



Interest Rate Modeling

The Potential Approach and Post-Crisis Multi-Curve Potential Models

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To Ha-My

With love from Papa

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Abstract

This thesis is concerned with interest rate modeling by means of the potential approach. The contribution of this work is twofold. First, by making use of the potential approach and the theory of affine Markov processes, we develop a general class of rational models to the term structure of interest rates which we refer to as *the affine rational potential model*. These models feature positive interest rates and analytical pricing formulae for zero-coupon bonds, caps, swaptions, and European currency options. We present some concrete models to illustrate the scope of the affine rational potential model and calibrate a model specification to real-world market data. Second, we develop a general family of *multi-curve potential models* for post-crisis interest rates. Our models feature positive stochastic basis spreads, positive term structures, and analytic pricing formulae for interest rate derivatives. This modeling framework is also flexible enough to accommodate negative interest rates and positive basis spreads.

Zusammenfassung

Diese Arbeit befasst sich mit der Modellierung von Zinsstrukturkurven mit Hilfe des Potential-Ansatzes. Zunächst konstruieren wir mittels des Potential-Ansatzes und der Theorie der affinen Markov-Prozesse eine allgemeine Klasse von rationalen Zinsstrukturmodellen, die wir als *The Affine Rational Potential Model* bezeichnen. Diese Modelle bieten positive Zinssätze und analytische Formeln für Zero-Coupon Bonds, Caps, Swaptions, und europäische Währungsoptionen. Zur Illustration betrachten wir einige spezielle Modelle in dieser allgemeinen Klasse und kalibrieren ein spezifisches Modell an empirische Marktdaten. Der zweite zentrale Beitrag dieser Dissertation ist die Entwicklung der *Multi-Curve Potential* Modelle. Die Multi-Curve Potential Modelle verfügen über zahlreiche wünschenswerte Eigenschaften wie zum Beispiel positive stochastische Spreads, positive Zinssätze und analytische Bewertungsformeln für Zinsderivate. Unser Modellierungsansatz kann jedoch auch negative Zinssätze und positive Spreads abbilden.

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Chapter 1

Introduction

The primary concern of this thesis is interest rate modeling using the potential approach, with a particular focus on post-crisis multi-curve models. In this introductory chapter, we generally outline the necessary background and provide an overview of the thesis.

Interest rate models and the potential approach. Fixed-income instruments such as bonds, forward rate agreements, swaps, caps/floors, and swaptions make up the largest portion of the global financial market. Therefore, modeling the term structure of interest rates is of paramount importance. The vast majority of the classical interest rate models can be divided into three classes: the *short rate models*, the *forward rate models*, and the *market models*.

In the short rate models, one specifies directly the spot short rate process and then derives pricing formulae for zero-coupon bonds and other interest rate derivatives from the short rate dynamics. A huge number of research papers have been devoted to developing various models for the short rate process including, among many others, Vasicek (1977), Cox, Ingersoll & Ross (1985), Schaefer & Schwartz (1987), Black, Derman & Toy (1990), Hull & White (1990), Longstaff & Schwartz (1991), and Duffie & Kan (1996).

In the forward rate models, the instantaneous forward rate process is modeled directly. This modeling approach was pioneered by Ho & Lee (1986) who study

the evolution of forward rates in a discrete-time setting. It was then thoroughly extended in a continuous-time setting by Heath, Jarrow & Morton (1992).

The so-called market models are based on direct specification of observable market rates such as LIBOR and swap rates. These models are easy to calibrate to market data and have quickly become popular among practitioners. Important contributions in this direction are Brace, Gatarek & Musiela (1997), Jamshidian (1997), and Miltersen, Sandmann & Sondermann (1997).

In contrast to the above interest models, the basic idea of the *potential approach* is to consider the state-price deflator process, D say, as the modeling primitive and to express the time t -price of any contingent claim C_T settled at time $T \geq t$ through the formula:

$$C_t = \frac{\mathbb{E}_t[C_T D_T]}{D_t},$$

where the conditional expectation $\mathbb{E}_t[\cdot]$ is taken under a common reference probability measure \mathbb{P} . To our knowledge, the earliest paper on the potential approach is Constantinides (1992), who specifies the state-price deflator as an exponential quadratic function of a Gaussian process. This modeling methodology was further analyzed by Flesaker & Hughston (1996a), and was formalized by Rogers (1997); see also Rutkowski (1997), Goldberg (1998), Nakamura & Yu (2000), Jin & Glasserman (2001), and Yao (2001). The potential approach offers a great simplification when it comes to modeling in a cross-currency setting: the no-arbitrage spot exchange rate process between two economies is determined as the quotient of their state-price deflators. This important advantage of the potential approach in multi-currency modeling has also been highlighted by Rogers (1997):

"[...] if one has adopted the potential approach to term-structure modeling, then once the term structure has been modeled in two countries, the exchange rate between them is determined; no further Brownian motions are needed. [...] By modeling a single Markov process, and then defining a yield curve model in terms of a function of that Markov process, we are able to extend our model to incorporate additional countries, simply by taking a new function of the Markov process."

In Chapter 4 of this thesis we will employ this practical feature to model multiple term structures after making a quanto interpretation of the LIBOR rate which renders an analogy between multi-curve and multi-currency modeling.

Multi-curve framework. Prior to the recent financial crisis, LIBOR¹ was considered as a proxy for the riskless interest rate. Since many interest rate derivatives depend on LIBOR, only one curve is needed to be constructed for valuation purposes when discounting is based on LIBOR. This single-curve modeling framework was typically consistent with the pre-crisis market observations as the basis spreads were just about some basis points and thus could be safely regarded as negligible. However, after the financial crisis started in the second half of 2007, the traditional fixed-income pricing methodology of using a unique curve for both discounting and generating future cash flows has been called into question when substantial spreads emerged between rates that used to track each other: In post-crisis interest rate markets, overnight indexed swap rates generally deviate from swap rates of the same maturities, as do swap rates with different floating legs and identical fixed legs; see Figure 1.1 for an illustration. Therefore, the multi-curve approach has been adopted by academics and practitioners alike to deal consistently with the new market realities. This has quickly become the standard pricing approach for interest rate derivatives: In the multi-curve framework, one curve is used for discounting, and for each given market tenor (e.g., 3m) a different curve is constructed for generating forward rates corresponding to that tenor. For a brief review of the recent literature concerning multi-curve modeling we refer to Section 4.1.

Outline of the thesis. The purpose of this thesis is twofold. On the one hand, we develop a class of tractable interest rate models using the potential methodology and the theory of affine Markov processes. On the other hand, we establish a general family of multi-curve potential models for post-crisis interest rates.

¹LIBOR stands for London Interbank Offered Rate, the rate at which panel banks lend to each other. In chapter 4, we will elaborate upon LIBOR as well as other important interest rates.

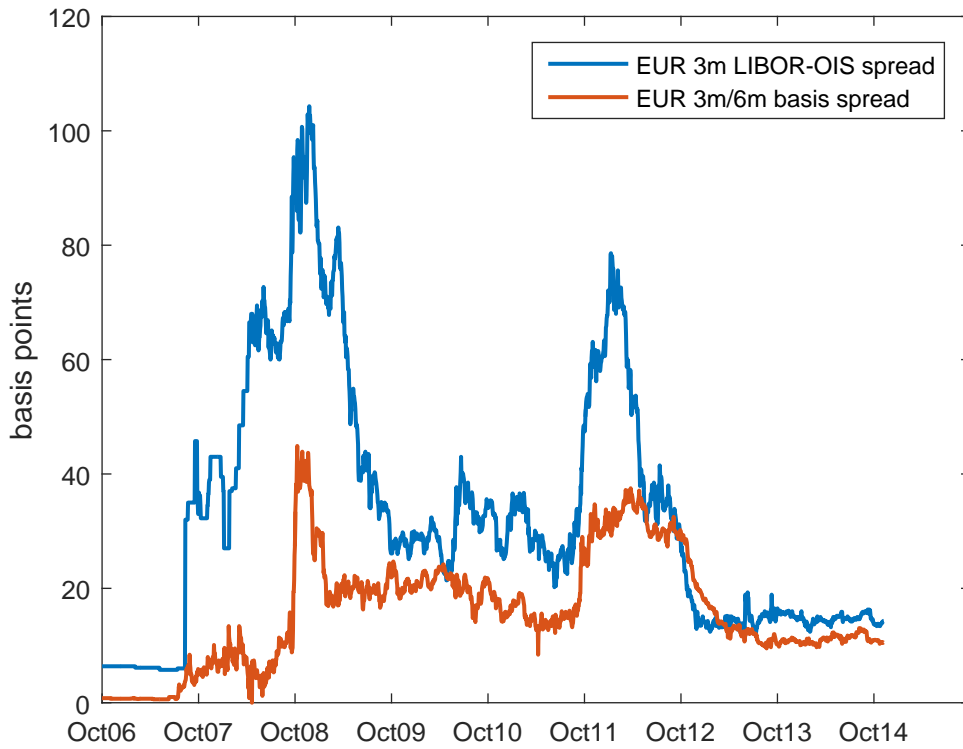


Figure 1.1: Historical basis spreads from October 2006 to October 2014 with one year maturity.

The thesis is structured as follows: Chapter 2 briefly provides the necessary background on interest rate modeling and the potential approach. We discuss in detail the special advantage of the potential approach for multi-currency modeling since this will be exploited to construct our multi-curve potential models in Chapter 4.

In Chapter 3, we construct a positive supermartingale from an affine Markov process and use it to model the state-price deflator. We derive closed-form expressions for bond prices and semi-closed form formulae for caps, swaptions, and European currency options. Moreover, we present some specific rational term structure models within our modeling framework and provide some numerical

examples at the end of the chapter.

In Chapter 4, we develop a general multi-curve potential model which can provide stochastic positive basis spreads and analytical formulae for important interest rate derivatives. We also present four concrete model specifications and calibrate the multi-curve rational lognormal models to swap and swaption market data.

Chapter 5 concludes with a review of the main contributions of this thesis and the important directions for future research.

Chapter 2

The Potential Approach

In this chapter we present the definitions and basic facts of the potential approach to interest rate modeling, with a particular focus on its application in a cross-currency setting. For a more detailed account of the issues discussed here, we refer to Flesaker & Hughston (1996a,b, 1997), Rogers (1997), Rutkowski (1997), and Jin & Glasserman (2001).

2.1 Basics of Interest Rate Modeling

This section recalls some basic definitions of interest rate modeling. We refer to Brigo & Mercurio (2006) for a full treatment of classical interest rate models and related issues.

2.1.1 Bonds and interest rates

To begin with, we recall the definition of zero coupon bonds, the fundamental building blocks of interest rate products.

Definition 2.1. A **zero (coupon) bond** with maturity T , also called a T -**bond** or **pure discount bond**, is a default-free contract that guarantees its holder the payment of one unit of currency at time T , without intermediate coupon payments. We denote by $p(t, T)$ the time t -price of a zero bond with maturity T . Obviously, $p(t, t) = 1$ for all t .

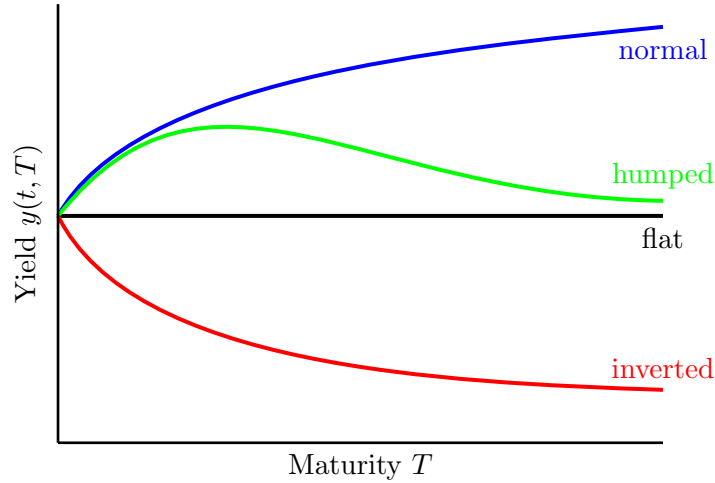


Figure 2.1: Term structure of interest rates.

For a fixed maturity $T > 0$, $p(\cdot, T)$ is a scalar stochastic process of prices at different times prior to maturity of the T -bond. Hence, its trajectory is very irregular. For a fixed time t , the function $p(t, \cdot)$ gives time t -prices of zero bonds with different maturities and therefore its graph is typically rather smooth.

Spot rates

Definition 2.2. The (zero coupon) **yield**, $y(t, T)$, is defined as the fixed rate for the time interval $[t, T]$ at which an investment of $p(t, T)$ at time t accrues continuously to 1 at maturity T , i.e.

$$p(t, T)e^{(T-t)y(t, T)} = 1,$$

or

$$y(t, T) = -\frac{\ln p(t, T)}{T - t}.$$

For a fixed t , the curve $(y(t, T))_{T>t}$ is called the **term structure of interest rates** or **yield curve** as of time t .

The term structure is said to be *normal* if it is increasing, *inverted* if it is decreasing and *flat* if it is constant; see Figure 2.1. The shape of a yield curve

often provides valuable insights as to what is believed to take place in the fixed income market in the future. A normal yield curve indicates that the economy is expected to grow in the future and this growth may lead to higher inflation and higher interest rates. A normal yield curve typically takes place when central banks (such as European Central Bank) ease monetary policy, thereby increasing money supply in the economy. By contrast, an inverted yield curve implies that investors expect the economy to slow or even decline in the future and this might lead to lower inflation and lower interest rates.

Definition 2.3. The **LIBOR rate**, $L(t, T)$, for the time period $[t, T]$ is defined as the simple compounding counterpart of the yield $y(t, T)$, i.e.

$$p(t, T)(1 + (T - t)L(t, T)) = 1,$$

or

$$L(t, T) = \frac{1}{T - t} \left(\frac{1}{p(t, T)} - 1 \right). \quad (2.1)$$

Note that in an infinitesimal time period, the above definitions of yield and LIBOR rate are equivalent. This leads to the following:

Definition 2.4. The **short rate**, r_t , is the limit of the yield $y(t, T)$ and LIBOR rate $L(t, T)$ as maturity T approaches t , i.e.

$$r_t = \lim_{T \rightarrow t^+} y(t, T) = \lim_{T \rightarrow t^+} L(t, T).$$

A simple computation yields that

$$r_t = -\partial_T \ln p(t, T)|_{T=t} = -\partial_T p(t, T)|_{T=t}.$$

Definition 2.5. The **money market account** B is defined by

$$B_t = e^{\int_0^t r_s ds}.$$

Clearly, B satisfies

$$dB_t = r_t B_t dt, \quad B_0 = 1.$$

From the above dynamics we have

$$\frac{B_{t+dt} - B_t}{B_t} \approx r_t dt,$$

for an infinitesimal dt . Hence, at each time instant t , the money market account B grows at a rate of r_t .

Forward rates

Definition 2.6. A **forward rate agreement** for the time interval $[T, T + \Delta]$ is a contract in which two parties agree at a time t prior to T that, at the maturity time $T + \Delta$

- one party pays a fixed amount of ΔK and receives a LIBOR-payment of $\Delta L(T, T + \Delta)$,
- the other party pays $\Delta L(T, T + \Delta)$ and receives ΔK .

The party that pays the fixed rate (here, ΔK) is called the payer of the contract, the other the receiver.

The value of the contract to the payer at maturity is therefore

$$V(T + \Delta) = \Delta(L(T, T + \Delta) - K).$$

From Equation (2.1) we further have

$$V(T + \Delta) = \frac{1}{p(T, T + \Delta)} - (1 + \Delta K).$$

Its time T -value is therefore given by

$$V(T) = 1 - (1 + \Delta K)p(T, T + \Delta).$$

Hence, the time t -value of the forward contract is

$$V(t) = p(t, T) - (1 + \Delta K)p(t, T + \Delta).$$

We then define the time t -forward LIBOR rate as the fixed rate K that renders the forward contract fair at time t . Formally, we have the following:

Definition 2.7. The time t -forward **LIBOR rate** on $[T, T + \Delta]$, denoted by $F(t; T, T + \Delta)$, is defined by

$$F(t; T, T + \Delta) := \frac{1}{\Delta} \left(\frac{p(t, T)}{p(t, T + \Delta)} - 1 \right). \quad (2.2)$$

The time t -**instantaneous forward rate** for investment at time T is defined as

$$f(t, T) := \lim_{\Delta \downarrow 0} F(t; T, T + \Delta). \quad (2.3)$$

It can be interpreted as the time t -forward LIBOR rate for the infinitesimal period $[T, T + dT]$.

A more explicit formula for instantaneous forward rate is derived as follows

$$f(t, T) = \lim_{\Delta \downarrow 0} \frac{1}{p(t, T + \Delta)} \frac{p(t, T) - p(t, T + \Delta)}{\Delta} = -\frac{\partial_T p(t, T)}{p(t, T)} = -\frac{\partial \ln p(t, T)}{\partial T}.$$

It then follows immediately that bond prices relate to forward rates via

$$p(t, T) = e^{-\int_t^T f(t, s) ds}.$$

2.1.2 Interest rate derivatives

We briefly present the definitions of some important interest rate derivatives. More details on these instruments will be provided as we proceed.

Definition 2.8. An **interest rate swap** is a contract in which two parties exchange (swap) a stream of fixed rate payments for a stream of floating rate (e.g., LIBOR, EURIBOR) payments over the payment period. Maturity, notional amount and tenor are agreed in the contract. The **par swap rate** is the value of the fixed rate that renders the value of the contract zero at inception. The swap which pays (receives) the fixed leg is called a *payer* (*receiver*) swap.

As an example, we consider an interest rate swap with two different tenor structures:

$$0 < T_0 < T_1 < \dots < T_N \text{ and } T_0 = \tau_0 < \tau_1 < \dots < \tau_M = T_N.$$

We denote by $\Delta_i := T_i - T_{i-1}$ and $\delta_i := \tau_i - \tau_{i-1}$. At each time T_i one party pays $\Delta_i C$, $i = 1, \dots, N$, for a fixed rate C , and the other party pays the floating amount of $\delta_i L(\tau_{i-1}, \tau_i)$ at every time τ_i , $i = 1, \dots, M$, where

$$L(\tau_{i-1}, \tau_i) = \frac{1}{\delta_i} \left(\frac{1}{p(\tau_{i-1}, \tau_i)} - 1 \right).$$

By a simple calculation, we obtain today's values of the fixed leg and the floating leg as follows

$$Fi(0) = C \sum_{i=1}^N \Delta_i p(0, T_i),$$

$$Fl(0) = \sum_{i=1}^M \delta_i p(t, \tau_i) F(t; \tau_{i-1}, \tau_i) = p(0, \tau_0) - p(0, \tau_M).$$

By definition, the par swap rate is the fixed rate C which solves the equation $Fi(0) = Fl(0)$, and it is thus given by

$$C = \frac{p(0, \tau_0) - p(0, \tau_M)}{\sum_{i=1}^N \Delta_i p(0, T_i)}.$$

In addition to bonds and swaps, there are two classes of highly liquid interest rate derivatives, which are caps and floors, and options on swaps, termed swaptions.

Definition 2.9. Consider a tenor structure $T = T_0 < T_1 < \dots < T_N$. A **payer (receiver) swaption** with strike rate C , maturity $T = T_0$ and payment dates $T_1 < \dots < T_N$ is an option that gives its holder the right (but not obligation) to enter a payer (receiver) interest rate swap with fixed leg rate C and payment dates T_1, \dots, T_N at time T . A swaption is said to be *at-the-money* if the strike rate C coincides with the par swap rate of the associated interest rate swap.

The time T -value of a payer swaption is thus given by

$$(S(T))^+ := \text{Max}(S(T), 0),$$

where $S(T)$ denotes the time T -value of the associated payer swap.

Definition 2.10. A **caplet** with strike rate K on the time interval $[S, T]$ is a call option on the future spot LIBOR rate $L(S, T)$ which guarantees its holder the payment of

$$(T - S)(L(S, T) - K)^+$$

at maturity T . A **floorlet** with strike K on $[S, T]$ is a put option on the future spot LIBOR rate $L(S, T)$ with payoff

$$(T - S)(K - L(S, T))^+$$

at time T . A **cap (floor)** is simply a portfolio of caplets (floorlets).

2.2 The Potential Approach

The potential approach to the term structure of interest rates, set forth by Constantinides (1992), Flesaker & Hughston (1996a,b, 1997), and Rogers (1997), has been around for years. In contrast to alternative interest rate models, which consider basic interest rates such as short rates, instantaneous forward rates, and spot LIBOR rates as the modeling primitives, the key element of the potential approach is to model directly the state-price deflator process and to express prices of interest rate derivatives in terms of this process.

2.2.1 Generalities

Our starting point is a probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ satisfying the usual conditions. We assume that the market is free of arbitrage opportunities and denote by \mathbb{Q} a fixed equivalent martingale measure with the money market account B as the numéraire. Then the time- t no-arbitrage price $\Pi^Y(t, T)$ of any contingent claim Y settled at time T is given by

$$\Pi^Y(t, T) = \mathbb{E}_t^{\mathbb{Q}} \left[\frac{B_t}{B_T} Y \right],$$

where $\mathbb{E}_t^{\mathbb{Q}}$ indicates conditional expectation, given the σ -field \mathcal{F}_t , with respect to the measure \mathbb{Q} . We denote by Z the Radon-Nikodym density process of \mathbb{Q} with

respect to \mathbb{P} , i.e.

$$\left. \frac{d\mathbb{Q}}{d\mathbb{P}} \right|_{\mathcal{F}_t} = Z_t.$$

Applying Bayes' rule for conditional expectations, we can represent the price of the contingent claim Y under the reference measure \mathbb{P} as follows:

$$\Pi^Y(t, T) = \mathbb{E}_t^{\mathbb{Q}} \left[\frac{B_t}{B_T} Y \right] = \frac{\mathbb{E}_t[Z_T \frac{B_t}{B_T} Y]}{Z_t} = \frac{\mathbb{E}_t[\frac{Z_T}{B_T} Y]}{\frac{Z_t}{B_t}},$$

where \mathbb{E}_t denotes conditional expectation with respect to \mathbb{P} . Setting

$$D_t = \frac{Z_t}{B_t}$$

we obtain

$$\Pi^Y(t, T) = \frac{\mathbb{E}_t[D_T Y]}{D_t}. \quad (2.4)$$

We refer to D as the **state-price deflator** process (D is also known as the state-price density process or pricing kernel process). As a special case of formula (2.4), we obtain the no-arbitrage price of a T -bond at time t prior to T :

$$p(t, T) = \frac{\mathbb{E}_t[D_T]}{D_t}. \quad (2.5)$$

The following result gives a necessary and sufficient condition that ensures that the implied interest rates remain non-negative.

Proposition 2.11. *The term structure $p(t, \cdot)$ implied by (2.5) is decreasing for all $t \geq 0$ if and only if the state-price deflator D is a positive supermartingale.*

Proof. Suppose that $p(t, \cdot)$ is decreasing and observe that in that case

$$1 = p(t, t) \geq p(t, T) = \frac{\mathbb{E}_t[D_T]}{D_t}.$$

Hence

$$D_t \geq \mathbb{E}_t[D_T] \text{ for all } t \leq T,$$

or D is a supermartingale. Conversely, if D is a positive super-martingale, then for any $t \leq T \leq S$ we have

$$p(t, S) = \frac{\mathbb{E}_t[D_S]}{D_t} = \frac{\mathbb{E}_t[\mathbb{E}_T[D_S]]}{D_t} \leq \frac{\mathbb{E}_t[D_T]}{D_t} = p(t, T)$$

by the tower property of conditional expectations. It follows that $p(t, \cdot)$ is decreasing for all $t \geq 0$. \square

Obviously, p satisfies in addition the natural condition $\lim_{T \rightarrow \infty} p(0, T) \rightarrow 0$ if and only if $\lim_{T \rightarrow \infty} \mathbb{E}[D_T] = 0$. A positive supermartingale D with this asymptotic property is called a **potential** (Meyer 1966, Protter 1990).

The basic idea of the state-price deflator pricing methodology is to model the market under the reference measure \mathbb{P} together with the state-price deflator process D , and to use Equations (2.4), (2.5) for pricing. The following result shows that this ensures the model is free of arbitrage opportunities.

Proposition 2.12. *Let D be a strictly positive semimartingale with $D_0 = 1$ and $\mathbb{E}[D_t] < \infty$ for all $t \geq 0$. Suppose that for any $T \geq 0$ the price $p(\cdot, T)$ of the T -bond is given by (2.5). Then, for any $T^* > 0$, there are no arbitrage opportunities in the bond market $(p(t, T))_{t \in [0, T^*], T \geq 0}$. In particular, (2.4) yields a no-arbitrage price for any contingent claim Y settled at time T .*

Proof. It suffices to show that there exists an equivalent martingale measure along with a suitable numéraire for any fixed $T^* > 0$. By (2.5) the process $(D_t p(t, T))_{t \in [0, T]}$ is a \mathbb{P} -martingale for each $T \geq 0$. In particular, the process Z given by

$$Z_t := D_t \frac{p(t, T^*)}{p(0, T^*)}, \quad t \in [0, T^*]$$

is a strictly positive martingale with $Z_0 = 1$. Thus, we can define an equivalent probability measure \mathbb{P}^* on $(\Omega, \mathcal{F}_{T^*})$ via

$$\left. \frac{d\mathbb{P}^*}{d\mathbb{P}} \right|_{\mathcal{F}_t} = Z_t, \quad t \in [0, T^*].$$

Denote by $\mathbb{E}_u^*[\cdot]$ the conditional expectation, given \mathcal{F}_u , under the measure \mathbb{P}^* . For any $0 \leq s \leq t \leq T^*$, Bayes' rule yields

$$\mathbb{E}_s^* \left[\frac{p(t, T)}{p(t, T^*)} \right] = \frac{\mathbb{E}_s[Z_t \frac{p(t, T)}{p(t, T^*)}]}{Z_s} = \frac{\mathbb{E}_s[p(t, T)D_t]}{p(s, T^*)D_s} = \frac{p(s, T)}{p(s, T^*)}.$$

Hence, $\frac{p(\cdot, T)}{p(\cdot, T^*)}$ is a \mathbb{P}^* -martingale on $[0, T^*]$ for any $T \geq 0$, and \mathbb{P}^* is an equivalent martingale measure for the bond market with associated numéraire $p(\cdot, T^*)$. \square

Remark 2.13. If instead of the base measure \mathbb{P} , we work with another base measure \mathbb{P}° , which is equivalent to \mathbb{P} , then there exists a \mathbb{P}° -positive supermartingale D° such that

$$p(t, T) = \frac{\mathbb{E}_t^\circ[D_T^\circ]}{D_t^\circ} \text{ for all } t \in [0, T],$$

where \mathbb{E}_t° denotes the conditional expectation given the filtration \mathcal{F}_t , taken with respect to \mathbb{P}° . Indeed, denote by V the Radon-Nikodym density process of \mathbb{P}° with respect to \mathbb{P} , i.e.

$$\left. \frac{d\mathbb{P}^\circ}{d\mathbb{P}} \right|_{\mathcal{F}_t} = V_t,$$

and define the process D° as $D_t^\circ = \frac{D_t}{V_t}$ for all $t \geq 0$. Then, by virtue of the Bayes' rule for conditional expectation, we have

$$\frac{\mathbb{E}_t^\circ[D_T^\circ]}{D_t^\circ} = \frac{\mathbb{E}_t[D_T V_T]}{D_t V_t} = \frac{\mathbb{E}_t[D_T]}{D_t} = p(t, T), \text{ for all } t \in [0, T].$$

Moreover, for any $0 \leq s < t$, the Bayes' rule yields that

$$\mathbb{E}_s^\circ[D_t^\circ] = \frac{\mathbb{E}_s[D_t V_t]}{V_s} = \frac{\mathbb{E}_s[D_t]}{V_s} \leq \frac{D_s}{V_s} = D_s^\circ,$$

whence D° is a supermartingale with respect to \mathbb{P}° .

2.2.2 Connection to multi-currency modeling

The potential approach outlined above is particularly well-suited for multi-currency settings because the exchange rate is determined as the quotient of the associated state-price deflators; see Flesaker & Hughston (1996a,b) and Rogers (1997). Hence, if the term structures in each currency are modeled using the potential approach, we have direct access to the associated exchange rate. This in particular implies that, in contrast to alternative models, it is not necessary to postulate separate dynamics for the spot exchange rate process. The following key result formalizes this insight:

Proposition 2.14 (Exchange rate in the potential framework). *Consider a domestic market in the currency $\d , and a foreign market with currency $\f , and assume that each market is free of arbitrage. Let D^d and D^f denote the associated state-price deflators, and denote by x_{fd} the spot exchange rate between $\d and $\f : At time t , one unit of the foreign currency is equivalent to $x_{fd}(t)$ units of the domestic currency. Then the overall market excludes arbitrage opportunities if*

$$x_{fd}(t) = \frac{D_t^f}{D_t^d} x_{fd}(0). \quad (2.6)$$

Proof. Let S^f be the price process of a traded asset in the foreign currency $\f . By construction of D^f , $D^f S^f$ is a \mathbb{P} -local martingale. The price of the same asset in the domestic currency is given by

$$S^f x_{fd} = \frac{D_t^f S_t^f}{D_t^d}$$

and hence $D^d S^f x_{fd}$ is a \mathbb{P} -local martingale as well. An analogous argument applies to the price process of any asset traded in the domestic currency. \square

Remark 2.15. If the (multi-currency) market is complete, then the process x_{fd} given by (2.6) is the only exchange rate process that excludes arbitrage opportunities; see, e.g., Rogers (1997) and Björk (2009).

Proposition 2.16. *Let X_{fd} denote the forward exchange rate between the currencies $\d and $\f . Thus, as of time $t \leq T$, one unit of the foreign currency delivered at time T is equivalent to $X_{fd}(t, T)$ units of the domestic currency deliverable at T . Then we have the no-arbitrage relation*

$$X_{fd}(t, T) p_d(t, T) = x_{fd}(t) p_f(t, T), \quad (2.7)$$

where p_f and p_d are the discount curves in the foreign and domestic currency. Moreover, the forward exchange rate process X_{fd} satisfies

$$X_{fd}(t, T) = \frac{x_{fd}(t) p_f(t, T)}{p_d(t, T)} = x_{fd}(0) \frac{\mathbb{E}_t[D_T^f]}{\mathbb{E}_t[D_T^d]}. \quad (2.8)$$

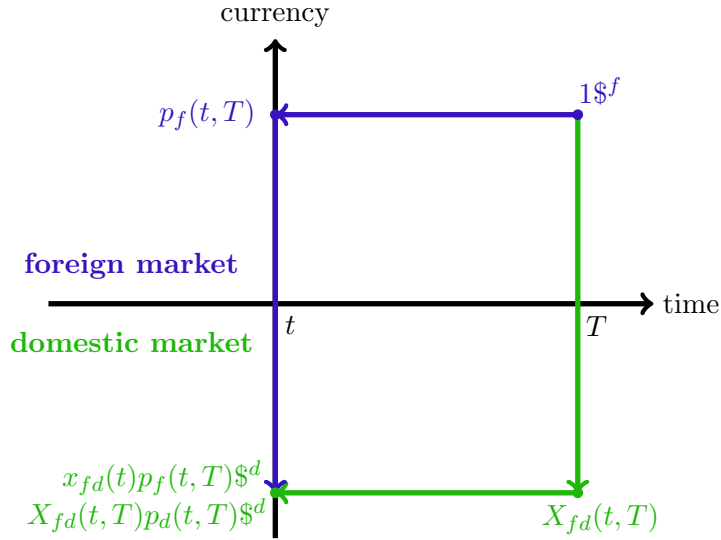


Figure 2.2: Illustration of the proof of Proposition 2.16: Discounting in the foreign market with $p_f(t, T)$ and currency conversion at spot rate $x_{fd}(t)$ (blue) versus conversion with time- t forward exchange rate $X_{fd}(t, T)$ and discounting in the domestic market with $p_d(t, T)$ (green).

