gamma = (\Gamma_0, \Gamma_1)$ consists of a set of vertices Γ_0 and a set of arrows $\Gamma_1 \subseteq \Gamma_0 \times \Gamma_0$ together with maps $d_0, d_1 : \Gamma_1 \to \Gamma_0$, where d_0 maps an arrow of Γ_1 to its endpoint and d_1 maps an arrow of Γ_1 to its starting point.
A *valuation* on a quiver $\Gamma = (\Gamma_0, \Gamma_1)$ is a map

$$\nu : \Gamma_1 \to \mathbb{Z}_{>0} \times \mathbb{Z}_{>0}.$$

We denote the image of an arrow $\alpha : x \to y$ in Γ_1 under ν by (v_α, v'_α). If a quiver Γ comes equipped with a valuation ν, then we say that (Γ, ν) is a *valued quiver*.
A morphism between valued quivers (Γ, ν) and (Γ', μ) is a morphism $f = (f_0, f_1) : \Gamma \to \Gamma'$ of quivers such that if $\alpha : x \to y$ is an arrow in Γ_1 with valuation $(\nu_\alpha, \nu'_\alpha)$, then $f_1(\alpha) : f_0(x) \to f_0(y)$ has valuation $(\mu_{f_1(\alpha)}, \mu'_{f_1(\alpha)}) = (\nu_\alpha, \nu'_\alpha)$.

As before, Λ denotes a finite-dimensional $\Bbbk$-algebra. We associate to Λ a valued quiver $\Gamma(\Lambda)$ as follows. The vertices of $\Gamma(\Lambda)$ are in one-to-one correspondence with the objects of $\Lambda - \mathrm{ind}$, and are denoted by $[M]$ for M in $\Lambda - \mathrm{ind}$.
There is an arrow $[M] \to [N]$ if and only if there is an irreducible homomorphism $M \to N$ in $\Lambda - \mathrm{mod}$.
The arrow has valuation (b, a) if there is a minimal right almost split homomorphism $U \oplus X \to N$ such that U is isomorphic to a direct sum of a copies of M and X has no direct summand isomorphic to M, and a minimal left almost split morphism $M \to V \oplus Y$ such that V is isomorphic to a direct sum of b copies of N, where Y has no direct summand isomorphic to N. The resulting valued quiver is called the *Auslander–Reiten quiver* (AR-quiver for short) of Λ.
If we denote by T_X the division algebra $\mathrm{End}_\Lambda(X)/\mathrm{rad}(\mathrm{End}_\Lambda(X))$ for each X in $\Lambda - \mathrm{ind}$, then for indecomposable modules A and B in $\Lambda - \mathrm{mod}$, the $\Bbbk$-vector space $\mathrm{Irr}(A, B) := \mathrm{rad}_\Lambda(A, B)/\mathrm{rad}^2_\Lambda(A, B)$ can be viewed as a $T_B - T_A$-bimodule. In fact, one can show that a equals the dimension of $\mathrm{Irr}(A, B)$ as a T_A^{op}-vector space, and that b equals the dimension of $\mathrm{Irr}(A, B)$ as a T_B-vector space, see [7, VII, Proposition 1.3]. It follows that if $\Bbbk$ is algebraically closed then $a = b$.
We may impose an equivalence relation on the objects of $\Lambda - \mathrm{ind}$. Two modules in $\Lambda - \mathrm{ind}$ are said to be *related* by an irreducible homomorphism if there exists an irreducible homomorphism $f : A \to B$. An equivalence class under the equivalence relation generated by this relation is called a *component* of $\Lambda - \mathrm{ind}$. Then A and B are in the same component if and only if there exists a positive integer m, indecomposable modules X_i, $1 \le i \le m$, and for each i either an irreducible homomorphism $f_i : X_i \to X_{i+1}$ or an irreducible homomorphism $g_i : X_{i+1} \to X_i$ with $X_1 = A$ and $X_m = B$.
Moreover, the equivalence relation on $\Lambda - \mathrm{ind}$ induces an equivalence relation on $\Gamma(\Lambda)$. Then, an equivalence class under this equivalence relation is called a component of $\Gamma(\Lambda)$. Note that we may view a component as a full subquiver of $\Gamma(\Lambda)$, which is connected, when considered as an undirected graph.

In the following we use the notation of [49] and [77]. A *representation quiver* (Γ, τ) is a quiver $\Gamma = (\Gamma_0, \Gamma_1)$ together with a subset $P_0 \subseteq \Gamma_0$ and an injective map $\tau : P_0 \to \Gamma_0$ such that

(a) Γ neither contains multiple arrows nor loops.

(b) $(\tau(x))^+ = x^-$ for all $x \in P_0$.

For a vertex $x \in \Gamma_0$, we denote by x^- the set consisting of all vertices of Γ_0 which are starting points of arrows of Γ_1 ending in x and by x^+ the set of all vertices of Γ_0 which are end points of arrows of Γ_1 starting in x.

For each arrow $\alpha : x \to y$ with $y \in P_0$, there is precisely one arrow $\beta : \tau y \to x$. If we denote $\sigma\alpha = \beta$, we obtain a bijective map

$$\sigma : \{\alpha \in \Gamma_1 \mid d_0(\alpha) \in P_0\} \longrightarrow \{\beta \in \Gamma_1 \mid d_1(\beta) \in \tau(P_0)\}.$$

If P and τP denote the full subquivers of Γ with vertex sets P_0, $(\tau P)_0 = \tau(P_0)$ and arrow sets $P_1 = d_0^{-1}(P_0) \cap d_1^{-1}(P_0)$, $(\tau P)_1 = d_0^{-1}(\tau(P_0)) \cap d_1^{-1}(\tau(P_0))$, then there is a unique isomorphism $\pi : P \to \tau P$ of quivers extending τ by setting $\pi(\alpha) = \sigma^2(\alpha)$.
A morphism of representation quivers (Γ, τ) and (Γ', τ') is a morphism $f : \Gamma \to \Gamma'$ of quivers such that $f_0(\tau(x)) = \tau'(f_0(x))$ for all $x \in P_0$.
A *valued representation quiver* is a representation quiver (Γ, τ) together with a valuation ν such that $\nu(\sigma\alpha) = (v'_\alpha, v_\alpha)$ for all $\alpha : x \to y$ in Γ_1 with $y \in P_0$ and $\nu(\alpha) = (\nu_\alpha, \nu'_\alpha)$.
A *morphism* of valued representation quivers is both a morphism of valued quivers and a morphism of representation quivers.
A representation quiver (Γ, τ) is connected if $\Gamma_0 \neq \emptyset$ and it cannot be written as the disjoint union of two representation quivers. Observe that this does not imply that the underlying quiver is connected.
A representation quiver (Γ, τ) is called *stable* if $P_0 = \Gamma_0$ (i.e., τ is defined everywhere on Γ_0) and τ is surjective.
A vertex x of a representation quiver (Γ, τ) is called *τ-periodic* if $\tau^n(x) = x$ for some positive integer n.

Example 2.2.1. (1) The AR-quiver of a finite-dimensional algebra Λ is a valued representation quiver in the following way: If we define P_0 to be the set of vertices of $\Gamma(\Lambda)$ corresponding to the isomorphism classes of indecomposable Λ-modules that are not projective, then the AR-translate $\tau = D\mathrm{Tr}$ induces an injective map $\tau : P_0 \to (\Gamma(\Lambda))_0$, $[M] \mapsto [D\mathrm{Tr}(M)]$.
(2) Another important example of a representation quiver is the following. Given an valued oriented tree (T, ν) (a valued quiver whose underlying graph is a tree), denote by $\mathbb{Z}T$ the quiver with vertex set $(\mathbb{Z}T)_0 = \mathbb{Z} \times T_0$. Whenever there is an arrow $\alpha : x \to y$ in T_1, there are arrows $(n, \alpha) : (n, x) \to (n, y)$ and $\sigma(n, \alpha) : (n + 1, y) \to (n, x)$ in $(\mathbb{Z}T)_1$. Define a valuation μ on $\mathbb{Z}T$ by setting $\mu_{(n,\alpha)} = \mu'_{\sigma(n,\alpha)} = \nu_\alpha$ and $\mu'_{(n,\alpha)} = \mu_{\sigma(n,\alpha)} = \nu'_\alpha$. If we set $\tau((n, x)) = (n + 1, x)$, then $\mathbb{Z}T$ becomes a valued stable representation quiver.

Let Γ be a (valued) representation quiver. A subgroup $G \subseteq \mathrm{Aut}(\Gamma)$ of Γ is called *admissible* if each orbit of G in Γ_0 intersects a set of the form $x \cup x^+$ or $x^- \cup x$, $x \in \Gamma_0$, in at most one element.
One has the following structure theorem for stable valued representation quivers, which is due to C. Riedtmann:

Theorem 2.2.2. *(Riedtmann structure theorem) Let T and T' be valued oriented trees. Then $\mathbb{Z}T$ and $\mathbb{Z}T'$ are isomorphic as stable valued representation quivers if and only if the undirected graphs $\bar{T}$ and $\bar{T}'$ obtained from T and T' are isomorphic.*

Furthermore, if Γ *is a connected stable valued representation quiver, there is a valued oriented tree and an admissible group* G *of automorphism of* $\mathbb{Z}T$ *such that* Γ *is isomorphic to* $\mathbb{Z}T/G$ *as stable valued representation quivers.*

Proof. The statement can be found in §2 of [49] and it follows from the main theorem of [77, Struktursatz], where the result is shown for Γ a connected stable representation quiver without valuation. □

Remark 2.2.3. Up to conjugation in $\mathrm{Aut}(\mathbb{Z}T)$, the group G of the above theorem is uniquely determined by Γ. Moreover, the undirected tree $\bar{T}$ is uniquely determined up to isomorphism by Γ.

Definition 2.2.4. Let Γ be a connected stable valued representation quiver. The isomorphism type of the undirected tree $\bar{T}$ of Theorem 2.2.2 is called the *tree class* of Γ.

Recall from Example 2.2.1 (1) that the AR-quiver $\Gamma(\Lambda)$ of a finite-dimensional $\Bbbk$-algebra Λ is a valued representation quiver, which is not stable since the translation τ, which is induced from the AR-translation DTr, is not defined on projective vertices.
The full subquiver of $\Gamma(\Lambda)$ obtained from $\Gamma(\Lambda)$ by deleting the projective and injective vertices together with all τ-orbits starting and ending in such vertices, yields a stable valued representation quiver, called the *stable part* of $\Gamma(\Lambda)$ and is denoted by $\Gamma_s(\Lambda)$. Moreover, if $\mathcal{C}$ is a component of $\Gamma(\Lambda)$, we may also remove the τ-orbits of the projective and injective vertices in $\mathcal{C}$. The resulting quiver is denoted by $\mathcal{C}^s$ and is called the *stable part* of $\mathcal{C}$. The latter is then a stable valued representation quiver.

Remark 2.2.5. For arbitrary Artin algebras, one loses much information of $\Gamma(\Lambda)$ by only considering its stable part since the latter might even be empty.

By a component $\mathcal{C}$ of $\Gamma_s(\Lambda)$ we mean a full subquiver of $\Gamma_s(\Lambda)$, which is a stable valued representation quiver. In particular, if $x \in \mathcal{C}$, then also $\tau(x) \in \mathcal{C}$.
There is the following important result about the structure of a connected component of $\Gamma_s(\Lambda)$ containing τ-periodic modules, see [49, Main theorem].

Theorem 2.2.6. *Let* $\mathcal{C}$ *be a connected component of* $\Gamma_s(\Lambda)$*. Suppose that* $\mathcal{C}$ *contains a* τ*-periodic vertex. Then the tree class of* $\mathcal{C}$ *is either a finite Dynkin diagram or* A_∞*.*

Suppose now that Λ is a finite-dimensional selfinjective $\Bbbk$-algebra, i.e., the regular left Λ-module Λ is also an injective Λ-module. Thus, every finite-dimensional projective indecomposable Λ-module P is also injective, and it follows that the modules $P/\mathrm{rad}(P)$ and $\mathrm{soc}(P)$ are irreducible. Moreover, it follows that there are no τ-orbits in the AR-quiver of Λ starting or ending in an indecomposable projective (injective) module. Therefore, the subset $P_0 \subseteq (\Gamma(\Lambda))_0$ consists of vertices corresponding to the isomorphism classes of indecomposable Λ-modules that are neither projective nor injective. We may define a map

$$\tau^{-1} : P_0 \to P_0, \quad [M] \mapsto [\mathrm{Tr}D(M)].$$

Then, $\tau^{-1}(\tau([M])) = [M] = \tau(\tau^{-1}([M]))$ for $[M] \in P_0$.
Furthermore, to obtain the stable part $\Gamma_s(\Lambda)$ of $\Gamma(\Lambda)$ we just have to delete the vertices corresponding to the isomorphism classes of indecomposable projective (injective) modules and all arrows attached to them. If P is such a module with $l(P) \geq 2$, that is to say, P does not belong to a semisimple block of Λ, we know from Proposition 2.1.10 that the only AR-sequence where P occurs is isomorphic to the sequence

$$\delta : 0 \longrightarrow \mathrm{rad}(P) \xrightarrow{\binom{-i}{p}} P \oplus \mathrm{rad}(P)/\mathrm{soc}(P) \xrightarrow{(q,j)} P/\mathrm{soc}(\mathrm{P}) \longrightarrow 0.$$

The vertices corresponding to $\mathrm{rad}(P)$ and $P/\mathrm{soc}(P)$ are visible in $\Gamma_s(\Lambda)$ and so one can attach P there. Hence, we may reconstruct $\Gamma(\Lambda)$ from $\Gamma_s(\Lambda)$ in the selfinjective case. Therefore, in this case the shape of $\Gamma(\Lambda)$ can be determined from that of $\Gamma_s(\Lambda)$.

It will be also useful to know when the middle term of an AR-sequence is projective:

Proposition 2.2.7. *Let Λ be an indecomposable selfinjective $\Bbbk$-algebra. Then the following are equivalent:*

(a) *There is an AR-sequence with projective middle term.*

(b) *All AR-sequences have projective middle term.*

(c) Λ *is a Nakayama algebra of Loewy length two.*

(d) Λ *has Loewy length two.*

Proof. This is [7, X, Proposition 1.8]. □

Remark 2.2.8. (1) Note that if Λ is indecomposable selfinjective and one of the equivalent statements of the last proposition holds, then Λ is of finite representation type.
(2) It follows that if Λ is indecomposable selfinjective and not of finite representation type, then every connected component of $\Gamma_s(\Lambda)$ is also connected as a quiver. Thus, such a component can also be considered as the stable part of a component of $\Gamma(\Lambda)$.

2.3 The syzygy functor and stable equivalence

Next, recall that for a finite-dimensional selfinjective $\Bbbk$-algebra Λ one has the functors

$$\Omega_\Lambda : \Lambda - \underline{\mathrm{mod}} \to \Lambda - \underline{\mathrm{mod}}, \tag{2.4}$$

and

$$\Omega_\Lambda^{-1} : \Lambda - \underline{\mathrm{mod}} \to \Lambda - \underline{\mathrm{mod}}, \tag{2.5}$$

called the *syzygy functor* and *cosyzygy functor*. On objects, Ω_Λ is defined as follows: For each M in $\Lambda - \mathrm{mod}$, we fix a projective cover

$$P \xrightarrow{p} M \longrightarrow 0$$

in $\Lambda - \mathrm{mod}$, and define $\Omega_\Lambda(M)$ to be the Λ-module $\mathrm{Ker}(p)$. Dually, for each M in $\Lambda - \mathrm{mod}$, let us fix an injective envelope

$$0 \longrightarrow M \xrightarrow{j} I$$

in $\Lambda - \mathrm{mod}$. Then $\Omega_\Lambda^{-1}(M)$ is defined as the module $\mathrm{Coker}(j)$. For the definition of Ω_Λ and Ω_Λ^{-1} on homomorphisms, see [7, IV.3]. As the notation suggests, one has the following:

Proposition 2.3.1. *Suppose that Λ is selfinjective. Then the functors $\Omega_\Lambda : \Lambda - \underline{\mathrm{mod}} \to \Lambda - \underline{\mathrm{mod}}$ and $\Omega_\Lambda^{-1} : \Lambda - \underline{\mathrm{mod}} \to \Lambda - \underline{\mathrm{mod}}$ are inverse equivalences.*

Proof. This is [7, IV, Proposition 3.5]. □

Remark 2.3.2. (1) Note that if Λ is selfinjective, the functor

$$\mathrm{Hom}_\Lambda(-,\Lambda) : \Lambda - \mathrm{mod} \longrightarrow \Lambda^{\mathrm{op}} - \mathrm{mod}$$

is a duality with dual inverse $\mathrm{Hom}_{\Lambda^{\mathrm{op}}}(-,\Lambda) : \Lambda^{\mathrm{op}} - \mathrm{mod} \longrightarrow \Lambda - \mathrm{mod}$, see [7, IV, Proposition 3.4].
(2) One can show that the functor Ω_Λ^{-1} is isomorphic to the functor

$$\mathrm{Hom}_{\Lambda^{\mathrm{op}}}(-,\Lambda)\Omega_{\Lambda^{\mathrm{op}}}\mathrm{Hom}_\Lambda(-,\Lambda) : \Lambda - \underline{\mathrm{mod}} \longrightarrow \Lambda - \underline{\mathrm{mod}},$$

see [7, IV.3].

It follows from the remark that the composition of the functors $\mathrm{Hom}_\Lambda(-,\Lambda)$ and D yields an equivalence

$$\mathcal{N} := D\mathrm{Hom}_\Lambda(-,\Lambda) : \Lambda - \mathrm{mod} \longrightarrow \Lambda - \mathrm{mod}, \tag{2.6}$$

which is called the *Nakayama automorphism* on $\Lambda - \mathrm{mod}$. Then the following holds:

Proposition 2.3.3. *Let Λ be a selfinjective $\Bbbk$-algebra. Then:*

(a) *The functors $D\mathrm{Tr}$, $\Omega_\Lambda^2\mathcal{N}$, $\mathcal{N}\Omega_\Lambda^2 : \Lambda - \underline{\mathrm{mod}} \to \Lambda - \underline{\mathrm{mod}}$ are isomorphic.*

(b) *The functors $\mathrm{Tr}D$, $\Omega_\Lambda^{-2}\mathcal{N}^{-1}$, $\mathcal{N}^{-1}\Omega_\Lambda^{-2} : \Lambda - \underline{\mathrm{mod}} \to \Lambda - \underline{\mathrm{mod}}$ are isomorphic.*

Proof. This is the content of [7, IV, Proposition 3.7]. □

Remark 2.3.4. In the case when Λ is symmetric, i.e., $\Lambda \cong D(\Lambda)$ as two-sided Λ-modules, we have that $\mathcal{N} \cong 1_{\Lambda-\mathrm{mod}}$ as functors and, thus, $D\mathrm{Tr} \cong \Omega_\Lambda^2$ and $\mathrm{Tr}D \cong \Omega_\Lambda^{-2}$ as functors from $\Lambda - \underline{\mathrm{mod}}$ to $\Lambda - \underline{\mathrm{mod}}$, see [7, IV, Proposition 3.8].

Next, we describe another crucial concept, which will be important in our considerations later on.

Definition 2.3.5. Two finite-dimensional $\Bbbk$-algebras Λ and Λ' are said to be *stably equivalent* if there is an equivalence $F : \Lambda - \underline{\mathrm{mod}} \to \Lambda' - \underline{\mathrm{mod}}$.

In this case, F is called a *stable equivalence*. Given a stable equivalence $F : \Lambda - \underline{\mathrm{mod}} \to \Lambda' - \underline{\mathrm{mod}}$, one also has an induced correspondence between $\Lambda - \mathrm{mod}_{\mathcal{P}}$ and $\Lambda' - \mathrm{mod}_{\mathcal{P}}$, which is also denoted by F.

Example 2.3.6. If Λ is selfinjective, by Proposition 2.3.1, the syzygy functor Ω_Λ defines a stable equivalence $\Lambda - \underline{\mathrm{mod}} \to \Lambda - \underline{\mathrm{mod}}$.

In the following we denote by $\underline{f}$ the image of a homomorphism f in $\Lambda - \mathrm{mod}$ under the natural functor $\Lambda - \mathrm{mod} \to \Lambda - \underline{\mathrm{mod}}$. We then have the following statement, which is [7, Lemma 1.2, X].

Proposition 2.3.7. *Let $F : \Lambda - \underline{\mathrm{mod}} \to \Lambda' - \underline{\mathrm{mod}}$ be a stable equivalence between finite-dimensional $\Bbbk$-algebras and A, B in $\Lambda - \mathrm{mod}_{\mathcal{P}}$. If A and B are indecomposable, then F induces an isomorphism $\mathrm{Irr}(A, B) \cong \mathrm{Irr}(F(A), F(B))$ of $\Bbbk$-vector spaces.*

Moreover, one has the following:

Proposition 2.3.8. *Let $F : \Lambda - \underline{\mathrm{mod}} \to \Lambda' - \underline{\mathrm{mod}}$ be a stable equivalence between finite-dimensional $\Bbbk$-algebras. If $f : A \to B$ is a homomorphism in $\Lambda - \mathrm{mod}_{\mathcal{P}}$ with A or B indecomposable, let $f' : F(A) \to F(B)$ be such that $F(\underline{f}) = \underline{f'}$. Then the following hold:*

(a) *$f : A \to B$ is irreducible in $\Lambda - \mathrm{mod}$ if and only if $f' : F(A) \to F(B)$ is irreducible in $\Lambda' - \mathrm{mod}$.*

(b) *If B is indecomposable in $\Lambda - \mathrm{mod}$, then the following are equivalent:*

  (i) *There exists a homomorphism $g : P \to B$ with P projective in $\Lambda - \mathrm{mod}$ such that $(f, g) : A \oplus P \to B$ is minimal right almost split.*

  (ii) *There exists a homomorphism $h : Q \to F(B)$ with Q projective in $\Lambda' - \mathrm{mod}$ such that $(f', h) : F(A) \oplus Q \to F(B)$ is minimal right almost split.*

(c) *If A is indecomposable in $\Lambda - \mathrm{mod}$, then the following are equivalent:*

  (i) *There exists a homomorphism $g : A \to P$ with P projective in $\Lambda - \mathrm{mod}$ such that $\binom{f}{g} : A \to B \oplus P$ is minimal left almost split.*

(ii) *There exists a homomorphism* $h : F(A) \to Q$ *with* Q *projective in* $\Lambda' -$ mod *such that* $\binom{f'}{h} : F(A) \to F(B) \oplus Q$ *is minimal left almost split.*

Proof. This is [7, X, Proposition 1.3]. ☐

Remark 2.3.9. Let Λ be selfinjective, and consider the stable equivalence

$$\Omega = \Omega_\Lambda : \Lambda - \underline{\mathrm{mod}} \to \Lambda - \underline{\mathrm{mod}}.$$

Let $0 \to A \to B \to C \to 0$ be an exact sequence. Then there exists an exact commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \Omega(A) \arrow[r] \arrow[d] & \Omega(B) \oplus Q \arrow[r] \arrow[d] & \Omega(C) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & P(A) \arrow[r] \arrow[d] & P(A) \oplus P(C) \arrow[r] \arrow[d] & P(C) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & 0, \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

where $P(A) \to A$ and $P(C) \to C$ are projective covers in $\Lambda-$mod. If $0 \to A \to B \to C \to 0$ is an AR-sequence, then the modules A and C are indecomposable and thus, by [7, IV, Proposition 3.6], so are $\Omega(A)$ and $\Omega(C)$. Moreover, from Proposition 2.3.8, we infer that the top row of the diagram above is an AR-sequence. In other words, if C in $\Lambda -$ mod is indecomposable and not projective, then $\tau(\Omega(C)) \cong \Omega(\tau(C))$.

The following proposition describes what happens to the stable AR-quiver of a finite-dimensional selfinjective $\Bbbk$-algebra under stable equivalences. This will be essential in our further investigations.

Proposition 2.3.10. *Suppose that* Λ *and* Λ' *are finite-dimensional selfinjective* $\Bbbk$-*algebras and let* $F : \Lambda - \underline{\mathrm{mod}} \to \Lambda' - \underline{\mathrm{mod}}$ *be an equivalence. Moreover, assume that* Λ *and* Λ' *have no block of Loewy length* 2. *Then* F *induces an isomorphism of stable valued representation quivers between* $\Gamma_s(\Lambda)$ *and* $\Gamma_s(\Lambda')$.

Proof. This is precisely statement (b) of [7, X, Corollary 1.9]. ☐

Chapter 3

Stable Auslander–Reiten components for skew group algebras

The goal of this chapter is to introduce an important class of algebras, called *skew group algebras*. These algebras can be thought of as generalizations of group rings, and they will become important to us in the investigation of the possible shapes of the stable AR-components of Hecke algebras of type A in characteristic zero.
First of all, we will give a rough survey of the representation theory of the latter, in particular, we will explain how the representation theory of the latter and that of the underlying algebra are intertwined. Afterwards, we will prove a result, giving a criterion when certain tree classes of stable AR-components are preserved under the skew group construction. For the definitions and basic results concerning AR-sequences, we refer to Chapter 2.
Throughout this chapter, Λ denotes a finite-dimensional selfinjective $\Bbbk$-algebra over a fixed field $\Bbbk$. Furthermore, G will denote a finite group.

3.1 Skew group algebras

We will state the crucial properties of skew group algebras that are needed for our purposes. Most of the results stated in this section can be found in [79].

Recall that a $\Bbbk$-algebra action of G on Λ is given by a map

$$\phi : G \times \Lambda \longrightarrow \Lambda,$$

satisfying the following conditions, where we write $\phi(g, \lambda) = g(\lambda)$, for $g \in G$ and $\lambda \in \Lambda$.

(i) The map $g : \Lambda \to \Lambda$ is a $\Bbbk$-algebra automorphism for all $g \in G$.

(ii) $(g_1g_2)(\lambda) = g_1(g_2(\lambda))$, for $g_1, g_2 \in G$ and $\lambda \in \Lambda$.

(iii) $e(\lambda) = \lambda$ for all $\lambda \in \Lambda$, where e is the identity element in G.

Remark 3.1.1. From this definition, it follows that ϕ induces a group homomorphism $G \to \mathrm{Aut}_\Lambda(\Lambda)$.

Definition 3.1.2. Given a $\Bbbk$-algebra action ϕ of G on Λ, the *skew group algebra* of G over Λ, which is denoted by $\Lambda \star G$, is the free left Λ-module with the elements of G as a basis. Furthermore, the multiplication on $\Lambda \star G$ is defined by the rule

$$(\lambda_g \star g)(\lambda_h \star h) = (\lambda_g g(\lambda_h)) \star (gh),$$

for all λ_g, λ_h in Λ and g, h in G.

Remark 3.1.3. From the definition it follows immediately that $\Lambda \star G$ is a finite-dimensional unitary $\Bbbk$-algebra with unit $1 \star e$. If $\{b_1, \ldots, b_r\}$ is a basis of Λ as a $\Bbbk$-vector space, then the elements $\{b_j \star g \mid 1 \leq j \leq r, g \in G\}$ form a basis of $\Lambda \star G$ as a $\Bbbk$-vector space.

Next we want to describe important functors that relate the representation theory of Λ to that of $\Lambda \star G$. First of all there is a natural monomorphism of $\Bbbk$-algebras

$$i : \Lambda \longrightarrow \Lambda \star G,$$

given by $i(\lambda) = \lambda \star e$. Throughout we will consider Λ as a unitary $\Bbbk$-subalgebra of $\Lambda \star G$ with respect to i. With respect to this embedding, we then get a functor

$$F : \Lambda - \mathrm{mod} \longrightarrow \Lambda \star G - \mathrm{mod}, \tag{3.1}$$

defined on objects by $F(M) = \Lambda \star G \otimes_\Lambda M$, for all M in $\Lambda-$mod, called the *induction* from Λ to $\Lambda \star G$.
If $f : M \to N$ is a morphism in $\Lambda -$ mod, $F(f)$ is defined as the map

$$\mathrm{id}_{\Lambda \star G} \otimes f : \Lambda \star G \otimes_\Lambda M \to \Lambda \star G \otimes_\Lambda N,$$

which then becomes a morphism in $\Lambda \star G -$ mod.
Moreover, we define a functor

$$H : \Lambda \star G - \mathrm{mod} \longrightarrow \Lambda - \mathrm{mod}, \tag{3.2}$$

by setting $H(N) = \mathrm{Res}_\Lambda^{\Lambda \star G}(N)$, for all N in $\Lambda \star G -$ mod. It is called the *restriction* from $\Lambda \star G$ to Λ.
If $f : X \to Y$ is a morphism in $\Lambda \star G -$ mod, then $H(f)$ is the same as f considered as $\Bbbk$-linear homomorphism, and since we consider Λ as a $\Bbbk$-subalgebra of $\Lambda \star G$ through i, it then becomes a morphism in $\Lambda -$ mod.

Assumption. Throughout this chapter, we assume that the order $|G|$ of G is invertible in $\Bbbk$.

Under this additional assumption, the functors defined above have nice properties, and we are going to record them in the sequel.

Proposition 3.1.4. *Let F and H be as above. Then the following hold:*

(a) (H, F) *is an adjoint pair of functors.*

(b) *The natural morphism $I \to HF$ of functors is a split monomorphism, where I is the identity functor of* $\Lambda - \text{mod}$.

(c) *The natural morphism $FH \to J$ of functors is a split epimorphism, where J is the identity functor of* $\Lambda \star G - \text{mod}$.

(d) $\text{rad}(\Lambda)(\Lambda \star G) = (\Lambda \star G)\text{rad}(\Lambda) = \text{rad}(\Lambda \star G)$.

Proof. This is [79, Theorem 1.1]. □

From the last proposition, one gets the following consequences:

Proposition 3.1.5. *Under our assumptions on Λ and G, the following hold.*

(i) $\Lambda \star G$ *is free as a left and right Λ-module.*

(ii) Λ *is of finite representation type if and only if $\Lambda \star G$ is.*

(iii) $\Lambda \star G$ *is a selfinjective $\Bbbk$-algebra.*

Proof. This is [79, Theorem 1.3] and [7, III, Lemma 4.5]. □

With this information, one is now able to transfer properties of $\Lambda - \text{mod}$ to $\Lambda \star G - \text{mod}$. In particular, one is interested in what happens to almost split sequences, almost split maps and irreducible morphisms under the functors F and H.

Through the action of G on Λ we also obtain an action of G on $\Lambda - \text{mod}$, which we will describe next. If g is an element of G and X is in $\Lambda - \text{mod}$, we define ${}^{g}X$ as the Λ-module that has the same underlying $\Bbbk$-vector space as X together with the Λ-module structure given by $\lambda \centerdot x := g^{-1}(\lambda)x$.
Observe that the subset $g \otimes X := (1 \star g) \otimes X = \{(1 \star g) \otimes x | \; x \in X\}$ of $\Lambda \star G \otimes_{\Lambda} X$ carries a Λ-module structure via

$$\begin{aligned}(\lambda \star e)((1 \star g) \otimes x) &= (\lambda \star g) \otimes x \\ &= [(1 \star g)(g^{-1}(\lambda) \star e)] \otimes x \\ &= (1 \star g) \otimes g^{-1}(\lambda)x.\end{aligned}$$

It follows that ${}^{g}X$ and $g \otimes X$ are isomorphic as Λ -modules. If we have a morphism $f : X \to Y$ in $\Lambda - \text{mod}$, then there is an induced morphism ${}^{g}f : {}^{g}X \to {}^{g}Y$ in $\Lambda - \text{mod}$ defined by ${}^{g}f(x) = f(x)$, for all $x \in X$. Thus, considered as $\Bbbk$-vector space homomorphisms, ${}^{g}f$ equals f. To prove that it is a Λ-homomorphism, let $\lambda \in \Lambda$. Then

$$\begin{aligned}{}^{g}f(\lambda \centerdot x) &= {}^{g}f(g^{-1}(\lambda)x) = f(g^{-1}(\lambda)x) \\ &= g^{-1}(\lambda)f(x) = g^{-1}(\lambda)({}^{g}f(x)) \\ &= \lambda \centerdot ({}^{g}f(x)).\end{aligned}$$

With this definition, for a given $g \in G$, we get a $\Bbbk$-linear functor $\mathcal{G} : \Lambda - \mathrm{mod} \to \Lambda - \mathrm{mod}$ defined by

$$\mathcal{G}(X) = {}^{g}X \text{ and } \mathcal{G}(f) = {}^{g}f.$$

We have the following:

Lemma 3.1.6. *For all $g \in G$, the functor $\mathcal{G} : \Lambda - \mathrm{mod} \to \Lambda - \mathrm{mod}$ induces an equivalence of $\Bbbk$-categories.*

Proof. By [7, II, Theorem 1.2], it is enough to show that $\mathcal{G}$ is full, faithful and dense. Since ${}^{g}f$ and f coincide as $\Bbbk$-vector space homomorphisms it follows that $\mathcal{G}$ is faithful. To show that $\mathcal{G}$ is full, let $h : {}^{g}X \to {}^{g}Y$ be a morphism in $\Lambda - \mathrm{mod}$. Now define $f : X \to Y$ to be the same $\Bbbk$-vector space homomorphism as h. Then, with λ in Λ, we have

$$f(\lambda x) = f(g^{-1}(g(\lambda))x) = h(g(\lambda) \centerdot x) = g(\lambda) \centerdot h(x) = \lambda f(x).$$

So f is a morphism in $\Lambda - \mathrm{mod}$, and it follows that ${}^{g}f = h$, i.e., $\mathcal{G}$ is full. Since $\mathcal{G}({}^{g^{-1}}X) = {}^{g}({}^{g^{-1}}X) \cong {}^{e}X \cong X$, the functor $\mathcal{G}$ is also dense. Hence, the claim follows. □

Remark 3.1.7. In particular, we infer that $\mathcal{G}$ sends irreducible morphisms to irreducible morphisms, and almost split sequences to almost split sequences.

The following proposition is crucial in our investigations.

Proposition 3.1.8. *Let X and Y be indecomposable modules in $\Lambda - \mathrm{mod}$. With the notation as above the following holds for the functors H and F:*

(a) *$HF(X) \cong \bigoplus_{g \in G} {}^{g}X$ as Λ-modules.*

(b) *$FX \cong FY$ if and only if $X \cong {}^{g}Y$, for some g in G.*

Proof. This is part of [79, Proposition 1.8]. □

As a consequence, one obtains the following remarkable properties of the functors F and H.

Proposition 3.1.9. *The functors F and H preserve projectivity, projective covers and semisimple modules.*

Proof. Since, by Proposition 3.1.5 (a), $\Lambda \star G$ is free as a right Λ-module, the functor F preserves projectivity. Now let P in $\Lambda \star G - \mathrm{mod}$ be projective. Then P is isomorphic to a direct summand of a finite direct sum of copies of the regular $\Lambda \star G$-module $\Lambda \star G$. Again by Proposition 3.1.5 (a), $\Lambda \star G$ is free as a left Λ-module, hence, by the additivity of H, the Λ-module $H(P)$ is a direct summand of a direct sum of copies of the regular Λ-module Λ, thus projective.
To show that F preserves projective covers assume that X is an indecomposable Λ-module and let $P \to X$ be a projective cover of X in $\Lambda - \mathrm{mod}$. Applying F

yields an epimorphism $FP \to FX$ in $\Lambda \star G - \text{mod}$. Then we choose a projective cover $Y \to FX$ of FX in $\Lambda \star G - \text{mod}$. It follows that Y is isomorphic to a direct summand of FP, and applying H affords an epimorphism $HY \to HFX$ in $\Lambda - \text{mod}$, with HY projective since H preserves projectivity. Now, by Proposition 3.1.8 (a), the Λ-module HFX is isomorphic to $\bigoplus_{g\in G} {}^gX$. Since conjugation is an equivalence of $\Lambda - \text{mod}$ to itself, the projective cover of a conjugate ${}^gX$ is just ${}^gP$, and, hence, $HFP = \bigoplus_{g\in G} {}^gP$ is a projective cover of HFX. Since HY is isomorphic to a direct summand of HFP, it follows that $HY \cong HFP$. Now, since Y is a direct summand of FP it follows by counting dimensions that $Y \cong FP$, so FP is a projective cover of FX, and the claim follows.
The corresponding property for H follows from Proposition 3.1.4 (d) and [7, I, Proposition 4.3]. □

The next proposition shows how almost split sequence behave under induction and restriction.

Proposition 3.1.10. *With the notation as above we have:*

(a) *If $0 \to X \to Y \to Z \to 0$ is an almost split sequence in $\Lambda - \text{mod}$, then the exact sequence $0 \to FX \to FY \to FZ \to 0$ is a direct sum of almost split sequences in $\Lambda \star G - \text{mod}$.*

(b) *If $X \to Y$ is a minimal left or minimal right almost split map in $\Lambda - \text{mod}$, then the map $FX \to FY$ is a direct sum of minimal left or minimal right almost split maps in $\Lambda \star G - \text{mod}$.*

(c) *If $0 \to X \to Y \to Z \to 0$ is an almost split sequence in $\Lambda \star G - \text{mod}$, then the exact sequence $0 \to HX \to HY \to HZ \to 0$ is a direct sum of almost split sequences in $\Lambda - \text{mod}$.*

(d) *If $X \to Y$ is a minimal left or minimal right almost split map in $\Lambda \star G - \text{mod}$, then the map $HX \to HY$ is a direct sum of minimal left or minimal right almost split maps in $\Lambda - \text{mod}$.*

Proof. This is Theorem 3.8 in [79] in the more general context of dualizing $\Bbbk$-varieties, but we will give the arguments in our case. By Proposition 3.1.8 we know that $HF(M) \cong \bigoplus_{g\in G} {}^gM$, for $M \in \Lambda - \text{mod}$. Moreover, if $f : X \to Y$ is a homomorphism in $\Lambda - \text{mod}$, from the definition of $F(f)$ it is easy to see that $HF(f)$ decomposes as

$$HF(f) = (f, {}^{g_2}f, \dots, {}^{g_n}f) : {}^eX \oplus {}^{g_2}X \oplus \cdots \oplus {}^{g_n}X \longrightarrow {}^eY \oplus {}^{g_2}Y \oplus \cdots \oplus {}^{g_n}Y,$$

where $G = \{g_1 = e, g_2, \dots, g_n\}$. It follows from [79, Theorem 3.6] that the induced functors

$$\begin{aligned} \hat{F} &: (\Lambda - \text{mod}) - \text{mod} \longrightarrow (\Lambda \star G - \text{mod}) - \text{mod}, \\ \hat{H} &: (\Lambda \star G - \text{mod}) - \text{mod} \longrightarrow (\Lambda - \text{mod}) - \text{mod}, \end{aligned}$$

defined by

$$\hat{F}(\mathrm{Hom}_\Lambda(-, M)) = \mathrm{Hom}_{\Lambda\star G}(-, FM),$$

and

$$\hat{H}(\mathrm{Hom}_{\Lambda\star G}(-, N)) = \mathrm{Hom}_\Lambda(-, HN),$$

for M in $\Lambda - \mathrm{mod}$ and N in $\Lambda \star G - \mathrm{mod}$, preserve semisimple objects and projective covers. Using the functorial description of AR-sequences and minimal left or right almost split maps, see for example [8, §4.12], the claim follows. $\square$

Next, let X be an indecomposable Λ-module. We denote by $[X]$ the Λ-modules in the G-orbit of X, i.e., $[X] = \{{}^gX \mid g \in G\}$. For an indecomposable module Z in $\Lambda \star G - \mathrm{mod}$ we choose an indecomposable module X in $\Lambda - \mathrm{mod}$ such that Z is a direct summand of $\Lambda \star G \otimes_\Lambda X$. This is possible since the natural morphism $FH \to J$ is a split epimorphism, following Proposition 3.1.4 (c). Denote by $[Z]$ the set of isomorphism classes of non-isomorphic indecomposable direct summands of $\Lambda \star G \otimes_\Lambda X$. In this situation we write $[\widetilde{X}] = [Z]$. Then we have: $[\widetilde{X}] = [Z]$ if and only if $Z \mid FX$ if and only if ${}^gX \mid HZ$, for some g in G. For the last equivalence, let W be in $\Lambda - \mathrm{mod}$ indecomposable such that $Z \mid FW$. Then we have ${}^gX \mid HFW$ and since HFW is a direct sum of conjugates of the Λ-module W, we must have $W \cong {}^hX$ for some h in G, and so $FW \cong FX$, by 3.1.8 (b).

Notation 3.1.11. If there is an irreducible morphism $X' \to Y'$ in $\Lambda - \mathrm{mod}$, for objects X' in $[X]$ and Y' in $[Y]$, then we say that there is an irreducible morphism $[X] \to [Y]$ between $[X]$ and $[Y]$. Similarly, we say that there is an irreducible map $[\tilde{X}] \to [\tilde{Y}]$ if there is an irreducible map $Z \to U$ in $\Lambda \star G - \mathrm{mod}$, for $Z \in [\tilde{X}]$ and $U \in [\tilde{Y}]$.

With this notation the following holds, see [79, Lemma 4.1].

Lemma 3.1.12. *If X and Y are indecomposable objects in $\Lambda - \mathrm{mod}$, then the following statements are equivalent:*

(a) *There is an irreducible map $[X] \to [Y]$.*

(b) *Given X' in $[X]$, there are some Y' in $[Y]$ and an irreducible map $X' \to Y'$.*

(c) *Given Y' in $[Y]$, there are some X' in $[X]$ and an irreducible map $X' \to Y'$.*

(d) *There is an irreducible map $[\widetilde{X}] \to [\widetilde{Y}]$.*

(e) *Given Z in $[\widetilde{X}]$, there are some U in $[\widetilde{Y}]$ and an irreducible map $Z \to U$.*

(f) *Given U in $[\widetilde{Y}]$, there are some Z in $[\widetilde{X}]$ and an irreducible map $Z \to U$.*

Proof. First, we prove that (a) implies (b). By definition, there is an irreducible morphism $X'' \to Y''$ with X'' in $[X]$ and Y'' in $[Y]$. Since there is some g in G with $X' = {}^gX''$ and since conjugation by g affords an equivalence of categories, there is an irreducible morphism $X' \to {}^gY''$. The converse is just the definition of an irreducible morphism between $[X]$ and $[Y]$. In the same way, the equivalence of (b) and (c) follows.
Now let there be an irreducible morphism $[X] \to [Y]$. Therefore we have an irreducible morphism $X' \to Y'$ with X' in $[X]$ and Y' in $[Y]$. This map can be completed to a minimal right almost split map $X' \oplus W \to Y'$, with W in $\Lambda - \mathrm{mod}$. By Proposition 3.1.10, the map $FX' \oplus FW \to FY'$ is a direct sum of minimal right almost split maps in $\Lambda \star G - \mathrm{mod}$. Since, by definition, every Z in $[\widetilde{X'}]$ is a direct summand of FX', from Theorem 2.1.9, we infer that there is an irreducible morphism $Z \to U$ with U in $[\widetilde{Y'}]$, i.e., we have an irreducible morphism $[\widetilde{X}] \to [\widetilde{Y}]$, hence (a) implies (d).
For the converse, let there be an irreducible morphism $[\widetilde{X}] \to [\widetilde{Y}]$, i.e., there is an irreducible map $Z \to U$ with Z in $[\widetilde{X}]$ and U in $[\widetilde{Y}]$. This map can be completed to a minimal right almost split map $Z \oplus Z' \to U$, and by Proposition 3.1.10, the map $HZ \oplus HZ' \to HU$ is a direct sum of minimal right almost split morphisms. Since ${}^gX$ is a direct summand of HZ, for some g in G, by Theorem 2.1.9, there is an irreducible morphism ${}^gX \to Y'$, with Y' a direct summand of HU. Since HU is a direct summand of HFY, we have that Y' is isomorphic to a conjugate of Y, and the claim follows. The remaining equivalences are shown by using dual arguments. □

Remark 3.1.13. Denote by τ the AR-translation of Λ. For $g \in G$, observe that if

$$0 \longrightarrow \tau X \xrightarrow{f_1} E \xrightarrow{f_2} X \longrightarrow 0$$

is an almost split sequence in $\Lambda - \mathrm{mod}$, where X is not projective, then

$$0 \longrightarrow {}^g(\tau X) \xrightarrow{{}^gf_1} {}^gE \xrightarrow{{}^gf_2} {}^gX \longrightarrow 0$$

is an almost split sequence as well. We conclude that $\tau({}^gX) \cong {}^g(\tau X)$.
This yields that if $[X] = [X']$, then we have $[\tau X] = [\tau X']$. Thus, it is reasonable to define $\tau[X] := [\tau X]$.
Moreover, one defines

$$\tau[\widetilde{X}] = \{\tau Z \mid Z \in [\widetilde{X}]\}.$$

The next lemma is [79, Lemma 4.2], but we will give the arguments here since they become important in the next section.

Lemma 3.1.14. *If X is indecomposable and non-projective in $\Lambda-\mathrm{mod}$, then $\tau[\widetilde{X}] = [\widetilde{\tau X}]$.*

Proof. Let Z be in $[\widetilde{X}]$. Note that Z is not projective. Otherwise, we have that ${}^gX \mid HZ$, for some $g \in G$, and since H preserves projectivity, ${}^gX$ is projective.

Since conjugation by an element $g \in G$ induces an equivalence on $\Lambda - \text{mod}$, it follows that X is projective.
Next, consider the almost split sequences $0 \to \tau Z \to U \to Z \to 0$ and $0 \to \tau X \to E \to X \to 0$. By Proposition 3.1.10, the induced sequence $0 \to F(\tau X) \to FE \to FX \to 0$ is a direct sum of almost split sequences, and so $F(\tau X) \cong \tau FX$. By definition, the indecomposable summands of $F(\tau X)$ are those in $[\widetilde{\tau X}]$ and since Z is a direct summand of FX, it follows that τZ is in $[\widetilde{\tau X}]$. □

3.2 Stable AR-components

In the following let $\mathcal{D}$ be a component of the AR-quiver $\Gamma(\Lambda \star G)$ of $\Lambda \star G$ and $\mathcal{D}^s$ its stable part, see Section 2.2 for definitions. Let Z be a non-projective indecomposable $\Lambda \star G$-module belonging to the component $\mathcal{D}$ and choose an indecomposable Λ-module X such that Z is isomorphic to a direct summand of $\Lambda \star G \otimes_\Lambda X$. By our assumptions on Λ and G this is always possible. Now consider the component $\mathcal{C}$ of the AR-quiver that contains X, and observe that since Z is not projective, neither is X. In the following we will assume that the stable part $\mathcal{C}^s$ of $\mathcal{C}$ has tree class A_∞. Then we can choose an infinite sectional path

$$\ldots \to X_n \to \ldots \to X_1 \to X_0 \tag{3.3}$$

in $\mathcal{C}$, i.e., $\tau X_i \ncong X_{i+2}$ for all i, such that $X \cong X_n$ for some $n \geq 0$. Then, by assumption, all the modules occurring in (3.3) are not projective. We also may assume that X_0 lies at the end of $\mathcal{C}^s$, i.e., X_0 has exactly one predecessor in $\mathcal{C}^s$.
Now, since we have irreducible morphisms, $X_{n+1} \to X$ and $X \to X_{n-1}$, by Lemma 3.1.12, there are irreducible morphisms $Z_{n+1} \to Z$ and $Z \to Z_{n-1}$ with $Z_{n+1} \in [\widetilde{X_{n+1}}]$ and $Z_{n-1} \in [\widetilde{X_{n-1}}]$. It follows that we get a path

$$\ldots \to Z_n \to \ldots \to Z_1 \to Z_0 \tag{3.4}$$

in $\mathcal{D}$ with $Z \cong Z_n$ such that $Z_i \in [\widetilde{X_i}]$ for all i. Again, non of the modules occurring in (3.4) are projective. Now, we have the following result:

Proposition 3.2.1. *The path (3.4) in $\mathcal{D}$ is a sectional path.*

Proof. If not, then there is an index $i \geq 0$ such that $\tau Z_i \cong Z_{i+2}$. Since Z_i is in $[\widetilde{X_i}]$, it follows, by Lemma 3.1.14, that $Z_{i+2} \cong \tau Z_i$ is in $[\widetilde{\tau X_i}]$. But then it follows that Z_{i+2} is isomorphic to a direct summand of $\Lambda \star G \otimes_\Lambda \tau X_i$, and so HZ_{i+2} is isomorphic to a direct summand of $HF(\tau X_i) \cong \bigoplus_{g \in G} {}^g(\tau X_i)$. On the other hand, since Z_{i+2} is in $[\widetilde{X_{i+2}}]$, a conjugate ${}^hX_{i+2}$ of X_{i+2}, $h \in G$, is isomorphic to a direct summand of HZ_{i+2} and so ${}^hX_{i+2} \cong {}^p(\tau X_i)$, for some $p \in G$, by the Krull–Schmidt Theorem. This implies that ${}^lX_{i+2} \cong \tau X_i$, for $l = p^{-1}h \in G$. Since $\mathcal{C}$ is connected and conjugation by l is an equivalence on $\Lambda - \text{mod}$, it induces a graph automorphism ${}^l(-) : \mathcal{C} \to \mathcal{C}$. Furthermore, by restriction, we obtain an isomorphism ${}^l(-) : \mathcal{C}^s \to \mathcal{C}^s$ of stable translation quivers, which implies that ${}^lX_{i+2}$ must be located in the same row as X_{i+2} in $\mathcal{C}^s$. To this end, take a sectional path from the end of $\mathcal{C}^s$ to X_{i+2}. The length of such a path is uniquely determined by X_{i+2} since the tree class of $\mathcal{C}^s$ is A_∞, and this determines the row of X_{i+2}. Since conjugation is an equivalence, we obtain a sectional path from the end of $\mathcal{C}^s = {}^l(\mathcal{C}^s)$ to ${}^lX_{i+2}$ of the same length. By our assumption, X_{i+2} and τX_i are in different rows of $\mathcal{C}^s$, and so we have a contradiction. Therefore, the above path is a sectional path in $\mathcal{D}$. □

Corollary 3.2.2. *Under our assumptions, the component $\mathcal{D}$ does not belong to a block of $\Lambda \star G$ of finite representation type.*

Proof. Suppose $\mathcal{D}$ belongs to a block of finite representation type. Then, in the path of (3.4), only finitely many isomorphism classes of indecomposable modules occur. Since the latter path is infinite, there exists some i such that Z_i occurs in (3.4) infinitely often. In particular, we have a sequence

$$T \to Z_i \to Z_n \to \ldots \to Z_{i+1} \to Z_i$$

of irreducible maps in $\mathcal{D}$ that is sectional. Since the block is of finite representation type, we may assume that $T \cong Z_{i+1}$. We conclude that there is a sectional cycle in $\mathcal{D}$, contradicting the fact that there cannot exist a sectional cycle in the AR-quiver of an Artin algebra, see [7, VII, Corollary 2.6]. □

Remark 3.2.3. If we assume $\mathcal{C}$ to be non-periodic, that is to say, no module in $\mathcal{C}$ is τ-periodic, then it follows from [79, Theorem 4.3] that no module in the path of 3.4 occurs more than once.

For a given $W \in \Lambda - \text{mod}$ define a function $d = d_W : \Lambda - \text{mod} \to \mathbb{N} \cup \{0\}$ by

$$d(M) = d_W(M) = \dim_{\Bbbk} \underline{\mathrm{Hom}}_\Lambda(W, M).$$

Recall from Section 2.3 that for M in $\Lambda - \text{mod}$, the Λ-module $\Omega_\Lambda(M)$ is defined to be the kernel of a projective cover of M in $\Lambda - \text{mod}$. Since Λ is selfinjective, by Proposition 2.3.1, the induced functor

$$\Omega_\Lambda : \Lambda - \underline{\text{mod}} \longrightarrow \Lambda - \underline{\text{mod}}$$

is an equivalence of $\Bbbk$-categories. The inverse Ω_Λ^{-1} of Ω_Λ is defined on objects to be the cokernel of an injective envelope of M in $\Lambda - \text{mod}$. We will need the following lemma.

Remark 3.2.4. For a connected component Θ of $\Gamma_s(\Lambda)$, we call a function $d : \Theta \to \mathbb{N} \cup \{0\}$ additive if it is additive on AR-sequences, i.e., if M is in Θ, then

$$d(E) = d(M) + d(\tau M),$$

where

$$0 \longrightarrow \tau M \longrightarrow E \longrightarrow M \longrightarrow 0$$

is an AR-sequence.

Lemma 3.2.5. *Suppose that Θ is a connected component of $\Gamma_s(\Lambda)$ such that no indecomposable summand of W belongs to Θ or $\Omega_\Lambda(\Theta)$. Then d is an additive function on Θ.*

Proof. This is [37, Lemma 3.2]. □

Moreover, if we assume that W is τ-periodic, i.e, $\tau W \cong W$, we have isomorphisms of $\mathbb{k}$-vector spaces

$$\underline{\mathrm{Hom}}_\Lambda(W, \tau M) \cong \underline{\mathrm{Hom}}_\Lambda(\tau W, \tau M) \cong \underline{\mathrm{Hom}}_\Lambda(W, M), \tag{3.5}$$

and thus, d is constant on τ-orbits. Note that τ defines an equivalence of $\mathbb{k}$-categories from $\Lambda - \underline{\mathrm{mod}}$ to itself, see [7, IV, Proposition 1.9].
Suppose now that we have a non-zero additive function d on Θ that is constant on τ-orbits. Recall that a connected component Θ of $\Gamma_s(\Lambda)$ is called τ-periodic if each Λ-module in Θ is τ-periodic. The following is a consequence of [49, §2, Theorem].

Lemma 3.2.6. *For a non-τ-periodic connected component Θ of $\Gamma_s(\Lambda)$ we have the following:*

(i) *The tree class of Θ is either one of the infinite Dynkin diagrams A_∞, A_∞^∞, B_∞, C_∞, D_∞, or a Euclidean diagram.*

(ii) *If Θ has tree class A_∞, then d is unbounded.*

Proof. Let T be a directed tree and let Π be an admissible group of automorphisms of $\mathbb{Z}T$ such that $\mathbb{Z}T/\Pi \cong \Theta$ as stable translation quivers. By composition with the above isomorphism, d induces an additive function on $\mathbb{Z}T$ which is constant on τ-orbits. Thus, d induces an additive function on $\bar{T}$, the undirected tree obtained from T. Now the first part follows by [8, Theorem 4.5.8 (iii)].
For the second part, take a sectional path $\mathcal{P}$ in Θ and denote by s_i the value of d at the ith node of $\mathcal{P}$, where s_1 corresponds to the value of d on the node at the end of Θ. Since d is supposed to be non-zero, we have that $s_1 \neq 0$. We proceed by induction. Since d is additive on the associated tree of Θ, we have $2s_1 = s_2$, which shows that $s_1 < s_2$. For an arbitrary $n > 1$, we have

$$2s_n = s_{n-1} + s_{n+1},$$

and then

$$s_{n+1} - s_n = s_n - s_{n-1}.$$

Since, by the inductive hypothesis, $s_n - s_{n-1} > 0$, we conclude from the last equation that $s_{n+1} > s_n$. □

Remark 3.2.7. (1) Suppose that Θ does not belong to a block of Loewy length two. Then, if d is a non-zero additive function on Θ, that is to say, there is some $M \in \Lambda - \mathrm{mod}$ such that $d(M) \neq 0$, then it is non-zero everywhere. To this end,

suppose that $d(N) = 0$ for some $N \in \Lambda - \text{mod}$. Since d is constant on τ-orbits, d is zero on the whole τ-orbit of N. This implies that $d(E) = 0$, with E being the middle of the almost split sequence ending in N. This implies that d is zero on all τ-orbits joined to that of N. Inductively, we see that d must be zero on the whole of Θ since Θ is connected by [7, X, Proposition 1.8]. This shows that d is an additive function in the sense of [49].
(2) If we assume $\Bbbk$ to be algebraically closed, then the valuation (a, b) of an edge in the AR-quiver of Λ satisfies $a = b$. Therefore, the infinite Dynkin trees B_∞ and C_∞ do not occur in this case.

We return to the situation of the beginning of this section. From now on we will assume that the connected component $\mathcal{C}^s$ of $\Gamma_s(\Lambda)$ is not τ-periodic. Moreover, we assume that there exists a module $W \in \Lambda - \text{mod}$ such that $\tau W \cong W$ and

$$\underline{\text{Hom}}_\Lambda(W, M) \neq 0,$$

for some M in $\mathcal{C}^s$. Since $\mathcal{C}^s$ is not τ-periodic, no indecomposable direct summand of W belongs to $\mathcal{C}^s$ since direct summands of τ-periodic modules are again τ-periodic and components containing τ-periodic modules are τ-periodic. By [7, Chapter X, Corollary 1.9], the functor Ω_Λ induces an isomorphism $\mathcal{C}^s \to \Omega_\Lambda(\mathcal{C}^s)$ of stable translation quivers, and thus, by the same argument, also $\Omega_\Lambda(\mathcal{C}^s)$ cannot contain an indecomposable direct summand of W. Hence, by Lemma 3.2.5, the function d_W is additive and also constant on τ-orbits by (3.5).
Observe that a conjugate ${}^gW$ of W, $g \in G$, satisfies $\tau({}^gW) \cong {}^gW$, and hence, defines an additive function $d_{{}^gW}$ on the conjugate component ${}^g(\mathcal{C}^s)$, which is constant on τ-orbits. Then we define $V = \bigoplus_{g \in G} {}^gW$, and it follows that $\tau V \cong V$. From the proof of Lemma 3.1.14, we get that

$$\tau FV \cong F(\tau V) \cong FV,$$

which shows that FV is a τ-periodic $\Lambda \star G$-module.
Recall from Lemma 3.2.1 that given a sectional path of the form (3.3) of $\mathcal{C}$ we can construct a sectional path of the form (3.4). Moreover, we may assume that $\mathcal{D}^s$ is not τ-periodic. In fact, using Proposition 3.1.10, it is easy to see that $\mathcal{D}^s$ is τ-periodic if and only if $\mathcal{C}^s$ is.
Suppose that we have chosen W, V and FV as above. Then we define a function

$$d_{FV} : \Lambda \star G - \text{mod} \longrightarrow \mathbb{N} \cup \{0\}, \;\; M \mapsto \dim_{\Bbbk} \underline{\text{Hom}}_{\Lambda \star G}(FV, M). \tag{3.6}$$

Recall form [6] that for a finite-dimensional selfinjective $\Bbbk$-algebra A and finite-dimensional A-modules X, Y we have an isomorphism

$$D\underline{\text{Hom}}_A(X, Y) \cong \text{Ext}^1_A(Y, \tau X) \tag{3.7}$$

of $\Bbbk$-vector spaces, where D denotes the usual duality on $\Bbbk - \text{mod}$. We then get the following result.

Theorem 3.2.8. *The function d_{FV} of (3.6) is a non-zero additive function on $\mathcal{D}^s$. Moreover, it is unbounded on the tree associated to $\mathcal{D}^s$, and thus, $\mathcal{D}^s$ has tree class A_∞.*

Proof. Recall that, for every Z_i in the sectional path (3.4) of $\mathcal{D}$, the restriction HZ_i has, by construction, a direct summand isomorphic to ${}^gX_i$, for some $g \in G$. Now for such Z_i on the path in $\mathcal{D}^s$ we have

$$\begin{aligned} d_{FV}(Z_i) &= \dim_{\Bbbk}\underline{\mathrm{Hom}}_{\Lambda\star G}(FV, Z_i) \\ &= \dim_{\Bbbk}\underline{\mathrm{Hom}}_{\Lambda}(V, HZ_i) \\ &\geq \dim_{\Bbbk}\underline{\mathrm{Hom}}_{\Lambda}(V, {}^gX_i) \\ &\geq \dim_{\Bbbk}\underline{\mathrm{Hom}}_{\Lambda}({}^gW, {}^gX_i) \\ &= \dim_{\Bbbk}\underline{\mathrm{Hom}}_{\Lambda}(W, X_i) \\ &= d_W(X_i). \end{aligned} \tag{3.8}$$

Observe that we have used the identity (3.7), the Eckmann–Shapiro Lemma (see for example [8, Corollary 2.8.4]) and the fact that H is left and right adjoint to F, see Proposition 3.1.4. By our assumption, we have that $d_{{}^gW}({}^gX_0) \neq 0$, for all $g \in G$. From (3.8) we infer that $d_{FV}(Z_0) \neq 0$, hence d_{FV} is non-zero since, by assumption, $\mathcal{D}^s$ is connected. Also, by our assumption on $\mathcal{D}^s$, we have that no indecomposable direct summand of FV belongs to $\mathcal{D}^s$ or $\Omega_{\Lambda\star G}(\mathcal{D}^s)$. Therefore, by Lemma 3.2.5, d_{FV} is a non-zero additive function on $\mathcal{D}^s$. Moreover, since $\tau FV \cong FV$, it is constant on τ-orbits. By Lemma 3.2.6, we conclude that the tree class of $\mathcal{D}^s$ is either an infinite Dynkin diagram or a Euclidean diagram.
Now, by Lemma 3.2.6 (ii), d_V is unbounded on every conjugate component of $\mathcal{C}^s$ since all components of $\Gamma_s(\Lambda)$ conjugate to $\mathcal{C}^s$ have tree class A_∞. Thus, we can find, for every $r \in \mathbb{N}$, an index u, such that $d_V({}^hX_u) > r$, for all h in G. But then it follows from the inequality in (3.8) that also $d_{FV}(Z_u) > r$ holds, thus, d_{FV} is unbounded. By [8, Theorem 4.5.8], $\mathcal{D}^s$ has tree class A_∞. □

We say that Λ has enough τ-periodic modules if for every non-projective M in $\Lambda-\mathrm{mod}$ there exists W in $\Lambda-\mathrm{mod}$ with $\tau W \cong W$ such that $\dim_{\Bbbk}\underline{\mathrm{Hom}}_{\Lambda}(W, M) \neq 0$. Then we obtain the following theorem:

Theorem 3.2.9. *Suppose that Λ has enough τ-periodic modules and let the order of G be invertible in $\Bbbk$. Then the following holds: If every non-periodic connected component of $\Gamma_s(\Lambda)$ has tree class A_∞, then every non-periodic connected component of $\Gamma_s(\Lambda\star G)$ has tree class A_∞.*

Proof. This follows immediately from Theorem 3.2.8. □

Chapter 4

The stable Auslander–Reiten quiver of a quantum complete intersection

The goal of this chapter is to describe the shape of the stable AR-quiver of a quantum complete intersection. These algebras occur naturally in the representation theory of Hecke algebras of type A with defining parameter -1. In this case, the algebra $H_2^f(-1)$ is isomorphic to the truncated polynomial ring $\Bbbk[X]/(X^2)$, a quantum complete intersection with all parameters equal to 1. More generally, the outer tensor product of n copies of the algebra $H_2^f(-1)$ will then be isomorphic to the algebra $\Bbbk[X_1, \ldots, X_n]/(X_1^2, \ldots, X_n^2)$.
In [11, Theorem 3.6] the stable AR-quiver of these algebras was determined, but some details in the proof are missing. In this chapter we will give a complete and detailed proof of this theorem.
Throughout, $\Bbbk$ denotes a field of characteristic $p \geq 0$. Moreover, we will assume that $\Bbbk$ is a splitting field for all algebras and groups that occur.

4.1 A skew group algebra construction

In this section we show that a quantum complete intersection whose parameters are arbitrary roots of unity is a truncation of a skew group algebra over a homogeneous quantum complete intersection. The construction relies on a construction given in [9, §4.2].

Let $\mathbf{q} = (q_{ij}) \in \mathrm{Mat}_m(\Bbbk)$ be a commutation matrix, i.e., $q_{ii} = 1$ and $q_{ij}q_{ji} = 1$ for all i, j. We will assume that q_{ij} is a (not necessarily primitive) root of unity in $\Bbbk$ for all i, j. For $n > 1$ and $m \geq 1$ we then define

$$A^n_{\mathbf{q},m} := \Bbbk\langle Z_1, \ldots, Z_m\rangle/(Z_i^n, Z_iZ_j - q_{ij}Z_jZ_i,\ 1 \leq i < j \leq m),$$

which is a finite-dimensional selfinjective $\Bbbk$-algebra of dimension n^m, see for example [10, §2]. The strategy here is to relate $A := A^n_{\mathbf{q},m}$ to the $\Bbbk$-algebra

$$A^n_m := \Bbbk\langle X_1, \dots, X_m\rangle/(X_i^n, X_iX_j - \zeta X_jX_i,\ 1 \leq i < j \leq m),$$

with ζ a primitive sth root of unity, where $n = p^a s$ and $(p, s) = 1$, i.e., s is the p'-part of n. If $p = 0$, we set $s = n$. Note that A^n_m is a special instance of a quantum complete intersection, where we have chosen $\mathbf{q}$ to be the commutation matrix with $q_{ij} = \zeta$, for all $i < j$. The algebra A^n_m is called a *homogeneous* quantum complete intersection.

Remark 4.1.1. Observe that if $\mathrm{char}(\Bbbk) = p > 0$ and $n = p^a$, then A^n_m is isomorphic to the group algebra of a homocyclic group, i.e., $A^n_m \cong \Bbbk G$, with $G \cong (\mathbb{Z}/p^a\mathbb{Z})^m$.

From [11, Theorem 3.3 and Theorem 3.5], we know that if $m \geq 3$ or $n \geq 3$ then all the connected components of the stable AR-quiver of A^n_m have tree class A_∞.

To see how the two classes of algebras are related, we recall a construction given in [9, §4]. Let $\mathbf{u} = (u_{ij}) \in \mathrm{Mat}_m(\Bbbk)$ be a commutation matrix. We then choose $r \geq 1$ such that $u^r_{ij} = 1$, for all $1 \leq i, j \leq m$ and p does not divide r. Since $\mathrm{char}(\Bbbk) = p$, this is possible. Define the finite group $E_{\mathbf{u}}$ as follows: $E_{\mathbf{u}}$ is the central extension given by

$$1 \longrightarrow \langle \nu\rangle \longrightarrow E_{\mathbf{u}} \longrightarrow (\mathbb{Z}/r\mathbb{Z})^m \longrightarrow 1, \tag{4.1}$$

where $\langle \nu\rangle \cong \mathbb{Z}/r\mathbb{Z}$ and with relations as follows. Denote by e_i the preimage of a generator of the ith factor appearing on the right-hand side of (4.1). Then we require:

(1) $\nu^r = 1$ and $e^r_i = 1$ for $1 \leq i \leq m$,

(2) $e_ie_j = \nu_{ij}e_je_i$ for $1 \leq i < j \leq m$,

(3) $e_i\nu = \nu e_i$, $1 \leq i \leq m$,

where we specify the elements $\nu_{ij} \in \langle \nu\rangle$ as follows. We fix a group isomorphism

$$\phi : \langle \nu\rangle \longrightarrow U_r,$$

with U_r the group of rth roots of unity in $\Bbbk$, i.e., we fix a character $\phi \in \mathrm{Irr}(\langle \nu\rangle)$. Then ν is sent to a primitive rth root of unity and we define ν_{ij} by $\phi(\nu_{ij}) = u_{ij}$, $1 \leq i < j \leq m$.
Directly from the definition we see that $|E_{\mathbf{u}}| = r^{m+1}$, and since $(p, r) = 1$, the associated group algebra $\Bbbk E_{\mathbf{u}}$ is split semisimple.
As a next step we define an appropriate central idempotent e in the group algebra $\Bbbk E_{\mathbf{u}}$. To this end, let $E_0 := Z(E)$ be the center of $E := E_{\mathbf{u}}$. We define a form

$$\begin{aligned}\beta : E/E_0 \times E/E_0 &\longrightarrow \Bbbk^\times,\\ (aE_0, bE_0) &\mapsto \phi([a, b]),\end{aligned}$$

on E/E_0. Here, for elements $a, b \in E$, the element $[a, b] = a^{-1}b^{-1}ab$ denotes their commutator in E. It is easy to see that β is bilinear, $\beta(a, a) = 1$ and $\beta(a, b) = \beta(b, a)^{-1}$ for all $a, b \in E$, that is to say, β is a *symplectic* form. Then, for a subgroup $B \subseteq E/E_0$ we set

$$B^{\perp} = \{a \in E \mid \beta(a, b) = 1, \text{ for all } b \in B\}.$$

If $B \subseteq B^{\perp}$, then B is called *isotropic*.
Since ϕ is an isomorphism of abelian groups, we see that β is non-singular. By [73, Corollary 5.7], it follows that E/E_0 admits a decomposition $E/E_0 \cong B \times B'$, where B is a maximal isotropic subgroup of E/E_0 and B' denotes the dual group of B. Since $|B| = |B'|$, we see that $|E/E_0| = d^2$, for some positive integer d.
We will recall some facts about characters of finite groups and their associated group algebras. Our main reference for all this will be [25, §9 and §10]. For a finite-dimensional $\Bbbk$-algebra C and a finite-dimensional C-module M, the trace function

$$\mu : C \to \Bbbk, \; \mu(a) = \mathrm{trace}(a, M)$$

defines the *character* afforded by M. The *degree* of the character μ, denoted by $\deg(\mu)$, is defined as $\mu(1)$.
Let E_1 be the preimage of a maximal isotropic subgroup of E/E_0 in E. Note that E_1 can be chosen to be any maximal abelian subgroup of E. We construct an irreducible character χ of E_1 that extends ϕ. To this end, let S be the irreducible $\Bbbk\langle\nu\rangle$-module corresponding to the irreducible character ϕ. By Mackey's Theorem, $\mathrm{Res}^{\Bbbk E_1}_{\Bbbk\langle\nu\rangle}(\mathrm{Ind}^{\Bbbk E_1}_{\Bbbk\langle\nu\rangle}(S))$ has a direct summand isomorphic to S. Therefore, there is some irreducible $\Bbbk E_1$-module T whose restriction to $\Bbbk\langle\nu\rangle$ has a direct summand isomorphic to S. Since E_1 is abelian, and $\Bbbk$ is a splitting field for E_1, we have that $\dim_{\Bbbk}(T) = 1$, and, thus, $\mathrm{Res}^{\Bbbk E_1}_{\Bbbk\langle\nu\rangle}(T) \cong S$. It follows that if χ denotes the irreducible character of E_1 afforded by T, then $\mathrm{Res}^{E_1}_{\langle\nu\rangle}(\chi) = \phi$.
We set $\chi_0 = \mathrm{Res}^{E_1}_{E_0}(\chi)$. Let $\mathcal{R} := \{g_1, \ldots, g_s\}$ be a set of left coset representatives of E_1 in E. We claim that the set

$$\{{}^{g_i}\chi \mid g_i \in \mathcal{R}\} \tag{4.2}$$

enumerates all the irreducible characters of E_1 that restrict to χ_0. Note that since E_1 is normal in E the definition of the character ${}^{g_i}\chi$ makes sense. Suppose that ${}^{g}\chi = \chi$, for some $g \in E$, i.e., $\chi(g^{-1}hg) = \chi(h)$, for all $h \in E_1$. Since χ is a group homomorphism, we infer that

$$\phi([h, g]) = \chi([h, g]) = \chi(h^{-1}g^{-1}hg) = 1,$$

for all $h \in E_1$. But this implies that $[h, g] = 1$ for all $h \in E_1$. Since E_1 is the preimage of a maximal isotropic subgroup of E/E_0, we conclude that $g \in E_1$. Therefore, the characters given in (4.2) are pairwise distinct. On the other hand, if T_i denotes the irreducible $\Bbbk E_1$-module corresponding to ${}^{g_i}\chi$, and T_0 the irreducible

$\Bbbk E_0$-module corresponding to χ_0, then by Frobenius Reciprocity, each T_i occurs with multiplicity one in the induced module $\mathrm{Ind}_{\Bbbk E_0}^{\Bbbk E_1}(T_0)$. Similarly, each irreducible $\Bbbk E_1$-module whose restriction to $\Bbbk E_0$ is isomorphic to T_0 occurs with multiplicity one in the latter. Since

$$\dim_{\Bbbk} \mathrm{Ind}_{\Bbbk E_0}^{\Bbbk E_1}(T_0) = [E_1 : E_0] = [E : E_1],$$

we infer that the set in (4.2) gives all the irreducible characters of E_1 that restrict to ϕ.
Note that, by [25, §10A], the character of $\mathrm{Ind}_{\Bbbk E_0}^{\Bbbk E_1}(T_0)$ is given by

$$\mathrm{Ind}_{E_0}^{E_1}(\chi_0) := \frac{1}{|E_0|} \sum_{x \in E_1} {}^{x}\dot{\chi_0},$$

where $\dot{\chi_0}$ is the extension of χ_0 to E_1, and is defined as

$$\dot{\chi_0}(h) = \begin{cases} \chi_0(h), & h \in E_0, \\ 0, & h \neq E_0. \end{cases}$$

We then see that

$$\mathrm{Ind}_{E_0}^{E_1}(\chi_0) = \sum_{g_i \in \mathcal{R}} {}^{g_i}\chi. \tag{4.3}$$

Since the ${}^{g_i}\chi$, $g_i \in \mathcal{R}$ are pairwise distinct, by [26, Corollary 45.5], the $\Bbbk E$-module $\mathrm{Ind}_{\Bbbk E_1}^{\Bbbk E}(T)$ is irreducible. If ψ denotes the corresponding irreducible character of E, then, by [25, Proposition 9.17], the associated central primitive idempotent e has the form

$$e = \frac{\psi(1)}{|E|} \sum_{g \in E} \psi(g) g^{-1} \in \Bbbk E_{\mathbf{u}}. \tag{4.4}$$

Since e is a block idempotent, we obtain an isomorphism

$$\rho : e\Bbbk E_{\mathbf{u}} \longrightarrow \mathrm{Mat}_d(\Bbbk),$$

of $\Bbbk$-algebras. Moreover, using (4.3), we get that

$$e = \frac{1}{|E_0|} \sum_{g \in E_0} \chi_0(g) g^{-1}. \tag{4.5}$$

Remark 4.1.2. Observe that the block algebra $e\Bbbk E_{\mathbf{u}}$ is generated by the elements ee_i, $1 \leq i \leq m$ such that

(a) $ee_iee_j = u_{ij}ee_jee_i$, $1 \leq i < j \leq m$, and

(b) $(ee_i)^r = e$, $1 \leq i \leq m$.

Relation (a) is due to the fact that $e\nu_{ij} = u_{ij}e$ for all i, j.

Now we go back to the situation of the beginning of this chapter. For the given commutation matrix $\mathbf{q}$ let us define a new commutation matrix $\mathbf{q}' \in \mathrm{Mat}_m(\Bbbk)$ via the equations

$$q'_{ij}q_{ij} = \zeta, \tag{4.6}$$

for all $1 \leq i < j \leq m$. Then the elements q'_{ij} are again roots of unity. Recall that ζ was chosen to be a primitive sth root of unity in $\Bbbk$. We define

$$r = s\prod_{i<j} \mathrm{ord}(q_{ij}), \tag{4.7}$$

where $\mathrm{ord}(q_{ij})$ the order of q_{ij} in $\Bbbk^\times$. Clearly we have that $r \geq s$, and p does not divide r. We then obtain a finite group $E_{\mathbf{q}'}$ as in (4.1) together with the associated group algebra $\Bbbk E_{\mathbf{q}'}$ and central primitive idempotent e as in (4.5).
Next consider the algebra $B := A \otimes_\Bbbk e\Bbbk E_{\mathbf{q}'}$, the outer tensor product of A and the split semisimple algebra $e\Bbbk E_{\mathbf{q}'}$. Denote by S the unique irreducible $e\Bbbk E_{\mathbf{q}'}$-module. Recall that, as $\Bbbk$-algebras, $e\Bbbk E_{\mathbf{q}'}$ is isomorphic to $\mathrm{Mat}_d(\Bbbk)$, where d is such that $|E_{\mathbf{q}'}/Z(E_{\mathbf{q}'})| = d^2$. We then have the following:

Lemma 4.1.3. *The algebras A and B are Morita equivalent.*

Proof. Consider the (B, A)-bimodule $P = A\otimes_\Bbbk S$. Since $B \cong (P)^d$, P is a projective generator in $B-\mathrm{mod}$. By [25, Lemma 10.37] we have that as $\Bbbk$-algebras,

$$\begin{aligned}
\mathrm{End}_B(A \otimes_\Bbbk S)^{\mathrm{op}} &\cong (\mathrm{End}_A(A) \otimes_\Bbbk \mathrm{End}_{e\Bbbk E_{\mathbf{q}'}}(S))^{\mathrm{op}} \\
&\cong (A^{op} \otimes_\Bbbk \Bbbk)^{\mathrm{op}} \\
&\cong A.
\end{aligned}$$

By [25, Theorem 3.54], the $\Bbbk$-algebras A and B are Morita equivalent. □

As a next step we consider the $\Bbbk$-subalgebra $\tilde{R}$ of B generated by the elements $Z_i \otimes ee_i$, $1 \leq i \leq m$. We claim that the latter is isomorphic to $R := A^n_m$ as a $\Bbbk$-algebra. To this end, we will denote by I^n_m the set of tuples $\underline{i} = (i_1, \ldots, i_c)$, $c \geq 1$, such that $1 \leq i_1 < i_2 < \ldots < i_c \leq m$. Then, as a $\Bbbk$-vector space, R has as basis the set

$$\{X_{i_1}^{n_{i_1}} \cdots X_{i_c}^{n_{i_c}} \mid \underline{i} \in I^n_m, 1 \leq n_{i_j} < n, 1 \leq j \leq c\} \cup \{1\}.$$

We define a $\Bbbk$-homomorphism $\phi : R \to \tilde{R}$ by

$$\begin{aligned}
X_{i_1}^{n_{i_1}} \cdots X_{i_c}^{n_{i_c}} &\mapsto (Z_{i_1}^{n_{i_1}} \cdots Z_{i_c}^{n_{i_c}}) \otimes e(e_{i_1}^{n_{i_1}} \cdots e_{i_c}^{n_{i_c}}),\ \underline{i} \in I^n_m, \\
1 &\mapsto 1 \otimes e,
\end{aligned} \tag{4.8}$$

and extending linearly. Note that since the element $e_{i_1}^{n_{i_1}} \cdots e_{i_c}^{n_{i_c}}$, $\underline{i} \in I^n_m$, is invertible in $\Bbbk E_{\mathbf{q}'}$, the element $e(e_{i_1}^{n_{i_1}} \cdots e_{i_c}^{n_{i_c}})$ must be invertible in $e\Bbbk E_{\mathbf{q}'}$. Thus, all the elements on the right-hand side are different from zero. It then follows that these elements are linearly independent over $\Bbbk$, and since they generate $\tilde{R}$, one sees, using the relations given in Remark 4.1.2 and the defining relations of A, that they form a basis of $\tilde{R}$ over $\Bbbk$. Hence, ϕ is an isomorphism of $\Bbbk$-vector spaces. Furthermore, for $1 \leq i < j \leq m$, we have that

$$\begin{aligned}
\phi(X_j)\phi(X_i) &= (Z_j \otimes ee_j)(Z_i \otimes ee_i) \\
&= Z_jZ_i \otimes ee_je_i \\
&= (q_{ij}^{-1}Z_iZ_j) \otimes e\nu_{ij}^{-1}e_ie_j \\
&= q_{ij}^{-1}q_{ij}'^{-1}(Z_iZ_j \otimes ee_ie_j) \\
&= \phi(\zeta^{-1}X_iX_j) \\
&= \phi(X_jX_i),
\end{aligned}$$

using again the defining relations of Remark 4.1.2 and the defining relations of R. Therefore, the map ϕ is a $\Bbbk$-algebra isomorphism.

Next we want to define a $\Bbbk$-algebra action $E_{\mathbf{q}'} \times \tilde{R} \to \tilde{R}$. To this end, for an element $\sigma \in E_{\mathbf{q}'}$, we first define a map on the elements of the basis of $\tilde{R}$ given in (4.8) by setting

$$f_\sigma((Z_{i_1}^{n_{i_1}} \cdots Z_{i_c}^{n_{i_c}}) \otimes e(e_{i_1}^{n_{i_1}} \cdots e_{i_c}^{n_{i_c}})) = (Z_{i_1}^{n_{i_1}} \cdots Z_{i_c}^{n_{i_c}}) \otimes e(\sigma(e_{i_1}^{n_{i_1}} \cdots e_{i_c}^{n_{i_c}})\sigma^{-1}),$$

for all $\underline{i} \in I_m^n$, and $f_\sigma(1 \otimes e) = 1 \otimes e$. Then, we extend this map linearly to a map of $\tilde{R}$, which we also denote by f_σ. It is easy to see that $f_{\sigma_1\sigma_2}(r) = f_{\sigma_1}(f_{\sigma_2}(r))$ for all $\sigma_1, \sigma_2 \in E_{\mathbf{q}'}$, $r \in \tilde{R}$, and $f_1 = 1$. Hence, the group $E_{\mathbf{q}'}$ acts via $\Bbbk$-linear endomorphisms on $\tilde{R}$. Directly from the definition, we have that $f_\tau = 1$ for all $\tau \in Z(E_{\mathbf{q}'})$. Moreover, using the relations of Remark 4.1.2 and the defining relations of A, we get that

$$f_\sigma(b_1b_2) = f_\sigma(b_1)f_\sigma(b_2),$$

for basis elements b_1, b_2 of (4.8), and all $\sigma \in E_{\mathbf{q}'}$. Thus, the $\Bbbk$-endomorphisms are actually $\Bbbk$-algebra endomorphisms of $\tilde{R}$. Moreover, writing σ as a product of the generators $\nu, e_1, \ldots, e_m$ of $E_{\mathbf{q}'}$, we see that $f_\sigma(b) = \lambda b$, where b is a basis element of (4.8), and $\lambda \in \Bbbk$ is a product of roots of unity in $\Bbbk$, i.e., $\lambda \neq 0$. It follows that each f_σ is a $\Bbbk$-algebra automorphism of $\tilde{R}$, and we obtain a $\Bbbk$-algebra action of $E_{\mathbf{q}'}$ on $\tilde{R}$. In particular,

$$e_j(Z_i \otimes ee_i) = \begin{cases} q_{ji}'(Z_i \otimes ee_i), & i \neq j, \\ Z_i \otimes ee_i, & i = j, \end{cases} \tag{4.9}$$

for all $1 \leq i \leq m$.

Through the $\Bbbk$-algebra isomorphism ϕ, we then obtain a $\Bbbk$-action of $E_{\mathbf{q}'}$ on R as well. In particular, we get that

$$e_j(X_i) = \begin{cases} q_{ji}'X_i, & i \neq j, \\ X_i, & i = j, \end{cases} \tag{4.10}$$

which is a direct consequence of (4.9).

Next we want to give another description of the algebra B. To this end, consider the skew group algebra $R \star E_{\mathbf{q}'}$ with $R = A_m^n$, which, as a $\Bbbk$-vector space, is isomorphic to $R \otimes_{\Bbbk} \Bbbk E_{\mathbf{q}'}$. Following Chapter 3, the multiplication in $R \star E_{\mathbf{q}'}$ is given by the rule

$$(a \star \sigma)(b \star \tau) = (a\sigma(b)) \star (\sigma\tau),$$

where $\sigma(b) := \phi^{-1}(\sigma\phi(b))$ is the action of $E_{\mathbf{q}'}$ on R according to the action of $E_{\mathbf{q}'}$ on $\tilde{R}$ defined above.
As mentioned above, every element in the center of $E_{\mathbf{q}'}$ acts trivially on R. Set $E_0 = Z(E_{\mathbf{q}'})$ and $\tilde{e} = \frac{1}{|E_0|} \sum_{g \in E_0} \chi_0(g^{-1}) \star g \in \Bbbk E_{\mathbf{q}'}$. Here, $\Bbbk E_{\mathbf{q}'}$ is considered as a unitary $\Bbbk$-subalgebra of $R \star E_{\mathbf{q}'}$ via the canonical $\Bbbk$-algebra embedding

$$\sum_{g \in E_{\mathbf{q}'}} \lambda_g g \mapsto \sum_{g \in E_{\mathbf{q}'}} \lambda_g \star g, \quad \lambda_g \in \Bbbk.$$

Thus, $\tilde{e}$ is the image of our block idempotent e in $\Bbbk E_{\mathbf{q}'}$ and, therefore, is an idempotent in $R \star E_{\mathbf{q}'}$.

Lemma 4.1.4. *The idempotent $\tilde{e}$ is central in $R \star E_{\mathbf{q}'}$.*

Proof. If $s \star h$ denotes an arbitrary basis element in $R \star E_{\mathbf{q}'}$, we then have

$$\begin{aligned}
\tilde{e}(s \star h) &= \frac{1}{|E_0|} \sum_{g \in E_0} (\chi_0(g^{-1}) \star g)(s \star h) \\
&= \frac{1}{|E_0|} \sum_{g \in E_0} (\chi_0(g^{-1})s) \star (gh) \\
&= \frac{1}{|E_0|} \sum_{g \in E_0} (s[h(\chi_0(g^{-1}))]) \star (hg) \\
&= \frac{1}{|E_0|} \sum_{g \in E_0} (s \star h)(\chi_0(g^{-1}) \star g) \\
&= (s \star h)\tilde{e}.
\end{aligned}$$

We conclude that $\tilde{e}$ is a central idempotent in $R \star E_{\mathbf{q}'}$. ☐

Next, we will construct an isomorphism of $\Bbbk$-algebras between the algebras $\tilde{e}(R \star E_{\mathbf{q}'})\tilde{e} = \tilde{e}(R \star E_{\mathbf{q}'}) = (R \star E_{\mathbf{q}'})\tilde{e}$ and B. To do this, we first choose a $\Bbbk$-basis of $\tilde{e}(R \star E_{\mathbf{q}'})$. For R choose the canonical basis given by the subset

$$\left\{ \prod_{i \in I} X_i^{v_i} \right\}_{I \subseteq \{1,\dots,m\}}$$

of R, with $1 \leq v_i < n$ for all $i \in I$. Here, we only consider subsets $I = \{i_1, \dots, i_k\} \subseteq \{1, \dots, m\}$ such that $i_1 < \dots < i_k$. Then we let b_I^v be the basis element corresponding to the subset I and the exponent vector $v = (v_1, \dots, v_m) \in \mathbb{N}^m$, where we set

$v_i = 0$ if $i \notin I$. By convention, the empty product corresponds to the identity in R. We then get a $\Bbbk$-basis of $R \star E_{\mathbf{q}'}$ that is given by the set

$$\{b_I^v \star \sigma\}_{I \subseteq \{1,\ldots,m\},\, v \in \mathbb{N}^m,\, \sigma \in E_{\mathbf{q}'}}.$$

From this it follows that $\tilde{e}(R \star E_{\mathbf{q}'})$ is spanned by the set

$$\{\tilde{e}(b_I^v \star \sigma)\}_{I \subseteq \{1,\ldots,m\},\, v \in \mathbb{N}^m,\, \sigma \in E_{\mathbf{q}'}}.$$

Suppose that $\tilde{e}(b_I^v \star \gamma)$ is such that $\gamma = \tau\sigma$, for elements $\tau \in E_0$ and $\sigma \in E_{\mathbf{q}'}$. We calculate:

$$\begin{aligned}\tilde{e}(b_I^v \star \gamma) &= \tilde{e}(b_I^v \star 1)\tilde{e}(1 \star \tau)\tilde{e}(1 \star \sigma)\\ &= \tilde{e}(1 \star \tau)\tilde{e}(b_I^v \star \sigma)\\ &= \chi_0(\tau)\tilde{e}(b_I^v \star \sigma),\end{aligned}$$

which shows that it is enough to choose one element from each right (or left) coset of E_0 in $E_{\mathbf{q}'}$ to obtain a spanning set

$$\mathcal{B} := \{\tilde{e}(b_I^v \star \sigma)\}_{I \subseteq \{1,\ldots,m\},\, v \in \mathbb{N}^m,\, \sigma \in \mathcal{T}}$$

for $\tilde{e}(R \star E_{\mathbf{q}'})$, where $\mathcal{T}$ denotes a set of coset representatives of E_0 in $E_{\mathbf{q}'}$. We may assume that the identity 1 of $E_{\mathbf{q}'}$ is in $\mathcal{T}$. Expanding the elements in $\mathcal{B}$, we see that all the occurring basis elements of $R \star E_{\mathbf{q}'}$ are pairwise distinct. Therefore, the elements of $\mathcal{B}$ are linearly independent over $\Bbbk$ and, thus, form a basis of $\tilde{e}(R \star E_{\mathbf{q}'})$ as a $\Bbbk$-vector space. Then $\dim_\Bbbk \tilde{e}(R \star E_{\mathbf{q}'}) = (\dim_\Bbbk R)|E_{\mathbf{q}'}/E_0|$, and so, $\tilde{e}(R \star E_{\mathbf{q}'})$ and $R \otimes_\Bbbk e\Bbbk E_{\mathbf{q}'}$ are isomorphic as $\Bbbk$-vector spaces.
As a next step we define a map $\psi : \mathcal{B} \to B$ by setting

$$\psi(\tilde{e}(b_I^v \star \sigma)) = \prod_{i \in I} Z_i^{v_i} \otimes e(\prod_{i \in I} e_i^{v_i})\sigma.$$

Note that if $I = \emptyset$, we set $\prod_{i \in I} e_i^{v_i} = 1$. Then ψ induces a $\Bbbk$-linear map

$$\tilde{e}(R \star E_{\mathbf{q}'}) \to B,$$

which also will be denoted by ψ. Directly from the definition it follows that $\psi(\tilde{e}) = 1 \otimes e$. To show that ψ is a homomorphism of $\Bbbk$-algebras, we choose two basis elements $\tilde{e}(b_I^v \star \sigma)$ and $\tilde{e}(b_J^{v'} \star \tau)$ with subset $I, J \subseteq \{1, \ldots, m\}$ and exponent vectors v and v'. Then we have

$$(\tilde{e}(b_I^v \star \sigma))(\tilde{e}(b_J^{v'} \star \tau)) = \tilde{e}(b_I^v \sigma(b_J^{v'}) \star \sigma\tau).$$

Since every element in $\langle \nu \rangle$ acts trivially on R, we may assume that $\sigma = \prod_{u \in U} e_u^{c_u}$, for an ordered subset $U \subseteq \{1, \ldots, m\}$, with generators e_u of $E_{\mathbf{q}'}$ and $c_u \in \{0, \ldots, r-1\}, u \in U$. By (4.10), we then have:

$$\sigma(b_J^{v'}) = (\prod_{u \in U} e_u^{c_u})(b_J^{v'}) = (\prod_{u \in U}\prod_{j \in J}(q'_{uj})^{c_u v'_j})b_J^{v'}. \tag{4.11}$$

In the following let $a_\sigma = \prod_{u\in U}\prod_{j\in J}(q'_{uj})^{c_u v'_j}$. For the product $b_I^v b_J^{v'}$ we have:

$$b_I^v b_J^{v'} = (\prod_{j\in J}\prod_{i\in I, i>j}\zeta^{-v_i v'_j})b^w_{I\cup J}, \tag{4.12}$$

where w is such that $w_i = v_i + v'_i$ for all $1 \leq i \leq m$. The set $I \cup J \subseteq \{1,\dots,m\}$ is again ordered. Note that this product equals zero whenever $w_i \geq n$ for some $i \in I \cup J$.

In the following, we write a^w_{IJ} for the exponent of ζ in (4.12). Let $\sigma\tau = z\gamma$, for $z \in E_0$ and $\gamma \in \mathcal{T}$. By definition of ψ, we then get

$$\begin{aligned}
\psi(\tilde{e}(b_I^v\sigma(b_J^{v'})\star\sigma\tau)) &= \psi(\tilde{e}(b_I^v\sigma(b_J^{v'})\star z\gamma))\\
&= \psi(\chi_0(z)\tilde{e}(b_I^v\sigma(b_J^{v'})\star\gamma))\\
&= \chi_0(z)\psi(\tilde{e}(b_I^v\sigma(b_J^{v'})\star\gamma))\\
&= \chi_0(z)a_\sigma\zeta^{a^w_{IJ}}(\prod_{i\in I\cup J}Z_i^{w_i}\otimes e(\prod_{i\in I\cup J}e_i^{w_i})\gamma)\\
&= a_\sigma\zeta^{a^w_{IJ}}(\prod_{i\in I\cup J}Z_i^{w_i}\otimes ez(\prod_{i\in I\cup J}e_i^{w_i})\gamma)\\
&= a_\sigma\zeta^{a^w_{IJ}}(\prod_{i\in I\cup J}Z_i^{w_i}\otimes e(\prod_{i\in I\cup J}e_i^{w_i})\sigma\tau),
\end{aligned}$$

which is equal to zero whenever $w_i \geq n$ for some $i \in I \cup J$.

On the other hand we have

$$\begin{aligned}
\psi(\tilde{e}(b_I^v\star\sigma))\psi(\tilde{e}(b_J^{v'}\star\tau)) &= (\prod_{i\in I}Z_i^{v_i}\otimes e(\prod_{i\in I}e_i^{v_i})\sigma)(\prod_{j\in J}Z_j^{v'_j}\otimes e(\prod_{j\in J}e_j^{v'_j})\tau)\\
&= (\prod_{i\in I}Z_i^{v_i})(\prod_{j\in J}Z_j^{v'_j})\otimes e((\prod_{i\in I}e_i^{v_i})\sigma)((\prod_{j\in J}e_j^{v'_j})\tau).
\end{aligned}$$

We first transform the element $\sigma(\prod_{j\in J}e_j^{v'_j})$. So let $\sigma = \prod_{u\in U}e_u^{c_u}$ as before and take the last generator e_t in this product. Then we have

$$e_t(\prod_{j\in J}e_j^{v'_j}) = (\prod_{j>t}\nu_{tj}^{v'_j})(\prod_{j<t}\nu_{jt}^{-v'_j})(\prod_{j\in J}e_j^{v'_j})e_t.$$

Write $\nu_{e_t} := (\prod_{j>t}\nu_{tj}^{v'_j})(\prod_{j<t}\nu_{jt}^{-v'_j})$. Next, take the last generator in σe_t^{-1} and repeat this procedure. Then we get

$$\sigma(\prod_{j\in J}e_j^{v'_j}) = (\prod_{u\in U}\nu_{e_u}^{c_u})(\prod_{j\in J}e_j^{v'_j})\sigma.$$

Since ν_{ij} is central in $E_{\mathbf{q}'}$, for all $1 \leq i < j \leq m$, we have that $e\nu_{ij} = q'_{ij}e$ and, equivalently, $e\nu_{ij}^{-1} = (q_{ij})'^{-1}e = q'_{ji}e$. We get

$$e((\prod_{i\in I}e_i^{v_i})\sigma)((\prod_{j\in J}e_j^{v'_j})\tau) = (\prod_{u\in U}\prod_{j\in J}(q'_{uj})^{c_u v'_j})e((\prod_{i\in I}e_i^{v_i})(\prod_{j\in J}e_j^{v'_j})\sigma\tau)). \tag{4.13}$$

Next we want to analyze the product $(\prod_{i\in I} e_i^{v_i})(\prod_{j\in J} e_j^{v'_j})$. But with the relations $e_ie_j = \nu_{ij}e_je_i$ in $E_{\mathbf{q}'}$, for $1 \le i < j \le m$, we calculate:

$$(\prod_{i\in I} e_i^{v_i})(\prod_{j\in J} e_j^{v'_j}) = (\prod_{j\in J}\prod_{i\in I, i>j} \nu_{ji}^{-v_iv'_j}) \prod_{i\in I\cup J} e_i^{w_i}. \tag{4.14}$$

Finally, we obtain

$$e((\prod_{i\in I} e_i^{v_i})(\prod_{j\in J} e_j^{v'_j})) = (\prod_{j\in J}\prod_{i\in I, i>j} (q'_{ij})^{v_iv'_j})e \prod_{i\in I\cup J} e_i^{w_i}. \tag{4.15}$$

As a last step we transform the product $(\prod_{i\in I} Z_i^{v_i})(\prod_{j\in J} Z_j^{v'_j})$ by using the defining relations in $A = A^n_{\mathbf{q}',m}$:

$$\begin{aligned}(\prod_{i\in I} Z_i^{v_i})(\prod_{j\in J} Z_j^{v'_j}) &= (\prod_{j\in J}\prod_{i\in I, i>j} (q_{ji})^{-v_iv'_j}) \prod_{i\in I\cup J} Z_i^{w_i} \\ &= (\prod_{j\in J}\prod_{i\in I, i>j} (q_{ij})^{v_iv'_j}) \prod_{i\in I\cup J} Z_i^{w_i}.\end{aligned} \tag{4.16}$$

Recall from (4.6) that $q'_{uv}q_{uv} = \zeta$ for all $1 \le u < v \le m$, i.e., $q'_{vu}q_{vu} = \zeta^{-1}$. Therefore, the product of the scalars in (4.15) and (4.16) equals

$$\prod_{j\in J}\prod_{i\in I, i>j} \zeta^{-v_iv'_j}. \tag{4.17}$$

Comparing the coefficients of (4.11) and (4.12) with the ones calculated in (4.13) and (4.17) we obtain:

$$\psi((\tilde{e}(b_I \star \sigma))(\tilde{e}(b_J \star \tau))) = \psi(\tilde{e}(b_I \star \sigma))\psi(\tilde{e}(b_J \star \tau)).$$

Now that we have that $\psi(bb') = \psi(b)\psi(b')$ for basis elements $b, b' \in \mathcal{B}$, it is easy to show that this holds for arbitrary elements in $\tilde{e}(R \star E_{\mathbf{q}'})$.
Next we want to show that ψ is surjective. To do this, note that the elements $e\sigma$, $\sigma \in \mathcal{T}$, form a basis of $e\Bbbk E_{\mathbf{q}'}$ as a $\Bbbk$-vector space. By definition of ψ, $\psi(\tilde{e}(1\star\sigma)) = 1 \otimes e\sigma$, and thus, the $\Bbbk$-subalgebra $1 \otimes e\Bbbk E_{\mathbf{q}'}$ of B is contained in the image of ψ. Since $Z_i \otimes e = (Z_i \otimes ee_i)(1 \otimes ee_i^{-1})$, for all $1 \le i \le m$, we see that also all the elements $Z_i \otimes e$, $1 \le i \le m$, are in the image of ψ. Since these elements generate the $\Bbbk$-subalgebra $A \otimes e$ of B, the latter algebra must be contained in the image of ψ. But, as a $\Bbbk$-algebra, B is generated by $A \otimes e$ and $1 \otimes e\Bbbk E_{\mathbf{q}'}$ and hence, ψ is surjective. Since $\dim_\Bbbk \tilde{e}(R \star E_{\mathbf{q}'}) = \dim_\Bbbk B$, it must be an isomorphism. Therefore, we have shown:

Theorem 4.1.5. *The $\Bbbk$-algebras B and $\tilde{e}(R \star E_{\mathbf{q}'})$ are isomorphic.*

With Lemma 4.1.3, we obtain the following corollary.

Corollary 4.1.6. *The* $\Bbbk$*-algebra* $\tilde{e}(R \star E_{\mathbf{q}'})$ *is indecomposable.*

Proof. By Lemma 4.1.3, $\tilde{e}(R \star E_{\mathbf{q}'})$ is Morita equivalent to A, and therefore, by [1, Proposition 21.10], have isomorphic centers. Since A is a local algebra, the only idempotents that occur are 0 and 1 and, hence, the claim follows. □

4.2 Stable components of a quantum complete intersection

In this section we finally describe the shape of the stable AR-quiver of a quantum complete intersection.

Let C be a finite-dimensional $\Bbbk$-algebra, and let $\Omega := \Omega_C$ be the syzygy functor associated to C. Recall from [9] C is said to have enough Ω-periodic modules if for every non-projective M in $C-\mathrm{mod}$ there exists some $W \in C-\mathrm{mod}$ such that

(a) $W \cong \Omega_C^r(W) \oplus P$, for some $r > 0$, and some projective C-module P,

(b) $\underline{\mathrm{Hom}}_C(W, M) \neq 0$.

We now have the following:

Corollary 4.2.1. *With the notation as in Section 4.1, the algebra* $\tilde{e}(R \star E_{\mathbf{q}'})$ *has enough* Ω*-periodic modules. Furthermore,* A *has enough* Ω*-periodic modules.*

Proof. By [9, Lemma 3.2], $R \star E_{\mathbf{q}'}$ has enough Ω-periodic modules since R has, see [9, Corollary 2.16]. Thus, $\tilde{e}(R \star E_{\mathbf{q}'})$ has enough Ω-periodic modules. By Theorem 4.1.5, we infer, using the Morita equivalence given in Lemma 4.1.3, that A has enough Ω-periodic modules. □

Remark 4.2.2. In [9] a theory of rank varieties for the algebra $R = A_m^n$ was developed. To each M in $R-\mathrm{mod}$ one can associate a set

$$V_R^r(M) = \{0\} \cup \{0 \neq \lambda \in \Bbbk^m | \mathrm{Res}^R_{k[u_\lambda]}(M) \text{ is not a projective } \Bbbk[u_\lambda]\text{-module}\},$$

where $u_\lambda = \sum_i \lambda_i X_i$, and $k[u_\lambda]$ denotes the $\Bbbk$-subalgebra generated by u_λ. Then, by [9, Theorem 2.6], M is projective if and only if $V_R^r(M) = 0$. Therefore, by [9, Lemma 2.15], if M is not projective, then there is some $0 \neq \lambda \in \Bbbk^m$ such that the $\Bbbk$-vector spaces

$$\underline{\mathrm{Hom}}_R(Ru_\lambda, M) \quad \text{and} \quad \underline{\mathrm{Hom}}_R(Ru_\lambda^{n-1}, M)$$

are both non-zero. Moreover, by [9, Lemma 2.14], we have that

$$\Omega(Ru_\lambda) = Ru_\lambda^{n-1},$$

i.e., the indecomposable R-modules Ru_λ and Ru_λ^{n-1} have Ω-period two and are syzygies of each other. Then, consider the R-modules $\tau(Ru_\lambda)$ and $\tau(Ru_\lambda^{n-1})$, where

τ denotes the AR-translate of R, see Chapter 2, Section 2.1, for definitions. Since R is selfinjective, we know from Proposition 2.3.3 that $\tau \cong \Omega^2\mathcal{N}$ as functors from $R - \underline{\text{mod}}$ to itself, where $\mathcal{N}$ denotes the Nakayama automorphism of R. By the same proposition, we also have that $\Omega^2\mathcal{N} \cong \mathcal{N}\Omega^2$. It follows that

$$\tau(Ru_\lambda) \cong \mathcal{N}(Ru_\lambda) \quad \text{and} \quad \tau(Ru_\lambda^{n-1}) \cong \mathcal{N}(Ru_\lambda^{n-1}).$$

Set $W_0 = \tau(Ru_\lambda)$ or $W_0 = \tau(Ru_\lambda^{n-1})$. Then, for any integer $a \neq 0$, we have that $\tau^a(W_0) \cong \mathcal{N}^a(W_0)$. From [10, Lemma 3.1], we know that $\mathcal{N}$ has finite order, which implies that every Ω-periodic R-module is also τ-periodic. Hence, there exists a non-zero integer a such that $\tau^a(W_0) \cong W_0$. If we set

$$W = W_0 \oplus \tau(W_0) \oplus \cdots \oplus \tau^{a-1}(W_0),$$

then we see that $\tau(W) \cong W$. Therefore, if M is a non-projective R-module, we have constructed an R-module W such that $\tau(W) \cong W$ and $\underline{\text{Hom}}_R(W, \tau(M)) \neq 0$. Note that if M is non-projective, then neither is $\tau(M)$ since R is selfinjective. Therefore, we have that $\underline{\text{Hom}}_R(W, M) \neq 0$.

As a consequence of Theorem 4.1.5, we get the following result.

Theorem 4.2.3. *Let $\Bbbk$ be algebraically closed and suppose that $m, n \geq 2$. If $m \geq 3$ or $n \geq 3$, then every connected component of the stable AR-quiver $\Gamma_s(A)$ of the $\Bbbk$-algebra A has tree class A_∞.*

Proof. We first show that every connected component of the stable AR-quiver of the skew group algebra $R \star E_{\mathbf{q}'}$ has tree class A_∞. Note that all the assumptions needed in the proof of Theorem 3.2.8 in Chapter 3 are satisfied, i.e., $E_{\mathbf{q}'}$ is a finite group such that the order $|E_{\mathbf{q}'}|$ of $E_{\mathbf{q}'}$ is invertible in $\Bbbk$. Moreover, by Remark 4.2.2, R has enough τ-periodic modules, i.e., for each non-projective M in $R - \text{mod}$ there exists an R-module W such that $\tau(W) \cong W$ and $\underline{\text{Hom}}_R(W, \tau(M)) \neq 0$. Furthermore, we know from [11, Theorem 3.3 and Theorem 3.5] that all the connected components of the stable AR-quiver of R for $m \geq 3$ or $n \geq 3$ have tree class A_∞. Now we can proceed as in Section 3.2, and, thus, by Theorem 3.2.9, we get that every connected component of $\Gamma_s(R \star E_{\mathbf{q}'})$ that is not τ-periodic has tree class A_∞. From this it follows that every non-τ-periodic connected component of the stable AR-quiver of any block of $R \star E_{\mathbf{q}'}$ must have tree class A_∞.
On the other hand, if a connected component of the stable AR-quiver of $R \star E_{\mathbf{q}'}$ is τ-periodic, then it follows from Theorem 2.2.6 that it must have tree class either a finite Dynkin diagram or A_∞. In the first case the corresponding component of the stable AR-quiver is finite, thus belongs to a block of $R \star E_{\mathbf{q}'}$ of finite representation type. By Proposition 3.2.1 and Corollary 3.2.2 this cannot be the case. Hence, it must have tree class A_∞.
By Theorem 4.1.5 we have a $\Bbbk$-algebra isomorphism between B and $\tilde{e}(R \star E_{\mathbf{q}'})$, and, by Lemma 4.1.3, a Morita equivalence between A and B. Thus, A and $\tilde{e}(R \star E_{\mathbf{q}'})$ are Morita equivalent $\Bbbk$-algebras and the claim follows. $\square$

Recall that a finite-dimensional algebra A over an algebraically closed field is of finite, tame or wild representation type, and these types are mutually exclusive. If A is of finite representation type, there are only finitely many isomorphism classes of indecomposable A-modules, whereas in the tame and wild cases there are infinitely many. If A is tame, then one can classify the isomorphism classes of indecomposable A-modules, whereas in the wild case, in general, no such classification exists.

The next proposition completes the picture for the quantum complete intersections.

Theorem 4.2.4. *Suppose that* $\Bbbk$ *is algebraically closed. Then the following hold for the algebra* $A = A^n_{\mathbf{q},m}$.

(a) *If* $m = 1$*, then* A *is of finite representation type and the connected component of the stable AR-quiver has tree class* A_{n-1}.

(b) *If* $m = n = 2$*, then* A *is of tame representation type and there is one connected component of the stable AR-quiver of tree class* $\tilde{A}_{12}$ *and infinitely many* 1*-tubes, which have tree class* A_∞.

(c) *In all other cases,* A *is of wild representation type and all the connected components of* $\Gamma_s(A)$ *have tree class* A_∞.

Proof. If $m = 1$, then, as a $\Bbbk$-algebra, A is isomorphic to $\Bbbk[Z]/(Z^n)$, the truncated commutative polynomial ring. For the latter algebra, the modules $\Bbbk[Z]/(Z^i)$, $1 \leq i \leq n$, give a complete list of non-isomorphic indecomposable modules, so A is of finite representation type. For each $1 \leq i \leq n-2$ we have the following AR-sequence:

$$0 \longrightarrow \Bbbk[Z]/(Z^i) \xrightarrow{f_i} \Bbbk[Z]/(Z^{i-1}) \oplus \Bbbk[Z]/(Z^{i+1}) \xrightarrow{g_i} \Bbbk[Z]/(Z^i) \longrightarrow 0,$$

where the homomorphism f_i is induced by the canonical inclusion $\Bbbk[Z]/(Z^i) \to \Bbbk[Z]/(Z^{i+1})$ and the canonical epimorphism $\Bbbk[Z]/(Z^i) \to \Bbbk[Z]/(Z^{i-1})$, the homomorphism g_i is induced by the canonical inclusion $\Bbbk[Z]/(Z^{i-1}) \to \Bbbk[Z]/(Z^i)$ and the canonical epimorphism $\Bbbk[Z]/(Z^{i+1}) \to \Bbbk[Z]/(Z^i)$ with switched sign. Note that the module $\Bbbk[Z]/(Z^n)$ is the unique projective and injective indecomposable A-module. From this it follows that the stable part of the AR-quiver of A is a 1-tube of tree class A_{n-1}.
Let $m = n = 2$. In this case, A is a factor algebra of the algebra

$$\Bbbk\langle Z_1, Z_2\rangle / \langle Z_1^2, Z_2^2\rangle,$$

which is of tame representation type by [78, 1.3]. Using the methods of [8, §4.3], one can show that A is special biserial and domestic. By [37, Theorem 2.1], $\Gamma_s(A)$ consists of infinitely many tubes that have tree class A_∞ and finitely many components of the form $\mathbb{Z}\tilde{A}_{p,q}$ for positive integers p, q. For $pq \geq 2$, the component $\mathbb{Z}\tilde{A}_{p,q}$ has tree class A^∞_∞ and if $pq = 1$, it has tree class $\tilde{A}_{12}$. Since $\mathrm{rad}(A)/\mathrm{soc}(A)$ is indecomposable, it must have tree class $\tilde{A}_{12}$.

If $m \geq 3$, then it follows from [14, Theorem 4.1] that A has wild representation type. If $m = 2$ and $n \geq 3$, then the algebra $\Bbbk\langle X, Y\rangle/(X^2, XY - qYX, Y^3, Y^2X)$ is a factor algebra of A. By [78, 3.4], this factor is wild. It follows that A is wild. The statement about the components of $\Gamma_s(A)$ follows from Theorem 4.2.3.

□

Remark 4.2.5. As mentioned in the beginning of this chapter, the case when A is a homogeneous quantum complete intersection was already treated in [11]. The general case, i.e., if all the entries of the commutation matrix are arbitrary roots of unity, was also stated there but a thorough proof was missing. Theorem 4.2.4 is filling the gap.

In the following we list the Dynkin and Euclidean diagrams occurring as tree classes in Theorem 4.2.4, where the diagram of type A_n consists of n nodes.

$$A_n : \quad \bullet - \bullet - \bullet \cdots \bullet - \bullet$$

$$A_\infty : \quad \bullet - \bullet - \bullet \cdots \bullet - \bullet \cdots$$

$$\tilde{A}_{12} : \quad \bullet \xrightarrow{(2,2)} \bullet$$

Chapter 5

The stable Auslander–Reiten quiver of blocks of e-parabolic Hecke algebras

In this chapter we will describe the shape of the stable AR-quiver of outer tensor products of Brauer tree algebras whose Brauer tree is given by a line with no exceptional vertex. These algebras naturally occur as blocks of Hecke algebras of type A of finite representation type.
Therefore, as a result, we may describe the stable AR-quivers of blocks of parabolic Hecke algebras of the form $H_e^f(q)^{\otimes k}$, $k \geq 1$, where the defining parameter q is a primitive eth root of unity, $e \geq 3$. The latter algebras are also called e-parabolic Hecke algebras and if $\mathrm{char}(\Bbbk) = 0$, they occur as vertices of indecomposable $H_n^f(q)$-modules, see [32].
First of all, we will give some important definitions and constructions concerning finite-dimensional algebras.
Afterwards, we recollect the definition of a *Brauer tree algebra*. In particular we will be interested in Brauer tree algebras whose Brauer tree is a star.
After that we will give an alternative description of such an algebra via a skew group construction. Together with results of J. Rickard this will enable us to describe the shape of the stable AR-quiver of outer tensor products of Brauer tree algebras.
Throughout, we denote by $\Bbbk$ a fixed field.

5.1 Preliminaries

In this section we briefly recall some fundamental definitions and constructions in the theory of finite-dimensional algebras.

To a finite-dimensional $\Bbbk$-algebra B one can associate a quiver in the following way:

Definition 5.1.1. Let B be a finite-dimensional $\Bbbk$-algebra. Denote by

$$S_0, S_1, \ldots, S_m$$

the irreducible B-modules, corresponding to the projective indecomposable modules $P_i = Be_i$, $0 \leq i \leq m$, where $e_0, \dots, e_m$ are pairwise orthogonal primitive idempotents in B. The *Ext-quiver* $Q(B)$ of B has vertices the set $\{v_0, v_1, \dots, v_m\}$, and the number of edges from v_x to v_y, $0 \leq x, y \leq m$, is the same as

$$\dim_{\Bbbk} \operatorname{Ext}^1_B(S_x, S_y) = [\operatorname{rad}(P_x)/\operatorname{rad}^2(P_x) : S_y]. \tag{5.1}$$

Remark 5.1.2. Note that the non-negative integer $\dim_{\Bbbk} \operatorname{Ext}^1_B(S_x, S_y)$ also coincides with the $\Bbbk$-dimension of the spaces

$$\operatorname{Hom}_B(P_y, \operatorname{rad}(P_x))/\operatorname{Hom}_B(P_y, \operatorname{rad}^2(P_x)) = e_y\operatorname{rad}(B)e_x/e_y\operatorname{rad}^2(B)e_x,$$

see [8, Proposition 2.4.3].

Recall that a finite-dimensional $\Bbbk$-algebra B is called *basic* if all the irreducible B-modules are one-dimensional. It is well known that for each finite-dimensional $\Bbbk$-algebra B there exists a basic algebra that is Morita equivalent to B.

There is the following important theorem, which is due to P. Gabriel:

Theorem 5.1.3. *(P. Gabriel) Assume $\Bbbk$ to be algebraically closed. Let B be a finite-dimensional basic $\Bbbk$-algebra, and $Q := Q(\Lambda)$ be its* Ext*-quiver. Then there exists a surjective $\Bbbk$-algebra homomorphism $\psi : \Bbbk Q \to B$ such that the kernel of ψ is contained in the ideal of $\Bbbk Q$ generated by paths of length at least two.*

Proof. See [41, §4]. ☐

5.2 Brauer tree algebras

A *Brauer tree* T is a finite connected undirected tree where around each vertex of T there is a cyclic ordering of the adjacent edges. Moreover, to each vertex u one assigns a positive integer $m(u)$, called the *multiplicity* of u, such that at most one vertex of T has multiplicity greater than one. A vertex u with $m(u) > 1$ is called *exceptional*.
A finite-dimensional $\Bbbk$-algebra A is called a *Brauer tree algebra for T* if there is a bijection between the edges $1, \dots, r$ of T and the irreducible A-modules $S_1, \dots, S_r$ such that the structure of the projective indecomposable A-modules $P_1, \dots, P_r$ is given in the following way: $P_i/\operatorname{rad}(P_i) \cong S_i \cong \operatorname{soc}(P_i)$ and $\operatorname{rad}(P_i)/\operatorname{soc}(P_i)$ is the sum of two not necessarily non-zero uniserial modules corresponding to the endpoints of the edge i. If $i = i_0, i_1, \dots, i_s, i_0$ is the cyclic ordering around one of the endpoints u of i, then the corresponding uniserial module has composition factors from top to socle

$$S_{i_1}, S_{i_2}, \dots, S_{i_s}, S_i, S_{i_1}, S_{i_2}, \dots, S_{i_s}, S_i, \dots, \dots, S_{i_1}, S_{i_2}, \dots, S_{i_s},$$

where S_i appears $m(u) - 1$ times and S_{i_j} for $j \neq 0$ appears $m(u)$-times.

Every Brauer tree determines a corresponding Brauer tree algebra that is symmetric and unique up to Morita equivalence.

Next, we want to discuss a very important instance of a Brauer tree algebra: Let T be the Brauer tree that is a star with r edges and (possibly) exceptional vertex in the center of T with multiplicity m. Denote by $B(r,m)$ the corresponding Brauer tree algebra.

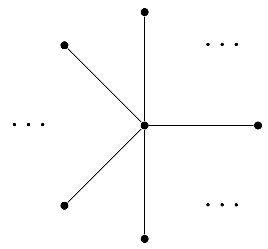


Figure 5.1: The Brauer tree $B(r,m)$

Consider the path algebra $\Bbbk Q$ of the circular quiver Q with r vertices $v_0,\dots,v_{r-1}$, and arrows $\alpha_i : v_i \to v_{i+1}$ for all $0 \le i < r-1$, and $\alpha_r : v_{r-1} \to v_0$. Recall that $\Bbbk Q$ is the $\Bbbk$-algebra with basis the set of paths of Q, and multiplication is given by the composition of paths, see for example [7, III, §1]. For each $1 \le i \le r$ we denote by p_i the path $\alpha_{i-1}\cdots\alpha_{i+1}\alpha_i$, with the convention that $\alpha_j = \alpha_k$, whenever $j-k$ is divisible by r.
The basic algebra of $B(r,m)$ is then isomorphic as a $\Bbbk$-algebra to the algebra $\Bbbk Q/J$, where J is the two-sided ideal of $\Bbbk Q$ generated by the paths $\alpha_i p_i^m$, for all $0 \le i \le r-1$, see for example [46, §2].

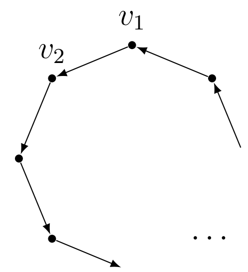


Figure 5.2: The circular quiver Q

5.3 A skew group construction for $B(r,1)$

In the following let $G \cong C_r$ be the cyclic group of order r, and we fix a generator g of G. Moreover, let $\Lambda = \Bbbk[t]/(t^{r+1})$, an $r+1$-dimensional local $\Bbbk$-algebra, and set $\bar{t} := t + (t^{r+1}) \in \Lambda$.
Let $I = \mathbb{Z}/r\mathbb{Z} = \{0,\dots,r-1\}$ and fix a primitive rth root of unity ζ in $\Bbbk$. From now on we assume that r is invertible in $\Bbbk$. Then, by Maschke's Theorem, the group

algebra $\Bbbk G$ is semisimple. Fix a complete set of non-isomorphic irreducible $\Bbbk G$-modules $\{T_0, \ldots, T_{r-1}\}$ such that for each $i \in I$, the irreducible character afforded by T_i is given by

$$\chi_i : G \to \Bbbk, \ h = g^a \mapsto \zeta^{ai}.$$

For each $i \in I$, we define

$$e_i = r^{-1} \sum_{h \in G} \chi_i(h^{-1})h. \tag{5.2}$$

Remark 5.3.1. An easy calculation shows that $ge_i = \zeta^i e_i$. Moreover, the set $\{e_i \mid i \in I\}$ is a complete set of primitive orthogonal central idempotents in $\Bbbk G$.

Define a $\Bbbk$-algebra action

$$\varphi : G \times \Lambda \to \Lambda \tag{5.3}$$

of G on Λ, by defining $\varphi(g^i, \bar{t}^j) = \zeta^{ij}\bar{t}^j$, for $0 \leq j \leq r$, $i \in I$, with the convention that $\bar{t}^0 = 1$, and extending linearly. It is easy to see that this defines a group homomorphism

$$G \longrightarrow \mathrm{Aut}_\Lambda(\Lambda).$$

Let $A := \Lambda \star G$ be the skew group algebra of G over Λ, and consider Λ as a unitary $\Bbbk$-subalgebra of A via the monomorphism $\Lambda \to A$, given by $\lambda \to \lambda \star 1$. By [79, Theorem 1.3], since Λ is selfinjective, so is A.
As a first step, we want to describe the Loewy structure of the indecomposable projective A-modules. To this end, for each $i \in I$, define

$$\tilde{e}_i = (r^{-1} \star 1) \sum_{h \in G} \chi_i(h^{-1}) \star h, \tag{5.4}$$

the image of the idempotent e_i under the canonical monomorphism $\Bbbk G \to A$ given by $bh \mapsto b \star h$, $b \in \Bbbk$. Again we see that the elements $\tilde{e}_i$, $i \in I$, are mutually orthogonal idempotents in A. For each $i \in I$ define $P_i = A\tilde{e}_i$. Then each P_i is a projective A-module. We have the following:

Lemma 5.3.2. *For each $i \in I$, the set*

$$\{\tilde{e}_i, (\bar{t} \star 1)\tilde{e}_i, \ldots, (\bar{t}^r \star 1)\tilde{e}_i\} \subseteq P_i$$

is a basis of P_i considered as a $\Bbbk$-vector space.

Proof. Let $\lambda \star h$, $\lambda \in \Lambda, h \in G$, be an arbitrary basis element of A. Suppose that $h = g^j$, $j \in \mathbb{Z}$. Then

$$(\lambda \star h)\tilde{e}_i = (\zeta^{ij} \star 1)(\lambda \star 1)\tilde{e}_i,$$

by Remark 5.3.1. Therefore, the given set is a spanning set for P_i. Next, assume that

$$\sum_{j=0}^{r} \mu_j(\bar{t}^j \star 1)\tilde{e}_i = 0,$$

for coefficients $\mu_j \in \Bbbk$. After reordering, we see that this is a sum of pairwise different basis elements for A, which implies that $\mu_j = 0$ for all $0 \leq j \leq r$. Thus, the set is a $\Bbbk$-basis of P_i. $\square$

Lemma 5.3.3. *As A-modules we have that*

$$A \cong \bigoplus_{i \in I} P_i.$$

Proof. The claim follows immediately with Lemma 5.3.2 since $\dim_{\Bbbk} A = r(r+1)$. $\square$

For any $i \in I$, the restriction $\mathrm{Res}^A_\Lambda(P_i)$ of P_i is isomorphic to Λ as a Λ-module. Since the latter is indecomposable, $\mathrm{Res}^A_\Lambda(P_i)$ is indecomposable as a Λ-module, and thus, P_i is indecomposable as an A-module. Therefore, each P_i is an indecomposable projective A-module. Then, for each $i \in I$, define S_i to be the irreducible A-module corresponding to P_i, i.e.,

$$S_i \cong P_i/\mathrm{rad}(P_i).$$

In the sequel let $\mathfrak{r} = \mathrm{rad}(\Lambda)$ be the Jacobson radical of Λ. Since Λ is a local algebra, we have that $\mathfrak{r} = (\bar{t})$, the unique maximal ideal of Λ. Moreover, since r is invertible in $\Bbbk$, we know from [79, Theorem 1.3] that $\mathfrak{r}A = A\mathfrak{r} = \mathrm{rad}(A)$. Then, for each $i \in I$, we see that

$$\mathrm{rad}(P_i) = \mathrm{rad}(A)P_i = \mathfrak{r}P_i. \tag{5.5}$$

Furthermore, for all $j > 0$, we define the A-submodule $\mathrm{rad}^j(P_i)$ of P_i inductively as follows: $\mathrm{rad}^j(P_i) = \mathrm{rad}(\mathrm{rad}^{j-1}(P_i))$, where we set $\mathrm{rad}^0(P_i) = P_i$.

Remark 5.3.4. By (5.5), for $i \in I$, the irreducible A-module S_i is generated by $\tilde{e}_i + \mathrm{rad}(P_i)$ as an A-module. Moreover, $(1 \star g)v = \zeta^i v$ for all $v \in S_i$, by Remark 5.3.1.

Recall that a finite-dimensional $\Bbbk$-algebra B is called a *Nakayama algebra* if the indecomposable projective B-modules as well as the indecomposable injective B-modules are uniserial.

Proposition 5.3.5. *For each $i \in I$, the indecomposable projective A-module P_i has radical series*

$$S_i, S_{i+1}, S_{i+2}, \ldots, S_{i+r-1}, S_i. \tag{5.6}$$

This is already a composition series of P_i and it follows that A is a Nakayama algebra.

Proof. Let $i \in I$. From (5.5) we infer that $\mathrm{rad}(P_i)$ is spanned by the set $\{(\bar{t} \star 1)\tilde{e}_i, \ldots, (\bar{t}^r \star 1)\tilde{e}_i\}$ as a $\Bbbk$-vector space. Iterating this procedure, we see that for all $0 \leq j \leq r$, $\mathrm{rad}^j(P_i)$ is spanned by $\{\bar{t}^j \star 1)\tilde{e}_i, \ldots, (\bar{t}^r \star 1)\tilde{e}_i\}$, and $\mathrm{rad}^{r+1}(P_i) = 0$.

From this we see that for all $0 \leq j \leq r$, $\mathrm{rad}^j(P_i)/\mathrm{rad}^{j+1}(P_i)$ is a non-zero A-module, spanned by $u_i := (\bar{t}^j \star 1)\tilde{e}_i + \mathrm{rad}^{j+1}(P_i)$. Then we have that $(1 \star g)u_i = \zeta^{i+j}u_i$, and hence, $(1 \star g)v = \zeta^{i+j}v$ for all $v \in \mathrm{rad}^j(P_i)/\mathrm{rad}^{j+1}(P_i)$. From Lemma 5.3.3 we know that $\dim_{\Bbbk} P_i = r + 1$. It follows that the modules $\mathrm{rad}^j(P_i)/\mathrm{rad}^{j+1}(P_i)$, $0 \leq j \leq r$, are one-dimensional, and thus, irreducible. By Remark 5.3.4 we have that $\mathrm{rad}^j(P_i)/\mathrm{rad}^{j+1}(P_i) \cong S_{i+j}$, and so, we conclude that (5.6) is the radical series of P_i, and also the composition series of P_i.
From [7, IV, Lemma 2.1], we have that P_i is a uniserial A-module, and therefore, A is a Nakayama algebra. □

As we have the composition series of the projective indecomposable A-modules at our disposal, we may try to give a presentation of A by a quiver with relations.
To this end note that the algebra homomorphism

$$\psi : \Bbbk Q \longrightarrow B$$

given in Theorem 5.1.3 is constructed as follows: Write $1 = e_1 + \cdots + e_m$ as a sum of pairwise orthogonal primitive idempotents such that e_i corresponds to the indecomposable projective B-module P_i, for all $0 \leq i \leq m$. Since B is basic, we may define a map $\psi' : \{v_0, \dots, v_m\} \to B$, $v_i \mapsto e_i$, for all $0 \leq i \leq m$. Note that in $\Bbbk Q$, the elements v_i, $0 \leq i \leq m$, are mutually orthogonal idempotents. Next, choose a complement K to $e_j\mathrm{rad}^2(B)e_i$ in $e_j\mathrm{rad}(B)e_i$ as a $\Bbbk$-vector space and choose a $\Bbbk$-basis for K. For each (i, j), $0 \leq i, j \leq m$, choose a bijective map between the arrows starting in v_i and ending in v_j and the basis elements of K. Then extend the map ψ' to a map $Q \to B$ by sending the arrows from v_i to v_j to their images under the chosen maps.
Any relation in $\Bbbk Q$ is given by the non-composability of paths. But these relations are satisfied by corresponding products in B since $e_ie_j = e_je_i = 0$, for $i \neq j$. Therefore, we obtain a well-defined $\Bbbk$-algebra homomorphism $\psi : \Bbbk Q \to B$. By construction, $B = \mathrm{Im}(\psi) + \mathrm{rad}^2(B)$. But then it follows that $B = \mathrm{Im}(\psi)$, see for example [8, Proposition 1.2.8]. Hence, ψ is surjective.

We want to apply this to A. By Proposition 5.3.5, A is a basic $\Bbbk$-algebra, and from the structure of the projective indecomposable modules we see that

$$\dim_k \mathrm{Ext}^1_A(S_i, S_j) = \begin{cases} 1 & \text{if } j = i + 1 \bmod r, \\ 0 & \text{if } j \neq i + 1 \bmod r. \end{cases}$$

Therefore, we get that the Ext-quiver $Q = Q(A)$ is the circular quiver with r vertices given at the end of Section 5.2.
By construction, $1 = \tilde{e}_0 + \cdots + \tilde{e}_{r-1}$ and $\tilde{e}_i\tilde{e}_j = \tilde{e}_j\tilde{e}_i = 0$, $i \neq j$. Fix $i \in I$. Straightforward calculations show that $\tilde{e}_{i+1}(\bar{t}\star 1) = (\bar{t}\star 1)\tilde{e}_i$, for all $i \in I$. Therefore, we get that

$$(\bar{t} \star 1)\tilde{e}_i = (\bar{t} \star 1)\tilde{e}_i^2 = \tilde{e}_{i+1}(\bar{t} \star 1)\tilde{e}_i,$$

for each $i \in I$. Moreover, one sees that $\tilde{e}_j(\bar{t}\star 1)\tilde{e}_i = 0$ for all $j \neq i+1$. By the proof of Proposition 5.3.5, we have that a complement of $\tilde{e}_{i+1}\mathrm{rad}^2(A)\tilde{e}_i$ in $\tilde{e}_{i+1}\mathrm{rad}(A)\tilde{e}_i$ is spanned by $(\bar{t}\star 1)\tilde{e}_i = \tilde{e}_{i+1}(\bar{t}\star 1)\tilde{e}_i$. We now define ψ by sending the vertex v_i to $\tilde{e}_i$ and the arrow between v_i and v_{i+1} to $(\bar{t}\star 1)\tilde{e}_i$. Then, by Theorem 5.1.3 and the discussion afterwards, ψ extends to a well-defined surjective $\Bbbk$-algebra homomorphism

$$\Bbbk Q \to A,$$

which we also denote by ψ.
Let $J = \operatorname{Ker}(\psi)$. We want to give a precise description of J. First, observe that for $i \in I$, we have

$$((\bar{t}\star 1)\tilde{e}_i)((\bar{t}\star 1)\tilde{e}_{i+r-1})\cdots((\bar{t}\star 1)\tilde{e}_{i+1})((\bar{t}\star 1)\tilde{e}_i) = (\bar{t}^{r+1}\star 1)\tilde{e}_i = 0.$$

Therefore, for any $i \in I$, $\psi(\alpha_i p_i) = 0$, i.e., $\alpha_i p_i \in J$. Let $\tilde{J}$ be the two-sided ideal of $\Bbbk Q$ generated by the paths $\alpha_i p_i$, $i \in I$. Then $\tilde{J} \subseteq J$. Note that

$$\psi(\alpha_{i+j}\alpha_{i+j-1}\dots\alpha_i) = (\bar{t}^j \star 1)\tilde{e}_i \neq 0,$$

for all $i \in I$ and $0 \leq j \leq r-1$, since the elements $(\bar{t}^j \star 1)\tilde{e}_i$ form a basis of P_i. Therefore, the subset of paths

$$\mathcal{P} = \{p_{ij} = \alpha_{i+j}\alpha_{i+j-1}\dots\alpha_i \mid i \in I, 0 \leq j \leq r-1\} \cup \{v_j \mid 0 \leq j \leq r-1\}$$

in Q is such that $\psi(p_{ij}) \neq 0$ for $p_{ij} \in \mathcal{P}$. On the other hand, if p is a path in Q not contained in $\mathcal{P}$, then from the structure of Q we infer that $p = p'\alpha_i p_i p''$, for some $i \in I$ and paths p', p''. Then $\psi(p) = 0$, i.e., $p \in J$. Since $\Bbbk Q$ has $\Bbbk$-basis the set of paths of Q, we see that a complement K of the $\Bbbk$-subspace of $\Bbbk Q$ generated by $\mathcal{P}$ is spanned by such paths and is contained in $\tilde{J}$. But also $\tilde{J} \subseteq K$, and therefore, $K = \tilde{J}$. Since $|\mathcal{P}| = r(r+1)$, it follows that

$$\dim_{\Bbbk} \Bbbk Q/\tilde{J} = r(r+1).$$

Now, $A \cong \Bbbk Q/J$ is a factor algebra of $\Bbbk Q/\tilde{J}$, and since they have the same dimension we must have $J = \tilde{J}$. We have proved the following.

Theorem 5.3.6. *Let r be a positive integer that is invertible in the algebraically closed field $\Bbbk$. Let $\Lambda = \Bbbk[t]/(t^{r+1})$, $G = C_r$ be the cyclic group of order r, and φ as in (5.3). Then, as $\Bbbk$-algebras, the skew group algebra $\Lambda \star G$ and the basic Brauer tree algebra $B(r,1)$ are isomorphic.*

We will also need the following:

Lemma 5.3.7. *Let Λ and Γ be two finite-dimensional $\Bbbk$-algebras, and G, H be two finite groups. Suppose that there is a $\Bbbk$-algebra action of G on Λ and a $\Bbbk$-algebra action of H on Γ. Let $\Lambda \star G$ and $\Gamma \star H$ be the corresponding skew group algebras. Then, as $\Bbbk$-algebras,*

$$(\Lambda \star G) \otimes_{\Bbbk} (\Gamma \star H) \cong (\Lambda \otimes_{\Bbbk} \Gamma) \star (G \times H),$$

where $G \times H$ is the direct product of G and H and the $\Bbbk$-algebra action of $G \times H$ on $\Lambda \otimes_{\Bbbk} \Gamma$ is defined by

$$(g,h)(a \otimes b) = ga \otimes hb,$$

for elements $(g,h) \in G \times H$ and $a \otimes b \in \Lambda \otimes_{\Bbbk} \Gamma$.

Proof. Choose a $\Bbbk$-basis $\{\lambda_1, \ldots, \lambda_n\}$ of Λ and a $\Bbbk$-basis $\{\gamma_1, \ldots, \gamma_m\}$ of Γ. Moreover, suppose that $G = \{g_1 = \mathrm{id}_G, g_2, \ldots, g_r\}$ and $H = \{h_1 = \mathrm{id}_H, h_2, \ldots, h_s\}$. Then the sets $\{\lambda_i \star g_j \mid 1 \leq i \leq n, 1 \leq j \leq r\}$ and $\{\gamma_i \star h_j \mid 1 \leq i \leq m, 1 \leq j \leq s\}$ are $\Bbbk$-bases of $\Lambda \star G$ and $\Gamma \star H$, respectively. We define a map

$$\psi : (\Lambda \star G) \otimes_{\Bbbk} (\Gamma \star H) \longrightarrow (\Lambda \otimes_{\Bbbk} \Gamma) \star (G \times H),$$

by setting $(\lambda_i \star g_j) \otimes (\gamma_u \star h_v) \mapsto (\lambda_i \otimes \gamma_u) \star (g_j, h_v)$, for admissible indices i, j, u, v, and extending linearly. It is easy to see that ψ is an isomorphism of $\Bbbk$-vector spaces. Also, straightforward calculations show that ψ is a $\Bbbk$-algebra homomorphism. $\square$

5.4 Derived categories, derived and stable equivalences

For the convenience of the reader, in the following, we will recall the basic notions of derived categories and derived equivalences. After that we will state the crucial results due to J. Rickard giving a criterion when two rings are derived equivalent.
In the following, we denote by R a unital commutative ring. As before, for an R-algebra A, we denote by $A - \mathrm{Mod}$ the category of all left A-modules, by $A - \mathrm{mod}$ the category of finitely presented left A-modules, by $A - \mathrm{Proj}$ the category of all projective left A-modules, and by $A - \mathrm{proj}$ the category of finitely generated projective left A-modules. Recall that an A-module M is called *finitely presented* if there is an exact sequence of A-modules

$$A^m \longrightarrow A^n \longrightarrow M \longrightarrow 0,$$

for positive integers m, n.
All categories are assumed to be additive R-categories and all functors are R-linear. Moreover, if A and B are two R-algebras and M is an $A - B$-bimodule, then we assume that the two actions of R coincide, i.e., M is a left $A \otimes_R B^{\mathrm{op}}$-module.

Let $\mathcal{A}$ be an additive category. By $\mathrm{Kom}(\mathcal{A})$ we denote the category of differential complexes

$$A^\bullet = (\ldots \to A^r \overset{d_A^r}{\to} A^{r+1} \overset{d_A^{r+1}}{\to} A^{r+2} \to \ldots),$$

with objects A^r and morphisms d_A^r in $\mathcal{A}$, $r \in \mathbb{Z}$, such that $d_A^{r+1} d_A^r = 0$, for all r. A morphism $f : A^\bullet \to B^\bullet$ in $\mathrm{Kom}(\mathcal{A})$ is a family $\{f^i\}_{i \in \mathbb{Z}}$ of morphisms in $\mathcal{A}$ such that $f^{i+1} \circ d_A^i = d_B^{i+1} \circ f^i$, for all $i \in \mathbb{Z}$. We will denote by $\mathrm{Hom}_{\mathrm{Kom}(\mathcal{A})}(A^\bullet, B^\bullet)$ the set of morphism for objects $A^\bullet, B^\bullet$ in $\mathrm{Kom}(\mathcal{A})$, which is an abelian group with respect to the componentwise addition.
We may define several subcategories of $\mathrm{Kom}(\mathcal{A})$ satisfying certain finiteness conditions. Let $\mathrm{Kom}^+(\mathcal{A})$ be the full subcategory of $\mathrm{Kom}(\mathcal{A})$ consisting of complexes $A^\bullet$ over $\mathcal{A}$ such that there exists an integer $s = s(A^\bullet)$ with $A^i = 0$ for all $i \leq s$. Similarly define $\mathrm{Kom}^-(\mathcal{A})$ to be the full subcategory of $\mathrm{Kom}(\mathcal{A})$ consisting of complexes

$A^\bullet$ such that there is some integer $s = s(A^\bullet)$ with $A^i = 0$ for $i \geq s$. Then we define $\mathrm{Kom}^{\mathrm{b}}(\mathcal{A}) = \mathrm{Kom}^+(\mathcal{A}) \cap \mathrm{Kom}^-(\mathcal{A})$. Note that these categories are again additive.
A morphism $f : A^\bullet \to B^\bullet$ is called *null-homotopic* if there exists a family of morphisms $h^i : A^i \to B^{i-1}$ in $\mathcal{A}$ such that $f^i = d_B^{i-1} h^i + h^{i+1} d_A^i$. Two morphisms f and g are called *homotopic* if $f - g$ is null-homotopic.
Clearly, the set of null-homotopic morphisms between objects $A^\bullet$ and $B^\bullet$ forms a $\mathbb{Z}$-submodule $\mathcal{N}(A^\bullet, B^\bullet)$ of $\mathrm{Hom}_{\mathrm{Kom}(\mathcal{A})}(A^\bullet, B^\bullet)$. The *homotopy category* $\mathrm{K}(\mathcal{A})$ has the same objects as $\mathrm{Kom}(\mathcal{A})$ and the morphisms from $A^\bullet$ to $B^\bullet$ are defined to be the elements of $\mathrm{Hom}_{\mathrm{Kom}(\mathcal{A})}(A^\bullet, B^\bullet)/\mathcal{N}(A^\bullet, B^\bullet)$.
Note that for each $i \in \mathbb{Z}$ we have the *homology functor*

$$H^i : \mathrm{Kom}(\mathcal{A}) \longrightarrow \mathcal{A},$$

given by $H^i(A^\bullet) = \mathrm{Ker}(d_A^i)/\mathrm{Im}(d_A^{i-1})$. If $f : A^\bullet \to B^\bullet$ is a morphism, the corresponding morphism $H^i(f)$ in $\mathcal{A}$ is called the *homology morphism*. The homotopy category of an additive category has the following structure.

Proposition 5.4.1. *Let $f : A^\bullet \to B^\bullet$ in $\mathrm{Kom}(\mathcal{A})$ be null-homotopic. Then $H^i(f) = 0$ for all $i \in \mathbb{Z}$. In particular, if f is homotopic to g, then $H^i(f) = H^i(g)$ for all $i \in \mathbb{Z}$.*

Proof. See [60, Chapter 1, Lemma 1.3.1]. □

The proposition shows that for every $i \in \mathbb{Z}$ the homology $H^i : \mathrm{Kom}(\mathcal{A}) \to \mathcal{A}$ induces a well-defined functor $\bar{H}^i : \mathrm{K}(\mathcal{A}) \to \mathcal{A}$.
A *quasi-isomorphism* in $\mathrm{K}(\mathcal{A})$ is a morphism $s : A^\bullet \to B^\bullet$ of $\mathrm{K}(\mathcal{A})$ such that the homology morphisms $H^i(s) : H^i(A^\bullet) \to H^i(B^\bullet)$ are isomorphisms for all $i \in \mathbb{Z}$. Denote by Σ the class of all quasi-isomorphisms in $\mathrm{K}(\mathcal{A})$. One has the following important theorem:

Theorem 5.4.2. *Let $\mathcal{A}$ be an abelian category, $\mathrm{K}(\mathcal{A})$ the homotopy category of $\mathcal{A}$. Then there exists a category $D(\mathcal{A})$ and a functor $Q : \mathrm{K}(\mathcal{A}) \to D(\mathcal{A})$ such that the following properties are satisfied:*

(i) *If s is a quasi-isomorphism in $\mathrm{K}(\mathcal{A})$, then $Q(s)$ is an isomorphism in $D(\mathcal{A})$.*

(ii) *If $F : \mathrm{K}(\mathcal{A}) \to \mathcal{B}$ is a functor sending quasi-isomorphisms to isomorphisms, then there exists a unique functor $G : D(\mathcal{A}) \to \mathcal{B}$ with $F = G \circ Q$.*

Proof. See [60, Chapter 4, Definition–Theorem 1.3]. □

The category $D(\mathcal{A})$ is called the *derived category* of the abelian category $\mathcal{A}$. In what follows we will also denote the latter category by $D^{\mathrm{ub}}(\mathcal{A})$. By Theorem 5.4.2, it can be thought of as the localization of $\mathrm{K}(\mathcal{A})$ at the class of quasi-isomorphisms in $\mathrm{K}(\mathcal{A})$. Similarly, localizing at the class of quasi-isomorphisms in $\mathrm{Kom}^+(\mathcal{A})$ (resp. $\mathrm{Kom}^-(\mathcal{A})$, $\mathrm{Kom}^{\mathrm{b}}(\mathcal{A})$) we obtain the derived categories $D^+(\mathcal{A})$ (resp. $D^-(\mathcal{A})$, $D^{\mathrm{b}}(\mathcal{A})$), see [55, §13.1].

Note that up to this point, it is not clear whether $D(\mathcal{A})$ is additive. Actually, this is the case, and a proof can be found in [60, Chapter 4, §2.8]. On the other hand, the category $D(\mathcal{A})$ will almost never be abelian.

Let $A^\bullet$ be a complex over $\mathcal{A}$. For an integer r, define a new complex $A^\bullet[r]$ by setting $(A[r])^i = A^{r+i}$, and $d^i_{A[r]} = (-1)^r d^i_A$ for all $i \in \mathbb{Z}$.
For a morphism $f : A^\bullet \to B^\bullet$ in $\mathrm{Kom}(\mathcal{A})$ define $f[r] : A^\bullet[r] \to B^\bullet[r]$ via $f[r]^i = f^{r+i}$ for all $i \in \mathbb{Z}$.
This defines a *translation functor*

$$T^r : \mathrm{Kom}(\mathcal{A}) \to \mathrm{Kom}(\mathcal{A}),\ T^r(A^\bullet) = A^\bullet[r],\ T^r(f) = f[r]. \tag{5.7}$$

Clearly, T^r is an auto-equivalence on $\mathrm{Kom}(\mathcal{A})$. Also it induces an auto-equivalence of the categories $\mathrm{Kom}^+(\mathcal{A})$, $\mathrm{Kom}^-(\mathcal{A})$, $\mathrm{Kom}^\mathrm{b}(\mathcal{A})$ and on the corresponding derived categories $D^+(\mathcal{A})$, $D^-(\mathcal{A})$, $D^b(\mathcal{A})$. If not stated otherwise, by a morphism f of complexes we mean a morphism in either of the categories introduced above.
Let $f : A^\bullet \to B^\bullet$ be a morphism of complexes. Then the complex $C(f)$ defined as

$$C(f)^i = A[1]^i \oplus B^i,\quad d^i_{C(f)}(a^{i+1}, b^i) = (-d^{i+1}_A(a^{i+1}), f^{i+1}(k^{i+1}) + d^i_B(b^i)),$$

for elements $a_{i+1} \in A[1]^i, b^i \in B^i$, and all $i \in \mathbb{Z}$, is called the *mapping cone* of f.
A diagram in the category $\mathrm{Kom}(\mathcal{A})$ (resp. $\mathrm{K}(\mathcal{A})$, $D(\mathcal{A})$, $D^+(\mathcal{A})$, ...) of the form

$$A^\bullet \xrightarrow{a} B^\bullet \xrightarrow{b} C^\bullet \xrightarrow{c} T(A^\bullet) \tag{5.8}$$

is called a *triangle*. A *morphism of triangles* is a commutative diagram of the form

$$\begin{tikzcd} A^\bullet \arrow[r, "a"] \arrow[d, "f"] & B^\bullet \arrow[r, "b"] \arrow[d, "g"] & C^\bullet \arrow[r, "c"] \arrow[d, "h"] & A^\bullet[1] \arrow[d, "f[1]"] \\ A^\bullet_1 \arrow[r, "a_1"] & B^\bullet_1 \arrow[r, "b_1"] & C^\bullet_1 \arrow[r, "c_1"] & A^\bullet_1[1]. \end{tikzcd}$$

A morphism of triangles is called an *isomorphism* if the morphisms f, g, h are all isomorphisms in the corresponding category.
Let $f : A^\bullet \to B^\bullet$ be a morphism of complexes. Then we define the morphisms

$$\alpha(f) : B^\bullet \longrightarrow C(f),\ \alpha(f) = 0 \oplus \mathrm{id}_{B^\bullet},$$

and

$$\beta(f) : C(f) \longrightarrow T(A^\bullet),\ \beta(f) = (\mathrm{id}_{T(A^\bullet)}, 0).$$

We obtain a triangle

$$A^\bullet \xrightarrow{f} B^\bullet \xrightarrow{\alpha(f)} C(f) \xrightarrow{\beta(f)} T(A^\bullet), \tag{5.9}$$

the *mapping cone triangle*.

Next we consider the homotopy category $\mathrm{K}(\mathcal{A})$ of $\mathcal{A}$. A triangle in $\mathrm{K}(\mathcal{A})$ is called *distinguished* if it is isomorphic to a mapping cone triangle (5.9). Recall that a *triangulated category* $\mathcal{D}$ is a category together with a translation functor T giving an equivalence $T : \mathcal{D} \to \mathcal{D}$ of categories. Furthermore, there is a family of triangles, called the *distinguished triangles*, and satisfying certain axioms, see for example [55, Definition 10.1.6]. For the homotopy category of an additive category we have the following:

Theorem 5.4.3. *The homotopy category* $\mathrm{K}(\mathcal{A})$ *of an additive category* $\mathcal{A}$ *together with translation (5.7) and the family of distinguished triangles of the form (5.9) is a triangulated category.*

Proof. This is [55, Proposition 11.2.8]. □

Corollary 5.4.4. *The derived category* $D(\mathcal{A})$ *of an abelian category* $\mathcal{A}$ *is a triangulated category.*

Proof. This follows from Theorem 5.4.3 and [55, Theorem 10.2.3] □

Let $\mathcal{D}$ and $\mathcal{D}'$ be triangulated categories with translations T and T'. A *triangle functor* $F : \mathcal{D} \to \mathcal{D}'$ is a functor of additive categories such that

(i) There is a functorial isomorphism $\phi_X : F(T(X)) \cong T'(F(X))$, for every object X of $\mathcal{D}$.

(ii) If $X \stackrel{u}{\to} Y \stackrel{v}{\to} Z \stackrel{w}{\to} T(X)$ is a triangle in $\mathcal{D}$, then the sequence $F(X) \stackrel{F(u)}{\to} F(Y) \stackrel{F(v)}{\to} F(Z) \stackrel{\phi_X(F(w))}{\to} T'(F(X))$ is a triangle in $\mathcal{D}'$.

A triangle functor F is said to be a *triangle equivalence* if F is an equivalence of additive categories. In this case the categories $\mathcal{D}$ and $\mathcal{D}'$ are equivalent as triangulated categories.

The following theorem of J. Rickard is the full analogue of Morita's Theorem in the situation of derived categories of module categories.

Theorem 5.4.5. (J. Rickard) *Let A, B be R-algebras. Then the following are equivalent:*

(i) *The categories* $D^{-}(A-\mathrm{Mod})$ *and* $D^{-}(B-\mathrm{Mod})$ *are equivalent as triangulated categories.*

(ii) *The categories* $D^{\mathrm{b}}(A-\mathrm{Mod})$ *and* $D^{\mathrm{b}}(B-\mathrm{Mod})$ *are equivalent as triangulated categories.*

(iii) *The categories* $\mathrm{K}^{\mathrm{b}}(A-\mathrm{Proj})$ *and* $\mathrm{K}^{\mathrm{b}}(B-\mathrm{Proj})$ *are equivalent as triangulated categories.*

(iv) *The categories* $\mathrm{K}^{\mathrm{b}}(A-\mathrm{proj})$ *and* $\mathrm{K}^{\mathrm{b}}(B-\mathrm{proj})$ *are equivalent as triangulated categories.*

(v) *B is isomorphic to* $\mathrm{End}_{\mathrm{D^b}(A-\mathrm{Mod})}(T^\bullet)$*, where $T^\bullet$ is a complex of* $\mathrm{K^b}(A-\mathrm{proj})$ *such that*

(a) $\mathrm{Hom}_{D^{\mathrm{b}}(A-\mathrm{Mod})}(T^\bullet, T^\bullet[r]) = 0$*, for $r \neq 0$.*

(b) *The category* $\mathrm{add}(T^\bullet)$ *of direct summands of finite direct sums of $T^\bullet$ generates the category* $\mathrm{K^b}(A-\mathrm{proj})$ *as a triangulated category.*

Proof. See [74, Theorem 6.4] and [76, Theorem 1.1]. □

If one of the equivalent conditions of the last theorem are satisfied, then the algebras A and B are said to be *derived equivalent.* The complex $T^\bullet$ of part (v) is called a *tilting complex.*

Remark 5.4.6. (1) If R is a field and A and B are finite-dimensional algebras R-algebras, then the statements of the theorem are equivalent to the statement that there is a triangle equivalence $\mathrm{D^b}(A-\mathrm{mod}) \to \mathrm{D^b}(B-\mathrm{mod})$.
(2) If A and B are Morita equivalent, then they are derived equivalent. The converse is not true and counterexamples can be found in the case where A and B are blocks of finite groups.

We will also need the following result, which is [76, Theorem 2.1], where we assume from now on that all R-algebras are projective when considered as R-modules.

Theorem 5.4.7. *Let A_1 be an R-algebra, with tilting complex $T^\bullet_{A_1}$ such that the endomorphism algebra* $\mathrm{End}_{\mathrm{D^b}(A_1-\mathrm{Mod})}(T^\bullet_{A_1})$ *is isomorphic to an R-algebra B_1. Moreover, let A_2 be an R-algebra with tilting complex $T^\bullet_{A_2}$ such that* $\mathrm{End}_{\mathrm{D^b}(A_2-\mathrm{Mod})}(T^\bullet_{A_2})$ *is isomorphic to an R-algebra B_2. Then $T^\bullet_{A_1} \otimes_R T^\bullet_{A_2}$ is a tilting complex for the algebra $A_1 \otimes_R A_2$ such that* $\mathrm{End}_{\mathrm{D^b}((A_1\otimes_R A_2)-\mathrm{Mod})}(T^\bullet_{A_1} \otimes_R T^\bullet_{A_2})$ *is isomorphic to $B_1 \otimes_R B_2$ as R-algebras, i.e., the algebras $A_1 \otimes_R A_2$ and $B_1 \otimes_R B_2$ are derived equivalent.*

In our situation we will also need the following crucial results of J. Rickard, see [75, Corollary 2.2] and [75, Theorem 4.2].

Theorem 5.4.8. *Let Λ and Γ be two finite-dimensional selfinjective $\Bbbk$-algebras. If Λ and Γ are derived equivalent, then they are stably equivalent.*

Theorem 5.4.9. *Let B be a Brauer tree algebra with r edges and let m be the multiplicity of the exceptional vertex. Then B is derived equivalent to $B(r, m)$.*

Next, we apply these theorems in our context. In what follows, let $\Lambda = \Bbbk[t]/(t^{r+1})$ and $G = C_r$ be the cyclic group of order r. We let G act on Λ as in Section 5.2, and we will denote the corresponding $\Bbbk$-algebra action by φ. We then obtain the skew group algebra $\Lambda \star G$. Furthermore, for a $\Bbbk$-algebra A and a positive integer n, we will denote by $A^{\otimes n}$ the n-fold outer tensor product of A. Note that $\Lambda^{\otimes n}$ is canonically isomorphic to $\Bbbk[t_1, \ldots, t_n]/(\{t_i^{r+1}\}_{1\le i\le n})$ as $\Bbbk$-algebras. In this case, by Lemma 5.3.7, we see that $H := G \times \cdots \times G$ (n-times), the direct product of n copies of G, defines a $\Bbbk$-algebra action through φ. We then obtain the skew group algebra $\Lambda^{\otimes n} \star H$. We have the following:

Theorem 5.4.10. *Let B be a Brauer tree algebra with r edges and multiplicity $m = 1$ over an algebraically closed field $\Bbbk$. Let n be a positive integer and suppose that r is invertible in $\Bbbk$. Then the algebras $B^{\otimes n}$ and $\Lambda^{\otimes n} \star H$ are stably equivalent.*

Proof. From Theorem 5.3.6 we have that the basic algebra of $B(r, 1)$ is isomorphic to $\Lambda \star G$ as a $\Bbbk$-algebra. By Theorem 5.4.9, we know that B is derived equivalent to $B(r, 1)$, and thus, derived equivalent to $\Lambda \star G$. Then, by Theorem 5.4.7, the algebra $B^{\otimes n}$ is derived equivalent to the algebra $(\Lambda \star G)^{\otimes n}$. By Lemma 5.3.7, the latter algebra is isomorphic to $\Lambda^{\otimes n} \star H$. From [51, Theorem 2.1] we get that $B^{\otimes n}$ and $\Lambda^{\otimes n} \star H$ are symmetric $\Bbbk$-algebras and, thus, selfinjective. The claim now follows from Theorem 5.4.8. $\square$

5.5 Components of the stable AR-quiver

In this section we assume that $\Bbbk$ is an algebraically closed field and q is a primitive eth root of unity in $\Bbbk$, $e \geq 3$. Moreover, we assume throughout that $e-1$ is invertible in $\Bbbk$.

For $e = 2$ we proved in Chapter 4 that a component of the stable Auslander–Reiten quiver of an e-parabolic subalgebra $H_e^f(q)^{\otimes r}$ of the Hecke algebra $H_n^f(q)$ of type A, n a positive integer, has tree class A_∞ when $r > 2$.

The aim now is to prove a similar result for e-parabolic Hecke algebras, when $e > 2$. In this case the algebra $H_e^f(q)$ is of finite representation type with a unique non-semisimple block B. This block is a Brauer tree algebra with $e - 1$ irreducible modules, and whose associate Brauer tree is a line with no exceptional vertex, see [42, Theorem 11.4] and [72, Theorem 1.4]. Therefore, any block of an e-parabolic subalgebra of $H_n^f(q)$ is an outer tensor product of copies of B and a number of semisimple blocks. The latter are split semisimple algebras, hence, such a product is Morita equivalent to an outer tensor product of Brauer tree algebras. We will summarize this in a proposition.

Proposition 5.5.1. *Suppose that $e > 2$. Let C be a non-semisimple block of an e-parabolic Hecke algebra $H_e^f(q)^{\otimes r}$. Then C is Morita equivalent to $B^{\otimes u}$, for some $u > 0$.*

Since the Brauer tree of B has no exceptional vertex, we know by Theorem 5.4.9 that B is derived equivalent to the Brauer tree algebra $B(e - 1, 1)$ corresponding to the star with $e - 1$ edges. By Remark 5.4.6 and Theorem 5.4.10, we get the following:

Theorem 5.5.2. *Keep the assumptions of Proposition 5.5.1. Then C is stably equivalent to a skew group algebra $\Lambda^{\otimes u} \star H$, for some $u > 0$, where $\Lambda = \Bbbk[t]/(t^e)$, $H = C_{e-1} \times \cdots \times C_{e-1}$ (u times) and C_{e-1} (resp. H) acts in the usual way on Λ (resp. $\Lambda^{\otimes u}$).*

By Theorem 2.3.10, we know that if Λ_0 and Λ_1 are stably equivalent selfinjective $\Bbbk$-algebras such that Λ_0 and Λ_1 have no block of Loewy length two, then the stable AR-quivers $\Gamma_s(\Lambda_0)$ and $\Gamma_s(\Lambda_1)$ are isomorphic as stable translation quivers.

Remark 5.5.3. Assume that Λ_0 is an indecomposable selfinjective algebra such that there exists an indecomposable projective module P of Loewy length one. Then P is irreducible, and since Λ_0 is selfinjective, P is the only projective indecomposable Λ_0-module. Thus, Λ_0 is a semisimple algebra.

In the situation where $e > 2$, we have that the Loewy length of a non-semisimple block C is different from two, and therefore, by Theorem 5.5.2 we get:

Theorem 5.5.4. *Keep the notation of Theorem 5.5.2. The stable AR-quivers* $\Gamma_s(C)$ *and* $\Gamma_s(\Lambda^{\otimes u} \star H)$ *are isomorphic as stable translation quivers.*

As a next step, we want to determine the structure of the stable AR-quiver $\Gamma_s(\Lambda^{\otimes u} \star H)$, for every $u > 0$. If $u = 1$, we know that $\Lambda \star C_{e-1}$ has finite representation type, and $\Gamma_s(\Lambda \star C_{e-1})$ has tree class A_{e-1}, see [85].
For $u > 1$, we know from Chapter 4 that every connected component of $\Gamma_s(\Lambda^{\otimes u})$ has tree class A_∞. We will need the following.

Lemma 5.5.5. *Let $u > 0$. Then the algebra $\Lambda^{\otimes u}$ has enough Ω-periodic modules.*

Proof. Recall that there is a canonical $\Bbbk$-algebra isomorphism

$$\Lambda^{\otimes u} \cong \Bbbk[t_1, \ldots, t_u]/(\{t_i^e\}_{1 \leq i \leq u}).$$

In Chapter 4, Theorem 4.1.5, it was shown that $\Lambda^{\otimes u}$ is Morita equivalent to a block D of a skew group algebra with enough τ-periodic modules. Since D is a symmetric algebra, we conclude that D has enough Ω-periodic modules. Since $\Lambda^{\otimes u}$ is Morita equivalent to D, we infer that also $\Lambda^{\otimes u}$ has enough Ω-periodic modules. $\square$

We now get the following result.

Theorem 5.5.6. *Let $u > 1$. Then every connected component of $\Gamma_s(\Lambda^{\otimes u} \star H)$ has tree class A_∞. In particular, if C is a block of an e-parabolic Hecke algebra $H_e^f(q)^{\otimes u}$, $e > 2$, that is not of finite representation type, then every connected component of $\Gamma_s(C)$ has tree class A_∞.*

Proof. By Lemma 5.5.5 we see that we can apply Theorem 3.2.9 of Chapter 3 to conclude that every component of $\Gamma_s(\Lambda^{\otimes u} \star H)$ has tree class A_∞. In particular, every component of $\Gamma_s(C)$ has tree class A_∞ since, by Theorem 5.5.2, C and $\Lambda^{\otimes u} \star H$ are stably equivalent. $\square$

Chapter 6

Finite general linear groups in non-defining characteristic and related algebras

The goal of this chapter is to state and explain a theorem of J. Chuang and H. Miyachi, giving a Morita equivalence between a Rouquier block of a Hecke algebra of type A and a wreath product constructed over an outer tensor product of Brauer tree algebras. This theorem will be a key ingredient in determining the tree classes of stable components of the AR-quiver of blocks of Hecke algebras of type A in characteristic zero.
To this end, we first recall several definitions and results concerning the representation theory of the general linear groups in non-defining characteristic. Afterwards, we explain the link between the representation theory of the finite general linear groups and the associated Hecke algebras as far as it is needed in the sequel. Then, we state the above-mentioned theorem, where we focus on the bimodule giving the Morita equivalence. We also mention that recently, A. Evseev has given another proof of the theorem using methods of the representation theory of KLR-algebras.

6.1 Blocks and characters

In the following we let $q > 1$ be a prime power and ℓ be a prime not dividing q. Moreover, we let F be an algebraically closed field of characteristic ℓ. By e we denote the least positive integer such that

$$1 + q + \cdots + q^{e-1} = 0$$

in $\mathbb{F}_\ell$, and set $e = \infty$ if no such integer exists. Throughout this chapter, we will assume that e is finite.
Let V be an $\mathbb{F}_q$-vector space of dimension $n > 0$, and let us fix a basis of V. Then the set of $\mathbb{F}_q$-automorphisms $GL(V)$ on V may be identified with $G := \mathrm{GL}_n(q)$,

the group of invertible $n \times n$-matrices with entries in $\mathbb{F}_q$. Moreover, we shall also consider the following subgroups of G: The subgroup

$$B = \{X \in G \mid x_{ij} = 0, i > j\}$$

of upper triangular matrices, a *Borel subgroup* of G, and the subgroup

$$T = \{X \in G \mid x_{ij} = 0, i \neq j\}$$

of diagonal matrices, a *maximal torus* in G.
For a composition $\mu = (\mu_1, \dots, \mu_r)$ of n, the subgroup

$$L_\mu = \left\{ \left(\begin{matrix} A_{\mu_1} & 0 & \dots & 0 \\ 0 & \ddots & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \dots & 0 & A_{\mu_r} \end{matrix} \right) \middle| \, A_{\mu_i} \in \mathrm{GL}_{\mu_i}(q), 1 \leq i \leq r \right\} \tag{6.1}$$

is the *Levi subgroup* of G associated to μ. The *parabolic subgroup* P_μ of G,

$$P_\mu = \left\{ \left(\begin{matrix} A_{\mu_1} & * & \dots & * \\ 0 & \ddots & & \vdots \\ \vdots & & \ddots & * \\ 0 & \dots & 0 & A_{\mu_r} \end{matrix} \right) \middle| \, A_{\mu_i} \in \mathrm{GL}_{\mu_i}(q), 1 \leq i \leq r \right\}, \tag{6.2}$$

contains L_μ as a subgroup, which has complement U_μ, called the *Levi complement*, a *unipotent subgroup* in G, given by

$$U_\mu = \left\{ \left(\begin{matrix} I_{\mu_1} & * & \dots & * \\ 0 & \ddots & & \vdots \\ \vdots & & \ddots & * \\ 0 & \dots & 0 & I_{\mu_r} \end{matrix} \right) \right\}. \tag{6.3}$$

Here, for a positive integer k, the symbol I_k denotes the $k \times k$-matrix with diagonal entries 1 and zeros otherwise. Note that for the normalizer $\mathrm{N}_G(U_\mu)$ of U_μ in G one has that

$$\mathrm{N}_G(U_\mu) = P_\mu \tag{6.4}$$

such that $P_\mu = L_\mu U_\mu = U_\mu L_\mu$. In particular, L_μ normalizes U_μ.
Next, we will give the definition of important functors ubiquitous in the representation theory of general linear groups.

Definition 6.1.1. Let $\mu \models n$ be a composition of n. Since U_μ is normal in P_μ, we may inflate an FL_μ-module M to P_μ with respect to the exact sequence

$$0 \longrightarrow U_\mu \longrightarrow P_\mu \longrightarrow L_\mu \cong P_\mu/U_\mu \longrightarrow 0$$

of finite groups. If we induce this FP_μ-module to FG, we obtain a functor

$$\mathrm{R}_{L_\mu}^G = \mathrm{Ind}_{P_\mu}^G \circ \mathrm{infl}_{L_\mu}^{P_\mu} : FL_\mu - \mathrm{mod} \longrightarrow FG - \mathrm{mod}, \tag{6.5}$$

which is called the *Harish-Chandra induction.* On the other hand, if M is an FG-module, we may restrict it to the subalgebra FP_μ and then take U_μ-invariants to obtain a functor

$$^*\mathrm{R}_{L_\mu}^G : FG - \mathrm{mod} \longrightarrow FL_\mu - \mathrm{mod}, \tag{6.6}$$

called *Harish-Chandra restriction.*

Remark 6.1.2. Since q and ℓ are coprime, these functors are exact and left and right adjoint to each other, see [28, §1].

In the following, we fix an ℓ-modular system $(K, \mathcal{O}, F)$, that is to say, $\mathcal{O}$ denotes a complete discrete valuation ring with residue field F of characteristic ℓ, and K is its field of fractions of characteristic zero. Also, we assume that K is a splitting field for all groups that occur.

Next, we recall the parametrization of ordinary characters and blocks of G. Fortunately, all this is well known due to work of P. Fong and B. Srinivasan [39]. To explain these various parametrizations, we have to introduce some more notation, which follows the lines of [17] and [84].
Let σ be an element of the multiplicative group $\bar{\mathbb{F}}_q^\times$ of an algebraic closure $\bar{\mathbb{F}}_q$ of $\mathbb{F}_q$. Suppose that σ has degree d_σ, i.e., the algebra $\mathbb{F}_q[\sigma]$ has dimension d_σ as an $\mathbb{F}_q$-vector space. We denote by (σ) the matrix of $\mathrm{GL}_{d_\sigma}(q)$ which describes the left multiplication of σ on $\mathbb{F}_q[\sigma]$ with respect to the basis $\{1, \sigma, \dots, \sigma^{d_\sigma - 1}\}$. For a positive integer k, we let

$$(\sigma)^k = \begin{pmatrix} (\sigma) & 0 & \dots & 0 \\ 0 & \ddots & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \dots & 0 & (\sigma) \end{pmatrix}, \tag{6.7}$$

a matrix of $\mathrm{GL}_{kd_\sigma}(q)$. If the elements $\sigma, \tau \in \bar{\mathbb{F}}_q^\times$ have the same degree over $\mathbb{F}_q$, we write $\sigma \sim \tau$ if σ and τ have the same minimal polynomial over $\mathbb{F}_q$.

Definition 6.1.3. (a) Let $\tilde{\mathcal{C}}_{ss}$ be the set of tuples

$$\{((\sigma_1)^{k_1}, \dots, (\sigma_t)^{k_t}) \mid \sigma_i \in \bar{\mathbb{F}}_q^\times, \sigma_i \not\sim \sigma_j, 1 \leq i, j \leq t, i \neq j, \sum_{i=1}^t k_i d_{\sigma_i} = n\},$$

where d_{σ_i} denotes the degree of σ_i, for all i. In the following we identify the elements of $\tilde{\mathcal{C}}_{ss}$ with block diagonal matrices in $\mathrm{GL}_n(q)$ in the obvious way.

(b) Define an equivalence relation $\sim$ on $\tilde{\mathcal{C}}_{ss}$ as follows: For two elements $u = ((\sigma_1)^{k_1}, \dots, (\sigma_t)^{k_t})$ and $v = ((\tau_1)^{m_1}, \dots, (\tau_r)^{m_r}) \in \tilde{\mathcal{C}}_{ss}$ we write $u \sim v$ if

(i) $t = r$, and

(ii) there is a permutation w of the set $\{1, \ldots, t\}$ such that $k_{wi} = m_i$ and $\sigma_{wi} \sim \tau_i$ for all $1 \leq i \leq t$.

Then we define $\mathcal{C}_{ss} = \tilde{\mathcal{C}}_{ss}/\sim$. Moreover, we define $\mathcal{C}_{ss,\ell'}$ to be the subset of $\mathcal{C}_{ss}$ consisting of elements of order coprime to ℓ.

For an element $u = ((\sigma_1)^{k_1}, \ldots, (\sigma_t)^{k_t})$ in $\mathcal{C}_{ss}$, define $\kappa(u) = (k_1, \ldots, k_t)$. If $\underline{\lambda} = (\lambda_1, \ldots, \lambda_t)$ is a sequence of partitions such that $|\lambda_i| = k_i$ for all i, then we write $\underline{\lambda} \vdash \kappa(u)$.