Proof. Consider a claim to one unit of the foreign currency $\f at time T ; see Figure 2.2. Discounting with the foreign discount curve p_f and then converting to the domestic currency, we obtain its time t -value

$$p_f(t, T)\$^f = p_f(t, T)x_{fd}(t)\$^d.$$

On the other hand, by definition of the forward exchange rate, at time t the time- T value of $1\f in the domestic currency is $X_{fd}(t, T)\d . Discounting with the domestic discount curve p_d we have that $1\f at time T is worth $X_{fd}(t, T)p_d(t, T)\d at time t . Hence, by absence of arbitrage, we have

$$X_{fd}(t, T)p_d(t, T) = x_{fd}(t)p_f(t, T).$$

Recall that $p_f(t, T) = \frac{\mathbb{E}_t[D_T^f]}{D_t^f}$ and $p_d(t, T) = \frac{\mathbb{E}_t[D_T^d]}{D_t^d}$. Using Proposition 2.14 it

then follows that

$$X_{fd}(t, T) = \frac{x_{fd}(t)p_f(t, T)}{p_d(t, T)} = x_{fd}(0) \frac{\mathbb{E}_t[D_T^f]}{\mathbb{E}_t[D_T^d]}.$$

□

2.2.3 Flesaker-Hughston framework

We briefly focus on a specific model in the class of the potential approach that was proposed by Flesaker & Hughston (1996a). Let $\Phi(\cdot)$ be a positive deterministic function such that $\int_0^\infty \Phi(s)ds = 1$ and let $\{M_s(t)\}_{0 \leq t \leq s}$ be a family of positive martingale indexed by $s \in [0, \infty)$ with $M_s(0) = 1$ for all $s \geq 0$. Then the process D given by

$$D_t = \int_t^\infty \Phi(s)M_s(t)ds$$

is a positive supermartingale with $D_0 = 1$. Indeed, for any $t \leq T$ we have

$$\begin{aligned} \mathbb{E}_t[D_T] &= \int_T^\infty \mathbb{E}_t[\Phi(s)M_s(T)]ds = \int_T^\infty \Phi(s)\mathbb{E}_t[M_s(T)]ds \\ &= \int_T^\infty \Phi(s)M_s(t)ds \leq \int_t^\infty \Phi(s)M_s(t)ds = D_t. \end{aligned}$$

Therefore, D can be used as a state-price deflator in the potential approach. With this specification, bond prices are given by

$$p(t, T) = \frac{\mathbb{E}_t[D_T]}{D_t} = \frac{\int_T^\infty \Phi(s)M_s(t)ds}{\int_t^\infty \Phi(s)M_s(t)ds}.$$

and a straightforward calculation gives the short rates

$$r_t = \frac{\Phi(t)M_t(t)}{\int_t^\infty \Phi(s)M_s(t)ds}, \quad t \geq 0.$$

This model is commonly known as the Flesaker-Hughston model. It guarantees positive interest rates and is not subsumed by the standard classical short rate models. Moreover, it is sufficiently flexible to provide a perfect fit to the initial

term structure of interest rates: Indeed, by choosing $\Phi(s) = -\partial_s p(0, s)$, $s \geq 0$, we obtain

$$p(0, T) = \frac{\int_T^\infty \Phi(s) M_s(0) ds}{\int_0^\infty \Phi(s) M_s(0) ds} = \int_T^\infty \Phi(s) ds, \text{ for all } T \geq 0.$$

The most popular example of the Flesaker-Hughston class is the so-called *rational lognormal model*. This model has many appealing features, including positive interest rates, perfect fit to the initial yield curve, and analytic formulas for caps and swaptions. The rational lognormal model is obtained by specifying the state-price deflator

$$D_t = \alpha(t) + \beta(t)M_t, \quad t \in [0, \infty)$$

where α, β are positive, decreasing deterministic functions and M is a lognormal martingale,

$$M_t = e^{\int_0^t \sigma(s) dW_s - \frac{1}{2} \int_0^t \sigma^2(s) ds}, \quad t \geq 0,$$

with W a standard Brownian motion and σ a deterministic function. Bond prices are then obtained immediately from (2.4) via

$$p(t, T) = \frac{\mathbb{E}_t[D_T]}{D_t} = \frac{\alpha(T) + \beta(T)M_t}{\alpha(t) + \beta(t)M_t}$$

and the short rate is given by

$$r_t = -\frac{\alpha'(t) + \beta'(t)M_t}{\alpha(t) + \beta(t)M_t} \geq 0, \quad t \in [0, \infty).$$

To fit the rational lognormal model to any given initial term structure, it suffices to choose α, β in such a way that

$$\alpha(0) + \beta(0) = 1 \quad \text{and} \quad \alpha(T) + \beta(T) = p(0, T) \text{ for } T \in [0, \infty).$$

For further details of the Flesaker-Hughston approach and the rational lognormal model see, e.g., Hunt & Kennedy (2000), Brody & Hughston (2004), Cairns (2004), Björk (2009), and Musiela & Rutkowski (2011).

2.2.4 Rogers' Markov potential approach

Rogers (1997) presents a generic approach to model the state-price deflator using the classical theory of Markov processes. The key elements of the approach are a continuous-time Markov process X taking values in \mathbb{R}^d and a positive function f with domain \mathbb{R}^d . Rogers then defines the state-price deflator D in terms of f and X as follows:

$$D_t = e^{-\alpha t} f(X_t), \quad t \geq 0,$$

for some parameter $\alpha \in \mathbb{R}$. Then the time t -price, C_t , of any contingent claim C_T settled at time $T > t$ is determined by the formula:

$$C_t = \frac{\mathbb{E}[D_T C_T]}{D_t} = e^{-\alpha(T-t)} \frac{\mathbb{E}_t[C_T f(X_T)]}{f(X_t)}.$$

In particular, the price of a zero-coupon bond with maturity T is given by

$$p(t, T) = e^{-\alpha(T-t)} \frac{\mathbb{E}_t[f(X_T)]}{f(X_t)}, \quad t \in [0, T].$$

If the function f is given by

$$f(x) = R_\alpha[g](x) := \mathbb{E} \left[\int_0^\infty e^{-\alpha s} g(X_s) ds \mid X_0 = x \right],^1$$

for some bounded, measurable function $g : \mathbb{R}^d \rightarrow [0, \infty)$ and $\alpha > 0$, then the state-price deflator

$$D_t = e^{-\alpha t} f(X_t) = e^{-\alpha t} R_\alpha[g](X_t)$$

is a positive supermartingale. Moreover, the corresponding short rate process r is given by

$$r_t = \frac{g(X_t)}{f(X_t)}, \quad t \geq 0.$$

For any given positive function f , one can show that

$$f = R_\alpha[g] \quad \text{if and only if} \quad g = \alpha f - \mathbb{G}f,$$

¹ R_α is called *resolvent* of the Markov process X .

where \mathbb{G} is the *infinitesimal generator* of the Markov process X , which is defined by

$$\mathbb{G}f(x) := \lim_{t \rightarrow 0} \frac{\mathbb{E}[f(X_t)|X_0 = x] - f(x)}{t}.$$

We refer to Rogers (1997), Rogers & Yousaf (2002), and Björk (2009) for further details on the Rogers' Markov potential approach.

Chapter 3

The Affine Rational Potential Models

We develop a class of rational term structure models in the framework of the potential approach, based upon a family of positive supermartingales that are driven by an affine Markov process. These models generally feature non-negative interest rates and analytic pricing formulae for zero bonds, caps, swaptions, and European currency options, even in the presence of multiple factors. Moreover, in a model specification, the short rate stays near the zero lower bound for an extended period.

This chapter is based on a joint work with Frank Thomas Seifried: Nguyen & Seifried (2015b).

3.1 Introduction

The main advantage of the potential approach, as discussed in Section 2.2.2, is the ease in multi-currency modeling since the arbitrage-free exchange rate between any two currencies is determined as the quotient of the associated state-price deflators and therefore there is no need to postulate a separate dynamics for the exchange rate process. However, as mentioned in Rogers & Yousaf (2002),

the potential approach is likely to suffer the drawback of analytical intractability since only one universal probability measure together with a state-price deflator is used for pricing *every* interest rate derivative. For instance, the generic models of Flesaker & Hughston (1996a) and Rogers (1997) do not provide analytic pricing formulae for important interest rate derivatives such as caps and swaptions; analytical tractability is only obtained in some specific models.

In view of the issue discussed above, in this chapter we propose a class of interest rate models in the potential framework, which overcomes the limitation of analytical intractability. Using an affine Markov process we construct (strictly) positive supermartingales to be used as the state-price deflator. The rich structural properties of affine Markov processes allow us to derive analytic pricing formulae for important interest rate derivatives including zero bonds, caps, swaptions, and European currency options. Moreover, as the state-price deflator is a supermartingale, our models feature non-negative interest rates. In a specific model, where the underlying affine process is chosen to be a Cox-Ingersoll-Ross (CIR) process, we show that the short rate can stay near the zero lower bound for an extended period of time, which is a frequently observed feature in the low interest rate regime; see Section 3.6. In addition, bond prices and interest rates in our models are rational functions of the underlying state process, therefore our models belong to the broad class of rational term structure models including, among others, the models by Flesaker & Hughston (1996a), Brody & Hughston (2004), Hughston & Rafailidis (2005), Brody et al. (2012), Akahori et al. (2014), and Filipović et al. (2015). In particular, our models are not part of the classical short-rate ones.

The remainder of this chapter is structured as follows: Section 3.2 briefly introduces the affine Markov processes along with a construction of non-negative supermartingales, which are used to construct the state-price deflator. In Section 3.3, based upon the non-negative supermartingales constructed in Section 3.2, we develop a general affine rational potential model and general explicit formulae for zero bond prices. In relation to Filipović et al. (2015), we present a decom-

position of the underlying process into unspanned factors and term structure factors. In Section 3.4 we derive analytic pricing formulae for caps, swaptions, and European options on the exchange rate in the generic affine rational potential model. Section 3.5 presents some concrete models based on the specification of the affine process as a CIR process and a subordinator Lévy process. In Section 3.6, we calibrate the two-factor CIR specification of the affine rational potential model to market data. The stability of model parameters over the calibrated dates is analyzed. Moreover, using Monte Carlo estimation method we show that the short rate stays near the zero lower bound over an extended period.

3.2 Affine Processes

In this section we briefly review some key properties of affine Markov processes and present a general construction of non-negative supermartingales from an underlying affine process. These supermartingales will be used as the basic building block in our rational term structure model that is presented in Section 3.3. For further background on the theory of Markov affine processes and their applications in finance, we refer to Ethier & Kurtz (1986), Rogers & Williams (2000), Duffie et al. (2000), Duffie et al. (2003), Keller-Ressel (2008), and Keller-Ressel et al. (2013).

Let $(\Omega, \mathcal{F}, \mathbf{F})$ be a filtered space, where $\mathbf{F} = (\mathcal{F}_t)_{t \in [0, T^\infty]}$ denotes the associated filtration and $T^\infty > 0$ represents a fixed, finite time horizon. Suppose that $(X_t, \mathbb{P}^x)_{t \in [0, T^\infty], x \in E}$ is a stochastically continuous, time-homogeneous Markov process taking values in $E := \mathbb{R}_{\geq 0}^d$ and $(\mathbb{P}^x)_{x \in E}$ is a family of probability measures on (Ω, \mathcal{F}) such that $X_0 = x$ holds \mathbb{P}^x -almost surely. We assume that X is an affine process, i.e., there exist functions $\phi : [0, T^\infty] \times i\mathbb{R}^d \rightarrow \mathbb{C}$ and $\psi : [0, T^\infty] \times i\mathbb{R}^d \rightarrow \mathbb{C}^d$ such that

$$\mathbb{E}^x \left[e^{\langle u, X_t \rangle} \right] = \exp(\phi_t(u) + \langle \psi_t(u), x \rangle) \quad (3.1)$$

for all $x \in E$ and all $(t, u) \in [0, T^\infty] \times i\mathbb{R}^d$. Here, we denote by $\mathbb{E}^x[\cdot]$ the expectation with respect to probability measure \mathbb{P}^x and $\langle \cdot, \cdot \rangle$ the inner product

over \mathbb{C}^d . We set

$$\mathcal{J} := \left\{ u \in \mathbb{R}^d : \sup_{t \in [0, T^\infty]} \mathbb{E}^x [e^{\langle u, X_t \rangle}] < \infty \text{ for all } x \in E \right\}. \quad (3.2)$$

Assumption. We assume throughout that $0 \in \text{int}(\mathcal{J})$, where $\text{int}(\mathcal{J})$ denotes the interior of the set \mathcal{J} . We denote by $S(\mathcal{J})$ the strip

$$S(\mathcal{J}) = \{z \in \mathbb{C}^d : \Re z \in \mathcal{J}\}.$$

By Lemma 3.12 and Lemma 3.19 in Keller-Ressel (2008), there exists a unique continuous extension of $\phi_t(u)$ and $\psi_t(u)$ to $[0, T^\infty] \times S(\mathcal{J})$ such that (3.1) holds for all $(t, u) \in [0, T^\infty] \times S(\mathcal{J})$. By Theorem 3.18 in Keller-Ressel (2008), the affine process X is *regular* in the sense that the derivatives

$$F(u) := \left. \frac{\partial}{\partial t} \right|_{t=0+} \phi_t(u) \text{ and } R(u) := \left. \frac{\partial}{\partial t} \right|_{t=0+} \psi_t(u)$$

exist for all $u \in S(\mathcal{J})$ and are analytic on $\text{int}(S(\mathcal{J}))$. The functions ϕ and ψ solve the generalized Riccati equations

$$\frac{\partial}{\partial t} \phi_t(u) = F(\psi_t(u)), \quad \phi_0(u) = 0 \quad (3.3a)$$

$$\frac{\partial}{\partial t} \psi_t(u) = R(\psi_t(u)), \quad \psi_0(u) = u \quad (3.3b)$$

for all $(t, u) \in [0, T^\infty] \times S(\mathcal{J})$, where F and R have the Lévy-Khintchine form

$$F(u) = \langle b, u \rangle + \int_E \left(e^{\langle \xi, u \rangle} - 1 \right) m(d\xi), \quad (3.4a)$$

$$R_i(u) = \langle \beta_i, u \rangle + \frac{1}{2} \langle \alpha_i u, u \rangle + \int_E \left(e^{\langle \xi, u \rangle} - 1 - \langle u, h_i(\xi) \rangle \right) \mu_i(d\xi). \quad (3.4b)$$

Here, $(b, m, \alpha_i, \beta_i, \mu_i)_{1 \leq i \leq d}$ are *admissible parameters* in the sense of Duffie et al. (2003) that are uniquely determined by the dynamics of X , and $h_i : \mathbb{R}_{\geq 0}^d \rightarrow \mathbb{R}^d$, $i = 1, \dots, d$, are suitable truncation functions.

Remark 3.1. Since X is a time-homogeneous Markov process, it follows that

$$\mathbb{E}_t^x [e^{\langle u, X_T \rangle}] = \exp(\phi_{T-t}(u) + \langle \psi_{T-t}(u), X_t \rangle) \quad \text{for } 0 \leq t \leq T \leq T^\infty, \quad u \in S(\mathcal{J}).$$

Hence, for any $0 \leq t \leq t + s \leq T^\infty$ and for any $x \in E$ we have

$$\begin{aligned} \mathbb{E}^x[e^{\langle u, X_{t+s} \rangle}] &= \mathbb{E}^x[\mathbb{E}_s^x[e^{\langle u, X_{t+s} \rangle}]] \\ &= \mathbb{E}^x[e^{\phi_t(u) + \langle \psi_t(u), X_s \rangle}] \\ &= \exp(\phi_t(u) + \phi_s(\psi_t(u)) + \langle \psi_s(\psi_t(u)), x \rangle). \end{aligned} \quad (3.5)$$

On the other hand, the definitions of ϕ and ψ imply that

$$\mathbb{E}^x[e^{\langle u, X_{t+s} \rangle}] = \exp(\phi_{t+s}(u) + \langle \psi_{t+s}(u), x \rangle). \quad (3.6)$$

By comparing (3.5) and (3.6) we obtain the *semi-flow property* of ϕ and ψ :

$$\phi_{t+s}(u) = \phi_t(u) + \phi_s(\psi_t(u)) \quad (3.7a)$$

$$\psi_{t+s}(u) = \psi_s(\psi_t(u)), \quad (3.7b)$$

for $0 \leq t \leq t + s \leq T^\infty$ and $u \in S(\mathcal{J})$.

We now construct a family of non-negative supermartingales on $[0, T^\infty]$ based on the above affine Markov dynamics.

Theorem 3.2 (Construction of non-negative supermartingales). *For any $u \in E \cap \mathcal{J}$ the process M^u given by*

$$M_t^u = \phi_{T^\infty-t}(u) + \langle \psi_{T^\infty-t}(u), X_t \rangle$$

is a non-negative supermartingale (with respect to all $(\mathbb{P}^x)_{x \in E}$).

Proof. First, note that $\phi_t(u) \in \mathbb{R}$ and $\psi_t(u) \in \mathbb{R}^d$ for all $(t, u) \in [0, T^\infty] \times \mathcal{J}$. Therefore, the process M^u takes only real values. Further observe that the process $N_t^u = \exp(\phi_{T^\infty-t}(u) + \langle \psi_{T^\infty-t}(u), X_t \rangle)$ is a martingale since

$$N_t^u = \mathbb{E}_t^x[\exp(\langle u, X_{T^\infty} \rangle)].$$

Since $u \in E = \mathbb{R}_{\geq 0}^d$, it follows that $\langle u, X_{T^\infty} \rangle \geq 0$, and thus

$$N_t^u = \mathbb{E}_t^x[\exp(\langle u, X_{T^\infty} \rangle)] \geq 1.$$

Hence $M_t^u = \ln(N_t^u) \geq 0$ for all $t \in [0, T^\infty]$. Jensen's inequality for conditional expectations implies that

$$\mathbb{E}_s[M_t^u] = \mathbb{E}_s^x[\ln(N_t^u)] \leq \ln(\mathbb{E}_s^x[N_t^u]) = \ln(N_s^u) = M_s^u$$

for all $s, t \in [0, T^\infty]$ with $s \leq t$, so M^u is a supermartingale. This completes the proof. \square

Remark 3.3. In a related article, Keller-Ressel et al. (2013) propose a general LIBOR market model which produces non-negative LIBOR rates, where the quotients of zero bond prices are modeled using the martingales N^u in the above proof.

We conclude this section with an auxiliary result that we will need in Chapter 4.

Lemma 3.4. *The functions $\phi_t(\cdot)$ and $\psi_t(\cdot)$ are order-preserving in the sense that for any $(t, u), (t, v) \in [0, T^\infty] \times \mathcal{J}$ with $u \leq v$, it holds*

$$\phi_t(u) \leq \phi_t(v) \text{ and } \psi_t(u) \leq \psi_t(v).$$

Here, the inequality $u \leq v$ is interpreted component-wise.

Proof. See Lemma 4.2 in Keller-Ressel et al. (2013). \square

3.3 The Affine Rational Potential Model

In this section, based on the non-negative supermartingales in Theorem 3.2, we construct a general rational term structure model which produces non-negative interest rates and closed-form formulae for zero bond prices. Moreover, in this general model we obtain analytic pricing formulae for caps, swaptions, and European currency options, even in the presence of multiple factors ; see Section 3.4. In connection to Filipović et al. (2015), a decomposition of the underlying state process into so-called *unspanned factors* and *terms structure factors* is also presented.

In all that follows we fix an initial value $x \in E$ of the process X and set $\mathbb{P} = \mathbb{P}^x$ and $\mathbb{E}[\cdot] = \mathbb{E}^x[\cdot]$. For each fixed $u \in E \cap \mathcal{J}$ we define

$$D_t^u := c + M_t^u = c + \phi_{T^\infty-t}(u) + \langle \psi_{T^\infty-t}(u), X_t \rangle, \quad t \in [0, T^\infty], \quad (3.8)$$

for some positive constant $c > 0$. Theorem 3.2 implies that D^u is a (strictly) positive supermartingale. Thus, D^u can act as a state-price deflator, and this specification defines a rational term structure model that we will henceforth refer to as the **affine rational potential model**. In this framework, zero-coupon bond prices are given by

$$p(t, T) = \frac{\mathbb{E}_t[D_T^u]}{D_t^u} \quad \text{for all } 0 \leq t \leq T \leq T^\infty, \quad (3.9)$$

and the time t -price C_t of any contingent claim C_T to be paid at time $T > t$ is given by

$$C_t = \frac{\mathbb{E}_t[C_T D_T^u]}{D_t^u}. \quad (3.10)$$

Remark 3.5. Propositions 2.11 and 2.12 imply that the affine rational potential model is free of arbitrage opportunities and it generally guarantees non-negative interest rates.

In order to obtain closed-form expressions for the bond prices (3.9), we next address the conditional expectation $\mathbb{E}_t[D_T^u]$.

Lemma 3.6 (Conditional expectations). *For any $u \in \mathcal{J}$ and $t \in [0, T]$, we have*

$$\mathbb{E}_t[X_T] = \frac{\partial \phi_{T-t}(0)}{\partial u} + \frac{\partial \psi_{T-t}(0)^\dagger}{\partial u} \cdot X_t \quad (3.11)$$

where¹

$$\frac{\partial \phi_t(u)}{\partial u} := \left(\frac{\partial \phi_t(u)}{\partial u_1}, \dots, \frac{\partial \phi_t(u)}{\partial u_d} \right)^\dagger,$$

and

$$\frac{\partial \psi_t(u)}{\partial u} := \begin{pmatrix} \frac{\partial \psi_t^1(u)}{\partial u_1} & \dots & \frac{\partial \psi_t^1(u)}{\partial u_d} \\ \vdots & \ddots & \vdots \\ \frac{\partial \psi_t^d(u)}{\partial u_1} & \dots & \frac{\partial \psi_t^d(u)}{\partial u_d} \end{pmatrix}.$$

¹Throughout this thesis, we use the notation A^\dagger to indicate the transpose of matrix A .

In particular, it follows that

$$\begin{aligned}\mathbb{E}_t[D_T^u] &= c + \phi_{T^\infty-T}(u) + \left\langle \psi_{T^\infty-T}(u), \frac{\partial \phi_{T-t}(0)}{\partial u} \right\rangle \\ &\quad + \left\langle \frac{\partial \psi_{T-t}(0)}{\partial u} \cdot \psi_{T^\infty-T}(u), X_t \right\rangle.\end{aligned}\quad (3.12)$$

Proof. Consider an arbitrary component X^j of the process X . Then for $v \in \mathbb{R}$ with $|v|$ sufficiently small we have

$$\mathbb{E}_t[\exp(vX_T^j)] = \exp(\phi_{T-t}(ve_j) + \langle \psi_{T-t}(ve_j), X_t \rangle)$$

where e_k denotes the k^{th} unit vector in \mathbb{R}^d . Applying Lemma A.1 we obtain

$$\mathbb{E}_t[X_T^j] = \frac{\partial \phi_{T-t}(0)}{\partial u_j} + \sum_{k=1}^d \frac{\partial \psi_{T-t}^k(0)}{\partial u_j} X_t^k.$$

Hence

$$\mathbb{E}_t[X_T] = \frac{\partial \phi_{T-t}(0)}{\partial u} + \frac{\partial \psi_{T-t}(0)^t}{\partial u} \cdot X_t$$

and it follows that

$$\begin{aligned}\mathbb{E}_t[M_T^u] &= \mathbb{E}_t[\phi_{T^\infty-T}(u) + \langle \psi_{T^\infty-T}(u), X_T \rangle] \\ &= \phi_{T^\infty-T}(u) + \langle \psi_{T^\infty-T}(u), \mathbb{E}_t[X_T] \rangle \\ &= \phi_{T^\infty-T}(u) + \left\langle \psi_{T^\infty-T}(u), \frac{\partial \phi_{T-t}(0)}{\partial u} + \frac{\partial \psi_{T-t}(0)^t}{\partial u} \cdot X_t \right\rangle \\ &= \phi_{T^\infty-T}(u) + \left\langle \psi_{T^\infty-T}(u), \frac{\partial \phi_{T-t}(0)}{\partial u} \right\rangle + \left\langle \frac{\partial \psi_{T-t}(0)}{\partial u} \cdot \psi_{T^\infty-T}(u), X_t \right\rangle.\end{aligned}$$

Since $\mathbb{E}_t[D_T^u] = \mathbb{E}_t[M_T^u] + c$, this completes the proof. \square

Corollary 3.7 (Bond prices). *In the affine rational potential model, zero-coupon bond prices are given by $p(t, T) = F(t, T, X_t)$ where the deterministic function F is defined via*

$$F(t, T, x) = \frac{c + \phi_{T^\infty-T}(u) + \left\langle \psi_{T^\infty-T}(u), \frac{\partial \phi_{T-t}(0)}{\partial u} \right\rangle + \left\langle \frac{\partial \psi_{T-t}(0)}{\partial u} \cdot \psi_{T^\infty-T}(u), x \right\rangle}{c + \phi_{T^\infty-t}(u) + \langle \psi_{T^\infty-t}(u), x \rangle}.\quad (3.13)$$

Proof. The representation (3.13) follows immediately from (3.9) and (3.12). \square

Following Filipović et al. (2015), we define the *term structure kernel* as the set

$$\mathcal{A} = \bigcap_{0 \leq t \leq T \leq T^\infty} \ker F(t, T, \cdot)$$

where

$$\ker F(t, T, \cdot) := \{\xi \in \mathbb{R}^d : \langle \nabla F(t, T, x), \xi \rangle = 0 \text{ for all } x \in E\}.$$

We next characterize the term structure kernel \mathcal{A} in terms of the underlying model coefficients:

Proposition 3.8. *Suppose that for each $T \in (0, T^\infty]$ there exist $t \leq T$ and $x, y \in E$ such that $F(t, T, x) \neq F(t, T, y)$. Then the term structure kernel \mathcal{A} is given by*

$$\mathcal{A} = \text{span} \left\{ \frac{\partial \psi_{T-t}(0)}{\partial u} \cdot \psi_{T^\infty-T}(u) : 0 \leq t \leq T \leq T^\infty \right\}^\perp.$$

Proof. A direct computation using (3.13) shows that

$$\nabla F(t, T, x) = \frac{\frac{\partial \psi_{T-t}(0)}{\partial u} \cdot \psi_{T^\infty-T}(u) - F(t, T, x) \psi_{T^\infty-T}(u)}{c + \phi_{T^\infty-t}(u) + \langle \psi_{T^\infty-t}(u), x \rangle}.$$

Obviously we have

$$\mathcal{A}^\perp = \text{span} \left\{ \nabla F(t, T, x) : 0 \leq t \leq T \leq T^\infty, x \in \mathbb{R}_{\geq 0}^d \right\}.$$

Hence, the above representation of $\nabla F(t, T, x)$ implies that \mathcal{A}^\perp is given by the linear span of the set

$$\left\{ \frac{\partial \psi_{T-t}(0)}{\partial u} \cdot \psi_{T^\infty-T}(u) - F(t, T, x) \psi_{T^\infty-T}(u) : 0 \leq t \leq T \leq T^\infty, x \in \mathbb{R}_{\geq 0}^d \right\}.$$

For each $0 < T \leq T^\infty$, we choose $t \in [0, T]$, and $x, y \in \mathbb{R}_{\geq 0}^d$ such that $F(t, T, x) \neq F(t, T, y)$. Since $(F(t, T, x) - F(t, T, y)) \cdot \psi_{T^\infty-T}(u) \in \mathcal{A}^\perp$ and $F(t, T, x) - F(t, T, y) \neq 0$, it follows that $\psi_{T^\infty-T}(u) \in \mathcal{A}^\perp$, and thus

$$\frac{\partial \psi_{T-s}(0)}{\partial u} \cdot \psi_{T^\infty-T}(u) \in \mathcal{A}^\perp \text{ for all } s \in [0, T].$$

On the other hand we have the trivial identity

$$\psi_{T^\infty-T}(u) = \frac{\partial\psi_0(0)}{\partial u} \cdot \psi_{T^\infty-T}(u).$$

Hence, we obtain

$$\mathcal{A}^\perp = \text{span} \left\{ \frac{\partial\psi_{T-t}(0)}{\partial u} \cdot \psi_{T^\infty-T}(u) : t \in [0, T], T \in (0, T^\infty] \right\}$$

as asserted. \square

In an important special case, there are no unspanned factors:

Corollary 3.9. *Suppose the matrix $\frac{\partial\psi_t(0)}{\partial u}$ is diagonal, i.e.*

$$\frac{\partial\psi_t(0)}{\partial u} = \text{diag}(a_1(t), \dots, a_d(t))$$

and that the functions a_1, \dots, a_d are linearly independent. Then the term structure kernel \mathcal{A} is trivial, i.e. $\mathcal{A} = \{0\}$, if and only if the parameter $u = (u_1, \dots, u_d)$ satisfies $u_i > 0$ for all $i = 1, \dots, d$.

Proof. Necessity is obvious. Conversely, for $T = T^\infty$ we get $\psi_{T^\infty-T}(u) = \psi_0(u) = u$. Thus, by Proposition 3.8 we have

$$\text{span} \left\{ \frac{\partial\psi_t(0)}{\partial u} \cdot u : 0 \leq t \leq T^\infty \right\} \subseteq \mathcal{A}^\perp.$$

Hence, for any $z \in \mathcal{A}$ and all $t \in [0, T^\infty]$ we have $\langle z, \frac{\partial\psi_t(0)}{\partial u} \cdot u \rangle = 0$, or equivalently $\sum_{i=1}^d z_i u_i a_i(t) = 0$. Since a_1, \dots, a_d are linearly independent and $u_i > 0$ for all $i = 1, \dots, d$, it follows that $z_i = 0$ for all $i = 1, \dots, d$. Hence $z = 0$ and $\mathcal{A} = \{0\}$. \square

Let $n = \dim \mathcal{A}$ and $m = d - n = \dim \mathcal{A}^\perp$. We now use some linear algebra to obtain an equivalent representation where the unspanned dimensions are contained in the last components of the state vector.

Lemma 3.10. *There exists an invertible linear transformation S on \mathbb{R}^d such that $S(\mathcal{A}) = \{0\} \times \mathbb{R}^n$ and $S^{-t}(\mathcal{A}^\perp) = \mathbb{R}^m \times \{0\}$.*

Proof. Denote by $\{e_1, \dots, e_d\}$ the canonical basis of \mathbb{R}^d and by $\{\epsilon_1, \dots, \epsilon_d\}$ another basis of \mathbb{R}^d , where $\{\epsilon_1, \dots, \epsilon_m\}$ spans \mathcal{A}^\perp and $\{\epsilon_{m+1}, \dots, \epsilon_d\}$ spans \mathcal{A} . Set $B = (b_{ij})_{i,j=1,\dots,d}$ where b_{ij} are given by $\epsilon_i = \sum_{j=1}^d b_{ij} e_j$ and let $S = B^{-t}$. We first show that $\{0\} \times \mathbb{R}^n \subseteq S(\mathcal{A})$. For any

$$z = \sum_{k=m+1}^d c_k e_k \in \{0\} \times \mathbb{R}^n$$

we have

$$S^{-1}z = \sum_{k=m+1}^d c_k B^t e_k = \sum_{k=m+1}^d c_k \epsilon_k,$$

so for every $i \in \{1, \dots, m\}$ we have

$$\langle S^{-1}z, \epsilon_i \rangle = \sum_{k=m+1}^d c_k \langle \epsilon_k, \epsilon_i \rangle = 0.$$

Hence $S^{-1}z \in \mathcal{A}$, i.e. $z \in S(\mathcal{A})$, and $\{0\} \times \mathbb{R}^n \subseteq S(\mathcal{A})$. To show the converse, let

$$y = \sum_{k=m+1}^d d_k \epsilon_k \in \mathcal{A}$$

and observe that

$$Sy = \sum_{k=m+1}^d d_k B^{-t} \epsilon_k = \sum_{k=m+1}^d d_k e_k \in \{0\} \times \mathbb{R}^n.$$

The second part of the assertion is immediate from the first one. \square

We now decompose the transformed vector process $\bar{X}_t = SX_t$ into two sub-vectors via

$$\bar{X}_t = SX_t = \begin{pmatrix} Z_t \\ U_t \end{pmatrix} \in \mathbb{R}^m \times \mathbb{R}^n.$$

Then it follows that the term structure depends only on the sub-vector Z_t . Indeed, since $\psi_t(u), \frac{\partial \psi_{T-t}(0)}{\partial u} \cdot \psi_{T-t}(u) \in \mathcal{A}^\perp$, Lemma 3.10 implies

$$S^{-t} \psi_t(u) = \begin{pmatrix} \overline{\psi_t(u)}^Z \\ 0 \end{pmatrix} \in \mathbb{R}^m \times \{0\},$$

and

$$S^{-t} \cdot \frac{\partial \psi_{T-t}(0)}{\partial u} \cdot \psi_{T^\infty-T}(u) = \begin{pmatrix} \overline{\frac{\partial \psi_{T-t}(0)}{\partial u} \cdot \psi_{T^\infty-T}(u)}^Z \\ 0 \end{pmatrix} \in \mathbb{R}^m \times \{0\}.$$

Therefore, we have

$$\begin{aligned} p(t, T) &= \frac{c + \phi_{T^\infty-T}(u) + \left\langle \psi_{T^\infty-T}(u), \frac{\partial \phi_{T-t}(0)}{\partial u} \right\rangle + \left\langle \frac{\partial \psi_{T-t}(0)}{\partial u} \cdot \psi_{T^\infty-T}(u), S^{-1} \bar{X}_t \right\rangle}{c + \phi_{T^\infty-t}(u) + \left\langle \psi_{T^\infty-t}(u), S^{-1} \bar{X}_t \right\rangle} \\ &= \frac{c + \phi_{T^\infty-T}(u) + \left\langle \psi_{T^\infty-T}(u), \frac{\partial \phi_{T-t}(0)}{\partial u} \right\rangle + \left\langle S^{-t} \frac{\partial \psi_{T-t}(0)}{\partial u} \cdot \psi_{T^\infty-T}(u), \bar{X}_t \right\rangle}{c + \phi_{T^\infty-t}(u) + \left\langle S^{-t} \psi_{T^\infty-t}(u), \bar{X}_t \right\rangle} \\ &= \frac{c + \phi_{T^\infty-T}(u) + \left\langle \psi_{T^\infty-T}(u), \frac{\partial \phi_{T-t}(0)}{\partial u} \right\rangle + \left\langle \overline{\frac{\partial \psi_{T-t}(0)}{\partial u} \cdot \psi_{T^\infty-T}(u)}^Z, Z_t \right\rangle}{c + \phi_{T^\infty-t}(u) + \left\langle \overline{\psi_{T^\infty-t}(u)}^Z, Z_t \right\rangle}. \end{aligned}$$

Thus, in the terminology of Filipović et al. (2015), the components of U_t are *unspanned factors* and the components of Z_t are *term structure factors*.