From the Jordan decomposition of an $\mathbb{F}_q$-endomorphism of V we obtain:

Theorem 6.1.4. *The conjugacy classes of G are parametrized by the set*

$$\{(u, \underline{\lambda}) \mid u \in \mathcal{C}_{ss}, \underline{\lambda} \vdash \kappa(u)\}.$$

Proof. See [67, Theorem 2.5]. ☐

Next, define $X(G)$ to be the character ring of G over $\bar{\mathbb{Q}}_\ell$, an algebraic closure of the ℓ-adic number field $\mathbb{Q}_\ell$. To describe the parametrization of irreducible characters of G given in [39], we need to fix an embedding $\bar{\mathbb{F}}_q^\times \to \bar{\mathbb{Q}}_\ell^\times$. Then, by using the theory of Deligne–Lusztig, P. Fong and B. Srinivasan reproved the following result, which was first stated by J. A. Green in [43].

Theorem 6.1.5. *(Green [44], Fong–Srinivasan [39]) The irreducible characters of $X(G)$ can be parametrized by the set*

$$\{(u, \underline{\lambda}) \mid u \in \mathcal{C}_{ss}, \underline{\lambda} \vdash \kappa(u)\}.$$

Proof. See [39, §1]. ☐

In the following we will write $\chi_{u,\underline{\lambda}}$ for the character of $X(G)$ labelled by the pair $(u, \underline{\lambda}) \in \mathcal{C}_{ss}$. The next theorem is very important.

Theorem 6.1.6. *(Fong–Srinivasan [39]) The group algebra FG decomposes as*

$$FG \cong \bigoplus_{u \in \mathcal{C}_{ss,\ell'}} FB_u$$

into a direct sum of two-sided ideals. The set of characters in FB_u coincides with the set

$$\{\chi_{v,\underline{\lambda}} \mid \underline{\lambda} \vdash \kappa(v), v \in \mathcal{C}_{ss} \textit{ has } \ell\textit{-regular part conjugate to } u\}.$$

Proof. This follows from [39, Theorem 7A]. ☐

Definition 6.1.7. The characters of the form $\chi_{1,\lambda}$, where λ is a composition of n, are called the *unipotent* characters of G. The indecomposable direct summands of the $FG - FG$-bimodule FB_1 are the *unipotent blocks* of G.

Let λ be a partition of a non-negative integer m, and r be a positive integer. By successively removing r-hooks from the diagram of λ, we obtain a partition of an integer $m' \leq m$ from which no r-hooks can be removed. Suppose that the number of removed r-hooks is w. Then $m' = m - wr$ and w is called the *weight* of λ. A partition from which no r-hooks can be removed is called an *r-core*.

For our considerations the following is important.

Proposition 6.1.8. *The blocks of FB_1 are parametrized by pairs (τ, w), where w is a non-negative integer and τ is an e-core of size $n - ew$.*

Proof. This is again a corollary of [39, Theorem 7A]. $\square$

6.2 The Hecke algebras associated to G

In this section we will assume that $\ell \nmid q, q-1$. Let B be the standard Borel subgroup of G, i.e., B is the set of upper triangular matrices in G. It is easy to see that

$$|B| = q^{\frac{n(n-1)}{2}}(q-1)^{n-1},$$

thus, by our assumption on ℓ and q, we have that $|B|$ is invertible in F. Therefore, the element

$$e_B = \frac{1}{|B|}\sum_{b \in B} b$$

is defined and is an idempotent in FG. In fact, it is the central primitive idempotent of FB corresponding to the trivial FB-module, denoted by 1_B. Then,

$$M_B := \mathrm{Ind}_B^G(1_B) = FGe_B \in FG$$

is the permutation representation of G on the left cosets of B in G, i.e., the representation of FG obtained by letting G act from the left on the cosets of B in G. Note that $M \cong \mathrm{R}_T^G(1_T)$ as FG-modules, where we view $1_T \cong 1_{\mathrm{GL}_1(q)} \boxtimes \cdots \boxtimes 1_{\mathrm{GL}_1(q)}$ as a cuspidal $F(\mathrm{GL}_1(q) \times \cdots \times \mathrm{GL}_1(q))$-module. Recall that if L is a standard Levi subgroup of G, then an FL-module N is called *cuspidal* if ${}^*\mathrm{R}_{L'}^L(N) = 0$, for all proper standard Levi subgroups L' of L.
Letting endomorphisms act on the right, it is well known that

$$\mathrm{End}_{FG}(M) \cong e_B FG e_B$$

as F-algebras, see for example [8, Lemma 1.3.3]. On the other hand, there is a right action of the Hecke algebra $H_n^f(q)$ defined over F on M, see for example [17, §2.5]. In fact one gets an embedding of $H_n^f(q)$ into $\mathrm{End}_{FG}(M)$ which is actually an isomorphism of F-algebras, see [17, 2.5(b)]. Therefore, through the above isomorphisms we may identify $H_n^f(q)$ with $e_B FG e_B$. Moreover, left multiplication with the idempotent e_B gives a functor

$$H : FG - \mathrm{mod} \longrightarrow H_n^f(q) - \mathrm{mod}, \tag{6.8}$$

defined on objects by $H(X) = e_B X$, and on morphisms by restriction. The functor H is called *Schur functor*. Since M_B is a projective FG-module, the above functor coincides with the functor defined in [31, §2]. The functor H has the following remarkable property:

Proposition 6.2.1. *The functor H induces a bijection between the irreducible $H_n^f(q)$-modules and the irreducible FG-modules occurring in the head of M_B. In particular, if S is an irreducible FG-module, then $H(S) = e_B S \neq 0$ if and only if S is a constituent of* $\mathrm{hd}(M_B)$.

Proof. See [17, Corollary 3.2e]. □

Note that we may define H also over $\mathcal{O}$ or K. In the following let $R \in \{K, F\}$. Recall from [50, §6] that given an element s of the form (6.7) of degree d and a partition $\lambda \vdash k$, where $n = kd$, one can construct a certain RG-module $M_R(s, \lambda)$. The latter module contains a distinguished RG-submodule $S_R(s, \lambda)$ that contains a unique maximal submodule with factor module an irreducible RG-module, denoted by $D_R(s, \lambda)$. As it turns out, these modules give a complete list of non-isomorphic irreducible RG-modules. In the case $R = K$, the modules $S_K(s, \lambda)$ are already irreducible, and the modules $S_F(s, \lambda)$ are ℓ-modular reductions of the modules $S_K(s, \lambda)$, see [31, 3.1]. In particular, in the case $s = 1$, the set

$$\{D_R(1, \lambda) \mid \lambda \vdash n\}$$

is a complete set of non-isomorphic irreducible modules for RB_1. Note that for a partition $\lambda \vdash n$, the module $M_R(1, \lambda)$ can also be viewed as the permutation module of G on the standard parabolic subgroup associated to λ.

Let A be a block of FG. By multiplication with e_B, we obtain an algebra $e_B A e_B$, which can be thought of as a non-unitary subalgebra of $H_n^f(q) = e_B FG e_B$. The next proposition identifies these algebras in $H_n^f(q)$.

Proposition 6.2.2. *Let A be a block of FG. If A is not a direct summand of FB_1, then $e_B A e_B = 0$. Otherwise, $e_B A e_B$ is a block of $H_n^f(q)$.*

Proof. By [17, Lemma 2.4c], all the composition factors of $M = FGe_B$ belong to FB_1. Hence, if A is not a direct summand of FB_1, then, by Proposition 6.2.1, we get that $e_B A e_B = 0$.
For λ a partition of n and $R \in \{K, F\}$, we denote by S_R^λ the Specht module defined by Dipper and James in [29] for the Hecke algebra $RH_n^f(q) = R \otimes_{\mathcal{O}} H_n^f(q)$. Then, from [31, §3.1] we know that $H(S_K(1, \lambda)) = S_K^\lambda$. Since $S_F(1, \lambda)$ and S_F^λ are ℓ-modular reductions of the modules $S_K(1, \lambda)$ and S_K^λ, we conclude that if A is a unipotent block of FG, then $e_B A e_B$ is non-zero. By [30, Corollary 4.4], the blocks of $H_n^f(q)$ are parametrized in the same way as the blocks of FB_1. □

Remark 6.2.3. It follows immediately from Proposition 6.2.2 that if A is a unipotent block of FG, which is labelled by (w, τ), then the corresponding block $e_B A e_B$ of $H_n^f(q)$ is labelled with (w, τ).

6.3 A Morita equivalence for Rouquier blocks

Recall from Section 6.1 and Section 6.2 that both, the unipotent blocks of FG and the blocks of $H_n^f(q)$, can be parametrized by pairs (w, τ), where w is a non-negative integer and τ is an e-core.
For the next definition we recall the notion of an abacus display of a partition of n, see also Chapter 17 of Part II.
Let d be a positive integer. Then to a partition $\lambda = (\lambda_1, \ldots, \lambda_m)$ of n one associates a sequence of non-negative integers β_i, $1 \leq i \leq d$, by setting

$$\beta_i = \lambda_i - i + d,$$

if $1 \leq i \leq m$, and

$$\beta_i = d - i,$$

if $i > m$. These integers are displayed on an abacus with e runners, where for each $1 \leq i \leq d$ a bead is inserted on runner $a + 1$ and row $b + 1$ if $\beta_i = eb + a$.

Definition 6.3.1. (Rouquier core, Rouquier block) Let w be a non-negative integer. An e-core ρ is said to be a *Rouquier core* with respect to w if there is some positive integer d such that in the d-abacus representation of ρ, for any pair of adjacent runners there are at least $w - 1$ more beads on the right-hand runner. A unipotent block of G is said to be a *Rouquier block* if the e-core associated to the latter is a Rouquier core. Similarly, we define Rouquier blocks for the algebras $F\mathfrak{S}_n$ and $H_n^f(q)$.

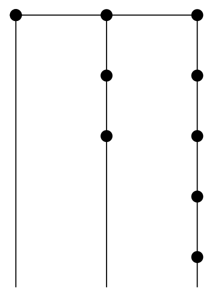


Figure 6.1: Abacus display of the 3-core $\rho = (11, 4, 2^2, 1^2)$.

In what follows we assume that $\mathrm{char}(F) = \ell > w > 0$, and q is a prime power such that $\ell \nmid q, q-1$. Then e is the multiplicative order of q in $\mathbb{F}_\ell$. Furthermore, we denote by L_λ the standard Levi subgroup of $\mathrm{GL}_{|\lambda|}(q)$ associated to the partition λ, see (6.1). The following was proved independently by W. Turner in [83] and H. Miyachi in [71]. We state the version of [21].

Theorem 6.3.2. *Let ρ be a Rouquier e-core with respect to w. Let $n = ew + |\rho|$. Let $\lambda = (e^w, |\rho|)$, a composition of n, and let $L := L_\lambda$ be the corresponding standard Levi subgroup of $G = \mathrm{GL}_n(q)$. Also, let $P = LU$ be the standard parabolic subgroup in G determined by L with complement U. Set $I := \mathrm{N}_G(L) \cong L \rtimes \mathfrak{S}_w$. Thus,*

$FI \cong F[L \rtimes \mathfrak{S}_w]$. Then there exists an (FG, FI)-bimodule M such that the following hold:

(a) *M is a direct summand of $\mathrm{R}_L^G(FI)$ as an (FG, FI)-bimodule.*

(b) *The functor $M \otimes_{A^*} - : A^* - \mathrm{mod} \to A - \mathrm{mod}$ induces a Morita equivalence between the principal block A^* of FI and the Rouquier unipotent block A of FG.*

Proof. See [21, Theorem 13]. □

By applying Schur functors we may also obtain a Hecke algebra version of the latter theorem. This is given in [84, Theorem 80], and we will sketch the arguments here. First of all we need an analogue of the semi-direct product for Hecke algebras. Here, the appropriate construction is the wreath product of an algebra with the symmetric group $\mathfrak{S}_w$, which we now define.
Let C be an F-algebra. Then the symmetric group $\mathfrak{S}_w$ acts on the w-fold tensor product $C^{\otimes w}$ of C over F by permuting the factors. More precisely, an element $\sigma \in \mathfrak{S}_w$ defines an F-algebra automorphism on $C^{\otimes w}$ by setting

$$\sigma(c_1 \otimes \cdots \otimes c_w) = c_{\sigma^{-1}1} \otimes \cdots \otimes c_{\sigma^{-1}w},$$

for homogeneous elements $c_1 \otimes \cdots \otimes c_w \in C^{\otimes w}$, and extending linearly. Then the *wreath product* $C \wr \mathfrak{S}_w$ is defined to be the skew group algebra of $\mathfrak{S}_w$ over $C^{\otimes w}$. Note that $C \wr \mathfrak{S}_w$ may be identified with the F-algebra whose underlying vector space is $C^{\otimes w} \otimes_F F\mathfrak{S}_w$, and where the multiplication is given by

$$(c \otimes \sigma)(d \otimes \tau) = c\sigma(d) \otimes \sigma\tau, \tag{6.9}$$

for elements $c, d \in C^{\otimes w}$ and elements $\sigma, \tau \in \mathfrak{S}_w$. For more details concerning the representation theory of wreath products, the reader is referred to [23].

Recall the definition of the idempotent e_B of FG from Section 6.2. Let B_e and $B_{|\rho|}$ be the standard Borel subgroups of $\mathrm{GL}_e(q)$ and $\mathrm{GL}_{|\rho|}(q)$, respectively, and denote by B_L the group

$$\underbrace{B_e \times \cdots \times B_e}_{w \text{ times}} \times B_{|\rho|},$$

considered as a subgroup of the Levi subgroup $L := L_\lambda$, $\lambda = (e^w, |\rho|)$, defined in Theorem 6.3.2 in the obvious way. Let P be the parabolic subgroup defined by L with Levi complement U. Then we define the element

$$f_{B_L} = \frac{1}{|B_e|^w |B_{|\rho|}|} \sum_{x \in B_L} x,$$

an idempotent in FL. If $f^i_{B_e}$, $1 \leq i \leq w$, denotes the idempotent

$$\frac{1}{|B_e|} \sum_{y \in B_e} y$$

in FB_e, and $f_{B_{|\rho|}}$ denotes the idempotent

$$\frac{1}{|B_{|\rho|}|}\sum_{z\in B_{|\rho|}} z$$

in $FB_{|\rho|}$, then we have that $f_{B_L} = (\prod_{i=1}^{w} f^i_{B_e})f_{B_{|\rho|}}$. Here, we consider B_e as the ith factor and $B_{|\rho|}$ as the $(w+1)$th factor embedded in B_L. Clearly, $f^i_{B_e}f^j_{B_e} = f^j_{B_e}f^i_{B_e}$ and $f^i_{B_e}f_{B_{|\rho|}} = f_{B_{|\rho|}}f^i_{B_e}$, for all $1 \leq i,j \leq w$. Then we get that as F-algebras

$$f_{B_L}FLf_{B_L} \cong (f^1_{B_e}F\mathrm{GL}_e(q)f^1_{B_e}) \otimes_F \cdots \otimes_F (f^w_{B_e}F\mathrm{GL}_e(q)f^w_{B_e}) \otimes_F (f_{B_{|\rho|}}F\mathrm{GL}_{|\rho|}(q)f_{B_{|\rho|}}).$$

In Section 6.2 we have seen that we may identify the algebra $f^i_{B_e}F\mathrm{GL}_e(q)f^i_{B_e}$ with the algebra $H^f_e(q)$, for all $1 \leq i \leq w$. Similarly, we can identify the algebra $f_{B_{|\rho|}}F\mathrm{GL}_{|\rho|}(q)f_{B_{|\rho|}}$ with the algebra $H^f_{|\rho|}(q)$. Therefore, we get an isomorphism of F-algebras

$$f_{B_L}FLf_{B_L} \cong H^f_e(q) \otimes_F \cdots \otimes_F H^f_e(q) \otimes_F H^f_{|\rho|}(q), \tag{6.10}$$

which is isomorphic to H_λ, the parabolic subalgebra of $H^f_n(q)$ associated to λ. We view this isomorphism as an identification.
Next, let $M^e = \mathrm{Ind}^{\mathrm{GL}_e(q)}_{B_e}(1_{B_e})$ be the permutation module on the Borel subgroup B_e of $\mathrm{GL}_e(q)$, see Section 6.2. Similarly, we define $M^{|\rho|} := \mathrm{Ind}^{\mathrm{GL}_{|\rho|}(q)}_{B_{|\rho|}}(1_{B_{|\rho|}})$. As in Section 6.2, the module M^e becomes an $(F\mathrm{GL}_e(q), H^f_e(q))$-bimodule and $M^{|\rho|}$ is an $(F\mathrm{GL}_{|\rho|}(q), H^f_{|\rho|}(q))$-bimodule. If we identify $\mathrm{GL}_e(q)$ as the ith factor of L, $1 \leq i \leq w$, and $\mathrm{GL}_{|\rho|}(q)$ as the $(w+1)$th factor of L, then we have that $M^e \cong F\mathrm{GL}_e(q)f^i_{B_e}$, and $M^{|\rho|} \cong F\mathrm{GL}_{|\rho|}(q)f_{B_{|\rho|}}$. With these identifications, the outer tensor product

$$M^\lambda := \underbrace{M^e \boxtimes \cdots \boxtimes M^e}_{w \text{ times}} \boxtimes M^{|\rho|}$$

is an (FL, H_λ)-bimodule, and as such, isomorphic to FLf_{B_L}. It follows that $\mathrm{R}^G_L(M^\lambda)$ is an (FG, H_λ)-bimodule.
Recall the definition of the $(FG, H^f_n(q))$-bimodule $M_B = \mathrm{Ind}^G_B(1_B) = FGe_B$, given in Section 6.2. If H_λ is the parabolic subalgebra of $H^f_n(q)$ corresponding to the partition $\lambda \vdash n$, then M_B becomes an (FG, H_λ)-bimodule. One has the following:

Lemma 6.3.3. *As (FG, H_λ)-bimodules, we have that $M_B \cong \mathrm{R}^G_L(M^\lambda)$.*

Proof. This is [17, Lemma 3.2f]. □

Remark 6.3.4. Let $U^+ = \frac{1}{|U|}\sum_{u\in U} u$, an idempotent in U. The isomorphism of the previous lemma is given by the map

$$FGU^+ \otimes_{FL} FLf_{B_L} \longrightarrow FGe_B,\ a\otimes b \mapsto ab.$$

Note that this is well defined since L normalizes U and $U^+f_{B_L} = e_B$.

Next we want to apply the Schur functors from (6.8):

$$H_B : FG - \text{mod} \longrightarrow H_n^f(q) \text{ and } H_{B_L} : FL - \text{mod} \longrightarrow H_\lambda - \text{mod}. \tag{6.11}$$

Recall the definition of the (FL, FI)-bimodule $FI = F[L \rtimes \mathfrak{S}_w]$ from Theorem 6.3.2. Since FI may be identified with the wreath product $FL \wr \mathfrak{S}_w$, we have that as left FL-modules

$$FI \cong \bigoplus_{\sigma \in \mathfrak{S}_w} FL \otimes \sigma \cong \bigoplus_{\sigma \in \mathfrak{S}_w} {}_{FL}FL.$$

Since the action of $\mathfrak{S}_w$ on L leaves the idempotent f_{B_L} invariant, we get an isomorphism

$$FIf_{B_L} \cong \bigoplus_{\sigma \in \mathfrak{S}_w} FLf_{B_L} \otimes \sigma \tag{6.12}$$

of left FL-modules. As Harish-Chandra induction is an exact functor, together with Lemma 6.3.3 we infer that

$$\mathrm{R}_L^G(FIf_{B_L}) \cong \bigoplus_{\sigma \in \mathfrak{S}_w} \mathrm{R}_L^G(FLf_{B_L} \otimes \sigma) \cong \bigoplus_{\sigma \in \mathfrak{S}_w} M_B \otimes \sigma \tag{6.13}$$

as left FG-modules. Observe that by using the isomorphism given in Remark 6.3.4, we get a natural right $e_{B_L}FIe_{B_L}$-module structure on $\bigoplus_{\sigma \in \mathfrak{S}_w} M_B \otimes \sigma$.
On the other hand, let C^w be the skew group algebra of $\mathfrak{S}_w$ over H_λ, where $\mathfrak{S}_w$ acts on the latter algebra by permuting the first w factors. By the discussion after Theorem 6.3.2, we have that $C^w \cong (H_e^f(q) \wr \mathfrak{S}_w) \otimes_F H_{|\rho|}^f(q)$ as F-algebras.
Consider now the $(H_n^f(q), C^w)$-bimodule $H_n^f(q) \otimes_{H_\lambda} C^w$, where the left $H_n^f(q)$-module and the right C^w-module structure is given by left and right multiplication with the corresponding algebras. Since $C^w \cong \bigoplus_{\sigma \in \mathfrak{S}_w} H_\lambda \otimes \sigma$ as left H_λ-modules, we get that

$$H_n^f(q) \otimes_{H_\lambda} C^w \cong \bigoplus_{\sigma \in \mathfrak{S}_w} H_n^f(q) \otimes_{H_\lambda} (H_\lambda \otimes \sigma) \cong \bigoplus_{\sigma \in \mathfrak{S}_w} H_n^f(q) \otimes \sigma$$

as left $H_n^f(q)$-modules. Identifying $H_n^f(q)$ with e_BFGe_B and H_λ with $f_{B_L}FLf_{B_L}$, by using (6.13), we have that

$$H_n^f(q) \otimes_{H_\lambda} C^w \cong e_B\mathrm{R}_L^G(FIf_{B_L}) \cong e_B\mathrm{R}_L^G(FI)f_{B_L}$$

as $(H_n^f(q), C^w)$-bimodules.
Next, set $N = e_BMf_{B_L}$, where M is the (FG, FI)-bimodule of Theorem 6.3.2. Then N becomes an $(H_n^f(q), C^w)$-bimodule and, as such, it is a direct summand of $H_n^f(q) \otimes_{H_\lambda} C^w$, see Theorem 6.3.2 (a). Combining this with the arguments of W. Turner, see [84, Theorem 80], we get:

Theorem 6.3.5. *Keep the notation from above. Let ρ be a Rouquier e-core with respect to $w > 0$ and $n = ew + |\rho|$. Then we have the following:*

(a) *N is a direct summand of $H_n^f(q) \otimes_{H_\lambda} C^w$ as an $(H_n^f(q), C^w)$-bimodule.*

(b) *The functor $N \otimes_{C^w} - : C^w - \mathrm{mod} \to D_F - \mathrm{mod}$ induces a Morita equivalence between the algebra C^w and the Rouquier block D_F of $H_n^f(q)$.*

Since $H_{|\rho|}^f(q)$ is a split semisimple algebra, we get the following corollary:

Corollary 6.3.6. *Keep the notation of Theorem 6.3.5. The Rouquier block D_F of $H_n^f(q)$ is Morita equivalent to the algebra $B_0(H_e^f(q)) \wr \mathfrak{S}_w$.*

By a lifting argument due to J. Chuang, see [21, §6], we also get a version of the last corollary in the characteristic zero case.

Theorem 6.3.7. *Let $1 \neq \zeta_e \in \mathbb{C}$ be a primitive eth root of unity, and ρ, w, n be as above. Moreover, suppose that $\Bbbk$ is a field of characteristic 0 containing ζ_e. Then the Rouquier block $D_\Bbbk$ of $H_n^f(\zeta_e)$ and the $\Bbbk$-algebra $B_0(H_e^f(\zeta_e)) \wr \mathfrak{S}_w$ are Morita equivalent.*

Proof. See [21, Theorem 18 and §6]. □

Remark 6.3.8. Recently, in [38], A. Evseev deduced a generalization of Theorem 6.3.7 using methods from the representation theory of KLR-algebras.

Chapter 7

Stable components for blocks of Hecke algebras of type A

In this section, we will finally derive the main result of the chapter, which classifies the tree classes of connected components of the stable AR-quiver of blocks of Hecke algebras of type A in characteristic zero. To this end we shall need the famous theorem of J. Chuang and R. Rouquier giving derived equivalences between blocks of the same weight of Hecke algebras of type A, defined over the same field $\Bbbk$ with parameter q that is invertible in $\Bbbk$.
In the following let e be the multiplicative order of the defining parameter q in $\Bbbk$, i.e., the smallest positive integer e such that $q^e = 1$. We assume throughout that $2 \leq e < \infty$.

Theorem 7.1.9. *(Chuang–Rouquier [22]) Let A be a block of $H_n^f(q)$ and B be a block of $H_m^f(q)$. Then A and B are derived equivalent if and only if A and B have the same weight.*

Proof. This is [22, Theorem 7.12]. □

Remark 7.1.10. The equivalence constructed in the proof of Theorem 7.1.9 is induced by a complex of exact functors that are direct summands of compositions of refinements of the restriction and induction functors. Therefore, they induce a $\Bbbk$-linear triangulated equivalence.

Using [75, Corollary 2.2] and [7, X, Corollary 1.9], we can immediately deduce the following corollary:

Corollary 7.1.11. *If A and B have the same weight w, then A and B are stably equivalent. Furthermore, the stable AR-quivers $\Gamma_s(A)$ and $\Gamma_s(B)$ are isomorphic as stable translation quivers.*

Remark 7.1.12. If A is a block of weight $w = 1$, then A is a Brauer tree algebra, whose Brauer tree is a line with no exceptional vertex, see [42, Theorem 11.4] and [72, Theorem 1.4].

Next recall the notions of a Rouquier block and a Rouquier core from Definition 6.3.1. It is easy to see that for a given weight w, it is always possible to find a Rouquier core ρ and a positive integer r such that $r = ew + |\rho|$, i.e., there is a Rouquier block of weight w. This and Theorem 7.1.9 yield:

Lemma 7.1.13. *Every block of weight w of some Hecke algebra of type A is derived equivalent to a Rouquier block of weight w defined over the same field.*

In the sequel, we will denote by B the principal block of the Hecke algebra $H_e^f(q)$. To state our main result, we will need one further result concerning the finite generation of the Hochschild cohomology ring of outer tensor products of B. Recall that a $\Bbbk$-algebra A is called *separable* if for every extension field $\mathbb{E}$ of $\Bbbk$ the $\mathbb{E}$-algebra $\mathbb{E} \otimes_{\Bbbk} A$ is semisimple.

Lemma 7.1.14. *Suppose that $B/\mathrm{rad}(B)$ is a separable $\Bbbk$-algebra. Then for any $k \geq 0$, the algebra $B^{\otimes k}$ has enough Ω-periodic modules.*

Proof. The principal block B of $H_e^f(q)$ is a Brauer tree algebra, hence, by [63, Proposition 4.9] and the remark after, the Hochschild cohomology ring $HH^*(B)$ is noetherian and the $\mathrm{Ext}^*_{B\otimes_k B^{\mathrm{op}}}(B, U)$ is noetherian as a $HH^*(B)$-module for every finitely generated $B - B$-bimodule U. By [35, Proposition 1.4] the latter is equivalent to the statement that $\mathrm{Ext}^*_B(B/\mathrm{rad}(B), B/\mathrm{rad}(B))$ is finitely generated as an $HH^*(B)$-module. Then by [81, Proposition 5.7] we have that B satisfies the finite generation hypotheses (Fg1) and (Fg2) of [35]. It follows from [13, Corollary 4.8], by imposing the trivial grading on B, that the algebra $B^{\otimes k}$, $k \geq 1$, satisfies the finite generation hypotheses as well. Then, by [35, Theorem 4.5], $B^{\otimes k}$ has enough Ω-periodic modules. ☐

We can now state the main theorem of the first part of this thesis:

Theorem 7.1.15. *Let $\Bbbk$ be an algebraically closed field of characteristic zero, and A be a block of $H_n^f(q)$ of e-weight w. Then we have the following:*

(a) *If $w = 1$, then A is of finite representation type. In this case the stable AR-quiver $\Gamma_s(A)$ has just one component which is of tree class A_{e-1}.*

(b) *If $e = 2$ and $w = 2$, then A is of tame representation type. The stable AR-quiver of A consists of*

  (i) *infinitely many tubes of rank one,*

  (ii) *two tubes of rank two,*

  (iii) *one component of the form $\mathbb{Z}\tilde{A}_{2,2}$.*

(c) *If $e = 2$ and $w \geq 3$, or $e \geq 3$ and $w \geq 2$, then A is wild and all components of the stable AR-quiver $\Gamma_s(A)$ of A have tree class A_∞.*

Proof. The statement about the representation type is [36, Theorem 3.1] together with [40, Theorem 3.3], and statement (a) can be found in [85].
For (b) we use the Morita equivalence given in Theorem 6.3.7. In this case, the principal block B of $H_e^f(q)$ is isomorphic to the algebra $\Bbbk[X]/(X^2)$, and hence, $B \otimes_{\Bbbk} B \cong \Bbbk[X,Y]/(X^2,Y^2)$. Then we have an isomorphism of $\Bbbk$-algebras between the algebra $B^{\otimes 2} \star \mathfrak{S}_2$ and the skew group algebra of $\mathfrak{S}_2$ over $\Lambda := \Bbbk[X,Y]/(X^2,Y^2)$, where $\mathfrak{S}_2$ acts on Λ by sending X to Y and Y to X. Let $C := \Lambda \star \mathfrak{S}_2$. Note that $\mathfrak{S}_2$ is cyclic of order two, generated by the basic transposition $g = (1,2)$. By Corollary 7.1.11, it is enough to show the statement for the Rouquier block of weight two, which in turn is Morita equivalent to C by Theorem 6.3.7.
First, we will determine a quiver with relations for C. To this end we have to determine the Loewy structures of the projective indecomposable modules of C. Let

$$f := \frac{1}{2}(1 \star 1 + 1 \star g),$$

an idempotent in C. Then, we get a decomposition

$$C \cong Cf \oplus C(1-f)$$

of the left regular module C. In the following we set $f_1 := f$, $f_2 = 1 - f$, and $P_i := Cf_i$, $i = 1,2$. For $i = 1,2$, the projective module P_i has $\Bbbk$-basis the set

$$\{f_i, Xf_i := (X \star 1)f_i, Yf_i := (Y \star 1)f_i, XYf_i := (XY \star 1)f_i\},$$

and it follows that $\mathrm{Res}_{\Lambda}^{C}(P_i) \cong \Lambda$ as left Λ-modules. Thus, P_i is indecomposable as a C-module. In the following, we will denote by S_i the irreducible C-module corresponding to the indecomposable projective module P_i, $i = 1,2$.
Since $|\mathfrak{S}_2|$ is invertible in $\Bbbk$, by [79, Theorem 1.3], we have that $\mathrm{rad}(C) = \mathrm{rad}(\Lambda)C$. It follows that

$$\mathrm{rad}(P_i) = \mathrm{rad}(C)P_i = \mathrm{rad}(\Lambda)P_i,$$

for $i = 1,2$. Note that, as a Λ-module, $\mathrm{rad}(\Lambda)$ is generated by the set $\{X, Y, XY\}$. Therefore,

$$\begin{aligned} \mathrm{rad}(P_i) &= \mathrm{span}(\{Xf_i, Yf_i, XYf_i\}), \\ \mathrm{rad}^2(P_i) &= \mathrm{span}(\{XYf_i\}), \\ \mathrm{rad}^3(P_i) &= 0. \end{aligned} \tag{7.1}$$

Next, consider the subspaces $\Bbbk(Xf_i + Yf_i)$ and $\Bbbk(Xf_i - Yf_i)$ of $\mathrm{rad}(P_i)$. Since $\mathrm{char}(\Bbbk) \neq 2$, the latter intersect trivially. By (7.1), the sum of these spaces form a complement of $\mathrm{rad}^2(P_i)$ in $\mathrm{rad}(P_i)$, hence, the elements $(Xf_i + Yf_i) + \mathrm{rad}^2(P_i)$ and $(Xf_i - Yf_i) + \mathrm{rad}^2(P_i)$ form a basis of the vector space $\mathrm{rad}(P_i)/\mathrm{rad}^2(P_i)$. Moreover, the latter generate two irreducible C-submodules S_{i1} and S_{i2} of $\mathrm{rad}(P_i)/\mathrm{rad}^2(P_i)$. By direct computation, we have that

$$\begin{aligned} f_i(Xf_i + Yf_i) &= Xf_i + Yf_i, \\ f_i(Xf_i - Yf_i) &= 0, \end{aligned}$$

so that $S_{11} \cong S_{22} \cong S_1$ and $S_{12} \cong S_{21} \cong S_2$. Therefore, for $i = 1, 2$, the radical layers of the projective indecomposable C-modules P_i have the following form:

$$\begin{aligned} P_i/\mathrm{rad}(P_i) &\cong \mathrm{rad}^2(P_i) \cong S_i, \\ \mathrm{rad}(P_i)/\mathrm{rad}^2(P_i) &\cong S_1 \oplus S_2. \end{aligned} \tag{7.2}$$

From this information, we may draw the Ext-quiver Q of C, which is given in the figure below.

$$\begin{tikzcd} \bullet \arrow[loop left, "\varepsilon"] \arrow[r, shift left, "\alpha"] & \bullet \arrow[l, shift left, "\beta"] \arrow[loop right, "\gamma"] \end{tikzcd}$$

Figure 7.1: The Ext-quiver of C.

Denote by x_1 (resp. x_2) the vertex of Q in which the arrow α (resp. β) starts. We want to construct a surjective $\Bbbk$-algebra homomorphism $\Bbbk Q \to C$. To this end, we use Theorem 5.1.3. By explicit calculations, we get the following:

$$\begin{aligned} f_1\mathrm{rad}(C)f_1 &= \mathrm{span}(f_1Xf_1, f_1XYf_1), \\ f_1\mathrm{rad}^2(C)f_1 &= \mathrm{span}(f_1XYf_1), \\ f_2\mathrm{rad}(C)f_1 &= \mathrm{span}(f_2Xf_1), \\ f_2\mathrm{rad}^2(C)f_1 &= 0, \\ f_2\mathrm{rad}(C)f_2 &= \mathrm{span}(f_2Yf_2, f_2XYf_2), \\ f_2\mathrm{rad}^2(C)f_2 &= \mathrm{span}(f_2XYf_2), \\ f_1\mathrm{rad}(C)f_2 &= \mathrm{span}(f_1Yf_2), \\ f_1\mathrm{rad}^2(C)f_2 &= 0. \end{aligned} \tag{7.3}$$

Therefore, by choosing an appropriate basis of a complement of $f_j\mathrm{rad}^2(C)f_k$ in $f_j\mathrm{rad}(C)f_k$, $j, k = 1, 2$, and setting

$$x_1 \mapsto f_1,\ x_2 \mapsto f_2,\ \alpha \mapsto f_2Xf_1,\ \beta \mapsto f_1Yf_2,\ \varepsilon \mapsto f_1Xf_1,\ \gamma \mapsto f_2Yf_2,$$

by Theorem 5.1.3, we get a surjective $\Bbbk$-algebra homomorphism $\psi : \Bbbk Q \to C$. Set $I := \mathrm{Ker}(\psi)$. Then immediately from the above assignment we get that the two-sided ideal

$$J := \langle \varepsilon^k,\ k \geq 3,\ \gamma^j,\ j \geq 3,\ \alpha\varepsilon,\ \varepsilon\beta,\ \gamma\alpha,\ \beta\gamma,\ \gamma^2 - \alpha\beta,\ \varepsilon^2 - \beta\alpha\rangle_{\Bbbk Q} \tag{7.4}$$

of $\Bbbk Q$ is contained in I. On the other hand, we see directly from (7.4), that the set of paths

$$\mathcal{P} := \{x_1, x_2, \alpha, \beta, \varepsilon, \varepsilon^2, \gamma, \gamma^2\}$$

is contained in the set-theoretic complement of I in $\Bbbk Q$. Also from (7.3) and the definition of ψ, we have that $\psi(\mathcal{P})$ is a basis of C. Therefore, the subspace U of $\Bbbk Q$ spanned by the elements in $\mathcal{P}$ intersects I trivially, thus, is a complement of I

in $\Bbbk Q$. Since all paths in Q are linear combinations of the paths in $\mathcal{P}$ and elements of $\Bbbk Q$ generated by J, we must have $I = J$.
In other words, C is given by the quiver Q of Figure 7 together with the relations

$$\alpha\varepsilon = 0 = \varepsilon\beta,\ \gamma\alpha = 0 = \beta\gamma,\ \varepsilon^2 = \beta\alpha,\ \gamma^2 = \alpha\beta.$$

From the structure of Q and the relations, we see that $C \cong \Bbbk Q/I$ is special biserial and, hence, tame, by [24].
To determine the structure of $\Gamma_s(C)$, we use [34, Chapter II] as a background material. Instead of C, one works with the algebra $C' := C/\mathrm{soc}(C)$, which is a string algebra. Note that C' is given by the same quiver as C but with relations

$$\alpha\varepsilon = 0 = \varepsilon\beta,\ \gamma\alpha = 0 = \beta\gamma,\ \beta\alpha = 0 = \alpha\beta.$$

Moreover, since C is selfinjective, we have that $\Gamma_s(C) = \Gamma(C')$, hence, to determine $\Gamma_s(C)$ we may use techniques to compute AR-sequences over string algebras.
First, note that the word $w = \varepsilon\alpha^{-1}\gamma\beta^{-1}$ is the only band, and contributes infinitely many tubes of rank one to $\Gamma_s(C)$. Given an arrow in Q, one can construct an Auslander–Reiten sequence of C' with indecomposable middle term. By [34, II.6.2], these are given in the following list:

$$\begin{aligned}
&0 \longrightarrow M(\gamma^{-1}) \longrightarrow M(\gamma^{-1}\alpha\varepsilon^{-1}) \longrightarrow M(\varepsilon^{-1}) \longrightarrow 0,\\
&0 \longrightarrow M(\varepsilon^{-1}) \longrightarrow M(\varepsilon^{-1}\beta\gamma^{-1}) \longrightarrow M(\gamma^{-1}) \longrightarrow 0,\\
&0 \longrightarrow M(\beta^{-1}) \longrightarrow M(\beta^{-1}\varepsilon\alpha^{-1}) \longrightarrow M(\alpha^{-1}) \longrightarrow 0,\\
&0 \longrightarrow M(\alpha^{-1}) \longrightarrow M(\alpha^{-1}\gamma\beta^{-1}) \longrightarrow M(\beta^{-1}) \longrightarrow 0.
\end{aligned}$$

Here, $M(D)$ denotes the string module of C' associated to the string D. Therefore, we get that $\Gamma_s(C)$ has two components that are tubes of rank two. It now follows from [37, Theorem 2.1] that $\Gamma_s(C)$ also contains a component of the form $\mathbb{Z}\tilde{A}_{2,2}$. This already determines the structure of $\Gamma_s(C)$ completely and finishes the proof of (b).

Again, to prove (c), it is enough to consider Rouquier blocks. Let B be the principal block of $H_e^f(q)$. We know from Theorem 4.2.4 and Theorem 5.5.6 that in the case where $e = 2$ and $w \geq 3$, or $e \geq 3$ and $w \geq 2$, every connected component of the stable AR-quiver $\Gamma_s(B^{\otimes w})$ of $B^{\otimes w}$ has tree class A_∞. Since $\mathrm{char}(\Bbbk) = 0$, we have by Lemma 7.1.14 that the algebra $B^{\otimes w}$ has enough periodic modules. Now, the claim follows from Theorem 3.2.9. $\square$

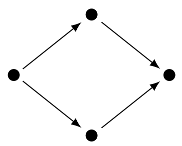
Remark 7.1.16. (i) A block A of $H_n^f(q)$ is of weight zero if and only if A is semisimple.
(ii) Compare part (b) of the theorem with [3, Lemma 6.5], where the stable AR-quiver of $H_4^f(-1)$ was computed.

In the following we list the Dynkin diagrams occurring as tree classes in Theorem 7.1.15, where the diagram of type A_n has n nodes.

A_n :

A_∞ :

The quiver $\tilde{A}_{2,2}$ is given as follows:

$\tilde{A}_{2,2}$: 

Part II

Divided power functors for cyclotomic Hecke algebras with an application to the Dipper–Du Conjecture

Chapter 8

Preliminaries

8.1 Goal

In the second part of the thesis we will describe and analyze certain functors first defined by I. Grojnowski in [44] in the context of cyclotomic Hecke algebras (the non-degenerate case) and later by A. Kleshchev in [59] in the case of degenerate cyclotomic Hecke algebras (the degenerate case). These functors were used to describe the Lie-theoretic structure of the irreducible modules of cyclotomic Hecke algebras of degree n, where n ranges over all non-negative integers. This was first discovered by S. Ariki in [2] in the case where the ground field is $\mathbb{C}$. Later, by using a different approach, this was generalized by I. Grojnowski to cyclotomic Hecke algebras defined over arbitrary fields.

One goal of the second part of the thesis is to explain and understand the theory developed by S. Ariki and I. Grojnowski explicitly.

Afterwards, as an application, we will deduce new results on the structure of vertices of Hecke algebras of the symmetric group. In particular we will verify a conjecture of R. Dipper and J. Du stated in [27] in the case where the modules under consideration lie in blocks of finite representation type. The conjecture states that the vertices of indecomposable modules are $l-p$-parabolic, with l the order of the defining parameter and p the characteristic of the ground field.

The following is intended to be self-contained. Most of the time, we will follow the lines of [59], where a good exposition of the material is given in the case of degenerate cyclotomic Hecke algebras.

8.2 Notation

Throughout this part, if not stated otherwise, F will denote an algebraically closed field of characteristic $p \geq 0$.

For an F-algebra A we denote by $A-\mathrm{Mod}$ (resp. $\mathrm{Mod}-A$) the abelian category of all left (resp. right) A-modules. By $A-\mathrm{mod}$ (resp. $\mathrm{mod}-A$) we will denote the abelian category of finite-dimensional left (resp. right) A-modules. Moreover, by

$A-\mathrm{proj}$ (resp. $\mathrm{proj}-A$) we mean the full subcategory of $A-\mathrm{mod}$ (resp. $\mathrm{mod}-A$) consisting of projective modules.
For F-algebras A, B we write $A \otimes B$ for the tensor product $A \otimes_F B$ of A and B over F, which is considered as an F-algebra in the usual way. If M is a left A-module and N is a left B-module, we denote by $M \boxtimes N$ the outer tensor product of M and N, i.e., as vector space $M \boxtimes N \cong M \otimes_F N$, and is considered as a left $A \otimes B$ - module by defining $(a \otimes b)(m \boxtimes n) = (am) \boxtimes (bn)$, for $a \in A$, $b \in B$, $m \in M$, $n \in N$.
If A is an F-algebra and $B \subseteq A$ is a subalgebra of A, we denote by

$$\mathrm{Ind}_B^A : B - \mathrm{mod} \longrightarrow A - \mathrm{mod},\ \mathrm{Ind}_B^A(N) = A \otimes_B N,$$

the induction functor,

$$\widehat{\mathrm{Ind}}_B^A : B - \mathrm{mod} \longrightarrow A - \mathrm{mod},\ \widehat{\mathrm{Ind}}_B^A(N) = \mathrm{Hom}_B(A, N),$$

the coinduction functor, and by

$$\mathrm{Res}_B^A : A - \mathrm{mod} \longrightarrow B - \mathrm{mod}$$

the restriction functor, where $\mathrm{Res}_B^A(M)$ is the left A-module M considered as a B-module by restriction of scalars. Note that Ind_B^A is a left adjoint of Res_B^A and $\widehat{\mathrm{Ind}}_B^A$ is a right adjoint.
Let $\mathcal{C}$ be an exact category, that is, an additive category together with a class of distinguished sequences of morphisms

$$L \longrightarrow M \longrightarrow N,$$

called exact sequences. Then we write $\mathcal{K}(\mathcal{C})$ for the Grothendieck group of $\mathcal{C}$, that is, the quotient of the free abelian group generated by the objects $M \in \mathrm{Ob}(\mathcal{C})$ by the ideal generated by the elements $L - M + N$ for every distinguished sequence $L \to M \to N$.
Throughout, if $\mathcal{C}$ is abelian, then we take the usual short exact sequences in $\mathcal{C}$ as the class of distinguished sequences.
Moreover, if $\mathcal{C}$ is supposed to be additive, then the class of distinguished sequences will consist of exact sequences of the form

$$0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0,$$

with $M \cong L \oplus N$.

Remark 8.2.1. If A is a finite-dimensional F-algebra, then the category $\mathcal{C}$ of finite-dimensional left A-modules is abelian. In this case, $\mathcal{K}(\mathcal{C})$ is a free abelian group with basis the set of isomorphism classes of irreducible A-modules.
Let $\mathcal{D}$ be the full subcategory of $\mathcal{C}$ consisting of projective A-modules. Then $\mathcal{D}$ is an additive category and the class of distinguished sequences is given as above.

Chapter 9

Affine Hecke algebras of type A

The intention of this chapter is to give the definition of the affine Hecke algebra in type A and state basic representation-theoretic results on the latter. We will also focus on the subalgebra of Laurent polynomials since this commutative subalgebra is essential in understanding the representation theory of affine Hecke algebras.
Later on, we define important functors on the module categories of the latter, which will play a decisive role in our further investigations.

9.1 Basic definitions

Throughout, we will denote by $W(n)$ the symmetric group acting on $\{1, \dots, n\}$ from the *left*, that is to say, the element $(1,2)(2,3)$ equals $(1,2,3)$. Also, we denote by $\mathbb{Z}_+$ the set of non-negative integers.
As a group, $W(n)$ is generated by the set $\mathfrak{B} = \{s_1 = (1,2), \dots, s_{n-1} = (n-1,n)\}$ of basic transpositions. Moreover, we denote by $H_n^f(q)$ the Iwahori–Hecke algebra over $W(n)$ with non-zero parameter $q \in F$. Recall from Section 1 in Part I that, as an F-algebra, the latter is generated by the symbols T_i, $1 \leq i \leq n-1$, together with the following relations:

(H1) $(T_i - q)(T_i + 1) = 0$.

(H2) $T_i T_j = T_j T_i$, for $1 \leq i < j-1 \leq n-2$.

(H3) $T_{i+1} T_i T_{i+1} = T_i T_{i+1} T_i$, $1 \leq i \leq n-2$.

Remark 9.1.1. From relation (H1) we deduce that $T_i^2 = (q-1)T_i + q$.

If $w = s_{i_1} \cdots s_{i_k}$, $1 \leq i_j \leq n-1$, is a reduced expression of w in $W(n)$, we define $T_w := T_{i_1} \cdots T_{i_k}$, an element of $H_n^f(q)$. For $w = 1$ we set T_w to be the unit element of $H_n^f(q)$. Note that the definition of T_w does not depend on the chosen reduced expression. Moreover, by $l(w)$ we denote the *length* of $w \in W(n)$, i.e., the minimal number of basic transpositions in a reduced expression of w.
Then, the following holds for the multiplication in $H_n^f(q)$, see [29, Lemma 2.1]:

$$T_s T_w = \begin{cases} T_{sw} & \text{if } l(sw) = l(w) + 1, \\ qT_{sw} + (q-1)T_w & \text{otherwise.} \end{cases} \tag{9.1}$$

Recall from [68, Theorem 1.13] that, as an F-vector space, $H_n^f(q)$ has basis the set

$$\{T_w \mid w \in W(n)\},$$

hence, $\dim_F(H_n^f(q)) = |W(n)| = n!$.

In the following, let

$$P_n = F[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$$

be the F-algebra of Laurent polynomials in the indeterminates $x_1, \dots, x_n$. If $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}^n$, we write

$$x^\alpha = x_n^{\alpha_n} \dots x_1^{\alpha_1}.$$

Note that for P_n we have the relation

$$x_i x_j = x_j x_i, \tag{9.2}$$

for $1 \le i \le j \le n$, together with

$$x_i x_i^{-1} = x_i^{-1} x_i = 1, \tag{9.3}$$

for $1 \le i \le n$.

Furthermore, $W(n)$ acts on P_n via

$$w \cdot x_i = x_{wi},$$

for all $w \in W(n)$ and $1 \le i \le n$.

Definition 9.1.2. Let $q \neq 1$. The affine Hecke algebra $H_n(q)$ is the associative F-algebra given by generators $x_1^{\pm 1}, \dots, x_n^{\pm 1}$ and $T_1, \dots, T_{n-1}$, subject to the relations (9.2), (9.3) and (H1) – (H3), together with

$$T_i x_j = x_j T_i, \tag{9.4}$$

$$T_i x_i = x_{i+1} T_i - (q-1) x_{i+1}, \tag{9.5}$$

$$T_i x_{i+1} = x_i T_i + (q-1) x_{i+1}, \tag{9.6}$$

for $j \neq i, i+1$, $1 \le i \le n-1$. Note that by using relation (H1), (9.5) and (9.6) are equivalent to

$$T_i x_i T_i = q x_{i+1}. \tag{9.7}$$

9.2 The ring of Laurent polynomials

In the previous section we defined the affine Hecke algebra $H_n(q)$, $n \geq 0$, together with the subalgebras $H_n^f(q)$ and P_n. Let

$$Z_n := F[x_1^{\pm 1}, \dots, x_n^{\pm 1}]^{W(n)} = \{f \in P_n \mid w \cdot f = f, \text{ for all } w \in W(n)\}, \tag{9.8}$$

the set of symmetric Laurent polynomials. Moreover, we set

$$P_n^+ := F[x_1, \dots, x_n] \text{ and } Z_n^+ := F[x_1, \dots, x_n]^{W(n)},$$

which we consider as subalgebras of $H_n(q)$ in the obvious way. The goal of this section is to describe the structure of P_n considered as a Z_n-module. The results that are proven here will be needed in Chapter 9.8.

The first lemma below can be found in [59, Theorem 1.0.2], and was proved by R. Steinberg in [82].

Lemma 9.2.1. *P_n^+ is free as a Z_n^+-module of rank $n!$. A Z_n^+-basis is given by the subset*

$$\{x_1^{a_1} \cdots x_n^{a_n} \mid 0 \leq a_i < i \text{ for all } 1 \leq i \leq n\}$$

of P_n^+.

From the previous lemma we can conclude the following fact about P_n:

Theorem 9.2.2. *P_n is free as a Z_n-module of rank $n!$. A Z_n-basis is given by the subset*

$$\mathcal{B}_n = \{x_1^{a_1} \cdots x_n^{a_n} \mid 0 \leq a_i < i \text{ for all } 1 \leq i \leq n\}$$

of P_n.

Proof. Let $z_1 b_1 + \dots + z_r b_r = 0$, for some $1 \leq r \leq n!$, elements $z_i \in Z_n$, and distinct elements $b_i \in \mathcal{B}_n$, $1 \leq i \leq r$. Let $t \in \mathbb{N}$ be the highest power of some x_i^{-1}, $1 \leq i \leq n$, occurring among $z_1, \dots, z_n$. Setting $X := \prod_{j=1}^n x_j^t$, we conclude that $X z_i \in Z_n^+$, for all i. But then $(Xz_1)b_1 + \ldots + (Xz_n)b_n = 0$, and it follows by Lemma 9.2.1 that $Xz_i = 0$, for all i. Since P_n is a domain, and $X \neq 0$, we conclude that $z_i = 0$, for all i. Therefore, $\mathcal{B}_n$ is linearly independent.
Next, let x^α, $\alpha \in \mathbb{Z}^n$, be a monomial in P_n. Denote by $Y^+ = \{i \mid 1 \leq i \leq n \text{ and } \alpha_i \geq 0\}$, where α_i is the ith component of the vector α. Similarly, we define $Y^- = \{j \mid 1 \leq j \leq n \text{ and } \alpha_j < 0\}$. Then we have:

$$\begin{aligned} x^\alpha &= \prod_{i \in Y^+} x_i^{\alpha_i} \prod_{j \in Y^-} (\prod_{k=1}^n x_k^{\alpha_j})(\prod_{k=1, k \neq j}^n x_k^{-\alpha_j}) \\ &= (\prod_{j \in Y^-} \prod_{k=1}^n x_k^{\alpha_j})(\prod_{i \in Y^+} x_i^{\alpha_i})(\prod_{j \in Y^-} \prod_{k=1, k \neq j}^n x_k^{-\alpha_j}). \end{aligned}$$

Now, $\prod_{j \in Y^-} \prod_{k=1}^n x_k^{\alpha_j} \in Z_n$, and $(\prod_{i \in Y^+} x_i^{\alpha_i})(\prod_{j \in Y^-} \prod_{k=1, k \neq j}^n x_k^{-\alpha_j}) \in P_n^+$. Hence, by Lemma 9.2.1, we can write x^α as a sum of elements in $\mathcal{B}_n$ with coefficients in Z_n. This finishes the proof. ☐

Corollary 9.2.3. *Let $a \in F$. The subset*

$$\mathcal{B}_n^a = \{(x_1 - a)^{a_1} \cdots (x_n - a)^{a_n} \mid 0 \leq a_i < i \text{ for all } 1 \leq i \leq n\}$$

of P_n is a Z_n-basis for P_n.

Proof. By Theorem 9.2.2 we know that the set

$$\mathcal{B}_n = \{x_1^{a_1} \cdots x_n^{a_n} \mid 0 \leq a_i < i \text{ for all } 1 \leq i \leq n\}$$

is a basis of P_n considered as Z_n-module. Let $(x_1 - a)^{a_1} \cdots (x_n - a)^{a_n} \in \mathcal{B}_n^a$. By binomial expansion, we see that

$$(x_1 - a)^{a_1} \cdots (x_n - a)^{a_n} = \sum_{\alpha \in \mathbb{Z}^n} \lambda_\alpha x^\alpha, \quad \lambda_\alpha \in F, \tag{9.9}$$

where for $\alpha = (\alpha_1, \ldots, \alpha_n)$, we have that $0 \leq \alpha_i < i$, for all $1 \leq i \leq n$. It is easy to see that for $\alpha_{\max} := (a_1, \ldots, a_n)$, we have $\lambda_{\alpha_{\max}} \neq 0$. Therefore, if we exchange $x_1^{a_1} \cdots x_n^{a_n}$ by the element $w := (x_1 - a)^{a_1} \cdots (x_n - a)^{a_n}$, the new set remains linearly independent over Z_n. To this end suppose that

$$z_w w \; + \sum_{\substack{\alpha \in \mathbb{Z}^n, \\ \alpha \neq \alpha_{\max}}} z_\alpha x^\alpha = 0, \quad z_w, z_\alpha \in Z_n,$$

where for all $\alpha = (\alpha_1, \ldots, \alpha_n)$, we have that $0 \leq \alpha_i < i$, $1 \leq i \leq n$. Then, if we plug in the expression (9.9) for w, we obtain

$$(\lambda_{\alpha_{\max}} z_w) x^{\alpha_{\max}} + \sum_{\substack{\alpha \in \mathbb{Z}^n, \\ \alpha \neq \alpha_{\max}}} (\lambda_\alpha z_w + z_\alpha) x^\alpha = 0.$$

Since $\mathcal{B}_n$ is linearly independent over Z_n, we infer that $\lambda_{\alpha_{\max}} z_w = 0$. Since $\lambda_{\alpha_{\max}} \neq 0$, we see that $z_w = 0$. But then it follows that $z_\alpha = 0$, for all $\alpha \in \mathbb{Z}^n$. Therefore, if we replace the elements $x_1^{b_1} \cdots x_n^{b_n}$ in $\mathcal{B}_n$ successively by the elements $(x_1 - a)^{b_1} \cdots (x_n - a)^{b_n}$ according to the lexicographic order on $\mathbb{Z}_+^n$ beginning with the element $\alpha = (0, \ldots, 0)$, we see that $\mathcal{B}_n^a$ is linearly independent over Z_n.
On the other hand, via induction on the lexicographic order, it is easy to see that each element $x_1^{a_1} \cdots x_n^{a_n} \in \mathcal{B}_n$ can be written as an F-linear combination of the elements in $\mathcal{B}_n^a$. E.g., if $n = 3$, we have for $\alpha = (0, 1, 1)$:

$$x_2 x_3 = (x_2 - a)(x_3 - a) + a(x_2 - a) + a(x_3 - a) + a^2.$$

Thus, the elements of $\mathcal{B}_n^a$ span P_n as a Z_n-module, i.e., $\mathcal{B}_n^a$ is a Z_n-basis of P_n. $\square$

Let $a \in F^\times$. In what follows, we will denote by $Z_{(a^n)}$ the subset of P_n consisting of all symmetric polynomials in the $x_i - a$ and $x_i^{-1} - a^{-1}$ without constant term.

Corollary 9.2.4. *As an F-vector space, the F-algebra $P_n / P_n Z_{(a^n)}$ has dimension $n!$, and a basis is given by the cosets of the elements in $\mathcal{B}_n^a$.*

Proof. Denote by ϕ the F-algebra epimorphism $P_n \to P_n/P_nZ_{(a^n)}$. Let $c_1\phi(b_1) + \ldots + c_r\phi(b_r) = 0$ in $P_n/P_nZ_{(a^n)}$, for some $1 \leq r \leq n!$, some distinct elements b_i in $\mathcal{B}_n^a$, and $c_i \in F$. This is equivalent to

$$c_1b_1 + \ldots + c_rb_r = \sum_j p_jz_j \in P_n,$$

for elements $p_j \in P_n$ and elements $z_j \in Z_{(a^n)}$. Since, by Corollary 9.2.3, $\mathcal{B}_n^a$ is a Z_n-basis of P_n, we can write p_j as linear combinations of elements in $\mathcal{B}_n^a$ with coefficients in Z_n, i.e.,

$$\sum_j p_jz_j = \sum_j(\sum_k w_{k,j}b_k)z_j = \sum_k(\sum_j w_{k,j}z_j)b_k,$$

for elements $w_{k,j} \in Z_n$, and $b_k \in \mathcal{B}_n^a$. If the right-hand side of the above equation is zero, then $c_i = 0$ immediately, for all i. Otherwise, there is some k such that $\sum_j w_{k,j}z_j \neq 0$. Note that the ideal $P_nZ_{(a^n)}$ is contained in the kernel of the algebra homomorphism $P_n \to F$, $f \mapsto f(a,\ldots,a)$. Hence, each element of $P_nZ_{(a^n)}$ is not invertible in P_n. In particular, the element $\sum_j w_{k,j}z_j \in P_nZ_{(a^n)}$ is not invertible. Since $\mathcal{B}_n^a$ is a basis of P_n, considered as Z_n-module, and the linear combinations have coefficients in Z_n, the linear combinations on both sides of the equation must coincide. But this is impossible since on the left-hand side of the equation only constant terms occur. This shows that the subset $\phi(\mathcal{B}_n^a)$ of $P_n/P_nZ_{(a^n)}$ is linearly independent over F.
Next, let $u \in P_n/P_nZ_{(a^n)}$, and choose an element $v \in \phi^{-1}(u)$. Then, by Corollary 9.2.3, $v = \sum_j z_jb_j$, with $z_i \in Z_n$, and distinct elements $b_i \in \mathcal{B}_n^a$. Now, for each i, we can write z_i as a symmetric polynomial in the $x_j - a$ and $x_k^{-1} - a^{-1}$,$1 \leq j,k \leq n$, by substituting x_j by $(x_j - a) + a$ and x_k^{-1} by $(x_k^{-1} - a^{-1}) + a^{-1}$. Thus, by binomial expansion, for all i, z_i is a sum of an element of $Z_{(a^n)}$ and some constant term c_{z_i}. It follows that $u = \phi(v) = \sum_j c_{z_j}\phi(b_j)$, and hence, $\phi(\mathcal{B}_n^a)$ spans $P_n/P_nZ_{(a^n)}$ as an F-vector space. □

9.3 A basis for $H_n(q)$

In order to describe F-subalgebras of $H_n(q)$ in a convenient way, we want to determine a basis for $H_n(q)$ as an F-vector space. The main result of this section is well known, but we intend to give a detailed proof here as it is of great importance in our further investigations. We will follow the lines of [59, Theorem 3.2.2], where a basis for the degenerate affine Hecke algebra is given. The result is as follows:

Theorem 9.3.1. *The set $\{x^\alpha T_w \mid \alpha \in \mathbb{Z}^n, w \in W(n)\}$ forms an F-basis of $H_n(q)$.*

Proof. The proof will cover all of the rest of this section. A major tool will be Bergman's diamond lemma, which is [12, Theorem 1.2]. First, we will recall some definitions of [12, §1]:

Let R be a commutative ring with 1, X be a set, and denote by $\langle X\rangle$ the free semigroup with 1 on X. Moreover, let $R\langle X\rangle$ be the free associative R-algebra on X, which is the same as the semigroup algebra of X. Let $\mathcal{S}$ be a set consisting of pairs of the form $\sigma = (W_\sigma, f_\sigma)$, with $W_\sigma \in \langle X\rangle$ and $f_\sigma \in R\langle X\rangle$. Such a set is called a *reduction set.* For an element $\sigma = (W_\sigma, f_\sigma) \in \mathcal{S}$, and elements $A, B \in \langle X\rangle$, we get an R-endomorphism $r_{A\sigma B} : R\langle X\rangle \to R\langle X\rangle$, defined by

$$r_{A\sigma B}(x) = \begin{cases} Af_\sigma B & \text{if } x = AW_\sigma B, \\ x & \text{otherwise.} \end{cases}$$

Such an endomorphism is called a *reduction.* A reduction $r_{A\sigma B}$ is said to act *trivially* on $a \in R\langle X\rangle$ if the coefficient of $A\sigma B$ in a is zero. An element $a \in R\langle X\rangle$ is called *irreducible* if each reduction acts trivially on a. The R-submodule of $R\langle X\rangle$ consisting of irreducible elements is denoted by $R\langle X\rangle_{\mathrm{irred}}$.
A 5-tupel (σ, τ, A, B, C), with elements $\sigma, \tau \in \mathcal{S}$, $A, B, C \in \langle X\rangle$ is called an *overlap ambiguity* if $W_\sigma = AB$, and $W_\tau = BC$. An overlap ambiguity is called *resolvable* if there exist compositions of reductions r, r' such that $r(f_\sigma C) = r'(Af_\tau)$. Similarly, such a 5-tupel (σ, τ, A, B, C) will be called an *inclusion ambiguity* if $W_\sigma = B$ and $W_\tau = ABC$. Such an ambiguity will be called resolvable if there are compositions of reductions r, r' such that $r(Af_\sigma C) = r'(f_\tau)$.
A partial order $\le$ on $\langle X\rangle$ is called a *semigroup partial order* if $B < B'$ implies $ABC < AB'C$, for elements $A, B, B', C \in \langle X\rangle$. A partial order on $\langle X\rangle$ is called *compatible* with $\mathcal{S}$ if for all $\sigma \in \mathcal{S}$, the element f_σ is a linear combination of elements that are strictly smaller than W_σ with respect to the partial order.

Next, let $R = F$ and denote by $\tilde{H}_n(q)$ the F-algebra given by generators $\tilde{x}_j^{\pm 1}$, $\tilde{T}_i$, $1 \le j \le n$, $1 \le i \le n-1$, subject to the relations (H1) and (9.2)-(9.6), i.e., the same relations as for $H_n(q)$, except the commutation relations for T_i, T_j, $|i-j| > 1$, and the braid relations. Set $X := \{x_1^{\pm 1}, \ldots, x_n^{\pm 1}, T_1, \ldots, T_{n-1}\}$. Consider the following elements of $\langle X\rangle \times F\langle X\rangle$, which will form the reduction system in Bergman's diamond lemma:

$$\sigma_{x_i x_j} = (x_i x_j, x_j x_i), \tag{9.10}$$

$$\sigma_{x_i^{-1} x_j^{-1}} = (x_i^{-1} x_j^{-1}, x_j^{-1} x_i^{-1}), \tag{9.11}$$

where $1 \le i < j \le n$,

$$\sigma_{x_i} = (x_i x_i^{-1}, 1), \tag{9.12}$$

$$\sigma_{x_i^{-1}} = (x_i^{-1} x_i, 1), \tag{9.13}$$

for $1 \leq i \leq n$,

$$\sigma_{x_i x_j^{-1}} = (x_i x_j^{-1}, x_j^{-1} x_i), \tag{9.14}$$

$$\sigma_{x_i^{-1} x_j} = (x_i^{-1} x_j, x_j x_i^{-1}), \tag{9.15}$$

where $1 \leq i < j \leq n$,

$$\sigma_{T_i} = (T_i^2, (q-1)T_i + q), \tag{9.16}$$

with $1 \leq i \leq n-1$,

$$\sigma_{T_i, x_j} = (T_i x_j, x_j T_i), \tag{9.17}$$

$$\sigma_{T_i, x_j^{-1}} = (T_i x_j^{-1}, x_j^{-1} T_i), \tag{9.18}$$

for $1 \leq i, j \leq n$ and $j \neq i, i+1$,

$$\sigma_{T_i x_i} = (T_i x_i, x_{i+1} T_i - (q-1) x_{i+1}), \tag{9.19}$$

$$\sigma_{T_i x_{i+1}} = (T_i x_{i+1}, x_i T_i + (q-1) x_{i+1}), \tag{9.20}$$

$$\sigma_{T_i x_i^{-1}} = (T_i x_i^{-1}, x_{i+1}^{-1} T_i + (q-1) x_i^{-1}), \tag{9.21}$$

$$\sigma_{T_i x_{i+1}^{-1}} = (T_i x_{i+1}^{-1}, x_i^{-1} T_i - (q-1) x_i^{-1}), \tag{9.22}$$

where $1 \leq i \leq n-1$.

Denote by $\mathcal{S}$ the set of all these elements. Next we define a partial order $\leq_{\langle X \rangle}$ on $\langle X \rangle$ that is compatible with the semigroup operation. If $y, y' \in \langle X \rangle$, we set $y \leq_{\langle X \rangle} y'$ if $|y| \leq |y'|$, where we denote by $|y|$ the length of a word $y \in \langle X \rangle$, and $\leq$ denotes the total order on $\mathbb{N}$. If $|y| = |y'|$, we set $y \leq_{\langle X \rangle} y'$ if $y \leq_{\text{lex}} y'$, where $\leq_{\text{lex}}$ denotes the lexicographical order on $\langle X \rangle$ with respect to

$$(T_1, \ldots, T_{n-1}, x_1, x_1^{-1}, \ldots, x_n, x_n^{-1}).$$

We want to show the following:

Lemma 9.3.2. *The relation $\leq_{\langle X\rangle}$ defines a semigroup partial order on $\langle X\rangle$. Moreover, every descending sequence*

$$y_1 \geq_{\langle X\rangle} y_2 \geq_{\langle X\rangle} \cdots$$

in $\langle X\rangle$ eventually becomes stationary.

Proof. Let $a, b, c \in \langle X\rangle$. Then $|a| = |a|$ and $a \leq_{\text{lex}} a$, i.e., $a \leq_{\langle X\rangle} a$. Next, let $a \leq_{\langle X\rangle} b$ and $b \leq_{\langle X\rangle} a$. This implies $|a| = |b|$ immediately. But then we have $a \leq_{\text{lex}} b$ and $b \leq_{\text{lex}} a$. Since $\leq_{\text{lex}}$ is a partial order, we have that $a = b$. If $a \leq_{\langle X\rangle} b$ and $b \leq_{\langle X\rangle} c$ holds in $\langle X\rangle$, then we have that $|a| \leq |b|$ and $|b| \leq |c|$. This implies $|a| \leq |c|$. Suppose that $|a| = |c|$. Then $|a| = |b|$ and $|b| = |c|$, and we get that $a \leq_{\text{lex}} b$ and $b \leq_{\text{lex}} c$. Since $\leq_{\text{lex}}$ is a partial order, we infer that $a \leq_{\langle X\rangle} c$.
It is clear from the definition of $\leq_{\langle X\rangle}$ that $a \leq_{\langle X\rangle} b$, for elements $a, b \in \langle X\rangle$, implies that $XaY \leq_{\langle X\rangle} XbY$, for all $X, Y \in \langle X\rangle$. Hence $\leq_{\langle X\rangle}$ is a semigroup partial order. Note that for any element $y \in \langle X\rangle$ there are only finitely many $y' \in \langle X\rangle$ such that $|y| > |y'|$. Since X is finite, for all $r \in \mathbb{N}$, there are only finitely many elements in $\langle X\rangle$ of length r that are pairwise different. Therefore, the partial order $\leq_{\langle X\rangle}$ satisfies the descending chain condition. $\square$

Let $\sigma = (X, Y)$ be an element of $\mathcal{S}$. Then we see from the list above that $Y <_{\langle X\rangle} X$, and hence $\leq_{\langle X\rangle}$ is compatible with $\mathcal{S}$. Therefore, all the conditions in [12, Theorem 1.2] are satisfied. To obtain our desired result, we have to check that all ambiguities of $\mathcal{S}$ are resolvable, and the corresponding calculation, which are given in Appendix A, show that all the ambiguities are resolvable.

Let I be the two-sided ideal of $F\langle X\rangle$ generated by the elements $\sigma(1) - \sigma(2)$, $\sigma \in \mathcal{S}$. Also let $\tilde{I}$ be the two-sided ideal of $F\langle X\rangle$ generated by the relations given in (H1) and (9.2)–(9.6). Then we have an isomorphism of F-algebras

$$\phi : F\langle X\rangle/I \longrightarrow F\langle X\rangle/\tilde{I} = \tilde{H}_n(q),$$

given by $x_i \mapsto \tilde{x}_i$, $x_i^{-1} \mapsto \tilde{x}_i^{-1}$, $1 \leq i \leq n$, and $T_i \mapsto \tilde{T}_i$, $1 \leq i \leq n-1$, which we view as an identification. To see this, observe that the relations coming from (9.11), (9.14), (9.15), (9.18), (9.21) and (9.22), can be obtained by applying the relation (9.3) to (9.10), (9.17), (9.19) and (9.20). In the following we will write $\bar{x}$ for the image of an element $x \in F\langle X\rangle$ under ϕ.
By [12, Theorem 1.2], a set of representatives of $\tilde{H}_n(q)$ is given by the images of the elements in $F\langle X\rangle_{\text{irred}}$, under ϕ. In other words, the images of the $\mathcal{S}$-irreducible monomials span $\tilde{H}_n(q)$ as an F-vector space. To see that these elements are linearly independent over F, let $p_1, \ldots, p_n$ be pairwise different $\mathcal{S}$-irreducible monomials in $\langle X\rangle$, and denote by $\bar{p}_1, \ldots, \bar{p}_n$ their images in $\tilde{H}_n(q)$. Suppose that

$$\sum_{i=1}^{n} \lambda_i \bar{p}_i = 0 \in \tilde{H}_n(q),$$

with elements $\lambda_i \in F$, $1 \leq i \leq n$. This implies, that we have

$$\sum_{i=1}^{n} \lambda_i p_i = m \in F\langle X \rangle,$$

for some $m \in I$. Then, we can write m as a polynomial in $F\langle\{\sigma(1) - \sigma(2)\}_{\sigma\in\mathcal{S}}\rangle$. Since, by assumption, the p_i are irreducible, we have $r(m) = m$, for all reductions r induced by $\mathcal{S}$. But this implies that $m = 0$. Since a set of pairwise different monomials is linearly independent in $F\langle X \rangle$, we infer that $\lambda_i = 0$, for all i. Therefore, the images of the $\mathcal{S}$-irreducible monomials in $\langle X \rangle$ are linear independent, thus form a basis of $\tilde{H}_n(q)$ as an F-vector space.
Next we want to determine the $\mathcal{S}$-irreducible monomials in $\langle X \rangle$. Suppose m is an $\mathcal{S}$-irreducible monomial in $\langle X \rangle$. In what follows, the set

$$\sqrt{\langle T_1, \ldots, T_{n-1} \rangle}$$

will denote the subset of $\langle X \rangle$ consisting of elements $T \in \langle T_1, \ldots, T_{n-1} \rangle$ such that T does not involve a subword of the form T_i^2, for some $1 \leq i \leq n - 1$.

Lemma 9.3.3. *An element $m \in \langle X \rangle$ is $\mathcal{S}$-irreducible if and only if m has the form*

$$x_n^{\alpha_n} \cdots x_1^{\alpha_1} T, \tag{9.23}$$

where $\alpha_i \in \mathbb{Z}$, $1 \leq i \leq n$, and $T \in \sqrt{\langle T_1, \ldots, T_{n-1} \rangle}$.

Proof. If m has the given form, we see $r(m) = m$, for all reductions induced by $\mathcal{S}$. Hence m is $\mathcal{S}$-irreducible.
On the other hand, suppose that m is not of the form given in (9.23). If m has a factor of the form T_i^2, for some $1 \leq i \leq n-1$, we can apply reduction (9.16), hence, m is not $\mathcal{S}$-irreducible. If there are i, j, $1 \leq i \leq n - 1$, $1 \leq j \leq n$ such that T_i and x_j (or x_j^{-1}) are factors of m, and T_i is to the left of x_j (or x_j^{-1}), then we can apply one of the reduction (9.17)–(9.22). Hence m is not $\mathcal{S}$-irreducible. Suppose m is of the form XT, with $X \in \langle x_1^{\pm 1}, \ldots, x_n^{\pm 1} \rangle$ and $T \in \langle T_1, \ldots, T_{n-1} \rangle$. If there are $1 \leq i < j \leq n$ such that x_i (or x_i^{-1}) and x_j (or x_j^{-1}) are factors of m and x_i (or x_i^{-1}) is to the left of x_j (or x_j^{-1}), then we can apply one of the reductions (9.10), (9.11), (9.14) or (9.15). Therefore, m is not irreducible. Otherwise, X is of the form $x_n^{\pm 1} \cdots x_n^{\pm 1} \cdots x_1^{\pm 1} \cdots x_n^{\pm 1}$, where at least one x_i^{-1}, $1 \leq i \leq n$ occurs. But in this case we can apply (9.12) or (9.13), to see that m is not $\mathcal{S}$-irreducible. This shows that if m is $\mathcal{S}$-irreducible, then m must be of the form given in (9.23). □

Bergman's Theorem ([12, Theorem 1.2]) now shows that the set

$$\phi(\{x^\alpha T \mid \alpha \in \mathbb{Z}^n, T \in \sqrt{\langle T_1, \ldots, T_{n-1} \rangle}\}) \subseteq \tilde{H}_n(q)$$

is a basis of $\tilde{H}_n(q)$.
The subalgebra $\tilde{P}_n$ of $\tilde{H}_n(q)$ generated by the elements $\bar{x}_j^{\pm 1}$, $1 \leq j \leq n$, is isomorphic to P_n. Moreover, denote by $\tilde{H}_n^f(q)$ the subalgebra of $\tilde{H}_n(q)$ generated by the

elements $\bar{T}_i$, $1 \leq i \leq n-1$, which is isomorphic to the algebra on generators T_i, $1 \leq i \leq n-1$, subject to the relation $T_i^2 = (q-1)T_i + q$, for all i. If there is no danger of confusion, we will also write $x_j^{\pm 1}$ and T_i for the elements $\bar{x}_j^{\pm 1}$ and $\bar{T}_i$, $1 \leq j \leq n$, $1 \leq i \leq n-1$. Note that $H_n(q)$ is the quotient of $\tilde{H}_n(q)$ by the two-sided ideal $\tilde{J}$ generated by the elements

$$a_{i,j} = T_iT_j - T_jT_i, \tag{9.24}$$

for $1 \leq i, j \leq n-1$, $|i-j| > 1$, and

$$b_i = T_{i+1}T_iT_{i+1} - T_iT_{i+1}T_i, \tag{9.25}$$

for $1 \leq i \leq n-2$. Denote by J the two-sided ideal of $\tilde{H}_n^f(q)$ generated by the same elements. We want to show that $\tilde{J} = \tilde{P}_nJ$. Let p be an element in $\tilde{P}_n$. Also, let $T'yT'' \in J$, where $y \in \{a_{i,j}, b_k \mid 1 \leq i,j \leq n-1,\ 1 \leq k \leq n-2\}$ and $T, T' \in \sqrt{\langle T_1, \ldots, T_{n-1}\rangle}$. Then we see that $p(TyT') = (pT)yT' \in \tilde{J}$, hence $\tilde{P}_nJ \subseteq \tilde{J}$. On the other hand let $x^\alpha T$ and $x^\beta T'$ be a basis elements in $\tilde{H}_n(q)$, where $x^\alpha, x^\beta \in \tilde{P}_n$, and $T, T' \in \sqrt{\langle T_1, \ldots, T_{n-1}\rangle}$. Moreover, let $y \in \{a_{i,j}, b_k \mid 1 \leq i, j \leq n-1,$ $1 \leq k \leq n-2\}$. Then $(x^\alpha T)y(x^\beta T') \in \tilde{P}_nJ$ if we can show that $yx^\beta \in \tilde{P}_ny$, or equivalently, if $yx_i^{\pm 1} \in \tilde{P}_ny$, for all $1 \leq i \leq n$. Note that by the discussion above, the element Tx^β may be written as a linear combinations of elements $x^\gamma U$, for $\gamma \in \mathbb{Z}^m$ and $U \in \sqrt{\langle T_1, \ldots, T_{n-1}\rangle}$. Suppose that $1 \leq i \leq n-2$. Setting $y := b_i = T_{i+1}T_iT_{i+1} - T_iT_{i+1}T_i$, we have:

$$\begin{aligned}
&(T_{i+1}T_iT_{i+1})x_{i+2} \\
&= T_{i+1}T_i(x_{i+1}T_{i+1} + (q-1)x_{i+2}) \\
&= T_{i+1}(T_ix_{i+1})T_{i+1} + (q-1)T_{i+1}T_ix_{i+2} \\
&= T_{i+1}(x_iT_i + (q-1)x_{i+1})T_{i+1} + (q-1)T_{i+1}T_ix_{i+2} \\
&= x_iT_{i+1}T_iT_{i+1} + (q-1)(T_{i+1}x_{i+1}T_{i+1}) + (q-1)T_{i+1}(T_ix_{i+2}) \\
&= x_iT_{i+1}T_iT_{i+1} + (q-1)qx_{i+2} + (q-1)(T_{i+1}x_{i+2})T_i \\
&= x_iT_{i+1}T_iT_{i+1} + (q-1)x_{i+1}T_{i+1}T_i + (q-1)^2x_{i+2}T_i + (q-1)qx_{i+2},
\end{aligned}$$

using the relations in $\tilde{H}_n(q)$ induced from (9.20), (9.17), and relation (9.7). On the other hand, we have

$$\begin{aligned}
&(T_iT_{i+1}T_i)x_{i+2} \\
&= T_i(T_{i+1}x_{i+2})T_i \\
&= T_i(x_{i+1}T_{i+1} + (q-1)x_{i+2})T_i \\
&= (T_ix_{i+1})T_{i+1}T_i + (q-1)(T_ix_{i+2})T_i \\
&= (x_iT_i + (q-1)x_{i+1})T_{i+1}T_i + (q-1)x_{i+2}(T_i^2) \\
&= x_iT_iT_{i+1}T_i + (q-1)x_{i+1}T_{i+1}T_i + (q-1)x_{i+2}((q-1)T_i + q) \\
&= x_iT_iT_{i+1}T_i + (q-1)x_{i+1}T_{i+1}T_i + (q-1)^2x_{i+2}T_i + (q-1)qx_{i+2},
\end{aligned}$$

by (9.17), (9.20) and (9.16). Now we see that

$$(T_{i+1}T_iT_{i+1} - T_iT_{i+1}T_i)x_{i+2} = x_i(T_{i+1}T_iT_{i+1} - T_iT_{i+1}T_i).$$

Next consider the following:

$$\begin{aligned}
&(T_{i+1}T_iT_{i+1})x_{i+1}\\
&= T_{i+1}T_i(x_{i+2}T_{i+1} - (q-1)x_{i+2})\\
&= T_{i+1}(T_ix_{i+2})T_{i+1} - (q-1)T_{i+1}T_ix_{i+2}\\
&= (T_{i+1}x_{i+2})T_iT_{i+1} - (q-1)T_{i+1}(T_ix_{i+2})\\
&= (x_{i+1}T_{i+1} + (q-1)x_{i+2})T_iT_{i+1} - (q-1)(T_{i+1}x_{i+2})T_i\\
&= x_{i+1}T_{i+1}T_iT_{i+1} + (q-1)x_{i+2}T_iT_{i+1} - (q-1)(x_{i+1}T_{i+1} + (q-1)x_{i+2})T_i\\
&= x_{i+1}T_{i+1}T_iT_{i+1} + (q-1)x_{i+2}T_iT_{i+1} - (q-1)x_{i+1}T_{i+1}T_i - (q-1)^2x_{i+2}T_i,
\end{aligned}$$

using (9.19), (9.17) and (9.20). Also, we get

$$\begin{aligned}
&(T_iT_{i+1}T_i)x_{i+1}\\
&= T_iT_{i+1}(x_iT_i + (q-1)x_{i+1})\\
&= T_i(T_{i+1}x_i)T_i + (q-1)T_i(T_{i+1}x_{i+1})\\
&= (T_ix_i)T_{i+1}T_i + (q-1)T_i(x_{i+2}T_{i+1} - (q-1)x_{i+2})\\
&= (x_{i+1}T_i - (q-1)x_{i+1})T_{i+1}T_i + (q-1)(T_ix_{i+2})T_{i+1} - (q-1)^2(T_ix_{i+2})\\
&= x_{i+1}T_iT_{i+1}T_i + (q-1)x_{i+2}T_iT_{i+1} - (q-1)x_{i+1}T_{i+1}T_i - (q-1)^2x_{i+2}T_i,
\end{aligned}$$

by (9.20), (9.17) and (9.19). Therefore,

$$(T_{i+1}T_iT_{i+1} - T_iT_{i+1}T_i)x_{i+1} = x_{i+1}(T_{i+1}T_iT_{i+1} - T_iT_{i+1}T_i).$$

Next, consider the following calculations:

$$\begin{aligned}
&(T_{i+1}T_iT_{i+1})x_i\\
&= T_{i+1}(T_ix_i)T_{i+1}\\
&= T_{i+1}(x_{i+1}T_i - (q-1)x_{i+1})T_{i+1}\\
&= (T_{i+1}x_{i+1})T_iT_{i+1} - (q-1)(T_{i+1}x_{i+1}T_{i+1})\\
&= (x_{i+2}T_{i+1} - (q-1)x_{i+2})T_iT_{i+1} - (q-1)qx_{i+2}\\
&= x_{i+2}T_{i+1}T_iT_{i+1} - (q-1)x_{i+2}T_iT_{i+1} - (q-1)qx_{i+2},
\end{aligned}$$

using (9.17), (9.19) and (9.7). Also, we have

$$\begin{aligned}
&(T_iT_{i+1}T_i)x_i\\
&= T_iT_{i+1}(x_{i+1}T_i - (q-1)x_{i+1})\\
&= T_i(T_{i+1}x_{i+1})T_i - (q-1)T_i(T_{i+1}x_{i+1})\\
&= T_i(x_{i+2}T_{i+1} - (q-1)x_{i+2})T_i - (q-1)T_i(x_{i+2}T_{i+1} - (q-1)x_{i+2})\\
&= (T_ix_{i+2})T_{i+1}T_i - (q-1)(T_ix_{i+2})T_i - (q-1)(T_ix_{i+2})T_{i+1} + (q-1)^2(T_ix_{i+2})\\
&= x_{i+2}T_iT_{i+1}T_i - (q-1)x_{i+2}(T_i^2) - (q-1)x_{i+2}T_iT_{i+1} + (q-1)^2x_{i+2}T_i\\
&= x_{i+2}T_iT_{i+1}T_i - (q-1)x_{i+2}((q-1)T_i + q) - (q-1)x_{i+2}T_iT_{i+1} + (q-1)^2x_{i+2}T_i\\
&= x_{i+2}T_iT_{i+1}T_i - (q-1)x_{i+2}T_iT_{i+1} - (q-1)qx_{i+2},
\end{aligned}$$

using (9.19), (9.17) and (9.16). Therefore, it follows that

$$(T_{i+1}T_iT_{i+1} - T_iT_{i+1}T_i)x_i = x_{i+2}(T_{i+1}T_iT_{i+1} - T_iT_{i+1}T_i).$$

If $j \neq i, i+1, i+2$, then by (9.17) it follows that

$$(T_{i+1}T_iT_{i+1} - T_iT_{i+1}T_i)x_j = x_j(T_{i+1}T_iT_{i+1} - T_iT_{i+1}T_i).$$

Hence the claim holds for the elements b_i, $1 \leq i \leq n-2$.
Next, we consider the case where $y = a_{i,j} = T_iT_j - T_jT_i$, for some $1 \leq i, j \leq n-1$, $|i-j| > 1$. We may assume that $i < j$. Then

$$\begin{aligned}(T_iT_j)x_j &= T_i(x_{j+1}T_j - (q-1)x_{j+1}) \\ &= (T_ix_{j+1})T_j - (q-1)(T_ix_{j+1}) \\ &= x_{j+1}T_iT_j - (q-1)x_{j+1}T_i,\end{aligned}$$

by (9.19) and (9.17). On the other hand,

$$\begin{aligned}(T_jT_i)x_j &= (T_jx_j)T_i \\ &= (x_{j+1}T_j - (q-1)x_{j+1})T_i \\ &= x_{j+1}T_jT_i - (q-1)x_{j+1}T_i,\end{aligned}$$

using (9.17) and (9.19). Therefore,

$$(T_iT_j - T_jT_i)x_j = x_{j+1}(T_iT_j - T_jT_i).$$

Furthermore, we compute that

$$\begin{aligned}(T_iT_j)x_{j+1} &= T_i(x_jT_j + (q-1)x_{j+1}) \\ &= (T_ix_j)T_j + (q-1)(T_ix_{j+1}) \\ &= x_jT_iT_j + (q-1)x_{j+1}T_i,\end{aligned}$$

by (9.20) and (9.17). Also, we get that

$$\begin{aligned}(T_jT_i)x_{j+1} &= (T_jx_{j+1})T_i \\ &= (x_jT_j + (q-1)x_{j+1})T_i \\ &= x_jT_jT_i + (q-1)x_{j+1}T_i,\end{aligned}$$

using (9.17) and (9.20). Hence,

$$(T_iT_j - T_jT_i)x_{j+1} = x_j(T_iT_j - T_jT_i).$$

Moreover,

$$\begin{aligned}(T_iT_j)x_i &= (T_ix_i)T_j \\ &= (x_{i+1}T_i - (q-1)x_{i+1})T_j \\ &= x_{i+1}T_iT_j - (q-1)x_{i+1}T_j,\end{aligned}$$

using (9.17) and (9.19). On the other hand:

$$\begin{aligned}(T_jT_i)x_i &= T_j(x_{i+1}T_i - (q-1)x_{i+1})\\ &= (T_jx_{i+1})T_i - (q-1)(T_jx_{i+1})\\ &= x_{i+1}T_jT_i - (q-1)x_{i+1}T_j,\end{aligned}$$

by (9.19) and (9.17). Therefore,

$$(T_iT_j - T_jT_i)x_i = x_{i+1}(T_iT_j - T_jT_i).$$

Furthermore,

$$\begin{aligned}(T_iT_j)x_{i+1} &= (T_ix_{i+1})T_j\\ &= (x_iT_i + (q-1)x_{i+1})T_j\\ &= x_iT_iT_j + (q-1)x_{i+1}T_j,\end{aligned}$$

using (9.17) and (9.20). Also, we have that

$$\begin{aligned}(T_jT_i)x_{i+1} &= T_j(x_iT_i + (q-1)x_{i+1})\\ &= (T_jx_i)T_i + (q-1)(T_jx_{i+1})\\ &= x_iT_jT_i + (q-1)x_{i+1}T_j,\end{aligned}$$

using (9.20) and (9.17). And hence,

$$(T_iT_j - T_jT_i)x_{i+1} = x_i(T_iT_j - T_jT_i).$$

If $1 \leq k \leq n$, and $k \notin \{i, i+1, j, j+1\}$, then, by (9.17):

$$(T_iT_j - T_jT_i)x_k = x_k(T_iT_j - T_jT_i).$$

The corresponding equations for the elements x_i^{-1}, $1 \leq i \leq n$, are obtained by using the relations in $\tilde{H}_n(q)$ induced from (9.12) and (9.13). For example, the identity

$$(T_{i+1}T_iT_{i+1} - T_iT_{i+1}T_i)x_i^{-1} = x_{i+2}^{-1}(T_{i+1}T_iT_{i+1} - T_iT_{i+1}T_i)$$

follows from the equation

$$(T_{i+1}T_iT_{i+1} - T_iT_{i+1}T_i)x_i = x_{i+2}(T_{i+1}T_iT_{i+1} - T_iT_{i+1}T_i)$$

by multiplication of x_i^{-1} from the right, and x_{i+2}^{-1} from the left.
Therefore we have proved that $yx^\beta \in \tilde{P}_ny$, for all $y \in \{a_{i,j}, b_k \mid 1 \leq i,j \leq n-1, 1 \leq k \leq n-2\}$, and all $\beta \in \mathbb{Z}^n$, using only the relations in $\tilde{H}_n(q)$. As mentioned before, it now follows that $\tilde{J} = \tilde{P}_nJ$.
Recall that $H_n(q)$ is the quotient of $\tilde{H}_n(q)$ by the two-sided ideal $\tilde{J}$. Denote the corresponding projection homomorphism by p. Since the set $\{x^\alpha T \mid \alpha \in \mathbb{Z}^n, T \in \sqrt{\langle T_1, \ldots, T_{n-1}\rangle}\}$ is a basis of $\tilde{H}_n(q)$, the images in $H_n(q)$ of these elements under

the surjection p certainly span $H_n(q)$. Let $\alpha_1, \dots, \alpha_m \in \mathbb{Z}^n$, and $T^{(1)}, \dots, T^{(m)} \in \sqrt{\langle T_1, \dots, T_{n-1}\rangle}$ such that

$$\sum_{i=1}^{m} \lambda_i p(x^{\alpha_i} T^{(i)}) = 0 \in H_n(q),$$

Equivalently,

$$\sum_{i=1}^{m} \lambda_i x^{\alpha_i} T^{(i)} = c \in \tilde{H}_n(q). \tag{9.26}$$

for some $c \in \tilde{J}$. By the calculations above, we know that $\tilde{J} = \tilde{P}_n J$, thus, we may write

$$c = \sum_{s=1}^{t} \mu_s q_s j_s, \tag{9.27}$$

for $q_s \in \tilde{P}_n$, $j_s \in J$ and $\mu_s \in F$, $1 \leq s \leq t$. By rewriting the elements j_s as sums of products of the T_k, $1 \leq k \leq n-1$, and applying relation coming from (9.16) if necessary, we may assume that the right-hand side of the last equation is a sum over elements of $\{x^\alpha T \mid \alpha \in \mathbb{Z}^n, T \in \sqrt{\langle T_1, \dots, T_{n-1}\rangle}\}$. But the latter set forms an F-basis of $\tilde{H}_n(q)$, so the sums of (9.26) and (9.27) must coincide. As for $\sqrt{T \in \langle T_1, \dots, T_{n-1}\rangle}$, we have that $p(T) \in \sqrt{\langle T_1, \dots, T_{n-1}\rangle}/J$, we conclude that the set

$$\{p(x^\beta T) \mid \beta \in \mathbb{Z}^n, p(T) \in \sqrt{\langle T_1, \dots, T_{n-1}\rangle}/J\} \subseteq H_n(q)$$

forms a basis of $H_n(q)$. Note that the subalgebra of $H_n(q)$ generated by the set $\sqrt{\langle T_1, \dots, T_{n-1}\rangle}/J$ is isomorphic to $H_n^f(q)$. Therefore, we can identify each $p(T) \in \sqrt{\langle T_1, \dots, T_{n-1}\rangle}/J$ with some basis element $T_w \in H_n^f(q)$, for some $w \in W(n)$. Therefore, by abuse of notation, if we write also x^β for the element $p(x^\beta)$ of $H_n(q)$, $\beta \in \mathbb{Z}^n$, then the set

$$\{x^\beta T_w \mid \beta \in \mathbb{Z}^n,\ w \in W(n)\}$$

is a basis of $H_n(q)$ as an F-vector space. This finishes the proof. □

Corollary 9.3.4. *The set $\{T_w x^\alpha \mid \alpha \in \mathbb{Z}^n, w \in W(n)\}$ forms a basis of $H_n(q)$ as an F-vector space.*

Proof. Changing the reduction system, and the lexicographical order in the appropriate way, the result follows from Theorem 9.3.1. □

9.4 Parabolic subalgebras

In this section, we recall the notion of a parabolic subalgebra of the finite-dimensional Hecke algebra $H_n^f(q)$ and state the definition of the corresponding analogues of the algebra $H_n(q)$. These subalgebras play an important role in the following as we may consider induced modules from those smaller (affine) Hecke algebras, and may study restrictions of modules to smaller ones. Moreover, we deduce an analogue of Theorem 9.3.1 for parabolic subalgebras.