3.4 Valuation of Interest Rate Derivatives

In this section we derive semi-closed form pricing formulae for caps, swaptions, and European options on foreign exchange rate in the generic affine rational potential model established in Section 3.3.

3.4.1 Caps

Since a cap is simply a linear portfolio of caplets, it suffices to consider caplets. We consider a caplet with strike rate K and tenor structure $0 < T < T + \Delta \leq T^\infty$ for some $\Delta > 0$. Thus, at time $T + \Delta$ the buyer of the caplet receives the payoff

$$C(T + \Delta) = \Delta(L(T, T + \Delta) - K)^+,$$

where the LIBOR rate $L(T, T + \Delta)$ settled at time T is given by

$$L(T, T + \Delta) = \frac{1}{\Delta} \left(\frac{1}{p(T, T + \Delta)} - 1 \right).$$

Using (3.9) it follows that the time- T price of the caplet is

$$C(T) = p(T, T + \Delta)C(T + \Delta) = \left(1 - \mathcal{K} \frac{\mathbb{E}_T[D_{T+\Delta}^u]}{D_T^u}\right)^+,$$

where $\mathcal{K} := 1 + \Delta K$. For $t \leq T$ the time- t -price of the caplet is thus given by

$$C(t) = \frac{\mathbb{E}_t[D_T^u C(T)]}{D_t^u} = \frac{1}{D_t^u} \mathbb{E}_t \left[(D_T^u - \mathcal{K} \mathbb{E}_T[D_{T+\Delta}^u])^+ \right]. \quad (3.14)$$

By Theorem 3.6 and by the definition of D^u , we have

$$D_T^u = c + \phi_{T\infty-T}(u) + \langle \psi_{T\infty-T}(u), X_T \rangle, \quad (3.15)$$

$$\begin{aligned} \mathbb{E}_T[D_{T+\Delta}^u] &= c + \phi_{T\infty-T-\Delta}(u) + \left\langle \psi_{T\infty-T-\Delta}(u), \frac{\partial \phi_\Delta(0)}{\partial u} \right\rangle \\ &\quad + \left\langle \frac{\partial \psi_\Delta(0)}{\partial u} \cdot \psi_{T\infty-T-\Delta}(u), X_T \right\rangle. \end{aligned} \quad (3.16)$$

Plugging (3.15) and (3.16) into (3.14) yields

$$C(t) = \frac{1}{D_t^u} \mathbb{E}_t \left[(\langle a, X_T \rangle + b)^+ \right], \quad (3.17)$$

where

$$a := \psi_{T\infty-T}(u) - \mathcal{K} \frac{\partial \psi_\Delta(0)}{\partial u} \cdot \psi_{T\infty-T-\Delta}(u), \quad (3.18a)$$

$$\begin{aligned} b &:= c(1 - \mathcal{K}) + \phi_{T\infty-T}(u) - \mathcal{K} \phi_{T\infty-T-\Delta}(u) \\ &\quad - \mathcal{K} \left\langle \psi_{T\infty-T-\Delta}(u), \frac{\partial \phi_\Delta(0)}{\partial u} \right\rangle. \end{aligned} \quad (3.18b)$$

Proposition 3.11. *The time- t price of a Δ -tenor caplet with strike rate K maturing at time T is given by the formula*

$$C(t) = \frac{\int_0^\infty \Re \left[\frac{\exp(b(\mu+iy) + \phi_{T\infty-t}((\mu+iy)a) + \langle \psi_{T\infty-t}((\mu+iy)a), X_t \rangle)}{(\mu+iy)^2} \right] dy}{\pi (c + \phi_{T\infty-t}(u) + \langle \psi_{T\infty-t}(u), X_t \rangle)}, \quad (3.19)$$

where $\mu > 0$ is a real number such that $\mu a \in \mathcal{J}$, and a, b are given in (3.18).

Proof. Since $\mu a \in \mathcal{J}$, the conditional expectation $\mathbb{E}_t[e^{\langle(\mu+iy)a, X_T\rangle}]$ exists and is given by

$$\mathbb{E}_t \left[e^{\langle(\mu+iy)a, X_T\rangle} \right] = \exp(\phi_{T-t}((\mu+iy)a) + \langle\psi_{T-t}((\mu+iy)a), X_t\rangle).$$

Hence

$$\begin{aligned} \mathbb{E}_t \left[e^{(\mu+iy)\langle a, X_T\rangle + b} \right] \\ = \exp((\mu+iy)b + \phi_{T-t}((\mu+iy)a) + \langle\psi_{T-t}((\mu+iy)a), X_t\rangle). \end{aligned} \quad (3.20)$$

Applying Lemma A.2 and combining (3.17) with (3.20) we obtain the caplet pricing formula (3.19). \square

3.4.2 Swaptions

Consider a payer swaption with strike rate K , maturity $T = T_0$ and payment dates

$$T_1 < \dots < T_N.$$

Setting $\Delta_i := T_i - T_{i-1}$, $i = 1, \dots, N$, the payoff of the swaption at maturity T is given by

$$S(T) = \left(\sum_{i=1}^N \Delta_i p(T, T_i) (L(T; T_{i-1}, T_i) - K) \right)^+.$$

Substituting for

$$L(T; T_{i-1}, T_i) = \frac{1}{\Delta_i} \left(\frac{p(T, T_{i-1})}{p(T, T_i)} - 1 \right), \quad i = 1, \dots, N,$$

and for $p(T, T_k) = \mathbb{E}_T[D_{T_k}^u]/D_T^u$, $k = 1, \dots, N$, yields

$$S(T) = \frac{1}{D_T^u} \left(\sum_{i=1}^N \mathbb{E}_T \left[D_{T_{i-1}}^u \right] - \mathcal{K}_i \mathbb{E}_T \left[D_{T_i}^u \right] \right)^+,$$

where $\mathcal{K}_i := 1 + \Delta_i K$. Hence, the time- t price of the swaption is

$$S(t) = \frac{\mathbb{E}_t[S(T)D_T^u]}{D_t^u} = \frac{1}{D_t^u} \mathbb{E}_t \left[\left(\sum_{i=1}^N \mathbb{E}_T \left[D_{T_{i-1}}^u \right] - \mathcal{K}_i \mathbb{E}_T \left[D_{T_i}^u \right] \right)^+ \right].$$

By Theorem 3.6, for $i = 0, \dots, N$, we have

$$\begin{aligned} \mathbb{E}_T [D_{T_i}^u] &= c + \phi_{T^\infty - T_i}(u) + \left\langle \psi_{T^\infty - T_i}(u), \frac{\partial \phi_{T_i - T}(0)}{\partial u} \right\rangle \\ &\quad + \left\langle \frac{\partial \psi_{T_i - T}(0)}{\partial u} \cdot \psi_{T^\infty - T_i}(u), X_T \right\rangle. \end{aligned}$$

So we get

$$S(t) = \frac{1}{D_t^u} \mathbb{E}_t \left[(\langle \alpha, X_T \rangle + \beta)^+ \right], \quad (3.21)$$

where

$$\alpha := \sum_{i=1}^N \frac{\partial \psi_{T_{i-1} - T}(0)}{\partial u} \cdot \psi_{T^\infty - T_{i-1}}(u) - \mathcal{K}_i \frac{\partial \psi_{T_i - T}(0)}{\partial u} \cdot \psi_{T^\infty - T_i}(u), \quad (3.22a)$$

$$\begin{aligned} \beta &:= \sum_{i=1}^N c(1 - \mathcal{K}_i) + \phi_{T^\infty - T_{i-1}}(u) + \left\langle \psi_{T^\infty - T_{i-1}}(u), \frac{\partial \phi_{T_{i-1} - T}(0)}{\partial u} \right\rangle \\ &\quad - \mathcal{K}_i \phi_{T^\infty - T_i}(u) - \mathcal{K}_i \left\langle \psi_{T^\infty - T_i}(u), \frac{\partial \phi_{T_i - T}(0)}{\partial u} \right\rangle. \end{aligned} \quad (3.22b)$$

Note that (3.21) is of the same form as (3.17). Thus, similarly as in the proof of Theorem 3.11, we obtain the following no-arbitrage pricing formula:

Proposition 3.12. *The time- t no-arbitrage price of the swaption is given by*

$$S(t) = \frac{\int_0^\infty \Re \left[\frac{\exp(\beta(\mu + iy) + \phi_{T-t}((\mu + iy)\alpha) + (\psi_{T-t}((\mu + iy)\alpha), X_t))}{(\mu + iy)^2} \right] dy}{\pi (c + \phi_{T^\infty - t}(u) + \langle \psi_{T^\infty - t}(u), X_t \rangle)}, \quad (3.23)$$

where $\mu > 0$ is such that $\mu\alpha \in \mathcal{J}$, and α, β are given in (3.22).

Remark 3.13. From (3.19) and (3.23), we see that if we transform the underlying process X via the transformation S discussed at the end of Section 3.3, then the unspanned sub-vector U_t does not appear in the pricing formulae for bonds, caplets or swaptions. Therefore, the unspanned factors are, in fact, irrelevant for the term structure model.

3.4.3 Currency options

We consider two economies with two currencies $\i and $\j , whose state-price deflators, D^{u_i} and D^{u_j} , are given respectively by

$$D_t^{u_i} = c_i + \phi_{T^\infty-t}(u_i) + \langle \psi_{T^\infty-t}(u_i), X_t \rangle, \quad (3.24)$$

$$D_t^{u_j} = c_j + \phi_{T^\infty-t}(u_j) + \langle \psi_{T^\infty-t}(u_j), X_t \rangle, \quad (3.25)$$

for some parameters $u_i, u_j \in \mathcal{J} \cap E$, and some positive parameters $c_i > 0$, $c_j > 0$. Denote by x_{ij} the exchange process between these two currencies, so one $\i can be exchanged for $x_{ij}(t)$ units of $\j at time t . Proposition 2.14 then implies that the following specification of x_{ij} in terms of the two state-price deflators excludes arbitrage opportunities:

$$x_{ij}(t) = \frac{D_t^{u_i}}{D_t^{u_j}} x_{ij}(0). \quad (3.26)$$

We now consider a European call option on x_{ij} with strike rate k and maturity $T \in [0, T^\infty]$. Its payoff, in currency $\j , is given by

$$O(T) := (x_{ij}(T) - k)^+.$$

It follows from Equation (3.26) that the time t -value of the option is provided by

$$O(t) := \frac{1}{D_t^{u_j}} \mathbb{E}_t [D_T^{u_j} (x_{ij}(T) - k)^+] = \frac{1}{D_t^{u_j}} \mathbb{E}_t [(x_{ij}(0) D_T^{u_i} - k D_T^{u_j})^+].$$

Substituting (3.24) and (3.25) in the above equation yields

$$O(t) = \frac{\mathbb{E}_t [(\langle A, X_T \rangle + B)^+]}{c_j + \phi_{T^\infty-t}(u_j) + \langle \psi_{T^\infty-t}(u_j), X_t \rangle},$$

where

$$A := x_{ij}(0) \psi_{T^\infty-T}(u_i) - k \psi_{T^\infty-T}(u_j), \quad (3.27a)$$

$$B := x_{ij}(0) c_i - k c_j + x_{ij}(0) \phi_{T^\infty-T}(u_i) - k \phi_{T^\infty-T}(u_j). \quad (3.27b)$$

As in the preceding subsection, this immediately yields the following pricing formula for the European currency option:

Proposition 3.14 (Currency option). *In the affine rational potential model, the time- t no-arbitrage price of a European call option on the foreign exchange rate with strike k and maturity $T \geq t$ is given by*

$$O(t) = \frac{\int_0^\infty \Re \left[\frac{\exp((\mu+iy)B + \phi_{T-t}((\mu+iy)A) + \langle \psi_{T-t}((\mu+iy)A), X_t \rangle)}{(\mu+iy)^2} \right] dy}{c_j + \pi(\phi_{T^\infty-t}(u_j) + \langle \psi_{T^\infty-t}(u_j), X_t \rangle)}, \quad (3.28)$$

where $\mu > 0$ is such that $\mu A \in \mathcal{J}$, and A, B are given in (3.27).

3.5 Examples

In this section the underlying process in the general affine rational potential model is specified as a one-factor CIR process, a multi-factor CIR process and a subordinator Lévy process. This in turn yields three concrete rational term structure models with non-negative interest rates.

3.5.1 CIR processes

Consider a one factor CIR process X given by

$$dX_t = \lambda(\theta - X_t)dt + 2\eta\sqrt{X_t}dW_t, \quad X_0 = x \in \mathbb{R}_{\geq 0},$$

where $\lambda > 0$ and $\theta, \eta \in \mathbb{R}_{\geq 0}$. This is an affine process with

$$\mathbb{E}[e^{uX_t}] = \exp(\phi_t(u) + \psi_t(u)x),$$

where

$$\phi_t(u) = -\frac{\lambda\theta}{2\eta^2} \ln \left(1 - \frac{2\eta^2 u}{\lambda} (1 - e^{-\lambda t}) \right), \quad \psi_t(u) = \frac{ue^{-\lambda t}}{1 - \frac{2\eta^2 u}{\lambda} (1 - e^{-\lambda t})}, \quad (3.29)$$

and u satisfies

$$u \in \left[0, \frac{\lambda}{2\eta^2(1 - e^{-\lambda T^\infty})} \right);$$

see also Section 8 in Keller-Ressel et al. (2013).

From (3.8) the state-price deflator D^u is given in explicit form by

$$D_t^u = c - \frac{\lambda\theta}{2\eta^2} \ln \left(1 - \frac{2\eta^2 u}{\lambda} (1 - e^{-\lambda(T^\infty-t)}) \right) + \frac{ue^{-\lambda(T^\infty-t)} X_t}{1 - \frac{2\eta^2 u}{\lambda} (1 - e^{-\lambda(T^\infty-t)})}.$$

A simple computation yields

$$\frac{\partial \phi_t(0)}{\partial u} = \theta(1 - e^{-\lambda t}) \quad \text{and} \quad \frac{\partial \psi_t(0)}{\partial u} = e^{-\lambda t}. \quad (3.30)$$

Plugging (3.29) and (3.30) into (3.13) we obtain an explicit representation of bond prices via

$$p(t, T) = \frac{G(t, T)X_t + H(t, T) + c}{G(t, t)X_t + H(t, t) + c} \quad \text{for } t \leq T \leq T^\infty,$$

where

$$G(t, T) = \frac{ue^{-\lambda(T^\infty - t)}}{1 - \frac{2\eta^2 u}{\lambda}(1 - e^{-\lambda(T^\infty - T)})},$$

and

$$H(t, T) = \frac{\theta ue^{-\lambda(T^\infty - T)}(1 - e^{-\lambda(T - t)})}{1 - \frac{2\eta^2 u}{\lambda}(1 - e^{-\lambda(T^\infty - T)})} - \frac{\lambda\theta}{2\eta^2} \ln \left(1 - \frac{2\eta^2 u}{\lambda}(1 - e^{-\lambda(T^\infty - T)}) \right).$$

It follows that the short rate is given by

$$r_t = -\partial_T p(t, T)|_{T=t} = \frac{-\partial_T G(t, t)X_t}{G(t, t)X_t + H(t, t) + c}, \quad (3.31)$$

where

$$-\partial_T G(t, t) = \frac{2\eta^2 \lambda^2 u^2 e^{-2\lambda(T^\infty - t)}}{(\lambda - 2\eta^2 u(1 - e^{-\lambda(T^\infty - t)}))^2}.$$

3.5.2 Multi-factor CIR process

We extend the one-factor CIR process in the previous example to the case of d independent CIR processes. Consider a process X consisting of d factors X^1, \dots, X^d given, for each $i \in \{1, \dots, d\}$, by

$$dX_t^i = \lambda_i(\theta_i - X_t^i)dt + 2\eta_i \sqrt{X_t^i} dW_t^i, \quad X_0^i = x_i \in \mathbb{R}_{\geq 0},$$

where $\lambda_i > 0$, $\theta_i, \eta_i \in \mathbb{R}_{\geq 0}$, and W^1, \dots, W^d are independent Brownian motions. This process is an affine process satisfying

$$\mathbb{E}[e^{\langle u, X_t \rangle}] = \exp(\phi_t(u) + \langle \psi_t(u), x \rangle),$$

where

$$\phi_t(u) = \sum_{i=1}^d -\frac{\lambda_i \theta_i}{2\eta_i^2} \ln \left(1 - \frac{2\eta_i^2 u_i}{\lambda_i} (1 - e^{-\lambda_i t}) \right), \quad (3.32a)$$

$$\psi_t(u) = \left(\frac{u_1 e^{-\lambda_1 t}}{1 - \frac{2\eta_1^2 u_1}{\lambda_1} (1 - e^{-\lambda_1 t})}, \dots, \frac{u_d e^{-\lambda_d t}}{1 - \frac{2\eta_d^2 u_d}{\lambda_d} (1 - e^{-\lambda_d t})} \right)^t, \quad (3.32b)$$

and $u = (u_1, \dots, u_d)$ such that

$$u_i \in \left[0, \frac{\lambda_i}{2\eta_i^2 (1 - e^{-\lambda_i T^\infty})} \right), \quad i = 1, 2, \dots, d.$$

As in the previous example, we have

$$\frac{\partial \phi_t(0)}{\partial u} = \left(\theta_1 (1 - e^{-\lambda_1 t}), \dots, \theta_d (1 - e^{-\lambda_d t}) \right)^t \quad (3.33a)$$

$$\frac{\partial \psi_t(0)}{\partial u} = \text{diag} \left(e^{-\lambda_1 t}, \dots, e^{-\lambda_d t} \right). \quad (3.33b)$$

Now using (3.32) and (3.33) in (3.13) we obtain the closed-form bond price formula

$$p(t, T) = \frac{\langle A(t, T), X_t \rangle + B(t, T) + c}{\langle A(t, T), X_t \rangle + B(t, t) + c}, \quad 0 \leq t \leq T \leq T^\infty, \quad (3.34)$$

where

$$A(t, T) = \left(\frac{u_1 e^{-\lambda_1 (T^\infty - t)}}{1 - \frac{2\eta_1^2 u_1}{\lambda_1} (1 - e^{-\lambda_1 (T^\infty - T)})}, \dots, \frac{u_d e^{-\lambda_d (T^\infty - t)}}{1 - \frac{2\eta_d^2 u_d}{\lambda_d} (1 - e^{-\lambda_d (T^\infty - T)})} \right),$$

and

$$B(t, T) = \sum_{i=1}^d \frac{\theta_i u_i e^{-\lambda_i (T^\infty - T)} (1 - e^{-\lambda_i (T - t)})}{1 - \frac{2\eta_i^2 u_i}{\lambda_i} (1 - e^{-\lambda_i (T^\infty - T)})} - \frac{\lambda_i \theta_i}{2\eta_i^2} \ln \left(1 - \frac{2\eta_i^2 u_i}{\lambda_i} (1 - e^{-\lambda_i (T^\infty - T)}) \right).$$

A straightforward computation yields the short rate via

$$r_t = -\partial_T p(t, T)|_{T=t} = \frac{-\langle \partial_T A(t, t), X_t \rangle}{\langle A(t, t), X_t \rangle + B(t, t) + c}, \quad (3.35)$$

where

$$\begin{aligned} & \partial_T A(t, t) \\ &= \left(\frac{-2\eta_1^2 \lambda_1^2 u_1^2 e^{-2\lambda_1(T^\infty-t)}}{(\lambda_1 - 2\eta_1^2 u_1(1 - e^{-\lambda_1(T^\infty-t)}))^2}, \dots, \frac{-2\eta_d^2 \lambda_d^2 u_d^2 e^{-2\lambda_d(T^\infty-t)}}{(\lambda_d - 2\eta_d^2 u_d(1 - e^{-\lambda_d(T^\infty-t)}))^2} \right). \end{aligned}$$

3.5.3 Lévy processes

Suppose the affine process X is a Lévy subordinator, with cumulant generating function κ . Then we know that

$$\phi_t(u) = t\kappa(u) \text{ and } \psi_t(u) = u \text{ for } u \geq 0.$$

The supermartingale D^u in (3.8) takes the form

$$D_t^u = c + \phi_{T^\infty-t}(u) + \psi_{T^\infty-t}(u)X_t = c + (T^\infty - t)\kappa(u) + uX_t. \quad (3.36)$$

We have

$$\frac{\partial \phi_t(u)}{\partial u} = t\kappa'(u) \quad \text{and} \quad \frac{\partial \psi_t(u)}{\partial u} = 1.$$

Substituting these derivatives in Equation (3.13) we obtain

$$p(t, T) = \frac{uX_t + (T^\infty - T)\kappa(u) + (T - t)\kappa'(0)u + c}{uX_t + (T^\infty - t)\kappa(u) + c} \quad \text{for all } t \leq T \leq T^\infty. \quad (3.37)$$

The corresponding short rate process is given by

$$r_t = \frac{\kappa(u) - u\kappa'(0)}{uX_t + (T^\infty - t)\kappa(u) + c}. \quad (3.38)$$

Remark 3.15. We observe that bond prices and the short rate in the above models are quotients of affine functions of the underlying state process. This in particular implies that our models do not belong to the classical short-rate world. Moreover the short rate is bounded from below by zero and $r_t \rightarrow 0$ as $X_t \rightarrow 0$ in the CIR-process specifications, while in the subordinator Lévy model we have $r_t \rightarrow 0$ as $X_t \rightarrow \infty$.

3.6 Numerical Examples and Calibration

We calibrate the two-factor CIR process specification of the affine rational potential model to market zero-coupon yields of several dates. We then numerically show that the short rate in this specific model stays in the vicinity of the zero lower bound for an extended time period, which is a desired feature of interest rate models in the low interest rate environment.

3.6.1 Calibration to zero yield curves

We calibrate the above mentioned specific model to market yields, $\bar{y}(0, T)$, of the first business days of six months ranging from March 2014 to August 2014, where time-to-maturity T takes values in the set $\{1/12, \dots, 11/12, 1, 2, 3, \dots, 20\}$. The market yields are bootstrapped from Overnight Indexed Swap rates in the EUR market collected from Thomson-Reuters. We then use the log-linear interpolation method to construct the corresponding market yield curves. It follows from (3.34) that the model yield curve is provided by

$$y(0, T) = -\frac{1}{T} \ln p(0, T) = -\frac{1}{T} \ln \left(\frac{\langle A(0, T), X_0 \rangle + B(0, T) + c}{\langle A(0, 0), X_0 \rangle + B(0, 0) + c} \right),$$

where

$$A(0, T) = \left(\frac{u_1 e^{-\lambda_1 T^\infty}}{1 - \frac{2\eta_1^2 u_1}{\lambda_1} (1 - e^{-\lambda_1 (T^\infty - T)})}, \frac{u_2 e^{-\lambda_2 T^\infty}}{1 - \frac{2\eta_2^2 u_2}{\lambda_2} (1 - e^{-\lambda_2 (T^\infty - T)})} \right),$$

and

$$B(0, T) = B_1(0, T) + B_2(0, T),$$

with

$$B_i(0, T) = \frac{\theta_i u_i e^{-\lambda_i (T^\infty - T)} (1 - e^{-\lambda_i T})}{1 - \frac{2\eta_i^2 u_i}{\lambda_i} (1 - e^{-\lambda_i (T^\infty - T)})} - \frac{\lambda_i \theta_i \ln \left(1 - \frac{2\eta_i^2 u_i}{\lambda_i} (1 - e^{-\lambda_i (T^\infty - T)}) \right)}{2\eta_i^2}.$$

In this specification we fix the time horizon $T^\infty = 30$ and the positive constant $c = 10^{-12}$.

	March	April	May	June	July	August
λ_1	0.549276	0.497267	0.500330	0.573803	0.607132	0.581543
λ_2	0.790560	0.774901	0.773401	0.777001	0.778531	0.778708
θ_1	0.013684	0.015599	0.013054	0.014416	0.012805	0.014741
θ_2	0.938575	0.950647	0.949184	0.952366	0.955570	0.955717
η_1	0.997704	1.001440	0.998511	1.004880	1.072343	1.063727
η_2	0.759785	0.759879	0.763593	0.756466	0.760272	0.752361
u_1	0.275902	0.247914	0.250907	0.284121	0.263989	0.256974
u_2	0.050178	0.047340	0.044421	0.059910	0.053436	0.063993
X_0^1	5.000358	4.276595	3.928803	6.495502	7.500005	6.952000
X_0^2	2.525113	1.924437	1.244368	0.244368	4.599037	6.153459

Table 3.1: Calibrated parameters.

We calibrate the model for the following set of parameters:

$$\Theta = \{\lambda_1, \lambda_2, \theta_1, \theta_2, \eta_1, \eta_2, u_1, u_2, X_0^1, X_0^2\},$$

where we consider X_0^1 and X_0^2 as extra parameters of our model. We use the least squares method for the calibration, i.e., we minimize the squared differences between market yields and model yields

$$\sum_{i=1}^{31} (y(0, T_i) - \bar{y}(0, T_i))^2,$$

where

$$\{T_1, T_2, \dots, T_{31}\} \equiv \{1/12, \dots, 11/12, 1, 2, 3, \dots, 20\}.$$

The calibrated parameters for the selected dates are reported in Table 3.1. The calibration results are rather satisfactory as displayed in Figure 3.1. Moreover, since the parameters $\lambda_1, \lambda_2, \theta_1, \theta_2, \eta_1, \eta_2, u_1, u_2$ are constant, they are expected not to fluctuate too much over time. As shown in Figure 3.2, the calibrated parameters of our model are relatively stable over the six-month period of calibration.

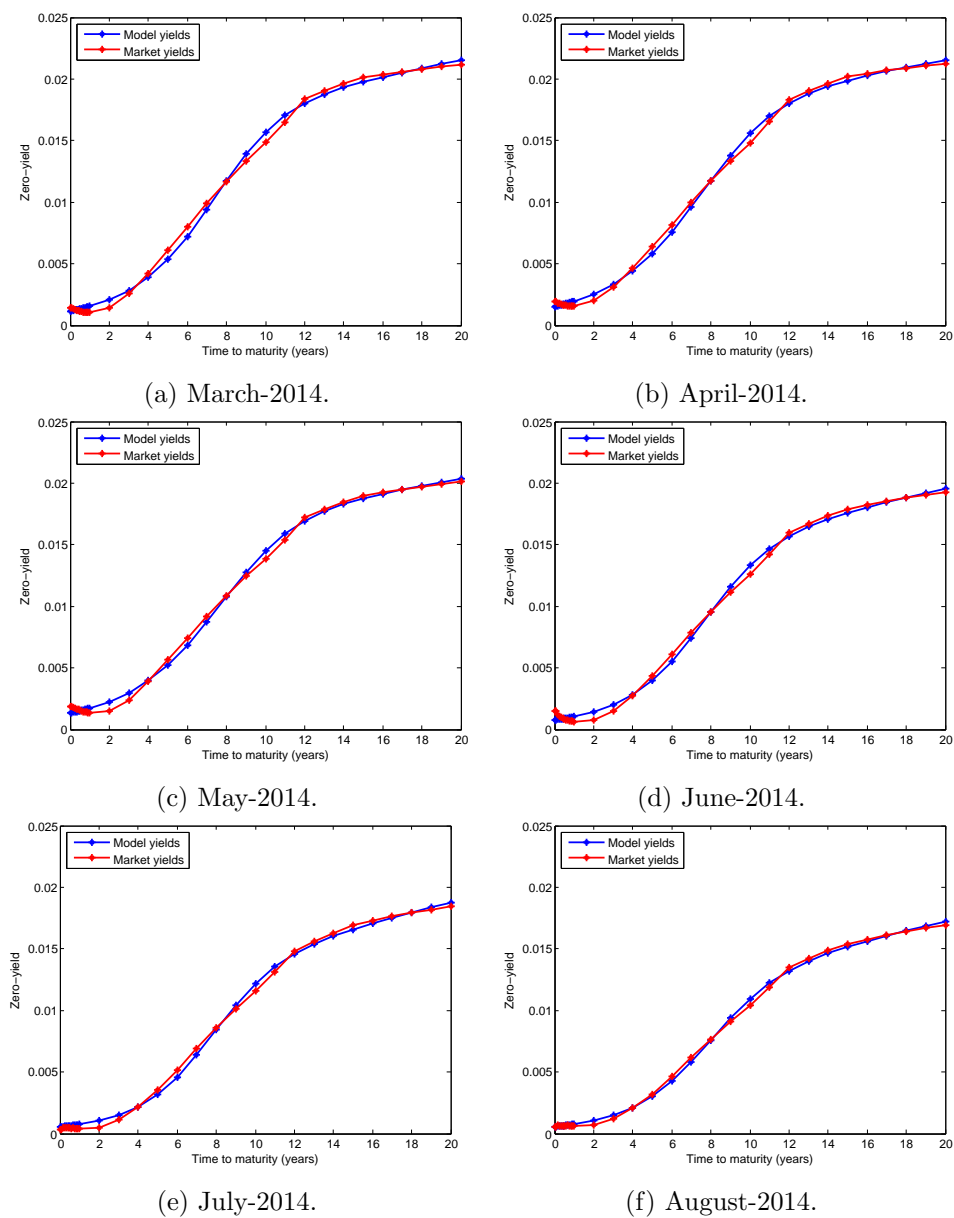


Figure 3.1: Market versus model implied zero-yields.

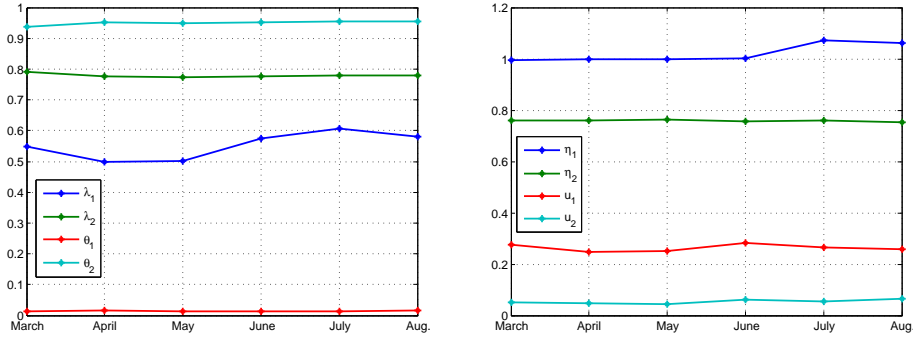


Figure 3.2: Stability of calibrated parameters over time.

3.6.2 Short rate

We consider the short rate in the two-factor CIR specific model with the calibrated parameters from August 2014. Specifically, we analyze the evolution of the short rate in a period of one year starting from the calibrated date. We will be interested in the event that the short rate stays below a given level during that period. Hence, we consider for each level $L > 0$ a quantity

$$q_L := \mathbb{P} \left(\max_{t \in [0,1]} r_t \leq L \right),$$

where the short rate r_t is given by (3.35) with the number of factors $d = 2$. Given that the current short rate r_0 stays near the zero lower bound, we want to see how likely the short rate will stay in a small neighborhood of r_0 in a period of one year.

For each level L , we estimate the corresponding probability q_L using Monte Carlo method via

$$\hat{q}_L := \frac{\#\{\max_{t \in [0,1]} r_t \leq L\}}{N},$$

where N is the number of trials and $\#S$ denotes the cardinality of the set S . These estimates are plotted in Figure 3.3 with $N = 100000$ and 95% confidence intervals.

We observe that with the current short rate r_0 being 5 basis points (bps), the probabilities that the short rate will stay below 10bps, 15bps, and 25bps in one year ahead are about 42%, 76%, and 97 % respectively, while the probabilities that it stays below 25bps, 30bps, and 40bps are respectively 38%, 60%, and 96% when the current short rate r_0 is 15bps.

As an illustration, Figure 3.4 displays 5 simulated paths of the short rate with the current short rate r_0 being 5bps and 15bps, where we use the exact simulation scheme for CIR processes in Glasserman (2003), p. 124.