Let $(\mu_1, \ldots, \mu_r)$ be a composition of n, and let

$$W_\mu \cong W_{\mu_1} \times \ldots \times W_{\mu_r}$$

be the corresponding parabolic subgroup of $W(n)$. The subalgebra of $H_n^f(q)$ generated by the elements T_j, where $s_j \in W_\mu$, $1 \leq j \leq n-1$, will be denoted by $H_\mu^f(q)$. The algebra $H_\mu^f(q)$ is also called the parabolic subalgebra of $H_n^f(q)$ corresponding to W_μ.
In a similar way, the subalgebra $H_\mu(q)$ of $H_n(q)$ generated by P_n and the elements T_j, for $s_j \in W_\mu$, $1 \leq j \leq n-1$ is called a parabolic subalgebra of $H_n(q)$ corresponding to the parabolic subgroup W_μ. In view of Theorem 9.3.1 one has the following:

Proposition 9.4.1. *As an F-vector space, the parabolic subalgebra $H_\mu(q)$ of $H_n(q)$ has basis the set $\{x^\alpha T_w \mid \alpha \in \mathbb{Z}^n, w \in W_\mu\}$.*

Proof. By Theorem 9.3.1, the elements of $\mathcal{B} := \{x^\alpha T_w \mid \alpha \in \mathbb{Z}^n, w \in W_\mu\} \subseteq H_\mu(q)$ are linearly independent.
By definition of $H_\mu(q)$, every element in $H_\mu(q)$ can be expressed as an F-polynomial in the elements of P_n and $H_\mu^f(q)$. Looking at the relations given by the pairs in (9.19)-(9.22), we see that applying a relation to such an expression produces again a polynomial in the elements P_n and $H_\mu^f(q)$. Therefore, by applying such relations, we can transform an expression given above into a polynomial in the elements of $\mathcal{B}$. Thus, the elements of $\mathcal{B}$ are a spanning set for $H_\mu(q)$ over F. $\square$

Remark 9.4.2. (1) For μ a composition of n, it follows from the last proposition that the parabolic subalgebra $H_\mu(q)$ is a free P_n-module of rank $|W_\mu|$, and $H_n(q)$ is a free $H_\mu(q)$-module of rank $[W(n) : W_\mu]$.
(2) The parabolic subalgebra $H_{(1,\ldots,1)}(q)$ is nothing else but the subalgebra P_n.
(3) As F-algebras, $H_\mu(q) \cong H_{\mu_1}(q) \otimes_F \ldots \otimes_F H_{\mu_r}(q)$.

9.5 The center of affine Hecke algebras and generalized eigenspaces

Here, we will state the important theorem due to Bernstein, describing the center of $H_n(q)$. Moreover, we recall some basic statements about the representation theory of P_n-modules. To this end we consider generalized eigenspaces corresponding to eigenvalues of the commuting operators $x_1, \ldots, x_n$ of P_n. In particular we recall the decomposition of a P_n-module into simultaneous generalized eigenspaces with respect to the latter operators. As we will see later on, this allows one to partition the category $H_n(q) - \text{mod}$ into blocks.
Furthermore, we give the definition of formal character of $H_n(q)$-modules, and state several lemmas that are needed later on.

Also, we will give a proof of the well-known fact that every irreducible $H_n(q)$-module is finite dimensional.

From now on, set $S := F^{\times} = F \setminus \{0\}$. The following is fundamental in the theory of affine Hecke algebras:

Theorem 9.5.1. *(Bernstein) Let $Z(H_n(q))$ denote the center of $H_n(q)$. Then*

$$Z(H_n(q)) = F[x_1^{\pm 1}, \dots, x_n^{\pm 1}]^{W(n)},$$

the set of symmetric Laurent polynomials in P_n.

Proof. See [19, Proposition 7.1.14]. □

For $a \in S$, let us denote by $L(a)$ the one-dimensional irreducible P_1-module such that x_1 acts with eigenvalue a, i.e., $x_1 v = av$, for a basis vector $v \in L(a)$. Moreover, if $\underline{a} = (a_1, \dots, a_n) \in S^n$, then the P_n-module $L(\underline{a}) := L(a_1) \boxtimes \dots \boxtimes L(a_n)$ is a one-dimensional, and thus, an irreducible P_n-module. Since F is assumed to be algebraically closed, and since P_n is commutative, every finite-dimensional irreducible P_n-module is one-dimensional, and arises in this way. Thus, the set

$$\{L(\underline{a}) \mid \underline{a} \in S^n\}$$

forms a complete set of representatives of isomorphism classes of irreducible P_n-modules.

Remark 9.5.2. Let M be a finite-dimensional P_n-module. Then M may be viewed as a vector space together with invertible linear transformations $f_1, \dots, f_n$ such that $f_i f_j = f_j f_i$, for all $1 \leq i, j \leq n$.

Let $\underline{a} \in S^n$. In what follows, we denote by $M_{\underline{a}}$ the largest submodule of M, all of whose composition factors are isomorphic to $L(\underline{a})$. We have the following:

Lemma 9.5.3. *Let $\underline{a} \in S^n$ and $M \in P_n - \mathrm{mod}$. The subspace $M_{\underline{a}}$ of M equals the simultaneous generalized eigenspace of the commuting operators $x_1, \dots, x_n$ on M, corresponding to the eigenvalues $a_1, \dots, a_n$, i.e.,*

$$M_{\underline{a}} = \{m \in M \mid (x_i - a_i)^{k_i} m = 0, \text{ for some } k_i > 0, 1 \leq i \leq n\}.$$

Proof. Denote by H the simultaneous generalized eigenspace of $x_1, \dots, x_n$ on M corresponding to the eigenvalues $a_1, \dots, a_n$. Since the x_i commute with each other, H is a P_n-submodule of M. Let C be a composition factor of H as a P_n-module, i.e., there exist P_n-submodules $V \subsetneq U \subseteq H$ such that $U/V \cong C$. Since C is an irreducible P_n-module, $\dim_F C = 1$, hence is isomorphic to $L(\underline{a}')$, for some $\underline{a}' \in S^n$. Denote by v a basis vector of $L(\underline{a}')$, and let p be the corresponding surjection $p : U \to C$. Also take $w \in U$ such that $p(w) = v$. Since $w \in H$, there is some $k \in \mathbb{N}$, $k \neq 0$, such that $(x_i - a_i)^k w = 0$, for all i. But then

$(x_i - a_i)^k v = (x_i - a_i)^k p(w) = p((x_i - a_i)^k w) = 0$. Since C is one-dimensional, it follows that $x_i v = a_i v$, for all i. We infer that $\underline{a}' = \underline{a}$ and, thus, $C \cong L(\underline{a})$.
On the other hand, let U be the largest submodule of M such that if C is a composition factor of U, then $C \cong L(\underline{a})$. Let $u \in U$, and

$$0 = U_0 \subsetneq U_1 \subsetneq \ldots \subsetneq U_r = U$$

be a composition series of U such that, for some $j > 0$, $u \in U_j$ and $u \notin U_{j-1}$. Denote by $p : U_j \to U_j/U_{j-1}$ the canonical surjection. We argue by induction on the composition length r of U to show that $U \subseteq H$.
If $r = 1$, then, since $U_1 \cong L(\underline{a})$, we see that $x_i u = a_i u$, for all i, and therefore $u \in H$. Now, let $r > 1$. Note that, by assumption, we have that $U_r/U_{r-1} \cong L(\underline{a})$. Then $0 = (x_i - a_i)p(u) = p((x_i - a_i)u)$, i.e., $(x_i - a_i)u \in U_{j-1}$, for all i. By the induction hypothesis, $U_{r-1} \subseteq H$, and, hence, we have that

$$(x_i - a_i)^s((x_i - a_i)u) = (x_i - a_i)^{s+1}u = 0,$$

for some $s \in \mathbb{N}$, and all i. We conclude that $u \in H$. $\square$

Lemma 9.5.4. *Let $M \in P_n - \text{mod}$. Then $M \cong \bigoplus_{\underline{a} \in S^n} M_{\underline{a}}$ as P_n-modules.*

Proof. For an F-vector space V, a map $f \in \mathrm{End}_F(V)$ and an element $\lambda \in F$, in the following, we will denote by $\mathrm{Gen}(f, \lambda)$ the generalized eigenspace of f with respect to λ.
Then, by basic linear algebra, for each $1 \leq i \leq n$, we have a decomposition

$$M \cong \bigoplus_{j=1}^{r_i} \mathrm{Gen}(x_i, a_{i,j})$$

of M into generalized eigenspaces corresponding to the eigenvalues $a_{i,j} \in S$ of x_i on M. Let $1 \leq k \leq n$ be such that $i \neq k$. Furthermore, let $v \in X := \mathrm{Gen}(x_i, a_{i,j})$, for some $1 \leq j \leq r_i$. Then there is some $r \in \mathbb{Z}^+$ such that $(x_i - a_{i,j})^r v = 0$. Since x_i commutes with x_k, we get that $(x_i - a_{i,j})^r(x_k v) = x_k(x_i - a_{i,j})^r v = 0$. Hence, $x_k v \in X$. In other words, X is an $F[x_k^{\pm 1}]$-module. Again, we get a decomposition

$$X \cong \bigoplus_{j=1}^{s_k} \mathrm{Gen}(x_{k|X}, b_{k,j})$$

into generalized eigenspaces corresponding to the eigenvalues $b_{k,j}$ of $(x_k)_{|X}$ on the space $\mathrm{Gen}(x_i, a_{i,j})$. Here, $(x_k)_{|X}$ denotes the restriction of x_k to X. Clearly, the eigenvalues must be contained in the set of eigenvalues of x_k on M. This implies that $\mathrm{Gen}((x_k)_{|X}, b_{k,j}) \subseteq \mathrm{Gen}(x_k, b_{k,j}) \cap X$, for all j. On the other hand, if $u \in \mathrm{Gen}(x_k, b_{k,j}) \cap X$, then

$$(x_k - b_{k,j})^s u = ((x_k - b_{k,j})_{|X})^s u = ((x_k)_{|X} - b_{k,j})^s u = 0,$$

for some $s \in \mathbb{Z}^+$, which shows that $u \in \mathrm{Gen}((x_k)_{|X}, b_{k,j})$. Therefore, we get that $\mathrm{Gen}((x_k)_{|X}, b_{k,j}) = X \cap \mathrm{Gen}(x_k, b_{k,j})$.
Next, consider the decomposition

$$M \cong \bigoplus_{j=1}^{r_1} \mathrm{Gen}(x_1, a_{1,j}),$$

for $1 \leq j \leq r_1$. If we consider the action of x_2 on the generalized eigenspace $\mathrm{Gen}(x_1, a_{1,j})$, $1 \leq j \leq r_1$, we get, by the discussion above, that

$$M \cong \bigoplus_{j=1}^{r_1} \bigoplus_{k \in K_j} \mathrm{Gen}(x_1, a_{1,j}) \cap \mathrm{Gen}(x_2, a_{2,k}),$$

for some $K_j \subseteq \{1, \dots, r_2\}$, $1 \leq j \leq r_1$. If we continue in the same way with the operators $x_3, \dots, x_n$, we eventually obtain the desired decomposition, using Lemma 9.5.3. □

In view of Lemma 9.5.4, if $M \in P_n - \mathrm{mod}$, we can expand M as

$$[M] = \sum_{\underline{a} \in S^n} r_{\underline{a}} [L(\underline{a})],$$

in the Grothendieck group $\mathcal{K}(P_n - \mathrm{mod})$ with respect to the basis $\{[L(\underline{a})] \mid \underline{a} \in S^n\}$. Note that $r_{\underline{a}} = \dim_F M_{\underline{a}}$, for all $\underline{a} \in S^n$.

Definition 9.5.5. Let $M \in H_n(q) - \mathrm{mod}$. Then we define the *formal character* of M as

$$\mathrm{ch}(M) = [\mathrm{Res}_{P_n}^{H_n(q)}(M)] \in \mathcal{K}(P_n - \mathrm{mod}).$$

Since the functor $\mathrm{Res}_{P_n}^{H_n(q)}$ is exact, we get a homomorphism

$$\Psi : \mathcal{K}(H_n(q) - \mathrm{mod}) \longrightarrow \mathcal{K}(P_n - \mathrm{mod})$$

of abelian groups. We obtain the following, which can be found in [45, Lemma 2.4]. Note that $W(n)$ acts on S^n by place permutation, i.e., $w \cdot \underline{a} = (a_{w^{-1}1}, \dots, a_{w^{-1}n})$, for $\underline{a} \in S^n$ and $w \in W(n)$.

Lemma 9.5.6. *For $\underline{a} \in S^n$ we have:*

$$\mathrm{ch}(\mathrm{Ind}_{P_n}^{H_n(q)}(L(\underline{a}))) = \sum_{w \in W(n)} [L(w \cdot \underline{a})].$$

We will also need the following statement, which describes the character of a module induced from a parabolic subalgebra. It is also called the *Shuffle Lemma.*

Lemma 9.5.7. *If $M \in H_n(q) - \mathrm{mod}$, and $N \in H_m(q) - \mathrm{mod}$, then*

$$\mathrm{ch}(\mathrm{Ind}_{H_{(n,m)}(q)}^{H_{n+m}(q)}(M \boxtimes N)) = \sum_{\underline{s'} \in S^n, \underline{s''} \in S^m} (\dim_F M_{\underline{s'}} \cdot \dim_F N_{\underline{s''}})[L(\underline{s})],$$

where $\underline{s} = (s_1, \dots, s_{n+m})$ is such that there are $i_1, \dots, i_n \in \{1, \dots, n+m\}$ with $\underline{s'} = (s_{i_1}, \dots, s_{i_n})$, and $\underline{s''}$ is obtained from s by deleting the substring s'.

Proof. This can be found in [44, Lemma 5.5], and follows from the Mackey Formula for affine Hecke algebras, see [44, Corollary 5.4]. □

Finally, let us state the remarkable result about irreducible $H_n(q)$-modules, which says that any irreducible $H_n(q)$-module is finite dimensional, see also [45, Proposition 2.2].

Proposition 9.5.8. *Let U be an irreducible $H_n(q)$-module. Then U is finite dimensional and* $\dim_F U \leq n!$.

Proof. Let T be an irreducible P_n-submodule of the head of $\mathrm{Res}_{P_n}^{H_n(q)}(U)$. Then there exists a surjective map $\mathrm{Res}_{P_n}^{H_n(q)}(U) \to T$ of P_n-modules. By adjointness of coinduction and restriction, we have that

$$\begin{aligned} \mathrm{Hom}_{P_n}(\mathrm{Res}_{P_n}^{H_n(q)}(U), T) &\cong \mathrm{Hom}_{H_n(q)}(U, \widehat{\mathrm{Ind}}_{P_n}^{H_n(q)}(T)) \\ &= \mathrm{Hom}_{H_n(q)}(U, \mathrm{Hom}_{P_n}(H_n(q), T)), \end{aligned}$$

where the latter term is non-zero. Since U is irreducible, we infer that there exists a monomorphism $U \to \mathrm{Hom}_{P_n}(H_n(q), T)$. Since $H_n(q)$ is a free P_n-module of rank $n!$, and every irreducible P_n-module is finite dimensional, the F-vector space $\mathrm{Hom}_{P_n}(H_n(q), T)$ is finite dimensional. Thus, also U is finite dimensional. □

Remark 9.5.9. Observe that the previous proposition has the following consequence: If $M \in H_n(q) \otimes H_m(q) - \mathrm{mod}$ is irreducible and finite dimensional, then M is isomorphic to $V \boxtimes W$, for irreducible $V \in H_n(q) - \mathrm{mod}$ and irreducible $W \in H_m(q) - \mathrm{mod}$, where the latter are uniquely determined by M up to isomorphism.

9.6 Central characters

Using the results of the latter section, we will now give a partition of the finite-dimensional $H_n(q)$-modules. To do this, we will label the latter by certain F-valued functions defined on the center of $H_n(q)$.

Recall that, by Theorem 9.5.1, the center $Z(H_n(q))$ of $H_n(q)$ coincides with the set $F[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]^{W(n)}$ of symmetric Laurent polynomials. For $\underline{a} \in S^n$, we define the *central character* of $H_n(q)$ associated with $\underline{a}$ as the map

$$\chi_{\underline{a}} : Z(H_n(q)) \longrightarrow F, \ f \mapsto f(\underline{a}),$$

where by $f(\underline{a})$, we mean $f(a_1, \ldots, a_n)$, for $f = f(x_1, \ldots, x_n) \in F[x_1^{\pm 1}, \ldots x_n^{\pm 1}]$.
Note that $W(n)$ acts on S^n by place permutation, see the definition before Lemma 9.5.6. Hence, under our assumptions, we easily see that $\chi_{\underline{a}} = \chi_{\underline{b}}$ if $\underline{a}, \underline{b} \in S^n$ belong to the same $W(n)$-orbit. Denote by $\sim$ the equivalence relation on S^n induced by the action of $W(n)$. Then, for $\underline{a} \in S^n$ belonging to the $W(n)$-orbit γ, we will write χ_γ for the central character $\chi_{\underline{a}}$ given by $\underline{a}$.

Then, for $M \in H_n(q) - \text{mod}$, we denote by

$$M[\gamma] := \{x \in M \mid (z - \chi_\gamma(z))^k x = 0, \text{ for all } z \in Z(H_n(q)), \text{ and some } k \in \mathbb{Z}^+\}$$

the simultaneous generalized eigenspace of the elements $z \in Z(H_n(q))$ of M. Since the elements z are central elements, $M[\gamma]$ is actually an $H_n(q)$-submodule of M: Let $h \in H_n(q)$, $z \in Z(H_n(q))$ and $x \in M[\gamma]$. Then, for some $k \in \mathbb{Z}^+$, we have that

$$(z - \chi_\gamma(z))^k(hx) = h((z - \chi_\gamma(z))^k x) = 0,$$

thus, $hx \in M[\gamma]$.
In view of Lemma 9.5.3, we see that as P_n-modules

$$M[\gamma] \cong \bigoplus_{\underline{a} \in \gamma} M_{\underline{a}}. \tag{9.28}$$

One has the following:

Lemma 9.6.1. *Let* $M \in H_n(q) - \text{mod}$. *Then* $M \cong \bigoplus_{\gamma \in S^n/\sim} M[\gamma]$ *as* $H_n(q)$-*modules.*

Proof. By Lemma 9.5.4, we have an isomorphism

$$M \cong \bigoplus_{\underline{a} \in S^n} M_{\underline{a}}$$

of P_n-modules. Reordering the sum, we get that

$$M \cong \bigoplus_{\gamma \in S^n/\sim} \bigoplus_{\underline{a} \in \gamma} M_{\underline{a}}$$

as P_n-modules. On the other hand, by (9.28), we have

$$M[\gamma] \cong \bigoplus_{\underline{a} \in \gamma} M_{\underline{a}}$$

as P_n-modules, for $\gamma \in S^n/\sim$. Since $M[\gamma]$ is actually an $H_n(q)$-submodule of M, the claim follows. ☐

The above decomposition of $M \in H_n(q) - \text{mod}$ is called the *block decomposition* of M. For $\gamma \in S^n/\sim$, we denote by $H_n(q) - \text{mod}[\gamma]$ the full subcategory of $H_n(q) - \text{mod}$, consisting of all modules such that $M[\gamma] = M$. The previous lemma then implies that there is an equivalence of categories

$$H_n(q) - \text{mod} \cong \bigoplus_{\gamma \in S^n/\sim} H_n(q) - \text{mod}[\gamma]. \tag{9.29}$$

The category $H_n(q) - \text{mod}[\gamma]$ is called the *block* of $H_n(q) - \text{mod}$ corresponding to γ. If $M \in H_n(q) - \text{mod}[\gamma]$, then one says that M *belongs* to the block determined by γ.
The next proposition shows that every finite-dimensional indecomposable $H_n(q)$-module has a central character, and thus, belongs to a block, which is uniquely determined by the latter.

Proposition 9.6.2. *Let $M \in H_n(q) - \mathrm{mod}$ be indecomposable. If $M_{\underline{a}} \neq 0$ and $M_{\underline{a}'} \neq 0$, for $\underline{a}, \underline{a}' \in S^n$, then there exists some $w \in W(n)$ such that $w \cdot \underline{a} = \underline{a}'$.*

Proof. Note that for some $\underline{a} \in S^n$, $M_{\underline{a}} \neq 0$ if and only if the simultaneous eigenspace of the $x_1, \ldots, x_n$ does not vanish. Since F is algebraically closed, the eigenspace $\mathrm{Eig}(f, \lambda)$ corresponding to the eigenvalue $\lambda \in F$ is non-zero, for some $f \in Z(H_n(q))$. Since f is a central element in $H_n(q)$, the subspace $\mathrm{Eig}(f, \lambda)$ of M is an $H_n(q)$-submodule of M. Since eigenspaces corresponding to different eigenvalues intersect in zero, by the indecomposability of M, we conclude that $M \cong \mathrm{Eig}(f, \lambda)$.
Now, if $M_{\underline{a}} \neq 0$ and $M_{\underline{a}'} \neq 0$, for some $\underline{a}, \underline{a}' \in S^n$, then $fv = f(\underline{a})v$ and $fv' = f(\underline{a}')v'$, for some non-zero vectors $v \in M_{\underline{a}}$ and $v' \in M_{\underline{a}'}$. It follows that $\lambda = f(\underline{a}) = f(\underline{a}')$, for all $f \in F[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]^{W(n)}$.
For $\lambda \in F$, consider the polynomial

$$\prod_{j=1}^{n} (X_j + \lambda) = \lambda^n + \mathrm{e}_1(x_1, \ldots, x_n)\lambda^{n-1} + \ldots + \mathrm{e}_n(x_1, \ldots, x_n),$$

where for $1 \leq i \leq n$, the polynomial $\mathrm{e}_i(x_1, \ldots, x_n)$ denotes the ith elementary symmetric polynomial in the variables $x_1, \ldots, x_n$. Thus, if we set $\lambda = -a_k$, for some $1 \leq k \leq n$, we see that a_k is a root of the polynomial

$$(-1)^n \lambda^n + (-1)^{n-1}\mathrm{e}_1(a_1, \ldots, a_n)\lambda^{n-1} + \ldots + \mathrm{e}_n(a_1, \ldots, a_n).$$

On the other hand, since, by the discussion above, $\mathrm{e}_i(a_1, \ldots, a_n) = \mathrm{e}_i(a_1', \ldots, a_n')$, also $-a_k'$ is a root of the polynomial above, $1 \leq k \leq n$. Thus, there exists some $w \in W(n)$ such that $w \cdot \underline{a} = \underline{a}'$, and the claim follows. $\square$

Remark 9.6.3. Therefore, in what follows, if M is an indecomposable finite-dimensional $H_n(q)$-module such that for some $\underline{a} \in S^n$, $M_{\underline{a}} \neq 0$, then we say that M has *central character* $\chi_{\underline{a}}$.

9.7 The Kato module

In the previous section, we defined the central character of an irreducible $H_n(q)$-module. Here, we will be mainly concerned with the central character associated to $(a, \ldots, a)$, $a \in S$, of $H_n(q)$. We will state the remarkable result of Kato that, up to isomorphism, the block of $H_n(q) - \mathrm{mod}$ corresponding to this character contains a unique irreducible module, the famous Kato module.

Definition 9.7.1. For $a \in S$, the *Kato module* corresponding to a is defined to be the $H_n(q)$-module $L(a^n) := \mathrm{Ind}_{P_n}^{H_n(q)}(L(\underline{a}))$.

Remark 9.7.2. Using Lemma 9.5.6, we see that

$$\mathrm{ch}(L(a^n)) = n![L(a) \boxtimes \cdots \boxtimes L(a)],$$

thus $\dim_F L(a^n) = n!$. Moreover, for all $1 \leq i \leq n$, the only eigenvalue of x_i on $L(a^n)$ is a.

The following theorem, which is [45, Proposition 3.3 (1)], describes the crucial properties of $L(a^n)$.

Theorem 9.7.3. *(Kato's Theorem) The $H_n(q)$-module $L(a^n)$ is irreducible. It is the unique irreducible $H_n(q)$-module with central character $\chi_{(a,\ldots,a)}$.*

For the rest of this section, we will fix an element $a \in S$. Denote by $\mathcal{J}(a^n)$ the annihilator of $L(a^n)$ in $H_n(q)$, which is a two-sided ideal in $H_n(q)$. Then, for each $m \geq 1$, we get a corresponding quotient algebra

$$\mathcal{R}_m(a^n) := H_n(q)/\mathcal{J}(a^n)^m. \tag{9.30}$$

Proposition 9.7.4. *For each $m \geq 1$, the algebra $\mathcal{R}_m(a^n)$ is finite dimensional. Furthermore, up to isomorphism, $L(a^n)$ is the unique irreducible $\mathcal{R}_m(a^n)$-module .*

Proof. The ideal $\mathcal{J}(a^n)^m$ contains the elements $(x_i - a)^{n!m}$ and $(x_i^{-1} - a^{-1})^{n!m}$, for all $1 \leq i \leq n$. Then, Theorem 9.3.1 shows that $\mathcal{R}_m(a^n)$ is finite dimensional. The second statement follows from Theorem 9.7.3, together with the fact that $(x_i - a)^{n!m}$ acts as zero on $\mathcal{R}_m(a^n)$ for all i. $\square$

For $m \geq 1$, we will denote by $\mathcal{L}_m(a^n)$ the projective cover of $L(a^n)$ in $\mathcal{R}_m(a^n)-$mod. We have:

Lemma 9.7.5. *Let $m \geq 1$. Then $\mathcal{R}_m(a^n) \cong \bigoplus_{n!} \mathcal{L}_m(a^n)$ as $\mathcal{R}_m(a^n)$-modules.*

Proof. For a finite-dimensional F-algebra A, and a complete set $\{S_1, \ldots, S_k\}$ of representatives of isomorphism classes of irreducible A-modules, we have that

$$A \cong \bigoplus_{i=1}^{k} P_i^{(\dim_F S_i)},$$

where P_i denotes the projective cover of S_i in $A-$mod, for all i. To see this, write $A \cong \bigoplus_{i=1}^k P_i^{t_i}$, for some $t_i \in \mathbb{N}$. Then $A/\mathrm{rad}(A) \cong \bigoplus_{i=1}^k (P_i/\mathrm{rad}(P_i)^{t_i}$. Hence, t_i equals the number of times S_i occurs as a direct summand of the semisimple algebra $A/\mathrm{rad}(A)$. By the theory of finite-dimensional semisimple algebras, this number is the same as $\dim_F S_i$ since F is algebraically closed (see, for example, [25, Theorem 3.22, Theorem 3.28]). Now, since, by Proposition 9.7.4, $L(a^n)$ is the unique irreducible $\mathcal{R}_m(a^n)$-module up to isomorphism, the claim follows from this and the fact that $\dim_F L(a^n) = n!$. $\square$

Next, we try to describe the annihilator $\mathcal{J}(a^n)$ of $L(a^n)$ in $H_n(q)$ in more detail. Thus, we will consider the case $m = 1$. Note that $\mathcal{R}_1(a^n)$ is a left primitive ring since $L(a^n)$ is a faithful $\mathcal{R}_1(a^n)$-module. Furthermore, since $\mathcal{R}_1(a^n)$ is finite dimensional over F, by [61, Proposition 11.7], this is equivalent to $\mathcal{R}_1(a^n)$ being a semisimple ring. Therefore, by Lemma 9.7.5, $\mathcal{L}_1(a^n) \cong L(a^n)$ and $\dim_F \mathcal{R}_1(a^n) = (n!)^2$.
In the following we will write $\chi_{(a^n)}$ instead of $\chi_{(a,\ldots,a)}$. Also, recall the definition of the subset $Z_{(a^n)}$ of P_n from Section 2. We note the following:

Lemma 9.7.6. *The kernel of $\chi_{(a^n)}$ coincides with $Z_{(a^n)} \cup \{0\}$.*

Proof. Denote by $\mathrm{Ker}(\chi_{(a^n)})$ the kernel of $\chi_{(a^n)}$. If $f \in Z_{(a^n)}$, then $f(a, \ldots, a) = 0$, hence $Z_{(a^n)} \subseteq \mathrm{Ker}(\chi_{(a^n)})$. Conversely, let $f \in \mathrm{Ker}(\chi_{(a^n)})$. As in the proof of Corollary 9.2.4, we may write f as a sum of a symmetric polynomial in the $x_j - a$ and $x_k^{-1} - a^{-1}$,$1 \leq j, k \leq n$ plus a constant term $c \in F$. Since $f(a, \ldots, a) = 0$, we see that $c = 0$, thus, the result follows. □

We now can deduce the following. Note that this is stated in [59, §4] in the degenerate case.

Theorem 9.7.7. *The ideal $\mathcal{J}(a^n)$ is the same as the ideal $H_n(q)Z_{(a^n)}$.*

Proof. By Corollary 9.2.4, we have that $\dim_F P_n/P_nZ_{(a^n)} = n!$. Moreover, a basis of $P_n/P_nZ_{(a^n)}$ is given by the cosets of

$$\mathcal{B}_n^a = \{(x_1 - a)^{a_1} \cdots (x_n - a)^{a_n} \mid 0 \leq a_i < i \text{ for all } 1 \leq i \leq n\}.$$

We first show that $H_n(q)$ is a free Z_n-module of rank $(n!)^2$. To see this, let

$$\mathcal{B} := \{T_w x^\alpha \mid \alpha \in \mathbb{Z}^n, w \in W(n)\},$$

which, by Corollary 9.2.3, is a basis of $H_n(q)$, considered as an F-vector space. Let $T_w x^\alpha \in \mathcal{B}$. Since, by Corollary 9.2.3, P_n is free as Z_n-module with basis $\mathcal{B}_n^a$, we can write $x^\alpha = \sum_j z_j b_j$, for elements $z_j \in Z_n$ and $b_j \in \mathcal{B}_n^a$. Hence,

$$T_w x^\alpha = T_w(\sum_j z_j b_j) = \sum_j z_j(T_w b_j).$$

The second equality follows from Theorem 9.5.1, stating that Z_n equals the center of $H_n(q)$. This shows that the set

$$\mathcal{C}_n := \{T_w b \mid w \in W(n), b \in \mathcal{B}_n^a\}$$

spans $H_n(q)$ as a Z_n-module.

Suppose that $\sum_k z_k c_k = 0$, for pairwise different elements $c_k \in \mathcal{C}_n$, and $z_k \in Z_n$. Let $c_k = T_{w_k} b_k$, for $w_k \in W(n)$, $b_k \in \mathcal{B}_n^a$. Again by Theorem 9.5.1, we can write $z_k c_k = z_k(T_{w_k} b_k) = T_{w_k}(z_k b_k)$, for all k. Now we reorder the elements in the above sum to obtain

$$\sum_{w_k} T_{w_k} p_{w_k} = 0,$$

where, by assumption on the elements c_k, p_{w_k} is a Z_n-linear combination of pairwise different elements $b_k' \in \mathcal{B}_n^a$. Since $H_n(q)$ is a free P_n-module with basis $\{T_w \mid w \in W(n)\}$, by Corollary 9.2.3, if some of the p_{w_k} are different from zero, we get a contradiction immediately. But if $p_{w_k} = 0$ for all k, we conclude that $z_k = 0$ for all k since the elements $\mathcal{B}_n^a$ are linearly independent over Z_n. Therefore, $\mathcal{C}_n$ is a basis for $H_n(q)$ as Z_n-module.

Next, let $q : H_n(q) \to H_n(q)/H_n(q)Z_{(a^n)}$ be the quotient of $H_n(q)$ by the ideal $H_n(q)Z_{(a^n)}$. Let $d_1q(c_1) + \ldots + d_rq(c_r) = 0 \in H_n(q)/H_n(q)Z_{(a^n)}$, for some $1 \leq r \leq (n!)^2$, distinct elements $c_i \in \mathcal{C}_n$, and $d_i \in F$. This is equivalent to

$$d_1c_1 + \ldots + d_rc_r = \sum_j h_jz_j \in H_n(q),$$

for elements $h_j \in H_n(q)$, and some $z_j \in Z_{(a^n)}$. As $\mathcal{C}_n$ is a Z_n-basis of $H_n(q)$, we may write each h_j as a sum of the form $\sum_k v_{k_j}c'_k$, with elements $v_{k_j} \in Z_n$ and $c'_k \in \mathcal{C}_n$. Therefore, we can write the right-hand side of the equation as $\sum_p s_pc'_p$, where $s_p \in Z_nZ_{(a^n)}$, for all p. Since $s_p \in H_n(q)Z_{(a^n)}$, it cannot be invertible. But all the coefficients on the left-hand side are either zero or invertible, thus, we must have $s_p = 0$ for all p. This also implies that $d_i = 0$ for all i, hence, $q(\mathcal{C}_n)$ is linearly independent over F.

Next, let $u \in H_n(q)/H_n(q)Z_{(a^n)}$, and choose a representative $v \in q^{-1}(u) \subseteq H_n(q)$. Since $\mathcal{C}_n$ is a basis for $H_n(q)$ considered as Z_n-module, we can write

$$v = \sum_i z_ic_i,$$

for elements $z_i \in Z_n$, and $c_i \in \mathcal{C}_n$. As in the proof of Corollary 9.2.4, write each z_i as a sum of an element in $Z_{(a^n)}$ plus a constant term $d_i \in F$. Since $Z_{(a^n)} \subseteq Z(H_n(q))$, we conclude that

$$u = q(v) = \sum_i d_iq(c_i).$$

Therefore, $q(\mathcal{C}_n)$ spans $H_n(q)/H_n(q)Z_{(a^n)}$. This immediately implies that

$$\dim_F H_n(q)/H_n(q)Z_{(a^n)} = (n!)^2.$$

Let $\tilde{v}$ be a simultaneous eigenvector of the operators $x_1, \ldots, x_n$ on $L(a^n)$. Since $L(a^n)$ is an irreducible $H_n(q)$-module, it is cyclic. Hence, $H_n(q)\tilde{v} = L(a^n)$. Moreover, $Z_{(a^n)} \subseteq Z(H_n(q))$, and we conclude that

$$(H_n(q)Z_{(a^n)})L(a^n) = (H_n(q)Z_{(a^n)})(H_n(q)\tilde{v}) = (H_n(q)Z_{(a^n)})\tilde{v} = 0.$$

Therefore, $H_n(q)Z_{(a^n)} \subseteq \mathcal{J}(a^n)$. Following the discussion prior to Lemma 9.7.6, we infer that $\mathcal{R}_1(a^n) = H_n(q)/\mathcal{J}(a^n) = (n!)^2$. By counting dimensions, we conclude that $H_n(q)Z_{(a^n)} = \mathcal{J}(a^n)$. □

Remark. (1) Since $Z_{(a^n)}$ is contained in the center of $H_n(q)$, we have that $\mathcal{J}(a^n)^m = H_n(q)^m(Z_{(a^n)})^m = H_n(q)(Z_{(a^n)})^m$, for all $m \geq 1$.
(2) The previous theorem was deduced in [59, §4] in the case for degenerate affine Hecke algebras. In the non-degenerate case, this seems to be new.

9.8 Refinements of restriction functors

This section may be viewed as a key section in our further investigations. We will define various refinements of the restriction functors, and prove several important properties of these. It should be noted that these functors were originally defined by I. Grojnowski in [44] in the non-degenerate case, where for the degenerate case, this was done by A. Kleshchev in [59], using the ideas of the former author. In the sequel we will follow the lines of Kleshchev, giving an explicit way of defining these functors in the non-degenerate case. In particular, we give a construction analogous to that of [59, §4.4].
Note that all the results stated in previous sections will be, at least implicitly, applied in this section.

In the following, for a composition $\mu = (\mu_1, \ldots, \mu_k)$ of n, we will write $\mathrm{Res}^n_{\mu_1,\ldots,\mu_k}$ for the restriction functor $\mathrm{Res}^{H_n(q)}_{H_\mu(q)}$.
Let $M \in H_n(q)-\mathrm{mod}$, and $a \in S$. We denote by $\Delta_a(M)$ the generalized eigenspace of x_n on M, corresponding to the eigenvalue a. In view of Lemma 9.5.3, we have that

$$\Delta_a(M) \cong \bigoplus_{\underline{a}\in S^n,\ a_n=a} M_{\underline{a}}. \tag{9.31}$$

Recall the definition of the parabolic subalgebra $H_{(n-1,1)}(q)$ from Section 9.4. Since x_n is central in $H_{(n-1,1)}(q)$, on restriction, $\Delta_a(M)$ becomes an $H_{(n-1,1)}(q)$-submodule of M. Since a homomorphism $M \to N$ in $H_n(q)-\mathrm{mod}$ maps $\Delta_a(M)$ to $\Delta_a(N)$, we get a functor

$$\Delta_a : H_n(q) - \mathrm{mod} \longrightarrow H_{(n-1,1)}(q) - \mathrm{mod} \tag{9.32}$$

being defined on morphism as restriction. More generally, if $m \geq 1$, define $\Delta_{a^m}(M)$ as the simultaneous generalized eigenspace of the operators $x_{n-m+1}, \ldots, x_n$ corresponding to the eigenvalue a. In the same way as above, we obtain a functor

$$\Delta_{a^m} : H_n(q) - \mathrm{mod} \longrightarrow H_{(n-m,m)}(q) - \mathrm{mod}. \tag{9.33}$$

If we consider $H_{n-1}(q)$ as a subalgebra of $H_{(n-1,1)}(q)$ in the natural way, we obtain the functor

$$e_a^{n,1} := \mathrm{Res}^{H_{(n-1,1)}(q)}_{H_{n-1}(q)} \circ \Delta_a : H_n(q) - \mathrm{mod} \longrightarrow H_{n-1}(q) - \mathrm{mod}. \tag{9.34}$$

By iterating this procedure, for $1 \leq r \leq n$, we also get a functor

$$e_a^{n,r} : H_n(q) - \mathrm{mod} \longrightarrow H_{n-r}(q) - \mathrm{mod}. \tag{9.35}$$

In the following we will write, e_a (resp. e_a^r) for the functor $e_a^{n,1}$ (resp. $e_a^{n,r}$) and all $n \geq 1$. With this notation, we have that $e_a^r = e_a \circ \ldots \circ e_a$ (r times).

For $M \in H_n(q)-\mathrm{mod}$, by Lemma 9.6.1, we have an isomorphism of $H_{n-1}(q)$-modules

$$\mathrm{Res}^{H_n(q)(q)}_{H_{n-1}(q)}(M) \cong \bigoplus_{a \in S} e_a(M). \tag{9.36}$$

Let $\mathcal{J}(a^n)$ be the annihilator of the Kato module $L(a^n)$ in $H_n(q)$, see Section 9.7 for definitions. From the chain of ideals

$$\mathcal{J}(a^n) \supseteq \mathcal{J}(a^n)^2 \supseteq \ldots \supseteq \mathcal{J}(a^n)^m \supseteq \ldots$$

in $H_n(q)$, we obtain surjective F-algebra homomorphisms

$$\ldots \xrightarrow{p_m} \mathcal{R}_m(a^n) \longrightarrow \ldots \xrightarrow{p_2} \mathcal{R}_2(a^n) \xrightarrow{p_1} \mathcal{R}_1(a^n), \tag{9.37}$$

given by $p_k(h + \mathcal{J}(a^n)^{k+1}) = h + \mathcal{J}(a^n)^k$, for all $h \in H_n(q)$, and all $k \geq 1$. Note that for all $m \geq 1$, we consider $\mathcal{R}_m(a^n)$ as an $H_n(q)$-module via inflation. With this, the above epimorphisms p_k become surjective $H_n(q)$-homomorphisms. By Lemma 9.7.5, we have that $\mathcal{R}_m(a^n) \cong \bigoplus_{n!} \mathcal{L}_m(a^n)$, as $\mathcal{R}_m(a^n)$-modules, and hence as $H_n(q)$-modules. Since, for $k \leq m$, $\mathcal{J}(a^n)^k / \mathcal{J}(a^n)^m$ is a nilpotent ideal in $\mathcal{R}_m(a^n)$, and thus nil, we have that idempotents lift along the above algebra epimorphisms, see [1, Proposition 27.1]. Let $m \geq 1$, and $1 = e_1^m + \ldots + e_{n!}^m$ be the decomposition of 1 in orthogonal primitive idempotents e_i, corresponding to the decomposition $\mathcal{R}_m(a^n) \cong \bigoplus_{n!} \mathcal{L}_m(a^n)$. By lifting, we get a decomposition $1 = e_1^{m+1} + \ldots + e_{n!}^{m+1}$ in $\mathcal{R}_{m+1}(a^n)$, see [1, Proposition 27.4].
For $m \geq 1$ and $1 \leq i \leq n!$, denote by $r_m^i : \mathcal{R}_m(a^n) \to \mathcal{R}_m(a^n)e_i^m$ and $j_m^i : \mathcal{R}_m(a^n)e_i^m \to \mathcal{R}_m(a^n)$ the retraction and section corresponding to the decomposition

$$\mathcal{R}_m(a^n) \cong \bigoplus_{n!} \mathcal{L}_m(a^n) \cong \mathcal{R}_m(a^n)e_1^m \oplus \ldots \oplus \mathcal{R}_m(a^n)e_{n!}^m.$$

Then the ring epimorphism $p_m : \mathcal{R}_{m+1}(a^n) \to \mathcal{R}_m(a^n)$ defines a surjective $H_n(q)$-module homomorphism

$$q_m^i = r_m^i p_m j_{m+1}^i : \mathcal{L}_{m+1}(a^n) \cong \mathcal{R}_{m+1}(a^n)e_i^{m+1} \longrightarrow \mathcal{R}_m(a^n)e_i^m \cong \mathcal{L}_m(a^n),$$

for all $1 \leq i \leq n!$. Therefore, fixing such an idempotent decomposition for $m = 1$ yields a chain of $H_n(q)$-homomorphisms

$$\ldots \xrightarrow{q_m^i} \mathcal{R}_m(a^n)e_i^m \xrightarrow{q_{m-1}^i} \ldots \xrightarrow{q_2^i} \mathcal{R}_2(a^n)e_i^2 \xrightarrow{q_1^i} \mathcal{R}_1(a^n)e_i^1, \tag{9.38}$$

for all $1 \leq i \leq n!$.
Furthermore, by the definition of the maps q_m^i, $m \geq 1$, we have the following commutative diagram:

$$\begin{tikzcd}
\ldots \arrow[r, "p_m"] & \mathcal{R}_m(a^n) \arrow[r, "p_{m-1}"] \arrow[d, "r_m^i"'] & \ldots \arrow[r, "p_2"] & \mathcal{R}_2(a^n) \arrow[r, "p_1"] \arrow[d, "r_2^i"'] & \mathcal{R}_1(a^n) \arrow[d, "r_1^i"'] \\
\ldots \arrow[r, "q_m^i"] & \mathcal{R}_m(a^n)e_i^m \arrow[r, "q_{m-1}^i"] & \ldots \arrow[r, "q_2^i"] & \mathcal{R}_2(a^n)e_i^2 \arrow[r, "q_1^i"] & \mathcal{R}_1(a^n)e_i^1
\end{tikzcd}$$

Similarly, the diagram corresponding to the inclusion homomorphisms j_m^i, $m \geq 1$, $1 \leq i \leq n!$, is commutative. We have the following analogue of [59, Lemma 4.4.2] in the non-degenerate case:

Lemma 9.8.1. *Let M be an $H_n(q)$-module, and suppose that $\mathcal{J}(a^n)^k M = 0$, for some $k \in \mathbb{N}$. Then for all $m \geq k$, there exists an isomorphism of $H_n(q)$-modules*

$$\mathrm{Hom}_{H_n(q)}(\mathcal{R}_m(a^n), M) \cong M.$$

Furthermore, there is an isomorphism of functors

$$\varinjlim_m \mathrm{Hom}_{H_n(q)}(\mathcal{R}_m(a^n), -) \cong \varinjlim_m \bigoplus_{n!} \mathrm{Hom}_{H_n(q)}(\mathcal{L}_m(a^n), -)$$

from the category of $H_n(q)$-modules annihilated by some power of $\mathcal{J}(a^n)$ to the category of vector spaces.

Proof. Since $\mathcal{J}(a^n)^k M = 0$, M is the inflation of an $\mathcal{R}_m(a^n)$-module. Hence,

$$\mathrm{Hom}_{H_n(q)}(\mathcal{R}_m(a^n), M) \cong \mathrm{Hom}_{\mathcal{R}_m(a^n)}(\mathcal{R}_m(a^n), M) \cong M.$$

The isomorphism can be derived from Lemma 9.7.5, and the discussion prior to this lemma. ☐

Next, we try to describe the connection between these various functors. In the following, we will denote by $H_r(q)'$, $1 \leq r \leq n$ the subalgebra of $H_{(n-r,r)} \subseteq H_n(q)$ generated by the elements

$$x_{n-r+1}^{\pm 1}, \dots, x_n^{\pm 1}, T_{n-r+1}, \dots, T_{n-1}.$$

Note that, as F-algebras, $H_r(q)' \cong H_r(q)$, which we will use as an identification. Observe that for $M \in H_n(q) - \mathrm{mod}$, we may define an $H_{n-r}(q)$-module structure on the F-vector space $\mathrm{Hom}_{H_r(q)'}(\mathcal{R}_m(a^r), M)$, by setting $(hf)(v) = h(f(v))$, for $f \in \mathrm{Hom}_{H_r(q)'}(\mathcal{R}_m(a^r), M)$, $h \in H_{n-r}(q)$, and $v \in \mathcal{R}_m$.
Also, if $M, N \in H_n(q) - \mathrm{mod}$, and $f : M \to N$ is a homomorphism of $H_n(q)$-modules, we obtain a map

$$f^* : \mathrm{Hom}_{H_r(q)'}(\mathcal{R}_m(a^r), M) \longrightarrow \mathrm{Hom}_{H_r(q)'}(\mathcal{R}_m(a^r), N) \; g \mapsto f \circ g.$$

Moreover, for $h \in H_{n-r}(q)$, we have that

$$f^*(hg)(x) = f((hg)(x)) = f(h(g(x)) = hf(g(x)) = h(f^*(g))(x)$$

since f is $H_{n-r}(q)$-linear. Therefore, we get a functor

$$\varinjlim_m \mathrm{Hom}_{H_r(q)}(\mathcal{R}_m(a^r), -) : H_n(q) - \mathrm{mod} \longrightarrow H_{n-r}(q) - \mathrm{mod}. \tag{9.39}$$

Theorem 9.8.2. *Let $M \in H_n(q) - \mathrm{mod}$, $a \in S$. Then*

$$e_a^r(M) \cong \varinjlim_m \mathrm{Hom}_{H_r(q)'}(\mathcal{R}_m(a^r), M) \tag{9.40}$$

as $H_{n-r}(q)$-modules. Moreover, the above isomorphism is natural, and, thus, induces an isomorphism of functors from the category $H_n(q)-\mathrm{mod}$ to the category $H_{n-r}(q)-\mathrm{mod}$.

Proof. First, we show that $e_a^r(M) = \Delta_{a^r}(M)$ as $H_{n-r}(q)$-modules. Let $r = 1$. Then the statement follows from the definition of e_a given in (9.34). Next, let $r > 1$. If $v \in \Delta_{a^r}(M)$, then $v \in \Delta_{a^{r-1}}(M)$, and $(x_{n-r+1} - a)^k v = 0$, for some $k \geq 0$. Hence $\Delta_{a^r}(M) \subseteq \Delta_a(\Delta_{a^{r-1}}(M))$. On the other hand, if $v \in \Delta_a(\Delta_{a^{r-1}}(M))$, then $(x_i - a)^k v = 0$, for all $n - r + 1 \leq i \leq n$ and some $k \geq 0$. Thus, $v \in \Delta_{a^r}(M)$, hence $\Delta_a(\Delta_{a^{r-1}}(M)) = \Delta_{a^r}(M)$.
Now, by definition, $e_a^r(M) = e_a(e_a^{r-1}(M))$. By the induction hypothesis, we may assume that $e_a^{r-1}(M) = \Delta_{a^{r-1}}(M)$ as $H_{n-r+1}(q)$-modules. As we have seen above, the statement holds for $r = 1$, therefore:

$$\begin{aligned} e_a^r(M) &= e_a(e_a^{r-1}(M)) = e_a(\Delta_{a^{r-1}}(M)) \\ &= \Delta_a(\Delta_{a^{r-1}}(M)) = \Delta_{a^r}(M), \end{aligned}$$

as $H_{n-r}(q)$-modules.
Next, we want to show that

$$X := \Delta_{a^r}(M) \cong \varinjlim_m \mathrm{Hom}_{H_r(q)'}(\mathcal{R}_m(a^r), M)$$

as $H_{n-r}(q)$-modules. To this end, we use the canonical isomorphism $H_{(n-r,r)} \cong H_{n-r} \otimes_F H_r' \cong H_{n-r} \otimes_F H_r$, and throughout, we will consider M as an $H_{n-r} \otimes_F H_r$-module with respect to this isomorphism.
Recall that, for $m \geq 1$, $\mathcal{R}_m(a^r) = H_r(q)/\mathcal{J}(a^r)^m$, and we will consider it as an $H_r(q)$-module via inflation along the canonical F-algebra epimorphism $p : H_r(q) \to H_r(q)/\mathcal{J}(a^r)^m$. Let $Y_m(M) := \{y \in M \mid \mathcal{J}(a^r)^m y = 0\}$. For $y \in Y_m(M)$, we get a homomorphism $f_y \in \mathrm{Hom}_{H_r(q)}(\mathcal{R}_m(a^r), M)$, given by the following diagram:

$$\begin{tikzcd} \mathcal{J}(a^r)^m \arrow[r, "i"] & H_r(q) \arrow[r, "p"] \arrow[d, "\cdot y"'] & \mathcal{R}_m(a^r) = H_r(q)/\mathcal{J}(a^r)^m \arrow[dl, dotted, "f_y"] \\ & M & \end{tikzcd}$$

Since $\mathcal{J}(a^r)^m y = 0$, this map exists and is unique by the cokernel property. Hence, we obtain a well-defined map

$$\psi_m : Y_m(M) \longrightarrow \mathrm{Hom}_{H_r(q)}(\mathcal{R}_m(a^r), M), \quad y \mapsto f_y.$$

Since the elements of $H_{n-r}(q)$ commute with the elements of $H_r(q)$, $Y_m(M)$ is an $H_{n-r}(q)$-submodule of M. For every $t \in \mathcal{R}_m(a^r)$, it follows from the definition of f_y that if $s \in p^{-1}(t)$, then $f_y(t) = sy$. Also, since cokernels are unique, the map ψ_m is a homomorphism of F-vector spaces. Moreover, for $h \in H_{n-r}(q)$,

$$\begin{aligned} \psi_m(hy)(t) &= f_{hy}(t) = s(hy) \\ &= h(sy) = h\psi_m(y)(t), \end{aligned}$$

for all $t \in \mathcal{R}_m(a^r)$, since h commutes with all elements in $H_r(q)$. This shows that ψ_m is even an $H_{n-r}(q)$-homomorphism.

On the other hand, let $f \in \mathrm{Hom}_{H_r(q)}(\mathcal{R}_m(a^r), M)$, for some $m \geq 1$. Since we view $\mathcal{R}_m(a^r)$ as an $H_r(q)$-module via inflation along p, f is determined by the element $f(1) \in M$. Here, by 1 we mean the unit in the ring $\mathcal{R}_m(a^r)$. To see this, take an element $t \in \mathcal{R}_m(a^r)$, and some $s \in p^{-1}(t)$. Hence we can write $t = t1 = p(s)1 = s \cdot 1$, where we indicate the action of $H_r(q)$ on $\mathcal{R}_m(a^r)$ by " $\cdot$ ". Then it follows that

$$f(t) = f(p(s)1) = f(s \cdot 1) = sf(1),$$

where the third equality comes from the fact that f is an $H_r(q)$-homomorphism. Moreover, we have that

$$\mathcal{J}(a^r)^m f(1) = f(p(\mathcal{J}(a^r)^m)1) = 0,$$

and hence, $f(1) \in Y_m(M)$. Therefore, we get a map

$$\phi_m : \mathrm{Hom}_{H_r(q)}(\mathcal{R}_m(a^r), M) \longrightarrow Y_m(M),$$

given by $\phi_m(f) = f(1)$. It is easy to see that ϕ_m is a homomorphism of F-vector spaces. Furthermore, by the definition of the $H_{n-r}(q)$-module structure on $\mathrm{Hom}_{H_r(q)}(\mathcal{R}_m(a^r), M)$:

$$\begin{aligned} \phi_m(hf) &= (hf)(1) = h(f(1)) \\ &= h\phi_m(f), \end{aligned}$$

for all $h \in H_{n-r}(q)$, and $f \in \mathrm{Hom}_{H_r(q)}(\mathcal{R}_m(a^r), M)$, showing that ϕ_m is an $H_{n-r}(q)$-homomorphism.

On the one hand, since $1_{H_r(q)} \in p^{-1}(1)$, we have for all $y \in Y_m(M)$:

$$\phi_m(\psi_m(y)) = \phi_m(f_y) = f_y(1) = y.$$

On the other hand, for all $t \in \mathcal{R}_m(a^r)$:

$$\begin{aligned} \psi_m(\phi_m(f))(t) &= \psi_m(f(1))(t) = sf(1) \\ &= f(s \cdot 1) = f(p(s)1) \\ &= f(t), \end{aligned}$$

where $f \in \mathrm{Hom}_{H_r(q)}(\mathcal{R}_m(a^r), M)$, $t \in \mathcal{R}_m(a^r)$, and some $s \in p^{-1}(t)$. This shows that ψ_m and ϕ_m are mutually inverse bijections and, hence,

$$Y_m(M) \cong \mathrm{Hom}_{H_r(q)}(\mathcal{R}_m(a^r), M)$$

as $H_{n-r}(q)$-modules.
Next, let $M, N \in H_n(q) - \mathrm{mod}$. For $m \geq 1$, and a homomorphism $f : M \to N$ in $H_n(q) - \mathrm{mod}$, we get a homomorphism

$$f^* : \mathrm{Hom}_{H_r(q)}(\mathcal{R}_m(a^r), M) \longrightarrow \mathrm{Hom}_{H_r(q)}(\mathcal{R}_m(a^r), N),$$

of $H_{n-r}(q)$-modules, given by $f^*(g) = fg$, for $g \in \mathrm{Hom}_{H_r(q)}(\mathcal{R}_m(a^r), M)$. Let $y \in Y_m(M)$. Then

$$\mathcal{J}(a^r)^m f(y) = f(\mathcal{J}(a^r)^m y) = 0,$$

thus, $f(y) \in Y_m(N)$, and we get a homomorphism

$$\tilde{f} : Y_m(M) \longrightarrow Y_m(N)$$

of $H_{n-r}(q)$-modules. Moreover,

$$\begin{aligned} \phi_m(f^*(g)) &= \phi_m(fg) = (fg)(1) \\ &= f(g(1)) = f(\phi_m(g)) \\ &= \tilde{f}(\phi_m(g)). \end{aligned}$$

We consider the following diagram:

$$\begin{tikzcd} \mathrm{Hom}_{H_r(q)}(\mathcal{R}_m(a^r), M) \arrow[r, hook, "j_m"] \arrow[d, "\phi_m"'] & \mathrm{Hom}_{H_r(q)}(\mathcal{R}_{m+1}(a^r), M) \arrow[d, "\phi_{m+1}"] \\ Y_m \arrow[r, hook, "i_m"] & Y_{m+1} \end{tikzcd}$$

Here, i_m denotes the $H_{n-r}(q)$-monomorphism induced by the inclusion of the sets $Y_m \subseteq Y_{m+1}$. Moreover, j_m is induced from the homomorphism $p_m : \mathcal{R}_{m+1}(a^r) \to \mathcal{R}_m(a^r)$ of (9.37), and applying the left exact functor $\mathrm{Hom}_{H_r(q)}(-, M)$. More precisely, if $f \in \mathrm{Hom}_{H_r(q)}(\mathcal{R}_m(a^r), M)$, then $j_m(f) = fp_m$. From this one can see that j_m is a homomorphism of $H_{n-r}(q)$-modules. Furthermore,

$$\begin{aligned} \phi_{m+1}(j_m(f)) &= (fp_m)(1_{\mathcal{R}_{m+1}(a^r)}) = f(p_m(1_{\mathcal{R}_{m+1}(a^r)})) \\ &= f(1_{\mathcal{R}_m(a^r)}) = i_m \phi_m(f), \end{aligned}$$

which shows that the diagram commutes. With this, it follows that

$$\begin{aligned} &Y := \{y \in M \mid \mathcal{J}(a^r)^m y = 0, \text{ for some } m \geq 1\} \\ &= \varinjlim_m Y_m \\ &\cong \varinjlim_m \mathrm{Hom}_{H_r(q)'}(\mathcal{R}_m(a^r), M), \end{aligned}$$

as $H_{n-r}(q)$-modules. The above calculations show that this isomorphism is functorial.

To finish the proof, we show that $Y = X$. For, let $y \in Y$, i.e., there is some $m \geq 1$ such that $\mathcal{J}(a^r)^m y = 0$. Since $(x_i - a)^{r!} \in \mathcal{J}(a^r)$, it follows that $(x_i - a)^{r!m} \in \mathcal{J}(a^r)^m$. Thus, $(x_i - a)^{r!m} y = 0$, for all $n - r + 1 \leq i \leq n$, thus, $y \in X$.

On the other hand, X can be characterized as the largest submodule of $\mathrm{Res}^n_{n-r,r}(M)$ all of whose composition factors are of the form $N \boxtimes L(a^r)$, for an irreducible $H_{n-r}(q)$-module N. Therefore, if we restrict X to $H_r(q) \cong 1 \otimes H_r(q) \hookrightarrow H_{n-r}(q) \otimes H_r(q)$, all composition factors are isomorphic to $L(a^r)$. This implies that the restriction must belong to the block of $H_r(q)$ indexed by the central character $\gamma := (a, \ldots, a)$ (r times). We show by induction on the composition length that all modules in this block are annihilated by some power of $\mathcal{J}(a^r)$.

Let $W \in H_r(q) - \mathrm{mod}$ with central character γ, and composition length 1. By Theorem 9.7.3, up to isomorphism, $L(a^r)$ is the only irreducible module in its block. Hence, $W \cong L(a^r)$, and so, by definition of $\mathcal{J}(a^r)$, $\mathcal{J}(a^r)W = 0$. Now suppose that W has composition length greater than 1. Then take some maximal submodule $W' \subset W$ which is non-zero by assumption on W. Consider the exact sequence

$$0 \to W' \to W \to W/W' \to 0$$

of $H_r(q)$-modules given by the quotient module W/W'. Again, by Theorem 9.7.3, $W/W' \cong L(a^r)$, and we get that $\mathcal{J}(a^r)(W/W') = 0$. By definition of the action of $H_r(q)$ on W/W', it follows that $\mathcal{J}(a^r)W \subseteq W'$. Now by the induction hypothesis, some power of $\mathcal{J}(a^r)$ annihilates W'. We conclude that some power of $\mathcal{J}(a^r)$ must annihilate W.

It follows that $\mathcal{J}(a^r)^k X = 0$, for some $k > 0$, so $X \subseteq Y$. Therefore, since X and Y are both subsets of M, we infer that $X = Y$ as $H_{n-r}(q)$-modules.

From the above discussion, we now obtain an isomorphism of functors, given by the isomorphisms constructed in the proof. $\square$

Next, for $a \in S$, $r \geq 1$, and $M \in H_n(q) - \mathrm{mod}$, we define

$$e_a^{(r)}(M) = \varinjlim_m \mathrm{Hom}_{H_r(q)'}(\mathcal{L}_m(a^r), M).$$

Note that we get an $H_{n-r}(q)$-module structure on $\mathrm{Hom}_{H_r(q)'}(\mathcal{L}_m(a^r), M)$, by defining $(hf)(v) = h(f(v))$, for $f \in \mathrm{Hom}_{H_r(q)'}(\mathcal{L}_m(a^r), M)$, $h \in H_{n-r}(q)$, and $v \in \mathcal{L}_m(a^r)$. Also, if $M, N \in H_n(q) - \mathrm{mod}$, and $f : M \to N$ is a homomorphism of $H_n(q)$-modules, we obtain a map

$$f^* : \mathrm{Hom}_{H_r(q)'}(\mathcal{L}_m(a^r), M) \longrightarrow \mathrm{Hom}_{H_r(q)'}(\mathcal{L}_m(a^r), N),\ g \mapsto f \circ g.$$

For $h \in H_{n-r}(q)$ and $x \in \mathcal{L}_m(a^r)$, we have that

$$f^*(hg)(x) = f((hg)(x)) = f(h(g(x)) = hf(g(x)) = h(f^*(g))(x)$$

since f is $H_{n-r}(q)$-linear. Thus, we obtain a functor

$$e_a^{(r)} : H_n(q) - \mathrm{mod} \longrightarrow H_{n-r}(q) - \mathrm{mod}. \tag{9.41}$$

We have the following:

Theorem 9.8.3. *There is an isomorphism*

$$e_a^r \cong \bigoplus_{r!} e_a^{(r)}$$

of functors from the category $H_n(q) - \text{mod}$ *to the category* $H_{n-r}(q) - \text{mod}$.

Proof. By Theorem 9.8.2, for $M \in H_n(q) - \text{mod}$, we have an isomorphism

$$e_a^r(M) \cong \varinjlim_t \mathrm{Hom}_{H_r(q)}(\mathcal{R}_t(a^r), M)$$

of $H_{n-r}(q)$-modules that is natural in M. Since M is finite-dimensional, the direct limit above stabilizes after finitely many steps. Therefore, we may assume that for m large enough:

$$\varinjlim_t \mathrm{Hom}_{H_r(q)}(\mathcal{R}_t(a^r), M) = \mathrm{Hom}_{H_r(q)}(\mathcal{R}_m(a^r), M).$$

By Lemma 9.8.1, we have that

$$\mathrm{Hom}_{H_r(q)}(\mathcal{R}_m(a^r), M) \cong \bigoplus_{r!} \mathrm{Hom}_{H_r(q)}(\mathcal{L}_m(a^r), M),$$

where the isomorphism ψ_M is defined by

$$f \mapsto (f \circ i_1, \dots, f \circ i_{r!}),$$

for $f \in \mathrm{Hom}_{H_r(q)}(\mathcal{R}_m(a^r), M)$ and i_k, $1 \leq k \leq r!$, denotes the split monomorphism given by the direct sum decomposition of $\mathcal{R}_m(a^r)$, see the discussion prior to Lemma 9.8.1. Moreover, for $h \in H_{n-r}(q)$, we have that

$$\begin{aligned}
\psi_M(hf)(x) &= ((hf) \circ i_1(x), \dots, (hf) \circ i_{r!}(x)) \\
&= ((hf(i_1(x)), \dots, hf(i_{r!}(x)) \\
&= (h(f(i_1(x))), \dots, h(f(i_{r!}(x))),
\end{aligned}$$

thus, by the definition of the $H_{n-r}(q)$-module structure on $\mathrm{Hom}_{H_r(q)}(\mathcal{L}_m(a^r), M)$, we see that ψ_M is an isomorphism of $H_{n-r}(q)$-modules. Next, let $M, N \in H_n(q) - \text{mod}$ and $f : M \to N$ be an $H_n(q)$-homomorphism. Choose m large enough such that

$$\begin{aligned}
\varinjlim_t \mathrm{Hom}_{H_r(q)}(\mathcal{R}_t(a^r), M) &= \mathrm{Hom}_{H_r(q)}(\mathcal{R}_m(a^r), M), \quad \text{and} \\
\varinjlim_t \mathrm{Hom}_{H_r(q)}(\mathcal{R}_t(a^r), N) &= \mathrm{Hom}_{H_r(q)}(\mathcal{R}_m(a^r), N).
\end{aligned}$$

We obtain a diagram

$$\begin{tikzcd}
\mathrm{Hom}_{H_r(q)}(\mathcal{R}_m(a^r), M) \arrow[r, "\psi_M"] \arrow[d, "\eta_{M,N}"'] & \bigoplus_{r!} \mathrm{Hom}_{H_r(q)}(\mathcal{L}_m(a^r), M) \arrow[d, "\nu_{M,N}"'] \\
\mathrm{Hom}_{H_r(q)}(\mathcal{R}_m(a^r), N) \arrow[r, "\psi_N"] & \bigoplus_{r!} \mathrm{Hom}_{H_r(q)}(\mathcal{L}_m(a^r), N)
\end{tikzcd}$$

where $\eta_{M,N}(h) = f \circ h$, and $\nu_{M,N}(k_1, \ldots, k_{r!}) = (f \circ k_1, \ldots, f \circ k_{r!})$. For $h \in \mathrm{Hom}_{H_r(q)}(\mathcal{R}_m(a^r), M)$, we get that

$$\begin{aligned}
\nu_{M,N}(\psi_M(h)) &= \nu_{M,N}(h \circ i_1, \ldots, h \circ i_{r!}) \\
&= (f \circ (h \circ i_1), \ldots, f \circ (h \circ i_{r!})) \\
&= ((f \circ h) \circ i_1, \ldots, (f \circ h) \circ i_{r!})) \\
&= \psi_N(\eta_{M,N}(h)).
\end{aligned}$$

Hence, the diagram commutes, i.e., ψ determines a natural transformation between the functors

$$\mathrm{Hom}_{H_r(q)}(\mathcal{R}_m(a^r), -) \quad \text{and} \quad \bigoplus_{r!} \mathrm{Hom}_{H_r(q)}(\mathcal{L}_m(a^r), -)$$

from the category $H_n(q) - \mathrm{mod}$ to the category $H_{n-r}(q) - \mathrm{mod}$. The claim now follows since, for $M \in H_n(q) - \mathrm{mod}$, ψ_M is an isomorphism of $H_{n-r}(q)$-modules. □

Next, we want to give another description of the $H_n(q)$-modules $\mathcal{L}_m(a^n)$, $m \geq 1$. Consider the finite-dimensional Hecke algebra $H_n^f(q)$, i.e., the subalgebra of $H_n(q)$ generated by $T_1, \ldots, T_{n-1}$. Let λ be a composition of n. Then the element

$$x_\lambda = \sum_{w \in W_\lambda} T_w \in H_n^f(q)$$

affords the permutation module $M^\lambda := H_n^f(q)x_\lambda$ of $H_n^f(q)$ corresponding to the composition λ. For example, if $\lambda = (n)$, we obtain the trivial module for $H_n^f(q)$, denoted by $\mathbf{1}$. Note that for all $w \in W(n)$, we have that

$$T_w x_{(n)} = q^{\mathrm{l}(w)} x_{(n)},$$

see [68, Lemma 3.2].
Set $P(n) := \mathrm{Ind}_{H_n^f(q)}^{H_n(q)}(\mathbf{1})$. For the next proposition, recall that by Theorem 9.3.1, $H_n(q)$ is a free P_n-module with basis $\{T_w \mid w \in W(n)\}$. It follows that

$$L(a^n) = \mathrm{Ind}_{P_n}^{H_n(q)} L(a) \boxtimes \ldots \boxtimes L(a) = \bigoplus_{w \in W(n)} F(T_w \otimes_{P_n} b), \tag{9.42}$$

where b is a fixed basis vector of the one-dimensional P_n-module $L(a) \boxtimes \ldots \boxtimes L(a)$. Thus we see that, considered as $H_n^f(q)$-module, $L(a^n)$ is isomorphic to the left regular module $H_n^f(q)$. Then we have an analogue of [59, Lemma 4.4.3] in the non-degenerate case:

Proposition 9.8.4. *We have that* $\mathcal{L}_m(a^n) \cong P(n)/\mathcal{J}(a^n)^m P(n)$ *as* $H_n(q)$*-modules.*

Proof. Let $p_m : P(n) \to P(n)/\mathcal{J}(a^n)^m P(n)$ denote the epimorphism corresponding to the quotient. Then we have an exact sequence

$$\begin{aligned} 0 &\longrightarrow \mathrm{Hom}_{H_n(q)}(P(n)/\mathcal{J}(a^n)^m P(n), L(a^n)) \longrightarrow \mathrm{Hom}_{H_n(q)}(P(n), L(a^n)) \\ &\longrightarrow \mathrm{Hom}_{H_n(q)}(\mathcal{J}(a^n)^m P(n), L(a^n)). \end{aligned}$$

Since $\mathcal{J}(a^n)$ annihilates $L(a^n)$, the last term of the sequence above equals zero, thus, we have an isomorphism of F-vector spaces

$$\mathrm{Hom}_{H_n(q)}(P(n)/\mathcal{J}(a^n)^m P(n), L(a^n)) \cong \mathrm{Hom}_{H_n(q)}(P(n), L(a^n)).$$

Using Frobenius Reciprocity, we get that

$$\begin{aligned} \mathrm{Hom}_{H_n(q)}(P(n)/\mathcal{J}(a^n)^m P(n), L(a^n)) &\cong \mathrm{Hom}_{H_n(q)}(P(n), L(a^n)) \\ &\cong \mathrm{Hom}_{H_n^f(q)}(\mathbf{1}, \mathrm{Res}^{H_n(q)}_{H_n^f(q)}(L(a^n))). \end{aligned}$$

From (9.42) it is easy to see that $\mathrm{Res}^{H_n(q)}_{H_n^f(q)}(L(a^n)) \cong H_n^f(q)$. Since the trivial module occurs precisely once in the socle of $H_n^f(q)$, we conclude that

$$\mathrm{Hom}_{H_n(q)}(P(n)/\mathcal{J}(a^n)^m P(n), L(a^n)) \cong F.$$

Therefore, the $\mathcal{R}_m(a^n)$-module $P(n)/\mathcal{J}(a^n)^m P(n)$ has irreducible head $L(a^n)$, and hence is a quotient of $\mathcal{L}_m(a^n)$.
Consider the projective cover $\pi : \mathcal{L}_m(a^n) \to L(a^n)$ in $\mathcal{R}_m(a^n) - \mathrm{mod}$. If we consider this epimorphism as a homomorphism in $H_n(q) - \mathrm{mod}$ and restrict it to $H_n^f(q) - \mathrm{mod}$, we see that it must be split since $\mathrm{Res}^{H_n(q)}_{H_n^f(q)}(L(a^n)) \cong H_n^f(q)$. Moreover, $\mathrm{Res}^{H_n(q)}_{H_n^f(q)}(L(a^n))$ contains a non-zero vector x such that $T_w x = q^{l(w)} x$, for all $w \in W(n)$. Let $v \in \mathrm{Res}^{H_n(q)}_{H_n^f(q)}(\mathcal{L}_m(a^n))$ be such that $\mathrm{Res}^{H_n(q)}_{H_n^f(q)}(\pi)(v) = x$. It follows that we have a non-zero $H_n^f(q)$-homomorphism

$$f : \mathbf{1} \to \mathrm{Res}^{H_n(q)}_{H_n^f(q)}(\mathcal{L}_m(a^n)).$$

Again, by Frobenius Reciprocity, we get a non-zero homomorphism $\tilde{f} : P(n) \to \mathcal{L}_m(a^n)$ in $H_n(q) - \mathrm{mod}$ such that v is contained in the image of $\tilde{f}$. This induces a non-zero $\mathcal{R}_m(a^n)$-homomorphism $g : P(n)/\mathcal{J}(a^n)^m P(n) \to \mathcal{L}_m(a^n)$, because $\mathcal{J}(a^n)^m \mathcal{L}_m(a^n) = 0$. Since $\mathcal{L}_m(a^n)$ is a projective cover of $L(a^n)$ as $\mathcal{R}_m(a^n)$-module, it is cyclic, generated by v. It follows that g must be surjective. The result now follows. $\square$

Since, by Theorem 9.3.1, $H_n(q)$ is a free right $H_n^f(q)$-module with basis $\{x^\alpha \mid \alpha \in \mathbb{Z}^n\}$, we have

$$P(n) = \bigoplus_{\alpha \in \mathbb{Z}^n} F(x^\alpha \otimes x_{(n)}),$$

with $\mathbf{1} = Fx_{(n)}$. From this we can see that $P(n)$ is a cyclic left $H_n(q)$-module, generated by $1 \otimes x_{(n)}$. Moreover, we get that

$$T_w(1 \otimes x_{(n)}) = T_w \otimes x_{(n)} = 1 \otimes T_w x_{(n)} = q^{\mathrm{l}(w)}(1 \otimes x_{(n)}),$$

for all $w \in W(n)$.
Let $h \in H_n(q)$. By Theorem 9.3.1, we may write $h = \sum_{i=1}^{k} \lambda_i x^{\alpha_i} T_{w_i}$, for some $\alpha_i \in \mathbb{Z}^n, \lambda_i \in F$, and $w_i \in W(n)$. Given the element $1 \otimes x_{(n)} \in P(n)$, we see that h acts as

$$\begin{aligned} h(1 \otimes x_{(n)}) = h \otimes x_{(n)} &= (\sum_{i=1}^{k} \lambda_i x^{\alpha_i} T_{w_i}) \otimes x_{(n)} \\ &= \sum_{i=1}^{k} \lambda_i x^{\alpha_i} \otimes T_{w_i} x_{(n)} = \sum_{i=1}^{k} \lambda_i q^{l(w_i)} x^{\alpha_i} \otimes x_{(n)} \\ &= (\sum_{i=1}^{k} \lambda_i q^{l(w_i)} x^{\alpha_i}) \otimes x_{(n)}. \end{aligned} \tag{9.43}$$

Let $M \in H_n(q)-\mathrm{mod}$, and U be a subgroup of $W(n)$, generated by basic transpositions of $W(n)$. Then, denote by $M^U := \{v \in M \mid T_w v = q^{\mathrm{l}(w)} v \text{ for all } w \in U\} \subseteq M$, which is a subspace of M. Note that U acts via the T_w on M^U. To this end, let $w \in U$, $s \in \mathfrak{B} \cap U$ and $m \in M^U$. If $l(ws) = l(w) + 1$, then

$$(T_w T_s)m = T_{ws} m = q^{\mathrm{l}(w)+1} m = q^{\mathrm{l}(w)}(qm) = T_w(T_s m).$$

On the other hand, if $l(ws) < l(w)$, we have by the multiplication rule for $H_n^f(q)$, see (9.1), that

$$\begin{aligned} (T_w T_s)m &= (qT_{ws} + (q-1)T_w)m \\ &= q^{\mathrm{l}(w)} m + (q-1) q^{\mathrm{l}(w)} m \\ &= q^{\mathrm{l}(w)+1} m \\ &= T_w(T_s m). \end{aligned}$$

Recall the definition of the subalgebra $H_r(q)'$ of $H_n(q)$ from the discussion after Lemma 9.8.1. In the following, we will denote by $W(r)'$, for $r > 0$, the subgroup of $W(n)$ generated by the elements

$$s_{n-r+1}, s_{n-r+2}, \ldots, s_{n-1} \in W(n).$$

Clearly $W(r)' \cong W(r)$. We get the following:

Theorem 9.8.5. *Let $M \in H_n(q) - \mathrm{mod}$. Then we have a functorial isomorphism*

$$e_a^{(r)}(M) \cong (\Delta_{a^r}(M))^{W(r)'}$$

of $H_{n-r}(q)$-modules.

Proof. First note that the subspace $(\Delta_{a^r}(M))^{W(r)'}$ is indeed an $H_{n-r}(q)$-submodule of M since $H_{n-r}(q)$ commutes with all the elements of $H_r(q)'$.
By Proposition 9.8.4, we have an isomorphism

$$e_a^{(r)}(M) = \varinjlim_m \mathrm{Hom}_{H_r(q)'}(\mathcal{L}_m(a^r), M) \cong \varinjlim_m \mathrm{Hom}_{H_r(q)'}(P(r)/\mathcal{J}(a^r)^m P(r), M),$$

which is clearly functorial. Since $P(r)$ is generated by $1 \otimes x_{(r)}$ as $H_r(q)'$-module, $P(r)/\mathcal{J}(a^r)^m P(r)$ is generated by $u := \pi(1 \otimes x_{(r)})$ as a $H_r(q)'$-module, where

$$\pi : P(r) \to P(r)/\mathcal{J}(a^r)^m P(r)$$

denotes the natural epimorphism.
Let $f \in \mathrm{Hom}_{H_r(q)}(P(r)/\mathcal{J}(a^r)^m P(r), M)$. Set $v = f(u)$. Then we get that

$$\begin{aligned} \mathcal{J}(a^r)^m v &= \mathcal{J}(a^r)^m f(u) \\ &= f(\mathcal{J}(a^r)^m u) \\ &= 0. \end{aligned}$$

Therefore, $v \in \Delta_{a^r}(M)$. Moreover, for $w \in W(r)'$, we have

$$\begin{aligned} T_w v &= T_w f(u) = f(T_w u) \\ &= f(\pi(T_w(1 \otimes x_{(r)}))) = q^{\mathrm{l}(w)} v, \end{aligned}$$

thus $v \in (\Delta_{a^r}(M))^{W(r)'}$.
On the other hand, let $v \in (\Delta_{a^r}(M))^{W(r)'}$. Then, $\mathcal{J}(a^r)^m v = 0$, for some $m \geq 1$. We now define a homomorphism $f_v \in \mathrm{Hom}_{H_r(q)'}(P(r)/\mathcal{J}(a^r)^m P(r), M)$. For $x \in P(r)/\mathcal{J}(a^r)^m P(r)$, we know from the discussion prior to this theorem that $x = hu$, for some $h \in H_r(q)'$. We define a map

$$f_v : P(r)/\mathcal{J}(a^r)^m P(r) \to M$$

by setting $f_v(x) = hv$. We must show that this is well defined. To do this, let $h \in H_r(q)'$ be such that $hu = 0 \in P(r)/\mathcal{J}(a^r)^m P(r)$. In other words, $h(1 \otimes x_{(r)}) \in \mathcal{J}(a^r)^m P(r)$ in $P(r)$.
Suppose first that $h(1 \otimes x_{(r)}) = 0$. By (9.43), if $h = \sum_{i=1}^k \lambda_i x^{\alpha_i} T_{w_i}$, then

$$h(1 \otimes x_{(r)}) = (\sum_{i=1}^k \lambda_i q^{\mathrm{l}(w_i)} x^{\alpha_i}) \otimes x_{(r)}.$$

Since the elements $x^\alpha \otimes x_{(r)}, \alpha \in \mathbb{Z}^r$, form a basis of $P(r)$ as an F-vector space, we must have that $\lambda_k = 0$, whenever $x^{\alpha_k} \neq 0$. This shows that $h = \sum_{i=1}^k \lambda_k T_{w_k}$, i.e., $h \in H_r^f(q)'$, the subalgebra of $H_r(q)'$ generated by $T_{n-r+1}, \dots, T_{n-1}$. But considered as an $H_r^f(q)'$-module, the subspace $F(1 \otimes x_{(r)})$ of $P(r)$ is isomorphic to the trivial representation of $H_r^f(q)'$. It is easy to see that the two-sided ideal

$$\bigoplus_{w \in W(r)', w \neq 1} F(T_w - q^{l(w)})$$

is the kernel of the F-algebra homomorphism $H_r^f(q)' \to \mathrm{End}_F(F(1 \otimes x_{(r)})$ given by the trivial representation. Therefore, we have that $h = \sum \mu_j (T_{w_j} - q^{l(w_j)})$, for some $\mu_j \in F$.
Now, let $h(1 \otimes x_{(r)}) \in \mathcal{J}(a^r)^m P(r)$ in $P(r)$, then, by the alternative description of the ideal $\mathcal{J}(a^r)$ in Theorem 9.7.7 and the action of $H_r(q)'$ on $1 \otimes x_{(r)}$, we may write

$$h(1 \otimes x_{(r)}) = Q(1 \otimes x_{(r)}),$$

with some element $Q \in P_r'(Z_{(a^r)})^m$. But then $(h - Q)(1 \otimes x_{(r)}) = 0$, which shows that h has the form

$$h = (\sum \mu_j (T_{w_j} - q^{l(w_j)})) + Q,$$

for elements $w_j \in W(r)'$, and $\mu_j \in F$. Since, by assumption, $v \in (\Delta_{a^r}(M))^{W(r)'}$, it is now easy to see that $hv = 0$, and therefore, f_v is well defined. Also, it is not hard to show that f_v is a homomorphism of F-vector spaces: If $x, y \in P(r)/\mathcal{J}(a^r)^m P(r)$, and $\mu \in F$, then there are $h, h' \in H_r(q)'$ such that

$$\mu x + y = \mu(hu) + h'u = (\mu h + h')u.$$

It follows that

$$f_v(\mu x + y) = (\mu h + h')v = \mu(hv) + h'v = \mu f_v(x) + f_v(y).$$

Let $h'' \in H_r(q)'$, $x \in P(r)/\mathcal{J}(a^r)^m P(r)$, and choose $h \in H_r(q)'$ such that $x = hu$. Then

$$f_v(h''x) = f_v((h''h)u) = (h''h)v = h''(hv) = h'' f_v(x),$$

and so, $f_v \in \mathrm{Hom}_{H_r(q)'}(P(r)/\mathcal{J}(a^r)^m P(r), M)$.
Therefore, we have constructed two maps:

$$\psi : (\Delta_{a^r}(M))^{W(r)'} \longrightarrow \varinjlim_m \mathrm{Hom}_{H_r(q)'}(P(r)/\mathcal{J}(a^r)^m P(r), M),$$

given by $\psi(v) = f_v$, and

$$\phi : \varinjlim_m \mathrm{Hom}_{H_r(q)'}(P(r)/\mathcal{J}(a^r)^m P(r), M) \longrightarrow (\Delta_{a^r}(M))^{W(r)'},$$

given by $\phi(f) = f(u)$.
We now show that the latter are inverse to each other. On the one hand,

$$\phi(\psi(v)) = \phi(f_v) = f_v(u) = v,$$

and on the other hand,

$$\begin{aligned} \psi(\phi(f))(t) &= \psi(f(u))(t) = f_{f(u)}(t) \\ &= hf(u) = f(hu) \\ &= f(t), \end{aligned}$$

where $t = hu \in P(r)/\mathcal{J}(a^r)^m P(r)$, for some $h \in H_r(q)'$. Therefore, ψ and ϕ are mutually inverse bijections.
Next, fix an element $h \in H_{n-r}(q)$. Then, for $t = h'u \in P(r)/\mathcal{J}(a^r)^m P(r)$, $h' \in H_r(q)'$, we have that

$$\begin{aligned} \psi(hv)(t) &= f_{hv}(t) = h'(hv) \\ &= h(h'v) = hf_v(t) \\ &= h\psi(v) \end{aligned}$$

since the elements of $H_{n-r}(q)$ commute with those of $H_r(q)'$.
Moreover, for an element $f \in \mathrm{Hom}_{H_r(q)'}(P(r)/\mathcal{J}(a^r)^m P(r), M)$, we compute:

$$\begin{aligned} \phi(hf) &= (hf)(u) = h(f(u)) \\ &= h\phi(f). \end{aligned}$$

Thus, we obtain an isomorphism

$$(\Delta_{a^r}(M))^{W(r)'} \cong \varinjlim_m \mathrm{Hom}_{H_r(q)}(P(r)/\mathcal{J}(a^r)^m P(r), M)$$

of $H_{n-r}(q)$-modules. The above construction shows that this isomorphism is functorial. □

Remark 9.8.6. The definition of the functors $e_i^{(r)}$, $r \geq 1$, are based on ideas of I. Grojnowski, see [44, §8]. The explicit description of these in the degenerate case, as well as the alternative description, is due to A. Kleshchev, see [59, §8]. The alternative description of Theorem 9.8.5 of the functors in the non-degenerate case seems to be new to us.

Chapter 10

Kac–Moody algebras

In this chapter, we will briefly describe the theory of Kac–Moody algebras and their representation theory.

First we state the basic definitions of these algebras, some of them are needed in the definition of the cyclotomic Hecke algebras, given in Chapter 11. Afterwards, we will discuss the basic notions of the representation theory of a Kac–Moody algebra, which will be necessary in Chapter 13. There, we will mainly work with the derived algebra $\mathfrak{g}'(A)$ of the Kac–Moody algebra $\mathfrak{g}(A)$ of a generalized Cartan matrix of type A. Therefore, we will explain how the representation theory of $\mathfrak{g}'(A)$ is related to that of $\mathfrak{g}(A)$. This is done in the last section.

All the results stated here are well known and can mostly be found in [53].

10.1 Basic definitions

If not otherwise stated, in the following, K will denote a field of characteristic 0. Let I be a finite set, say $I = \{0, \ldots, n\}$, for some $n \in \mathbb{N}$, $n > 0$.

A *generalized Cartan matrix* is given by a matrix $A \in \mathrm{Mat}(n \times n, \mathbb{Z})$, $n \geq 1$, such that

(1) $a_{ii} = 2$, for all $0 \leq i \leq n$,

(2) $a_{ij} = 0$ if and only if $a_{ji} = 0$, for all $0 \leq i, j \leq n$,

(3) $a_{ij} \leq 0$, for all $0 \leq i, j \leq n$ and $i \neq j$.

Example 10.1.1. Let $n \geq 1$. Then the generalized Cartan matrix of type $A_n^{(1)}$ is the matrix $(a_{ij})_{0 \leq i,j \leq n}$, of the form

$$\begin{pmatrix} 2 & -1 & 0 & \cdots & 0 & 0 & -1 \\ -1 & 2 & -1 & \cdots & 0 & 0 & 0 \\ 0 & -1 & 2 & \cdots & 0 & 0 & 0 \\ & & & \ddots & & & \\ 0 & 0 & 0 & \cdots & 2 & -1 & 0 \\ 0 & 0 & 0 & \cdots & -1 & 2 & -1 \\ -1 & 0 & 0 & \cdots & 0 & -1 & 2 \end{pmatrix},$$

if $n > 1$, and

$$\begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix},$$

if $n = 1$.

Recall that an $n \times n$ matrix A is called symmetrizable if there exists an invertible diagonal $n \times n$ matrix D and a symmetric $n \times n$ matrix B such that $A = DB$.

To a generalized Cartan matrix A one associates a Lie algebra as follows: Let l denote the rank of the matrix A, and let $\mathfrak{h}$ be a K-vector space with $\dim_K \mathfrak{h} = 2n - l$. Moreover, choose linearly independent subsets $\Pi = \{\alpha_0, \dots, \alpha_n\} \subseteq \mathfrak{h}^*$, and $\Pi^* = \{h_0, \dots, h_n\} \subseteq \mathfrak{h}$, such that $\alpha_j(h_i) = a_{ji}$. The 3-tuple $(\mathfrak{h}, \Pi, \Pi^*)$ is called a *realization* of the matrix A. Then, the *Kac–Moody algebra* $\mathfrak{g}(A)$ associated to A is defined to be the Lie algebra on generators e_i, f_i, $i \in I$, and $h \in \mathfrak{h}$ subject to the following relations:

(L1) $[e_i, f_j] = \delta_{ij} h_i$,

(L2) $[h, e_i] = \alpha_i(h) e_i$,

(L3) $[h, f_i] = -\alpha_i(h) f_i$,

(L4) $[h, h'] = 0$, for all $h, h' \in \mathfrak{h}$,

(L5) $(\mathrm{ad}\ e_i)^{1-a_{ij}} e_j = 0$, for $i \neq j$,

(L6) $(\mathrm{ad}\ f_i)^{1-a_{ij}} f_j = 0$, for $i \neq j$.

Moreover, we will also work with the derived Lie algebra $\mathfrak{g}'(A) = [\mathfrak{g}(A), \mathfrak{g}(A)]$ of $\mathfrak{g}(A)$. Then, by [53, §9.11], $\mathfrak{g}'(A)$ is generated by the elements e_i, f_i, h_i, $i \in I$, together with the *Chevalley relations*

(D1) $[e_i, f_j] = \delta_{ij} h_i$,

(D2) $[h_j, e_i] = \alpha_i(h_j) e_i$,

(D3) $[h_j, f_i] = -\alpha_i(h_j) f_i$,

(D4) $[h_i, h_j] = 0$, for all $i, j \in I$,

(D5) $(\mathrm{ad}\ e_i)^{1-a_{ij}} e_j = 0$, for $i \neq j$,

(D6) $(\mathrm{ad}\ f_i)^{1-a_{ij}} f_j = 0$, for $i \neq j$.

The relations (L1)–(L4) (resp. (D1)–(D4)) are called the *Weyl relations* and the relations (L5)–(L6) (resp. (D5)–(D6)) are the *Serre relations*. Note that, by [53, §1.3], we have $\mathfrak{g}(A) = \mathfrak{g}'(A) + \mathfrak{h}$. If we set $\mathfrak{h}' = \sum_{i=0}^{n} K h_i$, then $\mathfrak{g}'(A) \cap \mathfrak{h} = \mathfrak{h}'$.

Remark 10.1.2. The elements of the set $\Pi = \{\alpha_0, \ldots, \alpha_n\}$ are the *simple roots* and the elements of the set $\Pi^* = \{h_0, \ldots, h_n\}$ are the *simple coroots* of $\mathfrak{g}(A)$. One may define a bilinear form

$$\bigoplus_{i \in I} \mathbb{Z}\alpha_i \times \bigoplus_{i \in I} \mathbb{Z}h_i \longrightarrow \mathbb{Z}$$

by setting $\langle \alpha_i, h_j \rangle = \alpha_i(h_j) = a_{ij}$, for $0 \leq i, j \leq n$.

Let L be a Lie algebra. Suppose that U is an associative unitary K-algebra and $\iota : L \to U$ is a K-linear map with

$$\iota([x, y]) = \iota(x)\iota(y) - \iota(y)\iota(x),$$

for all $x, y \in L$. The pair (U, ι) is called a *universal enveloping algebra* if it has the following universal property: For any associative unitary K-algebra A, and any linear map $j : L \to A$ such that

$$j([x, y]) = j(x)j(y) - j(y)j(x),$$

for all $x, y \in L$, there exists a unique homomorphism of associative algebras $\varphi : U \to A$ such that $j = \varphi \circ \iota$. Hence, if U exists, then it is unique up to isomorphism of associative algebras, and is called the universal enveloping algebra of L denoted by $U(L)$.
For the existence, consider the tensor algebra $\mathcal{T}(L) := \bigoplus_{k=0}^{\infty} L^{\otimes k}$, together with the factor algebra

$$\mathcal{T}(L)/\mathcal{I}, \tag{10.1}$$

where $\mathcal{I}$ is the two-sided ideal of $\mathcal{T}(L)$ generated by the elements of the form $x \otimes y - y \otimes x - [x, y]$. If $\iota : L \to \mathcal{T}(L)/\mathcal{I}$ denotes the natural map, one can show that the pair $(\mathcal{T}(L)/\mathcal{I}, \iota)$ satisfies the universal property above.

Theorem 10.1.3. *(Poincaré–Birkhoff–Witt) Let L be a Lie algebra, and $(U(L), \iota)$ its universal enveloping algebra. Then:*

(i) *The map $\iota : L \to U(L)$ is injective.*

(ii) *Let* $\{x_\alpha \mid \alpha \in \Omega\}$ *be a basis of* L, *where* Ω *is a partially ordered set. Then, all the elements of the form* $x_{\alpha_1} \cdots x_{\alpha_n}$ *such that* $\alpha_1 \leq \ldots \leq \alpha_n$ *together with* 1 *form a basis of* $U(L)$.

Proof. See [47, Theorem 1.2.4]. □

Next, let V be an L-module for the Lie algebra L. From the corresponding representation over K, we obtain, by the universal property of $U(L)$, an $U(L)$-module structure on V. Conversely, via the map ι of Theorem 10.1.3, we can give a $U(L)$-module W an L-module structure. This viewpoint will become important later on.

In the sequel denote by $U(\mathfrak{g}(A))$ the universal enveloping algebra associated to the Lie algebra $\mathfrak{g}(A)$. By the construction of the universal enveloping algebra given in (10.1), $U(\mathfrak{g}(A))$ has a presentation given by generators e_i, f_i, $i \in I$, and $h \in \mathfrak{h}$, subject to the following relations:

(U1) $e_i f_j - f_j e_i = \delta_{ij} h_i$,

(U2) $he_i - e_i h = \alpha_i(h) e_i$,

(U3) $hf_i - f_i h = -\alpha_i(h) f_i$,

(U4) $hh' = h'h$, for all $h, h' \in \mathfrak{h}$,

(U5) $\sum_{k=0}^{1-a_{ij}} (-1)^k \binom{1-a_{ij}}{k} e_i^{1-a_{ij}-k} e_j e_i^k = 0$, for $i \neq j$,

(U6) $\sum_{k=0}^{1-a_{ij}} (-1)^k \binom{1-a_{ij}}{k} f_i^{1-a_{ij}-k} f_j f_i^k = 0$, for $i \neq j$.

Furthermore, we denote by $U(\mathfrak{g}'(A))$ the universal enveloping algebra of the Lie algebra $\mathfrak{g}'(A)$. It is generated by elements e_i, f_i, h_i, $i \in I$, subject to the relations

(UD1) $e_i f_j - f_j e_i = \delta_{ij} h_i$,

(UD2) $h_j e_i - e_i h_j = \alpha_i(h_j) e_i$,

(UD3) $h_j f_i - f_i h_j = -\alpha_i(h_j) f_i$,

(UD4) $[h_i, h_j] = 0$, for all $i, j = 1, \ldots, n$,

(UD5) $\sum_{k=0}^{1-a_{ij}} (-1)^k \binom{1-a_{ij}}{k} e_i^{1-a_{ij}-k} e_j e_i^k = 0$, for $i \neq j$,

(UD6) $\sum_{k=0}^{1-a_{ij}} (-1)^k \binom{1-a_{ij}}{k} f_i^{1-a_{ij}-k} f_j f_i^k = 0$, for $i \neq j$.

To compare the Lie algebras $\mathfrak{g}(A)$ and $\mathfrak{g}'(A)$, we describe $\mathfrak{g}(A)$ in more detail.

Proposition 10.1.4. (i) *The center of* $\mathfrak{g}(A)$ *is given by*

$$Z(\mathfrak{g}(A)) = \{h \in \mathfrak{h} \mid \alpha_i(h) = 0 \textit{ for all } i \in I\}.$$

Therefore, $\dim_K Z(\mathfrak{g}(A)) = \dim_K \mathfrak{h} - |I| = \operatorname{corank}(A)$.

(ii) *Suppose that A is indecomposable. Then every ideal of $\mathfrak{g}(A)$ either contains $\mathfrak{g}'(A)$ or is contained in $Z(\mathfrak{g}(A))$.*

Proof. This is [53, Proposition 1.6 and Proposition 1.7 (b)]. □

From now on, we assume that the matrix A is of affine type, i.e., $\det(A) = 0$. Then it follows from Proposition 10.1.4 that the dimension of the center of $\mathfrak{g}(A)$ equals one, and is spanned by the vector

$$C := \sum_{i=0}^{n} \tilde{a}_i h_i,$$

where the $\tilde{a}_i$, $i \in I$, are the labels of the Dynkin diagram obtained from the Dynkin diagram associated to A by reversing the arrows and keeping the same enumeration of the vertices, see [53, §6.2]. For example, if A is of type $A_n^{(1)}$, then $\tilde{a}_i = 1$, for all i. Since we are considering the affine case, by [53, Theorem 5.6], the imaginary roots of $\mathfrak{g}(A)$ are given by the set

$$\{k\delta \mid k \in \mathbb{Z}\},$$

where $\delta = \sum_{i=0}^{n} a_i \alpha_i$, and the a_i, $i \in I$, are the labels of the Dynkin diagram associated to A. Observe that $\delta(h_i) = 0$, for all $i \in I$. Fix an element $d \in \mathfrak{h}$ such that $\alpha_i(d) = 0$, for all $i \in I$, $i \neq 0$, and $\alpha_0(d) = 1$. Then the elements $h_0, \ldots, h_n, d$ form a basis of $\mathfrak{h}$. Moreover, we have that

$$\mathfrak{g}(A) = \mathfrak{g}'(A) + Kd.$$

In $\mathfrak{h}^*$, we define elements Λ_i by $\Lambda_i(h_j) = \delta_{ij}$, and $\Lambda_i(d) = 0$, for $i, j \in I$. Note that the set $\{\Lambda_0, \ldots, \Lambda_n, \delta\}$ forms a basis of $\mathfrak{h}^*$ dual to the basis $\{h_0, \ldots, h_n, d\}$ of $\mathfrak{h}$. In particular, if A is of type $A_n^{(1)}$, then we have that

$$\alpha_i = \begin{cases} 2\Lambda_0 - \Lambda_n - \Lambda_1 + \delta & \text{if } i = 0, \\ 2\Lambda_i - \Lambda_{i-1} - \Lambda_{i+1} & \text{if } i > 0. \end{cases}$$

Note that, if we restrict a weight $\lambda \in \mathfrak{h}^*$ to $\mathfrak{g}'(A)$, we obtain a weight $\lambda_{|\mathfrak{h}'} \in (\mathfrak{h}')^*$ of $\mathfrak{g}'(A)$. In particular, $(\alpha_0)_{|\mathfrak{h}'} = (\Lambda_0)_{|\mathfrak{h}'} - (\Lambda_n)_{|\mathfrak{h}'} - (\Lambda_1)_{|\mathfrak{h}'}$ since $\delta_{|\mathfrak{h}'} = 0$. On the other hand, given a weight $\mu \in (\mathfrak{h}')^*$, we can extend it to a weight $\mu_u^{\mathfrak{h}}$ of $\mathfrak{g}(A)$, by setting $\mu_u^{\mathfrak{h}}(h_i) = \mu(h_i)$, for $i \in I$, $\mu_u^{\mathfrak{h}}(d) = u$, for $u \in K$, and extending linearly.

Remark 10.1.5. Note that the set $\{(\alpha_0)_{|\mathfrak{h}'}, \ldots, (\alpha_n)_{|\mathfrak{h}'}\}$ is in general not linearly independent. For example, in type $A_n^{(1)}$, we have that $\sum_{i=0}^{n} (\alpha_i)_{|\mathfrak{h}'} = 0$. This is one of the reasons why it is easier to work with the Lie algebra $\mathfrak{g}(A)$ instead of $\mathfrak{g}'(A)$, although the Lie algebra $\mathfrak{g}'(A)$ is the algebra originally investigated by V. Kac and R.V. Moody. It mimics the situation of a finite-dimensional Lie algebra, where the simple roots are linearly independent. On the other hand, the presentation by generators and relations of $\mathfrak{g}'(A)$ is more natural compared to that of $\mathfrak{g}(A)$, having in mind the definition of a finite-dimensional Lie algebra by the Chevalley generators.

For each $i \in I$, we define the *fundamental reflection* r_i of the space $\mathfrak{h}^*$ by setting

$$r_i(\lambda) = \lambda - \lambda(h_i)\alpha_i, \quad \lambda \in \mathfrak{h}^*.$$

The subgroup W of $\mathrm{GL}_{n+2}(\mathfrak{h}^*)$ generated by all fundamental reflections is called the *Weyl group* of $\mathfrak{g}(A)$.

10.2 Representation theory of Kac–Moody algebras

Keep the notation of the last section. In this section we will describe the fundamental notions of the representation theory of $\mathfrak{g}(A)$. From now on we assume that all the generalized Cartan matrices are symmetrizable.
For $n \geq 1$, let $I = \{0, \dots, n\}$. The free abelian group $Q := \bigoplus_{i \in I} \mathbb{Z}\alpha_i$ is called the *root lattice* of $\mathfrak{g}$, and $Q_+ := \sum_{i \in I} \mathbb{Z}_{\geq 0}\alpha_i$ is called the *positive root lattice*. We have a partial ordering on $\mathfrak{h}^*$ as follows: For $\lambda, \mu \in \mathfrak{h}^*$, we set $\lambda \geq \mu$ if $\lambda - \mu \in Q_+$.
Furthermore, let

$$P = \{\Lambda \in \mathfrak{h}^* \mid \Lambda(h_i) \in \mathbb{Z} \text{ for all } i \in I\},$$

and denote by P_+ the subset

$$\{\Lambda \in P \mid \Lambda(h_i) \geq 0 \text{ for all } i \in I\}$$

of P. The set P is called the weight lattice of $\mathfrak{g}(A)$, the elements from P (resp. P_+) are called *integral weights* (resp. *dominant integral weights*).
For $\alpha \in Q$, let

$$\mathfrak{g}_\alpha = \{x \in \mathfrak{g}(A) \mid [h, x] = \alpha(h)x, \text{ for all } h \in \mathfrak{h}\}.$$

By [53, Theorem 1.2], we have the following:

Proposition 10.2.1. (i) *The Lie algebra $\mathfrak{g}(A)$ admits the triangular decomposition*

$$\mathfrak{g}(A) \cong \mathfrak{g}_- \oplus \mathfrak{h} \oplus \mathfrak{g}_+,$$

where the subalgebra $\mathfrak{g}_+$ (resp. $\mathfrak{g}_-$) is the subalgebra of $\mathfrak{g}(A)$ generated by the elements e_i, ($i \in I$) (resp. f_i ($i \in I$)), with the defining relations $(\mathrm{ad}\ e_i)^{1-a_{ij}} e_j = 0$ *($i, j \in I$; $i \neq j$) (resp.* $(\mathrm{ad}\ f_i)^{1-a_{ij}} f_j = 0$, *($i, j \in I$; $i \neq j$)).*

(ii) *We have the root space decomposition*

$$\mathfrak{g}(A) \cong \bigoplus_{\alpha \in Q} \mathfrak{g}_\alpha,$$

with $\dim_K \mathfrak{g}_\alpha < \infty$, *for all $\alpha \in Q$.*

(iii) *There exists an involution* $w : \mathfrak{g}(A) \to \mathfrak{g}(A)$*, the Chevalley involution, defined by* $e_i \mapsto -f_i$*,* $f_i \mapsto -e_i$*, and* $h \mapsto -h$*,* $i \in I$*,* $h \in \mathfrak{h}$*.*

Remark 10.2.2. (i) Note that Proposition 10.2.1 is still true if we replace $\mathfrak{g}(A)$ by $\mathfrak{g}'(A)$, and $\mathfrak{h}$ by $\mathfrak{h}'$. Then we have a decomposition

$$\mathfrak{g}'(A) \cong \bigoplus_{\alpha \in Q} \mathfrak{g}'_\alpha,$$

where $\mathfrak{g}'_\alpha = \mathfrak{g}'(A) \cap \mathfrak{g}_\alpha$.

(ii) Sometimes it is also useful to consider the anti-involution $w^* : \mathfrak{g}(A) \to \mathfrak{g}(A)$ defined by $e_i \mapsto f_i$, $f_i \mapsto e_i$, and $h \mapsto h$, $i \in I$, $h \in \mathfrak{h}$. It is called the *Chevalley anti-involution.*

For the universal enveloping algebra $U := U(\mathfrak{g}(A))$ of $\mathfrak{g}(A)$, we define U^+ (resp. U^0, U^-) as the subalgebra of U generated by the elements e_i (resp. the elements of $\mathfrak{h}$, f_i), $i \in I$. Moreover, we define the *root spaces* to be

$$U_\beta = \{u \in U \mid [h,u] = hu - uh = \beta(h)u \text{ for all } h \in \mathfrak{h}\}, \ \beta \in \mathfrak{h}^*.$$

By Theorem 10.1.3 we get the following:

Proposition 10.2.3. *For the universal enveloping algebra* $U = U(\mathfrak{g}(A))$ *of* $\mathfrak{g}(A)$ *the following hold.*

(i) $U \cong U^- \otimes U^0 \otimes U^+$.

(ii) $U = \bigoplus_{\beta \in \mathfrak{h}^*} U_\beta$.

Remark 10.2.4. For the derived algebra $\mathfrak{g}'(A)$ of $\mathfrak{g}(A)$, we set for its universal enveloping algebra $U' = U(\mathfrak{g}'(A))$, $U'_\beta = U' \cap U_\beta$, $\beta \in Q$. Thus, from Proposition 10.2.3 we have that

$$U(\mathfrak{g}'(A)) = \bigoplus_{\beta \in Q} U'_\beta.$$

Moreover, if we denote by $(U')^+$ (resp. $(U')^0$, $(U')^-$) the subalgebra of $U(\mathfrak{g}'(A))$ generated by the elements e_i (resp. the elements of $\mathfrak{h}'$, f_i), $i \in I$, we get the triangular decomposition

$$U' \cong (U')^- \otimes (U')^0 \otimes (U')^+$$

of U'.

A $\mathfrak{g}(A)$-module V is called a *weight module* if it has a *weight space decomposition*

$$V \cong \bigoplus_{\mu \in \mathfrak{h}^*} V_\mu,$$

where $V_\mu = \{v \in V \mid hv = \mu(h)v \text{ for all } h \in \mathfrak{h}\}$. If $V_\mu \neq 0$, then μ is called a *weight* of V and V_μ is the *weight space* corresponding to μ. The dimension of V_μ as

a K-vector space is called the *weight multiplicity* of μ. We denote by $\mathrm{wt}(V)$ the set consisting of elements $\lambda \in \mathfrak{h}^*$, such that $V_\lambda \neq 0$.
An element $v \in V_\mu$ is called a *weight vector* of weight μ. If $e_i v = 0$ for all $i \in I$, then v is called a *maximal vector* of weight μ. An element $v \in V_\mu$ is called *primitive* if there exists a $\mathfrak{g}(A)$-submodule U of V such that $v \notin U$, and $\mathfrak{g}_+(v) \subseteq U$. In this case, μ is called a *primitive weight*.

The category $\mathcal{O}$ is defined in the following way. Its objects are the weight modules V over $\mathfrak{g}(A)$ with finite-dimensional weight spaces such that there exists a finite number of weights $\lambda_1, \ldots, \lambda_s \in \mathfrak{h}^*$, with

$$\mathrm{wt}(V) \subseteq D(\lambda_1) \cup \ldots \cup D(\lambda_s),$$

where $D(\lambda_j) = \{\mu \in \mathfrak{h}^* \mid \mu \leq \lambda_j\}$, for $j = 1, \ldots, s$. The morphisms are $\mathfrak{g}(A)$-module homomorphisms.

A weight module V is called a *highest-weight module* with *highest weight* $\Lambda \in \mathfrak{h}^*$ if there exists a non-zero vector $v_\Lambda \in V$, called a *highest-weight vector*, if the following is true:

(i) $e_i v_\Lambda = 0$, for all $i \in I$.

(ii) $h v_\Lambda = \Lambda(h) v_\Lambda$, for all $h \in \mathfrak{h}$.

(iii) $V = U(\mathfrak{g}(A)) v_\Lambda$.

It follows from condition (i) and (iii) that $V = U^- v_\Lambda$. Since, for $i \in I$, and $v \in V_\mu$, $\mu \in \mathfrak{h}^*$, we have that $f_i v \in V_{\mu - \alpha_i}$. It follows that V has a weight space decomposition of the form

$$V \cong \bigoplus_{\Lambda \geq \mu} V_\mu, \tag{10.2}$$

and, thus, $V \in \mathcal{O}$.

Let $\Lambda \in \mathfrak{h}^*$. Then a $\mathfrak{g}(A)$-module $M(\Lambda)$ with highest weight Λ is called a *Verma module* if every $\mathfrak{g}(A)$-module with highest weight Λ is a quotient of $M(\Lambda)$.
Define $J(\Lambda)$ to be the left ideal of $U(\mathfrak{g}(A))$ generated by the elements e_i, $i \in I$, and $h - \Lambda(h)1$, $h \in \mathfrak{h}$. By [53, Proposition 9.2], for every $\Lambda \in \mathfrak{h}^*$ there exists a Verma module, which is unique up to isomorphism. Moreover, there is an isomorphism of $\mathfrak{g}(A)$-modules

$$M(\Lambda) \cong U(\mathfrak{g}(A))/J(\Lambda).$$

Furthermore, $M(\Lambda)$ has a unique maximal submodule $N(\Lambda)$. Therefore, the $\mathfrak{g}(A)$-module $L(\Lambda) := M(\Lambda)/N(\Lambda)$ is irreducible, and is called the *irreducible highest-weight module* of highest weight Λ.

There is the following crucial fact:

Theorem 10.2.5. *Every irreducible $\mathfrak{g}(A)$-module in the category $\mathcal{O}$ is isomorphic to $L(\Lambda)$, for some $\Lambda \in \mathfrak{h}^*$.*

Proof. This is [53, Proposition 9.3]. □

A weight module V is called *integrable* if e_i and f_i, $i \in I$, are locally nilpotent on V, i.e., for all $v \in V$ there exists a positive integer k such that $e_i^k v = 0$ and $f_i^k v = 0$, for all $i \in I$.
The full subcategory of the category $\mathcal{O}$ consisting of integrable $\mathfrak{g}(A)$-modules is denoted by $\mathcal{O}_{\text{int}}$.

Proposition 10.2.6. *Let $L(\Lambda)$ be the irreducible highest-weight $\mathfrak{g}(A)$-module of highest weight $\Lambda \in \mathfrak{h}^*$. Then the following hold:*

(i) $L(\Lambda) \in \mathcal{O}_{\text{int}}$ *if and only if* $\Lambda \in P_+$.

(ii) *Let v_Λ be a highest-weight vector of $L(\Lambda)$. Then for all $i \in I$, we have that* $f_i^{\Lambda(h_i)+1} v_\Lambda = 0$.

Proof. See [53, Lemma 10.1]. □

We also note the following analogue of 10.2.5 in $\mathcal{O}_{\text{int}}$:

Proposition 10.2.7. *Every irreducible highest-weight $\mathfrak{g}(A)$-module in the category $\mathcal{O}_{\text{int}}$ is isomorphic to some $L(\Lambda)$, with $\Lambda \in P_+$.*

Proof. This can be found in [53, Theorem 10.7]. □

Remark 10.2.8. Note that in [53] the field K is the field of complex numbers. However, all the results stated here are also true over arbitrary fields of characteristic zero, see [47].

10.3 Relating the representation theory of $\mathfrak{g}'(A)$ to that of $\mathfrak{g}(A)$

For the derived algebra $\mathfrak{g}'(A) = [\mathfrak{g}(A), \mathfrak{g}(A)]$, we may define weight modules and integrable modules in the same fashion as for $\mathfrak{g}(A)$, see [53, §9.10]: A $\mathfrak{g}'(A)$-module V is called a *highest-weight module* with *highest weight* $\Lambda \in (\mathfrak{h}')^*$ if it admits a Q_+-grading $V = \bigoplus_{\alpha \in Q_+} V_{\Lambda-\alpha}$ such that the following conditions are satisfied:

(i) $\mathfrak{g}'_\beta(V_{\Lambda-\alpha}) \subseteq V_{\Lambda-\alpha+\beta}$, for all $\beta \in Q$.

(ii) $\dim_K V_\Lambda = 1$.

(iii) $hv = \Lambda(h)v$, for $h \in \mathfrak{h}'$, $v \in V_\Lambda$.

(iv) $V = U(\mathfrak{g}'(A))V_\Lambda$.

In the same way as in the preceding section, for $\Lambda \in (\mathfrak{h}')^*$, we define the Verma module $M(\Lambda)$ over $\mathfrak{g}'(A)$ as the $\mathfrak{g}'(A)$-module with highest weight Λ such that every highest-weight $\mathfrak{g}'(A)$-module with highest weight Λ is a quotient of $M(\Lambda)$. Then $M(\Lambda)$ has a unique maximal graded $\mathfrak{g}'(A)$-submodule, and we set $L(\Lambda) = M(\Lambda)/N(\Lambda)$. We have the following:

Proposition 10.3.1. *The $\mathfrak{g}'(A)$-module $L(\Lambda)$ is irreducible.*

Proof. See [53, Lemma 9.10]. □

Note that we can consider a highest-weight $\mathfrak{g}'(A)$-module V with highest weight $\Lambda \in (\mathfrak{h}')^*$ as a restriction of a highest-weight module over $\mathfrak{g}(A)$. For the sake of simplicity, in the following we assume that $\operatorname{corank}(A) = 1$.

Let $V = \bigoplus_{\alpha \in Q_+} V_{\Lambda-\alpha}$ be a Q_+-grading of V as above. Since $\{h_0, \ldots, h_n, d\}$ is a basis of $\mathfrak{h}$, we can define a weight $\tilde{\Lambda}_a \in \mathfrak{h}^*$, by setting $\tilde{\Lambda}_a(h_i) = \Lambda(h_i)$, for all $i \in I$, and $\tilde{\Lambda}_a(d) = a$, for some $a \in K$, and extending linearly. Clearly we have that $(\tilde{\Lambda}_a)_{|\mathfrak{h}'} = \Lambda$. Also, we define a $\mathfrak{g}(A)$-module $\tilde{V}_a$, by setting $\tilde{V}_a = V$ as a K-vector space, letting e_i, f_i, h_i act on $\tilde{V}_a$ as on V, for all $i \in I$. For $\alpha \in Q_+$ and $x \in V_{\Lambda-\alpha}$, we define $dx = (\tilde{\Lambda}_a - \alpha)(d)x$. It is easy to verify that this defines a $\mathfrak{g}(A)$-module structure on V. For example, if $x \in V_{\Lambda-\alpha}$, then

$$\begin{aligned}(de_i)x &= (e_i d + \alpha_i(d)e_i)x = (\tilde{\Lambda}_a - \alpha)(d)(e_i x) + \alpha_i(d)(e_i x)\\ &= (\tilde{\Lambda}_a - \alpha + \alpha_i)(d)(e_i x) = d(e_i x)\end{aligned}$$

since $e_i x \in V_{\Lambda-\alpha+\alpha_i}$.

Then we have that $(\tilde{V}_a)_{\tilde{\Lambda}_a-\alpha} = V_{\Lambda-\alpha}$, and hence $\tilde{V}_a = \bigoplus_{\alpha \in Q_+} (\tilde{V}_a)_{\tilde{\Lambda}_a-\alpha}$. From condition (i), we infer that $e_i V_\Lambda = 0$, for all $i \in I$. Together with (iii) and (iv), we conclude that $\tilde{V}_a$ is a highest-weight $\mathfrak{g}(A)$-module with highest weight $\tilde{\Lambda}_a$. Moreover, it is easy to see that the restriction of $\tilde{V}_a$ to $\mathfrak{g}'(A)$ equals V.

If $\Lambda \in \mathfrak{h}^*$, we may describe it by the labels $\Lambda(h_i)$, $i \in I$. Then, if $\Lambda, \Phi \in \mathfrak{h}^*$ have the same labels, they differ only off $\mathfrak{h}'$. In this case, if $L(\Lambda)$ and $L(\Phi)$ are irreducible $\mathfrak{g}(A)$-modules, then on restriction to $\mathfrak{g}'(A)$, they are isomorphic as irreducible $\mathfrak{g}'(A)$-modules.

Example 10.3.2. Let Λ_i, $i \in I$, denote the fundamental weights in $(\mathfrak{h}')^*$. Then we consider $\widetilde{(\Lambda_i)}_0 \in \mathfrak{h}^*$. From our definitions of $\widetilde{(\Lambda_i)}_0$ we see that $\widetilde{(\Lambda_i)}_0 = \Lambda_i$, a fundamental weight in $\mathfrak{h}^*$. If $L(\Lambda_i)$ denotes the irreducible highest-weight module of highest weight $\Lambda_i \in \mathfrak{h}^*$ over $\mathfrak{g}(A)$, then its restriction to $\mathfrak{g}'(A)$ equals the irreducible highest-weight module of highest weight Λ_i over $\mathfrak{g}'(A)$.

The next proposition gives a characterization of the module $L(\Lambda)$, $\Lambda \in P_+$.

Proposition 10.3.3. *Let $\Lambda \in (\mathfrak{h}')^*$ be such that $\Lambda(h_i) \in \mathbb{Z}_+$, for all $i \in I$. Then the $\mathfrak{g}'(A)$-module $L(\Lambda)$ is characterized by the property that it is irreducible and that there exists a non-zero vector $v \in L(\lambda)$ with $h_i v = \Lambda(h_i)v$ and $e_i v = 0$, for all $i \in I$.*

Proof. This is [53, Proposition 10.4]. □

As for $\mathfrak{g}(A)$, we define the category $\mathcal{O}$ for $\mathfrak{g}'(A)$-modules. We have the following:

Proposition 10.3.4. *Let V be a non-zero module from the category $\mathcal{O}$. Then:*

(i) *V contains a non-zero weight vector such that $\mathfrak{g}_+(v) = 0$.*

(ii) *The following conditions are equivalent:*

  (a) *V is irreducible.*

  (b) *V is a highest-weight module and any primitive vector of V is a highest-weight vector.*

  (c) *$V \cong L(\Lambda)$ for some $\Lambda \in \mathfrak{h}'$.*

(iii) *V is generated by its primitive vectors as a $\mathfrak{g}'(A)$-module.*

Proof. The proof of [53, Proposition 9.3] for $\mathfrak{g}(A)$-modules carries over. □

Therefore, Proposition 10.3.4 gives a bijection between isomorphism classes of irreducible modules in $\mathcal{O}$ and elements $\Lambda \in (\mathfrak{h}')^*$.

By $\mathcal{O}_{\mathrm{int}}$ we denote the category of all integrable $\mathfrak{g}'(A)$-modules V with the property that for every $v \in V$ there is some $m \in \mathbb{N}$ such that $e_{i_1} \cdots e_{i_s} v = 0$, for $i_j \in I$, whenever $s \geq m$. Of course, the category $\mathcal{O}_{\mathrm{int}}$ is a full subcategory of the category $\mathcal{O}$.

Remark 10.3.5. Note that if V is a highest-weight module of highest weight Λ, such that $\Lambda(h_i) \geq 0$, for all $i \in I$, then it follows from (10.2) that V has the latter property.

The next theorem states that every module in the category $\mathcal{O}_{\mathrm{int}}$ is semisimple.

Theorem 10.3.6. *Every $\mathfrak{g}'(A)$-module in $\mathcal{O}_{\mathrm{int}}$ decomposes into a direct sum of irreducible $\mathfrak{g}'(A)$-modules $L(\Lambda)$ such that $\Lambda(h_i) \in \mathbb{Z}_+$, for all $i \in I$.*

Proof. See [53, Theorem 10.7]. □

Next, let $L(\Lambda)$ denote an irreducible $\mathfrak{g}'(A)$-module in $\mathcal{O}_{\mathrm{int}}$, and let

$$L(\Lambda)^* = \prod_{\lambda \leq \Lambda} (L(\Lambda)_\lambda)^*$$

be the $\mathfrak{g}'(A)$-module contragredient to $L(\Lambda)$. The subspace $L^*(\Lambda) := \bigoplus_{\lambda \leq \Lambda} (L(\Lambda)_\lambda)^*$ of the latter is then a $\mathfrak{g}'(A)$-submodule of $L(\Lambda)^*$. The module $L^*(\Lambda)$ is again irreducible, and for $v \in (L(\Lambda)_\Lambda)^*$, we have that

(1) $\mathfrak{g}_-(v) = 0$,

(2) $hv = -\Lambda(h)v$, for $h \in \mathfrak{h}'$.

We shall refer to such a module as an *irreducible module with lowest weight* $-\Lambda$. Then, Proposition 10.3.4 gives a bijection between the elements of $(\mathfrak{h}')^*$ and the irreducible lowest weight modules of $\mathfrak{g}'(A)$.
Recall the definition of the Chevalley involution w and the Chevalley anti-involution w^* of $\mathfrak{g}'(A)$ from Proposition 10.2.1 and Remark 10.2.2. Denote by π_Λ the action of $\mathfrak{g}'(A)$ on $L(\Lambda)$. We introduce a new action π_Λ^* on the vector space $L(\Lambda)$, by defining

$$\pi_\Lambda^*(u)v = \pi_\Lambda(w(u))v,$$

for $u \in \mathfrak{g}'(A)$ and $v \in L(\Lambda)$. This new action affords an irreducible $\mathfrak{g}'(A)$-module with lowest weight $-\Lambda$. By Proposition 10.3.4, this module must be isomorphic to $L(\Lambda)^*$. Under this identification, the duality pairing between $L(\Lambda)^*$ and $L(\Lambda)$ induces a non-degenerate bilinear form B on $L(\Lambda)$ with the property

$$B(ux, y) = -B(x, w(u)y), \quad u \in \mathfrak{g}'(A), \quad x, y \in L(\Lambda). \tag{10.3}$$

A bilinear form on $L(\Lambda)$ satisfying (10.3), is called a *contravariant form*. One has the following:

Proposition 10.3.7. *On every irreducible highest-weight $\mathfrak{g}'(A)$-module $L(\Lambda)$ one may define a non-degenerate contravariant bilinear form that is unique up to a constant factor. It is symmetric and $L(\Lambda)$ decomposes into an orthogonal direct sum of weight spaces with respect to this form.*

Proof. This is [53, Proposition 9.4]. □

One can construct such a contravariant form explicitly. Let V be a highest-weight $\mathfrak{g}'(A)$-module with highest-weight vector v_Λ. For $v \in V$, we define an element $\langle v\rangle \in K$ via the expression

$$v = \langle v\rangle v_\Lambda + \sum_{\alpha\in Q_+\setminus\{0\}} v_{\Lambda-\alpha},$$

where $v_{\Lambda-\alpha} \in V_{\Lambda-\alpha}$. Using the triangular decomposition of $\mathfrak{g}'(A)$, one sees that

$$\langle w^*(a)v_\Lambda\rangle = \langle av_\Lambda\rangle,$$

for all $a \in \mathfrak{g}'(A)$. Hence, for $a, a' \in \mathfrak{g}'(A)$, we get that

$$\begin{aligned} \langle w^*(a)a'v_\Lambda\rangle &= \langle w^*(a)w^*(w^*(a'))v_\Lambda\rangle \\ &= \langle w^*(w^*(a')a)v_\Lambda\rangle \\ &= \langle w^*(a')av_\Lambda\rangle. \end{aligned}$$

Therefore, if we set

$$B(av_\Lambda, a'v_\Lambda) = \langle w^*(a)a'v_\Lambda\rangle, \tag{10.4}$$

we obtain a well-defined symmetric bilinear form on V, which is contravariant and normalized, that is to say,

$$B(v_\Lambda, v_\Lambda) = 1.$$

Chapter 11

Cyclotomic quotients, cyclotomic functors and their adjoints

In this chapter we will discuss important factor algebras of $H_n(q)$, called *cyclotomic* Hecke algebras, and define the cyclotomic analogues of the functors defined in Chapter 9. Furthermore, we will define their adjoints as well, and give crucial properties of these, which will become important in later sections. The definitions of the functors are based on the ideas of I. Grojnowski [44].

11.1 Cyclotomic quotients

In the following we use the notation given in Chapter 10. Let $l > 1$ be the order of the parameter $q \in F^\times$, where we assume that l is finite. Denote by $\mathfrak{g}(A)$ the affine Kac–Moody algebra associated to the generalized Cartan matrix A of type $A_{l-1}^{(1)}$ over $\mathbb{Q}$. We label the Dynkin diagram of $A_{l-1}^{(1)}$ by the index set $I := \mathbb{Z}/l\mathbb{Z} = \{0, \ldots, l-1\}$. Let $\{\alpha_i \mid i \in I\}$ be the set of simple roots, and $\{h_i \mid i \in I\}$ the set of simple coroots of $\mathfrak{g}$. Also, recall from Remark 10.1.2 the definition of the bilinear form

$$\langle ., . \rangle : \bigoplus_{i \in I} \mathbb{Z}\alpha_i \times \bigoplus_{i \in I} \mathbb{Z}h_i \longrightarrow \mathbb{Z},$$

given by $\langle \alpha_i, h_j \rangle = a_{ij}$, for $0 \leq i, j \leq l-1$. Moreover, recall from Chapter 10 the definition of the root lattice $Q = \bigoplus_{i \in I} \mathbb{Z}\alpha_i$, the positive root lattice $Q_+ = \bigoplus_{i \in I} \mathbb{Z}_{\geq 0}\alpha_i$, the weight lattice $P = \{h \in \mathfrak{h}^* \mid \langle h, h_i \rangle \in \mathbb{Z}, i \in I\}$, the fundamental weights $\{\Lambda_i \mid i \in I\} \subseteq P$, and the set of dominant integral weights $P_+ = \{h \in P \mid \langle h, h_i \rangle \geq 0, i \in I\} \subseteq P$. Note that if $h \in P_+$, then we may also identify h with a function $\lambda_h : I \to \mathbb{Z}_{\geq 0}$. As in Chapter 10, we set

$$c = \sum_{i=0}^{l-1} h_i, \qquad \delta = \sum_{i=0}^{l-1} \alpha_i.$$

The element δ is the basic imaginary root of $\mathfrak{g}(A)$. Then $\Lambda_0, \ldots, \Lambda_{l-1}, \delta$ form a $\mathbb{Z}$-basis of P, and $\langle \alpha_i, c \rangle = \langle \delta, h_i \rangle = 0$. Observe that $\alpha_i = \sum_{j=0}^{l-1} a_{ji}\Lambda_j$. This equals

$2\Lambda_i - \Lambda_{i-1} - \Lambda_{i+1}$ if $0 < i < l-1$, $2\Lambda_0 - \Lambda_1 - \Lambda_{l-1}$ if $i = 0$, and $2\Lambda_{l-1} - \Lambda_{l-2} - \Lambda_0$ if $i = l-1$.

Let $\Lambda \in P_+$, and denote by J_Λ the two-sided ideal in $H_n(q)$ generated by the element $\prod_{i\in I}(x_1 - q^i)^{\langle \Lambda, h_i\rangle}$. Then the quotient

$$H_n^\Lambda(q) = H_n(q)/J_\Lambda \tag{11.1}$$

is an F-algebra, called the *cyclotomic Hecke algebra* corresponding to $\Lambda \in P_+$. If d is the degree of the monic polynomial $\prod_{i\in I}(x_1 - q^i)^{\langle \Lambda, h_i\rangle}$, then by [44, Proposition 4.5] we have that the images of the elements

$$x^\alpha T_w \in H_n(q),$$

where $w \in W(n)$, and $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n$ with $0 \le \alpha_i < d, 1 \le i \le n$, form a basis of $H_n^\Lambda(q)$ as an F-vector space. Note that, for $\Lambda = \Lambda_0$, this is just the finite-dimensional Hecke algebra $H_n^f(q)$. Henceforth, we will write T_i for the images of the elements $T_i \in H_n(q)$, $1 \le i \le n-1$, under this quotient. Also write T_0 for the image of x_1.

We can also follow the lead of Ariki and Koike, see [4], who defined $H_n^\Lambda(q)$ by generators and relations: As an F-algebra, $H_n^\Lambda(q)$ is generated by $T_0, T_1, \ldots, T_{n-1}$ together with the relations

(1) $\prod_{i\in I}(T_0 - q^i)^{\langle \Lambda, h_i\rangle} = 0$.

(2) $T_0T_1T_0T_1 = T_1T_0T_1T_0$.

(3) $T_i^2 = (q-1)T_i + q$.

(4) $T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1}$, and $T_iT_j = T_jT_i$ if $|i-j| > 1$.

Throughout, we denote by ev_Λ the algebra homomorphism corresponding to the quotient of (11.1). Then one can show the following important fact.

Proposition 11.1.1. *For all $2 \le i \le n$, the image of the element x_i under* ev_Λ *equals*

$$L_i := q^{1-i}T_{i-1}\cdots T_1T_0T_1\cdots T_{i-1}.$$

Proof. This can be proven by induction using the defining relation $T_ix_iT_i = qx_{i+1}$ of $H_n(q)$, for $1 \le i \le n-1$. ∎

The elements L_i, $1 \le i \le n$, are the cyclotomic analogues of the Jucys–Murphy elements in the case of a finite-dimensional Hecke algebra of type A, see [68, Chapter 3, §3]. With this we get the following:

Theorem 11.1.2. *Let $\Lambda \in P_+$, and write $r = \sum_{i\in I}\langle \Lambda, h_i\rangle$. The algebra $H_n^\Lambda(q)$ is free as an F-vector space with basis*

$$\{L_1^{a_1}\cdots L_n^{a_n}T_w \mid w \in W(n), 0 \le a_i < r, 1 \le i \le n\}. \tag{11.2}$$

In particular, $\dim_F H_n^\Lambda(q) = r^n n!$.

Proof. See [4, Theorem 3.10]. □

In the special case when $r = 1$, we obtain the well-known fact that the set

$$\{T_w \mid w \in W(n)\}$$

forms an F-basis of $H_n^f(q)$.

Next, we want to define some important subcategories of the category $H_n(q) - \text{mod}$. Let $H_n(q) - \text{mod}_q$ be the full subcategory of $H_n(q) - \text{mod}$ consisting of modules M such that x_1 has eigenvalue a power of q. Note that for an exact sequence

$$0 \longrightarrow M_1 \longrightarrow M_2 \longrightarrow M_3 \longrightarrow 0$$

in $H_n(q) - \text{mod}$ the following holds: If two modules occurring in the sequence are in $H_n(q) - \text{mod}_q$, then so is the third. Thus, $H_n(q) - \text{mod}_q$ is closed under submodules, quotients and extensions. Moreover, one has the following amazing property:

Proposition 11.1.3. *If $M \in H_n(q) - \text{mod}$, and if there is some $1 \leq i \leq n$ such that the only eigenvalues of x_i on M are powers of q, then for all $1 \leq j \leq n$, the only eigenvalues of x_j on M are powers of q.*

Proof. See [44, Lemma 4.7]. □

In the following, we will denote by

$$\text{ev}_\Lambda^* : H_n^\Lambda(q) - \text{mod} \longrightarrow H_n(q) - \text{mod}$$

the inflation along the surjective ring homomorphism ev_Λ. Since, by definition of $H_n^\Lambda(q)$, $\prod_{i \in I}(T_0 - q^i)^{\langle \Lambda, h_i \rangle} = 0$, we have that if $M \in H_n^\Lambda(q) - \text{mod}$, then $\text{ev}_\Lambda^*(M) \in H_n(q) - \text{mod}_q$.
Next, denote by $H_n(q) - \text{mod}_q^\Lambda$ the full subcategory of $H_n(q) - \text{mod}$ whose objects are the modules M such that the Jordan blocks of the operator x_1 corresponding to the eigenvalue q^i have size $\leq (\Lambda, \alpha_i)$, for all $i \in I$, and there are no other eigenvalues. In other words, $H_n(q) - \text{mod}_q^\Lambda$ consists of modules that are annihilated by the element $\prod_{i \in I}(x_1 - q^i)^{\langle \Lambda, h_i \rangle}$. We have the following:

Proposition 11.1.4. *For all $\Lambda \in P_+$, the functor ev_Λ^* induces an equivalence*

$$\text{ev}_\Lambda^* : H_n^\Lambda(q) - \text{mod} \longrightarrow H_n(q) - \text{mod}_q^\Lambda \tag{11.3}$$

of categories.

Proof. By [5, Lemma 1.2], we see that the elements L_i, $i \in I$, act with eigenvalues a power of q on each $M \in H_n^\Lambda(q) - \text{mod}$. Hence, via restriction, ev_Λ^* is a well-defined functor. On the other hand, let $M \in H_n(q) - \text{mod}_q^\Lambda$. We want to define an action of $H_n^\Lambda(q)$ on M. Let $h^\Lambda \in H_n^\Lambda(q)$. Then, for some representative $h \in \text{ev}_\Lambda^{-1}(h^\Lambda)$, we define $h^\Lambda m = hm$. If h' is another representative in $\text{ev}_\Lambda^{-1}(h^\Lambda)$, then $h - h' \in \text{Ker}(\text{ev}_\Lambda)$. Since $\prod_{i \in I}(x_1 - q^i)^{\langle \Lambda, h_i \rangle}$ annihilates M, we see that h and h' act in the same way on M. It is clear that this gives mutually inverse equivalences between $H_n^\Lambda(q) - \mathrm{mod}$ and $H_n(q) - \mathrm{mod}_q^\Lambda$. $\square$

If $N \in H_n(q) - \mathrm{mod}$, then $N/J_\Lambda N$ is an $H_n(q)$-module on which J_Λ acts trivially, i.e., $N/J_\Lambda N$ is an $H_n^\Lambda(q)$-module. If $M \in H_n^\Lambda(q) - \mathrm{mod}$, then we have isomorphisms of F-vector spaces

$$\mathrm{Hom}_{H_n^\Lambda(q)}(N/J_\Lambda N, M) \cong \mathrm{Hom}_{H_n(q)}(N/J_\Lambda N, \mathrm{ev}_\Lambda^*(M)) \cong \mathrm{Hom}_{H_n(q)}(N, \mathrm{ev}_\Lambda^*(M)),$$

where we have used the fact that, by assumption, J_Λ annihilates $\mathrm{ev}_\Lambda^*(M)$. Therefore, if we define the functor

$$\mathrm{pr}_\Lambda : H_n(q) - \mathrm{mod} \longrightarrow H_n^\Lambda(q) - \mathrm{mod} \tag{11.4}$$

by $\mathrm{pr}_\Lambda(N) = N/J_\Lambda N$, then the exact functor ev_Λ^* is right adjoint to pr_Λ. Similarly, if we define the functor

$$\widehat{\mathrm{pr}}_\Lambda : H_n(q) - \mathrm{mod} \longrightarrow H_n^\Lambda(q) - \mathrm{mod}, \tag{11.5}$$

via $\widehat{\mathrm{pr}}_\Lambda(N) = N^{J_\Lambda}$, where N^{J_Λ} denotes the largest submodule of N such that $J_\Lambda N = 0$, then ev_Λ^* is a left adjoint of $\widehat{\mathrm{pr}}_\Lambda$.

Remark 11.1.5. By Proposition 11.1.4, we know that

$$\mathrm{ev}_\Lambda^* : H_n^\Lambda(q) - \mathrm{mod} \longrightarrow H_n(q) - \mathrm{mod}_q^\Lambda$$

is an equivalence of categories. If we denote by

$${\mathrm{ev}_\Lambda^*}^{-1} : H_n(q) - \mathrm{mod}_q^\Lambda \longrightarrow H_n^\Lambda(q) - \mathrm{mod}$$

a quasi inverse of ev_Λ^*, then it is left adjoint to ev_Λ^*. On the other hand, on restriction to $H_n(q) - \mathrm{mod}_q^\Lambda$ we see that also pr_Λ is left adjoint to ev_Λ^*. By uniqueness of adjoints, it follows that both functors must be isomorphic.

11.2 Cyclotomic functors

Here, we want to define the cyclotomic analogues of the functors given in Section 9.8 as well as their adjoints. For $i \in I$, define the functors

$$\begin{aligned} e_i &: H_n(q) - \mathrm{mod}_q \longrightarrow H_{n-1}(q) - \mathrm{mod}_q, \\ e_i^\Lambda &: H_n^\Lambda(q) - \mathrm{mod} \longrightarrow H_{n-1}^\Lambda(q) - \mathrm{mod} \end{aligned} \tag{11.6}$$

as follows. If $M \in H_n(q) - \mathrm{mod}$, then $e_i(M)$ is the generalized eigenspace of x_n with respect to the eigenvalue q^i. Since x_n commutes with $H_{n-1}(q)$, considered as subalgebra of $H_n(q)$ in the natural way, $e_i(M)$ is easily seen to be an $H_{n-1}(q)$-module. In other words, this is just the restriction of the functor e_i from Section 9.8 to the category $H_n(q) - \mathrm{mod}_q$.

Since x_1 acts in the same way on $e_i(M)$ as on M, $e_i(\mathrm{ev}_\Lambda(M)) \in H_{n-1}(q) - \mathrm{mod}_q^\Lambda$ if $M \in H_n^\Lambda(q) - \mathrm{mod}$. Thus, for $M \in H_n^\Lambda(q) - \mathrm{mod}$, if we define $e_i^\Lambda(M) = \mathrm{pr}_\Lambda \circ e_i \circ \mathrm{ev}_\Lambda^*(M)$, we get a functor

$$e_i^\Lambda : H_n^\Lambda(q) - \mathrm{mod} \longrightarrow H_{n-1}^\Lambda(q) - \mathrm{mod}. \tag{11.7}$$

Remark 11.2.1. In other words, for $i \in I$, the functor e_i^Λ can be identified with the restriction of e_i from $H_n(q) - \mathrm{mod}$ to the subcategory $H_n(q) - \mathrm{mod}_q^\Lambda$.

The next lemma, which is taken form [44], says that these functors are exact.

Lemma 11.2.2. *The functors $e_i : H_n(q) - \mathrm{mod}_q \to H_{n-1}(q) - \mathrm{mod}_q$, and $e_i^\Lambda : H_n^\Lambda(q) - \mathrm{mod} \to H_{n-1}^\Lambda(q) - \mathrm{mod}$ are exact.*

Proof. See [44, Lemma 8.1]. □

Next, we consider the F-algebra $F[x_n^{\pm 1}]$. This is a principal ideal domain, where each element $x_n - a$, $a \in F^\times$, is a prime element. The quotient $F[x_n^{\pm 1}]/\langle x_n - a\rangle$ is isomorphic to F, hence an integral domain. For $m \geq 1$ and $a \in F^\times$, the F-algebras $\mathcal{R}_m(a) = F[x_n^{\pm 1}]/\langle (x_n - a)^m\rangle$ are all finite dimensional of dimension m. Moreover, for any $m \geq 2$ we denote by

$$p_m : \mathcal{R}_m(a) \longrightarrow \mathcal{R}_{m-1}(a)$$

the natural algebra epimorphism with kernel $\langle (x_n - a)^{m-1}\rangle/\langle (x_n - a)^m\rangle$, see Section 9.8. From these epimorphisms, we then get an inverse system

$$\ldots \longrightarrow \mathcal{R}_m(a) \longrightarrow \mathcal{R}_{m-1}(a) \longrightarrow \ldots \longrightarrow \mathcal{R}_1(a) \longrightarrow 0. \tag{11.8}$$

Remark 11.2.3. Note that as an F-vector space, $\mathcal{R}_m(a)$ has as basis the image of the set $\{1, x_n - a, \ldots, (x_n - a)^{m-1}\}$ under p_m. If we consider $\mathcal{R}_m(a)$ as an $F[x_n^{\pm 1}]$-module in the natural way, we see that, with respect to this basis, x_n acts on $\mathcal{R}_m(a)$ as the matrix

$$\begin{pmatrix} a & 1 & & & \\ & a & 1 & & \\ & & \ddots & & \\ & & & a & 1 \\ & & & & a & 1 \end{pmatrix}.$$

Therefore, as an $F[x_n^{\pm 1}]$-module, $\mathcal{R}_m(a)$ is isomorphic to the Jordan block $\mathcal{L}_m(a)$ of size m corresponding to the eigenvalue a.

Next we recall the definition of the functors f_i^Λ and $\hat{f}_i^\Lambda$ given in [44, §8].

Definition 11.2.4. Let $n \geq 1$. For $i \in I$, define the functor

$$f_i^\Lambda : H_{n-1}^\Lambda(q) - \mathrm{mod} \longrightarrow H_n^\Lambda(q) - \mathrm{Mod} \tag{11.9}$$

by setting $f_i^\Lambda(M) = \varprojlim_m \mathrm{pr}_\Lambda(\mathrm{Ind}_{H_{(n-1,1)}(q)}^{H_n(q)}(\mathrm{ev}_\Lambda^*(M) \boxtimes \mathcal{R}_m(q^i)))$, for $M \in H_{n-1}^\Lambda(q) -$ mod, where the inverse limit is taken over the inverse system in (11.8).

From the remark above, we see that, as F-spaces, $\mathcal{R}_j(a)$ can be naturally embedded into $\mathcal{R}_m(a)$, for all $m \geq 1$, and $1 \leq j \leq m$. If we denote by i_j the embedding $\mathcal{R}_j(a) \to \mathcal{R}_{j+1}(a)$, which then is even an $F[x_n^{\pm 1}]$-homomorphism, we obtain a directed system

$$\mathcal{R}_1(a) \xrightarrow{i_1} \mathcal{R}_2(a) \xrightarrow{i_2} \dots \xrightarrow{i_{m-1}} \mathcal{R}_m(a) \xrightarrow{i_m} \dots . \tag{11.10}$$

Therefore, for $i \in I$, we may also define a functor

$$\hat{f}_i^\Lambda : H_{n-1}^\Lambda(q) - \mathrm{mod} \longrightarrow H_n^\Lambda(q) - \mathrm{Mod} \tag{11.11}$$

given by $\hat{f}_i^\Lambda(M) = \varinjlim_m \widehat{\mathrm{pr}}_\Lambda(\widehat{\mathrm{Ind}}_{H_{(n-1,1)}(q)}^{H_n(q)}(\mathrm{ev}_\Lambda^*(M) \boxtimes \mathcal{R}_m(q^i)))$, $M \in H_{n-1}^\Lambda(q) - \mathrm{mod}$, with respect to the above direct system.
Up to now, we do not know whether f_i^Λ and $\hat{f}_i^\Lambda$ send finite-dimensional modules to finite-dimensional ones. The next proposition clarifies the situation.

Proposition 11.2.5. *Let $M \in H_{n-1}(q) - \mathrm{mod}$. Then:*

(i) *The inverse system $\mathrm{pr}_\Lambda(\mathrm{Ind}_{H_{(n-1,1)}(q)}^{H_n(q)}(M \boxtimes \mathcal{R}_m(q^i)))$ stabilizes after finitely many steps.*

(ii) *The direct system $\widehat{\mathrm{pr}}_\Lambda(\widehat{\mathrm{Ind}}_{H_{(n-1,1)}}^{H_n(q)}(M \boxtimes \mathcal{R}_m(q^i)))$ stabilizes after finitely many steps.*

In particular, for any $i \in I$, the functors f_i^Λ and $\hat{f}_i^\Lambda$ send finite-dimensional modules to finite-dimensional ones, hence, give well-defined functors $f_i^\Lambda, \hat{f}_i^\Lambda : H_{n-1}^\Lambda(q) -$ mod $\to H_n^\Lambda(q) -$ mod.

Proof. We show (i), the argumentation for (ii) being similar. To this end we show that the dimension of $\mathrm{pr}_\Lambda(\mathrm{Ind}_{H_{(n-1,1)}(q)}^{H_n(q)}(M \boxtimes \mathcal{R}_m(q^i)))$ as an F-vector space is bounded by a natural number, which is independent of m. To this end, let v be a vector which generates the Jordan block $\mathcal{R}_m(q^i)$ as an $F[x_n^{\pm 1}]$-module. Let $V = Fv$ be the one-dimensional subspace of $\mathcal{R}_m(q^i)$ spanned by v. It follows that $\mathrm{Ind}_{H_{(n-1,1)}(q)}^{H_n(q)}(M \boxtimes \mathcal{R}_m(q^i))$ is generated as an $H_n(q)$-module by the F-subspace $W = 1 \otimes (M \otimes V)$, and this has F-dimension independent of m. But $\mathrm{pr}_\Lambda(\mathrm{Ind}_{H_{(n-1,1)}(q)}^{H_n(q)}(M \boxtimes \mathcal{L}_m(q^i)))$ is a quotient of the vector space $H_n^\Lambda(q) \otimes_F W$, whose dimension is again independent of m. Thus, (i) follows. $\square$

The next theorem, which is [44, Proposition 8.4] and is a consequence of the previous proposition, is fundamental in our further considerations. By [44, §8, Remark 3], the functor e_i can be alternatively described as

$$e_i(M) = \varinjlim_m \mathrm{Hom}_{F[x_n^{\pm 1}]}(\mathcal{R}_m(q^i), M), \tag{11.12}$$

for $M \in H_n(q) - \mathrm{mod}_q$, where the direct limit is taken over the inverse system in (11.8). This is just another way of expressing the generalized eigenspace of the operator x_n on M corresponding to the eigenvalue q^i. Moreover, we need the following:

Lemma 11.2.6. *Let A and B be F-algebras, L an A-module, M a B-module, and N an $A \otimes B$-module. Then there is an isomorphism*

$$\mathrm{Hom}_{A\otimes B}(L \boxtimes M, N) \cong \mathrm{Hom}_A(L, \mathrm{Hom}_B(M, N))$$

of F-vector spaces, which is natural in L and N.

Proof. It is straightforward to check that the map

$$\psi : \mathrm{Hom}_{A\otimes B}(L \boxtimes M, N) \longrightarrow \mathrm{Hom}_A(L, \mathrm{Hom}_B(M, N)),$$

defined by $f \mapsto \tilde{f} : l \mapsto (m \mapsto f(l \boxtimes m))$ gives an isomorphism of F-vector spaces. $\square$

Theorem 11.2.7. *Let $i \in I$. The functor $f_i^\Lambda : H_{n-1}^\Lambda(q) - \mathrm{mod} \to H_n^\Lambda(q) - \mathrm{mod}$ is left adjoint to the functor $e_i^\Lambda : H_n^\Lambda(q) - \mathrm{mod} \to H_{n-1}^\Lambda(q) - \mathrm{mod}$. Similarly, $\hat{f}_i^\Lambda$ is right adjoint to e_i^Λ.*

Proof. Let $M \in H_n^\Lambda(q) - \mathrm{mod}$, and $N \in H_{n-1}^\Lambda(q) - \mathrm{mod}$. Then

$$\begin{aligned}
&\mathrm{Hom}_{H_{n-1}^\Lambda(q)}(N, e_i^\Lambda(M)) \\
&= \mathrm{Hom}_{H_{n-1}^\Lambda(q)}(N, \mathrm{pr}_\Lambda \circ e_i \circ \mathrm{ev}_\Lambda^*(M))) \\
&\cong \mathrm{Hom}_{H_{n-1}(q)}(\mathrm{ev}_\Lambda^*(N), e_i(\mathrm{ev}_\Lambda^*(M))) \\
&= \mathrm{Hom}_{H_{n-1}(q)}(\mathrm{ev}_\Lambda^*(N), \varinjlim_m \mathrm{Hom}_{F[x_n^{\pm 1}]}(\mathcal{R}_m(q^i), \mathrm{ev}_\Lambda^*(M))) \\
&\cong \varinjlim_m \mathrm{Hom}_{H_{n-1}(q)}(\mathrm{ev}_\Lambda^*(N), \mathrm{Hom}_{F[x_n^{\pm 1}]}(\mathcal{R}_m(q^i), \mathrm{ev}_\Lambda^*(M))) \\
&\cong \varinjlim_m \mathrm{Hom}_{H_{(n-1,1)}(q)}(\mathrm{ev}_\Lambda^*(N) \boxtimes \mathcal{R}_m(q^i), \mathrm{ev}_\Lambda^*(M)),
\end{aligned}$$

by Lemma 11.2.6, and since the first direct system stabilizes after finitely many terms. Also, since $\mathrm{Ind}_{H_{(n-1,1)}(q)}^{H_n(q)}$ is left adjoint to $\mathrm{Res}_{H_{(n-1,1)}(q)}^{H_n(q)}$, as well as pr_Λ is left adjoint to ev_Λ^*, we have:

$$\begin{aligned}
&\mathrm{Hom}_{H_{n-1}^\Lambda(q)}(N, e_i^\Lambda(M)) \\
&\cong \varinjlim_m \mathrm{Hom}_{H_{(n-1,1)}(q)}(\mathrm{ev}_\Lambda^*(N) \boxtimes \mathcal{R}_m(q^i), \mathrm{ev}_\Lambda^*(M)) \\
&\cong \varinjlim_m \mathrm{Hom}_{H_n(q)}(\mathrm{Ind}_{H_{(n-1,1)}(q)}^{H_n(q)}(\mathrm{ev}_\Lambda^*(N) \boxtimes \mathcal{R}_m(q^i)), \mathrm{ev}_\Lambda^*(M)) \\
&\cong \varinjlim_m \mathrm{Hom}_{H_n^\Lambda(q)}(\mathrm{pr}_\Lambda(\mathrm{Ind}_{H_{(n-1,1)}(q)}^{H_n(q)}(\mathrm{ev}_\Lambda(N) \boxtimes \mathcal{R}_m(q^i))), M) \\
&\cong \mathrm{Hom}_{H_n^\Lambda(q)}(\varprojlim_m \mathrm{pr}_\Lambda(\mathrm{Ind}_{H_{(n-1,1)}(q)}^{H_n(q)}(\mathrm{ev}_\Lambda(N) \boxtimes \mathcal{L}_m(q^i))), M) \\
&= \mathrm{Hom}_{H_n^\Lambda(q)}(f_i^\Lambda(N), M),
\end{aligned}$$

where we have used that, by Proposition 11.2.5, the inverse limit stabilizes after finitely many terms. Note that all the isomorphisms above are natural. It follows that f_i^Λ is left adjoint to e_i^Λ. The proof for $\hat{f}_i^\Lambda$ is similar. $\square$

Remark 11.2.8. From (11.12) we also obtain an alternative description of the functor e_i^Λ, namely:

$$e_i^\Lambda(M) = \varinjlim_m \mathrm{pr}_\Lambda \mathrm{Hom}_{F[x_n^{\pm 1}]}(\mathcal{R}_m(q^i), \mathrm{ev}_\Lambda^*(M)),$$

for $M \in H_n^\Lambda(q) - \mathrm{mod}$.

Recall from Lemma 9.6.1 that we have a decomposition

$$H_n(q) - \mathrm{mod} \cong \bigoplus_{\gamma \in S^n/\sim} H_n(q) - \mathrm{mod}[\gamma]$$

of categories. By Proposition 9.6.2 and the Krull–Schmidt Theorem, the full subcategory $H_n(q)-\mathrm{mod}[\gamma]$ can be characterized as the full subcategory of $H_n(q)-\mathrm{mod}$ consisting of all finite-dimensional $H_n(q)$-modules whose central character is γ. Moreover, these subcategories are precisely the blocks of $H_n(q)$. For $\Lambda \in P_+$, we infer from Theorem 9.5.1 and Proposition 11.1.1 that the image of the center of $H_n(q)$ under ev_Λ is contained in the center of $H_n^\Lambda(q)$. Therefore, we also may decompose $H_n^\Lambda(q) - \mathrm{mod}$ as

$$H_n^\Lambda(q) - \mathrm{mod} \cong \bigoplus_{\gamma \in S^n/\sim} H_n^\Lambda(q) - \mathrm{mod}[\gamma]. \tag{11.13}$$

The fact that the categories $H_n^\Lambda(q)-\mathrm{mod}[\gamma]$ are actually the blocks of $H_n^\Lambda(q)-\mathrm{mod}$ is non-trivial and a proof of this is given in [64].

In the following, let $\gamma := (\gamma_1, \ldots, \gamma_n) \in S^n/\sim$, see Section 9.6, and suppose that $q^i \in \{\gamma_1, \ldots, \gamma_n\}$. We may assume that $\gamma_n = q^i$. We will write $\gamma \setminus q^i$ to denote the unique $W(n-1)$-orbit obtained by deleting q^i from γ. Moreover, write $\gamma + q^i$ for the $W(n+1)$-orbit of $(\gamma_1, \ldots, \gamma_n, q^i)$ in $S^{n+1}/\sim$. We then obtain a refinement of the functors e_i^Λ and f_i^Λ, see also [44, Lemma 8.6].

Proposition 11.2.9. *Let $n \geq 1$, and let $i \in I$. The functors e_i and e_i^Λ descend to functors*

$$\begin{aligned} H_n(q) - \mathrm{mod}[\gamma] &\longrightarrow H_{n-1}(q) - \mathrm{mod}[\gamma \setminus q^i] \quad \textit{and} \\ H_n^\Lambda(q) - \mathrm{mod}[\gamma] &\longrightarrow H_{n-1}^\Lambda(q) - \mathrm{mod}[\gamma \setminus q^i], \end{aligned}$$

where for $M \in H_n(q)-\mathrm{mod}$ (resp. $M \in H_n^\Lambda(q)-\mathrm{mod}$), $e_i(M) = 0$ (resp. $e_i^\Lambda(M) = 0$), whenever $q^i \notin \{\gamma_1, \ldots, \gamma_n\}$. Similarly, the functors f_i^Λ and $\hat{f}_i^\Lambda$ restrict to functors

$$H_n^\Lambda(q) - \mathrm{mod}[\gamma] \to H_{n+1}^\Lambda(q) - \mathrm{mod}[\gamma + q^i].$$

Proof. Since on objects, $e_i(M)$ is just the generalized eigenspace of the operator x_n with respect to the eigenvalue q^i, we see that $e_i(M) = 0$ if $q^i \notin \gamma$.

For the functor f_i^Λ, we may assume that M is indecomposable. Then $M = M_{\underline{s}}$, for some $s \in S^n$. Thus, by Lemma 9.5.7, we see that the possible central characters for $\mathrm{Ind}_{H_{(n,1)}(q)}^{H_{n+1}(q)}(\mathrm{ev}_\Lambda(M) \boxtimes \mathcal{L}_m(q^i))$ are obtained by shuffling $\underline{s}$ and q^i. Thus $\mathrm{Ind}_{H_{(n,1)}(q)}^{H_{n+1}(q)}(\mathrm{ev}_\Lambda(M) \boxtimes \mathcal{L}_m(q^i)) \in H_{n+1}(q) - \mathrm{mod}[\gamma + q^i]$. But if a module N is in $H_n(q) - \mathrm{mod}[\gamma + q^i]$, then $\mathrm{pr}_\Lambda(N) \in H_{n+1}^\Lambda(q) - \mathrm{mod}[\gamma + q^i]$. Therefore, $f_i^\Lambda(M) \in H_{n+1}^\Lambda(q) - \mathrm{mod}[\gamma + q^i]$. The proof for $\hat{f}_i^\Lambda$ is similar. ☐

We also get a refinement of Theorem 11.2.7, see [44, Proposition 8.7].

Proposition 11.2.10. *Let $i \in I$. The functor*

$$f_i^\Lambda : H_{n-1}^\Lambda(q) - \mathrm{mod}[\gamma] \to H_n^\Lambda(q) - \mathrm{mod}[\gamma + q^i]$$

is left adjoint to the functor

$$e_i^\Lambda : H_n^\Lambda(q) - \mathrm{mod}[\gamma + q^i] \to H_{n-1}^\Lambda(q) - \mathrm{mod}[\gamma].$$

Similarly, $\hat{f}_i^\Lambda$ is a right adjoint of e_i^Λ.

Proof. This follows from Proposition 11.2.9 since, by Theorem 11.2.7, f_i^Λ is left adjoint to e_i^Λ on the whole of $H_{n-1}^\Lambda(q) - \mathrm{mod}$. ☐

We also have the following, which is [44, Lemma 8.8]:

Proposition 11.2.11. *Let $M \in H_n^\Lambda(q) - \mathrm{mod}$, then:*

(i) $\mathrm{Res}_{H_{n-1}^\Lambda(q)}^{H_n^\Lambda(q)}(M) \cong \bigoplus_{i\in I} e_i^\Lambda(M)$ *as $H_{n-1}^\Lambda(q)$-modules.*

(ii) $\mathrm{Ind}_{H_n^\Lambda(q)}^{H_{n+1}^\Lambda(q)}(M) \cong \bigoplus_{i\in I} f_i^\Lambda(M) \cong \bigoplus_{i\in I} \hat{f}_i^\Lambda(M)$ *as $H_{n+1}^\Lambda(q)$-modules.*

Proof. Part (i) follows from the theory developed in Section 9.6. Part (ii) follows from Propositions 11.2.9 and 11.2.10. ☐

From this we obtain the following corollary, see [44, Corollary 8.9]:

Corollary 11.2.12. *The following hold for the functors $f_i^\Lambda, \hat{f}_i^\Lambda : H_{n-1}^\Lambda(q) - \mathrm{mod} \to H_n(q) - \mathrm{mod}$:*

(i) *As functors, $f_i^\Lambda \cong \hat{f}_i^\Lambda$.*

(ii) *f_i^Λ and $\hat{f}_i^\Lambda$ are exact.*

Proof. The first part follows from Proposition 11.2.11 (ii) and Proposition 11.2.9. The second part follows from (i), because from the latter we infer that f_i^Λ is both a left and a right adjoint, and as such an exact functor. ☐

As a consequence of the preceding results, one gets:

Proposition 11.2.13. *If $M \in H_n^\Lambda(q) - \mathrm{mod}$ is projective, then so are $e_i^\Lambda(M)$ and $f_i^\Lambda(M)$.*

Proof. Since restriction and induction take free modules to free modules, the result follows from Proposition 11.2.11. □

Recall the definition of the functor $e_i^{(r)} : H_n(q) - \text{mod} \to H_{n-r}(q) - \text{mod}$, for $i \in I$, and $1 \le r \le n$, from Section 9.8. As $e_i^{(r)}$ takes finite-dimensional $H_n(q)$-modules to finite-dimensional $H_{n-r}(q)$-modules, we have a functor

$$e_i^{(r)} \circ \text{ev}_\Lambda : H_n^\Lambda(q) - \text{mod} \longrightarrow H_{n-r}(q) - \text{mod}.$$

Moreover, from the definition of $e_i^{(r)}$, we see that, for $M \in H_n^\Lambda(q)-\text{mod}$, the element $\prod_{i\in I}(x_1 - q^i)^{\langle\Lambda,h_i\rangle}$ must annihilate $e_i^{(r)}(M)$ since it annihilates M. This shows that we get a well-defined functor

$$(e_i^\Lambda)^{(r)} := \text{pr}_\Lambda \circ e_i^{(r)} \circ \text{ev}_\Lambda^* : H_n^\Lambda(q) - \text{mod} \longrightarrow H_{n-r}^\Lambda(q) - \text{mod}. \tag{11.14}$$

With Theorem 9.8.3, we obtain:

Theorem 11.2.14. *Let $i \in I$. There is an isomorphism of functors*

$$(e_i^\Lambda)^r \cong \bigoplus_{r!} (e_i^\Lambda)^{(r)}$$

from $H_n^\Lambda(q) - \text{mod}$ to $H_{n-r}^\Lambda(q) - \text{mod}$.

Proof. Recall from the beginning of this section that on restriction to the category $H_{n-r}(q) - \text{mod}_q^\Lambda$, the functor pr_Λ gives an inverse equivalence to the equivalence

$$\text{ev}_\Lambda^* : H_{n-r}^\Lambda(q) - \text{mod} \longrightarrow H_{n-r}(q) - \text{mod}_q^\Lambda.$$

Thus, if $M \in H_n^\Lambda(q) - \text{mod}$, we obtain the desired result by using Theorem 9.8.3. □

Recall the $H_r(q)'$ denotes the subalgebra of $H_{n+r}(q)$ generated by

$$x_{n+1}^{\pm 1}, \dots, x_{n+r}^{\pm 1}, T_{n+1}, \dots, T_{n+r-1}.$$

Furthermore, in Section 9.8 we considered the $H_r(q)'$-modules $\mathcal{L}_m((q^i)^r)$, for $m \ge 1$, and the inverse system associated with them. Then, for $r \ge 1$ and $M \in H_n^\Lambda(q) - \text{mod}$, we define a functor

$$(f_i^\Lambda)^{(r)} : H_n^\Lambda(q) - \text{mod} \longrightarrow H_{n+r}^\Lambda(q) - \text{mod}$$

by setting

$$(f_i^\Lambda)^{(r)}(M) = \varprojlim_m \text{pr}_\Lambda(\text{Ind}_{H_{(n,r)}(q)}^{H_{n+r}(q)}(\text{ev}_\Lambda(M) \boxtimes \mathcal{L}_m((q^i)^r))). \tag{11.15}$$

Observe that $\mathcal{L}_m((q^i)^r)$ is a cyclic $H_r(q)'$-module. Therefore, with the same argumentation as in Proposition 11.2.5, we see that the inverse limit above stabilizes after finitely many terms, and thus, $(f_i^\Lambda)^{(r)}(M)$ is indeed finite dimensional.