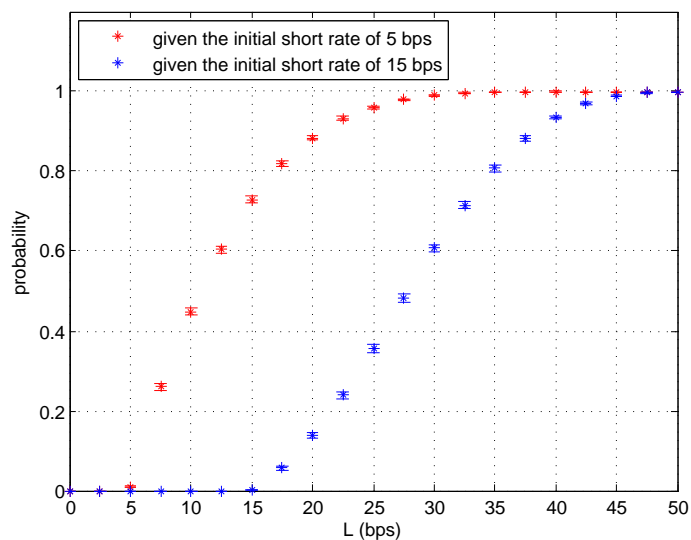


Figure 3.3: Estimated probabilities $L \mapsto q_L = \mathbb{P}(\max_{t \in [0,1]} r_t \leq L)$ with 95%-confidence intervals.

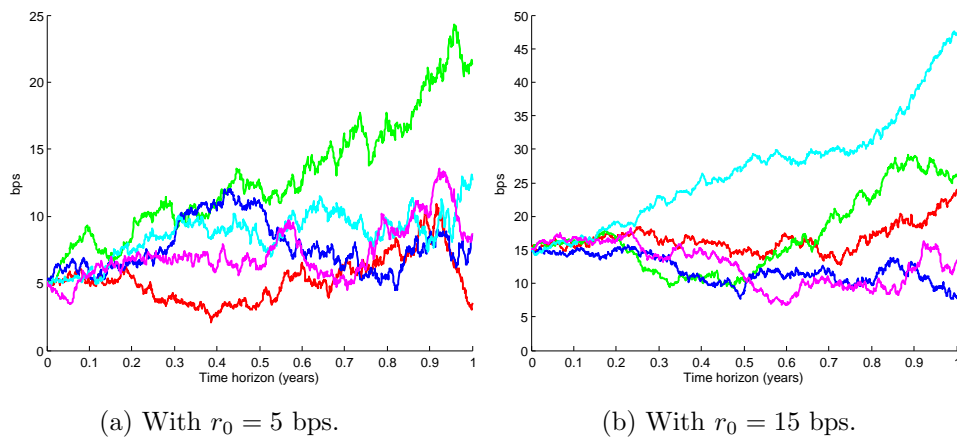


Figure 3.4: Simulated paths of the short rate in one year ahead.

Chapter 4

The Multi-Curve Potential Models

We develop a general class of multi-curve term structure models for post-crisis interest rates using the potential approach. Our models feature positive stochastic basis spreads, positive term structures, and analytic pricing formulae for interest rate derivatives. Our modeling framework is also flexible enough to accommodate negative rates and positive basis spreads. Making a quanto interpretation of LIBOR lending transactions, we use a multi-currency analogy to model multiple term structures and formulate a general, tractable model of multiple term structures.

This chapter is an extended version of the article Nguyen & Seifried (2015a), which has been published in *International Journal of Theoretical and Applied Finance*, Vol. 18, No. 07.

4.1 Introduction

Motivated by the considerations discussed in Chapter 1, a rapidly growing literature on multi-curve modeling has evolved recently. From a methodological perspective, the majority of existing multi-curve models can be broadly classified

into the following three categories, for each of which we provide an incomplete set of references:

Short Rate Models. Kijima et al. (2009) apply short rate models to rates of various qualities. Kenyon (2010) uses a Hull-White model for the short rate processes driving the OIS and the LIBOR curves. Morini & Runggaldier (2014) model the risk-free short rate jointly with the short rate spread between the LIBOR and the OIS term structure, based on Vasicek and CIR processes.

Heath-Jarrow-Morton (HJM) Approaches. Fujii et al. (2011), Crépey et al. (2012), Cuchiero et al. (2015), Moreni & Pallavicini (2014), and Crépey et al. (2015a), among others, use HJM methodology in the multi-curve framework. Fujii et al. (2011) propose an HJM approach to model simultaneously the instantaneous OIS forward rate and the spread between the FRA rate and the OIS forward rate in a cross-currency setup. Crépey et al. (2012) apply a defaultable HJM methodology to model the term structure of multiple interest rate curves, and Crépey et al. (2015a) consider a Lévy-driven HJM model for credit valuation adjustments. Moreni & Pallavicini (2014) extend the classical HJM framework to the multi-curve setting and model FRA rates using a common family of Markov processes. Cuchiero et al. (2015) provide an HJM approach in a general semimartingale setting to model the joint evolution of the term structures of OIS bond prices and of the term structures of multiplicative spreads between FRA rates and OIS forward rates.

Libor Market Models (LMM). Mercurio (2009, 2010b), Bianchetti (2010), Grbac et al. (2015), among others, construct multi-curve models in the LMM setup. Mercurio (2009, 2010b) extends the classical LMM to the multi-curve framework. Bianchetti (2010) exploits a foreign-currency analogy to derive double-curve market pricing formulae for plain vanilla interest rate derivatives. Grbac et al. (2015) extend the affine LIBOR models proposed by Keller-Ressel et al. (2013) to the multi-curve setting.

The articles referred to above all take a reduced-form approach: They acknowledge the market segmentation across tenors, and construct consistent pricing models that reflect the relevant basis spreads. The spreads are not, however, attributed to fundamental risk factors. By contrast, Filipović & Trolle (2013)

analyze the structure of interbank rates and decompose the LIBOR-OIS spread into default and liquidity components. In a recent paper, Gallitschke et al. (2014) construct a structural model for interbank money market rates that endogenously generates post-crisis basis spreads from fundamental risk factors including credit risk and liquidity risk.

In this chapter, we adopt the reduced-form perspective and develop a general reduced-form multi-curve model in a potential framework. Following Constantinides (1992) and Rogers (1997), the general potential approach is based on the direct specification of the relevant state-price deflator. Both Rogers (1997) and Flesaker & Hughston (1996a,b, 1997) point out that this methodology is particularly suited for cross-currency modeling: The arbitrage-free exchange rate between two currencies is uniquely determined as the ratio of their state-price deflators; see Chapter 2. To the best of our knowledge, this work is the first to make use of potential methods in a multi-curve setup.

Following the multi-currency analogy of Bianchetti (2010), in the multi-curve potential approach the OIS term structure is modeled as the discount (equivalently, domestic) curve, and the spot multiplicative spread between LIBOR and OIS is modeled as the quotient between the state-price deflator in a hypothetical forward (equivalently, foreign) currency and the state-price deflator in the domestic currency. LIBOR rates then emerge as the interest rates implicit in quanto borrowing transactions, where debt is issued in the domestic currency and repaid in the foreign currency. In this setting, multiplicative LIBOR-OIS spreads can be identified with forward exchange rates. In particular, the multi-curve potential approach is able to generate stochastic spreads. By an appropriate specification of the relevant state-price deflators, we are able to construct multi-curve term structure models that offer both positive spreads and tractable pricing formulae.

The remainder of this chapter is structured as follows: Section 4.2 presents the basic post-crisis concepts and the notation we use. Section 4.3 presents the general multi-curve potential model, based on the interpretation of the spot mul-

tiplicative spread as an exchange rate between two currencies. We construct a general framework with pricing formulae for basic interest rate derivatives. In Section 4.4 we analyze a specific multi-curve potential model that generalizes the single-curve rational lognormal model of Flesaker & Hughston (1996a) to the multi-curve framework. In this model we obtain stochastic spreads as well as explicit valuation formulae for interest rate derivatives. In Section 4.5 we extend the affine rational potential model in Chapter 3 and the linear rational model in Filipović et al. (2015) to the multi-curve framework by using the general multi-curve potential model established in Section 4.3. We show that the analytical tractability of the single-curve models carries over to these multi-curve extensions. Section 4.6 presents another tractable specification of the multi-curve potential model in which the state-price deflators are constructed from a common Gaussian dynamics. Finally, in Section 4.7 we calibrate the multi-curve rational lognormal models to EUR market data.

4.2 Multi-Curve Definitions and Notation

In this section we briefly introduce the most important definitions and notation for post-crisis multi-curve interest rate markets.

4.2.1 Interbank interest rates

The main reference rates for an enormous number of fixed income contracts are the London Interbank Offered Rate (LIBOR) in the USD fixed income market and the Euro Interbank Offered Rate (EURIBOR) in the EUR fixed income market. LIBOR is the average rate at which a contributor bank can obtain unsecured funding in the London interbank market. It is produced every business day for 5 currencies: CHF (Swiss Franc), EUR (Euro), GBP (Pound Sterling), JPY (Japanese Yen), and USD (US Dollar), with 7 maturities: 1 day, 1 week, and 1, 2, 3, 6, and 12 months. Every LIBOR contributor bank is asked to base their submissions to the IntercontinentalExchange (ICE) on the following question: *”At what rate could you borrow funds, were you to do so by asking for and then accepting interbank offers in a reasonable market size just prior to 11 am*

London time? Every ICE LIBOR rate is calculated as a trimmed average: the highest and lowest 25 % of the rates submitted are excluded, the mean of the rest is published as ICE LIBOR rate. EURIBOR is calculated every business day by Global Rate Set Systems (GRSS) for 8 maturities: 1, and 2 weeks, and 1, 2, 3, 6, 9, and 12 months. Each panel bank submits the rates that it believes one prime bank is quoting to another prime bank for interbank term deposits within the Euro zone. The published rate is the trimmed average of the quoted rates: the highest and lowest 15% of the quotes are eliminated, the remainder are averaged. Thus, EURIBOR is an average of the rates at which contributor banks believe a prime bank can access unsecured funding in the euro interbank market. In the following we will use the term LIBOR to refer to any of these two interbank rates.

Another important reference rate is the overnight rate: the effective federal funds (FF) rate in the USD market, and the Euro overnight index average (EONIA) rate in the EUR market. The FF rate is the weighted average of rates of overnight unsecured transactions of reserve balances at Federal Reserve Bank of New York that depository institutions make to one another. The rate is calculated daily by the Federal Reserve Bank of New York based on data provided by the depository institutions. The day count convention is ACT/365. EONIA is the reference rate for overnight unsecured transactions in the EUR market, calculated on the basis of actual transactions. Each contributing bank reports daily data on the total volume of transactions in unsecured overnight money and the weighted average of interest rates for these transactions to the European Central Bank, which calculates a weighted average interest rate. The day count convention is ACT/360. EONIA and EURIBOR are the two most important benchmarks for interest rates in the EUR markets.

4.2.2 Interest rate swaps and spreads

Overnight Indexed Swap. An overnight indexed swap is a fixed-for-floating interest rate swap (see Definition 2.8) whose floating rate is the geometric average of an overnight index, defined as the FF rate in the USD market and the EONIA

rate in the EUR market. The par rate of this swap is called the OIS rate. OIS rates are widely considered as the best proxy for the risk-free term structure. In addition, OIS rates have also become market standard as the reference rates in collateralization agreements; see Mercurio (2010a).

LIBOR-OIS Spread. In a LIBOR-based interest rate swap, the floating rate is the LIBOR rate of the corresponding tenor. The LIBOR-OIS spread is then defined as the difference between the par swap rates of a LIBOR-based swap and an overnight indexed swap with the same maturities.

Basis Spread. The basis spread is the difference between the par swap rates of two LIBOR-based interest rate swaps with different floating legs and the same fixed leg and identical maturities.

For further information on various types of interest rate swaps, spreads, and market quoting conventions, we refer to Filipović & Trolle (2013).

4.2.3 Notation

We consider a given domestic currency $\d , which could be either USD or EUR. We identify the OIS term structure with the discount curve in this domestic currency. As above, we denote by $p(t, T)$ the discount factor at time t for maturity T . Based on market data, the discount curve p can be bootstrapped from quotes of OIS rates using classical single-curve methods.

4.2.4 Forward rate agreements and FRA rates

In all that follows, we denote by $L^\Delta(T, T + \Delta)$ the spot LIBOR rate determined at time T for the time interval $[T, T + \Delta]$. We recall from Definition 2.6 that a forward rate agreement for the future time interval $[T, T + \Delta]$ is a contract where two parties agree at time $t \leq T$ that, at the maturity time $T + \Delta$, one party pays a fixed amount of ΔK and receives $\Delta L^\Delta(T, T + \Delta)$, and the other party pays $\Delta L^\Delta(T, T + \Delta)$ and receives ΔK . The cashflow to the payer is given by

$$\text{FRA}(T + \Delta) = \Delta(L^\Delta(T, T + \Delta) - K).$$

By absence of arbitrage, the time t -FRA rate $L^\Delta(t; T, T + \Delta)$ is given by

$$L^\Delta(t; T, T + \Delta) = \mathbb{E}_t^{T+\Delta}[L^\Delta(T, T + \Delta)],$$

where \mathbb{E}^S denotes expectation under the S -forward measure, with the OIS-based zero-coupon bond p as the numéraire. Obviously we have

$$L^\Delta(T; T, T + \Delta) = L^\Delta(T, T + \Delta).$$

In the single-curve setting, we would have

$$L^\Delta(t; T, T + \Delta) = F(t; T, T + \Delta) \tag{4.1}$$

where $F(t; T, T + \Delta)$ denotes the time- t OIS forward rate for the time interval $[T, T + \Delta]$, which is defined via

$$F(t; T, T + \Delta) := \frac{1}{\Delta} \left(\frac{p(t, T)}{p(t, T + \Delta)} - 1 \right);$$

see Definition 2.7. The identity (4.1), however, no longer holds true in the multi-curve setting; in post-crisis markets one generally observes that

$$L^\Delta(t; T, T + \Delta) > F(t; T, T + \Delta) \text{ for all } t \in [0, T].$$

Intuitively, this can be attributed to increased credit and liquidity risks in the interbank market; see, e.g., Mercurio (2009), Filipović & Trolle (2013), and Gallitschke et al. (2014).

Definition 4.1. The (additive) **FRA spread** between the FRA rate and the OIS forward rate is defined as

$$S^\Delta(t, T) := L^\Delta(t; T, T + \Delta) - F(t; T, T + \Delta),$$

and the associated **multiplicative FRA spread** is given by

$$S_m^\Delta(t, T) := \frac{1 + \Delta L^\Delta(t; T, T + \Delta)}{1 + \Delta F(t; T, T + \Delta)}.$$

In the literature, there exist two main approaches to spread modeling: *deterministic spreads* as in Henrard (2010), and *stochastic spreads* as in Mercurio & Xie (2012) or Henrard (2013). Henrard (2010) assumes the multiplicative spread $S_m^\Delta(t, T)$ to be constant over time, i.e.

$$S_m^\Delta(t, T) = S_m^\Delta(0, T) \text{ for all } t \in [0, T].$$

Note that the multiplicative spread S_m^Δ is not constant, but a deterministic function of the relevant maturity. Mercurio & Xie (2012) construct a model with a stochastic basis using an additive FRA spread S^Δ , modeled as a function of the corresponding OIS forward rate and an independent martingale. This approach is also used by Henrard (2013) to model the multiplicative spread S_m^Δ .

In this thesis, we choose the spot multiplicative FRA spread $S_m^\Delta(t, t)$, or equivalently, the hypothetical forward (foreign) state-price deflator D_t^Δ , as a building block for our model. $S_m^\Delta(t, t)$ can be interpreted easily as a spot exchange rate between two currencies (see below). Thus, modeling spot multiplicative spreads corresponds to modeling exchange rates in a cross-currency setup, which fits naturally into the potential approach introduced in Chapter 2. Furthermore, we also show that the multiplicative spread $S_m^\Delta(t, T)$ can be interpreted as a forward exchange rate and is therefore, in general, stochastic.

4.3 The Multi-Curve Potential Model

4.3.1 Motivation

In the single-curve setting we have

$$L^\Delta(t, t + \Delta) = F(t, t + \Delta) = \frac{1}{\Delta} \left(\frac{1}{p(t, t + \Delta)} - 1 \right)$$

and thus $L^\Delta(t, t + \Delta)$ is equivalently characterized as the simply compounded rate of the following simple investment:

- invest $1\d at time t ,

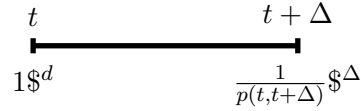


Figure 4.1: Quanto investment underlying the LIBOR-OIS spread.

- receive $\frac{1}{p(t, t+\Delta)}\d at maturity $t + \Delta$.

By contrast, as discussed in Section 4.2.4, in the post-crisis framework we generally have

$$L^\Delta(t, t + \Delta) > F(t, t + \Delta).$$

The implied positive LIBOR-OIS spread can be regarded as a premium that the lending bank requires from its counterparty due to credit and liquidity risk concerns. Equivalently, we can think of a typical financial institution as borrowing in the currency $\d , but repaying its lenders in a different currency $\$^\Delta$ that reflects the level of credit, liquidity and other interbank market risks. Then we identify the LIBOR rate as the interest rate paid on this type of transaction. To formalize this, let us consider the following quanto investment:

- invest $1\d at time t ,
- receive $\frac{1}{p(t, t+\Delta)}\$^\Delta$ at maturity $t + \Delta$, in a foreign currency $\$^\Delta$;

see Figure 4.1. Thus, the lender invests $\d , but receives $\$^\Delta$. Note that, in this reduced-form approach, the forward (foreign) currency $\$^\Delta$ is a theoretical object. Let us denote by x^Δ the exchange rate process between $\d and $\$^\Delta$: $1\$^\Delta$ at time t is equivalent to $x^\Delta(t)$ units of $\d . Then the above quanto transaction is equivalent to investing $\frac{1}{x^\Delta(t)}\$^\Delta$ at time t and receiving $\frac{1}{p(t, t+\Delta)}\$^\Delta$ at maturity $t + \Delta$. Note that, if we consider $L^\Delta(t, t + \Delta)$ as the simply compounded rate of this quanto investment, in units of the currency $\$^\Delta$, i.e.

$$\frac{1}{x^\Delta(t)}(1 + \Delta L^\Delta(t, t + \Delta)) = \frac{1}{p(t, t + \Delta)},$$

then the spot multiplicative spread $S_m^\Delta(t, t)$ is identical to the spot exchange rate process $x^\Delta(t)$:

$$S_m^\Delta(t, t) = \frac{1 + \Delta L^\Delta(t, t + \Delta)}{1 + \Delta F(t, t + \Delta)} = x^\Delta(t).$$

4.3.2 Definition of the multi-curve potential model

Since the spot multiplicative spread $S_m^\Delta(t, t)$ can be regarded as a spot exchange rate process, we can use Proposition 2.14 to construct a general, arbitrage-free multi-curve potential model by simultaneously modeling the OIS curve p and the spot multiplicative spread process $(S_m^\Delta(t, t))_{t \geq 0}$: First, choose a (strictly) positive semimartingale D as the state-price deflator in the domestic currency $\d , and construct the OIS term structure via

$$p(t, T) = \frac{\mathbb{E}_t[D_T]}{D_t} \text{ for } t \leq T.$$

Second, model the spot multiplicative spread process $(S_m^\Delta(t, t))_{t \geq 0}$ as

$$S_m^\Delta(t, t) = \frac{D_t^\Delta}{D_t}$$

for some positive, semimartingale D^Δ . The spot LIBOR rate $L^\Delta(t, t + \Delta)$ is then given by

$$L^\Delta(t, t + \Delta) = \frac{1}{\Delta} \left(\frac{1}{p(t, t + \Delta)} \frac{D_t^\Delta}{D_t} - 1 \right). \quad (4.2)$$

A few remarks are in order. First, it is easy to see that we obtain positive LIBOR rates if and only if $D_t^\Delta \geq p(t, t + \Delta)D_t$. Second, note that our approach is not symmetric with respect to the domestic and forward currency: The domestic currency $\d and the associated deflator D are distinguished as the unique state-price deflator to be used in the general risk-neutral pricing formula

$$\Pi(t) = \frac{\mathbb{E}_t[\Pi(T)D_T]}{D_t},$$

which is valid for any price process Π . Third, this asymmetry also becomes apparent from the Δ -LIBOR rate (4.2), which involves both the corresponding deflator D^Δ and the domestic deflator D . Equivalently, although the normalized process D_t^Δ could be interpreted as the state-price deflator in the currency $\$^\Delta$, the simply compounded interest rate in that market is *not* the Δ -LIBOR rate. Thus, in this respect our approach differs from that of Bianchetti (2010), where the Δ -LIBOR rate is defined as the simply compounded rate implied by the $\$^\Delta$ -term structure. Finally, we recover the single-curve potential model when $D^\Delta \equiv D$, i.e., the two currencies $\$$ and $\$^\Delta$ are identical.

4.3.3 General pricing formulae

In this section we present general clean valuation formulae for important interest rate derivatives in the general multi-curve potential framework. Here “clean” means that we consider contracts that are fully collateralized in cash, and where the collateral rate coincides with the OIS rate. In that case, it is clear from arbitrage arguments (see Piterbarg 2010) that the OIS curve is the correct discount curve; this is also the widely accepted market standard.

Forward rate agreements

We consider a forward rate agreement for the time interval $[T, T + \Delta]$ with cash flow at time $T + \Delta$:

$$\text{FRA}(T + \Delta) = \Delta(L^\Delta(T, T + \Delta) - K).$$

Discounting with the OIS curve p from $T + \Delta$ back to time T we have

$$\begin{aligned} \text{FRA}(T) &= p(T, T + \Delta)\Delta(L^\Delta(T, T + \Delta) - K) \\ &= p(T, T + \Delta) \left(\frac{S_m^\Delta(T, T)}{p(T, T + \Delta)} - (1 + \Delta K) \right) \\ &= S_m^\Delta(T, T) - (1 + \Delta K)p(T, T + \Delta). \end{aligned}$$

Therefore, the time- t price of the forward rate agreement is given by

$$\begin{aligned} \text{FRA}(t) &= \frac{1}{D_t} \mathbb{E}_t[D_T \text{FRA}(T)] \\ &= \frac{1}{D_t} \mathbb{E}_t [D_T S_m^\Delta(T, T)] - (1 + \Delta K) \frac{1}{D_t} \mathbb{E}_t[D_T p(T, T + \Delta)] \\ &= \frac{1}{D_t} \mathbb{E}_t [D_T^\Delta] - (1 + \Delta K) p(t, T + \Delta). \end{aligned}$$

Solving the equation $\text{FRA}(t) = 0$ for K we obtain the time- t FRA rate

$$L^\Delta(t; T, T + \Delta) = \frac{1}{\Delta} \left(\frac{p(t, T)}{p(t, T + \Delta)} \frac{\mathbb{E}_t[D_T^\Delta]}{\mathbb{E}_t[D_T]} - 1 \right). \quad (4.3)$$

We thus obtain the additive and multiplicative FRA spreads as follows.

Theorem 4.2 (FRA spread in the multi-curve potential model). *The FRA spread in the multi-curve potential model is given by*

$$\begin{aligned} S^\Delta(t, T) &= L^\Delta(t; T, T + \Delta) - F(t; T, T + \Delta) \\ &= \frac{p(t, T)}{\Delta p(t, T + \Delta)} \left(\frac{\mathbb{E}_t[D_T^\Delta]}{\mathbb{E}_t[D_T]} - 1 \right), \end{aligned} \quad (4.4)$$

and the multiplicative FRA spread $S_m^\Delta(t, T)$ is provided by

$$S_m^\Delta(t, T) = \frac{1 + \Delta L^\Delta(t; T, T + \Delta)}{1 + \Delta F(t; T, T + \Delta)} = \frac{\mathbb{E}_t[D_T^\Delta]}{\mathbb{E}_t[D_T]}. \quad (4.5)$$

In particular, all FRA spreads are positive (equivalently, all multiplicative FRA spreads are greater than 1) if the condition $D_t^\Delta \geq D_t$ holds for all $t \geq 0$.

Remark 4.3. By comparing Equations (4.5) and (2.8) in the general case, we see that the multiplicative spread S_m^Δ is the forward exchange rate process between the domestic currency \mathbb{S}^d and the foreign currency \mathbb{S}^Δ . Thus, the interpretation of the spot multiplicative spread $S_m^\Delta(t, t)$ as an exchange rate process is consistent and the model produces, in general, stochastic spreads. In particular, if $D_t^\Delta = \varphi^\Delta(t) D_t$ for some deterministic function φ^Δ , then the multiplicative spread S_m^Δ is deterministic:

$$S_m^\Delta(t, T) = \frac{\mathbb{E}_t[\varphi^\Delta(T) D_T]}{\mathbb{E}_t[D_T]} = \varphi^\Delta(T) \text{ for all } t \in [0, T], \text{ and } T \geq 0.$$

We thus recover the deterministic spread assumption in Henrard (2010).

Overnight indexed swaps

We consider an overnight indexed swap that exchanges a fixed rate $K > 0$ for the floating compound overnight rate $\bar{L}(t, T)$ that is defined by

$$\bar{L}(t, T) = \frac{1}{T-t} \left(e^{\int_t^T r_s ds} - 1 \right)$$

where r is the relevant overnight rate (FF in the USD market and EONIA in the EUR market). We further assume that the swap has the tenor structure

$$0 \leq T_0 < T_1 \dots < T_N = T \text{ for some } N \in \mathbb{N}, \text{ where } T_i - T_{i-1} = \bar{\Delta}, i = 1, \dots, N.$$

At each time T_i , $i = 1, \dots, N$, the fixed leg pays $\bar{\Delta}K$ and the floating leg pays $\bar{\Delta}\bar{L}(T_{i-1}, T_i)$. The fair values of the fixed leg and the floating leg at any time $t \leq T_0$ are given by

$$\text{OIS}^{\text{fix}}(t) = \frac{1}{D_t} \sum_{i=1}^N \mathbb{E}_t[\bar{\Delta}K D_{T_i}] = \sum_{i=1}^N \bar{\Delta}K p(t, T_i)$$

and by

$$\text{OIS}^{\text{flo}}(t) = \frac{1}{D_t} \sum_{i=1}^N \mathbb{E}_t[\bar{\Delta}\bar{L}(T_{i-1}, T_i) D_{T_i}] = p(t, T_0) - p(t, T).$$

By solving the equation $\text{OIS}^{\text{fix}}(t) = \text{OIS}^{\text{flo}}(t)$ we obtain the following classical single-curve formula for the OIS rate:

$$\text{OIS}^{\bar{\Delta}}(t; T_0, T) = \frac{p(t, T_0) - p(t, T)}{\sum_{i=1}^N \bar{\Delta} p(t, T_i)}. \quad (4.6)$$

Remark 4.4. By the standard market convention, payments for both legs of an overnight indexed swap take place at a one-year frequency, i.e., $\bar{\Delta} = 1$. For overnight indexed swaps with maturities less than one year ($T < 1$), there is only one payment, which occurs at maturity.

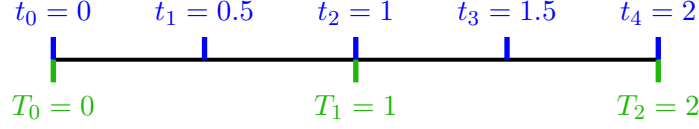


Figure 4.2: Two underlying tenor structures with maturity $T = 2y$, fixed leg tenor $\bar{\Delta} = 1y$, and floating leg tenor $\Delta = 6m$.

LIBOR-swaps and LIBOR-OIS spreads

We consider a LIBOR-swap with the following two underlying tenor structures:

$$0 \leq T_0 < T_1 \dots < T_N = T \quad \text{and} \quad 0 \leq t_0 = T_0 < t_1 < \dots < t_M = T.$$

Let $\bar{\Delta} = T_i - T_{i-1}$ and $\Delta = t_i - t_{i-1}$; see Figure 4.2 for an illustration. At each time T_i , $i = 1, \dots, N$, one party pays $\bar{\Delta}K$, whereas at each time t_i , $i = 1, \dots, M$, the other party pays $\Delta L^\Delta(t_{i-1}, t_i)$. The time- t values of the fixed and the floating leg are given respectively by

$$\text{IRS}^{\text{fix}}(t) = \sum_{i=1}^N \bar{\Delta} K p(t, T_i)$$

and

$$\text{IRS}^{\text{flo}}(t) = \frac{1}{D_t} \sum_{i=1}^M \mathbb{E}_t[\Delta L^\Delta(t_{i-1}, t_i) D_{t_i}] = \sum_{i=1}^M \Delta p(t, t_i) L^\Delta(t; t_{i-1}, t_i).$$

Hence, the swap rate $\text{IRS}_{\bar{\Delta}}^\Delta(t; T_0, T)$, i.e., the value of K that ensures

$$\text{IRS}^{\text{fix}}(t) = \text{IRS}^{\text{flo}}(t),$$

is given by

$$\text{IRS}_{\bar{\Delta}}^\Delta(t; T_0, T) = \frac{\sum_{i=1}^M \Delta p(t, t_i) L^\Delta(t; t_{i-1}, t_i)}{\sum_{i=1}^N \bar{\Delta} p(t, T_i)}. \quad (4.7)$$

Combining Equations (4.6) and (4.7), and noting that for the discount curve we have the classical telescoping sum formula

$$p(t, t_0) - p(t, t_M) = \sum_{i=1}^M \Delta p(t, t_i) F(t; t_{i-1}, t_i),$$

we obtain the following representation of the LIBOR-OIS spread.

Theorem 4.5 (LIBOR-OIS spreads in the multi-curve potential model).
The Δ -tenor LIBOR-OIS spread (with maturity T) in the multi-curve potential model is provided by

$$\begin{aligned}
 S_{\Delta}^{\Delta}(t; T_0, T) &:= IRS_{\Delta}^{\Delta}(t; T_0, T) - OIS^{\Delta}(t; T_0, T) \\
 &= \frac{\sum_{i=1}^M \Delta p(t, t_i) [L^{\Delta}(t; t_{i-1}, t_i) - F(t; t_{i-1}, t_i)]}{\sum_{i=1}^N \bar{\Delta} p(t, T_i)} \\
 &= \frac{\sum_{i=1}^M \Delta p(t, t_i) S^{\Delta}(t, t_{i-1})}{\sum_{i=1}^N \bar{\Delta} p(t, T_i)}. \tag{4.8}
 \end{aligned}$$

If the condition $D_t^{\Delta} \geq D_t$ for all $t \geq 0$ is satisfied, then the Δ -tenor LIBOR-OIS spread is also positive.

Proof. The first part of the claim is clear from the preceding discussion. Positivity of the LIBOR-OIS spread follows from the fact that, under the stated condition, each FRA spread $S^{\Delta}(t, t_{i-1})$, $i = 1, 2, \dots, M$, is positive. \square

We next address the most important nonlinear products in the interest rate market and derive general pricing formulae for caps and swaptions. These products are among the most liquid interest rate derivatives, and are typically used as calibration instruments. Hence, it is crucial to have tractable valuation formulae for these options, so that the model can be easily calibrated to market data; see Section 4.7 below for an illustration with real-world market data.

Caplets

As a cap is nothing but a linear portfolio of caplets, valuation of a cap boils down to pricing its component caplets. We thus consider a caplet with strike rate K for the time interval $[T, T + \Delta]$, whose payoff is given by

$$C(T + \Delta) = \Delta(L^{\Delta}(T, T + \Delta) - K)^+.$$

The time- T price $C(T)$ of the caplet therefore satisfies

$$\begin{aligned} C(T) &= \Delta p(T, T + \Delta) (L^\Delta(T, T + \Delta) - K)^+ \\ &= \Delta p(T, T + \Delta) \left(\frac{1}{\Delta} \left(\frac{D_T^\Delta}{D_T p(T, T + \Delta)} - 1 \right) - K \right)^+ \\ &= \left(\frac{D_T^\Delta}{D_T} - (1 + \Delta K) p(T, T + \Delta) \right)^+. \end{aligned}$$

Discounting with the OIS curve p , we obtain the time- t price $C(t)$ of the caplet:

$$C(t) = \frac{\mathbb{E}_t[D_T C(T)]}{D_t} = \frac{1}{D_t} \mathbb{E}_t[(D_T^\Delta - (1 + \Delta K) D_T p(T, T + \Delta))^+].$$

By plugging $p(T, T + \Delta) = \frac{\mathbb{E}_T[D_{T+\Delta}]}{D_T}$ into the above equation we further have

$$C(t) = \frac{1}{D_t} \mathbb{E}_t[(D_T^\Delta - (1 + \Delta K) \mathbb{E}_T[D_{T+\Delta}])^+]. \quad (4.9)$$

Swaptions

Consider a payer swaption with strike rate K , maturity $T = T_0$ and payment dates

$$T_1 < \dots < T_N \text{ where } T_i - T_{i-1} = \Delta, \quad i = 1, \dots, N.$$

The payoff of the swaption at T is given by

$$S(T) = \Delta \left(\sum_{i=1}^N p(T, T_i) (L^\Delta(T; T_{i-1}, T_i) - K) \right)^+.$$

Substituting for

$$L^\Delta(T; T_{i-1}, T_i) = \frac{1}{\Delta} \left(\frac{p(T, T_{i-1}) \mathbb{E}_T[D_{T_{i-1}}^\Delta]}{p(T, T_i) \mathbb{E}_T[D_{T_{i-1}}]} - 1 \right), \quad i = 1, \dots, N,$$

and for $p(T, T_k) = \mathbb{E}_T[D_{T_k}]/D_T$, $k = 1, \dots, N$ in the above equation yields

$$S(T) = \frac{1}{D_T} \left(\mathbb{E}_T \left[\sum_{i=1}^N D_{T_{i-1}}^\Delta - (1 + \Delta K) D_{T_i} \right] \right)^+.$$

Hence, the time- t price of the swaption is given by

$$\begin{aligned} S(t) &= \frac{\mathbb{E}_t[S(T)D_T]}{D_t} \\ &= \frac{1}{D_t} \mathbb{E}_t \left[\left(\sum_{i=1}^N \mathbb{E}_T[D_{T_i}^\Delta] - (1 + \Delta K) \mathbb{E}_T[D_{T_i}] \right)^+ \right]. \end{aligned} \quad (4.10)$$

Remark 4.6. As we have seen, the two state-price deflator processes D and D^Δ are the basic quantities in all the interest rate derivatives considered above. Thus, once these quantities are specified, the whole model is constructed with pricing formulae for all relevant products, represented in terms of the two deflators. Therefore, the tractability of the model depends crucially on the dynamics of these processes. The most important criterion for the specification of the general multi-curve potential model is the availability of analytic formulae for basic interest rate products such as forward rate agreements and swaps, and this requires the computation of the conditional distributions of D and D^Δ .

4.4 Multi-Curve Rational Lognormal Models

In this section we consider a specification of the general multi-curve potential model developed in Section 4.3. The specific potential model that we consider here is a multi-curve extension of the rational lognormal model proposed in Flesaker & Hughston (1996a) (see also Section 2.2.3). In a single-curve setting, the classical rational lognormal model guarantees positive interest rates and closed-form formulae for both caps and swaptions (see Rutkowski 1997). We demonstrate that the analytic tractability of the model carries over to our multi-curve extension. As in the single-curve setting, in the case of a one-factor model we obtain closed-form formulae for both caps and swaptions. In the case of a two-factor model, we obtain semi-closed form cap and swaption formulae that are explicit modulo a one-dimensional integration.

4.4.1 One-factor model

In the one-factor multi-curve rational lognormal model, the state-price deflator processes D and D^Δ are given by

$$D_t = f(t) + g(t)M_t \quad \text{and} \quad D_t^\Delta = f^\Delta(t) + g^\Delta(t)M_t$$

where

- f and g are deterministic functions with $f(0) + g(0) = 1$,
- f^Δ and g^Δ are positive deterministic functions,
- and the positive martingale M follows the lognormal dynamics

$$M_t = e^{\int_0^t \sigma(s) dW_s - \frac{1}{2} \int_0^t \sigma^2(s) ds}$$

for some deterministic function $\sigma : [0, \infty) \rightarrow \mathbb{R}$, and a standard Brownian motion W .

Remark 4.7. If the conditions $f^\Delta(t) \geq f(t)$, $g^\Delta(t) \geq g(t)$ are satisfied for all $t \geq 0$ then $D_t^\Delta \geq D_t$ for all $t \geq 0$, which in turn guarantees positivity of the FRA spreads and the LIBOR-OIS spreads generated by the model.

As discussed in Section 4.3.3, the conditional distributions of the processes D and D^Δ play a crucial role to the availability of closed-form pricing formulae for the prices of interest rate derivatives. Specifically, FRA rates, FRA spreads and basis spreads can only have explicit formulae if the conditional expectations $\mathbb{E}_t[D_T]$ and $\mathbb{E}_t[D_T^\Delta]$ can be calculated analytically. With the above specification of D and D^Δ , these conditional expectations are given explicitly:

$$\mathbb{E}_t[D_T] = \mathbb{E}_t[f(T) + g(T)M_T] = f(T) + g(T)M_t, \quad (4.11)$$

$$\mathbb{E}_t[D_T^\Delta] = \mathbb{E}_t[f^\Delta(T) + g^\Delta(T)M_T] = f^\Delta(T) + g^\Delta(T)M_t. \quad (4.12)$$

The OIS discount curve is then given, as in the single-curve setting, by the following explicit formula:

$$p(t, T) = \frac{\mathbb{E}_t[D_T]}{D_t} = \frac{f(T) + g(T)M_t}{f(t) + g(t)M_t} \quad \text{for all } 0 \leq t \leq T. \quad (4.13)$$

We now proceed to work out analytic expressions for the interest rate products from the general formulae derived in Section 4.3.3.

FRA rates and FRA spreads

Plugging (4.13), (4.11), and (4.12) into the general formula (4.3), we obtain the following explicit formula for the time- t FRA rate:

$$L^\Delta(t; T, T + \Delta) = \frac{1}{\Delta} \left(\frac{f^\Delta(T) + g^\Delta(T)M_t}{f(T + \Delta) + g(T + \Delta)M_t} - 1 \right). \quad (4.14)$$

The model thus implies a stochastic FRA spread given by

$$\begin{aligned} S^\Delta(t, T) &= L(t; T, T + \Delta) - F(t; T, T + \Delta) \\ &= \frac{1}{\Delta} \frac{f^\Delta(T) - f(T) + (g^\Delta(T) - g(T))M_t}{f(T + \Delta) + g(T + \Delta)M_t}. \end{aligned} \quad (4.15)$$

Swap rates, LIBOR-OIS spreads

Having closed-form formulae for the OIS discount curve p , FRA rates, and FRA spreads, it is straightforward to calculate the swap rates and Δ -tenor LIBOR-OIS spreads in Equations (4.7) and (4.8). We have

$$\text{IRS}_\Delta^\Delta(t; T_0, T) = \frac{\sum_{i=1}^M f^\Delta(t_{i-1}) - f(t_i) + (g^\Delta(t_{i-1}) - g(t_i))M_t}{\bar{\Delta} \sum_{i=1}^N f(T_i) + g(T_i)M_t}, \quad (4.16)$$

$$S_\Delta^\Delta(t; T_0, T) = \frac{\sum_{i=1}^M f^\Delta(t_{i-1}) - f(t_{i-1}) + (g^\Delta(t_{i-1}) - g(t_{i-1}))M_t}{\bar{\Delta} \sum_{i=1}^N f(T_i) + g(T_i)M_t}. \quad (4.17)$$

In the following we focus on deriving closed-form pricing formulae for caplets and swaptions.

Valuation of caplets

We consider the caplet as specified in Section 4.3.3. Since

$$D_T p(T, T + \Delta) = \mathbb{E}_T[D_{T+\Delta}] = f(T + \Delta) + g(T + \Delta)M_T$$

and

$$D_T^\Delta = f^\Delta(T) + g^\Delta(T)M_T$$

we obtain from Equation (4.9)

$$C(t) = \frac{1}{D_t} \mathbb{E}_t [(AM_T - B)^+],$$

where

$$\begin{aligned} A &:= g^\Delta(T) - (1 + \Delta K)g(T + \Delta), \\ B &:= (1 + \Delta K)f(T + \Delta) - f^\Delta(T). \end{aligned}$$

We consider the following cases:

Case 1. Assume

$$\frac{g^\Delta(T)}{g(T + \Delta)} > \frac{f^\Delta(T)}{f(T + \Delta)}.$$

Then there are three possibilities:

Case 1.1. We have

$$K \in \left(\frac{f^\Delta(T)}{\Delta f(T + \Delta)} - \frac{1}{\Delta}, \frac{g^\Delta(T)}{\Delta g(T + \Delta)} - \frac{1}{\Delta} \right)$$

and in that case it follows that $A > 0$, $B > 0$. We have

$$\mathbb{E}[(AM_T - B)^+ | \mathcal{F}_t] = \mathbb{E}[(AM_t e^Y - B)^+ | \mathcal{F}_t],$$

where

$$Y := \int_t^T \sigma(s) dW_s - \frac{1}{2} \int_t^T |\sigma(s)|^2 ds. \quad (4.18)$$

Note that the conditional distribution of Y given \mathcal{F}_t is normal,

$$Y | \mathcal{F}_t \sim \mathcal{N} \left(-\frac{1}{2} \nu^2(t, T), \nu^2(t, T) \right) \quad \text{with} \quad \nu^2(t, T) = \int_t^T |\sigma(s)|^2 ds,$$

as Y is independent of \mathcal{F}_t . Thus, applying Lemma A.4 with $a = AM_t$, $b = B$ and $\sigma = \nu(t, T)$ we get

$$C(t) = (f(t) + g(t)M_t)^{-1} (AM_t \Phi(d_1^\Delta) - B \Phi(d_2^\Delta)),$$

where

$$d_{1,2}^\Delta = \frac{1}{\nu(t, T)} \left(\ln c_t^\Delta \pm \frac{1}{2} \nu^2(t, T) \right), \quad (4.19)$$

$$c_t^\Delta = \frac{AM_t}{B} = M_t \frac{g^\Delta(T) - (1 + \Delta K)g(T + \Delta)}{(1 + \Delta K)f(T + \Delta) - f^\Delta(T)}. \quad (4.20)$$

Case 1.2. We have

$$K \geq \frac{g^\Delta(T)}{\Delta g(T + \Delta)} - \frac{1}{\Delta}$$

in which case $A \leq 0$, $B \geq 0$ and thus $(AM_T - B)^+ = 0$. Hence $C(t) = 0$.

Case 1.3. We have

$$K \leq \frac{f^\Delta(T)}{\Delta f(T + \Delta)} - \frac{1}{\Delta}$$

so $A \geq 0$, $B \leq 0$. Hence $(AM_T - B)^+ = AM_T - B$ and we get

$$C(t) = \frac{1}{D_t} \mathbb{E}_t[AM_T - B] = \frac{AM_t - B}{f(t) + g(t)M_t}.$$

Case 2. Assuming

$$\frac{g^\Delta(T)}{g(T + \Delta)} \leq \frac{f^\Delta(T)}{f(T + \Delta)},$$

we also have the following three possibilities:

Case 2.1. We have

$$K \in \left(\frac{g^\Delta(T)}{\Delta g(T + \Delta)} - \frac{1}{\Delta}, \frac{f^\Delta(T)}{\Delta f(T + \Delta)} - \frac{1}{\Delta} \right).$$

This implies that $A < 0$ and $B < 0$. Then we have

$$(AM_T - B)^+ = (-(-AM_T + B))^+ = (-AM_T + B)^+ + AM_T - B.^1$$

¹Here, we use the identity $(-x)^+ = x^+ - x, \forall x \in \mathbb{R}$.

It follows that

$$\begin{aligned}\mathbb{E}[(AM_T - B)^+ | \mathcal{F}_t] &= \mathbb{E}[(-AM_t e^Y + B)^+ | \mathcal{F}_t] + AM_t \mathbb{E}_t[e^Y] - B \\ &= \mathbb{E}[(-AM_t e^Y + B)^+ | \mathcal{F}_t] + AM_t - B,\end{aligned}\quad (4.21)$$

since $\mathbb{E}_t[e^Y] = 1$, where Y is given in (4.18). Applying Lemma A.4 with

$$a = -AM_t > 0, b = -B > 0 \text{ and } \sigma = \nu(t, T)$$

yields

$$\mathbb{E}[(-AM_t e^Y + B)^+ | \mathcal{F}_t] = -AM_t \Phi(d_1^\Delta) + B \Phi(d_2^\Delta), \quad (4.22)$$

where $d_{1,2}^\Delta$ and c_t^Δ are defined in (4.19) and (4.20). Combining Equations (4.21) and (4.22), we obtain

$$\begin{aligned}\mathbb{E}[(AM_T - B)^+ | \mathcal{F}_t] &= -AM_t \Phi(d_1^\Delta) + B \Phi(d_2^\Delta) + AM_t - B \\ &= AM_t(1 - \Phi(d_1^\Delta)) - B(1 - \Phi(d_2^\Delta)) \\ &= AM_t \Phi(-d_1^\Delta) - B \Phi(-d_2^\Delta).\end{aligned}$$

Therefore, in this case we have

$$C(t) = (f(t) + g(t)M_t)^{-1} (AM_t \Phi(-d_1^\Delta) - B \Phi(-d_2^\Delta)).$$

Case 2.2. We have

$$K \geq \frac{f^\Delta(T)}{\Delta f(T + \Delta)} - \frac{1}{\Delta}.$$

This implies $A \leq 0$ and $B \geq 0$ and thus $(AM_T - B)^+ = 0$. Hence, we have $C(t) = 0$.

Case 2.3. We have

$$K \leq \frac{g^\Delta(T)}{\Delta g(T + \Delta)} - \frac{1}{\Delta}$$

which implies $A \geq 0$ and $B \leq 0$. It follows then that $(AM_T - B)^+ = AM_T - B$ and we obtain

$$C(t) = \frac{1}{D_t} \mathbb{E}_t[AM_T - B] = \frac{AM_t - B}{f(t) + g(t)M_t},$$

since M is a martingale.

We summarize the above results in the following:

Theorem 4.8 (Caplets in the one-factor rational lognormal model).

Consider a caplet based on the LIBOR rate for the future time interval $[T, T + \Delta]$ with strike rate K , whose payoff is given by

$$C(T + \Delta) = \Delta(L^\Delta(T, T + \Delta) - K)^+.$$

Then its time- t price $C(t)$ is given by

$$C(t) = \begin{cases} 0 & \text{if } K \geq K^* \\ \frac{AM_t - B}{f(t) + g(t)M_t} & \text{if } K \leq K_* \\ (f(t) + g(t)M_t)^{-1} (AM_t \Phi(wd_1^\Delta) - B \Phi(wd_2^\Delta)) & \text{otherwise} \end{cases}$$

where

$$\begin{aligned} K^* &:= \frac{f^\Delta(T)}{\Delta f(T + \Delta)} - \frac{1}{\Delta} \vee \frac{g^\Delta(T)}{\Delta g(T + \Delta)} - \frac{1}{\Delta} \\ K_* &:= \frac{f^\Delta(T)}{\Delta f(T + \Delta)} - \frac{1}{\Delta} \wedge \frac{g^\Delta(T)}{\Delta g(T + \Delta)} - \frac{1}{\Delta} \\ A &= g^\Delta(T) - (1 + \Delta K)g(T + \Delta), \\ B &= (1 + \Delta K)f(T + \Delta) - f^\Delta(T), \\ d_{1,2}^\Delta &= \frac{1}{\nu(t, T)} \left(\ln c_t^\Delta \pm \frac{1}{2} \nu^2(t, T) \right), \\ c_t^\Delta &= \frac{AM_t}{B} = M_t \frac{g^\Delta(T) - (1 + \Delta K)g(T + \Delta)}{(1 + \Delta K)f(T + \Delta) - f^\Delta(T)}, \\ \nu^2(t, T) &= \int_t^T |\sigma(s)|^2 ds, \end{aligned}$$

and

$$w := \begin{cases} 1 & \text{if } \frac{g^\Delta(T)}{g(T + \Delta)} \geq \frac{f^\Delta(T)}{f(T + \Delta)}, \\ -1 & \text{if } \frac{g^\Delta(T)}{g(T + \Delta)} < \frac{f^\Delta(T)}{f(T + \Delta)}. \end{cases}$$

Valuation of swaptions

We next determine the fair price of the swaption given in Section 4.3.3. The time- t price of the swaption is obtained from the general formula (4.10) via

$$S(t) = \frac{\mathbb{E}_t[D_T S(T)]}{D_t} = \frac{1}{D_t} \mathbb{E}_t [(pM_T - q)^+],$$

where

$$p := \sum_{i=1}^N g^\Delta(T_{i-1}) - (1 + \Delta K)g(T_i), \quad (4.23)$$

$$q := \sum_{i=1}^N (1 + \Delta K)f(T_i) - f^\Delta(T_{i-1}). \quad (4.24)$$

We further define

$$K_{\min} := \frac{\sum_{i=1}^N f^\Delta(T_{i-1})}{\Delta \sum_{i=1}^N f(T_i)} - \frac{1}{\Delta} \wedge \frac{\sum_{i=1}^N g^\Delta(T_{i-1})}{\Delta \sum_{i=1}^N g(T_i)} - \frac{1}{\Delta}, \quad (4.25)$$

$$K_{\max} := \frac{\sum_{i=1}^N f^\Delta(T_{i-1})}{\Delta \sum_{i=1}^N f(T_i)} - \frac{1}{\Delta} \vee \frac{\sum_{i=1}^N g^\Delta(T_{i-1})}{\Delta \sum_{i=1}^N g(T_i)} - \frac{1}{\Delta}. \quad (4.26)$$

We are now in exactly the same situation as in the valuation of caplets. Hence, following similar steps as above, we obtain:

Theorem 4.9 (Swaptions in the one-factor rational lognormal model).

The time- t price $S(t)$ of the swaption is provided by

$$S(t) = \begin{cases} 0 & \text{if } K \geq K_{\max} \\ \frac{pM_t - q}{f(t) + g(t)M_t} & \text{if } K \leq K_{\min} \\ \frac{pM_t \Phi(w \tilde{d}_1^\Delta) - q \Phi(w \tilde{d}_2^\Delta)}{f(t) + g(t)M_t} & \text{if } K \in (K_{\min}, K_{\max}) \end{cases}$$

where

$$\begin{aligned} \tilde{d}_{1,2}^\Delta &:= \frac{1}{\nu(t, T)} \left(\ln \tilde{c}_t^\Delta \pm \frac{1}{2} \nu^2(t, T) \right), \\ \tilde{c}_t^\Delta &:= \frac{pM_t}{q} = M_t \frac{\sum_{i=1}^N g^\Delta(T_{i-1}) - (1 + \Delta K)g(T_i)}{\sum_{i=1}^N (1 + \Delta K)f(T_i) - f^\Delta(T_{i-1})}, \\ \nu^2(t, T) &= \int_t^T |\sigma(s)|^2 ds, \end{aligned}$$

and

$$w := \begin{cases} 1 & \text{if } \frac{\sum_{i=1}^N f^\Delta(T_{i-1})}{\Delta \sum_{i=1}^N f(T_i)} - \frac{1}{\Delta} \leq \frac{\sum_{i=1}^N g^\Delta(T_{i-1})}{\Delta \sum_{i=1}^N g(T_i)} - \frac{1}{\Delta}, \\ -1 & \text{if } \frac{\sum_{i=1}^N f^\Delta(T_{i-1})}{\Delta \sum_{i=1}^N f(T_i)} - \frac{1}{\Delta} > \frac{\sum_{i=1}^N g^\Delta(T_{i-1})}{\Delta \sum_{i=1}^N g(T_i)} - \frac{1}{\Delta}. \end{cases}$$

4.4.2 Two-factor model

We next address a multi-factor extension of the model discussed above. In the two-factor rational lognormal model, the state-price deflators D and D^Δ are specified as follows:

$$\begin{aligned} D_t &= f(t) + g(t)M_t + h(t)N_t, \\ D_t^\Delta &= f^\Delta(t) + g^\Delta(t)M_t + h^\Delta(t)N_t, \end{aligned}$$

where

- f , g and h are deterministic functions with $f(0) + g(0) + h(0) = 1$,
- f^Δ , g^Δ and h^Δ are positive deterministic functions,
- M and N are two positive martingales with $M_0 = N_0 = 1$.

In this setting, the conditional expectations $\mathbb{E}_t[D_T]$ and $\mathbb{E}_t[D_T^\Delta]$ are given by

$$\begin{aligned} \mathbb{E}_t[D_T] &= f(T) + g(T)M_t + h(T)N_t, \\ \mathbb{E}_t[D_T^\Delta] &= f^\Delta(T) + g^\Delta(T)M_t + h^\Delta(T)N_t. \end{aligned}$$

In particular, the OIS term structure is given in closed form via

$$p(t, T) = \frac{\mathbb{E}_t[D_T]}{D_t} = \frac{f(T) + g(T)M_t + h(T)N_t}{f(t) + g(t)M_t + h(t)N_t}.$$

Proceeding similarly as in Section 4.4.1, we obtain the following formulae for FRA rates, FRA spreads, and LIBOR-OIS spreads.

FRA rates, FRA spreads, and LIBOR-OIS spreads

In the two-factor multi-curve rational lognormal model, the FRA rate, FRA spread, and Δ -tenor LIBOR-OIS spread are given respectively by:

$$L^\Delta(t; T, T + \Delta) = \frac{1}{\Delta} \left(\frac{f^\Delta(T) + g^\Delta(T)M_t + h^\Delta(T)N_t}{f(T + \Delta) + g(T + \Delta)M_t + h(T + \Delta)N_t} - 1 \right), \quad (4.27)$$

$$S^\Delta(t, T) = \frac{f^\Delta(T) - f(T) + (g^\Delta(T) - g(T))M_t + (h^\Delta(T) - h(T))N_t}{\Delta (f(T + \Delta) + g(T + \Delta)M_t + h(T + \Delta)N_t)}, \quad (4.28)$$

and

$$\begin{aligned} S_{\Delta}^\Delta(t; T_0, T) \\ = \frac{\sum_{i=1}^M f^\Delta(t_{i-1}) - f(t_{i-1}) + (g^\Delta(t_{i-1}) - g(t_{i-1}))M_t + (h^\Delta(t_{i-1}) - h(t_{i-1}))N_t}{\Delta \sum_{i=1}^N [f(T_i) + g(T_i)M_t + h(T_i)N_t]}. \end{aligned} \quad (4.29)$$

Remark 4.10. If $f^\Delta(t) \geq f(t)$, $g^\Delta(t) \geq g(t)$ and $h^\Delta(t) \geq h(t)$ for all $t \geq 0$, then the FRA spreads and the LIBOR-OIS spreads generated by the two-factor rational lognormal model are positive.

In what follows, we again assume lognormal dynamics for M and N , i.e.

$$dM_t = M_t \sigma_1(t) d\tilde{W}_1(t), \quad dN_t = N_t \sigma_2(t) d\tilde{W}_2(t),$$

where \tilde{W}_1 and \tilde{W}_2 are possibly correlated Brownian motions with instantaneous correlation ρ , i.e. $\rho dt = d\tilde{W}_1(t)d\tilde{W}_2(t)$, and σ_1, σ_2 are two deterministic functions. We can represent \tilde{W}_1, \tilde{W}_2 in terms of independent Brownian motions W_1 and W_2 via

$$\tilde{W}_1(t) = W_1(t), \quad \tilde{W}_2(t) = \rho W_1(t) + \sqrt{1 - \rho^2} W_2(t).$$

Then M and N satisfy

$$M_t = \exp \left(\int_0^t \sigma_1(u) dW_1(u) - \frac{1}{2} \int_0^t \sigma_1^2(u) du \right), \quad (4.30)$$

$$N_t = \exp \left(\rho \int_0^t \sigma_2(u) dW_1(u) + \sqrt{1 - \rho^2} \int_0^t \sigma_2(u) dW_2(u) - \frac{1}{2} \int_0^t \sigma_2^2(u) du \right). \quad (4.31)$$

We now proceed to derive caplet and swaption pricing formulae.

Valuation of caplets

We consider again the caplet in Section 4.3.3. Plugging

$$\mathbb{E}_T[D_{T+\Delta}] = f(T + \Delta) + g(T + \Delta)M_T + h(T + \Delta)N_T$$

and

$$D_T^\Delta = f^\Delta(T) + g^\Delta(T)M_T + h^\Delta(T)N_T$$

into Equation (4.9), we get

$$C(t) = \frac{1}{D_t} \mathbb{E}_t \left[\left(A^\Delta(T)M_T + B^\Delta(T)N_T + C^\Delta(T) \right)^+ \right], \quad (4.32)$$

where

$$\begin{aligned} A^\Delta(T) &:= g^\Delta(T) - (1 + \Delta K)g(T + \Delta), \\ B^\Delta(T) &:= h^\Delta(T) - (1 + \Delta K)h(T + \Delta), \\ C^\Delta(T) &:= f^\Delta(T) - (1 + \Delta K)f(T + \Delta). \end{aligned}$$

Combining (4.30), (4.31), and (4.32) yields

$$C(0) = \mathbb{E} \left[\left(A^\Delta(T)e^{X - \frac{1}{2}v_1^2} + B^\Delta(T)e^{Y - \frac{1}{2}v_2^2} + C^\Delta(T) \right)^+ \right], \quad (4.33)$$

where

$$\begin{aligned} X &:= \int_0^T \sigma_1(u) dW_1(u) \sim \mathcal{N}(0, v_1^2), \\ Y &:= \rho \int_0^T \sigma_2(u) dW_1(u) + \sqrt{1 - \rho^2} \int_0^T \sigma_2(u) dW_2(u) \sim \mathcal{N}(0, v_2^2), \\ v_1^2 &:= \int_0^T \sigma_1^2(u) du, \quad v_2^2 := \int_0^T \sigma_2^2(u) du. \end{aligned}$$

Set

$$\theta := \frac{\text{Cov}[X, Y]}{\sqrt{\text{Var}[X]}\sqrt{\text{Var}[Y]}} = \frac{\rho \int_0^T \sigma_1(u)\sigma_2(u) du}{v_1 v_2}.$$

We then have $(X, Y) \sim \mathcal{N}(0, \Sigma)$, where

$$\Sigma := \begin{pmatrix} v_1^2 & \theta v_1 v_2 \\ \theta v_1 v_2 & v_2^2 \end{pmatrix}.$$

Hence, (X, Y) has the density

$$\begin{aligned} f(x, y) &= \frac{1}{2\pi v_1 v_2 \sqrt{1 - \theta^2}} \exp \left\{ -\frac{1}{2(1 - \theta^2)} \left[\left(\frac{x}{v_1}\right)^2 - 2\theta \frac{xy}{v_1 v_2} + \left(\frac{y}{v_2}\right)^2 \right] \right\} \\ &= \lambda e^E e^{Fy - Gy^2}, \end{aligned}$$

where

$$\begin{aligned} \lambda &= \frac{1}{2\pi v_1 v_2 \sqrt{1 - \theta^2}}, & F &= \frac{\theta}{1 - \theta^2} \frac{x}{v_1 v_2}, \\ E &= -\frac{1}{2(1 - \theta^2)} \frac{x^2}{v_1^2}, & G &= \frac{1}{2(1 - \theta^2) v_2^2}. \end{aligned}$$

Assuming $B^\Delta(T) \neq 0$, we consider the following possibilities:

Case 1: $B^\Delta(T) > 0$. It follows immediately from Equation (4.33) that

$$\begin{aligned} C(0) &= \int_{\mathbb{R}^2} \left(A^\Delta(T) e^{x - \frac{1}{2}v_1^2} + B^\Delta(T) e^{y - \frac{1}{2}v_2^2} + C^\Delta(T) \right)^+ f(x, y) dx dy \\ &= \int_{\mathbb{R}} \int_{\bar{y}(x)}^{+\infty} \left(A^\Delta(T) e^{x - \frac{1}{2}v_1^2} + B^\Delta(T) e^{y - \frac{1}{2}v_2^2} + C^\Delta(T) \right) f(x, y) dy dx, \end{aligned}$$

where $\bar{y}(x)$ is given by

$$\bar{y}(x) := \begin{cases} \ln \left(\frac{-C^\Delta(T) - A^\Delta(T) e^{x - \frac{1}{2}v_1^2}}{B^\Delta(T) e^{-\frac{1}{2}v_2^2}} \right) & \text{if } C^\Delta(T) + A^\Delta(T) e^{x - \frac{1}{2}v_1^2} < 0 \\ -\infty & \text{otherwise.} \end{cases}$$

We find

$$C(0) = \int_{\mathbb{R}} \left[\left(C^\Delta(T) + A^\Delta(T) e^{x - \frac{1}{2}v_1^2} \right) I_1(x) + \left(B^\Delta(T) e^{-\frac{1}{2}v_2^2} \right) I_2(x) \right] dx, \quad (4.34)$$

where

$$I_1(x) := \int_{\bar{y}(x)}^{\infty} \lambda e^E e^{Fy - Gy^2} dy, \quad I_2(x) := \int_{\bar{y}(x)}^{\infty} \lambda e^E e^{(F+1)y - Gy^2} dy.$$

Applying Lemma A.3, we obtain

$$\begin{aligned} I_1(x) &= \lambda \frac{\sqrt{\pi}}{\sqrt{G}} e^{E + \frac{F^2}{4G}} \left[1 - \Phi \left(\bar{y}(x) \sqrt{2G} - \frac{F}{\sqrt{2G}} \right) \right], \\ I_2(x) &= \lambda \frac{\sqrt{\pi}}{\sqrt{G}} e^{E + \frac{(F+1)^2}{4G}} \left[1 - \Phi \left(\bar{y}(x) \sqrt{2G} - \frac{F+1}{\sqrt{2G}} \right) \right]. \end{aligned}$$

Here, as above, Φ is the distribution function of the standard normal distribution. By noting that

$$\begin{aligned} \lambda \frac{\sqrt{\pi}}{\sqrt{G}} &= \frac{1}{v_1 \sqrt{2\pi}}, & E + \frac{F^2}{4G} &= -\frac{1}{2} \left(\frac{x}{v_1} \right)^2, \\ \frac{F}{\sqrt{2G}} &= \frac{\theta x}{v_1 \sqrt{1-\theta^2}}, & \sqrt{2G} &= \frac{1}{v_2 \sqrt{1-\theta^2}}, \end{aligned}$$

and

$$\frac{2F+1}{4G} = \frac{(1-\theta^2)v_2^2}{2} + \frac{v_2\theta x}{v_1},$$

we further have

$$I_1(x) = \frac{e^{-\frac{1}{2}\left(\frac{x}{v_1}\right)^2}}{v_1 \sqrt{2\pi}} \Phi \left(\frac{\theta x}{v_1 \sqrt{1-\theta^2}} - \frac{\bar{y}(x)}{v_2 \sqrt{1-\theta^2}} \right), \quad (4.35)$$

$$I_2(x) = \frac{e^{-\frac{1}{2}\left(\frac{x}{v_1}\right)^2}}{v_1 \sqrt{2\pi}} e^{\frac{(1-\theta^2)v_2^2}{2} + \frac{v_2\theta x}{v_1}} \Phi \left(\frac{\theta x}{v_1 \sqrt{1-\theta^2}} - \frac{\bar{y}(x)}{v_2 \sqrt{1-\theta^2}} + v_2 \sqrt{1-\theta^2} \right). \quad (4.36)$$

Using (4.35) and (4.36) in Equation (4.34), we arrive at the following formula:

$$\begin{aligned} C(0) &= \int_{-\infty}^{+\infty} \frac{e^{-\frac{1}{2}\left(\frac{x}{v_1}\right)^2}}{v_1 \sqrt{2\pi}} \left[\left(A^\Delta(T) e^{x - \frac{v_1^2}{2}} + C^\Delta(T) \right) \Phi(\bar{h}_1(x)) \right. \\ &\quad \left. + B^\Delta(T) e^{\frac{v_2\theta x}{v_1} - \frac{\theta^2 v_2^2}{2}} \Phi(\bar{h}_2(x)) \right] dx, \end{aligned}$$

where

$$\bar{h}_1(x) := \frac{\theta x}{v_1 \sqrt{1-\theta^2}} - \frac{\bar{y}(x)}{v_2 \sqrt{1-\theta^2}}, \quad \bar{h}_2(x) := \bar{h}_1(x) + v_2 \sqrt{1-\theta^2}.$$

Case 2: $B^\Delta(T) < 0$. In this case we obtain from Equation (4.33) that

$$\begin{aligned} C(0) &= \int_{\mathbb{R}^2} \left(A^\Delta(T)e^{x-\frac{1}{2}v_1^2} + B^\Delta(T)e^{y-\frac{1}{2}v_2^2} + C^\Delta(T) \right)^+ f(x,y) dx dy \\ &= \int_{\mathbb{R}} \int_{-\infty}^{\tilde{y}(x)} \left(A^\Delta(T)e^{x-\frac{1}{2}v_1^2} + B^\Delta(T)e^{y-\frac{1}{2}v_2^2} + C^\Delta(T) \right) f(x,y) dy dx, \end{aligned}$$

where $\tilde{y}(x)$ is given by

$$\tilde{y}(x) := \begin{cases} \ln \left(\frac{-C^\Delta(T) - A^\Delta(T)e^{x-\frac{1}{2}v_1^2}}{B^\Delta(T)e^{-\frac{1}{2}v_2^2}} \right) & \text{if } C^\Delta(T) + A^\Delta(T)e^{x-\frac{1}{2}v_1^2} > 0 \\ -\infty & \text{otherwise.} \end{cases}$$

Similarly as above, we have

$$C(0) = \int_{\mathbb{R}} \left[\left(C^\Delta(T) + A^\Delta(T)e^{x-\frac{1}{2}v_1^2} \right) J_1(x) + \left(B^\Delta(T)e^{-\frac{1}{2}v_2^2} \right) J_2(x) \right] dx, \quad (4.37)$$

where

$$J_1(x) := \int_{-\infty}^{\tilde{y}(x)} \lambda e^E e^{Fy-Gy^2} dy, \quad J_2(x) := \int_{-\infty}^{\tilde{y}(x)} \lambda e^E e^{(F+1)y-Gy^2} dy.$$

Applying Lemma A.3 and following similar steps as in the above case, we further obtain

$$\begin{aligned} J_1(x) &= \frac{e^{-\frac{1}{2}\left(\frac{x}{v_1}\right)^2}}{v_1\sqrt{2\pi}} \Phi \left(\frac{\tilde{y}(x)}{v_2\sqrt{1-\theta^2}} - \frac{\theta x}{v_1\sqrt{1-\theta^2}} \right), \quad (4.38) \\ J_2(x) &= \frac{e^{-\frac{1}{2}\left(\frac{x}{v_1}\right)^2}}{v_1\sqrt{2\pi}} e^{\frac{(1-\theta^2)v_2^2}{2} + \frac{v_2\theta x}{v_1}} \Phi \left(\frac{\tilde{y}(x)}{v_2\sqrt{1-\theta^2}} - \frac{\theta x}{v_1\sqrt{1-\theta^2}} - v_2\sqrt{1-\theta^2} \right). \quad (4.39) \end{aligned}$$

Plugging Equations (4.38) and (4.39) into Equation (4.37), we obtain the following caplet formula:

$$\begin{aligned} C(0) &= \int_{-\infty}^{+\infty} \frac{e^{-\frac{1}{2}\left(\frac{x}{v_1}\right)^2}}{v_1\sqrt{2\pi}} \left[\left(A^\Delta(T)e^{x-\frac{1}{2}v_1^2} + C^\Delta(T) \right) \Phi(\tilde{h}_1(x)) \right. \\ &\quad \left. + B^\Delta(T)e^{\frac{v_2\theta x}{v_1} - \frac{\theta^2 v_2^2}{2}} \Phi(\tilde{h}_2(x)) \right] dx, \end{aligned}$$

where

$$\tilde{h}_1(x) := \frac{\tilde{y}(x)}{v_2\sqrt{1-\theta^2}} - \frac{\theta x}{v_1\sqrt{1-\theta^2}}, \quad \tilde{h}_2(x) := \tilde{h}_1(x) - v_2\sqrt{1-\theta^2}.$$

As a combination of the above two cases, we have the following result:

Theorem 4.11 (Caplets in the two-factor rational lognormal model).