Theorem 11.2.15. *For all $i \in I$, and $r \geq 1$, the functor $(f_i^\Lambda)^{(r)}$ is left adjoint to the functor $(e_i^\Lambda)^{(r)}$.*

Proof. Let $M \in H_n^\Lambda(q) - \mathrm{mod}$, and $N \in H_{n+r}^\Lambda(q) - \mathrm{mod}$. Then

$$\begin{aligned}
&\mathrm{Hom}_{H_n^\Lambda(q)}(M, (e_i^\Lambda)^{(r)}(N)) \\
&\cong \mathrm{Hom}_{H_n(q)}(\mathrm{ev}_\Lambda^*(M), \varinjlim_m \mathrm{Hom}_{H_r(q)'}(\mathcal{L}_m((q^i)^r), \mathrm{ev}_\Lambda^*(N)) \\
&\cong \varinjlim_m \mathrm{Hom}_{H_n(q)}(\mathrm{ev}_\Lambda^*(M), \mathrm{Hom}_{H_r(q)'}(\mathcal{L}_m((q^i)^r), \mathrm{ev}_\Lambda^*(N)) \\
&\cong \varinjlim_m \mathrm{Hom}_{H_{(n,r)}(q)}(\mathrm{ev}_\Lambda^*(M) \boxtimes \mathcal{L}_m((q^i)^r), \mathrm{ev}_\Lambda^*(N)) \\
&\cong \varinjlim_m \mathrm{Hom}_{H_{n+r}(q)}(\mathrm{Ind}_{H_{(n,r)}(q)}^{H_{n+r}(q)}(\mathrm{ev}_\Lambda^*(M) \boxtimes \mathcal{L}_m((q^i)^r)), \mathrm{ev}_\Lambda^*(N)) \\
&\cong \varinjlim_m \mathrm{Hom}_{H_{n+r}^\Lambda(q)}(\mathrm{pr}_\Lambda \mathrm{Ind}_{H_{(n,r)}(q)}^{H_{n+r}(q)}(\mathrm{ev}_\Lambda^*(M) \boxtimes \mathcal{L}_m((q^i)^r)), N) \\
&\cong \mathrm{Hom}_{H_{n+r}^\Lambda(q)}(\varprojlim_m \mathrm{pr}_\Lambda \mathrm{Ind}_{H_{(n,r)}(q)}^{H_{n+r}(q)}(\mathrm{ev}_\Lambda^*(M) \boxtimes \mathcal{L}_m((q^i)^r)), N) \\
&= \mathrm{Hom}_{H_{n+r}^\Lambda(q)}((f_i^\Lambda)^{(r)}(M), N),
\end{aligned}$$

where we have used that the direct and inverse limit stabilizes after finitely many steps and Lemma 11.2.6. Furthermore, all the above isomorphisms are natural in M and N, thus the claim follows. □

Let $M \in H_n^\Lambda(q) - \mathrm{mod}[\gamma]$, for a central character $\gamma = (\gamma_1, \ldots, \gamma_n)$ of $H_n(q)$. In view of Proposition 11.2.9 and Proposition 11.2.11, for $i \in I$, we a have that

$$e_i^\Lambda(M) = \begin{cases} (\mathrm{Res}_{H_{n-1}^\Lambda(q)}^{H_n^\Lambda(q)}(M))[\gamma \setminus q^i] & \text{if } q^i \in \{\gamma_1, \ldots, \gamma_n\}, \\ 0 & \text{if } q^i \notin \{\gamma_1, \ldots, \gamma_n\}, \end{cases} \tag{11.16}$$

on the one hand, and

$$f_i^\Lambda(M) = (\mathrm{Ind}_{H_n^\Lambda(q)}^{H_{n+1}^\Lambda(q)}(M))[\gamma + q^i]$$

on the other. From Propositions 11.2.10 and 11.2.11 we already know that the functors $\mathrm{Ind}_{H_{n-1}^\Lambda(q)}^{H_n^\Lambda(q)}$ and $\mathrm{Res}_{H_{n-1}^\Lambda(q)}^{H_n^\Lambda(q)}$ are left and right adjoint to one another. From the above description of e_i^Λ and f_i^Λ, we get another proof of this fact.
This has also consequences for the functors $(e_i^\Lambda)^{(r)}$ and $(f_i^\Lambda)^{(r)}$, for $i \in I$, and $r > 0$. Recall that by Theorem 9.8.5, we have a functorial isomorphism

$$(e_i^\Lambda)^{(r)}(M) \cong (\Delta_{(q^i)^r}(M))^{W(r)'}, \tag{11.17}$$

for $M \in H_n^\Lambda(q) - \mathrm{mod}$. From the definition of $e_i^\Lambda(M)$, we have that

$$(e_i^\Lambda)^r(M) = \Delta_{(q^i)^r}(M)$$

as $H_{n-r}^\Lambda(q)$-modules. Hence, with our new description of the functor e_i^Λ we get that

$$(e_i^\Lambda)^{(r)}(M) \cong ((\mathrm{Res}_{H_{n-r}^\Lambda(q)}^{H_n^\Lambda(q)}(M))[\gamma \setminus rq^i])^{W(r)'}. \tag{11.18}$$

For notational reasons, in the following, we will also write $M^{H_n^f(q)}$ for the subset $M^{W(n)}$ of an $H_n(q)$-module M.
Next, we will recall some facts about the finite-dimensional Hecke-algebra $H_n^f(q)$, $n \geq 0$. Recall that $H_n^f(q)$ becomes an augmented algebra via the surjective F-algebra homomorphism

$$\epsilon : H_n^f(q) \to F \tag{11.19}$$

defined by $\epsilon(\sum r_w T_w) = \sum q^{l(w)} r_w$. Considering F as a left $H_n^f(q)$-module via ϵ, we obtain the *trivial* left module for $H_n^f(q)$. We could also view F as a right $H_n^f(q)$-module via ϵ, and obtain the trivial right $H_n^f(q)$-module. Moreover, if M is an $H_n^f(q)$-module such that $T_w m = q^{l(w)} m$, for all $w \in W(n)$ and $m \in M$, we say that $H_n^f(q)$ acts *trivially* on M.
In the following we will denote by Aug the kernel of ϵ. It is easy to see that the elements of $H_n^f(q)$ of the form $T_w - q^{l(w)}$, $w \in W(n)$, $w \neq 1$, are contained in Aug. Since the set $\{T_w \mid w \in W(n)\}$ forms a basis of $H_n^f(q)$ as an F-vector space, we see that these elements are linearly independent over F. Considering dimensions, we have that $\dim_F \mathrm{Aug} = n! - 1$. Therefore, the two-sided ideal Aug is spanned by the set $\{T_w - q^{l(w)} \mid w \in W(n), w \neq 1\}$.
Next, consider the exact sequence

$$0 \longrightarrow \mathrm{Aug} \xrightarrow{\ker(\epsilon)} H_n^f(q) \xrightarrow{\epsilon} F \longrightarrow 0 \tag{11.20}$$

of right $H_n^f(q)$-modules afforded by ϵ. Tensoring with a finite-dimensional left $H_n^f(q)$-module M gives an exact sequence

$$\mathrm{Aug} \otimes_{H_n^f(q)} M \xrightarrow{\ker(\epsilon)\otimes 1} H_n^f(q) \otimes_{H_n^f(q)} M \xrightarrow{\epsilon\otimes 1} F \otimes_{H_n^f(q)} M \longrightarrow 0 \tag{11.21}$$

of left $H_n^f(q)$-modules, where $1 = \mathrm{id}_M$. Of course, $H_n^f(q) \otimes_{H_n^f(q)} M \cong M$ as left $H_n^f(q)$-modules, where the isomorphism is given by the map determined by $h \otimes m \mapsto hm$. Under this isomorphism, the image of $\ker(\epsilon) \otimes 1$ equals

$$\mathrm{Aug} \cdot M = \{\sum a_i m_i \mid a_i \in \mathrm{Aug}, m_i \in M\}.$$

Then, by the exactness of the sequence in (11.21), we get an isomorphism

$$F \otimes_{H_n^f(q)} M \cong M/\mathrm{Aug} \cdot M \tag{11.22}$$

of left $H_n^f(q)$-modules. Note that $M/\mathrm{Aug}\cdot M$ can also be characterized as the largest factor module of M on which $H_n^f(q)$ acts trivially. For a left $H_n^f(q)$-module M, let us write $M_{H_n^f(q)}$ for the factor $M/\mathrm{Aug} \cdot M$.
We will need the following:

Lemma 11.2.16. *Let A be an F-algebra, $n \geq 0$, and $\mathcal{C}$ be the full subcategory of $A \otimes H_n^f(q) - \mathrm{mod}$ that consists of modules whose restrictions to $1 \otimes H_n^f(q)$ are free. Then the functors $(-)^{H_n^f(q)}$ and $(-)_{H_n^f(q)}$ from $\mathcal{C}$ to $A - \mathrm{mod}$ are isomorphic.*

Proof. In the following we will identify the subalgebras $A \otimes 1$ and $1 \otimes H_n^f(q)$ of $A \otimes H_n^f(q)$ with A and $H_n^f(q)$, respectively. We set $x := \sum_{w \in W(n)} T_w \in H_n^f(q)$. Then it is not hard to show that $T_w x = x T_w = q^{l(w)} x$ for all $w \in W(n)$, see for example [68, Lemma 3.2]. Let $M \in \mathcal{C}$. Then, by assumption on M, considered as $H_n^f(q)$-module, $M \cong (H_n^f(q))^b$, for some $b > 0$. Suppose first that $b = 1$. We also may assume that $M = H_n^f(q)$. Since the trivial module occurs precisely once in the socle of the left regular module $H_n^f(q)$, we conclude that $\dim_F(M^{H_n^f(q)}) = 1$. Since $x \neq 0$, and $xM \subseteq M^{H_n^f(q)}$, we must have that $xM = M^{H_n^f(q)}$. From this it follows that $M^{H_n^f(q)} = xM$ for arbitrary b. Now define a map

$$\psi_M : M_{H_n^f(q)} \longrightarrow M^{H_n^f(q)},\ m + \mathrm{Aug} \cdot M \mapsto xm.$$

We have to show that this is well defined. Suppose that $m + \mathrm{Aug} \cdot M = m' + \mathrm{Aug} \cdot M$, for $m, m' \in M$. Then $m - m' \in \mathrm{Aug} \cdot M$. Since $x\mathrm{Aug} = 0$, we see that ψ_M is well defined.

Moreover, from the discussion above, we know that ψ_M is surjective. As A commutes with $H_n^f(q)$, $M_{H_n^f(q)}$ and $M^{H_n^f(q)}$ are A-modules and the map ψ_M is a homomorphism of A-modules. Since the trivial module occurs precisely once in the head of the left regular module $H_n^f(q)$, we have that $\dim_F(M_{H_n^f(q)}) = \dim_F(M^{H_n^f(q)})$. We conclude that ψ_M is an isomorphism of A-modules.

If $f : M \to N$ is a homomorphism in $\mathcal{C}$, then f induces a map

$$f^{H_n^f(q)} : M^{H_n^f(q)} \to N^{H_n^f(q)},$$

which is just the restriction of f to $M^{H_n^f(q)}$. This is because of $f^{H_n^f(q)}(xm) = x f^{H_n^f(q)}(m)$, for all $m \in M$. Furthermore, since $f(\mathrm{Aug} \cdot M) \subseteq \mathrm{Aug} \cdot N$, the homomorphism f induces a well-defined map

$$f_{H_n^f(q)} : M_{H_n^f(q)} \to N_{H_n^f(q)}.$$

Clearly, both maps are homomorphisms of A-modules. Then it is easy to see that $\psi_N \circ f_{H_n^f(q)} = f^{H_n^f(q)} \circ \psi_M$, thus, the constructed isomorphism is functorial in M. $\square$

Next, let $\Lambda \in P_+$. Then we define a functor $H_n^\Lambda(q) - \mathrm{mod} \to H_{n+r}^\Lambda(q) - \mathrm{mod}$ as follows: For $M \in H_n^\Lambda(q)$ with central character γ and $i \in I$, we set

$$\widetilde{(f_i^\Lambda)}^{(r)}(M) = (\mathrm{Ind}_{H_n^\Lambda(q) \otimes H_r^f(q)}^{H_{n+r}^\Lambda(q)}(M \boxtimes \mathbf{1}))[\gamma + rq^i],\ r \geq 1, \tag{11.23}$$

where $\mathbf{1}$ stands for the trivial $H_r^f(q)$-module. Then, we extend this additively to a functor on $H_n^\Lambda(q) - \mathrm{mod}$. Here we identify $H_r^f(q)$ with the subalgebra of $H_{n+r}^\Lambda(q)$ generated by $T_{n+1}, \ldots, T_{n+r-1}$. We have the following:

Proposition 11.2.17. *Let $\Lambda \in P_+$, $M \in H_n^\Lambda(q) - \mathrm{mod}$ and $r \geq 1$. There is a functorial isomorphism*

$$\widetilde{(f_i^\Lambda)}^{(r)}(M) \cong (f_i^\Lambda)^{(r)}(M)$$

of $H_{n+r}^\Lambda(q)$-modules.

Proof. We show that $\widetilde{(f_i^\Lambda)}^{(r)}$ is a left adjoint of $(e_i^\Lambda)^{(r)}$. To this end, we use the alternative description of $(e_i^\Lambda)^{(r)}$ given in (11.17). Let $M \in H_n^\Lambda(q) - \text{mod}$ and $N \in H_{n+r}^\Lambda(q) - \text{mod}$. Without loss of generality, we may assume that M has central character γ, and N has central character $\gamma + rq^i$. Then

$$\begin{aligned}
\mathrm{Hom}_{H_{n+r}^\Lambda(q)}(\widetilde{(f_i^\Lambda)}^{(r)}(M), N) &\cong \mathrm{Hom}_{H_{n+r}^\Lambda(q)}((\mathrm{Ind}_{H_n^\Lambda(q)\otimes H_r^f(q)}^{H_{n+r}^\Lambda(q)}(M \boxtimes \mathbf{1}))[\gamma + rq^i], N) \\
&\cong \mathrm{Hom}_{H_{n+r}^\Lambda(q)}(\mathrm{Ind}_{H_n^\Lambda(q)\otimes H_r^f(q)}^{H_{n+r}^\Lambda(q)}(M \boxtimes \mathbf{1}), N) \\
&\cong \mathrm{Hom}_{H_n^\Lambda(q)\otimes H_r^f(q)}(M \boxtimes \mathbf{1}, \mathrm{Res}_{H_n^\Lambda(q)\otimes H_r^f(q)}^{H_{n+r}^\Lambda(q)}(N)) \\
&\cong \mathrm{Hom}_{H_n^\Lambda(q)}(M, \mathrm{Hom}_{H_r^f(q)}(\mathbf{1}, \mathrm{Res}_{H_n^\Lambda(q)\otimes H_r^f(q)}^{H_{n+r}^\Lambda(q)}(N))) \\
&\cong \mathrm{Hom}_{H_n^\Lambda(q)}(M, (\mathrm{Res}_{H_n^\Lambda(q)}^{H_{n+r}^\Lambda(q)}(N))^{H_r^f(q)}) \\
&\cong \mathrm{Hom}_{H_n^\Lambda(q)}(M, ((\mathrm{Res}_{H_n^\Lambda(q)}^{H_{n+r}^\Lambda(q)}(N))^{H_r^f(q)})[\gamma]) \\
&\cong \mathrm{Hom}_{H_n^\Lambda(q)}(M, ((\mathrm{Res}_{H_n^\Lambda(q)}^{H_{n+r}^\Lambda(q)}(N))[\gamma])^{H_r^f(q)}) \\
&\cong \mathrm{Hom}_{H_n^\Lambda(q)}(M, (e_i^\Lambda)^{(r)}(N)),
\end{aligned}$$

where all the above isomorphisms are natural. Hence, by uniqueness of adjoints, we get an isomorphism $\widetilde{(f_i^\Lambda)}^{(r)} \cong (f_i^\Lambda)^{(r)}$ of functors. $\square$

Finally, we are in the position to deduce crucial properties of the functors introduced so far:

Proposition 11.2.18. *Let $i \in I$, and $r \geq 1$. Then the following hold:*

(i) *As functors, $(f_i^\Lambda)^r \cong \bigoplus_{r!}(f_i^\Lambda)^{(r)}$.*

(ii) *$(f_i^\Lambda)^{(r)}$ is right adjoint to $(e_i^\Lambda)^{(r)}$.*

(iii) *$(f_i^\Lambda)^{(r)}$ is exact.*

Proof. Note that, by Theorem 11.2.7, f_i^Λ is left adjoint to e_i^Λ. Since $(e_i^\Lambda)^r \cong \bigoplus_{r!}(e_i^\Lambda)^{(r)}$ as functors, we can construct, with the help of Theorem 11.2.15, an adjunction between $\bigoplus_{r!}(f_i^\Lambda)^{(r)}$ and $(e_i^\Lambda)^r$ such that the functor $\bigoplus_{r!}(f_i^\Lambda)^{(r)}$ is left adjoint to $(e_i^\Lambda)^r$. By uniqueness of adjoints, we must have that $(f_i^\Lambda)^r \cong \bigoplus_{r!}(f_i^\Lambda)^{(r)}$ as functors, thus, (i) holds.
Next we show (ii). To this end, we use the alternative descriptions of the functors $(e_i^\Lambda)^{(r)}$ and $(f_i^\Lambda)^{(r)}$ given in (11.17) and (11.23). We let $M \in H_n^\Lambda(q) - \text{mod}$ and $N \in H_{n+r}^\Lambda(q) - \text{mod}$, and assume that M has central character γ and N has central character $\gamma + rq^i$. We have:

$$\begin{aligned}
&\mathrm{Hom}_{H_{n+r}^\Lambda(q)}(N, (\mathrm{Ind}_{H_n^\Lambda(q)\otimes H_r^f(q)}^{H_{n+r}^\Lambda(q)}(M \boxtimes \mathbf{1}))[\gamma + rq^i]) \\
&\cong \mathrm{Hom}_{H_n^\Lambda(q)\otimes H_r^f(q)}((\mathrm{Res}_{H_n^\Lambda(q)\otimes H_r^f(q)}^{H_{n+r}^\Lambda(q)}(N))[\gamma], M \boxtimes \mathbf{1}),
\end{aligned}$$

where we take the block component with respect to the subalgebra $H_n^\Lambda(q) \otimes 1$. Note that if $X \in H_n^\Lambda(q) \otimes H_r^f(q) - \text{mod}$, and $f \in \operatorname{Hom}_{H_n^\Lambda(q)\otimes H_r^f(q)}(X, M \boxtimes \mathbf{1})$, then $(1 \otimes \mathrm{Aug}) \cdot f(x) = 0$, for all $x \in X$. Since $H_n^\Lambda(q)$ and $H_r^f(q)$, considered as subalgebras of $H_n^\Lambda(q) \otimes H_r^f(q)$ in the natural way, commute with each other, and Aug is a two-sided ideal in $H_r^f(q)$, we have that $(1 \otimes \mathrm{Aug}) \cdot X$ is an $H_n^\Lambda(q) \otimes H_r^f(q)$-submodule of X. Then, from the exact sequence

$$0 \longrightarrow (1 \otimes \mathrm{Aug}) \cdot X \longrightarrow X \longrightarrow X/(1 \otimes \mathrm{Aug}) \cdot X \longrightarrow 0,$$

we get that $\operatorname{Hom}_{H_n^\Lambda(q)\otimes H_r^f(q)}(X, M \boxtimes \mathbf{1}) \cong \operatorname{Hom}_{H_n^\Lambda(q)\otimes H_r^f(q)}(X/(1 \otimes \mathrm{Aug}) \cdot X, M \boxtimes \mathbf{1})$. Considered as $H_r^f(q)$-module, we have seen in the discussion before Lemma 11.2.16 that $H_r^f(q)$ acts trivially on $X/(1 \otimes \mathrm{Aug}) \cdot X$. Then there is an isomorphism

$$\operatorname{Hom}_{H_n^\Lambda(q)\otimes H_r^f(q)}(X/(1 \otimes \mathrm{Aug}) \cdot X, M \boxtimes \mathbf{1}) \cong \operatorname{Hom}_{H_n^\Lambda(q)}(X/(1 \otimes \mathrm{Aug}) \cdot X, M),$$

which is functorial in both variables. Next, suppose that on restriction to $H_r^f(q)$, X is free. Then, by Lemma 11.2.16, we have a functorial isomorphism

$$\operatorname{Hom}_{H_n^\Lambda(q)}(X/(1 \otimes \mathrm{Aug}) \cdot X, M) \cong \operatorname{Hom}_{H_n^\Lambda(q)}(X^{H_r^f(q)}, M).$$

In particular, if we set

$$X := (\operatorname{Res}^{H_{n+r}^\Lambda(q)}_{H_n^\Lambda(q)\otimes H_r^f(q)}(N))[\gamma],$$

then, on restriction to $H_r^f(q)$, X is free. Note that the restriction of the Kato module $L((q^i)^r)$ from $H_r(q)$ to $H_r^f(q)$ is isomorphic to the left regular $H_r^f(q)$-module, see the paragraph before Proposition 9.8.4. Thus, together with the discussion at the beginning of the proof, we get a functorial isomorphism

$$\begin{aligned} &\operatorname{Hom}_{H_{n+r}^\Lambda(q)}(N, (\operatorname{Ind}^{H_{n+r}^\Lambda(q)}_{H_n^\Lambda(q)\otimes H_r^f(q)}(M \boxtimes \mathbf{1}))[\gamma + rq^i]) \\ &\cong \operatorname{Hom}_{H_n^\Lambda(q)}(((\operatorname{Res}^{H_{n+r}^\Lambda(q)}_{H_n^\Lambda(q)}(N))[\gamma])^{H_r^f(q)}, M), \end{aligned}$$

showing that $(f_i^\Lambda)^{(r)}$ is right adjoint to $(e_i^\Lambda)^{(r)}$.

Since, $(f_i^\Lambda)^{(r)}$ is a left and a right adjoint, it must be exact. Thus, (iii) follows from (ii). □

Remark 11.2.19. As in Section 9.8, we note that the ideas for the definition of the various functors given in this section are based on the ideas of I. Grojnowski, where we have tried to be as explicit as possible. In the degenerate case this is due to A. Kleshchev, and the definition of our divided power functors is similar to that given in [59, §8.3].
The proof of part (ii) of Proposition 11.2.18 is based on ideas of A. Kleshchev, see [59, Lemma 8.3.1], and seems to be new in the non-degenerate case.

11.3 Block parametrization for cyclotomic Hecke algebras

Recall from Lemma 9.6.1 that by considering central characters, we can decompose the category $H_n(q)-\text{mod}$ into blocks, i.e., every $M \in H_n(q)-\text{mod}$ can be written as

$$M \cong \bigoplus_{\gamma \in S^n/\sim} M[\gamma].$$

As mentioned in the previous section, for all $\Lambda \in P_+$, we also obtain a decomposition

$$H_n^\Lambda(q)-\text{mod} \cong \bigoplus_{\gamma \in S^n/\sim} H_n^\Lambda(q)-\text{mod}[\gamma].$$

Then [64, Theorem A] asserts that this is already the block decomposition of $H_n^\Lambda(q)-$ mod.

Note that, by [30, Theorem 2.14], for $\Lambda = \Lambda_0$ the homomorphism

$$Z(H_n(q)) \longrightarrow Z(H_n^\Lambda(q))$$

induced by ev_Λ is actually onto, giving another proof of this fact in the case when $\Lambda = \Lambda_0$.

Next, we state a convenient way of parametrizing the blocks of $H_n(q)$. Here, we are interested in blocks, where the central character γ only contains powers of q. In the language developed so far, we consider the block decomposition of $H_n(q) - \text{mod}_q$. Then, the central characters are labelled by the $W(n)$-orbits on I^n. If $\underline{i} \in I^n$, define its *content* $\text{cont}(\underline{i}) \in P$ by

$$\text{cont}(\underline{i}) = \sum_{i \in I} \gamma_i \alpha_i, \tag{11.24}$$

where $\gamma_i = |\{j \mid j = 1, \ldots, n, i_j = i\}|$, for all $i \in I$. Also, by definition, $\sum_{i\in I} \gamma_i = n$. Denote by Γ_n the set of all non-negative integral linear combinations $\gamma = \sum_{i\in I} \gamma_i\alpha_i$ of the simple roots α_i, $i \in I$, such that $\sum_{i\in I} \gamma_i = n$.

The $W(n)$-orbit of some $\underline{i} \in I^n$ is uniquely determined by the content of $\underline{i}$. Hence we may label these orbits by elements of Γ_n.

Chapter 12

Multiplicity-one results

In this chapter we will recall a famous result of I. Grojnowski and M. Vazirani given in [45], which states that the socle of the restriction of an irreducible $H_n(q)$-module to the subalgebra $H_{n-1}(q)$ is multiplicity free.
Also, we recall the definition of an antiautomorphism of $H_n(q)$, which gives a duality on $H_n(q)-\text{mod}$. We give a proof of the well-known fact that the irreducible $H_n(q)$-modules are self-dual.
Moreover, we define important operators on the Grothendieck groups of $H_n(q)$-modules and $H_n^\Lambda(q)$-modules, $\Lambda \in P_+$, respectively, and state crucial properties of these.
In what follows, we will present here the viewpoint of [59] in the non-degenerate case.

12.1 Multiplicity-free socles

Recall from Section 9.5 that for $M \in H_n(q) - \text{mod}$ and some $a \in S := F^\times$, we denote by $M_{\underline{a}}$ the simultaneous generalized eigenspace of the commuting operators $x_1, \ldots, x_n$ corresponding to the eigenvalues $\underline{a} = (a_1, \ldots, a_n)$. Moreover, we denote by $\Delta_a(M)$ the generalized eigenspace of x_n on M corresponding to the eigenvalue $a \in S$. Recall the definition of the F-subalgebra $H_{(n-1,1)}(q)$ of $H_n(q)$ from Section 9.4, and observe that it is generated by the elements $x_1, \ldots, x_{n-1}, T_1, \ldots, T_{n-2}$ of $H_n(q)$. Then, the space $\Delta_a(M)$ can be considered as an $H_{(n-1,1)}(q)$-module in the natural way, hence affording a functor

$$\Delta_a : H_n(q) - \text{mod} \longrightarrow H_{(n-1,1)}(q) - \text{mod},$$

see (9.32). Recall that for a composition $\mu = (\mu_1, \ldots, \mu_k)$ of n we write $\text{Res}^n_{\mu_1,\ldots,\mu_k}$ for the restriction functor $\text{Res}^{H_n(q)(q)}_{H_\mu(q)}$. Then, via the natural embedding of $H_{n-1}(q) \subseteq H_{(n-1,1)}(q)$, we also have the functor

$$e_a := \text{Res}^{n-1,1}_{n-1} \circ \Delta_a : H_n(q) - \text{mod} \longrightarrow H_{n-1}(q) - \text{mod}.$$

More generally, define $\Delta_{a^m}(M)$ to be the simultaneous generalized eigenspace of the operators $x_{n-m+1}, \ldots, x_n$. We view M as an $H_m(q)$-module via the canonical embedding $H_m(q) \cong 1 \otimes H_m(q) \subseteq H_{n-m}(q) \otimes H_m(q) \cong H_{(n-m,m)}(q)$. Then M decomposes as

$$M \cong \bigoplus_{\gamma \in S^m} M[\gamma],$$

where $M[\gamma] = \bigoplus_{\underline{a} \in \gamma} M_{\underline{a}}$ is actually an $H_m(q)$-submodule of M. Note that if $\gamma = (a, \ldots, a)$, we have that $M[\gamma] = M_{(a,\ldots,a)}$, and this equals $\Delta_{a^m}(M)$. We will consider $\Delta_{a^m}(M)$ as an $H_m(q)$-module in this way. Moreover, we set $\Delta_{a^0}(M) := M$. Therefore, we again have a functor

$$\Delta_{a^m} : H_n(q) - \text{mod} \longrightarrow H_{n-m,m}(q) - \text{mod},$$

as well as a functor

$$e_{a^m} : H_n(q) - \text{mod} \longrightarrow H_{n-m}(q) - \text{mod},$$

if we consider the natural embedding $H_{n-m}(q) \subseteq H_{(n-m,m)}(q)$. Note that $e_{a^m}(M) = e_a^m(M) = e_a \circ \ldots \circ e_a(M)$ (m times).
Applying Kato's Theorem, see Theorem 9.7.3, we infer that $\Delta_{a^m}(M)$ is the largest $H_{(n-m,m)}(q)$-submodule of M all of whose composition factors are isomorphic to a module $N \boxtimes L(a^m)$, for some irreducible $H_{n-m}(q)$-module N. Also by adjointness of induction and restriction, we have that

$$\begin{aligned} &\mathrm{Hom}_{H_{(n-m,m)}(q)}(N \boxtimes L(a^m), \Delta_{a^m}(M)) \\ &\cong \mathrm{Hom}_{H_n(q)}(\mathrm{Ind}^n_{n-m,m}(N \boxtimes L(a^m)), M), \end{aligned} \tag{12.1}$$

where we set $\mathrm{Ind}^n_{n-m,m}(-) := \mathrm{Ind}^{H_n(q)}_{H_{(n-m,m)}(q)}(-)$.

The proof of the following lemmas are analogous to those of [59, §5.1].

Lemma 12.1.1. *Let $M \in H_n(q) - \text{mod}$. If*

$$\mathrm{ch}(M) = \sum_{\underline{a} \in S^n} \lambda_{\underline{a}} [L(a_1) \boxtimes \ldots \boxtimes L(a_n)],$$

then we have that

$$\mathrm{ch}(\Delta_{a^m}(M)) = \sum_{\underline{b} \in S^n} \lambda_{\underline{b}} [L(b_1) \boxtimes \ldots \boxtimes L(b_n)],$$

where $\underline{b} \in S^n$ is such that $b_{n-m+1} = \ldots = b_n = a$.

Proof. Note that $\Delta_{a^m}(M)$ is the direct sum of those subspaces $M_{\underline{a}}$ of M for which $a_{n-m+1} = \ldots = a_n = a$. The claim now follows. ☐

Definition 12.1.2. For $a \in S$, and $M \in H_n(q) - \text{mod}$, we define

$$\varepsilon_a(M) = \max\{m \geq 0 \mid \Delta_{a^m}(M) \neq 0\}. \tag{12.2}$$

Lemma 12.1.3. *Let $M \in H_n(q) - \mathrm{mod}$ be irreducible, $\varepsilon := \varepsilon_a(M)$. Let $N \boxtimes L(a^m)$ be an irreducible $H_{(n-m,m)}(q)$-submodule of $\Delta_{a^m}(M)$. Then $\varepsilon_a(N) = \varepsilon - m$.*

Proof. By Lemma 12.1.1, ε is the longest a-tail occurring in the expansion of $\mathrm{ch}(M)$. Therefore, we must have $\varepsilon_a(N) + m \leq \varepsilon$. Since M is irreducible, by (12.1), we see that M is a quotient of $\mathrm{Ind}^n_{n-m,m}(N \boxtimes L(a^m))$. By exactness of the functor Δ_{a^ε}, we obtain an epimorphism $\Delta_{a^\varepsilon}(\mathrm{Ind}^n_{n-m,m}(N \boxtimes L(a^m))) \to \Delta_{a^\varepsilon}(M)$ of $H_{(n-m,m)}(q)$-modules. Since $\Delta_{a^\varepsilon}(M) \neq 0$, it follows that $\Delta_{a^\varepsilon}(\mathrm{Ind}^n_{n-m,m}(N \boxtimes L(a^m))) \neq 0$. By Lemma 12.1.1 and the Shuffle Lemma, see Lemma 9.5.7, we infer $\varepsilon_a(N) + m \geq \varepsilon$. □

Lemma 12.1.4. *Let $N \in H_n(q) - \mathrm{mod}$ be irreducible, $a \in S$, and $\varepsilon_a(N) = 0$. Set $M := \mathrm{Ind}^{n+m}_{n,m}(N \boxtimes L(a^m))$. Then:*

(i) $\Delta_{a^m}(M) \cong N \boxtimes L(a^m)$.

(ii) $\mathrm{hd}(M)$ *is irreducible, and* $\varepsilon_a(\mathrm{hd}(M)) = m$.

(iii) *For all other composition factors L of M, we get that $\varepsilon_a(L) < m$.*

Proof. By (12.1), $N \boxtimes L(a^m)$ is certainly a submodule of $\Delta_{a^m}(M)$. By the Shuffle Lemma, Lemma 12.1.1, and our hypothesis on N we have that

$$\dim_F(\Delta_{a^m}(M)) = \dim_F(N \boxtimes L(a^m)).$$

Thus, (i) follows. The remaining parts follow from [45, Lemma 3.5]. □

Lemma 12.1.5. *Let $M \in H_n(q) - \mathrm{mod}$ be irreducible, $a \in S$, and set $\varepsilon := \varepsilon_a(M)$. Then $\Delta_{a^\varepsilon}(M)$ is isomorphic to $N \boxtimes L(a^\varepsilon)$ as an $H_{(n-\varepsilon,\varepsilon)}(q)$-module, for some irreducible $H_{n-\varepsilon}(q)$-module N with $\varepsilon_a(N) = 0$.*

Proof. Pick some irreducible $H_{(n-\varepsilon,\varepsilon)}(q)$-module in the socle of $\Delta_{a^\varepsilon}(M)$, which, by Kato's Theorem, has the form $N \boxtimes L(a^\varepsilon)$, for some irreducible $H_{n-\varepsilon}(q)$-module N. By Lemma 12.1.3, we have that $\varepsilon_a(N) = 0$. Then, by (12.1), and the irreducibility of M we have that M is a quotient of $\mathrm{Ind}^n_{n-\varepsilon,\varepsilon}(N \boxtimes L(a^\varepsilon))$. By exactness of Δ_{a^ε} and the first part of Lemma 12.1.4, we see that $\Delta_{a^\varepsilon}(M)$ is irreducible, and the claim follows. □

Lemma 12.1.6. *Let $m \geq 0$, $a \in S$, and $N \in H_n(q) - \mathrm{mod}$ be irreducible. Set $M := \mathrm{Ind}^{n+m}_{n,m}(N \boxtimes L(a^m))$. Then $\mathrm{hd}(M)$ is irreducible with $\varepsilon_a(\mathrm{hd}(M)) = \varepsilon_a(N) + m$, and for all other composition factors L of M we have $\varepsilon_a(L) < \varepsilon_a(N)$.*

Proof. This is [45, Lemma 3.5, (3)]. □

Theorem 12.1.7. *Let $M \in H_n(q) - \mathrm{mod}$ be irreducible, $a \in S$. Then for $0 \leq m \leq \varepsilon := \varepsilon_a(M)$, the socle of $\Delta_{a^m}(M)$ is an irreducible $H_{(n-m,m)}(q)$-module of the form $L \boxtimes L(a^m)$ such that $\varepsilon_a(L) = \varepsilon_a(M) - m$.*

Proof. Note that $\Delta_{a^\varepsilon}(M) \subseteq \Delta_{a^m}(M)$ for all $0 \leq m \leq \varepsilon$. As before, we consider $\Delta_{a^m}(M)$ as an $H_{(n-m,m)}(q)$-module. If we consider $\Delta_{a^m}(M)$ as an $H_{n-m}(q)$-module in the usual way, we see that

$$\Delta_{a^\varepsilon}(M) = \Delta_{a^{\varepsilon-m}}(\Delta_{a^m}(M)),$$

considered as $H_{(n-\varepsilon,\varepsilon-m,m)}(q)$-modules.
Suppose that $L \boxtimes L(a^m)$ is an irreducible constituent of $\mathrm{soc}(\Delta_{a^m}(M))$. By Lemma 12.1.3, $\varepsilon_a(L) = \varepsilon - m$. Thus, via restriction, L determines a non-trivial submodule of $\mathrm{Res}^{n-\varepsilon,\varepsilon}_{n-\varepsilon,\varepsilon-m,m}(\Delta_{a^\varepsilon}(M))$. By Lemma 12.1.5, $\Delta_{a^\varepsilon}(M)$ is an irreducible $H_{(n-\varepsilon,\varepsilon)}(q)$-module of the form $N \boxtimes L(a^\varepsilon)$. Let T be an irreducible $H_{(n-\varepsilon,\varepsilon-m,m)}(q)$-submodule of $\mathrm{Res}^{n-\varepsilon,\varepsilon}_{n-\varepsilon,\varepsilon-m,m}(N \boxtimes L(a^\varepsilon))$. Then $T \cong X \boxtimes Y \boxtimes Z$, with an irreducible $H_{n-\varepsilon}(q)$-module X, an irreducible $H_{\varepsilon-m}(q)$-module Y, and an irreducible $H_m(q)$-module Z. Since $\mathrm{Res}^{n-\varepsilon,\varepsilon}_{n-\varepsilon,\varepsilon-m,m}(N \boxtimes L(a^\varepsilon)) = N \boxtimes \mathrm{Res}^{\varepsilon}_{\varepsilon-m,m}(L(a^\varepsilon))$, it follows that $X \cong N$ as $H_{n-\varepsilon}$-modules. Moreover, if we apply [45, Lemma 3.4], we see that the socle of $\mathrm{Res}^{\varepsilon}_{\varepsilon-m,m}(L(a^\varepsilon))$ is isomorphic to $L(a^{\varepsilon-m}) \boxtimes L(a^m)$, hence $Y \cong L(a^{\varepsilon-m})$, and $Z \cong L(a^m)$. We infer that the socle of $\mathrm{Res}^{n-m,m}_{n-\varepsilon,\varepsilon-m,m}(\Delta_{a^\varepsilon}(M))$ is isomorphic to $N \boxtimes L(a^{\varepsilon-m}) \boxtimes L(a^m)$. By exactness of restriction, it follows that the socle of $\Delta_{a^m}(M)$ must equal $L \boxtimes L(a^m)$, and therefore is irreducible as claimed. □

The next theorem is the famous "Multiplicity one result" given in [45]. We include a proof here, which is analogous to that of [59, Corollary 5.1.7].

Theorem 12.1.8. *Let $M \in H_n(q)-\mathrm{mod}$ be irreducible, $a \in S$, and $\varepsilon_a(M) > 0$. Then $\mathrm{soc}(e_a(M))$ is irreducible.*

Proof. Let L be an irreducible constituent of the socle of $e_a(M)$. Since $\varepsilon_a(M) > 0$, such an L exists. By Schur's Lemma, the central element $z := x_1 + \ldots + x_n \in H_n(q)$ acts as a scalar on the whole of M. With the same argument, the element $z' := x_1 + \ldots + x_{n-1} \in Z(H_{n-1}(q))$ acts via some scalar on L. Hence $x_n = z - z'$ acts as a scalar on L. Since $e_a(M) = \mathrm{Res}^{n-1,1}_{n-1}(\Delta_a(M))$, this scalar must equal a. Therefore, L contributes an $H_{(n-1,1)}(q)$-submodule of the form $L \boxtimes L(a)$ to the socle of $\Delta_a(M)$. But the latter is irreducible by Theorem 12.1.7. Thus, the claim follows. □

Corollary 12.1.9. *For $M \in H_n(q)-\mathrm{mod}$ irreducible, the socle of $\mathrm{Res}^n_{n-1}(M)$ is multiplicity-free.*

Proof. From (9.36) we know that $\mathrm{Res}^n_{n-1}(M) = \bigoplus_{a\in S} e_a(M)$. By Theorem 12.1.8, for all $a \in S$, the socle of $e_a(M)$ is either zero or irreducible. For $a \neq b$, the $H_{n-1}(q)$-modules $e_a(M)$ and $e_b(M)$ are in different blocks, thus their socles are not isomorphic. The claim now follows. □

Definition 12.1.10. For $M \in H_n(q)-\mathrm{mod}$, denote by $\tilde{e}_a(M) := \mathrm{soc}(e_a(M))$, and $\tilde{f}_a(M) := \mathrm{hd}(\mathrm{Ind}^{n+1}_{n,1}(M \boxtimes L(a)))$.

Remark 12.1.11. (1) If we set $m = 1$ in Lemma 12.1.6, then it follows that $\varepsilon_a(\tilde{f}_a(M)) = \varepsilon_a(M) + 1$.

(2) In view of Theorem 12.1.8, for an irreducible $M \in H_n(q) - \mathrm{mod}$, we could also write $\varepsilon_a(M) = \max\{m \geq 0 \mid \tilde{e}_a^m(M) \neq 0\}$.

For the next lemma we need the following, which can be found in [25, Lemma 10.17]. For F-algebras A, B together with F-subalgebras $A' \subseteq A$, $B' \subseteq, B$, we have that

$$\mathrm{Ind}_{A' \otimes B'}^{A \otimes B}(M \boxtimes N) \cong \mathrm{Ind}_{A'}^{A}(M) \boxtimes \mathrm{Ind}_{B'}^{B}(N), \tag{12.3}$$

for an A'-module M and a B'-module N.

The next lemma is an analogue of [59, Lemma 5.2.1] in the non-degenerate case.

Lemma 12.1.12. *Let* $M \in H_n(q) - \mathrm{mod}$ *be irreducible,* $a \in S$*, and* $m \geq 0$*. Then the following hold:*

(i) $\mathrm{soc}(\Delta_{a^m}(M)) \cong \tilde{e}_a^m(M) \boxtimes L(a^m)$.

(ii) $\mathrm{hd}(\mathrm{Ind}_{n,m}^{n+m}(M \boxtimes L(a^m))) \cong \tilde{f}_a^m(M)$.

Proof. By Theorem 12.1.7, we have that $\mathrm{soc}(\Delta_a(M)) \cong N \boxtimes L(a)$, for some irreducible $N \in H_{n-1}(q) - \mathrm{mod}$. By Theorem 12.1.8, $N \cong \tilde{e}_a(M)$. By applying this m times, we see that $\tilde{e}_a^m(M) \boxtimes L(a)^{\boxtimes m}$ is a submodule of $\mathrm{Res}_{n-m,1,\ldots,1}^{n-m,m}(\Delta_{a^m}(M))$. By Frobenius Reciprocity and Kato's Theorem, $\tilde{e}_a^m(M) \boxtimes L(a^m)$ is an $H_{(n-m,m)}(q)$- submodule of $\Delta_{a^m}(M)$. But the socle of $\Delta_{a^m}(M)$ is an irreducible $H_{(n-m,m)}(q)$-module by Theorem 12.1.7. This finishes the proof of (i).
If $m = 1$, the statement in (ii) is just the definition of $\tilde{f}_a(M)$. By definition, we have an $H_{n+1}(q)$-epimorphism

$$\mathrm{Ind}_{n,1}^{n+1}(M \boxtimes L(a)) \to \tilde{f}_a(M).$$

From this we get an $H_{n+2}(q)$-epimorphism

$$\mathrm{Ind}_{n+1,1}^{n+2}((\mathrm{Ind}_{n,1}^{n+1}(M \boxtimes L(a))) \boxtimes L(a)) \to \tilde{f}_a^2(M),$$

by exactness of induction and right exactness of taking tensor products. By (12.3), and transitivity of induction, we have

$$\begin{aligned}\mathrm{Ind}_{n+1,1}^{n+2}((\mathrm{Ind}_{n,1}^{n+1}(M \boxtimes L(a))) \boxtimes L(a)) &\cong \mathrm{Ind}_{n+1,1}^{n+2}((\mathrm{Ind}_{n,1,1}^{n+1,1}((M \boxtimes L(a))) \boxtimes L(a)) \\ &\cong \mathrm{Ind}_{n,1,1}^{n+2}(M \boxtimes L(a)^{\boxtimes 2})\end{aligned}$$

as $H_{n+2}(q)$-modules. Inductively, we conclude that for $m \geq 1$, $\tilde{f}_a^m(M)$ is a quotient of $\mathrm{Ind}_{n,1,\ldots,1}^{n+m}(M \boxtimes L(a)^{\boxtimes m})$. Since

$$\begin{aligned}\mathrm{Ind}_{n,1,\ldots,1}^{n+m}(M \boxtimes L(a)^{\boxtimes m}) &\cong \mathrm{Ind}_{n,m}^{n+m}(\mathrm{Ind}_{n,1\ldots,1}^{n,m}(M \boxtimes L(a)^{\boxtimes m})) \\ &\cong \mathrm{Ind}_{n,m}^{n+m}(M \boxtimes L(a^m)),\end{aligned}$$

we infer that $\tilde{f}_a^m(M)$ is a quotient of $\mathrm{Ind}_{n,m}^{n+m}(M \boxtimes L(a^m))$, where the last equivalence is again due to (3) and Kato's Theorem. By Lemma 12.1.6, we know that the head of $\mathrm{Ind}_{n,m}^{n+m}(M \boxtimes L(a^m))$ is irreducible, hence the result. $\square$

Theorem 12.1.13. *Let $M \in H_n(q) - \mathrm{mod}$ be irreducible, $a \in S$, and $m \geq 0$. Then the socle of e_a^m is isomorphic to $\bigoplus_{m!} \tilde{e}_a^m(M)$.*

Proof. Denote by Z_m the center of $H_m(q)$. We know that P_m is a free Z_m-module of rank $m!$. Moreover, $H_m(q)$ is a free P_m-module of rank $m!$. Hence, $H_m(q)$ is free of rank $(m!)^2$ considered as a Z_m-module. In the following denote by χ the character of Z_m obtained from the action on the Kato module $L(a^m)$. Recall that, by Schur's Lemma, each $z \in Z_m$ acts by a scalar on the irreducible $H_m(q)$-module $L(a^m)$. We may view χ as an irreducible Z_m-module.

Consider $U := \mathrm{Ind}_{Z_m}^{H_m(q)}(\chi)$. By Frobenius Reciprocity we see that

$$\dim_F \mathrm{Hom}_{H_m(q)}(U, L(a^m)) = m!.$$

Thus, U is a non-zero $H_m(q)$-module. Furthermore, $\dim_F U = (m!)^2$, by the discussion at the beginning of the proof, and the fact that χ is a one-dimensional Z_m-module. It follows that $U \cong \bigoplus_{m!} L(a^m)$.

Next, let L be an irreducible submodule of $e_a^m(M)$. By Schur's Lemma, every symmetric polynomial in the variables $x_1^{\pm 1}, \ldots, x_n^{\pm 1}$ acts as a scalar on the whole of M. By the same argument, any symmetric polynomial in the $x_1^{\pm 1}, \ldots, x_{n-m}^{\pm 1}$ acts as a scalar on L. It follows that any symmetric function in the variables $x_{n-m+1}^{\pm 1}, \ldots, x_n^{\pm 1}$ acts on L with scalars. To this end, one shows that any symmetric polynomial in the variables $x_{n-m+1}, \ldots, x_n$ can be expressed in terms of symmetric polynomials in the $x_1, \ldots, x_n$ and of symmetric polynomials in the $x_1, \ldots, x_{n-m}$. Then use the fact that, for $k \in \mathbb{N}$, Z_k is generated by the elements $t_1, \ldots, t_{k-1}, t_k^{\pm 1}$, where t_i denotes the ith elementary symmetric polynomial in the variables $x_1, \ldots, x_k$. This is easy to see since for every Laurent polynomial g in the $x_1, \ldots, x_k$, we can find an $m \in \mathbb{Z}$ such that $t_k^m g$ is a symmetric polynomial in the $x_1, \ldots, x_k$, and every such polynomial can be written as a polynomial expression in the elementary symmetric polynomials $t_1, \ldots, t_k$.

Now we get that the center Z_m of the subalgebra $H_m(q) \cong 1 \otimes H_m(q) \subset H_{n-m}(q) \otimes H_m(q)$ acts on L with scalars. Since $L \subseteq e_a^m(M) \subseteq \Delta_{a^m}(M)$, and the central character of $\Delta_{a^m}(M)$ is χ, the central character (over Z_m) of L must equal χ as well. This affords a non-zero $H_{n-m}(q) \otimes Z_m$-homomorphism from $L \boxtimes \chi$ to M, whose image is L. Frobenius Reciprocity gives us a non-zero homomorphism $\varphi : L \boxtimes U \cong \mathrm{Ind}_{H_{n-m}(q) \otimes Z_m}^{H_{n-m}(q) \otimes H_m(q)}(L \boxtimes \chi) \to M$, whose image contains L. As observed at the beginning, $U \cong \bigoplus_{m!} L(a^m)$, which implies that the $H_{n-m}(q) \otimes H_m(q)$-module $L \boxtimes U$ is a direct sum of copies of the irreducible $H_{n-m}(q) \otimes H_m(q)$-module $L \boxtimes L(a^m)$. Therefore, all the composition factors of the image of φ are isomorphic to $L \boxtimes L(a^m)$. Since L is contained in the image of φ, an $H_{n-m}(q) \otimes H_m(q)$-submodule of M, the smallest $H_{n-m}(q) \otimes H_m(q)$-submodule of M that contains L is also contained in the image of φ. It follows that the $H_{n-m}(q) \otimes H_m(q)$-submodule of M generated by L is isomorphic to $L \boxtimes L(a^m)$. By Lemma 12.1.12, we conclude that $L \cong \tilde{e}_a^m(M)$.

Note that we consider $H_{n-m}(q)$ as a subalgebra of $H_{n-m}(q) \otimes H_m(q)$ via the embedding $H_{n-m}(q) \cong H_{n-m}(q) \otimes 1 \subseteq H_{n-m}(q) \otimes H_m(q)$. Therefore, if we restrict $\tilde{e}_a^m(M) \boxtimes L(a^m)$ to $H_{n-m}(q)$ we get that

$$\begin{aligned}
&\mathrm{Res}^{H_{n-m}(q)\otimes H_m(q)}_{H_{n-m}(q)\otimes 1}(\tilde{e}_a^m(M) \boxtimes L(a^m)) \\
&= (\mathrm{Res}^{H_{n-m}(q)}_{H_{n-m}(q)}(\tilde{e}_a^m(M))) \boxtimes (\mathrm{Res}^{H_m(q)}_{F}(L(a^m))) \\
&= \tilde{e}_a^m(M) \boxtimes \bigoplus_{m!} F \\
&\cong \bigoplus_{m!} \tilde{e}_a^m(M).
\end{aligned}$$

Hence, by exactness of restriction, $\bigoplus_{m!} \tilde{e}_a^m(M)$ is contained in the socle of $e_a^m(M)$. On the other hand the above argument shows that if L' is contained in the socle of $e_a^m(M)$, then it is contained in

$$\mathrm{Res}^{H_{n-m}(q)\otimes H_m(q)}_{H_{n-m}(q)\otimes 1}(\mathrm{soc}(\Delta_{a^m}(M)) \subseteq \mathrm{soc}(e_a^m(M)).$$

Therefore, $\mathrm{soc}(e_a^m(M)) \cong \bigoplus_{m!} \tilde{e}_a^m(M)$. □

Remark 12.1.14. All the results of this section except possibly Theorem 12.1.13 can be found in [45]. Note that most of the proofs are different from [45], following the lines of [59, §5]. Theorem 12.1.13 seems to be new in the non-degenerate case.

12.2 Self-duality of irreducible modules

In this section we will state several useful properties of irreducible modules of affine Hecke algebras and their cyclotomic quotients. We recall the definition of an important antiautomorphism of $H_n(q)$. This gives a duality on $H_n(q) - \mathrm{mod}$, which reduces to a duality for its cyclotomic quotients. In particular we give a proof of the well-known fact that the irreducible modules of $H_n(q)$, as well as those of its quotients, are self-dual with respect to this duality.

One defines an antiautomorphism τ on $H_n(q)$ as follows. On the generators of $H_n(q)$, we set:

$$T_i \mapsto T_i, \quad x_j \mapsto x_j,$$

for $1 \le i \le n-1$, and $1 \le j \le n$. Using Theorem 9.3.1, it is easy to check that this defines an antiautomorphism of $H_n(q)$. Note that for $w \in W(n)$ and $f \in P_n$ we have that $\tau(T_w) = T_{w^{-1}}$ and $\tau(f) = f$.
Let $M \in H_n(q)-\mathrm{mod}$. Then we can make the dual $M^* := \mathrm{Hom}_F(M, F) \in H_n(q)^{\mathrm{op}}-$mod into an $H_n(q)$-module by defining $(hf)(m) = f(\tau(h)m)$, for all $f \in M^*$ and all $h \in H_n(q)$. We will denote this module by M^τ and call it the *dual* of M. In this way we obtain a duality

$$\tau^* : H_n(q) - \mathrm{mod} \longrightarrow H_n(q) - \mathrm{mod}, \quad M \mapsto M^\tau, \tag{12.4}$$

such that $\tau^* \circ \tau^* = 1_{H_n(q)-\mathrm{mod}}$.

The following is a key observation, and can be found in [86, Section 5.5].

Theorem 12.2.1. *The map*

$$\mathrm{ch} : \mathcal{K}(H_n(q) - \mathrm{mod}) \longrightarrow \mathcal{K}(P_n - \mathrm{mod}),$$

defined by $[M] \mapsto \sum_{\underline{a} \in S^n} r_{\underline{a}}[L(a_1) \boxtimes \ldots \boxtimes L(a_n)]$*, is injective.*

This already gives the following:

Corollary 12.2.2. *If* $M \in H_n(q) - \mathrm{mod}$ *is irreducible, then* $M^\tau \cong M$*.*

Proof. Since $\tau(x_i) = x_i$, $1 \leq i \leq n$, it leaves characters invariant. Since, by Theorem 12.2.1, the irreducible modules of $H_n(q)$ are determined up to isomorphism by their characters, it leaves irreducible modules invariant. □

From now on we will assume that all eigenvalues of the operators $x_1, \ldots, x_n$ are in the set $\{q^i \mid i \in I\}$, where $I = \{0, \ldots, l-1\}$ and $l \in \mathbb{Z}_+$ denotes the order of q. If $a = q^i$, for some $i \in I$, we will refer to it just as i.
For $i \in I$, recall the definition of the functors e_i^Λ and f_i^Λ from Chapter 11. There it was shown that for $\Lambda \in P^+$, the functors e_i^Λ and f_i^Λ are left and right adjoint to each other.
Next, we want to state another important property of these functors. As we have defined them, the parameters of the cyclotomic Hecke algebras $H_n^\Lambda(q)$ are all different from zero. It is shown in [66, Corollary 5.5] that in this case, $H_n^\Lambda(q)$ is a symmetric algebra.
As defined in Chapter 11, we denote by $\mathrm{ev}_\Lambda : H_n(q) \to H_n^\Lambda(q)$ the corresponding quotient map. Furthermore, for $M \in H_n^\Lambda(q) - \mathrm{mod}$, we will denote by $\mathrm{ev}_\Lambda^*(M)$ the inflation of M along ev_Λ. For the next proposition, observe that the anti-automorphism τ leaves the kernel $\ker(\mathrm{ev}_\Lambda)$ of ev_Λ invariant.

Proposition 12.2.3. *The functors* e_i*,* e_i^Λ *and* f_i^Λ *commute with the duality given by* τ*.*

Proof. Note that from the duality τ^*, we obtain a duality $\mathrm{pr}_\Lambda \circ \tau^* \circ \mathrm{ev}_\Lambda^* : H_n^\Lambda(q) - \mathrm{mod} \to H_n^\Lambda(q) - \mathrm{mod}$, for each $\Lambda \in P^+$. We will also denote the latter duality by τ^*, and refer to it as the cyclotomic duality. From the usual embedding of $H_{n-1}(q)$ into $H_n(q)$, it follows that the following diagram commutes:

$$\begin{array}{ccc} H_n(q) - \mathrm{mod} & \xrightarrow{\tau^*} & H_n(q) - \mathrm{mod} \\ \Big\downarrow \mathrm{Res} & & \Big\downarrow \mathrm{Res} \\ H_{n-1}(q) - \mathrm{mod} & \xrightarrow{\tau^*} & H_{n-1}(q) - \mathrm{mod}. \end{array}$$

Hence, the functors $\mathrm{Res}^{H_n(q)}_{H_{n-1}(q)}$ and $\mathrm{Res}^{H^\Lambda_n(q)}_{H^\Lambda_{n-1}(q)}$ commute with duality. Moreover, if we replace $H_n(q)$ by $H^\Lambda_n(q)$, and $H_{n-1}(q)$ by $H^\Lambda_{n-1}(q)$, and use the cyclotomic duality in the above diagram, then this new diagram remains commutative.

By (9.36), $e_i(M)$ is a direct summand of $\mathrm{Res}^{H_n(q)}_{H_{n-1}(q)}(M)$. Moreover, the duality τ^* leaves central characters invariant. Therefore, by the Krull–Schmidt Theorem, we get an isomorphism $e_i(M^\tau) \cong (e_i(M))^\tau$. The same argument holds for the cyclotomic functors.

Note that $\mathrm{Res}^{H^\Lambda_n(q)}_{H^\Lambda_{n-1}(q)}$ is right adjoint to $\mathrm{Ind}^{H^\Lambda_n(q)}_{H^\Lambda_{n-1}(q)}$ and left adjoint to the functor $\mathrm{Hom}_{H^\Lambda_{n-1}(q)}(H^\Lambda_n(q), -)$. From Chapter 11 we know that as functors

$$\mathrm{Ind}^{H^\Lambda_n(q)}_{H^\Lambda_{n-1}(q)}(-) \cong \mathrm{Hom}_{H^\Lambda_{n-1}(q)}(H^\Lambda_n(q), -).$$

Using the fact that τ^* is a duality, we see that

$$\begin{aligned}
\mathrm{Hom}_{H^\Lambda_n(q)}((\tau^* \circ \mathrm{Ind}^{H^\Lambda_n(q)}_{H^\Lambda_{n-1}(q)} \circ \tau^*)(X), Y) &\cong \mathrm{Hom}_{H^\Lambda_n(q)}(\tau^*(Y), \mathrm{Ind}^{H^\Lambda_n(q)}_{H^\Lambda_{n-1}(q)}(\tau^*(X))) \\
&\cong \mathrm{Hom}_{H^\Lambda_{n-1}(q)}(\mathrm{Res}^{H^\Lambda_n(q)}_{H^\Lambda_{n-1}(q)}(\tau^*(Y)), \tau^*(X)) \\
&\cong \mathrm{Hom}_{H^\Lambda_{n-1}(q)}(X, \tau^*(\mathrm{Res}^{H^\Lambda_n(q)}_{H^\Lambda_{n-1}(q)}(\tau^*(Y)))) \\
&\cong \mathrm{Hom}_{H^\Lambda_{n-1}(q)}(X, \mathrm{Res}^{H^\Lambda_n(q)}_{H^\Lambda_{n-1}(q)}(Y)),
\end{aligned}$$

using again the commutativity of the above diagram. Hence, the functor $\tau^* \circ \mathrm{Ind}^{H^\Lambda_n(q)}_{H^\Lambda_{n-1}(q)} \circ \tau^*$ is also a left adjoint of $\mathrm{Res}^{H^\Lambda_n(q)}_{H^\Lambda_{n-1}(q)}$, and so, by uniqueness of adjoints, there is a natural equivalence of functors $\mathrm{Ind}^{H^\Lambda_n(q)}_{H^\Lambda_{n-1}(q)} \cong \tau^* \circ \mathrm{Ind}^{H^\Lambda_n(q)}_{H^\Lambda_{n-1}(q)} \circ \tau^*$. The result now follows since τ^* preserves central characters. □

Corollary 12.2.4. *The functors $(e^\Lambda_i)^r$ and $(f^\Lambda_i)^r$ are left and right adjoint to one another, and commute with duality.*

Proof. For $r = 1$, the statement is precisely Theorem 11.2.7. For an integer $r > 1$, $M \in H^\Lambda_n(q) - \mathrm{mod}$, and $N \in H^\Lambda_{n+r}(q) - \mathrm{mod}$, it follows from this that

$$\begin{aligned}
\mathrm{Hom}_{H^\Lambda_{n+r}(q)}((f^\Lambda_i)^r(M), N) &= \mathrm{Hom}_{H^\Lambda_{n+r}(q)}(f^\Lambda_i((f^\Lambda_i)^{r-1}(M)), N) \\
&\cong \mathrm{Hom}_{H^\Lambda_{n+r-1}(q)}((f^\Lambda_i)^{r-1}(M), (e^\Lambda_i)(N)).
\end{aligned}$$

Inductively, we get that

$$\mathrm{Hom}_{H^\Lambda_{n+r}(q)}((f^\Lambda_i)^r(M), N) \cong \mathrm{Hom}_{H^\Lambda_n(q)}(M, (e^\Lambda_i)^r(N)).$$

This proves that $(f^\Lambda_i)^r$ is left adjoint to $(e^\Lambda_i)^r$. Right adjointness is proven similarly. For the second part, observe that for $r = 1$, this is the content of the previous proposition. Let $r > 1$, and suppose that the statement is true for all $t < r$. Then for some $M \in H^\Lambda_n(q) - \mathrm{mod}$, we get

$$\begin{aligned}
(e^\Lambda_i)^r(M^\tau) &= e^\Lambda_i((e^\Lambda_i)^{r-1}(M^\tau)) \\
&\cong e^\Lambda_i(((e^\Lambda_i)^{r-1}(M))^\tau) \\
&\cong (e^\Lambda_i((e^\Lambda_i)^{r-1}(M)))^\tau \\
&= ((e^\Lambda_i)^r(M))^\tau
\end{aligned}$$

by Proposition 12.2.3. The corresponding result for $(f_i^\Lambda)^r$ is proven analogously. $\square$

12.3 Operators on the Grothendieck groups

The goal of this section is to investigate the behaviour of the operators on the Grothendieck groups coming from the functors e_i, e_i^Λ and f_i^Λ, which will give important numerical information. As a consequence we also obtain new information on the operators on the Grothendieck groups induced by the functors $(e_i^\Lambda)^{(r)}$ and $(f_i^\Lambda)^{(r)}$, $r \geq 1$.
We will also give several interpretations of the integer $\varepsilon_a(M)$, for an irreducible $M \in H_n(q) - \mathrm{mod}$, defined in Chapter 12.1. We keep the notation of the previous sections.

Recall the definitions of the full subcategories $H_n(q) - \mathrm{mod}_q$ and $H_n(q) - \mathrm{mod}_q^\Lambda$ of $H_n(q) - \mathrm{mod}$ given in Chapter 11, and note that the functor ev_Λ^* induces an equivalence of categories between $H_n^\Lambda(q) - \mathrm{mod}$ and $H_n(q) - \mathrm{mod}_q^\Lambda$, see Proposition 11.1.4. Then, for $M \in H_n^\Lambda(q) - \mathrm{mod}$ irreducible, we define

$$\tilde{e}_i^\Lambda(M) = (\mathrm{pr}_\Lambda \circ \tilde{e}_i \circ \mathrm{ev}_\Lambda^*)(M) \in H_{n-1}^\Lambda(q) - \mathrm{mod}, \tag{12.5}$$
$$\tilde{f}_i^\Lambda(M) = (\mathrm{pr}_\Lambda \circ \tilde{f}_i \circ \mathrm{ev}_\Lambda^*)(M) \in H_{n+1}^\Lambda(q) - \mathrm{mod}, \tag{12.6}$$

where we assume that $n \geq 1$ in the first case.
Henceforth, we will denote by $B(\infty)$ (resp. $B(\Lambda)$) the set of isomorphism classes of irreducible modules in $H_n(q) - \mathrm{mod}_q$ (resp. $H_n^\Lambda(q) - \mathrm{mod}$). Observe that we may view the set $B(\Lambda)$ as a subset of $B(\infty)$ via the functor ev_Λ^*. Furthermore, for all $i \in I$, we obtain maps

$$\begin{aligned}
\tilde{e}_i &: B(\infty) \longrightarrow B(\infty) \cup \{0\}, \quad [M] \mapsto [\tilde{e}_i(M)],\\
\tilde{f}_i &: B(\infty) \longrightarrow B(\infty), \quad [M] \mapsto [\tilde{f}_i(M)],\\
\tilde{e}_i^\Lambda &: B(\Lambda) \longrightarrow B(\Lambda) \cup \{0\}, \quad [M] \mapsto [\tilde{e}_i^\Lambda(M)],\\
\tilde{f}_i^\Lambda &: B(\Lambda) \longrightarrow B(\Lambda) \cup \{0\}, \quad [M] \mapsto [\tilde{f}_i^\Lambda(M)].
\end{aligned}$$

By [44, Lemma 9.3], $\tilde{e}_i^\Lambda$ is the restriction of $\tilde{e}_i$ from $B(\infty)$ to $B(\Lambda)$, i.e.,

$$(\mathrm{pr}_\Lambda \circ \tilde{e}_i \circ \mathrm{ev}_\Lambda^*)(M) = (\tilde{e}_i \circ \mathrm{ev}_\Lambda^*)(M)$$

for an irreducible $M \in H_n^\Lambda(q) - \mathrm{mod}$. Unfortunately, this is not the case for $\tilde{f}_i^\Lambda$. The problem here is that $\tilde{f}_i$ does not leave $B(\Lambda)$ invariant: For an irreducible $M \in H_n^\Lambda(q) - \mathrm{mod}$, it can happen that $\tilde{f}_i^\Lambda(M) = 0$, whereas $\tilde{f}_i(M)$ is never zero. However, there is the following result due to I. Grojnowski complementing Chapter 12.1:

Theorem 12.3.1. *Let M be irreducible in $H_n(q) - \mathrm{mod}_q$. Then $\tilde{e}_i(M)$ and $\tilde{f}_i(M)$ are either zero or irreducible. Moreover, if $N \neq 0$, then $\tilde{e}_i(M) \cong N$ if and only if $\tilde{f}_i(N) \cong M$. The same holds for the maps $\tilde{e}_i^\Lambda$ and $\tilde{f}_i^\Lambda$.*

Proof. This is [44, Theorem 9.4]. □

Note that for M in $H_n^\Lambda(q)-\mathrm{mod}$, it follows from Theorem 12.1.8 that $\mathrm{soc}(e_i^\Lambda(M))$ is either zero or isomorphic to $\tilde{e}_i^\Lambda(M)$. Moreover, by [44, Lemma 9.2, Theorem 9.9], also $\mathrm{hd}(f_i^\Lambda(M))$ is either zero or isomorphic to $\tilde{f}_i^\Lambda(M)$. Then we have:

Corollary 12.3.2. *For $M \in H_n^\Lambda(q)-\mathrm{mod}$ irreducible, we have that $e_i^\Lambda(M)$ and $f_i^\Lambda(M)$ are indecomposable or zero.*

Proof. This is clear since by Theorem 12.3.1, the modules $\tilde{e}_i^\Lambda(M)$ and $\tilde{f}_i^\Lambda(M)$ are either irreducible or zero, hence, $e_i^\Lambda(M)$ and $f_i^\Lambda(M)$ are indecomposable. □

Corollary 12.3.3. *Suppose that $M \in H_n^\Lambda(q)-\mathrm{mod}$ is irreducible. Then the head of $\mathrm{Ind}_{H_n^\Lambda(q)}^{H_{n+1}^\Lambda(q)}(M)$ and the socle of $\mathrm{Res}_{H_n^\Lambda(q)}^{H_{n+1}^\Lambda(q)}(M)$ are multiplicity free.*

Proof. See [44, Corollary 9.10]. For e_i^Λ this follows from Corollary 12.1.9. □

One also has the following remarkable fact:

Proposition 12.3.4. *Let $M \in H_n(q)-\mathrm{mod}_q$, and $N \in H_n^\Lambda(q)-\mathrm{mod}$ be irreducible. Then:*

(i) $\mathrm{soc}(e_i(M)) \cong \mathrm{hd}(e_i(M))$.

(ii) $\mathrm{soc}(f_i^\Lambda(M)) \cong \mathrm{hd}(f_i^\Lambda(M))$.

Proof. This is [44, Proposition 9.12]. □

For an irreducible $M \in H_n^\Lambda(q)-\mathrm{mod}$, we define:

$$\varepsilon_i^\Lambda(M) = \max\{m \geq 0 \mid (\tilde{e}_i^\Lambda)^m(M) \neq 0\}, \tag{12.7}$$
$$\varphi_i^\Lambda(M) = \max\{m \geq 0 \mid (\tilde{f}_i^\Lambda)^m(M) \neq 0\}. \tag{12.8}$$

Next, we recall from Chapter 11 that, for a positive integer r, and $i \in I$,

$$e_i^r \cong \bigoplus_{r!} e_i^{(r)} \tag{12.9}$$

as functors from the category of finite-dimensional $H_n(q)$-modules to the category of finite-dimensional $H_{n-r}(q)$-modules. Moreover, if we descend to some quotient $H_n^\Lambda(q)$, $\Lambda \in P^+$, the same statement for the functors $(e_i^\Lambda)^r$ and $(e_i^\Lambda)^{(r)}$ holds true. By adjointness, for a positive integer r, we also get that

$$(f_i^\Lambda)^r \cong \bigoplus_{r!} (f_i^\Lambda)^{(r)} \tag{12.10}$$

as functors from the category of finite dimensional $H_n^\Lambda(q)$-modules to the category of finite dimensional $H_{n+r}^\Lambda(q)$-modules, see Chapter 11.

One defines the abelian groups

$$\mathcal{K}(\infty) := \bigoplus_{n\geq 0} \mathcal{K}(H_n(q) - \mathrm{mod}_q),$$

and

$$\mathcal{K}(\Lambda) := \bigoplus_{n\geq 0} \mathcal{K}(H_n^\Lambda(q) - \mathrm{mod}).$$

Then the exact functors e_i, $e_i^{(r)}$, e_i^Λ, f_i^Λ, $(e_i^\Lambda)^{(r)}$, $(f_i^\Lambda)^{(r)}$, $i \in I$, $r > 0$, induce linear operators on these groups in the obvious way and one has the following:

Lemma 12.3.5. *Let $i \in I$. As operators on $\mathcal{K}(\infty)$ and $\mathcal{K}(\Lambda)$, we have that $e_i^r = (r!)e_i^{(r)}$, $(e_i^\Lambda)^r = (r!)(e_i^\Lambda)^{(r)}$ and $(f_i^\Lambda)^r = (r!)(f_i^\Lambda)^{(r)}$, for all positive integers r.*

We now focus on the behaviour of the functors e_i^Λ and f_i^Λ on the Grothendieck group. The first result to mention is the following, which is due to I. Grojnowski and M. Vazirani:

Theorem 12.3.6. *Let $M \in H_n(q) - \mathrm{mod}_q$ be irreducible and $i \in I$. Then:*

(i) $[e_i(M)] = \varepsilon_i(M)[\tilde{e}_i(M)] + \sum u_k[N_k]$, *where $N_k \in H_{n-1}(q)-\mathrm{mod}$ are irreducible such that $\varepsilon_i(N_k) < \varepsilon_i(\tilde{e}_i(M)) = \varepsilon_i(M) - 1$, for all k.*

(ii) *$\varepsilon_i(M)$ is the maximal size of a Jordan block of x_n on M with eigenvalue i.*

(iii) *The algebra $\mathrm{End}_{H_{n-1}(q)}(e_i(M))$ is isomorphic to the algebra $F[x]/(x^{\varepsilon_i(M)})$, and thus,*

$$\varepsilon_i(M) = \dim_F \mathrm{End}_{H_{n-1}(q)}(e_i(M)).$$

Proof. This is [44, Theorem 9.13], together with Claim 6.5 in [86]. ☐

We have the following corollary:

Corollary 12.3.7. *Let $M, N \in H_n(q) - \mathrm{mod}$ be irreducible and $M \ncong N$. Then, for every $i \in I$, we have $\mathrm{Hom}_{H_{n-1}(q)}(e_i(M), e_i(N)) = 0$.*

Proof. Suppose there is some non-zero element in $\mathrm{Hom}_{H_{n-1}(q)}(e_i(M), e_i(N))$. By Theorem 12.1.8 combined with Proposition 12.3.4, we see that the head of $e_i(M)$ is irreducible. Thus, $\tilde{e}_i(M)$ is a composition factor of $e_i(N)$. By part (i) of the previous theorem, $\varepsilon_i(N) \geq \varepsilon_i(M)$. On the other hand, again by Theorem 12.1.8, the socle of $e_i(N)$ is irreducible, hence a composition factor of $e_i(M)$. Again, by the previous theorem, we conclude that $\varepsilon_i(M) \geq \varepsilon_i(N)$, and so $\varepsilon_i(M) = \varepsilon_i(N)$ which implies $\tilde{e}_i(M) \cong \tilde{e}_i(N)$. From Theorem 12.3.1 it follows that $\tilde{e}_i(M) \cong \tilde{e}_i(N)$ if and only if $M \cong N$. This gives the result. ☐

Remark 12.3.8. Since, on finite-dimensional $H_n^\Lambda(q)$-modules, e_i^Λ is just the restriction of e_i, Proposition 12.3.4, Theorem 12.3.6 and Corollary 12.3.7 are still true if one replaces $e_i, \tilde{e}_i, \varepsilon_i$ by $e_i^\Lambda, \tilde{e}_i^\Lambda, \varepsilon_i^\Lambda$ in the corresponding statements.

Also, from (12.9) we obtain more information about the $H_{n-r}(q)$-module $e_i^{(r)}(M)$ in the case where M is an irreducible $H_n^\Lambda(q)$-module.

Proposition 12.3.9. *Let $M \in H_n(q) - \mathrm{mod}$, $N \in H_n^\Lambda(q) - \mathrm{mod}_q$ be irreducible and $i \in I$. Then:*

(i) $e_i^{(r)}(M) \neq 0$ *if and only if* $\tilde{e}_i^r(M) \neq 0$. *In that case,* $e_i^{(r)}(M)$ *is an indecomposable self-dual* $H_{n-r}^\Lambda(q)$*-module with irreducible head and socle isomorphic to* $\tilde{e}_i^r(M)$.

(ii) $(f_i^\Lambda)^{(r)}(N) \neq 0$ *if and only if* $(\tilde{f}_i^\Lambda)^r(N) \neq 0$. *In that case,* $(f_i^\Lambda)^{(r)}(N)$ *is an indecomposable self-dual* $H_{n-r}^\Lambda(q)$*-module with irreducible head and socle isomorphic to* $(\tilde{f}_i^\Lambda)^r(N)$.

Proof. From (12.9), we see that $e_i^r(M) \neq 0$ if and only if $e_i^{(r)}(M) \neq 0$. Furthermore, by Theorem 12.1.13, the socle of $e_i^r(M)$ is isomorphic to $\bigoplus_{r!} \tilde{e}_i^r(M)$. Thus, $\mathrm{soc}(e_i^{(r)}(M)) \cong \tilde{e}_i^r(M)$, and $e_i^{(r)}(M)$ is indecomposable. Moreover, by Corollary 12.2.2, M is a self-dual $H_n(q)$-module. Then, by Corollary 12.2.4, we have that $e_i^r(M)$ is a self-dual $H_{n-r}(q)$-module. By the Krull–Schmidt Theorem, it follows that $e_i^{(r)}(M)$ is a self-dual module. This immediately implies that $\mathrm{hd}(e_i^{(r)}(M)) \cong \tilde{e}_i^r(M)$, and (i) follows.
For (ii), the first part is proven similarly as the first part of (i). By adjointness of $(f_i^\Lambda)^r$ and $(e_i^\Lambda)^r$, see Corollary 12.2.4, we have for an irreducible $H_{n+r}^\Lambda(q)$-module N that

$$\mathrm{Hom}_{H_{n+r}^\Lambda(q)}((f_i^\Lambda)^r(M), N) \cong \mathrm{Hom}_{H_n^\Lambda(q)}(M, (e_i^\Lambda)^r(N)).$$

By Theorem 12.1.13 the latter is non-zero if and only if $M \cong (\tilde{e}_i^\Lambda)^r(N)$, which is equivalent to $N \cong (\tilde{f}_i^\Lambda)^r(M)$, by Theorem 12.3.1. Therefore,

$$\mathrm{hd}((f_i^\Lambda)^r(M)) \cong \bigoplus_{r!} (\tilde{f}_i^\Lambda)^r(M),$$

and then $\mathrm{hd}((f_i^\Lambda)^{(r)}(M)) \cong (\tilde{f}_i^\Lambda)^r(M)$. By the self-duality of $(f_i^\Lambda)^{(r)}(M)$, we also see that the socle of $(f_i^\Lambda)^{(r)}(M)$ is isomorphic to $(\tilde{f}_i^\Lambda)^r(M)$. □

The next result is a direct consequence of Theorem 12.3.6.

Proposition 12.3.10. *Let $M, N \in H_n(q) - \mathrm{mod}_q$, such that $M \ncong N$, $i \in I$, and set $\varepsilon := \varepsilon_i(M)$. Then for a positive integer $r \leq \varepsilon$ the following hold:*

(i) $[(e_i)^{(r)}(M)] = \binom{\varepsilon}{r}[\tilde{e}_i^r(M)] + \sum u_r[N_r]$, *where the N_r are irreducible $H_{n-r}(q)$-modules with $\varepsilon_i(N_r) < \varepsilon_i(\tilde{e}_i^r(M)) = \varepsilon - r$.*

(ii) $\mathrm{Hom}_{H_{n-r}(q)}(e_i^{(r)}(M), e_i^{(r)}(N)) = 0$.

Proof. To show (i), we proceed by induction. For $r = 1$, this is Theorem 12.3.6 (i) since $(e_i)^{(1)} \cong e_i$. Suppose the statement is true for all $1 \leq t < r$. To avoid confusion, in the following we will write $\hat{e}_i$ for the operator on $\mathcal{K}(\infty)$ induced by e_i. Then, using Lemma 12.3.5, we have:

$$(r!)[e_i^{(r)}(M)] = [e_i^r(M)] = [e_i(e_i^{r-1}(M))] = \hat{e}_i([e_i^{r-1}(M)]).$$

Again, by Lemma 12.3.5 and the inductive hypothesis, we see that

$$\begin{aligned} \hat{e}_i([e_i^{r-1}(M)]) &= \hat{e}_i((r-1)!\binom{\varepsilon}{r-1}[\tilde{e}_i^{r-1}(M)] + \sum u_{r-1}[N_{r-1}]) \\ &= (r-1)!\binom{\varepsilon}{r-1}[e_i(\tilde{e}_i^{r-1}(M))] + \sum u_{r-1}[e_i(N_{r-1})], \end{aligned}$$

where the $N_{r-1} \in H_{n-r+1}(q) - \mathrm{mod}_q$ are irreducible with $\varepsilon_i(N_{r-1}) < \varepsilon_i(\tilde{e}_i^{r-1}(M)) = \varepsilon - r + 1$. By part (i) of Theorem 12.3.6, the second summand in the equation above ranges over summands $[N'_k] \in \mathcal{K}(H_{n-r}(q) - \mathrm{mod})$ with $\varepsilon_i(N'_k) < \varepsilon - r$. Moreover, $\tilde{e}_i^{r-1}(M)$ is an irreducible $H_{n-r+1}(q)$-module, and hence, we see that

$$(r-1)!\binom{\varepsilon}{r-1}[e_i(\tilde{e}_i^{r-1}(M))] = (r-1)!\binom{\varepsilon}{r-1}\varepsilon_i(\tilde{e}_i^{r-1}(M))[\tilde{e}_i^r(M)] + \sum v_k[L_k],$$

where the L_k are irreducible $H_{n-r}(q)$-modules such that $\varepsilon_i(L_k) < \varepsilon_i(\tilde{e}_i^r(M)) = \varepsilon - r$. Now $\varepsilon_i(\tilde{e}_i^{r-1}(M)) = \varepsilon - r + 1$, and therefore

$$(r-1)!\binom{\varepsilon}{r-1}\varepsilon_i(\tilde{e}_i^{r-1}(M)) = r!\binom{\varepsilon}{r}.$$

From this, part (i) follows.

The proof of (ii) is similar to Corollary 12.3.7, using (i) and the fact that, by Proposition 12.3.9 (i), the socle of $e_i^{(r)}(M)$ (resp. $e_i^{(r)}(N)$) is isomorphic to $\tilde{e}_i^r(M)$ (resp. $\tilde{e}_i^r(N)$). $\square$

In the rest of this section, we want to establish analogues of Proposition 12.3.6 and 12.3.10 for the functors f_a^Λ and $(f_a^\Lambda)^{(r)}$, for all integers $r > 0$. We will need the following, which is the statement of [44, Theorem 9.15].

Theorem 12.3.11. *Let $M \in H_n^\Lambda(q) - \mathrm{mod}$ be irreducible, $i \in I$. Then the following hold:*

(i) $[f_i^\Lambda(M)] = \varphi_i^\Lambda(M)[\tilde{f}_i^\Lambda(M)] + \sum u_r N_r$, *where the $H_{n+1}^\Lambda(q)$-modules N_r are irreducible such that $\varepsilon_i^\Lambda(N_r) < \varepsilon_i^\Lambda(\tilde{f}_i^\Lambda(M)) = \varepsilon_i^\Lambda(M) + 1$.*

(ii) $\varphi_i^\Lambda(M)$ *is the smallest integer m such that $f_i^\Lambda(M) = \mathrm{pr}_\Lambda(\mathrm{Ind}_{n,1}^{n+1}(M \boxtimes L_m(i)))$.*

(iii) $\varphi_i^\Lambda(M) = \dim_F \mathrm{Hom}_{H_{n+1}^\Lambda(q)}(f_i^\Lambda(M), f_i^\Lambda(M))$.

Proof. See [44, Theorem 9.15]. $\square$

In order to obtain a full analogue of Theorem 12.3.6 for the operator f_i^Λ, we have to show that in (i) we may write φ_i^Λ instead of ε_i^Λ. For this, we use the following lemma, which is [44, Theorem 12.6]. To state it, we need some new notation. For an irreducible $H_n^\Lambda(q)$-module M, we define

$$\begin{aligned} \mathrm{wt}_i(M) &:= \varphi_i^\Lambda(M) - \varepsilon_i^\Lambda(M), & (12.11)\\ \mathrm{wt}(M) &:= \sum_{i\in I} \mathrm{wt}_i(M)\Lambda_i \in P, & (12.12)\end{aligned}$$

see Chapter 11 for definitions. In particular we identify a central character γ with the element $\gamma = \sum_{i\in I}\gamma_i\alpha_i \in \Gamma_n$.

Lemma 12.3.12. *Let $M \in H_n^\Lambda(q) - \mathrm{mod}$ be irreducible with central character γ, and let $i \in I$. Then:*

(i) *If $f_i^\Lambda(M) \neq 0$, then $\mathrm{wt}(\tilde{f}_i^\Lambda(M)) = \mathrm{wt}(M) - \alpha_i$.*

(ii) *$\mathrm{wt}_i(\mathbf{1}_\Lambda) = \langle \Lambda, h_i\rangle$, for the simple coroot h_i. Here, $\mathbf{1}_\Lambda$ denotes the unique irreducible $H_0^\Lambda(q)$-module.*

(iii) *$\mathrm{wt}_i(M) = \varphi_i^\Lambda(M) - \varepsilon_i^\Lambda(M) = \langle \Lambda - \gamma, h_i\rangle$.*

Proof. This is [44, Theorem 12.6]. □

As a consequence, we can complete our argument. Suppose that we are in the situation of Theorem 12.3.11. The module $\tilde{f}_i^\Lambda(M)$, and all the irreducible modules N_r are composition factors of $f_i^\Lambda(M)$, thus, by Theorem 12.3.1 and the indecomposability of $f_i^\Lambda(M)$, we see that they have the same central character. Therefore, by Lemma 12.3.12 (iii),

$$\begin{aligned} \varphi_i^\Lambda(N_r) &= \langle \Lambda - \gamma, h_i\rangle + \varepsilon_i^\Lambda(N_r),\\ \varphi_i^\Lambda(\tilde{f}_i^\Lambda(M)) &= \langle \Lambda - \gamma, h_i\rangle + \varepsilon_i^\Lambda(\tilde{f}_i^\Lambda(M)).\end{aligned}$$

Therefore,

$$\varphi_i^\Lambda(\tilde{f}_i^\Lambda(M)) - \varphi_i^\Lambda(N_r) = \varepsilon_i^\Lambda(\tilde{f}_i^\Lambda(M)) - \varepsilon_i^\Lambda(N_r) > 0,$$

by part (i) of Theorem 12.3.11. Furthermore, by the proof of [44, Theorem 12.6], we have that

$$\mathrm{wt}_i(\tilde{f}_i^\Lambda(N)) = \mathrm{wt}_i(N) - 2,$$

for all irreducible $H_n^\Lambda(q)$-modules N. Thus,

$$\varphi_i^\Lambda(\tilde{f}_i^\Lambda(M)) - \varepsilon_i^\Lambda(\tilde{f}_i^\Lambda(M)) = \varphi_i^\Lambda(M) - \varepsilon_i^\Lambda(M) - 2,$$

and, hence,

$$\varphi_i^\Lambda(\tilde{f}_i^\Lambda(M)) = \varphi_i^\Lambda(M) - 1$$

since $\varepsilon_i^\Lambda(\tilde{f}_i^\Lambda(M)) = \varepsilon_i^\Lambda(M) + 1$, by Theorem 12.3.11. Then, as a consequence, we can restate Theorem 12.3.11:

Theorem 12.3.13. *Let $M \in H_n^\Lambda(q)-\mathrm{mod}$ be irreducible. Then the following hold:*

(i) $[f_i^\Lambda(M)] = \varphi_i^\Lambda(M)[\tilde{f}_i^\Lambda(M)] + \sum u_r N_r$, *for irreducible $H_{n+1}^\Lambda(q)$-modules N_r satisfying $\varphi_i^\Lambda(N_r) < \varphi_i^\Lambda(\tilde{f}_i^\Lambda(M)) = \varphi_i^\Lambda(M) - 1$.*

(ii) $\varphi_i^\Lambda(M)$ *is the smallest integer m such that $f_i^\Lambda(M) = \mathrm{pr}_\Lambda(\mathrm{Ind}_{n,1}^{n+1}(M \boxtimes L_m(i)))$.*

(iii) $\varphi_i^\Lambda(M) = \dim_F \mathrm{Hom}_{H_{n+1}^\Lambda(q)}(f_i^\Lambda(M), f_i^\Lambda(M))$.