In the two-factor multi-curve rational lognormal model, the price at time $t = 0$ of the caplet specified in Section 4.3.3 is given by

$$\begin{aligned} C(0) = \int_{-\infty}^{+\infty} \frac{e^{-\frac{1}{2}\left(\frac{x}{v_1}\right)^2}}{v_1\sqrt{2\pi}} & \left[\left(A^\Delta(T)e^{x-\frac{v_1^2}{2}} + C^\Delta(T) \right) \Phi(wh_1(x)) \right. \\ & \left. + B^\Delta(T)e^{\frac{v_2\theta x}{v_1} - \frac{\theta^2 v_2^2}{2}} \Phi(wh_2(x)) \right] dx, \end{aligned} \quad (4.40)$$

where we assume $B^\Delta(T) \neq 0$ and define

$$\begin{aligned} v_1 &= \sqrt{\int_0^T \sigma_1^2(u) du}, \quad v_2 = \sqrt{\int_0^T \sigma_2^2(u) du}, \quad \theta = \frac{\rho \int_0^T \sigma_1(u)\sigma_2(u) du}{v_1 v_2}, \\ h_1(x) &= \frac{\theta x}{v_1\sqrt{1-\theta^2}} - \frac{y^*(x)}{v_2\sqrt{1-\theta^2}}, \quad h_2(x) = h_1(x) + v_2\sqrt{1-\theta^2}, \\ y^*(x) &= \begin{cases} \ln \left(-\frac{C^\Delta(T) + A^\Delta(T)e^{x-\frac{1}{2}v_1^2}}{B^\Delta(T)e^{-\frac{1}{2}v_2^2}} \right) & \text{if } B^\Delta(T) \left(C^\Delta(T) + A^\Delta(T)e^{x-\frac{1}{2}v_1^2} \right) < 0 \\ -\infty & \text{otherwise,} \end{cases} \end{aligned}$$

and

$$w = \frac{B^\Delta(T)}{|B^\Delta(T)|} = \begin{cases} 1 & \text{if } B^\Delta(T) > 0 \\ -1 & \text{if } B^\Delta(T) < 0. \end{cases}$$

Remark 4.12. If $B^\Delta(T) = 0$, the price of the caplet becomes

$$C(0) = \mathbb{E}[(A^\Delta(T)M_T + C^\Delta(T))^+],$$

where the martingale M is given by (4.30). Then there are four possible situations:

Case 1: $A^\Delta(T) < 0$ and $C^\Delta(T) < 0$. Then trivially $(A^\Delta(T)M_T + C^\Delta(T))^+ = 0$ and therefore $C(0) = 0$.

Case 2: $A^\Delta(T) > 0$ and $C^\Delta(T) > 0$. In this case we have

$$(A^\Delta(T)M_T + C^\Delta(T))^+ = A^\Delta(T)M_T + C^\Delta(T)$$

and thus

$$\begin{aligned} C(0) &= \mathbb{E}[A^\Delta(T)M_T + C^\Delta(T)] = A^\Delta(T)M_0 + C^\Delta(T) \\ &= A^\Delta(T) + C^\Delta(T). \end{aligned}$$

Case 3: $A^\Delta(T) > 0$ and $C^\Delta(T) < 0$. Applying Lemma A.4 with $a = A^\Delta(T)$, $b = -C^\Delta(T)$ and $\sigma = v_1 = \sqrt{\int_0^T \sigma_1^2(u)du}$, we obtain

$$C(0) = A^\Delta(T)\Phi(d_1^\Delta) + C^\Delta(T)\Phi(d_2^\Delta),$$

where

$$d_{1,2}^\Delta := \frac{1}{v_1} \left(\ln \left(-\frac{A^\Delta(T)}{C^\Delta(T)} \right) \pm \frac{v_1^2}{2} \right).$$

Case 4: $A^\Delta(T) < 0$ and $C^\Delta(T) > 0$. Applying Lemma A.4 again with $a = -A^\Delta(T)$, $b = C^\Delta(T)$ and $\sigma = v_1$, we get

$$\mathbb{E}[(-A^\Delta(T)M_T - C^\Delta(T))^+] = -A^\Delta(T)\Phi(d_1^\Delta) - C^\Delta(T)\Phi(d_2^\Delta).$$

Using the identity $(-x)^+ = x^+ - x$ we further have

$$\begin{aligned} C(0) &= \mathbb{E}[(A^\Delta(T)M_T + C^\Delta(T))^+] \\ &= \mathbb{E}[(-A^\Delta(T)M_T - C^\Delta(T))^+ + A^\Delta(T)M_T + C^\Delta(T)] \\ &= A^\Delta(T)(1 - \Phi(d_1^\Delta)) + C^\Delta(T)(1 - \Phi(d_2^\Delta)) \\ &= A^\Delta(T)\Phi(-d_1^\Delta) + C^\Delta(T)\Phi(-d_2^\Delta). \end{aligned}$$

Valuation of swaptions

Finally, we reconsider the swaption given in Section 4.3.3. With the two-factor specification of D and D^Δ , it is not difficult to verify that the time- t price of the swaption is given by

$$S(t) = \frac{1}{D_t} \mathbb{E}_t \left[\left(O^\Delta(T)M_T + P^\Delta(T)N_T + Q^\Delta(T) \right)^+ \right],$$

where

$$\begin{aligned} O^\Delta(T) &:= \sum_{i=1}^N g^\Delta(T_{i-1}) - (1 + \Delta K)g(T_i), \\ P^\Delta(T) &:= \sum_{i=1}^N h^\Delta(T_{i-1}) - (1 + \Delta K)h(T_i), \\ Q^\Delta(T) &:= \sum_{i=1}^N f^\Delta(T_{i-1}) - (1 + \Delta K)f(T_i). \end{aligned}$$

Note that $S(t)$ has the same form as $C(t)$. Thus, similarly as above, we obtain

Theorem 4.13 (Swaptions in the two-factor rational lognormal model).

The fair price at time $t = 0$ of the swaption specified in Section 4.3.3 is given by

$$\begin{aligned} S(0) = \int_{-\infty}^{+\infty} \frac{e^{-\frac{1}{2}\left(\frac{x}{v_1}\right)^2}}{v_1\sqrt{2\pi}} &\left[\left(O^\Delta(T)e^{x-\frac{v_1^2}{2}} + Q^\Delta(T) \right) \Phi(w\ell_1(x)) \right. \\ &\left. + P^\Delta(T)e^{\frac{v_2\theta x}{v_1} - \frac{\theta^2 v_2^2}{2}} \Phi(w\ell_2(x)) \right] dx, \end{aligned} \quad (4.41)$$

where we assume $P^\Delta(T) \neq 0$ and set

$$\begin{aligned} v_1 &= \sqrt{\int_0^T \sigma_1^2(u)du}, \quad v_2 = \sqrt{\int_0^T \sigma_2^2(u)du}, \quad \theta = \frac{\rho \int_0^T \sigma_1(u)\sigma_2(u)du}{v_1 v_2}, \\ \ell_1(x) &= \frac{\theta x}{v_1 \sqrt{1 - \theta^2}} - \frac{\bar{z}(x)}{v_2 \sqrt{1 - \theta^2}}, \quad \ell_2(x) = \ell_1(x) + v_2 \sqrt{1 - \theta^2}, \\ \bar{z}(x) &= \begin{cases} \ln \left(-\frac{Q^\Delta(T) + O^\Delta(T)e^{x-\frac{1}{2}v_1^2}}{P^\Delta(T)e^{-\frac{1}{2}v_2^2}} \right) & \text{if } P^\Delta(T) \left(Q^\Delta(T) + O^\Delta(T)e^{x-\frac{1}{2}v_1^2} \right) < 0 \\ -\infty & \text{otherwise,} \end{cases} \end{aligned}$$

and

$$w = \frac{P^\Delta(T)}{|P^\Delta(T)|} = \begin{cases} 1 & \text{if } P^\Delta(T) > 0 \\ -1 & \text{if } P^\Delta(T) < 0. \end{cases}$$

Remark 4.14. In the case where $P^\Delta(T) = 0$, similarly as in Remark 4.12, we obtain:

$$S(0) = \begin{cases} 0 & \text{if } O^\Delta(T) < 0, \quad Q^\Delta(T) < 0, \\ O^\Delta(T) + Q^\Delta(T) & \text{if } O^\Delta(T) > 0, \quad Q^\Delta(T) > 0, \\ O^\Delta(T)\Phi(\tilde{d}_1^\Delta) + Q^\Delta(T)\Phi(\tilde{d}_2^\Delta) & \text{if } O^\Delta(T) > 0, \quad Q^\Delta(T) < 0, \\ O^\Delta(T)\Phi(-\tilde{d}_1^\Delta) + Q^\Delta(T)\Phi(-\tilde{d}_2^\Delta) & \text{otherwise,} \end{cases}$$

where

$$\tilde{d}_{1,2}^\Delta := \frac{1}{v_1} \left(\ln \left(-\frac{O^\Delta(T)}{Q^\Delta(T)} \right) \pm \frac{v_1^2}{2} \right).$$

4.5 Multi-Curve Affine Rational Models

We present two explicit specifications of the general multi-curve potential model based on the Markov affine process considered in Chapter 3. The first specification is an extension of *the affine rational potential model* established in Chapter 3 to the multi-curve framework. We show that the analytical tractability of the single-curve model carries over to this multi-curve extension. In the second specific multi-curve potential model, we extend the single-curve rational model proposed by Filipović et al. (2015) to the multi-curve setting, where the original underlying process is replaced by the Markov affine dynamics. We show that the analytical formulae for linear products, and caps and swaptions are obtained in this model. Upon appropriate selection of relevant parameters we are able to obtain positive stochastic basis spreads in both models. Moreover, while the first concrete model guarantees interest rate positivity, which is consistent with typical market observations, the second model can produce negative OIS rates, which is in line with the current observations in the European- and the Japanese markets. In this section we work exclusively in the context of Chapter 3.

4.5.1 Multi-curve affine rational potential model

We specify the state-price deflators D and D^Δ as follows:

$$\begin{aligned} D_t &= c + \phi_{T^\infty-t}(u) + \langle \psi_{T^\infty-t}(u), X_t \rangle, \\ D_t^\Delta &= c_\Delta + \phi_{T^\infty-t}(u^\Delta) + \langle \psi_{T^\infty-t}(u^\Delta), X_t \rangle, \end{aligned}$$

where $c, c_\Delta \in \mathbb{R}_{>0}$, $u, u^\Delta \in \mathcal{J} \cap E$, and the underlying state process X is the affine Markov process considered in Chapter 3; recall that $E := \mathbb{R}_{\geq 0}^d$ and \mathcal{J} is defined by (3.2).

Remark 4.15. One sufficient condition for positive basis spreads in the model is that $D_t^\Delta \geq D_t$ for all $t \in [0, T^\infty]$; see Theorem 4.2 and Theorem 4.5. This condition is fulfilled if $c_\Delta \geq c$ and $u^\Delta \geq u$. Indeed, from Lemma 3.4 it then follows that

$$\phi_{T^\infty-t}(u^\Delta) \geq \phi_{T^\infty-t}(u) \text{ and } \psi_{T^\infty-t}(u^\Delta) \geq \psi_{T^\infty-t}(u), \text{ for all } t \in [0, T^\infty].$$

On the other hand, the process X takes values in $E = \mathbb{R}_{\geq 0}^d$. We thus obtain

$$D_t^\Delta = c_\Delta + \phi_{T^\infty-t}(u^\Delta) + \langle \psi_{T^\infty-t}(u^\Delta), X_t \rangle \geq c + \phi_{T^\infty-t}(u) + \langle \psi_{T^\infty-t}(u), X_t \rangle = D_t.$$

Therefore, the multi-curve affine rational potential model can produce positive stochastic basis spreads.

Lemma 3.6 implies that the conditional expectations $\mathbb{E}_t[D_T]$ and $\mathbb{E}_t[D_T^\Delta]$ have the following closed-form expressions:

$$\begin{aligned} \mathbb{E}_t[D_T] &= c + \phi_{T^\infty-T}(u) + \left\langle \psi_{T^\infty-T}(u), \frac{\partial \phi_{T-t}(0)}{\partial u} \right\rangle \\ &\quad + \left\langle \frac{\partial \psi_{T-t}(0)}{\partial u} \cdot \psi_{T^\infty-T}(u), X_t \right\rangle, \end{aligned} \tag{4.42}$$

and

$$\begin{aligned} \mathbb{E}_t[D_T^\Delta] &= c_\Delta + \phi_{T^\infty-T}(u^\Delta) + \left\langle \psi_{T^\infty-T}(u^\Delta), \frac{\partial \phi_{T-t}(0)}{\partial u} \right\rangle \\ &\quad + \left\langle \frac{\partial \psi_{T-t}(0)}{\partial u} \cdot \psi_{T^\infty-T}(u^\Delta), X_t \right\rangle. \end{aligned} \tag{4.43}$$

OIS discount curve

Using Equation (4.42), we find that the OIS discount curve in the multi-curve affine rational model is given, as in the single-curve model, by

$$\begin{aligned} p(t, T) &= \frac{\mathbb{E}_t[D_T]}{D_t} \\ &= \frac{\langle A(t, T), X_t \rangle + B(t, T)}{\langle A(t, t), X_t \rangle + B(t, t)}, \quad t \in [0, T], \end{aligned} \quad (4.44)$$

where

$$A(t, T) := \frac{\partial \psi_{T-t}(0)}{\partial u} \cdot \psi_{T^\infty - T}(u), \quad (4.45)$$

$$B(t, T) := c + \phi_{T^\infty - T}(u) + \left\langle \psi_{T^\infty - T}(u), \frac{\partial \phi_{T-t}(0)}{\partial u} \right\rangle, \quad (4.46)$$

and $\frac{\partial \psi_{T-t}(0)}{\partial u}$, $\frac{\partial \phi_{T-t}(0)}{\partial u}$ are defined in Lemma 3.6.

Remark 4.16. Note that for any fixed $t \in [0, T^\infty]$, the discount curve $p(t, \cdot)$ is a decreasing function since, by construction, the state-price deflator D is a positive supermartingale; see Theorem 3.2. Thus, the multi-curve affine rational model guarantees interest rate positivity.

FRA rates and FRA spreads

We consider the FRA rate $L^\Delta(t; T, T + \Delta)$ of a forward rate agreement for the future time interval $[T, T + \Delta]$. Substituting Equations (4.42) and (4.43) in the general formula (4.5), we find that the multiplicative FRA spread $S_m^\Delta(t, T)$ is given by

$$\begin{aligned} S_m^\Delta(t, T) &= \frac{\mathbb{E}_t[D_T^\Delta]}{\mathbb{E}_t[D_T]} \\ &= \frac{\langle A^\Delta(t, T), X_t \rangle + B^\Delta(t, T)}{\langle A(t, T), X_t \rangle + B(t, T)}, \end{aligned} \quad (4.47)$$

where

$$A^\Delta(t, T) := \frac{\partial \psi_{T-t}(0)}{\partial u} \cdot \psi_{T^\infty - T}(u^\Delta), \quad (4.48)$$

$$B^\Delta(t, T) := c_\Delta + \phi_{T^\infty - T}(u^\Delta) + \left\langle \psi_{T^\infty - T}(u^\Delta), \frac{\partial \phi_{T-t}(0)}{\partial u} \right\rangle, \quad (4.49)$$

and $A(t, T)$ and $B(t, T)$ are given in (4.45) and (4.46). Using Equation (4.3), we then obtain the following closed-form formula for the FRA rate $L^\Delta(t; T, T + \Delta)$:

$$\begin{aligned} L^\Delta(t; T, T + \Delta) &= \frac{1}{\Delta} \left(\frac{p(t, T)}{p(t, T + \Delta)} S_m^\Delta(t, T) - 1 \right) \\ &= \frac{1}{\Delta} \left(\frac{\langle A^\Delta(t, T), X_t \rangle + B^\Delta(t, T)}{\langle A(t, T + \Delta), X_t \rangle + B(t, T + \Delta)} - 1 \right). \end{aligned} \quad (4.50)$$

Remark 4.17. Having closed-form expressions for OIS discount curve and FRA rates, we easily obtain closed-form formula for LIBOR-OIS spreads using Theorem 4.5.

We now focus on the derivation of pricing formulae for caplets and swaptions.

Caplets

We consider the caplet in Section 4.3.3. Substituting

$$\begin{aligned} D_T^\Delta &= c_\Delta + \phi_{T^\infty - T}(u^\Delta) + \langle \psi_{T^\infty - T}(u^\Delta), X_T \rangle, \\ \mathbb{E}_T[D_{T+\Delta}] &= c + \phi_{T^\infty - T - \Delta}(u) + \left\langle \psi_{T^\infty - T - \Delta}(u), \frac{\partial \phi_\Delta(0)}{\partial u} \right\rangle \\ &\quad + \left\langle \frac{\partial \psi_\Delta(0)}{\partial u} \cdot \psi_{T^\infty - T - \Delta}(u), X_T \right\rangle \end{aligned}$$

in the general caplet pricing formula (4.9) we obtain the time t -price, $C(t)$, of the caplet:

$$C(t) = \frac{1}{D_t} \mathbb{E}_t [(\langle a, X_T \rangle + b)^+], \quad (4.51)$$

where

$$a := \psi_{T^\infty - T}(u^\Delta) - (1 + \Delta K) \frac{\partial \psi_\Delta(0)}{\partial u} \cdot \psi_{T^\infty - T - \Delta}(u), \quad (4.52a)$$

$$\begin{aligned} b &:= c_\Delta - (1 + \Delta K)c + \phi_{T^\infty - T}(u^\Delta) - (1 + \Delta K)\phi_{T^\infty - T - \Delta}(u) \\ &\quad - (1 + \Delta K) \left\langle \psi_{T^\infty - T - \Delta}(u), \frac{\partial \phi_\Delta(0)}{\partial u} \right\rangle. \end{aligned} \quad (4.52b)$$

We are now in a position to derive caplet pricing formula:

Proposition 4.18 (Caplets in the multi-curve affine rational potential model). *The time- t price of the Δ -tenor caplet considered in Section 4.3.3 is provided by*

$$C(t) = \frac{\int_0^\infty \Re \left[\frac{\exp(b(\mu+iy) + \phi_{T-t}((\mu+iy)a) + \langle \psi_{T-t}((\mu+iy)a), X_t \rangle)}{(\mu+iy)^2} \right] dy}{\pi (c + \phi_{T-t}(u) + \langle \psi_{T-t}(u), X_t \rangle)}, \quad (4.53)$$

where $\mu > 0$ is a real number such that $\mu a \in \mathcal{J}$, and a, b are given in (4.52).

Proof. It follows from the condition $\mu a \in \mathcal{J}$ that the conditional expectation $\mathbb{E}_t[e^{\langle (\mu+iy)a, X_T \rangle}]$ exists. Moreover, it is given by

$$\mathbb{E}_t \left[e^{\langle (\mu+iy)a, X_T \rangle} \right] = \exp(\phi_{T-t}((\mu+iy)a) + \langle \psi_{T-t}((\mu+iy)a), X_t \rangle).$$

Hence, we obtain

$$\begin{aligned} & \mathbb{E}_t \left[e^{(\mu+iy)(\langle a, X_T \rangle + b)} \right] \\ &= \exp((\mu+iy)b + \phi_{T-t}((\mu+iy)a) + \langle \psi_{T-t}((\mu+iy)a), X_t \rangle). \end{aligned} \quad (4.54)$$

The pricing formula (4.53) now follows readily from Equations (4.51), (4.54), and Lemma A.2. \square

Swaptions

We consider the swaption considered in Section 4.3.3. From the general pricing formula (4.10), it follows that the time t -price of the swaption is given by

$$S(t) = \frac{1}{D_t} \mathbb{E}_t \left[(\langle \alpha, X_T \rangle + \beta)^+ \right], \quad (4.55)$$

where

$$\alpha := \sum_{i=1}^N \frac{\partial \psi_{T_{i-1}-T}(0)}{\partial u} \cdot \psi_{T^\infty-T_{i-1}}(u^\Delta) - (1 + \Delta K) \frac{\partial \psi_{T_i-T}(0)}{\partial u} \cdot \psi_{T^\infty-T_i}(u), \quad (4.56a)$$

$$\begin{aligned} \beta := & N(c_\Delta - (1 + \Delta K)c) + \sum_{i=1}^N \phi_{T^\infty-T_{i-1}}(u^\Delta) - (1 + \Delta K)\phi_{T^\infty-T_i}(u) \\ & + \left\langle \psi_{T^\infty-T_{i-1}}(u^\Delta), \frac{\partial \phi_{T_{i-1}-T}(0)}{\partial u} \right\rangle - (1 + \Delta K) \left\langle \psi_{T^\infty-T_i}(u), \frac{\partial \phi_{T_i-T}(0)}{\partial u} \right\rangle. \end{aligned} \quad (4.56b)$$

Similarly as in the above derivation of caplet formula, we obtain the following semi-closed form formula for the swaption:

Proposition 4.19 (Swaptions in the multi-curve affine rational potential model). *The time- t price of the swaption in Section 4.3.3 is given by*

$$S(t) = \frac{\int_0^\infty \Re \left[\frac{\exp(\beta(\mu+iy) + \phi_{T-t}((\mu+iy)\alpha) + \langle \psi_{T-t}((\mu+iy)\alpha), X_t \rangle)}{(\mu+iy)^2} \right] dy}{\pi (c + \phi_{T^\infty-t}(u) + \langle \psi_{T^\infty-t}(u), X_t \rangle)}, \quad (4.57)$$

where $\mu > 0$ is such that $\mu\alpha \in \mathcal{I}$, and α, β are defined in (4.56).

4.5.2 An affine rational model with negative rates and positive spreads

Following Filipović et al. (2015), we define the state-price deflators D and D^Δ as follows:

$$\begin{aligned} D_t &= e^{-\alpha t} (\langle \gamma, X_t \rangle + c), \\ D_t^\Delta &= e^{-\alpha^\Delta t} (\langle \gamma^\Delta, X_t \rangle + c^\Delta), \end{aligned}$$

where $\alpha, \alpha^\Delta \in \mathbb{R}, \gamma, \gamma^\Delta \in E = \mathbb{R}_{\geq 0}^d, c, c^\Delta \in \mathbb{R}_{> 0}$ are parameters of the model and X is the affine process considered in Chapter 3. Applying Lemma 3.6, we have

$$\begin{aligned} \mathbb{E}_t[D_T] &= \mathbb{E}_t \left[e^{-\alpha T} (\langle \gamma, X_T \rangle + c) \right] \\ &= e^{-\alpha T} (\langle \gamma, \mathbb{E}_t[X_T] \rangle + c) \\ &= e^{-\alpha T} \left(\left\langle \gamma, \frac{\partial \phi_{T-t}(0)}{\partial u} + \frac{\partial \psi_{T-t}(0)}{\partial u} \cdot X_t \right\rangle + c \right) \\ &= e^{-\alpha T} \left(\left\langle \frac{\partial \psi_{T-t}(0)}{\partial u} \cdot \gamma, X_t \right\rangle + \left\langle \frac{\partial \phi_{T-t}(0)}{\partial u}, \gamma \right\rangle + c \right). \end{aligned} \quad (4.58)$$

Similarly, we have

$$\mathbb{E}_t[D^\Delta] = e^{-\alpha^\Delta T} \left(\left\langle \frac{\partial \psi_{T-t}(0)}{\partial u} \cdot \gamma^\Delta, X_t \right\rangle + \left\langle \frac{\partial \phi_{T-t}(0)}{\partial u}, \gamma^\Delta \right\rangle + c^\Delta \right). \quad (4.59)$$

OIS-discount curve

OIS-discount curve is given in closed-form expression:

$$\begin{aligned} p(t, T) &= \frac{\mathbb{E}_t[D_T]}{D_t} \\ &= e^{-\alpha(T-t)} \frac{\left\langle \frac{\partial \psi_{T-t}(0)}{\partial u} \cdot \gamma, X_t \right\rangle + \left\langle \frac{\partial \phi_{T-t}(0)}{\partial u}, \gamma \right\rangle + c}{\langle \gamma, X_t \rangle + c} \end{aligned} \quad (4.60)$$

Using the formula $r_t = -\partial_T p(t, T)|_{T=t}$, we find that the short rate is given by

$$r_t = \alpha - \frac{\left\langle \frac{\partial^2 \psi_{T-t}(0)}{\partial u \partial T} \Big|_{T=t} \cdot \gamma, X_t \right\rangle + \left\langle \gamma, \frac{\partial^2 \phi_{T-t}(0)}{\partial u \partial T} \Big|_{T=t} \right\rangle}{\langle \gamma, X_t \rangle + c}. \quad (4.61)$$

From this we see that the role of the parameter α is to control the level of the short rate. For sufficiently small α , the short rate can take negative values. This is also clear from the fact that D is not a supermartingale; see Proposition 2.11.

FRA rate and spreads

We next derive closed-form expressions for FRA rates and FRA spreads. From Equations (4.58) and (4.59), we obtain the following explicit formula for the

multiplicative FRA spread:

$$\begin{aligned} S_m^\Delta(t, T) &= \frac{\mathbb{E}_t[D_T^\Delta]}{\mathbb{E}_t[D_T]} \\ &= e^{-(\alpha^\Delta - \alpha)T} \frac{\left\langle \frac{\partial \psi_{T-t}(0)}{\partial u} \cdot \gamma^\Delta, X_t \right\rangle + \left\langle \frac{\partial \phi_{T-t}(0)}{\partial u}, \gamma^\Delta \right\rangle + c^\Delta}{\left\langle \frac{\partial \psi_{T-t}(0)}{\partial u} \cdot \gamma, X_t \right\rangle + \left\langle \frac{\partial \phi_{T-t}(0)}{\partial u}, \gamma \right\rangle + c}. \end{aligned}$$

Then, a closed-form formula for the FRA rate $L^\Delta(t; T, T + \Delta)$ follows immediately:

$$\begin{aligned} L^\Delta(t; T, T + \Delta) &= \frac{1}{\Delta} \left(\frac{p(t, T)}{p(t, T + \Delta)} S_m^\Delta(t, T) - 1 \right) \\ &= \frac{1}{\Delta} \left(e^{\alpha^\Delta - (\alpha^\Delta - \alpha)T} \frac{\left\langle \frac{\partial \psi_{T-t}(0)}{\partial u} \cdot \gamma^\Delta, X_t \right\rangle + \left\langle \frac{\partial \phi_{T-t}(0)}{\partial u}, \gamma^\Delta \right\rangle + c^\Delta}{\left\langle \frac{\partial \psi_{T+\Delta-t}(0)}{\partial u} \cdot \gamma, X_t \right\rangle + \left\langle \frac{\partial \phi_{T+\Delta-t}(0)}{\partial u}, \gamma \right\rangle + c} - 1 \right). \end{aligned}$$

The following proposition shows that, under certain condition, our model can guarantee positivity of basis spreads.

Proposition 4.20. *Suppose that the parameters $\alpha, \alpha^\Delta, \gamma, \gamma^\Delta, c, c^\Delta$ satisfy the condition:*

$$\alpha^\Delta \leq \alpha, \quad \gamma^\Delta \geq \gamma, \quad \text{and} \quad c^\Delta \geq c, \quad (4.62)$$

where the second inequality is interpreted component-wise. Then the FRA- and LIBOR-OIS spreads in the model are positive.

Proof. Under condition (4.62), it is obvious that the state-price deflators D and D^Δ satisfy $D_t^\Delta \geq D_t$ for all $t \geq 0$. The claim then follows readily from Theorem 4.8. \square

In the following we show that semi-closed form expressions for caps and swaptions are also obtained easily in this multi-curve model.

Valuation of Caplets

We consider again the caplet in Section 4.3.3. Substituting for

$$D_T^\Delta = e^{-\alpha^\Delta T} (\langle \gamma^\Delta, X_T \rangle + c^\Delta),$$

and

$$\mathbb{E}_T[D_{T+\Delta}] = e^{-\alpha(T+\Delta)} \left(\left\langle \frac{\partial \psi_\Delta(0)}{\partial u} \cdot \gamma, X_T \right\rangle + \left\langle \frac{\partial \phi_\Delta(0)}{\partial u}, \gamma \right\rangle + c \right)$$

in the general formula (4.9), we find that the time t -price of the caplet is given by

$$C(t) = \frac{1}{D_t} \mathbb{E}_t [(\langle a, X_T \rangle + b)^+],$$

where

$$a := e^{-\alpha^\Delta T} \gamma^\Delta - (1 + \Delta K) e^{-\alpha(T+\Delta)} \frac{\partial \psi_\Delta(0)}{\partial u} \cdot \gamma, \quad (4.63a)$$

$$b := e^{-\alpha^\Delta T} c^\Delta - (1 + \Delta K) e^{-\alpha(T+\Delta)} \left(\left\langle \frac{\partial \phi_\Delta(0)}{\partial u}, \gamma \right\rangle + c \right). \quad (4.63b)$$

We are now in exactly the same situation as in the valuation of caplets in the multi-curve affine rational potential model. Therefore, we obtain immediately the following result:

Proposition 4.21. *The time t -price of the caplet in our model is given by the following semi-closed form formula:*

$$C(t) = \frac{\int_0^\infty \Re \left[\frac{\exp(b(\mu+iy) + \phi_{T-t}((\mu+iy)a) + \langle \psi_{T-t}((\mu+iy)a), X_t \rangle)}{(\mu+iy)^2} \right] dy}{\pi e^{-\alpha t} (\langle \gamma, X_t \rangle + c)},$$

where $\mu > 0$ satisfies $\mu a \in \mathcal{J}$ and a, b are given in (4.63).

Valuation of Swaptions

We consider the swaption in Section 4.3.3. By plugging

$$\mathbb{E}_T[D_{T_i-1}^\Delta] = e^{-\alpha^\Delta T_{i-1}} \left(\left\langle \frac{\partial \psi_{T_{i-1}-T}(0)}{\partial u} \cdot \gamma^\Delta, X_T \right\rangle + \left\langle \frac{\partial \phi_{T_{i-1}-T}(0)}{\partial u}, \gamma^\Delta \right\rangle + c^\Delta \right),$$

and

$$\mathbb{E}_T[D_{T_i}] = e^{-\alpha T_i} \left(\left\langle \frac{\partial \psi_{T_i-T}(0)}{\partial u} \cdot \gamma, X_T \right\rangle + \left\langle \frac{\partial \phi_{T_i-T}(0)}{\partial u}, \gamma \right\rangle + c \right)$$

into the formula (4.10), we obtain the time t -price of the swaption:

$$S(t) = \frac{1}{D_t} \mathbb{E}_t [(\langle A, X_T \rangle + B)^+],$$

where

$$A := \sum_{i=1}^N e^{-\alpha \Delta T_{i-1}} \frac{\partial \psi_{T_{i-1}-T}(0)}{\partial u} \cdot \gamma^\Delta - (1 + \Delta K) e^{-\alpha T_i} \frac{\partial \psi_{T_i-T}(0)}{\partial u} \cdot \gamma, \quad (4.64a)$$

$$B := \sum_{i=1}^N e^{-\alpha \Delta T_{i-1}} \left(\left\langle \frac{\partial \phi_{T_{i-1}-T}(0)}{\partial u}, \gamma^\Delta \right\rangle + c^\Delta \right) - (1 + \Delta K) e^{-\alpha T_i} \left(\left\langle \frac{\partial \phi_{T_i-T}(0)}{\partial u}, \gamma \right\rangle + c \right). \quad (4.64b)$$

Similarly as in the valuation of caplets, we arrive at the following result:

Proposition 4.22. *The time t -price of the swaption considered above is provided by:*

$$S(t) = \frac{\int_0^\infty \Re \left[\frac{\exp(B(\mu+iy) + \phi_{T-t}((\mu+iy)A) + \langle \psi_{T-t}((\mu+iy)A), X_t \rangle)}{(\mu+iy)^2} \right] dy}{\pi e^{-\alpha t} (\langle \gamma, X_t \rangle + c)}, \quad (4.65)$$

where $\mu > 0$ is such that $\mu A \in \mathcal{J}$, and A, B are defined in (4.64).

4.6 Multi-Curve Exponential Affine Model

In this section we consider another specification of the generic multi-curve potential model, in which the state-price deflator processes D and D^Δ are specified as exponential functions of a common Gaussian process X whose dynamics is given by

$$dX_t = \kappa(\theta_t - X_t)dt + C dW_t, \quad (4.66)$$

where

- $W = (W^1, \dots, W^d)$ is a d -dimensional Brownian motion,
- $\kappa = \text{diag}(\kappa_1, \dots, \kappa_d) \in \mathbb{R}^d$ is a diagonal matrix,
- $\theta_t = (\theta_t^1, \dots, \theta_t^d)$ is a vector of deterministic functions,
- $C = (c_{ij}) \in \mathbb{R}^{d \times d}$ is a $d \times d$ real matrix.

We first work out the distribution of the Gaussian process X .

Lemma 4.23. *For any $0 \leq t \leq T$, the conditional distribution of X_T given X_t is normal with mean*

$$\mu(t, T) := \mathbb{E}[X_T | X_t] = e^{-\kappa(T-t)} X_t + \kappa \int_t^T e^{-\kappa(T-u)} \theta_u du, \quad (4.67)$$

and covariance matrix

$$\Sigma(t, T) := \left(\frac{\rho_{ij}}{\kappa_i + \kappa_j} (1 - e^{-(\kappa_i + \kappa_j)(T-t)}) \right)_{i,j=1,\dots,d}, \quad (4.68)$$

where

$$\rho_{ij} := \sum_{l=1}^d c_{il} c_{jl}, \text{ i.e. } CC^t = (\rho_{ij})_{i,j=1,\dots,d},$$

and

$$e^{-\kappa z} := \text{diag}(e^{-\kappa_1 z}, \dots, e^{-\kappa_d z}) \quad \text{for any } z \in \mathbb{R}.$$

In particular, for any $\beta \in \mathbb{R}^d$, it holds

$$\mathbb{E}_t \left[e^{\langle \beta, X_T \rangle} \right] = \exp \left(\langle \beta, \mu(t, T) \rangle + \frac{1}{2} \beta^t \Sigma(t, T) \beta \right). \quad (4.69)$$

Proof. Since (X_t, X_T) is normally distributed, so is $X_T | X_t$. By Lemma A.6, we have

$$\begin{aligned} & \mathbb{E}[X_T^i | X_t] \\ &= \mathbb{E} \left[X_t^i e^{-\kappa_i(T-t)} + \kappa_i \int_t^T e^{-\kappa_i(T-u)} \theta_u^i du + \sum_{j=1}^d c_{ij} \int_t^T e^{-\kappa_i(T-u)} dW_u^j \middle| X_t \right] \\ &= X_t^i e^{-\kappa_i(T-t)} + \kappa_i \int_t^T e^{-\kappa_i(T-u)} \theta_u^i du, \end{aligned}$$

for all $i = 1, \dots, d$. Hence

$$\mathbb{E}[X_T|X_t] = e^{-\kappa(T-t)}X_t + \kappa \int_t^T e^{-\kappa(T-u)}\theta_u du.$$

Moreover, for any $i, j \in \{1, \dots, d\}$, we have

$$\begin{aligned} & \text{Cov}[X_T^i, X_T^j|X_t] \\ &= \mathbb{E}[(X_T^i - \mathbb{E}[X_T^i|X_t])(X_T^j - \mathbb{E}[X_T^j|X_t])|X_t] \\ &= \mathbb{E}\left[\left(\sum_{l=1}^d c_{il} \int_t^T e^{-\kappa_i(T-u)} dW_u^l\right)\left(\sum_{k=1}^d c_{jk} \int_t^T e^{-\kappa_j(T-u)} dW_u^k\right)\middle|X_t\right] \\ &= \sum_{l=1}^d c_{il}c_{jl} \int_t^T e^{-(\kappa_i+\kappa_j)(T-u)} du \\ &= \frac{\rho_{ij}}{\kappa_i + \kappa_j} \left(1 - e^{-(\kappa_i+\kappa_j)(T-t)}\right). \end{aligned}$$

All in all, $X_T|X_t$ is normally distributed with mean and covariance matrix given by $\mu(t, T)$ and $\Sigma(t, T)$ defined in (4.67) and (4.68), respectively. Finally, (4.69) follows immediately from the fact that

$$\langle \beta, X_T \rangle | X_t \sim \mathcal{N}(\langle \beta, \mu(t, T) \rangle, \beta^t \Sigma(t, T) \beta).$$

□

We next model the state-price deflator processes D and D^Δ in the (multi-curve) exponential affine model as follows:

$$\begin{aligned} D_t &= \exp(-\alpha t + \langle \gamma, X_t - X_0 \rangle), \\ D_t^\Delta &= \exp(-\alpha_\Delta t + \langle \gamma_\Delta, X_t \rangle), \end{aligned}$$

for some parameters $\alpha, \alpha_\Delta \in \mathbb{R}$, and $\gamma, \gamma_\Delta \in \mathbb{R}^d$. As a direct consequence of Lemma 4.23, we have the following:

Corollary 4.24. *In the exponential affine model, the conditional expectations $\mathbb{E}_t[D_T]$ and $\mathbb{E}_t[D_T^\Delta]$ are provided by*

$$\mathbb{E}_t[D_T] = \exp\left(-\alpha T - \langle \gamma, X_0 \rangle + \langle \gamma, \mu(t, T) \rangle + \frac{1}{2} \gamma^t \Sigma(t, T) \gamma\right), \quad (4.70)$$

$$\mathbb{E}_t[D_T^\Delta] = \exp\left(-\alpha_\Delta T + \langle \gamma_\Delta, \mu(t, T) \rangle + \frac{1}{2} \gamma_\Delta^t \Sigma(t, T) \gamma_\Delta\right), \quad (4.71)$$

where $\mu(t, T)$ and $\Sigma(t, T)$ are given in (4.67) and (4.68). In particular, the OIS discount bond has the closed-form formula:

$$p(t, T) = \exp\left(\left\langle e^{-\kappa(T-t)} \gamma - \gamma, X_t \right\rangle - \alpha(T-t) + \frac{1}{2} \gamma^t \Sigma(t, T) \gamma + \left\langle \gamma, \kappa \int_t^T e^{-\kappa(T-u)} \theta_u du \right\rangle\right), \quad (4.72)$$

and the FRA rate $L^\Delta(t; T, T + \Delta)$ for a future period $[T, T + \Delta]$ is given by

$$\begin{aligned} L^\Delta(t; T, T + \Delta) &= \frac{1}{\Delta} \left(\frac{p(t, T)}{p(t, T + \Delta)} \frac{\mathbb{E}_t[D_T^\Delta]}{\mathbb{E}_t[D_t]} - 1 \right) \\ &= \frac{1}{\Delta} \left(\frac{p(t, T)}{p(t, T + \Delta)} K^\Delta(t, T) - 1 \right), \end{aligned} \quad (4.73)$$

where the convexity adjustment $K^\Delta(t, T)$ is defined as

$$\begin{aligned} K^\Delta(t, T) &:= \exp\left((\alpha - \alpha_\Delta)T + \langle \gamma, X_0 \rangle + \langle \gamma_\Delta - \gamma, \mu(t, T) \rangle \right. \\ &\quad \left. + \frac{1}{2} \gamma_\Delta^t \Sigma(t, T) \gamma_\Delta - \frac{1}{2} \gamma^t \Sigma(t, T) \gamma\right). \end{aligned} \quad (4.74)$$

Proof. (4.70) and (4.71) follow immediately from (4.69) since

$$\begin{aligned} \mathbb{E}_t[D_T] &= e^{-\alpha T - \langle \gamma, X_0 \rangle} \mathbb{E}_t[e^{\langle \gamma, X_T \rangle}], \\ \mathbb{E}_t[D_T^\Delta] &= e^{-\alpha_\Delta T} \mathbb{E}_t[e^{\langle \gamma_\Delta, X_T \rangle}]. \end{aligned}$$

The closed-form formula (4.72) follows then from (4.70) and the general formula

$$p(t, T) = \frac{\mathbb{E}_t[D_T]}{D_t}.$$

Finally, using (4.70) and (4.71), we obtain the closed-form formula (4.73) for FRA rate $L^\Delta(t; T, T + \Delta)$. \square

Remark 4.25. From (4.73), we obtain the multiplicative FRA spread

$$S_m^\Delta(t, T) = \frac{1 + \Delta L^\Delta(t; T, T + \Delta)}{1 + \Delta F(t; T, T + \Delta)} = K^\Delta(t, T),$$

where $K^\Delta(t, T)$ is given in (4.74).

Having a Gaussian dynamics as the underlying process, our exponential affine model is analytically very tractable. In the following, we derive closed-form pricing formula for caplets and semi-closed form formula for swaptions.

Valuation of Caplets

In the exponential affine model, we obtain closed-form pricing formula for caplets. Formally, we have the following:

Proposition 4.26 (Black-type caplet prices in the exponential affine model). *Consider a caplet for the future period $[T, T + \Delta]$ with strike rate K , whose cashflow at maturity $T + \Delta$ is provided by*

$$C(T + \Delta) = \Delta(L^\Delta(T, T + \Delta) - K)^+.$$

Then its time t -price is given by the following Black-type formula:

$$C(t) = p(t, T + \Delta) [(1 + \Delta L^\Delta(t; T, T + \Delta))\Phi(d_1) - (1 + \Delta K)\Phi(d_2)], \quad (4.75)$$

where

$$d_1 = \frac{\ln\left(\frac{1 + \Delta L^\Delta(t; T, T + \Delta)}{1 + \Delta K}\right) + \frac{1}{2}(\gamma_\Delta - e^{-\kappa\Delta}\gamma)^\dagger \Sigma(t, T)(\gamma_\Delta - e^{-\kappa\Delta}\gamma)}{\sqrt{(\gamma_\Delta - e^{-\kappa\Delta}\gamma)^\dagger \Sigma(t, T)(\gamma_\Delta - e^{-\kappa\Delta}\gamma)}},$$

$$d_2 = \frac{\ln\left(\frac{1 + \Delta L^\Delta(t; T, T + \Delta)}{1 + \Delta K}\right) - \frac{1}{2}(\gamma_\Delta - e^{-\kappa\Delta}\gamma)^\dagger \Sigma(t, T)(\gamma_\Delta - e^{-\kappa\Delta}\gamma)}{\sqrt{(\gamma_\Delta - e^{-\kappa\Delta}\gamma)^\dagger \Sigma(t, T)(\gamma_\Delta - e^{-\kappa\Delta}\gamma)}},$$

and $\mu(\cdot, \cdot)$, $\Sigma(\cdot, \cdot)$ are given by (4.67) and (4.68).

Proof. From Equation (4.9), we get

$$C(t) = \frac{1}{D_t} \mathbb{E}_t[D_T^\Delta 1_A] - (1 + \Delta K) \frac{1}{D_t} \mathbb{E}_t[\mathbb{E}_T[D_{T+\Delta}^\Delta] 1_A], \quad (4.76)$$

where

$$A := \{D_T^\Delta \geq (1 + \Delta K)\mathbb{E}_T[D_{T+\Delta}]\}. \quad (4.77)$$

Plugging

$$\begin{aligned} D_T^\Delta &= \exp(-\gamma_\Delta T + \langle \gamma_\Delta, X_T \rangle), \\ \mathbb{E}_T[D_{T+\Delta}] &= \exp(-\alpha(T + \Delta) - \langle \gamma, X_0 \rangle + \langle \gamma, \mu(T, T + \Delta) \rangle \\ &\quad + \frac{1}{2}\gamma^t \Sigma(T, T + \Delta) \gamma) \end{aligned}$$

into Equation (4.77), we obtain

$$A = \{ \langle \gamma_\Delta - e^{-\kappa\Delta}\gamma, X_T \rangle \geq v \}, \quad (4.78)$$

where

$$\begin{aligned} v &:= \alpha_\Delta T - \alpha(T + \Delta) - \langle \gamma, X_0 \rangle + \frac{1}{2}\gamma^t \Sigma(T, T + \Delta) \gamma + \ln(1 + \Delta K) \\ &\quad + \left\langle \gamma, \kappa \int_T^{T+\Delta} e^{-\kappa(T+\Delta-s)} \theta_s ds \right\rangle. \end{aligned} \quad (4.79)$$

Using Lemma A.5 and the fact that $X_T | X_t \sim \mathcal{N}(\mu(t, T), \Sigma(t, T))$, we can compute the first term of $C(t)$ in Equation (4.76) explicitly as follows

$$\begin{aligned} \frac{1}{D_t} \mathbb{E}_t[D_T^\Delta 1_A] &= \frac{e^{-\alpha_\Delta T}}{D_t} \mathbb{E} \left[e^{\langle \gamma_\Delta, X_T \rangle} 1_{\{\langle \gamma_\Delta - e^{-\kappa\Delta}\gamma, X_T \rangle \geq v\}} \right] \\ &= \frac{1}{D_t} \exp \left(-\alpha_\Delta T + \langle \gamma_\Delta, \mu(t, T) \rangle + \frac{1}{2}\gamma_\Delta^t \Sigma(t, T) \gamma_\Delta \right) \Phi(d_1), \end{aligned} \quad (4.80)$$

where

$$d_1 = \frac{\gamma_\Delta^t \Sigma(t, T) (\gamma_\Delta - e^{-\kappa\Delta}\gamma) + \langle \gamma_\Delta - e^{-\kappa\Delta}\gamma, \mu(t, T) \rangle - v}{\sqrt{(\gamma_\Delta - e^{-\kappa\Delta}\gamma)^t \Sigma(t, T) (\gamma_\Delta - e^{-\kappa\Delta}\gamma)}}.$$

A simple computation using (4.79) yields

$$d_1 = \frac{\ln \left(\frac{1 + \Delta L^\Delta(t; T, T + \Delta)}{1 + \Delta K} \right) + \frac{1}{2}(\gamma_\Delta - e^{-\kappa\Delta}\gamma)^t \Sigma(t, T) (\gamma_\Delta - e^{-\kappa\Delta}\gamma)}{\sqrt{(\gamma_\Delta - e^{-\kappa\Delta}\gamma)^t \Sigma(t, T) (\gamma_\Delta - e^{-\kappa\Delta}\gamma)}},$$

and

$$\begin{aligned} & \frac{1}{D_t} \exp \left(-\alpha_\Delta T + \langle \gamma_\Delta, \mu(t, T) \rangle + \frac{1}{2} \gamma_\Delta^\top \Sigma(t, T) \gamma_\Delta \right) \\ & = p(t, T + \Delta) (1 + \Delta L^\Delta(t; T, T + \Delta)). \end{aligned}$$

Therefore, we obtain from Equation (4.80) that

$$\frac{1}{D_t} \mathbb{E}_t [D_T^\Delta 1_A] = p(t, T + \Delta) (1 + \Delta L^\Delta(t; T, T + \Delta)) \Phi(d_1). \quad (4.81)$$

Similarly, applying Lemma A.5, it is not hard to verify that

$$\frac{1}{D_t} \mathbb{E}_t [\mathbb{E}_T [D_{T+\Delta}^\Delta 1_A]] = p(t, T + \Delta) \Phi(d_2), \quad (4.82)$$

where

$$d_2 = \frac{\ln \left(\frac{1 + \Delta L^\Delta(t; T, T + \Delta)}{1 + \Delta K} \right) - \frac{1}{2} (\gamma_\Delta - e^{-\kappa \Delta} \gamma)^\top \Sigma(t, T) (\gamma_\Delta - e^{-\kappa \Delta} \gamma)}{\sqrt{(\gamma_\Delta - e^{-\kappa \Delta} \gamma)^\top \Sigma(t, T) (\gamma_\Delta - e^{-\kappa \Delta} \gamma)}}.$$

Finally, (4.75) follows by combining Equations (4.76), (4.81), and (4.82). \square

Valuation of Swaptions

In order to obtain analytic pricing formulae for swaptions, we restrict ourselves to the special case in which $d = 2$, $\gamma = (\gamma_1, 0)$, and $\gamma_\Delta = (\gamma_{\Delta 1}, \gamma_{\Delta 2})$ with

$$\gamma_1, \gamma_{\Delta 1}, \gamma_{\Delta 2} \in \mathbb{R}, \quad \gamma_{\Delta 2} \neq 0.$$

The following result gives a semi-closed form formula for a payer swaption in the exponential affine model.

Proposition 4.27 (Swaption prices in the exponential affine model).

With the above restriction of the exponential affine Gaussian model, the price at time $t = 0$ of the payer swaption considered in Section 4.3.3 is provided by

$$S(0) = \int_{-\infty}^{\infty} \frac{e^{-\frac{1}{2} \left(\frac{x - \mu_1}{\sigma_1} \right)^2}}{\sigma_1 \sqrt{2\pi}} \delta(x) \left[\sum_{i=1}^N \lambda_i(x) e^{\psi_i(x)} \Phi(\omega h_i(x)) - \Phi(\omega h(x)) \right] dx, \quad (4.83)$$

where

$$\begin{aligned}\omega &= \frac{\gamma_{\Delta 2}}{|\gamma_{\Delta 2}|}, \\ h(x) &= \frac{\rho}{\sqrt{1-\rho^2}} \frac{x-\mu_1}{\sigma_1} - \frac{\bar{y}(x)-\mu_2}{\sigma_2\sqrt{1-\rho^2}}, \\ h_i(x) &= h(x) + \gamma_{\Delta 2}\sigma_2\sqrt{1-\rho^2}e^{-\kappa_2(T_{i-1}-T)}, \\ \delta(x) &= (1+\Delta K) \sum_{i=1}^N \exp\left(\gamma_1 e^{-\kappa_1(T_i-T)}x + G_i(T)\right), \\ \lambda_i(x) &= \frac{1}{\delta(x)} \exp(\gamma_{\Delta 1}e^{-\kappa_1(T_{i-1}-T)}x + H_i(T)), \\ \psi_i(x) &= \gamma_{\Delta 2}e^{-\kappa_2(T_{i-1}-T)} \left(\mu_2 + \rho\sigma_2 \frac{x-\mu_1}{\sigma_1} + \frac{1}{2}(1-\rho^2)\sigma_2^2\gamma_{\Delta 2}e^{-\kappa_2(T_{i-1}-T)}\right), \\ G_i(T) &= -\alpha T_i - \gamma_1 X_0^1 + \gamma_1 \kappa_1 \int_T^{T_i} e^{-\kappa_1(T_i-u)}\theta_u^1 du + \frac{\rho_{11}\gamma_1^2}{4\kappa_1}(1-e^{-2\kappa_1(T_i-T)}), \\ H_i(T) &= -\alpha_{\Delta} T_{i-1} + \gamma_{\Delta 1}\kappa_1 \int_T^{T_{i-1}} e^{-\kappa_1(T_{i-1}-u)}\theta_u^1 du \\ &\quad + \gamma_{\Delta 2}\kappa_2 \int_T^{T_{i-1}} e^{-\kappa_2(T_{i-1}-u)}\theta_u^2 du + \frac{\gamma_{\Delta 1}^2\rho_{11}}{4\kappa_1}(1-e^{-2\kappa_1(T_{i-1}-T)}) \\ &\quad + \frac{\gamma_{\Delta 2}^2\rho_{22}}{4\kappa_2}(1-e^{-2\kappa_2(T_{i-1}-T)}) + \frac{\gamma_{\Delta 1}\gamma_{\Delta 2}\rho_{12}}{\kappa_1+\kappa_2}(1-e^{-(\kappa_1+\kappa_2)(T_{i-1}-T)}),\end{aligned}$$

$\bar{y}(x)$ is the unique solution of the equation

$$\begin{aligned}&\sum_{i=1}^N \exp\left(\gamma_{\Delta 2}e^{-\kappa_2(T_{i-1}-T)}y + \gamma_{\Delta 1}e^{-\kappa_1(T_{i-1}-T)}x + H_i(T)\right) \\ &= (1+\Delta K) \sum_{i=1}^N \exp\left(\gamma_1 e^{-\kappa_1(T_i-T)}x + G_i(T)\right),\end{aligned}$$

and

$$\begin{aligned}\mu_1 &= e^{-\kappa_1 T} X_0^1 + \kappa_1 \int_0^T e^{-\kappa_1(T-u)}\theta_u^1 du, \\ \mu_2 &= e^{-\kappa_2 T} X_0^2 + \kappa_2 \int_0^T e^{-\kappa_2(T-u)}\theta_u^2 du,\end{aligned}$$

$$\begin{aligned}\sigma_1 &= \sqrt{\frac{\rho_{11}}{2\kappa_1}(1 - e^{-2\kappa_1 T})}, \\ \sigma_2 &= \sqrt{\frac{\rho_{22}}{2\kappa_2}(1 - e^{-2\kappa_2 T})}, \\ \rho &= \frac{\rho_{12}}{(\kappa_1 + \kappa_2)\sigma_1\sigma_2}(1 - e^{-(\kappa_1 + \kappa_2)T}).\end{aligned}$$

Proof. We first note that the state-price deflators D and D^Δ are given by

$$\begin{aligned}D_t &= \exp(-\alpha t + \gamma_1(X_t^1 - X_0^1)), \\ D_t^\Delta &= \exp(-\alpha_\Delta t + \gamma_{\Delta 1}X_t^1 + \gamma_{\Delta 2}X_t^2).\end{aligned}$$

As a special case of Corollary 4.24, we obtain

$$\mathbb{E}_T[D_{T_i}] = \exp\left(\gamma_1 e^{-\kappa_1(T_i - T)} X_T^1 + G_i(T)\right), \quad (4.84)$$

$$\mathbb{E}_T[D_{T_{i-1}}^\Delta] = \exp\left(\gamma_{\Delta 1} e^{-\kappa_1(T_{i-1} - T)} X_T^1 + \gamma_{\Delta 2} e^{-\kappa_2(T_{i-1} - T)} X_T^2 + H_i(T)\right). \quad (4.85)$$

Pugging Equations (4.84) and (4.85) into the general formula (4.10), we have

$$\begin{aligned}S(0) &= \mathbb{E} \left[\left(\sum_{i=1}^N \exp\left(\gamma_{\Delta 1} e^{-\kappa_1(T_{i-1} - T)} X_T^1 + \gamma_{\Delta 2} e^{-\kappa_2(T_{i-1} - T)} X_T^2 + H_i(T)\right) \right. \right. \\ &\quad \left. \left. - (1 + \Delta K) \exp\left(\gamma_1 e^{-\kappa_1(T_i - T)} X_T^1 + G_i(T)\right) \right)^+ \right] \\ &= \int_{\mathbb{R}^2} \left(\sum_{i=1}^N \exp\left(\gamma_{\Delta 1} e^{-\kappa_1(T_{i-1} - T)} x + \gamma_{\Delta 2} e^{-\kappa_2(T_{i-1} - T)} y + H_i(T)\right) \right. \\ &\quad \left. - (1 + \Delta K) \exp\left(\gamma_1 e^{-\kappa_1(T_i - T)} x + G_i(T)\right) \right)^+ f(x, y) dy dx,\end{aligned}$$

where f is the density function of (X_T^1, X_T^2) :

$$f(x, y) = \frac{\exp\left(\frac{-1}{2(1-\rho^2)} \left[\left(\frac{x-\mu_1}{\sigma_1}\right)^2 - 2\rho \frac{(x-\mu_1)(y-\mu_2)}{\sigma_1\sigma_2} + \left(\frac{y-\mu_2}{\sigma_2}\right)^2 \right]\right)}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}}.$$

We can rewrite f as follows

$$f(x, y) = \lambda e^{E + F(y - \mu_2) - G(y - \mu_2)^2},$$

where

$$\lambda := \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}}, \quad E := \frac{-1}{2(1-\rho^2)} \left(\frac{x-\mu_1}{\sigma_1} \right)^2$$

$$F := \frac{\rho}{1-\rho^2} \frac{x-\mu_1}{\sigma_1\sigma_2}, \quad G := \frac{1}{2(1-\rho^2)\sigma_2^2}.$$

Set

$$\varphi(x, y) := \sum_{i=1}^N \exp \left(\gamma_{\Delta 1} e^{-\kappa_1(T_{i-1}-T)} x + \gamma_{\Delta 2} e^{-\kappa_2(T_{i-1}-T)} y + H_i(T) \right)$$

$$- (1 + \Delta K) \exp \left(\gamma_1 e^{-\kappa_1(T_i-T)} x + G_i(T) \right).$$

We have

$$\partial_y \varphi(x, y)$$

$$= \gamma_{\Delta 2} \sum_{i=1}^N \exp \left(\gamma_{\Delta 1} e^{-\kappa_1(T_{i-1}-T)} x + \gamma_{\Delta 2} e^{-\kappa_2(T_{i-1}-T)} y + H_i(T) + e^{-\kappa_2(T_{i-1}-T)} \right).$$

Therefore, $\varphi(x, \cdot)$ is monotonic in y for each $x \in \mathbb{R}$: it is increasing if $\gamma_{\Delta 2} > 0$ and decreasing if $\gamma_{\Delta 2} < 0$. Moreover, observe that

$$\lim_{y \rightarrow -\infty} \varphi(x, y) \lim_{y \rightarrow \infty} \varphi(x, y) < 0.$$

Hence, for any fixed $x \in \mathbb{R}$, the equation $\varphi(x, y) = 0$ has a unique solution $\bar{y}(x)$.

We next consider two possible cases:

Case 1: $\gamma_{\Delta 2} > 0$. Then,

$$S(0) = \int_{\mathbb{R}^2} (\varphi(x, y))^+ dy dx$$

$$= \int_{\mathbb{R}} \int_{\bar{y}(x)}^{\infty} \varphi(x, y) dy dx$$

$$= \int_{\mathbb{R}} \sum_{i=1}^N P_i(x) - (1 + \Delta K) Q_i(x) dx, \quad (4.86)$$

where

$$P_i(x) := \int_{\bar{y}(x)}^{\infty} \exp\left(\gamma_{\Delta 1} e^{-\kappa_1(T_{i-1}-T)}x + \gamma_{\Delta 2} e^{-\kappa_2(T_{i-1}-T)}y + H_i(T)\right) dy,$$

$$Q_i(x) := \int_{\bar{y}(x)}^{\infty} \exp\left(\gamma_1 e^{-\kappa_1(T_i-T)}x + G_i(T)\right) dy.$$

Applying Lemma A.3 yields

$$Q_i(x) = \lambda \frac{\sqrt{\pi}}{\sqrt{G}} e^{E + \frac{F^2}{4G}} \exp\left(\gamma_1 e^{-\kappa_1(T_i-T)}x + G_i(T)\right) \left[1 - \Phi\left(\frac{(\bar{y}(x) - \mu_2)\sqrt{2G} - \frac{F}{2G}}{\sigma_1\sqrt{2\pi}}\right)\right].$$

By noting that

$$\lambda \frac{\sqrt{\pi}}{\sqrt{G}} = \frac{1}{\sigma_1\sqrt{2\pi}}, \quad E + \frac{F^2}{4G} = -\frac{1}{2} \left(\frac{x - \mu_1}{\sigma_1}\right)^2,$$

$$\frac{F}{\sqrt{2G}} = \frac{\rho(x - \mu_1)}{\sigma_1\sqrt{1 - \rho^2}}, \quad \sqrt{2G} = \frac{1}{\sigma_2\sqrt{1 - \rho^2}},$$

and $1 - \Phi(z) = \Phi(-z)$, we further obtain

$$Q_i(x) = \frac{e^{-\frac{1}{2}\left(\frac{x - \mu_1}{\sigma_1}\right)^2}}{\sigma_1\sqrt{2\pi}} \exp\left(\gamma_1 e^{-\kappa_1(T_i-T)}x + G_i(T)\right) \Phi(h(x)). \quad (4.87)$$

Similarly, we arrive at the following explicit formula for $P_i(x)$:

$$P_i(x) = \frac{e^{-\frac{1}{2}\left(\frac{x - \mu_1}{\sigma_1}\right)^2}}{\sigma_1\sqrt{2\pi}} \exp(\gamma_{\Delta 1} e^{-\kappa_1(T_{i-1}-T)}x + H_i(T)) e^{\psi_i(x)} \Phi(h_i(x)). \quad (4.88)$$

A combination of Equations (4.86), (4.87), and (4.88) yields

$$S(0) = \int_{-\infty}^{\infty} \frac{e^{-\frac{1}{2}\left(\frac{x - \mu_1}{\sigma_1}\right)^2}}{\sigma_1\sqrt{2\pi}} \delta(x) \left[\sum_{i=1}^N \lambda_i(x) e^{\psi_i(x)} \Phi(h_i(x)) - \Phi(h(x)) \right] dx.$$

Case 2: $\gamma_{\Delta 2} < 0$. Following similar steps as in case 1, we obtain:

$$S(0) = \int_{-\infty}^{\infty} \frac{e^{-\frac{1}{2}\left(\frac{x - \mu_1}{\sigma_1}\right)^2}}{\sigma_1\sqrt{2\pi}} \delta(x) \left[\sum_{i=1}^N \lambda_i(x) e^{\psi_i(x)} \Phi(-h_i(x)) - \Phi(-h(x)) \right] dx.$$

□

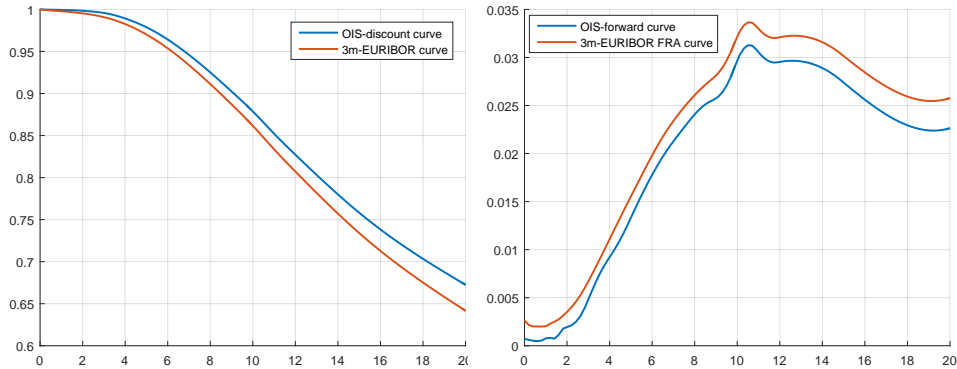
4.7 Model Calibration

In this section we illustrate the scope of the multi-curve potential model by calibrating the rational lognormal specification of our model to real-world market data. We fix June 10, 2014 as our reference date and use data collected from Thomson-Reuters. What we have at our disposal are EONIA swap rates, 3m-EURIBOR swap data and at-the-money swaption prices, where swaptions are written on the 3m-EURIBOR. Using cubic spline interpolation we are able to construct the OIS-discount curve p^* and the 3m-pseudo discount curve p_*^{3m} ; see Figure 4.3a. The implied OIS-forward rates $F^*(0; t, t + 0.25)$ and the 3m-FRA rates $L_*^{3m}(0; t, t + 0.25)$ are then given by

$$F^*(0; t, t + 0.25) = 4 \left(\frac{p^*(0, t)}{p^*(0, t + 0.25)} - 1 \right),$$

$$L_*^{3m}(0; t, t + 0.25) = 4 \left(\frac{p_*^{3m}(0, t)}{p_*^{3m}(0, t + 0.25)} - 1 \right)$$

and are displayed in Figure 4.3b.



(a) OIS and 3m-EURIBOR curves.

(b) OIS-forward and 3m-FRA rates.

Figure 4.3: EONIA and 3m-EURIBOR term structures as of June 10, 2014.

We assume that $h(t) = 0$ for all $t \geq 0$, so the state-price deflator process D

is given by

$$D_t = f(t) + g(t)M_t.$$

The OIS curve is therefore exclusively driven by the martingale M , and we have

$$p(0, t) = \frac{f(t) + g(t)M_0}{f(0) + g(0)} = f(t) + g(t) \quad (4.89)$$

since $M_0 = 1$ and $f(0) + g(0) = 1$. Given the market-consistent OIS curve p^* , bootstrapped from EONIA swaps, we follow Nakamura & Yu (2000) and choose the parametrizations

$$\begin{aligned} g(t) &= \frac{\alpha}{\beta + 1} (p^*(0, t))^{\beta+1}, \\ f(t) &= p^*(0, t) - g(t) = p^*(0, t) - \frac{\alpha}{\beta + 1} (p^*(0, t))^{\beta+1}. \end{aligned}$$

Here, α and β are parameters satisfying $\alpha > 0$ and $\beta > 0$.