Finally, we can give the full analogue of Proposition 12.3.10:

Proposition 12.3.14. *Let $M, N \in H_n^\Lambda(q) - \mathrm{mod}$, such that $M \ncong N$, and set $\varphi := \varphi_i^\Lambda(M)$. Then for a positive integer $r \leq \varphi$ the following hold:*

(i) $[(f_i^\Lambda)^{(r)}(M)] = \binom{\varphi}{r}[(\tilde{f}_i^\Lambda)^r(M)] + \sum u_r[N_r]$, *where the $H_{n+r}^\Lambda(q)$-modules N_r are irreducible with $\varphi_i^\Lambda(N_r) < \varphi_i^\Lambda((\tilde{f}_i^\Lambda)^r(M)) = \varphi - r$.*

(ii) $\mathrm{Hom}_{H_{n+r}^\Lambda(q)}((f_i^\Lambda)^{(r)}(M), (f_i^\Lambda)^{(r)}(N)) = 0$.

Proof. For $r = 1$, this is the content of Theorem 12.3.13 (i) since $(f_i^\Lambda)^{(1)} \cong f_i^\Lambda$. Let $1 < r \leq \varphi$, and suppose that the statement is true for all $1 \leq t < r$. Denote by $\hat{f}_i^\Lambda$ the operator on $\mathcal{K}(\Lambda)$ induced by f_i^Λ. With Lemma 12.3.5, we get that

$$r![(f_i^\Lambda)^{(r)}(M)] = [(f_i^\Lambda)^r(M)] = [f_i^\Lambda((f_i^\Lambda)^{r-1}(M))] = \hat{f}_i^\Lambda([(f_i^\Lambda)^{r-1}(M)]).$$

Then, by the inductive hypothesis, we have that

$$\begin{aligned}\hat{f}_i^\Lambda([(f_i^\Lambda)^{r-1}(M)]) &= \hat{f}_i^\Lambda((r-1)!\binom{\varphi}{r-1}[(\tilde{f}_i^\Lambda)^{r-1}(M)] + \sum u_{r-1}[N_{r-1}]) \\ &= (r-1)!\binom{\varphi}{r-1}[f_i^\Lambda((\tilde{f}_i^\Lambda)^{r-1}(M))] + \sum u_{r-1}[f_i^\Lambda(N_{r-1})],\end{aligned}$$

with irreducible $H_{n+r-1}^\Lambda(q)$-modules N_{r-1} such that $\varphi_i^\Lambda(N_{r-1}) < \varphi_i^\Lambda((\tilde{f}_i^\Lambda)^{r-1}(M)) = \varphi - r + 1$. By part (i) of Theorem 12.3.13, the last summand ranges over elements $[N_k'] \in \mathcal{K}(H_{n+r}^\Lambda(q) - \mathrm{mod})$ with $\varphi_i^\Lambda(N_k') < \varphi - r$. Moreover, by Theorem 12.3.1, we have that $(\tilde{f}_i^\Lambda)^{r-1}(M)$ is irreducible since, by assumption, $(\tilde{f}_i^\Lambda)^{r-1}(M) \neq 0$. Using Theorem 12.3.13 again, we have that the first term of the last sum equals

$$(r-1)!\binom{\varphi}{r-1}\varphi_i^\Lambda((\tilde{f}_i^\Lambda)^{r-1}(M))[(\tilde{f}_i^\Lambda)^r(M)] + \sum v_k[L_k],$$

where all irreducible $H_{n+r}^\Lambda(q)$-modules L_k satisfy $\varphi_i^\Lambda(L_k) < \varphi - r$. Since

$$(r-1)!\binom{\varphi}{r-1}(\varphi - r + 1) = r!\binom{\varphi}{r},$$

part (i) follows.
For (ii), suppose that $f \in \mathrm{Hom}_{H_{n+r}^\Lambda(q)}((f_i^\Lambda)^{(r)}(M), (f_i^\Lambda)^{(r)}(N))$. By Proposition 12.3.9 (ii), we know that $(f_i^\Lambda)^{(r)}(M)$ and $(f_i^\Lambda)^{(r)}(N)$ are self-dual modules with

irreducible head and socle isomorphic to $(\tilde{f}_i^\Lambda)^r(M)$ and $(\tilde{f}_i^\Lambda)^r(N)$. If f is non-zero, $(\tilde{f}_i^\Lambda)^r(M)$ is a composition factor of $(f_i^\Lambda)^{(r)}(N)$, and thus, by part (i),

$$\varphi_i^\Lambda((\tilde{f}_i^\Lambda)^r(M)) \leq \varphi_i^\Lambda((\tilde{f}_i^\Lambda)^r(N)).$$

Similarly, $(\tilde{f}_i^\Lambda)^r(N)$ must be a composition factor of $(\tilde{f}_i^\Lambda)^{(r)}(M)$, hence, we may conclude that

$$\varphi_i^\Lambda((\tilde{f}_i^\Lambda)^r(N)) \leq \varphi_i^\Lambda((\tilde{f}_i^\Lambda)^r(M)),$$

which implies

$$\varphi_i^\Lambda((\tilde{f}_i^\Lambda)^r(M)) = \varphi_i^\Lambda((\tilde{f}_i^\Lambda)^r(N)).$$

Again by (i), this is only possible if $(\tilde{f}_i^\Lambda)^r(M) \cong (\tilde{f}_i^\Lambda)^r(N)$. Then, Theorem 12.3.1 implies $M \cong N$. $\square$

Remark 12.3.15. Note that Proposition 12.3.10 and Proposition 12.3.14 are analogues of the results given in [59, Proposition 8.5.10].

Chapter 13

Action of $\mathfrak{g}'(A)$ on the Grothendieck group

The aim of this chapter is to explain how the sum over the Grothendieck groups of finite-dimensional $H_n^\Lambda(q)$-modules, $n \geq 0$, can be given the structure of a $\mathfrak{g}'(A)$-module, where A denotes the generalized Cartan matrix of type $A_{l-1}^{(1)}$, $l \geq 2$. Moreover, we explain the main result given in [44], which states that this module is precisely the irreducible highest-weight module over $\mathfrak{g}'(A)$ associated to the highest weight Λ.

We will follow the lines of [59, §9], where the result was established in the degenerate case.

As in previous chapters, $H_n(q)$ denotes the affine Hecke algebra over the field F, where the defining parameter q is invertible in F. Moreover, we assume that $q^l = 1$ for some positive integer $l \geq 2$, where l is minimal with this property. We set $I := \{0, \ldots, l-1\}$.

Furthermore, we denote by A the generalized Cartan matrix of type $A_{l-1}^{(1)}$, and let $\mathfrak{g}'(A)$ be the derived algebra of the Kac–Moody algebra defined as in Chapter 10 over the field $K = \mathbb{Q}$ of rational numbers. Moreover, we denote by U' the universal enveloping algebra $U(\mathfrak{g}'(A))$ of $\mathfrak{g}'(A)$. Also we denote by $U'_{\mathbb{Z}}$ the subalgebra of U' generated by the *divided powers* $e_i^{(r)} := \frac{e_i^r}{r!}$ and $f_i^{(r)} := \frac{f_i^r}{r!}$, $i \in I$.

13.1 Shapovalov form

For $\Lambda \in P_+$, recall from (11.1) of Chapter 11 that $H_n^\Lambda(q)$ is the factor algebra of $H_n(q)$ by the two-sided ideal of $H_n(q)$ generated by the element $\prod_{i \in I}(x_1 - q^i)^{\langle \Lambda, h_i \rangle}$. Moreover, recall from Section 12.3 the definition of the abelian groups $\mathcal{K}(\infty)$ and $\mathcal{K}(\Lambda)$, as well as the sets $B(\Lambda)$ and $B(\infty)$, the union of the sets of isomorphism classes of irreducible $H_n^\Lambda(q)$-modules and irreducible $H_n(q)$-modules in $H_n(q) - \mathrm{mod}_q$, respectively, $n \geq 0$.

In this section, we recall the definition of the *Shapovalov* form, an important bilinear form on $\mathcal{K}(\Lambda)$, $\Lambda \in P_+$. Denote by $H_n^\Lambda(q)-\text{proj}$ the full subcategory of $H_n^\Lambda(q)-\text{mod}$ consisting of projective left $H_n^\Lambda(q)$-modules.
If $P \in H_n^\Lambda(q) - \text{proj}$, then the functor

$$\mathrm{Hom}_{H_n^\Lambda(q)}(P, -) : H_n^\Lambda(q) - \text{mod} \longrightarrow F - \text{mod}$$

is exact. Therefore, we obtain a well-defined bilinear mapping

$$(-,-) : \mathcal{K}(H_n^\Lambda(q) - \text{proj}) \times \mathcal{K}(H_n^\Lambda(q) - \text{mod}) \longrightarrow \mathbb{Z} \tag{13.1}$$

by setting $([P],[M]) = \dim_F \mathrm{Hom}_{H_n^\Lambda(q)}(P, M)$. If $N \in H_n^\Lambda(q) - \text{mod}$ is irreducible, and P_N denotes its projective cover, then $([P_N],[M])$ can also be interpreted as the multiplicity of N in a composition series of M. If N runs through the isomorphism classes of irreducible $H_n^\Lambda(q)$-modules, then the $\mathbb{Z}$-linear maps

$$\delta_N := (P_N, -) : \mathcal{K}(H_n^\Lambda(q) - \text{mod}) \to \mathbb{Z}$$

give a basis of $\mathcal{K}(H_n^\Lambda(q) - \text{mod})^*$ that is dual to the basis of $\mathcal{K}(H_n^\Lambda(q) - \text{mod})$ consisting of the elements $[N]$, $N \in H_n^\Lambda(q) - \text{mod}$ irreducible, one for each isomorphism class. We infer that the bilinear form $(-,-)$ then gives an isomorphism of abelian groups

$$\mathcal{K}(H_n^\Lambda(q) - \text{proj}) \longrightarrow \mathcal{K}(H_n^\Lambda(q) - \text{mod})^*, \tag{13.2}$$

which is determined by $P_N \mapsto \delta_N$, for $N \in H_n^\Lambda(q) - \text{mod}$ irreducible.
Next consider the map

$$\omega_n : \mathcal{K}(H_n^\Lambda(q) - \text{proj}) \to \mathcal{K}(H_n^\Lambda(q) - \text{mod}), \quad [P] \mapsto \sum [P : N][N], \tag{13.3}$$

where the sum on the right-hand side is taken over all isomorphism classes of irreducible $H_n^\Lambda(q)$-modules. Note that this map is injective if and only if it becomes an isomorphism of $\mathbb{Q}$-vector spaces after extending scalars. And this happens precisely when the bilinear form given in (13.1) is non-degenerate when restricted to

$$\mathcal{K}(H_n^\Lambda(q) - \text{proj}) \times \mathcal{K}(H_n^\Lambda(q) - \text{proj}).$$

The last statement holds as the following theorem shows.

Theorem 13.1.1. *The pairing*

$$(P, Q) \mapsto \dim_F \mathrm{Hom}_{H_n^\Lambda(q)}(P, Q),$$

where $P, Q \in H_n^\Lambda(q) - \text{proj}$*, induces a non-degenerate symmetric bilinear form*

$$(-,-)^* : \mathcal{K}(H_n^\Lambda(q) - \text{mod})^* \times \mathcal{K}(H_n^\Lambda(q) - \text{mod})^* \longrightarrow \mathbb{Z}.$$

Proof. This is precisely [44, Theorem 11.1]. □

Denote by $\mathcal{K}^*(\Lambda)$ the *restricted dual* of $\mathcal{K}(\Lambda)$, i.e., the set of the $\mathbb{Z}$-linear maps

$$f : \mathcal{K}(\Lambda) \to \mathbb{Z}$$

such that f is zero on all but finitely many classes $[N]$, where $N \in H_n^\Lambda(q) - \text{mod}$ is irreducible and $n \geq 0$. Note that this set can be identified with the subset $\bigoplus_{n\geq 0} \mathcal{K}(H_n^\Lambda(q) - \text{mod})^*$ of the dual of $\mathcal{K}(\Lambda)$. Then, under the isomorphism given in (13.2), we also have an identification

$$\mathcal{K}^*(\Lambda) = \bigoplus_{n\geq 0} \mathcal{K}(H_n^\Lambda(q) - \text{proj})$$

in such a way that for each irreducible $H_n^\Lambda(q)$-module N, $n \geq 0$, the basis element δ_N corresponds to the isomorphism class $[P_N]$. Moreover from the map given in (13.3), we obtain a homogeneous map $\omega : \mathcal{K}^*(\Lambda) \to \mathcal{K}(\Lambda)$ with respect to the natural grading of $\mathcal{K}^*(\Lambda)$ and $\mathcal{K}(\Lambda)$. Then one has:

Corollary 13.1.2. *The map $\omega : \mathcal{K}^*(\Lambda) \to \mathcal{K}(\Lambda)$ is injective.*

Proof. This follows from Theorem 13.1.1 and the previous discussion. □

Therefore, we can identify $\mathcal{K}^*(\Lambda)$ with its image under ω, and thus we may view $\mathcal{K}^*(\Lambda) \subseteq \mathcal{K}(\Lambda)$ as $\mathbb{Z}$-sublattices in $\mathcal{K}(\Lambda)_\mathbb{Q} := \mathbb{Q} \otimes_\mathbb{Z} \mathcal{K}(\Lambda)$. By extension of scalars, we obtain an isomorphism

$$\widehat{\omega} := \text{id}_\mathbb{Q} \otimes \omega : (\mathcal{K}^*(\Lambda))_\mathbb{Q} \longrightarrow \mathcal{K}(\Lambda)_\mathbb{Q}$$

of $\mathbb{Q}$-vector spaces. Furthermore, the form given in Theorem 13.1.1 induces a bilinear form

$$(-,-) : \mathcal{K}(H_n^\Lambda(q) - \text{mod})_\mathbb{Q} \times \mathcal{K}(H_n^\Lambda(q) - \text{mod})_\mathbb{Q} \longrightarrow \mathbb{Q}. \tag{13.4}$$

It is non-degenerate and symmetric. The form $(-,-)$, or the form $(-,-)^*$ is called the *Shapovalov form*. Moreover, these forms induce symmetric non-degenerate bilinear forms on $(\mathcal{K}^*(\Lambda))_\mathbb{Q}$ and $\mathcal{K}(\Lambda)_\mathbb{Q}$, respectively, which we will also refer to as the Shapovalov form.

13.2 Action of $\mathfrak{g}'(A)$

For $i \in I$ and $r \geq 1$, recall from Chapter 9 and Chapter 11 the definitions of the various functors $e_i, e_i^\Lambda, f_i^\Lambda, e_i^{(r)}, (e_i^\Lambda)^{(r)}$, and $(f_i^\Lambda)^{(r)}$. They are exact, and therefore additive. Thus, on the level of Grothendieck groups, they descend to homomorphisms of abelian groups, and define actions on $\mathcal{K}(\infty)$ and $\mathcal{K}(\Lambda)$, respectively.
Furthermore, by Proposition 11.2.13, for $i \in I$, the functors e_i^Λ and f_i^Λ take projective modules to projective modules, their action on $\mathcal{K}^*(\Lambda)$ is defined. More precisely,

if $\phi \in \mathcal{K}(H_n^\Lambda(q) - \text{mod})^*$ is given by the map $(P, -)$ with $P \in H_n^\Lambda(q) - \text{proj}$, then we define $e_i^\Lambda(\phi)$ (resp. $f_i^\Lambda(\phi)$) as the map in $\mathcal{K}(H_{n-1}^\Lambda(q) - \text{mod})^*$ (resp. $\mathcal{K}(H_{n+1}^\Lambda(q) - \text{mod})^*$) given by the map $(e_i^\Lambda([P]), -)$ (resp. $(f_i^\Lambda([P]), -)$), where $e_i^\Lambda([P]) \in \mathcal{K}(H_{n-1}^\Lambda(q) - \text{proj})$ (resp. $f_i^\Lambda([P]) \in \mathcal{K}(H_{n+1}^\Lambda(q) - \text{proj})$). By adjointness of e_i^Λ and f_i^Λ, $i \in I$, we have that

$$\phi \circ f_i^\Lambda([M]) = ([P], f_i^\Lambda(M)) = (e_i^\Lambda([P]), [M]),$$

for $[M] \in \mathcal{K}(H_{n-1}^\Lambda(q) - \text{mod})$. Therefore, the action of e_i^Λ (resp. f_i^Λ) on $\mathcal{K}^*(\Lambda)$ is just the transpose of the action of f_i^Λ (resp. e_i^Λ) on $\mathcal{K}(\Lambda)$.
In the sequel we state crucial relations among these various operators. The first of these is the following:

Lemma 13.2.1. *The operators $e_i : \mathcal{K}(\infty) \to \mathcal{K}(\infty)$, and $e_i^\Lambda : \mathcal{K}(\Lambda) \to \mathcal{K}(\Lambda)$, $i \in I$, satisfy the Serre relations:*

(i) *If $q^{i-j} \neq q^{\pm 1}$, then $e_i e_j = e_j e_i$.*

(ii) *If $q^{i-j} \neq q^{\pm 1}$ and $q \neq q^{-1}$, then $e_i^2 e_j + e_j e_i^2 = 2 e_i e_j e_i$.*

(iii) *If $q^{i-j} \neq q^{\pm 1}$ and $q = q^{-1}$, i.e., $q = -1$, then $e_i^3 e_j + 3 e_i e_j e_i^2 = 3 e_i^2 e_j e_i + e_j e_i^3$.*

Proof. This is [44, Proposition 12.1]. □

From the fact that the functors e_i^Λ and f_i^Λ are both left and right adjoint to one another, and $(-, -)$ is non-degenerate, we also obtain an analogue of the previous lemma for the operators f_i^Λ.

Lemma 13.2.2. *The operators $f_i^\Lambda : \mathcal{K}(\Lambda) \to \mathcal{K}(\Lambda)$, $i \in I$, satisfy the Serre relations.*

By Theorem 11.2.14 and Proposition 11.2.18, also the functors $(e_i^\Lambda)^{(r)}$ and $(f_i^\Lambda)^{(r)}$, $i \in I$, $r \geq 1$, take projective $H_n^\Lambda(q)$-modules to projective $H_{n-r}^\Lambda(q)$-modules (resp. $H_{n+r}^\Lambda(q)$-modules), and so, their action on $\mathcal{K}^*(\Lambda)$ is defined as well. Moreover, as shown in Chapter 11, the functors $(e_i^\Lambda)^{(r)}$ and $(f_i^\Lambda)^{(r)}$ are adjoint to one another and, thus, we get:

Lemma 13.2.3. *For all $i \in I$, the operators $(e_i^\Lambda)^{(r)}$ and $(f_i^\Lambda)^{(r)}$ on $\mathcal{K}^*(\Lambda)$ and $\mathcal{K}(\Lambda)$ satisfy*

$$((e_i^\Lambda)^{(r)}([P]), [N]) = ([P], (f_i^\Lambda)^{(r)}([N])), \quad ((f_i^\Lambda)^{(r)}([P]), [N]) = ([P], (e_i^\Lambda)^{(r)}([N])),$$

for all $[P] \in \mathcal{K}^*(\Lambda)$ and $[N] \in \mathcal{K}(\Lambda)$.

Proof. This follows from (13.1) and the adjointness of the functors $(e_i^\Lambda)^{(r)}$ and $(f_i^\Lambda)^{(r)}$. □

From this we immediately obtain the following corollary.

Corollary 13.2.4. *Assume that, for $r \geq 1$,*

$$(e_i^\Lambda)^{(r)}([M]) = \sum t_{M,N}[N], \quad (f_i^\Lambda)^{(r)}([M]) = \sum s_{M,N}[N],$$

where the first sum is taken over the classes $[N] \in \mathcal{K}(\Lambda)$, with $N \in H_{n-r}^\Lambda(q) - \mathrm{mod}$ irreducible, $n \geq r$, and the second is taken over the classes $[N] \in \mathcal{K}(\Lambda)$, with $N \in H_{n+r}^\Lambda(q) - \mathrm{mod}$ irreducible, $n \geq 0$. Then

$$(e_i^\Lambda)^{(r)}([P_N]) = \sum s_{M,N}[P_M], \quad (f_i^\Lambda)^{(r)}([P_N]) = \sum t_{M,N}[P_M],$$

where both sums are taken over the classes $[P_M] \in \mathcal{K}^*(\Lambda)$ such that P_M is the projective cover of $M \in H_n^\Lambda(q) - \mathrm{mod}$, M irreducible.

Proof. This follows from the previous lemma and the fact that for M irreducible one has that $([P_N],[M]) = \delta_{[N],[M]}$. $\square$

Let $i \in I$, and let $M \in H_n^\Lambda(q) - \mathrm{mod}$ be irreducible. Recall the definitions of the functions ε_i^Λ and φ_i^Λ from (12.7). Then we define a linear operator h_i^Λ on $\mathcal{K}(\Lambda)$ by setting

$$h_i^\Lambda([M]) = (\varphi_i^\Lambda(M) - \varepsilon_i^\Lambda(M))[M]. \tag{13.5}$$

Equivalently, by Lemma 12.3.12, we have that

$$h_i^\Lambda([M]) = (\Lambda - \gamma)(h_i)[M], \tag{13.6}$$

where M has central character χ_γ. More generally, one defines

$$\binom{h_i^\Lambda}{r} : \mathcal{K}(\Lambda) \to \mathcal{K}(\Lambda), \quad [M] \mapsto \binom{\varphi_i^\Lambda(M) - \varepsilon_i^\Lambda(M)}{r}([M]).$$

From (13.6), we immediately obtain the following relations:

Lemma 13.2.5. *As operators on $\mathcal{K}(\Lambda)$,*

$$[h_i^\Lambda, e_j^\Lambda] = \alpha_j(h_i)e_j^\Lambda, \tag{13.7}$$

and

$$[h_i^\Lambda, f_j^\Lambda] = -\alpha_j(h_i)f_j^\Lambda, \tag{13.8}$$

for all $i, j \in I$.

Proof. This follows from (13.6): If $M \in H_n^\Lambda(q) - \mathrm{mod}$ is irreducible with central character χ_γ, $\gamma = (\gamma_0, \ldots, \gamma_{l-1})$, we get that

$$\begin{aligned}
[h_i^\Lambda, e_j^\Lambda]([M]) &= (h_i^\Lambda e_j^\Lambda - e_j^\Lambda h_i^\Lambda)([M]) = h_i^\Lambda e_j^\Lambda([M]) - e_j^\Lambda h_i^\Lambda([M]) \\
&= (\Lambda - \gamma')(h_i)e_j^\Lambda([M]) - (\Lambda - \gamma)(h_i)e_j^\Lambda([M]) \\
&= ((\Lambda - \gamma') - (\Lambda - \gamma))(h_i)e_j^\Lambda([M]) \\
&= (\gamma - \gamma')(h_i)e_j^\Lambda([M]) \\
&= \alpha_j(h_i)e_j^\Lambda([M])
\end{aligned}$$

since all the composition factors of $e_j^\Lambda([M])$ have central character $\chi_{\gamma'}$, with $\gamma' = (\gamma_0, \ldots, \gamma_{j-1}, \gamma_j - 1, \gamma_{j+1}, \ldots, \gamma_{l-1})$. Since h_i^Λ is linear, the result follows. □

Moreover, the following holds true.

Lemma 13.2.6. *As operators on $\mathcal{K}(\Lambda)$,*

$$[e_i^\Lambda, f_j^\Lambda] = \delta_{i,j} h_i^\Lambda, \tag{13.9}$$

for all $i, j \in I$.

Proof. See [44, Theorem 12.4]. □

In the following, we set $(e_i^\Lambda)^{(0)} = (f_i^\Lambda)^{(0)} = (e_i^\Lambda)^{(0)} = (f_i^\Lambda)^{(0)} = \mathrm{id}_{\mathcal{K}(\Lambda)}$ as operators on $\mathcal{K}(\Lambda)$. Then, for $\Lambda \in P_+$, Lemma 13.2.1, 13.2.2, 13.2.5 and 13.2.6 show that the abelian group $\mathcal{K}(\Lambda)$ carries the structure of a U'-module:

Theorem 13.2.7. *The action of the operators e_i^Λ, f_i^Λ and h_i^Λ on $\mathcal{K}(\Lambda)$, for all $i \in I$, satisfy the Chevalley relations (D1)–(D6) of Section 10.1. Furthermore, the operators $(e_i^\Lambda)^{(r)}$, $(f_i^\Lambda)^{(r)}$ and $\binom{h_i^\Lambda}{r}$, $i \in I$, and $r \geq 1$, define a U'-module structure on $\mathcal{K}(\Lambda)_\mathbb{Q}$ such that $\mathcal{K}^*(\Lambda)$ and $\mathcal{K}(\Lambda)$ are $U'_\mathbb{Z}$-submodules.*

Proof. This follows from Lemma 13.2.1, 13.2.2, 13.2.5 and 13.2.6 together with the definitions in Section 10.1. □

Now that we have a U'-module structure on $\mathcal{K}(\Lambda)_\mathbb{Q}$, it is natural to ask whether $\mathcal{K}(\Lambda)_\mathbb{Q}$ possesses even more structure. By definition, the elements h_i^Λ, $i \in I$, are semisimple operators. Moreover, from (13.6) we see that

$$\mathcal{K}(\Lambda)_\mathbb{Q} = \bigoplus_{\gamma \in Q_+} (\mathcal{K}(\Lambda)_\mathbb{Q})_{\Lambda - \gamma},$$

and, hence, $\mathcal{K}(\Lambda)_\mathbb{Q}$ is a weight module over $\mathfrak{g}'(A)$, see Chapter 10 for definitions. In the following, we will denote by $[1_\Lambda]$ the class of the irreducible $H_0^\Lambda(q)$-module in $\mathcal{K}(H_0^\Lambda(q) - \mathrm{mod})$. For the next theorem we will need the following lemma, which is an analogue of [59, Lemma 9.3.3] in the non-degenerate case.

Lemma 13.2.8. *Suppose that $M \in H_n^\Lambda(q) - \mathrm{mod}$ is irreducible. Let $i \in I$, and set $\varepsilon := \varepsilon_i^\Lambda(M)$, and $\varphi := \varphi_i^\Lambda(M)$. For $m \geq 0$, we get that*

$$(e_i^\Lambda)^{(m)}([P_M]) = \sum_{[N],\ \varepsilon_i^\Lambda(N) \geq m} a_N [P_{(\tilde{e}_i^\Lambda)^m}(N)],$$

for non-negative integers a_N. Furthermore, if $m = \varepsilon$, then

$$(e_i^\Lambda)^{(\varepsilon)}([P_M]) = \binom{\varepsilon + \varphi}{\varepsilon} [P_{(\tilde{e}_i^\Lambda)^\varepsilon}(M)] + \sum_{[N],\ \varepsilon_i^\Lambda(N) > m} a_N [P_{(\tilde{e}_i^\Lambda)^m}(N)].$$

Proof. From Corollary 13.2.4 we infer that

$$(e_i^\Lambda)^{(m)}([P_M]) = \sum_{[L]\in B(\Lambda)} [(f_i^\Lambda)^{(m)}(L) : M][P_L].$$

By Proposition 12.3.9 and the definition of φ_i^Λ, the condition $[(f_i^\Lambda)^{(m)}(L) : M] > 0$ implies that $\varphi_i^\Lambda(L) \geq m$. Otherwise, we must have that $(f_i^\Lambda)^{(m)}(L) = 0$. In the first case, we denote by N the non-zero module $(\tilde{f}_i^\Lambda)^m(L)$. Together with Theorem 12.3.1 we have:

$$(e_i^\Lambda)^{(m)}([P_M]) = \sum_{[N]\in B(\Lambda),\ \varepsilon_i^\Lambda(N)\geq m} [(f_i^\Lambda)^{(m)}((\tilde{e}_i^\Lambda)^m(N)) : M]([P_{(\tilde{e}_i^\Lambda)^m(N)}].$$

This gives the first part of the lemma.
Next, let $m = \varepsilon$. By the definition of ε_i^Λ, if $\varepsilon_i^\Lambda(N) = \varepsilon$, then $\varepsilon_i^\Lambda((\tilde{e}_i^\Lambda)^\varepsilon(N)) = 0$. By Theorem 12.3.11, the only composition factor L of $(f_i^\Lambda)^{(m)}((\tilde{e}_i^\Lambda)^m(N))$ with $\varepsilon_i^\Lambda(L) = \varepsilon$ is $(\tilde{f}_i^\Lambda)^m((\tilde{e}_i^\Lambda)^m(N) = N$. This implies that $N \cong M$, and, by Proposition 12.3.14 (i), we see that

$$[(f_i^\Lambda)^{(\varepsilon)}((\tilde{e}_i^\Lambda)^\varepsilon(M)) : M] = \binom{\varepsilon+\varphi}{\varepsilon}.$$

□

We are now ready to state the following:

Theorem 13.2.9. *Let $M \in H_n^\Lambda(q)-\mathrm{mod}$ be irreducible. Then every element $[P_M] \in \mathcal{K}(H_n^\Lambda(q)-\mathrm{proj})$ can be written as an integral linear combination of monomial words of the form*

$$(f_{i_1}^\Lambda)^{(r_1)} \ldots (f_{i_k}^\Lambda)^{(r_k)}([1_\Lambda]),$$

with $r_j \geq 0$, and $i_j \in I$, for all j.

Proof. This is [44, Lemma 11.4], but we will give the arguments here, following the lines of [59, Theorem 9.3.4]. We argue by induction on n. For $n = 0$ the statement is obviously true. Thus, let $n > 0$, and suppose that the statement holds for all smaller non-negative integers. Assume for a contradiction that there is an irreducible $H_n^\Lambda(q)$-module M for which the statement is false. Let $i \in I$ such that $\varepsilon := \varepsilon_i^\Lambda(M) > 0$. Choose M so that the claim holds for all irreducible $H_n^\Lambda(q)$-modules L with $\varepsilon_i^\Lambda(L) > \varepsilon$. This is always possible since there are only finitely many isomorphism classes of irreducible $H_n^\Lambda(q)$-modules. Then we may write

$$(f_i^\Lambda)^{(\varepsilon)}([P_{(\tilde{e}_i^\Lambda)^\varepsilon(M)}] = \sum_{[N]\in B(\Lambda)} a_N[P_N],$$

for non-negative integers a_N. By Corollary 13.2.4, we have that

$$a_N = [(e_i^\Lambda)^{(\varepsilon)}(N) : (\tilde{e}_i^\Lambda)^\varepsilon(M)].$$

Suppose that $a_N > 0$, where N is such that $\varepsilon_i^\Lambda(N) = \varepsilon$. Then

$$(e_i^\Lambda)^{(\varepsilon)}(N) \cong (\tilde{e}_i^\Lambda)^\varepsilon(N) \cong (\tilde{e}_i^\Lambda)^\varepsilon(M),$$

by Proposition 12.3.10. From Theorem 12.3.1 we conclude that $N \cong M$ and $a_M = 1$. This yields

$$[P_M] = (f_i^\Lambda)^{(\varepsilon)}([P_{(\tilde{e}_i^\Lambda)^\varepsilon(M)}] - \sum_{[N]\in B(\Lambda),\ \varepsilon_i^\Lambda(N)>\varepsilon} a_N[P_N],$$

with non-negative integers a_N. By the inductive hypothesis, the first term of the right-hand side can be written in the desired form. Furthermore, by our choice of M, we may assume that the same holds for the second term. This contradicts our assumption on M.

□

Corollary 13.2.10. *For all $n \geq 0$, every element $y \in \mathcal{K}(H_n^\Lambda(q) - \mathrm{mod})_\mathbb{Q}$ can be written as a $\mathbb{Q}$-linear combination of elements of the form $f_i^\Lambda(x)$, for elements $x \in \mathcal{K}(H_{n-1}^\Lambda(q) - \mathrm{mod})_\mathbb{Q}$.*

Proof. Let P be a projective $H_{n-r}^\Lambda(q)$-module, $r \geq 0$, $n - r \geq 0$. Then, as stated prior to Lemma 13.2.3, $(f_i^\Lambda)^{(r)}(P)$ is a projective $H_n^\Lambda(q)$-module. Suppose that

$$[P] = \sum [P : M_j][M_j] \in \mathcal{K}(H_{n-r}^\Lambda(q) - \mathrm{mod}),$$

for irreducible $H_{n-r}^\Lambda(q)$-modules M_j. Then

$$[(f_i^\Lambda)^{(r)}(P)] = \sum [P : M_j][(f_i^\Lambda)^{(r)}(M_i)] \in \mathcal{K}(H_n^\Lambda(q) - \mathrm{mod})$$

since $(f_i^\Lambda)^{(r)}$ is an exact functor. The statement now follows from the injectivity of the map ω, Theorem 13.2.9 and the fact that, as operators on $\mathcal{K}(\Lambda)$, $(f_i^\Lambda)^{(r)} = \frac{(f_i^\Lambda)^r}{r!}$, see Lemma 12.3.5. □

We are now in a position to describe the structure of $\mathcal{K}(\Lambda)$ as a U'-module completely. Observe that the next theorem was first stated in [44]. We give another proof here, following the lines of [59, Theorem 9.5.1]. The necessary definitions and results concerning the representation theory of Kac–Moody algebras used here can be found in Section 10.2 and Section 10.3.

Theorem 13.2.11. *For $\Lambda \in P_+$ the following hold:*

(i) *As a U'-module, $\mathcal{K}(\Lambda)_\mathbb{Q}$ is isomorphic to the irreducible integral highest-weight U'-module $L(\Lambda)$ of highest weight Λ and highest-weight vector $[1_\Lambda]$.*

(ii) *The bilinear form $(-,-)$ of (13.4) on $\mathcal{K}(\Lambda)_\mathbb{Q}$, coincides with the contravariant form provided by the irreducible highest-weight U'-module $L(\Lambda)$, satisfying $([1_\Lambda],[1_\Lambda]) = 1$.*

(iii) $\mathcal{K}^*(\Lambda) \subseteq \mathcal{K}(\Lambda)$ *are integral forms of* $\mathcal{K}(\Lambda)_{\mathbb{Q}}$ *containing the highest-weight vector* $[1_\Lambda]$. *Moreover,* $\mathcal{K}^*(\Lambda)$ *is the minimal lattice* $(U'_{\mathbb{Z}})^-[1_\Lambda] \subseteq \mathcal{K}(\Lambda)_{\mathbb{Q}}$ *with* $\mathcal{K}(\Lambda)$ *being its dual under the contravariant form.*

(iv) *The classes of the irreducible* $H_n^\Lambda(q)$*-modules with central character* γ *form a basis of the* $(\Lambda-\gamma)$*-weight spaces* $(\mathcal{K}(\Lambda)_{\mathbb{Q}})_{\Lambda-\gamma}$. *The same statement holds for the classes* $[P_M]$ *of projective indecomposable modules in* $H_n^\Lambda(q)-\mathrm{mod}$ *with central character* γ.

Proof. By Theorem 13.2.7, we have already seen that $\mathcal{K}(\Lambda)_{\mathbb{Q}}$ is a U'-module. By Theorem 12.3.6 and Theorem 12.3.13, we see that the elements $e_i, f_i \in U'$, $i \in I$, act locally nilpotently on $\mathcal{K}(\Lambda)_{\mathbb{Q}}$. The action of the elements $h_i \in U'$, $i \in I$, on $\mathcal{K}(\Lambda)_{\mathbb{Q}}$ is diagonal by definition, see (13.6). It follows that $\mathcal{K}(\Lambda)_{\mathbb{Q}}$ is an integrable U'-module. Moreover, again by (13.6), we see that $[1_\Lambda]$ is a highest-weight vector of highest weight Λ. From Corollary 13.2.10, we infer that $\mathcal{K}(\Lambda)_{\mathbb{Q}} = (U')^-[1_\Lambda]$. This finishes the proof of (i).
From Lemma 13.2.3 we know that $(-,-)$ is non-degenerate and contravariant. By Proposition 10.3.7 we have that the bilinear form associated to an irreducible highest-weight U'-module is unique up to a constant factor. Since $([1_\Lambda],[1_\Lambda]) = 1$, we see that this form must coincide with $(-,-)$. Thus, (ii) follows.
By the discussion prior to Lemma 13.2.3, we know that $U'_{\mathbb{Z}}$ leaves the lattices $\mathcal{K}^*(\Lambda)$ and $\mathcal{K}(\Lambda)$ invariant. Moreover, the duality pairing given in (13.1) shows that they are dual to each other under the contravariant form. Also, Theorem 13.2.9 states that $(U'_{\mathbb{Z}})^-[1_\Lambda] = \mathcal{K}^*(\Lambda)$. Hence, (iii) follows.
Finally, part (iv) is just a reformulation of (13.6). $\square$

Chapter 14

Crystal structure

In this chapter we briefly recall the concept of crystals associated to highest-weight modules over enveloping algebras of Kac–Moody algebras.
Moreover, we recall work of I. Grojnowski [44], where he gave the sets $B(\infty)$ and $B(\Lambda)$, $\Lambda \in P_+$, the structure of crystals in the sense of M. Kashiwara. The crystal structure will become important in later chapters.
The aim of this chapter is to understand the proof of the following, which is one of the main results of [44]:

Theorem 14.0.12. *Let A be the generalized Cartan matrix of type $A_{l-1}^{(1)}$, $l \geq 2$. Then, for all $\Lambda \in P_+$, the crystal $B(\Lambda)$ is isomorphic to the crystal $B(\Lambda)_{\mathrm{Kas}}$ of the irreducible highest-weight module $L(\Lambda)$ of highest weight Λ defined over the quantized enveloping algebra $U_v(\mathfrak{g}(A))$ over $\mathbb{Q}(v)$.*

For the proof of this theorem we will follow the lines of [59, §10], where the theorem was established in the degenerate case.

14.1 Crystals

The theory of crystals was mainly developed by M. Kashiwara in the context of integrable highest-weight modules for quantized enveloping algebras $U_v(\mathfrak{g}(A))$ of a Kac–Moody algebra $\mathfrak{g}(A)$, see for example [54]. Due to this work, for a generalized Cartan matrix A, each irreducible highest-weight module $L(\Lambda)$ of highest weight Λ over the $\mathbb{Q}(v)$-algebra $U_v(\mathfrak{g}(A))$ admits a certain basis, called the *crystal basis*, which is determined uniquely by $L(\Lambda)$. To each crystal bases one can associate a graph $B(\Lambda)_{\mathrm{Kas}}$, which is called the *crystal graph* of $L(\Lambda)$. Moreover, via the latter crystal bases one also can associate a crystal basis to the subalgebra $U_v(\mathfrak{g}(A))^-$ of $U_v(\mathfrak{g}(A))$ with crystal graph denoted by B_{Kas}.

Recall the definition of the set P from Chapter 11. We now state the necessary definitions, which can be found in [54].

Definition 14.1.1. Let A be a generalized Cartan matrix. A *crystal* of type A is a set B together with maps

$$\begin{aligned} \varphi_i, \varepsilon_i &: B \longrightarrow \mathbb{Z} \cup \{-\infty\} \ (i \in I), \\ \tilde{e}_i, \tilde{f}_i &: B \longrightarrow B \cup \{0\} \ (i \in I), \\ \mathrm{wt} &: B \longrightarrow P, \end{aligned}$$

such that:

(C1) $\varphi_i(b) = \varepsilon_i(b) + \mathrm{wt}(b)(h_i)$, for all $i \in I$, $b \in B$.

(C2) If $b \in B$ satisfies $\tilde{e}_i(b) \neq 0$, then $\varepsilon_i(\tilde{e}_i(b)) = \varepsilon_i(b) - 1$, $\varphi_i(\tilde{e}_i(b)) = \varphi_i(b) + 1$, and $\mathrm{wt}(\tilde{e}_i(b)) = \mathrm{wt}(b) + \alpha_i$, $i \in I$.

(C3) If $b \in B$ satisfies $\tilde{f}_i(b) \neq 0$, then $\varepsilon_i(\tilde{f}_i(b)) = \varepsilon_i(b) + 1$, $\varphi_i(\tilde{f}_i(b)) = \varphi_i(b) - 1$, and $\mathrm{wt}(\tilde{f}_i(b)) = \mathrm{wt}(b) - \alpha_i$, $i \in I$.

(C4) For elements $b_1, b_2 \in B$, $b_2 = \tilde{f}_i(b_1)$ holds if and only if $b_1 = \tilde{e}_i(b_2)$, $i \in I$.

(C5) If $\varphi_i(b) = -\infty$, then $\tilde{e}_i(b) = \tilde{f}_i(b) = 0$, $i \in I$.

Example 14.1.2. (i) For every $i \in I$ one can define a crystal B_i as follows. As a set,

$$B_i = \{b_i(n) \mid n \in \mathbb{Z}\},$$

with symbols $b_i(n)$, together with the maps

$$\varepsilon_j(b_i(n)) = \begin{cases} -n & \text{if } j = i, \\ -\infty & \text{if } j \neq i, \end{cases}$$

$$\varphi_j(b_i(n)) = \begin{cases} n & \text{if } j = i, \\ -\infty & \text{if } j \neq i, \end{cases}$$

$$\tilde{e}_j(b_i(n)) = \begin{cases} b_i(n+1) & \text{if } j = i, \\ 0 & \text{if } j \neq i, \end{cases}$$

$$\tilde{f}_j(b_i(n)) = \begin{cases} b_i(n-1) & \text{if } j = i, \\ 0 & \text{if } j \neq i. \end{cases}$$

Finally, we set $\mathrm{wt}(b_i(n)) = n\alpha_i$, for all $n \in \mathbb{Z}$.
(ii) For $\Lambda \in P$, we define the crystal T_Λ as the set $\{t_\Lambda\}$, together with $\varepsilon_i(t_\Lambda) = \varphi_i(t_\Lambda) = -\infty$, $\tilde{e}_i(t_\Lambda) = \tilde{f}_i(t_\Lambda) = 0$, and $\mathrm{wt}(t_\Lambda) = \Lambda$.

A *morphism* $\psi : B \to B'$ of crystals B and B' is a map $\psi : B \cup \{0\} \to B' \cup \{0\}$ such that:

(M1) $\psi(0) = 0$.

(M2) Let $b \in B$. If $\psi(b) \neq 0$, then $\mathrm{wt}(\psi(b)) = \mathrm{wt}(b)$, $\varepsilon_i(\psi(b)) = \varepsilon_i(b)$, and $\varphi_i(\psi(b)) = \varphi_i(b)$.

(M3) For all $b \in B$ with $\psi(b) \neq 0$ and $\psi(\tilde{e}_i(b)) \neq 0$, we have that $\psi(\tilde{e}_i(b)) = \tilde{e}_i\psi(b)$.

(M4) For all $b \in B$ with $\psi(b) \neq 0$ and $\psi(\tilde{f}_i(b)) \neq 0$, we have that $\psi(\tilde{f}_i(b)) = \tilde{f}_i\psi(b)$.

A morphism ψ of crystals is called an *embedding* if ψ is injective. It is called *strict* if ψ commutes with the maps $\tilde{e}_i, \tilde{f}_i$, for all $i \in I$.
Given two crystals B and B', we may define their tensor product as follows. As a set, $B \otimes B'$ is defined as $B \times B'$, where we denote an element $(b, b') \in B \times B'$ by $b \otimes b'$. This set can be given the structure of a crystal, by setting

$$\begin{aligned} \varepsilon_i(b \otimes b') &= \max(\varepsilon_i(b), \varepsilon_i(b') - \mathrm{wt}(b)(h_i)), \\ \varphi_i(b \otimes b') &= \max(\varphi_i(b) + \mathrm{wt}(b')(h_i), \varphi_i(b')), \end{aligned}$$

$$\tilde{e}_i(b \otimes b') = \begin{cases} \tilde{e}_i(b) \otimes b' & \text{if } \varphi_i(b) \geq \varepsilon_i(b'), \\ b \otimes \tilde{e}_i(b') & \text{if } \varphi_i(b) < \varepsilon_i(b'), \end{cases}$$

$$\tilde{f}_i(b \otimes b') = \begin{cases} \tilde{f}_i(b) \otimes b' & \text{if } \varphi_i(b) > \varepsilon_i(b'), \\ b \otimes \tilde{f}_i(b') & \text{if } \varphi_i(b) \leq \varepsilon_i(b'), \end{cases}$$

$$\mathrm{wt}(b \otimes b') = \mathrm{wt}(b) + \mathrm{wt}(b'),$$

for all $i \in I$, and $b \otimes b' \in B \otimes B'$. Also, we set $b \otimes 0 = 0 = 0 \otimes b$.

14.2 Crystals associated to cyclotomic Hecke algebras

Next, recall the definitions $B(\infty)$ and $B(\Lambda)$, for $\Lambda \in P_+$, as well as the maps

$$\tilde{e}_i, \tilde{f}_i : B(\infty) \to B(\infty) \cup \{0\}$$

and

$$\tilde{e}_i^{\Lambda}, \tilde{f}_i^{\Lambda} : B(\Lambda) \to B(\Lambda) \cup \{0\},$$

for $i \in I$. We want to give the sets $B(\infty)$ and $B(\Lambda)$ the structures of crystals. The main source of reference for this will be Section 12.3. From now on, A will denote the generalized Cartan matrix of type $A_{l-1}^{(1)}$, $l \geq 2$. Furthermore, by σ we will denote the *diagram automorphism* of $H_n(q)$, which is defined by

$$x_i \mapsto x_{n+1-i}, \quad T_i \mapsto -(T_{n-i} + 1 - q).$$

If $M \in H_n(q) - \mathrm{mod}$, via σ, we get another $H_n(q)$-action on M and obtain a new $H_n(q)$-module that we will denote by M^σ. For $i \in I$, we define

$$\begin{aligned} \varepsilon_i^*(M) &= \varepsilon_i(M^\sigma), && (14.1) \\ \tilde{e}_i^*(M) &= (\tilde{e}_i(M^\sigma))^\sigma, && (14.2) \\ \tilde{f}_i^*(M) &= (\tilde{f}_i(M^\sigma))^\sigma, && (14.3) \end{aligned}$$

for an irreducible $M \in H_n(q) - \mathrm{mod}_q$.

Remark 14.2.1. From the action, we can interpret the integer $\varepsilon_i^*(M)$ as the maximal $t \in \mathbb{Z}_+$ such that $[L(i)^{\boxtimes t} \boxtimes \ldots]$ appears in the character $\mathrm{ch}(M)$ of M. Moreover, in view of Theorem 12.3.6, the integer $\varepsilon_i^*(M)$ is the maximal size of a Jordan block of x_1 with eigenvalue q^i.

As a direct consequence, we obtain the next lemma. Recall the definition of the functor

$$\mathrm{pr}_\Lambda : H_n(q) - \mathrm{mod} \longrightarrow H_n^\Lambda(q) - \mathrm{mod},$$

and the algebra homomorphism

$$\mathrm{ev}_\Lambda : H_n(q) \to H_n^\Lambda(q),$$

defined in Chapter 11.

Lemma 14.2.2. *Let $\Lambda \in P_+$, and let $M \in H_n(q) - \mathrm{mod}_q$ be irreducible. Then $\mathrm{pr}_\Lambda(M) = M$ if and only if $\varepsilon_i^*(M) \leq \Lambda(h_i)$, for all $i \in I$.*

Proof. Let $I_\Lambda = \ker(\mathrm{ev}_\Lambda)$. Since M is irreducible, $\mathrm{pr}_\Lambda(M) = M/I_\Lambda M$ is either zero or $I_\Lambda M = 0$. By the definition of I_Λ, $I_\Lambda M = 0$ if and only if the largest size of a Jordan block of x_1 on M corresponding to the eigenvalue q^i is less than or equal $\Lambda(h_i)$. By Remark 14.2.1, $\varepsilon_i^*(M)$ is the maximal size of a Jordan block of x_1 with eigenvalue q^i. Hence, the result follows. □

In the following, we will state some rather technical results. These will be necessary for giving the sets $B(\infty)$ and $B(\Lambda)$, $\Lambda \in P_+$, the structure of a crystal. The proof of the following lemma is similar to that of [59, Lemma 10.1.1].

Lemma 14.2.3. *Let $M \in H_n(q) - \mathrm{mod}_q$ be irreducible. Then the following hold:*

(i) *For all $i \in I$, either $\varepsilon_i^*(\tilde{f}_i(M)) = \varepsilon_i^*(M)$ or $\varepsilon_i^*(\tilde{f}_i(M)) = \varepsilon_i^*(M) + 1$.*

(ii) *For all $i, j \in I$ with $i \neq j$, we have that $\varepsilon_i^*(\tilde{f}_j(M)) = \varepsilon_i^*(M)$.*

Proof. By definition, $\tilde{f}_i(M) = \mathrm{hd}(\mathrm{Ind}_{H_{(n,1)}(q)}^{H_{n+1}(q)}(M \boxtimes L(i)))$, hence, by the Shuffle Lemma and Remark 14.2.1, we infer that $\varepsilon_i^*(\tilde{f}_i(M)) \leq \varepsilon_i^*(M) + 1$. Moreover, let $N := \tilde{f}_i(M)$. Then clearly, $\varepsilon_i^*(\tilde{e}_i(N)) \leq \varepsilon_i^*(N)$. Therefore, $\varepsilon_i^*(M) = \varepsilon_i^*(\tilde{e}_i(N)) \leq \varepsilon_i^*(\tilde{f}_i(M))$, by Theorem 12.3.1. This proves the first statement.
For the second statement, suppose that $i \neq j$, and write

$$\tilde{f}_j(M) = \mathrm{hd}(\mathrm{Ind}_{n,1}^{n+1}(M \boxtimes L(j))).$$

Then, by the Shuffle Lemma, we directly see that $\varepsilon_i^*(\tilde{f}_j(M)) = \varepsilon_i^*(M)$. □

Proposition 14.2.4. *Let $M \in H_n(q) - \mathrm{mod}_q$ be irreducible, and let $i, j \in I$. Moreover, write $a = \varepsilon_i^*(M)$. Then:*

(i) *If $\varepsilon_i^*(\tilde{f}_j(M)) = a$, then $(\tilde{e}_i^*)^a(\tilde{f}_j(M)) \cong \tilde{f}_j((\tilde{e}_i^*)^a(M))$.*

(ii) *If* $\varepsilon_i^*(\tilde{f}_i(M)) = a+1$, *then* $\tilde{e}_i^*(\tilde{f}_i(M)) \cong M$.

Proof. This is [44, Proposition 13.1], where the proof of statement (i) is given only in the case $i = j$. But also for $i \neq j$ the same argument applies. $\square$

If we replace in Lemma 14.2.3 and Proposition 14.2.4 M by M^σ, then we obtain the following:

Lemma 14.2.5. *Let* $M \in H_n(q) - \mathrm{mod}_q$ *be irreducible. Then the following hold:*

(i) *For all* $i \in I$, *either* $\varepsilon_i(\tilde{f}_i^*(M)) = \varepsilon_i(M)$ *or* $\varepsilon_i(\tilde{f}_i^*(M)) = \varepsilon_i(M) + 1$.

(ii) *For all* $i, j \in I$ *with* $i \neq j$, *we have that* $\varepsilon_i(\tilde{f}_j^*(M)) = \varepsilon_i(M)$.

Proposition 14.2.6. *Let* $M \in H_n(q) - \mathrm{mod}_q$ *be irreducible, and let* $i, j \in I$. *Moreover, write* $a = \varepsilon_i(M)$. *Then:*

(i) *If* $\varepsilon_i(\tilde{f}_j^*(M)) = a$, *then* $\tilde{e}_i^a(\tilde{f}_j^*(M)) \cong \tilde{f}_j^*(\tilde{e}_i^a(M))$.

(ii) *If* $\varepsilon_i(\tilde{f}_i^*(M)) = a+1$, *then* $\tilde{e}_i(\tilde{f}_i^*(M)) \cong M$.

As stated earlier, for $\Lambda \in P_+$, we want to make the sets $B(\infty)$ and $B(\Lambda)$ into crystals. For $B(\Lambda)$, we use the operators $\tilde{e}_i^\Lambda, \tilde{f}_i^\Lambda$ together with the functions $\varepsilon_i^\Lambda, \varphi_i^\Lambda$ defined in Section 12.3 to define the maps $\tilde{e}_i, \tilde{f}_i, \varepsilon_i, \varphi_i$, for all $i \in I$.
For $B(\infty)$, we use the maps $\tilde{e}_i, \tilde{f}_i$, and the function ε_i from the same section. The function φ_i is defined below.
On $B(\infty)$ and $B(\Lambda)$, we define the weight functions as

$$\mathrm{wt}([M]) = -\gamma, \tag{14.4}$$

for an irreducible $M \in H_n(q) - \mathrm{mod}_q$ with central character γ, and

$$\mathrm{wt}^\Lambda([N]) = \Lambda - \gamma, \tag{14.5}$$

for an irreducible $N \in H_n^\Lambda(q) - \mathrm{mod}$ with central character γ. Finally, for $[M] \in B(\infty)$, we set

$$\varphi_i([M]) = \varepsilon_i([M]) + \mathrm{wt}([M])(h_i). \tag{14.6}$$

We can now state the following:

Theorem 14.2.7. *The set* $B(\infty)$ *together with the maps* $\varepsilon_i, \varphi_i, \tilde{e}_i, \tilde{f}_i, \mathrm{wt}$, *as well as the set* $B(\Lambda)$, $\Lambda \in P_+$, *together with the maps* $\varepsilon_i^\Lambda, \varphi_i^\Lambda, \tilde{e}_i^\Lambda, \tilde{f}_i^\Lambda, \mathrm{wt}^\Lambda$, *are crystals.*

Proof. Property (C1) for $B(\Lambda)$ is Lemma 12.3.12 (iii), and equation (14.6) for $B(\infty)$. The property (C4) is Theorem 12.3.1, for both $B(\infty)$ and $B(\Lambda)$.
By Lemma 12.1.3, we see that for $M \in H_n(q) - \mathrm{mod}_q$ irreducible, $\varepsilon_i(\tilde{e}_i([M])) = \varepsilon_i([M]) - 1$, for $i \in I$. Moreover, by Theorem 12.3.13, we have that $\varphi_i(\tilde{f}_i([M])) = \varphi_i([M]) - 1$, for $M \in H_n^\Lambda(q) - \mathrm{mod}$ irreducible. Using Theorem 12.3.1, we see that

$\varphi_i(\tilde{e}_i([M]) = \varphi_i([M]) + 1$. For $M \in H_n(q) - \mathrm{mod}_q$, by definition, $\varphi_i(\tilde{e}_i([M]) = \varepsilon_i(\tilde{e}_i([M])) + \mathrm{wt}(\tilde{e}_i([M]))(h_i)$. Moreover, since $\tilde{e}_i([M])$ has central character $\gamma - \alpha_i$, for $M \in H_n^\Lambda(q) - \mathrm{mod}$ irreducible, we infer that

$$\begin{aligned} \mathrm{wt}^\Lambda(\tilde{e}_i([M])) &= \Lambda - (\gamma - \alpha_i) \\ &= \Lambda - \gamma + \alpha_i \\ &= \mathrm{wt}^\Lambda([M]) + \alpha_i. \end{aligned}$$

By (14.4), we have that $\mathrm{wt}(\tilde{e}_i([M])) = -\gamma + \alpha_i$. It follows by Theorem 12.3.1 and Remark 12.1.11 that

$$\varphi_i(\tilde{e}_i([M]) = \varepsilon_i([M]) - 1 + \mathrm{wt}(\tilde{e}_i([M]))(h_i).$$

Since $\alpha_i(h_i) = 2$, we infer that

$$\begin{aligned} \varphi_i(\tilde{e}_i([M]) &= \varepsilon_i([M]) + \mathrm{wt}([M])(h_i) + 1 \\ &= \varphi_i([M]) + 1. \end{aligned}$$

Now we see that $B(\infty)$ and $B(\Lambda)$ satisfy property (C2).
By Remark 12.1.11, we immediately have that $\varepsilon_i(\tilde{f}_i([M])) = \varepsilon_i([M]) + 1$, for irreducible $M \in H_n^\Lambda(q) - \mathrm{mod}$ (resp. $H_n(q) - \mathrm{mod}_q$). Furthermore, Theorem 12.3.13 shows that $\varphi_i(\tilde{f}_i^\Lambda(M)) = \varphi_i(M) - 1$, and hence, $\varphi_i(\tilde{f}_i([M]) = \varphi_i([M]) - 1$ holds for $B(\Lambda)$. Let $M \in H_n(q) - \mathrm{mod}_q$ be irreducible with central character γ. By (14.4), together with Remark 12.1.11, we see that

$$\varphi_i(\tilde{f}_i([M])) = \varepsilon_i([M]) + 1 + \mathrm{wt}(\tilde{f}_i([M]))(h_i).$$

Since $\mathrm{wt}(\tilde{f}_i([M]))(h_i) = -\gamma - \alpha_i$, we have that

$$\varphi_i(\tilde{f}_i([M])) = \varepsilon_i([M]) + \mathrm{wt}([M])(h_i) - 1,$$

and thus, $\varphi_i(\tilde{f}_i([M])) = \varphi_i([M]) - 1$. From the definition of wt and wt^Λ we easily see that $\mathrm{wt}(\tilde{f}_i([M])) = \mathrm{wt}([M]) - \alpha_i$ (resp. $\mathrm{wt}^\Lambda(\tilde{f}_i([M])) = \mathrm{wt}^\Lambda([M]) - \alpha_i$), for irreducible $M \in H_n(q) - \mathrm{mod}_q$ (resp. $H_n^\Lambda(q) - \mathrm{mod}$). This shows that (C3) holds for $B(\infty)$ and $B(\Lambda)$.
Suppose that $\tilde{e}_i(b) \neq 0$. Then, for $M \in H_n^\Lambda(q) - \mathrm{mod}$ irreducible, $\varphi_i([M])$ is finite, by Theorem 12.3.13 (iii). For $M \in H_n(q) - \mathrm{mod}_q$ this follows form (14.4) and the fact that $\varepsilon_i([M])$ is finite. Moreover, the same argument shows that if $\tilde{f}_i([M]) \neq 0$, then $\varphi_i([M])$ is finite for irreducible $M \in H_n(q) - \mathrm{mod}_q$ (resp. $H_n^\Lambda(q) - \mathrm{mod}$). This finishes the proof. $\square$

Via inflation we obtain an embedding $\mathrm{infl}^\Lambda : B(\Lambda) \cup \{0\} \to B(\infty) \cup \{0\}$. We have the following.

Lemma 14.2.8. *The map $\psi^\Lambda : B(\Lambda) \to B(\infty) \otimes T_\Lambda$, given by $[M] \mapsto \mathrm{infl}^\Lambda([M]) \otimes t_\Lambda$, is an embedding of crystals. Its image equals the subset*

$$\{[M] \otimes t_\Lambda \in B(\infty) \otimes T_\Lambda \mid \varepsilon_i^*(M) \leq \Lambda(h_i) \text{ for all } i \in I\}$$

of $B(\infty) \otimes T_\Lambda$.

Proof. Clearly, $\psi^\Lambda(0) = 0$. Let $[M] \in B(\Lambda)$ with $\psi^\Lambda([M]) \neq 0$, $M \in H_n^\Lambda(q) - \mathrm{mod}$ irreducible. Suppose that $\mathrm{infl}^\Lambda(M)$ has central character γ. Then

$$\begin{aligned} \mathrm{wt}(\psi^\Lambda([M])) &= \mathrm{wt}(\mathrm{infl}^\Lambda([M]) \otimes t_\Lambda) \\ &= \mathrm{wt}(\mathrm{infl}^\Lambda([M])) + \mathrm{wt}(t_\Lambda) \\ &= \Lambda - \gamma, \end{aligned}$$

see Example 14.1.2 (ii). Thus, $\mathrm{wt}(\psi^\Lambda([M])) = \mathrm{wt}([M])$. Next, let $i \in I$. Then:

$$\varepsilon_i(\psi^\Lambda([M])) = \max(\varepsilon_i(\mathrm{infl}^\Lambda([M])), \varepsilon_i(t_\Lambda) - \mathrm{wt}(\mathrm{infl}^\Lambda([M]))(h_i)).$$

Since, by definition, $\varepsilon_i(t_\Lambda) = -\infty$, we have that $\varepsilon_i(\psi^\Lambda([M])) = \varepsilon_i([M])$. Similarly, since $\varphi_i(t_\Lambda) = -\infty$, we have that $\varphi(\psi^\Lambda([M])) = \varphi_i([M])$. This shows that ψ^Λ satisfies (M2).
Next, suppose that $\psi^\Lambda([M]) \neq 0$ and $\psi^\Lambda(\tilde{e}_i([M])) \neq 0$. Recall from [44, Lemma 9.3] that $\mathrm{infl}^\Lambda(\tilde{e}_i^\Lambda([M])) = \tilde{e}_i(\mathrm{infl}^\Lambda([M]))$. Therefore,

$$\psi^\Lambda(\tilde{e}_i([M])) = \mathrm{infl}^\Lambda(\tilde{e}_i([M])) \otimes t_\Lambda = (\tilde{e}_i(\mathrm{infl}^\Lambda([M]))) \otimes t_\Lambda.$$

Since $\varphi_i(\mathrm{infl}^\Lambda([M])) \geq \varepsilon_i(t_\Lambda)$, we see that $\psi^\Lambda(\tilde{e}_i([M])) = \tilde{e}_i(\psi^\Lambda([M]))$. Hence, ψ^Λ satisfies (M3).
Assume $\psi^\Lambda([M]) \neq 0$ and $\psi^\Lambda(\tilde{f}_i([M])) \neq 0$. Property (M1) implies that $\tilde{f}_i([M]) \neq 0$. Then, by [44, Lemma 9.3], we have that $\mathrm{infl}^\Lambda(\tilde{f}_i([M])) = \tilde{f}_i(\mathrm{infl}^\Lambda([M]))$, and the same argumentation as above shows that $\psi^\Lambda(\tilde{f}_i([M])) = \tilde{f}_i(\psi^\Lambda([M]))$, hence property (M4).
The second statement follows from Lemma 14.2.2. This finishes the proof. ☐

The proof of the next lemma is analogous to that of [59, Lemma 10.3.1].

Lemma 14.2.9. *Let $M \in H_n(q) - \mathrm{mod}_q$ be irreducible, and $i, j \in I$ such that $i \neq j$. We set $c = \varepsilon_i^*(M)$. Then the following hold:*

(i) $\varepsilon_j(M) = \varepsilon_j((\tilde{e}_i^*)^c(M))$.

(ii) *Suppose that $\varepsilon_j(M) > 0$. Then $\varepsilon_i^*(\tilde{e}_j(M)) = \varepsilon_i^*(M)$ and $(\tilde{e}_i^*)^c(\tilde{e}_j(M)) = \tilde{e}_j((\tilde{e}_i^*)^c(M))$.*

Proof. For (i), we apply Lemma 14.2.5 (ii) c times:

$$\varepsilon_j((\tilde{e}_i^*)^c(M)) = \varepsilon_j(\tilde{f}_i^*((\tilde{e}_i^*)^c(M))) = \cdots = \varepsilon_j((\tilde{f}_i^*)^c((\tilde{e}_i^*)^c(M))) = \varepsilon_j(M).$$

By Lemma 14.2.3 (ii) we have that $\varepsilon_i^*(\tilde{f}_j(N)) = \varepsilon_i^*(N)$, for irreducible $N \in H_n(q) - \mathrm{mod}_q$. If we set $N = \tilde{e}_j(M)$, we get the first part of (ii). Moreover, by Proposition 14.2.4 (i), we have that $(\tilde{e}_i^*)^c(\tilde{f}_j(M')) = \tilde{f}_j((\tilde{e}_i^*)^c(M'))$, for irreducible $M' \in H_n(q) - \mathrm{mod}_q$ with $\varepsilon_i^*(\tilde{f}_j(M')) = \varepsilon_i^*(M')$. If we set $M' = \tilde{e}_j(M)$, then the last property holds by the first part, thus

$$(\tilde{e}_i^*)^c(M) = (\tilde{e}_i^*)^c(\tilde{f}_j(\tilde{e}_j(M))) = \tilde{f}_j((\tilde{e}_i^*)^c(\tilde{e}_j(M))).$$

If we apply $\tilde{e}_j$ on both sides, the claim follows. $\square$

The following, which is taken form [44], will be crucial in the proof of the next proposition.

Lemma 14.2.10. *Let $M \in H_n(q) - \mathrm{mod}_q$ be irreducible and $i \in I$. We set $c = \varepsilon_i^*(M)$, and let $L = (\tilde{e}_i^*)^c(M)$. Then:*

(i) $\varepsilon_i(M) = \max(\varepsilon_i(L), c - \mathrm{wt}(L)(h_i))$.

(ii) *If $\varepsilon_i(M) > 0$, then*

$$\varepsilon_i^*(\tilde{e}_i(M)) = \begin{cases} c & \text{if } \varepsilon_i(L) \geq c - \mathrm{wt}(L)(h_i), \\ c-1 & \text{if } \varepsilon_i(L) < c - \mathrm{wt}(L)(h_i). \end{cases}$$

(iii) *Suppose that $\varepsilon_i(M) > 0$. Then*

$$(\tilde{e}_i^*)^b(\tilde{e}_i(M)) \cong \begin{cases} \tilde{e}_i(L) & \text{if } \varepsilon_i(L) \geq c - \mathrm{wt}(L)(h_i), \\ L & \text{if } \varepsilon_i(L) < c - \mathrm{wt}(L)(h_i), \end{cases}$$

where $b = \varepsilon_i^*(\tilde{e}_i(M))$.

Proof. This is [44, Proposition 13.2]. $\square$

Next, recall the definition of the crystal B_i, $i \in I$ from Example 14.1.2. In what follows we denote by b_i the element $b_i(0)$ in B_i. Then we define a map

$$\Psi_i : B(\infty) \longrightarrow B(\infty) \otimes B_i, \tag{14.7}$$

given by $[M] \mapsto [(\tilde{e}_i^*)^c(M)] \otimes \tilde{f}_i^c(b_i)$, where $c = \varepsilon_i^*(M)$.
In the sequel, we will write $[\mathbf{1}]$ for the isomorphism class of the irreducible $H_0(q)$-module in $B(\infty)$. The following is [59, Lemma 10.3.3] in the non-degenerate case:

Proposition 14.2.11. *We have the following:*

(i) *For all $[M] \in B(\infty)$, $\mathrm{wt}(M)$ is a negative sum of simple roots.*

(ii) $[\mathbf{1}]$ *is the unique element of $B(\infty)$ with weight* 0.

(iii) $\varepsilon_i([\mathbf{1}]) = 0$, *for every $i \in I$.*

(iv) $\varepsilon_i([M]) \in \mathbb{Z}$, *for all $[M] \in B(\infty)$ and every $i \in I$.*

(v) *For all $i \in I$, the above defined map Ψ_i is a strict embedding of crystals.*

(vi) $\Psi_i(B(\infty)) \subseteq B(\infty) \times \{\tilde{f}_i^n(b_i) \mid n \geq 0\}$.

(vii) *For any $[M] \in B(\infty)$ different from $[\mathbf{1}]$, there is some $i \in I$ such that $\Psi_i([M]) = [N] \otimes \tilde{f}_i^n(b_i)$ for some $[N] \in B(\infty)$ and $n > 0$.*

Proof. Properties (i)-(iv) follow directly from the construction of the crystal $B(\infty)$. Property (v) follows from Lemma 14.2.9 and Lemma 14.2.10. Moreover, (vi) follows from the definition of the map Ψ_i, and property (vii) holds since for $[M] \in B(\infty)$ different from $[\mathbf{1}]$, we have that $\varepsilon_i^*(M) > 0$ for at least one $i \in I$. $\square$

Henceforth, we denote by $U_v(\mathfrak{g}(A))^-$ the subalgebra of $U_v(\mathfrak{g}(A))$ generated by the symbols f_i, $i \in I$.
By [56, Proposition 3.2.3] the conditions of Proposition 14.2.11 determine the crystal B_{Kas} of the $\mathbb{Q}(v)$-algebra $U_v(\mathfrak{g}(A))^-$ uniquely up to isomorphism:

Theorem 14.2.12. *The crystal $B(\infty)$ is isomorphic to the crystal B_{Kas} associated to the crystal basis of the $\mathbb{Q}(v)$-algebra $U_v(\mathfrak{g}(A))$.*

From [54, Theorem 8.2] we know the following. Note that in the notation of [54], the element $u_\infty \in B_{\mathrm{Kas}}$ corresponds to $1 \in U_v(\mathfrak{g}(A))^-$.

Proposition 14.2.13. *For all $i \in I$ there exists a unique strict embedding of crystals*

$$\Theta_i : B_{\mathrm{Kas}} \longrightarrow B_{\mathrm{Kas}} \otimes B_i$$

such that $\Theta_i(u_\infty) = u_\infty \otimes b_i$.

Now Theorem 14.2.12 gives us an isomorphism $f : B_{\mathrm{Kas}} \to B(\infty)$ of crystals. Therefore, for all $i \in I$, we have the following commutative diagram:

$$\begin{tikzcd} B(\infty) \arrow[r, "\Psi_i"] & B(\infty) \otimes B_i \\ B_{\mathrm{Kas}} \arrow[u, "f"] \arrow[r, "\Theta_i"] & B_{\mathrm{Kas}} \otimes B_i \arrow[u, "f \otimes \mathrm{id}_{B_i}"'] \end{tikzcd}$$

By Proposition 14.2.11, for $i \in I$, the map Ψ_i is a strict embedding of crystals and $\Psi_i([\mathbf{1}]) = [\mathbf{1}] \otimes b_i$. Since $f(u_\infty) = [\mathbf{1}]$, we see that we may identify Ψ_i with the map Θ_i, for all $i \in I$.
In [54, §8.3], an anti-automorphism $*$ is defined on $U_v(\mathfrak{g}(A))$, satisfying $v^* = v$, $e_i^* = e_i, f_i^* = f_i$, for all $i \in I$, and $(v^h)^* = v^{-h}$, for all $h \in \mathfrak{h}$. Moreover, functions $(\tilde{e}_i^{\mathrm{Kas}})^*, (\tilde{f}_i^{\mathrm{Kas}})^*, (\varepsilon_i^{\mathrm{Kas}})^*, (\varphi_i^{\mathrm{Kas}})^*$ are defined giving another crystal structure on B_{Kas}. The result of [54, Proposition 8.1] describes this structure via the maps Θ_i:

Proposition 14.2.14. *Let $i \in I$. Suppose that for $b \in B_{\mathrm{Kas}}$ we have $\Theta_i(b) = b' \otimes \tilde{f}_i^m(b_i)$, $b' \in B_{\mathrm{Kas}}$, $m \geq 0$. Then the following hold:*

(i) $(\varepsilon_i^{\mathrm{Kas}})^*(b') = 0$.

(ii) $(\varepsilon_i^{\mathrm{Kas}})^*(b) = m$.

(iii) $\Theta_i((\tilde{f}_i^{\mathrm{Kas}})^*(b)) = b' \otimes \tilde{f}_i^{m+1}(b_i)$.

(iv)
$$\Theta_i((\tilde{e}_i^{\mathrm{Kas}})^*(b)) = \begin{cases} b' \otimes \tilde{f}_i^{m-1}(b_i) & \text{if } m \geq 0, \\ 0 & \text{if } m = 0. \end{cases}$$

It follows from the previous proposition that

$$\mathrm{Im}(\Theta_i) = \{b \otimes \tilde{f}_i^m(b_i) \in B_{\mathrm{Kas}} \otimes B_i \mid m \in \mathbb{Z}_+, (\varepsilon_i^{\mathrm{Kas}})^*(b) = 0\}.$$

Comparing this with the definition of the map Ψ_i, and taking into account that we have identified the maps Ψ_i and Θ_i, we see that we also may identify the maps ε_i^* with the maps $(\varepsilon_i^{\mathrm{Kas}})^*$, for all $i \in I$.
Following the discussion after [54, Theorem 8.1], for all $\Lambda \in P_+$, we have an embedding

$$\iota_\Lambda : B(\Lambda)_{\mathrm{Kas}} \to B_{\mathrm{Kas}} \otimes T_\Lambda, \quad b \mapsto b \otimes t_\Lambda$$

of crystals, where $B(\Lambda)_{\mathrm{Kas}}$ denotes the crystal associated to the irreducible highest-weight module of highest weight Λ over $U_v(\mathfrak{g}(A))$. In [54, Proposition 8.2] a description of the image of the this map is given in terms of the maps $(\varepsilon_i^{\mathrm{Kas}})^*$:

Proposition 14.2.15. *For any $\Lambda \in P_+$, the image of ι_Λ equals the set*

$$\{b \otimes t_\Lambda \in B_{\mathrm{Kas}} \otimes T_\Lambda \mid (\varepsilon_i^{\mathrm{Kas}})^*(b) \leq \Lambda(h_i) \text{ for all } i \in I\}.$$

Therefore, in view of Lemma 14.2.8 and our identifications made above, we can conclude that for $\Lambda \in P_+$ we have an isomorphism of crystals between $B(\Lambda)$ and $B(\Lambda)_{\mathrm{Kas}}$. This establishes Theorem 14.0.12 of the beginning of this chapter.

Chapter 15

The theorem of Misra–Miwa

A major problem in the representation theory of quantum groups $U_v(\mathfrak{g}(A))$ is to describe explicitly the structure of the crystal graph associated with an irreducible highest-weight module $L(\Lambda)$ of highest weight $\Lambda \in P_+$. In the case when A is a generalized Cartan matrix of type $A_n^{(1)}$, $n \geq 1$, this problem was solved in [70], where it is shown that the crystal graph $B(\Lambda_0)$ of the basic representation $L(\Lambda_0)$ can be described in terms of Young diagrams. In this chapter we explain this parametrization of the latter crystal and deduce branching rules for the finite Hecke algebra $H_n^f(q)$, $n \geq 0$. The notation used in this chapter can be found in [68].

Let $\mu = (\mu_1, \dots, \mu_r)$ be a composition of the positive integer n, and l be a positive integer. Then μ is called *l-restricted* if $|\mu_j - \mu_{j+1}| < l$ for all $j \in \mathbb{Z}_+$, where we set $\mu_j = 0$ for $j > k$. For a composition μ, we denote by $[\mu]$ its associated diagram. For each node x of the diagram of μ, we shall define its *l-residue* as the coset $(j - i) \bmod l$, where x is located in the ith row and jth column of the diagram $[\mu]$. In what follows we will write $I = \{0, \dots, l-1\}$ for the elements in $\mathbb{Z}/l\mathbb{Z}$. Also, define $\mathrm{cont}(\mu) = (\gamma_i)_{i \in I}$, where $\gamma_i = |\{x \in [\mu] \mid x \text{ has } l - \text{content } i\}|$, for all i.

Now suppose that λ is a partition of n. A node x of $[\lambda]$ with l-content i is called *i-removable* if the diagram $[\lambda] \setminus \{x\}$ is a diagram of a partition of $n-1$. In this case we write λ_x for the partition $\lambda \setminus x$. On the other hand a node x is called *i-addable* if there is a partition ν of $n+1$ such that $[\nu] = [\lambda] \cup \{x\}$, and x has l-content i. In this case, we will also write λ^x for the partition ν. It is clear that if a node x of $[\lambda]$ is i-removable (resp. i-addable), then it lies on the rim of $[\lambda]$ (resp. $[\lambda^x]$).

If we read the i-addable and i-removable nodes in $[\lambda]$ from the bottom up, and write the symbol A (resp. R) each time we have an i-addable (resp. i-removable) node, we get a sequence consisting of the symbols A and R, the *i-signature* of λ. If we delete all of the substrings of the form AR, we obtain the *reduced i-signature* of λ. The reduced i-signature looks like a sequence of Rs followed by As. A node corresponding to an R in the reduced i-signature are called *i-normal*, a node corresponding to an A in the reduced i-signature is called *i-conormal*.

The rightmost i-normal node in $[\lambda]$ is called *i-good*, the leftmost i-conormal node in $[\lambda]$ is called *i-cogood*.

A node of $[\lambda]$ is called *normal* (resp. *conormal*, *good*, *cogood*) if it is i-normal (resp. i-conormal, i-good, i-cogood) for some $i \in I$. Note that a good (resp. cogood) node is, if it exists, unique.

For λ a partition of n and μ a partition of $n+1$ we write $\lambda \overset{\text{norm}}{\longrightarrow} \mu$ if $[\mu]$ is obtained from $[\lambda]$ by adding a normal node. Similarly write $\lambda \overset{\text{good}}{\longrightarrow} \mu$ if $[\mu]$ is obtained from $[\lambda]$ by adding a good node.

Let λ be a partition of some $n \in \mathbb{Z}_+$. For any $i \in I$ we shall define

$$\begin{aligned} \varepsilon_i(\lambda) &= |\{i - \text{normal nodes in } [\lambda]\}|, && (15.1)\\ \varphi_i(\lambda) &= |\{i - \text{conormal nodes in } [\lambda]\}|. && (15.2) \end{aligned}$$

Moreover, we set

$$\tilde{e}_i(\lambda) = \begin{cases} \lambda_x & \text{if } \varepsilon_i(\lambda) > 0 \text{ and } x \text{ is } i - \text{good},\\ 0 & \text{if } \varepsilon_i(\lambda) = 0, \end{cases} \tag{15.3}$$

$$\tilde{f}_i(\lambda) = \begin{cases} \lambda^x & \text{if } \varphi_i(\lambda) > 0 \text{ and } x \text{ is } i - \text{cogood},\\ 0 & \text{if } \varphi_i(\lambda) = 0. \end{cases} \tag{15.4}$$

In the following, for a non-negative integer n, we will write $\mathcal{P}_l(n)$ for the set of all l-restricted partitions of n. Also, set $\mathcal{P}_l = \bigcup_{n\geq 0} \mathcal{P}_l(n)$. Note that we write $\oslash$ for the empty partition, which is obviously l-restricted. We have the following lemma:

Lemma 15.1.16. *For any $i \in I$, $\tilde{e}_i(\lambda)$ and $\tilde{f}_i(\lambda)$ are l-restricted (or zero) if λ is l-restricted.*

Proof. Suppose that $\lambda = (\lambda_1, \ldots, \lambda_k)$ is l-restricted with an i-good node $x = (r, s)$, $i \in I$. If λ_x is not l-restricted, then $\lambda_{r-1} - \lambda_r = l - 1$. Since the node $(r-1, s+l-1)$ on the rim of $[\lambda]$ has the same l-content as the node (r, s), we see that this node is also an i-removable node. But this is a contradiction to the definition of an i-good node. Thus, if λ is l-restricted, then $\tilde{e}_i(\lambda)$ is l-restricted or zero. Similarly one shows that $\tilde{f}_i(\lambda)$ is l-restricted (or zero) if λ is. $\square$

If λ is a partition of n with $\mathrm{cont}(\lambda) = (\gamma_i)_{i\in I}$, define

$$\mathrm{wt}(\lambda) = \Lambda_0 - \sum_{i\in I} \gamma_i \alpha_i. \tag{15.5}$$

With these data, we have the following theorem, which is due to K. C. Misra and T. Miwa:

Theorem 15.1.17. *The datum $(\mathcal{P}_l, \varepsilon_i, \varphi_i, \tilde{e}_i, \tilde{f}_i, \mathrm{wt})$ determines a crystal. As a crystal it is isomorphic to $B(\Lambda_0)_{\mathrm{Kas}}$, the crystal graph of the irreducible highest-weight module $L(\Lambda_0)$ of highest weight Λ_0 over $U_v(\mathfrak{g}(A^{(1)}_{l-1}))$.*

Proof. This is [70, Theorem 4.7]. □

Remark 15.1.18. For $\lambda \in \mathcal{P}_l(n)$, by the previous theorem, we may write $\lambda = \tilde{f}_{i_n} \cdots \tilde{f}_{i_1}(\oslash)$, for some sequence $(i_1, \ldots, i_n)$ of elements of I. In view of Theorem 14.0.12, we may identify the crystal $B(\Lambda_0)$ with the crystal $\mathcal{P}_l$, and the operators $\tilde{e}_i$ and $\tilde{e}_i^{\Lambda_0}$ as well as $\tilde{f}_i$ and $\tilde{f}_i^{\Lambda_0}$, for all $i \in I$. Thus, to such λ as above, there corresponds an irreducible $H_n^f(q)$-module, denoted by C_λ, which is given by

$$C_\lambda = \tilde{f}_{i_n} \cdots \tilde{f}_{i_1}(1_{\Lambda_0}). \tag{15.6}$$

Conversely, if C is an irreducible $H_n^f(q)$-module, we can use the identification of $\mathcal{P}_l$ and $B(\Lambda_0)$ to label C by a partition in $\mathcal{P}_l(n)$. In this way, we obtain a labelling for the irreducible $H_n^f(q)$-modules, $n \geq 0$.

Taking into account the results from the former chapters, we obtain a branching rule for irreducible modules for the finite-dimensional Hecke algebras $H_n^f(q) = H_n^{\Lambda_0}(q)$. In what follows we write e_i, f_i for the cyclotomic functors $e_i^{\Lambda_0}, f_i^{\Lambda_0}$, for all $i \in I$.

Theorem 15.1.19. *Let $n \geq 0$ and $\lambda \in \mathcal{P}_l(n)$. Then*

$$\mathrm{Res}_{n-1}^n(C_\lambda) \cong e_0(C_\lambda) \oplus \cdots \oplus e_{l-1}(C_\lambda),$$

where for every $i \in I$, $e_i(C_\lambda) \neq 0$ if and only if λ has an i-good node x. In this case, $e_i(C_\lambda)$ is an indecomposable self-dual $H_{n-1}^f(q)$-module with irreducible head and socle isomorphic to C_{λ_x}. Moreover, one has:

(i) *The multiplicity of C_{λ_x} in $e_i(C_\lambda)$ equals $\varepsilon_i(\lambda)$, $\varepsilon_i(\lambda_x) = \varepsilon_i(\lambda) - 1$, and all other composition factors C_μ of $e_i(C_\lambda)$ satisfy $\varepsilon_i(\mu) < \varepsilon_i(\lambda) - 1$.*

(ii) *The algebra $\mathrm{End}_{H_{n-1}^f(q)}(e_i(C_\lambda))$ is isomorphic to the F-algebra $F[x]/(x^{\varepsilon_i(\lambda)})$. Hence,*

$$\varepsilon_i(\lambda) = \dim_F \mathrm{End}_{H_{n-1}^f(q)}(e_i(C_\lambda)).$$

(iii) *If $\lambda, \nu \in \mathcal{P}_l(n)$ are such that $\lambda \neq \nu$, then $\mathrm{Hom}_{H_{n-1}^f(q)}(e_i(C_\lambda), e_i(C_\nu)) = 0$.*

(iv) *$e_i(C_\lambda)$ is irreducible if and only if $\varepsilon_i(\lambda) = 1$. In particular, $\mathrm{Res}_{n-1}^n(C_\lambda)$ is completely reducible if and only if $\varepsilon_i(\lambda) \leq 1$ for all $i \in I$, and is irreducible if and only if $\sum_{i \in I} \varepsilon_i(\lambda) = 1$.*

Proof. The first statement follows from Proposition 11.2.11 together with the labelling given in Remark 15.1.18. The fact that $e_i(C_\lambda)$ is indecomposable follows from Theorem 12.1.8.
By Proposition 12.2.3, it follows that $e_i(C_\lambda)$ is a self-dual $H_{n-1}^f(q)$-module. From Proposition 12.3.4 (ii) it follows that $\mathrm{soc}(e_i(C_\lambda)) \cong \mathrm{hd}(e_i(C_\lambda))$. Since $\mathrm{soc}(e_i(C_\lambda)) = \tilde{e}_i(C_\lambda)$ by definition of $\tilde{e}_i(C_\lambda)$ and $\tilde{e}_i(C_\lambda) = C_{\lambda_x}$ by our identification, we see that $\mathrm{soc}(e_i(C_\lambda)) \cong \mathrm{hd}(e_i(C_\lambda)) \cong C_{\lambda_x}$.
The statement in (i) is Theorem 12.3.6 (i) together with our identification of $\mathcal{P}_l$ and $B(\Lambda_0)$. Similarly, (ii) is Theorem 12.3.6 (ii). Statement (iii) is Corollary 12.3.7

again combined with the identification of $\mathcal{P}_l$ and $B(\Lambda_0)$. Since $e_i(C_\lambda)$ has irreducible socle and head, the statement in (iv) follows from (i). $\square$

Also the following is true.

Theorem 15.1.20. *Let $n \geq 0$ and $\lambda \in \mathcal{P}_l(n)$. Then*

$$\mathrm{Ind}_n^{n+1}(C_\lambda) \cong f_0(C_\lambda) \oplus \cdots \oplus f_{l-1}(C_\lambda),$$

where for every $i \in I$, $f_i(C_\lambda) \neq 0$ if and only if λ has an i-cogood node x. In this case, $f_i(C_\lambda)$ is an indecomposable self-dual $H^f_{n+1}(q)$-module with irreducible head and socle isomorphic to C_{λ^x}. Moreover, one has:

(i) *The multiplicity of C_{λ^x} in $f_i(C_\lambda)$ equals $\varphi_i(\lambda)$, $\varphi_i(\lambda^x) = \varphi_i(\lambda) - 1$, and all other composition factors C_μ of $f_i(C_\lambda)$ satisfy $\varphi_i(\mu) < \varphi_i(\lambda) - 1$.*

(ii) *The algebra $\mathrm{End}_{H^f_{n+1}(q)}(f_i(C_\lambda))$ is isomorphic to the F-algebra $F[x]/(x^{\varphi_i(\lambda)})$, which implies that*

$$\varphi_i(\lambda) = \dim_F \mathrm{End}_{H^f_{n+1}(q)}(f_i(C_\lambda)).$$

(iii) *We have that $\mathrm{Hom}_{H^f_{n+1}(q)}(f_i(C_\lambda), f_i(C_\nu)) = 0$ for all $\lambda, \nu \in \mathcal{P}_l(n)$ such that $\lambda \neq \nu$.*

(iv) *$f_i(C_\lambda)$ is irreducible if and only if $\varphi_i(\lambda) = 1$. In particular, $\mathrm{Ind}_n^{n+1}(C_\lambda)$ is completely reducible if and only if $\varphi_i(\lambda) \leq 1$ for all $i \in I$, and is irreducible if and only if $\sum_{i\in I} \varphi_i(\lambda) = 1$.*

Proof. This is proven similarly as Theorem 15.1.19, using Theorem 12.3.13 for part (i). Part (ii) can be deduced from the proof of [44, Theorem 9.15]. Part (iii) is proven similarly as Corollary 12.3.7, and part (iv) is a consequence of (i). $\square$

We also can state results about the functors $(e_i^{\Lambda_0})^{(r)}, (f_i^{\Lambda_0})^{(r)}$. $i \in I$. For convenience, we will write $e_i^{(r)}, f_i^{(r)}$ instead of $(e_i^{\Lambda_0})^{(r)}, (f_i^{\Lambda_0})^{(r)}$.

Theorem 15.1.21. *Let $n \geq 0$ and $\lambda \in \mathcal{P}_l(n)$. Moreover, let $i \in I$, and $r \geq 1$. Then*

$$e_i^r(C_\lambda) \cong \bigoplus_{r!} e_i^{(r)}(C_\lambda),$$

where $e_i^{(r)}(C_\lambda) \neq 0$ if and only if λ has at least r i-normal nodes. In this case, $e_i^{(r)}(C_\lambda)$ is a self-dual indecomposable $H^f_{n-r}(q)$-module with irreducible head and socle isomorphic to C_μ, where μ is obtained from λ by removing r top i-normal nodes. Furthermore,

(i) *The multiplicity of C_μ in $e_i^{(r)}(C_\lambda)$ is $\binom{\varepsilon_i(\lambda)}{r}$, $\varepsilon_i(\mu) = \varepsilon_i(\lambda) - r$, and $\varepsilon_i(\nu) < \varepsilon_i(\lambda) - r$ for all other composition factors C_ν of $e_i^{(r)}(C_\lambda)$.*

(ii) $\mathrm{Hom}_{H^f_{n-r}(q)}(e_i^{(r)}(C_\lambda), e_i^{(r)}(C_\nu)) = 0$, *for all $\nu \in \mathcal{P}_l(n)$ with $\nu \neq \lambda$.*

(iii) $e_i^{(r)}(C_\lambda)$ *is irreducible if and only if* $r = \varepsilon_i(\lambda)$.

Proof. The first part follows from Proposition 12.3.9, and the labelling of Remark 15.1.18. Statement (i) is Proposition 12.3.10 (i).
Part (ii) is the statement of Proposition 12.3.10 (ii) together with our labelling. Finally, (iii) follows from (i). □

Theorem 15.1.22. *Let* $n \geq 0$ *and* $\lambda \in \mathcal{P}_l(n)$. *Moreover, let* $i \in I$, *and* $r \geq 1$. *Then*

$$f_i^r(C_\lambda) \cong \bigoplus_{r!} f_i^{(r)}(C_\lambda),$$

where $f_i^{(r)}(C_\lambda) \neq 0$ *if and only if* λ *has at least* r i*-conormal nodes. In this case,* $f_i^{(r)}(C_\lambda)$ *is a self-dual indecomposable* $H_{n+r}^f(q)$*-module with irreducible head and socle isomorphic to* C_μ*, where* μ *is obtained from* λ *by adding* r *bottom* i*-conormal nodes. Moreover:*

(i) *The multiplicity of* C_μ *in* $f_i^{(r)}(C_\lambda)$ *is* $\binom{\varphi_i(\lambda)}{r}$, $\varphi_i(\mu) = \varphi_i(\lambda) - r$, *and* $\varphi_i(\nu) < \varphi_i(\lambda) - r$ *for all other composition factors* C_ν *of* $f_i^{(r)}(C_\lambda)$.

(ii) $\mathrm{Hom}_{H_{n+r}^f(q)}(f_i^{(r)}(C_\lambda), f_i^{(r)}(C_\nu)) = 0$, *for all* $\nu \in \mathcal{P}_l(n)$ *with* $\nu \neq \lambda$.

(iii) $f_i^{(r)}(C_\lambda)$ *is irreducible if and only if* $r = \varphi_i(\lambda)$.

Proof. As in the proof of Theorem 15.1.21, the first part follows from Proposition 12.3.9, and the labelling of Remark 15.1.18. Statement (i) is Proposition 12.3.14 (i).
Part (ii) is the statement of Proposition 12.3.14 (ii) together with our labelling. Also, (iii) can be deduced from (i). □

Remark 15.1.23. The corresponding branching rules in the degenerate case can be found in [59, §11].

Chapter 16

The Brundan–Kleshchev branching rules

In this chapter we recall the modular branching rules for Hecke algebras of type A, obtained by J. Brundan in [16], generalizing the results of [58] for the symmetric groups. We will also explain how this is related to the labelling obtained in the last chapter.

Recall that for a non-negative integer n, we denote the finite Hecke algebra of the symmetric group on n letters with parameter $q \in F$ by $H_n^f(q)$. We assume that $q \neq 1$ has finite order l in F.
In what follows, for a partition λ of $n \in \mathbb{Z}_+$, we will denote by S^λ the q-Specht module defined by Dipper and James in [29, §4]. Following [16], in our context here, we will use a different parametrization as follows: Denote by S_μ the q-Specht module $S^{\mu'}$ of Dipper and James, where by μ' we mean the partition conjugate to μ, i.e., if $\mu = (\mu_1, \ldots, \mu_k)$, then $\mu' = (\mu'_1, \ldots, \mu'_{k'})$ is such that $\mu'_j = |\{s \in \{1, \ldots, k\} \mid \mu_s \geq j\}|$, which again is a partition of n. In [29] it was shown that if λ is an l-regular partition of n, then S^λ has an irreducible head denoted by D^λ. Thus, if μ is an l-restricted partition of n, then S_μ has an irreducible head denoted by D_μ. It is easy to see that μ is l-restricted if and only if μ' is l-regular. Hence, if μ is l-restricted, it follows that D_μ equals $D^{\mu'}$.
The following are the main results of [16].

Theorem 16.1.24. *Let μ be an l-restricted partition of $n \in \mathbb{Z}_+$ and λ be an l-restricted partition of $n+1$. Then:*

$$\operatorname{Hom}_{H_n^f(q)}(S_\mu, \operatorname{Res}_n^{n+1}(D_\lambda)) \cong \begin{cases} F & \textit{if } \mu \stackrel{\text{norm}}{\longrightarrow} \lambda, \\ 0 & \textit{otherwise} . \end{cases} \tag{16.1}$$

Proof. This is [16, Theorem 2.5]. □

Theorem 16.1.25. *Let μ be an l-restricted partition of $n \in \mathbb{Z}_+$ and λ be an l-restricted partition of $n+1$. Then:*

$$\operatorname{Hom}_{H_n^f(q)}(D_\mu, \operatorname{Res}_n^{n+1}(D_\lambda)) \cong \begin{cases} F & \textit{if } \mu \stackrel{\text{good}}{\longrightarrow} \lambda, \\ 0 & \textit{otherwise} . \end{cases} \tag{16.2}$$

In particular we have that

$$\mathrm{soc}(\mathrm{Res}_n^{n+1}(D_\lambda)) \cong \bigoplus_{\mu \xrightarrow{\text{good}} \lambda} D_\mu. \tag{16.3}$$

Proof. This is [16, Theorem 2.6]. ☐

Remark 16.1.26. By Frobenius Reciprocity, we see that

$$\mathrm{Hom}_{H^f_{n+1}(q)}(\mathrm{Ind}_n^{n+1}(D_\mu), D_\lambda) \cong \begin{cases} F & \text{if } \mu \xrightarrow{\text{good}} \lambda, \\ 0 & \text{otherwise .} \end{cases} \tag{16.4}$$

Hence,

$$\mathrm{hd}(\mathrm{Ind}_n^{n+1}(D_\mu)) \cong \bigoplus_{\mu \xrightarrow{\text{good}} \lambda} D_\lambda. \tag{16.5}$$

In particular, from these theorems, we obtain a branching rule for the irreducible $H^f_n(q)$-modules, $n \geq 0$. It is natural to ask whether it coincides with the branching rule given by Theorem 15.1.17. We have the following:

Proposition 16.1.27. *The labelling of Chapter 15 of the irreducible $H^f_n(q)$-modules, $n \geq 0$, coincides with that induced by Theorem 16.1.25.*

Proof. We argue by induction. For $n = 0$, there is only one irreducible $H^f_0(q)$-module, hence $D_\oslash \cong C_\oslash$. Let $n > 0$, and choose a partition $\lambda \in \mathcal{P}_l(n)$. Suppose that $D_\lambda \cong C_\mu$, for some l-restricted partition μ of n. By Theorem 16.1.25, we know that

$$\mathrm{soc}(\mathrm{Res}_{n-1}^{n}(D_\lambda)) \cong \bigoplus_{\nu \xrightarrow{\text{good}} \lambda} D_\nu. \tag{16.6}$$

On the other hand, by the first part of Theorem 15.1.19, we have that

$$\mathrm{soc}(\mathrm{Res}_{n-1}^{n}(D_\lambda)) \cong \bigoplus_{i \in I} \tilde{e}_i(D_\lambda). \tag{16.7}$$

Suppose that $\tilde{e}_i(D_\lambda) \neq 0$, for some $i \in I$. By Theorem 15.1.19, we have that $\tilde{e}_i(D_\lambda) \cong C_{\mu_x}$ for some i-good node x of μ. Moreover, by the inductive hypothesis, $C_{\mu_x} \cong D_{\mu_x}$. Then, again by Theorem 16.1.25, we get that that $\mu_x \xrightarrow{\text{good}} \lambda$. The structure of the crystal graph $\mathcal{P}_l$ ensures that for each $i \in I$ there exists at most one partition of n that is obtained from μ_x by adding an i-cogood node. Since μ is such a partition, it follows that $\lambda = \mu$. ☐

Chapter 17

Scopes equivalence

In the sequel we will recall the fundamental result concerning Morita equivalent blocks of symmetric groups and their associated Hecke algebras, respectively. This was done in [80] and [52].
First, we give the necessary definitions of the combinatorics which are needed to obtain these results. After that we may restate the latter equivalence in terms of the functors which we defined in the previous chapters. As a consequence we may drop the restrictions on F and q of [52].
We shall assume throughout that $q \neq 1$ has finite order in F.

17.1 Combinatorics

We will turn our attention to the finite-dimensional Hecke algebra $H_n^f(q)$, for some $n \geq 0$. In [52] it was shown that the blocks B and $\bar{B}$ of $H_n^f(q)$ and $H_{n-k}^f(q)$ of weight w are Morita equivalent if they form a $[w:k]$-pair, where $k \geq w$. There, a bijection is given between the l-regular partitions of n lying in B and those l-regular partitions of $n-k$ lying in $\bar{B}$.
In view of Chapter 15, we want to translate this bijection to the l-restricted partitions of n resp. $n-k$. Recall that if λ is a partition of n, we denote by S_λ the q-Specht module defined by R. Dipper and G. James in [29], which they indexed by λ', the partition conjugate to λ. Therefore, if λ is l-restricted, by D_λ we mean the irreducible $H_n^f(q)$-module defined by Dipper–James and indexed by λ'. Note that λ is l-restricted if and only if λ' is l-regular.
First of all we want to explain the combinatorics used in [80] and [52]: Let $\lambda = (\lambda_1, \ldots, \lambda_s)$ be a partition of n. A sequence of *β-numbers* for λ is given by non-negative integers $\beta_1 > \beta_2 > \ldots > \beta_t, t \geq s$, satisfying

$$\beta_i = \begin{cases} \lambda_i - i + t & \text{if } i \leq s, \\ -i + t & \text{if } i > s. \end{cases} \tag{17.1}$$

Such a sequence is usually displayed on an *abacus* with a bead at each position β_i on the latter. The number of runners $l > 0$ relates to the order of q in F. If we use a display with l runners, we will say that the numbers $i, l+i, 2l+i, \ldots$ belong to

the $(i+1)$st column, for $i = 0, \dots l-1$, and that $m-1$ is *one place to the left* of m for $m \geq 1$.
We need the following two operations on the abacus of a β-set: First, if possible, we may slide a bead up along its runner. This corresponds to removing a rim l-hook from $[\lambda]$. When all the beads are as high as possible, we have the β-set for the corresponding l-core of λ.
Second, we consider the operation of sliding a bead one place to the left, from position m to $m-1$. This corresponds to removing a node from $[\lambda]$.
Next, let B denote a block of $H_n^f(q)$ with l-weight w and l-core $b = (b_1, \dots, b_r)$, which is a partition of $n - lw$. Let $\Gamma = \{\gamma_1, \dots, \gamma_{r+lw}\}$ be the $r + lw$-element β-set for b. Denote by Γ_j, $1 \leq j \leq l$, the number of beads in the jth column of the associated abacus with l runners. Note that if λ is a partition of n such that S_λ belongs to the block B, we always can represent λ by an $r + lw$-element β-set.
Suppose that for some $i \geq 2$, $\Gamma_i = \Gamma_{i-1} + k$, where $k \geq w$. Let $\bar{B}$ be the block of $H_{n-k}^f(q)$ of weight w and core $\bar{b}$, where $\bar{b}$ has an $r + lw$-element β-set that can be obtained from that of b by sliding the bottom k beads of column i to column $i-1$. Let $\bar{\Gamma}$ be the β-set of $\bar{b}$, and $\bar{\Gamma}_j$ the number of beads in the jth column of $\bar{\Gamma}$. Then we have that

$$\begin{aligned} \bar{\Gamma}_j &= \Gamma_j, \ j \neq i, i-1, \\ \bar{\Gamma}_i &= \Gamma_{i-1}, \\ \bar{\Gamma}_{i-1} &= \Gamma_i. \end{aligned}$$

In this situation we say that the blocks B and $\bar{B}$ form a $[w:k]$-*pair*.

We note the modular branching theorem for q-Specht modules.

Theorem 17.1.1. *Let $n \geq 1$, λ be a partition of n. There exists a filtration of $\mathrm{Res}_{n-1}^n(S_\lambda)$ by q-Specht modules of $H_{n-1}^f(q)$. The factors occurring in the latter are of the form S_μ, where μ is a partition of $n-1$, and $[\mu]$ can be obtained from $[\lambda]$ by removing a node. Moreover, each factor appears with multiplicity* 1.
Similarly, $\mathrm{Ind}_n^{n+1}(S_\lambda)$ has a filtration of q-Specht modules for $H_{n+1}^f(q)$, where each factor is isomorphic to some S_ν, ν a partition of $n+1$, where $[\nu]$ is obtained from $[\lambda]$ by adding a node. Each factor occurs precisely once.

Proof. This is [52, Theorem 3.4]. □

Now, if two blocks B and $\bar{B}$ form a $[w:k]$-pair, it was shown in [80, Lemma 2.1] for blocks of symmetric groups and [52, Lemma 5.1] for blocks of Hecke algebras of type A that there is a natural correspondence between partitions of n whose q-Specht modules belong to B and those of $n-k$ whose q-Specht modules belong to $\bar{B}$. To be more precise:

Proposition 17.1.2. *Let λ be a partition of n and suppose that S_λ belongs to B. Then there exists a partition $\bar{\lambda}$ of $n-k$ such that $S_{\bar{\lambda}}$ belongs to $\bar{B}$ and $e_{\bar{B}}\mathrm{Res}_{n-k}^n(S_\lambda)$*

has a filtration such that each subquotient is isomorphic to $S_{\bar{\lambda}}$, which occurs $k!$ times. Here $e_{\bar{B}}$ denotes the block idempotent of $H^f_{n-k}(q)$ corresponding to $\bar{B}$.
Similarly, if e_B denotes the block idempotent of $H^f_n(q)$ corresponding to B, then $e_B\mathrm{Ind}^n_{n-k}(S_{\bar{\lambda}})$ has a filtration such that each subquotient is isomorphic to S_λ, appearing $k!$ times in the latter.

Proof. The first part is similar to [80, Lemma 2.1]. For the statement about the induced module, one can use [69, Theorem A]. □

Remark 17.1.3. Note that Proposition 17.1.2 gives a bijective map Θ between the set of partitions of n for which the corresponding q-Specht modules belong to B and the set of partitions of $n-k$ for which the corresponding q-Specht modules belong to $\bar{B}$. Moreover, if λ is a partition of n belonging to B, then the abacus of $\Theta(\lambda)$ is obtained from that of λ by sliding k beads from runner i to runner $i-1$, that is to say, by interchanging runner i and runner $i-1$.
The crucial fact here is that whenever there is a bead in row j in the $(i-1)$st column of the abacus of λ, there is a bead in the same row in the ith column, see the proof of [80, Lemma 2.1]. This determines which beads can be moved. Let r be the number of parts of the l-core parametrizing the block B. In the language of Chapter 15, this means that if we set $a = (i-r-1) \bmod l$, then the number of a-removable nodes of $[\lambda]$ is k, whereas $[\lambda]$ has no a-addable nodes. It follows that all the a-removable nodes of $[\lambda]$ are a-normal.

The following two lemmas are important.

Lemma 17.1.4. *The map Θ preserves the lexicographic order of partitions.*

Proof. The argument of [80, Lemma 2.2] also applies in our context as the proof of this result is purely combinatorial. □

Lemma 17.1.5. *If λ is a an l-restricted partition lying in B, then $\Theta(\lambda)$ is also l-restricted.*

Proof. This follows from [80, Lemma 2.3], again the proof of the latter is purely combinatorial. □

As a consequence, Θ induces a bijective map between the l-restricted partitions of B and the l-restricted partitions of $\bar{B}$.

17.2 The equivalence

By combining Proposition 17.1.2, Lemma 17.1.5, together with Theorems 15.1.21, 15.1.22, we obtain the following reformulation of equivalence given by J. Scopes and T. Jost:

Theorem 17.2.1. *Let B a block of $H^f_n(q)$ and $\bar{B}$ be a block of $H^f_{n-k}(q)$ so that B and $\bar{B}$ form a $[w:k]$-pair, with $k \geq w$. Assume that the nodes of a partition λ of*

n that are removed to obtain $\Theta(\lambda)$ have l-content i. Then the functors $e_i^{(k)}$ and $f_i^{(k)}$ induce mutually inverse equivalences between the category $B-\text{mod}$ and the category $\bar{B}-\text{mod}$.

Proof. Proposition 17.1.2, 16.1.27, Lemma 17.1.5, and Theorems 15.1.21, 15.1.22, show that the functors $e_i^{(k)}$ and $f_i^{(k)}$ induce mutually inverse bijections between the irreducible $H_n^f(q)$-modules lying in B and those irreducible $H_{n-k}^f(q)$-modules lying in $\bar{B}$. Recall from Theorem 11.2.15 that the functor $S := f_i^{(k)}$ is left adjoint to the functor $T := e_i^{(k)}$. For $M \in \bar{B}-\text{mod}$ and $N \in B-\text{mod}$, let $\eta_{M,N}$ be the isomorphism of F-vector spaces

$$\text{Hom}_{H_n^f(q)}(S(M), N) \longrightarrow \text{Hom}_{H_{n-r}^f(q)}(M, T(N))$$

given by the adjunction. Recall that an adjunction comes with natural transformations

$$\xi : ST \to \text{id}_{H_n^f(q)} \text{ and } \zeta : \text{id}_{H_{n-k}^f(q)} \to TS$$

given by $\xi_N = \eta^{-1}_{T(N),N}(\text{id}_{T(N)}) : ST(N) \to N$, and $\zeta_M = \eta_{M,S(M)}(\text{id}_{S(M)}) : M \to TS(M)$, where $\text{id}_{H_n^f(q)}$ and $\text{id}_{H_{n-k}^f(q)}$ denote the identity functors on $H_n^f(q)-\text{mod}$ and $H_{n-k}^f(q)-\text{mod}$, respectively.
Suppose first that M is irreducible. By choosing $N = S(M)$, we get the non-zero map

$$M \xrightarrow{\zeta_M} TS(M).$$

Since $e_i^{(k)}(f_i^{(k)}(M)) \cong M$, we see that this map must be an isomorphism of $H_{n-k}^f(q)$-modules. Now assume that M in $\bar{B}-\text{mod}$ has composition length greater than one. Then we may consider an exact sequence of $\bar{B}$-modules

$$0 \longrightarrow M_1 \longrightarrow M \longrightarrow M_2 \longrightarrow 0,$$

such that M_1 and M_2 have composition length less than M. By the naturality of ζ, we get a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M_1 & \longrightarrow & M & \longrightarrow & M_2 & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \zeta_{M_1}} & & \downarrow{\scriptstyle \zeta_M} & & \downarrow{\scriptstyle \zeta_{M_2}} & & \\
0 & \longrightarrow & TS(M_1) & \longrightarrow & TS(M) & \longrightarrow & TS(M_2) & \longrightarrow & 0
\end{array}$$

Arguing by induction on the composition length, we may assume that ζ_{M_1} and ζ_{M_2} are isomorphisms of $\bar{B}$-modules. By the five lemma, see for example [65, VIII.4, Lemma 4], it follows that ζ_M must be an isomorphism of $\bar{B}$-modules too. This provides a natural isomorphism between TS and the identity functor on $\bar{B}-\text{mod}$. On the other hand, via the natural transformation ξ, we obtain a non-zero map

$$ST(N) \xrightarrow{\xi_N} N.$$

If we assume N in $B-\text{mod}$ to be irreducible, this map is an isomorphism of $H_n^f(q)$-modules as $ST(N) \cong N$. If N has composition length greater than one, there is an exact sequence of B-modules

$$0 \longrightarrow N_1 \longrightarrow N \longrightarrow N_2 \longrightarrow 0,$$

where the composition length of N_1 and N_2 is less than N. The naturality of ξ affords a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & TS(N_1) \arrow[r] \arrow[d, "\xi_{N_1}"] & TS(N) \arrow[r] \arrow[d, "\xi_N"] & TS(N_2) \arrow[r] \arrow[d, "\xi_{N_2}"] & 0 \\
0 \arrow[r] & N_1 \arrow[r] & N \arrow[r] & N_2 \arrow[r] & 0
\end{tikzcd}$$

As above, we may assume that ξ_{N_1} and ξ_{N_2} are isomorphisms of B-modules. Then, by the five lemma again, we infer that ξ_N is an isomorphism of B-modules. This shows that we get a natural isomorphism between TS and the identity functor on $B-\text{mod}$, showing that $B-\text{mod}$ and $\bar{B}-\text{mod}$ are equivalent categories. □

Remark 17.2.2. (i) Actually, we have shown in the Chapter 11 that the functors $e_i^{(k)}$ and $f_i^{(k)}$ are both left and right adjoint to one another, whereas for the proof of the previous theorem we only need the adjunction on one side.
(ii) Note that the restatement of the Scopes equivalence in the case of blocks of symmetric groups is [59, Theorem 11.2.28].

Note that we can partition the set of blocks of Hecke algebras $H_n^f(q)$, $n \geq 0$, into families. Two blocks B and B' belong to the same family if and only if there exists a sequence $B = B_0, B_1, \ldots, B_e = B'$, $e \geq 1$, of blocks of weight w such that for all $1 \leq j \leq e$, the blocks B_{j-1} and B_j or B_j and B_{j-1} form a $[w : k_j]$-pair for some $k_j \geq w$. Note that, by Theorem 17.2.1, all blocks in one family are Morita equivalent. The next theorem is a consequence of this fact.

Theorem 17.2.3. *The blocks of Hecke algebras of type A with parameter $q \in F$ of a given weight w can be collected into families as described above. Each family consists of Morita equivalent blocks. The number of such families is at most $\prod_{j=1}^{l}[(j-1)(w-1)+1]$, and each family contains a block of some $H_N^f(q)$, where N is at most $l^2(l-1)^2(w-1)^2/4 + wl$.*

Proof. The statement of the number of such families and the least element contained in such a family can be found in [80, Section 5] as the proof is purely combinatorial. □

Remark 17.2.4. Compare the previous theorem with [52, Theorem 8.2]. Note that in [52] there is a restriction on the parameter q if the ground field F has characteristic $p > 0$. Our statement is valid over an arbitrary field of characteristic p. Note that F need not to be algebraically closed since every field is a splitting field for $H_n^f(q)$, $n \geq 0$, and all the eigenvalues lie in F.

As a consequence of the last theorem we get the following corollary, see also [80, Example 1].

Corollary 17.2.5. *There is only one family of Morita equivalent blocks of weight* $w = 1$*. Moreover, all these blocks are Scopes equivalent to the principal block of* $H_l^f(q)$*.*

Chapter 18

The application

In this chapter we will prove the main results of the second part of the thesis. Namely we show that two indecomposable modules of Hecke algebras of type A that correspond under the Scopes equivalence of Chapter 17 have common vertex. As a consequence, this gives a proof of a conjecture stated by R. Dipper and J. Du in [27] in the case where the modules lie in blocks of finite representation type.
For the convenience of the reader, we will recall the statement of the conjecture and summarize the necessary tools developed in previous chapters.

In the sequel, we refer to the characteristic of the ground field F of $H_n^f(q)$ as p, and assume that $p \geq 0$. Moreover, we keep the assumptions on q of Chapter 17.
Recall that a vertex of an indecomposable finite-dimensional $H_n^f(q)$-module M is a minimal element of the set

$$\mathcal{V}(M) := \{W_\lambda \mid \lambda \text{ composition of } n, W_\lambda \subseteq W(n), M \text{ is } H_\lambda(q) - \text{projective}\}.$$

Next, we recall the conjecture of [27], see [27, Conjecture 1.9]. To this end, we need some more notation. Consider the set

$$\mathcal{N}_n = \{(n_{-1}, n_0, \ldots, n_t) \mid n = n_{-1} + n_0 l + n_1 pl + \ldots + n_t p^t l\}. \tag{18.1}$$

To each $\vec{n} = (n_{-1}, n_0, \ldots, n_t)$ we can associate a standard parabolic subgroup W_λ of $W(n)$ with

$$\lambda = (1, \ldots, 1, l, \ldots, l, \ldots, p^t l, \ldots, p^t l),$$

where each $p^j l$ occurs precisely n_j times for any $0 \leq j \leq t$, and 1 is repeated n_{-1} times.

Definition 18.1.6. Suppose that W is a standard parabolic subgroup of $W(n)$. Then W is called *l-p-parabolic* if it is conjugate to some W_λ, where λ is defined by an element $\vec{n} \in \mathcal{N}_n$.

In the case when $p = 0$, it is shown in [32] that the vertices of indecomposable modules are l-parabolic. As a natural extension of this result, the following conjecture was stated in [27]:

Conjecture 18.1.7. *Let F be a field of characteristic $p > 0$, and let $q \in F$ be a primitive lth root of unity with $l > 1$. Then the vertices of indecomposable $H_n^f(q)$-modules are $l - p$-parabolic.*

Next, we will use the theory developed in the former chapters to derive results of the structure of vertices for modules lying in Scopes equivalent blocks. First of all, we will recall the needed results obtained in the last chapters.
In the following we denote the functors

$$e_i^{\Lambda_0}, f_i^{\Lambda_0}, (e_i^{\Lambda_0})^{(r)}, (f_i^{\Lambda_0})^{(r)}$$

by $e_i, f_i, e_i^{(r)}, f_i^{(r)}$, for $i \in I$ and $r \geq 1$.

Lemma 18.1.8. *Let $i \in I$, and $r \geq 1$. For an $H_n^f(q)$-module M we have:*

(i) $e_i^r(M) \cong \bigoplus_{r!} e_i^{(r)}(M)$.

(ii) $f_i^r(M) \cong \bigoplus_{r!} f_i^{(r)}(M)$.

Lemma 18.1.9. *Let $i \in I$ and $M \in H_n^f(q) - \mathrm{mod}$. Then:*

(i) $\mathrm{Res}_{H_{n-1}^f(q)}^{H_n^f(q)}(M) \cong \bigoplus_{i \in I} e_i(M)$.

(ii) $\mathrm{Ind}_{H_n^f(q)}^{H_{n+1}^f(q)}(M) \cong \bigoplus_{i \in I} f_i(M)$.

For the rest of this section we will fix a block B of $H_n^f(q)$ and a block $\bar{B}$ of $H_{n-k}^f(q)$ forming a $[w : k]$-pair such that $k \geq w$, that is to say, B and $\bar{B}$ are Scopes equivalent. In this situation, there is some $i \in I$ such that the functors $e_i^{(k)}$ and $f_i^{(k)}$ induce mutually inverse equivalences of categories between $B - \mathrm{mod}$ and $\bar{B} - \mathrm{mod}$ as seen in Theorem 17.2.1. Then Lemma 18.1.8 has the following consequence:

Proposition 18.1.10. *Let $M \in B - \mathrm{mod}$, and $N \in \bar{B} - \mathrm{mod}$. Then the following hold:*

(i) $f_i^k(e_i^k(M)) \cong \bigoplus_{(k!)^2} M$.

(ii) $e_i^k(f_i^k(N)) \cong \bigoplus_{(k!)^2} N$.

Proof. We prove (i), then part (ii) is proven similarly. As $e_i^{(k)}$ and $f_i^{(k)}$ induce mutually inverse equivalences between $B - \mathrm{mod}$ and $\bar{B} - \mathrm{mod}$, we have by Lemma 18.1.8 that

$$\begin{aligned} f_i^k(e_i^k(M)) &\cong f_i^k(\bigoplus_{k!} e_i^{(k)}(M)) \\ &\cong \bigoplus_{k!} f_i^k(e_i^{(k)}(M)) \\ &\cong \bigoplus_{k!}\bigoplus_{k!} f_i^{(k)}(e_i^{(k)}(M)) \\ &\cong \bigoplus_{(k!)^2} M, \end{aligned}$$

where in the second equivalence we have used that the functor f_i, and thus f_i^k, is an exact functor. $\square$

In the following, for finite-dimensional $H_n^f(q)$-modules L and K, we write $L \mid K$ if there is an $H_n^f(q)$-module L' such that

$$K \cong L \oplus L'.$$

From Lemma 18.1.9 we obtain the next result.

Lemma 18.1.11. *Keep the notation of Proposition 18.1.10. Then we have that*

$$f_i^k(N) \mid \mathrm{Ind}_{H_{n-k}^f(q)}^{H_n^f(q)}(N),$$

and

$$e_i^k(M) \mid \mathrm{Res}_{H_{n-k}^f(q)}^{H_n^f(q)}(M).$$

Proof. Consider $N \in \bar{B}-\mathrm{mod}$ as an $H_{n-k}^f(q)$-module. Then it is clear from Lemma 18.1.9 (ii) that

$$f_i(N) \mid \mathrm{Ind}_{H_{n-k}^f(q)}^{H_{n-k+1}^f(q)}(N).$$

Inductively we see that $f_i^k(N) \mid \mathrm{Ind}_{H_{n-k}^f(q)}^{H_n^f(q)}(N)$.

Similarly, we have that $e_i^k(M) \mid \mathrm{Res}_{H_{n-k}^f(q)}^{H_n^f(q)}(M)$, for $M \in B-\mathrm{mod}$. $\square$

In the following we will identify the subalgebra $H_{n-1}^f(q)$ with the parabolic subalgebra $H_{(n-1,1)}(q)$ of $H_n^f(q)$, corresponding to the partition $(n-1,1)$ of n. Moreover, if $\mu = (\mu_1, \dots, \mu_t)$ is a composition of $n-1$ and $H_\mu(q)$ the parabolic subalgebra of $H_{n-1}^f(q)$, we identify $H_\mu(q)$ with the parabolic subalgebra $H_{(\mu_1,\dots,\mu_t,1)}(q) \subseteq H_{(n-1,1)}(q) \subseteq H_n^f(q)$. We will use the same identification on the level of parabolic subgroups of $W(n-k)$ and $W(n)$.
Next, we state one of the main statements of Part II, which is the key to verify the conjecture in the representation-finite case.

Theorem 18.1.12. *Let B be a block of $H_n^f(q)$, $\bar{B}$ be a block of $H_{n-k}^f(q)$ forming a $[w:k]$-pair such that $k \geq w$. Let $M \in B-\mathrm{mod}$ indecomposable, and $N = e_i^{(k)}(M)$. Then $\mathrm{vx}(M) =_{W(n)} \mathrm{vx}(N)$.*

Proof. Fix an indecomposable $M \in B-\mathrm{mod}$, and let $N := e_i^{(k)}(M) \in \bar{B}-\mathrm{mod}$. Let μ be a composition of $n-k$ and $W_\mu \subseteq W(n-k)$ be a vertex of N. Furthermore, we choose a source S of N, i.e., an indecomposable $H_\mu(q)$-module S such that

$$N \mid \mathrm{Ind}_{H_\mu(q)}^{H_{n-k}^f(q)}(S).$$

By Lemma 18.1.11 and transitivity of induction, it follows that $f_i^k(N)$ is a direct summand of $\mathrm{Ind}_{H_\mu(q)}^{H_n^f(q)}(S)$. From Proposition 18.1.10 it follows that $f_i^k(N) \cong \bigoplus_{k!} M$, and thus,

$$M \mid \mathrm{Ind}_{H_\mu(q)}^{H_n^f(q)}(S).$$

From Higman's criterion, see for example [27, Theorem 1.4], we infer that M is $H_\mu(q)$-projective, and therefore, a vertex of M is contained in W_μ.
Next, we choose a vertex $W_\lambda \subseteq W(n)$ of M, for a composition λ of n. Moreover, denote by T a source of M, i.e., an indecomposable H_λ-module such that

$$M \mid \mathrm{Ind}_{H_\lambda(q)}^{H_n^f(q)}(T).$$

Together with Lemma 18.1.11, we conclude that

$$e_i^k(M) \mid \mathrm{Res}_{H_{(n-k,1^k)}(q)}^{H_n^f(q)}(\mathrm{Ind}_{H_\lambda}^{H_n^f(q)}(T)),$$

where we identify $H_{n-k}^f(q)$ with the subalgebra $H_{(n-k,1^k)}(q)$ of $H_n^f(q)$. From Proposition 18.1.10 it follows that

$$N \mid \mathrm{Res}_{H_{(n-k,1^k)}(q)}^{H_n^f(q)}(\mathrm{Ind}_{H_\lambda(q)}^{H_n^f(q)}(T)).$$

Set $\lambda' = (n-k, 1^k)$. If we apply the Mackey Decomposition Theorem, see [29, Theorem 2.7], we see that

$$N \mid \bigoplus_{d \in \mathcal{D}_{\lambda'\lambda}} [H_{\lambda'}(q) \otimes_{H_{\nu(d)}(q)} (T_d \otimes_{H_\lambda(q)} T)],$$

where $\nu(d)$ is defined by $W_{\nu(d)} = {}^dW_\lambda \cap W_{\lambda'}$, for all $d \in \mathcal{D}_{\lambda'\lambda}$.
Since N is supposed to be indecomposable, it follows by the Krull–Schmidt Theorem that

$$N \mid H_{\lambda'}(q) \otimes_{H_{\nu(d)}(q)} (T_d \otimes_{H_\lambda(q)} T),$$

for some $d \in \mathcal{D}_{\lambda'\lambda}$. By Higman's Criterion, it follows that N is $H_{\nu(d)}(q)$-projective. But $W_{\nu(d)} \subseteq_{W(n)} W_\lambda$, and therefore, a vertex of N is contained in W_λ.
By [27, Proposition 1.7], a vertex is uniquely determined up to conjugacy in $W(n)$. It follows that $W_\lambda =_{W(n)} W_\mu$. □

Finally, we obtain our main result, which now is a consequence of Theorem 18.1.12.

Theorem 18.1.13. *Let B be a block of $H_n^f(q)$ of finite representation type, and $M \in B - \mathrm{mod}$. Then the vertex of M is $l - p$-parabolic.*

Proof. By Corollary 17.2.5 and Theorem 18.1.12, we can reduce this question to $H_l^f(q)$. If $M \in H_l^f(q) - \mathrm{mod}$ is non-projective it must have vertex W_l since all other parabolic subalgebras of $H_l^f(q)$ are semisimple. Hence, the claim follows. □

Remark 18.1.14. In [48] a supposed counterexample is given in the case where the ground field is of characteristic 2, $n = 3$, and q is a primitive 3th root of unity. There, a certain indecomposable $H_3^f(q)$-module S is constructed as a restriction of a Specht module over $H_4^f(q)$. Then it is claimed that S is $H_\mu(q)$-projective, with $\mu = (2, 1)$, whereas S itself is not a projective $H_4^f(q)$-module. But this cannot be the case as $H_\mu(q)$ is a semisimple algebra, and, thus, every module over $H_\mu(q)$ is projective, and so is S.

Part III

Appendix

Appendix A

Resolving ambiguities

In this chapter, we try to resolve the ambiguities in the reduction system of Theorem 9.3.1 in Chapter 9 of Part II.
A closer look at the pairs (9.10)-(9.22) given in Section 9.3 shows that there are no inclusion ambiguities, but overlap ambiguities. In the following we try to resolve these, where by $\Rightarrow_{\mathcal{S}}$ we will denote a reduction step.

1. Let $i < j$, and consider the pair in (9.10). If $j < k$, then we see that we have an overlap ambiguity with the pair $(x_j x_k, x_k x_j)$. We try to resolve:

$$(x_i x_j)x_k \Rightarrow_{\mathcal{S}} x_j(x_i x_k) \Rightarrow_{\mathcal{S}} (x_j x_k)x_i \Rightarrow_{\mathcal{S}} x_k x_j x_i.$$

If we reduce the other way, we get

$$x_i(x_j x_k) \Rightarrow_{\mathcal{S}} (x_i x_k)x_j \Rightarrow_{\mathcal{S}} x_k(x_i x_j) \Rightarrow_{\mathcal{S}} x_k x_j x_i,$$

using only the reduction given in (9.10). Hence, we have obtained the same element.
Next we look at the overlap ambiguity with the pair in (9.12). We have that

$$x_i(x_j x_j^{-1}) \Rightarrow_{\mathcal{S}} x_i 1.$$

On the other hand

$$(x_i x_j)x_j^{-1} \Rightarrow_{\mathcal{S}} x_j(x_i x_j^{-1}) \Rightarrow_{\mathcal{S}} (x_j x_j^{-1})x_i \Rightarrow_{\mathcal{S}} 1x_i,$$

using the reductions in (9.10) and (9.14). Hence the calculation shows that we obtain the same element. Similar calculations using the reductions in (9.10) and (9.15) show that the overlap ambiguity between $x_i^{-1}x_i$ and $x_i x_j$ is resolvable.
We try to resolve the ambiguity with the pair in (9.14). To this end, let $1 \leq i < j < k \leq n$. We compute:

$$(x_i x_j)x_k^{-1} \Rightarrow_{\mathcal{S}} x_j(x_i x_k^{-1}) \Rightarrow_{\mathcal{S}} (x_j x_k^{-1})x_i \Rightarrow_{\mathcal{S}} x_k^{-1}x_j x_i,$$

and

$$x_i(x_j x_k^{-1}) \Rightarrow_{\mathcal{S}} (x_i x_k^{-1})x_j \Rightarrow_{\mathcal{S}} x_k^{-1}(x_i x_j) \Rightarrow_{\mathcal{S}} x_k^{-1}x_j x_i,$$

using the reductions in (9.10) and (9.14). Therefore, the ambiguity is resolved. The same calculations using the reductions given in (9.10) and (9.15) show that the ambiguity between $x_i^{-1}x_j$ and x_jx_k can be resolved.
Next, let $1 \leq i, j, k \leq n$, $j \neq i, i+1$, and $j < k$. Then we have an ambiguity between T_ix_j and x_jx_k. By using reduction (9.10), we get that

$$T_i(x_jx_k) \Rightarrow_{\mathcal{S}} (T_ix_k)x_j.$$

Now there are several cases. First, assume that $k \neq i, i+1$. Then we have:

$$(T_ix_k)x_j \Rightarrow_{\mathcal{S}} x_k(T_ix_j) \Rightarrow_{\mathcal{S}} x_kx_jT_i,$$

using reduction (9.17). If $k = i$, then:

$$\begin{aligned} (T_ix_k)x_j &\Rightarrow_{\mathcal{S}} (x_{i+1}T_i - (q-1)x_{i+1})x_j = x_{i+1}(T_ix_j) - (q-1)x_{i+1}x_j \\ &\Rightarrow_{\mathcal{S}} x_{i+1}x_jT_i - (q-1)x_{i+1}x_j, \end{aligned}$$

using reductions (9.19) and (9.17). If $k = i+1$, we compute

$$\begin{aligned} (T_ix_k)x_j &\Rightarrow_{\mathcal{S}} (x_iT_i + (q-1)x_{i+1})x_j = x_i(T_ix_j) + (q-1)x_{i+1}x_j \\ &\Rightarrow_{\mathcal{S}} x_ix_jT_i + (q-1)x_{i+1}x_j, \end{aligned}$$

using reductions (9.20) and (9.17).
If we try to resolve the other way, we get:

$$(T_ix_j)x_k \Rightarrow_{\mathcal{S}} x_j(T_ix_k),$$

using reduction (9.17). Again we have to distinguish different cases. If $k \neq i, i+1$, then

$$x_j(T_ix_k) \Rightarrow_{\mathcal{S}} (x_jx_k)T_i \Rightarrow_{\mathcal{S}} x_kx_jT_i,$$

using reductions (9.17) and (9.10). If $k = i$, we have that

$$\begin{aligned} x_j(T_ix_k) &\Rightarrow_{\mathcal{S}} x_j(x_{i+1}T_i - (q-1)x_{i+1}) = (x_jx_{i+1})T_i - (q-1)(x_jx_{i+1}) \\ &\Rightarrow_{\mathcal{S}} x_{i+1}x_jT_i - (q-1)x_{i+1}x_j, \end{aligned}$$

by applying the reductions (9.19) and (9.10). If $k = i+1$, we compute that

$$\begin{aligned} x_j(T_ix_k) &\Rightarrow_{\mathcal{S}} x_j(x_iT_i + (q-1)x_{i+1}) = (x_jx_i)T_i + (q-1)(x_jx_{i+1}) \\ &\Rightarrow_{\mathcal{S}} x_ix_jT_i + (q-1)x_{i+1}x_j, \end{aligned}$$

using (9.20) and (9.10). Now we see that in both reductions we obtain the same element, i.e., the ambiguity is resolvable.
Next we investigate the ambiguity between T_ix_i and x_ix_j, where $1 \leq i \leq n-1$ and $1 \leq i < j \leq n$. We have:

$$T_i(x_ix_j) \Rightarrow_{\mathcal{S}} (T_ix_j)x_i,$$

using (9.10). Now we have to distinguish several cases. If $j \neq i+1$, we get

$$(T_i x_j)x_i \Rightarrow_{\mathcal{S}} x_j(T_i x_i) \Rightarrow_{\mathcal{S}} x_j(x_{i+1}T_i - (q-1)x_{i+1}) = x_j x_{i+1} T_i - (q-1)x_j x_{i+1},$$

by applying (9.17) and (9.19). If $j = i+1$, we have:

$$\begin{aligned} (T_i x_j)x_i &\Rightarrow_{\mathcal{S}} (x_i T_i + (q-1)x_{i+1})x_i = x_i(T_i x_i) + (q-1)x_{i+1}x_i \\ &\Rightarrow_{\mathcal{S}} x_i(x_{i+1}T_i - (q-1)x_{i+1}) + (q-1)x_{i+1}x_i \\ &= (x_i x_{i+1})T_i - (q-1)x_i x_{i+1} + (q-1)x_{i+1}x_i \\ &\Rightarrow_{\mathcal{S}} x_{i+1}x_i T_i - (q-1)x_{i+1}x_i + (q-1)x_{i+1}x_i \\ &= x_{i+1}x_i T_i, \end{aligned}$$

using (9.20), (9.19), and (9.10). Reducing the other way, we obtain

$$(T_i x_i)x_j \Rightarrow_{\mathcal{S}} (x_{i+1}T_i - (q-1)x_{i+1})x_j = x_{i+1}T_i x_j - (q-1)x_{i+1}x_j,$$

applying reduction (9.19). Again we have to consider two cases. First, assume that $j \neq i+1$. Then we get

$$\begin{aligned} x_{i+1}(T_i x_j) - (q-1)x_{i+1}x_j &\Rightarrow_{\mathcal{S}} (x_{i+1}x_j)T_i - (q-1)(x_{i+1}x_j) \\ &\Rightarrow_{\mathcal{S}} x_j x_{i+1} T_i - (q-1)x_j x_{i+1}, \end{aligned}$$

using (9.17) and (9.10). If $j = i+1$, we have

$$\begin{aligned} x_{i+1}(T_i x_j) - (q-1)x_{i+1}x_j &\Rightarrow_{\mathcal{S}} x_{i+1}(x_i T_i + (q-1)x_{i+1}) - (q-1)x_{i+1}x_j \\ &= x_{i+1}x_i T_i + (q-1)x_{i+1}^2 - (q-1)x_{i+1}x_j \\ &= x_{i+1}x_i T_i, \end{aligned}$$

applying (9.19). Now we see that the reduced elements coincide, hence the ambiguity is resolved.
Now we look at the overlap ambiguity between $T_i x_{i+1}$ and $x_{i+1}x_j$, where $1 \leq i \leq n-1$ and $1 < i+1 < j \leq n$. We have

$$\begin{aligned} T_i(x_{i+1}x_j) &\Rightarrow_{\mathcal{S}} (T_i x_j)x_{i+1} \Rightarrow_{\mathcal{S}} x_j(T_i x_{i+1}) \\ &\Rightarrow_{\mathcal{S}} x_j(x_i T_i + (q-1)x_{i+1}) = x_j x_i T_i + (q-1)x_j x_{i+1}, \end{aligned}$$

by applying the reductions (9.10), (9.17) and (9.20). On the other hand we get that

$$\begin{aligned} (T_i x_{i+1})x_j &\Rightarrow_{\mathcal{S}} (x_i T_i + (q-1)x_{i+1})x_j = x_i(T_i x_j) + (q-1)x_{i+1}x_j \\ &\Rightarrow_{\mathcal{S}} (x_i x_j)T_i + (q-1)(x_{i+1}x_j) \Rightarrow_{\mathcal{S}} x_j x_i T_i + (q-1)x_j x_{i+1}, \end{aligned}$$

using (9.20) and (9.10). Now we see that the ambiguity is resolvable.

2. For the pair in (9.11) one proceeds in a similar manner as for the pair in (9.10), using the appropriate reductions.

3. Here we consider the element given in (9.12). We see that we have an overlap ambiguity with the element in (9.13). We compute

$$(x_i x_i^{-1})x_i \Rightarrow_{\mathcal{S}} 1x_i,$$

using only reduction (9.12). Reducing the other way, we get

$$x_i(x_i^{-1}x_i) \Rightarrow_{\mathcal{S}} x_i 1,$$

applying reduction given in (9.13). Hence, the ambiguity is resolved. A similar calculation shows that the ambiguity between (9.13) and (9.12) is resolvable.
Next we investigate the ambiguity between (9.12) and (9.15). Suppose that $1 \leq i < j \leq n$. Then we get

$$x_i(x_i^{-1}x_j) \Rightarrow_{\mathcal{S}} (x_i x_j)x_i^{-1} \Rightarrow_{\mathcal{S}} x_j(x_i x_i^{-1}) \Rightarrow_{\mathcal{S}} x_j 1,$$

using (9.15) and (9.12). On the other hand

$$(x_i x_i^{-1})x_j \Rightarrow_{\mathcal{S}} 1x_j,$$

using only (9.12). Now we see that the ambiguity is resolvable.
Looking at the ambiguity between (9.15) and (9.12), we compute for $1 \leq i < j \leq n$:

$$(x_i^{-1}x_j)x_j^{-1} \Rightarrow_{\mathcal{S}} x_j(x_i^{-1}x_j^{-1}) \Rightarrow_{\mathcal{S}} (x_j x_j^{-1})x_i^{-1} \Rightarrow_{\mathcal{S}} 1x_i^{-1},$$

applying the reductions (9.15), (9.11) and (9.12). Moreover, we get

$$x_i^{-1}(x_j x_j^{-1}) \Rightarrow_{\mathcal{S}} x_i^{-1}1,$$

using (9.12), and showing that the ambiguity is resolvable.
Next, we investigate the ambiguity between (9.12) and (9.17). Let $1 \leq i \leq n-1$ and $j \neq i, i+1$. We get that

$$(T_i x_j)x_j^{-1} \Rightarrow_{\mathcal{S}} x_j(T_i x_j^{-1}) \Rightarrow_{\mathcal{S}} (x_j x_j^{-1})T_i \Rightarrow_{\mathcal{S}} 1T_i,$$

using (9.17), (9.18) and (9.12). The other way is easy, using (9.12):

$$T_i(x_j x_j^{-1}) \Rightarrow_{\mathcal{S}} T_i 1.$$

Therefore, the ambiguity is resolved.
Now, consider the pairs (9.12) and (9.19). We have for $1 \leq i \leq n-1$:

$$T_i(x_i x_i^{-1}) \Rightarrow_{\mathcal{S}} T_i 1,$$

applying (9.12), and

$$\begin{aligned}
(T_i x_i)x_i^{-1} &\Rightarrow_{\mathcal{S}} (x_{i+1}T_i - (q-1)x_{i+1})x_i^{-1} = x_{i+1}(T_i x_i^{-1}) - (q-1)x_{i+1}x_i^{-1} \\
&\Rightarrow_{\mathcal{S}} x_{i+1}(x_{i+1}^{-1}T_i + (q-1)x_i^{-1}) - (q-1)x_{i+1}x_i^{-1} \\
&= (x_{i+1}x_{i+1}^{-1})T_i + (q-1)x_{i+1}x_i^{-1} - (q-1)x_{i+1}x_i^{-1} \\
&\Rightarrow_{\mathcal{S}} 1T_i + (q-1)x_{i+1}x_i^{-1} - (q-1)x_{i+1}x_i^{-1} = 1T_i,
\end{aligned}$$

using (9.19), (9.21) and (9.12). Again we see that the ambiguity is resolvable.
Next we look at the ambiguity of (9.12) and (9.20). Let $1 \leq i \leq n-1$. Then we get that

$$T_i(x_{i+1}x_{i+1}^{-1}) \Rightarrow_{\mathcal{S}} T_i 1,$$

applying (9.12), and

$$\begin{aligned}(T_ix_{i+1})x_{i+1}^{-1} &\Rightarrow_{\mathcal{S}} (x_iT_i + (q-1)x_{i+1})x_{i+1}^{-1} = x_i(T_ix_{i+1}^{-1}) + (q-1)(x_{i+1}x_{i+1}^{-1}) \\ &\Rightarrow_{\mathcal{S}} x_i(x_i^{-1}T_i - (q-1)x_i^{-1}) + (q-1)1 \\ &= (x_ix_i^{-1})T_i - (q-1)(x_ix_i^{-1}) + (q-1)1 \\ &\Rightarrow_{\mathcal{S}} 1T_i - (q-1)1 + (q-1)1 = 1T_i,\end{aligned}$$

using (9.20), (9.22) and (9.12). Hence the ambiguity is resolved.

4. The calculations for the pair in (9.13) are similar to that in (3.), using the appropriate reductions.

5. Next we consider the pair given in (9.14). We see that we have an overlap ambiguity with the pair in (9.15). Let $1 \leq i < j < k \leq n$. Trying to resolve gives:

$$(x_ix_j^{-1})x_k \Rightarrow_{\mathcal{S}} x_j^{-1}(x_ix_k) \Rightarrow_{\mathcal{S}} (x_j^{-1}x_k)x_i \Rightarrow_{\mathcal{S}} x_kx_j^{-1}x_i,$$

and

$$x_i(x_j^{-1}x_k) \Rightarrow_{\mathcal{S}} (x_ix_k)x_j^{-1} \Rightarrow_{\mathcal{S}} x_k(x_ix_j^{-1}) \Rightarrow_{\mathcal{S}} x_kx_j^{-1}x_i,$$

using (9.14) and (9.10). Hence, the ambiguity is resolved.
Next, we consider the ambiguity between (9.14) and (9.17). Assume that $1 \leq i \leq n-1$, $j \neq i, i+1$ and $1 \leq j < k \leq n$. We get

$$T_i(x_jx_k^{-1}) \Rightarrow_{\mathcal{S}} (T_ix_k^{-1})x_j,$$

by reduction (9.14). We have to distinguish several cases. First, let $k \neq i, i+1$. Then

$$(T_ix_k^{-1})x_j \Rightarrow_{\mathcal{S}} x_k^{-1}(T_ix_j) \Rightarrow_{\mathcal{S}} x_k^{-1}x_jT_i,$$

using (9.18). If $k = i$, we get

$$\begin{aligned}(T_ix_k^{-1})x_j &\Rightarrow_{\mathcal{S}} (x_{i+1}^{-1}T_i + (q-1)x_i^{-1})x_j = x_{i+1}^{-1}(T_ix_j) + (q-1)x_i^{-1}x_j \\ &\Rightarrow_{\mathcal{S}} x_{i+1}^{-1}x_jT_i + (q-1)x_i^{-1}x_j,\end{aligned}$$

applying (9.21) and (9.17). If $k = i+1$, we have that

$$\begin{aligned}(T_ix_k^{-1})x_j &\Rightarrow_{\mathcal{S}} (x_i^{-1}T_i - (q-1)x_i^{-1})x_j = x_i^{-1}(T_ix_j) - (q-1)x_i^{-1}x_j \\ &\Rightarrow_{\mathcal{S}} x_i^{-1}x_jT_i - (q-1)x_i^{-1}x_j,\end{aligned}$$

using (9.22) and (9.17). On the other hand, we get:

$$(T_ix_j)x_k^{-1} \Rightarrow_{\mathcal{S}} x_j(T_ix_k^{-1}),$$

by (9.17). Again we have to distinguish. If $k \neq i, i+1$, then

$$x_j(T_ix_k^{-1}) \Rightarrow_{\mathcal{S}} (x_jx_k^{-1})T_i \Rightarrow_{\mathcal{S}} x_k^{-1}x_jT_i,$$

using (9.18) and (9.14). If $k = i$, we have

$$\begin{aligned} x_j(T_ix_k^{-1}) &\Rightarrow_{\mathcal{S}} x_j(x_{i+1}^{-1}T_i + (q-1)x_i^{-1}) = (x_jx_{i+1}^{-1})T_i + (q-1)(x_jx_i^{-1}) \\ &\Rightarrow_{\mathcal{S}} x_{i+1}^{-1}x_jT_i + (q-1)x_i^{-1}x_j, \end{aligned}$$

by (9.21) and (9.14). If $k = i+1$, we get

$$\begin{aligned} x_j(T_ix_k^{-1}) &\Rightarrow_{\mathcal{S}} x_j(x_i^{-1}T_i - (q-1)x_i^{-1}) = (x_jx_i^{-1})T_i - (q-1)(x_jx_i^{-1}) \\ &\Rightarrow_{\mathcal{S}} x_i^{-1}x_jT_i - (q-1)x_i^{-1}x_j, \end{aligned}$$

using (9.22) and (9.14). Now we see that the ambiguity is resolved.
Next, consider the ambiguity between (9.14) and (9.19). Let $1 \leq i \leq n-1$ and $1 \leq i < j \leq n$. Then we have

$$T_i(x_ix_j^{-1}) \Rightarrow_{\mathcal{S}} (T_ix_j^{-1})x_i,$$

by (9.14). We have two cases. First, let $j \neq i+1$. Then

$$\begin{aligned} (T_ix_j^{-1})x_i &\Rightarrow_{\mathcal{S}} x_j^{-1}(T_ix_i) \\ &\Rightarrow_{\mathcal{S}} x_j^{-1}(x_{i+1}T_i - (q-1)x_{i+1}) = x_j^{-1}x_{i+1}T_i - (q-1)x_j^{-1}x_{i+1}, \end{aligned}$$

using (9.18) and (9.19). If $j = i+1$, then

$$\begin{aligned} (T_ix_j^{-1})x_i &\Rightarrow_{\mathcal{S}} (x_i^{-1}T_i - (q-1)x_i^{-1})x_i = x_i^{-1}(T_ix_i) - (q-1)(x_i^{-1}x_i) \\ &\Rightarrow_{\mathcal{S}} x_i^{-1}(x_{i+1}T_i - (q-1)x_{i+1}) - (q-1)1 \\ &= (x_i^{-1}x_{i+1})T_i - (q-1)(x_i^{-1}x_{i+1}) - (q-1)1 \\ &\Rightarrow_{\mathcal{S}} x_{i+1}x_i^{-1}T_i - (q-1)x_{i+1}x_i^{-1} - (q-1)1, \end{aligned}$$

by (9.22), (9.19), (9.13) and (9.15). On the other hand, we get

$$(T_ix_i)x_j^{-1} \Rightarrow_{\mathcal{S}} (x_{i+1}T_i - (q-1)x_{i+1})x_j^{-1} = x_{i+1}(T_ix_j^{-1}) - (q-1)x_{i+1}x_j^{-1},$$

using (9.19). If $j \neq i+1$, we have

$$\begin{aligned} x_{i+1}(T_ix_j^{-1}) - (q-1)x_{i+1}x_j^{-1} &\Rightarrow_{\mathcal{S}} (x_{i+1}x_j^{-1})T_i - (q-1)(x_{i+1}x_j^{-1}) \\ &\Rightarrow_{\mathcal{S}} x_j^{-1}x_{i+1}T_i - (q-1)x_j^{-1}x_{i+1}, \end{aligned}$$

by (9.18) and (9.14). If $j = i+1$, we get

$$\begin{aligned} x_{i+1}(T_ix_j^{-1}) - (q-1)(x_{i+1}x_j^{-1}) &\Rightarrow_{\mathcal{S}} x_{i+1}(x_i^{-1}T_i - (q-1)x_i^{-1}) - (q-1)1 \\ &= x_{i+1}x_i^{-1}T_i - (q-1)x_{i+1}x_i^{-1} - (q-1)1 \end{aligned}$$

using (9.22) and (9.12), showing that the ambiguity is resolvable.
Next, lets consider the pairs (9.14) and (9.20). To this end, let $1 \leq i \leq n-1$ and $1 < i+1 < j \leq n$. Then

$$\begin{aligned} T_i(x_{i+1}x_j^{-1}) &\Rightarrow_{\mathcal{S}} (T_i x_j^{-1})x_{i+1} \Rightarrow_{\mathcal{S}} x_j^{-1}(T_i x_{i+1}) \\ &\Rightarrow_{\mathcal{S}} x_j^{-1}(x_i T_i + (q-1)x_{i+1}) = x_j^{-1}x_i T_i + (q-1)x_j^{-1}x_{i+1}, \end{aligned}$$

using (9.14), (9.18) and (9.20). Also, we have

$$\begin{aligned} (T_i x_{i+1})x_j^{-1} &\Rightarrow_{\mathcal{S}} (x_i T_i + (q-1)x_{i+1})x_j^{-1} \\ &= x_i(T_i x_j^{-1}) + (q-1)x_{i+1}x_j^{-1} \\ &\Rightarrow_{\mathcal{S}} (x_i x_j^{-1})T_i + (q-1)(x_{i+1}x_j^{-1}) \\ &\Rightarrow_{\mathcal{S}} x_j^{-1}x_i T_i + (q-1)x_j^{-1}x_{i+1}, \end{aligned}$$

by (9.20), (9.18) and (9.14). Therefore, the ambiguity is resolved.

6. If we consider the pair given in (9.15), we see that we get similar calculations as in (5.), using the appropriate reductions.

7. Next let us consider the pair given in (9.16). Let $1 \leq i \leq n-1$. We see that we have an ambiguity with the pair in (9.17). For $1 \leq j \leq n$ and $j \neq i, i+1$, we get

$$\begin{aligned} T_i(T_i x_j) &\Rightarrow_{\mathcal{S}} (T_i x_j)T_i \Rightarrow_{\mathcal{S}} x_j(T_i T_i) \\ &\Rightarrow_{\mathcal{S}} x_j((q-1)T_i + q) = (q-1)x_j T_i + q x_j, \end{aligned}$$

by (9.17) and (9.16). Also, we have that

$$\begin{aligned} (T_i T_i)x_j &\Rightarrow_{\mathcal{S}} ((q-1)T_i + q)x_j = (q-1)T_i x_j + q x_j \\ &\Rightarrow_{\mathcal{S}} (q-1)x_j T_i + q x_j, \end{aligned}$$

using reductions (9.16) and (9.17). Therefore, the ambiguity is resolvable.
Using (9.16) and (9.18), a similar calculation shows that the ambiguity coming from (9.16) and (9.18) is resolvable.
Next, consider the ambiguity (9.16) and (9.19). We compute

$$\begin{aligned} T_i(T_i x_i) &\Rightarrow_{\mathcal{S}} T_i(x_{i+1}T_i - (q-1)x_{i+1}) = (T_i x_{i+1})T_i - (q-1)(T_i x_{i+1}) \\ &\Rightarrow_{\mathcal{S}} (x_i T_i + (q-1)x_{i+1})T_i - (q-1)(x_i T_i + (q-1)x_{i+1}) \\ &= x_i(T_i T_i) + (q-1)x_{i+1}T_i - (q-1)x_i T_i - (q-1)^2 x_{i+1} \\ &\Rightarrow_{\mathcal{S}} x_i((q-1)T_i + q) + (q-1)x_{i+1}T_i - (q-1)x_i T_i - (q-1)^2 x_{i+1} \\ &= (q-1)x_i T_i + q x_i + (q-1)x_{i+1}T_i - (q-1)x_i T_i - (q-1)^2 x_{i+1} \\ &= (q-1)x_{i+1}T_i - (q-1)^2 x_{i+1} + q x_i, \end{aligned}$$

using (9.19), (9.20) and (9.16). On the other hand, we have that

$$\begin{aligned} (T_i T_i)x_i &\Rightarrow_{\mathcal{S}} ((q-1)T_i + q)x_i = (q-1)(T_i x_i) + q x_i \\ &\Rightarrow_{\mathcal{S}} (q-1)(x_{i+1}T_i - (q-1)x_{i+1}) + q x_i \\ &= (q-1)x_{i+1}T_i - (q-1)^2 x_{i+1} + q x_i, \end{aligned}$$

by (9.16) and (9.19). Hence, the ambiguity is resolvable.
Now we look at the pairs (9.16) and (9.20). We get

$$
\begin{aligned}
&T_i(T_ix_{i+1})\\
&\Rightarrow_{\mathcal{S}} T_i(x_iT_i+(q-1)x_{i+1}) = (T_ix_i)T_i + (q-1)(T_ix_{i+1})\\
&\Rightarrow_{\mathcal{S}} (x_{i+1}T_i-(q-1)x_{i+1})T_i + (q-1)(x_iT_i+(q-1)x_{i+1})\\
&= x_{i+1}(T_iT_i) - (q-1)x_{i+1}T_i + (q-1)x_iT_i + (q-1)^2x_{i+1}\\
&\Rightarrow_{\mathcal{S}} x_{i+1}((q-1)T_i+q) - (q-1)x_{i+1}T_i + (q-1)x_iT_i + (q-1)^2x_{i+1}\\
&= (q-1)x_{i+1}T_i + qx_{i+1} - (q-1)x_{i+1}T_i + (q-1)x_iT_i + (q-1)^2x_{i+1}\\
&= (q-1)x_iT_i + (q-1)^2x_{i+1} + qx_{i+1},
\end{aligned}
$$

using (9.20), (9.19) and (9.16). Moreover, we have

$$
\begin{aligned}
(T_iT_i)x_{i+1} &\Rightarrow_{\mathcal{S}} ((q-1)T_i+q)x_{i+1} = (q-1)(T_ix_{i+1}) + qx_{i+1}\\
&\Rightarrow_{\mathcal{S}} (q-1)(x_iT_i+(q-1)x_{i+1}) + qx_{i+1}\\
&= (q-1)x_iT_i + (q-1)^2x_{i+1} + qx_{i+1},
\end{aligned}
$$

by (9.16) and (9.20). Now we see that the ambiguity is resolved.
Next consider the ambiguity given by (9.16) and (9.21). We have

$$
\begin{aligned}
&T_i(T_ix_i^{-1})\\
&\Rightarrow_{\mathcal{S}} T_i(x_{i+1}^{-1}T_i+(q-1)x_i^{-1}) = (T_ix_{i+1}^{-1})T_i + (q-1)(T_ix_i^{-1})\\
&\Rightarrow_{\mathcal{S}} (x_i^{-1}T_i-(q-1)x_i^{-1})T_i + (q-1)(x_{i+1}^{-1}T_i+(q-1)x_i^{-1})\\
&= x_i^{-1}(T_iT_i) - (q-1)x_i^{-1}T_i + (q-1)x_{i+1}^{-1}T_i + (q-1)^2x_i^{-1}\\
&\Rightarrow_{\mathcal{S}} x_i^{-1}((q-1)T_i+q) - (q-1)x_i^{-1}T_i + (q-1)x_{i+1}^{-1}T_i + (q-1)^2x_i^{-1}\\
&= (q-1)x_i^{-1}T_i + qx_i^{-1} - (q-1)x_i^{-1}T_i + (q-1)x_{i+1}^{-1}T_i + (q-1)^2x_i^{-1}\\
&= (q-1)x_{i+1}^{-1}T_i + (q-1)^2x_i^{-1} + qx_i^{-1},
\end{aligned}
$$

by (9.21), (9.22) and (9.16). On the other hand we get

$$
\begin{aligned}
(T_iT_i)x_i^{-1} &\Rightarrow_{\mathcal{S}} ((q-1)T_i+q)x_i^{-1} = (q-1)(T_ix_i^{-1}) + qx_i^{-1}\\
&\Rightarrow_{\mathcal{S}} (q-1)(x_{i+1}^{-1}T_i+(q-1)x_i^{-1}) + qx_i^{-1}\\
&= (q-1)x_{i+1}^{-1}T_i + (q-1)^2x_i^{-1} + qx_i^{-1},
\end{aligned}
$$

using (9.16) and (9.21). Hence, the ambiguity is resolvable.
Now we look at the ambiguity coming from (9.16) and (9.22).

$$
\begin{aligned}
&T_i(T_ix_{i+1}^{-1})\\
&\Rightarrow_{\mathcal{S}} T_i(x_i^{-1}T_i-(q-1)x_i^{-1}) = (T_ix_i^{-1})T_i - (q-1)(T_ix_i^{-1})\\
&\Rightarrow_{\mathcal{S}} (x_{i+1}^{-1}T_i+(q-1)x_i^{-1})T_i - (q-1)(x_{i+1}^{-1}T_i+(q-1)x_i^{-1})\\
&= x_{i+1}^{-1}(T_iT_i) + (q-1)x_i^{-1}T_i - (q-1)x_{i+1}^{-1}T_i - (q-1)^2x_i^{-1}\\
&\Rightarrow_{\mathcal{S}} x_{i+1}^{-1}((q-1)T_i+q) + (q-1)x_i^{-1}T_i - (q-1)x_{i+1}^{-1}T_i - (q-1)^2x_i^{-1}\\
&= (q-1)x_{i+1}^{-1}T_i + qx_{i+1}^{-1} + (q-1)x_i^{-1}T_i - (q-1)x_{i+1}^{-1}T_i - (q-1)^2x_i^{-1}\\
&= (q-1)x_i^{-1}T_i + qx_{i+1}^{-1} - (q-1)^2x_i^{-1},
\end{aligned}
$$

using (9.22), (9.21) and (9.16). Also, we see that

$$\begin{aligned}(T_iT_i)x_{i+1}^{-1} &\Rightarrow_{\mathcal{S}} ((q-1)T_i+q)x_{i+1}^{-1} = (q-1)T_ix_{i+1}^{-1}+qx_{i+1}^{-1}\\ &\Rightarrow_{\mathcal{S}} (q-1)(x_i^{-1}T_i-(q-1)x_i^{-1})+qx_{i+1}^{-1}\\ &= (q-1)x_i^{-1}T_i+qx_{i+1}^{-1}-(q-1)^2x_i^{-1},\end{aligned}$$

by (9.16) and (9.22), showing that the ambiguity is resolvable.
Since there are no more ambiguities in $\mathcal{S}$, we have proved that all ambiguities occurring in $\mathcal{S}$ are resolvable.

Bibliography

[1] F. W. Anderson, K. R. Fuller, Rings and categories of modules, Springer-Verlag, 1974.

[2] S. Ariki, On the decomposition numbers of the Hecke algebra $G(m, 1, n)$, J. Math. Kyoto Univ. **36** (1996), 789–808.

[3] S. Ariki, K. Iijima, E. Park, Representation type of finite quiver Hecke algebras of type $A_l^{(1)}$ for arbitrary parameters, Int. Math. Res. Notices **15** (2015), 6070–6135.

[4] S. Ariki, K. Koike, A Hecke algebra of $(\mathbb{Z}/r\mathbb{Z}) \wr S_n$ and construction of its irreducible representations, Adv. Math. **106** (1994), no. 2, 216–243.

[5] S. Ariki, A. Mathas, The number of simple modules of the Hecke algebras of type G(r, 1, n), Math. Z. **233** (2000), 601–623.

[6] M. Auslander, I. Reiten, Representation theory of Artin algebras. III. Almost split sequences, Comm. Algebra **3** (1975), 239–294.

[7] M. Auslander, I. Reiten, S. O. Smalø, Representation Theory of Artin Algebras, Cambridge University Press (1995).

[8] D. J. Benson, Representations and cohomology I: Basic representation theory of finite groups and associative algebras, Cambridge University Press (1991).

[9] D. J. Benson, K. Erdmann, M. Holloway, Rank varieties for a class of finite-dimensional local algebras, J. Pure Appl. Algebra **211** (2007), 497–510.

[10] P. A. Bergh, Ext-symmetry over quantum complete intersections, Arch. Math. 92 (2009), 566–573.

[11] P. A. Bergh, K. Erdmann, The stable Auslander–Reiten quiver of a quantum complete intersection, Bull. London Math. Soc. **43** (2011), 44–56.

[12] G. Bergman, The Diamond Lemma for Ring Theory, Advances Math. **29** (1978), 178–218.

[13] P. A. Bergh, S. Oppermann, Cohomology of twisted tensor products, J. Algebra **320** (2008), 3327–3338.

[14] P. A. Bergh, Ø. Solberg, Relative support varieties, Q. J. Math. **61** (2010), 171–182.

[15] I. N. Bernstein, A.V. Zelevinsky, Induced representations of reductive p-adic groups, I. Ann. Sci. École Norm. Sup. (4) **10** (1977), no. 4, 441–472.

[16] J. Brundan, Modular branching rules and the Mullineux map for Hecke algebras of type A, Proc. London Math. Soc. (3) **77** (1998), 551–581.

[17] J. Brundan, R. Dipper, A. Kleshchev, Quantum linear groups and representations of $\mathrm{GL}_n(\mathbb{F}_q)$, Mem. Amer. Math. Soc. **149** (2001), no. 706.

[18] J. Brundan, A. Kleshchev, Blocks of cyclotomic Hecke algebras and Khovanov–Lauda algebras, Invent. Math. **178** (2009), 451–484.

[19] N. Chriss, V. Ginzburg, Representation Theory and Complex Geometry, Birkhäuser, Basel, 1997.

[20] H. Cartan, S. Eilenberg, Homological Algebra, Princeton University Press, 1956.

[21] J. Chuang, H. Miyachi, Runner removal Morita equivalences, in: A. Gyoja, et al. (Eds.), Representation Theory of Algebraic Groups and Quantum Groups, in: Progr. Math., vol. 284, Birkhäuser, 2010, 55–79.

[22] J. Chuang, R. Rouquier, Derived equivalences for symmetric groups and $\mathfrak{sl}_2$-categorification, Ann. of Math. **167** (2008), 245–298.

[23] J. Chuang, K. M. Tan, Representations of wreath products of algebras, Math. Proc. Cambridge Philos. Soc. **135** (2003), no. 3, 395–411.

[24] W. Crawley-Boevey, Tameness of biserial algebras, Arch. Math. **65** (1995), 399–407.

[25] C. W. Curtis, I. Reiner, Methods of representation theory with applications to finite groups and orders, Vol. 1, John Wiley & Sons, Inc., 1990.

[26] C. W. Curtis, I. Reiner, Representation theory of finite groups and associative algebras, Pure and Appl. Math., vol. 11, Interscience, New York, 1962; 2nd ed., 1966.

[27] R. Dipper, J. Du, Trivial and alternating source modules of Hecke algebras of Type A, Proc. London Math. Soc. (3) **66** (1993), 479–506.

[28] R. Dipper and P. Fleischmann, Modular Harish-Chandra theory I, Math. Z. **211** (1992), 49–71.

[29] R. Dipper, G. James, Representations of Hecke algebras of general linear groups, Proc. London Math. Soc. **52** (1986), 20–52.

[30] R. Dipper, G. James, Blocks and idempotents of Hecke algebras of general linear groups, Proc. London Math. Soc. **54** (1987), 57–82.

[31] R. Dipper, G. James, The q-Schur algebra, Proc. London Math. Soc. (1) **59** (1989), 23–50.

[32] J. Du, The Green correspondence for the representations of Hecke algebras of Type A_{r-1}, Trans. Amer. Math. Soc. (1) **329** (1992), 273–287.

[33] K. Erdmann, On Auslander–Reiten components for group algebras, J. Pure Appl. Alg. **104** (1995), 149–160.

[34] K. Erdmann, Blocks of tame representation type and related algebras, Lecture Notes in Mathematics, Springer-Verlag (1990).

[35] K. Erdmann, M. Holloway, N. Snashall, Ø. Solberg, R. Taillefer, Support varieties for selfinjective algebras, K-Theory **33** (2004) 67–87.

[36] K. Erdmann, D. K. Nakano: Representation type of Hecke algebras of type A, Trans. Amer. Math. Soc. **354** (2002), no. 1, 275–285.

[37] K. Erdmann, A. Skowroński, On Auslander–Reiten components of blocks of self-injective biserial algebras, Trans. Amer. Math. Soc. 361 (2009), 4195–4214.

[38] A. Evseev, Rock blocks, wreath products and KLR algebras, preprint, arXiv:1511.08004 [math.RT], 2015

[39] P. Fong, B. Srinivasan, The blocks of finite general linear and unitary groups, Invent. Math. **69** (1982), 109–153.

[40] J. Feldvoss, S. Witherspoon, Support varieties and representation type of self-injective algebras. Homology Homotopy Appl. **13** (2011), no. 2, 197–215.

[41] P. Gabriel, Auslander–Reiten sequences and representation-finite algebras, Representation Theory I, Ottawa 1979, Lecture Notes in Mathematics **831**, Springer-Verlag, Berlin/New-York 1980.

[42] M. Geck, Brauer trees of Hecke algebras, Comm. Algebra (10) **20** (1992), 2937–2973.

[43] J. A. Green, The characters of the finite general linear groups, Trans. Amer. Math. Soc. **80** (1955), 402–447.

[44] I. Grojnowski, Affine $\hat{\mathfrak{sl}}_p$ controls the modular representation theory of the symmetric group and related Hecke algebras, preprint, math.RT/9907129, 1999.

[45] I. Grojnowski, M. Vazirani, Strong multiplicity one theorems for affine Hecke algebras of type A, Transform. Groups **6** (2001), 143–155.

[46] T. Holm, Hochschild cohomology of Brauer tree algebras, Comm. Algebra (11) **26** (1998), 3625–3646.

[47] J. Hong, S.-J. Kang, Introduction to Quantum Groups and Crystal Bases, Graduate Studies in Mathematics, Volume 42, 2000.

[48] J. Hu, A counter-example of Dipper–Du's conjecture, Arch. Math. **78** (2002), 449–453.

[49] D. Happel, U. Preiser, C. M. Ringel, Vinberg's characterization of Dynkin diagrams using subbadditive functions with application to DTr-periodic modules, Proceedings of the Second International Conference on Representation Theory, II, Carleton University, Ottawa, ON, 1979, Lecture Notes in Mathematics 832, Springer-Verlag (1980), 280–294.

[50] G. James, The irreducible representations of the finite general linear groups, Proc. London Math. Soc (3) **52** (1986), 236–268.

[51] J. P. Jans, Some remarks on symmetric and Frobenius algebras. Nagoya Math. J. **16** (1960), 65–71.

[52] T. Jost, Morita equivalence for blocks of Hecke algebras of symmetric groups, J. Algebra **194** (1997), 201–223.

[53] V. Kac, Infinite dimensional Lie algebras, Cambridge University Press, third edition, 1990.

[54] M. Kashiwara, On crystal bases, in Representations of groups (Banff 1994), CMS Conf. Proc. **16** (1995), 155–197.

[55] M. Kashiwara, P. Schapira, Categories and Sheaves, Grundlehren der mathematischen Wissensschaft 332, Springer-Verlag, Berlin Heidelberg, 2006

[56] M. Kashiwara, Y. Saito, Geometric constructions of crystal bases, Duke Math. J. **89** (1997), 9–36.

[57] O. Kerner, D. Zacharia, Auslander–Reiten theory for modules of finite complexity over self-injective algebras, Bull. London Math. Soc. **43** (2011), 44–56.

[58] A. Kleshchev, Branching rules for modular representations of symmetric groups II, J. reine angew. Math. **459** (1995), 163–212.

[59] A. Kleshchev, Linear and Projective Representations of Symmetric Groups, Cambridge University Press, 2005.

[60] A. I. Kostrikin, I. R. Shafarevich (Eds.), Algerba V: Homological algebra, with contributions by S.I. Gelfand and Y. I. Manin, Encyclopaedia of Mathematical Sciences **38**, Springer, 1994.

[61] T.-Y. Lam, A first course in noncommutative rings (2nd ed.), Springer-Verlag, 2001.

[62] A. Lascoux, B. Leclerc, J.-Y. Thibon, Hecke algebras at roots of unity and crystal bases of quantum affine algebras, Comm. Math. Phys. **181** (1996), 205–263.

[63] M. Linckelmann, Finite generation of Hochschild cohomology of Hecke algebras of finite classical type in characteristic zero, Bull. London Math. Soc. **43** (5) (2011), 871–885.

[64] S. Lyle, A. Mathas, Blocks of cyclotomic Hecke algebras, Advances in Mathematics **216** (2007), 854–878.

[65] S. Mac Lane, Categories for the Working Mathematician, Graduate Texts in Mathematics, Vol. 5, 2nd ed., Springer-Verlag, New York, 1998.

[66] G. Malle, A. Mathas, Symmetric cyclotomic Hecke algebras, J. Algebra **205** (1998), 275–293.

[67] G. Malle, D. Testermann, Linear Algebraic Groups and Finite Groups of Lie Type, Cambridge University Press, Cambridge, 2011.

[68] A. Mathas, Iwahori–Hecke algebras and Schur algebras of the Symmetric group, University lecture series (15), 1999.

[69] A. Mathas, A Specht filtration of an induced Specht module, J. Algebra **322** (2009), 893–902.

[70] K. C. Misra, T. Miwa, Crystal base for the basic representation of $U_q(\hat{\mathfrak{sl}}(n))$, Commun. Math. Phys. **134** (1990), 79–88.

[71] H. Miyachi, Unipotent Blocks of Finite General Groups in Non-defining Characteristic. PhD thesis, Chiba univ., March 2001.

[72] H. Miyachi, Uno's conjecture for the exceptional Iwahori-Hecke algebras, New trends in combinatorial representation theory, 155–177, RIMS Kkyroku Bessatsu, B11, Res. Inst. Math. Sci. (RIMS), Kyoto, 2009.

[73] A. Prasad, I. Shapiro, M.K. Vermuri, Locally compact abelian groups with symplectic self-duality, Adv. Math. **225** (2010), no. 5, 2429–2454.

[74] J. Rickard, Morita theory for derived categories, J. London. Math. Soc (2) **39** (1989), 436–456.

[75] J. Rickard, Derived categories and stable equivalence, J. Pure Appl. Alg. **61** (1989), 303–317.

[76] J. Rickard, Derived equivalences as derived functors, J. London Math. Soc. (2) **43** (1991), 37–48.

[77] C. Riedtmann, Algebren, Darstellungsköcher, Überlagerungen und zurück, Comment. Math. Helv. **55** (1980), 199–224.

[78] C. M. Ringel, The representation type of local algebras, Representations of Algebras, Springer-Verlag, 1975.

[79] I. Reiten, C. Riedtmann, Skew group algebras in the representation theory of artin algebras, J. Algebra **92** (1985), 224–282.

[80] J. Scopes, Cartan Matrices and Morita Equivalence for Blocks of the Symmetric Groups, J. Algebra **142** (1991), 441–455.

[81] Ø. Solberg, Support varieties for modules and complexes, in: Trends in Representation Theory of Algebras and Related Topics, in: Contemp. Math., vol. 406, Amer. Math. Soc., Providence, RI, 2006, pp. 239–270.

[82] R. Steinberg, On a theorem of Pittie, Topology **14** (1975), 173–177.

[83] W. Turner, Equivalent blocks of finite general linear groups, J. Algebra **247** (2002), no. 1, 244–267.

[84] W. Turner, Rock blocks, Mem. Amer. Math. Soc. **202** (2009), no. 947.

[85] K. Uno, On representations of non-semisimple specialized Hecke algebras, J. Algebra **149** (1992), 287–312.

[86] M. Vazirani, Irreducible modules over the affine Hecke algebra: a strong multiplicity one result, Ph.D. thesis, UC Berkeley, 1999.

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