We calibrate our multi-curve rational lognormal models to a matrix of market prices of the at-the-money swaptions with 6 maturities: 1y, 2y, 3y, 4y, 5y, 7y and 10 tenors ranging from 1y to 10y. We recall that for the swaption considered in Section 4.3.3, $T = T_0$ is called *maturity* and the difference $T_N - T$ is the *tenor* or *swaption length*. The market prices of the at-the-money swaptions are given by the following Black formula:²

$$\begin{aligned} S^{Market}(0; m, n, K_{atm}(m, n)) \\ = \Delta \sum_{i=1}^{n/\Delta} p^*(0, m + i\Delta) K_{atm}(m, n) \left(2\Phi \left(\frac{\sigma(m, n)\sqrt{m}}{2} \right) - 1 \right), \end{aligned}$$

where

- $\Delta = 0.25$ is the tenor of the underlying swap which is written on 3m-EURIBOR,
- m, n are maturity and tenor of the swaption,
- $\sigma(m, n)$ is the market volatility of the swaption,

²It is market practice to price a swaption using a Black-type formula.

- $K_{atm}(m, n)$ is the strike rate which renders the value of the underlying swap zero at inception and is given by

$$K_{atm}(m, n) = \frac{\sum_{i=1}^{n/\Delta} p^*(0, m + i\Delta) L_*^{3m}(0; m + (i-1)\Delta, m + i\Delta)}{\sum_{i=1}^{n/\Delta} p^*(0, m + i\Delta)}.$$

We calibrate the model parameters by minimizing the sum of the squared differences between model and market swaption prices. So the objective function is defined as

$$\sum_{m, n} \left(S^{Model}(0; m, n, K_{atm}(m, n)) - S^{Market}(0; m, n, K_{atm}(m, n)) \right)^2,$$

where $S^{Model}(0; m, n, K_{atm}(m, n))$ denotes the model price of a swaption with maturity m , tenor n , and strike price $K_{atm}(m, n)$. The absolute *calibration errors* are then defined as

$$|\sigma^{implied}(m, n) - \sigma(m, n)|,$$

where $\sigma^{implied}(m, n)$ is the *implied volatility* of the model with the associated calibrated parameters. We recall that $\sigma^{implied}(m, n)$ is the value of σ that solves the equation

$$\begin{aligned} S^{Model}(0; m, n, K_{atm}(m, n)) \\ = \Delta \sum_{i=1}^{n/\Delta} p^*(0, m + i\Delta) K_{atm}(m, n) \left(2\Phi\left(\frac{\sigma\sqrt{m}}{2}\right) - 1 \right), \end{aligned}$$

and it is given explicitly by

$$\sigma^{implied}(m, n) = \frac{2}{\sqrt{m}} \Phi^{-1} \left(\frac{1}{2} \left(1 + \frac{S^{Model}(0; m, n, K_{atm}(m, n))}{\Delta \sum_{i=1}^{n/\Delta} p^*(0, m + i\Delta) K_{atm}(m, n)} \right) \right).$$

4.7.1 Calibration of the one-factor model

From Equations (4.14) and (4.89) and the fact that $M_0 = 1$, we obtain

$$L^{3m}(0; t, t + 0.25) = 4 \left(\frac{f^{3m}(t) + g^{3m}(t)}{p(0, t + 0.25)} - 1 \right).$$

Hence

$$f^{3m}(t) + g^{3m}(t) = (1 + 0.25L^{3m}(0; t, t + 0.25))p(0, t + 0.25). \quad (4.90)$$

We construct the functions f^{3m} and g^{3m} from the bootstrapped 3m-EURIBOR FRA curve $L_*^{3m}(0; t, t + 0.25)$ and the given OIS curve p^* via

$$\begin{aligned} f^{3m}(t) &= f(t) + \gamma e^{-\kappa t} s(t), \\ g^{3m}(t) &= g(t) + (1 - \gamma e^{-\kappa t}) s(t), \end{aligned}$$

where

$$s(t) := (1 + 0.25L_*^{3m}(0; t, t + 0.25))p^*(0, t + 0.25) - p^*(0, t), \quad (4.91)$$

and γ, κ are parameters satisfying $\gamma > 0$ and $\kappa > 0$. This choice of f, g, f^{3m} , and g^{3m} satisfies Equations (4.89) and (4.90), i.e., it guarantees a perfect fit of the one-factor multi-curve rational lognormal model to the initial EONIA and 3m-EURIBOR term structures p^* and L_*^{3m} . The volatilities $\sigma(t)$ of the martingale M is chosen to be a positive constant $\sigma(t) = \sigma$ for all $t \geq 0$. Therefore, the calibration output is the following set of 5 parameters:

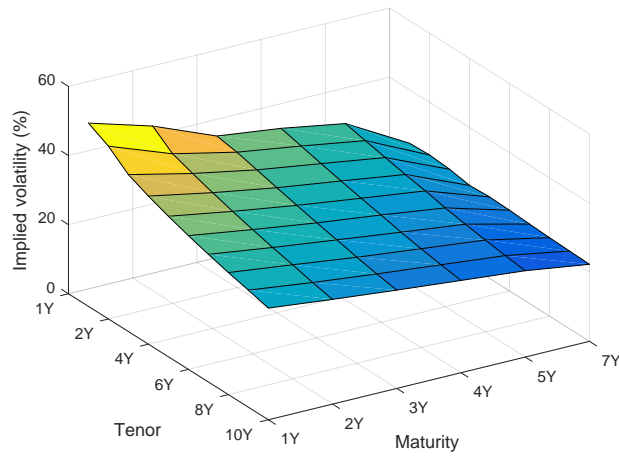
$$\Theta_1 = \{\alpha, \beta, \gamma, \kappa, \sigma\}.$$

Upon minimizing the sum of the squared differences between market and model swaption prices, we obtain the calibrated parameters reported in Table 4.1.

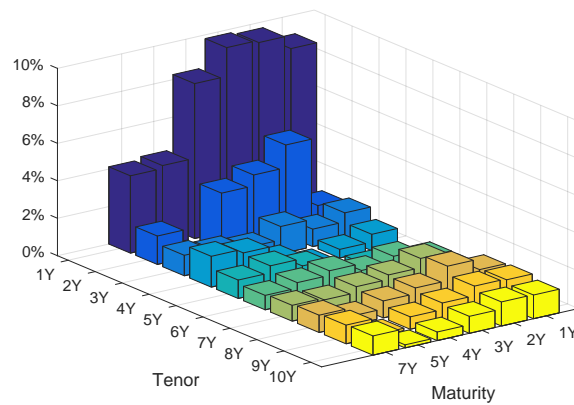
α	β	γ	κ	σ
0.638423	3.270780	0.010787	0.189737	0.767376

Table 4.1: Calibrated parameters of the one-factor rational lognormal multi-curve potential model.

The implied volatility surface is displayed in Figure 4.4a. In addition, Figure 4.4b shows the associated calibration errors. We observe that the calibration of the one-factor rational lognormal multi-curve potential model is rather satisfactory for swaptions with medium and long maturities and tenors. Moreover, with the



(a) Model-implied volatility surface.



(b) Calibration errors (differences in implied volatilities).

Figure 4.4: Calibration of the one-factor rational lognormal multi-curve potential model.

calibrated parameters in Table 4.1 we have $f^{3m}(t) \geq f(t)$ and $g^{3m}(t) \geq g(t)$ for all $t \geq 0$; see Figure 4.5. It follows then from Remark 4.7 that the calibrated parameters guarantee positive basis spreads.

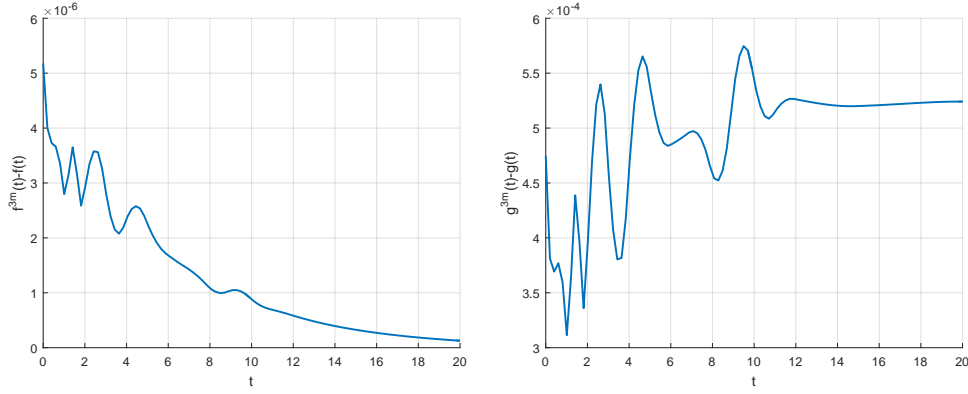


Figure 4.5: Differences between functions f^{3m} and f , and g^{3m} and g in the one-factor rational lognormal multi-curve potential model.

4.7.2 Calibration of the two-factor model

We specify the functions f^{3m} , g^{3m} , and h^{3m} in the two-factor multi-curve rational lognormal model as follows

$$\begin{aligned} f^{3m}(t) &= f(t) + \gamma_1 e^{-\kappa_1 t} s(t), \\ h^{3m}(t) &= \gamma_2 e^{-\kappa_2 t} s(t), \\ g^{3m}(t) &= g(t) + (1 - \gamma_1 e^{-\kappa_1 t} - \gamma_2 e^{-\kappa_2 t}) s(t), \end{aligned}$$

where $s(t)$ is defined in (4.91). It is obvious that the above construction guarantees a perfect fit of the model to the initial term structures. Moreover, we assume the volatilities $\sigma_1(t)$ and $\sigma_2(t)$ of the martingales M and N are constant, i.e., $\sigma_1(t) = \sigma_1 > 0$ and $\sigma_2(t) = \sigma_2 > 0$ for all $t \geq 0$. So the output of the calibration is the set of the following 9 parameters:

$$\Theta_2 = \{\alpha, \beta, \gamma_1, \kappa_1, \gamma_2, \kappa_2, \sigma_1, \sigma_2, \rho\}.$$

The calibrated parameters are reported in Table 4.2. The implied volatility surface $\{\sigma^{implied}(m, n)\}_{m, n}$ and the absolute calibration errors are displayed in Figure 4.6a and Figure 4.6b, respectively.

α	β	γ_1
0.571529	3.141539	0.009872
γ_2	κ_1	κ_2
0.254420	0.132585	0.000058
σ_1	σ_2	ρ
0.819876	1.702198	0.436887

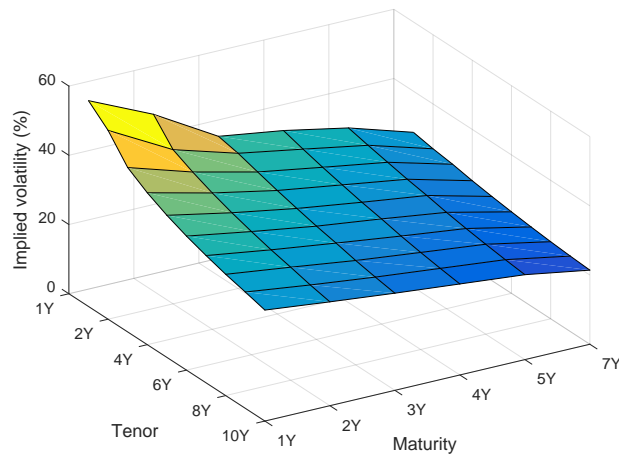
Table 4.2: Calibrated parameters of the two-factor rational lognormal multi-curve potential model.

With the exception of some outliers at short maturities and tenors, which might be due to misalignments in market data,³ the absolute errors are relatively small. As in the calibration of the one-factor model, we also have

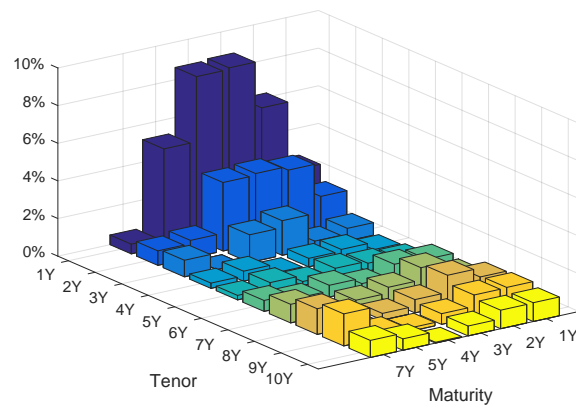
$$f^{3m}(t) \geq f(t), \quad g^{3m}(t) \geq g(t), \quad \text{and} \quad h^{3m}(t) \geq h(t), \quad \text{for all } t \geq 0;$$

see Figure 4.7. Hence, with the calibrated parameters in Table 4.2, the two-factor rational lognormal multi-curve potential model produces positive basis spreads; see Remark 4.10. In summary, it appears fair to say that the rational lognormal multi-curve potential models calibrate well to EONIA and EURIBOR swap and swaptions with medium and long maturities and tenors.

³It is well-known that, due to different levels of liquidity, swaption prices are not always updated simultaneously, which may lead to inconsistencies in market quotes (see, e.g., Brigo & Mercurio 2006, p. 288).



(a) Model-implied volatility surface.



(b) Calibration errors (differences in implied volatilities).

Figure 4.6: Calibration of the two-factor rational lognormal multi-curve potential model.

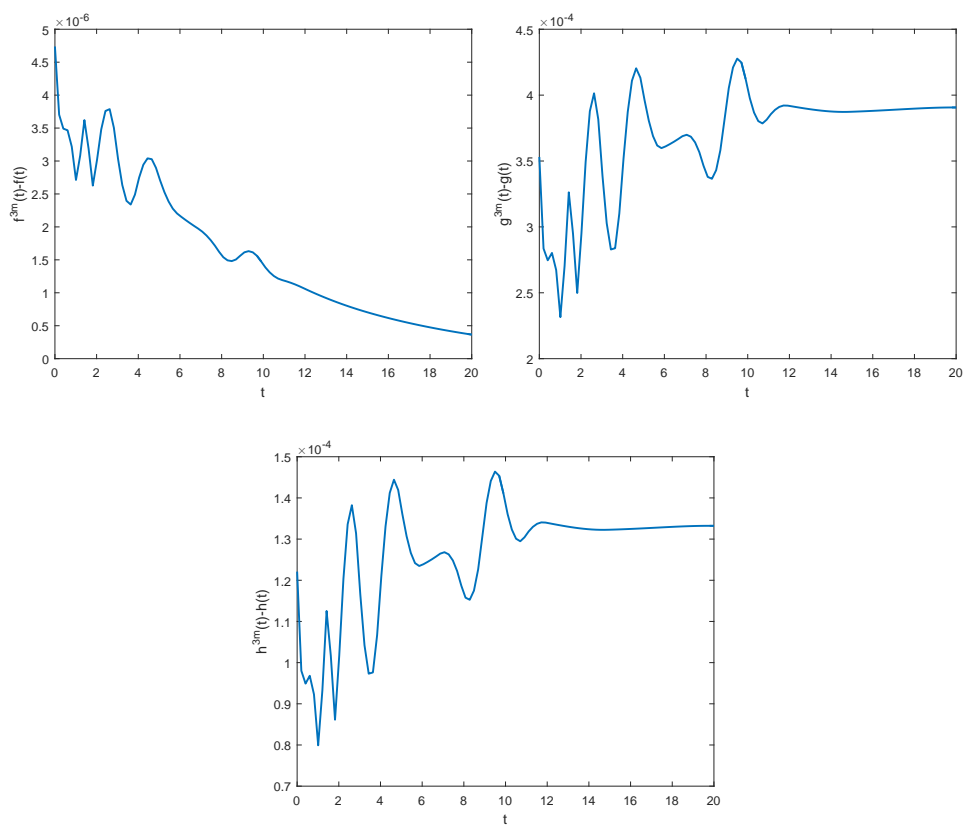


Figure 4.7: Differences between functions f^{3m} and f , g^{3m} and g , and h^{3m} and h in the two-factor rational lognormal multi-curve potential model.

Chapter 5

Conclusion and Outlook

In this concluding chapter we summarize the main contributions of this thesis as well as discuss some important directions for future research.

We establish in Chapter 3 a general class of rational models to the term structure of interest rates using the potential approach. As the basic building block of our models, we construct in Theorem 3.2 a general non-negative supermartingale from an affine Markov process. By simply adding a strictly positive constant to this supermartingale, we further obtain a (strictly) positive supermartingale which we use to model the state-price deflator.

With the help of the rich structural properties of affine processes, our models are analytically very tractable. Indeed, Lemma 3.6 shows that the conditional expectations of the state-price deflator are given in closed-form expressions. Based on this fact, we further show that our modeling framework guarantees: i) non-negative interest rates, ii) closed form formulae for bond prices, and iii) semi-closed form pricing formulae for important interest rate derivatives such as caps, swaptions, and European currency options.

A perfect fit of the affine rational potential model to the initial term structure can be obtained by adding a positive deterministic function f to the non-negative supermartingale M^u in Theorem 3.2. The state-price deflator D^u then becomes

$$D_t^u = f(t) + \phi_{T^\infty-t}(u) + \langle \psi_{T^\infty-t}(u), X_t \rangle, \quad t \in [0, T^\infty].$$

However, D^u might no longer be a supermartingale and this accommodates negative interest rates.

In Chapter 4, we extend the general single-curve potential model to a multi-curve setting. The starting point of this extension is the interpretation of the LIBOR rate as the simply compounded rate of a quanto investment in which borrowing and repayment are made in two different currencies; see Section 4.3.1. Therefore, the spot multiplicative spread between LIBOR and OIS curves can be interpreted as the spot exchange rate between these two currencies. On the other hand, as discussed in Section 2.2.2, the potential approach is well-suited for exchange rate as well as multi-currency modeling. These facts justify our use of the potential approach to model basis spreads and multiple term structures.

In the general multi-curve potential model, we simultaneously model the OIS-term structure via a "domestic" state-price deflator and model the multiplicative FRA spread as the quotient of the domestic- and a "foreign" state-price deflator. We show that our model can produce stochastic positive basis spreads and guarantee positive OIS-term structure; negative OIS-interest rates and positive spreads can also be obtained simultaneously in this modeling framework.

To illustrate our modeling framework, we present four concrete multi-curve potential models. The first specific model extends the classical rational lognormal model by Flesaker & Hughston (1996a) to the multi-curve framework. The second one is a multi-curve extension of our affine rational potential model developed in Chapter 3. The third model extends the single-curve linear rational model in Filipović et al. (2015) to a multi-curve setting. Finally, we present a specific multi-curve potential model based on a Gaussian process. We remark that these model specifications are very tractable: we obtain closed-form expressions for linear products such as zero-bonds, forward rate agreements, swaps, and semi-closed form formulae for important interest rate derivatives like caps and swaptions.

As a showcase example, we calibrate the multi-curve rational lognormal models to EUR swap and swaption market data. Our models perfectly fit to the initial EONIA and EURIBOR term structures and calibrate well to swaptions

with medium and long tenors and maturities.

As directions for future research, our multi-curve potential model can be extended to a multi-curve multi-currency setting as in the work by Fujii et al. (2011). Furthermore, it can be applied to computation of value adjustments (XVA) for interest rate derivatives as in Crépey et al. (2015a,b).

Appendix A

Auxiliary Results

We provide some auxiliary results that are used to derive pricing formulae in our models.

Lemma A.1. *Let Y be a random variable and assume there exists $\delta > 0$ such that $M^Y(v) := \mathbb{E}[e^{vY}] < \infty$ for all $v \in (-\delta, \delta)$. Then $\mathbb{E}[|Y|^k] < \infty$ for all $k \in \mathbb{N}$ and*

$$\mathbb{E}[Y^k] = D^k M^Y(0),$$

where $D^k M^Y(0)$ represents the k^{th} derivative of $M^Y(v)$ at $v = 0$.

Proof. See Theorem 4.25 in von Weizsäcker (2012). □

Lemma A.2. *Let Y be a random variable on the probability space $(\Omega, \mathcal{F}, \mathbf{F}, \mathbb{P})$ and fix a time $t \in [0, T^\infty]$. Define $\varphi(z) := \mathbb{E}_t[e^{zY}]$ for all $z \in \mathbb{C}$ such that the conditional expectation exists. Assume $\mu > 0$ is a positive number such that $\varphi(\mu) < \infty$. Then the expectation of Y^+ conditioned on the filtration \mathcal{F}_t is given by*

$$\mathbb{E}_t[Y^+] = \frac{1}{\pi} \int_0^\infty \Re \left[\frac{\varphi(\mu + iy)}{(\mu + iy)^2} \right] dy. \tag{A.1}$$

Proof. See Theorem 2.6 in Filipović et al. (2015). □

Lemma A.3. For any real constants a, b, A, B with $A > 0$ we have

$$\int_a^b e^{-Ax^2+Bx} dx = \frac{\sqrt{\pi}}{\sqrt{A}} e^{\frac{B^2}{4A}} \left[\Phi \left(b\sqrt{2A} - \frac{B}{\sqrt{2A}} \right) - \Phi \left(a\sqrt{2A} - \frac{B}{\sqrt{2A}} \right) \right],$$

where Φ denotes the distribution function of the standard normal distribution.

Proof. We have

$$e^{-Ax^2+Bx} = \sqrt{\frac{\pi}{A}} e^{\frac{B^2}{4A}} \sqrt{\frac{A}{\pi}} e^{-A(x-\frac{B}{2A})^2}.$$

On the other hand, note that $\sqrt{\frac{A}{\pi}} e^{-A(x-\frac{B}{2A})^2}$ is the density function of the random variable $Z \sim \mathcal{N}(\frac{B}{2A}, \frac{1}{2A})$. Therefore, we obtain

$$\begin{aligned} \int_a^b e^{-Ax^2+Bx} dx &= \sqrt{\frac{\pi}{A}} e^{\frac{B^2}{4A}} \int_a^b \sqrt{\frac{A}{\pi}} e^{-A(x-\frac{B}{2A})^2} dx \\ &= \sqrt{\frac{\pi}{A}} e^{\frac{B^2}{4A}} \mathbb{P}(a \leq Z \leq b) \\ &= \sqrt{\frac{\pi}{A}} e^{\frac{B^2}{4A}} \mathbb{P} \left(a\sqrt{2A} - \frac{B}{\sqrt{2A}} \leq \sqrt{2A} \left(Z - \frac{B}{2A} \right) \leq b\sqrt{2A} - \frac{B}{\sqrt{2A}} \right) \\ &= \frac{\sqrt{\pi}}{\sqrt{A}} e^{\frac{B^2}{4A}} \left[\Phi \left(b\sqrt{2A} - \frac{B}{\sqrt{2A}} \right) - \Phi \left(a\sqrt{2A} - \frac{B}{\sqrt{2A}} \right) \right], \end{aligned}$$

since $\sqrt{2A} \left(Z - \frac{B}{2A} \right) \sim \mathcal{N}(0, 1)$. □

Lemma A.4. Let σ, a, b be positive real numbers, and Y be a normally distributed random variable with mean $-\sigma^2/2$ and variance σ^2 , i.e. $Y \sim \mathcal{N}(-\frac{\sigma^2}{2}, \sigma^2)$. Then we have

$$\mathbb{E}[(ae^Y - b)^+] = a\Phi(h_1) - b\Phi(h_2),$$

where Φ is the distribution function of the standard normal distribution and

$$h_{1,2} = \frac{1}{\sigma} \left(\ln\left(\frac{a}{b}\right) \pm \frac{\sigma^2}{2} \right).$$

Proof. Let $Z \sim \mathcal{N}(\lambda, \nu^2)$ for some $\lambda \in \mathbb{R}$, $\nu > 0$, and let m be any real number. Then we have

$$\mathbb{E} [e^{mZ} 1_{\{Z \geq \gamma\}}] = \exp \left(m\lambda + \frac{m^2 \nu^2}{2} \right) \Phi \left(\frac{m\nu^2 + \lambda - \gamma}{\nu} \right). \quad (\text{A.2})$$

Indeed,

$$\begin{aligned} \mathbb{E} [e^{mZ} 1_{\{Z \geq \gamma\}}] &= \int_{\gamma}^{\infty} \frac{1}{\sqrt{2\pi\nu}} e^{mx} e^{-\frac{1}{2\nu^2}(x-\lambda)^2} dx \\ &= \exp \left(m\lambda + \frac{m^2 \nu^2}{2} \right) \int_{\gamma}^{\infty} \frac{1}{\sqrt{2\pi\nu}} e^{-\frac{1}{2\nu^2}[x-(m\nu^2+\lambda)]^2} dx \\ &= \exp \left(m\lambda + \frac{m^2 \nu^2}{2} \right) \mathbb{P}(U \geq \gamma), \end{aligned}$$

where $U \sim \mathcal{N}(m\nu^2 + \lambda, \nu^2)$. Since $\frac{U - m\nu^2 - \lambda}{\nu} \sim \mathcal{N}(0, 1)$ and $\mathbb{P}(X \geq u) = \Phi(-u)$ for any $X \sim \mathcal{N}(0, 1)$ and $u \in \mathbb{R}$, we have

$$\mathbb{P}(U \geq \gamma) = \mathbb{P} \left(\frac{U - m\nu^2 - \lambda}{\nu} \geq \frac{\gamma - m\nu^2 - \lambda}{\nu} \right) = \Phi \left(\frac{m\nu^2 + \lambda - \gamma}{\nu} \right).$$

Putting all together we obtain (A.2). Returning to the lemma, we have

$$\begin{aligned} \mathbb{E}[(ae^Y - b)^+] &= \mathbb{E} \left[(ae^Y - b) 1_{\{Y \geq \ln(\frac{b}{a})\}} \right] \\ &= a\mathbb{E} \left[e^Y 1_{\{Y \geq \ln(\frac{b}{a})\}} \right] - b\mathbb{P} \left(Y \geq \ln \left(\frac{b}{a} \right) \right). \end{aligned} \quad (\text{A.3})$$

Applying (A.2) with $m = 1$, $\lambda = -\sigma^2/2$, and $\nu = \sigma$, we have

$$\mathbb{E} \left[e^Y 1_{\{Y \geq \ln(\frac{b}{a})\}} \right] = \Phi \left(\frac{\ln(\frac{a}{b}) + \frac{\sigma^2}{2}}{\sigma} \right). \quad (\text{A.4})$$

Note that $\frac{Y + \frac{\sigma^2}{2}}{\sigma} \sim \mathcal{N}(0, 1)$. Hence

$$\mathbb{P} \left(Y \geq \ln \left(\frac{b}{a} \right) \right) = \mathbb{P} \left(\frac{Y + \frac{\sigma^2}{2}}{\sigma} \geq \frac{\ln(\frac{b}{a}) + \frac{\sigma^2}{2}}{\sigma} \right) = \Phi \left(\frac{\ln(\frac{a}{b}) - \frac{\sigma^2}{2}}{\sigma} \right). \quad (\text{A.5})$$

The result now follows readily from (A.3), (A.4), and (A.5). \square

Lemma A.5. *Let Z be a d -dimensional normal random variable with $Z \sim \mathcal{N}(\mu, \Sigma)$ for some $\mu \in \mathbb{R}^d$ and $\Sigma \in \mathbb{R}^{d \times d}$. Then for any $a, b \in \mathbb{R}^d$ we have*

$$\mathbb{E} \left[e^{\langle a, Z \rangle} 1_{\{\langle b, Z \rangle \geq u\}} \right] = \exp \left(\langle a, \mu \rangle + \frac{1}{2} a^t \Sigma a \right) \Phi \left(\frac{a^t \Sigma b + \langle b, \mu \rangle - u}{\sqrt{b^t \Sigma b}} \right), \quad (\text{A.6})$$

for any $u \in \mathbb{R}$.

Proof. Observe that $(\langle a, Z \rangle, \langle b, Z \rangle)$ is normally distributed with

$$(\langle a, Z \rangle, \langle b, Z \rangle) \sim \mathcal{N} \left(\begin{pmatrix} \langle a, \mu \rangle \\ \langle b, \mu \rangle \end{pmatrix}, \begin{pmatrix} a^t \Sigma a & a^t \Sigma b \\ a^t \Sigma b & b^t \Sigma b \end{pmatrix} \right).$$

Hence, the conditional distribution of $\langle a, Z \rangle$ given $\langle b, Z \rangle$ is normal:

$$\langle a, Z \rangle | \langle b, Z \rangle \sim \mathcal{N}(\bar{\mu}, \bar{\sigma}^2),$$

where

$$\begin{aligned} \bar{\mu} &:= \langle a, \mu \rangle + \frac{a^t \Sigma b}{b^t \Sigma b} (\langle b, Z \rangle - \langle b, \mu \rangle), \\ \bar{\sigma}^2 &:= a^t \Sigma a - \frac{(a^t \Sigma b)^2}{b^t \Sigma b}. \end{aligned}$$

It follows that

$$\begin{aligned} \mathbb{E}[e^{\langle a, Z \rangle} | \langle b, Z \rangle] &= \exp \left(\bar{\mu} + \frac{\bar{\sigma}^2}{2} \right) \\ &= \exp \left(\langle a, \mu \rangle + \frac{a^t \Sigma b}{b^t \Sigma b} (\langle b, Z \rangle - \langle b, \mu \rangle) + \frac{1}{2} \left(a^t \Sigma a - \frac{(a^t \Sigma b)^2}{b^t \Sigma b} \right) \right). \end{aligned} \quad (\text{A.7})$$

Using tower property of conditional expectation and Equation (A.7), we have

$$\begin{aligned} \mathbb{E} \left[e^{\langle a, Z \rangle} 1_{\{\langle b, Z \rangle \geq u\}} \right] &= \mathbb{E} \left[1_{\{\langle b, Z \rangle \geq u\}} \mathbb{E}[e^{\langle a, Z \rangle} | \langle b, Z \rangle] \right] \\ &= \exp \left(\langle a, \mu \rangle - \frac{a^t \Sigma b}{b^t \Sigma b} \langle b, \mu \rangle + \frac{1}{2} \left(a^t \Sigma a - \frac{(a^t \Sigma b)^2}{b^t \Sigma b} \right) \right) \\ &\quad \times \mathbb{E} \left[e^{\frac{a^t \Sigma b}{b^t \Sigma b} \langle b, Z \rangle} 1_{\{\langle b, Z \rangle \geq u\}} \right]. \end{aligned} \quad (\text{A.8})$$

Since $\langle b, Z \rangle \sim \mathcal{N}(\langle b, \mu \rangle, b^t \Sigma b)$, it follows from (A.2) that

$$\mathbb{E} \left[e^{\frac{a^t \Sigma b}{b^t \Sigma b} \langle b, Z \rangle} \mathbf{1}_{\{\langle b, Z \rangle \geq u\}} \right] = \exp \left(\frac{a^t \Sigma b}{b^t \Sigma b} \langle b, \mu \rangle + \frac{1}{2} \frac{(a^t \Sigma b)^2}{b^t \Sigma b} \right) \Phi \left(\frac{a^t \Sigma b + \langle b, \mu \rangle - u}{\sqrt{b^t \Sigma b}} \right). \quad (\text{A.9})$$

The equality in the lemma then follows readily from Equations (A.8) and (A.9). \square

Lemma A.6. *The stochastic differential equation (4.66) has a unique solution $X = (X^1, \dots, X^d)$ with*

$$X_t^i = X_s^i e^{-\kappa_i(t-s)} + \kappa_i \int_s^t e^{-\kappa_i(t-u)} \theta_u^i du + \sum_{j=1}^d \int_s^t e^{-\kappa_i(t-u)} c_{ij} dW_u^j,$$

for any $0 \leq s \leq t$.

Proof. Consider a fixed index $i \in \{1, \dots, d\}$. Applying Itô formula we have

$$\begin{aligned} d(e^{\kappa_i t} X_t^i) &= (\kappa_i e^{\kappa_i t} X_t^i + \kappa_i e^{\kappa_i t} (\theta_t^i - X_t^i)) dt + \sum_{j=1}^d c_{ij} e^{\kappa_i t} dW_t^j \\ &= \kappa_i e^{\kappa_i t} \theta_t^i dt + \sum_{j=1}^d c_{ij} e^{\kappa_i t} dW_t^j. \end{aligned}$$

It follows that

$$e^{\kappa_i t} X_t^i = e^{\kappa_i s} X_s^i + \kappa_i \int_s^t e^{\kappa_i u} \theta_u^i du + \sum_{j=1}^d c_{ij} \int_s^t e^{\kappa_i u} dW_u^j,$$

so that

$$X_t^i = X_s^i e^{-\kappa_i(t-s)} + \kappa_i \int_s^t e^{-\kappa_i(t-u)} \theta_u^i du + \sum_{j=1}^d \int_s^t e^{-\kappa_i(t-u)} c_{ij} dW_u^j.$$

\square

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Publications

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1. T. Nguyen & F. Seifried (2015a): The Multi-Curve Potential Model. *International Journal of Theoretical and Applied Finance*, Vol. 18, No. 7, 150049.
2. T. Nguyen & F. Seifried (2015b): The Affine Rational Potential Model. Available at SSRN: <http://ssrn.com/abstract=2707081>.

Declaration

I hereby declare that this doctoral thesis has been composed originally by myself under the supervision of Prof. Dr. Frank Thomas Seifried. Furthermore, I confirm that no sources have been used in the preparation of this work other than those indicated in the text or references.

To the best of my knowledge, the results stated in this work were not known or considered previously if not indicated otherwise.

This thesis has not been submitted or published anywhere else before.

Kaiserslautern, March 2016

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